The Effect of Shear on the Stability and Dynamic Properties of Elastic Bodies

by

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Preface

This thesis has been submitted in accordance with the rules of the Faculty of Science at the University of Glasgow for the degree of Doctor of Philosophy.

The body of work it contains is the result of research undertaken by myself but must also stand as a record of collective input.

My deepest thanks go to Professor Ray Ogden for his constant guidance and ever-present support throughout the course of this research in both his supervisory capacity and as Head of the Department. As his successor to the latter position, I also thank Professor Ken Brown.

For additional assistance with the research herein I also wish to thank Dr. David Haughton, Dr. Ken Lindsay and Dr. David Roxburgh.

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By way of closure to this section, and notwithstanding the extent of my gratitude to the above named, my final words of thanks go to my family — to whom this thesis is dedicated.

Abstract

The rôle that shear plays in the dynamical response and associated stability of elastic bodies is investigated within this thesis from two perspectives. Forming the major part of the study is the investigation of infinitesimal wave propagation within elastic material which has been subjected to a static pre-strain corresponding to simple shear.

Initially we consider a prototype problem wherein the theory of incremental motions provides the mechanism for analyzing Rayleigh waves propagating along the surface of an incompressible elastic half-space. This is looked at from a plane strain point of view but, significantly, the direction of propagation is not along a principal axis. Using co-ordinates measured relative to the Eulerian axes in the governing equation and boundary conditions, corresponding to the vanishing of incremental tractions, we derive the secular equation for infinitesimal waves in terms of wavespeed, shear and hydrostatic stress parameters for a *particular class* of materials. The dependence of the wavespeed on these parameters is illustrated and bifurcation criteria are found through setting the wavespeed to zero, this corresponding to quasi-static incremental displacements. For a general form of incompressible, isotropic strain-energy function we are also able to provide the bifurcation criteria incorporating an additional, material parameter. We also consider the compressible counterpart to this problem and follow the same approach, where possible, in establishing the secular equation for compressible materials.

This approach is also adopted for the next problem in which an infinite layer of incompressible elastic material, having uniform width, is pre-strained and within which infinitesimal waves are propagated along the layer. Owing to the layer width the waves are now dispersive and for three types of incremental boundary conditions we provide dispersion equations involving wavespeed, shear, hydrostatic stress and layer thickness (wavelength) parameters for the same particular class of materials. The interdependence of these parameters is comprehensively detailed for this class along with the bifurcation analysis which is again extended so that it be applicable to a general incompressible, isotropic material.

An alternative insight into the influence of shear is found in the study of the so-called effective shear modulus of a rectangular block. This modulus is a notional coefficient providing a linear relationship between certain shear strain and shear stress parameters, and is thus a generalization of Hooke's Law. Here, we utilize energy methods based in the non-linear theory to establish bounds on the effective shear modulus that incorporate both material and geometric parameters through the Lamé constants and aspect ratio respectively. Both analytic and numerical results are shown.

Chapter 1

Introduction

For this thesis, we have investigated two broad aspects of the influence of shear on elastic bodies. The first aspect to be discussed is that of the propagation of elastic waves within a body that has been subjected to a pre-stressed configuration, or underlying deformation, corresponding to simple shear. The incremental theory of non-linear elasticity is employed to seek solutions, fitting both the equations of incremental motion and the boundary conditions, corresponding to such wave propagation. Having established dynamical results, the specialization to the static case opens the problem to discussion on bifurcation criteria and, subsequently, the stability of the underlying deformation.

The study of these waves is divided into two further categories, each classified through the geometry of the elastic body. In the first, we consider a half-space supporting surface waves; and in the second category we have a strip of elastic material infinite in length but finite in width. In both, the basic geometry of the material determines the type of solution that is sought.

From the perspective of the techniques employed in this thesis, surface waves on a (pre-stressed) half-space were first examined by Hayes and Rivlin (1961). A feature in this, as is commonly found throughout the literature, is that the direction of the wave propagation is along a principal axis. Flavin (1963) developed the approach by considering a general direction of propagation along the surface of the material (one principal axis being perpendicular to this surface). Likewise, Chadwick and Jarvis (1979) also considered the general direction, their study adapted from the working of Barnett and Lothe (1973, 1974) in anisotropic solids, providing a general uniqueness theorem for surface waves along with the identification of the boundary for their domain of existence. More recently, Dowaikh and Ogden (1990, 1991) have investigated the effect of homogeneous pure strain on incompressible and compressible materials respectively. In specializing the results there to the undeformed and unstressed configuration we establish links with the results of this thesis (with the same specialization) and the classical study of surface waves originating with Rayleigh (1889).

Further contributions to the field include a sequence of publications by Willson (1973a, 1974a,b), who considers a variety of compressible and incompressible materials, and who also directs attention to dispersive waves in pre-strained homogeneous plates in Willson (1973b, 1977a,b). Recent contributions to the study of such rectangular plates include Ogden and Roxburgh (1993) for incompressible materials and Roxburgh and Ogden (1994) for the compressible case.

The second aspect to this thesis is again concerned with shear although in a very different context. We employ energy arguments to obtain bounds on the effective shear modulus of a rectangular block, this modulus being a measure of shear stress against shear strain for a deformed configuration — the classical shear modulus, in both linear and non-linear theory, being a quantity measured in the *undeformed* configuration and which is thus also known as the ground state shear modulus.

The approach to the effective shear modulus problem herein used owes much to that taken by Haddow and Ogden (1988) who sought bounds for the allied quantity of the effective Young's modulus. In this, both the non-linear and the linear theory are utilized — the latter being a specialization of the former. In this approach we acknowledge the fundamental contribution of Prager and Synge

(1947) who obtain approximated solutions to boundary value problems having used a geometrical approach to maximize and minimize energy functionals. This approach required the linear theory and it is within this setting that Greenberg and Truell (1948) applied the Prager-Synge method to the approximation of the effective Young's modulus.

For the effective shear modulus, there is considerable interest in composite materials, and where this is not the case then the majority of attention is in the the numerical calculation of the ratio of effective shear modulus to the classical shear modulus using finite element methods. In evaluation of the effective shear modulus of an elastic plate, Paik (1995) has developed an empirical formula which is based on maximum initial deflection (normalized for the plate thickness) along with the ratio of average shear stress to the shear buckling stress, although we do not include comparative results here. Furthermore, the approaches taken are usually relevant to the linear theory, see Gent and Meinecke (1970) and Lindley (1966) for examples, and we note that Gent and Lindley (1959) have found that for filled rubbers "the stress-strain relation in simple shear was markedly non-linear".

We now detail the structure of the work of this thesis along with some results contained therein.

Chapter 2 opens with a description of elastic deformation followed by a brief section on the theory of stress. After introducing the notion of constitutive relations we then use them in establishing the incremental equations of elasticity. With the formulation of the incremental theory and its corresponding notation we discuss the strong ellipticity condition. These latter few sections are relevant for the elastic waves problems whereas in consideration of the effective shear modulus we next set down the variational principles for boundary value problems. Finally, since the underlying motivation for this thesis is the effect of shear, we spend some time in setting down those quantities generally stated or derived in the preceding sections for when the deformation is taken to be simple shear.

In Chapter 3 we address the problem of surface waves on a half-space of homogeneous, isotropic elastic material. The body is pre-strained with the simple shear deformation of Section 2.6 and the equations for plane incremental motion are utilized with the important feature being that we must refer all quantities to the Eulerian axes if we wish to capitalize on existing information such as the compact expressions for the components of the fourth-order tensor of instantaneous elastic moduli $\mathcal{A}_0$. The main field equations are therefore transformed via a rotation of axes so as to comply with the requirements of the incremental theory. It is emphasized that the angle of rotation, θ_E, is intrinsically dependent on the magnitude of the deformation.

Taking the half-space to occupy the region $x_2 \leq 0$ in (x_1, x_2)-space, the surface waves sought are considered as propagating in the x_1-direction with the material having the underlying deformation

$$\left.\begin{aligned} x_1 &= X_1 + \varepsilon X_2, \\ x_2 &= X_2, \end{aligned}\right\} \tag{1.1}$$

where (X_1, X_2) are the co-ordinates for material points prior to the deformation and ε is the amount of shear.

Initially we consider incompressible materials and seek incremental displacements v_1, v_2. The incompressibility constraint, when applied to the incremental displacements, suggests the existence of a potential function $\psi(x_1, x_2, t)$ for which we propose

$$\psi(x_1, x_2, t) = Ae^{ik(ct-x_1)}e^{iksx_2} \tag{1.2}$$

as being suitable for wave propagation. The transformation of the original co-ordinate system to co-ordinates associated with the Eulerian axes, $(x_1, x_2) \mapsto (x_1', x_2')$, is invoked leading to a potential function $\psi'(x_1', x_2', t)$ and it is this function which is used to establish the governing equation of motion

$$\alpha\psi'_{,1111} + 2\beta\psi'_{,1122} + \gamma\psi'_{,2222} = \rho(\ddot{\psi}'_{,11} + \ddot{\psi}'_{,22}) \tag{1.3}$$

where α, β, γ are prescribed material parameters, $()_{,i}$ denotes $\partial/\partial x_i$ and the double dots denote $\partial^2/\partial t^2$ — this does not rely on (1.2).

In (1.2) we have the wavespeed c, the wavenumber k, an arbitrary constant A, and the variable s which depends on c through the governing equation. Extracting this dependency is unproductive in the main though specialization to materials forming a particular class produces a sufficiently simple dependency such that we can proceed. Details are provided of this following discussion and derivation of the boundary conditions which are taken here to be the vanishing of incremental traction rates on the surface $x_2 = 0$. The derivation is established from the perspective of the transformed co-ordinates (x_1', x_2').

There are two possible solutions from the equation of motion corresponding to surface waves, entailing modification of the potential function ψ. Using two boundary conditions, we present the distillation of the condition ensuring non-trivial solutions into the secular equation

$$\eta^3 + \eta^2 + (1 + \varepsilon^2 + 2\overline{p})\eta - \overline{p}^2 = 0 \tag{1.4}$$

forming the main result within the chapter in the sense that it yields connections with previous publications both immediately and in subsequent application. The secular equation establishes the functional dependency of the wavespeed c, a known function of η, on the deformation parameter ε and a (non-dimensionalized) hydrostatic pressure $\overline{p}$. Analysis of it reveals conditions on the existence of surface waves — namely the deformation dependent interval for $\overline{p}$

$$-\sqrt{4+\varepsilon^2} \leq \overline{p} - 1 \leq \sqrt{4+\varepsilon^2}. \tag{1.5}$$

This dependence of the wavespeed on the two parameters is then displayed graphically with a variety of parameter choices.

Returning to the governing equation (1.3), we proceed more generally with the introduction of a further material parameter δ generated through α, β, γ; and the vanishing of which corresponds to our previously adopted material class. Further results are only found for the static case $c = 0$, it being a simplified case. This enables us to establish the generalized bifurcation criterion

$$(\overline{p} - 1)^2 = \frac{\varepsilon^2 + 4}{1 + \delta} \tag{1.6}$$

indicating the region in $(\varepsilon, \overline{p}, \delta)$-space where there is a loss of stability in the deformation giving rise, potentially, to quasi-static surface deformations. Note that this corresponds with the endpoints of the interval (1.5) in the special case $\delta = 0$ and we conclude that the underlying deformation is incrementally stable provided (1.5) holds for the strengthened condition of *strict* inequality — the analogous generalized condition given by replacing the equality in (1.6) with strict inequality in a parallel fashion.

As a final installment to this chapter, we provide a further generalization — that of extending the analysis to compressible materials. No similar potential function exists in this setting but the governing equations decouple symmetrically for the incremental displacements v_i', $i \in \{1, 2\}$, as

$$\begin{aligned} a'v'_{i,1111} &+ 2b'v'_{i,1122} + c'v'_{i,2222} \\ &= \rho(\alpha_{11} + \gamma_1)v'_{i,11tt} + \rho(\alpha_{22} + \gamma_2)v'_{i,22tt} + \rho^2 v'_{i,tttt}, \end{aligned} \tag{1.7}$$

where all quantities excepting v_i' and the material density ρ are constitutive material parameters.

For wave propagation, each of the two increments v_j, $j \in \{1, 2\}$, are taken as having the same structure as the right-hand side of (1.2) with the equations of motion providing a compatibility link in the otherwise arbitrary constants. Once established, we have a problem which has been reduced to only involving two arbitrary constants in the incremental solutions. From here, the approach follows exactly that of the previous sections on incompressible materials to result in an unspecialized secular equation for compressible materials. In this thesis at least, we choose to leave the secular equation for compressible materials as a problem for future development noting the increased structural complexity in a system already having necessitated simplification to achieve meaningful results.

Chapter 4 deals with the same basic problem albeit embellished with a distinct initial geometry, i.e. that of a layer, and is combined with three sets of boundary conditions to extend its application.

Before embarking on detailed analyses of the problems for the layer we include a discussion on shear band phenomena drawing on several sources to provide an overview of the topic and linking the theory, such as it stands, to elements of the subsequent problems.

The material here occupies the region $\{(x_1, x_2) : x_1 \in \mathbb{R}, -h \leq x_2 \leq 0\}$ and is once again pre-strained via the simple shear deformation (1.1) to a point where incremental solutions corresponding to wave propagation in the x_1-direction are sought.

In contrast with the previous chapter, we are no longer restricted to surface waves but can introduce solutions from a wider field, the finite width of the layer being sufficient to enable bounds to be placed on the growth of the incremental displacements.

With the material being incompressible, the potential function ψ reappears as

$$\psi(x_1, x_2, t) = \sum_{j=1}^{2} \left(A_j e^{iks_j x_2} + B_j e^{ik\overline{s}_j x_2} \right) e^{ik(ct - x_1)}, \tag{1.8}$$

where $\overline{s}_j$, $j \in \{1, 2\}$, is the complex conjugate of s_j, a solution to the governing equation of motion, this being exactly the same as that of the half-space problem.

Three boundary value problems are now proposed: one being where the traction rates vanish on $x_2 = 0, -h$; the next where the traction rates vanish on one surface while the incremental displacements vanish on the other; the last being where the displacement increments vanish on both surfaces.

The common thread through each of these is that through two conditions (two traction rates *or* two displacements) on each of the two surfaces, we arrive at a homogeneous system of linear equations in the four unknowns A_1, A_2, B_1, B_2. The matrices of coefficients are able to be put in a simple enough style as to facilitate direct calculation, in the same style, of the determinants — their vanishing being the aforementioned condition ensuring non-trivial solutions. These conditions have the nomenclature dispersion equations since they provide the necessary conditions for the propagation of dispersive waves.

A clear feature of these dispersion equations is their possession of components reminiscent of the half-space secular equation. This is useful as a device for setting down the dispersion equations in detail. In each of the dispersion equations from those problems involving traction boundary conditions, the limiting case of large kh yields the secular equation for the half-space. It is worth noting here that increased constraints, in the form of bonding surfaces to rigid plates say, reduces the complexity of the resultant dispersion equations.

The following sections then either describe the dynamic case for the special class of strain-energy functions, or the bifurcation criteria for a general material. Owing to the complexity of the dispersion (or bifurcation) equations we choose not to record them here but we do note some of their resulting effects.

For the incremental traction problem we find that the bifurcation equation is expressible as

$$f(\overline{\beta}, \overline{p}, kh, \delta) = \pm g(\overline{\beta}, \overline{p}, kh, \delta, \varepsilon), \tag{1.9}$$

where $\overline{\beta}$ is the non-dimensionalized version of β, and both f and g are, for the moment, unspecified functions of their arguments. The salient point here is the appearance of the $(\pm)$ sign; each possibility of which gives rise to a different family of bifurcation modes through varying a controlling parameter such as $\overline{\beta}$. We identify these families as being of either flexural $(+)$ or barreling $(-)$ type. In the reference configuration, for example, the flexural modes correspond to displacements v_1, v_2 where v_1 is an even function of x_2 and v_2 an odd function of x_2. By contrast the barreling modes are such that v_1 is an odd function of x_2

and v_2 an even function of x_2.

No expression such as (1.9) is possible in the mixed boundary value problem though we generate one family of bifurcation curves. With the third, displacement, boundary value problem we obtain a rearrangement of the bifurcation equation in a manner such as (1.9) which is easily shown never to hold. The overall trend is to reduce the bifurcation potential of the layer with successively restrictive boundary conditions.

The effect of this restriction also manifests itself within the dynamical context wherein the bifurcation results provide insight into the stability of the layer. Similarly to these bifurcation results, we find that in measuring wavespeed against $\overline{p}$ there are solutions in the first (traction) problem corresponding to both ($\pm$) modes — albeit with a slightly different interpretation owing to principal axes rotation; to only one of the modes in the second (mixed) problem; and no such possible solution at all in the third problem.

When plotting wavespeed against kh, all three problems show evidence of behaving similarly for sufficiently large kh and c, these results being identified with higher order modes of solution, these modes being associated with the periodicity of trigonometrical components within the dispersion equations. It is in the lower range for c that the bifurcation results are significant in describing the dynamical aspect of the deformation. In particular, the third problem will only admit stable solutions and its dynamical behaviour is characterized by the total absence of all lowest order modes.

Our penultimate part, Chapter 5, is the chapter given over to the effective shear modulus problem. Principally this is a straightforward application of the theory of minimum potential energy and its counterpart, the theory of maximum complementary energy. These state that solutions to boundary value problems, carrying all of the constitutive relationships, produce the appropriate extrema in certain energy functionals (which are both equal when evaluated for such a solution). Whilst this is guaranteed in linear elasticity, we describe arguments for extending its use to the non-linear setting.

The second section deals with how the chain of inequalities is constructed from the variational principles outlined in Chapter 2 — the essential task being to take the energy functionals as one parameter functions of either the deformation, using ε here; or the stress s which is, in this case, the resultant shear stress per unit surface area along the top surface of the block.

The third section details the application of the theory by using artificially contrived deformation and stress fields for the upper and lower bounds respectively. The stress fields need fulfil three conditions, one of which is a further assumption, and we could generate an infinite family of fields satisfying them all. We have elected to look at one small class and show how the resultant lower bound varies within this class. Finally, the results obtained with our stress fields are compared with the numerical results of Lindley and Teo (1978) for a variety of aspect ratios and material parameters.

The final chapter is a brief discussion on the direction of further investigation.

Chapter 2

Basic Equations

In this chapter we shall introduce the equations of elasticity using the notation of Truesdell and Noll (1965), Wang and Truesdell (1973) and Ogden (1984).

2.1 Kinematics and Deformations

The motion of an elastic body, $\mathcal{B}$, consisting of continuously distributed material, is described by first choosing an initial geometric configuration and analyzing the geometry of the body, through time, relative to this choice. To this end we take as our choice the *reference configuration* B_0, which is a region in three-dimensional Euclidean space. While it is not strictly necessary to insist that the elastic body occupies the region B_0 at any time, we do require that material points of $\mathcal{B}$ can be put into a bijective mapping with points of B_0. As the body moves in time we identify it at time t with a *current configuration* B_t.

In order to establish a link between B_0 and B_t we choose Cartesian co-ordinate systems with bases $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ for B_0 and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for B_t so that a material point P of $\mathcal{B}$ will have position vectors $\mathbf{X} \in B_0$ and $\mathbf{x} \in B_t$ given by $\mathbf{X} = X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3$ and $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. Since we again insist upon a bijective correspondence between points of B_t and $\mathcal{B}$ we can define the mapping for P,

$$\mathrm{x} = \chi_t(\mathrm{X}), \quad \forall \mathrm{X} \in B_0, \tag{2.1}$$

and its unique inverse

$$\mathrm{X} = \chi_t^{-1}(\mathrm{x}), \quad \forall \mathrm{x} \in B_t. \tag{2.2}$$

We adopt the convention that χ is a a smooth, twice-continuously differentiable function of space and time.

At this point we remark that using $\mathrm{X} \in B_0$ and time as parameters in describing physical characteristics of the motion of the body is said to be the Lagrangean description whereas adopting $\mathrm{x} \in B_t$ and t is the Eulerian description.

The *deformation gradient* tensor $\mathbf{A}$ is given the meaning

$$\begin{aligned} \mathrm{A} &= \operatorname{Grad} \chi_t(\mathrm{X}) \\ &= \operatorname{Grad} \chi(\mathrm{X}, t) \end{aligned} \tag{2.3}$$

and is used to provide a measure of the deformation of line, surface and volume elements with the motion of the body. We have used the convention that $\chi_t(\mathrm{X}) = \chi(\mathrm{X}, t)$ since $\chi_t(\mathrm{X})$ describes the motion of the point P, initially at $\mathrm{X} \in B_0$, with t acting as a parameter. In addition we have adopted the convention that Grad denotes the gradient operator in the reference configuration and so (2.3) may be expressed in Cartesian components as

$$A_{ij} = \frac{\partial \chi_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j}, \quad i, j \in \{1, 2, 3\}. \tag{2.4}$$

The deformation is said to be *homogeneous* when the components of $\mathbf{A}$ are independent of $\mathbf{X}$.

The assumed smoothness and regularity conditions on χ allows for the existence of a local inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ which has Cartesian components

$$A_{ij}^{-1} = \frac{\partial X_i}{\partial x_j}, \quad i, j \in \{1, 2, 3\}. \tag{2.5}$$

Local invertibility of $\mathbf{A}$ is succinctly characterized by the condition

$$J := \det A_{ij} \neq 0 \tag{2.6}$$

which we shall return to later.

Line elements $\mathbf{dX}$ of B_0 will deform into line elements $\mathbf{dx}$ of B_t and are linked through

$$\mathbf{dx} = \mathbf{AdX}, \tag{2.7}$$

or in component form,

$$dx_i = A_{ij}dX_j, \quad i \in \{1,2,3\}. \tag{2.8}$$

We note that (2.8) uses the summation convention and that it will be employed throughout this thesis unless specifically stated otherwise.

Using (2.7) and a result from tensor theory we may establish *Nanson's Formula*,

$$\mathbf{n}da = J\mathbf{A}^{-T}\mathbf{N}dA, \tag{2.9}$$

where da and dA are area elements of B_t and B_0 respectively, taking $\mathbf{n}$ and $\mathbf{N}$ as respective unit outward normals and $\mathbf{A}^{-T}$ is the transpose of $\mathbf{A}^{-1}$.

Further application of (2.7) results in the relation

$$dv = JdV \tag{2.10}$$

between volume elements dv of B_t and dV of B_0.

The invertibility condition $J \neq 0$ also shows that there does not exist $\mathbf{dX} \neq \mathbf{0}$ such that $\mathbf{AdX} = \mathbf{0}$ which, through (2.7), (2.9) and (2.10), has the physical interpretation that line, area and volume elements cannot be annihilated. Furthermore, since volume is taken as a positive quantity then (2.6) is strengthened through (2.10) to $J > 0$, for all deformations χ.

If the deformation is volume preserving then it is said to be *isochoric* and $J \equiv 1$ in that case. If the material itself will only admit isochoric deformations then it is said to be *incompressible* and $J = 1$, which holds at all points P, is referred to as the *incompressibility constraint.*

The second order tensors $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are clearly symmetric and positive definite and so the *Polar Decomposition Theorem*

$$\mathbf{A} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{2.11}$$

follows, where $\mathbf{U}$ and $\mathbf{V}$ are unique positive definite, symmetric tensors and $\mathbf{R}$ is a unique tensor satisfying $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$.

Physically (2.11) represents the decomposition of a given deformation $\mathbf{A}$ into a rotation $\mathbf{R}$ coupled with a strain $\mathbf{U}$ or $\mathbf{V}$ with $\mathbf{R}\mathbf{U}$ representing a strain followed by a rotation and $\mathbf{V}\mathbf{R}$ representing a rotation followed by a strain. Both $\mathbf{U}$ and $\mathbf{V}$, called the *right* and *left stretch tensors*, may be calculated from the spectral decomposition of

$$\left.\begin{aligned} \mathbf{B} &= \mathbf{V}^2 = \mathbf{A}\mathbf{A}^T \\ \mathbf{C} &= \mathbf{U}^2 = \mathbf{A}^T\mathbf{A}. \end{aligned}\right\} \tag{2.12}$$

The *stretch* of a line element $\mathbf{dX}$ after deformation into the line element $\mathbf{dx}$ is denoted by $\lambda(\mathbf{M})$, where $\mathbf{M}$ is a unit vector in the direction of $\mathbf{dX}$ and has the specific form

$$\lambda(\mathbf{M}) = [\mathbf{M} \cdot \mathbf{A}^T\mathbf{A}\mathbf{M}]^{1/2} = [\mathbf{M} \cdot \mathbf{U}^2\mathbf{M}]^{1/2}. \tag{2.13}$$

By considering the eigenvalues λ_i and their associated eigenvectors $\mathbf{u}^{(i)}$, $i \in \{1, 2, 3\}$, of $\mathbf{U}$ we see from (2.13) that $\lambda_i = \lambda(\mathbf{u}^{(i)})$ and so the λ_i are referred to as the *principal stretches* with the $\mathbf{u}^{(i)}$ referred to as the *principal directions* of $\mathbf{U}$.

It follows from (2.11) that $\mathbf{V}\mathbf{R}\mathbf{u}^{(i)} = \mathbf{R}\mathbf{U}\mathbf{u}^{(i)} = \lambda_i\mathbf{R}\mathbf{u}^{(i)}$, thus showing that the principal stretches λ_i are also eigenvalues of $\mathbf{V}$ corresponding to eigenvectors $\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}$. This may be interpreted as the deformation rotating the principal directions of $\mathbf{U}$ into the principal directions of $\mathbf{V}$ which we will now call the *Lagrangean (principal) axes* and *Eulerian (principal) axes* respectively to indicate the configurations in which they arise.

2.2 Elastic Stress

The stress, or traction, t acting on the surface of the elastic material with unit normal n in the current configuration is

$$\mathrm{t} = \boldsymbol{\sigma}^T \mathrm{n}, \tag{2.14}$$

which introduces the *Cauchy Stress Tensor* $\boldsymbol{\sigma}$.

Use of Nanson's Formula (2.9) on the total traction acting on the surface of the arbitrary region R_t gives

$$\begin{aligned} \int_{\partial R_t} \mathrm{t}\, da &= \int_{\partial R_t} \boldsymbol{\sigma}^T \mathrm{n}\, da \\ &= \int_{\partial R_0} J\boldsymbol{\sigma}^T \mathrm{A}^{-T} \mathrm{N}\, dA \end{aligned} \tag{2.15}$$

with which we define the *Nominal Stress Tensor*

$$\mathrm{S} = J\mathrm{A}^{-1}\boldsymbol{\sigma} \tag{2.16}$$

so that the traction t on a surface at x may again be written as the product of a (transposed) stress tensor and the unit normal to the surface, but now using Lagrangean co-ordinates, to provide the Lagrangean analogue to (2.14)

$$\mathrm{t} = \mathrm{S}^T \mathrm{N}. \tag{2.17}$$

Through the balance of angular momentum we find that $\boldsymbol{\sigma}$ is a symmetric tensor but we see from (2.16) that S, because of the presence of A, is not symmetric in general. We can, however, express angular momentum balance in terms of S by

$$\mathrm{AS} = \mathrm{S}^T \mathrm{A}^T. \tag{2.18}$$

The Eulerian equation of motion is

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathrm{b} = \rho \ddot{\mathrm{x}}, \tag{2.19}$$

where ρ is the material density, b is the body force per unit volume at x and div is the divergence operator in B_t.

The equation of motion (2.19) can also be recast in terms of the Lagrangean co-ordinates. We first record the result

$$\rho = J^{-1}\rho_0 \tag{2.20}$$

linking the density ρ at x and the density ρ_0 at $\mathbf{X}$ which is a result of the conservation of mass. The Lagrangean equation of motion is then

$$\operatorname{Div}\mathrm{S} + \rho_0\mathrm{b} = \rho_0\ddot{\mathrm{x}} \tag{2.21}$$

where Div is the divergence operator in B_0.

2.3 Constitutive Relations

The field equations governing the motion of an elastic body do not in themselves provide enough scope for determining all of the unknown quantities. The equation of motion (2.19), the mass continuity equation

$$\dot{\rho} + \rho\operatorname{div}\mathrm{v} = 0 \tag{2.22}$$

(which is equivalent to (2.20)), and the balance of angular momentum taken together comprise 7 scalar equations for the 13 unknowns $\rho, \mathrm{v}, \boldsymbol{\sigma}$ leading us to introduce *constitutive equations* to complete the system. We remark that in general the field equations also include the (scalar) thermal energy equation, and through this 4 more unknowns, but we consider only isothermal deformations and therefore do not incorporate this into our system.

We assume the constitutive equation

$$\boldsymbol{\sigma} = \mathrm{g}(\mathrm{A}) \tag{2.23}$$

to indicate that the stress depends only on the deformation (and not on the history of the deformation since we are applying this to elastic materials). The tensor valued function g is known as the *response function* relative to B_0 and we impose the condition that there is no stress in the undeformed configuration so that $\mathbf{g}(\mathbf{I}) = \mathbf{0}$. If the body undergoes a rigid body rotation and translation we expect the stress-deformation relation to be unchanged, i.e. for the response function to be *objective*, and so

$$\mathbf{g}(\mathbf{QA}) = \mathbf{Qg}(\mathbf{A})\mathbf{Q}^T \tag{2.24}$$

for all rotations $\mathbf{Q}$. In particular we note that setting $\mathbf{Q} = \mathbf{R}^T$ gives

$$\mathbf{g}(\mathbf{A}) = \mathbf{Rg}(\mathbf{U})\mathbf{R}^T. \tag{2.25}$$

An elastic body has *material symmetry* when, for any given deformation, the stress responses relative to the distinct reference configurations B_0 and B_0' are equal — the greater the degree of symmetry inherent in the body then the larger the set of reference configurations B_0' for which this holds. If the current state has deformation gradients $\mathbf{A}$ relative to B_0, and $\mathbf{A}'$ relative to B_0', then there exists an invertible tensor $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{A}'\mathbf{P}, \tag{2.26}$$

where $\mathbf{P}$ may be treated as the deformation gradient of B_0' relative to B_0.

As the stress response is the same then the response function $\mathbf{g}$ relative to B_0 or B_0' is the same and so we are justified in writing

$$\mathbf{g}(\mathbf{A}) = \mathbf{g}(\mathbf{A}') = \mathbf{g}(\mathbf{AP}^{-1}). \tag{2.27}$$

The *symmetry group* $\mathcal{G}$ is thus defined to be the set of all invertible second-order tensors $\mathbf{P}$ such that

$$\mathbf{g}(\mathbf{A}) = \mathbf{g}(\mathbf{AP}) \tag{2.28}$$

for all deformations $\mathbf{A}$.

We make the further definitions that, for $\mathcal{G}$ relative to some reference configuration, an elastic *material* is *isotropic* if $\mathcal{G}$ contains the set $\mathcal{Q}$ of all proper orthogonal tensors and an elastic *solid* is isotropic if $\mathcal{G} = \mathcal{Q}$.

By using $\mathbf{A} = \mathbf{V}\mathbf{R}$ and setting $\mathbf{P} = \mathbf{R}^T$ in (2.28) we see that for isotropic elastic solids

$$\boldsymbol{\sigma} = \mathrm{g}(\mathbf{V}) \tag{2.29}$$

where both $\boldsymbol{\sigma}$ and $\mathbf{V}$ are symmetric tensors and hence co-axial. The stress $\boldsymbol{\sigma}$ may then be put into the form

$$\boldsymbol{\sigma} = \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.30}$$

similarly to the spectral decomposition of $\mathbf{V}$, where the σ_i's are the principal values (eigenvalues) of $\boldsymbol{\sigma}$.

A further constitutive relation is provided by assuming the existence of a *strain-energy function* W which satisfies

$$\frac{d}{dt} W(\mathbf{A}) = J \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L}), \tag{2.31}$$

where $\mathbf{L} = \dfrac{\partial \mathbf{v}}{\partial \mathbf{x}}$ is the velocity gradient. Such a function then represents, per unit reference volume, the internal potential energy (or strain energy) of the body caused by the deformation $\mathbf{A}$. Elastic materials supporting a strain-energy function are called *hyperelastic* or *Green Elastic* materials.

By considering isotropic, homogeneous (objective) hyperelastic materials it is possible to write

$$W(\mathbf{A}) = W(\mathbf{Q}\mathbf{R}\mathbf{U}) = W(\mathbf{R}^T\mathbf{R}\mathbf{U}) = W(\mathbf{U}) \tag{2.32}$$

and

$$W(\mathbf{A}) = W(\mathbf{V}\mathbf{R}\mathbf{Q}) = W(\mathbf{V}\mathbf{R}\mathbf{R}^T) = W(\mathbf{V}), \tag{2.33}$$

or quite generally

$$W(\mathbf{Q}\mathbf{V}\mathbf{Q}^T) = W(\mathbf{V}) \tag{2.34}$$

for all proper orthogonal $\mathbf{Q}$. For this case, it follows that we must have

$$W = W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_1, \lambda_3) = W(\lambda_3, \lambda_1, \lambda_2), \tag{2.35}$$

where we have made no distinction in the notation *here* between the different domains upon which W is defined. Throughout this thesis we take the scalar quantity W to be defined over *either* a tensor valued function $\mathbf{T}$ or the principal values of the tensor $(\mathbf{T}\mathbf{T}^T)^{1/2}$, with the appropriate definition being clear from the context.

An immediate consequence of (2.31) is the stress-deformation relation for compressible materials

$$J\sigma = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}}, \tag{2.36}$$

where we adopt the notation $\left(\frac{\partial}{\partial \mathbf{T}}\right)_{ij} = \frac{\partial}{\partial T_{ji}}$ for any second-order tensor $\mathbf{T}$. From using (2.16) we also see that

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}}. \tag{2.37}$$

In the case of incompressible materials the incompressibility constraint

$$J = \lambda_1\lambda_2\lambda_3 = 1 \tag{2.38}$$

removes the independence of the principal stretches and so we introduce the Lagrange multiplier p and write

$$\sigma = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}} - p\mathbf{I} \tag{2.39}$$

in place of (2.36) with (2.37) being replaced with

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} - p\mathbf{A}^{-1}. \tag{2.40}$$

We see from (2.39) that p may be treated as a hydrostatic pressure term.

Since we have the decompositions $\mathbf{V} = \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$ and $\sigma = \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$ we have, from starting with (2.36), that relative to the principal axes

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}, \tag{2.41}$$

with the incompressible counterpart

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p. \tag{2.42}$$

In (2.35) we define W symmetrically in terms of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and it would therefore be possible, at least in principle, to determine W in terms of 3 independent and symmetric functions of $\lambda_1, \lambda_2, \lambda_3$. A common approach is to define W in terms of the *principal invariants* I_1, I_2, I_3 given by

$$\left.\begin{array}{rclcl} I_1 & = & \operatorname{tr}\mathbf{B} & = & \lambda_1^2+\lambda_2^2+\lambda_3^2, \\ I_2 & = & \frac{1}{2}[(\operatorname{tr}\mathbf{B})^2-\operatorname{tr}(\mathbf{B}^2)] & = & \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2, \\ I_3 & = & \det\mathbf{B} & = & \lambda_1^2\lambda_2^2\lambda_3^2, \end{array}\right\} \tag{2.43}$$

where $\operatorname{tr}\mathbf{B}$ denotes the trace of $\mathbf{B}$, so that

$$W(\lambda_1, \lambda_2, \lambda_3) = \widehat{W}(I_1, I_2, I_3). \tag{2.44}$$

The nominal stress $\mathbf{S}$, for example, in an unconstrained material could be directly calculated from a specific form of $\widehat{W}$ through the stress-deformation relation

$$\mathbf{S} = 2\frac{\partial\widehat{W}}{\partial I_1}\mathbf{A}^T + 2\frac{\partial\widehat{W}}{\partial I_2}\{I_1\mathbf{I} - \mathbf{A}^T\mathbf{A}\}\mathbf{A}^T + 2I_3\frac{\partial\widehat{W}}{\partial I_3}\mathbf{A}^{-1}. \tag{2.45}$$

The derivation of this relation uses (2.37) as a starting point, by starting with (2.40) we can extend (2.45) to the incompressible case by including the extra term $-p\mathbf{A}^{-1}$ and setting $I_3 = 1$.

2.4 Incremental Elasticity

2.4.1 Incremental Equations

We consider the deformation χ of a body into the configuration B_t relative to B_0 followed by a further incremental deformation $\delta\chi$ with the implication in the term incremental being that $\delta\chi$ is sufficiently small, relative to the underlying deformation χ, as to allow us to neglect quantities of order $|\delta\chi|^2$.

The change brought about in the deformation gradient is given exactly by

$$\delta\mathbf{A} = \operatorname{Grad}(\chi + \delta\chi) - \operatorname{Grad}(\chi) = \operatorname{Grad}(\delta\chi) \tag{2.46}$$

and the corresponding change in the determinant is, to first order,

$$\delta J = J\,\mathrm{tr}\,((\delta \mathrm{A})\mathrm{A}^{-1}). \tag{2.47}$$

The subsequent increment in S for unconstrained materials is shown, with the alternative notation $\dot{\mathrm{S}} = \delta \mathrm{S}$, as

$$\dot{\mathrm{S}} = \frac{\partial \mathrm{S}}{\partial \mathrm{A}}\dot{\mathrm{A}}. \tag{2.48}$$

We use this as motive to define the fourth-order tensor $\mathcal{A}$ which has components

$$\mathcal{A}_{ijkl} = \frac{\partial S_{ij}}{\partial A_{lk}}. \tag{2.49}$$

and is referred to as the first-order elastic moduli tensor. In particular we note that for hyperelastic materials, $\mathcal{A}$ is expressible as

$$\left.\begin{aligned} \mathcal{A} &= \frac{\partial^2 W}{\partial \mathrm{A} \partial \mathrm{A}} \\ \text{or} \quad \mathcal{A}_{ijkl} &= \frac{\partial^2 W}{\partial A_{ji} \partial A_{lk}}. \end{aligned}\right\} \tag{2.50}$$

In the case of incompressible materials the constraint $J \equiv 1$ changes (2.47) into

$$\mathrm{tr}\,(\dot{\mathrm{A}}\mathrm{A}^{-1}) = 0 \tag{2.51}$$

and the additional terms present in the stress-deformation relations give

$$\dot{\mathrm{S}} = \mathcal{A}\dot{\mathrm{A}} - \dot{p}\mathrm{A}^{-1} + p\mathrm{A}^{-1}\dot{\mathrm{A}}\mathrm{A}^{-1}. \tag{2.52}$$

If we now make the choice that B_t is the reference configuration then the increment in the deformation gradient is

$$\dot{\mathrm{A}}_0 = \mathrm{grad}\,(\dot{\chi}) = \dot{\mathrm{A}}\mathrm{A}^{-1} \tag{2.53}$$

where the subscript $_0$ denotes a quantity evaluated in B_t. We also have the relation for the nominal stress increments

$$\dot{\mathrm{S}}_0 = J^{-1}\mathrm{A}\dot{\mathrm{S}}. \tag{2.54}$$

The analogous relation to (2.48), taken with (2.49), of

$$\dot{\mathbf{S}}_0 = \mathcal{A}_0 \dot{\mathbf{A}}_0 \tag{2.55}$$

yields the link

$$\mathcal{A}_{0ijkl} = J^{-1} A_{i\alpha} A_{k\beta} \mathcal{A}_{\alpha j \beta l} \tag{2.56}$$

when we compare the components of (2.48) and (2.55) after substitution of (2.53) and (2.54). The fourth-order tensor $\mathcal{A}_0$ is called the tensor of instantaneous elastic moduli to show its having been evaluated in the current configuration.

