Geometric Methods in the Study of Pride Groups and Relative Presentations

by

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I should have written about pigeons. Everyone is interested in pigeons.

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Statement

Chapter 1 covers the basic material used throughout this thesis. The majority of this is standard and can be found in the numerous references listed therein. The proofs of Proposition 1.5.2, and Theorems 1.6.2 and 1.6.3 are my own.

Chapter 2 is an introduction to Pride groups. Most of the material in §2.1 is taken from [82, 83, 85]. Section 2.2 is a survey of known results and §2.3 contains statements of the original results that appear in this thesis.

Chapter 3 contains various technical results. The results in §3.1 are already known - see [35, 82, 85]. However, my treatment of this material is different than that found in the literature. See in particular §3.1.1. Section 3.2 is my own work.

Chapter 4 is my own work. The proofs of Theorems 4.2.1 and 4.2.2 are based on similar proofs found in [66, V.5 & V.7].

Chapter 5 is my own work, except for §5.1 and the proof of Lemma 5.2.2. These are taken from [31].

Chapter 6 is an introduction to relative presentations. The material is standard and can be found in the many references listed therein.

Chapter 7 is my own work. The spherical pictures illustrated in Figs. 7.19, 7.20 and 7.21 are based on similar pictures constructed in [2].

Appendix A is my own work.

Acknowledgements

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Abstract

Combinatorial group theory is the study of groups given by presentations. Algebraic and geometric methods pervade this area of mathematics and it is the latter which forms the main theme of this thesis. In particular, we use diagrams and pictures over presentations to study problems in the domain of finitely presented groups. Our thesis is split into two distinct halves, though the techniques used in each are very similar. In Chapters 2 - 4 we study Pride groups with the aim to solve their word and conjugacy problems. We also study the second homotopy module of a natural presentation of a Pride group. Chapters 6 and 7 are devoted to the study of relative presentations, with particular attention being paid to those of the form $\langle H, t\,; t^n a t^{-1} b\rangle$. Determining when such presentations are aspherical is our main objective.

Chapter 1 covers the basic material that is used throughout this thesis. The main topics of interest are free groups; presentations of groups; the word, conjugacy, and isomorphism problems for finitely presented groups; first and second order Dehn functions of finitely presented groups; diagrams and pictures over finite presentations; and the second homotopy module of a finite presentation. The reader may skip Chapter 1 if they are familiar with this material.

A Pride group is a finitely presented group which can be defined by means of a finite simplicial graph; this is done in Chapter 2. Examples of Pride groups are given in §2.1. This section also contains the statements of Conditions (I), (II), (H-I), (H-II), and the asphericity condition. We will always assume that a Pride group satisfies one of these conditions. In §2.2 we survey the known results that appear in the literature, while in §2.3 we present our original results. We obtain isoperimetric functions for a vertex-finite Pride group G which satisfies (I), (II), (H-I) or (H-II). Sufficient conditions are then obtained for G to have a soluble word problem. Solutions of the conjugacy problem for G are also obtained. However, we require that G satisfies some extra conditions. We calculate a generating set for the second homotopy module of the natural

presentation of a non-spherical Pride group, i.e. one which satisfies the asphericity condition. Using this generating set, we obtain an upper bound for the second order Dehn function of a non-spherical vertex-free Pride group. We also obtain information about the second order Dehn function of an arbitrary non-spherical Pride group.

Chapter 3 contains various technical results that are needed in Chapter 4. The main focus is that of diagrams over the standard presentation of a vertex-finite Pride group. We study simply-connected **r**-diagrams in §3.1 and in §3.2 we study annular **r**-diagrams. Propositions 3.1.1, 3.2.1, 3.2.2, and Theorems 3.2.1 and 3.2.2 are the main results of this chapter.

Chapters 4 and 5 are devoted to the proofs of our main results. Proofs of our results for the word and conjugacy problems of a vertex-finite Pride group are contained in Chapter 4, while Chapter 5 contains proofs of our results about the second homotopy module of a non-spherical Pride group. Chapter 5 also contains a study of pictures over the natural presentation of such a group.

In Chapter 6, we turn our attention to relative presentations. Our interest lies in determining when such presentations are aspherical. Relevant background material and definitions are given in this chapter and pictures over relative presentations are also studied. Five tests which are used to determine whether or not a relative presentation is aspherical are given in §6.4. Chapter 6 also contains a brief survey of known results in this area.

In Chapter 7, the final chapter of this thesis, we present our original contribution to the area of aspherical relative presentations. In particular, we determine when the relative presentation $\langle H, t\,; t^n a t^{-1} b\rangle$ is aspherical where $n \geqslant 4$ and a, b are elements of H each of order at least 3. There are four exceptional cases for which asphericity cannot be determined.

Notation

After the description of each piece of notation we list the page number in which that notation is first used. If no page number is given, then it is assumed that the notation is well-known.

Sets and number systems

X, Y	sets.
$X \cup Y$	the union of X and Y.
$X \cap Y$	the intersection of X and Y.
$X \subseteq Y$	X is a subset of Y.
$X - Y$	the difference between X and Y.
$\lvert X \rvert$	the cardinality of X.
$x \in X$	x is an element of X.
$\emptyset$	the empty set.

$\mathbb{N}$	the set of natural numbers $\{0, 1, 2, \ldots\}$.
$\mathbb{Z}$	the set of integers.
$\mathbb{Q}$	the set of rational numbers.
$\mathbb{R}$	the set of real numbers.

Functions

f, g	functions.
$f \simeq g$	the inequalities $f \preccurlyeq g$ and $g \preccurlyeq f$ are satisfied for f and g (p. 1).
$\bar{f}$	the subnegative closure of f (p. 1).

Groups

G, H	groups.
$H \leqslant G$	H is a subgroup of G.
$G * H$	the free product of G and H.
$G \times H$	the direct product of G and H.
$H^n(G, A)$	the n-th cohomology group of G with coefficients in A (p. 40).
$H_n(G, B)$	the n-th homology group of G with coefficients in B (p. 40).
$\mathbb{Z}G$	the integral group ring of G (p. 32).
IG	the augmentation ideal of G (p. 41).
M	the relation module corresponding to a presentation of G (p. 41).
$o(g)$	the order of $g \in G$ (p. 110).
$[g_1, g_2]$	the commutator of $g_1, g_2 \in G$ (p. 39).

Free groups

$\mathbf{x}$	a set (p. 2).
$\mathbf{x}^{-1}$	the set of formal inverses of $\mathbf{x}$ (p. 2).
$\mathbf{x}^{\pm 1}$	the union $\mathbf{x} \cup \mathbf{x}^{-1}$ (p. 2).
W	a word on $\mathbf{x}^{\pm 1}$ (p. 2).
$\|W\|$	the length of W (p. 2).
$W_1 \equiv W_2$	W_1 and W_2 are equal as words (p. 2).
$[W]$	the $\sim$-equivalence class containing W (p. 2).
ϵ	the empty word (p. 2).
$(\mathbf{x}^{\pm 1})^*$	the set of all words on $\mathbf{x}^{\pm 1}$ (p. 2).
$F(\mathbf{x})$	the free group on $\mathbf{x}$ (p. 2).

Presentations

$\mathbf{r}$	a set of words on $\mathbf{x}^{\pm 1}$ (p. 2).
$\mathbf{r}^s$	the symmetric closure of $\mathbf{r}$ (p. 22).
$<< \mathbf{r} >>$	the normal closure of $\mathbf{r}$ (p. 4).
$\langle \mathbf{x}\,; \mathbf{r} \rangle$	a presentation (p. 2).
$[W]_{\mathbf{r}}$	the $\sim_{\mathbf{r}}$-equivalence class containing the word W (p. 3).
$G(\mathcal{P})$	the group defined by $\mathcal{P}$ (p. 3).
$\overline{W}$	the element of $G(\mathcal{P})$ represented by the word W (p. 3).
$d_{\mathbf{x}}$	the word metric with respect to $\mathbf{x}$ (p. 3).
$\vert\vert \cdot \vert\vert_{\mathbf{x}}$	the norm corresponding to $d_{\mathbf{x}}$ (p. 3).
Δ_H^G	the length distortion function of a subgroup H in a group G (p. 4).
$\Lambda_{\mathcal{Q}}^{\mathcal{R}}$	the area distortion function of a presentation $\mathcal{Q}$ relative to a presentation $\mathcal{R}$ (p. 104).

First order Dehn function

$\mathrm{Area}_{\mathcal{P}}(W)$	the area of a word W with respect to $\mathcal{P}$ (p. 8).
$\delta_{\mathcal{P}}$	the first order Dehn function of a finite presentation $\mathcal{P}$ (p. 8).
δ_G	the first order Dehn function of a finitely presented group G (p. 9).

Diagrams

$\mathbb{E}^2$	the Euclidean plane (p. 10).
∂S	the boundary of $S \subseteq \mathbb{E}^2$ (p. 10).
$\overline{S}$	the topological closure of S (p. 10).
$-S$	the subset $\mathbb{E}^2 - S$ (p. 10).

$\mathcal{D}$	a diagram (p. 10).
$\mathcal{D}^*$	the dual of $\mathcal{D}$ (p. 14).
$\mathrm{Area}(\mathcal{D})$	the area of $\mathcal{D}$ (p. 11).
$\mathcal{H}$	a hole in $\mathcal{D}$ (p. 13).
ν	a vertex of $\mathcal{D}$ (p. 10).
$d(\nu)$	the degree of ν (p. 11).
ε	an edge of $\mathcal{D}$ (p. 10).
Δ	a region of $\mathcal{D}$ (p. 10).
$d(\Delta)$	the degree of Δ (p. 11).
$i(\Delta)$	the number of interior edges of $\mathcal{D}$ contained in $\partial\Delta$ (p. 12).
$\mathcal{S}$	a simply-connected diagram (p. 13).
$d(\mathcal{S})$	the degree of $\mathcal{S}$ (p. 13).
$\mathcal{A}$	an annular diagram (p. 13).
$d(\mathcal{A})$	the degree of $\mathcal{A}$ (p. 13).
$\beta(\mathcal{A})$	the number of regions in the boundary layer of $\mathcal{A}$ (p. 22).

$\sum_{\mathcal{D}}$	summation over the vertices or regions of $\mathcal{D}$ (p. 15).
$\sum_{\mathcal{D}}^{\bullet}$	summation over the boundary vertices or boundary regions of $\mathcal{D}$ (p. 15).
$\sum_{\mathcal{D}}^{\circ}$	summation over the interior vertices or interior regions of $\mathcal{D}$ (p. 15).
$\sum_{\mathcal{D}}^{*}$	summation over the simple boundary regions of $\mathcal{D}$ (p. 16).

Pictures

$\mathbb{P}$	a picture (p. 26).
$\partial\mathbb{P}$	the boundary of $\mathbb{P}$ (p. 26).
$-\mathbb{P}$	the mirror image of $\mathbb{P}$ (p. 31).
$\mathrm{Area}(\mathbb{P})$	the area of $\mathbb{P}$ (p. 26).
D	a disc of $\mathbb{P}$ (p. 26).
κ	a corner of a disc (p. 26).
α	an arc of $\mathbb{P}$ (p. 26).
F	a region of $\mathbb{P}$ (p. 26).
$W(\mathbb{P})$	the boundary label of a simply-connected **r**-picture $\mathbb{P}$ (p. 27).
$W(\beta)$	the label of a transverse path β in $\mathbb{P}$ (p. 28).
$\mathbb{M}^c$	the complement of a simply-connected subpicture $\mathbb{M}$ (p. 28).

Second homotopy module

$\langle\mathbb{P}\rangle$	the equivalence class containing the spherical **r**-picture $\mathbb{P}$ (p. 31).
$\pi_2(\mathcal{P})$	the second homotopy module of a finite presentation $\mathcal{P}$ (p. 31).
ξ	an element of $\pi_2(\mathcal{P})$ (p. 32).
$\mathrm{Area}(\xi)$	the area of ξ (p. 32).
X	a generating set of $\pi_2(\mathcal{P})$ (p. 33).
$V_X(\xi)$	the volume of ξ with respect to X (p. 34).
$\delta^{(2)}_{\mathcal{P},X}$	the second order Dehn function of $\mathcal{P}$ with respect to X (p. 34).
$\delta^{(2)}_{G}$	the second order Dehn function of a group G which is of type F_3 (p. 35).

Pride groups

G	a Pride group (p. 37).
$\{V, E\}$	the underlying graph of G (p. 36).
$\{u, v\}$	an edge of E with endpoints $u, v \in V$ (p. 36).
G_v	the vertex group corresponding to $v \in V$ (p. 36).
G_V	the free product $*_{v \in V} G_v$ (p. 37).
$\widetilde{G}_e$	the free product $G_u * G_v$ where $e = \{u, v\}$ (p. 36).
G_e	the edge group corresponding to $e \in E$ (p. 36).
$-G_E$	the set $G - \bigcup_{e \in E} G_e$ (p. 45).
G_Ω	the subgraph group corresponding to the full subgraph Ω (p. 37).
ψ_e	the natural epimorphism $\widetilde{G}_e \to G_e$ (p. 38).
m_e	the length of a shortest non-identity element of ψ_e (p. 38).
$\widehat{\mathbf{r}}_e$	the set of all words that represent a non-identity element of ψ_e (p. 50).
$\mathcal{P}_s$	the standard presentation of a vertex-finite Pride group (p. 48).
$\mathcal{P}$	the natural presentation of a non-spherical Pride group (p. 93).

Diagrams over vertex-finite Pride groups

$E(\mathcal{S})$	the set of regions of an $\mathbf{r}$-diagram $\mathcal{S}$ whose labels are elements of $(\mathbf{r}')^s$ (p. 49).
$E(\mathcal{A})$	the set of regions of an annular $\mathbf{r}$-diagram $\mathcal{A}$ whose labels are elements of $(\mathbf{r}')^s$ (p. 56).
$\Sigma(\Delta)$	the edge $e \in E$ such that $\phi(\partial\Delta) \in (\mathbf{r}'_e)^s$, where $\Delta \in E(\mathcal{S})$ (p. 49).
$t(\alpha)$	the set of all $v \in V$ for which some element of $\mathbf{x}_v$ appears in the label of α (p. 49).
$\mathcal{F}$	a federation of a simply-connected $\mathbf{r}$-diagram or of an annular $\mathbf{r}$-diagram (p. 49).
$\mathcal{F}_\mathcal{S}$	a federal subdivision of a simply-connected $\mathbf{r}$-diagram $\mathcal{S}$ (p. 50).
$\mathcal{F}_\mathcal{A}$	a federal subdivision of an annular $\mathbf{r}$-diagram $\mathcal{A}$ (p. 56).
$\widehat{\mathcal{S}}$	the derived diagram of a simply-connected $\mathbf{r}$-diagram $\mathcal{S}$ (p. 53).
$\widehat{\mathcal{A}}$	the derived diagram of an annular $\mathbf{r}$-diagram $\mathcal{A}$ (p. 57).

Pictures over non-spherical Pride groups

$E(\mathbb{P})$ the set of discs of an $\mathbf{r}$-picture $\mathbb{P}$ whose labels are elements of $(\mathbf{r}')^s$ (p. 56).

$\mathbb{F}$ a federation of an $\mathbf{r}$-picture (p. 94).

$\mathbb{F}_{\mathbb{P}}$ a federal subdivision of an $\mathbf{r}$-picture $\mathbb{P}$ (p. 94).

$\widehat{\mathbb{P}}$ the derived picture of an $\mathbf{r}$-picture $\mathbb{P}$ (p. 95).

Relative presentations

$\langle H, \mathbf{t}\,; \mathbf{r}\rangle$ a relative presentation (p. 107).

$\widehat{\mathcal{P}}$ the lifted presentation of $\mathcal{P}$ (p. 108).

$\widehat{G}$ the group defined by $\widehat{\mathcal{P}}$ (p. 108).

$\mathcal{P}^{st}$ the star graph of $\mathcal{P}$ (p. 113).

Pictures over relative presentations

$\mathbb{P}$ a picture over a relative presentation (p. 110).

$\widehat{\mathbb{P}}$ a lifted picture of $\mathbb{P}$ (p. 111).

$W(\kappa)$ the label of a disc associated with the corner κ (p. 110).

$\mathcal{F}$ the set of regions of $\mathbb{P}$ (p. 115).

θ an angle function on $\mathbb{P}$ (p. 114).

γ the curvature function associated with θ (p. 114).

η a distribution scheme on $\mathbb{P}$ (p. 115).

θ^* the angle function induced by η (p. 115).

γ^* the distributed curvature function associated with θ^* (p. 115).

Chapter 1

Preliminaries

We introduce the main concepts and definitions that are used throughout this thesis. Our primary references for this chapter are [17, 66, 67].

1.1 Equivalence and subnegativity of functions

Let f, g be functions from $\mathbb{N}$ to $\mathbb{N}$. We write $f \preccurlyeq g$ if there exist positive integers K_1, K_2, K_3 such that $f(n) \leqslant K_1 g(K_2 n) + K_3 n$ for all $n \in \mathbb{N}$. We say that f and g are *equivalent*, and write $f \simeq g$, if and only if $f \preccurlyeq g$ and $g \preccurlyeq f$. Note that $\simeq$ is an equivalence relation on the set of all functions from $\mathbb{N}$ to itself.

Following Brick [22, p. 378], we say that a function $f : \mathbb{N} \to \mathbb{N}$ is *subnegative* whenever

$$f(m) + f(n) \leqslant f(m+n)$$

for all $m, n \in \mathbb{N}$. For every function $f : \mathbb{N} \to \mathbb{N}$ one can define a function $\bar{f} : \mathbb{N} \to \mathbb{N}$ by the formula

$$\bar{f}(n) = \max\{f(n_1) + \ldots + f(n_r)\},$$

where the maximum is taken for all $r \geqslant 1$ and all $n_1, \ldots, n_r \in \mathbb{N} - \{0\}$ such that $n_1 + \ldots + n_r = n$. This function is the smallest subnegative function which is greater than or equal to f. The function $\bar{f}$ is said to be the *subnegative closure* of f. Note, if $f \preccurlyeq g$, then $\bar{f} \preccurlyeq \bar{g}$. Also, if $f(n) \preccurlyeq l^n$, then $\bar{f}(n) \preccurlyeq l^n$ $(n \in \mathbb{N})$.

1.2 Free groups

Let $\mathbf{x}$ be a set. The set of formal inverses of $\mathbf{x}$ will be denoted by $\mathbf{x}^{-1}$ and $\mathbf{x}^{\pm 1}$ will denote $\mathbf{x} \cup \mathbf{x}^{-1}$. A *word* W on $\mathbf{x}^{\pm 1}$ is a sequence of symbols

$$x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$$

where $n \geqslant 0$, $x_i \in \mathbf{x}$, and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$. The symbols $x_i^{\varepsilon_i}$ are the *letters* of W. A *subword* of W is any consecutive sequence of letters of W. The *length* $|W|$ of W is the integer n. If $|W| = 0$, then we say that W is the *empty word* and denote it by ϵ. If $x_i^{\varepsilon_i} \neq x_{i+1}^{-\varepsilon_{i+1}}$ for $i = 1, \ldots, n-1$, then W is *reduced*; W is *cyclically reduced* if in addition $x_1^{\varepsilon_1} \neq x_n^{-\varepsilon_n}$. The *inverse* of W is the word

$$W^{-1} = x_n^{-\varepsilon_n} \ldots x_1^{-\varepsilon_1},$$

and it is clear that $|W^{-1}| = |W|$. If $W_1 = x_{i_1}^{\varepsilon_{i_1}} \ldots x_{i_m}^{\varepsilon_{i_m}}$, $W_2 = x_{j_1}^{\varepsilon_{j_1}} \ldots x_{j_n}^{\varepsilon_{j_n}}$ are words on $\mathbf{x}^{\pm 1}$, then the product of W_1 and W_2 is the word

$$W_1 W_2 = x_{i_1}^{\varepsilon_{i_1}} \ldots x_{i_m}^{\varepsilon_{i_m}} x_{j_1}^{\varepsilon_{j_1}} \ldots x_{j_n}^{\varepsilon_{j_n}}.$$

Clearly $|W_1 W_2| = |W_1| + |W_2|$. We say that W_1, W_2 are equal as words and write $W_1 \equiv W_2$ if $m = n$ and $x_{i_k}^{\varepsilon_{i_k}} = x_{j_k}^{\varepsilon_{j_k}}$ for $k = 1, \ldots, n$.

Let $(\mathbf{x}^{\pm 1})^*$ denote the set of all words on $\mathbf{x}^{\pm 1}$. Define a relation $\sim$ on $(\mathbf{x}^{\pm 1})^*$ as follows: $W_1 \sim W_2$ if and only if W_1 can be transformed into W_2 by adding and/or deleting finitely many pairs $x_i^{\varepsilon_i} x_i^{-\varepsilon_i}$. It is clear that $\sim$ is an equivalence relation on $(\mathbf{x}^{\pm 1})^*$, and if two words W_1, W_2 are equivalent, then we say that W_1 and W_2 are *freely equal*. Let $[W]$ denote the $\sim$-equivalence class of the word W. The *free group* $F(\mathbf{x})$ *on* $\mathbf{x}$ is the set of equivalence classes $\{[W] : W \in (\mathbf{x}^{\pm 1})^*\}$ with multiplication given by $[W_1][W_2] = [W_1 W_2]$ for all $W_1, W_2 \in (\mathbf{x}^{\pm 1})^*$. The identity element of $F(\mathbf{x})$ is $[\epsilon]$ and $[W]^{-1} = [W^{-1}]$ for all $W \in (\mathbf{x}^{\pm 1})^*$. We will sometimes write $W \in F(\mathbf{x})$ or say that W is an element of $F(\mathbf{x})$ with the understanding that we mean $[W] \in F(\mathbf{x})$.

1.3 Presentations

A *presentation* $\mathcal{P}$ is given by $\langle \mathbf{x} \,; \mathbf{r} \rangle$ where $\mathbf{x}$ is a non-empty set and $\mathbf{r}$ is a (possibly empty) set of words on $\mathbf{x}^{\pm 1}$. We say that $\mathbf{x}$ is the set of *generators* of $\mathcal{P}$ and $\mathbf{r}$ is the set of *defining relators* of $\mathcal{P}$. The elements of $\mathbf{r}$ are usually assumed to be cyclically reduced. The presentation is *finite* if both $\mathbf{x}$

and $\mathbf{r}$ are finite sets. Given a presentation $\mathcal{P}$ we can construct a group $G(\mathcal{P})$. First, define a relation $\sim_{\mathbf{r}}$ as follows: $W_1 \sim_{\mathbf{r}} W_2$ if and only if W_1 can be transformed into W_2 by adding and/or deleting finitely pairs $x_i^{\varepsilon_i} x_i^{-\varepsilon_i}$ together with adding and/or deleting finitely many occurrences of elements of $\mathbf{r}^{\pm 1}$. The relation $\sim_{\mathbf{r}}$ is an equivalence relation on $(\mathbf{x}^{\pm 1})^*$ and we denote the equivalence class of W by $[W]_{\mathbf{r}}$. The group $G(\mathcal{P})$ *defined by* $\mathcal{P}$ is then the set of equivalence classes $\{[W]_{\mathbf{r}} : W \in (\mathbf{x}^{\pm 1})^*\}$ with multiplication defined by $[W_1]_{\mathbf{r}}[W_2]_{\mathbf{r}} = [W_1 W_2]_{\mathbf{r}}$ for all $W_1, W_2 \in (\mathbf{x}^{\pm 1})^*$. The identity element of $G(\mathcal{P})$ is $[\epsilon]_{\mathbf{r}}$ and $[W]_{\mathbf{r}}^{-1} = [W^{-1}]_{\mathbf{r}}$ for all $W \in (\mathbf{x}^{\pm 1})^*$.

We say $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$ is a *presentation of a group* G if $G \cong G(\mathcal{P})$. A group is *finitely generated* (respectively, *finitely related*, *finitely presented*) if it has a presentation $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$ with $\mathbf{x}$ (respectively, $\mathbf{r}, \mathbf{x} \cup \mathbf{r}$) finite.

Let $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$ be a presentation of a group G and let $\overline{W} = [W]_{\mathbf{r}}$ where W is any word on $\mathbf{x}^{\pm 1}$. We say that W *represents* $g \in G$ if $g = \overline{W}$. For $g, h \in G$, define $d_{\mathbf{x}}(g,h)$ to be the length of a *shortest* word on $\mathbf{x}^{\pm 1}$ that represents $g^{-1}h$. It is not hard to show that $d_{\mathbf{x}}(g,h) = 0$ if and only if $g = h$; $d_{\mathbf{x}}(g,h) = d_{\mathbf{x}}(h,g)$; and $d_{\mathbf{x}}(g,k) \leqslant d_{\mathbf{x}}(g,h) + d_{\mathbf{x}}(h,k)$ for all $g,h,k \in G$. It follows that $d_{\mathbf{x}}$ is a *metric* on G, so G can be viewed as a *metric space*. The metric $d_{\mathbf{x}}$ is called the *word metric* on G with respect to $\mathbf{x}$. Given the word metric $d_{\mathbf{x}}$ on G, we define a corresponding *norm* by the formula

$$||g||_{\mathbf{x}} = d_{\mathbf{x}}(1, g).$$

Recall, a non-negative function $g \to ||g||$ on a group G is a *norm* if it satisfies the following conditions: for any $g, h \in G$ one has $||g|| = ||g^{-1}||$, $||gh|| \leqslant ||g|| + ||h||$, and $||g|| = 0$ implies $g = 1$.

The word metric clearly depends on the choice of generating set $\mathbf{x}$. Thus so does the norm $||\cdot||_{\mathbf{x}}$. However, if $\mathbf{x}$ is a finite generating set and if we choose another presentation $\mathcal{Q} = \langle \mathbf{y}\,;\mathbf{s}\rangle$ of G where $\mathbf{y}$ is also finite, then the metric spaces $(G, d_{\mathbf{x}})$ and $(G, d_{\mathbf{y}})$ can be shown to be *quasi-isometric*.

Definition 1.3.1. ([33, p. 85]) A map $\phi : X \to X'$ between two metric spaces (X,d) and (X', d') is a *quasi-isometric embedding* if there are constants $\epsilon \geqslant 0$ and $\lambda > 0$ such that for any two elements $x, y \in X$,

$$\frac{1}{\lambda} d(x,y) - \epsilon \leqslant d'(\phi(x), \phi(y)) \leqslant \lambda d(x,y) + \epsilon.$$

It is a *quasi-isometry* if there exists a constant $c \geqslant 0$ such that for all $x' \in X'$, there is an element $x \in X$ such that $d'(\phi(x), x') \leqslant c$. The metric spaces X and X' are then *quasi-isometric*.

Equivalently, X and X' are quasi-isometric if there exist two mappings $\phi : X \to X'$, $\psi : X' \to X$ and constants $\epsilon \geqslant 0$, $\lambda > 0$, $c \geqslant 0$ such that

$$d'(\phi(x), \phi(y)) \leqslant \lambda d(x, y) + \epsilon \quad d(\psi(x'), \psi(y')) \leqslant \lambda d'(x', y') + \epsilon$$

$$d(\psi\phi(x), x) \leqslant c \quad d(\phi\psi(x'), x') \leqslant c$$

for all $x, y \in X$ and $x', y' \in X'$.

Let G be a finitely generated group with finite generating set $\mathbf{x}$ and let H be a finitely generated subgroup of G. We can assume that a generating set $\mathbf{y}$ of H is contained in $\mathbf{x}$. Then for $h \in H$, $||h||_{\mathbf{y}} \geqslant ||h||_{\mathbf{x}}$ where $||\cdot||_{\mathbf{x}}, ||\cdot||_{\mathbf{y}}$ are the norms associated with the word metrics $d_{\mathbf{x}}$ and $d_{\mathbf{y}}$, respectively. The *length distortion function* of H in G is the function $\Delta_H^G : \mathbb{N} \to \mathbb{N}$ given by

$$\Delta_H^G(n) = \max\{||h||_{\mathbf{y}} : h \in H \text{ and } ||h||_{\mathbf{x}} \leqslant n\}.$$

It is not hard to show that, up to $\simeq$-equivalence, the length distortion function is independent of the choice of word metrics $d_{\mathbf{x}}$ and $d_{\mathbf{y}}$.

If $\Delta_H^G(n) \preccurlyeq n$, then H is said to be *undistorted in* G, or simply *undistorted*. Subgroups of finitely generated free groups are undistorted. Any subgroup of a finitely generated nilpotent group has quadratic distortion [78, Theorem 2.2] (for example, consider the (cyclic) centre of the 3-dimensional Heisenberg group given by the presentation $\langle a, b, c\,; [a, b] = c, [a, c], [b, c]\rangle$). There exist 2-dimensional CAT(-1) groups which contain free subgroups with arbitrary iterated exponential distortion and with distortion higher than any iterated exponential (see [11]). We refer to [41,71,77] and [52, §3] for more on length distortion functions.

Let $\mathcal{P} = \langle \mathbf{x}\,; \mathbf{r}\rangle$ be a presentation. Each element $R \in \mathbf{r}$ gives rise to an element $[R] \in F(\mathbf{x})$, where $F(\mathbf{x})$ is the free group on $\mathbf{x}$. The *normal closure* $<< \mathbf{r} >>$ of $\mathbf{r}$ in $F(\mathbf{x})$ is the smallest normal subgroup of $F(\mathbf{x})$ that contains the elements $\{[R] : R \in \mathbf{r}\}$. It is not hard to see that $<< \mathbf{r} >>$ consists of all elements of the form

$$\prod_{i=1}^{k} [W_i R_i^{\varepsilon_i} W_i^{-1}], \tag{1.1}$$

where $W_i \in (\mathbf{x}^{\pm 1})^*$, $R_i \in \mathbf{r}$, and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, k$.

Let W be a reduced word on $\mathbf{x}^{\pm 1}$ which is of the form (1.1). If V is obtained from W by adding or deleting an element of $\mathbf{r}^{\pm 1}$, then it is clear that $[V] \in << \mathbf{r} >>$. Therefore, if $V \sim_{\mathbf{r}} U$ where U

is any word on $\mathbf{x}^{\pm 1}$ such that $[U] \in << \mathbf{r} >>$, then $[V] \in << \mathbf{r} >>$. We use this simple observation in the proof of the following lemma.

Lemma 1.3.1. *If $\mathcal{P} = \langle \mathbf{x} ; \mathbf{r} \rangle$ is a presentation, then $G(\mathcal{P})$ is isomorphic to the quotient group $F(\mathbf{x})/ << \mathbf{r} >>$.*

Proof. Define a function $\theta : F(\mathbf{x}) \to G(\mathcal{P})$ by $[W] \mapsto [W]_{\mathbf{r}}$. Then θ is a homomorphism whose image is equal to $G(\mathcal{P})$ and whose kernel contains the elements $[R]$ for all $R \in \mathbf{r}$. Since $<< \mathbf{r} >>$ is the smallest normal subgroup of $F(\mathbf{x})$ containing all such elements, we deduce that $<< \mathbf{r} >> \subseteq \ker \theta$. If $[W] \in \ker \theta$, then $[W]_{\mathbf{r}} = [\epsilon]_{\mathbf{r}}$ and so $W \sim_{\mathbf{r}} \epsilon$. Since $[\epsilon] \in << \mathbf{r} >>$, we deduce that that $[W] \in << \mathbf{r} >>$. Therefore, $\ker \theta \subseteq << \mathbf{r} >>$ and so $\ker \theta = << \mathbf{r} >>$. The result then follows from the first isomorphism theorem. □

Let $\mathcal{P} = \langle \mathbf{x} ; \mathbf{r} \rangle$ be a finite presentation of a group G. In general, G will have infinitely many different presentations; however, each presentation can be obtained from $\mathcal{P}$ by a series of *Tietze transformations* [98]. The following are the Tietze transformations one can apply to $\mathcal{P}$, as given in [66].

(T1) Add a word $W \in (\mathbf{x}^{\pm 1})^*$ to $\mathbf{r}$ if $[W] \in << \mathbf{r} >>$.

(T2) The inverse of a (T1) transformation.

(T3) Add a symbol y to the set of generators $\mathbf{x}$ and add a new relator yW^{-1} to $\mathbf{r}$, where W is some word on $\mathbf{x}^{\pm 1}$.

(T4) The inverse of a (T3) transformation.

Theorem 1.3.1. ([66, Proposition II.2.1]) *Two finite presentations define isomorphic groups if and only if it is possible to pass from one to the other by a finite sequence of Tietze transformations.*

We end this section with a definition. Let $\mathcal{P} = \langle \mathbf{x} ; \mathbf{r} \rangle$ be a presentation.

Definition 1.3.2. A word W on $\mathbf{x}^{\pm 1}$ is *injective relative to* $\mathcal{P}$ if no proper subword of W represents the identity element of $G(\mathcal{P})$. We say that W is *cyclically injective relative to* $\mathcal{P}$ if every cyclic permutation of W is injective relative to $\mathcal{P}$.

If there is no confusion over which presentation we are working with, then we will say that W is *injective* or *cyclically injective*. Note that a cyclically injective word of length strictly greater than 2 is necessarily cyclically reduced.

1.4 Dehn's fundamental algorithmic problems

In 1912, Max Dehn formulated three algorithmic problems for groups that lie at the heart of combinatorial group theory. They are: the *Word Problem*, the *Conjugacy Problem*, and the *Isomorphism Problem*. Let us first state the word and conjugacy problems for a finite presentation $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$.

Word Problem. Is there an algorithm that will decide, given any word $W \in (\mathbf{x}^{\pm 1})^*$, whether or not $W \sim_{\mathbf{r}} \epsilon$?

Conjugacy Problem. Is there an algorithm that will decide, given any two words $W_1, W_2 \in (\mathbf{x}^{\pm 1})^*$, whether or not there exists a word $U \in (\mathbf{x}^{\pm 1})^*$ such that $W_1 \sim_{\mathbf{r}} UW_2U^{-1}$?

Theorem 1.4.1. ([66, Proposition II.2.2] & [67, §1.5, Problem 11]) *Let $\mathcal{P}_1, \mathcal{P}_2$ be two finite presentations of a finitely presented group G. If the word problem (respectively, conjugacy problem) is soluble for $\mathcal{P}_1$, then the word problem (respectively, conjugacy problem) is soluble for $\mathcal{P}_2$.*

Remark 1.4.1. To prove Theorem 1.4.1 it is enough, by Theorem 1.3.1, to show that if we apply a single Tietze transformation to $\mathcal{P}_1$, the resulting presentation has a soluble word problem (respectively, soluble conjugacy problem).

Thus, having soluble word or conjugacy problem is a *group property*, in the sense that it is independent of the choice of finite presentation. If we can solve the word (respectively, conjugacy) problem for one finite presentation of a group G, then we can solve it for all finite presentations of G.

The word problem has been solved for many different classes of groups. For example: free groups [67, Corollary 1.2.2]; one-relator groups (i.e. groups that have a presentation in which there is exactly one defining relator) [67, Theorem 4.14]; small cancellation groups [50, 66, 97]; residually finite groups [66, Theorem IV.4.6]; automatic groups [39, Theorem 2.3.10]; and word hyperbolic groups [52]. Word hyperbolic groups can be characterized as those that have presentations which have a *Dehn Algorithm*. If a finite presentation has such an algorithm, then an extremely efficient solution exists for its word problem (see [66, Chapter IV] for details). The word problem is by no means soluble for all groups. A fundamental result of Novikov [74] and, independently, Boone [18] (see also [90, Chapter 12]) exhibits a finite presentation with unsolvable word problem. An elementary construction of finitely presented groups with unsolvable word problems is given in [68]. We note that it is unknown whether or not the word problem for an arbitrary two-relator group (i.e. a group that has a presentation in which there are exactly two defining relators) is soluble.

It is clear that a solution of the conjugacy problem contains a solution of the word problem (select W_2 to be the empty word). Thus, the conjugacy problem may be viewed as a more difficult problem than the word problem. Indeed, there exist finitely presented groups for which the word problem is soluble but whose conjugacy problem is unsolvable [29,46,70,73]. The conjugacy problem is soluble for free groups [67, Theorem 1.3], small cancellation groups [51,66], and strongly relatively hyperbolic groups (in the sense of Farb), provided that the parabolic subgroup has soluble conjugacy problem [26, Theorem 1.1]. The conjugacy problem is also soluble for one-relator groups with torsion [72] and Pride [86] has recently shown that it is soluble for a special class of one-relator groups. However, it is unknown whether or not the conjugacy problem is soluble for an arbitrary *torsion-free* one-relator group [13, Problem O5].

We are primarily interested in the word and conjugacy problems for finitely presented groups. However, there is another problem, the *generalized word problem* or *membership problem*, that we will also consider.

Generalized word problem (relative to $\mathbf{u}$). Given a finite presentation $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$ and a finite subset $\mathbf{u}$ of $(\mathbf{x}^{\pm 1})^*$, is there an algorithm that will decide, given any word $W \in (\mathbf{x}^{\pm 1})^*$, whether or not there exists a word Z on $\mathbf{u}^{\pm 1}$ such that $[W]_{\mathbf{r}} = [Z]_{\mathbf{r}}$?

The presentation is said to have a soluble *generalized word problem* if the generalized word problem is soluble for every finite subset $\mathbf{u}$ of $(\mathbf{x}^{\pm 1})^*$. Unsurprisingly, a solution of the generalized word problem contains a solution of the word problem; take $\mathbf{u}$ to be empty. We will only be interested in the generalized word problem for a specific choice of $\mathbf{u}$, so we will not consider this problem under the effect of changing presentation. We note that Farb [41] has studied the generalized word problem for finitely presented groups.

The isomorphism problem is the hardest of Dehn's fundamental problems.

Isomorphism Problem. Given a recursively enumerable set X of finite presentations $\mathcal{P}_i = \langle \mathbf{x}_i\,;\mathbf{r}_i\rangle$ ($i \in \mathbb{N}$), is there an algorithm that will decide, given two arbitrary presentations $\mathcal{P}_i, \mathcal{P}_j$ from X, whether or not $G(\mathcal{P}_i) \cong G(\mathcal{P}_j)$?

Adyan [1] and Rabin [89] proved that there is no such algorithm for an arbitrary set of recursively enumerable finite presentations. (In fact, they proved that there is no algorithm that will decide, given a finite presentation, whether or not the group it defines is trivial. This despite the fact that there is a procedure to enumerate all presentations of the trivial group (via Tietze transformations).) One may then ask whether there is a solution within a particular class $\mathcal{C}$ of groups. That is, is there an algorithm which, given any two finite presentations from a set X of recursively enumerable finite presentations and the knowledge that each presentation from X defines a group in $\mathcal{C}$, decides whether or not the presentations define isomorphic groups? It is usually assumed that a positive solution to this question should not require that the presentations are given along with a proof that the groups which they define lie in $\mathcal{C}$, merely the knowledge that they do should suffice.

The isomorphism problem has been solved for some classes of groups. Free groups have soluble isomorphism problem [67], as do polycyclic-by-finite groups [93]. Recently, Dahmani and Groves [32] have solved the isomorphism problem for torsion-free relatively hyperbolic groups with abelian parabolics. As a special case of this, they recover solutions of the isomorphism problem for torsion-free hyperbolic groups [94] and fully residually free groups [27]. Also, Kapovich and Schupp [59] have shown that the isomorphism problem is soluble for an exponentially generic class of one-relator groups.

