Mathematical Modelling of Non-linear Magneto- and Electro-active Rubber-like Materials

by

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Abstract In this thesis we study several different problems concerning the mathematical modelling of non-linear magneto- and electro-active elastomers. Three main problems have been addressed: universal relations, the modelling of transversely isotropic magneto- and electro-active elastomers, and the variational formulation.

The complete set of linear universal relations was found for isotropic magneto- and electro-active elastomers. Some universal relations for some special simplified cases of the constitutive equations were also found. Two non-linear universal relations were studied, for the helical shear and for the anti-plane shear deformations.

Two boundary value problems were solved using the finite difference method: one of them was the inflation and extension of a tube of finite length under the influence of a uniform axial magnetic field applied far away, and the other was the uniform extension of a cylinder with an electric field applied far away.

The constitutive equations for transversely isotropic magneto- and electro-active elastomers were developed, and several simple boundary value problems were solved. For the case of transversely isotropic magneto-active elastomers a preliminary form for the energy function was proposed.

Finally simple variational formulations for the magneto-elastic problem were found, and an extension of these formulations, which takes into account the interaction with a rigid semi-infinite body was proposed.

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Chapter 1

Introduction

Magneto- and electro-active elastomers are smart materials, which are composed basically of a rubber-like matrix, filled with magneto- or electro-active particles, which in the presence of an external magnetic or electric field, may develop large non-linear elastic deformations.

The rapid response, the high level of deformations that may be achieved, and the possibility of controlling these deformations by varying an external field, either magnetic or electric, make these materials of special interest for vibration and noise suppression and in the design of robots.

In the case of magneto-active elastomers, we may cite the paper by Farshad and Le Roux [44], where a new design for a window has been proposed, which makes use of an isotropic [1] magneto-active elastomer in order to generate a controlled vibration in the window such that the noise may be reduced or suppressed.

The papers by Yalcintas and Dai [120], and Kari and Blom [62] also address the issue of the application of magneto-active elastomers (MS) [2] for noise and vibration reduction; in this last paper, experimental results have been obtained for the shear modulus from a dynamic test; as in the paper by Farshad and Le Roux [44], only the isotropic case was considered.

An example of a smart actuator based in the use of a transversely isotropic MS [3] has

[1] Isotropic means with a random distribution of particles.

[2] The abbreviations MS and ES will be used in order to speak about the magneto- and electro-active elastomers respectively. These names are originated from the phrases magneto-sensitive and electro-sensitive elastomers, which are alternative names for these smart materials.

[3] Transversely isotropic means a magneto-active elastomer in which the particles have a preferred alignment, which has been obtained by applying a field during the curing process.

been given by Zhou and Wang [129].

In the case of electro-active elastomers (ES), a comprehensive account of some of the capabilities of these materials may be found in the paper by Bar-Cohen [3], where, in particular, applications of these materials in the development of a new generation of robots have been addressed.

Most of the experimental data available have been obtained for MS elastomers. The most important class of MS elastomers studied in this thesis is basically made with a rubber-like matrix filled with magneto-active particles (see Ginder [48]).

Regarding ES elastomers, besides the ES elastomer studied in this thesis, which is made in a similar way to the MS elastomers mentioned above (see, for example, Bossis et al. [11], Varga et al. [115] and Fehér et al. [46]), we have an additional class of elastomers, which react to the presence of an electric field due only to their particular composition; for these materials it is unnecessary to add electro-active particles (see, for example, Nam at al. [76]).

Some of the papers cited above deal actually more with the characterization of fluids and gels filled with either magneto- or electro-active particles. Magneto- and electro-active fluids and gels have attracted the attention of researchers for a longer time than the magneto- and electro-active elastomers, which have been studied in detail only during the last ten years, although a class of rigid polymers filled with magnetic particles has been in use for a long time (see, for example, [115]).

A 'type' of material, which might be difficult to classify as a different class, corresponds to a kind of electro-active elastomer that is presented as thin membranes composed of one layer of normal rubber-like material covered by two layers of electrodes, which with the application of an electric potential provokes large, elastic and 'controllable' [4] deformations for the elastomer; see, for example, Costen et al. [26], and also Mazzoldi et al. [71] for a short yet complete summary about these electro-active elastomers.

Regarding the composition, in the case of MS elastomers the matrix is made of a highly elastic material with a low magnetic permeability, filled with particles with high permeability, low residual magnetization [5] (see Lokander and Stenberg [66]), and high

[4]Here the word controllable means that the form and 'amount' of deformation may be controlled by an appropriate intensity and distribution of the electric field in the electrodes.

[5]Low permanent magnetization is required in order to be able to vary the magnetic field inside the MS elastomer quickly, since if there is too much residual magnetization, the response of the material would become 'slow'. Note that the above requirement would imply a negligible magnetic hysteresis.

magnetic saturation. Among the different choices for the matrix, natural rubber [6] and silicone elastomer [7] are the most widely used. Regarding the magneto-active particles, Carbonyl iron has been the principal choice [8], followed by pure iron (see, for example, Shen et al. [97] and Blom and Kari [8]); other materials used are cobalt and nickel alloys.

The size of the particles used in the papers cited above is highly variable; in most of them, spherical particles of about 2 to 4 [μm] have been used; however, experiments with larger particles with irregular shape have been carried out as well; see, for example, Blom and Kari [8], and Kari and Blom [62], where particles of about 60 [μm] have been considered.

The proportion of particles in the matrix is highly variable as well; in the above references it ranges from 10% to a 30% per volume.

The particles are added to the matrix during the curing process, where it is necessary to apply temperature and pressure; for example, in Lokander and Stenberg [66] a temperature of 150[°C] and a pressure of 12[MPa] were used for the curing process, which lasted for 30 minutes. The temperature differs depending on whether we are working with a natural rubber or a silicone elastomer [8].

An external magnetic field (or magnetic induction) may be applied during the curing process, which has the purpose of generating a preferred alignment for the particles, which as we will see later on, may enhance the properties of the MS elastomer significantly in comparison with the isotropic case; a range of values for this external field might be 0.5[T] to 0.8[T] (see Shen at al. [97], and Ginder at al. [49,50]).

Much less information is available for ES elastomers. We only study the class of electro-active elastomers that are made by filling a rubber-like matrix with electro-active particles. From the paper by Bossis et al. [11] we have data for an ES elastomer whose structure is basically the same as the structure of some of the MS elastomers described previously. From that paper, we find that the matrix corresponds to a silicone elastomer, which is filled with carbonyl iron particles of an average size of 2[μm], and a 30% fraction per volume of particles in the matrix.

[6]Lokander and Stenberg [66], Shen at al. [97], Ginder et al. [49], Blom and Kari [8], Kari and Blom [62], and Ginder et al. [50].

[7]Jolly et al. [59], Bellan and Bossis [7], Varga et al. [117], Zhou [128], and Farshad and Le Roux [44].

[8]See, for example, Lokander and Stenberg [66], Shen et al. [97], Ginder et al. [49], Zhou [128], Farshad and Le Roux [44], Varga et al. [115,117], Bellan and Bossis [7], Jolly et al. [59], and Ginder et al. [50].

Carbonyl iron corresponds to a powder composed of particles with a proportion of iron of about 97%, with traces of carbon, oxygen and nitrogen.

We have discussed the applications and the composition of the MS and the ES elastomers; let's now speak about their properties, and let's give a short review of the experimental data available.

There are three factors that are especially important regarding the behaviour of these materials; the average size of the particles, the total proportion of particles per volume, and the existence of a preferred alignment for the particles.

The inclusion of the magneto- or electro-active particles in the matrix has as a first consequence an increase of the stiffness of the material, but since the particles are almost rigid in comparison with the matrix, we have that the capacity of the material to deform elastically diminishes in comparison with the original pure elastomer. As can be expected, more particles would imply the stronger the magneto- or electrostriction effect is; then there should be an 'optimal' proportion of particles depending on the application; Davis [27] estimated it as about 27% of particles per volume in order to have a maximum change in the shear modulus as a function of the magnetic field.

The majority of the applications have considered 'small' particles with an average size of $3[\mu\text{m}]$; however, Blom and Kari [8] studied MS elastomers made with much larger and more irregular particles (of an average size of $40[\mu\text{m}]$). They showed that in this case, with a random distribution of particles, is possible to achieve a larger magnetostriction effect than for the case with small particles [9].

Let us discuss the above statement in more detail. As we will see later on, when the particles are added to the matrix, after the curing they become trapped; as a result, when a field is applied, they tend to displace thereby provoking the deformation of the body. Now, sometimes this effect may be enhanced significantly by applying a field during the curing process, which permits the particles to adopt a preferred alignment; once cured, this alignment has been shown to enhance the magnetostriction significantly in comparison with the random distribution (see, for example, [7]); in fact, most of the experimental researchers have studied this particular kind of MS elastomer instead of the randomly distributed one, as we will see in the following.

Let us now discuss briefly the available experimental data. First, we point out again that most of the data available correspond to MS elastomers, and that the information available for ES elastomers is very scarce. Regarding the kind of mechanical and magnetic tests that have been done in order to characterize the behaviour of these materials, we

[9] Our mathematical models are based on the assumption of 'small' particles.

can expect many practical problems, in particular about how to measure the magnetic field for a body, and about how to apply a field for a body under deformation. Due to these reasons, as far as it is known, only simple tests have been done, like the traction of a cylindrical bar, the simple shear of a block, and the compression of a cube or a short cylinder.

The paper by Bellan and Bossis [7] was one of the important sources of information for part of this research. They investigated the behaviour of a MS elastomer by using the traction test for a cylindrical bar. Two main problems were treated (maintaining the same proportion of particles): the case of a random distribution of particles, and the case of a preferred alignment for the particles (in the axial direction). These two materials were studied with and without the application of a magnetic field. They also explored the dynamical response of this material. More details of these results will be shown in Section 5.3.

Another important paper whose results were used in this research is that by Ginder et al. [50], who investigated the behaviour of a MS elastomer with a preferred alignment for the particles by working with a block under simple shear deformation. Two main experiments were done, by using either a particle alignment in the direction of the shear, or in a direction perpendicular to the shear. They also studied the behaviour of this material under dynamical conditions.

The paper by Jolly et al. [59], which has been used as a reference in many theoretical researches [12,30,31], shows also some results for a shear test of a block, which was made using a transversely isotropic MS elastomer. They only considered the case in which the particles are aligned in the same direction as the field, which is perpendicular to the direction of the shear. Results for the 'shear modulus' as a function of the magnetic induction were obtained.

In a series of papers Varga et al. [115–117] show some experimental results for a cube under compression, working with gels and MS elastomers. It is interesting to study what they did with the transversely isotropic material, where they studied a cube made of this material, under compression, for several different alignments for the particles and the external field. An interesting phenomenon they found is a sort of 'collapse' for the case in which there is compression in the direction of the particle alignment, when the proportion of particles is 'high' (see Figure 12, page 7786, of [116]).

Kari and Blom [62], and Lokander and Stenberg [66] studied the properties of isotropic

MS elastomers, with special attention to the dynamical response of these materials, using the shear test. Lokander and Stenberg [66] were especially interested in studying MS elastomers made with large irregular particles. Shen at al. [97] also used the shear test but in this case they studied the behaviour of the transversely isotropic MS elastomers. Farshad and Le Roux [45], and Zhou [128] used the compression test for a 'short' cylinder and a plate respectively; Farshad and Le Roux [45] worked with magneto-active gels and elastomers, while Zhou [128] only studied transversely MS elastomers.

The traction test was used by Rigby and Jilkén [88]. A 'bending' test was used by Yalcintas and Dai [120], and Farshad and Benine [43]. Other experimental results that may be mentioned as well correspond to the results obtained by Ginder et al. [49] for transversely isotropic MS elastomers, who studied with particular attention the dynamical behaviour of these materials.

The mathematical modelling of MS and ES elastomers is a difficult task. The main problem, as we will discuss in detail in Section 2.3.1, is to work with large deformations and magnetic or electric fields. A series of simplifications must be adopted; nevertheless, the final models are in general highly complex.

There are basically two different approaches in order to develop a mathematical model for these materials. One them we may call 'micro-mechanical' model, also called sometimes 'structural' model, and the other, the 'continuum' or phenomenological model. Let us speak about the first model. In micro-mechanics a model may be obtained by considering one magneto- or electro-active particle as a rigid sphere (rigid in comparison with the surrounding elastomer matrix), and then by studying what happens with this particle in the presence of a field, when it is surrounded by other particles and the host rubber-like material. It is possible to obtain an approximation for the average field in the material, and for some other 'properties', such as the 'shear modulus'. The magnetic or electric interaction between the particles in the material is in general the most difficult part of the model; a common approximation is to assume that the field in the neighborhood of a particle is affected only by the presence of the field due to the closest particles. The distribution of particles might be assumed random with a low density of particles per volume [10], or it might be assumed with a preferred alignment for the particles, in which case we speak about 'chains of particles' [124]. An advantage of working with micro-mechanical models is, for example, that it enables us to work with mathematical expressions where the constants and coefficients that appear have a clear physical meaning. About some of the disadvantages; the most important that may be mentioned is the complexity of the mathematical expressions that usually appear, especially, for example, when the results for a particle or chain are extrapolated in order to obtain 'global' expressions for the body, when many simplifications are usually made in order to obtain these 'global expressions'.

Borcea and Bruno [10] developed a mathematical micro-mechanical model for a MS elastomer. They considered the particles as rigid 'small' non-sliding uniformly magnetized spheres. Regarding the distribution of particles, a random distribution with a low fraction of particles was assumed. This last assumption is important, because an approximation of the behaviour of the material was developed by studying what happens with one particle, by assuming that only the closest particles had an effect on the magnetic behaviour of this particle. Another important simplification used by Borcea and Bruno [10] was to consider the rubber-like matrix as a linear elastic material. The basic idea of the model was to obtain an approximate expression for the elastic and magnetic energies for the material by calculating the energy accumulated by one particle and its surrounding. Using statistical averaging, the assumption of a random distribution of particles, and the existence of an applied external load and field far away, Borcea and Bruno [10] obtained an expression for the energy for the complete body, which was used in order to calculate 'average' expressions for the stress and the deformation, which depend, among other variables, on the external field.

Davis [27] considered a transversely isotropic MS elastomer; as in [10], the particles were considered as rigid spheres; however, the rubber-like matrix was assumed as an hyperelastic material, whose behaviour was approximated by the 'Ogden' model [77]. Davis only studied the behaviour of one particle and the surrounding rubber-like material around it, which was assumed of a cubic shape. The particles were aligned in chains, as an approximation, only the other particles in the chain were assumed to have an influence in the magnetic field of the particle; the field in the surrounding cubic host material was assumed to be uniform, no interaction between chains was considered, and the particles were supposed to be evenly distributed in the chains; finally, the particles were assumed to be completely magnetically saturated. With these assumptions, using the finite element method in order to model the mechanical behaviour of the rubber-like material, Davis [27] obtained an approximation for the behaviour of a chain under shear deformation, with a uniform field applied far away; with this approximation he calculated the 'shear modulus', and in particular he studied the behaviour of the shear modulus as a function of the percentage of particles per volume, and the external applied field. Davis estimated that the maximum change in the shear modulus due to the external field was about 50%, and that the 'optimal' proportion of particles was about 27% [10].

Shen et al. [97], and Jolly et al. [59] also worked with transversely isotropic MS elastomers. They were mainly concerned with the calculation of approximate expressions for the shear modulus. In the case of Shen et al. [97], they used an approximate analytical expression for the magnetic field, valid for an isolated chain composed of rigid spherical particles, assuming, as Davis [27], that the magnetic interaction is only important inside the chain, and disregarding the magnetic interaction between chains. The 'energy' of the chain was calculated by using the angular momentum exerted by the particles due to the presence of an external magnetic field; then a shear deformation is assumed, where the 'shear force' was calculated from an appropriate variation of the energy. The elastic energy of the surrounding rubber-like matrix was calculated assuming it to be a hyperelastic material. With the shear deformation and stress, Shen et al. [97] obtained an expression for the shear modulus as a function of, for example, the particle proportion per volume and the external field.

Jolly et al. [59] used a similar procedure in order to calculate the shear modulus. A difference is that they also worked with particles partially magnetically saturated.

Another paper on micro-mechanics modelling that we can mention is that by Simon et al. [98]. They did not work actually with MS elastomers, but with magnetorheological materials, which are highly viscous fluids filled with magneto-active particles. This kind of material has attracted the attention of researchers for a long time, and as a result there is much more experimental and theoretical information about these materials than for MS or ES elastomers. Simon et al. [98] were interested in the magnetic response of this material; in order to model it, they worked with one particle, and they developed the magnetic equations for the particle and the surrounding material. These equations were solved analytically, and then, by using an asymptotic approximation, they used a method, called 'the linear homogenization method', in order to extrapolate the results of the particle for the whole material.

Armstrong [1] worked on an interesting model of a non-magnetic composite matrix filled

[10] As was mentioned earlier, there should be an optimal proportion of particles in order to obtain the 'strongest' magneto- or electrostriction effect. Too few particles implies low electric or magnetic 'forces', but too many particles would imply too little rubber-like matrix material, which would reduce the capacity of deformation for the MS or ES elastomer.

with magneto-active particles. In this case the particles were considered as ellipses. It was shown that the magnetostrictive effect may be enhanced by an appropriate selection of the relation between the major and minor axes of the ellipses. Interestingly, this is similar somehow to the results of Kari and Blom [62], who worked with irregular relatively large particles. The calculation of the mean stress and strain was first done by computing an approximation of the energy associated with one particle and its surroundings, for both the magnetic and elastic part of the energies, and then the stress and strain were approximated using such an energy, assuming a random distribution of particles in the body.

A last series of papers that it is necessary to mention is by Yin and associates. Yin et al. [125] studied the behaviour of a non-magnetic matrix filled with randomly distributed particles; the stress was calculated by using an approximation for the energy and Eshelby's equivalent inclusion method. Both the linear and non-linear cases were considered.

Yin and Sun [123] studied the same problem as above, but in this case they were concerned with the behaviour of the magnetic permeability. The models developed considered just the linear case; the magnetic field inside the material was found by solving Laplace's equation for the magnetic scalar potential for a particle and its surrounding by using the Green's function method, and by dividing the total magnetic field inside the material in two parts, one being a uniform applied field, and the other being a perturbed local field. This field was found by using the above solution for the Laplace's equation, and by expanding it in a Taylor series, keeping only two terms (linear approximation). With the above results, a model for the interaction of two particles was developed, which, by using the statistical average method in order to obtain the behaviour of the random distribution of particles, permitted the obtention of an expression for the permeability. No study of the strain or stress was carried out in this paper.

Yin and Sun [124], and Yin et al. [126] also studied the case of transversely isotropic MS elastomers. Their model only considered the linear case, assuming the magnetic and mechanical properties as constant, which means that they do not depend on the deformation. The chain particles were considered randomly distributed but aligned in the same direction. The Green's function method was used in both cases in order to solve the boundary value problem for one chain, assuming that the chains do not interact with each other. With the above results they calculated expressions for the magnetic field, stress and strain. In [124] they were concerned, in particular, with the behaviour of the shear modulus and Young's modulus; they were able to show the existence of an optimum for the fraction of particles per volume. In [126] they obtained the same expressions as in [124], but they also studied the effect of working with different host materials with different Young's moduli.

The second kind of model, which is the theoretical basis for what has been done in this thesis, corresponds to the so-called phenomenological or continuum model. For MS and ES elastomers, we assume that the magneto- or electro-active particles are very small (in comparison with the overall size of the bodies), such that we can assume a 'continuous' distribution of particles in the elastomeric matrix. Global balances of linear and angular momentum, energy and entropy production, plus the Maxwell equations, are used in order to find a system of partial differential equations, from which we can obtain, for example, the stress, the displacement field, the strain, and the magnetic or electric fields.

In our particular case, one of the most important assumptions was to consider the material as Green elastic, which means we assume the existence of a free energy function, which takes account of both the elastic and the magnetic (or electric) energies accumulated in a body. The main problem then is to find from experiments such an energy function; these experiments must be designed in a 'rational way' by an appropriate use of analytical solutions of boundary value problems.

The main difficulty at the present moment is the lack of enough experimental data in order to propose realistic forms for the energy function, which, as well as this, must be 'mathematically consistent', which basically means that this energy function should lead to a 'well posed' boundary value problem.

Details about this formulation and a review of the most significant references in the area are given in the Section 2.3.

Besides the introduction, this thesis is divided in nine chapters.

Chapter 2 provides a review of the theory of non-linear elasticity and the theory of electromagnetism, and a detailed review of the most important references in the theory of deformable magneto- and electro-elastic solids. Particular attention and detail are provided in order to show the theory developed by Dorfmann and Ogden, which is the basis of what has been done in this thesis.

In Chapter 3 we review the basic equations for MS elastomers.

Chapters 4 and 5 are concerned with isotropic and transversely isotropic materials respectively. For isotropic MS elastomers (Chapter 4), linear and non-linear universal relations were found, and the results for a prototype boundary value problem are shown, which was solved with the finite difference method. For transversely isotropic materials (Chapter 5), some boundary value problems were solved by the inverse method for some problems with homogeneous deformations. As well as this, a prototype energy function was proposed, which was used in order to solve some boundary value problems for non-homogeneous deformations. The problem of determining the linear and non-linear universal relations for this particular kind of energy function was also treated.

Chapters 6, 7 and 8 show similar results as in Chapters 3, 4 and 5 respectively, but in this case applied to ES elastomers.

Finally, in Chapter 9, some results about the variational formulation are shown, with a discussion about the boundary conditions for the mixed boundary value problem. Chapter 10 contains the conclusions and comments.

The main results shown in this thesis are based on the following published and unpublished papers.

In [22] Bustamante and Ogden studied the problems of finding all the linear universal relations for non-linear electro-elastic solids; different particular cases for the constitutive equations were studied. The extension of these results for magneto-elastic solids was addressed by Bustamante et al. in [18].

Bustamante and Ogden studied the problem of non-linear universal relations for the purely elastic problem in [21]. These results have been extended to the magneto- and electro-elastic cases as it is shown in Subsections 4.1.2 and 7.1.2.

Some results for a boundary value problem are shown in [15]. In this paper Bustamante et al. investigated the magnetic behaviour of a tube of finite length under inflation and extension with an axial uniform magnetic field. The problem was solved using the finite difference method.

Bustamante et al. [17] provided two equivalent variational formulations for magneto-elastic materials. An extension of those results for the case of a mixed boundary value problem is given in [16].

The particular case of transversely isotropic magneto- and electro-active elastomers is treated by Bustamante and Ogden in [20] and [19] respectively.

Chapter 2

Electromagnetic fields and deformable media

In this chapter a detailed review of the continuum theory for magneto- and electro-elastic deformable bodies is presented. First, a brief review of some important concepts in continuum mechanics and non-linear elasticity is given. In Section 2.2 we review the basic aspects of the theory of electromagnetism. In Section 2.3 a detailed account of the theory of non-linear magneto- and electro-elasticity is given.

2.1 Basic concepts of continuum mechanics and non-linear elasticity

Much of this section is based on the book by Ogden [78], where a complete account of the continuum mechanics theory with especial applications to non-linear elasticity theory can be found. Four other important reference books on continuum mechanics that can be also mentioned are the two comprehensive articles by Truesdell and Toupin [113], and Truesdell and Noll [112], and the introductory books by Chadwick [23], and Gurtin [52]. An updated account of different problems in non-linear elasticity may be found, for example, in the book by Fu and Ogden [47].

2.1.1 Elements of tensor theory

Vectors and tensors will be denoted in general by bold lower case and capital Latin letters respectively; there are, however, some exceptions as we will see later on[1]. Scalars will be denoted in general by lower case Greek letters. Bodies will be denoted by calligraphic letters.

Let's denote by $\mathbb{E}$ the Euclidean real vector space, and let $\{\mathbf{e}_i\}$ denote an orthonormal basis for this space; let $\mathbf{u}$ and $\mathbf{v}$ be two vectors and u_i, v_i the components of these vectors in the basis $\{\mathbf{e}_i\}$; then we define the dot product $(\cdot)$ as[2] [23, 63, 78]

$$\mathbf{u} \cdot \mathbf{v} \equiv u_i v_i, \tag{2.1}$$

where $\mathbf{u} = u_i \mathbf{e}_i$ and $\mathbf{v} = v_i \mathbf{e}_i$

The vector product is defined as

$$(\mathbf{u} \times \mathbf{v})_k \equiv \varepsilon_{ijk} u_i v_j \mathbf{e}_k, \tag{2.2}$$

where ε_{ijk} is the permutation symbol.

Consider three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$; the tensor product $\otimes$ is defined as

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} \equiv (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \quad \forall \mathbf{w} \in \mathbb{E}. \tag{2.3}$$

Let $\mathbf{T}$ be a second order tensor, let $\{\mathbf{e}_i\}$ denotes an orthonormal basis for $\mathbb{E}$; the component T_{ij} of this tensor is obtained as $T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j$. Then $\mathbf{T}^T$ denotes the transpose of the tensor $\mathbf{T}$, and it has components T_{ji}. The trace of $\mathbf{T}$ is defined as

$$\operatorname{tr} \mathbf{T} \equiv T_{ii}. \tag{2.4}$$

[1] For example, the vector field that represents the position of each particle of a body in the reference configuration is denoted as $\mathbf{X}$. For the electromagnetic theory we use capital letters in order to denote the magnetic and the electric variables.

[2] We use the convention of summation of index, then, for example, if we have the expression

$$a_i b_i,$$

this means

$$a_i b_i \equiv \sum_i a_i b_i.$$

Another example is

$$Q_{ij} b_j \equiv \sum_j Q_{ij} b_j.$$

Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ be two different bases for $\mathbb{E}$, and $\mathbf{Q}$ be the matrix for the transformation $\{\mathbf{e}_i\} \to \{\mathbf{e}'_i\}$; the components Q_{ij} of $\mathbf{Q}$ are obtained as

$$Q_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}'_j. \tag{2.5}$$

It is possible to show that under this basis transformation the components of a tensor $\mathbf{T}$ transform as [23, 78]

$$T'_{ij} = Q_{ip} Q_{jq} T_{pq}. \tag{2.6}$$

In the following sections and chapters we will work mostly with Cartesian coordinates. Appendix A contains a summary of some useful expressions for cylindrical and spherical coordinates. An analysis of vectors and tensors in a general coordinate system may be found in the book by Sokolnikoff [100], and also in [113].

2.1.2 Kinematics

Consider a body in a reference configuration, denoted by $\mathcal{B}_r$, with boundary $\partial\mathcal{B}_r$. Let's use X as a name for each particle of the body, and let use the symbol $\mathbf{X}$ in order to denote the position of these particles in the reference configuration. This configuration may be chosen arbitrarily; we assume that it corresponds to the region occupied by the body at the beginning of the process, when it is unstressed [78, 111].

Now, let us assume that the body is deformed to a current configuration, called $\mathcal{B}$, with boundary $\partial\mathcal{B}$. Let us use $\mathbf{x}$ in order to denote the position of each particle in this current configuration (also called the Eulerian configuration). If there is no time dependence we have that (see Figure 2.1)

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \tag{2.7}$$

where $\boldsymbol{\chi}$ is a one-to-one, orientation-preserving mapping with suitable regularity properties. The above conditions for $\boldsymbol{\chi}$ are assumed in order to avoid problems such as self penetration (principle of impenetrability), and in order to avoid regions of finite volume to be deformed into regions of zero or infinite volume (permanence of matter) [113].

The displacement field $\mathbf{u}$ is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{2.8}$$

The deformation gradient $\mathbf{F}$ is defined as

$$\mathbf{F} \equiv \operatorname{Grad} \boldsymbol{\chi}, \tag{2.9}$$

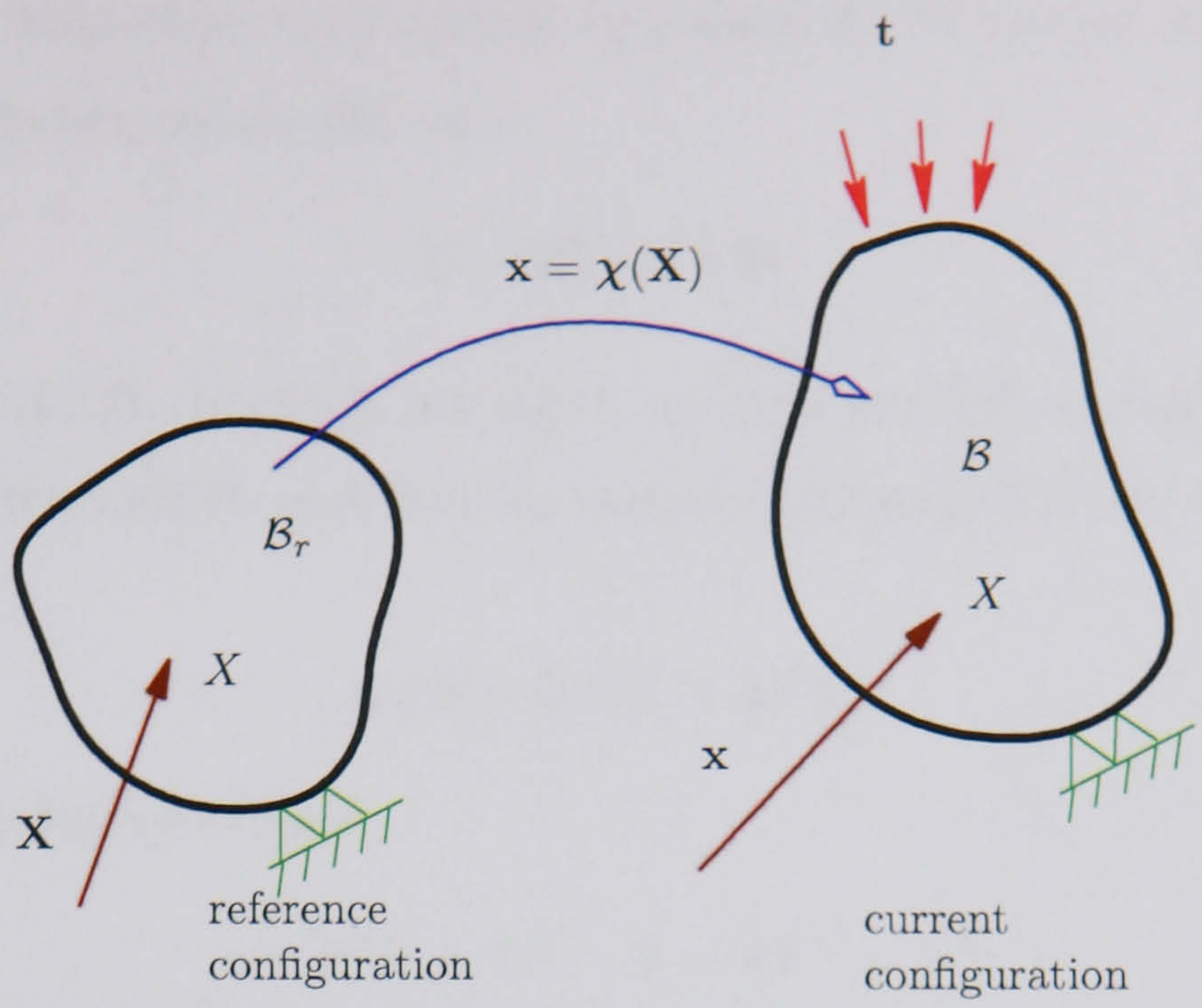


Figure 2.1: Current and reference configurations. The vector **t** represents an external surface load.

where the operator Grad is defined with respect to **X**. **F** is a second order tensor, its components F_{ij} in Cartesian coordinates are given as[3]

$$F_{ij} = \frac{\partial x_i}{\partial X_j}. \tag{2.10}$$

In terms of the displacement field we have

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{u} + \mathbf{I}, \tag{2.11}$$

where **I** is the identity tensor.

The Jacobian of the transformation χ is defined as

$$J \equiv \det \mathbf{F}. \tag{2.12}$$

We assume

$$J \neq 0 \quad \forall \mathbf{x}, \tag{2.13}$$

which implies that χ is locally invertible at each **X**.

As well as this, by convention the orientation of a line element is preserved (there is no inversion of an element), so [78]

$$J > 0 \quad \forall \mathbf{x}. \tag{2.14}$$

[3]See Appendix A.3 for some examples of **F** in other coordinates systems.

Now if $\det \mathbf{F} \neq 0$, then there exist symmetric tensors $\mathbf{U}$, $\mathbf{V}$ and an orthogonal tensor $\mathbf{R}$ such that (polar decomposition theorem)

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{2.15}$$

where $\mathbf{R}\mathbf{R}^T = \mathbf{I} = \mathbf{R}^T\mathbf{R}$. $\mathbf{U}$ and $\mathbf{V}$ are called the right and left stretch tensors. Note that $\mathbf{U}$ and $\mathbf{V}$ are positive definite and that the decompositions (2.15) are unique. We have as well

$$\det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V}. \tag{2.16}$$

Consider the following tensors

$$\mathbf{c} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2; \tag{2.17}$$

$\mathbf{c}$ and $\mathbf{b}$ are called the Cauchy-Green right and left deformation tensors respectively.

2.1.3 Balance laws. Stress vector and stress tensor.

The mass m of the body $\mathcal{B}$ in the current configuration is given as

$$m = \int_{\mathcal{B}} \rho(\mathbf{x})\, \mathrm{d}v \quad \rho \geq 0 \quad \forall \mathbf{x} \in \mathcal{B}, \tag{2.18}$$

where ρ is the density of mass per unit of volume in the current configuration. There is no change in the mass of the body, so the total mass is the same for the reference configuration; if we call ρ_o the density, we have

$$m = \int_{\mathcal{B}} \rho(\mathbf{x})\, \mathrm{d}v = \int_{\mathcal{B}_r} \rho_o(\mathbf{X})\, \mathrm{d}V, \tag{2.19}$$

but (see, for example, [78])

$$\mathrm{d}v = J\mathrm{d}V, \tag{2.20}$$

therefore from (2.19) we obtain

$$\rho = J^{-1}\rho_o. \tag{2.21}$$

In the previous subsection we reviewed briefly the kinematics of deformation for a body; in this subsection we concentrate our interest on the 'cause' of this 'deformation'.

The causes of the deformation in a body are the forces. We have two kind of forces, the body forces and the surface forces. By body forces we mean the forces that act at a distance on each point of the body, and their magnitude is usually given as force per unit of mass; some examples are the gravity force and the forces 'generated' by electromagnetic fields; we will study in detail this last kind of body forces in Section 2.3.

For surface forces we mean the forces that require the direct contact or interaction between the surfaces of two bodies. These forces are given as force per unit of area.

One of the most important assumptions in continuum mechanics is Cauchy's hypothesis for surface forces. Cauchy assumed that the surface forces, which we denote by the symbol $\mathbf{t}$, are function of the position, time, and a normal vector to the surface of the body, then[4]

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}), \tag{2.22}$$

which has units of force per unit of area. For the body force we use the symbol $\mathbf{b}$.

The total force and moment for a body (in the current configuration) are given respectively as

$$\int_{\mathcal{B}} \rho \mathbf{b}\, dv + \int_{\partial \mathcal{B}} \mathbf{t}\, da, \tag{2.23}$$

and

$$\int_{\mathcal{B}} \rho (\mathbf{x} - \mathbf{x}_o) \times \mathbf{b}\, dv + \int_{\partial \mathcal{B}} (\mathbf{x} - \mathbf{x}_o) \times \mathbf{t}\, da, \tag{2.24}$$

where ρ and $\mathbf{b}$ are functions of $\mathbf{x}$ and $\mathbf{t}$, and $\mathbf{t}$ is function of $\mathbf{x}$, $\mathbf{n}$ and the time t; while $\mathbf{x}_o$ is a given fixed point in the current configuration about which the moments are calculated.

Let $\mathbf{v}$ denote the velocity field for each particle of the body, and $\dot{(\ )}$ the material time derivative; then the balances of linear and rotational momentum, known as Euler's laws, are

$$\int_{\mathcal{B}} \rho \mathbf{b}\, dv + \int_{\partial \mathcal{B}} \mathbf{t}\, da = \left(\int_{\mathcal{B}} \mathbf{v}\, dm \right)^{\cdot} = \int_{\mathcal{B}} \dot{\mathbf{v}}\, dm, \tag{2.25}$$

and

$$\begin{aligned} \int_{\mathcal{B}} \rho (\mathbf{x} - \mathbf{x}_o) \times \mathbf{b}\, dv + \int_{\partial \mathcal{B}} (\mathbf{x} - \mathbf{x}_o) \times \mathbf{t}\, da &= \left(\int_{\mathcal{B}} \rho (\mathbf{x} - \mathbf{x}_o) \times \mathbf{v}\, dv \right)^{\cdot} \\ &= \int_{\mathcal{B}} \rho (\mathbf{x} - \mathbf{x}_o) \times \dot{\mathbf{v}}\, dv. \end{aligned} \tag{2.26}$$

For brevity in the notation we will omit the time dependence.

Theorem 1. *Cauchy's theorem: if* $\mathbf{t}(\mathbf{x}, \mathbf{n})$ *is continuous in* $\mathbf{x}$ *then there exists a second-order tensor* $\boldsymbol{\sigma}$ *such that*

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x}) \mathbf{n}, \tag{2.27}$$

where $\boldsymbol{\sigma}$ *is known as the Cauchy stress tensor.*

[4] For a full account of the Cauchy hypothesis see, for example, [113] Chapter D.

If ρ, $\mathbf{b}$ and $\dot{\mathbf{v}}$ are continuous, and $\boldsymbol{\sigma}$ is continuously differentiable, it is possible to prove that (2.25) and (2.26) become (see [78] or [113])

$$\mathrm{div}\boldsymbol{\sigma}^T + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \tag{2.28}$$

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}, \tag{2.29}$$

which are known as the first and second Cauchy's laws of motion (balance equations). Note that in all the previous expressions we have not considered the case of electromagnetic forces and body torques; it can be shown that in that case in general $\boldsymbol{\sigma}$ is not symmetric. We will show these results in Subsection 2.3.1.

The relation between the element of area $\mathrm{d}\mathbf{a}$ for the current configuration and $\mathrm{d}\mathbf{A}$ for the reference configuration is known as the Nanson's formula, and is given as follows [52,78]:

$$\mathrm{d}\mathbf{a} = J\mathbf{F}^{-T}\mathrm{d}\mathbf{A}. \tag{2.30}$$

Then, consider the total surface force for a body in the current configuration:

$$\int_{\mathcal{B}} \mathbf{t}\, \mathrm{d}a = \int_{\mathcal{B}} \boldsymbol{\sigma}\mathbf{n}\, \mathrm{d}a = \int_{\mathcal{B}} \boldsymbol{\sigma}\, \mathrm{d}\mathbf{a} = \int_{\mathcal{B}_r} J\boldsymbol{\sigma}\mathbf{F}^{-T}\, \mathrm{d}\mathbf{A}. \tag{2.31}$$

We define the nominal stress tensor $\mathbf{S}$ via

$$\mathbf{S}^T = J\boldsymbol{\sigma}\mathbf{F}^{-T}. \tag{2.32}$$

Since $\boldsymbol{\sigma}$ is symmetric we have

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{2.33}$$

Sometimes $\mathbf{S}^T$ is called the first Piola-Kirchhoff stress tensor.

2.1.4 Constitutive equations

By experience we know that two bodies of the same 'shape' and dimensions, with the same distribution of external load, but made of different materials will behave differently. So, in our previous equations we need to connect the 'deformation' $\mathbf{x}$ with the stresses $\boldsymbol{\sigma}$.

The constitutive equations correspond to these relations between the 'deformation' and the 'external loads'. The constitutive equations are simplified mathematical models used in order to represent approximately the behaviour of the material [112], and they do not intend to model this behaviour in all the possible external conditions. For example, the response of steel under small deformations may be appropriately approximated by a linear model for the 'stress-deformation' relation; however, that is not the case, for example, when we have large plastic deformations.

There are several assumptions that are necessary in order to find simple forms for the constitutive equations, the three principal assumptions are (see, for example, [113] Chapter G, and [112] Chapter III).

- **Principle of determinism**: we assume that, for example, the Cauchy stress tensor is a functional of the 'history of deformation', the 'particle' and time.

  The history of deformation is just the set of all the configurations χ of the body from the beginning of the process until the time t; it is denoted by the symbol χ^t, where the superscript t means the 'collection' of configurations from the beginning of the process until a time t. We have

$$\sigma(X,t) = \mathfrak{F}(\chi^t; X, t). \tag{2.34}$$

- **Principle of local action**: we assume that what happens for a small portion of material in the body is only affected by immediate surroundings.

- **Principle of material frame-indifference**: we can expect that the response of the material will be the same for two different observers. Consider the transformation

$$\chi^*(X,t^*) = \mathbf{p}(t) + \mathbf{Q}(t)(\chi(X,t) - \mathbf{x}_o), \tag{2.35}$$

$$t^* = t - a, \tag{2.36}$$

$$\sigma^*(X,t^*) = \mathbf{Q}(t)\sigma(X,t)\mathbf{Q}(t)^T. \tag{2.37}$$

  So we have that

$$\sigma^*(X,t^*) = \mathfrak{F}(\chi^{*t^*}; X, t^*). \tag{2.38}$$

Despite the above principles, the form of the functional $\mathfrak{F}$ is still too general for practical purposes. Most of the models are based on an extra assumption, the so-called 'simple materials'.

Definition 2.1. *If for a deformation we have that* $\mathbf{F}$ *does not depend on* $\mathbf{X}$ *for all time* t, *then the deformation is called homogeneous.*

Definition 2.2. *A material is called simple if for any point* $\mathbf{x} \in \mathcal{B}$ *the deformation is 'locally' homogeneous (see [78] section 4.1.2, and [112]).*

In the above definition the word 'locally' means that for any point $\mathbf{x}$ very close to a point $\mathbf{x}_o$ the following approximation is valid

$$\mathbf{x} \approx \mathbf{x}_o + \mathbf{F}(\mathbf{X} - \mathbf{X}_o), \tag{2.39}$$

where $\mathbf{X}_o$ is the position in the reference configuration of the particle, whose position in the current configuration is $\mathbf{x}_o$.

If a material is simple it is possible to show then that (2.38) becomes [78, 112]

$$\boldsymbol{\sigma}(X,t) = \mathfrak{F}(\mathbf{F}^t; \mathbf{X}, t), \tag{2.40}$$

where $\mathbf{F}^t$ is the history of the deformation gradient up until the time t.

In this thesis we do not work with materials with 'memory'; furthermore, we only consider quasi-static problems, so we do not consider dependence on time either. For simplicity we do not explicitly state the dependence on the position $\mathbf{X}$.

A material that depends on the 'instantaneous' value of $\mathbf{F}$ is called a Cauchy elastic material, and in this case we have [78, 111].

$$\boldsymbol{\sigma}(X,t) = \mathfrak{G}(\mathbf{F}). \tag{2.41}$$

It is possible to show that from the principle of material frame indifference we have

$$\mathfrak{G}(\mathbf{QF}) = \mathbf{Q}\mathfrak{G}(\mathbf{F})\mathbf{Q}^T \tag{2.42}$$

for all proper orthogonal tensors $\mathbf{Q}$.

We can show that (2.42) holds if $\mathfrak{G}$ is a function of $\mathbf{U}$ instead $\mathbf{F}$ (see, for example, [111]). Then an alternative form of writing (2.41) is

$$\boldsymbol{\sigma}(X,t) = \mathbf{R}\mathfrak{G}(\mathbf{U})\mathbf{R}^T, \tag{2.43}$$

and since $\mathbf{U}^2 = \mathbf{c}$, we have also

$$\boldsymbol{\sigma}(X,t) = \mathbf{R}\mathfrak{H}(\mathbf{c})\mathbf{R}^T, \tag{2.44}$$

where we defined $\mathfrak{H}(\mathbf{c}) \equiv \mathfrak{G}(\mathbf{U}^2)$.

There are two additional topics to treat in this section; material symmetry and internal constraints.

Regarding the material symmetry, consider the following example. Let assume that there is a body, which shows the same 'behaviour' independently of the 'orientation' of the undeformed material relative to the external load. In order to understand this, consider a cube made, for example, of rubber; it is known that under certain conditions for the external load, we can apply tractions on the three main faces of the cube, such that we obtain the same behaviour for the stress as a function of the stretch. A material like this is called isotropic material, and for the case of Cauchy elastic materials, it can be shown that the constitutive equation (2.41) may be written as ([78] section 4.2.6)

$$\boldsymbol{\sigma} = \phi_0 \mathbf{I} + \phi_1 \mathbf{b} + \phi_2 \mathbf{b}^2, \tag{2.45}$$

where ϕ_i, $i = 0, 1, 2$, are scalar functions of the invariants

$$I_1 = \operatorname{tr} \mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr} \mathbf{c})^2 - \operatorname{tr} \mathbf{c}^2], \quad I_3 = \det \mathbf{c}, \tag{2.46}$$

such that $\phi_i = \phi_i(I_1, I_2, I_3)$.

In the case of MS and ES elastomers the situation is more complex. We give more details of the theory of material symmetry in Chapters 4, 5, 7 and 8.

Let us now speak briefly about internal constraints. Some materials may present restrictions on the class of deformation they may undergo; consider, for example, 'incompressible' materials. If we have a body made of an incompressible material, then the only deformations that are possible for this body are the ones that keep the volume constant. As another example, consider the case of a composite material composed of an isotropic matrix filled with inextensible fibers, which may have a preferred alignment; then a body made with this material cannot stretch in that direction, and as a consequence, we have a 'restriction' in the class of admissible deformations for the body.

The above two examples are actually idealized situations; there is no material, for example, which is perfectly incompressible. For example, water is considered as an incompressible fluid, but it behaves as a compressible fluid for very high pressures. So, we must regard the constraints as mathematical idealizations, which model in an approximate way the behaviour of some materials under specific external conditions.

Since a constraint is a restriction on the class of deformations, a form to write it is given as

$$\gamma(\mathbf{F}) = 0, \tag{2.47}$$

where γ is a scalar function of the deformation gradient. This function should be also frame-indifferent, and in such a case it can be proved that (2.47) may be written alternatively as [112]

$$\lambda(\mathbf{c}) = 0. \tag{2.48}$$

Now, for a material with an internal constraint, we do not intend to find a particular constitutive equation for each different material for each different kind of constraint, rather we decompose the stress in two portions; one is calculated in the usual way with the constitutive equation, and the other part is calculated by assuming that it does not do any work for any deformation compatible with the constraint. This last statement is tricky; it would be actually valid only for elastic materials without internal dissipation; see, for example, [86].

The decomposition for the stress is

$$\boldsymbol{\sigma} = \mathbf{Z} + \mathfrak{G}(\mathbf{F}), \tag{2.49}$$

where $\mathbf{Z}$ is the part of the stress that does not do any work, and it is given as [112]

$$\mathbf{Z} = q\mathbf{F}\frac{\partial \lambda}{\partial \mathbf{c}}\mathbf{F}^T, \tag{2.50}$$

where q is an arbitrary scalar.

For example, in the case of incompressible materials we have

$$\lambda(\mathbf{c}) = \det \mathbf{c} - 1 = 0, \tag{2.51}$$

and as a result (see, for example, [23])

$$\mathbf{Z} = -p\mathbf{I}, \tag{2.52}$$

where $p = -q$.

2.1.4.1 Green elastic materials

Green elastic materials correspond to a subclass of the Cauchy elastic materials; the principal characteristic is the assumption of existence of an 'energy function' (scalar function), such that the 'stress' tensor may be calculated as the derivative of this function.

Consider the first Cauchy law of motion (2.28)

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}.$$

Taking the dot product of the above equation with $\mathbf{v}$, using $\operatorname{div}\boldsymbol{\sigma}\cdot\mathbf{v} = \operatorname{div}(\boldsymbol{\sigma}\mathbf{v}) - \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\Gamma})$, where $\boldsymbol{\Gamma} = \operatorname{grad}\mathbf{v}$, we have that (2.28) becomes

$$\operatorname{div}(\boldsymbol{\sigma}\mathbf{v}) - \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\Gamma}) + \rho\mathbf{b}\cdot\mathbf{v} = \rho\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\cdot\mathbf{v}). \tag{2.53}$$

Integrating over $\mathcal{B}$ (current configuration) and using the divergence theorem we have

$$\int_{\partial\mathcal{B}} (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{v}\, \mathrm{d}a + \int_{\mathcal{B}} \rho \mathbf{b} \cdot \mathbf{v}\, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{B}} \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v})\, \mathrm{d}v + \int_{\mathcal{B}} \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\Gamma})\, \mathrm{d}v. \tag{2.54}$$

Let's use the definition of the nominal stress tensor $\mathbf{S}$ (2.33), the identity $\operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\Gamma}) = \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} = \frac{1}{2}(\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^T)$, and

$$\operatorname{tr}(\mathbf{S}\dot{\mathbf{F}}) = J \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\Sigma}), \tag{2.55}$$

taking (2.54) back to the reference configuration it then becomes

$$\int_{\partial\mathcal{B}_r} (\mathbf{S}\mathbf{N}) \cdot \mathbf{v}\, \mathrm{d}A + \int_{\mathcal{B}_r} \rho_o \mathbf{b}_o \cdot \mathbf{v}\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{B}_r} \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v})\, \mathrm{d}V + \int_{\mathcal{B}_r} \operatorname{tr}(\mathbf{S}\dot{\mathbf{F}})\, \mathrm{d}V. \tag{2.56}$$

Definition 2.3. *A material is called Green elastic (or hyperelastic) if there exists a scalar function $W = W(\mathbf{F})$, called energy function, such that*

$$\dot{W} = \operatorname{tr}(\mathbf{S}\dot{\mathbf{F}}). \tag{2.57}$$

From the above definition it follows that

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}. \tag{2.58}$$

In components form (Cartesian coordinates) the above expression is $S_{ij} = \frac{\partial W}{\partial F_{ji}}$ (see [78]).

More details about the above expression and about the theory of non-linear elasticity may be found, for example, in [78].

Note that (2.54) is the balance of energy for an elastic material; in fact, in Subsection 2.3.1 we will derive expressions for the stress and the magnetic or electric fields by working with the balance of energy and the second law of thermodynamics, which we have not treated in this section.

2.2 Electromagnetic fields

In this section a brief summary of the basic aspects of the theory of electromagnetism is provided.

Much of the theory of electromagnetism is based on or may be found in the important work by Maxwell [69, 70]; another classical book that can be mentioned is the volume by Landau and Lifshitz [65]. A modern treatment of the subject may be found, for example, in the book by Kovetz [64].

Let's start with a brief review of the electric and magnetic properties of materials and some basic concepts.

2.2.1 Electric and magnetic properties of materials

There is an abundant bibliography on the electric and magnetic properties of materials. In the context of magneto- and electro-elasticity we can mention Chapter 1 of the book by Maugin [67], and Chapter 4 of the book by Eringen and Maugin [42]. In the particular case of magnetic materials, we mention the book by Spaldin [101], where there is a more detailed explanation of the magnetic properties of materials at the atomic and quantum levels. This section follows the simpler approach presented in [9].

2.2.1.1 Electric properties of materials

A basic and simple experiment that can be used in order to 'see' the presence of electric fields and forces consists of rubbing a piece of plastic against wool; due to this friction and the particular composition of the polymer, it will eventually lose some electrons, and as a result the piece of plastic has a distribution of 'positive charges', which generate an 'electric field'. The presence of this electric field may be detected by putting, for example, a small piece of paper close to the plastic body; there will be an attractive force that acts through the distance and it will depend among many other variables, especially on the 'strength' of the electric field generated by the positively charged plastic body, which, at the same time, will depend on the number of electrons lost during the rubbing of the plastic against the wool.

The basic concept here is the charge, and like the concept of mass, in a closed system it is assumed that the charge is conserved (see Chapter F of [113], and [64]). Charges are classified as positive or negative.

A body is composed of atoms, and each atom has a nucleus with protons, which have positive charge, and orbiting the nucleus (the orbits are called sometimes 'shells') we have electrons with negative charge. If a body loses electrons it is said that it is positively charged, and if a body gains electron it is said it is negatively charged.

Consider Figure 2.2 where we have a schematic representation of a positive and a negative charge, and the interaction (field) between them.

An electric field exists at a point if a charged object that is put at that point experiences a force.

Some materials, such as copper and gold, have an atomic structure where in the outer shell there is an electron that can be easily 'detached'; as a result we may have an 'electric current', in which we have a 'flux' of electrons moving through the body. These materials

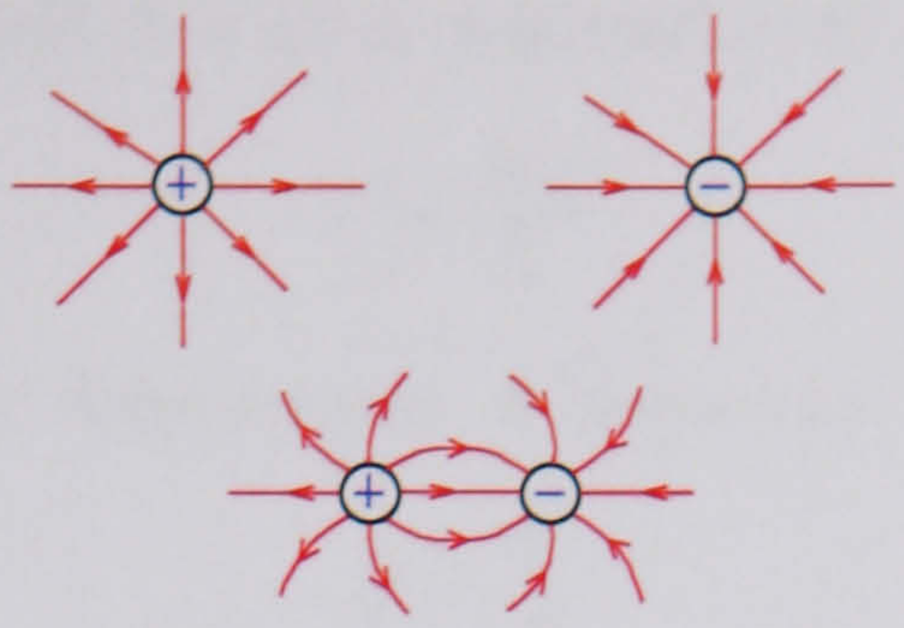


Figure 2.2: Positive, negative charges and dipole.

are called conductors.

The electric potential φ at a point P is defined as the work that it is necessary to do in order to move a unit charge from infinity to the point P [107].

Consider two parallel plate conductors (Figure 2.3), where we have applied a difference in the potential $\Delta\varphi$.

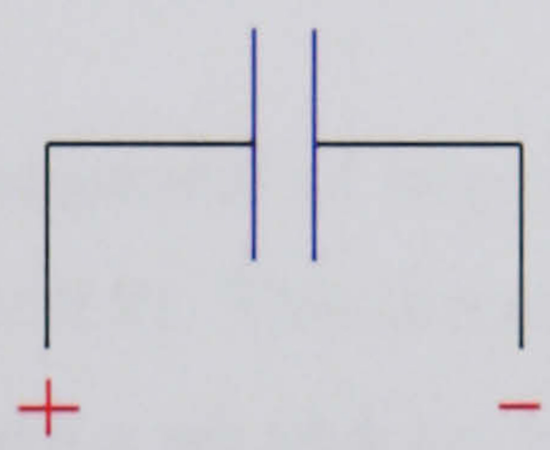


Figure 2.3: Scheme of two parallel plate conductors (blue lines) with an electric potential.

Imagine that in the space between the plates we put a thin piece of material; then the amount of charge Q which will be accumulated on the plates due to the presence of the difference in the potential is given as

$$Q = C\Delta\varphi, \tag{2.59}$$

where the constant C is called the capacitance. This is a property of the material and also depends on the geometry. It is given as

$$C = \varepsilon_r \varepsilon_o \frac{A}{d}, \tag{2.60}$$

where ε_o is the permittivity of the free space, ε_r is the relative permittivity of the material, A is the area of the plate, and d is the distance between the plates. We have that

$$\varepsilon_o = 8.85 * 10^{-12} \left[\frac{C^2}{Nm^2}\right]. \tag{2.61}$$

If d is small, the electric field E is given approximately as [5]

$$E = \frac{\Delta\varphi}{d}. \tag{2.62}$$

The flux D, also called electric displacement, is defined as

$$D = \frac{Q}{A} \tag{2.63}$$

from where we get for free space ($\varepsilon_r = 1$)

$$D = \varepsilon_o E, \tag{2.64}$$

and for the material

$$D = \varepsilon_o \varepsilon_r E. \tag{2.65}$$

Sometimes ε_r may be considered as a constant, but in general it depends on the temperature of the material, and, in particular, on the electric field E. A general relation will be shown as follows.

A dipole corresponds to an arrangement of two charges, positive and negative, separated by a finite distance (see Figure 2.2). This is a mathematical model of what happens for some materials, which do not have a net charge, but which react to the presence of an external field.

Imagine we have a body with no free charges but only dipoles; since a dipole is composed of a positive and a negative charge, the net charge in the body will be zero. But these dipoles will generate a field inside the body, which will modify the final total field, this phenomenon is known as polarization. So, for free space the relationship between $\mathbf{D}$ and $\mathbf{E}$ is

$$\mathbf{D} = \varepsilon_o \mathbf{E}. \tag{2.66}$$

But for condensed matter is given as

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}, \tag{2.67}$$

where $\mathbf{P}$ is known as the polarization vector.

There are different sources of polarization, which are listed as follows.

- **Electronic polarization**: The polarization is generated when, because of the external field, the electrons move and become more concentrated on one side of their orbits; as a result of this asymmetry we obtain a dipole.

[5] There would be a minus sign depending on the way $\Delta\varphi$ is defined.

- **Ionic polarization**: Ions are atoms or molecules that are electrically neutral, but which have lost or gained one or more electrons. The ionic polarization appears in ionic crystals due to the relative displacements between positive and negative ions.

- **Molecular polarization**: Some molecules have a non-symmetrical arrangement of electrons, so we have a permanent dipole.

- **Interfacial polarization**: Due to the frequent imperfections we find in the arrangements of atoms and molecules, we have for some materials the presence of a high density of gaps between the arrays; now, in some cases we may have a few free electrons moving freely around these gaps; as a result, when an external field is applied these electrons will move and become located on one face of the array, and we have an asymmetrical arrangement of electrons, which can be considered as a dipole.

A material may have more than one source of polarization, but for different materials some of the above mechanisms will be more important than others.

Consider the Figure 2.4, on the left side we have a body with a distribution of dipoles, the figure on the right side shows an alternative way to represent these dipoles.

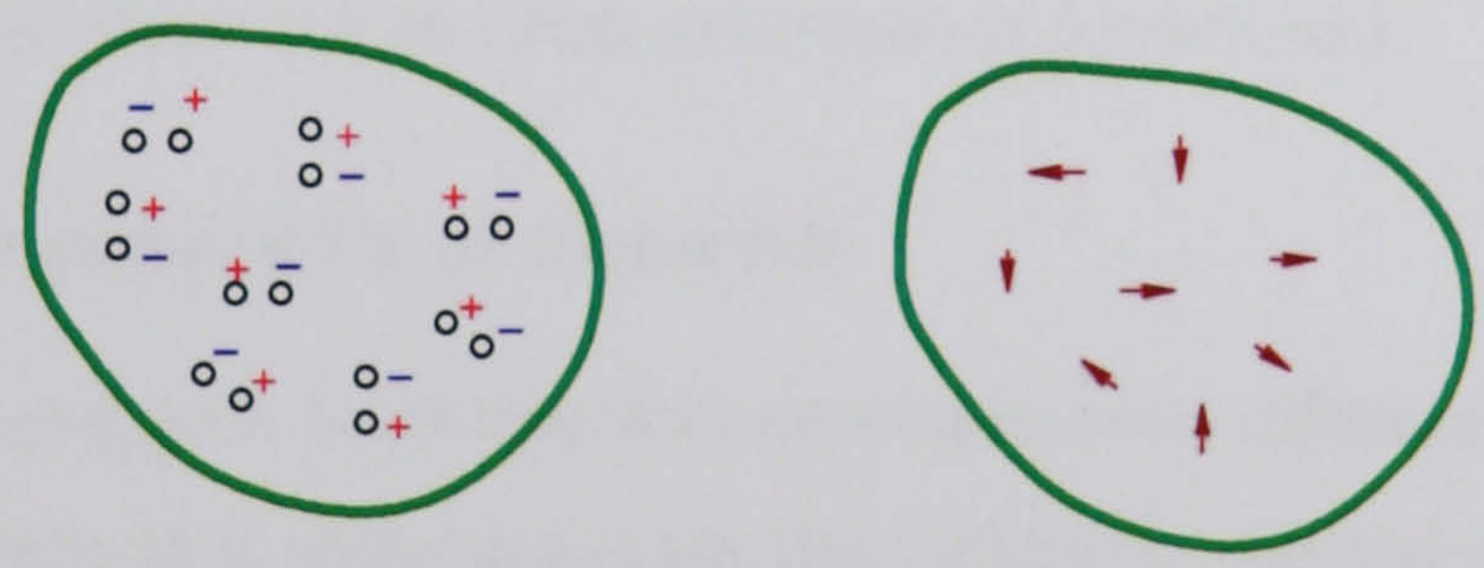


Figure 2.4: Two equivalent representations for the polarization field.

When an external field is applied the dipoles will start to appear and to become aligned with this external field; the stronger the field the more dipoles appear; this process continues until no more dipoles can be created, and in such a case we say that the material is saturated. A typical graph of the polarization as a function of the electric field is shown in the Figure 2.5.

Some materials remain polarized after the external field is removed, which is the case, for example, for ferroelectric materials (see Figure 2.6).

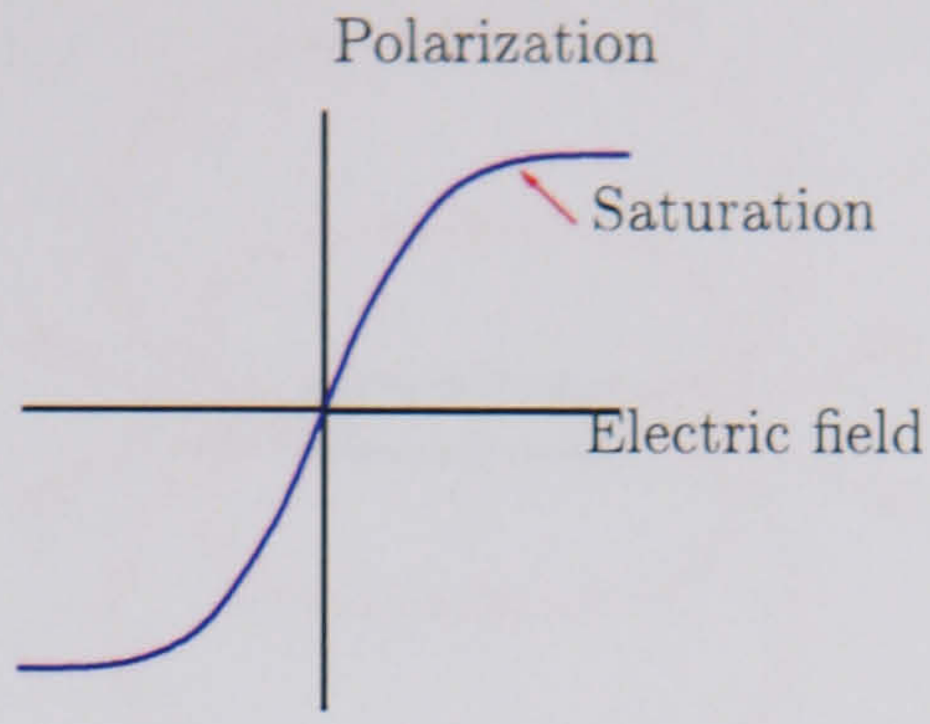


Figure 2.5: Normal behaviour for the polarization field as a function of the electric field.

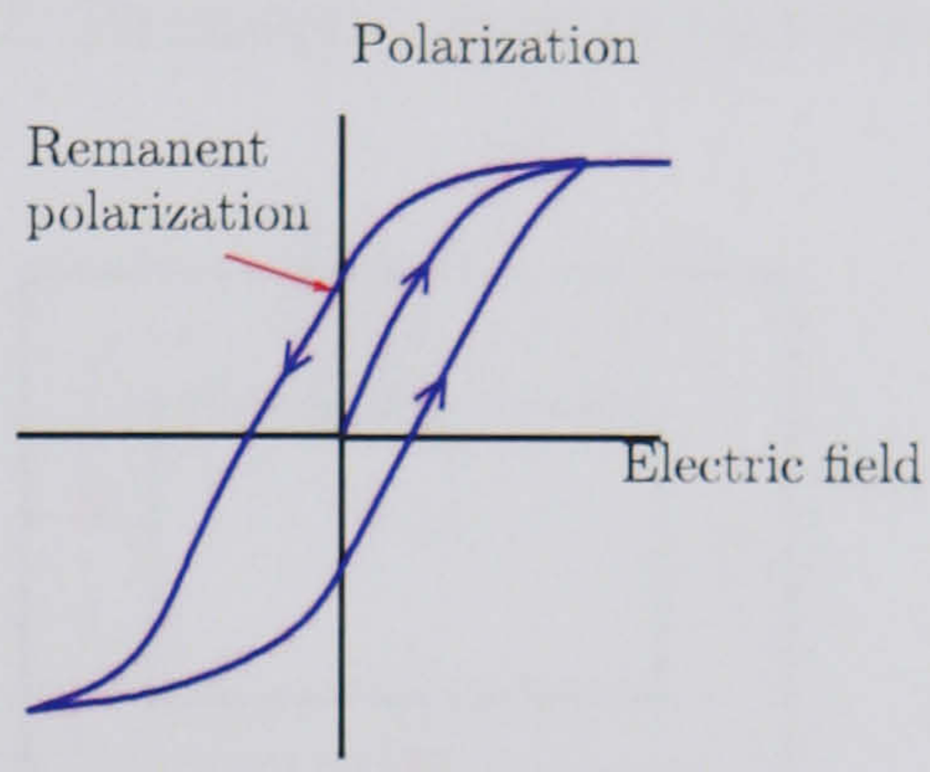


Figure 2.6: Behaviour of the polarization as a function of the electric field. When the external field vanishes there is a residual polarization (hysteresis).

2.2.1.2 Magnetic properties of materials

The phenomenon of magnetic fields was discovered from two different sources, one of them was the attractive force that some materials like lodestone exerted on iron particles, and the other was the magnetic flux generated by electric currents.

In the case of permanent magnets, we identify north and south 'poles'; a magnetic flux is generated whose field lines go (by convention) from the north to the south pole as is shown in Figure 2.7.

Figure (2.7) suggests a close resemblance between the phenomena of magnetization and polarization; however, in the case of polarization we work with two charges, positive and negative, separated by a small distance, but in the case of magnetic materials, it has been impossible to isolate north or south poles; it does not matter how small a piece of magnet is, it has always a north and a south pole.

The magnetic flux density per unit of area is denoted as $\mathbf{B}$ and is also called magnetic induction.

Consider Figure 2.8. In this figure (a) represents a wire where we have an electric

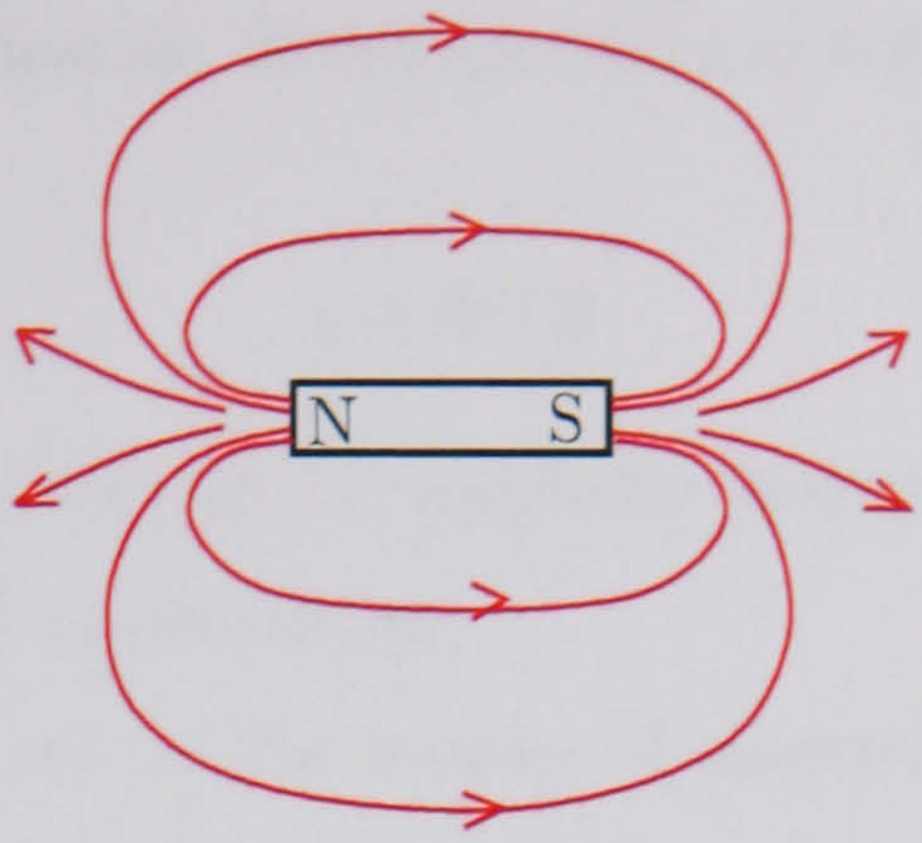


Figure 2.7: Permanent magnet and lines of flux.

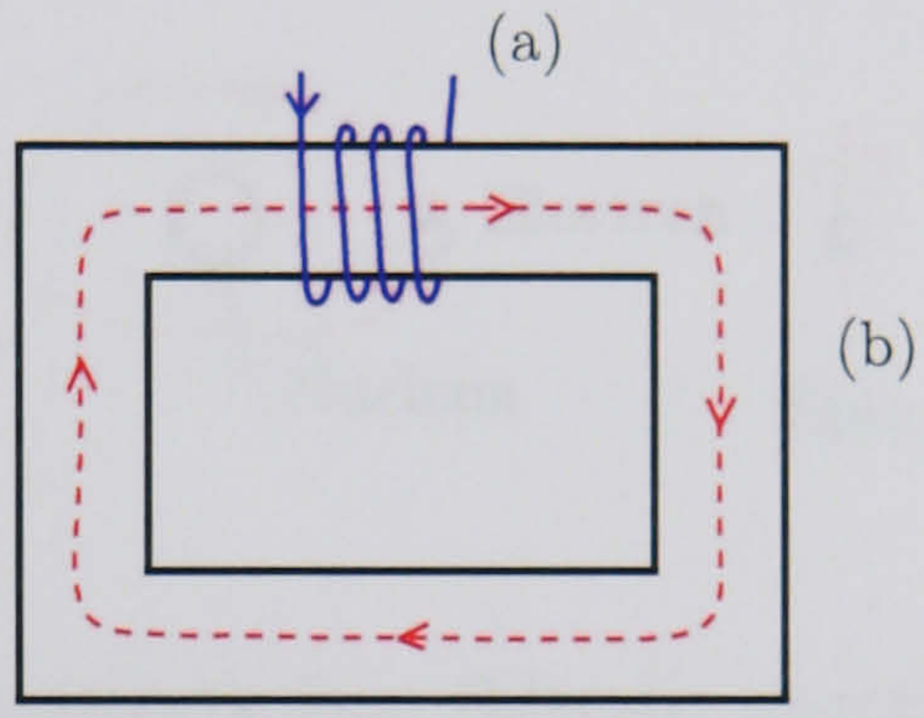


Figure 2.8: Magnetic flux due to an electric current

current, and (b) is a body. Experiments show that an electric current in (a) generates a magnetic flux through (b), as is shown by the dashed line. The body will show some 'resistance' to this magnetic flux. The body (b) would be the magnetic equivalent of an electric circuit, where **B** (the flux of magnetic induction) would be the equivalent of the current, the resistance of the body to the flux would be the same as the resistance of a body to the passage of electric current, and the equivalent of the difference of electric potential would be denoted as **H**, which is called the magnetic field or magnetic field intensity.

For free space we have the relation

$$\mathbf{B} = \mu_o \mathbf{H}, \tag{2.68}$$

where μ_o is the magnetic permeability for free space.

In the case of condensed matter, a linear model would be

$$\mathbf{B} = \mu_r \mu_o \mathbf{H}, \tag{2.69}$$

where μ_r is the relative permeability. For the general non-linear case we have

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}), \tag{2.70}$$

where **M** is called the magnetization. From here we can define the magnetic susceptibility χ (one dimensional case) as

$$\chi = M/H. \tag{2.71}$$

Before giving a short classification of magnetic materials, let us discuss briefly the 'atomic' cause of the magnetic phenomenon.

As we mentioned before, one of the sources of magnetic flux is the electric current. It is argued that all magnetic interactions are generated ultimately by the 'movement' of electric charges. In order to understand this consider Figure 2.9, which shows a very simplified model for an atom with one electron.

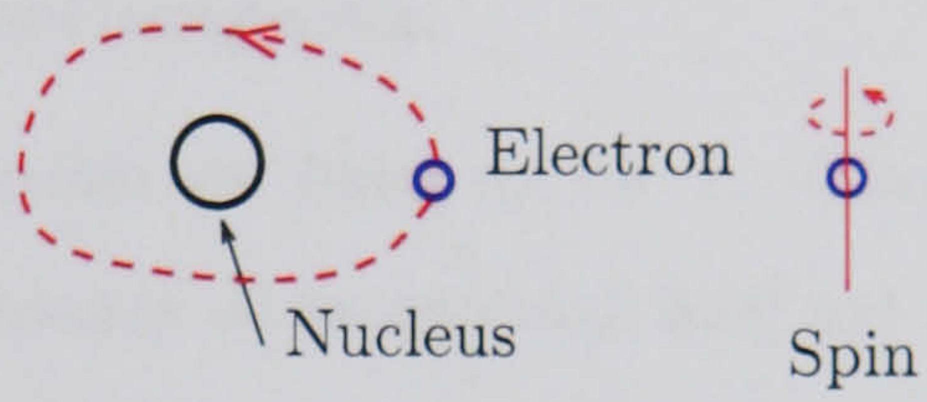


Figure 2.9: Atomic source of magnetic flux. 'Electric currents' due to the movement of an electron around its nucleus, and due to its spin.

Because of the rotation of the electron around the nucleus, we have a phenomenon analogous to the movement of an electric current, because the electron has a negative unit charge. This 'current' generates a magnetic flux.

There is another source of magnetic flux, which is analogous to the rotation around an axis of the electron; this phenomenon is called spin of the electron, and it may also be considered as an analogous of the circulating of an electric current.

A much more detailed explanation of the atomic basis of the magnetic phenomena may be found, for example, in [101] as mentioned previously.

The materials may be classified in the following three categories accordingly with their magnetic response.

- **Diamagnetic**

  A material for which χ is slightly less than 1 is called a diamagnetic materials (**M** is in the opposite direction as **H**).

  For these materials the main source of magnetization (also called magnetic dipoles) comes from the orbits of the electrons around the nucleus of the atoms. When a field is applied far away, it will generate an induced 'current' for these electrons, which at the same time will generate an induced magnetic field that opposes this external field. Examples of these materials are copper and mercury.

- **Paramagnetic**

  For these materials there are atoms or ions with permanent magnetic dipoles, which are randomly oriented. When a field is applied some of these magnetic dipoles become aligned with the external field, and as a result $\mathbf{M}$ is aligned with $\mathbf{H}$; the field inside is greater than 'outside' (the applied field) and χ is slightly greater than 1. An example of these materials is aluminium.

- **Ferromagnetic and ferrimagnetic**

  For ferromagnetic materials we have $\mu_r \gg 1$. These materials have permanent dipoles, which in the presence of an external field are all aligned. Examples of these materials are iron and cobalt.

  In some cases half of the dipoles will align in the direction of the field, and the other half of them in the opposite direction. These materials are called antiferrimagnetic.

  Finally, for some materials some dipoles align in the direction of the field, and some of them in the opposite direction, but not in the same proportion and with the same total 'strength'; as a result the magnetization is not zero. These materials are called ferrimagnetic; an example is the ferrite iron oxide.

The phenomena described previously are actually common to all materials, but for some of them one of these phenomena may be more important than the others.

When it is said that a material has permanent magnetic dipoles, in general this means that the internal structure of the material is divided into many magnetic domains, inside each of which the dipoles are all aligned in the same direction; see Figure 2.10.

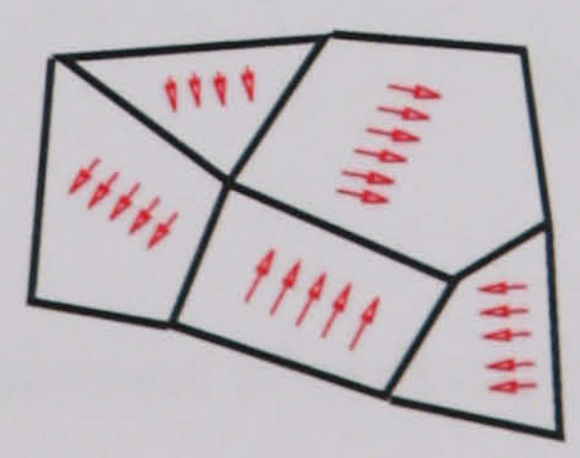


Figure 2.10: Representation of the magnetic domains for a material. Inside each domain the 'magnetic dipoles' are aligned in the same direction.

So, we may have that the total magnetization may be zero due to the randomness of the alignments of the different domains. Now, when an external field is applied, the domains that have alignments closest to the alignment of the external field will start growing at the expense of the other domains; little by little the magnetization is incremented until all the domains are aligned in the same direction, at which point we say the material is magnetically saturated. Figure 2.11 shows a plot of B against H which illustrates the saturation effect.

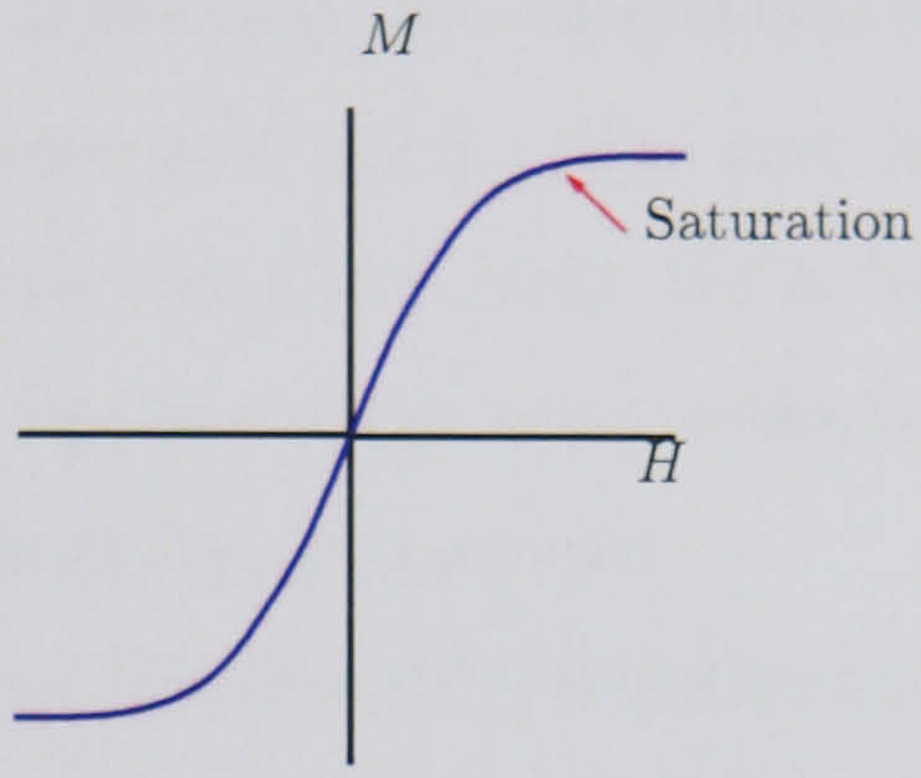


Figure 2.11: Magnetic saturation.

If the external field becomes zero, it may happen that more or less domains remain in the direction of the field and we would have some residual magnetization. A soft magnetic material is a material with low permanent magnetization, and vice versa a hard magnetic material is a material with high permanent magnetization. See Figure 2.12.

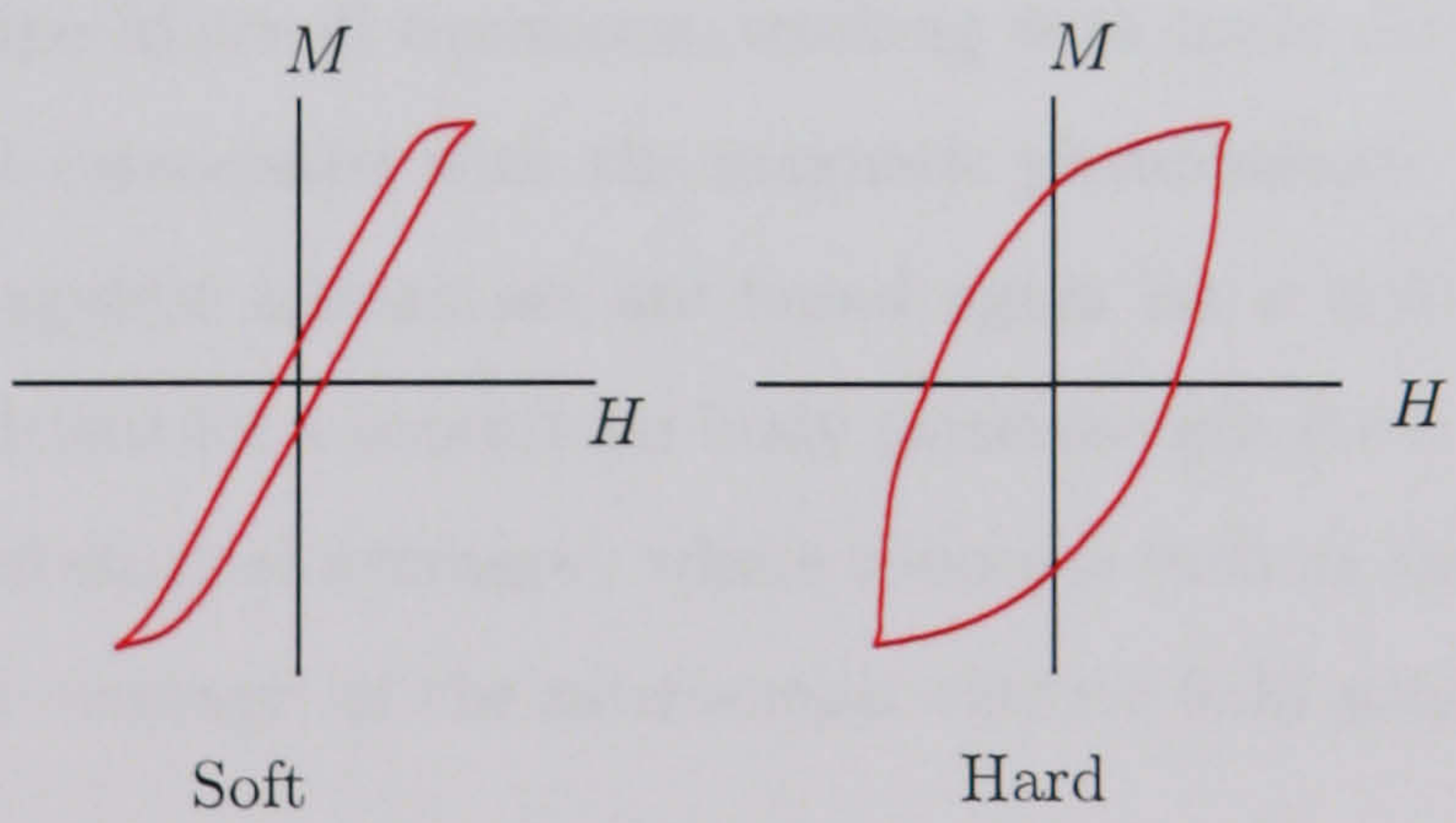


Figure 2.12: Magnetic behaviour of soft and hard magnetic materials.

2.2.2 Maxwell's equations

The purpose of this subsection is to show a brief summary of the most important aspects of the Maxwell equations.

There are different approaches that are used in order to obtain the Maxwell equations. A simple method of deduction is presented, for example, by Tiersten [107] for the special case of electro- and magnetostatics problems. In the electrostatics case Tiersten studied the electric interaction for one charge, defining the concept of electric field and electric potential; then these results are extended for a finite group of charges. The charges and dipoles are defined as in Subsection 2.2.1. In order to extend these results for the continuum case, Tiersten first studied the field and electric interaction due to a charged body for a point outside the body, by transforming the expressions for the discrete distribution of charges to a continuum distribution of charge. The field inside the body is calculated in the same way, by taking a point inside the body and a 'small' neighborhood around it, and then applying the above results for the body without this portion of material, and taking the limit when this volume becomes a point.

In the case of magnetostatics Tiersten [107] followed a similar procedure as in the case of electrostatics, but in this case the basic element used in the formulation was to consider that all the magnetic interactions were caused by an electric current moving in a small loop. The same method was used by Brown in his classical treatise on magnetoelasticity [13].

Eringen and Maugin [42] used a more sophisticated method in order to find the Maxwell equations and in order to study the electromagnetic interactions in continuous media. This formulation is more general since it considers the dynamic case as well (and as a result there is an interaction between the magnetic and electric fields). Basically they started with the microscopic form of the Maxwell equations, working with basic concepts of a charge and a microscopic current (associated with the magnetic phenomena); the 'forces' (Lorentz's forces) and electromagnetic interaction are found again for a system of charges, but in order to find the equations for a continuum body (macroscopic form of the equation), they used the method of 'statistical averages', where concepts such as the macroscopic electric field $\mathbf{E}$ appears as an 'average' of the microscopic electric field generated for each charge in the body.

The summary shown in this section follows the approach presented by Kovetz [64], which is based on the principle of conservation of charge and conservation of flux (see also [113]).

One of the basic concepts here is the charge, which like the mass is a property of the body, and it may be positive, zero or negative. The charge is conserved, which means for a given region if the total amount of charges changes this is because charges pass into the region through its boundary.

If we call q the 'density' of charge per unit of volume and α the amount of charge per unit of area passing into the region $\mathcal{V}$ with boundary $\partial\mathcal{V}$, for an interval of time $[t_1.t_2]$ we have

$$\Delta \int_{\mathcal{V}} q \, \mathrm{d}v + \int_{t_1}^{t_2} \int_{\partial\mathcal{V}} \alpha \, \mathrm{d}a = 0, \tag{2.72}$$

where $Q = \int_{\mathcal{V}} q \, \mathrm{d}v$ is the total charge in the region and $\Delta \int_{\mathcal{V}} q \, \mathrm{d}v = Q(t_2) - Q(t_1)$. The flux of charge α may be expressed as $\alpha = \mathbf{J} \cdot \mathbf{n}$ for $\partial\mathcal{V}$, where $\mathbf{n}$ is the outward normal vector in $\partial\mathcal{V}$. By assuming q and $\mathbf{J}$ differentiable and using the divergence theorem, (2.72) becomes

$$\int_{t_1}^{t_2} \int_{\mathcal{V}} \left(\frac{\partial q}{\partial t} + \operatorname{div} \mathbf{J} \right) \mathrm{d}v \, \mathrm{d}t = 0. \tag{2.73}$$

The idea now is to write the above equation in a four dimensional space, where this space would include the three dimensions of the body plus an extra dimension equivalent to the time. By defining $\boldsymbol{s}$ as

$$\boldsymbol{s} = (\mathbf{J}, q), \tag{2.74}$$

$\frac{\partial q}{\partial t} + \operatorname{div} \mathbf{J}$ is equivalent to $\partial_\gamma s^\gamma$, where $\gamma = 1, 2, 3, 4$. It is possible to show that (2.73) is equal to (see [64] and the chapter F of [113])

$$\oint \boldsymbol{s} \cdot \mathbf{n} \, \mathrm{d}_3 v = 0, \tag{2.75}$$

where the integral is calculated over the 'surface' of a four dimensional cylinder shown in Figure (2.13). The vector $\mathbf{n}$ is the normal to this 'surface' (it is a four dimensional vector too), and $\mathrm{d}_3 v$ is the element of surface in this four dimensional space. For the top and bottom of this body $\mathrm{d}_3 v$ is equal to $\mathrm{d}v$, and for the side of this 'cylinder' is equal to $\mathrm{d}a \, \mathrm{d}t$.

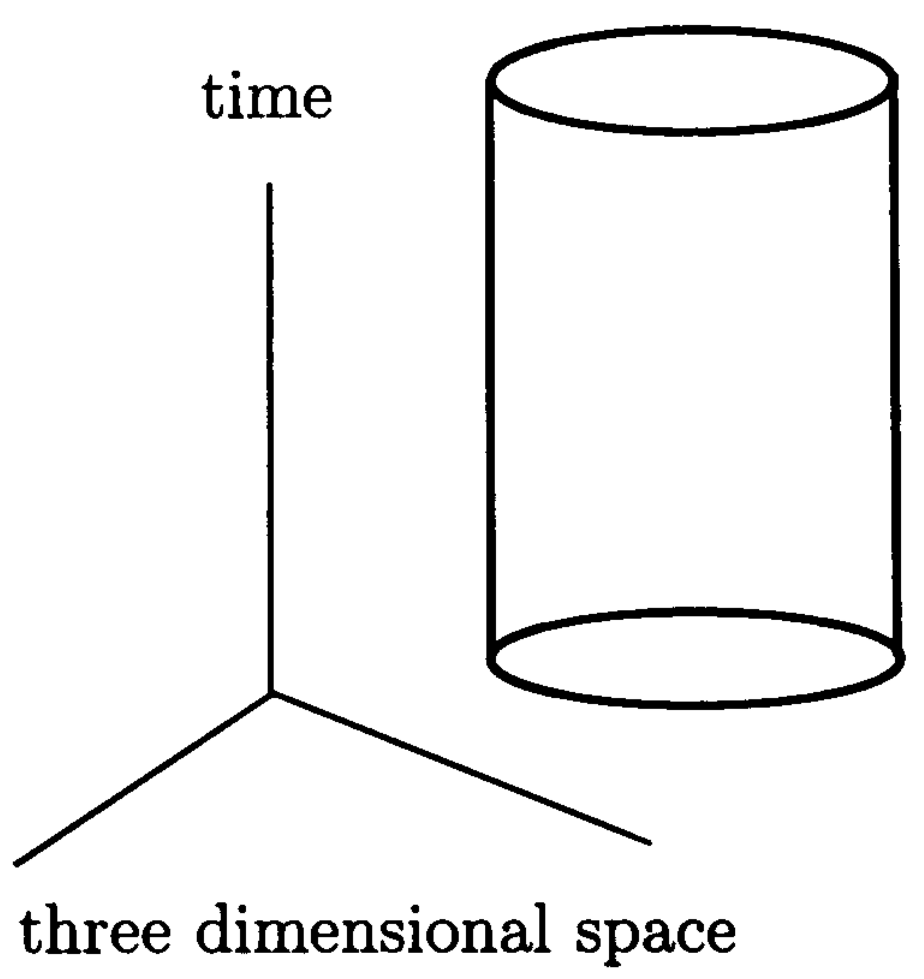


Figure 2.13: Four dimensional space

In order to deal with (2.75) is necessary to extend the normal theorems of vector calculus for n dimensional spaces, where appear concepts like rot and dual of a field. For details see parts I and II of Chapter F of [113] and Chapter 2 of [64].

For an integral like (2.75) it is possible to show that there exists an antisymmetric tensor $\boldsymbol{f}$ such that

$$s = \mathrm{div}\boldsymbol{f}, \tag{2.76}$$

$\boldsymbol{f}$ is called charge-current potential. For an Euclidean space $\boldsymbol{f}$ is usually given as

$$f = \begin{pmatrix} 0 & H_3 & -H_2 & -D^1 \\ -H_3 & 0 & H_1 & -D^2 \\ H_2 & -H_1 & 0 & -D^3 \\ D^1 & D^2 & D^3 & 0 \end{pmatrix}. \tag{2.77}$$

In Cartesian coordinates we would have (there is no distinction between covariant or contravariant components)

$$s_i = f_{ij,j},$$

then

$$s_1 = \frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} - \frac{\partial D_1}{\partial t}, \tag{2.78}$$
$$s_2 = -\frac{\partial H_3}{\partial x_1} + \frac{\partial H_1}{\partial x_3} - \frac{\partial D_2}{\partial t}, \tag{2.79}$$
$$s_3 = \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} - \frac{\partial D_3}{\partial t}, \tag{2.80}$$
$$s_4 = \frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3}. \tag{2.81}$$

Since from (2.74) we had $s_1 = J_1$, $s_2 = J_2$, $s_3 = J_3$ and $s_4 = q$, from (2.78)-(2.81) we get

$$\mathrm{curl}\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{2.82}$$
$$\mathrm{div}\mathbf{D} = q. \tag{2.83}$$

These two equations correspond to the first pair of Maxwell equations. The global forms of (2.82) and (2.83) are

$$\oint_{\mathcal{C}} \mathbf{H} \cdot \mathrm{d}\mathbf{r} - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}} \mathbf{D} \cdot \mathbf{n}\, \mathrm{d}a = \int_{\mathcal{S}} \mathbf{J} \cdot \mathbf{n}\, \mathrm{d}a, \tag{2.84}$$
$$\int_{\mathcal{S}} \mathbf{D} \cdot \mathbf{n}\, \mathrm{d}a = \int_{\mathcal{V}} q\, \mathrm{d}v, \tag{2.85}$$

where $\mathcal{V}$ is a region in $\mathbb{R}^3$. In the first equation $\mathcal{S}$ is an open surface with boundary $\mathcal{C}$, and in the second equation $\mathcal{S}$ is a closed surface that is boundary of $\mathcal{V}$.

Let $\mathcal{S}$ be a surface of discontinuity or jump for $\mathbf{H}$ and $\mathbf{D}$: following the standard procedure of working with a pillbox across $\mathcal{S}$ and using (2.84) and (2.85), it is possible to show that the jump conditions for $\mathbf{H}$ and $\mathbf{D}$ across $\mathcal{S}$ are [42, 64, 113]

$$[\![\mathbf{D}]\!] \cdot \mathbf{n} = \sigma, \quad [\![\mathbf{H}]\!] \times \mathbf{n} + v_n [\![\mathbf{D}]\!] = \mathbf{K}, \tag{2.86}$$

where σ is a surface density of charge on $\mathcal{S}$, v_n is the normal component of the velocity of the interface, and $\mathbf{K}$ is a surface density of current. For a function f, the double square brackets $[\![f]\!]$ means the difference for f approaching from the two sides of the interface.

The other pair of Maxwell equations can be found by working with the 'principle of conservation of flux'. This principle can be written in a four dimensional space as in the previous case as [64]

$$\int_{R_2} \mathcal{F} \cdot \mathrm{d}\tau_{(2)} = 0. \tag{2.87}$$

Here $\mathcal{F}$ is called the electromagnetic field tensor, R_2 is a closed surface in this four dimensional space, and $\mathrm{d}\tau_{(2)}$ is the element of surface in the four dimensional space. $\mathcal{F}$ is antisymmetric; let's assume that it can be written as

$$\mathcal{F} = \begin{pmatrix} 0 & B^3 & -B^2 & E_1 \\ -B^3 & 0 & B^1 & E_2 \\ B^2 & -B^1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}. \tag{2.88}$$

From (2.87) it is possible to show that [113]

$$\int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n}\, \mathrm{d}a = 0, \tag{2.89}$$

$$\Delta \int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n}\, \mathrm{d}a = -\int_{t_1}^{t_2} \oint_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{r}\, \mathrm{d}t, \tag{2.90}$$

where $\Delta \int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n}\, \mathrm{d}a = \left(\int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n}\, \mathrm{d}a\right)_{t_2} - \left(\int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n}\, \mathrm{d}a\right)_{t_1}$. In (2.89) $\mathcal{S}$ is a closed surface, and in (2.90) $\mathcal{S}$ is an open surface with boundary $\mathcal{C}$.

From (2.89), (2.90) and (2.90) we have the local form of the conservation of flux

$$\operatorname{div}\mathbf{B} = 0, \tag{2.91}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{E}, \tag{2.92}$$

which form the second pair of Maxwell equations. From (2.89) and (2.90) the jump conditions for a surface with discontinuity in $\mathbf{E}$ and $\mathbf{B}$ are

$$[\![\mathbf{B}]\!] \cdot \mathbf{n} = 0, \quad [\![\mathbf{E}]\!] \times \mathbf{n} - v_n [\![\mathbf{B}]\!] = \mathbf{0}. \tag{2.93}$$

As a summary we have the Maxwell equations

$$\mathrm{curl}\,\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{2.94}$$
$$\mathrm{div}\,\mathbf{D} = q, \tag{2.95}$$
$$\mathrm{div}\,\mathbf{B} = 0, \tag{2.96}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\mathrm{curl}\,\mathbf{E}, \tag{2.97}$$

with boundary conditions

$$[\![\mathbf{D}]\!] \cdot \mathbf{n} = \sigma, \quad [\![\mathbf{H}]\!] \times \mathbf{n} + v_n [\![\mathbf{D}]\!] = \mathbf{K}, \tag{2.98}$$

$$[\![\mathbf{B}]\!] \cdot \mathbf{n} = 0, \quad [\![\mathbf{E}]\!] \times \mathbf{n} - v_n [\![\mathbf{B}]\!] = \mathbf{0}. \tag{2.99}$$

The equation $\mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ is also called Faraday's law, the equation $\mathrm{curl}\,\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$ is called Ampere's law, the equations $\mathrm{div}\,\mathbf{D} = q$ and $\mathrm{div}\,\mathbf{B} = 0$ are called the Gauss' law and the Gauss' law for magnetism respectively.

We can solve (2.96) and (2.97) by using the vector and scalar potentials $\mathbf{A}$ and φ as

$$\mathbf{B} = \mathrm{curl}\,\mathbf{A}, \tag{2.100}$$
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\varphi. \tag{2.101}$$

In general the physical meaning of φ is very clear; however that is not the case with $\mathbf{A}$; a discussion of the physical meaning of $\mathbf{A}$ is presented, for example, in [96].

The boundary or jump conditions for φ and $\mathbf{A}$ will be discussed later on.

We work with the quasi-static case; also for the problems under our consideration we assume that $\mathbf{J} = \mathbf{0}$ and $q = 0$, so (2.94)-(2.97) become

$$\mathrm{curl}\,\mathbf{H} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{B} = 0, \tag{2.102}$$
$$\mathrm{curl}\,\mathbf{E} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{D} = 0, \tag{2.103}$$

with boundary conditions

$$[\![\mathbf{H}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{B}]\!] \cdot \mathbf{n} = 0, \tag{2.104}$$

$$[\![\mathbf{E}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{D}]\!] \cdot \mathbf{n} = 0. \tag{2.105}$$

An important principle in electromagnetism is to assume that there exists an inertial frame such that

$$\mathbf{D} = \varepsilon_o \mathbf{E}, \quad \mathbf{B} = \mu_o \mathbf{H}. \tag{2.106}$$

The above relations are valid for vacuum, μ_o and ε_o are known as the susceptibility and permittivity of free space; their numerical values are

$$\mu_o = 4\pi * 10^{-7} \left[\frac{N}{A^2}\right],$$
$$\varepsilon_o = \frac{1}{36\pi} * 10^{-9} \left[\frac{C^2}{Nm^2}\right].$$

For the case of condensed matter we have

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}), \tag{2.107}$$

where $\mathbf{P}$ and $\mathbf{M}$ are known as the polarization and magnetization respectively. $\mathbf{P}$ and $\mathbf{M}$ are in general non-linear functions of, for example, the fields $\mathbf{E}$ and $\mathbf{H}$ respectively.

2.3 Deformable media

In the previous two sections we reviewed some basic topics in continuum mechanics and electromagnetism. The natural step now it is to present the continuum theory of deformable media and the interaction with electromagnetic fields. However, at this step we may find that there are in fact many different theories for electromagnetism and deformable continua. The reason for this multitude of seemingly different theories is because of the essential nature of these two phenomena; the theory of electromagnetic fields has been laid down within a relativistic framework, while, on the other hand, continuum mechanics is based on the principles of classical Newtonian mechanics; therefore what has been done is to obtain a theory of deformable media by approximating the case when the velocities are 'small' in comparison with the velocity of light. This approximation process has been argued as one of the reasons why there are several different theories for electromagnetism and deformable continua (see Chapter F of [113]).

Let us review some of the important literature in the area. Two important references with a complete discussion of the different theories for electromagnetism and deformable media are the review paper by Pao [80], and the monograph by Hutter and van de Ven [57]; here it is possible to find the comparison of the different theories and, for example, expressions for the 'body' force due to electromagnetic fields. In particular, the paper by Pao [80] has been the basis of many subsequent works in the area.

The first paper in electroelasticity, where the case of finite deformations was considered, is the paper by Toupin [109], who obtained the equations for the case of a dielectric material using the principle of virtual work.

In a series of papers Tiersten developed his own theory for the interaction of electromagnetic fields and deformable media. In [104] a theory for magnetoelastic phenomena in solids was developed, by assuming that the behaviour of the material could be described by working with two 'continua', one of them called the 'spin continuum', where all the purely magnetic interactions happen, and the other called the normal 'lattice continuum', which is the continuum used in the theory of finite elasticity; these two continua interact with each other, and in the case of the spin continuum, each point there translates with the lattice continuum. For the quasi-static problem, using the second law of thermodynamics and the concept of free energy function, Tiersten found an appropriate set of balance equations, boundary conditions, and a general form for the constitutive relations for the magnetoelastic problem.

The same procedure was used by Tiersten in [105] in order to obtain the balance equations, and a general form for the constitutive relations for the case of thermo-electroelastic problems (also for the quasi-static case). In [108] Tiersten and Tsai generalized the above results for the dynamic problem with magnetic and electric fields, considering also thermal effects.

An important reference that is necessary to mention again is the monograph by Brown [13], which, as with the article by Pao [80], has been used as a main source of reference for many subsequent researchers in the theory of magneto-elastic interactions.

A work that is contemporary with the papers by Tiersten [104] and the monograph by Brown [13], is the paper by Jordan and Eringen [60], who developed a constitutive theory for the interaction of electromagnetic fields and thermal gradients, assuming the material to be Cauchy elastic, i.e. they did not use the second law of thermodynamics. This work, which is based on the representation theory for tensor and vector functions, provided a complete yet extremely complex general form for the constitutive equations, which, for example, in the case of the stress consists of 100 parameters!

In subsequent works Eringen has provided a complete theory for the electromagnetic interaction with deformable bodies. In [68] Maugin and Eringen deduced the general balance and constitutive equations for the electroelastic problem, by working with the Lorentz's theory of electrons, and using the same average procedure in order to go from the microscopic to the macroscopic description as mentioned previously in Subsection 2.2.2. This work is one the sources of their book [42].

Coleman and Dill [24] studied the case of a material with memory, electric and magnetic fields, and thermal effects as well. They only studied coupling between electromagnetic phenomena and thermal effects, and did not consider deformations.

In the case of the present thesis, we consider magneto- and electro-active elastomers as continua, and we do not study the microstructure of the material. There are, however, applications where it is necessary to do a more detailed analysis of the microstructure; this is the case, for example, for ferromagnetic materials, which as we mentioned in Subsection 2.2.1, posses the characteristic of having magnetic domains that evolve due to changes in the external field. We can mention the papers by Romano [92], Pao [81], De Simone and Podio-Guidugli [29], and James [29]. In this last two papers we have a study of the microstructure of ferromagnetic materials, which is based mainly in the theory of Brown [13].

Let's now speak about recent references more connected with magneto- and electro-active elastomers.

For the case of non-linear electro-elasticity Yang and Batra [122] proposed a variational formulation for the mixed boundary value problem. More details about this paper are given in [14].

Rajagopal and Wineman [87] studied the problem of finding appropriate forms for the constitutive equations for the electroelastic problem. They considered the material as Cauchy elastic, and obtained a general form for the constitutive equation, which they used in order to study some simple boundary value problems, such as the triaxial extension of a cube and the simple shear of a slab.

McMeeking and Landis [72] developed their own theory for electro-active elastomers. In their formulation they considered the interaction of the body and the free space (which is something that many other researchers do not do, as we see in Chapter 9). The constitutive equations were found by working with the free energy function and the second law of thermodynamics. From their general theory, McMeeking and Landis [72] also considered many simplified cases, such as linear elastic dielectrics (small deformations), and in particular, a kind of material mentioned in the introduction [26, 71], called here 'compliant isotropic dielectric', which corresponds to a thin membrane of elastomer, sandwiched between two electrodes, and as a result by applying an electric potential we may have large elastic deformations for the elastomer. The modelling of 'compliant isotropic dielectric' has been also treated by Goulbourne et al. [51], who explored especially the possible applications of these materials to the development of small pumps.

Voltairas et al. [118] developed a model for electro-active gels, which are gels filled with electro-active particles. They assumed that the behaviour of the gels is similar to that of an electro-active elastomer, and proposed a general form for the constitutive equation based on the use of a free energy function. They also solved a boundary value problem, the flexure of a slab.

Steigmann [103] developed a theory for magnetic elastomers with especial applications in membranes. Steigmann's work was based partly on the works by Brown [13] and Kovetz [64]; he worked with the decomposition of the magnetic field into an applied and a self field, and the use of the free energy function and the magnetic field as the independent magnetic variable. The weak form was also provided, and it was used in order to obtain an appropriate theory for membranes from the original three-dimensional theory. In [4] this theory is developed further and some results of a boundary value problem for membranes are shown.

Kankanala and Triantafyllidis [61] also developed a theory for magneto-sensitive elastomers. They used the magnetization as the independent magnetic variable, and they put especial attention on the development of a suitable variational formulation. In this paper an example of a closed form for the energy function is provided, which was obtained using appropriate experimental data. The concept of quasiconvexity was studied and its extension to the field of magneto-elasticity was addressed by working with the variational formulation; with this, Kankanala and Triantafyllidis [61] obtained restrictions in the form of the energy function, which if they hold, are supposed to lead to well posed boundary value problems. Two boundary value problems were studied, the uniaxial stretching and torsion of a cylinder.

In [40, 41] Ericksen developed a theory for magnetic effects in elastic materials using the principle of virtual work.

More details of the papers by Steigmann [103], Kankanala and Triantafyllidis [61], and Ericksen [40, 41] are given in the following sections, especially in Chapter 9.

The results obtained in this thesis are based on the theory for magneto- and electro-sensitive elastomers developed by Dorfmann, Ogden and Brigadnov [12, 31] (see also [30] and [34]), where they obtained the basic equations and general forms for the constitutive equations for the magneto-elastic problem, based on the use of the free energy function and the second law of thermodynamics. The solution of some boundary value problems such as the shear of a slab and a problem with cylindrical symmetry are also provided.

In [33] Dorfmann and Ogden introduced the important concept of the 'total energy function', from which it is possible to obtain very simple forms for the constitutive equations. In the same paper Dorfmann and Ogden solved some boundary value problems such as the helical shear and the extension and torsion of a circular cylinder. An additional boundary value problem, the steady rotation of a tube, is presented in [38].

For the electro-elastic problem Dorfmann and Ogden [32] developed a similar theory (quasi-static case), defining the total energy function and solving some boundary value problems, such as the simple and axial shear. In [36] more results for boundary value problems are presented, such as the azimuthal shear, the extension and inflation of a tube, and the inflation of a spherical shell.

2.3.1 Basic equations

As has been mentioned previously, the article by Pao [80] is the basis for the review provided in this section (see also [12, 31]).

The local form of the balance laws for the case of a deformable solid and electromagnetic fields is given as [80] (see also Kovetz [64] for an equivalent formulation).

- **Conservation of mass**

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \quad J\rho = \rho_o, \tag{2.108}$$

where $\mathbf{v}$ is the velocity field, ρ_o is the density in the reference configuration and $J = \det \mathbf{F} > 0$.

- **Balance of linear momentum**

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{f} + \mathbf{f}_e = \rho \dot{\mathbf{v}}, \tag{2.109}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{f}$ is the body force per unit of mass due to non-electromagnetic effects, and $\mathbf{f}_e$ is the electro-magnetic body force.

From [80] $\mathbf{f}_e$ is given as

$$\mathbf{f}_e = q\mathbf{E} + \mathbf{J} \times \mathbf{B} + \mu_o^{-1} (\operatorname{grad} \mathbf{B})^T \mathbf{M} + (\operatorname{grad} \mathbf{E})^T \mathbf{P} + \frac{\partial}{\partial t} (\mathbf{P} \times \mathbf{B}) + \operatorname{div} (\mathbf{v} \otimes (\mathbf{P} \times \mathbf{B})). \tag{2.110}$$

q represents a distribution of free charge, $\mathbf{J}$ corresponds to a conduction current, $\mathbf{M}$ is the magnetization, and $\mathbf{P}$ is the polarization.

- **Balance of angular momentum**

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + (\mu_o^{-1}\mathbf{M} + \mathbf{v} \times \mathbf{P}) \times \mathbf{B} + \mathbf{P} \times \mathbf{E} = \mathbf{0}, \tag{2.111}$$

where we have the notation $(\boldsymbol{\varepsilon} : \boldsymbol{\sigma})_i = \varepsilon_{ijk}\sigma_{jk}$, where ε_{ijk} is the permutation tensor.

- **Balance of energy (first law of thermodynamics)**

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(U + \frac{1}{2}|\mathbf{v}|^2\right) + \mathrm{div}\,\mathbf{Q} = \mathrm{div}\,(\boldsymbol{\sigma}\mathbf{v}) + \rho\mathbf{f}\cdot\mathbf{v} + \rho R + w_e, \tag{2.112}$$

where U is the internal energy per unit of mass, the term $\frac{1}{2}|\mathbf{v}|^2$ is the kinetic energy per unit of mass, $\mathbf{Q}$ is the heat flux, R is the 'radiant heating' [75], and w_e is the electromagnetic power given as [80]

$$w_e = \mathbf{f}_e\cdot\mathbf{v} + \mathbf{J}_e\cdot\mathbf{E}_e - \mathbf{M}_e\cdot\dot{\mathbf{B}} + \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\mathbf{P}\right)\cdot\mathbf{E}_e. \tag{2.113}$$

$\mathbf{J}_e$, $\mathbf{M}_e$ and $\mathbf{E}_e$ are called the effective conduction current, the effective magnetization, and effective electric current respectively.

For a body with a distribution of charge q, the movement of a point with velocity $\mathbf{v}$ can be interpreted as an 'induced current', and as a result the 'effective current' can be defined as

$$\mathbf{J}_e = \mathbf{J} - q\mathbf{v}. \tag{2.114}$$

Likewise, from the Lorentz's theory of electrons it is possible to show that the effective magnetization and electric field are given respectively as

$$\mathbf{M}_e = \mu_o^{-1}\mathbf{M} + \mathbf{v}\times\mathbf{P}, \quad \mathbf{E}_e = \mathbf{E} + \mathbf{v}\times\mathbf{B}. \tag{2.115}$$

Note that the first equation says that a polarized body in movement will induce the appearance of a magnetization; regarding the second equation we see the induced effect on the electric field due to a magnetic induction, however, we do not see a symmetric effect of the electric field or electric displacement on the magnetic field or magnetic induction.

- **Second law of thermodynamics**

The Clausius-Duhem inequality in its local form is

$$\rho\dot{s} + \mathrm{div}\left(\frac{1}{T}\mathbf{Q}\right) - \rho\frac{R}{T} \geq 0, \tag{2.116}$$

s is the specific entropy and T is the absolute temperature.

Let us expand (2.112); then we have

$$\rho\dot{U} + \rho\mathbf{v}\cdot\dot{\mathbf{v}} + \mathrm{div}\mathbf{Q} = \mathrm{div}\boldsymbol{\sigma}\cdot\mathbf{v} + \boldsymbol{\sigma} : \mathrm{grad}\,\mathbf{v} + \rho\mathbf{f}\cdot\mathbf{v} + \rho R + w_e, \tag{2.117}$$

but the first term of the right side $\mathrm{div}\boldsymbol{\sigma}\cdot\mathbf{v}$ can be replaced from the balance of linear momentum (2.109) as $\mathrm{div}\boldsymbol{\sigma} = \rho\dot{\mathbf{v}} - \rho\mathbf{f} - \mathbf{f}_e$; then, replacing in (2.117) and using (2.113) we have

$$\rho\dot{U} + \mathrm{div}\mathbf{Q} = \boldsymbol{\sigma} : \boldsymbol{\Gamma} + \rho R + \mathbf{J}_e\cdot\mathbf{E}_e - \mathbf{M}_e\cdot\dot{\mathbf{B}} - \frac{\dot{\rho}}{\rho}\mathbf{P}\cdot\mathbf{E}_e + \dot{\mathbf{P}}\cdot\mathbf{E}_e. \tag{2.118}$$

In the above equations we have used the definition

$$\boldsymbol{\Gamma} \equiv \mathrm{grad}\,\mathbf{v}. \tag{2.119}$$

Let us define the Helmholtz free energy function ψ as (see, for example, [80] part 12.2, and see also [64] section 55 for an alternative definition of the 'specific free energy')

$$\psi = U - Ts - \frac{1}{\rho}\mathbf{E}_e\cdot\mathbf{P}. \tag{2.120}$$

Let's expand (2.116); then, we get

$$\rho T\dot{s} - \frac{1}{T}\mathrm{grad}T\cdot\mathbf{Q} + \mathrm{div}\mathbf{Q} - \rho R \geq 0, \tag{2.121}$$

and from (2.120) and (2.118) we have respectively

$$T\rho\dot{s} = \rho\dot{U} - \rho\dot{\psi} - \rho\dot{T}s + \frac{\dot{\rho}}{\rho}\mathbf{E}_e\cdot\mathbf{P} - \dot{\mathbf{E}}_e\cdot\mathbf{P} - \mathbf{E}_e\cdot\dot{\mathbf{P}}, \tag{2.122}$$

$$\rho R = \rho\dot{U} + \mathrm{div}\mathbf{Q} - \boldsymbol{\sigma} : \boldsymbol{\Gamma} + \mathbf{M}_e\cdot\dot{\mathbf{B}} - \mathbf{J}_e\cdot\mathbf{E}_e - \cdot\mathbf{P}\cdot\mathbf{E}_e - (\mathbf{P}\cdot\mathbf{E}_e)\mathrm{div}\mathbf{v}. \tag{2.123}$$

Using these in (2.121) and $\frac{\dot{\rho}}{\rho} = -\mathrm{div}\mathbf{v}$ from (2.108), we obtain the inequality (see [12, 30, 31, 34] and [80])

$$-\rho(\dot{\psi} + \dot{T}s) + \boldsymbol{\sigma} : \boldsymbol{\Gamma} - \mathbf{M}_e\cdot\dot{\mathbf{B}} - \frac{1}{T}\mathrm{grad}T\cdot\mathbf{Q} + \mathbf{J}_e\cdot\mathbf{E}_e - \dot{\mathbf{E}}_e\cdot\mathbf{P} \geq 0. \tag{2.124}$$

For most materials the heat flux $\mathbf{Q}$ is given by the Fourier law

$$\mathbf{Q} = -k\mathrm{grad}T. \tag{2.125}$$

Let assume that ψ is a function of T, $\mathbf{F}$, $\mathbf{E}_e$ and $\mathbf{B}$ (an alternative formulation is shown in [30])

$$\psi = \psi(T, \mathbf{F}, \mathbf{E}_e, \mathbf{B}), \tag{2.126}$$

then by the chain rule we have

$$\dot{\psi} = \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \psi}{\partial \mathbf{E}_e} \cdot \dot{\mathbf{E}}_e + \frac{\partial \psi}{\partial \mathbf{B}} \cdot \dot{\mathbf{B}}. \tag{2.127}$$

As a result, in (2.124) we have

$$\begin{aligned}-\rho\dot{T}\left(\frac{\partial \psi}{\partial T} + s\right) - \rho\frac{\partial \psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} - \rho\frac{\partial \psi}{\partial \mathbf{E}_e} \cdot \dot{\mathbf{E}}_e - \rho\frac{\partial \psi}{\partial \mathbf{B}} \cdot \dot{\mathbf{B}} + \boldsymbol{\sigma} : \boldsymbol{\Gamma} - \mathbf{M}_e \cdot \dot{\mathbf{B}} \\ -\frac{1}{T}\mathrm{grad}T \cdot \mathbf{Q} + \mathbf{J} \cdot \mathbf{E}_e - \dot{\mathbf{E}}_e \cdot \mathbf{P} \geq 0.\end{aligned} \tag{2.128}$$

Using (2.125) and the identity [78]

$$\boldsymbol{\sigma} : \boldsymbol{\Gamma} = (\mathbf{F}^{-1}\boldsymbol{\sigma})^T : \dot{\mathbf{F}}, \tag{2.129}$$

we finally have

$$\begin{aligned}-\rho\dot{T}\left(\frac{\partial \psi}{\partial T} + s\right) - \left(\rho\frac{\partial \psi}{\partial \mathbf{F}} - (\mathbf{F}^{-1}\boldsymbol{\sigma})^T\right) : \dot{\mathbf{F}} - \left(\rho\frac{\partial \psi}{\partial \mathbf{E}_e} + \mathbf{P}\right) \cdot \dot{\mathbf{E}}_e \\ - \left(\rho\frac{\partial \psi}{\partial \mathbf{B}} + \mathbf{M}_e\right) \cdot \dot{\mathbf{B}} + \frac{k}{T}|\mathrm{grad}T|^2 + \mathbf{J}_e \cdot \mathbf{E}_e \geq 0.\end{aligned} \tag{2.130}$$

By a standard argument, since T, $\mathbf{F}$, $\mathbf{E}_e$ and $\mathbf{B}$ are independent variables, the above inequality holds if and only if [25]

$$s = -\frac{\partial \psi}{\partial T}, \quad (\mathbf{F}^{-1}\boldsymbol{\sigma})^T = \rho\frac{\partial \psi}{\partial \mathbf{F}}, \quad \mathbf{P} = -\rho\frac{\partial \psi}{\partial \mathbf{E}_e}, \quad \mathbf{M}_e = -\rho\frac{\partial \psi}{\partial \mathbf{B}}, \tag{2.131}$$

and

$$\frac{k}{T}|\mathrm{grad}T|^2 + \mathbf{J}_e \cdot \mathbf{E}_e \geq 0. \tag{2.132}$$

Chapter 3

Basic equations for magneto-active elastomers

In this chapter we show some results for magneto-active elastomers. We do not consider the effect of electric fields; we assume that there is neither a distribution of charge q nor a current $\mathbf{J}$. We disregard any thermal effect, and finally we restrict our analysis to the quasi-static case.

