Generalising the Cohen-Macaulay condition and other homological properties

by

Marjory Jane Macleod

A thesis submitted to the

Faculty of Information and Mathematical Sciences

at the University of Glasgow

for the degree of

Doctor of Philosophy

April 2010

Abstract

This thesis concerns some homological properties for noetherian rings which are finite modules over their centres, but we look most particularly at the Cohen-Macaulay property. We look at ways if generalising these homological properties, either from the commutative to the noncommutative case, or from finite dimensional k-algebras to rings which are finite modules over a central subring. Chapter 1 contains known background material as well as some preliminary results to be used in later proofs. We consider, in Chapter 2, some generalisations of the well-known Cohen-Macaulay property for commutative rings. We focus on the centrally-Macaulay property as defined by Brown, Hajarnavis and MacEacharn in [17] and what we call Krull-Macaulay, a stronger and more homological condition. Centrally-Macaulay rings are defined in terms of a central subring and we consider the extent to which the property is dependent on the choice of central subring. We then consider generalisations of a commutative result relating the Cohen-Macaulay property to freeness over a regular subring before applying the Cohen-Macaulay condition to reconstruction algebras, as defined recently by Wemyss and Craw. Given that there is more than one generalisation of the Cohen-Macaulay property, we seek, in Chapter 3, to find the best generalisation to the particular class of rings we study. That is, noetherian rings which are finite modules over their centre. Thus we compare the two properties, centrally-Macaulay and Krull-Macaulay, and variants of them. In particular, we combine centrally-Macaulay with the symmetry of homological grade to obtain a property which is equivalent to Krull-Macaulay for equidimensional rings. We suggest that this property, which we call symmetrically-Macaulay, is the best generalisation in this case. We then go on to consider, in Chapter 4, generalisations of related properties for commutative rings, regular and Gorenstein and generalise the commutative hierarchy: regular implies Gorenstein which implies Cohen-Macaulay. Finally, in Chapter 5, we demonstrate the significance of the module $\mathrm{Hom}_C(R, C)$ for any central subring C over which R is finitely

generated. This leads us to generalise the definition of a symmetric algebra and using our generalisation we are able to generalise some results of Braun in [9]. We finish with showing when skew group algebras are symmetric.

Acknowledgements

First of all I acknowledge all the support and advice which I received from my supervisor, Professor Ken Brown. I couldn't have asked for a more obliging, approachable and helpful supervisor. I would also like to thank Amiram Braun, Alastair Craw, Ulrich Kraemar, Monica Macaulay and Alexander Quintero Velez for helpful mathematical discussions and the many others in the Maths Department who have been kind to me over the years. Thanks are especially due to my office mates: Ehsan, Monica, Robert, Susan, and Yunfei for all their help and distractions throughout the past three and a half years. I also acknowledge the financial support of the EPSRC and the Glasgow Maths Department.

I acknowledge that what I owe to my parents is more than I can say. I would like to thank my Dad for passing on to me his mathematical abilities and for providing me with accommodation in Glasgow but more importantly I am very much indebted to my parents' Christian example, teaching and prayers as a major influence in making me who I am. The rest of my family are also worthy of thanks, particularly my brothers and sisters-in-law, for their help and advice in many things and their hospitality; Aunty Alice and Uncle Jan, for providing a home from home in Glasgow and many meals, lifts and favours; and Aunty Catherine, for everything she has done and for being a lovely second mum to me. I also thank my niece Julia for being a most welcome distraction from maths during the past 20 months. Thanks are due to my friends for their support and encouragement, particularly Joanna who, through experience, understood the ups and downs of a PhD; Jacqueline, my best friend, who also shares my love of maths; Naomi, a good friend; and Cathie Chapman for providing many delicious "Tuesday fun lunches".

But above all I have to acknowledge the help of God, "who doth all things for me perform most perfectly". As one of my undergraduate lecturers told me, the best research tool is prayer, and I have made much use of it. "Without me ye can do nothing" (John 15:5).

Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

No part of this thesis has previously been submitted by me for a degree at this or any other university.

Contents

Chapter 1

Background Material

Throughout this thesis, all rings have identity, all modules are unital and unless explicitly stated, all modules are assumed to be right modules.

1.1 Commutative Cohen-Macaulay rings

We first discuss the well-known theory of commutative Cohen-Macaulay rings and modules, regular rings and Gorenstein rings. A good basic reference for this is [35], while [20] gives a very thorough treatment of the subject, focusing on local rings, and [25] is also very useful. Throughout this section R denotes a commutative noetherian ring and Krull $\dim(R) = n < \infty$.

Definition 1.1.1. Let R be a commutative noetherian ring and X an R-module.
(i) A sequence $\{x_1, \ldots, x_t\} \subseteq R$ is an *R-sequence on X* if

- $X / \sum_{i=1}^{t} x_i X \neq 0$ and
- for all $1 \leq i \leq t$, x_i is a non zero divisor on $X / \sum_{j=1}^{i-1} x_j X$.

(ii) Let I be an ideal of R. An R-sequence contained in I is *maximal* if it cannot be extended to a longer R-sequence.

From [35, Theorem 121] we see that all maximal R-sequences in an ideal I have the same length.

Definition 1.1.2. Let R be a commutative noetherian ring.
(i) We define the *grade* of an ideal I to be the maximal length of an R-sequence in I on

R and denote it $G_R(I)$. Similarly for an R-module X we let $G_R(I, X)$ be the length of a maximal R-sequence in I on X. If R is local with maximal ideal $\mathfrak{m}$ then we define the *depth* of X to be $G_R(\mathfrak{m}, X)$.
(ii) If P is a prime ideal we define the *height* of P, $\mathrm{ht}(P)$ to be the maximal length of a chain of prime ideals descending from P. If I is any ideal, $\mathrm{ht}(I) := \min\{\mathrm{ht}(P) : I \subseteq P, P \text{ prime}\}$.

Now since R is noetherian, [20, Theorem A1] shows that $\mathrm{ht}(I) < \infty$ for any proper ideal I. Also,

$$G(I) \leq \mathrm{ht}(I) \tag{1.1}$$

for all ideals I by [35, Theorem 132]. This leads us to consider the case where these values coincide and thus we have the following definition.

Definition 1.1.3. Let R be a commutative noetherian ring. Then R is *Cohen-Macaulay* if

$$G(\mathfrak{m}) = \mathrm{ht}(\mathfrak{m})$$

for all maximal ideals $\mathfrak{m}$ of R.

It turns out that in a Cohen-Macaulay ring, $G(p) = \mathrm{ht}(\mathfrak{p})$ for all ideals $\mathfrak{p}$ of R. See [35, Theorem 136]. In order to give the definition of a Cohen-Macaulay module we need to define Krull dimension.

Definition 1.1.4. Let R be a commutative noetherian ring.
(i) The *Krull dimension* of R, Krull $\dim(R)$ is the supremum of all lengths of chains of prime ideals of R.
(ii) For a finitely generated R-module X, Krull $\dim(X) =$ Krull $\dim(R/\operatorname{Ann}(X))$.

We note that for $R = k[X_1, \ldots, X_n]$, k a field, we have Krull $\dim(R) = n$ by [40, Corollary 5.6].

Definition 1.1.5. Let R be a commutative noetherian ring.
(i) An R-module X is a *Cohen-Macaulay R-module* if $G(\mathfrak{m}, X) =$ Krull $\dim_{R_\mathfrak{m}}(X_\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.
(ii) A Cohen-Macaulay module X is *maximal* if it is Cohen-Macaulay and $G(\mathfrak{m}, X) =$ Krull $\dim(R_\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.

Note that, since $\mathrm{ht}(\mathfrak{m}) = \mathrm{ht}(\mathfrak{m}_\mathfrak{m}) = \text{Krull } \dim(R_\mathfrak{m})$ for any maximal ideal $\mathfrak{m}$, a commutative noetherian ring R is Cohen-Macaulay if and only if it is a maximal Cohen-Macaulay R-module.

Definition 1.1.6. Let R be a commutative ring and X an R-module. Then

$$j(X) := \inf\{i : \mathrm{Ext}^i(X, R) \neq 0\}.$$

Proposition 1.1.7. *Let R be a commutative ring. Then $G(I) = j(R/I)$ for all ideals I of R.*

Proof. Follows from [25, Proposition 18.4] with $M = R/I$ and $N = R$. □

Definition 1.1.8. A ring R is catenary if, for any two primes $P \subseteq Q$ of R, all maximal chains of prime ideals from Q to P have the same length.

Proposition 1.1.9. *[25, Corollary 18.10] A commutative Cohen-Macaulay ring is* catenary.

We sometimes restrict to the equidimensional case, so here is what we mean by equidimensional.

Definition 1.1.10. A ring R is *equidimensional* if all maximal ideals have the same height.

Note that a local ring is equidimensional as is any commutative affine domain (see [25, Chapter 8, Theorem A]).

Now suppose R is equidimensional. Then, since a Cohen-Macaulay ring is catenary, for any prime ideal P of R we have

$$\text{Krull } \dim(R) = \text{Krull } \dim(R/P) + \mathrm{ht}(P)$$

and we get the following:

Theorem 1.1.11. *Let R be a commutative equidimensional Cohen-Macaulay ring. Then for any finitely generated module X*

$$\text{Krull } \dim(R) = \text{Krull } \dim(X) + j(X).$$

Proof. For $X = R/P$ where P is a prime ideal, the equation follows easily from the equation $\text{Krull } \dim(R) = \text{Krull } \dim(R/P) + \mathrm{ht}(P)$ since $\mathrm{ht}(P) = j(R/P)$ by the Cohen-Macaulay property and Proposition 1.1.7. Induction on Krull dimension gives the equation for arbitrary X. (See Proposition 1.4.13 for a more general version of the induction.) □

Other properties, connected to but stronger than Cohen-Macaulay, are regular and Gorenstein which we now define.

Definition 1.1.12. Let R be a commutative noetherian ring. Then R is *regular* if it has finite global dimension.

Definition 1.1.13. A commutative noetherian ring is *Gorenstein* if it has finite injective dimension.

These three properties are local as we see in the following two results.

Proposition 1.1.14. *Let R be a commutative noetherian ring and $\mathfrak{p}$ any prime ideal of R. Then*

(i) If R is Cohen-Macaulay, then so is $R_{\mathfrak{p}}$;

(ii) If R is Gorenstein so is $R_{\mathfrak{p}}$;

(iii) If R is regular then so is $R_{\mathfrak{p}}$.

Proof. See [25, Proposition 18.8], [35, Exercise 4.5.12] and [48, Theorem 9.52]. □

Theorem 1.1.15. *Let R be a commutative noetherian ring.*

(i) R is Cohen-Macaulay if and only if $R_{\mathfrak{m}}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$ of R.

(iii) R is Gorenstein if and only if $R_{\mathfrak{m}}$ is Gorenstein for every maximal ideal $\mathfrak{m}$ of R.

(ii) R is regular if and only if $R_{\mathfrak{m}}$ is regular of bounded global dimension for every maximal ideal $\mathfrak{m}$ of R.

Proof. See [25, Proposition 18.8], [35, Exercise 4.5.17] and [48, Theorem 9.52]. □

Theorem 1.1.16. *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and n a positive integer. Then the following conditions are equivalent:*

(i) R is regular;

(ii) $\operatorname{gl.dim}(R) = n < \infty$*;*

(iii) $\operatorname{gl.dim}(R) = \text{Krull dim}(R) = n$*;*

(iv) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \text{Krull dim}(R) = n$*;*

(v) every set $\{x_1, \ldots, x_n\}$ of elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis for $\mathfrak{m}/\mathfrak{m}^2$ forms a regular sequence in $\mathfrak{m}$;

(vi) $\mathfrak{m}$ is generated by an R-sequence of length n.

Proof. See e.g. [40, Theorem 19.2] and [35, Theorem 169]. □

Again, we consider the polynomial algebra $R = k[X_1, \ldots, X_n]$ and note that from Theorems 1.1.15 and 1.1.16(ii) we can deduce that $\operatorname{gl.dim}(R) = n$.

Proposition 1.1.17. *Let R be a commuatative noetherian ring and let $\mathbf{x}$ be an R-sequence. Then if R is Gorenstein (respectively Cohen-Macaulay) so is $R/\langle \mathbf{x} \rangle$.*

Proof. This is [20], Theorem 2.1.3(a) and Proposition 3.1.19(b). □

We will sometimes use the notation $R, \mathfrak{m}, k$ to denote a local ring R with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$.

Definition 1.1.18. Let $(R, \mathfrak{m}, k)$ be a local ring. The type of an R-module M of depth t is

$$r(M) = \dim_k \operatorname{Ext}^t_R(k, M).$$

Theorem 1.1.19. *Let $(R, \mathfrak{m}, k)$ be a local noetherian commutative ring. Then R is Gorenstein if and only if R is Cohen-Macaulay of type 1 if and only if R is Cohen-Macaulay and $\dim_k(soc(R/\mathbf{x}R)) = 1$ for any maximal sequence $\mathbf{x}$.*

Proof. This is [20, Theorem 3.2.10 and Lemma 1.2.19]. □

Theorem 1.1.20. *Let R be a commutative noetherian ring. Then R regular implies R is Gorenstein which in turn implies that R is Cohen-Macaulay.*

Proof. The first implication is clear since $\operatorname{inj.dim}(R) \leq \operatorname{gl.dim}(R)$. The second is Theorem 1.1.19 which extends to the non-local case by Theorem 1.1.15. □

However, these implications cannot be reversed as can be seen from the following examples.

Example 1.1.21. A Gorenstein ring need not be regular. Consider the coordinate ring $R := k[x, y]/\langle y^2 - x^3 \rangle$ of the cusp $y^2 - x^3$. Then

(i) R is a Gorenstein ring;

(ii) R is not regular.

Proof. (i) First note that $k[x, y]$ is regular with global dimension 2 and hence is Gorenstein by Theorem 1.1.20. We then see from Proposition 1.1.17 that R is Gorenstein.

(ii) On the other hand, if we consider the maximal ideal $\mathfrak{m}_0 := \langle x, y \rangle / \langle y^2 - x^3 \rangle$ we see that

$$\dim_k(\mathfrak{m}_0 / \mathfrak{m}_0^2) = 2 > 1 = \text{Krull } \dim(R).$$

So by Theorem 1.1.16(iv) R is not regular. □

Example 1.1.22. A Cohen-Macaulay ring need not be Gorenstein. Let $S = \mathbb{C}[X, Y, Z]$, and $G = \langle g \rangle \subseteq GL(3, \mathbb{C})$ where $g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. If $R = S^G$ then

(i) R is a Cohen-Macaulay ring;

(ii) R is not Gorenstein.

Proof. (i) By [6, Theorem 1.3.1] $R := S^G$ is an affine $\mathbb{C}$-algebra (and hence noetherian) such that S is a finitely generated R-module. Thus R is Cohen-Macaulay by [6, Theorem 4.3.6].

(ii) On the other hand, [54, Theorem 1] says that if G acts linearly on the polynomial algebra $k[x_1, \ldots, x_n] = S$ and if G contains no pseudo-reflections then $R = S^G$ is Gorenstein if and only if $G \subseteq SL(n, k)$. But $G = \{1, g\}$ and g is not a pseudo-reflection so this theorem applies. Also $\det g = -1 \neq 0$ so R is not Gorenstein. □

Definition 1.1.23. (See [40, §14]) Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. Let X be a finitely generated R-module, Krull $\dim(X) = s$. Then there exist $x_1, \ldots, x_s \in \mathfrak{m}$ such that $X/\sum_{i=1}^{s} x_i X$ has dimension 0. Then $\{x_1, \ldots, x_s\}$ is a *system of parameters* for X.

Proposition 1.1.24. *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and X a Cohen-Macaulay R-module. Then the following are equivalent:*

(i) $\{x_1, \ldots, x_r\}$ is a regular sequence on X in $\mathfrak{m}$;

(ii) $\{x_1, \ldots, x_r\}$ is part of a system of parameters for X;

(iii) Krull $\dim(X/\sum_{i=1}^{r} x_i X) =$ Krull $\dim(X) - r$.

Proof. See [20, Theorem 2.1.2]. □

The following proposition follows from (i) implies (v) of Theorem 1.1.16.

Proposition 1.1.25. *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. If R is regular then there exists a system of parameters for R generating $\mathfrak{m}$ (called a* regular system of parameters*).*

Theorem 1.1.26. *(The Auslander-Buchsbaum formula) Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and let X be a finitely generated R-module with* $\operatorname{pr.dim}(X) < \infty$. *Then*

$$\operatorname{pr.dim}(X) + G(\mathfrak{m}, X) = G(\mathfrak{m}, R).$$

Proof. See [20, Theorem 1.3.3]. □

Proposition 1.1.27. *Let R be a commutative noetherian ring and X a finitely generated free R-module. Then if $\{x_1, \dots, x_s\}$ is an R-sequence on R it is also an R-sequence on X.*

Proof. Let $\{x_1, \dots, x_s\}$ be an R-sequence on R. First note that $X/\sum_{i=1}^{s} x_i X \neq 0$ since $R/\sum_{i=1}^{s} x_i R \neq 0$. Now let $X = R^{\oplus d}$ and suppose that there exists some $m \in X$ such that $x_i m \in \sum_{j=1}^{i-1} x_j X$. Then $m = (r_k)$ and $x_i m = (x_i r_k) \in \sum_{j=1}^{i-1} x_j R^{\oplus d} = (\sum_{j=1}^{i-1} x_j R)^{\oplus d}$. Thus for $1 \leq k \leq d$ we have $x_i r_k \in \sum_{j=1}^{i-1} x_j R$. But since $\{x_1, \dots, x_s\}$ is an R-sequence on R we must have $r_k \in \sum_{j=1}^{i-1} x_j R$. Hence $m \in \sum_{j=1}^{i-1} x_j X$ as required and $\{x_1, \dots, x_s\}$ is an R-sequence on X. □

A local ring R which is a finite module over a regular subring S is Cohen-Macaulay if and only if R is a free S-module. See [20, Proposition 2.2.11]. This generalises to R-modules as in the proposition below, which we prove, for lack of a suitable reference.

Proposition 1.1.28. *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{n}$ and S a regular local subring with maximal ideal $\mathfrak{m}$ such that R is a finitely generated S-module. Let X be a finitely generated R-module. Then X is a maximal Cohen-Macaulay R-module if and only if it is a free S-module.*

Proof. Since S is regular, $\operatorname{pr.dim}_S(X) < \infty$. Thus by the Auslander-Buchsbaum formula

$$\operatorname{pr.dim}_S(X) + G(\mathfrak{m}, X) = G(\mathfrak{m}, S), \tag{1.2}$$

where

$$G(\mathfrak{m}, S) = \text{Krull } \dim(S) = \text{Krull } \dim(R) =: n. \tag{1.3}$$

Here the first equality holds since S is Cohen-Macaulay, the second by [20, Corollary A8]. Since projective modules over the local ring S are free by [40, Theorem 2.5], X is S-free if and only if $G_S(\mathfrak{m}, X) = \text{Krull } \dim(S)$. Let $\mathbf{x} = \{x_1, \dots, x_n\}$ be a regular system of parameters for S generating $\mathfrak{m}$. Then $\mathbf{x}$ is also a system of parameters of R.

Suppose X is S-free. Then since by Proposition 1.1.24 $\mathbf{x}$ is a regular sequence for S it is also a regular sequence for X by Proposition 1.1.27. Thus $G_R(\mathfrak{n}, X) = n$ and X is a maximal Cohen-Macaulay R-module by (1.3).

Now suppose X is a maximal Cohen-Macaulay R-module. Then any system of parameters of R is a regular sequence on X by Proposition 1.1.24, and thus $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a regular sequence for X. Hence $G_S(\mathfrak{m}, X) = \text{Krull dim}(S)$ and so X is a free S-module by (1.2).

□

Definition 1.1.29. Let R be a commutative domain and S any ring containing R as a subring. An element $s \in S$ is *integral over* R if it satisfies a monic polynomial with coefficients in R. The ring of all elements of S integral over R is the *integral closure* or *normalisation* of R in S. If R is equal to its integral closure in $S = Q(R)$ then R is *integrally closed* or *normal.*

Theorem 1.1.30. *A regular noetherian domain is normal.*

Proof. By Serre's criterion [20, Theorem 2.2.22] R is normal if and only if:
(1) for every prime ideal $\mathfrak{p}$, $G(\mathfrak{p}) \geq \min\{\text{ht}\,\mathfrak{p}, 2\}$.
(2) for every prime $\mathfrak{p}$ of height less than or equal to 1, $R_\mathfrak{p}$ is regular.
The first condition is satisfied since R is Cohen-Macaulay by Theorem 1.1.20 and so grade is equal to height for all ideals of R [35, Theorem 136]. The second condition is clear by Proposition 1.1.14. □

From this we get that every regular local ring is normal since by [35, Theorem 164] it is a domain. But this tells us, by [35, Theorem 168], that every regular ring is a finite direct sum of normal domains.

On the other hand, note that this doesn't generalise to Gorenstein domains. Take Example 1.1.21 and note that R is Gorenstein but is not integrally closed. For $R \cong k[t^2, t^3] \subseteq k[t] = Q(R)$ the element $t \notin R$ satisfies the monic polynomial $x^2 - t^2 = 0$ so is integral over R.

1.2 Rings which are finite modules over their centres

We are particularly interested in noncommutative rings R which satisfy the following hypothesis:

$$R \text{ is noetherian with Krull dim}(R) = n < \infty \text{ and is a finite module over its centre } Z. \tag{H}$$

We refer the reader to [41, Chapter 6] for a general definition of Krull dimension. However, for FBN rings, Krull dimension is equivalent to classical Krull dimension, which we will use here. We refer the reader to [31, Chapter 9] for the definition of an FBN ring and note that [31, Proposition 9.1(i)] shows that rings satisfying (H) are FBN.

Definition 1.2.1. Let R be a noetherian ring. The *classical Krull dimension* of R is the supremum of the lengths of chains of prime ideals in R.

Then by [41, Theorem 6.4.8] we can take this as our definition of Krull dimension for rings, which coincides with the definition for commutative rings. For finitely generated R-modules X we note that since X is a faithful $R/\operatorname{Ann}(X)$-module, [41, Proposition 6.4.12] shows that $\text{Krull dim}(X) = \text{Krull dim}(R/\operatorname{Ann}(X))$, which again gives us a nice way of thinking of the Krull dimension of modules, for rings satisfying hypothesis (H).

The fact that, when (H) holds, Z, or in fact any central subring over which R is a finite module, is also noetherian is seen from the following result:

Proposition 1.2.2. *[41, Cor 10.1.11(ii)] Let R, S be rings with R a finite normalising extension of S. Then R is right noetherian if and only if S is right noetherian and then* $\text{Krull dim}(R) = \text{Krull dim}(S)$.

We note that R is a finite normalising extension of S if R_S has a finite generating set $a_1, \ldots, a_t$ where each a_i normalises S. That is, $a_iS = Sa_i$. In particular, this holds if R is a finitely generated S-module with S central in R.

Also important in this context is the Artin-Tate Lemma.

Lemma 1.2.3. *[41, Lemma 13.9.10] Let $A \subseteq B \subseteq S$ be rings such that A, B are central subrings of S with S being an affine A-algebra and a finitely generated B-module.*
(i) There exists an affine A-subalgebra B' of B such that S is a finitely generated B'-module.
(ii) If either A is noetherian or B is a direct summand of S_B, then B is a finitely generated B'-module and an affine A-algebra.

If R satisfies (H) and the centre Z is affine, then we can use Noether normalisation, as follows, to find a central polynomial subring Z_0 such that R is a finitely generated Z_0-module.

Theorem 1.2.4. *[25, Chapter 8, Theorem A1] Let Z be a commutative affine ring over a field k. Then there is a subring Z_0 of Z with $\mathrm{Z}_0 = k[x_1, \ldots, x_r]$ such that Z is a finitely generated S-module.*

Note that throughout this thesis we will use Roman capitals such as M, P, Q to denote ideals of a ring R and the corresponding fraktur letters $\mathfrak{m}$, $\mathfrak{p}$, $\mathfrak{q}$ to denote the intersection of the ideal with either $Z(R)$ or the relevant central subring C. Then for a semiprime ideal $\mathfrak{s}$ of C we will write $R_{\mathfrak{s}}$ for the localisation of R at $C\backslash\mathfrak{s}$. For a prime ideal P we will use $\mathcal{C}(P)$ to denote the set of elements of R which are regular in R/P.

Other useful properties of rings which are finite modules over their centres are lying over, going up and incomparability. We say that a prime ideal P of R *lies over* a prime ideal $\mathfrak{p}$ of Z if $\mathfrak{p} = P \cap Z$.

Theorem 1.2.5. *Let R be a ring which is a finite module over its centre Z (or equivalently, some central subring). Let l be the minimal number of generators of R as a Z-module.*
Lying over: *If $\mathfrak{p}$ is a prime ideal of Z then there are finitely many primes $P_1, \ldots, P_s$ of R such that P_i lies over $\mathfrak{p}$. Here $1 \leq s \leq l$.*
Incomparability: *Let $P \subsetneq I$ be ideals of R with P prime. Then $P \cap Z \subsetneq I \cap Z$.*
Going up: *If P is a prime ideal of R and $\mathfrak{p} \subset \mathfrak{q}$ primes of Z such that P lies over $\mathfrak{p}$. Then there exists a prime ideal Q of R such that $P \subset Q$ and Q lies over $\mathfrak{q}$.*

Proof. Follows from [45, Theorem 16.9] since R is a finite normalising extension of Z. □

Note that going up and incomparability imply that M is a maximal ideal of R if and only if $\mathfrak{m}$ is a maximal ideal of Z.

Corollary 1.2.6. *Let R be a ring which is a finite module over a central subring C. Let P be a prime ideal of R and $\mathfrak{p} = P \cap C$. Then* $\mathrm{ht}(P) \leq \mathrm{ht}(\mathfrak{p})$.

Proof. Suppose we have a chain of prime ideals of R

$$P = P_t \supset \cdots \supset P_0.$$

Then incomparability gives a chain of prime ideals of C

$$\mathfrak{p} = \mathfrak{p}_t \supset \cdots \supset \mathfrak{p}_0$$

where each P_i lies over $\mathfrak{p}_i$. □

Corollary 1.2.7. *Let R be a ring which is a finite module over a central subring C. Let $\mathfrak{m}$ be a maximal ideal of C. Then there exists a maximal ideal M of R lying over $\mathfrak{m}$ such that* $\mathrm{ht}(M) = \mathrm{ht}(\mathfrak{m})$.

Proof. Let $\mathfrak{m}_0 \subset \mathfrak{m}_1 \subset \ldots \subset \mathfrak{m}_n = \mathfrak{m}$ be a maximal chain of primes descending from $\mathfrak{m}$. There exists a prime ideal M_0 of R lying over $\mathfrak{m}_0$ and going up and incomparability give a chain of primes $M_0 \subset M_1 \subset \ldots \subset M_n = M$ with each M_i lying over $\mathfrak{m}_i$. Thus M is a maximal ideal of height at least n lying over $\mathfrak{m}$. That M has height n follows from Corollary 1.2.6. □

In the previous section we explained what we mean by an equidimensional commutative ring. We apply the same definition to noncommutative rings as follows:

Definition 1.2.8. Let R be any ring. Then R is *equidimensional* if all maximal ideals of R have the same height.

Corollary 1.2.9. *Let $C \subset R$ be noetherian rings, with C central, such that R is a finite C-module. If R is equidimensional then so is C and $\mathrm{ht}(M) = \mathrm{ht}(M \cap C)$ for all maximal ideals M of R.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of C. Then by Corollary 1.2.7 there exists a maximal ideal M of R such that $M \cap C = \mathfrak{m}$ and $\mathrm{ht}(M) = \mathrm{ht}(\mathfrak{m})$. But $\mathrm{ht}(M) = \mathrm{Krull\ dim}(R)$ and thus C must be equidimensional with $\mathrm{Krull\ dim}(C) = \mathrm{Krull\ dim}(R)$. □

Definition 1.2.10. For a ring R satisfying (H) we define the *support* of a module X to be

$$Supp(X) = \{P \lhd R : P \text{ is prime and } X_{\mathfrak{p}} \neq 0\}.$$

Lemma 1.2.11. *For a non zero R-module X, $Supp(X) \cap \mathrm{maxspec}(R) \neq 0$.*

Proof. Adapt the proof of [48, Theorem 3.80(i)] which is for a commutative ring. □

Proposition 1.2.12. *Let R be a noetherian ring which is a finite module over its centre. Let V be a simple R-module. Then $R/\operatorname{Ann}_R(V)$ is isomorphic to the direct sum of a number of copies of V.*

Proof. Clearly V is a simple R/M-module, where $M = \operatorname{Ann}_R(V)$. Then by Kaplansky's theorem (see [13, Theorem I.13.3]) R/M is a central simple algebra and hence a matrix ring over a division ring. Thus R/M is a direct sum of a number of copies of the unique simple module. □

We now discuss links between prime ideals and defined in [31].

Definition 1.2.13. Let P and Q be primes in a noetherian ring R. There is a *link from P to Q*, $P \rightsquigarrow Q$ if there is an ideal A of R such that $P \cap Q \supseteq A \supseteq PQ$ and $(P \cap Q)/A$ is nonzero and torsion-free both as a left R/P-module and as a right R/Q-module.

Definition 1.2.14. Let P be a prime of R. The *clique* of P, $Clq(P)$ is defined to be the set of primes Q such that there exist primes $P = P_1, P_2, \ldots, P_t = Q$ with either $P_i \rightsquigarrow P_{i+1}$ or $P_{i+1} \rightsquigarrow P_i$ for all $1 \leq i \leq t-1$.

The following is Müller's theorem which gives a very useful description of a clique, for rings of the type we are studying.

Theorem 1.2.15. *[31, Theorem 13.10] Let R be a ring satisfying (H). Then, for any prime ideal P of R*

$$Clq(P) = \{Q \in \operatorname{spec}(R) : Q \cap Z = P \cap Z\}.$$

We finish this section with a couple of well-known results on quotient rings. But first we give the proof of an easy and well-known result, for which we have been unable to find a reference.

Lemma 1.2.16. *Let R be an artinian ring. Then every regular element is a unit.*

Proof. Let c be a regular element in R. We consider the following chain of non-zero ideals:

$$cR \supseteq c^2R \supseteq c^3R \supseteq \cdots .$$

Then since R is artinian, there exists some $n \in \mathbb{N}$ such that $c^nR = c^{n+1}R$. Thus $c^n = c^{n+1}r$ for some $r \in R$ and $c^n(cr-1) = 0$. But by regularity of c, $cr = 1$ and hence c is a unit. □

Proposition 1.2.17. *Let R be a prime noetherian ring which is a finite module over a central subring C. Then $Q(R) = R[\mathcal{C}]^{-1}$ where $\mathcal{C} := C\backslash\{0\}$.*

Proof. Since R is prime, $\mathcal{C} := C\backslash\{0\}$ consists of regular elements of R and can be inverted to give the partial quotient ring $R[\mathcal{C}]^{-1} \subseteq Q(R)$. Now $R[\mathcal{C}]^{-1}$ is a finite module over the subfield $C[\mathcal{C}]^{-1}$ so it must be artinian. Thus by Lemma 1.2.16 all regular elements in $R[\mathcal{C}]^{-1}$ are units and hence $R[\mathcal{C}]^{-1} = Q(R)$. □

Lemma 1.2.18. *Let R be a prime noetherian ring which is a finite module over its centre Z and let $Q := Q(R)$. Then*

$$Z(Q) = Q(Z).$$

Proof. We know from Proposition 1.2.17 that $Q = R[Z\backslash\{0\}]^{-1}$, so for $c, d \in Z\backslash\{0\}$,

$$\begin{aligned} Z(Q) &= \{rc^{-1} : sd^{-1}rc^{-1} = rc^{-1}sd^{-1} \;\; \forall \;\; sd^{-1} \in Q\} \\ &= \{rc^{-1} : srd^{-1}c^{-1} = rsc^{-1}d^{-1} \;\; \forall \;\; sd^{-1} \in Q\} \\ &= \{rc^{-1} : sr = rs \;\; \forall \;\; s \in R\} \\ &= Q(Z). \end{aligned}$$

□

1.3 Some homological algebra

Here we consider some homological definitions and results. In Definition 1.1.6 we defined $j(X)$, where X is a module over a commutative ring. We now look at this concept for noncommutative rings.

Definition 1.3.1. Let M be a right R-module and N a left R-module. Then we define the *right (homological) grade* of M, $j^r_R(M) = \min\{i : \mathrm{Ext}^i(M_R, R_R) \neq 0\}$, or $j^r_R(M) = \infty$ if no such i exists. Similarly the *left (homological) grade* of N, $j^l_R(N) = \min\{i : \mathrm{Ext}^i({}_RN, {}_RR) \neq 0\}$ or $j^l_R(M) = \infty$ if no such i exists.

Definition 1.3.2. A *central R-bimodule* X is an R-R-bimodule such that $zx = xz$ for all $x \in X$ and $z \in Z(R)$.

Definition 1.3.3. We say that R is *grade-symmetric* if $j^r_R(X) = j^l_R(X)$ for all finitely generated central R-bimodules X.

In this case, or if it is clear from the context which side we are working on, we will drop the superscripts and write $j_R(X)$. If R is clear we also drop the subscript.

Proposition 1.3.4. *Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence of left (or right) R-modules with $i = j(X') = j(X'')$. Then $j(X) = i$.*

Proof. We have the long exact sequence,

$$0 = \mathrm{Ext}^{i-1}(X', R) \to \mathrm{Ext}^i(X'', R) \to \mathrm{Ext}^i(X, R) \to \cdots$$

with $\mathrm{Ext}^i(X'', R) \neq 0$. Then we must have $\mathrm{Ext}^i(X, R) \neq 0$, so that $j(X) \leq i$. A similar argument shows that $j(X) \geq i$. □

Consequently, for R satisfying hypothesis (H) and V a simple R-module, Proposition 1.2.12 and the proposition above show that $j(V) = j(R/\operatorname{Ann}_R(V))$.

Theorem 1.3.5. *[48, Thm 9.37] Let R be any ring. Assume $x \in R$ is a central element that is neither a unit or a zero divisor, and let $R^* = R/xR$; assume B is a right R-module for which multiplication by x is monic. Then for every R^*-module A*

$$\operatorname{Ext}^i_{R^*}(A, B/xB) \cong \operatorname{Ext}^{i+1}_R(A, B).$$

Corollary 1.3.6. *Let I be an ideal of R and let C be a central subring of R. Then $G_C(I \cap C) \leq j(R/I)$.*

Proof. Suppose $G_C(I \cap C) = t$ and $\{x_1, \ldots, x_t\}$ is a maximal C-sequence in $I \cap C$. Then set $T := \sum_{i=1}^{t} x_i R$. By Theorem 1.3.5, $\operatorname{Ext}^{m-t}_{R/T}(R/I, R/T) \cong \operatorname{Ext}^m_R(R/I, R)$ for all $m \geq 0$, so that if $i < t$ we have $\operatorname{Ext}^i_R(R/I, R) = 0$. □

Lemma 1.3.7. *Let R be a ring satisfying hypothesis (H). Let M a maximal ideal of R and $0 \neq X$ a finitely generated R/M-module. Then X is torsion-free as an R/M-module and $X \otimes R_{\mathfrak{m}} \neq 0$ where $\mathfrak{m} = M \cap Z$.*

Proof. Since R/M is artinian, then for all $c \in \mathcal{C}(M)$, $c + M$ is a unit in R/M by Lemma 1.2.16. Thus X is R/M-torsion-free and hence $X \otimes R_{\mathfrak{m}} \neq 0$. □

Corollary 1.3.8. *Let R be a ring satisfying hypothesis (H). Let M be a maximal ideal of R and $\mathfrak{m} = M \cap Z$. Then*

$$j^r_R(R/M) = j^r_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/M_{\mathfrak{m}}) \text{ and } j^l_R(R/M) = j^l_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/M_{\mathfrak{m}}).$$

Proof. By [19, Proposition 1.6] we get

$$\operatorname{Ext}^i_R(R/M, R) \otimes_R R_{\mathfrak{m}} \cong \operatorname{Ext}^i_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/M_{\mathfrak{m}}, R_{\mathfrak{m}}).$$

Hence $j^r_R(R/M) \leq j^r_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/M_{\mathfrak{m}})$. But $\operatorname{Ext}^{j(R/M)}(R/M, R)$ is a non-zero right R/M-module so Lemma 1.3.7 shows that $\operatorname{Ext}^{j(R/M)}_R(R/M, R) \otimes_R R_{\mathfrak{m}} \neq 0$. Thus $j^r_R(R/M) = j^r_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/M_{\mathfrak{m}})$. Similarly, considering R/M as a left module gives the second statement. □

In Theorem 1.1.11 we saw that finitely generated modules over equidimensional commutative Cohen-Macaulay rings satisfy the following equation:

$$\text{Krull dim}(R) = \text{Krull dim}(X) + j(X). \tag{1.4}$$

We now consider this as an additional hypothesis on noncommutative rings satisfying (H).

Lemma 1.3.9. *Let R be a ring satisfying hypothesis (H) and such that (1.4) holds for all finitely generated right R-modules. Let P be any prime ideal of R. Then the R-R/P-bimodule* $\mathrm{Ext}^{j(R/P)}(R/P_R, R_R)$ *is not torsion as a right R/P-module.*

Before proving Lemma 1.3.9 we give the following well known lemma used in the proof.

Lemma 1.3.10. *Let R, S be rings such that R is prime noetherian. Let ${}_ST_R$ be an S-R-bimodule which is finitely generated as a left S-module and torsion as a right R-module. Then there exists a regular element c of R such that $Tc = 0$.*

Proof. Let $T = \sum_{i=1}^{m} St_i$ where $t_ic_i = 0$ for some $c_i \in \mathcal{C}_R(0)$. By Goldie's Theorem $\mathcal{C}_R(0)$ is an Ore set in R, so there exist $y_1 \in \mathcal{C}(0)$ and $r_2 \in R$ such that $c_1y_1 = c_2r_2$. Then taking $d_1 = c_1y_1$ gives $t_1d_1 = 0 = t_2d_1 = 0$ and $d_1 \in \mathcal{C}_R(0)$. In the same way we find $d_2 \in \mathcal{C}_R(0)$ such that $t_1d_2 = t_2d_2 = t_3d_2 = 0$. Iterating the process gives $c = d_{m-1}$ as required. □

We now give the proof of Lemma 1.3.9.

Proof. If P is maximal then we can use Corollary 1.3.7. Thus we can assume that P is not maximal.

Let x be any element in $Z \backslash Z \cap P$ whose image is not a unit in R/P. Consider the following short exact sequence:

$$0 \to R/P \xrightarrow{\times x} R/P \to R/I \to 0,$$

where $I := xR + P$.

Let $i := j(R/P)$. Since $P \subsetneq I$ we have $\mathrm{Krull\ dim}(R/I) < \mathrm{Krull\ dim}(R/P)$ by [31, Ex 15F] so by (1.4), $j(R/I) > j(R/P) = i$. Thus, from the long exact sequence we have

$$0 = \mathrm{Ext}^i(R/I, R) \to \mathrm{Ext}^i(R/P, R) \xrightarrow{x^*} \mathrm{Ext}^i(R/P, R)\ , \tag{1.5}$$

where by [10, Lemma 2.1(ii)] x^* is multiplication by x. Clearly, the map x^* is an injection.

Let $E := \mathrm{Ext}^i(R/P, R)$. Then by [10, Lemma 2.1(i)] E is finitely generated as a left and right R-module. We assume for a contradiction that E is right R/P-torsion. Then E is $Z/P \cap Z$-torsion so by Lemma 1.3.10 we have some $0 \neq x \in \mathcal{C}_Z(P)$ such that $Ex = 0$. Hence $P \cap Z \subsetneq \mathrm{Ann}_Z(E)$ which is a two sided ideal of Z. Now $x \in \mathrm{Ann}_R(E)$ cannot be a unit in R and thus the map $x^* : E \to E$ is injective by (1.5). But this is a contradiction since $Ex = 0$. So $\mathrm{Ext}^i(R/P, R)$ cannot be R/P-torsion. □

Lemma 1.3.11. *Let R be a noetherian ring which is a finite module over a central subring C, such that all finitely generated R-modules satisfy (1.4). Then for a prime ideal P with $\mathfrak{p} = P \cap C$ we have*

$$j^r_R(R/P) = j^r_{R_\mathfrak{p}}(R_\mathfrak{p}/P_\mathfrak{p}) \text{ and } j^l_R(R/P) = j^l_{R_\mathfrak{p}}(R_\mathfrak{p}/P_\mathfrak{p}).$$

Proof. Apply [19, Prop 1.6] to get

$$\mathrm{Ext}^i_R(R/P, R) \otimes_R R_\mathfrak{p} \cong \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/P_\mathfrak{p}, R_\mathfrak{p}).$$

Hence $j^r_R(R/P) \leq j^r_{R_\mathfrak{p}}(R_\mathfrak{p}/P_\mathfrak{p})$. Now by Lemma 1.3.9 we know that $\mathrm{Ext}^{j(R/P)}(R/P, R)$ is not torsion as a right R/P-module giving $\mathrm{Ext}^{j(R/P)}_R(R/P, R) \otimes_R R_\mathfrak{p} \neq 0$. Hence $j^r_R(R/P) = j^r_{R_\mathfrak{p}}(R_\mathfrak{p}/P_\mathfrak{p})$. The second statement follows from considering R/P as a left module. □

Lemma 1.3.12. *Let R be a noetherian ring which is a finite module over a central subring C. If $R_\mathfrak{m}$ is grade-symmetric for all maximal ideals $\mathfrak{m}$ of C then R is grade-symmetric*

Proof. Suppose that $R_\mathfrak{m}$ is grade-symmetric for all maximal ideals $\mathfrak{m}$ of C. Let X be a finitely generated central R-R-bimodule. Then by Lemma 1.2.11 there exists some maximal ideal $M \in Supp(X)$ such that $X_m \neq 0$, for $\mathfrak{m} = M \cap C$. By hypothesis, $j^l_{R_\mathfrak{m}}(X_\mathfrak{m}) = j^r_{R_\mathfrak{m}}(X_\mathfrak{m})$. Thus

$$j^l_R(X) = j^l_{R_\mathfrak{m}}(X_\mathfrak{m}) = j^r_{R_\mathfrak{m}}(X_\mathfrak{m}) = j^r_R(X).$$

□

We are as yet unsure as to whether or not the converse holds in general.

Lemma 1.3.13. *Let R satisfy hypothesis (H) and P be a prime ideal of R that is not maximal. Let $0 \neq x \in Z\backslash(P \cap Z)$ which is not a unit modulo P and consider the short exact sequence*

$$0 \to R/P \xrightarrow{x\times} R/P \to R/I \to 0$$

where $I = xR + P$. Then
(i) $\mathrm{Ext}^{j(R/I)-1}(R/P, R) \neq 0$.
(ii) if x is in the Jacobson radical of R then $j(R/P) = j(R/I) - 1$.