1.5 The first order Dehn function

Let $\mathcal{P} = \langle \mathbf{x} \,;\mathbf{r}\rangle$ be a finite presentation of a finitely presented group G and let W be a word on $\mathbf{x}^{\pm 1}$. If W represents the identity element of G, then $[W] \in << \mathbf{r} >>$ and so is equal to an element of the form (1.1). The smallest natural number k among all expressions of the form (1.1) is the *area* of W, which we denote by $\mathrm{Area}_{\mathcal{P}}(W)$. We will write $\mathrm{Area}(W)$ for the area of W if there is no confusion over which presentation we are working with.

Definition 1.5.1. The *first order Dehn function* of a finite presentation $\mathcal{P}$ is the function $\delta_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ given by $\delta_{\mathcal{P}}(n) = \max\{\mathrm{Area}_{\mathcal{P}}(W) : W \in << \mathbf{r} >> \text{ and } |W| \leqslant n\}$.

Different presentations can have different Dehn functions even if they define the same group. Consider, for example, the following presentations of the infinite cyclic group: $\mathcal{P}_1 = \langle x \,; \,\rangle$ and $\mathcal{P}_2 = \langle x, y \,; y\rangle$. It is easy to see that $\delta_{\mathcal{P}_1}(n) = 0$ for all $n \in \mathbb{N}$, whereas $\delta_{\mathcal{P}_2}(n) = n$ for all $n \in \mathbb{N}$. Now $\mathcal{P}_2$ can obtained from $\mathcal{P}_1$ by applying the Tietze transformation (T3), so we see that passage from one presentation to another via Tietze transformations has an effect on the Dehn function.

Proposition 1.5.1. ([22, Proposition 1.1]) *If $\mathcal{P}_1, \mathcal{P}_2$ are two finite presentations of the same finitely presented group, then $\delta_{\mathcal{P}_1} \simeq \delta_{\mathcal{P}_2}$.*

Since we shall not distinguish between $\simeq$-equivalent functions, we may speak of *the* first order Dehn function δ_G of a finitely presented group G. Work carried out in [21] and [91] has lead to a fairly complete understanding of which functions arise as Dehn functions. In particular, it is known that the *isoperimetric spectrum of first order Dehn functions*

$$\mathrm{IP} = \{\alpha \in [1, \infty) : f(x) = x^\alpha \text{ is a Dehn function}\}$$

is dense in the range $[4, \infty)$. A celebrated result of Gromov (see [19, 76] for concise proofs) showed that $\mathrm{IP} \cap (1, 2)$ is empty, and it was later shown in [21] that this is the only gap in IP. Thus, $\mathbb{Q} \cap (2, \infty) \subset \mathrm{IP}$. We note that, previous to this result, the snowflake construction of [20] was used to provide a dense set of exponents in $\mathrm{IP} \cap [2, \infty)$.

An *isoperimetric function* for a finitely presented group G is any function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $\delta_G(n) \preccurlyeq f(n)$ for all $n \in \mathbb{N}$. Thus, the Dehn function δ_G is the smallest isoperimetric function for G. The following theorem provides a link between the word problem and the Dehn function of a finitely presented group.

Theorem 1.5.1. ([48, Theorem 2.1]) *A finitely presented group has a recursive isoperimetric function (in which case, the Dehn function itself is recursive) if and only if it has a soluble word problem.*

The following theorem provides a second characterization of word hyperbolic groups. A comprehensive proof of this result can be found in [4].

Theorem 1.5.2. ([48, Theorem 3.1]) *A finitely presented group G has a linear isoperimetric function if and only if G is word hyperbolic.*

A consequence of Theorem 1.5.2 is the fact that the finitely presented group $\mathbb{Z} \times \mathbb{Z}$ is not word hyperbolic; its Dehn function is quadratic [12, pp. 551-552]. Automatic groups [39, Theorem 2.3.12] and, more generally, combable groups [48, Theorem 3.3] & [95, Theorem 2.3.4] provide further examples of groups which admit a quadratic isoperimetric function.

We end this section by proving a simple result regarding the Dehn function of a free product of groups.

Proposition 1.5.2. *Let $G = G_1 * \ldots * G_n$ ($n \geqslant 2$) be a free product of groups, where each G_i is finitely presented. Then $\delta_G \simeq \max\{\bar{\delta}_{G_i} : i = 1, \ldots, n\}$.*

Proof. We proceed by induction on n. For $n = 2$, the result holds by a theorem of Guba and Sapir [54]. For $n > 2$ write $G = G' * G_n$, where $G' = G_1 * \ldots * G_{n-1}$. Then $\delta_G \simeq \max(\bar{\delta}_{G'}, \bar{\delta}_{G_n})$ and by induction, we have

$$\delta_{G'} \simeq \max\{\bar{\delta}_{G_j} : j = 1, \ldots, n-1\}.$$

Since $\bar{\delta}_{G'}$ is the smallest subnegative function which is greater than or equal to $\delta_{G'}$, it follows that $\bar{\delta}_{G'} \simeq \max\{\bar{\delta}_{G_j} : j = 1, \ldots, n-1\}$. Thus, $\delta_G \simeq \max\{\bar{\delta}_{G_i} : i = 1, \ldots, n\}$. □

1.6 Diagrams

The following treatment of diagrams is mostly taken from [66, Chapter V]. Let $\mathbb{E}^2$ denote the Euclidean plane. If $S \subseteq \mathbb{E}^2$, then ∂S will denote the *boundary* of S. The topological closure of S will be denoted by $\overline{S}$, and $-S$ will denote $\mathbb{E}^2 - S$. A *vertex* is a point of $\mathbb{E}^2$ and an *edge* is a bounded subset of $\mathbb{E}^2$ that is homeomorphic to the open unit interval. A region is a bounded set homeomorphic to the open unit disc.

A *diagram* $\mathcal{D}$ is a finite collection of vertices, edges and regions that are pairwise disjoint and satisfy the following two conditions:

(i) If ε is an edge of $\mathcal{D}$, then there are vertices ν_1 and ν_2 (not necessarily distinct) in $\mathcal{D}$ such that $\overline{\varepsilon} = \varepsilon \cup \{\nu_1\} \cup \{\nu_2\}$.

(ii) The boundary $\partial\Delta$ of each region Δ of $\mathcal{D}$ is connected and there is a set of edges $\varepsilon_1, \ldots, \varepsilon_n$ in $\mathcal{D}$ such that $\partial\Delta = \overline{\varepsilon}_1 \cup \ldots \cup \overline{\varepsilon}_n$.

If ε is an edge with $\overline{\varepsilon} = \varepsilon \cup \{\nu_1\} \cup \{\nu_2\}$, then the vertices ν_1 and ν_2 are called the *endpoints* of ε. Each edge of $\mathcal{D}$ may be given an orientation to obtain an *oriented diagram.* Let f be an arbitrary homeomorphism of the interval $(0, 1)$ to itself. Since f is monotone and bijective, there are two possibilities: either (1) $\lim f(x) = 0$ as $x \to 0$ and $\lim f(x) = 1$ as $x \to 1$, or (2) $\lim f(x) = 1$ as $x \to 0$ and $\lim f(x) = 0$ as $x \to 1$. It follows that f extends to a homeomorphism $[0, 1] \to [0, 1]$. Following Ol'shanskii [75, p. 101], in case (1) we say that f *preserves* the orientation of $(0, 1)$ (and of $[0, 1]$), and in case (2) that f *reverses* the orientation.

Now let ε be an edge of a diagram $\mathcal{D}$. We say that homeomorphisms $f_1 : (0,1) \to \varepsilon$ and $f_2 : (0,1) \to \varepsilon$ give ε the same orientation if $f_1^{-1} f_2$ preserves the orientation of $(0,1)$, and they give ε opposite orientations if $f_1^{-1} f_2$ reverses the orientation of $(0,1)$. We can then endow the set of all homeomorphisms $f : (0,1) \to \varepsilon$ with an equivalence relation which splits it into two classes. Assigning an orientation to ε amounts to choosing one of these two classes of homeomorphisms. It is natural to indicate the choice of orientation of an edge by a small arrow on that edge. We refer to [75, Chapter 3, §10.3] for more on orientable surfaces.

If ε is an oriented edge running from endpoint ν_1 to endpoint ν_2, then ν_1 is the *initial* vertex of ε and ν_2 is the *terminal* vertex of ε. The oppositely oriented edge, or *inverse* of ε, is denoted by $\varepsilon^{-1}$and runs from ν_2 to ν_1.

A *path* is a sequence of oriented edges $\varepsilon_1, \ldots, \varepsilon_n$ such that the initial vertex of ε_{i+1} is the terminal vertex of ε_i for $1 \leqslant i \leqslant n-1$. We also allow the empty path. The endpoints of a path are the initial and terminal vertices of ε_1 and ε_n, respectively. A *closed path* or *cycle* is a path such that the initial vertex of ε_1 is the terminal vertex of ε_n. A path is *reduced* if it does not contain a successive pair of edges $\varepsilon^{\delta}\varepsilon^{-\delta}$ $(\delta = \pm 1)$, and it is *simple* if, for $j \neq i$, ε_i and ε_j have different initial vertices.

Since $\mathcal{D}$ is planar, it is possible to orient the regions of $\mathcal{D}$ and the components of $-\mathcal{D}$ so that in traversing the boundaries of regions of $\mathcal{D}$ and the components of $-\mathcal{D}$, each edge of $\mathcal{D}$ is traversed twice, once in each of its possible orientations. If Δ is a region of $\mathcal{D}$ with a given orientation, then any cycle of minimal length which includes all the edges of $\partial\Delta$ and in which the edges are oriented in accordance with the orientation of Δ is a *boundary cycle* of Δ. The *degree* $d(\Delta)$ of Δ is the number of edges in a boundary cycle of Δ, with an edge counted twice if it appears twice in the cycle.

The *degree* $d(\nu)$ of a vertex ν of $\mathcal{D}$ is the number of unoriented edges incident with ν. Thus, if an unoriented edge has both endpoints at ν we count it twice. The *area* of $\mathcal{D}$ is the number of regions contained in $\mathcal{D}$ and we denote it by $\text{Area}(\mathcal{D})$.

A vertex is a *boundary vertex* of $\mathcal{D}$ if it is contained in $\partial\mathcal{D}$. Similarly, an edge is a *boundary edge* of $\mathcal{D}$ if it is contained in $\partial\mathcal{D}$. We say that a region Δ is a *boundary region* of $\mathcal{D}$ if $\partial\Delta \cap \partial\mathcal{D} \neq \emptyset$. Thus, if Δ is a boundary region of $\mathcal{D}$, $\partial\Delta \cap \partial\mathcal{D}$ need not contain an edge, but may consist only of one or more vertices. A vertex, edge or region of $\mathcal{D}$ that is not a boundary vertex, edge or region is called *interior*. In Fig. 1.1 below ν_1 (respectively, ε_2) is a boundary vertex (respectively, edge), whereas ν_2 (respectively, ε_1) is an interior vertex (respectively, edge). Regions $\Delta_1, \Delta_2, \Delta_3$ are boundary regions, whereas Δ_4, Δ_5 are interior regions.

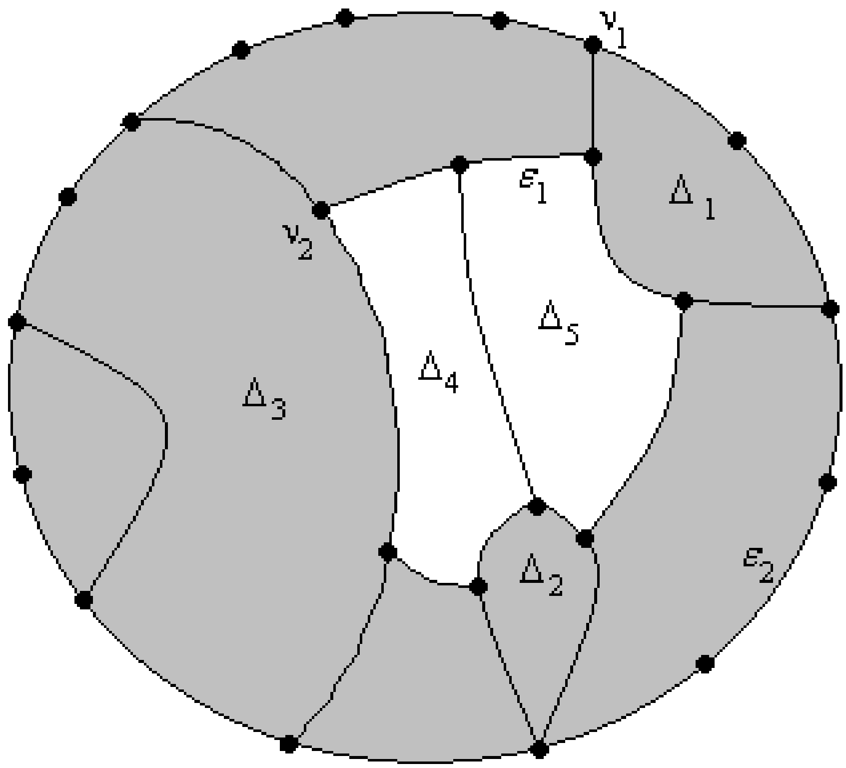


Figure 1.1: A diagram displaying boundary vertices, edges and regions.

If Δ is a region of $\mathcal{D}$, the number of interior edges of $\mathcal{D}$ that are contained in $\partial\Delta$ is denoted by $i(\Delta)$, again with an edge counted twice if it appears twice in a boundary cycle of Δ. In Fig. 1.1, $i(\Delta_1) = 3$, $i(\Delta_2) = 4$ and $i(\Delta_3) = 4$. Note that $i(\Delta) = d(\Delta)$ if and only if $\partial\Delta \cap \partial\mathcal{D}$ does not contain an edge.

Definition 1.6.1. The *boundary layer* of a diagram $\mathcal{D}$ consists of all boundary vertices of $\mathcal{D}$, all edges of $\mathcal{D}$ incident with boundary vertices, and all boundary regions of $\mathcal{D}$.

The shaded regions in Fig. 1.1 are the regions contained in the boundary layer of $\mathcal{D}$.

Definition 1.6.2. Let $\mathcal{D}$ be a diagram. If each interior vertex of $\mathcal{D}$ has degree at least p and *every* region of $\mathcal{D}$ has degree at least q (where p and q are positive integers), then $\mathcal{D}$ is a $[p, q]$-*diagram.*

If each interior vertex of $\mathcal{D}$ has degree at least p and each *interior* region of $\mathcal{D}$ has degree at least q, then $\mathcal{D}$ is a (p, q)-*diagram*.

A *subdiagram* $\mathcal{B}$ of a diagram $\mathcal{D}$ is a sub-collection of the vertices, edges and regions of $\mathcal{D}$ such that if ε is an edge of $\mathcal{B}$, then the endpoints of ε are in $\mathcal{B}$, and if Δ is a region of $\mathcal{B}$, then $\partial\Delta$ is in $\mathcal{B}$. An orientation of $\mathcal{D}$ induces an orientation of $\mathcal{B}$. Boundary vertices (respectively, edges, regions), and interior vertices (respectively, edges, regions) of $\mathcal{B}$ are defined in the obvious way. The degree of a vertex ν of $\mathcal{B}$ is denoted $d_{\mathcal{B}}(\nu)$. If Δ is a region of $\mathcal{B}$, then $i_{\mathcal{B}}(\Delta)$ denotes the number of interior edges of $\mathcal{B}$ that are contained in $\partial\Delta$, and $d_{\mathcal{B}}(\Delta)$ denotes the degree of Δ.

A *hole* $\mathcal{H}$ in a diagram $\mathcal{D}$ is a bounded component of $-\mathcal{D}$. The *components* of $\mathcal{D}$ are the connected components of

$$\bigcup_i \nu_i \cup \bigcup_j \varepsilon_j \cup \bigcup_k \Delta_k,$$

and we say that $\mathcal{D}$ is *connected* if it has at most one component.

Definition 1.6.3. A diagram is *simply-connected* if it is connected and if it is does not contain a hole.

Let $\mathcal{S}$ be a simply-connected diagram with boundary $\partial\mathcal{S}$. A *boundary cycle* of $\mathcal{S}$ is a closed path of minimal length that contains all the edges of $\partial\mathcal{S}$. The *degree* $d(\mathcal{S})$ of $\mathcal{S}$ is the number of edges contained in a boundary cycle of $\mathcal{S}$, counted with multiplicity. Let Δ be a boundary region of $\mathcal{S}$. We say that $\partial\Delta \cap \partial\mathcal{S}$ is a *consecutive part* of $\partial\mathcal{S}$ if $\partial\Delta \cap \partial\mathcal{S}$ is the union of a sequence of edges $\varepsilon_1, \ldots, \varepsilon_n$ $(n \geqslant 1)$ that occur consecutively in a boundary cycle of Δ, and in some boundary cycle of $\mathcal{S}$. We say Δ is a *simple* boundary region if $\partial\Delta \cap \partial\mathcal{S}$ is a consecutive part of $\partial\mathcal{S}$, and Δ is a *non-simple* boundary region otherwise. In Fig. 1.1, Δ_1 is a simple boundary region, whereas Δ_2, Δ_3 are non-simple boundary regions.

Definition 1.6.4. An *annular diagram* $\mathcal{A}$ is a connected diagram such that $-\mathcal{A}$ contains exactly two components. Equivalently, $\mathcal{A}$ is connected and contains exactly one hole.

Let $\mathcal{A}$ be an annular diagram. Let $\mathcal{K}$ be the unbounded component of $-\mathcal{A}$ and let $\mathcal{H}$ be the bounded component of $-\mathcal{A}$. We call $\sigma = \partial\mathcal{A} \cap \partial\mathcal{K}$ the *outer boundary* of $\mathcal{A}$ and $\tau = \partial\mathcal{A} \cap \partial\mathcal{H}$ the *inner boundary* of $\mathcal{A}$. A cycle of minimal length which contains all the edges in the outer (respectively, inner) boundary of $\mathcal{A}$ is an *outer* (respectively, *inner*) *boundary cycle* of $\mathcal{A}$. The *degree* $d(\mathcal{A})$ of $\mathcal{A}$ is the number of edges contained in an outer boundary cycle of $\mathcal{A}$ plus the number of edges contained in an inner boundary cycle of $\mathcal{A}$ (counted with appropriate multiplicities).

Let Δ be a region of $\mathcal{A}$. Then Δ is an *outer* boundary region if $\partial\Delta\cap\sigma \neq \emptyset$ and $\partial\Delta\cap\tau = \emptyset$. It is an *inner* boundary region if $\partial\Delta\cap\sigma = \emptyset$ and $\partial\Delta\cap\tau \neq \emptyset$. We say that Δ is a *simple* outer boundary region if it is an outer boundary region and if $\partial\Delta \cap \sigma$ is a consecutive part of σ. We define *simple* inner boundary regions in an analogous way. We say that Δ is an *almost* simple boundary region of $\mathcal{A}$ if $\partial\Delta \cap \sigma$ and $\partial\Delta \cap \tau$ are consecutive parts of σ and τ, respectively (see Fig. 1.2). Finally, we say that Δ is a *non-simple* boundary region if at least one of $\partial\Delta \cap \sigma$ or $\partial\Delta \cap \tau$ is non-empty and is not a consecutive part of σ or τ, respectively.

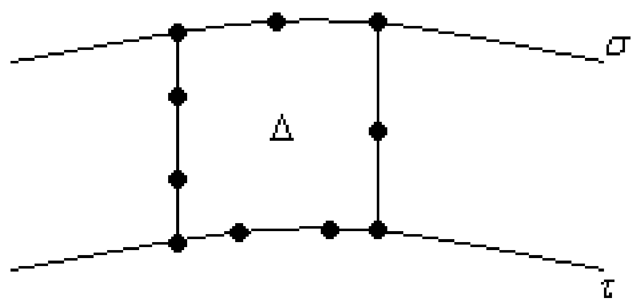


Figure 1.2: An almost simple boundary region.

Definition 1.6.5. Let $\mathcal{A}$ be an annular diagram with outer boundary σ and inner boundary τ. An edge ε of $\mathcal{A}$ is a *bridge* if $\varepsilon \subseteq \sigma$ and $\varepsilon \subseteq \tau$. A vertex ν is a *pinch* if $\nu \subseteq \sigma$, $\nu \subseteq \tau$ and ν is not the endpoint of a bridge.

1.6.1 Dual diagrams

Let $\mathcal{D}$ be an unoriented diagram. We construct the *dual diagram* $\mathcal{D}^*$ as follows: pick a point ν_i^* in each region Δ_i of $\mathcal{D}$. The collection of the ν_i^*'s are the vertices of $\mathcal{D}^*$. If Δ_1 and Δ_2 are distinct regions of $\mathcal{D}$ having an edge ε in common, then an edge ε^* is drawn from ν_1^* to ν_2^* crossing ε but no other edges of $\mathcal{D}$ or edges of $\mathcal{D}^*$ already constructed. Since $\varepsilon \subseteq \partial\Delta_1 \cap \partial\Delta_2$, ε is an interior edge of $\mathcal{D}$. If a region Δ_i of $\mathcal{D}$ contains an edge ε in its boundary such that Δ_i lies on both sides of ε, then a loop is draw at ν_i^* crossing ε but no other edges. The edges and vertices of $\mathcal{D}^*$ form a graph Γ^*. The regions of $\mathcal{D}^*$ are the regions bounded by Γ^* that contain an interior vertex of $\mathcal{D}$. It is easy to see that $\mathcal{D}^*$ has the following properties:

(1) The vertices of $\mathcal{D}^*$ are in one-to-one correspondence with the regions of $\mathcal{D}$. If ν^* corresponds to Δ, then $d(\nu^*) = i(\Delta)$.

(2) The edges of $\mathcal{D}^*$ are in one-to-one correspondence with the *interior* edges of $\mathcal{D}$.

(3) The regions of $\mathcal{D}^*$ are in one-to-one correspondence with the *interior* vertices of $\mathcal{D}$. If ν is an interior vertex of $\mathcal{D}$, then there are $d(\nu)$ edges at ν. Each of these edges is crossed by an edge ε^* of $\mathcal{D}^*$, and the collection of these edges in $\mathcal{D}^*$ form a region Δ^* of $\mathcal{D}^*$ with $d(\Delta^*) = d(\nu)$.

(4) The boundary vertices of $\mathcal{D}^*$ are in one-to-one correspondence with the boundary regions of $\mathcal{D}$.

(5) If $\mathcal{D}$ has h holes, then $\mathcal{D}^*$ has at most h holes.

(6) If $\mathcal{D}$ is a (p, q)-diagram, then $\mathcal{D}^*$ is a $[q, p]$-diagram.

1.6.2 The basic formulas

Throughout this section p and q will denote positive integers such that $1/p + 1/q = 1/2$.

Let $\mathcal{D}$ be a diagram. Summation signs $\sum_{\mathcal{D}}$ will denote summations over vertices or regions of $\mathcal{D}$. The notation $\sum_{\mathcal{D}}^{\bullet}$ denotes summation restricted to boundary vertices or boundary regions while $\sum_{\mathcal{D}}^{\circ}$ denotes summation over interior vertices or interior regions. The number of vertices of $\mathcal{D}$ will be denoted by V. The number of unoriented edges of $\mathcal{D}$ will be denoted by E and F will denote the area of $\mathcal{D}$. Let $V^{\bullet}$, $E^{\bullet}$ and $F^{\bullet}$ denote the number of boundary vertices, edges and regions of $\mathcal{D}$, respectively. Note that $E^{\bullet}$ is counted *with multiplicity*. Let Q be the number of components of $\mathcal{D}$ and h be the number of holes in $\mathcal{D}$.

Lemma 1.6.1. ([66, Lemma V.3.2]) *If $\mathcal{D}$ is a diagram with no isolated vertices, then $V^{\bullet} \leqslant E^{\bullet}$.*

Theorem 1.6.1. ([66, Theorem V.3.1]) *Let $\mathcal{D}$ be an arbitrary diagram. Then:*

$$p(Q-h) = \sum_{\mathcal{D}}^{\bullet}[p - d(\nu)] + \sum_{\mathcal{D}}^{\circ}[p - d(\nu)] + \frac{p}{q}\sum_{\mathcal{D}}[q - d(\Delta)] - \frac{p}{q}E^{\bullet}; \tag{1.2}$$

$$p(Q-h) = \sum_{\mathcal{D}}^{\bullet}[\frac{p}{q} + 2 - d(\nu)] + \sum_{\mathcal{D}}^{\circ}[p - d(\nu)] + \frac{p}{q}\sum_{\mathcal{D}}[q - d(\Delta)] + \frac{p}{q}(V^{\bullet} - E^{\bullet}). \tag{1.3}$$

Corollary 1.6.1. ([66, Corollary V.3.4]) *Let $\mathcal{S}$ be a simply-connected (q,p)-diagram which contains more than one region. Then*

$$\sum_{\mathcal{S}}^{\bullet}[\frac{p}{q}+2-i(\Delta)] \geqslant p.$$

Definition 1.6.6. An *extremal disc* of a diagram $\mathcal{D}$ is a simply-connected subdiagram $\mathcal{K}$ that has a boundary cycle $\varepsilon_1 \ldots \varepsilon_n$ such that the edges $\varepsilon_1, \ldots, \varepsilon_n$ occur in order in the boundary of $\mathcal{D}$.

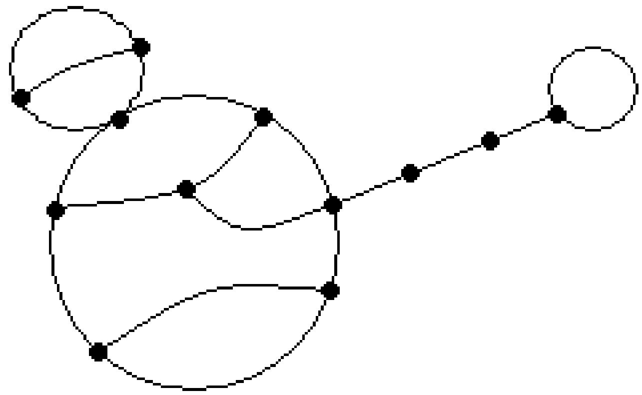


Figure 1.3: A simply-connected diagram containing two extremal discs.

Lemma 1.6.2. ([66, Lemma V.4.2]) *Let $\mathcal{S}$ be a simply-connected diagram that contains no vertices of degree* 1. *If $\partial\mathcal{S}$ is not a simple closed path, then $\mathcal{S}$ contains at least two extremal discs.*

The notation $\sum_{\mathcal{D}}^{*}$ will be used to denote summation over the *simple* boundary regions of a diagram $\mathcal{D}$. The following result whose proof can be found in [66, Page 248], strengthens the inequality stated in Corollary 1.6.1. The proof of Theorem 1.6.2 given below is more comprehensive than the one given in [66]. Also, we have weakened one of the assumptions in the statement of the result.

Theorem 1.6.2. *Let $\mathcal{S}$ be a simply-connected (q,p)-diagram. Suppose that $q \geqslant 3$ and that* $\mathrm{Area}(\mathcal{S}) \geqslant 2$. *Furthermore, assume that if Δ is a region of $\mathcal{S}$ such that $\partial\Delta \cap \partial\mathcal{S} = \nu$ for some boundary vertex ν, then $d(\Delta) \geqslant p/q+2$. Then*

$$\sum_{\mathcal{S}}^{*}[\frac{p}{q}+2-i(\Delta)] \geqslant p.$$

Proof. First, suppose $\partial\mathcal{S}$ is a simple closed path. We proceed by induction on Area$(\mathcal{S})$. If Area$(\mathcal{S}) = 2$, then $\mathcal{S}$ consists of two regions Δ_1 and Δ_2 that have a single edge in common. Therefore, $i(\Delta_j) = 1$ for $j = 1$ or 2 and

$$\sum_{\mathcal{S}}^{*}[\frac{p}{q}+2-i(\Delta)] = 2[\frac{p}{q}+1] = p.$$

Assume the result holds for all diagrams satisfying $2 \leqslant \text{Area}(\mathcal{S}) \leqslant k$. By Corollary 1.6.1, we have

$$\sum_{\mathcal{S}}^{\bullet}[\frac{p}{q}+2-i(\Delta)] \geqslant p.$$

If every boundary region of $\mathcal{S}$ is simple, then

$$\sum_{\mathcal{S}}^{*}[\frac{p}{q}+2-i(\Delta)] = \sum_{\mathcal{S}}^{\bullet}[\frac{p}{q}+2-i(\Delta)] \geqslant p,$$

as required. Suppose $\mathcal{S}$ contains a non-simple boundary region Δ such that $\partial\Delta \cap \partial\mathcal{S} = \nu$ for some boundary vertex ν, and call such a region an *almost interior region*. By our hypotheses, the almost interior regions have degree at least $p/q+2$ and so make a non-positive contribution to the sum $\sum_{\mathcal{S}}^{\bullet}[p/q+2-i(\Delta)]$. Let $\sum_{\mathcal{S}}^{!}$ denote summation over the boundary regions of $\mathcal{S}$ which *exclude* the almost interior regions. Then

$$\sum_{\mathcal{S}}^{!}[\frac{p}{q}+2-i(\Delta)] \geqslant \sum_{\mathcal{S}}^{\bullet}[\frac{p}{q}+2-i(\Delta)].$$

Therefore, if the only non-simple boundary regions of $\mathcal{S}$ are precisely the almost interior regions, then

$$\sum_{\mathcal{S}}^{*}[\frac{p}{q}+2-i(\Delta)] = \sum_{\mathcal{S}}^{!}[\frac{p}{q}+2-i(\Delta)] \geqslant \sum_{\mathcal{S}}^{\bullet}[\frac{p}{q}+2-i(\Delta)] \geqslant p,$$

as required.

Now suppose $\mathcal{S}$ contains a non-simple boundary region Δ' which is not an almost interior region. Then $\mathcal{S} - \Delta'$ has at least two components, $\mathcal{C}_1$ and $\mathcal{C}_2$ say, which each contain at least one region (see Fig. 1.4). Let $\mathcal{S}_1 = \mathcal{C}_1 \cup \Delta'$ and $\mathcal{S}_2 = \mathcal{C}_2 \cup \Delta'$.

If Δ is a region of $\mathcal{S}_j$ $(j = 1, 2)$, then $\partial\Delta \cap \partial\mathcal{S}_j = \partial\Delta \cap \partial\mathcal{S}$ unless $\Delta = \Delta'$. The only region common to $\mathcal{S}_1$ and $\mathcal{S}_2$ is Δ' and $i(\Delta') \geqslant 1$ in both $\mathcal{S}_1$ and $\mathcal{S}_2$. Applying the inductive hypothesis to $\mathcal{S}_1$ and $\mathcal{S}_2$ gives

$$\sum_{\mathcal{S}_1}^{*}[\frac{p}{2}+1-i(\Delta)] + \sum_{\mathcal{S}_2}^{*}[\frac{p}{2}+1-i(\Delta)] \geqslant 2p.$$

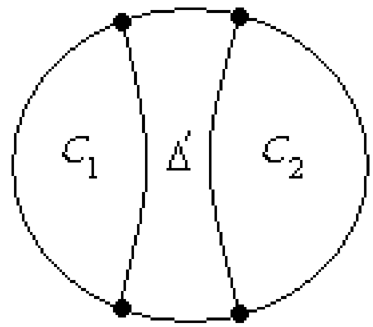


Figure 1.4: Components $\mathcal{C}_1$ and $\mathcal{C}_2$.

Now Δ' appears in both sums and, at worst, has $i(\Delta') = 1$ in each; thus contributing $p/2$ to both. It follows that

$$\sum_{\mathcal{S}}^{*}[\frac{p}{q}+2-i(\Delta)] = \sum_{\substack{\mathcal{S}_1 \\ \Delta\neq\Delta'}}^{*}[\frac{p}{2}+1-i(\Delta)] + \sum_{\substack{\mathcal{S}_2 \\ \Delta\neq\Delta'}}^{*}[\frac{p}{2}+1-i(\Delta)] \geqslant p.$$

This concludes the proof for the case when $\partial\mathcal{S}$ is a simple closed path.

If the boundary of $\mathcal{S}$ is not a simple closed path, then by Lemma 1.6.2, $\mathcal{S}$ contains at least two extremal discs, say $\mathcal{K}_1$ and $\mathcal{K}_2$. If $\mathcal{K}_j$ $(j = 1, 2)$ consists of a single region Δ_j, then $\partial\Delta_j \cap \partial\mathcal{S}$ is a consecutive part of $\partial\mathcal{S}$ and

$$\sum_{\mathcal{K}_j}^{*}[\frac{p}{2}+1-i(\Delta_j)] = \frac{p}{2}+1.$$

If $\mathcal{K}_j$ contains more than one region, then

$$\sum_{\mathcal{K}_j}^{*}[\frac{p}{2}+1-i(\Delta)] \geqslant p$$

by the first part of the proof. Since an extremal disc is connected to the rest of $\mathcal{S}$ by a single vertex, each $\mathcal{K}_j$ $(j = 1, 2)$ can contain at most one region Δ'_j such that $\partial\Delta'_j \cap \partial\mathcal{K}_j$ is a consecutive part of $\partial\mathcal{K}_j$ but $\partial\Delta'_j \cap \partial\mathcal{S}$ is not a consecutive part of $\partial\mathcal{S}$. Since $i(\Delta'_j) \geqslant 1$, Δ'_j can contribute at most $p/2$ to the sum $\sum^{*}_{\mathcal{K}_j}[\frac{p}{2}+1-i(\Delta)]$, so

$$\sum_{\substack{\mathcal{K}_j \\ \Delta\neq\Delta'_j}}^{*}[\frac{p}{2}+1-i(\Delta)] \geqslant \frac{p}{2}.$$

If $\mathcal{K}_1$ and $\mathcal{K}_2$ both contain a single region, then

$$\sum_{\mathcal{K}_1}^{*}[\frac{p}{2}+1-i(\Delta)] + \sum_{\mathcal{K}_2}^{*}[\frac{p}{2}+1-i(\Delta)] = 2(\frac{p}{2}+1) \geqslant p.$$

If $\mathcal{K}_1$ consists of a single region and $\mathcal{K}_2$ contains more than one region, then

$$\sum_{\mathcal{K}_1}^{*}[\frac{p}{2}+1-i(\Delta)]+\sum_{\substack{\mathcal{K}_2\\ \Delta\neq\Delta_2'}}^{*}[\frac{p}{2}+1-i(\Delta)]\geqslant(\frac{p}{2}+1)+\frac{p}{2}>p.$$

Clearly, the same is true if we interchange $\mathcal{K}_1$ and $\mathcal{K}_2$. Finally, if $\mathcal{K}_1$ and $\mathcal{K}_2$ both contain more than one region, then

$$\sum_{\substack{\mathcal{K}_1\\ \Delta\neq\Delta_1'}}^{*}[\frac{p}{2}+1-i(\Delta)]+\sum_{\substack{\mathcal{K}_2\\ \Delta\neq\Delta_2'}}^{*}[\frac{p}{2}+1-i(\Delta)]\geqslant\frac{p}{2}+\frac{p}{2}=p.$$

Hence in all cases, we have

$$\sum_{\mathcal{K}_1}^{*}[\frac{p}{q}+2-i(\Delta)]+\sum_{\mathcal{K}_2}^{*}[\frac{p}{q}+2-i(\Delta)]\geqslant p,$$

from which we deduce

$$\sum_{\mathcal{S}}^{*}[\frac{p}{q}+2-i(\Delta)]\geqslant p,$$

as required. $\square$

We conclude from Theorem 1.6.2 that $\mathcal{S}$ contains at least two simple boundary regions which contain at most $p/q+1$ interior edges in their boundaries. Note that $p/q+1=3$ if $\mathcal{S}$ is a $(3,6)$-diagram and $p/q+1=2$ if $\mathcal{S}$ is a $(4,4)$-diagram.

We now prove some results for simply-connected $[p,q]$-diagrams.

Theorem 1.6.3. *Let $\mathcal{S}$ be a simply-connected diagram which does not contain any vertices of degree 1 and where $d(\mathcal{S})\leqslant n$.*

(i) *If $\mathcal{S}$ is a $[3,6]$- or a $[4,4]$-diagram, then $d(\Delta)\leqslant 2n$ for each region Δ of $\mathcal{S}$.*

(ii) *If $\mathcal{S}$ is a $[3,8]$- or a $[4,6]$-diagram, then Area$(\mathcal{S})\leqslant n$.*

Proof. Let $\mathcal{S}$ be a $[3,6]$-diagram. To prove Statement (i) we proceed by induction on Area$(\mathcal{S})$. If Area$(\mathcal{S})=1$, then $\mathcal{S}$ consists of a single region Δ with $d(\Delta)\leqslant n$.

The boundary of $\mathcal{S}$ is either a simple closed path or, by Lemma 1.6.2, $\mathcal{S}$ contains at least two extremal discs. In either case $\partial\mathcal{S}$ will contain a simple closed subpath γ. First, suppose γ is the boundary of a single region. Let $\mathcal{S}_1$ be the subdiagram of $\mathcal{S}$ which is bounded by $\partial\mathcal{S}-\gamma$. Now $\mathcal{S}_1$

may contain vertices of degree 1, but we can remove, one at a time, any vertices of degree 1 and the edges incident to such vertices. This process yields a subdiagram $\mathcal{S}_1'$ of $\mathcal{S}$ to which the induction hypothesis applies. If Δ' is a region of $\mathcal{S}_1'$, then $d(\Delta') \leqslant 2d(\mathcal{S}_1') < 2n$. Furthermore, $d(\Delta) < n$ where Δ is the region bounded by γ.

Now suppose γ bounds more than one region. In this case the subdiagram $\mathcal{S}_2$ of $\mathcal{S}$ which is bounded by γ is a $[3,6]$-diagram that does not contain any vertices of degree 1. From Theorem 1.6.2 we deduce that $\mathcal{S}_2$ must contain at least two simple boundary regions Δ_1, Δ_2 with $i_{\mathcal{S}_2}(\Delta_j) \leqslant 3$ $(j = 1, 2)$. Now $\mathcal{S}_2$ can contain at most one region Δ' which is a simple boundary region of $\mathcal{S}_2$ but which is not a simple boundary region of $\mathcal{S}$. Therefore, we may assume that Δ_1 is a simple boundary region of both $\mathcal{S}_2$ and $\mathcal{S}$. That is $\alpha = \partial\Delta_1 \cap \partial\mathcal{S}$ is a consecutive part of γ and $\partial\mathcal{S}$. Since $d(\Delta_1) \geqslant 6$ and since $i_{\mathcal{S}_2}(\Delta_1) \leqslant 3$, we deduce that α must contain at least three edges. Delete α (but not its endpoints) from γ to obtain a $[3,6]$-diagram $\mathcal{S}_2'$. Since $\mathcal{S}_2'$ does not contain any vertices of degree 1, and since $\mathrm{Area}(\mathcal{S}_2') < \mathrm{Area}(\mathcal{S})$ and $d(\mathcal{S}_2') \leqslant d(\mathcal{S})$, we may apply the inductive hypothesis to $\mathcal{S}_2'$. Therefore, if Δ is a region of $\mathcal{S}_2'$, then $d(\Delta) \leqslant 2n$. Also, $d(\Delta_1) \leqslant n + 3$. Thus every region of $\mathcal{S}$ has degree at most $2n$. This completes the proof for the case when $\mathcal{S}$ is a $[3,6]$-diagram.

