Conic Structures in Differential Geometry

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Abstract

The interrelations between projective and conic connections on G_m structures are studied. An Einstein quaternionic manifold of rank p is associated with a twistor space equipped with holomorphic distribution, whose Fröbenius form has rank precisely p. This construction is inverted, establishing one-to-one correspondence between local Einstein quaternionic manifolds of given rank and the twistorial data consisting of twistor space, distribution on the twistor space and its Fröbenius form of the same rank.

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Statement

Chapter 1 covers some basic material such as sheaf theory, cohomology, analytic family and moduli space. This material can be found, for example, in [Mer00], [Kod62], [Kod80], [Wel91], [WW90].

Chapter 2 covers conic structures and conic connections, which can be found, for example, in [Man97]. Section 3.2 of Chapter 3 is the joint work of the author and S. A. Merkulov. The rest of the Chapter includes material on conformal 3-manifolds, which can be found, for example, in [Bry91], [CT96], [HM99], [Tod92]. Chapter 4 covers quaternionic structures. The material can be found elsewhere, such as in [BE91] and [PR84].

Chapters 5 and 6 are the original work of the author, with the exception of the instances indicated within the text as well as the material in the Section 5.1 and Theorem 5.2.1, which can be found in [Man97].

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0.1 Overview

Let M be an n-dimensional complex manifold and V a fixed n-dimensional complex vector space (typically, $V = \mathbb{C}^n$). Let $\pi : \mathcal{L}^*M \longrightarrow M$ be the holomorphic bundle of V-valued coframes, whose fibres $\pi^{-1}(t)$ consist, by definition, of all $\mathbb{C}$-linear isomorphisms $e : T_tM \longrightarrow V$, where T_tM is the tangent space at $t \in M$. The space $\mathcal{L}^*M$ is a principal right GL(V)-bundle with the right action given by $R_g(e) = g^{-1} \circ e$. If G is a closed complex subgroup of GL(V), then a complex G-structure on M is a principal holomorphic subbundle $\mathcal{G}$ of $\mathcal{L}^*M$ with structure group G. Given an affine connection ∇ on a simply connected complex manifold M, the set $\mathcal{G}_u$ of all points in the bundle of V-valued coframes $\mathcal{L}^*M$ which can be connected to a fixed point $u \in \mathcal{L}^*M$ by a horizontal curve is a principal right subbundle of $\mathcal{L}^*M$ whose structure group G_u is a Lie subgroup of GL(V), called the (restricted) holonomy group of ∇ at u. The conjugacy class of G_u in GL(V) does not depend, in fact, on the choice of u, and any representative $G \subset \mathrm{GL}(V)$ of this conjugacy class is called, by abuse of language, the *holonomy group of* ∇.The holonomy group G can also be represented at any particular point $p \in M$ as the set of all linear automorphisms of the associated tangent space T_pM which are induced by parallel translation along p-based loops.

Holonomy turns out to be one of the most informative characteristics of an affine connection on a smooth connected manifold M. The notion of holonomy group was introduces by Elie Cartan in 1920s [Car26] who used it to classify all Riemannian locally symmetric spaces.

If a connection is (locally) symmetric, then its holonomy group, if irreducible, equals essentially the (local) isotropy group. In 1955 Berger [Ber55] gave necessary conditions for irreducible Lie subgroup $G \subset \mathrm{End}(V)$, where V is real or complex finite dimensional vector space, to occur as the holonomy group of a torsion-free affine connection which is not symmetric.[1] The case of locally symmetric connections is equivalent to the classification problem of symmetric spaces which was solved long

[1] 1956 Hano and Ozeki [HO56] showed that *any* (closed) Lie subgroup $G \subset \mathrm{End}(V)$ can occur as the holonomy of some affine connection, generally with torsion;

time ago (see [Car26] [Ber57]) . Berger then classified the groups satisfying this conditions. The first ("metric") part of this classification consists of all holonomy groups which leave some symmetric bilinear form invariant: $\mathrm{SO}(n)$, $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, $\mathrm{Sp}(k)\mathrm{Sp}(1)$, G_2, Spin(7). The connections with one of these holonomies are always Levi-Civita connections of a (pseudo-)Riemannian metric on M.

The second (" non-metric") part of Berger's classification was stated to contain all remaining holonomy groups, up to a finite number of missing terms. These "missing terms" were later called exotic. Non-emptiness of the list of the exotic holonomies, was established by Bryant [Bry91]. In [CMS96] Chi et al. discovered an infinite family of exotic holonomies, thus showing the incompleteness of non-metric part of Berger's list. Finally, the holonomy problem was solved by S. Merkulov and L. Schwachhofer in [MS99].

Any Riemannian manifold is locally the product of symmetric spaces and/or manifolds with holonomy groups appearing on the metric part of Berger's list. The case of $\mathrm{SO}(n)$ corresponds to "generic" geometry. Of the remaining six types of Riemannian geometry, three

$$[\mathrm{U}(m), \mathrm{SU}(m), \mathrm{Sp}(k)]$$

correspond to Kähler manifolds of varying degrees of speciality, while G_2 and Spin(7) only occur in dimensions 7 and 8 respectively. Which leaves the very interesting family of *quaternionic-Kähler manifolds*, i.e. $4n$-manifolds, $n \geq 2$ with holonomy group

$$\mathrm{Sp}(k)\mathrm{Sp}(1) := \mathrm{Sp}(k) \times \mathrm{Sp}(1)/\mathbb{Z}_2.$$

In 1982 Salamon has shown that if M is a quaternionic manifold [2], the total space of the associated bundle Z is a complex manifold, thus constructing a twistor space.(see [Sal86])

When can we reverse this construction?

Let Z be a complex $(2n+1)$-dimensional manifold equipped with a holomorphic

[2]A quaternionic manifold is defined by a G-structure admitting a torsion-free connection, where G denotes the maximal subgroup $\mathrm{GL}(n,\mathbb{H})\mathrm{GL}(1,\mathbb{H})$ of $\mathrm{GL}(4n,\mathbb{R})$

contact structure which is a maximally non-degenerate rank $2n$ distribution $D \subset TZ$. Let X be a rational curve embedded into Z transversely to D and with normal bundle $N = \mathbb{C}^{2n} \otimes \mathcal{O}(1)$.Then the $4n$-dimensional Kodaira moduli space M comes equipped with the induced torsion-free affine connection satisfying natural integrability conditions. Ward (1981) showed that in the case $n = 1$ the Kodaira moduli space M has an induced complex Riemannian metric satisfying the self-dual Einstein equation with non-zero scalar curvature. The case $n \geq 2$ has been investigated by LeBrun (1989), Pedersen and Poon (1989), and Bailey and Eastwood (1991) who proved that the Kodaira moduli space M comes equipped canonically with a torsion-free connection compatible with the induced complexified quaternionic-Kähler structure on M. This inverts the construction of Salamon's (1982) in quaternionic-Kähler case, who associated a contact $(2n+1)$-dimensional manifold Z to any quaternionic-Kähler $4n$-fold M.

What happens if we lose the condition for the structure to be contact?

It is also well-known that hyperKähler manifolds (i.e. the ones equipped with a torsion-free $\mathrm{Sp}(n)$-structure) give rise to *integrable* codimension 1 distributions on the associated twistor spaces and vice versa, see N. Hitchin, A. Karlhede, U. Lindström and M. Roŏek [HKLR87].

Let Z be a $(2n+1)$-complex manifold equipped with a codimension 1 holomorphic distribution, $D \subset TZ$. Assume also that Z contains a rational curve X, which is transversal to D and

$$D_{|X} \simeq \mathbb{C}^{2n}(1).$$

Consider the associated Fröbenius form $\Phi_D : \wedge^2 D \longrightarrow TZ/D$. In general it has rank p, $0 \leq p \leq 2n$. From the discussion above, we know that the case $p = 2n$ corresponds the quaternionic-Kähler geometry, while the case $p = 0$ corresponds to hyperKähler geometry.

One of the main questions which we address and answer in this thesis is which geometry corresponds to the generic value of p?

We begin this work by recalling some basic facts, theorems and definitions in

Chapter 1. Section 1.1 consists of detailed introduction to the language of the sheaf theory. Apart from definitions and basic facts about sheaf mappings and exact sequences of sheaves, we construct space etalé,quotient sheaf and look at the notions of direct and inverse image of sheaves. In **Section 1.2**, we recall some definitions and theorems of cohomology theory. There are no proofs given. **Section 1.3** contains the statement of Kodaira's seminal theorem. In this chapter we used the book by S. Merkulov, [Mer00].

In **Chapter 2** we give the definition of a connection on a fibration, comparing it with connections on a vector bundles and connection along the base and looking at its curvature and integrability conditions. Then, in **Section 2.2**, we discuss the definition and examples of conic structures.

Another theme of our studies is the conic and projective geometry of Bryant's exotic G_3-connections, and their natural generalization G_m, for any $m \geq 3$. In **Chapter 3** we apply the language of conic structures developed in the previous chapter to the research of geometry of conformal 3-manifolds. Then these methods also used to look at G_m structures, which are natural "torsion" generalizations of exotic G_3-structures.

The main result of **Chapter 3** is **Theorem 3.2.1**, which studies interrelations between projective and conic connections.

In **Chapter 4** we turn to quaternionic geometry and study basic invariants of associated conic structures. This chapter is of an auxiliary nature, and most of the material is well-known, though in a differential language.

In **Chapter 5** we prove two new results. The first one is **Theorem 5.2.1**, which identifies Manin's obstructions [Man97] with certain torsion invariants of a general almost quaternionic structure. The result is then used in proving **Theorem 5.3.1**, which is one of the central results of the thesis. This theorem associates to an Einstein quaternionic manifold with rank $\Lambda_{AB} = p$, a twistor space Z, equipped with distribution $D \subset TZ$, whose Fröbenius form, $\Phi_D : \wedge^2 D \longrightarrow TZ/D$, has rank precisely p.

In **Chapter 6** we invert the construction of Theorem 5.3.1, thus establishing a one-to-one correspondence between local quaternionic Einstein manifolds with rank $\Lambda_{AB} = p$ and the twistorial data $(Z, D \subset TZ, \text{rank } \Phi_D = p)$.

Chapter 1

Basic Facts And Definitions

As a language of sheaves is widely used throughout this work, it is logical to start by recalling some useful definitions, facts and notations of the sheaf theory. It is followed by a brief overview of homological algebra of sheaves. The Chapter ends with a short list of various algebraic and geometric facts and propositions, including Kodaira's analytic family theorem, which are going to be used through out the thesis. As the material is standard, no proofs are given, although the necessary cross references provided.

1.1 Sheaf Theory

Here we used some material from the forthcoming book [Mer00]. For more detailed look into the sheaf theory, see also [WW90] and [Wel91]

1.1.1 Presheaf

A *presheaf* on a topological space M is a covariant functor

$$\mathcal{F} : Top(M) \longrightarrow Ab,$$

from the category of open sets of M to the category of Abelian groups. Equivalently, one can write this down as follows

Definition 1.1.1 *A* presheaf $\mathcal{F}$ *over a topological space* M *is*

(a) an assignment to each non-empty open set $U \subseteq M$ *of a set* $\mathcal{F}(U)$

(b) a collection of mappings (called restriction homomorphisms)

$$r_V^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

for each pair of open sets U *and* V *such that* $V \subseteq U$, *satisfying*

1. $r_U^U = \mathrm{id}$ *on* U

2. *for* $W \subseteq V \subseteq U, r_W^U = r_W^V \circ r_V^U$

Visually, this structure can be represented by a diagram

$$\begin{tikzcd} W \arrow[r, "\subseteq"] \arrow[d, "\mathcal{F}"'] & V \arrow[r, "\subseteq"] \arrow[d, "\mathcal{F}"'] & U \arrow[d, "\mathcal{F}"'] \\ \mathcal{F}(W) & \mathcal{F}(V) \arrow[l, "r_W^V"'] & \mathcal{F}(U). \arrow[l, "r_V^U"'] \end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are presheaves over M, then a *morphism* (of presheaves) $h : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of maps $h_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for each open set U in X such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "r_V^U"] & \mathcal{G}(U) \arrow[d, "r_V^U"] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd} \quad V \subseteq U \subseteq M.$$

$\mathcal{F}$ is said to be a subpresheaf of $\mathcal{G}$ if the maps h_U above are inclusions.

Example

Let M be a complex manifold, and assume that $\mathcal{O}(U)$ is the space of all holomorphic functions defined on an open subset $\mathcal{U} \subseteq M$. Take $r_V^U : \mathcal{O}(U) \longrightarrow \mathcal{O}(V)$ to be the usual restriction of a holomorphic function to an open subset $V \subseteq U$. The result is a presheaf $\mathcal{O}_M$ of holomorphic functions on M.

1.1.2 Sheaf

Definition 1.1.2 *A presheaf $\mathcal{F}$ on a topological space M is called a sheaf if for every open set $U \subseteq M$ and every collection U_i of open subsets of M with $U = \bigcup U_i$, $\mathcal{F}$ satisfies*

1. *Axiom s_1: If $s, t \in \mathcal{F}(U)$ and $r^U_{U_i}(s) = r^U_{U_i}(t)$ for all i, then $s = t$.*

2. *Axiom s_2: If $s_i \in \mathcal{F}(U_i)$ and if for $U_i \cap U_j \neq \emptyset$ we have*

$$r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$$

   for all i then there exists an $s \in \mathcal{F}(U)$ such that $r^U_{U_i} = s_i$ for all i.

Morphisms of sheaves (or *sheaf mappings*) are simply morphisms of the underlying presheaves. When a subpresheaf of a sheaf $\mathcal{F}$ is also a sheaf, then it is called a subsheaf of $\mathcal{F}$.

1.1.3 Stalks

Let $\mathcal{F}$ be a presheaf of Abelian groups on a topological space M. Consider an arbitrary point $x \in M$ and a system, $\mathcal{I}$, of all open subsets of M which contain x. $\mathcal{I}$ is a partially ordered set with respect to the following relation: $U \leq V \Leftrightarrow U \subseteq V$. The direct limit,

$$\mathcal{F}_x = \varinjlim_{U \in \mathcal{I}} \mathcal{F}(U),$$

is an Abelian group called the *stalk* of the presheaf $\mathcal{F}$ at x. In other words, $\mathcal{F}_x$ is the quotient of the disjoint union of Abelian groups,

$$\bigcup_{U \in \mathcal{I}} \mathcal{F}(U) / \sim_x,$$

with respect to the following equivalence relation: $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ are equivalent, $f \sim_x g$, if and only if there is an open set W such that $x \in W \subseteq U \cap V$ and $f|_W = g|_W$.

For an open set $U \subseteq M$ and a point $x \in U$ there is a canonical homomorphism of Abelian groups, $r_x : \mathcal{F}(U) \longrightarrow \mathcal{F}_x$, which associates to an element $f \in \mathcal{F}(U)$ its equivalence class with respect to the equivalence relation $\sim_x$. The image $r_x(f)$ is called the *germ* of f at x.

1.1.4 Exact sequences

Let $\tau : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves of Abelian groups on a topological space M. For each point $x \in M$, it induces a map of stalks

$$\tau_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

which is a homomorphism of Abelian groups.

A sequence of morphisms of sheaves on M,

$$\mathcal{F} \xrightarrow{\tau} \mathcal{G} \xrightarrow{\sigma} \mathcal{H},$$

is called *exact* if, for every $x \in M$, the associated sequence of homomorphisms of Abelian groups,

$$\mathcal{F}_x \xrightarrow{\tau_x} \mathcal{G}_x \xrightarrow{\sigma_x} \mathcal{H}_x,$$

is exact, i.e. $\mathrm{Ker}\sigma_x = \mathrm{Im}\tau_x$.

A sequence of morphisms of sheaves on M,

$$\mathcal{F}_1 \xrightarrow{\tau_1} \mathcal{F}_2 \xrightarrow{\tau_2} \ldots \xrightarrow{\tau_{n-2}} \mathcal{F}_{n-1} \xrightarrow{\tau_{n-1}} \mathcal{F}_n, \quad n > 3,$$

is called

1. *exact at the term* $\mathcal{F}_i, i \in \{2, \ldots, n-1\}$ if the sequence

$$\mathcal{F}_{i-1} \xrightarrow{\tau_{i-1}} \mathcal{F}_i \xrightarrow{\tau_i} \mathcal{F}_{i+1};$$

is exact.

2. *exact* if it is exact at every term $\mathcal{F}_i$ with $i \in \{2, \ldots, n-1\}$.

An exact sequence of the form

$$0 \longrightarrow \mathcal{F} \xrightarrow{\tau} \mathcal{G} \xrightarrow{\sigma} \mathcal{H} \longrightarrow 0$$

is called a *short exact sequence.*

1.1.5 Space etalé

There is a canonical construction which associates to any presheaf $\mathcal{F}$ on a topological space a sheaf, denoted by $\tilde{\mathcal{F}}$ and called *space etalé.*

First, let us consider a set,

$$|\mathcal{F}| = \bigcup_{x \in U} \mathcal{F}_x,$$

the disjoint union of stalks. It comes equipped with the natural projection,

$$\pi : |\mathcal{F}| \longrightarrow M$$
$$f_x \longrightarrow x.$$

For each open set $U \subseteq M$ and an element $f \in \mathcal{F}(U)$, let us construct a set

$$[U, f] := \{r_x(f) | x \in U\} \subseteq |\mathcal{F}|.$$

A topology can be introduced on $|\mathcal{F}|$ by declaring a subset of $\mathcal{F}$ open if and only if it is a union[1] or a finite intersection of subsets of the form $[U, f]$. The topology is well-defined. Clearly, every element of $|\mathcal{F}|$ is contained in, at least, one subset of the form $[U, f]$. Assuming that $e \in [U, f] \cap [V, g]$ one has that if $x = \pi(e)$, then $x \in U \cap V$ and $e = r_x(f) = r_x(g)$. In other words, $f \sim_x g$ means that there is an open neighbourhood $W \subseteq U \cap V$ of the point x and an element $h \in \mathcal{F}(W)$ such that $f|_W = g|_W = h$. Thus, $e \in [W, h] \subseteq [U, f] \cap [V, g]$.

Therefore, $\mathcal{F}$ now can be regarded as a topological space with the topology described above.

Proposition 1.1.1 *The natural projection, $\pi : |\mathcal{F}| \longrightarrow M$ is a local homeomorphism.*

[1]possibly infinite

Proof. For any $e \in |\mathcal{F}|$ there is an open set $[U, f] \subset |\mathcal{F}|$ containing e. The map π, as follows from definitions, is open and continuous. Since $\pi([U, f]) = U$, then it was shown that e has an open neighbourhood $\hat{U} = [U, f]$ such that $\pi : \hat{U} \longrightarrow U$ is a homeomorphism onto its image. □

A (continuous) *section* of a covering space $\pi : |\mathcal{F}| \longrightarrow M$ over an open subset $U \subseteq M$ is, by definition, a (continuous) map $\sigma : U \longrightarrow |\mathcal{F}|$ such that $\pi \circ \sigma = \mathrm{Id}$. Let $\Gamma(U, |\mathcal{F}|)$ denote the set of all continuous sections of $|\mathcal{F}|$. If $|\mathcal{F}|$ is space etalé associated to a presheaf of Abelian groups $\mathcal{F}$, then $\Gamma(U, |\mathcal{F}|)$ is an Abelian group.

The sheaf, $\tilde{\mathcal{F}}$, *associated*, to a presheaf of Abelian groups $\mathcal{F}$ is defined as follows:

(i) for an open subset $U \subseteq M, \tilde{\mathcal{F}}(U) := \Gamma(U, |\mathcal{F}|)$;

(ii) for every pair of open subsets $v \subseteq U, r_V^U : \tilde{\mathcal{F}}(U) \longrightarrow \tilde{\mathcal{F}}(V)$ is the usual restriction of maps.

This gives rise to a functor,

$$\begin{array}{rccc} \Phi : & \text{Presheaves}(M) & \longrightarrow & \text{Sheaves}(M) \\ & \mathcal{F} & \longrightarrow & \tilde{\mathcal{F}}, \end{array}$$

from the category of presheaves on a topological space M to the category of sheaves on the same space.

Theorem 1.1.1 *If $\mathcal{F}$ is a sheaf on a topological space, then $\Phi : \mathcal{F} \longrightarrow \tilde{\mathcal{F}}$ is an isomorphism.*

Proof. It has just to be checked that, for any open subset $U \subseteq M$, the map $\Phi_U : \mathcal{F}(U) \longrightarrow \tilde{\mathcal{F}}(U)$ is a bijection.

At first, it can be shown that Φ_U is injective. Let assume $f_1, f_2 \in \mathcal{F}(U)$ to be such that $\Phi_U(f_1) = \Phi_U(f_2)$. The map Φ_U sends f_1, f_2 into a section, σ_1, σ_2, of the associated space etalé,

$$\begin{array}{rccc} \sigma_k : & U & \longrightarrow & |\mathcal{F}| \\ & x & \longrightarrow & r_x(f_k), \end{array} \quad k = 1, 2,$$

and, therefore, the equality $\Phi_U(f_1) = \Phi_U(f_2)$ implies $r_x(f_1) = r_x(f_2)$ for all $x \in U$. This means that $f_1 \sim_x f_2$ and, hence, there is an open neighbourhood $V \subseteq U$ of the point x such that $f_1|_V = f_2|_V$. Since this is true for every $x \in U$, we can cover U by such sets V_i that $f_1|_{V_i} = f_2|_{V_i}$. By axiom (s_1) (see 1.1.2), this means that $f_1 = f_2$.

Next, it has to be shown that Φ_U is surjective. Assume $\sigma \in \Gamma(U, |\mathcal{F}|)$. Then for every $x \in U$ there is an open neighbourhood $V \subseteq U$ of x and an element $f \in \mathcal{F}(V)$ such that $\sigma(x) = r_x(f)$. Thus we get two local sections, σ_V and $\Phi_V(F)$, of $|\mathcal{F}|$ over V which coincide at x. Since both sections are locally inverse to the local homeomorphism $\pi : |\mathcal{F}| \longrightarrow M$, they must coincide in some open neighbourhood $W \subseteq V$ of x,

$$\sigma|_W = \Phi_W(f|_W).$$

This is true for any $x \in U$, therefore, there is a covering U by open sets W_i and a family of elements $f_i \in \mathcal{F}(W_i)$ such that

$$\sigma|_{W_i} = \Phi_{W_i}(f|_i).$$

Then

$$\Phi_{W_i \cap W_j}(f_i|_{W_i \cap W_j}) = \Phi_{W_i \cap W_j}(f_j|_{W_i \cap W_j}) = \sigma_{W_i \cap W_j},$$

and due to the fact that the maps $\Phi_{W_i \cap W_j}$ are injective,

$$f_i|_{W_i \cap W_j} = f_j|_{W_i \cap W_j}.$$

By axiom (s_2) (see 1.1.2), there is an element $f \in \mathcal{F}(U)$ such that $f|_{w_i} = f_i$. Therefore,

$$\Phi_U(f)|_{W_i} = \Phi_{W_i}(f|_{W_i}) = \sigma_{W_i},$$

which finally implies that $\Phi_U(f) = \sigma$. □

1.1.6 Kernels and quotients

Let $\tau : \mathcal{F} \longrightarrow \mathcal{G}$ be a homomorphism of sheaves of Abelian groups on a topological space M. For any open subset $U \subseteq M$, let us define an Abelian group

$$\mathcal{K} := \text{Ker } \tau_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U).$$

The family of such subgroups together with the restrictions maps $r_V^U : \mathcal{K}(U) \longrightarrow \mathcal{K}(V)$ induced by restriction maps on $\mathcal{F}$ forms a presheaf of Abelian groups. This presheaf is actually a sheaf and it's called the *kernel* of the morphism τ.