Proof. (i) Let $j(R/I) = t$, so that we want to show that $\mathrm{Ext}^{t-1}(R/P, R) \neq 0$. We obtain the long exact sequence

$$\cdots \to \mathrm{Ext}^i(R/P, R) \xrightarrow{x\times} \mathrm{Ext}^i(R/P, R) \to \mathrm{Ext}^{i+1}(R/I, R) \to \mathrm{Ext}^{i+1}(R/P, R) \to \cdots.$$

Consider $\mathrm{Ext}^{t-1}(R/P, R)$. Suppose for a contradiction that this is zero and consider the following part of the long exact sequence:

$$0 = \mathrm{Ext}^{t-1}(R/P, R) \to \mathrm{Ext}^{t}(R/I, R) \to \mathrm{Ext}^{t}(R/P, R) \to \mathrm{Ext}^{t}(R/P, R) \qquad (1.6)$$

where $\mathrm{Ext}^{t}(R/I, R) \neq 0$ since $j(R/I) = t$. Then the non-zero submodule $\mathrm{Ext}^{t}(R/I, R)$ embeds in $\mathrm{Ext}^{t}(R/P, R)$ and so is isomorphic to a submodule of $\mathrm{Ext}^{t}(R/P, R)$. If this embedding was an isomorphism then the last term in (1.6) would be zero, which is impossible. Thus $\mathrm{Ext}^{t}(R/I, R) \subsetneq \mathrm{Ext}^{t}(R/P, R)$. Note that we can also take x^2, x^3 and so on instead of x, giving the short exact sequence

$$0 \to R/P \xrightarrow{x^j \times} R/P \to R/I_j \to 0$$

where $I_j := P + x^j R$ for $j = 1, 2, \ldots$. Hence we have $\mathrm{Ext}^{t}(R/I_j, R)$ isomorphic to a proper submodule of $\mathrm{Ext}^{t}(R/P, R)$ for all $j > 0$.

Now notice that $j(R/I_j) = j(R/I) = t$. We see this by induction on j. First of all note that $I_j/I_{j+1} \cong R/I$ as R-R-bimodules for all $j \geq 1$. We see this by defining the isomorphism

$$\Theta : I_j/I_{j+1} \to R/I$$
$$\overline{x^j r} \to \bar{r}.$$

For $j = 2$ we have the short exact sequence

$$0 \to I/I_2 \to R/I_2 \to R/I,$$

where $I/I_2 \cong R/I$, giving $j(R/I_2) = j(R/I)$. Then for higher values of j we have

$$0 \to I_{j-1}/I_j \to R/I_j \to R/I_{j-1} \to 0$$

so assuming by induction that $j(R/I_{j-1}) = j(R/I)$ we deduce that $j(R/I_j) = j(R/I)$.

Furthermore we can take the short exact sequence

$$0 \to R/I \xrightarrow{x^{j-1} \times} R/I_j \to R/I_{j-1} \to 0,$$

for $j \geq 2$, giving

$$\mathrm{Ext}^{t-1}(R/I, R) \to \mathrm{Ext}^{t}(R/I_{j-1}, R) \to \mathrm{Ext}^{t}(R/I_j, R) \to \mathrm{Ext}^{t}(R/I, R) \to \cdots . \qquad (1.7)$$

Now $j(R/I) = t$ so $\mathrm{Ext}^{t-1}(R/I, R) = 0$ and thus $\mathrm{Ext}^{t}(R/I_{j-1}, R)$ is isomorphic to a submodule of $\mathrm{Ext}^{t}(R/I_j, R)$. As before, we must have $\mathrm{Ext}^{t}(R/I_{j-1}, R) \subsetneq \mathrm{Ext}^{t}(R/I_j, R)$

as otherwise (1.7) would give $\mathrm{Ext}^t(R/I, R) = 0$. Hence we get the following ascending chain

$$E_1 \subsetneq E_2 \subsetneq E_3 \subsetneq \cdots \subsetneq \mathrm{Ext}^{i+1}(R/P, R),$$

where $E_j \cong \mathrm{Ext}^t(R/I_j, R)$. But this contradicts the fact that $\mathrm{Ext}^{i+1}(R/P, R)$ and all its submodules are finitely generated. Hence we must have $\mathrm{Ext}^{t-1}(R/P, R) \neq 0$ as required.
(ii) Suppose now that $x \in J(R)$. If we consider our long exact sequence

$$\cdots \to \mathrm{Ext}^i(R/I, R) \to \mathrm{Ext}^i(R/P, R) \to \mathrm{Ext}^i(R/P, R) \to \mathrm{Ext}^{i+1}(R/I, R) \to \cdots$$

with $i < t-1$ then $\mathrm{Ext}^i(R/I, R) = \mathrm{Ext}^{i+1}(R/I, R) = 0$ and for $E^i := \mathrm{Ext}^i(R/P, R)$ we get $E^i = xE^i$. But since $x \in J(R)$ Nakayama's lemma shows that $E^i = 0$. Thus $j(R/P) \geq t-1$. Together with (i) this gives $j(R/P) = t-1$. □

We finish this section with some well known theorems and corollaries on projective and injective dimension.

Theorem 1.3.14. *[35, Theorem 205] Let R be a ring and x a central element in R. Let $R^* = R/\langle x \rangle$. Let A be an R-module and suppose x is a non-zero-divisor on both R and A. Then*

$$\mathrm{inj.\,dim}_{R^*}(A/xA) \leq \mathrm{inj.\,dim}_R(A) - 1$$

except when A is R-injective (in which case $A = xA$).

Corollary 1.3.15. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring. Then* $\mathrm{inj.\,dim}(R) = n$.

Proof. First note that $\mathrm{inj.\,dim}(R) \leq \mathrm{gl.\,dim}(R)$ which equals n by Hilbert's syzygy Theorem [48, Corollary 9.35]. On the other hand Theorem 1.3.14 shows that $\mathrm{inj.\,dim}(R) \geq \mathrm{inj.\,dim}(k) + n = n$. □

Theorem 1.3.16. *[35, Theorem C] Let R be a ring, x a central element of R which is a non zero divisor. Let $R^* := R/\langle x \rangle$. Let A be a non-zero R^*-module with $\mathrm{pd}_{R^*}(A) = n < \infty$. Then $\mathrm{pd}_R(A) = n + 1$.*

Theorem 1.3.17. *[35, Theorem E] Let R be a left noetherian ring, x a central element in the Jacobson radical of R. Let $R^* := R/\langle x \rangle$ and let A be a finitely generated R-module. Assume x is a non zero divisor on both R and A. Then*

$$\mathrm{pd}_{R^*}(A/xA) = \mathrm{pd}_R(A).$$

1.4 Dimension functions

Definition 1.4.1. Let R be a noetherian ring. A *dimension function* γ for R is a function which assigns a value $\gamma(X)$ in the set of all ordinals including 0 together with $\pm\infty$ to each finitely generated module X satisfying the following properties:
(i) $\gamma(0) = -\infty$;
(ii) If $0 \to X' \to X \to X'' \to 0$ is exact then $\gamma(X) \geq \sup\{\gamma(X'), \gamma(X'')\}$ with equality if the sequence is split;
(iii) If XP=0 for some prime P of R and X is a torsion R/P-module then $\gamma(X) + 1 \leq \gamma(R/P)$.
The dimension function is *exact* if equality always holds in condition (ii).

Well known examples of dimension functions are Krull dimension and Gelfand-Kirillov dimension. The latter is often abbreviated to GK-dimension. We noted at the start of section 1.2 that given our hypothesis (H), we can take the Krull dimension of a ring to be the maximal length of a chain of prime ideals of R and Krull $\dim(X) =$ Krull $\dim(R/\operatorname{Ann}_R(X))$ for a finitely generated R-module X. That Krull dimension is an exact dimension function is observed in [41, 6.8.5].

We refer the reader to [41, Chapter 8] for the definition of Gelfand-Kirillov dimension, which is defined for finite dimensional algebras over a field. For affine algebras satisfying (H), Gelfand-Kirillov dimension is equivalent to Krull dimension as noted in [13, I.15.4].

Definition 1.4.2. A dimension function is *symmetric* if $\gamma({}_RX) = \gamma(X_R)$ for all R-bimodules X.

We now focus on the dimension function Krull dimension.

Proposition 1.4.3. *[41, Corollary 6.4.13] For an FBN ring, R, Krull dimension is a symmetric dimension function. That is, for a bimodule X we have*

$$\text{Krull } \dim({}_RX) = \text{Krull } \dim(X_R).$$

Lemma 1.4.4. *[31, Proposition 15.4] Let R be a noetherian ring and X a finitely generated R-module. Then* Krull $\dim(X) \leq$ Krull $\dim(R)$.

The following two lemmas are proved for lack of a suitable reference.

Lemma 1.4.5. *Let R be a noetherian ring and $A \subseteq B \subseteq C$ finitely generated R-modules. Then* Krull $\dim(C/B) \leq$ Krull $\dim(C/A)$.

Proof. We consider the submodule B/A of C/A and apply [41, Lemma 6.2.4] to get that Krull $\dim(C/A) = \sup\{\text{Krull } \dim(B/A), \text{Krull } \dim(C/B)\}$. □

Lemma 1.4.6. *Let R be a noetherian ring, X a finitely generated R-module and A, B submodules of X. Then*

$$\text{Krull } \dim(X/A \cap B) = \max\{\text{Krull } \dim(X/A), \text{Krull } \dim(X, B)\}.$$

Proof. From the short exact sequence

$$0 \to A/A \cap B \to X/A \cap B \to X/A \to 0,$$

[41, Lemma 6.2.4] shows that Krull $\dim(X/A\cap B) = \max\{\text{Krull } \dim(A/A\cap B), \text{Krull } \dim(X/A)\}$. Similarly we can show that Krull $\dim(X/A\cap B) = \max\{\text{Krull } \dim(B/A\cap B), \text{Krull } \dim(X/B)\}$. But $A/A \cap B \cong A + B/B \subset X/B$ so that Krull $\dim(A/A \cap B) \leq$ Krull $\dim(X/B)$ and similarly Krull $\dim(B/A \cap B) \leq$ Krull $\dim(X/A)$. Thus Krull $\dim(X/A \cap B) = \max\{\text{Krull } \dim(X/A), \text{Krull } \dim(X, B)\}$. □

Definition 1.4.7. Let R be a ring with a dimension function γ. Then an R-module X is d-pure if every non-zero submodule Y of X has $\gamma(Y) = d$.

Lemma 1.4.8. *Let R be a noetherian ring, $\gamma(-)$ an exact dimension function and X a finitely generated right R-module with $\gamma(X) = d$. Let*

$$T := \sum\{A : A \subseteq X, \gamma(A) < \gamma(X)\}.$$

Then X/T is d-pure.

Proof. First $\gamma(T) < \gamma(X)$ by condition (ii) of Definition 1.4.1. Then $\gamma(X/T) = d$ by (ii) again. Let Y/T be a non-zero submodule of X/T. Then if $\gamma(Y/T) < d$, we must have $\gamma(Y) < d$. But this is impossible since $T \subsetneq Y$ but T is the largest submodule of X with γ less than d. Hence $\gamma(Y/T) = d$ and X/T is d-pure. □

We have been unable to find a reference for the following result, so a proof is given below.

Proposition 1.4.9. *Let R satisfy hypothesis (H) and X be a finitely generated d-pure right R-module with respect to Krull dimension. Then there exists a finite chain*

$$0 = X_0 \subset X_1 \subset \cdots \subset X_t = X \tag{1.8}$$

such that for all $1 \leq l \leq t$:

- $\operatorname{Ann}_r(X_l/X_{l-1}) =: P_l$ *a prime ideal of* R*, and* $\operatorname{Ann}(Y) = P_l$ *for all* $0 \neq Y \subseteq X_l/X_{l-1}$.

- Krull $\dim(R/P_l) =$ Krull $\dim(X_l/X_{l-1}) = d$

Proof. We suppose by Noetherian induction that the result holds for all proper d-pure factors of X. Let I be maximal among the annihilators of non-zero submodules of X. Let $I = \operatorname{Ann}_R(Y)$, for $0 \neq Y \subseteq X$. Then we can show that I is a prime ideal of R.

Let $0 \neq A = \{x \in X : xI = 0\}$. By definition of A, we have $I \subseteq \operatorname{Ann}_R(A)$ so the maximality of I gives $I = \operatorname{Ann}_R(A)$. Then since X is d-pure, we see that $d =$ Krull $\dim(A) =$ Krull $\dim(R/\operatorname{Ann}_R(A))$. Thus we have the submodule A of X with

- Krull $\dim(A) = d$, $\operatorname{Ann}_R(A) = I$ with Krull $\dim(R/I) = d$.

- If $0 \neq B \subseteq A$ then $I = \operatorname{Ann}_R(B)$.

We now show that it is enough to show that X/A is d-pure. So if this is the case then, by our induction hypothesis, we have a series like (1.8):

$$0 = A/A \subseteq X_1/A \subseteq \cdots \subseteq X_t/A = X/A,$$

where each $X_l/A/X_{l-1}/A \cong X_l/X_{l-1}$ has the required properties. This gives the series

$$0 \subseteq A \subseteq X_1 \subseteq \cdots \subseteq X_l = X$$

which we require.

It remains to show that X/A is d-pure. So suppose that there exists some $A \subsetneq \hat{A} \subseteq X$ with Krull $\dim(\hat{A}/A) = d' < d$. Then $\hat{A} = \sum_{i=1}^{m} b_j R$ for some $b_j \in \hat{A}$. Let $L_j := \{r \in R : b_j r \in A\}$. Then $R/L_j \cong b_j R + A/A$ so that Krull $\dim(R/L_j) =$ Krull $\dim(b_j R + A/A) \leq d' < d$. Let $L := \bigcap_{i=1}^{m} L_j$. Then Krull $\dim(R/L) \leq d' < d$ by Lemma 1.4.6. Now, $\hat{A}(L \cap Z) \subseteq A$ giving $\hat{A}I(L \cap Z) = \hat{A}(L \cap Z)I \subseteq AI = 0$. Thus $\hat{A}I$ is a $Z/L \cap Z$-module and

$$\text{Krull } \dim(\hat{A}I) \leq \text{Krull } \dim(Z/L \cap Z) = \text{Krull } \dim(R/L) < d.$$

Note that Krull $\dim(Z/L \cap Z) =$ Krull $\dim(R/L)$ by Proposition 1.2.2 since R/L is a finitely generated $Z/(L \cap Z)$-module. Now $\hat{A}I \subseteq X$ has Krull dimension less than d, so it must be zero as X is d-pure. Then $\hat{A} \subseteq \operatorname{Ann}_X(I) = A$, so that $\hat{A} = A$, giving the contradiction we require. □

For the following two lemmas we briefly consider uniform dimension and Goldie rings. By [31, Corollary 5.18], for any noetherian module S, the injective hull $E(X)$ is a finite direct sum of t indecomposable submodules, where t is the *uniform dimension* of X. We refer the reader to [31, Chapter 5] for further details. A ring R is *right Goldie* if R_R has finite uniform dimension and R satisfies the ascending chain condition on right annihilators. In particular, it is clear from the definition that a noetherian ring is right Goldie.

Lemma 1.4.10. *[31, Ex 7K] Let R be a prime right Goldie ring with uniform dimension t. If X is a finitely generated torsion-free right R-module then X^t is isomorphic to an essential submodule of a finitely generated free module F.*

Lemma 1.4.11. *[31, Cor 7.26] Let R a prime right Goldie ring with uniform dimension t. If X is a finitely generated torsion-free right R-module then X^t has an essential finitely generated free submodule.*

We again return to equation (1.4), stated just before Lemma 1.3.9, where X denotes a finitely generated right R-module:

$$\text{Krull dim}(R) = \text{Krull dim}(X) + j(X).$$

Lemma 1.4.12. *Suppose the ring R satisfies hypothesis (H). Let X be a module with Krull dimension d. If (1.4) holds for all finitely generated modules Y with* Krull dim$(Y) < d$ *and also for all R/P, P prime, such that* Krull dim $R/P = d$ *then (1.4) holds for X.*

Proof. We first show that it will be enough to prove that (1.4) holds for all d-pure modules. By Lemma 1.4.8 X/T is d-pure where

$$T := \sum\{A : A \subseteq X, \text{Krull dim}(A) < \text{Krull dim}(X)\}.$$

We then have the short exact sequence

$$0 \to T \to X \to \overline{X} \to 0,$$

where $\overline{X} = X/T$ is d-pure. Suppose (1.4) holds for $\overline{X}$, that is

$$j(\overline{X}) = i := n - d \tag{1.9}$$

for $n = \text{Krull dim}(R)$. The short exact sequence above gives the following long exact sequence:

$$\cdots \to \text{Ext}^{j-1}(T, R) \to \text{Ext}^j(\overline{X}, R) \to \text{Ext}^j(X, R) \to \text{Ext}^j(T, R) \to \cdots.$$

We now show that $j(X) = i$. We have $j(X) \geq i$ since, for all $j < i$ we have the following part of the long exact sequence,

$$0 = \operatorname{Ext}^j(\overline{X}, R) \to \operatorname{Ext}^j(X, R) \to \operatorname{Ext}^j(T, R) = 0,$$

where the first equality holds by (1.9) and the second holds since Krull $\dim(T) < d$ and (1.4) applied to T gives $j(T) > n - d = i > j$. Then for $j = i$ we have

$$\cdots \operatorname{Ext}^{i-1}(T, R) \to \operatorname{Ext}^i(\overline{X}, R) \to \operatorname{Ext}^i(X, R) \cdots$$

where $\operatorname{Ext}^{i-1}(T, R) = 0$ since $j(T) > n - d = i$. Then since $\operatorname{Ext}^i(\overline{X}, R) \neq 0$ we get $\operatorname{Ext}^i(X, R) \neq 0$, and hence $j(X) = i$. Thus (1.4) holds for X if it holds for all d-pure modules.

To prove (1.4) for every d-pure module, we first show that it holds for the steps in the chain in Proposition 1.4.9. So consider a finitely generated R-module X with Krull dimension d and $\operatorname{Ann}(X) = P$, a prime ideal of R, such that for all non-zero $Y \subseteq X$ we have Krull $\dim(Y) = d$, $\operatorname{Ann}(Y) = P$ and Krull $\dim(R/P) = d$.

Now let $\overline{R} = R/P$. Then X is a torsion-free right $\overline{R}$-module since if we have $0 \neq x \in X$ such that $xc = 0$ for some $c \in \mathcal{C}(P)$ then Krull $\dim(R/(cR + P)) <$ Krull $\dim(R/P) = d$ by [41, Lemma 6.3.9]. Now $xR \cong R/\operatorname{Ann}_R(x)$ as right modules so that Krull $\dim(xR) =$ Krull $\dim(R/\operatorname{Ann}_R(x))$ and since $cR + P \subseteq \operatorname{Ann}_R(x)$ we see from Lemma 1.4.5 that Krull $\dim(R/\operatorname{Ann}_R(x)) \leq$ Krull $\dim(R/cR+P)$. Thus Krull $\dim(xR) \leq$ Krull $\dim(R/cR+P) < d =$ Krull $\dim(X)$ which is a contradiction since X is d-pure.

By Lemma 1.4.10 we have a finitely generated free $\overline{R}$-module F with an essential submodule isomorphic to X^t, where t is the uniform dimension of R/P. Hence we have the short exact sequence

$$0 \to X^t \to F \to T \to 0$$

where T is finitely generated and a torsion $\overline{R}$-module.

Since $T = \sum_i t_i\overline{R}$ is torsion there exist $c_i \in \mathcal{C}(P)$ such that $t_ic_i \in P$. Thus $c_iR + P \subseteq \operatorname{Ann}(t_i)$ and so Krull $\dim(t_iR) \leq$ Krull $\dim(R/c_iR + P) <$ Krull $\dim(R/P)$. Hence Krull $\dim(T) <$ Krull $\dim(R/P) = d$, and since (1.4) holds for T by our induction hypothesis, $j(T) > n - d$. Also since (1.4) holds for $X \cong R/P$ by hypothesis for the right R-module R/P we see from Proposition 1.3.4 that $j(F) = n - d$.

Now consider the long exact sequence

$$\cdots \to \operatorname{Ext}^j(T, R) \to \operatorname{Ext}^j(F, R) \to \operatorname{Ext}^j(X^t, R) \to \operatorname{Ext}^{j+1}(T, R) \to \cdots.$$

If $j < n-d$ then $\mathrm{Ext}^j(F,R) = 0 = \mathrm{Ext}^{j+1}(T,R)$, so that $\mathrm{Ext}^j(X^t,R) = 0$ and hence $j(X^t) \geq n-d$. And if $j = n-d$ then $\mathrm{Ext}^j(T,R) = 0$ and $\mathrm{Ext}^j(F,R) \neq 0$ so that $\mathrm{Ext}^j(X^t,R) \neq 0$. Hence $j(X^t) = n-d$ so by Proposition 1.3.4 again we have $j(X) = n-d$. Hence (1.4) holds for each step in the chain.

Now use induction to show that (1.4) holds for an arbitrary d-pure module C. We have the chain

$$0 = X_0 \subset X_1 \subset \cdots \subset X_t = X$$

and know that (1.4) holds for each factor X_l/X_{l-1}, $1 \leq l \leq t$, so $j(X_l/X_{l-1}) = n-d$. We want to show that $j(X) = n-d$. We have

$$0 \to X_1 \to X_2 \to X_2/X_1 \to 0$$

with $j(X_1) = j(X_2/X_1) = n-d$ so Proposition 1.3.4 shows that $j(X_2) = n-d$. By induction we can then show in the same way that $j(X) = n-d$ as required. □

Proposition 1.4.13. *Suppose the ring R satisfies hypothesis (H). To prove equation (1.4) for all finitely generated R-modules X it is enough to prove it for all modules of the form R/P where P is a prime ideal of R.*

Proof. Suppose that (1.4) is known for all modules of the form R/P where P is a prime ideal of R. We want to show (1.4) for all finitely generated (left) R-modules X and do this by induction on the Krull dimension d of X.

$d = 0$: If $d = 0$ then X has finite length, so we want to show that $j(X) = \text{Krull } \dim(R) = n$ for all finite length modules X. To do this induct on the composition length c of X. For the case $c = 1$, we just need to show that (1.4) holds for simple modules, but this is true because, by Proposition 1.2.12, a simple module is a direct summand of R/M for some maximal ideal M and we know that (1.4) holds for R/M. Let

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_c = X.$$

Then we can obtain the short exact sequence

$$0 \to A \to X \to B \to 0$$

where A, B have composition series of shorter length than c (i.e take $A = X_1, B = X/X_1$). Then from the long exact sequence

$$\cdots \to \mathrm{Ext}^i(B,R) \to \mathrm{Ext}^i(X,R) \to \mathrm{Ext}^i(A,R) \to \cdots ,$$

$i < n$ gives $\mathrm{Ext}^i(B, R) = 0$ and $\mathrm{Ext}^i(A, R) = 0$, so $\mathrm{Ext}^i(X, R) = 0$ and $i = n$ gives a monomorphism from $0 \neq \mathrm{Ext}^i(B, R) \hookrightarrow \mathrm{Ext}^i(X, R)$. Hence $j(X) = n$.

The induction step is Lemma 1.4.12. □

1.5 Polynomial identity rings

Here we introduce polynomial identity rings, trace rings and Azumaya algebras.

Definition 1.5.1. A ring R is a *polynomial identity ring* or a *PI ring* if R satisfies a monic polynomial $f \in \mathbb{Z}\langle x_1, \dots, x_m \rangle$. That is, $f(r_1, \dots, r_m) = 0$ for all $r_1, \dots, r_m \in R$. The *minimal degree* of R is the least degree of a monic polynomial identity of R.

We note that if R is a finite module over its centre then [41, Corollary 13.1.13] shows that R is a PI ring.

Let R be a prime ring which is a finite module over its centre Z. Let $Q := Q(R)$ be the quotient ring of R, which by Proposition 1.2.17 is $R[Z\backslash 0]^{-1} = RQ(Z)$. By Posner's Theorem, Q is a central simple algebra over its centre, K and Lemma 1.2.18 shows that $K = Q(Z)$.

Definition 1.5.2. We define the *PI degree* of R to be m, where $\dim_K(Q) = m^2$.

Each element $r \in R$ acts by left multiplication on Q giving a linear transformation $\lambda(r)$ of Q over K. This transformation has a characteristic polynomial, say $\chi_r(X)$ with coefficients in K. Then

$$\chi_r(X) = X^m - \mathrm{Tr}(r)X^{m-1} + \cdots + (-1)^m N(r),$$

where $\mathrm{Tr}(r)$ is the *trace map* and $N(r)$ is the *norm map* of r. See [47, Chapter 9] for more details on this.

For a central simple algebra A with centre K we also have a reduced trace map, which we now define. As in [47, Chapter 9] we see that there exists an extension field E of K with an isomorphism

$$h : E \otimes_K A \cong M_n(E)$$

where $n^2 := \dim_K A$. Then for $a \in A$ the *reduced characteristic polynomial* of a is the characteristic polynomial of $h(1 \otimes a)$. By [47, Theorem 9.3] the characteristic polynomial is independent of the choice of the splitting field E.

Definition 1.5.3. For a central simple algebra A we define the *reduced trace map* $\mathrm{tr} : A \to K$ to be the trace of the reduced characteristic polynomial. That $\mathrm{Im}(\mathrm{tr})$ is indeed K is shown by [47, Theorem 9.3].

Note that if R is prime, we can take A to be the central simple algebra $Q(R)$ and an element of R can be considered as an element of $Q(R)$. This gives the map $\mathrm{tr} : R \to K$. By [47, 9.8] tr is a $Z(Q)$-module homomorphism and hence a Z-module homomorphism and by [47, 9.7] we have $\mathrm{Tr}(r) = m.\,\mathrm{tr}(r)$ for $r \in R$.

Definition 1.5.4. Let R be a prime PI ring with centre Z. Let T be the subring of $Z(Q)$ generated over Z by all the coefficients of $\chi_r(X)$, letting r vary throughout R. Then the *trace ring*, $T(R) := TR$. Thus

$$R \subseteq T(R) \subseteq Q(R).$$

Proposition 1.5.5. *Let R be a prime noetherian ring which is a finite module over its centre Z. Suppose that the PI degree of R is invertible in R and that $T(R) = R$. Then,*
(i) the image of the reduced trace map on R is Z.
(ii) Z is a direct summand of R as a Z-module.

Proof. (i) The trace ring is generated by all the coefficients of the charactaristic polynomials of $\lambda(r)$ where $r \in R$. But $\mathrm{Tr}(r) = m.\,\mathrm{tr}(r)$ is the coefficient of X^{m-1} so that (since m is a unit in R) for all $r \in R$ we can see that $\mathrm{tr}(r)$ is in the trace ring, that is, $\mathrm{tr}(r) \in R$ by hypothesis. Thus $\mathrm{Im}(\mathrm{tr}) \subseteq Z(Q) \cap R = Z$. On the other hand, we note that $m = \mathrm{Tr}(1) = m\,\mathrm{tr}(1)$ so that $1 = \mathrm{tr}(1) \in \mathrm{Im}(\mathrm{tr})$. Then since tr is a Z-module homomorphism we have $\mathrm{Im}(\mathrm{tr}) = Z$.
(ii) By (i) we have the following short exact sequence of Z-modules,

$$0 \to \ker \mathrm{tr} \to R \to Z \to 0$$

which splits since Z is a projective Z-module. Thus $R = Z \oplus \ker \mathrm{tr}$. □

Proposition 1.5.6. *Let R be a prime noetherian ring which is a finite module over its centre Z. If Z is integrally closed then $T(R) = R$.*

Proof. By [41, Proposition 13.9.11(i)], TR is a finitely generated R-module and hence a finitely generated Z-module. Thus T is a finitely generated Z-module, so that $T = \sum t_i Z$ for some $t_i \in T \subseteq Q(Z)$. Let $t_i = z_i c_i^{-1}$ for some $z_i, c_i \in Z$. Then there exists some $c \in Z$ such that $t_i = z_i' c^{-1}$ for some $z_i' \in Z$. Then $T \cong Tc \subseteq Z$ so that T is integral over

Z. Since Z is integrally closed, T must be contained in Z. Then $T(R) = TR \subseteq R$, so $T(R) = R$. □

We now come to Azumaya algebras and we refer the reader to [13, III.1] for further details and proofs.

Definition 1.5.7. A ring R is an *Azumaya algebra over its centre* Z if
(i) R is a finitely generated projective Z-module; and
(ii) The ring homomorphism

$$\Theta : R \otimes_Z R^{op} \to \text{End}_Z(R)$$

$$a \otimes b \mapsto (x \mapsto axb)$$

is an isomorphism.

Example 1.5.8. If Z is a commutative ring and $n \geq 1$ then $M_n(Z)$ ia an Azumaya algebra over Z.

Definition 1.5.9. A prime ideal of a PI ring R is *regular* if the PI degree of R/P equals the PI degree of R.

Theorem 1.5.10. *Let R be a prime affine algebra over a field k and a finite module over a central subalgebra Z. The following are equivalent:*
(i) R is an Azumaya algebra of rank m^2 over Z;
(ii) R is a PI ring of PI degree m whose prime ideals are all regular;
(iii) R is a PI ring of PI degree m whose maximal ideals are all regular;
(iv) R is a PI ring of PI degree m whose simple modules all have k-dimension m.

Proposition 1.5.11. *R is Azumaya over Z if and only if $R_\mathfrak{m}$ is Azumaya over $Z_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of Z.*

We now note a connection between the Azumaya property and trace rings.

Proposition 1.5.12. *Suppose that $R_\mathfrak{q}$ is Azumaya for all height one primes of $Z(R)$ and $R = \cap\{R_\mathfrak{q} : \mathfrak{q}$ a height 1 prime of $Z(R)\}$. Then $T(R) = R$.*

Proof. We know from [41, Proposition 13.9.8(ii)] that $T(R_\mathfrak{q}) = R_\mathfrak{q}$, so that

$$T(R) \subseteq \cap_\mathfrak{q} T(R_\mathfrak{q}) = \cap_\mathfrak{q} R_\mathfrak{q} = R.$$

□

1.6 Invertible modules

We now consider invertible modules and give some basic results which we will use in later chapters.

Definition 1.6.1. Let R, S be rings. A bimodule ${}_RI_S$ is *invertible* if there exists a bimodule ${}_SJ_R$ and a Morita context giving a Morita equivalence of R and S. In other words, there exist bimodule isomorphisms

$$I \otimes_S J \cong R \text{ and } J \otimes_R I \cong S$$

making the following diagrams commute:

$$\begin{CD} I \otimes_S J \otimes_R I @>>> R \otimes_R I \\ @VVV @VVV \\ I \otimes_S S @>>> I \end{CD} \quad \text{and} \quad \begin{CD} J \otimes_R I \otimes_S J @>>> S \otimes_S J \\ @VVV @VVV \\ J \otimes_R R @>>> J \end{CD} .$$

We call J the inverse of I and denote it I^{-1}.

It is in fact enough to have

$$I \otimes_S J \cong R \text{ and } J \otimes_R I \cong S,$$

respectively as R-R- and S-S-bimodules. To see this, we use the following proposition, which is [27, Proposition 12.13].

Proposition 1.6.2. *The following are equivalent for an R-S-bimodule I and an S-R-bimodule J:*

(a) The functor $- \otimes_R I : \text{mod-}R \to \text{mod-}S$ *is an equivalence;*

(b) $I \otimes_S J \cong R$ *and* $J \otimes_R I \cong S$;

(c) The functor $I \otimes_S - : S\text{-mod} \to R\text{-mod}$ *is an equivalence.*

Then by [27, Theorem 12.12] we can always choose isomorphisms making the above diagrams commute.

Also, [47, Theorem 16.14] shows that $I^{-1} = J \cong \mathrm{Hom}_R({}_RI, {}_RR) \cong \mathrm{Hom}_S(I_S, S_S)$ and $J^{-1} = I \cong \mathrm{Hom}_R(J_R, R_R) \cong \mathrm{Hom}_S({}_SJ, {}_SS)$.

Definition 1.6.3. We say that an R-module X is a *generator* if there exists some $l \in \mathbb{N}$ such that $X^l \twoheadrightarrow R$. Also, X is a *progenerator* if it is projective and a generator.

To show modules are invertible we will sometimes use the following theorem:

Theorem 1.6.4. *Let R be any ring and I a finitely generated central R-bimodule. Then*

$$\begin{aligned} {}_RI_R \text{ is invertible} &\Leftrightarrow I_R \text{ is a progenerator and } \operatorname{End}_R(I_R) \cong R \\ &\Leftrightarrow {}_RI \text{ is a progenerator and } \operatorname{End}_R({}_RI) \cong R, \end{aligned}$$

where the isomorphism is a ring isomorphism.

We prove this theorem using the following two results:

Theorem 1.6.5. *[47, Corollary 16.9]. Let I be a nonzero right R-module and let $S = \operatorname{End}_R(I)$. Let*

$$\begin{aligned} \mu : I \otimes_R \operatorname{Hom}_R(I, R) &\to S \\ a \otimes f &\mapsto (a' \mapsto a.f(a')) \end{aligned}$$

and

$$\begin{aligned} \tau : \operatorname{Hom}_R(I, R) \otimes_S I &\to R \\ f \otimes a &\mapsto f(a). \end{aligned}$$

Then I_R is a progenerator if and only if both μ and τ are isomorphisms. If I_R is a progenerator then the Morita context derived from I gives a Morita equivalence between the rings R and S.

Theorem 1.6.6. *[47, Theorem 16.14] Let the rings R and S be Morita equivalent relative to a Morita context ${}_SI_R$, ${}_RJ_S$. Then*
(i) I is a progenerator in the categories of left S-modules and right R-modules and J is a progenerator in the categories of left R-modules and right S-modules.
(ii) There are bimodule isomorphisms

$$J \cong \operatorname{Hom}_R(I, R) \cong \operatorname{Hom}_S(I, S)$$

$$I \cong \operatorname{Hom}_S(J, S) \cong \operatorname{Hom}_R(J, R).$$

(iii) There are ring isomorphisms

$$S \cong \operatorname{End}_R(I) \cong \operatorname{End}_R(J)$$

$$R \cong \operatorname{End}_S(I) \cong \operatorname{End}_S(J).$$

We can now prove Theorem 1.6.4.

Proof. We prove that ${}_RI_R$ is invertible if and only if I_R is a progenerator and $\mathrm{End}_R(I_R) \cong R$, as the other equivalence is similar.

$\Rightarrow$: Suppose ${}_RI_R$ is invertible. Then there exists a bimodule ${}_RJ_R \cong \mathrm{Hom}_R(I, R)$ and a morita context such that the maps $I \otimes_R J \to R$ and $J \otimes_R I \to R$ are isomorphisms. Thus Theorem 1.6.6(iii) gives

$$R \cong \mathrm{End}_R(I)$$

and I_R is a progenerator by Theorem 1.6.6(i).

$\Leftarrow$: Suppose that I_R is a progenerator and $\mathrm{End}_R(I_R) \cong R$. Then by Theorem 1.6.5

$$I \otimes \mathrm{Hom}_R(I, R) \cong \mathrm{End}_R(I) \cong R$$

and

$$\mathrm{Hom}_R(I, R) \otimes I \cong R,$$

so taking $J = \mathrm{Hom}_R(I, R)$ we get the invertibility of ${}_RI_R$. □

Definition 1.6.7. We define the *Picard group*, $\mathrm{Pic}(R)$ to be the group of isomorphism classes of invertible R-R-bimodules, with operation $[I][J] = [I \otimes_R J]$.

We are interested in invertible central R-R-bimodules. That is, for all $a \in I$ and all $z \in Z = Z(R)$ we have $a.z = z.a$. The subgroup of $\mathrm{Pic}(R)$, consisting of central invertible bimodules is called $\mathrm{Picent}(R)$. We now assume hypothesis (H).

Proposition 1.6.8. *Let R be a ring which is a finite module over a central subring C. Let I be an invertible central R-R-bimodule. Then $I_{\mathfrak{m}}$ is an invertible central $R_{\mathfrak{m}}$-$R_{\mathfrak{m}}$-bimodule for any maximal ideal $\mathfrak{m}$ of C.*

Proof. Since I is invertible we have $I \otimes_R J \cong R$ for some R-R-bimodule J. Then we tensor with $R_{\mathfrak{m}}$ to get

$$R_{\mathfrak{m}} \cong R_{\mathfrak{m}} \otimes_R I \otimes_R J \cong I \otimes_R R_{\mathfrak{m}} \otimes_R J \cong (I \otimes R_{\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} (R_{\mathfrak{m}} \otimes J).$$

Similarly we can show that $R_{\mathfrak{m}} \cong (J \otimes R_{\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} (R_{\mathfrak{m}} \otimes I)$, and thus Proposition 1.6.2 and the remark following it apply. □

Now we assume that R is prime and show that if an invertible central R-R-bimodule I embeds into the quotient ring then it can be considered as an ideal of R.

Lemma 1.6.9. *Let R be a prime noetherian ring which is a finite module over a central subring C. Let I be an invertible R-submodule of $Q := Q(R)$. Then I is isomorphic to an ideal of R.*

Proof. Since I is finitely generated there exist $q_1, \dots, q_n \in I \subseteq Q$ such that

$$I = \sum_{i=1}^{n} Cq_i.$$

There exists $c \in \mathcal{C}$ and $b_i \in R$ such that $q_i = b_j c^{-1}$ for all $1 \leq i \leq n$. So

$$I \cong Ic = \sum_{i=1}^{n} Cb_i \subseteq R,$$

the isomorphism being a bimodule isomorphism since c is central. Thus I is isomorphic to an ideal of R. □

Lemma 1.6.10. *Let R be a prime noetherian ring and I an ideal of R such that $I = aR = Rb$ for some $a, b \in R$. Then $I = Ra = aR$.*

Proof. This follows from the proof of [41, Lemma 5.2.8]. □

1.7 Frobenius and Symmetric algebras

Frobenius and Symmetric algebras have been studied since the early 1900s. Here we introduce these finite dimensional algebras and give some basic examples and properties, referring the reader to [57] for further details. We start with some preliminary definitions. Let A be an artinian ring. Then we can decompose A as

$$A = A_1 \oplus \cdots \oplus A_t,$$

where each A_i is an indecomposable artinian ring. Then taking f_i to be the identity of A_i we have $1_A = f_i + \cdots + f_t$, where the f_i are central *pairwise orthogonal primitive idempotents*. That is, $f_i f_j = 0$ for $i \neq j$, and each f_i cannot be written as the sum of two non-zero orthogonal central idempotents. We say that f_i and f_j are *isomorphic* if $Af_i \cong Af_j$. We can then choose representatives for the isomorphism classes to get non-isomorphic idempotents, which we denote by, $e_1, \dots, e_n$ for some $n \leq t$.

Definition 1.7.1. Let A be a finite dimensional k-algebra. A bilinear form $\beta : A \times A \to k$ is *non-degenerate* if for all $x \in A$ there exists some $y \in A$ such that $\beta(x, y) \neq 0$. The bilinear form is *associative* if $\beta(xy, z) = \beta(x, yz)$ for all $x, y, z \in A$ and is *symmetric* if $\beta(x, y) = \beta(y, x)$ for all $x, y \in A$.

Let A be a finite dimensional k algebra. Let $e := \sum_{i=1}^{n} e_i$ be the sum of the non-isomorphic orthogonal primitive idempotents in A. Then we give the following conditions on our algebra A.

Definition 1.7.2. A is *quasi-Frobenius* if the following equivalent conditions hold:
(i) A is injective as a left or right A-module;
(ii) $Ae \cong \mathrm{Hom}_k(eA, k)$ as left A-modules or $eA \cong \mathrm{Hom}_k(Ae, k)$ as right A-modules;
(iii) There's a permutation π on $\{1, \ldots, n\}$ such that $\mathrm{soc}(Ae_i) \cong \mathrm{top}(Ae_{\pi(i)})$;
(iv) There's a permutation π on $\{1, \ldots, n\}$ such that $\mathrm{soc}(e_iA) \cong \mathrm{top}(e_{\pi(i)}A)$;
(v) There's a permutation π on $\{1, \ldots, n\}$ such that ${}_AAe_i \cong {}_A\mathrm{Hom}(e_{\pi(i)}A, k)$;
(vi) There's a permutation π on $\{1, \ldots, n\}$ such that $e_iA_A \cong \mathrm{Hom}(Ae_{\pi(i)}, k)_A$.

Definition 1.7.3. A is *Frobenius* if the following equivalent conditions hold:
(i) $A \cong \mathrm{Hom}_k(A, k)$ as left or right A-modules;
(ii) $\mathrm{Hom}_k(A, k)$ is cyclic as a left or right A-module;
(iii) A has a non-degenerate associative bilinear form.

Definition 1.7.4. A is *symmetric* if the following equivalent conditions hold;
(i) $A \cong \mathrm{Hom}_k(A, k)$ as an A-bimodule;
(ii) A has a symmetric non-degenerate associative bilinear form.

We refer to [57] for why these conditions are equivalent. An important example is the group algebra of a finite group.

Example 1.7.5. Let k be a field and G a finite group. Then the group algebra kG is a symmetric Frobenius algebra.

Proof. We have the linear functional $\lambda : kG \to k$ where $\lambda(\sum_{g\in G} \alpha_g g) = \alpha_1$. This gives the non-degenerate bilinear form β where

$$\beta(x, y) = \lambda(xy)$$

for all $x, y \in kG$. We also note that β is symmetric. To see this, let $x = \sum_{g\in G} \alpha_g g$ and $y = \sum_{h\in G} \beta_h h$. Then $\lambda(xy) = \alpha_g \beta_{g^{-1}} = \lambda(yx)$. □

It is clear from the definitions that

$$\{\text{quasi-Frobenius algebras}\} \subseteq \{\text{Frobenius algebras}\} \subseteq \{\text{symmetric algebras}\}.$$

However, these inequalities are strict. In chapter 4 we will see an example which is quasi-Frobenius but not Frobenius (Example 5.2.8). To see that Frobenius algebras may not be symmetric, consider the following example.

Example 1.7.6. Let k be a field, and $0 \neq \alpha \in k$. Then let $A = k\langle x, y\rangle / \langle x^2, y^2, yx - \alpha xy\rangle$. Then A is Frobenius; it is symmetric if and only if $\alpha = 1$.

Proof. The set $\{1, x, y, xy\}$ is a k-basis for A and we define $\lambda : A \to k$ to be the linear map sending an element of A to the coefficient of xy. Then we define the bilinear form to be $\beta(a, b) = \lambda(ab)$ for all $a, b \in A$. To show that β is non-degenerate it is enough to show that for all a in the basis, we have some $b \in A$ such that $\lambda(ab) \neq 0$. But for $a = 1, x, y, xy$ we can take $b = xy, y, x, 1$ respectively. Also, β is clearly associative so A is Frobenius.

However, if $\alpha \neq 1$ then β is not symmetric, since

$$\beta(x, y) = \lambda(xy) = 1 \neq \alpha = \lambda(yx) = \beta(y, x).$$

□

If A is a Frobenius algebra then A has a Frobenius form β. From condition (i) of the definition we see that $A \cong \mathrm{Hom}_k(A, k)$ as left modules and as right modules. The isomorphisms are given by

$$x \mapsto \beta(-, x) \text{ and } x \mapsto \beta(x, -).$$

Thus, $\beta(-, x) = \beta(\hat{x}, -)$ for some unique $\hat{x} \in A$. We can then define $\nu : A \to A$ by $\nu(x) = \hat{x}$. To see that this is an algebra homomorphism, let $x, y \in A$. Then for any $a \in A$,

$$\beta(\nu(xy), a) = \beta(a, xy) = \beta(ax, y) = \beta(\nu(y), ax),$$

while

$$\beta(\nu(x)\nu(y), a) = \beta(\nu(x), \nu(y)a) = \beta(\nu(y)a, x) = \beta(\nu(y), ax).$$

Thus $\beta(\nu(xy), -) = \beta(\nu(x)\nu(y), -)$ so that $\nu(xy) = \nu(x)\nu(y)$.

Definition 1.7.7. The automorphism ν is called the *Nakayama automorphism* of A.

Note that the Nakayama automorphism is uniquely determined up to inner automorphism as shown by [57, Theorem 2.1.2]. We now explain some notation used in the following well-known result. Let X be a right R-module and let μ be an automorphism of X. Then we denote by X^μ the right R-module X with action $x.r = x\mu(r)$, for all $x \in X$ and $r \in R$. We make a similar definition for a left R-module.