The proof for the case when $\mathcal{S}$ is a $[4,4]$-diagram differs only in the numbers used. This completes the proof of Statement (i).

Now let $\mathcal{S}$ be a $[4,6]$-diagram. To prove Statement (ii) we proceed by induction on $d(\mathcal{S})$.

The boundary of $\mathcal{S}$ is either a simple closed path or, by Lemma 1.6.2, $\mathcal{S}$ contains at least two extremal discs. In either case $\partial\mathcal{S}$ will contain a simple closed subpath γ. First, suppose γ is the boundary of a single region. Let $\mathcal{S}_1$ be the subdiagram of $\mathcal{S}$ which is bounded by $\partial\mathcal{S} - \gamma$. Now $\mathcal{S}_1$ may contain vertices of degree 1, but we can remove, one at a time, any vertices of degree 1 and the edges incident to such vertices. This process yields a subdiagram $\mathcal{S}_1'$ of $\mathcal{S}$ to which the induction hypothesis applies. Therefore, $\mathrm{Area}(\mathcal{S}) \leqslant n$.

Now suppose that γ bounds more than one region. In this case the subdiagram $\mathcal{S}_2$ of $\mathcal{S}$ which is bounded by γ is a $[4,6]$-diagram, which does not contain any vertices of degree 1. In particular, $\mathcal{S}_2$ is a $(4,4)$-diagram and so, from Theorem 1.6.2, we deduce that $\mathcal{S}_2$ must contain at least two simple boundary regions Δ_1, Δ_2 with $i_{\mathcal{S}_2}(\Delta_j) \leqslant 2$ $(j = 1, 2)$. Now $\mathcal{S}_2$ can contain at most one region Δ' which is a simple boundary region of $\mathcal{S}_2$ but which is not a simple boundary region of $\mathcal{S}$. Therefore, we may assume that Δ_1 is a simple boundary region of both $\mathcal{S}_2$ and $\mathcal{S}$. That is $\alpha = \partial\Delta_1 \cap \partial\mathcal{S}$ is a consecutive part of γ and $\partial\mathcal{S}$. Since $d(\Delta_1) \geqslant 6$ and since $i_{\mathcal{S}_2}(\Delta_1) \leqslant 2$, we deduce that α must contain at least four edges. Delete α (but not its endpoints) from γ to obtain a $[4,6]$-diagram $\mathcal{S}_2'$. Since $\mathcal{S}_2'$ does not contain any vertices of degree 1 and since $d(\mathcal{S}_2') < d(\mathcal{S})$, we may apply the inductive hypothesis to $\mathcal{S}_2'$. Therefore, $\text{Area}(\mathcal{S}_2') < n$ and it follows that $\text{Area}(\mathcal{S}) \leqslant n$. This completes the proof for the case when $\mathcal{S}$ is a $[4,6]$-diagram.

The proof for the case when $\mathcal{S}$ is a $[3,8]$-diagram differs only in the numbers used. This completes the proof of Statement (ii). □

Statement (ii) of Theorem 1.6.3 gives a bound on the area of a simply-connected $[3,8]$- or $[4,6]$-diagram. The following result applies to the more general case of a (p,q)-diagram where $1/p + 1/q = 1/2$.

Theorem 1.6.4. ([95, Theorem 3.6, p. 182]) *For any simply-connected (p,q)-diagram $\mathcal{S}$ of degree d, there is a number $k > 0$ such that Area($\mathcal{S}$) $\leqslant kd^2$.*

Remark 1.6.1. In [66, Theorem V.6.2] a formula is given for the area of a simply-connected $(3,6)$- or $(4,4)$-diagram. However, the statement of this result does not appear to be correct. (What appears to be a corrected version of the formula is later used in the proof of Theorem V.6.3.)

We end this section with some general results concerning annular diagrams. Let $\mathcal{A}$ be an annular diagram and let $\mathcal{B}$ be the boundary layer of $\mathcal{A}$ (recall Definition 1.6.1). The diagram $\mathcal{C} = \mathcal{A} - \mathcal{B}$ may have several components; however, $\mathcal{C}$ contains at most one annular component. A simply-connected component of $\mathcal{C}$ is called a *gap* of $\mathcal{B}$. Thus, a gap is a simply-connected subdiagram of $\mathcal{A}$ that contains only interior regions and which is entirely surrounded by boundary regions. Let $\mathcal{K}_1, \ldots, \mathcal{K}_n$ be the gaps of $\mathcal{B}$ and let $\mathcal{B}' = \mathcal{B} \cup \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_n$. Then $\mathcal{A}' = \mathcal{A} - \mathcal{B}'$ is the annular component of $\mathcal{A} - \mathcal{B}$ and we say that $\mathcal{A}'$ is *obtained from $\mathcal{A}$ by removing the boundary layer and its gaps.*

Definition 1.6.7. A pair (Δ_1, Δ_2) of (not necessarily distinct) regions of $\mathcal{A}$ is called a *boundary linking pair* if $\sigma \cap \partial\Delta_1 \neq \emptyset$, $\partial\Delta_1 \cap \partial\Delta_2 \neq \emptyset$ and $\partial\Delta_2 \cap \tau \neq \emptyset$.

Lemma 1.6.3. ([66, Lemma V.7.5]) *Let $\mathcal{A}$ be an annular diagram containing at least one region, and let $\mathcal{A}'$ be obtained from $\mathcal{A}$ by removing the boundary layer and its gaps. If there are no boundary linking pairs in $\mathcal{A}$, then $\mathcal{A}'$ is an annular diagram containing at least one region.*

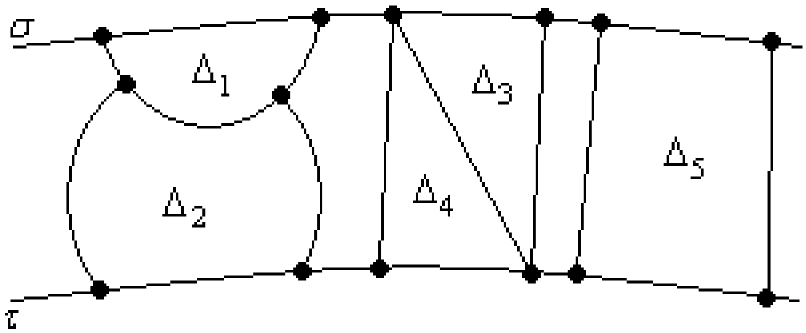


Figure 1.5: Examples of boundary linking pairs.

Let $\beta(\mathcal{A})$ denote the number of regions contained in the boundary layer of $\mathcal{A}$.

Theorem 1.6.5. ([66, Theorem V.7.4]) *Let $\mathcal{A}$ be an annular (q,p) diagram. Consider the sequence of diagrams $\mathcal{A} = \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k$ where $\mathcal{A}_i$ is obtained by deleting the boundary layer of $\mathcal{A}_{i-1}$, and the process is continued until the boundary layer of $\mathcal{A}_k$ is equal to $\mathcal{A}_k$. Then*

$$\frac{q}{p} \dot{\sum_{\mathcal{A}}} [p - i(\Delta)] \geqslant \max\{\beta(\mathcal{A}_i) : i = 0, \ldots, k\}.$$

1.7 r-diagrams

Let $\mathcal{P} = \langle \mathbf{x} \,; \mathbf{r} \rangle$ be a finite presentation of a group G.

Definition 1.7.1. The *symmetric closure* $\mathbf{r}^s$ of $\mathbf{r}$ is the set of all cyclic permutations of elements of $\mathbf{r}^{\pm 1}$. If $\mathbf{r}^s = \mathbf{r}$, then $\mathbf{r}$ is *symmetric*.

Let $\mathcal{D}$ be an oriented diagram and let ϕ be a function which assigns to each oriented edge ε of $\mathcal{D}$ an element $\phi(\varepsilon) \in \mathbf{x}$ such that if ε^{-1} is the oppositely oriented edge, then $\phi(\varepsilon^{-1}) = \phi(\varepsilon)^{-1}$. We call ϕ the *labelling function* of $\mathcal{D}$ and $\phi(\varepsilon)$ the *label* of ε. If $\alpha = \varepsilon_1 \ldots \varepsilon_n$ is a path in $\mathcal{D}$, then the *label* of α is $\phi(\alpha) = \phi(\varepsilon_1) \ldots \phi(\varepsilon_n)$. A *label* of a region Δ of $\mathcal{D}$ is a word $\phi(\delta)$ where δ is a boundary cycle of $\mathcal{D}$. Similarly, a *label* of a simply-connected subdiagram $\mathcal{B}$ of $\mathcal{D}$ is a word $\phi(\beta)$ where β is a boundary cycle of $\mathcal{B}$. If we wish to denote a label of Δ or $\mathcal{B}$ without giving reference to any particular boundary cycle, then we will write $\phi(\partial\Delta)$ or $\phi(\partial\mathcal{B})$, respectively.

Definition 1.7.2. We say that an oriented diagram $\mathcal{D}$ is an $\mathbf{r}$*-diagram*, or is a *diagram over* $\mathcal{P}$, if for any boundary cycle δ of a region of $\mathcal{D}$, $\phi(\delta) \in \mathbf{r}^s$.

An $\mathbf{r}$-diagram $\mathcal{D}$ is *reduced* if whenever one has regions Δ_1, Δ_2 meeting along a common path α, then the labels of the boundary cycles $\alpha\delta_1$ and $\alpha\delta_2$ (see Fig. 1.6) are distinct. If $\mathcal{D}$ is not reduced, then it is an *unreduced* diagram. It is conceivable that $\Delta_1 = \Delta_2$ (such a region is often called *self-identified*). In this case if $\phi(\alpha\delta_1) \equiv W$, then the label of $\alpha\delta_2$ would be a cyclic permutation of W^{-1} (one reads $\alpha\delta_2$ around the boundary of Δ in the opposite direction to the way one reads $\alpha\delta_1$). Thus, the label on $\alpha\delta_2$ would be $W_1^{-1}W_2^{-1}$ say, where $W \equiv W_1W_2$. However, it is impossible for $W_1^{-1}W_2^{-1} \equiv W_1W_2$. Thus, one never actually has to consider self-identified regions.

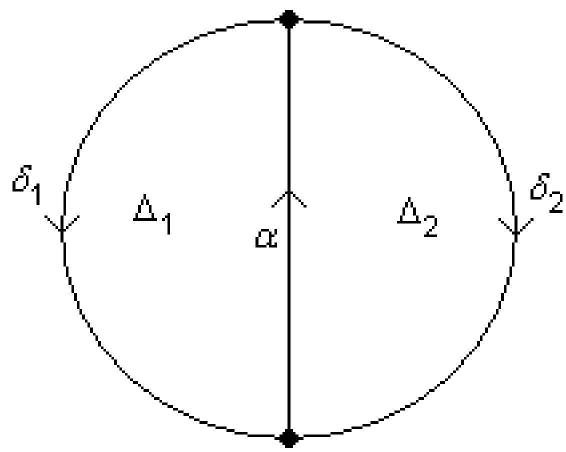


Figure 1.6: Regions Δ_1 and Δ_2.

A *label* of a simply-connected $\mathbf{r}$-diagram $\mathcal{S}$ is a word $\phi(\alpha)$ where α is a boundary cycle of $\mathcal{S}$. Let 0 be a distinguished boundary vertex of $\mathcal{S}$; the *basepoint* of $\mathcal{S}$. Let W be a non-empty word on $\mathbf{x}^{\pm 1}$ and let α_0 be the anticlockwise boundary cycle of $\mathcal{S}$ that starts and ends at 0. We say that $\mathcal{S}$ *is an* $\mathbf{r}$*-diagram for* W if $\phi(\alpha_0) \equiv W$.

A proof of the following result, most commonly referred to as *Van Kampen's Lemma*, can be found in [75, pp. 188 - 119] and [66, Chapter V].

Theorem 1.7.1 (van Kampen's Lemma). *Let $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r} \rangle$ be a presentation defining a group G and let W be a non-empty word on $\mathbf{x}^{\pm 1}$. Then W represents the identity element of G if and only if there exists a simply-connected $\mathbf{r}$-diagram for W.*

Recall that in §1.5 the area of a word W was defined to be the minimum number of relators in any expression of the form (1.1) for $[W]$. Due to van Kampen's Lemma, we can re-define the area of W in a geometric way.

Definition 1.7.3. The *area* of W is equal to $\min\{\text{Area}(\mathcal{S})\}$ where the minimum is taken over all simply-connected $\mathbf{r}$-diagrams $\mathcal{S}$ for W.

We say that $\mathcal{S}$ is a *minimal* $\mathbf{r}$-diagram for W if $\mathcal{S}$ is an $\mathbf{r}$-diagram for W and $\text{Area}(\mathcal{S}) = \text{Area}(W)$.

Lemma 1.7.1. ([66, Lemma V.2.1]) *If $\mathcal{S}$ is a minimal $\mathbf{r}$-diagram for a word W, then $\mathcal{S}$ is reduced.*

Let W be a word on $\mathbf{x}^{\pm 1}$ of length n that represents the identity element of G. Then W is $\sim_{\mathbf{r}}$-equivalent to the empty word. Recall that the Dehn function of $\mathcal{P}$ measures the maximum number of relators needed to "prove" this equivalence. Following Definition 1.7.3, we can now view the Dehn function in a geometric setting: the Dehn function of $\mathcal{P}$ measures the maximum number of regions needed to construct a simply-connected $\mathbf{r}$-diagram for W. Thus, one may view the Dehn function as measuring the number of discs needed to fill a closed curve of length n.

Just as simply-connected $\mathbf{r}$-diagrams play a role in the study of the word problem, *annular* $\mathbf{r}$-diagrams play a role in the study of the conjugacy problem. Let $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$ be a presentation of a group G. Let $\mathcal{A}$ be an annular $\mathbf{r}$-diagram with two distinguished boundary vertices: a vertex o contained in σ and a vertex ι contained in τ. We call o and ι the outer and inner basepoints of $\mathcal{A}$, respectively. Let W and Z be two non-empty words on $\mathbf{x}^{\pm 1}$, and let σ_o be the anticlockwise boundary cycle of $\mathcal{A}$ that starts and ends at o, and τ_ι be the clockwise boundary cycle of $\mathcal{A}$ that starts and ends at ι. If $\phi(\sigma_o) \equiv W$ and $\phi(\tau_\iota) \equiv Z^{-1}$, then we say that $\mathcal{A}$ *is an annular $\mathbf{r}$-diagram for the pair* (W, Z^{-1}).

Lemma 1.7.2. ([75, Lemma 11.2] & [66, Lemmas V.5.1, V.5.2]) *Let W and Z be two non-empty words on $\mathbf{x}^{\pm 1}$. Then W, Z represent conjugate elements of G if and only if there exists an annular $\mathbf{r}$-diagram for the pair* (W, Z^{-1}).

1.8 Pictures and the second homotopy module

Our aim in this section is to introduce the *second homotopy module* $\pi_2(\mathcal{P})$ of a presentation $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$. The elements of $\pi_2(\mathcal{P})$ can be represented by equivalence classes of *spherical* $\mathbf{r}$-diagrams. Let $\mathcal{S}$ be a simply-connected $\mathbf{r}$-diagram, which can be viewed as the planar projection of a tessellation of the two-sphere. If the label of *every* region of this tessellation is an element of $\mathbf{r}^s$, then the tessellated two-sphere is an example of a spherical $\mathbf{r}$-diagram. Note that in this case the label of $\mathcal{S}$

is an element of $\mathbf{r}^s$. For the general definition of a spherical $\mathbf{r}$-diagram we refer to [30, pp. 159-160] and [83].

It is more convenient to represent the elements of $\pi_2(\mathcal{P})$ by equivalence classes of spherical $\mathbf{r}$-*pictures*; one of the main reasons being that we can always work with spherical $\mathbf{r}$-pictures in the plane. A spherical $\mathbf{r}$-picture $\mathbb{P}$ is basically the dual of a spherical $\mathbf{r}$-diagram. Let $\mathcal{S}$ be the planar projection of a spherical $\mathbf{r}$-diagram. Insert a disc D_i in the interior of each region of $\mathcal{S}$ and insert a disc D_∞ in the unbounded region $-\mathcal{S}$. The collection of the D_i's together with D_∞ are the *discs* of $\mathbb{P}$. If Δ_1, Δ_2 are distinct region of $\mathcal{S}$ having an edge ε in common, then an *arc* α is drawn from D_1 to D_2 which crosses ε but no other edge of $\mathcal{S}$, nor arc of $\mathbb{P}$ already drawn. If a region Δ_i of $\mathcal{S}$ contains an edge ε in its boundary such that Δ_i lies on both sides of ε, then a loop is drawn at D_i, crossing ε, but no other edge of $\mathcal{S}$, nor arc of $\mathbb{P}$. If Δ_i is a boundary region of $\mathcal{S}$, then for each boundary edge ε of Δ_i, an arc is drawn from D_i to D_∞ which crosses ε but no other edge of $\mathcal{S}$, nor arc of $\mathbb{P}$. Finally, if $\mathcal{S}$ contains an edge ε which is not contained in the boundary of any region of $\mathcal{S}$, then an arc is drawn from D_∞ to itself, crossing ε but no other edges of $\mathcal{S}$. The collection of discs and arcs form the spherical $\mathbf{r}$-picture $\mathbb{P}$. The labelling and orientation of the edges of $\mathcal{S}$ induce a transverse labelling and orientation of the arcs of $\mathbb{P}$, the result of which is that each disc of $\mathbb{P}$ is labelled by an element of $\mathbf{r}^s$. We note that the construction of $\mathbb{P}$ is similar to the construction of the dual diagram $\mathcal{S}^*$ of $\mathcal{S}$. Also, $\mathbb{P}$ can be supported on the two-sphere by identifying D_∞ with the arctic circle.

Although we have defined by means of an example a spherical $\mathbf{r}$-picture as deriving from a spherical $\mathbf{r}$-diagram, it is possible to formulate the notion of a spherical $\mathbf{r}$-picture, and more generally a *picture*, directly.

1.8.1 Pictures

The definition of a picture given in this section is more general than the standard one found in the literature (see for example [17, 30, 42, 58, 84]). This more general definition will be particularly useful for our work in Chapter 5.

A closed punctured disc Π with n ($\geqslant 0$) holes is the closure of

$$D - \bigcup_{i=1}^{n} \text{interior}(B_i),$$

where D is a closed disc and $B_1, \ldots, B_n$ are disjoint closed discs lying inside D. The *boundary* $\partial\Pi$ of Π is defined to be

$$\partial D \cup \bigcup_{i=1}^{n} \partial B_i,$$

where ∂D and ∂B_i $(i = 1, \ldots, n)$ are the boundaries of D and B_i, respectively.

Definition 1.8.1. A *picture* $\mathbb{P}$ is a geometric configuration consisting of a finite collection of pairwise disjoint closed discs $D_1, \ldots, D_m$ in a closed punctured disc Π with n $(\geqslant 0)$ holes, together with a finite collection of pairwise disjoint compact one-manifolds $\alpha_1, \ldots, \alpha_k$ (the *arcs* of $\mathbb{P}$) properly embedded in $\Pi - \bigcup_{i=1}^{m} \text{interior}(D_i)$. The punctured disc Π has a basepoint 0 on $\partial\Pi$, each disc B_i has a basepoint b_i on ∂B_i, and each disc D_i has a basepoint 0_i on ∂D_i. Each arc is either a simple closed curve having trivial intersection with $\partial\Pi \bigcup_{i=1}^{m} \partial D_i$, or is a simple curve which joins two points of $\partial\Pi \bigcup_{i=1}^{m} \partial D_i$, neither point being a basepoint.

By the *discs* of $\mathbb{P}$ we mean the discs $D_1, \ldots, D_m$ and not the ambient punctured disc Π. If Π does not contain any holes, then we say that $\mathbb{P}$ is *simply-connected*. Otherwise, the picture is *non-simply-connected*. The *area* $\text{Area}(\mathbb{P})$ of $\mathbb{P}$ is equal to the number of discs of $\mathbb{P}$. The boundary $\partial\mathbb{P}$ of $\mathbb{P}$ is defined to be $\partial\Pi$. The *corners* κ of a disc D_i are the closures of the connected components of $\partial D_i - \bigcup_{j=1}^{k} \alpha_j$, and the *regions* F of $\mathbb{P}$ are the closures of the connected components of $\Pi - ((\bigcup_{i=1}^{m} D_i) \cup (\bigcup_{j=1}^{k} \alpha_j))$. A region is a *boundary region* if it meets $\partial\mathbb{P}$ and it is an *interior region* otherwise. The *degree* of a region is equal to the number of corners contained in its boundary. An arc is a *boundary arc* of $\mathbb{P}$ if it has at least one endpoint on $\partial\mathbb{P}$. The *components* of $\mathbb{P}$ are the connected components of $(\bigcup_{i=1}^{m} D_i) \cup (\bigcup_{j=1}^{k} \alpha_j)$ and we say that $\mathbb{P}$ is *connected* if it has at most one component. The *empty picture* is the picture which does not contain any arcs or discs.

Definition 1.8.2. A picture $\mathbb{P}$ is *spherical* if it is simply-connected and if no arc of $\mathbb{P}$ meets $\partial\mathbb{P}$.

1.8.2 r-pictures

Let $\mathcal{P} = \langle \mathbf{x} \,;\mathbf{r} \rangle$ be a presentation defining a group G. An $\mathbf{r}$-*picture*, or a *picture over* $\mathcal{P}$, is a picture $\mathbb{P}$ that satisfies the following two conditions:

(i) Each arc has a normal orientation indicated by a short arrow meeting the arc transversely and is labelled by an element of $\mathbf{x}$.

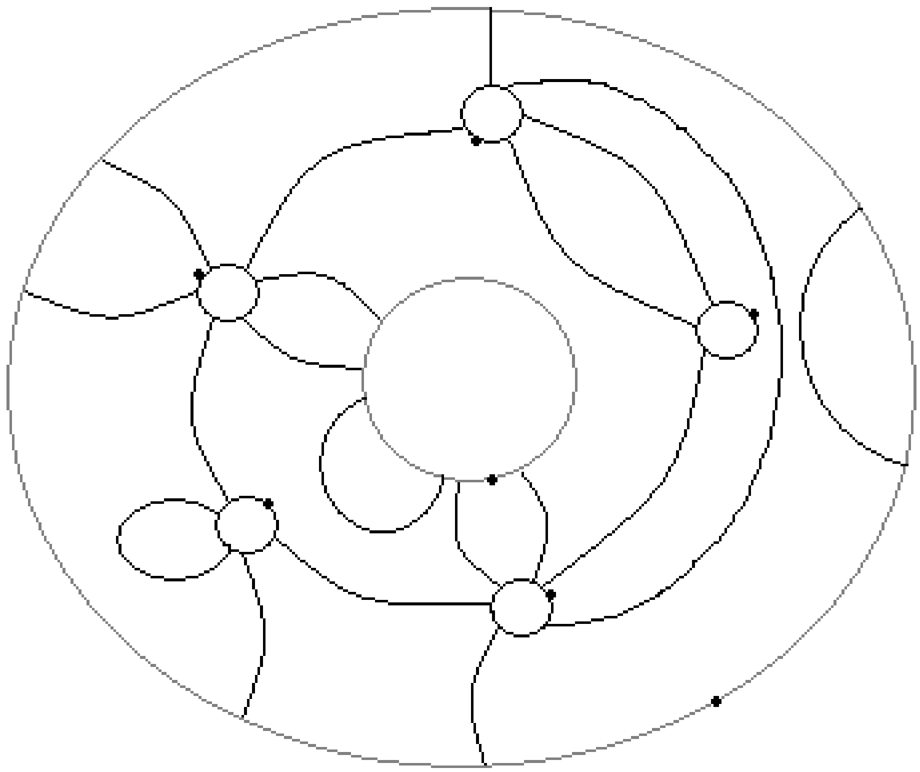


Figure 1.7: A picture in which the ambient punctured disc contains one hole.

(ii) If we travel around ∂D_i, where D_i is a disc of $\mathbb{P}$, once in a clockwise direction starting from 0_i and read off the labels on arcs encountered (with the understanding that we read x if we cross an arc labelled x in the direction of its normal orientation, and we read x^{-1} otherwise), then we obtain an element of $\mathbf{r}^s$. We call this word the *label* of D_i.

The *boundary label* of a simply-connected $\mathbf{r}$-picture $\mathbb{P}$ is the word $W(\mathbb{P})$ obtained by reading the labels of arcs that are encountered in a walk around $\partial\mathbb{P}$ in anticlockwise direction, starting and ending at 0.

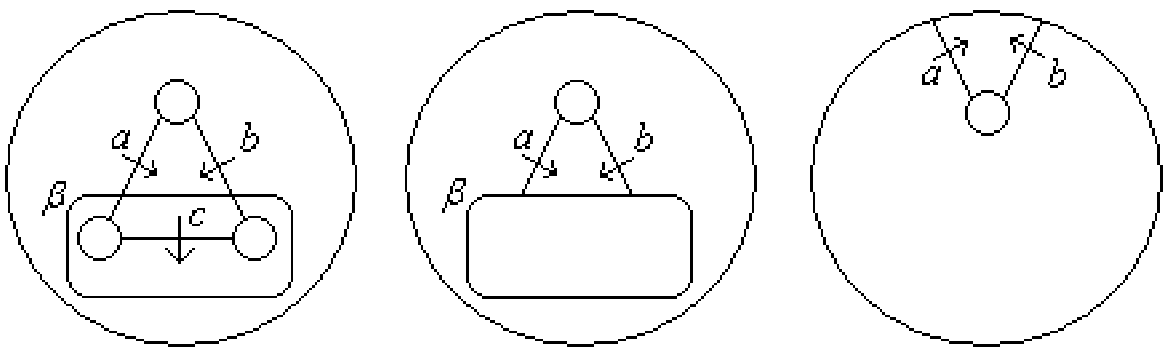
Definition 1.8.3. A path β in a picture $\mathbb{P}$ that does not meet the interior of any disc of $\mathbb{P}$ is called a *transverse path* if: (i) whenever β meets $\partial\mathbb{P}$ or a disc of $\mathbb{P}$ it does so only at its endpoints; (ii) no endpoint of β touches any arc of $\mathbb{P}$; (iii) β meets the arcs of $\mathbb{P}$ in just finitely many transverse intersections.

A *label* $W(\beta)$ of an oriented transverse path β is the word obtained by reading the labels of arcs that are encountered in a walk from the initial point of β to the terminal point of β.

Definition 1.8.4. A *subpicture* of an $\mathbf{r}$-picture $\mathbb{P}$ is an $\mathbf{r}$-picture $\mathbb{M}$ together with an embedding $\mathbb{M} \hookrightarrow \mathbb{P}$ such that the boundary of $\mathbb{M}$ is the union of a collection of closed transverse paths β_λ $(\lambda \in \Lambda)$ in $\mathbb{P}$.

A subpicture is *simply-connected* (respectively, *non-simply-connected*) if it is a simply-connected (respectively, non-simply-connected) picture. The *label* $W(\mathbb{M})$ of a simply-connected subpicture $\mathbb{M}$ is the label of the transverse path that forms the boundary of $\mathbb{M}$.

When $\mathbb{P}$ is a spherical $\mathbf{r}$-picture and when $\mathbb{M}$ is a simply-connected subpicture of $\mathbb{P}$ with boundary β, the *complement* $\mathbb{M}^c$ of $\mathbb{M}$ in $\mathbb{P}$ is defined as follows. Delete the interior of $\mathbb{M}$ to form an oriented annulus (see Fig. 1.8(b)). Identification of $\partial\mathbb{P}$ to a point produces an oriented disc that has boundary β and which supports a new $\mathbf{r}$-picture over $\mathcal{P}$. The complement of $\mathbb{M}$ in $\mathbb{P}$ is obtained from this new picture by a planar reflection. The complement has the same boundary label as $\mathbb{M}$ and its discs are those of $\mathbb{P} - \mathbb{M}$.



(a) $\mathbb{P}$ (b) Deleting $\mathbb{M}$ (c) $\mathbb{M}^c$

Figure 1.8: The complement of $\mathbb{M}$ in $\mathbb{P}$.

In the introduction to this section we described how a spherical $\mathbf{r}$-picture is basically the dual of a spherical $\mathbf{r}$-diagram. If $\mathcal{D}$ is an arbitrary $\mathbf{r}$-diagram, which is not the planar projection of a spherical $\mathbf{r}$-diagram, then we can obtain from $\mathcal{D}$ an $\mathbf{r}$-picture $\mathbb{P}$ by a similar dual construction. First, surround $\mathcal{D}$ by a circle S^1 and insert a disc B with basepoint b in the interior of each hole of $\mathcal{D}$. In the interior of each region Δ_i of $\mathcal{D}$ insert a disc D_i with basepoint 0_i. The collection of the D_i's form the discs of the $\mathbf{r}$-picture $\mathbb{P}$. The construction of $\mathbb{P}$ now follows as in the spherical case; however, we treat boundary regions of $\mathcal{D}$ in a slightly different way. If Δ_i is a boundary region of $\mathcal{D}$, then for each boundary edge ε of Δ_i an arc is drawn from D_i to S^1, or from D_i to B (if Δ_i

meets the boundary of a hole in $\mathcal{D}$), which crosses ε but no other edge of $\mathcal{D}$, nor arc of $\mathbb{P}$ already drawn. If $\mathcal{D}$ contains an edge ε which is not contained in the boundary of any region of $\mathcal{D}$, then an arc is drawn from S^1 to itself, which crosses ε but no other edge of $\mathcal{D}$. Label and orient these arcs according to the orientation and label of the original edges. The boundary $\partial\mathbb{P}$ of the resulting $\mathbf{r}$-picture $\mathbb{P}$ is defined to be

$$\partial S^1 \cup \bigcup_i \partial B_i,$$

and we choose a basepoint 0 on ∂S^1. We say that $\mathbb{P}$ is the *corresponding* $\mathbf{r}$*-picture of* $\mathcal{D}$, or simply the *corresponding picture*. Note that the area of $\mathbb{P}$ is equal to the area of the diagram $\mathcal{D}$. In Chapter 4, there will be occasions when we have to consider the corresponding $\mathbf{r}$-picture of an $\mathbf{r}$-diagram. We do so to ensure that we can always work in the plane (see Remark 4.1.1 proceeding the proof of Theorem 4.1.1).

If $\mathcal{S}$ is a simply-connected $\mathbf{r}$-diagram for a word W, then the corresponding picture $\mathbb{P}$ is a simply-connected $\mathbf{r}$-picture that satisfies $W(\mathbb{P}) \equiv W$ (for some appropriate choice of basepoint 0). This observation leads us to the following pictorial version of Theorem 1.7.1.

Theorem 1.8.1. *Let $\mathcal{P} = \langle \mathbf{x}\,; \mathbf{r}\rangle$ be a presentation of a group G and let W be a non-empty word on $\mathbf{x}^{\pm 1}$. Then W represents the identity element of G if and only if there exists a simply-connected* $\mathbf{r}$*-picture $\mathbb{P}$ such that $W(\mathbb{P}) \equiv W$.*

We say that $\mathbb{P}$ is a *simply-connected* $\mathbf{r}$*-picture for the word W* (or simply an $\mathbf{r}$*-picture for W*) if $W(\mathbb{P}) \equiv W$. Using this terminology, we can now define the area of a word W in terms of the area of an $\mathbf{r}$-picture for W:

$$\text{Area}(W) = \min\{\text{Area}(\mathbb{P}) : \mathbb{P} \text{ is a simply connected } \mathbf{r}\text{-picture for } W\}.$$

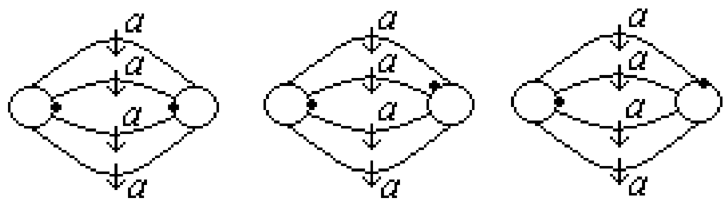
We say that $\mathbb{P}$ is a *minimal* $\mathbf{r}$-picture for W if $\mathbb{P}$ is an $\mathbf{r}$-picture for W and $\text{Area}(\mathbb{P}) = \text{Area}(W)$. The Dehn function of $\mathcal{P}$ can now be viewed in terms of pictures: the Dehn function of $\mathcal{P}$ measures the maximum number of discs needed to construct a simply-connected $\mathbf{r}$-picture for W.

If $\mathcal{A}$ is an annular $\mathbf{r}$-diagram for the pair (W, Z^{-1}), then the corresponding $\mathbf{r}$-picture is an *annular* $\mathbf{r}$-picture for the pair (W, Z^{-1}) (again for some appropriate choice of basepoints). This leads us to the following pictorial version of Lemma 1.7.2

Lemma 1.8.1. *Let W and Z be two non-empty words on $\mathbf{x}^{\pm 1}$. Then W, Z represent conjugate elements of G if and only if there exists an annular* $\mathbf{r}$*-picture for the pair (W, Z^{-1}).*

1.8.3 The second homotopy module

Let $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$ be a finite presentation of a finitely presented group G and let $\mathbb{P}$ be an $\mathbf{r}$-picture. A closed arc which encircles no disc or arc of $\mathbb{P}$ is called a *floating circle*. A *cancelling pair* in $\mathbb{P}$ is a connected spherical $\mathbf{r}$-subpicture that contains exactly two discs where the basepoints of each disc lie in the same region. Furthermore, each arc in the cancelling has an endpoint on each disc. Clearly each disc in a cancelling pair is labelled by the same element of $\mathbf{r}^s$.



(a) (b) (c)

Figure 1.9: A cancelling pair.

If $R \in \mathbf{r}$ is a proper power, then we need to be careful. The picture in Fig .1.9(a) is a cancelling pair but those in Figs. 1.9(b) and 1.9(c) are not.

The following operations can be applied to an arbitrary $\mathbf{r}$-picture.

BRIDGE Perform a bridge move (see Fig .1.10).

FLOAT Insert or delete a floating circle.

CANCEL Insert or delete a cancelling pair.

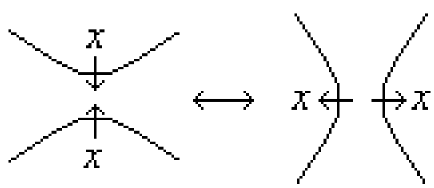
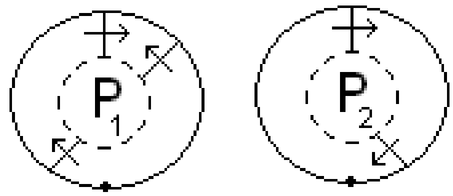


Figure 1.10: The bridge moves.

Two *spherical* pictures are said to be *equivalent* if one can be transformed to the other by a finite number of the operations BRIDGE, FLOAT and CANCEL.

Remark 1.8.1. Bridge moves highlight another advantage of using spherical **r**-pictures over spherical **r**-diagrams. *Diamond moves* [30, pp. 160-164] are the operations dual to bridge moves that one can perform on a spherical **r**-diagram. Special care is needed when using diamond moves and there are various cases one must consider. Unlike diamond moves, bridge moves can always be realised in the plane.

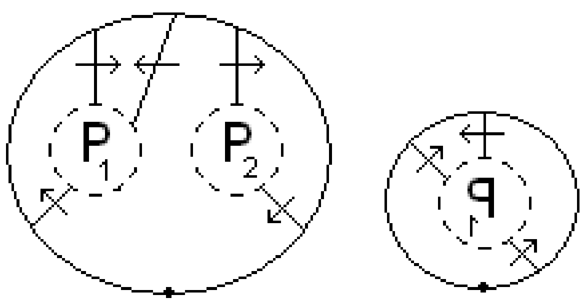
Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two **r**-pictures represented schematically in Fig. 1.11.



(a) $\mathbb{P}_1$ (b) $\mathbb{P}_2$

Figure 1.11: Schematics for $\mathbb{P}_1$ and $\mathbb{P}_2$.

Two new pictures $\mathbb{P}_1 + \mathbb{P}_2$ and $-\mathbb{P}_1$ are constructed in Fig. 1.12. Thus $-\mathbb{P}_1$ is a mirror image of $\mathbb{P}_1$ obtained by a planar reflection and $\mathbb{P}_1 + \mathbb{P}_2$ is obtained by identifying an anticlockwise segment of $\partial\mathbb{P}_1$ with a clockwise segment of $\partial\mathbb{P}_2$ of the same length. (Each segment starts at the corresponding basepoint and does not intersect any arc of the picture.) The identified segment is then deleted to obtain the boundary of $\mathbb{P}_1 + \mathbb{P}_2$. We will write $\mathbb{P}_1 - \mathbb{P}_2$ for $\mathbb{P}_1 + (-\mathbb{P}_2)$. Clearly for any **r**-picture $\mathbb{P}$, $\mathbb{P} - \mathbb{P}$ is equivalent to the empty picture and if $\mathbb{P}_1, \mathbb{P}_2$ are both spherical, then $\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{P}_2 + \mathbb{P}_1$.



(a) $\mathbb{P}_1 + \mathbb{P}_2$ (b) $-\mathbb{P}_1$

Figure 1.12: The pictures $\mathbb{P}_1 + \mathbb{P}_2$ and $-\mathbb{P}_1$.

Let $\langle\mathbb{P}\rangle$ denote the equivalence class that contains the spherical **r**-picture $\mathbb{P}$. The set of all equivalence classes of all spherical **r**-pictures form an abelian group, denoted $\pi_2(\mathcal{P})$, under the binary operation $\langle \mathbb{P}_1 \rangle + \langle \mathbb{P}_2 \rangle = \langle \mathbb{P}_1 + \mathbb{P}_2 \rangle$. The identity element of $\pi_2(\mathcal{P})$ is the equivalence class containing the empty picture and the inverse of an element $\langle \mathbb{P} \rangle \in \pi_2(\mathcal{P})$ is $\langle -\mathbb{P} \rangle$.

Let W be a word on $\mathbf{x}^{\pm 1}$ and let $\mathbb{P}$ be a spherical $\mathbf{r}$-picture. The spherical picture illustrated in Fig. 1.13 obtained by surrounding $\mathbb{P}$ with a succession of concentric closed arcs with total label W is denoted by $W \cdot \mathbb{P}$.

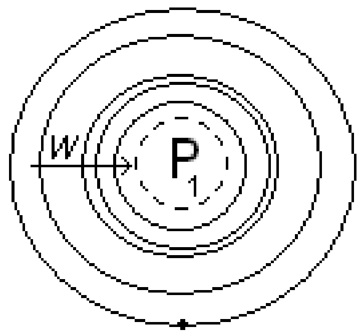


Figure 1.13: The spherical picture $W \cdot \mathbb{P}$.