Similarly, one constructs another family of Abelian groups,

$$\mathcal{I} := \mathcal{F}(U)/\tau_U(\mathcal{G}(U)), \quad U \text{ is open in } M,$$

which together with the restriction maps induced from $\mathcal{G}$, forms a presheaf of Abelian groups. Although, this presheaf may not, in general, be a sheaf, this problem can be easily fixed via space etalé. Thus one gets a sheaf $\tilde{\mathcal{I}}$ on M, called *quotient sheaf* and denoted by $\mathcal{G}/\mathcal{F}$.

The morphism $\tau : \mathcal{F} \longrightarrow \mathcal{G}$ gives rise to an exact sequence,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \xrightarrow{\tau} \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0,$$

which is a visual summary of both constructions.

1.1.7 Direct and inverse images of sheaves

Let $f : M \longrightarrow N$ be a continuous map of topological spaces.

Given a sheaf $\mathcal{F}$ of Abelian groups on M, we define the family of Abelian groups,

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)), \ U \text{ is open in } N,$$

which together with restriction maps induced from $\mathcal{F}$, forms a presheaf of Abelian groups on N. It can be easily checked that $f_*(\mathcal{F})$ satisfies both sheaf axioms S_1 and S_2 and, hence, defines a sheaf on N called *the direct image* or *pushforward* of $\mathcal{F}$.

Given a sheaf $\mathcal{G}$ of Abelian groups on N, we construct a sheaf $f^*(\mathcal{G})$, of Abelian groups on M in four steps:

a. to any open subset $U \subseteq M$ we associate a partially ordered set $\mathcal{I}_U$ whose elements are open subsets V of N such that $f^{-1}(V) \supseteq U$; the relation is defined by $V_1 \leq V_2 \Leftrightarrow V_1 \subseteq V_2$. There is a natural contravariant functor F from the category $\mathcal{I}_U$ to the category of Abelian groups represented by a commutative diagram

$$\begin{array}{ccccc} V_1 & \xrightarrow{\subseteq} & V_2 & \xrightarrow{\subseteq} & V_3 \\ {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow \\ \mathcal{G}(V_1) & \xleftarrow{r^{V_2}_{V_1}} & \mathcal{G}(V_2) & \xleftarrow{r^{V_3}_{V_2}} & \mathcal{G}(V_3). \end{array}$$

b. using $(\mathcal{I}_U, F)$ and the direct limit procedure we associate to U an Abelian group

$$f^{-1}(\mathcal{G})(U) := \overrightarrow{\lim_{V \in \mathcal{I}_U}} \mathcal{G}(V)$$

c. the family of Abelian groups $f^{-1}(\mathcal{G})(U)$ together with the restriction maps induced form $\mathcal{G}$ form a presheaf of Abelian groups on M which, in general, is not a sheaf.

d. applying the functor Φ to the presheaf defined in c. we obtain a sheaf on M called the *inverse image* or *pullback* of the sheaf $\mathcal{G}$ and denoted by $f^*(\mathcal{G})$.

Therefore, in this section the definitions of presheaf and sheaf were given; construction of a sheaf from a presheaf via space etalé was described; stalks, exact sequences of sheaves and presheaves were considered as well as their kernels and quotients were presented along with the direct and inverse images of sheaves.

1.2 Cohomology

As it was shown in the sections above, to a certain extent, sheaves can be viewed as a next step generalization of such concepts as vector bundles, inheriting most of its structure. In this section the key theorems concerning homological algebra of sheaves are recalled. The material of the section constitutes the main results of cohomology theory of sheaves. The detailed view of the theory can be found in various textbooks on Cohomology, while the material of the section is based on [Kod80].

Let R be a ring with unity. The *chain complex* $(\underline{\underline{c}}, \partial)$ consists of a collection $\{c_i\}_{i \in \mathbb{Z}}$ of left R-modules and R-homomorphisms $c_{i-1} \xleftarrow{\partial_i} c_i$ such that $\partial_{i-1}\partial_i = 0$.

$$\underline{\underline{c}} : \ldots \longleftarrow c_{i-2} \xleftarrow{\partial_{i-1}} c_{i-1} \xleftarrow{\partial_i} c_i \xleftarrow{\partial_{i+1}} c_{i+1},$$
$$\partial^2 = 0,$$

where ∂_i is called a *boundary map.*

We say that $\underline{\underline{c}}$ is concentrated on $[k, l]$ if $c_i = 0$ for $i < k, i > l$. Kernel of the boundary map, Ker $\partial_i = Z_i(\underline{\underline{c}}) = Z_i$, is called an *i-cycle.* Image, Im $\partial_{i+1} = B_i(\underline{\underline{c}}) = B_i$, is referred to as an *i-boundary.*

Since $\partial^2 = 0$, we have $B_i \subseteq Z_i$. Hence we can define $H_i(\underline{\underline{c}}) = Z_i/B_i$, called the *i-th homology group (module/sheaf) of the chain complex.* $\underline{\underline{c}}$ is said to be *exact* at c_i if $H_i(\underline{\underline{c}}) = 0$, i.e. $Z_i = B_i$. If $\underline{\underline{c}}$ is exact everywhere it is said to be *acyclic.*

A *chain map* $\phi : (\underline{\underline{c}}, \partial) \longrightarrow (\underline{\underline{c}}', \partial')$ is a collection $\{\phi_i\}_{i \in \mathbb{Z}}$ of R-homomorphisms $\phi_i : c_i \longrightarrow c_i'$ such that the following diagram commutes for any i:

$$\begin{array}{ccccccc}
\ldots & \longleftarrow & c_{i_1} & \xleftarrow{\partial_i} & c_i & \longleftarrow & \ldots \\
 & & \downarrow{\scriptstyle \phi_{i-1}} & & \downarrow{\scriptstyle \phi_i} & & \\
\ldots & \longleftarrow & c_{i_1}' & \xleftarrow{\partial_i'} & c_i' & \longleftarrow & \ldots .
\end{array}$$

Here some theorems linking the exact sequences and the cohomology theory can be recalled. Proves can be found in [Kod80].

Theorem 1.2.1 *The exact sequence of sheaves* $0 \longrightarrow \mathcal{L}' \xrightarrow{i} \mathcal{L} \xrightarrow{h} \mathcal{L}'' \longrightarrow 0$ *induces the exact sequence of cohomology groups*

$$\begin{array}{l}
0 \longrightarrow H^0(\mathcal{L}') \xrightarrow{i} H^0(\mathcal{L}) \xrightarrow{h} H^0(\mathcal{L}'') \xrightarrow{\delta^*} H^1(\mathcal{L}') \xrightarrow{i} \ldots \\
\xrightarrow{\delta^*} H^q(\mathcal{L}') \xrightarrow{i} H^q(\mathcal{L}) \xrightarrow{h} H^q(\mathcal{L}'') \xrightarrow{\delta^*} H^{q+1}(\mathcal{L}') \xrightarrow{i} \ldots
\end{array} \tag{1.1}$$

Theorem 1.2.2 *An exact commutative diagram of sheaves*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}' & \xrightarrow{i} & \mathcal{L} & \xrightarrow{h} & \mathcal{L}'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi''} & & \\
0 & \longrightarrow & \mathcal{T}' & \xrightarrow{j} & \mathcal{T} & \xrightarrow{k} & \mathcal{T}'' & \longrightarrow & 0
\end{array}$$

induces an exact commutative diagram of cohomology groups:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^0(\mathcal{L}') & \xrightarrow{i} & H^0(\mathcal{L}) & \xrightarrow{h} & H^0(\mathcal{L}'') & \xrightarrow{\delta^*} & H^1(\mathcal{L}') & \xrightarrow{i} & \dots \\ & & \psi'\big\downarrow & & \psi\big\downarrow & & \psi''\big\downarrow & & \psi'\big\downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{T}') & \xrightarrow{j} & H^0(\mathcal{T}) & \xrightarrow{k} & H^0(\mathcal{T}'') & \xrightarrow{\delta^*} & H^1(\mathcal{T}') & \xrightarrow{j} & \dots \end{array}$$

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H^{q-1}(\mathcal{L}) & \xrightarrow{h} & H^{q-1}(\mathcal{L}'') & \xrightarrow{\delta^*} & H^q(\mathcal{L}') & \xrightarrow{i} & H^q(\mathcal{L}) & \xrightarrow{h} & \dots \\ & & \psi\big\downarrow & & \psi''\big\downarrow & & \psi'\big\downarrow & & \psi\big\downarrow & & \\ \dots & \longrightarrow & H^{q-1}(\mathcal{T}) & \xrightarrow{k} & H^{q-1}(\mathcal{T}'') & \xrightarrow{\delta^*} & H^q(\mathcal{T}') & \xrightarrow{j} & H^q(\mathcal{T}) & \xrightarrow{k} & \dots \end{array}.$$

1.3 Analytic family and moduli space

Let Y and M be complex manifolds and let $\pi_1 : Y \times M \longrightarrow Y$ and $\pi_2 : Y \times M \longrightarrow M$ be the natural projections. An analytic family of compact submanifolds of the complex manifold Y with the moduli space M is a complex submanifold $F \hookrightarrow Y \times M$ such that the restrictions of the projection π_2 on F is a proper regular map (regularity means that the rank of the differential of $\nu := \pi_2|_F : F \longrightarrow M$ is equal to dim M at every point). Thus the family F has the structure of double fibration

$$\begin{array}{ccc} & F & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ Y & & M \end{array}$$

where $\mu \equiv \pi_1 |_F$. For each $t \in M$ the compact complex submanifold $X_t = \mu \circ \nu^{-1}(t) \hookrightarrow Y$ is said to belong to the family M.

Let us denote the normal bundle $TA|_B/TB$ of a complex submanifold B of a complex manifold A by $N_{B|A}$.

If $F \hookrightarrow Y \times M$ is an analytic family of compact submanifolds, then, for any $t \in M$, there is a natural linear map (Kodaira 1962),

$$k_t : T_tM \longrightarrow H^0(X_t, N_{X_t|Y}),$$

from the tangent space at t to the vector space of global holomorphic sections of the normal bundle $N_{X_t|Y} = TY|_{X_t}/TX_t$ to the submanifold $X_t \hookrightarrow Y$, which can be described as follows. First, one can note that the normal bundle of the embedding $\nu^{-1}(t) \hookrightarrow F$ is trivial and thus there is a canonical map $\rho_t : T_tM \longrightarrow H^0(\nu^{-1}(t), N_{\nu^{-1}(t)|F})$. Then a composition $d\nu \circ \rho_t$ gives the desired map k_t for the differential of ν maps global sections of $N_{\nu^{-1}(t)|F}$ to global sections of $N_{X_t|Y}$.

An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds is called *complete* if the Kodaira map k_t is an isomorphism at each point t in the moduli space M. It is called *maximal* if for any other analytic family $\tilde{F} \hookrightarrow Y \times \tilde{M}$ such that $\nu^{-1}(t) = \tilde{\nu}^{-1}(\tilde{t})$ for some points $t \in M$ and $\tilde{t} \in \tilde{M}$ there is a neighbourhood $\tilde{U} \subset \tilde{M}$ of the point $\tilde{t}$ and a holomorphic map $f : \tilde{U} \longrightarrow M$ such that $\tilde{\nu}^{-1}(\tilde{t}') = \nu^{-1}\left(f(\tilde{t}')\right)$ for every $\tilde{t}' \in \tilde{U}$. Here the equality $\nu^{-1}(t) = \tilde{\nu}^{-1}(\tilde{t})$ means that $\mu \circ \nu^{-1}(t)$ and $\tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t})$ are the same submanifolds of Y.

In 1962 Kodaira [Kod62] proved the following important theorem.

Theorem 1.3.1 (Kodaira) *If $X \hookrightarrow Y$ is a compact complex submanifold with normal bundle N such that $H^1(X, N) = 0$, then X belongs to the complete analytic family $\{X_t : t \in M\}$ of compact submanifolds X_t of Y. The family is maximal and its moduli space of complex dimension* $\dim_{\mathbb{C}} H^0(X, N)$.

Chapter 2

Conic Structures and Conic Connections

2.1 Connection on a fibration

In this Chapter a (holomorphic) distribution is defined and, subsequently, the notion of a connection on a fibration is introduced. The conditions for existence of such a connection are considered. Following these, connections on vector bundles and connections along a distribution on the base are both defined. It is followed by definitions of curvature and integrability of a distribution. These are our basic technological tools.

The material of this Chapter based mainly on [Man97].

2.1.1 Distributions and connections on a fibration

Definition 2.1.1 A (holomorphic) distribution *on a manifold F is a subsheaf D of tangent sheaf TF which is a locally direct summand.*

Distribution is said to be *integrable* if D is a subsheaf of Lie algebras, i.e. for any $X, Y \in D$

$$[X, Y] \bmod D = 0.$$

Being locally free D has an associated vector bundle which we sometimes denote by the same symbol, D. The rank of this vector bundle is called the rank of the distribution D. A local section is called a vector field tangent to the distribution.

Definition 2.1.2

1. *(a) By a* fibration $\pi : F \longrightarrow M$ *we mean a morphism which is a submersion of complex manifolds.*

2. *(b) By a* connection on a fibration (F, π) *we mean a distribution $D \subset TF$ for which the morphism $d\pi$ in the exact sequence*

$$0 \longrightarrow TF/M \longrightarrow TF \xrightarrow{d\pi} \pi^*(TM) \longrightarrow 0 \tag{2.1}$$

induces an isomorphism $D \xrightarrow{\sim} \pi^*(TM)$.

From the above exact sequence (2.1) it follows that a connection on a fibration is equivalent to giving a direct sum decomposition

$$TF = TF/M \oplus \pi^*(TM),$$

and its dual

$$\Omega^1 F = \Omega^1 F/M \oplus \pi^*(\Omega^1 M).$$

Such a decomposition corresponds to a splitting of $d : \mathcal{O}_F \longrightarrow \Omega^1 F$ into the differentials in the horizontal and vertical directions:

$$d_h = \partial : \mathcal{O}_F \longrightarrow \pi^*(\Omega^1 M), d_v : \mathcal{O}_F \longrightarrow \Omega^1 F/M.$$

From the geometrical point of view, this can be interpreted in the following terms: if $n = \dim M$, then at each point $x \in F$ a connection (on a fibration) singles out a d-dimensional tangent subspace of horizontal directions, which $d\pi$ projects isomorphically onto the tangent space at $\pi(x) \in M$.

2.1.2 Obstruction to the existence of a connection

In general, let

$$\aleph : 0 \longrightarrow S \xrightarrow{i} D \xrightarrow{j} \hat{S} \longrightarrow 0$$

be an exact sequence of sheaves on complex manifold F. By a splitting of the above sequence one means a morphism $h : \hat{S} \longrightarrow D$ such that its composition with j gives an identity on $\hat{S}$, $j \circ h = \mathrm{id}_{\hat{S}}$. Then $D = S \oplus h(\hat{S})$, and j is an isomorphism on $h(\hat{S})$. The difference of two splittings,

$$h_1 - h_2 : \hat{S} \longrightarrow D,$$

maps $\hat{S}$ to the kernel of j. If this kernel is identified with S, then

$$h_1 - h_2 \in \mathrm{Hom}(\hat{S}, S).$$

Conversely, if h is such a splitting and $f \in \mathrm{Hom}(\hat{S}, S)$, then $h + i \circ f$ is another splitting. Thus, the set of splittings is either empty or is, in fact, a principal homogeneous space for the group $\mathrm{Hom}(\hat{S}, S)$.

Clearly, these notions can be localised. If the morphism i is a direct sum imbedding, then there is a sheaf of splittings which is a principal homogeneous space for the sheaf $\mathit{Hom}(\hat{S}, S)$. This sheaf can be used to construct the characteristic class

$$c(\aleph) \in H^1(F, \mathit{Hom}(\hat{S}, S)),$$

which is the obstruction for a global splitting of $\aleph$.

Proposition 2.1.1

1. *The obstruction to the existence of a connection on the fibration (F, π) on the complex manifold M is the class*

$$c(F, \pi) \in H^1(F, \pi^*\Omega^1 M \otimes TF/M).$$

2. *If $c(F,\pi)=0$, then the group*

$$H^0(F,\pi^*\Omega^1 M\otimes TF/M) = H^0(M,\Omega^1 M\otimes \pi_*(TF/M))$$

acts transitively and effectively on the set of all connections.

Proof. Explicitly, let us consider the sequence of sheaves

$$Hom(\hat{S},\aleph): 0 \longrightarrow Hom(\hat{S},S) \longrightarrow Hom(\hat{S},D) \longrightarrow Hom(\hat{S},\hat{S}) \longrightarrow 0.$$

This sequence is exact when $\aleph$ splits locally. Let us set $c(\aleph)=\delta(\mathrm{id}_{\hat{S}})$, where δ is the boundary homomorphism,

$$\delta : H^0(F, Hom(\hat{S},\hat{S})) \longrightarrow H^1(F, Hom(\hat{S},S)).$$

Choose an open covering on $F=\cup U_i$. If we have splittings

$$h_i : \hat{S}|_{U_i} \longrightarrow D|_{U_i}$$

on the pieces of this covering, then the Cech cocycle

$$(h_i|_{U_i\cap U_j} - h_j|_{U_i\cap U_j})$$

represents the class of $c(\aleph)$. If $c(\aleph)=0$ then $H^0(M, Hom(\hat{S},S))$ acts transitively and effectively on the settings. $\square$

2.1.3 Connections on a vector bundle

Suppose that $\pi : F \longrightarrow M$ is a vector bundle. Let $\mathcal{F}$ be the locally free sheaf of holomorphic sections of π. Then the sections of $\mathcal{F}^*$ are functions on F. At every point of F we have a local coordinate system, part of which is lifted from M. The other part of this local coordinates system consists of a basis of sections of $\mathcal{F}^*$ which are linearly independent at the point. On F we consider the sheaf $S(\mathcal{F}^*)$ of functions which are polynomial along the fibres of π. Any connection on F is uniquely determined by its action on $S(\mathcal{F}^*)$. We say that the connection is compatible with the vector bundle structure if any local vector field X of the connection takes $S^i(\mathcal{F})^*$ to $S^i(\mathcal{F}^*)$ for all $i\geq 0$.

2.1.4 Connections on a fibration along a distribution on the base

Let $\pi : F \longrightarrow M$ be a fibration, let $D \in TM$ be a distribution on the base, and set $T_0F = (d\pi)^{-1}(\pi^* D)$. A connection on F along D is defined as a splitting of the exact sequence

$$0 \longrightarrow TF/M \longrightarrow T_0F \xrightarrow{d\pi} D \longrightarrow 0. \tag{2.2}$$

Similar to Section 2.1.3, one can introduce the sheaf of coefficients of such a connection, i.e., $\pi_*(D^* \otimes TF/M)$, and also define a compatibility with various additional structures, such as a vector bundle structure on F.

Suppose that $F \longrightarrow M$ is a vector bundle, $\mathcal{F}^*$ is the dual sheaf of holomorphic sections on F, and $\mathbb{P}(\mathcal{F}^*)$ is the corresponding relative projective space. Further, let us assume that we have local coordinates (x^a) on M, and that in the domain of the definition of these coordinates the sheaf $\mathcal{F}^*$ is trivialised by a basis of sections (ω^a). A trivialisation of any vector bundle, i.e., a choice of isomorphism $F \xrightarrow{\sim} F_0 \times M$ compatible with π, automatically determines a connection $D = TF/F_0$ on the bundle (these are the vector fields which are vertical relative to projection onto the fibre). Using this connection as our "origin" , we can describe all of the other connections by giving a section of the sheaf of coefficients. In this situation, the sheaf of coefficients is also trivialised by the choice of (x^a) and ω^a, and it is the resulting expansion which leads to the generalised Christoffel symbols. Thus the following proposition holds:

Proposition 2.1.2 ([Man97]) *The following structures are equivalent:*
(a) A connection on a vector bundle $F \longrightarrow M$ which is compatible with the vector bundle structure.
(b) A covariant differential $\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1 M$, i.e. a $\mathbb{C}$-linear morphism of sheaves satisfying Leibnitz' formula $\nabla(af) = a \nabla f + f \otimes da$, where a is a local function and f is a local section of $\mathcal{F}$.
(c) A pair consisting of a connection D on the bundle $\mathbb{P}_M(\mathcal{F}^*) \longrightarrow M$ and a connection on the vector bundle $F \longrightarrow \mathbb{P}_M(\mathcal{F}^*)$ along the distribution D.

2.1.5 Integrability and Curvature

The Fröbenius form of a distribution $D \subset TF$ is defined as the map

$$\Phi : D \times D \longrightarrow TF/D,$$

which is given by

$$\Phi(X,Y) = [X,Y]\text{mod}D.$$

Obviously, one gets $\Phi(X,Y) = -\Phi(Y,X)$. Also, using the Leibnitz formula a bilinearity of Φ can be deduced as follows:

$$\begin{aligned} [aX,Y] &= aXY - Y(aX) \\ &= aXY - aYX - (Ya)X \\ &\equiv a[X,Y]\text{mod}D \end{aligned}$$

Hence, the Fröbenius form, Φ, can be regarded as a mapping from $\wedge^2 D$ to TF/D, or as a section of the corresponding sheaf:

$$\Phi \in H^0(F, \wedge^2 D^* \otimes TF/D).$$

If D is a connection on $\pi : F \longrightarrow M$, then we call $\pi_*(\wedge^2 D^* \otimes TF/D)$ *the curvature sheaf*, and we call $\pi_*(\Phi)$ the curvature of D.

Integrability of D is equivalent to the vanishing of Φ. Locally, the integrability of the distribution is given via Fröbenius theorem:

Theorem 2.1.1 Holomorphic Fröbenius theorem

The following conditions are equivalent:

(a) The distribution $D \subset TF$ is integrable.

(b) Each point $x \in F$ has a neighbourhood with local coordinate system (x^a), $a = 1, ..., m$, such that D is freely generated in this neighbourhood by a subset of the coordinate vector fields (i.e. by $\partial/\partial x^a, a = 1, ..., d = \text{rank}D$).

(The proof of the above can be found in [Ste65]).