This chapter is based on the theory for MS elastomers developed by Dorfmann and Ogden [33].

With the simplifications mentioned previously, from Subsection 2.3.1 we have that the Helmoholtz free energy function (2.126) is only a function of $\mathbf{F}$ and $\mathbf{B}$, and hence

$$\psi = \psi(\mathbf{F}, \mathbf{B}), \tag{3.1}$$

and from $(2.131)_2$ and $(2.131)_4$ we have[1]

$$\boldsymbol{\sigma} = \rho \mathbf{F} \left(\frac{\partial \psi}{\partial \mathbf{F}} \right)^T, \quad \mathbf{M} = -\rho \frac{\partial \psi}{\partial \mathbf{B}}. \tag{3.2}$$

Now, let us use the following convention for the derivative of a scalar function on the gradient of deformation (this convention is usually used by Ogden [78])

$$\left(\frac{\partial \psi}{\partial \mathbf{F}} \right)_{ij} \equiv \frac{\partial \psi}{\partial F_{ji}}. \tag{3.3}$$

With this convention $(3.2)_1$ can be written as

$$\boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}}. \tag{3.4}$$

[1]For the quasi-static case there is no difference between the effective and the laboratory frame fields.

In magnetoelastostatics we work with the three magnetic variables, $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{M}$, the magnetic field, the magnetic induction, and the magnetization, respectively. We regard $\mathbf{H}$ and $\mathbf{B}$ as the primary fields, and $\mathbf{M}$ only as an auxiliary field [64, 113], which is defined in terms of $\mathbf{H}$ and $\mathbf{B}$ via $(2.107)_2$.

3.1 Lagrangian forms of the fields

The simplified global forms of the Maxwell equations (2.102) in this case are

$$\int_S \mathbf{B}\cdot\mathbf{n}\,da = 0, \quad \oint_C \mathbf{H}\cdot d\mathbf{r} = 0. \tag{3.5}$$

In magnetostatics we work with the Eulerian frame, which means $\mathbf{H}$ and $\mathbf{B}$ are related to the current configuration. Let us use (3.5) in order to determine the 'Lagrangian' or pull back version of these fields [33–35]. For $(3.5)_1$ from Nanson's formula we have [78]

$$\begin{aligned}\int_S \mathbf{B}\cdot\mathbf{n}\,da &= \int_{S_r} \mathbf{B}\cdot J\mathbf{F}^{-T}\mathbf{N}\,dA,\\ &= \int_{S_r} J\mathbf{F}^{-1}\mathbf{B}\cdot\mathbf{N}\,dA,\end{aligned} \tag{3.6}$$

and we can define the Lagrangian magnetic induction $\mathbf{B}_l$ as [33–35]

$$\mathbf{B}_l \equiv J\mathbf{F}^{-1}\mathbf{B}. \tag{3.7}$$

From $(3.5)_2$ using $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ (see Section 2.1.2), we have

$$\begin{aligned}\oint_C \mathbf{H}\cdot d\mathbf{r} &= \oint_{C_r} \mathbf{H}\cdot\mathbf{F}d\mathbf{R},\\ &= \oint_{C_r} \mathbf{F}^T\mathbf{H}\cdot d\mathbf{R},\end{aligned} \tag{3.8}$$

and we can define the Lagrangian magnetic field $\mathbf{H}_l$ as [33–35]

$$\mathbf{H}_l \equiv \mathbf{F}^T\mathbf{H}. \tag{3.9}$$

Assuming appropriate regularity of the deformation, the standard kinematical identities (see, for example, [17])

$$\mathrm{Div}(J\mathbf{F}^{-1}\mathbf{B}) = J\mathrm{div}\mathbf{B}, \quad \mathbf{F}\mathrm{Curl}(\mathbf{F}^T\mathbf{H}) = J\mathrm{curl}\mathbf{H} \tag{3.10}$$

ensure that the equations (2.102) are equivalent to the pair

$$\mathrm{Curl}\mathbf{H}_l = \mathbf{0}, \quad \mathrm{Div}\mathbf{B}_l = 0. \tag{3.11}$$

Let's give the proof of the identities (3.10).

Proof. Consider (3.7), applying the divergence theorem to both sides of the equation we obtain

$$\int_{\mathcal{V}} \operatorname{div}\mathbf{B}\,\mathrm{d}v = \int_{\mathcal{V}_r} \operatorname{Div}(J\mathbf{F}^{-1}\mathbf{B})\,\mathrm{d}V.$$

Let's transform the integral of the left side to the reference configuration (we use $\mathrm{d}v = J\mathrm{d}V$); we obtain

$$\int_{\mathcal{V}_r} J\operatorname{div}\mathbf{B}\,\mathrm{d}V = \int_{\mathcal{V}_r} \operatorname{Div}(J\mathbf{F}^{-1}\mathbf{B})\,\mathrm{d}V.$$

We have not specified the form of $\mathcal{V}_r$, which is arbitrary, and so the above equation holds if only if

$$J\operatorname{div}\mathbf{B} = \operatorname{Div}(J\mathbf{F}^{-1}\mathbf{B}).$$

Consider now (3.8), using the Stokes theorem (for an open surface $\mathcal{S}$ with boundary $\mathcal{C}$) we get

$$\int_{\mathcal{S}} \operatorname{curl}\mathbf{H}\cdot\mathrm{d}\mathbf{a} = \int_{\mathcal{S}_r} \operatorname{Curl}(\mathbf{F}^T\mathbf{H})\cdot\mathrm{d}\mathbf{A}.$$

Using Nanson's formula to write the integral of the left side of the above equation in the reference configuration we obtain

$$\int_{\mathcal{S}_r} \operatorname{curl}\mathbf{H}\cdot J\mathbf{F}^{-T}\,\mathrm{d}\mathbf{A} = \int_{\mathcal{S}_r} \operatorname{Curl}(\mathbf{F}^T\mathbf{H})\cdot\mathrm{d}\mathbf{A}.$$

This holds for any $\mathcal{S}_r$, and we therefore obtain

$$J\operatorname{curl}\mathbf{H} = \mathbf{F}\operatorname{Curl}(\mathbf{F}^T\mathbf{H}).$$

□

It is not possible to derive an equivalent Lagrangian form for $\mathbf{M}$ in such a simple and unique way as for $\mathbf{B}$ or $\mathbf{H}$. From $(2.107)_2$ we had that

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}). \tag{3.12}$$

Since $\mathbf{H}$ and $\mathbf{M}$ are being added in the above expression, we can assume that $\mathbf{M}$ transforms in the same way as $\mathbf{H}$ [33–35] (note that this definition is not unique), so a definition for the Lagrangian form of $\mathbf{M}$ may be

$$\mathbf{M}_l \equiv \mathbf{F}^T\mathbf{M}, \tag{3.13}$$

from where we have for (3.12) from (3.7), (3.9) and (3.13)

$$J^{-1}\mathbf{c}\mathbf{B}_l = \mu_o(\mathbf{H}_l + \mathbf{M}_l). \tag{3.14}$$

3.2 Initial field and Eulerian forms

Consider the situation where there is no deformation, and let's denote the magnetic induction, the magnetic field, and the magnetization in such a situation as $\mathbf{B}_o$, $\mathbf{H}_o$ and $\mathbf{M}_o$, respectively. In this case, from (3.12), we have

$$\mathbf{B}_o = \mu_o(\mathbf{H}_o + \mathbf{M}_o). \tag{3.15}$$

Assume now that the body deforms such that the gradient of deformation is $\mathbf{F}$; we could obtain the 'push forward' versions of $\mathbf{B}_o$, $\mathbf{H}_o$ and $\mathbf{M}_o$ by using (3.7), (3.9) and (3.13) [33], then by denoting $\mathbf{B}_f$, $\mathbf{H}_f$ and $\mathbf{M}_f$ in order to speak about such fields, we would have

$$\mathbf{B}_f = J^{-1}\mathbf{F}\mathbf{B}_o, \quad \mathbf{H}_f = \mathbf{F}^{-T}\mathbf{H}_o, \quad \mathbf{M}_f = \mathbf{F}^{-T}\mathbf{M}_o, \tag{3.16}$$

and substituting in (3.15) we get

$$\mathbf{B}_f = \mu_o J^{-1}\mathbf{b}(\mathbf{H}_f + \mathbf{M}_f). \tag{3.17}$$

Thus, the form (3.15) is not preserved under the deformation. The same can be concluded from our previous expressions when we started with the Eulerian form of the equation $\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})$ and then we found that the Lagrangian form of that equation was $J^{-1}\mathbf{c}\mathbf{B}_l = \mu_o(\mathbf{H}_l + \mathbf{M}_l)$ (equation (3.14)). As a result of this lack of invariance we have to be careful regarding the transformation of the magnetic variables; if we choose to work with $\mathbf{B}$ as the independent magnetic variable, then we will assume that $\mathbf{B}_l = \mathbf{B}_o$, noting that in this case in general $\mathbf{H}_l \neq \mathbf{H}_o$. Likewise if we work with $\mathbf{H}$ as the independent magnetic variable, then $\mathbf{H}_l = \mathbf{H}_o$, and in general $\mathbf{B}_l \neq \mathbf{B}_o$ [33].

3.3 Constitutive equations and the total energy function

The concept of the 'total energy function' was introduced by Dorfmann and Ogden [33–35] in order to obtain a simpler representation for the problem in magnetostatic than, for example, the formulations developed by Kankanala and Triantafyllidis [61] or Steigmann [103].

Let us define the function Φ by

$$\Phi(\mathbf{F}, \mathbf{B}_l) \equiv \psi(\mathbf{F}, J^{-1}\mathbf{F}\mathbf{B}_l). \tag{3.18}$$

From the principle of material frame-indifference [78, 112] for Φ we must have

$$\Phi(\mathbf{F}, \mathbf{B}_l) = \Phi(\mathbf{Q}\mathbf{F}, \mathbf{B}_l), \tag{3.19}$$

where $\mathbf{Q}$ is a proper orthogonal tensor. In the above expression $\mathbf{B}_l$ is a Lagrangian vector and as a result is not affected by the rotation $\mathbf{Q}$ in the deformed or current configuration (see Section 4 of [103]).

From (3.4) we had

$$\sigma = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}}.$$

Let us calculate $\frac{\partial \psi}{\partial \mathbf{F}}$ in terms of Φ from (3.18) in component form (Cartesian coordinates). Remembering the convention for the derivative (3.3), we have

$$\left(\frac{\partial \Phi}{\partial \mathbf{F}}\right)_{ij} \equiv \frac{\partial \Phi}{\partial F_{ji}} = \frac{\partial \psi}{\partial F_{ji}} + \frac{\partial \psi}{\partial B_k} \frac{\partial B_k}{\partial F_{ji}}. \tag{3.20}$$

But from (3.7) we have

$$\frac{\partial B_l}{\partial F_{ji}} = -J^{-2} \frac{\partial J}{\partial F_{ji}} F_{km} B_{l_m} + J^{-1} \delta^{ji}_{km} B_{l_m}, \tag{3.21}$$

where B_{l_m} is the component m of the vector $\mathbf{B}_l$ and the meaning of the symbol δ^{ji}_{km} is given as follows

$$\delta^{ji}_{km} \equiv \begin{cases} 1 & \text{if} \quad j = k \quad \text{and} \quad i = m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.22}$$

We have that $\frac{\partial J}{\partial F_{ji}} = J \overset{-1}{F}_{ij}$ [78] (where $\overset{-1}{F}_{ij}$ is the component ij of the tensor $\mathbf{F}^{-1}$), then

$$\frac{\partial \psi}{\partial B_k} \frac{\partial B_k}{\partial F_{ji}} = - \overset{-1}{F}_{ij} \frac{\partial \psi}{\partial B_k} J^{-1} F_{km} B_{l_m} + J^{-1} \delta^{ji}_{km} \frac{\partial \psi}{\partial B_k} B_{l_m}. \tag{3.23}$$

But $J^{-1} F_{km} B_{l_m} \equiv B_k$ and $\delta^{ji}_{km} \frac{\partial \psi}{\partial B_k} B_{l_m} = \frac{\partial \psi}{\partial B_j} B_{l_i}$. As a result (3.20) is equivalent to

$$\frac{\partial \Phi}{\partial \mathbf{F}} = \frac{\partial \psi}{\partial \mathbf{F}} - \mathbf{F}^{-1} \left(\frac{\partial \psi}{\partial \mathbf{B}} \cdot \mathbf{B} \right) + J^{-1} \mathbf{B}_l \otimes \frac{\partial \psi}{\partial \mathbf{B}}, \tag{3.24}$$

and then

$$\rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}} = \rho \mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}} + \left(\rho \frac{\partial \psi}{\partial \mathbf{B}} \cdot \mathbf{B} \right) \mathbf{I} - J^{-1} \mathbf{F} \mathbf{B}_l \otimes \rho \frac{\partial \psi}{\partial \mathbf{B}}. \tag{3.25}$$

From $(3.2)_1$ we finally have

$$\sigma = \rho \mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}} - (\mathbf{M} \cdot \mathbf{B}) \mathbf{I} + \mathbf{B} \otimes \mathbf{M}. \tag{3.26}$$

Also, for the magnetization from (3.18) we have

$$\frac{\partial \psi}{\partial B_i} = \frac{\partial \Phi}{\partial B_{l_k}} \frac{\partial B_{l_k}}{\partial B_i}. \tag{3.27}$$

But

$$\frac{\partial \mathbf{B}_l}{\partial \mathbf{B}} = J \mathbf{F}^{-1}, \tag{3.28}$$

and therefore

$$\frac{\partial \psi}{\partial \mathbf{B}} = J\mathbf{F}^{-T}\frac{\partial \Phi}{\partial \mathbf{B}_l}. \tag{3.29}$$

As a result, from $(3.2)_2$,

$$\mathbf{M} = -\rho J\mathbf{F}^{-T}\frac{\partial \Phi}{\partial \mathbf{B}_l}. \tag{3.30}$$

But from (2.21) $\rho = J^{-1}\rho_o$, using (3.13) we finally get [33]

$$\mathbf{M}_l = -\rho_o\frac{\partial \Phi}{\partial \mathbf{B}_l}. \tag{3.31}$$

With all the simplification stated in the introduction of this chapter, we have that the particular form of the balance of linear momentum (2.109) is (remember we do not consider non-magnetic body forces and we work with the quasi-static case)

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}_e = \mathbf{0}, \tag{3.32}$$

where in this case the magnetic force is given as (2.110)

$$\mathbf{f}_e = \mu_o^{-1}(\operatorname{grad}\mathbf{B})^T\mathbf{M}. \tag{3.33}$$

The factor μ_o^{-1} is sometimes incorporated in the definition of the magnetization, in which case we speak about the 'effective magnetization' $\mathbf{M}_e$, such that $\mathbf{M}_e \equiv \mu_o\mathbf{M}$ [33], then (3.33) can be replaced by (we do not use the notation $\mathbf{M}_e$)

$$\mathbf{f}_e = (\operatorname{grad}\mathbf{B})^T\mathbf{M}. \tag{3.34}$$

Proposition 3.1. *The magnetic body force* $\mathbf{f}_e$ *can be written as the divergence of the following second order tensor [33, 103]*

$$\mu_o^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right] + (\mathbf{M}\cdot\mathbf{B})\mathbf{I} - \mathbf{B}\otimes\mathbf{M}. \tag{3.35}$$

Proof. We want to show that [2]

$$\mathbf{f}_e = \operatorname{div}\left\{\mu_o^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right] + (\mathbf{M}\cdot\mathbf{B})\mathbf{I} - \mathbf{B}\otimes\mathbf{M}\right\}. \tag{3.36}$$

In index notation and expanding we for (3.36) (Cartesian coordinates)

$$f_{e_j} = \mu_o^{-1}(B_{i,i}B_j + B_iB_{j,i} - B_iB_{i,j}) + M_{i,j}B_i + M_iB_{i,j} - B_{i,i}M_j - B_iM_{j,i}, \tag{3.37}$$

[2]As with the case of the definition of the derivative of ψ with respect to $\mathbf{F}$, here for the divergence operator we adopt the convention of taking the derivative with respect to the first index [78]. Note that in Subsection 2.2.2 we used a different convention for the divergence operator.

but using the Gauss law of magnetism $(2.102)_2$ and after some manipulations we get

$$f_{e_j} = B_i \mu_o^{-1}(B_{j,i} - B_{i,j}) + M_{i,j}B_i - B_i M_{j,i} + M_i B_{i,j}. \tag{3.38}$$

From (3.12) we can replace the term $\mu_o^{-1}(B_{j,i} - B_{i,j})$ of the right side of the above equations as

$$f_{e_j} = B_i(M_{j,i} - M_{i,j} + H_{j,i} - H_{i,j}) + M_{i,j}B_i - B_i M_{j,i} + M_i B_{i,j}, \tag{3.39}$$

which by using the simplified form of the Ampere law $\text{curl}\mathbf{H} = \mathbf{0}$ (see equation $(2.102)_1$), and after some manipulations becomes

$$f_{e_i} = B_{i,j} M_i, \tag{3.40}$$

which in index notation is the same as (3.34).

□

Since we can express $\mathbf{f}_e$ as the divergence of a tensor, then from the equilibrium equation (3.32) we can define the 'total stress tensor' $\boldsymbol{\tau}$ as [33]

$$\boldsymbol{\tau} \equiv \boldsymbol{\sigma} + \mu_o^{-1}\left[\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}\right] + (\mathbf{M} \cdot \mathbf{B})\mathbf{I} - \mathbf{B} \otimes \mathbf{M}, \tag{3.41}$$

and from (3.26) we finally get

$$\boldsymbol{\tau} = \rho \mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}} + \mu_o^{-1}\left[\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}\right]. \tag{3.42}$$

Definition 3.1. *In magnetostatics the Maxwell stress tensor* $\boldsymbol{\tau}_m$ *is defined as [69, 70]*

$$\boldsymbol{\tau}_m = \mu_o^{-1}\left[\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}\right]. \tag{3.43}$$

For vacuum (free space) we have the linear relation $(2.106)_2$ $\mathbf{B} = \mu_o \mathbf{H}$, and we have the alternative expression for $\boldsymbol{\tau}_m$

$$\boldsymbol{\tau}_m = \mu_o\left[\mathbf{H} \otimes \mathbf{H} - \frac{1}{2}(\mathbf{H} \cdot \mathbf{H})\mathbf{I}\right]. \tag{3.44}$$

From (3.42) if there is no material we conclude that the total stress is equivalent to the Maxwell stress.

Remark The concepts of the Maxwell and total stresses are controversial [89]; from the mathematical point of view there is no problem with this definition, but, what is the physical meaning of the Maxwell stress especially outside the body?. As we will see in the next chapter, the Maxwell stress calculated using the field outside the body (just over the surface of it) must be added as an external load [17]. The Maxwell stress only has a meaning when it is used in order to calculate the surface traction on the surface of a body, and then the total force is computed by integrating this vector field over the total surface of the body. The question is, how do we formulate the boundary value problem when there are magnetic fields, mechanical loads and restrictions on the displacement (mixed boundary condition)?; the answer is not so simple. From the variational formulation of Bustamante et al. [17] we have to consider the Maxwell stress as an external load for the boundary condition for the stress, but that formulation was based on the assumption of a body completely surrounded by a free space; but, what is the real situation when we have a mechanical surface traction and a restriction on the displacement?. The answer is that in such cases we have that our body is interacting directly with the surface of an 'external body', and as a result the assumption of a body completely surrounded by a free space does not hold anymore. For the sake of simplicity for all the boundary value problems presented in this thesis we assume that the bodies are totally surrounded by a free space. A discussion on the boundary condition for the magneto- and electro-elastic problem has been presented in [16].

3.3.1 Symmetry condition for Φ and balance of angular momentum

From (2.111) the local form for the balance of angular momentum in magnetostatics is [3]

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + \mathbf{M} \times \mathbf{B} = \mathbf{0}. \tag{3.45}$$

From here we see that in general the Cauchy stress tensor is not symmetric. Let us study the 'symmetry condition' for the function Φ. From (3.26) replacing the expression for $\boldsymbol{\sigma}$ in terms of Φ in (3.45) we get

$$\boldsymbol{\varepsilon} : \left(\rho \mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}} - (\mathbf{M} \cdot \mathbf{B}) \mathbf{I} + \mathbf{B} \otimes \mathbf{M} \right) + \mathbf{M} \times \mathbf{B} = \mathbf{0}, \tag{3.46}$$

but $\boldsymbol{\varepsilon} : \mathbf{I} = \mathbf{0}$ and $\boldsymbol{\varepsilon} : (\mathbf{B} \otimes \mathbf{M}) = -\mathbf{M} \times \mathbf{B}$, and as a result we have

$$\boldsymbol{\varepsilon} : \left(\mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}} \right) = \mathbf{0}, \tag{3.47}$$

which means that $\mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}}$ is symmetric. It follows from (3.42) that (unlike $\boldsymbol{\sigma}$) $\boldsymbol{\tau}$ is symmetric; see the paper by Steigmann [103] for a discussion of the symmetry and restrictions on Φ in the context of his theory of MS elastomers.

[3] The factor μ_o^{-1} is included in the definition; see page 58 for a discussion about this.

3.3.2 The total energy function

From the definition (3.42) we have that the total nominal stress tensor $\mathbf{T}$ associated with $\boldsymbol{\tau}$ is given by (equation (2.33))

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau} = \rho_o\frac{\partial\Phi}{\partial\mathbf{F}} - \mu_o^{-1}\left[J\mathbf{F}^{-1}\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})J\mathbf{F}^{-1}\right]. \tag{3.48}$$

Proposition 3.2. *The following identity holds*

$$\frac{\partial}{\partial\mathbf{F}}(J\mathbf{B}\cdot\mathbf{B}) = 2J\mathbf{F}^{-1}\mathbf{B}\otimes\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})J\mathbf{F}^{-1}, \tag{3.49}$$

where $\frac{\partial}{\partial\mathbf{F}}$ *is at fixed* $\mathbf{B}_l$.

Proof. In component form the left side of (3.49) is

$$\begin{aligned}\left[\frac{\partial}{\partial\mathbf{F}}(J\mathbf{B}\cdot\mathbf{B})\right]_{ij} &\equiv \frac{\partial}{\partial F_{ji}}(JB_kB_k),\\ &= \frac{\partial J}{\partial F_{ji}}B_kB_k + 2JB_k\frac{\partial B_k}{\partial F_{ji}},\end{aligned} \tag{3.50}$$

but from [78] we have that

$$\frac{\partial J}{\partial F_{ji}} = J\overset{-1}{F}_{ij}, \tag{3.51}$$

where $\overset{-1}{F}_{ij}$ is the component ij os the tensor $\mathbf{F}^{-1}$. Also, from the definition (3.7), $\mathbf{B} = J^{-1}\mathbf{F}\mathbf{B}_l$, substituting in the derivative $\frac{\partial B_k}{\partial F_{ji}}$ of the right side of (3.50), we have

$$\begin{aligned}\frac{\partial B_k}{\partial F_{ji}} &= \frac{\partial}{\partial F_{ji}}(J^{-1}F_{km}B_{l_m}),\\ &= -J^{-1}\overset{-1}{F}_{ij}F_{km}B_{l_m} + J^{-1}\delta^{ji}_{km}B_{l_m}.\end{aligned} \tag{3.52}$$

Then for (3.50) we get (for the meaning of the symbol δ^{ji}_{km} see (3.22))

$$\frac{\partial}{\partial F_{ji}}(JB_kB_k) = -J\overset{-1}{F}_{ij}B_kB_k + 2B_{l_i}B_j, \tag{3.53}$$

and we have that

$$\frac{\partial}{\partial\mathbf{F}}(J\mathbf{B}\cdot\mathbf{B}) = -J\mathbf{F}^{-1}(\mathbf{B}\cdot\mathbf{B}) + 2\mathbf{B}_l\otimes\mathbf{B}, \tag{3.54}$$

$$= J\mathbf{F}^{-1}[-(\mathbf{B}\cdot\mathbf{B})\mathbf{I} + 2\mathbf{B}\otimes\mathbf{B}], \tag{3.55}$$

and so we have proved (3.49).

□

From (3.49) we have in (3.48)

$$\mathbf{T} = \frac{\partial}{\partial\mathbf{F}}\left(\rho_o\Phi + \frac{\mu_o^{-1}}{2}J\mathbf{B}\cdot\mathbf{B}\right). \tag{3.56}$$

Definition 3.2. *The amended free energy function Ω (sometimes called the total energy function) is defined as [33]*

$$\Omega \equiv \rho_o \Phi + \frac{\mu_o^{-1}}{2} J\mathbf{B} \cdot \mathbf{B}. \tag{3.57}$$

From the above definition we have that

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}. \tag{3.58}$$

Then from (3.48)

$$\tau = J^{-1}\mathbf{F}\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}. \tag{3.59}$$

For the magnetization, from (3.31) we had $\mathbf{M}_l = -\rho_o \frac{\partial \Phi}{\partial \mathbf{B}_l}$, and from (3.57) and (3.7)

$$-\rho_o \frac{\partial \Phi}{\partial \mathbf{B}_l} = -\frac{\partial \Omega}{\partial \mathbf{B}_l} - \frac{\mu_o^{-1}}{2}\frac{\partial}{\partial \mathbf{B}_l}\left[J^{-1}\mathbf{B}_l \cdot (\mathbf{c}\mathbf{B}_l)\right]. \tag{3.60}$$

Now, in the second term of the right side of the above equation we have that $\mathbf{F}$ and $\mathbf{B}_l$ are independent variables. Therefore, from (3.60), we get

$$\mathbf{M}_l = -\frac{\partial \Omega}{\partial \mathbf{B}_l} + \mu_o^{-1} J^{-1}\mathbf{c}\mathbf{B}_l, \tag{3.61}$$

but from (3.14) we can replace $\mathbf{M}_l$ by $\mathbf{M}_l = \mu_o^{-1} J^{-1}\mathbf{c}\mathbf{B}_l - \mathbf{H}_l$, and as a result in (3.61) we have

$$\mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}. \tag{3.62}$$

Together, equations (3.58) and (3.62) constitute the constitutive equations for magnetoelastic material.

3.3.3 An alternative formulation

In the previous formulation for the energy, $\mathbf{B}_l$ was the independent magnetic variable. Now if we choose $\mathbf{H}_l$ as the independent variable, we can define the complementary energy function $\Omega^*(\mathbf{F}, \mathbf{H}_l)$ through the partial Legendre transformation as

$$\Omega^*(\mathbf{F}, \mathbf{H}_l) = \Omega(\mathbf{F}, \mathbf{B}_l) - \mathbf{H}_l \cdot \mathbf{B}_l. \tag{3.63}$$

Let's take the derivative of (3.63) with respect to $\mathbf{F}$ at fixed $\mathbf{H}_l$ and $\mathbf{B}_l$, we obtain $\frac{\partial \Omega^*}{\partial \mathbf{F}} = \frac{\partial \Omega}{\partial \mathbf{F}}$, and we have that (see equation (3.58))

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}. \tag{3.64}$$

If we take now the derivative of (3.63) with respect to $\mathbf{H}_l$ at $\mathbf{F}$ and $\mathbf{B}_l$ fixed we get

$$\mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}. \tag{3.65}$$

3.4 Boundary conditions

As was discussed previously (see the remark on page 60), we assume for the boundary value problem that the body is completely surrounded by a free space. The external magnetic induction or magnetic field is applied far away on the external boundary of this free space. From (2.104) the boundary conditions in the current configuration for the magnetic variables are[4]

$$[\![\mathbf{H}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{B}]\!] \cdot \mathbf{n} = 0. \tag{3.66}$$

With the global forms of the Maxwell equations in the reference configuration $\int_{\mathcal{S}_r} \mathbf{B}_l \cdot \mathbf{N}\, \mathrm{d}A = 0$ and $\oint_{\mathcal{C}_r} \mathbf{H}_l \cdot \mathrm{d}\mathbf{R} = 0$ it is possible to show that the Lagrangian counterpart of (3.66) are [33, 107]

$$[\![\mathbf{H}_l]\!] \times \mathbf{N} = \mathbf{0}, \quad [\![\mathbf{B}_l]\!] \cdot \mathbf{N} = 0, \tag{3.67}$$

where $\mathbf{n}$ and $\mathbf{N}$ are the outward normal vectors to the surface of the body in the current and reference configurations.

Regarding the boundary condition for the stress we have

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \quad [\![\mathbf{T}]\!]\mathbf{N} = \mathbf{0}, \tag{3.68}$$

in the current and the reference configurations respectively.

As was discussed on pages 60 and 61, the boundary condition (3.68) must include the Maxwell stresses (3.43) in the external load; therefore, for the current configuration we would have from $(3.68)_1$

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t} + \boldsymbol{\tau}_m \mathbf{n}, \tag{3.69}$$

where $\mathbf{t}$ is the purely mechanical contribution to the surface traction, and $\boldsymbol{\tau}_m$ was the notation for the Maxwell stresses (see equations (3.43) and (3.44)). The boundary condition (3.69) is assumed valid only if the body is completely surrounded by a free space [17] (see also Chapter 9).

[4] The double brackets $[\![f]\!]$ means the difference of the function at the boundary approaching from outside and inside; this means if f^o is the function evaluated at the surface from outside, and f^i is the function evaluated from inside, then

$$[\![f]\!] \equiv f^o - f^i.$$

3.5 The magnetic scalar and vector potentials in magnetostatics

In Subsection 2.2.2 we discussed the vector and scalar potentials in the general context of electromagnetic fields. In magnetostatics (when there is no electric field, no free charge or current, and no time dependence), the Maxwell equations are (2.102):

$$\operatorname{curl}\mathbf{H} = \mathbf{0}, \quad \operatorname{div}\mathbf{B} = 0. \tag{3.70}$$

A solution of the equation $(3.70)_1$ is

$$\mathbf{H} = -\operatorname{grad}\varphi, \tag{3.71}$$

where φ is a scalar potential. This solution is not unique since $\varphi + c_o$ where c_o is a constant is also a solution for $(3.70)_1$. The boundary condition $(3.66)_1$ implies that

$$[\![\operatorname{grad}\varphi]\!] \cdot \mathbf{r} = 0, \tag{3.72}$$

where $\mathbf{r}$ is a vector tangent to the surface at the point where $\operatorname{grad}\varphi$ is evaluated. The condition (3.72) implies that the directional derivative of $[\![\operatorname{grad}\varphi]\!]$ in the direction $\mathbf{r}$ is zero, which since $\mathbf{r}$ is arbitrary, is equivalent to

$$[\![\varphi]\!] = K, \tag{3.73}$$

where K is a constant and it may be assumed to be zero, then we finally have the boundary condition (see, for example, [107])

$$[\![\varphi]\!] = 0. \tag{3.74}$$

A solution of $(3.70)_2$ is

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{3.75}$$

where $\mathbf{A}$ is called the vector potential. It is a vector field, therefore is affected by a change in the reference configuration. In order to see how this vector changes with respect to a pull back to the reference configuration, let's consider the following surface integral, where the surface S is assumed open with boundary C

$$\int_S \mathbf{B} \cdot \mathbf{n}\,\mathrm{d}a = \int_S \operatorname{curl}\mathbf{A} \cdot \mathbf{n}\,\mathrm{d}a = \oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{r}. \tag{3.76}$$

where use has been made of Stokes' theorem. Now, consider the transformation of the last integral of the above expression back to the reference configuration, using $\mathrm{d}\mathbf{r} = \mathbf{F}\mathrm{d}\mathbf{R}$, and the symbol $\mathcal{C}_r$ in order to denote this closed curve in the reference configuration. We have

$$\oint_{\mathcal{C}} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = \oint_{\mathcal{C}_r} \mathbf{A} \cdot \mathbf{F}\mathrm{d}\mathbf{R} = \oint_{\mathcal{C}_r} \mathbf{F}^T\mathbf{A} \cdot \mathrm{d}\mathbf{R} = \int_{\mathcal{S}_o} \mathrm{Curl}\,\mathbf{A}_l \cdot \mathbf{N}\,\mathrm{d}A. \tag{3.77}$$

where we have defined the Lagrangian form of $\mathbf{A}$ as

$$\mathbf{A}_l = \mathbf{F}^T\mathbf{A}, \tag{3.78}$$

and it is easy to show that

$$\mathbf{B}_l = \mathrm{Curl}\,\mathbf{A}_l. \tag{3.79}$$

There is an important question now, What is the continuity condition for $\mathbf{A}$? And this question brings one more question. Is it necessary then to work with two different vector potentials, one for the body and one for the free surrounding space?.

Consider the small pillbox of Figure 3.1, which surrounds a portion of the body and the free space just in the boundary of the body.

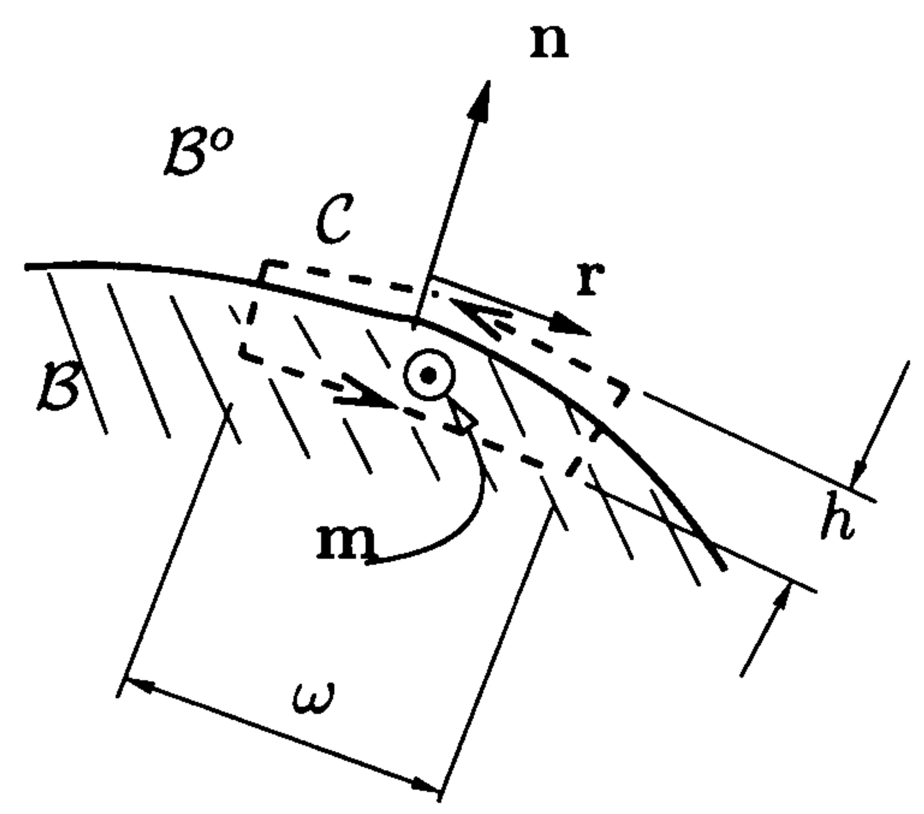


Figure 3.1: 'Pillbox' used in the determination of the boundary condition for the vector potential.

Consider now the following surface integral over $\mathcal{S}$, which is the open surface surrounded by the curve $\mathcal{C}$

$$\int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = \int_{\mathcal{S}} \mathrm{curl}\,\mathbf{A} \cdot \mathrm{d}\mathbf{a} = \oint_{\mathcal{C}} \mathbf{A} \cdot \mathrm{d}\mathbf{r}, \tag{3.80}$$

where again we have used Stokes' theorem. For a small circuit such as the one shown in Figure 3.1, the integral on the left side of the above expression may be approximated as $h\omega\mathbf{B} \cdot \mathbf{m}$, where $\mathbf{m}$ is a vector normal to $\mathcal{S}$. Meanwhile the integral on the right side may be approximated as $-\mathbf{A}^o \cdot \omega\mathrm{r} + \mathbf{A} \cdot \omega\mathbf{r}$, then we have

$$h\omega\mathbf{B} \cdot \mathbf{m} \approx \omega(\mathbf{A} - \mathbf{A}^o) \cdot \mathbf{r}, \tag{3.81}$$

which, as $h \to 0$, implies that

$$[\![\mathbf{A}]\!] \cdot \mathbf{r} = 0, \tag{3.82}$$

which means that the tangential component of $\mathbf{A}$ must be continuous.

An additional condition (which does not hold for all potentials) may be obtained by studying further the equation $\mathbf{B} = \operatorname{curl}\mathbf{A}$. This equation is also satisfied by a field $\mathbf{A}'$ such that $\mathbf{A}' = \mathbf{A} + \operatorname{grad}\xi$, where ξ is a scalar field. Let assume now that

$$\operatorname{div}\mathbf{A}' = 0. \tag{3.83}$$

The above equation holds if only if

$$\operatorname{div}(\operatorname{grad}\xi) = -\operatorname{div}\mathbf{A}. \tag{3.84}$$

It can be proved that the above partial differential equation has always a solution for ξ. Then for a given magnetic induction $\mathbf{B}$, we can always find a vector potential $\mathbf{A}$, such that $\mathbf{B} = \operatorname{curl}\mathbf{A}$ and $\operatorname{div}\mathbf{A} = 0$. It is necessary to point out that it is not mandatory to work with such a potential, but, as we will see, this assumption permits us to work with only one potential for the body and the free space.

The condition $\operatorname{div}\mathbf{A} = 0$ implies that for any volume (for example the circuit in Figure 3.1 but now considered as a three dimensional object) with the use of the divergence theorem we have

$$\int_{\mathcal{V}} \operatorname{div}\mathbf{A}\, dv = \int_{\partial\mathcal{V}} \mathbf{A} \cdot \mathbf{da} = 0, \tag{3.85}$$

and from this last condition, it is easy to show that

$$[\![\mathbf{A}]\!] \cdot \mathbf{n} = 0. \tag{3.86}$$

So with (3.82) and the above equation we have that the vector potential is continuous:

$$[\![\mathbf{A}]\!] = \mathbf{0}. \tag{3.87}$$

3.6 The boundary value problem

Consider the following summary of the main results of the Dorfmann and Ogden's theory for MS elastomers presented in this section.

We have to solve the following system of partial differential equations (in the current configuration)

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{curl}\mathbf{H} = \mathbf{0}, \quad \operatorname{div}\mathbf{B} = 0, \tag{3.88}$$

with boundary conditions

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \quad [\![\mathbf{H}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{B}]\!] \cdot \mathbf{n} = 0. \tag{3.89}$$

The body is assumed to be completely surrounded by a free space (see Figure 3.2).

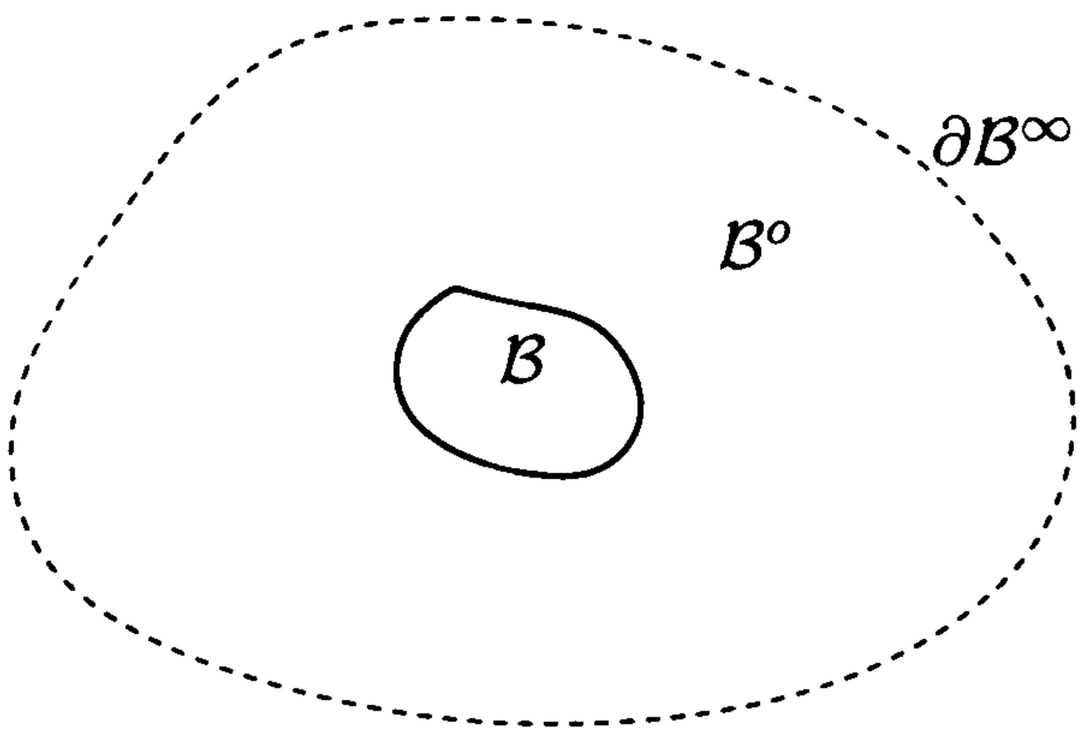


Figure 3.2: The magneto-elastic problem

We need to solve $(3.88)_2$ and $(3.88)_3$ for the body $\mathcal{B}$ and the free space $\mathcal{B}^o$, and we only need to solve $(3.88)_1$ for the body since for $\mathcal{B}^o$ we have $\boldsymbol{\tau} = \boldsymbol{\tau}_m$, and if $(3.88)_2$ and $(3.88)_3$ hold then it is easy to prove that $\operatorname{div}\boldsymbol{\tau}_m = \mathbf{0}$ also holds.

The partial differential equations (3.88) are coupled. If we choose to work with $\mathbf{B}$ as the magnetic independent variable, then from Section 3.5, we have that $\mathbf{B} = \operatorname{curl}\mathbf{A}$ was a solution for $(3.88)_3$, and as a result we would need to solve

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{curl}\mathbf{H} = \mathbf{0}, \tag{3.90}$$

where from (3.59) $\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}$ and from (3.62) and (3.9) $\mathbf{H} = \mathbf{F}^{-T}\frac{\partial\Omega}{\partial\mathbf{B}_l}$. With the boundary conditions (we may assume that the vector potential is continuous across $\partial\mathcal{B}$ and we use the same notation for the whole space)

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \quad [\![\mathbf{H}]\!] \times \mathbf{n} = \mathbf{0}. \tag{3.91}$$

Therefore, as a summary, we look for $\boldsymbol{\chi}$ and $\mathbf{A}$ (vector fields) such that

$$\operatorname{div}\left(J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}\right) = \mathbf{0}, \quad \operatorname{curl}\left(\mathbf{F}^{-T}\frac{\partial\Omega}{\partial\mathbf{B}_l}\right) = \mathbf{0}, \quad \mathbf{x} \in \mathcal{B}, \tag{3.92}$$

and[5]

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \mathbf{0}, \quad \mathbf{x} \in \mathcal{B}^o, \tag{3.93}$$

where $\mathbf{F} \equiv \frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}$, $\mathbf{B} \equiv \operatorname{curl}\mathbf{A}$, $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$ and $\mathbf{B}_l \equiv J\mathbf{F}^{-1}\mathbf{B}$.

[5]For free space we had the linear relation $\mathbf{B}^o = \mu_o\mathbf{H}^o$.

In the case we choose to work with $\mathbf{H}$ as the independent magnetic variable, if φ is the scalar potential for $\mathbf{H}$, it is easy to show we would need to solve the system

$$\operatorname{div}\left(J^{-1}\mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{F}}\right)=\mathbf{0},\quad \operatorname{div}\left(J^{-1}\mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{H}_l}\right)=0,\quad \mathbf{x}\in\mathcal{B}, \tag{3.94}$$

and

$$\operatorname{div}\operatorname{grad}\varphi=0,\quad \mathbf{x}\in\mathcal{B}^o, \tag{3.95}$$

where $\mathbf{H}\equiv-\operatorname{grad}\varphi$, $\Omega^*=\Omega^*(\mathbf{F},\mathbf{H}_l)$ and $\mathbf{H}_l\equiv\mathbf{F}^T\mathbf{H}$. The scalar potential φ is continuous and we use the notation φ for the whole space.

To work with $\mathbf{B}_l$ or $\mathbf{H}_l$ as the independent magnetic variables is not the same from the point of view of the restrictions we would have to impose on Ω and Ω^* in order to have a solution for the equilibrium and the Maxwell equations [33–35].

We will discuss more about the boundary value problem in the following chapter.

Chapter 4

Isotropic magneto-active elastomers

In this chapter we restrict our attention to the case where the magneto-active particles are distributed randomly in the elastomer [62, 120]. These materials are called isotropic MS elastomers.

Despite the above name and the random distribution of particles, the application of an external magnetic field implies that in some respects these materials behave like a 'normal' (non-magnetic) transversely isotropic solid [12, 30–34].

Consider the total energy function $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$ (3.57) and the constitutive equations (3.59) and (3.62) (in this last case for $\mathbf{H}$ given in (3.9))

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-T}\frac{\partial\Omega}{\partial\mathbf{B}_l}. \tag{4.1}$$

For the case of an incompressible material we would have for the stress [33]

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{I}. \tag{4.2}$$

In the case of an isotropic MS elastomer we have that Ω depends on six invariants (compressible case) [33, 102, 127], i.e.

$$\Omega = \Omega(I_1, I_2, I_3, I_4, I_5, I_6), \tag{4.3}$$

where the invariants I_k, $k = 1, \ldots, 6$ are given as

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det\mathbf{c}, \tag{4.4}$$

$$I_4 = \mathbf{B}_l \cdot \mathbf{B}_l, \quad I_5 = \mathbf{B}_l \cdot \mathbf{c}\mathbf{B}_l, \quad I_6 = \mathbf{B}_l \cdot \mathbf{c}^2\mathbf{B}_l. \tag{4.5}$$

Consider the following derivatives of the invariants in terms of the deformation gradient (remember the convention for the derivative (3.3))

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{F}\mathbf{F}^T), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{4.6}$$

$$\frac{\partial I_5}{\partial \mathbf{F}} = 2\mathbf{B}_l \otimes \mathbf{F}\mathbf{B}_l, \quad \frac{\partial I_6}{\partial \mathbf{F}} = 2(\mathbf{B}_l \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{B}_l + \mathbf{F}^T\mathbf{F}\mathbf{B}_l \otimes \mathbf{F}\mathbf{B}_l), \tag{4.7}$$

and the derivatives of the invariants in terms of the Lagrangian magnetic induction

$$\frac{\partial I_4}{\partial \mathbf{B}_l} = 2\mathbf{B}_l, \quad \frac{\partial I_5}{\partial \mathbf{B}_l} = 2\mathbf{c}\mathbf{B}_l, \quad \frac{\partial I_6}{\partial \mathbf{B}_l} = 2\mathbf{c}^2\mathbf{B}_l. \tag{4.8}$$

Then from the chain rule, we have that $\frac{\partial \Omega}{\partial \mathbf{F}}$ and $\frac{\partial \Omega}{\partial \mathbf{B}_l}$ are given as

$$\frac{\partial \Omega}{\partial \mathbf{F}} = \sum_{k=1}^{6} \frac{\partial \Omega}{\partial I_k}\frac{\partial I_k}{\partial \mathbf{F}}, \quad \frac{\partial \Omega}{\partial \mathbf{B}_l} = \sum_{k=1}^{6} \frac{\partial \Omega}{\partial I_k}\frac{\partial I_k}{\partial \mathbf{B}_l}, \tag{4.9}$$

and as a result, from (4.1) and the above derivatives, we have the explicit forms of $\boldsymbol{\tau}$ and[1] $\mathbf{H}$

$$\boldsymbol{\tau} = 2J^{-1}[\mathbf{b}\Omega_1 + (I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 + I_3\Omega_3 + J^2\mathbf{B}\otimes\mathbf{B}\Omega_5 \\ + J^2(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B})\Omega_6], \tag{4.10}$$

$$\mathbf{H} = 2J(\mathbf{b}^{-1}\mathbf{B}\Omega_4 + \mathbf{B}\Omega_5 + \mathbf{b}\mathbf{B}\Omega_6). \tag{4.11}$$

In the case of an incompressible material (2.1.4) we have $J = 1$ ($I_3 = 1$), and we omit this invariant in this case. From (4.2) we have (see, for example, [33]) for the stress and the magnetic field the following expressions

$$\boldsymbol{\tau} = 2\mathbf{b}\Omega_1 + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 - p\mathbf{I} + 2\mathbf{B}\otimes\mathbf{B}\Omega_5 \\ + 2(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B})\Omega_6, \tag{4.12}$$

$$\mathbf{H} = 2(\mathbf{b}^{-1}\mathbf{B}\Omega_4 + \mathbf{B}\Omega_5 + \mathbf{b}\mathbf{B}\Omega_6). \tag{4.13}$$

Note the similarity of the above equation for the stress with the case of a simple transversely isotropic material [102] (see also [73,74]), the difference in this case is that in general $\mathbf{B}_l \cdot \mathbf{B}_l$ (the fourth invariant) is not unity.

Consider now that we work with the complementary energy function $\Omega^*(\mathbf{F}, \mathbf{H}_l)$. In such a case for the stress and the magnetic induction from (3.64), (3.65) and (3.9) we have [33]

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B} = -J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{H}_l}, \tag{4.14}$$

[1] We use the notation

$$\Omega_k \equiv \frac{\partial \Omega}{\partial I_k} \quad k = 1, ..., 6.$$

and for an incompressible material

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}. \tag{4.15}$$

In the case of an isotropic MS elastomer we have that Ω^* depends on six invariants

$$\Omega^* = \Omega^*(I_1, I_2, I_3, K_4, K_5, K_6), \tag{4.16}$$

where I_2, I_2 and I_3 are given in (4.4), and the new invariants K_4, K_5 and K_6 are defined as [33]

$$K_4 = \mathbf{H}_l \cdot \mathbf{H}_l, \quad K_5 = \mathbf{H}_l \cdot \mathbf{c}\mathbf{H}_l, \quad K_6 = \mathbf{H}_l \cdot \mathbf{c}^2\mathbf{H}_l. \tag{4.17}$$

Consider the derivatives

$$\frac{\partial K_5}{\partial \mathbf{F}} = 2\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l, \quad \frac{\partial K_6}{\partial \mathbf{F}} = 2(\mathbf{H}_l \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{H}_l + \mathbf{F}^T\mathbf{F}\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l). \tag{4.18}$$

and

$$\frac{\partial K_4}{\partial \mathbf{H}_l} = 2\mathbf{H}_l, \quad \frac{\partial K_5}{\partial \mathbf{H}_l} = 2\mathbf{c}\mathbf{H}_l, \quad \frac{\partial K_6}{\partial \mathbf{H}_l} = 2\mathbf{c}^2\mathbf{H}_l. \tag{4.19}$$

As a result, from the chain rule, we obtain for $\boldsymbol{\tau}$ and $\mathbf{B}$ from (4.14) the following explicit expressions in terms of Ω_k^* (here Ω_k^* means a partial derivative on I_k if $k = 1, 2, 3$, or K_k if $k = 4, 5, 6$) [33]

$$\begin{aligned}\boldsymbol{\tau} = 2J^{-1}[\mathbf{b}\Omega_1^* + (I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* + I_3\Omega_3^* + \mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H}\Omega_5^* \\ +2(\mathbf{b}\mathbf{H} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{b}\mathbf{H})\Omega_6^*],\end{aligned} \tag{4.20}$$

$$\mathbf{B} = -2J^{-1}(\mathbf{b}\mathbf{H}\Omega_4^* + \mathbf{b}^2\mathbf{H}\Omega_5^* + \mathbf{b}^3\mathbf{H}\Omega_6^*), \tag{4.21}$$

and for an incompressible material, from (4.15), we get

$$\begin{aligned}\boldsymbol{\tau} = 2\mathbf{b}\Omega_1^* + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* - p^*\mathbf{I} + 2\mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H}\Omega_5^* \\ +2(\mathbf{b}\mathbf{H} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{b}\mathbf{H})\Omega_6^*,\end{aligned} \tag{4.22}$$

$$\mathbf{B} = -(2\mathbf{b}\mathbf{H}\Omega_4^* + 2\mathbf{b}^2\mathbf{H}\Omega_5^* + 2\mathbf{b}^3\mathbf{H}\Omega_6^*). \tag{4.23}$$

4.1 Universal relations

In magneto-elasticity, universal relations are equations connecting the components of the stress, the deformation and the magnetic fields, which hold independently of the specific choice of constitutive law for a family of materials. Such relations provide guidelines for the experimenters, since at the moment of proposing a constitutive equation for a material.

it is possible, through experiment, to test whether a material can be included in a family, as was emphasized by Beatty [6].

In non-linear elasticity (purely elastic context), the universal relation between the shear and normal components of the stress for a cube under simple shear deformation found by Rivlin [90] is well known.

The first systematic way of generating such relations were provided by Hayes and Knops [54], and more recently by Beatty [6]. This last paper is one basis for the general analysis of Pucci and Saccomandi [85] (see also the review articles [93, 94]). Another interesting and recent review has been presented by Rivlin [91].

The papers mentioned above are mainly restricted to the case of isotropic materials, although the paper by Pucci and Saccomandi [85] present a theory that can be applied to other classes of materials. Batra [5] presented some results for universal relations for transversely isotropic materials, which as we will see in this subsection resemble the results found in magneto-elasticity.

Dorfmann et al. [37] and more recently Bustamante et al. [18] have studied the problem of finding universal relations in the context of magneto-elasticity.

4.1.1 Linear universal relations

Universal relations are classified as either linear or non-linear. In the first case the components of the stress tensor appear as linear combinations. Here we treat the problem of finding linear universal relations for our formulation of the MS elastomers; this section is mainly based on the results given in [18].

In order to derive linear universal relations, consider the constitutive equations (4.12) and (4.13), and let us introduce the notation

$$\gamma_1 = 2(\Omega_1 + \Omega_2 I_1), \quad \gamma_2 = -2\Omega_2, \quad \gamma_4 = 2\Omega_4, \quad \gamma_5 = 2\Omega_5, \quad \gamma_6 = 2\Omega_6. \tag{4.24}$$

As a result (4.12) and (4.13) can be written in the compact forms

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1\mathbf{b} + \gamma_2\mathbf{b}^2 + \gamma_5\mathbf{B}\otimes\mathbf{B} + \gamma_6(\mathbf{B}\otimes\mathbf{bB} + \mathbf{bB}\otimes\mathbf{B}), \tag{4.25}$$

$$\mathbf{H} = \gamma_4\mathbf{b}^{-1}\mathbf{B} + \gamma_5\mathbf{B} + \gamma_6\mathbf{bB}. \tag{4.26}$$

Following Dorfmann at al. [37] we form the antisymmetric tensor

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \gamma_5(\mathbf{B}\otimes\mathbf{bB} - \mathbf{bB}\otimes\mathbf{B}) + \gamma_6(\mathbf{B}\otimes\mathbf{b}^2\mathbf{B} - \mathbf{b}^2\mathbf{B}\otimes\mathbf{B}), \tag{4.27}$$

noting that this vanishes when $\mathbf{B}$ is an eigenvector of $\mathbf{b}$.

For any antisymmetric second-order tensor an associated axial vector can be defined. For example, for the tensor $\mathbf{u}\otimes\mathbf{v}-\mathbf{v}\otimes\mathbf{u}$, where $\mathbf{u}$ and $\mathbf{v}$ are two vectors. the axial vector is $\mathbf{v}\times\mathbf{u}$. More generally, if $\mathbf{W}$ is an antisymmetric second-order tensor, we denote by $(\mathbf{W})_\times$ its axial vector[2].

Therefore, the axial vector corresponding to the expression in equation (4.27) has the form

$$(\boldsymbol{\tau}\mathbf{b}-\mathbf{b}\boldsymbol{\tau})_\times = (\gamma_5\mathbf{b}\mathbf{B}+\gamma_6\mathbf{b}^2\mathbf{B})\times\mathbf{B}, \tag{4.28}$$

from which we obtain the universal relation

$$(\boldsymbol{\tau}\mathbf{b}-\mathbf{b}\boldsymbol{\tau})_\times\cdot\mathbf{B}=0. \tag{4.29}$$

This is identical to the universal relation found by Dorfmann at al. in [37].

For the alternative formulation (4.22), (4.23) we use the notation

$$\gamma_1^* = 2(\Omega_1^*+\Omega_2^* I_1),\quad \gamma_2^*=-2\Omega_2^*,\quad \gamma_4^*=2\Omega_4^*,\quad \gamma_5^*=2\Omega_5^*,\quad \gamma_6^*=2\Omega_6^*, \tag{4.30}$$

as a result (4.22) and (4.23) can be rewritten as

$$\boldsymbol{\tau} = -p^*\mathbf{I}+\gamma_1^*\mathbf{b}+\gamma_2^*\mathbf{b}^2+\gamma_5^*(\mathbf{bH}\otimes\mathbf{bH})+\gamma_6^*(\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H}+\mathbf{b}^2\mathbf{H}\otimes\mathbf{bH}), \tag{4.31}$$

$$\mathbf{B} = -(\gamma_4^*\mathbf{bH}+\gamma_5^*\mathbf{b}^2\mathbf{B}+\gamma_6^*\mathbf{b}^3\mathbf{B}), \tag{4.32}$$

from where we form the antisymmetric tensor

$$\boldsymbol{\tau}\mathbf{b}-\mathbf{b}\boldsymbol{\tau} = \gamma_5^*(\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H}-\mathbf{b}^2\mathbf{H}\otimes\mathbf{bH})+\gamma_6^*(\mathbf{bH}\otimes\mathbf{b}^3\mathbf{H}-\mathbf{b}^3\mathbf{H}\otimes\mathbf{bH}), \tag{4.33}$$

and its axial vector

$$(\boldsymbol{\tau}\mathbf{b}-\mathbf{b}\boldsymbol{\tau})_\times = (\gamma_5^*\mathbf{b}^2\mathbf{H}+\gamma_6^*\mathbf{b}^3\mathbf{H})\times\mathbf{bH}, \tag{4.34}$$

from where we obtain the universal relation

$$(\boldsymbol{\tau}\mathbf{b}-\mathbf{b}\boldsymbol{\tau})_\times\cdot(\mathbf{bH})=0. \tag{4.35}$$

[2] For an antisymmetric tensor $\mathbf{W}$ with components W_{ij}, we define the axial vector $(\mathbf{W})_\times$ such that for any vector $\mathbf{a}$ we have

$$\mathbf{Wa}=(\mathbf{W})_\times\times\mathbf{a}.$$

From this definition we have that the vector of components of $(\mathbf{W})_\times$ is given as

$$(\mathsf{W})_\times = (-W_{23}, W_{13}, -W_{12})^T.$$

Proposition 4.1. *The relations (4.29) and (4.35) are equivalent, and can be obtained from [18]*

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \mathbf{B} \otimes \mathbf{bH} - \mathbf{bH} \otimes \mathbf{B}. \tag{4.36}$$

Proof. Consider the right-side of (4.36), using (4.26) we obtain

$$\mathbf{B} \otimes \mathbf{bH} - \mathbf{bH} \otimes \mathbf{B} = \gamma_5(\mathbf{B} \otimes \mathbf{bB} - \mathbf{bB} \otimes \mathbf{B}) + \gamma_6(\mathbf{B} \otimes \mathbf{b}^2\mathbf{B} - \mathbf{b}^2\mathbf{B} \otimes \mathbf{B}), \tag{4.37}$$

which is equal to the right-side of (4.27). Similarly, from (4.32) we have for (4.36)

$$\mathbf{B} \otimes \mathbf{bH} - \mathbf{bH} \otimes \mathbf{B} = \gamma_5^*(\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} - \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH}) + \gamma_6^*(\mathbf{bH} \otimes \mathbf{b}^3\mathbf{H} - \mathbf{b}^3\mathbf{H} \otimes \mathbf{bH}), \tag{4.38}$$

which is equal to the right-side of (4.33).

□

In order to show that the universal relation obtained previously (in either of the two forms presented) is unique for this class of materials, consider, for example, the representation (4.25) [22]

$$\boldsymbol{\tau} = \mathcal{M}^T \boldsymbol{\gamma}, \tag{4.39}$$

where [3]

$$\boldsymbol{\tau} \equiv (\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33})^T, \quad \boldsymbol{\gamma} = (-p, \gamma_1, \gamma_2, \gamma_5, \gamma_6)^T, \tag{4.40}$$

and

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ b_{11} & b_{22} & b_{13} & b_{22} & b_{23} & b_{13} \\ b_{11}^{(2)} & b_{12}^{(2)} & b_{13}^{(2)} & b_{22}^{(2)} & b_{23}^{(2)} & b_{33}^{(2)} \\ \mathbb{B}_{11} & \mathbb{B}_{12} & \mathbb{B}_{13} & \mathbb{B}_{22} & \mathbb{B}_{23} & \mathbb{B}_{33} \\ \mathbb{M}_{11} & \mathbb{M}_{12} & \mathbb{M}_{13} & \mathbb{M}_{22} & \mathbb{M}_{23} & \mathbb{M}_{33} \end{pmatrix}, \tag{4.41}$$

where $b_{ij}^{(2)}$ is the component ij of the tensor $\mathbf{b}^2$, $\mathbb{B}_{ij} \equiv B_iB_j$ is the component ij of the tensor $\mathbf{B} \otimes \mathbf{B}$, and $\mathbb{M}_{ij} \equiv B_ib_{jk}^{(2)}B_k - b_{ik}^{(2)}B_kB_j$ is the component ij of $\mathbf{B} \otimes \mathbf{b}^2\mathbf{B} - \mathbf{b}^2\mathbf{B} \otimes \mathbf{B}$.

Pucci and Saccomandi [85] proved that the number of independent linear universal relations that can be found for a given constitutive law is equal to the number of linearly independent equations (in our case six), minus the rank of the matrix $\mathcal{M}$, which in our case is five. Thus, only one linear universal relation is to be expected in the general case.

[3]We will use a slightly different notation in Section 5.4.

4.1.1.1 Special cases

As we will see in Section 5.3, the problem of finding an appropriate form for Ω from experimental data is in general highly difficult. Two main reasons are the lack of enough experimental data, and the complexity of the function Ω that depends on five invariants (incompressible case). In general it is necessary to propose a simplified form for Ω (see [73, 74] for the equivalent case for pure elastic transversely isotropic materials), which means we work with an energy function Ω that would depend on fewer invariants.

From the point of view of the linear universal relations, the above assumption would mean in some cases that the rank of the matrix $\mathcal{M}$ would be less than five, and as a result we could find more linear universal relations for these special cases; this is very important, because, as mentioned previously, we need criteria in order to know in advance from experiments if the simplifications we have made for Ω are or are not realistic, and we would like to do this without giving more detailed information for the form of the energy function [6].

Here we show universal relations for some simplified forms for Ω; for brevity we do not list the similar results that can be obtained from the alternative formulation based on Ω^*.

Case 1: $\Omega = \Omega(I_1, I_2, I_4, I_5)$

This is the case where the free energy function Ω does not depend on I_6, which is to say $\gamma_6 = 0$ (that is equivalent to $\frac{\partial \Omega}{\partial I_6} = 0$). Equation (4.27) reduces to

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \gamma_5(\mathbf{B} \otimes \mathbf{bB} - \mathbf{bB} \otimes \mathbf{B}), \tag{4.42}$$

from which we obtain the two universal relations

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot \mathbf{B} = 0, \quad (\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot (\mathbf{bB}) = 0. \tag{4.43}$$

Similarly, from equation (4.26) with $\gamma_6 = 0$ we have the expression for the magnetic field $\mathbf{H} = \gamma_4 \mathbf{b}^{-1}\mathbf{B} + \gamma_5 \mathbf{B}$. We obtain

$$(\mathbf{H} \times \mathbf{b}^{-1}\mathbf{B}) \cdot \mathbf{B} = 0, \tag{4.44}$$

which is a relation between the deformation $\mathbf{b}$ and the magnetic field quantities $\mathbf{H}$ and $\mathbf{B}$ and does not involve the stress $\boldsymbol{\tau}$.

Let's now examine some other subcases (see [22]).

(a) $\Omega = \Omega(I_1, I_4, I_5)$

In this case for the total stress, from (4.25), we have

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1\mathbf{b} + \gamma_5\mathbf{B}\otimes\mathbf{B}, \quad (4.45)$$

from where we obtain

$$\boldsymbol{\tau}\mathbf{B}\times\mathbf{B} = \gamma_1\mathbf{b}\mathbf{B}\times\mathbf{B}, \quad (4.46)$$

and we get the extra linear universal relation

$$\boldsymbol{\tau}\mathbf{B}\cdot(\mathbf{B}\times\mathbf{b}\mathbf{B}) = 0. \quad (4.47)$$

(b) $\Omega = \Omega(I_2, I_4, I_5)$

From (4.25) we have

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1\mathbf{b}^2 + \gamma_5\mathbf{B}\otimes\mathbf{B}, \quad (4.48)$$

and following a similar procedure as before we get the linear universal relation

$$\boldsymbol{\tau}\mathbf{B}\cdot(\mathbf{B}\times\mathbf{b}^2\mathbf{B}) = 0. \quad (4.49)$$

(c) $\Omega = \Omega(I_1, I_2, I_5)$

In this case there is no new relation for the stress, but for the magnetic field, from (4.26), we have

$$\mathbf{H} = \gamma_5\mathbf{B}, \quad (4.50)$$

from which we obtain the simple universal relation

$$\mathbf{H}\times\mathbf{B} = \mathbf{0}. \quad (4.51)$$

(d) $\Omega = \Omega(I_1, I_2, I_4)$

For this last situation we have new universal relations for both the stress and the magnetic field. From (4.25), we get

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1\mathbf{b}, \quad (4.52)$$

and we get the universal relations

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \mathbf{0}, \quad \boldsymbol{\tau}\mathbf{B}\cdot(\mathbf{B}\times\mathbf{b}\mathbf{B}) = 0. \quad (4.53)$$

The relation $(4.53)_1$ is the classical result found by Beatty [6], and $(4.53)_2$ is the same as (4.26).

For the magnetic field, from (4.26), we have

$$\mathbf{H} = \gamma_4 \mathbf{b}^{-1}\mathbf{B}, \tag{4.54}$$

and we obtain the relation

$$\mathbf{H} \times \mathbf{b}^{-1}\mathbf{B} = \mathbf{0}. \tag{4.55}$$

Case 2: $\Omega = \Omega(I_1, I_2, I_4, I_6)$

We consider the special case where the energy function Ω does not depend on I_5. From equation (4.27) we obtain

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \gamma_6(\mathbf{B} \otimes \mathbf{b}^2\mathbf{B} - \mathbf{b}^2\mathbf{B} \otimes \mathbf{B}), \tag{4.56}$$

and the corresponding axial vector has the form

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times = \gamma_6(\mathbf{b}^2\mathbf{B}) \times \mathbf{B}. \tag{4.57}$$

We get only two independent universal relations [85], which have the form

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot \mathbf{B} = 0, \quad (\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot (\mathbf{b}^2\mathbf{B}) = 0. \tag{4.58}$$

For this special case, the magnetic field $\mathbf{H}$ in terms of the magnetic induction vector $\mathbf{B}$ is given by

$$\mathbf{H} = \gamma_4 \mathbf{b}^{-1}\mathbf{B} + \gamma_6 \mathbf{b}\mathbf{B}, \tag{4.59}$$

and the additional universal relation, not involving the components of the stress $\boldsymbol{\tau}$, has the form

$$(\mathbf{H} \times \mathbf{b}^{-1}\mathbf{B}) \cdot (\mathbf{b}\mathbf{B}) = 0. \tag{4.60}$$

As before we consider some subcases.

(a) $\Omega = \Omega(I_1, I_4, I_6)$

From (4.26), we have

$$\mathbf{H} = \gamma_6 \mathbf{b}\mathbf{B}, \tag{4.61}$$

and we get the linear universal relation

$$\mathbf{H} \times \mathbf{b}\mathbf{B} = \mathbf{0}. \tag{4.62}$$

(b) $\Omega = \Omega(I_1, I_2, I_6)$

In this case for the total stress we have from (4.25)

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1 \mathbf{b} + \gamma_6(\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B}), \tag{4.63}$$

and it is easy to prove that the extra linear universal relation obtained here is the same as (4.47).

(c) $\Omega = \Omega(I_2, I_4, I_6)$

This case is more complex. From (4.25), we have

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_2 \mathbf{b}^2 + \gamma_6(\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B}), \tag{4.64}$$

from where with the use of the Cayley-Hamilton theorem we can prove that

$$\begin{aligned} \boldsymbol{\tau}(\mathbf{bB}) = -p\mathbf{bB} + \gamma_2(I_1 \mathbf{b}^2\mathbf{B} - I_2 \mathbf{bB} + \mathbf{B}) \\ + \gamma_6(|\mathbf{bB}|^2\mathbf{B} + [\mathbf{B} \cdot (\mathbf{bB})]\mathbf{bB}), \end{aligned} \tag{4.65}$$

and we get

$$(\boldsymbol{\tau}\mathbf{bB} \times \mathbf{B}) \cdot \mathbf{bB} = \gamma_2 I_1 (\mathbf{b}^2\mathbf{B} \otimes \mathbf{B}) \cdot \mathbf{bB}. \tag{4.66}$$

Also

$$\boldsymbol{\tau}\mathbf{B} = -p\mathbf{B} + \gamma_2 \mathbf{b}^2\mathbf{B} + \gamma_6([(\mathbf{bB}) \cdot \mathbf{B}]\mathbf{B} + |\mathbf{B}|^2\mathbf{bB}), \tag{4.67}$$

and we get

$$(\boldsymbol{\tau}\mathbf{B} \times \mathbf{bB}) \cdot \mathbf{B} = \gamma_2(\mathbf{b}^2\mathbf{B} \times \mathbf{B}) \cdot \mathbf{bB}. \tag{4.68}$$

Then, from (4.66) and (4.68), we get the universal relation

$$(\boldsymbol{\tau}\mathbf{bB} - I_1 \boldsymbol{\tau}\mathbf{B}) \cdot (\mathbf{B} \times \mathbf{bB}) = 0. \tag{4.69}$$

Case 3: $\Omega = \Omega(I_1, I_2, I_5, I_6)$

The free energy formulation Ω does not depend on I_4 and the constitutive equation for the magnetic field $\mathbf{H}$ reduces to

$$\mathbf{H} = \gamma_5 \mathbf{B} + \gamma_6 \mathbf{bB}. \tag{4.70}$$

The additional universal relation, again involving the magnetic fields and the deformation $\mathbf{b}$, is given by

$$(\mathbf{H} \times \mathbf{B}) \cdot (\mathbf{bB}) = 0. \tag{4.71}$$

No additional relation is found for the stress components.

For additional simplification of this energy, we found that the only interesting sub-cases are $\Omega = \Omega(I_1, I_5, I_6)$ and $\Omega = \Omega(I_2, I_5, I_6)$; for these simplified forms of the energy it is easy to prove that the additional linear universal relations found are (4.47) and (4.69) respectively.

Case 4: $\Omega = \Omega(I_1, I_4, I_5, I_6)$.

For an energy formulation independent of I_2, the expression for the total stress $\boldsymbol{\tau}$ reduces to

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1 \mathbf{b} + \gamma_5 \mathbf{B} \otimes \mathbf{B} + \gamma_6(\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B}), \tag{4.72}$$

and no additional universal relation can be obtained starting from equation (4.72). An additional non-trivial approach is by considering the antisymmetric tensor $\boldsymbol{\tau}\mathbf{B} \otimes \mathbf{B} - \mathbf{B} \otimes \boldsymbol{\tau}\mathbf{B}$, which leads to

$$\boldsymbol{\tau}\mathbf{B} \otimes \mathbf{B} - \mathbf{B} \otimes \boldsymbol{\tau}\mathbf{B} = \gamma_1(\mathbf{bB} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{bB}) + \gamma_6 \mathbf{B} \cdot \mathbf{B}(\mathbf{bB} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{bB}). \tag{4.73}$$

The corresponding axial vector has the form

$$\begin{aligned}(\boldsymbol{\tau}\mathbf{B} \otimes \mathbf{B} - \mathbf{B} \otimes \boldsymbol{\tau}\mathbf{B})_\times = \boldsymbol{\tau}\mathbf{B} \times \mathbf{B} &= \gamma_1(\mathbf{bB} \times \mathbf{B}) + \gamma_6 \mathbf{B} \cdot \mathbf{B}(\mathbf{bB} \times \mathbf{B}) \\ &= (\gamma_1 + \gamma_6 \mathbf{B} \cdot \mathbf{B})(\mathbf{bB} \times \mathbf{B}),\end{aligned} \tag{4.74}$$

and the additional corresponding universal relation is given by

$$(\boldsymbol{\tau}\mathbf{B} \times \mathbf{B}) \cdot (\mathbf{bB}) = 0, \tag{4.75}$$

which is the same as (4.47).

Case 5: $\Omega = \Omega(I_2, I_4, I_5, I_6)$.

This case is similar to Case 4. For convenience, we write the reduced form of the total stress tensor $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = -p\mathbf{I} + \tilde{\gamma}_2(I_1\mathbf{b} - \mathbf{b}^2) + \gamma_5 \mathbf{B} \otimes \mathbf{B} + \gamma_6(\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B}), \tag{4.76}$$

where we defined $\tilde{\gamma}_2 = 2\Omega_2$. Starting with equation $\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau}$, we obtain the universal relation shown previously in equation (4.29). However, an additional universal relation can be found by considering the expression

$$\begin{aligned}&\boldsymbol{\tau}\mathbf{bB} \otimes \mathbf{bB} - \mathbf{bB} \otimes \boldsymbol{\tau}\mathbf{bB} = \\ &= \tilde{\gamma}_2 \left[I_1(\mathbf{b}^2\mathbf{B} \otimes \mathbf{bB} - \mathbf{bB} \otimes \mathbf{b}^2\mathbf{B}) - (\mathbf{b}^3\mathbf{B} \otimes \mathbf{bB} - \mathbf{bB} \otimes \mathbf{b}^3\mathbf{B})\right] \\ &+ \left[\gamma_5(\mathbf{B} \cdot \mathbf{bB}) + \gamma_6(\mathbf{bB}) \cdot (\mathbf{bB})\right](\mathbf{B} \otimes \mathbf{bB} - \mathbf{bB} \otimes \mathbf{B}).\end{aligned} \tag{4.77}$$

The corresponding axial vector has the form

$$\begin{aligned}\boldsymbol{\tau}\mathbf{bB}\times\mathbf{bB} &= \tilde{\gamma}_2\left[I_1(\mathbf{b}^2\mathbf{B}\times\mathbf{bB})-\mathbf{b}^3\mathbf{B}\times\mathbf{bB}\right]\\ &\quad+\left[\gamma_5(\mathbf{B}\cdot\mathbf{bB})+\gamma_6(\mathbf{bB})\cdot(\mathbf{bB})\right](\mathbf{B}\times\mathbf{bB}),\end{aligned}\tag{4.78}$$

and it follows, on the use of the Cayley-Hamilton theorem in the form

$$\mathbf{b}^3-I_1\mathbf{b}^2+I_2\mathbf{b}-\mathbf{I}=\mathbf{0},\tag{4.79}$$

that

$$\boldsymbol{\tau}\mathbf{bB}\times\mathbf{bB}=\tilde{\gamma}_2(-\mathbf{B}\times\mathbf{bB})+\left[\gamma_5(\mathbf{B}\cdot\mathbf{bB})+\gamma_6(\mathbf{bB})\cdot(\mathbf{bB})\right](\mathbf{B}\times\mathbf{bB}).\tag{4.80}$$

The additional universal relation is then given by

$$(\boldsymbol{\tau}\mathbf{bB}\times\mathbf{bB})\cdot\mathbf{B}=0.\tag{4.81}$$

Remark The linear universal relations shown previously were obtained for the case of an incompressible material (4.12); it is easy to show that the same relations hold for the general case of a compressible (unconstrained) MS elastomer (4.10).

In the case of a constrained material (like an incompressible solid), Pucci and Saccomandi [83] have shown that is possible to find additional universal relations; this is done by an appropriate manipulation of the constraint and by using the 'controllable' solutions for the problem. Consider, for example, the case of an incompressible material; the parameter p is not a property of the material, but a quantity that may be found from the boundary conditions of a boundary value problem. In some cases (see, for example, [83]) p may be found as a linear combination of the material parameters γ_i (equation (4.25)), and as a result the structure of the matrix $\mathcal{M}$ (equation (4.41)) would change, and we would have a matrix of rank less than five. From the theory of Pucci and Saccomandi [85], we would have then an extra linear universal relation.

The theory of Pucci and Saccomandi [83] has been applied, for example, by Saccomandi and Batra [95] for the case of transversely isotropic elastic materials.

We do not explore the possibility of finding more linear universal relations following this theory for MS materials in this thesis.

4.1.1.2 Applications

We have mentioned that the linear universal relations can be used as criteria in order to know whether a material belongs to a particular family of constitutive laws. In order to do so, we need to do experiments and to determine the stresses, the deformation, and the magnetic field, and then to use these results in order to evaluate the universal relations to check whether they hold. These experiments should be done in a rational way, which means we should use 'exact' solutions of the boundary value problem in order to design these experiments. From Section 3.6 we see that the partial differential equations are highly non-linear and that the solution will in general depend strongly on the particular form of the energy function. We need to work with particular kinds of exact solution, which do not depend on the particular form of the energy function for a given family of materials; these solutions or 'deformations' should be obtained by only applying a surface traction, and in the magneto-elastic case a magnetic field, and this is the reason this particular kind of solutions are called 'universal solutions' [4].

The problem of finding all the universal solutions in the purely elastic context was first treated by Ericksen [39] (see also [112]), who found five 'families' of solutions. For the case of non-linear electro-elasticity, Singh and Pipkin [99] found the complete set of solutions[5] for a constitutive law not derived from an energy function, but considering the material as Cauchy-elastic [87]. Pucci and Saccomandi [82] studied the case of universal solutions in magneto-elasticity and electro-elasticity (but assuming now the existence of an energy function); they found that the only universal solutions for these two cases are the same as those listed by Singh and Pipkin [99]. For the case of magneto-elastic solids it is only necessary to replace the electric field and electric displacement, by the magnetic field and the magnetic induction respectively.

Note that in all the examples considered by Singh and Pipkin [99] the matrix of com-

[4] In the theory of non-linear elasticity there are two definitions that deserve an explanation in this thesis, these two concepts are the 'universal solutions' and the 'controllable solutions'.

By 'universal solutions' we mean solutions of the boundary value problem whose forms do not depend on the particular form of the constitutive equation. The partial differential equations in non-linear elasticity are highly non-linear, and the form of the solutions in general depend strongly on the constitutive equations.

By 'controllable solution' we mean a solution of the boundary value problem that can be produced by only applying a surface load. With the semi-inverse method we can find such solutions by solving the partial differential equations for specific forms of the constitutive equation; therefore not all controllable solutions are universal.

[5] The problem of finding all the universal solutions for the purely elastic case is still open; see, for example, [55] and the references therein.

ponents of the left Cauchy-Green tensor $\mathbf{b}$, denoted b, has the form

$$\mathsf{b} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \tag{4.82}$$

or its equivalent obtained by re-ordering the axes, where $*$ denotes a nonzero entry. Thus, the implications of the universal relations for each of the Singh and Pipkin solutions are rather similar in structure and it therefore suffices here to examine a limited number of the Singh-Pipkin solutions in order to illustrate the new results obtainable by invoking the universal relations.

We must remark that the solutions found by Singh and Pipkin [99] are not actually exact. The boundary conditions (3.66) are quite severe, and in order to obtain some exact solutions Singh and Pipkin had to assume, for example, slabs or cylinders very 'large' or 'long'. They called this problem the 'fringe effect'. We will speak in detail about this phenomenon in Section 4.2.

Consider the following two examples.

Homogeneous deformation in a uniform field We consider a slab of uniform thickness limited by top and bottom faces normal to the X_3 direction and with unlimited extent in the X_1 and X_2 directions, where (X_1, X_2, X_3) define the rectangular Cartesian coordinates in the reference configuration $\mathcal{B}_o$. The universal relation in equation (4.29) has been applied to a similar geometry subject to triaxial stretch and simple shear in [37]. We also recall that the solution of the corresponding boundary value problem with an applied magnetic field normal to the top and bottom faces was given in [35].

Here we assume that the slab is subjected to a uniform magnetic field and stretched along the three coordinate axes with stretch ratios μ_1, μ_2, μ_3, respectively and then sheared by amounts κ_1 and κ_2 along the two in-plane directions. The combined triaxial stretch and shear deformation is given by

$$x_1 = \mu_1 X_1 + \kappa_1 \mu_3 X_3, \quad x_2 = \mu_2 X_2 + \kappa_2 \mu_3 X_3, \quad x_3 = \mu_3 X_3, \tag{4.83}$$

where (x_1, x_2, x_3) are the rectangular Cartesian coordinates in the deformed configuration of the material point initially located at (X_1, X_2, X_3) and μ_1, μ_2, μ_3 and κ_1, κ_2 are constants. For this homogeneous deformation and uniform applied magnetic field all strain components are constant and the field equations (3.88) are satisfied.

The matrix of the Cartesian components F of the deformation gradient tensor $\mathbf{F}$ is

$$\mathsf{F} = \begin{pmatrix} \mu_1 & 0 & \kappa_1\mu_3 \\ 0 & \mu_2 & \kappa_2\mu_3 \\ 0 & 0 & \mu_3 \end{pmatrix}. \tag{4.84}$$

The uniform magnetic field vector $\mathbf{B}$ in the deformed configuration $\mathcal{B}$, related through equation (3.7) to its Lagrangian counterparts, has the Cartesian components $\mathsf{B} = (B_1, B_2, B_3)^T$.

The universal relation (4.29) of the form $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot \mathbf{B} = 0$ is given in component form

$$\left[\tau_{13}\kappa_1\kappa_2\mu_3^2 + \tau_{23}(\mu_2^2 + \mu_3^2(\kappa_2^2 - 1)) + \tau_{33}\kappa_2\mu_3^2 - \tau_{12}\kappa_1\mu_3^2 - \tau_{22}\kappa_2\mu_3^2\right] B_1 +$$

$$\left[\tau_{11}\kappa_1\mu_3^2 + \tau_{12}\kappa_2\mu_3^2 - \tau_{13}(\mu_1^2 + \mu_3^2(\kappa_1^2 - 1)) - \tau_{23}\kappa_1\kappa_2\mu_3^2 - \tau_{33}\kappa_1\mu_3^2\right] B_2 +$$

$$\left[\tau_{12}(\mu_1^2 - \mu_2^2 + \mu_3^2(\kappa_1^2 - \kappa_2^2)) + \mu_3^2(\tau_{22}\kappa_1\kappa_2 + \tau_{23}\kappa_1 - \tau_{11}\kappa_1\kappa_2 - \tau_{13}\kappa_2)\right] B_3 = 0. \tag{4.85}$$

For illustration, consider the special case of simple shear along the x_1 direction only and rename, for simplicity, the amount of shear κ_1 by $\kappa_1 = \kappa$. Suppose further that the applied magnetic field vector is oriented along the x_2 direction with components $(0, B_2, 0)$. Then, equation (4.85) reduces to the universal relation

$$\kappa\mu_3^2(\tau_{11} - \tau_{33}) = \tau_{13}[\mu_1^2 + \mu_3^2(\kappa^2 - 1)], \tag{4.86}$$

which for the case of $\mu_1 = \mu_3 = 1$, reduces to the well known correlation $\tau_{11} - \tau_{33} = \kappa\,\tau_{13}$. Note that the structure of this relation coincides with the corresponding formulation found in the purely elastic case [90].

An additional special case is obtained by considering triaxial stretches μ_1, μ_2, μ_3 and no shear. We therefore set in equation (4.83) $\kappa_1 = \kappa_2 \equiv 0$ and obtain the relation

$$\tau_{23}(\mu_2^2 - \mu_3^2)\, B_1 + \tau_{13}(\mu_3^2 - \mu_1^2)\, B_2 + \tau_{12}(\mu_1^2 - \mu_2^2)\, B_3 = 0. \tag{4.87}$$

We conclude this example by considering a special case of the energy Ω as outlined previously. Consider the case where Ω is independent of I_6, which is equivalent to $\gamma_6 = 0$. This provides the additional universal relation $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot (\mathbf{b}\mathbf{B}) = 0$, shown by equation $(4.43)_2$. From equation (4.83), assuming $\kappa_2 = 0$ and $B_1 = B_3 \equiv 0$, we can show that the universal relation $(4.43)_2$ reduces to the same expression as given by equation (4.86).

The specialization of the universal relations $(4.43)_2$ (for the case where Ω is independent of I_6) requires the computing of $(\mathbf{bB})$ and $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times$. By using the particular form of $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ with (4.84), the vector $(\mathbf{bB})$ is given by

$$\begin{aligned}&\left[(\mu_1^2 + \kappa_1^2\mu_3^2)B_1 + \kappa_1\kappa_2\mu_3^2 B_2 + \kappa_1\mu_3^2 B_3\right]\mathbf{i}_1 + \\ &\left[\kappa_1\kappa_2\mu_3^2 B_1 + (\mu_2^2 + \kappa_2^2\mu_3^2)B_2 + \kappa_2\mu_3^2 B_3\right]\mathbf{i}_2 + \\ &\left[\kappa_1\mu_3^2 B_1 + \kappa_2\mu_3^2 B_2 + \mu_3^2 B_3\right]\mathbf{i}_3,\end{aligned} \tag{4.88}$$

where $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ are the unit base vectors along the x_1, x_2, x_3 directions. The universal relation $(4.43)_2$ is obtained by performing the scalar product of the axial vector $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times$ and the vector (4.88). Thus we have

$$\begin{aligned}&\{\tau_{23}[\kappa_1^2\mu_2^2\mu_3^2 + \mu_1^2(\mu_2^2 + (\kappa_2^2 - 1)\mu_3^2)] - \mu_3^2[\kappa_1\tau_{12}\mu_2^2 + \kappa_1\kappa_2^2\tau_{12}\mu_3^2 + \kappa_2(\tau_{22}\mu_1^2 \\ &-\tau_{33}\mu_1^2 - \kappa_1\tau_{12}\kappa_2\mu_3^2)]\}B_1 + \{[\kappa_2\tau_{12}\mu_1^2 + \kappa_1(\tau_{11} - \tau_{33})\mu_2^2]\mu_3^2 - \tau_{13}[(\kappa_1^2 - 1)\mu_2^2\mu_3^2 \\ &+\mu_1^2(\mu_2^2 + \kappa_2^2\mu_3^2)]\}B_2 + [\kappa_1\tau_{23}\mu_2^2 - \kappa_2\tau_{13}\mu_1^2 + \tau_{12}(\mu_1^2 - \mu_2^2)]\mu_3^2 B_3 = 0.\end{aligned} \tag{4.89}$$

An extra couple of connections can be derived by using $(4.43)_1$ and $(4.43)_2$. Consider the particular form of (4.43), which is given by (4.85) and (4.89). If we eliminate B_2 from this pair of equations, and then if we assume $\kappa_2 = 0$ (with $\mu_2 \neq 0$), we obtain

$$(\tau_{23}B_1 - \tau_{12}B_3)(-\mu_2^4 + \mu_2^2\mu_3^2 + \mu_2^2\kappa_1^2\mu_3^2 + \mu_2^2\mu_1^2 - \mu_3^2\mu_1^2) = 0. \tag{4.90}$$

Since this condition must be satisfied for all deformations, we deduce the connection

$$\tau_{23}B_1 - \tau_{12}B_3 \equiv 0. \tag{4.91}$$

Meanwhile, if we eliminate B_1 from (4.85) and (4.89) instead, assuming $\kappa_1 = 0$, we can get the following relation

$$(\tau_{12}B_3 - \tau_{13}B_2)(\mu_1^4 + \mu_2^2\mu_3^2 - \mu_1^2(\mu_2^2 + (1 + \kappa_2^2)\mu_3^2)) = 0. \tag{4.92}$$

which, as in the above case, must also be satisfied for all deformations. We deduce the connection

$$\tau_{12}B_3 - \tau_{13}B_2 \equiv 0. \tag{4.93}$$

Extension and torsion of a circular cylinder Consider an infinitely long solid circular cylinder whose reference geometry, using cylindrical polar coordinates (R, Θ, Z), is given by

$$0 \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -\infty \leq Z \leq \infty. \tag{4.94}$$

Combined torsion and axial extension is defined by the equations

$$r = \lambda_z^{-1/2} R, \quad \theta = \Theta + \lambda_z \tau Z, \quad z = \lambda_z Z, \tag{4.95}$$

where τ is the amount of torsional twist per unit deformed length, λ_z is the constant axial stretch and (r, θ, z) are cylindrical polar coordinates in the deformed configuration. For details of the solution of this boundary value problem in the context of magneto-sensitive materials, we refer to [33].

The components of the deformation gradient $\mathbf{F}$, referred to the two sets of cylindrical polar coordinate axes, are represented by the matrix F and given by

$$\mathsf{F} = \begin{pmatrix} \lambda_z^{-1/2} & 0 & 0 \\ 0 & \lambda_z^{-1/2} & \lambda_z \gamma \\ 0 & 0 & \lambda_z \end{pmatrix}, \tag{4.96}$$

where the notation $\gamma = \tau r$ has been defined. The corresponding matrix for the left Cauchy-Green deformation tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^T$, written b, is given by

$$\mathsf{b} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} + \lambda_z^2\gamma^2 & \lambda_z^2\gamma \\ 0 & \lambda_z^2\gamma & \lambda_z^2 \end{pmatrix}. \tag{4.97}$$

The matrix of the square of the left Cauchy-Green tensor needed to evaluate the stress components has the form

$$\mathsf{b}^2 = \begin{pmatrix} \lambda_z^{-2} & 0 & 0 \\ 0 & (\lambda_z^{-1} + \lambda_z^2\gamma^2)^2 + \lambda_z^4\gamma^2 & \lambda_z^2\gamma(\lambda_z^{-1} + \lambda_z^2\gamma^2 + \lambda_z^2) \\ 0 & \lambda_z^2\gamma(\lambda_z^{-1} + \lambda_z^2\gamma^2 + \lambda_z^2) & \lambda_z^4(1+\gamma^2) \end{pmatrix}, \tag{4.98}$$

where, depending on the selection of the independent magnetic field quantity, the formulation given by equation (4.25) or (4.31) can be used. The consequences of using one or the other formulation have been discussed in detail in [33] and will therefore not be repeated here. The corresponding invariants I_1, I_2, assuming an incompressible material, are given by

$$I_1 = 2\lambda_z^{-1} + \lambda_z^2(1+\gamma^2), \quad I_2 = 2\lambda_z + \lambda_z^{-2} + \lambda_z\gamma^2. \tag{4.99}$$

Following the development in [33], it is convenient to select the formulation based on Ω^* with $\mathbf{H}_l$ as the independent variable. The corresponding constitutive equations are (4.31) and (4.32) for the total stress components $\boldsymbol{\tau}$ and for the magnetic induction field $\mathbf{B}$, respectively.

Consider an axial magnetic field given in the reference configuration by the only non-zero component denoted by H_Z. From the corresponding Maxwell field equation $(3.88)_2$ we concluded that the axial component H_Z is constant. In the current configuration, the magnetic field is given by $\mathbf{H} = \mathbf{F}^{-T}\mathbf{H}_l$ and has components along the three cylindrical polar axes

$$H_r = 0, \quad H_\theta = 0, \quad H_z = \lambda_z^{-1} H_Z. \tag{4.100}$$

From equation (4.17) the corresponding invariants are

$$K_4 = H_Z^2, \quad K_5 = (1+\gamma^2)\lambda_z^2 K_4, \quad K_6 = [\gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4]K_4. \tag{4.101}$$

The non-zero components of the stress $\boldsymbol{\tau}$ are given by equation (4.31) and have the explicit forms

$$\tau_{rr} = -p^* + \gamma_1^*\lambda_z^{-1} + \gamma_2^*\lambda_z^{-2}, \tag{4.102}$$

$$\begin{aligned}\tau_{\theta\theta} = & -p^* + \gamma_1^*(\lambda_z^{-1} + \lambda_z^2\gamma^2) + \gamma_2^*(I_1\lambda_z^2\gamma^2 + \lambda_z^{-2}) \\ & + \gamma_5^*\gamma^2\lambda_z^2 K_4 + 2\gamma_6^*\gamma^2\lambda_z^2[\lambda_z^{-1} + \lambda_z^2(1+\gamma^2)]K_4,\end{aligned} \tag{4.103}$$

$$\tau_{zz} = -p^* + \gamma_1^*\lambda_z^2 + \gamma_2^*\lambda_z^4(1+\gamma^2) + \gamma_5^*\lambda_z^2 K_4 + 2\gamma_6^*\lambda_z^4(1+\gamma^2)K_4, \tag{4.104}$$

$$\tau_{\theta z} = \gamma_1^*\lambda_z^2\gamma + \gamma_2^*\lambda_z\gamma(1 + \lambda_z^3 + \lambda_z^3\gamma^2) + \gamma_5^*\gamma\lambda_z^2 K_4 + \gamma_6^*\gamma\lambda_z[1 + 2\lambda_z^3(1+\gamma^2)]K_4. \tag{4.105}$$

From equation (4.32), the components of the magnetic induction vector $\mathbf{B}$ are

$$B_r = 0 \tag{4.106}$$

$$\begin{aligned}B_\theta = & -[\gamma_4^*\lambda_z^2 + \gamma_5^*[\lambda_z + \lambda_z^4(1+\gamma^2)] + \gamma_6^*[1 + \lambda_z^3(1+2\gamma^2) \\ & + \lambda_z^6(1+\gamma^2)^2]]\gamma H_z,\end{aligned} \tag{4.107}$$

$$B_z = -[\gamma_4^* + \gamma_5^*\lambda_z^2(1+\gamma^2) + \gamma_6^*\lambda_z[\gamma^2 + \lambda_z^3(1+\gamma^2)^2]]\lambda_z^2 H_z. \tag{4.108}$$

Now, by using (4.33) we calculate the axial tensor $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times$ as

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times = (\kappa\lambda_z^2 H_z^2[\gamma_6^*(1 + \lambda_z^3(1+\kappa^2)) + \gamma_5^*], 0, 0)^T \tag{4.109}$$

The components of the vector $\mathbf{bH}$ are $\lambda_z^2 H_z(0, \kappa, 1)^T$. Referring to (4.35) it follows that, for the considered combination of deformation and magnetic field, the universal relation (4.35) is satisfied identically.

4.1.2 Non-linear universal relations

The linear universal relation (4.29) is the only linearly independent relation for the full form of the constitutive equation for MS elastomers (4.25) [85]. In this case is not necessary to provide any particular form for the 'deformation'; but if we want to use (4.29), we would need to work with the 'universal solutions', which are solutions of the boundary value problem that do not depend on the particular form of the energy function [6].

Here we show two examples of non-linear universal relations. These relations are non-linear in term of the components of the stress tensor; they are found by studying some non-universal solutions, and cannot be generated from the linear universal relations [85].

When we work with non-universal solutions, the components of the stress and magnetic field depend on some unknown functions (semi-inverse method), which must be found by solving the partial differential equations (3.92), (3.93) for a given specific form of the constitutive equation. However, there might be some non-linear combinations of the components of the stress that do not depend explicitly on these unknown functions, and which as a result are universal relations in the sense described previously.

More details about non-linear universal relations in the purely elastic context can be found in the papers by Pucci and Saccomandi [84, 85], and Bustamante and Ogden [21].

4.1.2.1 Helical shear

In the purely elastic context Ogden et al. [79] derived a non-linear universal relation for this deformation (see [21] for a more general perspective). For the magneto-elastic case Bustamante et al. [18] showed that the same relation holds with some restrictions on the form of the magnetic field. We describe these results in the following.

Consider the problem of helical shear for a right circular tube with internal and external radii R_i and R_e, respectively, in the reference configuration. Consider the following deformation in cylindrical coordinates (see [79] and [33])

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R), \tag{4.110}$$

and

$$0 < R_i \leq R \leq R_e, \quad 0 \leq \Theta \leq 2\pi, \quad -\infty \leq Z \leq \infty, \tag{4.111}$$

Note that the length of the tube has been assumed infinite so as to avoid problems with the boundary conditions (3.66) (see [15, 99]).

[6] These deformations must be produced only by applying a surface traction (and an external field), because of this reason they are called 'controllable' solutions [99]. However, it is possible to have non-universal solutions that are also controllable for some particular constitutive equations.

The deformation functions $g(R)$ and $w(R)$ should be determined by solution of the boundary value problem. Consider the notation

$$\kappa_\theta = rg'(r), \quad \kappa_z = w'(r), \quad \kappa = \sqrt{\kappa_\theta^2 + \kappa_z^2}. \tag{4.112}$$

With this notation the deformation gradient and the left and right Cauchy-Green tensors are given as (Appendix A.3)

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1+\kappa_\theta^2 & \kappa_\theta\kappa_z \\ \kappa_z & \kappa_\theta\kappa_z & 1+\kappa_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1+\kappa^2 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}. \tag{4.113}$$

We work with the magnetic field as the independent magnetic variable. Consider the field [21] $\mathsf{H}_l = (0, H_{l\Theta}, H_{lZ})^T$; from (3.9) and (4.113)$_1$, the Eulerian form of the field is $\mathsf{H} = (-\bar{H}, H_{l\Theta}, H_{lZ})^T$, where $\bar{H} = \kappa_\theta H_{l\Theta} + \kappa_z H_{lZ}$.

From (4.31) the components of the total stress and the magnetic induction are

$$\tau_{rr} = -p^* + 2(\Omega_1^* + 2\Omega_2^*), \tag{4.114}$$

$$\tau_{\theta\theta} = -p^* + 2\Omega_1^*(1+\kappa_\theta^2) + 2\Omega_2^*(2+\kappa^2) + 2\Omega_5^* H_\theta^2 + 4\Omega_6^* H_\theta(H_\theta + \kappa_\theta \bar{H}), \tag{4.115}$$

$$\tau_{zz} = -p^* + 2\Omega_1^*(1+\kappa_z^2) + 2\Omega_2^*(2+\kappa^2) + 2\Omega_5^* H_z^2 + 2\Omega_6^* H_z(H_z + \kappa_z \bar{H}), \tag{4.116}$$

$$\tau_{r\theta} = 2(\Omega_1^* + \Omega_2^*)\kappa_\theta + 2\Omega_6^* H_\theta \bar{H}, \tag{4.117}$$

$$\tau_{rz} = 2(\Omega_1^* + \Omega_2^*)\kappa_z + 2\Omega_6^* H_z \bar{H}, \tag{4.118}$$

$$\tau_{\theta z} = 2\Omega_1^*\kappa_\theta\kappa_z + 2\Omega_5^* H_\theta H_z + 2\Omega_6^*[(2+\kappa^2)H_\theta H_z + (H_{l\Theta}^2 + H_{lZ}^2)\kappa_\theta\kappa_z]. \tag{4.119}$$

and

$$B_r = -2[\Omega_5^* + \Omega_6^*(2+\kappa^2)]\bar{H}, \tag{4.120}$$

$$B_\theta = -2[(\Omega_4^* + \Omega_5^* + \Omega_6^*)H_\theta + (\Omega_5^* + [3+\kappa^2]\Omega_6^*)\kappa_\theta \bar{H}], \tag{4.121}$$

$$B_z = -2[(\Omega_4^* + \Omega_5^* + \Omega_6^*)H_z + (\Omega_5^* + [3+\kappa^2]\Omega_6^*)\kappa_z \bar{H}]. \tag{4.122}$$

Dorfmann and Ogden [33] found the necessary condition $B_r = 0$ in order to avoid a singularity at $r = 0$, this condition appears from the Gauss conservation law (3.70). From (4.120) this implies

$$\Omega_5^* + \Omega_6^*(2+\kappa^2) = 0 \quad \text{or} \quad \bar{H} = 0. \tag{4.123}$$

Let us assume that $\bar{H} \neq 0$, which means that (4.123)$_1$ holds, and we have a restriction on the form of Ω^* (see [18]).

Using the components of the total stress (4.114)-(4.119) we can show that the following non-linear universal relation holds

$$\tau_{\theta z}(\tau_{r\theta}^2 - \tau_{rz}^2) - \tau_{r\theta}\tau_{rz}(\tau_{\theta\theta} - \tau_{zz}) = 0. \tag{4.124}$$

4.1.2.2 Anti-plane shear

In non-linear elasticity the anti-plane shear has been studied by, for example, Gurtin and Temam [53], and more recently in the review article by Horgan [56]. A non-linear universal relation was found by Bustamante and Ogden [21]. For the magneto-elastic case we want to check in which cases this relation also holds.

The anti-plane shear (of, for example, a slab) is described in Cartesian coordinates as

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + \varphi(X_1, X_2), \tag{4.125}$$

where φ is an unknown function, which in general depends on the constitutive law. The deformation gradient and the left Cauchy-Green tensor have matrix representations

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_1 & \varphi_2 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1 & 0 & \varphi_1 \\ 0 & 1 & \varphi_2 \\ \varphi_1 & \varphi_2 & 1+\varphi_1^2+\varphi_2^2 \end{pmatrix}, \tag{4.126}$$

where we have defined $\varphi_i \equiv \partial\varphi/\partial X_i$, $i = 1, 2$.

Let us work first with the magnetic induction as the magnetic independent variable, and furthermore let's assume that $\mathsf{B}_l = (0, 0, B_o)^T$; then from (4.12) and (4.13), with the notation $\vartheta = \sqrt{1+\varphi_1^2+\varphi_2^2}$, we have

$$\tau_{11} = -p + 2\Omega_1 + 2(2+\varphi_2^2)\Omega_2, \tag{4.127}$$

$$\tau_{12} = -2\varphi_1\varphi_2\Omega_2, \tag{4.128}$$

$$\tau_{13} = 2\varphi_1(\Omega_1 + \Omega_2 + B_o^2\Omega_6), \tag{4.129}$$

$$\tau_{22} = -p + 2\Omega_2 + 2(2+\varphi_1^2)\Omega_2, \tag{4.130}$$

$$\tau_{23} = 2\varphi_2(\Omega_1 + \Omega_2 + B_o^2\Omega_6), \tag{4.131}$$

$$\tau_{33} = -p + 2\vartheta^2\Omega_1 + 2(1+\vartheta^2)\Omega_2 + 2B_o^2(\Omega_5 + 2\vartheta^2\Omega_6), \tag{4.132}$$

and

$$H_1 = 2B_o\varphi_1(\Omega_6 - \Omega_4), \tag{4.133}$$

$$H_2 = 2B_o\varphi_2(\Omega_6 - \Omega_4), \tag{4.134}$$

$$H_3 = 2B_o(\Omega_4 + \Omega_5 + \vartheta^2\Omega_6). \tag{4.135}$$

With the above components of the total stress we can show that the following non-linear universal relation holds [21]:

$$\tau_{13}\tau_{23}(\tau_{22} - \tau_{11}) + \tau_{12}(\tau_{13}^2 - \tau_{23}^2) = 0. \tag{4.136}$$

If we work with the magnetic field as the independent magnetic variable. with $\mathsf{H}_l = (0, 0, H_o)^T$, from (4.22) and (4.23) we have

$$\tau_{11} = -p^* + 2\Omega_1^* + 2(2+\varphi_2^2)\Omega_2^* + 2H_o^2\varphi_1^2[\Omega_5^* + 2(1+\vartheta^2)\Omega_6^*], \quad (4.137)$$

$$\tau_{12} = 2\varphi_1\varphi_2\{-\Omega_2^* + H_o^2[\Omega_5^* + 2(1+\vartheta^2)\Omega_6^*]\}, \quad (4.138)$$

$$\tau_{13} = 2\varphi_1\{\Omega_1^* + \Omega_2^* + H_o^2[\vartheta^2\Omega_5^* + (3 + 2(\varphi_1^2+\varphi_2^2)(2+\vartheta^2))\Omega_6^*]\}, \quad (4.139)$$

$$\tau_{22} = -p^* + 2\Omega_1^* + 2(2+\varphi_1^2)\Omega_2^* + 2H_o^2\varphi_2^2[\Omega_5^* + 2(1+\vartheta^2)\Omega_6^*], \quad (4.140)$$

$$\tau_{23} = 2\varphi_2\{\Omega_1^* + \Omega_2^* + H_o^2[\vartheta^2\Omega_5^* + (3 + 2(\varphi_1^2+\varphi_2^2)(2+\vartheta^2))\Omega_6^*]\}, \quad (4.141)$$

$$\tau_{33} = -p^* + 2\vartheta^2\Omega_1^* + 2(1+\vartheta^2)\Omega_2^* + 2H_o^2\Omega_5^* + 2H_o^2\{2\Omega_6^* + (\varphi_1^2+\varphi_2^2)(1 +\vartheta^2)[\Omega_5^* + 2(1+\vartheta^2)\Omega_6^*]\}, \quad (4.142)$$

and

$$B_1 = -2H_o\varphi_1[\Omega_4^* + (1+\vartheta^2)\Omega_5^* + \vartheta^2(2+\vartheta^2)\Omega_6^*], \quad (4.143)$$

$$B_2 = -2H_o\varphi_2[\Omega_4^* + (1+\vartheta^2)\Omega_5^* + \vartheta^2(2+\vartheta^2)\Omega_6^*], \quad (4.144)$$

$$B_3 = -2H_o\{\vartheta^2\Omega_4^* + \Omega_5^* + \Omega_6^* + (\varphi_1^2+\varphi_2^2)(2+\vartheta^2)[\Omega_5^* + (1+\vartheta^2)\Omega_6^*]\}. \quad (4.145)$$

It is possible to show that in this case (4.136) also holds.

4.2 Numerical solution of a boundary value problem

In Section 4.1.1.2 it was mentioned that the set of universal solutions found by Singh and Pipkin [99] is only valid for infinite geometries. This assumption is important in order to meet the boundary conditions for the electric part of the problem. This set of universal solutions is the same in magneto-elasticity [82] and we have the same situation with the boundary conditions for the magnetic part of the problem (3.66).

Few boundary value problems have been solved in non-linear magneto-elasticity [4], and, as far as it is known, no analytical or exact solution of a boundary value problem considering a finite geometry has been found yet. However, as mentioned in [15], it is important to obtain solutions of the boundary value problem (Section 3.6) in order to make the appropriate connections between the theory and experiment, especially regarding the development of realistic forms for the constitutive relations. Due to the difficulty of finding an exact solution for a finite geometry, we have to solve the boundary value problem (Section 3.6) by using numeric methods (see [4] and [119] for the electroelastic counterpart).

In this section we show some results for a tube under extension and inflation. For this problem, Singh and Pipkin had to assume a tube of 'infinite' length, such that we do not need to consider the boundary conditions for the magnetic part of the problem over the end surfaces of the cylinder [35], in which case it can be proved that for an external axial uniform magnetic field, the field inside is also uniform. For a tube of finite length it is assumed that the exact solution will be very close to the 'real' solution for the case in which the length of the tube is much greater than, for example, its external radius; then the field would be almost uniform inside the tube wall except near the ends of the tube, where there should be some variation in the value of the field; this phenomenon was called 'fringe effect' by Sing and Pipkin. By solving a boundary value problem we want to have a first look at such phenomena. The results shown in this section are an extension of the results given in the paper by Bustamante et. al. [15]. Finally, we point out that most of the boundary value problems solved by numerical methods in non-linear magneto- and electro-elasticity [4, 119, 122] have not considered the interaction between the body and the free space around it.

4.2.1 Extension and inflation of a cylindrical tube

Dorfmann and Ogden [35] considered the extension and inflation of an infinitely long circular cylindrical tube subjected to an axial and a circumferential magnetic field. Equivalent solutions for the change in radius as a function of the applied pressure and for the corresponding resultant axial load were given using first the magnetic induction $\mathbf{B}$ and subsequently the magnetic field vector $\mathbf{H}$ as the independent magnetic field variable.

Here we again consider the extension and inflation of a cylindrical tube with circular cross section subjected to an axial magnetic field $\mathbf{H}$. In close proximity of the material interface, the magnetic field lines deviate from the axial direction due to interaction with the finite length cylindrical tube. Therefore, no closed form analytical solutions are available to determine the distributions of the magnetic fields inside the body and in the surrounding space. Numerical methods are used instead to determine the approximate distribution of the magnetic field and the magnetic induction inside the body and in the surroundings.

Using cylindrical polar coordinates (R, Θ, Z), associated with the unit basis vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$, the undeformed reference configuration is given by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \tag{4.146}$$

where the interior and exterior radii are denoted by A and B and the total length by L. The tube is deformed by applying a resultant tensile load $\mathcal{N}$, with same magnitude but with opposite directions, to the top and bottom surface located respectively at $Z = 0$ and $Z = L$, and by an interior positive pressure P. The deformed configuration maintains a circular cylindrical shape and, using cylindrical polar coordinates (r, θ, z), is defined by

$$a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq l, \tag{4.147}$$

where r is the current value of the radius, l the current total length and a and b the inner and outer radii. For an incompressible material, the deformation is given by [82, 99]

$$r = \sqrt{cR^2 + d}, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{4.148}$$

and the axial stretch, denoted by λ_z, is

$$\lambda_z = \frac{1}{c}, \tag{4.149}$$

where c and $d = a^2 - cA^2$ are constants.

The position vector in the deformed configuration may be written conveniently as

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z, \tag{4.150}$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ denote the unit basis vectors.

Using the position vector (4.150) and the corresponding expressions in (2.9) the deformation gradient tensor in cylindrical coordinates is obtained from (Appendix A.3)

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial R} \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{E}_\Theta + \frac{\partial \mathbf{x}}{\partial Z} \otimes \mathbf{E}_Z \tag{4.151}$$

and has the form

$$\mathbf{F} = \lambda^{-1}\lambda_z^{-1}\mathbf{e}_r \otimes \mathbf{E}_R + \lambda\,\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z \mathbf{e}_z \otimes \mathbf{E}_Z, \tag{4.152}$$

where the notation λ is introduced and defined as $\lambda = r/R$. The corresponding principal stretches are then given by

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \lambda \equiv \frac{r}{R}, \quad \lambda_3 = \lambda_z, \tag{4.153}$$

where, due to the incompressibility condition, only two are independent.

Remark From Subsection 3.6 we see clearly an interaction between the magnetic field and the deformation of the body. The exact solution studied by Dorfmann and Ogden [35], which is shown in[7] Subsection 5.5.1 is based in the assumption of a tube of infinite length, and as a result a variation in the actual form of the field will affect the actual deformation of the body. We do not consider that effect here, which means we assume the solution for the deformation (4.148) as given, and then we calculate from $(3.94)_2$ and (3.95) the magnetic field and the magnetic induction [15]. Of course, this assumption would mean that our 'exact' numerical solution for the field is actually an approximation of the actual solution; nevertheless, it is a better approximation than the solution based on the assumption of a tube of infinite length. The model presented here then is a sort of semi-inverse approximation of the 'actual' solution, where the deformation is given but the field is calculated by solving numerically $(3.94)_2$ and (3.95).

4.2.2 Field equations and boundary conditions

An axisymmetric deformation applied to a circular tube with finite length implies that all fields are independent of θ but depend, in addition to the radius r, also on the axial coordinate z, i.e. $\mathbf{B} = \mathbf{B}(r,z)$, $\mathbf{H} = \mathbf{H}(r,z)$, $\mathbf{M} = \mathbf{M}(r,z)$ and $\boldsymbol{\tau} = \boldsymbol{\tau}(r,z)$. In the deformed configuration, the magnetic induction and the magnetic field have to satisfy the governing equations

$$\text{div}\mathbf{B} = 0, \quad \text{curl}\mathbf{H} = \mathbf{0}, \tag{4.154}$$

together with the equilibrium equation

$$\text{div}\boldsymbol{\tau} = \mathbf{0}. \tag{4.155}$$

For the numerical solution of this boundary value problem, for convenience, we select the magnetic field $\mathbf{H}$ as the independent variable. Following Brown [13] and Kovetz [64], we introduce a scalar magnetic potential φ such that the magnetic field can be expressed as

$$\mathbf{H} = -\text{grad}\varphi. \tag{4.156}$$

Outside the body in vacuum, for example, where the magnetic induction $\mathbf{B}$ is proportional to the magnetic field $\mathbf{H}$, or in a material with uniform magnetization $\mathbf{M}$, the

[7] There is a slight difference for the deformation presented here, which was taken from [99], and the form presented by Dorfmann and Ogden [35].

divergence of $\mathbf{H}$ is zero, which follows directly from equation $(4.154)_1$. Therefore, taking the divergence of both sides of (4.156) gives

$$\mathrm{div}\mathbf{H} = -\mathrm{div}(\mathrm{grad}\varphi) = -\nabla^2\varphi = 0, \tag{4.157}$$

which shows that the scalar potential φ obeys Laplace's equation.