Lemma 1.8.2. ([99, Lemma 1.3.3]) *There is a well-defined $G(\mathcal{P})$-action on $\pi_2(\mathcal{P})$ given by*

$$\overline{W} \cdot \langle \mathbb{P} \rangle = \langle W \cdot \mathbb{P} \rangle, \tag{1.4}$$

where W is a word on $\mathbf{x}^{\pm 1}$ and $\langle \mathbb{P} \rangle \in \pi_2(\mathcal{P})$.

The $G(\mathcal{P})$-action described in Lemma 1.8.2 gives $\pi_2(\mathcal{P})$ the structure of a left $\mathbb{Z}G(\mathcal{P})$-module.

Definition 1.8.5. The *second homotopy module of* $\mathcal{P}$ is the abelian group $\pi_2(\mathcal{P})$ with $G(\mathcal{P})$-action given in (1.4).

Let $\xi \in \pi_2(\mathcal{P})$. We say that a spherical $\mathbf{r}$-picture $\mathbb{P}$ *represents* ξ if $\xi = \langle \mathbb{P} \rangle$ and we define $\text{Area}(\xi) = \min\{\text{Area}(\mathbb{P}) : \langle \mathbb{P} \rangle = \xi\}$. If $\mathbb{P}$ represents ξ and satisfies $\text{Area}(\mathbb{P}) = \text{Area}(\xi)$, then $\mathbb{P}$ is said to be a *minimal* $\mathbf{r}$-picture that represents ξ.

Definition 1.8.6. A presentation $\mathcal{P}$ is *aspherical* if $\pi_2(\mathcal{P}) = 0$. A group G is *aspherical* if it has an aspherical presentation.

Definition 1.8.6 is equivalent to saying that every spherical $\mathbf{r}$-picture is equivalent to the empty picture. Knot groups have been shown to be aspherical by Papakyriakopoulos [79], as are Artin groups of large type [7, 8].

A famous open problem regarding aspherical presentations is the so called *Whitehead conjecture*. We refer to [15] for a general discussion of this conjecture.

Conjecture (Whitehead). *Is every subpresentation of an aspherical presentation aspherical?*

A presentation is *diagrammatically reducible* (also called *singularly aspherical* in [28]) if every non-empty spherical **r**-picture contains (after performing bridge moves) a cancelling pair. Thus, each **r**-picture can be reduced to the empty picture by performing bridge moves and deleting floating arc and cancelling pairs. We will sometimes abuse terminology and say a group G is *diagrammatically reducible* if it has a diagrammatically reducible presentation.

It is clear that every diagrammatically reducible group is aspherical. Weinbaum [102] proved that groups of prime alternating links are diagrammatically reducible. Later in this section we define *combinatorial asphericity*. There is yet another notion of asphericity, *Cohen-Lyndon asphericity*, which we will not define. For the definitions of all the different notions of asphericity and an illustration of how they are linked, we refer to [28]. See also [17].

Let X be a collection of spherical pictures over $\mathcal{P}$. We introduce a further operation that can be preformed on a spherical **r**-picture.

INSERT(X) Insert or delete an X-picture.

Two spherical pictures are said to be *equivalent (modulo X)* if one can be transformed to the other by a finite number of the operations BRIDGE, FLOAT, CANCEL and INSERT(X). We present the following result as a definition.

Definition 1.8.7. ([84, Theorem 2.5*]) The elements $\langle \mathbb{X} \rangle$ ($\mathbb{X} \in X$) *generate* $\pi_2(\mathcal{P})$ if and only if every spherical **r**-picture is equivalent (modulo X) to the empty picture. We say that X *generates* $\pi_2(\mathcal{P})$ if the elements $\{\langle \mathbb{X} \rangle : \mathbb{X} \in X\}$ generate $\pi_2(\mathcal{P})$.

In practice it is convenient to include the following derived operation as one of the basic operations one may perform on a spherical **r**-picture.

REPLACE(X) If an **r**-picture contains a subpicture $\mathbb{M}$ which is a copy of some subpicture of a picture $\mathbb{X} \in X$, then replace $\mathbb{M}$ by its complement $\mathbb{M}^c$ in $\mathbb{X}$.

The operation REPLACE(X) is a consequence of the operations INSERT(X), BRIDGE and CANCEL.

Lemma 1.8.3. ([17, Lemma 1.5]) *Let $\mathbb{A}$ be a spherical* $\mathbf{r}$*-picture. Let $\mathbb{B}$ be a subpicture of $\mathbb{A}$ and let $\mathbb{B}^c$ be the complement of $\mathbb{B}$ in $\mathbb{A}$. Suppose $\mathbb{P}$ is a spherical* $\mathbf{r}$*-picture having $\mathbb{B}$ as a subpicture and that $\mathbb{P}'$ is obtained from $\mathbb{P}$ by replacing $\mathbb{B}$ with $\mathbb{B}^c$. Then*

$$\langle \mathbb{P} \rangle - \langle \mathbb{P}' \rangle = \overline{W} \cdot \langle \mathbb{A} \rangle$$

for some word W on $\mathbf{x}^{\pm 1}$.

A presentation is said to be *combinatorially aspherical* if $\pi_2(\mathcal{P})$ is generated by *primitive* cancelling pairs (also called *primite dipoles* in [17]). Fig. 1.9(b) is an example of a primitive cancelling pair. We refer to [84] for an extensive catalogue of combinatorial aspherical presentations, which includes, for example, one-relator presentations [66] and presentations of groups which satisfy the small cancellation conditions $C(p)$, $T(q)$ (where $1/p + 1/q = 1/2$) [65].

Let X be a generating set for $\pi_2(\mathcal{P})$. For any element $\xi \in \pi_2(\mathcal{P})$ there exist elements $\xi_1, \ldots, \xi_m$ where each $\xi_i = \langle \mathbb{X}_i \rangle$ for some $\mathbb{X}_i \in X$ $(i = 1, \ldots, m)$; elements $g_1, \ldots, g_m \in G$, and $\varepsilon_i = \pm 1$ $(i = 1, \ldots, m)$ such that ξ can be written as a sum

$$\sum_{i=1}^{m} \varepsilon_i g_i \xi_i. \tag{1.5}$$

The *volume* $V_X(\xi)$ of ξ (with respect to X) is the minimum value of m amongst all sums of the form (1.5) that are equal to ξ. We will write $V(\xi)$ for the volume of ξ if there is no confusion over which set of module generators we are working with.

Definition 1.8.8. The *second order Dehn function of $\mathcal{P}$* with respect to a generating set X of $\pi_2(\mathcal{P})$ is the function $\delta^{(2)}_{\mathcal{P},X} : \mathbb{N} \to \mathbb{N}$ given by

$$\delta^{(2)}_{\mathcal{P},X}(n) = \max\{V_X(\xi) : \mathrm{Area}(\xi) \leqslant n\}.$$

It is not immediately obvious that the set $\chi_n = \{V_X(\xi) : \mathrm{Area}(\xi) \leqslant n\}$ is finite for all $n \in \mathbb{N}$. The following result confirms that this is indeed the case and hence proves that the definition of the second order Dehn function is well-defined.

Lemma 1.8.4. ([99, Lemma 2.1.3]) *The set χ_n is finite for all $n \in \mathbb{N}$.*

The second order Dehn function can be interpreted as measuring the number of balls needed to fill a 2-sphere of fixed area. Compare this with the interpretation of the first Dehn function measuring the number of discs needed to fill a closed curve of fixed length.

Definition 1.8.9. A finite presentation $\mathcal{P}$ is said to be of *type* F_3 if $\pi_2(\mathcal{P})$ is finitely generated as a $\mathbb{Z}G(\mathcal{P})$-module. A group G is of *type* F_3 if it has a finite presentation which is of type F_3.

The following theorem is the 2-dimensional analogue of Proposition 1.5.1.

Theorem 1.8.2. ([6] & [99, Corollary 2.3]) *If $\mathcal{P}_1, \mathcal{P}_2$ are two finite presentations of a group G where $\mathcal{P}_1$ is of type F_3, then $\mathcal{P}_2$ is of type F_3. In this case, we have $\delta^{(2)}_{\mathcal{P}_1} \simeq \delta^{(2)}_{\mathcal{P}_2}$.*

Theorem 1.8.2 allows us to speak, up to $\simeq$-equivalence, of *the* second order Dehn function $\delta^{(2)}_G$ of a group G which is of type F_3. In [5, Theorem 4.1] it was shown that word hyperbolic groups have linear second order Dehn functions. Recall that word hyperbolic groups can be characterized as the finitely presented groups which have linear first order Dehn functions. We note that the converse of Theorem 4.1 in [5] is false. Any group with a finite aspherical presentation has a linear second order Dehn function; however, such groups need not be word hyperbolic, as demonstrated by the free abelian group of rank two.

As for first order Dehn functions, one may consider the *isoperimetric spectrum of second order Dehn functions*, i.e.

$$\mathrm{IP}(2) = \{\alpha \in [1, \infty) : f(x) = x^\alpha \text{ is a second order Dehn function}\}.$$

Work carried out in [5] and [101] provided an infinite set of exponents in the range $[3/2, 3)$ and provided evidence for the existence of exponents in the range $[2, \infty)$. In particular, the results of [5] proved that there is no gap between 1 and 2 in IP(2). The snowflake construction of [20] provided a dense set of exponents in the interval $[3/2, 2)$, and in [24] it was shown that $2, 3 \in \mathrm{IP}(2)$. More recently, it was shown in [21] that $\mathbb{Q} \cap [3/2, \infty) \subset \mathrm{IP}(2)$.

Chapter 2

An introduction to Pride groups

This chapter serves as a general introduction to the class of Pride groups. In addition to defining the groups we are interested in, we shall present a survey of known results and state the original results which appear in this thesis.

2.1 Pride groups

Pride groups first appeared in [85] under the title *groups given by presentations in which each defining relator involves at most two types of generators.* The class of Pride groups is huge. It includes Coxeter groups, Artin groups, generalized tetrahedron groups, free products with amalgamation, and cyclic presentations in which each defining relator involves two types of generators.

Let $\Gamma = \{V, E\}$ be a finite simplicial graph with vertex set V and edge set E. Thus, Γ does not contain two edges which have the same endpoints, nor does Γ contain any edge that is a loop. We assume for convenience that Γ does not contain any isolated vertices. To each vertex $v \in V$ we assign a group G_v with a fixed finite presentation. Let $e = \{u, v\} \in E$ and let $\widetilde{G}_e$ denote the free product $G_u * G_v$. To each such edge we assign a set $\mathbf{t}_e$ that consists of cyclically reduced elements of $\widetilde{G}_e$, where each element of $\mathbf{t}_e$ involves at least one term from each of G_u and G_v. Associated to this edge, we define a group

$$G_e = \widetilde{G}_e / << \mathbf{t}_e >> .$$

Let

$$\mathbf{t} = \bigcup_{e \in E} \mathbf{t}_e.$$

The *Pride group* associated with the above data is the group

$$G = G_V / << \mathbf{t} >>,$$

where

$$G_V = \mathop{\ast}_{v \in V} G_v.$$

We say that Γ is the *underlying graph* of G. The groups G_v $(v \in V)$ are called the *vertex groups* of G and the groups G_e $(e \in E)$ are called the *edge groups* of G. More generally, if Ω is any full subgraph of Γ which is generated by a set of vertices $V(\Omega) \subseteq V$ and has an edge set $E(\Omega) \subseteq E$, then the *subgraph group* is

$$G_\Omega = \mathop{\ast}_{v \in V(\Omega)} G_v / << \{\mathbf{t_e} : e \in E(\Omega)\} >> .$$

(Recall, a subgraph Ω of Γ is *full* if $E(\Omega)$ contains *all* edges $\{u, v\}$ of Γ where $u, v \in V(\Omega)$.)

Example 2.1.1. Let $\mathbf{s}$ be a finite set. A *Coxeter group* is a group C that has a presentation of the form

$$\langle \mathbf{s}\,; s^2, (st)^{m_{st}}\ (s, t \in \mathbf{s})\rangle,$$

where $m_{st} \in \mathbb{N} \cup \{\infty\}$ and $m_{st} = m_{ts}$. If $m_{st} = \infty$, then there is no relation $(st)^{m_{st}}$. We see that C is a Pride group with finite vertex groups (cyclic of order 2) and dihedral edge groups.

Example 2.1.2. Let $\mathbf{a}$ be finite set. An *Artin group* is a group A that has a presentation of the form

$$\langle \mathbf{a}\,; < a, b >^{\mu_{ab}} = < b, a >^{\mu_{ba}}\ \ (a, b \in \mathbf{a})\rangle,$$

where $\mu_{ab} \in \{2, 3, \dots, \infty\}$, $\mu_{ab} = \mu_{ba}$, and $< a, b >^{\mu_{ab}}$ denotes the alternating product of a and b of length μ_{ab}, beginning with a. If $\mu_{ab} = \infty$, then there is no relation involving a and b. We see that A is a Pride group with infinite cyclic vertex groups and edge groups given by the presentation

$$\langle a, b\,;\ \underbrace{aba\dots}_{\mu_{ab}\text{ terms}} = \underbrace{bab\dots}_{\mu_{ba}\text{ terms}}\ \rangle.$$

Example 2.1.3. A *generalized tetrahedron group* is a group T that has a presentation of the form

$$\langle x, y, z\,; x^l, y^m, z^n, W_1(x, y)^p, W_2(y, z)^q, W_3(z, x)^r\rangle,$$

where each $W_i(a,b)$ $(i = 1,2,3)$ is a cyclically reduced word involving both a and b and all powers are integers greater than 1. Observe that T is a Pride group with finite vertex groups (cyclic of orders l, m, n, respectively) and whose edge groups are *generalized triangle groups*. For example, the edge group G_e where $e = \{x, y\}$, is given by the presentation $\langle x, y\,; x^l, y^m, W_1(x,y)^p \rangle$.

For each $v \in V$ and each $e \in E$ there are natural homomorphisms $G_v \to G$ and $G_e \to G$. In general, these homomorphisms are not injective. For example, consider the group given by the presentation $\mathcal{P} = \langle x, y, z\,; yxy^{-1} = x^2, zyz^{-1} = y^2, xzx^{-1} = z^2 \rangle$. This presentation defines a Pride group with infinite cyclic vertex groups and Baumslag-Solitar edge groups. It is well-known, however, that $G(\mathcal{P})$ is the trivial group, so the natural homomorphisms $G_v \to G(\mathcal{P})$ $(v \in V)$ and $G_e \to G(\mathcal{P})$ $(e \in E)$ are clearly not injective.

When the natural homomorphisms *are* injective, i.e. when the vertex and edge groups embed in G, the idea is then to try to describe the structure of G in terms of its edge groups; the philosophy being that the edge groups should "control" the structure of G. More generally, one may consider Pride groups in which for each full subgraph Ω the natural homomorphism $G_\Omega \to G$ is injective. Such Pride groups are said to be *developable*.

Let G be a Pride group with underlying graph Γ. For each edge $e = \{u, v\}$, we define ψ_e to be the natural epimorphism of $\widetilde{G}_e$ onto G_e, and we denote by m_e the length of a shortest *non-identity* element of $\ker \psi_e$. Note that either $m_e = 1$ (in which case one of the natural homomorphisms $G_u \to G_e$, $G_v \to G_e$ is not injective), or m_e is even. If $\ker \psi_e$ is trivial, then $m_e := \infty$.

Definition 2.1.1. An edge group G_e (or more precisely a given presentation of G_e) has *property-*W_k if and only if $m_e > 2k$.

Pride proved [85, Theorem 3] that the vertex and edge groups embed in G if either of the following two conditions is satisfied:

(I) For each $e \in E$, G_e has property-W_2;

(II) The graph Γ is triangle-free and for each $e \in E$, G_e has property-W_1.

It was later remarked in [85] that the vertex and edge groups should embed in G under the weaker *asphericity condition*:

(III) For each $e \in E$, G_e has property-W_1 and for any triangle in Γ (with edges e_1, e_2, e_3, say)

$$\frac{1}{m_{e_1}} + \frac{1}{m_{e_2}} + \frac{1}{m_{e_3}} \leqslant \frac{1}{2}.$$

Pride groups which satisfy the asphericity condition are said to be *non-spherical*. Corson confirmed Pride's prediction by proving that the vertex and edge groups embed in a non-spherical Pride group. In fact, he proved [31, p. 562] that a non-spherical Pride group is developable.

We note that Conditions (I) - (III) can be stated in terms of the *Gersten-Stallings angle* $(G_e; G_u, G_v)$ $(e = \{u, v\})$ between the vertex groups G_u and G_v in the edge group G_e. For $m_e > 1$ one defines $(G_e; G_u, G_v) = \frac{2\pi}{m_e}$ and if $m_e = \infty$, the angle is 0. See [62, 96] for more details.

Example 2.1.4. ([35, Example 4.1]) Let C be a Coxeter group (see Example 2.1.1) and let G_e be the edge group given by the presentation

$$\langle s, t\,; s^2, t^2, (st)^{m_{st}} \rangle.$$

Then G_e always have property-W_1 and has property-W_2 if $m_{st} > 2$.

Example 2.1.5. ([83, Example 2]) Let A be an Artin group (see Example 2.1.2) and let G_e be the edge group generated by a and b. Then G_e has property-$W_{\mu_{ab}-1}$ [8, Lemma 6].

Example 2.1.6. Let T be a generalized tetrahedron group (see Example 2.1.3) and let G_e be the edge group given by the presentation

$$\langle x, y\,; x^l, y^m, W(x, y)^p \rangle,$$

where $W \equiv x^{\alpha_1} y^{\beta_1} \ldots x^{\alpha_k} y^{\beta_k}$ $(1 \leqslant \alpha_i < l, 1 \leqslant \beta_i < m \text{ for } i = 1, \ldots, k)$. Then $m_e = pk$ [56, Theorem 3.2], so G_e has property-W_1 if $k \geqslant 2$ and G_e has property-W_2 if $k > 2$.

Example 2.1.7. ([44, Example 1]) For an edge $e = \{u, v\}$, let D_e denote the Cartesian subgroup of $G_u * G_v$ (i.e. D_e is the kernel of the natural epimorphism $G_u * G_v \to G_u \times G_v$). Then G_e has property-W_1 if $\mathbf{t}_e \subseteq D_e$ and G_e has property-W_2 if $\mathbf{t}_e \subseteq D_e^{p(e)} D_e'$ for some prime $p(e)$. This follows from the fact that $D_e / D_e^{p(e)} D_e'$ is an elementary abelian $p(e)$-group with basis $[x, y] D_e^{p(e)} D_e'$, where $1 \neq x \in G_u$ and $1 \neq y \in G_v$.

We will also be interested in Pride groups which satisfy stronger versions of (I) or (II):

(H-I) For each $e \in E$, G_e has property-W_3;

(H-II) The graph Γ is triangle-free and for each $e \in E$, G_e has property-W_2.

One may view Conditions (H-I) and (H-II) as hyperbolic analogues of (I) and (II), respectively. We will see that among Pride groups (of a particular special class) that have soluble word and conjugacy problems, the solutions of these problems for groups which satisfy (H-I) or (H-II) are simpler than those which satisfy (I) or (II).

When studying an arbitrary Pride group one may also wish to impose certain conditions on the vertex groups.

Definition 2.1.2. Let $\mathbf{P}$ be a property of groups (finite, free, hyperbolic, etc.). A Pride group is said to be *vertex*-$\mathbf{P}$ if each of its vertex groups has property $\mathbf{P}$.

We will be particularly interested in *vertex-finite* Pride groups, of which Coxeter groups and generalized tetrahedron groups are examples.

2.2 A survey of known results

A special class of Pride groups in which each defining relator involves *exactly* two types of generators first appeared in [82]. These are precisely the *vertex-free* Pride groups. We note that Meier [69] uses the terminology *simple* Pride groups for vertex-free Pride groups.

Theorem 2.2.1. ([82, Theorem 4]) *If a vertex-free Pride group satisfies* (I) *or* (II)*, then it is developable.*

The (co)homology of a vertex-free Pride group G was calculated in [87]. The authors proved [87, Theorem 1] that the relation module of G decomposes into a direct sum of relation modules of the edge groups. Using this, they were then able to express the (co)homology of G in terms of the (co)homology of the edge groups. In particular, they proved the following.

Theorem 2.2.2. ([87, Theorem 2]) *Let G be a vertex-free Pride group which satisfies* (I) *or* (II)*. Then for any left G-module A and any right G-module B,*

$$H^n(G, A) \cong \prod_{e \in E} H^n(G_e, A)$$

and

$$H_n(G,B) \cong \prod_{e\in E} H_n(G_e,B)$$

for all integers $n \geqslant 3$.

A consequence of this result [87, Theorem 3] is that any finite subgroup of a vertex-free Pride group is contained in a conjugate of one of the edge groups.

The next result gives necessary and sufficient conditions for a vertex-free Pride group to be *diagrammatically aspherical*. Essentially, a presentation is diagrammatically aspherical (also called *combinatorially reducible* in [17]) if every non-empty spherical picture over the presentation contains (after performing bridge moves) a spherical subpicture which contains exactly two discs. Note, diagrammatic asphericity is weaker than diagrammatic reducibility. We refer to [28] for the precise definition of a diagrammatically aspherical presentation.

Theorem 2.2.3. ([83, Theorem 1]) *Let G be a vertex-free Pride group which satisfies* (I) *or* (II)*, and let $\mathcal{P}$ be a presentation of G. Then $\mathcal{P}$ is diagrammatically aspherical if and only if each presentation of an edge group is diagrammatically aspherical.*

We note that Edjvet [35] has introduced a larger class of presentations that generalize vertex-free Pride groups. By replacing "graph" with "set of finite subsets of a given set" and "edge group" by "face group" he is able to prove analogous results to Theorems 2.2.1 and 2.2.3. Also, Benson [14] has studied the relative hyperbolicity of such groups.

Let G be an arbitrary Pride group which satisfies (I) or (II), and let Γ be the underlying graph of G. In [85, Theorem 1], Pride obtained the short exact sequence

$$0 \to \bigoplus_{v\in V} (\mathbb{Z}G \otimes_{G_v} M_v)^{|S(v)|-1} \to \bigoplus_{e\in E} (\mathbb{Z}G \otimes_{G_e} M_e) \to M \to 0,$$

where M_v, M_e, M are the relation modules corresponding to the presentations of G_v, G_e, G, respectively ($v \in V, e \in E$), and where $S(v)$ is the set of edges of Γ that are incident with the vertex v. From this Pride obtained [85, Corollary 1.1] the short exact sequence

$$0 \to \bigoplus_{v\in V} (\mathbb{Z}G \otimes_{G_v} IG_v)^{|S(v)|-1} \to \bigoplus_{e\in E} (\mathbb{Z}G \otimes_{G_e} IG_e) \to IG \to 0, \tag{2.1}$$

where for any group H, IH is the augmentation ideal. Some (co)homological consequences can be derived from (2.1). In particular, if there exists $n \geqslant 1$ such that G_v has cohomological dimension less than or equal to n for all $v \in V$, then any finite subgroup of G is contained in a conjugate of some subgroup of G_e ($e \in E$) [85, Corollary 1.3].

Theorem 2.2.4. ([85, Theorem 2]) *If G satisfies* (I) *or* (II) *and if each G_e is combinatorially aspherical, then G is combinatorially aspherical.*

Fennessey [44] gave sufficient conditions for a Pride group to be SQ-universal. A group G is said to be *SQ-universal* if every countable group can be embedded in some quotient group of G. We note that SQ-universality is a measure of the *largeness* of a group. See [38] and [81] for a general discussion of largeness in group theory.

Theorem 2.2.5. ([44, Theorem 2.2]) *Let G be a Pride group which satisfies* (I) *or* (II), *and let* Γ *be the underlying graph of G. Assume that there are vertices u, v of Γ satisfying the following: not both G_u, G_v have order 2; $\{u, v\}$ is not an edge of Γ; and if condition* (II) *holds (but* (I) *does not), then adjoining $\{u, v\}$ to Γ does not create a triangle. Then G is SQ-universal.*

The final result of this section concerns a non-spherical Pride group. A class of groups $\mathcal{C}$ satisfies the *Tits alternative* if each group in $\mathcal{C}$ contains a non-abelian free subgroup or has a soluble subgroup of finite index.

Theorem 2.2.6. ([62, Theorem 1]) *Every non-spherical Pride group G based on a graph with at least four vertices contains a non-abelian free subgroup unless G has the presentation*

$$\langle x_1, x_2, x_3, x_4 \,;\, x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_2x_3)^2, (x_3x_4)^2, (x_4x_1)^2 \rangle,$$

in which case G is virtually abelian.

2.3 A survey of original results

In Chapter 4 we solve the word and conjugacy problems for a *vertex-finite* Pride group and in Chapter 5 we study the second homotopy module of the natural presentation of a *non-spherical* Pride group. In the following three subsections we present our main results. The statement of each result contains a reference to the section in which the proof of that result may be found.

2.3.1 The word problem

In our first result we obtain isoperimetric functions for a vertex-finite Pride group.

Theorem 1. (§4.1) *Let G be a vertex-finite Pride group with underlying graph $\Gamma = \{V, E\}$, and let $\delta_E = \max\{\delta_{G_e} : e \in E\}$.*

(1) *If G satisfies* (I) *or* (II)*, then $\delta_G(n) \preccurlyeq n^2\delta_E(n)$ for all $n \in \mathbb{N}$.*

(2) *If G satisfies* (H-I) *or* (H-II)*, then $\delta_G(n) \preccurlyeq n\delta_E(n)$ for all $n \in \mathbb{N}$.*

Thus, if G satisfies (I) or (II), then it has a quadratic isoperimetric inequality (modulo δ_E). Whereas, if G satisfies (H-I) or (H-II), it has a linear isoperimetric inequality (modulo δ_E). This fact adds weight to the claim that Conditions (H-I) and (H-II) are hyperbolic analogues of (I) and (II), respectively.

The following corollary of Theorem 1 provides a solution of the word problem for a vertex-finite Pride group which satisfies (I) or (II).

Corollary 1. (§4.1) *If each edge group has a soluble word problem, then G has a soluble word problem.*

We suspect that an isoperimetric function similar to the one that appears in Statement (1) of Theorem 1 could be obtained for a Pride group in which each vertex group is not necessarily finite. The following theorem provides examples of such groups for which this can be done. Let G be a vertex-finite Pride group which satisfies (II), and let $\Gamma = \{V, E\}$ be the underlying graph of G.

Theorem 2. (§4.1.1) *Given G one can construct a Pride group $Q(G)$ which satisfies* (II) *and which contains an infinite cyclic vertex group. The remaining vertex groups are finite. Moreover, the first order Dehn function δ_Q of $Q(G)$ satisfies*

$$n^2 \preccurlyeq \delta_Q(n) \preccurlyeq n^3\delta_E(n).$$

Remark 1. The Pride group $Q(G)$ is a trivial HNN-extension of G.

Under suitable conditions on the edge groups, we obtain a *lower* bound for the first order Dehn function of a *vertex-free* Pride group.

Theorem 3. (§4.1.1) *Let G be a vertex-free Pride group which satisfies* (I) *or* (II)*, and let $\Gamma = \{V, E\}$ be the underlying graph of G. Suppose each edge group is diagrammatically reducible. Then for all $n \in \mathbb{N}$,*

$$\delta_G(n) \succcurlyeq \delta_E(n)$$

where $\delta_E = \max\{\delta_{G_e} : e \in E\}$.

2.3.2 The conjugacy problem

Let G be a vertex-finite Pride group which satisfies (I) or (II), and let $\Gamma = \{V, E\}$ be the underlying graph of G. Also, let $\mathcal{P}_s = \langle \mathbf{x}\,; \mathbf{r}\rangle$ be the standard presentation of G (see the introduction of Chapter 3). In Theorem 4 we obtain necessary and sufficient conditions for two cyclically injective words W and Z on $\mathbf{x}^{\pm 1}$ to represent conjugate elements of G. In addition to (I) or (II), we require that G satisfies Conditions (1) - (6), which are stated at the end of this section. We are then able to prove Lemmas 4.2.2 - 4.2.7 (see §4.2) which reduce the problem of deciding whether or not two words on $\mathbf{x}^{\pm 1}$ represent conjugate elements of G to a "nice" set of words. In particular, we may assume that W and Z satisfy Conditions (i) - (iv) (these are also stated at the end of this section).

For any two words U, V on $\mathbf{x}^{\pm 1}$, we will write $U \overset{n}{\sim} V$ if there exists a word Y on $\mathbf{x}^{\pm 1}$ of length at most n such that $UYV^{-1}Y^{-1}$ represents the identity element of G. If $|U| = n$, then $U \overset{n}{\sim} U'$ for any cyclic permutation U' of U.

Theorem 4. (§4.2) *Let G satisfy Conditions* (1) - (6) *and assume that W and Z satisfy Conditions* (i) - (iv)*. Let $n = |W| + |Z|$. Then W, Z represent conjugate elements of G if and only if there exist words $W_1, \dots, W_l$, $Z_1, \dots, Z_l$ on $\mathbf{x}^{\pm 1}$ such that*

$$W \overset{n}{\sim} W' \overset{10n}{\sim} W_1 \overset{10n}{\sim} \dots \overset{10n}{\sim} W_l \overset{20n}{\sim} Z_l \overset{10n}{\sim} \dots \overset{10n}{\sim} Z_1 \overset{10n}{\sim} Z' \overset{n}{\sim} Z \tag{2.2}$$

where $|W_i|, |Z_i| \leqslant 10qn^2$ $(i = 1, \dots, l)$ for $q = 3$ or 4 (depending if (I) *or* (II) *holds, respectively), and where W', Z' are cyclic permutations of W and Z, respectively.*

Given Theorem 4, we can write down an algorithm that solves the conjugacy problem for G. First note that for any two words $U, V \in (\mathbf{x}^{\pm 1})^*$ and any $n \in \mathbb{N}$, the relation $U \overset{n}{\sim} V$ is decidable (G is finitely generated and has a soluble word problem). Also, given any word U on $\mathbf{x}^{\pm 1}$ we can decide whether or not U is cyclically injective. Let W, Z be cyclically injective words on $\mathbf{x}^{\pm 1}$ which satisfy Conditions (i) - (iv), and where $|W| + |Z| = n$. Write $W \sim\sim Z$ if there exist $W_1, \dots, W_l$

and $Z_1, \ldots, Z_l$ such that (2.2) holds. Let $\mathcal{C} = \{U : U \in (\mathbf{x}^{\pm 1})^* \text{ and } |U| \leqslant 10qn^2\}$, where $q = 3$ or 4 (depending if G satisfies (I) or (II), respectively). Theorem 4 states that W, Z represent conjugate elements of G if and only if $W \sim\sim Z$ with all the intermediate W_i's and Z_j's in $\mathcal{C}$. Since G is finitely generated, $\mathcal{C}$ is a finite set. All words U on $\mathbf{x}^{\pm 1}$ such that $U \in \mathcal{C}$ and $W \sim\sim U$ with all the intermediate $W_i, Z_j \in \mathcal{C}$, can be enumerated in a finite number of steps. Then W and Z represent conjugate elements of G if and only if Z appears in the list of the U's.

Remark 2. The above algorithm is essentially Schupp's algorithm [92] which solves the conjugacy problem for presentations satisfying the small cancellation conditions $C(6)$, $C(4)$ and $T(4)$, or $C(3)$ and $T(6)$.

Now suppose G satisfies (H-I) or (H-II) and Conditions (1) - (6). Let W, Z be cyclically injective words on $\mathbf{x}^{\pm 1}$. As above, we may assume that W and Z satisfy (i) - (iv). In this case it is convenient to further assume that:

(v) W and Z are cyclically $\widehat{\mathbf{r}}$-reduced words on $\mathbf{x}^{\pm 1}$.

We may assume this by Lemma 4.2.9. (See Definition 4.2.1 for the meaning of a cyclically $\widehat{\mathbf{r}}$-reduced word.)

Theorem 5. (§4.2.1) *Let G satisfy Conditions* (1) - (6) *and assume that W and Z satisfy Conditions* (i) - (v). *Let $n = |W| + |Z|$. Then W, Z represent conjugate elements of G if and only if one of the following two conditions holds:*

(C1) *There exist cyclic permutations W' and Z' of W and Z, respectively, such that*

$$W \overset{n}{\sim} W' \overset{3}{\sim} Z' \overset{n}{\sim} Z.$$

(C2) *There exist cyclic permutations W' and Z' of W and Z, respectively, such that $\overline{W'} = \overline{Z'}$.*

From Theorem 5 we obtain a particularly simple algorithm that solves the conjugacy problem for G. Let W, Z be cyclically injective words on $\mathbf{x}^{\pm 1}$ which satisfy Conditions (i) - (v), and let $|W| + |Z| = n$. Write down all cyclic permutations W', Z' of W and Z, respectively, and write down all words on $\mathbf{x}^{\pm 1}$ of length at most 3. Since G is finitely generated, there are finitely many such words. Now use the solution of the word problem for G to test whether Condition (C1) or (C2) holds. If one of (C1) or (C2) holds, then W and Z represent conjugate elements of G. Otherwise, they do not.

Let us now state Conditions (1) - (6). In what follows

$$-G_E := G - \bigcup_{e \in E} G_e.$$

Also, for each $e \in E$, $\mathbf{x}_e$ denotes the generators of a particular presentation of the edge group G_e.

(1) For each $e \in E$, G_e has a soluble conjugacy problem.

(2) For each $e \in E$, G_e is malnormal in G.

(3) For each pair of distinct edges $e, f \in E$, we can decide, given a word W on $\mathbf{x}_e^{\pm 1}$ and a word Z on $\mathbf{x}_f^{\pm 1}$, whether or not W, Z represent conjugate elements of G.

(4) For each $e \in E$, the generalized word problem (relative to $\mathbf{x}_e^{\pm 1}$) is soluble.

(5) For each $e \in E$, we can decide, given a word U on $\mathbf{x}_e^{\pm 1}$ and a word W on $\mathbf{x}^{\pm 1}$ that represents an element of $-G_E$, whether or not U, W represent conjugate elements of G.

(6) For each $e \in E$, we can decide, given words W, Z on $\mathbf{x}^{\pm 1}$ that represent elements of $-G_E$, whether or not there exists a word U on $\mathbf{x}_e^{\pm 1}$ such that W, U and U, Z represent conjugate elements of G.

A subgroup H of a group K is said to be *malnormal* if for each $k \in K$,

$$H \cap kHk^{-1} \neq \{1\} \Leftrightarrow k \in H.$$

Thus, if $h_1, h_2 \in H/\{1\}$ are conjugate in H, then any conjugating element for h_1 and h_2 must also be in H. In this case we say that h_1 *and* h_2 *are conjugate in* K *if and only if they are conjugate in* H.

Conditions (i) - (iv) are as follows:

(i) W and Z both represent non-identity elements of G;

(ii) W and Z do not represent conjugate elements of G_V;

(iii) W and Z are words on $\mathbf{x}^{\pm 1}$ that represent (distinct) elements of $-G_E$;

(iv) For each $e \in E$, there does not exist a word U on $\mathbf{x}_e^{\pm 1}$ such that W, U and U, Z represent conjugate elements of G.

2.3.3 The second homotopy module

Let G be a non-spherical Pride group with underlying graph $\Gamma = \{V, E\}$, and let $\mathcal{P} = \langle \mathbf{x}\,; \mathbf{r} \rangle$ be the natural presentation for G (see the introduction of Chapter 5).

Theorem 6. (§5.3) *For each $e \in E$, let X_e be a generating set for $\pi_2(\mathcal{P}_e)$. Then*

$$X = \bigcup_{e \in E} X_e$$

is a generating set for $\pi_2(\mathcal{P})$. In particular, if each G_e is of type F_3, then G is of type F_3.

The following corollary of Theorem 6 generalizes Pride's result [85, Theorem 2] concerning the combinatorial asphericity of Pride groups which satisfy Conditions (I) or (II).

Corollary 2. (§5.3) *Let G be a non-spherical Pride group. If each G_e is combinatorially aspherical, then G is combinatorially aspherical. In particular, if each G_e is aspherical, then G is aspherical.*

If we assume each vertex group is free, then we obtain necessary and sufficient conditions for G to be diagrammatically reducible.

Theorem 7. (§5.2) *Let G be a non-spherical vertex-free Pride group with underlying graph $\Gamma = \{V, E\}$. Then G is diagrammatically reducible if and only if each edge group is diagrammatically reducible.*

Remark 3. One may replace "reducible" with "aspherical" in the statement of Theorem 7 to obtain a generalization of Pride's result [83, Theorem 1] concerning the diagrammatic asphericity of vertex-free Pride groups.

The proof of Theorem 6 provides us with a method for calculating an upper bound for the second order Dehn function $\delta_G^{(2)}$ of a non-spherical Pride group G. If each vertex group is free, then we have the following.

Theorem 8. (§5.4) *Let G be a non-spherical vertex-free Pride group with underlying graph $\Gamma = \{V, E\}$. Assume for each $e \in E$ that G_e is of type F_3 and let $\delta_E^{(2)} = \max\{\bar{\delta}_{G_e}^{(2)} : e \in E\}$. Then*

$$\delta_G^{(2)}(n) \preccurlyeq \delta_E^{(2)}(n)$$

for all $n \in \mathbb{N}$.

The general case is more complicated. In order to obtain an upper bound for $\delta_G^{(2)}$ we need to define a new *area distortion* function (see Definition 5.4.1 in §5.4). This function captures the key information which one must obtain to calculate an upper bound for $\delta_G^{(2)}$; however, it is not well-defined in general. With this is mind, we prove the following.

Proposition 1. (§5.4) *Let G be a non-spherical Pride group with underlying graph $\Gamma = \{V, E\}$, and let $\mathcal{P}$ be the natural presentation of G. For each $e = \{u, v\} \in E$, let $\widetilde{\mathcal{P}}_e$ be a presentation of $\widetilde{G}_e$ and let Λ_e be the area distortion function of $\widetilde{\mathcal{P}}_e$ relative to $\mathcal{P}_e$. Assume each G_e is of type F_3 and set $\Lambda = \max\{\bar{\Lambda}_e : e \in E\}$. Then for all $n \in \mathbb{N}$,*

$$\delta_{\mathcal{P}}^{(2)}(n) \leqslant \delta_E^{(2)}(n + 2n\Lambda^n(n^2 + n)) + \delta_V^{(2)}(n)$$

where $\delta_V^{(2)} = \max\{\bar{\delta}_{\mathcal{P}_v}^{(2)} : v \in V\}$, $\delta_E^{(2)} = \max\{\bar{\delta}_{\mathcal{P}_e}^{(2)} : e \in E\}$ and Λ^n is the n-th power of Λ.