2.2 Conic Structures and Conic Connections

The second section is dedicated to the conic structures and conic connections. Some useful examples are presented in detail. The curvature and integrability conditions are considered. The material of this section is based on a book by I. Manin [Man97].

2.2.1 Introduction

Conic structures is a very useful language for the description of various classic geometry structures. They also turn out to be the only effective means of describing supergravity models and its cousins in various dimensions, i.e. when moving from classical geometry to supergeometry. Moreover, conic structures are essential for quantizing Hamilton systems with constraints. It turns out that Dirac's constraints can be explicitly solved, when lifted from the base to an appropriate conic structure. Such a trick of lifting was crucial in theories like Quantum Twistor Particle Theory and Quantum Twistor String Theory.

2.2.2 Definition of a Conic Structure

Let M be a complex manifold, TM its tangent bundle and let $d \geq 0$ an integer.

Definition 2.2.1 *A complex closed submanifold $F \subset G_M(d;TM)$, such that the projection $\pi : F \longrightarrow M$ is a submersion, is called* d-conic structure.

In other words, for any $x \in M$, F determines a set of distinguished d-dimensional (complex) tangent directions in T_xM corresponding to the points $\pi^{-1}(x) \subset G_M(d;T_xM)$.

Examples:

1. Full conic structure

Full conic structure by definition is a relative Grassmanian

$$F = G_M(d;TM).$$

It is clear to see that the fibre of the natural projection $\pi : F \longrightarrow M$ is just the usual Grassmanian $G(d, T_x M)$.

2. Almost quaternionic structure

Let M be a complex manifold of dimension $2k$. Assume k is even. An almost quaternionic structure on M is an isomorphism defined as follows:

$$\varphi : TM \longrightarrow S \otimes \hat{S},$$

where S and $\hat{S}$ are holomorphic vector bundles of rank k and 2, respectively. Now it will be shown that the relative projective line $\mathbb{P}_M(S)$ has a canonical k-conic structure.

Take

$$F = P_M(\hat{S}) \overset{i}{\hookrightarrow} \mathrm{G}(k; TM); F = P_M(\hat{S}) \xrightarrow{\pi} M.$$

A single point of F is a one dimensional subspace (line) in $\hat{S}(x), x \in M$. Its tensor product with $S(x)$ is a k-dimensional subspace which lies in $S(x) \otimes \hat{S}(x) = TM(x)$. This determines a k-conic structure, which plays a very important role in the quaternionic geometry (see Chapter 4).

Explicitly, let $x^1, \ldots, x^{2k}$ be coordinates in some neighbourhood $U \subset M$, and let $\{e_A\}$ be a local frame of S, $\{e_{\dot{A}}\}$ a local frame of $\hat{S}$, π^A the associated coordinates in S, $\pi^{\dot{A}}$ the associated coordinates in $\hat{S}$, $A = 1, \ldots, k; \dot{A} = 1, 2$.

Then, one has

$$\varphi(\partial_a) = \varphi_a^{A\dot{A}}(x) e_A \otimes e_{\dot{A}},$$

and

$$\varphi^{-1}(e_A \otimes e_{\dot{A}}) = \varphi^a_{A\dot{A}} \partial_a,$$

for some smooth functions $\varphi_a^{A\dot{A}}, \varphi^a_{A\dot{A}}$, where $\partial_a = \frac{\partial}{\partial x^a}$.

For any $p \in P_M(\hat{S})$:

$$\pi(p) = \{x_0^1, \ldots, x_0^{2k}\} \in M$$

and

$$\pi^{-1}(x) = \mathbb{P}(\hat{S}_p) \simeq \mathbb{P}^1.$$

The coordinates $\pi^{\dot{A}}$ defined above serve as homogeneous coordinates on $\mathbb{P}^1$, so $p \in \mathbb{P}_M(\hat{S})$ can be described (up to scalar multiplication) as

$$p = [\sum \pi^{\dot{A}} e_{\dot{A}}].$$

Then, the map $F \hookrightarrow G(k, TM)$ is explicitly given by:

$$\begin{array}{rcl} i : F & \hookrightarrow & G(k, TM), \\ p = [\sum \pi^{\dot{A}} e_{\dot{A}}] & \longrightarrow & \mathrm{span}[\pi^{\dot{A}} \varphi^a_{A\dot{A}} \partial_a]. \end{array}$$

Thus, the mapping

$$i : p \longrightarrow S \otimes p \longrightarrow [\sum \pi^{\dot{A}} e_A \otimes e_{\dot{A}}]$$

can be described as follows

$$\begin{aligned} i(p) &= \mathrm{span}[\varphi^{-1}(\pi^{\dot{A}} e_A \otimes e_{\dot{A}})] \\ &= \mathrm{span}[\pi^{\dot{A}} \varphi^{-1}(e_A \otimes e_{\dot{A}})] \\ &= \mathrm{span}(\pi^{\dot{A}} \varphi^a_{A\dot{A}} \partial_a), \end{aligned}$$

where $A = 1, \ldots, k; \dot{A} = 1, 2$.

In other words, $i(p) = \mathrm{span}\{\alpha_1, \ldots, \alpha_k\}$, where

$$\alpha_A = \sum \pi^{\dot{A}} \varphi^a_{A\dot{A}} \partial_a.$$

Let us note, that in a specific neighbourhood $\mathcal{U} = \{\pi^{\dot{0}} \neq 0\}$,

$$\alpha_A = [\varphi^a_{A\dot{0}} + \frac{\pi^{\dot{1}}}{\pi^{\dot{0}}} \varphi^a_{A\dot{1}}] \partial_a.$$

Also it can be easily seen, that $D_{A\dot{A}} = \sum \varphi^a_{A\dot{A}} \partial_a$ form basis in TM, so α_A are linearly independent. Therefore, $i(p)$ is k-dimensional subspace of TM, $i(p) \hookrightarrow \mathrm{G}(k, TM)$. Thus, F is a conic structure.

3. G_3 structure

Let $\mathbb{S}$ be a standard 2-dimensional representation of space $\mathrm{GL}(2,\mathbb{C})$. Then $\mathrm{GL}(2,\mathbb{C})$ naturally acts on symmetric tensor product $\odot^3\mathbb{S}$. If $\rho : \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(4,\mathbb{C})$ is the associated representation, we can define a subgroup $G_3 = \rho(\mathrm{GL}(2,\mathbb{C}))$ of $\mathrm{GL}(4,\mathbb{C})$. Let M be a complex 4-manifold and $\pi : L^*M \longrightarrow M$ the holomorphic coframe bundle whose fibres $L_t^* = \pi^{-1}(t)$ consist of all $\mathbb{C}$-linear isomorphisms $e : \mathbb{C}^4 \longrightarrow \Omega_t^1 M$. The space L^*M is naturally a principal right $\mathrm{GL}(4,\mathbb{C})$-bundle, where the right action $R_g : L^*M \longrightarrow L^*M$ is given by $R_g(e) = e \circ g$. A *G_3-structure* on M is, by definition, a principal subbundle of L^*M with the group G_3. It is clear that G_3-structure is equivalent to a local factorisation of the tangent bundle into symmetric cube

$$TM = \odot^3 S$$

of locally defined vector bundle S of rank 2. Though such a vector bundle may fail to exist on the whole of M, the projectivised vector bundle $\mathbb{P}_M(S)$ is well-defined *globally*. This G_3 structure has been very popular recently in connection with holonomy problem (cf. Bryant [Bry91], Hugget and Merkulov [HM99]).

Let us assume that M is a complex manifold with G_3-structure such that S exists on the whole of M; it is called a *spinor* bundle on M. A linear connection ∇ on S is called a *spinor connection* on M. Any spinor connection on M induces, via isomorphism $TM = \otimes^3 S$, an affine connection with holonomy in G_3; moreover, any affine connection on M with holonomy in G_3 arises in this way, at least locally. By a torsion tensor of a spinor connection we mean the torsion tensor of the associated affine connection.

There is a canonical injective bundle map, $i : \mathbb{P}_M(S^*) \longrightarrow \mathrm{Gr}(2, \Omega^1 M)$, which can be unambiguously neighbourhood by the isomorphism $i^*(U) = \nu^*(S^*)(-2) \equiv \nu^*(S^*) \otimes \mathcal{O}_F(-2)$, where $\mathcal{O}_F(-2) = [\mathcal{O}_F(-1)]^{\otimes^2}$, $\mathcal{O}_F(-1)$ stands for the tautological sheaf on $\mathbb{P}_M(S^*)$ and U is the relative tautological vector bundle on $\mathrm{Gr}\,(2, \Omega^1 M)$. Thus $\mathbb{P}_M(S^*)$ *is naturally a* 2*-conic structure* F *on* M.

This 2-conic structure plays a crucial role in the understanding of G_3-exotic holonomies (see Chapter 3).

2.2.3 Conic connection

Let F be a d-conic structure on the manifold M.

Definition 2.2.2 *A distribution of c-dimensional tangent planes in F is called* tangent to the conic structure *if for any $x \in F$ the projection of the tangent plane at this point onto $TM(x)$ is the d-dimensional subspace corresponding to x.*

Definition 2.2.3 *By a* conic connection *on F one calls a distribution of d-dimensional tangent planes which is tangent to the conic structure.*

From the definitions it is obvious that conic connection is given by distribution $D \subset TF$. A conic connection, $D \subset TF$, is said to be *integrable* if it is integrable as a distribution, i.e. the Fröbenius map,

$$\begin{aligned} D \otimes D &\longrightarrow \quad TF/D, \\ X \otimes Y &\longrightarrow [X, Y]\mathrm{mod}D, \end{aligned}$$

is zero.

Suppose that S is a tautological sheaf on $G_M(d; TM)$ and S_F is its restriction to F. $S_F \subset \pi^*(TM)$, where $\pi : F \longrightarrow M$. Starting with the exact sequence

$$0 \longrightarrow TF/M \longrightarrow TF \longrightarrow \pi^*(TM) \longrightarrow 0$$

one shall compute the coefficient sheaf of conic connections. For any point $p \in F$ one has an exact sequence of linear spaces:

$$0 \longrightarrow TF/M|_p \longrightarrow T_pF \xrightarrow{\pi^*} T_{\pi(p)}M \longrightarrow 0. \tag{2.3}$$

The point p can be considered as a d-dimensional subspace S_p of $T_{\pi(p)}M$. Then from formulae (2.3) the exact sequence of linear spaces can be obtained

$$0 \longrightarrow TF/M|_p \longrightarrow \pi_*^{-1}(S_p) \longrightarrow S_p \longrightarrow 0,$$

and, hence, the associated sequence of locally free sheaves

$$0 \longrightarrow TF/M \longrightarrow \pi_*^{-1}(S_F) \longrightarrow S_F \longrightarrow 0.$$

Denote $\pi_*^{-1}(S_F)$ as T_cF, then one can write the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TF/M & \longrightarrow & TF & \xrightarrow{\pi_*} & \pi^*(TM) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & TF/M & \longrightarrow & T_cF & \xrightarrow{\pi_*} & S_F & \longrightarrow & 0. \end{array}$$

A conic connection on F is a splitting of the lower horizontal exact sequence. Therefore, if ∇_1 and ∇_2 are two different conic connections, then, see Section 2.1.2,

$$\nabla_1 - \nabla_2 = H^0(F, TF/M \otimes S_F^*).$$

A choice of local coordinates (x^a) in M trivialises TM, F, and the coefficient sheaves. The sheaf of coefficients of conic connections is $\pi_*^0(TF/M \otimes S_F^*)$.

For computational reasons it is useful to compare this sheaf, $\pi_*^0(TF/M \otimes S_F^*)$, with the connection coefficient sheaf on the full Grassmanian $G_M(d;TM) \longrightarrow M$.

Proposition 2.2.1 *(a) The sheaf of connection coefficients on the fibration $G_M(d;TM) \longrightarrow M$ is $\Omega^1 M \otimes sl(TM) = \Omega^1 M \otimes (\Omega^1 M \otimes TM)_0$. Its local sections are characterised by the Christoffel symbols "with no second trace":*

$$(\Gamma^c_{ab}) = \Gamma^c_{ab} \otimes dx^b \otimes \frac{\partial}{\partial x^c}, \quad \Gamma^b_{ab} = 0.$$

(b) The map

$$\{\text{connections on the fibration}\} \longrightarrow \{\text{full conic connections}\}$$

is surjective and has the following appearance on the coefficients:

$d = 1$

$$\Omega^1 M \otimes (\Omega^1 M \otimes TM)_0 \longrightarrow (\odot^2(\Omega^1 M) \otimes TM)_0;$$

$d > 1$

$$\Omega^1 M \otimes (\Omega^1 M \otimes TM)_0 \longrightarrow (\odot^2(\Omega^1 M) \otimes TM)_0 \oplus (\Lambda^2(\Omega^1 M) \otimes TM)_0.$$

Proof. As in Manin [Man97].

2.2.4 Curvature and curvature sheaves

Connection on a fibration

Let $F \xrightarrow{\pi} M$ be a fibration, and $D \subset TF$ be a connection on a fibration. In other words, one has an isomorphism $d\pi : D \tilde{\longrightarrow} \pi^*TM$. The curvature Φ of D is an element of $\pi_*(\wedge^2 D^* \otimes TF/D)$, which can be decomposed as follows:

$$\begin{aligned}\pi_*(\wedge^2 D^* \otimes TF/D) &= \pi_*(\wedge^2 \pi^*(\Omega^1 M) \otimes TF/D) \\ &= \Omega^2 M \otimes \pi_* TF/D.\end{aligned}$$

So, the curvature is a 2-form on the base M with values in the sheaf TF/D of the vertical vector fields of the fibration.

Full conic connections

Let $G = G_M(d; TM) \longrightarrow M$ be the relative Grassmanian on a complex manifold M, and let S be tautological sheaf. The sheaf $\wedge^2 S^* \otimes TG/D$ which contains the curvature of the full conic connection $D \subset TG$ seems explicitly to depend on D. However, it can be represented as an extension of two sheaves which no longer depend on D: after factoring $TG \xrightarrow{d\pi} \pi^*(TM)$ by $D \xrightarrow{d\pi} S$, one obtains the following exact sequence (zero if $d = 1$):

$$0 \longrightarrow \wedge^2 S^* \otimes TG/M \longrightarrow \wedge^2 S^* \otimes TG/D \xrightarrow{\beta} \wedge^2 S^* \otimes \hat{S}^* \longrightarrow 0,$$

where $\hat{S}^* = \pi^*(TM)/D$.

For $d > 1$ the integrability condition splits into two parts. Let $\Phi_0 \in H^0(M, \wedge^2 S^* \otimes \hat{S}^*) = (\Omega^2 M \otimes TM)_0$ be the $\pi_*(\beta)$-image of the Fröbenius form. The first condition for integrability is that $\Phi_0 = 0$.

If this condition holds, then

$$\Phi \in H^0(M, \pi_*(\wedge^2 S^* \otimes TG/M)) \subset \Omega^2 M \otimes sl(TM).$$

This element, which is true curvature, must also vanish.

General conic connection

Let $F \longrightarrow M$ be a d-conic structure on a complex manifold M. Let $D \subset TF$ be a d-conic connection on F. Its curvature lies in the sheaf $\wedge^2 S^* \otimes TF/D$. Again, $\wedge^2 S^* \otimes TF/D$ can be represented as an extension of two sheaves which no longer depend on D: after factoring $TF \xrightarrow{d\pi} \pi^*(TM)$ by $D \xrightarrow{d\pi} S_F$, the following exact sequence (zero if $d = 1$) is obtained:

$$0 \longrightarrow \wedge^2 S_F^* \otimes TF/M \longrightarrow \wedge^2 S_F^* \otimes TF/D \xrightarrow{\beta} \wedge^2 S^* \otimes \hat{S}_F^* \longrightarrow 0$$

For $d > 1$ the integrability condition splits into two parts. Let $\Phi_0 \in H^0(M, \wedge^2 S_F^* \otimes \hat{S}_F^*) = (\Omega^2 M \otimes TM)_0$ be the $\pi_*(\beta)$-image of the Fröbenius form. The first condition for integrability is that $\Phi_0 = 0$. If $\Phi_0 = 0$, then the true curvature Φ lies in $H^0(M, \pi_*(\wedge^2 \hat{S}_F \otimes TF/M))$.

2.2.5 An example of conic connection

G_3-structure

Here we use the notation introduced in Section 2.2.2, Example 3. Since

$$S_F^* = \nu^*(S \otimes (\wedge^2 S)^{\otimes^2})(-2)$$

and

$$TF/M = \nu^*(\wedge^2 S^*)(2),$$

the conic connection coefficient sheaf on F is isomorphic to

$$\nu_*^0(TF/M \otimes S_F^*) = \odot^5 S^* \otimes \wedge^2 S \oplus \odot^3 S^*.$$

A projective connection on $F = \mathbb{P}_M(S^*)$ determines a splitting of the exact sequence

$$0 \longrightarrow TF/M \longrightarrow TF \xrightarrow{\nu} \nu^*(TM) \longrightarrow 0,$$

that is a morphism $\gamma : \nu^*(TM) \longrightarrow TF$ such that $d\nu \circ \gamma = id$. Then, restricting γ to the subsheaf $S_F^* \subset \nu^*(TM)$, one gets a conic connection on F. From the exact sequence which relates coefficient sheaves of projective and 2-conic connections,

$$0 \longrightarrow \nu^*(S^* \otimes \wedge^2 S^*) \longrightarrow TF/M \otimes \nu^*(\Omega^1 M) \longrightarrow TF/M \otimes S^*_F \longrightarrow 0,$$

it is clear that the map

$$\{\text{projective connections on } \mathbb{P}_M(S^*)\} \longrightarrow \{\text{conic connections on } F\}$$

is surjective with its kernel given by sections of $S^* \otimes \wedge^2 S^*$. Hence, the kernel of the surjection

$$\{\text{linear connections on } S\} \xrightarrow{pr} \{\text{conic connections on } F\}$$

consists of arbitrary sections of $S^* \otimes \wedge^2 S^* \oplus \odot^3 S^*$. Using this freedom, together with (*), it is not hard to check that there exists a unique 2-conic connection D on F, called the *distinguished 2-conic connection*, such that the set $pr^{-1}(D)$ contains a (necessarily unique) linear connection ∇ whose torsion tensor is a section of $\odot^7 S^* \otimes (\wedge^2 S)^{\otimes^2} \subset TM \otimes \Omega^2 M$. Then, this section is nothing but the invariant torsion of the G_3-structure. The vanishing of this section is a necessary and sufficient condition for the G_3-structure to admit a torsion-free connection.

Chapter 3

Conic connections on G_m-structures

Here we study first applications of the conic geometry. We start with a conic interpretation of the well-known conformal geometry in 3 dimensions and then use it as a launching pad for investigation of some new geometric structures.

3.1 Conic Connections On Conformal 3-manifold

The section considers the canonical 2-conic structure on conformal 3-manifold, distinguishing a 2-conic connection and calculating its curvature sheaf. Also a correspondence between the conic and projective connections is considered in Section 3.1.6. The definitions, theorems and computations in paragraphs 3.1.1 and 3.1.2 are based on papers of K.P. Tod [Tod92] and Cheng-chih Tsai [CT96].

3.1.1 Weyl and Einstein-Weyl Manifold

A *Weyl space* is a smooth manifold M equipped with

(1) a conformal class of metrics, $[g_{ab}]$,

(2) affine torsion-free connection D (called the *Weyl connection*),

which are compatible in the sense that the connection preserves the conformal class of metrics, i.e. in local coordinates, a chosen representative g for the class of conformal metrics is written as g_{ab} and the Weyl covariant derivative is written as D_a. Then, the compatibility condition becomes

$$D_a g_{bc} = \omega_a g_{bc}$$

for some 1-form $\omega = \omega_a dx^a$.

The Ricci tensor for Weyl connection is defined as

$$W_{ab} = W^d_{adb},$$

where W^d_{abc} is the curvature of the Weyl connection.

Weyl manifold M is *Einstein-Weyl* if and only if the symmetric part of the Ricci tensor is proportional to conformal metric, i.e.

$$W_{(ab)} = \frac{1}{3} W g_{ab}.$$

Proposition 3.1.1

$$W_{(ab)} = R_{ab} + \nabla_{(a}\omega_{b)} - \omega_a\omega_b + g_{ab}(\nabla_k\omega^k + \omega_k\omega^k),$$

where R_{ab} is the Ricci curvature for metric connection ∇_a.

Proof. Standard calculations following [CT96].

3.1.2 Conformal 3-manifold

Let S be a standard 2-dimensional representation of space $\mathrm{GL}(2,\mathbb{C})$. Then $\mathrm{GL}(2,\mathbb{C})$ naturally acts on 3-dimensional $\odot^2 S$, thereby defining the homomorphism:

$$\rho : \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(\odot^2 S) \cong \mathrm{GL}(3,\mathbb{C}).$$

It is a well known fact that $\mathrm{Im}\rho$ in $\mathrm{GL}(3,\mathbb{C})$ is precisely the 3-dimensional conformal group $\mathrm{CO}(3,\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C}) \times \mathbb{C}^*$.

Therefore, if M is 3-manifold equipped with a conformal structure $[g_{ab}]$, then there exists, at least *locally*, a rank 2 vector bundle, $S \longrightarrow M$, with the following isomorphism:

$$\varphi : TM \longrightarrow \odot^2 S.$$

Though such a vector bundle may fail to exist on the whole of M, the projectivised vector bundle $\mathbb{P}_M(S)$ is well-defined *globally*. So if one is interested in a local geometry of M, it can be assumed, by shrinking M as necessary, that S and isomorphism $\varphi : TM \longrightarrow \odot^2 S$ are defined globally on M.

Note, that conformal class of metrics $[g_{ab}]$ can be easily reconstructed from the data $(S, \varphi : TM \longrightarrow \odot^2 S)$.

Indeed, there is a canonical decomposition:

$$\odot^2 \Omega^1 M \overset{\hat{\varphi}}{\simeq} \odot^2(\odot^2 S^*) = \odot^4 S^* \bigoplus (\wedge^2 S^*)^{\otimes^2},$$

where $\hat{\varphi}$ is the dual of $\odot^2 \varphi : \odot^2 TM \longrightarrow \odot^2(\odot^2 S)$. Since $\dim S = 2$, the bundle $(\wedge^2 S^*)^{\otimes^2}$ has rank 1. Hence, its image in $\odot^2 \Omega^1 M$ under the inverse map,

$$(\wedge^2 S^*)^{\otimes^2} \longrightarrow \odot^2 \Omega^1 M,$$

is a line bundle. This is precisely an equivalence class (conformal) metrics $[g_{ab}]$. It is easy to check that $[g_{ab}]$ are non-degenerate.