The corresponding updated relation for incompressible materials is

$$\dot{\mathbf{S}}_0 = \mathcal{A}_0 \dot{\mathbf{A}}_0 - \dot{p}\mathbf{I} + p\dot{\mathbf{A}}_0. \tag{2.57}$$

The effect of the updating on the increment in the determinant (2.47) produces

$$\dot{J} = J \mathrm{tr}\,(\dot{\mathbf{A}}_0) \tag{2.58}$$

with the incompressible counterpart being

$$\mathrm{tr}\,(\dot{\mathbf{A}}_0) = \mathrm{div}\,\dot{\mathbf{x}} = 0, \tag{2.59}$$

where we write $\dot{\chi} = \dot{\mathbf{x}}$. We now record that the non-zero components of $\mathcal{A}_0$ *relative to the Eulerian principal axes* take the form

$$\mathcal{A}_{0iijj} = J^{-1}\lambda_i\lambda_j W_{ij}, \tag{2.60}$$

$$\mathcal{A}_{0ijij} = \begin{cases} \dfrac{J^{-1}(\lambda_i W_i - \lambda_j W_j)\lambda_i^2}{\lambda_i^2 - \lambda_j^2} & i \neq j, \lambda_i \neq \lambda_j, \\ \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + J^{-1}\lambda_i W_i) & i \neq j, \lambda_i = \lambda_j, \end{cases} \tag{2.61}$$

$$\mathcal{A}_{0ijji} = \mathcal{A}_{0jiij} = \mathcal{A}_{0ijij} - J^{-1}\lambda_i W_i \qquad i \neq j, \tag{2.62}$$

where $i, j \in \{1, 2, 3\}$, $W_i := \dfrac{\partial W}{\partial \lambda_i}$, $W_{ij} := \dfrac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}$, and the summation convention does not apply.

With the principal stretches being dependent on $\mathbf{A}$ through (2.12) it can readily be seen that the components of $\mathcal{A}_0$ are constant in the case of a homogeneous deformation and take the particular values

$$\left.\begin{array}{c}\mathcal{A}_{0iiii} = \lambda + 2\mu, \quad \mathcal{A}_{0iijj} = \lambda, \\ \mathcal{A}_{0ijij} = \mathcal{A}_{0ijji} = \mu,\end{array}\right\} \quad i \neq j, \tag{2.63}$$

in the unstressed configuration. The equations in (2.63) are consistent with the classical theory of linear elasticity and λ and μ are the classical Lamé constants.

Equations (2.60)–(2.63) are correct for unconstrained materials and the specialization to incompressible materials provides

$$\mathcal{A}_{0iijj} = \lambda_i \lambda_j W_{ij}, \tag{2.64}$$

$$\mathcal{A}_{0ijij} = \begin{cases} \dfrac{(\lambda_i W_i - \lambda_j W_j)\lambda_i^2}{\lambda_i^2 - \lambda_j^2} & i \neq j, \lambda_i \neq \lambda_j, \\ \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + \lambda_i W_i) & i \neq j, \lambda_i = \lambda_j, \end{cases} \tag{2.65}$$

$$\mathcal{A}_{0ijji} = \mathcal{A}_{0jiij} = \mathcal{A}_{0ijij} - J^{-1}\lambda_i W_i \qquad i \neq j, \tag{2.66}$$

along with replacing (2.63) with, for example,

$$\left.\begin{array}{ccccc}\mathcal{A}_{0iiii} & = & \mathcal{A}_{0ijij} & = & \mu, \\ \mathcal{A}_{0iijj} & = & \mathcal{A}_{0ijji} & = & 0,\end{array}\right\} \quad i \neq j, \tag{2.67}$$

and μ is now the shear modulus, we note the non-uniqueness of (2.67). In the absence of body forces, the equations of motion, when put into incremental form and setting $\dot{\mathbf{x}} = \mathbf{v}$, are

$$\operatorname{Div} \dot{\mathbf{S}} = \rho_0 \mathbf{v}_{,tt} \tag{2.68}$$

which, when the reference configuration is updated to the current configuration, become

$$\operatorname{div} \dot{\mathbf{S}}_0 = \rho \mathbf{v}_{,tt}. \tag{2.69}$$

On making use of (2.55) we write this equation of motion in component form as

$$\frac{\partial}{\partial x_i}\left(\mathcal{A}_{0ijkl}\frac{\partial v_l}{\partial x_k}\right) = \rho v_{j,tt} \tag{2.70}$$

which has the simplified form

$$\mathcal{A}_{0ijkl}\frac{\partial^2 v_l}{\partial x_i \partial x_k} = \rho v_{j,tt} \tag{2.71}$$

when the underlying deformation is homogeneous.

The corresponding equations for incompressible materials are, with use of (2.57),

$$\frac{\partial}{\partial x_i}\left(\mathcal{A}_{0ijkl}\frac{\partial v_l}{\partial x_k} - \dot{p}\delta_{ij} + p\frac{\partial v_i}{\partial x_j}\right) = \rho v_{j,tt} \tag{2.72}$$

and, for a homogeneous underlying deformation, p is also constant so we have

$$\mathcal{A}_{0ijkl}\frac{\partial^2 v_l}{\partial x_i \partial x_k} + p\frac{\partial^2 v_i}{\partial x_j \partial x_i} - \frac{\partial \dot{p}}{\partial x_j} = \rho v_{j,tt} \tag{2.73}$$

which should both be coupled with the constraint (2.59) – this has the effect of setting to zero the coefficient of p in (2.73).

2.4.2 Strong Ellipticity Condition

When we take the rank-one tensor $\mathbf{m} \otimes \mathbf{N}$, with $\mathbf{m}$ an Eulerian and $\mathbf{N}$ a Lagrangean vector, then the *Strong Ellipticity condition* is that

$$\operatorname{tr}[(\mathcal{A}(\mathbf{m} \otimes \mathbf{N}))(\mathbf{m} \otimes \mathbf{N})] > 0 \tag{2.74}$$

and should hold for all $\mathbf{m} \otimes \mathbf{N} \neq \mathbf{0}$. If this inequality does hold then the equilibrium equations for either the global or the incremental problems produce a strongly elliptic system of partial differential equations, the left-hand side being dependent upon the particular form of strain-energy function used in the constitutive equation (2.50) due to the presence of $\mathcal{A}$. We note that when strict inequality in (2.74) is relaxed to include equality then this is the Legendre-Hadamard condition.

It is useful to introduce the *Acoustic Tensor* $\mathbf{Q}(\mathbf{N})$ as

$$Q_{ij} = \mathcal{A}_{\alpha i \beta j} N_\alpha N_\beta, \tag{2.75}$$

allowing strong ellipticity to have the equivalent form

$$[\mathbf{Q}(\mathbf{N})\mathbf{m}] \cdot \mathbf{m} > 0. \tag{2.76}$$

In terms of the acoustic tensor we see that for strong ellipticity to hold then $\mathbf{Q}$ must be positive definite and, on making use of the fact that $\mathbf{Q}$ is symmetric for hyperelastic materials, that necessary and sufficient conditions are

$$
\begin{aligned}
Q_{ii}(\mathrm{N}) &> 0, \quad i \in \{1,2,3\}, & (2.77)\\
Q_{ii}(\mathrm{N})Q_{jj}(\mathrm{N}) - Q_{ij}(\mathrm{N})Q_{ij}(\mathrm{N}) &> 0, \quad i \neq j \in \{1,2,3\}, & (2.78)\\
\det \mathbf{Q}(\mathrm{N}) &> 0, & (2.79)
\end{aligned}
$$

for all $\mathrm{N} \neq 0$.

These conditions may equally be given in terms of $\mathcal{A}$ and we record that for the two-dimensional case only, to reduce the algebraic complexity, strong ellipticity holds if and only if

$$
\left.\begin{array}{c}
\mathcal{A}_{01111} > 0, \quad \mathcal{A}_{02222} > 0, \quad \mathcal{A}_{01212} > 0, \quad \mathcal{A}_{02121} > 0, \\
(\mathcal{A}_{01111}\mathcal{A}_{2222})^{1/2} + (\mathcal{A}_{01212}\mathcal{A}_{02121})^{1/2} \pm (\mathcal{A}_{01122} + \mathcal{A}_{02112}) > 0
\end{array}\right\} \qquad (2.80)
$$

in the compressible case, while for the incompressible case we impose the constraint $\mathbf{m} \cdot \mathrm{A}^{-T}\mathrm{N} = 0$ and have

$$
\left.\begin{array}{c}
\mathcal{A}_{01212} > 0, \quad \mathcal{A}_{02121} > 0, \\
\mathcal{A}_{01111} + \mathcal{A}_{02222} + 2(\mathcal{A}_{01212}\mathcal{A}_{02121})^{1/2} - 2(\mathcal{A}_{01122} + \mathcal{A}_{02112}) > 0.
\end{array}\right\} \qquad (2.81)
$$

2.5 Boundary Value Problems

2.5.1 Variational Principles

We formulate the boundary value problem which has as its governing equations

$$
\begin{aligned}
\mathrm{Div}\,\mathrm{S} + \rho_0 \mathrm{b} &= \mathbf{0}, & (2.82)\\
\mathrm{S} &= \frac{\partial W}{\partial \mathrm{A}}, & (2.83)\\
\mathrm{A} &= \mathrm{Grad}\,\chi, & (2.84)
\end{aligned}
$$

for all $\mathrm{X} \in \mathcal{B}_0$. Implicit in this setting is that the material is hyperelastic and the body is in equilibrium.

The boundary conditions themselves are of both displacement and stress, each being specified on distinct regions of the boundary ∂B_0. We define the part of the boundary ∂B_0^x to be where the displacements are prescribed and similarly ∂B_0^τ to be where the tractions are prescribed. In addition to having $\partial B_0^x \cap \partial B_0^\tau = \emptyset$, we have $\partial B_0^x \cup \partial B_0^\tau = \partial B_0$.

The boundary conditions are then

$$\mathbf{x} = \boldsymbol{\xi} \quad \text{on } \partial B_0^x, \tag{2.85}$$

$$\mathbf{S}^T\mathbf{N} = \boldsymbol{\tau} \quad \text{on } \partial B_0^\tau, \tag{2.86}$$

with $\boldsymbol{\xi}$ being a known function on ∂B_o^x and $\boldsymbol{\tau}$ having functional dependency on both $\boldsymbol{\chi}$ and $\mathbf{A}$ on ∂B_0^τ.

A *Kinematically Admissible Deformation* $\boldsymbol{\chi}$ is one which satisfies the boundary condition of place (2.85). It is emphasized that the stress associated with this deformation, through (2.84) and (2.83), need not satisfy the equilibrium equation (2.82) or the boundary condition (2.86).

Likewise, a *Statically Admissible Stress Field* $\mathbf{S}$ is one which satisfies the boundary condition of stress (2.86) and, furthermore, satisfies the equilibrium equations (2.82).

The effect of the deformation field $\boldsymbol{\chi}$ on $\mathbf{S}$ is through the body force $\mathbf{b}$ and prescribed traction $\boldsymbol{\tau}$ wherein we may choose any kinematically admissible deformation.

On using the equilibrium equations (2.82), boundary conditions (2.85), (2.86) and the divergence theorem we have, for statically admissible stress fields and kinematically admissible deformation fields,

$$\left.\begin{aligned}\int_{\partial B_0^\tau} \boldsymbol{\tau}\cdot\boldsymbol{\chi}\,dA + \int_{\partial B_0^x} \mathbf{S}^T\mathbf{N}\cdot\boldsymbol{\xi}\,dA &= \int_{\partial B_0} \mathbf{N}\cdot(\mathbf{S}\boldsymbol{\chi})\,dA \\ &= \int_{B_0} \mathrm{Div}\,(\mathbf{S}\boldsymbol{\chi})\,dV \\ &= \int_{B_0} [\mathrm{tr}\,(\mathbf{S}\mathbf{A} - \rho_0\mathbf{b}\cdot\boldsymbol{\chi}]\,dV.\end{aligned}\right\} \tag{2.87}$$

Considering the kinematically admissible deformation χ^* and adopting the same working as in (2.87) gives

$$\int_{\partial B_0^\tau} \tau \cdot \chi^* \, dA + \int_{\partial B_0^x} \mathbf{S}^T\mathbf{N} \cdot \xi \, dA \\ = \int_{B_0} \operatorname{tr}(\mathbf{S}\mathbf{A}^*) \, dV - \int_{B_0} \rho_0 \mathbf{b} \cdot \chi^* \, dV, \tag{2.88}$$

where we have introduced the notation $\mathbf{A}^* = \operatorname{Grad}\chi^*$.

We define the quantity $\delta\chi = \chi^* - \chi$ and subtract (2.87) from (2.88) to get

$$\int_{\partial B_0^\tau} \tau \cdot \delta\chi \, dA + \int_{B_0} \rho_0 \mathbf{b} \cdot \delta\chi \, dV = \int_{B_0} \operatorname{tr}[\mathbf{S}\operatorname{Grad}\delta\chi] \, dV \tag{2.89}$$

The integrals on the left-hand side represent the work done by the, unchanging, tractions and body forces in the *virtual displacement* $\delta\chi$ while the right hand side corresponds to the change in the stress work. This is the *Principle of Virtual Work.*

We now turn our attention to deriving a complementary *Principle of Virtual Stress.* Consider a deformation χ and stress $\mathbf{S}$ satisfying all of the governing equations (2.82)–(2.84) and boundary conditions (2.85) and (2.86). Now take a statically admissible stress field $\mathbf{S}^*$ associated with χ (which is necessarily kinematically admissible) and define $\delta\mathbf{S} = \mathbf{S}^* - \mathbf{S}$. From the definition of statically admissible we can see that

$$\operatorname{Div}\delta\mathbf{S} = \mathbf{0} \quad \text{in } B_0 \tag{2.90}$$

and

$$\delta\mathbf{S}^T\mathbf{N} = \mathbf{0} \quad \text{on } \partial B_0^\tau. \tag{2.91}$$

As in the derivation of the result in (2.89), we may also derive

$$\int_{\partial B_0^x} \mathbf{N} \cdot (\delta\mathbf{S}\chi) \, dA = \int_{B_0} \operatorname{tr}(\mathbf{A}\delta\mathbf{S}) \, dV \tag{2.92}$$

which is the *Principle of Virtual Complementary Work.*

Since we are working in the context of hyperelastic materials, the first order approximation

$$\delta W = \operatorname{tr}\left(\frac{\partial W}{\partial \mathbf{A}}\delta\mathbf{A}\right) \tag{2.93}$$

follows. Thus the integral on the right-hand side of (2.89) becomes $\int_{B_0} \delta W \, dV$ and we now seek to rewrite (2.89) as the variation of a scalar functional.

Setting $\mathrm{b} = -\operatorname{grad}\phi$, $\boldsymbol{\tau} = -\operatorname{grad} v$ in the special case where $\boldsymbol{\tau} = \boldsymbol{\tau}(\chi)$ reduces equation (2.89) to

$$\delta\left\{\int_{B_0} (W + \rho_0\phi)\, dV + \int_{\partial B_0^\tau} v \, dA\right\} = 0. \tag{2.94}$$

More generally, $\boldsymbol{\tau} = \boldsymbol{\tau}(\chi, \mathrm{A})$ and we write

$$\boldsymbol{\tau} = \boldsymbol{\Sigma}^T \mathrm{N} \tag{2.95}$$

where $\boldsymbol{\Sigma}$ is known on ∂B_0^τ. Use of the divergence theorem then gives

$$\delta\left\{\int_{B_0} (W + \rho_0\phi - \omega)\, dV\right\} = 0 \tag{2.96}$$

where, now,

$$\delta\omega = (\operatorname{Div}\boldsymbol{\Sigma}) \cdot \delta\chi + \operatorname{tr}(\boldsymbol{\Sigma}\delta\mathrm{A}) \tag{2.97}$$

represents the variation through the tractions and $\boldsymbol{\Sigma}$ is assumed to be implicitly defined throughout B_0.

We now define the functional $E\{\chi\}$ by

$$E\{\chi\} = \int_{B_0} [W(\operatorname{Grad}\chi) + \rho_0(\mathrm{X})\phi(\chi)]\, dV + \int_{\partial B_0^\tau} v(\mathrm{X}, \chi)\, dA \tag{2.98}$$

or

$$E\{\chi\} = \int_{B_0} [W(\operatorname{Grad}\chi) + \rho_0(\mathrm{X})\phi(\chi) - \omega(\mathrm{X}, \chi, \operatorname{Grad}\chi)]\, dV, \tag{2.99}$$

as appropriate, which both give

$$\delta E = \int_{\partial B_0^\tau} (\mathrm{S}^T\mathrm{N} - \boldsymbol{\tau}) \cdot \delta\chi \, dA - \int_{B_0} (\operatorname{Div}\mathbf{S} + \rho_0\mathrm{b}) \cdot \delta\chi \, dV \tag{2.100}$$

with $\delta\chi$ an arbitrary variation.

Thus the variational principle is given and is taken to mean that the deformation χ is a solution to the boundary value problem if and only if it produces a stationary value for the functional $E\{\chi\}$ for any variation $\delta\chi$ satisfying the condition $\delta\chi = \mathbf{0}$ on ∂B_0^x. See, for example Ogden (1984).

This statement is the *Principle of Stationary Potential Energy,* since $E\{\chi\}$ represents the potential energy, and may be further strengthened in certain cases. One such context is that of *linear elasticity* where we have the following

Theorem (Minimum Potential Energy) *Of all displacements satisfying the given boundary conditions, those which satisfy the equilibrium equations make the potential energy an absolute minimum.*

Proof of this theorem is omitted here but can be found in texts such as Sokolnikoff (1956) from where we have taken the statement of the theorem. We merely remark that the proof given in this text relies upon W being of positive definite form.

In contrast with the potential energy functional $E\{\chi\}$, we also have

$$\delta\left\{\int_{\partial B_0^x} (\mathbf{N}\cdot\mathbf{S}\boldsymbol{\xi})\,dA - \int_{B_0} [\operatorname{tr}(\mathbf{AS}) - W(\mathbf{A})]\,dV\right\} = 0 \tag{2.101}$$

from where we have made use of

$$\operatorname{tr}(\mathbf{A}\delta\mathbf{S}) = \delta\{\operatorname{tr}(\mathbf{AS}) - W(\mathbf{A})\} \tag{2.102}$$

with the help of (2.83) and (2.93). We note in particular that we rely upon $\mathbf{S}$ and $\mathbf{A}$ to be actual stress and deformation fields and that the functional in (2.101) has dependence on $\mathbf{S}$ and χ.

Define now the *Complementary Energy Density* $W_c(\mathbf{S},\mathbf{A})$ by the Legendre transformation

$$W_c(\mathbf{S},\mathbf{A}) = \operatorname{tr}(\mathbf{AS}) - W(\mathbf{A}). \tag{2.103}$$

We make the assumption that $\mathbf{S} = \partial W/\partial\mathbf{A}$ is invertible,thus allowing $\mathbf{A}$ to be regarded as a function of $\mathbf{S}$, and thereby establish the connection

$$\mathbf{A} = \frac{\partial W_c}{\partial \mathbf{S}} \tag{2.104}$$

from (2.103).

Formally we define the complementary energy functional $E_c\{\mathbf{S}\}$ by

$$E_c\{\mathbf{S}\} = \int_{\partial B_0^{x}} \mathbf{N} \cdot (\mathbf{S}\xi)\, dA - \int_{B_0} W_c(\mathbf{S})\, dV \tag{2.105}$$

which holds for statically admissible stress fields $\mathbf{S}$ and kinematically admissible deformation fields χ.

Taking, now, the variation δE_c with (2.88) and (2.89) being applied we can reduce it to

$$\delta E_c = \int_{B_0} \operatorname{tr}\left[(\operatorname{Grad}\chi - \mathbf{A})\delta\mathbf{S}\right] dV \tag{2.106}$$

where $\mathbf{A} = \partial W_c/\partial \mathbf{S}$ need not be a deformation field. However $\delta E_c = 0$ if and only if $\mathbf{A} = \operatorname{Grad}\chi$.

Thus, the energy functional $E_c\{\mathbf{S}\}$ is stationary within the class of statically admissible stress fields if and only if $\mathbf{S}$ is associated with a solution of the boundary value problem, and this is a statement of the *Principle of Stationary Complementary Energy.*

In respect of the difficulties mentioned in inverting $\mathbf{S}$ and $\mathbf{A}$ we note that such difficulties are considerably reduced when we consider the conjugate variables $(\mathbf{T},\mathbf{U})$ instead of the conjugate variables $(\mathbf{S},\mathbf{A})$. In particular, as $W(\mathbf{A}) = W(\mathbf{U})$ we define $\mathbf{T}$ by

$$\mathbf{T} = \frac{\partial W(\mathbf{U})}{\partial \mathbf{U}} \tag{2.107}$$

and the complementary energy density $W_c(\mathbf{T})$ by

$$W(\mathbf{U}) + W_c(\mathbf{T}) = \operatorname{tr}(\mathbf{TU}). \tag{2.108}$$

Notably, if $W(\mathbf{U})$ is a strictly convex function of $\mathbf{U}$ then (2.107) is uniquely invertible and $W_c(\mathbf{T})$ is a strictly convex function of $(\mathbf{T})$.

2.5.2 Stability

We return to the boundary value problem set out in (2.82)–(2.86). We suppose that a solution χ is known and allow the boundary conditions to vary incrementally according as

$$\mathrm{v} = \dot{\mathrm{x}} \;=\; \dot{\xi} \quad \text{on } \partial B_0^x, \tag{2.109}$$

$$\dot{\mathrm{S}}^T\mathrm{N} \;=\; \dot{\tau} \quad \text{on } \partial B_0^\tau. \tag{2.110}$$

The incremental form of the equilibrium equations are

$$\mathrm{Div}\,\dot{\mathrm{S}} + \rho_0\dot{\mathrm{b}} = \mathbf{0}, \tag{2.111}$$

where $\dot{\mathrm{b}}$ represents the increment in the body force per unit reference volume. If the reference configuration is updated to the current configuration then (2.111) becomes

$$\mathrm{div}\,\dot{\mathrm{S}}_0 + \rho_0\dot{\mathrm{b}} = \mathbf{0}, \tag{2.112}$$

the rotational balance equations becoming

$$\dot{\mathrm{A}}_0\sigma + \dot{\mathrm{S}}_0 = \dot{\mathrm{S}}_0^T + \sigma\dot{\mathrm{A}}_0^T. \tag{2.113}$$

Suppose χ and χ' are two solutions to (2.82)–(2.86) with the corresponding deformation gradients A, A'; nominal stresses S, S'; body forces b, b' in B_0; and surface tractions τ, τ' on ∂B_0^τ. We take the vector dot product of (2.82), in each of the forms given by using (S, b) and $(\mathrm{S}', \mathrm{b}')$, with each of the solutions χ, χ'; then on integrating over the volume and using the divergence theorem we have

$$\begin{aligned}&\int_{B_0} \mathrm{tr}\,[(\mathrm{S}' - \mathrm{S})(\mathrm{A}' - \mathrm{A})]\,dV \\ &\quad= \int_{\partial B_0^\tau} (\tau' - \tau)\cdot(\chi' - \chi)\,dA + \int_{B_0} \rho_0(\mathrm{b}' - \mathrm{b})\cdot(\chi' - \chi)\,dV.\end{aligned} \tag{2.114}$$

In the case of the body forces being independent of χ and, likewise, the surface tractions being independent of χ then we have $\mathrm{b}' = \mathrm{b}$ and $\tau' = \tau$, leading to

$$\int_{B_0} \mathrm{tr}\,[(\mathrm{S}' - \mathrm{S})(\mathrm{A}' - \mathrm{A})]\,dV = 0. \tag{2.115}$$

Since (2.115) holds in the case of two solutions χ, χ' it follows that a sufficient condition for uniqueness of χ is

$$\int_{B_0} \mathrm{tr}\,[(\mathrm{S}' - \mathrm{S})(\mathrm{A}' - \mathrm{A})]\,dV > 0. \tag{2.116}$$

Consider now the inequality for hyperelastic materials

$$W(\mathbf{A}') - W(\mathbf{A}) - \operatorname{tr}[\mathbf{S}(\mathbf{A}' - \mathbf{A})] > 0 \tag{2.117}$$

for all $\mathbf{A}, \mathbf{A}' \neq \mathbf{A}$; where W is a convex function. Notice that interchanging $\mathbf{A}$ with $\mathbf{A}'$ and adding the two inequalities implies (2.116) pointwise. Integration of (2.117) with use of the divergence theorem leads to

$$\begin{aligned} &\int_{B_0} [W(\mathbf{A}' - \rho_0 \mathbf{b} \cdot \chi'] \, dV - \int_{\partial B_0^\tau} \tau \cdot \chi' \, dA \\ &> \int_{B_0} [W(\mathbf{A} - \rho_0 \mathbf{b} \cdot \chi] \, dV - \int_{\partial B_0^\tau} \tau \cdot \chi \, dA \end{aligned} \tag{2.118}$$

which, through (2.98), is also written as

$$E\{\chi'\} > E\{\chi\}\,. \tag{2.119}$$

Thus, if (2.119) holds for χ a solution and $\chi' \neq \chi$ a kinematically admissible deformation then χ is unique.

Rearranging (2.118) into

$$\begin{aligned} &\int_{B_0} [W(\mathbf{A}' - W(\mathbf{A})] \, dV \\ &> \int_{\partial B_0^\tau} \tau \cdot (\chi' - \chi) \, dA + \int_{B_0} \rho_0 \mathbf{b} \cdot (\chi' - \chi) \, dV \end{aligned} \tag{2.120}$$

shows that the increase in the internal stored energy is greater than the work done by the body force and the surface tractions in moving from χ to χ'. It therefore follows that the equilibrium solution χ is a stable solution.

We can weaken the inequality (2.120) to include equality with zero, so defining points that are neutrally stable. We thus have a global condition, which is sufficient, for stability of χ as being

$$E\{\chi'\} \geq E\{\chi\}\,. \tag{2.121}$$

The restrictions placed on the development of this condition mean that it is unsuitable as a global necessary condition although we can take (2.121) as necessary and sufficient for local stability by restricting χ' to some neighbourhood of χ. This criterion is referred to as the *infinitesimal stability criterion.*

Turning now to incremental uniqueness and stability we consider two incremental solutions $\dot{\chi}$ and $\dot{\chi}'$ to the incremental boundary value problem (2.109)–(2.111) with the corresponding incremental quantities of nominal stress $\dot{\mathbf{S}}, \dot{\mathbf{S}}'$; body forces $\dot{\mathbf{b}}, \dot{\mathbf{b}}'$ in B_0; and incremental tractions $\dot{\tau}, \dot{\tau}'$ on ∂B_0^{τ}.

Defining $\Delta(\dot{\chi}) = (\dot{\chi})' - (\dot{\chi})$, and generally $\Delta(\cdot) = (\cdot)' - (\cdot)$, it follows from the boundary conditions and equilibrium equations that

$$\begin{aligned} \Delta(\dot{\chi}) &= 0 \qquad \text{on } \partial B_0^{x}, & (2.122)\\ \Delta(\dot{\mathbf{S}}^T)\mathbf{N} &= \Delta(\dot{\tau}) \quad \text{on } \partial B_0^{\tau}, & (2.123)\\ \mathrm{Div}\,(\Delta(\dot{\mathbf{S}})) + \rho_0\Delta(\dot{\mathbf{b}}) &= 0 \qquad \text{in } B_0. & (2.124) \end{aligned}$$

Integrating the vector dot product of $\Delta(\dot{\chi})$ and the left-hand side of (2.124) over B_0 we obtain, upon use of the divergence theorem,

$$\begin{aligned} &\int_{B_0} \mathrm{tr}\,\left[\Delta(\dot{\mathbf{S}})\Delta(\dot{\mathbf{A}})\right] dV & (2.125)\\ &\quad = \int_{\partial B_0^{\tau}} \Delta(\dot{\tau})\cdot\Delta(\dot{\chi})\, dA + \int_{B_0} \rho_0\Delta(\dot{\mathbf{b}})\cdot\Delta(\dot{\chi})\, dV. \end{aligned}$$

Paralleling the derivation of (2.119), this time without restricting the body forces and surface tractions, we have the sufficient condition for uniqueness of the incremental solution $\dot{\chi}$ as

$$\begin{aligned} &\int_{B_0} \mathrm{tr}\,\left[\Delta(\dot{\mathbf{S}})\Delta(\dot{\mathbf{A}})\right] dV - \int_{\partial B_0^{\tau}} \Delta(\dot{\tau})\cdot\Delta(\dot{\chi})\, dA \\ &\qquad - \int_{B_0} \rho_0\Delta(\dot{\mathbf{b}})\cdot\Delta(\dot{\chi})\, dV \;>\; 0. & (2.126) \end{aligned}$$

With equality included in (2.126) we have the sufficient condition for stability.

2.6 Simple Shear

In this section we describe the simple shear deformation and record some of the main features of the deformation and stress fields arising.

2.6.1 Deformation and Stress

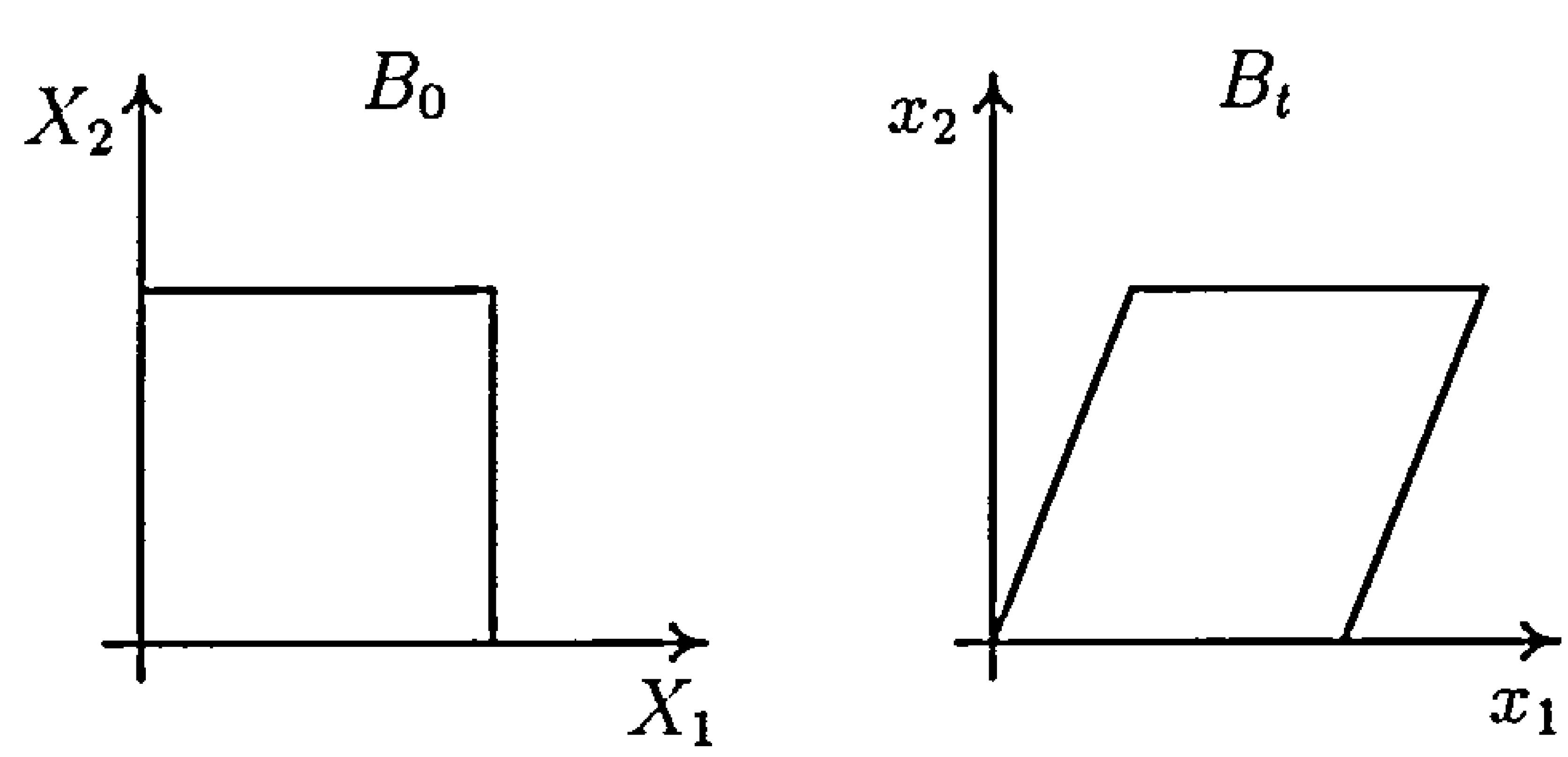
We consider a solid block which will be deformed from B_0 to B_t and take Cartesian co-ordinates (X_1, X_2, X_3) and (x_1, x_2, x_3) to represent a point P in each configuration. The co-ordinate systems will be aligned for this problem and and have the same origin so $\mathbf{e}_i = \mathbf{E}_i$, $i \in \{1, 2, 3\}$.

The simple shear deformation in the (1,2)-plane is given by the mapping

$$\left.\begin{aligned} x_1 &= X_1 + \varepsilon X_2, \\ x_2 &= X_2, \\ x_3 &= X_3, \end{aligned}\right\} \tag{2.127}$$

and is shown in Figure 2.1. The deformation parameter ε is known as the *amount of shear.*

Figure 2.1: Simple shear deformation



The deformation gradient is then

$$\mathbf{A} = \begin{bmatrix} 1 & \varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.128}$$

which we can easily verify as having $J = 1$ showing that the deformation is isochoric.

We can see that $\lambda_3 = 1$ is a principal stretch (reflecting no strain in the x_3 direction). If λ is an eigenvalue of $\mathbf{U}$ then λ^2 is an eigenvalue of $\mathbf{U}^2 = \mathbf{C} = \mathbf{A}^T\mathbf{A}$, and we calculate the other two stretches from the characteristic equation

$$\det(\mathbf{C} - \lambda^2 \mathbf{I}) = 0, \tag{2.129}$$

which gives the explicit results

$$\lambda = \left[\frac{2 + \varepsilon^2 \pm \varepsilon\sqrt{4 + \varepsilon^2}}{2}\right]^{1/2} \tag{2.130}$$

but is perhaps more useful in the form

$$\lambda^2 + \lambda^{-2} = 2 + \varepsilon^2 \tag{2.131}$$

or, upon rearrangement,

$$\lambda - \lambda^{-1} = \varepsilon. \tag{2.132}$$

We have made use of the isochoric nature of the deformation and set, without loss of generality in view of the symmetry of $W(\lambda_1, \lambda_2\lambda_3)$, $\lambda_1 = \lambda \geq 1$ so that $\lambda_2 = \lambda^{-1}$and $\varepsilon \geq 0$. Specifically,

$$\lambda = \frac{1}{2}\varepsilon + \sqrt{1 + \frac{1}{4}\varepsilon^2}. \tag{2.133}$$

Now consider the Eulerian axes $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\}$ which have as components, relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$,

$$\left.\begin{array}{rcl} \mathbf{v}^{(1)} &=& (\cos\theta, \sin\theta, 0), \\ \mathbf{v}^{(2)} &=& (-\sin\theta, \cos\theta, 0), \\ \mathbf{v}^{(3)} &=& (0, 0, 1). \end{array}\right\} \tag{2.134}$$

From here we restrict attention to the (1,2)-plane and suppress the dependence upon the x_3 direction for simplicity, there is no loss of generality in doing this.

We have the following connection

$$\mathbf{B} = \mathbf{V}^2 = \lambda^2 \mathbf{v}^{(1)} \otimes \mathbf{v}^{(1)} + \lambda^{-2} \mathbf{v}^{(2)} \otimes \mathbf{v}^{(2)} \tag{2.135}$$

which gives, once we calculate the product $\mathbf{B} = \mathbf{A}\mathbf{A}^T$,

$$\begin{bmatrix} 1+\varepsilon^2 & \varepsilon \\ \varepsilon & 1 \end{bmatrix} = \lambda^2 \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} + \lambda^{-2} \begin{bmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix}. \tag{2.136}$$

We then get the three equations linking the rotation of the Eulerian axes to the deformation through the stretch λ,

$$\lambda^2\cos^2\theta + \lambda^{-2}\sin^2\theta = 1+\varepsilon^2, \tag{2.137}$$

$$\lambda^2\cos\theta\sin\theta - \lambda^{-2}\cos\theta\sin\theta = \varepsilon, \tag{2.138}$$

$$\lambda^2\sin^2\theta + \lambda^{-2}\cos^2\theta = 1, \tag{2.139}$$

from which we derive the relationship

$$\tan 2\theta_E = \frac{2}{\varepsilon} \tag{2.140}$$

with the restriction $0 < \theta_E \leq \pi/4$, where we now take θ_E to be the angle made by $\mathbf{v}^{(1)}$ with the X_1 direction.

Similarly, we may establish for the rotation of the Lagrangean principal axes the relation

$$\tan 2\theta_L = -\frac{2}{\varepsilon} \tag{2.141}$$

with the restriction $\pi/4 \leq \theta_L \leq \pi/2$ by replacing $\mathbf{B}$ with $\mathbf{C}$ in the above calculation. It can be seen from equations (2.140) and (2.141) that the undeformed configuration $\varepsilon = 0$ finds the Eulerian and Lagrangean axes aligned with $\theta_E = \theta_L = \pi/4$.

In terms of the parameter ε we have the particular values for the principal invariants

$$\left.\begin{aligned} I_1 &= 3+\varepsilon^2, \\ I_2 &= 3+\varepsilon^2, \\ I_3 &= 1, \end{aligned}\right\} \tag{2.142}$$

the last being clear from the deformation being isochoric. We are then able to calculate the nominal stress $\mathbf{S}$ from (2.45) as

$$\mathbf{S} = 2\left(\frac{\partial\widehat{W}}{\partial I_1} + (3+\varepsilon^2)\frac{\partial\widehat{W}}{\partial I_2}\right)\begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix} + \left(2\frac{\partial\widehat{W}}{\partial I_3} - p\right)\begin{bmatrix} 1 & -\varepsilon \\ 0 & 1 \end{bmatrix} \tag{2.143}$$
$$-2\frac{\partial\widehat{W}}{\partial I_2}\begin{bmatrix} 1+\varepsilon^2 & \varepsilon \\ \varepsilon(2+\varepsilon) & 1+\varepsilon^2 \end{bmatrix}$$

where we have specialized this to the incompressible case but the compressible case is recovered by setting $p = 0$, with I_3 being unity in either case.

2.6.2 Incremental Relations

In this section we make explicit the relationship between the components of the tensor $\mathcal{A}_0$ with respect to the Eulerian principal axes and the corresponding components with respect to the chosen Cartesian co-ordinate system.