Chapter 3

Diagrams over the standard presentation of a vertex-finite Pride group

Let G be a vertex-finite Pride group with underlying graph $\Gamma = \{V, E\}$. We now fix a presentation of G that will be used throughout this chapter. For each $v \in V$, let $\mathbf{x}_v$ be a set that is in one-to-one correspondence with the elements of G_v and let

$$\mathbf{r}_v = \{x_1 x_2 x_3^{-1} : x_1, x_2, x_3 \in \mathbf{x}_v\},$$

where x_i $(i = 1, 2, 3)$ corresponds to the group element $g_i \in G_v$ and where g_3 is equal to the product $g_1 g_2$ in G_v. Then $\mathcal{P}_v = \langle \mathbf{x}_v \,;\mathbf{r}_v \rangle$ is a finite presentation of G_v where $\mathbf{r}_v$ is the multiplication table of G_v. For each $e = \{u, v\} \in E$, let $\mathbf{x}_e = \mathbf{x}_u \cup \mathbf{x}_v$ and let

$$\mathbf{r}_e = \mathbf{r}_u \cup \mathbf{r}_v \cup \mathbf{r}'_e,$$

where $\mathbf{r}'_e$ is a set of cyclically reduced words on $\mathbf{x}_e^{\pm 1}$ that represent the elements of $\mathbf{t}_e$. We denote the union of the $\mathbf{r}'_e$'s $(e \in E)$ by $\mathbf{r}'$. Then $\mathcal{P}_e = \langle \mathbf{x}_e \,;\mathbf{r}_e \rangle$ is a finite presentation of G_e. It follows that

$$\mathcal{P}_s = \langle \mathbf{x} \,;\mathbf{r} \rangle \tag{3.1}$$

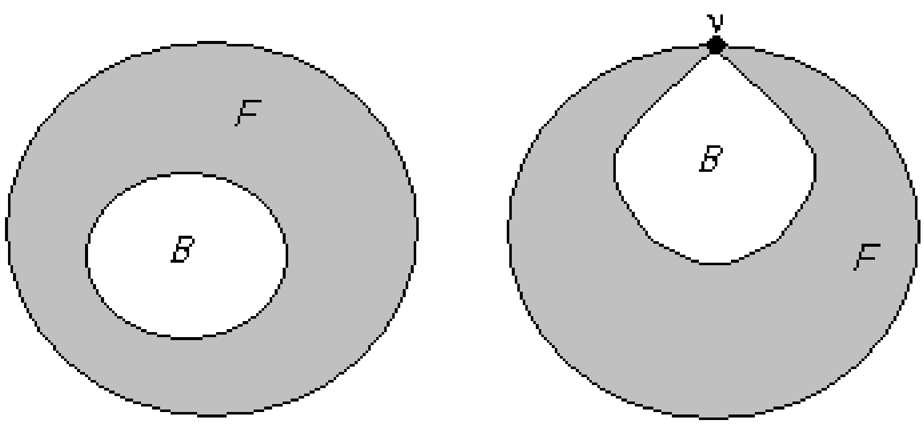
is a finite presentation of G where

$$\mathbf{x} = \bigcup_{v \in V} \mathbf{x}_v \text{ and } \mathbf{r} = \bigcup_{e \in E} \mathbf{r}_e.$$

We call $\mathcal{P}_s$ the *standard presentation* of G.

3.1 Simply-connected r-diagrams and federations

Let $\mathcal{S}$ be a simply-connected $\mathbf{r}$-diagram with labelling function ϕ, and let $E(\mathcal{S})$ denote the set of regions of $\mathcal{S}$ whose labels are elements of $(\mathbf{r}')^s$ (recall Definition 1.7.1). If $\Delta \in E(\mathcal{S})$ is such that $\phi(\partial\Delta) \in (\mathbf{r}'_e)^s$ for some $e \in E$, then we define $\Sigma(\Delta) = e$. For a simple path $\alpha \in \mathcal{S}$, let $t(\alpha)$ denote the set of all $v \in V$ for which some element of $\mathbf{x}_v$ occurs in the label of α. Assume $E(\mathcal{S}) \neq \emptyset$. Choose some region $\Delta \in E(\mathcal{S})$ and consider the collection of regions Δ' of $\mathcal{S}$ for which the following holds: *there exist regions $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ of $\mathcal{S}$ with $t(\partial\Delta_i) \subseteq \Sigma(\Delta)$ for $i = 1, \ldots, n$ and where Δ_i, Δ_{i+1} have an edge in common for $i = 0, \ldots, n-1$. Furthermore, $\{\Delta_0, \ldots, \Delta_n\}$ is a maximal set of regions that have this property.* Following [85], we call this collection of regions, and the subdiagram they determine, a *federation* and we denote it by $\mathcal{F}$. We define $\Sigma(\mathcal{F}) = \Sigma(\Delta)$. If $\Sigma(\mathcal{F}) = e$ for some $e \in E$, then $\mathcal{F}$ is an $\mathbf{r}_e$-diagram. If $\mathcal{F}$ is a simply-connected $\mathbf{r}_e$-diagram, then by Theorem 1.7.1 any label of $\mathcal{F}$ represents the identity element of G_e. Equivalently, $\phi(\partial\mathcal{F})$ represents an element of $\ker\psi_e : \widetilde{G}_e \to G_e$. If $\mathcal{F}$ is not simply-connected, then $\mathcal{S} - \mathcal{F}$ contains at least one non-empty bounded simply-connected $\mathbf{r}$-subdiagram $\mathcal{B}$ which contains at least one region Δ' with the property that $t(\partial\Delta') = f$ for some $f \in E/\{e\}$. Moreover, either $\partial\mathcal{F} \cap \partial\mathcal{B} = \emptyset$ or $\partial\mathcal{F} \cap \partial\mathcal{B} = \nu$ for some vertex ν of $\partial\mathcal{F}$ as shown in Fig. 3.1 below.



(a) $\partial\mathcal{F} \cap \partial\mathcal{B} = \emptyset$ (b) $\partial\mathcal{F} \cap \partial\mathcal{B} \neq \emptyset$

Figure 3.1: Non-simply-connected federations.

Let $\mathcal{F}_1$ be a federation of $\mathcal{S}$. If $\mathcal{F}_1 \neq \mathcal{S}$, then construct a federation $\mathcal{F}_2$ of $\mathcal{S} - \mathcal{F}_1$. If $\mathcal{F}_2 \neq \mathcal{S} - \mathcal{F}_1$, then construct a federation $\mathcal{F}_3$ of $\mathcal{S} - (\mathcal{F}_1 \cup \mathcal{F}_2)$, and so on. Eventually, we will end up with a collection of subdiagrams $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of $\mathcal{S}$ which cover $\mathcal{S}$ and satisfy the property that $\mathcal{F}_{i+1}$ is a federation of

$$\mathcal{S} - (\bigcup_{j=1}^{i} \mathcal{F}_j).$$

We call the collection of subdiagrams $\mathcal{F}_{\mathcal{S}} = \{\mathcal{F}_i\}_{i=1}^{n}$ a *federal subdivision* of $\mathcal{S}$. Although it has no bearing on what follows, we note that a federal subdivision is not necessarily unique.

For each edge $e \in E$, let $\widehat{\mathbf{r}}_e$ denote the set of all words on $\mathbf{x}_e^{\pm 1}$ that represent a *non-identity* element of $\ker \psi_e$ and let $\widehat{\mathbf{r}}$ be the union of the $\widehat{\mathbf{r}}_e$'s. Note that $\widehat{\mathbf{r}}$ is symmetrized. Let $\mathcal{F}_{\mathcal{S}} = \{\mathcal{F}_i\}_{i=1}^{n}$ be a federal subdivision of a simply-connected $\mathbf{r}$-diagram $\mathcal{S}$ that satisfies the following two conditions:

(i) Each $\mathcal{F}_i \in \mathcal{F}_{\mathcal{S}}$ is simply-connected ($i = 1, \ldots, n$);

(ii) $\phi(\partial \mathcal{F}_i) \in \widehat{\mathbf{r}}$ for $i = 1, \ldots, n$.

By deleting the interior vertices and interior edges of each federation $\mathcal{F}_i \in \mathcal{F}_s$ ($i = 1, \ldots, n$) we obtain an $\widehat{\mathbf{r}}$-diagram which we denote by $\mathcal{D}$. In the following subsection we describe two modifications which we make to $\mathcal{D}$. The fact that each vertex group is finite will play an important role in one of these modifications.

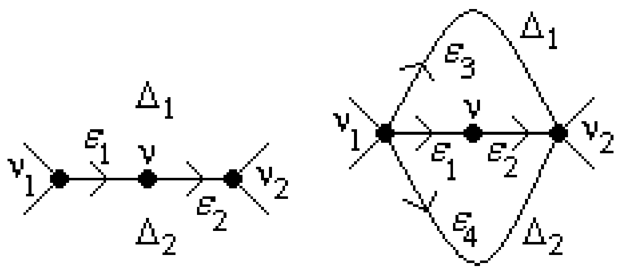
3.1.1 The importance of being finite

Suppose $\mathcal{D}$ contains an interior vertex ν of degree 2 such that the edges $\varepsilon_1, \varepsilon_2$ which are incident with ν have labels $x_1, x_2 \in \mathbf{x}_u$, respectively, for some $u \in V$. We wish to remove such a vertex. This is a standard modification to make and one normally proceeds as follows. Delete $\nu, \varepsilon_1, \varepsilon_2$ and in their place add a single edge ε which has the same initial (respectively, terminal) vertex as ε_1 (respectively, ε_2). One then assigns the label $x_1 x_2$ to this new edge. Thus interior edges become labelled by *words* on $\mathbf{x}_u^{\pm 1}$ ($u \in V$). We will proceed in a similar way; however, the label we assign to ε will be an element of $\mathbf{x}_u$. Thus interior edges will remain labelled by *elements* of $\mathbf{x}_u$ ($u \in V$). *We are able to do this by virtue of the fact that each vertex group is finite.*

Let ν_1 and ν_2 be the endpoints of ε_1 and ε_2, respectively, which are not equal to ν. There are two cases to consider: the case when $\nu_1 = \nu_2$ and the case when $\nu_1 \neq \nu_2$.

Let us first consider the case when $\nu_1 \neq \nu_2$ (see Fig. 3.2 (a)). Let α be the path $\varepsilon_1 \varepsilon_2$. On either side of α, draw edges $\varepsilon_3, \varepsilon_4$ which are "close" to α and which have endpoints ν_1, ν_2 (see Fig. 3.2 (b)).

We assign the same orientation to ε_3 and ε_4. The presentation $\mathcal{P}_u$ contains, as its defining relators, the multiplication table of G_u. We may use, therefore, the defining relator $x_1x_2x'^{-1}$ to assign the label x' to ε_3 and ε_4. Thus, ε_3 and ε_4 are both labelled by the same *element* of $\mathbf{x}_u$.



(a) $\nu_1 \neq \nu_2$ (b) Edges ε_3 and ε_4

Figure 3.2: A vertex of degree 2 with $\nu_1 \neq \nu_2$.

Next, delete $\nu, \varepsilon_1, \varepsilon_2$ and identify ε_3 with ε_4 (see Fig. 3.3). The labels of Δ_1 and Δ_2 have changed; however, $\phi(\partial\Delta_1)$ and $\phi(\partial\Delta_2)$ are still elements of $\widehat{\mathbf{r}}$. We proceed to remove other such vertices of degree 2 in this way.

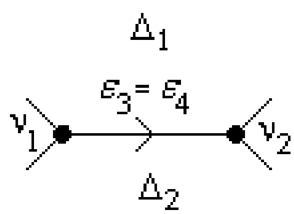
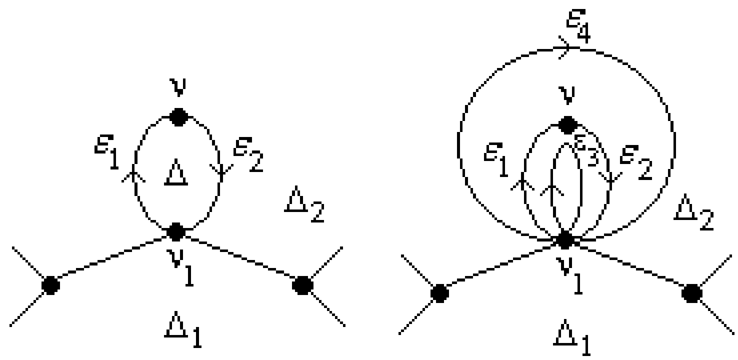


Figure 3.3: Identifying ε_3 with ε_4 in the case when $\nu_1 \neq \nu_2$.

Now suppose $\nu_1 = \nu_2$. Let α denote the (closed) path $\varepsilon_1\varepsilon_2$ and let Δ be the component of $\mathcal{D}$ bounded by α (see Fig. 3.4 (a)). In the interior of Δ draw a loop ε_3 at ν_1. Also, draw a loop ε_4 at ν_1 which lies in the region Δ_2 and which is "close" to the path $\varepsilon_1\varepsilon_2$ (see Fig. 3.4 (b)). We assign the same orientation to ε_3 and ε_4. As in the previous case, we label ε_3 and ε_4 by an element $x' \in \mathbf{x}_u$ which corresponds to the product of the group elements which correspond to x_1 and x_2, respectively. Next, delete $\nu, \varepsilon_1, \varepsilon_2$ and identity ε_3 with ε_4 (see Fig. 3.5). The label of Δ_2 has changed; however, $\phi(\partial\Delta_2)$ is still an element of $\widehat{\mathbf{r}}$. In this case we can delete the loop ε_3 $(= \varepsilon_4)$ as the resulting label of Δ_2 will still be an element of $\widehat{\mathbf{r}}$.

Remark 3.1.1. After removing all such interior vertices of degree 2, we observe that each interior



(a) $\nu_1 = \nu_2$ (b) Edges ε_3 and ε_4

Figure 3.4: A vertex of degree 2 with $\nu_1 = \nu_2$.

edge is labelled by an element of $\mathbf{x}_u$ $(u \in V)$. This fact will be crucial to our work. See in particular Lemma 3.1.1.

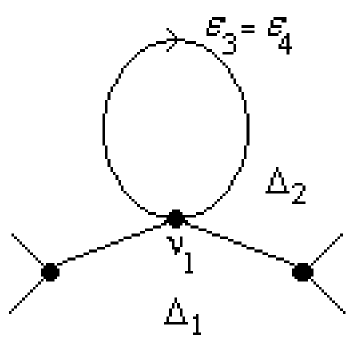


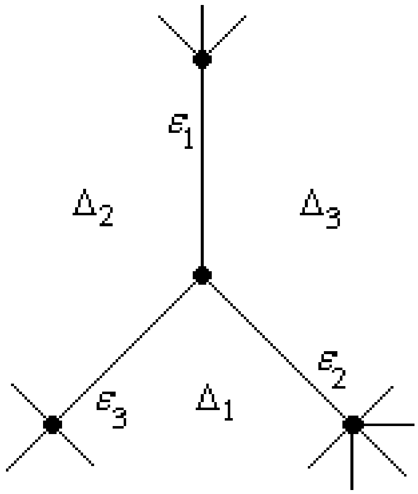
Figure 3.5: Identifying ε_3 with ε_4 in the case when $\nu_1 = \nu_2$.

The second modification removes particular interior vertices of degree 3. Suppose $\mathcal{D}$ contains an interior vertex of degree 3 (see Fig. 3.6(a)). Then $t(\partial\Delta_1) \cap t(\partial\Delta_2) \cap t(\partial\Delta_3)$ is either empty, or is a singleton set, i.e. equal to $\{u\}$ for some $u \in V$. Using the same technique as the one described in [82, p. 254], we will modify $\mathcal{D}$ so that the latter possibility does not occur.

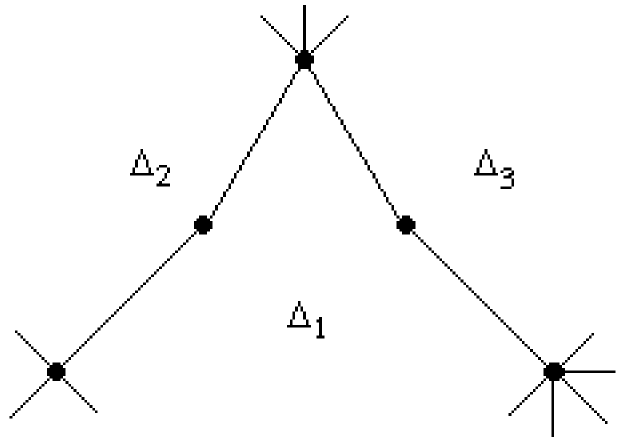
Suppose that

$$t(\partial\Delta_1) \cap t(\partial\Delta_2) \cap t(\partial\Delta_3) = \{u\}$$

for some $u \in V$. Then each of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is labelled by an element of $\mathbf{x}_u$. Cut along the edge ε_1 as in Fig. 3.6(b). Note that the boundary of Δ_1 has changed; however, $\phi(\partial\Delta_1)$ is still an element of $\widehat{\mathbf{r}}$. We have created new vertices of degree 2 which must be removed. After removing such vertices we proceed to examine other interior vertices of degree 3, and repeat the procedure if necessary.



(a) A vertex of degree 3



(b) The result of cutting along ε_1

Figure 3.6: Removing a vertex of degree 3.

3.1.2 The derived diagram

After making the modifications to $\mathcal{D}$ described in §3.1.1 we obtain an $\widehat{\mathbf{r}}$-diagram which we denote by $\widehat{\mathcal{S}}$. We say that $\widehat{\mathcal{S}}$ is the *derived diagram of* $\mathcal{S}$ *corresponding to* $\mathcal{F}_{\mathcal{S}}$. If there is no confusion over which diagram or federal subdivision we are working with, then we shall call $\widehat{\mathcal{S}}$ *the* derived diagram. Note that by construction of $\widehat{\mathcal{S}}$, $\partial\widehat{\mathcal{S}} = \partial\mathcal{S}$. Therefore, if $\mathcal{S}$ is a simply-connected $\mathbf{r}$-diagram for $W \in (\mathbf{x}^{\pm 1})^*$, then some boundary cycle of $\widehat{\mathcal{S}}$ has label W. The following lemma highlights two key properties of the derived diagram.

Lemma 3.1.1. *Let $\widehat{\mathcal{S}}$ be the derived diagram constructed above.*

(1) *If Δ_1, Δ_2 are distinct regions of $\widehat{\mathcal{S}}$ that have an edge in common, then $t(\partial\Delta_1) \cap t(\partial\Delta_2) = \{u\}$ for some $u \in V$.*

(2) *If α is a non-empty simple path in $\widehat{\mathcal{S}}$ with label W, then $|W|$ is equal to the number of edges contained in α. In particular, if Δ is a region of $\widehat{\mathcal{S}}$, then $|\phi(\partial\Delta)| = d(\Delta)$.*

Proof. Let Δ_1, Δ_2 be two distinct regions of $\widehat{\mathcal{S}}$ that have an edge in common, and suppose that $t(\partial\Delta_1) \cap t(\partial\Delta_2) = \{u, v\}$ for some $u, v \in V$. Let $\mathcal{F}_1, \mathcal{F}_2$ be the federations that correspond to Δ_1

and Δ_2, respectively. Then $\Sigma(\mathcal{F}_1) = \Sigma(\mathcal{F}_2) = \{u, v\}$ and $\mathcal{F}_1, \mathcal{F}_2$ have an edge in common. However, this contradicts the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ are federations (they should have been incorporated into one larger federation).

Property (2) follows easily from Remark 3.1.1. Let α be a non-empty path in $\widehat{\mathcal{S}}$ with label W. Since each edge of $\widehat{\mathcal{S}}$ is labelled by an element of $\mathbf{x}_u$ (for some $u \in V$) we conclude that $|W|$ is equal to the number of edges contained in α. $\square$

The following lemmas give $\widehat{\mathcal{S}}$ the structure of a $[3, 2k+2]$-diagram or, if Γ is triangle-free, the structure of a $[4, 2k+2]$-diagram.

Lemma 3.1.2. *If ν is an interior vertex of $\widehat{\mathcal{S}}$, then $d(\nu) \geqslant 3$, and if Γ is triangle-free, then each interior vertex of $\widehat{\mathcal{S}}$ has degree at least 4.*

Proof. It is clear that $d(\nu) > 1$. Suppose $d(\nu) = 2$ and let $\varepsilon_1, \varepsilon_2$ be the edges incident with ν. By construction of $\widehat{\mathcal{S}}$, $\phi(\varepsilon_1) \in \mathbf{x}_u$ and $\phi(\varepsilon_2) \in \mathbf{x}_v$ for some distinct $u, v \in V$. Since ν is an interior vertex there exist distinct regions Δ_1, Δ_2 of $\widehat{\mathcal{S}}$ such that $\varepsilon_1\varepsilon_2 \in \partial\Delta_1 \cap \partial\Delta_2$. Therefore, Δ_1 and Δ_2 have an edge in common and $t(\partial\Delta_1) \cap t(\partial\Delta_2) = \{u, v\}$, contradicting Property (1) of Lemma 3.1.1.

Now assume Γ is triangle-free and that $\widehat{\mathcal{S}}$ contains an interior vertex of degree 3 (see Fig. 3.6(a)). By Property (1) of Lemma 3.1.1,

$$|t(\partial\Delta_1) \cap t(\partial\Delta_2)| = |t(\partial\Delta_2) \cap t(\partial\Delta_3)| = |t(\partial\Delta_3) \cap t(\partial\Delta_1)| = 1$$

and by construction of $\widehat{\mathcal{S}}$, $t(\partial\Delta_1) \cap t(\partial\Delta_2) \cap t(\partial\Delta_3) = \emptyset$. Thus there exist distinct vertices $u, v, w \in V$ with $t(\partial\Delta_1) = \{u, v\}$, $t(\partial\Delta_2) = \{v, w\}$ and $t(\partial\Delta_3) = \{w, u\}$. Therefore, we have a triangle in Γ. $\square$

Lemma 3.1.3. *If each edge group has property-W_k, then $d(\Delta) \geqslant 2k+2$ for each region Δ of $\widehat{\mathcal{S}}$.*

Proof. Let Δ be a region of $\widehat{\mathcal{S}}$ with $t(\partial\Delta) = e$ for some $e \in E$. Since $\widehat{\mathcal{S}}$ is an $\widehat{\mathbf{r}}$-diagram, we have $\phi(\partial\Delta) \in \widehat{\mathbf{r}}_e$, i.e. $\phi(\partial\Delta)$ represents a non-identity element of $\ker\psi_e$. Since G_e has property-W_k, $|\phi(\partial\Delta)| \geqslant 2k+2$. Thus, $d(\Delta) \geqslant 2k+2$. $\square$

Let $\mathcal{S}$ be a simply-connected $\mathbf{r}$-diagram and let $\mathcal{F}_{\mathcal{S}}$ be a federal subdivision of $\mathcal{S}$ which satisfies Conditions (i) and (ii) of §3.1.

Lemma 3.1.4. *Suppose $\mathcal{F}_{\mathcal{S}}$ contains at least two federations. If* (I) *or* (II) *holds, then $\mathcal{S}$ contains a federation $\mathcal{F}$ such that $\Sigma(\mathcal{F}) \subseteq t(\partial\mathcal{S})$ and $\partial\mathcal{F} \cap \partial\mathcal{S}$ is a consecutive part of $\partial\mathcal{S}$.*

Proof. Let $\widehat{\mathcal{S}}$ be the derived diagram of $\mathcal{S}$ corresponding to $\mathcal{F}_{\mathcal{S}}$. If (I) holds, then $\widehat{\mathcal{S}}$ is a $[3,6]$-diagram and it follows from Theorem 1.6.2 that $\widehat{\mathcal{S}}$ contains a simple boundary region Δ with $i(\Delta) \leqslant 3$. Let $t(\partial\Delta) = \{u, v\}$ for some $u, v \in V$, so that the label of each interior edge of $\partial\Delta$ is an element of $\mathbf{x}_u$ or $\mathbf{x}_v$. Since $i(\Delta) \leqslant 3$, the W_2 condition implies that the label of $\alpha = \partial\Delta \cap \partial\widehat{\mathcal{S}}$ is a word which contains letters from both $\mathbf{x}_u$ and $\mathbf{x}_v$. Therefore, $t(\partial\Delta) = t(\alpha) \subseteq t(\partial\widehat{\mathcal{S}})$. Now, Δ arose from some federation $\mathcal{F}$ in $\mathcal{S}$. Since $\Sigma(\mathcal{F}) = t(\partial\Delta)$, we have

$$\Sigma(\mathcal{F}) = t(\partial\Delta) \subseteq t(\partial\widehat{\mathcal{S}}) = t(\partial\mathcal{S}).$$

Furthermore, since Δ is a simple boundary region of $\widehat{\mathcal{S}}$, $\partial\mathcal{F} \cap \partial\mathcal{S}$ is a consecutive part of $\partial\mathcal{S}$.

Now assume (II) holds. In this case $\widehat{\mathcal{S}}$ is a $[4,4]$-diagram and it follows from Theorem 1.6.2 that $\widehat{\mathcal{S}}$ contains a simple boundary region Δ with $i(\Delta) \leqslant 2$. In this case, if $t(\partial\Delta) = \{u, v\}$ for some $u, v \in V$, the W_1 condition implies that the label of $\alpha = \partial\Delta \cap \partial\widehat{\mathcal{S}}$ is a word which contains letters from both $\mathbf{x}_u$ and $\mathbf{x}_v$. Therefore, $t(\partial\Delta) = t(\alpha) \subseteq t(\partial\widehat{\mathcal{S}})$ and so $\mathcal{S}$ must contain a federation satisfying the desired conditions. □

Recall, $E(\mathcal{S})$ is the set of regions of a simply-connected $\mathbf{r}$-diagram $\mathcal{S}$ whose labels are elements of $(\mathbf{r}')^s$. We say that $\mathcal{S}$ is *minimal with respect to* $|E(\mathcal{S})|$ if the following is true:

$$|E(\mathcal{S})| = \min\{|E(\mathcal{S}')| : \ \mathcal{S}' \text{ is a simply-connected } \mathbf{r}\text{-diagram such that } \phi(\partial\mathcal{S}') \equiv \phi(\partial\mathcal{S})\}.$$

Proposition 3.1.1. *Let $\mathcal{F}_{\mathcal{S}}$ be a federal subdivision of a simply-connected $\mathbf{r}$-diagram $\mathcal{S}$ which is minimal with respect to $|E(\mathcal{S})|$ ($\geqslant 1$). If* (I) *or* (II) *holds, then $\mathcal{F}_{\mathcal{S}}$ satisfies the following two conditions:*

(i) *Each $\mathcal{F} \in \mathcal{F}_{\mathcal{S}}$ is simply-connected;*

(ii) $\phi(\partial\mathcal{F}) \in \widehat{\mathbf{r}}$ *for each $\mathcal{F} \in \mathcal{F}_{\mathcal{S}}$.*

Proof. Assume $\mathcal{F}_{\mathcal{S}}$ *does* satisfy Condition (i). We will show that $\mathcal{F}_{\mathcal{S}}$ must then satisfy Condition (ii). Let $\mathcal{F}$ be a federation with $\Sigma(\mathcal{F}) = e$ $(= \{u, v\})$ and let δ be a boundary cycle of $\mathcal{F}$ whose label W represents the identity element of $\ker\psi_e$. Let $\mathbb{P}$ be the simply-connected $\mathbf{r}$-picture corresponding to $\mathcal{S}$ and let $\mathbb{F}$ be the subpicture of $\mathbb{P}$ that corresponds to the federation $\mathcal{F}$. Note that $W(\mathbb{F}) \equiv W$. Since $\mathcal{S}$ is minimal with respect to $|E(\mathcal{S})|$, $\mathbb{P}$ is minimal with respect to $|E(\mathbb{P})|$, where $E(\mathbb{P})$ is the set of discs of $\mathbb{P}$ whose labels are elements of $(\mathbf{r}')^s$.

Suppose W is freely equal to the empty word. In this case we can perform bridge moves on the boundary arcs of $\mathbb{F}$ to separate $\mathbb{F}$ from the rest of $\mathbb{P}$. We can then delete $\mathbb{F}$ to obtain a simply-connected $\mathbf{r}$-picture $\mathbb{P}_1$ such that $W(\mathbb{P}_1) \equiv W(\mathbb{P})$ and $|E(\mathbb{P}_1)| < |E(\mathbb{P})|$, contradicting the minimality of $\mathbb{P}$.

Now suppose W is not freely equal to the empty word. Since W represents the identity element of $\widetilde{G}_e$ $(= G_u * G_v)$, it follows from Theorem 1.8.1 that there exists a simply-connected $(\mathbf{r}_u \cup \mathbf{r}_v)$-picture $\mathbb{P}_W$ for W. We can replace $\mathbb{F}$ with $\mathbb{P}_W$ in the following way. First, surround $\mathbb{F}$ with a circle S^1 such that S^1 is transverse to the boundary arcs of $\mathbb{F}$ and does not intersect any other arc of $\mathbb{F}$ or $\mathbb{P}$. Delete the discs and arcs of $\mathbb{F}$ that are contained in S^1, and in their place add $\mathbb{P}_W$. Next, join together the boundary arcs of $\mathbb{P}_W$ and the arcs of $\mathbb{P}$ which meet S^1 in such a way that no two arcs cross each other and only arcs of like label and orientation are joined. We obtain an $\mathbf{r}$-picture $\mathbb{P}_2$ such that $W(\mathbb{P}_2) \equiv W(\mathbb{P})$ and $|E(\mathbb{P}_2)| < |E(\mathbb{P})|$, contradicting the minimality of $\mathbb{P}$. Thus, W must represent a non-identity element of $\ker \psi_e$, i.e. $W \in \widehat{\mathbf{r}}$.

Now assume $\mathcal{F}_{\mathcal{S}}$ *does not* satisfy Condition (i). Let $\mathcal{F}_j$ be a non-simply-connected federation of $\mathcal{F}_{\mathcal{S}}$ and let $\mathcal{B}$ be a non-empty bounded simply-connected $\mathbf{r}$-subdiagram of $\mathcal{S} - \mathcal{F}_j$ (see Fig. 3.1). Note that $\mathcal{B}$ is minimal with respect to $|E(\mathcal{B})|$. We may choose $\mathcal{F}_j$ so that each federation contained in $\mathcal{B}$ is simply-connected. Thus, each federation contained in $\mathcal{B}$ satisfies Condition (ii), and $\mathcal{B}$ must contain at least two federations. It follows from Lemma 3.1.4 that $\mathcal{B}$ contains a federation $\mathcal{F}_i$ such that $\Sigma(\mathcal{F}_i) \subseteq t(\partial\mathcal{B})$ and $\partial\mathcal{F}_i \cap \partial\mathcal{B}$ is a consecutive part of $\partial\mathcal{B}$. Since $t(\partial\mathcal{B}) \subseteq \Sigma(\mathcal{F}_j)$ we deduce that $\Sigma(\mathcal{F}_i) = \Sigma(\mathcal{F}_j)$. However, $\mathcal{F}_i$ and $\mathcal{F}_j$ have an edge in common so this contradicts the fact that $\mathcal{F}_i$ and $\mathcal{F}_j$ are federations. We conclude that $\mathcal{F}_{\mathcal{S}}$ must satisfy Condition (i). This completes the proof of Proposition 3.1.1. $\square$

3.2 Annular r-diagrams

Let $\mathcal{A}$ be an annular $\mathbf{r}$-diagram with outer boundary σ and inner boundary τ, and with labelling function ϕ. The notation $E(\mathcal{A})$ and $t(\alpha)$ makes sense for $\mathcal{A}$, so we may define a *federation* of $\mathcal{A}$ and a *federal subdivision* $\mathcal{F}_{\mathcal{A}}$ of $\mathcal{A}$. Suppose $\mathcal{A}$ contains a non-simply-connected federation $\mathcal{F}$ such that $\mathcal{A} - \mathcal{F}$ contains exactly two annular components (see Fig. 3.7). In this case we say that $\mathcal{F}$ is an *annular* federation of $\mathcal{A}$. An annular federation has a unique outer boundary $\sigma_{\mathcal{F}}$ and a unique inner boundary $\tau_{\mathcal{F}}$. Note that $\sigma_{\mathcal{F}} \cap \sigma$ and $\tau_{\mathcal{F}} \cap \tau$ are not necessarily empty. Moreover, if $\Sigma(\mathcal{F}) = e$

for some $e = \{u, v\} \in E$, then $\mathcal{F}$ is not necessarily an annular $\mathbf{r}_e$-diagram. However, we shall never have to consider such federations. (Thus one may assume that the "holes" which appear in Fig. 3.7 are filled in.) If $\mathcal{F}$ is an annular $\mathbf{r}_e$-diagram and if W and Z^{-1} are the labels of, respectively, an outer and an inner boundary cycle of $\mathcal{F}$, then W and Z represent conjugate elements of G_e. Note that W (respectively, Z^{-1}) is either a word on $\mathbf{x}_u^{\pm 1}$, a word on $\mathbf{x}_v^{\pm 1}$, or is a word involving at least one $\mathbf{x}_u$-letter and at least one $\mathbf{x}_v$-letter.

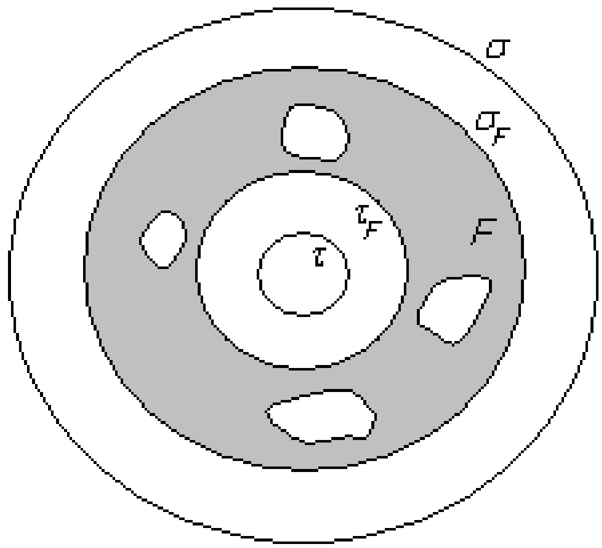


Figure 3.7: An annular federation $\mathcal{F}$.

Let $\mathcal{F}_{\mathcal{A}} = \{\mathcal{F}_i\}_{i=1}^{n}$ be a federal subdivision of $\mathcal{A}$ which satisfies the following two conditions:

(i) Each $\mathcal{F}_i \in \mathcal{F}_{\mathcal{A}}$ is simply-connected ($i = 1, \ldots, n$);

(ii) $\phi(\partial \mathcal{F}_i) \in \widehat{\mathbf{r}}$ for $i = 1, \ldots, n$.

In this case we can construct the *annular derived diagram* $\widehat{\mathcal{A}}$ *of* $\mathcal{A}$ *corresponding to* $\mathcal{F}$. The construction of $\widehat{\mathcal{A}}$ is the same as the construction of its simply-connected counterpart. Thus, $\widehat{\mathcal{A}}$ is an annular $\widehat{\mathbf{r}}$-diagram that satisfies Properties (1) and (2) of Lemma 3.1.1. Note that the outer boundary of $\widehat{\mathcal{A}}$ is equal to the outer boundary of $\mathcal{A}$, so the label of any outer boundary cycle of $\mathcal{A}$ is the label of an outer boundary cycle of $\widehat{\mathcal{A}}$. The same is true if we replace "outer" by "inner".

Replacing "$\widehat{\mathcal{S}}$" with "$\widehat{\mathcal{A}}$" in the proofs of Lemma 3.1.2 and Lemma 3.1.3 gives $\widehat{\mathcal{A}}$ the structure of an annular $[3, 2k+2]$-diagram. If Γ is triangle-free, then $\widehat{\mathcal{A}}$ has the structure of an annular $[4, 2k+2]$-diagram.

Lemma 3.2.1. *If ν is an interior vertex of $\widehat{\mathcal{A}}$, then $d(\nu) \geqslant 3$, and if Γ is triangle-free, then each interior vertex of $\widehat{\mathcal{A}}$ has degree at least 4.*

Lemma 3.2.2. *If each edge group has property-W_k, then each region of $\widehat{\mathcal{A}}$ has degree at least $2k+2$.*

Let $\mathcal{A}$ be an annular $\mathbf{r}$-diagram for the pair (W, Z^{-1}) where W, Z are words on $\mathbf{x}^{\pm 1}$. We say that $\mathcal{A}$ is *minimal with respect to* $|E(\mathcal{A})|$ if

$$|E(\mathcal{A})| = \min\{|E(\mathcal{A}')| : \mathcal{A}' \text{ in an annular } \mathbf{r}\text{-diagram for the pair } (W, Z^{-1})\}.$$

One may check that the proof of Proposition 3.1.1 did not depend on the fact that the $\mathbf{r}$-diagram in question was simply-connected.

Proposition 3.2.1. *Let $\mathcal{F}_{\mathcal{A}}$ be a federal subdivision of an annular $\mathbf{r}$-diagram $\mathcal{A}$, which is minimal with respect to $|E(\mathcal{A})|$ $(\geqslant 1)$. If $\mathcal{F}_{\mathcal{A}}$ does not contain any annular federations and if* (I) *or* (II) *holds, then $\mathcal{F}_{\mathcal{A}}$ satisfies Conditions* (i) *and* (ii).

Recall that the degree of an annular diagram is the number of edges contained in an outer boundary cycle plus the number of edges contained in an inner boundary cycle (counted with appropriate multiplicities).

The following result is crucial to our work. Let $\mathcal{A}$ be an annular $\mathbf{r}$-diagram which is minimal with respect to $|E(\mathcal{A})|$. Suppose σ and τ are simple closed paths and that $\mathcal{A}$ has degree n. Let $\mathcal{F}_{\mathcal{A}}$ be a federal subdivision of $\mathcal{A}$ which does not contain any annular federations.

Proposition 3.2.2. *If G satisfies* (I) *or* (II)*, then the degree of each region of the annular derived diagram $\widehat{\mathcal{A}}$ (corresponding to $\mathcal{F}_{\mathcal{A}}$) is at most $10n$.*

Proof. The proof is split into two cases: the case when G satisfies Condition (I) and the case when G satisfies Condition (II). The arguments used in both are similar.

Case 1. Suppose G satisfies (I). Then by Lemmas 3.2.1 and 3.2.2, $\widehat{\mathcal{A}}$ is an annular $[3,6]$-diagram. We proceed by induction on Area$(\widehat{\mathcal{A}})$. We note that Area$(\widehat{\mathcal{A}}) \neq 1$ for otherwise, $\mathcal{F}_{\mathcal{A}}$ would contain an annular federation. If Area$(\widehat{\mathcal{A}}) = 2$, then $\widehat{\mathcal{A}}$ can have at most two interior edges so each region has degree at most $2+n < 10n$. Suppose the result is true for all diagrams satisfying the hypotheses with area $k \geqslant 2$.

Suppose $\widehat{\mathcal{A}}$ contains a simple outer boundary region Δ that contains at most 3 interior edges in its boundary, i.e. $i(\Delta) \leqslant 3$. Let $\alpha = \partial\Delta \cap \sigma$. Since $d(\Delta) \geqslant 6$, α contains at least 3 edges.

Delete α from σ to obtain an annular diagram $\mathcal{A}_1$ to which the inductive hypothesis applies. Since $d(\mathcal{A}_1) \leqslant n$, each region of $\mathcal{A}_1$ has degree at most $10n$. Furthermore, $d(\Delta) \leqslant 3 + n < 10n$. Thus each region of $\widehat{\mathcal{A}}$ has degree at most $10n$.