3.1.3 A Canonical 2-conic Structure

Let M be a complex 3-manifold equipped with a conformal structure, in other words there is an isomorphism, $\varphi : TM \cong \odot^2 S$. Consider a relative projective line $F = \mathbb{P}(S)$, $\pi : F \longrightarrow M$. There is a canonical embedding $i : F \hookrightarrow \mathrm{Gr}_M(2; TM)$, which supplements F with a 2-conic structure. This embedding $i : F \hookrightarrow \mathrm{Gr}_M(2; TM)$ is uniquely neighbourhood by the condition that the pullback $i^*(S)$ is locally given by

$$i(\mathbb{S}) = \pi^*(S)(-1),$$

where $\mathbb{S}$ is the tautological bundle on $\mathrm{Gr}_M(2; TM)$.

A conic connection on F is a splitting of

$$0 \longrightarrow TF/M \longrightarrow T_cF \longrightarrow \mathcal{L} \longrightarrow 0,$$

where $\mathcal{L} = \pi^*(S)(-1)$.

Note that since fibres of π are projective lines, one has an isomorphism:

$$TF/M = \pi^*(\wedge^2 S)(2).$$

Hence, the coefficient sheaf of 2-conic connections given by

$$\pi_*(TF/M \otimes S_F^*) = \pi_*(\pi^*(\wedge^2 S) \otimes \pi^*(S^*)(3) \doteq \wedge^2 S \otimes S^* \otimes \odot^3 S^* = S \otimes \odot^3 S^*.$$

There is an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi^*(S) \longrightarrow TF/M(-1) \longrightarrow 0,$$

where $TF/M(-1) = \pi^*(\wedge^2 S)(1)$ and by $\mathcal{O}(-1)$ one means the tautological sheaf on $F = \mathbb{P}(S)$.

3.1.4 2-conic Structure in Local Coordinates

Let $\{x^a\}$ be the local coordinates on M, e_A denote a local frame of S, $A = 1, 2$. Thus, the isomorphism $\varphi : TM \cong \odot^2 S$ is given by:

$$\varphi(\partial_a) = \varphi_a^{AB}(x) e_A \odot e_B, \tag{3.1}$$

$$\varphi(e_A \odot e_B) = \varphi_{AB}^a \partial_a, \tag{3.2}$$

where functions φ_a^{AB} and φ_{AB}^a uniquely represent φ and its inverse φ^{-1}.

The embedding $i : F \longrightarrow \mathrm{Gr}_M(2; TM)$ is defined as follows:

$$[\pi^A] \longrightarrow \mathrm{span}[\pi^A \varphi_{AB}^a \partial_a].$$

A single point in F is a one dimensional subspace (line) in $S(x), x \in M$. The symmetric tensor product of this line and $S(x)$ is in $\odot^2 S(x) = TM(x)$ and 2-dimensional.

This realizes a point in F as a 2-plane in TM and, hence, determines a 2-conic structure.

Let $[\pi^A] = [\pi^0, \pi^1]$ be the homogeneous coordinates on $\mathbb{P}^1$. Explicitly, let $p \in S$. Then $p = \sum \pi_p^A e_A$ for some π^A. Now $[\pi_p^A]$ form homogeneous coordinates on $\mathbb{P}(S)$. Thus $[p] \in \mathbb{P}(S)$.

$$\pi([p]) = \{x_0^1, \dots, x_0^3\} \in M,$$

$$\pi^{-1}(x) = \mathbb{P}(S_p) \simeq \mathbb{P}^1.$$

So $[p]$ is determined by the numeric data: $(x_0^1, x_0^2, x_0^3, [\pi^0, \pi^1])$, where $[\pi^A]$.

$$[\pi^A] = [\pi^0, \pi^1] = [1, \pi^0/\pi^1] \cup [\pi^1/\pi^0, 1].$$

So, $p \in \mathbb{P}_M(S)$ can be described (up to multiplication) as

$$p = \sum \pi^A e_A.$$

The realization of p as a 2-plane, i.e.the inclusion

$$p \overset{i}{\hookrightarrow} \mathrm{Gr}(2, TM),$$

is then described as follows:

$$p \longrightarrow S \odot p \longrightarrow \mathrm{span}(\sum \pi^A e_A \odot e_B),$$

$$i(p) = \mathrm{span}[\varphi^{-1}(\pi^A e_A \odot e_B)] = \mathrm{span}[\pi^A \varphi^{-1}(e_A \odot e_B)] = \mathrm{span}(\pi^A \varphi^a_{AB} \partial_a),$$

where $A = 1, 2$.

So, $i(p) = \mathrm{span}\{\alpha_1, \alpha_2\}$, where

$$\alpha_A = \sum \pi^B \varphi^a_{AB} \partial_a.$$

In a specific neighbourhood $\mathcal{U} = \{\pi^0 \neq 0\}$,

$$\alpha_A = [\varphi^a_{A0} + \frac{\pi^1}{\pi^0} \varphi^a_{A1}] \partial_a.$$

Note, that $D_{(AB)} = \sum \varphi^a_{AB} \partial_a$ form basis in TM, so α_A are linearly independent. Therefore, $i(p)$ is a 2-dimensional subspace of TM, $i(p) \hookrightarrow \mathrm{G}(2, TM)$.

3.1.5 2-Conic Connection

The projection $\pi : F \longrightarrow M$ gives us the exact sequence

$$0 \longrightarrow TF/M \longrightarrow TF \longrightarrow \pi^*(TM) \longrightarrow 0,$$

where TF/M is the sheaf of vertical vector fields and $\pi^*(TM) = \pi^*(\odot^2 S)$. A projective connection on $F = \mathbb{P}(S)$ is a splitting of this extension. There is a canonical sequence:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi^*(S) \longrightarrow TF/M(-1) \longrightarrow 0. \tag{3.3}$$

Since the fibres of π are $\mathbb{CP}^1$, one can deduce that the sheaf of π-vertical vector-fields, TF/M, is isomorphic to $\pi^*(\wedge^2 S)(2)$. From this and (3.3):

$$0 \longrightarrow \pi^*(S)(-1) \xrightarrow{j^*} \pi^*(\odot^2 S) \longrightarrow \pi^*(\wedge^2 S)^{\otimes^2}(2) \longrightarrow 0.$$

The map j defines a subsheaf

$$T_c F \hookrightarrow TF$$

via the following commutative diagram:

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \pi^*(\wedge^2 S)^{\otimes 2}(2) \arrow[u] \arrow[r, equal] & \pi^*(\wedge^2 S)^{\otimes 2}(2) \arrow[u] & \\
0 \arrow[r] & TF/M \arrow[r] & TF \arrow[u] \arrow[r] & \pi^*(TM) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & TF/M \arrow[u, "="] \arrow[r] & T_c F \arrow[u] \arrow[r] & \pi^*(S)(-1) \arrow[u, "j^*"'] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

3.1.6 Conic vs Projective Connections

The main goal of this section is to understand a 2-conic connection in terms of projective line connections on S. The question to answer is which of this 2-conic connections come from the projective connections.

Consider the following exact sequence

$$0 \longrightarrow \pi^*(S)(-1) \longrightarrow \pi^*(\odot^2 S) \longrightarrow \pi^*(\wedge^2 S)^{\otimes 2}(2) \longrightarrow 0$$

or

$$0 \longrightarrow \pi^*(\wedge^2 S^*)^{\otimes 2}(-2) \longrightarrow \pi^*(S^*) \longrightarrow S_F^* = \pi^*(S^*)(1) \longrightarrow 0$$

or

$$0 \longrightarrow TF/M \otimes \pi^*(\wedge^2 S^*)^{\otimes 2}(-2) \longrightarrow TF/M \otimes \pi^*(\Omega^1 M)$$
$$\xrightarrow{\alpha} TF/M \otimes \pi^*(\mathcal{L}_F^*) \longrightarrow 0$$

Note that $TF/M \otimes \pi^*(\wedge^2 S^*)^{\otimes^2}(-2) = \pi^*(\wedge^2 S^*)$.

This exact sequence relates the coefficient sheaf, $TF/M \otimes \pi^*(\Omega^1 M)$, for projective connections on $\mathbb{P}(S) = F$, and the coefficient sheaf, $TF/M \otimes \pi^*(\mathcal{L}_F^*)$, for 2-conic connection on F.

Since $\pi_*^1(\pi^*(\wedge^2 S^*) = 0$, one has an exact sequence:

$$0 \longrightarrow \wedge^2 S^* \longrightarrow (S \otimes S^*)_0 \otimes \odot^2 S^* \xrightarrow{pr} \wedge^2 S \otimes S^* \otimes \odot^3 S^* \longrightarrow 0.$$

Let us note that

$$(S \otimes S^*)_0 \otimes \odot^2 S^* = \wedge^2 S \otimes \odot^2 S^* \otimes \odot^2 S^*, \tag{3.4}$$

which gives us the coefficients of projective connections. And also note that

$$\wedge^2 S \otimes S^* \otimes \odot^3 S^* \tag{3.5}$$

corresponds to the coefficients of the 2-conic connection.

It is clear that the dimension of (3.4) is equal to 9, while the dimension of (3.5) is 8. This implies that the map from projective spinor connections on S to 2-conic connection on F is *surjective.* Thus, the following theorem holds:

Theorem 3.1.1 *Any 2-conic connection on conformal 3-manifold M can be pulled back to a spinor connection. In other words, any such a connection is induced by a projective connection on S.*

3.1.7 Curvature of a 2-Conic Connection

Let $\mathcal{T} \subset TF$ be a 2-conic connection. Then $d\pi : TF \longrightarrow \pi^*(TM)$ and $d\pi : \mathcal{T} \longrightarrow \mathcal{L}_F = \pi^*(S)(1)$.

Curvature of this connection is a global section of $\wedge^2 \mathcal{L}_F^* \otimes TF/\mathcal{T}$, which fits in the exact sequence:

$$0 \longrightarrow TF/M \otimes \wedge^2 \mathcal{L}_F^* \longrightarrow \wedge^2 \mathcal{L}_F \otimes TF/\mathcal{T} \longrightarrow \wedge^2 \mathcal{L}_F \otimes \pi^*(\wedge^2 S)^{\otimes 2}(2) \longrightarrow 0.$$

Note that

$$TF/M \otimes \wedge^2 \mathcal{L}_F^* = \pi^*(\wedge^2(S)(2) \otimes \pi^*(\wedge^2(S^*))(2) = \mathcal{O}_F(4)$$

and

$$\pi^*(\wedge^2 S)^{\otimes 2}(2) = \pi^*(TM)/\mathcal{L}_F = \pi^*(\wedge^2(S^*))(2) \otimes \pi^*(\wedge^2(S^*))^{\otimes 2}(2).$$

Therefore, the sheaf of "curvature tensors" fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_F(4) \longrightarrow \wedge^2 \mathcal{L}_F^* \otimes TF/\mathcal{T} \longrightarrow \pi^*(\wedge^2(S)) \otimes \mathcal{O}_F(4) \longrightarrow 0,$$

which implies

$$0 \longrightarrow \odot^4 S^* \longrightarrow \pi_*^0(\wedge^2 \mathcal{L}_F^* \otimes TF/\mathcal{T}) \xrightarrow{\varphi_0} \wedge^2 S \otimes \odot^4 S^* \longrightarrow 0,$$

where $\pi_*^0(\wedge^2 \mathcal{L}_F^* \otimes TF/\mathcal{T})$ is a sheaf of 2-conic connection curvatures.

Hence, the obstruction for conic connection being integrable can be understood in two stages:

$$0 \longrightarrow \Phi_1(R^\nabla) \in \odot^4 S^* \longrightarrow R^\tau \longrightarrow \Phi_0(R^\nabla) \longrightarrow 0, \qquad (3.6)$$

$$\Phi_0(R^\tau) \in \wedge^2 S \otimes \odot^4 S^*.$$

If $\Phi_0(R^\tau) = 0$, the remaining obstruction will be, as follows from (3.6), an element $\Phi_1(R^\tau) \in \odot^4 S$.

Thus, the calculation above have proved the following:

Theorem 3.1.2 *The obstruction for the integrability of a distinguished 2-conic connection is the symmetric part of the Ricci tensor, W_{ABCD}.*

Let us note that the freedom of choice for a conic connection is the sheaf $\wedge^2 S \otimes S^* \odot^3 S^*$. Using some of this freedom one can make $\Phi_0(R^\nabla)$, which corresponds to torsion, disappear. But this condition, $\Phi_0(R^\nabla) = 0$ defines a whole class of 2-conic connection, *distinguished* connections, which differ from each other by the section of $\odot^2 S^* = \Omega^1 M$. In other words, this condition defines the family of Weyl-connection (torsion-free and conformal).

3.2 G_m-structures

Here the technique used for a conformal 3-manifold is applied to more general case.

3.2.1 Basic Definitions

Consider a complex manifold M, such that its tangent bundle TM is isomorphic to the m-th symmetric product of a rank 2 vector bundle S on M, i.e.

$$\varphi : TM \cong \odot^m S.$$

Note, that in this particular case $\dim M = m + 1$. If $m = 2$ this becomes a 3-dimensional conformal structure studied in previous section. If $m = 3$ this is a so called *exotic* G_3-structure, extensively studied in the context of holonomy problems. The case $m = 3$ shall be studied in more detail below/later.

This datum, $TM \cong \odot^m S$, gives a rise to a host of conic structures on M, which can be parameterised by an integer $k \in \{1, ..., m\}$.

If $\mathcal{U}_{m-k+1}$ is a tautological line bundle on $G_M(m-k+1, TM)$, then one canonically defines an embedding:

$$i_k : \mathbb{P}(S) \xrightarrow{i_k} G_M(m - k + 1, TM)$$

by the condition on the pullback,

$$i^*(\mathcal{U}_{m-k+1}) = \pi^*(\odot^{m-k}S)(-k).$$

Each such embedding i_k gives rise to an $m-k+1$-conical structure on the projective "spinor" bundle $\mathbb{P}(S)$.

3.2.2 m-conic connection

The purpose of this subsection is to understand m-conic ($k=1$) structure in more detail. In particular, one wants, if possible, to establish the relation between the m-conic connection on $\mathbb{P}(S)$ and a projective connection on the same space, $\mathbb{P}(S)$.

Theorem 3.2.1 *Let $\mathbb{P}(S) \in G(m, TM)$ be a m-conic structure on an $m+1$-dimensional manifold with an isomorphism $TM \cong \odot^m S$. Then*

1. *if $m = 3$ the sheaf of projective connection coefficients coincides with the coefficient sheaf of 3-conic connection,*

   $$\wedge^2 S \otimes \odot^3 S^* \otimes \odot^2 S^*,$$

   or, in other words, a 3-conic connection on F is equivalent to a projective connection on $\mathbb{P}(S)$.

2. *if $m \geq 4$ an m-conic connection is* not *equivalent to a projective connection on $\mathbb{P}(S)$; moreover, the obstruction for a conic connection is to be representable by a projective connection lies in*

   $$H^0(M, (\wedge^2 S^*)^{\otimes 2} \otimes \odot^{m-4} S^*).$$

Proof. Consider M with $TM = \odot^m S, \dim M = m+1$. Let $i : F = \mathbb{P}(S) \hookrightarrow \mathrm{Gr}_M(m; TM)$, be an m-conic structure. If $\mathbb{U}$ is a tautological bundle on $\mathrm{Gr}_M(m; TM)$, then

$$\mathcal{L}_F = i^*(\mathbb{U}) = \pi^*(\odot^{m-1}S)(-1).$$

Therefore, the coefficient sheaf of m-conic connection can be described as:

$$\pi_*(\mathcal{L}_F^* \otimes TF/M) = \pi_*(\pi^*(\odot^{m-1}S^*)(1) \otimes \pi^*(\wedge^2 S)(2)) = \wedge^2 S \otimes \odot^{m-1}S^* \otimes \odot^3 S^*.$$

Since

$$0 \longrightarrow \pi^*(\wedge^2 S^*)(-1) \longrightarrow \pi^*(S^*) \longrightarrow \mathcal{O}(1) \longrightarrow 0,$$

one has

$$0 \longrightarrow \pi^*(\wedge^2 S^*)^{\otimes m}(-m) \longrightarrow \pi^*(\odot^m S^*) \longrightarrow \pi^*(\odot^{m-1}S^*)(1) \longrightarrow 0$$

and hence

$$0 \longrightarrow \pi^*(\wedge^2 S^*)^{\otimes^{m-1}}(-m+2) \longrightarrow TF/M \otimes \pi^*(\Omega^1 M)$$
$$\longrightarrow TF/M \otimes \mathcal{L}_F^* \longrightarrow 0,$$

where $TF/M \otimes \pi^*(\Omega^1 M) = \pi^*(\wedge^2 S \otimes \odot^m S^*)(2)$ is the coefficient sheaf for projective connection, and $TF/M \otimes \mathcal{L}_F^* = \pi^*(\wedge^2 S \otimes \odot^{m-1}S^*)(3)$ is the sheaf of m-conic connection coefficients. The associated long exact sequence degenerates into

$$0 \longrightarrow \{\text{coefficient sheaf of proj. connection}\} \longrightarrow$$

$$\longrightarrow \{\text{coefficient sheaf of conic connection}\} \longrightarrow \odot^{m-4}S \otimes \wedge^2 S^{\otimes(m-2)} \longrightarrow \ldots,$$

where we used isomorphism

$$\pi_*^1(\wedge^2 S^{\otimes(m-1)}(-m+2)) = \odot^{m-4}S \otimes \wedge^2 S^{\otimes(m-2)}.$$

Finally, the isomorphism

$$\odot^{m-4}S \otimes \wedge^2 S^{\otimes(m-2)} = (\wedge^2 S^*)^{\otimes 2} \otimes \odot^{m-4}S^*$$

completes the proof.□

As in the previous case, G_2, the obstruction to integrability of m-conic connection can be described by two steps. The first obstruction is an element $\Phi_0(R^\tau) \in H^0(F, \pi^*(TM)/\mathcal{L}_F \otimes \wedge^2 \mathcal{L}_F^*)$, i.e.

$$\Phi_0(R^\tau) \in \Gamma(M, \pi^*(\wedge^2 S)^{\otimes m}(m) \otimes \wedge^2 \pi^*(\odot^{m-1}S^*)(2)),$$

i.e.

$$\Phi_0(R^\tau) \in (\wedge^2 S)^m \otimes \wedge^2(\odot^{m-1} S^*) \otimes \odot^{m+2} S^*,$$

In other words, the first obstruction lies in a " torsion " part.

3.3 G_3 structure

Let $\mathbb{S}$ be a standard 2-dimensional representation of space $\mathrm{GL}(2,\mathbb{C})$. Then $\mathrm{GL}(2,\mathbb{C})$ naturally acts on symmetric tensor product $\odot^3\mathbb{S}$. If $\rho : \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(4,\mathbb{C})$ is the associated representation, we can define a subgroup $G_3 = \rho(\mathrm{GL}(2,\mathbb{C}))$ of $\mathrm{GL}(4,\mathbb{C})$. Let M be a complex 4-manifold and $\pi : L^*M \longrightarrow M$ the holomorphic coframe bundle whose fibres $L_t^* = \pi^{-1}(t)$ consist of all $\mathbb{C}$-linear isomorphisms $e : \mathbb{C}^4 \longrightarrow \Omega_t^1 M$. The space L^*M is naturally a principal right $\mathrm{GL}(4,\mathbb{C})$-bundle, where the right action $R_g : L^*M \longrightarrow L^*M$ is given by $R_g(e) = e \circ g$. A *G_3-structure* on M is, by definition, a principal subbundle of L^*M with th group G_3. It is clear that G_3-structure is equivalent to a local factorisation of the tangent bundle into symmetric cube

$$TM = \odot^3 S$$

of locally defined vector bundle S of rank 2. Though such a vector bundle may fail to exist on the whole of M, the projectivised vector bundle $\mathbb{P}_M(S)$ is well-defined *globally*. These G_3 structures have been very popular recently in connection with the holonomy problem (cf. works by Bryant [Bry91] and Hugget and Merkulov [HM99].

Let us assume that M is a complex manifold with G_3-structure such that S exists on the whole of M; it is called a *spinor* bundle on M. A linear connection ∇ on S is called a *spinor connection* on M. Any spinor connection on M induces, via isomorphism $TM = \odot^3 S$, an affine connection with holonomy in G_3; moreover, any affine connection on M with holonomy in G_3 arises in this way, at least locally. By a torsion tensor of a spinor connection we mean the torsion tensor of the associated affine connection.

Chapter 4

Quaternionic Structures

In this Chapter the notion of a quaternionic structure is reminded and connections on quaternionic structures are investigated. Using the torsion and curvature tensors some of the invariants of quaternionic structures are established. The results of this chapter are to be used in Chapters 5 and 6.

4.1 Quaternionic Structures

In this section different ways of defining quaternionic and almost quaternionic structures are considered. Although both concepts were mentioned in the Section 0.1 and defined via the notions of holonomy and structure groups, it is useful to consider an equivalent definition, arising from the spinor structure of the tangent space of underlying manifold. It gives us almost direct access to the calculations of corresponding coefficient sheaves in local coordinates.

4.1.1 Definition

Let M be a $2k$-dimensional complex manifold, where $k > 1$.

Definition 4.1.1 *A* manifold with almost quaternionic structure *is a four-tuple* $(M, S, \hat{S}, \phi)$, *in which* M *is a complex manifold,* S *and* $\hat{S}$ *are locally free sheaves on* M *of ranks*

$k > 1$ and 2, respectively, and

$$\varphi : TM \longrightarrow S \otimes \hat{S}$$

is an isomorphism. [1]

4.1.2 Equivalent definition

This definition can be paraphrased as a reduction of the structure group of manifold M. Consider $2k$-dimensional vector space $\mathbb{C}^{2k} = \mathbb{C}^k \otimes \mathbb{C}^2$. Then $\mathrm{GL}(k, \mathbb{C})$ acts in the usual way on $\mathbb{C}^k$, and let $\mathrm{GL}(2, \mathbb{C})$ act on the right, by inverses, on $\mathbb{C}^2$. The combined induced action on $\mathbb{C}^{2k}$ will generate a subgroup $\mathrm{PC}\{k, 2\} \subset \mathrm{GL}(2k, \mathbb{C})$. Further, there will be a $(k+2)$-fold cover

$$S(\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})) \longrightarrow \mathrm{PC}\{k, 2\} \subset \mathrm{GL}(2k, \mathbb{C}),$$

where the ' S ' means that 2×2 and $k \times k$ matrices which follow are to have determinants, whose product is 1. In other words, $S(\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C}))$ is the subgroup of $\mathrm{SL}(p + q, \mathbb{C})$ consisting of matrices of the form:

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

The action on $\mathbb{C}^{2k}$ is exactly the adjoint action on matrices of the form

$$\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}.$$

A quaternionic structure as defined above is a reduction of the general $GL(2k, \mathbb{C})$-structure group to $\mathrm{PC}\{k, 2\}$, together with the lifting to the left-hand group above. In the case of four dimensions, the reduction is equivalent to a conformal structure,since

$$\mathrm{CO}(4, \mathbb{C}) \simeq \mathrm{GL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}),$$

while the lifting is connected with the existence of spin structure.