In the incremental equilibrium equations (2.70) and (2.72), the components of $\mathcal{A}_0$ are taken to be those given in (2.60)–(2.62) or (2.64)–(2.66). Therefore, for the equations to be consistent, both v_i and x_i, $i \in \{1,2,3\}$, should be measured along the principal axes. If we wish to measure them along the original choice of axes then we must derive another tensor $\mathcal{B}_0$, equivalent in its function to $\mathcal{A}_0$ but also measured with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

As the tensor $\mathcal{A}_0$ is measured along the principal axes we must necessarily have

$$\mathcal{A}_0 = \mathcal{A}_{0ijkl}\mathbf{v}^{(i)} \otimes \mathbf{v}^{(j)} \otimes \mathbf{v}^{(k)} \otimes \mathbf{v}^{(l)} \tag{2.144}$$

in general. The symmetries inherent in the components of $\mathcal{A}_0$ enable this to be expanded to the following simple form

$$\left.\begin{aligned}\mathcal{A}_0 \;=\;\; & \mathcal{A}_{01111}\{\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)}\} \\ & +\mathcal{A}_{01122}\{\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(2)} + \mathbf{v}^{(2)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)}\} \\ & +\mathcal{A}_{01221}\{\mathbf{v}^{(1)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(1)} + \mathbf{v}^{(2)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(2)}\} \\ & +\mathcal{A}_{01212}\{\mathbf{v}^{(1)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(2)} + \frac{\lambda_2^2}{\lambda_1^2}\mathbf{v}^{(2)}\otimes\mathbf{v}^{(1)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(1)}\} \\ & +\mathcal{A}_{02222}\{\mathbf{v}^{(2)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(2)}\otimes\mathbf{v}^{(2)}\}\end{aligned}\right\} \tag{2.145}$$

where for completeness we have kept the distinction between λ_1 and λ_2.

The equilibrium equations will now be taken to have the equivalent form

$$\operatorname{div}(\mathcal{B}_0\dot{\mathbf{A}}_0) = \mathbf{0} \tag{2.146}$$

which in component form is

$$\mathcal{B}_{0ijkl}v_{l,ik} = 0, \quad j \in \{1,2\}, \tag{2.147}$$

all quantities being relative to the chosen axes of B_t.

Substitution for the vectors $\mathbf{v}^{(i)}$, from (2.134), in the expansion of $\mathcal{A}_0$ will provide expressions for the components of $\mathcal{B}_0$ in terms of the $\mathcal{A}_{0ijkl}$, $\theta = \theta_E$ and λ.

We first record the symmetries that result from the transformation. We have for $i \neq j \in \{1,2\}$

$$\left.\begin{aligned}\mathcal{B}_{0ijji} = \mathcal{B}_{0jiij}, &\quad \mathcal{B}_{0iijj} = \mathcal{B}_{0jjii}, \\ \mathcal{B}_{0iiij} = \mathcal{B}_{0ijii}, &\quad \mathcal{B}_{0iiji} = \mathcal{B}_{0jiii}, \\ \mathcal{B}_{0ijij} = \mathcal{B}_{0jiji} + \mathcal{A}_{0jiji}&\left(\frac{\lambda_i^2}{\lambda_j^2}-1\right)(\cos^2\theta - \sin^2\theta).\end{aligned}\right\} \tag{2.148}$$

In the following expressions we remove the distinction between λ_1 and λ_2 without loss of generality since they only appear in the ratio $(\lambda_2/\lambda_1)^2 = \lambda^{-4}$.

The independent components of $\mathcal{B}_0$ are then

$$\left.\begin{aligned}
\mathcal{B}_{01111} &= \mathcal{A}_{01111}\cos^4\theta + \mathcal{A}_{02222}\sin^4\theta \\
&\quad + [2(\mathcal{A}_{01122} + \mathcal{A}_{01221}) + \mathcal{A}_{01212}(1+\lambda^{-4})]\cos^2\theta\sin^2\theta, \\
\mathcal{B}_{02222} &= \mathcal{A}_{01111}\sin^4\theta + \mathcal{A}_{02222}\cos^4\theta \\
&\quad + [2(\mathcal{A}_{01122} + \mathcal{A}_{01221}) + \mathcal{A}_{01212}(1+\lambda^{-4})]\cos^2\theta\sin^2\theta, \\
\mathcal{B}_{01221} &= [\mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - \mathcal{A}_{01212}(1+\lambda^{-4})]\cos^2\theta\sin^2\theta \\
&\quad + \mathcal{A}_{01221}(\cos^4\theta + \sin^4\theta), \\
\mathcal{B}_{01122} &= [\mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01221} - \mathcal{A}_{01212}(1+\lambda^{-4})]\cos^2\theta\sin^2\theta \\
&\quad + \mathcal{A}_{01122}(\cos^4\theta + \sin^4\theta), \\
\mathcal{B}_{01112} &= \cos\theta\sin\theta[\mathcal{A}_{01111}\cos^2\theta - \mathcal{A}_{02222}\sin^2\theta \\
&\quad + (\mathcal{A}_{01122} + \mathcal{A}_{01221})(\sin^2\theta - \cos^2\theta) \\
&\quad - \mathcal{A}_{01212}(\cos^2\theta - \lambda^{-4}\sin^2\theta)], \\
\mathcal{B}_{01121} &= \cos\theta\sin\theta[\mathcal{A}_{01111}\cos^2\theta - \mathcal{A}_{02222}\sin^2\theta \\
&\quad + (\mathcal{A}_{01122} + \mathcal{A}_{01221})(\sin^2\theta - \cos^2\theta) \\
&\quad - \mathcal{A}_{01212}(\sin^2\theta - \lambda^{-4}\cos^2\theta)], \\
\mathcal{B}_{01212} &= [\mathcal{A}_{01111} - 2(\mathcal{A}_{01122} + \mathcal{A}_{01221}) + \mathcal{A}_{02222}]\cos^2\theta\sin^2\theta \\
&\quad + \mathcal{A}_{01212}(\cos^4\theta + \lambda^{-4}\sin^4\theta).
\end{aligned}\right\} \tag{2.149}$$

The angle θ in equations (2.149) is the Eulerian angle θ_E.

Chapter 3

Surface Waves in an Elastic Half-Space

3.1 Formulation of Problem

In this section we consider an isotropic, hyperelastic and incompressible elastic half-space occupying the region $x_2 \leq 0$. The material is put into the quasi-static simple shear deformation upon which we look for incremental deformations corresponding to surface (Rayleigh) waves.

The direction of propagation of these waves will be taken to be the x_1 direction, again so that x_3 dependence is suppressed, in order that the algebraic complexity will be reduced.

The reference configuration of the problem will be updated to the current configuration and the incremental equations from (2.53) onwards will apply. We establish the incremental equations of motion appropriate to this problem and also consider boundary conditions corresponding to zero tractions, in the incremental sense, on the surface $x_2 = 0$.

3.1.1 Incremental Equations of Motion

The equations of incremental motion are recorded in (2.73) for an incompressible material. As remarked in Section 2.6 the components $\mathcal{A}_{0ijkl}$ given by (2.64)–(2.66) are with respect to the Eulerian axes. We adopt the approach of taking a co-ordinate system which is aligned to these axes in order that the other quantities in the equations may be evaluated relative to this system.

In B_t we therefore have the co-ordinates (x_1, x_2) as before but we also introduce the co-ordinates (x_1', x_2') which are associated with the $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ directions respectively. The two systems are linked through the transformation

$$\begin{aligned} x_1 &= x_1' \cos\theta - x_2' \sin\theta, \\ x_2 &= x_1' \sin\theta + x_2' \cos\theta, \end{aligned} \tag{3.1}$$

where again $\theta = \theta_E$. For simplicity, we no longer use the subscript $_E$ and let it be implicit that the angle θ is the Eulerian angle θ_E.

Similarly, we define (v_1', v_2') to be the increments in the deformation or, equivalently, the velocity components corresponding to (x_1', x_2'), and we also take $\dot{p}'$ to have dependence on x_1', x_2'.

We can therefore recast the equations of motion in component form as

$$\mathcal{A}_{0jilk} v'_{k,jl} - \dot{p}'_{,i} = \rho \ddot{v}'_i, \quad i \in \{1, 2\}, \tag{3.2}$$

where now the notation $()_{,i}$ is used to represent $\partial/\partial x_i'$.

Expansion of the equations of motion is facilitated by the symmetries of the components of $\mathcal{A}_0$ and yields

$$\begin{aligned} \mathcal{A}_{01111} v'_{1,11} + (\mathcal{A}_{01122} + \mathcal{A}_{01221}) v'_{2,12} & \qquad\qquad (3.3) \\ + \mathcal{A}_{02121} v'_{1,22} + \dot{p}'_{,1} &= \rho \ddot{v}'_1, \\ \mathcal{A}_{02222} v'_{2,22} + (\mathcal{A}_{01122} + \mathcal{A}_{01221}) v'_{1,12} & \qquad\qquad (3.4) \\ + \mathcal{A}_{01212} v'_{2,11} + \dot{p}'_{,2} &= \rho \ddot{v}'_2. \end{aligned}$$

Removal of the $\dot{p}'_{,i}$ is effected by differentiating both equations so as to obtain the term $\dot{p}_{,12}$ which can then be used for substituting. Thus we shall be left with a single equation involving the constants $\mathcal{A}_{0ijkl}$ and the third partial derivatives of v_i, $i \in \{1, 2\}$. In taking this approach we lose information about p through differentiating and note that it can only be determined to within an additve constant

At this stage we bring into play the incompressibility condition $\operatorname{div} \mathrm{v} = 0$ from equation (2.59). It is easy to verify, using the transformation given in (3.1), that $\dfrac{\partial v_i}{\partial x_i} = 0 \Longleftrightarrow \dfrac{\partial v_i'}{\partial x_i'} = 0$. Therefore for both co-ordinate systems we deduce the existence of the functions $\psi(x_1, x_2, t)$ and $\psi'(x_1', x_2', t)$ such that

$$\left.\begin{aligned} v_1 = \frac{\partial \psi}{\partial x_2}, \quad v_2 = -\frac{\partial \psi}{\partial x_1}, \\ \text{and} \quad v_1' = \psi'_{,2}, \quad v_2' = -\psi'_{,1}. \end{aligned}\right\} \tag{3.5}$$

Our governing equation thus transforms into a fourth-order partial differential equation in ψ' with constant coefficients.

For convenience we use the following definitions by Dowaikh and Ogden (1990)

$$\left.\begin{aligned} \alpha = \mathcal{A}_{01212}, \quad \gamma = \mathcal{A}_{02121}, \\ 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{01221}. \end{aligned}\right\} \tag{3.6}$$

These terms arise as the coefficients of the governing equation after making the above substitutions so that it simplifies as

$$\alpha\psi'_{,1111} + 2\beta\psi'_{,1122} + \gamma\psi'_{,2222} = \rho(\ddot{\psi}'_{,11} + \ddot{\psi}'_{,22}). \tag{3.7}$$

The form of the differential equation for ψ' is the same as that found by Dowaikh and Ogden (1990) with the modification here being through the choice of axes. In their paper the principal axes coincided with the co-ordinate axes whereas we have had to rotate our reference frame in order to recover the above form.

Alternatively, in the light of (3.5), the equations of motion (3.3), (3.4) may now be written in the form

$$
\begin{aligned}
(\mathcal{A}_{01111} - \mathcal{A}_{01122} + p)v'_{1,11} - \dot{p}'_{,1} + \mathcal{A}_{02121}v'_{1,22} & \\
+ (\mathcal{A}_{02121} - \sigma_2)v'_{2,12} &= \rho\ddot{v}'_1, \qquad (3.8)\\
(\mathcal{A}_{02222} - \mathcal{A}_{01122} + p)v'_{2,22} - \dot{p}'_{,2} + \mathcal{A}_{01212}v'_{2,11} & \\
+ (\mathcal{A}_{02121} - \sigma_2)v'_{1,21} &= \rho\ddot{v}'_2. \qquad (3.9)
\end{aligned}
$$

To show these are equivalent to (3.3) and (3.4) we first subtract (3.3) from (3.8) and get the remainder

$$
\begin{aligned}
(-\mathcal{A}_{01122} + p')v'_{1,11} + (\mathcal{A}_{02121} - \sigma_2)v'_{2,12} & \\
- (\mathcal{A}_{01122} + \mathcal{A}_{01221})v'_{2,12} &= (\mathcal{A}_{01221} - \mathcal{A}_{02121} + p' + \sigma_2)\psi'_{,112}.
\end{aligned}
$$

We now make use of the connection

$$
\mathcal{A}_{01221} + p' = \mathcal{A}_{02121} - \sigma_2 = \gamma - \sigma_2 \qquad (3.10)
$$

that is found from (2.42) and (2.65) to verify that the coefficient of $\psi'_{,112}$ is identically zero. By the same method we are able to verify the equivalence of (3.4) and (3.9).

The governing equation of motion is thus far general enough to represent any two dimensional incremental motion in an incompressible material. Here we consider surface waves propagating on the half-space in the x_1-direction. We thus choose that $\psi(x_1, x_2, t) = \psi'(x'_1, x'_2, t)$ is written

$$
\psi(x_1, x_2, t) = A\exp\{iksx_2\}\exp\{i(\omega t - kx_1)\} \qquad (3.11)
$$

where A is a constant, ω is the frequency, k is the wavenumber and s is to be determined. The wavespeed c appears implicitly in (3.11) through the connection $c = \omega/k$. We remark that, in general, s is complex valued.

We now have that ψ' is explicitly defined as

$$
\psi'(x'_1, x'_2, t) = A\exp\{ik(s\sin\theta - \cos\theta)x'_1 + ik(s\cos\theta + \sin\theta)x'_2 + i\omega t\}. \qquad (3.12)
$$

Substitution of this into the equation of motion will then produce a specific equation for s. From the definition in (3.12) it is clear that we have the following as the derivatives of ψ',

$$\left.\begin{aligned} \psi'_{,1111} &= k^4(s\sin\theta - \cos\theta)^4\psi', \\ \psi'_{,1122} &= k^4(s\sin\theta - \cos\theta)^2(s\cos\theta + \sin\theta)^2\psi', \\ \psi'_{,2222} &= k^4(s\cos\theta + \sin\theta)^4\psi', \\ \ddot{\psi}'_{,11} &= \omega^2k^2(s\sin\theta - \cos\theta)^2\psi', \\ \ddot{\psi}'_{,22} &= \omega^2k^2(s\cos\theta + \sin\theta)^2\psi'. \end{aligned}\right\} \tag{3.13}$$

We thus derive the quartic in s

$$\begin{aligned} &\{\alpha\sin^4\theta + 2\beta\cos^2\theta\sin^2\theta + \gamma\cos^4\theta\}s^4 \\ &+ 4\cos\theta\sin\theta\{\gamma\cos^2\theta - \beta\cos 2\theta - \alpha\sin^2\theta\}s^3 \\ &+ \{2\beta(1 - 6\cos^2\theta\sin^2\theta) \\ &\qquad\qquad + 6(\alpha+\gamma)\cos^2\theta\sin^2\theta - \rho c^2\}s^2 \\ &+ 4\cos\theta\sin\theta\{\gamma\sin^2\theta + \beta\cos 2\theta - \alpha\cos^2\theta\}s \\ &+ \gamma\sin^4\theta + 2\beta\cos^2\theta\sin^2\theta + \alpha\cos^4\theta - \rho c^2 \qquad = 0. \end{aligned} \tag{3.14}$$

The quartic for s will provide us with either 4 real solutions; 2 real solutions and 2 complex conjugate solutions; or 2 pairs of complex conjugate solutions. However, for ψ to represent surface waves we must insist that $e^{iksx_2} \to 0$ as $x_2 \to -\infty$. This means that s must be complex and, furthermore, have negative imaginary part so that only two of the complex solutions correspond to surface waves.

We now label the appropriate solutions as s_1 and s_2 and then take the general solution for ψ to be

$$\psi = (A_1 e^{iks_1x_2} + A_2 e^{iks_2x_2})e^{ik(ct-x_1)}. \tag{3.15}$$

In principle then, we must only find solutions to a quartic in s in order to determine the incremental deformations. From (3.14) it can be seen that the solutions for s depend on the amount of shear (through θ); the constitutive variables α, β, γ; and the wavespeed which is, as yet, undetermined. It is through the application of the boundary conditions that we determine the wavespeed, as we shall demonstrate.

3.1.2 Boundary Conditions

We take as our boundary conditions the vanishing of the increments in the traction, or *traction rates*, on the surface $x_2 = 0$. In particular, the components of the traction rates are to measured with respect to the Eulerian axes.

Recall that the traction $\mathbf{t}$ on a surface with unit outward normal $\mathbf{N}$ can be written $\mathbf{t} = \mathbf{S}^T\mathbf{N}$, we have that the traction rate will be

$$\dot{t} = \dot{\mathbf{S}}_0^T \mathbf{n} \tag{3.16}$$

where we have updated the reference configuration to coincide with B_t.

The outward unit normal $\mathbf{n} = \mathbf{e}_2$ is here taken as $\mathbf{n}' = \mathbf{v}^{(1)} \sin\theta + \mathbf{v}^{(2)} \cos\theta$. Also, we have the stress-deformation relation (2.57), which we use to evaluate the components of $\dot{\mathbf{S}}_0'$ in terms of ψ' as

$$\left.\begin{aligned}
\dot{S}_{011}' &= (\mathcal{A}_{01111} + p' - \mathcal{A}_{01122})\psi_{,12}' - \dot{p}, \\
\dot{S}_{012}' &= (\mathcal{A}_{01221} + p')\psi_{,22}' - \mathcal{A}_{01212}\psi_{,,11}, \\
\dot{S}_{021}' &= \mathcal{A}_{02121}\psi_{,22}' - (\mathcal{A}_{01221} + p')\psi_{,11}', \\
\dot{S}_{022}' &= (\mathcal{A}_{01122} - \mathcal{A}_{02222} - p')\psi_{,12}' - \dot{p}.
\end{aligned}\right\} \tag{3.17}$$

The prime on the components $\dot{S}_{0ij}'$ is to emphasize the Eulerian description.

The boundary conditions are then

$$\left.\begin{aligned}
\dot{t}_1' &= \dot{S}_{011}' \sin\theta + \dot{S}_{021}' \cos\theta \\
\dot{t}_2' &= \dot{S}_{012}' \sin\theta + \dot{S}_{022}' \cos\theta
\end{aligned}\right\} = 0 \quad \text{on } x_2 = 0, \tag{3.18}$$

with the appropriate substitutions being made from (3.17) for the $\dot{S}_{0ij}'$ terms.

As these boundary conditions feature $\dot{p}$ with its possible dependence upon x_1' and x_2', we again seek to remove it by differentiating the components of $\dot{\mathbf{t}}'$ with respect to x_1, and hence with respect to x_1' and x_2', and substituting from the equations of motion (3.8) and (3.9).

We thus arrive at the alternative boundary conditions, given only in terms of ψ', which are

$$\begin{aligned}
&\cos^2\theta\{\gamma\psi'_{,122} - (\gamma-\sigma_2)\psi'_{,111}\} \\
&+ \cos\theta\sin\theta\{\rho\ddot{\psi}'_{,2} - 2\gamma\psi'_{,222} + 2(\gamma-\sigma_2)\psi'_{,112}\} \\
&+ \sin^2\theta\{\rho\ddot{\psi}'_{,1} - (2\beta+\gamma-\sigma_2)\psi'_{,122} - \alpha\psi'_{111}\} \quad = \quad 0, \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
&\cos^2\theta\{\rho\ddot{\psi}'_{,2} - \gamma\psi'_{,222} - (2\beta+\gamma-\sigma_2)\psi'_{,112}\} \\
&+ \cos\theta\sin\theta\{\rho\ddot{\psi}'_{,1} - 2\alpha\psi'_{,111} + 2(\gamma-\sigma_2)\psi'_{,122}\} \\
&+ \sin^2\theta\{\alpha\psi'_{,112} - (\gamma-\sigma_2)\psi'_{,222}\} \quad = \quad 0, \qquad (3.20)
\end{aligned}$$

on $x_2 = 0$. The derivation of these conditions also makes use of the connection given by (3.10).

In view of the structure of the derivatives of ψ' from (3.13) and the expression for ψ in (3.15) we introduce the notation

$$\left.\begin{aligned}
m_i &= s_i\sin\theta - \cos\theta, \quad i = 1,2, \\
n_i &= s_i\cos\theta + \sin\theta, \quad i = 1,2.
\end{aligned}\right\} \qquad (3.21)$$

Substitution of ψ', taken from (3.15), into the boundary conditions (3.19) and (3.20) yields

$$\begin{aligned}
&\{\gamma(m_1n_1^2A_1 + m_2n_2^2A_2) \\
&\qquad - (\gamma-\sigma_2)(m_1^3A_1 + m_2^3A_2)\}\cos^2\theta \\
&+ \{\rho c^2(n_1A_1 + n_2A_2) - 2\gamma(n_1^3A_1 + n_2^3A_2) \\
&+ 2(\gamma-\sigma_2)(m_1^2n_1A_1 + m_2^2n_2A_2)\}\cos\theta\sin\theta \\
&+ \{\rho c^2(m_1A_1 + m_2A_2) - \alpha(m_1^3A_1 + m_2^3A_2) \\
&- (2\beta+\gamma-\sigma_2)(m_1n_1^2A_1 + m_2n_2^2A_2)\}\sin^2\theta \quad = \quad 0, \qquad (3.22)
\end{aligned}$$

$$
\begin{aligned}
&\{\rho c^2(n_1A_1+n_2A_2)-\gamma(n_1^3A_1+n_2^3A_2)\\
&-(2\beta+\gamma-\sigma_2)(m_1^2n_1A_1+m_2^2n_2A_2)\}\cos^2\theta\\
&+\{\rho c^2(m_1A_1+m_2A_2)-2\alpha(m_1^3A_1+m_2^3A_2)\\
&+2(\gamma-\sigma_2)(m_1n_1^2A_1+m_2n_2^2A_2)\}\cos\theta\sin\theta\\
&+\{\alpha(m_1^2n_1A_1+m_2^2n_2A_2)\\
&\qquad-(\gamma-\sigma_2)(n_1^3A_1+n_2^3A_2)\}\sin^2\theta \quad = \quad 0,
\end{aligned} \tag{3.23}
$$

where the m_i, n_i, $i \in \{1,2\}$, are known functions of the wavespeed through the dependence of s_1, s_2 on ρc^2.

3.2 Strain-Energy Functions

We recall from Section 2.4 that we assume the existence of a strain-energy function $W(\lambda_1, \lambda_2, \lambda_3)$ for our material. In the subsequent analysis of this problem we focus our attention in Section 3.3 on a particular class of strain-energy function in order to provide an insight into the dynamical behaviour and stability of the deformation without overly complicating the mathematics. After presenting the results for this class of materials we then provide the extension to a general strain-energy function in Section 3.4.

We choose to identify the class of materials at this stage, rather than Section 2.3 wherein we introduced constitutive relations, as it is at this point where it is helpful to make the specialization.

Using the notation of (3.6), we consider the class of strain-energy functions for which $2\beta = \alpha + \gamma$. We note that the strong ellipticity condition (2.81) is satisfied provided

$$\alpha > 0, \quad \beta > -\sqrt{\alpha\gamma}, \tag{3.24}$$

and that the second condition is automatically satisfied for this particular class. We henceforth assume that the strong ellipticity condition holds throughout this thesis.

Examples of materials with strain-energy functions belonging to this class are the neo-Hookean material

$$W = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right), \tag{3.25}$$

where μ is defined in (2.67), and the Mooney-Rivlin material

$$W = \frac{\mu_1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + \frac{\mu_2}{2}\left(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right). \tag{3.26}$$

We now demonstrate that for the considered simple shear deformation, where we have set $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$, $\lambda_3 = 1$ in (3.25) and (3.26), we need not distinguish between the two materials.

Consider, for any plane strain isochoric deformation, the strain-energy functions $W(\lambda^2, \lambda^{-2}, 1) = \widehat{W}(I_1, I_2, 1)$ where $I := I_1 = I_2 = \lambda^2 + \lambda^{-2} + 1$ in the notation of (2.43) and (2.44). The expressions above may be equally cast in terms of the principal invariants giving, for neo-Hookean materials,

$$\widehat{W} = \frac{\mu}{2}(I_1 - 3) \tag{3.27}$$

and, for Mooney-Rivlin materials,

$$\widehat{W} = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3). \tag{3.28}$$

Furthermore, for these types of deformation we may write $W = \widetilde{W}(\lambda)$ and $\widehat{W} = \overline{W}(I) = \frac{C}{2}(I - 3)$, where $C = \mu$ for neo-Hookean materials and $C = \mu_1 + \mu_2$ for Mooney-Rivlin materials. Hence, the two materials are effectively equivalent. That they belong to this class is straightforward as we can show that the relation $2\beta = \alpha + \gamma$ is equivalent to

$$\widehat{W}_{11} + 2\widehat{W}_{12} + \widehat{W}_{22} = \overline{W}'' = 0 \tag{3.29}$$

where now $\widehat{W}_j := \dfrac{\partial \widehat{W}}{\partial I_j}$.

In particular we have that α, β, γ may be defined by

$$\begin{gathered} \alpha = \frac{\lambda^3 \widetilde{W}'(\lambda)}{\lambda^2 - \lambda^{-2}}, \quad \gamma = \frac{\lambda^{-1}\widetilde{W}'(\lambda)}{\lambda^2 - \lambda^{-2}}, \\ 2\beta + 2\gamma = \lambda^2 \widetilde{W}''(\lambda). \end{gathered} \tag{3.30}$$

and that it therefore follows that the strong ellipticity condition holds for

$$\frac{\lambda^3 \widetilde{W}'(\lambda)}{\lambda^2 - \lambda^{-2}} > 0, \quad \lambda^2 \widetilde{W}''(\lambda) + \frac{2\lambda \widetilde{W}'(\lambda)}{\lambda^2 + 1} > 0. \tag{3.31}$$

In the reference configuration $\lambda = 1$ and we have that

$$\alpha = \beta = \gamma = \mu. \tag{3.32}$$

For the type of strain-energy function $\widetilde{W}(\lambda) = \frac{1}{2}C(\lambda^2 + \lambda^{-2} - 2)$ we have the identifications

$$\alpha = C\lambda^2, \quad \gamma = C\lambda^{-2}, \quad 2\beta + 2\gamma = C(\lambda^2 + 3\lambda^{-2}), \tag{3.33}$$

and the strong ellipticity conditions reduce to

$$C > 0. \tag{3.34}$$

3.3 Secular Equation

3.3.1 Derivation of the Secular Equation

We now regard the boundary conditions (3.22) and (3.23) as a homogeneous system of linear equations for the unknowns A_1, A_2. Non-trivial solutions for these constants may be found provided the determinant of the coefficient matrix is zero. It is in the setting of this determinant to zero that we derive the *secular equation.*

Consider (3.22) and (3.23). For convenience, in the calculation of the secular equation we express the coefficients of A_1 and A_2 as being sums over the variables $\alpha, \gamma, (\gamma - \sigma_2)$ and ρc^2, where we have taken our material to belong to the class $2\beta = \alpha + \gamma$ of strain-energy functions.

The respective coefficients of these four quantities are simplified further on making use of the following identities, which arise through the definitions (3.21), for m_1, m_2, n_1, n_2. We have, for $i = 1, 2$,

$$\left.\begin{array}{rcl} m_i \cos^2\theta - 2n_i \cos\theta \sin\theta - m_i \sin^2\theta & = & -(s_i \sin\theta + \cos\theta), \\ -n_i \cos^2\theta - 2m_i \cos\theta \sin\theta + n_i \sin^2\theta & = & (-s_i \cos\theta + \sin\theta), \\ -m_i^2 \cos^2\theta + 2m_i n_i \cos\theta \sin\theta - n_i^2 \sin^2\theta & = & -1, \\ n_i \cos\theta + m_i \sin\theta & = & s_i, \\ m_i \cos\theta - n_i \sin\theta & = & -1, \\ m_i^2 + n_i^2 & = & s_i^2 + 1, \end{array}\right\} \quad (3.35)$$

and, hence, find that the secular equation is

$$\begin{aligned} D(s_1, s_2) :=& \\ & \Big\{\Big[-(s_1^2+1)m_1 \sin^2\theta\Big]\alpha + \Big[-(s_1 \sin\theta + \cos\theta)n_1^2\Big]\gamma \\ & \qquad + [s_1 \sin\theta]\,\rho c^2 + [-m_1]\,(\gamma - \sigma_2)\Big\} \times \\ & \Big\{\Big[(-s_2 \cos\theta + \sin\theta)m_2^2\Big]\alpha + \Big[-(s_2^2+1)n_2 \cos^2\theta\Big]\gamma \\ & \qquad + [s_2 \cos\theta]\,\rho c^2 + [-n_2]\,(\gamma - \sigma_2)\Big\} \\ - & \Big\{\Big[-(s_2^2+1)m_2 \sin^2\theta\Big]\alpha + \Big[-(s_2 \sin\theta + \cos\theta)n_2^2\Big]\gamma \\ & \qquad + [s_2 \sin\theta]\,\rho c^2 + [-m_2]\,(\gamma - \sigma_2)\Big\} \times \\ & \Big\{\Big[(-s_1 \cos\theta + \sin\theta)m_1^2\Big]\alpha + \Big[-(s_1^2+1)n_1 \cos^2\theta\Big]\gamma \\ & \qquad + [s_1 \cos\theta]\,\rho c^2 + [-n_1]\,(\gamma - \sigma_2)\Big\} \\ =& \; 0. \end{aligned} \quad (3.36)$$

The introduction of the notation $D(s_1, s_2)$ to represent the above expression is made at this stage for future reference. Strictly, D should have been defined as a function of m_1, m_2, n_1, n_2 in (3.36); however, we implicitly acknowledge their respective dependence on s_1, s_2 in the above definition.

Clearly, the secular equation will, on expansion, be a quadratic form in the four parameters $\alpha, \gamma, \gamma - \sigma_2, \rho c^2$. After some algebraic manipulation it can be shown that

$$
\begin{aligned}
D(s_1, s_2) = \\
& (s_2 - s_1)\Big\{\alpha^2 m_1 m_2 \sin^2\theta(s_1 s_2 - 1) + \gamma^2 n_1 n_2 \cos^2\theta(s_1 s_2 - 1) \\
& + \rho c^2 s_1 s_2 \alpha \sin^2\theta + \rho c^2 s_1 s_2 \gamma \cos^2\theta \\
& + \rho c^2(\gamma - \sigma_2) - (\gamma - \sigma_2)^2 \\
& - \alpha\gamma\left[s_1 s_2 + (s_1{+}_2)(s_1 s_2 - 1)\cos\theta\sin\theta\cos 2\theta\right. \\
& \qquad\qquad\qquad \left. - 2(s_1 s_2 - 1)^2\cos^2\theta\sin^2\theta\right] \\
& - \alpha(\gamma - \sigma_2)\left[1 - 2(s_1 + s_2)\cos\theta\sin\theta + (s_1 + s_2)^2\sin^2\theta\right] \\
& -\gamma(\gamma - \sigma_2)\left[1 + 2(s_1 + s_2)\cos\theta\sin\theta + (s_1 + s_2)^2\cos^2\theta\right]\Big\}.
\end{aligned} \tag{3.37}
$$

There is the possibility of $s_1 = s_2$ arising, this being one solution of $D(s_1, s_2) = 0$. The general solution for ψ then becomes

$$
\psi = (A + Bx_2)e^{iksx_2}e^{i(\omega t - kx_1)},
$$

where $s = s_1$. We do not include details here but state that it can be shown that the same secular equation will result from use of this form of ψ.

Our boundary value problem is still generally posed inasmuch as no use has yet been made of information from the underlying deformation. From the definitions of α and γ, and using (2.65), we have that

$$
\alpha\lambda_2^2 = \gamma\lambda_1^2 \tag{3.38}
$$

in general, or $\sqrt{\alpha/\gamma} = \lambda^2$ for this deformation so that

$$
\alpha\sin^2\theta + \gamma\cos^2\theta = \sqrt{\alpha\gamma}\left[\lambda^2\sin^2\theta + \lambda^{-2}\cos^2\theta\right] = \sqrt{\alpha\gamma}, \tag{3.39}
$$

where we have used (2.139). We also have from (2.131) that

$$
\alpha + \gamma = (2 + \varepsilon^2)\sqrt{\alpha\gamma} \tag{3.40}
$$

along with

$$
(\alpha - \gamma)\cos\theta\sin\theta = \varepsilon\sqrt{\alpha\gamma} \tag{3.41}
$$

upon use of (2.138).

These relations are of considerable use in the simplification of the secular equation and this is illustrated by considering the terms involving α^2, γ^2, and $\alpha\gamma$. Following substitution for m_i, n_i, $i \in \{1,2\}$, these three terms are able to be rearranged as

$$\begin{aligned}
&s_1^2 s_2^2 \left[\alpha^2 \sin^4\theta + \gamma^2\cos^4\theta + 2\alpha\gamma\cos^2\theta\sin^2\theta\right] \\
&+ s_1 s_2 \left[-\alpha^2\sin^4\theta - \gamma^2\cos^4\theta - \alpha\gamma - 2\alpha\gamma\cos^2\theta\sin^2\theta\right] \\
&+ (s_1+s_2)(s_1s_2-1)\cos\theta\sin\theta\left[-\alpha^2\sin^2\theta + \gamma^2\cos^2\theta - \alpha\gamma\cos 2\theta\right] \\
&+ (s_1s_2-1)\left[\alpha^2\cos^2\theta\sin^2\theta + \gamma^2\cos^2\theta\sin^2\theta - 2\alpha\gamma\cos^2\theta\sin^2\theta\right].
\end{aligned}$$

It is now easy to verify that the considered deformation gives rise to the secular equation

$$\begin{aligned}
D(s_1,s_2) &= \alpha\gamma(s_2-s_1)\{s_1^2s_2^2 - 2s_1s_2 - (s_1s_2-1)(s_1+s_2)\varepsilon + (s_1s_2-1)^2\varepsilon \\
&\quad +\zeta s_1s_2 - (\overline{\gamma}-\overline{\sigma}_2)^2 + \zeta(\overline{\gamma}-\overline{\sigma}_2) - (\overline{\gamma}-\overline{\sigma}_2)(2+\varepsilon^2) \\
&\quad +2(s_1+s_2)(\overline{\gamma}-\overline{\sigma}_2)\varepsilon - (s_1+s_2)^2(\overline{\gamma}-\overline{\sigma}_2)\} \\
&= 0,
\end{aligned} \tag{3.42}$$

where

$$\zeta = \frac{\rho c^2}{\sqrt{\alpha\gamma}}, \quad \overline{\gamma} = \frac{\gamma}{\sqrt{\alpha\gamma}}, \quad \overline{\sigma}_2 = \frac{\sigma_2}{\sqrt{\alpha\gamma}}. \tag{3.43}$$

As the resulting secular equation came about through the specialization to the class of strain-energy functions $2\beta = \alpha+\gamma$ it is now appropriate to reconsider the equation of motion (3.14). Upon making the substitution for 2β and using (3.39)-(3.41) we find that the governing equation reduces to

$$s^4 - 2\varepsilon s^3 + (2+\varepsilon^2-\zeta)s^2 - 2\varepsilon s + 1 + \varepsilon^2 - \zeta = 0, \tag{3.44}$$

which has solutions $s = \pm i$ and $s = \varepsilon \pm i\sqrt{1-\zeta}$.

We make the definition of η as being

$$\eta^2 = 1 - \zeta \tag{3.45}$$

and take those solutions with negative imaginary part as

$$s_1 = -i, \quad s_2 = \varepsilon - \eta i. \tag{3.46}$$

In addition, for surface waves we must also have $0 \le \zeta < 1$ and hence $0 < \eta \le 1$.

Substitution of (3.46) into (3.42) then yields the equation

$$f(\eta) := \eta^3 + \eta^2 + (1 + \varepsilon^2 + 2\overline{p})\eta - \overline{p}^2 = 0, \tag{3.47}$$

where $\overline{p} = (\overline{\gamma} - \overline{\sigma}_2)$.

Solving (3.47) provides us with information on how the wavespeed depends upon the deformation, through ε, and the hydrostatic pressure, through $\overline{p}$.

3.3.2 Analysis of the Secular Equation

For the neo-Hookean strain-energy function,

$$W = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right), \tag{3.48}$$

we have $\overline{p} = p/\mu$, where p is the hydrostatic stress appearing in (2.42). Moreover, using (2.139), (3.48) and by the result (in Cartesian co-ordinates)

$$\left.\begin{aligned} \sigma_{11} &= \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta, \\ \sigma_{12} &= (\sigma_1 - \sigma_2)\cos\theta \sin\theta, \\ \sigma_{22} &= \sigma_1 \sin^2\theta + \sigma_2 \cos^2\theta, \end{aligned}\right\} \tag{3.49}$$

it can be shown that $\sigma_{22} = \mu - p$, so that the prescription of p may be thought of as equivalent to assignment of the normal stress on the surface $x_2 = 0$. The derivation of (3.49) parallels that of (2.135)–(2.141) with $\boldsymbol{\sigma}$ replacing $\mathbf{B}$ and σ_i replacing λ_i^2.

If $\overline{p} \neq 0$ we have $f(0) < 0$ and $f'(0) = 1 + \varepsilon^2 + 2\overline{p}$. If $f'(0) \ge 0$ then $f'(\eta) > 0$ for all $\eta > 0$ and hence $f(\eta)$ is strictly increasing for $0 < \eta \le 1$. If, on the other hand, $f'(0) < 0$ then $f(\eta)$ has a local minimum for $\eta > 0$ (and a local maximum for $\eta < 0$), and (3.47) therefore has at most one solution for $0 < \eta \leq 1$. In either case (3.47) has a unique solution for $0 < \eta \leq 1$ if and only if $f(1) \geq 0$, i.e

$$4 + \varepsilon^2 \geq (\overline{p} - 1)^2, \tag{3.50}$$

with equality corresponding to the solution $\eta = 1$ ($\zeta = 0$).

For $-1 \leq \overline{p} \leq 3$ the inequality (3.50) is satisfied for all ε. In particular, if the material is undeformed ($\varepsilon = 0$) surface waves are admitted for this range of values for $\overline{p}$ (which includes the stress-free configuration $\overline{p} = 1$). Note that $\overline{p} > 1$ (< 1) corresponds to hydrostatic compression (tension). For $\varepsilon = 0$ these results agree with those of Dowaikh and Ogden (1990) for the case of pure hydrostatic stress.

More generally, for any given $\varepsilon \neq 0$, a unique surface wave exists for all values of $\overline{p}$ such that

$$-\sqrt{4 + \varepsilon^2} \leq \overline{p} - 1 \leq \sqrt{4 + \varepsilon^2}. \tag{3.51}$$

The extremities of the range (3.51) identify points in $(\varepsilon, \overline{p})$-space for which quasi-static surface deformations (corresponding to $\zeta = 0$) can emerge on a path of loading from $(\varepsilon, \overline{p}) = (0, 1)$.