Inside the body the magnetic induction is determined by the constitutive law (4.23), which for convenience is repeated here:

$$\mathbf{B} = -2(\Omega_4^*\mathbf{b} + \Omega_5^*\mathbf{b}^2 + \Omega_6^*\mathbf{b}^3)\mathbf{H}. \tag{4.158}$$

Let the magnetic field $\mathbf{H}$ inside the material body be given in terms of a scalar potential φ. The magnetic induction $\mathbf{B}$ can now be written as

$$\mathbf{B} = 2(\Omega_4^*\mathbf{b} + \Omega_5^*\mathbf{b}^2 + \Omega_6^*\mathbf{b}^3)\mathrm{grad}\varphi, \tag{4.159}$$

which has to satisfy the field equation $\mathrm{div}\mathbf{B} = 0$. For convenience we introduce the tensor $\mathbf{C}$ defined as

$$\mathbf{C} = 2(\Omega_4^*\mathbf{b} + \Omega_5^*\mathbf{b}^2 + \Omega_6^*\mathbf{b}^3), \tag{4.160}$$

which allows the differential equations $(4.154)_1$ to be written in the simple form

$$\mathrm{div}(\mathbf{C}\,\mathrm{grad}\varphi) = 0. \tag{4.161}$$

For a given free energy function Ω^* and for a known deformation $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ (see the previous remark), equation (4.161) can be solved for the scalar potential φ and, as a consequence, the spatial distributions of the fields $\mathbf{H}$ and $\mathbf{B}$ and the magnetization $\mathbf{M}$ are known.

Let the magnetic far field condition of this problem be given by an applied constant field with nonzero component H_{0Z}, i.e. the applied field is parallel to the axial direction of the undeformed and deformed circular tube; see Figure 4.1. Due to interaction with the body the magnetic field lines deviate from the original direction in proximity to the boundary interface to satisfy jump conditions specified in equation (3.66). The field will therefore be dependent on both coordinates r and z.

A circular cylindrical tube subjected to axial extension and inflation maintains its original axisymmetric geometry. The numerical solution can therefore be reduced to a two-dimensional problem restricted to the $r-z$ plane as shown in Figure 4.2. The determination of the magnetic field distribution reduces, for the given deformation, to first defining a particular form of the energy function Ω^* in terms of the invariants $I_1, I_2, I_3, K_4, K_5, K_6$

and secondly finding a solution of equation (4.161) at any given point inside the body. In the surrounding space, the solution of the Laplace equation (4.157) determines the scalar potential φ.

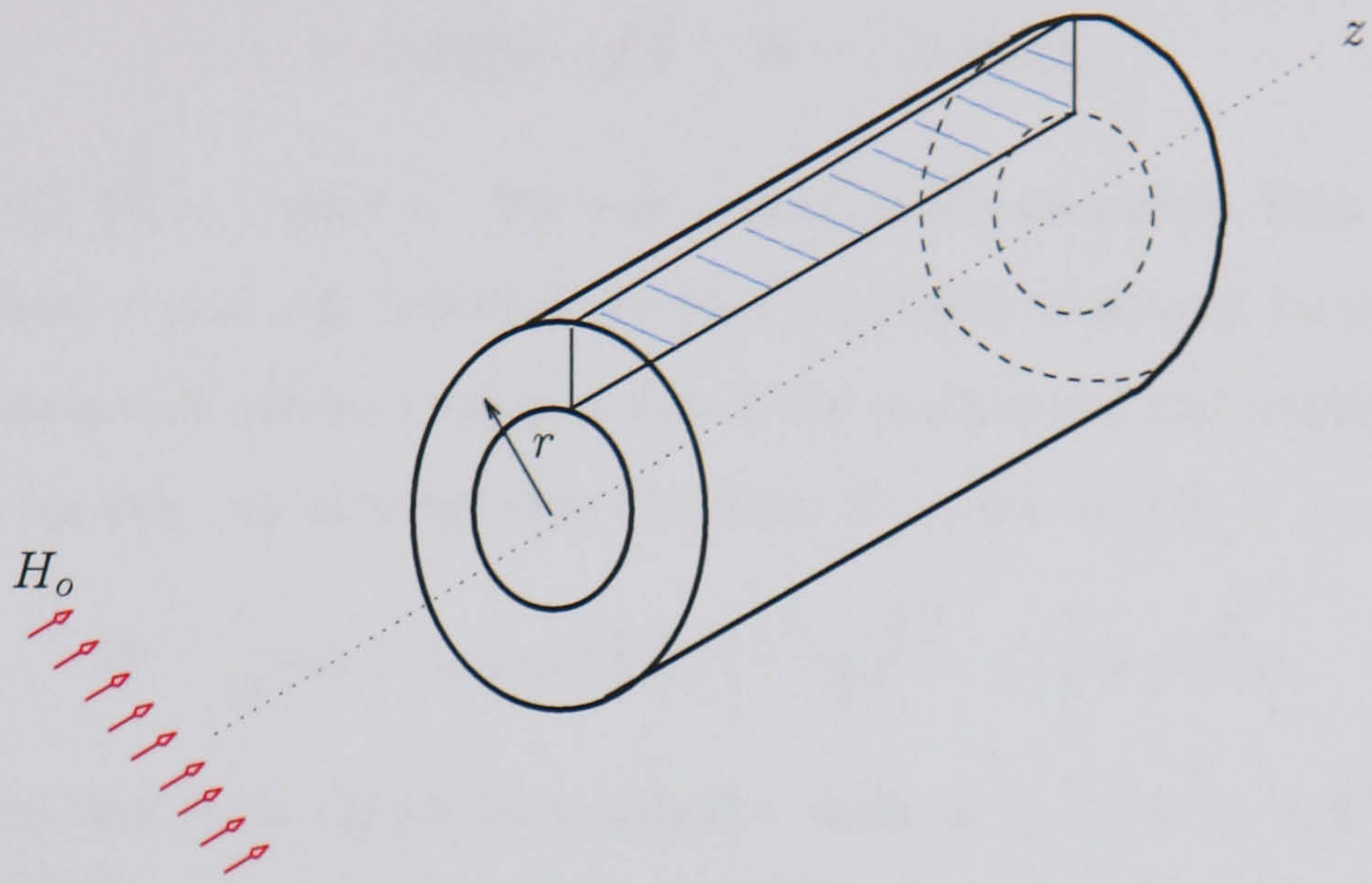


Figure 4.1: The three-dimensional problem of a deformed circular tube in an axial far field.

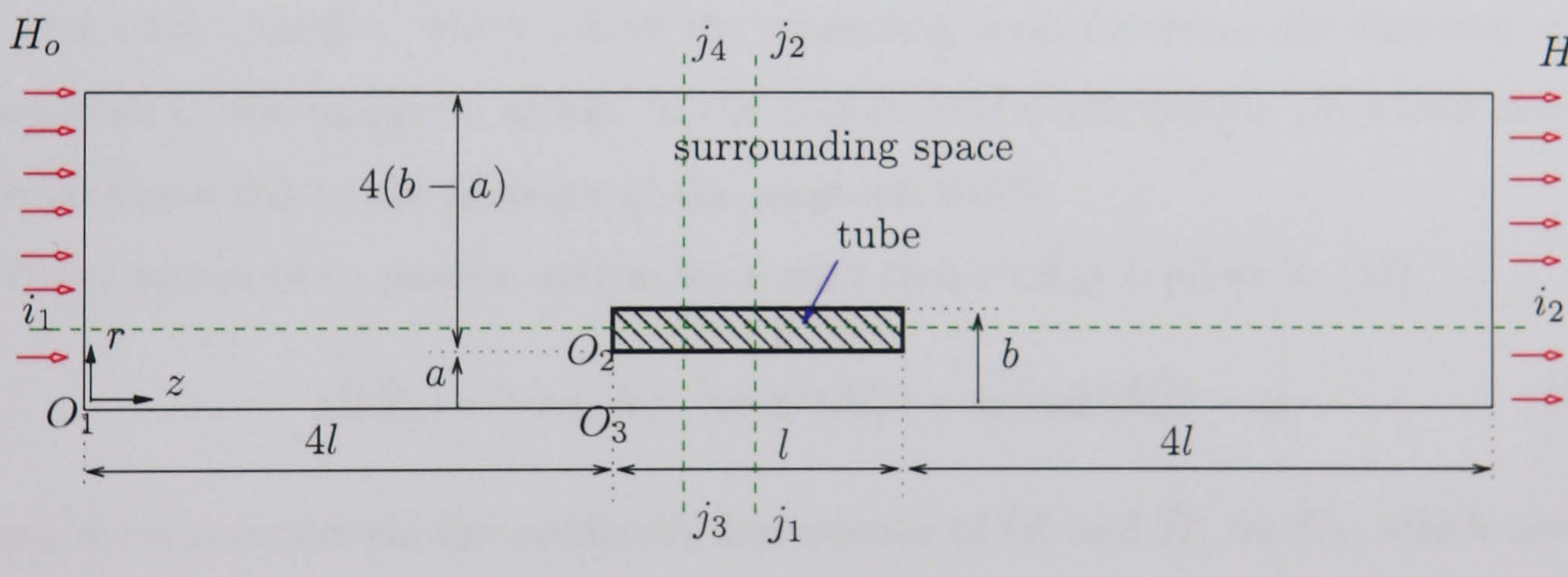


Figure 4.2: A section through the axis of the tube and its exterior corresponding to the domain of numerical computation in terms of coordinates (r, z).

The continuity condition on the material interfaces given by the equation $[\![\mathbf{H}]\!] \times \mathbf{n} = \mathbf{0}$ is satisfied automatically, since φ may be taken as continuous across these boundaries, as noted earlier. The continuity condition $[\![\mathbf{B}]\!] \cdot \mathbf{n} = 0$ requires that the radial component B_r is continuous across $r = a$ and $r = b$ and that the axial component B_z is continuous across the ends of the tube $z = 0$ and $z = l$ (here z is measured relative to the origin O_3 in Figure 4.2).

4.2.3 A particular form for the energy function

We consider a simplified form of Ω^* that depends only on I_1 and K_4. Then the stress (4.22) and magnetic induction (4.158) reduce to

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} - p^*\mathbf{I}, \quad \mathbf{B} = -2\Omega_4^*\mathbf{b}\mathbf{H}, \tag{4.162}$$

respectively, with $\mathbf{H} = -\text{grad}\varphi$. We also define a dimensionless form of K_4, namely $\bar{K}_4 = K_4/\kappa$, where κ is a constant that enables a suitable scaling to be made in order to produce the appropriate relative magnitudes of the mechanical and magnetic effects.

To be more specific, we now consider the form Ω^* given by [15]

$$\Omega^* = \frac{1}{\kappa}[\alpha_0 + \beta_0 \tanh(\bar{K}_4^n)]\left[\frac{(I_1-1)^k}{2^k} - 1\right] + \nu(\bar{K}_4), \tag{4.163}$$

where α_0, β_0, n and k are positive constants, with $n \geq 1$, $k \geq 1/2$. Note that α_0 corresponds to the shear modulus of the underlying elastic material in the absence of magnetic effects. The form of (4.163) is motivated by an elastic strain-energy function that has been used to solve a number of specific boundary value problems [58] and reduces to it in the absence of a magnetic field provided we take $\nu(0) = 0$. When there is no deformation, the first term vanishes, which allows the remaining term involving the function ν to be interpreted as the magnetic energy in the undeformed configuration (in which there is a residual stress due to the presence of the magnetic field).

It is convenient to assume a form for ν such that $\nu'(\bar{K}_4)$ is given by [15]

$$\nu'(\bar{K}_4) = -\kappa\gamma_0\bar{K}_4^{n-1}\text{sech}^2(\bar{K}_4^n) + \delta_0\tanh(\bar{K}_4^2) - \varepsilon_0. \tag{4.164}$$

These equations provide the nonlinear dependence of Ω_1^* and Ω_4^* on K_4, which are illustrated in the Figure 4.3.

The choice of the dependence of Ω_1^* and Ω_4^* on K_4 is motivated by the phenomenon of magnetic saturation (see Subsection 2.2.1.2). In Figure 4.3 the first value for I_{1_a} is the smallest value of I_1 through the wall, and the other two values are chosen in order to illustrate the dependence on the deformation.

The values of the different constants that appear in (4.163) and (4.164) are given in Table 4.1 [15].

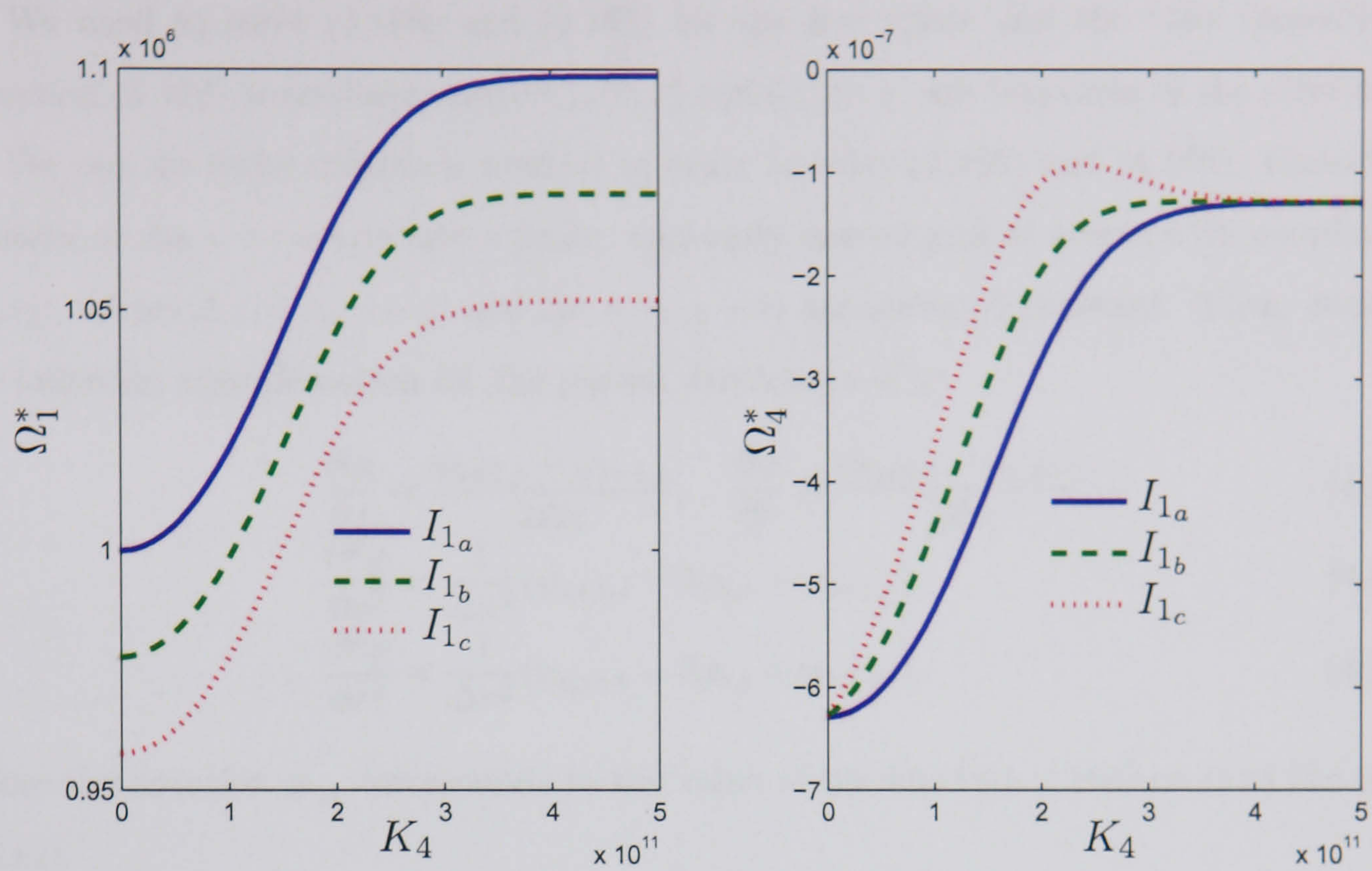


Figure 4.3: The nonlinear dependence of Ω_1^* and Ω_4^* on $K_4[\text{Amp}^2/\text{m}^2]$ for $I_1 = I_{1_a} = 3.1027$, $I_1 = I_{1_b} = 3.3$ and $I_1 = I_{1_c} = 3.5$.

α_0	2.205[MPa]
β_0	0.4[MPa]
γ_0	$9.49344 \times 10^{-8}[\text{N}/\text{Amp}^2]$
δ_0	$5 \times 10^{-7}[\text{N}/\text{Amp}^2]$
ε_0	$2\pi \times 10^{-7}[\text{N}/\text{Amp}^2]$
κ	$2.15 \times 10^{11}[\text{Amp}^2/\text{m}^2]$
k	3/4
n	2

Table 4.1: Values of constants for the energy function: isotropic MS elastomers.

4.2.4 Finite difference method

In cylindrical coordinates, considering only r and z, the equation (4.157) has the form (Appendix A.1)

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{4.165}$$

and equation (4.161) becomes

$$\left(\frac{C_{rr}}{r} + \frac{\partial C_{rr}}{\partial r}\right)\frac{\partial \varphi}{\partial r} + C_{rr}\frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial C_{zz}}{\partial z}\frac{\partial \varphi}{\partial z} + C_{zz}\frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{4.166}$$

We need to solve (4.165) and (4.166) for the free space and the tube respectively. Equation (4.166) is nonlinear since $C_{rr}(r,z)$ and $C_{zz}(r,z)$ are functions of the solution φ.

We use the finite difference method in order to solve (4.165) and (4.166). Consider a division of the $z-r$ space into a finite, uniformly spaced grid of points with coordinates (z_i, r_j), where $\Delta z = z_{i+1} - z_i$ and $\Delta r = r_{i+1} - r_i$ are assumed constant. Then, consider the following approximation for the partial derivatives of φ:

$$\frac{\partial \varphi}{\partial z} \approx \frac{\varphi_{i+1,j} - \varphi_{i-1,j}}{2\Delta z}, \quad \frac{\partial \varphi}{\partial r} \approx \frac{\varphi_{i,j+1} - \varphi_{i,j-1}}{2\Delta r}, \tag{4.167}$$

$$\frac{\partial^2 \varphi}{\partial z^2} \approx \frac{1}{\Delta z^2}(\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}), \tag{4.168}$$

$$\frac{\partial^2 \varphi}{\partial r^2} \approx \frac{1}{\Delta r^2}(\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}), \tag{4.169}$$

where the notation $\varphi_{i,j}$ corresponds to the value of the function φ evaluated at the point (z_i, r_j).

Using the above approximations in (4.165) and (4.166) we get, respectively,

$$\frac{1}{\Delta z^2}\varphi_{i-1,j} + \left(\frac{1}{\Delta r^2} - \frac{1}{2r\Delta r}\right)\varphi_{i,j-1} - 2\left(\frac{1}{\Delta r^2} - \frac{1}{2r\Delta r}\right)\varphi_{i,j-1}$$
$$-2\left(\frac{1}{\Delta r^2} + \frac{1}{\Delta z^2}\right)\varphi_{i,j} + \left(\frac{1}{\Delta r^2} + \frac{1}{2r\Delta r}\right)\varphi_{i,j+1} + \frac{1}{\Delta z^2}\varphi_{i+1,j} = 0, \tag{4.170}$$

and

$$(f_z - g_z)\varphi_{i-1,j} + (f_r - g_r)\varphi_{i,j-1} - 2(f_z + f_r)\varphi_{i,j} + (f_r + g_r)\varphi_{i,j+1}$$
$$+(f_z + g_z)\varphi_{i+1,j} = 0, \tag{4.171}$$

where

$$f_z = \frac{C_{zz}}{\Delta z^2}, \quad f_r = \frac{C_{rr}}{\Delta r^2}, \quad g_z = \frac{1}{2\Delta z}\frac{\partial C_{zz}}{\partial z}, \quad g_r = \frac{1}{2\Delta r}\left(\frac{C_{rr}}{r} + \frac{\partial C_{rr}}{\partial r}\right). \tag{4.172}$$

In equations (4.170) and (4.171) it is possible by setting $\Delta r = \Delta z$ to obtain 'normalized' forms for the difference equations, but we do not do that here (in order to have some degree of freedom regarding the relative size of Δr and Δz).

We must remember that in (4.171) f_z, f_r, g_z, and g_r are not constant; they depend on φ.

The difference equations applied for the complete space (4.2) produce a system of nonlinear algebraic equations, which has the form

$$\mathbf{Q}\varphi = \mathbf{h}, \tag{4.173}$$

where $\boldsymbol{\varphi}$ is a column vector formed with a rearrangement of $\varphi_{i,j}$, $\mathbf{Q}$ is a matrix and in general we have $\mathbf{Q} = \mathbf{Q}(\boldsymbol{\varphi})$, and $\mathbf{h}$ appears from the use of the boundary conditions.

Due to the continuity of φ at the boundary of the tube, it is possible to solve (4.170) and (4.171) considering only the boundary condition for the exterior surface of the surrounding free space (4.2); however, this would produce a highly ill-conditioned matrix $\mathbf{Q}$, and we opt to treat the equations (4.170) and (4.171) separately.

4.2.5 A numerical scheme

Consider Figure 4.4, where we have divided the surrounding free space into six parts; this division of the surrounding space is totally arbitrary, but we need to do this in order to transform (4.170) and (4.171) into a system of equations of the form (4.173). It has been found that different divisions of this surrounding space produce matrices $\mathbf{Q}$ of different qualities regarding the condition number, and the scheme shown in Figure 4.4 was the one with the smallest number among the different divisions we tried for this problem.

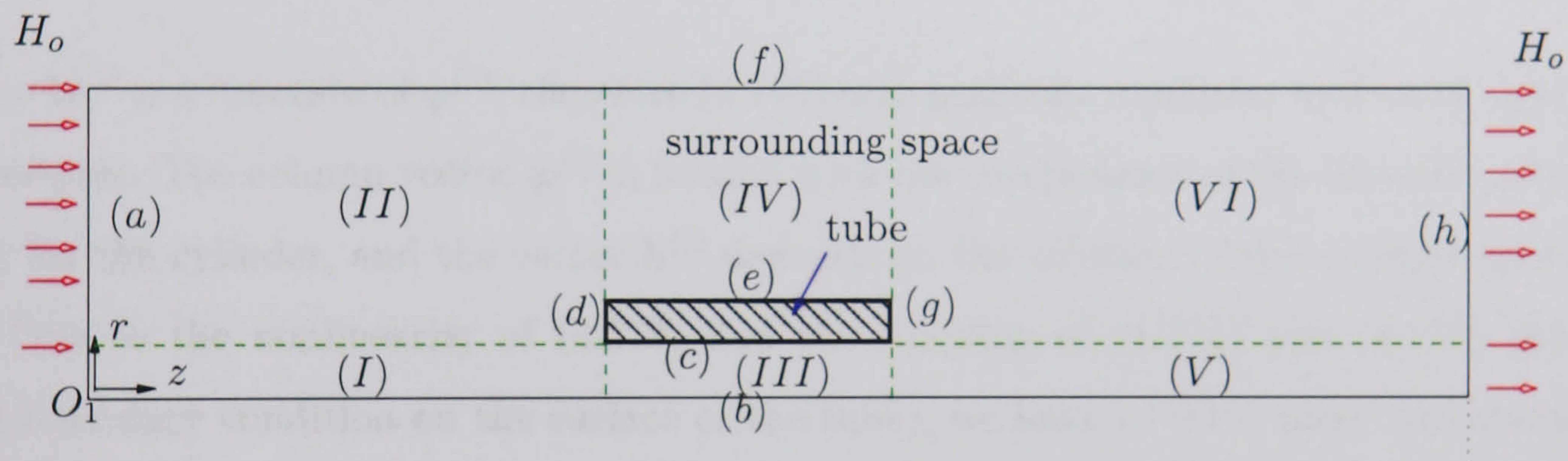


Figure 4.4: Division of the surrounding space for the finite difference method.

For (4.170) along the line (a) we have the boundary condition (3.66) for a uniform axial field $\mathsf{H} = (0, 0, H_o)^T$ with $\mathbf{B} = \mu_o \mathbf{H}_o$, and from $(4.167)_1$ we have $\varphi_{0,j} = \Delta z H_o - \varphi_{1,j}$; a similar condition can be found for the boundary (h). Regarding (b) we use the condition $\frac{\partial B_r}{\partial r} = 0$, which for the free space from $(4.167)_2$ means $\varphi_{i,0} = \varphi_{i,1}$, and for (f) we have a similar condition. In the interfaces (c), (d), (e) and (g) we use the continuity condition $(3.66)_2$; for (d), for example, we have $\varphi_{i^*,j} = \varphi_{i^*-1,j} + \frac{\Delta z}{\mu_o} B_j$, where i^* is the particular value of i at the line (d), and B_j is the value of the axial component of the magnetic induction from <u>inside</u> the cylinder, and as a result it depends on the solution of (4.171). Similar conditions may be found along (c), (e) and (g).

Therefore (4.170) can be rewritten as

$$\mathbf{Q}^{(o)}\boldsymbol{\varphi}^{(o)} = \mathbf{h}^{(o)}, \tag{4.174}$$

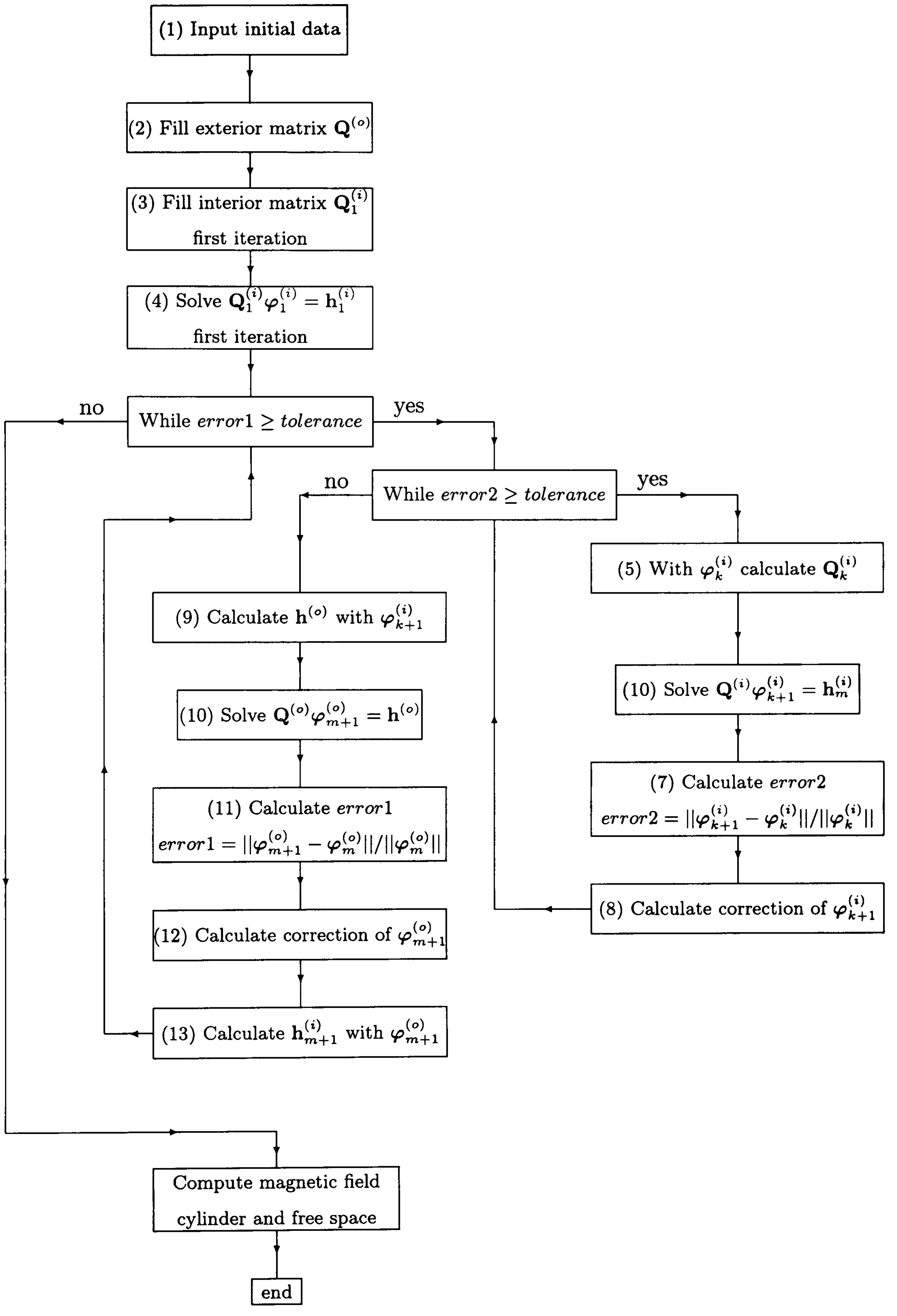
where $\mathbf{Q}^{(o)}$ only depends on Δz and Δr and as a result is constant, and (4.174) is just a linear system of equations. The vector $\boldsymbol{\varphi}^{(o)}$ is a column vector formed with the components $\varphi_{i,j}$ of the discrete scalar potential outside (o) the body (surrounding space Figure 4.2). The vector $\mathbf{h}^{(o)}$ is formed by using the boundary conditions mentioned previously for the lines (a), (b), (c), (e), (f), (g) and (h); $\mathbf{h}^{(o)}$ depends on the external field H_o and the magnetic induction from the tube.

For the difference equation (4.171) we use the boundary condition (3.74); for example, for the line (d) we have $\varphi_{0,j} = \varphi_j$, where φ_j is the value of the potential in (d) from outside the tube; then it depends on the result of (4.170) (or (4.174)) for the surrounding space (Figure 4.2). For the tube (superscript (i)) we have from (4.171)

$$\mathbf{Q}^{(i)}\boldsymbol{\varphi}^{(i)} = \mathbf{h}^{(i)}. \tag{4.175}$$

Here $\mathbf{Q}^{(i)}$ is a function of $\boldsymbol{\varphi}^{(i)}$; therefore (4.175) is in general a nonlinear system of algebraic equations. The column vector $\boldsymbol{\varphi}^{(i)}$ is formed with the components of the discrete potential $\varphi_{i,j}$ for the cylinder, and the vector $\mathbf{h}^{(i)}$ depends on the solution of the exterior problem.

Due to the nonlinearity of (4.175) and the coupling of (4.174) and (4.175) (due to the boundary condition on the surface of the tube), we have to solve these two system of equations with an iterative scheme. The figure on the next page shows this scheme; the problem was solved using MATLAB and the program can be obtained from the author if requested.


(1) Input initial data
(2) Fill exterior matrix $\mathbf{Q}^{(o)}$
(3) Fill interior matrix $\mathbf{Q}_1^{(i)}$ first iteration
(4) Solve $\mathbf{Q}_1^{(i)}\varphi_1^{(i)} = \mathbf{h}_1^{(i)}$ first iteration
no
While $error1 \geq tolerance$
yes
no
While $error2 \geq tolerance$
yes
(5) With $\varphi_k^{(i)}$ calculate $\mathbf{Q}_k^{(i)}$
(9) Calculate $\mathbf{h}^{(o)}$ with $\varphi_{k+1}^{(i)}$
(10) Solve $\mathbf{Q}^{(i)}\varphi_{k+1}^{(i)} = \mathbf{h}_m^{(i)}$
(10) Solve $\mathbf{Q}^{(o)}\varphi_{m+1}^{(o)} = \mathbf{h}^{(o)}$
(7) Calculate $error2$
$error2 = ||\varphi_{k+1}^{(i)} - \varphi_k^{(i)}||/||\varphi_k^{(i)}||$
(11) Calculate $error1$
$error1 = ||\varphi_{m+1}^{(o)} - \varphi_m^{(o)}||/||\varphi_m^{(o)}||$
(8) Calculate correction of $\varphi_{k+1}^{(i)}$
(12) Calculate correction of $\varphi_{m+1}^{(o)}$
(13) Calculate $\mathbf{h}_{m+1}^{(i)}$ with $\varphi_{m+1}^{(o)}$
Compute magnetic field cylinder and free space
end

Let us explain in more detail the steps of the numeric procedure shown above.

1. We give the data about the geometry of the tube (interior radius, thickness and length of the tube), the 'deformation parameters', the mesh parameters (number of points in the z and r directions), and the external magnetic field.

2. Fill the exterior matrix $\mathbf{Q}^{(o)}$ for the surrounding free space. This matrix is formed using the finite difference system of equations (4.170) with the modifications along the boundaries (a), (b), (c), (e), (f), (g) and (h) (Figure 4.4). The vector $\mathbf{h}^{(o)}$ with the boundary conditions for (4.165) is partially filled with the data along (a), (b), (f) and (h) (Figure 4.4), and for (c), (d), (e) and (g) we assume a first approximation for the boundary condition along the interface with the tube wall.

3. The matrix $\mathbf{Q}^{(i)}$, which is formed from the difference equation (4.171), depends on the solution of the problem for the tube. In this step we assume a solution for the potential in the tube wall, and we evaluate $\mathbf{Q}^{(i)}$ for this first iteration. The vector with boundary conditions for the interior (tube wall) problem $\mathbf{h}^{(i)}$ is filled with a first approximation of the potential φ from the surrounding space.

4. For a first iteration we solve the linear system $\mathbf{Q}_1^{(i)}\varphi_1^{(i)} = \mathbf{h}_1^{(i)}$, where the subscript 1 means the first iteration. We assume here that for one iteration $\mathbf{Q}^{(i)}$ is constant.

   Now we start an iteration procedure. Here we iterate in order to find the correct boundary condition for the interface between the tube and the surrounding free space (3.74), $(3.66)_2$. The criterion to finish this iteration loop is to make $error1$ less than a given tolerance, where $error1$ corresponds to the norm of the potential outside $\varphi_{m+1}^{(o)}$ for the step $m+1$ minus the potential outside in the previous step $\varphi_m^{(o)}$, normalized by the norm of this vector.

5. Inside the above iteration loop there is another loop in this case in order to solve the nonlinear equation $\mathbf{Q}^{(i)}\varphi^{(i)} = \mathbf{h}^{(i)}$. First with the potential for the tube from the previous step we calculate the coefficients for $\mathbf{Q}_k^{(i)}$.

6. Then assuming $\mathbf{Q}_k^{(i)}$ as a constant matrix we solve the now linear system $\mathbf{Q}_k^{(i)}\varphi_{k+1}^{(i)} = \mathbf{h}_m^{(i)}$ for the potential in the next iteration $\varphi_{k+1}^{(i)}$; here $\mathbf{h}_m^{(i)}$ is the vector with the boundary conditions from the solution of the problem in the free space.

7. We calculate a relative error $error2$ for the potential $\varphi^{(i)}$ in the tube wall.

8. Now the potential for the next iteration $\varphi_{k+1}^{(i)}$ is corrected following a procedure found, for example, in [28].

   If *error*2 is less than a tolerance the loop ends; if not, we start again with the steps (5), (6), (7) and (8), but now working with $\varphi_{k+1}^{(i)}$ corrected as mentioned before.

9. Once *error*2 is less than a given tolerance, with the potential for the tube $\varphi_{k+1}^{(i)}$ we calculate the components of the vector $\mathbf{h}_m^{(o)}$ for the exterior problem along the boundaries (c), (d), (e) and (g) (Figure 4.4).

10. We solve the linear system of equations $\mathbf{Q}^{(o)}\varphi_{m+1}^{(o)} = \mathbf{h}_m^{(o)}$.

11. The relative error *error*1 is calculated for the potential of the surrounding space.

12. As in (8), the potential for the next iteration $\varphi_{m+1}^{(o)}$ is corrected.

13. With the above corrected potential for the free space we calculate the vector $\mathbf{h}_{m+1}^{(i)}$ for the boundary conditions for the interior problem.

    If the relative error *error*1 is greater than or equal to the tolerance, then we repeat everything again; if the error is less than the tolerance we finish the iteration.

14. With the potential for the tube $\varphi^{(i)}$ and the surrounding space $\varphi^{(o)}$ we calculate the magnetic field and the magnetic induction for the problem.

4.2.6 Results

In this subsection, the numerical results obtained for the spatial distribution of the magnitudes of the magnetic field and magnetic induction are summarized.

We consider aspect ratios l/a of 4, 6 and 8 and describe the deformation through the constants c and d introduced in (4.148). Axial compression (extension) corresponds to $c > 1$ (< 1), and we consider values $c = 0.5, 1, 1.2, 2$ with[8] $d = 0$ or 0.0002. The magnetic-field distributions inside the tube wall in the deformed configuration are compared with those in the deformed configurations. For each calculation we set $b = 2a$. Note that for $d = 0$ the deformation is homogeneous, while for $d \neq 0$ the deformation is nonhomogeneous since λ then depends on r.

[8] In the original problem the dimension of the cylinder was given in meters (in order to be consistent with the values shown in Table 4.1), the actual size of the cylinder (length and, for example, internal radius) was of the order of the centimeters. The dimension of d was [m^2], and this is the reason this parameter seems to be too 'small'

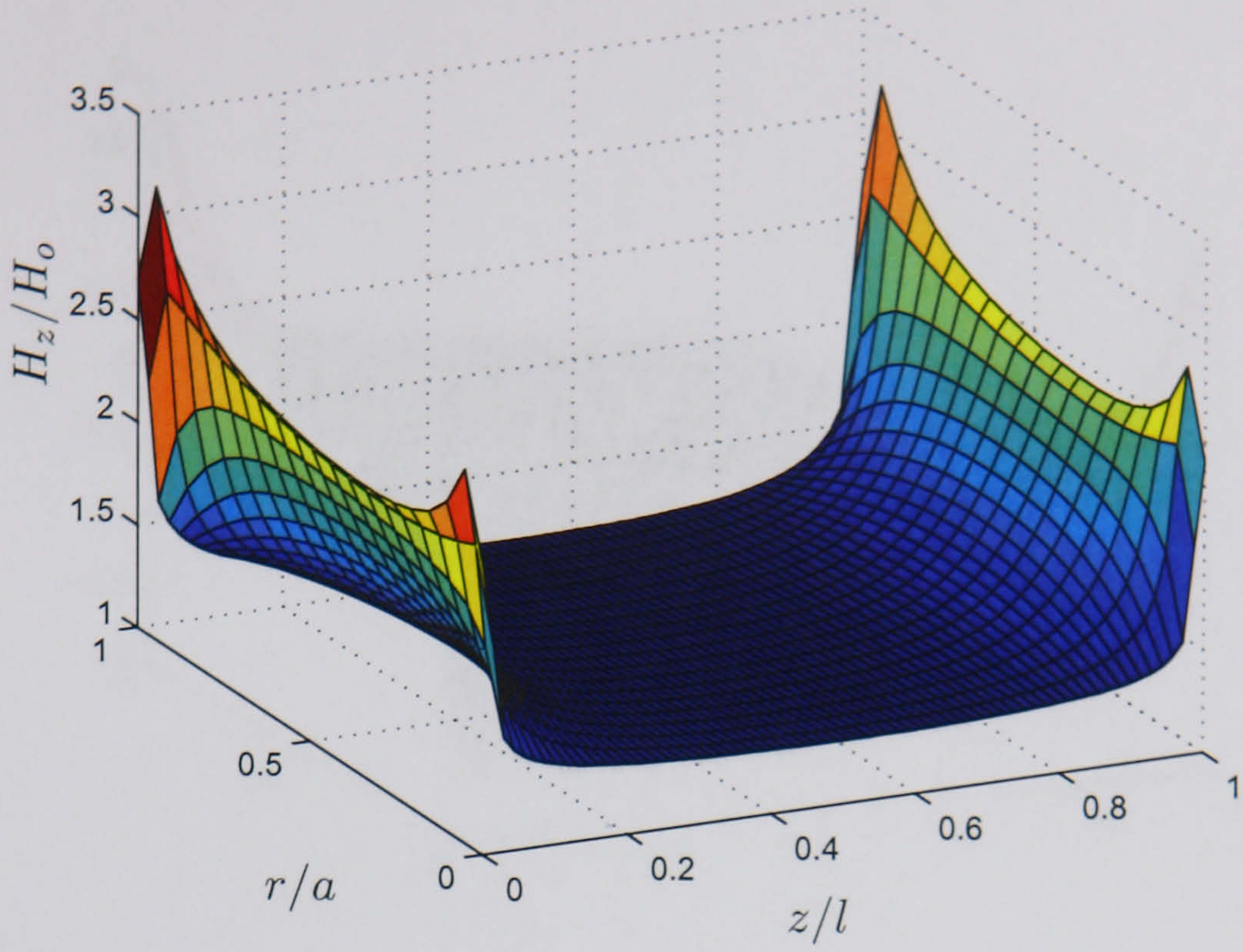


Figure 4.5: Magnitude of the dimensionless axial component of the magnetic field through the tube wall in an arbitrary (r, z) plane for aspect ratio $l/a = 4$, for a tube under compression ($c = 1.2$) with $d = 0.0002$.

Figure 4.5 shows the dimensionless magnitude of the axial component of the magnetic field through the material wall for an arbitrary (r, z) plane and for aspect ratio $l/a = 4$ [15]. For this illustration the values $c = 1.2$ and $d = 0.0002$ were chosen. The origin of the nondimensionalized coordinate system in Figure 4.5 is indicated by O_2 in Figure 4.2, so that r/a and z/l both run from 0 to 1. The results show that the magnetic field is essentially constant away from the boundary and symmetric about the centre $z/l = 0.5$ of the tube.

Similarly, for the same geometry and deformation, the dimensionless magnitude of the radial component of the magnetic field is shown in the Figure 4.6, which reveals that the radial component is antisymmetric with respect to $z/l = 0.5$ and $r/a = 0.5$. In each case the magnetic field is nondimensionalized with respect to the far field H_o, which is given the value 10^5[Amp/m].

We complement these results, which were shown in [15], with similar figures for the surrounding free space for the same problem. In order to facilitate the implementation of the numeric method the surrounding space was divided in six parts (see Figure 4.4); in Figure 4.7 we have the dimensionless magnitude of the axial component of the magnetic

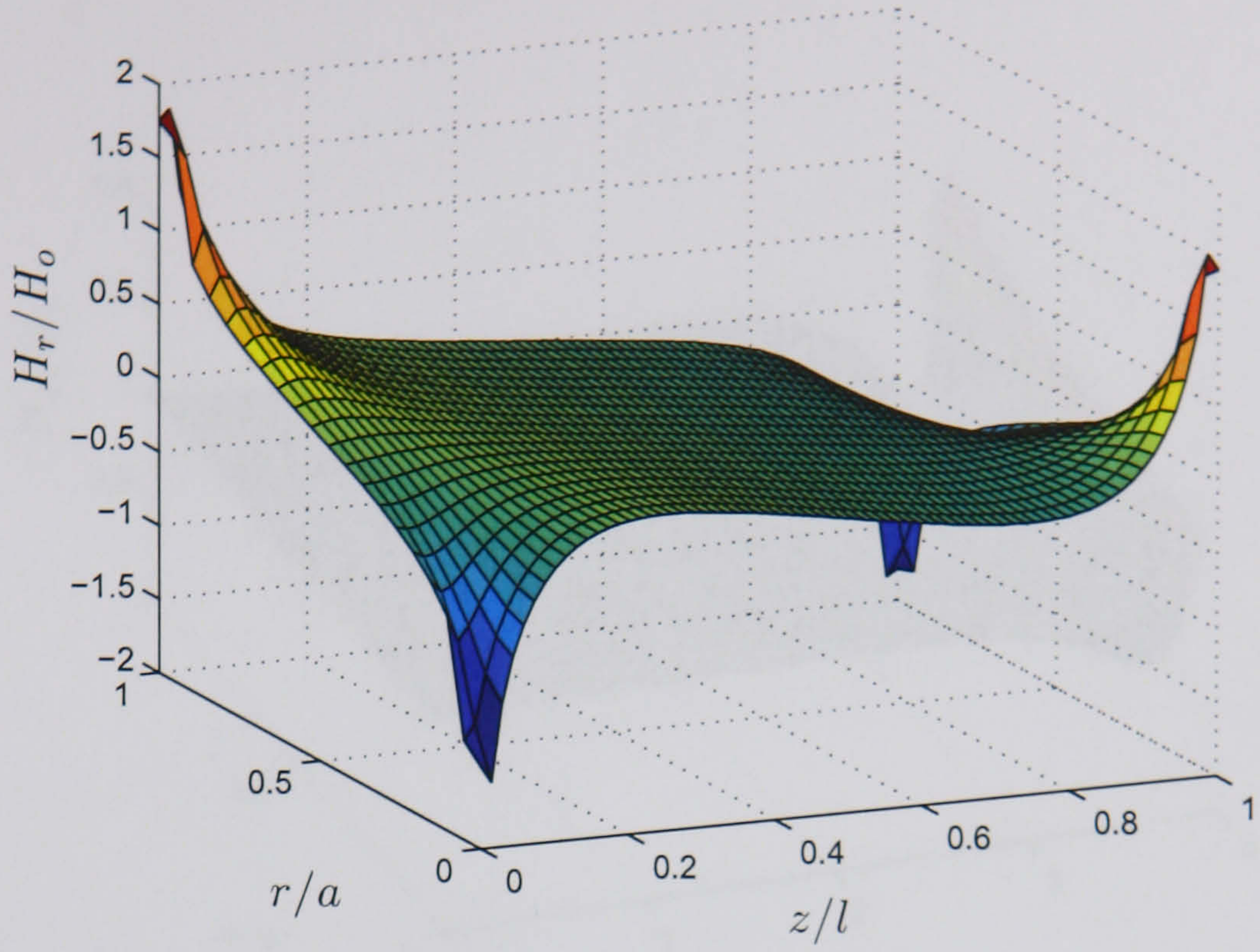


Figure 4.6: Magnitude of the dimensionless radial component of the magnetic field inside the material wall on an arbitrary (r, z) plane for aspect ratio $l/a = 4$, for a tube under compression $(c = 1.2)$ with $d = 0.0002$.

field for the region (I) in Figure 4.4.

Figures 4.8-4.12 show the same results for the region (II), (III), (IV), (V) and (VI) respectively.

Similar results for the radial component of the field are shown in the Figures 4.13-4.18.

From these figures for the behaviour of the field for the surrounding free space we see that far away from the tube the axial component is constant; in fact this component changes quickly only very close to the tube wall. As it can be expected, the axial component is symmetric with respect to the line $\overline{j_1 j_2}$ (Figure 4.2); compare, for example, Figures 4.7 and 4.11.

Regarding the radial component of the magnetic field, for example, from the Figures 4.13 and 4.16, we see that far away from the tube wall it is almost zero. This component is antisymmetric with respect to the line $\overline{j_1 j_2}$.

The magnitudes of the axial and radial components of the magnetic field within the material depend on the aspect ration l/a of the tube. The distribution of these values along the line $\overline{i_1 i_2}$ identified in Figure 4.2 is shown in Figure 4.19 for aspect ratios $l/a = 4, 6, 8$. The line $\overline{i_2 i_2}$ is located at a distance of $3a/2$ from the centreline (Figure 4.2).

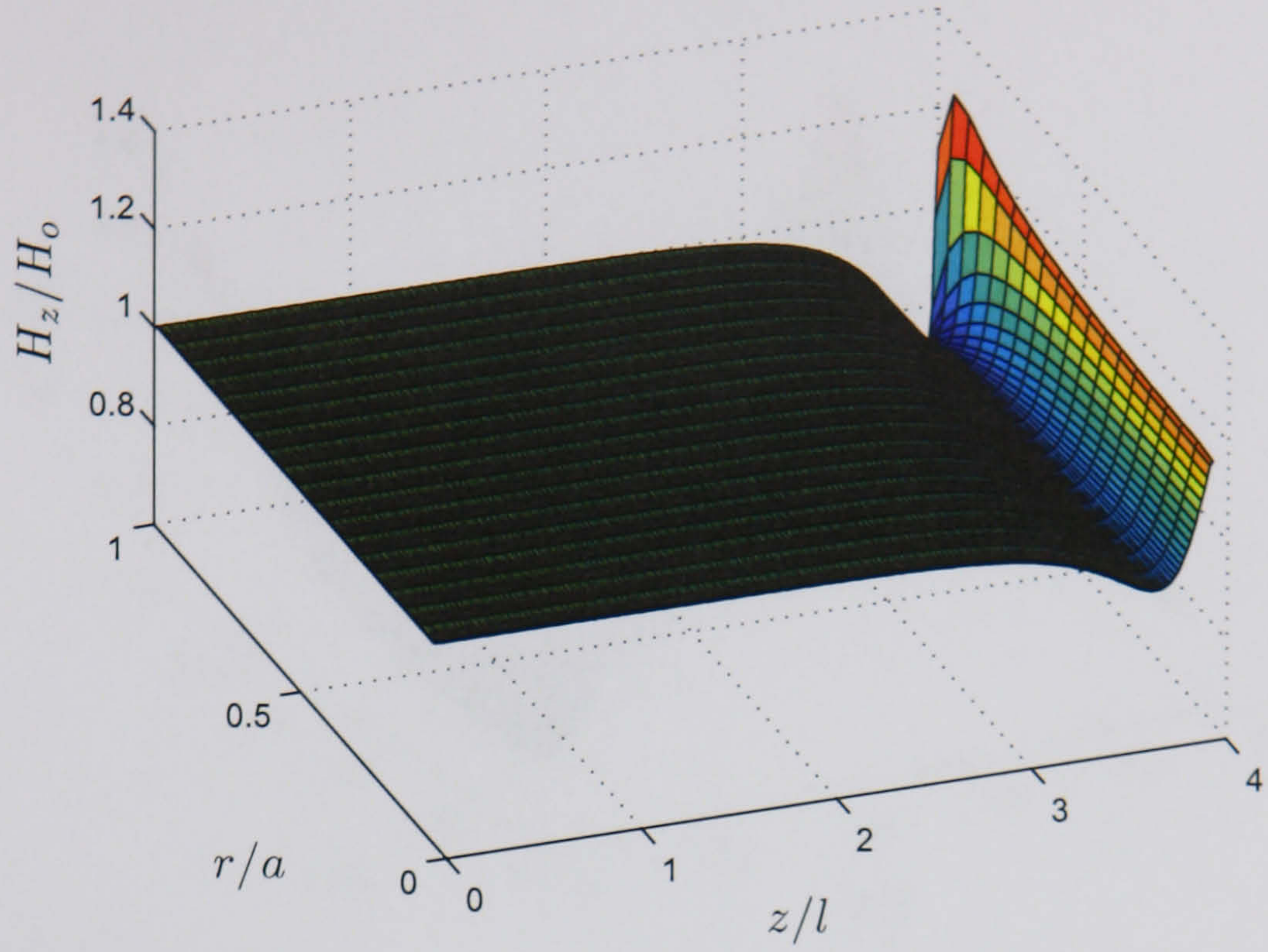


Figure 4.7: Magnitude of the dimensionless axial component of the magnetic field for the free surrounding space in (I) in an arbitrary (r, z), for the same data of Figure 4.5.

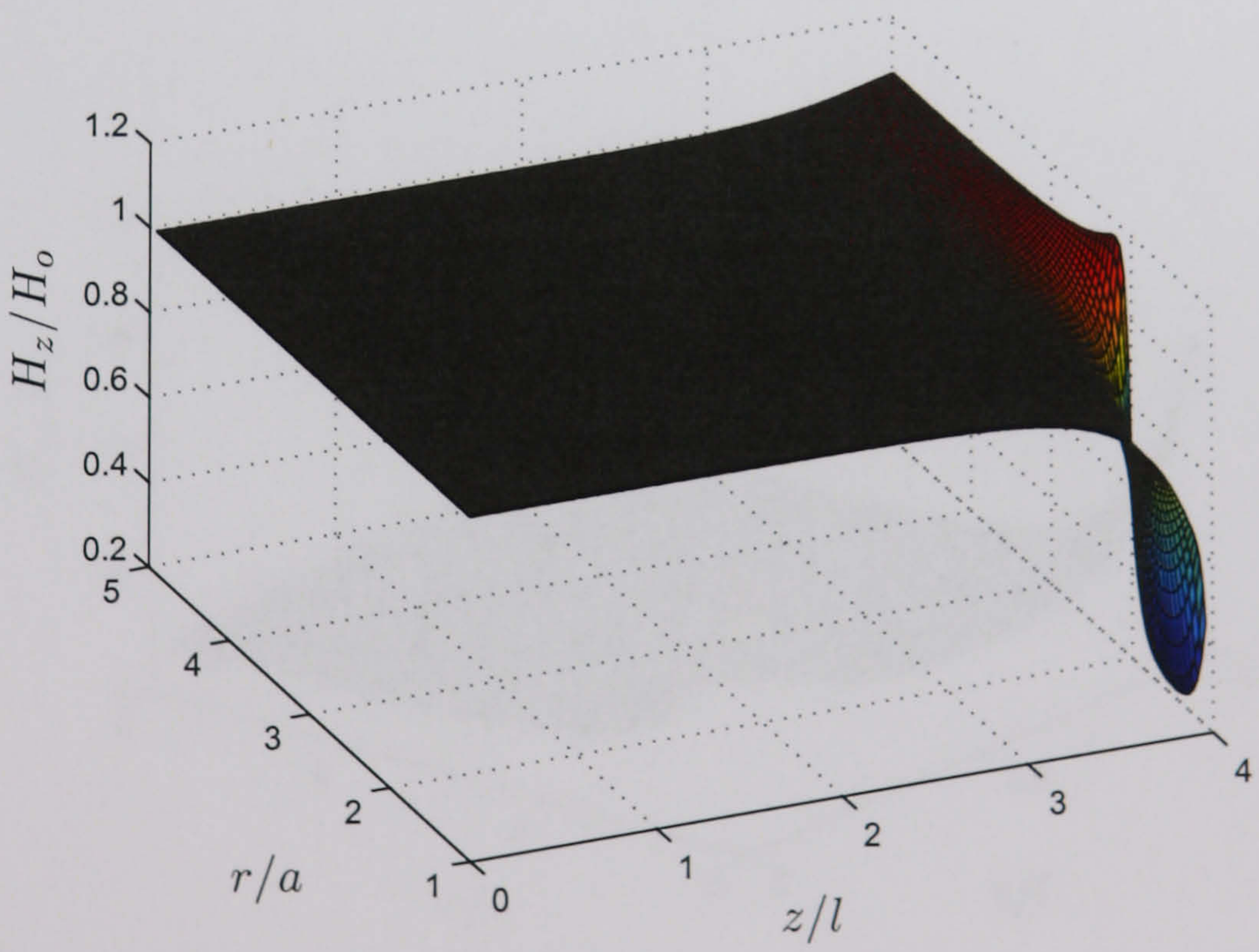


Figure 4.8: Magnitude of the dimensionless axial component of the magnetic field for the free surrounding space in (II) in an arbitrary (r, z).

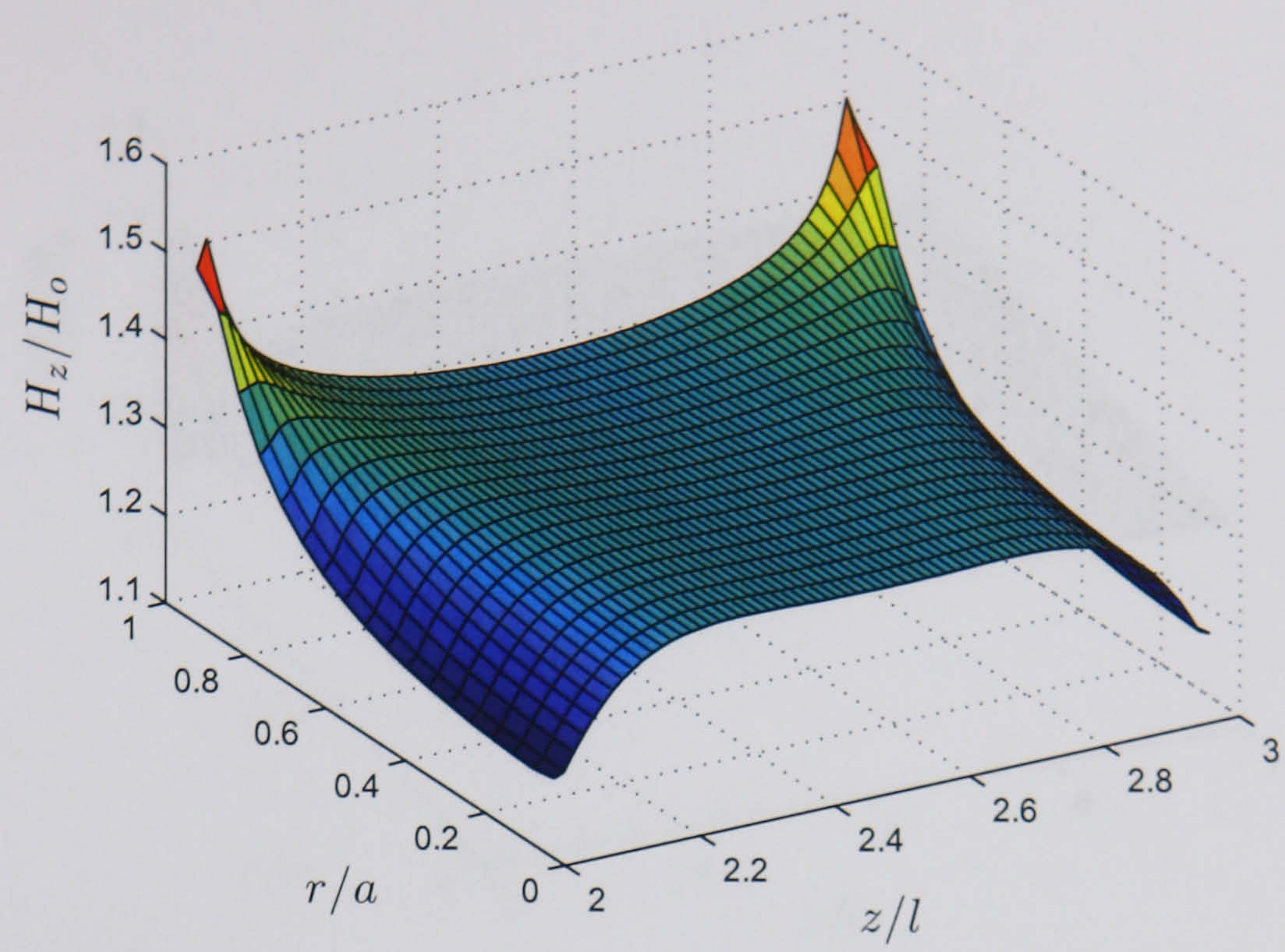


Figure 4.9: Magnitude of the dimensionless axial component of the magnetic field for the free surrounding space in (III) in an arbitrary (r, z).

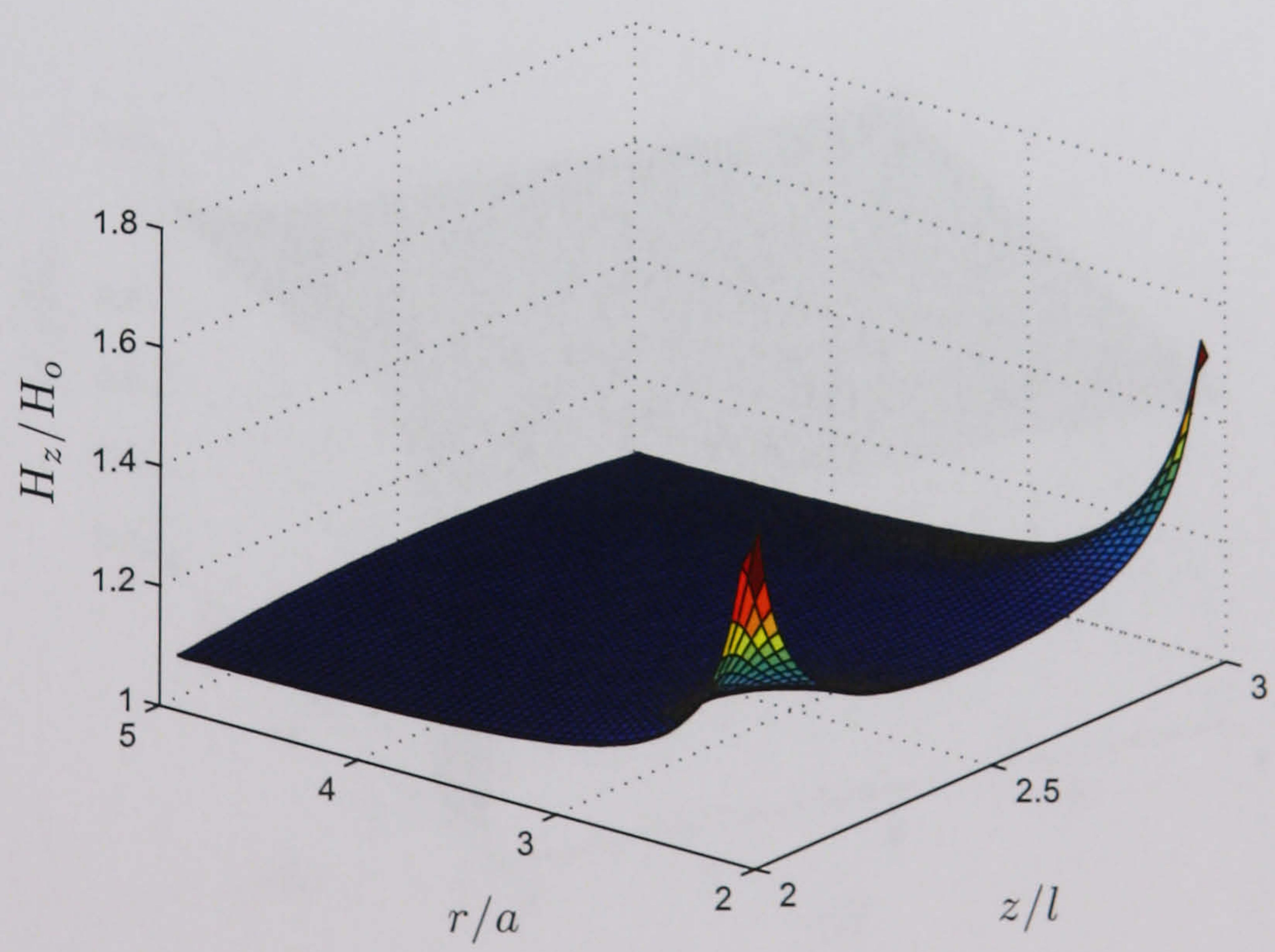


Figure 4.10: Magnitude of the dimensionless axial component of the magnetic field for the free surrounding space in (IV) in an arbitrary (r, z).

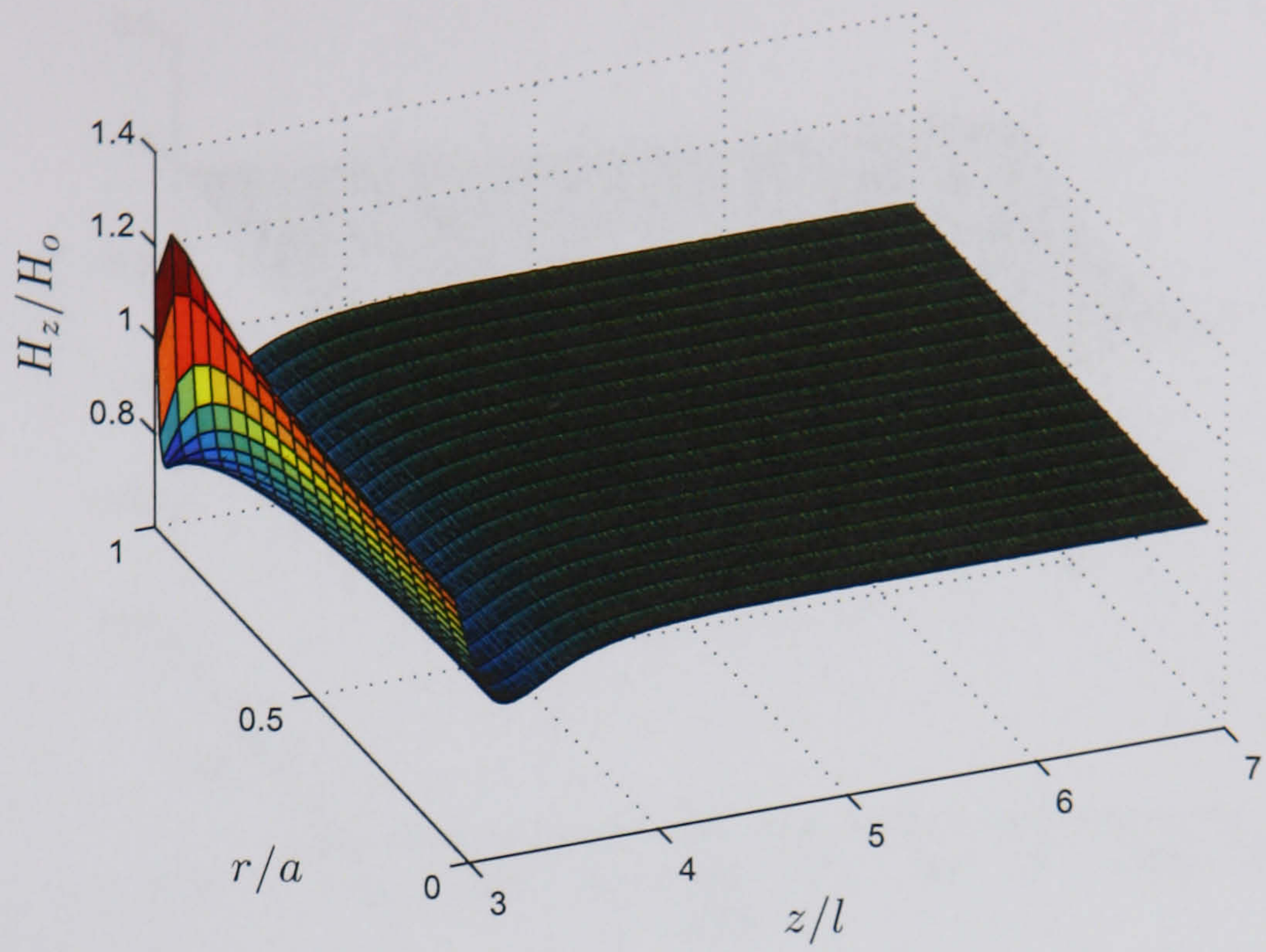


Figure 4.11: Magnitude of the dimensionless axial component of the magnetic field for the free surrounding space in (V) in an arbitrary (r, z).

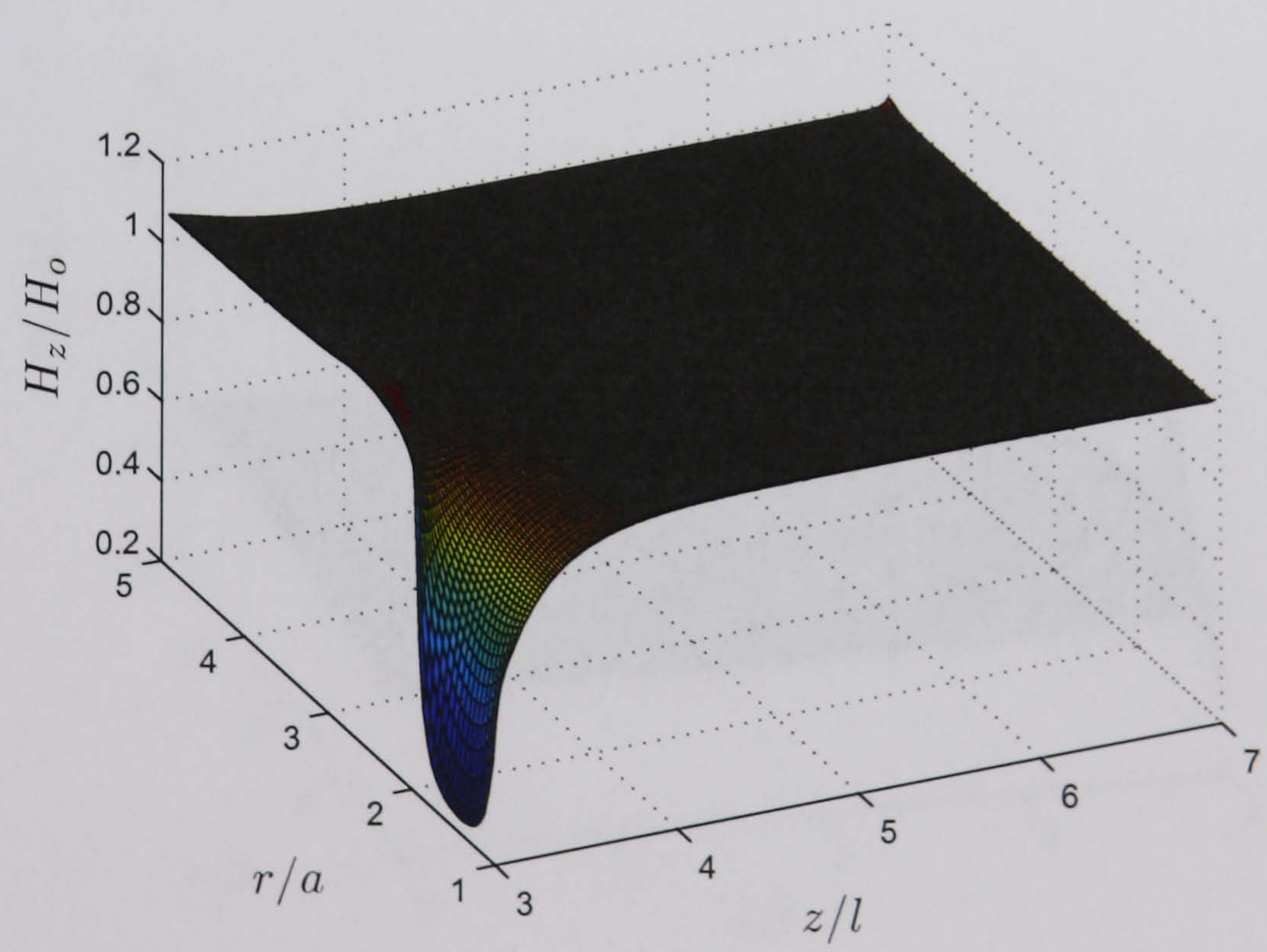


Figure 4.12: Magnitude of the dimensionless axial component of the magnetic field for the free surrounding space in (VI) in an arbitrary (r, z).

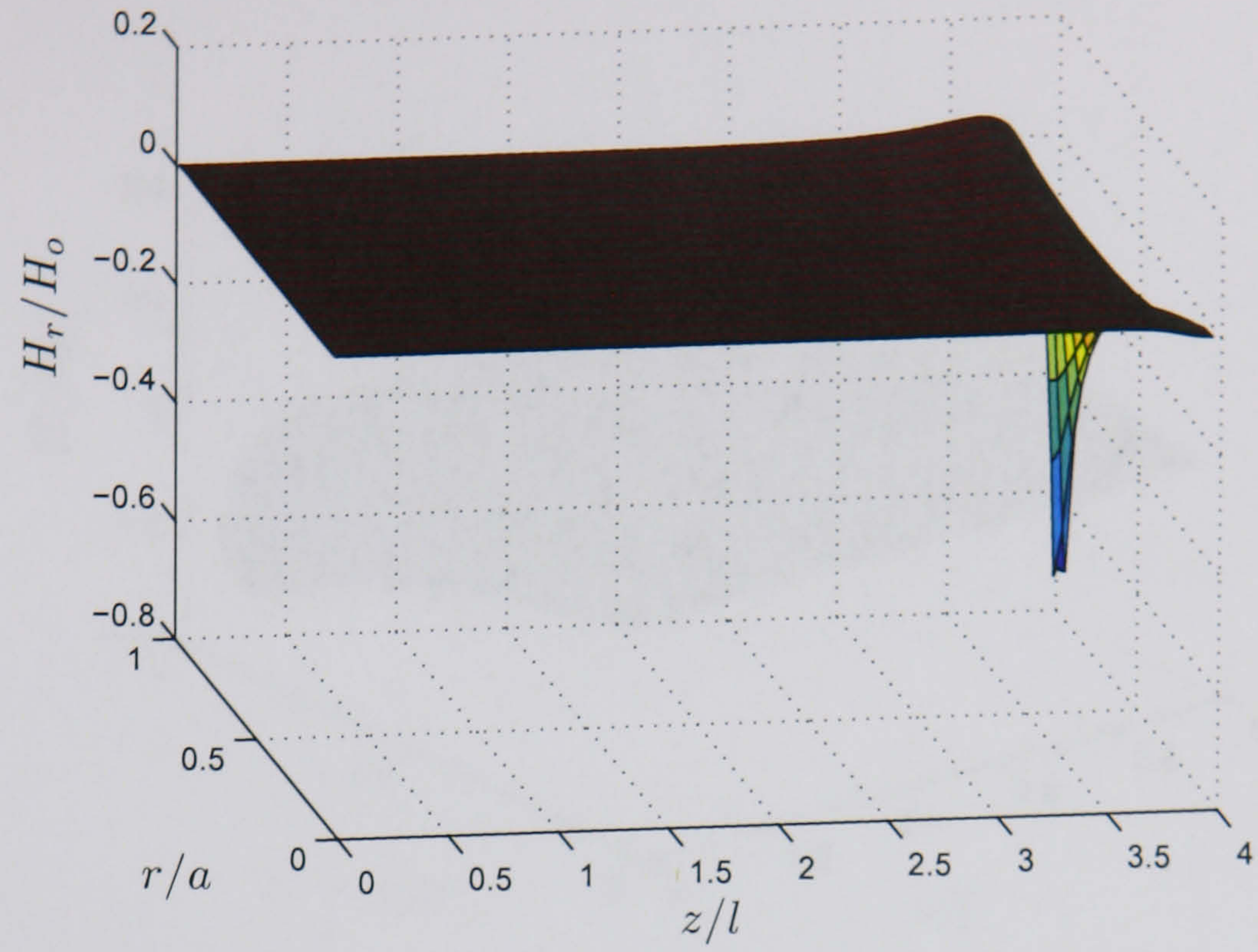


Figure 4.13: Magnitude of the dimensionless radial component of the magnetic field for the free surrounding space in (I) in an arbitrary (r, z).

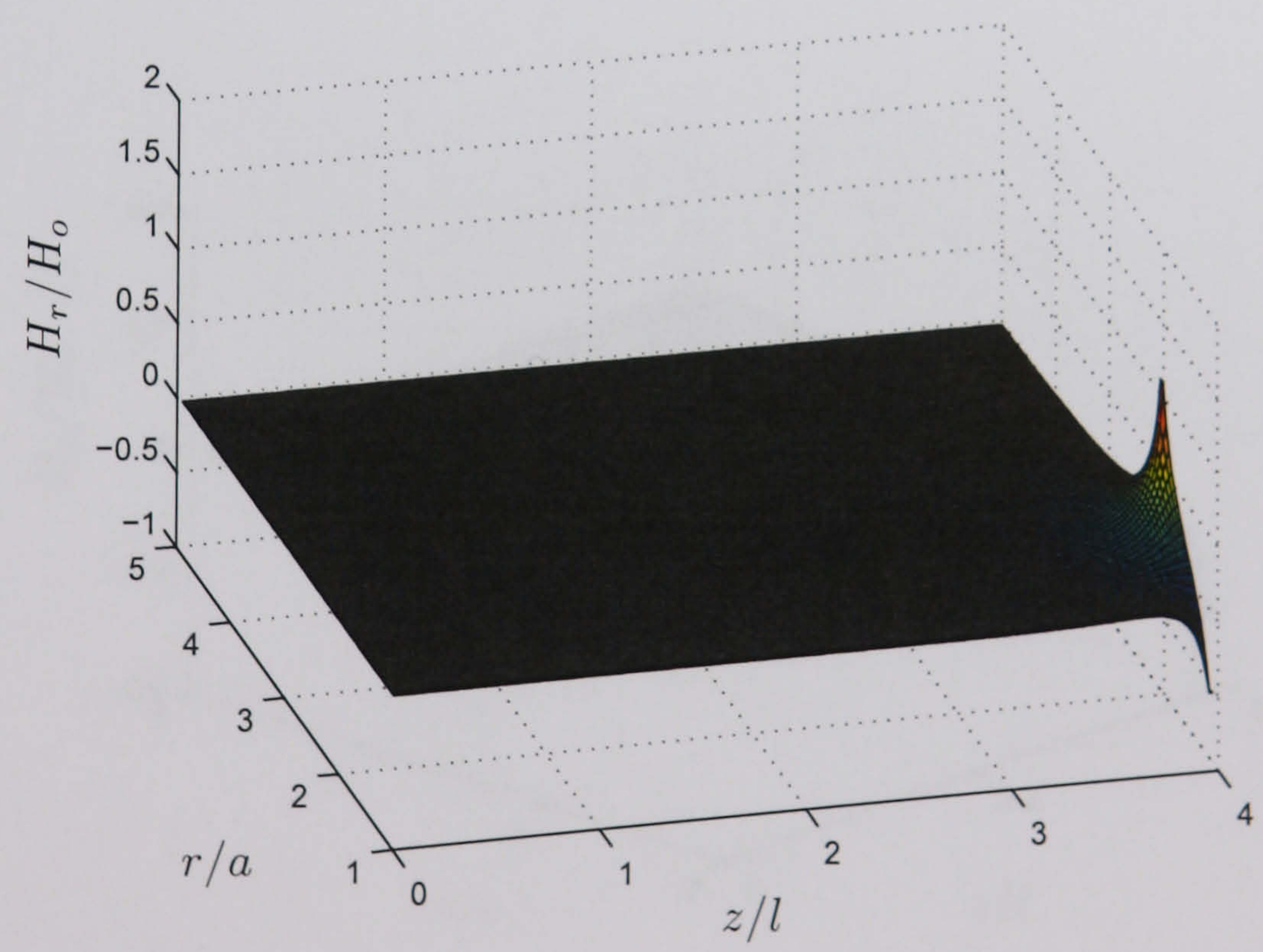


Figure 4.14: Magnitude of the dimensionless radial component of the magnetic field for the free surrounding space in (II) in an arbitrary (r, z).

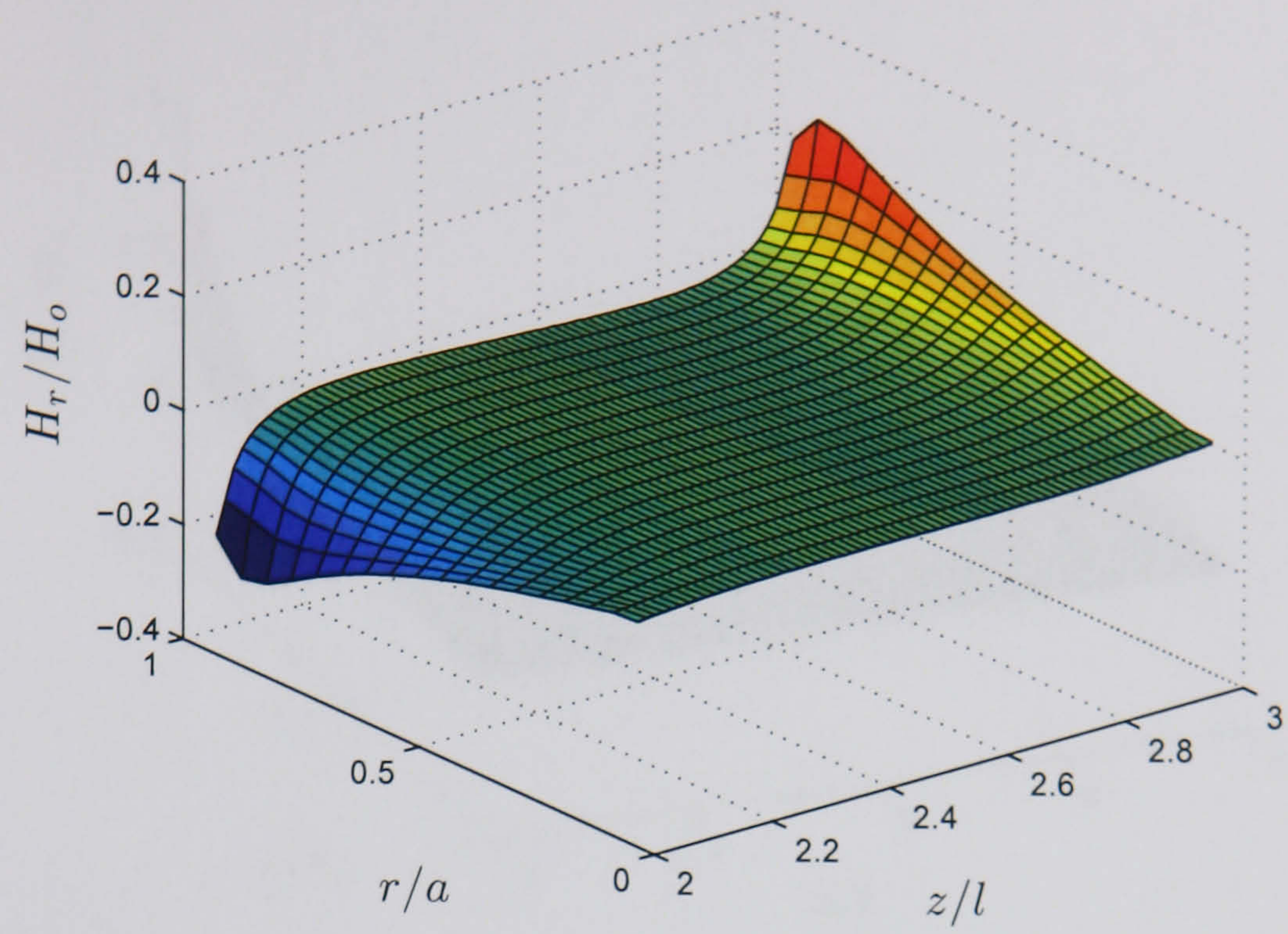


Figure 4.15: Magnitude of the dimensionless radial component of the magnetic field for the free surrounding space in (III) in an arbitrary (r, z).

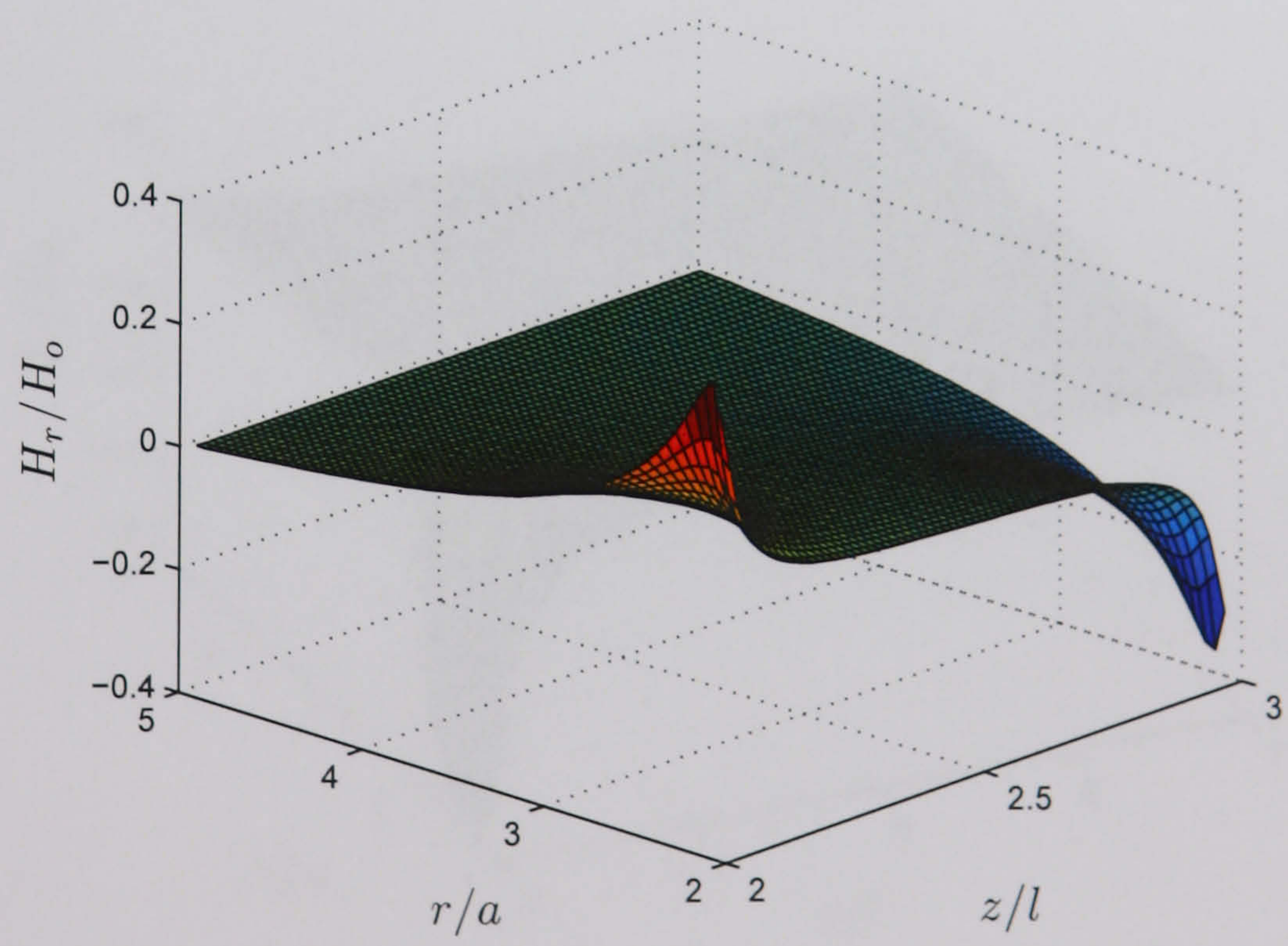


Figure 4.16: Magnitude of the dimensionless radial component of the magnetic field for the free surrounding space in (IV) in an arbitrary (r, z).

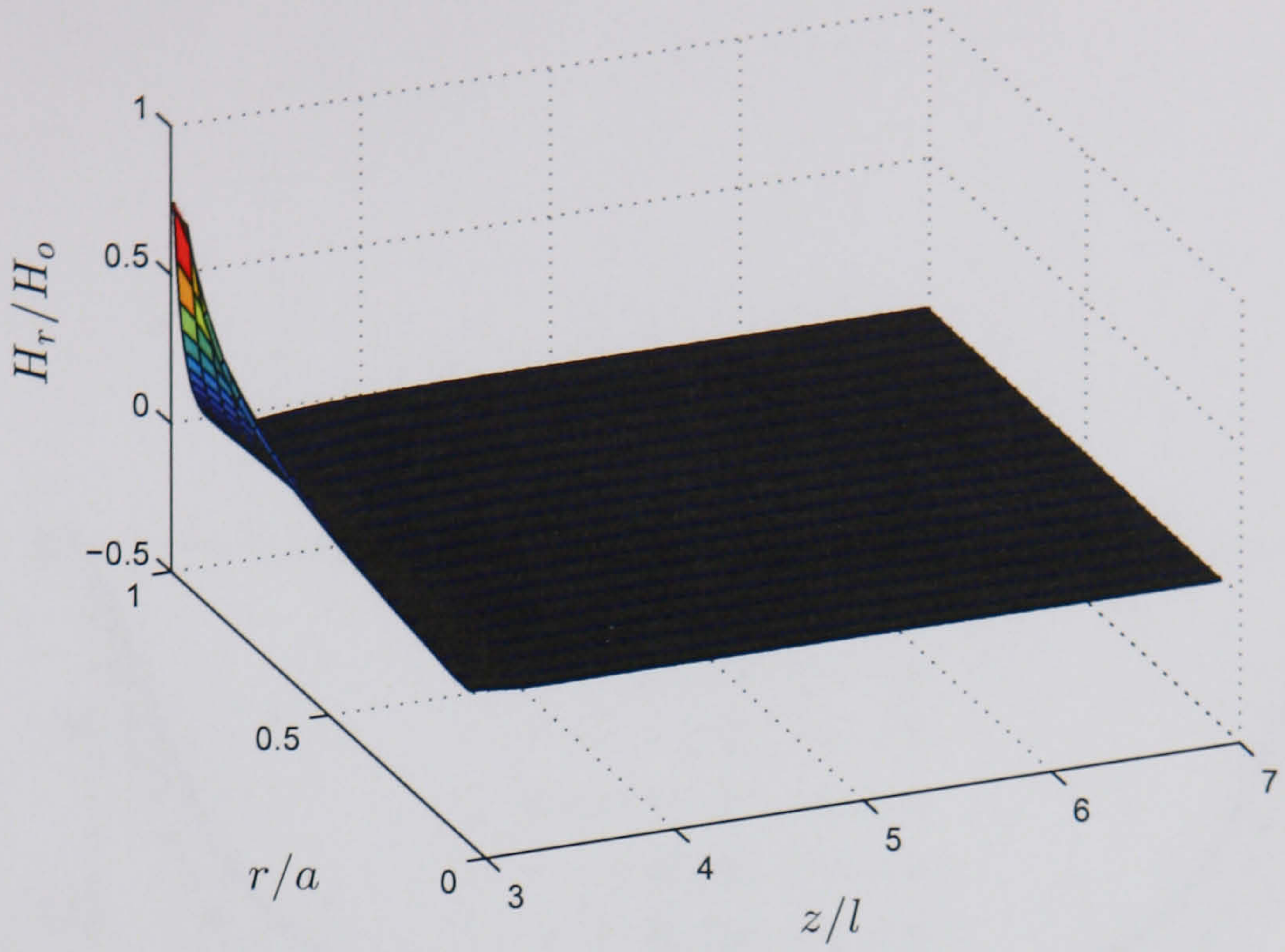


Figure 4.17: Magnitude of the dimensionless radial component of the magnetic field for the free surrounding space in (V) in an arbitrary (r, z).

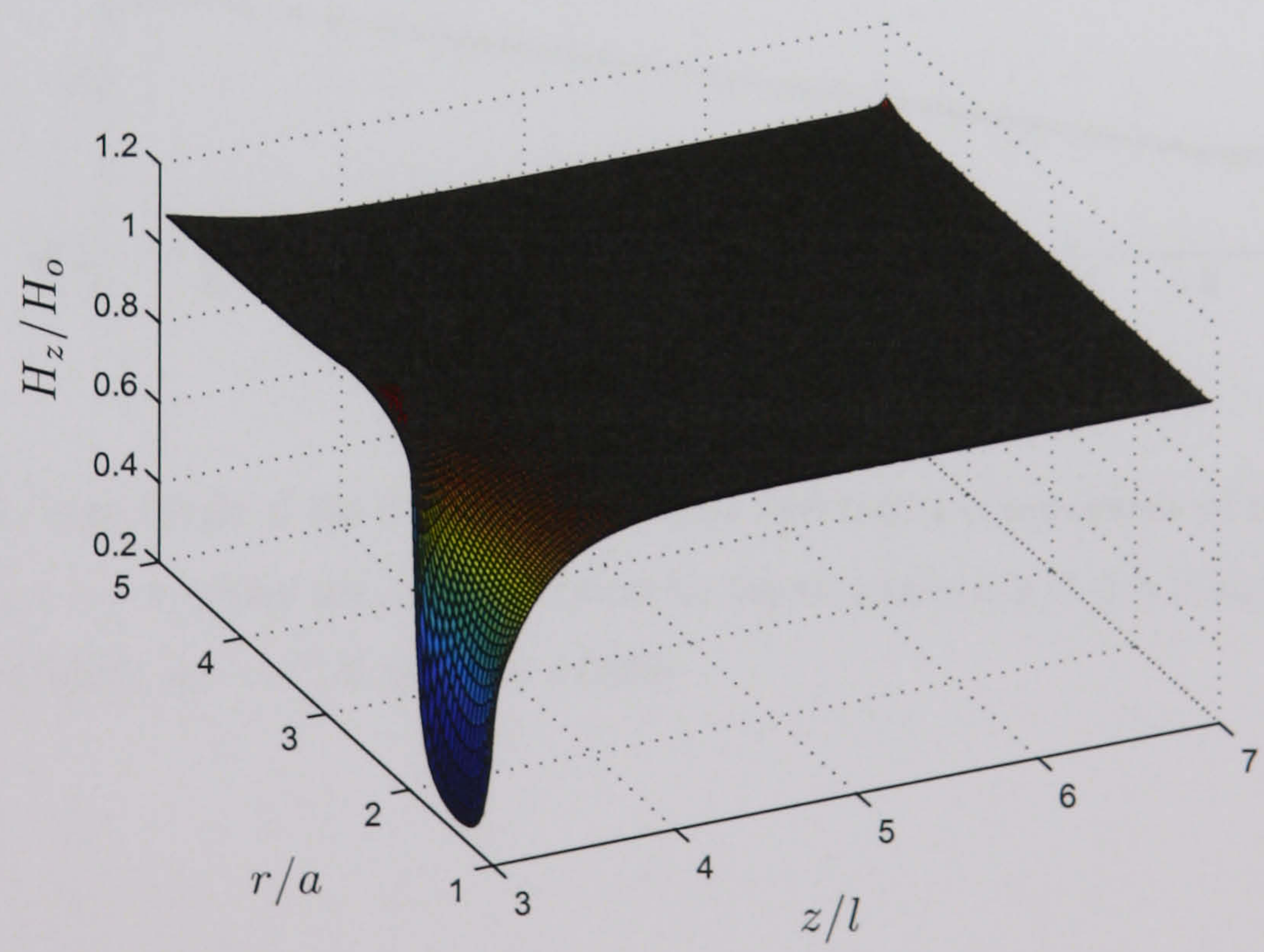


Figure 4.18: Magnitude of the dimensionless radial component of the magnetic field for the free surrounding space in (VI) in an arbitrary (r, z).

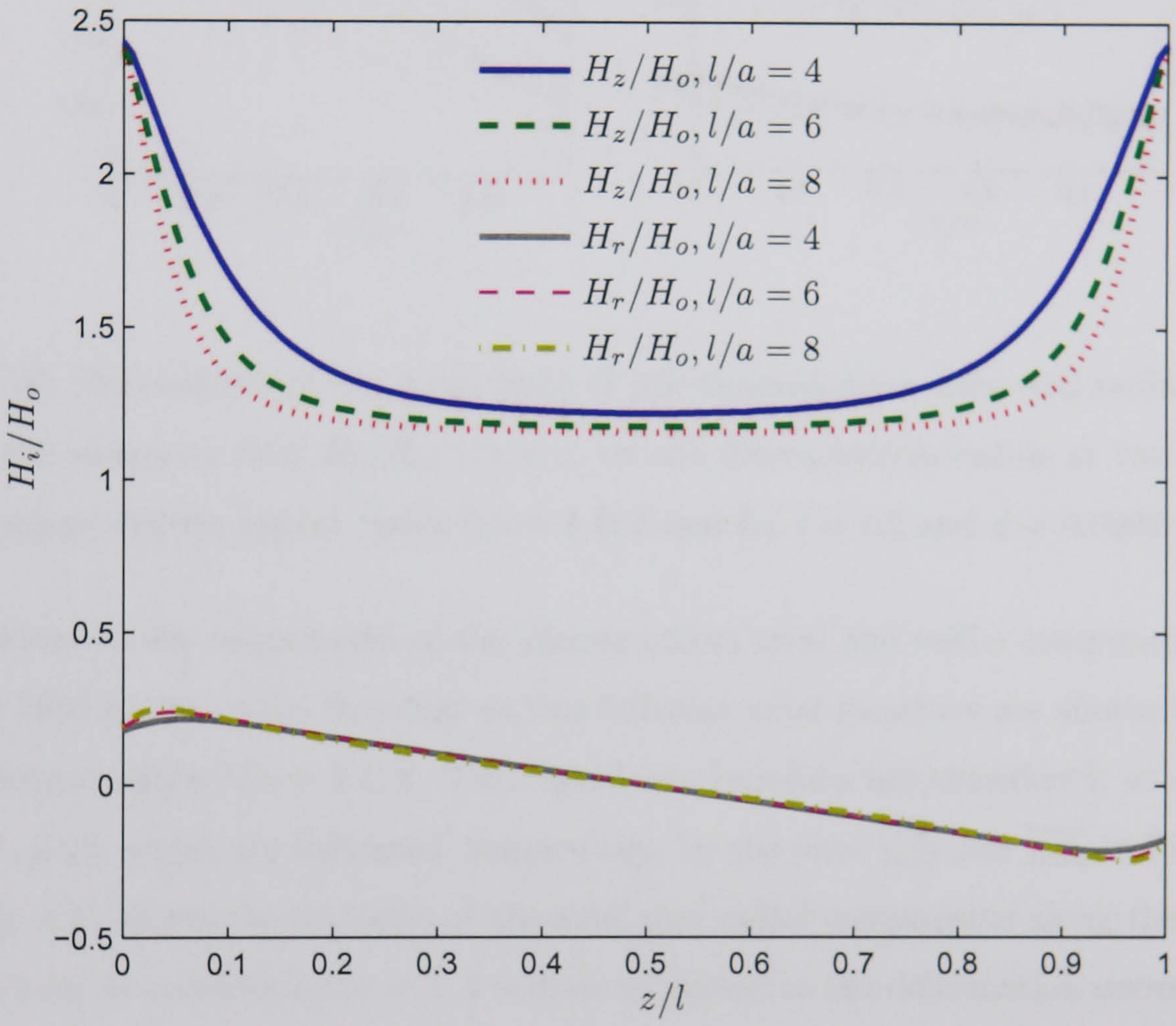


Figure 4.19: Magnitude of the dimensionless axial and radial components of the magnetic field H_i/H_o, $i = r, z$, along the axial direction for aspect ratios $l/a = 4, 6, 8$ and at location $\overline{i_1 i_2}$ in figure (4.2), for $c = 1.2$ and $d = 0.0002$.

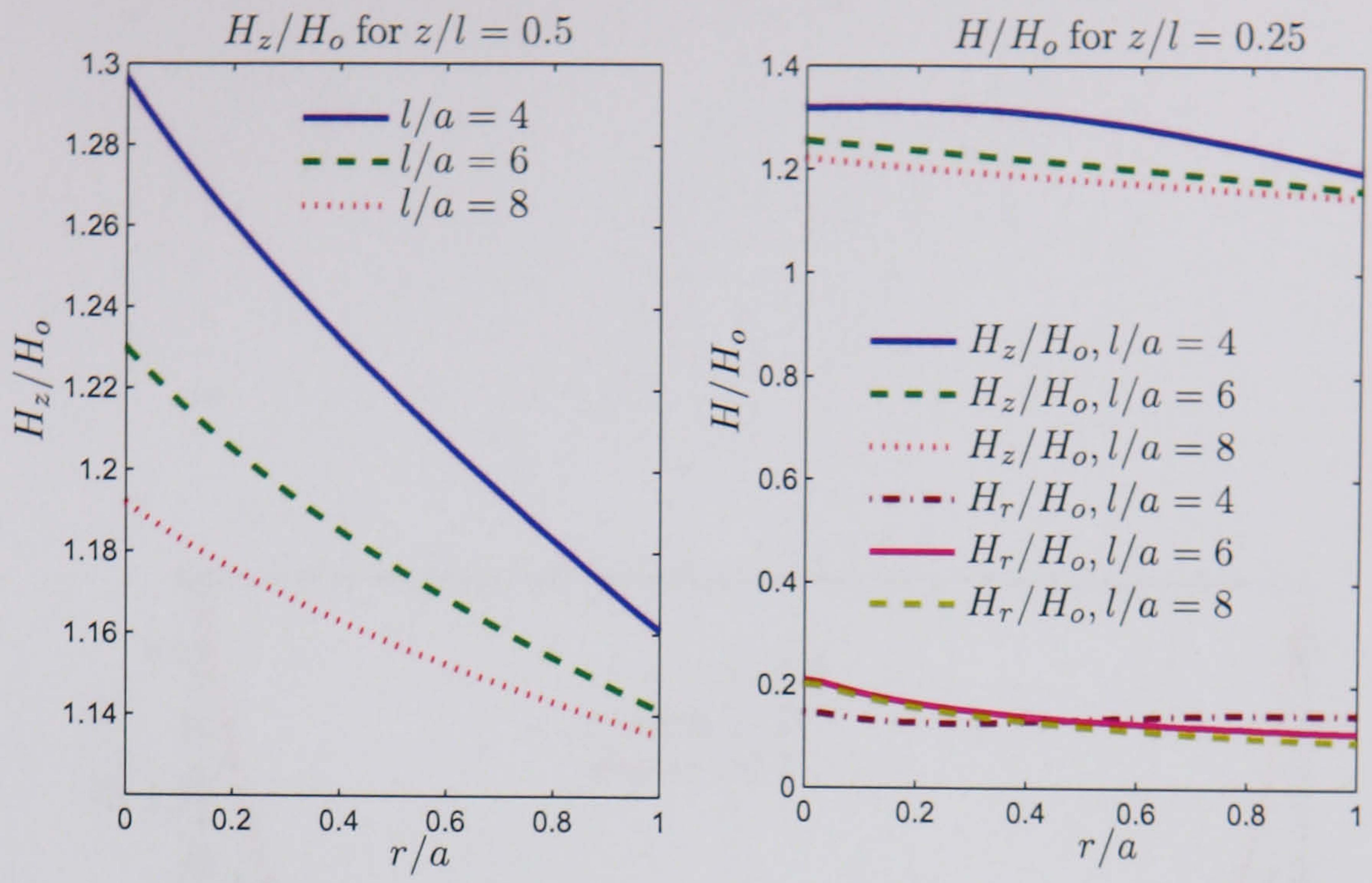


Figure 4.20: Dependence of the magnitude of the dimensionless axial and radial components of the magnetic field H_i/H_o, $i = r, z$, on the dimensionless radius at two different axial locations and for aspect ratios $l/a = 4, 6, 8$ and for $c = 1.2$ and $d = 0.0002$.

Variations in the magnitudes of the dimensionless axial and radial components of the magnetic field in the radial direction at two different axial locations are shown in Figure 4.20 for aspect ratios $l/a = 4, 6, 8$. Two significant locations are considered, at $z/l = 0.5$ and $z/l = 0.25$, which are indicated, respectively, by the lines $\overline{j_1 j_2}$ and $\overline{j_3 j_4}$ in Figure 4.2.

Figure 4.21 shows the variation of the axial and radial components along the line $\overline{i_1 i_2}$ when the tube is undeformed ($c = 1, d = 0$) or subjected to the deformation corresponding to the parameters $c = 0.5$ (extension), $c = 2$ (compression), with $d = 00002$ in each case. An aspect ratio of $l/a = 4$ is used here. Extension of a tube has, in particular, a tendency to make the axial field more uniform while compression has the opposite effect. Similarly, the variations of the same components along the radial direction, located at $z/l = 0.25$ and indicated by $\overline{j_3 j_4}$ in Figure 4.2, are shown in Figure 4.22 for the undeformed and deformed configurations.

In Figure 4.23, at the radial station corresponding to the line $\overline{i_1 i_2}$ (Figure 4.2), the axial component B_z and the radial component H_r are plotted in dimensionless forms (respectively B_z/B_o and H_r/H_o, with $B_o = \mu H_o$) for the whole axial range (z/l running from 0 to 9) in order to illustrate the continuity of the axial component of the magnetic induction $\mathbf{B}$ and the radial component of the magnetic field $\mathbf{H}$. The component B_z is

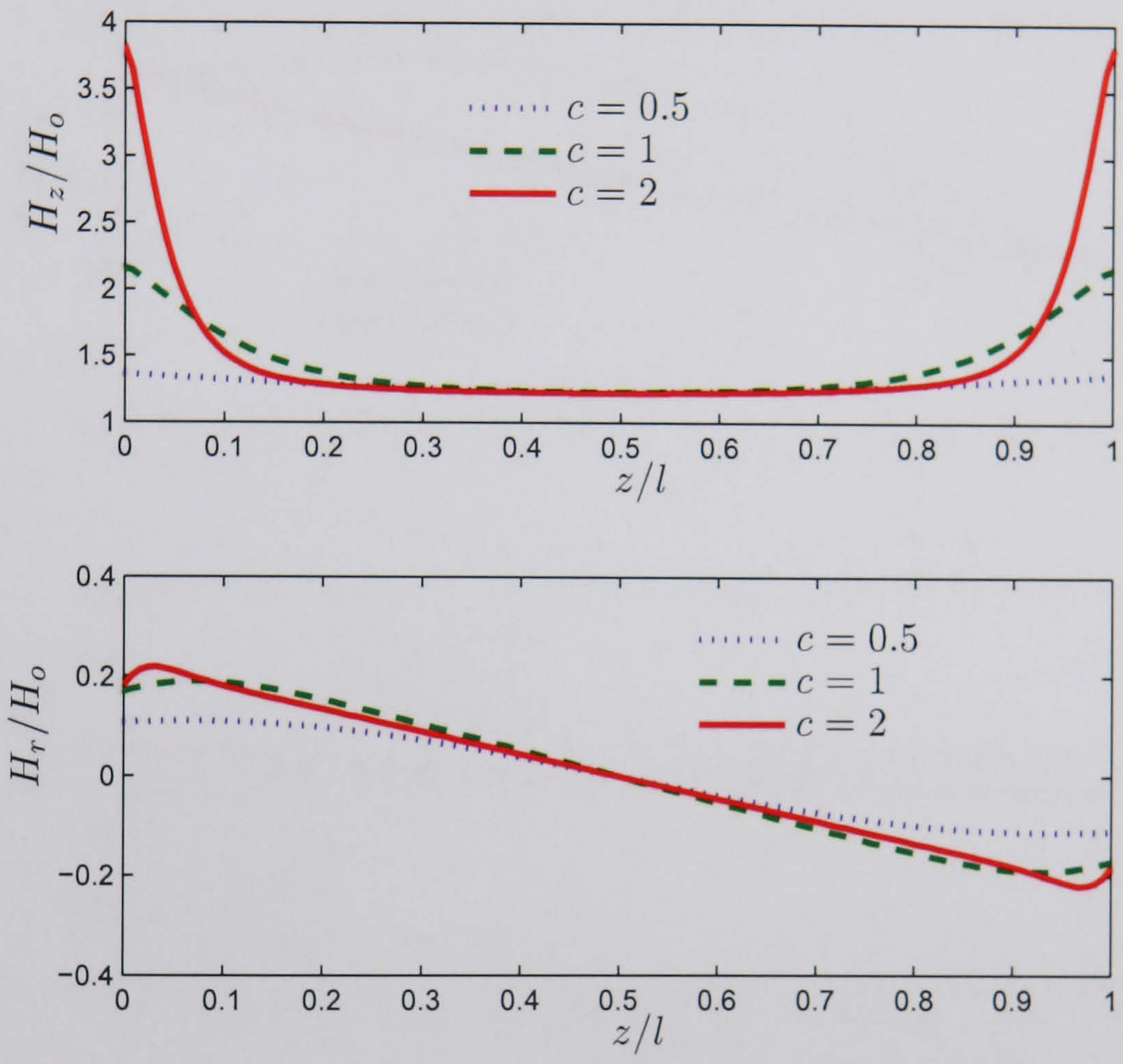


Figure 4.21: Variation of the dimensionless axial and radial components of the magnetic field for $l/a = 4$ along the axial direction $\overline{i_1 i_2}$ $(r/a = 3/2)$ for the undeformed configuration $(c = 1$, and $d = 0)$ and two deformed configurations corresponding to $c = 0.5$ and $c = 2$, with $d = 0.0002$.

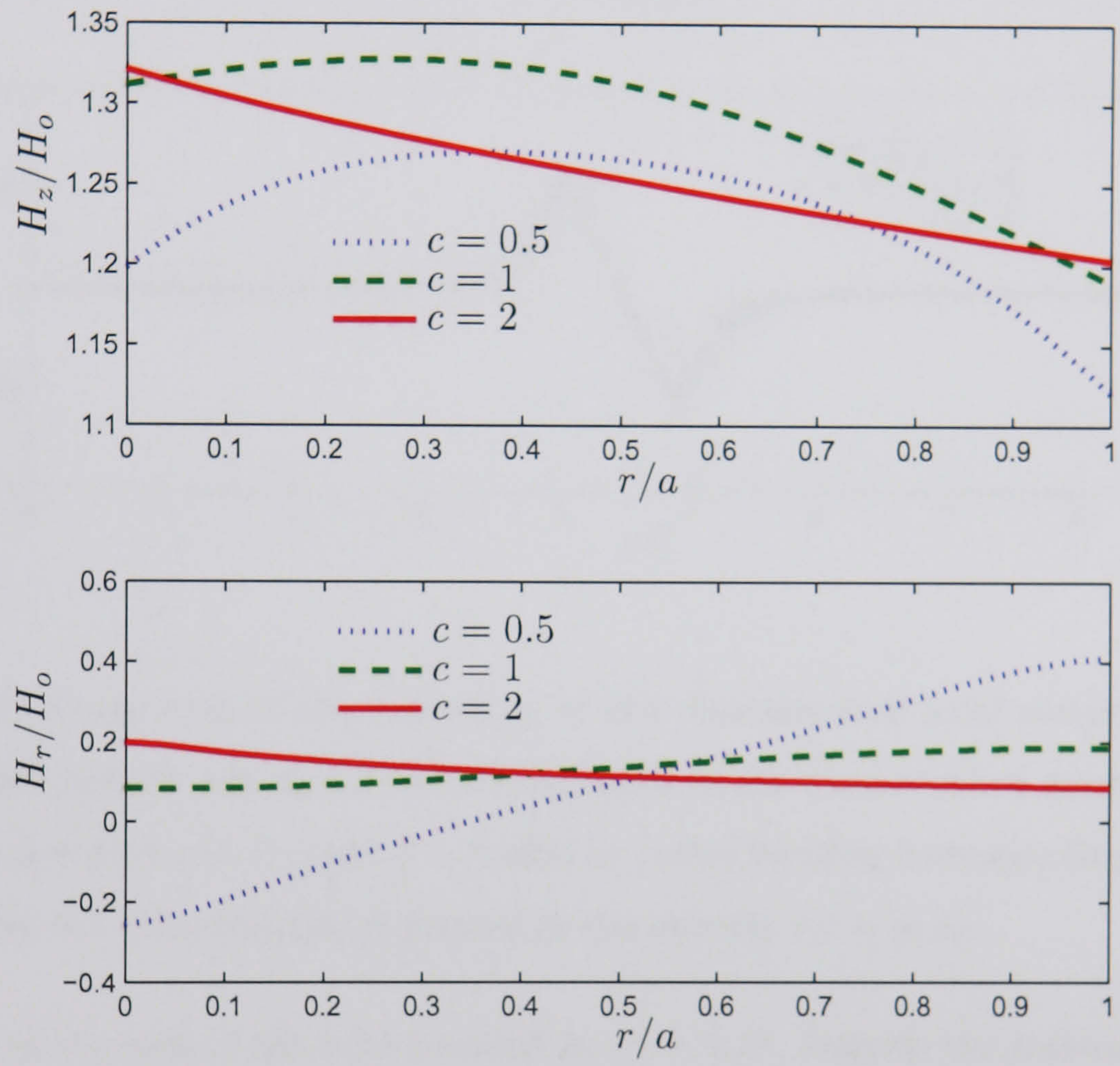


Figure 4.22: Variation of the dimensionless axial and radial components of the magnetic field for $l/a = 4$ along the radial direction at the axial location $z/l = 0.25$ (line $\overline{j_3 j_4}$) for the undeformed configuration ($c = 1$, and $d = 0$) and two deformed configurations corresponding to $c = 0.5$ and $c = 2$, with $d = 0.0002$.

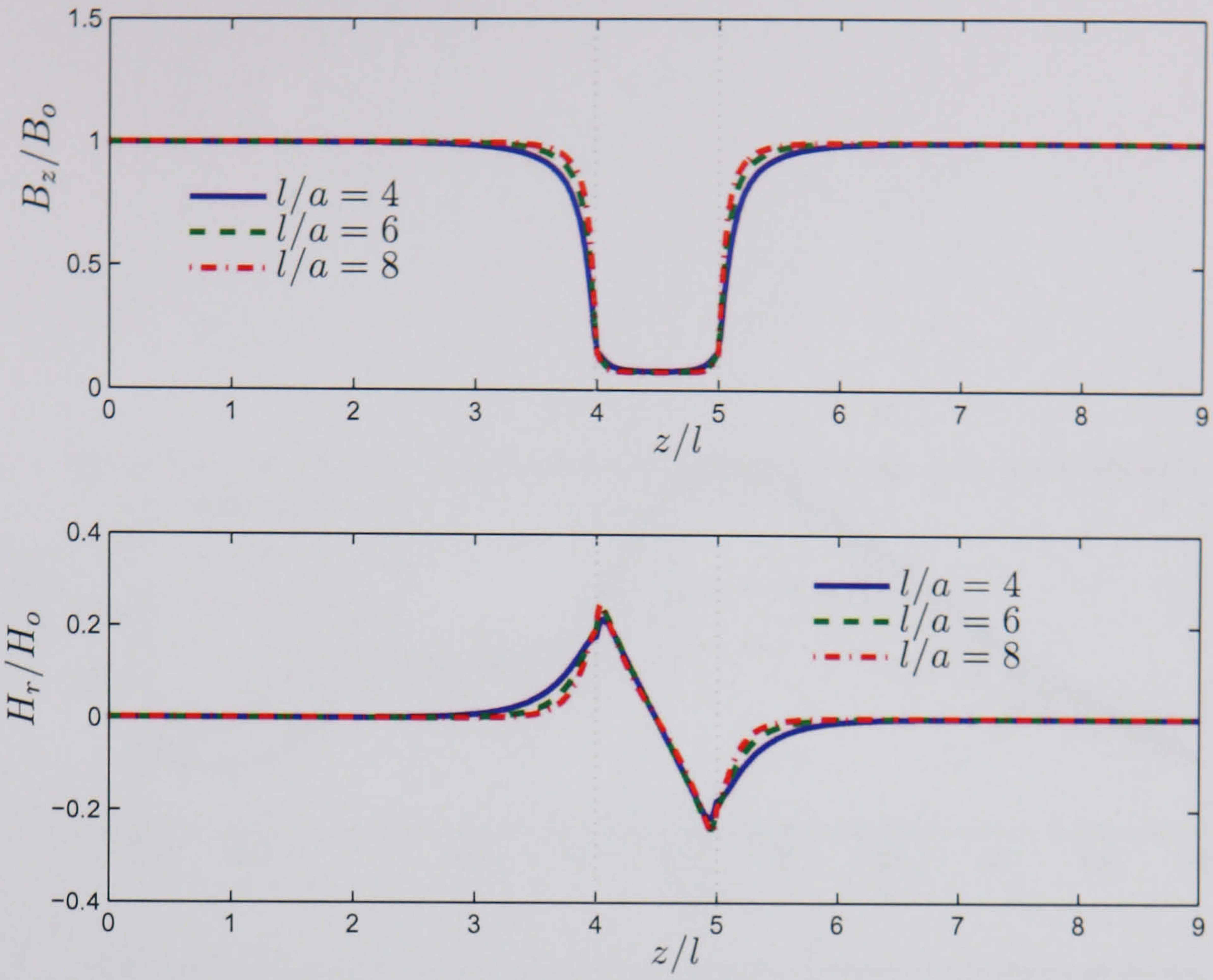


Figure 4.23: Illustration of the continuity of the dimensionless axial component of the magnetic induction $\mathbf{B}$ and on the radial component of the dimensionless magnetic field $\mathbf{H}$ for $l/a = 4, 6, 8$ with $c = 1.2$ and $d = 0.0002$ at radial location corresponding to the line $\overline{i_1 i_2}$ in Figure 4.2. The material is located in the interval $z/l = [4, 5]$.

continuous on the ends of the tube (located at $z/l = 4, 5$). Outside the material, the magnetic induction is obtained from the applied magnetic field by application of the standard equation $\mathbf{B} = \mu_o \mathbf{H}$. The continuity of the radial component H_r of the magnetic field on the same boundaries is illustrated in the lower graph in Figure 4.23. Outside the material, far from the tube the magnetic field reduces to a field with an axial components only.