[1]There is another definition of an almost quaternionic structure, see [BE91], where one can also ask for a fixed isomorphism $\alpha : \wedge^k S \longrightarrow \wedge^2 \hat{S}$.

4.1.3 Torsion of an almost quaternionic structure

Let M be a complex $2k$-dimensional manifold equipped with an almost quaternionic structure. Choose any linear connections, $\nabla_0, \tilde{\nabla}_0$, on the bundles S and $\hat{S}$:

$$\nabla_0 : S \longrightarrow S \otimes \Omega^1 M,$$

$$\tilde{\nabla}_0 : \hat{S} \longrightarrow \hat{S} \otimes \Omega^1 M.$$

Consider the induced tensor product connection on $TM, \nabla = \nabla_0 \otimes \tilde{\nabla}_0$. Locally, the isomorphism $\varphi : TM \longrightarrow S \otimes \hat{S}$ and its inverse $\varphi^{-1} : S \otimes \hat{S} \longrightarrow TM$ are given by

$$\varphi(\partial_a) = e_a^{A\dot{A}}(x) e_A \otimes e_{\dot{A}},$$

$$\varphi^{-1}(e_A \otimes e_{\dot{A}}) = e^a_{A\dot{A}} \partial_a,$$

where $\partial_a = \frac{\partial}{\partial x^a}$, $e_A, e_{\dot{A}}$ are local frames in $S, \hat{S}$ respectively and $e_a^{A\dot{A}}, e^{-1}_{A\dot{A}}$ are holomorphic functions.

Let $\nabla_0 : S \longrightarrow S \otimes \Omega^1 M$ be represented by functions Γ^A_{aB}:

$$\nabla_{0_{\partial_a}} e_A = -\Gamma^B_{aA}(x) e_B.$$

Similarly, let $\tilde{\nabla}_0 : \hat{S} \longrightarrow \hat{S} \otimes \Omega^1 M$ be represented by $\hat{\Gamma}^{\dot{B}}_{a\dot{A}}$.

The affine connection $\nabla : TM \longrightarrow TM \otimes \Omega^1 M$ induced by ∇_0 and $\tilde{\nabla}_0$ is described by following functions:

$$\begin{aligned}
\nabla_{\partial_a} \partial_b &:= \varphi^{-1}[(\nabla_0 \otimes \tilde{\nabla}_0)_{\partial_a} \varphi(\partial_b)] \\
&= \varphi^{-1}[\partial_a e_b^{A\dot{A}} e_A \otimes e_{\dot{A}} + e_b^{A\dot{A}} (\nabla_{\partial_a} e_A) \otimes e_{\dot{A}} + e_b^{A\dot{A}} e_A \otimes (\nabla_{\partial a} e_{\dot{A}})] \\
&= (\partial_a e_b^{A\dot{A}} e^d_{A\dot{A}} \partial_d - e_b^{A\dot{A}} \Gamma^B_{aA} e^d_{B\dot{A}} \partial_d - e_b^{A\dot{A}} \hat{\Gamma}^{\dot{B}}_{a\dot{A}} e^d_{A\dot{B}} \partial_d).
\end{aligned}$$

So, the coordinate symbols of the induced connection are

$$\Gamma^d_{ab} = (\partial_a e_b^{A\dot{A}}) e^d_{A\dot{A}} - e_b^{A\dot{A}} [\Gamma^B_{aA} e^d_{B\dot{A}} + \hat{\Gamma}^{\dot{B}}_{a\dot{A}} e^d_{A\dot{B}}]. \tag{4.1}$$

This connection on TM will generally have *torsion*, that is given a scalar field f on M, for which the following is valid:

$$2 \nabla_{[a} \nabla_{b]} f = T^c_{ab} \nabla_c f$$

for some tensor $T^c_{ab} = -T^c_{ba}$.

Also, one can use φ to move torsion from TM to $S \otimes \hat{S}$, meaning:

$$TM \otimes \Omega^2 M \ni T^d_{ab} \xrightarrow{\varphi} T^{D\dot{D}}_{A\dot{A}B\dot{B}} \in S \otimes \hat{S} \otimes \wedge^2(S^* \otimes \hat{S}^*).$$

So,

$$T^d_{ab} \longrightarrow e^a_{A\dot{A}} e^b_{B\dot{B}} e^c_{C\dot{C}} T^c_{ab} = T^{C\dot{C}}_{A\dot{A}B\dot{B}}.$$

Since, $\wedge^2(S^* \otimes \hat{S}^*) = \odot^2 S^* \otimes \wedge^2 \hat{S}^* \bigoplus \wedge^2 S^* \otimes \odot^2 \hat{S}^*$, this skew tensor can be decomposed into sum of two terms

$$T^{C\dot{C}}_{A\dot{A}B\dot{B}} = F^{C\dot{C}}_{ABA\dot{B}} + \tilde{F}^{C\dot{C}}_{\dot{A}\dot{B}AB},$$

where there are the symmetries

$$F^{C\dot{C}}_{AB\dot{A}\dot{B}} = F^{C\dot{C}}_{(AB)[\dot{A}\dot{B}]},$$

$$\hat{F}^{C\dot{C}}_{\dot{A}\dot{B}AB} = \tilde{F}^{C\dot{C}}_{(\dot{A}\dot{B})[AB]}.$$

The following theorem is due to Bailey and Eastwood [BE91].

Theorem 4.1.1 *The totally trace-free parts of $F^{C\dot{C}}_{AB\dot{A}\dot{B}}$ and $\hat{F}^{C\dot{C}}_{\dot{A}\dot{B}AB}$ are independent of the original choice of connections, and are, hence, invariants of the almost quaternionic structure.*

Let us note, that $F^{C\dot{C}}_{AB\dot{A}\dot{B}}$ is 'totally trace-free' if all possible traces on the upper indices, e.g. $F^{A\dot{C}}_{AB\dot{A}\dot{B}}$, vanish.

Proof. By definition:

$$T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}.$$

Using the formulae (4.1) $T^{C\dot{C}}_{A\dot{A}B\dot{B}}$ can be calculated as follows:

$$\begin{aligned}
T^{C\dot{C}}_{A\dot{A}B\dot{B}} &= e^a_{A\dot{A}}e^b_{B\dot{B}}e^c_{C\dot{C}}\{((\partial_a e^{D\dot{D}}_b)e^c_{D\dot{D}} - (\partial_b e^{D\dot{D}}_a)e^c_{D\dot{D}}) \\
&\quad -e^{D\dot{D}}_b(\Gamma^R_{aD}e^c_{R\dot{D}} + \hat{\Gamma}^{\dot{S}}_{a\dot{D}}e^c_{D\dot{S}}) + e^{D\dot{D}}_a(\Gamma^R_{bD}e^c_{R\dot{D}} + \hat{\Gamma}^{\dot{S}}_{b\dot{D}}e^c_{D\dot{S}})\} \\
&= e^a_{A\dot{A}}e^b_{B\dot{B}}(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) \\
&\quad -\delta^D_B\delta^{\dot{D}}_{\dot{B}}(\Gamma^R_{aD}e^a_{A\dot{A}}\delta^C_R\delta^{\dot{C}}_{ddD} + \hat{\Gamma}^{\dot{S}}_{a\dot{D}}e^a_{A\dot{A}}\delta^C_D\delta^{\dot{C}}_{\dot{S}}) \\
&\quad +\delta^D_A\delta^{\dot{D}}_{\dot{A}}(\Gamma^R_{bD}e^b_{B\dot{B}}\delta^C_R\delta^{\dot{C}}_{ddD} + \hat{\Gamma}^{\dot{S}}_{b\dot{D}}e^b_{B\dot{B}}\delta^C_D\delta^{\dot{C}}_{\dot{S}}) \\
&= I^{C\dot{C}}_{A\dot{A}B\dot{B}} - \Gamma^C_{aB}e^a_{A\dot{A}}\delta^{\dot{C}}_{\dot{B}} - \hat{\Gamma}^{\dot{C}}_{a\dot{B}}e^a_{A\dot{A}}\delta^C_B \\
&\quad +\Gamma^C_{bA}e^b_{B\dot{B}}\delta^{\dot{C}}_{\dot{A}} + \hat{\Gamma}^{\dot{C}}_{b\dot{A}}e^b_{B\dot{B}}\delta^C_A,
\end{aligned}$$

where $I^{C\dot{C}}_{A\dot{A}B\dot{B}} = e^a_{A\dot{A}}e^b_{B\dot{B}}(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)$.

Further on, it can be easily checked, that

$$\begin{aligned}
2F^{C\dot{C}}_{A\dot{A}B\dot{B}} &= 2F^{C\dot{C}}_{(AB)[\dot{A}\dot{B}]} \\
&= e^a_{A\dot{A}}e^b_{B\dot{B}}(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) + e^a_{B\dot{A}}e^b_{A\dot{B}}(\partial_a e^{C\dot{C}}_B - \partial_b e^{C\dot{C}}_a) \\
&\quad +\Gamma^C_{cA}(\delta^{\dot{C}}_{\dot{A}}e^c_{B\dot{B}} - \delta^{\dot{C}}_{\dot{B}}e^c_{B\dot{A}}) + \Gamma^C_{cB}(\delta^{\dot{C}}_{\dot{A}}e^c_{A\dot{B}} - \delta^{\dot{C}}_{\dot{B}}e^c_{A\dot{A}}) \\
&\quad +\hat{\Gamma}^{\dot{C}}_{c\dot{A}}(\delta^C_B e^c_{A\dot{B}} + \delta^C_A e^c_{B\dot{B}}) - \hat{\Gamma}^{\dot{C}}_{c\dot{B}}(\delta^C_B e^c_{A\dot{A}} + \delta^C_A e^c_{B\dot{A}})
\end{aligned}$$

and

$$\begin{aligned}
2\hat{F}^{C\dot{C}}_{A\dot{A}B\dot{B}} &= 2\hat{F}^{C\dot{C}}_{[AB](\dot{A}\dot{B})} \\
&= e^a_{A\dot{A}}e^b_{B\dot{B}}(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) - e^a_{B\dot{A}}e^b_{A\dot{B}}(\partial_a e^{C\dot{C}}_B - \partial_b e^{C\dot{C}}_a) \\
&\quad +\Gamma^C_{cA}(\delta^{\dot{C}}_{\dot{B}}e^c_{B\dot{A}} + \delta^{\dot{C}}_{\dot{A}}e^c_{B\dot{B}}) - \Gamma^C_{cB}(\delta^{\dot{C}}_{\dot{B}}e^c_{A\dot{A}} + \delta^{\dot{C}}_{\dot{A}}e^c_{A\dot{B}}) \\
&\quad +\hat{\Gamma}^{\dot{C}}_{c\dot{B}}(\delta^C_A e^c_{B\dot{A}} - \delta^C_B e^c_{A\dot{A}}) + \hat{\Gamma}^{\dot{C}}_{c\dot{A}}(\delta^C_A e^c_{B\dot{B}} - \delta^C_B e^c_{A\dot{B}}).
\end{aligned}$$

Using the symmetric/anti-symmetric notation, this can be rewritten as

$$2F^{C\dot{C}}_{A\dot{A}B\dot{B}} = (\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)e^a_{(A\dot{A}}e^b_{B)\dot{B}} - \Gamma^C_{c(A}e^c_{B)[\dot{A}}\delta^{\dot{C}}_{\dot{B}]} + \hat{\Gamma}^{\dot{C}}_{c[\dot{A}}e^c_{(A\dot{B}]}\delta^C_{B)}, \tag{4.2}$$

$$2\hat{F}^{C\dot{C}}_{A\dot{A}B\dot{B}} = (\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)e^a_{[A\dot{A}}e^b_{B]\dot{B}} + \Gamma^C_{c[A}e^c_{B](\dot{A}}\delta^{\dot{C}}_{\dot{B})} - \hat{\Gamma}^{\dot{C}}_{c(\dot{A}}e^c_{[A\dot{B})}\delta^C_{B]}, \tag{4.3}$$

where brackets and square brackets denote respectively symmetrisation and anti-symmetrisation on the corresponding indices.

According to formulas (4.2) and (4.3), the trace-free parts of $F^{C\dot{C}}_{A\dot{A}B\dot{B}}$ and $\hat{F}^{C\dot{C}}_{A\dot{A}B\dot{B}}$ would be the trace-free parts of

$$(e^a_{A\dot{A}}e^b_{B\dot{B}} + e^a_{B\dot{A}}e^b_{A\dot{B}})(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)$$

and

$$(e^a_{A\dot{A}}e^b_{B\dot{B}} - e^a_{B\dot{A}}e^b_{A\dot{B}})(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)$$

which are independent of the original connections.

Any change in the original connections can be written as $\tilde{\Gamma}^C_{aB} = \Gamma^C_{aB} + \Omega^C_{aB}$, where Ω^C_{aB} is an arbitrary tensor and coefficients with tilde represent coefficients after the coordinate change. Then, as can be seen from the formulas (4.2) and (4.3) above, the difference between $F^{C\dot{C}}_{A\dot{A}B\dot{B}}$ and $\tilde{F}^{C\dot{C}}_{A\dot{A}B\dot{B}}$ will lie in the trace parts, i.e. the change in connection will affect only the trace parts of torsion.

Therefore, the trace-free part of $F^{C\dot{C}}_{A\dot{A}B\dot{B}}$ is independent of the original choice of connections. □

Lemma 1 *Given an almost quaternionic manifold $(M, S, \hat{S}, \varphi)$, there exists a pair of connections on S and S^* such that the induced affine connection is torsion-free.*

Proof. Easily follows from the above calculations.

Definition 4.1.2 A scale *on an almost quaternionic quadruple $(M, S, \hat{S}, \varphi)$ is a choice of a pair $(\varepsilon, \hat{\varepsilon})$ of nowhere vanishing sections:*

$$\varepsilon = \Gamma(M, \wedge^k S^*), \hat{\varepsilon} = \Gamma(M, \wedge^2 \hat{S}^*).$$

The following theorem is due to Bailey and Eastwood, [BE91].

Theorem 4.1.2 *Given a scale on $(M, S, \hat{S}, \varphi)$, there are unique spinor connections:*

$$\nabla_0 : S \longrightarrow S \otimes \Omega^1 M,$$

$$\hat{\nabla}_0 : \hat{S} \longrightarrow \hat{S} \otimes \Omega^1 M.$$

such that

$$\nabla_0(\varepsilon) = \hat{\nabla}_0(\hat{\varepsilon}) = 0$$

and the torsion tensors, $F^{C\dot{C}}_{AB\dot{A}\dot{B}}$ *and* $\hat{F}^{C\dot{C}}_{AB\dot{A}\dot{B}}$*, of the induced affine connection are totally trace-free.*

Definition 4.1.3 A quaternionic structure *on a $2k$-dimensional complex manifold N is an almost quaternionic quadruple $(M, S, \hat{S}, \varphi)$, such that its invariant trace-free parts of* $F^{C\dot{C}}_{AB\dot{A}\dot{B}}$ *and* $\hat{F}^{C\dot{C}}_{AB\dot{A}\dot{B}}$ *are zero.*

4.1.4 Torsion and curvature tensors

Torsion

In local coordinates the torsion tensor can be expressed as follows

$$T^c_{bd} = (\partial_{[b} e^{A\dot{A}}_{d]} - \Gamma^A_{[bB} e^{B\dot{A}}_{d]} - \Gamma^{\dot{A}}_{[b\dot{B}} e^{A\dot{B}}_{d]}) e^c_{A\dot{A}}. \tag{4.4}$$

Using formulas (4.2) and (4.3) and the fact that

$$T^{C\dot{C}}_{A\dot{A}B\dot{B}} = F^{C\dot{C}}_{A\dot{A}B\dot{B}} + \tilde{F}^{C\dot{C}}_{A\dot{A}B\dot{B}},$$

torsion can be rewritten as

$$\begin{aligned} T^{C\dot{C}}_{A\dot{A}B\dot{B}} &= e^a_{A\dot{A}} e^b_{B\dot{B}} (\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) \\ &\quad - \Gamma^C_{A\dot{A}B} \delta^{\dot{C}}_{\dot{B}} - \Gamma^{\dot{C}}_{A\dot{A}\dot{B}} \delta^C_B + \Gamma^C_{B\dot{B}A} \delta^{\dot{C}}_{\dot{A}} + \Gamma^{\dot{C}}_{B\dot{B}\dot{A}} \delta^C_A \end{aligned} \tag{4.5}$$

Trace free parts of torsion

$$2F^{C\dot{C}}_{(AB)[\dot{A}\dot{B}]} = (\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) e^a_{(A[\dot{A}} e^b_{B)\dot{B}]} - \Gamma^C_{c(A} e^c_{B)[\dot{A}} \delta^{\dot{C}}_{\dot{B}]} - \tilde{\Gamma}^{\dot{C}}_{c[\dot{A}} e^c_{(A\dot{B}]} \delta^C_{B)}$$

$$2\tilde{F}^{C\dot{C}}_{[AB](\dot{A}\dot{B})} = (\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) e^a_{[A(\dot{A}} e^b_{B]\dot{B})} + \Gamma^C_{c[A} e^c_{B](\dot{A}} \delta^{\dot{C}}_{\dot{B})} - \tilde{\Gamma}^{\dot{C}}_{c(\dot{A}} e^c_{[A\dot{B})} \delta^C_{B]}$$

Trace free part of $F^{C\dot{C}}_{(AB)[\dot{A}\dot{B}]}$ is

$$\mathrm{Tr}_0(F^{C\dot{C}}_{(AB)[\dot{A}\dot{B}]}) = \mathrm{Tr}_0((\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) e^a_{[A(\dot{A}} e^b_{B]\dot{B})}) = 0,$$

since rank $\hat{S} = 2$. While

$$\mathrm{Tr}_0(\tilde{F}^{C\dot{C}}_{[AB](\dot{A}\dot{B})}) = \mathrm{Tr}_0((\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a) e^a_{[A(\dot{A}} e^b_{B]\dot{B})}). \tag{4.6}$$

4.1.5 The flat model

Let $\mathbf{T}$ be a $k+2$ dimensional complex vector space, together with a chosen volume form $e \in \wedge^{k+2}(\mathbf{T})$. Then $M = \mathrm{Gr}(2, \mathbf{T})$, the Grassmanian of 2-dimensional subspaces of $\mathbf{T}$, has a natural almost quaternionic structure which can be constructed as follows.

There is the trivial bundle $\mathcal{T} = \mathbf{T} \times M$ over M. Denote by $\hat{S}$ the tautological sub-bundle of $\mathcal{T}$, whose fibre is the subspace of $\mathbf{T}$ defined by the point in the base. One can now define the bundle S^A on M by the exactness of

$$0 \longrightarrow \hat{S} \longrightarrow \mathcal{T} \longrightarrow S \longrightarrow 0.$$

It is a well known fact that the tangent bundle $\mathcal{T}$ of a Grassmanian is canonically isomorphic to the bundle of homomorphisms from the tautological bundle to its complement, so it results in

$$\mathcal{T} = S^* \otimes \hat{S}.$$

Therefore, $Gr(2, T)$ has a canonical almost quaternionic structure.

Alternatively, one also can define M as the homogeneous space $\mathrm{SL}(\mathbf{T})/\mathrm{P}$ where P is the subgroup consisting of matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

By general arguments, the tangent bundle to any homogeneous space G/P is the homogeneous bundle induced from the Adjoint representation of P on $\mathbf{g}/\mathbf{p}$ where g and $\mathbf{p}$ are the Lie algebras of G and P respectively. This directly gives the quaternionic reduction of structure group discussed in Section 4.1.2.

It is now possible to set up a flat space twistor correspondence. There is a double fibration on the *flag manifold* $F = F(1, 2, \mathbf{T})$, the parameter space of one dimensional subspaces inside two dimensional subspaces of the $k+2$ dimensional complex vector space $\mathbf{T}$:

$$\begin{tikzcd} & F(1;2|T) \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ Y = \mathbb{P}(T) & & M = Gr(2,T), \end{tikzcd}$$

where P, the *twistor space* of M, is the projective space of $\mathbf{T}$, and the maps 'forget' the appropriate subspace.

4.1.6 The Curvature of an induced affine connection

Thinking behind the formulas and calculations in this Chapter is largely based on a book by W. Rindler and R. Penrose [PR84] and works of T. Bailey and M. Eastwood [BE91].

Let $(M, S, \hat{S}, \varphi)$ be an almost quaternionic manifold. And let $\nabla_0, \hat{\nabla}_0$ be spinor connections on S and $\hat{S}$, respectively, and also let ∇ be the associated induced affine connection. Its curvature tensor, in local coordinates, is given by

$$2\,\nabla_{\partial_{[a}}\nabla_{\partial_{b]}}\partial_c - T^d_{ab}\nabla_{\partial_d}\partial_c = R^d_{abc}\partial_d.$$

There is an associated "spinor" curvature:

$$R_{A\dot{A}B\dot{B}C\dot{C}}^{D\dot{D}} = \varphi^a_{A\dot{A}}\varphi^b_{B\dot{B}}\varphi^c_{C\dot{C}}\varphi^{D\dot{D}}_d R^d_{abc}.$$

Since the holonomy algebra of the affine connection ∇ lies in $sl(2,\mathbb{C}) \oplus gl(k,\mathbb{C})$, the spinor can be decomposed as a direct sum as follows

$$\begin{aligned} &R_{A\dot{A}B\dot{B}C\dot{C}}^{D\dot{D}} = \\ &[X^D_{[\dot{A}\dot{B}](AB)C} + H^D_{(\dot{A}\dot{B})[AB]C}]\delta^{\dot{D}}_{\dot{C}} \\ &+[\tilde{H}^{\dot{D}}_{(AB)[\dot{A}\dot{B}]\dot{C}} + \tilde{X}^{\dot{D}}_{[AB](\dot{A}\dot{B})\dot{C}}]\delta^D_C. \end{aligned}$$

Let us choose a scale, $\varepsilon = \varepsilon_{[A...B]}, \hat{\varepsilon} = \hat{\varepsilon}_{\dot{A}\dot{B}}$, and assume that the affine connection ∇ preserves them, so that

$$\nabla\varepsilon = 0, \nabla\hat{\varepsilon} = 0.$$

Then the its holonomy algebra of this affine connection lies in $sl(2,\mathbb{C}) \oplus sl(k,\mathbb{C})$. Hence

$$X^{D}_{[\dot{A}\dot{B}](AB)C} = \hat{\varepsilon}_{\dot{A}\dot{B}} X^{D}_{(AB)C},$$
$$\tilde{H}^{\dot{D}}_{(AB)[\dot{A}\dot{B}]\dot{C}} = \hat{\varepsilon}_{\dot{A}\dot{B}} \tilde{H}^{\dot{D}}_{(AB)\dot{C}},$$

for some tensors $X, \tilde{H}$.