Proposition 1.7.8. *Let A be a Frobenius algebra with Nakayama automorphism ν. Then* $\mathrm{Hom}_k(A, k) \cong {}^{\nu}A^1$ *as A-bimodules and A is symmetric if and only if ν is an inner automorphism.*

Proof. We know that $A \cong \mathrm{Hom}_k(A, k)$ as right modules, via the map $\theta : x \mapsto \beta(x, -)$. We show that θ is an A-A-bimodule homomorphism from ${}^{\nu}A^1$ to $\mathrm{Hom}_k(A, k)$. So let $a, b, x, y \in A$. Then

$$a.\theta(x).b(y) = \beta(x, -)(bya) = \beta(x, bya)$$

and

$$\theta(\nu(a)xb)(y) = \beta(\nu(a)xb, y) = \beta(\nu(a), xby) = \beta(xby, a) = \beta(x, bya).$$

Thus $a\theta(x)b = \theta(\nu(a)xb)$ and θ is an A-A-bimodule homomorphism.

If A is symmetric, then it is clear that ν is inner. For the converse, suppose that A has a non-degenerate bilinear form β with inner Nakayama automorphism ν. Then $\nu(x) = cxc^{-1}$ for some unit $c \in A$. We then define the bilinear form β' with $\beta'(x, y) = \beta(cx, y)$, which we claim is symmetric. To show this, let $x, y \in A$. Then

$$\beta'(x, y) = \beta(cx, y) = \beta(cyc^{-1}, cx) = \beta(cy, x) = \beta'(y, x).$$

Non-degeneracy and associativity follow from the non-degeneracy and associativity of β.

□

Proposition 1.7.9. *[24, Theorem 58.14] Let X be a finitely generated module over a quasi-Frobenius ring A. Then X is projective if and only if X is injective.*

We now prove an easy result which we will need in chapter 5. We remind the reader of the definition of a generator as in Definition 1.6.3. Note also the well known fact that the functor $(-)^* := \mathrm{Hom}_k(-, k)$ sends injectives to projectives and projectives to injectives.

Proposition 1.7.10. *A finite dimensional algebra A is quasi-Frobenius if and only if* $\mathrm{Hom}_k(A, k)$ *is projective if and only if* $\mathrm{Hom}_k(A, k)$ *is a progenerator.*

Proof. First suppose that A is quasi-Frobenius. Then since A is injective $\mathrm{Hom}_k(A, k) = A^*$ is projective. Now let

$$A = Ae_1 \oplus \cdots \oplus Ae_n$$

where $e_1, \ldots, e_n$ are the non-isomorphic primitive orthogonal idempotents, each occurring in the decomposition of A with multiplicity t_i. Then by condition (iv) of Definition 1.7.2

$$A^* = (Ae_1)^* \oplus \cdots \oplus (Ae_n)^* \cong e_{\pi(1)}A \oplus \cdots \oplus e_{\pi(n)}A.$$

Then taking $l = \sup\{t_i\}$ gives $(A^*)^l \twoheadrightarrow A$ so that A^* is a generator.

Conversely if A^* is projective it follows that $A = (A^*)^*$ is injective and hence A is quasi-Frobenius. □

Let A be a finite dimensional k-algebra with $e = \sum_{i=1}^{m} e_i$ where the e_i are the non-isomprphic primitive orthogonal idempotents. Then A is *basic* if $e = 1$.

Proposition 1.7.11. *A basic quasi-Frobenius algebra is Frobenius.*

Proof. This follows from condition (ii) of Definition 1.7.2 since $e = 1$. □

1.8 Some preliminary results

Here we give some preliminary results which are used in later chapters. We begin this section with Nakayama's lemma.

Lemma 1.8.1. *[41, Lemma 0.3.10] Let R be any ring and let $J(R)$ denote the Jacobson radical of R. If X is a non-zero finitely generated R-module then $X \neq XJ(R)$.*

Lemma 1.8.2. *[17, Prop 3.4(ii)] Let C be a central subring of a right noetherian ring R and X a non-zero finitely generated right R-module. If S is a subring of C consisting of zero divisors on X, then there exists a non-zero element $x \in X$ such that $xS = 0$.*

Lemma 1.8.3. *Let R be noetherian and X a right R-module. If J is an ideal of R and $0 \neq x \in X$ such that $xJ = 0$, then there exists a prime ideal Q, such that $J \subseteq Q$ and Q is maximal with respect to* $\mathrm{Ann}_X(Q) \neq 0$*.*

Proof. The ideal Q exists since R is noetherian and the set of ideals with non-zero annihilator in X is non-empty. Suppose that $A, B \lhd R$, $A \supseteq Q$, $B \supseteq Q$ and $AB \subseteq Q$. Let $\mathrm{Ann}_X(Q) = T$. Suppose $TA \neq 0$. Then $(TA)B \subseteq TQ = 0$, so $\mathrm{Ann}_X(B) \neq 0$. But $B \supseteq Q$ so $B = Q$. On the other hand, if $TA = 0$ then $A = Q$. □

The following is a noncommutative version of the principal ideal theorem, originally proved by Krull in 1928 for the commutative case.

Theorem 1.8.4. *[41, Thm 4.1.11] Let R be a right noetherian ring, a any normal element which is not a unit and Q a prime ideal of R minimal over aR. Then Q has height at most* 1*.*

Lemma 1.8.5. *Let R be a ring, M, N finitely generated right R-modules with M projective and x a central non zero divisor on both M and N. Consider the map,*

$$\theta : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{R/Rx}(M/Mx, N/Nx)$$

defined by $\theta(f)(m + Mx) = f(m) + Nx$.
(i) If M and N are central R-R-bimodules then $\mathrm{Hom}_R(M, N)$ becomes an R-R-bimodule with $r.f.s(m) = r.f(sm)$ for $r, s \in R, m \in M$ and $f \in \mathrm{Hom}_R(M, N)$ and

$$\mathrm{Hom}_{R/Rx}(M/Mx, N/Nx) \cong \mathrm{Hom}_R(M, N)/\mathrm{Hom}_R(M, N)x$$

as R-R-bimodules.
(ii) If $M = N$ then then $\mathrm{Hom}_R(M, M)$ is a ring and we have a ring isomorphism,

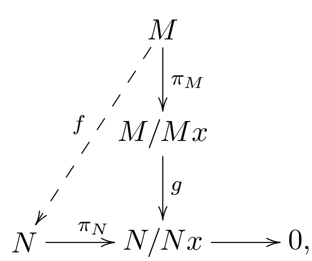
$$\mathrm{End}_{R/Rx}(M/Mx) \cong \mathrm{End}_R(M)/\rho_x \mathrm{End}_R(M),$$

where ρ_x is right multiplication by x.

Proof. (i) It is easy to see that $\theta(f)$ is well defined. Also, θ is a bimodule homomorphism since for $f : M \to N$, $r, s \in R$ and $\mathfrak{m} \in M$ we have

$$r.\theta(f).s(m+Mx) = r.\theta(f)(sm+Mx) = r.f(sm)+Nx = (r.f.s)(m)+Nx = \theta(r.f.s)(m+Mx).$$

For surjectivity, suppose that $g \in \mathrm{Hom}_{R/Rx}(M/Mx, N/Nx)$. Projectivity of M allows us to complete the following diagram.



where π_M and π_N are the obvious projection maps. Thus we have a right R-module homomorphism $f : M \to N$ such that $g(m + Mx) = f(m) + Nx = \theta(f)(m + Mx)$ as required.

Also, for $g \in \mathrm{Hom}_R(M, N)$ and $m \in M$, $gx(m) = g(xm) = g(mx) = g(m)x \in Nx$ since M is a central bimodule and g is a right homomorphism. Then,

$$\begin{aligned} \mathrm{Hom}_R(M, N)x &= \{g.x : g \in \mathrm{Hom}_R(M, N)\} \\ &= \{f \in \mathrm{Hom}_R(M, N) : f(m) \in Nx \ \forall \ m \in M\} \\ &= \ker \theta. \end{aligned}$$

So we now have $\operatorname{Hom}_{R/Rx}(M/Mx, N/Nx) \cong \operatorname{Hom}_R(M,N)/\operatorname{Hom}_R(M,N)x$.

(ii) Now suppose that $M = N$. We need to show that $\theta : \operatorname{End}_R(M) \to \operatorname{End}_{R/Rx}(M/Mx)$ is a ring homomorphism with kernel $\rho_x \operatorname{End}_R(M)$. So let $f,\ g \in \operatorname{End}_R(M)$ and $m \in M$. Then

$$\theta(f)\circ\theta(g)(m+Mx) = \theta(f)(g(m)+Mx) = f\circ g(m)+Mx = \theta(f\circ g)(m+Mx).$$

Also, since x is a non zero divisor on M, ρ_x is central and non-zero. Thus $\rho_x \operatorname{End}_R(M)$ is an ideal of $\operatorname{End}_R(M)$ and for $g \in \operatorname{End}_R(M)$ and $m \in M$ we have $\rho_x \circ g(m) = g(m)x \in Mx$. Thus

$$\rho_x \operatorname{End}_R(M) = \{f \in \operatorname{End}_R(M) : f(m) \in Mx \ \ \forall \ \ m \in M\} = \ker\theta.$$

□

Lemma 1.8.6. *Let R be a noetherian ring which is a finite module over a central subring C. Let X be a right R-module and M a maximal ideal in Supp(X). Then for $\mathfrak{m} = M \cap C$, $\operatorname{End}_R(X)\otimes_C C_{\mathfrak{m}} \cong \operatorname{End}_{R_{\mathfrak{m}}}(X\otimes_C C_{\mathfrak{m}})$ as rings.*

Proof. The map $\theta : \operatorname{End}_R(X)\otimes C_{\mathfrak{m}} \to \operatorname{End}_{R_{\mathfrak{m}}}(X\otimes C_{\mathfrak{m}})$ defined by $\theta(f\otimes x)(m\otimes a) = f(m)\otimes xa$ is a ring homomorphism since for $f, g \in \operatorname{End}_R(X)$, $x, y \in C_{\mathfrak{m}}$ and $m\otimes a \in X\otimes C_{\mathfrak{m}}$ we have

$$\begin{aligned} \theta[(f\otimes x)\circ(g\otimes y)](m\otimes a) &= \theta[f\circ g\otimes xy](m\otimes a) \\ &= f(g(m))\otimes xya \\ &= \theta(f\otimes x)(g(m)\otimes ya) \\ &= \theta(f\otimes x)\circ\theta(g\otimes y)(m\otimes a). \end{aligned}$$

It is easy to show that θ is an isomorphism. □

Lemma 1.8.7. *Let R be a noetherian ring which is a finite module over a central subring C. Then for any maximal ideal $\mathfrak{m}$ of C,*

$$\operatorname{Hom}_C(R,C)\otimes_C C_{\mathfrak{m}} \cong \operatorname{Hom}_{C_{\mathfrak{m}}}(R_{\mathfrak{m}}, C_{\mathfrak{m}})$$

as $R_{\mathfrak{m}}$-$R_{\mathfrak{m}}$-bimodules.

Proof. The map $\theta : \operatorname{Hom}_C(R,C)\otimes C_{\mathfrak{m}} \to \operatorname{Hom}_{C_{\mathfrak{m}}}(R_{\mathfrak{m}}, C_{\mathfrak{m}})$ defined by $\theta(f\otimes x)(ba^{-1}) = f(b)xa^{-1}$ is an R-$C_{\mathfrak{m}}$-module isomorphism by [19, Proposition 1.6]. However, it is also an

$R_{\mathfrak{m}}$-$R_{\mathfrak{m}}$-bimodule homomorphism as follows:

$$\begin{aligned}
\theta(rc^{-1}.f \otimes x.sd^{-1})(ba^{-1}) &= \theta(r.f.s \otimes c^{-1}xd^{-1})(ba^{-1}) \\
&= (r.f.s)(b)c^{-1}xd^{-1}a^{-1} \\
&= f(sbr)c^{-1}xd^{-1}a^{-1} \\
&= \theta(f \otimes x)(sd^{-1}.ba^{-1}.rc^{-1}) \\
&= rc^{-1}\theta(f \otimes x).sd^{-1}(ba^{-1}).
\end{aligned}$$

□

Chapter 2

The Cohen-Macaulay property and its generalisations

The notion of a ring being Cohen-Macaulay is well known in the commutative case as we have discussed in section 1.1. However, in the noncommutative case, there are already a number of possible definitions for the Cohen-Macaulay property which are not necessarily equivalent. We concentrate on noetherian rings which are finite modules over their centres and consider the question of how the Cohen-Macaulay property should be defined in this case. In this Chapter we look mainly at two generalisations, centrally-Macaulay as defined in Definition 2.1.1 and Krull-Macaulay as defined in Definition 2.2.1 and very briefly at GK-Macaulay as defined in 2.3.1. We focus more on centrally-Macaulay in this chapter, which is defined in terms of a central subring C. In section 2.1.1 we consider the question of whether or not the centrally-Macaulay property depends on the choice of C. We show by example that it does depend on C but give a number of situations where the centrally-Macaulay property is independent of the choice of central subring, for example, when R is equidimensional or prime. In section 2.1.2 we generalise a well known commutative result which says that an equidimensional ring R is Cohen-Macaulay if and only if there exists a central regular local subring C with R a free C-module. In section 2.1.3 we consider reconstruction algebras, which were recently invented by Wemyss and Craw, and prove that they satisfy the centrally-Macaulay property. Following on from this, we generalise a theorem of Brown and Goodearl in [12], in order to apply the result to reconstruction algebras. In section 2.2 we consider the Krull-Macaulay property, and prove some basic properties of Krull-Macaulay rings as well as suggesting some interesting questions. The final section of this chapter briefly deals with the GK-Macaulay property.

Throughout this chapter we assume the following hypothesis on R as defined in section 1.2:

R is noetherian with Krull $\dim(R) = n < \infty$ and a finite module over its centre Z. (H)

Recall that, by Proposition 1.2.2, Z is also noetherian and Krull $\dim(Z) = n$.

2.1 Centrally-Macaulay rings

The first generalisation we will look at is centrally-Macaulay as defined by Brown, Hajarnavis and MacEacharn in [17]. We will look at some properties of this condition, including the question of if and when it depends on the choice of central subring. Brown, Hajarnavis and MacEacharn have already proved many properties of centrally-Macaulay rings.

Definition 2.1.1. A noetherian ring R which is a finite module over a central subring C is *C-Macaulay* or *centrally Macaulay with respect to* C if R is a maximal Cohen-Macaulay C-module, as defined in Definition 1.1.5.

We give a second characterisation of this condition which shows how it is a generalisation of the usual commutative definition of Cohen-Macaulay. Note that we can allow C to be commutative but not necessarily central.

Proposition 2.1.2. *Let R be a noetherian ring which is a finite (left or right) module over a communtative subring C. Then R is a maximal Cohen-Macaulay C-module if and only if $G_C(\mathfrak{m}, R) = \text{ht}(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of C.*

Proof. Suppose R is a maximal Cohen-Macaulay (right) C module. That is, by definition,

$$G_C(\mathfrak{m}, R) = \text{Krull } \dim_{C_\mathfrak{m}}(C_\mathfrak{m} \otimes R) = \text{Krull } \dim_{C_\mathfrak{m}}(C_\mathfrak{m})$$

for all maximal ideals of C. Then since Krull $\dim_{C_\mathfrak{m}}(C_\mathfrak{m}) = \text{ht}(\mathfrak{m})$ we clearly have $G_C(\mathfrak{m}, R) = \text{ht}(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$.

On the other hand let $\mathfrak{m}$ be a maximal ideal of C and suppose that $G_C(\mathfrak{m}, R) = \text{ht}(\mathfrak{m})$. Then by [25, Lemma 18.1] and [15, Proposition 1.2.12] we have

$$G_C(\mathfrak{m}, R) = G_{C_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}, C_\mathfrak{m} \otimes_C R) \leq \text{Krull } \dim_{C_\mathfrak{m}}(C_\mathfrak{m} \otimes_C R).$$

Also, by [41, Corollary 6.2.18(ii)], Krull $\dim_{C_\mathfrak{m}}(C_\mathfrak{m} \otimes_C R) \leq$ Krull $\dim_{C_\mathfrak{m}}(C_\mathfrak{m})$ and thus

$$G_C(\mathfrak{m}, R) \leq \text{Krull } \dim_{C_\mathfrak{m}}(C_\mathfrak{m} \otimes R) \leq \text{Krull } \dim(C_\mathfrak{m}) = \text{ht}(\mathfrak{m}) = G_C(\mathfrak{m}, R).$$

So we must have equality throughout. Since the above applies for all maximal ideals $\mathfrak{m}$ of C, R is a maximal Cohen-Macaulay (right) C-module. □

Remarks. (i) We note that for a maximal ideal M of R and $\mathfrak{m} = M \cap C$,

$$G_C(\mathfrak{m}, R) \leq \mathrm{ht}(M) \leq \mathrm{ht}(\mathfrak{m}) \tag{2.1}$$

always holds. This follows from [17, §4.4] and Corollary 1.2.6.
(ii) For an ideal I of R, we will sometimes write $G(I)$ for $G_C(I \cap C, R)$.

We have the following examples of centrally Macaulay rings.

Example 2.1.3. (i) A commutative noetherian ring R is R-Macaulay if and only if it is Cohen-Macaulay in the usual sense, as given in Definition 1.1.3.
(ii) Every finite dimensional k-algebra A, where k is a field, is k-Macaulay.

The first of these is clear from Proposition 2.1.2. For the second note that since A is artinian, $\mathrm{ht}(M) = 0$ for all maximal ideals M of A. And since $M \cap k = 0$ we also have $G_k(M, A) = 0$.

A nice property of C-Macaulay rings is the equality of grade and height for all prime rings, as shown in the following theorem.

Theorem 2.1.4. *[17, Corollary 4.12] Let R be a right noetherian ring which is a finite module over a central subring C, such that R is C-Macaulay. Then for all primes P of R,*

$$\mathrm{ht}(P) = \mathrm{ht}(P \cap C) = G_C(P, R).$$

Another useful property is that the condition is stable under localisation at a maximal ideal of the central subring.

Proposition 2.1.5. *Let R be as above and C a central subring over which R is finitely generated. Then R is C-Macaulay if and only if $R_\mathfrak{m}$ is $C_\mathfrak{m}$-Macaulay for all maximal ideals $\mathfrak{m}$ of C.*

Proof. This follows from Proposition 2.1.2 since $\mathrm{ht}(\mathfrak{m}) = \mathrm{ht}(\mathfrak{m}_\mathfrak{m})$ by [31, Theorm 10.20] and $G_C(\mathfrak{m}, R) = G_{C_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}, R_\mathfrak{m})$ by [25, Lemma 18.1]. □

In Chapter 1 we observed that commutative Cohen-Macaulay rings are catenary. See Definition 1.1.8 for the definition of catenary. We now consider the catenarity of centrally Macaulay rings. Brown, Hajarnavis and MacEacharn stated in [17] that centrally Macaulay rings were catenary, but their proof contains a serious gap. Later, Goto and Nishida fixed the proof for semi-local rings and proved the following:

Theorem 2.1.6. *[32, Corollary 1.3] Let R be a ring satisfying (H) and C is a commutative subring of R such that R is a Cohen-Macaulay C-module. Assume that C is local. Then R is catenary and one has the equality*

$$\text{Krull dim}(R) = \text{Krull dim}(R/Q) + \text{ht}(Q)$$

for any prime ideal Q of R.

We now generalise this to non-local C.

Theorem 2.1.7. *Let R be a ring satisfying (H) and let C be a commutative subring with R a finitely generated left or right C-module. If R is a Cohen-Macaulay C-module then R is catenary. In particular, a centrally-Macaulay ring is catenary.*

Proof. Let R and C be as in the theorem. Suppose for a contradiction that R is not catenary. Then we have some primes $P \supset Q$ with two chains $P \supset X_1 \supset \cdots \supset X_n \supset Q$ and $P \supset Y_1 \supset \cdots \supset Y_m \supset Q$ of different lengths. Localise at a maximal ideal $\mathfrak{m}$ of C where $\mathfrak{m} \supseteq P \cap C$. Then by [31, Theorem 10.20] we get chains

$$P_\mathfrak{m} \supset (X_1)_\mathfrak{m} \supset \cdots \supset (X_n)_\mathfrak{m} \supset Q_\mathfrak{m}$$

and

$$P_\mathfrak{m} \supset (Y_1)_\mathfrak{m} \supset \cdots \supset (Y_m)_\mathfrak{m} \supset Q_\mathfrak{m}$$

which have different lengths. This contradicts the catenarity of $R_\mathfrak{m}$. And thus R is catenary. □

The following result is well known but we have not been able to find a reference for it.

Proposition 2.1.8. *Let R, S be rings satisfying (H). If R and S are both Z-Macaulay then so is $R \oplus S$.*

Proof. The ring $R \oplus S$ has maximal ideals $M \oplus S$ and $R \oplus N$ where M is a maximal ideal of R and N a maximal ideal of S. We will show that $G(M \oplus S) = \text{ht}(M \oplus S)$, as $R \oplus N$ works in exactly the same way. We first note that $\text{ht}(M \oplus S) = \text{ht}(M)$ since if

$$M = M_n \supset \cdots \supset M_0$$

is a chain of primes descending from M then we have the following chain descending from $M \oplus S$:

$$M \oplus S = M_n \oplus S \supset \cdots \supset M_0 \oplus S. \tag{2.2}$$

Thus $\mathrm{ht}(M) \leq \mathrm{ht}(M \oplus S)$. Now let $P_1, \ldots, P_t$ be the minimal primes of R and $Q_1, \ldots, Q_s$ the minimal primes of S. Then it is easy to check that $P_i \oplus S$ and $R \oplus Q_j$ are minimal primes of $R \oplus S$. Also,

$$\bigcap_i (P_i \oplus S) \cap \bigcap_j (R \oplus Q_j) = \bigcap_i P_i \oplus \bigcap_j Q_j = N(R) \oplus N(S),$$

where $N(R)$ and $N(S)$ are the prime radicals of R and S. By [31, Theorem 3.11] $N(R) \oplus N(S)$ is nilpotent and hence contained in $N(R \oplus S)$. Thus any minimal prime of $R \oplus S$ must be one of the $P_i \oplus S$ or the $R \oplus Q_j$. So any minimal prime contained in $M \oplus S$ is of the form $P_i \oplus S$ for some minimal prime P_i of R and so any maximal length chain of primes contained in $M \oplus S$ is of the same form as (2.2). Thus $\mathrm{ht}(M \oplus S) = \mathrm{ht}(M)$.

Also, if $\{x_1, \ldots, x_t\}$ is an R-sequence in $M \cap Z(R)$ then $\{(x_1, 1), \ldots, (x_t, 1)\}$ is an $R \oplus S$-sequence in $M \oplus S \cap Z(R \oplus S)$. Thus $G(M) \leq G(M \oplus S)$ and

$$G(M) \leq G(M \oplus S) \leq \mathrm{ht}(M \oplus S) = \mathrm{ht}(M) = G(M)$$

giving equality throughout. □

We also note the following proposition, which is already known.

Proposition 2.1.9. *Let R be a noetherian ring which is a finite module over its centre Z. Suppose that Z is a direct summand of R as Z-modules. Then if R is Z-Macaulay, Z is a Cohen-Macaulay ring.*

Proof. (i) Let $\mathfrak{m}$ be a maximal ideal of Z and suppose that $R = Z \oplus Y$ as Z-modules. We know that $G_Z(\mathfrak{m}, R) = \mathrm{ht}(\mathfrak{m})$ since R is Z-Macaulay and want to show that $G_Z(\mathfrak{m}, Z) = \mathrm{ht}(\mathfrak{m})$. We know from [17, §4.4] that

$$G_Z(\mathfrak{m}, Z) \leq \mathrm{ht}(\mathfrak{m}) = G_Z(\mathfrak{m}, R) \tag{2.3}$$

so we just need to show that $G_Z(\mathfrak{m}, R) \leq G_Z(\mathfrak{m}, Z)$. So let $\{x_1, \ldots, x_n\}$ be a Z-sequence on R. We show that it is a Z-sequence on Z. First note that $Z/\sum_{i=1}^{n} x_i Z \neq 0$ since $R/\sum_{i=1}^{n} x_i R \neq 0$. This follows because if $\sum_{i=1}^{n} x_i Z = Z$ then $1_R \in \sum_{i=1}^{n} x_i Z \subseteq \sum_{i=1}^{n} x_i R$ which is a contradiction. Now suppose that $x_i z \in \sum_{j=1}^{i-1} x_j Z$ for some $z \in Z$. Then since $\sum_{j=1}^{i-1} x_j Z \subseteq \sum_{j=1}^{i-1} x_j R$ and $\{x_1, \ldots, x_n\}$ is a Z-sequence on R we see that $z \in \sum_{j=1}^{i-1} x_j R$. Thus

$z = \sum_{j=1}^{i-1} x_j r_j$ for some $r_j \in R$. But $r_j = z_j + y_j$ for some $z_j \in Z$ and $y_j \in Y$. So $\sum_{j=1}^{i-1} x_j y_j = z - \sum_{j=1}^{i-1} x_j z_j \in Z \cap Y = 0$ and $z \in \sum_{j=1}^{i-1} x_j Z$ as required. Hence $\{x_1, \ldots, x_n\}$ is a Z-sequence on Z and $G_Z(\mathfrak{m}, R) \leq G_Z(\mathfrak{m}, R)$ giving equality in (2.3). □

We note that the above result will not always hold without the hypothesis that Z is a direct summand of R. This can be seen from [17, Example 7.3] which gives an example of a centrally-Macaulay ring whose centre is not Cohen-Macaulay.

2.1.1 Dependence on the central subring

We now turn to the question of dependence on the central subring C.

Proposition 2.1.10. *Suppose R is C-Macaulay for some central subring C such that R is a finite module over C. Then for a central subring A of R containing C we have that R is A-Macaulay. In particular, R is Z-Macaulay.*

Proof. Let I be a maximal ideal of R. From Theorem 1.2.5 we get the inequalities $\mathrm{ht}(I) \leq \mathrm{ht}(I \cap A) \leq \mathrm{ht}(I \cap C)$ and by Theorem 2.1.4 we have $\mathrm{ht}(I) = \mathrm{ht}(I \cap C)$ giving equality throughout. But also a C-sequence on R is a A-sequence on R so $G_C(I, R) \leq G_A(I, R)$. Thus

$$\mathrm{ht}(I \cap A) = \mathrm{ht}(I \cap C) = G_C(I, R) \leq G_A(I, R) \leq \mathrm{ht}(I \cap A)$$

and R must be A-Macaulay. □

However the converse is not always true as we see from the following example.

Example 2.1.11. Let $S = k[X] \oplus k$. Then

(i) S is S-Macaulay;

(ii) S is not C-Macaulay where C is the central subring $k[(X, 1)]$.

Proof. (i) The commutative ring S has maximal ideals of the form $J := k[X] \oplus 0$ and $I_\lambda := \langle X - \lambda \rangle \oplus k$. Now J has height and grade equal to zero and the ideals $\langle X - \lambda \rangle \oplus k$ have height 1 and grade 1 so that S is S-Macaulay (here $Z = S$ as S is commuative).

(ii) However we can consider S as a finite module over the subring $C := k[(X, 1)] = \{(f(X), f(1)) : f \in k[X]\}$. That is,

$$S = (1, 0)C + (0, 1)C.$$

We now focus on the ideal $I_1 = \langle X-1\rangle \oplus k$ which has height 1. $I_1 \cap C = \langle X-1\rangle \oplus 0$ consists of zero divisors on R, so we must have $G_C(I_1) = 0 \neq \mathrm{ht}(I_1)$. Thus S is not C-Macaulay. □

Remark. Thus we see that the definition of centrally Macaulay does depend on the choice of the central subring. This example also shows that [15, Proposition 2.7] is not true. The proposition says that a ring is homologically homogeneous over Z if and only if it is homologically homogeneous over any central subring, but the example above is a counter example as we will show in Chapter 4. The error in the proof of [15, Proposition 2.7] is the assumption that $\mathrm{ht}(M) = \mathrm{ht}(M \cap C)$ for any maximal ideal M of Z, Z being integral over C. We show in the following theorem that this is crucial to the dependence of the C-Macaulay property on the choice of central subring C.

Theorem 2.1.12. *Let R be a noetherian ring which is finitely generated as a Z-module. Let C be a subring of Z such that R is a finitely generated C-module. Suppose R is Z-Macaulay. Then R is C-Macaulay if and only if for all maximal ideals M_1 and M_2 of Z with $M_1 \cap C = M_2 \cap C$ we have* $\mathrm{ht}(M_1) = \mathrm{ht}(M_2)$.

Proof. We first assume that R is C-Macaulay and let M_1, M_2 be maximal ideals of Z with $\mathfrak{m} := M_1 \cap C = M_2 \cap C$. By Theorem 1.2.5, $\mathfrak{m}$ is a maximal ideal of C so we must have

$$G_C(\mathfrak{m}, R) = \text{Krull dim}(C_{\mathfrak{m}}). \tag{2.4}$$

Also since $R_{\mathfrak{m}}$ is a finitely generated $C_{\mathfrak{m}}$-module we have

$$\text{Krull dim}(C_{\mathfrak{m}}) = \text{Krull dim}(R_{\mathfrak{m}}). \tag{2.5}$$

Now let I, J be any two maximal ideals of R lying over $\mathfrak{m}$ such that $I_{\mathfrak{m}} \cap C_{\mathfrak{m}} = J_{\mathfrak{m}} \cap C_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}$. By (2.1), [31, Theorem 10.20] and the definition of Krull dimension, respectively, we have

$$G_C(\mathfrak{m}, R) \leq \mathrm{ht}(I) = \mathrm{ht}(I_{\mathfrak{m}}) \leq \text{Krull dim}(R_{\mathfrak{m}}),$$

and similarly for J. But equations (2.4) and (2.5) show that the upper and lower bounds in the inequality above are both equal to Krull dim$(C_{\mathfrak{m}})$ giving $\mathrm{ht}(I) = \mathrm{ht}(I_{\mathfrak{m}}) = \mathrm{ht}(J_{\mathfrak{m}}) = \mathrm{ht}(J)$. But by Corollary 1.2.7 we can choose I and J such that I lies over M_1 and J lies over M_2 with $\mathrm{ht}(I) = \mathrm{ht}(M_1)$ and $\mathrm{ht}(J) = \mathrm{ht}(M_2)$. Thus $\mathrm{ht}(M_1) = \mathrm{ht}(M_2)$ as required.

For the other implication we assume the equality of heights. Let M be a maximal ideal of R with $M \cap C = \mathfrak{n}$ and set $\mathfrak{m} := M \cap Z$. By Proposition 2.1.5 we can assume without loss of generality that C is local with $J(C) = \mathfrak{n}$. Since R is Z-Macaulay, Theorem 2.1.4 shows that

$$\mathrm{ht}(M) = \mathrm{ht}(\mathfrak{m}) = G_Z(\mathfrak{m}, R) =: t.$$

Now let $\{x_1, \dots, x_s\}$ be a maximal C-sequence in $\mathfrak{n}$ on R. Since $\mathfrak{n} \subseteq \mathfrak{m}$ we know that $s \leq t$ and we suppose for a contradiction that $s < t$. Let $I = \sum_{j=1}^{s} x_j R$. Then $\mathfrak{n} + I/I$ consists of zero-divisors in R/I and thus by [17, Proposition 3.4] there exists $0 \neq \bar{c} \in R/I$ such that $c.\,\mathfrak{n} \subseteq I$. Then since $R/\,\mathfrak{n}\,R$ is an artinian ring it follows that $cR + I/I$ is a non-zero artinian R-module contained in R/I. Thus R/I contains a simple R-module, say L/I such that $(L/I)M' = 0$ for some maximal ideal M' of R; that is, $L.M' \subseteq I$. Let $M' \cap Z = \mathfrak{m}'$ which consists of zero-divisors modulo I. Thus $\{x_1, \dots, x_s\}$ is a maximal Z-sequence in M'. But R is Z-Macaulay so by Theorem 2.1.4 again,

$$s = G_Z(\mathfrak{m}', R) = \mathrm{ht}(M') = \mathrm{ht}(\mathfrak{m}').$$

But by hypothesis, since $\mathfrak{m}' \cap C = \mathfrak{n} = \mathfrak{m} \cap C$, we also have

$$\mathrm{ht}(\mathfrak{m}') = \mathrm{ht}(\mathfrak{m}) = t$$

giving $s = t$, a contradiction. So we must have $s = t$ and R is C-Macaulay. □

From this theorem we get a number of corollaries. For the definition of equidimensional we refer the reader to Definition 1.2.8.

Corollary 2.1.13. *Let R be a noetherian ring which is a finitely generated module over its equidimensional centre Z. Let C be a subring of Z with R a finitely generated C-module. Then R is Z-Macaulay if and only if R is C-Macaulay.*

Proof. Suppose R is Z-Macaulay. Since Z is equidimensional we see immediately that the required condition in Theorem 2.1.12 holds and thus R is C-Macaulay. □

Corollary 2.1.14. *Let R be an equidimensional noetherian ring which is a finitely generated module over its centre Z. Let C be a subring of Z with R a finitely generated C-module. Then R is Z-Macaulay if and only if R is C-Macaulay.*

Proof. By Corollary 1.2.9 Z is equidimensional, so we can apply Corollary 2.1.13. □

Definition 2.1.15. A pair of rings $S \subset R$ has the *going down* property if the following two conditions hold:

(i) Let $P \subset Q$ be prime ideals of R and $\mathfrak{q}$ a prime ideal of S such that Q lies over $\mathfrak{q}$. Then the ideal $\mathfrak{p} = P \cap S$ is a prime ideal of S with $\mathfrak{p} \subset \mathfrak{q}$.
(ii) Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of S and Q a prime ideal of R lying over $\mathfrak{q}$. Then there exists a prime ideal $P \subset Q$ of R such that P lies over $\mathfrak{p}$.

We note that rings satisfying condition (H) always satisfy condition (i) of the going down property by [45, Theorem 16.9].

Corollary 2.1.16. *Let R be a noetherian ring which is a finitely generated Z-module. Let C be a subring of Z such that Z is a finitely generated C-module and going down holds for the pair $C \subseteq Z$. Then R is Z-Macaulay if and only if R is C-Macaulay.*

Proof. By Theorem 2.1.12 above it is enough to show that for all maximal ideals M of Z we have $\mathrm{ht}(M) = \mathrm{ht}(M \cap C)$. So let M be a maximal ideal of Z and let $\mathfrak{m} = M \cap C$. Then if

$$\mathfrak{m} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_d$$

is a chain of prime ideals of C then by going down we have a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_d$$

of prime ideals of Z with $M_i \cap C = \mathfrak{m}_i$. Thus $\mathrm{ht}(M) \geq \mathrm{ht}(\mathfrak{m})$. On the other hand, we know from Corollary 1.2.6 that $\mathrm{ht}(M) \leq \mathrm{ht}(\mathfrak{m})$. □

Corollary 2.1.17. *Let R be a noetherian ring which is a finite module over Z with Z a domain. Let C be an integrally closed subring of Z such that Z is a finitely generated C-module. Then R is Z-Macaulay if and only if R is C-Macaulay.*

Proof. By [25, Theorem 13.9] going down holds so Corollary 2.1.16 above applies. □

Corollary 2.1.18. *Let R be a noetherian ring with Z an affine domain over a field k and R a finitely generated Z-module. Let C be a subalgebra of Z such that Z is a finitely generated C-module. Then R is Z-Macaulay if and only if R is C-Macaulay.*

Proof. We see from Theorem A in chapter 8 of [25] that Z is equidimensional. Thus we can apply Corollary 2.1.13. □

The above corollary still holds if we localise Z at a finite set of primes of the same height.

Corollary 2.1.19. *Let R be a noetherian ring with R a finitely generated Z-module and Z the localisation of an affine domain over a field k at a finite set of primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ of the same height. That is, we invert the elements which are not contained in any of the primes $\mathfrak{p}_i$. Let C be a subalgebra of Z such that Z is a finitely generated C-module. Then R is Z-Macaulay if and only if R is C-Macaulay.*

Proof. Let t be the height of the primes in $\mathcal{P}$. Then Krull $\dim(Z) = t = \text{ht}(M)$ for all maximal ideals M of Z. That is, Z is equidimensional and we can again apply Theorem 2.1.14. □

We also have the following result as a corollary to Corollary 2.1.18.

Corollary 2.1.20. *Let R be a prime noetherian ring which is a finite module over its affine centre Z. Let C be a subalgebra of Z such that Z is a finitely generated C-module. Then R is Z-Macaulay if and only if R is C-Macaulay.*

Proof. Since R is prime, it follows that Z is a domain. To see this, let $z, y \in Z$ such that $zy = 0$. Then we have $zRyR = zyR = 0$ so by primeness of R, either $zR = 0$ or $yR = 0$. Thus, $z = 0$ or $y = 0$. We can therefore apply Corollary 2.1.18. □

2.1.2 Freeness and Projectivity

One characterisation of the commutative Cohen-Macaulay property for an equidimensional ring is that R is Cohen-Macaulay if and only if it is a free module over a regular local subring. See [25, Corollary 18.17]. This fails if R is not equidimensional, as can be seen from the following example.

Example 2.1.21. Let $S := k[X] \oplus k$ and $C := k[(X, 1)]$ and localise S and C at the maximal ideal $\mathfrak{m} := \langle X, 1 \rangle$ of C. Then
(i) $S_\mathfrak{m}$ is Cohen-Macaulay;
(ii) $S_\mathfrak{m}$ is a finite module over the regular local subring $C_\mathfrak{m}$, but is not a free $C_\mathfrak{m}$-module.

Proof. (i) Example 2.1.11 shows that S is Cohen-Macaulay and thus by Theorem 2.1.5 $S_\mathfrak{m}$ is Cohen-Macaulay.
(ii) $S_\mathfrak{m} = k[X]_{\langle X \rangle} \oplus 0$ and $C_\mathfrak{m} = k[(X, 1)]_{\langle (X,1) \rangle}$ so that $S_\mathfrak{m} = (1, 0)C_\mathfrak{m}$. But $(1, 0)$ is a zero-divisor on $C_\mathfrak{m}$ so that $S_\mathfrak{m}$ cannot be a free $C_\mathfrak{m}$-module. □

However, we can generalise this characterisation to the following:

Proposition 2.1.22. *Let R be a noetherian ring which is a finite module over its centre Z. Let C be a regular central subring of R with R a finitely generated C-module. Then R is C-Macaulay if and only if R is a projective C-module.*

Proof. R is C-Macaulay if and only if $R_{\mathfrak{m}}$ is $C_{\mathfrak{m}}$-Macaulay for all maximal ideals $\mathfrak{m}$ of C, if and only if $R_{\mathfrak{m}}$ is a maximal Cohen Macaulay $C_{\mathfrak{m}}$-module for all $\mathfrak{m}$. Now by Proposition 1.1.28 this is true if and only if $R_{\mathfrak{m}}$ is a free $C_{\mathfrak{m}}$-module for all $\mathfrak{m}$ if and only if R is a projective C-module. □

Corollary 2.1.23. *Let R be a noetherian ring which is a finite module over its centre Z. Let C be a polynomial algebra contained in R with R a finitely generated C-module. If R is C-Macaulay then R is a free C-module.*

Proof. Projective modules over polynomial algebras are free by [48, Theorem 4.5.9] so the result follows from Proposition 2.1.22 □

To see how Proposition 2.1.22 might be useful we couple it with results from the previous section.

Corollary 2.1.24. *Let R be an equidimensional noetherian ring which is a finite module over its centre Z. Suppose that R is Z-Macaulay. Then if C is any regular subring of Z over which R is finitely generated, R must be a projective C-module.*

Proof. Use Corollary 2.1.14 and Proposition 2.1.22. □

Note that this reduces to [25, Corollary 18.17] where R is commutative. It also follows fairly easily from the Auslander-Bachsbaum formula if C is local, i.e. a maximal CM-module over a regular local ring is free.

Corollary 2.1.25. *Let R be a noetherian ring which is a finite module over its centre Z. Suppose that R is Z-Macaulay. Let C be any regular subring of Z over which R is finitely generated such that any of the following hold:*

1. *Going down holds between Z and C;*
2. *Z and C are domains with C integrally closed;*
3. *Z is an affine domain over a field k;*
4. *R is prime with Z affine over a field k.*

Then R is a projective C-module.

Proof. Use Corollaries 2.1.16, 2.1.17, 2.1.18 and 2.1.20 together with Proposition 2.1.22. □

We now consider the question of whether or not the subring C has to be central, or can we instead consider the situation where R is a finite module over a subring C which is merely commutative? Note that in this case, C will still be noetherian by the following theorem.

Theorem 2.1.26. *[28, Theorem 3] Let C be a commutative subring of a right noetherian ring R such that R is finitely generated as a right C-module. Then C is a noetherian ring.*

Theorem 2.1.27. *Let R be an equidimensional noetherian ring which is a finite module over its centre Z. Suppose that R is Z-Macaulay. Now let C be a commutative regular subring of R such that R is a finitely generated module over C. Then R is C-projective.*

Proof. We will consider the commutative subring $A := \langle Z, C\rangle$ generated by Z and C. Clearly R is a finitely generated right module over A. First we show that R is a maximal Cohen-Macaulay A-module. Let N be a maximal ideal of A and let $\mathfrak{n} := N \cap Z$ which is a maximal ideal of Z by Theorem 1.2.5. Since R is Z-Macaulay it is a maximal Cohen-Macaulay Z-module and by Proposition 2.1.2 we have that

$$G_Z(\mathfrak{n}, R) = \operatorname{ht}(\mathfrak{n}) = \text{Krull dim}(Z) = \text{Krull dim}(R).$$

We know by (1.1) that $G_A(N, R) \leq \operatorname{ht}(N) \leq \text{Krull dim}_A(A)$. Also, since A is a commutative ring and R is a finitely generated A-module we have

$$\text{Krull dim}_A(R) = \text{Krull dim}_A(A/\operatorname{Ann}(R)) = \text{Krull dim}_A(A).$$

But since R is an A-R-bimodule, [41, Corollary 6.4.13] applies to give $\text{Krull dim}_A(R) = \text{Krull dim}_R(R)$. So

$$G_A(N, R) \leq \operatorname{ht}(N) \leq \text{Krull dim}_A(A) = \text{Krull dim}_R(R) = G_Z(\mathfrak{n}, R) \leq G_A(N, R) \quad (2.6)$$

where the last inequality is because any Z-sequence is an A-sequence. Thus we have equality throughout equation (2.6) and so R is a maximal Cohen-Macaulay A-module. We also see from equation (2.6) that A is equidimensional with $\text{Krull dim}(A) = \text{Krull dim}(R)$.

Now let $\mathfrak{m}$ be a maximal ideal of C and N a maximal ideal of A lying over $\mathfrak{m}$. So $N \cap C = \mathfrak{m}$ and by Corollary 1.2.9 $\operatorname{ht}(N) = \operatorname{ht}(\mathfrak{m}) = \text{Krull dim}(A)$. Then we have, by (2.6),

$$\text{Krull dim}(A) = G_A(N, R) = \operatorname{ht}(N) = \operatorname{ht}(\mathfrak{m}) \geq G_C(\mathfrak{m}, R) =: s. \quad (2.7)$$

We suppose for a contradiction that the inequality in (2.7) is strict and let $\{x_1, \dots, x_s\}$ be a maximal C-sequence in $\mathfrak{m}$. Let $I := \sum_{i=1}^{s} x_i R$ so that R/I is a finitely generated right $C/(I \cap C)$-module. Then $\mathfrak{m}/I \cap C$ consists of zero divisors on R/I and we can apply Proposition 1.8.2 to get $0 \neq \bar{a} \in R/I$ such that $a\,\mathfrak{m} \subseteq I$.