Figure 4.24 shows the dimensionless radial component of the magnetic induction and the axial component of the magnetic field for aspect ratios of 4, 6 and 8 along the line $\overline{j_3 j_4}$ located at $z/l = 0.25$ (relative to the origin O_2 in Figure 4.2). The cylindrical boundaries correspond to $r/a = 1, 2$. Note that the radial component of the magnetic induction vanishes on the tube axis, as expected.

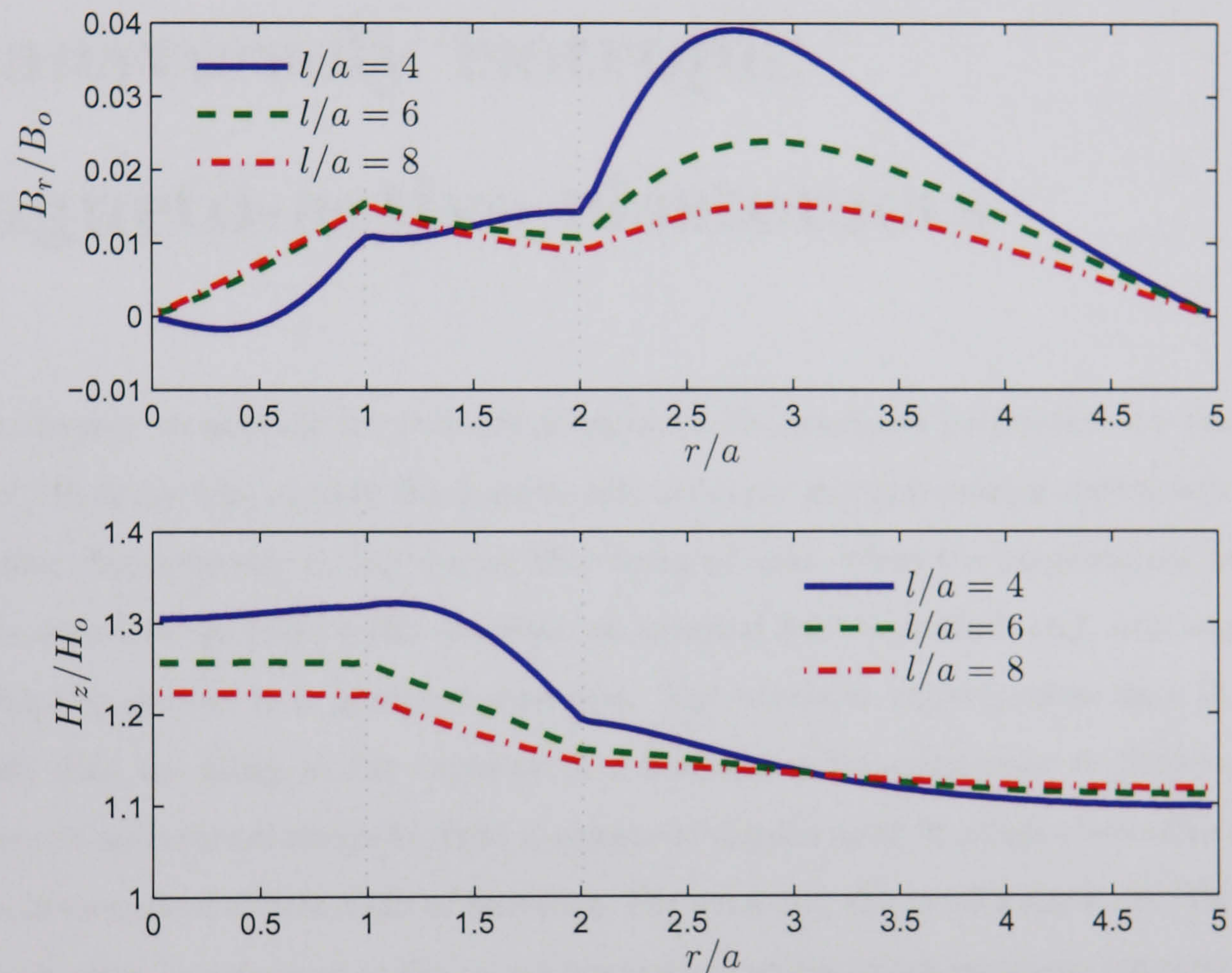


Figure 4.24: Variation of the radial component of the dimensionless magnetic induction and the axial component of the dimensionless magnetic field for aspect ratios $l/a = 4, 6, 8$ with $c = 1, 2$ and $d = 0.0002$ along the radial line $\overline{j_3 j_4}$ in Figure 4.2. The material is located in the interval $r/a = [1, 2]$

Chapter 5

Transversely isotropic magneto-active elastomers

In this chapter we provide the theoretical basis for the nonlinear properties of a particular class of MS materials, namely the transversely isotropic magneto-active elastomers, whose particular characteristic is that during the curing process, when the magneto-active particles are added to the rubber-like material, an external field is applied, and, as a result, the particles are aligned in a preferred direction. The available experimental data [7, 50, 59] suggests that by doing so the capacity of a magneto-active elastomer to deform in the presence of an external magnetic field is enhanced significantly in comparison with the situation of a random distribution of particles. For example, the results shown in [59], which have been used as reference in the recent theoretical works about isotropic magneto-active elastomers (see, for example, [30, 33–35]), were actually obtained for transversely isotropic magneto-active elastomers.

Three homogeneous boundary value problems are studied in Section 5.2. The first two of them, the simple shear of a block and the traction of a cylinder, are used in Section 5.3 in order to propose a first approximation for an energy potential, for the particular case of working with the magnetic field as the independent magnetic variable. Subsection 5.2.3 deals with the problem of the biaxial traction of a thin plate, which is used mainly in order to argue about the difficulties and the possibility of design of an experiment, which may allow us to find all the different derivatives of the energy function that appear in the constitutive equations (3.58), (3.62), (3.64) and (3.65).

As was mentioned above, the first two boundary value problems of Section 5.2 are used in order to propose a first approximation for an energy function in Section 5.3. Here, using as a reference the experimental data provided in [7,11] and [50], which were obtained essentially for the traction and the simple shear problems, a procedure in order to obtain the energy function is provided.

Since the particular form of the energy function given in Section 5.3 was found by appropriate simplifications of the general form of the energy function, a criterion, which should be independent of the particular form of the constitutive equation, must be provided, in order to know in advance whether these simplifications may be used for a particular material. This criterion is provided by the universal relations (see Section 4.1), which are relations that must hold for a given family of materials, independently of the particular form of the constitutive equation. One example of a linear universal relation is shown in Section 5.4.

Finally, in Section 5.5, some non-homogeneous boundary value problems are solved for two cases with cylindrical symmetry, the inflation and extension of a tube, and the torsion and extension of a cylinder (see [35] and [33] for the counterpart of these problems for isotropic materials). The particular form of the constitutive equation found in Section 5.3 is used in order to obtain closed form solutions for these problems.

Most of this chapter is based on a draft paper by Bustamante and Ogden [20].

5.1 Constitutive equation for transversely isotropic MS materials

Let us consider the case in which the magneto-active particles have a preferred alignment, which is caused by the presence of a magnetic field or a magnetic induction during the curing process. Let us denote by $\mathbf{a}_0$ the (unit) vector field associated with the alignment, and consider the following two cases for the independent magnetic variable.

5.1.1 The magnetic induction as the independent variable

In this case we have for the free energy function that

$$\Omega = \Omega(\mathbf{F}, \mathbf{B}_l, \mathbf{a}_0), \tag{5.1}$$

where the total stress was given by $(4.1)_1$ and (4.2) as

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}, \quad \boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I},$$

for a compressible and an incompressible material respectively. The magnetic field was given by (3.62)

$$\mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}.$$

As well as this, for the particle alignment field we have

$$\mathbf{a} = \mathbf{F}\mathbf{a}_0. \tag{5.2}$$

Now, for a transversely isotropic material, the energy potential Ω, which depends on a tensor field ($\mathbf{F}$) and two vector fields ($\mathbf{B}_l$ and $\mathbf{a}_0$), must be a function of the following ten invariants (see, for example, [102, 127]):

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2], \quad I_3 = \det \mathbf{c}, \tag{5.3}$$

$$I_4 = \mathbf{B}_l \cdot \mathbf{B}_l, \quad I_5 = \mathbf{B}_l \cdot \mathbf{c}\mathbf{B}_l, \quad I_6 = \mathbf{B}_l \cdot \mathbf{c}^2\mathbf{B}_l, \tag{5.4}$$

$$I_7 = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{a}_0, \quad I_8 = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{a}_0, \tag{5.5}$$

$$I_9 = \mathbf{a}_0 \cdot \mathbf{B}_l, \quad I_{10} = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{B}_l. \tag{5.6}$$

Then $\Omega = \Omega(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10})$.

Note that from the article by Zheng [127] we have that our function $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l, \mathbf{a}_0)$ should depend on eleven invariants; where the additional invariant would be $I_{11} = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{B}_l$; however, this invariant is not independent of the others. A proof is presented in Appendix B.

Now, in order to obtain appropriate expressions for the total stress and the magnetic field, consider the following derivatives

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{F}\mathbf{F}^T), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{5.7}$$

$$\frac{\partial I_5}{\partial \mathbf{F}} = 2\mathbf{B}_l \otimes \mathbf{F}\mathbf{B}_l, \quad \frac{\partial I_6}{\partial \mathbf{F}} = 2(\mathbf{B}_l \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{B}_l + \mathbf{F}^T\mathbf{F}\mathbf{B}_l \otimes \mathbf{F}\mathbf{B}_l), \tag{5.8}$$

$$\frac{\partial I_7}{\partial \mathbf{F}} = 2\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0, \quad \frac{\partial I_8}{\partial \mathbf{F}} = 2(\mathbf{a}_0 \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{a}_0 + \mathbf{F}^T\mathbf{F}\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0), \tag{5.9}$$

$$\frac{\partial I_{10}}{\partial \mathbf{F}} = \mathbf{a}_0 \otimes \mathbf{F}\mathbf{B}_l + \mathbf{B}_l \otimes \mathbf{F}\mathbf{a}_0. \tag{5.10}$$

Then, by using the chain rule (equation $(4.9)_1$ with $k = 1, ..., 10$) in $(4.1)_1$, we obtain for the total stress in the case of a compressible material[1]

$$\begin{aligned} \tau = {} & J^{-1}[2\mathbf{b}\Omega_1 + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 + 2I_3\Omega_3 + 2J^2\mathbf{B} \otimes \mathbf{B}\Omega_5 + 2J^2(\mathbf{B} \otimes \mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B} \otimes \mathbf{B})\Omega_6 \\ & + 2\mathbf{a} \otimes \mathbf{a}\Omega_7 + 2(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a} \otimes \mathbf{a})\Omega_8 + J(\mathbf{a} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{a})\Omega_{10}], \end{aligned} \tag{5.11}$$

[1] We use the notation $\Omega_i \equiv \frac{\partial \Omega}{\partial I_i}$ for $i = 1, ..., 10$.

where $\mathbf{a} = \mathbf{F}\mathbf{a}_0$. For an incompressible material we have from (4.2) that

$$\begin{aligned}\boldsymbol{\tau} =& 2\mathbf{b}\Omega_1 + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 - p\mathbf{I} + 2\mathbf{B}\otimes\mathbf{B}\Omega_5 + 2(\mathbf{B}\otimes\mathbf{bB} + \mathbf{bB}\otimes\mathbf{B})\Omega_6 \\ &+2\mathbf{a}\otimes\mathbf{a}\Omega_7 + 2(\mathbf{a}\otimes\mathbf{ba} + \mathbf{ba}\otimes\mathbf{a})\Omega_8 + (\mathbf{a}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{a})\Omega_{10}. \end{aligned} \quad (5.12)$$

Consider now the following derivatives of the invariants

$$\frac{\partial I_4}{\partial \mathbf{B}_l} = 2\mathbf{B}_l, \quad \frac{\partial I_5}{\partial \mathbf{B}_l} = 2\mathbf{c}\mathbf{B}_l, \quad \frac{\partial I_6}{\partial \mathbf{B}_l} = 2\mathbf{c}^2\mathbf{B}_l, \quad \frac{\partial I_9}{\partial \mathbf{B}_l} = \mathbf{a}_0, \quad (5.13)$$

$$\frac{\partial I_{10}}{\partial \mathbf{B}_l} = \mathbf{c}\mathbf{a}_0. \quad (5.14)$$

Using the chain rule in (3.62), we obtain for the magnetic field in the case of a compressible material

$$\mathbf{H} = 2J\mathbf{b}^{-1}\mathbf{B}\Omega_4 + 2J\mathbf{B}\Omega_5 + 2J\mathbf{bB}\Omega_6 + \mathbf{b}^{-1}\mathbf{a}\Omega_9 + \mathbf{a}\Omega_{10}, \quad (5.15)$$

and for an incompressible material (where $J \equiv 1$)

$$\mathbf{H} = 2\mathbf{b}^{-1}\mathbf{B}\Omega_4 + 2\mathbf{B}\Omega_5 + 2\mathbf{bB}\Omega_6 + \mathbf{b}^{-1}\mathbf{a}\Omega_9 + \mathbf{a}\Omega_{10}. \quad (5.16)$$

Some restrictions may be obtained on Ω, as was done, for example, in [19], by studying the particular case in which there is no external load and field. In such a case we have

$$\mathbf{F} = \mathbf{I}, \quad \mathbf{a} = \mathbf{a}_0,$$

and

$$I_1 = I_2 = 3, \quad I_3 = 1, \quad I_4 = I_5 = I_6 = 0, \quad I_7 = I_8 = 1, \quad I_9 = I_{10} = 0.$$

Let us use the symbol $\bar{f}$ in order to denote a function $f = f(I_k)$, $k = 1, ..., 10$, evaluated for the above values of I_k. Now, if there are no residual stresses and no residual field (which would mean we would not take account the hysteresis phenomenon for the magnetic field), we need $\boldsymbol{\tau} = \mathbf{0}$ and $\mathbf{H} = \mathbf{0}$. In the first case, from (5.11), this implies

$$(2\bar{\Omega}_1 + 4\bar{\Omega}_2 + 2\bar{\Omega}_3)\mathbf{I} + 2(\bar{\Omega}_7 + 2\bar{\Omega}_8)\mathbf{a}_0\otimes\mathbf{a}_0 = \mathbf{0}, \quad (5.17)$$

which should hold for any particular form of the free energy function and the field $\mathbf{a}_0$, therefore we have the following restrictions

$$\bar{\Omega}_1 + 2\bar{\Omega}_2 + \bar{\Omega}_3 = 0, \quad (5.18)$$

$$\bar{\Omega}_7 + 2\bar{\Omega}_8 = 0, \quad (5.19)$$

where in the case of an incompressible material from (5.12) we have that the equation (5.18) should be replaced by

$$2\bar{\Omega}_1 + 4\bar{\Omega}_2 - p = 0. \quad (5.20)$$

Finally, from $\mathbf{H} = \mathbf{0}$ and (5.15) we get

$$\bar{\Omega}_9 + \bar{\Omega}_{10} = 0. \tag{5.21}$$

Remark Consider the case when there is no field $\mathbf{B} = \mathbf{0}$, but there is deformation (due to some mechanical load) $\mathbf{F} \neq \mathbf{I}$, the question is: Is there any 'induced' magnetic field only due to the deformation as in the counterpart case of linear piezoelectric materials?.

For piezoelectric materials we have that due to the particular atomic structure, a deformation provokes an asymmetric rearrangement of charges, and as a result we have a non-zero polarization field (see, for example, [67]). However, these materials are very special, and there is no reason to expect something similar in the magnetic case; from the description of the basic properties of magnetic materials (Subsection 2.2.1.2) there seems to be no similar mechanism that might cause the presence of a magnetic field for an MS elastomer when there is deformation but no external field. Then, an additional restriction in the form of the constitutive equation would be to have $\mathbf{H} = \mathbf{0}$ whenever $\mathbf{B} = \mathbf{0}$; for the isotropic case (equation (4.13)) this does not introduce any additional restriction, but this is not the case for transversely MS materials. Let us denote by $\check{f}$ the function $f = f(I_k), \quad k = 1, ..., 10$ evaluated for $I_4 = I_5 = I_6 = I_9 = I_{10} = 0$, then from (5.15) the restriction $\mathbf{H} = \mathbf{0}$ if $\mathbf{B} = \mathbf{0}$ implies

$$\mathbf{b}^{-1}\check{\Omega}_9 + \mathbf{I}\check{\Omega}_{10} = \mathbf{0}, \tag{5.22}$$

which should holds for any tensor $\mathbf{b}$, therefore this implies

$$\check{\Omega}_9 = \check{\Omega}_{10} = 0. \tag{5.23}$$

Note that the above restrictions would not mean that there is no a coupling between the magnetic effect and the deformation, but it only means that is always necessary to have an external field for such a coupling to exist. As well as this, note that if (5.23) holds, this implies that (5.21) holds.

5.1.2 The magnetic field as the independent magnetic variable

In this second case from Subsection 3.3.3 we would have for the free energy function Ω^* that

$$\Omega^* = \Omega^*(\mathbf{F}, \mathbf{H}_l, \mathbf{a}_0), \tag{5.24}$$

where the total stress was given by $(4.14)_1$ and (4.15) as

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I},$$

for a compressible and an incompressible material respectively. The magnetic induction was given by (3.65) as

$$\mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}.$$

Similarly to the previous case, a transversely isotropic material must be a function of the following alternative set of ten invariants

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2], \quad I_3 = \det\mathbf{c}, \tag{5.25}$$

$$K_4 = \mathbf{H}_l \cdot \mathbf{H}_l, \quad K_5 = \mathbf{H}_l \cdot \mathbf{c}\mathbf{H}_l, \quad K_6 = \mathbf{H}_l \cdot \mathbf{c}^2\mathbf{H}_l, \tag{5.26}$$

$$I_7 = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{a}_0, \quad I_8 = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{a}_0, \tag{5.27}$$

$$K_9 = \mathbf{a}_0 \cdot \mathbf{H}_l, \quad K_{10} = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{H}_l. \tag{5.28}$$

Consider now the following derivatives

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{F}\mathbf{F}^T), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{5.29}$$

$$\frac{\partial K_5}{\partial \mathbf{F}} = 2\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l, \quad \frac{\partial K_6}{\partial \mathbf{F}} = 2(\mathbf{H}_l \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{H}_l + \mathbf{F}^T\mathbf{F}\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l), \tag{5.30}$$

$$\frac{\partial I_7}{\partial \mathbf{F}} = 2\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0, \quad \frac{\partial I_8}{\partial \mathbf{F}} = 2(\mathbf{a}_0 \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{a}_0 + \mathbf{F}^T\mathbf{F}\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0), \tag{5.31}$$

$$\frac{\partial K_{10}}{\partial \mathbf{F}} = \mathbf{a}_0 \otimes \mathbf{F}\mathbf{H}_l + \mathbf{H}_l \otimes \mathbf{F}\mathbf{a}_0. \tag{5.32}$$

Then, by using the chain rule for derivatives in $(4.14)_1$, we obtain for an unconstrained material the following expression[2]

$$\begin{aligned}\boldsymbol{\tau} = \; & J^{-1}[2\mathbf{b}\Omega_1^* + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* + 2I_3\Omega_3^* + 2\mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H}\Omega_5^* + 2(\mathbf{b}\mathbf{H} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{b}\mathbf{H})\Omega_6^* \\ & + 2\mathbf{a} \otimes \mathbf{a}\Omega_7^* + 2(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a} \otimes \mathbf{a})\Omega_8^* + (\mathbf{a} \otimes \mathbf{b}\mathbf{H} + \mathbf{b}\mathbf{H} \otimes \mathbf{a})\Omega_{10}^*,\end{aligned} \tag{5.33}$$

and for an incompressible material, from (4.15) we get

$$\begin{aligned}\boldsymbol{\tau} = \; & 2\mathbf{b}\Omega_1^* + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* - p^*\mathbf{I} + 2\mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H}\Omega_5^* + 2(\mathbf{b}\mathbf{H} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{b}\mathbf{H})\Omega_6^* \\ & + 2\mathbf{a} \otimes \mathbf{a}\Omega_7^* + 2(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a} \otimes \mathbf{a})\Omega_8^* + (\mathbf{a} \otimes \mathbf{b}\mathbf{H} + \mathbf{b}\mathbf{H} \otimes \mathbf{a})\Omega_{10}^*.\end{aligned} \tag{5.34}$$

Consider the following derivatives

$$\frac{\partial K_4}{\partial \mathbf{H}_l} = 2\mathbf{H}_l, \quad \frac{\partial K_5}{\partial \mathbf{H}_l} = 2\mathbf{c}\mathbf{H}_l, \quad \frac{\partial K_6}{\partial \mathbf{H}_l} = 2\mathbf{c}^2\mathbf{H}_l, \quad \frac{\partial K_9}{\partial \mathbf{H}_l} = \mathbf{a}_0, \tag{5.35}$$

$$\frac{\partial K_{10}}{\partial \mathbf{K}_l} = \mathbf{c}\mathbf{a}_0. \tag{5.36}$$

[2]Here the notation Ω_i^* means the partial derivative of Ω^* in I_i if $i = 1, 2, 3, 7, 8$, or in K_i if $i = 4, 5, 6, 9, 10$.

As a result, from (3.65) and using the chain rule, we get for the magnetic induction, in the case of an unconstrained material, that

$$\mathbf{B} = -J^{-1}(2\mathbf{bH}\Omega_4^* + 2\mathbf{b}^2\mathbf{H}\Omega_5^* + 2\mathbf{b}^3\mathbf{H}\Omega_6^* + \mathbf{a}\Omega_9^* + \mathbf{ba}\Omega_{10}^*), \tag{5.37}$$

and for an incompressible material we have

$$\mathbf{B} = -(2\mathbf{bH}\Omega_4^* + 2\mathbf{b}^2\mathbf{H}\Omega_5^* + 2\mathbf{b}^3\mathbf{H}\Omega_6^* + \mathbf{a}\Omega_9^* + \mathbf{ba}\Omega_{10}^*). \tag{5.38}$$

As in the previous case, some restrictions for Ω^* may be obtained if we study the case when there is no external loads or fields, assuming that there are no residual stresses and no residual fields. In such a case we have

$$I_1 = I_2 = 3, \quad I_3 = 1, \quad K_4 = K_5 = K_6 = 0, \quad I_7 = I_8 = 1, \quad K_9 = K_{10} = 0.$$

Again, let us use the symbol $\bar{f}$ in order to denote the function f evaluated with such values. Then, if there is no residual stresses, from (5.33) we have for an unconstrained material

$$\bar{\Omega}_1^* + 2\bar{\Omega}_2^* + \bar{\Omega}_3^* = 0, \tag{5.39}$$

$$\bar{\Omega}_7^* + 2\bar{\Omega}_8^* = 0, \tag{5.40}$$

and for an incompressible material (5.39) must be replaced by

$$2\bar{\Omega}_1^* + 4\bar{\Omega}_2^* - p^* = 0. \tag{5.41}$$

If there is no residual field, we have from (5.37) that

$$\bar{\Omega}_9^* + \bar{\Omega}_{10}^* = 0. \tag{5.42}$$

As in the previous subsection, if we assume that there is no induced magnetic induction when the external magnetic field is zero, and when there is deformation, then we would have the following extra restriction for Ω^*

$$\mathbf{I}\breve{\Omega}_9^* + \mathbf{b}\breve{\Omega}_{10}^* = \mathbf{0} \tag{5.43}$$

where $\breve{\Omega}_k^*$, $k = 9, 10$ is the function Ω_k^* evaluated for $K_4 = K_5 = K_6 = K_9 = K_{10} = 0$. Therefore we obtain the restrictions

$$\breve{\Omega}_9^* = \breve{\Omega}_{10}^* = 0. \tag{5.44}$$

5.2 Boundary value problems: homogeneous deformations

In this section, as was mentioned in the introduction, we will consider three simple boundary value problems, where the deformation is essentially homogeneous. These problems are the simple shear of a block, the simple tension of a cylinder and the biaxial tension of a thin plate.

The idea of the two first problems is the following. Little experimental information is available at the present moment for magneto-active elastomers. Due to the complexity of the phenomena involved, it is not as easy as in the pure non-linear elastic case to do experiments in order to find an appropriate form for the energy function. Moreover, the large number of invariants involved makes the analysis even more difficult. As we will see in detail in Section 5.3, most of these experimental researches have been done for rather simple problems. The idea is to have general results for these two homogeneous problems, and then to use them in the following section in order to look for a preliminary form for the energy function.

The idea of studying the biaxial tension problem is the following. As is well known for the pure non-linear elastic problem (see, for example, [78]), it is possible in the case of an incompressible rubber-like material to find the behaviour of the energy function for a wide range of values of the invariants by using only the biaxial tension experiment for a thin plate. It is not difficult to see that in our case such is not the situation; nevertheless, we study in which cases it would be possible at least theoretically to find the energy function as for the pure elastic case.

5.2.1 Simple shear

Simple shear for the magneto-elastic case has been treated several times in the literature; see, for example, [35] for the isotropic problem; see also [12] for isotropic materials but with a slightly different formulation for the energy function.

Consider the simple shear deformation

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3. \tag{5.45}$$

As a result, we obtain for the deformation gradient and the left and right Cauchy deformation tensors, respectively, the components

$$\mathsf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.46}$$

The two first invariants $(5.25)_1$ and $(5.25)_2$ are

$$I_1 = I_2 = 3 + \gamma^2. \tag{5.47}$$

We will consider the two cases for the independent magnetic variable separately.

5.2.1.1 The magnetic field as the independent variable

We will only consider the case of an external magnetic field that is uniform and parallel to the direction 2 (see [59] and [50] for the experimental counterpart of this problem, in particular regarding the orientation of the field and the particles). Then,

$$\mathsf{H}_l = (0, H_o, 0)^T, \tag{5.48}$$

where H_o is a constant. From (5.26) we have

$$K_4 = H_o^2, \quad K_5 = H_o^2(1+\gamma^2), \quad K_6 = H_o^2[\gamma^2 + (1+\gamma^2)^2]. \tag{5.49}$$

The Maxwell stress was given by (3.44) as

$$\tau_m = \mu_o \left[\mathbf{H} \otimes \mathbf{H} - \frac{1}{2}(\mathbf{H} \cdot \mathbf{H})\mathbf{I} \right], \tag{5.50}$$

where from (3.9) we had that $\mathbf{H} = \mathbf{F}^{-T}\mathbf{H}_l$; then in this case $\mathsf{H} = (0, H_o, 0)^T$, and as a result the non-zero components of the Maxwell stress are

$$\tau_{m11} = -\frac{H_o^2 \mu_o}{2}, \quad \tau_{m22} = \frac{H_o^2 \mu_o}{2}, \quad \tau_{m33} = -\frac{H_o^2 \mu_o}{2}. \tag{5.51}$$

Now, there are several options for particle alignments; let us consider the following two simple cases (see, in particular, [50] for experimental results obtained for a slab under shear with the two following particle alignments).

Particle alignment in the x_2 direction. In this case the initial particle alignment is given by the field

$$\mathsf{a}_0 = (0, 1, 0)^T; \tag{5.52}$$

as a result, from (5.2), we get

$$\mathsf{a} = (\gamma, 1, 0)^T, \tag{5.53}$$

and the invariants (5.27) and (5.28) are given by

$$I_7 = 1 + \gamma^2, \quad I_8 = \gamma^2 + (1+\gamma^2)^2, \quad K_9 = H_o, \quad K_{10} = H_o(1+\gamma^2). \tag{5.54}$$

Then, from (5.34) the components of the total stress are given as

$$\begin{aligned}
\tau_{11} &= -p^* + 2(1+\gamma^2)\Omega_1^* + 2(2+\gamma^2)\Omega_2^* + 2H_o^2\gamma^2\Omega_5^* + 2H_o^2\gamma^2(2+\gamma^2)\Omega_6^* \\
&\quad +2\gamma^2\Omega_7^* + 4\gamma^2(2+\gamma^2)\Omega_8^* + 2H_o\gamma^2\Omega_{10}^*, && (5.55)\\
\tau_{22} &= -p^* + 2\Omega_1^* + 4\Omega_2^* + 2H_o^2\Omega_5^* + 4H_o^2(1+\gamma^2)\Omega_6^* + 2\Omega_7^* + 4(1+\gamma^2)\Omega_8^* \\
&\quad +2H_o\Omega_{10}^*, && (5.56)\\
\tau_{33} &= -p^* + 2\Omega_1^* + 2(2+\gamma^2)\Omega_2^*, && (5.57)\\
\tau_{12} &= 2\gamma[\Omega_1^* + \Omega_2^* + H_o^2\Omega_5^* + H_o^2(3+2\gamma^2)\Omega_6^* + \Omega_7^* + (3+2\gamma^2)\Omega_8^* + H_o\Omega_{10}^*], && (5.58)\\
\tau_{23} &= \tau_{13} = 0. && (5.59)
\end{aligned}$$

Regarding the components of the magnetic induction we have from (5.38) that

$$\begin{aligned}
B_1 &= -\gamma[2H_o\Omega_4^* + 2H_o(2+\gamma^2)\Omega_5^* + 2H_o(3+4\gamma^2+\gamma^4)\Omega_6^* + \Omega_9^* + (2+\gamma^2)\Omega_{10}^*], && (5.60)\\
B_2 &= -[2H_o\Omega_4^* + 2H_o(1+\gamma^2)\Omega_5^* + 2H_o(1+3\gamma^2+\gamma^4)\Omega_6^* + \Omega_9^* + (1+\gamma^2)\Omega_{10}^*], && (5.61)\\
B_3 &= 0. && (5.62)
\end{aligned}$$

Particle alignment in the x_1 direction. In this case the initial particle alignment is given by the following field in the reference and the current configurations (from (5.2)), respectively, as

$$\mathsf{a}_0 = (1, 0, 0)^T, \quad \mathsf{a} = (1, 0, 0)^T. \tag{5.63}$$

The invariants we need to recalculate are (5.27) and (5.28) and they are given by

$$I_7 = 1, \quad I_8 = 1 + \gamma^2, \quad K_9 = 0, \quad K_{10} = H_o\gamma; \tag{5.64}$$

as a result, from (5.34), the components of the total stress are

$$\begin{aligned}
\tau_{11} &= -p^* + 2(1+\gamma^2)\Omega_1^* + 2(2+\gamma^2)\Omega_2^* + 2H_o^2\gamma^2\Omega_5^* + 4H_o^2\gamma^2(2+\gamma^2)\Omega_6^* + 2\Omega_7^* \\
&\quad +4(1+\gamma^2)\Omega_8^* + 2H_o\gamma\Omega_{10}^*, && (5.65)\\
\tau_{22} &= -p^* + 2\Omega_1^* + 4\Omega_2^* + 2H_o^2\Omega_5^* + 2H_o^2(1+\gamma^2)\Omega_6^*, && (5.66)\\
\tau_{33} &= -p^* + 2\Omega_1^* + 2(2+\gamma^2)\Omega_2^*, && (5.67)\\
\tau_{12} &= 2\gamma\Omega_1^* + 2\gamma\Omega_2^* + 2H_o^2\gamma\Omega_5^* + 2H_o^2\gamma(3+2\gamma^2)\Omega_6^* + 2\gamma\Omega_8^* + H_o\Omega_{10}^*, && (5.68)\\
\tau_{13} &= \tau_{23} = 0, && (5.69)
\end{aligned}$$

and the components of the magnetic induction are given by (5.38) as

$$B_1 = -[2H_o\gamma\Omega_4^* + 2H_o\gamma(2+\gamma^2)\Omega_5^* + 2H_o\gamma(3+4\gamma^2+\gamma^4)\Omega_6^* + \Omega_9^* + (1+\gamma^2)\Omega_{10}^*], \quad (5.70)$$

$$B_2 = -[2H_o\Omega_4^* + 2H_o(1+\gamma^2)\Omega_5^* + 2H_o(1+3\gamma^2+\gamma^4)\Omega_6^* + \gamma\Omega_{10}^*], \quad (5.71)$$

$$B_3 = 0. \quad (5.72)$$

5.2.1.2 The magnetic induction as the independent variable

For completeness we will consider the same problem, but now taking the magnetic induction as the independent magnetic variable (some of the results obtained by, for example, Jolly et al [59] correspond to this situation). As in the previous problem we consider only the case of an external magnetic induction, which is constant and parallel to the direction 2; then

$$\mathsf{B}_l = (0, B_o, 0)^T, \quad (5.73)$$

where B_o is constant. The fourth, fifth and sixth invariants are given by (5.4) as

$$I_4 = B_o^2, \quad I_5 = B_o^2(1+\gamma^2), \quad I_6 = B_o^2[\gamma^2 + (1+\gamma^2)^2]. \quad (5.74)$$

The Maxwell stress in this problem is given by (3.43) as

$$\tau_m = \mu_o^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right], \quad (5.75)$$

and in this case, from (3.7), $\mathbf{B} = \mathbf{F}\mathbf{B}_l$, and we have $\mathbf{B} = (\gamma, 1, 0)^T B_o$; hence, the non-zero components of the Maxwell stress are

$$\tau_{m11} = \mu_o^{-1}\frac{B_o^2}{2}(\gamma^2 - 1), \quad \tau_{m22} = \mu_o^{-1}\frac{B_o^2}{2}(1-\gamma^2), \quad \tau_{m33} = -\mu_o^{-1}\frac{B_o^2}{2}(1+\gamma^2), \quad (5.76)$$

$$\tau_{m12} = \mu_o^{-1}B_o^2\gamma. \quad (5.77)$$

We consider only the two following simple particle alignments.

Particle alignment in the x_2 direction. As in the previous problem let us consider the field (5.52) for the particle alignment; then the rest of the invariants are given by (5.5) and (5.6) as

$$I_7 = 1+\gamma^2, \quad I_8 = \gamma^2 + (1+\gamma^2)^2, \quad I_9 = B_o, \quad I_{10} = B_o(1+\gamma^2), \quad (5.78)$$

and the components of the total stress given by (5.12) are

$$\begin{aligned}\tau_{11} &= -p + 2(1+\gamma^2)\Omega_1 + 2(2+\gamma^2)\Omega_2 + 2B_o^2\gamma^2\Omega_5 + 4B_o^2\gamma^2(2+\gamma^2)\Omega_6 \\ &\quad +2\gamma^2\Omega_7 + 4\gamma^2(2+\gamma^2)\Omega_8 + 2B_o\Omega_{10}, &(5.79)\\ \tau_{22} &= -p + 2\Omega_1 + 4\Omega_2 + 2B_o^2\Omega_5 + 4B_o^2(1+\gamma^2)\Omega_6 + 2\Omega_7 + 4(1+\gamma^2)\Omega_8 + 2B_o\Omega_{10}. &(5.80)\\ \tau_{33} &= -p + 2\Omega_1 + 2(2+\gamma^2)\Omega_2, &(5.81)\\ \tau_{12} &= 2\gamma[\Omega_1 + \Omega_2 + B_o^2\Omega_5 + B_o^2(3+2\gamma^2)\Omega_6 + \Omega_7 + (3+2\gamma^2)\Omega_8 + B_o\Omega_{10}, &(5.82)\\ \tau_{13} &= \tau_{23} = 0. &(5.83)\end{aligned}$$

The components of the magnetic field, which are given by (5.16), are

$$\begin{aligned}H_1 &= \gamma[2B_o\Omega_5 + 2B_o(2+\gamma^2)\Omega_6 + \Omega_{10}], &(5.84)\\ H_2 &= 2B_o\Omega_4 + 2B_o\Omega_5 + 2B_o(1+\gamma^2)\Omega_6 + \Omega_9 + \Omega_{10}, &(5.85)\\ H_3 &= 0. &(5.86)\end{aligned}$$

Regarding the mechanical boundary conditions (3.68), let us denote by $\overset{\mathbf{n}}{\mathbf{t}}$ the external stress vector (current configuration), which corresponds only to the mechanical part of the external load; then the boundary condition (3.69) reads

$$\overset{\mathbf{n}}{\mathbf{t}} = (\tau - \tau_m)\cdot\mathbf{n}. \tag{5.87}$$

Let's assume that for the faces of the slab limited by the planes $x_3 = 0$ and $x_3 = L_3$ there is no external mechanical load; then the boundary condition (5.87) implies that

$$\tau_{33} = \tau_{m_{33}}, \tag{5.88}$$

and from (5.81) and (5.76)$_3$ we get

$$p = 2\Omega_1 + 2(2+\gamma^2)\Omega_2 + \mu_o^{-1}\frac{B_o^2}{2}(1+\gamma^2). \tag{5.89}$$

As a result, for (5.79) and (5.80) we have

$$\begin{aligned}\tau_{11} &= \gamma^2\Omega_1 + 2B_o^2\gamma^2\Omega_5 + 4B_o^2\gamma^2(2+\gamma^2)\Omega_6 + 2\gamma^2\Omega_7 + 4\gamma^2(2+\gamma^2)\Omega_8 \\ &\quad +2B_o\gamma^2\Omega_{10} - \mu_o^{-1}\frac{B_o^2}{2}(1+\gamma^2), &(5.90)\\ \tau_{22} &= -2\gamma^2\Omega_2 + 2B_o^2\Omega_5 + 4B_o^2(1+\gamma^2)\Omega_6 + 2\Omega_7 + 4(1+\gamma^2)\Omega_8 + 2B_o\Omega_{10} \\ &\quad -\mu_o^{-1}\frac{B_o^2}{2}(1+\gamma^2). &(5.91)\end{aligned}$$

Particle alignment in the x_1 direction. Let us work now with the field (5.63) in order to represent the particle alignment. In such a case we have

$$I_7 = 1, \quad I_8 = 1+\gamma^2, \quad I_9 = 0, \quad I_{10} = B_o\gamma. \tag{5.92}$$

For the components of the total stress from (5.12) we get

$$\begin{aligned}
\tau_{11} &= -p + 2(1+\gamma^2)\Omega_1 + 2(2+\gamma^2)\Omega_2 + 2B_o^2\gamma^2\Omega_5 + 4B_o^2\gamma^2(2+\gamma^2)\Omega_6 \\
&\quad +2\Omega_7 + 4(1+\gamma^2)\Omega_8 + 2B_o\gamma\Omega_{10}, &(5.93)\\
\tau_{22} &= -p + 2\Omega_1 + 4\Omega_2 + 2B_o^2\Omega_5 + 4B_o^2(1+\gamma^2)\Omega_6, &(5.94)\\
\tau_{33} &= -p + 2\Omega_1 + 2(2+\gamma^2)\Omega_2, &(5.95)\\
\tau_{12} &= 2\gamma\Omega_1 + 2\gamma\Omega_2 + 2B_o^2\gamma\Omega_5 + 2B_o^2\gamma(3+2\gamma^2)\Omega_6 + 2\gamma\Omega_8 + B_o\Omega_{10}, &(5.96)\\
\tau_{13} &= \tau_{23} = 0, &(5.97)
\end{aligned}$$

and the components of the magnetic field are given by (5.16) as

$$\begin{aligned}
H_1 &= 2B_o\gamma\Omega_5 + 2B_o\gamma(2+\gamma^2)\Omega_6 + \Omega_9 + \Omega_{10}, &(5.98)\\
H_2 &= 2B_o\Omega_4 + 2B_o\Omega_5 + 2B_o(1+\gamma^2)\Omega_6 - \gamma\Omega_9, &(5.99)\\
H_3 &= 0. &(5.100)
\end{aligned}$$

Regarding the mechanical boundary conditions (3.68), as in the previous case we assume no mechanical external load for the same planes; it is easy to show that the form of p is again given by (5.89) and then, from (5.93) and (5.94), we obtain

$$\begin{aligned}
\tau_{11} &= \gamma^2\Omega_1 + 2B_o^2\gamma^2\Omega_5 + 4B_o^2\gamma^2(2+\gamma^2)\Omega_6 + 2\Omega_7 + 4(1+\gamma^2)\Omega_8 \\
&\quad +2B_o\gamma\Omega_{10} - \mu_o^{-1}\frac{B_o^2}{2}(1+\gamma^2), &(5.101)\\
\tau_{22} &= -2\gamma^2\Omega_2 + 2B_o^2\Omega_5 + 4B_o^2(1+\gamma^2)\Omega_6 - \mu_o^{-1}\frac{B_o^2}{2}(1+\gamma^2). &(5.102)
\end{aligned}$$

5.2.2 Uniform extension of a bar

The simple tension of a cylindrical bar was used by Bellan and Bossis [7] and by Bossis et al. [11] in order to obtain some important experimental results, which will be used later on, in Section 5.3, in order to obtain some preliminary forms for the free energy function.

Consider the deformation

$$r = \lambda^{-1/2}R, \quad \theta = \Theta, \quad z = \lambda Z. \tag{5.103}$$

The components of the deformation gradient and the left and right Cauchy-Green tensors are given by

$$\mathsf{F} = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \mathsf{b} = \mathsf{c} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}. \tag{5.104}$$

and the first and second invariants are given from $(5.25)_1$ and $(5.25)_2$ as

$$I_1 = 2\lambda^{-1} + \lambda^2, \quad I_2 = \lambda^{-2} + 2\lambda. \tag{5.105}$$

As in the previous example, we will work with two cases, first taking the magnetic field as the independent magnetic variable, and then taking the magnetic induction as the independent magnetic variable. However, we only consider one example for the particle alignment, which corresponds to a uniform distribution in the axial direction.

5.2.2.1 The magnetic field as the independent variable

We consider only the axial magnetic field

$$\mathsf{H}_l = (0, 0, H_o)^T, \tag{5.106}$$

where H_o is constant. As was mentioned previously, let us consider the following uniform field representing the initial alignment of the magneto-active particles:

$$\mathsf{a}_0 = (0, 0, 1)^T. \tag{5.107}$$

As a result, from (5.2) we get $\mathsf{a} = (0, 0, \lambda)^T$. The rest of the invariants, which are given by (5.26), (5.27) and (5.28), are

$$K_4 = H_o^2, \quad K_5 = \lambda^2 H_o^2, \quad K_6 = \lambda^4 H_o^2, \quad I_7 = \lambda^2, \quad I_8 = \lambda^4, \quad K_9 = H_o, \tag{5.108}$$

$$K_{10} = \lambda^2 H_o. \tag{5.109}$$

From (5.25) the components of the stress are

$$\tau_{rr} = \tau_{\theta\theta} = -p^* + 2\lambda^{-1}\Omega_1^* + 2\lambda^{-2}(1 + \lambda^3)\Omega_2^*, \tag{5.110}$$

$$\tau_{zz} = -p^* + 2\lambda^2\Omega_1^* + 4\lambda\Omega_2^* + 2H_o^2\lambda^2\Omega_5^* + 4H_o^2\lambda^4\Omega_6^* + 2\lambda^2\Omega_7^* + 4\lambda^4\Omega_8^* + 2H_o\lambda^2\Omega_{10}^*, \tag{5.111}$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0, \tag{5.112}$$

and from (5.38) the components of the magnetic induction are

$$B_r = B_\theta = 0, \tag{5.113}$$
$$B_z = -\lambda(2H_o\Omega_4^* + 2H_o\lambda^2\Omega_5^* + 2H_o\lambda^4\Omega_6^* + \Omega_9^* + \lambda^2\Omega_{10}^*). \tag{5.114}$$

By using (5.106) in (3.9) it is easy to show that $\mathsf{H} = (0, 0, \lambda^{-1}H_o)^T$; then, from (3.44), the non-zero components of the Maxwell stress are given by

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{1}{2}\mu_o\lambda^{-2}H_o^2, \tag{5.115}$$
$$\tau_{m_{zz}} = \frac{1}{2}\mu_o\lambda^{-2}H_o^2. \tag{5.116}$$

As in the simple shear problem, let us study in more detail the mechanical boundary condition (3.69). If we assume no external mechanical surface load for the surface $r = a$, then we have

$$\tau_{rr} = \tau_{m_{rr}}. \tag{5.117}$$

From (5.110) this condition implies

$$p^* = 2\lambda^{-1}\Omega_1^* + 2(\lambda^{-2} + \lambda)\Omega_2^* + \frac{1}{2}\mu_o\lambda^{-2}H_o^2, \tag{5.118}$$

which is equivalent to the condition $\tau_{\theta\theta} = \tau_{m_{\theta\theta}}$. As a result we have, for (5.111),

$$\begin{aligned}\tau_{zz} = {} & 2(\lambda^2 - \lambda^{-1})\Omega_1^* + 2(\lambda - \lambda^{-2})\Omega_2^* + 2H_o^2\lambda^2\Omega_5^* + 4H_o^2\lambda^4\Omega_6^* + 2\lambda^2\Omega_7^* \\ & +4\lambda^4\Omega_8^* + 2H_o\lambda^2\Omega_{10}^* - \frac{1}{2}\mu_o\lambda^{-2}H_o^2.\end{aligned} \tag{5.119}$$

It is not difficult to show that the external load necessary to maintain this deformation is given by $t_z = \tau_{zz} - \tau_{m_{zz}}$.

5.2.2.2 The magnetic induction as the independent variable

As for the magnetic field, let us consider again a uniform axial magnetic induction given by

$$\mathsf{B}_l = (0, 0, B_o)^T. \tag{5.120}$$

As in the previous case, we consider the same initial particle alignment field. Then the invariants (5.4), (5.5) and (5.6) are given by

$$I_4 = B_o^2, \quad I_5 = \lambda^2B_o^2, \quad I_6 = \lambda^4B_o^4, \quad I_7 = \lambda^2, \quad I_8 = \lambda^4, \quad I_9 = B_o, \tag{5.121}$$
$$I_{10} = \lambda^2B_o. \tag{5.122}$$

The non-zero components of the total stress (5.12) are

$$\tau_{rr} = \tau_{\theta\theta} = -p + 2\lambda^{-1}\Omega_1 + 2(\lambda^{-2} + \lambda)\Omega_2, \tag{5.123}$$

$$\begin{aligned}\tau_{zz} = {} & -p + 2\lambda^2\Omega_1 + 4\lambda\Omega_2 + 2B_o^2\lambda^2\Omega_5 + 4B_o^2\lambda^4\Omega_6 + 2\lambda^2\Omega_7 + 4\lambda^4\Omega_8 \\ & +2B_o\lambda^2\Omega_{10},\end{aligned} \tag{5.124}$$

and the components of the magnetic field (5.16) are given by

$$H_r = H_\theta = 0, \tag{5.125}$$

$$H_z = \lambda^{-1}(2B_o\Omega_4 + 2B_o\lambda^2\Omega_5 + 2B_o\lambda^4\Omega_6 + \Omega_9 + \lambda^2\Omega_{10}). \tag{5.126}$$

With $\mathsf{B} = (0, 0, \lambda B_o)^T$ the non-zero components of the Maxwell stress (3.43) are

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{1}{2}\mu_o^{-1}\lambda^2 B_o^2, \tag{5.127}$$

$$\tau_{m_{zz}} = \frac{1}{2}\mu_o^{-1}\lambda^2 B_o^2. \tag{5.128}$$

Then, as in the above problem, if we assume no external load on the surface $r = a$, we get

$$p = 2\lambda^{-1}\Omega_1 + 2(\lambda^{-2} + \lambda)\Omega_2 + \frac{1}{2}\mu_o^{-1}\lambda^2 B_o^2, \tag{5.129}$$

and hence, from (5.124), we have

$$\begin{aligned}\tau_{zz} = {} & 2(\lambda^2 - \lambda^{-1})\Omega_1 + 2(\lambda - \lambda^{-2})\Omega_2 + 2B_o^2\lambda^2\Omega_5 + 4B_o^2\lambda^4\Omega_6 + 2\lambda^2\Omega_7 \\ & +4\lambda^4\Omega_8 + 2B_o\lambda^2\Omega_{10} - \frac{1}{2}\mu_o^{-1}\lambda^2 B_o^2.\end{aligned} \tag{5.130}$$

5.2.3 Biaxial tension for a thin plate

For the pure elastic non-linear problem, especially for incompressible materials, it is not difficult to show that the biaxial tension of a thin plate may theoretically give all the data necessary in order to find the form of the energy function (see, for example, [110] and [78]), without assuming any further simplification. This is not the case for transversely isotropic materials, and of course of our problem, in which, as we have seen in Subsections 5.1.1 and 5.1.2, the number of invariants involved is too large for this to be possible here. Nevertheless, it is interesting to study in which particular situations we would be able, at least theoretically, to find the complete form of the energy function with such a test.

Consider the uniform deformation given by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3. \tag{5.131}$$

For the incompressible case we have $\lambda_1\lambda_2\lambda_3 = 1$, so that

$$\lambda_3 = \frac{1}{\lambda_1\lambda_2}. \tag{5.132}$$

Consider now the following particular external magnetic field (for brevity we will not consider the case of the magnetic induction as the independent magnetic variable here)

$$\mathsf{H}_l = H_o(\cos\theta, \sin\theta, 0)^T, \tag{5.133}$$

where H_o is constant. There are several options for the initial alignment of the magneto-active particles, and we consider only two cases: a field parallel to the external magnetic field, and a field perpendicular (in the plane 1-2) to the external magnetic field (see Figure 5.1).

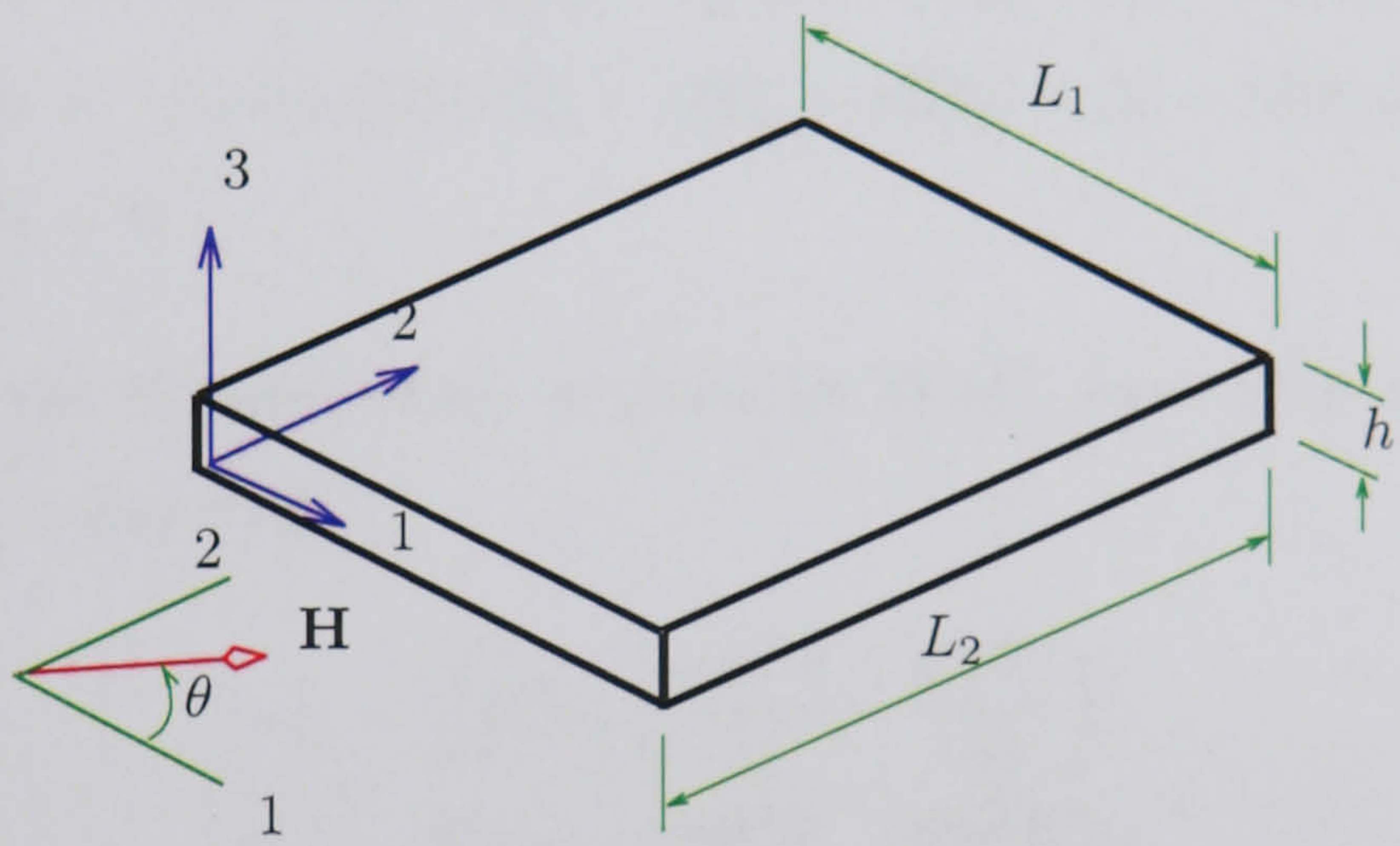


Figure 5.1: Thin plate for biaxial traction.

5.2.3.1 Parallel particle alignment

In this case we have

$$\mathsf{a}_0 = (\cos\theta, \sin\theta, 0)^T. \tag{5.134}$$

The invariants (5.25)-(5.28) are given by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{5.135}$$

$$K_4 = H_o^2, \quad K_5 = H_o^2(\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta), \tag{5.136}$$

$$K_6 = H_o^2(\lambda_1^4\cos^2\theta + \lambda_2^4\sin^2\theta), \tag{5.137}$$

$$I_7 = \lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta, \quad I_8 = \lambda_1^4\cos^2\theta + \lambda_2^4\sin^2\theta, \tag{5.138}$$

$$K_9 = H_o, \quad K_{10} = H_o(\lambda_1^2\cos^2\theta + \lambda_4^2\sin^2\theta), \tag{5.139}$$

and the components of the total stress (5.34) are

$$\tau_{11} = -p^* + 2\lambda_1^2\Omega_1^* + 2\lambda_1^2(\lambda_2^2 + \lambda_3^2)\Omega_2^* + 2H_o^2\cos^2\theta\lambda_1^2\Omega_5^* + 4H_o^2\cos^2\theta\lambda_1^4\Omega_6^*$$
$$+2\cos^2\theta\lambda_1^2\Omega_7^* + 4\cos^2\theta\lambda_1^4\Omega_8^* + 2H_o\cos^2\theta\lambda_1^2\Omega_{10}^*. \quad (5.140)$$
$$\tau_{22} = -p^* + 2\lambda_2^2\Omega_1^* + 2\lambda_2^2(\lambda_1^2 + \lambda_3^2)\Omega_2^* + 2H_o^2\sin^2\theta\lambda_2^2\Omega_5^* + 4H_o^2\sin^2\theta\lambda_2^4\Omega_6^*$$
$$+2\sin^2\theta\lambda_2^2\Omega_7^* + 4\sin^2\theta\lambda_2^4\Omega_8^* + 2H_o\sin^2\theta\lambda_2^2\Omega_{10}^*, \quad (5.141)$$
$$\tau_{33} = -p^* + 2\lambda_3^2\Omega_1^* + 2\lambda_3^2(\lambda_1^2 + \lambda_2^2)\Omega_2^*, \quad (5.142)$$
$$\tau_{12} = \sin(2\theta)\lambda_1\lambda_2\{[\Omega_5^* + \Omega_{10}^* + (\lambda_1^2 + \lambda_2^2)\Omega_6^*]H_o + \Omega_7^* + (\lambda_1^2 + \lambda_2^2)\Omega_8^*\}. \quad (5.143)$$
$$\tau_{13} = \tau_{23} = 0. \quad (5.144)$$

The components of the magnetic induction (5.38) are

$$B_1 = -\lambda_1\cos\theta[2H_o(\Omega_4^* + \lambda_1^2\Omega_5^* + \lambda_1^4\Omega_6^*) + \Omega_9^* + \lambda_1^2\Omega_{10}^*], \quad (5.145)$$
$$B_2 = -\lambda_2\sin\theta[2H_o(\Omega_4^* + \lambda_2^2\Omega_5^* + \lambda_2^4\Omega_6^*) + \Omega_9^* + \lambda_2^2\Omega_{10}^*], \quad (5.146)$$
$$B_3 = 0. \quad (5.147)$$

For this problem the Maxwell stress is given by (3.44), from (3.9) we have that $\mathsf{H} = H_o\left(\frac{\cos\theta}{\lambda_1}, \frac{\sin\theta}{\lambda_2}, 0\right)^T$, and then

$$\tau_{m_{11}} = \frac{1}{2}H_o^2\mu_o\left(\frac{\cos^2\theta}{\lambda_1^2} - \frac{\sin^2\theta}{\lambda_2^2}\right), \quad (5.148)$$
$$\tau_{m_{22}} = \frac{1}{2}H_o^2\mu_o\left(\frac{\sin^2\theta}{\lambda_2^2} - \frac{\cos^2\theta}{\lambda_1^2}\right), \quad (5.149)$$
$$\tau_{m_{33}} = -\frac{1}{2}H_o^2\mu_o\left(\frac{\cos^2\theta}{\lambda_1^2} + \frac{\sin^2\theta}{\lambda_2^2}\right), \quad (5.150)$$
$$\tau_{m_{12}} = \frac{H_o^2\mu_o}{\lambda_1\lambda_2}\cos\theta\sin\theta, \quad (5.151)$$
$$\tau_{m_{13}} = \tau_{m_{23}} = 0. \quad (5.152)$$

Regarding the mechanical boundary conditions, let us assume that there is no external load on the faces of the plate defined by the surfaces $X_3 = 0,\ h$, as in the above boundary value problems. Let's define $\overset{(n)}{\mathbf{t}}$ as the stress vector over the surface with unit normal $\mathbf{n}$, in this case let's call the surface $X_3 = h$ the surface 3. Then for this surface we have $\mathsf{n} = (0, 0, 1)^T$, and as a result from $(3.68)_1$ we have

$$\overset{(n)}{\mathbf{t}} = (\boldsymbol{\tau} - \boldsymbol{\tau}_m)\mathbf{n}. \quad (5.153)$$

If $\overset{(3)}{\mathbf{t}} = \mathbf{0}$ we obtain

$$\tau_{33} = \tau_{m_{33}}. \quad (5.154)$$

and hence

$$p^* = 2\lambda_3^2\Omega_1^* + 2\lambda_3^2(\lambda_1^2 + \lambda_2^2)\Omega_2^* + \frac{1}{2}H_o^2\mu_o\left(\frac{\cos^2\theta}{\lambda_1^2} + \frac{\sin^2\theta}{\lambda_2^2}\right). \tag{5.155}$$

Then for (5.140) and (5.141) we finally get

$$\begin{aligned}\tau_{11} &= 2(\lambda_1^2 - \lambda_3^2)\Omega_1^* + 2\lambda_2^2(\lambda_1^2 - \lambda_3^2)\Omega_2^* + 2H_o^2\cos^2\theta\lambda_1^2\Omega_5^* + 4H_o^2\cos^2\theta\lambda_1^4\Omega_6^* \\ &\quad +2\cos^2\theta\lambda_1^2\Omega_7^* + 4\cos^2\theta\lambda_1^4\Omega_8^* + 2H_o\cos^2\theta\lambda_1^2\Omega_{10}^* \\ &\quad -\frac{1}{2}H_o^2\mu_o\left(\frac{\cos^2\theta}{\lambda_1^2} + \frac{\sin^2\theta}{\lambda_2^2}\right), \qquad (5.156)\\ \tau_{22} &= 2(\lambda_2^2 - \lambda_3^2)\Omega_1^* + 2\lambda_1^2(\lambda_2^2 - \lambda_3^2)\Omega_2^* + 2H_o^2\sin^2\theta\lambda_2^2\Omega_5^* + 4H_o^2\sin^2\theta\lambda_2^4\Omega_6^* \\ &\quad +2\sin^2\theta\lambda_2^2\Omega_7^* + 4\sin^2\theta\lambda_2^4\Omega_8^* + 2H_o\sin^2\theta\lambda_2^2\Omega_{10}^* \\ &\quad -\frac{1}{2}H_o^2\mu_o\left(\frac{\cos^2\theta}{\lambda_1^2} + \frac{\sin^2\theta}{\lambda_2^2}\right). \qquad (5.157)\end{aligned}$$

For the edges of the plate, we define the plane (1) with normal vector $\mathsf{n} = (1,0,0)^T$, which corresponds to the plane defined by $X_1 = L_1$; equally we define the plane (2) with normal vector $\mathsf{n} = (0,1,0)^T$, which corresponds to the plane defined by $X_2 = L_2$ respectively (see Figure 5.1). For the plane (1) we have the external load

$$\overset{(1)}{\mathsf{t}} = (\tau_{11} - \tau_{m_{11}}, \tau_{12} - \tau_{m_{12}}, 0)^T, \tag{5.158}$$

and for the plane (2) we get

$$\overset{(2)}{\mathsf{t}} = (\tau_{12} - \tau_{m_{12}}, \tau_{22} - \tau_{m_{22}}, 0)^T. \tag{5.159}$$

Let $\overset{(i)}{t}_j$ be the component in the direction j of the stress vector on the plane i.

To summarize, from (5.145), (5.146), (5.158) and (5.159), we have

$$\begin{aligned}B_1 &= -\lambda_1\cos\theta[2H_o(\Omega_4^* + \lambda_1^2\Omega_5^* + \lambda_1^4\Omega_6^*) + \Omega_9^* + \lambda_1^2\Omega_{10}^*], \qquad (5.160)\\ B_2 &= -\lambda_2\sin\theta[2H_o(\Omega_4^* + \lambda_2^2\Omega_5^* + \lambda_2^4\Omega_6^*) + \Omega_9^* + \lambda_2^2\Omega_{10}^*], \qquad (5.161)\\ \overset{(1)}{t}_1 &= 2(\lambda_1^2 - \lambda_3^2)\Omega_1^* + 2\lambda_2^2(\lambda_1^2 - \lambda_3^2)\Omega_2^* + 2H_o^2\cos^2\theta\lambda_1^2\Omega_5^* + 4H_o^2\cos^2\theta\lambda_1^4\Omega_6^* \\ &\quad +2\cos^2\theta\lambda_1^2\Omega_7^* + 4\cos^2\theta\lambda_1^4\Omega_8^* + 2H_o\cos^2\theta\lambda_1^2\Omega_{10}^* - \frac{H_o^2}{\lambda_1^2}\cos^2\theta\mu_o, \qquad (5.162)\\ \overset{(2)}{t}_2 &= 2(\lambda_2^2 - \lambda_3^2)\Omega_1^* + 2\lambda_1^2(\lambda_2^2 - \lambda_3^2)\Omega_2^* + 2H_o^2\sin^2\theta\lambda_2^2\Omega_5^* + 4H_o^2\sin^2\theta\lambda_2^4\Omega_6^* \\ &\quad +2\sin^2\theta\lambda_2^2\Omega_7^* + 4\sin^2\theta\lambda_2^4\Omega_8^* + 2H_o\sin^2\theta\lambda_2^2\Omega_{10}^* - \frac{H_o^2}{\lambda_1^2}\sin^2\theta\mu_o, \qquad (5.163)\\ \overset{(1)}{t}_2 &= \overset{(2)}{t}_1 = \sin(2\theta)\lambda_1\lambda_2\{[\Omega_5^* + \Omega_{10}^* + (\lambda_1^2 + \lambda_2^2)\Omega_6^*]H_o + \Omega_7^* + (\lambda_1^2 + \lambda_2^2)\Omega_8^*\} \\ &\quad -\frac{H_o^2\mu_o}{\lambda_1\lambda_2}\cos\theta\sin\theta. \qquad (5.164)\end{aligned}$$

5.2.3.2 Perpendicular particle alignment

In this second case we have

$$\mathsf{a}_0 = (-\sin\theta, \cos\theta, 0)^T. \tag{5.165}$$

The invariants (5.25) and (5.26) are the same, given by (5.135), (5.136) and (5.137). As for the rest of the invariants, from (5.27) and (5.28) we have

$$I_7 = \sin^2\theta\lambda_1^2 + \cos^2\theta\lambda_2^2, \quad I_8 = \sin^2\theta\lambda_1^4 + \cos^2\theta\lambda_2^4, \quad K_9 = 0, \tag{5.166}$$

$$K_{10} = H_o\cos\theta\sin\theta(\lambda_2^2 - \lambda_1^2), \tag{5.167}$$

and from (5.34) the components of the stress are

$$\begin{aligned}\tau_{11} &= -p^* + 2\lambda_1^2\Omega_1^* + 2\lambda_1^2(\lambda_2^2+\lambda_3^2)\Omega_2^* + 2H_o^2\cos^2\theta\lambda_1^2\Omega_5^* + 4H_o^2\cos^2\theta\lambda_1^4\Omega_6^* \\ &\quad +2\sin^2\theta\lambda_1^2\Omega_7^* + 4\sin^2\theta\lambda_1^4\Omega_8^* - 2H_o\cos\theta\sin\theta\lambda_1^2\Omega_{10}^*, \end{aligned}\tag{5.168}$$

$$\begin{aligned}\tau_{22} &= -p^* + 2\lambda_2^2\Omega_1^* + 2\lambda_2^2(\lambda_1^2+\lambda_3^2)\Omega_2^* + 2H_o^2\cos^2\theta\lambda_2^2\Omega_5^* + 4H_o^2\cos^2\theta\lambda_2^4\Omega_6^* \\ &\quad +2\sin^2\theta\lambda_2^2\Omega_7^* + 4\sin^2\theta\lambda_2^4\Omega_8^* - 2H_o\cos\theta\sin\theta\lambda_2^2\Omega_{10}^*, \end{aligned}\tag{5.169}$$

$$\tau_{33} = -p^* + 2\lambda_3^2\Omega_1^* + 2\lambda_3^2(\lambda_1^2+\lambda_2^2)\Omega_2^*, \tag{5.170}$$

$$\begin{aligned}\tau_{12} &= \sin(2\theta)\lambda_1\lambda_2[H_o^2(\Omega_5^* + (\lambda_1^2+\lambda_2^2)\Omega_6^*) + \Omega_7^* - (\lambda_1^2+\lambda_2^2)\Omega_8^*] \\ &\quad +H_o\cos(2\theta)\lambda_1\lambda_2\Omega_{10}^*, \end{aligned}\tag{5.171}$$

$$\tau_{13} = \tau_{23} = 0. \tag{5.172}$$

And from (5.38) the components of the magnetic induction are given as

$$B_1 = -2H_o\cos\theta\lambda_1(\Omega_4^* + \lambda_1^2\Omega_5^* + \lambda_1^4\Omega_6^*) + \lambda_1\sin\theta(\Omega_9^* + \lambda_1^2\Omega_{10}^*), \tag{5.173}$$

$$B_2 = -[2H_o\sin\theta\lambda_2(\Omega_4^* + \lambda_2^2\Omega_5^* + \lambda_2^4\Omega_6^*) + \lambda_2\cos\theta(\Omega_9^* + \lambda_2^2\Omega_{10}^*)], \tag{5.174}$$

$$B_3 = 0. \tag{5.175}$$

Since (5.142) is the same as (5.169), and since the same happens with the Maxwell stress, then the expression for p^*, in the case we assume no external mechanical load on the plane 3, is given by (5.155), and we may derive five expressions, as in the parallel case, for the two components of the magnetic induction, and the three components of the external loads, which for brevity we do not list here.

So from the first problem (the parallel alignment) we have five equations and ten partial derivatives of Ω^* that must be found. It is clear we do not have a unique solution. But from the second case (the perpendicular alignment) we have obtained five more equations. The question is if they are independent or not. Nevertheless, we still would have more unknowns than equations, and it would be necessary to assume further simplifications in the form of Ω^* in order to be able to solve the above system of equations. Note that in the particular case of an isotropic material, where we would have that Ω^* would not depend on I_7, I_8, K_9 and K_{10}, we have that the system of five equation of the parallel case provide enough equations in order to find the partial derivatives of Ω^* in I_1, I_2, K_4, K_5 and K_6, but if and only if $\theta \neq 0$ and $\theta \neq \pi/2$.

Of course the above may work from the theoretical point of view, but from the practical point of view it may be very difficult to achieve a uniform biaxial extension, where besides the normal loads, we must provide a shear (5.164). As well as this, we would need to be able to measure somehow for a thin plate the components of the magnetic induction inside it.

The following variation of the original problem may be easier to implement from the experimental point of view.

5.2.3.3 An additional case

Let us consider a variation of the original problem, in which we assume now the presence of a uniform field in the direction 3 (see Figure 5.1):

$$\mathsf{H}_l = (0, 0, H_o)^T. \tag{5.176}$$

Let us also assume the particle alignment field

$$\mathsf{a}_0 = (\cos\theta, \sin\theta, 0)^T. \tag{5.177}$$

For this problem, the first two invariants are the same as (5.135), the same happens with K_4, I_7 and I_8, which are given by $(5.136)_1$ and (5.138); the rest of the invariants $(5.26)_2$, $(5.26)_3$, $(5.27)_2$ and (5.28) are given by

$$K_5 = \lambda_3^2 H_o^2, \quad K_6 = \lambda_3^2 H_o^2, \quad K_9 = K_{10} = 0. \tag{5.178}$$

The components of the total stress are (equation (5.34))

$$\tau_{11} = -p^* + 2\lambda_1^2\Omega_1^* + 2\lambda_1^2(\lambda_2^2 + \lambda_3^2)\Omega_2^* + 2\cos^2\theta\lambda_1^2\Omega_7^* + 4\cos^2\theta\lambda_1^4\Omega_8^*, \tag{5.179}$$

$$\tau_{22} = -p^* + 2\lambda_2^2\Omega_1^* + 2\lambda_2^2(\lambda_1^2 + \lambda_3^2)\Omega_2^* + 2\sin^2\theta\lambda_2^2\Omega_7^* + 4\sin^2\theta\lambda_2^4\Omega_8^*, \tag{5.180}$$

$$\tau_{33} = -p^* + 2\lambda_3^2\Omega_1^* + 2\lambda_3^2(\lambda_1^2 + \lambda_2^2)\Omega_2^* + 2H_o^2\lambda_3^2\Omega_5^* + 4H_o^2\lambda_3^4\Omega_6^*, \tag{5.181}$$

$$\tau_{12} = \sin(2\theta)\lambda_1\lambda_2[\Omega_7^* + (\lambda_1^2 + \lambda_2^2)\Omega_8^*], \tag{5.182}$$

$$\tau_{13} = H_o\cos\theta\lambda_1\lambda_3\Omega_{10}^*. \tag{5.183}$$

$$\tau_{23} = H_o\sin\theta\lambda_2\lambda_3\Omega_{10}^*. \tag{5.184}$$

As we can see from the above results, the shear components in the plane 3 are not zero, and as a result, we cannot in general obtain a free load condition by only manipulating p, as was done in the previous two problems. Let us assume then the particular case of an isotropic material, which means Ω^* would depend only on I_1, I_2, K_4, K_5 and K_6. In such a case, by assuming no load in the plane 3, as before, we have the condition $\tau_{33} = \tau_{m_{33}}$, where the non-zero components of the Maxwell stress are (equation (3.44))

$$\tau_{m_{11}} = \tau_{m_{22}} = -\tau_{m_{33}} = -\frac{H_o^2\mu_o}{2\lambda_3^2}, \tag{5.185}$$

where we have used $\mathsf{H} = (0, 0, H_o/\lambda_3)^T$. Then, from $\tau_{33} = \tau_{m_{33}}$ we obtain

$$p^* = 2\lambda_3^2\Omega_1^* + 2\lambda_3^2(\lambda_1^2 + \lambda_2^2)\Omega_2^* + 2H_o^2\lambda_3^2\Omega_5^* + 4H_o^2\lambda_3^4\Omega_6^* - \frac{H_o^2}{2\lambda_3^2}\mu_o. \tag{5.186}$$

As a result, we get for (5.179) and (5.180), respectively,

$$\tau_{11} = 2(\lambda_1^2 - \lambda_3^2)\Omega_1^* + 2\lambda_2^2(\lambda_1^2 - \lambda_3^2)\Omega_2^* - 2H_o^2\lambda_3^2\Omega_5^* - 4H_o^2\lambda_3^2\Omega_6^* + \frac{H_o^2\mu_o}{2\lambda_3^2}, \tag{5.187}$$

$$\tau_{22} = 2(\lambda_2^2 - \lambda_3^2)\Omega_1^* + 2\lambda_1^2(\lambda_2^2 - \lambda_3^2)\Omega_2^* - 2H_o^2\lambda_3^2\Omega_5^* - 4H_o^2\lambda_3^2\Omega_6^* + \frac{H_o^2\mu_o}{2\lambda_3^2}. \tag{5.188}$$

Additionally for the isotropic case, from (5.38), we have for the magnetic induction that

$$B_1 = B_2 = 0, \tag{5.189}$$

$$B_3 = -2H_o\lambda_3(\Omega_4^* + \lambda_3^2\Omega_5^* + \lambda_3^4\Omega_6^*). \tag{5.190}$$

As a result, we have three equations[3] (5.187), (5.188) and (5.190), and five unknowns, so again we would need to assume further simplifications in order to obtain from this experiment an appropriate form for the energy function.

5.3 A first approximation for an energy function

As was expected, the last problem of the previous section showed us that in general it is not possible, at least with the biaxial test, to find the complete form of the energy function. Although for the isotropic case it would be possible to do that at least theoretically. A more complex experiment, involving a uniform deformation might be proposed; this experiment could be the uniform traction of a cube, which is essentially a three-dimensional problem, in which we would have (eliminating p with one of the normal components) five equations

[3] It is not difficult to calculate with the components of the total and the Maxwell stresses the external load as in the previous problems.

for the rest of the components of the total stress, and three components of, for example, the magnetic induction, so in total we would have eight equations. Unfortunately to carry out such experiment (which must be done with a 'large' cube) seems not possible from the practical point of view.

The above considerations indicate that it is necessary to look for a way to simplify our general form of the constitutive equation; we do that in this section in several steps. We will only consider the case of an incompressible material.

Since most of the experimental data used in this section was taken from [7], in which the independent magnetic variable was the magnetic field, in the rest of this section we put our attention to the problem formulated with Ω^*.

Let us go back to Subsection 5.1.2, and let us study the meaning of the invariants, which are given in (5.25)-(5.28). In the classical theory of non-linear elasticity, in which we work essentially with the two invariants $(5.25)_1$, $(5.25)_2$ (incompressible case), some well known simple energy functions have been proposed; for example, the neo-Hookean, where the energy function only depends on I_1. Then as a fist approximation we will assume that Ω^* will not depend on I_2.

Regarding the invariants shown in (5.26), they are given as $K_4 = \mathbf{H}_l \cdot \mathbf{H}_l$, $K_5 = \mathbf{H}_l \cdot \mathbf{c}\mathbf{H}_l$ and $K_6 = \mathbf{H}_l \cdot \mathbf{c}^2\mathbf{H}_l$. The first of them just takes account for the 'magnitude' of the field in the response of the material, regarding K_5, it would correspond to the effect of the combination of the deformation and the magnetic field, it might not be considered as an important invariant, since an invariant with a similar 'property' appears also in the formulation of the transversely isotropic material, and since the experimental data suggest (as we will se later on [7]) that in comparison with the transversely isotropic material the magnetostriction is less significant in the isotropic case; however, for reasons that we will explain in detail later on, we will assume that Ω^* does depend on K_5, but as a first approximation we will not take account the invariant K_6.

Now, let us consider the two invariants that appear in (5.27). The first, $I_7 = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{a}_0$, takes account for the effect of the alignment of the particles in the material, but does not consider any field, so, this would correspond to the typical invariant that appears in the modelling of transversely isotropic materials (pure elastic case). We will consider this effect as important and we will consider it in our simplified energy function (see [7]). As a first approximation, however, we will not take account the invariant I_8.

So, finally we have to make a decision about the last two invariants in (5.28). The first of them K_9, given as $K_9 = \mathbf{a}_0 \cdot \mathbf{H}_l$, might be considered as important: the reason is simple, this invariant only takes account of the effect of the particle alignment and the field, which, as has been mentioned several times, seems to be an important factor in the behaviour of these materials. Moreover, it should be mentioned that some micro-mechanical models recently developed (see for example [10] and [126]) take in particular consideration, as an important factor in the overall behavior of the material, the alignment of the particles and their interaction with the magnetic field. Regarding the invariant K_{10}, given as $K_{10} = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{H}_l$, it will be considered for our model as well, since we would like to include the effect of the combination of the deformation (in this case given by $\mathbf{c}$) and the field. Notice that if δ corresponds to the order of magnitude of $\mathbf{H}_l$, then in terms of this vector, the invariants K_4, K_5 and K_6 are all of order δ^2, while only K_9 and K_{10} are of order δ.

So from the above considerations as a first approximation we have for the energy function that

$$\Omega^* = \Omega^*(I_1, K_4, K_5, I_7, K_9, K_{10}). \tag{5.191}$$

The particular forms of the stress and the magnetic induction in this case are

$$\boldsymbol{\tau} = 2\mathbf{b}\Omega_1^* - p^*\mathbf{I} + 2\mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H}\Omega_5^* + 2\mathbf{a} \otimes \mathbf{a}\Omega_7^* + (\mathbf{a} \otimes \mathbf{b}\mathbf{H} + \mathbf{b}\mathbf{H} \otimes \mathbf{a})\Omega_{10}^*, \tag{5.192}$$

and

$$\mathbf{B} = -(2\mathbf{b}\mathbf{H}\Omega_4^* + 2\mathbf{b}^2\mathbf{H}\Omega_5^* + \mathbf{a}\Omega_9^* + \mathbf{b}\mathbf{a}\Omega_{10}^*). \tag{5.193}$$

So, we have reduced our original problem to one where we have to look for a function of six variables. This is still a difficult task, as it is in the analogous case of pure elastic deformations for transversely isotropic materials; then, further simplifications may be necessary. Taking as an example from what has been done in the case of transversely isotropic materials (see [73, 74]), we split the energy function in two portions. One of them, denoted $\hat{\Omega}^*$, corresponds to the contribution to the total energy from the factors which do not depend on the orientation of the particles, which means this portion would be the 'isotropic' part of the energy. The other part, denoted $\tilde{\Omega}^*$, would correspond to the rest of the total energy function, meaning that this part of the energy would be the part due to the presence of the alignment of the particles. Then, as a second approximation we would have

$$\Omega^*(I_1, K_4, K_5, I_7, K_9, K_{10}) = \hat{\Omega}^*(I_1, K_4, K_5) + \tilde{\Omega}^*(I_7, K_9, K_{10}). \tag{5.194}$$

The simplifications (5.191) and (5.194) are strong, it could be interesting to study them in the context of some boundary value problems using the semi-inverse method. in order to find conditions for existence of solutions for Ω^*. The same might be done with the regularity conditions [61].

Even so, with the above simplifications for the energy function it may still be difficult to find an appropriate form for the energy function from the limited experimental data available. Thus, further assumptions are needed, but we must point out that the following simplifications do not intend to represent at all the whole behaviour of these materials, and that they must be taken only as a first attempt in order to propose a constitutive equation, which would be used to obtain some closed form solutions for some boundary value problems. Consider the following proposed form for $\hat{\Omega}^*$ (see, for example, [35])

$$\hat{\Omega}^*(I_1, K_4, K_5) = f(I_1)g(K_4) + \nu(K_4) + \vartheta(K_5). \tag{5.195}$$

In the above expression for the isotropic part of the energy, we have assumed that the energy is separable in the variables I_1 and K_4. The function $\nu(K_4)$ represents the energy that the body accumulates only due to the magnetic field, when there is no deformation. The function $\vartheta(K_5)$ has been introduced in order to deal with the presence of Maxwell stresses for the mechanical boundary conditions.

Regarding the function $\tilde{\Omega}^*$, we suggest the following form (see for example [73, 74])

$$\tilde{\Omega}^*(I_7, K_9, K_{10}) = h(I_7)\omega(K_9, K_{10}) + \eta(K_9). \tag{5.196}$$

The function $\eta(K_9)$ represents the magnetic energy that arises in a body as a result only of the magnetic field, and due in particular to the alignment of the magneto-sensitive particles.

The conditions of no residual stress and no residual magnetic induction (5.40) and (5.42) are, we recall,

$$\bar{\Omega}_7^* + 2\bar{\Omega}_8^* = 0, \tag{5.197}$$

$$\bar{\Omega}_9^* + \bar{\Omega}_{10}^* = 0, \tag{5.198}$$

from which we can obtain some restrictions for our functions h, η and ω. Remembering the values of the invariants for this case from (5.1.2) were given as $I_1 = 3$, $K_4 = 0$, $K_5 = 0$, $I_7 = 1$, $K_9 = 0$ and $K_{10} = 0$, we have

$$\frac{\mathrm{d}h}{\mathrm{d}I_7}(1)\omega(0,0) = 0, \tag{5.199}$$

$$h(1)\frac{\partial \omega}{\partial K_9}(0,0) + \frac{\mathrm{d}\eta}{\mathrm{d}K_9}(0) + h(1)\frac{\partial \omega}{\partial K_{10}}(0,0) = 0. \tag{5.200}$$

From the above relations we have several options in order to impose conditions on our functions. In order to study this, let us replace in (5.192) the particular form for the energy (5.194) and (5.195), (5.196). Then we get

$$\tau = 2\mathbf{b}\frac{\mathrm{d}f}{\mathrm{d}I_1}g - p^*\mathbf{I} + 2\mathbf{bH}\otimes\mathbf{bH}\frac{\mathrm{d}\vartheta}{\mathrm{d}K_5} + 2\mathbf{a}\otimes\mathbf{a}\frac{\mathrm{d}h}{\mathrm{d}I_7}\omega + (\mathbf{a}\otimes\mathbf{bH} + \mathbf{bH}\otimes\mathbf{a})h\frac{\partial\omega}{\partial K_{10}}. \quad (5.201)$$

Consider the case in which $\omega(0,0) = 0$. In such a case, for a pure elastic problem, in which we do not have a field[4], but in which we do have an effect because of the alignment of the particles, we could not distinguish from (5.201) an isotropic material from a transversely isotropic one; as a result, in order that (5.199) holds, we need $\frac{\mathrm{d}h}{\mathrm{d}I_7}(1) = 0$, and we also insist that $\omega(0,0) \neq 0$.

Now, consider the case in which $\frac{\partial\omega}{\partial K_{10}}(0,0) \neq 0$; then from (5.200), since each partial derivative of ω is independent, and the same happens with the derivative of the function η, in order that (5.200) holds we would need the condition $h(1) = 0$. But in such a case, let us study from (5.201) what could happen with the following experiment. Imagine the situation in which there is no deformation, but in which there is a magnetic field. Then we would have $h = h(1) = 0$ for any $\mathbf{H}$, but since from the above considerations we have $\frac{\mathrm{d}h}{\mathrm{d}I_7}(1) = 0$, then, from (5.201), we would not see again any distinction in the behavior of the stress between an isotropic and a transversely isotropic material, which is not the case (see [7]). Hence, we need to impose the condition $\frac{\partial\omega}{\partial K_{10}}(0,0) = 0$, and additionally $\frac{\partial\omega}{\partial K_9}(0,0) = 0$, $\frac{\mathrm{d}\eta}{\mathrm{d}K_9}(0) = 0$, and we want $h(1) \neq 0$.

In summary, we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}K_9}(0) = 0, \quad \omega(0,0) \neq 0, \quad \frac{\partial\omega}{\partial K_9}(0,0) = \frac{\partial\omega}{\partial K_{10}}(0,0) = 0, \quad (5.202)$$

$$\frac{\mathrm{d}h}{\mathrm{d}I_7}(1) = 0, \quad h(1) \neq 0. \quad (5.203)$$

5.3.1 Results for the simple traction problem of a cylinder

Consider the following set of experiments for the traction problem. Let us have a magneto-sensitive elastomer, and let us have two cylindrical samples of material, one with a random distribution of particles, and another with a preferred particle alignment, in this case in the axial direction. Assume that the proportion of particles for each cylinder is the <u>same</u>, and let us do the following experiment; first measure the axial stress as a function of the deformation (in this case, for example, the stretch λ) for the situations with and

[4] We have that $\mathbf{H} = \mathbf{0}$ and as a result $\omega = \omega(0,0)$ always.

without magnetic field (a uniform axial magnetic field only). In such an experiment we have (for one value of the magnetic field) four profiles for the stress as a function of the deformation. Of course more data may be added by doing the experiment for different values of the external magnetic field. This is what has been done in the paper of Bellan and Bossis [7] (see in particular the Figures 1 and 2 therein).

There are some problems in the case of our models. As has been mentioned before, there is a problem with the boundary conditions for the magnetic part of the problem (see [15]); the analytic solutions presented in Section 5.2 are valid only for semi-infinite geometries, in the case of the simple shear, assuming an infinite slab in the directions 1 and 3, and in the case of the traction problem, assuming an infinite cylinder in the axial direction (see Subsection (5.2.2)).

Another problem with the model is the treatment of the Maxwell stress. As it was been shown recently (see [17], also Chapter 9), the Maxwell stresses appear when a body is completely surrounded by a free space, and they are the result of the variation of the energy in the free space due to the deformation of the body. All the solutions of the previous sections were obtained assuming bodies surrounded completely by a free space. However, the real situation might be different. In the traction problem, for example, we know that the extremes of the cylinder are 'connected' with the test machine. How to model this situation in a real but simple way is not known yet, and as a result we will assume the cylinder surrounded by a free space, but, as we will see later on, this implies the existence of an extra factor for the stress, which will be appropriately handled by the use of the function $\vartheta(K_5)$.

From Section 5.2.2 we have that the total stress applied to a bar, in the particular case of using (5.191), is given as

$$t_z = 2(\lambda^2 - \lambda^{-1})\Omega_1^* + 2H_o^2\lambda^2\Omega_5^* + 2\lambda^2\Omega_7^* + 2H_o\lambda^2\Omega_{10}^* - \mu_o\lambda^{-2}H_o^2. \tag{5.204}$$

We notice that the last term in the above expression accounts for the effect of the Maxwell stress. The invariants of interest are given by (5.105), $(5.108)_1$, $(5.108)_2$, $(5.108)_4$, $(5.108)_6$ and (5.109) as

$$I_1 = 2\lambda^{-1} + \lambda^2, \quad K_4 = H_o^2, \quad K_5 = \lambda^2 H_o^2, \quad I_7 = \lambda^2, \quad K_9 = H_o, \quad K_{10} = \lambda^2 H_o. \tag{5.205}$$

Let us write down the experiment mentioned at the beginning of this subsection, in which we have two cylinders made of the same basis material, and basis magneto-active particles.

and with the same proportion of particles, but in one case with random distribution, and in the other case with a preferred alignment for the particles (in this case axial).

- **Isotropic case, $H_o = 0$**

  Consider an isotropic cylinder with no external magnetic field, from (5.204), and (5.195), (5.196) we have

$$t_z = 2(\lambda^2 - \lambda^{-1})\frac{\mathrm{d}f}{\mathrm{d}I_1}(I_1)g(0). \tag{5.206}$$

  Consider the following particular form for the function f

$$f(I_1) = \frac{1}{k}\left[\frac{(I_1 - 1)^k}{2^k} - 1\right], \tag{5.207}$$

  where k is a constant such that $k \geq 1/2$ (see [58] for the basis of this model in the non-linear elastic case).

- **Isotropic case, $H_o \neq 0$**

  For the isotropic case when $H_o \neq 0$, from (5.204), (5.195) and (5.196) we get

$$t_z = 2(\lambda^2 - \lambda^{-1})\frac{\mathrm{d}f}{\mathrm{d}I_1}g(K_4) + 2H_o^2\lambda^2\frac{\mathrm{d}\vartheta}{\mathrm{d}K_5} - \mu_o\lambda^{-2}H_o^2. \tag{5.208}$$

  A simple model for the function g might be the linear approximation

$$g(K_4) = g_0 + g_1 K_4, \tag{5.209}$$

  where g_0 and g_1 are constants.

  Regarding ϑ, from [7] it seems there is no an effect of the Maxwell stress at the end of the cylinder when $\lambda = 1$. From (5.208), we can see that the factor $-\mu_o\lambda^{-2}H_o^2$ would imply the presence of a compressive stress for $H_o \neq 0$, when $\lambda = 1$, which from [7] is not the case; then, we must find a rational way to deal with this problem. One way would be not to consider the Maxwell stress for the extremes of the cylinder; however, we would still have this stress around the lateral surface of the cylinder, and from the expression for p^* we would end up with a factor $-1/2\mu_o\lambda^{-2}H_o^2$ instead of $-\mu_o\lambda^{-2}H_o^2$ in (5.208); also, we do not know yet how to handle appropriately this kind of boundary condition in a general context [16]. This is the reason we included the invariant K_5 in our formulation. We assume that ϑ has the linear form in K_5

$$\vartheta(K_5) = \vartheta_o K_5, \tag{5.210}$$

where ϑ_o is a constant. We could 'eliminate' the Maxwell stress for $\lambda = 1$ if we set

$$\vartheta_o = \frac{\mu_o}{2}, \tag{5.211}$$

which finally implies

$$t_z = 2(\lambda^2 - \lambda^{-1})\frac{\mathrm{d}f}{\mathrm{d}I_1}g(K_4) + \mu_o H_o^2(\lambda^2 - \lambda^{-2}). \tag{5.212}$$

- **Transversely isotropic case, $H_o = 0$**

In the case of a transversely isotropic cylinder, with no external field as in the above case, we would get

$$t_z = 2(\lambda^2 - \lambda^{-1})\frac{\mathrm{d}f}{\mathrm{d}I_1}(I_1)g(0) + 2\lambda^2\frac{\mathrm{d}h}{\mathrm{d}I_7}(I_7)\omega(0,0). \tag{5.213}$$

A function h that may be compatible with the conditions (5.203), in part suggested from [73, 74], may be the taken as

$$h(I_7) = h_0 + h_1(I_7 - 1)^m, \quad m > 1, \tag{5.214}$$

where h_0, h_1 and m are constants, and the condition $m > 1$ is imposed in order to avoid problems with the evaluation of the derivative of h at[5] $I_7 = 1$.

- **Transversely isotropic case, $H_o \neq 0$**

In the previous cases the particular form given for the functions f, g and h were not proposed arbitrarily, but in order to fit the data provided in the Figure 2 of [7].

However, there is a more complex situation when we try to find an appropriate form for ω. The following expression is the external stress for the transversely isotropic case when there is magnetic field (remember (5.210))

$$\begin{aligned} t_z &= 2(\lambda^2 - \lambda^{-1})\frac{\mathrm{d}f}{\mathrm{d}I_1}(I_1)g(K_4) + 2\lambda^2\frac{\mathrm{d}h}{\mathrm{d}I_7}(I_7)\omega(K_9, K_{10}) \\ &+2H_o\lambda^2 h(I_7)\frac{\partial\omega}{\partial K_{10}}(K_9, K_{10}) - \mu_o H_o^2(\lambda^2 - \lambda^{-2}). \end{aligned} \tag{5.215}$$

[5]An additional condition for m may be the following. Consider the case in which $m < 2$, then it may not be convenient to work, for example, with $m = 3/2$, because in such a case we would have a problem for the function h and its derivative when we would like to evaluate them for values of I_7 less than 1 (compression). This situation, which may not appear problematic for the particular case of the tension problem, may generate problems at the moment of applying the final result with our energy function for more general problems in which we might have localized compression for a body.

The second derivative of h might be needed to calculate the moduli tensors, in such a case in order to avoid problems for $I_7 = 1$ we would need to impose the stronger restriction $m > 2$; as well as this, m should be chosen such that to avoid problems for the evaluation of this second derivative for $I_7 < 1$. We do not use this last restriction in this thesis.

Figure 4 of [7] shows the result for the difference of (5.215) and (5.213), for different values of the external magnetic field. The difference for these two cases is given as

$$\Delta t_z = 2(\lambda^2 - \lambda^{-1})\frac{\mathrm{d}f}{\mathrm{d}I_1}(I_1)(g(K_4) - g(0)) + 2\lambda^2\frac{\mathrm{d}h}{\mathrm{d}I_7}(I_7)(\omega(K_9, K_{10}) - \omega(0, 0))$$
$$+2H_o\lambda^2 h(I_7)\frac{\partial\omega}{\partial K_{10}}(K_9, K_{10}) - \mu_o H_o^2(\lambda^2 - \lambda^{-2}). \quad (5.216)$$

It has not been possible to find an appropriate and simple form for the function ω, such that $(5.202)_2$, $(5.202)_3$ and $(5.202)_4$ hold, and such that the behavior of Δt_z may be reproduced with accuracy. As a result, we have proposed a simple bi-quadratic form for ω, which nevertheless works very well for the data presented in Figure 2 of [7]. The following is the expression[6] proposed for ω

$$\omega(K_9, K_{10}) = \omega_0 + \omega_1 K_9^2 + \omega_2 K_{10}^2 + \omega_3 K_9 K_{10}, \quad (5.217)$$

where ω_i, with $i = 0, 1, 2, 3$, are constants.

This is the form for the energy function:

$$\Omega^* = \left(\frac{I_1 - 3}{2}\right)(g_0 + g_1 K_4) + \nu(K_4) + \frac{\mu_o}{2}K_5 + [h_0 + h_1(I_7 - 1)^m](\omega_0 + \omega_1 K_9^2$$
$$+\omega_2 K_{10}^2 + \omega_3 K_9 K_{10}) + \eta(K_9) + \Omega_o^*. \quad (5.218)$$

The values of the constants k, g_0, g_1, h_0, h_1, m, ω_0, ω_1, ω_2 and ω_3 are given in Table 5.1.

k	1		m	4/3	
g_0	95	kPa	ω_0	2000	kPa
g_1	0.00001	kPa/(kA/m)^2	ω_1	0.00323	kPa/(kA/m)^2
h_0	0.02		ω_2	-0.000475	kPa/(kA/m)^2
h_1	0.01		ω_3	0.020557	kPa/(kA/m)^2

Table 5.1: Constants for the energy function ('mechanical part' of the energy).

As well as this, we have used $\mu_o = 1.2566 * 10^{-3}$kN/kA^2. Figure 5.2 shows the graphs for the stress for the different situations mentioned above.

[6]The expression for the functions ω is just a truncated double Taylor expansion, where we have used the conditions (5.202). The problem with the Taylor expansions is that the choice of the optimal constants may not be unique, then there might be other values for ω_i in Table 5.1 that might also be acceptable.

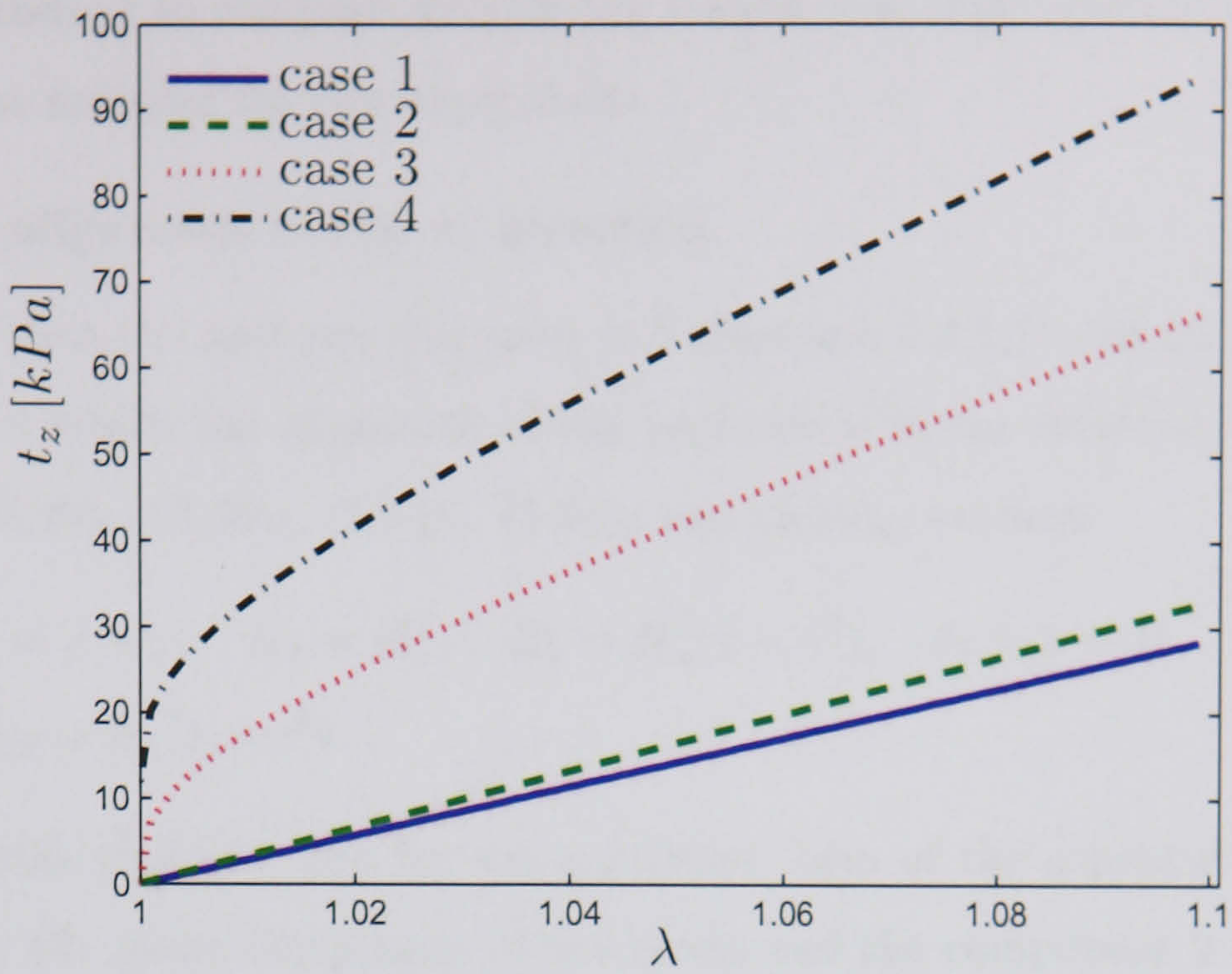


Figure 5.2: Results for the traction experiment. Case 1; isotropic with $H_o = 0$; case 2; isotropic $H_o = 123\ [kA/m]$; case 3; transversely isotropic $H_o = 0$, and case 4; transversely isotropic $H_o = 123\ [kA/m]$.

5.3.2 Results for the shear problem

Having found part of the form for the energy function with the results for the traction problem [7], there are still some parts of the energy function missing, in particular the functions ν and η.

Here, two additional papers with experimental results may seem useful, one of them by Jolly et al [59], from which in particular we mention Figure 7, where we have the 'shear modulus', as a function of the 'flux density' (magnetic induction), for a slab, which is a problem very similar to the problem described in Subsection 5.2.1.2.

Another important paper with experimental results for the shear problem is that by Ginder et al [50]. We mention especially Figure 4, where the magnetization M is shown as a function of the magnetic field for a slab under shear, for the two alignments of the particles described in Subsection 5.2.1.1.

Let us study our particular form for the energy function Ω^* in relation to the two papers mentioned above. In the first case, we want to calculate the 'shear modulus' for our problem, and then to study its behavior as a function of the magnetic field; unfortunately, Figure 7 of [59] was obtained assuming the magnetic induction $\mathbf{B}$ as the independent magnetic variable, and, moreover, the proportion of particles was different from that in [7], so we do not attempt to compare directly the results, but only to study qualitatively the behaviour of the material for this experiment.

- **Particle alignment in the x_2 direction**

Consider then the problem discussed in Subsection 5.2.1.1, and let us consider first the case in which the alignment of the particles is in the direction x_2. Then, from $(5.47)_1$, $(5.49)_1$, $(5.49)_2$, $(5.54)_1$, $(5.54)_3$ and $(5.54)_4$, we have

$$I_1 = 3+\gamma, \quad K_4 = H_o^2, \quad K_5 = H_o^2(1+\gamma^2), \quad I_7 = 1+\gamma^2, \quad K_9 = H_o,$$
$$K_{10} = H_o(1+\gamma^2).$$

Also for this problem, and for the particular form of the constitutive equation we are using, the shear component of the stress and the component 2 of the magnetic induction are given by (5.58) and (5.61) as

$$\tau_{12} = 2(\Omega_1^* + H_o^2\Omega_5^* + \Omega_7^* + H_o\Omega_{10}^*)\gamma, \tag{5.219}$$
$$B_2 = -[2H_o\Omega_4^* + 2H_o(1+\gamma^2)\Omega_5^* + \Omega_9^* + (1+\gamma^2)\Omega_{10}^*]. \tag{5.220}$$

In the linear theory of elasticity in order to have a shear deformation it is only necessary to apply a shear stress. This is not the case here. As it can be seen from subsection 5.2.1.1, normal components are also necessary. So, the results shown in Figure 7 of [59] must be valid only for small deformations. In such a case, the shear modulus, which we denote by G, is defined from $\tau = G\gamma$, where τ is the shear stress and γ is the 'amount' of shear. Then from (5.219) we get

$$G = 2(\Omega_1^* + H_o^2\Omega_5^* + \Omega_7^* + H_o\Omega_{10}^*). \tag{5.221}$$

Using (5.218) in the above equation, after some algebraic manipulations we obtain

$$G = \alpha_0 + \alpha_1 H_o^2, \tag{5.222}$$

where α_0 and α_1 are defined as

$$\alpha_0 = g_0 + 2m\gamma^{2(m-1)}h_1\omega_0, \tag{5.223}$$
$$\alpha_1 = g_1 + 2[h_0 + h_1\gamma^{2m}][2(1+\gamma^2)\omega_2 + \omega_3] + 2m\gamma^{2(m-1)}h_1[\omega_1 + (1+\gamma^2)((1+\gamma^2)\omega_2 + \omega_3)] + \mu_o. \tag{5.224}$$

The above parameters depend on the amount of shear. For the case of infinitesimal deformations they might be obtained by evaluating them for $\gamma \to 0$, but we do not do that here.

Figure 7 of [59] shows actually the difference in the shear modulus for the case in which we have an external field and for the case in which there is no field. So, let us calculate

$$\Delta G \equiv G(H_o) - G(H_o = 0) = \alpha_1 H_o^2. \tag{5.225}$$

Then, independently of the value of α_1, which as it was mentioned before depends on the amount of shear, the 'shape' of the curve $\Delta G(H_o)$ will be a quadratic function. The results shown in Figure 7 of [59] (see also Figure 3 of [50]) suggests that the difference in the shear modulus increases until reaching a value H_o' (probably associated with the saturation point of the magneto-active particles), and then it tends to remain constant. This is not of course the behaviour of (5.225), so in the light of this data our model does not seem to work.

In order to use the data provided by Ginder et al [50], let us determine the form of the component 2 of the magnetic induction as a function of the external magnetic field. In order to do so, let us use (5.218) in (5.220). After some manipulations we have

$$B_2 = -(\beta_0 + \beta_1 H_o), \tag{5.226}$$

where the constants β_0 and β_1 are given as

$$\beta_0 = \eta'(K_9), \tag{5.227}$$

$$\begin{aligned}\beta_1 &= \gamma^2 g_1 + (1+\gamma^2)(h_0 + h_1\gamma^{2m})[2(1+\gamma^2)\omega_2 + \omega_3] \\ &\quad +(h_0 + h_1\gamma^{2m})[2\omega_1 + (1+\gamma^2)\omega_3] + (1+\gamma^2)\mu_o + 2\nu'(K_4).\end{aligned} \tag{5.228}$$

The equation (5.226) indicates that there is a linear relation between this component of the field and the external field (for a fixed shear deformation).

- **Particle alignment in the x_1 direction**

It will be necessary to have results for the shear problem when the particles are aligned in the direction 1 (perpendicular to the original magnetic field). In such a case the only invariants that we need to recalculate are I_7, K_9 and K_{10}, which from $(5.64)_1$, $(5.64)_3$ and $(5.64)_4$ are given as

$$I_7 = 1, \quad K_9 = 0, \quad K_{10} = H_o\gamma.$$

From (5.68) and (5.71) we have

$$\tau_{12} = 2\gamma\Omega_1^* + 2H_o^2\gamma\Omega_5^* + H_o\Omega_{10}^*, \tag{5.229}$$

$$B_2 = -(2H_o\Omega_4^* + 2H_o(1+\gamma^2)\Omega_5^* + \gamma\Omega_{10}^*). \tag{5.230}$$

As in the above case, using (5.218) and after some manipulations, we may obtain the following expression for the shear stress

$$\tau_{12} = [g_0 + (g_1 + 2h_0\omega_2 + \mu_o)H_o^2]\gamma, \tag{5.231}$$

and for the component 2 of the magnetic displacement

$$B_2 = -[\gamma^2(g_1 + 2h_0\omega_2) + (1+\gamma^2)\mu_o + 2\nu'(K_4)]H_o. \tag{5.232}$$

5.3.2.1 A model for ν and η

Figure 4 of the paper by Ginder et al [50] presents the results for the magnetization $\mathbf{M}$ as a function of the magnetic field for the shear problem, considering the two particle alignments studied in subsection 5.2.1.1. The only problem with this data is that the material had a different proportion of particles as compared with the one used in [7]. Nevertheless, since we do not expect at this stage to obtain precise expressions for the energy function, but rather a first approximation (as good as possible from the qualitative point of view), we will use this data, in particular in order to obtain the functions ν and η. In order to do so, consider (3.12) and the results for the case where the particle alignment is in the direction 1. Then, using (3.12) and (5.232) we have

$$\mu_o M_2 = -[\gamma_o^2(g_1 + 2h_0\omega_2) + \mu_o(\gamma_o^2 + 2) + 2\nu'(K_4)]H_o. \tag{5.233}$$

Let us define

$$\zeta_0 = \gamma_o^2(g_1 + 2h_0\omega_2) + \mu_o(\gamma_o^2 + 2), \tag{5.234}$$

where γ_o is a given (small) value for the shear.

Now, the data of Figure 4 in [50] suggests, as must be expected, that the magnetization is an odd function, and for values greater than a given magnetic field, remains constant; this indicates that the magnetization has reached the saturation point for the magneto-active particles. We must expect to obtain a hysteresis effect as well, but we will assume that this effect is small.

Then, an appropriate function that may be used in order to model the behaviour of M_2 is given as

$$\mu_o M_2 = m_0 \tanh\left(\frac{H_o}{m_1}\right), \tag{5.235}$$

where the constant m_1 is related to the magnetic field necessary in order to reach the saturation point, while the constant m_0 corresponds to the value of the magnetization (times the magnetic constant μ_o) for that point of saturation.

From (5.233), (5.234), the particular value of K_4 for this problem (equation $(5.49)_1$), and (5.235), we get

$$\nu(K_4) = -\ln\left[\cosh\left(\frac{\sqrt{K_4}}{m_1}\right)\right] m_0 m_1 - \frac{K_4\zeta_0}{2} + \nu_o. \tag{5.236}$$

where ν_o is an arbitrary constant. Now, in order to obtain η we can do the following. Consider the case in which the particles are aligned in the direction 2, and let us define η' as (remembering that for this problem $K_9 = H_o$)

$$\eta'(K_9) = \tilde{\eta}(K_9)K_9. \tag{5.237}$$

Also, let us define the constant ζ_1 as

$$\begin{aligned}\zeta_1 &= \gamma_o^2 g_1 + (1+\gamma_o^2)(h_0 + h_1\gamma_o^{2m})[2(1+\gamma_o^2)\omega_2 + \omega_3] + (h_0 + h_1\gamma_o^{2m})[2\omega_1 \\ &\quad +(1+\gamma_o^2)\omega_3] + (1+\gamma_o^2)\mu_o.\end{aligned} \tag{5.238}$$

Then (5.226) becomes

$$B_2 = -[\tilde{\eta}(K_9) + \zeta_1 + 2\nu'(K_4)]H_o. \tag{5.239}$$

The magnetization M_2 is then given from (3.12) as

$$\mu_o M_2 = -[\tilde{\eta}(K_9) + \zeta_1 + \mu_o + 2\nu'(K_9^2)]K_9. \tag{5.240}$$

As in the case of the alignment in the direction 2, the experimental data may be fitted by the same kind of hyperbolic tangent function as above. Then we would have

$$\mu_o M_2 = m_0 \tanh\left(\frac{K_9}{m_2}\right), \tag{5.241}$$

where m_2 is a constant. Note that the experimental data suggest the same level of saturation for the magnetization for the two cases, which is of course expected if the proportion of particles is the same.

Then, using (5.241) and (5.236) in (5.240), we obtain

$$\tilde{\eta}(K_9) = \frac{m_0}{K_9}\left[\tanh\left(\frac{K_9}{m_2}\right) - \tanh\left(\frac{K_9}{m_1}\right)\right] - \zeta_1 - \mu_o + \zeta_0. \tag{5.242}$$

As a result, from (5.237), we have

$$\eta(K_9) = m_0 \ln\left[\frac{\cosh^{m_2}\left(\frac{K_9}{m_2}\right)}{\cosh^{m_1}\left(\frac{K_9}{m_1}\right)}\right] + (\zeta_0 - \zeta_1 - \mu_o)\frac{K_9^2}{2} + \eta_o, \tag{5.243}$$

where η_o is an arbitrary constant.

Figure 5.3 shows the results for the magnetization for our model and the data provided in Figure 4 of the paper by Ginder et al [50]. The values of the constants m_0, m_1 and m_2 are

$$m_0 = 0.4998[T], \quad m_1 = 309.3395[kA/m], \quad m_2 = 199.1828[kA/m]. \tag{5.244}$$

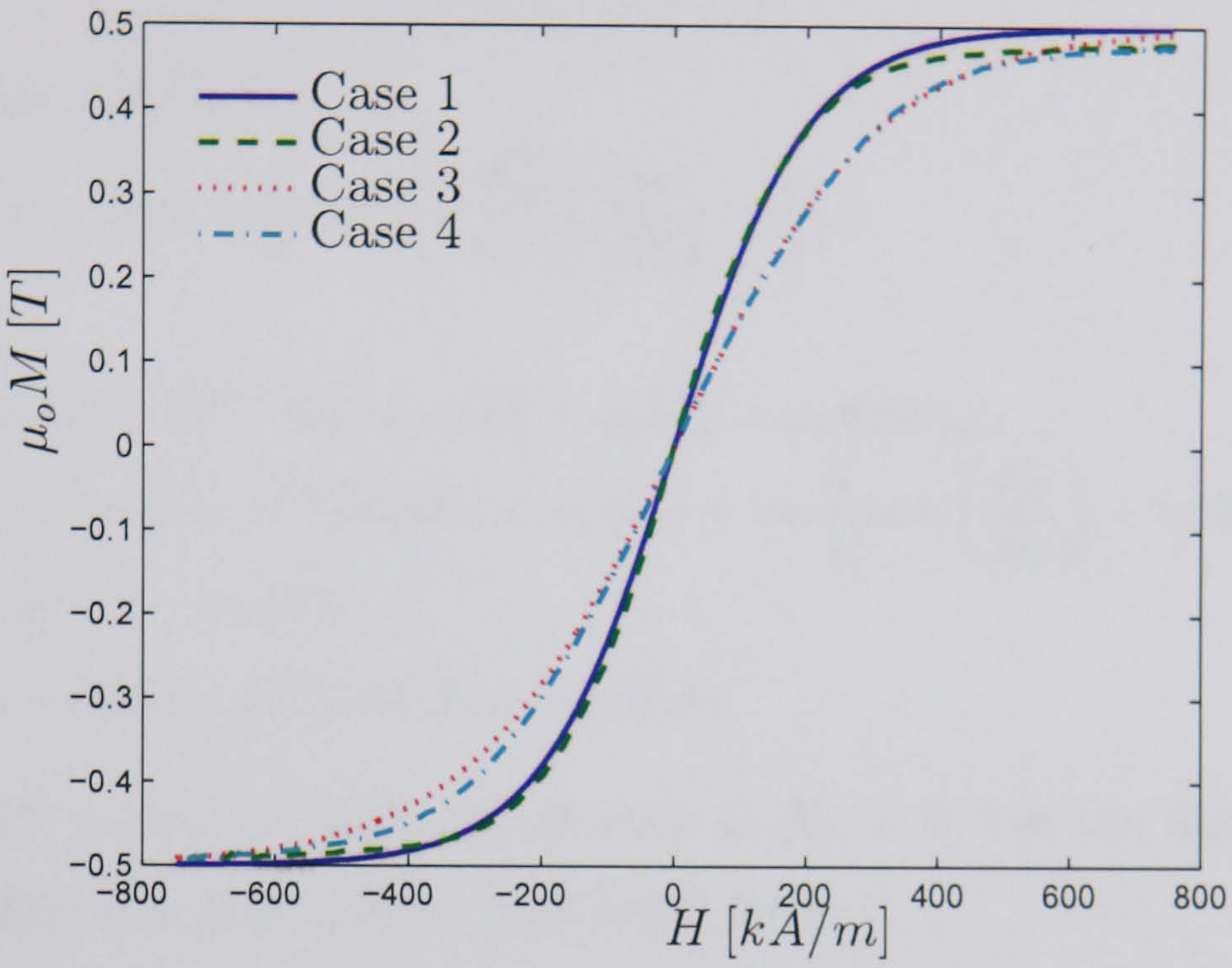


Figure 5.3: Results for the shear experiment. Magnetization as a function of the magnetic field. Case 1: parallel alignment (direction 2), theoretical model. Case 2: parallel alignment, experimental results. Case 3: perpendicular alignment (direction 1), theoretical model. Case 4: perpendicular alignment, experimental results.

5.3.3 Summary of results for the energy function

As a summary, we propose the following expression for our energy function Ω^*

$$\begin{aligned}\Omega^* = &\left(\frac{I_1-3}{2}\right)(g_0+g_1K_4) - \ln\left[\cosh\left(\frac{\sqrt{K_4}}{m_1}\right)\right] m_0 m_1 - \frac{K_4\zeta_0}{2} + \frac{\mu_o}{2}K_5 + [h_0 \\ &+h_1(I_7-1)^m](\omega_0+\omega_1K_9^2+\omega_2K_{10}^2+\omega_3K_9K_{10}) + m_0\ln\left[\frac{\cosh^{m_2}\left(\frac{K_9}{m_2}\right)}{\cosh^{m_1}\left(\frac{K_9}{m_1}\right)}\right] \\ &+(\zeta_0-\zeta_1-\mu_o)\frac{K_9^2}{2}+\Omega_o^*,\end{aligned} \tag{5.245}$$

where the numerical values of the different constants that appear in the above expression are given in Table 5.1, and in (5.244). Regarding the constants ζ_0 and ζ_1, they may be obtained from the above expressions, evaluating at a given shear γ_o, which may be chosen as $\gamma_o \to 0$. Finally, Ω_o^* is just an arbitrary constant that does not need to be specified here, but it may be chosen such that the energy, when there is no field and deformation, becomes zero.