Call a region Δ' of $\widehat{\mathcal{A}}$ an *almost simple outer boundary region* if $\partial\Delta' \cap \sigma$ is a consecutive part of σ and $\partial\Delta' \cap \tau = \nu$ for some vertex ν in τ. Suppose $\widehat{\mathcal{A}}$ contains an almost simple outer boundary region that contains at most 3 interior edges in its boundary. Then, arguing as in the previous paragraph, we have that each region of $\widehat{\mathcal{A}}$ has degree at most $10n$.

Now suppose $\widehat{\mathcal{A}}$ contains a simple inner boundary region or an almost simple inner boundary region that contains at most 3 interior edges in its boundary (where *almost simple inner boundary region* has its obvious meaning). Then, arguing as in the previous paragraphs, we have that each region of $\widehat{\mathcal{A}}$ has degree at most $10n$.

Now assume $\widehat{\mathcal{A}}$ does *not* contain a simple outer (respectively, inner) boundary region nor an almost simple outer (respectively, inner) boundary region which contains at most 3 interior edges in its boundary. We claim that in this case $\widehat{\mathcal{A}}$ *cannot* contain a non-simple boundary region Δ which satisfies:

(1) At least one of $\partial\Delta \cap \sigma$, $\partial\Delta \cap \tau$ is non-empty;

(2) If $\partial\Delta \cap \sigma \neq \emptyset$, then $\partial\Delta \cap \sigma \neq \nu$ for some ν in σ. Similarly, if $\partial\Delta \cap \tau \neq \emptyset$, then $\partial\Delta \cap \tau \neq \nu'$ for some ν' in τ.

In other words, $\widehat{\mathcal{A}}$ cannot contain a non-simple boundary region Δ with $\partial\Delta \cap \sigma$ or $\partial\Delta \cap \tau$ disconnected. Therefore, if Δ is a non-simple boundary region of $\widehat{\mathcal{A}}$, then either $\partial\Delta \cap \tau = \emptyset$ and $\partial\Delta \cap \sigma = \nu$ for some vertex ν in σ; $\partial\Delta \cap \sigma = \emptyset$ and $\partial\Delta \cap \tau = \nu'$ for some vertex ν' in τ; or $\partial\Delta$ intersects both σ and τ at a single vertex.

We now proceed with the proof of our claim. Let Δ be a non-simple boundary region of $\widehat{\mathcal{A}}$ and assume, without loss of generality, that $\partial\Delta\cap\sigma$ is non-empty and does not consist of a single vertex. Then $\widehat{\mathcal{A}} - \Delta$ contains at least one simply-connected component $\mathcal{B}$ (see Fig. 3.8). We will prove that $\mathcal{B}$ must contain a simple outer boundary region which contains at most 3 interior edges of $\widehat{\mathcal{A}}$ in its boundary, thus contradicting our assumptions.

Suppose $\mathcal{B}$ contains only one region Δ'. Since Δ' can have at most one edge in common with Δ, $i(\Delta') = 1$. Furthermore, $\partial\Delta' \cap \sigma$ is a consecutive part of σ and $\partial\Delta' \cap \tau$ is empty. Thus, Δ' is a simple outer boundary region of $\widehat{\mathcal{A}}$ with $i(\Delta') = 1$. Now suppose $\mathcal{B}$ contains at least two regions.

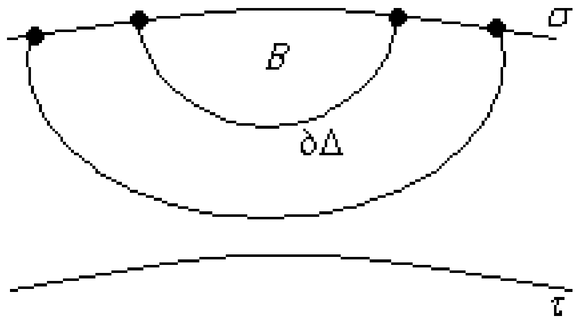


Figure 3.8: A simply-connected component of $\widehat{\mathcal{A}} - \Delta$.

Since $\widehat{A}$ is an annular $[3,6]$-diagram, $\mathcal{B}$ is a simply-connected $[3,6]$-diagram. From Theorem 1.6.2, we have

$$\sum_{\mathcal{B}}^{*} [4 - i_{\mathcal{B}}(\Phi)] \geqslant 6, \tag{3.2}$$

where the sum is taken over all simple boundary regions of $\mathcal{B}$. We deduce that $\mathcal{B}$ contains a simple boundary region Φ' with $i_{\mathcal{B}}(\Phi') \leqslant 3$. Let $\gamma = \partial\Phi' \cap \partial\mathcal{B}$ and assume that $i_{\mathcal{B}}(\Phi') < 3$. Since $d(\Phi') \geqslant 6$, we deduce that γ contains at least 4 edges. Now γ can either be: (a) a subpath of σ; (b) a subpath of $\partial\Delta$; or (c) equal to a path $\gamma_1\gamma_2$ where γ_1 and γ_2 are non-empty simple subpaths of σ and $\partial\Delta$, respectively. Each possibility corresponds to a specific position for Φ' in $\mathcal{B}$ (see Fig. 3.9).

Suppose γ is a subpath of σ. Then $\partial\Phi' \cap \sigma$ is a consecutive part of σ, $\partial\Phi' \cap \tau$ is empty, and $i(\Phi') \leqslant 2$. Thus, Φ' a simple outer boundary region of $\widehat{\mathcal{A}}$ with $i(\Phi') \leqslant 2$. Now suppose γ is a subpath of $\partial\Delta$. By construction of $\widehat{\mathcal{A}}$, Δ and Φ' can have at most one edge in common, contradicting the fact that γ must contain at least 4 edges. Finally, suppose $\gamma = \gamma_1\gamma_2$ where γ_1 and γ_2 are non-empty simple subpaths of σ and $\partial\Delta$, respectively. Since Δ and Φ' can have at most one edge in common, γ_2 contains exactly one edge. Recalling that $i_{\mathcal{B}}(\Phi') \leqslant 2$, we deduce that $i(\Phi') \leqslant 3$. Since γ_1 is a consecutive part of σ, it follows that Φ' is a simple outer boundary region of $\widehat{\mathcal{A}}$ that contains at most 3 interior edges in its boundary.

Now assume each simple boundary region Φ' of $\mathcal{B}$ satisfies $i_{\mathcal{B}}(\Phi') = 3$. Then from (3.2) we deduce that $\mathcal{B}$ must contain at least six simple boundary regions. Since each simple boundary region can have at most one edge in common with Δ, we deduce that no simple boundary region can have position (b) as illustrated in Fig. 3.9(b). Thus, each simple boundary region either has position (a) or (c). At most two simple boundary regions can have position (c) (one for each

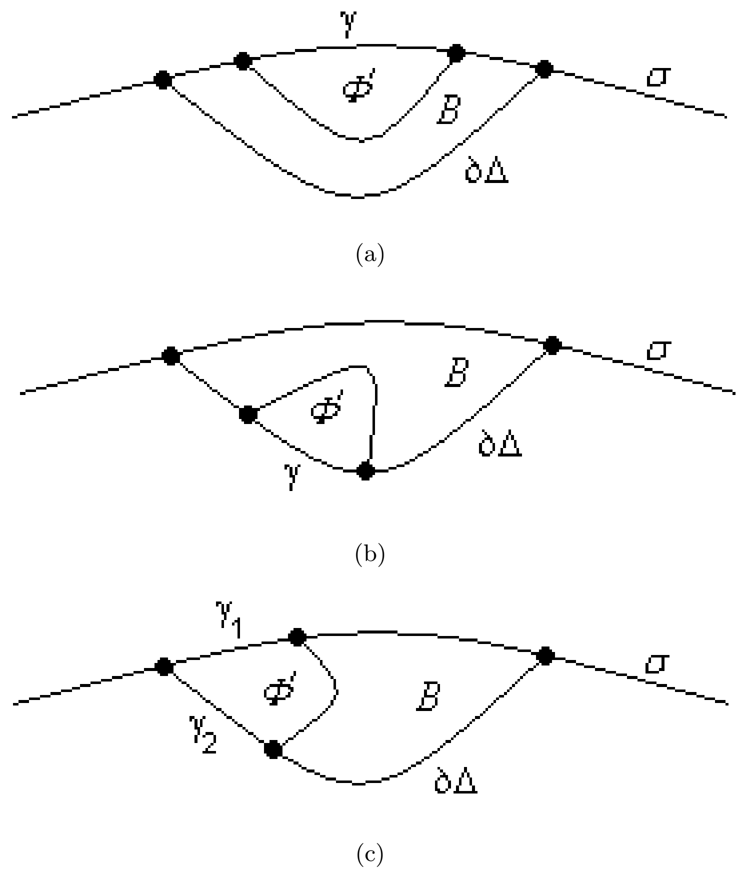


Figure 3.9: The possible positions of Φ' in $\mathcal{B}$.

"corner"). Therefore, $\mathcal{B}$ must contain a simple boundary region Λ that has position (a). Thus, $\widehat{\mathcal{A}}$ contains a region Λ such that $\partial\Lambda \cap \sigma$ is a consecutive part of σ, $\partial\Lambda \cap \tau_{\mathcal{A}}$ is empty, and $i(\Lambda') = 3$. This completes the proof of our claim.

Let us now delete the label and orientation of each edge of $\widehat{\mathcal{A}}$ to obtain an ordinary unoriented annular diagram $\mathcal{N}'$. If $\mathcal{N}'$ contains a bridge ε (recall Definition 1.6.5), then we may remove it by identifying its endpoints as in Fig. 3.10. Note, removing a bridge decreases the number of boundary edges by one. We may also remove a pinch (recall Definition 1.6.5) from $\mathcal{N}'$ as illustrated in Fig. 3.11. Note, removing a pinch increases the number of interior edges by one. By removing all bridges and pinches in this way we obtain an annular $[3,6]$-diagram $\mathcal{N}$ in which each region has at least one interior edge in its boundary.

Let $\sigma_{\mathcal{N}}$ be the outer boundary of $\mathcal{N}$ and let $\tau_{\mathcal{N}}$ be the inner boundary of $\mathcal{N}$. Since $d(\mathcal{A}) = n$, the total number of edges contained in $\sigma_{\mathcal{N}}$ and $\tau_{\mathcal{N}}$ is at most n. If Δ is a boundary region of $\widehat{\mathcal{A}}$,

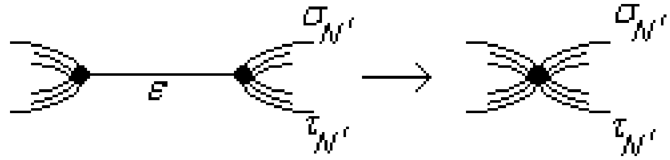


Figure 3.10: Removing a bridge.

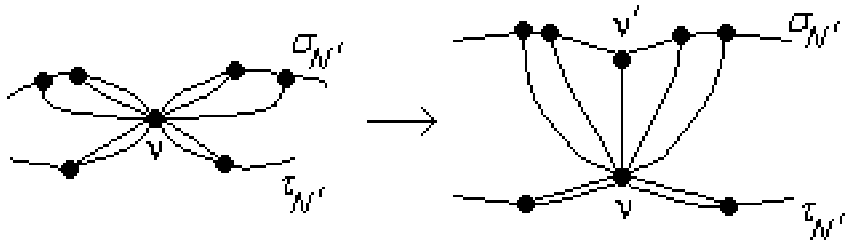


Figure 3.11: Removing a pinch.

then $i_{\widehat{\mathcal{A}}}(\Delta) \leqslant i_{\mathcal{N}}(\Delta)$ and so $d_{\widehat{\mathcal{A}}}(\Delta) \leqslant i_{\mathcal{N}}(\Delta) + n$. Thus, we can estimate $d_{\widehat{\mathcal{A}}}(\Delta)$ by calculating an upper bound for $i_{\mathcal{N}}(\Delta)$. Similarly, if Δ is an interior region of $\widehat{\mathcal{A}}$, then $d_{\widehat{\mathcal{A}}}(\Delta) \leqslant d_{\mathcal{N}}(\Delta)$. So we can estimate $d_{\widehat{\mathcal{A}}}(\Delta)$ by calculating an upper bound for $d_{\mathcal{N}}(\Delta)$.

Since $i_{\mathcal{N}}(\Delta) \geqslant i_{\widehat{\mathcal{A}}}(\Delta)$ for each region Δ of $\mathcal{N}$, each simple outer (respectively, inner) boundary region of $\mathcal{N}$ and each almost simple outer (respectively, inner) boundary region of $\mathcal{N}$ must contain at least 4 interior edges in its boundary.

Let $\mathcal{N}^*$ be the dual diagram of $\mathcal{N}$. Applying formula (1.3) of Theorem 1.6.1 to $\mathcal{N}^*$ gives

$$6(Q_{\mathcal{N}^*} - h_{\mathcal{N}^*}) = \sum_{\mathcal{N}^*}^{\bullet} [4 - d_{\mathcal{N}^*}(\nu)] + \sum_{\mathcal{N}^*}^{\circ} [6 - d_{\mathcal{N}^*}(\nu)] + 2\sum_{\mathcal{N}^*} [3 - d_{\mathcal{N}^*}(\Delta)] + 2(V^{\bullet}_{\mathcal{N}^*} - E^{\bullet}_{\mathcal{N}^*}).$$

Since $\mathcal{N}^*$ has at most one hole, $6(Q_{\mathcal{N}^*} - h_{\mathcal{N}^*}) \geqslant 0$. Since each region of $\mathcal{N}$ has at least one interior edge in its boundary, $\mathcal{N}^*$ has no isolated vertices and we deduce that $V^{\bullet}_{\mathcal{N}^*} \leqslant E^{\bullet}_{\mathcal{N}^*}$. Therefore,

$$0 \leqslant \sum_{\mathcal{N}^*}^{\bullet} [4 - d_{\mathcal{N}^*}(\nu)] + \sum_{\mathcal{N}^*}^{\circ} [6 - d_{\mathcal{N}^*}(\nu)] + 2\sum_{\mathcal{N}^*} [3 - d_{\mathcal{N}^*}(\Delta)].$$

Using the correspondence between $\mathcal{N}$ and $\mathcal{N}^*$, we have

$$0 \leqslant \sum_{\mathcal{N}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] + \sum_{\mathcal{N}}^{\circ} [6 - d_{\mathcal{N}}(\Delta)] + 2\sum_{\mathcal{N}}^{\circ} [3 - d_{\mathcal{N}}(\nu)]. \tag{3.3}$$

Since each interior region of $\mathcal{N}$ has degree at least six and since each interior vertex has degree at least 3, each term of $\sum^{\circ}_{\mathcal{N}} [6 - d_{\mathcal{N}}(\Delta)]$ and $\sum^{\circ}_{\mathcal{N}} [3 - d_{\mathcal{N}}(\nu)]$ is non-positive. Therefore,

$$\sum_{\mathcal{N}}^{\circ} [6 - d_{\mathcal{N}}(\Delta)] + 2\sum_{\mathcal{N}}^{\circ} [3 - d_{\mathcal{N}}(\nu)] \leqslant 0$$

and so

$$\sum_{\mathcal{N}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] \geqslant 0. \tag{3.4}$$

Boundary regions which contain at least 4 interior edges of $\mathcal{N}$ in their boundary make a non-positive contribution to the sum in (3.4). Therefore, each simple outer (respectively, inner) boundary region of $\mathcal{N}$ and each almost simple outer (respectively, inner) boundary region of $\mathcal{N}$ makes a non-positive contribution to the sum in (3.4).

If Δ is a non-simple boundary region of $\widehat{\mathcal{A}}$, and hence is a non-simple boundary region of $\mathcal{N}$, then $\partial\Delta \cap \sigma_{\mathcal{N}}$ or $\partial\Delta \cap \tau_N$ (or possibly both) is connected and consists of a single vertex. Therefore, $i_{\mathcal{N}}(\Delta) = d_{\mathcal{N}}(\Delta) \geqslant 6$, so Δ makes a negative contribution to the sum in (3.4).

From the two preceding paragraphs, we conclude that only the almost simple boundary regions of $\mathcal{N}$ which contain at most 3 interior edges in their boundaries (see Fig. 3.12) make a positive contribution to the sum in (3.4). Note that such regions contain *at least* 2 interior edges in their boundaries. For convenience, we will call an almost simple boundary region of $\mathcal{N}$ that satisfies $i_{\mathcal{N}}(\Delta) = 2$ or 3 an *AS-region.* Note that each AS-region has degree at most $3 + n < 10n$.

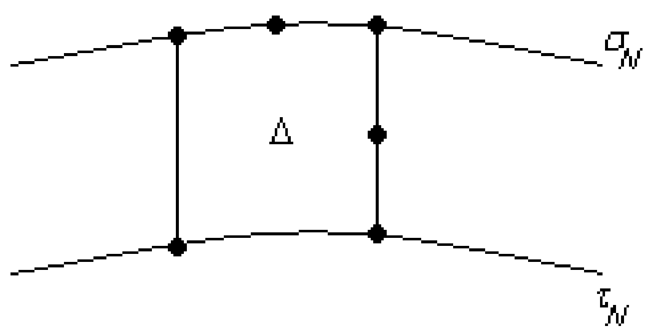


Figure 3.12: An AS-region.

Since $\mathcal{N}$ contains at most n boundary edges it contains at most $\frac{n}{2}$ AS-regions. Splitting (3.4) into terms that make a positive and a non-positive contribution gives

$$\begin{aligned}
0 &\leqslant \sum_{\mathcal{N}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] \\
&= \sum_{\substack{\mathcal{N} \\ i_{\mathcal{N}}(\Delta) \geqslant 4}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] + \sum_{\text{AS-regions}} [4 - i_{\mathcal{N}}(\Delta)] \\
&\leqslant \sum_{\substack{\mathcal{N} \\ i_{\mathcal{N}}(\Delta) \geqslant 4}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] + 2\frac{n}{2}.
\end{aligned}$$

Therefore,

$$\sum_{\substack{\mathcal{N} \\ i_{\mathcal{N}}(\Delta) \geqslant 4}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] \geqslant -n$$

where the sum is taken over all boundary regions of $\mathcal{N}$ that contain at least 4 interior edges in their boundary. From this sum, we have

$$\sum_{\substack{\mathcal{N} \\ i_{\mathcal{N}}(\Delta) \geqslant 4}}^{\bullet} [i_{\mathcal{N}}(\Delta) - 4] \leqslant n, \tag{3.5}$$

where each term is non-negative. If Δ is a boundary of $\mathcal{N}$ that satisfies $i_{\mathcal{N}}(\Delta) \geqslant 4$, then $i_{\mathcal{N}}(\Delta) - 4 \leqslant n$ and so $i_{\mathcal{N}}(\Delta) \leqslant 4 + n$. We deduce that $d_{\widehat{\mathcal{A}}}(\Delta) \leqslant 4 + 2n < 10$. Thus,

$$d(\Delta) \leqslant 10n \tag{3.6}$$

for each boundary region Δ of $\widehat{\mathcal{A}}$.

We now calculate an upper bound for $d_{\widehat{\mathcal{A}}}(\Delta)$ where Δ is an interior region of $\widehat{A}$. Note that

$$0 \leqslant \sum_{\mathcal{N}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] \leqslant \sum_{\text{AS-regions}} [4 - i_{\mathcal{N}}(\Delta)] \leqslant 2\frac{n}{2} = n.$$

Then from (3.3), we have

$$\begin{aligned}
0 &\leqslant \sum_{\mathcal{N}}^{\bullet} [4 - i_{\mathcal{N}}(\Delta)] + \sum_{\mathcal{N}}^{\circ} [6 - d_{\mathcal{N}}(\Delta)] + 2\sum_{\mathcal{N}}^{\circ} [3 - d_{\mathcal{N}}(\nu)] \\
&\leqslant n + \sum_{\mathcal{N}}^{\circ} [6 - d_{\mathcal{N}}(\Delta)] + 0
\end{aligned}$$

and so

$$\sum_{\mathcal{N}}^{\circ} [6 - d_{\mathcal{N}}(\Delta)] \geqslant -n.$$

Thus,

$$\sum_{\mathcal{N}}^{\circ} [d_{\mathcal{N}}(\Delta) - 6] \leqslant n. \tag{3.7}$$

Since each term of the sum in (3.7) is non-negative, each interior region of $\mathcal{N}$ must satisfy $d_{\mathcal{N}}(\Delta) - 6 \leqslant n$, that is $d_{\mathcal{N}}(\Delta) \leqslant 6 + n$. Therefore,

$$d(\Delta) \leqslant 10n \tag{3.8}$$

for each interior region Δ of $\widehat{\mathcal{A}}$. From (3.6) and (3.8) we conclude that $d(\Delta) \leqslant 10n$ for each region Δ of $\widehat{\mathcal{A}}$. This completes the proof of Case 1.

Case 2. Suppose G satisfies (II). In this case $\widehat{\mathcal{A}}$ is an annular $[4,4]$-diagram. We proceed by induction on Area($\widehat{\mathcal{A}}$). The result is clearly true for the initial case, so assume that the result holds for all diagrams satisfying the hypotheses with area $k \geqslant 2$.

Suppose $\widehat{\mathcal{A}}$ contains a simple outer boundary region Δ that contains at most two interior edges in its boundary, i.e. $i(\Delta) \leqslant 2$. Let $\alpha = \partial\Delta \cap \sigma$. Since $\sigma(\Delta) \geqslant 4$, α contains at least two interior edges. Delete α from σ to obtain an annular diagram $\mathcal{A}_2$ to which the inductive hypothesis applies. Since $d(\mathcal{A}_2) \leqslant n$, each region of $\mathcal{A}_2$ has degree at most $10n$. Furthermore, $d(\Delta) \leqslant 2 + n < 10n$. Thus, each region of $\widehat{\mathcal{A}}$ has degree at most $10n$.

If $\widehat{\mathcal{A}}$ contains a simple inner boundary region which contains at most two interior edges in its boundary or if $\widehat{\mathcal{A}}$ contains an almost simple outer (respectively, inner) boundary region that contains at most two interior edges in its boundary, then we argue as in the previous paragraph to show that each region of $\widehat{\mathcal{A}}$ has degree at most $10n$.

Now assume $\widehat{\mathcal{A}}$ does *not* contain a simple outer (respectively, inner) boundary region, nor an almost simple outer (respectively, inner) boundary region that contains at most two interior edges in its boundary. Following a similar argument to the one used in Case 1, we can show that if Δ is a non-simple boundary region of $\widehat{\mathcal{A}}$, then either $\partial\Delta \cap \tau = \emptyset$ and $\partial\Delta \cap \sigma = \nu$ for some vertex ν in σ; $\partial\Delta \cap \sigma = \emptyset$ and $\partial\Delta \cap \tau = \nu'$ for some vertex ν' in τ; or $\partial\Delta$ intersects both σ and τ at a single vertex. That is, $\widehat{\mathcal{A}}$ cannot contain a non-simple boundary region Δ with $\partial\Delta \cap \sigma$ or $\partial\Delta \cap \tau$ disconnected.

Delete the label and orientation of each edge of $\widehat{\mathcal{A}}$ and remove any bridges and pinches from the resulting diagram to obtain an ordinary unoriented annular $[4,4]$-diagram $\mathcal{N}$. Note that each region of $\mathcal{N}$ has at least one interior edge in its boundary.

Since $i_{\mathcal{N}}(\Delta) \geqslant i_{\widehat{\mathcal{A}}}(\Delta)$ for each region Δ of $\mathcal{N}$, each simple outer (respectively, inner) boundary region and each almost simple outer (respectively, inner) boundary region of $\mathcal{N}$ must contain at least 3 interior edges in its boundary.

Applying formula (1.3) of Theorem 1.6.1 to the dual $\mathcal{N}^*$ of $\mathcal{N}$ gives

$$4(Q_{\mathcal{N}^*} - h_{\mathcal{N}^*}) = \sum_{\mathcal{N}^*}^{\bullet}[3 - d_{\mathcal{N}^*}(\nu)] + \sum_{\mathcal{N}^*}^{\circ}[4 - d_{\mathcal{N}^*}(\nu)] + \sum_{\mathcal{N}^*}[4 - d_{\mathcal{N}^*}(\Delta)] + (V^{\bullet}_{\mathcal{N}^*} - E^{\bullet}_{\mathcal{N}^*}).$$

Since $\mathcal{N}^*$ has at most one hole, $4(Q_{\mathcal{N}^*} - h_{\mathcal{N}^*}) \geqslant 0$. Since each region of $\mathcal{N}$ contains at least one interior edge in its boundary, $\mathcal{N}^*$ has no isolated vertices and we deduce that $V^{\bullet}_{\mathcal{N}^*} \leqslant E^{\bullet}_{\mathcal{N}^*}$. Therefore,

$$0 \leqslant \sum_{\mathcal{N}^*}^{\bullet}[3 - d_{\mathcal{N}^*}(\nu)] + \sum_{\mathcal{N}^*}^{\circ}[4 - d_{\mathcal{N}^*}(\nu)] + \sum_{\mathcal{N}^*}[4 - d_{\mathcal{N}^*}(\Delta)].$$

Using the correspondence between $\mathcal{N}$ and $\mathcal{N}^*$, we have

$$0 \leqslant \sum_{\mathcal{N}}^{\bullet}[3 - i_{\mathcal{N}}(\Delta)] + \sum_{\mathcal{N}}^{\circ}[4 - d_{\mathcal{N}}(\Delta)] + \sum_{\mathcal{N}}^{\circ}[4 - d_{\mathcal{N}}(\nu)]. \tag{3.9}$$

Since each interior region of $\mathcal{N}$ has degree at least 4 and since each interior vertex has degree at least 4, each term of $\sum_{\mathcal{N}}^{\circ}[4 - d_{\mathcal{N}}(\Delta)]$ and $\sum_{\mathcal{N}}^{\circ}[4 - d_{\mathcal{N}}(\nu)]$ is non-positive. Therefore, $\sum_{\mathcal{N}}^{\circ}[4 - d_{\mathcal{N}}(\Delta)] + \sum_{\mathcal{N}}^{\circ}[4 - d_{\mathcal{N}}(\nu)] \leqslant 0$ and it follows that

$$\sum_{\mathcal{N}}^{\bullet}[3 - i_{\mathcal{N}}(\Delta)] \geqslant 0. \tag{3.10}$$

Boundary regions which contain at least 3 interior edges of $\mathcal{N}$ in their boundary make a non-positive contribution to the sum in (3.10). Therefore, each simple outer (respectively, inner) boundary region of $\mathcal{N}$ and each almost simple outer (respectively, inner) boundary region of $\mathcal{N}$ makes a non-positive contribution to the sum in (3.10). If Δ is a non-simple boundary region of $\widehat{\mathcal{A}}$, and hence is a non-simple boundary region of $\mathcal{N}$, then $\partial\Delta \cap \sigma_{\mathcal{N}}$ or $\partial\Delta \cap \tau_N$ (or possibly both) is connected and consists of a single vertex. Therefore, $i_{\mathcal{N}}(\Delta) = d_{\mathcal{N}}(\Delta) \geqslant 4$ and so Δ makes a negative contribution to the sum in (3.10). Thus, only the almost simple boundary regions that satisfy $i(\Delta) = 2$ make a positive contribution to the sum in (3.10) As in Case A, we call such a region an *AS-region*. Note that each AS-region has degree at most $2 + n < 10n$.

Splitting (3.10) into terms which make a positive and a non-positive contribution gives

$$\begin{aligned}
0 &\leqslant \sum_{\mathcal{N}}^{\bullet} [3 - i_{\mathcal{N}}(\Delta)] \\
&= \sum_{\substack{\mathcal{N} \\ i_{\mathcal{N}}(\Delta) \geqslant 3}}^{\bullet} [3 - i_{\mathcal{N}}(\Delta)] + \sum_{\text{AS-regions}} [3 - i_{\mathcal{N}}(\Delta)] \\
&\leqslant \sum_{\substack{\mathcal{N} \\ i_{\mathcal{N}}(\Delta') \geqslant 3}}^{\bullet} [3 - i_{\mathcal{N}}(\Delta)] + \frac{n}{2}.
\end{aligned}$$

Hence,

$$\sum_{\substack{\mathcal{N} \\ i(\Delta') \geqslant 3}}^{\bullet} [3 - i_{\mathcal{N}}(\Delta)] \geqslant -\frac{n}{2}$$

and so

$$\sum_{\substack{\mathcal{N} \\ i(\Delta') \geqslant 3}}^{\bullet} [i_{\mathcal{N}}(\Delta) - 3] \leqslant \frac{n}{2}, \tag{3.11}$$

where each term is non-negative. If Δ is a boundary of $\mathcal{N}$ that satisfies $i_{\mathcal{N}}(\Delta) \geqslant 3$, then $i_{\mathcal{N}}(\Delta) - 3 \leqslant \frac{1}{2}n$ and so $i_{\mathcal{N}}(\Delta) \leqslant 3 + \frac{1}{2}n$. We deduce that $d_{\widehat{\mathcal{A}}}(\Delta) \leqslant 3 + \frac{3}{2}n < 10n$. Thus,

$$d(\Delta) \leqslant 10n \tag{3.12}$$

for each boundary region Δ of $\widehat{\mathcal{A}}$.

We now calculate an upper bound for $d_{\widehat{\mathcal{A}}}(\Delta)$ where Δ is an interior region of $\widehat{A}$. Note that

$$0 \leqslant \sum_{\mathcal{N}}^{\bullet} [3 - i_{\mathcal{N}}(\Delta)] \leqslant \sum_{\text{AS-regions}} [3 - i_{\mathcal{N}}(\Delta)] \leqslant \frac{n}{2}.$$

Then from (3.9), we have

$$\begin{aligned}
0 &\leqslant \sum_{\mathcal{N}}^{\bullet} [3 - i_{\mathcal{N}}(\Delta)] + \sum_{\mathcal{N}}^{\circ} [4 - d_{\mathcal{N}}(\Delta)] + \sum_{\mathcal{N}}^{\circ} [4 - d_{\mathcal{N}}(\nu)] \\
&\leqslant \frac{n}{2} + \sum_{\mathcal{N}}^{\circ} [4 - d_{\mathcal{N}}(\Delta)] + 0 \\
&\leqslant \sum_{\mathcal{N}}^{\circ} [4 - d_{\mathcal{N}}(\Delta)] + \frac{n}{2},
\end{aligned}$$

and so

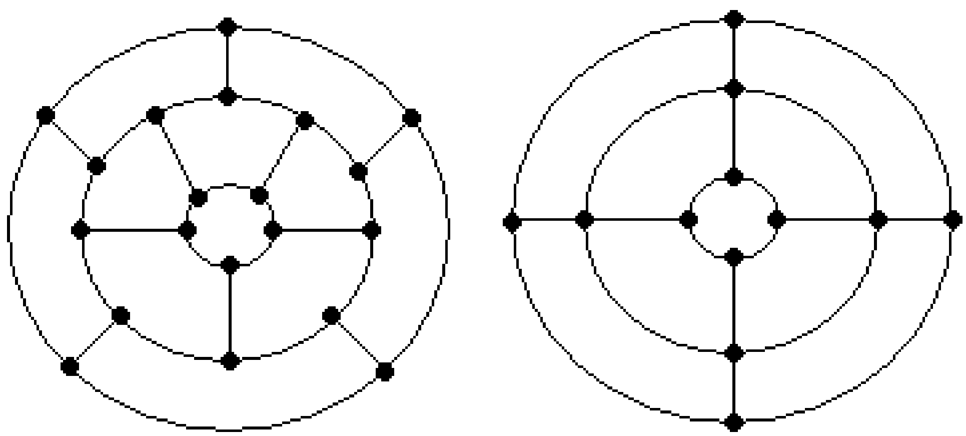
$$\sum_{\mathcal{N}}^{\circ} [d_{\mathcal{N}}(\Delta) - 4] \leqslant \frac{n}{2}. \tag{3.13}$$

Since each term of the sum in (3.13) is non-negative, each interior region of $\mathcal{N}$ must satisfy $d_{\mathcal{N}}(\Delta) - 4 \leqslant \frac{1}{2}n$, that is $d_{\mathcal{N}}(\Delta) \leqslant 4 + \frac{1}{2}n < 10n$. Therefore,

$$d(\Delta) \leqslant 10n \tag{3.14}$$

for each interior region Δ of $\widehat{\mathcal{A}}$. From (3.12) and (3.14) we conclude that $d(\Delta) \leqslant 10n$ for each region Δ of $\widehat{\mathcal{A}}$. This completes the proof of Case 2. The proof of Proposition 3.2.2 is now complete. $\square$

We now turn our attention to annular diagrams over vertex-finite Pride groups which satisfy (H-I) or (H-II). In Theorems 3.2.1 and 3.2.2 we obtain information about the structure of annular derived diagrams over the standard presentations of such groups. In particular, we prove that either all regions have edges on both boundaries, or that the diagram has a "thickness" of two regions as illustrated in Fig. 3.13. The diagram in Fig. 3.13(a) corresponds to the condition (H-I), while Fig. 3.13(b) corresponds to the condition (H-II). These structure theorems are based on analogous theorems [66, Theorems V.5.3, V.5.5] for presentations satisfying the small cancellation conditions $C'(\frac{1}{6})$, or $C'(\frac{1}{4})$ and $T(4)$.



(a) (H-I) (b) (H-II)

Figure 3.13: Annular derived diagrams for the Conditions (H-I) and (H-II).

Let $\mathcal{P}_s = \langle \mathbf{x}\,; \mathbf{r} \rangle$ be the standard presentation of a vertex-finite Pride group G and let $\Gamma = \{V, E\}$ be the underlying graph of G.

Definition 3.2.1. Let W be a non-empty word on $\mathbf{x}^{\pm 1}$. If $W \equiv W_1 W_2 \ldots W_l$ where each W_i $(i = 1, \ldots, l)$ is a non-empty word on $\mathbf{x}_v^{\pm 1}$ for some $v \in V$, and W_i, W_{i+1} are both not words on

$\mathbf{x}_v^{\pm 1}$ for $i = 1, \ldots, l-1$, then the *syllable length* of W, denoted $|W|_s$, is l. The *syllables* of W are the subwords $W_1, W_2, \ldots, W_l$.

Recall (Definition 2.1.1), each edge group G_e ($e \in E$) has property-W_k if and only if $m_e > 2k$, where m_e is the length of a shortest non-identity element of $\ker \psi_e$. We interpret this definition in terms of syllable length in the following lemma. Recall that $\widehat{\mathbf{r}}$ denotes the union of the $\widehat{\mathbf{r}}_e$'s, where for each $e \in E$, $\widehat{\mathbf{r}}_e$ is the set of all words on $\mathbf{x}_e^{\pm 1}$ that represent a non-identity element of $\ker \psi_e$.

Lemma 3.2.3. *If each edge group has property-W_k, then $|R|_s > 2k$ for all $R \in \widehat{\mathbf{r}}$.*

Let U be a word on $\mathbf{x}^{\pm 1}$, let a be a rational number, and let $e \in E$. We write $U > a\widehat{\mathbf{r}}_e$ if there exists an $R \in \widehat{\mathbf{r}}_e$ such that $R \equiv UZ$ where $|U|_s > a|R|_s$. We write $U > a\widehat{\mathbf{r}}$ if $U > a\widehat{\mathbf{r}}_e$ for some $e \in E$.

Let $\mathcal{A}$ be an annular $\mathbf{r}$-diagram which is minimal with respect to $|E(\mathcal{A})|$ ($\geqslant 1$) and let σ (respectively, τ) be the outer (respectively, inner) boundary of $\mathcal{A}$. Let $\mathcal{F}_{\mathcal{A}}$ be a federal subdivision of $\mathcal{A}$ which does not contain an annular federation and construct the annular derived diagram $\widehat{\mathcal{A}}$ of $\mathcal{A}$ corresponding to $\mathcal{F}_{\mathcal{A}}$.

Theorem 3.2.1. *Let G satisfy* (H-I) *or* (H-II) *and assume the following two hypotheses on $\widehat{\mathcal{A}}$.*

(A) *If Δ is a region of $\widehat{\mathcal{A}}$ with $\alpha = \partial\Delta \cap \sigma$ a consecutive part of σ, then $\phi(\alpha) \not> \frac{1}{2}\widehat{\mathbf{r}}$. Assume the same hypothesis replacing σ by τ.*

(B) *$\widehat{\mathcal{A}}$ does not contain a region Δ such that $\partial\Delta$ contains an edge of both σ and τ.*

Let $(q,p) = (3,6)$ or $(4,4)$ depending if (H-I) *or* (H-II) *holds, respectively. Then $\widehat{\mathcal{A}}$ satisfies the following three conditions:*

(i) *For each region Δ, $\partial\Delta$ contains a boundary edge of $\widehat{\mathcal{A}}$;*

(ii) *$i(\Delta) = p/q + 2$ for all regions Δ of $\widehat{\mathcal{A}}$;*

(iii) *$d(\nu) = q$ for all interior vertices ν of $\widehat{\mathcal{A}}$.*

Proof. Suppose G satisfies Condition (H-I). The proof for the case when G satisfies Condition (H-II) differs only in the numbers used.

Suppose σ is not a simple closed path. Then σ must contain a simple closed subpath η. Now η bounds a simply-connected subdiagram $\mathcal{B}$ of $\widehat{\mathcal{A}}$. If $\mathcal{B}$ consisted of a single region Δ, then Δ would contradict Hypothesis (A), so assume that Area($\mathcal{B}$) > 1. Since G satisfies (H-I), each edge group has property-W_3. It then follows from Lemmas 3.2.1 and 3.2.2 that $\mathcal{B}$ is a $[3, 8]$-diagram. From Theorem 1.6.2, we have

$$\sum_{\mathcal{B}}^{*} [4 - i(\Delta)] \geqslant 6,$$

so $\mathcal{B}$ must contain at least two simple boundary regions Δ_1, Δ_2 with $i_{\mathcal{B}}(\Delta_j) \leqslant 3$ $(j = 1, 2)$. Since $\mathcal{B}$ can contain at most one simple boundary region which is not a simple boundary region of $\widehat{\mathcal{A}}$, we may assume that Δ_1 is a simple boundary region of both $\mathcal{B}$ and $\widehat{\mathcal{A}}$. Let $\phi(\partial\Delta_1) \equiv W$ and let $\alpha = \partial\Delta_1 \cap \eta$. The W_3 condition implies that $|W|_s \geqslant 8$ and since $i_{\mathcal{B}}(\Delta_1) \leqslant 3$, we have $\phi(\alpha) > \frac{1}{2}\widehat{\mathbf{r}}$. Therefore, Δ_1 is a region which contradicts (A). We deduce that σ must be a simple closed path. The same remarks apply to τ.

We now show that $\widehat{\mathcal{A}}$ does not contain a region Δ such that $\partial\Delta \cap \sigma$ is disconnected. Suppose there is such a region. Then $\widehat{\mathcal{A}} - \Delta$ contains at least one component which is a non-empty simply-connected subdiagram $\mathcal{B}$ of $\widehat{\mathcal{A}}$. If $\mathcal{B}$ contained only one region, then this region would contradict (A). Therefore, Area($\mathcal{B}$) $\geqslant 2$.