We now show that $M = aA + I/I$ is an artinian A-module. Since R/I is a finitely generated A-module where A is noetherian, M is a non-zero finitely generated A-module. But $M(\mathfrak{m}\,A) = 0$ since A is commutative and $a\,\mathfrak{m} \subseteq I$ so M is an $A/\mathfrak{m}\,A$-module. Now $A/\mathfrak{m}\,A$ is an artinian ring because A is a finite module over C and thus $A/\mathfrak{m}\,A$ is a finite module over the field $C/\mathfrak{m}$. Thus M is an artinian A-module. Hence $\mathrm{soc}_A(M) \neq 0$ and there exists a maximal ideal N' of A with $UN' = I$ for some simple A-submodule U of M. Now let $x \in N' \backslash \sum_{i=1}^{s} x_i A$. Then there exists $0 \neq u + I \in U \subseteq M$ such that $ux \in I$. Then $u \in R$ which shows that x is a zero divisor on R/I and hence $\{x_1 \dots, x_s\}$ is a maximal A-sequence in N' on R.

Hence

$$s = G_A(N', R) = \mathrm{ht}(N') = \text{Krull}\ \dim(A),$$

where the second equality follows from (2.6) as we have shown that equality holds throughout. But this contradicts our assumption that the inequality in (2.7) was strict. So $G_C(\mathfrak{m}, R) = \mathrm{ht}(\mathfrak{m})$ and R is a maximal Cohen-Macaulay C-module.

Now we apply the Auslander-Buchsbaum formula, Theorem 1.1.26, to the local ring $C_{\mathfrak{m}}$ to get

$$\mathrm{pr.\,dim}_{C_{\mathfrak{m}}}(R \otimes C_{\mathfrak{m}}) + G_{C_{\mathfrak{m}}}(\mathfrak{m}, R \otimes C_{\mathfrak{m}}) = G_{C_{\mathfrak{m}}}(\mathfrak{m}, C_{\mathfrak{m}}).$$

Now by [25, Lemma 18.1] and the fact that C is regular and hence Cohen-Macaulay we have

$$G_{C_{\mathfrak{m}}}(\mathfrak{m}, R \otimes C_{\mathfrak{m}}) = G_C(\mathfrak{m}, R) = \mathrm{ht}(\mathfrak{m}) = G_C(\mathfrak{m}, C) = G_{C_{\mathfrak{m}}}(\mathfrak{m}_{\mathfrak{m}}, C_{\mathfrak{m}}).$$

Thus $\mathrm{pr.\,dim}_{C_{\mathfrak{m}}}(R \otimes C_{\mathfrak{m}}) = 0$. But since this is true for all maximal ideals $\mathfrak{m}$ of C we must have $\mathrm{pr.\,dim}_C(R) = 0$ as required. □

This theorem gives the following corollary:

Corollary 2.1.28. *Let R be a noetherian ring which is a finite module over its centre Z, where Z is an affine domain. Suppose that R is Z-Macaulay. Now let C be a commutative regular subring of R such that R is a finitely generated module over C. Then R is C-projective.*

Proof. Since Z is an affine domain, [25, Chapter 8, Theorem A] shows that Z is equidimensional. Then since R is Z-Macaulay, [17, Corollary 4.12] shows that R is equidimensional so that Theorem 2.1.27 applies. □

We now show by example that, in Theorem 2.1.27, we do need R to be a finitely generated C-module.

Example 2.1.29. Let R be the coordinate ring of $SL(2, \mathbb{C})$, so that

$$R = \mathbb{C}[X, Y, Z, U]/\langle XT - YZ - 1\rangle$$

and take $C = \mathbb{C}[X, Y] \subseteq R$. Then R is a commutative noetherian equidimensional ring (in fact, R is a Hopf algebra) such that

(i) C is a commutative regular subring of R;

(ii) R is not a finitely generated C-module;

(iii) R is Z-Macaulay;

(iv) R is not C-projective.

Proof. (i) This is clear since C is a polynomial ring.

(ii) Proposition 1.2.2 shows that R cannot be a finitely generated C-module as

$$\text{Krull dim}(R) = 3 > 2 = \text{Krull dim}(C).$$

(iii) R is a commutative affine Hopf algebra by [11, Examples 2.1.3] and since $\mathbb{C}$ has characteristic zero, we see from [11, §3.2.1] that R is regular. Thus it is Cohen-Macaulay by Theorem 1.1.20. Hence R is Z-Macaulay, since it is commutative.

(iv) Since C is a polynomial ring, C-projective is the same as C-free. Consider the maximal ideal $\mathfrak{m} := \langle X, Y\rangle$ of C. If R is C-free then $\mathfrak{m}R \subsetneq R$ as $\mathfrak{m}C \subsetneq C$, but we see that $1 = XU - YZ \in \mathfrak{m}R$ so that $\mathfrak{m}R = R$. Thus R cannot be C-projective. □

The question remains open of whether Theorem 2.1.27 can be generalised to noncommutative subrings C of R.

2.1.3 Reconstruction Algebras

In this section we consider Reconstruction algebras as examples of rings which are centrally-Macaulay. These were introduced by Wemyss in [55]. In [23], Craw considers the quotient algebras kQ/R of bound Special McKay quivers (Q,R), which turn out to be the same algebras. We will keep to the more algebraic approach of Wemyss.

Commutative crepant resolutions of singularities are well known in algebraic geometry. In [53], Van den Bergh introduced a noncommutative analogue to this - the idea of a noncommutative crepant resolution of a Gorenstein singularity. Van den Bergh's definition is as follows:

Definition 2.1.30. Let R be a normal Gorenstein domain and let $X = Spec(R)$. Then a *noncommutative crepant resolution* for R is a homologically homogeneous R-algebra of the form $A = \text{End}_R(M)$, where M is a reflexive R-module.

We refer the reader to either [53, §3] or Definition 4.1.2 for the definition of a homologically homogeneous algebra, but we observe that it is a generalisation of a commutative regular ring. A standard example of a noncommutative crepant resolution, as given in [53, Example 1.1] is the following:

Example 2.1.31. Let $G \subseteq SL(V)$ be a finite group and let V be a finite dimensional vector space on which G acts linearly. Let $S = S(V)$ and $R = S^G$. Then $A = \text{End}_R(S) \cong S * G$ is a noncommutative crepant resolution of R.

However, in [55] Wemyss looks at non-Gorenstein singularities and shows that they can be resolved by reconstruction algebras. These are noncommutative algebras, but they are not in general homologically homogeneous, which suggests that Van den Bergh's requirement that the noncommutative crepant resolution be homologically homogeneous may be too strong.

Here however, we simply show that reconstruction algebras are Z-Macaulay. Wemyss defines them in terms of quivers and relations and shows that they are isomorphic to the endomorphism ring of a Cohen-Macaulay module. We will define them in terms of the endomorphism ring and show that they are Z-Macaulay.

We consider the polynomial ring $\mathbb{C}[x, y]$ and the group $G = \frac{1}{r}(1, a)$ which we define, for $a, r \in \mathbb{N}$ with $\text{hcf}(a, r) = 1$, to be

$$G = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle \subseteq GL(2, \mathbb{C}),$$

where ε is a primitive r^{th} root of unity. Thus G is a cyclic group of order r. We have thus identified a particular embedding of G into $GL(2, \mathbb{C})$, and thus a linear action of G on $\mathbb{C}\,x + \mathbb{C}\,y$ which extends to an action of G by algebra automorphisms on $\mathbb{C}[x, y]$.

Definition 2.1.32. For $0 \leq t \leq r-1$, let

$$S_t := \{f \in \mathbb{C}[x,y] : gf = \varepsilon^t f \;\; \forall \;\; g \in G\}.$$

These are indecomposable Cohen-Macaulay $\mathbb{C}[x,y]^G$-modules as can be seen from [59, Corollary 10.10], which gives a one-to-one correspondence between the irreducible $\mathbb{C}\,G$-modules and the indecomposable $\mathbb{C}[x,y]^G$ Cohen-Macaulay modules. Notice also that $S_0 = \mathbb{C}[x,y]^G$.

Definition 2.1.33. Let $r/a = [\alpha_1, \ldots, \alpha_n]$ be the Jung-Hirzebruch continued fraction expansion of r/a, as defined in [55] immediately before Definition 5.2.11, and define the i-series by:

$$i_0 = r, \;\; i_1 = a \text{ and } i_t = \alpha_{t-1} i_{t-1} - i_{t-2} \text{ for } 2 \leq t \leq n+1.$$

Then the module S_t is *special* if $t = i_p$ for $1 \leq p \leq n$.

The concept of special modules is due to Wunram in [56]. We are now able to define reconstruction algebras.

Definition 2.1.34. The *reconstruction algebra of type A* is

$$A_{r,a} = \text{End}_{\mathbb{C}[x,y]^G}(M),$$

where M is the sum of the "special" Cohen-Macaulay modules.

Note that M is a direct summand of $\mathbb{C}[x,y]$ as a $\mathbb{C}[x,y]^G$-module. This is because $\mathbb{C}[x,y]$ is a $\mathbb{C}\,G$-module where $\mathbb{C}\,G$ is semisimple artinian by Masche's theorem. Therefore it must split as a direct sum of simple $\mathbb{C}\,G$-modules. In particular

$$\mathbb{C}[x,y] = S_0 \oplus S_1 \oplus \cdots \oplus S_{r-1},$$

where S_i is a direct sum of copies of a simple module, so that M is the direct sum of some of these summands.

We aim to show that $A_{r,a}$ is Z-Macaulay and in order to do so we first we establish that $\text{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])$ is Z-Macaulay. We now define maximal orders as we require them for the following theorem.

Definition 2.1.35. Let R be a subring of some quotient ring Q. Then
(i) R is a *right order* in Q if each $q \in Q$ has the form rs^{-1} for some $r, s \in R$. R is an *order* in Q if it is both a left order and a right order, where left order is defined analogously.

(ii) Two orders R_1 and R_2 in Q are *equivalent orders* if there exist units $a_1, a_2, b_1, b_2 \in Q$ such that $a_1 R_1 b_1 \subseteq R_2$ and $a_2 R_2 b_2 \subseteq R_1$.
(iii) R is a *maximal order* if it is maximal within its equivalence class.

The following result is well known, and can be found in the proof of Theorem 1.5 in [26].

Theorem 2.1.36. *Let $S(V)$ be a polynomial algebra and G a finite group acting linearly and faithfully on V such that the skew group algebra $S(V) * G$ is a maximal order. Then there is an isomorphism of algebras*

$$\mathrm{End}_{S(V)^G}(S(V)) \cong S(V) * G.$$

Proof. Define $\Phi : S(V) * G \to \mathrm{End}_{S(V)^G}(S(V))$ by

$$\sum_{g \in G} s_g g \mapsto (t \mapsto \sum_{g \in G} s_g g(t)).$$

Then Φ is an algebra homomorphism.

In order to show Φ is an isomorphism we first tensor with the quotient field $Q(S(V)^G)$ which gives

$$\phi : Q(S(V)^G) \otimes_{S(V)^G} (S(V) * G) \to Q(S(V)^G) \otimes_{S(V)^G} \mathrm{End}_{S(V)^G}(S(V)),$$

where $\phi = id \otimes \Phi$. For brevity, let

$$\mathrm{QS} := Q(S(V)^G) \otimes_{S(V)^G} (S(V) * G)$$

and

$$\mathrm{QEnd} := Q(S(V)^G) \otimes_{S(V)^G} \mathrm{End}_{S(V)^G}(S(V)).$$

Then by [25, Lemma 2.4 and Proposition 2.10] we have the $S(V)^G$-module isomorphisms

$$\begin{aligned} \mathrm{QS} &\cong [Q(S(V)^G) \otimes_{S(V)^G} S(V)] * G) \\ &\cong Q(S(V)) * G, \end{aligned}$$

and

$$\begin{aligned} \mathrm{QEnd} &\cong \mathrm{End}_{Q(S(V)^G)}[Q(S(V)^G) \otimes_{S(V)^G} S(V)] \\ &\cong \mathrm{End}_{Q(S(V)^G)}[Q(S(V))]. \end{aligned}$$

Now QS is simple by [41, Proposition 7.8.12] and the simplicity of the quotient field $Q(S(V))$. Hence ϕ is injective because it is non-zero. Now $Q(S(V))$ is a Galois extension of

$Q(S(V))^G = Q(S(V)^G)$ with Galois group G, by [6, Proposition 1.1.1] so that $[Q(S(V)) : Q(S(V)^G)] = |G|$. Hence as a vector space over $Q(S(V)^G)$ we have

$$\dim[Q(S(V)) * G] = |G|^2.$$

But also, as a vector space, $\operatorname{End}_{Q(S(V)^G)} Q(S(V)$ is isomorphic to $|G| \times |G|$ matrices over $Q(S(V)^G)$ so has dimension $|G|^2$. This shows that ϕ is surjective. So ϕ affords an isomorphism of algebras from QS to QEnd.

Now consider the commutative diagram

$$\begin{array}{ccc} \mathrm{QS} & \xrightarrow{\ \phi\ } & \mathrm{QEnd} \\ {\scriptstyle i_S}\uparrow & & \uparrow{\scriptstyle i_E} \\ S(V) * G & \xrightarrow{\ \Phi\ } & \operatorname{End}_{S(V)^G}(S(V)) \end{array}$$

where i_S and i_E are the obvious embeddings. Injectivity of Φ is clear from the diagram and we just need to show that it is surjective.

From the diagram, making the identifications permitted by the defined embeddings, we have

$$S(V) * G \subseteq \operatorname{End}_{S(V)^G} S(V) \hookrightarrow \mathrm{QEnd} = Q(S(V) * G).$$

The multiplicatively closed set $\mathcal{S} := S(V)^G \backslash \{0\}$ consists of central non-zero-divisors in $\operatorname{End}_{S(V)^G} S(V)$, which we can invert to get

$$A := \mathcal{S}^{-1} S(V) * G \subseteq \mathcal{S}^{-1} \operatorname{End}_{S(V)^G} S(V) = \mathrm{QEnd}\,.$$

Now $Q(S(V)^G) \otimes_{S(V)^G} S(V)$ and hence $A = Q(S(V)^G) \otimes_{S(V)^G} (S(V) * G)$ are finite dimensional vector spaces over $Q(S(V)^G)$. Thus A is artinian and so every non-zero-divisor is a unit. Hence A is the whole of the quotient ring of $S(V) * G$, i.e $A = \mathrm{QEnd}$. By hypothesis $S(V) * G$ is a maximal order and so $\operatorname{Im} \Phi$ is a maximal order in its quotient ring QEnd. Also since $\operatorname{Im} \Phi \subseteq \operatorname{End}_{S(V)^G}(S(V))$ we see that $\operatorname{End}_{S(V)^G}(S(V))$ is an order in QEnd.

Now by [6, Theorem 1.3.1], $S(V)$ is a finitely generated $S(V)^G$-module and so $\operatorname{End}_{S(V)^G} S(V)$ is a finitely generated $S(V)^G$-module by [22, Lemma 5.1.3]. Hence $\operatorname{End}_{S(V)^G}(S(V))$ is finitely generated over $S(V) * G \cong \operatorname{Im} \Phi$ and we will call the generators $x_1, \ldots, x_t \in \operatorname{End}_{S(V)^G}(S(V)) \subseteq \mathrm{QEnd} = \mathrm{QS}$. Each $x_i = d^{-1} y_i$ for some $y_i \in \operatorname{Im} \Phi$ and $d \in S(V)^G \backslash \{0\}$, by the common denominator property, and so

$$d \operatorname{End}_{S(V)^G}(S(V)) \subseteq \operatorname{Im} \Phi.$$

Hence $\operatorname{End}_{S(V)^G}(S(V))$ is equivalent to the maximal order $\operatorname{Im} \Phi$. By maximality $\operatorname{Im} \Phi = \operatorname{End}_{S(V)^G}(S(V))$, so Φ is surjective. □

The following two results are also well-known.

Corollary 2.1.37.

$$\mathrm{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y]) \cong \mathbb{C}[x,y] * G.$$

Proof. First of all, G acts faithfully on $\mathbb{C}[x,y]$. By [39, Theorem 4.6], $\mathbb{C}[x,y] * G$ is a maximal order if $\mathbb{C}[x,y]$ is integrally closed and if there exists no non-identity element $g \in G$ such that

$$I(g) := \{s - g(s) : s \in \mathbb{C}[x,y]\}\, \mathbb{C}[x,y] \subseteq \mathfrak{p}$$

for some height 1 prime $\mathfrak{p}$ of $\mathbb{C}[x,y]$. The first property is clear and by [39, Proposition 4.10] the second condition holds if each non-identity element of G acts non-trivially on $\mathbb{C}[x,y]$. So let $1 \neq g = \begin{pmatrix} \varepsilon^n & 0 \\ 0 & \varepsilon^{an} \end{pmatrix} \in G$, for some $1 \leq n \leq r-1$. Then the action of g on $\mathbb{C}[x,y]$ cannot be trivial as $g(x) = \varepsilon^n x \neq x$. Thus Theorem 2.1.36 applies to give the result. □

Lemma 2.1.38. *Let $S(V)$ be a polynomial algebra over a field and let G be a finite group acting linearly and faithfully on V. Then the centre of $S(V) * G$ is $S(V)^G$.*

Proof. First of all, let $s \in S(V)^G$ so that $g(s) = s$ for all $g \in G$. Then for any $\sum_{g\in G} t_g g \in S(V) * G$,

$$(\sum_{g\in G} t_g g)s = \sum_{g\in G} t_g g(s) g = \sum_{g\in G} t_g s g = s(\sum_{g\in G} t_g g).$$

So $s \in Z(S(V) * G)$.

Now let $s = \sum_{g\in G} s_g g \in Z(S(V) * G)$. Then s commutes with all $t \in S(V)$ so that

$$\sum_{g\in G} s_g g(t) g = \sum_{g\in G} s_g g t = \sum_{g\in G} t s_g g.$$

Hence, for all $g \in G$, since $S(V)$ is an integral domain, we have either $s_g = 0$ or $g(t) = t$ for all $t \in S(V)$. Since the action is faithful, the only group element for which the second option occurs is the identity, thus s must be in $S(V)$. Then for all $g \in G$,

$$g(s)g = gs = sg$$

so that $g(s) = s$ and thus $s \in S(V)^G$. □

Corollary 2.1.39. $\mathrm{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])$ *is* $\mathbb{C}[x,y]^G$*-Macaulay.*

Proof. By Corollary 2.1.37 $\operatorname{End}_{C[x,y]^G}(C[x,y])$ is isomorphic to the skew group algebra $\mathbb{C}[x,y] * G$. Now $\mathbb{C}[x,y]$ is commutative with finite injective dimension so it is injectively homogeneous (as we will define in Chapter 4) and hence $\mathbb{C}[x,y] * G$ is injectively homogeneous by [60, Proposition 2.8]. But by [16, Theorem 3.4], injectively homogeneity of $\mathbb{C}[x,y] * G$ implies that it is $Z(\operatorname{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y]))$-Macaulay. By Lemma 2.1.38 $Z(\operatorname{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])) = \mathbb{C}[x,y]^G$ and we are done. □

Now we relate $A_{r,a} = \operatorname{End}_{\mathbb{C}[x,y]^G}(M)$ to $\operatorname{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])$. To do this we need the following lemma.

Lemma 2.1.40. *Let R be a ring, $U = M \oplus M'$ an R-module. Let $E := \operatorname{End}_R(U)$ and $T := \operatorname{End}_R(M)$. Then*

$$T \cong eEe$$

where e is the idempotent endomorphism

$$M \oplus M' \to M \oplus M'$$

$$(m, m') \mapsto (m, 0).$$

Proof. Define $\phi : eEe \to T$, where ϕ sends efe to the map f restricted to M. Then ϕ is a bijective ring homomorphism. □

Thus we have

$$A_{r,a} \cong e \operatorname{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])e$$

where $e = e^2 \in \operatorname{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])$ such that $e \mid_M$ is the identity.

Lemma 2.1.41. *Let R be a commutative ring, M a finitely generated R-module and $T := \operatorname{End}_R(M)$. Then for any maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}T \neq T$.*

Proof. We first show that the ring T is finitely generated as an R-module. Let $m_1, \ldots, m_t$ be the generators of M as an R-module. For $1 \leq i, j \leq t$, we define $f_{ij} : M \to M$ by

$$f_{ij}(m_k) = \delta_{ik} m_j.$$

Then it is easy to check that the f_{ij} generate T as an R-module. It is also easy to see that R embeds into $Z(T)$ via the map

$$r \mapsto \lambda_r,$$

where λ_r is left multiplication by r. Thus we can consider R as a subring of T and Theorem 1.2.5 applies. Hence for any maximal ideal $\mathfrak{m}$ of R there is a maximal ideal M of T such that $\mathfrak{m} \subseteq M$ and $MT \neq T$. Thus $\mathfrak{m}T \neq T$. □

Lemma 2.1.42. *Let R be a commutative noetherian ring and $U = M \oplus M'$ a finitely generated R-module. Let $E = \operatorname{End}_R(U)$ and let $e \in E$ be the idempotent defined in Lemma 2.1.40. Then if E is a Cohen-Macaulay R-module then so is eEe.*

Proof. Let $T := eEe$ which is isomorphic to $\operatorname{End}_R(M)$ by Lemma 2.1.40. and let $\mathfrak{m}$ be a maximal ideal of R and $\{x_1, \dots, x_n\}$ an R-sequence in $\mathfrak{m}$ on E. We claim that $\{x_1, \dots, x_n\}$ is also an R-sequence on T.

That $T/\sum_{i=1}^{n} x_i T = 0$ follows from Lemma 2.1.41 as $\sum_{i=1}^{n} x_i \subseteq \mathfrak{m}$. Now suppose that there exists some $efe \in T$ such that $x_i(efe) \in \sum_{j=1}^{i-1} x_j(eEe) \subseteq \sum_{j=1}^{i-1} x_j E$. Then, since x_i is a non zero divisor on $E/\sum_{j=1}^{i-1} x_j E$, we must have $efe = \sum_{j=1}^{i-1} x_j f_j$ for some $f_j \in E$. Hence, using that fact that $x_j \in Z(E)$,

$$efe = e^2 f e^2 = \sum_{j=1}^{i-1} e x_j f_j e = \sum_{j=1}^{i-1} x_j e f_j e \in \sum_{j=1}^{i-1} x_j(eEe) = \sum_{j=1}^{i-1} x_j T.$$

Thus x_i is a non zero divisor in $T/\sum_{j=1}^{i-1} x_j T$ and $\{x_1, \dots, x_n\}$ is also an R-sequence on T.

This gives $G(\mathfrak{m}, E) \leq G(\mathfrak{m}, T)$ and by [25, Lemma 18.1], [20, Proposition 1.2.12], [31, Lemma 15.1] and Cohen-Macaulayness of E, respectively, we have

$$G(\mathfrak{m}, E) \leq G(\mathfrak{m}, T) = G(\mathfrak{m}_\mathfrak{m}, T_\mathfrak{m}) \leq \text{Krull } \dim_{R_\mathfrak{m}}(T_\mathfrak{m}) \leq \text{Krull } \dim_{R_\mathfrak{m}}(E_\mathfrak{m}) = G(\mathfrak{m}, E).$$

It follows that T is a Cohen-Macaulay R-module. □

We can now apply this to $A_{r,a}$ giving the following theorem:

Theorem 2.1.43. *The reconstruction algebra $A_{r,a}$ is Z-Macaulay.*

Proof. Since $\operatorname{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])$ is a Cohen-Macaulay $\mathbb{C}[x,y]^G$-module by Corollary 2.1.39, Lemmas 2.1.40 and 2.1.42 apply with $R = \mathbb{C}[x,y]^G$, $U = \mathbb{C}[x,y]$ and $E = \operatorname{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])$ to give $A_{r,a}$ a Cohen-Macaulay $\mathbb{C}[x,y]^G$-module. [55] shows that $Z(A_{r,a}) = \mathbb{C}[x,y]^G$ and thus $R_{r,a}$ is Z-Macaulay. □

Theorem 2.1.43 and its proof suggest the following question:

Question 1. Let A be a noetherian affine algebra which is a finite module over its centre and X a Cohen-Macaulay A-module. When is $\operatorname{End}_A(M)$ is a Cohen-Macaulay A-module?

Here A is not necessarily commutative, but by a Cohen-Macaulay A-module we just mean an A-module X such that $G_Z(\mathfrak{m}, X) = \text{Krull } \dim_{A_\mathfrak{m}}(X_\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of Z.

The following example shows that the answer is not always positive.

Example 2.1.44. There exists a prime noetherian affine algebra R which is a finite module over its centre and a finitely generated R-module S with the following properties:
(i) S is a Cohen-Macaulay R-module,
(ii) $\text{End}_R(S)$ is not Cohen-Macaulay.

Proof. Let k be the field of 2 elements, $S = k[X_1, X_2, X_3, X_4]$ and $\sigma : S \to S$ the automorphism sending

$$\begin{array}{rcl} X_1 & \mapsto & X_1 + X_2 \\ X_2 & \mapsto & X_2 + X_3 \\ X_3 & \mapsto & X_3 + X_4 \\ X_4 & \mapsto & X_4. \end{array}$$

Then let $R := S * G$ where $G = \langle g \rangle$ is the finite group of order 4 acting on S by

$$gs = \sigma(s)g.$$

Then by Lemma 2.1.38 R is a noetherian ring with centre $Z = S^G = \{s \in S : \sigma(s) = s\}$. By [6, Theorem 1.3.1] R is a finite module over $Z = S^G$. Also, since S is prime and G is X-outer, [42, Theorem 4.1] shows that R is prime.

Now consider the left ideal $M := R(\sum_{x \in G} x)$. Then $M = S(\sum_{x \in G} x)$ since $sg(\sum_{x \in G} x) = s(\sum_{x \in G} x)$. Thus $M \cong S$ and so S is a left R-module with action $g.s = \sigma(s)$.
(i) Now by [60, Proposition 2.8] R is injectively homogeneous (to be defined in Chapter 4) and so by [16, Theorem 3.4] is centrally-Macaulay.

Let $\mathfrak{m}$ be a maximal ideal of $Z(R)$ and $\{x_1, \dots, x_n\} \subseteq \mathfrak{m}$ a sequence on R. Then we claim that it is a sequence on S. First note that since $\sum_{i=1}^{n} x_i R \neq R$ we must have $\sum_{i=1}^{n} x_i S \neq S$ since $R = S * G$. Now suppose there exists some $s \in S$ such that $x_{i+1} s \in \sum_{j=1}^{i} x_i S$. Then $x_{i+1} s \in \sum_{j=1}^{i} x_i R$ so that $s \in \sum_{j=1}^{i} x_i R \cap S$ since $\{x_1, \dots, x_n\}$ is a Z-sequence on R. But

$R = \bigoplus_{g \in G} Sg = S \oplus \bigoplus_{\substack{g \in G \\ g \neq 1}} Sg$ as left S-modules, so that $\sum_{j=1}^{i} R \cap S = \sum_{j=1}^{i} x_j S$. It therefore follows that $\{x_1, \ldots, x_n\}$ is a $Z(R)$-sequence on S. Thus

$$\mathrm{ht}(\mathfrak{m}) = G_Z(\mathfrak{m}, R) \leq G_Z(\mathfrak{m}, S) \leq \mathrm{ht}(\mathfrak{m}).$$

Hence S is a Cohen-Macaulay R-module.

(ii) However, $\mathrm{End}_R(S)$ is not Cohen-Macaulay. For,

$$\mathrm{End}_R(S) \subseteq \mathrm{End}_S(S) \cong S.$$

So for any $f \in \mathrm{End}_R(S)$ with $f(1) = s$, $g \in G$

$$f(g.1) = g(f(1)) = g.s = \sigma(s)$$

and

$$f(g.1) = f(\sigma(1)) = f(1) = s.$$

Thus $\sigma(s) = s$ and so $s \in S^G$. Similarly if $s \in S^G$ the map $f : 1 \mapsto s$ is in $\mathrm{End}_R(S)$, so $\mathrm{End}_R(S) \cong S^G$. But S^G is not Cohen Macaulay by [29, Example 16.8] so $\mathrm{End}_R(S)$ is not Cohen-Macaulay. □

On the other hand if we restrict ourselves to Krull dimension 2 we have the following result, where a *Cohen-Macaulay singularity* is a commutative Cohen-Macaulay ring.

Proposition 2.1.45. *[21, Lemma 3.1] Let (A, m) be a semiprime local Cohen-Macaulay singularity of Krull dimension 2. Let N be a maximal Cohen-Macaulay A-module and M a noetherian A-module. Then the A-module $\mathrm{Hom}_A(M, N)$ is a maximal Cohen-Macaulay module.*

So if we take $M = N$ in the above proposition, then we have $\mathrm{End}_A(N)$ Cohen-Macaulay for a maximal Cohen-Macaulay module N, giving a positive answer to our question in this case.

2.1.4 The Azumaya Locus

In [12, Theorem 3.8] Brown and Goodearl proved that under certain conditions the Azumaya locus is the complement of the singular locus, in $\mathrm{maxspec}(Z)$. Here we weaken their hypotheses in order to prove that this holds for the reconstruction algebras considered in the previous section. In particular, Brown and Goodearl prove this for a prime noetherian ring, which is Auslander-regular, Krull-Macaulay and height 1 Azumaya. We refer the reader to Definitions 2.2.1 and 4.1.5 for the definitions of Auslander-regular and Krull-Macaulay, but note that, in general, these conditions do not all hold for reconstruction algebras. We replace them with the weaker assumption that the ring is prime, has finite global dimension and is Z-Macaulay.

Let R be a prime noetherian ring satisfying hypothesis (H) as defined at the start of section 1.2 We also assume that the centre Z of R is affine over a field k. Recall from section 1.5 the definition of an Azumya algebra.

Definition 2.1.46. The *Azumaya locus* of a ring R is

$$\mathcal{A}_R = \{\mathfrak{m} \in \max Z : R_{\mathfrak{m}} \text{ is Azumaya over } Z_{\mathfrak{m}}\}$$

and the *singular locus* of R is

$$\mathcal{S}_R = \{\mathfrak{m} \in \max Z : \operatorname{gl.dim}(Z_{\mathfrak{m}}) = \infty\}.$$

We note from [15, Theorem III.1.7] that $\mathcal{A}_R$ is a nonempty open subset of maxspec Z. It is also easy to see from Proposition 1.5.11 that R is Azumaya if and only if $\mathcal{A}_R =$ maxspec Z.

In order to prove the theorem we use the following lemmas.

Lemma 2.1.47. *[13, Lemma III.1.8] Let R be a prime noetherian ring which is a finite module over its centre Z. Suppose that* $\operatorname{gl.dim}(R) < \infty$. *Then $\mathcal{A}_R \subseteq \max Z \backslash \mathcal{S}_R$.*

The following Lemma is [12, Lemma 3.6], but we give the proof here for completeness.

Lemma 2.1.48. *Let R be a prime noetherian ring, finitely generated and projective over its centre Z. If R is height* 1 *Azumaya over Z then it is Azumaya over Z.*

Proof. By Proposition 1.5.11, it's enough to show that $R_{\mathfrak{m}}$ is Azumaya over $Z_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$. Therefore we can assume that Z is local and hence R is a free Z-module, of rank r say. Then $R \otimes_Z R^{\text{op}}$ and $E := \operatorname{End}_Z(R)$ are both free Z-modules of rank r^2.

Consider the ring homomorphism $f : R \otimes_Z R^{\text{op}} \to E$ defined by

$$a \otimes b \mapsto (x \mapsto axb).$$

Then for any height 1 prime $\mathfrak{p}$ we have the following commutative diagram, where the vertical maps are the obvious ones:

$$\begin{array}{ccc} R \otimes_Z R^{\mathrm{op}} & \xrightarrow{\quad f \quad} & \mathrm{End}_Z(R) \\ \Big\downarrow{\scriptstyle \lambda_{\mathfrak{p}}} & & \Big\downarrow \\ R_{\mathfrak{p}} \otimes_{Z_{\mathfrak{p}}} R_{\mathfrak{p}}^{\mathrm{op}} & \xrightarrow{\quad f_{\mathfrak{p}} \quad} & \mathrm{End}_{Z_{\mathfrak{p}}}(R_{\mathfrak{p}}) \end{array} .$$

Now $\lambda_{\mathfrak{p}}$ is injective because $R \otimes_Z R^{\mathrm{op}}$ is torsion free with respect to the Ore set $Z \backslash \mathfrak{p}$, and since R is height 1 Azumaya over Z the map $f_{\mathfrak{p}}$ is an isomorphism. Thus f is injective. Since $\mathrm{Im}\, f$ and E are both free of rank r^2 there exists a monomorphism $g : E \to E$ such that $\mathrm{Im}\, f = \mathrm{Im}\, g$. Then for any height 1 prime $\mathfrak{p}$ of Z, we have $g_{\mathfrak{p}} = Z_{\mathfrak{p}} \otimes g : E_{\mathfrak{p}} \to E_{\mathfrak{p}}$ such that $\mathrm{Im}\, g_{\mathfrak{p}} = \mathrm{Im}\, f_p = E_{\mathfrak{p}}$. Hence $g_{\mathfrak{p}}$ is an isomorphism.

But $E \cong M_r(Z)$ and g is equivalent to an $r^2 \times r^2$-matrix, G, with determinant in Z. Since $g_{\mathfrak{p}}$ is an isomorphism, $\det g \in Z$ is invertible in $Z_{\mathfrak{p}}$ so is not contained in $\mathfrak{p}$. This holds for all primes $\mathfrak{p}$ of height 1.

Now consider the principal ideal $I := (\det g)Z$ of Z. If $I \neq Z$ then the principal ideal theorem [35, Thm 142] applies so that any prime $\mathfrak{q}$ minimal over I has height 1 contradicting the fact that $\det g$ is not contained in any prime of height 1. Thus $I = Z$ and $\det g$ is a unit in Z so that g is an isomorphism. Then $\mathrm{Im}\, f = \mathrm{Im}\, g = E$, f is an isomorphism and R is Azumaya over Z. □

Lemma 2.1.49. *Let R be a prime noetherian ring. Suppose that R is a finite module over its centre Z and R is Z-Macaulay. If R is height 1 Azumaya over Z, then*

$$\mathcal{A}_R \supseteq \max Z \backslash \mathcal{S}_R.$$

Proof. Let $\mathfrak{m} \in \max Z \backslash \mathcal{S}_R$, i.e. $\mathrm{gl.dim}(Z_{\mathfrak{m}}) < \infty$. Since R is Z-Macaulay, $R_{\mathfrak{m}}$ is $Z_{\mathfrak{m}}$-Macaulay by Proposition 2.1.5 and so by the Auslander-Buchsbaum depth Theorem (Theorem 1.1.26) $R_{\mathfrak{m}}$ is a projective $Z_{\mathfrak{m}}$-module. Now Lemma 2.1.48 applies to give $R_{\mathfrak{m}}$ Azumaya over $Z_{\mathfrak{m}}$ and thus $\max Z \backslash \mathcal{S}_R \subseteq \mathcal{A}_R$. □

We can now state the Theorem, whose proof follows immediately from the lemmas above.

Theorem 2.1.50. *Let R be a prime noetherian ring. Suppose that R is a finite module over its centre Z, $\mathrm{gl.dim}(R) < \infty$, and that R is Z-Macaulay. If R is height 1 Azumaya over Z, then*

$$\mathcal{A}_R = \max Z \backslash \mathcal{S}_R.$$

We now apply this theorem to reconstruction algebras.

Lemma 2.1.51. *The reconstruction algebra $A_{r,a}$ is height 1 Azumaya.*

Proof. $A = A_{r,a} = \mathrm{End}_{\mathbb{C}[x,y]^G}(M)$ where M is a maximal Cohen-Macaulay Z-module and $Z = \mathbb{C}[x,y]^G$. Now Z is the invariant ring of an integrally closed domain so by [20, Proposition 6.4.1] is an integrally closed domain. Let $\mathfrak{p}$ be a height one prime of Z. Then $Z_{\mathfrak{p}}$ is an integrally closed local noetherian domain of Krull dimension 1. Hence $Z_{\mathfrak{p}}$ is a DVR and thus a PID. Also

$$A_{\mathfrak{p}} = \mathrm{End}_Z(M) \otimes_Z Z_{\mathfrak{p}} = \mathrm{End}_{Z_{\mathfrak{p}}}(M_{\mathfrak{p}}),$$

by [19, Proposition 1.6]. Since $M_{\mathfrak{p}}$ is a torsion-free $Z_{\mathfrak{p}}$-module, $M_{\mathfrak{p}}$ is free. Thus $A_{\mathfrak{p}} \cong M_n(Z_{\mathfrak{p}})$ for some n and hence is Azumaya. □

Corollary 2.1.52. *Let $R = A_{r,a}$ be a reconstruction algebra as defined in Definition 2.1.34. Then*

$$\mathcal{A}_R = \max Z \backslash \mathcal{S}_R.$$

Proof. By [55, Corollary 6.4 and Theorem 6.18] $A_{r,a}$ is a prime noetherian ring with finite global dimension. We showed in Theorem 2.1.43 that it is Z-Macaulay and in Lemma 2.1.51 that it is height 1 Azumaya. Thus $A_{r,a}$ satisfies the conditions required in Theorem 2.1.50. □

We also consider whether or not the hypotheses in Theorem 2.1.50 are necessary. That is, if we have a prime noetherian ring which is a finite module over its centre, then would the conclusion still hold if the following properties didn't all hold:

1. $\mathrm{gl.dim}(R) < \infty$;
2. R is height 1 Azumaya over Z;
3. R is a Cohen-Macaulay Z-module?

For hypothesis 1 consider the following commutative example.

Example 2.1.53. Let $R = \mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t]$, the coordinate ring of the cusp. We showed in Example 1.1.21 that R is Gorenstein and hence Cohen-Macaulay but has infinite global dimension. Since R is commutative it is Azumaya over R so $\mathcal{A}_R = \max Z \neq \max Z \backslash \mathcal{S}_R$.

The following example, which is the enveloping algebra of the 2-dimensional non-abelian Lie algebra in characteristic p, shows that the second property is necessary.

Example 2.1.54. Let R be the skew polynomial ring $k[x][y;x(d/dx)]$ where k is a field of characteristic $p>0$. Then R is a prime noetherian ring which is a finite module over its centre and,

(i) R has finite global dimension;

(ii) R is a Cohen-Macaulay Z-module;

(iii) $\mathcal{A}_R \neq \max Z \backslash \mathcal{S}_R$.

Proof. First note that one easily shows by direct calculation that the centre $Z=k[x^p,z]$, where $z=\prod_{i=0}^{p-1}(y-i)=y^p-y$. Thus R is Z-free of degree p^2 and by [41, Theorem 1.2.9] R is prime and noetherian.

(i) By [41, Theorem 7.5.3(i)] R has finite global dimension.

(ii) The maximal ideals of Z are $\mathfrak{m}=\langle x^p-\lambda, z-\mu\rangle$ which clearly have height 2. But $\{x^p-\lambda, z-\mu\}$ is a Z-sequence on R, so $G_Z(\mathfrak{m},R)=2$ for all maximal ideals $\mathfrak{m}$ of Z.

(iii) First note that the singular locus is empty as Z is regular, so we just need to show that R is not Azumaya. But, for each $\alpha\in k$, the simple module

$$V_\alpha := R/\langle x, y-\alpha\rangle \cong \frac{R/xR}{\langle x, y-\alpha\rangle/xR} \cong k[y]/\langle y-\alpha\rangle$$

is 1-dimensional, whereas R has PI-degree p. Thus R cannot be Azumaya. □

The third example shows the necessity of hypothesis 3.

Example 2.1.55. Let k be a field, let $Z=k[X,Y]$ with $M=\langle X,Y\rangle$ and let

$$R=\begin{pmatrix} Z & M \\ Z & Z \end{pmatrix}.$$

Then R is a prime noetherian ring which is a finite module over its centre such that

(i) R has global dimension 2;

(ii) R is height 1 Azumaya over Z;

(iii) $\mathcal{A}_R \neq \max Z \backslash \mathcal{S}_R$.

Proof. It is easy to check that $Z(R)=Z$ and that R is prime.

(i) The ring R is the idealizer of the maximal right ideal

$$A=\begin{pmatrix} M & M \\ Z & Z \end{pmatrix}$$

of $M_2(Z)$. Thus [41, Corollary 7.5.12] applies to give $\mathrm{gl.dim}(R)=\mathrm{gl.dim}(M_2(Z))=2$.

(ii) Let $\mathfrak{p}$ be any height 1 prime of Z. Then $M\cap(Z\backslash\mathfrak{p})\neq\emptyset$ so that $M\otimes Z_\mathfrak{p}=Z_\mathfrak{p}$. Hence

$R_{\mathfrak{p}} = M_2(Z_{\mathfrak{p}})$ which is Azumaya.

(iii) Again the singular locus is empty so we just need to show that R is not Azumaya. However, this is true as the maximal ideals $\begin{pmatrix} M & M \\ Z & Z \end{pmatrix}$ and $\begin{pmatrix} Z & M \\ Z & M \end{pmatrix}$ are not regular as the factor rings are isomorphic to k and so must have PI degree 1. □

2.2 Krull-Macaulay rings

We saw in Theorem 1.1.11 that for a commutative equidimensional Cohen-Macaulay ring we have the following equation for all finitely generated R-modules,

$$\text{Krull dim}(R) = \text{Krull dim}(M) + j(M).$$

This gives us another generalisation of the Cohen-Macaulay property.

Definition 2.2.1. Let R be a ring satisfying (H). Then R is *Krull-Macaulay* if

$$\text{Krull dim}(R) = \text{Krull dim}(M) + j(M) \tag{2.8}$$

for all finitely generated left and right R-modules M.

It is clear that commutative equidimensional Cohen-Macaulay rings give examples of Krull-Macaulay rings. And as we'll see in chapter 4, equidimensional injectively homogeneous rings are another class of examples (this follows from Theorem 4.2.2). Note that, as the following theorem shows, when proving a ring is Krull-Macaulay, it is enough to show that the equation (2.8) holds for all simple R modules. However we leave the proof till section 3.2.3.

Theorem 2.2.2. *Assume hypothesis (H). To show that R is Krull-Macaulay it is enough to show that equation (2.8) holds for all simple modules.*

We now consider some properties and open questions about Krull-Maculay rings. We will prove the following nice properties of Krull-Macaulay rings in part of Theorem 3.2.4:

Proposition 2.2.3. *Let R be a ring satisfying (H). If R is Krull-Macaulay then*

(i) R is Z-Macaulay and

(ii) R is equidimensional.

Proposition 2.2.4. *Let R be a ring satisfying (H). If R is Krull-Macaulay then R is locally Krull-Macaulay. That is, $R_{\mathfrak{m}}$ is Krull-Macaulay for all maximal ideals $\mathfrak{m}$ of Z.*

Proof. Let $\mathfrak{m}$ be any maximal ideal of Z. In view of Theorem 2.2.2, we just need to check equation (2.8) for the modules $R_{\mathfrak{m}}/M_{\mathfrak{m}}$ where M runs through the maximal ideals of R lying over $\mathfrak{m}$. So let M be a maximal ideal of R with $\mathfrak{m} = M \cap Z$. Then by Lemma 1.3.11 we have

$$j(R/M) = j(R_{\mathfrak{m}}/M_{\mathfrak{m}}).$$

On the other hand, Proposition 2.2.3 shows that R is equidimensional, so Krull $\dim(R) = \mathrm{ht}(M) = \mathrm{ht}(\mathfrak{m}) =$ Krull $\dim(R_{\mathfrak{m}})$. Thus we now have

$$\text{Krull}\ \dim(R_{\mathfrak{m}}) = j(R_{\mathfrak{m}}/M_{\mathfrak{m}})$$

as required. □

However we have the following counterexample to the converse.

Example 2.2.5. A locally Krull-Macaulay ring need not be Krull-Macaulay. Let $S = k[X] \oplus k$. Then

(i) S is not Krull-Macaulay but

(ii) S is locally Krull-Macaulay.