The partial derivatives of Ω^* will be used in the next section, so here we provide a summary of them

$$\Omega_1^* = \frac{1}{2}(g_0 + g_1 K_4), \tag{5.246}$$

$$\Omega_4^* = \frac{1}{2}(I_1 - 3)g_1 - \frac{1}{2}\tanh\left(\frac{\sqrt{K_4}}{m_1}\right)\frac{m_0}{\sqrt{K_4}} - \frac{\zeta_0}{2}, \tag{5.247}$$

$$\Omega_5^* = \frac{\mu_o}{2}, \tag{5.248}$$

$$\Omega_7^* = mh_1(I_7 - 1)^{m-1}[\omega_0 + \omega_1 K_9^2 + \omega_2 K_{10}^2 + \omega_3 K_9 K_{10}], \tag{5.249}$$

$$\Omega_9^* = [h_0 + h_1(I_7 - 1)^m][2\omega_1 K_9 + \omega_3 K_{10}] + m_0\left[\tanh\left(\frac{K_9}{m_2}\right) - \tanh\left(\frac{K_9}{m_1}\right)\right] + (\zeta_0 - \zeta_1 - \mu_o)K_9, \tag{5.250}$$

$$\Omega_{10}^* = [h_0 + h_1(I_7 - 1)^m](2\omega_2 K_{10} + \omega_3 K_9). \tag{5.251}$$

Note that (5.247) cannot be evaluated directly at $K_4 = 0$, but the limit exists. From (5.250) and (5.251) it is easy to show that (5.44) holds.

5.3.4 Magnetic permeability and the traction problem

Before finishing this section, we would like to study some results from the additional paper by Bossis et al [11]. This paper deals with electro and magneto elastomers, and it has in particular a result for the 'magnetic permeability', which may be of interest to study on the basis of our theory.

Figure 7 of [11] shows the results for the relative permeability for the traction experiment of a bar, for two situations, namely with and without external magnetic field, where in the last case a field $H_o = 123[kA/m]$ was applied. The permeability is plotted as a function in this case of the deformation.

In the linear theory of magnetism, the magnetic permeability is defined in the same way as for the free space (equation $(2.106)_2$), as $\mathbf{B} = \mu\mathbf{H}$, where in this case μ depends on the properties of the material. A more advanced non-linear model could be $\mathbf{B} = \mu(\lambda, H_o)\,\mathbf{H}$, where the permeability could in general be a non-linear function of the 'deformation' and the field, and eventually of other important parameters such as the temperature.

So, let's consider again the results of the traction problem for a cylindrical bar (Subsection (5.2.2)), in particular the results considering the magnetic field as the independent variable (Subsection (5.2.2.1)). Now, consider the result (5.114) for the axial component of the magnetic permeability. For our particular form of the energy function (5.245) we have

$$B_z = -\lambda(2H_o\Omega_4^* + 2H_o\lambda^2\Omega_5^* + \Omega_9^* + \lambda^2\Omega_{10}^*). \tag{5.252}$$

In particular, from (5.236), we had

$$\nu(K_4) = -\ln\left[\cosh\left(\frac{\sqrt{K_4}}{m_1}\right)\right] m_0 m_1 - \frac{K_4\zeta_0}{2} + \nu_o.$$

Hence,

$$\nu'(K_4) = -\frac{1}{2}\left[\frac{m_0}{\sqrt{K_4}}\tanh\left(\frac{\sqrt{K_4}}{m_1}\right) + \zeta_0\right], \tag{5.253}$$

and from (5.243) we had

$$\eta(K_9) = m_0 \ln\left[\frac{\cosh^{m_2}\left(\frac{K_9}{m_2}\right)}{\cosh^{m_1}\left(\frac{K_9}{m_1}\right)}\right] + (\zeta_0 - \zeta_1 - \mu_o)\frac{K_9^2}{2} + \eta_o,$$

and we obtain

$$\eta'(K_9) = m_0\left[\tanh\left(\frac{K_9}{m_2}\right) - \tanh\left(\frac{K_9}{m_1}\right)\right] + (\zeta_0 - \zeta_1 - \mu_o)K_9. \tag{5.254}$$

Now, what we want is to obtain an expression equivalent to $\mathbf{B} = \mu\mathbf{H}$ from (5.252) by using our particular model for Ω^*. Using the above expression for Ω^* (equation (5.245)), the derivatives of ν and η given above, along with $(5.105)_1$, $(5.108)_1$, $(5.108)_2$, $(5.108)_4$, $(5.108)_6$ and $(5.109)_1$, and after some manipulations, it can be proved that (5.252) may be rewritten as

$$B_z = \mu_\lambda(H_o)H_o, \tag{5.255}$$

where $\mu_\lambda(H_o)$ is given as

$$\begin{aligned}\mu_\lambda(H_o) &= -(\lambda-1)^2(2+\lambda)g_1 - \lambda\{(\lambda^2-1)\mu_o - \zeta_1 + [(\lambda^2-1)^m h_1 + h_0][2\omega_1 + \omega_3 \\ &\quad + \lambda^2(2\omega_2+\omega_3)] + \frac{m_0}{H_o}\left[\tanh\left(\frac{H_o}{m_2}\right) - 2\tanh\left(\frac{H_o}{m_1}\right)\right]\right\}.\end{aligned} \tag{5.256}$$

From the above expression we cannot evaluate μ directly at $H_o = 0$, but the limit exists and is given (for the case in which there is no deformation) as

$$\mu_{\lambda=1}(0) = \lim_{H_o\to 0}\mu_{\lambda=1}(H_o) = \zeta_1 - 2h_0(\omega_1+\omega_2+\omega_3) + m_o\left(\frac{2}{m_1} - \frac{1}{m_2}\right). \tag{5.257}$$

Also,

$$\mu_{\lambda=1}(H_o \neq 0) = \zeta_1 - 2h_0(\omega_1+\omega_2+\omega_3) + \frac{m_0}{H_o}\left[2\tanh\left(\frac{H_o}{m_1}\right) - \tanh\left(\frac{H_o}{m_2}\right)\right]. \tag{5.258}$$

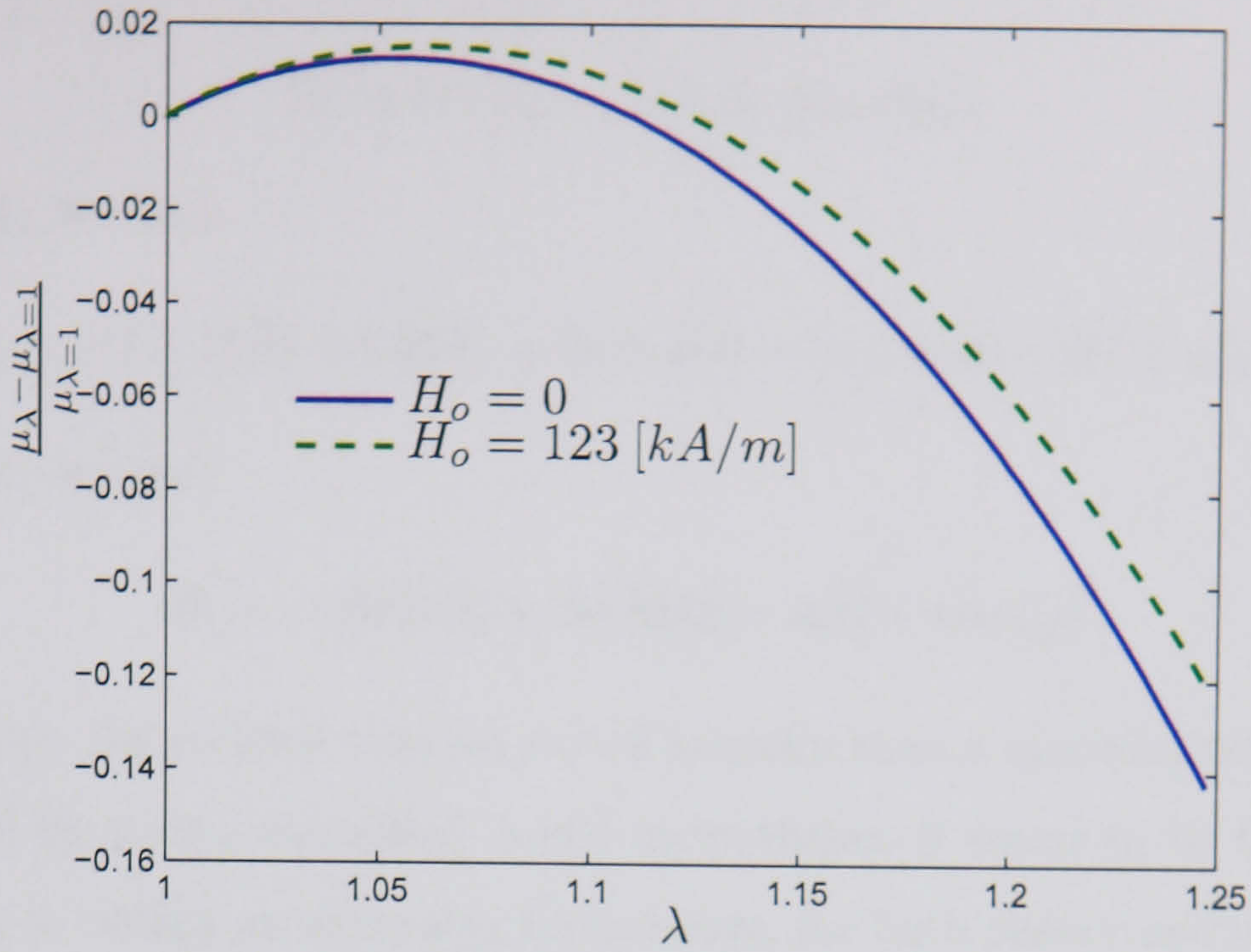


Figure 5.4: Relative magnetic permeability for the traction problem. Theoretical result.

The quantities that Figure 7 of [11] displays are the quotients $\frac{\mu_\lambda(H_o=0)-\mu_{\lambda=1}(H_o=0)}{\mu_{\lambda=1}(H_o=0)}$ and $\frac{\mu_\lambda(H_o)-\mu_{\lambda=1}(H_o)}{\mu_{\lambda=1}(H_o)}$. For our particular constitutive equation[7] we obtain the result shown in Figure 5.4.

The experimental results obtained by Bossis et al. [11] suggest that this relative magnetic permeability is negative for $1 \leq \lambda \leq 1.5$; from Figure 5.4 we see that is not the case for our model, where there is a small interval around $\lambda = 1.05$ where the relative permeability is positive. The experimental value for this permeability for $\lambda = 1.5$ and $H_o = 0$ is approximately -0.15, which is very close to the value obtained with our energy function. Overall our results for the relative magnetic permeability do not compare well with the results shown in Figure 7 of [11].

5.4 Linear universal relations for transversely isotropic MS elastomers

In the previous section we presented the difficulties of finding an appropriate model for the energy function. These difficulties are not only related to the lack of enough experimental data, but also to the complexity of the original problem, where we have to work with a function that depends on ten invariants. As a result, it was necessary to assume a simplified form for the energy function, which depends only on six of these ten invariants.

[7]Some numerical values, for example ζ_1, were obtained by assuming $\gamma_o = 0$.

We had

$$\Omega^* = \Omega^*(I_1, K_4, K_5, I_7, K_9, K_{10}).$$

and, from (5.34), we have

$$\tau = 2\mathbf{b}\Omega_1^* - p^*\mathbf{I} + 2\mathbf{bH}\otimes\mathbf{bH}\Omega_5^* + 2\mathbf{a}\otimes\mathbf{a}\Omega_7^* + (\mathbf{a}\otimes\mathbf{bH} + \mathbf{bH}\otimes\mathbf{a})\Omega_{10}^*. \tag{5.259}$$

and, from (5.38), we get

$$\mathbf{B} = -(2\mathbf{bH}\Omega_4^* + 2\mathbf{b}^2\mathbf{H}\Omega_5^* + \mathbf{a}\Omega_9^* + \mathbf{ba}\Omega_{10}^*). \tag{5.260}$$

It might be highly improbable that we would actually have a material that may be accurately described by such a simplified model; nevertheless it seems to be the best that we can do in order to obtain an amenable formulation, for both theory and experiments.

The question is how to know if these simplifications are actually valid, without having to give more detailed information about the form of the energy function. The solution is to use universal relations; as was emphasized by Beatty [6], these relations hold independently of the particular form of the constitutive equation for a family of materials.

Universal relations have been found for the particular case of isotropic magneto-elastic elastomers; see, for example, [18] and Section 4.1. The results presented in [18] follow closely the general method developed by Pucci and Saccomandi [85]. Unfortunately, in our problem it has not been possible to find the universal relations following such an elegant method; instead we use a more direct but less concise method developed originally only for non-linear elastic problems by Bustamante and Ogden [21]. This method, which has also been used in order to find universal relations for transversely electro-elastic problems [19] (see Section 8.4), is described as follows.

Consider the equation (5.259), rewritten in the form

$$\boldsymbol{\tau} = \mathbf{d}^{(0)}p^* + \mathbf{d}^{(1)}\Omega_1^* + \mathbf{d}^{(2)}\Omega_5^* + \mathbf{d}^{(3)}\Omega_7^* + \mathbf{d}^{(4)}\Omega_{10}^*, \tag{5.261}$$

where the vectors $\boldsymbol{\tau}$ and $\mathbf{d}^{(i)}$, with $i = 0, 1, 2, 3, 4$, are defined as follows

$$\boldsymbol{\tau} = (\tau_{11}, \tau_{22}, \tau_{33}, \tau_{23}, \tau_{13}, \tau_{12})^T, \tag{5.262}$$

$$\mathsf{d}^{(0)} = -(1, 1, 1, 0, 0, 0)^T, \tag{5.263}$$

$$\mathsf{d}^{(1)} = 2(b_{11}, b_{22}, b_{33}, b_{23}, b_{13}, b_{12})^T, \tag{5.264}$$

$$\mathsf{d}^{(2)} = 2(f_1^2, f_2^2, f_3^2, f_2f_3, f_1f_3, f_1f_2)^T, \tag{5.265}$$

$$\mathsf{d}^{(3)} = 2(a_1^2, a_2^2, a_3^2, a_2a_3, a_1a_3, a_1a_2)^T. \tag{5.266}$$

$$\mathsf{d}^{(4)} = (2a_1f_1, 2a_2f_2, 2a_3f_3, a_2f_3 + a_3f_2, a_1f_3 + a_3f_1, a_1f_2 + a_2f_1)^T, \tag{5.267}$$

where f_j are the components of the vector $\mathbf{f}$ defined as

$$\mathbf{f} = \mathbf{bH}. \tag{5.268}$$

It is necessary to remark that we will not make a distinction between constrained and unconstrained materials in order to look for universal relations[8].

Now, let us look for a vector $\mathbf{e}$

$$\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5, \mathrm{e}_6)^T,$$

such that the equation

$$\boldsymbol{\tau} \cdot \mathbf{e} = 0 \tag{5.269}$$

holds for <u>any</u> particular form of the energy function. In such a case from (5.261) we would need

$$\mathbf{d}^{(i)} \cdot \mathbf{e} = 0, \quad i = 0, 1, 2, 3, 4, \tag{5.270}$$

which may be written as

$$\boldsymbol{\mathcal{M}}\mathbf{e} = \mathbf{0}, \tag{5.271}$$

where $\boldsymbol{\mathcal{M}}$ is a matrix of six columns and five rows, which is formed with the vectors $\mathbf{d}^{(i)}$ as rows. The solution of (5.271) corresponds to the vectors of the null space of $\boldsymbol{\mathcal{M}}$, which in this case has dimension one (this is in concordance with the theory of Pucci and Saccomandi [85]). So, there is one linearly independent vector $\mathbf{e}$ that is solution of (5.269). This is the result (using the original notation for the stress)

$$\mathrm{e}_1\tau_{11} + \mathrm{e}_2\tau_{22} + \mathrm{e}_3\tau_{33} + \mathrm{e}_4\tau_{23} + \mathrm{e}_5\tau_{13} + \mathrm{e}_6\tau_{12} = 0, \tag{5.272}$$

where the constants e_j with $j = 1, ..., 6$ are given below as

$$\begin{aligned}
\mathrm{e}_1 = {} & (a_3 f_2 - a_2 f_3)\{a_3^2 f_1(b_{13} f_2 - b_{23} f_1) + a_2^2 f_1(b_{23} f_1 - b_{12} f_2) - a_1 a_3[b_{33} f_1 f_2 - b_{22} f_1 f_2 \\
& + b_{12} f_2^2 - 2b_{23} f_1 f_3 + b_{13} f_2 f_3] + a_1^2[(b_{33} - b_{22}) f_2 f_3 + b_{23}(f_2^2 - f_3^2)] \\
& + a_2[a_3 f_1(b_{33} f_1 - b_{22} f_1 + b_{12} f_2 - b_{13} f_3) + a_1(-2b_{23} f_1 f_2 + f_3(b_{22} f_1 \\
& - b_{33} f_1 + b_{12} f_2 + b_{13} f_3))]\},
\end{aligned} \tag{5.273}$$

[8]More universal relations might be found for the case of an incompressible material, but these relations must be found by studying particular boundary value problems; see, for example, [83]. See also the remark at the end of Subsection 4.1.1.2.

$$\begin{aligned}
\mathsf{e}_2 = {} & (a_3 f_1 - a_1 f_3)\{a_2^2[(b_{33} - b_{11}) f_1 f_3 + b_{13}(f_1^2 - f_3^2)] + f_2[a_3^2(b_{23} f_1 - b_{13} f_2) \\
& + a_1^2(b_{13} f_2 - b_{12} f_3) + a_1 a_3(b_{12} f_1 - b_{11} f_2 + b_{33} f_2 - b_{23} f_3)] + a_2[-a_3(b_{12} f_1^2 \\
& - b_{11} f_1 f_2 + b_{33} f_1 f_2 + b_{23} f_1 f_3 - 2b_{13} f_2 f_3) + a_1(-2b_{13} f_1 f_2 + f_3(b_{12} f_1 + b_{11} f_2 \\
& - b_{33} f_2 + b_{22} f_3))]\}, \qquad (5.274)
\end{aligned}$$

$$\begin{aligned}
\mathsf{e}_3 = {} & (a_2 f_1 - a_1 f_2)\{-a_3(a_2 f_1 - a_1 f_2)(b_{13} f_1 + b_{23} f_2) + a_3^2[(b_{22} - b_{11}) f_1 f_2 + b_{12}(f_1^2 - f_2^2)] \\
& + a_3[a_2(b_{11} f_1 - b_{22} f_1 + 2b_{12} f_2) + a_1(b_{11} f_2 - 2b_{12} f_1 - b_{22} f_2)] f_3 + f_3[-(a_1 b_{13} \\
& + a_2 b_{23})(a_1 f_2 - a_2 f_1) + (a_1^2 b_{12} - a_2^2 b_{12} + a_1 a_2(b_{22} - b_{11})) f_3]\}, \qquad (5.275)
\end{aligned}$$

$$\begin{aligned}
\mathsf{e}_4 = {} & a_3^2 f_1[b_{12}(f_2^2 - f_1^2) + (b_{11} - b_{22}) f_1 f_2] + a_1^3(f_2^2 - f_3^2)(b_{13} f_2 - b_{12} f_3) \\
& + a_1 a_3^2\{-b_{13} f_2(f_1^2 + f_2^2) + [3b_{12} f_1^2 - 2b_{11} f_1 f_2 + 2b_{22} f_1 f_2 - b_{12} f_2^2] f_3\} \\
& + a_1^2 a_3[f_2^2(b_{12} f_1 - b_{11} f_2 + b_{33} f_2) + 2b_{13} f_1 f_2 f_3 - (3b_{12} f_1 - b_{11} f_2 + b_{22} f_2) f_3^2] \\
& + a_2^3 f_1[(b_{11} - b_{33}) f_1 f_3 + b_{13}(f_3^2 - f_1^2)] + a_2\{a_3^2 f_1[b_{13}(f_1^2 + f_2^2) \\
& + (f_1(b_{22} - b_{33}) - 2b_{12} f_2) f_3] + 2a_1 a_3[-f_1 f_2(b_{12} f_1 + (b_{33} - b_{11}) f_2) - b_{13}(f_1^2 \\
& - f_2^2) f_3 + (f_1(b_{11} - b_{22}) + b_{12} f_2) f_3^2] + a_1^2[f_2(2b_{12} f_1 + (b_{11} - b_{33}) f_2) f_3 \\
& - (b_{11} - b_{22}) f_3^2 + b_{13} f_1(f_3^2 - 3f_2^2)]\} + a_2^2\{a_3 f_1[f_2(f_1(b_{33} - b_{11}) - 2b_{13} f_3) \\
& + b_{12}(f_1^2 + f_3^2)] + a_1(b_{13} f_2(3f_1^2 - f_3^2) - f_3(2(b_{11} - b_{33}) f_1 f_2 \\
& + b_{12}(f_1^2 + f_3^2)))\}, \qquad (5.276)
\end{aligned}$$

$$\begin{aligned}
\mathsf{e}_5 = {} & a_2^3(f_1^2 - f_3^2)(b_{22} f_1 - b_{12} f_3) + a_2^2\{a_3[f_1^2(f_1(b_{33} - b_{22}) + b_{12} f_2) + 2b_{23} f_1 f_2 f_3 \\
& + ((b_{22} - b_{11}) f_1 - 3b_{12} f_2) b_3^2] + a_1[f_1(f_1(b_{22} - b_{33}) + 2b_{12} f_2) f_3 + (b_{11} - b_{22}) f_3^2 \\
& + b_{23} f_2(f_3^2 - 3f_1^2)]\} + f_2\{a_3^2[(b_{22} - b_{11}) f_1 f_2 + b_{12}(f_1^2 - f_2^2)] + a_1 a_3^2[b_{23}(f_1^2 \\
& + f_2^2) + (f_2(b_{11} - b_{22}) - 2b_{12} f_1) f_3] + a_1^3[(b_{22} - b_{33}) f_2 f_3 + b_{23}(f_3^2 - f_2^2)] \\
& + a_1^2 a_3[f_1 f_2(b_{33} - b_{22}) - 2b_{23} f_1 f_3 + b_{12}(f_2^2 + f_3^2)]\} + a_2\{-a_3^2[b_{23} f_1(f_1^2 \\
& + f_2^2) + (2(b_{22} - b_{11}) f_1 f_2 + b_{12}(f_1^2 - 3f_2^2)) f_3] + 2a_1 a_3[f_1 f_2((b_{22} - b_{33}) f_1 - b_{12} f_2) \\
& b_{23}(f_1^2 - f_2^2) f_3 + (b_{12} f_1 + (b_{22} - b_{11}) f_2) f_3^2] + a_1^2[b_{23} f_1(3f_2^2 - f_3^2) \\
& - f_3(f_2(2f_1(b_{22} - b_{33}) + b_{12} f_2) + b_{12} f_3^2)]\}, \qquad (5.277)
\end{aligned}$$

$$\begin{aligned}
\mathsf{e}_6 = {} & a_3^3(f_1^2 - f_2^2) + (b_{23}f_1 - b_{13}f_2) + a_1a_3^2[b_{11}f_2^3 - b_{22}f_1^2f_2 + b_{33}(f_1^2 - f_2^2)f_2 \\
& -3b_{23}f_1^2f_3 + 2b_{13}f_1f_2f_3 + b_{23}f_2^2f_3] + a_1^2a_3[-f_2^2(b_{23}f_1 + b_{13}f_2) \\
& +2(b_{22} - b_{33})f_1f_2f_3 + (3b_{23}f_1 - b_{13}f_2)f_3^2] + a_2^2f_3[(b_{33} - b_{11})f_1f_3 \\
& b_{13}(f_1^2 - f_3^2)] + a_1^3f_3[(b_{33} - b_{22})f_2f_3 + b_{23}(f_2^2 - f_3^2)] + a_2\{a_3^2[(b_{22} \\
& -b_{33})f_1^3 + (b_{33} - b_{11})f_1f_2^2 + b_{13}f_1^2f_3 + 2b_{23}f_1f_2f_3 - 3b_{13}f_2^2f_3] \\
& +2a_1a_3[f_1f_2(b_{23}f_1 + b_{13}f_2) - ((b_{22} - b_{33})f_1^2 + (b_{11} - b_{33})f_2^2)f_3 \\
& -(b_{13}f_1 + b_{23}f_2)f_3^2] + a_1^2f_3[-2b_{23}f_1f_2 + (b_{22} - b_{33})f_1f_3 + b_{13}(f_2^2 \\
& +f_3^2)]\} + a_2^2\{a_1f_3[f_2((b_{11} - b_{33})f_3 - 2b_{13}f_1) + b_{23}(f_1^2 + f_3^2)] - a_3[b_{23}f_1(f_1^2 \\
& +f_3^2) + f_2(2(b_{33} - b_{11})f_1f_3 + b_{13}(f_1^2 - 3f_3^2))]\}. \qquad (5.278)
\end{aligned}$$

Equation (5.272) is linear (in the stress components) and holds independently of the particular form of the energy function, for the particular case in which we consider an energy function given by (5.191). The coefficients e_j do not depend on Ω^*, but only in the deformation, the magnetic field, and the particle alignment. What it is necessary to do now is to use (5.272) for some particular boundary value problems. Note that when the general form for the stress (5.34) is considered, then the matrix $\boldsymbol{\mathcal{M}}$ has an empty null space; in fact, if the energy function Ω^* is such that the number of rows of $\boldsymbol{\mathcal{M}}$ is equal or greater than six, then in general for unconstrained materials we do not have linear universal relations (see [83, 85]).

5.4.1 Application: the homogeneous deformation of a slab in a uniform field

The coefficients e_k of the linear universal relation (5.272) are complex and lengthy expressions. Here we show an example of application of the above relation.

For the simple shear (5.45) we have that $b_{11} = 1 + \gamma^2$, $b_{22} = 1$, $b_{33} = 1$, $b_{12} = \gamma$ and $b_{13} = b_{23} = 0$; if we use the external field (5.48) we have $\mathsf{H} = (0, H_o, 0)^T$, and as a result from (5.268) we get $\mathsf{f} = (\gamma H_o, H_o, 0)^T$. Regarding the particle alignment, from Subsection 5.2.1.1 we choose two cases, namely $\mathsf{a}_0 = (0,1,0)^T$ and $\mathsf{a}_0 = (1,0,0)^T$; from (5.2) we have $\mathsf{a} = (\gamma, 1, 0)^T$ and $\mathsf{a} = (1,0,0)^T$ respectively. It is possible to show that in both cases $\tau_{13} = \tau_{23} = 0$.

For simple shear with the above two different alignments for the particles it is straightforward to show that the linear universal relation (5.272) is satisfied trivially.

Consider now the homogeneous deformation of a slab in a uniform field (Subsection

(4.1.1.2)). For this example we have

$$\mathsf{F} = \begin{pmatrix} \mu_1 & 0 & \kappa_1\mu_3 \\ 0 & \mu_2 & \kappa_1\mu_3 \\ 0 & 0 & \mu_3 \end{pmatrix},$$

and for the matrix form of the left Cauchy-Green deformation tensor we have

$$\mathsf{b} = \begin{pmatrix} \mu_1^2 + (\kappa_1\mu_3)^2 & \kappa_1\kappa_2\mu_3^2 & \kappa_1\mu_3^2 \\ \kappa_1\kappa_2\mu_3^2 & \mu_2^2 + (\kappa_2\mu_3)^2 & \kappa_2\mu_3 \\ \kappa_1\mu_3^2 & \kappa_2\mu_3 & \mu_3^2 \end{pmatrix}. \tag{5.279}$$

Let's consider the case of a uniform field in the direction 3, such that $\mathsf{H}_l = (0, 0, H_o)^T$. As a result from (5.2) and (4.84) we have $\mathsf{H} = (0, 0, \mu_1\mu_2 H_o)^T$. Then from (5.268) and (5.279) we have

$$\mathsf{f} = \mu_3(\kappa_1 H_o, \kappa_2 H_o, 1)^T. \tag{5.280}$$

Consider the three following cases for the particle alignment:

Particle alignment in the X_1 direction. In this case we have $\mathsf{a}_0 = (1, 0, 0)^T$; then, from (5.2), $\mathsf{a} = (\mu_1, 0, 0)^T$, and hence for the linear universal relation (5.272) we get

$$\begin{aligned}\kappa_2\mu_1^3\mu_3^3\{\mu_3^2(\tau_{12} - \kappa_1\tau_{23}) + H_o\{\mu_1^2\tau_{12} + \mu_3^2[(\kappa_2^2 - 1)\tau_{12} + \kappa_1\tau_{23} \\ -\kappa_2(\tau_{13} + \kappa_1(\tau_{22} - \tau_{33}))] + \kappa_2 H_o[-\mu_2^2\tau_{13} + \mu_3^2(\tau_{13} + \kappa_1\tau_{22} \\ -\kappa_2(\tau_{12} + \kappa_2\tau_{13} - \kappa_1\tau_{23}) - \kappa_1\tau_{33}) + \kappa_2\mu_3^2(\kappa_2\tau_{13} - \kappa_1\tau_{23})H_o]\}\} = 0.\end{aligned} \tag{5.281}$$

If $\kappa_2\mu_1^3\mu_3^3 \neq 0$, then from the above relation we have

$$\begin{aligned}\mu_3^2(\tau_{12} - \kappa_1\tau_{23}) + H_o\{\mu_1^2\tau_{12} + \mu_3^2[(\kappa_2^2 - 1)\tau_{12} + \kappa_1\tau_{23} - \kappa_2(\tau_{13} + \kappa_1(\tau_{22} - \tau_{33}))] \\ +\kappa_2 H_o[-\mu_2^2\tau_{13} + \mu_3^2(\tau_{13} + \kappa_1\tau_{22} - \kappa_2(\tau_{12} + \kappa_2\tau_{13} - \kappa_1\tau_{23}) - \kappa_1\tau_{33}) \\ +\kappa_2\mu_3^2(\kappa_2\tau_{13} - \kappa_1\tau_{23})H_o]\} = 0.\end{aligned} \tag{5.282}$$

Particle alignment in the X_2 direction. In this case $\mathsf{a}_0 = (0, 1, 0)^T$ and $\mathsf{a} = (0, \mu_2, 0)^T$, and, from (5.272), with the condition $\kappa_1\mu_2^3\mu_3^3 \neq 0$ we have a similar relation

$$\begin{aligned}\mu_3^2(\tau_{12} - \kappa_2\tau_{23}) + H_o\{\mu_1^2\tau_{12} + \mu_3^2[(\kappa_1^2 - 1)\tau_{12} + \kappa_2\tau_{13} - \kappa_1(\tau_{23} + \kappa_2(\tau_{11} - \tau_{33}))] \\ +\kappa_1 H_o[-\mu_1^2\tau_{23} + \mu_3^2(\tau_{23} - \kappa_1(\tau_{12} + \kappa_1\tau_{23}) + \kappa_2(\tau_{11} + \kappa_1\tau_{13} - \tau_{33})) \\ +\kappa_1\mu_3^2(\kappa_1\tau_{23} - \kappa_2\tau_{13})H_o]\} = 0.\end{aligned} \tag{5.283}$$

Particle alignment in the X_3 direction. Here we have $a_0 = (0,0,1)^T$ and $a = \mu_3(\kappa_1, \kappa_2, 1)^T$. From (5.272), we have

$$\kappa_1\kappa_2\mu_3^6(H_o - 1)^3(\mu_1^2 - \mu_2^2)(\kappa_1\tau_{23} - \kappa_2\tau_{13}) = 0. \quad (5.284)$$

If $\kappa_1\kappa_2\mu_3^6(H_o - 1)^3(\mu_1^2 - \mu_2^2) \neq 0$ this implies that

$$\kappa_1\tau_{23} = \kappa_2\tau_{13}. \quad (5.285)$$

5.5 Boundary value problems: non-homogeneous deformations

The idea in this last section is to use the particular form of the energy function Ω^* (equation (5.245)), in order to obtain results for some controllable non-homogeneous deformations.

Two problems with cylindrical symmetry will be treated, namely the extension and inflation of a tube, and the extension and torsion of a cylinder. First we will study under what conditions for the magnetic field and the particle alignment the problems are controllable, and then for the particular energy function (5.245) closed form solutions for the boundary value problems will be presented.

The two problems presented in this section, theoretically speaking, would correspond to tubes and cylinders of 'infinite' length. This is in order to avoid problems with the magnetic boundary conditions (3.66), in the same way as it was done for the homogeneous problems of Section 5.2. In practice, what may be done is to use tubes and cylinders with a length much greater than their diameters. We do not discuss here in more detail this topic, but some preliminary results about the effect of the boundary conditions (3.66) on the magnetic field, for the particular case of isotropic magneto-elastic problems, may be found in [15] and Section 4.2.

In order to study the controllability of the solutions, let's consider the balance equations

$$\mathrm{div}\boldsymbol{\tau} = \mathbf{0}, \quad \mathrm{curl}\mathbf{H} = \mathbf{0}, \quad \mathrm{div}\mathbf{B} = 0,$$

which for the particular case of cylindrical coordinates become (see Appendix A.1)

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0, \quad (5.286)$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} = 0, \quad (5.287)$$

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} = 0, \quad (5.288)$$

$$\frac{\partial H_\theta}{\partial z}=0,\quad \frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}=0,\quad \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)=0, \tag{5.289}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r)+\frac{\partial B_z}{\partial z}=0, \tag{5.290}$$

respectively (in the absence of θ dependence).

5.5.1 Extension and inflation of a tube

Consider the problem of extension and inflation of a tube. The kinematics for this deformation are given as (see [35] for the magneto elastic isotropic counterpart of this problem)

$$r^2=a_i^2+\lambda_z^{-1}(R^2-A_i^2),\quad \theta=\Theta,\quad z=\lambda_z Z, \tag{5.291}$$

where $A_i\le R\le A_e$, $0\le\Theta<2\pi$ and $-\infty\le Z\le\infty$, and a_i and a_e correspond to the interior and exterior radii for the tube in the current configuration. The deformation gradient, and the left and right Cauchy-Green deformation tensors have components

$$\mathsf{F}=\begin{pmatrix}(\lambda_z\lambda)^{-1} & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda_z\end{pmatrix},\quad \mathsf{b}=\mathsf{c}=\begin{pmatrix}(\lambda_z\lambda)^{-2} & 0 & 0\\ 0 & \lambda^2 & 0\\ 0 & 0 & \lambda_z^2\end{pmatrix}, \tag{5.292}$$

where $\lambda=r/R$. The first and second invariants $(5.25)_1$ and $(5.25)_2$ are

$$I_1=\mathrm{tr}\,\mathbf{c}=(\lambda_z\lambda)^{-2}+\lambda^2+\lambda_z^2, \tag{5.293}$$

$$I_2=\frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2-\mathrm{tr}\,\mathbf{c}^2]=\lambda_z^{-2}+\lambda^{-2}+(\lambda\lambda_z)^2. \tag{5.294}$$

Now, there are several options for the magnetic field and the particle alignment field. We study the simplest cases.

5.5.1.1 Axial magnetic field and axial particle alignment

In this case we consider an external axial magnetic field and an axial initial alignment for the particles[9]. Then, the magnetic field will be given as

$$\mathsf{H}_l=(0,0,H_o)^T, \tag{5.295}$$

and from (3.9) and $(5.292)_1$ we have

$$\mathsf{H}=(0,0,\lambda_z^{-1}H_o)^T. \tag{5.296}$$

[9] It should not be difficult to obtain a real tube like this in order to do experiments.

Then the equation (5.289) is satisfied automatically. The invariants K_4, K_5 and K_6 are given from (5.26) as

$$K_4 = H_o^2, \quad K_5 = \lambda_z^2 H_o^2, \quad K_6 = \lambda_z^4 H_o^2. \tag{5.297}$$

From (5.296) and (3.44) the non-zero components of the Maxwell stress are[10]

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{\mu_o}{2}\lambda_z^{-2} H_o^2, \quad \tau_{m_{zz}} = \frac{\mu_o}{2}\lambda_z^{-2} H_o^2. \tag{5.298}$$

Consider now a uniform axial initial alignment for the particles, so that

$$\mathsf{a}_0 = (0, 0, 1)^T. \tag{5.299}$$

From (5.2) and $(5.292)_1$ we obtain

$$\mathsf{a} = (0, 0, \lambda_z)^T. \tag{5.300}$$

The rest of the invariants (5.27), (5.28) are

$$I_7 = \lambda_z^2, \quad I_8 = \lambda_z^4, \quad K_9 = H_o, \quad K_{10} = H_o \lambda_z^2. \tag{5.301}$$

The components of the total stress (5.34) are

$$\tau_{rr} = -p^* + 2(\lambda\lambda_z)^{-2}\Omega_1^* + 2(\lambda^{-2} + \lambda_z^{-2})\Omega_2^*, \tag{5.302}$$

$$\tau_{\theta\theta} = -p^* + 2\lambda^2\Omega_1^* + 2[\lambda_z^{-2} + (\lambda\lambda_z)^2]\Omega_2^*, \tag{5.303}$$

$$\begin{aligned}\tau_{zz} = & -p^* + 2\lambda_z^2\Omega_1^* + 2[\lambda^{-2} + (\lambda\lambda_z)^2]\Omega_2^* + 2H_o^2\lambda_z^2\Omega_5^* + 4H_o^2\lambda_z^4\Omega_6^* + 2\lambda_z^2\Omega_7^* \\ & +4\lambda_z^4\Omega_8^* + 2H_o\lambda_z^2\Omega_{10}^*,\end{aligned} \tag{5.304}$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0, \tag{5.305}$$

and the components of the magnetic induction (5.38) become

$$B_r = B_\theta = 0, \tag{5.306}$$

$$B_z = -(2H_o\lambda_z\Omega_4^* + 2H_o\lambda_z^3\Omega_5^* + 2H_o\lambda_z^5\Omega_6^* + \lambda_z\Omega_9^* + \lambda_z^3\Omega_{10}^*). \tag{5.307}$$

In order to study the controllability of the solution, we must study what happens with the balance equations. Consider the following decomposition of the total stress:

$$\tau_{rr} = \tilde{\tau}_{rr}(r) - p^*, \quad \tau_{\theta\theta} = \tilde{\tau}_{\theta\theta}(r) - p^*, \quad \tau_{zz} = \tilde{\tau}_{zz}(r) - p^*. \tag{5.308}$$

[10] It was mentioned previously that we will assume that the bodies will be completely surrounded by a free space; as a result we need to consider the presence of the Maxwell stresses for the extremes of the cylinder as external loads.

It is easy to see why $\tilde{\tau}_{rr}$, $\tilde{\tau}_{\theta\theta}$ and $\tilde{\tau}_{zz}(r)$ are only functions of r, this is because from the above results for the invariants it is clear that they only depend on r. and as a result Ω^* will only be a function of this variable.

Now, if we use the above decomposition of the stress in (5.286)-(5.288), remembering that $\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0$, then (5.287) is satisfied automatically, while (5.288) implies that $p^* = p^*(r)$, and as a result the only equation to solve is (5.286). which we now write as

$$\frac{\mathrm{d}\tilde{\tau}_{rr}}{\mathrm{d}r} - \frac{\mathrm{d}p^*}{\mathrm{d}r} + \frac{1}{r}(\tilde{\tau}_{rr} - \tilde{\tau}_{\theta\theta}) = 0. \tag{5.309}$$

Regarding the magnetic induction, from (5.307) and the same considerations as before, we have that $B_z = B_z(r)$, and as a result (5.290) is satisfied trivially. Hence, this solution is controllable.

Other possibilities for the magnetic field and the particle alignment might be considered. For example, one simple case may be to work with the same magnetic field $\mathbf{H}_l$, but with a radial uniform particle alignment field, given as $\mathsf{a}_0 = (1,0,0)^T$. But in this case, from (5.34) is not difficult to see that $\tau_{rz} \neq 0$ in general, and as a result, if we use the same decomposition for the normal components of the stress (5.308), from (5.288) we would find that p^* would be a function of r and z, and it cannot be obtained from (5.286). Thus, this solution would not be controllable in general.

Another simple possibility may be to work with a radial uniform particle alignment as before, and a radial uniform magnetic field $\mathsf{H}_l = (H_o, 0, 0)^T$. In such a case we would have that $\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0$, then p^* may be obtained by simple integration from (5.286), but from (5.38) we would have that $B_r = B_r(r)$, thus would give B_r singular at $r = 0$. This solution is not admissible either.

Other possibilities may arise. We could try to work with the magnetic induction as the independent magnetic variable, but we do not study the problem of finding more controllable solutions for this case further[11].

5.5.1.2 Boundary conditions

We study further now the boundary conditions for the above problem. First in a general context, and then for our particular form for the energy function.

From (5.309) we have that p^* may be calculated as

$$p^*(r) = \tilde{\tau}_{rr}(r) - \tilde{\tau}_{rr}(a_i) + \int_{a_i}^{r} \frac{1}{\bar{r}}(\tilde{\tau}_{rr}(\bar{r}) - \tilde{\tau}_{\theta\theta}(\bar{r}))\,\mathrm{d}\bar{r}. \tag{5.310}$$

[11]The problem of finding all the controllable solutions for the isotropic case was treated in [82].

As for the boundary conditions for the mechanical part of the problem (3.69). let us require a free stress condition for the outer surface of the tube, in such a case we would have

$$\tau_{rr}(a_e) - \tau_{m_{rr}}(a_e) = 0. \tag{5.311}$$

The radial component of the stress might be calculated directly from (5.286) without determining p^*. From (5.286) we have

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \frac{1}{r}(\tau_{\theta\theta} - \tau_{rr}), \tag{5.312}$$

with the boundary conditions

$$\tau_{rr}(a_i) = -P + \tau_{m_{rr}}(a_i), \tag{5.313}$$

$$\tau_{rr}(a_e) = \tau_{m_{rr}}(a_e). \tag{5.314}$$

In the above boundary conditions P is the internal pressure required to inflate the cylinder. With the boundary condition (5.314) and $(5.298)_1$, along with (5.312), we obtain

$$\tau_{rr}(r) = -P - \frac{\mu_o}{2}\lambda_z^{-2}H_o^2 + \int_{a_i}^{r} \frac{1}{\bar{r}}(\tilde{\tau}_{rr}(\bar{r}) - \tilde{\tau}_{\theta\theta}(\bar{r}))\,\mathrm{d}\bar{r}. \tag{5.315}$$

With the above solution and (5.313), P is expressed as

$$P = \int_{a_i}^{a_e} \frac{1}{\bar{r}}(\tilde{\tau}_{rr}(\bar{r}) - \tilde{\tau}_{\theta\theta}(\bar{r}))\,\mathrm{d}\bar{r}, \tag{5.316}$$

and hence, from (5.302) and (5.303), we obtain

$$P = \int_{a_i}^{a_e} \frac{2}{\bar{r}}[(\bar{\lambda}^2 - (\bar{\lambda}\lambda_z)^{-2})\Omega_1^* + ((\bar{\lambda}\lambda_z)^2 - \bar{\lambda}^{-2})\Omega_2^*]\mathrm{d}\bar{r}. \tag{5.317}$$

Note that the above result tells us that for the internal pressure there is no difference in the behaviour of a transversely isotropic material and an isotropic one.

The relation between p^* and P is given as follows

$$p^*(a_i) = P + \tilde{\tau}_{rr}(a_i) + \frac{\mu_o}{2}\lambda_z^{-2}H_o^2. \tag{5.318}$$

From the above results and from (5.310) we obtain for p^*

$$p^*(r) = \tilde{\tau}_{rr}(r) - \int_{r}^{a_e} \frac{1}{\bar{r}}(\tilde{\tau}_{rr}(\bar{r}) - \tilde{\tau}_{\theta\theta}(\bar{r}))\,\mathrm{d}\bar{r} + \frac{\mu_o}{2}\lambda_z^{-2}H_o^2. \tag{5.319}$$

In order to obtain the extension for the tube we need a traction force applied at the ends of the tube; this force, denoted $\mathcal{N}$, is given as $\mathcal{N} = \int_{a_i}^{a_e} t_z\,\mathrm{d}A$, where t_z is the component of the external vector stress applied at the end on the tube in the axial direction, and dA is the differential element of area. Using (3.69), $(5.298)_2$ and (5.304) we have

$$\begin{aligned}\mathcal{N} &= \int_{a_1}^{a_e} \tau_{zz} 2\pi r \,\mathrm{d}r - \frac{\mu_o}{2}\lambda_z^{-2} H_o^2 2\pi (a_e^2 - a_i^2),\\ &= \int_{a_1}^{a_e} [-p^* + 2\lambda_z^2 \Omega_1^* + 2(\lambda^{-2} + (\lambda\lambda_z)^2)\Omega_2^* + 2H_o^2\lambda_z^2\Omega_5^* + 4H_o^2\lambda_z^4\Omega_6^* + 2\lambda_z^2\Omega_7^*\\ &+4\lambda_z^4\Omega_8^* + 2H_o\lambda_z^2\Omega_{10}^*] 2\pi r \mathrm{d}r - \frac{\mu_o}{2}\lambda_z^{-2} H_o^2 2\pi (a_e^2 - a_i^2).\end{aligned} \quad (5.320)$$

5.5.1.3 Boundary conditions for a particular energy function

For the particular form of the energy function (5.245), we have

$$\Omega_1^* = \frac{1}{2}(g_0 + g_1 K_4), \quad \Omega_2^* = 0.$$

As a result, after integrating (5.317) and after some manipulations, we obtain

$$P = \frac{1}{2\lambda_z}(g_0 + g_1 H_o^2)\left\{\ln\left[\lambda_z\left(\frac{a_e^2}{A_i^2} - \frac{a_i^2}{A_i^2}\right) + 1\right] + \ln\left(\frac{a_i^2}{a_e^2}\right) + A_i^2\left(\frac{1}{\lambda_z} - \frac{a_i^2}{A_i^2}\right)\right\}, \quad (5.321)$$

where a_e is calculated as $a_e = \sqrt{a_i^2 + \lambda_z^{-1}(A_e^2 - A_i^2)}$. Let us define the non-dimensional parameters χ and ς as

$$\chi = \frac{a_i}{A_i}, \quad \varsigma = \frac{A_e}{A_i}. \quad (5.322)$$

Then (5.321) may be rewritten as

$$\begin{aligned}P = \frac{1}{2\lambda_z}(g_0 + g_1 H_o^2)&\left\{\ln \varsigma^2 + \ln\left[\frac{\chi^2}{\chi^2 + \lambda_z^{-1}(\varsigma^2 - 1)}\right]\right.\\ &\left.+\left(\frac{1}{\lambda_z} - \chi^2\right)\left[\frac{1}{\chi^2 + \lambda_z^{-1}(\varsigma^2 - 1)} - \chi^{-2}\right]\right\}.\end{aligned} \quad (5.323)$$

As was mentioned previously, the above expression is valid for both transversely isotropic and isotropic materials. Figure 5.5 shows the function (5.323) for different values of the parameters χ, ς, λ_z and H_o.

Regarding the normal force $\mathcal{N}$ necessary in order to have an extension for the tube, for the energy function (5.245), the equation (5.320) becomes

$$\begin{aligned}\mathcal{N} = \int_{a_i}^{a_e} [-p^* + 2\lambda_z^2\Omega_1^* + 2H_o^2\lambda_z^2\Omega_5^* + 2\lambda_z^2\Omega_7^* + 2H_o\lambda_z^2\Omega_{10}^*] 2\pi r \,\mathrm{d}r&\\ -\frac{\mu_o}{2}\lambda_z^{-2}H_o^2 2\pi(a_e^2 - a_i^2),&\end{aligned} \quad (5.324)$$

where p^* is given from (5.319) as

$$p^*(r) = 2(\lambda\lambda_z)^{-2}\Omega_1^* - \int_r^{a_e} \frac{2}{\bar{r}}[(\bar{\lambda}\lambda_z)^{-2} - \bar{\lambda}^2]\Omega_1^* \,\mathrm{d}\bar{r} + \frac{\mu_o}{2}\lambda_z^{-2}H_o^2, \quad (5.325)$$

where $\bar{\lambda} = \frac{\bar{r}}{R}$. From (5.324) we see now that there is a difference between the transversely isotropic and the isotropic materials, so we treat these two cases separately as follows.

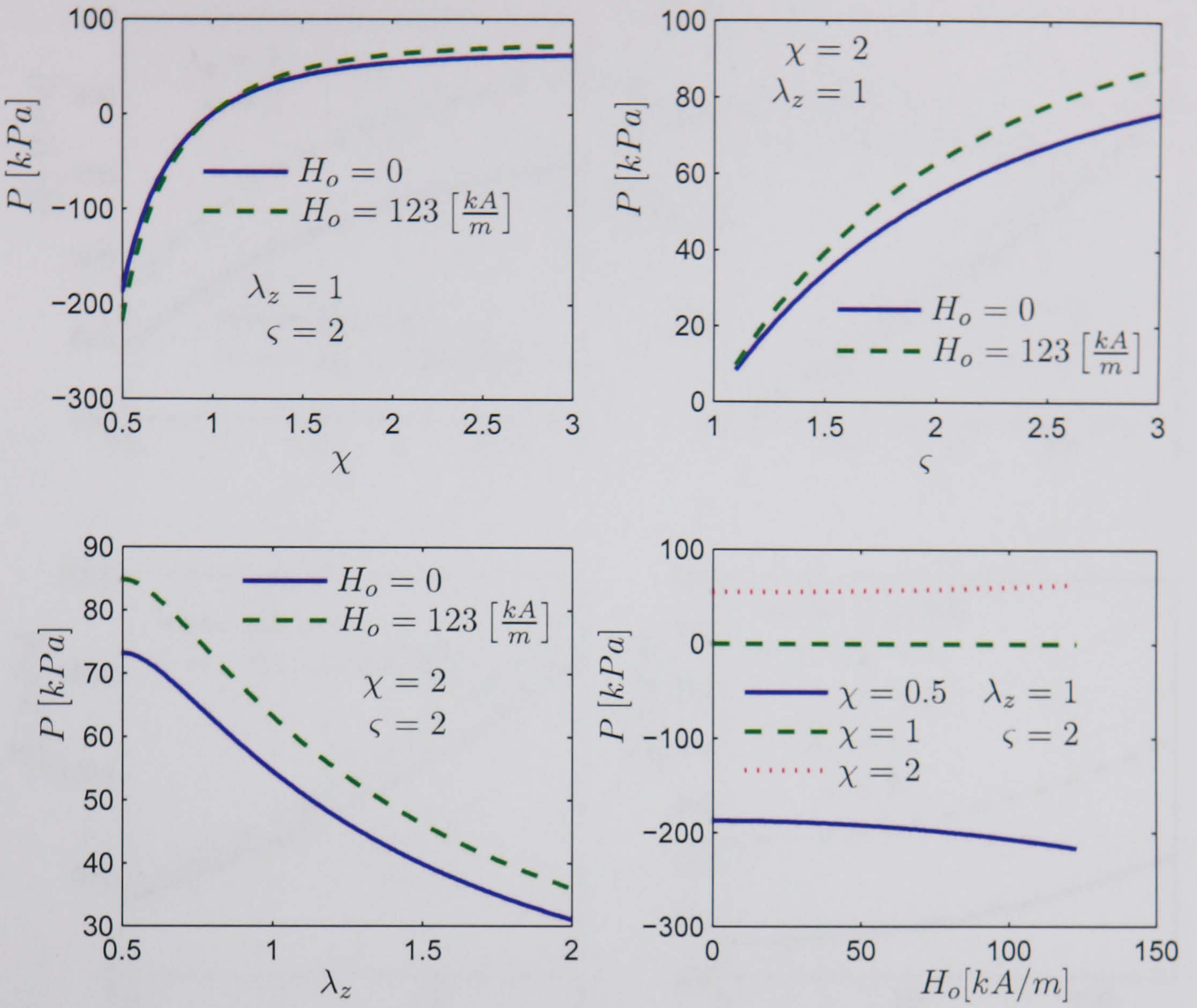


Figure 5.5: Pressure for different parameters for the tube with extension and expansion. The magnetic field H_o is given.

Isotropic case. Let's start with the simplest case where we only have isotropy. In such a case in (5.324) the only derivatives that appear are Ω_1^* and Ω_5^*, then we have

$$\mathcal{N} = \int_{a_i}^{a_e} [-p^* + 2\lambda_z^2\Omega_1^* + 2H_o^2\lambda_z^2\Omega_5^*]2\pi r\, \mathrm{d}r - \frac{\mu_o}{2}\lambda_z^{-2}H_o^2 2\pi(a_e^2 - a_i^2), \tag{5.326}$$

which by using (5.319) and after some manipulations implies that

$$\begin{aligned}\frac{\mathcal{N}}{A_i^2\pi} = \frac{1}{2\lambda_z^3}\Bigg\{&(g_0 + g_1 H_o^2)\left[\frac{\lambda_z^2}{1-\varsigma^2-\lambda_z\chi^2}\left[\chi^2(1-2\varsigma^2+2\lambda_z^3(\varsigma^2-1)+\lambda_z(2\lambda_z^3-1)\chi^2)\right.\right.\\ &+\frac{1-\varsigma^2+\lambda\chi^2}{\lambda}\left.\left.\left[(1-\lambda_z\chi^2)\log(\chi^2)-\lambda_z\chi^2\left(\log\left(\frac{1}{\varsigma^2}\right)+\log\left(1+\frac{\varsigma^2-1}{\lambda_z\chi^2}\right)\right)\right]\right]\right.\\ &+\lambda_z\left[1-2\lambda_z^3+2(\lambda_z^3-1)\varsigma^2+\lambda_z(2\lambda_z^3-1)\chi^2\right.\\ &\left.\left.+(1-\lambda_z\chi^2)\log\left(\frac{\varsigma^2\lambda_z}{-1+\varsigma^2+\lambda_z\chi^2}\right)\right]\right]\Bigg\}+\frac{1}{2\lambda_z^3}(2\lambda_z^4-3)(\varsigma^2-1)H_o^2\mu_o. \end{aligned} \tag{5.327}$$

Figure 5.6 shows the normal force for different values of the parameters.

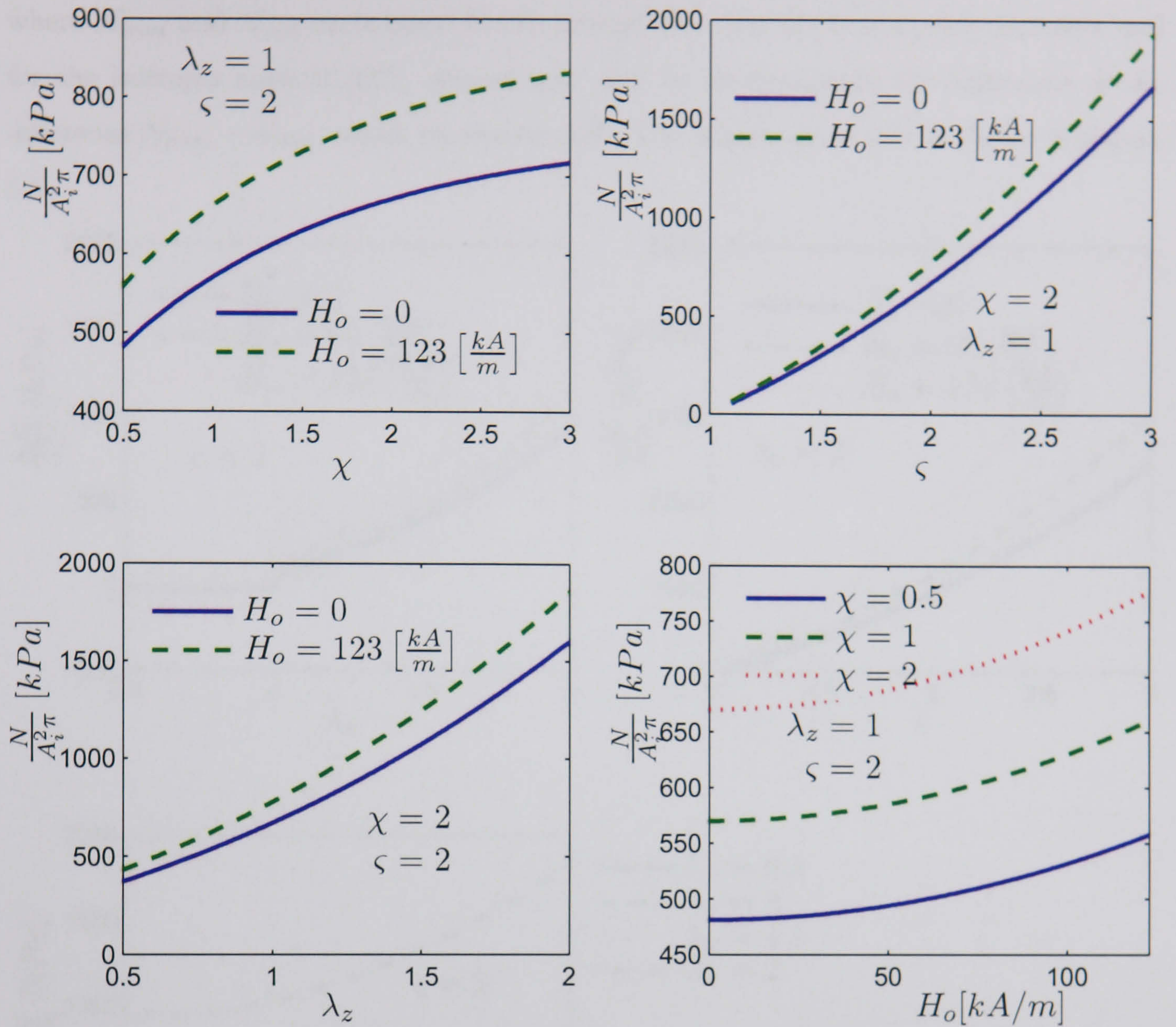


Figure 5.6: Normal force for different parameters for the tube with extension and expansion.

Transversely isotropic case. Regarding the transversely isotropic case, from (5.249) and (5.251) we have for the partial derivatives of Ω^* that

$$\Omega_7^* = mh_1(I_7 - 1)^{m-1}[\omega_0 + \omega_1 K_9^2 + \omega_2 K_{10}^2 + \omega_3 K_9 K_{10}],$$
$$\Omega_{10}^* = [h_0 + h_1(I_7 - 1)^m][2\omega_2 K_{10} + \omega_3 K_9],$$

where from $(5.301)_1$, $(5.301)_3$ and $(5.301)_4$ we have for the invariants that

$$I_7 = \lambda_z^2, \quad K_9 = H_o, \quad K_{10} = H_o\lambda_z^2.$$

These invariants do not depend on r. Then (5.324) may be written as

$$\frac{\mathcal{N}_{tran}}{\pi A_i^2} = \frac{\mathcal{N}_{isot}}{\pi A_i^2} + 2\lambda_z^2[\Omega_7^* + H_o\Omega_{10}^*](\varsigma^2 - 1), \tag{5.328}$$

where $\mathcal{N}_{tran}$ and $\mathcal{N}_{isot}$ correspond to the normal force for the transversely isotropic and for the isotropic cases (5.327), respectively. Let us study instead the behaviour of the difference $\mathcal{N}_{tran} - \mathcal{N}_{isot}$, which we denote $\Delta\mathcal{N}$. The behaviour of $\Delta\mathcal{N}$ is shown in Figure 5.7.

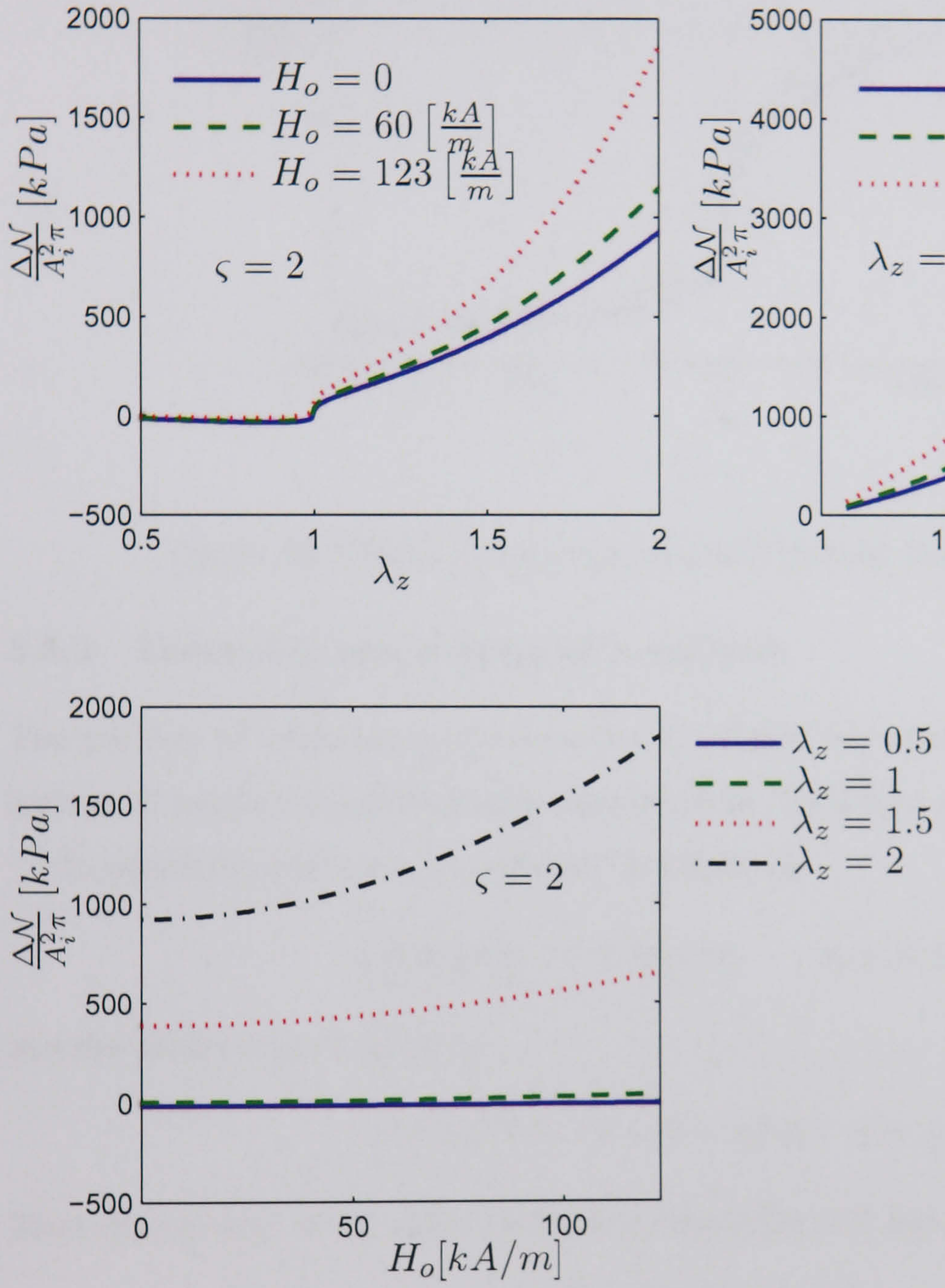


Figure 5.7: Difference for the normal force for different parameters for the tube with extension and expansion, between the transversely isotropic and the isotropic case.

A zoom of the behavior of the function $\Delta\mathcal{N}$ near zero, for the first case presented in Figure 5.7 is shown in Figure 5.8.

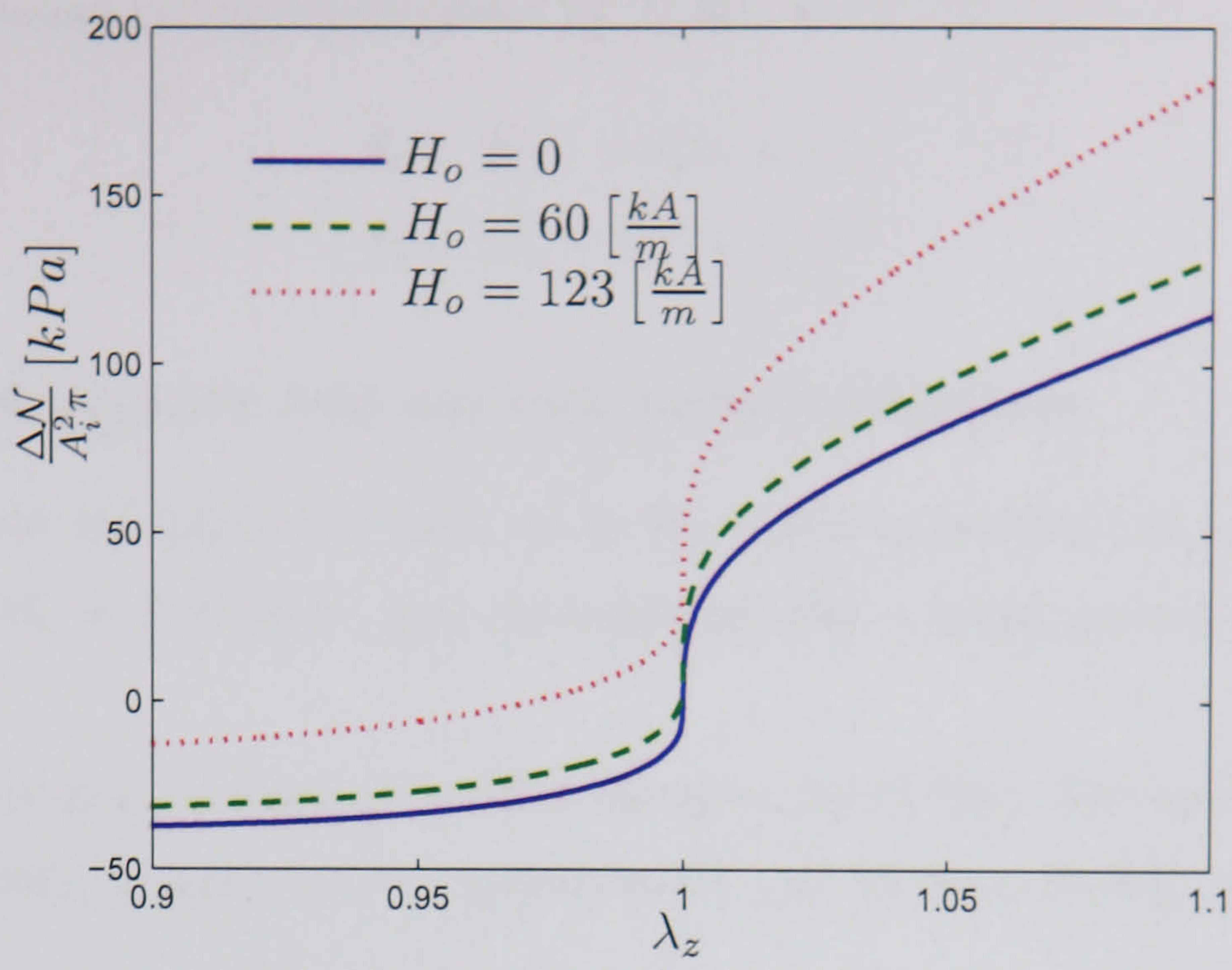


Figure 5.8: Detail of the behaviour of $\mathcal{N}$ for the first Figure of 5.7.

5.5.2 Extension and torsion of a cylinder

The problem of extension and torsion for a cylinder has been studied previously in the context of isotropic magneto-elastic elastomers by Dorfmann and Ogden [33].

In material coordinates the cylinder is defined as

$$0 \leq R \leq A, \quad 0 \leq \Theta \leq 2\pi, \quad -\infty \leq Z \leq \infty,$$

and the deformation is given by

$$r = \lambda_z^{-1/2} R, \quad \theta = \Theta + \lambda_z \tau Z, \quad z = \lambda_z Z. \tag{5.329}$$

Then the gradient of the deformation (see Appendix A.3) has components

$$\mathsf{F} = \begin{pmatrix} \lambda_z^{-1/2} & 0 & 0 \\ 0 & \lambda_z^{-1/2} & \lambda_z \gamma \\ 0 & 0 & \lambda_z \end{pmatrix}, \tag{5.330}$$

where γ is defined as

$$\gamma = \tau r. \tag{5.331}$$

The associated left and right Cauchy-Green deformation tensors have components

$$\mathsf{b} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} + \lambda_z^2 \gamma^2 & \lambda_z^2 \gamma \\ 0 & \lambda_z^2 \gamma & \lambda_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} & \lambda_z^{1/2} \gamma \\ 0 & \lambda_z^{1/2} \gamma & \lambda_z^2 (1 + \gamma^2) \end{pmatrix}. \tag{5.332}$$

The first and second invariants are given by $(5.25)_1$ and $(5.25)_2$ as

$$I_1 = 2\lambda_z^{-1} + \lambda_z^2(1+\gamma^2), \tag{5.333}$$

$$I_2 = 2\lambda_z + \lambda_z^{-2} + \lambda_z\gamma^2. \tag{5.334}$$

5.5.2.1 Axial magnetic field and axial particle alignment

We consider now the particular case, as in the previous problem, of an axial uniform magnetic field $\mathsf{H}_l = (0,0,H_o)^T$, and an axial and also uniform particle alignment field $\mathsf{a}_0 = (0,0,1)^T$.

In this case it may be shown that $\mathbf{H}$ is also given by (5.296). The fourth invariant K_4 is given by $(5.297)_1$ as well, and the invariants K_5 and K_6 from $(5.26)_2$ and $(5.26)_3$ as

$$K_5 = H_o^2\lambda_z^2(1+\gamma^2), \quad K_6 = H_o^2[\gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4]. \tag{5.335}$$

With $\mathsf{a}_0 = (0,0,1)^T$, along with (5.330) and (5.2), we have that

$$\mathsf{a} = (0, \gamma\lambda_z, \lambda_z)^T. \tag{5.336}$$

The rest of the invariants are given respectively from (5.27) and (5.28) as

$$I_7 = (1+\gamma^2)\lambda_z^2, \quad I_8 = \gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4, \tag{5.337}$$

$$K_{10} = H_o(1+\gamma^2)\lambda_z^2. \tag{5.338}$$

Since $\mathbf{H}$ is the same as in the previous problem, then the non-zero components of the Maxwell stress (3.44) are given by (5.298).

The components of the stress are given by (5.34) as

$$\tau_{rr} = -p^* + 2\lambda_z^{-1}\Omega_1^* + 2\lambda_z^{-2}[1+(1+\gamma^2)\lambda_z^3]\Omega_2^*, \tag{5.339}$$

$$\begin{aligned}\tau_{\theta\theta} = & -p^* + 2(\lambda_z^{-1} + \gamma^2\lambda_z^2)\Omega_1^* + 2\lambda_z^{-2}[1+(1+\gamma^2)\lambda_z^3]\Omega_2^* + 2H_o^2\gamma^2\lambda_z^2\Omega_5^* \\ & +4H_o^2\gamma^2\lambda_z[1+(1+\gamma^2)\lambda_z^3]\Omega_6^* + 2\gamma^2\lambda_z^2\Omega_7^* + 4\gamma^2\lambda_z[1+(1+\gamma^2)\lambda_z^3]\Omega_8^* \\ & +2H_o\gamma^2\lambda_z^2\Omega_{10}^*,\end{aligned} \tag{5.340}$$

$$\begin{aligned}\tau_{zz} = & -p^* + 2\lambda_z^2\Omega_1^* + 4\lambda_z\Omega_2 + 2H_o^2\lambda_z^2\Omega_5^* + 4H_o^2(1+\gamma^2)\lambda_z^4\Omega_6^* + 2\lambda_z^2\Omega_7^* \\ & +4(1+\gamma^2)\lambda_z^4\Omega_8^* + 2H_o\lambda_z^2\Omega_{10}^*,\end{aligned} \tag{5.341}$$

$$\tau_{r\theta} = \tau_{rz} = 0, \tag{5.342}$$

$$\begin{aligned}\tau_{\theta z} = & \; 2\gamma\lambda_z^2\Omega_1^* + 2\gamma\lambda_z\Omega_2^* + 2H_o^2\gamma\lambda_z^2\Omega_5^* + 2H_o^2\gamma\lambda_z[1+2(1+\gamma^2)\lambda_z^3]\Omega_6^* \\ & +2\gamma\lambda_z^2\Omega_7^* + 2\gamma\lambda_z[1+2(1+\gamma^2)\lambda_z^3]\Omega_8^* + 2H_o\gamma\lambda_z^2\Omega_{10}^*,\end{aligned} \tag{5.343}$$

and from (5.38) we obtain for the components of the magnetic induction

$$B_r = 0, \tag{5.344}$$

$$B_\theta = -\{2H_o\gamma\lambda_z\Omega_4^* + 2H_o\gamma[1+(1+\gamma^2)\lambda_z^3]\Omega_5^* + 2H_o\gamma[\lambda_z^{-1} + (1+2\gamma^2)\lambda_z^2 + (1+\gamma^2)^2\lambda_z^5]\Omega_6^* + \gamma\lambda_z\Omega_9^* + \gamma[1+(1+\gamma^2)\lambda_z^3]\Omega_{10}^*\}, \tag{5.345}$$

$$B_z = -\{2H_o\lambda_z\Omega_4^* + 2H_o(1+\gamma^2)\lambda_z^3\Omega_5^* + 2H_o\lambda_z^2[\gamma^2 + (1+\gamma^2)^2\lambda_z^3]\Omega_6^* + \lambda_z\Omega_9^* + \lambda_z^3(1+\gamma^2)\Omega_{10}^*\}. \tag{5.346}$$

Now, let us study the controllability of the above solution (we do not give all the details).

First, since the only parameter that appears in the invariants is γ, which depends on r, we have that Ω^* will only depend on r, then from the above solution for the stress, it is not difficult to show that (5.287) is satisfied trivially, and from (5.288) we conclude that $p^* = p^*(r)$. As a result p^* may be obtained by integration from (5.286).

Regarding the magnetic induction, we have that $B_\theta = B_\theta(r)$ and $B_z = B_z(r)$, so that (5.290) is also satisfied trivially. As a result this solution is controllable.

As in the expansion and inflation problem for a tube, other options may be chosen for the magnetic field and the particle alignment; however, as in that problem, the other simple cases that have been studied have been found to be not controllable, so we consider only this particular case.

5.5.2.2 Boundary conditions

Now, regarding the boundary conditions, we have essentially two quantities to calculate. One of them corresponds to the traction load applied on the ends of the cylinder, and the other to a torque, which we call $\mathcal{M}$, applied also on the ends of the cylinder. These loads are given by

$$\mathcal{N} = 2\pi\int_0^a \overset{z}{t}_z\, r\,\mathrm{d}r, \quad \mathcal{M} = 2\pi\int_0^a \tau_{\theta z} r^2\,\mathrm{d}r, \tag{5.347}$$

where $\overset{z}{t}_z$ is defined as the external axial load per unit area applied at the ends of the cylinder. From (3.69), as in the previous problem (assuming that the cylinder is completely surrounded by a free space), it is given by

$$\tau_{zz} = \overset{z}{t}_z + \tau_{m_{zz}}. \tag{5.348}$$

If we have free traction for the surface $r = a$, then from (3.69) this would imply that $\tau_{rr}(a) = \tau_{m_{rr}}(a)$, and as a result from (5.286) we have

$$p^*(r) = \tilde{\tau}_{rr}(r) - \int_r^a \frac{1}{\bar{r}}(\tau_{rr}(\bar{r}) - \tau_{\theta\theta}(\bar{r}))\mathrm{d}\bar{r} + \frac{\mu_o}{2}\lambda_z^{-2}H_o^2, \tag{5.349}$$

where $\tilde{\tau}_{rr}(r)$ is defined by $(5.308)_1$.

Then, using (5.341), along with $(5.298)_2$ and (5.348) in (5.347) we obtain for the normal traction

$$\mathcal{N} = 2\pi \int_0^a \Big\{ -p^*(r) + 2\lambda_z^2\Omega_1^* + 4\lambda_z\Omega_2^* + 2H_o^2\lambda_z^2\Omega_5^* + 4H_o^2(1+\gamma^2)\lambda_z^4\Omega_6^* \\ +2\lambda_z^2\Omega_7^* + 4(1+\gamma^2)\lambda_z^4\Omega_8^* + 2H_o\lambda_z^2\Omega_{10}^* - \frac{\mu_o}{2}\lambda_z^{-2}H_o^2 \Big\} r \, dr. \tag{5.350}$$

As well as this, using (5.343) in $(5.347)_2$, we have the following expression for the torque:

$$\mathcal{M} = 2\pi \int_0^a \{2\gamma\lambda_z^2\Omega_1^* + 2\gamma\lambda_z\Omega_2^* + 2H_o^2\gamma\lambda_z^2\Omega_5^* + 2H_o^2\gamma\lambda_z[1 + 2(1+\gamma^2)\lambda_z^3]\Omega_6^* \\ +2\gamma\lambda_z^2\Omega_7^* + 2\gamma\lambda_z[1 + 2(1+\gamma^2)\lambda_z^3]\Omega_8^* + 2H_o\gamma\lambda_z^2\Omega_{10}^*\} r^2 \, dr. \tag{5.351}$$

5.5.2.3 Boundary conditions for a particular energy function

For our particular energy function (5.245), the equation (5.350) for the normal force becomes

$$\mathcal{N} = 2\pi \int_0^a \Big\{ -p^*(r) + 2\lambda_z^2\Omega_1^* + 2H_o^2\lambda_z^2\Omega_5^* + 2\lambda_z^2\Omega_7^* + 2H_o\lambda_z^2\Omega_{10}^* - \frac{\mu_o}{2}\lambda_z^{-2}H_o^2 \Big\} r \, dr, \tag{5.352}$$

and the equation (5.351) for the torque is given as

$$\mathcal{M} = 2\pi \int_0^a \{2\gamma\lambda_z^2\Omega_1^* + 2H_o^2\gamma\lambda_z^2\Omega_5^* + 2\gamma\lambda_z^2\Omega_7^* + 2H_o\gamma\lambda_z^2\Omega_{10}^*\} r^2 \, dr. \tag{5.353}$$

We will study the isotropic case first, which is only a special case here.

Isotropic case. For the isotropic case we have the following simplification of (5.352)

$$\mathcal{N} = 2\pi \int_0^a \Big\{ -p^*(r) + 2\lambda_z^2\Omega_1^* + 2H_o^2\lambda_z^2\Omega_5^* - \frac{\mu_o}{2}\lambda_z^{-2}H_o^2 \Big\} r \, dr, \tag{5.354}$$

and from (5.349) we get (Ω_1^* and Ω_5^* are constant)

$$p^*(r) = 2\lambda_z^{-1}\Omega_1^* + \frac{\mu_o}{2}\lambda_z^{-2}H_o^2 + \tau^2\lambda_z^2(\Omega_1^* + H_o^2\Omega_5^*)(a^2 - r^2), \tag{5.355}$$

and

$$\frac{\mathcal{N}}{a^2\pi} = 2(\lambda_z^2 - \lambda_z^{-1})\Omega_1^* + 2H_o^2\lambda_z^2\Omega_5^* - \mu_o\lambda_z^{-2}H_o^2 - \frac{1}{2}(\tau a)^2\lambda_z^2(\Omega_1^* + H_o^2\Omega_5^*). \tag{5.356}$$

About the torque, from (5.353), for an isotropic material, we have the following simple expression

$$\frac{\mathcal{M}}{\pi a^3} = \lambda_z^2(\tau a)(\Omega_1^* + H_o^2\Omega_5^*). \tag{5.357}$$

The above two expressions are rather simple. Therefore we do not provide figures of the behaviors of $\mathcal{N}$ or $\mathcal{M}$ for this case.

Transversely isotropic case. The transversely isotropic case (5.352) and (5.353), using (5.349) along with (5.246)-(5.251) and (5.333), $(5.337)_1$, $(5.338)_1$ and $(5.301)_3$, is much more complicated than the above isotropic problem. Nevertheless for this particular form of the energy function, (5.352) and (5.353) can be obtained analytically. The following is the expression for the normal force, where the dimensionless parameter ξ has been defined as $\xi = \tau a$

$$\frac{\mathcal{N}}{a^2\pi} = \frac{K_1}{2D}, \tag{5.358}$$

and the expression for the torque is

$$\frac{\mathcal{M}}{a^3\pi} = \frac{K_2}{D}, \tag{5.359}$$

where K_1 and K_2 are given respectively as

$$\begin{aligned} K_1 = {} & 3(2+m)(3+m)\{2[(\lambda_z^2-1)^m(1-(3+2m)\lambda_z^2)-(\lambda_z^2(1+\xi^2)-1)^m(1+\lambda_z^2(m(\xi^2\\ & -2)-3))]h_1\omega_0-(1+m)\lambda_z\xi^2[4+\lambda_z^3(\xi^2-4)]\Omega_1^*\}+H_o^2\{-6(1+m)(2+m)(3\\ & +m)\xi^2\mu_o+6h_1[(2+m)(3+m)((\lambda_z^2-1)^m(1-(3+2m)\lambda_z^2)-(\lambda_z^2(1+\xi^2)-1)^m\\ & *(1+\lambda_z^2(m(\xi^2-2)-3)))\omega_1-(\lambda_z^2-1)^m((-2-2m\lambda_z^2-m(1+m)\lambda_z^4+(1+m)\\ & *(2+m)(7+2m)\lambda_z^6)\omega_2+(3+m)(-1-m\lambda_z^2+(1+m)(5+2m)\lambda_z^4)\omega_3)-(\lambda_z^2(1\\ & +\xi^2)-1)^m((2+\lambda_z^2(1+\xi^2)(2m+m\lambda_z^2+m^2\lambda_z^2-14\lambda_z^4-25m\lambda_z^4-13m^2\lambda_z^4\\ & -2m^3\lambda_z^4-(1+m)\lambda_z^2(-m+(2+m)(5+m)\lambda_z^2)\xi^2+(1+m)(2+m)^2\lambda_z^4\xi^4))\omega_2\\ & +(3+m)(1+\lambda_z^2(1+\xi^2)(m-5\lambda_z^2-m(7+2m)\lambda_z^2+(1+m)^2\lambda_z^2\xi^2))\omega_3)]\\ & -(1+m)(2+m)(3+m)\lambda_z^4\xi^2[h_0(2\lambda_z^2(-12-3\xi^2+2\xi^4)\omega_2+3(\xi^2-4)\omega_3)\\ & +3(\xi^2-4)\Omega_5^*]\}, \end{aligned} \tag{5.360}$$

$$\begin{aligned} K_2 = {} & 6(\lambda_z^2-1)^{1+m}h_1\{H_o^2[(2+m)(3+m)\omega_1+(2+(1+m)\lambda_z^2(2+(2+m)\lambda_z^2))\omega_2\\ & +(3+m)(1+(1+m)\lambda_z^2)\omega_3]+(2+m)(3+m)\omega_0\}+6[\lambda_z^2(1+\xi^2)-1]^m h_1\{H_o^2[(2\\ & +m)(3+m)(1+\lambda_z^2(m\xi^2-1))\omega_1+(2+2m\lambda_z^2+m(1+m)\lambda_z^4-(1+m)(2+m)\lambda_z^6\\ & +m\lambda_z^2(2+(1+m)\lambda_z^2(2+(2+m)\lambda_z^2))\xi^2+(1+m)\lambda_z^4(m+(2+m)(3+2m)\lambda_z^2)\xi^4\\ & +(1+m)(2+m)^2\lambda_z^6\xi^6)\omega_2+(3+m)(1+m\lambda_z^2-(1+m)\lambda_z^4+m\lambda_z^2(1+(1+m)\lambda_z^2)\xi^2\\ & +(1+m)^2\lambda_z^4\xi^4)\omega_3]+(2+m)(3+m)[1+\lambda_z^2(m\xi^2-1)]\omega_0\}+(1+m)(2+m)(3\\ & +m)\lambda_z^4\xi^4\{3\Omega_1^*+H_o^2[h_0(2\lambda_z^2(3+2\xi^2)\omega_2+3\omega_3)+3\Omega_5^*]\}, \end{aligned} \tag{5.361}$$

and

$$D = 3(1+m)(2+m)(3+m)\lambda_z^2\xi^3. \tag{5.362}$$

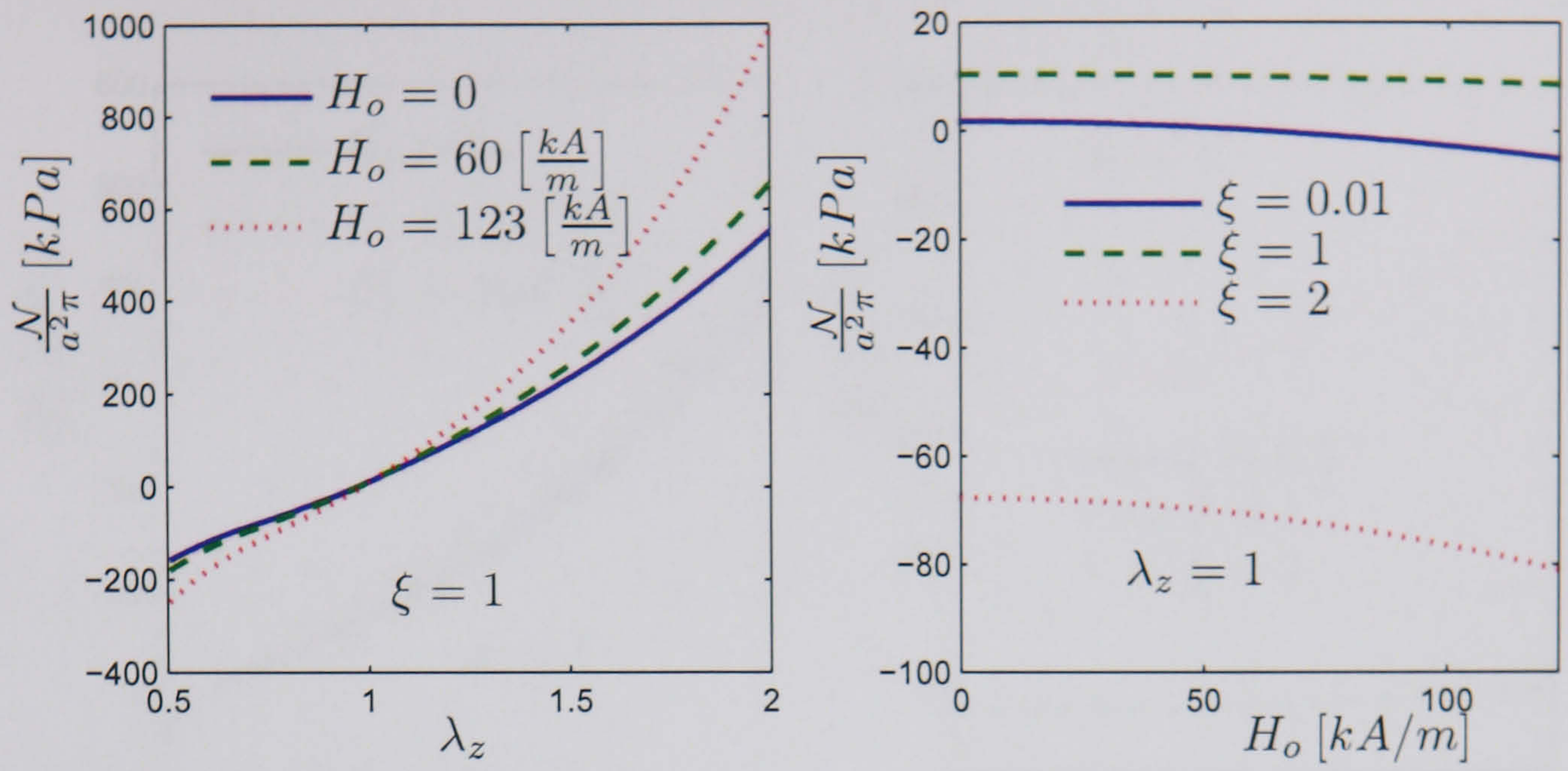


Figure 5.9: Normal force for the extension and torsion problem.

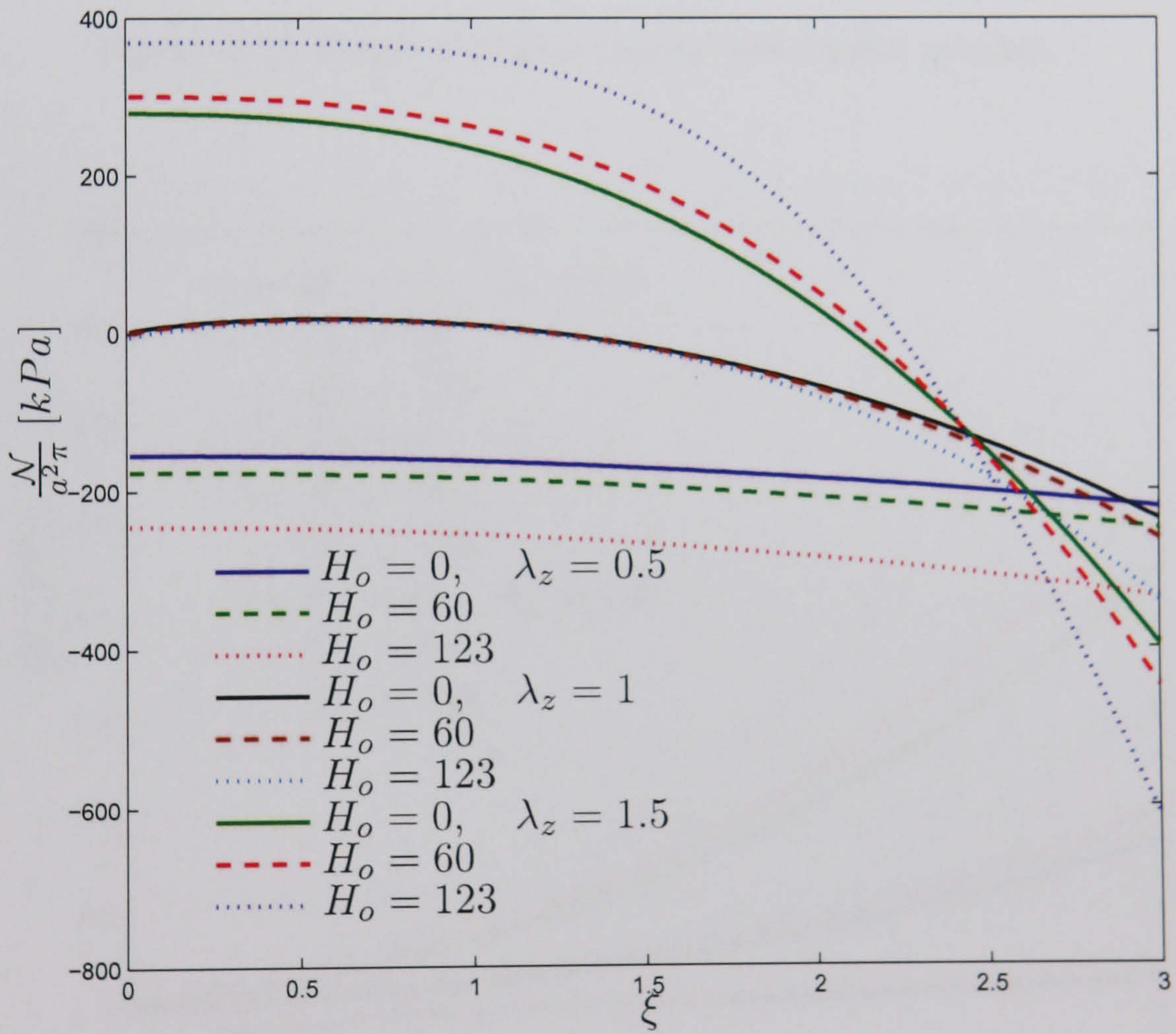


Figure 5.10: Normal force for the extension and torsion problem as function of ξ (the magnetic field is in kA/m).

Figures 5.9 and 5.10 show the results for the normal force, and Figures 5.11 and 5.12 the results for the torque.

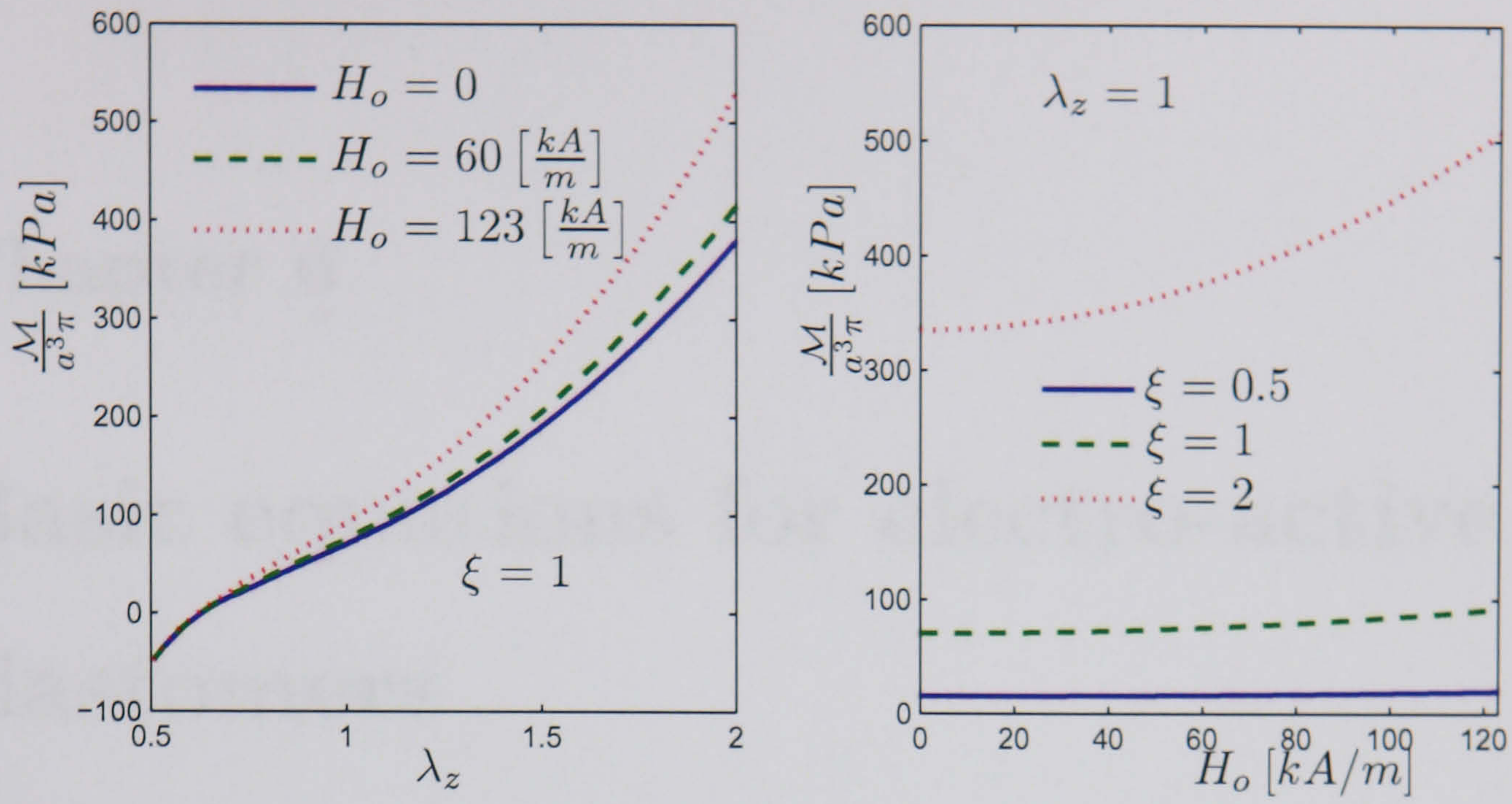


Figure 5.11: Torque for the extension and torsion problem.

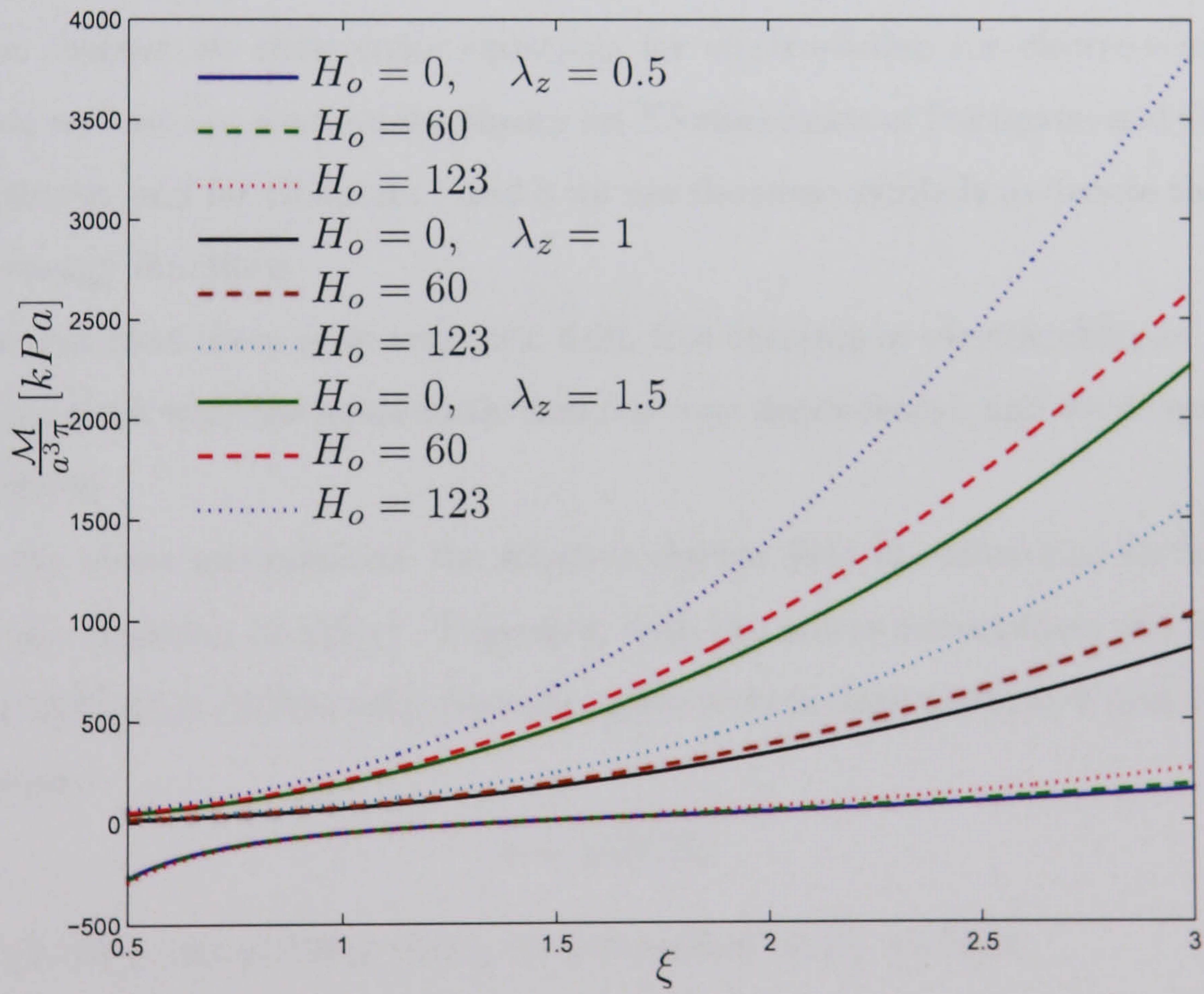


Figure 5.12: Torque for the extension and torsion problem as function of ξ (the magnetic field is in kA/m).

Chapter 6

Basic equations for electro-active elastomers

The theory for electro-active elastomers developed by Dorfmann and Ogden [32,36] follows closely the theory for MS elastomers presented in Chapters 3, 4 and 5. Thus, for this chapter, and for Chapters 7 and 8, we do not show the full proofs of some of the expressions.

In this chapter we show some equations for electro-active (or electro-sensitive ES) elastomers; we base our work on the theory for ES elastomers of Dorfmann and Ogden [32]. For this chapter and for Chapters 7 and 8 we use the same symbols to denote the free and the total energy functions.

We assume that there is no magnetic field, free currents or electric charges, as well as this, we only work with the quasi-static case (no time dependence), and we do not consider thermal effects.

With the above assumptions, the effective electric field $\mathbf{E}_e$ defined in Section 2.3 becomes $\mathbf{E}$ (see equation $(2.115)_2$). Therefore, with the above assumptions the free energy function ψ defined in (2.126) only depends on the deformation gradient $\mathbf{F}$ and the electric field $\mathbf{E}$; hence

$$\psi = \psi(\mathbf{F}, \mathbf{E}). \tag{6.1}$$

From $(2.131)_2$ and $(2.131)_3$ (using the convention (3.3)), we have

$$\sigma = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}}, \quad \mathbf{P} = -\rho \frac{\partial \psi}{\partial \mathbf{E}}, \tag{6.2}$$

for the Cauchy stress and the polarization $\mathbf{P}$ respectively.

6.1 Eulerian forms, Lagrangian forms and the initial forms of the fields

As for MS elastomers (see Section 3.1) we can define the Lagrangian forms of the electric field and the electric displacement by considering the global forms of the Maxwell equations for electrostatics (2.103)

$$\int_S \mathbf{D} \cdot \mathbf{n}\, da = 0, \quad \oint_C \mathbf{E} \cdot \mathbf{dr} = 0. \tag{6.3}$$

As in Section 3.1 from $(6.3)_1$ using the Nanson's formula [78], and from $(6.3)_2$ using $\mathbf{dx} = \mathbf{F}\mathrm{d}\mathbf{X}$, we can define the Lagrangian electric displacement and electric field as [32]

$$\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}, \quad \mathbf{E}_l = \mathbf{F}^T\mathbf{E}. \tag{6.4}$$

With the identities

$$\mathrm{Div}(J\mathbf{F}^{-1}\mathbf{D}) = J\mathrm{div}\mathbf{D}, \quad \mathbf{F}\mathrm{Curl}(\mathbf{F}^T\mathbf{E}) = J\mathrm{curl}\mathbf{E}, \tag{6.5}$$

we can prove that (2.103) are equivalent to

$$\mathrm{Curl}\mathbf{E}_l = \mathbf{0}, \quad \mathrm{Div}\mathbf{D}_l = 0. \tag{6.6}$$

From $(2.107)_1$ we had

$$\mathbf{D} = \varepsilon_o\mathbf{E} + \mathbf{P}. \tag{6.7}$$

We may assume that the polarization field $\mathbf{P}$ transforms in the same way as the electric displacement $\mathbf{D}$; this definition is not unique. We have [32]

$$\mathbf{P}_l = J\mathbf{F}^{-1}\mathbf{P}. \tag{6.8}$$

Then in (6.7) we have

$$J\mathbf{F}^{-1}\mathbf{D} = \varepsilon_o J\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{F}^T\mathbf{E} + J\mathbf{F}^{-1}\mathbf{P}; \tag{6.9}$$

as a result

$$\mathbf{D}_l = \varepsilon_o J\mathbf{c}^{-1}\mathbf{E}_l + \mathbf{P}_l. \tag{6.10}$$

For the case in which there is no deformation, let's denote by $\mathbf{E}_o$, $\mathbf{D}_o$ and $\mathbf{P}_o$ the electric field, electric displacement and the polarization, respectively. From (6.7) we have

$$\mathbf{D}_o = \varepsilon_o\mathbf{E}_o + \mathbf{P}_o. \tag{6.11}$$

If we assume that the body deforms, such that the deformation gradient is $\mathbf{F}$, from (6.4) and (6.8) we can define the push forward version (subscript f) of the fields $\mathbf{E}_o$, $\mathbf{D}_o$ and $\mathbf{P}_o$ as

$$\mathbf{D}_f = J^{-1}\mathbf{F}\mathbf{D}_o, \quad \mathbf{E}_f = \mathbf{F}^{-T}\mathbf{E}_o, \quad \mathbf{P}_f = J^{-1}\mathbf{F}\mathbf{P}_o, \tag{6.12}$$

and substituting in (6.11) we get

$$J\mathbf{F}^{-1}\mathbf{D}_f = \varepsilon_o\mathbf{F}^T\mathbf{E}_l + J\mathbf{F}^{-1}\mathbf{P}_f, \tag{6.13}$$

thus

$$\mathbf{D}_f = \varepsilon_o J^{-1}\mathbf{b}\mathbf{E}_f + \mathbf{P}_f. \tag{6.14}$$

Then, the form (6.11) is not preserved.

If we choose to work with $\mathbf{E}$ as the independent electric variable, we have $\mathbf{E}_l = \mathbf{E}_o$, but in general $\mathbf{D}_o \neq \mathbf{D}_l$. And conversely if we work with $\mathbf{D}$ as the independent electric variable we have $\mathbf{D}_l = \mathbf{D}_o$, but in general $\mathbf{E}_l \neq \mathbf{E}_o$.

6.2 Constitutive equation and the total energy function for ES elastomers

Let's define the function Φ as [32, 36]

$$\Phi(\mathbf{F}, \mathbf{E}_l) = \psi(\mathbf{F}, \mathbf{F}^T\mathbf{E}). \tag{6.15}$$

From the principle of material frame-indifference [32, 78, 112] for Φ we must have

$$\Phi(\mathbf{F}, \mathbf{E}_l) = \Phi(\mathbf{Q}\mathbf{F}, \mathbf{E}_l), \tag{6.16}$$

for all proper orthogonal tensors $\mathbf{Q}$. As for $\mathbf{E}_l$, the transformation for $\mathbf{F}$ and $\mathbf{E}$ is $\mathbf{F}' = \mathbf{Q}\mathbf{F}$ and $\mathbf{E}' = \mathbf{Q}\mathbf{E}$ respectively, then $\mathbf{E}_l' = \mathbf{F}'^T\mathbf{E}' = \mathbf{F}^T\mathbf{Q}^T\mathbf{Q}\mathbf{E} = \mathbf{F}^T\mathbf{E} = \mathbf{E}_l$. We have that $\mathbf{E}_l$ is a Lagrangian vector and is not affected by the rotation $\mathbf{Q}$.

Remembering the convention for the derivative (3.3), from the definition (6.15), following similar steps as in Section 3.3, we can show that

$$\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{F}} = \mathbf{F}\frac{\partial\psi}{\partial\mathbf{F}} - \frac{\partial\psi}{\partial\mathbf{E}} \otimes \mathbf{E}. \tag{6.17}$$

Thus for $(6.2)_1$ using $(6.2)_2$ we have

$$\boldsymbol{\sigma} = \rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{F}} - \mathbf{P} \otimes \mathbf{E}, \tag{6.18}$$

and

$$\mathbf{P} = -\rho\frac{\partial\psi}{\partial\mathbf{E}} = -\rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{E}_l}. \tag{6.19}$$

From Section 2.3 with the simplifications enunciated at the beginning of this chapter, the particular form of the balance of linear momentum, if we do not consider mechanical body forces, is (see equation (2.109))

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}_e = \mathbf{0}, \tag{6.20}$$

where, from (2.110), $\mathbf{f}_e$ is given as

$$\mathbf{f}_e = (\operatorname{grad}\mathbf{E})^T\mathbf{P}. \tag{6.21}$$

Proposition 6.1. *The electric body force $\mathbf{f}_e$ can be expressed as the divergence of the following second order tensor [32]*

$$\mathbf{D}\otimes\mathbf{E} - \frac{1}{2}\varepsilon_o(\mathbf{E}\cdot\mathbf{E})\mathbf{I}. \tag{6.22}$$

Proof. Let's work with Cartesian coordinates. We want to show that

$$\mathbf{f}_e = \operatorname{div}\left(\mathbf{D}\otimes\mathbf{E} - \frac{1}{2}\varepsilon_o(\mathbf{E}\cdot\mathbf{E})\mathbf{I}\right). \tag{6.23}$$

In index notation we have

$$\begin{aligned} f_{e_j} &= \left(D_iE_j - \frac{1}{2}\varepsilon E_kE_k\delta_{ij}\right)_{,i} \\ &= D_{i,i}E_j + D_iE_{j,i} - \varepsilon_oE_kE_{k,i}\delta_{ij}, \end{aligned} \tag{6.24}$$

but from $(2.103)_2$ we have that $D_{i,i} = 0$, and from $(2.103)_1$ we have $E_{j,i} = E_{i,j}$, therefore

$$f_{e_j} = (D_i - \varepsilon_oE_i)E_{i,j}, \tag{6.25}$$

which from (6.7) is equivalent to

$$\mathbf{f}_e = (\operatorname{grad}\mathbf{E})^T\mathbf{P}. \tag{6.26}$$

□

Then, we can define the total stress $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mathbf{D}\otimes\mathbf{E} - \frac{1}{2}\varepsilon_o(\mathbf{E}\otimes\mathbf{E})\mathbf{I}, \tag{6.27}$$

and (6.20) is equivalent to

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}. \tag{6.28}$$

Definition 6.1. *In electrostatics the Maxwell stress tensor* $\boldsymbol{\tau}_m$ *is defined as [32]*

$$\boldsymbol{\tau}_m = \mathbf{D} \otimes \mathbf{E} - \frac{1}{2}\varepsilon_o(\mathbf{E} \cdot \mathbf{E})\mathbf{I}. \tag{6.29}$$

For vacuum we have the linear relation $(2.106)_1$ $\mathbf{D} = \varepsilon_o \mathbf{E}$, and so $\boldsymbol{\tau}_m$ can be expressed in the two equivalent forms

$$\boldsymbol{\tau}_m = \varepsilon_o \left[\mathbf{E} \otimes \mathbf{E} - \frac{1}{2}(\mathbf{E} \cdot \mathbf{E})\mathbf{I} \right], \tag{6.30}$$

$$\boldsymbol{\tau}_m = \frac{1}{\varepsilon_o} \left[\mathbf{D} \otimes \mathbf{D} - \frac{1}{2}(\mathbf{D} \cdot \mathbf{D})\mathbf{I} \right]. \tag{6.31}$$

The final remark of Section 3.3 also applies here. In Chapters 7 and 8 we solve some boundary value problems and we assume a body totally surrounded by free space; in such a case the Maxwell stress defined above must be included as an external load.

Regarding the balance of angular momentum, from (2.111) for our particular electrostatic problem we have

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + \mathbf{P} \times \mathbf{E} = \mathbf{0}. \tag{6.32}$$

It is clear that in general $\boldsymbol{\sigma}$ is not symmetric. From (6.27) we have $\boldsymbol{\sigma} = \boldsymbol{\tau} - \mathbf{D} \otimes \mathbf{E} - \frac{1}{2}\varepsilon_o(\mathbf{E} \cdot \mathbf{E})\mathbf{I}$, and so from (6.32) since $\boldsymbol{\varepsilon} : \mathbf{I} = \mathbf{0}$ and we get

$$\boldsymbol{\varepsilon} : \boldsymbol{\tau} - \boldsymbol{\varepsilon} : (\mathbf{D} \otimes \mathbf{E}) + \mathbf{P} \times \mathbf{E} = \mathbf{0}. \tag{6.33}$$

But $\boldsymbol{\varepsilon} : (\mathbf{D} \otimes \mathbf{E}) = \mathbf{D} \times \mathbf{E}$, therefore the above expression is equal to

$$\boldsymbol{\varepsilon} : \boldsymbol{\tau} + (\mathbf{P} - \mathbf{D}) \times \mathbf{E} = \mathbf{0}, \tag{6.34}$$

which is equivalent to

$$\boldsymbol{\varepsilon} : \boldsymbol{\tau} = \mathbf{0}, \tag{6.35}$$

because $\mathbf{P} - \mathbf{D} = -\varepsilon_o \mathbf{E}$. As a result we have that $\boldsymbol{\tau}$ is a symmetric tensor.

6.2.1 The total energy function

As was done in the magnetostatic case, here we can define a total nominal stress tensor, denoted $\mathbf{T}$, associated with $\boldsymbol{\tau}$ (equation (6.27)) as[1]

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau} = \rho_o \frac{\partial \Phi}{\partial \mathbf{F}} + J\mathbf{F}^{-1} \left[-\mathbf{P} \otimes \mathbf{E} + \mathbf{D} \otimes \mathbf{E} - \frac{1}{2}\varepsilon_o(\mathbf{E} \cdot \mathbf{E})\mathbf{I} \right]. \tag{6.36}$$

[1]See equations (2.33) and (6.18).

but from (6.7) we have $(-\mathbf{P}+\mathbf{D})\otimes\mathbf{E}=\varepsilon_o\mathbf{E}\otimes\mathbf{E}$, thus

$$\mathbf{T}=\rho_o\frac{\partial\Phi}{\partial\mathbf{F}}+\varepsilon_o J\mathbf{F}^{-1}\left[\mathbf{E}\otimes\mathbf{E}-\frac{1}{2}(\mathbf{E}\cdot\mathbf{E})\mathbf{I}\right]. \tag{6.37}$$

From the definition of the Lagrangian electric field $(6.4)_2$ and $(2.17)_1$ we can prove that $\mathbf{E}\cdot\mathbf{E}=\mathbf{E}_l\cdot(\mathbf{c}^{-1}\mathbf{E}_l)$.

Proposition 6.2. *The following identity holds*

$$\frac{\partial}{\partial\mathbf{F}}(J\mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l)=-2J\mathbf{F}^{-1}\left[\mathbf{E}\otimes\mathbf{E}-\frac{1}{2}(\mathbf{E}\cdot\mathbf{E})\mathbf{I}\right], \tag{6.38}$$

where $\frac{\partial}{\partial\mathbf{F}}$ *is at fixed* $\mathbf{E}_l$.

The proof of the above proposition is not trivial and so it is given in full form as follows.

Proof. In index notation (Cartesian coordinates) we have

$$\left[\frac{\partial}{\partial\mathbf{F}}(J\mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l)\right]_{ij}=\frac{\partial}{\partial F_{ji}}(JE_{l_k}\overset{-1}{c}_{kr}E_{l_r}), \tag{6.39}$$

where $\overset{-1}{c}_{kr}$ is the component kr of the tensor $\mathbf{c}^{-1}$, and E_{l_k} is the component k of the vector $\mathbf{E}_l$. From (3.51) (see [78]) we have $\frac{\partial J}{\partial F_{ji}}=J\overset{-1}{F}_{ij}$, where we recall the notation $\overset{-1}{F}_{ij}$ for the component ij of the tensor $\mathbf{F}^{-1}$. Thus

$$\begin{aligned}\frac{\partial}{\partial F_{ji}}(JE_{l_k}\overset{-1}{c}_{kr}E_{l_r}) &= \frac{\partial J}{\partial F_{ji}}E_{l_k}\overset{-1}{c}_{kr}E_{l_r}+E_{l_k}\frac{\partial\overset{-1}{c}_{kr}}{\partial F_{ji}}E_{l_r},\\ &= J\overset{-1}{F}_{ij}E_{l_k}\overset{-1}{c}_{kr}E_{l_r}+E_{l_k}\frac{\partial\overset{-1}{c}_{kr}}{\partial F_{ji}}E_{l_r}.\end{aligned} \tag{6.40}$$

Let's calculate $\frac{\partial\overset{-1}{c}_{kr}}{\partial F_{ji}}$ from the expression $\mathbf{c}^{-1}\mathbf{c}=\mathbf{I}$; taking the derivative in $\mathbf{F}$, we have

$$\frac{\partial\overset{-1}{c}_{km}}{\partial F_{ji}}c_{mp}+\overset{-1}{c}_{km}\frac{\partial c_{mp}}{\partial F_{ji}}=0. \tag{6.41}$$

But $\mathbf{c}=\mathbf{F}^T\mathbf{F}$, therefore

$$\frac{\partial c_{mp}}{\partial F_{ji}}=\frac{\partial}{\partial F_{ji}}(F_{qm}F_{qp})=\delta^{qm}_{ji}F_{qp}+F_{qm}\delta^{qp}_{ji}, \tag{6.42}$$

where the definition of the symbol δ^{ji}_{km} is given in (3.22). From (6.41) we obtain (multiplying by $\overset{-1}{c}_{pr}$)

$$\frac{\partial\overset{-1}{c}_{km}}{\partial F_{ji}}\delta_{mr}=-\overset{-1}{c}_{km}\overset{-1}{c}_{pr}(\delta_{im}F_{jp}+\delta_{ip}F_{jm}), \tag{6.43}$$

and we get

$$\frac{\partial\overset{-1}{c}_{kr}}{\partial F_{ji}}=-(\overset{-1}{c}_{ki}\overset{-1}{c}_{pr}F_{jp}+\overset{-1}{c}_{kr}\overset{-1}{c}_{ir}F_{jp}). \tag{6.44}$$

As a result for the second term of the right side of (6.40), using $(6.4)_2$, we have

$$E_{l_k}\frac{\partial \overset{-1}{c}_{kr}}{\partial F_{ji}}E_{l_r} = -(E_sF_{sk}\overset{-1}{c}_{ki}\overset{-1}{c}_{pr}F_{jp}F_{tr}E_t + E_sF_{sk}\overset{-1}{c}_{kp}\overset{-1}{c}_{ir}F_{jp}F_{tr}E_t). \tag{6.45}$$

Using $\mathbf{c}^{-1} = \mathbf{F}^{-1}\mathbf{F}^T$ above, it is easy to prove that the right side of the above equation is equivalent to $-2\mathbf{F}^{-1}\mathbf{E}\otimes\mathbf{E}$.