Since $\mathcal{B}$ is a simply-connected $[3, 8]$-diagram, it follows from Theorem 1.6.2 that $\mathcal{B}$ must contain a simple boundary region Δ' with $i_{\mathcal{B}}(\Delta') \leqslant 3$. Suppose $i_{\mathcal{B}}(\Delta') < 3$ and let $\gamma = \partial\Delta' \cap \mathcal{B}$. Since $i_{\mathcal{B}}(\Delta') \leqslant 2$, the W_3 condition implies that $|\phi(\gamma)|_s \geqslant 6$. The only possibility is that $\gamma = \gamma_1\gamma_2$ where γ_1 is a non-empty simple subpath of $\partial\Delta$ and γ_2 is a non-empty simple subpath of σ (see Fig. 3.14). Since Δ' can have at most one edge in common with Δ, γ_1 contains exactly one edge. Therefore, γ_2 must contain at least 5 edges. The W_3 condition then implies that $|\phi(\gamma_2)|_s \geqslant 5$ and so $\phi(\gamma_2) > \frac{1}{2}\widehat{\mathbf{r}}$, which contradicts (A).

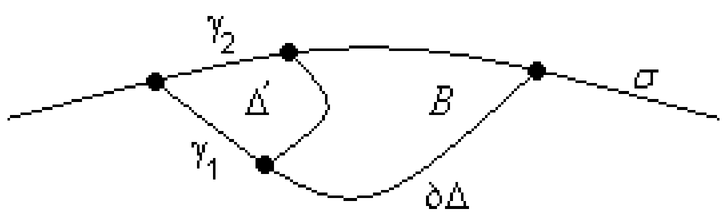


Figure 3.14: The simple paths γ_1 and γ_2.

If each simple boundary region of $\mathcal{B}$ satisfies $i_{\mathcal{B}}(\Delta) = 3$, then $\mathcal{B}$ must contain at least 6 such regions. It is then easy to show that $\mathcal{B}$ must contain a simple boundary region which is a simple boundary region of $\widehat{\mathcal{A}}$ and which contradicts (A). Thus, $\widehat{\mathcal{A}}$ cannot contain a region Δ such that $\partial\Delta \cap \sigma$ is disconnected. The same is true if we replace σ by τ.

Delete the label and orientation of each edge of $\widehat{\mathcal{A}}$, and remove any boundary vertices of degree two. Then each vertex of $\widehat{\mathcal{A}}$ has degree at least 3 and if Δ is any region of $\widehat{\mathcal{A}}$, $\partial\Delta \cap \widehat{\mathcal{A}}$ contains at most one boundary edge. If $\partial\Delta$ contains a boundary edge, then $i(\Delta) \geqslant 4$ by Hypothesis (A). If $\partial\Delta$ has no edge in $\partial\widehat{\mathcal{A}}$, then $d(\Delta) \geqslant 8$. Let $\widehat{\mathcal{A}}^*$ be the dual map of $\widehat{\mathcal{A}}$. Applying Theorem 1.6.1 to $\widehat{\mathcal{A}}^*$, we have

$$6(Q_{\widehat{\mathcal{A}}^*} - h_{\widehat{\mathcal{A}}^*}) = \sum_{\widehat{\mathcal{A}}^*}^{\bullet} [4 - d_{\widehat{\mathcal{A}}^*}(\nu)] + \sum_{\widehat{\mathcal{A}}^*}^{\circ} [6 - d_{\widehat{\mathcal{A}}^*}(\nu)] + 2\sum_{\widehat{\mathcal{A}}^*} [3 - d_{\widehat{\mathcal{A}}^*}(\Delta)] + 2(V^{\bullet}_{\widehat{\mathcal{A}}^*} - E^{\bullet}_{\widehat{\mathcal{A}}^*}).$$

Since $\widehat{\mathcal{A}}^*$ has at most one hole, $6(Q_{\widehat{\mathcal{A}}^*} - h_{\widehat{\mathcal{A}}^*}) \geqslant 0$. Each region of $\widehat{\mathcal{A}}$ has interior edges by Hypothesis (B), so $\widehat{\mathcal{A}}^*$ has no isolated vertices and $V^{\bullet}_{\widehat{\mathcal{A}}^*} \leqslant E^{\bullet}_{\widehat{\mathcal{A}}^*}$. Therefore,

$$0 \leqslant \sum_{\widehat{\mathcal{A}}^*}^{\bullet} [4 - d_{\widehat{\mathcal{A}}^*}(\nu)] + \sum_{\widehat{\mathcal{A}}^*}^{\circ} [6 - d_{\widehat{\mathcal{A}}^*}(\nu)] + 2\sum_{\widehat{\mathcal{A}}^*} [3 - d_{\widehat{\mathcal{A}}^*}(\Delta)].$$

Using the correspondence between $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^*$, we have

$$0 \leqslant \sum_{\widehat{\mathcal{A}}}^{\bullet} [4 - i_{\widehat{\mathcal{A}}}(\Delta)] + \sum_{\widehat{\mathcal{A}}}^{\circ} [6 - d_{\widehat{\mathcal{A}}}(\Delta)] + 2\sum_{\widehat{\mathcal{A}}}^{\circ} [3 - d_{\widehat{\mathcal{A}}}(\nu)].$$

Each term of the first sum is non-positive. Also, each term in the last sum is non-positive. If $\widehat{\mathcal{A}}$ contained an interior region, the second sum would be negative. Therefore, $\widehat{\mathcal{A}}$ cannot contain any interior regions. If some boundary region Δ of $\widehat{\mathcal{A}}$ had $i(\Delta) > 4$, then the first sum would be negative. Hence, $i(\Delta) = 4$ for all regions Δ of $\widehat{\mathcal{A}}$. Similarly, from the last term we conclude that $d(\nu) = 3$ for all vertices ν of $\widehat{\mathcal{A}}$. □

Theorem 3.2.2. *Let G satisfy* (H-I) *or* (H-II) *and assume the following two hypotheses on $\widehat{\mathcal{A}}$.*

(A) *If Δ is a region of $\widehat{\mathcal{A}}$ with $\alpha = \partial\Delta \cap \sigma$ a consecutive part of σ, then $\phi(\alpha) \not> \frac{1}{2}\hat{\mathbf{r}}$. Assume the same hypothesis replacing σ by τ.*

(B) *There is a region Λ of $\widehat{\mathcal{A}}$ such that $\partial\Lambda$ contains an edge of both σ and τ.*

Then every region Δ of $\widehat{\mathcal{A}}$ has edges on both $\sigma_{\mathcal{A}}$ and $\tau_{\mathcal{A}}$, and $i(\Delta) \leqslant 2$.

Proof. Suppose G satisfies Condition (H-I). The proof for the case when G satisfies Condition (H-II) differs only in the numbers used.

In view of Hypotheses (A) it follows, exactly as in the proof of Theorem 3.2.1, that $\widehat{\mathcal{A}}$ does not contain a region whose boundary has disconnected intersection with one of the boundaries of $\widehat{\mathcal{A}}$. Furthermore, σ and τ are both simple closed paths and if Δ is a region of $\widehat{\mathcal{A}}$ such that $\partial\Delta$ contains an edge of only one of σ or τ, then $i(\Delta) \geqslant 4$.

Delete the label and orientation of each edge of $\widehat{\mathcal{A}}$ to obtain an ordinary unoriented annular diagram $\mathcal{N}'$, which we may assume does not contain any vertices of degree two. Using the operations illustrated in Figs. 3.10 and 3.11, we may remove any bridges or pinches contained in $\mathcal{N}'$ to obtain an annular diagram $\mathcal{N}$ in which each region has at least one interior edge. Note that $\mathcal{N}$ is a $[3, 8]$-annular diagram and if Δ is a region of $\mathcal{N}$, then $i_{\mathcal{N}}(\Delta) \geqslant i_{\widehat{\mathcal{A}}}(\Delta)$.

Our aim is to show that for each region Δ of $\mathcal{N}$, $i_{\mathcal{N}}(\Delta) = 2$ and Δ has edges on both $\sigma_{\mathcal{N}}$ and $\tau_{\mathcal{N}}$. We proceed by induction on Area($\mathcal{N}$). It is clear that the result holds if Area($\mathcal{N}$) $= 2$, so assume Area($\mathcal{N}$) > 2.

By Hypothesis (B), $\mathcal{N}$ contains a region Λ such that $\partial\Lambda$ contains an edge of both $\sigma_{\mathcal{N}}$ and $\tau_{\mathcal{N}}$. Since $\mathcal{N}$ does not contain any pinches, $i_{\mathcal{N}}(\Delta) \geqslant 2$. It follows that there is a region Π on one side of Λ and a simple path β from $\sigma_{\mathcal{N}}$ to $\tau_{\mathcal{N}}$ such that $\beta \subseteq \partial\Lambda$ and β contains an edge of $\partial\Pi$. Cut $\mathcal{N}$ open along β as in Fig. 3.15 and adjoin a copy Λ_1 of Λ along the side of β that borders Π to obtain a simply-connected diagram $\mathcal{N}_1$.

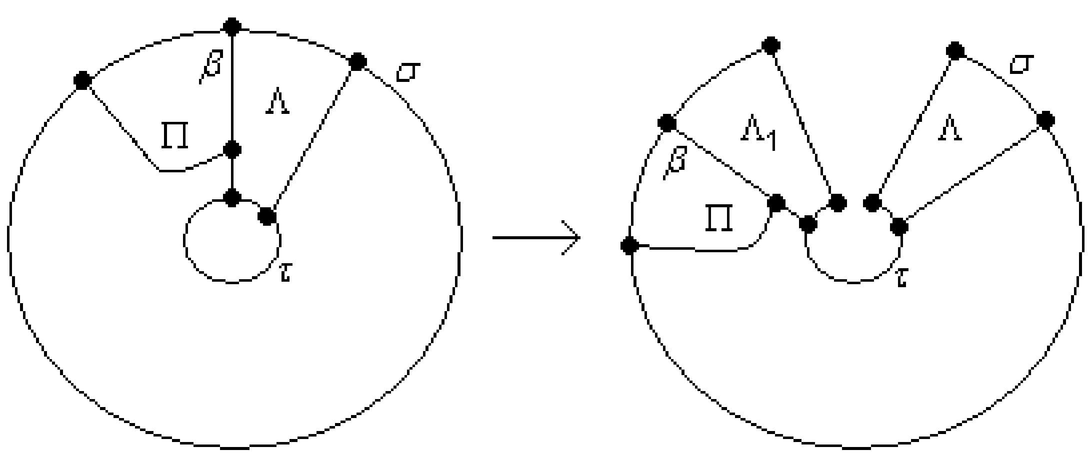


Figure 3.15: Cutting $\mathcal{N}$ open along β.

Note that if a region Δ' of $\mathcal{N}$ which is not equal to Λ has edges on both $\sigma_{\mathcal{N}}$ and $\tau_{\mathcal{N}}$, then $\partial\Delta' \cap \partial\mathcal{N}_1$ is not a consecutive part of $\partial\mathcal{N}_1$. Also, if $\partial\Delta'$ has edges on only one boundary of $\mathcal{N}$, then $i_{\mathcal{N}_1}(\Delta) \geqslant 4$. Now $\mathcal{N}_1$ is a $[3, 8]$-diagram so from Theorem 1.6.2, we have

$$\sum_{\mathcal{N}_1}^{*} [4 - i_{\mathcal{N}_1}(\Delta)] \geqslant 6.$$

By the above remarks, only Λ and Λ_1 can make a positive contribution to this sum. Since $i_{\mathcal{N}_1}(\Lambda_1) \geqslant 1$, one of the following holds:

(i) $i_{\mathcal{N}_1}(\Lambda_1) = 1$ and $i_{\mathcal{N}_1}(\Lambda) = 0$;

(ii) $i_{\mathcal{N}_1}(\Lambda_1) = 1$ and $i_{\mathcal{N}_1}(\Lambda) = 1$;

(iii) $i_{\mathcal{N}_1}(\Lambda_1) = 2$ and $i_{\mathcal{N}_1}(\Lambda) = 0$.

However, $\mathcal{N}$ does not contain any pinches, so $i_{\mathcal{N}_1}(\Lambda) \neq 0$. Therefore, $i_{\mathcal{N}_1}(\Lambda_1) = i_{\mathcal{N}_1}(\Lambda) = 1$. It follows that in $\mathcal{N}$, β is completely contained in $\partial\Pi$ (see Fig. 3.16). Furthermore, if there is a region Φ on the other side of Λ, then Φ is the only such region. We can now apply the same argument to Π and Φ. Thus working out from Λ, we can prove that each region of $\mathcal{N}$ has exactly two interior edges in its boundary. By reinstating any bridges and pinches that were deleted in the construction of $\mathcal{N}$, we deduce that each region of $\widehat{\mathcal{A}}$ has an edge on both $\sigma_{\mathcal{A}}$ and $\tau_{\mathcal{A}}$, and that each region satisfies $i(\Delta) \leqslant 2$. $\square$

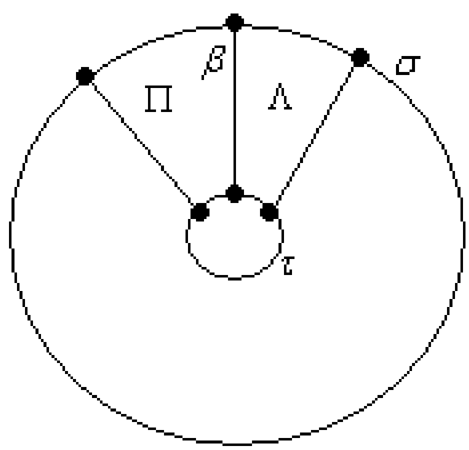


Figure 3.16: β completely contained in $\partial\Pi$.

The annular diagram illustrated in Fig. 3.17 satisfies the hypotheses of Theorem 3.2.2.

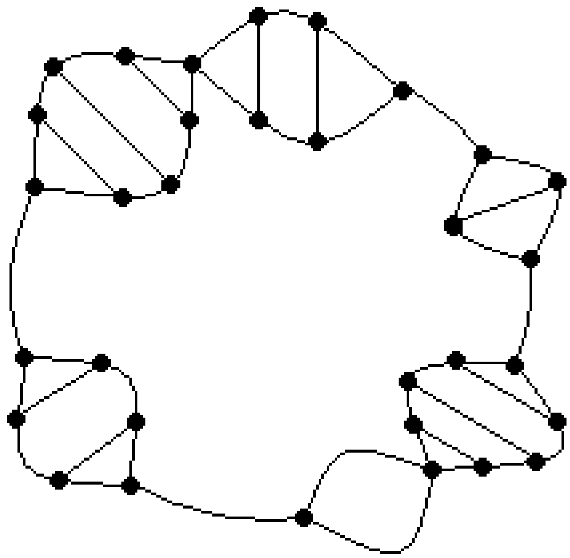


Figure 3.17: An annular derived diagram satisfying the hypotheses of Theorem 3.2.2.

Chapter 4

The word and conjugacy problems for a vertex-finite Pride group

This chapter contains the proofs of our results for the word and conjugacy problems for a vertex-finite Pride group.

4.1 The word problem for a vertex-finite Pride group

In this section we prove Theorem 1, which we now restate.

Theorem 4.1.1. *Let G be a vertex-finite Pride group with underlying graph $\Gamma = \{V, E\}$, and let $\delta_E = \max\{\delta_{G_e} : e \in E\}$.*

(1) *If G satisfies* (I) *or* (II)*, then $\delta_G(n) \preccurlyeq n^2\delta_E(n)$ for all $n \in \mathbb{N}$.*

(2) *If G satisfies* (H-I) *or* (H-II)*, then $\delta_G(n) \preccurlyeq n\delta_E(n)$ for all $n \in \mathbb{N}$.*

Proof. Let $\mathcal{P}_s = \langle \mathbf{x}\,;\mathbf{r}\rangle$ be the standard presentation of G. We first prove Statement (1). Statement (2) will then follow easily from the proof of Statement (1).

Suppose G satisfies (I) or (II) and let W be a non-empty word on $\mathbf{x}^{\pm 1}$ that represents the identity element of G. Let $|W| = n$ and assume, without loss of generality, that W is cyclically reduced. By Theorem 1.7.1, there exists a simply-connected $\mathbf{r}$-diagram $\mathcal{S}$ for W, which we may choose to be minimal with respect to $|E(\mathcal{S})|$. If $|E(\mathcal{S})| = 0$, then $\mathcal{S}$ is a simply-connected diagram over a presentation of G_V and we deduce that W represents the identity element of G_V. From

Proposition 1.5.2, we have $\mathrm{Area}(W) \leqslant \max\{\bar{\delta}_{G_v}(n) : v \in V\}$. Since each vertex group is finite, δ_{G_v} is linear for all $v \in V$. Therefore, $\delta_V := \max\{\bar{\delta}_{G_v}\}$ is linear. Since δ_{G_e} is at least linear for all $e \in E$, we deduce that $\delta_V \preccurlyeq \delta_E$. Thus, $\mathrm{Area}(W) \leqslant \delta_E(n)$.

Assume $|E(\mathcal{S})| \geqslant 1$ and let $\mathcal{F}_{\mathcal{S}}$ be a federal subdivision of $\mathcal{S}$ (see §3.1). If $\mathcal{F}_{\mathcal{S}}$ consists of a single federation, then $\mathcal{S}$ is an $\mathbf{r}_e$-diagram for some $e \in E$ and W is a word on $\mathbf{x}_e^{\pm 1}$ that represents the identity element of G_e. Therefore, $\mathrm{Area}(W) \leqslant \delta_{G_e}(n) \leqslant \delta_E(n)$. Now assume that $\mathcal{F}_{\mathcal{S}}$ contains more than one federation. Since G satisfies (I) or (II) and since $\mathcal{S}$ is minimal with respect to $|E(\mathcal{S})|$, it follows that $\mathcal{F}_{\mathcal{S}}$ satisfies Conditions (i) and (ii) of Proposition 3.1.1.

Let $\widehat{\mathcal{S}}$ be the derived diagram of $\mathcal{S}$ corresponding to $\mathcal{F}_{\mathcal{S}}$. If (I) holds, then each edge group has property-W_2, so from Lemmas 3.1.2 and 3.1.3 we have that $\widehat{\mathcal{S}}$ is a $[3,6]$-diagram. On the other hand, if (II) holds, each edge group has property-W_1 and Γ is triangle-free, so $\widehat{\mathcal{S}}$ is a $[4,4]$-diagram. In either case, by Theorem 1.6.4, there exists a number $k > 0$ such that

$$\mathrm{Area}(\widehat{\mathcal{S}}) \leqslant k d(\widehat{\mathcal{S}})^2$$

where $d(\widehat{\mathcal{S}})$ is the degree of $\widehat{\mathcal{S}}$. Since $d(\widehat{\mathcal{S}}) = |W| = n$, we have

$$\mathrm{Area}(\widehat{\mathcal{S}}) \leqslant kn^2. \tag{4.1}$$

Let Δ be a region of $\widehat{\mathcal{S}}$. Since $\widehat{\mathcal{S}}$ is either a $[3,6]$-diagram or a $[4,4]$-diagram which does not contain any vertices of degree 1, and since $d(\widehat{\mathcal{S}}) = n$, it follows from Theorem 1.6.3 that $d(\Delta) \leqslant 2n$. Let U be a label of Δ. Then U is a word on $\mathbf{x}_e^{\pm 1}$ ($e \in E$) that represents the identity element of G_e. Furthermore, $|U| \leqslant 2n$ by Property (2) of Lemma 3.1.1.

It is now convenient to change to pictures (see Remark 4.1.1 for explanation). Let $\widehat{\mathbb{P}}$ be the $\widehat{\mathbf{r}}$-picture corresponding to $\widehat{\mathcal{S}}$ and let D be the disc of $\widehat{\mathbb{P}}$ that corresponds to Δ. Since U represents the identity element of G_e, there exists a minimal simply-connected $\mathbf{r}_e$-picture $\mathbb{P}_D$ for U such that

$$\mathrm{Area}(\mathbb{P}_D) \leqslant \delta_{G_e}(|U|) = \delta_{G_e}(2n).$$

Replace D with $\mathbb{P}_D$ as follows. First, surround D with a circle S^1 such that S^1 is transverse to the arcs incident with D and does not intersect any other arc of $\mathbb{P}$. Next, delete D and the arcs contained in S^1 which are incident with D, and in their place add $\mathbb{P}_D$. Finally, join together the boundary arcs of $\mathbb{P}_D$ and the arcs of $\mathbb{P}$ which meet S^1 in such a way that no two arcs cross each other and only arcs of like label and orientation are joined. In the same way, we proceed to replace the remaining discs of $\widehat{\mathbb{P}}$ with appropriate minimal simply-connected $\mathbf{r}_e$-pictures. In doing so, we obtain a simply-connected $\mathbf{r}$-picture $\mathbb{P}'$ for W. From (4.1), we have

$$\text{Area}(\mathbb{P}') \leqslant M \cdot kn^2,$$

where $M = \max\{\text{Area}(\mathbb{P}_D)\}$ with the maximum taken over all discs D of $\widehat{\mathbb{P}}$. Since $M = \delta_E(2n)$, we have

$$\text{Area}(W) \leqslant \text{Area}(\mathbb{P}') \leqslant kn^2\, \delta_E(2n).$$

Statement (1) of Theorem 4.1.1 now follows.

Now suppose G satisfies (H-I) or (H-II). Statement (2) then follows from the proof of Statement (1) by noting that the derived diagram $\widehat{\mathcal{S}}$ is in this case a $[3,8]$-diagram or a $[4,6]$-diagram, so by Theorem 1.6.3 the bound in (4.1) can be replaced by kn. □

Remark 4.1.1. The reader might ask why in the proof of Theorem 4.1.1 did we not replace each region of the derived diagram $\widehat{\mathcal{S}}$ with a simply-connected $\mathbf{r}_e$-*diagram*? One could certainly fill each region of the derived diagram with such a diagram by gluing together boundary edges of like label and orientation. However, one must check that the resulting diagram is planar. Planarity is guaranteed by working with the corresponding $\mathbf{r}$-picture - any spherical diagram that may have been created during the "gluing in" process will correspond to a spherical subpicture, which may be deleted from the $\mathbf{r}$-picture.

The statement of Corollary 1 (restated below) now follows from Theorem 1.5.1 and Statement (1) of Theorem 4.1.1.

Corollary 4.1.1. *If each edge group has a soluble word problem, then G has a soluble word problem.*

4.1.1 The word problem for non-vertex-finite Pride groups.

In this section we prove Theorems 2 and 3. We begin with the proof of Theorem 2, which will follow from Propositions 4.1.1 and 4.1.2.

Let G be a vertex-finite Pride group which satisfies Condition (II) and let $\Gamma = \{V, E\}$ be the underlying graph of G. The underlying graph $Q(\Gamma)$ of $Q(G)$ is constructed from Γ as follows. Let v be a fixed but arbitrary vertex of Γ. Adjoin a new vertex ω to Γ and adjoin a new edge e_ω which has endpoints v and ω. Then $Q(\Gamma)$ is the graph with vertex set $V(\omega) = V \cup \{\omega\}$ and edge set

$E(\omega) = E \cup \{e_\omega\}$. Since Γ is triangle-free, $Q(\Gamma)$ is triangle-free. Assign the infinite cyclic group $F(\omega)$ to the vertex ω and let $\mathbf{t}_{e_\omega} = \{[g, \omega] : g \in G_v\}$. Then

$$Q(G) = \frac{G_V * F(\omega)}{<< \mathbf{t} \cup \mathbf{t}_{e_\omega} >>}.$$

Proposition 4.1.1. *The Pride group $Q(G)$ with underlying graph $Q(\Gamma)$ satisfies Condition* (II)*.*

Proof. Since G satisfies (II), G_e has property-W_1 for each $e \in E \subset E(\omega)$. Now consider the edge group G_{e_ω}. Let D_{e_ω} be the Cartesian subgroup of $G_v * G_\omega$. Since $\mathbf{t}_{e_\omega} \subseteq D_{e_\omega}$ it follows from Example 2.1.7 that G_{e_ω} has property-W_1. Therefore each edge group of $Q(G)$ has property-W_1. Since $Q(\Gamma)$ is triangle-free, it follows that $Q(G)$ satisfies Condition (II). $\square$

By Proposition 4.1.1 and Corson's result on developable Pride groups [31, p. 562], $Q = Q(G)$ is developable, i.e. each subgraph group embeds in Q. Now Γ is the full subgraph of $Q(\Gamma)$ generated by the vertices $V \subset V(\omega)$ and G is the subgraph group G_Γ. Thus, $G \hookrightarrow Q$. We can now view Q as the following trivial HNN-extension of G:

$$Q = \langle G, t\,; [g, t]\, \forall g \in G_v \rangle.$$

Proposition 4.1.2. *The first order Dehn function δ_Q of Q satisfies*

$$n^2 \preccurlyeq \delta_Q(n) \preccurlyeq n^3 \delta_E(n),$$

where $\delta_E = \max\{\delta_{G_e} : e \in E)\}$.

Proof. Since Q is an HNN-extension of G, it follows from [9, Theorem 1.1] that for all $n \in \mathbb{N}$,

$$\delta_Q(n) \preccurlyeq n \delta_G(\bar{\Delta}^Q_{G_v}(n))$$

where δ_G is the first order Dehn function of G and where $\Delta^Q_{G_v}$ is the length distortion function of G_v in Q (see §1.3 for the definition of this function). Since G_v is a finite subgroup of Q, we have $\Delta^Q_{G_v}(n) \simeq n$ for all $n \in \mathbb{N}$. Therefore, $\delta_Q(n) \preccurlyeq n\delta_G(n)$. We now use Statement (1) of Theorem 4.1.1 to obtain the desired upper bound.

For the lower bound, from [23, Proposition 3.4], we have

$$\delta_Q(n) \succcurlyeq n \Delta^G_{G_v}(n)$$

where $\Delta^G_{G_v}$ is the length distortion function of G_v in G. Since G_v is a finite subgroup of G, we have $\delta_Q(n) \succcurlyeq n^2$ as required. $\square$

We now prove Theorem 3. Recall:

Theorem 4.1.2. *Let G be a vertex-free Pride group which satisfies* (I) *or* (II)*, and let $\Gamma = \{V, E\}$ be the underlying graph of G. Suppose edge group is diagrammatically reducible. Then for all $n \in \mathbb{N}$, $\delta_G(n) \succcurlyeq \delta_E(n)$ where $\delta_E(n) = \max\{\delta_{G_e} : e \in E\}$.*

Proof. Let $\mathcal{P} = \langle \mathbf{x}\,; \mathbf{r}\rangle$ be the *natural* presentation of G defined at the start of Chapter 5. Choose some $e \in E$ and let $\mathcal{P}_e = \langle \mathbf{x}_e\,; \mathbf{r}_e\rangle$ be a diagrammatically reducible presentation of G_e. Let $W_1, W_2, \ldots, W_k$ be a sequence of words on $\mathbf{x}_e^{\pm 1}$ that satisfy the following three conditions:

(i) Each W_i represents the identity element of G_e;

(ii) $n_1 < n_2 < n_3 < \ldots$, where $n_i = |W_i|$;

(iii) $\mathrm{Area}(W_i) = \delta_{G_e}(n_i)$.

There exist, by Theorem 1.8.1, minimal simply-connected $\mathbf{r}_e$-pictures $\mathbb{P}_i$ for each W_i. Since each edge group is diagrammatically reducible, each $\mathbb{P}_i$ is the *unique* simply-connected $\mathbf{r}$-picture for W_i. To see this, suppose $\mathbb{M}$ is another simply-connected $\mathbf{r}$-picture for W which is not equal to $\mathbb{P}_i$. Since $W(\mathbb{M}) \equiv W(\mathbb{P}_i)$, we may identify $\partial\mathbb{P}_i$ with $\partial(-\mathbb{M})$ to obtain a non-empty spherical $\mathbf{r}$-picture $\mathbb{A}$. Since $\mathbb{P}_i$ and $\mathbb{M}$ are distinct, $\mathbb{A}$ is *not* equivalent to the empty picture via bridge moves and the deletion of floating arcs and cancelling pairs alone. This contradicts, however, the fact that G is diagrammatically reducible (G is diagrammatically reducible by Theorem 7). Thus, $\mathbb{M}$ cannot exist. Since each $\mathbb{P}_i$ is the unique minimal simply-connected $\mathbf{r}$-diagram for W_i, we have $\delta_G(n) \succcurlyeq \delta_{G_e}(n)$ for all $n \in \mathbb{N}$. However, this does not depend on the choice of e. Therefore,

$$\delta_G(n) \succcurlyeq \delta_{G_e}(n)$$

for all $n \in \mathbb{N}$ and all $e \in E$. The statement of the result now follows. $\square$

4.2 The conjugacy problem for a vertex-finite Pride group

In this section we prove Theorem 4. Let G be a vertex-finite Pride group which satisfies (I) or (II), and let $\Gamma = \{V, E\}$ be the underlying graph of G. Also, let $\mathcal{P}_s = \langle \mathbf{x}\,; \mathbf{r}\rangle$ be the standard presentation of G. Recall that

$$-G_E = G - \bigcup_{e \in E} G_e$$

and

$$G_V = \mathop{*}_{v \in V} G_v.$$

Assume G satisfies Conditions (1) - (6) of §2.3.2 and let W be a word on $\mathbf{x}^{\pm 1}$. Condition (4) states that for each $e \in E$, the generalized word problem (relative to $\mathbf{x}_e^{\pm 1}$) is soluble. Therefore, we can decide, for each $e \in E$, whether or not $\overline{W} \in G_e$.

Lemma 4.2.1. *If $\overline{W} \in G_e$ for some $e \in E$, then we can find a word U on $\mathbf{x}_e^{\pm 1}$ that represents $\overline{W}$.*

Proof. We can recursively enumerate all words U_i on $\mathbf{x}_e^{\pm 1}$ and using the solution of the word problem for G, we can decide, for each $i \geqslant 1$, whether or not $\overline{W} = \overline{U_i}$. Starting with U_1, we test whether or not $U_1, U_2, U_3, \ldots$ represents W. Since $\overline{W} \in G_e$, this algorithm must terminate by outputting a word U_j on $\mathbf{x}_e^{\pm 1}$ that represents $\overline{W}$. □

Let W, Z be words on $\mathbf{x}^{\pm 1}$. The following lemmas allows us to determine, in certain special cases, whether or not W and Z represent conjugate elements of G.

Lemma 4.2.2. *If W, Z both represent the identity element of G, then they represent conjugate elements of G. If one of W, Z represents the identity element of G while the other does not, then W, Z do not represent conjugate elements of G.*

Proof. Since G has a soluble word problem, we can determine whether or not W, Z represent the identity element of G. The statement of the lemma is trivially true. □

Lemma 4.2.3. *We can decide whether or not W, Z represent conjugate elements of G_V.*

Proof. Each G_v is finite and so has a soluble conjugacy problem. It then follows from the Conjugacy Theorem for Free Products [66, Corollary IV.1.5] that G_V has a soluble conjugacy problem. □

Lemma 4.2.4. *If W, Z represent elements of G_e for some $e \in E$, then we can decide whether or not W, Z represent conjugate elements of G.*

Proof. We can find, by Lemma 4.2.1, words U, V on $\mathbf{x}_e^{\pm 1}$ that represent $\overline{W}$ and $\overline{Z}$, respectively. Then W, Z represent conjugate elements of G if and only if U, V represent conjugate elements of G. Using the solution of the conjugacy problem for G_e (Condition (1)) we can decide whether or not U, V represent conjugate elements of G_e. Moreover, since G_e is malnormal in G (Condition (2)), U, V represent conjugate elements of G if and only if they represent conjugate elements of G_e. Thus, we can decide whether or not U, V represent conjugate elements of G. □

Lemma 4.2.5. *If W represents an element of G_e and Z represents an element of G_f, where $e, f \in E$ with $e \neq f$, then we can decide whether or not W, Z represent conjugate elements of G.*

Proof. We can find, by Lemma 4.2.1, a word U on $\mathbf{x}_e^{\pm 1}$ that represents $\overline{W}$ and a word V on $\mathbf{x}_f^{\pm 1}$ that represents $\overline{Z}$. Then W, Z represent conjugate elements of G if and only if U, V represent conjugate elements of G. The latter part of this statement is decidable by Condition (3). $\square$

Lemma 4.2.6. *If W represents an element of G_e $(e \in E)$ and Z represents an element of $-G_E$, then we can decide whether or not W, Z represent conjugate elements of G. The same is true if we interchange the roles of W and Z.*

Proof. We can find, by Lemma 4.2.1, a word U on $\mathbf{x}_e^{\pm 1}$ that represents W. Then W, Z represent conjugate elements of G if and only if U, Z represent conjugate elements of G. The latter part of this statement if decidable by Condition (5). $\square$

Lemma 4.2.7. *If W, Z represent elements of $-G_E$, then we can decide whether or not there exists an edge $e \in E$ and a word U on $\mathbf{x}_e^{\pm 1}$ such that W, U and U, Z represent conjugate element of G. If such a word exists, then W, Z represent conjugate elements of G.*

Proof. For each $e \in E$, we can decide, by Condition (6), whether or not such a word U on $\mathbf{x}_e^{\pm 1}$ exists. Since the number of edges is finite, we have only finitely many checks to make. The last sentence of the statement of the lemma is clearly true. $\square$

We can decide, by Condition (4), whether or not there exists an edge $e \in E$ such that W (respectively, Z) represents an element of G_e. Therefore, we can decide if the hypotheses of Lemmas 4.2.4 - 4.2.7 hold for W (respectively, Z). We can now assume that W and Z satisfy the following conditions:

(i) W and Z both represent non-identity elements of G;

(ii) W and Z do not represent conjugate elements of G_V;

(iii) W and Z are words on $\mathbf{x}^{\pm 1}$ that represent distinct elements of $-G_E$;

(iv) For each $e \in E$, there does not exist a word U on $\mathbf{x}_e^{\pm 1}$ such that W, U and U, Z represent conjugate elements of G.

Using the solution of the word problem for G, we can decide, given any proper subword of any cyclic permutation of W or Z, whether or not this subword represents the identity element of G. Thus, we may assume that W and Z are both cyclically injective words on $\mathbf{x}^{\pm 1}$ (recall Definition 1.3.2). We now proceed with the proof of Theorem 4.

Theorem 4.2.1. *Let W and Z be as above and let $n = |W| + |Z|$. Then W and Z represent conjugate elements of G if and only if there exist words $W_1, \dots, W_l$, $Z_1, \dots, Z_l$ on $\mathbf{x}^{\pm 1}$ such that*

$$W \overset{n}{\sim} W' \overset{10n}{\sim} W_1 \overset{10n}{\sim} \dots \overset{10n}{\sim} W_l \overset{20n}{\sim} Z_l \overset{10n}{\sim} \dots \overset{10n}{\sim} Z_1 \overset{10n}{\sim} Z' \overset{n}{\sim} Z$$

where $|W_i|, |Z_i| \leqslant 10qn^2$ $(i = 1, \dots, l)$ for $q = 3$ or 4 (depending if (I) or (II) holds, respectively), and where W', Z' are cyclic permutations of W and Z, respectively.

Proof. The "if" part of the statement is trivially true so assume that W, Z represent conjugate elements of G. There exists an annular $\mathbf{r}$-diagram $\mathcal{A}$ for the pair (W, Z^{-1}) which we may choose to be minimal with respect to $|E(\mathcal{A})|$. Since W, Z do not represent conjugate elements of G_V, we have $|E(\mathcal{A})| \geqslant 1$. Let σ be the outer boundary of $\mathcal{A}$ and let τ be the inner boundary of $\mathcal{A}$. Since W and Z are both cyclically injective words, σ and τ are simple closed paths.

Let $\mathcal{F}_{\mathcal{A}}$ be a federal subdivision of $\mathcal{A}$ and suppose $\mathcal{F}_{\mathcal{A}}$ contains an annular federation $\mathcal{F}$, where $\Sigma(\mathcal{F}) = e$ for some $e \in E$ (see Fig. 3.7). Let $\sigma_{\mathcal{F}}$ be the outer boundary of $\mathcal{F}$ and let U be the label of an outer boundary cycle of $\mathcal{F}$. Note that U is word on $\mathbf{x}_e^{\pm 1}$. If $\sigma_{\mathcal{F}} = \sigma$, then U is a cyclic permutation of W which contradicts the fact that W represents an element of $-G_E$. Therefore, $\sigma_{\mathcal{F}}$ and σ are distinct. Similarly, $\sigma_{\mathcal{F}}$ and τ are distinct. Let $\mathcal{A}_1$ be the annular subdiagram of $\mathcal{A}$ that contains $\sigma_{\mathcal{F}}$ and σ, and all of $\mathcal{A}$ between $\sigma_{\mathcal{F}}$ and σ. Then $\mathcal{A}_1$ is an annular $\mathbf{r}$-diagram for the pair (W, U^{-1}) and hence W, U represent conjugate elements of G. Let $\mathcal{A}_2$ be the annular subdiagram of $\mathcal{A}$ that contains $\sigma_{\mathcal{F}}$ and τ, and all of $\mathcal{A}$ between $\sigma_{\mathcal{F}}$ and τ. Then $\mathcal{A}_2$ is an annular $\mathbf{r}$-diagram for the pair (U, Z^{-1}) and hence U, Z represent conjugate elements of G. However, we have assumed that such a word U does not exist. Thus, $\mathcal{F}_{\mathcal{A}}$ cannot contain any annular federations.

Since G satisfies (I) or (II) it follows that $\mathcal{F}_{\mathcal{A}}$ satisfies Conditions (i) and (ii) of Proposition 3.2.1. We may then construct the derived diagram $\widehat{\mathcal{A}}$ of $\mathcal{A}$ corresponding to $\mathcal{F}_{\mathcal{A}}$. If G satisfies (I), then $\widehat{\mathcal{A}}$ is an annular $[3,6]$-diagram and if G satisfies (II), $\widehat{\mathcal{A}}$ is an annular $[4,4]$-diagram. Let $(q, p) = (3, 6)$ or $(4, 4)$, depending on the case. Since $\widehat{\mathcal{A}}$ has n boundary edges, there can be no more than n regions which have edges on $\partial\widehat{\mathcal{A}}$. If Δ is a boundary region of $\widehat{\mathcal{A}}$ such that $\partial\Delta \cap \partial\widehat{\mathcal{A}}$

does not contain an edge, then all edges of Δ are interior edges and $i(\Delta) \geqslant p$. Therefore,

$$\frac{q}{p} \sum_{\widehat{\mathcal{A}}}^{\bullet} [p - i(\Delta)] \leqslant \frac{q}{p} \cdot pn = qn.$$

Consider the sequence of diagrams $\widehat{\mathcal{A}} = \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_k$ where $\mathcal{R}_i$ is obtained from $\mathcal{R}_{i-1}$ by removing the boundary layer and gaps of $\mathcal{R}_{i-1}$, and $\mathcal{R}_k$ is the first diagram obtained by this process which contains a boundary linking pair. Let $\widehat{\mathcal{A}} = \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k$ be the sequence of diagrams where $\mathcal{A}_i$ is obtained from $\mathcal{A}_{i-1}$ by removing the boundary layer of $\mathcal{A}_{i-1}$, and the process is continued until the boundary layer of $\mathcal{A}_k$ is equal to $\mathcal{A}_k$. By construction, we have

$$\beta(\mathcal{R}_i) \leqslant \beta(\mathcal{A}_i)$$

for $i = 0, 1, \ldots, k$, where $\beta(\mathcal{R}_i)$ (respectively, $\beta(\mathcal{A}_i)$) is the number of boundary regions contained in the boundary layer of $\mathcal{R}_i$ (respectively, $\mathcal{A}_i$). Hence, from Theorem 1.6.5, we have

$$\beta(\mathcal{R}_i) \leqslant \beta(\mathcal{A}_i) \leqslant \frac{q}{p} \sum_{\widehat{\mathcal{A}}}^{\bullet} [p - i(\Delta)] \leqslant qn$$

for $i = 0, 1, \ldots, k$.