Proof. The ring S has maximal ideals $J := k[X] \oplus 0$ and $I_\lambda := \langle X - \lambda \rangle \oplus k$ for $\lambda \in k$. The ideal J has height zero while the ideals I_λ have height one as they all contain the minimal prime $I := k \oplus 0$.

(i) It's not true that all finitely generated S-modules satisfy (2.8). For, if we take the module $S/J \cong k$ then Krull $\dim(S) = 1$, Krull $\dim(S/J) =$ Krull $\dim(k) = 0$ and $j(S/J) = 0$ since we have the non-zero map $h : S/J \to S$ where $(0, 1) + J \mapsto (0, 1)$. Hence

$$\text{Krull}\ \dim(S/J) + j(S/J) = 0 + 0 \neq 1 = \text{Krull}\ \dim(S).$$

(ii) We first localise at the maximal ideal J to get $S_J \cong k$ and consider the simple module $(S/J)_J = S/J \cong k$. Then

$$\text{Krull}\ \dim(S_J) = 0$$

and

$$\text{Krull}\ \dim((S/J)_J) + j_{S_J}((S/J)_J) = 0 + 0 = 0.$$

For maximal ideals $I_\lambda := \langle X - \lambda \rangle \oplus k$ we have $S_{I_\lambda} \cong \{fg^{-1} : f, g \in k[X], (X - \lambda) \nmid g\}$ and the simple module $(S/I_\lambda)_{I_\lambda} = S/I_\lambda$. Thus

$$\text{Krull}\ \dim(S_{I_\lambda}) = \mathrm{ht}(I_\lambda) = 1$$

and

$$\text{Krull}\ \dim((S/I_\lambda)_{I_\lambda}) + j_{S_{I_\lambda}}((S/I_\lambda)_{I_\lambda}) = 0 + 1 = 1.$$

Hence the ring S is locally Krull-Macaulay. □

We now consider whether or not the Krull-Macaulay property is stable under factoring by central non zero divisors.

Lemma 2.2.6. *Let R be a ring satisfying (H). Let x be any central non zero divisor in R and let $\overline{\mathrm{R}} := R/xR$. Then*
(i) $\mathrm{ht}(\overline{M}) = \mathrm{ht}(M) - 1$ *for all maximal ideals $\overline{M} = M/xR$ of $\overline{\mathrm{R}}$, and*
(ii) assuming that R is equidimensional, Krull $\dim(\overline{\mathrm{R}}) =$ Krull $\dim(R) - 1$.

Proof. We first note that [41, Lemma 6.3.9] tells us that

$$\text{Krull}\ \dim(\overline{\mathrm{R}}) \leq \text{Krull}\ \dim(R) - 1. \tag{2.9}$$

(i) Applying the Principal Ideal Theorem, Theorem 1.8.4, shows that any prime Q of R minimal over xR has height at most one. If Q has height zero, this would contradict (2.9), so it must have height one and therefore $\mathrm{ht}(\overline{M}) = \mathrm{ht}(M) - 1$.
(ii) Equidimensionality of R gives $\mathrm{ht}(\overline{M}) =$ Krull $\dim(R) - 1$, which together with (2.9) shows that Krull $\dim(\overline{\mathrm{R}}) = \mathrm{ht}(\overline{M}) =$ Krull $\dim(R) - 1$. □

Proposition 2.2.7. *Let R be a ring satisfying (H). Let x be any central non zero divisor in R. If R is Krull-Macaulay then so is $\overline{R} := R/xR$.*

Proof. We assume that R is Krull-Macaulay and from Proposition 2.2.4 we see that R is also equidimensional. By Theorem 2.2.2 it is enough to show that (2.8) holds for the $\overline{\mathrm{R}}$-module $\overline{R}/\overline{M}$ where $\overline{M}$ is a maximal ideal of R/xR. So let $\overline{M}$ be a maximal ideal of $\overline{\mathrm{R}}$. Then $\overline{M} = M/xR$ for some maximal ideal M of R and $\overline{\mathrm{R}}/\overline{M} \cong R/M$. We see from Theorem 1.3.5 that

$$\mathrm{Ext}^n_{\overline{\mathrm{R}}}(R/M, R/xR) \cong \mathrm{Ext}^{n+1}_R(R/M, R)$$

and hence $j_{\overline{\mathrm{R}}}(R/M) = j_R(R/M) - 1$. Together with Lemma 2.2.6(ii) this shows that

$$\text{Krull}\ \dim(\overline{\mathrm{R}}/\overline{M}) = j_{\overline{\mathrm{R}}}(\overline{\mathrm{R}}/\overline{M})$$

since R is Krull-Macaulay. □

We have the following partial converse to this proposition.

Proposition 2.2.8. *Let R be a ring satisfying (H) such that Z is local. Let $x \in J(R)$ be a central non zero divisor. If $\overline{\mathrm{R}} := R/xR$ is Krull-Macaulay then so is R.*

Proof. We first show that R is Z-Macaulay. Since $\overline{\mathrm{R}}$ is Krull-Macaulay it follows from Proposition 2.2.3 that $\overline{\mathrm{R}}$ is $Z(\overline{\mathrm{R}})$-Macaulay. Then it is easy to see that R is Z-Macaulay. To see this, let M be a maximal ideal of R. Then since $x \in M$ and x is a central non zero divisor $G_Z(M) = G_{Z(R)}(\overline{M}) + 1$. Also, Lemma 2.2.6(i) tells us that $\mathrm{ht}(M) = \mathrm{ht}(\overline{M}) + 1$ and thus R is Z-Macaulay. Then since Z is local, Theorem 2.1.4 shows that R is equidimensional and Lemma 2.2.6(ii) applies to give Krull $\dim(\overline{\mathrm{R}}) = \mathrm{ht}(\overline{M}) =$ Krull $\dim(R) - 1$.

On the other hand, R/M is an $\overline{\mathrm{R}}$-module so we again have

$$\mathrm{Ext}^n_{\overline{\mathrm{R}}}(R/M, R/xR) \cong \mathrm{Ext}^{n+1}_R(R/M, R)$$

by Theorem 1.3.5. Hence $j_{\overline{\mathrm{R}}}(R/M) = j_R(R/M) - 1$, and the result now follows. □

However, the converse is not true in general as we see from the following example, where $R = Z$ is not local.

Example 2.2.9. Let $S = k[X] \oplus k$. Then

(i) S is not Krull-Macaulay but

(ii) $x = (X, 1)$ is a non zero divisor such that S/xS is Krull-Macaulay.

Proof. (i) We showed in Example 2.2.5 that S is not Krull-Macaulay.

(ii) We have the following ring isomorphisms:

$$S/xS = k[X] \oplus k/\langle X \rangle \oplus k \cong k \oplus 0 \cong k.$$

The field k is trivially Krull-Macaulay. □

Proposition 2.2.10. *Let R and S be Morita equivalent noetherian rings. Then R Krull-Macaulay if and only if S is Krull-Macaulay. In particular, if R is Krull-Macaulay and $n \geq 1$ then $M_n(R)$ is Krull-Macaulay.*

Proof. Since R and S are Morita equivalent, their module categories are equivalent. Suppose that R is Krull-Macaulay and let X be a finitely generated S-module. Then X corresponds to an R-module Y with Krull $\dim(R)$ = Krull $\dim(Y) + j_R(Y)$. Now, Krull $\dim(R)$ = Krull $\dim(S)$ by [41, Proposition 6.5.1], and the equivalence of the module categories gives Krull $\dim_S(X)$ = Krull $\dim_R(Y)$ and $j_S(X) = j_R(Y)$. Thus Krull $\dim(S) =$ Krull $\dim(X) + j_S(X)$. □

We now give some open questions on Krull-Macaulay, which would have been interesting to consider, had more time been available.

Question 2. If R is Krull-Macaulay is $R[X]$ Krull-Macaulay?

Question 3. If R is Krull-Macaulay are the skew polynomial rings $R[X;\sigma]$, $R[X;\delta]$ and $R[X;\sigma,\delta]$ Krull-Macaulay, assuming these extensions are finite modules over their centres?

Question 4. If R is Krull-Macaulay and $G \subseteq \mathrm{Aut}(R)$ with $|G| < \infty$, is the skew-group algebra $R * G$ Krull-Macaulay?

Question 5. Are reconstruction algebras Krull-Macaulay?

Question 6. If R is Krull-Macaulay and $e = e^2 \in R$ then is eRe Krull-Macaulay?

We have already seen one non-example, the ring $S = k[X] \oplus k$. To find more examples we concentrate on finite dimensional k-algebras. We will show in Chapter 3 that an equidimensional ring is Krull-Macaulay if and only if it is Z-Macaulay and j-symmetric (see Theorem 3.2.10). A finite dimensional k-algebra A is always equidimensional and Z-Macaulay so A is Krull-Macaulay if and only if it is j-symmetric.

Proposition 2.2.11. *Let A be a finite dimensional k-algebra. Then A is Krull-Macaulay if and only if $j(V) = 0$ for all simple left and right A-modules.*

Proof. It remains to show that j is symmetric if and only if $j(V) = 0$ for all simple left and right A-modules. So assume j is symmetric and let M be a maximal ideal of A. Then by Theorem 3.2.3 we have a maximal ideal Q with $Q \cap Z = M \cap Z$ such that $j(R/Q) = G(M) = 0$. Then Muller's Theorem (Theorem 1.2.15) shows that Q and M are in the same clique, as defined in Definition 1.2.14. Then, as we will show in Lemma 3.2.8, $j(R/M) = j(R/Q) = 0$.

Now suppose that $j(V) = 0$ for all simple left and right A-modules. Then since any finitely generated left or right A-module X has a simple homomorphic image we can see that $j(X) = 0$. Thus j is symmetric. □

Thus we are interested in finitely generated k-algebras A whose simple modules embed into A.

Definition 2.2.12. [37, Definition 8.26] A ring R is a *right Kasch ring* if every simple right R-module can be embedded in R_R. R is a *Kasch ring* if it is both left and right Kasch.

We now consider some examples which illustrate this property.

Example 2.2.13. Let A be a finite dimensional k-algebra.

(i) Clearly if A is commutative j is symmetric so this property is satisfied.

(ii) If A is quasi-Frobenius then $\operatorname{inj.dim}(A) = 0$ so $j(X) = 0$ for any module X.

(iii) If A is local then clearly A is Kasch.

(iv) A ring R is a cogenerator ring if the modules R_R and ${}_RR$ are cogenerators, where a module U is a cogenerator if $\operatorname{Hom}_R(-, U) : mod-R \to Ab$ is a faithful functor. [37, Proposition 19.16] shows that A is Kasch if it is a cogenerator ring.

(v) Dual rings are Kasch as shown in [43, Theorem 6.19], where R is a dual ring if $\operatorname{Ann}_r(\operatorname{Ann}_l(I)) = I$ for all right ideals I and $\operatorname{Ann}_l(\operatorname{Ann}_r(J)) = J$ for all left ideals J.

(vi) Let $S = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ where k is a field. It is easy to check that S has a simple left modules and a simple right module which do not embed into R and hence S is not a Kasch ring. We will return to this example in section 3.1.

We also note the necessity of requiring $j(V) = 0$ for all left and right simple modules as [37] gives an example of an artinian ring with is right Kasch but not left Kasch (see Example 8.29.6).

One property of Krull-Macaulay rings is that the function $-j(-)$ becomes a symmetric dimension function. If the equation

$$\text{Krull }\dim(R) = \text{Krull }\dim(M) + j(M)$$

holds then $-j$ becomes a shifted version of Krull dimension which we have already noted is a symmetric dimension function for rings satisfying (H), and so it is easy to see that the conditions for a dimension function hold. We could ask if the converse is true, i.e. if $-j(-)$ is a symmetric dimension function for a ring R is R Krull-Macaulay. However, this is not the case as we see from the following example.

Example 2.2.14. Let $S = k[X] \oplus k$. Then,

(i) $-j(-)$ is a symmetric dimension function for S, but

(ii) S is not Krull-Macaulay.

Proof. (i) Since R is a commutative Gorenstein ring it is Auslander-Gorenstein (see chapter 3 for the definition) and Levasseur proves in [38, Proposition 4.5] that $-j(-)$ is a dimension function in an Auslander-Gorenstein ring.

(ii) We showed this in Example 2.2.5. □

2.3 GK-Macaulay rings

We also note here that we could also generalise the Cohen-Macaulay property using GK dimension rather than Krull dimension. We again refer the reader to [41, Chapter 8] for the definition and properties of GK dimension.

Definition 2.3.1. Let R be a noetherian k-algebra with finite GK dimension. Then R is *GK-Macaulay* if

$$\text{GK}\ \dim(R) = \text{GK}\ \dim(M) + j_R(M)$$

for all finitely generated R-modules M.

However, we remind the reader that, as noted in section 1.4, Krull dimension is equivalent to GK dimension for affine algebras satisfying hypthesis (H). Therefore we focus our attention on Krull-Macaulay.

2.4 Notes

All the results in Chapter 2 are original unless explicitly referenced, with the exception of Propositions 2.1.2, 2.1.5, 2.1.8 and 2.1.9 which are almost certainly known but for which we were unable to find suitable references.

Chapter 3

Comparing different generalisations of Cohen-Macaulay

Throughout this chapter, R is a ring satisfying hypothesis (H) as defined in section 1.2. Here we define 10 different properties which are possible generalisations of the Cohen-Macaulay property to the noncommutative case. They are all related to either centrally-Macaulay or Krull-Macaulay as defined in chapter 2 and we will consider when and how these properties are equivalent. Recall the definition of grade symmetry as found in Definition 1.3.3. We suggest that the notion of the grade function $j(-)$ being symmetric is crucial in generalising Cohen-Macaulay to the noncommutative case.

Property 1. *R is Krull-Macaulay.*

Property 2. *For all simple left or right modules X,*

$$\text{Krull dim}(R) = j(X). \tag{3.1}$$

Property 3. *R is grade-symmetric and for all simple left modules X,*

$$\text{Krull dim}(R) = j(X).$$

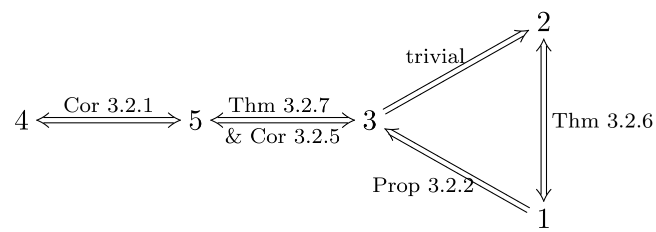
Note that it is enough to require the equation to hold for left modules as it then holds for all simple right module. For, if X is a simple right R-module then $M :=$ $\text{Ann}(X)$ is a maximal ideal and $j^r(X) = j^r(R/M) = j^l(R/M)$ by grade symmetry. But M is the annihilator of some simple left module Y so that $j^r(X) = j^l(R/M) = j^l(Y) =$ Krull dim(R).

Property 4. *R is grade-symmetric and for all prime ideals P of R,*

$$G_Z(P) = \mathrm{ht}(P).$$

Property 5. *R is grade-symmetric and Z-Macaulay.*

The question is, when and how are they equivalent? So we first try to show that at least for a ring which satisfies hypothesis (H) and is equidimensional these 5 properties are equivalent. Also, for an equidimensional ring R and a central subring C over which R is finitely generated, R is Z-Macaulay if and only if R is C-Macaulay, by Corollary 2.1.14. This means that there is no added generality in replacing Z by such a subring in Properties 4 or 5. The following diagram shows how we go about proving the equivalence.



Then, in order to get an equivalent set of properties for a ring that is not equidimensional we consider the following "local" versions of Properties 1-5 . Here C is an arbitrary central subring of R over which R is finitely generated.

Property 6. *R is locally Krull-Macaulay. That is, for all non-zero finitely generated left or right R-modules X and all maximal ideals $M \in Supp(X)$, with $\mathfrak{m} = M \cap C$*

$$\text{Krull dim}(R_{\mathfrak{m}}) = \text{Krull dim}(X_{\mathfrak{m}}) + j_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}). \tag{3.2}$$

Property 7. *For all simple left or right R-modules X,*

$$\text{Krull dim}(R_{\mathfrak{m}}) = j_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}).$$

where M is the maximal ideal $\mathrm{Ann}(X)$ *of R and $\mathfrak{m} = M \cap C$.*

Note that $\mathrm{Ann}(X) \in Supp(X)$ since in passing to $X_{\mathrm{Ann}(X)\cap Z}$ we only invert elements which do not annihilate X.

Property 8. *For all maximal ideals $\mathfrak{m}$ of C, $R_{\mathfrak{m}}$ is grade-symmetric and for all simple left or right $R_{\mathfrak{m}}$-modules X, we have*

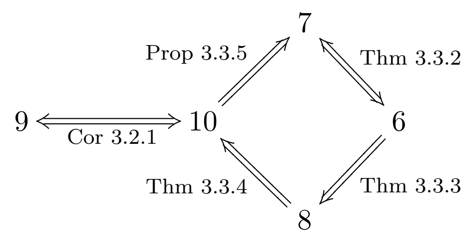
$$\text{Krull dim}(R_{\mathfrak{m}}) = j_{R_{\mathfrak{m}}}(X).$$

Property 9. *For all maximal ideals* $\mathfrak{m}$ *of* C, $R_{\mathfrak{m}}$ *is grade-symmetric, and for all prime ideals* P *of* $R_{\mathfrak{m}}$,

$$G_{C_m}(P) = \mathrm{ht}(P).$$

Property 10. *For all maximal ideals* $\mathfrak{m}$ *of* C, $R_{\mathfrak{m}}$ *is grade-symmetric and* $C_{\mathfrak{m}}$*-Macaulay.*

We will use the results we prove about the first 5 properties to prove that the second 5 properties are indeed equivalent. Again, we show in a diagram how we intend to do this.



We note that Properties 1-5 and 6-10 are equivalent if R is equidimensional as we will demonstrate in Theorem 3.3.1.

3.1 Examples

We consider some examples which illustrate the properties defined above.

Example 3.1.1. Z-Macaulay rings are not necessarily Krull-Macaulay, neither are all finite dimensional algebras grade-symmetric. Let $S = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ where k is a field. Then

(i) S is Z-Macaulay,

(ii) S is not Krull-Macaulay,

(iii) S is not grade-symmetric.

Proof. (i) Since S is a finite dimensional k-algebra with centre $Z \cong k$, Example 2.1.3 applies to show that S is Z-Macaulay.

(ii) Let

$$M = \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$$

be the two maximal ideals of S. We will consider S/M as a left S-module and check equation (2.8). First of all $S/M \hookrightarrow S$ if and only if S/M is isomorphic to a minimal left ideal. But the only minimal left ideals are $\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} =: I$, neither of which is isomorphic to S/M as a left S-module. So $\mathrm{Hom}_S({}_S(S/M), S) = 0$. However, it can be checked that $\mathrm{Ext}^1({}_S(S/M), S) \cong M_2(k)/S \neq 0$ as right R-modules, so $j^l(S/M) = 1$. However S is artinian so Krull $\dim(S/M) =$ Krull $\dim(S) = 0$ and (2.8) fails for the left module S/M. Hence S is not Krull-Macaulay.

(iii) If we now consider S/M as a right S-module we have $S/M \cong \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}$ which is a right ideal of S, so $\mathrm{Hom}_S((S/M)_S, S) \neq 0$. Thus $j^r(S/M) = 0$ and $j^l(S/M) = 1$ so that S is not grade-symmetric. □

Example 3.1.2. Z-Macaulay, grade-symmetric rings are not necessarily Krull-Macaulay. Let $S = k[X] \oplus k$. Then

(i) S is Z-Macaulay,

(ii) S is grade-symmetric,

(iii) S is not Krull-Macaulay.

Proof. We first consider the prime ideals of S. S has maximal ideals $J := k \oplus 0$ of height zero and $I_\lambda := \langle X - \lambda \rangle \oplus k$, for $\lambda \in k$, of height one.

(i) This is shown in Example 2.1.11.

(ii) It is clear that $j(-)$ is symmetric because S is commutative.

(iii) This was proved in Example 2.2.5. □

Note that the ring S in Example 3.1.2 is not equidimensional and because of this we restrict to the equidimensional case when trying to prove that property 5 implies property 1. However, we have already seen in Example 2.2.5 that S is locally Krull-Macaulay.

3.2 The properties are equivalent for equidimensional rings

3.2.1 Equivalence of 4 and 5

The equivalence of properties 4 and 5 is an easy corollary of Theorem 2.1.4.

Corollary 3.2.1. *For a ring satisfying hypothesis (H) Properties 4 and 5 are equivalent, that is,* $\mathrm{ht}(P) = G_Z(P, R)$ *for all prime ideals* P *of* R *if and only if* $\mathrm{ht}(M) = G_Z(M, R)$ *for all maximal ideals* M *of* R.

Proof. That Property 5 implies Property 4 follows from Theorem 2.1.4 while the other implication is trivial. □

3.2.2 Krull-Macaulay implies Z-Macaulay and j symmetric.

For this we will prove that 1 implies 3 which in turn implies 5.

1 implies 3

Here we assume that R is Krull-Macaulay. Then trivially (3.1) holds for all simple modules and we just need to show that R is grade-symmetric. But by Proposition 1.4.3 Krull dimension is symmetric on bimodules so by (2.8) $j(-)$ is symmetric. Hence we have

Proposition 3.2.2. *Assume hypothesis (H). Then property 1 implies property 3. That is, if R is Krull-Macaulay, (3.1) holds for all simple modules and R is grade-symmetric.*

3 implies 5

We now show that if property 3 holds then 5 holds. We already have $j(-)$ symmetric so we only need to show that R is Z-Macaulay.

Theorem 3.2.3. *Assume that R satisfies hypothesis (H). Let I be a maximal ideal of R. Then there exists a maximal ideal Q of R with $Q \cap Z = I \cap Z$ such that $G(Q) = j(R/Q)$.*

Proof. Let I be a maximal ideal of R, and let $G(I) = t$. Let $\{x_1, \dots, x_t\}$ be a maximal Z-sequence in I and set $T := \sum_{i=1}^{t} x_i R$. Then by Theorem 1.3.5, we get

$$\mathrm{Ext}^i_R(R/I, R/T) \cong \mathrm{Ext}^{i+t}_R(R/I, R).$$

Thus, if $k < t$ then $\mathrm{Ext}^k(R/I, R) = 0$ so that $j(R/I) \geq t$. And $j(R/I) = t \Leftrightarrow \mathrm{Hom}(R/I, R/T) \neq 0$.

Apply Lemma 1.8.2 with $S = ((I \cap Z) + T)/T$, which consists of zero-divisors on $M := R/T$ since the sequence is maximal. Then we have some $y \in R \backslash T$ such that $(I \cap Z)y \in T$. Now apply Lemma 1.8.3 to the ring R/T with $X = R/T$, $J = R(I \cap Z) + T/T$ and $m = y + T$. Then there exists some prime ideal Q/T of R/T, with $I \cap Z \subseteq Q$, such that Q/T is maximal with respect to $\mathrm{Ann}_M(Q/T) \neq 0$. But since $I \cap Z \subseteq Q \cap Z$ and $I \cap Z$ is maximal, we must have $I \cap Z = Q \cap Z$ and so Q lies over $I \cap Z$. Then by Theorem 1.2.5 Q must be maximal.

We then have some $0 \neq x + T \in R/T$ such that $Q(x + T) = 0$, which gives $0 \neq f \in \mathrm{Hom}(R/Q, R/T)$. So it follows that $j_{R/T}(R/Q) = 0$, and hence by Theorem 1.3.5 $j_R(R/Q) = t$. Since $Q \cap Z = I \cap Z$ we have $G(Q) = G(I) = j(R/Q)$ as required. □

Theorem 3.2.4. *Let R be a ring with hypothesis (H) which satisfies Property 2. Then*
(i) R is Z-Macaulay,
(ii) if I is an arbitrary maximal ideal,

$$\text{Krull }\dim(R) = j(R/I) = G(I) = \text{ht}(I)$$

and hence R is equidimensional,
(iii) if $\{x_1, \ldots, x_t\}$ is a maximal Z-sequence in I and $T = \sum_{i=1}^{t} x_i R$ then, $\text{Hom}_{R/T}(R/I, R/T) \neq 0$ *and I is an annihilator prime of R/T.*

Proof. Assume that R satisfies hypothesis (H) and that property 2 holds for R. Let I be a maximal ideal of R. By Theorem 3.2.3 we have some maximal ideal Q such that $Q \cap Z = I \cap Z$ and $G(Q) = j(R/Q)$. Now (3.1) gives $j(R/Q) = \text{Krull }\dim(R) \geq \text{ht}(I)$, so that

$$G(I) = G(Q) = j(R/Q) \geq \text{ht}(I). \tag{3.3}$$

But we know from [17, §4.4] that $G(I) \leq \text{ht}(I)$, so $G(I) = \text{ht}(I)$. Thus R is Z-Macaulay.

To see that (ii) is true, we note that (3.1) shows that Krull $\dim(R) = j(R/I) = j(R/Q)$ and then the remaining equalities follow from (3.3).

Now from Theorem 1.3.5, $\text{Hom}_{R/T}(R/I, R/T) \cong \text{Ext}^t(R/I, R) \neq 0$ so I is an annihilator prime of R/T. □

The following corollary of Theorem 3.2.4 is clear.

Corollary 3.2.5. *Assume hypothesis (H) and suppose that Property 3 holds, i.e. R is grade-symmetric and the equation*

$$\text{Krull }\dim(R) = j(M)$$

holds for all simple R-modules M. Then R is Z-Macaulay and grade-symmetric. That is, Property 5 holds for R.

3.2.3 Equivalence of 1 and 2

We now give the proof for Theorem 2.2.2 as stated in section 2.2.

Theorem 3.2.6. *Assume hypothesis (H). Then properties 1 and 2 are equivalent, i.e. (2.8) holds for all finitely generated R-modules if and only if it holds for all simple modules.*

Proof. Here we would like to show that if (2.8) holds for all simple modules then it holds for all finitely generated modules. So suppose that (2.8) holds for all simple modules. Then it holds for all artinian modules by the step $d = 0$ in the proof of Proposition 1.4.13. We want to show that (2.8) holds for all finitely generated modules X and do this by induction on the Krull dimension of X. So suppose that X has Krull dimension $d > 0$ and that (2.8) holds for all finitely generated modules of Krull dimension at most $d-1$. To show (2.8) holds for all finitely generated modules of Krull dimension d, by Lemma 1.4.12 it is enough to do so for $X = R/P$, where P is prime and Krull $\dim(R/P) = d$.

Since P is not maximal, there exists a proper prime ideal Q of R such that $P \subsetneq Q$ with $\mathrm{ht}(Q) = \mathrm{ht}(P) + 1$. Choose $x \in (Q \cap Z) \backslash (P \cap Z)$. Note that x exists by Theorem 1.2.5 since P is properly contained in Q. Then we get the following short exact sequence

$$0 \to R/P \xrightarrow{x\times} R/P \to R/I \to 0$$

where $I = xR + P$. We know from Theorem 3.2.4 that if all simple modules of R satisfy (2.8) then R is centrally Macaulay, so by Theorem 2.1.7 R is catenary. Thus

$$\text{Krull } \dim(R/Q) = \text{Krull } \dim(R/P) - 1 = d - 1.$$

Also Krull $\dim(R/I) <$ Krull $\dim(R/P)$ by [31, Ex 15F] so

$$\text{Krull } \dim(R/Q) \leq \text{Krull } \dim(R/I) < \text{Krull } \dim(R/P) = \text{Krull } \dim(R/Q) + 1.$$

Hence Krull $\dim(R/I) =$ Krull $\dim(R/Q) = d - 1$, so by our induction hypothesis we have $j(R/I) = \text{Krull } \dim(R) - \text{Krull } \dim(R/I) = n - d + 1$.

The short exact sequence above gives the following long exact sequence

$$\cdots \to \mathrm{Ext}^i(R/P, R) \xrightarrow{x\times} \mathrm{Ext}^i(R/P, R) \to \mathrm{Ext}^{i+1}(R/I, R) \to \mathrm{Ext}^{i+1}(R/P, R) \to \cdots. \tag{3.4}$$

Since R is centrally Macaulay, and hence $G(P) = \mathrm{ht}(P)$ for all primes P by Theorem 2.1.4, we know that

$$\mathrm{ht}(P) \leq j(R/P) \tag{3.5}$$

by Corollary 1.3.6.

Now let $i := \mathrm{ht}(P) = n - d$ by catenarity. To show (2.8) holds for R/P it's enough to show that $j(R/P) = i$. We know from (3.5) that $j(R/P) \geq i$ so it remains to show that $\mathrm{Ext}^i(R/P, R) \neq 0$. But since $j(R/I) = i + 1$ this is true by Lemma 1.3.13(i). □

3.2.4 Z-Macaulay and j symmetric implies Krull-Macaulay.

Recall that in Example 3.1.2 we showed that property 5 does not imply property 1 in a non-equidimensional ring so in this section we assume that R is equidimensional. Note that here we also see the significance of the symmetry of $j(-)$. We would like to show that property 1 holds if $G(\mathfrak{m}) = \mathrm{ht}(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R and to do this we assume that the function $j(-)$ is symmetric. We show that property 5 implies property 3 and then Theorem 3.2.6 shows that property 1 is satisfied.

Theorem 3.2.7. *Suppose that R is an equidimensional ring satisfying hypothesis (H). If R has property 5 then it has property 3. That is, assuming grade-symmetry, if R is Z-Macaulay then equation (3.1),*

$$\text{Krull dim}(R) = j(X),$$

holds for all simple R-modules X.

Recall from section 1.2 the definition of the clique of a prime ideal of R.

Lemma 3.2.8. *Assume that R satisfies hypothesis (H) and is grade-symmetricj. Let P, Q be maximal ideals of R belonging to the same clique. Then $j(R/P) = j(R/Q)$.*

Proof. Since P and Q are in the same clique, there exist $P = P_1, P_2, \dots, P_m = Q$ such that either $P_i \rightsquigarrow P_{i+1}$ or $P_{i+1} \rightsquigarrow P_i$. So it's enough to show that if $M \rightsquigarrow N$ then $j(R/M) = j(R/N)$. So assume $0 \neq M \cap N/MN$. Then it is a left R/M-module and a right R/N-module where R/M and R/N are simple artinian. Therefore, as left modules, $M \cap N/MN$ is isomorphic to a direct sum of copies of the irreducible left R/M-module so that

$$j^l(M \cap N/MN) = j(R/M),$$

and, as right modules, to a direct sum of copies of the irreducible right R/N-module so that

$$j^r(M \cap N/MN) = j(R/N).$$

Hence $j(R/M) = j(R/N)$ since j is symmetric. □

Now we prove Theorem 3.2.7.

Proof. Consider R/P where P is a maximal ideal. Then Krull $\dim(R/P) = 0$ and, since R is equidimensional and Z-Macaulay,

$$\text{Krull dim}(R) = \mathrm{ht}(P) = G(P)$$

so equation (3.1) holds if and only if

$$j(R/P) = G(P).$$

By Theorem 3.2.3 there exists a maximal ideal Q such that $Q \cap Z = P \cap Z$ and $j(R/Q) = G(Q) = G(P)$. By Müller's Theorem (Theorem 1.2.15) Q and P belong to the same clique. Thus $j(R/Q) = j(R/P)$ by Lemma 3.2.8. Hence $j(R/P) = G(P)$ so that (3.1) holds for R/P. Then it holds for any simple module V since $V \oplus \cdots \oplus V \cong R/P$ for some maximal ideal P. □

We have now shown that in the equidimensional case the Z-Macaulay property and the symmetry of $j(-)$ imply the Krull-Macaulay property. We also get the following corollary to Lemma 3.2.8.

Corollary 3.2.9. *Assume that R satisfies hypothesis (H) and is Krull-Macaulay. Let P and Q be prime ideals of R in the same clique. Then $j(R/P) = j(R/Q)$.*

Proof. Notice first that since R is Krull-Macaulay $j(-)$ is symmetric. Localise at $\mathfrak{p} := P \cap Z = Q \cap Z$. Then $P_\mathfrak{p} \cap Z_\mathfrak{p} = Q_\mathfrak{p} \cap Z_\mathfrak{p}$ so that $P_\mathfrak{p}$ and $Q_\mathfrak{p}$ are in the same clique by Theorem 1.2.15. But in $R_\mathfrak{p}$ the ideals $P_\mathfrak{p}$ and $Q_\mathfrak{p}$ are maximal so by Lemma 3.2.8 (which applies since R is grade-symmetric) they have the same homological grade.

Note that Lemma 1.3.11 applies because R is Krull-Macaulay. Applying it to P and Q gives

$$j(R/P) = j_{R_\mathfrak{p}}(R_\mathfrak{p}/P_\mathfrak{p})$$

and

$$j(R/Q) = j_{R_\mathfrak{p}}(R_\mathfrak{p}/Q_\mathfrak{p}).$$

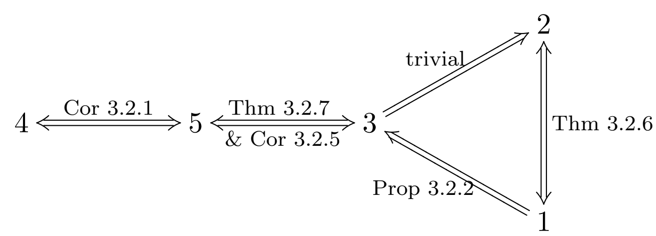
Hence it follows that

$$j(R/P) = j(R/Q).$$

□

3.2.5 Summary

We summarise progress in the following diagram.



We have therefore proved the following Theorem.

Theorem 3.2.10. *Let R be an equidimensional ring satisfying (H). Then the first five properties are equivalent for R.*

Only Theorem 3.2.7 uses equidimensionality and we note that Example 3.1.2 shows that the implication $5 \Rightarrow 3$ is not always valid for a ring which is not equidimensional.

3.3 The equivalence of properties for rings which are not equidimensional.

We now consider R as before but no longer assume that it is equidimensional. We fix a central subring C over which R is finitely generated. Here we mainly consider properties 6 to 10 but we use results we have proved about properties 1 to 5. Before doing so, we first show that the two different sets of properties are indeed equivalent for a ring which is equidimensional. This follows from the following result together with Theorem 3.2.10 and the parallel result Theorem 3.3.6.

Theorem 3.3.1. *Let R satisfy hypothesis (H) and suppose that R is equidimensional. Then the two sets of properties are equivalent.*

Proof. By Corollary 1.3.8 we have $j_R(R/M) = j_{R_\mathfrak{m}}(R_\mathfrak{m}/M_\mathfrak{m})$ for all maximal ideals M of R with $\mathfrak{m} = M \cap C$. Let X be any simple R-module and $M = \mathrm{Ann}_R(X)$. Then we have

$$j_R(X) = j_R(R/M) = j_{R_\mathfrak{m}}(R_\mathfrak{m}/M_\mathfrak{m}) = j_{R_\mathfrak{m}}(X).$$

Also, since R is equidimensional, Krull $\dim(R) =$ Krull $\dim(R_\mathfrak{m})$. Thus it is clear that in this case, properties 2 and 7 are equivalent. □

3.3.1 Equivalence of 6 and 7

We show here that property 7 implies 6. So suppose that for all simple R modules, X we have

$$\text{Krull dim}(R_{\mathfrak{m}}) = j_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}),$$

where $M = \text{Ann}_R(X)$ and $\mathfrak{m} = M \cap C$. Then for all maximal ideals M

$$\text{Krull dim}(R_{\mathfrak{m}}) = j_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/M_{\mathfrak{m}})$$

so that Property 2 holds for $R_{\mathfrak{m}}$. Then by Theorem 3.2.6 we know that (2.8) holds for all finitely generated $R_{\mathfrak{m}}$-modules. Now let X be any finitely generated R-module and M any maximal ideal in $Supp(X)$. Then for $\mathfrak{m} = M \cap C$, $X_{\mathfrak{m}}$ is an $R_{\mathfrak{m}}$-module and

$$\text{Krull dim}(R_{\mathfrak{m}}) = \text{Krull dim}(X_{\mathfrak{m}}) + j_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}).$$

Thus property 6 holds and we have

Theorem 3.3.2. *Properties 6 and 7 are equivalent. That is, if the equation*

$$\text{Krull dim}(R_{\mathfrak{m}}) = \text{Krull dim}(X_{\mathfrak{m}}) + j_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}),$$

holds for all simple R-modules (where $\mathfrak{m}$ is a maximal ideal of C such that $M \in Supp(X)$ for any M lying over $\mathfrak{m}$), then it holds for all finitely generated modules.

3.3.2 6 implies 8

Theorem 3.3.3. *Property 6 implies Property 8. That is, suppose that all finitely generated R-modules X satisfy equation (3.2):*

$$\text{Krull dim}(R_{\mathfrak{m}}) = \text{Krull dim}(X_{\mathfrak{m}}) + j_{R_{\mathfrak{m}}}(X_{\mathfrak{m}})$$

for every maximal ideal M of R in the support of X. Then for all maximal ideals M of R, $j(-)$ is symmetric on $R_{\mathfrak{m}}$-$R_{\mathfrak{m}}$-bimodules and (trivially) all simple modules satisfy (3.2).

Proof. Assume property 6. Let $\mathfrak{m}$ be a maximal ideal of C. We just need to show the symmetry of $j(-)$ on finitely generated central $R_{\mathfrak{m}}$-$R_{\mathfrak{m}}$-bimodules. So let $0 \neq X$ be a finitely generated central $R_{\mathfrak{m}}$-$R_{\mathfrak{m}}$-bimodule. Then $X = Y_{\mathfrak{m}}$ for some finitely generated central R-R-bimodule Y. Note that since $R_{\mathfrak{m}}$ is FBN, the right and left Krull dimensions of finitely generated $R_{\mathfrak{m}}$-$R_{\mathfrak{m}}$-bimodules coincide by Proposition 1.4.3. Thus, by the left and right versions of (3.2),

$$\text{Krull dim}(X) + j^l(X) = \text{Krull dim}(R_{\mathfrak{m}}) = \text{Krull dim}(X) + j^r(X)$$

and $j^l(X) = j^r(X)$ as required. □

3.3.3 8 implies 10

Here we assume property 8, i.e. $j(-)$ is symmetric on $R_\mathfrak{m}$-$R_\mathfrak{m}$-bimodules and for all simple modules X we have

$$\text{Krull dim}(R_\mathfrak{m}) = j_{R_\mathfrak{m}}(X_\mathfrak{m}),$$

for the maximal ideal $\mathfrak{m} = \text{Ann}(X) \cap C$ of C. We want to show Property 10, namely that $R_\mathfrak{m}$ is grade-symmetric and $C_\mathfrak{m}$-Macaulay.

Theorem 3.3.4. *Suppose that the ring R satisfies hypothesis (H) and C is a central subring of R with R a finitely generated C-module. Then Property 8 implies Property 10.*

Proof. Suppose that Property 8 holds for R. Let $\mathfrak{m}$ be a maximal ideal of C and let $X_\mathfrak{m}$ be a simple $R_\mathfrak{m}$-module. Then by Property 8 we have

$$\text{Krull dim}(R_\mathfrak{m}) = j_{R_\mathfrak{m}}(X_\mathfrak{m}).$$

which means that Property 3 holds in the semi-local ring $R_\mathfrak{m}$. Then by Corollary 3.2.5 $R_\mathfrak{m}$ is grade-symmetric and $C_\mathfrak{m}$-Macaulay. □

3.3.4 10 implies 7

It now remains to prove 10 implies 7.

Proposition 3.3.5. *Let R satisfy hypothesis (H) and let C be a central subring of R such that R is a finitely generated C-module. If $R_\mathfrak{n}$ is grade-symmetric and $C_\mathfrak{n}$-Macaulay for all maximal ideals $\mathfrak{n}$ of C then for all simple R-modules X with $\mathfrak{m} = \text{Ann}(X) \cap C$ we have*

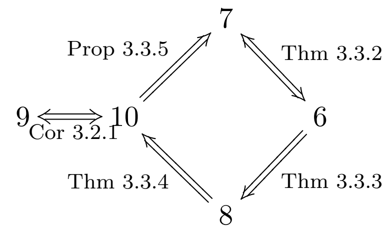
$$\text{Krull dim}(R_\mathfrak{m}) = j_{R_\mathfrak{m}}(X_\mathfrak{m}).$$

Proof. For all maximal ideals $\mathfrak{m}$ of C, property 5 holds for $R_\mathfrak{m}$. Now $R_\mathfrak{m}$ is semi-local and $C_\mathfrak{m}$-Macaulay so by [17, Theorem 4.11] is equidimensional. Thus Theorem 3.2.7 applies and property 1 holds for $R_\mathfrak{m}$. Now let X be a simple R-module. Then $X_\mathfrak{m}$ is a simple $R_\mathfrak{m}$-module and

$$\text{Krull dim}_{R_\mathfrak{m}} = j_{R_\mathfrak{m}}(X_\mathfrak{m}).$$

□

To summarise we draw another diagram.



Thus we have:

Theorem 3.3.6. *Let R be a ring satisfying (H). Then properties 6 to 10 are equivalent.*

3.4 What is the correct generalisation for the Cohen-Macaulay property?

This leads us to consider the question of which is the best generalisation of the commutative definition of Cohen-Macaulay.

First note that we think that the Krull-Macaulay property is too strong a definition as even in the commutative case, non-equidimensional Cohen-Macaulay rings are not necessarily Krull-Macaulay. So the Krull-Macaulay property does not reduce to the usual commutative definition.

We think that it should be Property 5, that is, R is Z-Macaulay and grade-symmetric.

Definition 3.4.1. We will call a ring satisfying Property 5 *symmetrically Macaulay.*

So what should be true of the "correct" definition? Well firstly, it should reduce to the usual definition in the commutative case, which is clearly true of symmetrically Macaulay. We should also be able to generalise the hierarchy which we have in Theorem 1.1.20:

$$\text{regular} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay},$$

and we will see in Chapter 4 that if we take symmetrically Macaulay as our generalisation of the Cohen-Macaulay property we get a corresponding hierarchy for rings which are finite modules over their centres. Also many results can be proved about centrally Macaulay rings (and hence about symmetrically Macaulay rings) such as the results in [17] and our Theorem 2.1.50.

We will return to this discussion in Chapter 4 to explain why symmetrically Macaulay is a better generalisation than centrally Macaulay.

We may also consider the use of the GK-Macaulay property as a generalisation for the Cohen-Macaulay property. For a connected graded k-algebra R, where k is a field, it is common to define R to be Cohen-Macaulay if it satisfies the GK-Macaulay property. This is done, for example, by Levasseur in [38] and by Stafford and Zhang in [51]. The main disadvantage of this approach is that many nice algebras have infinite GK dimension. For example, let k be a field and G a finitely generated group. Then [36, Theorem 1.1] shows that kG has finite GK-dimension if and only if G is has a nilpotent normal subgroup N such that G/N is finite. We note that there do exist examples of polycyclic by finite groups which are not nilpotent by finite. These give group algebras which are noetherian, but with infinite GK-dimension.

We note also an example of an algebra which is GK-Macaulay but not Krull-Macaulay, suggesting that Krull-Macaulay is not the best generalisation of the Cohen-Macaulay property.

Example 3.4.2. Let $A := A_2(\mathbb{C})$, be the second Weyl algebra over $\mathbb{C}$. Then $A_n(\mathbb{C})$ is GK-Macaulay but not Krull-Macaulay.