Then, using $\mathbf{E}\cdot\mathbf{E} = \mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l$, and the above results in (6.40), we finally obtain

$$\frac{\partial}{\partial \mathbf{F}}(J\mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l) = -2J\mathbf{F}^{-1}\left[\mathbf{E}\otimes\mathbf{E} - \frac{1}{2}(\mathbf{E}\cdot\mathbf{E})\mathbf{I}\right].$$

□

From (6.38), for the total nominal stress tensor $\mathbf{T}$ (6.37) we have

$$\mathbf{T} = \frac{\partial}{\partial \mathbf{F}}\left(\rho_o\Phi + \frac{\varepsilon_o}{2}J\mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l\right). \tag{6.46}$$

Definition 6.2. *The amended free energy function (electrostatics) Ω is defined as*

$$\Omega(\mathbf{F},\mathbf{E}_l) = \rho_o\Phi(\mathbf{F},\mathbf{E}_l) + \frac{\varepsilon_o}{2}J\mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l. \tag{6.47}$$

From the above definition we have [32]

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}. \tag{6.48}$$

Hence, from (6.36) we get

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}. \tag{6.49}$$

From (6.19) we have

$$\mathbf{P} = -\rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{E}_l} = -\frac{\rho}{\rho_o}\mathbf{F}\left[\frac{\partial\Omega}{\partial\mathbf{E}_l} - \frac{\partial}{\partial\mathbf{E}_l}\left(\frac{\varepsilon_o}{2}J\mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l\right)\right], \tag{6.50}$$

but $\frac{\partial}{\partial\mathbf{E}_l}\left(\frac{\varepsilon_o}{2}J\mathbf{E}_l\cdot\mathbf{c}^{-1}\mathbf{E}_l\right) = \varepsilon_oJ\mathbf{c}^{-1}\mathbf{E}_l$; using (2.21), we obtain

$$J\mathbf{F}^{-1}\mathbf{P} = -\frac{\partial\Omega}{\partial\mathbf{E}_l} + \varepsilon_oJ\mathbf{c}^{-1}\mathbf{E}_l, \tag{6.51}$$

which from (6.8) is equivalent to

$$\mathbf{P}_l - \varepsilon_oJ\mathbf{c}^{-1}\mathbf{E}_l = -\frac{\partial\Omega}{\partial\mathbf{E}_l}, \tag{6.52}$$

and from (6.10) we finally get [32]

$$\mathbf{D}_l = -\frac{\partial\Omega}{\partial\mathbf{E}_l}. \tag{6.53}$$

6.2.2 An alternative formulation

If we work with the Lagrangian electric displacement $\mathbf{D}_l$ as the independent electric variable, we can introduce the complementary energy function $\Omega^*(\mathbf{F}.\mathbf{D}_l)$ through the partial Legendre transformation as [32]

$$\Omega^*(\mathbf{F},\mathbf{D}_l) = \Omega(\mathbf{F},\mathbf{E}_l) + \mathbf{D}_l \cdot \mathbf{E}_l. \tag{6.54}$$

Then

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \tag{6.55}$$

and

$$\mathbf{E}_l = \frac{\partial \Omega^*}{\partial \mathbf{D}_l}. \tag{6.56}$$

6.3 Boundary conditions

As in the case of MS elastomers (Chapter 3), we assume here that the body is completely surrounded by a free space. From (2.105) the boundary conditions, in the current configuration for the electric variables (no distribution of surface charges), are

$$[\![\mathbf{E}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{D}]\!] \cdot \mathbf{n} = 0. \tag{6.57}$$

Using (6.3) in the reference configuration $\int_{\mathcal{S}_r} \mathbf{D}_l \cdot \mathbf{N}\, \mathrm{d}A = 0$, $\oint_{\mathcal{C}_r} \mathbf{E}_l \cdot \mathrm{d}\mathbf{R} = 0$, the above boundary conditions are equivalent to

$$[\![\mathbf{E}_l]\!] \times \mathbf{N} = \mathbf{0}, \quad [\![\mathbf{D}_l]\!] \cdot \mathbf{N} = 0. \tag{6.58}$$

Here the vectors $\mathbf{n}$ and $\mathbf{N}$ are the outward normal vectors to the surface of the body in the current and reference configurations.

The boundary conditions for the stress in the current and reference configurations are

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \quad [\![\mathbf{T}]\!]\mathbf{N} = \mathbf{0}, \tag{6.59}$$

where for a body completely surrounded by free space we must include the Maxwell stress (6.30) (or (6.31)) in the external load. For $(6.59)_1$, for example, we have

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t} + \boldsymbol{\tau}_m \mathbf{n}, \tag{6.60}$$

where $\mathbf{t}$ is the purely mechanical contribution to the surface traction.

6.4 The boundary value problem

This is a summary of the main results of the Dorfmann and Ogden's theory for ES elastomers [32, 36].

We have to solve the following system of partial differential equations in the current configuration

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{curl}\mathbf{E} = \mathbf{0}, \quad \operatorname{div}\mathbf{D} = 0, \tag{6.61}$$

with boundary conditions

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \quad [\![\mathbf{E}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{D}]\!] \cdot \mathbf{n} = 0. \tag{6.62}$$

The system of equations (6.61) is coupled. We work with a body $\mathcal{B}$ completely surrounded by a free space (vacuum) $\mathcal{B}^o$. We solve $(6.61)_1$ for $\mathcal{B}$ and $(6.61)_2$ and $(6.61)_3$ for $\mathcal{B}$ and $\mathcal{B}^o$ respectively. From the definition of the total stress (6.27), we have that $\boldsymbol{\tau} = \boldsymbol{\tau}_m$ in $\mathcal{B}^o$, where $\boldsymbol{\tau}_m$ is defined in (6.30) and (6.31); we can prove that if $(6.61)_2$ and $(6.61)_3$ hold, then $(6.61)_1$ is satisfied trivially for $\mathcal{B}^o$.

In analogy with what was done in Section 3.5, we can define the scalar and vector potentials for the electric field and the electric displacement respectively. We do not repeat the theory here.

Consider the case in which we work with the electric field $\mathbf{E}$ as the independent electric variable. A solution of $(6.61)_2$ is

$$\mathbf{E} = -\operatorname{grad}\varphi, \tag{6.63}$$

where φ is the scalar electric potential. This potential is very important in the theory of electrostatics [42, 107]; in practical problems an electric field is produced by a difference in the potential.

From (6.49), (6.53) and $(6.4)_1$, the system of equations we have to solve is $(6.61)_1$ and $(6.61)_3$, where we look for $\boldsymbol{\chi}$ and φ such that

$$\operatorname{div}\left(J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}\right) = \mathbf{0}, \quad \operatorname{div}\left(J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{E}_l}\right) = 0, \quad \mathbf{x} \in \mathcal{B}, \tag{6.64}$$

and

$$\operatorname{div}\operatorname{grad}\varphi = 0, \quad \mathbf{x} \in \mathcal{B}^o, \tag{6.65}$$

where $\mathbf{F} = \frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}$, $\mathbf{E} = -\operatorname{grad}\varphi$, $\Omega = \Omega(\mathbf{F}, \mathbf{E}_l)$ and $\mathbf{E}_l = \mathbf{F}^T\mathbf{E}$. The field φ is continuous across $\partial\mathcal{B}$, and we use the same notation for the scalar potential for $\mathcal{B}$ and $\mathcal{B}^o$.

In the case in which we work with the electric displacement $\mathbf{D}$ as the independent electric variable, a solution of $(6.61)_3$ is (see Section 3.5)

$$\mathbf{D} = \operatorname{curl}\mathbf{A}, \tag{6.66}$$

where $\mathbf{A}$ is known as the vector potential.

Therefore from $(6.61)_1$, $(6.61)_2$, (6.55), (6.56), the definition $\boldsymbol{\tau} = J^{-1}\mathbf{F}\mathbf{T}$ and $(6.4)_2$, we look for $\boldsymbol{\chi}$ and $\mathbf{A}$ (vectors fields) such that

$$\operatorname{div}\left(J^{-1}\mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{F}}\right) = \mathbf{0}, \quad \operatorname{curl}\left(\mathbf{F}^{-T}\frac{\partial\Omega^*}{\partial\mathbf{D}_l}\right) = \mathbf{0}, \quad \mathbf{x} \in \mathcal{B}, \tag{6.67}$$

and

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \mathbf{0}, \quad \mathbf{x} \in \mathcal{B}^o, \tag{6.68}$$

where $\mathbf{D} = \operatorname{curl}\mathbf{A}$, $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{D}_l)$ and $\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}$. The vector potential $\mathbf{A}$ is assumed continuous across $\partial\mathcal{B}$ (see Section 3.5), and we use the same notation for this potential for $\mathcal{B}$ and $\mathcal{B}^o$.

Chapter 7

Isotropic electro-active elastomers

In this chapter we study the case of electro-active elastomers with a random distribution of particles; these materials are called isotropic ES (electro-sensitive) elastomers. The distribution of particles is random but homogeneous.

Consider the case in which the electric field $\mathbf{E}$ is the independent electric variable; from (6.47) we have $\Omega = \Omega(\mathbf{F}, \mathbf{E}_l)$; from (6.49) and (6.53) we have

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{D}_l = -\frac{\partial \Omega}{\partial \mathbf{E}_l}. \tag{7.1}$$

For an incompressible material $J = 1$, and $(7.1)_1$ is replaced by

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}. \tag{7.2}$$

For the function $\Omega(\mathbf{F}, \mathbf{E}_l)$ in the isotropic case we have that it depends on six invariants [32, 102]

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det \mathbf{c}, \tag{7.3}$$

$$I_4 = \mathbf{E}_l \cdot \mathbf{E}_l, \quad I_5 = \mathbf{E}_l \cdot \mathbf{c}\mathbf{E}_l, \quad I_6 = \mathbf{E}_l \cdot \mathbf{c}^2\mathbf{E}_l. \tag{7.4}$$

For the fifth and sixth invariants I_5 and I_6 we have chosen the forms $I_5 = \mathbf{E}_l \cdot \mathbf{c}\mathbf{E}_l$ and $I_6 = \mathbf{E}_l \cdot \mathbf{c}^2\mathbf{E}_l$ instead the forms $I_5 = \mathbf{E}_l \cdot \mathbf{c}^{-1}\mathbf{E}_l$ and $I_6 = \mathbf{E}_l \cdot \mathbf{c}^{-2}\mathbf{E}_l$ used originally by Dorfmann and Ogden [32]; it can be shown that these two set of invariants are equivalent.

Consider the following derivatives of the invariants in terms of the gradient of deformation and the Lagrangian electric field:

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{F}\mathbf{F}^T), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{7.5}$$

$$\frac{\partial I_5}{\partial \mathbf{F}} = 2\mathbf{E}_l \otimes \mathbf{F}\mathbf{E}_l, \quad \frac{\partial I_6}{\partial \mathbf{F}} = 2(\mathbf{E}_l \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{E}_l) + \mathbf{F}^T\mathbf{F}\mathbf{E}_l \otimes \mathbf{F}\mathbf{E}_l. \tag{7.6}$$

and

$$\frac{\partial I_4}{\partial \mathbf{E}_l} = 2\mathbf{E}_l, \quad \frac{\partial I_5}{\partial \mathbf{E}_l} = 2\mathbf{c}\mathbf{E}_l, \quad \frac{\partial I_6}{\partial \mathbf{E}_l} = 2\mathbf{c}^2\mathbf{E}_l. \tag{7.7}$$

Thus, from (7.1) with the chain rule and $(6.4)_1$ we get[1] [32]

$$\begin{aligned} \boldsymbol{\tau} &= J^{-1}(2\mathbf{b}\Omega_1 + 2[I_1\mathbf{b} - \mathbf{b}^2]\Omega_2 + 2I_3\mathbf{I}\Omega_3 + 2\mathbf{b}\mathbf{E}\otimes\mathbf{b}\mathbf{E}\Omega_5 \\ &\quad +2[\mathbf{b}\mathbf{E}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{b}\mathbf{E}]\Omega_6), \qquad (7.8) \\ \mathbf{D} &= -2J^{-1}(\mathbf{b}\mathbf{E}\Omega_4 + \mathbf{b}^2\mathbf{E}\Omega_5 + \mathbf{b}^3\mathbf{E}\Omega_6). \qquad (7.9) \end{aligned}$$

In the case of an incompressible material $J = 1$, from (7.2) we have

$$\begin{aligned} \boldsymbol{\tau} &= 2\mathbf{b}\Omega_1 + 2[I_1\mathbf{b} - \mathbf{b}^2]\Omega_2 - p\mathbf{I} + 2\mathbf{b}\mathbf{E}\otimes\mathbf{b}\mathbf{E}\Omega_5 \\ &\quad +2[\mathbf{b}\mathbf{E}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{b}\mathbf{E}]\Omega_6; \end{aligned} \tag{7.10}$$

additionally,

$$\mathbf{D} = -2(\mathbf{b}\mathbf{E}\Omega_4 + \mathbf{b}^2\mathbf{E}\Omega_5 + \mathbf{b}^3\mathbf{E}\Omega_6). \tag{7.11}$$

If we work with the electric displacement $\mathbf{D}$ as the independent electric variable, we use the complementary function (6.54) $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{D}_l)$; from (6.55), (2.33) and (6.56) we have

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{E}_l = \frac{\partial \Omega^*}{\partial \mathbf{D}_l}. \tag{7.12}$$

As in (7.2), for an incompressible material $(7.12)_1$ is replaced by

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}. \tag{7.13}$$

For an isotropic ES elastomer Ω^* depends on six invariants; thus

$$\Omega^* = \Omega^*(I_1, I_2, I_3, K_4, K_5, K_6),$$

where I_1, I_2 and I_3 are given in (7.3), and K_4, K_5 and K_6 are defined as [32]

$$K_4 = \mathbf{D}_l \cdot \mathbf{D}_l, \quad K_5 = \mathbf{D}_l \cdot \mathbf{c}\mathbf{D}_l, \quad K_6 = \mathbf{D}_l \cdot \mathbf{c}^2\mathbf{D}_l. \tag{7.14}$$

Consider the derivatives of the invariants

$$\frac{\partial K_5}{\partial \mathbf{F}} = 2\mathbf{D}_l \otimes \mathbf{F}\mathbf{D}_l, \quad \frac{\partial K_6}{\partial \mathbf{F}} = 2(\mathbf{D}_l \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{D}_l + \mathbf{F}^T\mathbf{F}\mathbf{D}_l \otimes \mathbf{F}\mathbf{D}_l). \tag{7.15}$$

[1] Note that in these expressions $\mathbf{b}^3$ depends on $\mathbf{b}$ and $\mathbf{b}^2$; from the Cayley-Hamilton theorem we have $\mathbf{b}^3 = I_1\mathbf{b}^2 - I_2\mathbf{b} + I_3\mathbf{I}$.

The notation Ω_i means the derivative of Ω in I_i for $i = 1, ..., 6$.

and

$$\frac{\partial K_4}{\partial \mathbf{D}_l} = 2\mathbf{D}_l, \quad \frac{\partial K_5}{\partial \mathbf{D}_l} = 2\mathbf{c}\mathbf{D}_l, \quad \frac{\partial K_6}{\partial \mathbf{D}_l} = 2\mathbf{c}^2\mathbf{D}_l. \tag{7.16}$$

With the chain rule and $(6.4)_1$, from (7.12) we obtain[2]

$$\begin{aligned} \boldsymbol{\tau} &= J^{-1}(2\mathbf{b}\Omega_1^* + 2[I_1\mathbf{b} - \mathbf{b}^2]\Omega_2^* + 2I_3\mathbf{I}\Omega_3^* + 2J^2\mathbf{D}\otimes\mathbf{D}\Omega_5^* \\ &\quad + 2J^2[\mathbf{D}\otimes\mathbf{bD} + \mathbf{bD}\otimes\mathbf{D}]\Omega_6^*), \qquad (7.17) \\ \mathbf{E} &= 2J(\mathbf{b}^{-1}\mathbf{D}\Omega_4^* + \mathbf{D}\Omega_5^* + \mathbf{bD}\Omega_6^*), \qquad (7.18) \end{aligned}$$

and for an incompressible material

$$\begin{aligned} \boldsymbol{\tau} &= 2\mathbf{b}\Omega_1^* + 2[I_1\mathbf{b} - \mathbf{b}^2]\Omega_2^* - p^*\mathbf{I} + 2\mathbf{D}\otimes\mathbf{D}\Omega_5^* \\ &\quad + 2[\mathbf{D}\otimes\mathbf{bD} + \mathbf{bD}\otimes\mathbf{D}]\Omega_6^*, \qquad (7.19) \\ \mathbf{E} &= 2(\mathbf{b}^{-1}\mathbf{D}\Omega_4^* + \mathbf{D}\Omega_5^* + \mathbf{bD}\Omega_6^*). \qquad (7.20) \end{aligned}$$

7.1 Universal relations

The importance of universal relations was stressed at the beginning of Section 4.1. In this section we show some results for ES elastomers; we study in detail the theory of linear universal relations, and provide two examples of non-linear universal relations (see Subsection 4.1.2). The results of this section are based on the results shown in the paper by Bustamante and Ogden [22].

7.1.1 Linear universal relations

Let's work with the electric field as the independent electric variable, and let's work with an incompressible material. Consider the notation

$$\gamma_1 = 2(\Omega_1 + \Omega_2 I_1), \quad \gamma_2 = -2\Omega_2, \quad \gamma_4 = 2\Omega_4, \quad \gamma_5 = 2\Omega_5, \quad \gamma_6 = 2\Omega_6, \tag{7.21}$$

so (7.10) and (7.11) become

$$\begin{aligned} \boldsymbol{\tau} &= -p\mathbf{I} + \gamma_1\mathbf{b} + \gamma_2\mathbf{b}^2 + \gamma_5\mathbf{bE}\otimes\mathbf{bE} + \gamma_6(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{bE}), \qquad (7.22) \\ \mathbf{D} &= -(\gamma_4\mathbf{bE} + \gamma_5\mathbf{b}^2\mathbf{E} + \gamma_6\mathbf{b}^3\mathbf{E}). \qquad (7.23) \end{aligned}$$

Consider the skew-symmetric tensor

$$\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau} = \gamma_5(\mathbf{bE}\otimes\mathbf{E} - \mathbf{E}\otimes\mathbf{bE}) + \gamma_6(\mathbf{b}^2\mathbf{E}\otimes\mathbf{E} - \mathbf{E}\otimes\mathbf{b}^2\mathbf{E}). \tag{7.24}$$

[2]Here the notation Ω_i^* means the partial derivative of Ω^* in I_i if $i = 1, 2, 3$, or in K_i if $i = 4, 5, 6$.

Using the same representation for a skew-symmetric tensor as in Subsection 4.1.1, denoting $(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times$ its associated axial vector, we have

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times = \mathbf{E} \times (\gamma_5\mathbf{b}\mathbf{E} + \gamma_6\mathbf{b}^2\mathbf{E}), \tag{7.25}$$

from which we obtain the universal relation

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times \cdot \mathbf{E} = 0. \tag{7.26}$$

Let's use the formulation based on $\Omega^*(\mathbf{F}, \mathbf{D}_l)$ and let's introduce the notation

$$\gamma_1^* = 2(\Omega_1^* + \Omega_2^* I_1), \quad \gamma_2^* = -2\Omega_2^*, \quad \gamma_4^* = 2\Omega_4^*, \quad \gamma_5^* = 2\Omega_5^*, \quad \gamma_6^* = 2\Omega_6^*, \tag{7.27}$$

so that (7.19) and (7.20) become

$$\boldsymbol{\tau} = -p^*\mathbf{I} + \gamma_1^*\mathbf{b} + \gamma_2^*\mathbf{b}^2 + \gamma_5^*\mathbf{D}\otimes\mathbf{D} + \gamma_6^*(\mathbf{D}\otimes\mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D}\otimes\mathbf{D}), \tag{7.28}$$

$$\mathbf{E} = \gamma_4^*\mathbf{b}^{-1}\mathbf{D} + \gamma_5^*\mathbf{D} + \gamma_6^*\mathbf{b}\mathbf{D}, \tag{7.29}$$

and hence

$$\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau} = \gamma_5^*(\mathbf{D}\otimes\mathbf{b}^{-1}\mathbf{D} - \mathbf{b}^{-1}\mathbf{D}\otimes\mathbf{D}) + \gamma_6^*(\mathbf{b}\mathbf{D}\otimes\mathbf{b}^{-1}\mathbf{D} - \mathbf{b}^{-1}\mathbf{D}\otimes\mathbf{b}\mathbf{D}), \tag{7.30}$$

then

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times = \mathbf{b}^{-1}\mathbf{D} \times (\gamma_5^*\mathbf{D} + \gamma_6^*\mathbf{b}\mathbf{D}), \tag{7.31}$$

and we obtain the universal relation

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times \cdot \mathbf{b}^{-1}\mathbf{D} = 0. \tag{7.32}$$

As in Subsection 4.1.1 we have the proposition

Proposition 7.1. *The relations (7.26) and (7.32) are equivalent and can be obtained from*

$$\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau} = \mathbf{E}\otimes\mathbf{b}^{-1}\mathbf{D} - \mathbf{b}^{-1}\mathbf{D}\otimes\mathbf{E}. \tag{7.33}$$

Proof. Consider the right side of (7.33), from (7.23) we obtain

$$\begin{aligned}\mathbf{b}^{-1}\mathbf{D}\otimes\mathbf{E} - \mathbf{E}\otimes\mathbf{b}^{-1}\mathbf{D} &= (\gamma_4\mathbf{E} + \gamma_5\mathbf{b}\mathbf{E} + \gamma_6\mathbf{b}^2\mathbf{E})\otimes\mathbf{E} \\ &\quad -\mathbf{E}\otimes(\gamma_4\mathbf{E} + \gamma_5\mathbf{b}\mathbf{E} + \gamma_6\mathbf{b}^2\mathbf{E}), \\ &= \gamma_5(\mathbf{b}\mathbf{E}\otimes\mathbf{E} - \mathbf{E}\otimes\mathbf{b}\mathbf{E}) + \gamma_6(\mathbf{b}^2\mathbf{E}\otimes\mathbf{E} - \mathbf{E}\otimes\mathbf{b}^2\mathbf{E}), \end{aligned} \tag{7.34}$$

which is equal to the right side of (7.24). □

From the theory of Pucci and Saccomandi [85] we have that for an isotropic ES elastomer the number of independent universal relations is one. For an incompressible material we could find more universal relations by solving some specific boundary value problems, but we do not do that here (see remark at the end of Subsection 4.1.1).

7.1.1.1 Special cases

Sometimes it is necessary to propose simplified forms for Ω or Ω^*, which would mean we assume, for example, that Ω would only be a function of some of the invariants shown in (7.3) and (7.4). As we explained at the beginning of Subsection 4.1.1.1, to assume that Ω or Ω^* depend on fewer invariants means a modification of the matrix $\mathcal{M}$ of the Pucci and Saccomandi's theory (see [18, 22, 85]), and we can find more universal relations for these simplified cases.

Case 1: $\Omega = \Omega(I_1, I_2, I_4, I_5)$

In this case $\gamma_6 = 0$ and from (7.25) we have

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times = \gamma_5 \mathbf{E} \times \mathbf{bE}, \tag{7.35}$$

from which we find the two universal relations

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times \times \mathbf{E} = 0, \quad (\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times \times \mathbf{bE} = 0. \tag{7.36}$$

From (7.23) with γ_6 we have

$$\mathbf{D} \cdot (\mathbf{bE} \times \mathbf{b}^2\mathbf{E}) = 0, \tag{7.37}$$

provided $\mathbf{E}$ is not an eigenvalue of $\mathbf{b}$.

Consider the following subcases [22].

(a) $\Omega = \Omega(I_1, I_4, I_5)$

From (7.22) we have

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1 \mathbf{b} + \gamma_5 \mathbf{bE} \otimes \mathbf{bE}, \tag{7.38}$$

so

$$\boldsymbol{\tau}\mathbf{E} = -p\mathbf{E} + \gamma_1 \mathbf{bE} + \gamma_5 (\mathbf{bE} \cdot \mathbf{E})\mathbf{bE}, \tag{7.39}$$

and we get the universal relation

$$\boldsymbol{\tau}\mathbf{E} \cdot (\mathbf{E} \times \mathbf{bE}) = 0. \tag{7.40}$$

(b) $\Omega = \Omega(I_1, I_2, I_4)$

The total stress is given as (7.22)

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1 \mathbf{b} + \gamma_2 \mathbf{b}^2, \tag{7.41}$$

and we get the classical result of the theory of non-linear elasticity [6]

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \mathbf{0}. \tag{7.42}$$

Regarding the electric displacement, from (7.23) we have

$$\mathbf{D} = -\gamma_4 \mathbf{b}\mathbf{E}, \tag{7.43}$$

and we obtain the relation

$$\mathbf{D} \times \mathbf{b}\mathbf{E} = \mathbf{0}. \tag{7.44}$$

(c) $\Omega = \Omega(I_1, I_2, I_5)$

In this case we do not have any new relation involving the stress tensor. But for the electric displacement from (7.23) we have

$$\mathbf{D} = -\gamma_5 \mathbf{b}^2\mathbf{E}, \tag{7.45}$$

from where we get the relation

$$\mathbf{D} \times \mathbf{b}^2\mathbf{E} = \mathbf{0}. \tag{7.46}$$

(d) $\Omega = \Omega(I_2, I_4, I_5)$

In this case we do not have a new relation for the electric displacement, but for the total stress from (7.22) we have

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1 \mathbf{b} + \gamma_5 \mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E}, \tag{7.47}$$

and it is not difficult to prove that the following relation holds:

$$\boldsymbol{\tau}\mathbf{E} \cdot (\mathbf{E} \times \mathbf{b}\mathbf{E}) = 0. \tag{7.48}$$

Case 2: $\Omega = \Omega(I_1, I_2, I_4, I_6)$

In this case $\gamma_5 = 0$ and from (7.25) we have

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times = \gamma_6 \mathbf{E} \times \mathbf{b}^2\mathbf{E}, \tag{7.49}$$

from where we obtain

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times \cdot \mathbf{E} = 0, \quad (\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times \cdot (\mathbf{b}^2\mathbf{E}) = 0. \tag{7.50}$$

As for the electric displacement, from (7.23) with γ_6 we have

$$\mathbf{D} = -(\gamma_4 \mathbf{bE} + \gamma_6 \mathbf{b}^3\mathbf{E}), \tag{7.51}$$

and we get the relation

$$\mathbf{D} \cdot (\mathbf{bE} \times \mathbf{b}^3\mathbf{E}) = 0. \tag{7.52}$$

Again let's consider the following subcases.

(a) $\Omega = \Omega(I_1, I_2, I_6)$

There is not new relation for the stress. From (7.23) for the electric displacement we have

$$\mathbf{D} = -\gamma_6 \mathbf{b}^3\mathbf{E}, \tag{7.53}$$

and we obtain the universal relation

$$\mathbf{D} \times \mathbf{b}^3\mathbf{E} = \mathbf{0}. \tag{7.54}$$

(b) $\Omega = \Omega(I_1, I_4, I_6)$

From (7.22) we have for the total stress

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1 \mathbf{b} + \gamma_6(\mathbf{bE} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{bE}). \tag{7.55}$$

Let's take the product

$$\boldsymbol{\tau}\mathbf{bE} = -p\mathbf{bE} + \gamma_1 \mathbf{b}^2\mathbf{E} + \gamma_6[(\mathbf{b}^2\mathbf{E} \cdot \mathbf{bE})\mathbf{bE} + |\mathbf{bE}|^2\mathbf{b}^2\mathbf{E}], \tag{7.56}$$

and we get the relation

$$\boldsymbol{\tau}\mathbf{bE} \cdot (\mathbf{bE} \times \mathbf{b}^2\mathbf{E}) = 0. \tag{7.57}$$

There is no new relation for the electric displacement.

(c) $\Omega = \Omega(I_2, I_4, I_6)$

From (7.22) we have

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_2 \mathbf{b}^2 + \gamma_6(\mathbf{bE} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{bE}). \tag{7.58}$$

We can obtain the two following expressions

$$(\boldsymbol{\tau}\mathbf{E} \times \mathbf{E}) \cdot \mathbf{b}^2\mathbf{E} = \gamma_6(\mathbf{b}^2\mathbf{E} \cdot \mathbf{E})(\mathbf{bE} \times \mathbf{E}) \cdot \mathbf{b}^2\mathbf{E}, \tag{7.59}$$

$$(\boldsymbol{\tau}\mathbf{b}^{-1}\mathbf{E} \times \mathbf{b}^{-1}\mathbf{E}) \cdot \mathbf{bE} = \gamma_6(\mathbf{bE} \cdot \mathbf{b}^{-1}\mathbf{E})(\mathbf{b}^2\mathbf{E} \times \mathbf{b}^{-1}\mathbf{E}) \cdot \mathbf{bE}. \tag{7.60}$$

from where we have the relation[3]

$$(\mathbf{bE}\cdot\mathbf{b}^{-1}\mathbf{E})[(\mathbf{b}^2\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}][(\boldsymbol{\tau}\mathbf{E}\times\mathbf{E})\cdot\mathbf{b}^2\mathbf{E}] \\ =(\mathbf{b}^2\mathbf{E}\cdot\mathbf{E})[(\mathbf{bE}\times\mathbf{E})\cdot\mathbf{b}^2\mathbf{E}][(\boldsymbol{\tau}\mathbf{b}^{-1}\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}]. \quad (7.63)$$

Using the Cayley-Hamilton theorem in the form $\mathbf{b}^{-1}=\mathbf{b}^2-\mathbf{b}I_1+\mathbf{I}I_2$ we could rewrite the above expression only in terms of $\mathbf{I}$, $\mathbf{b}$ and $\mathbf{b}^2$.

Case 3: $\Omega=\Omega(I_1,I_4,I_5,I_6)$

Here from (7.22) for the stress we have

$$\boldsymbol{\tau}=-p\mathbf{I}+\gamma_1\mathbf{b}+\gamma_5\mathbf{bE}\otimes\mathbf{bE}+\gamma_6(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E}+\mathbf{b}^2\mathbf{E}\otimes\mathbf{bE}). \quad (7.64)$$

Multiplying by $\mathbf{bE}$ we get

$$\boldsymbol{\tau}\mathbf{bE}=-p\mathbf{bE}+\gamma_1\mathbf{b}^2\mathbf{E}+\gamma_5|\mathbf{bE}|^2\mathbf{bE}+\gamma_6[(\mathbf{bE}\cdot\mathbf{bE})\mathbf{bE}+|\mathbf{bE}|^2\mathbf{b}^2\mathbf{E}]. \quad (7.65)$$

With the above expression we can obtain the universal relation (7.57).

For the subcase $\Omega=\Omega(I_1,I_5,I_6)$, from (7.23) for the electric displacement we have

$$\mathbf{D}=-(\gamma_5\mathbf{b}^2\mathbf{E}+\gamma_6\mathbf{b}^3\mathbf{E}), \quad (7.66)$$

and we obtain the universal relation

$$\mathbf{D}\cdot(\mathbf{b}^2\mathbf{E}\times\mathbf{b}^3\mathbf{E})=0. \quad (7.67)$$

Case 4: $\Omega=\Omega(I_2,I_4,I_5,I_6)$

This case is more complicated. From (7.22) the total stress is given as

$$\boldsymbol{\tau}=-p\mathbf{I}+\gamma_2\mathbf{b}^2+\gamma_5\mathbf{bE}\otimes\mathbf{bE}+\gamma_6(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E}+\mathbf{b}^2\mathbf{E}\otimes\mathbf{bE}); \quad (7.68)$$

taking the product with $\mathbf{bE}$ and $\mathbf{b}^{-1}\mathbf{E}$ we can obtain respectively

$$(\boldsymbol{\tau}\mathbf{bE}\times\mathbf{bE})\cdot\mathbf{b}^3\mathbf{E}=\gamma_6|\mathbf{bE}|^2(\mathbf{b}^2\mathbf{E}\times\mathbf{bE})\cdot\mathbf{b}^3\mathbf{E}, \quad (7.69)$$

$$(\boldsymbol{\tau}\mathbf{b}^{-1}\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}=\gamma_6(\mathbf{bE}\cdot\mathbf{b}^{-1}\mathbf{E})(\mathbf{b}^2\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}, \quad (7.70)$$

[3]Here we do not consider the cases

$$(\mathbf{b}^2\mathbf{E}\cdot\mathbf{E})(\mathbf{bE}\times\mathbf{E})\cdot\mathbf{b}^2\mathbf{E}=0, \quad \text{or} \quad (\mathbf{bE}\cdot\mathbf{b}^{-1}\mathbf{E})(\mathbf{b}^2\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}=0, \quad (7.61)$$

where we would have the universal relations

$$(\boldsymbol{\tau}\mathbf{E}\times\mathbf{E})\cdot\mathbf{b}^2\mathbf{E}=0, \quad \text{or} \quad (\boldsymbol{\tau}\mathbf{b}^{-1}\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}=0 \quad (7.62)$$

respectively.

and we obtain

$$(\mathbf{bE}\cdot\mathbf{b}^{-1}\mathbf{E})[(\mathbf{b}^2\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}][(\tau\mathbf{bE}\times\mathbf{bE})\cdot\mathbf{b}^3\mathbf{E}]$$
$$=|\mathbf{bE}|^2[(\mathbf{b}^2\mathbf{E}\times\mathbf{bE})\cdot\mathbf{b}^3\mathbf{E}][(\tau\mathbf{b}^{-1}\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}]. \qquad (7.71)$$

As in the case 2 (c), when $(\mathbf{bE}\cdot\mathbf{b}^{-1}\mathbf{E})[(\mathbf{b}^2\mathbf{E}\times\mathbf{b}^{-1}\mathbf{E})\cdot\mathbf{bE}]=0$ or $|\mathbf{bE}|^2[(\mathbf{b}^2\mathbf{E}\times\mathbf{bE})\cdot\mathbf{b}^3\mathbf{E}]=0$ we obtain the universal relations $(7.62)_2$ and

$$(\tau\mathbf{bE}\times\mathbf{bE})\cdot\mathbf{b}^3\mathbf{E}=0. \qquad (7.72)$$

In the above expressions $\mathbf{b}^{-1}$ and $\mathbf{b}^3$ can be obtained with $\mathbf{b}$ and $\mathbf{b}^2$ from the Cayley-Hamilton theorem, but we do not do that here.

Case 5: $\Omega=\Omega(I_1,I_2,I_5,I_6)$

Finally for this energy function from (7.22) we do not have a new relation involving the total stress. As for the electric displacement, from (7.23) it is easy to show that we obtain the universal relation (7.67).

The remark at the end of Subsection 4.1.1.1 also applies here. The universal relations found in this subsection are valid for constrained and unconstrained ES elastomers. But for the case of constrained elastomers we could find more relations by studying the solutions of some boundary value problems and working with the constraint stress (in this case p); see, for example, [83].

7.1.1.2 Applications

The comments at the beginning of the Subsection 4.1.1.2 also apply here. If we want to use the linear universal relations found in the previous subsection we need to work with universal solutions. For electro-elastic materials these solutions were found by Singh and Pipkin [99].

We consider two solutions.

Homogeneous deformation in a uniform field This problem differs slightly from the problem (4.83) presented for MS elastomers. Here we use the same deformation for a slab presented by Bustamante and Ogden [22], which is defined as

$$x_1=\mu_1X_1+\kappa\mu_2X_2,\quad x_2=\mu_2X_2,\quad x_3=\mu_3X_3. \qquad (7.73)$$

The material is incompressible, so that

$$\mu_1\mu_2\mu_3 = 1. \tag{7.74}$$

We work with the electric field as the independent electric variable. We assume that in the reference configuration there is an external uniform field applied far away; thus,

$$\mathsf{E}_l = (E_{o_1}, E_{o_2}, E_{o_3})^T. \tag{7.75}$$

The matrix of Cartesian components of the deformation gradient and the left Cauchy-Green deformation tensors are

$$\mathsf{F} = \begin{pmatrix} \mu_1 & \kappa\mu_2 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} \mu_1^2 + \kappa^2\mu_2^2 & \kappa\mu_2^2 & 0 \\ \mu_2^2 & \kappa\mu_2^2 & 0 \\ 0 & 0 & \mu_3^2 \end{pmatrix}. \tag{7.76}$$

With $(6.4)_2$, $(7.76)_1$ and (7.75) the components of the electric field in the current configuration are given as

$$\mathsf{E} = (E_1, E_2, E_2)^T = (\mu_2\mu_3 E_{o_1}, -\kappa\mu_2\mu_3 E_{o_1} + \mu_1\mu_3 E_{o_2}, \mu_1\mu_2 E_{o_3})^T. \tag{7.77}$$

The first invariant $(7.3)_1$ has the form

$$I_1 = \mu_1^2 + \mu_2^2(1+\kappa^2) + \mu_3^2. \tag{7.78}$$

From (7.10) the components of the total stress are

$$\begin{aligned}
\tau_{11} = {} & -p + 2(\mu_1^2 + \kappa^2\mu_2^2)\Omega_1 + 2[\kappa^2\mu_2^2\mu_3^3 + \mu_1^2(\mu_2^2 + \mu_3^2)]\Omega_2 + 2[E_1\mu_2^2 + \kappa(E_2 \\
& + E_1\kappa)\mu_2^2]^2\Omega_5 + 4[E_1\mu_1^2 + \kappa(E_1 + E_1\kappa)\mu_2^2][E_1\mu_1^4 + \kappa(E_2 + 2E_1\kappa)\mu_1^2\mu_2^2 \\
& + (E_2 + E_1\kappa)(\kappa + \kappa^3)\mu_2^4]\Omega_6, && (7.79)\\
\tau_{22} = {} & -p + 2\mu_2^2\{\Omega_1 + \mu_3^2\Omega_2 + \mu_1^2[\Omega_2 + 2E_1\kappa(E_2 + E_1\kappa)\mu_2^2\Omega_6] \\
& + (E_2 + E_1\kappa)^2\mu_2^2[\Omega_5 + 2(1+\kappa^2)\mu_2^2\Omega_6]\}, && (7.80)\\
\tau_{33} = {} & -p + 2\mu_3^2\{\Omega_1 + [\mu_1^2 + (1+\kappa^2)\mu_2^2]\Omega_2 + E_3^2\mu_3^2(\Omega_5 + 2\mu_3^2\Omega_6)\}, && (7.81)\\
\tau_{12} = {} & 2\mu_2^2\{\kappa\Omega_1 + \kappa\mu_3^3\Omega_2 + (E_2 + E_1\kappa)[E_1\mu_1^2 + \kappa(E_2 + E_1\kappa)\mu_2^2]\Omega_5 \\
& + [E_1(E_2 + 2E_1\kappa)\mu_1^4 + (E_2 + E_1\kappa)(E_1 + E_2\kappa + 4E_1\kappa^2)\mu_1^2\mu_2^2 \\
& + 2(E_2 + E_1\kappa)^2(\kappa + \kappa^3)\mu_2^4]\Omega_6\}, && (7.82)\\
\tau_{13} = {} & 2E_3\mu_3^2\{E_1\mu_1^4\Omega_6 + \kappa(E_2 + E_1\kappa)\mu_2^2[\Omega_5 + ((1+\kappa^2)\mu_2^2 + \mu_3^2)\Omega_6] \\
& + \mu_1^2[E_1\Omega_5 + (\kappa(E_2 + 2E_1\kappa)\mu_1^2 + E_1\mu_3^2)\Omega_6]\}. && (7.83)\\
\tau_{23} = {} & 2E_3\mu_2^2\mu_3^2\{(E_2 + E_1\kappa)\Omega_5 + [E_1\kappa\mu_1^2 + (E_2 + E_1\kappa)((1+\kappa^2)\mu_2^2 + \mu_2^2)]\Omega_6\}. && (7.84)
\end{aligned}$$

For the electric displacement from (7.11) we obtain

$$\begin{aligned}D_1 &= -2\{E_1\mu_1^6\Omega_6 + \mu_1^4[E_1\Omega_5 + \kappa(E_2 + 3E_1\kappa)\mu_2^2\Omega_6] + \mu_1^2[E_1\Omega_4 + \kappa(E_2 \\ &\quad +2E_1\kappa)\mu_1^2\Omega_5 + \kappa(E_2 + 2E_1 + 2E_2\kappa^2 + 3E_1\kappa^2)\mu_2^4\Omega_6] + \kappa(E_1 + E_1\kappa)\mu_2^2[\Omega_4 \\ &\quad +(1+\kappa^2)\mu_2^2(\Omega_5 + (1+\kappa^2)\mu_2^2\Omega_6)]\}, \qquad (7.85)\\ D_2 &= -2\mu_2^2\{(E_1 + E_1\kappa)\Omega_4 + E_1\kappa\mu_1^4\Omega_6 + (E_2 + E_1\kappa)(1+\kappa^2)\mu_2^2[\Omega_5 \\ &\quad +(1+\kappa^2)\mu_2^2\Omega_6] + \kappa\mu_1^2[E_1\Omega_5 + (E_1 + E_2\kappa + 2E_1\kappa^2)\mu_2^2\Omega_6]\}, \qquad (7.86)\\ D_3 &= -2E_3\mu_3^3(\Omega_4 + \mu_3^2\Omega_5 + \mu_3^4\Omega_6). \qquad (7.87)\end{aligned}$$

In general all the components of the total stress are not zero. From (7.26) we obtain

$$\begin{aligned}&\{\tau_{12}\mu_1^2 + [(\tau_{22}-\tau_{11})\kappa + \tau_{12}(\kappa^2-1)]\mu_2^2\}\mu_3^2E_1 - [\tau_{13}\mu_1^2 + (\tau_{23}\kappa - \tau_{13})\mu_3^2]\mu_2^2E_2 \\ &\quad +\{[\tau_{12} + (\tau_{11}-\tau_{22})\kappa - \tau_{12}\kappa^2]\mu_2^2 - \tau_{12}\mu_1^2\}\mu_3^2E_3 = 0. \qquad (7.88)\end{aligned}$$

Let's consider the special case $\gamma_6 = 0$ ($\Omega_6 = 0$). We have the extra universal relation $(7.36)_2$. The component form of the vector $\mathbf{bE}$ is

$$\mathsf{bE} = (E_1\mu_1^2 + \kappa[E_2 + E_1\kappa]\mu_2^2, [E_2 + E_1\kappa]\mu_2^2, E_3\mu_3^2)^T, \qquad (7.89)$$

and $(7.36)_2$ becomes

$$\begin{aligned}&\{\tau_{12}\mu_1^2 + [(\tau_{22}-\tau_{11})\kappa + \tau_{12}(\kappa^2-1)]\mu_2^2\}(E_1\mu_1^2 + \kappa[E_2 + E_1\kappa]\mu_2^2) \\ &\quad -[\tau_{13}\mu_1^2 + (\tau_{23}\kappa - \tau_{13})\mu_3^2][E_2 + E_1\kappa]\mu_2^2 + \{[\tau_{12} + (\tau_{11}-\tau_{22})\kappa \\ &\quad -\tau_{12}\kappa^2]\mu_2^2 - \tau_{12}\mu_1^2\}E_3\mu_3^2 = 0. \qquad (7.90)\end{aligned}$$

For (7.37), since from (7.85)-(7.87) all the components of the electric displacement are in general not zero, we get

$$\begin{aligned}&D_1E_3\mu_2^2\mu_3^2\{E_1\kappa\mu_1^2 + (E_2 + E_1\kappa)[(1+\kappa^2)\mu_2^2 - \mu_3^2]\} - D_2E_3\mu_3^2\{E_1\mu_1^4 + \kappa(E_2 \\ &\quad +E_1\kappa)\mu_2^2[(1+\kappa^2)\mu_2^2 - \mu_3^2] + \mu_1^2[\kappa(E_2 + 2E_1\kappa)\mu_2^2 - E_1\mu_3^2]\} \\ &\quad +D_3\mu_1^2\mu_2^2[E_1E_2\mu_1^2 + (E_2 + E_1\kappa)(E_2\kappa - E_1)\mu_2^2] = 0. \qquad (7.91)\end{aligned}$$

In [22] some extra relations were obtained by manipulating the results for $(7.36)_1$ and $(7.36)_2$. Bustamante and Ogden [22] used a different form for the invariants I_5 and I_6, as was stated at the beginning of this chapter. In our case we have not found the same relations by manipulating (7.36).

Extension and torsion of a circular cylinder Here we use the same deformation presented in Subsection 4.1.1.2 for MS elastomers. Consider a cylinder of infinite length defined as

$$0 \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -\infty \leq Z \leq \infty. \tag{7.92}$$

The extension and torsion of a circular cylinder was defined by the equations

$$r = \lambda_z^{-1/2} R, \quad \theta = \Theta + \lambda_z \tau Z, \quad z = \lambda_z Z, \tag{7.93}$$

where τ is the amount of torsional twist per unit deformed length, λ_z is the constant axial stretch.

The deformation gradient, left Cauchy-Green deformation tensor, and the first invariant $(7.3)_1$ in this case are given by (4.96), (4.97) and $(4.99)_1$ respectively. The external field in the reference configuration has only axial component and it is uniform, let $\mathsf{E}_l = (0, 0, E_o)^T$ denotes the component form of this field. The electric field in the reference configuration is given by $(6.4)_2$ in component form as $\mathsf{E} = (0, 0, \lambda_z^{-1} E_o)^T$

From (7.10) the components of the total stress are

$$\begin{aligned}
\tau_{rr} &= -p + 2\lambda_z^{-1}\Omega_1 + (1 + 2\lambda_z^{-3/2} - \lambda_z^2)\Omega_2, && (7.94)\\
\tau_{\theta\theta} &= \lambda_z^{-2}\{2(\lambda_z + \gamma^2\lambda_z^4)\Omega_1 - 2[1 + \lambda_z^{1/2}(-2 + (\gamma^2 - 1)\lambda_z^{5/2})(1 + \gamma^2\lambda_z^3)]\Omega_2 \\
&\quad + \lambda_z^2[-p + 2E_o^2\gamma^4\lambda_z(\lambda_z\Omega_5 + 2(1 + (1 + \gamma^2)\lambda_z^3)\Omega_6)]\}, && (7.95)\\
\tau_{zz} &= -p + 2\lambda_z^2\Omega_1 + 2[2\lambda_z^{3/2} - \lambda_z^2 + (\gamma^4 - 1)\lambda_z^4]\Omega_2 + 2E_o^2\lambda_z^2[\Omega_5 + 2(1 + \gamma^4)\lambda_z^2\Omega_6], && (7.96)\\
\tau_{r\theta} &= \tau_{rz} = 0, && (7.97)\\
\tau_{\theta z} &= 2\gamma^2\lambda_z\{\lambda_z\Omega_1 + (2\lambda_z^{1/2} - 1)\Omega_2 + E_o^2[\lambda_z\Omega_5 + (1 + (2 + \gamma^2 + \gamma^4)\lambda_z^3)\Omega_6]\}. && (7.98)
\end{aligned}$$

Using (4.97) and taking in consideration the zero components of the stress tensor given above, the component form of the skew-symmetric tensor $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times$ is

$$(0, [\gamma^2(\gamma^2 - 1)\lambda_z^5 - \lambda_z^2]^{-1}\{\tau_{\theta z} + [(\tau_{\theta z} + \tau_{zz} - \tau_{\theta\theta})\gamma^2 - \tau_{\theta z}]\lambda_z^3\}, 0)^T.$$

Due to the form of the vector $\mathbf{E}$ presented above we have that the universal relation (7.26) is satisfied trivially.

7.1.2 Non-linear universal relations

We study briefly the two non-linear universal relations shown in Subsection 4.1.2.

7.1.2.1 Helical shear

A more detailed analysis of the helical shear for isotropic electroelastic elastomers can be found in [22], where, in particular, some restrictions on the deformation and the form of the energy function were found in order to ensure existence of the solution. In [22] was shown that from the point of view of these restrictions it is not the same to work with the electric field or the electric displacement as the electric independent variable.

Consider the helical shear defined in (4.110), applied to a semi-infinite tube as described in (4.111). With the notation (4.112) for the deformation gradient and the left Cauchy-Green deformation tensors we have the matrix representation (4.113)

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1+\kappa_\theta^2 & \kappa_\theta\kappa_z \\ \kappa_z & \kappa_\theta\kappa_z & 1+\kappa_z^2 \end{pmatrix}. \tag{7.99}$$

We work with the electric field as the independent electric variable; we assume that the initial field is only radial, such that $\mathsf{E}_l = (E_r, 0, 0)^T$, from $(6.4)_2$ with $(7.99)_1$ we have $\mathsf{E} = (E_r, 0, 0)^T$, and the equation $\text{curl}\mathbf{E} = \mathbf{0}$ is satisfied automatically.

From (7.10) and (7.11) the components of the stresses and the electric field are (we use the notation $\kappa = \sqrt{\kappa_\theta^2 + \kappa_z^2}$)

$$\begin{aligned}
\tau_{rr} &= -p + 2\Omega_1 + 4\Omega_2 + 2E_r^2[\Omega_5 + 2(1+\kappa^2)\Omega_6], && (7.100)\\
\tau_{\theta\theta} &= -p + 2(1+\kappa_\theta^2)\Omega_1 + 2(2+\kappa^2)\Omega_2 + 2E_r^2\kappa_\theta^2[\Omega_5 + 2(2+\kappa^2)\Omega_6], && (7.101)\\
\tau_{zz} &= -p + 2(1+\kappa_z^2)\Omega_1 + 2(2+\kappa^2)\Omega_2 + 2E_r^2\kappa_z^2[\Omega_5 + 2(2+\kappa^2)\Omega_6], && (7.102)\\
\tau_{r\theta} &= 2\kappa_z\{\Omega_1 + \Omega_2 + E_r^2[\Omega_5 + (3+2\kappa^2)\Omega_6]\}, && (7.103)\\
\tau_{rz} &= 2\kappa_\theta\{\Omega_1 + \Omega_2 + E_r^2[\Omega_5 + (3+2\kappa^2)\Omega_6]\}, && (7.104)\\
\tau_{\theta z} &= 2\kappa_2\kappa_\theta\{\Omega_1 + E_r^2[\Omega_5 + 2(2+\kappa^2)\Omega_6]\}, && (7.105)
\end{aligned}$$

and

$$\begin{aligned}
D_r &= -2E_r\{\Omega_4 + (1+\kappa^2)\Omega_5 + [1+\kappa^2(3+\kappa^2)]\Omega_6\}, && (7.106)\\
D_\theta &= -2E_r\kappa_\theta[\Omega_4 + (2+\kappa^2)\Omega_5 + (1+\kappa^2)(3+\kappa^2)\Omega_6], && (7.107)\\
D_z &= -2E_r\kappa_z[\Omega_4 + (2+\kappa^2)\Omega_5 + (1+\kappa^2)(3+\kappa^2)\Omega_6]. && (7.108)
\end{aligned}$$

In order to satisfy the equation div$\mathbf{D} = 0$ (see (A.2) for the form of this equation in cylindrical coordinates), the radial component of the electric displacement must have the form [22]

$$-2E_r\{\Omega_4 + (1+\kappa^2)\Omega_5 + [1+\kappa^2(3+\kappa^2)]\Omega_6\} = \frac{D_o}{r}. \tag{7.109}$$

This equation can be used in order to find E_r as a function of r for a given energy function Ω [22].

Using the components of the stress tensor given previously it is straightforward to show that the following non-linear universal relation holds [22, 79]

$$\tau_{\theta z}(\tau_{r\theta}^2 - \tau_{rz}^2) - \tau_{r\theta}\tau_{rz}(\tau_{\theta\theta} - \tau_{zz}) = 0. \tag{7.110}$$

We do not treat the case where the electric displacement is the independent electric variable.

7.1.2.2 Anti-plane shear

We study the anti-plane shear [53,56] in the context of electro-elasticity, in particular with respect to the existence of a non-linear universal relation found recently by Bustamante and Ogden [21] in the purely elastic context.

Consider the anti-plane deformation defined in Subsection 4.1.2.2 (equation (4.125)) in Cartesian coordinates. The matrix representation of the gradient of deformation and the left Cauchy-Green deformation tensors are (4.126)

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_1 & \varphi_2 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1 & 0 & \varphi_1 \\ 0 & 1 & \varphi_2 \\ \varphi_1 & \varphi_2 & 1+\varphi_1^2+\varphi_2^2 \end{pmatrix}. \tag{7.111}$$

Let's work with the electric field as the independent electric variable, and let's assume an initial field given in component form as $\mathsf{E}_l = (0, 0, E_o)^T$, with E_o constant. With $(7.111)_1$ and $(6.4)_2$ we have $\mathsf{E} = E_o(-\varphi_1, \varphi_2, 1)^T$; the equation curl$\mathbf{E} = \mathbf{0}$ is satisfied since $\varphi = \varphi(X_1, X_2)$, and we assume φ smooth enough such that $\frac{\partial^2\varphi}{\partial X_1 \partial X_2} = \frac{\partial^2\varphi}{\partial X_2 \partial X_1}$.

From (7.10) and (7.11) the components of the stress and the electric displacement are[4]

$$\tau_{11} = -p + 2\Omega_1 + 2(2 + \varphi_2^2)\Omega_2, \tag{7.112}$$

$$\tau_{22} = -p + 2\Omega_1 + 2(2 + \varphi_2^2)\Omega_2, \tag{7.113}$$

$$\tau_{33} = -p + 2\vartheta^2\Omega_1 + 2(1 + \vartheta^2)\Omega_2 + 2E_o^2(\Omega_5 + 2\vartheta^2\Omega_6), \tag{7.114}$$

$$\tau_{12} = -2\varphi_1\varphi_2\Omega_2, \tag{7.115}$$

$$\tau_{13} = 2\varphi_1(\Omega_1 + \Omega_2 + E_o^2\Omega_6), \tag{7.116}$$

$$\tau_{23} = 2\varphi_2(\Omega_1 + \Omega_2 + E_o^2\Omega_6). \tag{7.117}$$

The equation $\operatorname{div}\mathbf{D} = 0$ imposes restriction in the form of φ and/or Ω; we do not explore this further in this thesis.

With the above components for the stress tensor it is easy to show that the non-linear universal relation (4.136) holds, i.e.

$$\tau_{13}\tau_{23}(\tau_{22} - \tau_{11}) + \tau_{12}(\tau_{13}^2 - \tau_{23}^2) = 0. \tag{7.118}$$

Unlike the magneto-elastic case (Subsection 4.1.2.2) for brevity we do not explore the case where the electric displacement is the independent electric variable.

7.2 Numerical solution of a boundary value problem: uniform extension of a cylinder

In Section 4.2 we showed some numerical results for a boundary value problem for the magnetoelastic case, the extension and inflation of a tube. In this section we provide similar results, in this case for a cylinder made of an ES elastomer, under a uniform extension. As in Section 4.2, the objective is to have a first approach of the 'fringe' effect (see [99]), by studying the behaviour of the electric field for a cylinder of finite length (see the introduction of Section 4.2). For the problem of this section we assume again that the deformation is given and then we solve the equations $(6.64)_2$ and (6.65) for the cylinder and the free space around it; a full numerical solution should consider the interaction between the electric field and the deformation of the body, and we should also have to solve $(6.64)_1$. We do not do that here, as we mentioned in Section 4.2 such numerical problems will be solved in the future using the finite element method. This method has been recently used by Vu et al. [119] for a problem involving an electro-active elastomer;

[4] From Subsection 4.1.2.2 we recall the notation $\vartheta = \sqrt{1 + \varphi_1^2 + \varphi_2^2}$. As well as this, we had $I_1 = 2 + \vartheta^2$.

however, they considered the case of a body with a surface distribution of charges, and so they did not consider the free space surrounding the body (see [16]).

7.2.1 Uniform extension of a bar

Using cylindrical coordinates the undeformed reference configuration is given by

$$0 \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L. \tag{7.119}$$

The cylinder is deformed by applying a tensile load, the deformed configuration, using cylindrical coordinates (r, θ, z), is defined by

$$0 \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq l, \tag{7.120}$$

where

$$r = \lambda^{-1/2} R, \quad \theta = \Theta, \quad z = \lambda Z. \tag{7.121}$$

The matrix forms of the deformation gradient and the right and left Cauchy-Green deformation tensors are

$$\mathsf{F} = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \mathsf{c} = \mathsf{b} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}. \tag{7.122}$$

We work with the electric field as the independent electric variable, and we assume that the initial far field is uniform and has only an axial component; therefore $\mathsf{E}_l = (0, 0, E_o)^T$, with E_o constant.

From (7.3) and (7.4) the invariants are given as ($\det \mathbf{F} = 1$)

$$I_1 = 2\lambda^{-1} + \lambda^2, \quad I_2 = \lambda^{-2} + 2\lambda, \quad I_4 = E_o^2, \quad I_5 = \lambda^2 E_o^2, \quad I_6 = \lambda^4 E_o^2. \tag{7.123}$$

7.2.2 Field equation and boundary conditions

For the axisymmetric deformation (7.121), as in Subsection 4.2.2, we have that all fields are independent of θ, but for a finite cylinder in general they depend on the radius r and the axial coordinate z, i.e.

$$\mathbf{E} = \mathbf{E}(r, z), \quad \mathbf{D} = \mathbf{D}(r, z), \quad \mathbf{P} = \mathbf{P}(r, z), \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}(r, z). \tag{7.124}$$

Figure 7.1 shows the original three-dimensional problem.

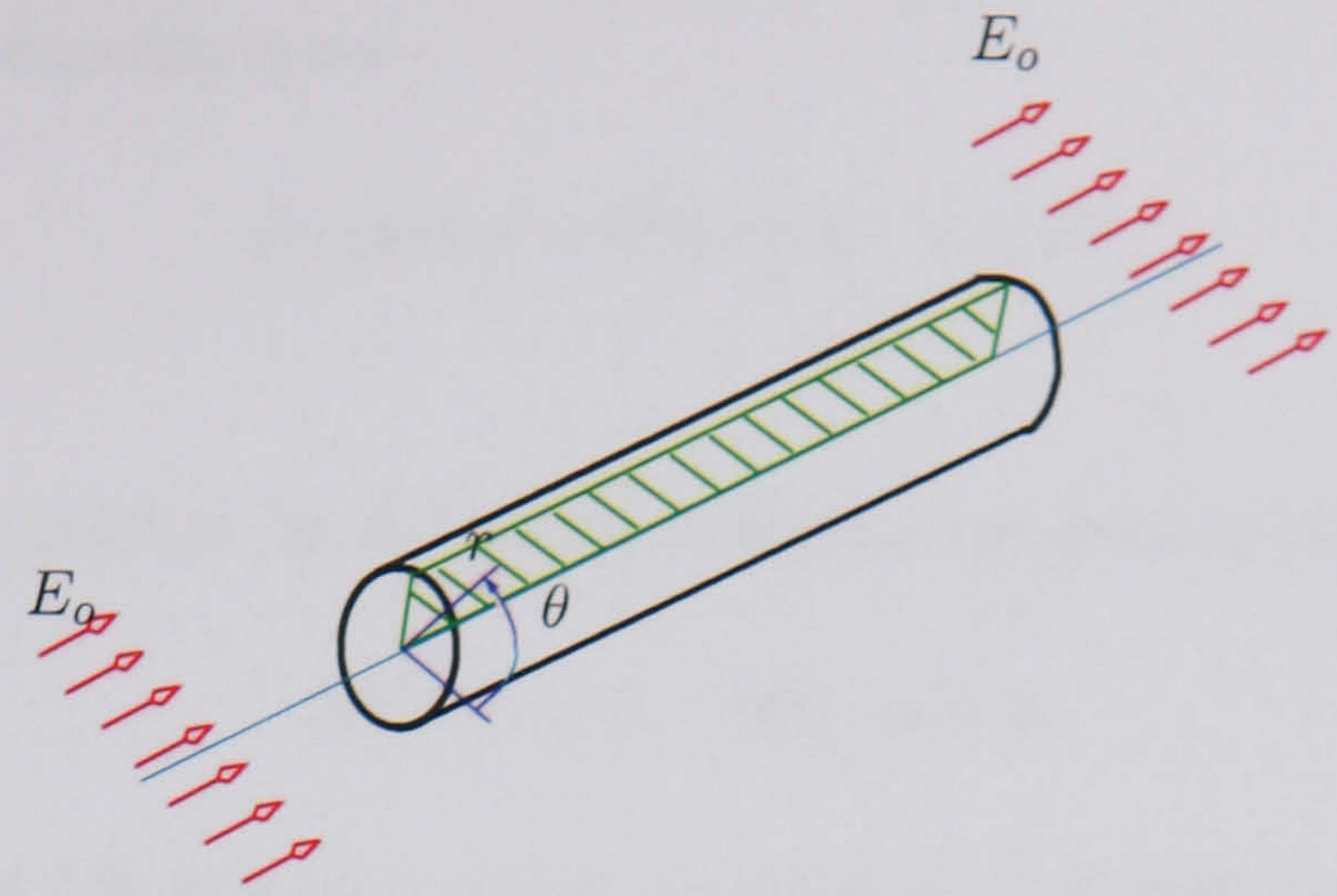


Figure 7.1: Scheme of the problem of a cylinder under uniform extension and an axial electric field applied far away.

For the cylinder and the free space we have to solve the equations $(6.61)_2$ and $(6.61)_3$

$$\operatorname{curl}\mathbf{E} = \mathbf{0}, \quad \operatorname{div}\mathbf{D} = 0. \tag{7.125}$$

For the present problem we do not solve the equilibrium equation $(6.61)_1$ $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$. As was mentioned before we assume that the deformation is given; it is easy to prove that for such deformation and for an infinitely long cylinder, with $\mathsf{E}_l = (0, 0, E_o)^T$, from (7.10) and (7.3), (7.4), the equation $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$ is satisfied trivially.

From (7.11) we have

$$\mathbf{D} = -2(\mathbf{b}\Omega_4 + \mathbf{b}^2\Omega_5 + \mathbf{b}^3\Omega_6)\mathbf{E}. \tag{7.126}$$

We introduce the tensor $\mathbf{C}$ defined as (see Subsection 4.2.2)

$$\mathbf{C} = 2(\mathbf{b}\Omega_4 + \mathbf{b}^2\Omega_5 + \mathbf{b}^3\Omega_6), \tag{7.127}$$

and so

$$\mathbf{D} = \mathbf{C}\mathbf{E}. \tag{7.128}$$

From (6.63) (see Section 6.4) a solution of $(7.125)_1$ is

$$\mathbf{E} = -\operatorname{grad}\varphi, \tag{7.129}$$

and as a result from $(7.125)_2$, (7.128) and the relation $\mathbf{D} = \varepsilon_o\mathbf{E}$ for free space, we have to solve (see (6.64) and (6.65)) for the body

$$\operatorname{div}(\mathbf{C}\operatorname{grad}\varphi) = 0, \quad \mathbf{x} \in \mathcal{B}, \tag{7.130}$$

and for the free surrounding space

$$\operatorname{div}\operatorname{grad}\varphi = \nabla^2\varphi = 0, \quad \mathbf{x} \in \mathcal{B}^o \tag{7.131}$$

respectively.

The continuity condition for the electric variables are given in $(6.62)_2$ and $(6.62)_3$ as

$$[\![\mathbf{E}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{D}]\!] \cdot \mathbf{n} = 0. \tag{7.132}$$

From Section 3.5 for the magnetic scalar potential we had that if φ was continuous on $\partial\mathcal{B}$, then the condition $[\![\mathbf{H}]\!] \times \mathbf{n} = \mathbf{0}$ was satisfied automatically. We can do the same in our case; we assume φ continuous on $\partial\mathcal{B}$, therefore we use only one symbol φ, in order to speak about the potential in the cylinder and in the free surrounding space.

The partial differential equations (7.130), (7.131) are solved for the 'plane' $r - z$ (axisymmetric problem); consider the figure 7.2 where we have a representation of the cylinder and a 'finite' portion of free space around it.

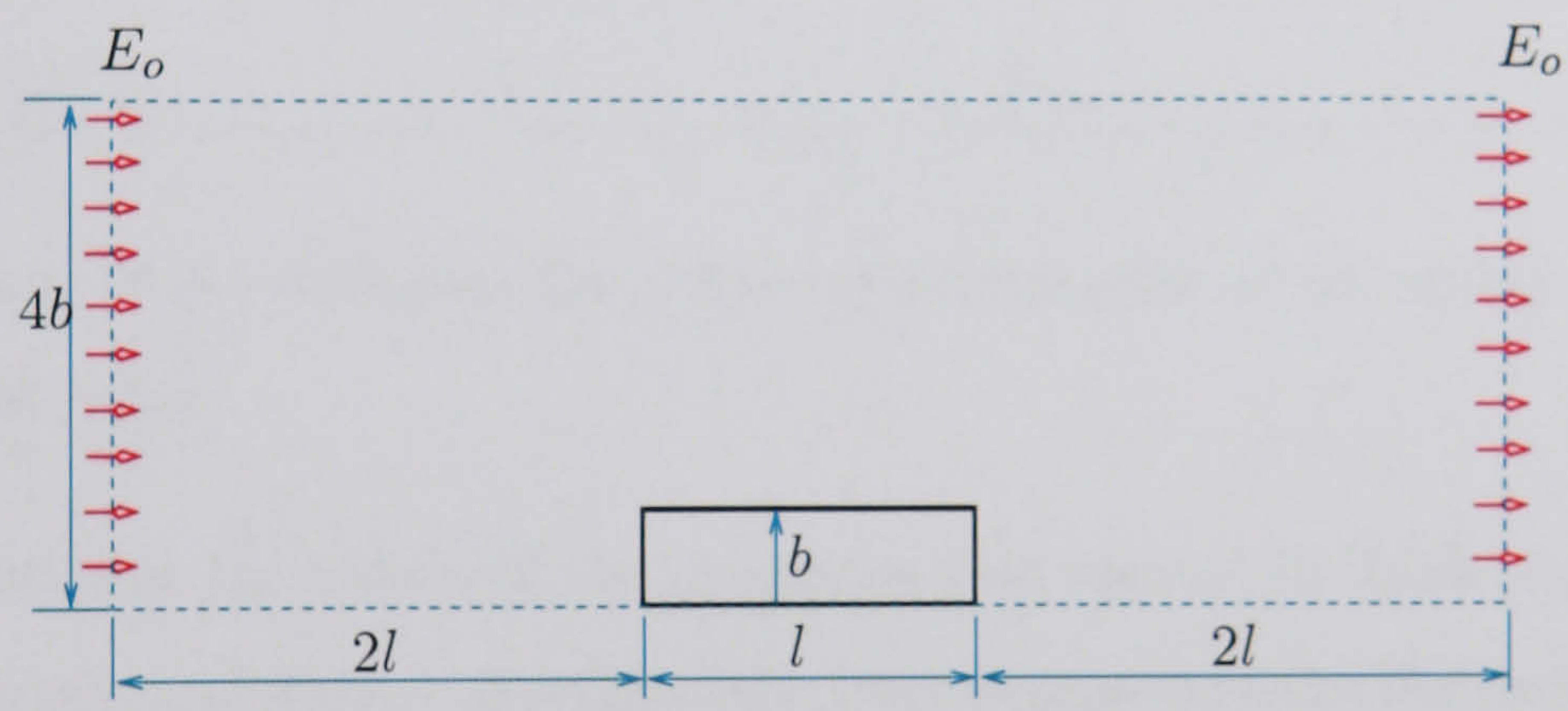


Figure 7.2: Two-dimensional simplification of the problem of a cylinder under uniform extension.

The boundary condition $(7.132)_2$ is satisfied if the radial component of the electric displacement is continuous across $r = b$, and if the axial component of the electric displacement is continuous across the ends of the cylinder $z = 0$ and $z = l$.

7.2.3 A first approximation for the energy function

In the introduction (Chapter 1) was mentioned that for ES elastomers there is very little experimental data; unlike for MS elastomers where the papers by Bellan and Bossis [7], and Ginder et al. [50] provided valuable information in order to propose a first approximation for the energy function, for ES elastomers it has not been possible to find similar data.

For this reason, and only as a first illustration we use an energy function of the form (see a similar problem in [119])

$$\Omega = \frac{1}{2}(I_1 - 3)(g_0 + g_1 I_4) - \ln\left[\cosh\left(\frac{\sqrt{I_4}}{m_1}\right)\right] m_1 m_0 - \frac{I_4}{2}\zeta_0 + \nu_0. \tag{7.133}$$

The above form for the energy function has been adopted from Section 5.3 (equation (5.245)) for the isotropic case, not considering the dependence in the fifth invariant I_5.

The derivative in I_4 is

$$\frac{\partial \Omega}{\partial I_4} = \frac{1}{2}(I_1 - 3)g_1 - \frac{1}{2}\tanh\left(\frac{\sqrt{I_4}}{m_1}\right)\frac{m_0}{\sqrt{I_4}} - \frac{\zeta_0}{2}. \tag{7.134}$$

The values of the different constants that appear in (7.134) are given in Table 7.1. The electric permittivity for free space is $\varepsilon_o = 8.8419 * 10^{-12}$[Coul^2/Nm^2].

g_1	10^{-3}[Coul^2/Nm^2]
m_1	10^4[V/m]
m_0	5*10^{-11} [Coul^2/Nm^2]
ζ_0	ε_o

Table 7.1: Values of the constants for a first approximation of an energy function for an isotropic ES elastomer.

We point out that the values of the constants that appear in Table 7.1 have not been found from experimental data, but rather have been suggested from the probable behaviour of the polarization as a function of the electric field (Figure 2.5), where we have assumed a value for the polarization for the saturation point 'close' in magnitude to the value of the electric permeability for free space.

7.2.4 Results

The equations (7.130) and (7.131) are solved using the finite difference method. The equation (7.130) is non-linear, because in general $\mathbf{C}$, which depends on Ω, is a function of the solution φ. We use the same method as was presented in Subsection 4.2.4, and so, for brevity, we do not repeat it here (note that for the problem solved in Section 4.2 we also used cylindrical coordinates).

Figure 7.2 shows the axil-symmetric simplification used in this problem. For definiteness the surrounding free space (which in theory is infinite) was assumed of a length five times the length of the cylinder and four times the radius of the cylinder. The external electric field is applied far away, as shown in Figure 7.2.

We consider three aspect ratios l/b equal 4, 8 and 12, and we work with three different values for the stretch λ of 1.3, 1.5 and 2.

Figures 7.3 and 7.4 show the results for the axial and radial component of the electric field, as functions of the coordinates r, z, for the region shown in the lower middle part of Figure 7.2. The origin of the system of coordinates for these two figures is located on the inferior corner of the left side of the axisymmetric representation of the cylinder. These results were obtained for $\lambda = 1.5$ and $l/b = 4$.

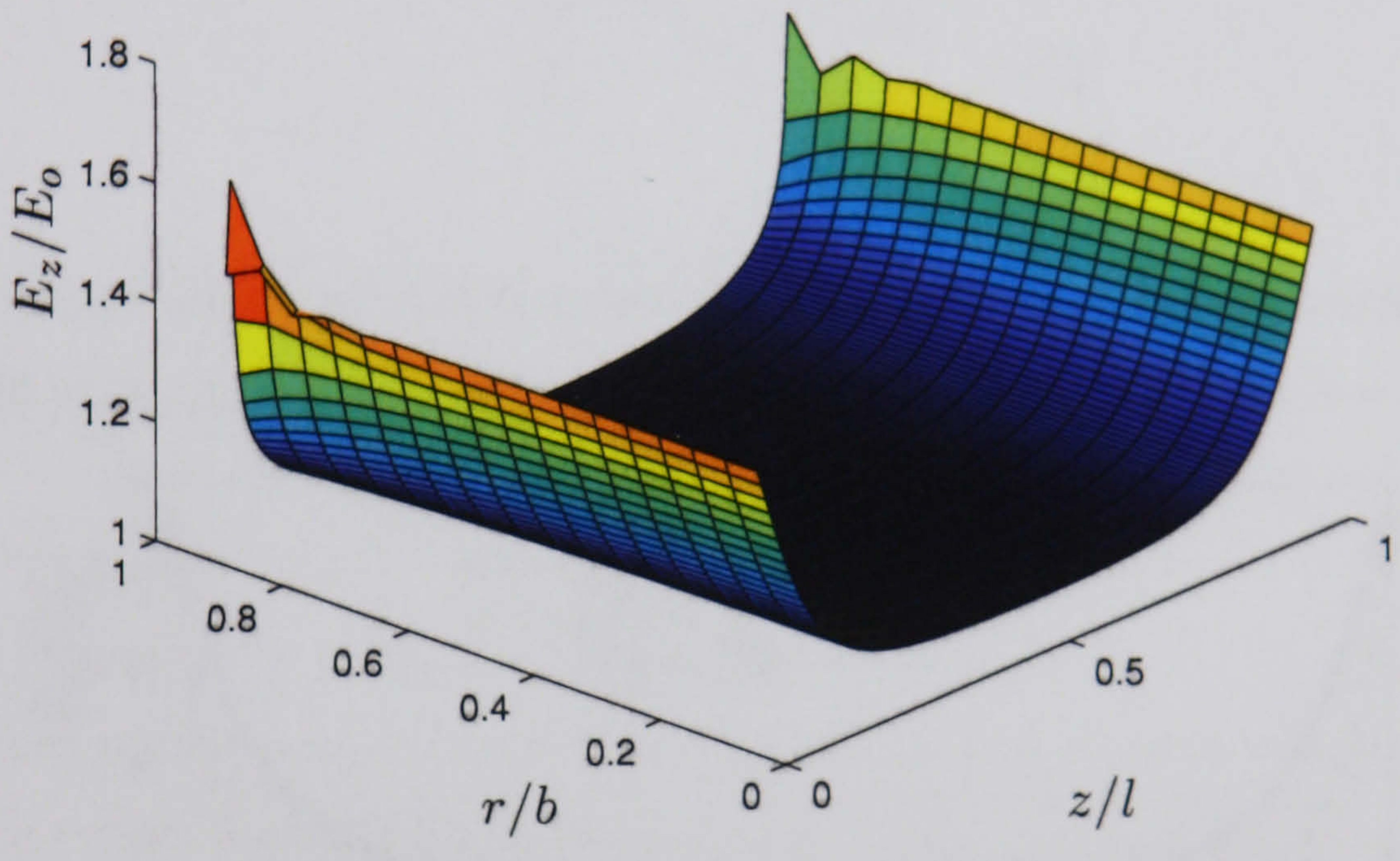


Figure 7.3: Magnitude of the dimensionless axial component of the electric field through the tube wall in an arbitrary (r, z) plane for aspect ratio $l/b = 4$ and for $\lambda = 1.5$.

As we can see from Figures 7.3 and 7.4, far away from the cylinder the field is almost uniform; the same phenomena was observed for a tube made of MS elastomer and under extension and inflation (see Section 4.2 and [15]).

In Figure 7.5 we have the comparison of the behaviour of the axial and radial components of the electric field, for three different aspect ratios l/b, as functions of the axial position z. In this case the graph was drawn for a line that passes through $r = b/2$ (see Figure 7.2), these results were obtained for $\lambda = 1.5$.

From Figure 7.5 we can see that for larger aspect ratios l/b, the change in the components of the field is located in a narrower zone near the ends of the cylinder (see Figure 4.19 for the magnetic counterpart of these results, for the case of the tube).

In Figure 7.6 we have the behaviour of the radial component of the electric field, and

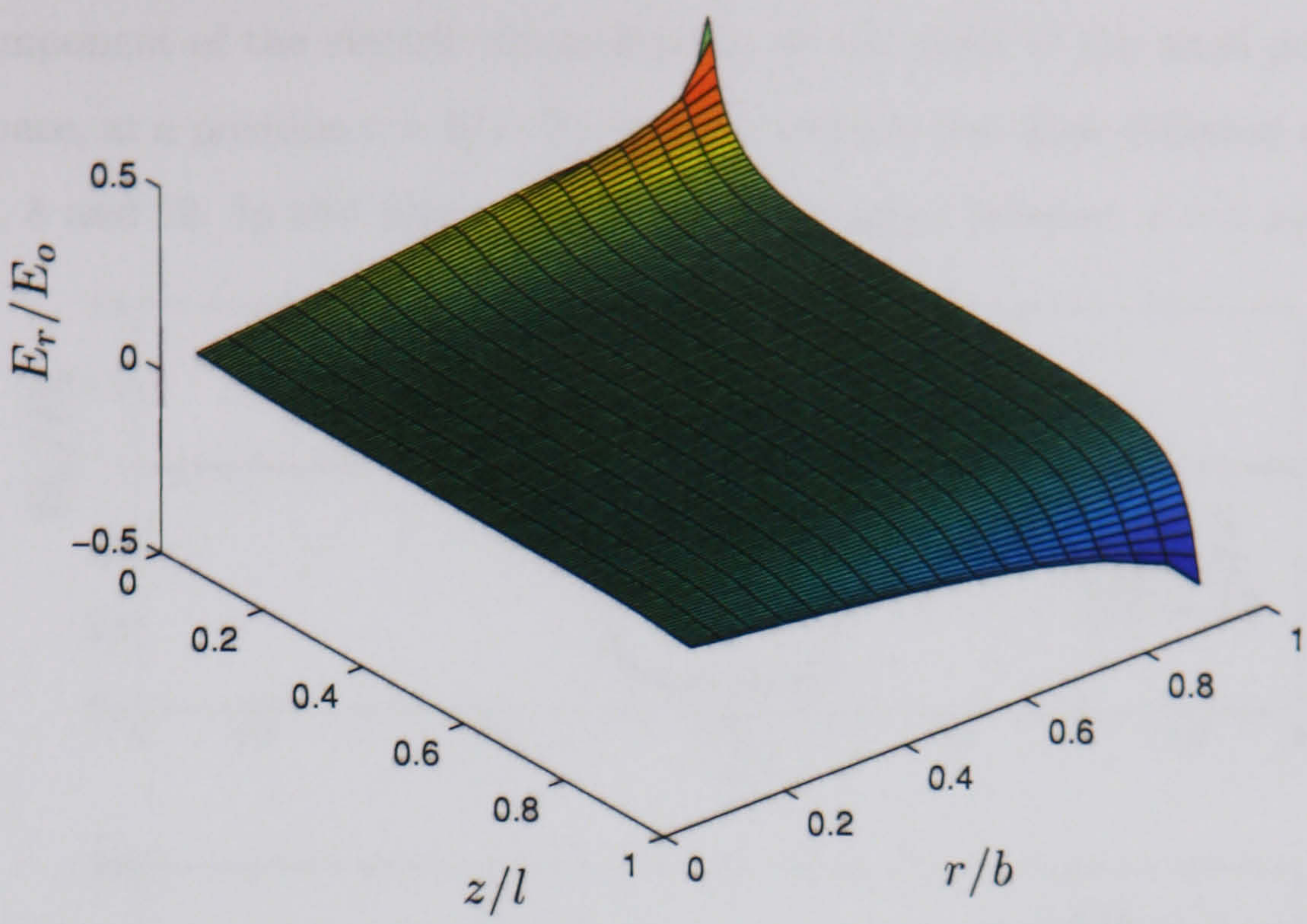


Figure 7.4: Magnitude of the dimensionless radial component of the electric field through the tube wall in an arbitrary (r, z) plane for aspect ratio $l/b = 4$ and for $\lambda = 1.5$.

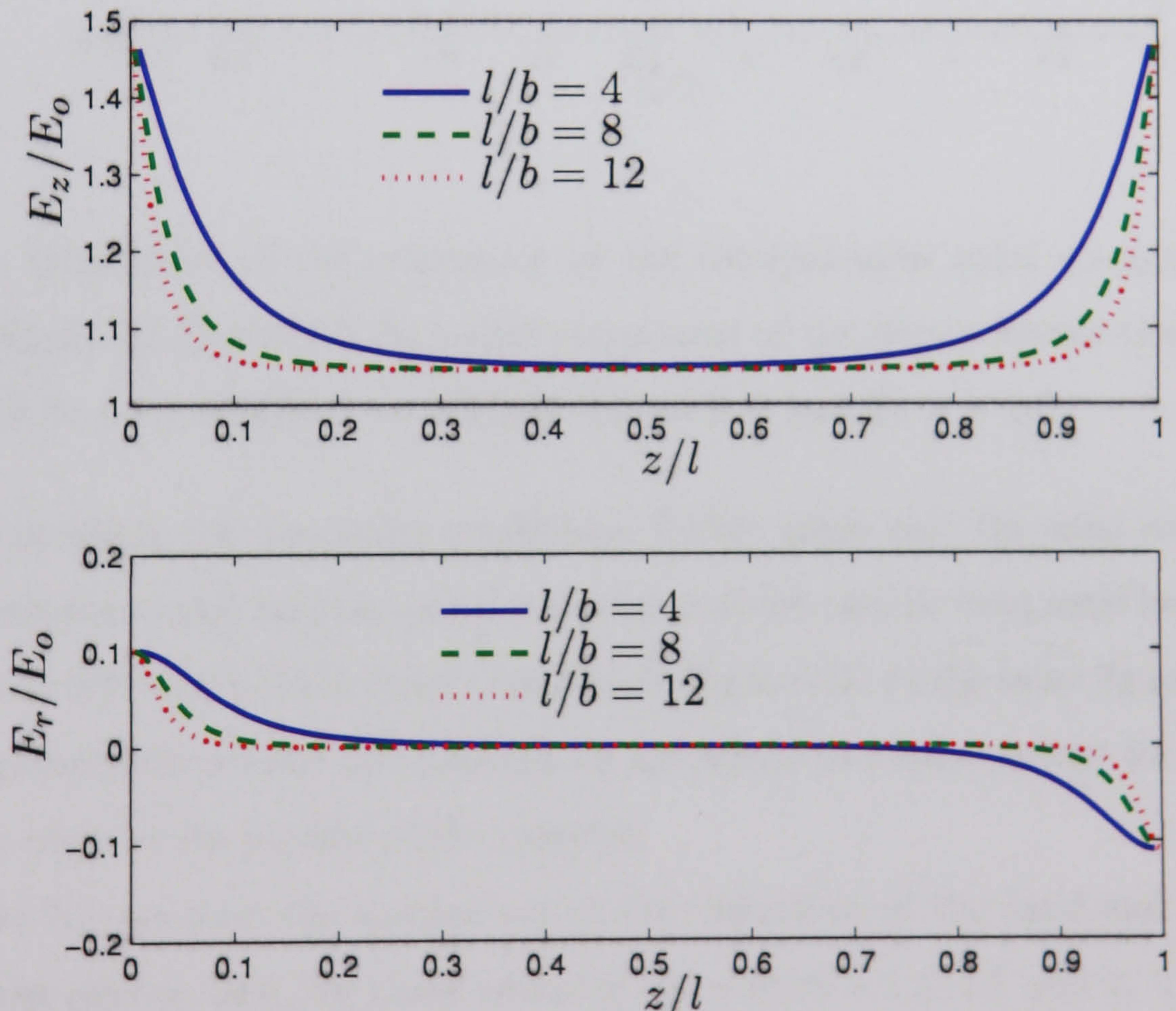


Figure 7.5: Magnitude of the dimensionless axial and radial components of the electric field E_i/E_o, $i = r, z$, along the axial direction for aspect ratios $l/b = 4, 8, 12$ for a location $r = b/2$ (see Figure 7.2) and for $\lambda = 1.5$.

the axial component of the electric displacement, as functions of the axial position z, for the whole space, at a position $r = b/2$, for $\lambda = 1.5$, and for the three different aspect ratios l/b equals 4, 8 and 12. In this figure the cylinder is located between $z = 2$ and $z = 3$.

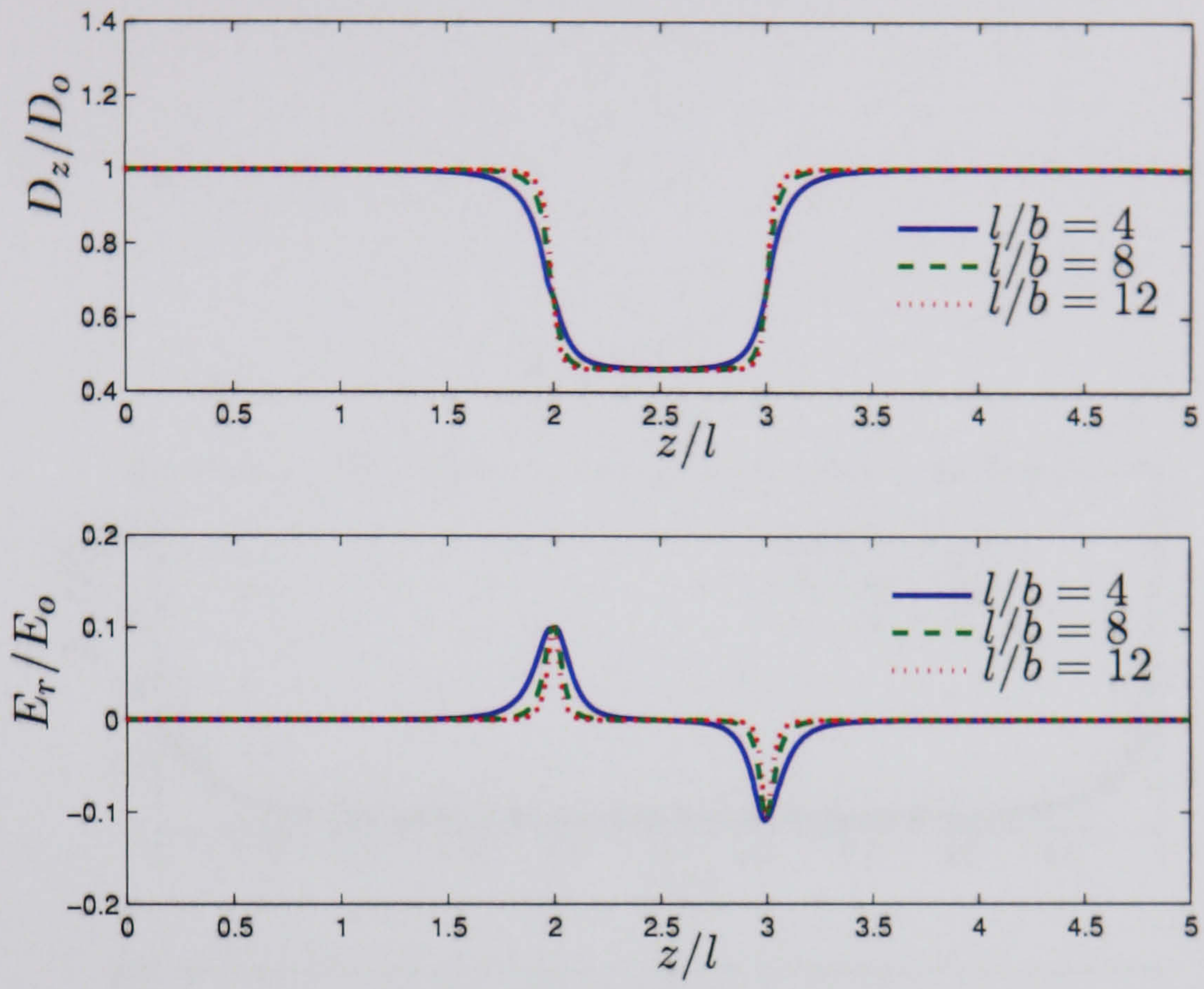


Figure 7.6: Illustration of the continuity of the dimensionless axial component of the electric displacement **D** and on the radial component of the dimensionless electric field **E** for $l/a = 4, 8, 12$ for a location $r = b/2$ (see Figure 7.2) and for $\lambda = 1.5$.

For this problem, the continuity conditions (7.132) imply that the axial component of the electric displacement, and the radial component of the electric field must be continuous for the line $r = b/2$, which is the result observed in Figure 7.6. In the same figure we can see that the external field is axial and uniform for the whole free space except for a relatively small region close to the surface of the cylinder.

In Figure 7.7, we have the comparison of the behaviour of the axial and radial components of the electric field, for three values of the stretch λ 1.3, 1.5 and 2. These results show the behaviour of the field only for the cylinder, for the line $r = b/2$, and for $l/b = 4$.

From this figure we see that for a larger value of λ the change in the field if restricted to a relative narrower zone near the ends of the cylinder, but the change in the magnitude of the field becomes more abrupt.

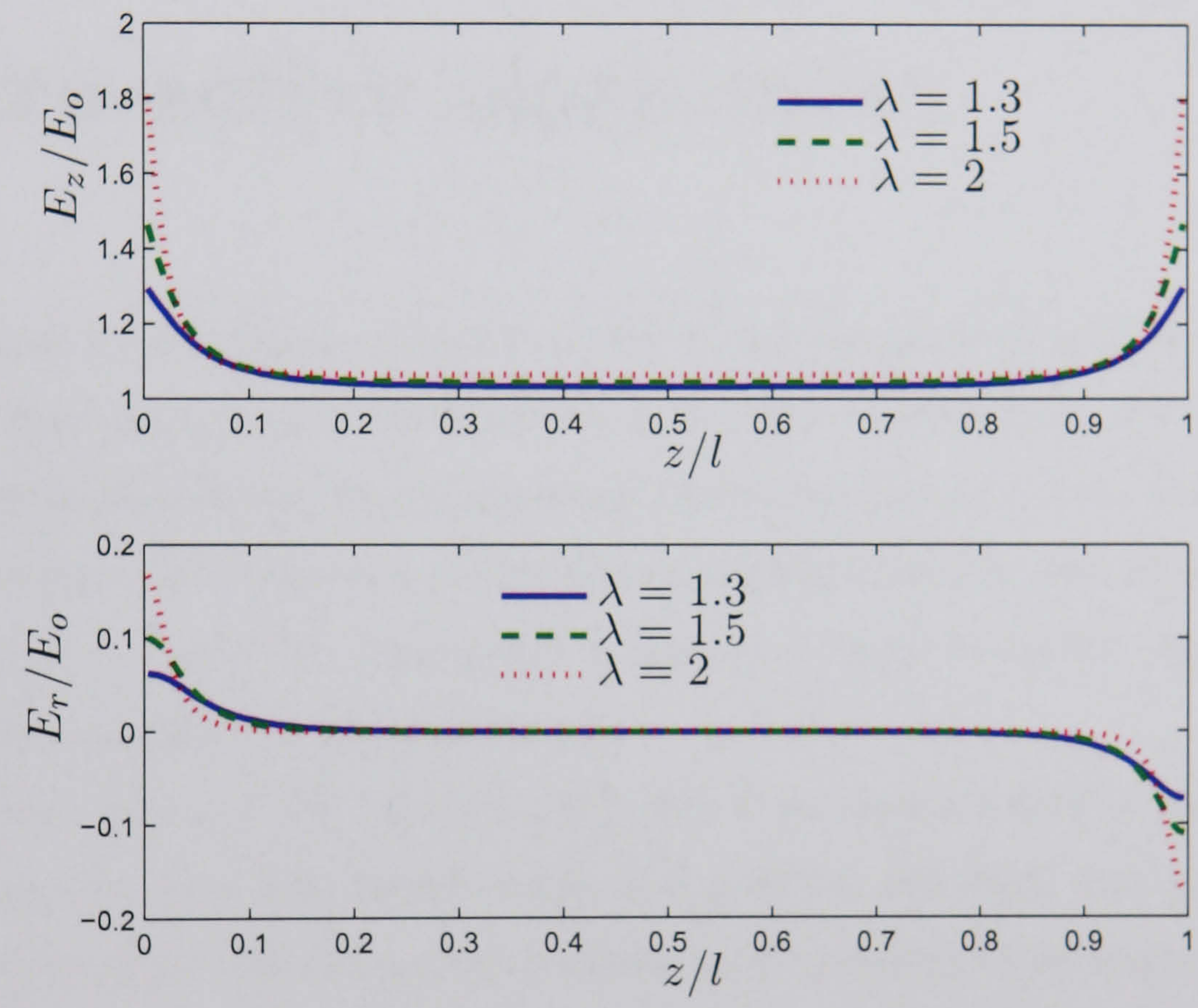


Figure 7.7: Variation of the dimensionless axial and radial components of the dimensionless electric field in the cylinder for different values of λ.

Chapter 8

Transversely isotropic electro-active elastomers

In this chapter we provide the theoretical basis of the non-linear properties for a special class of ES elastomers, namely the transversely isotropic electro-active elastomers, whose characteristic is that during the curing process due to the presence of an external applied field, the electro-active particles are aligned in a preferred direction (like with its magnetic counterpart, see Chapter 5). The theory is applied to some boundary value problems. Some universal relations are obtained as well.

It has been shown [7, 11] (see also [59]) that if an external field is applied during the curing process, then the particles align in a preferred direction, and as a result the capability of the ES or MS elastomers to deform in the presence of an external electric or magnetic field is enhanced significantly, in comparison with the same kind of material but with a random distribution of particles in the rubber-like matrix.

Dorfmann and Ogden have been working with isotropic electro- or magneto-active elastomers, which basically implies the assumption of a random distribution of the particles inside the rubber-like matrix material [31, 32, 34, 36]. For this particular case several boundary value problems have been solved. The complete set of controllable or universal solutions is also available [99].

The situation is not the same for the transversely isotropic electro-active elastomers. The general theory developed, for example, by Eringen and Maugin [42], or by Dorfmann and Ogden [32, 36] are good starting points in order to study this problem.

Using as a starting point the work by Dorfmann and Ogden [32] (Chapter 6), we develop the constitutive equations for transversely isotropic electro-elastic elastomers. In Subsection 8.1.1 we study the form of the constitutive equations using the electric field as the independent electric variable. In Subsection 8.1.2 an equivalent set of equations is found assuming the electric-displacement as the independent electric variable.

Most of the researches on electro-elasticity have focused mainly on the linear theory, which means the assumption of small deformations, displacement and electric field. The departure from the non-linear formulation to the linear one can be found for example in [42] and [121]. However, this process of linear approximation has been formulated starting from the general expression for the stress and the independent electric variable as derivatives of the energy function. In Section 8.2 we obtain a linear approximation in a different way. First, for the full non-linear formulation we compute the stress and the independent electric variable as functions of the invariants, and then, we approximate the expressions, thereby obtaining the same kind of linear constitutive equation as, for example, for some well known piezoelectric materials such as certain polarized ceramics (see, for example, [121]).

In Section 8.3 we study different boundary value problems. The simple shear, the uniform extension of a bar, the extension and inflation of a tube, the extension and torsion of a tube and helical shear. For some of them we study the effect of assuming different alignments for the particles in the reference configuration; we are especially interested in the controllability of the solutions. In the particular case of helical shear [79], which is not a controllable solution, we are interested mainly in finding a non-linear universal relation, like the one found for the isotropic case by Bustamante and Ogden [21] (see Subsections 4.1.2.1 and 7.1.2).

Finally, in Section 8.4, by recognizing that the experimental research with these kind of materials would be especially difficult in order to find realistic models for the energy function, we explore the situation of assuming a simplified form for the constitutive equation, and we obtain a linear universal relation for this particular simplified form of the energy function.

This chapter is based in the on a draft paper by Bustamante and Ogden [19].

8.1 Constitutive equations for transversely isotropic ES materials

8.1.1 The electric field as the independent electric variable

Consider the constitutive equations for a Green elastic electro-active solid in terms of the independent electric variable $\mathbf{E}_l$, which for a compressible and an incompressible material were given respectively by (6.49) and (7.2):

$$\tau = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{D}_l = -\frac{\partial\Omega}{\partial\mathbf{E}_l}, \tag{8.1}$$

$$\tau = \mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{I}. \tag{8.2}$$

We had the connections (6.4)

$$\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}, \quad \mathbf{E}_l = \mathbf{F}^T\mathbf{E}. \tag{8.3}$$

Consider now the case of a transversely isotropic electro-elastic solid where the energy function is given as

$$\Omega = \Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{a}_0), \quad |\mathbf{a}_0| = 1, \tag{8.4}$$

where $\mathbf{a}_0$ is a field that represents the particular alignment of the electro-active particles in the reference configuration. In the current configuration we have

$$\mathbf{a} = \mathbf{F}\mathbf{a}_0. \tag{8.5}$$

For the energy function $\Omega = \Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{a}_0)$, we have that Ω depends in the following set of invariants [1]

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det\mathbf{c} = J^2, \tag{8.6}$$

$$I_4 = \mathbf{E}_l\cdot\mathbf{E}_l, \quad I_5 = \mathbf{E}_l\cdot\mathbf{c}\mathbf{E}_l, \quad I_6 = \mathbf{E}_l\cdot\mathbf{c}^2\mathbf{E}_l, \tag{8.7}$$

$$I_7 = \mathbf{a}_0\cdot\mathbf{c}\mathbf{a}_0, \quad I_8 = \mathbf{a}_0\cdot\mathbf{c}^2\mathbf{a}_0, \tag{8.8}$$

$$I_9 = \mathbf{a}_0\cdot\mathbf{E}_l, \quad I_{10} = \mathbf{a}_0\cdot\mathbf{c}\mathbf{E}_l. \tag{8.9}$$

[1] In this case we use only 10 invariants, as mentioned for the magnetoelastic case (Subsection 5.1.1). there is an error in Zheng's paper in the theory of invariants [127], see Appendix B.

Then, $\Omega = \Omega(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10})$. Consider the derivatives

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{F}\mathbf{F}^T), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}. \tag{8.10}$$

$$\frac{\partial I_5}{\partial \mathbf{F}} = 2\mathbf{E}_l \otimes \mathbf{F}\mathbf{E}_l, \quad \frac{\partial I_6}{\partial \mathbf{F}} = 2(\mathbf{E}_l \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{E}_l) + \mathbf{F}^T\mathbf{F}\mathbf{E}_l \otimes \mathbf{F}\mathbf{E}_l. \tag{8.11}$$

$$\frac{\partial I_7}{\partial \mathbf{F}} = 2\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0, \quad \frac{\partial I_8}{\partial \mathbf{F}} = 2(\mathbf{a}_0 \otimes \mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{a}_0 + \mathbf{F}^T\mathbf{F}\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0), \tag{8.12}$$

$$\frac{\partial I_{10}}{\partial \mathbf{F}} = \mathbf{a}_0 \otimes \mathbf{F}\mathbf{E}_l + \mathbf{E}_l \otimes \mathbf{F}\mathbf{a}_0. \tag{8.13}$$

Using the chain rule and these derivatives in $(8.1)_1$, the total stress tensor is given as[2]

$$\begin{aligned}\boldsymbol{\tau} = {} & J^{-1}[2\mathbf{b}\Omega_1 + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 + 2I_3\mathbf{I}\Omega_3 + 2\mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E}\Omega_5 \\ & +2(\mathbf{b}\mathbf{E} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{b}\mathbf{E})\Omega_6 + 2\mathbf{a} \otimes \mathbf{a}\Omega_7 \\ & +2(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a} \otimes \mathbf{a})\Omega_8 + (\mathbf{a} \otimes \mathbf{b}\mathbf{E} + \mathbf{b}\mathbf{E} \otimes \mathbf{a})\Omega_{10}].\end{aligned} \tag{8.14}$$

For an incompressible material (8.2) we have

$$\begin{aligned}\boldsymbol{\tau} = {} & 2\mathbf{b}\Omega_1 + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 - p\mathbf{I} + 2\mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E}\Omega_5 \\ & +2(\mathbf{b}\mathbf{E} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{b}\mathbf{E})\Omega_6 + 2\mathbf{a} \otimes \mathbf{a}\Omega_7 \\ & +2(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a} \otimes \mathbf{a})\Omega_8 + (\mathbf{a} \otimes \mathbf{b}\mathbf{E} + \mathbf{b}\mathbf{E} \otimes \mathbf{a})\Omega_{10}.\end{aligned} \tag{8.15}$$

Consider the derivatives of the invariants in $\mathbf{E}_l$

$$\frac{\partial I_4}{\partial \mathbf{E}_l} = 2\mathbf{E}_l, \quad \frac{\partial I_5}{\partial \mathbf{E}_l} = 2\mathbf{c}\mathbf{E}_l, \quad \frac{\partial I_6}{\partial \mathbf{E}_l} = 2\mathbf{c}^2\mathbf{E}_l, \quad \frac{\partial I_9}{\partial \mathbf{E}_l} = \mathbf{a}_0, \quad \frac{\partial I_{10}}{\partial \mathbf{E}_l} = \mathbf{c}\mathbf{a}_0, \tag{8.16}$$

and so from $(8.1)_2$ with the chain rule we have

$$\mathbf{D} = -J^{-1}(2\mathbf{b}\mathbf{E}\Omega_4 + 2\mathbf{b}^2\mathbf{E}\Omega_5 + 2\mathbf{b}^3\mathbf{E}\Omega_6 + \mathbf{a}\Omega_9 + \mathbf{b}\mathbf{a}\Omega_{10}). \tag{8.17}$$

And for an incompressible material ($J = 1$)

$$\mathbf{D} = -(2\mathbf{b}\mathbf{E}\Omega_4 + 2\mathbf{b}^2\mathbf{E}\Omega_5 + 2\mathbf{b}^3\mathbf{E}\Omega_6 + \mathbf{a}\Omega_9 + \mathbf{b}\mathbf{a}\Omega_{10}). \tag{8.18}$$

As for the magnetoelastic case (Chapter 5), some restrictions on the energy function Ω can be obtained by considering the undeformed state.

If for the undeformed state with no external electric field there is no residual stresses and no residual polarization, then we have

$$\boldsymbol{\tau} = \mathbf{0}, \quad \mathbf{D} = \mathbf{0}. \tag{8.19}$$

[2]We use the notation $\Omega_i = \frac{\partial \Omega}{\partial I_i}$.

In this case the invariants (8.6)-(8.9) are given by

$$I_1 = I_2 = 3, \quad I_3 = 1, \quad I_4 = I_5 = I_6 = I_9 = I_{10} = 0, \quad I_7 = I_8 = \mathbf{a}_0 \cdot \mathbf{a}_0. \tag{8.20}$$

Let's denote by $\bar{\Omega}_i$ the function Ω_i evaluated with the above values. Remembering that $\mathbf{F} = \mathbf{I}$, $\mathbf{E}_l = \mathbf{0}$ and $\mathbf{a} = \mathbf{F}\mathbf{a}_0 = \mathbf{a}_0$, then (8.14) and (8.17) become

$$\tau = 2(\bar{\Omega}_1 + 2\bar{\Omega}_2 + \bar{\Omega}_3)\mathbf{I} + 2(\bar{\Omega}_7 + 2\bar{\Omega}_8)\mathbf{a}_0 \otimes \mathbf{a}_0. \tag{8.21}$$

$$\mathbf{D} = -(\bar{\Omega}_9 + \bar{\Omega}_{10})\mathbf{a}_0, \tag{8.22}$$

and in view of (8.19) we need

$$\bar{\Omega}_1 + 2\bar{\Omega}_2 + \bar{\Omega}_3 = 0, \tag{8.23}$$

$$\bar{\Omega}_7 + 2\bar{\Omega}_8 = 0, \tag{8.24}$$

$$\bar{\Omega}_9 + \bar{\Omega}_{10} = 0. \tag{8.25}$$

In the incompressible case (8.23) should be replaced by

$$2\bar{\Omega}_1 + 4\bar{\Omega}_2 - p = 0. \tag{8.26}$$

Piezoelectric materials produce polarization when deformed even when there is no external field [67, 121]; the reason why some materials like quartz produces a polarization field when deformed lies with its particular atomic structure; a deformation produces an asymmetric arrangement of charges, creating this polarization field. We cannot expect in general the same phenomenon for transversely isotropic ES materials.

Consider the case when there is deformation but no applied external field, in such a case if $\mathbf{E} = \mathbf{0}$ we have the extra restriction $\mathbf{D} = \mathbf{0}$. As in the magnetoelastic case (see remark at the final of Subsection 5.1.1), let $\breve{\Omega}_i$ denotes the function Ω_i evaluated for $I_4 = I_5 = I_6 = I_9 = I_{10} = 0$ (these values for the invariants are consequence of $\mathbf{E} = 0$). With $\mathbf{D} = \mathbf{0}$ from (8.17) we have the restriction

$$\mathbf{I}\breve{\Omega}_9 + \mathbf{b}\breve{\Omega}_{10} = \mathbf{0}, \tag{8.27}$$

which, if $\mathbf{b} \neq \mathbf{I}$, implies

$$\breve{\Omega}_9 = \breve{\Omega}_{10} = 0. \tag{8.28}$$

8.1.2 The electric displacement as the independent electric variable

If we choose to work with $\mathbf{D}_l$ as the independent variable, then by defining the energy potential Ω^* by using the partial Legrende transform

$$\Omega^*(\mathbf{F}, \mathbf{D}_l, \mathbf{a}_0) = \Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{a}_0) + \mathbf{D}_l \cdot \mathbf{E}_l, \tag{8.29}$$

it follows that

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{E}_l = \frac{\partial \Omega^*}{\partial \mathbf{D}_l}. \tag{8.30}$$

For an incompressible material we have

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}. \tag{8.31}$$

Consider the following set of invariants [102, 127]:

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det \mathbf{c}. \tag{8.32}$$

$$K_4 = \mathbf{D}_l \cdot \mathbf{D}_l, \quad K_5 = \mathbf{D}_l \cdot \mathbf{c}\mathbf{D}_l, \quad K_6 = \mathbf{D}_l \cdot \mathbf{c}^2\mathbf{D}_l, \tag{8.33}$$

$$I_7 = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{a}_0, \quad I_8 = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{a}_0, \tag{8.34}$$

$$K_9 = \mathbf{a}_0 \cdot \mathbf{D}_l, \quad K_{10} = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{D}_l. \tag{8.35}$$

Using a similar procedure as for Ω, the derivative $\frac{\partial \Omega^*}{\partial \mathbf{F}}$ can be calculated using the invariants I_i and K_i defined as above, in which case the expression for the total stress $(8.30)_1$ becomes[3]

$$\begin{aligned}\boldsymbol{\tau} = {} & J^{-1}[2\mathbf{b}\Omega_1^* + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* + 2I_3\mathbf{I}\Omega_3^* + 2J^2\mathbf{D}\otimes\mathbf{D}\Omega_5^* \\ & +2J^2(\mathbf{D}\otimes\mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D}\otimes\mathbf{D})\Omega_6^* + 2\mathbf{a}\otimes\mathbf{a}\Omega_7^* \\ & +2(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a}\otimes\mathbf{a})\Omega_8^* + J(\mathbf{a}\otimes\mathbf{D} + \mathbf{D}\otimes\mathbf{a})\Omega_{10}^*],\end{aligned} \tag{8.36}$$

where the connections $\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}$ and $\mathbf{a}_0 = \mathbf{F}^{-1}\mathbf{a}$ have been used. The corresponding expression for the incompressible case (8.31) is

$$\begin{aligned}\boldsymbol{\tau} = {} & 2\mathbf{b}\Omega_1^* + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* - p^*\mathbf{I} + 2\mathbf{D}\otimes\mathbf{D}\Omega_5^* \\ & +2(\mathbf{D}\otimes\mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D}\otimes\mathbf{D})\Omega_6^* + 2\mathbf{a}\otimes\mathbf{a}\Omega_7^* \\ & +2(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a}\otimes\mathbf{a})\Omega_8^* + (\mathbf{a}\otimes\mathbf{D} + \mathbf{D}\otimes\mathbf{a})\Omega_{10}^*.\end{aligned} \tag{8.37}$$

Finally, the expression for the electric field $(8.30)_2$ becomes

$$\mathbf{E} = 2J\mathbf{b}^{-1}\mathbf{D}\Omega_4^* + 2J\mathbf{D}\Omega_5^* + 2J\mathbf{b}\mathbf{D}\Omega_6^* + \mathbf{b}^{-1}\mathbf{a}\Omega_9^* + \mathbf{a}\Omega_{10}^*, \tag{8.38}$$

and the corresponding incompressible case is

$$\mathbf{E} = 2\mathbf{b}^{-1}\mathbf{D}\Omega_4^* + 2\mathbf{D}\Omega_5^* + 2\mathbf{b}\mathbf{D}\Omega_6^* + \mathbf{b}^{-1}\mathbf{a}\Omega_9^* + \mathbf{a}\Omega_{10}^*. \tag{8.39}$$

As in Subsection 8.1.1 we can find some restrictions on the form of the energy function if we assume that for the case when there is no deformation or external electric displacement,

[3] The notation Ω_i^* means the derivative of Ω^* in I_i if $i = 1, 2, 3, 7, 8$, or K_i if $i = 4, 5, 6, 9, 10$.

there is no residual stresses and residual electric field. Let $\bar{\Omega}_i^*$ denotes the function Ω_i^* evaluated with the invariants (8.32)-(8.35) calculated using $\mathbf{F} = \mathbf{I}$ and $\mathbf{D}_l = \mathbf{0}$. From (8.36) and (8.38) the conditions $\boldsymbol{\tau} = \mathbf{0}$ and $\mathbf{E} = 0$ imply

$$\bar{\Omega}_1^* + 2\bar{\Omega}_2^* + \bar{\Omega}_3^* = 0, \tag{8.40}$$

$$\bar{\Omega}_7^* + 2\bar{\Omega}_8^* = 0, \tag{8.41}$$

$$\bar{\Omega}_9^* + \bar{\Omega}_{10}^* = 0. \tag{8.42}$$

In the incompressible case (8.40) should be replaced by

$$2\bar{\Omega}_1^* + 4\bar{\Omega}_2^* - p = 0. \tag{8.43}$$

A different restriction (more general in a way) can be found if we assume now that whenever we have deformation but no external electric displacement then from (8.38) the electric field is zero. Let $\breve{\Omega}_i^*$ denotes the function Ω_i^* evaluated for $\mathbf{D}_l = \mathbf{0}$ but with $\mathbf{F}$ in general different to the identity tensor. From (8.38) we have

$$\mathbf{b}^{-1}\breve{\Omega}_9^* + \mathbf{I}\breve{\Omega}_{10}^* = \mathbf{0}, \tag{8.44}$$

which, if $\mathbf{b}^{-1} \neq \mathbf{I}$, implies

$$\breve{\Omega}_9^* = \breve{\Omega}_{10}^* = 0. \tag{8.45}$$

8.2 Derivation of the equations for the linear elastic case

To develop a linear theory through a linear expansion from the non-linear general formulation is a standard procedure. However, in electro-elasticity that has been made mainly by expanding directly, for example, the expressions (8.1) as Taylor series in $\mathbf{c}$ (instead of $\mathbf{F}$) and $\mathbf{E}_l$ (see, for example, [42, 106, 114, 121]). We have not found in the literature a linear approximate expression from, for example, (8.14) and (8.17), which would relate directly the different parameters and quantities that appear in the general non-linear formulation, and the parameters that appear in the classical linear theory; this is the reason this problem has been studied in this thesis.

The main assumption in order to obtain a linear approximation from the general non-linear constitutive equations, is to consider that the gradient of the displacement (2.8) and the external electric field are 'small', more precisely

$$|\mathrm{Grad}\,\mathbf{u}| \ll 1, \quad |\mathbf{E}_l| \ll 1. \tag{8.46}$$

It is not problematic to define what we means for 'small' in the case of the gradient of the displacement. However, the situation is more complicated for the electric field. In such a case, for small do we mean a field which is less than a given value in order to be able to approximate some non-linear expressions?. The concept of 'small' is relative, in the case of the gradient of the displacement this is not a problem because this gradient of the displacement is dimensionless. For the electric field we need to define the 'smallness' of the electric field $\mathbf{E}_l$ with respect to some 'reference value' for the field. Let's denote this value E_R, then the inequality $(8.46)_2$ should be understood as $|\mathbf{E}_l|/E_R \ll 1$ or $|\mathbf{E}_l| \ll E_R$. There is another remark; E_R has units of electric field, which is Volt per metre. When we propose a form for the function Ω, we may use in the formulation, for example, trigonometric functions; but from the point of view of the physical dimensions we cannot have an expression like $\tan(|\mathbf{E}_l|)$, we need in fact to divide $|\mathbf{E}_l|$ by a scalar with the same physical dimensions of the electric field, such that the tangent function can be evaluated with a dimensionless parameter (see, for example, the energy function (5.245) for the magnetoelastic problem, where we have the parameters m_1 and m_2.). For this section then we assume that $\mathbf{E}_l$ has been divided by E_R, and so $|\mathbf{E}_l| \ll 1$ actually means $|\mathbf{E}_l| \ll E_R$. We do not use a different notation for this dimensionless electric field. As for E_R, it might seem to be an arbitrary value, but in fact it should have a physical meaning related, for example, to the behaviour of the polarization, as in the magnetoelastic case with the parameters m_1 and m_2 in (5.245). Since we do not have enough experimental data for ES elastomers we do not discuss further about E_R.

Departing from the equations (8.14) and (8.17) is complex, and in order to avoid confusion the linearization process is done in three steps.