Of particular note is the rearrangement of the secular equation into a cubic in ζ,

$$\begin{aligned} \zeta^3 - 2\left[\varepsilon^2 + 2(\overline{p} + 1)\right]\zeta^2 + \left[6(\overline{p} + 1)^2 + \varepsilon^2\left(\varepsilon^2 + 2(\overline{p} + 3)\right)\right]\zeta \\ + \left((\overline{p} + 1)^2 + \varepsilon^2\right)\left[(\overline{p} - 1)^2 - (4 + \varepsilon^2)\right] = 0, \end{aligned} \tag{3.52}$$

since through the setting of $(\varepsilon, \overline{p}) = (0, 1)$ in the left-hand side we recover the classical equation for the (phase) velocity of Rayleigh waves in an undeformed and unstressed incompressible half-space

$$\zeta^3 - 8\zeta^2 + 24\zeta - 16 = 0, \tag{3.53}$$

as found in, for example, Ewing, Jardetzky and Press (1957). It is clear that the coefficient of ζ^0 in (3.52) vanishes at the extremities of the range (3.51), thus verifying $\zeta = 0$ as being a solution when these limits are attained.

In the special case $\overline{p} = 0$, (3.52) has, for any ε, only the solution $\eta = 0$ (corresponding to $\zeta = 1$) which is identified with the left-hand limit of the range $0 < \eta \leq 1$. This does not give rise to a pure surface wave. However, with $s_2 = \varepsilon$ the term in A_2 in (3.15) corresponds to a plane wave with direction of propagation $(1, -\varepsilon)$, inextricably linked, through the boundary conditions, to the surface wave associated with A_1. We note that $\overline{p} = 0$ satisfies (3.51). A similar limiting case, also associated with a plane wave, is found to arise (Dowaikh and Ogden, 1990) for particular forms of strain-energy function when the underlying deformation is a pure homogeneous strain, but, in contrast to the present situation, the surface and plane waves can exist independently.

3.3.3 Numerical Results

We now include some numerical results that illustrate the dependence of the wavespeed on both the deformation, through the amount of shear ε, and the non-dimensionalized hydrostatic pressure $\overline{p}$.

Figure 3.1 shows ζ, defined in (3.43), as a function of the pressure $\overline{p}$ for several values of ε. When the material is undeformed ($\varepsilon = 0$) the dependence of the wavespeed on the pressure is depicted by the inner arc which cuts $\zeta = 0$ at $\overline{p} = 1$ and $\overline{p} = 3$ corresponding to the end points of the interval (3.50) in this case. A unique wavespeed is obtained for each $\overline{p}$ between these values. As the material is progressively sheared, the range of values of $\overline{p}$ for which surface waves exist is increased in accordance with the inequality (3.50). As $\overline{p} \to 0$ we observe that for all ε the limiting case $\zeta = 1$ associated with the combined plane wave and surface wave arises.

An alternative view of the results is obtained by considering ζ as a function of ε for prescribed values of $\overline{p}$. For clarity we confine attention to two cases corresponding to two distinct ranges of values of $\overline{p}$. Accordingly, in Figure 3.2 we show the family of curves generated by values of $\overline{p} \geq 0$, with equality corresponding to the limiting case $\zeta = 1$. It can be seen that in the unstressed configuration ($\varepsilon = 0$) pressures in the range $0 \leq \overline{p} \leq 3$ admit surface waves, their speed decreasing as $\overline{p}$ increases, and that quasi-static modes can occur when $\overline{p} = 3$. Higher values of $\overline{p}$ must be accompanied by larger values of ε for surface waves to be admitted.

Figure 3.3 shows that for $\overline{p} \leq 0$ a similar family of curves arises but here they are obtained by *decreasing* the pressure term from $\overline{p} = 0$ through the threshold value $\overline{p} = -1$.

Finally, in Figure 3.4, curves in $(\varepsilon, \overline{p})$-space corresponding to prescribed values of ζ are plotted. Rearrangement of the secular equation (3.47) into the form

$$(\overline{p} - \eta)^2 - \eta\varepsilon^2 = \eta(\eta + 1)^2 \tag{3.54}$$

shows that the points $(\varepsilon, \overline{p})$ so defined by this equation for a given η lie on a hyperbola. We note, in particular, that the region between the branches of the hyperbola $(\overline{p} - 1)^2 - \varepsilon^2 = 4$ (corresponding to $\zeta = 0$) is the region of existence of surface waves. It may also be thought of as defining values of ε and $\overline{p}$ for which the pre-stressed configuration is infinitesimally stable.

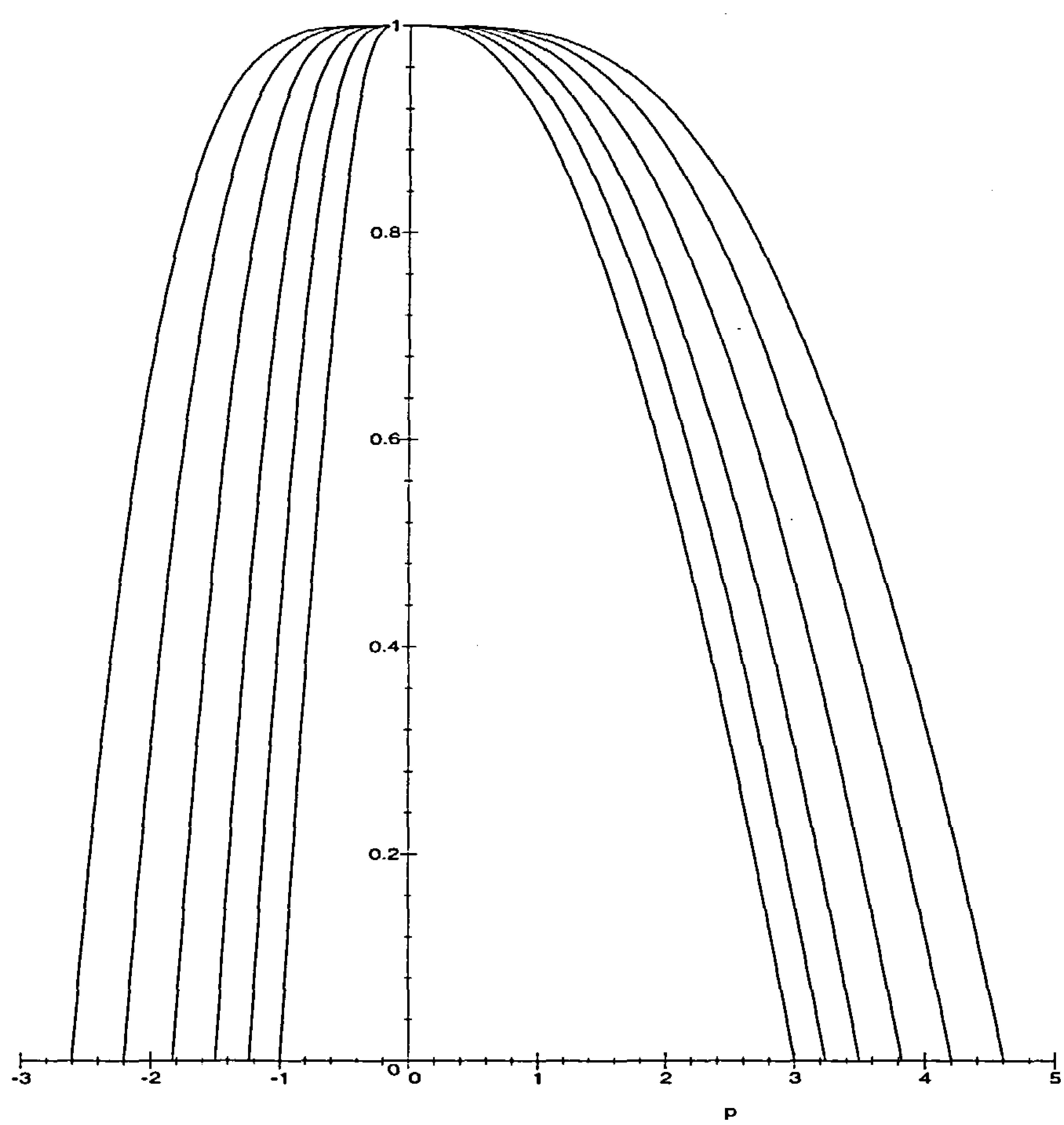


Figure 3.1: Plot of ζ (vertical scale) against non-dimensionalized $\overline{p}$ (horizontal scale) with pre-stress states shown for $\varepsilon = 0, 1, 1.5, 2, 2.5, ,4$. The inner arc corresponds to $\varepsilon = 0$, the arcs diverging with increasing ε.

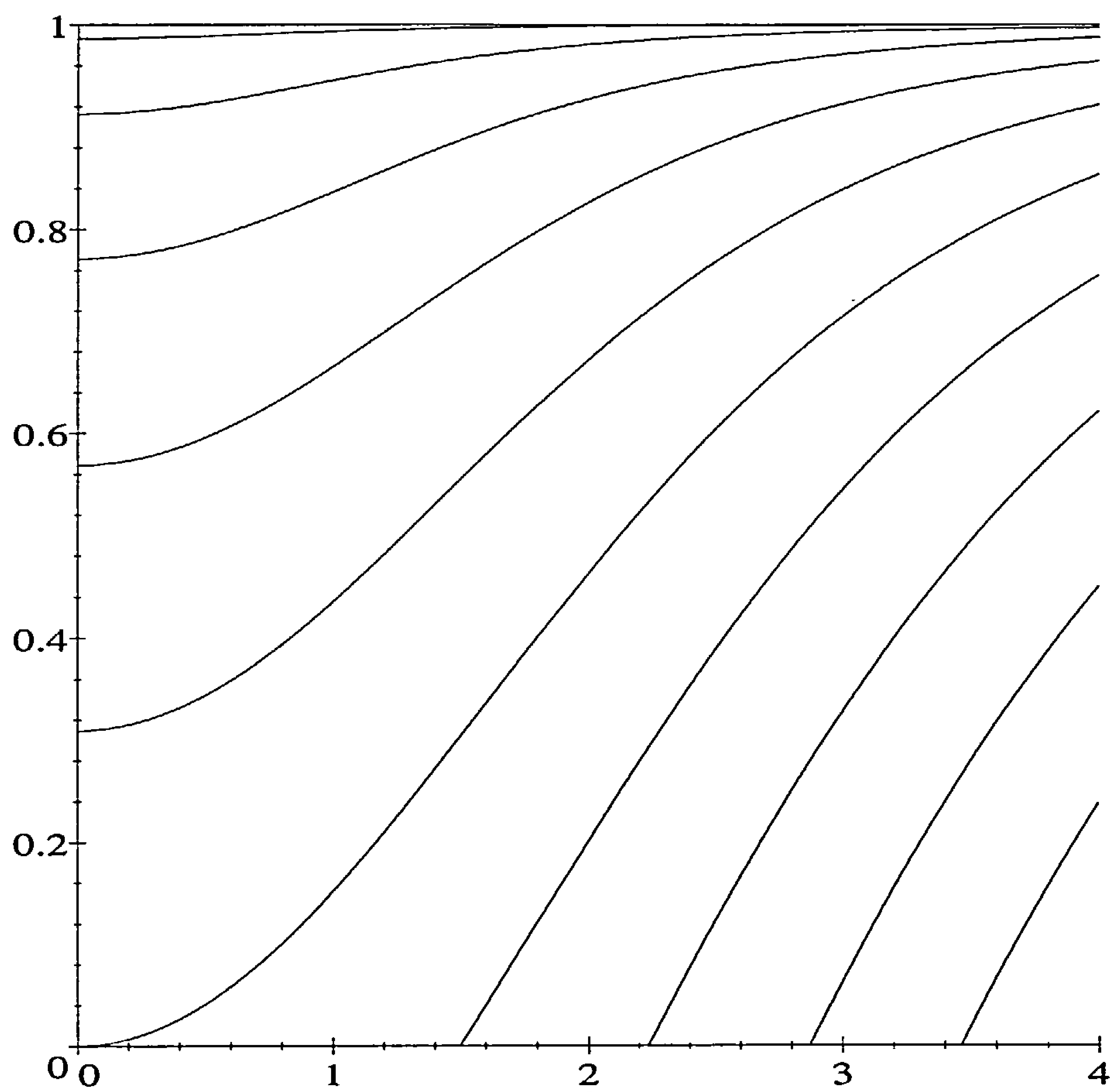


Figure 3.2: Plot of wavespeed (vertical scale) as a function of the deformation parameter ε (horizontal scale) for $\overline{p} = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5,\ 4,\ 4.5,\ 5$.

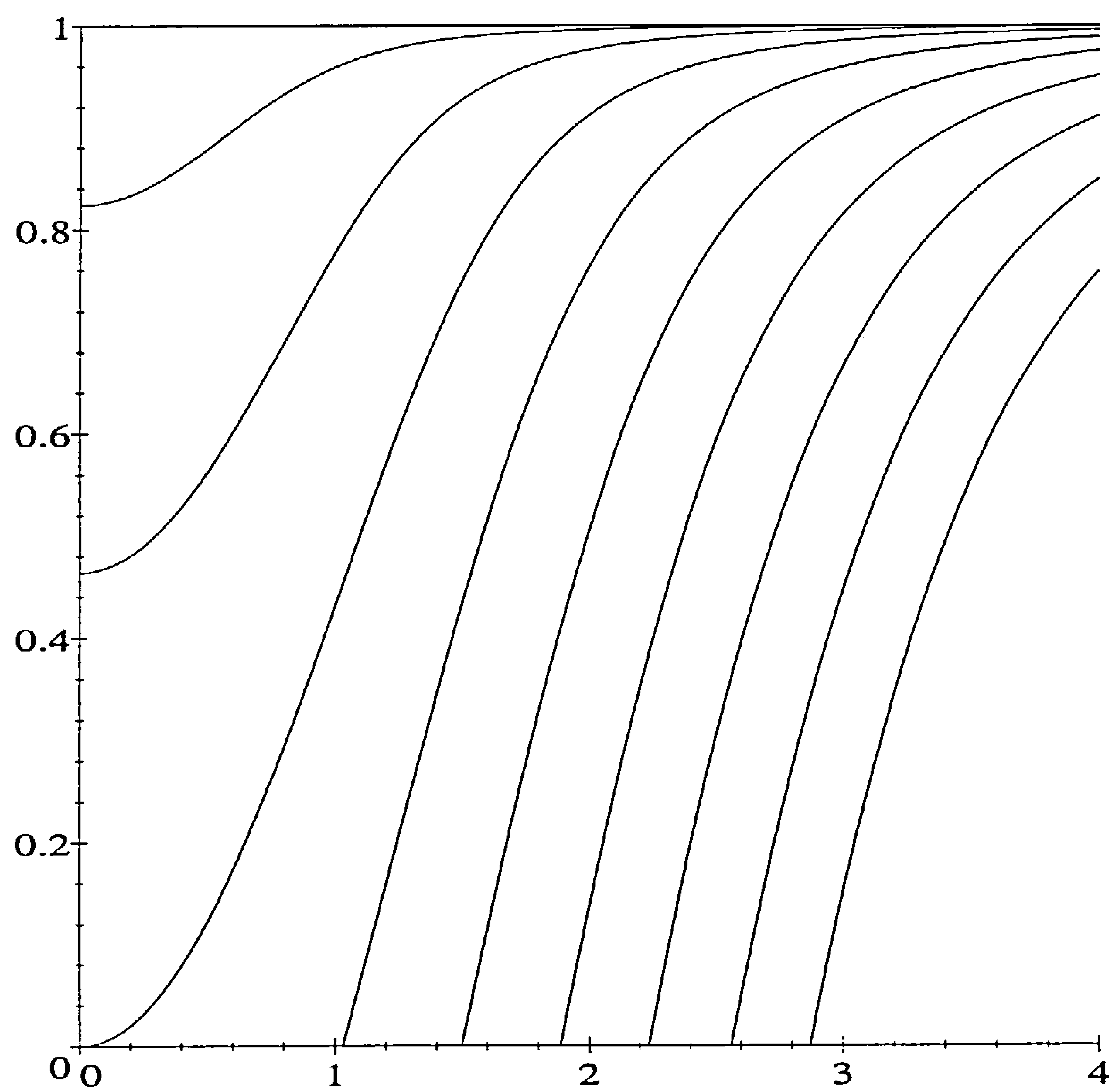


Figure 3.3: Plot of wavespeed (vertical scale) as a function of the deformation parameter ε (horizontal scale) for $\overline{p} = 0, -0.5, -0.75, -1, -1.25, -1.5, -1.75, -2, -2.25, -2.5$.

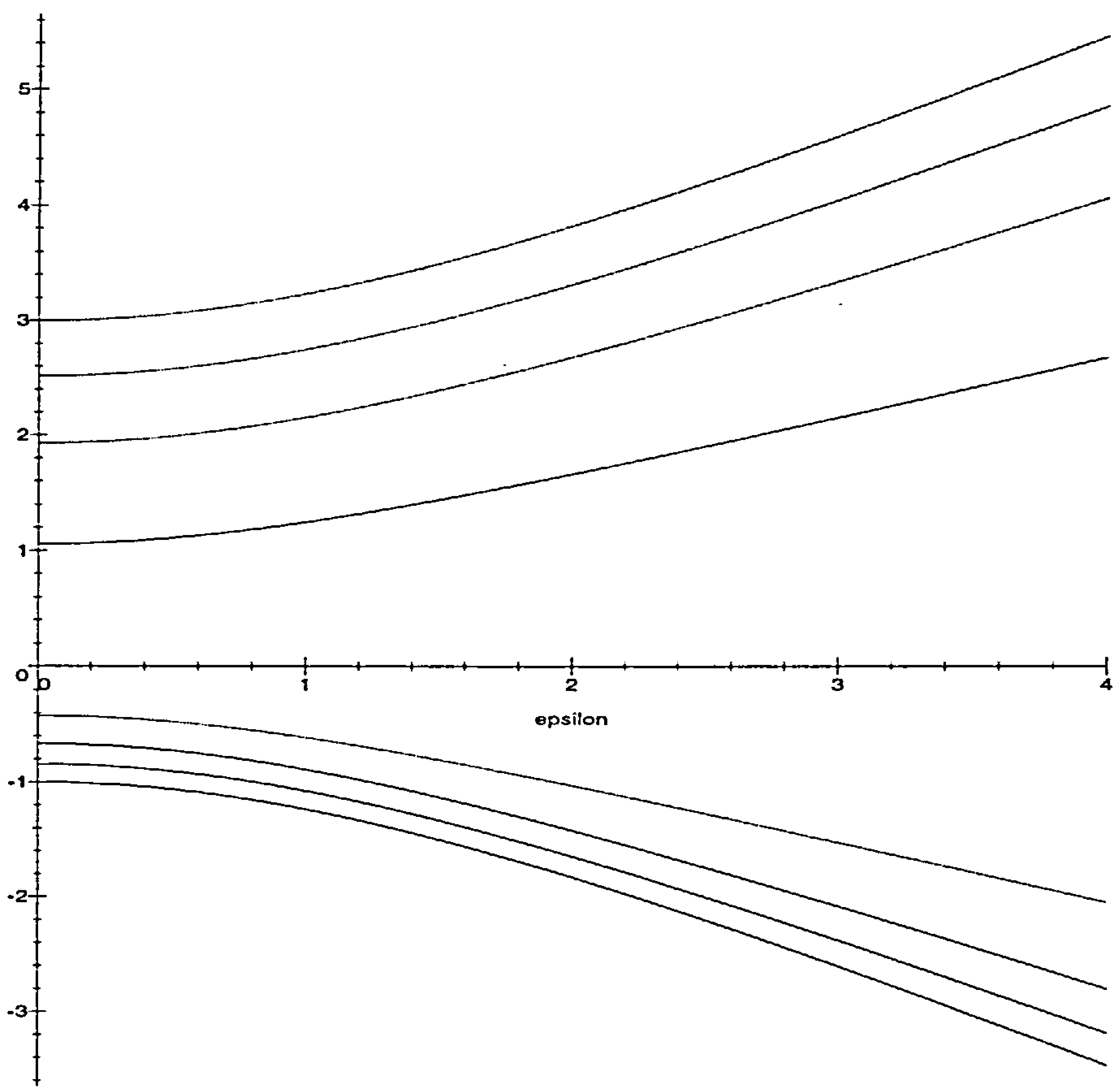


Figure 3.4: Plot of $\overline{p}$ (vertical scale) against ε (horizontal scale) for ζ = 0, 0.3, 0.6, 0.9.

3.4 Extension to General Materials

In this section we investigate certain features of the deformation in respect of a general incompressible isotropic elastic solid and also establish the equations of motion, boundary conditions and secular equation for surface waves when the considered material is compressible.

3.4.1 Incompressible Materials

We start by focusing on the governing equation (3.14) that was derived quite generally for this underlying deformation. Whereas in Section 3.3 we specialized to a particular class of strain-energy function, we now work with this general equation directly.

Upon simple rearrangement of (2.139) we express the trigonometric terms featured in (3.14) as functions of λ through the following connections

$$\cos^2\theta = \frac{\lambda^2}{\lambda^2+1}, \quad \sin^2\theta = \frac{1}{\lambda^2+1}, \quad \cos\theta\sin\theta = \frac{\lambda}{\lambda^2+1}. \tag{3.55}$$

Removal of a factor $\sqrt{\alpha\gamma}$ from the coefficients of the quartic in s, bearing in mind (3.38), and defining

$$\overline{\beta} = \frac{\beta}{\sqrt{\alpha\gamma}} \tag{3.56}$$

we see that the quartic in s becomes

$$\begin{aligned}
&2\lambda^2\left[\overline{\beta}+1\right]s^4 + 4\lambda\left(1-\lambda^2\right)\left[\overline{\beta}+1\right]s^3 \\
&+\left(\lambda^2+1\right)^2\left[2\overline{\beta} + \frac{6\lambda^2\left(\lambda^2+\lambda^{-2}-2\overline{\beta}\right)}{\left(\lambda^2+1\right)^2} - \zeta\right]s^2 \\
&+4\lambda\left(\lambda^2-1\right)\left[\overline{\beta}-\left(3+\varepsilon^2\right)\right]s + \left[\lambda^{-2}+2\overline{\beta}\lambda^2+\lambda^6-\zeta\left(\lambda^2+1\right)^2\right] = 0
\end{aligned} \tag{3.57}$$

where we have also removed a factor $(\lambda^2+1)^2$ and, in the coefficient of s, made use of $\lambda^2+\lambda^{-2} = 2+\varepsilon^2$.

After division by the coefficient of s^4 (which is non-zero by virtue of the strong ellipticity condition (3.24)), equation (3.57) may be written more compactly as

$$s^4 - 2\varepsilon s^3 + \left[4\delta + 2 + \varepsilon^2 - (1+\delta)\zeta\right] s^2 - 2(1+2\delta)\varepsilon s + 1 + \varepsilon^2 + \delta\varepsilon^2 - (1+\delta)\zeta = 0, \tag{3.58}$$

where we have defined the material parameter δ to be

$$\delta := \frac{\alpha + \gamma - 2\beta}{2\left(\beta + \sqrt{\alpha\gamma}\right)} = \frac{\lambda^2 + \lambda^{-2} - 2\overline{\beta}}{2\left(\overline{\beta} + 1\right)}. \tag{3.59}$$

Connection is made with the results of Section 3.3 when we set $\delta = 0$. Clearly, from the definition of δ, this corresponds to the setting of $2\beta = \alpha + \gamma$, and the resultant governing equation (3.58) recovers the left-hand side of (3.44) with this choice of δ.

Solutions of (3.58) may be found with the aid of a computer algebra package such as *Maple V* (Char *et al*, 1992) or *Mathematica* (Wolfram, 1993). However, as may be expected, the resulting expressions are too cumbersome to be of use analytically and we would rely on numerical procedures to provide us with any insight into the behaviour of the material.

It is the case, though, that (3.58) yields manageable solutions when we limit our scope to the static case, $\zeta = 0$, and thus establish bifurcation criteria for which the material will admit quasi-static incremental deformations.

When the wavespeed is zero, the governing equation (3.58) reduces to

$$s^4 - 2\varepsilon s^3 + \left[4\delta + 2 + \varepsilon^2\right] s^2 - 2(1+2\delta)\varepsilon s + 1 + (1+\delta)\varepsilon^2 = 0 \tag{3.60}$$

solutions of which fall into two distinct cases depending on the parameters ε, δ and, by implication, $\overline{\beta}$. With this in mind, we note some of the consequences of (3.59) and (3.24). Firstly, we have

$$(1+\delta) = \frac{\left(\sqrt{\alpha} + \sqrt{\gamma}\right)^2}{2\left(\beta + \sqrt{\alpha\gamma}\right)} = \frac{(\lambda + \lambda^{-1})^2}{2\left(\overline{\beta} + 1\right)} > 0, \tag{3.61}$$

from which we find that

$$\frac{\varepsilon^2+4}{1+\delta}=2\left(\overline{\beta}+1\right) \quad \text{and} \quad \frac{\varepsilon^2-4\delta}{1+\delta}=2\left(\overline{\beta}-1\right). \tag{3.62}$$

The solutions of (3.60) are then

$$\left.\begin{aligned} s_1 &= \tfrac{1}{2}\varepsilon-\tfrac{1}{2}\sqrt{\varepsilon^2-4\delta}-i\sqrt{\delta+1} \\ s_2 &= \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\sqrt{\varepsilon^2-4\delta}-i\sqrt{\delta+1} \end{aligned}\right\} \quad \left(\overline{\beta}>1\right) \tag{3.63}$$

or

$$\left.\begin{aligned} s_1 &= \tfrac{1}{2}\varepsilon-\tfrac{1}{2}i\sqrt{4\delta-\varepsilon^2}-i\sqrt{\delta+1} \\ s_2 &= \tfrac{1}{2}\varepsilon+\tfrac{1}{2}i\sqrt{4\delta-\varepsilon^2}-i\sqrt{\delta+1} \end{aligned}\right\} \quad \left(\overline{\beta}<1\right). \tag{3.64}$$

As before, we are only considering surface deformations and so we take the solutions s_1, s_2 to be those shown in (3.63) and (3.64) since they have negative imaginary part.

An observation to make regarding both sets of solutions is that $\overline{s}_1=\varepsilon-s_2$ when $\overline{\beta}>1$ while $\overline{s}_j=\varepsilon-s_j$, $j\in\{1,2\}$, when $\overline{\beta}<1$ so that, in either case, if s is a solution of (3.60) then so is $\varepsilon-s$. Furthermore, the solutions (3.63) reduce, when $\delta=0$, to the static case solutions ($\eta=1$) given in (3.46).

The working in Section 3.3 was appropriate only to the class $2\beta=\alpha+\gamma$ and is thus of little consequence here. Rather, we return to the boundary conditions (3.22) and (3.23), which we rewrite as

$$\left.\begin{aligned} \Phi(s_1)A_1+\Phi(s_2)A_2=0, \\ \Xi(s_1)A_1+\Xi(s_2)A_2=0, \end{aligned}\right\} \tag{3.65}$$

where

$$\left.\begin{aligned} \Phi(s_i) \;=\;& \{\gamma m_i n_i^2-(\gamma-\sigma_2)m_i^3\}\cos^2\theta \\ &+\{\rho c^2 n_i-2\gamma n_i^3+2(\gamma-\sigma_2)m_i^2 n_i\}\cos\theta\sin\theta \\ &+\{\rho c^2 m_i-\alpha m_i^3-(2\beta+\gamma-\sigma_2)m_i n_i^2\}\sin^2\theta, \\ \Xi(s_i) \;=\;& \{\rho c^2 n_i-\gamma n_i^3-(2\beta+\gamma-\sigma_2)m_i^2 n_i\}\cos^2\theta \\ &+\{\rho c^2 m_i-2\alpha m_i^3+2(\gamma-\sigma_2)m_i n_i^2\}\cos\theta\sin\theta \\ &+\{\alpha m_i^2 n_i-(\gamma-\sigma_2)n_i^3\}\sin^2\theta, \end{aligned}\right\} \tag{3.66}$$

with $i \in \{1,2\}$ and the ρc^2 terms are included for completeness.

We define the functions $\phi(s), \xi(s)$ *a posteriori* as

$$\phi(s)\sqrt{\alpha\gamma}\sin\theta = \Phi(s), \quad \xi(s)\sqrt{\alpha\gamma}\cos\theta = \Xi(s), \tag{3.67}$$

which allows us, in a similar fashion to the derivation of (3.57), to put the essentials of (3.66) as

$$\left.\begin{aligned} \phi(s) &= (1+\delta)^{-1}(s-\lambda)(s+\lambda^{-1})^2 + 3s - \lambda + 2\lambda^{-1} + \overline{p}(s-\lambda) - \zeta s, \\ \xi(s) &= (1+\delta)^{-1}(s+\lambda^{-1})(s-\lambda)^2 + 3s + \lambda^{-1} - 2\lambda + \overline{p}(s+\lambda^{-1}) - \zeta s. \end{aligned}\right\} \tag{3.68}$$

Therefore, we now see that non-trivial solutions for A_1, A_2 can be found provided

$$\phi(s_1)\xi(s_2) - \phi(s_2)\xi(s_1) = 0 \tag{3.69}$$

and so this becomes our bifurcation criterion on use of (3.63) and with $\zeta = 0$.

We do not include details at this stage of the calculation of (3.69) but we find that the bifurcation criterion provided by it is

$$\sqrt{\varepsilon^2 - 4\delta}\left[\frac{\varepsilon^2+4}{1+\delta} - (\overline{p}-1)^2\right] = 0. \tag{3.70}$$

Since the material parameter δ is a prescribed quantity then (3.70) identifies points in $(\varepsilon, \overline{p})$-space capable of admitting standing waves through the vanishing of the second factor. Using the notation of (3.62), this bifurcation criterion is

$$\overline{p} = 1 \pm \sqrt{2(\overline{\beta}+1)}, \tag{3.71}$$

which corresponds to the extremities of the interval (3.51) when generalized to the case $\delta \neq 0$.

The incremental stability, in respect of the appearance of quasi-static incremental waves, of the homogeneous deformation is guaranteed provided

$$1 - \sqrt{2(\overline{\beta}+1)} < \overline{p} < 1 + \sqrt{2(\overline{\beta}+1)}. \tag{3.72}$$

Implicit in (3.70) is the dependence of $\overline{p}$ on ε through δ which requires to be determined for each material being considered before any interpretation may be made.

One possibility arising from (3.70) is for $\varepsilon^2 = 4\delta$ — in particular, for $\varepsilon = 0$ all materials satisfy $2\beta = \alpha + \gamma$. More generally, for all ε such that $\varepsilon^2 = 4\delta$ we have that $\overline{\beta} = 1$ and, moreover, the solutions in (3.63) degenerate into the case $s_1 = s_2 = \frac{1}{2}[\varepsilon - i\sqrt{\varepsilon^2+4}]$ which, as previously mentioned, leads to the same conclusion as $s_1 \neq s_2$.

By way of illustrating the influence of δ on the stability of the deformation we introduce the class of strain-energy functions defined by

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{2\mu}{m^2}\left(\lambda_1^m + \lambda_2^m + \lambda_3^m - 3\right), \tag{3.73}$$

where μ is the shear modulus and m is free to be chosen. We then have, for our underlying deformation,

$$\widetilde{W}(\lambda) = \frac{2\mu}{m^2}\left(\lambda^m + \lambda^{-m} - 2\right), \tag{3.74}$$

and note that $\widetilde{W}$ is an even function with respect to the choice of m, implying that we need only consider $m > 0$. Through (3.31) it can be seen that strong ellipticity holds for $|m| \geq 1$ but cannot be guaranteed for $|m| < 1$.

For $m = 1$ the strain-energy function is that of a Varga material. We use (3.30) to calculate that $\overline{\beta} = 1$ for this material and so the deformation is stable for $-1 < \overline{p} < 3$.

The case $m = 2$ recovers the neo-Hookean strain-energy function with its corresponding result that stability holds whenever $(\overline{p}-1)^2 < \varepsilon^2 + 4$ as previously shown.

The case $m = 3$ yields $\overline{\beta} = 1 + \varepsilon^2 + \dfrac{\varepsilon^2}{3+\varepsilon^2}$ and the curve demarcating the region of stability is depicted in Figure 3.5.

For $m = 1/2$ a more complex situation emerges as here we have

$$\overline{\beta} = \frac{\lambda^{-1}}{4}\left[-\lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda - 1\right]. \tag{3.75}$$

One question here is, for what range of values for λ will the strong ellipticity condition hold? With reference to the two inequalities (3.31) that determine strongly elliptic regimes, we see that the first of these is automatically satisfied with this choice of strain-energy function. The second reduces to

$$\frac{2\mu\lambda^{1/2}}{\lambda^2+1}\left[-\lambda^3+3\lambda^2+3\lambda-1\right]>0, \tag{3.76}$$

from which we find that we require $1 \leq \lambda < 2+\sqrt{3}$ $\quad (0 \leq \varepsilon < 2\sqrt{3})$. We also find that we require λ to be bounded above by the same critical value $\lambda_c = 2+\sqrt{3}$ in order that $\overline{\beta}+1$ is non-negative.

We show the regions of stability of these materials in Figure 3.5.

3.4.2 Compressible Materials

For this discussion only we shall be re-defining some of the parameters previously used. As before, we consider an isotropic elastic material occupying the region $x_2 \leq 0$ and subject to the same underlying deformation as described in Section 2.6.

(I) Equations of Motion

Similarly as for incompressible materials, we use the Eulerian axes as our coordinate system and retain the notation for the increments. The incremental equations of motion for a compressible material are as given in (2.71) and provide

$$\begin{aligned} \alpha_{11}v'_{1,11}+\delta v'_{2,12}+\gamma_2 v'_{1,22} &= \rho\ddot{v}'_1, \\ \alpha_{22}v'_{2,22}+\delta v'_{1,12}+\gamma_1 v'_{2,11} &= \rho\ddot{v}'_2, \end{aligned} \tag{3.77}$$

where now

$$\begin{gathered} \alpha_{ij} = \mathcal{A}_{0iijj}, \quad \delta = (\mathcal{A}_{01221}+\mathcal{A}_{01122}), \\ \gamma_1 = \mathcal{A}_{01212}, \quad \gamma_2 = \mathcal{A}_{02121}. \end{gathered} \tag{3.78}$$

By combining these two equations for v'_1 or v'_2 it is possible to decouple them into

$$a'v'_{i,1111}+2b'v'_{i,1122}+c'v'_{2222} = \rho(\alpha_{11}+\gamma_1)v'_{i,11tt}+\rho(\alpha_{22}+\gamma_2)v'_{i,22tt}-\rho^2 v'_{i,tttt} \tag{3.79}$$

for $i \in \{1,2\}$ and where we now have

$$a' = \alpha_{11}\gamma_1, \quad 2b' = \alpha_{11}\alpha_{22} + \gamma_1\gamma_2 - \delta^2, \quad c' = \alpha_{22}\gamma_2. \tag{3.80}$$

We seek solutions for $\mathbf{v}$ of the type

$$v_j = A_j \exp\{iksx_2 + i\omega t - ikx_1\}, \quad j = 1, 2, \tag{3.81}$$

corresponding to incremental surface deformations and in keeping with the notation of Section 3.1.1. We therefore have the increments in terms of the coordinates associated with the Eulerian principal axes as

$$v'_j = A'_j \exp\{ikmx'_1 + iknx'_2 + i\omega t\}, \quad j = 1, 2, \tag{3.82}$$

where the connection between $\mathbf{A}$ and $\mathbf{A}'$ is the same as for $\mathbf{x}$ and $\mathbf{x}'$ in (3.1) and where the m and n are subscript-free versions of the m_i and n_i defined in (3.21).

Through putting (3.82) into either of the equations of motion in (3.77) it follows that the hitherto arbitrary constants A'_i are not independent. Rather, they are subject to the equivalent relationships

$$\begin{aligned} [\alpha_{11}m^2 + \gamma_2 n^2 - \rho c^2]\, A'_1 + [\delta mn]\, A'_2 &= 0, \\ [\alpha_{22}n^2 + \gamma_1 m^2 - \rho c^2]\, A'_2 + [\delta mn]\, A'_1 &= 0, \end{aligned} \tag{3.83}$$

and from which it is convenient to define the ratio r_i, $i \in \{1,2\}$ as

$$r_i := \frac{\rho c^2 - \alpha_{11}m_i^2 - \gamma_2 n_i^2}{\delta m_i n_i}, \tag{3.84}$$

so that $A'_2 = r_i A'_1$ when $s = s_i$ in (3.82).

With substitution of (3.82) into the equation of motion (3.79) we find that m, n and the wavespeed $c = \omega/k$ must satisfy

$$a'm^4 + 2b'm^2n^2 + c'n^4 = \rho(\alpha_{11} + \gamma_1)c^2m^2 + \rho(\alpha_{22} + \gamma_2)c^2n^2 - (\rho c^2)^2. \tag{3.85}$$

We thus require to find solutions for s from the governing quartic

$$
\begin{aligned}
&\left\{a' \sin^4\theta + 2b' \cos^2\theta \sin^2\theta + c' \cos^4\theta\right\} s^4 \\
&+ 4\cos\theta \sin\theta \left\{c' \cos^2\theta - b' \cos 2\theta - a' \sin^2\theta\right\} s^3 \\
&+ \Big\{2b'(1 - 6\cos^2\theta \sin^2\theta) + 6(a' + c')\cos^2\theta \sin^2\theta \\
&\qquad\qquad - \left[(\alpha_{11} + \gamma_1)\sin^2\theta + (\alpha_{22} + \gamma_2)\cos^2\theta\right]\rho c^2\Big\} s^2 \\
&+ 2\cos\theta \sin\theta \Big\{2c' \sin^2\theta + 2b' \cos 2\theta - 2a' \cos^2\theta \\
&\qquad\qquad\qquad\qquad + [(\alpha_{11} + \gamma_1) - (\alpha_{22} + \gamma_2)]\rho c^2\Big\} s \\
&+ a' \cos^4\theta + 2b' \cos^2\theta \sin^2\theta + c' \sin^4\theta \\
&\qquad - \left[(\alpha_{11} + \gamma_1)\cos^2\theta + (\alpha_{22} + \gamma_2)\sin^2\theta + \rho c^2\right]\rho c^2 = 0.
\end{aligned}
\tag{3.86}
$$

For this generally posed problem we are unable to proceed analytically with any great effect although solutions of (3.86) may be found in principle. Moreover, since the material constants featured above do not readily lend themselves to the type of combination used in the incompressible problem that would allow both a simplification of the problem as it stands at this stage and physical significance in respect of a class of strain-energy functions it would now be necessary to specialize to a particular material in order to continue.

For surface deformations we consider the two solutions s_1, s_2 of (3.86) having negative imaginary part so that our general solution for the increments (v_1', v_2') is now given by

$$
\left.
\begin{aligned}
v_1' &= \left[A' e^{ik(m_1 x_1' + n_1 x_2')} + B' e^{ik(m_2 x_1' + n_2 x_2')}\right] e^{i\omega t}, \\
v_2' &= \left[A' r_1 e^{ik(m_1 x_1' + n_1 x_2')} + B' r_2 e^{ik(m_2 x_1' + n_2 x_2')}\right] e^{i\omega t}
\end{aligned}
\right\}
\tag{3.87}
$$

with the constants A', B' being arbitrary.