With this in mind, one can write down some symmetries of the curvature quantities:

$$X^{D}_{\dot{A}\dot{B}ABC} = X^{D}_{[\dot{A}\dot{B}](AB)C}, \quad X^{C}_{\dot{A}\dot{B}ABC} = 0,$$
$$H^{D}_{\dot{A}\dot{B}ABC} = H^{D}_{(\dot{A}\dot{B})[AB]C}, \quad H^{C}_{\dot{A}\dot{B}ABC} = 0,$$

where the left-hand equations are immediate consequences of the definitions, and the other two follow from $\nabla_{A\dot{A}}\epsilon_{B\ldots M} = 0$.

These two quantities can now be decomposed into irreducibles. The conventions are as follows:

$$X^{D}_{ABC} = \Psi^{D}_{ABC} + Y^{D}_{ABC} + S_{AB}\delta^{D}_{C} - kS_{C(A}\delta^{D}_{B)} - 2\Lambda_{C(A}\delta^{D}_{B)},$$

$$H^{D}_{\dot{A}\dot{B}ABC} = \Sigma^{D}_{\dot{A}\dot{B}ABC} + \Theta^{D}_{\dot{A}\dot{B}ABC} + A_{\dot{A}\dot{B}AB}\delta^{D}_{C} + kA_{\dot{A}\dot{B}C[A}\delta^{D}_{B]} - 2\Phi_{\dot{A}\dot{B}C[A}\delta^{D}_{B]},$$

where the quantities introduced above have the property that if they have a possible trace, then they are totally trace-free. This also implies the following:

$$\Psi^{D}_{ABC} = \Psi^{D}_{(ABC)}$$

$$Y^{D}_{ABC} = Y^{D}_{(AB)C}; Y^{D}_{(ABC)} = 0$$

$$S_{AB} = S_{(AB)}$$

$$\Lambda_{AB} = \Lambda_{[AB]}$$

$$\Sigma^{D}_{\dot{A}\dot{B}ABC} = \Sigma^{D}_{(\dot{A}\dot{B})[ABC]}$$

$$\Theta^{D}_{\dot{A}\dot{B}ABC} = \Theta^{D}_{(\dot{A}\dot{B})[AB]C}; \Theta^{D}_{\dot{A}\dot{B}[ABC]} = 0$$

$$A_{\dot{A}\dot{B}AB} = A_{(\dot{A}\dot{B})[AB]}$$

$$\Phi_{\dot{A}\dot{B}AB} = \Phi_{(\dot{A}\dot{B})(AB)}.$$

There is the Bianchi symmetry for the induced curvature on $\mathcal{T}$,

$$R^d_{[abc]} + \nabla_{[a} T^d_{bc]} + T^e_{[ab} T^d_{c]e} = 0,$$

which provides numerous relationships between the above curvature quantities and two irreducible parts F and $\hat{F}$ of the torsion.

Curvature

Recalling formula (4.4), let us denote

$$\Delta^c_{bd} = (\partial_b e^{A\dot{A}}_d - \Gamma^A_{bB} e^{B\dot{A}}_d - \Gamma^{\dot{A}}_{b\dot{B}} e^{A\dot{B}}_d) e^c_{A\dot{A}},$$

or

$$\Delta^c_{bd} = X^{A\dot{A}}_{bd} e^c_{A\dot{A}}.$$

Thus $\nabla_b e_d = \Delta^c_{bd} e_c$ and $T^c_{bd} = \Delta^c_{[bd]}$. It follows that

$$\nabla_a(\nabla_b e_d) = \nabla_a(\Delta^c_{bd} e_c) = \partial_a \Delta^c_{bd} e_c + \Delta^c_{bd} \nabla_a e_c.$$

Curvature tensor is defined as

$$[\nabla_a, \nabla_b] e_d = \pm R^c_{abd} e_c.$$

Looking closer it can be seen that

$$\begin{aligned} [\nabla_a, \nabla_b] e_d &= \nabla_a(\Delta^f_{bd} e_f) - \nabla_b(\Delta^f_{ad} e_f) \\ &= \partial_a(\Delta^c_{bd}) e_c + \Delta^f_{bd} \Delta^c_{af} e_c - \partial_b(\Delta^c_{ad}) e_c - \Delta^f_{ad} \Delta^c_{bf} e_c \end{aligned}$$

This means that the tensor can be rewritten as

$$R^c_{abd} = I^c_{[ab]d} + \hat{I}^c_{[ab]d}.$$

where $I^c_{[ab]d} = \partial_{[a}(\Delta^c_{b]d})$ and $\hat{I}^c_{[ab]d} = \Delta^f_{[bd}\Delta^c_{a]f}$. Firstly, let us take a look at $\hat{I}^c_{[ab]d}$. For time being the anti-symmetrisation on indices a and b shall be omitted. Thus $\hat{I}^c_{[ab]d}$ can be written as

$$\hat{I}^c_{abd} = e^f_{A\dot{A}} e^c_{D\dot{D}} X^{A\dot{A}}_{bd} X^{D\dot{D}}_{ad}.$$

Using the fact that $e^f_{A\dot{A}} e^{N\dot{N}}_f = \delta^N_A \delta^{\dot{N}}_{\dot{A}}$ it's easy to check that

$$e^f_{A\dot{A}} X^{D\dot{D}}_{ad} = e^f_{A\dot{A}} \partial_a e^{D\dot{D}}_f - \Gamma^D_{aA} \delta^{\dot{D}}_{\dot{A}} - \Gamma^{\dot{D}}_{a\dot{A}} \partial^D_A.$$

Also note that

$$e^c_{D\dot{D}} e^f_{A\dot{A}} (\partial_a e^{D\dot{D}}_f) = -e^f_{A\dot{A}} e^{D\dot{D}}_f \partial_a (e^c_{D\dot{D}}) = -\delta^D_A \delta^{\dot{D}}_{\dot{A}} \partial_a (e^c_{D\dot{D}}),$$

enabling us to rewrite $\hat{I}_2$ as

$$\begin{aligned}
\hat{I}^c_{abd} &= -(e^c_{D\dot{D}} \Gamma^D_{aA} \delta^{\dot{D}}_{\dot{A}} X^{A\dot{A}}_{bd} + e^c_{D\dot{D}} \Gamma^{\dot{D}}_{a\dot{A}} \delta^D_A X^{A\dot{A}}_{bd} \\
&+ \partial_a (e^c_{A\dot{A}}) \partial_b (e^{A\dot{A}}_d) - \partial_a (e^c_{A\dot{A}}) \Gamma^A_{bB} e^{B\dot{A}}_d - \partial_a (e^c_{A\dot{A}}) \Gamma^{\dot{A}}_{b\dot{B}} e^{A\dot{B}}_d).
\end{aligned}$$

Now, let us consider I^c_{abd}:

$$\begin{aligned}
I^c_{[ab]d} &= \partial_a [e^c_{A\dot{A}} X^{A\dot{A}}_{bd}] - \partial_b [e^c_{A\dot{A}} X^{A\dot{A}}_{bd}] \\
&= \partial_{[a} (e^c_{A\dot{A}}) \partial_{b]} (e^{A\dot{A}}_d) - \partial_{[a} (e^c_{A\dot{A}}) \Gamma^A_{b]B} e^{B\dot{A}}_d - \partial_{[a} (e^c_{A\dot{A}}) \Gamma^{\dot{A}}_{b]\dot{B}} e^{A\dot{B}}_d \\
&- e^c_{A\dot{A}} e^{B\dot{A}}_d \partial_{[a} (\Gamma^A_{b]B}) - e^c_{A\dot{A}} e^{A\dot{B}}_d \partial_{[a} (\Gamma^{\dot{A}}_{b]\dot{B}}) \\
&- e^c_{A\dot{A}} \partial_{[a} (e^{B\dot{A}}_d) \Gamma^A_{b]B} - e^c_{A\dot{A}} \partial_{[a} (e^{A\dot{B}}_d) \Gamma^{\dot{A}}_{b]\dot{B}}.
\end{aligned}$$

Adding both $I^c_{[ab]d}$ and $\hat{I}^c_{[ab]d}$ (note that the last three terms in formula for $\hat{I}^c_{[ab]d}$ and the first three terms on the formula for $I^c_{[ab]d}$ cancel each other out) and slightly changing the notation we get,

$$\begin{aligned}
R^c_{abd} &= -(e^c_{C\dot{C}} \Gamma^C_{[aA} \delta^{\dot{C}}_{\dot{A}} X^{A\dot{A}}_{b]d} + e^c_{C\dot{C}} \Gamma^{\dot{C}}_{[a\dot{A}} \delta^C_A X^{A\dot{A}}_{b]d} \\
&+ e^c_{C\dot{C}} e^{B\dot{C}}_d \partial_{[a} (\Gamma^C_{b]B}) + e^c_{C\dot{C}} e^{C\dot{B}}_d \partial_{[a} (\Gamma^{\dot{C}}_{b]\dot{B}}) \\
&+ e^c_{C\dot{C}} \partial_{[a} (e^{B\dot{C}}_d) \Gamma^C_{b]B} + e^c_{C\dot{C}} \partial_{[a} (e^{C\dot{B}}_d) \Gamma^{\dot{C}}_{b]\dot{B}}).
\end{aligned}$$

Now consider $R^{C\dot{C}}_{abD\dot{D}} = e^d_{D\dot{D}} e^{C\dot{C}}_c R^c_{abd}$. It follows from above that

$$\begin{aligned}
R^{C\dot{C}}_{abD\dot{D}} = \delta^{\dot{C}}_{\dot{A}} \Gamma^C_{[aA} &(e^d_{D\dot{D}} \partial_{b]} (e^{A\dot{A}}_d) + \Gamma^A_{b]B} \delta^B_D \delta^{\dot{A}}_{\dot{D}} + \Gamma^{\dot{A}}_{b]\dot{B}} \delta^A_D \delta^{\dot{B}}_{\dot{D}}) \\
+\delta^C_A \Gamma^{\dot{C}}_{[a\dot{A}} &(e^d_{D\dot{D}} \partial_{b]} (e^{A\dot{A}}_d) + \Gamma^A_{b]B} \delta^B_D \delta^{\dot{A}}_{\dot{D}} + \Gamma^{\dot{A}}_{b]\dot{B}} \delta^A_D \delta^{\dot{B}}_{\dot{D}}) \\
&+\delta^B_D \delta^{\dot{C}}_{\dot{D}} \partial_{[a} (\Gamma^C_{b]B}) + \delta^C_D \delta^{\dot{B}}_{\dot{D}} \partial_{a]} (\Gamma^{\dot{C}}_{b]\dot{B}}) \\
&+e^d_{D\dot{D}} (\partial_{[a} (e^{B\dot{C}}_d) \Gamma^C_{b]B} + \partial_{[a} (e^{C\dot{B}}_d) \Gamma^{\dot{C}}_{b]\dot{B}}).
\end{aligned}$$

After some calculations it can be shown that

$$R^{C\dot{C}}_{abD\dot{D}} = \delta^{\dot{C}}_{\dot{D}}\Gamma^{C}_{[aA}\Gamma^{A}_{b]D} + \delta^{C}_{D}\Gamma^{\dot{C}}_{[a\dot{A}}\Gamma^{\dot{A}}_{b]\dot{D}} + \delta^{\dot{C}}_{\dot{D}}(\partial_{[a}\Gamma^{C}_{b]D}) + \delta^{C}_{D}(\partial_{[a}\Gamma^{\dot{C}}_{b]\dot{D}}), \tag{4.7}$$

which in turn can be rewritten as

$$R^{C\dot{C}}_{abD\dot{D}} = \delta^{C}_{D}R^{\dot{C}}_{ab\dot{D}} + \delta^{\dot{C}}_{\dot{D}}R^{C}_{abD},$$

for some suitable tensors $R^{\dot{C}}_{ab\dot{D}}$ and R^{C}_{abD}

Also, note that

$$R^{C\dot{C}}_{A\dot{A}B\dot{B}D\dot{D}} = e^{a}_{A\dot{A}}e^{b}_{B\dot{B}}R^{C\dot{C}}_{abD\dot{D}}$$

and

$$R^{C\dot{C}}_{A\dot{A}B\dot{B}D\dot{D}} = \delta^{C}_{D}R^{\dot{C}}_{A\dot{A}B\dot{B}\dot{D}} + \delta^{\dot{C}}_{\dot{D}}R^{C}_{A\dot{A}B\dot{B}D}.$$

Curvature quantities associated with conic and projective connections

When calculating the Fröbenius form for conic or projective distribution on quaternionic manifold (Chapter 5), curvature tensor arises naturally. It might be useful to write down some of these formulas at this stage.

If $e_a = \partial_a + \Gamma^{\dot{D}}_{a\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}}$, ($e^{a}_{A\dot{A}}e_a$ are the spanning vectors of the distribution associated with projective connection on S, see Chapter 5) it can be checked that

$$[e_a, e_b] = R^{\dot{D}}_{ab\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}}.$$

Consider

$$[e^{a}_{A\dot{A}}e_a, e^{b}_{B\dot{B}}e_b] = e^{a}_{A\dot{A}}e^{b}_{B\dot{B}}[e_a, e_b] + e^{a}_{A\dot{A}}e_a(e^{b}_{B\dot{B}})e_b - e^{b}_{B\dot{B}}e_b(e^{a}_{A\dot{A}})e_a,$$

which, with the help of the fact that $e_c = e^{C\dot{C}}_{c}e_{C\dot{C}}$, can be rewritten as

$$\begin{aligned}
[e^{a}_{A\dot{A}}e_a, e^{b}_{B\dot{B}}e_b] &= e^{a}_{A\dot{A}}\partial_a(e^{c}_{B\dot{B}})e_c - e^{b}_{B\dot{B}}\partial_b(e^{c}_{A\dot{A}})e_c + R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}} \\
&= -(e^{a}_{A\dot{A}}e^{b}_{B\dot{B}}\partial_a e^{C\dot{C}}_{b} - e^{b}_{B\dot{B}}e^{a}_{A\dot{A}}\partial_b e^{C\dot{C}}_{a})e_{C\dot{C}} \\
&\quad + R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}}
\end{aligned} \tag{4.8}$$

In case of zero-torsion according to (4.5)

$$(e^a_{A\dot{A}}e^b_{B\dot{B}}\partial_a e^{C\dot{C}}_b - e^b_{B\dot{B}}e^a_{A\dot{A}}\partial_b e^{C\dot{C}}_a)e_{C\dot{C}} = \Gamma^C_{A\dot{A}B}\delta^{\dot{C}}_{\dot{B}} + \Gamma^{\dot{C}}_{A\dot{A}\dot{B}}\delta^C_B - \Gamma^C_{B\dot{B}A}\delta^{\dot{C}}_{\dot{A}} - \Gamma^{\dot{C}}_{B\dot{B}\dot{A}}\delta^C_A \quad (4.9)$$

thus resulting the following formula for $[e^a_{A\dot{A}}e_a, e^b_{B\dot{B}}e_b]$,

$$\begin{aligned} [e^a_{A\dot{A}}e_a, e^b_{B\dot{B}}e_b] &= (\Gamma^C_{B\dot{B}A}\delta^{\dot{C}}_{\dot{A}} + \Gamma^{\dot{C}}_{B\dot{B}\dot{A}}\delta^C_A - \Gamma^C_{A\dot{A}B}\delta^{\dot{C}}_{\dot{B}} - \Gamma^{\dot{C}}_{A\dot{A}\dot{B}}\delta^C_B)e_{C\dot{C}} \\ &+ R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}} \end{aligned} \quad (4.10)$$

4.1.7 Curvature in the quaternionic case

Assume now $(M, S, \hat{S}, \varphi, \varepsilon, \hat{\varepsilon})$ is a quaternionic manifold with fixed scale, and ∇ is the unique affine connection on M which satisfies the conditions of Theorem 2 and has a zero torsion.

$$\begin{aligned} R^{D\dot{D}}_{A\dot{A}B\dot{B}C\dot{C}} = [(\Psi^D_{ABC} - 2\Lambda_{C(A}\delta^D_{B)})\hat{\varepsilon}_{\dot{A}\dot{B}} + 2\delta^D_{[A}\Phi_{B]C\dot{A}\dot{B}}]\delta^{\dot{D}}_{\dot{C}} \\ +[\Phi^{\dot{D}}_{AB\dot{C}}\hat{\varepsilon}_{\dot{A}\dot{B}} - 2\Lambda_{AB}\delta^{\dot{D}}_{(\dot{A}}\hat{\varepsilon}_{\dot{B})\dot{C}}]\delta^D_C, \end{aligned}$$

where

$$R^D_{A\dot{A}B\dot{B}C} = (\Psi^D_{ABC} - 2\Lambda_{C(A}\delta^D_{B)})\hat{\varepsilon}_{\dot{A}\dot{B}} + 2\delta^D_{[A}\Phi_{B]C\dot{A}\dot{B}} \quad (4.11)$$

and

$$R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}} = \Phi^{\dot{D}}_{AB\dot{C}}\hat{\varepsilon}_{\dot{A}\dot{B}} - 2\Lambda_{AB}\delta^{\dot{D}}_{(\dot{A}}\hat{\varepsilon}_{\dot{B})\dot{C}}. \quad (4.12)$$

The three curvature quantities,Ψ^D_{ABC}, $\Phi_{AB\dot{A}\dot{B}}$ and Λ_{AB} appearing above have the following properties:

$$\Psi^D_{ABC} = \Psi^D_{(ABC)}, \quad \Psi^C_{ABC} = 0, \quad (4.13)$$

$$\Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)(\dot{A}\dot{B})}, \quad \Lambda_{AB} = \Lambda_{[AB]}. \quad (4.14)$$

4.1.8 Einstein Quaternionic Structure

Definition 4.1.4 *A quaternionic manifold $(M, S, \hat{S}, \varphi, \varepsilon, \hat{\varepsilon})$ with a fixed scale is called* Einstein *if $\Phi = 0$.*

Lemma 2 $\nabla_{\dot{A}A}\Lambda_{BC} = 0$ *for an Einstein quaternionic structure.*

Definition 4.1.5 *A* Ricci flat scale *on a quaternionic manifold is a scale for which both* $\Phi = 0$ *and* $\Lambda = 0$.

Theorem 4.1.3 *Given a quaternionic manifold* $(M, S, \hat{S}, \varphi, \varepsilon, \hat{\varepsilon})$*, such that the associated affine connection,* ∇*, has* $\Lambda \neq 0$*, and if there exists* $e_{BC} \in \Gamma(M, \wedge^2 \hat{S}^*)$ *satisfying* $\nabla_{A\dot{A}} e_{BC} = 0$*, then* $(M, S, \hat{S}, \varphi, \varepsilon, \hat{\varepsilon})$ *is Einstein and* $e_{AB} = C\Lambda_{AB}$ *for some constant* C.

Proof. $\nabla_{A\dot{A}} e_{BC} = 0$ implies that

$$e_{D[A}\Phi_{B]C\dot{A}\dot{B}} - e_{C[A}\Phi_{B]D\dot{A}\dot{B}} = 0.$$

If it is antisymmetrized on indices DAB one obtains

$$e_{[DA}\Phi_{B]C\dot{A}\dot{B}} = 0,$$

which can only hold if $\Phi_{AB\dot{A}\dot{B}} = 0$. Thus $(M, S, \hat{S}, \varphi, \varepsilon, \hat{\varepsilon})$ is Einstein.

Also, the torsion-free condition and $\nabla_{A\dot{A}} e_{BC} = 0$ imply that

$$\Psi^D_{ABC} e_{ED} + \Psi^E_{ABD} e_{CE} - 2\Lambda_{C(A} e_{B)D} + 2\Lambda_{D(A} e_{B)C} = 0.$$

The terms involving Λ_{AB} in this equation are as a consequence of the symmetries listed in equations (4.13) and (4.14), in a different irreducible representation from the terms involving Ψ^D_{ABC}. Hence, the terms involving Λ_{AB} must vanish independently. But these constitute essentially just the antisymmetric product on the space of two index antisymmetric objects, and thus vanishes if and only if Λ_{AB} and e_{AB} are proportional. Since by hypotheses $\Lambda_{AB} \neq 0$ the proof is complete. $\Box$.

4.1.9 Quaternionic Kähler and hyper-Kähler manifolds

Definition 4.1.6 A complex quaternionic Kähler structure *on an almost quaternionic manifold* $(M, S^2, \hat{S}^{2k}, \varphi)$ *of dimension* $4k$ *is the following set of data:*

(a) a pair of non-degenerate sections $\varepsilon \in \wedge^2 S^*, \hat{\varepsilon} \in \wedge^2 \hat{S}^*$;

(b) a pair of linear connections

$$\nabla_0 : S \longrightarrow S \otimes \Omega^1 M,$$

$$\hat{\nabla}_0 : \hat{S} \longrightarrow \hat{S} \otimes \Omega^1 M$$

such that $\nabla_0(\varepsilon) = 0, \hat{\nabla}_0(\hat{\varepsilon}) = 0$, *and the torsion* T_{ab}^c *of the associated induced connection* $\nabla = \nabla_0 \otimes \hat{\nabla}_0$ *is zero.*

Theorem 4.1.4 ([BE91]) *On a quaternionic manifold, there is natural one to one correspondence between compatible quaternionic Kähler metrics that are not hyperKähler, and Einstein scales for which* Λ_{AB} *is of full rank.*

Conclusion

In this chapter a deeper look at conic structures in particular case of quaternionic and almost-quaternionic manifolds was taken. Invariants of the conic structures were investigated via considering the torsion and curvature tensors of a conic connection on the structure.

Chapter 5

Twistor geometry of quaternionic Einstein manifold

5.1 Coordinates and coefficients

The local coordinates (x^a) are chosen on $4k$-dimensional manifold M equipped with quaternionic structure, along with a local trivialisation of the sheaves S and $\hat{S}$ by sections π^A and $\pi^{\dot{A}}$, respectively, where $A = 1, \ldots, 2k$; $\dot{A} = 1, 2$. A quaternionic structure is determined by $16k^2$ functions e on M which describe the spinor decomposition:

$$\varphi^{-1}(dx^a) = e^a_{A\dot{A}} \pi^A \otimes \pi^{\dot{A}}$$

or, in dual bases,

$$e^a_{A\dot{A}} \partial_a = (\varphi^*)^{-1}(\pi_A \otimes \pi_{\dot{A}}).$$

The choice of coordinates trivialises several fibrations and gives us a reference point for describing all possible connections be means of their coefficients.

In particular, a covariant differential $\nabla : S \longrightarrow S \otimes \Omega^1 M$ on S can be described using the coefficients Γ^A_{Bc} or $\Gamma^A_{BC\dot{C}} = e^c_{C\dot{C}}\Gamma^A_{Bc}$, as follows:

$$\nabla w^A = \Gamma^A_{Bc}\pi^B \otimes dx^c,$$

$$(\mathrm{id}_S \otimes \varphi^{-1})(\nabla \pi^A) = \Gamma^A_{BC\dot{C}} \otimes \pi^B \otimes \pi^C \otimes \pi^{\dot{C}}.$$

The differential ∇ induces a projective connection on the fibration $F = \mathbb{P}_M(S^*) \xrightarrow{\pi} M$. This connection depends only on the traceless part of Γ^A_{Bc} (the convolution with respect to A and B). All connections on this fibration can be obtained in this way.