Proof. By [41, 1.6.13(iii)] the associated graded ring of A is $k[x_1, x_2, y_1, y_2]$, which is GK-Macaulay. Then from [51, Lemma 4.4] A is GK-Macaulay. It remains to show that the equation

$$\text{Krull dim}(A) = \text{Krull dim}(M) + j_A(M)$$

does not hold for all finitely generated A-modules. But by [49, Theorem 1.1] there exists a simple right A-module $V = A/M$ where M is a maximal right ideal of A which is principal. We can show that $j_A(V) = 1$. Also, Krull dim$(A) = 2$ by [36, Corollary 8.4]. This gives

$$\text{Krull dim}(A) = 2 \neq 0 + 1 = \text{Krull dim}(M) + j_A(M).$$

□

We leave open the question of how to generalise the Cohen-Macaulay property for rings which are not finite modules over their centres. However, we have shown the importance of the symmetry of the grade function $j(-)$ which suggests that the generalisation of the Cohen-Macaulay property should be grade symmetry plus some other condition. For rings satisfying hypothesis (H), that other condition is Z-Macaulay.

3.5 Notes

All results in Chapter 3 are original apart from Corollary 3.2.1.

Chapter 4

Generalising the commutative result: Gorenstein implies Cohen-Macaulay

It is a well known result in commutative ring theory that a Gorenstein ring is Cohen-Macaulay. See Theorem 1.1.20 for this. In Chapter 2 we looked at generalisations of the Cohen-Macaulay property to the noncommutative case but Gorenstein and regular rings have also been generalised and we now look at these generalisations and the connections between them. Throughout this chapter we will assume that R is a noetherian ring which is a finite module over its centre Z. Also, C will be a central subring of R such that R is a finitely generated C-module.

4.1 Some definitions

In this section we return to the noncommutative generalisations of the Cohen-Macaulay property discussed in Chapters 2 and 3 and also consider parallel noncommutative generalisations for Gorenstein and regular. At various points we have to restrict to the case where the ring R is equidimensional, as defined in Definition 1.2.8.

Recall the generalisations of the Cohen-Macaulay property from Chapters 2 and 3: centrally-Macaulay in Definition 2.1.1, Krull-Macaulay in Definition 2.2.1 and smmetrically Macaulay in Definition 3.4.1.

In chapter 3 we proved the following theorem, as Theorem 3.2.10:

Theorem 4.1.1. *Let R be an equidimensional noetherian ring which is a finite module over its centre Z. Then R is symmetrically Macaulay if and only if it is Krull-Macaulay.*

Now we turn to generalisations of the Gorenstein property and other related properties of commutative rings. Homologically homogeneous rings and injectively homogeneous rings were introduced by Brown and Hajarnavis in [15] and [16] and they generalise the commutative properties regular and Gorenstein, respectively. The following definitions are slightly more restrictive in their scope than the original definitions of Brown and Hajarnavis in [15] and [16].

Definition 4.1.2. Let R be a noetherian ring which is a finite module over a central subring C.
(i) Then R is *homologically homogeneous over C* if it has finite global dimension and for all irreducible right (or left) R-modules, V, W whose annihilators in C are equal, we have

$$\text{pr.}\dim(V) = \text{pr.}\dim(W).$$

(ii) We say that R is *homologically homogeneous* if it is homologically homogeneous over its centre Z.

Definition 4.1.3. Let R be a noetherian ring which is a finite module over a central subring C.
(i) The *upper grade* of an ideal I is defined by $\text{u.gr}(I) = \sup\{n : \text{Ext}^n_R(R/I, R) \neq 0\}$.
(ii) R is *injectively homogeneous* over C if R has finite right (or left) injective dimension and

$$\text{u.gr}(M) = \text{u.gr}(N)$$

for all maximal ideals M and N of R, with $M \cap C = N \cap C$.
(iii) We say that R is *injectively homogeneous* if it is injectively homogeneous over its centre Z. Similarly R is *injectively smooth* if R has finite right (or left) injective dimension and

$$\text{u.gr}(M) = \text{inj.}\dim(R)$$

for all maximal ideals M of R.

Note that where we don't specify the central subring in any of the above definitions, we assume it to be the centre. We also notice that it is easy to see that a ring R which is homologically homogeneous over a central subring C is homologically homogeneous over

Z. This follows because any two modules with equal annihilators in Z have equal annihilators in C. Similarly, injective homogeneity over C implies injective homogeneity over Z. Also, we omit reference to left or right injective dimension when discussing injectively homogenous rings as [16, Corollary 4.4] shows that these are equal.

In [16, Theorem 6.5] Brown and Hajarnavis generalise the commutative result that regular implies Gorenstein:

Theorem 4.1.4. *The noetherian ring R is homologically homogeneous if and only if it is injectively homogeneous and has finite global dimension.*

Alternatively, we can generalise the Gorenstein property using the Auslander condition, which was first studied for commutative rings by Bass in [4], and was later considered by Auslander for noncommutative rings.

Definition 4.1.5. Let R be a noetherian ring. Then
(i) R satisfies the *Auslander condition* if for every finitely generated left or right R-module M and for all $i \geq 0$, $j(N) \geq i$ for all submodules $N \subseteq \mathrm{Ext}^i(M, R)$.
(ii) R is *Auslander-Gorenstein* if it satisfies the Auslander condition and has finite left and right injective dimension.
(iii) R is *Auslander-regular* if it satisfies the Auslander condition and has finite global dimension.

Note that commutative Gorenstein rings are Auslander-Gorenstein as proved in the fundamental theorem in [4].

We also have another generalisation of the Gorenstein property. In [2], Artin and Schelter introduced a notion of regularity for connected $\mathbb{N}$-graded algebras, now known as AS-regular. Then there is the related notion of an AS-Gorenstein algebra, which we focus on here. It is easy to define AS-Gorenstein for rings with a "special" simple module, such as connected graded rings, local rings and Hopf algebras. However, AS-Gorenstein has been generalised to general noetherian rings, as in [52], giving the following definition.

Definition 4.1.6. Let R be a noetherian ring which is a finite module over its centre. R is left *AS-Gorenstein* if there exists some $n \in \mathbb{N}$ such that for all irreducible left or right R-modules V,

$$\mathrm{Ext}^i_R(V, R) \neq 0 \Leftrightarrow i = n.$$

Note that by [16, Lemma 3.1] this implies that R has finite injective dimension as $\mathrm{inj.dim}(R) = \sup\{\mathrm{u.gr}(V) : V \text{ a simple } R\text{-module}\}$.

4.2 Some characterisations of the injectively homogeneous property

We now concentrate on the injectively homogeneous property. First we give some properties of injectively homogeneous rings before giving, in Theorem 4.2.2, a number of characterisations of injectively homogeneous rings, which we will use in order to prove our generalisation of the commutative result that Gorenstein rings are Cohen-Macaulay.

Theorem 4.2.1. *Let R be a noetherian ring which is a finite module over its centre.*
(i) If R is injectively homogeneous over a central subring C then R is C-Macaulay and

$$\text{inj.}\dim(R) = \text{Krull }\dim(R) \text{ and } \text{inj.}\dim(R_{\mathfrak{m}}) = \text{Krull }\dim(R_{\mathfrak{m}})$$

for all maximal ideals $\mathfrak{m}$ of C.
(ii) If R is injectively smooth then R is Krull-Macaulay.
(iii) R is injectively smooth if and only if R is equidimensional and injectively homogeneous over a central subring C for which R is a finitely generated C-module.
(iv) If R is injectively homogeneous then R is Auslander-Gorenstein

Proof. (i) Let R be injectively homogeneous over C. Then R is C-Macaulay by [16, Theorem 3.4]. That $\text{inj.}\dim(R) = \text{Krull }\dim(R)$ follows from [16, Corollary 3.5] and since $R_{\mathfrak{m}}$ is injectively homogeneous over $C_{\mathfrak{m}}$ by [16, Lemma 3.3] we have that $\text{inj.}\dim(R_{\mathfrak{m}}) = \text{Krull }\dim(R_{\mathfrak{m}})$ for all maximal ideals $\mathfrak{m}$ of C.
(ii) This is [60, Proposition 3.2].
(iii) Let R be injectively homogeneous over C. Then for all maximal ideals M of R we have $\text{ht}(M) = \text{u.gr}(R/M)$ by [16, Corollary 3.5]. The result then follows from (i).
(iv) This follows from [60, Theorem 3.7]. □

We now prove some characterisations of injectively homogeneous. Note that the hypothesis that R is equidimensional is needed in Theorem 4.2.2 as we show in Example 4.2.9 below.

Theorem 4.2.2. *Let R be a noetherian ring which is a finite module over a central subring C. Assume that R is equidimensional. Then the following are equivalent:*

1. R is injectively homogeneous over C of injective dimension n.

2. R is injectively smooth of injective dimension n.

3. R is C-Macaulay and $\text{inj.}\dim(R_R) = \text{Krull }\dim(R) = n < \infty$.

4. R is C-Macaulay and grade-symmetric with $\text{inj.dim}(R_R) = n < \infty$.

5. R is Krull-Macaulay and $\text{inj.dim}(R_R) = n < \infty$.

6. R is C-Macaulay and there exists an invertible R-bimodule I with $\text{inj.dim}(I_R) = \text{Krull dim}(R) = n < \infty$.

7. R is C-Macaulay and $\text{inj.dim}(I_R) = \text{Krull dim}(R) = n < \infty$ *for every invertible R-bimodule I.*

8. Any of the corresponding properties for $_RR$ *hold.*

We split up the proof of Theorem 4.2.2 into several propositions. Note that the equivalence of 1 and 2 is Theorem 4.2.1(iii). First we prove the equivalence of 1 and 3, noting that 1 implies 3 holds even for a nonequidimensional ring.

Proposition 4.2.3. *Let R be a noetherian ring which is a finite module over a central subring C. We assume that R is equidimensional. Then R is injectively homogeneous over C of injective dimension n if and only if* $\text{inj.dim}(R_R) = \text{Krull dim}(R) = n < \infty$ *and R is C-Macaulay.*

Proof. The forward implication follows from Theorems 3.4 and 3.5 in [16] as recorded above in Theorem 4.2.1(i). Now we assume that R is C-Macaulay and $\text{inj.dim}(R_R) = \text{Krull dim}(R) = n < \infty$ and show that R is injectively homogeneous over C. Let M be a maximal ideal of R. In the notation introduced in Definition 4.1.3, $\text{u.gr}(M) \geq G(M) = \text{ht}(M)$ by [15, Corollary 3.1] and the fact that R is C-Macaulay. Since we are assuming that all maximal ideals have the same height we therefore have

$$\text{u.gr}(M) \geq \text{ht}(M) = \text{Krull dim}(R) = \text{inj.dim}(R) \geq \text{u.gr}(M),$$

giving the equality of the upper grade of all maximal ideals. □

To show that 1 implies 4 it is enough to show that injectively homogeneous implies that j is symmetric as the other properties in 4 follow from the definition and from Theorem 4.2.1(i). We first prove that j is symmetric for an equidimensional injectively homogeneous ring, but will show in Theorem 4.2.10 that this restriction is not necessary.

Proposition 4.2.4. *Let R be an equidimensional noetherian ring, which is a finite module over its centre Z and injectively homogeneous over a central subring C. Then the grade function* $j(-)$ *is symmetric on all R-R-bimodules.*

Proof. Since R is injectively homogeneous over C and equidimensional, it is injectively smooth by Theorem 4.2.1(iii). Thus, by (ii) of the same theorem, R is Krull-Macaulay, and hence j is symmetric by Proposition 3.2.2. □

For 4 implies 1, we do not assume that R is equidimensional.

Proposition 4.2.5. *If R is C-Macaulay and grade-symmetric and R has finite injective dimension then R is injectively homogeneous over C.*

Proof. Suppose that R is C-Macaulay and grade-symmetric with $\operatorname{inj.dim}(R) < \infty$. Let M be a maximal ideal of R and $\mathfrak{m} = M \cap C$. Then

$$G_C(M) = G_{C_\mathfrak{m}}(M_\mathfrak{m}) \leq j(R_\mathfrak{m}/M_\mathfrak{m}) \leq \operatorname{inj.dim}(R_\mathfrak{m})$$

by [25, Lemma 18.1] and Corollary 1.3.6, so by [16, Theorem 3.4] it is enough to show that $\operatorname{inj.dim}(R_\mathfrak{m}) = G(M)$ for all maximal ideals $\mathfrak{m}$ of C with $M \cap C = \mathfrak{m}$.

So let $\mathfrak{m}$ be a maximal ideal of C. Let $G(M) = \operatorname{ht}(\mathfrak{m}) = d$ and let $\{x_1, \ldots, x_d\}$ be a maximal C-sequence on R in $\mathfrak{m}$. Then, using hats to denote images in $C_\mathfrak{m}$, [17, Proposition 4.7] shows that $\{\widehat{x_1}, \ldots, \widehat{x_d}\}$ is a $C_\mathfrak{m}$-sequence on $R_\mathfrak{m}$ in $\mathfrak{m}_\mathfrak{m}$. Let $\overline{R_\mathfrak{m}} := R_\mathfrak{m}/\sum_{i=1}^{d} \widehat{x}_i R_\mathfrak{m}$. Now Krull $\dim(R_\mathfrak{m}) = \operatorname{ht}(\mathfrak{m})$ by [31, Theorem 10.20] and by [41, Lemma 6.3.9], Krull $\dim(\overline{R_\mathfrak{m}}) \leq$ Krull $\dim(R_\mathfrak{m}) - d = 0$. Hence $\overline{R_\mathfrak{m}}$ is artinian. Thus the right socle of $\overline{R_\mathfrak{m}}$ contains some simple right $R_\mathfrak{m}$-module V.

By Theorem 1.3.5 we have $\operatorname{Ext}^d_{R_\mathfrak{m}}(\overline{R_\mathfrak{m}}, R_\mathfrak{m}) \cong \operatorname{Hom}_{R_\mathfrak{m}}(\overline{R_\mathfrak{m}}, \overline{R_\mathfrak{m}}) \neq 0$ and so $j_{R_\mathfrak{m}}(\overline{R_\mathfrak{m}}) \leq d$. On the other hand, Theorem 1.3.5 also gives $\operatorname{Ext}^i_{R_\mathfrak{m}}(\overline{R_\mathfrak{m}}, R_\mathfrak{m}) = 0$ for $i < d$ so that

$$j_{R_\mathfrak{m}}(\overline{R_\mathfrak{m}}) = d.$$

Similarly $\operatorname{Ext}^d_{R_\mathfrak{m}}(V, R_\mathfrak{m}) \cong \operatorname{Hom}_{R_\mathfrak{m}}(V, \overline{R_\mathfrak{m}}) \neq 0$ so that $j_{R_\mathfrak{m}}(V) \leq d$ and $\operatorname{Ext}^i_{R_\mathfrak{m}}(V, R_\mathfrak{m}) = 0$ for $i < d$ so that $j_{R_\mathfrak{m}}(V) = d$. The maximal ideal $\operatorname{Ann}_{R_\mathfrak{m}}(V)$ is of the form $M_\mathfrak{m}$ for some maximal ideal M of R and we have $R_\mathfrak{m}/M_\mathfrak{m} \cong V^{\oplus t}$ so that

$$j(R_\mathfrak{m}/M_\mathfrak{m}) = j(V) = d.$$

Now, since R is grade-symmetric, Lemma 3.2.8 shows that all maximal ideals M' lying over $\mathfrak{m}$ of R have $j(R/M') = d$. Then Corollary 1.3.8 shows that

$$j(R_\mathfrak{m}/M'_\mathfrak{m}) = d$$

which, by Theorem 1.3.5 again, gives the fact that every simple left $R_\mathfrak{m}$-module occurs in the socle of $\overline{R_\mathfrak{m}}$.

Now let $\text{inj.}\dim(R_{\mathfrak{m}}) = n < \infty$. Clearly $n \geq d$ (since all simple $R_{\mathfrak{m}}$-modules have grade equal to d). We want to show that $n = d$, so we assume that $n > d$, and look for a contradiction. [16, Lemma 3.1] tells us that there exists a simple right $R_{\mathfrak{m}}$-module $\hat{V}$ with

$$\text{Ext}^n_{R_{\mathfrak{m}}}(\hat{V}, R_{\mathfrak{m}}) \neq 0.$$

Then since $\hat{V}$ occurs in the right socle of $\bar{R}$ we have the following short exact sequence

$$0 \to \hat{V} \to \overline{R_{\mathfrak{m}}} \to X \to 0$$

for some right $R_{\mathfrak{m}}$-module X, and hence

$$\text{Ext}^n_{R_{\mathfrak{m}}}(\overline{R_{\mathfrak{m}}}, R_{\mathfrak{m}}) \to \text{Ext}^n_{R_{\mathfrak{m}}}(\hat{V}, R_{\mathfrak{m}}) \to \text{Ext}^{n+1}_{R_{\mathfrak{m}}}(X, R_{\mathfrak{m}}).$$

Now this gives a contradiction since the middle term is non-zero, but the outer terms are zero since $\text{pr.}\dim_{R_{\mathfrak{m}}}(\overline{R_{\mathfrak{m}}}) = d < n$ by Theorem 1.3.16 and $\text{inj.}\dim(R_{\mathfrak{m}}) = n$. Thus we must have n equal to d and the result now follows. □

Thus we have proved that 1 is equivalent to 4. That 4 and 5 are equivalent follows from Theorem 3.2.10 together with Corollary 2.1.14. We now show that 3, 6 and 7 are all equivalent via the following proposition. We refer back to Definition 1.6.1 for the definition of an invertible bimodule.

Proposition 4.2.6. *Let R be a noetherian ring. Then* $\text{inj.}\dim(R_R) = \text{inj.}\dim(A_R)$ *for any invertible right R-bimodule A.*

Noting that R is invertible as an R-bimodule, the equivalence of 3, 6 and 7 is now clear. Before proving the proposition we give a preparatory but well-known lemma.

Lemma 4.2.7. *Let R be a ring, E an injective right R-module and A an invertible R-bimodule. Then $E \otimes_R A$ is an injective right R-module.*

Proof. By Proposition 1.6.2 tensoring with an invertible bimodule gives an equivalence of categories between the right R-modules. Thus injectives go to injectives and $E \otimes_R A$ is injective. □

Now we prove Proposition 4.2.6.

Proof. Suppose we have an injective resolution

$$0 \to R \to I_0 \to I_1 \to \cdots \to I_m \to 0$$

for R where each I_i is an injective right R-module. Then we tensor the resolution by A to get

$$0 \to R \otimes_R A \to I_0 \otimes_R A \to \cdots \to I_m \otimes_R A \to 0.$$

Now $I_i \otimes A$ is injective by Lemma 4.2.7 and the invertible module A is projective and hence flat, so the new complex is exact and hence is an injective resolution for $A = R \otimes_R A$. Thus we see that

$$\text{inj.}\dim(A_R) \leq \text{inj.}\dim(R_R).$$

Similarly suppose we have an injective resolution

$$0 \to A \to J_0 \to J_1 \to \cdots \to J_n \to 0$$

for A. Then we can tensor with A^{-1} to get

$$0 \to A \otimes_R A^{-1} \to J_0 \otimes_R A^{-1} \to \cdots \to J_n \otimes_R A^{-1} \to 0.$$

Thus

$$\text{inj.}\dim(R_R) \leq \text{inj.}\dim(A_R).$$

□

Finally, that the properties in Theorem 4.2.2 for R as a left module are equivalent to the corresponding properties for R as a right module is due to [16, Corollary 3.3].

Thus we have proved Theorem 4.2.2. Note that we used equidimensionality to prove that 3 implies 1, so conditions 3,6 and 7 may not be equivalent to 1, 2, 4 and 5 for a non-equidimensional ring. In fact, the theorem is not true in the non-equidimensional case as we have the following counter-example to 3 implies 1.

Example 4.2.8. Let k be a field and S the ring of upper triangular 2×2 matrices over k. Let $R = S \oplus k[x, y]$. Then

(i) S is Z-Macaulay with $\text{inj.}\dim(S) = 1$ and $\text{gl.}\dim(S) = 1$ but is not injectively homogeneous;

(ii) $k[x, y]$ is injectively homogeneous with injective dimension and global dimension 2;

(iii) R is Z-Macaulay with $\text{inj.}\dim(R) = \text{gl.}\dim(R) = \text{Krull}\dim(R) = 2$ but is not injectively homogeneous.

Proof. (i) The centre of S is k and S is k-Macaulay by Example 2.1.3(iii). We saw in Example 3.1.1 the two simple modules S/M and S/N. As left modules, S/N is projective with

$$M \cong S/N \oplus S/N. \tag{4.1}$$

The exact sequence of left S-modules

$$0 \to M \to S \to S/M \to 0 \tag{4.2}$$

shows that S/M has projective dimension at most 1; since there is no copy of ${}_S(S/M)$ in the left socle of S, (4.2) does not split and so ${}_S(S/M)$ has projective dimension 1. By [41, Proposition 7.5.1] S has finite global dimension and by [41, Corollary 7.1.14] $\mathrm{gl.dim}(S) = 1$. Also, $\mathrm{Hom}({}_S S/N, {}_S S) \neq 0$ by (4.1), and $\mathrm{Ext}^1({}_S S/M, {}_S S) \neq 0$ as the above non-split short exact sequence demonstrates, while $\mathrm{Hom}_S({}_S S/M, {}_S S) = 0$, as already noted. This gives $\mathrm{u.gr}(S/N) = 0$ and $\mathrm{u.gr}(S/M) = 1$. Hence by [16, Lemma 3.1] $\mathrm{inj.dim}(S) = 1$. But $M \cap Z = N \cap Z = 0$ so S is not injectively homogeneous.

(ii) $k[x, y]$ is a commutative Gorenstein ring of injective and global dimension 2, therefore it is injectively homogeneous.

(iii) Clearly, $Z(R) := Z = k \oplus k[x, y]$. Now R is Z-Macaulay since the direct sum of two Z-Macaulay rings is also Z-Macaulay by Proposition 2.1.8. Now $\mathrm{inj.dim}(R) = \sup\{\mathrm{inj.dim}(S), \mathrm{inj.dim}(k[x, y])\} = 2$ and by [41, Lemma 6.2.4] we have $\mathrm{Krull\ dim}(R) = \sup\{\mathrm{Krull\ dim}(S), \mathrm{Krull\ dim}(k[x, y])\} = 2$. To show R is not injectively homogeneous we will show that it is not homologically homogeneous. We note that R has $\mathrm{gl.dim}(R) = \sup\{\mathrm{gl.dim}(S), \mathrm{gl.dim}(k[x, y])\} = 2$. Consider the maximal ideals $M \oplus k[x, y]$ and $N \oplus k[x, y]$ of R which have equal intersection with $Z(R)$. Then, the left module $R/(N \oplus k[x, y])$ is projective, while $R/(M \oplus k[x, y])$ has projective dimension 1. Thus R cannot be homologically homogeneous and therefore cannot be injectively homogeneous. □

However, the following trivial example shows that non-equidimensional examples occur naturally and should therefore be considered.

Example 4.2.9. Let $R = k \oplus k[X]$ for any field k. Then R is commutative and affine but not a domain. R is Gorenstein with injective dimension 1 so it is injectively homogeneous. But it has maximal ideals of different heights so is not equidimensional.

Thus we give the following "local" version of Theorem 4.2.2.

Theorem 4.2.10. *Let R be a noetherian ring which is a finite module over a central subring C. Then the following conditions are equivalent:*

1. R is injectively homogeneous over C of injective dimension n.

2. R is C-Macaulay and for all maximal ideals $\mathfrak{m}$ of C we have $\text{inj.}\dim(R_{\mathfrak{m}R_{\mathfrak{m}}}) = \text{Krull }\dim(R_{\mathfrak{m}}) \leq n < \infty$ *with* $\sup\{\text{inj.}\dim(R_{\mathfrak{m}}) : \mathfrak{m}$ *a maximal ideal of* $C\} = n$.

3. R is C-Macaulay with $\text{inj.}\dim(R_R) = n < \infty$ *and for all maximal ideals $\mathfrak{m}$ of C, $R_{\mathfrak{m}}$ is grade-symmetric.*

4. R is C-Macaulay and grade-symmetric with $\text{inj.}\dim(R_R) = n < \infty$.

5. R is locally Krull-Macaulay and $\text{inj.}\dim(R_R) = n < \infty$.

6. R is C-Macaulay and there exists an invertible R-bimodule I with $\text{inj.}\dim_{R_{\mathfrak{m}}}(I_{\mathfrak{m}}) = \text{Krull }\dim(R_{\mathfrak{m}}) \leq n < \infty$ *for all maximal ideals $\mathfrak{m}$ of C and with* $\sup\{\text{inj.}\dim(I_{\mathfrak{m}}) : \mathfrak{m}$ *a maximal ideal of* $C\} = n$.

7. R is C-Macaulay and for all invertible R-bimodules I, $\text{inj.}\dim_{R_{\mathfrak{m}}}(I_{\mathfrak{m}}) = \text{Krull }\dim(R_{\mathfrak{m}}) \leq n < \infty$ *for all maximal ideals $\mathfrak{m}$ of C and* $\sup\{\text{inj.}\dim(I_{\mathfrak{m}}) : \mathfrak{m}$ *a maximal ideal of* $C\} = n$.

Proof. $1 \Leftrightarrow 2$: That 1 implies 2 follows from [16, Lemma 3.3] and Theorem 4.2.2. We now show that 2 implies 1 and so we assume that R is C-Macaulay and for all maximal ideals $\mathfrak{m}$ of C we have $\text{inj.}\dim(R_{\mathfrak{m}R_{\mathfrak{m}}}) = \text{Krull }\dim(R_{\mathfrak{m}}) \leq n < \infty$ with $\sup\{\text{inj.}\dim(R_{\mathfrak{m}}) : \mathfrak{m}$ a maximal ideal of $C\} = n$. Since $R_{\mathfrak{m}}$ is $C_{\mathfrak{m}}$-Macaulay it is equidimensional by Theorem 2.1.4 and we can apply Theorem 4.2.2 to the ring $R_{\mathfrak{m}}$. Thus for all maximal ideals $\mathfrak{m}$ of C, $R_{\mathfrak{m}}$ is injectively homogeneous and thus R is injectively homogeneous by [16, Lemma 3.3].

$1 \Rightarrow 3$: Let R be injectively homogeneous over C. As before we just need to show that $R_{\mathfrak{m}}$ grade-symmetric for all maximal ideals $\mathfrak{m}$ of R. But, for all maximal ideals $\mathfrak{m}$ of C, $R_{\mathfrak{m}}$ is injectively homogeneous and equidimensional. Thus by Theorem 4.2.4 $R_{\mathfrak{m}}$ is grade-symmetric.

$3 \Rightarrow 4$: We just need to show that if $R_{\mathfrak{m}}$ is grade-symmetric for all maximal ideals $\mathfrak{m}$ of C then R is grade-symmetric. But this is Lemma 1.3.12.

$4 \Rightarrow 1$: This is $4 \Rightarrow 1$ of Theorem 4.2.2 which did not require equidimensionality.

$3 \Leftrightarrow 5$: This follows from Theorem 3.3.6 and Proposition 2.1.5.

$2 \Leftrightarrow 6 \Leftrightarrow 7$: This follows from Proposition 4.2.6. □

Note that thanks to Theorem 4.1.4 we have a similar result for homologically homogeneous rings.

Theorem 4.2.11. *Let R be a noetherian ring which is a finite module over a central subring C. Then the following are equivalent:*

1. *R is homologically homogeneous over C of global dimension n.*
2. *R is C-Macaulay and for all maximal ideals $\mathfrak{m}$ of C we have* $\text{gl.}\dim({}_{R_\mathfrak{m}}R_\mathfrak{m}) = \text{Krull}\ \dim(R_\mathfrak{m}) \leq n < \infty$ *with* $\sup\{\text{gl.}\dim(R_\mathfrak{m}) : \mathfrak{m}$ *a maximal ideal of* $C\} = n$.
3. *R is C-Macaulay with* $\text{gl.}\dim(R_R) = n < \infty$ *and $R_\mathfrak{m}$ is grade-symmetric for all maximal ideals $\mathfrak{m}$ of C.*
4. *R is C-Macaulay and grade-symmetric with* $\text{gl.}\dim(R_R) = n < \infty$.
5. *R is locally Krull-Macaulay and* $\text{gl.}\dim(R_R) = n < \infty$.

Proof. Theorem 4.1.4 says that R is homologically homogeneous if and only if it is injectively homogeneous with finite global dimension. In their proof of this theorem, Brown and Hajarnavis show that if R satisfies either condition then $\text{pr.}\dim(R/M) = \text{u.}\,\text{gr}(R/M)$ for all maximal ideals M of R. Thus, by [46, Corollary 4.2] and [16, Lemma 3.1], $\text{gl.}\dim(R) = \text{inj.}\dim(R)$. The result now follows from Theorem 4.2.10. □

4.3 The generalisation of the commutative result

Having established different characterisations of the injectively homogeneous and homologically homogeneous properties we are now in a position to state the generalisation of the commutative result that Gorenstein implies Cohen-Macaulay. The proof has been done already. Recall that the definition of a symmetrically Macaulay ring is given in Definition 3.4.1.

Theorem 4.3.1. *Let R be a noetherian ring which is a finite module over its centre. Then*

(i) R is injectively homogeneous if and only if R has finite injective dimension and is symmetrically Macaulay;

(ii) R is homologically homogeneous if and only if R has finite global dimension and is symmetrically Macaulay.

Proof. (i) This is Theorem 4.2.10.

(ii) This is Theorem 4.2.11. □

Note that the same proof shows that R is injectively (respectively homologically) homogeneous over a central subring C if and only if R has finite injective (respectively global) dimension, is C-Macaulay and grade-symmetric.

We noted immediately after defining the injectively homogeneous and homologically homogeneous properties that if R is injectively (respectively homologically) homogeneous over a central subring C then R is injectively (respectively homologically) homogeneous over the centre Z. However, the converse to this is false. In Example 2.1.11 we demonstrated that a Z-Macaulay ring need not be C-Macaulay for all central subrings. The same example shows that injective and homological homogeneity are also dependant on the choice of central subring, contrary to [15, Proposition 2.7] and [16, Corollary 3.6].

Example 4.3.2. Let $S = k[X] \oplus k$. Then
(i) S is regular and Gorenstein;
(ii) S is neither homologically or injectively homogeneous over C where C is the central subring $k\langle (X,1)\rangle$.

Proof. (i) The ring S has global dimension $1 = \sup\{\text{gl.}\dim(k[X], \text{gl.}\dim(k))\}$.
(ii) We saw in Example 2.1.11 that S is not C-Macaulay. Thus by Theorems 4.2.10 and 4.2.11 S cannot be injectively homogeneous over C nor homologically homogeneous over C. □

Theorem 4.3.1 gives further justification for our choice of symmetrically Macaulay as a suitable generalisation of Cohen-Macaulay. We now have the following hierarchy, generalising the well-known commutative one:

homologically homogeneous ⇒ injectively homogeneous ⇒ symmetrically Macaulay.

Note that Krull-Macaulay would not work here as the example $S = k[X] \oplus k$ is injectively homogeneous but not Krull-Macaulay (see Examples 4.2.9 and 2.2.5 for the proof of this).

Theorem 4.3.1 also shows why the centrally Macaulay property is not strong enough to be the most useful generalisation of Cohen-Macaulay. It is clear that injectively homogeneous implies Z-Macaulay, however the backward implication in Theorem 4.3.1 above would not hold if we only considered Z-Macaulay, rather than symmetrically-Macaulay. We see this in the following example.

Example 4.3.3. Let S be the ring $S = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ where k is a field. Then
(i) S is Z-Macaulay but not grade-symmetric,
(ii) S has finite injective dimension but is not injectively homogeneous.

Proof. (i) is proved in Example 3.1.1 and (ii) in Example 4.2.8. □

Thus we believe that symmetrically Macaulay is the best generalisation of Cohen-Macaulay for rings which satisfy hypothesis (H).

4.4 AS-Gorenstein

Finally, we see that AS-Gorenstein is related to the injectively smooth property.

Theorem 4.4.1. *Let R be a noetherian ring which is a finite module over its centre Z. Then R is injectively smooth if and only if R is AS-Gorenstein.*

Proof. In [52, Proposition 2.2] Teo proves that injectively smooth rings are AS-Gorenstein.

Now assume that R is an AS-Gorenstein ring with $\operatorname{Ext}^n(V, R) \neq 0$ for every simple right R-module V. Then for any maximal ideal M we have $\operatorname{Ext}^i_R(R/M_R, R_R) \neq 0 \Leftrightarrow i = n$ and hence, $\operatorname{u.gr}(M) = n$. By [16, Lemma 3.1]

$$\operatorname{inj.dim}(R) = \sup\{\operatorname{u.gr}(M) : M \text{ a maximal ideal of } R\}.$$

Thus $\operatorname{inj.dim}(R) = n < \infty$ and R is injectively smooth. □

Note that Theorem 4.4.1 shows that the AS-Gorenstein property is stronger than injective homogeneity; a specific example is $S = k[X] \oplus k$.

A similar property to AS-Gorenstein but which is equivalent to injectively homogeneous is the following:

For all irreducible left R-modules V there exists some $n_V \in \mathbb{N}$ such that

$$\operatorname{Ext}^i_R(V, R) \neq 0 \Leftrightarrow i = n_V,$$

with $n_V = n_W$ if $\operatorname{Ann}(V) \cap Z = \operatorname{Ann}(W) \cap Z$. Note also that the example $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ shows that it's not enough to have the even weaker definition that for all V there exists some n such that $\operatorname{Ext}^i_R(V, R) \neq 0 \Leftrightarrow i = n$.

4.5 Notes

All results in Chapter 4 are original apart from Theorem 4.1.4 and Lemma 4.2.7.

Chapter 5

Invertibility and Symmetry

We looked briefly at invertible modules in chapter 4. Now we return to them in order to prove another characterisation of injectively homogeneous rings. We then suggest a way to generalise the property of a finite dimensional algebra being symmetric to a more general context. In [9] Braun looks at rings which are finite modules over a central subring and gives conditions for being a symmetric algebra. Using our proposed generalisation of symmetry we are able to generalise his results. We close the chapter with a discussion of when skew group algebras are symmetric.

Throughout this chapter R is a ring which is a finite module over its affine centre Z and C is a central subring of R over which R is finitely generated. We know from Noether normalisation that we can, if we wish, take the central subring C to be a polynomial algebra $k[x_1, \ldots, x_n]$ which we will denote by Z_0. We refer to section 1.6 for the definition and some basic properties of invertible modules.

5.1 The significance of $\mathrm{Hom}_C(R, C)$

We have a ring R which is a finite module over its affine centre Z. For any central subring C such that R is a finitely generated C-module we consider the R-R-bimodule $\mathrm{Hom}_C(R, C)$, where the actions are defined as follows, for $r \in R$ and $f \in \mathrm{Hom}_C(R, C)$:

$$r.f(a) = f(ar) \text{ and } f.r(a) = f(ra)$$

for all $a \in R$.

We look more particularly at the invertibility of the module $H := \mathrm{Hom}_C(R, C)$. We note the notation X^{μ} as explained immediately after Definition 1.7.7.

Now, notice that H embeds into the quotient ring if R is prime:

Proposition 5.1.1. *Let R be a prime noetherian ring which is a finite module over a central subring C. Then $\operatorname{Hom}_C(R, C)$ embeds into $Q := Q(R)$ as R-R-bimodules.*

Proof. First note that by Proposition 1.2.17 $Q = R\mathcal{C}^{-1}$ where $\mathcal{C} := C\backslash\{0\}$. Also $H := \operatorname{Hom}_C(R, C)$ is a finitely generated torsion-free C-module, so

$$\operatorname{Hom}_C(R, C) \hookrightarrow \operatorname{Hom}_C(R, C) \otimes_R Q.$$

Now let $K := Q(C)$ and consider the map

$$\theta : \operatorname{Hom}_C(R, C) \otimes_R Q \to \operatorname{Hom}_K(Q, K)$$

where for $h \in \operatorname{Hom}_C(R, C)$, and $p = rc^{-1}$, $q = sd^{-1} \in Q$ we have $\theta(h \otimes p)(q) = h(rs)c^{-1}d^{-1}$. We first check that $\theta(h \otimes p)$ is a K-homomorphism. So let $k = je^{-1} \in K$. Then

$$\theta(h \otimes p.k)(q) = h(rjs)(ced)^{-1} = h(rs)(cd)^{-1}je^{-1} = \theta(h \otimes p)k.$$

Now we show that θ is an R-R-bimodule homomorphism. Let $a, b \in R$. Then

$$\theta(a.h \otimes p.b)(q) = (a.h)(rbs)(cd)^{-1} = h(rbsa)(cd)^{-1}$$

while on the other hand

$$a.\theta(h \otimes p).b(q) = \theta(h \otimes p)(bqa) = h(rbsa)(cd)^{-1}.$$

Thus θ is an R-R-bimodule homomorphism and it is clear from the definition of θ that it is injective.

Now $H \otimes_R Q = H\mathcal{C}^{-1}$ and $\operatorname{Hom}_K(Q, K)$ have the same vector space dimension over K, which is $\dim_K(Q)$, showing that θ must be an isomorphism. To see this we note that if $R = \sum_{i=1}^{t} r_i C$ then H is generated by $\{f_i : 1 \leq i \leq t\}$ where $f_i(r_j) = \delta_{ij}$. Then $H\mathcal{C}^{-1} = \sum_{i=1}^{t} f_i K$. Similarly, if $Q = \sum_{i=1}^{t} r_i K$ and $\operatorname{Hom}_K(Q, K)$ is spanned by $\{g_i : 1 \leq i \leq t\}$ where $g_i(r_j) = \delta_{ij}$. Hence we now have

$$\operatorname{Hom}_C(R, C) \otimes Q \cong \operatorname{Hom}_K(Q, K),$$

as R-R-bimodules. Now, by [47, Theorem 9.9] Q is a Frobenius algebra and hence $\operatorname{Hom}_K(Q, K) \cong Q$ as a left and as a right Q-module. Then, as bimodules, $\operatorname{Hom}_K(Q, K) \cong$

${}^1Q^\mu$ for some automorphism μ of Q. But Q is a central simple algebra so by the Skolem-Noether Theorem, which is [47, Theorem 7.21], μ is inner and hence $\mathrm{Hom}_K(Q, K) \cong Q$ as Q-Q-bimodules. Combining these gives

$$\mathrm{Hom}_C(R, C) \hookrightarrow \mathrm{Hom}_C(R, C) \otimes Q \cong \mathrm{Hom}_K(Q, K) \cong Q,$$

where the isomorphisms are all R-R-bimodule isomorphisms. $\square$

We now show that the module $H := \mathrm{Hom}_C(R, C)$ is significant in determining whether or not R is injectively homogeneous. In [50] Stafford and van den Bergh showed that if R is homologically homogeneous then H is invertible. More specifically, they prove the following, as [50, Proposition 2.6] where D_R is a dualising complex of R, as defined by Yekutieli in [58, Definition 4.1], and $\Omega[d]$ is the object in the derived category $\mathcal{D}(\mathrm{mod}(R))$ which consists of the R-module Ω in the $-d^{th}$ place and zeros elsewhere.

Theorem 5.1.2. *Let k be an algebraically closed field of characteristic zero. Assume that R is a prime affine k-algebra of GK dimension d. Let C be a central subring such that R is a finite module over C. Then R is homologically homogeneous of dimension d if and only if* $\mathrm{gl.dim}(R) < \infty$ *and $D_R = \Omega[d]$ for some invertible R-module Ω. If this holds then $\Omega \cong H$.*

It is now clear that if R is homologically homogeneous then H is isomorphic to an invertible module and so must be invertible itself. We will extend this result to injectively homogeneous rings, while at the same time removing the hypotheses on the field k and replacing the hypothesis that R is prime by the weaker assumption that R is equidimensional.

First we consider an injectively homogeneous k-algebra of injective dimension 0, that is, a quasi-Frobenius finite dimensional k-algebra, and look at $H := \mathrm{Hom}_k(R, k)$. To show that H is invertible it would be enough, by Theorem 1.6.4, to show that H is a progenerator and $\mathrm{End}_R({}_RH) \cong R$.

The following is proved in Proposition 1.7.10, but we restate it here for convenience.

Proposition 5.1.3. *If R is a quasi-Frobenius finite dimensional k-algebra then the right (or left) module* $\mathrm{Hom}_k(R, k)$ *is a projective generator.*

We also recall the following well-known lemma.

Lemma 5.1.4. *Let R be a finite dimensional k-algebra. Then we have the following ring isomorphism,*

$$\mathrm{End}_R({}_R\mathrm{Hom}_k(R,k)) \cong R.$$

Proof. By [3, Theorem II.3.3] $D = \mathrm{Hom}_k(-,k)$ defines a duality from the category of right R-modules to the category of left R-modules. Since D is an additive functor it restricts to a ring homomorphism

$$D : \mathrm{End}_R(A_R) \to \mathrm{End}_R({}_R D(A))$$

for any right R-module A by [1, Lemma 20.3]. But D is a duality, so this must be an isomorphism. We now take $A = R$ and get the ring isomorphism

$$\mathrm{End}_R(R_R) \cong \mathrm{End}_R({}_R\mathrm{Hom}_k(R,k)).$$

However, we have a ring isomorphism, $\lambda : R \to \mathrm{End}_R(R_R)$ where $\lambda(r)$ is left multiplication by r. Thus

$$R \cong \mathrm{End}_R(R_R) \cong \mathrm{End}_R({}_R\mathrm{Hom}_k(R,k)).$$

□

So together with Proposition 5.1.3 and Theorem 1.6.4 this gives

Theorem 5.1.5. *If R is a quasi-Frobenius finite dimensional k-algebra then* $\mathrm{Hom}_k(R,k)$ *is an invertible bimodule.*

We now consider an arbitrary injectively homogeneous ring R which is a finite module over its affine centre, but we assume that R is equidimensional. Let Z_0 be a polynomial algebra in $Z(R)$ over which R is finitely generated. We shall show that the right R-module $\mathrm{Hom}_{\mathrm{Z}_0}(R,\mathrm{Z}_0)$ is invertible. We will use Theorem 1.6.4 so we want to show that $H_R := \mathrm{Hom}_{\mathrm{Z}_0}(R,\mathrm{Z}_0)_R$ is projective, a generator and has endomorphism ring isomorphic to R.

Since R is injectively homogeneous, it is Z-Macaulay by Theorem 4.2.1(i). Then since R is equidimensional, Corollary 2.1.14 shows that R is Z_0-Macaulay and hence R is a free Z_0-module by Corollary 2.1.23. Then for any maximal ideal $\mathfrak{m}$ of Z_0, we localise at $\mathfrak{m}$ so that $\mathrm{Z}_{0\mathfrak{m}}$ is a regular local ring. Thus $\mathfrak{m}_\mathfrak{m}$ is generated by a $\mathrm{Z}_{0\mathfrak{m}}$-sequence $f_1,\ldots,f_n$ and since $R_\mathfrak{m}$ is a free $\mathrm{Z}_{0\mathfrak{m}}$-module $f_1,\ldots,f_n$ is a $\mathrm{Z}_{0\mathfrak{m}}$-sequence on $R_\mathfrak{m}$. Now, $R_\mathfrak{m}$ is injectively homogeneous by [16, Lemma 3.3] so that, by [16, Corollary 3.5(ii)],

$\operatorname{inj.dim}(R_{\mathfrak{m}}) = \text{Krull } \dim(R_{\mathfrak{m}}) = \text{Krull } \dim(Z_{\mathfrak{m}}) = n$. Thus we see from Theorem 1.3.14 that $\overline{\mathrm{R}} := R_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}}\,R_{\mathfrak{m}}$ is quasi-Frobenius. Then, since $\overline{\mathrm{R}}$ is quasi-Frobenius, Theorem 5.1.5 shows that

$$\overline{H}_{\mathfrak{m}} := \operatorname{Hom}_{\mathrm{Z}_{0\mathfrak{m}}\,/\,\mathfrak{m}_{\mathfrak{m}}}(R_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}}\,R_{\mathfrak{m}}, \mathrm{Z}_{0\mathfrak{m}}\,/\,\mathfrak{m}_{\mathfrak{m}}) \text{ is } \overline{\mathrm{R}}\text{-invertible,} \tag{5.1}$$

which, by Theorem 1.6.4, implies that the right $\overline{\mathrm{R}}$-module $\overline{H}_{\mathfrak{m}}$ is a progenerator and we have a ring isomorphism $\operatorname{End}_{\overline{\mathrm{R}}}(\overline{H}_{\mathfrak{m}}) \cong \overline{\mathrm{R}}$.