(II) Boundary Conditions

We again take the vanishing of the traction rates $\dot{\mathbf{t}}'$ on the surface $x_2 = 0$ as our boundary conditions and recall their component form as given in terms of $\dot{S}'_{0ij}$

from Section 3.1.2 with the components of $\dot{\mathbf{S}}_0'$ now being taken from (2.55) giving

$$\left.\begin{aligned}\dot{S}_{011}' &= \alpha_{11}v_{1,1}' + \alpha_{12}v_{2,2}',\\ \dot{S}_{012}' &= \mathcal{A}_{01221}v_{1,2}' + \gamma_1 v_{2,1}',\\ \dot{S}_{021}' &= \gamma_2 v_{1,2}' + \mathcal{A}_{01221}v_{2,1}',\\ \dot{S}_{022}' &= \alpha_{12}v_{1,1}' + \alpha_{22}v_{2,2}',\end{aligned}\right\} \tag{3.88}$$

and where we have used $\mathcal{A}_{01221} = \mathcal{A}_{02112}$.

While there are no pressure terms requiring removal from these expressions it is still useful to differentiate the traction rates with respect to x_1 so that the boundary conditions become

$$\begin{aligned}&\cos^2\theta\left[\gamma_2 v_{1,12}' + \mathcal{A}_{01221}v_{2,11}'\right]\\ &+\cos\theta\sin\theta\left[\alpha_{11}v_{1,11}' - \gamma_2 v_{1,22}' + (\alpha_{12} - \mathcal{A}_{01221})v_{2,12}'\right]\\ &-\sin^2\theta\left[\alpha_{11}v_{1,12}' + \alpha_{12}v_{2,22}'\right] \qquad = 0,\end{aligned} \tag{3.89}$$

$$\begin{aligned}&\cos^2\theta\left[\alpha_{12}v_{1,11}' + \alpha_{22}v_{2,12}'\right]\\ &+\cos\theta\sin\theta\left[(\mathcal{A}_{01221} - \alpha_{12})v_{1,12}' + \gamma_1 v_{2,11}' - \alpha_{22}v_{2,22}'\right]\\ &-\sin^2\theta\left[\mathcal{A}_{01221}v_{1,22}' + \gamma_1 v_{2,12}'\right] \qquad = 0\end{aligned} \tag{3.90}$$

on $x_2 = 0$.

We substitute the expressions (3.87) for $\mathbf{v}'$ into the boundary conditions (3.89), (3.90) to find their being expressible as

$$\left.\begin{aligned}\Phi(s_1)A' + \Phi(s_2)B' = 0,\\ \Xi(s_1)A' + \Xi(s_2)B' = 0\end{aligned}\right\} \tag{3.91}$$

where now we have taken

$$\left.\begin{aligned}\Phi(s_i) \;=\;& \cos^2\theta\left[\gamma_2 m_i n_i + \mathcal{A}_{01221}r_1 m_i^2\right]\\ &+\cos\theta\sin\theta\left[\alpha_{11}m_i^2 + (\alpha_{12} - \mathcal{A}_{01221})r_i m_i n_i - \gamma_2 n_i^2\right]\\ &-\sin^2\theta\left[\alpha_{11}m_i n_i + \alpha_{12}r_i n_i^2\right],\\ \Xi(s_i) \;=\;& \cos^2\theta\left[\alpha_{12}m_i^2 + \alpha_{22}r_i m_i n_i\right]\\ &+\cos\theta\sin\theta\left[\gamma_1 r_i m_i^2 + (\mathcal{A}_{01221} - \alpha_{12})m_i n_i - \alpha_{22}r_i n_i^2\right]\\ &-\sin^2\theta\left[\mathcal{A}_{01221}n_i^2 + \gamma_1 r_i m_i n_i\right].\end{aligned}\right\} \tag{3.92}$$

Setting to zero the determinant of coefficients of A' and B' yields the secular equation for compressible materials. Of note here is the absence, explicitly at least, of terms involving the wavespeed through ρc^2 but it is through the s_i terms that the wavespeed features in this problem. It is also notable that the above approach is also applicable to any surface with x_2 constant.

In the calculation of the determinant we now have to contend with 6 material parameters α_{11}, α_{22}, α_{12}, γ_1, γ_2 and $\mathcal{A}_{01221}$ giving rise to a quadratic form of up to 21 terms. We find, however, that there are only 10 non-zero terms and that the following identity

$$\lambda m_i - n_i = -(\lambda^2 + 1)\sin\theta, \quad i = 1, 2, \tag{3.93}$$

which makes use of (3.55), is a feature throughout. It is then a relatively straightforward matter to show that the secular equation is

$$\begin{aligned}
&[s_2 - s_1]\,[\lambda^2(\alpha_{12}\gamma_2) - (\alpha_{11}\mathcal{A}_{01221}) \\
&\qquad\qquad + r_1 r_2\,[(\alpha_{12}\gamma_1) - \lambda^2(\alpha_{22}\mathcal{A}_{01221})]] \\
&+ [r_2 - r_1]\,[m_1 m_2(\alpha_{11}\gamma_1) + \lambda^2 n_1 n_2(\alpha_{22}\gamma_2)] \\
&+ [\lambda(r_1 m_2 n_1 - r_2 m_1 n2)]\,(\alpha_{12}^2 - \alpha_{11}\alpha_{22}) \\
&+ [\lambda(r_1 m_1 n_2 - r_2 m_2 n_1)]\,(\mathcal{A}_{01221}^2 - \gamma_1\gamma_2) \qquad = 0.
\end{aligned} \tag{3.94}$$

Once again the factor $s_2 - s_1$ appears in the final expression although for clarity it is perhaps better to leave it unfactorized since despite (3.94) being relatively simple as it appears here, it is worth recalling the definition of r_1, r_2 in (3.84).

To conclude, we reiterate that the complexity of the equations thus far derived is substantial and, while certainly deserving of investigation, may not provide much more insight than that already established and so we do not continue with them here.

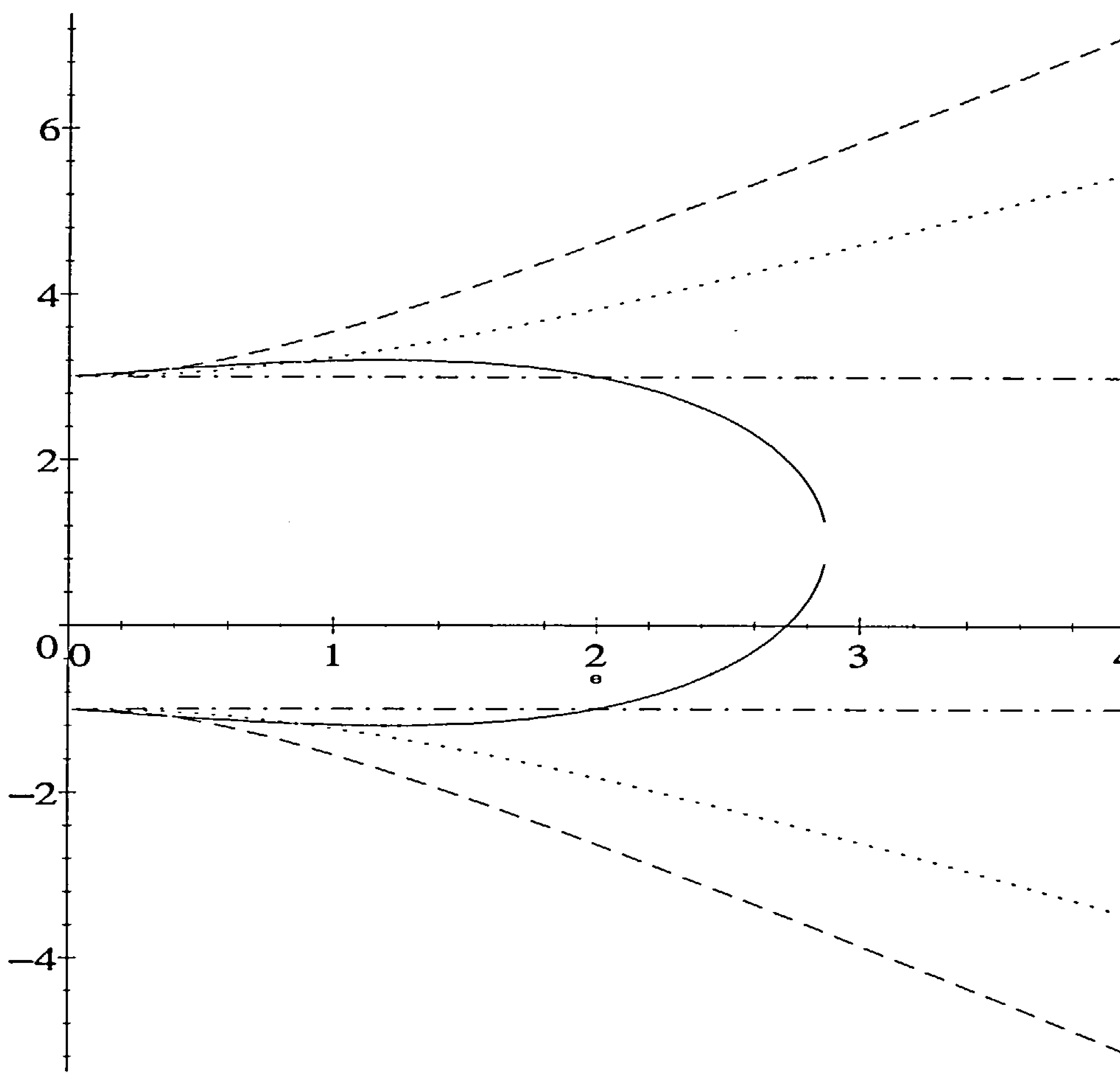


Figure 3.5: Plot of $\overline{p}$ (vertical scale) against ε (horizontal scale) showing the stable regions for strain-energy functions with $m = 0.5, 1, 2, 3$. The regions of stability are bounded by the two branches shown for each case.

Chapter 4

Waves in an Incompressible Elastic Layer

This chapter deals with applying the techniques of incremental elasticity, as used in the previous chapter, to a layer of elastic material having finite width and infinite length which has been pre-sheared. The layer, if subjected to appropriate boundary conditions, could then be thought of as modelling a shear band.

The formation of such a shear band is not within the scope of the work within this chapter but it is hoped that it contributes towards the understanding of the stability and dynamical nature of the material within the shear band.

Shear bands are frequently observed in metals and rock undergoing an otherwise smooth deformation path. They are characterized by the appearance of a strip of material of finite width whose boundaries with the surrounding material slide parallel to each other.

The onset of this localization has received a lot of attention and is understood to be associated with the transition from an elastic deformation to a plastic deformation; such localization is discussed in the review by Rice (1977). A feature of the shear band phenomenon is that it is often found that both within the band and in the surrounding material, the deformation is homogeneous, or at least smooth. The boundary of the shear band is thus a region of discontinuity in some deformation field. Material continuity ensures that the mapping $x = \chi(X)$ is continuous but it is in the deformation gradient that the discontinuities at the boundaries are to be found, this has the subsequent effect of producing discontinuities within the stress tensor and the stress rates.

The analysis of the shear band within the larger body is therefore suited to the approach used in analyzing acceleration waves within elasto-plastic solids as considered by Hill (1962). In contrast, within the context of elasticity the analysis of Abeyaratne (1980)' deals with the existence of elastostatic shocks in plane finite materials, this situation may be thought of as a precursor to the plastic stage shear band deformation. On these boundaries, the governing equations are found to have an associated loss of ellipticity (the partial differential equations becoming of hyperbolic type) and several authors have investigated the link between the loss of ellipticity and the incipient discontinuities/localization. In addition, such localization is linked with the existence of a local maximum of the shear stress against shear strain; see for example, Abeyaratne (1980), and Rudnicki and Rice (1975). Detailed discussion on the ellipticity of the equations of elasticity can be found in Knowles and Sternberg (1978).

Leroy and Molinari (1993) investigated the effect of a pre-stressed band and also found the significance of there being a maximum in the stress-strain relationship in identifying the onset of bifurcation although the model used there, based on a geophysical perspective which takes microstructural effects into account, does not require the loss of ellipticity.

It is also commonly found that in such discontinuities as may arise causing the localization, the incremental displacement is continuous and differentiable along the boundary (Leroy and Molinari (1993), Hill and Hutchinson (1975)) and it is with this in mind that the boundary conditions which we introduce are consistent with the existence of shear bands.

Examples of shear band effects can be found in Anand and Spitzig (1982) who investigate the effect of compression on the orientation of the shear bands in metals, polymers and sand; and also Williams and Price (1990) who conduct experiments on rock by putting it into plane strain (simple shear) and observing the shear band growth.

4.1 Formulation of Problem

We now consider an elastic layer having finite width h in the x_2 direction, with boundaries on $x_2 = 0, -h$, and extending to infinity in both the x_1 and x_3 directions.

The material is again taken to be isotropic, hyperelastic and incompressible, and we initially focus on the same class of materials for which $2\beta = \alpha + \gamma$.

The material will again be subjected to the quasi-static simple shear deformation and we employ the incremental theory in determining plane wave incremental deformations.

Effects such as reflection at the boundaries are neglected so that we seek steady state solutions corresponding to waves propagating in the x_1 direction, as discussed in Achenbach (1984), Chapter 6, for example.

The deformation being confined to the (x_1, x_2)-plane means that the results given in Section 2.6 governing the deformation and Section 3.1.1 concerning the incremental motions are also relevant here. We do note one fundamental distinction in that our four solutions for s of (3.14) — or, more generally, (3.58) — are all permitted in the general solution for ψ since we no longer need to make the restriction that the incremental displacements decay in the limit $x_2 \to -\infty$. We recall the solutions (3.46) for materials in this class and so find that ψ is now given by

$$\psi = \left(A_1 e^{iks_1x_2} + A_2 e^{iks_2x_2} + B_1 e^{ik\bar{s}_1x_2} + B_2 e^{ik\bar{s}_2x_2}\right) e^{ik(ct-x_1)}, \qquad (4.1)$$

where the same notation has been kept, the overbar denotes the complex conjugate and the A_1, A_2, B_1, B_2 are arbitrary constants.

With four constants to account for, the system will need four further equations from the boundary conditions; with the proviso that A_1, A_2, B_1, B_2 be non-trivial leading to a *dispersion equation* combining deformation and material parameters, hydrostatic pressure, wavespeed, wavelength and layer thickness. We introduce three such sets of boundary conditions in the next section and subsequently examine the resulting dispersion equations.

4.2 Dispersion Equations

For the layer problem we consider three sets of boundary conditions corresponding to; in the first case, the vanishing of the traction rates on the surfaces $x_2 = 0, -h$; in the second case, the vanishing of the traction rates on $x_2 = 0$ coupled with the vanishing of the incremental displacements on $x_2 = -h$; and in the third case, the vanishing of the incremental displacements on both surfaces.

In each of the first two cases, we require to establish expressions for the traction rates — or, more precisely, their derivatives with respect to x_1. We recall the expressions for the components of $\dfrac{\partial \dot{t}}{\partial x_1}$ given by the left-hand side of (3.19) and (3.20). For ψ' now being taken from (4.1), and with the notation of (3.66) being employed, the traction rate derivatives are given by

$$t'_{11} = \Big\{\Phi(s_1)e^{iks_1x_2}A_1 + \Phi(s_2)e^{iks_2x_2}A_2 \\ + \Phi(\bar{s}_1)e^{ik\bar{s}_1x_2}B_1 + \Phi(\bar{s}_2)e^{ik\bar{s}_2x_2}B_2\Big\}\, e^{ik(ct-x_1)}, \quad (4.2)$$

$$t'_{21} = \Big\{\Xi(s_1)e^{iks_1x_2}A_1 + \Xi(s_2)e^{iks_2x_2}A_2 \\ + \Xi(\bar{s}_1)e^{ik\bar{s}_1x_2}B_1 + \Xi(\bar{s}_2)e^{ik\bar{s}_2x_2}B_2\Big\}\, e^{ik(ct-x_1)}, \quad (4.3)$$

where we have introduced t'_{11}, t'_{21} as notation defining the traction rate derivatives.

It is easy to see from (3.5) that the incremental displacements are given by

$$\left.\begin{aligned} v_1 &= ik\left[A_1 s_1 e^{iks_1x_2} + A_2 s_2 e^{iks_2x_2} + B_1\overline{s}_1 e^{ik\overline{s}_1x_2} + B_1\overline{s}_2 e^{ik\overline{s}_2x_2}\right] e^{ik(ct-x_1)}, \\ v_2 &= ik\psi. \end{aligned}\right\} \tag{4.4}$$

Problem 1: Incremental Traction Boundary Conditions

Here we set the traction rates $\dot{S}_{021} = \dot{S}_{022} = 0$ on $x_2 = 0,\ -h$ which is equivalent to $t'_{11} = t'_{21} = 0$ on $x_2 = 0 - h$. We thus arrive at the four conditions

$$\begin{bmatrix} \Phi(s_1) & \Phi(s_2) & \Phi(\overline{s}_1) & \Phi(\overline{s}_2) \\ \Phi(s_1)e^{-iks_1h} & \Phi(s_2)e^{-iks_2h} & \Phi(\overline{s}_1)e^{-ik\overline{s}_1h} & \Phi(\overline{s}_2)e^{-ik\overline{s}_2h} \\ \Xi(s_1) & \Xi(s_2) & \Xi(\overline{s}_1) & \Xi(\overline{s}_2) \\ \Xi(s_1)e^{-iks_1h} & \Xi(s_2)e^{-iks_2h} & \Xi(\overline{s}_1)e^{-ik\overline{s}_1h} & \Xi(\overline{s}_2)e^{-ik\overline{s}_2h} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} = \mathbf{0}. \tag{4.5}$$

On introducing, for $\alpha \in \{1,2\}$, the convention that $\phi_\alpha = \phi(s_\alpha)$, $\overline{\phi}_\alpha = \phi(\overline{s}_\alpha)$, $\xi_\alpha = \xi(s_\alpha)$ and $\overline{\xi}_\alpha = \xi(\overline{s}_\alpha)$ along with the notation $a_\alpha = e^{-ikhs_\alpha}$, $\overline{a}_\alpha = e^{ikh\overline{s}_\alpha}$ this system is rewritten as

$$\begin{bmatrix} \phi_1 & \phi_2 & \overline{\phi}_1 & \overline{\phi}_2 \\ \phi_1 a_1 & \phi_2 a_2 & \overline{\phi}_1/\overline{a}_1 & \overline{\phi}_2/\overline{a}_2 \\ \xi_1 & \xi_2 & \overline{\xi}_1 & \overline{\xi}_2 \\ \xi_1 a_1 & \xi_2 a_2 & \overline{\xi}_1/\overline{a}_1 & \overline{\xi}_2/\overline{a}_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} = \mathbf{0}. \tag{4.6}$$

The dispersion equation resulting from this system is then

$$\begin{aligned} &(\phi_1\overline{\xi}_1 - \overline{\phi}_1\xi_1)(\phi_2\overline{\xi}_2 - \overline{\phi}_2\xi_2)(1 - a_1\overline{a}_2 - \overline{a}_1 a_2 + a_1 a_2\overline{a}_1\overline{a}_2) \\ &+ (\phi_1\overline{\xi}_2 - \overline{\phi}_2\xi_1)(\overline{\phi}_1\xi_2 - \phi_2\overline{\xi}_1)(1 - a_1\overline{a}_1 - a_2\overline{a}_2 + a_1 a_2\overline{a}_1\overline{a}_2) = 0. \end{aligned} \tag{4.7}$$

We immediately note that this representation of the dispersion equation makes it clear that the left-hand side of (4.7) is purely real.

Problem 2: Mixed Boundary Conditions

Here we specify that the traction rates $\dot{S}_{021}$, $\dot{S}_{022}$ vanish on the boundary $x_2 = 0$ while the incremental displacements v_1, v_2 vanish on the boundary $x_2 = -h$. These boundary conditions could be used to describe a layer of material bonded to a rigid plate on one side only with the other side being a free surface.

Use of (4.2)–(4.4) in the boundary conditions leads to the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & \overline{s}_1 & \overline{s}_2 \\ \phi_1 a_i & \phi_2 a_2 & \overline{\phi}_1/\overline{a}_1 & \overline{\phi}_2/\overline{a}_2 \\ \xi_1 a_i & \xi_2 a_2 & \overline{\xi}_1/\overline{a}_1 & \overline{\xi}_2/\overline{a}_2 \end{bmatrix} \tag{4.8}$$

for the constants A_1, A_2, B_1, B_2 which leads to the dispersion equation

$$\begin{aligned} &\left(\phi_1\overline{\xi}_1 - \overline{\phi}_1\xi_1\right)(s_2 - \overline{s}_2)\frac{a_1}{\overline{a}_1} + \left(\phi_2\overline{\xi}_2 - \overline{\phi}_2\xi_2\right)(s_1 - \overline{s}_1)\frac{a_2}{\overline{a}_2} \\ &+ (\phi_1\xi_2 - \phi_2\xi_1)(\overline{s}_2 - \overline{s}_1)a_1a_2 + \left(\overline{\phi}_1\overline{\xi}_2 - \overline{\phi}_2\overline{\xi}_1\right)(s_2 - s_1)\frac{1}{\overline{a}_1\overline{a}_2} \\ &+ \left(\phi_1\overline{\xi}_2 - \overline{\phi}_2\xi_1\right)(\overline{s}_1 - s_2)\frac{a_1}{\overline{a}_2} + \left(\overline{\phi}_1\xi_2 - \phi_2\overline{\xi}_1\right)(s_1 - \overline{s}_2)\frac{a_2}{\overline{a}_1} = 0. \end{aligned} \tag{4.9}$$

Problem 3: Incremental Displacement Boundary Conditions

The third problem we consider is one where the incremental displacements v_1, v_2 vanish on both surfaces $x_2 = 0, -h$. This could be the type of problem encountered in, for example, layered structures such as vibration isolators that are constructed by bonding alternate layers of elastic material and rigid metal plates together.

Setting the expressions (4.4) to zero on the surfaces $x_2 = 0, -h$ produces the system of equations that takes the matrix of coefficients

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & \overline{s}_1 & \overline{s}_2 \\ a_1 & a_2 & 1/\overline{a}_1 & 1/\overline{a}_2 \\ s_1a_1 & s_2a_2 & \overline{s}_1/\overline{a}_1 & \overline{s}_2/\overline{a}_2 \end{bmatrix}. \tag{4.10}$$

Establishing the determinant for this matrix is made considerably easier by comparing (4.8) and (4.10) where we find the same structure in evidence. Indeed, in setting $\phi_1 = \phi_2 = 1$ and $\xi_1 = s_1$, $\xi_2 = s_2$ we can simply write down the dispersion equation for this problem from (4.9) as

$$(s_1 - \bar{s}_1)(\bar{s}_2 - s_2)\left(\frac{a_1}{\bar{a}_1} + \frac{a_2}{\bar{a}_2}\right) + (s_1 - s_2)(\bar{s}_1 - \bar{s}_2)\left(a_1 a_2 + \frac{1}{\bar{a}_1 \bar{a}_2}\right) + (s_1 - \bar{s}_2)(s_2 - \bar{s}_1)\left(\frac{a_1}{\bar{a}_2} + \frac{a_2}{\bar{a}_1}\right) = 0. \quad (4.11)$$

4.3 Analysis of the Dispersion Equations

This section will cover two aspects of the three problems. The first part will be concerned with the dynamic behaviour of materials belonging to the class of strain-energy function for which $2\beta = \alpha + \gamma$ and it therefore parallels Section 3.3. The second part will, as in Section 3.4.1, focus on the static case for a general strain-energy function with a view to determining bifurcation criteria and identifying regions of stability for the body.

4.3.1 Class $2\beta = \alpha + \gamma$

Having established the dispersion equations in terms of ϕ and ξ we record here that, for $2\beta = \alpha + \gamma$,

$$\left.\begin{aligned}
\phi_1 &= -\lambda(\bar{p} + \lambda^{-2}) - i(\bar{p} + \lambda^{-2} - \zeta), \\
\xi_1 &= \lambda^{-1}(\bar{p} + \lambda^{2}) - i(\bar{p} + \lambda^{-2} - \zeta), \\
\phi_2 &= -\lambda^{-1}(\bar{p} + \lambda^{2}\eta^2) - i\eta(\bar{p} + \lambda^{2}), \\
\xi_2 &= \lambda(\bar{p} + \lambda^{-2}\eta^2) - i\eta(\bar{p} + \lambda^{-2}),
\end{aligned}\right\} \quad (4.12)$$

calculated from substitution of (3.46) and $\delta = 0$ into (3.68).

Problem 1

In progressing further it is useful to examine the structure of the factors involving ϕ and ξ. Recalling that we are currently considering the class of strain-energy functions for which $2\beta = \alpha + \gamma$, we refer back to the notation of (3.42). The dispersion equation is then put as

$$\begin{aligned} &D(s_1,\overline{s}_1)D(s_2,\overline{s}_2)(1 - a_1\overline{a}_2 - \overline{a}_1 a_2 + a_1 a_2 \overline{a}_1 \overline{a}_2) \\ &+ D(s_1,\overline{s}_2)D(\overline{s}_1,s_2)(1 - a_1\overline{a}_1 - a_2\overline{a}_2 + a_1 a_2 \overline{a}_1 \overline{a}_2) = 0. \end{aligned} \tag{4.13}$$

Written in this fashion, and with the solutions (3.46) for s, we find it straightforward to produce

$$\begin{aligned} &[\varepsilon^2 + (\eta+1)^2]\,[\eta^3 - \eta^2 + (1+\varepsilon^2+2\overline{p})\eta + \overline{p}^2]^2 \left(1 - e^{2kh}\right)\left(1 - e^{2kh\eta}\right) \\ &- 4\eta\,[(1+\overline{p})\eta^2 + \overline{p}^2 + (\varepsilon^2+1)\overline{p}]^2 \left(1 - e^{kh(1+\eta+i\varepsilon)}\right)\left(1 - e^{kh(1+\eta-i\varepsilon)}\right) = 0. \end{aligned} \tag{4.14}$$

The similarity between the second factor and the secular equation (3.47) produced in Section 3.3 is worth noting.

An interesting aspect of this approach lies in the arbitrary choice of labels for the solutions of the governing equation. If we retain $s_1 = -i$, but instead choose $s_2 = \varepsilon + \eta i$, we find that substitution of these (and their conjugates) yields the alternative dispersion equation

$$\begin{aligned} &[\varepsilon^2 + (\eta-1)^2]\,[\eta^3 + \eta^2 + (1+\varepsilon^2+2\overline{p})\eta - \overline{p}^2]^2 \left(1 - e^{2kh}\right)\left(1 - e^{-2kh\eta}\right) \\ &+ 4\eta\,[(1+\overline{p})\eta^2 + \overline{p}^2 + (\varepsilon^2+1)\overline{p}]^2 \left(1 - e^{kh(1-\eta+i\varepsilon)}\right)\left(1 - e^{kh(1-\eta-i\varepsilon)}\right) = 0, \end{aligned} \tag{4.15}$$

in which we find the left-hand side of the secular equation for the half-space problem (3.47) appearing explicitly.

An important connection between the factors found in both (4.14) and (4.15) is the following,

$$\begin{aligned} &\left[\varepsilon^2 + (\eta-1)^2\right]\left[\eta^3 + \eta^2 + (1+\varepsilon^2+2\overline{p})\eta - \overline{p}^2\right]^2 \\ &\quad = [\varepsilon^2 + (\eta+1)^2]\,[\eta^3 - \eta^2 + (1+\varepsilon^2+2\overline{p})\eta + \overline{p}^2]^2 \\ &\quad\quad - 4\eta\,[(1+\overline{p})\eta^2 + \overline{p}^2 + (\varepsilon^2+1)\overline{p}]^2 . \end{aligned} \tag{4.16}$$

Through use of (4.16) it can be established that (4.14) and (4.15) are equivalent in that they can both be rearranged as

$$\begin{aligned}
&[\varepsilon^2+(\eta-1)^2]\,[\eta^3+\eta^2+(1+\varepsilon^2+2\overline{p})\eta-\overline{p}^2]^2\cosh kh(1+\eta)\\
&-[\varepsilon^2+(\eta+1)^2]\,[\eta^3-\eta^2+(1+\varepsilon^2+2\overline{p})\eta+\overline{p}^2]^2\cosh kh(1-\eta)\\
&+4\eta\,[(1+\overline{p})\eta^2+\overline{p}^2+(\varepsilon^2+1)\overline{p}]^2\cos kh\varepsilon \qquad =0, \qquad (4.17)
\end{aligned}$$

which we henceforth refer to as the *dispersion equation for Problem 1.*

The leading term in (4.17) is $\cosh kh(1+\eta)$ so that for large kh the leading coefficient is

$$\left[\varepsilon^2+(\eta-1)^2\right]\left[\eta^3+\eta^2+(1+\varepsilon^2+2\overline{p})\eta-\overline{p}^2\right]^2. \qquad (4.18)$$

Thus the limiting case $kh\to\infty$ recovers the secular equation (3.47) for surface waves on a a half space. Equally, the limit $kh\to\infty$ is sometimes interpreted as the short wavelength limit since the wavelength is inversely proportional to the wavenumber k.

Unlike the half-space problem, we do not need to confine the wavespeed to the interval $[0,1]$ to ensure that η be real and positive. For $\zeta>1$, $\eta=\sqrt{1-\zeta}$ is purely imaginary. By writing the dispersion equation (4.17) , with the help of (4.10), as

$$\begin{aligned}
&\left[q\left(\overline{p}^2-\eta^2\right)^2+q\eta^2\left(q+2\overline{p}\right)^2+4\eta^2\left(\overline{p}^2-\eta^2\right)\left(q+2\overline{p}\right)\right]\sinh kh\sinh kh\eta\\
&\qquad+2\eta\left(\overline{p}q+\overline{p}^2+\eta^2\right)^2\left(\cos kh\varepsilon-\cosh kh\cosh kh\eta\right) \;=\; 0, \qquad (4.19)
\end{aligned}$$

where

$$q=1+\varepsilon^2+\eta^2, \qquad (4.20)$$

it can be seen that the left-hand side of (4.17) is purely real or purely imaginary according as η is itself purely real or purely imaginary respectively.

Problem 2

With our solutions for s substituted into the expression (4.9) and equating to zero, we arrive at the *dispersion equation for Problem 2*

$$\begin{aligned}
&[\varepsilon^2 + (\eta-1)^2]\,[\eta^3 + \eta^2 + (1+\varepsilon^2+2\overline{p})\eta - \overline{p}^2]\cosh kh(1+\eta) \\
&\quad + [\varepsilon^2 + (\eta+1)^2]\,[\eta^3 - \eta^2 + (1+\varepsilon^2+2\overline{p})\eta + \overline{p}^2]\cosh kh(1-\eta) \\
&\qquad\qquad - 4\eta\,[(1+\overline{p})\eta^2 + \overline{p}^2 + (\varepsilon^2+1)\overline{p}]\cos kh\varepsilon \;= 0, \qquad (4.21)
\end{aligned}$$

within which we observe the marked similarities with (4.17).

As was the case for Problem 1, we find the same leading coefficient (4.18) appearing in the dispersion equation. This is as would be expected from the choice of boundary conditions in both problems in that they share the same condition as the half-space problem on the common boundary $x_2 = 0$.

We also find that the dispersion equation arrived at in (4.21) is unaffected by setting the traction rates to be zero on $x_2 = -h$ and the incremental displacements to be zero on $x_2 = 0$. Details are omitted here.

Also as for Problem 1, we draw the same conclusion regarding the purely real or purely imaginary nature of the left-hand side of the dispersion equation after rearranging (4.21) as

$$\begin{aligned}
&\eta\,[q(q+2\overline{p}) - 2(\eta^2 - \overline{p}^2)]\cosh kh \cosh kh\eta \\
&\quad - [q(\eta^2+\overline{p}^2) + 4\eta^2\overline{p}]\sinh kh \sinh kh\eta \qquad (4.22)\\
&\qquad - 2\eta\,[(\eta^2+\overline{p}^2) + \overline{p}q]\cos kh\varepsilon \;=\; 0,
\end{aligned}$$

with q being defined as in (4.20).

Problem 3

Finally, the introduction of the solutions for s into (4.11) provides us with the *dispersion equation for Problem 3*

$$\begin{aligned}
&4\eta\cos kh\varepsilon + [\varepsilon^2 + (\eta^2-1)^2]\cosh kh(1+\eta) \\
&\quad - [\varepsilon^2 + (\eta^2+1)^2]\cosh kh(1-\eta) \;=\; 0. \qquad (4.23)
\end{aligned}$$

Notice that since no information about stress is introduced, there is no pressure parameter involved in (4.23) and that the leading term is no longer that of the above two problems but rather it reduces to $s_1 = s_2$ in the half-space limit.

The left-hand side of the dispersion equation for this problem is also a purely real (imaginary) expression dependent upon the nature of η, equation (4.23) being rewritten

$$q \sinh kh \sinh kh\eta + 2\eta(\cos kh\varepsilon - \cosh kh \cosh kh\eta) = 0 \qquad (4.24)$$

as corroboration.

4.3.2 Bifurcation Results

The wavespeed ζ is now set to zero ($\eta = 1$), this prescription enabling the extension of the theory to a general incompressible elastic material. The notation and results of Section 3.4.1 are relevant here, primarily (3.63), (3.64) and (3.68).

Looking at (3.68), and with $\zeta = 0$, we see that

$$\xi(s) = \phi(\varepsilon - s), \qquad (4.25)$$

so that the following connections hold

$$\left.\begin{array}{rclrcll} \xi_2 &=& -\overline{\phi}_1, & \phi_2 &=& -\overline{\xi}_1, & (\overline{\beta} > 1), \\ \overline{\phi}_\alpha &=& -\phi_\alpha, & \overline{\xi}_\alpha &=& -\xi_\alpha, & (\overline{\beta} < 1), \end{array}\right\} \qquad (4.26)$$

where $\alpha = 1, 2$.

We do not provide details of the calculation involved but rather we record the factors appearing in the dispersion equations of Problems 1 and 2 modified according to (4.26). For $\overline{\beta} > 1$ we have

$$\left.\begin{array}{rcl} \phi_1\overline{\phi}_1 - \xi_1\overline{\xi}_1 &=& \sqrt{4+\varepsilon^2}\sqrt{\varepsilon^2 - 4\delta}\left[(\overline{p}-1)^2 - \dfrac{4+\varepsilon^2}{1+\delta}\right], \\ \phi_1\overline{\xi}_1 - \overline{\phi}_1\xi_1 &=& -2i\sqrt{4+\varepsilon}\sqrt{1+\delta}\left[(\overline{p}+1)^2 + p\dfrac{\varepsilon^2 - 4\delta}{1+\delta}\right], \\ \phi_1^2 - \xi_1^2 &=& \sqrt{4+\varepsilon^2}\left[\sqrt{\varepsilon^2 - 4\delta} + 2i\sqrt{1+\delta}\right]\left[(\overline{p}+1)^2 + \dfrac{\varepsilon^2 - 4\delta}{1+\delta}\right], \end{array}\right\} \qquad (4.27)$$

whereas for $\overline{\beta} < 1$ we have

$$\left.\begin{aligned}
\phi_1^2 - \xi_1^2 &= i\sqrt{4+\varepsilon^2}\left[\sqrt{4\delta-\varepsilon^2}+2\sqrt{1+\delta}\right]\left[(\overline{p}+1)^2+\frac{\varepsilon^2-4\delta}{1+\delta}\right], \\
\phi_2^2 - \xi_2^2 &= i\sqrt{4+\varepsilon^2}\left[2\sqrt{1+\delta}-\sqrt{4\delta-\varepsilon^2}\right]\left[(\overline{p}+1)^2+\frac{\varepsilon^2-4\delta}{1+\delta}\right], \\
\phi_1\phi_2 - \xi_1\xi_2 &= 2i\sqrt{4+\varepsilon}\sqrt{1+\delta}\left[(\overline{p}+1)^2+p\frac{\varepsilon^2-4\delta}{1+\delta}\right], \\
\phi_1\xi_2 - \phi_2\xi_1 &= -i\sqrt{4+\varepsilon^2}\sqrt{1+\delta}\left[(\overline{p}-1)^2-\frac{4+\varepsilon^2}{1+\delta}\right].
\end{aligned}\right\} \tag{4.28}$$

For illustration, we assume that $\overline{\beta} > 1$. Justification for this lies in the fact that both regimes for $\overline{\beta}$ (along with equality with unity) produce essentially the same bifurcation equations although any differences arising will be noted.

Problem 1

With $\zeta = 0$ and the connections (4.25 and (4.26), the dispersion equation can initially be put as

$$\begin{aligned}
&(\phi_1\overline{\phi}_1 - \xi_1\overline{\xi}_1)^2\cosh[2kh\sqrt{1+\delta}] \\
&\quad - (\phi_1\overline{\xi}_1 - \overline{\phi}_1\xi_1)^2\cos[kh\sqrt{\varepsilon^2-4\delta}] \\
&\quad\quad -(\phi_1^2-\xi_1^2)(\overline{\phi}_1^2-\overline{\xi}_1^2) = 0,
\end{aligned} \tag{4.29}$$

whereupon use of a relation akin to (4.16) allows the rearrangement

$$(\phi_1\overline{\phi}_1 - \xi_1\overline{\xi}_1)\sinh\left[kh\sqrt{1+\delta}\right] = \pm i(\phi_1\overline{\xi}_1 - \overline{\phi}_1\xi_1)\sin\left[\frac{1}{2}kh\sqrt{\varepsilon^2-\delta}\right]. \tag{4.30}$$

Referring back to (4.27) and (3.62), the above bifurcation equations (4.29) and (4.30) are explicitly written as

$$\begin{aligned}
&(\overline{\beta}-1)(\overline{p}^2-2\overline{p}-2\overline{\beta}-1)^2\cosh[2kh\sqrt{1+\delta}] \\
&\quad + 2(\overline{p}^2+2\overline{\beta}\overline{p}+1)^2\cos[kh\sqrt{\varepsilon^2-4\delta}] \\
&\quad\quad - (\overline{\beta}+1)(\overline{p}^2+2\overline{p}+2\overline{\beta}-1)^2 = 0
\end{aligned} \tag{4.31}$$

and

$$(1+2\overline{\beta}+2\overline{p}-\overline{p}^2)\frac{\sinh[kh\sqrt{1+\delta}]}{\sqrt{1+\delta}} = \pm(\overline{p}^2+2\overline{\beta}\overline{p}+1)\frac{\sin[\frac{1}{2}kh\sqrt{\varepsilon^2-4\delta}]}{\frac{1}{2}\sqrt{\varepsilon^2-4\delta}}. \tag{4.32}$$

It is at this point that we note the effect of $\overline{\beta}$ on the bifurcation equation. When $\overline{\beta} < 1$ we find that the term

$$\frac{\sin[\frac{1}{2}kh\sqrt{\varepsilon^2 - 4\delta}]}{\frac{1}{2}\sqrt{\varepsilon^2 - 4\delta}}$$

in (4.32) would be replaced by its hyperbolic counterpart

$$\frac{\sinh[\frac{1}{2}kh\sqrt{4\delta - \varepsilon^2}]}{\frac{1}{2}\sqrt{4\delta - \varepsilon^2}},$$

whereas for $\overline{\beta} = 1$ we replace it with kh. We remark that replacement by kh is a consequence of both the result $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ as well as the necessary modification to (4.1) for the case of repeated roots.