F has a $2k$-conic structure, $c : F \longrightarrow G_M(2k, TM)$, uniquely defined by the fact that

$$c^*(S_{taut}) = \pi^*(\hat{S}^*)(-1).$$

A connection on F, in turn, induces a $2k$-conic connection on F:

$$\pi^*(\hat{S}^*)(-1) \subset \pi^*(\hat{S}^*) \otimes \pi^*(S^*) = \pi^*(TM).$$

The conic connection is the lifting to TF of the subsheaf $\pi^*(\hat{S}^*)(-1)$:

$$0 \longrightarrow TF/M \longrightarrow TF \xrightarrow{d\pi} \pi^*(TM) \longrightarrow 0.$$

(Note that $\pi^*(TM)$ contains $\pi^*(\hat{S}^*)(-1)$.)

Thus, the projective connection on F, which gives a lifting of all of $\pi^*(TM)$, in particular gives a lifting of the subsheaf. The corresponding mapping on connection coefficient sheaves is surjective; it can be described as a symmetrisation with respect to BC in the coordinates. In fact, this mapping is actually $\pi_*(\alpha)$, where α is the morphism in the exact sequence

$$\begin{aligned} 0 \longrightarrow TF/M \otimes \Omega^1 F/M(1) \otimes \pi^*(\hat{S}) &\longrightarrow TF/M \otimes \pi^*(S \otimes \hat{S}) \xrightarrow{\alpha} \\ TF/M \otimes \pi^*(\hat{S})(1) &\longrightarrow 0. \end{aligned}$$

The typical fibre of π is $C\mathbb{P}^1$ and

$$TF/M \otimes \Omega^1 F/M(1)|_{C\mathbb{P}^1} = \mathcal{O}(2) \otimes \mathcal{O}(-2) \otimes \mathcal{O}(1) \cong \mathcal{O}(1).$$

Which implies that

$$H^1(C\mathbb{P}^1, \mathcal{O}(1)) = 0.$$

The surjectivity of $\pi_*(\alpha)$ follows because $\pi^1_*(TF/M \otimes \Omega^1 F/M(1)) = 0$. This, in turn, follows from the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{\cdot} \pi^*(\hat{S}) \longrightarrow TF/M(-1) \longrightarrow 0$$

tensor multiplied by $\Omega^1 F/M(2)$:

$$0 \longrightarrow \Omega^1 F/M(1) \longrightarrow \pi^*(S^*) \otimes \Omega^1 F/M(2) \longrightarrow TF/M \otimes \Omega^1 F/M(1) \longrightarrow 0.$$

Since $\pi^0_*(\Omega^1/M(1)) = 0$ and $\pi^1_*(TF/M \otimes \Omega^1 F/M(1)) = 0$ one has

$$0 \longrightarrow S^* \otimes \pi_*(\Omega^1 F/M(2)) \xrightarrow{\sim} \pi_*(TF/M \otimes \Omega^1 F/M(1)) \longrightarrow 0,$$

and then, using the sequence

$$0 \longrightarrow \Omega^1 F/M(2) \longrightarrow \pi^*(S)(1) \longrightarrow \mathcal{O}(2) \longrightarrow 0,$$

one obtains

$$\pi_*(\Omega^1 F/M(2)) = \mathrm{Ker}(S \otimes S \longrightarrow \odot^2(S)) = \wedge^2 S.$$

Finally, the kernel of $\pi_*(\alpha)$ consists of the traceless symbols $\Gamma^A_{BC\dot{C}}$, which are anti-symmetric in BC, and this means that $\pi_*(\alpha)$ can be identified with symmetrisation.

Let us now summarise the description derived above. Stipulating the obstructions to the existence of the connections vanish, the following surjective maps are obtained, all of which are defined invariantly:

$$\{\text{connections on } S\} \longrightarrow \{\text{connections on the fibration } \mathbb{P}_M(S^*)\} \longrightarrow$$

$$\{c - \text{conic connection on } M.\}$$

$$(\Gamma^A_{BC\dot{C}}) \longrightarrow (\Gamma^A_{BC\dot{C}} \mathrm{mod}\ \delta^A_B \Gamma_{C\dot{C}}) \longrightarrow (\Gamma^A_{BC\dot{C}} \mathrm{mod}\ \delta^A_B \Gamma_{C\dot{C}} + \Gamma^A_{[BC]\dot{C}}).$$

However, the arrows in the reverse direction (for example if one wants to describe c-conic connections by choosing traceless anti-symmetric symbols which determine a connection on S) depend on a coordinate system.

5.2 Manin obstructions and torsion

Let M be a $2k$-dimensional (complex) manifold , such that $TM \longleftrightarrow S \otimes \hat{S}$, where $S, \hat{S}$ are of dimensions k and 2. A local trivialisation of the sheaves S and $\hat{S}$ is chosen by the choice of sections π^A and $\pi^{\dot{A}}$, respectively, where $A = 1, \ldots, k; \dot{A} = 1, 2$. A quaternionic structure is determined, as above, by functions e on M:

$$\varphi^{-1}(dx^a) = e^a_{A\dot{A}} \pi^A \otimes \pi^{\dot{A}}$$

or, in dual bases,

$$e^a_{A\dot{A}} \partial_a = (\varphi^*)^{-1}(\pi_A \otimes \pi_{\dot{A}}).$$

The choice of coordinates trivialises several fibrations and gives us a reference point for describing all possible connections be means of their coefficients.

Let $h : \pi^*(S^*)(-1) \longrightarrow TF$ be a k-conic connection. Then, its Fröbenius form Φ has a canonically defined quotient $\Phi_0(h) = d\pi(\Phi(h))$. It can be seen as

$$\Phi_0(h) : \quad \pi^*(\wedge^2 S^*(-2)) \longrightarrow \pi^*(TM)/\pi^*(S^*)(-1).$$

So, if X and Y are two local sections of $S^*(-1)$,

$$\Phi_0(h)(X, Y) = d\pi[h(X), h(Y)] \text{mod } \pi^*(S^*)(-1).$$

$\Phi_0(h)$ can be split into two irreducible components:

$$\Phi_0(h) = \Phi_0^1(h) + \Phi_0^2(h),$$

$$\Phi_0^1(h) \in (S^* \otimes \wedge^2 S)_0 \otimes (\odot^2 \hat{S} \otimes \hat{S}^*)_0,$$

$$\Phi_0^2(h) \in i(S) \otimes (\odot^2 \hat{S} \otimes S)_0$$

where $i : S \longrightarrow S^* \otimes \wedge^2 S$ is determined by the formula $i(\pi^A) = \pi_B \otimes \pi^B \wedge \pi^A$.

Here Φ_0^1 and Φ_0^2 are called *the first and second Manin obstructions*, respectively.

Theorem 5.2.1 *(a) The first Manin obstruction, $\Phi_0^1(h)$, does not depend on the choice of h. (b) There exists a unique $2k$-conic connection h for which the second Manin obstruction is zero, i.e. $\Phi_0^2 = 0$. It is the only $2k$-conic connection which can be integrable. Such a connection will be called a* distinguished *conic connection.*

Proof. Let dim $M = 2k$ and TM be isomorphic to $S \otimes \hat{S}$ (rank$S = k$, rank$\hat{S} = 2$): $\varphi : TM \longrightarrow S \otimes \hat{S}$.

Explicitly, let $x^1, \ldots, x^{2k}$ be coordinates in some characterised $U \subset M$, and let $\{e_A\}$ be local frame of S, $\{e_{\dot{A}}\}$ local frame of $\hat{S}$, π^A coordinates in S, $\pi^{\dot{A}}$ coordinates in $\hat{S}$.

Also,

$$\begin{aligned} \varphi(\frac{\partial}{\partial x^a}) = \quad \varphi(\partial_a) = & \quad e_a^{A\dot{A}}(x) e_A \otimes e_{\dot{A}}, \\ \varphi^{-1}(e_A \otimes e_{\dot{A}}) \quad = & \quad e^a_{A\dot{A}} \partial_a. \end{aligned}$$

If $h : \pi^*(S)(-1) \longrightarrow TF$ is a k-conic connection.

$$h(\pi_A)(1) = \pi^{\dot{A}} e_{A\dot{A}}, \tag{5.1}$$

where

$$e_{A\dot{A}} = e^a_{A\dot{A}} \partial_a + \Gamma^{\dot{D}}_{A\dot{A}\dot{C}} \pi^{\dot{C}} \frac{\partial}{\partial \pi^{\dot{D}}}.$$

By definition,

$$\Phi_0(h) = d\pi(\Phi(h)) = d\pi[h(X), h(Y)] \mathrm{mod} \pi^*(S)(-1).$$

Then,

$$\Phi_0(h)(2)(\pi_A, \pi_B) = d\pi[h(\pi_A)(1), h(\pi_B)(1)] \mathrm{mod} \pi^*(S)(1).$$

Taking (5.1) into account, let us consider

$$\begin{aligned} d\pi[h(\pi_A)(1), h(\pi_B)(1)] \quad = & \quad d\pi[\pi^{\dot{A}} e_{A\dot{A}}, \pi^{\dot{B}} e_{B\dot{B}}] \\ = & \quad d\pi(\pi^{\dot{A}}(e_{A\dot{A}} \pi^{\dot{B}}) e_{B\dot{B}} - \pi^{\dot{B}}(e_{B\dot{B}} \pi^{\dot{A}}) e_{A\dot{A}} \\ & + \pi^{\dot{A}} \pi^{\dot{B}} [e_{A\dot{A}}, e_{B\dot{B}}]). \end{aligned} \tag{5.2}$$

From (4.8) it follows that

$$\begin{aligned} d\pi[h(\pi_A)(1), h(\pi_B)(1)] \quad = & \quad \pi^{\dot{A}} \pi^{\dot{C}} \delta^{\dot{B}}_{\dot{D}} \Gamma^{\dot{D}}_{A\dot{A}\dot{C}} e^c_{B\dot{B}} \partial_c - \pi^{\dot{B}} \pi^{\dot{C}} \delta^{\dot{A}}_{\dot{D}} \Gamma^{\dot{D}}_{B\dot{B}\dot{C}} e^c_{A\dot{A}} \partial_c \\ & - \pi^{\dot{A}} \pi^{\dot{B}} (e^a_{A\dot{A}} e^b_{B\dot{B}} \partial_a e^{C\dot{C}}_b - e^b_{B\dot{B}} e^a_{A\dot{A}} \partial_b e^{C\dot{C}}_a) e^c_{C\dot{C}} \partial_c. \end{aligned}$$

It is clear then that

$$d\pi[h(\pi_A)(1), h(\pi_B)(1)] \bmod \pi^*(S)(1) = \pi^{\dot{A}}\pi^{\dot{B}}(e^a_{A\dot{A}}\partial_a e^c_{B\dot{B}} - e^b_{B\dot{B}}\partial_b e^c_{A\dot{A}})\partial_c$$

$$+\pi^{\dot{A}}\pi^{\dot{C}}\delta^{\dot{B}}_{\dot{D}}\Gamma^{\dot{D}}_{A(\dot{A}\dot{C})}e^c_{B\dot{B}}\partial_c - \pi^{\dot{B}}\pi^{\dot{C}}\delta^{\dot{A}}_{\dot{D}}\Gamma^{\dot{D}}_{B(\dot{B}\dot{C})}e^c_{A\dot{A}}\partial_c.$$

Making the substitution $\partial_c = e^{C\dot{C}}_c e_C \otimes e_{\dot{C}}$, it can be checked that

$$\Phi_0(h)(2) = \pi^{\dot{A}}\pi^{\dot{B}}[(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)e^a_{A\dot{A}}e^b_{B\dot{B}} + \Gamma^{\dot{C}}_{A(\dot{A}\dot{B})}\delta^C_B - \Gamma^{\dot{C}}_{B(\dot{A}\dot{B})}\delta^C_A]e_C \otimes e_{\dot{C}},$$

or

$$\Phi_0(h)(2) = \pi^{\dot{A}}\pi^{\dot{B}}\Xi^{C\dot{C}}_{[AB](\dot{A}\dot{B})}e_C \otimes e_{\dot{C}}, \tag{5.3}$$

where

$$2\Xi^{C\dot{C}}_{[AB](\dot{A}\dot{B})} = e^a_{[A(\dot{A}}e^b_{B]\dot{B})}\partial_{[a}e^{C\dot{C}}_{b]} + \Gamma^{\dot{C}}_{[A(\dot{A}\dot{B})}\delta^C_{B]}. \tag{5.4}$$

It is manifest from the formulas (5.3) and (5.4) that $\Phi^{(1)}_0(h)$, i.e. the totally trace free part of $\Phi_0(h)$, does not depend on variation of the original spinor connection, $\Gamma^{\dot{C}}_{A\dot{A}\dot{B}}$, and hence is an invariant of quaternionic structure. Moreover, the totally trace free part of $\Phi_0(h)$ will be

$$\Phi^{(1)}_0(h) = \mathrm{Tr}_0((\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)e^a_{[A(\dot{A}}e^b_{B]\dot{B})}).$$

In Chapter 4 one of the invariants of quaternionic structure was established, namely totally trace free parts of torsion. Referring to formula (4.6), we see that

$$\Phi^{(1)}_0(h) = \mathrm{Tr}_0(\tilde{F}^{C\dot{C}}_{[AB](\dot{A}\dot{B})}) \tag{5.5}$$

Now,

$$\Phi^{(2)}_0(h) = \delta^B_C[(\partial_a e^{C\dot{C}}_b - \partial_b e^{C\dot{C}}_a)e^a_{A\dot{A}}e^b_{B\dot{B}} + \Gamma^{\dot{C}}_{A(\dot{A}\dot{B})}\delta^C_B - \Gamma^{\dot{C}}_{B(\dot{A}\dot{B})}\delta^C_A]$$

$$\Phi^{(2)}_0(h)(2) = e^a_{A\dot{A}}e^b_{B\dot{B}}(\partial_a e^{B\dot{C}}_b - \partial_a e^{B\dot{C}}_a) + k\Gamma^{\dot{C}}_{A\dot{A}\dot{B}} - \Gamma^{\dot{C}}_{A\dot{A}\dot{B}}$$

since $\delta^B_C\delta^C_B = \mathrm{rank}\ S = k$.

Therefore, the integrability condition $\Phi_0^{(2)}(h)(2) = 0$ holds if and only if

$$(1-k)\Gamma^{\dot{C}}_{A\dot{A}\dot{B}} = e^a_{A\dot{A}} e^b_{B\dot{B}} (\partial_a e^{B\dot{C}}_b - \partial_a e^{B\dot{C}}_a).$$

The conic connection,

$$h(\pi_A)(1) = e^a_{A\dot{A}} \pi^{\dot{A}} \partial_a + \Gamma^{\dot{D}}_{A\dot{A}\dot{C}} \pi^{\dot{A}} \pi^{\dot{C}} \frac{\partial}{\partial \pi^{\dot{D}}},$$

such that

$$\Gamma^{\dot{D}}_{A\dot{A}\dot{C}} = \frac{1}{1-k} e^a_{A\dot{A}} e^b_{B\dot{C}} (\partial_a e^{B\dot{D}}_b - \partial_a e^{B\dot{D}}_a) \tag{5.6}$$

is the *distinguished (integrable)* connection in the body of the theorem 5.2.1. □

5.3 From Einstein quaternionic manifold to distributions on twistor space

Let us consider a quaternionic structure on M, $\dim M = 4k$. $F = \mathbb{P}(\hat{S})$, given a suitable covering F can be described in homogeneous coordinates $[\pi^{\dot{A}}]$. Any connection $\nabla : \hat{S} \longrightarrow \hat{S} \otimes \Omega^1 M$ defines a projective connection on $\mathbb{P}(\hat{S})$. In other words, it defines a distribution of rank $4k$:

$$D^{4k}_{pr} \subset TF^{4k+1},$$

or, equivalently, can be described via the operator

$$\tilde{\nabla} : \mathcal{O}_F(1) \longrightarrow \mathcal{O}_F(1) \otimes D^*_{pr}.$$

As it was shown above (theorem 5.2.1), F also has a distinguished conic connection, $D^{2k}_F \subset D^{4k}_{pr} \subset TF$, such that D^{2k}_F is integrable. Consider the following composition:

$$\nabla' : \mathcal{O}_F(1) \longrightarrow \mathcal{O}(1) \otimes D^*_{pr} \xrightarrow{\mathrm{id}\,\otimes\alpha} \mathcal{O}_F(1) \otimes D^*_F,$$

where α is a surjection in

$$D^*_{pr} \xrightarrow{\alpha} D^*_F \longrightarrow 0$$

Locally, a conic connection can be described as

$$D_F := \text{span } \{\pi^{\dot{A}}(e^a_{A\dot{A}}\partial_a + e^a_{A\dot{A}}\Gamma^{\dot{C}}_{a\dot{B}}(x)\pi^{\dot{B}}\frac{\partial}{\partial\pi^{\dot{C}}})\}$$

Let us denote:

$$e_{A\dot{A}} := (e^a_{A\dot{A}}\partial_a + e^a_{A\dot{A}}\Gamma^{\dot{C}}_{a\dot{B}}(x)\pi^{\dot{B}}\frac{\partial}{\partial\pi^{\dot{C}}})$$

In such a notation $D_{pr} = \text{span } (e_{A\dot{A}})$.

Since D_F is integrable, by Fröbenius theorem there exists a twistor space Z, dim $Z = 2k+1$, parameterizing its foliation leaves, i.e. we have a double fibration :

$$\begin{array}{ccc} & F = \mathbb{P}(\hat{S}) & \\ {}^{\mu}\swarrow & & \searrow^{\pi} \\ Z & & M, \end{array}$$

Let us define a rank $2k$ locally free sheaf, E^{2k}_F, on F by the exact sequence

$$0 \longrightarrow D^{2k}_F \longrightarrow D^{4k}_{pr} \longrightarrow E^{2k}_F \longrightarrow 0.$$

For our purposes we need a subsheaf of Abelian groups on E^{2k}_F given by,

$$Y = \{v \in E^{2k}_F | \mathcal{L}_w\tilde{v} \in D^{2k}_F, \text{for all } w \in D^{2k}_F\},$$

where $\tilde{v}$ is an arbitrary representative of v in D^{2k}_{pr}. Y is defined properly since the distribution D^{2k}_F is integrable.

Proposition 5.3.1 *Y is a sheaf of $\mathcal{O}_Z$-modules.*

Proof. Suppose, that $w \in D^{4k}_{pr}$ is such that

$$\mathcal{L}_p w \in D^{2k}_F \tag{5.7}$$

for any $p \in D^{2k}_F$. We claim that w gives rise to a vector field v on Z^{2k+1}. Indeed for any $f \in \mathcal{O}_Z$ we have

$$(pw - wp) \circ \mu^{-1}(f) = s \circ \mu^{-1}(f),$$

where $s = \mathcal{L}_p w \in D_F^{2k}$. But $s \circ \mu^{-1}(f) = 0$ and $p \circ \mu^{-1}(f) = 0$, since $s, p \in D_F$, meaning that $p \circ (w \circ \mu^{-1}(f)) = 0$. This on the other hand implies that there exists $g \in \mathcal{O}_Z$ such that $\mu^{-1}(g) = w\mu^{-1}(f)$. It is easy to check that this is a vector field, i.e. Leibnitz rule is satisfied. □

Theorem 5.3.1 *Let M be an Einstein quaternionic manifold with*

$$\text{rank } \Lambda_{AB} = p.$$

Then there exists a rank $2k$ distribution $D \subset TZ$ on the associated twistor space Z such that

1. $Y = \mu^{-1}(D)$;

2. The rank of Fröbenius form of D is equal to p.

Proof. To check when (5.7) holds it is appropriate look at the Lie derivative of a general vector $Q^{A\dot{A}}e_{A\dot{A}} \in D_{pr}$ along basis vectors, $\pi^{\dot{A}}e_{A\dot{A}}$, of D_F, : $[\pi^{\dot{A}}e_{A\dot{A}}, Q^{B\dot{B}}e_{B\dot{B}}]$.