First step: first we determine the approximation of $\mathbf{b}$, $\mathbf{b}^2$, $\mathbf{bE}$ and $\mathbf{b}^2\mathbf{E}$. From the definitions $(2.17)_2$, $(6.4)_2$, and (2.11) we have

$$\begin{aligned} \mathbf{b} &= \mathbf{F}\mathbf{F}^T = \mathbf{I} + \mathrm{Grad}\,\mathbf{u} + \mathrm{Grad}\,\mathbf{u}^T + \mathrm{Grad}\,\mathbf{u}\mathrm{Grad}\,\mathbf{u}^T \\ &\approx \mathbf{I} + \mathrm{Grad}\,\mathbf{u} + \mathrm{Grad}\,\mathbf{u}^T, \qquad (8.47) \\ \mathbf{E}_l &= \mathbf{F}^T\mathbf{E} = \mathbf{E} + \mathrm{Grad}\,\mathbf{u}^T\mathbf{E} \approx \mathbf{E} \qquad (8.48) \\ \mathbf{bE} &= \mathbf{E} + (\mathrm{Grad}\,\mathbf{u} + \mathrm{Grad}\,\mathbf{u}^T + \mathrm{Grad}\,\mathbf{u}\mathrm{Grad}\,\mathbf{u}^T)\mathbf{E} \approx \mathbf{E}. \qquad (8.49) \end{aligned}$$

The linear deformation tensor $\mathbf{e}$ is defined as [52]

$$\mathbf{e} \equiv \frac{1}{2}(\mathrm{Grad}\,\mathbf{u} + \mathrm{Grad}\,\mathbf{u}^T). \qquad (8.50)$$

Then we have

$$\mathbf{b} \approx \mathbf{I} + 2\mathbf{e}. \tag{8.51}$$

As well as this, it is not difficult to show that

$$\mathbf{b}^2 \approx \mathbf{I} + 4\mathbf{e}, \quad \mathbf{b}^2\mathbf{E} \approx \mathbf{E}. \tag{8.52}$$

Finally we also have

$$I_3 = J^2 \approx 1. \tag{8.53}$$

Using the fact that $|\mathbf{b}| \sim \delta$ and $|\mathbf{E}| \sim \delta$, where $\delta \ll 1$, neglecting the terms of order δ^2, for (8.14) we get

$$\begin{aligned}\boldsymbol{\tau} \approx\ & 2(\mathbf{I} + 2\mathbf{e})\Omega_1 + 2[I_1(\mathbf{I} + 2\mathbf{e}) - (\mathbf{I} + 4\mathbf{e})]\Omega_2 + 2\mathbf{I}\Omega_3 + 2\mathbf{a} \otimes \mathbf{a}\Omega_7 \\ & + 2[\mathbf{a} \otimes \mathbf{a}(\mathbf{I} + 2\mathbf{e}) + (\mathbf{I} + 2\mathbf{e})\mathbf{a} \otimes \mathbf{a}]\Omega_8 + [\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}]\Omega_{10};\end{aligned} \tag{8.54}$$

thus

$$\begin{aligned}\boldsymbol{\tau} \approx\ & 2[\Omega_1 + (I_1 - 1)\Omega_2 + \Omega_3]\mathbf{I} + 4[\Omega_1 + (I_1 - 2)\Omega_2]\mathbf{e} + 2[\Omega_7 + \Omega_8]\mathbf{a} \otimes \mathbf{a} \\ & + 4\Omega_8(\mathbf{a} \otimes \mathbf{ea} + \mathbf{ea} \otimes \mathbf{a}) + \Omega_{10}(\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}).\end{aligned} \tag{8.55}$$

For the electric displacement we have

$$\mathbf{D} \approx -(2[\Omega_4 + \Omega_5 + \Omega_6]\mathbf{E} + [\Omega_9 + \Omega_{10}]\mathbf{a} + 2\Omega_{10}\mathbf{ea}). \tag{8.56}$$

Second step: now we approximate Ω_i in the variables $\mathbf{F}$ and $\mathbf{E}$ (actually, we use the tensor $\mathbf{c}$ instead the gradient of deformation, which is the actual variable for Ω to be frame-indifferent). We have

$$\Omega_i = \bar{\Omega}_i + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{c}} : (\mathbf{c} - \mathbf{I}) + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{E}} \cdot \mathbf{E} + \dots,$$

where we remember that $\bar{f}$ means that the function f is evaluated at the reference configuration with zero electric field. Using the definition of the linear deformation tensor (8.50), we have the following approximation

$$\Omega_i \approx \bar{\Omega}_i + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{c}} : 2\mathbf{e} + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{E}} \cdot \mathbf{E}. \tag{8.57}$$

As a result, after neglecting the terms δ^2, we obtain for (8.55)

$$\begin{aligned}\boldsymbol{\tau} \approx\ & 2(\bar{\Omega}_1 + 2\bar{\Omega}_2 + \bar{\Omega}_3)\mathbf{I} + 4[(\bar{\boldsymbol{\Omega}}_{1c} + 2\bar{\boldsymbol{\Omega}}_{2c} + \bar{\boldsymbol{\Omega}}_{3c}) : \mathbf{e}]\mathbf{I} \\ & + 2[(\bar{\boldsymbol{\Omega}}_{1E} + 2\bar{\boldsymbol{\Omega}}_{2E} + \bar{\boldsymbol{\Omega}}_{3E}) \cdot \mathbf{E}]\mathbf{I} + 4(\bar{\Omega}_1 + \bar{\Omega}_2)\mathbf{e} \\ & + 2(\bar{\Omega}_7 + 2\bar{\Omega}_8)\mathbf{a} \otimes \mathbf{a} + 4[(\bar{\boldsymbol{\Omega}}_{7c} + 2\bar{\boldsymbol{\Omega}}_{8c}) : \mathbf{e}]\mathbf{a} \otimes \mathbf{a} \\ & + 2[(\bar{\boldsymbol{\Omega}}_{7E} + 2\bar{\boldsymbol{\Omega}}_{8E})]\mathbf{a} \otimes \mathbf{a} + 4\bar{\Omega}_8(\mathbf{a} \otimes \mathbf{ea} + \mathbf{ea} \otimes \mathbf{a}) \\ & + \bar{\Omega}_{10}(\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}),\end{aligned} \tag{8.58}$$

where we have used the notation $\mathbf{\Omega}_{ic} \equiv \frac{\partial \Omega_i}{\partial \mathbf{c}}$, $\mathbf{\Omega}_{iE} \equiv \frac{\partial \Omega_i}{\partial \mathbf{E}}$. Using (8.23) and (8.24) the above equation simplifies to

$$\begin{aligned}\boldsymbol{\tau} \approx\ & 4[(\bar{\mathbf{\Omega}}_{1c} + 2\bar{\mathbf{\Omega}}_{2c} + \bar{\mathbf{\Omega}}_{3c}) : \mathbf{e}]\mathbf{I} + 2[(\bar{\mathbf{\Omega}}_{1E} + 2\bar{\mathbf{\Omega}}_{2E} + \bar{\mathbf{\Omega}}_{3E}) \cdot \mathbf{E}]\mathbf{I} \\ & +4(\bar{\Omega}_1 + \bar{\Omega}_2)\mathbf{e} + 4[(\bar{\mathbf{\Omega}}_{7c} + 2\bar{\mathbf{\Omega}}_{8c}) : \mathbf{e}]\mathbf{a} \otimes \mathbf{a} + 2[(\bar{\mathbf{\Omega}}_{7E} + 2\bar{\mathbf{\Omega}}_{8E})]\mathbf{a} \otimes \mathbf{a} \\ & +4\bar{\Omega}_8(\mathbf{a} \otimes \mathbf{ea} + \mathbf{ea} \otimes \mathbf{a}) + \bar{\Omega}_{10}(\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}). \quad (8.59)\end{aligned}$$

For the electric part of the constitutive equation (8.59) we get

$$\begin{aligned}\mathbf{D} \approx -\{& 2(\bar{\Omega}_4 + \bar{\Omega}_5 + \bar{\Omega}_6)\mathbf{E} + (\bar{\Omega}_9 + \bar{\Omega}_{10})\mathbf{a} + 2[(\bar{\mathbf{\Omega}}_{9c} + \bar{\mathbf{\Omega}}_{10c}) : \mathbf{e}]\mathbf{a} \\ & +[(\bar{\mathbf{\Omega}}_{9E} + \bar{\mathbf{\Omega}}_{10E}) \cdot \mathbf{E}]\mathbf{a} + 2\bar{\Omega}_{10}\mathbf{ea}\}. \quad (8.60)\end{aligned}$$

which in view of (8.25) reduces to

$$\begin{aligned}\mathbf{D} \approx -\{& 2(\bar{\Omega}_4 + \bar{\Omega}_5 + \bar{\Omega}_6)\mathbf{E} + 2[(\bar{\mathbf{\Omega}}_{9c} + \bar{\mathbf{\Omega}}_{10c}) : \mathbf{e}]\mathbf{a} \\ & +[(\bar{\mathbf{\Omega}}_{9E} + \bar{\mathbf{\Omega}}_{10E}) \cdot \mathbf{E}]\mathbf{a} + 2\bar{\Omega}_{10}\mathbf{ea}\}. \quad (8.61)\end{aligned}$$

Third step: the final step consists in the calculation of $\bar{\mathbf{\Omega}}_{ic}$ and $\bar{\mathbf{\Omega}}_{iE}$ in terms of the derivatives in the invariants, which are evaluated at the reference configuration for $\mathbf{E} = \mathbf{0}$. We have

$$\mathbf{\Omega}_{ic} = \frac{\partial^2 \Omega}{\partial I_i \partial I_j} \frac{\partial I_j}{\partial \mathbf{c}}, \quad \mathbf{\Omega}_{iE} = \frac{\partial^2 \Omega}{\partial I_i \partial I_j} \frac{\partial I_j}{\partial \mathbf{E}}.$$

Consider the following derivatives of the invariants

$$\frac{\partial I_1}{\partial \mathbf{c}} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \mathbf{c}} = I_1 \mathbf{I} - \mathbf{c}, \quad \frac{\partial I_3}{\partial \mathbf{c}} = I_3 \mathbf{c}^{-1}, \quad \frac{\partial I_4}{\partial \mathbf{c}} = \mathbf{0}, \quad \frac{\partial I_5}{\partial \mathbf{c}} = \mathbf{E} \otimes \mathbf{E}, \quad (8.62)$$

$$\frac{\partial I_6}{\partial \mathbf{c}} = \mathbf{E} \otimes \mathbf{cE} + \mathbf{cE} \otimes \mathbf{E}, \quad \frac{\partial I_7}{\partial \mathbf{c}} = \mathbf{a}_0 \otimes \mathbf{a}_0, \quad \frac{\partial I_8}{\partial \mathbf{c}} = \mathbf{a}_0 \otimes \mathbf{ca}_0 + \mathbf{ca}_0 \otimes \mathbf{a}_0, \quad (8.63)$$

$$\frac{\partial I_9}{\partial \mathbf{c}} = \mathbf{0}, \quad \frac{\partial I_{10}}{\partial \mathbf{c}} = \frac{1}{2}(\mathbf{a}_0 \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}_0). \quad (8.64)$$

As well as this

$$\frac{\partial I_1}{\partial \mathbf{E}} = \frac{\partial I_2}{\partial \mathbf{E}} = \frac{\partial I_3}{\partial \mathbf{E}} = \mathbf{0}, \quad \frac{\partial I_4}{\partial \mathbf{E}} = 2\mathbf{E}, \quad \frac{\partial I_5}{\partial \mathbf{E}} = 2\mathbf{cE}, \quad \frac{\partial I_6}{\partial \mathbf{E}} = 2\mathbf{c}^2\mathbf{E}. \quad (8.65)$$

$$\frac{\partial I_7}{\partial \mathbf{E}} = \frac{\partial I_8}{\partial \mathbf{E}} = \mathbf{0}, \quad \frac{\partial I_9}{\partial \mathbf{E}} = \mathbf{a}_0, \quad \frac{\partial I_{10}}{\partial \mathbf{E}} = \mathbf{ca}_0, \quad (8.66)$$

so we have

$$\bar{\mathbf{\Omega}}_{ic} = (\bar{\Omega}_{i,1} + 2\bar{\Omega}_{i,2} + \bar{\Omega}_{i,3})\mathbf{I} + (\bar{\Omega}_{i,7} + 2\bar{\Omega}_{i,8})\mathbf{a} \otimes \mathbf{a}, \quad (8.67)$$

where we have used the notation $\Omega_{i,j} \equiv \frac{\partial^2 \Omega}{\partial I_i \partial I_j}$. As well as this,

$$\bar{\mathbf{\Omega}}_{iE} = (\bar{\Omega}_{i,9} + \bar{\Omega}_{i,10})\mathbf{a}. \quad (8.68)$$

8.2.1 Approximation for the stress

Let's calculate separately the different terms that appear in (8.59). Since $\Omega_{i,j} = \Omega_{j,i}$, we have the expressions

$$4(\bar{\mathbf{\Omega}}_{1c} + 2\bar{\mathbf{\Omega}}_{2c} + \bar{\mathbf{\Omega}}_{3c}) = 4[(\bar{\Omega}_{1,1} + 4\bar{\Omega}_{1,2} + 2\bar{\Omega}_{1,3} + 4\bar{\Omega}_{2,2} + 4\bar{\Omega}_{2,3} + \bar{\Omega}_{3,3})\mathbf{I} + (\bar{\Omega}_{1,7} + 2\bar{\Omega}_{1,8} + 2\bar{\Omega}_{2,7} + 4\bar{\Omega}_{2,8} + \bar{\Omega}_{3,7} + 2\bar{\Omega}_{3,8})\mathbf{a}\otimes\mathbf{a}]. \quad (8.69)$$

$$2(\bar{\mathbf{\Omega}}_{1E} + 2\bar{\mathbf{\Omega}}_{2E} + \bar{\mathbf{\Omega}}_{3E}) = 2(\bar{\Omega}_{1,9} + \bar{\Omega}_{1,10} + 2\bar{\Omega}_{2,9} + 2\bar{\Omega}_{2,10} + \bar{\Omega}_{3,9} + \bar{\Omega}_{3,10})\mathbf{a}, \quad (8.70)$$

$$4(\bar{\mathbf{\Omega}}_{7c} + 2\bar{\mathbf{\Omega}}_{8c}) = 4[(\bar{\Omega}_{7,1} + 2\bar{\Omega}_{7,2} + \bar{\Omega}_{7,3} + 2(\bar{\Omega}_{8,1} + 2\bar{\Omega}_{8,2} + \bar{\Omega}_{8,3}))\mathbf{I} + (\bar{\Omega}_{7,7} + 4\bar{\Omega}_{7,8} + 4\bar{\Omega}_{8,8})\mathbf{a}\otimes\mathbf{a}], \quad (8.71)$$

$$2(\bar{\mathbf{\Omega}}_{7E} + 2\bar{\mathbf{\Omega}}_{8E}) = 2[\bar{\Omega}_{7,9} + \bar{\Omega}_{7,10} + 2(\bar{\Omega}_{8,9} + \bar{\Omega}_{8,10})]\mathbf{a}. \quad (8.72)$$

Let's define

$$\alpha_1 = \bar{\Omega}_{1,1} + 4\bar{\Omega}_{1,2} + 2\bar{\Omega}_{1,3} + 4\bar{\Omega}_{2,2} + 4\bar{\Omega}_{2,3} + \bar{\Omega}_{3,3}, \quad (8.73)$$

$$\alpha_2 = \bar{\Omega}_{1,7} + 2\bar{\Omega}_{1,8} + 2\bar{\Omega}_{2,7} + 4\bar{\Omega}_{2,8} + \bar{\Omega}_{3,7} + 2\bar{\Omega}_{3,8}, \quad (8.74)$$

$$\alpha_3 = 2(\bar{\Omega}_{1,9} + \bar{\Omega}_{1,10} + 2\bar{\Omega}_{2,9} + 2\bar{\Omega}_{2,10} + \bar{\Omega}_{3,9} + \bar{\Omega}_{3,10}), \quad (8.75)$$

$$\alpha_4 = \bar{\Omega}_{7,7} + 4\bar{\Omega}_{7,8} + 4\bar{\Omega}_{8,8}, \quad (8.76)$$

$$\alpha_5 = 2[\bar{\Omega}_{7,9} + \bar{\Omega}_{7,10} + 2(\bar{\Omega}_{8,9} + \bar{\Omega}_{8,10})], \quad (8.77)$$

then (8.59) becomes

$$\boldsymbol{\tau} \approx 4[(\alpha_1\mathbf{I} + \alpha_2\mathbf{a}\otimes\mathbf{a}) : \mathbf{e}]\mathbf{I} + \alpha_3(\mathbf{a}\cdot\mathbf{E})\mathbf{I} + 4(\bar{\Omega}_1 + \bar{\Omega}_2)\mathbf{e} + 4[(\alpha_2\mathbf{I} + \alpha_4\mathbf{a}\otimes\mathbf{a}) : \mathbf{e}]\mathbf{a}\otimes\mathbf{a} + \alpha_5(\mathbf{a}\cdot\mathbf{E})\mathbf{a}\otimes\mathbf{a} + 4\bar{\Omega}_8(\mathbf{a}\otimes\mathbf{ea} + \mathbf{ea}\otimes\mathbf{a}) + \bar{\Omega}_{10}(\mathbf{a}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{a}). \quad (8.78)$$

By defining $\beta_1 \equiv 4(\bar{\Omega}_1 + \bar{\Omega}_2)$ and by considering the particular case $\mathbf{a} = \hat{\imath}_3$ we obtain

$$\tau_{11} = (4\alpha_1 + \beta_1)e_{11} + 4\alpha_1 e_{22} + 4(\alpha_1 + \alpha_2)e_{33} + \alpha_3 E_3, \quad (8.79)$$

$$\tau_{22} = 4\alpha_1 e_{11} + (4\alpha_1 + \beta_1)e_{22} + 4(\alpha_1 + \alpha_2)e_{33} + \alpha_3 E_3, \quad (8.80)$$

$$\tau_{33} = 4(\alpha_1 + \alpha_2)(e_{11} + e_{22}) + [4(\alpha_1 + 2\alpha_2 + \alpha_4) + \beta_1 + 8\bar{\Omega}_8]e_{33} + (\alpha_3 + \alpha_5 + 2\bar{\Omega}_{10})E_3, \quad (8.81)$$

$$\tau_{23} = (\beta_1 + 4\bar{\Omega}_8)e_{23} + \bar{\Omega}_{10}E_2, \quad (8.82)$$

$$\tau_{13} = (\beta_1 + 4\bar{\Omega}_8)e_{13} + \bar{\Omega}_{10}E_1, \quad (8.83)$$

$$\tau_{12} = \beta_1 e_{12}. \quad (8.84)$$

Finally by defining $\gamma_1 = 4(\alpha_1 + 2\alpha_2 + \alpha_4) + \beta_1 + 8\bar{\Omega}_8$, and by using the following vector notation for the stress and the deformation

$$\mathsf{T} = (T_1, T_2, T_3, T_4, T_5, T_6)^T = (\tau_{11}, \tau_{22}, \tau_{33}, \tau_{23}, \tau_{13}, \tau_{12})^T, \quad (8.85)$$

$$\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \mathcal{E}_6)^T = (e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{13}, 2e_{12})^T, \quad (8.86)$$

we can rewrite (8.79)-(8.84) as (see, for example, the form of the linear constitutive equations for a polarized ceramic [121])

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} (4\alpha_1+\beta_1) & 4\alpha_1 & 4(\alpha_1+\alpha_2) & 0 & 0 & 0 \\ 4\alpha_1 & (4\alpha_1+\beta_1) & 4(\alpha_1+\alpha_2) & 0 & 0 & 0 \\ 4(\alpha_1+\alpha_2) & 4(\alpha_1+\alpha_2) & \gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\beta_1+4\bar{\Omega}_8) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(\beta_1+4\bar{\Omega}_8) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\beta_1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & \alpha_3 \\ 0 & 0 & \alpha_3 \\ 0 & 0 & (\alpha_3+\alpha_5+2\bar{\Omega}_{10}) \\ 0 & \bar{\Omega}_{10} & 0 \\ \bar{\Omega}_{10} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \quad (8.87)$$

8.2.2 Approximation for the electric displacement

We can repeat the same procedure used in order to obtain an approximation for the stress but for the electric displacement. In the present case consider the equation (8.61). We have

$$2(\bar{\mathbf{\Omega}}_{9c}+\bar{\mathbf{\Omega}}_{10c}) = 2[(\bar{\Omega}_{9,1}+2\bar{\Omega}_{9,2}+\bar{\Omega}_{9,3}+\bar{\Omega}_{10,1}+2\bar{\Omega}_{10,2}+\bar{\Omega}_{10,3})\mathbf{I} + (\bar{\Omega}_{9,7}+2\bar{\Omega}_{9,8}+\bar{\Omega}_{10,7}+2\bar{\Omega}_{10,8})\mathbf{a}\otimes\mathbf{a}] \equiv \alpha_3\mathbf{I}+\alpha_5\mathbf{a}\otimes\mathbf{a}, \quad (8.88)$$

$$\bar{\mathbf{\Omega}}_{9E}+\bar{\mathbf{\Omega}}_{10E} = (\bar{\Omega}_{9,10}+\bar{\Omega}_{9,9}+\bar{\Omega}_{10,10}+\bar{\Omega}_{10,9})\mathbf{a}, \quad (8.89)$$

where in (8.88) and (8.89) we have used the fact that $\Omega_{i,j}=\Omega_{j,i}$. Finally, let's define

$$\beta_2 = \bar{\Omega}_{9,9}+\bar{\Omega}_{10,10}+2\bar{\Omega}_{9,10}, \quad (8.90)$$

$$\varepsilon_1 = 2(\bar{\Omega}_4+\bar{\Omega}_5+\bar{\Omega}_6). \quad (8.91)$$

We obtain for (8.61) (taking account of (8.25))

$$D_1 = -[\varepsilon_1 E_1 + 2\bar{\Omega}_{10}\mathcal{E}_5], \quad (8.92)$$

$$D_2 = -[\varepsilon_1 E_2 + 2\bar{\Omega}_{10}\mathcal{E}_4], \quad (8.93)$$

$$D_3 = -[\varepsilon_1 E_3 + \beta_2 E_3 + \alpha_3\mathcal{E}_1 + \alpha_3\mathcal{E}_2 + (\alpha_3+\alpha_5)\mathcal{E}_3 + 2\bar{\Omega}_{10}\mathcal{E}_3]. \quad (8.94)$$

which can be rewritten as

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = -\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_1+\beta_2 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & \bar{\Omega}_{10} & 0 \\ 0 & 0 & 0 & \bar{\Omega}_{10} & 0 & 0 \\ \alpha_3 & \alpha_3 & (\alpha_3+\alpha_5+2\bar{\Omega}_{10}) & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \end{pmatrix}. \tag{8.95}$$

8.3 Boundary value problems

For transversely isotropic electro-active elastomers we do not have a complete set of universal solutions as in the isotropic case [82, 99]. In the next examples, in particular for the problems with cylindrical symmetry, it is shown that the controllability will depend strongly on the particular alignment of the electro-active particles with respect to the given external electric field or electric displacement.

One important characteristic of the geometry of the bodies for the following problems, for example for the case of the simple traction or for the case of extension, inflation and torsion of a tube, is that one of the dimensions, generally the length of the tube or cylinder, should be in theory infinite. This restriction, which in practice would mean a 'long' cylinder, is a theoretical trick necessary in order to be able to work with the electric boundary conditions (6.57) (see Sections 4.2 and 7.2). A full theoretical solution of the boundary value problem would require not only looking for a solution of (6.61) inside the body, with boundary conditions (6.62), and with constitutive equations (8.1) (or (8.30)). but also to solve the equations $(6.61)_2$ and $(6.61)_3$ using $(2.106)_1$ for the exterior space surrounding the body. This problem is not easy, and at the present moment there is no general theory that would allow us to find theoretical solutions for any given kind of energy potential Ω.

All the problems considered below correspond to incompressible materials.

8.3.1 Simple Shear

Consider the simple shear deformation defined as

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{8.96}$$

which is applied to a slab of initial dimensions $0 \leq X_1 \leq A$, $0 \leq X_2 \leq B$ and $0 \leq X_3 \leq C$. We apply the following external field in component form

$$\mathsf{E}_o = \mathsf{E}_l = (0, E_o, 0)^T. \tag{8.97}$$

We assume that the particles of electro-active material are aligned in the X_2 direction in the reference configuration, which means [4]

$$\mathsf{a}_0 = (0, 1, 0)^T. \tag{8.98}$$

The matrix forms of the deformation gradient and the left and right Cauchy-Green deformation tensors are given as

$$\mathsf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.99}$$

We have $\det \mathbf{F} = 1$. Other useful expressions are

$$\mathsf{b}^2 = \begin{pmatrix} \gamma^2 + (1+\gamma^2)^2 & \gamma(2+\gamma^2) & 0 \\ \gamma(2+\gamma^2) & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{c}^2 = \begin{pmatrix} 1+\gamma^2 & \gamma(2+\gamma^2) & 0 \\ \gamma(2+\gamma^2) & \gamma^2 + (1+\gamma^2)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.100}$$

and

$$\mathsf{E} = (0, E_o, 0)^T, \quad \mathsf{a} = (\gamma, 1, 0)^T. \tag{8.101}$$

The invariants I_i are given by (8.6)-(8.9) as

$$I_1 = 3 + \gamma^2 = I_2, \quad I_4 = E_o^2, \quad I_5 = (1+\gamma^2)E_o^2, \quad I_6 = [\gamma^2 + (1+\gamma^2)]E_o^2, \tag{8.102}$$

$$I_7 = 1 + \gamma^2, \quad I_8 = \gamma^2 + (1+\gamma^2)^2, \quad I_9 = E_o, \quad I_{10} = (1+\gamma^2)E_o, \tag{8.103}$$

and from (8.15) and (8.18) we obtain

$$\begin{aligned} \tau_{11} &= 2(1+\gamma^2)\Omega_1 + 2(2+\gamma^2)\Omega_2 - p + 2(\gamma E_o)^2\Omega_5 + 4\gamma^2(2+\gamma^2)E_o^2\Omega_6 + 2\gamma^2\Omega_7 \\ &\quad + 4\gamma^2(2+\gamma^2)\Omega_8 + 2\gamma^2 E_o\Omega_{10}, \end{aligned} \tag{8.104}$$

$$\tau_{22} = 2\Omega_1 + 4\Omega_2 - p + 2E_o^2\Omega_5 + 4(1+\gamma^2)E_o^2\Omega_6 + 2\Omega_7 + 4(1+\gamma^2)\Omega_8 + 2E_o\Omega_{10}, \tag{8.105}$$

$$\tau_{33} = 2\Omega_1 + 2(2+\gamma^2)\Omega_2 - p, \tag{8.106}$$

$$\begin{aligned} \tau_{12} &= 2\gamma\Omega_1 + 2\gamma\Omega_2 + 2\gamma E_o^2\Omega_5 + 2\gamma(3+2\gamma^2)E_o^2\Omega_6 + 2\gamma\Omega_7 + 2\gamma(3+2\gamma^2)\Omega_8 \\ &\quad + 2\gamma E_o\Omega_{10}, \end{aligned} \tag{8.107}$$

$$\tau_{13} = \tau_{23} = 0, \tag{8.108}$$

[4]The reason for this particular alignment of the particles, and the form of the external electric field, is to reproduce theoretically what happens with the shear of a transversely isotropic slab, which has been studied experimentally for the magneto-elastic problem by Jolly et. al [59].

and

$$D_1 = -\{2 + \gamma E_o\Omega_4 + 2\gamma(2+\gamma^2)E_o\Omega_5 + 2\gamma(1+\gamma^2)(3+\gamma^2)E_o\Omega_6 + \gamma\Omega_9 + \gamma(2+\gamma^2)\Omega_{10}\}, \quad (8.109)$$

$$D_2 = -\{2E_o\Omega_4 + 2(1+\gamma^2)E_o\Omega_5 + 2(1+3\gamma^2+\gamma^4)E_o\Omega_6 + \Omega_9 + (1+\gamma^2)\Omega_{10}\}, \quad (8.110)$$

$$D_3 = 0. \quad (8.111)$$

By defining

$$\omega(\gamma, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10 \quad (8.112)$$

and using (8.102)-(8.103) and the chain rule we show that

$$\frac{\partial\omega}{\partial\gamma} = 2\gamma\Omega_1 + 2\gamma\Omega_2 + 2\gamma\Omega_5 E_o^2 + \Omega_6 2\gamma(3+\gamma^2)E_o^2 + 2\gamma\Omega_7 + \Omega_8 2\gamma(3+\gamma^2) + 2\gamma\Omega_{10}E_o, \quad (8.113)$$

$$\frac{\partial\omega}{\partial E_o} = 2\Omega_4 E_o + 2\Omega_5(1+\gamma^2)E_o + \Omega_6 2[\gamma^2 + (1+\gamma^2)^2]E_o + \Omega_9 + \Omega_{10}(1+\gamma^2)], \quad (8.114)$$

from where we get the connections

$$\tau_{12} = \frac{\partial\omega}{\partial\gamma}, \quad D_2 = -\frac{\partial\omega}{\partial E_o}. \quad (8.115)$$

An alternative expression for the electric displacement can be obtained if we consider ω as a function of I_4 instead of E_o, then

$$\omega = \omega(\gamma, I_4) \quad \Rightarrow \quad D_2 = -2E_o\frac{\partial\omega}{\partial I_4}. \quad (8.116)$$

The stress, the electric field, and the electric displacement are constant, and as a result, they satisfy automatically (6.61). However, as was mentioned in the introduction, the situation is not so simple with the boundary conditions (6.57). If we consider a 'finite' slab, then it is not difficult to see that in order to satisfy simultaneously the two boundary conditions (6.57), we would need a non-uniform field, which in general will depend strongly on the particular form of Ω. As a result, for the above solution to be valid we would need at least two of the three dimensions of the slab to be infinite. Consider for example the following initial dimensions for the slab, $-\infty \leq X_1 \leq \infty$, $0 \leq X_2 \leq B$ and $-\infty \leq X_3 \leq \infty$, this is actually the geometry of a infinite wall of width B. In this case the only surfaces where it is necessary to check the boundary conditions, are the surfaces $X_2 = 0$ and $X_2 = B$; for a uniform external electric field of the form (8.97), the boundary conditions (6.57) are satisfied automatically. In experiments what is done is to work with a slab such that $B \ll A$ and $B \ll C$.

8.3.2 Uniform extension of a bar

We consider now the uniform extension of a cylindrical bar. This problem has been studied because in two of the papers that have been mentioned about experimental researches on electro and magneto sensitive elastomers [7,11], the tension of a cylinder was used in order to obtain some important characteristics of these materials, in particular regarding the difference in the response for isotropic and transversely isotropic electro- and magneto-active elastomers.

Consider a cylinder whose length is assumed much larger than its diameter. Its initial dimensions are $0 \leq R \leq R_o$ and $0 \leq Z \leq L$, where $R_o \ll L$. In cylindrical coordinates the deformation is given as (see Section 7.2)

$$r = \lambda^{-1/2} R, \quad \theta = \Theta, \quad z = \lambda Z. \tag{8.117}$$

The external axial applied field is

$$\mathsf{E}_o = \mathsf{E}_l = (0, 0, E_o)^T, \tag{8.118}$$

and the orientation of the particles of electro-active material in the reference configuration is given as

$$\mathsf{a}_0 = (0, 0, 1)^T. \tag{8.119}$$

The matrix forms of the deformation gradient and left and right Cauchy-Green tensors are given by

$$\mathsf{F} = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \mathsf{b} = \mathsf{c} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}, \tag{8.120}$$

then the invariants are, from (8.15)-(8.18), given as

$$I_1 = 2\lambda^{-1} + \lambda^2, \quad I_2 = \lambda^{-2} + 2\lambda, \quad I_4 = E_o^2, \quad I_5 = \lambda^2 E_o^2, \quad I_6 = \lambda^4 E_o^2, \tag{8.121}$$

$$I_7 = \lambda^2, \quad I_8 = \lambda^4, \quad I_9 = E_o, \quad I_{10} = \lambda^2 E_o. \tag{8.122}$$

The components of the stress and the electric displacement (8.15) and (8.18) are

$$\tau_{rr} = \tau_{\theta\theta} = 2\lambda^{-1}\Omega_1 + 2(\lambda^{-2} + \lambda)\Omega_2 - p, \tag{8.123}$$

$$\tau_{zz} = 2\lambda^2\Omega_1 + 4\lambda\Omega_2 - p + 2\lambda^2 E_o^2 \Omega_5 + 4\lambda^4 E_o^2 \Omega_6 + 2\lambda^2\Omega_7 + 4\lambda^4\Omega_8 + 2\lambda^2 E_o \Omega_{10}. \tag{8.124}$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0, \tag{8.125}$$

and

$$D_r = D_\theta = 0, \tag{8.126}$$

$$D_z = -(2\lambda E_o\Omega_4 + 2\lambda^3 E_o\Omega_5 + 2\lambda^5 E_o\Omega_6 + \lambda\Omega_9 + \lambda^3\Omega_{10}). \tag{8.127}$$

Now, the components of the Maxwell stress in the radial and the azimuthal directions are given by (6.30)

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{\varepsilon_o}{2}\lambda^{-2}E_o^2. \tag{8.128}$$

If we want the external mechanical load in the radial and azimuthal directions to vanish, then from $(6.61)_1$ we have

$$\tau_{rr} - \tau_{m_{rr}} = 0, \quad \tau_{\theta\theta} - \tau_{m_{\theta\theta}} = 0,$$

and as a result we obtain

$$p = 2\lambda^{-1}\Omega_1 + 2(2 + \lambda^{-2})\Omega_2 + \frac{\varepsilon_o}{2}\lambda^{-2}E_o^2. \tag{8.129}$$

Therefore, for (8.124) we get

$$\begin{aligned}\tau_{zz} = {} & 2(\lambda^2 - \lambda^{-1})\Omega_1 + 2(\lambda - \lambda^{-2})\Omega_2 + 2\lambda^2 E_o^2\Omega_5 + 4\lambda^4 E_o^2\Omega_6 + 2\lambda^2\Omega_7 + 4\lambda^4\Omega_8 \\ & +2\lambda^2 E_o\Omega_{10} - \frac{\varepsilon_o}{2}\lambda^{-2}E_o^2. \end{aligned} \tag{8.130}$$

As for simple shear, we can define

$$\omega(\lambda, E_o) = \Omega(I_i), \quad i = 1, 2 \ldots, 10, \tag{8.131}$$

and by using the chain rule with (8.121) and (8.122) we can show that

$$\begin{aligned}\frac{\partial\omega}{\partial\lambda} = {} & 2\Omega_1(\lambda - \lambda^{-2}) + 2\Omega_2(1 - \lambda^{-3}) + 2\Omega_5\lambda E_o^2 + 4\Omega_6\lambda^3 E_o^2 + 2\Omega_7\lambda + 4\Omega_8\lambda^3 \\ & +2\Omega_{10}\lambda E_o, \end{aligned} \tag{8.132}$$

$$\frac{\partial\omega}{\partial E_o} = 2\Omega_4 E_o + 2\Omega_5\lambda^2 E_o + 2\Omega_6\lambda^4 E_o + \Omega_9 + \Omega_{10}\lambda^2. \tag{8.133}$$

Thus

$$\tau_{zz} = \lambda\frac{\partial\omega}{\partial\lambda} - \frac{\varepsilon_o}{2}\lambda^{-2}E_o^2, \tag{8.134}$$

and

$$D_z = -\lambda\frac{\partial\omega}{\partial E_o}. \tag{8.135}$$

Or, if we use instead $\omega = \omega(\lambda, I_4)$ we would get

$$D_z = -2E_o\lambda\frac{\partial\omega}{\partial I_4}. \tag{8.136}$$

As in the simple shear problem, since the components of the stress and the electric field and electric displacement are constant, they satisfy automatically (6.61). Regarding the boundary conditions (6.62), if the cylinder has an infinite length ($L = \infty$), then the only surface where we would need to check the boundary conditions (6.62) would be the surface $R = R_o$; in such a case, since the electric field is uniform in the axial direction, we would not have a component for the electric displacement in the radial direction (as has been shown here), and we would not need to check $(6.61)_3$; and in order to satisfy $(6.62)_2$ we would only need to have the same uniform electric field outside the body, which is the condition used here. In the case where $R \ll L$, as in the simple shear, the above solution would be valid far from the ends of the cylinder at $Z = 0$ and $Z = L$ (see Section 7.2).

8.3.3 Problems with cylindrical symmetry

The boundary value problem presented previously has cylindrical symmetry. In order to study the following boundary value problems it is necessary to consider in more detail the different characteristics of this kind of problem (the same problems have been studied by Dorfmann and Ogden [32,36] in the context of isotropic electro-active elastomers). In the current configuration the equilibrium equation (without non-electric body force) was

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \tag{8.137}$$

and the simplified forms of the electro-magnetic equations were

$$\operatorname{curl}\mathbf{E} = \mathbf{0}, \quad \operatorname{div}\mathbf{D} = 0. \tag{8.138}$$

If we assume that $\boldsymbol{\tau} = \boldsymbol{\tau}(r, z)$, then in cylindrical coordinates the equation (8.137) becomes (see Section A.1)

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0, \tag{8.139}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} = 0, \tag{8.140}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} = 0. \tag{8.141}$$

If we assume $\mathbf{E} = \mathbf{E}(r, z)$ then the simplified form of $(8.138)_1$ in cylindrical coordinates is

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) = 0, \tag{8.142}$$

and finally, by assuming $\mathbf{D} = \mathbf{D}(r, z)$, the simplified form of $(8.138)_2$ is

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{\partial D_z}{\partial z} = 0. \tag{8.143}$$

We study basically three problems, the extension and inflation of a tube. the extension and torsion of a tube, and helical shear. For the first two of these problems we want to find universal solutions, which will depend among other factors on the particular form of the field $\mathbf{a}_0$, which corresponds to the alignment of the electro-active particles in the reference configuration. From the practical point of view, the manufacturing of these materials at the moment would permit us to make a tube with the particles uniformly aligned in the axial direction [7,11]. A radially uniform alignment might be possible as well. So we will consider the following two cases, $\mathbf{a}_0 = (0,0,1)^T$ and $\mathbf{a}_0 = (1,0,0)^T$.

8.3.3.1 Extension and inflation of a tube.

Consider the following deformation given in cylindrical coordinates [36]

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{8.144}$$

where $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$ and[5] $-\infty \leq z \leq \infty$.

The matrix forms of the deformation gradient and the left and right Cauchy-Green tensors are given by

$$\mathsf{F} = \begin{pmatrix} (\lambda_z\lambda)^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}, \quad \mathsf{b} = \mathsf{c} = \begin{pmatrix} (\lambda_z\lambda)^{-2} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{pmatrix}, \tag{8.145}$$

where we have used the definition $\lambda = r/R$.

The first two invariants $(8.6)_1$ and $(8.6)_2$ are

$$I_1 = \operatorname{tr}\mathbf{c} = (\lambda_z\lambda)^{-2} + \lambda^2 + \lambda_z^2, \tag{8.146}$$

$$I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2] = \lambda_z^{-2} + \lambda^{-2} + (\lambda\lambda_z)^2. \tag{8.147}$$

Now, it is necessary to consider a particular form for the external applied electric field, and also for the initial alignment of the electro-active particles. Consider the following cases:

1. **Axial electric field**

   Let's assume that the external electric field is $\mathsf{E}_l = (0,0,E_o)^T$, where E_o is constant. From $(6.4)_2$ we have that $\mathsf{E} = (0,0,\lambda_z^{-1}E_o)^T$, and equations (8.142) are satisfied

[5] Note that here we have an infinitely long tube for the same reasons already discussed for the uniform extension of a bar and the electric boundary conditions.

automatically. As well as this, from (8.7) we have

$$I_4 = E_o^2, \quad I_5 = \lambda_z^2 E_o^2, \quad I_6 = \lambda_z^4 E_o^2. \tag{8.148}$$

With the above electric field, the non-zero components of the Maxwell stress tensor (6.30) are

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{\varepsilon_o}{2}\lambda^{-2}E_o^2, \quad \tau_{m_{zz}} = \frac{\varepsilon_o}{2}\lambda^{-2}E_o^2. \tag{8.149}$$

Now, regarding the initial alignment of the electro-active particles, we will consider two cases, an axial and a radial alignment.

(a) **Axial alignment**

In this case we assume that the particles are aligned uniformly in the axial direction in the reference configuration; as a result we have $\mathsf{a}_0 = (0,0,1)^T$, and the remaining invariants are given by (8.8) and (8.9) as

$$I_7 = \lambda_z^2, \quad I_8 = \lambda_z^4, \quad I_9 = E_o, \quad I_{10} = \lambda_z^2 E_o. \tag{8.150}$$

From (8.5) we obtain $\mathsf{a} = (0,0,\lambda_z)^T$. As well as this

$$\mathsf{bE} = (0,0,\lambda_z E_o)^T, \quad \mathsf{b}^2\mathsf{E} = (0,0,\lambda_z^3 E_o), \quad \mathsf{ba} = (0,0,\lambda^3)^T.$$

We finally obtain from (8.15)

$$\tau_{rr} = 2(\lambda_z\lambda)^{-2}\Omega_1 + 2(\lambda_z^{-2} + \lambda^{-2})\Omega_2 - p, \tag{8.151}$$

$$\tau_{\theta\theta} = 2\lambda^2\Omega_1 + 2(\lambda_z^{-2} + (\lambda_z\lambda)^2)\Omega_2 - p, \tag{8.152}$$

$$\tau_{zz} = 2\lambda_z^2\Omega_1 + 2(\lambda^{-2} + (\lambda_z\lambda)^2)\Omega_2 - p + 2(\lambda_z E_o)^2\Omega_5 + 4(\lambda_z^2 E_o)^2\Omega_6$$
$$+2\lambda_z^2\Omega_7 + 4\lambda_z^4\Omega_8 + 2\lambda_z^2 E_o\Omega_{10}, \tag{8.153}$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0, \tag{8.154}$$

and from (8.18) we have

$$D_r = D_\theta = 0, \tag{8.155}$$

$$D_z = -(2\lambda_z E_o\Omega_4 + 2\lambda_z^3 E_o\Omega_5 + 2\lambda_z^5 E_o\Omega_6 + \lambda_z\Omega_9 + \lambda_z^3\Omega_{10}). \tag{8.156}$$

Now, since by definition $\lambda = r/R$, and since R may be seen as a function of r, we have that $\lambda = \lambda(r)$. Thus the different invariants are function of r. Consider the following decomposition of the components of the stress τ_{rr}, $\tau_{\theta\theta}$ and τ_{zz} given by the equations (8.151)-(8.153) (see Section 5.5)

$$\tau_{rr} = \tilde{\tau}_{rr} - p, \quad \tau_{\theta\theta} = \tilde{\tau}_{\theta\theta} - p, \quad \tau_{zz} = \tilde{\tau}_{zz} - p.$$

with $\tilde{\tau}_{rr} = \tilde{\tau}_{rr}(r)$, $\tilde{\tau}_{\theta\theta} = \tilde{\tau}_{\theta\theta}(r)$ and $\tilde{\tau}_{zz} = \tilde{\tau}_{zz}(r)$. As a result, the equation (8.140) is satisfied automatically. From (8.141) we have

$$\frac{\partial}{\partial z}(\tilde{\tau}_{zz}(r) - p) = 0, \quad \Rightarrow \quad \frac{\partial p}{\partial z} = 0 \quad \Leftrightarrow \quad p = p(r). \tag{8.157}$$

and then from (8.139) we have

$$\frac{\mathrm{d}\tilde{\tau}_{rr}}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r} + \frac{1}{r}(\tilde{\tau}_{rr} - \tilde{\tau}_{\theta\theta}) = 0, \tag{8.158}$$

from where p can be obtained as

$$p(r) = \tilde{\tau}_{rr}(r) - \tilde{\tau}_{rr}(a) + \int_a^r \tilde{\tau}_{rr}(\eta) - \tilde{\tau}_{\theta\theta}(\eta)\,\mathrm{d}\eta. \tag{8.159}$$

Regarding the electric displacement, from (8.156) we have that $D_z = D_z(r)$, and using this and (8.155) we conclude that (8.143) is satisfied trivially. As a result for this electric field and initial orientation of the electro-active particles we conclude that (8.144) is universal (for an analysis of universal solutions in the context of electro- and magneto-elastic problems see [82, 99]).

Finally, let's consider the simplified form for the energy function

$$\omega(\lambda_z, \lambda, E_o) = \Omega(I_i), \quad i = 1, 2, \dots 10,$$

where from (8.146)-(8.147) and (8.148), (8.150) we have that in general $I_i = I_i(\lambda_z, \lambda, E_o)$. We have the partial derivatives

$$\begin{aligned}\frac{\partial \omega}{\partial \lambda_z} &= 2\Omega_1(\lambda_z - \lambda_z^{-3}\lambda^{-2}) + 2\Omega_2(\lambda^2\lambda_z - \lambda_z^{-3}) + 2\Omega_5\lambda_z E_o^2 + 4\Omega_6\lambda_z^3 E_o^2 \\ &\quad + 2\Omega_7\lambda_z + 4\Omega_8\lambda_z^3 + 2\Omega_{10}\lambda_z E_o, && (8.160)\\ \frac{\partial \omega}{\partial \lambda} &= 2\Omega_1(\lambda - \lambda_z^{-2}\lambda^{-3}) + 2\Omega_2(\lambda\lambda_z^2 - \lambda^{-3}), && (8.161)\\ \frac{\partial \omega}{\partial E_o} &= 2\Omega_4 E_o + 2\Omega_5\lambda_z^2 E_o + 2\Omega_6\lambda_z^4 E_o + \Omega_9 + \Omega_{10}\lambda_z^2, && (8.162)\end{aligned}$$

and from the above relations it is easy to prove that

$$\tau_{zz} - \tau_{\theta\theta} = \lambda_z\frac{\partial \omega}{\partial \lambda_z} - \lambda\frac{\partial \omega}{\partial \lambda}, \tag{8.163}$$

and

$$D_z = -\lambda_z\frac{\partial \omega}{\partial E_o}. \tag{8.164}$$

(b) **Radial orientation**

An additional case for this external electric field might be considered. Let's assume that the electro-active particles are initially aligned uniformly in the radial direction. Consider $a_0 = (1,0,0)^T$, in which case $a = ((\lambda_z\lambda)^{-1}, 0, 0)^T$. A straightforward calculation shows that the components of the stress tensor are

$$\tau_{rr} = 2(\lambda_z\lambda)^{-2}\Omega_1 + 2(\lambda_z^{-2} + \lambda^{-2})\Omega_2 - p + 2(\lambda_z\lambda)^{-2}\Omega_7 + 4(\lambda_z\lambda)^{-4}\Omega_8, \quad (8.165)$$

$$\tau_{\theta\theta} = 2\lambda^2\Omega_1 + 2(\lambda_z^{-2} + (\lambda\lambda_z)^2)\Omega_2 - p, \quad (8.166)$$

$$\tau_{zz} = 2\lambda_z^2\Omega_1 + 2(\lambda^{-2} + (\lambda\lambda_z)^2)\Omega_2 - p + 2(\lambda_z E_o)^2\Omega_5 + 4(\lambda_z^2 E_o)^2\Omega_6, \quad (8.167)$$

$$\tau_{rz} = \lambda^{-1}E_o\Omega_{10}, \quad (8.168)$$

$$\tau_{r\theta} = \tau_{\theta z} = 0. \quad (8.169)$$

Since in general $\tau_{rz} \neq 0$, from the equation (8.141) we would have that $\frac{\partial p}{\partial z} = -\frac{1}{r}\tau_{rz}(r)$, which along with (8.139) show us that this deformation is not controllable, and so we do not consider this case further.

2. **Radial electric displacement**

Consider now an external electric displacement in vector form given by $D_l = (D_o/R, 0, 0)^T$, where D_o is constant. Then from $(6.4)_1$ for an incompressible material we have $D = (\lambda_z^{-1}D_o/r, 0, 0)^T$. For this particular form of the electric displacement, the equation (8.143) is satisfied automatically.

Again, let's consider two possibilities for the particle alignment.

(a) **Axial orientation**

As in the previous case, let's take $a_0 = (0,0,1)^T$, then from (8.37) we obtain

$$\tau_{rr} = 2(\lambda_z\lambda)^{-2}\Omega_1^* + 2(\lambda_z^{-2} + \lambda^{-2})\Omega_2^* - p^* + 2\lambda_z^{-2}\left(\frac{D_o}{r}\right)^2\Omega_5^* + 4(\lambda_z\lambda)^{-4}\left(\frac{D_o}{r}\right)^2\Omega_6^*, \quad (8.170)$$

$$\tau_{\theta\theta} = 2\lambda^2\Omega_1^* + 2(\lambda_z^{-2} + (\lambda_z\lambda)^2)\Omega_2^* - p^*, \quad (8.171)$$

$$\tau_{zz} = 2\lambda_z^2\Omega_1^* + 2(\lambda^{-2} + (\lambda_z\lambda)^2)\Omega_2^* - p^* + 2\lambda_z^2\Omega_7^* + 4\lambda_z^4\Omega_8^*, \quad (8.172)$$

$$\tau_{rz} = \frac{D_o}{r}\Omega_{10}^*, \quad (8.173)$$

$$\tau_{r\theta} = \tau_{\theta z} = 0, \quad (8.174)$$

which because $\tau_{rz} \neq 0$, it means that the deformation in this case is not controllable, as in the case 1.b.

(b) **Radial orientation**

Consider a radial uniform orientation for the electro-active particles. So $a_0 = (1,0,0)^T$, and as a result $a = ((\lambda_z\lambda)^{-1}, 0, 0)^T$. The invariants are given by

(8.32)-(8.35), and we have

$$K_4 = (D_o/R)^2, \quad K_5 = (D_o/R)^2(\lambda_z\lambda)^{-2}, \quad K_6 = (D_o/R)^2(\lambda_z\lambda)^{-4}, \tag{8.175}$$

$$I_7 = (\lambda_z\lambda)^{-2}, \quad I_8 = (\lambda_z\lambda)^{-4}, \quad K_9 = D_o/R, \tag{8.176}$$

$$K_{10} = (D_o/R)(\lambda_z\lambda)^{-2}. \tag{8.177}$$

As well as this, consider the vectors

$$\mathsf{bD} = ((\lambda_z\lambda)^{-3}D_o/R, 0, 0)^T, \quad \mathsf{ba} = ((\lambda_z\lambda)^{-3}, 0, 0)^T.$$

From (8.37) we obtain

$$\begin{aligned}\tau_{rr} &= 2(\lambda_z\lambda)^{-2}\Omega_1^* + 2(\lambda^{-2} + \lambda_z^{-2})\Omega_2^* - p^* + 2\lambda_z^{-2}\left(\frac{D_o}{r}\right)^2\Omega_5^* \\ &\quad +4(\lambda_z\lambda)^{-4}\left(\frac{D_o}{R}\right)^2\Omega_6^* + 2(\lambda_z\lambda)^{-2}\Omega_7^* + 4(\lambda_z\lambda)^{-4}\Omega_8^* \\ &\quad +2(\lambda_z\lambda)^{-2}\frac{D_o}{R}\Omega_{10}^*, \end{aligned} \tag{8.178}$$

$$\tau_{\theta\theta} = 2\lambda^2\Omega_1^* + 2(\lambda_z^{-2} + (\lambda_z\lambda)^2)\Omega_2^* - p^*, \tag{8.179}$$

$$\tau_{zz} = 2\lambda_z^2\Omega_1^* + 2(\lambda^{-2} + (\lambda_z\lambda)^2)\Omega_2^* - p^*, \tag{8.180}$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0. \tag{8.181}$$

and from (8.39) we get for the electric field

$$\begin{aligned}E_r &= 2\lambda_z\lambda\frac{D_o}{R}\Omega_4^* + 2(\lambda_z\lambda)^{-1}\frac{D_o}{R}\Omega_5^* + 2(\lambda_z\lambda)^{-3}\frac{D_o}{R}\Omega_6^* + \lambda_z\lambda\Omega_9^* \\ &\quad (\lambda_z\lambda)^{-1}\Omega_{10}^*, \end{aligned} \tag{8.182}$$

$$E_\theta = E_z = 0. \tag{8.183}$$

The invariants are function of λ_z, λ and D_o/R. The above electric field then satisfies (8.142). As well as this, by an argument similar to the one used in case 1.a, it can be proved that this deformation is controllable, and that p^* can be calculated from (8.159) using the above components of the stress and the same decomposition used for the stress in 1.a.

Let's define $\xi = D_o/R$, and the simplified energy function

$$\omega(\lambda_z, \lambda, \xi) = \Omega^*(I_i, K_j),$$

and consider the derivatives

$$\frac{\partial\omega}{\partial\lambda_z} = 2\Omega_1^*(\lambda_z - \lambda_z^{-3}\lambda^{-2}) + 2\Omega_2^*(\lambda_z\lambda^2 - \lambda_z^{-3}) - 2\Omega_5^*\lambda_z^{-3}\lambda^{-2}\xi^2$$
$$-4\Omega_6^*\lambda_z^{-5}\lambda^{-4}\xi^2 - 2\Omega_7^*\lambda_z^{-3}\lambda^{-2} - 4\Omega_8^*\lambda^{-5}\lambda^{-4} - 2\Omega_{10}^*\lambda_z^{-3}\lambda^{-2}\xi, \quad (8.184)$$
$$\frac{\partial\omega}{\partial\lambda} = 2\Omega_1^*(\lambda - \lambda_z^{-2}\lambda^{-3}) + 2\Omega_2^*(\lambda\lambda_z^2 - \lambda^{-3}) - 2\Omega_5^*\lambda^{-3}\lambda_z^{-2}\xi^2$$
$$-4\Omega_6^*\lambda^{-5}\lambda_z^{-4}\xi^2 - 2\Omega_7^*\lambda_z^{-2}\lambda^{-3} - 4\Omega_8^*\lambda_z^{-4}\lambda^{-5} - 2\Omega_{10}^*\lambda_z^{-2}\lambda^{-3}\xi, \quad (8.185)$$
$$\frac{\partial\omega}{\partial\xi} = 2\Omega_4^*\xi + 2\Omega_5^*\xi(\lambda_z\lambda)^{-2} + 2\Omega_6^*\xi(\lambda_z\lambda)^{-4} + \Omega_9^* + \Omega_{10}^*(\lambda_z\lambda)^{-2}. \quad (8.186)$$

It is easy to show that

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\frac{\partial\omega}{\partial\lambda}, \quad \tau_{zz} - \tau_{rr} = \lambda_z\frac{\partial\omega}{\partial\lambda_z}, \quad (8.187)$$

and

$$\tau_{zz} - \tau_{\theta\theta} = \lambda_z\frac{\partial\omega}{\partial\lambda_z} - \lambda\frac{\partial\omega}{\partial\lambda}, \quad (8.188)$$

which is the same relation found in case 1.a. Finally, for the electric field we have

$$E_r = \lambda_z\lambda\frac{\partial\omega}{\partial\xi}. \quad (8.189)$$

8.3.3.2 Extension and torsion of a tube

For this problem, consider the deformation [33] (see Subsection 5.5.2)

$$r = \lambda_z^{-1/2}R, \quad \theta = \Theta + \lambda_z\tau Z, \quad z = \lambda_z Z, \quad (8.190)$$

where λ_z and τ are constants, and $a \leq r \leq b$, $0 \leq \theta < 2\pi$, and[6] $-\infty \leq z \leq \infty$. Let's define $\gamma = \tau r$, then the matrix representation of the deformation gradient is given as

$$\mathsf{F} = \begin{pmatrix} \lambda_z^{-1/2} & 0 & 0 \\ 0 & \lambda_z^{-1/2} & \gamma\lambda_z \\ 0 & 0 & \lambda_z \end{pmatrix}. \quad (8.191)$$

We have that $\det\mathbf{F} = 1$. Also, the matrix representations of the left and right Cauchy-Green tensors are

$$\mathsf{b} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} + \gamma^2\lambda_z^2 & \gamma\lambda_z^2 \\ 0 & \gamma\lambda_z^2 & \lambda_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} & \gamma\lambda_z^{1/2} \\ 0 & \gamma\lambda_z^{1/2} & (1+\gamma^2)\lambda_z^2 \end{pmatrix}. \quad (8.192)$$

[6]The reason of this conditions for the axial dimension of the tube has been explained already.

The following tensors (matrix form) are useful

$$\mathsf{b}^2 = \begin{pmatrix} \lambda_z^{-2} & 0 & 0 \\ 0 & \lambda_z^{-2} + 2\gamma^2\lambda_z + (\gamma^2+\gamma^4)\lambda_z^4 & \gamma\lambda_z(1+(1+\gamma^2)\lambda_z^3) \\ 0 & \gamma\lambda_z(1+(1+\gamma^2)\lambda_z^3) & (1+\gamma^2)\lambda_z^4 \end{pmatrix}.$$

$$\mathsf{c}^2 = \begin{pmatrix} \lambda_z^{-2} & 0 & 0 \\ 0 & \lambda_z^{-2} + \gamma^2\lambda_z & \lambda_z^{-1/2}\gamma(1+(1+\gamma^2)\lambda_z^3) \\ 0 & \lambda_z^{-1/2}\gamma(1+(1+\gamma^2)\lambda_z^3) & \gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4 \end{pmatrix}.$$

The first and second invariants given by $(8.6)_1$ and $(8.6)_2$ are

$$I_1 = 2\lambda_z^{-1} + (1+\gamma^2)\lambda_z^2, \quad I_2 = \lambda_z^{-2} + (2+\gamma^2)\lambda_z. \tag{8.193}$$

As in the previous boundary value problem, we must choose a field and an alignment for the electro-active particles. For brevity we will consider only two cases, as follows:

1. **Axial uniform electric field and axial alignment for the electro-active particles.**

   For this case we consider the external electric field $\mathsf{E}_l = (0, 0, E_o)^T$, and the field $\mathsf{a}_0 = (0, 0, 1)^T$ that represents the alignment of the particles in the reference configuration. The rest of the invariants (8.7)-(8.9) are

$$I_4 = E_o^2, \quad I_5 = E_o^2(1+\gamma^2)\lambda_z^2, \quad I_6 = E_o^2[\gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4]. \tag{8.194}$$

$$I_7 = (1+\gamma^2)\lambda_z^2, \quad I_8 = \gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4, \quad I_9 = E_o, \tag{8.195}$$

$$I_{10} = E_o(1+\gamma^2)\lambda_z^2. \tag{8.196}$$

   The non-zero components of the Maxwell stress (6.30) are

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{\varepsilon_o}{2}\lambda_z^{-2}E_o^2, \quad \tau_{m_{zz}} = \frac{\varepsilon_o}{2}\lambda_z^{-2}E_o^2. \tag{8.197}$$

   Consider the vectors

$$\mathsf{bE} = \begin{pmatrix} 0 \\ \gamma \\ 1 \end{pmatrix} E_o\lambda_z, \quad \mathsf{b}^2\mathsf{E} = \begin{pmatrix} 0 \\ \gamma[1+(1+\gamma^2)\lambda_z^3] \\ (1+\gamma^2)\lambda_z^3 \end{pmatrix} E_o, \quad \mathsf{ba} = \begin{pmatrix} 0 \\ \gamma[1+(1+\gamma^2)\lambda_z^3] \\ (1+\gamma^2)\lambda_z^3 \end{pmatrix}.$$

The components of the stress (8.15) are

$$\tau_{rr} = -p + 2\lambda_z^{-1}\Omega_1 + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2, \tag{8.198}$$

$$\begin{aligned}\tau_{\theta\theta} = & -p + 2(\lambda_z^{-1} + \gamma^2\lambda_z^2)\Omega_1 + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2 + 2E_o^2\gamma^2\lambda_z^2\Omega_5 \\ & +4E_o^2\gamma^2\lambda_z[1 + (1+\gamma^2)\lambda_z^3]\Omega_6 + 2\gamma^2\lambda_z^2\Omega_7 + 4\gamma^2\lambda_z[1 + (1+\gamma^2)\lambda_z^3]\Omega_8 \\ & +2E_o\gamma^2\lambda_z^2\Omega_{10},\end{aligned} \tag{8.199}$$

$$\begin{aligned}\tau_{zz} = & -p + 2\lambda_z^2\Omega_1 + 4\lambda_z\Omega_2 + 2E_o^2\lambda_z^2\Omega_5 + 4E_o^2\lambda_z^4(1+\gamma^2)\Omega_6 + 2\lambda_z^2\Omega_7 \\ & +4\lambda_z^4(1+\gamma^2)\Omega_8 + 2E_o\lambda_z^2\Omega_{10},\end{aligned} \tag{8.200}$$

$$\begin{aligned}\tau_{\theta z} = & \, 2\gamma\lambda_z\{\lambda_z\Omega_1 + \Omega_2 + E_o^2\lambda_z\Omega_5 + E_o^2[1 + 2(1+\gamma^2)\lambda_z^3]\Omega_6 + \lambda_z\Omega_7 \\ & +[1 + 2(1+\gamma^2)\lambda_z^3]\Omega_8 + E_o\lambda_z\Omega_{10}\},\end{aligned} \tag{8.201}$$

$$\tau_{r\theta} = \tau_{rz} = 0. \tag{8.202}$$

And the components of the electric displacement (8.18) are

$$D_r = 0, \tag{8.203}$$

$$\begin{aligned}D_\theta = & -\gamma\{2E_o\lambda_z\Omega_4 + 2E_o[1 + (1+\gamma^2)\lambda_z^3]\Omega_5 + 2E_o[\lambda_z^{-1} + (1+2\gamma^2)\lambda_z^2 \\ & +(1+2\gamma^2+\gamma^4)\lambda_z^5]\Omega_6 + \lambda_z\Omega_9 + [1 + (1+\gamma^2)\lambda_z^3]\Omega_{10}\},\end{aligned} \tag{8.204}$$

$$\begin{aligned}D_z = & -\lambda_z\{2E_o\Omega_4 + 2E_o\lambda_z^2(1+\gamma^2)\Omega_5 + 2E_o\lambda_z[\gamma^2 + (1+2\gamma^2+\gamma^4)\lambda_z^5]\Omega_6 \\ & +\Omega_9 + \lambda_z^2(1+\gamma^2)\Omega_{10}\}.\end{aligned} \tag{8.205}$$

We prove that the above deformation is controllable. As in the problem of Subsection 8.3.3.1, if we decompose τ_{rr}, $\tau_{\theta\theta}$ and τ_{zz} as $\tau_{rr} = -p + \tilde{\tau}_{rr}$, $\tau_{\theta\theta} = -p + \tilde{\tau}_{\theta\theta}$ and $\tau_{zz} = -p + \tilde{\tau}_{zz}$; remembering that $\gamma = \tau r$, and considering (8.193)-(8.196), we can show that $\tilde{\tau}_{rr} = \tilde{\tau}_{rr}(r)$, $\tilde{\tau}_{\theta\theta} = \tilde{\tau}_{\theta\theta}(r)$, $\tilde{\tau}_{zz} = \tilde{\tau}_{zz}(r)$ and $\tau_{\theta z} = \tau_{\theta z}(r)$ as well. Then, (8.140) is satisfied automatically, and from (8.139) and (8.141) we have that

$$-\frac{\partial p}{\partial r} + \frac{\mathrm{d}\tilde{\tau}_{rr}}{\mathrm{d}r} + \frac{1}{r}(\tilde{\tau}_{rr} - \tilde{\tau}_{\theta\theta}) = 0, \quad \frac{\partial p}{\partial z} = 0,$$

from where it is easy to see that p is a function of r, and that it can be calculated directly from (8.139).

As well as this, from (8.203)-(8.205) we have that $D_\theta = D_\theta(r)$ and $D_z = D_z(r)$, and as a result (8.143) is also satisfied, thus this deformation is universal. Also, since $-\infty \leq z \leq \infty$, it can be proved easily, as in the previous problems, that the boundary conditions (6.57) are satisfied.

Consider now the simplified form for the energy function

$$\omega = \omega(\lambda_z, \gamma, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10.$$

From (8.193)-(8.196) we have

$$\frac{\partial \omega}{\partial \lambda_z} = 2\Omega_1[(1+\gamma^2)\lambda_z - \lambda_z^{-2}] + \Omega_2(2+\gamma^2 - 2\lambda_z^{-3}) + 2\Omega_5 E_o^2(1+\gamma^2)\lambda_z + \Omega_6 E_o^2[\gamma^2 + 4(1+\gamma^2)^2\lambda_z^3] + 2\Omega_7(1+\gamma^2)\lambda_z + \Omega_8[\gamma^2 + 4(1+\gamma^2)^2\lambda_z^3] + 2\Omega_{10}E_o(1+\gamma^2)\lambda_z, \tag{8.206}$$

$$\frac{\partial \omega}{\partial \gamma} = 2\gamma\lambda_z\{\Omega_1\lambda_z + \Omega_2 + \Omega_5 E_o^2\lambda_z + \Omega_6 E_o^2[1+2(1+\gamma^2)\lambda_z^3] + \Omega_7\lambda_z + \Omega_8[1+2(1+\gamma^2)\lambda_z^3] + \Omega_{10}E_o\lambda_z\}, \tag{8.207}$$

$$\frac{\partial \omega}{\partial E_o} = 2\Omega_4 E_o + 2\Omega_5 E_o(1+\gamma^2)\lambda_z^2 + 2\Omega_6 E_o[\gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4] + \Omega_9 + \Omega_{10}(1+\gamma^2)\lambda_z^2, \tag{8.208}$$

and then it is possible to derive the simple connections

$$\tau_{\theta z} = \frac{\partial \omega}{\partial \gamma}, \quad D_z = -\lambda_z \frac{\partial \omega}{\partial E_o}. \tag{8.209}$$

2. **Radial electric displacement and radial orientation for the electro-active particles**

Now, consider the case where the external electric displacement has the vector form $\mathsf{D}_l = (D_o/R, 0, 0)^T$ in the reference configuration; as a result $\mathsf{D} = (\lambda_z^{-1/2}D_o/R, 0, 0)^T$; but $r = \lambda^{-1/2}R$, so that $\mathsf{D} = (\lambda_z^{-1}D_o/r, 0, 0)^T$, which satisfies (8.143). From (8.33) we have

$$K_4 = \frac{D_o^2\lambda_z^{-1}}{r^2}, \quad K_5 = \frac{D_o^2\lambda_z^{-2}}{r^2}, \quad K_6 = \frac{D_o^2\lambda_z^{-3}}{r^2}. \tag{8.210}$$

If we consider a radially uniform alignment for the particles in the reference configuration given by $\mathsf{a}_0 = (1,0,0)^T$, then $\mathsf{a} = (\lambda_z^{-1/2}, 0, 0)^T$, and the rest of the invariants (8.34)-(8.35) are (the first and second invariants are given in (8.19))

$$I_7 = \lambda_z^{-1}, \quad I_8 = \lambda_z^{-2}, \quad K_9 = \frac{D_o}{R}, \quad K_{10} = \frac{\lambda_z^{-1}D_o}{R}. \tag{8.211}$$

Using $R = \lambda^{1/2}r$, the components of the total stress (8.37) and the electric field are (8.39)

$$\tau_{rr} = -p^* + 2\lambda_z^{-1}\Omega_1^* + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2^* + 2\lambda_z^{-2}\left(\frac{D_o}{r}\right)^2\Omega_5^* + 4\lambda_z^{-3}\left(\frac{D_o}{r}\right)^2\Omega_6^* + 2\lambda_z^{-1}\Omega_7^* + 4\lambda_z^{-2}\Omega_8^* + 2\lambda^{-3/2}\frac{D_o}{r}\Omega_{10}^*, \tag{8.212}$$

$$\tau_{\theta\theta} = -p^* + 2(\lambda_z^{-1} + \gamma^2\lambda_z^2)\Omega_1^* + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2^*, \tag{8.213}$$

$$\tau_{zz} = -p^* + 2\lambda_z^2\Omega_1^* + 4\lambda_z\Omega_2^*, \tag{8.214}$$

$$\tau_{r\theta} = \tau_{rz} = 0, \tag{8.215}$$

$$\tau_{\theta z} = 2\gamma\lambda_z^2\Omega_1^* + 2\gamma\lambda_z\Omega_2^*, \tag{8.216}$$

and

$$E_r = 2\frac{D_o}{r}\Omega_4^* + 2\frac{D_o}{r}\lambda_z^{-1}\Omega_5^* + 2\frac{D_o}{r}\lambda_z^{-2}\Omega_6^* + \lambda_z^{1/2}\Omega_9^* + \lambda_z^{-1/2}\Omega_{10}^*. \quad (8.217)$$

$$E_\theta = E_z = 0, \quad (8.218)$$

which by the same reasons described in the previous problems is also universal and satisfies (8.142). Define $\xi = D_o/R$, and let's consider the simplified form for the energy function

$$\omega = \omega(\lambda_z, \gamma, \xi) = \Omega^*(I_i, K_j).$$

Then we have (6.57)

$$\frac{\partial\omega}{\partial\lambda_z} = 2[(1+\gamma^2)\lambda_z - \lambda_z^{-2}]\Omega_1^* + \Omega_2^*(2+\gamma^2 - 2\lambda_z^{-3}) - \Omega_5^*\lambda_z^{-2}\xi^2 - 2\Omega_6^*\lambda_z^{-3}\xi^2$$
$$-\Omega_7^*\lambda_z^{-2} - 2\Omega_8^*\lambda_z^{-3} - \Omega_{10}^*\lambda_z^{-2}\xi, \quad (8.219)$$

$$\frac{\partial\omega}{\partial\gamma} = 2\Omega_1^*\gamma\lambda_z^2 + 2\Omega_2^2\gamma\lambda_z, \quad (8.220)$$

$$\frac{\partial\omega}{\partial\xi} = 2\Omega_4^*\xi + 2\Omega_5^*\lambda_z^{-1}\xi + 2\Omega_6^*\lambda_z^{-2}\xi + \Omega_9^* + \Omega_{10}^*\lambda_z^{-1}, \quad (8.221)$$

from which it follows that

$$E_r = \lambda^{1/2}\frac{\partial\omega}{\partial\xi}, \quad \tau_{\theta z} = \frac{\partial\omega}{\partial\gamma}. \quad (8.222)$$

There are two extra possibilities that might be included. One is to consider a uniform axial electric field with a uniform radial alignment for the electro-active particles, and the other is to consider a radial electric displacement as in the above problem, but with a uniform axial alignment field for the electro-active particles. In any of these two extra cases is not difficult to show that a shear in the radial direction appears, which implies the arbitrary pressure p cannot be assumed to be a function of r only. As a result these cases are not controllable, and we do not consider them here.

8.3.3.3 Helical shear

Helical shear [22,79] has been studied in the context of isotropic ES elastomers (Subsection 7.1.2.1), and for MS elastomers as well (Subsection 4.1.2.1). In this subsection we want to find some connections, and to check in which situation the non-linear universal relation (7.110) holds.

From [79] helical shear was defined in cylindrical coordinates by

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R), \quad (8.223)$$

where g and w are unknown functions of R, and $A \leq R \leq B$, $0 \leq \Theta < 2\pi$ and $-\infty \leq Z \leq \infty$. The matrix forms of the deformation gradient and the left and right Cauchy-Green tensors are respectively

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1+\kappa_\theta^2 & \kappa_\theta\kappa_z \\ \kappa_z & \kappa_\theta\kappa_z & 1+\kappa_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1+\kappa^2 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}. \tag{8.224}$$

where $\kappa_\theta = rg'(r)$, $\kappa_z = w'(r)$ and $\kappa^2 = \kappa_\theta^2 + \kappa_z^2$.

There are many possibilities for the external electric field or electric displacement, and for the alignment of the electro-active particles as well. We consider only two cases:

1. **A uniform axial electric field, and an axial orientation for the particles**[7].

   As in the prior examples we work with $\mathsf{E}_l = (0, 0, E_o)^T$ and $\mathsf{a}_0 = (0, 0, 1)^T$, where E_o is constant. Then, from $(6.4)_2$ and (8.5) we get $\mathsf{E} = (-\kappa_z, 0, 1)^T E_o$ and $\mathsf{a} = (0, 0, 1)^T$. The following vectors are useful:

$$\mathsf{bE} = \begin{pmatrix} 0 \\ 0 \\ E_o \end{pmatrix}, \quad \mathsf{b}^2\mathsf{E} = \begin{pmatrix} \kappa_z \\ \kappa_z\kappa_\theta \\ 1+\kappa_z^2 \end{pmatrix} E_o, \quad \mathsf{ba} = \begin{pmatrix} \kappa_z \\ \kappa_z\kappa_\theta \\ 1+\kappa_z^2 \end{pmatrix}.$$

   The invariants (8.6)-(8.9) are

$$I_1 = I_2 = 3 + \kappa^2, \quad I_4 = I_5 = E_o^2, \quad I_6 = E_o^2(1+\kappa_z^2), \quad I_7 = 1, \tag{8.225}$$
$$I_8 = 1 + \kappa_z^2, \quad I_9 = I_{10} = E_o. \tag{8.226}$$

   The components of the stress tensor (8.15) and the electric displacement (8.18) are

$$\tau_{rr} = -p + 2\Omega_1 + 4\Omega_2, \tag{8.227}$$
$$\tau_{\theta\theta} = -p + 2(1+\kappa_\theta^2)\Omega_1 + 2(2+\kappa^2)\Omega_2, \tag{8.228}$$
$$\tau_{zz} = -p + 2(1+\kappa_z^2)\Omega_1 + 2(2+\kappa^2)\Omega_2 + 2E_o^2\Omega_5 + 4E_o^2(1+\kappa^2)\Omega_6 + 2\Omega_7 + 4(1+\kappa_z^2)\Omega_8 + 2E_o\Omega_{10}, \tag{8.229}$$
$$\tau_{r\theta} = 2\kappa_\theta(\Omega_1 + \Omega_2), \tag{8.230}$$
$$\tau_{rz} = 2\kappa_z(\Omega_1 + \Omega_2 + E_o^2\Omega_6 + \Omega_8), \tag{8.231}$$
$$\tau_{\theta z} = 2\kappa_z\kappa_\theta(\Omega_1 + E_o^2\Omega_6 + \Omega_8), \tag{8.232}$$

[7] For the isotropic electroelastic case see [22].

and

$$D_r = -\kappa_z[2E_o\Omega_5 + 2E_o(2+\kappa^2)\Omega_6 + \Omega_{10}], \tag{8.233}$$

$$D_\theta = -\kappa_z\kappa_\theta[2E_o\Omega_5 + 2E_o(3+\kappa^2)\Omega_6 + \Omega_{10}], \tag{8.234}$$

$$D_z = -[2E_o\Omega_4 + 2E_o(1+\kappa_z^2)\Omega_5 + 2E_o(1+3\kappa_z^2+\kappa_z^4+\kappa_z^2\kappa_\theta^2)\Omega_6 + \Omega_9 + (1+\kappa_z^2)\Omega_{10}]. \tag{8.235}$$

If we define the energy function ω as

$$\omega = \omega(\kappa_\theta, \kappa_z, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10,$$

we get

$$\frac{\partial\omega}{\partial\kappa_\theta} = 2\kappa_\theta(\Omega_1 + \Omega_2), \tag{8.236}$$

$$\frac{\partial\omega}{\partial\kappa_z} = 2\kappa_z(\Omega_1 + \Omega_2 + E_o^2\Omega_6 + \Omega_8), \tag{8.237}$$

$$\frac{\partial\omega}{\partial E_o} = 2E_o\Omega_4 + 2E_o\Omega_5 + 2E_o\Omega_6(1+\kappa_z^2)\Omega_9 + \Omega_{10}, \tag{8.238}$$

from which we obtain

$$\tau_{r\theta} = \frac{\partial\omega}{\partial\kappa_\theta}, \quad \tau_{rz} = \frac{\partial\omega}{\partial\kappa_z}. \tag{8.239}$$

2. **A uniform radial electric field, and a radial alignment for the particles.**

In this case the external electric field is $\mathsf{E}_l = (E_o, 0, 0)^T$, where E_o is constant, and the alignment in the reference configuration of the electro-active particles is $\mathsf{a}_0 = (1, 0, 0)^T$. Then the electric field and the particle orientation in the current configuration are

$$\mathsf{E} = (E_o, 0, 0)^T, \quad \mathsf{a} = (1, \kappa_\theta, \kappa_z)^T.$$

The following vectors are useful:

$$\mathsf{bE} = E_o\begin{pmatrix} 1 \\ \kappa_\theta \\ \kappa_z \end{pmatrix}, \quad \mathsf{b}^2\mathsf{E} = E_o\begin{pmatrix} 1+\kappa^2 \\ \kappa_\theta(2+\kappa^2) \\ \kappa_z(2+\kappa^2) \end{pmatrix}, \quad \mathsf{ba} = \begin{pmatrix} 1+\kappa^2 \\ \kappa_\theta(2+\kappa^2) \\ \kappa_z(2+\kappa^2) \end{pmatrix}.$$

The invariants are given by (8.6)-(8.9)

$$I_1 = I_2 = 3 + \kappa^2, \quad I_4 = E_o^2, \quad I_5 = E_o^2(1+\kappa^2), \tag{8.240}$$

$$I_6 = E_o^2[\kappa^2 + (1+\kappa^2)^2], \quad I_7 = 1 + \kappa^2, \quad I_8 = \kappa^2 + (1+\kappa^2)^2, \tag{8.241}$$

$$I_9 = E_o, \quad I_{10} = E_o(1+\kappa^2). \tag{8.242}$$

The components of the stress and the electric displacement are (8.15)

$$\begin{aligned}
\tau_{rr} &= -p + 2\Omega_1 + 4\Omega_2 + 2E_o^2\Omega_5 + 4E_o^2(1+\kappa^2)\Omega_6 + 2\Omega_7 + 4(1+\kappa^2)\Omega_8 \\
&\quad + 2E_o\Omega_{10}, && (8.243) \\
\tau_{\theta\theta} &= -p + 2(1+\kappa_\theta^2)\Omega_1 + 2(2+\kappa^2)\Omega_2 + 2E_o^2\kappa_\theta^2\Omega_5 + 4E_o^2\kappa_\theta^2(2+\kappa^2)\Omega_6 \\
&\quad + 2\kappa_\theta^2\Omega_7 + 4\kappa_\theta^2(2+\kappa^2)\Omega_8 + 2E_o\kappa_\theta^2\Omega_{10}, && (8.244) \\
\tau_{zz} &= -p + 2(1+\kappa_z^2)\Omega_1 + 2(2+\kappa^2)\Omega_2 + 2E_o^2\kappa_z^2\Omega_5 + 4E_o^2\kappa_z^2(2+\kappa^2)\Omega_6 \\
&\quad + 2\kappa_z^2\Omega_7 + 4\kappa_z^2(2+\kappa^2)\Omega_8 + 2E_o\kappa_z^2\Omega_{10}, && (8.245) \\
\tau_{r\theta} &= 2\kappa_\theta[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2(3+2\kappa^2)\Omega_6 + \Omega_7 + (3+2\kappa^2)\Omega_8 \\
&\quad + E_o\Omega_{10}], && (8.246) \\
\tau_{rz} &= 2\kappa_z[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2(3+2\kappa^2)\Omega_6 + \Omega_7 + (3+2\kappa^2)\Omega_8 \\
&\quad + E_o\Omega_{10}], && (8.247) \\
\tau_{\theta z} &= 2\kappa_z\kappa_\theta[\Omega_1 + E_o^2\Omega_5 + 2E_o^2(2+\kappa^2)\Omega_6 + \Omega_7 + 2(2+\kappa^2)\Omega_8 + E_o\Omega_{10}], && (8.248)
\end{aligned}$$

and (8.18)

$$\begin{aligned}
D_r &= -[2E_o\Omega_4 + 2E_o(1+\kappa^2)\Omega_5 + 2E_o(1+3\kappa^2+\kappa^4)\Omega_6 + \Omega_9 \\
&\quad + (1+\kappa^2)\Omega_{10}], && (8.249) \\
D_\theta &= -\kappa_\theta[2E_o\Omega_4 + 2E_o(2+\kappa^2)\Omega_5 + 2E_o(3+4\kappa^2+\kappa^4)\Omega_6 + \Omega_9 \\
&\quad + (2+\kappa^2)\Omega_{10}], && (8.250) \\
D_z &= -\kappa_z[2E_o\Omega_4 + 2E_o(2+\kappa^2)\Omega_5 + 2E_o(3+4\kappa^2+\kappa^4)\Omega_6 + \Omega_9 \\
&\quad + (2+\kappa^2)\Omega_{10}]. && (8.251)
\end{aligned}$$

Regarding the stress, we can prove that the components of the stress satisfy the following non-linear universal relation (see for example [22])

$$(\tau_{\theta\theta} - \tau_{zz})\tau_{rz}\tau_{r\theta} = \tau_{\theta z}(\tau_{r\theta}^2 - \tau_{rz}^2). \tag{8.252}$$

This relation is also satisfied by the components of the stress if they are calculated from the constitutive equation (8.37) for the electric displacement $\mathsf{D}_l = (D_o/R, 0, 0)^T$, and for a uniform radial field orientation for the particles $\mathsf{a}_0 = (1, 0, 0)^T$.

Let's define

$$\omega = \omega(\kappa_\theta, \kappa_z, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10.$$

Then

$$\frac{\partial\omega}{\partial\kappa_\theta} = 2\kappa_\theta[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2\Omega_6(1+2\kappa^2) + \Omega_7 + \Omega_8(1+2\kappa^2) \\ E_o\Omega_{10}], \qquad (8.253)$$

$$\frac{\partial\omega}{\partial\kappa_z} = 2\kappa_z[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2\Omega_6(1+2\kappa^2) + \Omega_7 + \Omega_8(1+2\kappa^2) \\ E_o\Omega_{10}], \qquad (8.254)$$

$$\frac{\partial\omega}{\partial E_o} = 2E_o\Omega_4 + 2E_o\Omega_5(1+\kappa^2) + 2E_o\Omega_6(1+3\kappa^2+\kappa^4) + \Omega_9 \\ +\Omega_{10}(1+\kappa^2), \qquad (8.255)$$

from which we recover the connections (8.239) plus the following additional connection for the radial component of the electric displacement:

$$D_r = -\frac{\partial\omega}{\partial E_o}. \qquad (8.256)$$

The boundary conditions (6.57) are satisfied trivially if $-\infty \leq Z \leq \infty$.

8.4 Universal relations

For a transversely isotropic electro-active elastomer the number of invariants involved in the energy function implies that it is difficult to find this function from experiments (see Section 5.3 for the same problem with transversely isotropic MS elastomer). What is usually done is to assume a simplified form for the general constitutive equation, assuming that the energy function depends only on some of the invariants (8.6)-(8.9). Consider the constitutive equation (8.15)

$$\boldsymbol{\tau} = 2\mathbf{b}\Omega_1 + 2[I_1\mathbf{b} - \mathbf{b}^2]\Omega_2 - p\mathbf{I} + 2\mathbf{bE}\otimes\mathbf{bE}\Omega_5 + 2[\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{bE}]\Omega_6 \\ +2\mathbf{a}\otimes\mathbf{a}\Omega_7 + 2[\mathbf{a}\otimes\mathbf{ba} + \mathbf{ba}\otimes\mathbf{a}]\Omega_8 + [\mathbf{a}\otimes\mathbf{bE} + \mathbf{bE}\otimes\mathbf{a}]\Omega_{10}.$$

Let assume now that[8] $\Omega = \Omega(I_1, I_4, I_7, I_8, I_9, I_{10})$; then this reduces to

$$\boldsymbol{\tau} = 2\mathbf{b}\Omega_1 - p\mathbf{I} + 2\mathbf{a}\otimes\mathbf{a}\Omega_7 + 2[\mathbf{a}\otimes\mathbf{ba} + \mathbf{ba}\otimes\mathbf{a}]\Omega_8 + [\mathbf{a}\otimes\mathbf{bE} + \mathbf{bE}\otimes\mathbf{a}]\Omega_{10}. \qquad (8.257)$$

[8] In this section we assume a different simplification for the energy function as in Section 5.3; the reason is that the constitutive equations (5.34) and (8.15) are identical in form (changing $\mathbf{E}$ by $\mathbf{H}$); therefore if we assume a form for the constitutive equation for the electric case similar to the one presented in (5.191) for the magnetic problem, we would end up with the same linear universal relation presented in Section 5.4 (equation (5.272)). In order to expand more our results here we have assumed a simplified form for the energy function that depends on a different set of invariants. The purpose is just to illustrate a different case; we do not intend to provide a physical explanation for this choice of invariants (as we did in the introduction of Section 5.3).

One criterion that may be used in order to determine the validity of a given constitutive equation corresponds to universal relations. Universal relations are relations that hold independently of the particular form of the parameters of the constitutive equation for a given family of materials. If they do not hold, this means that is not possible to use that particular constitutive equation for the material under consideration [6] (see the introduction of Section 4.1). Then a method to know whether the above simplification is valid is to find one or more of these universal relations (we have already found one non-linear universal relation for the particular case of helical shear, which is valid for the full constitutive equation (8.15)). Regarding the linear universal relations, these can only be found when the number of parameters of the equation is less than the number of independent components of the stress, which in this case and in general is six; then for the simplified form (8.257) it is possible to find one such relation [85].

In order to look for a universal relation from (8.257) we use the same method as presented in Section 5.4 (see [21]). We repeat the main steps of this method here.

The method consists essentially in the following steps. Consider the notation

$$\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)^T \equiv (\tau_{11}, \tau_{22}, \tau_{33}, \tau_{23}, \tau_{31}, \tau_{12})^T. \tag{8.258}$$

Then (8.257) can be written alternatively as

$$\boldsymbol{\tau} = \mathbf{d}^{(o)}p + \mathbf{d}^{(1)}\Omega_1 + \mathbf{d}^{(7)}\Omega_7 + \mathbf{d}^{(8)}\Omega_8 + \mathbf{d}^{(10)}\Omega_{10}, \tag{8.259}$$

where $\mathbf{d}^{(i)}$ are vectors defined in the same way as $\boldsymbol{\tau}$, from in this case, the tensors $\mathbf{b}$, $\mathbf{I}$ and the components of the vectors $\mathbf{a}$ and $\mathbf{E}$. For the simplified form of the constitutive equation (8.259), the theory of Pucci and Saccomandi [85] predicts only one linear universal relation, which by using the above notation may be found by solving the following problem. Let's look for a vector $\mathbf{e}$ such that

$$\mathbf{e} \cdot \boldsymbol{\tau} = 0 \tag{8.260}$$

for any particular form of p, Ω_1, Ω_7, Ω_8 and Ω_{10}. Using (8.259) in (8.260) the problem is reduced to finding $\mathbf{e}$ from the linear system of equations

$$\mathcal{M}\mathbf{e} = \mathbf{0}, \tag{8.261}$$

where $\mathcal{M}$ is a 6×5 matrix formed from the components of the vectors $\mathbf{d}^{(i)}$. The solution of the above linear system of equations is the null space of this matrix, which in this problem has one element. The components of this element are

$$\begin{aligned}
\mathsf{e}_1 &= a_2^3 b_1 + a_3^2(a_1 b_{23} - a_3 b_{12}) - a_2^2(a_3 b_{12} + a_1 b_{23}) + a_2 a_3[a_3 b_{13} + a_1(b_{22} - b_{33})],\\
\mathsf{e}_2 &= a_3^2(a_3 b_{12} - a_2 b_{13}) + a_1^2(a_3 b_{12} + a_2 b_{13}) - a_1^3 b_{23} - a_1 a_3[a_3 b_{23} + a_2(b_{11} - b_{33})],\\
\mathsf{e}_3 &= a_2^2(a_3 b_{12} - a_2 b_{13}) - a_1^2(a_3 b_{12} + a_2 b_{13}) + a_1^3 b_{23} + a_1 a_2[a_3(b_{11} - b_{22}) + a_2 b_{23}],\\
\mathsf{e}_4 &= 2a_2 a_3(a_2 b_{13} - a_3 b_{12}) + 2a_1^2(a_3 b_{13} - a_2 b_{12}) + a_1[a_3^2(b_{22} - b_{11}) + a_2^2(b_{11} - b_{33})]\\
&\quad + a_1^3(b_{22} - b_{33}),\\
\mathsf{e}_5 &= 2a_1 a_3(a_3 b_{12} - a_1 b_{23}) + 2a_2^2(a_1 b_{12} - a_3 b_{23}) + a_2[a_3^2(b_{22} - b_{11}) + a_1^2(b_{33} - b_{22})]\\
&\quad + a_2^3(b_{33} - b_{11}),\\
\mathsf{e}_6 &= 2a_2(a_1^2 + a_3^2)b_{23} + a_2^2[a_3(b_{11} - b_{33}) - 2a_1 b_{13}] + a_3[a_3^2(b_{11} - b_{22}) - 2a_1 a_3 b_{13}\\
&\quad + a_1^2(b_{33} - b_{22})].
\end{aligned}$$

Note that the components do not depend on $\mathbf{E}$. For the linear universal relation (8.260) we have

$$\mathsf{e}_1\tau_{11} + \mathsf{e}_2\tau_{22} + \mathsf{e}_3\tau_{33} + \mathsf{e}_4\tau_{23} + \mathsf{e}_5\tau_{31} + \mathsf{e}_6\tau_{12} = 0. \tag{8.262}$$

The above linear universal relation is valid for any material described by (8.257), and for any particular form of $\mathbf{a}$. The application of this linear universal relation requires the use of universal solutions.

Consider now a special but important problem, in which the alignment of the electro-active particles in the reference configuration is the same as the orientation of the applied electric field (see the boundary value problems of Section 8.3). We have

$$\mathbf{a}_0 = \beta\mathbf{E}_l, \quad \text{where} \quad \beta = \frac{1}{|\mathbf{E}_l|}. \tag{8.263}$$

Then from the connections $\mathbf{E}_l = \mathbf{F}^T\mathbf{E}$ and $\mathbf{a} = \mathbf{F}\mathbf{a}_0$ we get

$$\mathbf{a} = \beta\mathbf{b}\mathbf{E}. \tag{8.264}$$

We work with the full constitutive equation (8.15), using (8.264) we get

$$\begin{aligned}
\boldsymbol{\tau} = 2\mathbf{b}\Omega_1 + 2[I_1\mathbf{b} - \mathbf{b}^2]\Omega_2 - p\mathbf{I} + 2\mathbf{bE}\otimes\mathbf{bE}(\Omega_5 + \beta^2\Omega_7 + \beta\Omega_{10})\\
+2(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{bE})(\Omega_6 + \beta^2\Omega_8).
\end{aligned} \tag{8.265}$$

From the theory developed by Pucci and Saccomandi [85] the above equation may generate one independent linear universal relation. Consider the notation

$$\gamma_1 = 2(\Omega_1 + I_1\Omega_2), \quad \gamma_2 = -2\Omega_2, \quad \gamma_3 = 2(\Omega_5 + \beta^2\Omega_7 + \beta\Omega_{10}),$$
$$\gamma_4 = 2(\Omega_6 + \beta^2\Omega_8).$$

Then we have

$$\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau} = \gamma_3(\mathbf{bE} \otimes \mathbf{E} - \mathbf{E} \otimes \mathbf{bE}) + \gamma_4(\mathbf{b}^2\mathbf{E} \otimes \mathbf{E} - \mathbf{E} \otimes \mathbf{b}^2\mathbf{E}). \tag{8.266}$$

Consider the following property for two vectors $\mathbf{u}$ and $\mathbf{v}$

$$(\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u})_\times = \mathbf{v} \times \mathbf{u}, \tag{8.267}$$

where the subscript $\times$ means the axial vector of an antisymmetric tensor. From (8.266) we finally obtain

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times = \gamma_3\mathbf{E} \times \mathbf{bE} + \gamma_4\mathbf{E} \times \mathbf{b}^2\mathbf{E}, \tag{8.268}$$

from where we get the universal relation (see (7.26))

$$(\boldsymbol{\tau}\mathbf{b}^{-1} - \mathbf{b}^{-1}\boldsymbol{\tau})_\times \cdot \mathbf{E} = 0. \tag{8.269}$$

Chapter 9

Variational formulations

The boundary value problem in non-linear magnetoelasticity (see Section 3.6) and in non-linear electroelasticity (see Section 6.4) involves seeking solutions of a system of non-linear partial differential equations; we need to consider the body and the free space surrounding it, and we need to use the boundary conditions (2.104) (or (2.105)). This problem is highly difficult, and there is little prospect of obtaining analytical solutions, except for cases with 'semi-infinite' geometries [99].

Therefore it is necessary to develop numerical methods of solution (see [4]). Bustamante et al. [15] solved a boundary value problem using the finite difference method (see Section 4.2); the geometry of the problems was simple and they only solved the magnetic part of the boundary value problem. More generally, to solve realistic boundary value problems of practical interest, a finite element approach is desirable. A prerequisite for such a formulation is a suitable variational principle or, at least, a virtual work principle.

In this chapter we develop variational principles for the non-linear magnetoelastic problem. We do not treat the closely similar problem for electro-active elastomers in this thesis; this will be done in a future paper in preparation [14], though some remarks, which concern the electroelastic problem can be found in the Conclusions.

Regarding past researchers on the variational formulation, the most important reference is the book by Brown [13], who provides a partial variational principle based on use of the magnetization as the independent magnetic variable together with the deformation function. This has been developed into a full variational principle by Kankanala and Triantafyllidis [61], who use a magnetic vector potential as a third variable. A variational principle equivalent to that of Brown but based on the magnetic field rather than the magnetization has been stated by Steigmann [103], and a different but essentially equivalent formulation of the theory is contained in the recent work of Ericksen [41].

This chapter is divided in 3 sections. In Section 9.1 we summarize the basic equations of non-linear magnetoelasticity (see Subsection 2.3.1 and Chapter 3), in particular we review different constitutive laws based on different choices of independent magnetic variables. Section 9.2 is devoted to establishing connections between different energy expressions that are used in variational formulations of the non-linear magnetoelastostatic problem. Two new variational formulations are then derived, one based on the scalar magnetic potential and one on the vector potential. In each case all the relevant governing equations, boundary and continuity conditions are derived. In Section 9.3 we propose a variational formulation for a problem of a body interacting with a rigid semi-infinite body, which would be a first attempt to explore a formulation for a mixed boundary value problem.

This chapter is based on the results presented in [17] and [16].

9.1 Basic equations

In this section we review briefly the main results for the theory of magneto-active elastomers presented in Section 3, and then we present different possibilities for constitutive equations depending on the independent magnetic variable chosen for each case.

9.1.1 The equations of magnetostatics

From Sections 2.2 and 3 we had that the fields $\mathbf{H}$ and $\mathbf{B}$ satisfy the field equations (2.102)

$$\mathrm{curl}\mathbf{H} = \mathbf{0}, \quad \mathrm{div}\mathbf{B} = 0. \tag{9.1}$$

For condensed matter we introduced the magnetization vector $\mathbf{M}$, which is defined in terms of $\mathbf{H}$ and $\mathbf{B}$ via $(2.107)_2$

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}). \tag{9.2}$$

In vacuum or in non-magnetizable materials $\mathbf{M} = \mathbf{0}$, and we have $(2.106)_2$

$$\mathbf{B} = \mu_o\mathbf{H}. \tag{9.3}$$

From (2.104) the continuity condition for $\mathbf{H}$ and $\mathbf{B}$ across the boundary $\partial\mathcal{B}$ of the body $\mathcal{B}$ were

$$[\![\mathbf{H}]\!] \times \mathbf{n} = \mathbf{0}, \quad [\![\mathbf{B}]\!] \cdot \mathbf{n} = 0. \tag{9.4}$$

Proposition 9.1. *The boundary conditions (9.4) are equivalent to (see, for example, [103])*

$$[\![\mathbf{H}]\!] = (\mathbf{M}\cdot\mathbf{n})\mathbf{n}, \quad [\![\mathbf{B}]\!] = \mu_o[(\mathbf{M}\cdot\mathbf{n})\mathbf{n} - \mathbf{M}]. \tag{9.5}$$

Proof. Let $\mathbf{H}^i$, $\mathbf{B}^i$ denote the fields inside the body close to $\partial\mathcal{B}$, and let $\mathbf{H}^o$, $\mathbf{B}^o$ denote the fields outside the body (vacuum) close to $\partial\mathcal{B}$. The open square brackets in (9.4) designates the jump of the quantity in passing from the inside to the outside of the body (see Subsection 2.2.2). Thus,

$$[\![\mathbf{H}]\!] = \mathbf{H}^o - \mathbf{H}^i, \quad [\![\mathbf{B}]\!] = \mathbf{B}^o - \mathbf{B}^i. \tag{9.6}$$

With the above notation (9.2) and (9.3) can be rewritten respectively as (the magnetization only exists inside the body)

$$\mathbf{B}^i = \mu_o(\mathbf{H}^i + \mathbf{M}), \quad \mathbf{B}^o = \mu_o\mathbf{H}^o; \tag{9.7}$$

therefore

$$[\![\mathbf{B}]\!] = \mu_o([\![\mathbf{H}]\!] - \mathbf{M}), \tag{9.8}$$

and so

$$0 = [\![\mathbf{B}]\!]\cdot\mathbf{n} = \mu_o([\![\mathbf{H}]\!] - \mathbf{M})\cdot\mathbf{n}, \tag{9.9}$$

as a result

$$[\![\mathbf{H}]\!]\cdot\mathbf{n} = \mathbf{M}\cdot\mathbf{n}. \tag{9.10}$$

Let's decompose $[\![\mathbf{H}]\!]$ as $[\![\mathbf{H}]\!] = [\![\mathbf{H}]\!]_T + [\![\mathbf{H}]\!]_P$, where $[\![\mathbf{H}]\!]_P$ is the component of $[\![\mathbf{H}]\!]$ parallel to $\mathbf{n}$, and $[\![\mathbf{H}]\!]_T$ is the component normal to $\mathbf{n}$ (which is tangential to the surface of the body $\partial\mathcal{B}$); the boundary condition $(9.4)_1$ means that $[\![\mathbf{H}]\!]_T = \mathbf{0}$, and so $[\![\mathbf{H}]\!] = [\![\mathbf{H}]\!]_P$. The norm of $[\![\mathbf{H}]\!]_P$ is the absolute value of $[\![\mathbf{H}]\!]\cdot\mathbf{n}$, and so from (9.10) it is easy to show that

$$[\![\mathbf{H}]\!] = [\![\mathbf{H}]\!]_P = (\mathbf{M}\cdot\mathbf{n})\mathbf{n}. \tag{9.11}$$

From (9.8) with the above result we have

$$[\![\mathbf{B}]\!] = \mu_o[(\mathbf{M}\cdot\mathbf{n})\mathbf{n} - \mathbf{M}], \tag{9.12}$$

and it is easy to prove that $(9.4)_2$ holds.

□

9.1.2 Equilibrium, stress and constitutive laws

As was mentioned in the introduction of Section 2.3, for deformable media it is possible to find many different definitions of 'stress' tensors that can be included in the equilibrium equation. In this thesis we have used the so-called 'total stress tensor' (see, for example, [32, 33]), which has been denoted $\boldsymbol{\tau}$. This tensor is symmetric (see Sections 3.3 and 6.2), and is the analogue of the Cauchy stress tensor arising in elasticity theory.

In terms of $\boldsymbol{\tau}$ the equilibrium equation has the form

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0}, \tag{9.13}$$

where $\mathbf{f}$ is the 'mechanical' body force per unit of mass and ρ is the mass density of the material in the configuration $\mathcal{B}$.

As was emphasized in the remark of Section 3.3, $\boldsymbol{\tau}$ incorporates terms that may be considered as magnetic body forces rather than stresses; while from the mathematical point of view there is no essential difference in treating the magnetic contribution as a body force vector or as a stress tensor there are differences in the resulting physical interpretations (see [89]).

We assume that the material is not subject to any internal mechanical constraint.

9.1.2.1 Formulation based on the magnetization

Consider the classical formulation of Brown [13]. This is based on use of $\bar{\mathbf{M}} \equiv \frac{1}{\rho}\mathbf{M}$ as the independent magnetic variable and a free energy function per unit mass; in this chapter this energy is denoted $\chi(\mathbf{F}, \bar{\mathbf{M}})$. This energy does not include the magnetic self energy. With this function the 'Cauchy stress', denoted $\bar{\boldsymbol{\sigma}}$, and the magnetic field are given as

$$\bar{\boldsymbol{\sigma}} = \rho\mathbf{F}\frac{\partial\chi}{\partial\mathbf{F}}, \quad \mu_o\mathbf{H} = \frac{\partial\chi}{\partial\bar{\mathbf{M}}}. \tag{9.14}$$

Note that in general $\bar{\boldsymbol{\sigma}}$ is not symmetric.

The equilibrium equation (9.13) can then be expressed in the form

$$\operatorname{div}\bar{\boldsymbol{\sigma}} + \mu_o(\mathbf{M}\cdot\operatorname{grad})\mathbf{H} + \rho\mathbf{f} = \mathbf{0}, \tag{9.15}$$

the term $\mu_o(\mathbf{M}\cdot\operatorname{grad})\mathbf{H}$ having the role of a magnetic body force relative to the stress tensor $\bar{\boldsymbol{\sigma}}$.

Proposition 9.2. *The stress $\bar{\boldsymbol{\sigma}}$ is related to $\boldsymbol{\tau}$ by*

$$\boldsymbol{\tau} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\tau}_m, \tag{9.16}$$

where $\boldsymbol{\tau}_m$ is referred as the Maxwell stress tensor, and is defined by[1]

$$\boldsymbol{\tau}_m = \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}\mu_o(\mathbf{H} \cdot \mathbf{H})\mathbf{I}. \tag{9.17}$$

Proof. We have to show that

$$\operatorname{div} \boldsymbol{\tau}_m = \mu_o(\mathbf{M} \cdot \operatorname{grad} \mathbf{H}). \tag{9.18}$$

The component form of $\boldsymbol{\tau}_m$ in Cartesian coordinates is

$$\tau_{m_{ij}} = B_i H_j - \frac{1}{2}\mu_o H_k H_k \delta_{ij}. \tag{9.19}$$

and so $\operatorname{div} \boldsymbol{\tau}_m$ is equivalent to

$$\tau_{m_{ij,i}} = B_{i,i} H_j + B_i H_{i,j} - \mu_o H_{k,i} H_k \delta_{ij}. \tag{9.20}$$

But (9.1) in component form are equivalent to

$$H_{i,j} = H_{j,i}, \quad B_{i,i} = 0, \tag{9.21}$$

thus from (9.20) we get

$$\tau_{m_{ij,i}} = (B_i - \mu_o H_i) H_{j,i}, \tag{9.22}$$

which by using (9.20) is equivalent to $\mu_o(\mathbf{M} \cdot \operatorname{grad} \mathbf{H})$. □

On the boundary $\partial\mathcal{B}$ of the body the traction (per unit area) associated with $\bar{\boldsymbol{\sigma}}$ is given by (see [13,61,103])

$$\bar{\boldsymbol{\sigma}}^T \mathbf{n} = \mathbf{t}_a + \frac{1}{2}\mu_o(\mathbf{M} \cdot \mathbf{n})^2 \mathbf{n}, \tag{9.23}$$

where $\mathbf{t}_a$ is the applied mechanical traction (per unit area) and $\mathbf{n}$ is again the unit outward normal to $\partial\mathcal{B}$.

It follows from (9.16), (9.17) and (9.23) that

$$\boldsymbol{\tau}\mathbf{n} = \bar{\boldsymbol{\sigma}}^T \mathbf{n} + \bar{\boldsymbol{\tau}}_m^T \mathbf{n} = \mathbf{t}_a + \frac{1}{2}(\mathbf{M} \cdot \mathbf{n})^2 \mathbf{n} + (\mathbf{B} \cdot \mathbf{n})\mathbf{H} - \frac{1}{2}\mu_o(\mathbf{H} \cdot \mathbf{H})\mathbf{n}. \tag{9.24}$$

[1] From the definition 3.1 we had seemingly different expressions for the Maxwell stress (equations (3.43) and (3.44)). For free space with (9.3), however, we see that (9.17) is equivalent to (3.43) and (3.44). These differences arise from the different definitions of stresses and body forces we see in this section, which is equivalent, to say, on which magnetic variable is taken to be independent and on the associated stress tensor. See Table 1 in Kankanala and Triantafyllidis [61] for a list of some different Maxwell stress and magnetic body force expressions.

Proposition 9.3. *Let's define the (symmetric) Maxwell stress outside the material (close to* $\partial\mathcal{B}$*) as*

$$\boldsymbol{\tau}_m^o = \mathbf{B}^o \otimes \mathbf{H}^o - \frac{1}{2}\mu_o(\mathbf{H}^o \cdot \mathbf{H}^o)\mathbf{I}, \tag{9.25}$$

with $\mathbf{B}^o = \mu_o \mathbf{H}^o$*. Then, we have*

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_a + \boldsymbol{\tau}_m^o \mathbf{n}, \tag{9.26}$$

i.e. the traction calculated from the total stress in the body is balanced, on the relevant part of the boundary, by the applied mechanical tractions on the exterior of the body together with the effect of the Maxwell stress exterior to the body.

Proof. We show now that $\frac{1}{2}\mu_o(\mathbf{M}\cdot\mathbf{n})^2\mathbf{n} + (\mathbf{B}\cdot\mathbf{n})\mathbf{H} - \frac{1}{2}\mu_o(\mathbf{H}\cdot\mathbf{H})\mathbf{n}$ is equivalent to[2] $\boldsymbol{\tau}_m^o\mathbf{n}$.

The total traction on the boundary of the body due to $\boldsymbol{\tau}_m^o$ is

$$\int_{\partial\mathcal{B}} \boldsymbol{\tau}_m^o \mathbf{n}\, da = \int_{\partial\mathcal{B}} \left[(\mathbf{B}^o\cdot\mathbf{n})\mathbf{H}^o - \frac{1}{2}(\mathbf{H}^o\cdot\mathbf{B}^o)\mathbf{n}\right] da. \tag{9.27}$$

Using (9.5) we have

$$\mathbf{H}^o = \mathbf{H}^i + (\mathbf{M}\cdot\mathbf{n})\mathbf{n}, \qquad \mathbf{B}^o = \mu_o[\mathbf{H}^i + (\mathbf{M}\cdot\mathbf{n})\mathbf{n}]. \tag{9.28}$$

With (9.28) in (9.25) after some algebraic manipulations we obtain

$$\int_{\partial\mathcal{B}} \boldsymbol{\tau}_m \mathbf{n}\, da = \int_{\partial\mathcal{B}} \left[\frac{1}{2}\mu_o(\mathbf{M}\cdot\mathbf{n})^2\mathbf{n} + (\mathbf{B}^i\cdot\mathbf{n})\mathbf{H}^i - \frac{1}{2}\mu_o(\mathbf{H}^i\cdot\mathbf{H}^i)\mathbf{n}\right] da. \tag{9.29}$$

Dropping the index i we see that indeed $\frac{1}{2}\mu_o(\mathbf{M}\cdot\mathbf{n})^2\mathbf{n} + (\mathbf{B}\cdot\mathbf{n})\mathbf{H} - \frac{1}{2}\mu_o(\mathbf{H}\cdot\mathbf{H})\mathbf{n}$ represents the Maxwell stress times the normal vector.

□

9.1.2.2 Formulation based on the magnetic field

A formulation of the equations based on use of $\mathbf{H}$ as the independent magnetic variable has been used by Steigmann [103]. In this, the energy function $\psi(\mathbf{F}, \mathbf{H})$ is adopted, from which we obtain

$$\bar{\boldsymbol{\sigma}} = \rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{F}}, \quad \mu_o\bar{\mathbf{M}} = -\frac{\partial\psi}{\partial\mathbf{H}}. \tag{9.30}$$

This yield the same stress tensor $\bar{\boldsymbol{\sigma}}$ as does χ. The functions χ and ψ are related by the partial Legendre transform

$$\chi = \psi + \mu_o\bar{\mathbf{M}}\cdot\mathbf{H}. \tag{9.31}$$

The equilibrium equations and traction boundary conditions are the same as for χ.

[2] An important observation about this expression is that the magnetic field, magnetic induction (and of course the magnetization) are evaluated on the boundary of the body approaching from inside.

9.1.2.3 Formulation based on the magnetic induction

Another option is to base the formulation on the magnetic vector $\mathbf{B}$ and to define an energy function[3] $\phi(\mathbf{F}, \mathbf{B})$. This yields the stress tensor, denoted $\boldsymbol{\sigma}$, and the magnetization $\mathbf{M}$ in the forms (see equations (2.121) and (3.4))

$$\boldsymbol{\sigma} = \rho \mathbf{F}\frac{\partial \phi}{\partial \mathbf{F}}, \quad \mathbf{M} = -\rho\frac{\partial \phi}{\partial \mathbf{B}}, \tag{9.32}$$

while the equilibrium equations becomes[4] [80]

$$\mathrm{div}\boldsymbol{\sigma} + (\mathrm{grad}\mathbf{B})^T\mathbf{M} + \rho\mathbf{f} = \mathbf{0}. \tag{9.33}$$

In this case the term $(\mathrm{grad}\mathbf{B})^T\mathbf{M}$ has the role of a magnetic body force (in respect to the stress $\boldsymbol{\sigma}$).

Proposition 9.4. *We have the connections*

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mathbf{B}\otimes\mathbf{H} - (\mathbf{H}\cdot\mathbf{B})\mathbf{I} + \frac{1}{2}\mu_o^{-1}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}, \quad \bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \frac{1}{2}\mu_o(\mathbf{M}\cdot\mathbf{M})\mathbf{I}. \tag{9.34}$$

Proof. Let's prove $(9.34)_1$. Consider (3.41)

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mu_o^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right] + (\mathbf{M}\cdot\mathbf{B})\mathbf{I} - \mathbf{B}\otimes\mathbf{M},$$

using (9.2) $\mu_o\mathbf{M} = \mathbf{B} - \mu_o\mathbf{H}$ in the above equation we have

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mu_o^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right] + [(\mu_o^{-1}\mathbf{B} - \mathbf{H})\cdot\mathbf{B}]\mathbf{I} - \mathbf{B}\otimes(\mu_o^{-1}\mathbf{B} - \mathbf{H}), \tag{9.35}$$

and after some manipulations we obtain

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mathbf{B}\otimes\mathbf{H} - (\mathbf{H}\cdot\mathbf{B})\mathbf{I} + \frac{1}{2}\mu_o^{-1}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}.$$

We prove now $(9.34)_2$. From (9.16) we have

$$\boldsymbol{\tau} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\tau}_m = \bar{\boldsymbol{\sigma}} + \mathbf{B}\otimes\mathbf{H} - \frac{1}{2}\mu_o(\mathbf{H}\cdot\mathbf{H})\mathbf{I},$$

and with $(9.34)_1$ we have

$$\begin{aligned}\bar{\boldsymbol{\sigma}} &= \boldsymbol{\sigma} - (\mathbf{H}\cdot\mathbf{B})\mathbf{I} + \frac{1}{2}\mu_o^{-1}(\mathbf{B}\cdot\mathbf{B})\mathbf{I} + \frac{1}{2}\mu_o(\mathbf{H}\cdot\mathbf{H})\mathbf{I},\\ &= \boldsymbol{\sigma} + \frac{1}{2}[\mathbf{B}\cdot(\mu_o^{-1}\mathbf{B} - \mathbf{H})]\mathbf{I} + \frac{1}{2}\mu_o[\mathbf{H}\cdot(\mathbf{H} - \mu_o^{-1}\mathbf{B})]\mathbf{I},\end{aligned} \tag{9.36}$$

[3]See Subsection 2.3.1, where we use a different notation for this energy function.

[4]See equations (2.109), (2.110), (3.32) and (3.33).

using (9.2) we have

$$\begin{aligned}\bar{\sigma} &= \sigma + \frac{1}{2}[(\mathbf{B} - \mu_o\mathbf{H}) \cdot \mathbf{M}]\mathbf{I}, \\ &= \sigma + \frac{1}{2}\mu_o(\mathbf{M} \cdot \mathbf{M})\mathbf{I}. \end{aligned} \tag{9.37}$$

□

We also note that [103]

$$\rho\psi = \rho\phi + \frac{1}{2}\mu_o\mathbf{M} \cdot \mathbf{M}. \tag{9.38}$$

This is not a standard partial Legendre transform but can be converted into one by defining ϕ^* and ψ^* by

$$\rho\phi^*(\mathbf{F},\mathbf{B}) = \rho\phi(\mathbf{F},\mathbf{B}) + \frac{1}{2}\mu_o^{-1}\mathbf{B} \cdot \mathbf{B}, \quad \rho\psi^*(\mathbf{F},\mathbf{H}) = \rho\psi(\mathbf{F},\mathbf{H}) - \frac{1}{2}\mu_o\mathbf{H} \cdot \mathbf{H}, \tag{9.39}$$

so that

$$\rho\phi^*(\mathbf{F},\mathbf{B}) = \rho\psi^*(\mathbf{F},\mathbf{H}) + \mathbf{B} \cdot \mathbf{H}. \tag{9.40}$$

Proposition 9.5. *The stress tensors associated with ϕ^* and ψ^*, denoted σ^* and $\bar{\sigma}^*$ respectively , are given by*

$$\sigma^* = \rho\mathbf{F}\frac{\partial\phi^*}{\partial\mathbf{F}} = \sigma + \frac{1}{2}\mu_o^{-1}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}, \quad \bar{\sigma}^* = \rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{F}} = \bar{\sigma} - \frac{1}{2}\mu_o(\mathbf{H} \cdot \mathbf{H})\mathbf{I}. \tag{9.41}$$

Proof. Let's prove $(9.41)_1$. From $(9.39)_1$ we have $\rho\phi^*(\mathbf{F},\mathbf{B}) - \frac{1}{2}\mu_o^{-1}\mathbf{B} \cdot \mathbf{B} = \rho\phi(\mathbf{F},\mathbf{B})$; differentiating in $\mathbf{F}$ and after some rearrangements we get

$$\rho\frac{\partial\phi^*}{\partial\mathbf{F}} = \rho\frac{\partial\phi}{\partial\mathbf{F}} + \frac{\partial\rho}{\partial\mathbf{F}}(\phi - \phi^*), \tag{9.42}$$

but $\frac{\partial\rho}{\partial\mathbf{F}} = -\rho\mathbf{F}^{-1}$ (see, for example, [78]); as a result, multiplying (9.42) from the left by $\mathbf{F}$ we have

$$\rho\mathbf{F}\frac{\partial\phi^*}{\partial\mathbf{F}} = \rho\mathbf{F}\frac{\partial\phi}{\partial\mathbf{F}} + \rho(\phi - \phi^*)\mathbf{I}, \tag{9.43}$$

while from $(9.39)_1$, defining $\sigma^* = \rho\mathbf{F}\frac{\partial\phi^*}{\partial\mathbf{F}}$, and using $(9.32)_1$, from (9.43) we get

$$\sigma^* = \sigma + \frac{1}{2}\mu_o^{-1}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}.$$

We can prove in the same way $(9.41)_2$. □

We also have

$$\rho\frac{\partial\phi^*}{\partial\mathbf{B}} = \mathbf{H}, \quad \rho\frac{\partial\psi^*}{\partial\mathbf{H}} = -\mathbf{B}. \tag{9.44}$$

From $(9.41)_1$ we have

$$\begin{aligned}\operatorname{div}\boldsymbol{\sigma} &= \operatorname{div}\boldsymbol{\sigma}^* - \frac{1}{2}\mu_o^{-1}\operatorname{div}[(\mathbf{B}\cdot\mathbf{B})\mathbf{I}],\\ &= \operatorname{div}\boldsymbol{\sigma}^* - \mu_o^{-1}(\operatorname{grad}\mathbf{B})^T\mathbf{B}.\end{aligned} \tag{9.45}$$

As a result in (9.33) we get

$$\operatorname{div}\boldsymbol{\sigma}^* - (\operatorname{grad}\mathbf{B})^T(\mathbf{M} - \mu_o^{-1}\mathbf{B}) + \rho\mathbf{f} = \mathbf{0}, \tag{9.46}$$

which from (9.2) is equivalent to

$$\operatorname{div}\boldsymbol{\sigma}^* - (\operatorname{grad}\mathbf{B})^T\mathbf{H} + \rho\mathbf{f} = \mathbf{0}. \tag{9.47}$$

If we work with $\bar{\boldsymbol{\sigma}}^*$ we can prove in the same way that the equilibrium equation becomes

$$\operatorname{div}\bar{\boldsymbol{\sigma}}^* + (\mathbf{B}\cdot\operatorname{grad})\mathbf{H} + \rho\mathbf{f} = \mathbf{0}. \tag{9.48}$$

We see from (9.47) and (9.48) that the magnetic 'body force' term is different for each choice of 'stress' tensor.