In the bifurcation equation (4.32), the appearance of the $(\pm)$ sign indicates two modes of bifurcation with the parameters kh, δ, $\overline{\beta}$ and $\overline{p}$ determining which one occurs at the onset of instability. The modes are referred to as *antisymmetric*, or *flexural*, when the $(+)$ sign is taken; and *symmetric*, or *barreling*, in the case of the $(-)$ sign. The distinction is best visualized in the undeformed configuration as, there, the (anti)symmetry is with respect to the (x_1, x_2) axes whereas for $\varepsilon \neq 0$ the interpretation remains the same although the axes will have changed. For the undeformed configuration, symmetric incremental displacements are those for which

$$\left.\begin{array}{rcl} v_1(x_1, -h/2 - x_2) & = & v_1(x_1, -h/2 + x_2), \\ v_2(x_1, -h/2 - x_2) & = & -v_2(x_1, -h/2 + x_2), \end{array}\right\} \qquad (4.33)$$

whereas antisymmetric incremental displacements are such that

$$\left.\begin{array}{rcl} v_1(x_1, -h/2 - x_2) & = & -v_1(x_1, -h/2 + x_2), \\ v_2(x_1, -h/2 - x_2) & = & v_2(x_1, -h/2 + x_2). \end{array}\right\} \qquad (4.34)$$

By recasting the potential function ψ into a combination of trigonometric or hyperbolic components we could identify those components that are symmetric or antisymmetric with respect to x_2. If we then choose those components that correspond to, say, symmetric displacements and substitute for ψ into the boundary conditions, we find that the bifurcation criterion is given by (4.30), when $\varepsilon = 0$, with the right-hand side taking the $(-)$ sign. Conversely, choosing those components associated with antisymmetric displacements yields the case where the right-hand side of (4.30) takes the $(+)$ sign.

With $\varepsilon \neq 0$ we no longer have v_1, v_2 as clearly described, however motivation for referring to the two $(\pm)$ cases as flexural or barreling modes is to maintain consistency with the use of the terms made above. Detailed discussion on antisymmetric and symmetric modes can be found in Ogden and Roxburgh (1993) for the case of the bifurcation of a homogeneously pure strained elastic block linking, here, with the particular case of the undeformed configuration.

There exist certain configurations wherein both bifurcation criteria are met, arising through the vanishing of the both sides of (4.32). The factor $(\overline{p}^2 - 2\overline{p} - 2\overline{\beta} - 1)$ vanishes, for some $\overline{p}$, if and only if $\overline{\beta} \geq 1$. When equality holds we observe the vanishing of both sides when $\overline{p} = -1$. When $\overline{\beta} > 1$, both types of bifurcation occur provided that

$$\overline{p} = 1 \pm \sqrt{2(\overline{\beta} + 1)} \tag{4.35}$$

and that $\overline{\beta}, \delta, kh$ take values such that

$$(\overline{\beta} - 1)(1 + \delta) = \frac{2n^2\pi^2}{k^2h^2}, \quad n \in \mathbb{N}. \tag{4.36}$$

Recalling that $\delta = 0$ is implied when $\varepsilon = 0$, we examine (4.32) and find that, with kh prescribed, the underlying deformation remains stable provided

$$(2\overline{\beta} + 1 + 2\overline{p} - \overline{p}^2)\frac{\sinh[kh\sqrt{1+\delta}]}{\sqrt{1+\delta}} > \pm 2(\overline{p}^2 + 2\overline{\beta}\overline{p} + 1)\frac{\sin\left[\frac{1}{2}kh\sqrt{\varepsilon^2 - 4\delta}\right]}{\sqrt{\varepsilon^2 - 4\delta}} \tag{4.37}$$

as $(\varepsilon, \overline{p})$ evolve on a path of loading from $(0, 1)$.

We require to treat (4.37) separately for $\overline{\beta} > 1$, $\overline{\beta} < 1$ and $\overline{\beta} = 1$.

$\boldsymbol{\overline{\beta} > 1}$ **:** For $-1 < p < 1$, (4.37) holds for all ε and all kh. At the endpoints $\overline{p} = \pm 1$ it holds for all ε with $kh \neq 0$.

$\boldsymbol{\overline{\beta} < 1}$ **:** For $-\overline{\beta} < \overline{p} < 1$, (4.37) holds for all ε and all kh. Again, at the endpoints $\overline{p} = 1, -\overline{\beta}$, it holds for all ε with $kh \neq 0$.

$\overline{\beta} = 1$

This is the special case $\varepsilon^2 = 4\delta$ and the bifurcation equation (4.37) decouples into two equations :-

$$\overline{p} + 1 = 0, \tag{4.38}$$

as already noted, or

$$\frac{\sinh[kh\sqrt{1+\delta}]}{kh\sqrt{1+\delta}} = \pm\frac{\overline{p}+1}{3-\overline{p}}. \tag{4.39}$$

The stability inequalities may be further refined to

$$(3-\overline{p})\frac{\sinh[kh\sqrt{1+\delta}]}{kh\sqrt{1+\delta}} > \overline{p} + 1 > 0 \tag{4.40}$$

by merging the stability inequalities resulting from the bifurcation equations (4.38) and (4.39).

For $-1 < p < 1$ the inequalities (4.40) hold for all ε and for all kh. When $\overline{p} = 1$ this is modified to all ε but with $kh \neq 0$; the inequalities fail at $\overline{p} = -1$. Connection is made with a result in Ogden and Roxburgh (1993) when $\delta = 0$ in (4.39).

Consideration of the leading term in the bifurcation equation (4.32) shows that in the limit $kh \to \infty$ we recover

$$\overline{p}^2 - 2\overline{p} - 2\overline{\beta} - 1 = 0, \tag{4.41}$$

which is the bifurcation criterion (3.70) for a half-space and we also recover the stability inequality (3.72)

Problem 2

For this problem the dispersion equation becomes

$$\begin{aligned}(\overline{\beta}-1)(\overline{p}^2 - 2\overline{p} - 2\overline{\beta} - 1)\cosh[2kh\sqrt{1+\delta}] \\ + 2(\overline{p}^2 + 2\overline{\beta}\overline{p} + 1)\cos[kh\sqrt{\varepsilon^2 - 4\delta}] \\ - (\overline{\beta}+1)(\overline{p}^2 + 2\overline{p} + 2\overline{\beta} - 1) &= 0\end{aligned} \tag{4.42}$$

when we use (4.27), the similarity with (4.31) again being noted. As with the dynamic case, the half-space limit $kh \to \infty$ produces (4.41) through the leading term.

Unlike Problem 1, though, there is no rearrangement, such as the step in going from (4.29) to (4.30), that will decouple (4.42); however, it can be put as

$$\begin{aligned}(2\overline{\beta}+1+2\overline{p}-\overline{p}^2)&\frac{\sinh^2[kh\sqrt{1+\delta}]}{1+\delta}\\ &+4(\overline{p}^2+2\overline{\beta}\overline{p}+1)\frac{\sin^2[\frac{1}{2}kh\sqrt{\varepsilon^2-4\delta}]}{\varepsilon^2-4\delta}+\frac{4+\varepsilon^2}{(1+\delta)^2} = 0.\end{aligned} \tag{4.43}$$

We note that the underlying deformation is stable provided that the left-hand side is positive. For the particular case $\delta = 0$ the coefficient of the hyperbolic term remains non-negative for $-1 \leq \overline{p} \leq 3$, so that, since the coefficient of the trigonometric term is also positive here, stability is guaranteed for a wider range of pressure values than found in Problem 1.

Problem 3

The bifurcation equation resulting from (4.11) is

$$(\varepsilon^2-4\delta)\cosh[2kh\sqrt{1+\delta}]+4(1+\delta)\cos[kh\sqrt{\varepsilon^2-4\delta}]-(\varepsilon^2+4)=0, \tag{4.44}$$

which lends itself to the rearrangement

$$\frac{\sinh[kh\sqrt{1+\delta}]}{\sqrt{1+\delta}} = \pm\frac{\sin\left[\frac{1}{2}kh\sqrt{\varepsilon^2-4\delta}\right]}{\frac{1}{2}\sqrt{\varepsilon^2-4\delta}}. \tag{4.45}$$

Since $\dfrac{\sinh x}{x} \geq 1$ for all x and $\left|\dfrac{\sin x}{x}\right| \leq 1$ for all x, equality only holding when $x = 0$ in both cases, we see that (4.45) must have no solutions for $kh \neq 0$, recall $1+\delta > 0$ from (3.61), and consequently the material is stable under these boundary conditions.

4.4 Numerical Results

4.4.1 Bifurcation Results

Problem 1

The bifurcation equation for Problem 1 is tidied up as

$$(1+2\overline{\beta}+2\overline{p}-\overline{p}^2)\sinh\kappa = \pm(\overline{p}^2+2\overline{p}+2\overline{\beta}-1)\frac{\sin\left[\kappa\sqrt{\frac{1}{2}(\overline{\beta}-1)}\right]}{\sqrt{\frac{1}{2}(\overline{\beta}-1)}} \tag{4.46}$$

to reduce the degrees of freedom with $\kappa := kh\sqrt{1+\delta}$. It is thus viewed as an implicit equation in $\overline{p}$ and κ being parameterized by $\overline{\beta}$, itself being controlled by the deformation and material parameters.

We display the effect of κ on the pressure $\overline{p}$ in Figures 4.1–4.3 for various choices of $\overline{\beta}$. In doing so we shall be bearing in mind strain-energy functions of the type defined in (3.74). In particular, the Varga material of the case $m = 1$ takes $\overline{\beta} = 1$ for all strains ε as previously noted. The discussion of Page 87 is relevant to this case.

In Figure 4.1 we show the solutions to the bifurcation equations (4.38) and (4.39). Indicated on the diagram are the branches corresponding to the antisymmetric (+) and symmetric (−) modes. The overall region of stability is bounded by the line $\overline{p} = -1$ and the antisymmetric mode. For this material, then, we have $\varepsilon^2 = 4\delta$ so that the independent variable $\kappa = kh\sqrt{1+\delta}$ may equally be thought of as a measure of the strain; Figure 4.1 thus serving to show the dependence of the bifurcation on the deformation. Aside from the Varga material, Figure 4.1 is also appropriate for any strain-energy function satisfying $\varepsilon^2 = 4\delta$.

In Figure 4.2 we plot $\overline{p}$ against κ for a selection of $\overline{\beta}$. Figure 4.2(a) corresponds to the antisymmetric modes while Figure 4.2(b) displays the symmetric bifurcation modes. It can be seen from the two that, generally speaking, the effect of increasing $\overline{\beta}$ is to enlarge the region of stability with the single exception of small values of κ in respect of the symmetric modes. Having previously shown the dependence of $\overline{\beta}$ on ε, at least for a particular class of material, the net result is that increased deformation leads to a more stabilized regime.

Figure 4.3 provides an alternative viewpoint on the results shown in Figure 4.2. Each panel now refers to a particular value of $\overline{\beta}$ with both the symmetric and antisymmetric modes being plotted together. Features such as were discussed in Section 4.3 and which led to (4.35) and (4.36) are herein evident. The two modes have curves which are seen to intersect along the horizontal lines of constant pressure indicated by (4.35) with the periodicity of their crossing being governed by (4.36).

These same lines of constant pressure are also horizontal asymptotes corresponding to the layer thickness limit $\kappa \to \infty$ and is a consequence of setting to zero the coefficient of the leading term from the bifurcation equation (4.31). In the other limiting case, $\kappa \to 0$, there are three possible limits; for antisymmetric modes there is $\overline{p} = 1, -\overline{\beta}$ whereas for symmetric modes there is the unique (finite) limit $\overline{p} = -1$.

Problem 2

In view of the boundary conditions being employed, the material is imbued with more constraints than those of Problem 1. The results depicted here will reflect these constraints in there being fewer solutions present. That having been said, the similarities between the dispersion/bifurcation equations of the two problems are sufficient enough to produce similarities in the graphical solutions.

The analogous results to Figure 4.1–4.2 are pictured in Figures 4.4–4.5, the bifurcation equation (4.43) having been rearranged as

$$\left(1 + 2\overline{\beta} + 2\overline{p} - \overline{p}^2\right)\sinh^2\kappa + 2\left(1 + 2\overline{\beta}\overline{p} + \overline{p}^2\right)\frac{\sin^2\left[\kappa\sqrt{\frac{1}{2}(\overline{\beta}-1)}\right]}{(\overline{\beta}-1)} + 2(\overline{\beta}+1) = 0; \tag{4.47}$$

these figures clearly showing the reduction of solutions due to the extra constraints. Figure 4.4 shows the special case $\overline{\beta} = 1$ while the subsequent plot,

Figure 4.5, displays the effect of varying $\overline{\beta}$. In this we see effects similar to the symmetric modes of Problem 1 in respect of positive values of $\overline{p}$.

The comments made regarding constraints on the solutions through the boundary conditions are equally applicable to Problem 3 as we have seen there is no possible bifurcation for those boundary conditions.

4.4.2 Wavespeed Results

Here we present the results for the wavespeed of propagated waves in materials such that $\delta = 0$. The dispersion equations under consideration are therefore (4.17), (4.21) and (4.23) for the three posed problems. We remark that, for this class of material, $\overline{\beta} = 1 + \frac{1}{2}\varepsilon^2$.

Before presenting some of the results, some slight discussion of the structure of the dispersion equations may be included where it is useful in demonstrating expediency in the numerical analysis.

Problem 1

We are concerned here with exhibiting ζ as a function of the deformation, the layer thickness/wavelength, or the hydrostatic pressure.

In this problem, the dispersion equation (4.17) can be decoupled *when* $\varepsilon = 0$, similarly to the bifurcation equation (4.32), as

$$\begin{aligned}[]&[\eta^3 + \eta^2 + (1 + 2\overline{p})\eta - \overline{p}^2]\,\frac{\sinh \frac{1}{2}kh(1 + \eta)}{1 + \eta} \\ &\qquad = \pm\,[\eta^3 - \eta^2 + (1 + 2\overline{p})\eta + \overline{p}^2]\,\frac{\sinh \frac{1}{2}kh(1 - \eta)}{1 - \eta}.\end{aligned} \tag{4.48}$$

Equation (4.48) establishes another connection with a result, specialized to the undeformed configuration, in Ogden and Roxburgh (1993) albeit in different notation. As it appears here, the two signs are associated with symmetric $(-)$ and antisymmetric $(+)$ modes just as in the quasi-static case.

In the following we refer to lowest-order and higher order modes. The lowest-order modes are those which exist in the region $\zeta < 1$ and may be either of symmetric or antisymmetric type. In particular, wherever we have a curve in the region $\zeta < 0$ then the solutions are unstable as we describe below. When a curve intersects the axis $\zeta = 0$ the wavespeed is zero and this corresponds to a point of bifurcation. With $\zeta > 1$ the dispersion equations include trigonometric terms and there exists an infinite number of solutions available due to the periodic nature of these terms. As the arguments of the trigonometric terms increase through multiples of 2π, so we have distinct modes of solutions.

Solutions of (4.48) are shown in Figure 4.6; the material parameter $\overline{\beta} = 1$ providing the link with Figure 4.1 since bifurcation results are embodied along the axes $\zeta = 0$. In Figure 4.6(a), the pressure is $\overline{p} = -2$ which lies outside the domain of stability, see (4.40) for example. The lowest of the flexural and barreling modes are not supported here since the only solutions available here require $\zeta = \dfrac{\rho c^2}{\sqrt{\alpha\gamma}} < 0$ resulting in v_1, v_2 becoming unbounded thereby exceeding the limits of the incremental theory. The higher-order modes are possible, though. The next two Figures 4.6(b) and 4.6(c) are for pressures within the stable region for all non-zero kh. With $\overline{p} = 2$ in 4.6(d) the $(-)$ modes are stable for all kh whereas the lowest $(+)$ mode is unstable for small kh. This feature is also evident in Figure 4.1. With Figure 4.6(e) we arrive at a transitional phase, $\overline{p} = 3$, since this is an unstable regime for the lowest $(+)$ mode (excepting the infinite limit for kh) while the lowest $(-)$ mode is stable for finite kh — the higher-order modes are not affected. Further increasing the pressure, $\overline{p} = 4$, Figure 4.6(f) forces the lowest $(-)$ mode to lie in the unstable region *except* for small kh. Of particular note is that the onset of instability is only associated with the lowest $(+)$ and $(-)$ modes; the higher-order modes are such that the $(+)$ and $(-)$ branches are alternately distributed, as found for a homogeneously pure strained configuration by Ogden and Roxburgh (1993). Furthermore, the nature of the higher-order modes is little affected by changes in the pressure.

In Figures 4.7 and 4.8 we examine the effect of the deformation on these solutions. The corresponding analogues to Figure 4.1 are found from interpolating for $\overline{\beta} = 3$, 9 ($\varepsilon = 2$, 4 respectively) in Figure 4.2. The decoupling of (4.17) does not apply here, the graphs being calculated through (4.17) directly; the results, though, reveal a similar structure to those of Figure 4.6, including the appearance of analogues to the ($\pm$) modes.

In Figure 4.7(a) we have $\overline{p} = -2$ which lies within the domain of stability for the lowest (+) mode only when kh is small. With the following three figures the pressure ($\overline{p} = -1, 0, 1$) lies within the stable region for all kh. With $\overline{p} = 3$, Figure 4.7(e), the lowest (+) mode and (−) mode are unstable for small kh and increasing this to $\overline{p} = 4$, Figure 4.7(f) has the effect of making the lowest (+) mode unstable for all kh while the lowest (−) mode remains unstable except at small kh.

Throughout Figure 4.8 we observe the same qualitative behaviour in the solutions, especially in their respective figures (b)–(e). In Figure 4.8(a), $\overline{p} = -2$, the lowest (+) mode is supported whereas the (−) mode is unstable for small kh; conversely with $\overline{p} = 4$ in Figure 4.8(f) it is the (−) mode that is stable while the (+) mode is unstable for small kh.

In each of the two Figures 4.7 and 4.8, the higher-order modes are qualitatively the same as those found in Figure 4.6; the increased deformation thus having little effect on their nature.

In the alternative perspective of the dependency of ζ on the pressure, we consider the following rearrangement

$$\left[\frac{(1+\overline{p})\eta^2 + p^2 + (\varepsilon^2+1)\overline{p}}{\eta^3 - \eta^2 + (1+2\overline{p}+\varepsilon^2)\eta + \overline{p}^2}\right]^2 = \frac{[\varepsilon^2 + (\eta+1)^2][\sinh kh \sinh kh\eta]}{2\eta[\cosh kh(1+\eta) - \cos kh\varepsilon]} = \Gamma^2 \tag{4.49}$$

say. This is now square rooted and rearranged as a quadratic in $\overline{p}$ which has as its discriminant

$$\Delta = [\varepsilon^2 + (\eta-1)^2][\varepsilon^2 + (\eta+1)^2 - 4\eta\Gamma^2] \tag{4.50}$$

for either of the choices $\pm\Gamma$. The first factor is clearly non-negative, the second factor is seen to follow suit by referring to (4.49) and noting that

$$\frac{4\eta\Gamma^2}{\varepsilon^2+(\eta+1)^2}=1+\frac{\cos kh\varepsilon-\cosh kh(1-\eta)}{\cosh kh(1+\eta)-\cos kh\varepsilon}\leq 1. \tag{4.51}$$

Plotting ζ against $\overline{p}$ may now be effected by an inverse approach of selecting ζ and calculating the corresponding pressure values according to

$$\overline{p}=\frac{\varepsilon^2+\eta^2+1-2\hat{\Gamma}\eta\pm\sqrt{\Delta}}{2(1+\hat{\Gamma})}, \tag{4.52}$$

giving rise to up to 4 values of $\overline{p}$ corresponding to any prescribed ζ. Due to the presence of the $\pm$ sign connected with the discriminant, we have introduced the parameter $\hat{\Gamma}$ to mean $\pm\Gamma$ to remove ambiguity.

In Figures 4.9–4.10 we show the effect of $\overline{p}$ on the wavespeed ζ for a variety of ε values. Within each separate figure we have plotted a family of curves corresponding to distinct kh values. Reference should be made with Figure 3.1 in interpreting each. While the resolution of the curves is unsatisfactory for $\zeta < 1$, the traits discussed below are still in evidence. Numerical accuracy can, however, be achieved with the inverse method given above.

Figure 4.9 represents the case of the undeformed configuration. For each kh we find that there is a pair of arcs which may be thought of as comprising an inner and an outer arc. For negative pressure values, in all of Figures 4.9(a)–(f), both arcs are convergent to a single curve taking the limit $\overline{p}\to -1$ as $\zeta\to 0$, there is not necessarily such a limiting curve for $\overline{p}>0$. With small values of kh the two arcs are quite separated and it is only as kh is increased that the curves are again convergent, this time as $\overline{p}\to 3$, $\zeta\to 0$. The effect of increasing kh is to broaden out the inner arc and contract the outer arc so that eventually, in the limit $kh\to\infty$, they are coincident — with this limiting curve being that shown in Figure 3.1 for $\varepsilon=0$. The presence of the two arcs is consistent with the the number of pressure values arising through (4.52).

As was observed in the half-space problem, the case $\overline{p}=0$ produces the solution $\zeta=1$. This represents a plane shear wave propagating in the x_1 direction and is non-dispersive. It is associated with the repeated roots $s_2 = \bar{s}_2 = \varepsilon$ for which a similar case arose in the problems considered by Ogden and Roxburgh (1993).

In Figure 4.10 we depict how the dependence on $\bar{p}$ has evolved as the deformation increases to $\varepsilon = 1$. We observe the same qualitative nature in the figure in respect of a pair of arcs associated with each kh that converge, through increasing kh, towards the single arc corresponding to the half-space problem. The only significant difference is in the greater initial separation of the arcs in the range $\bar{p} < 0$.

Both Figures 4.9 and 4.10 show that, for $\zeta > 1$, where η is complex valued, there is a change in the nature of the curves. Each individual curve is associated with a particular mode number $n \in \mathbb{N}$ with the maximum value of ζ for some n being the minimum value of ζ for the $n+1$ mode. The mode number is due to the conversion of $\sinh kh\sqrt{1-\zeta}$ into its trigonometric counterpart $i \sin kh\sqrt{\zeta - 1}$. For example, with

$$\sqrt{\zeta - 1} := \eta^* = -i\eta = \frac{n\pi}{kh}, \tag{4.53}$$

the parameter Γ in (4.49) is equal to zero and this can be interpreted as finding wavespeeds at which solution branches of (4.52) connect. When this occurs here, the separation of the extrema is given by $2\sqrt{\Delta}$. Furthermore, equation (4.19) shows that the dispersion equation is

$$n\left[\bar{p}^2 + (1 + \varepsilon^2 - \eta^{*2})\bar{p} - \eta^{*2}\right]^2 \left[\cos kh\varepsilon - (-1)^n \cosh kh\right] = 0. \tag{4.54}$$

The property that $\left|\dfrac{\cos kh\varepsilon}{\cosh kh}\right| < 1$ for all $kh > 0$ ensures that it is through the quadratic in $\bar{p}$ that there are exactly two $\bar{p}$ satisfying (4.54) with this choice of η In particular, when $\varepsilon = 0$ then $\bar{p} = -1$ or $\dfrac{n^2\pi^2}{k^2h^2}$ and this is reflected in Figure 4.9.

We call the first mode that part of the solution curve which exists when ζ first exceeds unity. The n^{th} then lies in the range

$$1 + \left(\frac{(n-1)\pi}{kh}\right)^2 \leq \zeta \leq 1 + \left(\frac{n\pi}{kh}\right)^2$$

and has a single horizontal asymptote for ζ as $\overline{p} \to \pm\infty$.

Problem 2

We initially look at the dependence of ζ on $\overline{p}$. The lack of the algebraic rearrangement available to Problem 1 shields the fact that we no longer have the four potential solutions for $\overline{p}$ with respect to given ζ. In Figures 4.11–4.12 we show the dependence for a selection of values of the strain parameter ε and layer thickness kh. The solutions for $\zeta \leq 1$ are found on one single arc which is comparable with the outer arc of the corresponding plots of Problem 1. When $\zeta > 1$ the similarity with those plots again exists. Unlike Figures 4.9–4.10, however, the clear demarcation of the regions for the solution modes is different in that there is a degree of separation between the modes.This observed mode separation is caused by the presence of the extra constraints imposed through the boundary conditions.

With increasing kh, the arc for $kh \leq 1$ tends towards the limiting curve of the half-space problem while the modes for $\zeta > 1$ appear with greater frequency. The latter effect is in keeping with the trigonometric influence noted in Problem 1.

In view of the slight distinctions found in Figures 4.6–4.8 we herein show only the results for the case $\varepsilon = 0$ when considering ζ as a function of kh. A similar structure appears with the expected feature, in light of the discussion on the effect of $\overline{p}$ on ζ, of there being only one mode present. Reference may be made to Figure 4.4 in considering the stability arguments.

For small values of kh in Figure 4.13(a) ($\overline{p} = -2$) the lowest mode is stable. When $\overline{p} = -1$, as in Figure 4.13(b), there is a transitional effect through there being a horizontal asymptote to the boundary of the stability region. Subsequent figures 4.13(c)–(e) all lie within the stable region so that the lowest modes are supported and it is with $\overline{p} = 3$ in Figure 4.21(f) that we again reach a transitional phase. Further increases in $\overline{p}$ then result in stability being achieved in the lowest mode only for small kh.

Problem 3

The wavespeed results for various amounts of shear in Problem 3 are depicted in Figure 4.14, there being no pressure variation accommodated in the formulation of the problem. The results indicate a stable system with only the higher order modes in existence through the increased constraints. The effect of the change in deformation is seen to be only of a marginal quantitative nature with these modes behaving in a similar fashion to those of Problems 1 and 2.

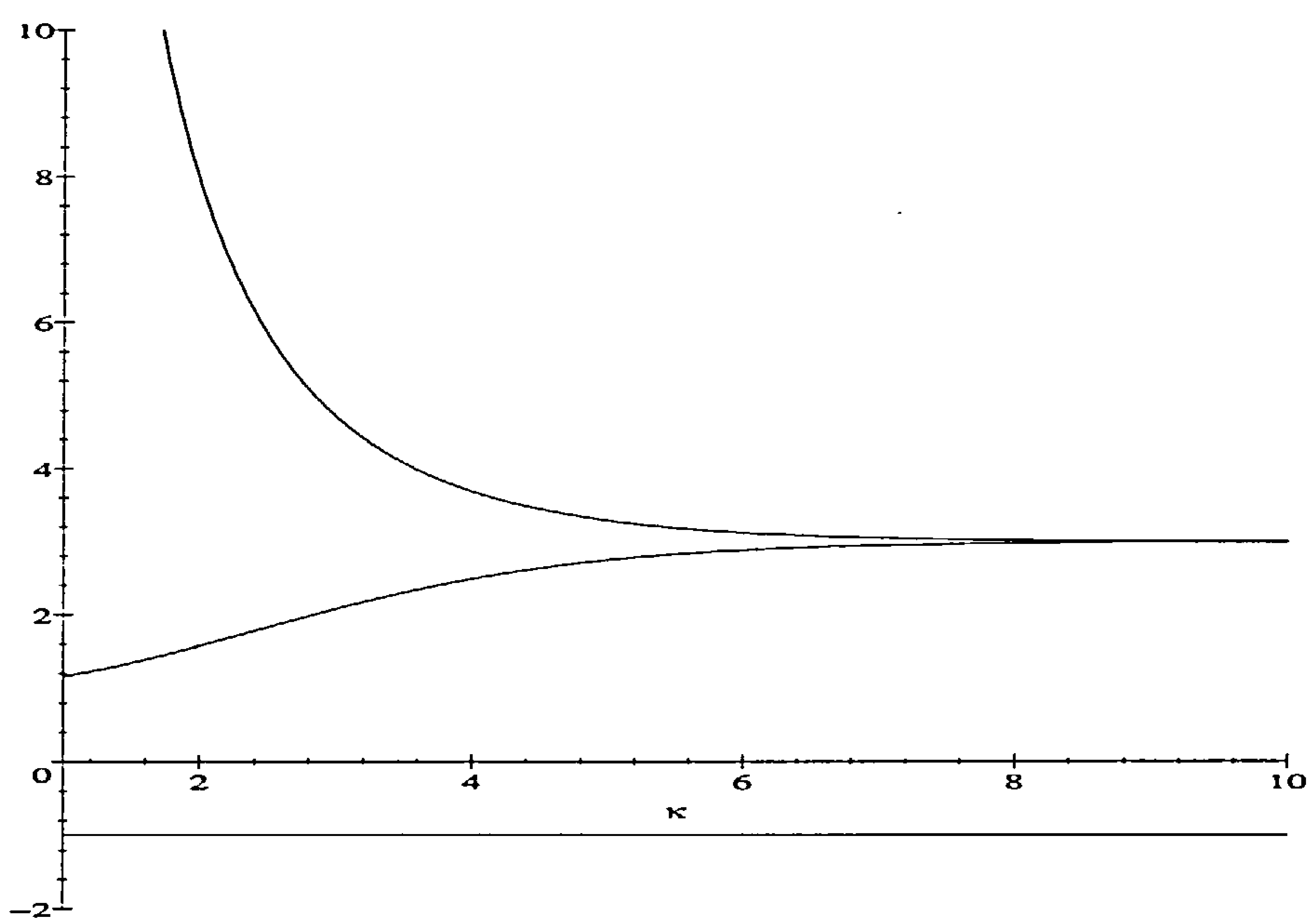
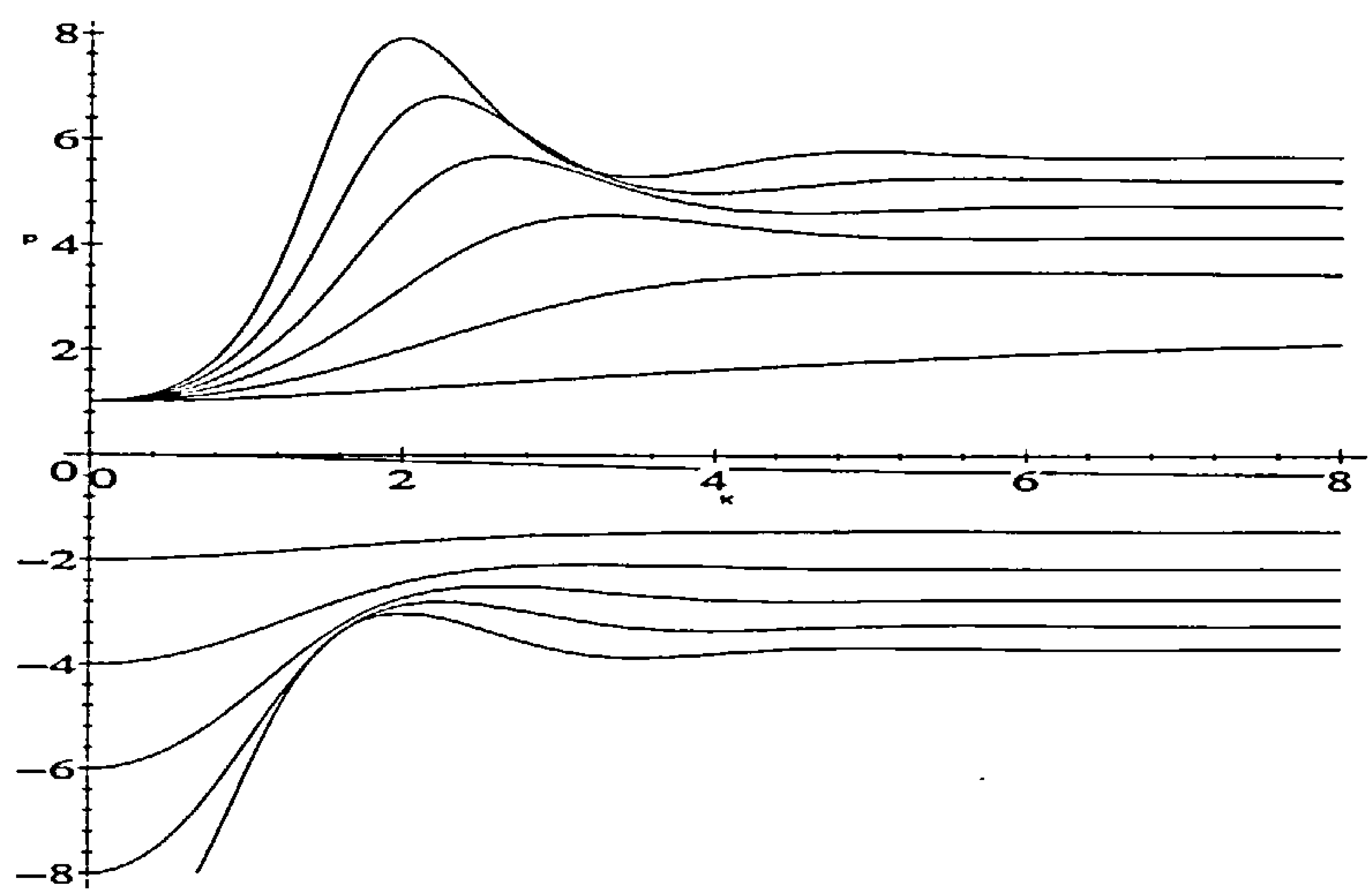
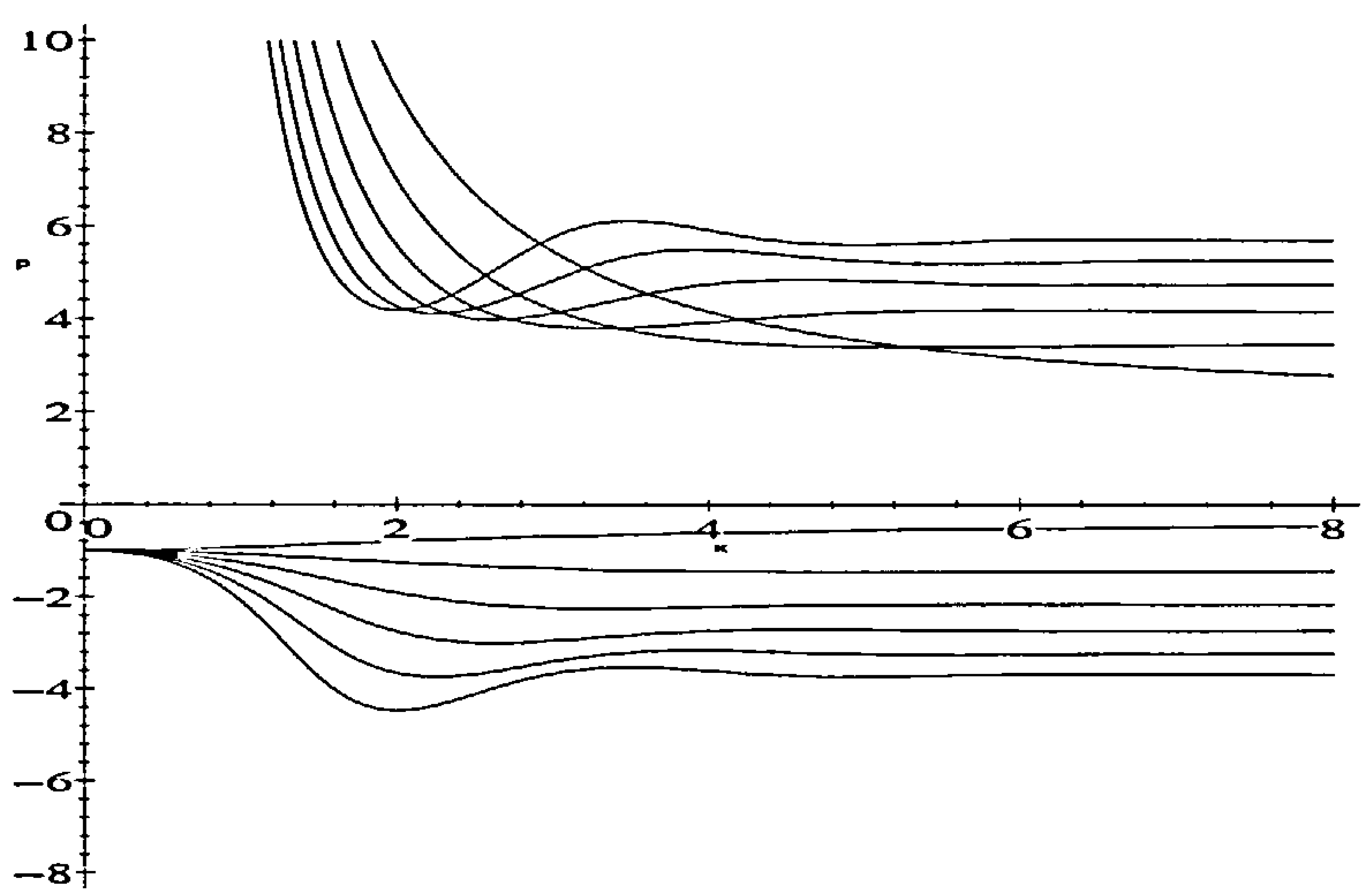


Figure 4.1: Plot of $\overline{p}$ (vertical scale) against κ (horizontal scale) for $\overline{\beta} = 1$ in Problem 1. The $(-)$ mode corresponds to the upper curve and the $(+)$ mode corresponds to the lower curve in the region $\overline{p} > 0$.



(a)



(b)

Figure 4.2: Plot of $\overline{p}$ (vertical scale) against κ (horizontal scale) for Problem 1 with $\overline{\beta}$ = 0, 2, 4, 6, 8, 10: (a) antisymmetric (+) modes, (b) symmetric (-) modes.