We first relate $H_{\mathfrak{m}}$ to $\overline{H}_{\mathfrak{m}}$. Lemma 1.8.7 shows that

$$H_{\mathfrak{m}} = \operatorname{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0) \otimes \mathrm{Z}_{0\mathfrak{m}} \cong \operatorname{Hom}_{\mathrm{Z}_{0\mathfrak{m}}}(R_{\mathfrak{m}}, \mathrm{Z}_{0\mathfrak{m}}). \tag{5.2}$$

Since $\mathfrak{m}_{\mathfrak{m}}$ is generated by a $\mathrm{Z}_{0\mathfrak{m}}$-sequence, repeated application of Lemma 1.8.5, with $M = R_{\mathfrak{m}}$ and $N = \mathrm{Z}_{0\mathfrak{m}}$, shows that

$$\begin{aligned} \overline{H}_{\mathfrak{m}} &= \operatorname{Hom}_{\mathrm{Z}_{0\mathfrak{m}}\,/\,\mathfrak{m}_{\mathfrak{m}}}(R_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}}\,R_{\mathfrak{m}}, \mathrm{Z}_{0\mathfrak{m}}\,/\,\mathfrak{m}_{\mathfrak{m}}) \\ &\cong \operatorname{Hom}_{\mathrm{Z}_{0\mathfrak{m}}}(R_{\mathfrak{m}}, \mathrm{Z}_{0\mathfrak{m}})/\,\mathfrak{m}_{\mathfrak{m}} \operatorname{Hom}_{\mathrm{Z}_{0\mathfrak{m}}}(R_{\mathfrak{m}}, \mathrm{Z}_{0\mathfrak{m}}) \\ &= H_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}}\,H_{\mathfrak{m}}. \end{aligned} \tag{5.3}$$

Since the elements in the Ore set $Z\backslash\,\mathfrak{m}$ are zero divisors in $R/\,\mathfrak{m}\,R$ we note that we have

$$R_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}}\,R_{\mathfrak{m}} = R/\,\mathfrak{m}\,R \otimes_R R_{\mathfrak{m}} = R/\,\mathfrak{m}\,R \tag{5.4}$$

and

$$H_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}}\,H_{\mathfrak{m}} = H/\,\mathfrak{m}\,H \otimes_R R_{\mathfrak{m}} = H/\,\mathfrak{m}\,H. \tag{5.5}$$

The following lemma is well-known.

Lemma 5.1.6. *Let A be a commutative noetherian ring and F a free A-module. If $\{x_1, \ldots, x_n\}$ is an A-sequence then it is an A-sequence on* $\operatorname{Hom}_A(F, A)$.

Proof. Let $F \cong A^{\oplus t}$ for some t. Then $\operatorname{Hom}_A(F, A) \cong \operatorname{Hom}_A(A, A)^{\oplus t} \cong A^{\oplus t}$ as A-modules. Thus Proposition 1.1.27 completes the proof. □

Now, we examine the three desired properties of H_R. First consider projectivity.

Proposition 5.1.7. *Let R be an equidimensional noetherian injectively homogeneous ring which is module finite over its affine centre. Let Z_0 be a polynomial algebra in $Z(R)$ over which R is finitely generated. Then $H := \operatorname{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0)$ is a projective right R-module.*

Proof. By [18, Proposition 2.4], which applies to rings satisfying (H) by the comment after the proof, it is enough to show that $H_{\mathfrak{m}}$ is a projective right $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of Z_0. So we fix a maximal ideal $\mathfrak{m}$ of Z_0 and localise at $\mathfrak{m}$.

Let $m_{\mathfrak{m}} = \langle f_1, \ldots, f_n \rangle$ where $\{f_1, \ldots, f_n\}$ is a $\mathrm{Z}_{0\mathfrak{m}}$-sequence. Since $R_{\mathfrak{m}}$ is a free $\mathrm{Z}_{0\mathfrak{m}}$-module, Proposition 1.1.27 and Lemma 5.1.6 show that $\{f_1, \ldots, f_n\}$ is a $\mathrm{Z}_{0\mathfrak{m}}$-sequence on $R_{\mathfrak{m}}$ and $H_{\mathfrak{m}}$.

By (5.1), $\overline{H}_{\mathfrak{m}}$ is an invertible $R_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}} R_{\mathfrak{m}}$-module and hence is a projective right $R_{\mathfrak{m}}/\,\mathfrak{m}\, R_{\mathfrak{m}}$-module. Then, since $f_1, \ldots, f_n$ are central elements in the Jacobson radical of $R_{\mathfrak{m}}$, Theorem 1.3.17 together with (5.3) show that $H_{\mathfrak{m}}$ is a projective right $R_{\mathfrak{m}}$-module. And since this holds for all maximal ideals $\mathfrak{m}$ of Z_0, H is a projective right R-module. □

Now we show that H_R is a generator. First of all, note that, for all maximal ideals $\mathfrak{m}$ of Z_0, $H_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}} H_{\mathfrak{m}}$ is $R_{\mathfrak{m}}/\,\mathfrak{m}_{\mathfrak{m}} R_{\mathfrak{m}}$-invertible by (5.3) and (5.1). Then (5.5) and (5.4) show that $H/\,\mathfrak{m}\, H$ is $R/\,\mathfrak{m}\, R$-invertible and hence is an $R/\,\mathfrak{m}\, R$-generator. We consider trace ideals.

Definition 5.1.8. Let R be a ring and ${}_RP$ a finitely generated projective right module. Let

$$\Phi : P \otimes_R \mathrm{Hom}_R(P, R) \to R$$

be defined by $p \otimes f \mapsto f(p)$. Then the *trace ideal of* P is $\mathrm{Im}\,\Phi$ and is denoted $\mathrm{Tr}(P)$.

Note that $\mathrm{Tr}(P)$ is a two-sided ideal of R since for $f \in \mathrm{Hom}_R(P, R)$ and $r \in R$ we have $f(p).r = f(pr)$ since f is a right R-module homomorphism and $r.f(p) = (r.f)(p)$ by the left R-module structure of $\mathrm{Hom}_R(P, R)$. Also, P is a generator if and only if $\mathrm{Tr}(P) = R$. This is because P is a generator if and only if there exists some $f : P^n \twoheadrightarrow R$, if and only if $1 = f(p_1, \ldots, p_n) \in \mathrm{Im}\,\Phi$, if and only if $1 = \sum_{i=1}^{n} f_i(p_i)$ for some $f_i : P \to R$, if and only if $\mathrm{Im}\,\Phi = R$.

We now consider the following well-known result.

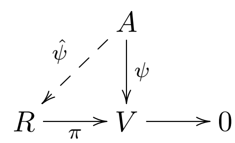
Lemma 5.1.9. *Let R be a ring and let A be a finitely generated projective R-module such that for all simple modules V*

$$A \twoheadrightarrow V.$$

Then A is a generator.

Proof. By the preceding comment it is enough to show that $\mathrm{Tr}(A) = R$, so suppose for a contradiction that $\mathrm{Tr}(A) \subsetneq R$. Then there exists a maximal ideal M of R with $\mathrm{Tr}(A) \subseteq M$

and a simple module V with $VM = 0$. Thus by hypothesis there exists an R-module epimorphism ψ from A to V. By projectivity of A there exists a map $\hat{\psi}$



such that the diagram commutes. The map π is the composite map $R \longrightarrow R/M \xrightarrow{\alpha} V$ where α is any epimorphism from R/M to V, so $M \subseteq \ker \pi$. Then $\operatorname{Im} \hat{\psi} \subseteq \operatorname{Tr}(A) \subseteq M \subseteq \ker \pi$ which is a contradiction. $\square$

Proposition 5.1.10. *Let R be a noetherian ring, finitely generated over its affine centre. Let Z_0 be a central polynomial algebra such that R is a finitely generated Z_0-module. Assume that $H_R = \operatorname{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0)$ is projective and that $H/\mathfrak{m}H$ is a generator for all maximal ideals $\mathfrak{m}$ of Z_0. Then H_R is a generator.*

Proof. By Lemma 5.1.9 it is enough to show that for all simple right R-modules V there exists some map

$$f : H \twoheadrightarrow V.$$

But for any simple right R-module V there exists a maximal ideal $\mathfrak{m}$ of Z_0 such that $V\mathfrak{m} = 0$. Then V is an $R/\mathfrak{m}R$-module. Since $H/\mathfrak{m}H$ is a generator we have $(H/\mathfrak{m}H)^{\oplus t} \twoheadrightarrow V$ for some t while the simplicity of V implies that $t = 1$. But $H \twoheadrightarrow H/\mathfrak{m}H$. $\square$

It only remains to show that $E := \operatorname{End}_R(H_R) \cong R$.

Proposition 5.1.11. *Let R be a noetherian ring finitely generated over its affine centre. Let Z_0 be a central polynomial algebra such that R is a finitely generated free Z_0-module. If the right R-module $H = \operatorname{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0)$ is a progenerator then $E := \operatorname{End}_R(H_R) \cong R$ as rings.*

Proof. Note that $R \hookrightarrow E$ via the ring homomorphism Φ where $\Phi(r)$ is left multiplication by r. To see that Φ is injective note that since H_R is a finitely generated projective generator, we have $H_R^{\oplus n} \twoheadrightarrow R_R$ for some $n \geq 1$. Now if r-Ann$(H_R) \neq 0$ then r-Ann$(R_R) \neq 0$ which is impossible. Thus we can consider R as a subring of E.

Note also that ${}_RE$ is a finitely generated module. For, $E \subseteq \operatorname{End}_{\mathrm{Z}_0}(H)$ as Z_0-modules, where $H = \operatorname{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0) \cong {\mathrm{Z}_0}^{\oplus m}$, m being the rank of R as a free Z_0-module. So E is a finitely generated and torsion-free Z_0-module and hence has to be a finitely generated R-module as the R action restricts to the Z_0-action on E.

Let $\mathfrak{m}$ be a maximal ideal of Z_0 and pass to the localised set-up, with $Z_{0\mathfrak{m}} \subseteq R_\mathfrak{m}$, with $H_\mathfrak{m} \cong \mathrm{Hom}_{Z_{0\mathfrak{m}}}(R_\mathfrak{m}, Z_{0\mathfrak{m}})$ and $E_\mathfrak{m} \cong \mathrm{End}_{R_\mathfrak{m}}(H_\mathfrak{m})$ by Lemmas 1.8.7 and 1.8.6. Then $\mathfrak{m}_\mathfrak{m}$ is generated by a $Z_{0\mathfrak{m}}$-sequence on $Z_{0\mathfrak{m}}$ which, since $R_\mathfrak{m}$ is free over $Z_{0\mathfrak{m}}$ is also a $Z_{0\mathfrak{m}}$-sequence on $R_\mathfrak{m}$. Then, by (5.3) we have $\overline{H}_\mathfrak{m} \cong H_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} H_\mathfrak{m}$. Then Lemma 5.1.4 gives

$$R_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} R_\mathfrak{m} \cong \mathrm{End}_{R_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} R_\mathfrak{m}}(H_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} H_\mathfrak{m})$$

and Lemma 1.8.5 shows that this is isomorphic to $E_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} E_\mathfrak{m}$. Thus we have

$$R_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} R_\mathfrak{m} \cong E_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} E_\mathfrak{m}. \tag{5.6}$$

We will now show that $E_\mathfrak{m} \cong R_\mathfrak{m}$.

Now $R_\mathfrak{m} \subseteq E_\mathfrak{m}$ and the isomorphisms in (5.6) actually give us, $R_\mathfrak{m} + \mathfrak{m}_\mathfrak{m} E_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} E_\mathfrak{m} = E_\mathfrak{m}/\,\mathfrak{m}_\mathfrak{m} E\mathfrak{m}$. Thus

$$R_\mathfrak{m} + \mathfrak{m}_\mathfrak{m} E_\mathfrak{m} = E_\mathfrak{m}. \tag{5.7}$$

Now suppose that $R_\mathfrak{m} \subsetneq E_\mathfrak{m}$. Then since $E_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module it has a composition series so there exists a simple $R_\mathfrak{m}$-module V such that $E_\mathfrak{m}/R_\mathfrak{m} \twoheadrightarrow V$. Also, there exists an $R_\mathfrak{m}$-module I such that $R_\mathfrak{m} \subseteq I \subsetneq E_\mathfrak{m}$ such that $E_\mathfrak{m}/I \cong V$. Thus $\mathfrak{m}_\mathfrak{m} E_\mathfrak{m} \subseteq I$. So $R_\mathfrak{m} + \mathfrak{m}_\mathfrak{m} E_\mathfrak{m} \subseteq I \subsetneq E_\mathfrak{m}$, contradicting (5.7). Hence $R_\mathfrak{m} = E_\mathfrak{m}$.

We thus have $R \subseteq E$ and $R_\mathfrak{m} = E_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of Z and so [25, Corollary 2.9] completes the proof. □

Thus we now have the following

Theorem 5.1.12. *Let R be a prime noetherian injectively homogeneous ring which is module finite over its affine centre. Let Z_0 be a polynomial algebra in $Z(R)$ over which R is finitely generated. Then* $\mathrm{Hom}_{Z_0}(R, Z_0)$ *is an invertible right R-module.*

Proof. That $\mathrm{Hom}_{Z_0}(R, Z_0)$ is a progenerator is proved in Lemma 5.1.7 and Corollary 5.1.10. Then Proposition 5.1.11 applies to give $\mathrm{End}_R(H, R) \cong R$. □

We now show by example that it is necessary to include the hypotheses that R is prime.

Example 5.1.13. Let $R = k[X] \oplus k$ and $Z_0 = k[(X, 1)] = \{(f, f(1)) : f \in k[X]\}$. Then
(i) R is injectively homogeneous;
(ii) $\mathrm{Hom}_{Z_0}(R, Z_0)$ is not an invertible R-module.

Proof. (i) That R is injectively homogeneous is shown in Example 4.3.2.

(ii) Consider $H := \mathrm{Hom}_{Z_0}(R, Z_0)$. Note that

$$R = (1,0)\,Z_0 + (0,1)\,Z_0$$

where $(1,0)\,Z_0$ is free and $(0,1)\,Z_0$ is the Z_0-simple module V with annihilator $\langle X-1\rangle \oplus 0$. Clearly, $\mathrm{Hom}_{Z_0}(V, Z_0) = 0$ so

$$\mathrm{Hom}_{Z_0}(R, Z_0) \cong \mathrm{Hom}_{Z_0}(Z_0, Z_0) \cong Z_0\,.$$

More precisely, writing e for $(0,1) \in R$, we see that, for all $h \in H$ and for all $r \in R$,

$$he(r) = h(er) = 0.$$

That is, $He = 0$ and hence H cannot be a generator as then $H^t \twoheadrightarrow R$ so that R is killed by the non zero idempotent e. Thus $\mathrm{Hom}_{Z_0}(R, Z_0)$ is not an invertible R-module. □

Now we consider the converse to Theorem 5.1.12. For this we use the characterisations of injective homogeneity in chapter 4, in particular $6 \Leftrightarrow 1$ of Theorem 4.2.2. Before giving the converse we need a couple of lemmas.

Lemma 5.1.14. *Let R be a noetherian ring which is a finite module over a central polynomial subring Z_0 and suppose that R is Z_0-Macaulay. Then the functor* $\mathrm{Hom}_{Z_0}(R, -) : Z_0 - mod \to R - mod$ *is exact.*

Proof. Since R is Z_0-Macaulay it is a projective Z_0-module by Proposition 2.1.22. Thus the projectivity of R gives the exactness of the functor. □

Lemma 5.1.15. *Let R be a noetherian ring which is a finite module over a central subring C. Let I be an injective C-module. Then*

(i) $\mathrm{Hom}_C(R, I)$ *is an injective (left) R-module;*

(ii) for any C-modules X and Y and a C-module homomorphism $f : X \to Y$ the map $f^* : \mathrm{Hom}_C(R, X) \to \mathrm{Hom}_C(R, Y)$ *is an R-module homomorphism.*

Proof. (i) Let $A := \mathrm{Hom}_C(R, I)$ which is a left R-module with $r.f(s) = f(sr)$ for all $r, s \in R$ and $f \in A$. Then A is R-injective if the functor $\mathrm{Hom}_R(-, A)$ is exact. By adjointness we have, for any left R-module M,

$$\mathrm{Hom}_R({}_RM, \mathrm{Hom}_C({}_CR_R, {}_CI)) \cong \mathrm{Hom}_C({}_CR_R \otimes_R M, {}_CI) = \mathrm{Hom}_C({}_CM, {}_CI).$$

This exactness of $\mathrm{Hom}_C(-, I)$ gives us the exactness of $\mathrm{Hom}_R(-, A)$.

(ii) Let $g \in \mathrm{Hom}_C(R, X)$ and $r, s \in R$. Then

$$f^*(r.g)(s) = f \circ (r.g)(s) = f \circ g(sr) = r.(f \circ g)(s) = r.f^*(g)(s)$$

so f^* is an R-module homomorphism. □

Lemma 5.1.16. *Let R be a noetherian Z-Macaulay ring which is a finite module over its centre Z. Then* Krull dim$(R) \leq$ inj. dim(R).

Proof. Let Krull dim$(R) = n$ and let M be a maximal ideal of R of height n. Then because R is Z-Macaulay M contains a Z-sequence $\{x_1, \ldots, x_n\}$ of length n. By Theorem 1.3.14 and induction we have

$$\text{inj. dim}(R/\sum x_i R) \leq \text{inj. dim}(R) - n.$$

Thus

$$\text{inj. dim}(R) \geq \text{inj. dim}(R/\sum x_i R) + n \geq n = \text{Krull dim}(R).$$

□

Theorem 5.1.17. *Let R be an equidimensional Z-Macaulay ring which is module finite over its affine centre. Let Z_0 be a polynomial algebra in $Z(R)$ over which R is finitely generated. Then R is injectively homogeneous if and only if $H := \mathrm{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0)$ is an invertible module.*

Proof. The forward implication is done in Theorem 5.1.12 so we assume that H is invertible and show that property 6 in Theorem 4.2.2 holds. That is, we show that R is Z_0-Macaulay and inj. dim$({}_R H) =$ Krull dim$(R) = n < \infty$ for the invertible module $H = \mathrm{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0)$. Let $\mathrm{Z}_0 = k[x_1, \ldots, x_n]$ where $n =$ Krull dim(R) so that inj. dim$_{\mathrm{Z}_0}(\mathrm{Z}_0) = n < \infty$ by Corollary 1.3.15. That R is Z_0-Macaulay follows from Corollary 2.1.14 since R is equidimensional. It remains to show that inj. dim$_{\mathrm{Z}_0}(H) = n$.

Now, we have an injective resolution of Z_0-modules

$$0 \to \mathrm{Z}_0 \to I_0 \to I_1 \to \cdots \to I_k \to 0.$$

Apply $\mathrm{Hom}_{\mathrm{Z}_0}(R, -)$, which is exact by Lemma 5.1.14, to get the exact sequence of R-modules

$$0 \to \mathrm{Hom}_{\mathrm{Z}_0}(R, \mathrm{Z}_0) \to \mathrm{Hom}_{\mathrm{Z}_0}(R, I_0) \to \cdots \to \mathrm{Hom}_{\mathrm{Z}_0}(R, I_k) \to 0$$

where the homomorphisms are R-module homomorphisms by Lemma 5.1.15(ii). Each $\operatorname{Hom}_{Z_0}(R, I_i)$ is R-injective by Lemma 5.1.15(i) and hence we see that $\operatorname{inj.dim}({}_R H) \leq n$. But by Lemma 5.1.16 and Proposition 4.2.6 we have

$$n = \text{Krull } \dim(R) \leq \operatorname{inj.dim}(R) = \operatorname{inj.dim}(H) \leq n.$$

Thus $\operatorname{inj.dim}(H) = n$ and R is injectively homogeneous over Z_0 by Theorem 4.2.10. □

5.2 A new definition of symmetry

Throughout this section, R is a noetherian ring which is a finite module over its affine centre Z. We let C be any central subring of R such that R is a finitely generated torsion-free C-module. In section 1.7 we briefly discussed finite dimensional Frobenius algebras and symmetric algebras. These have been generalised to rings which are finite modules over a central subring and are considered in [9], [5] and [14] for example. One important aspect of symmetric algebras algebras is their connection with Calabi-Yau algebras. The usual definition of a Calabi-Yau algebra is in terms of an equivalence of derived categories. Much could be said on the subject of Calabi-Yau algebras, but we will confine ourselves to the connection with symmetric algebras. Let R be a noetherian ring with a central subring C of Krull-dimension d, such that R is a finite module over C. The following definition is as in [33].

Definition 5.2.1. We say that R is a *Calabi-Yau algebra of dimension* d or d-CY if, for all $X, Y \in \mathcal{D}(\operatorname{mod}(fl - R))$, the bounded derived category of finite length R-modules, there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{D}(\operatorname{mod}(R))}(X, Y[n]) \cong D \operatorname{Hom}_{\mathcal{D}(\operatorname{mod}(R))}(Y, X).$$

Here, D is the *Matlis duality* functor $\operatorname{Hom}_C(-, E)$, where E is the direct sum of the C-injective hulls of the simple C-modules.

There are many examples of Calabi-Yau algebras as is demonstrated in, for example, [30] and [33].

The following proposition, which is [14, Proposition 2.4] but follows from [33, Propositions 3.1 and 3.2], shows the connection between symmetric algebras and Calabi-Yau algebras. In particular,

Proposition 5.2.2. *Let R, C be as above with* Krull $\dim(C) = d$ *and suppose that C is a regular domain. Then R is a Calabi-Yau algebra of dimension n if and only if $n = d$, R has finite global dimension, R is a projective C-module and R is a symmetric C-algebra. In this case, R has global dimension d.*

Thus we see that if C is a regular domain then being a symmetric algebra over C is a necessary condition for being Calabi-Yau.

We now define generalisations of finite dimensional Frobenius and symmetric algebras as given in [14, Definition 2.1] among other places.

Definition 5.2.3. R is a *Frobenius extension* of C if the following equivalent conditions hold:
(i) There is an R-C-bimodule isomorphism

$$F : R \to \mathrm{Hom}_C(R, C);$$

(ii) there is a C-R-bimodule isomorphism

$$G : R \to \mathrm{Hom}_C(R, C);$$

(iii) there is a non-degenerate associative C-bilinear form given by

$$\beta(r, s) = F(s)(r)$$

for all $r, s \in R$, or similarly for G.

Definition 5.2.4. (i) R is a *symmetric C-algebra* if R is a finitely generated C-module and

$$\mathrm{Hom}_C(R, C) \cong R$$

as R-R-bimodules.
(ii) Let R be a finitely generated C-module. Then R is locally symmetric if $R_{\mathfrak{m}}$ is a symmetric $C_{\mathfrak{m}}$-algebra for all maximal ideals $\mathfrak{m}$ of C.

Note that injectively homogeneous is a generalisation for quasi-Frobenius, and assuming that R is equidimensional and Z-Macaulay we can generalise the hierarchy in section 1.7 to get:

$$\{\text{symmetric algebra}\} \Rightarrow \{\text{Frobenius extension}\} \Rightarrow \{\text{injectively homogeneous}\}.$$

The first implication is clear from the definitions and the second follows from Theorem 5.1.17 since in a Frobenius extension the module $\mathrm{Hom}_C(R, C) \cong R$ is clearly invertible and hence R is injectively homogeneous.

We now give a couple of easy properties of symmetric C-algebras.

Proposition 5.2.5. *Let R be a finitely generated C-module. If R is symmetric then R is locally symmetric.*

Proof. Suppose $\mathrm{Hom}_C(R, C) \cong R$ as an R-R-bimodule and let $\mathfrak{m}$ be any maximal ideal of C. Then, by Lemma 1.8.7 and the definition of a symmetric algebra, we have the following R-R-bimodule isomorphisms,

$$\begin{aligned} \mathrm{Hom}_{C_\mathfrak{m}}(R_\mathfrak{m}, C_\mathfrak{m}) &\cong \mathrm{Hom}_C(R, C) \otimes_C C_\mathfrak{m} \\ &\cong R \otimes_C C_\mathfrak{m} \\ &\cong R_\mathfrak{m}. \quad \square \end{aligned}$$

Lemma 5.2.6. *Let R be a free module over a central subring C and let $a \in C$, with a a non zero divisor in R. If R is a symmetric C-algebra then R/aR is a symmetric C/aC-algebra.*

Proof. By Lemma 1.8.5 and by symmetry of R we have

$$\mathrm{Hom}_{C/aC}(R/aR, C/aC) \cong \frac{\mathrm{Hom}_C(R, C)}{a\,\mathrm{Hom}_C(R, C)} \cong R/aR$$

as R-R-bimodules. □

Similarly we can prove the following corollary.

Corollary 5.2.7. *Let R be a free module over a central subring C and let $a \in C$, with a a non zero divisor in R. If R is a Frobenius extension of C then R/aR is a Frobenius extension of C/aC.*

Definition 5.2.4 makes sense for rings where $H \cong R$ as a left and a right module, such as Frobenius extensions, but we would like a sensible definition of the symmetric property for the case where H is merely invertible, that is, for any equidimensional injectively homogeneous ring. The following example shows that, even if we insist our ring is prime and indeed homologically homogeneous, there do exist rings where H is invertible but not isomorphic to R as a left or right module, and hence this is a genuine issue.

Example 5.2.8. Let $D = \mathbb{C}[X]$ and $\mathfrak{m} = \langle X \rangle$ and define

$$S := \begin{pmatrix} D & \mathfrak{m} & \mathfrak{m} \\ D & D & D \\ D & D & D \end{pmatrix} \subseteq M_3(D).$$

Then S is a prime noetherian ring which is a finite module over its centre D such that

(i) S is a homologically homogeneous ring;

(ii) $\mathrm{Hom}_D(S, D)$ is an invertible S-module;

(iii) $\mathrm{Hom}_D(S, D)$ is not isomorphic to S as left or right S-modules.

Proof. (i) S has two "special" maximal ideals, namely

$$M := \begin{pmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ D & D & D \\ D & D & D \end{pmatrix} \text{ and } N := \begin{pmatrix} D & \mathfrak{m} & \mathfrak{m} \\ D & \mathfrak{m} & \mathfrak{m} \\ D & \mathfrak{m} & \mathfrak{m} \end{pmatrix}.$$

The remaining maximal ideals of S are the ideals $\{\mathfrak{n}\, S : \mathfrak{n} = \langle X - \lambda \rangle \lhd D = Z(S), 0 \neq \lambda \in \mathbb{C}\}$. Each of these is clearly S-free and $S/\,\mathfrak{n}\, S \cong M_3(\mathbb{C})$.

We first show that S is hereditary. Let $R = M_3(D)$. We consider M as a right ideal of R. Now, M is *generative* if $RM = R$ and *isomaximal* if R/M is a finite direct sum of isomorphic simple modules. Both these properties are easy to check so by [41, Theorem 5.5.10] $\mathbb{I}(M) = \{r \in R : rM \subseteq M\} = S$ is hereditary.

To show that S is homologically homogeneous it remains to show that S/M and S/N have the same projective dimension. Now, S is a free D-module so if S/M and S/N are projective S-modules they must be projective, and hence free, D-modules. But this is impossible so they cannot be projective. Then since S is hereditary they both have projective dimension 1.

(ii) This follows from (i) by Theorems 4.1.4 and 5.1.12.

(iii) Let $H := \mathrm{Hom}_D(S, D)$ We first consider the factor ring $\overline{\mathrm{S}} := S/XS$. Now

$$\overline{\mathrm{S}} = \begin{pmatrix} D & \mathfrak{m} & \mathfrak{m} \\ D & D & D \\ D & D & D \end{pmatrix} \Bigg/ \begin{pmatrix} \mathfrak{m} & \mathfrak{m}^2 & \mathfrak{m}^2 \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \end{pmatrix}$$

has maximal ideals

$$\begin{pmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ D & D & D \\ D & D & D \end{pmatrix} + XS/XS \text{ and } \begin{pmatrix} D & \mathfrak{m} & \mathfrak{m} \\ D & \mathfrak{m} & \mathfrak{m} \\ D & \mathfrak{m} & \mathfrak{m} \end{pmatrix} + XS/XS.$$

Thus $\overline{\mathrm{S}}/J(\overline{\mathrm{S}}) \cong \mathbb{C} \oplus \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$, giving simple left modules $V := \mathbb{C}$ and $W := \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$.

Now $\overline{\mathrm{S}}$ has orthogonal primitive idempotents

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + XS, \;\; e_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + XS, \;\; e_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + XS,$$

where the left modules generated by e_2 and e_3 are isomorphic. We then have

$$\overline{\mathrm{S}}\,e_1 := \begin{pmatrix} D & \mathfrak{m}^2 & \mathfrak{m}^2 \\ D & \mathfrak{m} & \mathfrak{m} \\ D & \mathfrak{m} & \mathfrak{m} \end{pmatrix} + XS, \;\; \overline{\mathrm{S}}\,e_2 := \begin{pmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m}^2 \\ \mathfrak{m} & D & \mathfrak{m} \\ \mathfrak{m} & D & \mathfrak{m} \end{pmatrix} + XS, \;\; \overline{\mathrm{S}}\,e_3 := \begin{pmatrix} \mathfrak{m} & \mathfrak{m}^2 & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & D \\ \mathfrak{m} & \mathfrak{m} & D \end{pmatrix} + XS.$$

Now, $\mathrm{soc}(\overline{\mathrm{S}}\,e_1) \cong W$ with $\mathrm{top}(\overline{\mathrm{S}}\,e_1) \cong V$ and $\mathrm{soc}(\overline{\mathrm{S}}\,e_2) \cong \mathrm{soc}(\overline{\mathrm{S}}\,e_3) \cong V$ with $\mathrm{top}(\overline{\mathrm{S}}\,e_2) \cong \mathrm{top}(\overline{\mathrm{S}}\,e_3) \cong W$. Thus property (iii) of Definition 1.7.2 is satisfied so that $\overline{\mathrm{S}}$ is quasi-Frobenius. Now property (v) of Definition 1.7.2 tells us that $\overline{\mathrm{S}}\,e_1 \cong \mathrm{Hom}_{\mathbb{C}}(e_2\,\overline{\mathrm{S}}, \mathbb{C}) =: (\overline{\mathrm{S}}\,e_2)^*$ and $\overline{\mathrm{S}}\,e_2 \cong \mathrm{Hom}_{\mathbb{C}}(e_1\,\overline{\mathrm{S}}, \mathbb{C}) =: (\overline{\mathrm{S}}\,e_1)^*$. Thus we have,

$$\begin{aligned} \overline{\mathrm{S}} &\cong \overline{\mathrm{S}}\,e_1 \oplus \overline{\mathrm{S}}\,e_2 \oplus \overline{\mathrm{S}}\,e_2 \\ &\cong (\overline{\mathrm{S}}\,e_2)^* \oplus (\overline{\mathrm{S}}\,e_1)^* \oplus (\overline{\mathrm{S}}\,e_1)^* \\ &\neq (\overline{\mathrm{S}}\,e_1)^* \oplus (\overline{\mathrm{S}}\,e_2)^* \oplus (\overline{\mathrm{S}}\,e_2)^* \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\overline{\mathrm{S}}, \mathbb{C}). \end{aligned}$$

Thus $\overline{\mathrm{S}}$ is not a Frobenius algebra. Now, Lemma 5.2.7 shows that S cannot be a Frobenius extension of D. □

Remark. The factor ring $\overline{\mathrm{S}}$ in the proof of (iii) is an example of a finite dimensional algebra which is quasi-Frobenius but not Frobenius.

We now give a definition for symmetric which applies to all injectively homogeneous rings.

Definition 5.2.9. Let R be a noetherian ring which is a finite module over a central subring C. Let $H = \mathrm{Hom}_C(R, C)$. We say that R is *dual-symmetric* if $H = RW = WR$, where $W := \{f \in H : rf = fr \;\; \forall \;\; r \in R\}$.

We note that $RW = WR$ is always true by definition of W and that since H is a finitely generated R-module, if $H = RW$ then there must then be some $f_1, \ldots f_n \in W$ which generate H.

We now check that dual-symmetric is a generalisation of the symmetric property defined in Definition 5.2.4. So suppose that R is symmetric in the usual sense. Then there is some isomorphism of R-R-bimodules, $\theta : R \to H$ such that $f := \theta(1)$ generates H. Now $f \in W$ since $rf = r\theta(1) = \theta(r) = \theta(1)r = f.r$, so we have $H = RW$ and hence R is dual-symmetric. Thus, our definition generalises the usual definition of symmetric.

Another condition we could impose on a ring R where H is invertible is to require that R be a Frobenius extension as defined in Definition 5.2.3. This suggests the following obvious question to which we do not know the answer.

Question 7. Suppose that R is Frobenius and dual-symmetric. Then is R symmetric?

Let $Z := Z(R)$. We note that W is a C-submodule of H and thus it makes sense to localise W at maximal ideals $\mathfrak{m}$ of C. So let $W_{\mathfrak{m}} = \{f \in H_m : rf = fr \ \forall \ r \in R_{\mathfrak{m}}\}$. Then

Lemma 5.2.10. *Let H and W be as above. Then $H = RW$ if and only if $H_{\mathfrak{m}} = R_{\mathfrak{m}}W_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of C.*

Proof. First note that $RW \otimes R_{\mathfrak{m}} = R_{\mathfrak{m}}W_{\mathfrak{m}}$ since $f \in W$ if and only if $f \otimes 1 \in W_{\mathfrak{m}}$. Then if $H = RW$ we have

$$H_{\mathfrak{m}} = H \otimes R_{\mathfrak{m}} = RW \otimes R_{\mathfrak{m}} = R_{\mathfrak{m}}W_{\mathfrak{m}}.$$

Conversely suppose that $R_{\mathfrak{m}}W_{\mathfrak{m}} = H_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of C. Now since $RW \subseteq H$ we can apply [25, Corollary 2.9] to see that $RW = H$. □

Thus we see that R is dual-symmetric if and only if $R_{\mathfrak{m}}$ is dual-symmetric for all maximal ideals $\mathfrak{m}$ of C.

Now we look at the connections between a ring R being symmetric or dual-symmetric and the centre of R being Gorenstein. In the commutative local case it is true that R is symmetric if and only if it is Gorenstein.

Proposition 5.2.11. *[9, Proposition 2.6] Let $C \subset Z$ be a finite extension of Noetherian commutative rings. Assume that C is local Gorenstein and that Z is Cohen-Macaulay and equidimensional. Then the following are equivalent:*
(i) Z is a Gorenstein ring,
(ii) $\mathrm{Hom}_C(Z, C) \cong Z$ *as Z-modules.*

This generalizes for C not local thanks to the following proposition which is [25, Theorem 11.6(a)]. Recall that a projective module over a commutative ring Z has rank n if $M_{\mathfrak{m}}$ is a free $Z_{\mathfrak{m}}$-module of rank n for all maximal ideals $\mathfrak{m}$ of Z.

Proposition 5.2.12. *[7, §II.5.6] Let Z be a commutative ring. A Z-module M is invertible if and only if it is finitely generated and projective of rank* 1*.*

We also need the following well-known lemma.

Lemma 5.2.13. *Let Z be a commutative ring. A Z-module M is invertible if and only if $M_{\mathfrak{m}}$ is invertible for all maximal ideals $\mathfrak{m}$ of Z.*

Proof. Use Proposition 5.2.12 and show that M is projective of rank 1 if and only if $M_{\mathfrak{m}}$ is projective of rank 1 for all maximal ideals $\mathfrak{m}$. □

Thus we get the following generalisation of Proposition 5.2.11.

Corollary 5.2.14. *Let $C \subset Z$ be a finite extension of Noetherian commutative rings. Assume that C is Gorenstein and that Z is Cohen-Macaulay. Then the following are equivalent:*
(i) Z is a Gorenstein ring,
(ii) $\mathrm{Hom}_C(Z, C)$ *is an invertible Z-module.*

Proof. By Theorem 1.1.15 and Lemma 5.2.13 we can reduce to the local case, which is Proposition 5.2.11 since for a local ring, free of rank 1 is the same as projective of rank 1. □

We now turn to the noncommutative case. In [9, Theorem 2.10] Braun effectively proves the following theorem.

Theorem 5.2.15. *Let R be a prime ring which is a finite module over a central regular local subring C. Suppose that*
(1) the PI degree of R is invertible in R,
(2) R_C is free,
(3) $Z(R)$ is a Gorenstein ring,
(4) $R_{\mathfrak{q}}$ is an Azumaya algebra for every height 1 *prime $\mathfrak{q}$ in $Z(R)$.*
Then R is a symmetric C-algebra.

Remarks. 1. Braun uses slightly different hypotheses. In particular, he assumes that $\mathrm{Hom}_C(Z, C) \cong Z$ which is equivalent to Z being Gorenstein since C is local. He also assumes that R is a maximal order and C is normal which we replace with R being a free module and C-being regular. However, the proofs are very similar, so we do not give the proof here.

2. We consider whether or not all the hypotheses are necessary. We are not sure about hypotheses (1) and (2). Hypothesis (3) must be necessary because if we take R to be commutative then it is necessary by Proposition 5.2.11. And finally hypothesis (4) is necessary as shown by $R = \mathbb{C}G$, where G is the infinite dihedral group. We now consider this example in more detail.

Example 5.2.16. Let $R = \mathbb{C}G$ where $G = \langle a, b : b^2 = 1, bab = a^{-1}\rangle$ and take $C = Z = \mathbb{C}[a + a^{-1}]$. Then R is a prime noetherian homologically homogeneous ring of Krull dimension 1 such that

(1) R is a free Z-module,

(2) The PI degree of R is invertible in R,

(3) Z is Gorenstein,

(4) R is not height 1 Azumaya,

(5) $\mathrm{Hom}_Z(R, Z) \cong R$ as left and right R-modules, but not as bimodules.

Proof. First note that R is prime by [45, Theorem 5.5]. By [15, Proposition 7.5] R is homologically homogeneous if it has an abelian normal subgoup A of finite index. We can take $A = \langle a\rangle$ which is normal of index 2. Also, Krull $\dim(R) = \mathrm{gl.dim}(R)$ which by the proof of [15, Proposition 7.5] is equal to 1, the Hirch number of G.

(1) Since R is homologically homogeneous is is Z-Macaulay by Theorem 4.3.1(ii). Then, equidimensionality of R and Corollary 2.1.23 show that R is a free Z-module.

(2) By (1) and [13, Corollary III.1.6] the PI degree of R is 2 which is clearly invertible.

(3) Z is Gorenstein since it is a polynomial algebra.

(4) Since R has Krull dimension 1, height 1 Azumaya is the same as Azumaya. Now if R is Azumaya then all the simple modules have dimension 2 but there are simple modules of dimension 1, namely $R/\langle a-1, b-1\rangle$ and $R/\langle a-1, b+1\rangle$. So R is not height 1 Azumaya.

(5) We use [14, Proposition 2.3] to show that R is a Frobenius extension of Z. We consider R as having the Z-basis $\{1, a, b, ab\}$ and have the linear functional $\Phi : R \to A$ which sends an element in R to the coefficient of 1. Let $0 \neq r \in R$. We need to show that there exists some $s \in R$ such that $\Phi(sr) = ux$ for some basis element x. Let $r = z_1 + az_2 + bz_3 + abz_4$. Then $z_i \neq 0$ for some $1 \leq i \leq 4$. If $z_1 \neq 0$ we just take $s = 1$. Otherwise, take s to be a^{-1}, b or ba^{-1} for $z_2 \neq 0$, $z_3 \neq 0$ and $z_4 \neq 0$ respectively. Thus R is a Frobenius extension of Z.

This gives the bilinear form $\beta(x, y) = \Phi(xy)$ for $x, y \in R$. Now this is not symmetric since taking $x = b$ and $y = ab$ gives $\beta(b, ab) = \Phi(a^{-1}) = \Phi(a + a^{-1} - a) = a + a^{-1}$ while

$\beta(ab, b) = \Phi(a) = 0$. Thus $\mathrm{Hom}_Z(R, Z) \cong R$ as left and right R-modules, but not as bimodules. □

We can extend Theorem 5.2.15 to the situation where C is not local, but showing that R is dual-symmetric rather than symmetric.

Corollary 5.2.17. *Let R be a prime ring which is a finite module over a central regular subring C. Suppose that*

(1) the PI degree of R is invertible in R,

(2) R_C is projective,

(3) $Z(R)$ is a Gorenstein ring,

(4) $R_{\mathfrak{q}}$ is an Azumaya algebra for every height 1 prime $\mathfrak{q}$ in $Z(R)$.

Then R is injectively homogeneous and dual-symmetric.

Proof. Localising R at maximal ideals of C allows us to apply Theorem 5.2.15 to $R_{\mathfrak{m}}$ giving $H_{\mathfrak{m}}$ symmetric for all maximal ideals $\mathfrak{m}$ of C. Then we see that for all maximal ideals $\mathfrak{m}$ of C, $H_{\mathfrak{m}}$ is invertible and hence $R_{\mathfrak{m}}$ is injectively homogeneous by Theorem 5.1.17. Also, $R_{\mathfrak{m}}$ being symmetric gives $H_{\mathfrak{m}} = R_{\mathfrak{m}} W_{\mathfrak{m}}$ for all $\mathfrak{m}$. Now [16, Lemma 3.3] and Lemma 5.2.10 gives the required result. □

This shows that the dual-symmetric property is a sensible generalisation of the symmetric property.

Remark. We again consider the necessity of the hypotheses. We suggest that it may be possible to remove hypothesis (1) by using the reduced trace map, but as yet have not succeeded in doing so. Hypotheses (2) and (3) are both consequences of the conclusion. The necessity of (4) is shown by Example 5.2.16, as we now demonstrate.

Example 5.2.18. Let R be the ring in Example 5.2.16. That is, the group algebra $\mathbb{C}G$ where $G = \langle a, b : b^2 = 1, bab = a^{-1} \rangle$. We take $C = Z = \mathbb{C}[a + a^{-1}]$. Then R is a prime noetherian homologically homogeneous ring of Krull dimension 1 satisfying the first three hypotheses in Corollary 5.2.17, but not hypothesis (4) nor the conclusion. That is, R is not dual-symmetric.

Proof. It remains to show that R is not dual-symmetric as the hypotheses were considered in Example 5.2.16. We showed in Example 5.2.16 that R has Z-basis $\{1, a, b, ab\}$ and that $H := \mathrm{Hom}_Z(R, Z) \cong R$ where the isomorphism $\theta : R \to \mathrm{Hom}_Z(R, Z)$ comes from the bilinear form β. That is, for all $r \in R$,

$$\theta(r) = \beta(-, r) = \Phi(-r),$$

where $\Phi(z_1 + z_2 a + z_3 b + z_4 ab) = z_1$. We show that $W := \{f \in H : xf = fx \ \forall \ x \in R\}$ cannot generate H. So let $f = \Phi(-r) \in W$. Then $x.\Phi(-r) = \Phi(r-).x$ for all $x \in R$. That is,

$$\Phi(yxr) = \Phi(xyr)$$

for all $x, y \in R$. This can only happen if $r = 0$. To see this, it is enough to check for $r = 1, a, b, ab$, the basis elements of R as a free Z-module. We leave the reader to check that for $r = 1$, $x = b$ and $y = ab$, $\Phi(yxr) \neq \Phi(xyr)$. Similarly we can take $r = a$, $x = ba$, $y = b$; $r = b$, $x = b$, $y = a$ and $r = ab$, $x = a$, $y = b$. Thus $W = 0$ and R cannot be dual-symmetric. □

Using Theorem 5.2.15 Braun proves the following theorem, which implies Corollary 5.2.17. However, our result above gives a different approach.