The formulation listed in the above sections are all equivalent, but they are not the only possible ones. The concept of 'stress', 'Maxwell stress' and 'magnetic body force' inside the material are clearly not uniquely defined. The formulation based on the 'total stress' [33,34] is simplest mathematically and avoids the need for defining either a magnetic body force or a Maxwell stress within a magnetizable material.

9.1.3 Lagrangian formulation and the total energy function

Now we recall some of the concepts seen in Sections 3.1 and 3.3.

We defined the Lagrangian counterparts of $\mathbf{H}$ and $\mathbf{B}$, denoted $\mathbf{H}_l$ and $\mathbf{B}_l$ respectively, as

$$\mathbf{H}_l = \mathbf{F}^T\mathbf{H}, \quad \mathbf{B}_l = J\mathbf{F}\mathbf{B}. \tag{9.49}$$

Using the identities (3.10), the field equations (9.1) are equivalent to

$$\operatorname{Curl}\mathbf{H}_l = \mathbf{0}, \quad \operatorname{Div}\mathbf{B}_l = 0. \tag{9.50}$$

with the jump conditions (3.67)

$$[\![\mathbf{H}_l]\!] \times \mathbf{N} = \mathbf{0}, \quad [\![\mathbf{B}_l]\!]\cdot\mathbf{N} = 0. \tag{9.51}$$

The equilibrium equation (9.13) in the reference configuration is [33, 34]

$$\mathrm{Div}\,\mathbf{T} + \rho_o \mathbf{f} = \mathbf{0}, \tag{9.52}$$

where $\mathbf{T}$ is the nominal stress tensor associated with the total stress tensor $\boldsymbol{\tau}$ (see (3.48)).

In Section 3.3 and Subsection 9.1.2.3 we introduced the energy function $\phi(\mathbf{F}, \mathbf{B})$ (in Section 3.3 we used a different notation for this function). From (3.18) with $(9.49)_2$ we introduced the function Φ as

$$\Phi(\mathbf{F}, \mathbf{B}_l) \equiv \phi(\mathbf{F}, J^{-1}\mathbf{F}\mathbf{B}_l). \tag{9.53}$$

And we could define the total energy function Ω as (Section 3.3, [33])

$$\Omega = \rho_o \Phi + \frac{1}{2} J^{-1} \mathbf{B}_l \cdot (\mathbf{c}\mathbf{B}_l). \tag{9.54}$$

This enables $\mathbf{T}$ and $\boldsymbol{\tau}$ to be given in the simple forms (equations (3.58) and (3.59))

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}, \tag{9.55}$$

and the Lagrangian and Eulerian forms $\mathbf{H}_l$ and $\mathbf{H}$ of the magnetic field are correspondingly (equations (3.62) and $(4.1)_2$)

$$\mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}, \quad \mathbf{H} = \mathbf{F}^{-T}\frac{\partial \Omega}{\partial \mathbf{B}_l}. \tag{9.56}$$

If, instead of $\mathbf{B}_l$, we wish to use $\mathbf{H}_l$ as the independent magnetic variable then this can be done, for example, by defining the complementary version of $\Omega(\mathbf{F}, \mathbf{B}_l)$, denoted $\Omega^*(\mathbf{F}, \mathbf{H}_l)$, through the partial Legendre transformation (see Subsection 3.3.3)

$$\Omega^*(\mathbf{F}, \mathbf{H}_l) = \Omega(\mathbf{F}, \mathbf{B}_l) - \mathbf{H}_l \cdot \mathbf{B}_l, \tag{9.57}$$

so that

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}, \tag{9.58}$$

which have Eulerian counterparts (see equations (3.64) and (3.65))

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B} = -J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{H}_l}. \tag{9.59}$$

From (9.54) multiplying by J^{-1} and with $(9.49)_2$ we get

$$J^{-1}\Omega = J^{-1}\rho_o\Phi + \frac{1}{2}\mu_o^{-1}J^{-2}(J\mathbf{F}^{-1}\mathbf{B}) \cdot (\mathbf{c}J\mathbf{F}^{-1}\mathbf{B}), \tag{9.60}$$

and from $(9.39)_1$ we get the connection

$$J^{-1}\Omega = \rho\phi + \frac{1}{2}\mu_o^{-1}\mathbf{B} \cdot \mathbf{B} = \rho\phi^*. \tag{9.61}$$

From (9.57) and $(9.39)_2$ we can found in the same way the connection

$$J^{-1}\Omega^*(\mathbf{F}, \mathbf{H}_l) = \rho\psi^*(\mathbf{F}, \mathbf{H}). \tag{9.62}$$

9.2 Energy and variational formulations

In the literature it is customary to consider the magnetic field as consisting of two contributions [13, 61]: an applied field $\mathbf{H}_a$ in the absence of material, and an additional 'self' field $\mathbf{H}_s$ generated by the presence of a magnetic material body that leaves $\mathbf{H}_a$ unchanged. These and the total field $\mathbf{H}$ each satisfies equation $(9.1)_1$ and the associated magnetostatic potentials are denoted φ_a, φ_s and φ, so that

$$\mathbf{H}_a = -\mathrm{grad}\varphi_a, \quad \mathbf{H}_s = -\mathrm{grad}\varphi_s, \quad \mathbf{H} = -\mathrm{grad}\varphi, \tag{9.63}$$

where $\varphi = \varphi_a + \varphi_s$ and $\mathbf{H} = \mathbf{H}_a + \mathbf{H}_s$. The magnetic induction is $\mathbf{B} = \mathbf{B}_a + \mathbf{B}_s$, where $\mathbf{B}_a = \mu_o \mathbf{H}_a$ everywhere, $\mathbf{B}_s = \mu_o \mathbf{H}_s$ outside the material, and $\mathbf{B}_s = \mu_o(\mathbf{H}_s + \mathbf{M})$ and $\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})$ inside the material, $\mathbf{M}$ being the magnetization. Moreover,

$$\mathrm{div}\mathbf{B} = \mathrm{div}\mathbf{B}_a = \mathrm{div}\mathbf{B}_s = 0 \tag{9.64}$$

both inside and outside the material.

The energy formulation of Brown [13] uses $\mathbf{x}$ and $\bar{\mathbf{M}} = \frac{1}{\rho}\mathbf{M}$ as the independent variables and the associated functional may be written as

$$\Pi\{\mathbf{x}, \bar{\mathbf{M}}\} = E\{\mathbf{x}, \bar{\mathbf{M}}\} - L\{\mathbf{x}\}, \tag{9.65}$$

where $L\{\mathbf{x}\}$ is the work of the mechanical loading, which consists of both body forces and boundary tractions. In [13] the load were taken of 'dead' type. Here, we consider the body force to be conservative, such that $\mathbf{f} = -\mathrm{grad}U$, where $U = U(\mathbf{x})$ is the associated potential, and the traction to be a dead load. We therefore write

$$L\{\mathbf{x}\} = -\int_{\mathcal{B}} \rho U \,\mathrm{d}v + \int_{\partial\mathcal{B}} \mathbf{t}_a \cdot \mathbf{x} \,\mathrm{d}a, \tag{9.66}$$

where $\mathbf{t}_a$ is the traction per unit current area.

The energy E may be written in terms of an energy density function $e(\mathbf{F}, \bar{\mathbf{M}})$ as (see [13] Chapter II, page 73, equation (7.13))

$$E = \int_{\mathcal{B}} \rho e \,\mathrm{d}v, \tag{9.67}$$

where

$$e(\mathbf{F}, \bar{\mathbf{M}}) = \chi(\mathbf{F}, \bar{\mathbf{M}}) - \frac{1}{2}\mu_o \bar{\mathbf{M}} \cdot \mathbf{H}_s - \mu_o \bar{\mathbf{M}} \cdot \mathbf{H}_a, \tag{9.68}$$

in which χ is the free energy, the second term is the magnetic self energy and the third term is the external work of the applied magnetic field (all per unit mass).

From (9.31), we see that e may also be written

$$e(\mathbf{F}, \bar{\mathbf{M}}) = \psi(\mathbf{F}, \mathbf{H}) + \frac{1}{2}\mu_o \bar{\mathbf{M}} \cdot \mathbf{H}_a, \tag{9.69}$$

which is the form used by Steigmann [103] when adopting $\mathbf{H}$ as the independent magnetic variable. Also, from (9.38) and (9.39), we have

$$\rho e(\mathbf{F}, \bar{\mathbf{M}}) = \rho\psi^*(\mathbf{F}, \mathbf{H}) + \frac{1}{2}\mathbf{B} \cdot \mathbf{H} - \frac{1}{2}\mu_o \mathbf{M} \cdot \mathbf{H}_a \tag{9.70}$$

$$= \rho\phi^*(\mathbf{F}, \mathbf{B}) - \frac{1}{2}\mathbf{B} \cdot \mathbf{H} - \frac{1}{2}\mu_o \mathbf{M} \cdot \mathbf{H}_a. \tag{9.71}$$

Consider the integral (9.67), which on use, for example, of (9.70) can be written as

$$E = \int_{\mathcal{B}} \left(\rho\psi^* + \frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) dv - \frac{1}{2}\mu_o \int_{\mathcal{B}} \mathbf{M} \cdot \mathbf{H}_a \, dv. \tag{9.72}$$

By using $\mathbf{H} = -\text{grad}\varphi$, $\text{div}\mathbf{B} = 0$, the continuity of φ and $\mathbf{B} \cdot \mathbf{n}$ across $\partial\mathcal{B}$ and the divergence theorem (once for the body and once for its exterior) we obtain, for the second term of the first integral in (9.72),

$$\frac{1}{2}\int_{\mathcal{B}} \mathbf{B} \cdot \mathbf{H} \, dv = -\frac{1}{2}\mu_o \int_{\mathcal{B}^o} \mathbf{H} \cdot \mathbf{H} \, dv - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi\mathbf{B} \cdot \mathbf{n} \, da, \tag{9.73}$$

where $\mathcal{B}^o$ is the complement of $\mathcal{B} \cup \partial\mathcal{B}$ (i.e. the exterior of the body) and $\partial\mathcal{B}^\infty$ is the boundary of $\mathcal{B}^o$ far from the body, as depicted in Figure 9.1.

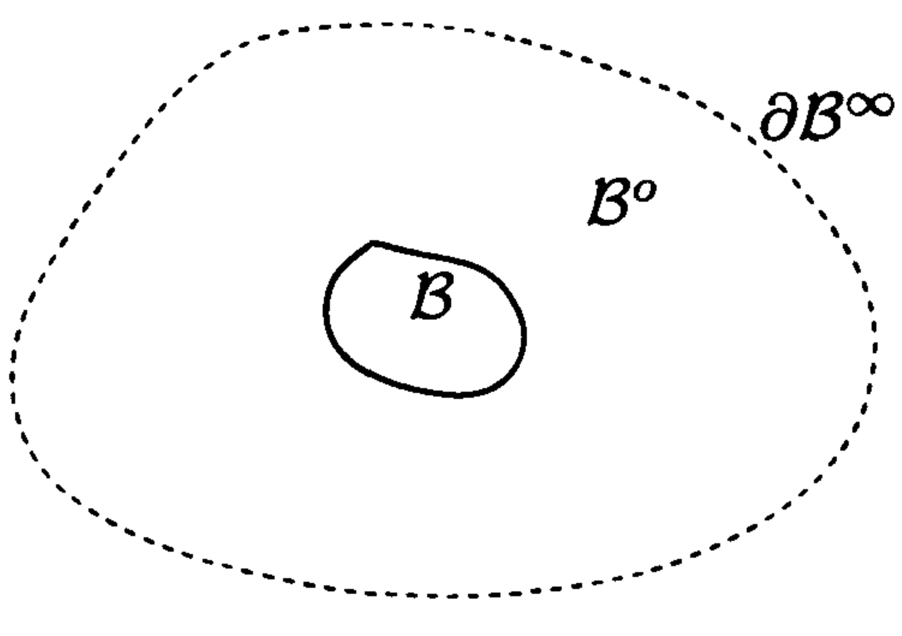


Figure 9.1: Depiction of the material body $\mathcal{B}$ in the deformed configuration and the surrounding space $\mathcal{B}^o$ with exterior boundary $\partial\mathcal{B}^\infty$.

Similarly, for the second integral in (9.72), by first using (9.2) to replace $\mathbf{M}$ and also using (9.63) and (9.64), we obtain

$$-\frac{1}{2}\mu_o \int_{\mathcal{B}} \mathbf{M} \cdot \mathbf{H}_a = -\frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi\mathbf{B}_a \cdot \mathbf{n} \, da + \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_a\mathbf{B} \cdot \mathbf{n} \, da$$

$$= -\frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_s\mathbf{B}_a \cdot \mathbf{n} \, da + \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_a\mathbf{B}_s \cdot \mathbf{n} \, da. \tag{9.74}$$

Hence we have, for example, either

$$E = \int_{\mathcal{B}} \rho\phi^* \, dv + \frac{1}{2}\mu_o \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H} \, dv + \int_{\partial\mathcal{B}^\infty} \varphi_a \mathbf{B}_s \cdot \mathbf{n} \, da + \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_a \mathbf{B}_a \cdot \mathbf{n} \, da + \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_s \mathbf{B}_s \cdot \mathbf{n} \, da, \quad (9.75)$$

or

$$E = \int_{\mathcal{B}} \rho\psi^* \, dv - \frac{1}{2}\mu_o \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H} \, dv - \int_{\partial\mathcal{B}^\infty} \varphi_s \mathbf{B}_a \cdot \mathbf{n} \, da - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_a \mathbf{B}_a \cdot \mathbf{n} \, da - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_s \mathbf{B}_s \cdot \mathbf{n} \, da. \quad (9.76)$$

Since φ_a and $\mathbf{B}_a$ are not affected by any variation in $\mathbf{x}$ or φ_s the integral of $\varphi_a \mathbf{B}_a$ over $\partial\mathcal{B}^\infty$ can be omitted (or absorbed as a constant into E). Since, in $\mathcal{B}^o$, φ_s must satisfy Laplace's equation, we may assume that $\varphi_s \sim \frac{1}{|\mathbf{x}|}$ as $|\mathbf{x}| \to \infty$, so that $|\mathbf{B}_s| \sim \frac{1}{|\mathbf{x}|^2}$ (see Section 1 of [103]) and the integral of $\varphi_s \mathbf{B}_s$ over $\partial\mathcal{B}^\infty$ therefore vanishes. The above two expressions for E then reduce to

$$E = \int_{\mathcal{B}} \rho\phi^* \, dv + \frac{1}{2}\mu_o \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H} \, dv + \int_{\partial\mathcal{B}^\infty} \varphi_a \mathbf{B}_s \cdot \mathbf{n} \, da, \quad (9.77)$$

and

$$E = \int_{\mathcal{B}} \rho\psi^* \, dv - \frac{1}{2}\mu_o \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H} \, dv - \int_{\partial\mathcal{B}^\infty} \varphi_s \mathbf{B}_a \cdot \mathbf{n} \, da. \quad (9.78)$$

Since the difference is a constant (and may be absorbed into the definition of E), we may replace $\mathbf{B}_s$ by $\mathbf{B}$ in (9.75) and φ_s by φ in (9.76), and we consider the two alternative but (apart from an additive constant) equivalent expressions

$$E = \int_{\mathcal{B}} \rho\phi^* \, dv + \frac{1}{2}\mu_o \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H} \, dv + \int_{\partial\mathcal{B}^\infty} \varphi_a \mathbf{B} \cdot \mathbf{n} \, da, \quad (9.79)$$

and

$$E = \int_{\mathcal{B}} \rho\psi^* \, dv - \frac{1}{2}\mu_o \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H} \, dv - \int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B}_a \cdot \mathbf{n} \, da, \quad (9.80)$$

instead of (9.75) and (9.76), respectively. The 'energy' (9.79) can be considered as a functional of $\mathbf{x}$ and a vector potential $\mathbf{A}$, with $\mathbf{B} = \operatorname{curl}\mathbf{A}$ (see Section 3.5), while (9.80) depends only on $\mathbf{x}$ and φ, with $\mathbf{H} = -\operatorname{grad}\varphi$.

9.2.1 Kankanala and Triantafyllidis' variational formulation

Kankanala and Triantafyllidis [61] used a functional with three indepedent variables, $\mathbf{x}$, $\bar{\mathbf{M}}$ and a vector potential, which at first sight is different from (9.65) and essentially amounts to replacing the magnetic self-energy term by the volume integral of $\frac{\mu_o}{2}\mathbf{H}\cdot\mathbf{H}$ over the whole body and its exterior. We show in this subsection that under the assumptions that they adopt their functional can be re-cast as (9.65).

The difference between the functional of Kankanala and Triantafyllidis [61] (equation (3.8) page 2886), and that of Brown[5] [13] is, in the present notation. given by

$$\frac{1}{2}\mu_o\int_{\mathcal{B}}\mathbf{M}\cdot\mathbf{H}_s\,dv+\frac{1}{2}\mu_o\int_{\mathcal{B}\cup\mathcal{B}^o}\mathbf{H}\cdot\mathbf{H}\,dv. \tag{9.81}$$

which, on use of $\mathbf{H}=\mathbf{H}_a+\mathbf{H}_s$ and $\mathbf{B}=\mu_o(\mathbf{H}+\mathbf{M})$, becomes

$$\frac{1}{2}\int_{\mathcal{B}}\mathbf{B}\cdot\mathbf{H}_s\,dv+\frac{1}{2}\mu_o\int_{\mathcal{B}}\mathbf{H}\cdot\mathbf{H}_a\,dv+\frac{1}{2}\mu_o\int_{\mathcal{B}^o}\mathbf{H}\cdot\mathbf{H}\,dv. \tag{9.82}$$

Next, on use of $\mathbf{H}_s=-\text{grad}\varphi_s$, $\mathbf{H}=-\text{grad}\varphi$, $\text{div}\mathbf{B}=0$, $\mathbf{B}_a=\mu_o\mathbf{H}_a$ and $\text{div}\mathbf{B}_a=0$, this gives

$$-\frac{1}{2}\int_{\mathcal{B}}\text{div}\,(\varphi_s\mathbf{B}+\varphi\mathbf{B}_a)\,dv+\frac{1}{2}\mu_o\int_{\mathcal{B}^o}\mathbf{H}\cdot\mathbf{H}\,dv. \tag{9.83}$$

and hence, by the divergence theorem,

$$-\frac{1}{2}\int_{\partial\mathcal{B}}(\varphi_s\mathbf{B}+\varphi\mathbf{B}_a)\cdot\mathbf{n}\,da+\frac{1}{2}\mu_o\int_{\mathcal{B}^o}\mathbf{H}\cdot\mathbf{H}\,dv. \tag{9.84}$$

Application of the divergence theorem over $\mathcal{B}^o$ and use of $\mathbf{B}=\mathbf{B}_a+\mathbf{B}_s$ then yields

$$-\frac{1}{2}\int_{\partial\mathcal{B}^\infty}(\varphi_s\mathbf{B}+\varphi\mathbf{B}_a)\cdot\mathbf{n}\,da+\frac{1}{2}\int_{\mathcal{B}^o}(\mathbf{H}\cdot\mathbf{B}_s-\mathbf{B}\cdot\mathbf{H}_s)\,dv. \tag{9.85}$$

But, in $\mathcal{B}^o$, we have $\mathbf{B}=\mu_o\mathbf{H}$, $\mathbf{B}_s=\mu_o\mathbf{H}_s$ so that the volume integral in the above vanishes. The remaining (surface) integral can be split up into three separate terms, namely

$$\int_{\partial\mathcal{B}^\infty}\varphi_s\mathbf{B}_a\cdot\mathbf{n}\,da+\frac{1}{2}\int_{\partial\mathcal{B}^\infty}\varphi_s\mathbf{B}_s\cdot\mathbf{n}\,da+\frac{1}{2}\int_{\partial\mathcal{B}^\infty}\varphi_a\mathbf{B}_a\cdot\mathbf{n}\,da. \tag{9.86}$$

The third term is not affected by any variation and is easily seen, by the divergence theorem, to be equal to

$$-\frac{1}{2}\mu_o\int_{\mathcal{B}\cup\mathcal{B}^\infty}\mathbf{H}_a\cdot\mathbf{H}_a\,dv, \tag{9.87}$$

which is a term dropped by Kankanala and Triantafyllidis [61]. They also omitted the first term by assuming rapid enough decay of φ_s at infinity. This assumption also ensures that the second term vanishes. Thus, their functional is equal to that of Brown apart from a non-essential constant. If we assume that φ_s decays rapidly enough for the first integral in (9.86) to vanish, then this would require that $\mathbf{B}_s$ behaves in such a way as infinity is

[5] Equation (9.68), see also the original expression (7.13), page 73 Chapter III.

approached that the final integral on the right side of (9.74) also vanishes, and hence from (9.74) we have that

$$\int_{\mathcal{B}} \mathbf{M}\cdot\mathbf{H}_a\,\mathrm{d}v \tag{9.88}$$

vanishes, which is untenable.

9.2.2 Ericksen's variational formulation

The energy functional used by Ericksen ([41], equation (3.8)) has, in the present notation, the form

$$\int_{\mathcal{B}}\left[\frac{1}{2}\mu_o^{-1}\mathbf{B}\cdot\mathbf{B}-(\mathbf{H}_a+\mathbf{M})\cdot\mathbf{B}+\rho\chi+\frac{1}{2}\mu_o\mathbf{M}\cdot\mathbf{M}\right]\mathrm{d}v \\ +\int_{\mathcal{B}^o}\left(\frac{1}{2}\mu_o^{-1}\mathbf{B}\cdot\mathbf{B}-\mu_o^{-1}\mathbf{B}\cdot\mathbf{B}_a\right)\mathrm{d}v, \tag{9.89}$$

where we have replaced the term $\hat{\varphi}$ in [41] by its equivalent $\rho\chi+\frac{1}{2}\mu_o\mathbf{M}\cdot\mathbf{M}$. This comes from the connections

$$\frac{\partial\hat{\varphi}}{\partial\mathbf{M}}=\mathbf{B},\quad \frac{\partial\chi}{\partial\bar{\mathbf{M}}}=\mu_o\mathbf{H},\quad \frac{\partial}{\partial\mathbf{M}}(\hat{\varphi}-\rho\chi)=\mu_o\mathbf{M}. \tag{9.90}$$

After use of (9.2) Ericksen's energy can be rearranged as the sum of E, given by the Brown form (9.67) with (9.68), and the sum

$$\int_{\mathcal{B}}\left(\frac{1}{2}\mu_o\mathbf{M}\cdot\mathbf{H}_s+\frac{1}{2}\mu_o\mathbf{H}\cdot\mathbf{H}-\mu_o\mathbf{H}\cdot\mathbf{H}_a\right)\mathrm{d}v+\int_{\mathcal{B}^o}\left(\frac{1}{2}\mu_o\mathbf{H}\cdot\mathbf{H}-\mu_o\mathbf{H}\cdot\mathbf{H}_a\right)\mathrm{d}v, \tag{9.91}$$

the latter two integrals representing the difference between Ericksen's and Brown's energies.

Next, the following steps show that application of (9.2) again, together with use of $\mathbf{H}=\mathbf{H}_a+\mathbf{H}_s$, $\mathbf{B}_a=\mu_o\mathbf{H}_a$, $\mathbf{H}_s=-\mathrm{grad}\varphi_s$ and $\mathbf{H}=-\mathrm{grad}\varphi$, followed by $\mathrm{div}\mathbf{B}=0$, $\mathrm{div}\mathbf{B}_a=0$ and an applications of the divergence theorem, leads the integral over $\mathcal{B}$ to become a surface integral. Thus,

$$\int_{\mathcal{B}}\left(\frac{1}{2}\mu_o\mathbf{M}\cdot\mathbf{H}_s+\frac{1}{2}\mu_o\mathbf{H}\cdot\mathbf{H}-\mu_o\mathbf{H}\cdot\mathbf{H}_a\right)\mathrm{d}v=\frac{1}{2}\int_{\mathcal{B}}(\mathbf{B}\cdot\mathbf{H}_s-\mu_o\mathbf{H}\cdot\mathbf{H}_a)\,\mathrm{d}v \\ =-\frac{1}{2}\int_{\mathcal{B}}\mathrm{div}\,(\varphi_s\mathbf{B}-\varphi\mathbf{B}_a)\,\mathrm{d}v=-\frac{1}{2}\int_{\partial\mathcal{B}}(\varphi_s\mathbf{B}-\varphi\mathbf{B}_a)\cdot\mathbf{n}\,\mathrm{d}a. \tag{9.92}$$

Converting the latter to an integral over $\partial\mathcal{B}^o$ by invoking continuity of φ, φ_s, $\mathbf{B}\cdot\mathbf{n}$ and $\mathbf{B}_a\cdot\mathbf{n}$ and application of the divergence theorem to its exterior yields

$$\frac{1}{2}\int_{\mathcal{B}^o}\mathrm{div}\,(\varphi_s\mathbf{B}-\varphi\mathbf{B}_a)\,\mathrm{d}v-\frac{1}{2}\int_{\partial\mathcal{B}^\infty}(\varphi_s\mathbf{B}-\varphi\mathbf{B}_a)\cdot\mathbf{n}\,\mathrm{d}a, \tag{9.93}$$

and hence

$$\frac{1}{2}\int_{\mathcal{B}^o}(\mathbf{H}\cdot\mathbf{B}_a-\mathbf{B}\cdot\mathbf{H}_s)\,\mathrm{d}v-\frac{1}{2}\int_{\partial\mathcal{B}^\infty}(\varphi_s\mathbf{B}_s-\varphi_a\mathbf{B}_a)\cdot\mathbf{n}\,\mathrm{d}a, \tag{9.94}$$

where we have used $\varphi=\varphi_a+\varphi_s$ and $\mathbf{B}=\mathbf{B}_a+\mathbf{B}_s$ in the surface integral.

By combining this with previous integral over $\mathcal{B}^o$ in (9.91) we find that the difference between Ericksen's and Brown's energies is

$$\frac{1}{2}\int_{\mathcal{B}^o}(\mu_o\mathbf{H}\cdot\mathbf{H}-2\mu_o\mathbf{H}\cdot\mathbf{H}_a+\mathbf{H}\cdot\mathbf{B}_a-\mathbf{B}\cdot\mathbf{H}_a)\,\mathrm{d}v$$
$$-\frac{1}{2}\int_{\partial\mathcal{B}^\infty}(\varphi_s\mathbf{B}_s-\varphi_a\mathbf{B}_a)\cdot\mathbf{n}\,\mathrm{d}a. \tag{9.95}$$

The integral over $\mathcal{B}^o$ vanishes since $\mathbf{H}=\mathbf{H}_a+\mathbf{H}_s$ and, in $\mathcal{B}^o$, $\mathbf{B}=\mu_o\mathbf{H}$, $\mathbf{B}_a=\mu_o\mathbf{H}_a$, while the integral of $\varphi_s\mathbf{B}_s\cdot\mathbf{n}$ over $\partial\mathcal{B}^\infty$ vanishes by the arguments used earlier. Thus, the difference is the inessential constant

$$\frac{1}{2}\int_{\partial\mathcal{B}^\infty}\varphi_a\mathbf{B}_a\cdot\mathbf{n}\,\mathrm{d}a. \tag{9.96}$$

9.2.3 Formulation in terms of the scalar potential

We now construct a variational principle that does not involve the magnetization or separate applied and self fields, one that produces the mechanical equations of equilibrium and boundary conditions together with the appropriate magnetic field equations and continuity conditions. In this subsection we construct such principle based on the magnetic field, while in Subsection 9.2.4 we work with the magnetic induction.

Consider the magnetic field expressed in terms of the potential φ (see Section 3.5), regarded as a function of $\mathbf{x}$:

$$\mathbf{H}=-\mathrm{grad}\varphi. \tag{9.97}$$

Without loss of generality we can extend φ continuously from $\mathcal{B}$ into $\mathcal{B}^o$ so that (9.97) holds in both $\mathcal{B}$ and $\mathcal{B}^o$. Since $\mathrm{curl}\mathbf{H}=\mathbf{0}$ the continuity condition $(9.4)_1$ is then automatically satisfied.

The potential φ may also be treated as a function of $\mathbf{X}$ via the connection $\varphi(\mathbf{x})=\varphi(\boldsymbol{\chi}(\mathbf{X}))$ and, in view of $(9.49)_1$, the Lagrangian version $\mathbf{H}_l$ of $\mathbf{H}$ is then given by

$$\mathbf{H}_l=-\mathrm{Grad}\varphi \tag{9.98}$$

inside the material. While there is no material outside $\mathcal{B}$ and hence no physical deformation to enable Lagrangian quantities to be defined naturally, one can define a fictitious deformation function that extends $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ smoothly from $\mathcal{B}$ into its exterior, as was done by Toupin [109].

We now rewrite equation (9.78) in Lagrangian form using (9.62) form as

$$E\{\mathbf{x}, \varphi\} = \int_{\mathcal{B}_r} \Omega^*(\mathbf{F}, \mathbf{H}_l)\,\mathrm{d}V - \frac{1}{2}\mu_o \int_{\mathcal{B}_r^o} J(\mathbf{F}^{-T}\mathbf{H}_l)\cdot(\mathbf{F}^{-T}\mathbf{H}_l)\,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty} \varphi\mathbf{B}_a\cdot\mathbf{n}\,\mathrm{d}a, \quad (9.99)$$

where $\mathcal{B}_r^o$ is the exterior of $\mathcal{B}_r \cup \partial\mathcal{B}_r$. It is assumed that the boundary at infinity is fixed so that the latter integral need not be converted to Lagrangian form.

In place of (9.65) we define the functional

$$\Pi\{\mathbf{x}, \varphi\} = E\{\mathbf{x}, \varphi\} - L\{\mathbf{x}\}. \quad (9.100)$$

Variations in $\mathbf{x}$ and φ may be considered as independent, with the proviso that since φ depends on $\mathbf{x}$ any variation in $\mathbf{x}$ induces a variation in φ. We begin by considering the variation of E with respect to φ at fixed $\mathbf{x}$. A variation is denoted by a superposed dot. Then, $\dot{\mathbf{H}}_l = -\mathrm{Grad}\,\dot{\varphi}$ and $\mathbf{F}^{-T}\dot{\mathbf{H}}_l = -\mathrm{grad}\,\dot{\varphi}$. The first variation $\dot{E}_\varphi$ of E with respect to φ is

$$\dot{E}_\varphi = \int_{\mathcal{B}_r} -\frac{\partial\Omega^*}{\partial\mathbf{H}_l}\cdot\mathrm{Grad}\,\dot{\varphi}\,\mathrm{d}V - \int_{\mathcal{B}_r^o} \mu_o J(\mathbf{F}^{-T}\mathbf{H}_l)\cdot(\mathbf{F}^{-T}\dot{\mathbf{H}}_l)\,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\mathbf{B}_a\cdot\mathbf{n}\,\mathrm{d}a, \quad (9.101)$$

In anticipation of equation $(9.58)_2$ we write $\frac{\partial\Omega^*}{\partial\mathbf{H}_l} = -\mathbf{B}_l$. Then for the term in the first integral of the right side of (9.101) we get

$$-\mathbf{B}_l\cdot\dot{\mathbf{H}}_l = \mathrm{Div}\,(\dot{\varphi}\mathbf{B}_l) - \dot{\varphi}\mathrm{Div}\,\mathbf{B}_l = J[\mathrm{div}\,(\dot{\varphi}\mathbf{B}) - \dot{\varphi}\mathrm{div}\,\mathbf{B}], \quad (9.102)$$

where we have written $\mathbf{B} = J^{-1}\mathbf{F}\mathbf{B}_l$ (in respect of $\mathcal{B}$). Also for the term in the second integral of the right side of (9.101) we have

$$-\mu_o J(\mathbf{F}^{-T}\mathbf{H}_l)\cdot(\mathbf{F}^{-T}\dot{\mathbf{H}}_l) = \mu_o J\mathbf{H}\cdot\mathrm{grad}\,\dot{\varphi} = J[\mathrm{div}\,(\dot{\varphi}\mathbf{B}) - \dot{\varphi}\mathrm{div}\,\mathbf{B}], \quad (9.103)$$

in which we have set $\mathbf{B} = \mu_o\mathbf{H}$ (in respect of $\mathcal{B}^o$). As a result for (9.101) we obtain

$$\dot{E}_\varphi = \int_{\mathcal{B}_r} J[\mathrm{div}\,(\dot{\varphi}\mathbf{B}) - \dot{\varphi}\mathrm{div}\,\mathbf{B}]\,\mathrm{d}V + \int_{\mathcal{B}_r^o} J[\mathrm{div}\,(\dot{\varphi}\mathbf{B}) - \dot{\varphi}\mathrm{div}\,\mathbf{B}]\,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\mathbf{B}_a\cdot\mathbf{n}\,\mathrm{d}a. \quad (9.104)$$

Writing the first two integrals on the right side in the current configuration, using $\mathrm{d}v = J\mathrm{d}V$, and after some rearrangements we have

$$\dot{E}_\varphi = -\int_{\mathcal{B}\cup\mathcal{B}^o} \dot{\varphi}\mathrm{div}\,\mathbf{B}\,\mathrm{d}v + \int_{\mathcal{B}} \mathrm{div}\,(\dot{\varphi}\mathbf{B})\,\mathrm{d}v + \int_{\mathcal{B}^o} \mathrm{div}\,(\dot{\varphi}\mathbf{B})\,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\mathbf{B}_a\cdot\mathbf{n}\,\mathrm{d}a. \quad (9.105)$$

On use of the divergence theorem applied separately to $\mathcal{B}$ and $\mathcal{B}^o$ (second and third integral of the right side of (9.105)), considering that $\partial\mathcal{B}^o = \partial\mathcal{B}\cup\partial\mathcal{B}^\infty$, and that the normal vector

$\mathbf{n}$ on $\partial\mathcal{B}$ is $-\mathbf{n}$ if seen as outward vector for $\mathcal{B}^o$, the first variation can be rearranged in the form

$$\dot{E}_\varphi = -\int_{\mathcal{B}\cup\mathcal{B}^o} \dot{\varphi}\,\mathrm{div}\,\mathbf{B}\,\mathrm{d}v - \int_{\partial\mathcal{B}} \dot{\varphi}[\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\infty} \dot{\varphi}(\mathbf{B}-\mathbf{B}_a)\cdot\mathbf{n}\,\mathrm{d}a. \tag{9.106}$$

The final term vanishes since $\mathbf{B}-\mathbf{B}_a = \mathbf{B}_s$ decays as $\frac{1}{|\mathbf{x}|^2}$ as $|\mathbf{x}| \to \infty$ and $\dot{\varphi}$ may be taken to decay like $\frac{1}{|\mathbf{x}|}$. Thus, E is stationary with respect to φ if and only if

$$\mathrm{div}\,\mathbf{B} = 0 \quad \text{in}\,\mathcal{B}\cup\mathcal{B}^o, \quad [\![\mathbf{B}]\!]\cdot\mathbf{n} = 0 \quad \text{on}\,\partial\mathcal{B}. \tag{9.107}$$

We turn next to the variation of E with $\mathbf{x}$, which we denote by $\dot{E}_\mathbf{x}$. Since φ depends on $\mathbf{x}$, when the latter is varied so is φ, and we denote the induced variation by $\dot{\varphi}_{\mathrm{ind}}$. The corresponding variation in $\mathbf{H}_l = -\mathrm{Grad}\varphi$ is $\dot{\mathbf{H}}_l = -\mathrm{Grad}\dot{\varphi}_{\mathrm{ind}}$ and equations (9.102) and (9.103) again hold, but with $\dot{\varphi}$ replaced by $\dot{\varphi}_{\mathrm{ind}}$. It then follows from the results (9.107) that the terms involving the induced variation do not contribute to $\dot{E}_\mathbf{x}$.

Consider the variation of E. From (9.99) we have

$$\dot{E}_\mathbf{x} = \int_{\mathcal{B}_r} \frac{\partial\Omega^*}{\partial\mathbf{F}} : \dot{\mathbf{F}}\,\mathrm{d}V - \mu_o\int_{\mathcal{B}_r^o} \frac{1}{2}\dot{J}(\mathbf{F}^{-T}\mathbf{H}_l)\cdot(\mathbf{F}^{-T}\mathbf{H}_l) + J(\mathbf{F}^{-T}\mathbf{H}_l)\cdot(\dot{\overline{\mathbf{F}^{-T}}}\mathbf{H}_l)\,\mathrm{d}V, \tag{9.108}$$

where $\dot{\mathbf{F}} = \mathrm{Grad}\dot{\mathbf{x}}$. Bearing in mind $(9.58)_1$ we set

$$\frac{\partial\Omega^*}{\partial\mathbf{F}} = \mathbf{T}, \tag{9.109}$$

then for the term in the first integral of the right side of (9.108) we have

$$\frac{\partial\Omega^*}{\partial\mathbf{F}} : \dot{\mathbf{F}} \equiv \mathrm{tr}(\mathbf{T}\dot{\mathbf{F}}) = \mathrm{Div}(\mathbf{T}\dot{\mathbf{x}}) - (\mathrm{Div}\mathbf{T})\cdot\dot{\mathbf{x}}. \tag{9.110}$$

Consider the results (see, for example, [78])

$$\dot{J} = J\mathrm{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}}), \quad (\dot{\overline{\mathbf{F}^{-T}}}) = -\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}. \tag{9.111}$$

Using (9.110) and (9.111) in (9.108), with $\mathbf{H}_l = \mathbf{F}^T\mathbf{H}$ we have

$$\dot{E}_\mathbf{x} = \int_{\mathcal{B}_r} \mathrm{Div}(\mathbf{T}\dot{\mathbf{x}}) - (\mathrm{Div}\mathbf{T})\cdot\dot{\mathbf{x}}\,\mathrm{d}V - \int_{\mathcal{B}_r^o} \mu_o\left[\frac{1}{2}J\mathrm{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}})\mathbf{H}\cdot\mathbf{H} - J\mathbf{H}\cdot(\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{H})\right]\mathrm{d}V. \tag{9.112}$$

For the first integral on the right side of (9.112), by applying the divergence theorem to $\mathcal{B}_r$, we get

$$-\int_{\mathcal{B}_r} (\mathrm{Div}\mathbf{T})\cdot\dot{\mathbf{x}}\,\mathrm{d}V + \int_{\partial\mathcal{B}_r} (\mathbf{T}^T\mathbf{N})\cdot\dot{\mathbf{x}}\,\mathrm{d}A. \tag{9.113}$$

Using $\mathbf{B} = \mu_o \mathbf{H}$, which is valid in $\mathcal{B}_r^o$, the second integral in (9.112) can be expressed as

$$\int_{\mathcal{B}_r^o} \mathrm{tr}\left\{\left[\mathbf{B}\otimes\mathbf{H} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{H})\mathbf{I}\right]\dot{\mathbf{F}}\mathbf{F}^{-1}\right\} J\,\mathrm{d}V, \tag{9.114}$$

which in the current configuration, in terms of the Maxwell stress (9.25) (with superscripts ^{o} dropped) can be rewritten as

$$\int_{\mathcal{B}^o} \mathrm{tr}\,(\boldsymbol{\tau}_m \dot{\mathbf{F}}\mathbf{F}^{-1})\,\mathrm{d}v. \tag{9.115}$$

Using $\dot{\mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ (see, for example, [78]) (9.115) becomes

$$\int_{\mathcal{B}^o} [\mathrm{div}\,(\boldsymbol{\tau}_m \dot{\mathbf{x}}) - (\mathrm{div}\,\boldsymbol{\tau}_m)\cdot\dot{\mathbf{x}}]\,\mathrm{d}v, \tag{9.116}$$

and an application of the divergence theorem to $\mathcal{B}^o$ yields

$$-\int_{\mathcal{B}^o} (\mathrm{div}\,\boldsymbol{\tau}_m)\cdot\dot{\mathbf{x}}\,\mathrm{d}v + \int_{\partial\mathcal{B}^o} (\boldsymbol{\tau}_m \mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\infty} (\boldsymbol{\tau}_m \mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}a, \tag{9.117}$$

and we note that

$$\int_{\partial\mathcal{B}^o} (\boldsymbol{\tau}_m \mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}a = -\int_{\partial\mathcal{B}} (\boldsymbol{\tau}_m \mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}a. \tag{9.118}$$

With (9.113), (9.117) and (9.118) for $\dot{E}_{\mathbf{x}}$ we obtain

$$\begin{aligned}\dot{E}_{\mathbf{x}} = -\int_{\mathcal{B}_r} (\mathrm{Div}\,\mathbf{T})\cdot\dot{\mathbf{x}}\,\mathrm{d}V + \int_{\partial\mathcal{B}_r} (\mathbf{T}^T\mathbf{N})\cdot\dot{\mathbf{x}}\,\mathrm{d}A - \int_{\mathcal{B}^o} (\mathrm{div}\,\boldsymbol{\tau}_m)\cdot\dot{\mathbf{x}}\,\mathrm{d}v \\ - \int_{\partial\mathcal{B}} (\boldsymbol{\tau}_m \mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\infty} (\boldsymbol{\tau}_m \mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}a.\end{aligned} \tag{9.119}$$

From (9.66) consider the variation of L (which has been written in the reference configuration)

$$\dot{L}_{\mathbf{x}} = \int_{\mathcal{B}_r} \rho_o \mathbf{f}\cdot\dot{\mathbf{x}}\,\mathrm{d}V + \int_{\partial\mathcal{B}_r} \mathbf{t}_A\cdot\dot{\mathbf{x}}\,\mathrm{d}A, \tag{9.120}$$

where $\mathbf{t}_A$ is the pull-back version of the mechanical traction $\mathbf{t}_a$.

Finally, from (9.100) using (9.119) and (9.120) we obtain

$$\begin{aligned}\dot{\Pi}_{\mathbf{x}} = \dot{E}_{\mathbf{x}} - \dot{L}_{\mathbf{x}} = -\int_{\mathcal{B}_r} (\mathrm{Div}\,\mathbf{T} + \rho_o\mathbf{f})\cdot\dot{\mathbf{x}}\,\mathrm{d}V - \int_{\mathcal{B}^o} (\mathrm{div}\,\boldsymbol{\tau}_m)\cdot\dot{\mathbf{x}}\,\mathrm{d}v \\ + \int_{\partial\mathcal{B}_r} (\mathbf{T}^T\mathbf{N} - \mathbf{t}_A - \mathbf{t}_m)\cdot\dot{\mathbf{x}}\,\mathrm{d}A + \int_{\partial\mathcal{B}^\infty} (\boldsymbol{\tau}_m \mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}a,\end{aligned} \tag{9.121}$$

where $\mathbf{t}_m = \mathbf{T}_m^T\mathbf{N}$ and $\mathbf{T}_m = J\mathbf{F}^{-1}\boldsymbol{\tau}_m$. The final term in (9.121) can be taken to vanish by assuming that $\dot{\mathbf{x}}$ decays sufficiently rapidly as $|\mathbf{x}| \to \infty$.

We conclude that Π is stationary with respect to $\mathbf{x}$ if and only if

$$\mathrm{Div}\,\mathbf{T} + \rho_o\mathbf{f} = \mathbf{0}, \quad \text{in } \mathcal{B}_r, \quad \mathbf{T}^T\mathbf{N} = \mathbf{t}_A + \mathbf{t}_m \quad \text{on } \partial\mathcal{B}_r. \tag{9.122}$$

and

$$\operatorname{div} \boldsymbol{\tau}_m = \mathbf{0}, \quad \text{in} \mathcal{B}^o. \tag{9.123}$$

The latter equation, however, follows from the definition $\mathbf{B} = \mu_o \mathbf{H}$ and the equations (9.1) in $\mathcal{B}^o$.

In summary, Π is stationary with respect to variations in $\mathbf{x}$ and φ if and only if (9.107) and (9.122) hold.

9.2.4 Variation in terms of the vector potential

An alternative formulation is now considered based on equation $(9.1)_2$

$$\operatorname{div} \mathbf{B} = 0, \tag{9.124}$$

a solution of which is given in terms of the vector potential $\mathbf{A}$ as (see Section 3.5)

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \tag{9.125}$$

Recall the Lagrangian version of $\mathbf{A}$ (equation (3.78)):

$$\mathbf{A}_l = \mathbf{F}^T \mathbf{A}. \tag{9.126}$$

Then it is easy to show that

$$\mathbf{F} \operatorname{Curl} \mathbf{A}_l = J \operatorname{curl} \mathbf{A}, \tag{9.127}$$

and

$$\mathbf{B}_l = \operatorname{Curl} \mathbf{A}_l. \tag{9.128}$$

From Section 3.5 one can take $\mathbf{A}$ to be continuous across $\partial\mathcal{B}$ (see equation (3.87)), and hence $\mathbf{A}_l$ is continuous across $\partial\mathcal{B}_r$.

Proposition 9.6. *The identity*

$$\int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B} \cdot \mathbf{n} \, da = -\int_{\partial\mathcal{B}^\infty} (\mathbf{H} \times \mathbf{A}) \cdot \mathbf{n} \, da \tag{9.129}$$

follows from (9.97) and (9.125).

Proof. For the left side of (9.129) with (9.125) and the divergence theorem we have

$$\begin{aligned} \int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B} \cdot \mathbf{n} \, da &= \int_{\partial\mathcal{B}^\infty} \varphi \operatorname{curl} \mathbf{A} \cdot \mathbf{n} \, da, \\ &= \int_{\mathcal{B}\cup\mathcal{B}^o} \operatorname{div} (\varphi \operatorname{curl} \mathbf{A}) \, dv, \end{aligned} \tag{9.130}$$

and from (9.97) we get

$$\int_{\partial\mathcal{B}^\infty} \varphi\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a = \int_{\mathcal{B}\cup\mathcal{B}^o} -\mathbf{H}\cdot\mathrm{curl}\,\mathbf{A}\,\mathrm{d}v. \tag{9.131}$$

Consider the identity

$$\mathrm{div}(\mathbf{H}\times\mathbf{A}) = \mathbf{H}\cdot\mathrm{curl}\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\mathbf{H}. \tag{9.132}$$

But from $(9.1)_1$ we have $\mathrm{curl}\mathbf{H} = \mathbf{0}$, therefore in (9.131) with the divergence theorem we get

$$\begin{aligned}\int_{\partial\mathcal{B}^\infty} \varphi\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a &= \int_{\partial\mathcal{B}^\infty} -\mathrm{div}(\mathbf{H}\times\mathbf{A})\,\mathrm{d}v,\\ &= -\int_{\partial\mathcal{B}^\infty} (\mathbf{H}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}a.\end{aligned} \tag{9.133}$$

□

Consider the expression for the energy (9.77). Using (9.61) and (9.129), it can be rewritten as (in this subsection we use the notation E^*)

$$E^*\{\mathbf{x},\mathbf{A}\} = \int_{\mathcal{B}_r} \Omega(\mathbf{F},\mathbf{B}_l)\,\mathrm{d}V + \frac{1}{2}\mu_o^{-1}\int_{\mathcal{B}^o} \mathbf{B}\cdot\mathbf{B}\,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} (\mathbf{H}_a\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}a, \tag{9.134}$$

with $\mathbf{B}_l = \mathrm{Curl}\,\mathbf{A}_l$.

Let's define the functional $\Pi^*\{\mathbf{x},\mathbf{A}\}$ as

$$\Pi^*\{\mathbf{x},\mathbf{A}\} = E^*\{\mathbf{x},\mathbf{A}\} - L\{\mathbf{x}\}, \tag{9.135}$$

where $L\{\mathbf{x}\}$ has been defined in (9.66).

Consider the first variation of E^* in $\mathbf{A}$ for fixed $\mathbf{x}$, which we denote $\dot{E}^*_{\mathbf{A}}$. Noting that $\dot{\mathbf{B}}_l = \mathrm{Curl}\,\dot{\mathbf{A}}_l$, where $\dot{\mathbf{A}}_l = \mathbf{F}^T\dot{\mathbf{A}}$, we have

$$\dot{E}^*_{\mathbf{A}} = \int_{\mathcal{B}_r} \frac{\partial\Omega}{\partial\mathbf{B}_l}\cdot\mathrm{Curl}\,\dot{\mathbf{A}}_l\,\mathrm{d}V + \mu_o^{-1}\int_{\mathcal{B}^o} \mathbf{B}\cdot\mathrm{curl}\dot{\mathbf{A}}\,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} (\mathbf{H}_a\times\dot{\mathbf{A}})\cdot\mathbf{n}\,\mathrm{d}a. \tag{9.136}$$

Let's write $\frac{\partial\Omega}{\partial\mathbf{B}_l} = \mathbf{H}_l$ in $\mathcal{B}_r$, and using the identity (9.132), with $\mathbf{B} = \mu_o\mathbf{H}$ in $\mathcal{B}^o$, for $\dot{E}^*_{\mathbf{A}}$ in (9.136) we obtain

$$\begin{aligned}\dot{E}^*_{\mathbf{A}} = \int_{\mathcal{B}_r} \mathrm{Div}(\mathbf{H}_l\times\dot{\mathbf{A}}_l) + \dot{\mathbf{A}}_l\cdot\mathrm{Curl}\mathbf{H}_l\,\mathrm{d}V + \int_{\mathcal{B}^o} \mathrm{div}(\mathbf{H}\times\dot{\mathbf{A}}) + \dot{\mathbf{A}}\cdot\mathrm{curl}\mathbf{H}\,\mathrm{d}v\\ - \int_{\partial\mathcal{B}^\infty} (\mathbf{H}_a\times\dot{\mathbf{A}})\cdot\mathbf{n}\,\mathrm{d}a.\end{aligned} \tag{9.137}$$

Using the divergence theorem we get

$$\begin{aligned}\dot{E}^*_{\mathbf{A}} = \int_{\mathcal{B}_r} \dot{\mathbf{A}}_l\cdot\mathrm{Curl}\mathbf{H}_l\,\mathrm{d}V + \int_{\partial\mathcal{B}_r} (\mathbf{H}_l\times\dot{\mathbf{A}}_l)\cdot\mathbf{N}\,\mathrm{d}A + \int_{\mathcal{B}^o} \dot{\mathbf{A}}\cdot\mathrm{curl}\mathbf{H}\,\mathrm{d}v\\ + \int_{\partial\mathcal{B}^o} (\mathbf{H}\times\dot{\mathbf{A}})\cdot\mathbf{n}\,\mathrm{d}a - \int_{\partial\mathcal{B}^\infty} (\mathbf{H}_a\times\dot{\mathbf{A}})\cdot\mathbf{n}\,\mathrm{d}a.\end{aligned} \tag{9.138}$$

But $\partial\mathcal{B}^o = \partial\mathcal{B} \cup \partial\mathcal{B}^\infty$ therefore

$$\dot{E}^*_{\mathbf{A}} = \int_{\mathcal{B}_r} \dot{\mathbf{A}}_l \cdot \operatorname{Curl}\mathbf{H}_l \, dV + \int_{\partial\mathcal{B}_r} (\mathbf{N} \times \mathbf{H}_l) \cdot \dot{\mathbf{A}}_l \, dA + \int_{\mathcal{B}^o} \dot{\mathbf{A}} \cdot \operatorname{curl}\mathbf{H} \, dv - \int_{\partial\mathcal{B}} (\mathbf{n} \times \mathbf{H}) \cdot \dot{\mathbf{A}} \, da + \int_{\partial\mathcal{B}^\infty} [\mathbf{n} \times (\mathbf{H} - \mathbf{H}_a)] \cdot \dot{\mathbf{A}} \, da, \quad (9.139)$$

where in the fourth integral of the right side $\mathbf{H}$ is evaluated from outside the body.

We have the identity based in Nanson's formula

$$(\mathbf{H}_l \times \dot{\mathbf{A}}_l) \cdot \mathbf{N} \, dA = (\mathbf{H} \times \dot{\mathbf{A}}) \cdot \mathbf{n} \, da. \quad (9.140)$$

Proof. If we work with Cartesian coordinates, the expression $(\mathbf{H}_l \times \dot{\mathbf{A}}_l) \cdot \mathbf{N} \, dA$ becomes

$$\epsilon_{ijk} H_{l_i} \dot{A}_{l_j} N_k \, dA, \quad (9.141)$$

where ϵ_{ijk} is the permutation symbol.

From $(9.49)_1$, (9.126) and Nanson's formula we have

$$\mathbf{H}_l = \mathbf{F}^T\mathbf{H}, \quad \dot{\mathbf{A}} = \mathbf{F}^T\dot{\mathbf{A}}, \quad \mathbf{N} \, dA = J^{-1}\mathbf{F}^T\mathbf{n} \, da,$$

and so (9.141) becomes

$$\epsilon_{ijk} F_{mi} H_m F_{nj} \dot{A}_n J^{-1} F_{pk} n_p \, da. \quad (9.142)$$

Consider the identity [78] $J\epsilon_{mnp} = \epsilon_{ijk} F_{mi} F_{nj} F_{pk}$, so that (9.142) becomes

$$\epsilon_{mnp} H_m \dot{A}_n n_p \, da, \quad (9.143)$$

which is equivalent to

$$(\mathbf{H} \times \dot{\mathbf{A}}) \cdot \mathbf{n} \, da. \quad (9.144)$$

□

With (9.127) and (9.140) in (9.139) for the variation of E^* we have

$$\dot{E}^*_{\mathbf{A}} = \int_{\mathcal{B}\cup\mathcal{B}^o} \dot{\mathbf{A}} \cdot \operatorname{curl}\mathbf{H} \, dv - \int_{\partial\mathcal{B}} (\mathbf{n} \times [\![\mathbf{H}]\!]) \cdot \dot{\mathbf{A}} \, da + \int_{\partial\mathcal{B}^\infty} [\mathbf{n} \times (\mathbf{H} - \mathbf{H}_a)] \cdot \dot{\mathbf{A}} \, da. \quad (9.145)$$

The final term vanishes since $\mathbf{H} - \mathbf{H}_a = \mathbf{H}_s$ behaves like $\frac{1}{|\mathbf{x}|^2}$ at infinity and[6] $\dot{\mathbf{A}} = \dot{\mathbf{A}}_s$ can be assumed to decay as $\frac{1}{|\mathbf{x}|}$, where we have set $\mathbf{B}_s = \operatorname{curl}\mathbf{A}_s$.

Therefore, from (9.135) and (9.145), $\Pi^*\{\mathbf{x}, \mathbf{A}\}$ is stationary with respect to $\mathbf{A}$ if only if

$$\operatorname{curl}\mathbf{H} = \mathbf{0} \quad \text{in}\, \mathcal{B}\cup\mathcal{B}^o, \quad \mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0} \quad \text{on}\, \partial\mathcal{B}. \quad (9.146)$$

[6] Remember that $\mathbf{A}_a$ is not affected by the variation.

The variation of E^* with respect to $\mathbf{x}$ follows a similar pattern to that in respect of E (Subsection 9.2.3) and leads to

$$\dot{\Pi}^*_{\mathbf{x}} = \dot{E}^*_{\mathbf{x}} - \dot{L}_{\mathbf{x}} = \dot{E}_{\mathbf{x}} - \dot{L}_{\mathbf{x}}. \tag{9.147}$$

exactly as given in (9.121). The variation in $\mathbf{A}$ induced by that in $\mathbf{x}$, just as for φ previously does not contribute to $\dot{\Pi}^*_{\mathbf{x}}$. Thus Π^* is stationary with respect to $\mathbf{x}$ if and only if equations (9.122) hold.

9.3 A variational formulation for a boundary value problem with mixed boundary conditions

There are several unresolved questions relating to the application of mixed boundary conditions where both mechanical and electromagnetic quantities are prescribed. For example, the appropriate boundary conditions for a body in direct contact with another body are unclear in this context. In [7] results for the uniaxial tension problem were obtained by attaching the material specimen directly to the traction device. Other experiments involving interaction between mechanical and magnetic or electric effects in elastomers are described in [59, 62] and [44], while some specific applications are discussed in [48] and [44]. The incorporation of the 'Maxwell stresses' exterior to the deforming body in the boundary conditions is also problematic [89].

In the previous section (see also [17]) a variational formulation for a body completely surrounded by free space was developed (see Figure 9.1); the magnetic field is applied far away, and a surface mechanical traction (dead load) is applied on a portion of the boundary of $\mathcal{B}$. In Section 9.2 there is no discussion of how a mechanical surface traction or a restriction on the displacement might be applied. In this section we therefore explore the possibility of extending the aforementioned results for the following problem. This section is based on the results presented in [16].

Consider Figure 9.2, which shows a body $\mathcal{B}$ with part of its boundary bonded to a semi-infinite rigid body $\tilde{\mathcal{B}}$.

The body $\tilde{\mathcal{B}}$ may displace and rotate. Exterior to these bodies is free space $\mathcal{B}^o$. The surface of $\mathcal{B}$ is divided in two disjoint and complementary parts, $\partial\mathcal{B} = \partial\mathcal{B}^\alpha \cup \partial\mathcal{B}^\beta$, where $\partial\mathcal{B}^\alpha$ adjoins the free space and $\partial\mathcal{B}^\beta$ is in contact with the surface of $\tilde{\mathcal{B}}$. The body $\tilde{\mathcal{B}}$ and the free space $\mathcal{B}^o$ are separated by the surface $\mathcal{S}$. The boundaries of $\mathcal{B}^o$ and $\tilde{\mathcal{B}}$ far away

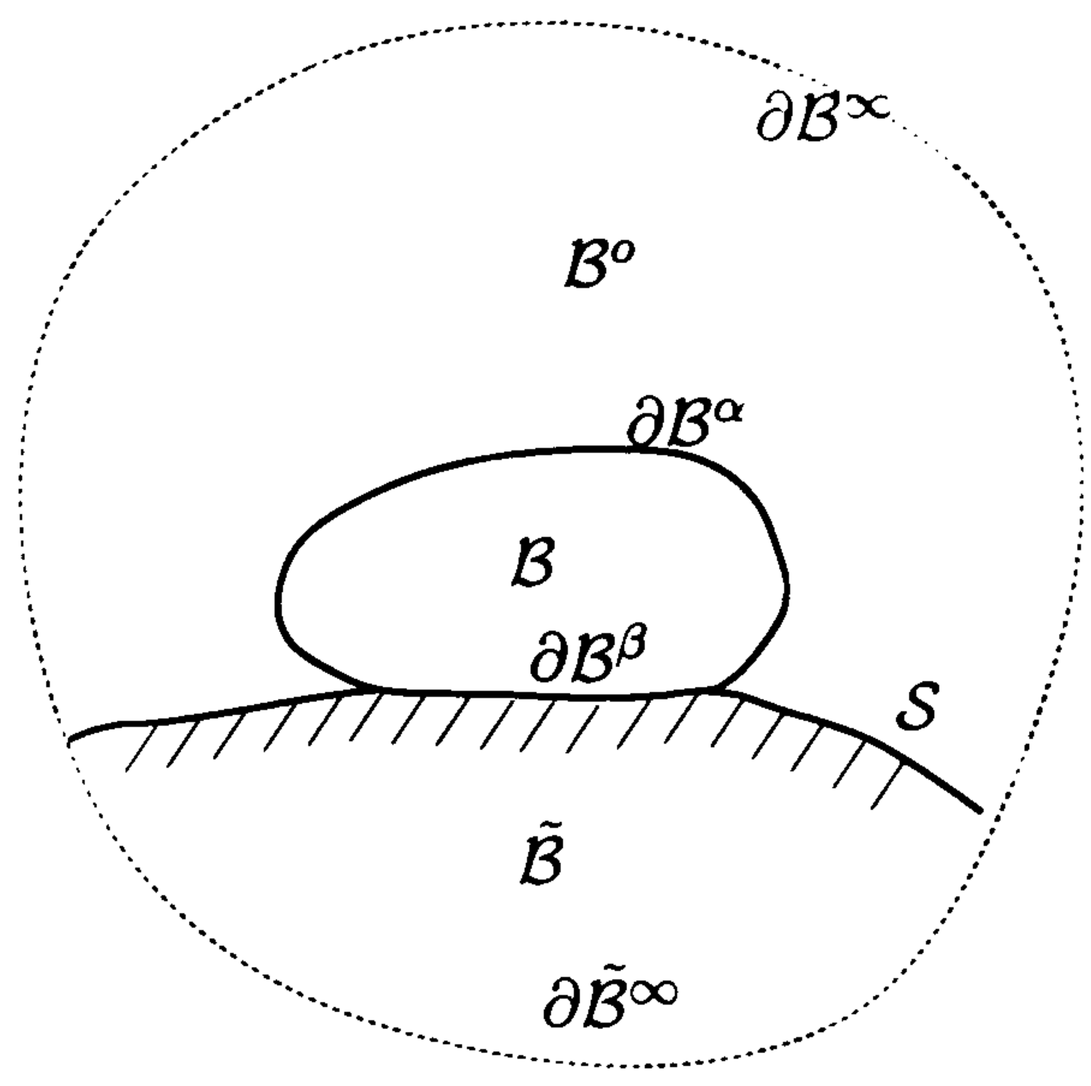


Figure 9.2: A mixed boundary value problem.

are denoted by $\partial\mathcal{B}^\infty$ and $\partial\tilde{\mathcal{B}}^\infty$, respectively. The normal vectors on the boundaries of $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\mathcal{B}^o$ are denoted $\mathbf{n}$, $\tilde{\mathbf{n}}$ and $\mathbf{n}'$, respectively, directed outwards from the region in each case; note that, for example, on $\partial\mathcal{B}^\alpha$ we have $-\mathbf{n}' = \mathbf{n}$, and on $\mathcal{S}$ we have $-\tilde{\mathbf{n}} = \mathbf{n}'$.

The body $\mathcal{B}$ is magnetoelastic. We work here with the magnetic field as the independent magnetic variable [33, 34] (see Subsection 9.2.3), and so the free energy function for $\mathcal{B}$ depends on the deformation gradient and the magnetic field (9.62). We assume that the (rigid) body $\tilde{\mathcal{B}}$ is magnetizable, and that the energy function depends only on the magnetic field $\mathbf{H}$.

For the boundary conditions on $\partial\mathcal{B}$ we prescribe a displacement on $\partial\mathcal{B}^\beta$ (the rigid displacement of $\tilde{\mathcal{B}}$), and we denote its position vector by $\hat{\mathbf{x}}$, while the boundary $\partial\mathcal{B}^\alpha$ is taken to be free of mechanical traction. We also apply an external magnetic induction far away on $\partial\mathcal{B}^\infty$ and/or $\partial\tilde{\mathcal{B}}^\infty$.

It is possible to show that the above model may be used to describe very well some real experiments, such as the uniaxial tension of a bar [7] and the shear of a slab [59] when both the magnetic interactions in the body and the surrounding free space are accounted for, and, importantly, the interaction with some external 'machine' that generates the applied magnetic induction or magnetic field is incorporated.

Consider the expression

$$\bar{E} = \int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \rho\Upsilon \,\mathrm{d}v + \frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\mu_0\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{M}\cdot\mathbf{H}_a\,\mathrm{d}v \tag{9.148}$$

for the energy of the bodies $\mathcal{B}$ and $\tilde{\mathcal{B}}$, where ρ is the mass density and Υ the energy per unit mass. This is a modification of the energy (9.72) [17] and is based on the classical formulation of Brown [13]. In the latter formulation the fields **H** and **B** are each decomposed as the sum of an 'applied' field in the absence of material and an additional 'self' field generated by the presence of the magnetic material body (9.63), (9.64) (Section 9.2).

The energy function appearing (9.148) is defined as

$$\Upsilon = \begin{cases} \psi^* & \mathbf{x} \in \mathcal{B}, \\ \tilde{\psi}^* & \mathbf{x} \in \tilde{\mathcal{B}}, \end{cases} \tag{9.149}$$

where $\psi^* = \psi^*(\mathbf{F}, \mathbf{H})$ (see $(9.39)_2$ and (9.62)), and $\tilde{\psi}^* = \tilde{\psi}^*(\mathbf{H})$.

The first term of the right side of (9.148) can be decomposed as

$$\int_{\mathcal{B}} \rho\psi^* \, dv + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\psi}^* \, dv, \tag{9.150}$$

ρ and $\tilde{\rho}$ being the mass densities associated with $\mathcal{B}$ and $\tilde{\mathcal{B}}$, respectively. For the second integral in (9.148), on use of (9.97), $(9.1)_2$ and the divergence theorem, we obtain [17]

$$-\frac{1}{2}\int_{\partial\mathcal{B}^\alpha \cup \mathcal{S}} \varphi \mathbf{B} \cdot \mathbf{n} \, da - \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} \varphi \mathbf{B} \cdot \tilde{\mathbf{n}} \, da. \tag{9.151}$$

The first integral in the above expression can be replaced by

$$-\frac{1}{2}\mu_0 \int_{\mathcal{B}^o} \mathbf{H} \cdot \mathbf{H} \, dv - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B} \cdot \mathbf{n}' \, da, \tag{9.152}$$

where again we have used the divergence theorem and the relation (9.3) appropriate for the free space $\mathcal{B}^o$.

The third integral in (9.148), on use of (9.2) and the fact that $\mathbf{B}_a = \mu_0 \mathbf{H}_a$ for the whole space $\mathcal{B} \cup \tilde{\mathcal{B}}$, can be rewritten as

$$-\frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{B} \cdot \mathbf{H}_a \, dv + \frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{H} \cdot \mathbf{B}_a \, dv. \tag{9.153}$$

Following a similar procedure as for the second integral in (9.148), on use of the divergence theorem, and with reference to Figure 9.2, the first term in the above equation can be written as

$$\frac{1}{2}\int_{\partial\mathcal{B}^\alpha \cup \mathcal{S}} \varphi_a \mathbf{B} \cdot \mathbf{n} \, da + \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} \varphi_a \mathbf{B} \cdot \tilde{\mathbf{n}} \, da, \tag{9.154}$$

and the first term in (9.154) is equivalent to

$$\frac{1}{2}\mu_0 \int_{\mathcal{B}^o} \mathbf{H} \cdot \mathbf{H}_a \, dv + \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_a \mathbf{B} \cdot \mathbf{n}' \, da. \tag{9.155}$$

The second integral in (9.153) is easily shown to be equal to

$$-\frac{1}{2}\mu_0 \int_{\mathcal{B}^o} \mathbf{H}_a \cdot \mathbf{H} \, dv - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B}_a \cdot \mathbf{n}' \, da - \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} \varphi \mathbf{B}_a \cdot \tilde{\mathbf{n}} \, da. \tag{9.156}$$

Using (9.154), (9.155) and (9.156) in (9.153), and (9.152) in (9.151), and then combining these results in (9.148) we can express $\bar{E}$ as

$$\bar{E} = \int_{\mathcal{B}} \rho\psi^* \,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\psi}^* \,\mathrm{d}v - \frac{1}{2}\mu_0 \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} (\varphi_s \mathbf{B} + \varphi\,\mathbf{B}_a)\cdot\mathbf{n}'\,\mathrm{d}a, \tag{9.157}$$

where, for brevity, we have written $\tilde{\mathbf{n}} = \mathbf{n}'$ on $\partial\tilde{\mathcal{B}}^\infty$. The latter integral in (9.157) can be decomposed as

$$-\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} \varphi_s \mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \frac{1}{2}\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} \varphi_a \mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \frac{1}{2}\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} \varphi_s \mathbf{B}_s\cdot\mathbf{n}'\,\mathrm{d}a. \tag{9.158}$$

The second integral in (9.158) is not affected by any variation [17] (see (9.76) and (9.78)), so we can omit it from our formulation. In the third integral in (9.158), as in Section 9.2, we assume that $\varphi_s \sim \frac{1}{|\mathbf{x}|}$ as $|\mathbf{x}| \to \infty$. For $\partial\mathcal{B}^\infty$ it follows that $|\mathbf{B}_s| \sim \frac{1}{|\mathbf{x}|^2}$, and the associated integral vanishes. We assume that the behaviour of $\mathbf{B}_s$ is such that the integral over $\partial\tilde{\mathcal{B}}^\infty$ also vanishes. For the first integral in (9.158) we can replace φ_s by φ (since the difference is a constant). Hence, equation (9.157) can be written

$$\bar{E} = \int_{\mathcal{B}} \rho\psi^* \,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\psi}^* \,\mathrm{d}v - \frac{1}{2}\mu_0 \int_{\mathcal{B}^o} \mathbf{H}\cdot\mathbf{H}\,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} \varphi\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \int_{\partial\tilde{\mathcal{B}}^\infty} \varphi\,\mathbf{B}_a\cdot\tilde{\mathbf{n}}\,\mathrm{d}a. \tag{9.159}$$

For the mechanical boundary conditions we have $\mathbf{x} = \hat{\mathbf{x}}$ on $\partial\mathcal{B}^\beta$. Also, we have $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^\beta$ (and on $\mathcal{S}$). On $\partial\mathcal{B}^\alpha$ there is no mechanical load or restriction on the displacement. We assume, as in Section 9.2, that the mechanical body force $\mathbf{f}$ is conservative and equal to $-\mathrm{grad}\,U$, where U is the associated potential. Then, since there is no prescribed mechanical traction on $\partial\mathcal{B}^\alpha$ and $\mathbf{x}$ is prescribed on $\partial\mathcal{B}^\beta$, the work $\bar{L}$ of the mechanical loading (9.66) is simply

$$\bar{L} = -\int_{\mathcal{B}} \rho U\,\mathrm{d}v. \tag{9.160}$$

We define the functional Ξ as

$$\Xi\{\mathbf{x},\varphi\} = \bar{E}\{\mathbf{x},\varphi\} - \bar{L}\{\mathbf{x}\}. \tag{9.161}$$

9.3.1 Variation in the magnetic potential

Let $\dot{\Xi}_\varphi$ denote the variation of the functional Ξ with respect to the variation $\dot{\varphi}$ in φ. From (9.159), (9.161) and (9.97) we have

$$\dot{\Xi}_\varphi = -\int_{\mathcal{B}} \rho\frac{\partial\psi^*}{\partial\mathbf{H}}\cdot\mathrm{grad}\dot{\varphi}\,\mathrm{d}v - \int_{\tilde{\mathcal{B}}} \tilde{\rho}\frac{\partial\tilde{\psi}^*}{\partial\mathbf{H}}\cdot\mathrm{grad}\dot{\varphi}\,\mathrm{d}v + \mu_0\int_{\mathcal{B}^o}\mathbf{H}\cdot\mathrm{grad}\dot{\varphi}\,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \int_{\partial\tilde{\mathcal{B}}^\infty} \dot{\varphi}\,\mathbf{B}_a\cdot\tilde{\mathbf{n}}\,\mathrm{d}a. \tag{9.162}$$

Following Bustamante et al. [17] (see equation $(9.44)_2$) we consider the connection $-\frac{\rho\partial\psi^*}{\partial\mathbf{H}} = \mathbf{B}$ in $\mathcal{B}$, and we also write $-\frac{\tilde{\rho}\partial\tilde{\psi}^*}{\partial\mathbf{H}} = \mathbf{B}$ in $\tilde{\mathcal{B}}$. Then, by using equation $(9.1)_2$, the divergence theorem, and the decomposition of the boundary $\partial\mathcal{B}$, the first integral in (9.162) is seen to be equivalent to

$$\int_{\partial\mathcal{B}^\alpha} \dot{\varphi}\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\beta} \dot{\varphi}\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a - \int_{\mathcal{B}} \dot{\varphi}\,\mathrm{div}\mathbf{B}\,\mathrm{d}v. \tag{9.163}$$

Similar expressions can be found for the second and third integrals in (9.162), but for brevity we do not include them here. Taking account of the decompositions $\partial\tilde{\mathcal{B}} = \partial\mathcal{B}^\beta \cup \mathcal{S} \cup \partial\tilde{\mathcal{B}}^\infty$ and $\partial\mathcal{B}^o = \partial\mathcal{B}^\alpha \cup \mathcal{S} \cup \partial\mathcal{B}^\infty$ (see Figure 9.2), and remembering the rule for the sign of the normal vectors, we can show that (9.162) can be written as

$$\dot{\Xi}_\varphi = \int_{\mathcal{B}\cup\tilde{\mathcal{B}}\cup\mathcal{B}^o} \dot{\varphi}\,\mathrm{div}\mathbf{B}\,\mathrm{d}v + \int_{\partial\mathcal{B}^\alpha} \dot{\varphi}\,[\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\beta} \dot{\varphi}\,[\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}a + \int_{\mathcal{S}} \dot{\varphi}\,[\![\mathbf{B}]\!]\cdot\tilde{\mathbf{n}}\,\mathrm{d}a$$
$$+ \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}'\,\mathrm{d}a + \int_{\partial\tilde{\mathcal{B}}^\infty} \dot{\varphi}\,\mathbf{B}_s\cdot\tilde{\mathbf{n}}\,\mathrm{d}a. \tag{9.164}$$

In the above expression the last two integrals vanish when $|\mathbf{x}| \to \infty$. It follows that Ξ is stationary with respect to φ if and only if

$$\mathrm{div}\mathbf{B} = 0 \quad \text{in} \quad \mathcal{B}\cup\tilde{\mathcal{B}}\cup\mathcal{B}^o \tag{9.165}$$

and

$$[\![\mathbf{B}]\!]\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}^\alpha, \tag{9.166}$$

$$[\![\mathbf{B}]\!]\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}^\beta, \tag{9.167}$$

$$[\![\mathbf{B}]\!]\cdot\tilde{\mathbf{n}} = 0 \quad \text{on} \quad \mathcal{S}. \tag{9.168}$$

9.3.2 Variation with x

From Subsection 9.1.3 we have the following connection between the function ψ^* and the complementary form of the energy function Ω^* [33, 34] (equation (9.62)):

$$J^{-1}\Omega^*(\mathbf{F},\mathbf{H}_l) = \rho\psi^*(\mathbf{F},\mathbf{H}). \tag{9.169}$$

Let us rewrite some of the integrals in (9.159) with respect to the reference configuration:

$$\bar{E} = \int_{\mathcal{B}_r} \Omega^*\,\mathrm{d}V + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\psi}^*\,\mathrm{d}v - \frac{1}{2}\mu_0 \int_{\mathcal{B}_r^o} J(\mathbf{F}^{-T}\mathbf{H}_l)\cdot(\mathbf{F}^{-T}\mathbf{H}_l)\,\mathrm{d}V$$
$$- \int_{\partial\mathcal{B}^\infty} \varphi\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \int_{\partial\tilde{\mathcal{B}}^\infty} \varphi\,\mathbf{B}_a\cdot\tilde{\mathbf{n}}\,\mathrm{d}a. \tag{9.170}$$

We did not have to modify the expression for the second, fourth or fifth integrals because the body $\tilde{\mathcal{B}}$ is rigid and $\partial\mathcal{B}^\infty$ and $\partial\tilde{\mathcal{B}}^\infty$ are assumed fixed. In $\mathcal{B}_r^o$ the deformation gradient is 'fictitious' in the sense that it is obtained arbitrarily from a smooth extension of the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in $\mathcal{B}_r$ to $\mathcal{B}_r^o$.

Next, we calculate the variation of $\bar{E}$ with respect to $\mathbf{x}$, which we denote by $\dot{\bar{E}}_{\mathbf{x}}$. For this purpose, we use the connection $(9.58)_1$ $\frac{\partial\Omega^*}{\partial\mathbf{F}} = \mathbf{T}$, where $\mathbf{T}$ is the total nominal stress, and recall that $\tilde{\psi}^*$ is a function of $\mathbf{H}$ only. We also use the definition (9.25) $\boldsymbol{\tau}_m = \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{H})\mathbf{I}$ of the Maxwell stress (see, for example, Kovetz [64]), and following a procedure similar to that used in [17] (see Subsection 9.2.3), the variation of $\bar{E}$ with respect to $\mathbf{x}$ can be written

$$\dot{\bar{E}}_{\mathbf{x}} = \int_{\mathcal{B}_r} \operatorname{tr}(\mathbf{T}\dot{\mathbf{F}})\,\mathrm{d}V + \int_{\mathcal{B}^o} \operatorname{div}(\boldsymbol{\tau}_m\dot{\mathbf{x}})\,\mathrm{d}v - \int_{\mathcal{B}'} (\operatorname{div}\boldsymbol{\tau}_m) \cdot \dot{\mathbf{x}}\,\mathrm{d}v. \tag{9.171}$$

On using the divergence theorem in the above expression, making some rearrangements, referring to (9.160), and noting that $\dot{\mathbf{x}} = \mathbf{0}$ on $\mathcal{S}$ and $\partial\mathcal{B}^\beta$, we obtain

$$\dot{\Xi}_{\mathbf{x}} = -\int_{\mathcal{B}_r} (\operatorname{Div}\mathbf{T} + \rho_o\mathbf{f}) \cdot \dot{\mathbf{x}}\,\mathrm{d}V - \int_{\mathcal{B}^o} (\operatorname{div}\boldsymbol{\tau}_m) \cdot \dot{\mathbf{x}}\,\mathrm{d}v + \int_{\partial\mathcal{B}_r^\alpha} (\mathbf{T}^T\mathbf{N} - \mathbf{t}_m) \cdot \dot{\mathbf{x}}\,\mathrm{d}A + \int_{\partial\mathcal{B}^\infty} \boldsymbol{\tau}_m\mathbf{n}' \cdot \dot{\mathbf{x}}\,\mathrm{d}a, \tag{9.172}$$

where $\rho_o = \rho J$ is the density in $\mathcal{B}_r$, $\mathbf{t}_m = \mathbf{T}_m^T\mathbf{N}$ and $\mathbf{T}_m = J\mathbf{F}^{-1}\boldsymbol{\tau}_m$.

In the above equation we can assume that the fourth integral vanishes for $|\mathbf{x}| \to \infty$. As a result Ξ is stationary with respect to $\mathbf{x}$ if and only if

$$\operatorname{Div}\mathbf{T} + \rho_r\mathbf{f} = \mathbf{0} \quad \text{in} \quad \mathcal{B}_r, \tag{9.173}$$

$$\mathbf{T}^T\mathbf{N} = \mathbf{t}_m \quad \text{on} \quad \partial\mathcal{B}_r^\alpha \tag{9.174}$$

and

$$\operatorname{div}\boldsymbol{\tau}_m = \mathbf{0} \quad \text{in} \quad \mathcal{B}^o. \tag{9.175}$$

Note that this last equation holds for free space if the field equations (9.1) are satisfied.

To summarize, the functional Ξ is stationary with respect to both φ and $\mathbf{x}$ if and only if equations (9.165)-(9.168), (9.173) and (9.174) hold.

Chapter 10

Conclusions

The final aim of the research on the mathematical modelling of MS and ES elastomers is to predict the behaviour of these materials, and so to help in the development of new applications where smart materials are required. There are two important steps in order to achieve this; one is to propose simple and realistic forms for the constitutive equations, and the other is to set up a numerical method to solve the boundary value problem (due to the highly non-linear nature of the problem, analytical methods of solution have a limited applicability). Several steps towards the above two objectives have been achieved in this thesis.

To propose simple and realistic forms for the constitutive equations is a very important part of our future research. Unfortunately, as stated in the Introduction, currently there is very little experimental data for the mechanical behaviour, especially for ES elastomers. The complexity of the problem, where we have to work with energy functions that may depends on five, six or even ten invariants (see, for example, (4.12) and (5.11)), implies that we have to propose at some moment simplified forms for these constitutive equations (see, for example, Section 5.3). It is necessary to have a criterion in order to know from experiments whether such simplifications are realistic, and this criterion is provided by the universal relations. For MS and ES materials universal relations are shown in Sections 4.1 and 7.1. In the case of MS materials, the results shown in this thesis correspond to an extension of the results shown in [18] and [37]. We found universal relations for some simplified cases of the constitutive equation, and these results will be important in the future when more forms for the constitutive equation are proposed.

As mentioned in the introduction, and also at the beginning of Chapter 5. it seems that most of the experimental data available [7, 50, 59] have been obtained actually for a particular class of materials, which we have described as transversely isotropic magneto- and electro-active elastomers (Chapters 5 and 8). Experimental data for the uniform extension of a cylinder [7] suggest that, for example, the magnetostriction effect is much stronger for these materials than for the isotropic ones.

In this thesis we have provided the basic forms of the constitutive equations for transversely isotropic MS and ES elastomers, based on the theory developed by Dorfmann and Ogden [32–34]. For transversely isotropic MS elastomers a preliminary form for the energy function was proposed (5.245), and several simple boundary value problems were solved (Sections 5.2 and 5.5). The idea of solving some simple boundary value problems such as the shear of a slab, was to use these results in order to propose a form for the energy function, and then to use this function to obtain solutions for some non-homogeneous deformations. The procedure presented in Chapter 5, can be seen as a prototype method to handle the search of constitutive equations for transversely isotropic MS and ES elastomers, when more experimental data becomes available.

Regarding the second main objective stated at the beginning of these conclusions, the developing of numerical methods, in this thesis we have obtained important results towards the implementation of the finite element method.

In Sections 4.2 and 7.2 we obtained some numerical results for a tube under extension and inflation, and a cylinder under uniform extension, where an external uniform axial magnetic and electric field was applied respectively. These results showed that the boundary conditions (2.104) and (2.105) imply sometimes a rapid change in the form of the fields near the boundary of a body of 'finite size'. The method used in order to solve these problems was the finite difference method, which is limited regarding the kind of problems we can solve. This was one of the reasons we explores the possibility of obtaining a simple variational formulation, as a first step in order to develop a finite element formulation for the problem. Another reason was the doubts regarding the use of the Maxwell stresses [89], which appeared when we looked for a form for the energy function from the experimental data for the uniform tension of a bar [7]. It was not clear if it was necessary or not to include the Maxwell stresses [64] as external load.

In order to answer these questions, and to develop a numerical method we proposed simple variational formulations for MS elastomers (see Chapter 9). In particular the simple forms for the total energy of the system (9.79) and (9.80) are considered as the most important contribution of this thesis, together with the extension of the variational formulation presented in Section 9.3, where we studied the problem of the interaction of a MS elastomer with a rigid semi-infinite body, which would represent some sort of external 'machine' such as the traction and shear devices used in [7] and [59].

Regarding future projects, there are several possibilities. First, as was mentioned already, with the variational formulation and the energy function proposed in (5.245) we can solve some boundary value problems using the finite element method. This would be very important step on the modelling of these materials.

There are many other things that can be done with the variational formulation. In [4] Barham et al. mentioned regarding their variational formulation, that if the functional is constructed based on the magnetic field or the magnetic induction as the magnetic independent variable, then the stationary point for the functional is not a minimum but a saddle point. It would be very interesting then to study this issue for our formulation; in order to do so we would need to work with the second variation.

Once the above questions are answered, a natural but very difficult next step would be to extend the analysis developed by Ball [2], in order to look for conditions on the energy function such that we would avoid solutions of the boundary value problem with discontinuities. With the variational formulation we can attempt to answer several questions such as existence of solution, and conditions in order to have either a 'stable' or an 'unstable' solution. From Ball's analysis [2] several concepts appear, such as quasi-convexity, and poly-convexity, which we would need to extend for our problem, where we have non-linear elastic deformations and magnetic fields (see [61]).

Of course the variational formulation presented in Chapter 9 can be used in order to find a similar formulation for ES elastomers. This is being done by the author, and it will appear in a future publication [14].

Regarding ES elastomers, there are a couple of boundary conditions we did not consider in this thesis. One of them would be to assume the scalar electric potential given for a portion of the surface of a body [122], or a distribution of surface charge [119]. The variational formulation of Yang and Batra [122], and Vu et al. [119] did not consider the exterior free space, and so we would need to modify their formulation accordingly with the method presented in Chapter 9 and [14], in order to include the free space for their formulations. Another important future topic of research would be to propose restrictions (inequalities) of the energy function, in order to have solutions of the boundary value problem with physical meaning. The idea would be to explore in detail the theory for the pure elastic case [112], and to extend it to the magneto- and electro-elastic cases.

The formulation of Dorfmann and Ogden (see, for example, [32–34]), which has been the basis of the results presented in this thesis, is based in the important assumption of 'small' magneto- and electro-active particles. Kari and Blom [62] and Armstrong [1] studied the case of 'large' particles. In the case of Kari and Blom, they showed that to work with comparatively 'larger' irregular shaped particles (random distribution) produces a magnetostriction effect as strong as for a transversely isotropic MS elastomer with 'small' particles. An interesting study would be to add to our model some 'microscopic' parameter, which would take account for the effect of the 'shape' of the particles, in the case we would assume not to work with perfectly spherical particles.

As well as this, we assumed that neither for the magnetization nor the polarization there was 'hysteresis', however, real materials do show some residual magnetization or polarization, and so it would be necessary at some moment to take account of this phenomenon in our models. As what happens with plastic deformations, we would probably need to work with an 'implicit' constitutive equation to take account the phenomenon properly.

Another very important but difficult future line of research would be to extend the formulation of Dorfmann and Ogden [32–34] to the dynamic case. The application mentioned in this thesis, for example, the paper by Farshad and Le Roux [44], involve the use of MS and ES elastomers for vibration and noise reduction, which are phenomena where it is clear the time dependence is important.

A full analysis of the dynamic problem would in general imply the coupling of electric and magnetic fields (see Subsection 2.2.2), which as happened with the analysis for transversely isotropic MS and ES elastomers, would mean most probably very complex mathematical expressions.

Appendix A

Cylindrical and spherical coordinate systems

In this appendix we show some expressions such as the gradient and the divergence operators, applied to a scalar function, a vector field and a tensor field, in cylindrical and spherical coordinates. As well as this, some examples of the calculation of the gradient of deformation are shown, for the case where either the current and/or the reference configurations are given in cylindrical or spherical coordinates. We do not show the full proofs of the following expressions, which may be found, for example, in [78, 113] (see also [100]).

A.1 Cylindrical coordinates

Consider the cylindrical coordinate system (r, θ, z), where the unit basis vectors are $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ respectively. Consider a scalar field Φ, and the vector field $\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$; we have the following results:

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\Phi}{\partial z}\mathbf{e}_z, \tag{A.1}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}, \tag{A.2}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\mathbf{e}_z, \tag{A.3}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}. \tag{A.4}$$

In the above expressions the operator nabla ∇ has been defined in terms of the coordinate system (r, θ, z).

Consider the Cauchy stress tensor $\boldsymbol{\sigma}$ with physical components in the cylindrical coordinate system

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{pmatrix}, \tag{A.5}$$

then the Cauchy-Euler first law of equilibrium is given in component form as (with no time dependence)

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho f_r = 0, \tag{A.6}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + \rho f_\theta = 0. \tag{A.7}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + \rho f_z = 0, \tag{A.8}$$

where $\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z$ is the body force in cylindrical coordinates.

A.2 Spherical coordinates

In the case of spherical coordinates system (r, θ, ϕ), where the unit basis vectors are $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$, we have

$$\nabla\Phi = \frac{\partial \Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\mathbf{e}_\phi, \tag{A.9}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}, \tag{A.10}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_\phi, \tag{A.11}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}, \tag{A.12}$$

where $\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\phi\mathbf{e}_\phi$.

Consider the Cauchy stress tensor $\boldsymbol{\sigma}$ with physical components

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix}, \tag{A.13}$$

then the Cauchy-Euler first law of equilibrium is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta) + \rho f_r = 0, \tag{A.14}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta) + \rho f_\theta = 0, \tag{A.15}$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta) + \rho f_\phi = 0. \tag{A.16}$$

A.3 Some examples for the deformation gradient in curvilinear coordinates

In this section we show some examples of how to calculate the deformation gradient for different situations involving cylindrical, spherical and Cartesian coordinates [78].

- **Current configuration in cylindrical coordinates and reference configuration in Cartesian coordinates.**

  In this case we have $\mathbf{x} = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z$ and $\mathbf{X} = X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3$, where $\{\mathbf{E}_i\}$ is the system of unit basis vectors in the Cartesian coordinate system for the reference configuration.

  We have

$$r = r(X_i), \quad \theta = \theta(X_i), \quad z = z(X_i) \quad i = 1, 2, 3 \tag{A.17}$$

  Then from the definition $\mathbf{F} = \text{Grad}\mathbf{x}$, where in this case the gradient operator is given as

$$\text{Grad} = \mathbf{E}_1\frac{\partial}{\partial X_1} + \mathbf{E}_2\frac{\partial}{\partial X_2} + \mathbf{E}_3\frac{\partial}{\partial X_3}. \tag{A.18}$$

  We have that [78]

$$\mathbf{F} = \frac{\partial r}{\partial X_1}\mathbf{e}_r \otimes \mathbf{E}_1 + \frac{\partial r}{\partial X_2}\mathbf{e}_r \otimes \mathbf{E}_2 + \frac{\partial r}{\partial X_3}\mathbf{e}_r \otimes \mathbf{E}_3 + r\frac{\partial \theta}{\partial X_1}\mathbf{e}_\theta \otimes \mathbf{E}_1 + r\frac{\partial \theta}{\partial X_2}\mathbf{e}_\theta \otimes \mathbf{E}_2 + r\frac{\partial \theta}{\partial X_3}\mathbf{e}_\theta \otimes \mathbf{E}_3 + \frac{\partial z}{\partial Z}\mathbf{e}_z \otimes \mathbf{E}_1 + \frac{\partial z}{\partial X_2}\mathbf{e}_z \otimes \mathbf{E}_2 + \frac{\partial z}{\partial X_3}\mathbf{e}_z \otimes \mathbf{E}_3. \tag{A.19}$$

- **Current configuration in Cartesian coordinates and reference configuration in cylindrical coordinates.**

  In this case we have $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and $\mathbf{X} = R\mathbf{E}_R + \Theta\mathbf{E}_\Theta + Z\mathbf{E}_Z$, where

$$x_i = x_i(R, \Theta, Z), \quad i = 1, 2, 3 \tag{A.20}$$

  and $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$ is the system of unit basis vectors in cylindrical coordinates for the reference configuration.

  The gradient operator Grad is defined as

$$\text{Grad} = \mathbf{E}_R\frac{\partial}{\partial R} + \frac{1}{R}\mathbf{E}_\Theta\frac{\partial}{\partial \Theta} + \mathbf{E}_Z\frac{\partial}{\partial Z}. \tag{A.21}$$

  Then for the deformation gradient we have

$$\mathbf{F} = \frac{\partial x_1}{\partial R}\mathbf{e}_1 \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial x_1}{\partial \Theta}\mathbf{e}_1 \otimes \mathbf{E}_\Theta + \frac{\partial x_1}{\partial Z}\mathbf{e}_1 \otimes \mathbf{E}_Z + \frac{\partial x_2}{\partial R}\mathbf{e}_2 \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial x_2}{\partial \Theta}\mathbf{e}_2 \otimes \mathbf{E}_\Theta + \frac{\partial x_2}{\partial Z}\mathbf{e}_2 \otimes \mathbf{E}_Z + \frac{\partial x_3}{\partial R}\mathbf{e}_3 \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial x_3}{\partial \Theta}\mathbf{e}_3 \otimes \mathbf{E}_\Theta + \frac{\partial x_3}{\partial Z}\mathbf{e}_3 \otimes \mathbf{E}_Z. \tag{A.22}$$

- **Current and reference configurations in cylindrical coordinates.**

  In this case we have $\mathbf{x} = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z$ and as before $\mathbf{X} = R\mathbf{E}_R + \Theta\mathbf{E}_\Theta + Z\mathbf{E}_Z$, where

$$r = r(R, \Theta, Z), \quad \theta = \theta(R, \Theta, Z), \quad z = z(R, \Theta, Z). \tag{A.23}$$

  then

$$\begin{aligned}\mathbf{F} = {} & \frac{\partial r}{\partial R}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial r}{\partial \Theta}\mathbf{e}_r \otimes \mathbf{E}_\Theta + \frac{\partial r}{\partial Z}\mathbf{e}_r \otimes \mathbf{E}_Z + r\frac{\partial \theta}{\partial R}\mathbf{e}_\theta \otimes \mathbf{E}_R + \frac{r}{R}\frac{\partial \theta}{\partial \Theta}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta \\ & + r\frac{\partial \theta}{\partial Z}\mathbf{e}_\theta \otimes \mathbf{E}_Z + \frac{\partial z}{\partial R}\mathbf{e}_z \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial z}{\partial \Theta}\mathbf{e}_z \otimes \mathbf{E}_\Theta + \frac{\partial z}{\partial Z}\mathbf{e}_z \otimes \mathbf{E}_Z.\end{aligned} \tag{A.24}$$

- **Current and reference configurations in spherical coordinates**

  This is the only case we treat for spherical coordinates. Here we have $\mathbf{x} = r\mathbf{e}_r$ and $\mathbf{X} = R\mathbf{E}_R$, where $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ and $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_\Phi\}$ are the systems of unit basis vectors in the current and reference configurations respectively.

  For this case the gradient operator in the reference configuration Grad is given as

$$\text{Grad} = \mathbf{E}_R\frac{\partial}{\partial R} + \frac{1}{R}\mathbf{E}_\Theta\frac{\partial}{\partial \Theta} + \frac{1}{R\sin\Theta}\mathbf{E}_\Phi\frac{\partial}{\partial \Phi}. \tag{A.25}$$

  Then for the deformation gradient we have

$$\mathbf{F} = \frac{\partial r}{\partial R}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial r}{\partial \Theta}\mathbf{e}_r \otimes \mathbf{E}_\Theta + \frac{1}{R\sin\Theta}\frac{\partial r}{\partial \Phi}\mathbf{e}_r \otimes \mathbf{E}_\Phi. \tag{A.26}$$

Appendix B

Note on the invariants for MS elastomers

Let us recall the form of the invariants for a transversely isotropic MS elastomer, for the case the magnetic induction is the independent magnetic variable. The ten invariants were

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det\mathbf{c}, \tag{B.1}$$

$$I_4 = \mathbf{B}_l \cdot \mathbf{B}_l, \quad I_5 = \mathbf{B}_l \cdot \mathbf{c}\mathbf{B}_l, \quad I_6 = \mathbf{B}_l \cdot \mathbf{c}^2\mathbf{B}_l, \tag{B.2}$$

$$I_7 = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{a}_0, \quad I_8 = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{a}_0, \tag{B.3}$$

$$I_9 = \mathbf{a}_0 \cdot \mathbf{B}_l, \quad I_{10} = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{B}_l. \tag{B.4}$$

For a function $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l, \mathbf{a}_0)$, which depends on one tensor and two vectors field, the theory presented by Zheng [127] includes an extra invariant, defines as

$$I_{11} = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{B}_l. \tag{B.5}$$

We prove here that I_{11} in fact depends on the rest of the invariants (B.1)-(B.2). Consider the Cayley-Hamilton theorem

$$\mathbf{G}^3 - I_1^G\mathbf{G}^2 + I_2^G\mathbf{G} - I_3^G\mathbf{I} = \mathbf{0}, \tag{B.6}$$

where

$$I_1^G = \operatorname{tr}\mathbf{G}, \tag{B.7}$$

$$I_2^G = \frac{1}{2}[(\operatorname{tr}\mathbf{G}^2) - \operatorname{tr}\mathbf{G}^2], \tag{B.8}$$

$$I_3^G = \det\mathbf{G} = \frac{1}{3}\operatorname{tr}\mathbf{G}^3 - \frac{1}{2}\operatorname{tr}\mathbf{G}\operatorname{tr}\mathbf{G}^2 + \frac{1}{6}(\operatorname{tr}\mathbf{G})^3. \tag{B.9}$$

Let's replace $\mathbf{G}$ by $\mathbf{c}+\lambda\mathbf{A}$. From (B.6) we have

$$(\mathbf{c}+\lambda\mathbf{A})^3 - I_1^G(\mathbf{c}+\lambda\mathbf{A})^2 + I_2^G(\mathbf{c}+\lambda\mathbf{A}) - I_3^G\mathbf{I} = \mathbf{0}, \tag{B.10}$$

and from (B.7)-(B.9)

$$I_1^G = I_1^c + \lambda I_1^A, \tag{B.11}$$

$$I_2^G = I_2^c + \lambda^2 I_2^A + \lambda\{\mathrm{tr}\,\mathbf{c}\,\mathrm{tr}\,\mathbf{A} - \mathrm{tr}\,(\mathbf{cA})\}, \tag{B.12}$$

$$I_3^G = I_3^c + \lambda^3 I_3^A + \Lambda, \tag{B.13}$$

where

$$I_1^c \equiv \mathrm{tr}\,\mathbf{c}, \quad I_2^c \equiv \frac{1}{2}\{(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2\}, \quad I_3^c \equiv \frac{1}{3}\mathrm{tr}\,\mathbf{c}^3 - \frac{1}{2}\mathrm{tr}\,\mathbf{c}\,\mathrm{tr}\,\mathbf{c}^2 + \frac{1}{6}(\mathrm{tr}\,\mathbf{c})^3, \tag{B.14}$$

$$I_1^A \equiv \mathrm{tr}\,\mathbf{A}, \quad I_2^A \equiv \frac{1}{2}\{(\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}\,\mathbf{A}^2\}, \quad I_3^A \equiv \frac{1}{3}\mathrm{tr}\,\mathbf{A}^3 - \frac{1}{2}\mathrm{tr}\,\mathbf{A}\,\mathrm{tr}\,\mathbf{A}^2 + \frac{1}{6}(\mathrm{tr}\,\mathbf{A})^3, \tag{B.15}$$

and

$$\begin{aligned}\Lambda = {}& \frac{\lambda}{3}\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) + \frac{\lambda}{3}\mathrm{tr}\,(\mathbf{cAc}) + \frac{\lambda^2}{3}\mathrm{tr}\,(\mathbf{cA}^2) + \frac{\lambda}{3}\mathrm{tr}\,(\mathbf{Ac}^2) + \frac{\lambda^2}{2}\mathrm{tr}\,(\mathbf{AcA}) + \frac{\lambda^2}{3}\mathrm{tr}\,(\mathbf{A}^2\mathbf{c}) \\ & -\lambda\mathrm{tr}\,\mathbf{c}\,\mathrm{tr}\,(\mathbf{cA}) - \frac{\lambda^2}{2}\mathrm{tr}\,\mathbf{c}\,\mathrm{tr}\,\mathbf{A}^2 - \frac{\lambda}{2}\mathrm{tr}\,\mathbf{A}\,\mathrm{tr}\,\mathbf{c}^2 - \lambda^2\mathrm{tr}\,\mathbf{A}\,\mathrm{tr}\,(\mathbf{cA}) + \frac{\lambda}{2}(\mathrm{tr}\,\mathbf{c})^2\mathrm{tr}\,\mathbf{A} \\ & + \frac{\lambda^2}{2}\mathrm{tr}\,\mathbf{c}(\mathrm{tr}\,\mathbf{A})^2, \end{aligned} \tag{B.16}$$

as well as this

$$\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) = \mathrm{tr}\,(\mathbf{cAc}) = \mathrm{tr}\,(\mathbf{Ac}^2), \quad \mathrm{tr}\,(\mathbf{cA}^2) = \mathrm{tr}\,(\mathbf{AcA}) = \mathrm{tr}\,(\mathbf{A}^2\mathbf{c}). \tag{B.17}$$

Also

$$(\mathbf{c}+\lambda\mathbf{A})^2 = \mathbf{c}^2 + \lambda\mathbf{cA} + \lambda\mathbf{Ac} + \lambda^2\mathbf{A}^2, \tag{B.18}$$

$$(\mathbf{c}+\lambda\mathbf{A})^3 = \mathbf{c}^3 + \lambda\mathbf{c}^2\mathbf{A} + \lambda\mathbf{cAc} + \lambda^2\mathbf{cA}^2 + \lambda\mathbf{Ac}^2 + \lambda^2\mathbf{AcA} + \lambda^2\mathbf{A}^2\mathbf{c} + \lambda^2\mathbf{A}^3. \tag{B.19}$$

From (B.11)-(B.19) after some manipulations we obtain for (B.10)

$$\mathbf{c}^3 - I_1^c\mathbf{c}^2 + I_2^c\mathbf{c} - I_3^c\mathbf{I} + \lambda^3(\mathbf{A}^3 - I_1^A\mathbf{A}^2 + I_2^A\mathbf{A} - I_3^A\mathbf{I}) + \lambda\mathbf{P}^{(1)} + \lambda^2\mathbf{P}^{(2)} = \mathbf{0}, \tag{B.20}$$

where the tensors $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ are defined as

$$\begin{aligned}\mathbf{P}^{(1)} = {}& \mathbf{c}^2\mathbf{A} + \mathbf{cAc} + \mathbf{Ac}^2 - (\mathrm{tr}\,\mathbf{c})\mathbf{cA} - (\mathrm{tr}\,\mathbf{c})\mathbf{Ac} - (\mathrm{tr}\,\mathbf{A})\mathbf{c}^2 + \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2]\mathbf{A} \\ & + [\mathrm{tr}\,\mathbf{c}\,\mathrm{tr}\,\mathbf{A} - \mathrm{tr}\,(\mathbf{cA})]\mathbf{c} - \left[\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - \mathrm{tr}\,\mathbf{c}\,\mathrm{tr}\,(\mathbf{cA}) - \frac{1}{2}\mathrm{tr}\,\mathbf{A}\,\mathrm{tr}\,\mathbf{c}^2 + \frac{1}{2}(\mathrm{tr}\,\mathbf{c})^2\mathrm{tr}\,\mathbf{A}\right]\mathbf{I}. \end{aligned} \tag{B.21}$$

and

$$\mathbf{P}^{(2)} = \mathbf{A}^2\mathbf{c} + \mathbf{AcA} + \mathbf{cA}^2 - (\mathrm{tr}\,\mathbf{A})\mathbf{Ac} - (\mathrm{tr}\,\mathbf{A})\mathbf{cA} - (\mathrm{tr}\,\mathbf{c})\mathbf{A}^2 + \frac{1}{2}[(\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}\,\mathbf{A}^2]\mathbf{c}$$
$$+[\mathrm{tr}\,\mathbf{A}\mathrm{tr}\,\mathbf{c} - \mathrm{tr}\,(\mathbf{Ac})]\mathbf{c} - \left[\mathrm{tr}\,(\mathbf{cA}^2) - \mathrm{tr}\,\mathbf{A}\mathrm{tr}\,(\mathbf{cA}) - \frac{1}{2}\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,\mathbf{A}^2 + \frac{1}{2}(\mathrm{tr}\,\mathbf{A})^2\mathrm{tr}\,\mathbf{c}\right]\mathbf{I}. \quad (B.22)$$

From the Cayley-Hamilton theorem we have that

$$\mathbf{c}^3 - I_1^c\mathbf{c}^2 + I_2^c\mathbf{c} - I_3^c\mathbf{I} = \mathbf{0}, \quad \mathbf{A}^3 - I_1^A\mathbf{A}^2 + I_2^A\mathbf{A} - I_3^A\mathbf{I} = \mathbf{0}, \quad (B.23)$$

therefore from (B.20) we get

$$\lambda\mathbf{P}^{(1)} + \lambda^2\mathbf{P}^{(2)} = \mathbf{0}. \quad (B.24)$$

It is possible to prove that $\mathrm{tr}\,\mathbf{P}^{(1)} = 0$ and that $\mathrm{tr}\,\mathbf{P}^{(2)} = 0$.

Let's multiply (B.24) by $\mathbf{c}$ and take the trace, we have

$$\lambda\mathrm{tr}\,(\mathbf{cP}^{(1)}) + \lambda^2\mathrm{tr}\,(\mathbf{cP}^{(2)}) = 0, \quad (B.25)$$

which must hold for any λ, and as a result is equivalent to

$$\mathrm{tr}\,(\mathbf{cP}^{(1)}) = 0 \quad \text{and} \quad \mathrm{tr}\,(\mathbf{cP}^{(2)}) = 0. \quad (B.26)$$

We prove now that the equation $\mathrm{tr}\,(\mathbf{cP}^{(1)}) = 0$ is satisfied trivially. From the definition (B.21) we have

$$\mathrm{tr}\,(\mathbf{cP}^{(1)}) = 3\mathrm{tr}\,(\mathbf{c}^3\mathbf{A}) - 2\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - \mathrm{tr}\,\mathbf{A}\mathrm{tr}\,\mathbf{c}^3 + \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2]\mathrm{tr}\,(\mathbf{Ac})$$
$$+[\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,\mathbf{A} - \mathrm{tr}\,(\mathbf{cA})]\mathrm{tr}\,\mathbf{c}^2 - \left[\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - \mathrm{tr}\,\mathbf{c}\mathrm{tr}\,(\mathbf{cA}) - \frac{1}{2}\mathrm{tr}\,\mathbf{A}\mathrm{tr}\,\mathbf{c}^2 + \frac{1}{2}(\mathrm{tr}\,\mathbf{c})^2\mathrm{tr}\,\mathbf{A}\right]\mathrm{tr}\,\mathbf{c}. \quad (B.27)$$

Let's use again the Cayley-Hamilton theorem in order to obtain the expression $\mathbf{c}^3 = I_1^c\mathbf{c}^2 - I_2^c\mathbf{c} + I_3^c\mathbf{I}$; then for (B.27) we have

$$\mathrm{tr}\,(\mathbf{cP}^{(1)}) = 3[I_1^c\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - I_2^c\mathrm{tr}\,(\mathbf{cA}) + I_3^c\mathrm{tr}\,\mathbf{A}] - 2\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - \mathrm{tr}\,\mathbf{A}[I_1^c\mathrm{tr}\,\mathbf{c}^2$$
$$-I_2^c\mathrm{tr}\,\mathbf{c} + 3I_3^c] + \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2]\mathrm{tr}\,(\mathbf{Ac}) + [\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,\mathbf{A} - \mathrm{tr}\,(\mathbf{cA})]\mathrm{tr}\,\mathbf{c}^2$$
$$-\left[\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - \mathrm{tr}\,\mathbf{c}\mathrm{tr}\,(\mathbf{cA}) - \frac{1}{2}\mathrm{tr}\,\mathbf{A}\mathrm{tr}\,\mathbf{c}^2 + \frac{1}{2}(\mathrm{tr}\,\mathbf{c})^2\mathrm{tr}\,\mathbf{A}\right]\mathrm{tr}\,\mathbf{c}. \quad (B.28)$$

Using the definition for I_1^c and I_2^c from $(B.14)_1$, $(B.14)_2$, it is easy to show from above that $\mathrm{tr}\,(\mathbf{cP}^{(1)}) = 0$ is satisfied trivially. As a result, from (B.26) the only equation left is

$$\mathrm{tr}\,(\mathbf{cP}^{(2)}) = 0. \quad (B.29)$$

From the definition (B.22) the above equation is equivalent to

$$\begin{aligned} \mathrm{tr}\,(\mathbf{c}\mathbf{P}^{(2)}) &= \mathrm{tr}\,(\mathbf{c}^2\mathbf{A}^2) + \mathrm{tr}\,(\mathbf{cAcA}) + \mathrm{tr}\,(\mathbf{cA}^2\mathbf{c}) - \mathrm{tr}\,\mathbf{A}\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - \mathrm{tr}\,\mathbf{A}\mathrm{tr}\,(\mathbf{cAc}) \\ &-\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,(\mathbf{cA}^2) + \frac{1}{2}(\mathrm{tr}\,\mathbf{A})^2\mathrm{tr}\,\mathbf{c}^2 - \frac{1}{2}\mathrm{tr}\,\mathbf{A}^2\mathrm{tr}\,\mathbf{c}^2 + \mathrm{tr}\,\mathbf{c}\mathrm{tr}\,\mathbf{A}\mathrm{tr}\,(\mathbf{cA}) - (\mathrm{tr}\,(\mathbf{cA}))^2 \\ &- \left[\mathrm{tr}\,(\mathbf{cA}^2) - \mathrm{tr}\,\mathbf{A}\mathrm{tr}\,(\mathbf{cA}) - \frac{1}{2}\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,\mathbf{A}^2 + \frac{1}{2}(\mathrm{tr}\,\mathbf{A})^2\mathrm{tr}\,\mathbf{c}\right]\mathrm{tr}\,\mathbf{c} = 0. \quad \text{(B.30)} \end{aligned}$$

We have that $\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}^2) = \mathrm{tr}\,(\mathbf{cA}^2\mathbf{c})$. After some manipulations (B.30) becomes

$$\begin{aligned} &2\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}^2) + \mathrm{tr}\,(\mathbf{cAcA}) - 2\mathrm{tr}\,\mathbf{A}\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) - 2\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,(\mathbf{cA}^2) - (\mathrm{tr}\,\mathbf{A})^2\frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2] \\ &+\mathrm{tr}\,\mathbf{A}^2\frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2] + 2\mathrm{tr}\,\mathbf{A}\mathrm{tr}\,\mathbf{c}\mathrm{tr}\,(\mathbf{cA}) - (\mathrm{tr}\,(\mathbf{cA}))^2 = 0. \quad \text{(B.31)} \end{aligned}$$

Let's assume the following form for the tensor $\mathbf{A}$.

$$\mathbf{A} = \frac{1}{2}(\mathbf{a}_0 \otimes \mathbf{B}_l + \mathbf{B}_l \otimes \mathbf{a}_0). \quad \text{(B.32)}$$

From the theory for transversely-isotropic magneto-sensitive elastomers, we identify the invariants $(\mathrm{B}.1)_1$, $I_1 \equiv \mathrm{tr}\,\mathbf{c}$ and $(\mathrm{B}.1)_2$ $I_2 \equiv \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2]$. For the rest of the expressions that appear in (B.31) we have (remember that in this case $\mathbf{c}$ is symmetric)

$$\mathrm{tr}\,\mathbf{A} = a_{0_i}B_{l_i} = \mathbf{a}_0 \cdot \mathbf{B}_l \equiv I_9, \quad \text{(B.33)}$$

$$\mathrm{tr}\,(\mathbf{cA}) = \frac{1}{2}(c_{ij}a_{0_j}B_{l_i} + c_{ij}B_{l_j}a_{0_i}) = \mathbf{a}_0 \cdot (\mathbf{c}\mathbf{B}_l) \equiv I_{10}, \quad \text{(B.34)}$$

$$\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}) = \mathbf{a}_0 \cdot (\mathbf{c}^2\mathbf{B}_l) \equiv I_{11}, \quad \text{(B.35)}$$

$$\begin{aligned} \mathrm{tr}\,\mathbf{A}^2 &= \frac{1}{2}(\mathbf{a}_0 \cdot \mathbf{B}_l)\mathrm{tr}\left[\frac{1}{2}(\mathbf{a}_0 \otimes \mathbf{B}_l + \mathbf{B}_l \otimes \mathbf{a}_0)\right] + \frac{|\mathbf{B}_l|^2}{4}\mathrm{tr}\,(\mathbf{a}_0 \otimes \mathbf{a}_0) \\ &+\frac{1}{4}\mathrm{tr}\,(\mathbf{B}_l \otimes \mathbf{B}_l) = \frac{1}{2}(I_9^2 + I_4), \quad \text{(B.36)} \end{aligned}$$

where we have used $(\mathrm{B}.2)_1$ $\mathrm{tr}\,(\mathbf{B}_l \otimes \mathbf{B}_l) = |\mathbf{B}_l|^2 \equiv I_4$, and $\mathrm{tr}\,(\mathbf{a}_0 \otimes \mathbf{a}_0) = |\mathbf{a}_0|^2 = 1$. As well as this, from the relations $(\mathrm{B}.3)_1$ $\mathrm{tr}\,(\mathbf{c}\mathbf{a}_0 \otimes \mathbf{a}_0) = \mathbf{a}_0 \cdot (\mathbf{c}\mathbf{a}_0) \equiv I_7$, and $(\mathrm{B}.2)_2$ $\mathrm{tr}\,(\mathbf{c}\mathbf{B}_l \otimes \mathbf{B}_l) = \mathbf{B}_l \cdot (\mathbf{c}\mathbf{B}_l) \equiv I_5$ we obtain

$$\begin{aligned} \mathrm{tr}\,(\mathbf{cA}^2) &= \mathrm{tr}\left\{\frac{I_9}{2}\mathbf{c}\frac{1}{2}[\mathbf{a}_0 \otimes \mathbf{B}_l + \mathbf{B}_l \otimes \mathbf{a}_0] + \frac{I_4}{4}\mathbf{c}\mathbf{a}_0 \otimes \mathbf{a}_0 + \frac{1}{4}\mathbf{c}\mathbf{B}_l \otimes \mathbf{B}_l\right\} \\ &= \frac{I_9 I_{10}}{2} + \frac{I_4}{4}\mathrm{tr}\,(\mathbf{c}\mathbf{a}_0 \otimes \mathbf{a}_0) + \frac{1}{4}\mathrm{tr}\,(\mathbf{c}\mathbf{B}_l \otimes \mathbf{B}_l) \\ &= \frac{1}{4}(2I_9 I_{10} + I_4 I_7 + I_5). \quad \text{(B.37)} \end{aligned}$$

Consider the term $\mathrm{tr}\,(\mathbf{c}^2\mathbf{A}^2)$, using the definition $(\mathrm{B}.3)_2$ $\mathrm{tr}\,(\mathbf{c}^2\mathbf{a}_0 \otimes \mathbf{a}_0) = \mathbf{a}_0 \cdot (\mathbf{c}^2\mathbf{a}_0) \equiv I_8$, from (B.32) we have

$$\begin{aligned} \mathrm{tr}\,(\mathbf{c}^2\mathbf{A}^2) &= \frac{I_9}{2}\mathrm{tr}\left\{\mathbf{c}^2\frac{1}{2}[\mathbf{a}_0 \otimes \mathbf{B}_l + \mathbf{B}_l \otimes \mathbf{a}_0]\right\} + \frac{I_4}{2}\mathrm{tr}\,(\mathbf{c}^2\mathbf{a}_0 \otimes \mathbf{a}_0) + \frac{1}{4}\mathrm{tr}\,(\mathbf{c}^2\mathbf{B}_l \otimes \mathbf{B}_l) \\ &= \frac{1}{4}(2I_9 I_{10} + I_4 I_8 + I_6). \quad \text{(B.38)} \end{aligned}$$

Finally for the term $\operatorname{tr}(\mathbf{cAcA})$ we obtain

$$\begin{aligned}\operatorname{tr}(\mathbf{cAcA}) &= c_{ij}A_{jk}c_{km}A_{mi} = \frac{1}{4}[c_{ij}(a_{0_j}B_{l_k} + B_{l_j}a_{0_k})c_{km}(a_{0_m}B_{l_i} + B_{l_m}a_{0_i})] \\ &= \frac{1}{4}[B_{l_i}c_{ij}a_{0_j}B_{l_k}c_{km}a_{0_m} + a_{0_i}c_{ij}a_{0_j}B_{l_k}c_{km}B_{l_m} + B_{l_i}c_{ij}B_{l_j}a_{0_k}c_{km}a_{0_m} \\ &\quad + a_{0_i}c_{ij}B_{l_j}a_{0_k}c_{km}B_{l_m}] = \frac{1}{2}(I_{10}^2 + I_5I_7). \end{aligned} \tag{B.39}$$

Using (B.33)-(B.39) in (B.31) we get

$$\begin{aligned} I_9I_{11} + \frac{1}{2}I_4I_8 + \frac{1}{2}I_6 + \frac{1}{2}(I_{10}^2 + I_5I_7) - 2I_9I_{11} - \frac{I_1}{2}(2I_9I_{10} + I_4I_7 + I_5) - I_9^2I_2 \\ + \frac{I_2}{2}(I_9^2 + I_4) + 2I_9I_1I_{10} - I_{10}^2 = 0, \end{aligned} \tag{B.40}$$

from where we obtain

$$I_{11} = \frac{1}{2I_9}[I_4I_8 + I_6 - I_{10}^2 + I_5I_7 + I_1(2I_9I_{10} - I_4I_7 - I_5) - I_2(I_9^2 - I_4)], \tag{B.41}$$

and so I_{11} is not an independent invariant.

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