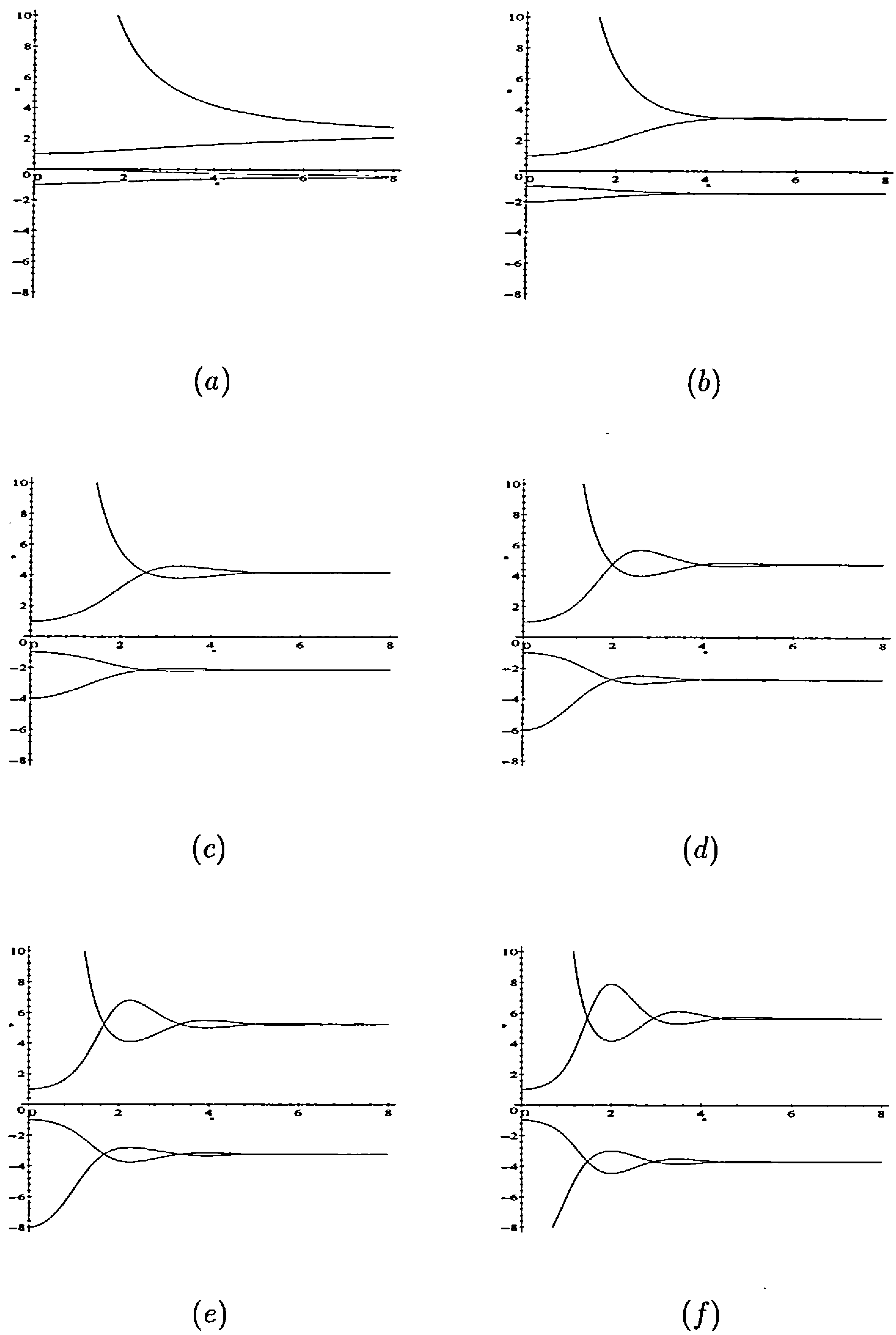


Figure 4.3: Plot of $\overline{p}$ (vertical scale) against κ (horizontal scale) showing anti-symmetric and symmetric modes for Problem 1: (a) $\overline{\beta} = 0$, (b) $\overline{\beta} = 2$, (c) $\overline{\beta} = 4$, (d) $\overline{\beta} = 6$, (e) $\overline{\beta} = 8$, (f) $\overline{\beta} = 10$.

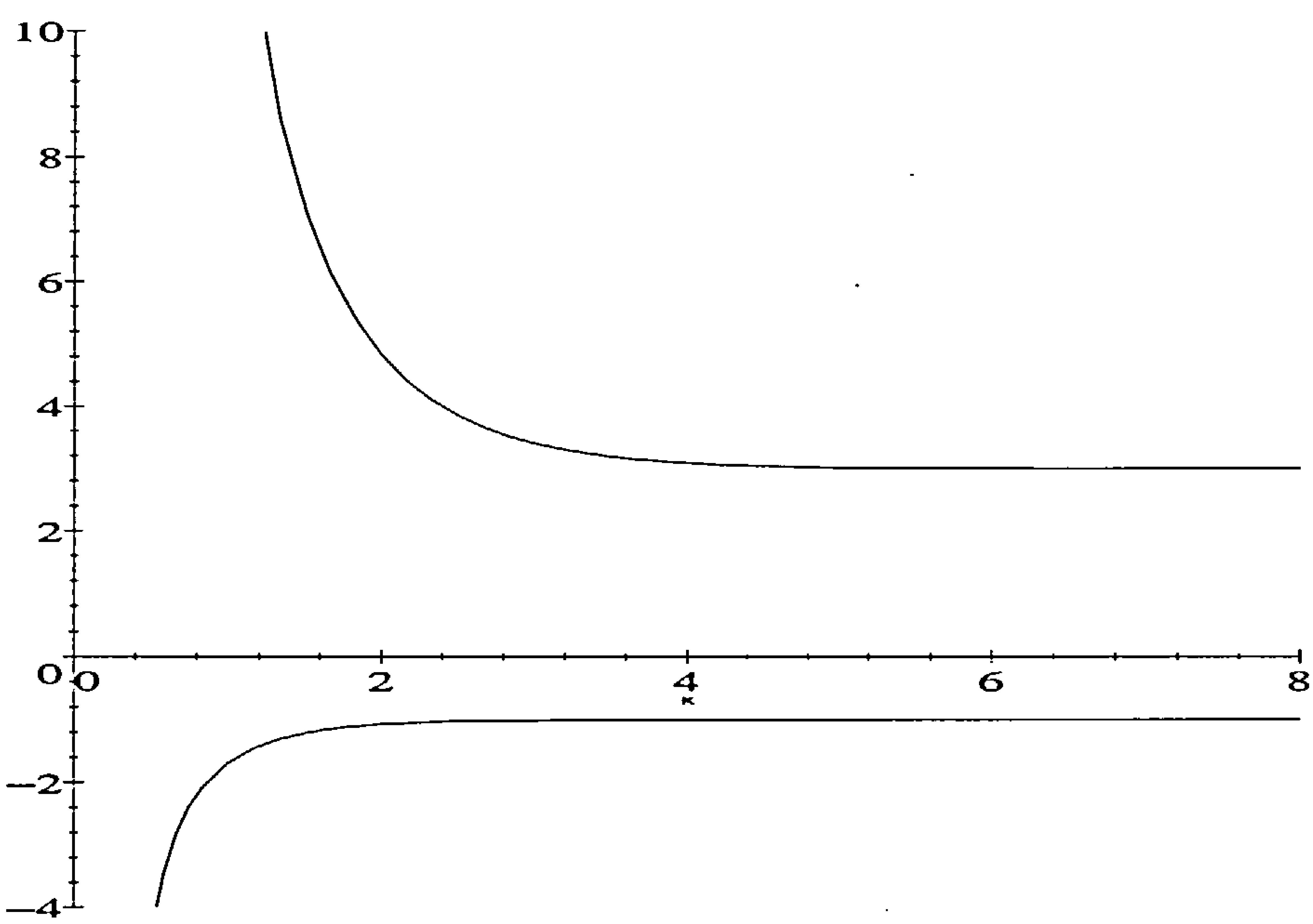


Figure 4.4: Plot of $\overline{p}$ (vertical scale) against κ (horizontal scale) for $\overline{\beta} = 1$ in Problem 2.

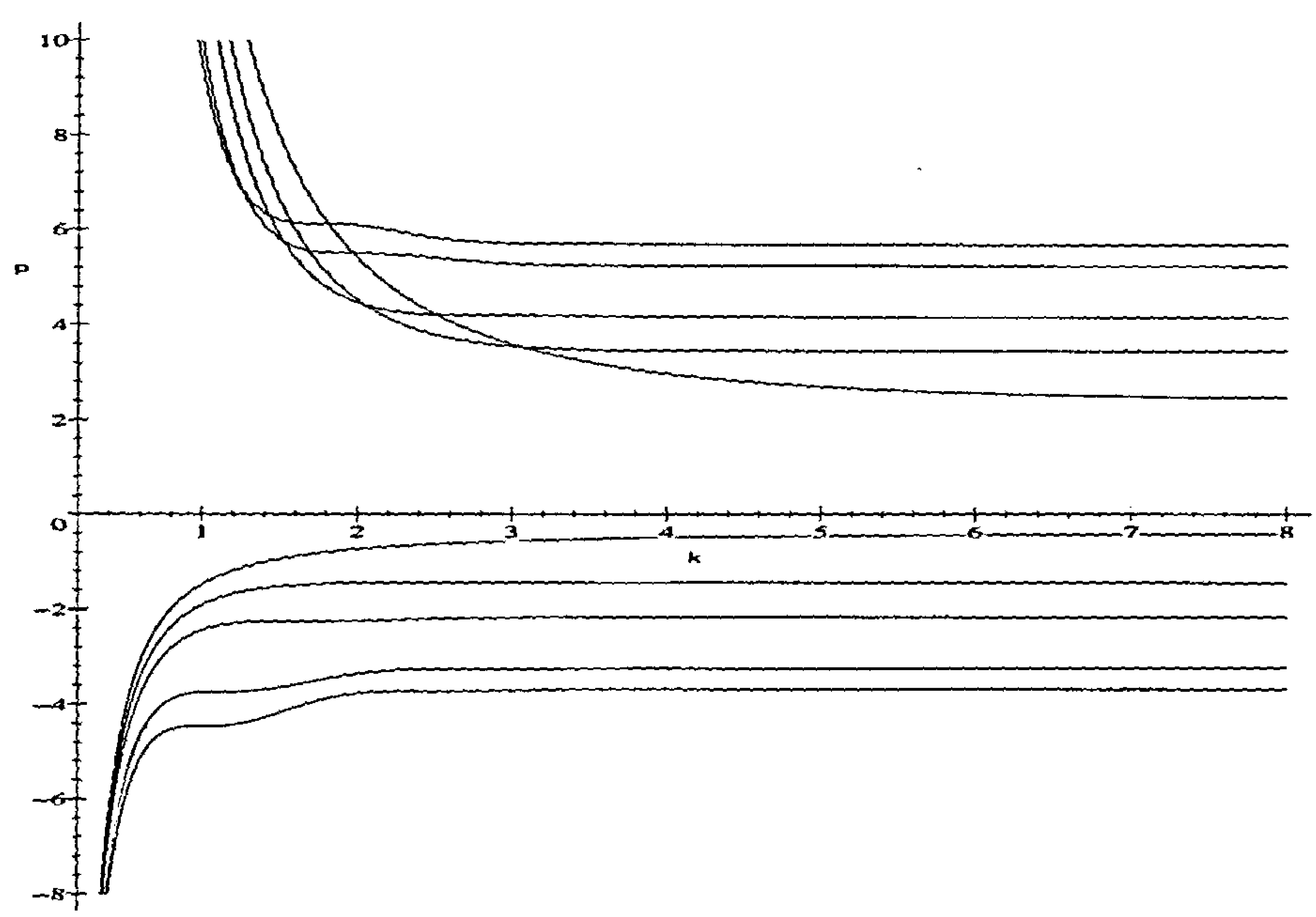


Figure 4.5: Plot of $\overline{p}$ (vertical scale) against κ (horizontal scale) for Problem 2 with $\overline{\beta} = 0,\ 2,\ 4,\ 6,\ 8,\ 10$.

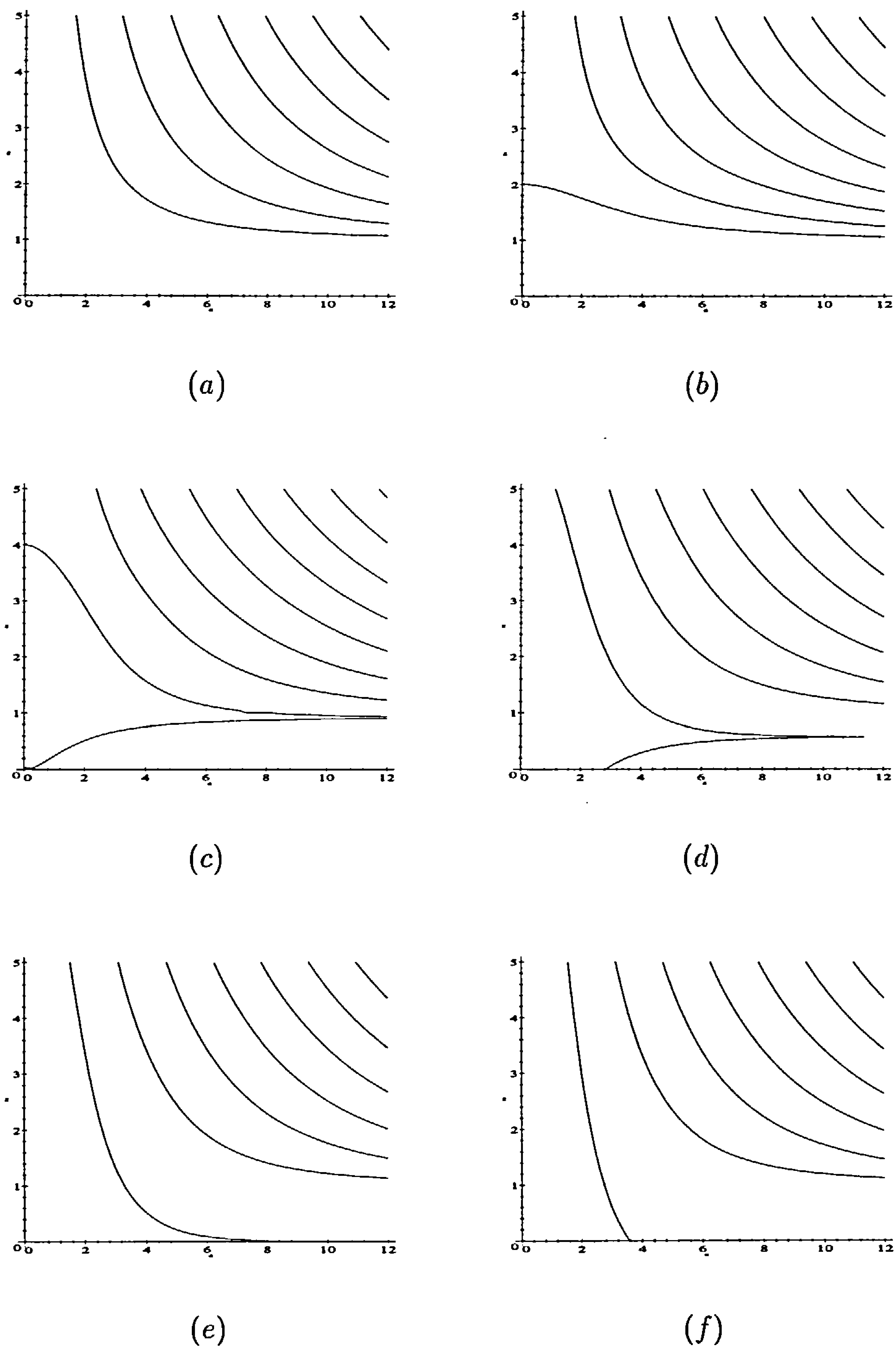


Figure 4.6: Plot of ζ (vertical scale) against kh (horizontal scale) for Problem 1 with $\varepsilon = 0$: (a) $\overline{p} = -2$, (b) $\overline{p} = 0$, (c) $\overline{p} = 1$, (d) $\overline{p} = 2$, (e) $\overline{p} = 3$, (f) $\overline{p} = 4$.

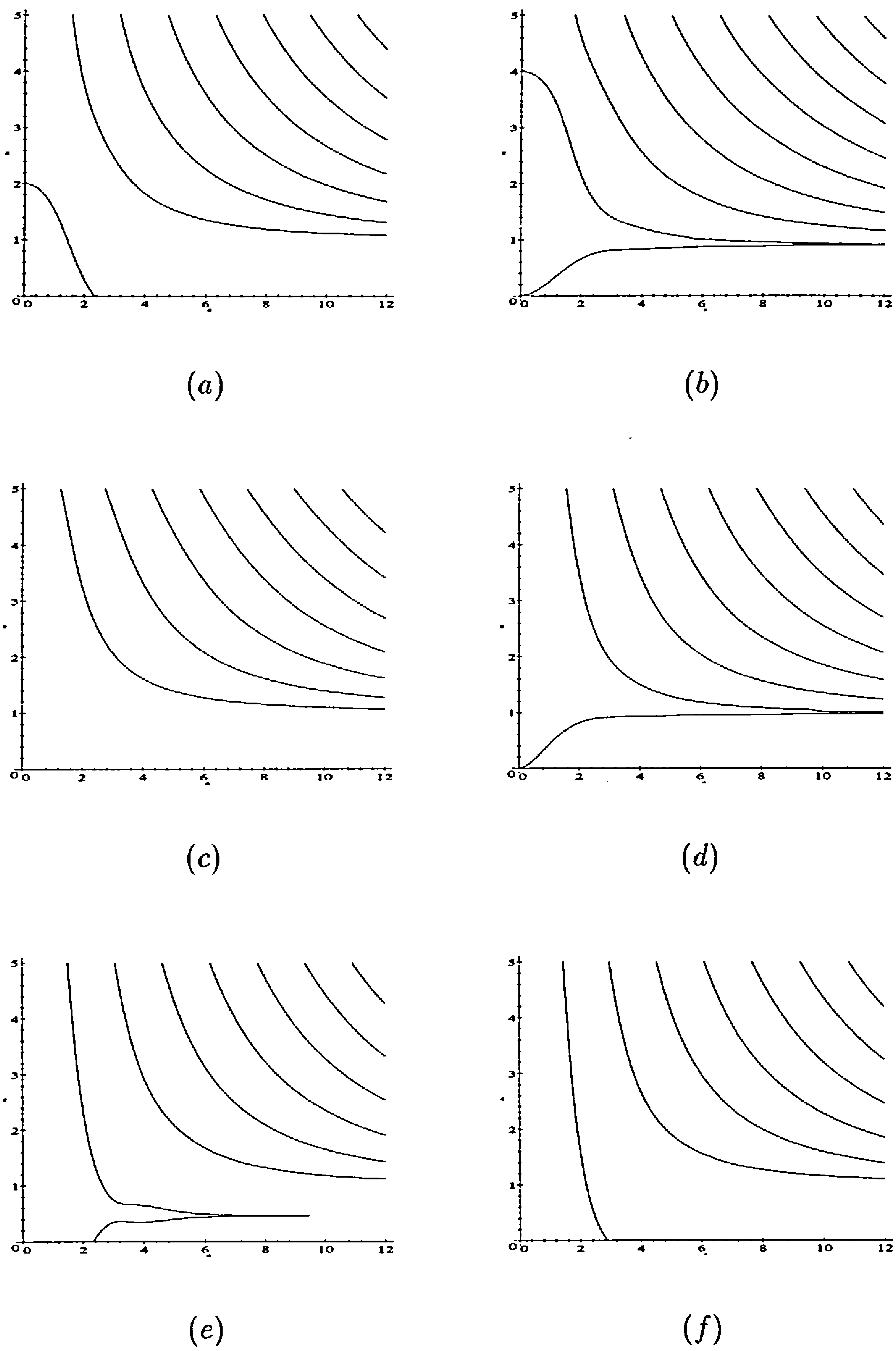


Figure 4.7: Plot of ζ (vertical scale) against kh (horizontal scale) for Problem 1 with $\varepsilon = 2$: (a) $\overline{p} = -2$, (b) $\overline{p} = -1$, (c) $\overline{p} = 0$, (d) $\overline{p} = 1$, (e) $\overline{p} = 3$, (f) $\overline{p} = 4$.

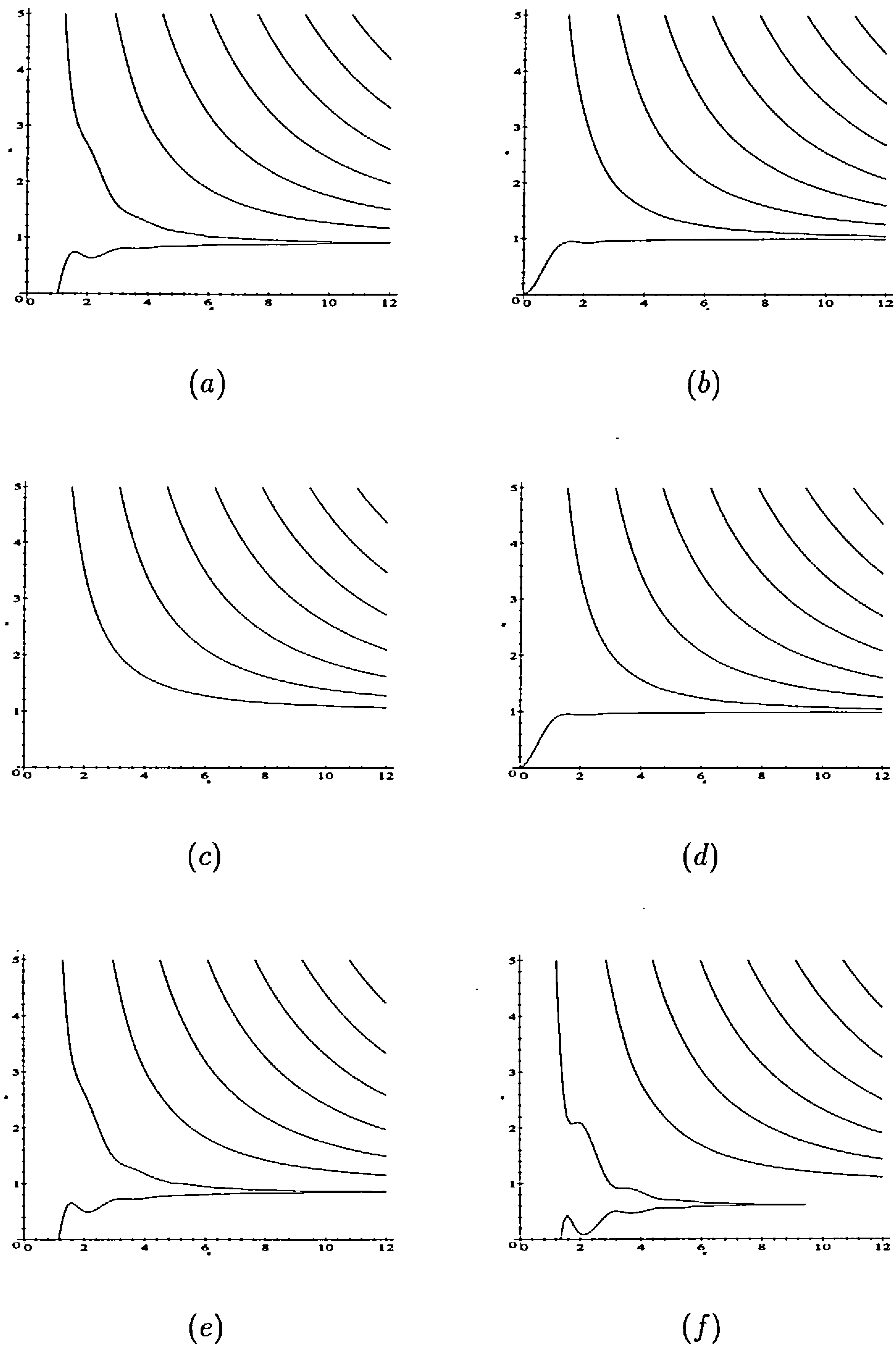


Figure 4.8: Plot of ζ (vertical scale) against kh (horizontal scale) for Problem 1 with $\varepsilon = 4$: (a) $\overline{p} = -2$, (b) $\overline{p} = -1$, (c) $\overline{p} = 0$, (d) $\overline{p} = 1$, (e) $\overline{p} = 3$, (f) $\overline{p} = 4$.

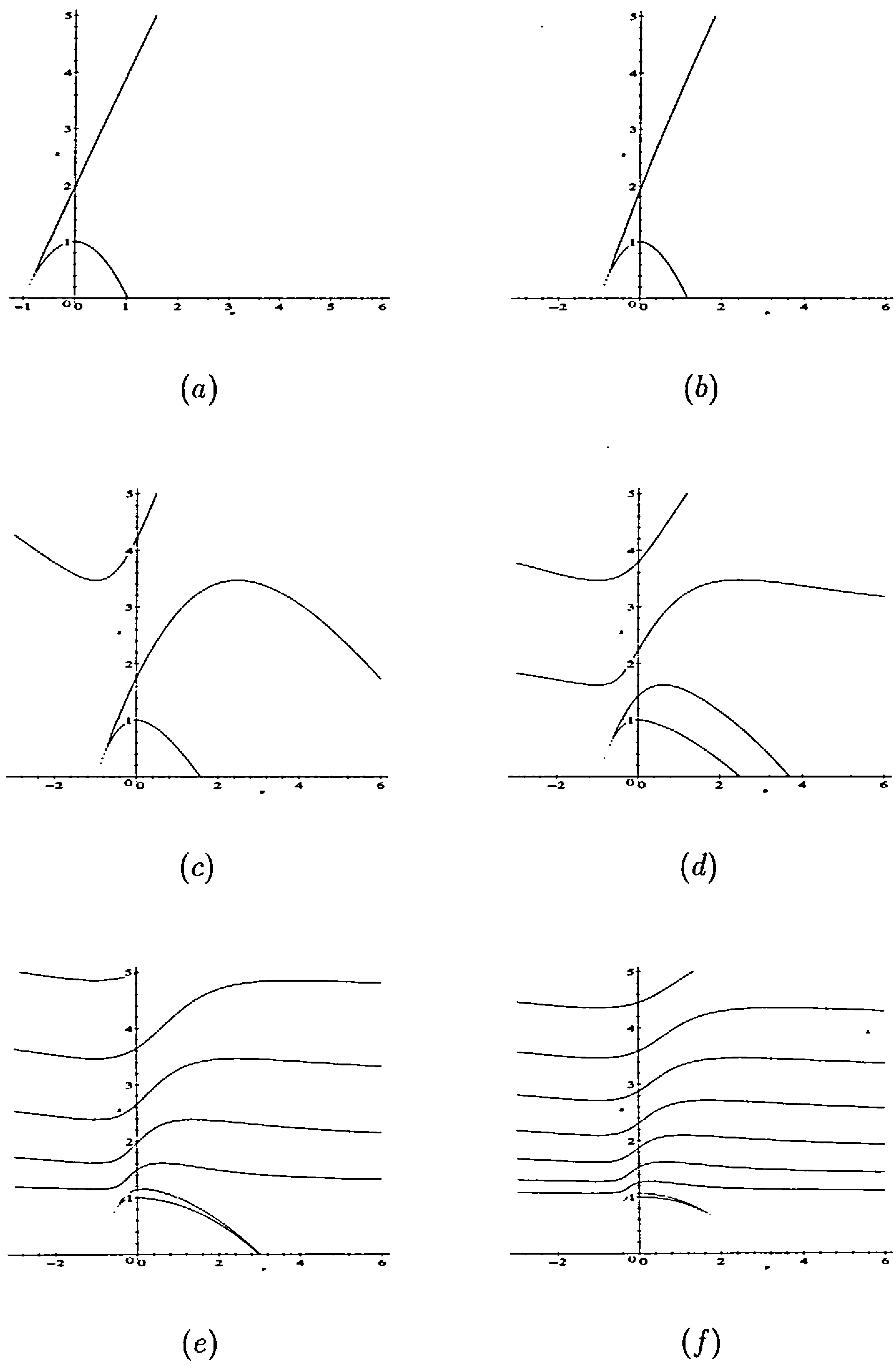


Figure 4.9: Plot of ζ (vertical scale) against $\overline{p}$ (horizontal scale) for $\varepsilon = 0$ in Problem 1: (a) $kh = 0.5$, (b) $kh = 1$, (c) $kh = 2$, (d) $kh = 4$, (e) $kh = 8$, (f) $kh = 12$.

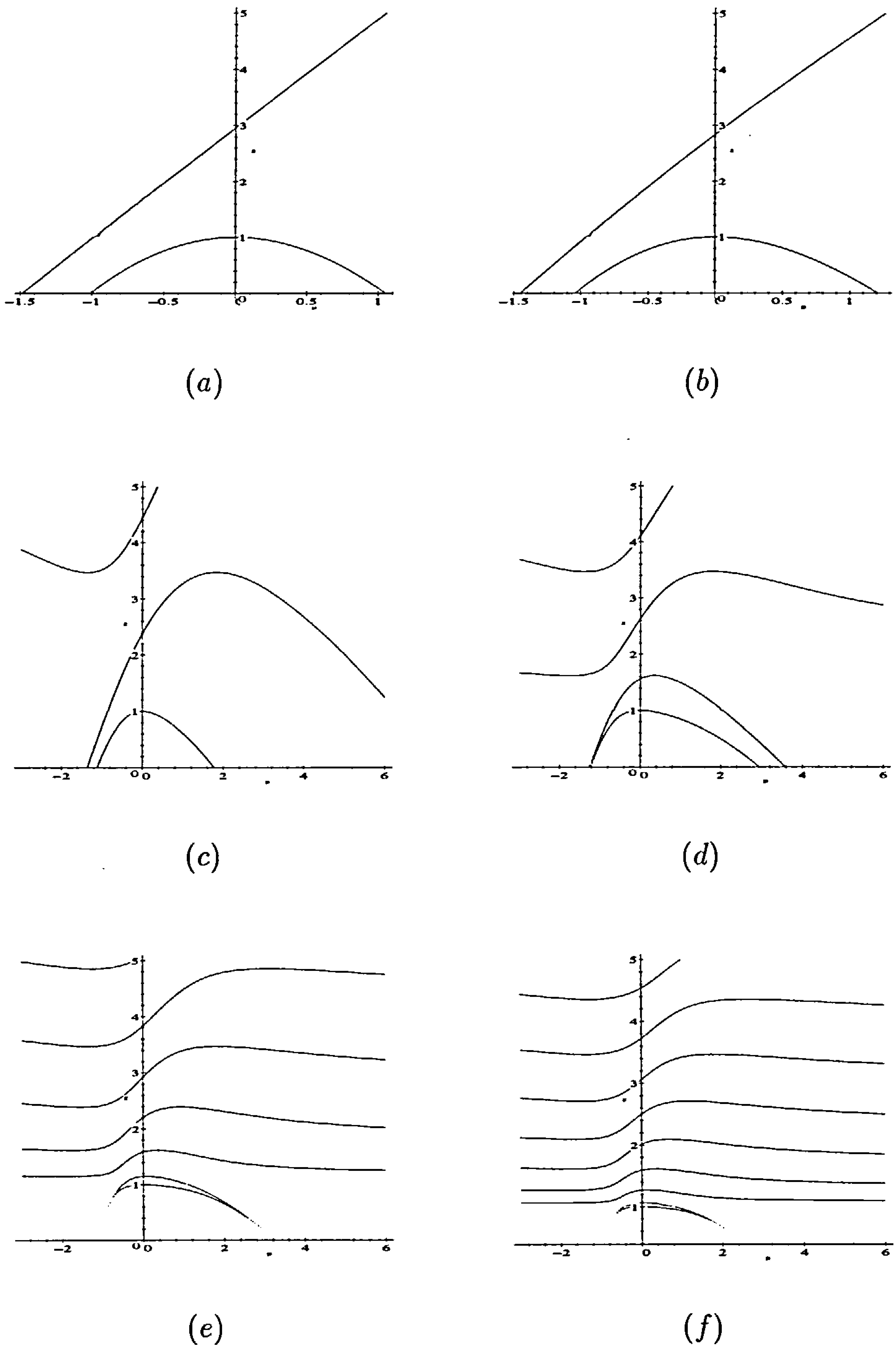


Figure 4.10: Plot of ζ (vertical scale) against $\overline{p}$ (horizontal scale) for $\varepsilon = 1$ in Problem 1: (a) $kh = 0.5$, (b) $kh = 1$, (c) $kh = 2$, (d) $kh = 4$, (e) $kh = 8$, (f) $kh = 12$.

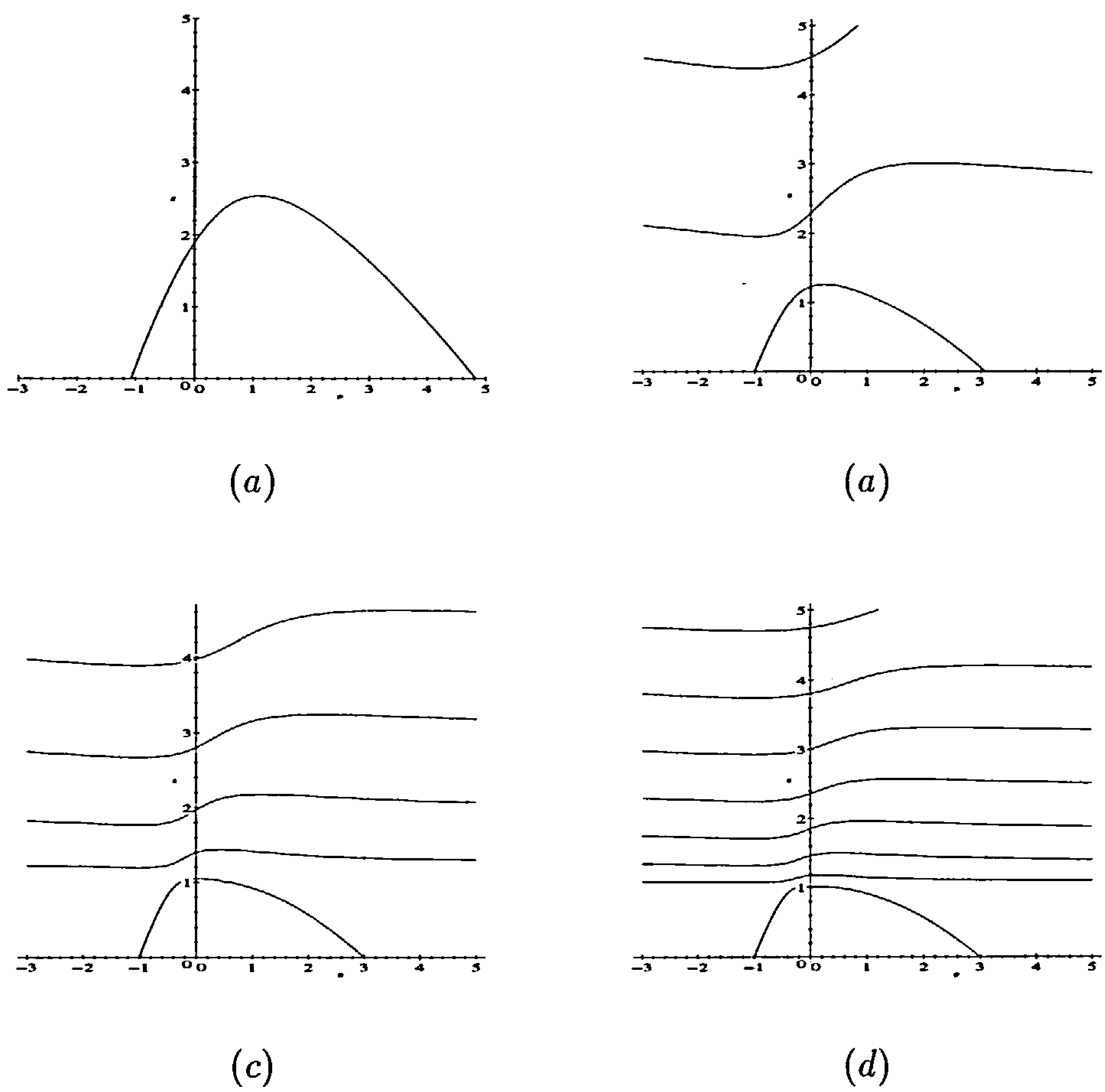


Figure 4.11: Plot of ζ (vertical scale) against $\overline{p}$ (horizontal scale) for $\varepsilon = 0$ in Problem 2: (a) $kh = 2$, (b) $kh = 4$, (c) $kh = 8$, (d) $kh = 12$.

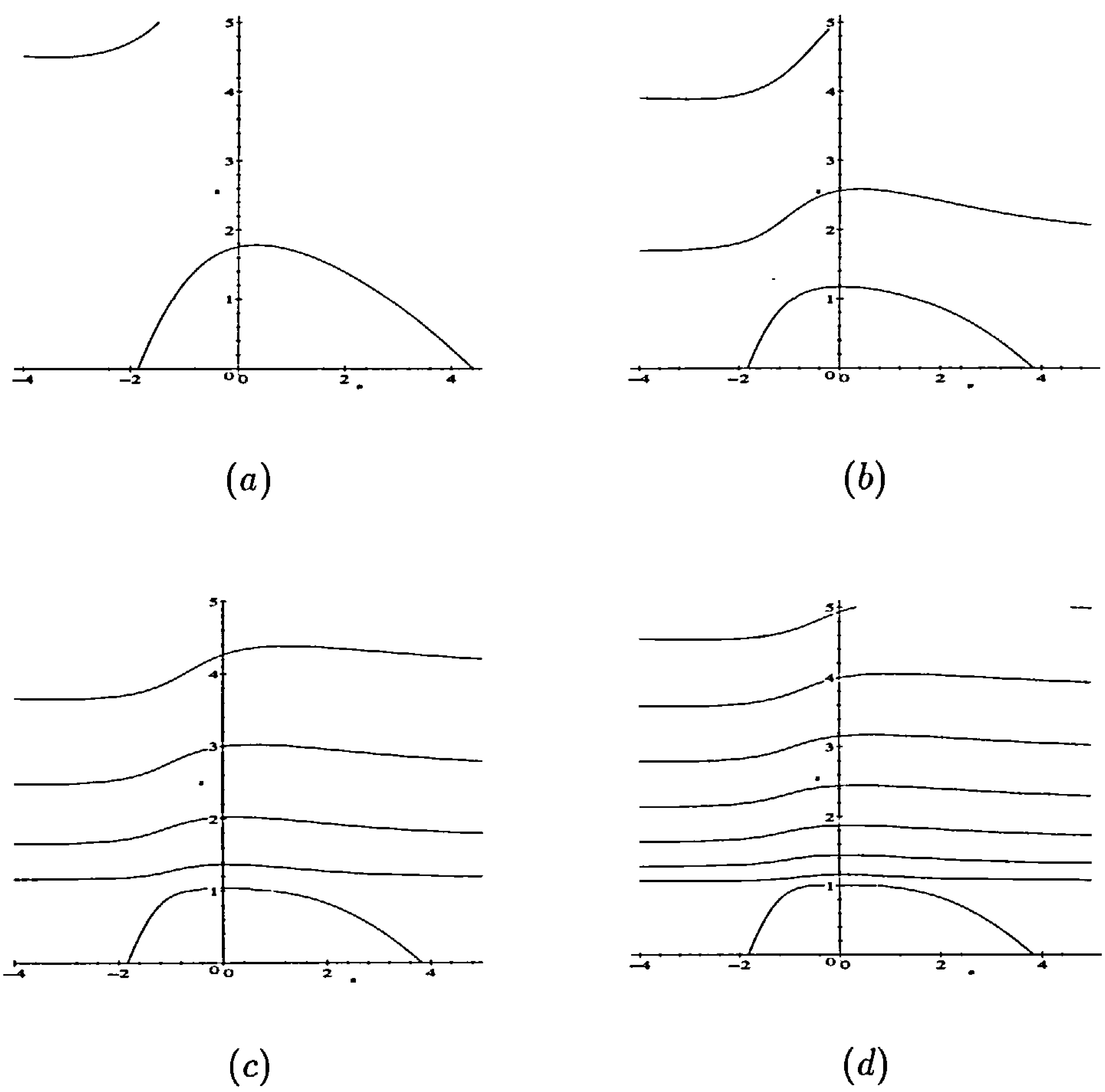


Figure 4.12: Plot of ζ (vertical scale) against $\overline{p}$ (horizontal scale) for $\varepsilon = 2$ in Problem 2: (a) $kh = 2$, (b) $kh = 4$, (c) $kh = 8$, (d) $kh = 12$.