So, take any $v \in E_F$ and let $\tilde{v} = Q^{A\dot{A}}e_{A\dot{A}}$ be its arbitrary lift to D_{pr}. In order to get explicit description of Y it is sufficient to solve the following equation

$$[\pi^{\dot{A}}e_{A\dot{A}}, Q^{B\dot{B}}e_{B\dot{B}}] \text{mod } (\text{span } \pi^{\dot{C}}e_{C\dot{C}}, \pi^{\dot{A}}\frac{\partial}{\partial \pi^{\dot{A}}}) = 0. \tag{5.8}$$

First,

$$[\pi^{\dot{A}}e_{A\dot{A}}, Q^{B\dot{B}}e_{B\dot{B}}] = (\pi^{\dot{A}}e_{A\dot{A}}Q^{B\dot{B}})e_{B\dot{B}} - Q^{B\dot{B}}(e_{B\dot{B}}\pi^{\dot{A}})e_{A\dot{A}} + \pi^{\dot{A}}Q^{B\dot{B}}[e_{A\dot{A}}, e_{B\dot{B}}].$$

Recalling (4.10), $[\pi^{\dot{A}}e_{A\dot{A}}, Q^{B\dot{B}}e_{B\dot{B}}]$ can be rewritten as

$$\begin{aligned}
[\pi^{\dot{A}}e_{A\dot{A}}, Q^{B\dot{B}}e_{B\dot{B}}] &= (\pi^{\dot{A}}e_{A\dot{A}}Q^{C\dot{C}})e_{C\dot{C}} - (Q^{B\dot{B}}\Gamma^{\dot{C}}_{B\dot{B}\dot{D}}\pi^{\dot{D}}\delta^{C}_{A})e_{C\dot{C}} \\
&+ \pi^{\dot{A}}Q^{B\dot{B}}(\delta^{\dot{C}}_{\dot{A}}\Gamma^{C}_{B\dot{B}A} + \delta^{C}_{A}\Gamma^{\dot{C}}_{B\dot{B}\dot{A}} - \delta^{\dot{C}}_{\dot{B}}\Gamma^{C}_{A\dot{A}B} - \delta^{C}_{B}\Gamma^{\dot{C}}_{A\dot{A}\dot{B}})e_{C\dot{C}} \\
&+ \pi^{\dot{A}}Q^{B\dot{B}}R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial \pi^{\dot{D}}} \\
&= A^{C\dot{C}}_{A}e_{C\dot{C}} + \pi^{\dot{A}}Q^{B\dot{B}}R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial \pi^{\dot{D}}},
\end{aligned} \tag{5.9}$$

where

$$A^{C\dot{C}}_{A} = \pi^{\dot{A}}e_{A\dot{A}}Q^{C\dot{C}} + \pi^{\dot{A}}Q^{B\dot{B}}(\delta^{\dot{C}}_{\dot{A}}\Gamma^{C}_{B\dot{B}A} - \delta^{\dot{C}}_{\dot{B}}\Gamma^{C}_{A\dot{A}B} - \delta^{C}_{B}\Gamma^{\dot{C}}_{A\dot{A}\dot{B}}).$$

Recalling 4.12, i.e.

$$R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}} = \Phi^{\dot{D}}_{AB\dot{C}}\hat{\varepsilon}_{\dot{A}\dot{B}} - 2\Lambda_{AB}\delta^{\dot{D}}_{(\dot{A}}\hat{\varepsilon}_{\dot{B})\dot{C}},$$

and the fact that M is Einstein, i.e. the curvature tensor $\Phi^{\dot{D}}_{AB\dot{C}}$ vanishes by definition, the last term in the (5.9) can be rewritten as

$$-2\pi^{\dot{A}}Q^{B\dot{B}}\Lambda_{AB}\delta^{\dot{D}}_{\dot{A}}\hat{\varepsilon}_{\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}} - 2\pi^{\dot{A}}Q^{B\dot{B}}\Lambda_{AB}\delta^{\dot{D}}_{\dot{B}}\hat{\varepsilon}_{\dot{A}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}} =$$

$$-2Q^{B\dot{B}}\Lambda_{AB}\hat{\varepsilon}_{\dot{B}\dot{C}}\pi^{\dot{C}}\pi^{\dot{D}}\frac{\partial}{\partial\pi^{\dot{D}}} - 2Q^{B\dot{B}}\Lambda_{AB}\hat{\varepsilon}_{\dot{A}\dot{C}}\pi^{\dot{A}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}} =$$

$$-2Q^{B\dot{B}}\Lambda_{AB}\hat{\varepsilon}_{\dot{B}\dot{C}}\pi^{\dot{C}}\pi^{\dot{D}}\frac{\partial}{\partial\pi^{\dot{D}}}.$$

In other words,

$$\pi^{\dot{A}}Q^{B\dot{B}}R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}}\text{mod (span } \pi^{\dot{C}}e_{C\dot{C}}, \pi^{\dot{A}}\frac{\partial}{\partial\pi^{\dot{A}}}) = 0.$$

Now, let us consider the equation,

$$A^{C\dot{C}}_{A}e_{C\dot{C}}\text{mod (span } \pi^{\dot{C}}e_{C\dot{C}}, \pi^{\dot{A}}\frac{\partial}{\partial\pi^{\dot{A}}}) = 0. \tag{5.10}$$

Note that if $A^{C\dot{C}}_{A}$ satisfies the equation above, then any expression of the form $A^{C\dot{C}}_{A} + \omega^{C}_{A}\pi^{\dot{C}}$ would also be a solution. Therefore solving (5.10) is equivalent to funding a solution for

$$A^{C\dot{C}}_{A}\pi_{\dot{C}}\text{mod } (\pi^{\dot{A}}\frac{\partial}{\partial\pi^{\dot{A}}}) = 0. \tag{5.11}$$

To simplify the left-hand side of this equation one might use the fact that

$$(e_{A\dot{A}}Q^{C\dot{C}})\pi_{\dot{C}} = e_{A\dot{A}}(Q^{C\dot{C}}\pi_{\dot{C}}) - Q^{C\dot{C}}e_{A\dot{A}}\pi_{\dot{C}}.$$

Thus (5.11) becomes

$$0 = \pi^{\dot{A}}e_{A\dot{A}}(Q^{C\dot{C}}\pi_{\dot{C}}) - \pi^{\dot{A}}(Q^{B\dot{C}}\pi_{\dot{C}})\Gamma^{C}_{A\dot{A}B} - \pi^{\dot{A}}Q^{C\dot{B}}\Gamma^{\dot{C}}_{A\dot{A}\dot{B}}\pi_{\dot{C}} - \pi^{\dot{A}}Q^{C\dot{C}}\Gamma^{\dot{D}}_{A\dot{A}\dot{E}}\pi^{\dot{E}}\hat{\varepsilon}_{\dot{D}\dot{C}}$$

The last two terms can be rewritten as

$$\begin{aligned}
-\pi^{\dot{A}}Q^{C\dot{B}}\Gamma^{\dot{C}}_{A\dot{A}\dot{B}}\pi^{\dot{D}}\tilde{\varepsilon}_{\dot{D}\dot{C}} - \pi^{\dot{A}}Q^{C\dot{C}}\Gamma^{\dot{D}}_{A\dot{A}\dot{E}}\pi^{\dot{E}}\tilde{\varepsilon}_{\dot{D}\dot{C}} &= \pi^{\dot{A}}Q^{C\dot{C}}\Gamma_{A\dot{A}\dot{C}\dot{D}}\pi^{\dot{D}} - \pi^{\dot{A}}Q^{C\dot{C}}\Gamma_{A\dot{A}\dot{D}\dot{C}}\pi^{\dot{D}} \\
&= \pi^{\dot{A}}Q^{C\dot{C}}\Gamma^{\dot{E}}_{A\dot{A}\dot{E}}\tilde{\varepsilon}_{\dot{C}\dot{D}}\pi^{\dot{D}}
\end{aligned}$$

$$= -\pi^{\dot{A}}Q^{C\dot{C}}\Gamma^{\dot{D}}_{A\dot{A}\dot{D}}\pi_{\dot{C}}.$$

Thus (5.11) can be rewritten as

$$0 = \pi^{\dot{A}}e_{A\dot{A}}(Q^{C\dot{C}}\pi_{\dot{C}}) - \pi^{\dot{A}}(Q^{B\dot{C}}\pi_{\dot{C}})\Gamma^{C}_{A\dot{A}B} - \pi^{\dot{A}}(Q^{C\dot{C}}\pi_{\dot{C}})\Gamma^{\dot{D}}_{A\dot{A}\dot{D}}. \tag{5.12}$$

It is clear from (5.12) that the space of solutions for (5.11), and thus for (5.8), will be of rank $2k$.

Now, assume that Q_1 and Q_2 are such that $[Q_1, X] \in D_F, [Q_2, X] \in D_F$, for any $X \in D_F$. On the other hand we have an identity

$$[[Q_1, Q_2], X] + [[X, Q_1], Q_2] + [[Q_2, X], Q_1] = 0.$$

Since $[X, Q_1] \in D_F$, it follows that $[[X, Q_1], Q_2] \in D_F$. Similarly, $[[Q_2, X], Q_1] \in D_F$. Therefore, $[[Q_1, Q_2], X] \in D_F$ for any $X \in D_F$.

So, if $Q^{C\dot{C}}\pi_{\dot{C}}$ satisfies the equation above, then $[D_F, Q]$ is in D_F But then $[[Q_1, Q_2], D_F]$ is also in D_F, so if Q_1, Q_2 satisfy the above condition, then $[Q_1, Q_2]$ also satisfies,

$$[D_F, Q] \mathrm{mod}(\pi\frac{\partial}{\partial\pi})$$

is in D_F.

Thus it was shown that Y is a locally free $\mu^{-1}(\mathcal{O}_Z)$ module. of rank $2k$. It follows from Proposition 5.3.1 that there exists a rank $2k$ distribution $D \subset TZ$ on the associated twistor space Z such that $Y = \mu^{-1}(D)$, thus proving the first part of the theorem.

Now, let $f_a, a = 1, \ldots, 2k$ be a basis of local sections of D^{2k} over Z^{2k+1}. On F it can be interpreted as vector fields $Q_a^{A\dot{A}}e_{A\dot{A}}$ with $(Q_a^{A\dot{A}}e_{A\dot{A}})$ satisfying (5.12). The operator which computes the Fröbenius form of D

$$\begin{array}{ccc} D \otimes D & \longrightarrow & L := TZ/D \\ f_a \otimes f_b & \longrightarrow & \Phi_{ab} = [f_a, f_b] \mathrm{mod}\ D, \end{array}$$

when lifted to F, corresponds to computing $[Q_a^{A\dot{A}}e_{A\dot{A}}, Q_b^{B\dot{B}}e_{B\dot{B}}] \in TF$and then projecting the result to TF/M along the sheaf of vertical fields,

$$\Phi_{ab} = [Q_a^{A\dot{A}}e_{A\dot{A}}, Q_b^{B\dot{B}}e_{B\dot{B}}] \mathrm{mod}\ D^{4k}_{pr}. \tag{5.13}$$

Since

$$[Q_a^{A\dot{A}}e_{A\dot{A}}, Q_b^{B\dot{B}}e_{B\dot{B}}] = (Q_a^{A\dot{A}}e_{A\dot{A}}Q_b^{C\dot{C}})e_{C\dot{C}} - (Q_b^{B\dot{B}}e_{B\dot{B}}Q_a^{C\dot{C}})e_{C\dot{C}} + Q_a^{A\dot{A}}Q_b^{B\dot{B}}[e_{A\dot{A}}, e_{B\dot{B}}],$$

it follows from (4.10) that

$$\Phi_{ab} = Q_a^{A\dot{A}}Q_b^{B\dot{B}}R^{\dot{D}}_{A\dot{A}B\dot{B}\dot{C}}\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}}.$$

Again using (4.12) and the fact that the curvature tensor $\Phi^{\dot{D}}_{AB\dot{C}}$ vanishes for Einstein manifold, we get

$$\Phi_{ab} = -2Q_a^{A\dot{A}}Q_b^{B\dot{B}}\Lambda_{AB}(\delta^{\dot{D}}_{\dot{A}}\tilde{\varepsilon}_{\dot{B}\dot{C}} + \delta^{\dot{D}}_{\dot{B}}\tilde{\varepsilon}_{\dot{A}\dot{C}})\pi^{\dot{C}}\frac{\partial}{\partial\pi^{\dot{D}}},$$

or

$$\Phi_{ab} = 2Q_a^{A\dot{A}}Q_b^{B\dot{B}}\Lambda_{AB}\delta^{\dot{D}}_{(\dot{A}}\pi_{\dot{B})}\frac{\partial}{\partial\pi^{\dot{D}}}.$$

It is clear from the formulae above that

$$\operatorname{rank}\Phi_{ab} = \operatorname{rank}\Lambda_{AB},$$

thus finishing the proof. □

Chapter 6

Inverse Construction

Theorem 6.0.2 *Let Y be a $(2n+1)$-dimensional complex manifold equipped with a rank $2n$ distribution $D \subset TY$ such that*

1. the Fröbenius form

$$\Phi : \wedge^2 D \longrightarrow L \cong TY/D \tag{6.1}$$

$$X \otimes Y \longrightarrow [X, Y] \mathrm{mod} D \tag{6.2}$$

has rank p, $0 \leq p \leq 2n$,

2. $\wedge^{2n} D \simeq L^{\otimes n}$

Assume that $X \hookrightarrow Y$ is a rational curve ($\mathbb{CP}^1$) embedded into Y with the normal bundle $N = \mathbb{C}^{2n} \otimes \mathcal{O}(1)$ transversally to D.

Then the associated Kodaira moduli space M of all deformations of X inside Y (while remaining transversal to D) has canonically a structure of an Einstein quaternionic manifold with the curvature Λ_{AB} such that

$$\mathrm{rank}\Lambda_{AB} = \mathrm{rank}\Phi = p.$$

Proof. The cases $p = 0$ (hyperKähler structure) and $p = 2n$ (quaternionic Kähler structure) are well known ([HKLR87] and [LeB89], respectively). Thus from now on one may assume that $0 < p < 2n$.

Since X is transversal to D, the quotient map θ,

$$\theta : TY \longrightarrow L,$$

provides an isomorphism $L_{|X} \simeq TX$. Hence

$$L|_X = \mathcal{O}(2).$$

Assume, there exists a line bundle $L^{1/2}$ on Y such that

$$(L^{1/2})^{\otimes 2} = L.$$

The assumption is weak. Such a bundle always exists at least on any sufficiently small tubular unambiguously of X inside Y (see [LeB89]) . Then,

$$L^{1/2}|_X = \mathcal{O}(1).$$

Consider the following double fibration:

$$\begin{tikzcd} & F \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ Y & & M, \end{tikzcd}$$

associated with Kodaira deformation problem, see Introduction, Section 1.3, and let us define the following locally free sheaves on the moduli space M,

$$\tilde{S} =: \nu^0_* \mu^*(L^{1/2}),$$
$$S =: \nu^0_* \mu^*(D \otimes L^{1/2}).$$

By definition of ν^0_*:

1. The fibre of $\tilde{S}$ over $t \in M$ is

$$\tilde{S}_t = H^0(\nu^{-1}(t), \mu^*(L^{1/2})) \simeq \mathbb{C}^2,$$

for any $t \in M$. Thus $\tilde{S}$ is a rank 2 vector bundle on M.

2. The fibre of S over $t \in M$ is

$$S_t = H^0(\nu^{-1}(t), \mu^*(D \otimes L^{-1/2}))$$

$$= H^0(\mathbb{CP}^1, \mathbb{C}^{2n} \otimes \mathcal{O}(1) \otimes \mathcal{O}(-1))$$

$$= H^0(\mathbb{CP}^1, \mathbb{C}^{2n}) = \mathbb{C}^{2n},$$

since $D_{|_{\mathbb{CP}^1}} = \mathbb{C}^{2n} \otimes \mathcal{O}(1)$ and $L^{-1/2}_{|_{\mathbb{CP}^1}}$, dual of $L^{1/2}_{|_{\mathbb{CP}^1}}$, is isomorphic to $\mathcal{O}(-1)$. Thus S is a rank $2k$ vector bundle on M. Also, by the Kodaira theorem see Section 1.3,

$$\begin{aligned} T_tM &= H^0(X_t, N_t) \\ &= H^0(X_t, D_{|X_t}) \\ &= H^0(X_t, D \otimes L^{-1/2}_{|X_t} \otimes L^{1/2}_{|X_t}) \\ &= S_t \otimes H^0(X_t, L^{1/2}_{|X_t}) \\ &= S_t \otimes \tilde{S}_t. \end{aligned}$$

Thus the moduli space M comes canonically equipped with an almost quaternionic structure.

Now, there is the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mu^*(L^*) & \xrightarrow{=} & \Omega^1 F/M & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mu^*(\Omega^1 Y) & \longrightarrow & \Omega^1 F & \longrightarrow & \Omega^1 F/Y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mu^*(D^*) & \longrightarrow & \nu^*(\Omega^1 M) & \longrightarrow & \Omega^1 F/Y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The lower exact sequence implies the long exact sequence:

$$\begin{array}{l} 0 \longrightarrow \nu^0_*\mu^*(D^*) \longrightarrow \nu^0_*\mu^*(\Omega^1 M) \longrightarrow \nu^0_*(\Omega^1 F/Y) \longrightarrow \\ \longrightarrow \nu^1_*\mu^*(D^*) \longrightarrow \qquad \dots \end{array}$$

But

$$\nu^0_*\mu^*(D^*) = 0, \quad \nu^1_*\mu^*(D^*) = 0, \tag{6.3}$$

since the typical fibres of those sheaves, respectively, are given by

$$\begin{array}{l} H^0(\mathbb{P}^1, D^*|_{\mathbb{P}}) = 0, \quad H^0(\mathbb{P}^1, \mathbb{C}^{2n}(-1)) = 0, \\ H^1(\mathbb{P}^1, D^*|_{\mathbb{P}}) = 0, \quad H^0(\mathbb{P}^1, \mathbb{C}^{2n}(-1)) = 0. \end{array}$$

Thus,

$$\Omega^1 M = \nu^0_*(\Omega^1 F/Y).$$

There is a relative de Rham differential on F, $d_{F/Y}$:

$$d_{F/Y} : \mu^*(D \otimes L^{-1/2}) \xrightarrow{d_F} \mu^*(D \otimes L^{-1/2}) \otimes \Omega^1 F/Y$$

Note, that its kernel is given by:

$$\text{Ker } d_F = \mu^{-1}(D \otimes L^{-1/2})$$

Taking the 0-th direct image of $d_{F/Y}$ and using (6.3) one obtains the derivation ∇:

$$\nabla : \nu^*_0\mu^*(D \otimes L^{-1/2}) \longrightarrow \nu_0\mu^*(D \otimes L^{-1/2}) \otimes \nu^*_0(\Omega^1 F/Y)$$

or, equivalently,

$$\nabla : S \longrightarrow S \otimes \Omega^1 M,$$

i.e. a linear connection on S.

Note that the sheaf $\nu^0_*(\wedge^2 D^* \otimes L)$ is isomorphic to $\wedge^2 S$. Indeed, the typical fibre of $\nu^0_*\mu^*(\wedge^2 D^* \otimes L)$ is given by

$$H^0(\mathbb{P}, \wedge^2(\mathbb{C}^{2n}(1))^* \otimes \mathcal{O}(2)) = (\wedge^2\mathbb{C}^{2n})^*.$$

The Fröbenius form Φ defines a global section of the sheaf $\wedge^2 D^* \otimes L$, and hence of the subsheaf

$$\mu^{-1}(\wedge^2 D^* \otimes L) \subset \mu^*(\wedge^2 D^* \otimes L).$$

Since this subsheaf is precisely the kernel of $d_{F/Y}$, one concludes that

$$\lambda \equiv \nu_0^* \mu^*(\Phi) \in \Gamma(M, \wedge^2 S^*)$$

is covariantly constant, i.e.

$$\nabla_S \lambda = 0.$$

Now, let us consider the sheaf $\wedge^{2n}(D \otimes L^{-1/2})^*$. It is easy to check that it is actually a trivial vector bundle on Y:

$$\wedge^{2n}(D \otimes L^{-1/2}) = \wedge^{2n} D \otimes L^{-n} = L^n \otimes L^{-n} = \mathcal{O}_Y.$$

Thus, there exists a non-vanishing section $\varepsilon_Y \in \Gamma(Y, \wedge^{2n}(D \otimes L^{-1/2}))$. It is defined up to a multiplication by a global function $f \in \Gamma(Y, \mathcal{O}_Y)$. However, $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}$ since $f_{|X_t} = \text{const}$ and, for any $y \in Y$, tangents to X_t passing through y span the whole tangent space $T_y Y$.

Thus, ε_Y is well defined up to a constant. Then ε_Y gives rise to a non-degenerate section ε of

$$\wedge^{2n} S^* = \nu_*^0 \mu^*(\wedge^{2n}(D \otimes L^{-1/2})^*).$$

Since $d_{F/Y}\mu^*(\varepsilon_Y) = 0$, it is clear that

$$\nabla \varepsilon = 0.$$

In conclusion, we have shown that rank $2n$ bundle S comes naturally equipped with a triple $(\nabla, \lambda, \varepsilon)$, where ∇ is a linear connection, $\lambda \in \Gamma(M, \wedge^2 S), \varepsilon \in \Gamma(M.\wedge^{2n} S^*)$ are such that

$$\nabla \varepsilon = \nabla \lambda = 0.$$

Let us now study the geometric structure induced on the rank 2 vector bundle $\tilde{S}$. Firstly, we notice that the fibration on F can be identified with the relative projective line $\mathbb{P}_M(\tilde{S}^*)$. There is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \hat{L} = \mu^*(L) & \xrightarrow{=} & \mu^*(L) & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & TF/Y & \longrightarrow & TF & \longrightarrow & \mu^*(TY) & \longrightarrow & 0 \\
 & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & TF/Y & \longrightarrow & \hat{D} & \longrightarrow & \mu^*(D) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

which defines rank $4n$ distribution $\hat{D} \subset TF$ and a line bundle $\hat{L} \longrightarrow F$.

Since for any $t \in M$, X_t is transversal to D, the distribution $\hat{D}$ is horizontal. Thus $\hat{D}$ defines a projective connection on $\tilde{S}$, $\{\tilde{\nabla}_{pr}\}$. Note, that the bundle $\wedge^2\tilde{S}^*$ comes canonically equipped with non-degenerate holomorphic section $\tilde{\varepsilon}$, constructed by LeBrun [LeB89]. There is a $\tilde{S}$-parallel holomorphic section $\tilde{\varepsilon}$ of $\wedge^2\tilde{S}^*$ constructed as follows: the Wronskian

$$W : \mathcal{O}(\hat{L}^{1/2}) \times \mathcal{O}(\hat{L}^{1/2}) \longrightarrow \mathcal{O}$$

$$(s_1, s_2) \longrightarrow s_1 \otimes d_p(s_2) - s_2 \otimes d_p(s_1),$$

where $d_p : \mathcal{O} \longrightarrow \mathcal{O}(\hat{L}^{1/2})$ is differentiation up the fibres of ν is independent of the choice of local trivialisation used for differentiating sections of $\hat{L}^{1/2}$. Taking direct image sheaves, we obtain an object:

$$\tilde{\varepsilon} : \mathcal{O}(\tilde{S}) \times \mathcal{O}(\tilde{S}) \longrightarrow \mathcal{O}$$

which may be interpreted as $\tilde{\varepsilon} \in \Gamma(M, \wedge^2\tilde{S}^2)$.

Here, the projective connection ∇_{pr} has a unique lift to linear connection $\tilde{\nabla}$ on $\tilde{S}$, such that the covariant derivative of $\tilde{\varepsilon}$ vanishes, i.e.

$$\tilde{\nabla}\tilde{\varepsilon} = 0.$$

In summary it was shown that moduli space M comes equipped with the following properties:

1. an almost quaternionic structure

$$TM = S \otimes \tilde{S},$$

where $\mathrm{rank} S = 2n, \mathrm{rank}\tilde{S} = 2$;

2. the scale

$$(\varepsilon \in \Gamma(M, \wedge^{2n} S^*), \tilde{\varepsilon} \in \Gamma(M, \wedge^2 \tilde{S}^*));$$

3. a rank $0 < p < n$ section

$$\lambda \in \Gamma(M, \wedge^2 S^*);$$

4. a pair of linear connection on S and $\tilde{S}$ which satisfy

$$\nabla \varepsilon = 0, \tilde{\nabla} \tilde{\varepsilon} = 0, \nabla \lambda = 0.$$

The data 4 induces an affine connection on TM. The torsion of this induced affine connection on TM, as it was shown independently by LeBrun [LeB89] and Merkulov [Mer97], is determined by the second order infinitesimal neighbourhood of the embedding $X \hookrightarrow Y$.

The latter in turn is controlled by the cohomology group $H^1(X, N \otimes \odot^2 N^*)$ (see Merkulov [Mer97]). As $N = \mathbb{C}^{2n}(1)$ in our case, this group vanishes, which implies that the induced connection is torsion-free. Therefore, the curvature tensor for this connection is completely described by three tensor fields:

$$\Psi_{ABC}^{D} = \Psi_{(ABC)}^{D}, \quad \Psi_{ABC}^{C} = 0,$$
$$\Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)(\dot{A}\dot{B})}, \quad \Lambda_{AB} = \Lambda_{[AB]}.$$

as described in section 4.1.8.

Since $\nabla \lambda = 0$, and by theorem 4.1.3

$$\Phi_{AB\dot{A}\dot{B}} = 0,$$

in other words, M is Einstein, and

$$\lambda_{AB} = c\Lambda_{AB}$$

for some constant c. This completes the proof. $\square$

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