Theorem 5.2.19. *Let R be a prime ring which is a finite module over a central regular subring C. Suppose that*

(1) The PI degree of R is invertible,

(2) R is a maximal Cohen-Macaulay $Z(R)$-module,

(3) $Z(R)$ is a Gorenstein ring,

(4) $R_{\mathfrak{q}}$ is an Azumaya algebra for every height one prime $\mathfrak{q}$ in $Z(R)$.

Then R is a locally symmetric C-algebra and R is injectively homogeneous.

Theorem 5.2.15 and Corollary 5.2.17 generalise (i) implies (ii) of Proposition 5.2.14. We can also generalise the other implication but before doing so we give some preparatory results. Let M be an R-R-bimodule. Then

$$M^R := \{m \in M : r.m = m.r \ \forall \ r \in R\}.$$

Recall from Definition 1.5.4 that, if R is a prime PI ring, $T(R)$ denotes the trace ring of R.

Lemma 5.2.20. *[9, Proposition 2.3] Let R be a prime noetherian ring which is a finite module over the central subring C. Suppose that $T(R) = R$ and $d =$PI deg R is invertible in R. Let*

$$W := \{f \in \mathrm{Hom}_C(R, C) : f([x, y]) = 0 \ \forall \ x, y \in R\}.$$

Then $W \cong \mathrm{Hom}_C(Z, C)$ as Z-modules.

Lemma 5.2.21. *Let R be a prime noetherian ring which is a finite module over its centre Z such that the PI degree of R is invertible and the trace ring of R is equal to R. Let I be an invertible R-submodule of Q which is centrally generated, i.e. $I = XR = RX$ for $X = I \cap Z$ then X is an invertible Z-module.*

Proof. First note that by Lemma 1.6.9 we can consider I as an ideal of R. We consider the reduced trace map $T : R \to Q(Z)$ as defined in Definition 1.5.3. By Lemma 1.5.5 $\mathrm{Im}\, T = Z$.

Now let $J = I^{-1}$. By hypothesis $I = XR$ so $R = IJ = XRJ = XJ$. Apply T to this to get

$$XT(J) = T(XJ) = T(R) = Z.$$

Thus X is an invertible Z-module and $X^{-1} = T(J)$. □

We can now state and prove the generalisation of (ii) implies (i) in Proposition 5.2.14.

Theorem 5.2.22. *Let R be a prime, noetherian ring, finitely generated over its centre Z such that the PI degree of R is invertible and $T(R) = R$. Suppose R is also injectively homogeneous, so that H is invertible by Theorem 5.1.17. Then if R is dual-symmetric, Z is Gorenstein.*

Proof. By Lemma 5.2.20, the PI degree of R being invertible and $T(R) = R$ give us that $W \cong \mathrm{Hom}_C(Z, C)$ as Z-modules. In order to apply Corollary 5.2.14 we want to show that Z is Cohen-Macaulay. Now, Lemma 1.5.5(ii) shows that Z is a direct summand of R which is Z-Macaulay, and hence Z is Cohen-Macaulay by Proposition 2.1.9.

Then by Corollary 5.2.14 and Lemma 5.2.20 Z is Gorenstein if and only if $\mathrm{Hom}_C(Z, C)$ is invertible, if and only if W is invertible. So it is enough to show that W is invertible. This follows from the invertibility of H, which is an R-submodule of Q by Proposition 5.1.1, via Lemma 5.2.21. □

Lemma 5.2.23. *Let R be a prime homologically homogeneous ring. Then $T(R) = R$.*

Proof. Since R is homologically homogeneous ring [15, Theorem 6.1] shows that Z is a Krull domain and hence is integrally closed. Then Proposition 1.5.6 applies and $T(R) = R$. □

We can now give the following corollary to Theorem 5.2.22.

Corollary 5.2.24. *Let R be a prime, noetherian ring, finitely generated over its centre Z such that the PI degree of R is invertible. Suppose R is also homologically homogeneous. Then if R is dual-symmetric, Z is Gorenstein.*

Proof. Follows from Theorem 5.2.22 and Lemma 5.2.23. □

The upshot of these results is that for a ring R satisfying certain conditions R is "symmetric" if and only if $Z(R)$ is Gorenstein.

5.3 Symmetry of skew group algebras

We now consider the question of whether or not skew group algebras are symmetric. So let k be a field of arbitrary characteristic and $S = S(V)$ where $V = \sum_{i=1}^{n} kX_i$. Let G be a finite group acting linearly on V. Then we define $T := S * G$ to be the skew group ring which is injectively homogeneous by [60, Proposition 2.8] and so, by Theorem 4.2.1(i), is Z-Macaulay. We note that S is a graded algebra, and denote the i^{th} homogeneous component by S_i. Thus $S = \bigoplus_{i=0}^{\infty} S_i$, where $S_0 = k$. The following known proposition shows that Z is equidimensional.

Proposition 5.3.1. *Let V and G be as above and let T be the skew group ring $S(V) * G$. Then the centre Z of T is equidimensional.*

Proof. We first note that $Z \supseteq S(V)^G$. Now let $P_1, \ldots, P_t$ be the minimal primes of Z. We claim that $P_j \cap S(V)^G = 0$ for all $1 \leq j \leq t$. This is because each $s \in S(V)^G$ is a non zero divisor on T while P_j consists of zero-divisors. To justify this last claim, we note that $P_j \bigcap_{j \neq i} P_i \subseteq \bigcap_{i=1}^{t} P_i$ which is nilpotent by [31, Proposition 3.6]. But $\bigcap_{j \neq i} P_i \neq 0$ otherwise $\bigcap_{i \neq j} P_i = 0 \subseteq P_j$ which contradicts the fact that $P_1, \ldots, P_t$ are distinct minimal primes. We choose d to be minimal such that $[P_j \bigcap_{j \neq i} P_i]^d = 0$, but $\bigcap_{j \neq i} P_i^{d-1} \neq 0$, which shows that P_j consists of zero-divisors.

Thus, since $P_j \cap S(V)^G = 0$,

$$S(V)^G \hookrightarrow Z/P_j$$

via the map $s \mapsto s + P_j$. And hence Z/P_j is a finitely generated torsion free $S(V)^G$-module. Thus, Krull $\dim(Z/P_j) =$ Krull $\dim(S(V)^G) = \dim_k(V) = n$, the second equality being by [6, Corollary 1.4.6]. Now let $\mathfrak{m}$ be any maximal ideal of Z, and choose j, $1 \leq j \leq t$ such that $P_j \subseteq \mathfrak{m}$ and $\mathrm{ht}(\mathfrak{m}/P_j) = \mathrm{ht}(\mathfrak{m})$. Thus $\mathrm{ht}(\mathfrak{m}) =$ Krull $\dim(Z/P_j) = n$. □

We also see from Noether normalisation (see [25, Theorem 13.3]) that there exists polynomial subalgebra $\mathrm{Z}_0 = k[f_1, \ldots, f_n]$ of the centre of T with $Z(T)$, and hence T, finitely generated over Z_0 and where the f_i are homogeneous elements of S with degree k_i. We can assume, by raising f_i to the power $|G|$ if necessary, that $|G|$ divides $\deg(f_i) = k_i$. Now, since Z is equidimensional, Corollary 2.1.13 shows that T is Z_0-Macaulay. Thus by Corollary 2.1.23, T and hence S are finitely generated free Z_0-modules.

We consider a result in [14] on Frobenius extensions where the authors consider a ring T which is a finitely generated free module over an affine central subalgebra Z_0 with basis $\mathcal{B}$ and which satisfies the following hypothesis.

Hypothesis 1. *There exists a Z_0-linear functional $\Phi : T \to \mathrm{Z}_0$ such that for any non-zero $a = \sum_{b \in \mathcal{B}} z_b b \in T$ there exists $x \in T$ with $\Phi(xa) = uz_b$ for some unit $u \in \mathrm{Z}_0$ and some non-zero $z_b \in \mathrm{Z}_0$.*

Proposition 5.3.2. *[14, Proposition 2.3] Let T be a finitely generated free Z_0-module with a basis $\mathcal{B}$ which satisfies the above hypothesis. Then T is a free Frobenius extension of Z_0 and for any maximal ideal $\mathfrak{m}$ of Z_0 the finite dimensional quotient $T/\mathfrak{m}T$ is a Frobenius algebra.*

We check that our algebras satisfy the hypothesis. Let M be the maximal ideal $\langle X_1, \ldots X_n \rangle = \sum_{i=1}^{\infty} S_i$ of S. Then we let $\mathfrak{m} = M \cap \mathrm{Z}_0 = \sum_{i=1}^{\infty} \mathrm{Z}_{0i} = \langle f_1, \ldots, f_n \rangle$. This gives the finite dimensional k-algebra $\overline{\mathrm{S}} = S/\mathfrak{m}S$. Since Z_0 is a polynomial algebra, $\{f_i\}$ is a maximal Z_0-sequence, which is also a Z_0-sequence on S since S is free over Z_0. Since S is a polynomial algebra in n variables, it has injective dimension n. Thus $\overline{\mathrm{S}} = S/\sum f_i S$ has injective dimension 0 by Theorem 1.3.14 and hence is a quasi-Frobenius algebra.

Now $\overline{\mathrm{S}}$ is a local ring by the following known lemma.

Lemma 5.3.3. *Let S be an $\mathbb{N}$-graded ring and R a graded subalgebra of S with $R_0 = S_0 = k$ and $\dim_k R_i,\ \dim_k S_i < \infty$. Let $\mathfrak{m} := \bigoplus_{i \geq 1} R_i$ be the augmentation ideal of R. Then if $\overline{\mathrm{S}} := S/\mathfrak{m}S$ is finite dimensional it is local.*

Proof. If $\overline{\mathrm{S}}$ is finite dimensional then there exists some $i_0 > 0$ such that $\overline{\mathrm{S}}_i = 0$ for all $i > i_0$. Then $(\mathfrak{m}S)_i = S_i$ for all $i > i_0$. Now take $M := \sum_{i \geq 1} S_i$. For any $j \geq 1$ and $x \in S_j$, we have $x^{i_0+1} \in S_{j(i_0+1)} = (\mathfrak{m}S)_{j(i_0+1)} \subseteq \mathfrak{m}S$. So $M/\mathfrak{m}S$ is generated by nilpotent elements and hence is nilpotent. So $M/\mathfrak{m}S = J(\overline{\mathrm{S}})$. □

The following Lemma is proved in [34, Proposition 20.3A]

Lemma 5.3.4. *Let $\overline{\mathrm{S}}$ be as above. Then the highest degree component is 1-dimensional.*

We also note that the highest degree component is the socle of $\overline{\mathrm{S}}$.

Proposition 5.3.5. *A local, quasi-Frobenius, finite dimensional commutative k-algebra is Frobenius.*

Proof. Let A be a local, quasi-Frobenius, finite dimensional k-algebra. Since A is finite dimensional we can express A as $Ae_1 \oplus \cdots \oplus Ae_t$ where the e_i are orthogonal primitive idempotents. Since A is commutative, Ae_i is scalar local and hence basic. But since A is local, $t = 1$ and hence A is a basic algebra. Now Proposition 1.7.11 shows that A is Frobenius.

□

Now $\overline{\mathrm{S}}$ is a local quasi-Frobenius ring, so is Frobenius by the preceding proposition. Let the 1-dimensional socle, $\overline{\mathrm{S}}_N$ say, of $\overline{\mathrm{S}}$ be generated by the element $\overline{b_0}$. We extend this to a homogeneous basis of $\overline{\mathrm{S}}$ by taking $\overline{b_1}, \ldots, \overline{b_u}$ to be a k-basis of $\overline{\mathrm{S}}_{N-1}$, $\overline{b_{u+1}}, \ldots, \overline{b_v}$ to be a k-basis of $\overline{\mathrm{S}}_{N-2}$ and so on. This gives us a basis $\{\overline{b_0}, \overline{b_1} \ldots, \overline{b_t}\}$ for $\overline{\mathrm{S}}$ where each $\overline{b_i}$ is homogeneous. Let $h_0, \ldots, h_t$ be the basis of the dual algebra $\overline{\mathrm{S}}^*$ such that $h_i(\overline{b_j}) = \delta_{ij}$. Now $\overline{\mathrm{S}}$ is a Frobenius algebra, and we can take the bilinear form to be $\langle a, b\rangle = \phi(ab)$ where $\phi(x)$ is the coefficient of $\overline{b_0}$ in x. Thus $h_i = \langle -, \overline{b^i}\rangle = \phi(-.\overline{b^i})$ for some homogeneous element $\overline{b^i}$ in $\overline{\mathrm{S}}$. Then since $h_i(\overline{b_j}) = \delta_{ij}$ we have $\phi(\overline{b_i}\overline{b^i}) = 1$ and hence, since the $\overline{b_i}$ and $\overline{b^i}$ are homogeneous, $\overline{b_i}\overline{b^i} = \overline{b_0}$. Thus $\overline{b^i} \in S_{N-d}$ where $d = \deg(\overline{b_i})$.

We would like to lift our bases to Z_0-bases of homogeneous elements for S. We can do this by the following well-known graded version of Nakayama's lemma.

Lemma 5.3.6. *Let $R = \bigoplus_{i\geq 0} R_i$ be a $\mathbb{N}$-graded ring with $R_0 = k$ and $\dim_k R_i < \infty$. Let $\mathfrak{m} := \bigoplus_{i\geq 1} R_i$ be the augmentation ideal and let X be a finitely generated graded R-module. So $X/\mathfrak{m}X = \bigoplus_{i\geq 1} X_i/\mathfrak{m}X_i$ is a finite dimensional vector space with a homogeneous basis. Then X is generated by any choice of homogeneous preimages of the specified basis of $X/\mathfrak{m}X$.*

Proof. Let $X/\mathfrak{m}X$ have homogeneous basis $\{\overline{x_1}, \ldots, \overline{x_d}\}$ and choose any homogeneous elements x_i such that $x_i + \mathfrak{m}X = \overline{x_i}$. Now let $Y := \sum_i Rx_j$ and suppose for a contradiction that $Y \subsetneq X$. Then X/Y is a non-zero finitely generated graded module and therefore has a simple graded factor module,which must be annihilated by $\mathfrak{m}$. Therefore $\mathfrak{m}X + Y \subsetneq X$, which is a contradiction. □

Proposition 5.3.7. *Let G be a finite group acting linearly on a finite dimensional vector space V. Let Z_0 be the central polynomial subalgebra of $T := S\ (V) * G$ defined at the start of section 5.3. Then T is a Frobenius extension of Z_0.*

Proof. We retain the notation used above. Thus we have dual bases $\{\overline{b_0}, \overline{b_1} \dots, \overline{b_t}\}$ and $\{\overline{b^0}, \overline{b^1} \dots, \overline{b^t}\}$ for $\overline{\mathrm{S}}$ which we can lift, by Lemma 5.3.6, to homogeneous generating sets $\{b_0, b_1 \dots, b_t\}$ and $\{b^0, b^1 \dots, b^t\}$ of S. However, $\mathrm{rank}_{\mathrm{Z}_0} S = \mathrm{rank}_k \overline{\mathrm{S}} = t+1$ so the homogeneous generating sets $\{b_0, b_1 \dots, b_t\}$ and $\{b^0, b^1 \dots, b^t\}$ are bases of S as a free Z_0-module. Now we can take the basis of T over Z_0 to be $\{b_i g : 0 \leq i \leq t, g \in G\}$ and we define a Z_0-linear functional

$$\Phi : T \to \mathrm{Z}_0$$

$$\sum_{i,g} z_{ig} b_i g \mapsto z_{01}.$$

We will show that Φ satisfies Hypothesis 1 but first, we consider $\Phi(b_i b^j)$. Since $\overline{b_i}\overline{b^i} = \overline{b_0}$ we get

$$\overline{b_i}\overline{b^j} = \delta_{ij}\overline{b_0} + \sum_{l=1}^{t} \lambda_{lij}\overline{b_l},$$

where $\lambda_{lii} = 0$ for all l. Then

$$b_i b^j = \delta_{ij} b_0 + \sum_{l=0}^{t} r_{lij} b_l \tag{5.8}$$

where, for $0 \leq l, i, j \leq t$, $r_{lij} \in \mathrm{Z}_0$, $r_{0ij} \in \mathfrak{m}$ and $r_{lii} \in \mathfrak{m}$. Now

$$\Phi(b_i b^j) = \Phi(\delta_{ij} b_0) + \Phi(\sum_{l=0}^{t} r_l i j b_k) = \delta_{ij} + r_{0ij}.$$

Now to show that Φ satisfies Hypothesis 1, let $0 \neq a = \sum_{g \in G} z_{ig} b_i g$. Then we choose $b_j h$ with b_j being of maximal degree such that $z_{jh} \neq 0$. We let $x = h^{-1} b^j$ and would like to show that $\Phi(ax) = z_{jh}$. We have

$$\Phi(ax) = \Phi(\sum_{\substack{0 \leq i \leq t \\ g \in G}} z_{ig} b_i g h^{-1} b^j) = \Phi(\sum_{0 \leq i \leq t} z_{ih} b_i b^j) = \sum_{0 \leq i \leq t} z_{ih} \Phi(b_i b^j), \tag{5.9}$$

where the second equality is because, for $s \in S$, $\Phi(sgh^{-1}) \neq 0$ only when $g = h$.

Thus

$$\Phi(ax) = \sum_{0 \le i \le t} z_{ih}(\delta_{ij} + r_{0ij})$$

and we want to show that $z_{ih} r_{0ij} = 0$ for all $1 \le i \le t$. As before, let $N = \deg(b_0)$, which is the highest degree among the b_i. Let $\deg(b_j) = m$ so that $\deg(b^j) = N - m$. We compare degrees in (5.8).

The left hand side of (5.8) is a homogeneous element, so the right hand side must be homogeneous of the same degree. We are interested in r_{0ij}, and it can only be non-zero if $\deg(b_i b^j) \ge N$, which is the case if $\deg(b_i) \ge m$. We have chosen j with b_j of maximal degree such that $z_{jh} \ne 0$, so for b_i of greater degree, $z_{ih}\Phi(b_i b^j) = 0$ and makes no contribution to $\Phi(ax)$ in (5.9). Thus we need only consider the b_i with $\deg(b_i b^j) = N$. This is the case if either, $i = j$ or if there is some other basis element b_{i_0} with degree m. Suppose we are in the latter instance. Then $\overline{b_{i_0}}\,\overline{b^j} = 0$ giving

$$b_{i_0} b^j = r_{0i_0j} b_0 + \sum_{l=1}^{t} r_{li_0j} b_l$$

where $r_{li_0j} \in \mathfrak{m}$ for all l. Now the left hand side has degree N and $\deg(r_{0i_0j} b_0) > N$ if $r_{0i_0j} b_0 \ne 0$, giving $r_{0i_0j} = 0$. Similarly, if $i = j$ we have

$$b_j b^j = b_0 + r_{0i_0j} b_0 + \sum_{l=1}^{t} r_l b_l$$

with $r_{0i_0j} = 0$ so in this case $\Phi(b_j b^j) = 1$. Thus we must have $\Phi(z_{ih} b_i b^j) = \delta_{ij} z_{ih}$.

Thus

$$\Phi(ax) = \Phi(\sum_{0 \le i \le t} z_{ih} b_i b^j) = z_{jh},$$

the hypothesis is satisfied and Proposition 5.3.2 tells us that T is a Frobenius extension of Z_0. □

We can therefore define the Frobenius form to be

$$\langle a, b \rangle = \Phi(ab)$$

for $a, b \in T$. To see whether or not this is symmetric, let $t \in T$ and consider $t.\Phi$ and $\Phi.t$. So for $a \in T$, $t.\Phi(a) = \Phi(at)$ and $\Phi.t(a) = \Phi(ta)$. Let $a = \sum z_{ig} b_i g$ where $z_{ig} \in \mathrm{Z}_0$. We first take $t = s$ for any $s \in S$. Then

$$\Phi(as) = \Phi(\sum_{\substack{0 \le i \le t \\ g \in G}} z_{ig} b_i g s) = \Phi(\sum_{\substack{0 \le i \le t \\ g \in G}} z_{ig} b_i s^g g) = \Phi \sum_{0 \le i \le t} (z_{i1} b_i s),$$

while

$$\Phi(sa) = \Phi(\sum_{\substack{0\leq i\leq t\\ g\in G}} sz_{ig}b_ig) = \Phi(\sum_{\substack{0\leq i\leq t\\ g\in G}} z_{ig}b_isg) = \Phi(\sum_{0\leq i\leq t} z_{i1}b_is).$$

So $s.\Phi = \Phi.s$ for all $s \in S$.

Now take $t = h \in G$. Note that there is an induced action of G on the graded ring $\overline{\mathrm{S}} = S/\mathfrak{m}S$, preserving degree. Therefore, since the highest degree component $\overline{S}_N$ of $\overline{\mathrm{S}}$ is one-dimensional, with $\overline{\mathrm{S}}_N = k\overline{b_0}$, we have $\overline{b_0}^g = \lambda_g\overline{b_0}$ for all $g \in G$, where λ_g is some non-zero element in k. Then $b_0^g = \lambda_g b_0 + m$ for some $m \in \mathfrak{m}S$. We can express m as $\sum_{i=0}^{t} d_ib_i$ where each $d_i \in \mathfrak{m}$ and thus has degree greater than zero if $d_i \neq 0$. But because the action of G preserves degree, $d_0 = 0$. Thus

$$\Phi(b_0^g) = \lambda_g.$$

Hence, for $h \in G$ and $a = \sum_{\substack{0\leq i\leq t\\ g\in G}} z_{ig}b_ig \in T$ as before,

$$\Phi(ah) = \Phi(\sum_{\substack{0\leq i\leq t\\ g\in G}} z_{ig}b_igh) = \Phi(\sum_{0\leq i\leq t} z_{ih^{-1}}b_i) = z_{0h^{-1}},$$

and

$$\Phi(ha) = \Phi(\sum_{\substack{0\leq i\leq t\\ g\in G}} hz_{ig}b_ig) = \Phi(\sum_{0\leq i\leq t} z_{ih^{-1}}b_i^h) = z_{0h^{-1}}\lambda_h.$$

So $\Phi(ha) = \Phi(ah)$ if and only if $\lambda_h = 1$ for all $h \in G$, that is, if and only if $\overline{b_0}^h = \overline{b_0}$ for all $h \in G$. In fact, the Nakayama automorphism ν, as defined in Definition 1.7.7, is the map such that

$$\nu(s) = s \text{ and } \nu(g) = \lambda_g g \tag{5.10}$$

for all $s \in S$ and $g \in G$, and where λ_g is the element in k such that $\overline{b_0}^g = \lambda_g\overline{b_0}$. We note that this is indeed a Z_0-automorphism since for $z \in \mathrm{Z}_0$ and $b_ig \in T$ we have

$$\nu(zb_ig) = zb_i\lambda_g g = z\nu(b_ig).$$

Thus T is symmetric if and only if $\overline{b_0}^h = \overline{b_0}$ for all $h \in G$.

To investigate when this condition holds, we first of all assume that k has characteristic p and G is a p-group. In this case we have the following well-known lemma.

Lemma 5.3.8. *Let k be a field with characteristic p and G a p-group. Then kG is local.*

Proof. By [44, Lemma 3.1.6] the augmentation ideal, $\mathfrak{g}$ is nilpotent. Hence $\mathfrak{g} = J(kG)$. □

Then since kG is local, the 1-dimensional socle $k\overline{b_0}$ of $\overline{\mathrm{S}}$, which is a kG-module, must be isomorphic to the trivial kG-module k. Thus for $g \in G$, $\overline{b_0}^g = \overline{b_0}$ and $\overline{b_0}$ is invariant under the action of G. Thus we have proved the following:

Theorem 5.3.9. *Let k be a field of characteristic p, and G a finite p-group acting linearly on a finite dimensional vector space V. Let Z_0 be a graded polynomial subalgebra of $Z(T) = S(V)^G$. Then $T := S(V) * G$ is a symmetric Z_0-algebra.*

Now we consider the case where the order of G is a unit in k. From above we know that $S * G$ is symmetric if and only if $\overline{b_0}^g = \overline{b_0}$ for all $g \in G$. Notice that the action of G on V gives a group homomorphism $\rho : G \to GL(V)$. We will show that symmetry depends on $\det(g) := \det \rho(g)$.

For a vector space V acted on by a group element g we denote by $\mathrm{Tr}(g, V)$ the trace of the matrix recording the action of g on V. We note the following easy observations, which we will use in proving the next lemma, which is effectively proved in [6, page 59].

(i) If g acts trivially on V then $\mathrm{Tr}(g, V) = \dim V$.

(ii) If $V = W \oplus X$, a sum of kG-modules, then $\mathrm{Tr}(g, V) = \mathrm{Tr}(g, W) + \mathrm{Tr}(g, X)$.

Lemma 5.3.10. *Let G be a finite group acting linearly on a k-vector space V such that the characteristic of k does not divide $|G|$. Let $S = S(V) = k[X_1, \dots, X_n]$ and let $\{f_1, \dots, f_n\}$ be homogeneous elements of S with $k_i = \deg f_i$, $1 \le i \le n$, such that $\mathrm{Z}_0 = k[f_1, \dots, f_n]$ is a polynomial algebra contained in S^G over which S is a finitely generated module. Let $\overline{\mathrm{S}} = S/\mathfrak{m} S$ where $\mathfrak{m}$ is $\langle X_1, \dots, X_n\rangle \cap \mathrm{Z}_0$. Let $\mathrm{Tr}(g, S_j)$ and $\mathrm{Tr}(g, \overline{\mathrm{S}}_j)$ be the traces of the matrices denoting the action of g on S_j and $\overline{\mathrm{S}}_j$ respectively. Assume that, for all $g \in G$, k contains a primitive $|\langle g\rangle|^{th}$ root of unity. Let $g \in G$. Then,*

$$\sum_{j=0}^{\infty} t^j \, \mathrm{Tr}(g, S_j) = \frac{\sum_{j=0}^{\infty} \mathrm{Tr}(g, \overline{S_j}) t^j}{\prod_{i=1}^{n} (1 - t^{k_i})}.$$

Proof. First note that extending the field doesn't affect the formula, so we can assume that k is algebraically closed. Since S_j and $\overline{\mathrm{S}}_j$ are vector spaces on which g acts, they are $k\langle g\rangle$-modules. Let $d := |\langle g\rangle|$. Also $k\langle g\rangle$ is semi-simple by Maschke's Theorem, so the irreducible $k\langle g\rangle$-modules correspond to $\lambda_1, \dots, \lambda_n$ where the λ_i are the d^{th} roots of unity. We can choose our bases $\{\overline{b_i}\}$ of $\overline{\mathrm{S}}$ and $\{b_i\}$ of S to be eigenvectors, where $\overline{b_i}$ and b_i correspond to some eigenvalue $\lambda_i \in \{\lambda_1, \dots, \lambda_n\}$.

Since $k\langle g\rangle$ is semi-simple, $S_j = (\mathfrak{m} S \cap S_j) \oplus A_j$ for some $k\langle g\rangle$-module A_j. Then

$A_j \cong S_j/(\mathfrak{m}\, S \cap S_j) = \overline{\mathrm{S}}_j$, giving

$$\mathrm{Tr}(g, S_j) = \mathrm{Tr}(g, \overline{\mathrm{S}}_j) + \mathrm{Tr}(g, \mathfrak{m}\, S \cap S_j).$$

Now, for $d_i = \deg(b_i)$,

$$\begin{aligned} \mathfrak{m}\, S \cap S_j &= \{\sum_{i=0}^{t} a_i b_i : a_i \in \mathfrak{m} \cap S_{j-d_i}\} \\ &= \oplus_{i=0}^{t} \mathfrak{m} \cap S_{j-d_i} b_i, \end{aligned}$$

where $\mathfrak{m} \cap S_{j-d_i} = 0$ if $j - d_i = 0$ and $\mathfrak{m} \cap S_{j-d_i} = Z_{0,j-d_i}$ if $j - d_i > 0$, where Z_{0i} is the i^{th} degree part of Z_0. Thus,

$$\mathrm{Tr}(g, \mathfrak{m}\, S \cap S_j) = \sum_{i=0}^{t} \mathrm{Tr}(g, Z_{0,j-d_i} b_i) = \sum_{i=1}^{j} \dim(\mathrm{Z}_{0i})\, \mathrm{Tr}(g, \overline{S_{j-i}}).$$

The second equality follows because g acts diagonally on the $\overline{b_i}$ and b_i giving $\mathrm{Tr}(g, Z_{0,j-d_i} b_i) = \dim(Z_{0,j-d_i})\lambda_i$ for some λ_i and $\mathrm{Tr}(g, \overline{S_{j-i}})$ is also a sum of λ_is.

Then,

$$\begin{aligned} \mathrm{Tr}(g, S_j) &= \mathrm{Tr}(g, \overline{\mathrm{S}}_j) + \sum_{i=1}^{j} \dim(\mathrm{Z}_{0i})\, \mathrm{Tr}(g, S_{j-i}) \\ &= \sum_{i=0}^{j} \dim Z_i\, \mathrm{Tr}(g, \overline{\mathrm{S}}_{j-i}). \end{aligned}$$

Expressing this as a power series and noting that $\mathrm{Tr}(g, \overline{\mathrm{S}}_{j-i}) = 0$ for $i > j$ gives

$$\begin{aligned} \sum_{j=0}^{\infty} t^j\, \mathrm{Tr}(g, S_j) &= \sum_{j=0}^{\infty} t^j \sum_{i=0}^{j} \dim Z_{0i}\, \mathrm{Tr}(g, \overline{\mathrm{S}}_{j-i}) \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \dim Z_{0i} t^i\, \mathrm{Tr}(g, \overline{\mathrm{S}}_{j-i}) t^{j-i} \\ &= \sum_{j=0}^{\infty} (\sum_{i=0}^{\infty} \dim Z_{0i} t^i)\, \mathrm{Tr}(g, \overline{\mathrm{S}}_j) t^j. \end{aligned}$$

Then for $k_i = deg(f_i)$, $\sum_{i=0}^{\infty} \dim Z_{0i} t^i = \dfrac{1}{\prod_{i=1}^{n}(1 - t^{k_i})}$ by [34, §17.1, Example 4] and we have

$$\sum_{j=0}^{\infty} t^j\, \mathrm{Tr}(g, S_j) = \frac{\sum_{j=0}^{\infty} \mathrm{Tr}(g, \overline{S_j}) t^j}{\prod_{i=1}^{n}(1 - t^{k_i})}.$$

□

Proposition 5.3.11. *Let G be a finite group acting linearly on a k-vector space V such that the characteristic of k does not divide $|G|$. Let $S = S(V) = k[X_1, \ldots, X_n]$ and let $\{f_1, \ldots, f_n\}$ be homogeneous elements of S with $k_i = \deg f_i$, $1 \le i \le n$, such that $\mathrm{Z}_0 = k[f_1, \ldots, f_n]$ is a polynomial algebra contained in S^G over which S is a finitely generated module. Assume that each f_i is homogeneous of degree divisible by $|G|$. Let $\overline{\mathrm{S}} = S/\mathfrak{m}\,S$ where $\mathfrak{m}$ is $\langle X_1, \ldots, X_n\rangle \cap \mathrm{Z}_0$. Let $\overline{b_0} \in \overline{\mathrm{S}}$ be the generator of the 1-dimensional socle of $\overline{\mathrm{S}}$. Then for any $g \in G$ we have $\overline{b_0}^g = (\det g)^{-1}\overline{b_0}$.*

Proof. We can again assume that k is algebraically closed. Let $g \in G$. As in the preceding lemma we see that we can choose bases such that g acts on V by $diag(\lambda_1, \ldots, \lambda_n)$ where each λ_i is a d^{th} root of unity (here $d = |\langle g\rangle|$). Then by Lemma 5.3.10 we have,

$$\sum_{i=0}^{\infty} \mathrm{Tr}(g, \overline{\mathrm{S}}_i)t^i = \prod_{j=1}^{n}(1 - t^{k_j}) \sum_{i=0}^{\infty} \mathrm{Tr}(g, S_i)t^i,$$

where k_j is the degree of f_j. By [6, Proposition 2.5.1] we have

$$\sum_{i=0}^{\infty} \mathrm{Tr}(g, S_i)t^i = \prod_{j=1}^{n} \frac{1}{1 - \lambda_j t}.$$

Also, since $\lambda_j^d = 1$ and $|G|$ divides k_i we have $\lambda_j^k = 1$ for all j, $1 \le j \le n$. Hence,

$$\sum_{i=0}^{\infty} \mathrm{Tr}(g, \overline{\mathrm{S}}_i)t^i = \prod_{j=1}^{n} \frac{1 - t^{k_j}}{1 - \lambda_j t} = \prod_{j=1}^{n}(1 + \lambda_j t + \cdots + \lambda_j^{k_j - 1} t^{k_j - 1}).$$

Now for $N = \deg(\overline{b_0})$, $\overline{\mathrm{S}}_N$ is the highest degree part of $\overline{\mathrm{S}}$ and hence $\mathrm{Tr}(g, \overline{\mathrm{S}}_N) = \prod_{j=1}^{n} \lambda_j^{k_j - 1} = \prod_{j=1}^{n} \lambda_j^{-1} = (\det g)^{-1}$. Hence, since $\overline{\mathrm{S}}_N$ is one dimensional and generated by $\overline{b_0}$ we must have

$$\overline{b_0}^g = (\det g)^{-1}\overline{b_0}.$$

□

Thus, for finite groups whose order is a unit in the ground field, T is symmetric if and only if $\rho(G) \subseteq SL(V)$ where $\rho : G \to GL(V)$ is the representation of G as linear automorphisms of V. And to show that this agrees with the case where G is a p-group, the following well-known lemma shows that in this case $\rho(G) \subseteq SL(V)$.

Lemma 5.3.12. *Let k be a field of characteristic p and G a finite p group acting linearly on a finite dimensional vector space V. Then for any $g \in G$, the matrix giving the action of g has determinant 1.*

Proof. The map $\det : G \to k$ is a k-homomorphism. Let $q := |G|$. Then for any $g \in G$, $\det(g)^q = \det(g^q) = 1$. But k has no non-identity elements of prime power order, so $\det(g) = 1$. □

We can now combine these results in the following theorem.

Theorem 5.3.13. *Let k be a field and G a finite group acting linearly on a finite dimensional vector space V. Let $\rho : G \to GL(V)$ be the representation of G as linear automorphisms of V and let Z_0 be any central polynomial subalgebra generated by homogeneous elements $f_1, \ldots, f_n$ of S with each $\deg(f_i)$ divisible by $|G|$ and such that $T := S(V) * G$ is a finitely generated Z_0-module. Then T is a symmetric Z_0-algebra if and only if $\rho(G) \subseteq SL(V)$.*

Proof. If char k does not divide the order of the group, then Proposition 5.3.11 gives the result. Similarly if char $k = p$ and G is a p group then Theorem 5.3.9 together with Lemma 5.3.12 give the result. For the remaining case, suppose that $p =$ char k divides the order of G and choose any element $g \in G$ of order d and consider the group $\langle g \rangle$. Let $d = p^n r$ where $(p, r) = 1$. Let $h = g^{p^n}$. Then $\langle h \rangle$ is a finite group where char k does not divide the order, so that Proposition 5.3.11 tells us that $\lambda_h = (\det h)^{-1}$ where λ_h is as defined in (5.10). Then since $g = h^r$ we have $\lambda_g = \lambda_h^r = ((\det h)^{-1})^r = (\det g)^{-1}$. Thus, T is symmetric if and only if $\det g = 1$ for all $g \in G$. □

We note that Braun has proved the same result in [8]. Our work is independent of his and uses somewhat different methods for part of the proof, but for the very last step the methods are the same so for convenience we follow his account.

5.4 Notes

Proposition 5.1.3 is taken from Chapter 1 and Lemmas 5.1.4, 5.1.6 and 5.1.9 are all known but proved here for lack of a suitable reference. The remaining results in section 5.1 are original except Theorem 5.1.2. The properties of symmetric and Frobenius algebras in Propositions 5.2.5, 5.2.6 and 5.2.7 are almost certainly already well-known, as are Lemmas 5.2.13 and 5.2.23, but the remaining unreferenced results in section 5.2 are original. Section 5.3 contains a number of known results, some of which are proved here, but Proposition 5.3.5, Proposition 5.3.7, Theorem 5.3.9, Proposition 5.3.11 and Theorem 5.3.13 are original.

References

[1] F. W. Anderson and K. R. Fuller. *Rings and categories of modules*. Springer-Verlag, New York, 1974. Graduate Texts in Mathematics, Vol. 13.

[2] M. Artin and W. F. Schelter. Graded algebras of global dimension 3. *Adv. in Math.*, 66(2):171–216, 1987.

[3] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[4] H. Bass. On the ubiquity of Gorenstein rings. *Math. Z.*, 82:8–28, 1963.

[5] A. D. Bell and R. Farnsteiner. On the theory of Frobenius extensions and its application to Lie superalgebras. *Trans. Amer. Math. Soc.*, 335(1):407–424, 1993.

[6] D. J. Benson. *Polynomial invariants of finite groups*, volume 190 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[7] N. Bourbaki. *Elements of mathematics. Commutative algebra*. Hermann, Paris, 1972. Translated from the French.

[8] A. Braun. On the gorenstein property for modular invariants. Preprint.

[9] A. Braun. On symmetric, smooth and Calabi-Yau algebras. *J. Algebra*, 317(2):519–533, 2007.

[10] K. A. Brown. Fully bounded Noetherian rings of finite injective dimension. *Quart. J. Math. Oxford Ser. (2)*, 41(161):1–13, 1990.

[11] K. A. Brown. Noetherian Hopf algebras. *Turkish J. Math.*, 31, 2007.

[12] K. A. Brown and K. R. Goodearl. Homological aspects of Noetherian PI Hopf algebras of irreducible modules and maximal dimension. *J. Algebra*, 198(1):240–265, 1997.

[13] K. A. Brown and K. R. Goodearl. *Lectures on algebraic quantum groups*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2002.

[14] K. A. Brown, I. G. Gordon, and C. H. Stroppel. Cherednik, Hecke and quantum algebras as free Frobenius and Calabi-Yau extensions. *J. Algebra*, 319(3):1007–1034, 2008.

[15] K. A. Brown and C. R. Hajarnavis. Homologically homogeneous rings. *Trans. Amer. Math. Soc.*, 281(1):197–208, 1984.

[16] K. A. Brown and C. R. Hajarnavis. Injectively homogeneous rings. *J. Pure Appl. Algebra*, 51(1-2):65–77, 1988.

[17] K. A. Brown, C. R. Hajarnavis, and A. B. MacEacharn. Rings of finite global dimension integral over their centres. *Comm. Algebra*, 11(1):67–93, 1983.

[18] K. A. Brown, Charudatta R. Hajarnavis, and A. B. MacEacharn. Noetherian rings of finite global dimension. *Proc. London Math. Soc. (3)*, 44(2):349–371, 1982.

[19] K. A. Brown and T. Levasseur. Cohomology of bimodules over enveloping algebras. *Math. Z.*, 189(3):393–413, 1985.

[20] W. Bruns and J. Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[21] I. Burban and Y. Drozd. Maximal Cohen-Macaulay modules over surface singularities.

[22] K. Changtong. *Symplectic Reflection Algebras in positive charactaristic*. PhD thesis, University of Glasgow, 2006.

[23] A. Craw. The special Mckay correspondence as an equivalence of derived categories. arXiv:0704.3627v3.

[24] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[25] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[26] P. Etingof and V. Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.*, 147(2):243–348, 2002.

[27] C. Faith. *Algebra: rings, modules and categories. I.* Springer-Verlag, New York, 1973. Die Grundlehren der mathematischen Wissenschaften, Band 190.

[28] E. Formanek and A. V. Jategaonkar. Subrings of Noetherian rings. *Proc. Amer. Math. Soc.*, 46:181–186, 1974.

[29] R. M. Fossum. *The divisor class group of a Krull domain.* Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74.

[30] V. Ginzburg. Calabi-yau algebras. arXiv:math.AG/0612139.

[31] K. R. Goodearl and R. B. Warfield, Jr. *An introduction to noncommutative Noetherian rings*, volume 61 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 2004.

[32] S. Goto and K. Nishida. Catenarity in module-finite algebras. *Proc. Amer. Math. Soc.*, 127(12):3495–3502, 1999.

[33] O. Iyama and I. Reiten. Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras. *Amer. J. Math.*, 130(4):1087–1149, 2008.

[34] R. Kane. *Reflection groups and invariant theory.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 5. Springer-Verlag, New York, 2001.

[35] I. Kaplansky. *Commutative rings.* Allyn and Bacon Inc., Boston, Mass., 1970.

[36] G. R. Krause and T. H. Lenagan. *Growth of algebras and Gelfand-Kirillov dimension*, volume 22 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2000.

[37] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[38] T. Levasseur. Some properties of noncommutative regular graded rings. *Glasgow Math. J.*, 34(3), 1992.

[39] R. Martin. Skew group rings and maximal orders. *Glasgow Math. J.*, 37(2):249–263, 1995.

[40] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.

[41] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1987. With the cooperation of L. W. Small, A Wiley-Interscience Publication.

[42] S. Montgomery. Outer automorphisms of semi-prime rings. *J. London Math. Soc. (2)*, 18(2):209–220, 1978.

[43] W. K. Nicholson and M. F. Yousif. *Quasi-Frobenius rings*, volume 158 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003.

[44] D. S. Passman. *The Algebraic Structure of Group Rings*. Pure and Applied Mathematics. Wiley-Interscience, New York, 1977.

[45] D. S. Passman. *Infinite crossed products*, volume 135 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1989.

[46] J. Rainwater. Global dimension of fully bounded noetherian rings. *Comm. Algebra*, 15:2143–2156, 1987.

[47] I. Reiner. *Maximal Orders*, volume 28 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, Oxford, 2003. Corrected reprint of the 1975 original, With a foreword by M. J. Taylor.

[48] J. J. Rotman. *An introduction to homological algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1979.

[49] J. T. Stafford. Nonholonomic modules over Weyl algebras and enveloping algebras. *Invent. Math.*, 79(3):619–638, 1985.

[50] J. T. Stafford and M. Van den Bergh. Noncommutative resolutions and rational singularities. *Michigan Math. J.*, 57, 2008. Special volume in honor of Melvin Hochster.

[51] J. T. Stafford and J. J Zhang. Homological properties of (graded) noetherian PI rings. *J. Algebra*, 168:988–1026, 1994.

[52] K.-M. Teo. Homological properties of fully bounded Noetherian rings. *J. London Math. Soc. (2)*, 55(1), 1997.

[53] M. van den Bergh. Non-commutative crepant resolutions. In *The legacy of Niels Henrik Abel*, pages 749–770. Springer, Berlin, 2004.

[54] K. Watanabe. Certain invariant subrings are Gorenstein. II. *Osaka J. Math.*, 11:379–388, 1974.

[55] M. Wemyss. Reconstruction algebras of type A. arXiv:0704.3693v3.

[56] J. Wunram. Reflexive modules on quotient surface singularities. *Math. Ann.*, 279(4):583–598, 1988.

[57] K. Yamagata. Frobenius algebras. In *Handbook of algebra, Vol. 1*, pages 841–887. North-Holland, Amsterdam, 1996.

[58] A. Yekutieli. Dualizing complexes, Morita equivalence and the derived Picard group of a ring. *J. London Math. Soc. (2)*, 60(3):723–746, 1999.

[59] Y. Yoshino. *Cohen-Macaulay modules over Cohen-Macaulay rings*, volume 146 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

[60] Y. Zhong. Injective homogeneity and the Auslander-Gorenstein property. *Glasgow Maths. J.*, 37:191–204, 1995.