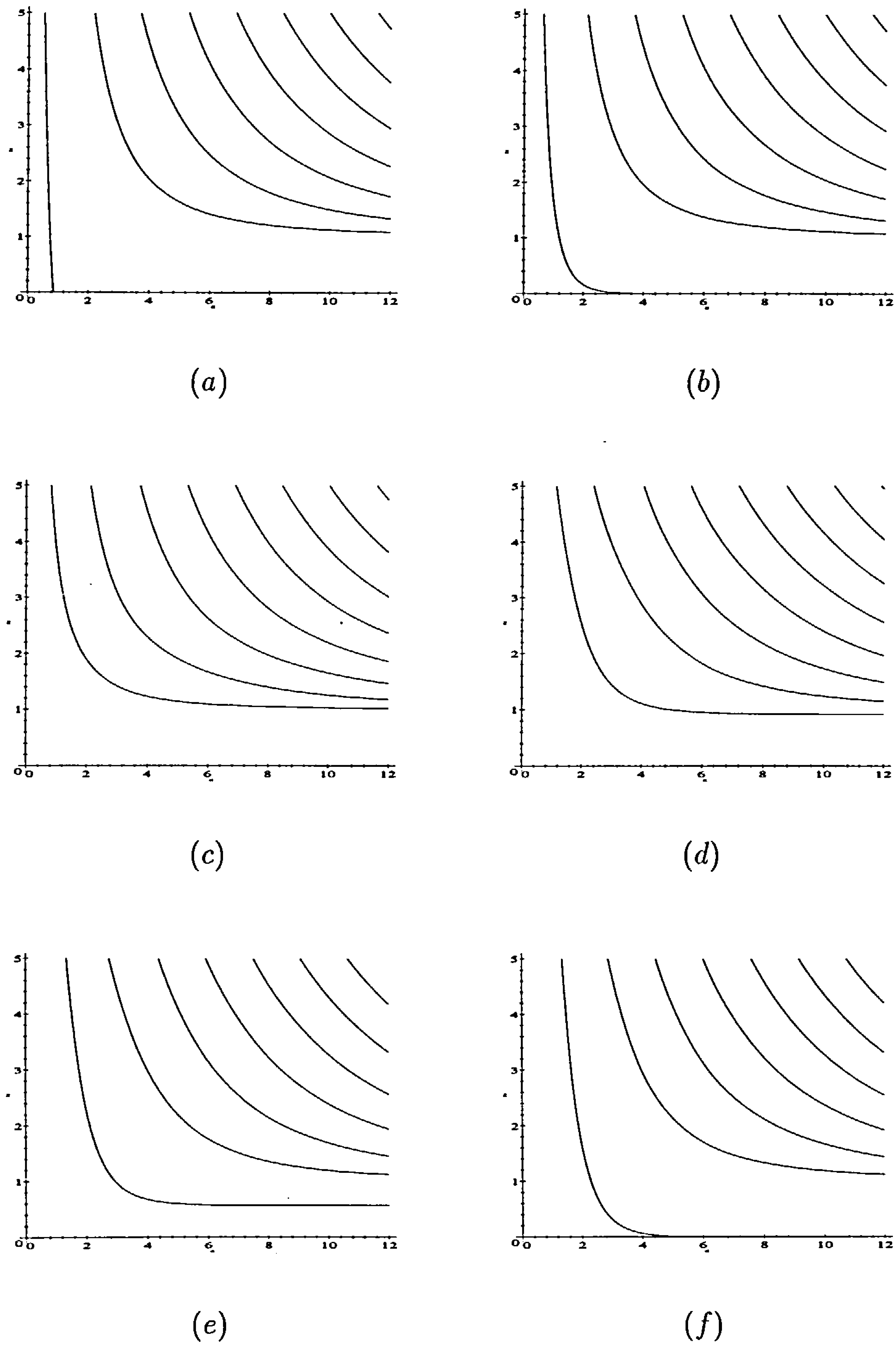


Figure 4.13: Plot of ζ (vertical scale) against kh (horizontal scale) for Problem 2 with $\varepsilon = 0$: (a) $\overline{p} = -2$, (b) $\overline{p} = -1$, (c) $\overline{p} = 0$, (d) $\overline{p} = 1$, (e) $\overline{p} = 2$, (f) $\overline{p} = 3$.

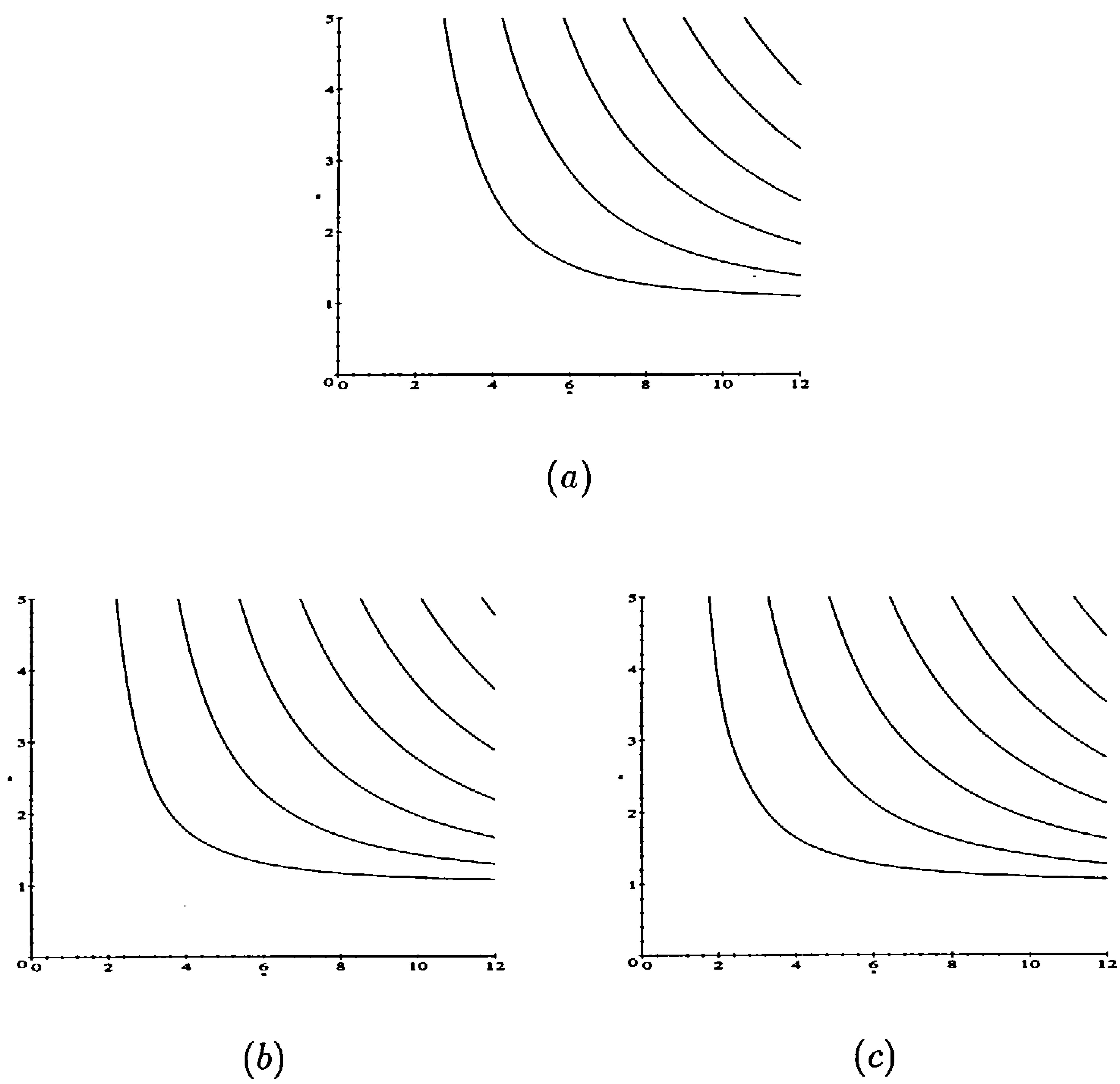


Figure 4.14: Plot of ζ (vertical scale) against kh (horizontal scale) for Problem 3: (a) $\varepsilon = 0$, (b) $\varepsilon = 2$, (c) $\varepsilon = 4$.

Chapter 5

Effective Shear Modulus of a Rectangular Block

5.1 Theory of Maximum and Minimum Energy

Recalling the concepts of Section 2.5.1 we have from (2.89) through (2.100) that a solution to the boundary value problem (2.82)–(2.86) makes stationary the functional

$$E\{\chi\} = \int_{B_0} W(\mathrm{Grad}\,\chi)\,dV - \int_{\partial B_0^{\tau}} \tau \cdot \chi\,dA, \tag{5.1}$$

where we have excluded the effects of body forces (as we shall be doing throughout this chapter). Within the context of linear elasticity we understand this solution, χ, to be a minimizer of the functional so that for $\chi^* \in \mathcal{K}$, the set of kinematically admissible deformation fields

$$\mathcal{K} = \{\chi : \chi \in C^2(B_0),\ \chi = \xi \text{ on } \partial B_0^{x}\},$$

we have

$$E\{\chi^*\} \geq E\{\chi\}. \tag{5.2}$$

It is possible to make the same inference in the context of non-linear elasticity but there are limitations which must be taken into consideration. One such limitation is that χ need not be a unique solution although identifying χ with a global minimizer is sufficient to validate (5.2); existence of such a minimizer being conditional upon particular properties — *polyconvexity* coupled with growth conditions is a sufficient condition — of the strain-energy function, details for which are to be found in the analysis of Ball (1977). Indeed, the difference between the two functional values shows the limitations directly since

$$E\{\chi^*\} - E\{\chi\} = \int_{B_0} \{W(\mathbf{A}^*) - W(\mathbf{A}) - \mathrm{tr}\,[\mathbf{S}(\mathbf{A}^* - \mathbf{A})]\}\, dV, \tag{5.3}$$

where we have used (2.89) along with $\mathbf{A}^* := \mathrm{Grad}\,\chi^*$, and non-convexity of W means that the integrand need not be strictly positive. A counterexample to the supposition of convexity is to be found in taking $\mathbf{A}^*$ to be $\mathbf{A}$ followed by a rigid rotation $\mathbf{Q}$, objectivity of W then ensuring that $W(\mathbf{A}^*) - W(\mathbf{A})$ vanishes in the integrand. The remaining term in the integrand can then be put as $-J\mathrm{tr}\,[\sigma(\mathbf{Q} - \mathbf{I})]$ by virtue of (2.16). Decomposition of this on the principal axes $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\}$ shows the integrand to be

$$J[\sigma_1(1 - Q_{11}) + \sigma_2(1 - Q_{22}) + \sigma_3(1 - Q_33)].$$

If we take $\mathbf{Q}$ to be a rotation about $\mathbf{v}^{(3)}$ the integrand becomes $\sigma_1 + \sigma_2$, for which we cannot determine the sign for arbitrary $\mathbf{A}$. Whilst such difficulties are to be found in general, they do not affect the problem such as it is posed in this chapter.

We now consider the alternative viewpoint of the complementary functional $E_c\{\mathbf{S}\}$ defined in (2.105). With $\mathbf{S}$ corresponding, through (2.82)-(2.84) and (2.86), to the deformation χ that minimizes $E\{\chi\}$ and $\mathbf{S}^*$ taken to belong to the set $\mathcal{S}$ of statically admissible stress fields

$$\mathcal{S} = \{\mathbf{S} : \mathbf{S} \in C^1(B_0),\ \mathrm{Div}\,\mathbf{S} = \mathbf{0} \text{ in } B_0,\ \mathbf{S}^T\mathbf{N} = \sigma \text{ on } \partial B_0^\tau\},$$

the difference in the complementary energy functionals is

$$E_c\{\mathbf{S}\} - E_c\{\mathbf{S}^*\} = \int_{B_0} \{W_c(\mathbf{S}^*) - W_c(\mathbf{S}) - \mathrm{tr}\,[\mathbf{A}(\mathbf{S}^* - \mathbf{S})]\}\, dV. \tag{5.4}$$

Recalling the conjugate variables $(\mathbf{T},\mathbf{U})$, (5.3) and (5.4) are also seen to be given as

$$\left.\begin{aligned} E\{\chi^*\}-E\{\chi\} &= \int_{B_0}\{W(\mathbf{U}^*)-W(\mathbf{U})-\operatorname{tr}[\mathbf{S}(\mathbf{A}^*-\mathbf{A})]\}\,dV, \\ E_c\{\mathbf{S}\}-E_c\{\mathbf{S}^*\} &= \int_{B_0}\{W_c(\mathbf{T}^*)-W_c(\mathbf{T})-\operatorname{tr}[\mathbf{A}(\mathbf{S}^*-\mathbf{S})]\}\,dV. \end{aligned}\right\} \tag{5.5}$$

The similarities in the integrands, along with the fact that strict convexity in $W(\mathbf{U})$ entails strict convexity in $W_c(\mathbf{T})$, lead us to suppose that both integrals are non-negative and, therefore, that

$$E_c\{\mathbf{S}\} \geq E_c\{\mathbf{S}^*\} \tag{5.6}$$

for all $\mathbf{S}^* \in \mathcal{S}$. We take this as the *Principle of Maximum Complementary Energy.*

The inequalities thus stand as

$$E\{\chi^*\} \geq E\{\chi\} = E_c\{\mathbf{S}\} \geq E_c\{\mathbf{S}^*\}, \quad \chi^* \in \mathcal{K},\ \mathbf{S}^* \in \mathcal{S}. \tag{5.7}$$

For an isotropic material, $W(\mathbf{U})$ depends symmetrically on its principal stretches $\lambda_1,\lambda_2,\lambda_3$ as does $W_c(\mathbf{T})$ depend on its principal stresses t_1,t_2,t_3. In terms of these we have, from (2.107) and (2.108), that

$$W_c(t_1,t_2,t_3) = \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3 - W(\lambda_1,\lambda_2,\lambda_3), \tag{5.8}$$

and

$$t_i = \frac{\partial W}{\partial \lambda_i}, \quad \lambda_i = \frac{\partial W_c}{\partial t_i}, \quad i \in \{1,2,3\}. \tag{5.9}$$

In keeping with the closing remark of Section 2.5.1, if W is a strictly convex function[1] then $W_c(t_1,t_2,t_3)$ may be determined uniquely from inversion of $(5.9)_1$.

For the purposes of this chapter we shall be considering the *semi-linear material*, the strain-energy function for which possesses the property that its complementary energy density can be derived explicitly through the aforementioned inversion. The strain-energy function for this material is given as

$$W(\lambda_1,\lambda_2,\lambda_3) = \mu\left\{(\lambda_1-1)^2+(\lambda_2-1)^2+(\lambda_3-1)^2\right\} + \frac{\lambda}{2}\{\lambda_1+\lambda_2+\lambda_3-3\}^2, \tag{5.10}$$

[1] $W(\mathbf{U})$ strictly convex $\Leftrightarrow W(\lambda_1,\lambda_2,\lambda_3)$ strictly convex.

where μ, λ are the Lamé constants. Hence, we may establish that

$$t_i = 2\mu(\lambda_i - 1) + \lambda\{\lambda_1 + \lambda_2 + \lambda_3 - 3\}, \quad i \in \{1,2,3\}, \tag{5.11}$$

and

$$W_c(t_1,t_2,t_3) = \frac{1}{4\mu}(t_1^2+t_2^2+t_3^2) - \frac{\lambda}{4\mu(3\lambda+2\mu)}(t_1+t_2+t_3)^2 + (t_1+t_2+t_3). \tag{5.12}$$

5.2 Applications to Bounds on Effective Shear Modulus

The principles of maximum and minimum energy may be utilized in providing both upper and lower bounds for the stored energy per unit volume and, hence, the effective shear modulus of a rectangular block through the inequalities (5.7) by choosing suitable fields $\chi^* \in \mathcal{K}$, $\mathbf{S}^* \in \mathcal{S}$.

Throughout we shall be considering the shearing of the block in either plane strain or plane stress. The reference configuration is the region

$$B_0 = \{(X_1, X_2) : -A \leq X_1 \leq A,\ 0 \leq X_2 \leq H\},$$

and the upper face is deflected by an amount δ where, for small deflections, the deformation will be described by the 'amount of shear' parameter defined as $\varepsilon = \delta/H$. Notice that whilst we retain ε as the parameter measuring the extent of the deflection, its use is distinct from that of earlier chapters in that no assumptions are yet made on its embodiment within the mapping $\mathbf{x} = \chi(\mathbf{X})$. The material is bonded to rigid plates on the surfaces $X_2 = 0, H$ so the deformation is assumed to be of an in-plane shear type with no accompanying compression.

The boundary conditions for this problem are

$$\left.\begin{array}{llllll} \partial B_0^x & \mathbf{x} & = & \mathbf{X} & \text{on } X_2 & = 0, \\ & \mathbf{x} & = & \mathbf{X} + \delta\mathbf{E}_1 & \text{on } X_2 & = H, \\ \partial B_0^\tau & \sigma & = & 0 & \text{on } X_1 & = \pm A. \end{array}\right\} \tag{5.13}$$

The effective shear modulus, μ_e, is introduced as a factor of proportionality in a stress-strain relationship so that

$$s = \mu_e \varepsilon, \tag{5.14}$$

where s is the resultant shear stress per unit area on the surface $X_2 = H$; i.e.

$$s := \frac{1}{2A} \int_{-A}^{A} S_{21}\, dX_1 \bigg|_{X_2 = H} . \tag{5.15}$$

Treating the functional $E\{\chi\}$ as a function of the strain parameter ε, we define the total energy per unit reference volume, said volume being denoted $v(B_0)$, to be

$$\overline{W}(\varepsilon) = \frac{1}{v(B_0)} E\{\chi(\varepsilon)\} = \frac{1}{2AH} \int_{B_0} W(\mathrm{Grad}\, \chi(\varepsilon))\, dV, \tag{5.16}$$

recalling $\boldsymbol{\sigma} = \mathbf{0}$ on ∂B_0^{τ}. Differentiating with respect to ε, indicated here by the prime, yields

$$\left. \begin{aligned} \overline{W}'(\varepsilon) &= s, \\ \overline{W}''(\varepsilon) &= \frac{1}{2AH} \int_{B_0} \mathcal{A}_{ijkl} \frac{\partial}{\partial X_j}(x_i') \frac{\partial}{\partial X_l}(x_k')\, dV, \end{aligned} \right\} \tag{5.17}$$

where $\mathcal{A}$ is as defined in (2.49) and (2.50). Local stability ensures that the integral on the right-hand side of $(5.17)_2$ is positive, meaning that $\overline{W}'(\varepsilon)$ is a monotonic increasing function of ε. Such monotonicity validates the employment of the Legendre transformation as a means of defining $\overline{W}_c(s)$, the complementary energy per unit reference volume, through

$$\overline{W}(\varepsilon) + \overline{W}_c(s) = \varepsilon s, \tag{5.18}$$

this being a parametric analogue to either (2.108) or (5.8). We can explicitly represent $\overline{W}_c(s)$ as

$$\overline{W}_c(s) = \frac{1}{2AH} \int_{B_0} \{W_c(\mathrm{T}) - \mathrm{tr}\, \mathbf{S}\}\, dV. \tag{5.19}$$

The chain of inequalities (5.7) then becomes

$$\overline{W}(\varepsilon^*) \geq \overline{W}(\varepsilon) = \varepsilon s - \overline{W}_c(s) \geq \varepsilon s^* - \overline{W}_c(s^*). \tag{5.20}$$

Thus, through suitable choice of admissible fields, we establish bounds on the stored energy. Likewise, for the effective shear modulus, we rearrange (5.14) as

$$\mu_e = \frac{\overline{W}'(\varepsilon)}{\varepsilon} = \frac{s}{\overline{W}_c'(s)}, \tag{5.21}$$

where we have made use of $(5.17)_1$ and

$$\overline{W}_c'(s) = \varepsilon \tag{5.22}$$

which arises from (5.18).

For results appropriate to the linear theory we evaluate these quantities in the reference configuration where L'Hôpital's Rule reveals

$$\overline{W}''(0) = \mu_e = \frac{1}{\overline{W}_c''(0)}, \tag{5.23}$$

and applying this with (5.20) in the reference configuration gives us the required bounds

$$\overline{W}''^*(0) \geq \mu_e \geq \frac{1}{\overline{W}_c''^*(0)}, \tag{5.24}$$

where we say $\overline{W}''^*(0) = \overline{W}''(\varepsilon^*)\big|_{\varepsilon^*=0}$ with a similar convention for $\overline{W}_c''$.

5.3 A Particular Example

5.3.1 Upper Bound

For the upper bound we take $\chi^*(\mathrm{X})$ to be the mapping

$$\left.\begin{aligned} x_1 &= X_1 + \varepsilon^* X_2, \\ x_2 &= X_2, \\ x_3 &= X_3, \end{aligned}\right\} \tag{5.25}$$

corresponding to simple shear, which we say takes the principal stretches $\lambda_3 = 1, \lambda_2 = \lambda_1^{-1}$ and $\lambda_1 = \lambda$ as given by (2.133). Viewing W as a function of λ_1, λ_2 and $\lambda_3 \equiv 1$ (and ε^* through them), then

$$\overline{W}''(\varepsilon^*) = \frac{1}{2AH}\int_{B_0} [W_1\lambda_1'' + W_2\lambda_2''+ \qquad\qquad W_{11}(\lambda_1')^2 + 2W_{12}\lambda_1'\lambda_2' + W_{22}(\lambda_2')^2\Big] \, dV. \tag{5.26}$$

Using the strain-energy function for the semi-linear material and $\lambda_1 = \lambda_2^{-1}$ defined as the right-hand side of (2.133), we find that $\overline{W}''(\varepsilon^*)$ is calculated explicitly as being

$$\overline{W}''(\varepsilon^*) = \left[\varepsilon^{*2} + 4\right]^{-\frac{3}{2}} \left\{(\lambda + 2\mu)\left[2\varepsilon^{*2} + (\varepsilon^{*2} + 4)^{\frac{3}{2}}\right] - 2(\lambda + \mu)(\varepsilon^{*2} + 4)\right\}. \tag{5.27}$$

We note the following values in the undeformed configuration

$$\lambda_1 = \lambda_2 = 1; \quad \lambda_1' = -\lambda_2' = \frac{1}{2}; \quad \lambda_1'' = \lambda_2'' = \frac{1}{4} \tag{5.28}$$

along with

$$\left.\begin{aligned} W(1,1,1) &= 0, & \\ \frac{\partial W}{\partial \lambda_i}(1,1,1) &= 0, & i \in \{1,2,3\}, \\ \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}(1,1,1) &= \lambda + 2\mu\delta_{ij}, & i,j \in \{1,2,3\}, \end{aligned}\right\} \tag{5.29}$$

which are requirements on the strain-energy function that ensure consistency between the general theory of isotropic elasticity and the linear approximation of the classical theory.

Taking these together, the upper bound evaluates as

$$\mu_e \leq \overline{W}''^*(0) = \mu. \tag{5.30}$$

5.3.2 Lower Bounds

We choose now a statically admissible stress field in the (X_1, X_2)-plane, noting that it need not correspond to a deformation satisfying the displacement boundary conditions. The stress we take is to be symmetric so that we can immediately say

$$\mathbf{S}^* = \mathbf{T}^*. \tag{5.31}$$

The conditions that $\mathbf{T}^*$ must satisfy are

$$\left.\begin{array}{llcll}
\text{(i)} & \operatorname{Div}\mathbf{T}^* & = & \mathbf{0}, & \mathbf{X}\in B_0,\\
\text{(ii)} & T^*_{11} = T^*_{12} & = & 0, & X_1 = \pm A,\\
\text{(iii)} & T^*_{ij}(X_1,X_2) & = & T^*_{ij}(-X_1, H-X_2), & \mathbf{X}\in B_0;
\end{array}\right\} \tag{5.32}$$

(i) being the equilibrium equations, (ii) the boundary conditions of stress, and (iii) an assumed condition based on the expected symmetry of the deformation.

In constructing $\mathbf{T}^*$, we first account for (i) by introducing an *Airy stress function* $\phi(X_1, X_2)$ which is defined so that

$$T^*_{11} = \phi_{,22}\,, \quad T^*_{22} = \phi_{,11}\,, \quad T^*_{12} = -\phi_{,12}\,, \tag{5.33}$$

and for definiteness we take

$$T^*_{12} = -s^*\alpha(X_1)\beta(X_2), \tag{5.34}$$

where $\alpha(\pm A) = 0$ is a necessary condition for (ii). One choice for α could be

$$\alpha(X_1) = p(X_1^2 - A^2) + qA^{-2}(X_1^4 - A^4) \tag{5.35}$$

where p and q are simply weighting factors.

Identifying s^* as being the resultant shear stress per unit area on $X_2 = H$, we have

$$\begin{aligned}
s^* &= \frac{1}{2A}\int_{-A}^{A} T^*_{12}\,dX_1\bigg|_{X_2=H} = -\frac{1}{2A}s^*\beta(H)\int_{-A}^{A}\alpha(X_1)\,dX_1 && (5.36)\\
\Rightarrow \beta(H) &= \frac{15}{2A^2(5p+6q)}. && (5.37)
\end{aligned}$$

The components $\phi_{,ij}$, $i,j \in \{1,2\}$, compute as

$$\left.\begin{array}{rrcl}
 & \phi_{,12} & = & s^*\alpha(X_1)\beta(X_2),\\
\text{so} & \phi_{,11} & = & s^*\alpha'(X_1)\displaystyle\int\beta(X_2)\,dX_2 + f'(X_1),\\
\text{and} & \phi_{,22} & = & s^*\beta'(X_2)\displaystyle\int\alpha(X_1)\,dX_1 + g'(X_2),
\end{array}\right\} \tag{5.38}$$

where $f(X_1)$ and $g(X_2)$ are arbitrary functions of integration.

Fulfilling the one outstanding boundary condition requires

$$\beta'(X_2) \times h(\pm A) + g'(X_2) = 0$$

for all $0 \leq X_2 \leq H$, from which we conclude that $g'(X_2) \equiv 0$ and $\beta'(X_2) = 0$ provided

$$h(\pm A) := \int \alpha(X_1)\, dX_1 \bigg|_{X_1=\pm A} \neq 0.$$

In fact, our choice for $\alpha(X_1)$ will only violate this latter proviso in the trivial case $p = q = 0$. Thus we take $\beta(X_2)$ to be a constant with $\beta = \beta(H)$ as in (5.37).

Now, with this choice of $\alpha(X_1)$ and with $\beta(X_2) = \beta$, we have

$$T^*_{22} = \phi_{,11} = s^*\beta(2pX_1 + 4qA^{-2}X_1^3)X_2 + f'(X_1), \tag{5.39}$$

and with our final condition imposed on the stress (iii), along with being free to take f' to be an odd function of X_1, we calculate T^*_{22} as being

$$T^*_{22} = s^*\beta X_1(p + 2qA^{-2}X_1^2)(2X_2 - H). \tag{5.40}$$

Our constructed stress field is thus

$$\mathbf{T}^* = s^*\beta \begin{bmatrix} 0 & p(A^2 - X_1^2) + qA^{-2}(A^4 - X_1^4) \\ p(A^2 - X_1^2) + qA^{-2}(A^4 - X_1^4) & X_1(p + 2qA^{-2}X_1^2)(2X_2 - H) \end{bmatrix} \tag{5.41}$$

so that the complementary function $\overline{W}_c(s^*)$ is derived from the invariant form of (5.12)

$$\overline{W}_c(s^*) = \frac{1}{4\mu AH}\int_{B_0} \left\{ \left(\frac{\lambda + \mu}{3\lambda + 2\mu}\right)(T_{11}^2 + T_{22}^2) + T_{12}^2 \right\} dV. \tag{5.42}$$

After some calculation we find that

$$\overline{W}_c(s^*) = \frac{5s^{*2}}{56\mu(5p + 6q)^2}\Big\{8(21p^2 + 48pq + 28q^2) \tag{5.43}$$
$$+ (35p^2 + 84pq + 60q^2)\frac{\lambda + \mu}{3\lambda + 2\mu}\left(\frac{H}{A}\right)^2\Big\}$$

and hence

$$\overline{W}''^{*}_{c}(0) = \frac{5C(p,q)}{28\mu}, \tag{5.44}$$

say.

Defining the aspect ratio $\eta = \dfrac{H}{A}$ and the material parameter $L = \dfrac{\lambda+\mu}{3\lambda+2\mu}$, then the bound obtained, $\dfrac{1}{\overline{W}''^{*}_{c}(0)}$, is of the form $\dfrac{c_1\mu}{c_2L\eta^2+c_3}$ where c_1, c_2, c_3 are constants determined by the weighting factors p, q. With these parameters prescribed, this lower bound has its maximum value when we take the limit $\eta \to 0$. Alternatively, with L and η prescribed, the bound will be maximized when we take p, q in the ratio

$$(p,q) = \left((L\eta^2 - \frac{2}{3})q, q\right), \tag{5.45}$$

the global maximum occuring on the line $\left(-\frac{2}{3}q, q, 0\right)$ in (p, q, η)-space.

5.3.3 Comparison of Results

We now display the relative effects of choosing the weight parameters for the evaluation of the lower bounds, and also provide comparison of these results with the results of Lindley and Teo (1978). Both analytic results and tables, offering various samples of material parameters and aspect ratios, will be given.

We take three basic choices for the parameters (p, q):

$\mathbf{(p, q) = (1, 0)}$

This corresponds to the shear stress being

$$T^{*}_{12} = s^{*}\beta(A^2 - X_1^2)$$

for which the lower bound turns out as

$$\frac{1}{\overline{W}''^{*}_{c}(0)} = \frac{28\mu}{5\times C(1,0)} = \frac{5\times 28\mu}{35L\eta^2+168} \to \frac{5\mu}{6} \text{ as } \eta \to 0. \tag{5.46}$$

$(p,q) = (0,1)$

This corresponds to

$$T^*_{12} = s^*\beta A^{-2}(A^4 - X_1^4)$$

giving the bound

$$\frac{1}{\overline{W}_c''^*(0)} = \frac{28\mu}{5 \times C(0,1)} = \frac{36 \times 28\mu}{5(60L\eta^2 + 224)} \to \frac{9\mu}{10} \text{ as } \eta \to 0. \tag{5.47}$$

$(p,q) = (-2,3)$

This refers to the maximizing ratio taken in the limit $\eta \to 0$ with its corresponding bound

$$\frac{1}{\overline{W}_c''^*(0)} = \frac{14\mu}{15}. \tag{5.48}$$

In conclusion, the above discussion reveals that for this example of a stress field, the effective shear modulus has its maximum lower bound given by (5.48).

In general, we find μ_e to lie in the range

$$\frac{28\mu}{5 \times C(p,q)} \leq \mu_e \leq \mu \tag{5.49}$$

with our best possible case being

$$\frac{14\mu}{15} \leq \mu_e \leq \mu. \tag{5.50}$$

In the Tables 5.1 – 5.4 we have elected to show the lower bound for μ_e/μ found from the stress fields given above. Each table will correspond to a particular material parameterized through the ratio λ/μ of the Lamé constants, and within each there will be a selection of aspect ratios. We record the inversion $\dfrac{\lambda}{\mu} = \dfrac{1-2L}{3L-1}$ for convenience and note that the columns with the heading C_{max} refer to the ratio $(p,q) = ((L\eta^2 - 2/3)q, q)$. In each table we see that the case $\eta = 0$ always produces the same values since, in this limit, the material effect is removed from the expression for the bound — these values agree with the analytic results.

The choice of material and geometric parameters has been influenced by the results of Lindley and Teo (1978), those chosen here match their choices for the purposes of direct comparison. Whereas we have chosen to seek *bounds* on the (relative) value of the effective shear modulus, they sought to determine the coefficient itself. In the tables, we have identified three classes of results — technically four by virtue of those results that are excluded from all three classes.

- The class of results consistent with the solutions obtained by Lindley and Teo, i.e. their solution lies within the range we obtain. Indicated by $^{\#}$.

- The class of results for which there is no corresponding solution offered by Lindley and Teo. Indicated by †. Note that there are no solutions for the limiting case $\eta \to 0$ at all.

- The class of results for which the corresponding solutions of Lindley and Teo may be in slight error due to inadequate convergence. Indicated by *.

As can be readily seen from this sample, the third column in each section of the tables provides a bound which is at least as big as the those in the other two columns. The more relaxed bounds are also seen to have a greater incidence of consistency with the results of Lindley and Teo which is to be expected.

	$\lambda/\mu = 1/3$			$\lambda/\mu = 1$		
η	$C(1,0)$	$C(0,1)$	$C_{\max}$	$C(1,0)$	$C(0,1)$	$C_{\max}$
4	0.3358	0.3098	0.3368	0.3571	0.3316	0.3584
2	0.6081*	0.6097*	0.6193	0.6250	0.6300	0.6378
4/3	0.7155*	0.7428*	0.7431*	0.7258*	0.7560*	0.7560*
1	0.7627	0.8043	0.8058*	0.7692	0.8129	0.8153
2/3	0.8004*	0.8548*	0.8654*	0.8036	0.8591	0.8711
1/2	0.8145#	0.8740#	0.8918	0.8163	0.8765	0.8955
1/4	0.8285#	0.8934#	0.9219	0.8290#	0.8940#	0.9230
1/8	0.8321#*	0.8983#*	0.9304#*	0.8322#	0.8985#	0.9307#
0	0.8333†	0.9000†	0.9333†	0.8333†	0.9000†	0.9333†

Table 5.1: Table of lower bounds for μ_e/μ with $\lambda/\mu = 1/3$, 1.

	$\lambda/\mu = 2$			$\lambda/\mu = 5$		
η	$C(1,0)$	$C(0,1)$	$C_{\max}$	$C(1,0)$	$C(0,1)$	$C_{\max}$
4	0.3714	0.3452	0.3718	0.3829	0.3582	0.3845
2	0.6349*	0.6420*	0.6488*	0.6439	0.6530	0.6589
4/3	0.7317	0.7636	0.7636	0.7370	0.7705	0.7705
1	0.7729	0.8178	0.8208	0.7763	0.8223	0.8259
2/3	0.8054	0.8615	0.8744	0.8070#	0.8637	0.8773
1/2	0.8174#	0.8780	0.8976	0.8183#	0.8792	0.8995
1/4	0.8293#	0.8944#	0.9236#	0.8295#	0.8947#	0.9242#
1/8	0.8323#	0.8986#	0.9308#	0.8324†	0.8987†	0.9310†
0	0.8333†	0.9000†	0.9333†	0.8333†	0.9000†	0.9333†

Table 5.2: Table of lower bounds for μ_e/μ with $\lambda/\mu = 2$, 5.

	$\lambda/\mu = 10$			$\lambda/\mu = 30$		
η	$C(1,0)$	$C(0,1)$	$C_{\max}$	$C(1,0)$	$C(0,1)$	$C_{\max}$
4	0.3883	0.3639	0.3901	0.3925	0.3682	0.3943
2	0.6478	0.6577	0.6632	0.6506†	0.6613†	0.6664†
4/3	0.7392	0.7734	0.7735	0.7409†	0.7756†	0.7757†
1	0.7776	0.8241	0.8280	0.7787	0.8255	0.8296
2/3	0.8076†	0.8646†	0.8785†	0.8081†	0.8653†	0.8795†
1/2	0.8187#	0.8797	0.9003	0.8190#	0.8801#	0.9009#
1/4	0.8296#	0.8949#	0.9244#	0.8297#	0.8950#	0.9246#
1/8	0.8324†	0.8987†	0.9311†	0.8324†	0.8987†	0.9311†
0	0.8333†	0.9000†	0.9333†	0.8333†	0.9000†	0.9333†

Table 5.3: Table of lower bounds for μ_e/μ with $\lambda/\mu = 10$, 30.

	$\lambda/\mu = 100$			$\lambda/\mu = 300$		
η	$C(1,0)$	$C(0,1)$	$C_{\max}$	$C(1,0)$	$C(0,1)$	$C_{\max}$
4	0.3941	0.3699	0.3959	0.3945	0.3703	0.3963
2	0.6517	0.6626	0.6677	0.6520	0.6630	0.6680
4/3	0.7415	0.7764	0.7765	0.7417	0.7766	0.7767
1	0.7791	0.8260	0.8302	0.7792	0.8262	0.9303
2/3	0.8083#	0.8655	0.8798	0.8084#	0.8656	0.8799
1/2	0.8191#	0.8803#	0.9011	0.8191#	0.8803#	0.9012
1/4	0.8297#	0.8950#	0.9246#	0.8297#	0.8950#	0.9247#
1/8	0.8324#	0.8987#	0.9311#	0.8324#	0.8987#	0.9311#
0	0.8333†	0.9000†	0.9333†	0.8333†	0.9000†	0.9333†

Table 5.4: Table of lower bounds for μ_e/μ with $\lambda/\mu = 100$, 300.

Chapter 6

Future Developments

As is often the case, answers to one set of questions lead to a further round of questions. An extreme example of this being the erstwhile industry that grew up following Fermat's infamous claim and its present day incarnation in either verifying or improving upon Andrew Wile's proof. We conclude this thesis by giving an overview of the questions that arise through the contents of the preceding pages.

The most glaring omission from these pages must be a full treatment of the compressible case. An obvious development, therefore, would be to seek solutions to the secular equation for a compressible elastic half space. As has been seen, however, the incompressible simplification is in itself insufficient for allowing a complete picture (within the incompressible context, of course) to emerge. Nonetheless, numerical work may extract certain information thus far lacking. Echoing the concluding remark of Chapter 3, we point out that the underlying deformation is isochoric and so superposed incremental displacements could not constitute a significant volume change. The incompressible theory is thus ideally suited as a first step and it may prove beneficial to adapt to the compressible case by considering it as a nearly isochoric deformation (using the terminology of Ogden (1978)).

Should such advances be made then the subsequent consideration would be the problem of a compressible elastic layer. Note that we have not derived even a general form of a compressible dispersion equation. The derivation is likely to follow precisely the same track as that of the incompressible layer with the left-hand side of the secular equation (generalized to the four equations of motion solutions $s_1, s_2, \bar{s}_1, \bar{s}_2$) appearing as factors within the dispersion equations. Any such resulting equations would certainly require numerical schemes to be used in their investigation.

Of a more fundamental nature, a further development would be the study of the effect of shear on a finite rectangular block. This would require an adjustment to the boundary conditions so that there was one condition per surface. Taking the example of Chapter 5, these could be the absence of incremental displacements on two opposite faces coupled with the absence of incremental tractions on the remaining two surfaces.

The mixture of conditions would be similarly found in Problem 2 of Chapter 4 but this development has its foundation in Problem 3 of the same chapter. A rectangular block subjected to such boundary conditions would then serve as a prototype problem for looking at vibration isolators, the block being but one of the bonded components within such a structure. The question of stability is thus, in this context, of paramount importance.

It is also within the context of vibration isolators that the effective shear modulus is relevant and worth developing. One may visualize a building, located in a region prone to earthquakes, that has been constructed with this type of vibration isolator embedded throughout its base. In an ensuing earthquake it is the lateral deflections of the earth that cause the greatest damage in that they induce flexural motions on otherwise rigid buildings. The question of how deflection and stress relate in a multi-layered component is therefore of significance. Thus we may wish to extend the effective shear modulus problem to that of two (or more) bonded blocks.

The amount of shear involved in such applications can be quite large, as has been permitted throughout this thesis, but is not necessarily isolated in applications. Given the amount of pre-stress already present in such foundations through the weight of the building it is natural to investigate the combined problem of compression and shear — both in the effective shear modulus of the rectangular block and the wave propagation through, or bifurcation of, the block — as the next step towards a more realistic model.

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