By Proposition 3.2.2, the degree of each region of $\widehat{\mathcal{A}}$ is at most $10n$. Therefore, the degree of each region of $\mathcal{R}_i$ ($i = 0, 1, \ldots, k$) is at most $10n$. If $i > 0$, then each boundary edge of $\mathcal{R}_i$ is an edge in the boundary of a boundary region of $\mathcal{R}_{i-1}$. It follows that the number of edges contained in the outer (respectively, inner) boundary of $\mathcal{R}_i$ is at most $10n \cdot qn = 10qn^2$. Thus, if U is the label of a boundary cycle of $\mathcal{R}_i$, then $|U| \leqslant 10qn^2$ by Property (2) of Lemma 3.1.1.

Let σ_i be the outer boundary of $\mathcal{R}_i$ and let τ_i be the inner boundary of $\mathcal{R}_i$. Let $\mathcal{J}_i$ ($i = 0, \ldots, k-1$) be the subdiagram of $\widehat{\mathcal{A}}$ consisting of σ_i and σ_{i+1}, and all of $\widehat{\mathcal{A}}$ between σ_i and σ_{i+1}. Let $\mathcal{K}_i$ ($i = 0, \ldots, k-1$) be the subdiagram of $\widehat{\mathcal{A}}$ consisting of τ_i and τ_{i+1}, and all of $\widehat{\mathcal{A}}$ between τ_i and τ_{i+1} (see Fig. 4.1). Because of the nature of the subdiagrams $\mathcal{J}_i$, each $\mathcal{J}_i$ must contain a region Δ_i that intersects both the inner and the outer boundary of $\mathcal{J}_i$. Note, $d(\Delta_i) \leqslant 10n$. Therefore, there is a simple path γ_i from σ_i to σ_{i+1} with $|\phi(\gamma_i)| \leqslant 10n$. Let J_i and J_{i+1}^{-1} be the labels of an outer and an inner boundary cycle of $\mathcal{J}_i$, respectively, and let $|\phi(\gamma_i)| \equiv V_i$. Cut $\mathcal{J}_i$ open along γ_i to obtain a simply-connected $\widehat{\mathbf{r}}$-diagram $\mathcal{S}_i$ with boundary label $J_i V_i J_{i+1}^{-1} V_i^{-1}$.

Once again it is convenient to change to pictures (see Remark 4.1.1 for explanation). Let $\mathbb{M}_i$ be the $\widehat{\mathbf{r}}$-picture corresponding to $\mathcal{S}_i$. Replace each disc of $\mathbb{M}_i$ with an appropriate simply-connected

$\mathbf{r}_e$-picture to obtain a simply-connected $\mathbf{r}$-picture $\mathbb{P}_i$ (see the proof of Theorem 4.1.1 for details). Now, $W(\mathbb{P}_i) \equiv J_i V_i J_{i+1}^{-1} V_i^{-1}$. Therefore, J_i and J_{i+1} represent conjugate elements of G, and V_i represents a conjugating element for $\overline{J}_i, \overline{J}_{i+1}$. Thus,

$$J_i \overset{10n}{\sim} J_{i+1}. \tag{4.2}$$

Now J_{i+1} will be a label of an outer boundary cycle of $\mathcal{J}_{i+1}$, so we can repeat the above argument with $\mathcal{J}_{i+1}$ in place of $\mathcal{J}_i$ to show that

$$J_{i+1} \overset{10n}{\sim} J_{i+2}.$$

Similarly, we have

$$K_i \overset{10n}{\sim} K_{i+1}, \tag{4.3}$$

where K_i^{-1} and K_{i+1} are the labels of an inner and an outer boundary cycle of $\mathcal{K}_i$, respectively.

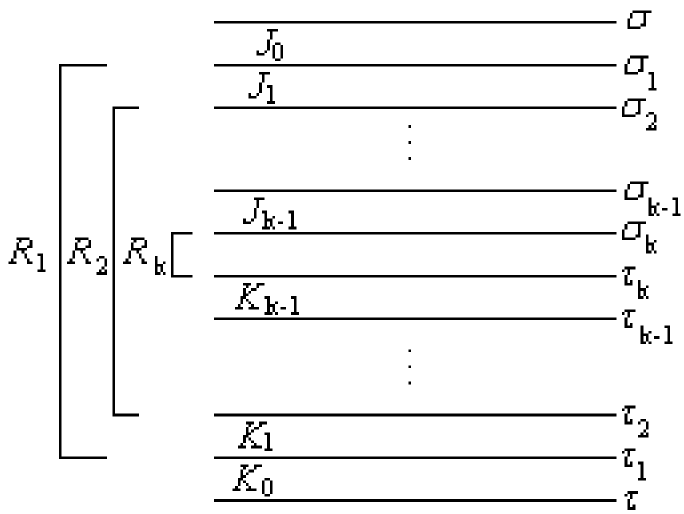


Figure 4.1: The diagrams $\mathcal{J}_i$ and $\mathcal{K}_i$.

There is a boundary linking pair (Δ_1, Δ_2) in $\mathcal{R}_k$ so there are vertices ν_0 in $\sigma_k \cap \partial\Delta_1$, ν_1 in $\partial\Delta_1 \cap \partial\Delta_2$ and ν_2 in $\partial\Delta_2 \cap \tau_k$. Therefore, there is a simple path $\beta_1 \subseteq \partial\Delta_1$ from ν_0 to ν_1 and a simple path $\beta_2 \subseteq \partial\Delta_2$ from ν_1 to ν_2 (see Fig. 4.2). Note that β_1 and β_2 both contain at most $10n$ edges. Let $\beta = \beta_1\beta_2$, and let $\phi(\beta_1) \equiv B_1$ and $\phi(\beta_2) \equiv B_2$. Then $B :\equiv \phi(\beta) \equiv B_1B_2$ and $|B| = |B_1| + |B_2| \leqslant 20n$. Let J_k be the label of the outer boundary cycle of $\mathcal{R}_k$ that starts and ends at ν_0 and let K_k^{-1} be the label of the inner boundary cycle of R_k that starts and ends at ν_2. Cut $\mathcal{R}_k$ open along β to obtain a simply-connected $\widehat{\mathbf{r}}$-diagram $\mathcal{S}_k$ with boundary label $J_kBK_k^{-1}B^{-1}$.

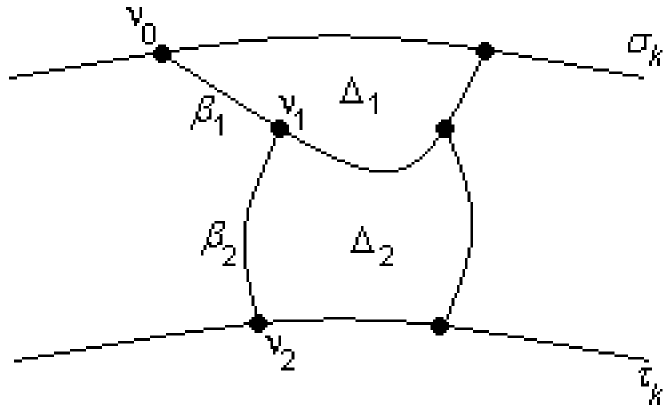


Figure 4.2: The boundary linking pair (Δ_1, Δ_2).

Let $\mathbb{M}_k$ be the simply-connected $\widehat{\mathbf{r}}$-picture corresponding to $\mathcal{S}_k$ and let $\mathbb{P}_k$ be the simply-connected $\mathbf{r}$-picture obtained from $\mathbb{M}_k$ by replacing each disc of $\mathbb{M}_k$ with an appropriate simply-connected $\mathbf{r}_e$-picture. Since $W(\mathbb{P}_k) \equiv J_k B K_k^{-1} B^{-1}$, it follows that J_k and K_k represent conjugate elements of G, with B representing a conjugating element for $\overline{J_k}, \overline{K_k}$. Thus,

$$J_k \overset{20n}{\sim} K_k. \tag{4.4}$$

From (4.2), (4.3) and (4.4), and noting that J_0, K_0 are cyclic permutations of W and Z, respectively, we have

$$W \overset{n}{\sim} J_0 \overset{10n}{\sim} J_1 \overset{10n}{\sim} \ldots \overset{10n}{\sim} J_{k-1} \overset{10n}{\sim} J_k \overset{20n}{\sim} K_k \overset{10n}{\sim} K_{k-1} \overset{10n}{\sim} \ldots \overset{10n}{\sim} K_1 \overset{10n}{\sim} K_0 \overset{n}{\sim} Z.$$

Furthermore, $|J_i|, |K_i| \leqslant 10qn^2$ for $i = 1, \ldots, k$. □

4.2.1 The conjugacy problem for a vertex-finite Pride group satisfying (H-I) or (H-II)

In this section we prove Theorem 5. Let $\mathcal{P}_s = \langle \mathbf{x} \,;\, \mathbf{r} \rangle$ be the standard presentation of a vertex-finite Pride group G, and let $\Gamma = \{V, E\}$ be the underlying graph of G. We now define the concept of an $\widehat{\mathbf{r}}$-reduced word. Recall the notation $U > \frac{1}{2}\widehat{\mathbf{r}}$ introduced in §3.2.

Definition 4.2.1. Let W be a word on $\mathbf{x}^{\pm 1}$. Then W is $\widehat{\mathbf{r}}$-*reduced* if it does not contain a subword U such that $U > \frac{1}{2}\widehat{\mathbf{r}}$. We say that W is *cyclically* $\widehat{\mathbf{r}}$-*reduced* if each cyclic permutation of W is $\widehat{\mathbf{r}}$-reduced.

Let W and Z be words on $\mathbf{x}^{\pm 1}$. The proof of Theorem 5 depends on Theorems 3.2.1 and 3.2.2, which give precise information about the structure of possible annular diagrams for the pair (W, Z^{-1}). In order to use these structure theorems we require that W and Z are both cyclically $\widehat{\mathbf{r}}$-reduced. However, it is not immediately clear that we can assume such a condition. Lemma 4.2.9 allows us to do precisely this. We require a new definition before we can prove this lemma.

Definition 4.2.2. A word is $\mathbf{x}$-*reduced* if successive letters a, b are such that $a \in \mathbf{x}_u$ and $b \in \mathbf{x}_v$ for some $u, v \in V$ with $u \neq v$. A word is *cyclically* $\mathbf{x}$-*reduced* if each cyclic permutation is $\mathbf{x}$-reduced.

If W is $\mathbf{x}$-reduced, then $|W|_s = |W|$. This fact will play an important role in the proof of Lemma 4.2.9. We also make use of the following observation. Suppose W is not cyclically $\widehat{\mathbf{r}}$-reduced. Then some cyclic permutation of W contains a subword U such that $U > \frac{1}{2}\widehat{\mathbf{r}}$, i.e. there exists some $R \in \widehat{\mathbf{r}}_e$ $(e \in E)$ such that $R \equiv UZ$ with $|U|_s > \frac{1}{2}|R|_s$. Now, Z is not $\mathbf{x}$-reduced in general; however, we can use the defining relators $\mathbf{r}$ to obtain an $\mathbf{x}$-reduced word Z_1 that represents $\overline{Z}$. Let $R_1 \equiv UZ_1$. Then $R_1 \in \widehat{\mathbf{r}}_e$ and $|U| > \frac{1}{2}|R_1|_s$. Therefore, if W is not cyclically $\widehat{\mathbf{r}}$-reduced, then there exists a subword U of a cyclic permutation of W and a word $R \in \widehat{\mathbf{r}}$ such that $R \equiv UZ$ where Z *is* $\mathbf{x}$-reduced and $|U|_s > \frac{1}{2}|R|_s$.

Lemma 4.2.8. *If each G_e $(e \in E)$ has a soluble word problem, then we can decide, given a word W on $\mathbf{x}^{\pm 1}$, whether or not W is cyclically $\widehat{\mathbf{r}}$-reduced.*

Proof. We can list all subwords U of all cyclic permutations of W. There are finitely many such subwords and each subword satisfies $|U|_s \leqslant |W|$. Since G is finitely generated, there are finitely many $\mathbf{x}$-reduced words Z on $\mathbf{x}^{\pm 1}$ that satisfy $|Z|_s < |U|_s$. Therefore, there are finitely many words $R \equiv UZ$ where Z is $\mathbf{x}$-reduced and $|U|_s > \frac{1}{2}|R|_s$. For each $e \in E$, we use the solution of the word problem for G_e to decide whether or not each R is an element of $\widehat{\mathbf{r}}_e$. If $R \in \widehat{\mathbf{r}}_e$ for some $e \in E$, then W is not cyclically $\widehat{\mathbf{r}}$-reduced. Otherwise, W is cyclically $\widehat{\mathbf{r}}$-reduced. $\square$

We now prove Lemma 4.2.9.

Lemma 4.2.9. *Given any non-empty word W on $\mathbf{x}^{\pm 1}$ we can write down a cyclically $\widehat{\mathbf{r}}$-reduced word W_1 on $\mathbf{x}^{\pm 1}$ such that W_1 and W represent conjugate elements of G.*

Proof. We proceed by induction on $|W|$. If $|W| = 1$, then W is cyclically $\widehat{\mathbf{r}}$-reduced, so take $W_1 = W$. Assume the results holds for all words of length less than k and let $|W| = k$.

By Lemma 4.2.8, we can decide whether or not W is cyclically $\widehat{\mathbf{r}}$-reduced. If it is, then take $W_1 = W$. Suppose W is not cyclically $\widehat{\mathbf{r}}$-reduced. Then there exists a cyclic permutation W_3 of W, a subword U of W_3, and a word $R \in \widehat{\mathbf{r}}$ such that $R \equiv UZ$ where Z is $\mathbf{x}$-reduced and $|U|_s > \frac{1}{2}|R|_s$. Since Z is $\mathbf{x}$-reduced, $|Z| = |Z|_s < |U|_s \leqslant |U|$. Replace U by Z^{-1} in W_3 to obtain a word W_2 which satisfies $|W_2| < |W_3| = |W|$. Therefore, the inductive hypothesis applies to W_2. We can write down a cyclically $\widehat{\mathbf{r}}$-reduced word W_1 on $\mathbf{x}^{\pm 1}$ such that W_1 and W_2 represent conjugate elements of G. Since $\overline{W_2} = \overline{W_3}$, and since W_3, W represent conjugate elements of G, we deduce that W_1 and W represent conjugate elements of G. □

Now let G satisfy (H-I) or (H-II) and assume G satisfies Conditions (1) - (6) of §2.3.2. Let W, Z be cyclically injective words on $\mathbf{x}^{\pm 1}$. By Lemmas 4.2.2 - 4.2.7, we may assume that W and Z satisfy Conditions (i) - (iv) of §4.2. Furthermore, by Lemma 4.2.9, we can assume the additional condition:

(v) W and Z are cyclically $\widehat{\mathbf{r}}$-reduced words on $\mathbf{x}^{\pm 1}$.

Theorem 4.2.2. *Let W and Z be as above and let $n = |W| + |Z|$. Then W, Z represent conjugate elements of G if and only if one of the following two conditions holds:*

(C1) *There exist cyclic permutations W' and Z' of W and Z, respectively, such that*

$$W \overset{n}{\sim} W' \overset{3}{\sim} Z' \overset{n}{\sim} Z.$$

(C2) *There exist cyclic permutations W' and Z' of W and Z, respectively, such that $\overline{W'} = \overline{Z'}$.*

Proof. The "if" part of the statement is trivially true so assume that W, Z represent conjugate elements of G. Let $\mathcal{A}$ be an annular $\mathbf{r}$-diagram for the pair (W, Z^{-1}), which is minimal with respect to $|E(\mathcal{A})|$ $(\geqslant 1)$. Let σ (respectively, τ) be the outer (respectively, inner) boundary of $\mathcal{A}$. Arguing as in the proof of Theorem 4.2.1, we may assume that $\mathcal{A}$ does not contain any annular federations. Let $\mathcal{F}_{\mathcal{A}}$ be a federal subdivision of $\mathcal{A}$. Since G satisfies (H-I) or (H-II), $\mathcal{F}_{\mathcal{A}}$ satisfies Conditions (i) and (ii) of Proposition 3.2.1. Let $\widehat{\mathcal{A}}$ be the annular derived diagram of $\mathcal{A}$ corresponding to $\mathcal{F}_{A}$. The proof is split into two cases: the case when G satisfies (H-I) and the case when G satisfies (H-II).

Case 1. Suppose G satisfies (H-I). Suppose $\widehat{A}$ does not contain a region Λ such that $\partial\Lambda$ has an edge on both σ and τ. Since W and Z are both cyclically $\widehat{\mathbf{r}}$-reduced, it follows that $\widehat{\mathcal{A}}$ satisfies Hypotheses (A) and (B) of Theorem 3.2.1. Let $\mathcal{B}_1$ be the union of σ and all regions of $\widehat{\mathcal{A}}$ whose boundaries contain an edge of σ. Define $\mathcal{B}_2$ similarly, replacing σ by τ. Then $\mathcal{B}_1 \cup \mathcal{B}_2 = \widehat{\mathcal{A}}$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ are both annular $\widehat{\mathbf{r}}$-diagrams. Let Δ_1, Δ_2 be regions of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, which have an edge in common. There exists an edge $\varepsilon_1 \subseteq \partial\Delta_1$ that connects a vertex ν_σ on σ to a vertex ν_1 on the inner boundary of $\mathcal{B}_1$. Similarly, there exists an edge $\varepsilon_2 \subseteq \partial\Delta_2$ that connects a vertex ν_τ on τ to a vertex ν_2 on the outer boundary of $\mathcal{B}_2$. Moreover, we may choose ε_2 so that ν_1 and ν_2 are joined by an edge $\varepsilon \subseteq \partial\Delta_1 \cap \partial\Delta_2$ (see Fig. 4.3). Therefore, we can construct a simple path $\beta = \varepsilon_1\varepsilon\varepsilon_2$ from ν_σ to ν_τ. Let $\phi(\beta) \equiv B$. By Property (2) of Lemma 3.1.1, $|B| = 3$. Let W_1 be the label of the outer boundary cycle of $\widehat{\mathcal{A}}$ that starts and ends at ν_σ, and let Z_1^{-1} be the label of the inner boundary cycle of $\widehat{\mathcal{A}}$ that starts and ends at ν_τ. Cut $\widehat{\mathcal{A}}$ open along β to obtain a simply-connected $\widehat{\mathbf{r}}$-diagram $\mathcal{S}$ with boundary label $W_1BZ_1^{-1}B^{-1}$.

As in previous proofs, we now change to pictures. Let $\mathbb{M}$ be the simply-connected $\widehat{\mathbf{r}}$-diagram corresponding to $\mathcal{S}$. Replace each disc of $\mathbb{M}$ with an appropriate simply-connected $\mathbf{r}_e$-picture to obtain a simply-connected $\mathbf{r}$-picture $\mathbb{P}$ (see the proof of Theorem 4.1.1 for details). Since $W(\mathbb{P}) \equiv W_1BZ_1^{-1}B^{-1}$, we have

$$W \overset{n}{\sim} W_1 \overset{3}{\sim} Z_1 \overset{n}{\sim} Z$$

where W_1 and Z_1 are cyclic permutations of W and Z, respectively.

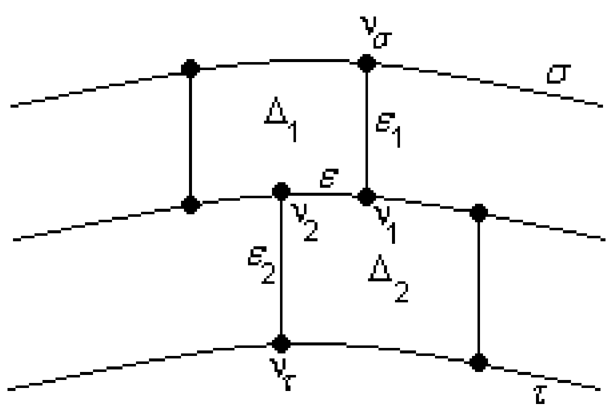


Figure 4.3: Edges ε_1, ε_2 and ε.

Now suppose $\widehat{\mathcal{A}}$ does contain a region Λ that has an edge on both σ and τ. Since W and Z are both cyclically $\widehat{\mathbf{r}}$-reduced, it follows that $\widehat{\mathcal{A}}$ satisfies Hypotheses (A) and (B) of Theorem 3.2.2. Therefore, $i(\Lambda) \leqslant 2$. Suppose $i(\Lambda) \neq 0$. Then there exists an edge ε in the boundary of Λ that connects σ to τ. The label of ε is an element of $\mathbf{x}_u^{\pm 1}$ for some $u \in V$. Cut $\widehat{\mathcal{A}}$ open along ε to obtain a simply-connected $\widehat{\mathbf{r}}$-diagram $\mathcal{S}$ with boundary label $W_2\phi(\varepsilon)Z_2^{-1}\phi(\varepsilon)^{-1}$, where W_2 and Z_2 are cyclic permutations of W and Z, respectively. Let $\mathbb{M}$ be the simply-connected $\widehat{\mathbf{r}}$-picture corresponding to $\mathcal{S}$ and let $\mathbb{P}$ be the simply-connected $\mathbf{r}$-picture obtained from $\mathbb{M}$ by replacing each disc with an appropriate $\mathbf{r}_e$-picture. Since $W(\mathbb{P}) \equiv W_2\phi(\varepsilon)Z_2^{-1}\phi(\varepsilon)^{-1}$, it follows that

$$W \overset{n}{\sim} W_2 \overset{1}{\sim} Z_2 \overset{n}{\sim} Z$$

as required.

Now suppose $i(\Lambda) = 0$. Then $\partial\Lambda$ must contain a pinch ν. Let σ_ν be the outer boundary cycle of $\widehat{\mathcal{A}}$ that starts and ends at ν, and let τ_ν be the inner boundary cycle of $\widehat{\mathcal{A}}$ that starts and ends at ν. Let W_ν be the label of σ_ν and let Z_ν be the label of τ_ν. Note that W_ν is a cyclic permutation of W and Z_ν is a cyclic permutation of Z^{-1}. Break $\widehat{\mathcal{A}}$ open at ν to obtain a simply-connected $\widehat{\mathbf{r}}$-diagram $\mathcal{S}$ with boundary label $W_\nu Z_\nu$ (see Fig. 4.4).

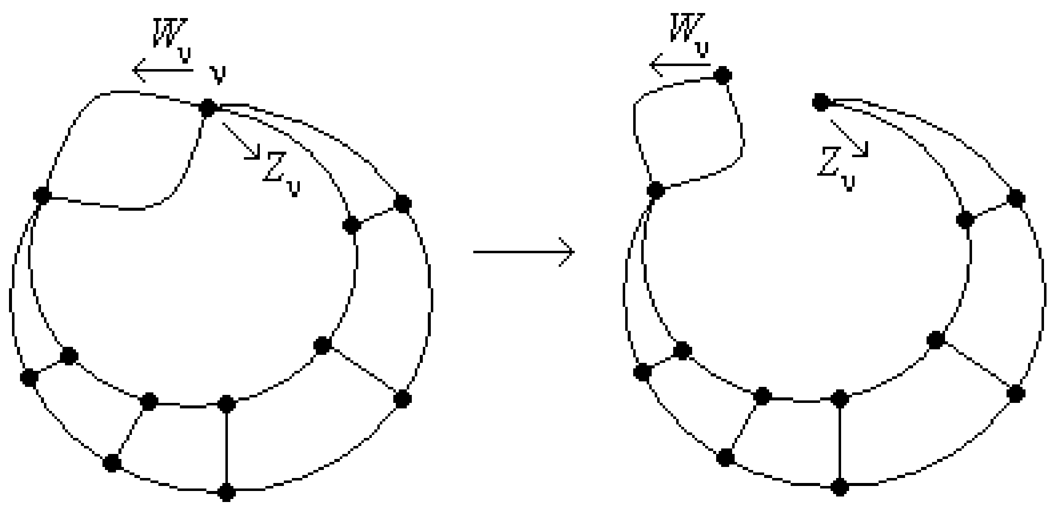


Figure 4.4: Breaking $\widehat{\mathcal{A}}$ open at ν.

Let $\mathbb{M}$ be the simply-connected $\widehat{\mathbf{r}}$-diagram corresponding to $\mathcal{S}$ and let $\mathbb{P}$ be the simply-connected $\mathbf{r}$-picture obtained from $\mathbb{M}$ by replacing each disc with an appropriate simply-connected $\mathbf{r}_e$-picture. Since $W(\mathbb{P}) \equiv W_\nu Z_\nu$, we have $\overline{W_\nu} = \overline{Z_\nu^{-1}}$ where W_ν and Z_ν^{-1} are cyclic permutations of W and Z, respectively.

Case 2. Suppose G satisfies (H-II). Suppose $\widehat{A}$ does not contain a region Λ such that $\partial\Lambda$ has an edge on both σ and τ. Since W and Z are both cyclically $\widehat{\mathbf{r}}$-reduced, it follows that $\widehat{\mathcal{A}}$ satisfies

Hypotheses (A) and (B) of Theorem 3.2.1. Let $\mathcal{B}_1$ be the union of σ and all regions of $\widehat{\mathcal{A}}$ whose boundaries contain an edge of σ. Define $\mathcal{B}_2$ similarly, replacing σ by τ. Then $\mathcal{B}_1 \cup \mathcal{B}_2 = \widehat{\mathcal{A}}$ where $\mathcal{B}_1$ and $\mathcal{B}_2$ are both annular $\widehat{\mathbf{r}}$-diagrams. Let Δ_1, Δ_2 be regions of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, which have a edge in common. There exists an edge $\varepsilon_1 \subseteq \partial\Delta_1$ that connects a vertex ν_σ on σ to a vertex ν_1 on the inner boundary of $\mathcal{B}_1$. Similarly, there exists an edge $\varepsilon_2 \subseteq \partial\Delta_2$ that connects a vertex ν_τ on τ to a vertex ν_2 on the outer boundary of $\mathcal{B}_2$. Moreover, we can choose ε_2 so that $\nu_1 = \nu_2$ as illustrated in Fig. 4.5 below.

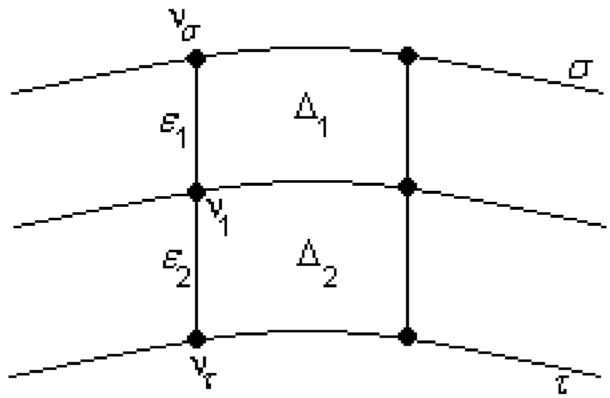


Figure 4.5: Edges ε_1 and ε_2.

We can construct a simple path $\beta = \varepsilon_1\varepsilon_2$ from ν_σ to ν_τ. Let $\phi(\beta) \equiv B$. By Property (2) of Lemma 3.1.1, $|B| = 2$. Let W_3 be the label of the outer boundary cycle of $\widehat{\mathcal{A}}$ that starts and ends at ν_σ, and let Z_3^{-1} be the label of the inner boundary cycle of $\widehat{\mathcal{A}}$ that starts and ends at ν_τ. Cut $\widehat{\mathcal{A}}$ open along β to obtain a simply-connected $\widehat{\mathbf{r}}$-diagram $\mathcal{S}$ with boundary label $W_3BZ_3^{-1}B^{-1}$. Let $\mathbb{M}$ be the simply-connected $\widehat{\mathbf{r}}$-diagram corresponding to $\mathcal{S}$. Replace each disc of $\mathbb{M}$ with an appropriate simply-connected $\mathbf{r}_e$-picture to obtain a simply-connected $\mathbf{r}$-picture $\mathbb{P}$. Since $W(\mathbb{P}) \equiv W_3BZ_3^{-1}B^{-1}$, we have

$$W \overset{n}{\sim} W_3 \overset{2}{\sim} Z_3 \overset{n}{\sim} Z$$

where W_3 and Z_3 are cyclic permutations of W and Z, respectively.

If $\widehat{\mathcal{A}}$ contains a region Λ such that $\partial\Lambda$ contains an edge of both σ and τ, then we argue as in Case 1 to show that either (C1) or (C2) holds. This completes the proof of Theorem 4.2.2. $\square$

4.2.2 Concluding remarks

We have solved the conjugacy problem for a vertex-finite Pride group which satisfies Condition (I) or (II), or Condition (H-I) or (H-II). However, our solutions depend on the assumption that the Pride group also satisfies Conditions (1) - (6). What can be said about these conditions?

Let us first suppose (H-I) or (H-II) holds. Condition (1) states that each edge group has a soluble conjugacy problem and, clearly, this assumption cannot be dropped.

Theorems 3.2.1 and 3.2.2 give us precise information about the structure of annular derived diagrams. We suspect these results can also help in determining the structure of annular **r**-diagrams which contain at least one annular federation. The following should be true. If $\mathcal{A}$ is an annular **r**-diagram, then: either (i) each federation in $\mathcal{A}$ is simply-connected, or (ii) each federation is annular. This result would be a major step forward in solving the decision problems stated in Conditions (3), (5) and (6).

There is no reason to suspect that Condition (2) will be true in general, i.e. that G_e will be malnormal in G for each $e \in E$. However, if the result stated in the previous paragraph is true, then it should be relatively straightforward to determine when two elements of G_e ($e \in E$) are conjugate.

Condition (4) is an interesting question in its own right and deserves further study.

Very little can be said about the structure of annular derived diagrams in the case when (I) or (II) holds. Despite this it is reasonable to expect that the decision problems stated in Conditions (3), (4) and (6) are soluble. Such solutions may also help in determining when Condition (2) holds. We cannot say anything more about Conditions (2) - (6) at this time.

Chapter 5

The second homotopy module of a non-spherical Pride group

In this chapter we prove Theorems 6, 7 and 8, and Proposition 1. Let G be a non-spherical Pride group (recall §2.1, Condition (III)) with underlying graph $\Gamma = \{V, E\}$. We now fix a presentation of G that will be used throughout this chapter. For each $v \in V$, let $\mathcal{P}_v = \langle \mathbf{x}_v \,;\, \mathbf{r}_v \rangle$ be a finite presentation of G_v. For each $e = \{u, v\} \in E$, let $\mathbf{x}_e = \mathbf{x}_u \cup \mathbf{x}_v$ and let $\mathbf{r}_e = \mathbf{r}_u \cup \mathbf{r}_v \cup \mathbf{r}'_e$, where $\mathbf{r}'_e$ is a set of cyclically reduced words on $\mathbf{x}_e^{\pm 1}$ that represent the elements of $\mathbf{t}_e$. (Recall, $\mathbf{t}_e$ is a set of cyclically reduced elements of $\widetilde{G}_e = G_u * G_v$.) Let $\mathbf{r}'$ be the union of the $\mathbf{r}'_e$'s. Then $\mathcal{P}_e = \langle \mathbf{x}_e \,;\, \mathbf{r}_e \rangle$ is a finite presentation of G_e and $\mathcal{P} = \langle \mathbf{x} \,;\, \mathbf{r} \rangle$ is a finite presentation of G, where

$$\mathbf{x} = \bigcup_{v \in V} \mathbf{x}_v \text{ and } \mathbf{r} = \bigcup_{e \in E} \mathbf{r}_e.$$

We call $\mathcal{P}$ the *natural* presentation of G.

5.1 Technicolor pictures

Let I be a fixed set whose elements shall be referred to as *colours* and let $\{m_{ij} : i, j \in I\}$ be a fixed family of elements of $\mathbb{N} \cup \{\infty\}$ such that $m_{ij} = m_{ji}$ and $m_{ij} \geqslant 4$ for $i \neq j$. Following [31], we say that a triple of distinct colours $i, j, k \in I$ is a *spherical* triple if

$$\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ki}} > \frac{1}{2},$$

where $1/\infty := 0$.

Definition 5.1.1. A *colouring* of a picture $\mathbb{P}$ by I is an I-valued function on the set of arcs of $\mathbb{P}$.

A picture together with a colouring function into I is called an *I-coloured* picture.

Lemma 5.1.1. ([31, Lemma 2.2]) *Suppose I does not contain any spherical triples and let $\mathbb{P}$ be a non-spherical simply-connected I-coloured picture satisfying:*

(i) *No arc is a floating circle nor has both endpoints on the same disc enclosing a region of degree 1;*

(ii) *Associated to each disc D are two distinct colours $i, j \in I$ (with $m_{ij} \neq \infty$) such that each arc incident with D is coloured either i or j and there are at least m_{ij} corners of D joining one arc of each colour;*

(iii) *No interior region has more than one corner in its boundary joining arcs of the same two distinct colours.*

If under the above conditions some arc of $\mathbb{P}$ is coloured k, then some arc meeting $\partial\mathbb{P}$ is coloured k.

5.2 Pictures over the natural presentation of a non-spherical Pride group

Let G be a non-spherical Pride group with underlying graph $\Gamma = \{V, E\}$, and let $\mathbb{P}$ be a picture over the natural presentation $\mathcal{P} = \langle \mathbf{x}\,; \mathbf{r}\rangle$ of G. There exists an obvious colouring of $\mathbb{P}$ (by V): arcs labelled by an element of $\mathbf{x}_v$ ($v \in V$) are coloured v. For each pair of vertices $u, v \in V$, we define

$$m_{uv} = \begin{cases} m_e & \text{if } \{u, v\} = e \text{ for some } e \in E; \\ \infty & \text{if } \{u, v\} \notin E. \end{cases}$$

Clearly $m_{uv} = m_{vu}$ and since G is non-spherical, $m_{uv} \geqslant 4$ for $u \neq v$. Thus, we may view $\mathbb{P}$ as a V-coloured picture.

Lemma 5.2.1. *When viewed as a set of colours, V cannot contain a spherical triple.*

Proof. Let u, v, w be distinct elements of V and let

$$T = \frac{1}{m_{uv}} + \frac{1}{m_{vw}} + \frac{1}{m_{wu}}.$$

Now T is maximal when $\{u,v\} = e_1$, $\{v,w\} = e_2$ and $\{w,u\} = e_3$ for some $e_1, e_2, e_3 \in E$. It follows that e_1, e_2, e_3 are the edges of a triangle in Γ. Since G is non-spherical,

$$T \leqslant \frac{1}{m_{e_1}} + \frac{1}{m_{e_2}} + \frac{1}{m_{e_3}} \leqslant \frac{1}{2}.$$

Thus, V does not contain any spherical triples. $\square$

Let $\mathbb{P}$ be an $\mathbf{r}$-picture and let $u, v \in V$. A (u,v)-*subpicture* of $\mathbb{P}$ is a subpicture in which each arc has colour u or colour v.

Definition 5.2.1. A *federation* is a maximal (u,v)-subpicture $\mathbb{F}$ such that $\{u,v\} = e$ for some edge $e \in E$, and such that $\mathbb{F}$ contains at least one disc whose label is an element of $\mathbf{r}'_e$. It is maximal in the sense that $\partial\mathbb{F}$ cannot be extended to include any other disc of $\mathbb{P}$ whose label is an element of $\mathbf{r}_e$. We define $\Sigma(\mathbb{F})$ to be e.

A federation is *simply-connected* if it is a simply-connected (u,v)-subpicture. Otherwise, it is *non-simply-connected*. If $\mathbb{F}$ is a simply-connected federation with $\Sigma(\mathbb{F}) = e$, then by Theorem 1.8.1 the label $W(\mathbb{F})$ of $\mathbb{F}$ represents the identity element of G_e. Equivalently, $W(\mathbb{F})$ represents an element of $\ker\psi_e$.

Let $\mathbb{F}_1$ be a federation of $\mathbb{P}$. If $\mathbb{F}_1 \neq \mathbb{P}$, then construct a federation $\mathbb{F}_2$ of $\mathbb{P} - \mathbb{F}_1$. If $\mathbb{F}_2 \neq \mathbb{P}_1 - \mathbb{F}_1$, then construct a federation $\mathbb{F}_3$ of $\mathbb{P}_1 - (\mathbb{F}_1 \cup \mathbb{F}_2)$, and so. Eventually, we will end up with a collection of subpictures $\mathbb{F}_1, \ldots, \mathbb{F}_n$ of $\mathbb{P}$ that cover $\mathbb{P}$ and satisfy the property that $\mathbb{F}_{i+1}$ is a federation of

$$\mathbb{P} - (\bigcup_{j=1}^{i} \mathbb{F}_j).$$

We call the collection of subpictures $\mathbb{F}_{\mathbb{P}} = \{\mathbb{F}_i\}_{i=1}^{n}$ a *federal subdivision* of $\mathbb{P}$.

Recall, for each $e \in E$, $\widehat{\mathbf{r}}_e$ denotes the set of all words on $\mathbf{x}_e^{\pm 1}$ that represent a *non-identity* element of $\ker\psi_e$, and that the union of the $\widehat{\mathbf{r}}_e$'s is denoted by $\widehat{\mathbf{r}}$.

Let $\mathbb{P}$ be a non-empty connected $\mathbf{r}$-picture and let $\mathbb{F}_{\mathbb{P}} = \{\mathbb{F}_i\}_{i=1}^{n}$ be a federal subdivision of $\mathbb{P}$ which satisfies the following two conditions:

(i) Each $\mathbb{F}_i \in \mathbb{F}_{\mathbb{P}}$ is simply-connected ($i = 1, \ldots, n$);

(ii) $W(\mathbb{F}_i) \in \widehat{\mathbf{r}}$ for $i = 1, \ldots, n$.

The *derived picture* $\widehat{\mathbb{P}}$ of $\mathbb{P}$ corresponding to $\mathbb{F}_{\mathbb{P}}$ is obtained from $\mathbb{P}$ by deleting the arcs and discs which are contained in each $\mathbb{F}_i$. If α is a boundary arc of $\mathbb{F}_i$, then we only delete the portion of α which is contained in $\mathbb{F}_i$. The boundary of $\mathbb{F}_i$ is then identified as the boundary of a disc of $\widehat{\mathbb{P}}$. By construction of $\widehat{\mathbb{P}}$, $W(\widehat{\mathbb{P}}) \equiv W(\mathbb{P})$. Also, $\widehat{\mathbb{P}}$ inherits a colouring by the elements of V. Furthermore, if D is a disc of $\widehat{\mathbb{P}}$ obtained from a federation $\mathbb{F}$, then the label of D is identical to $W(\mathbb{F})$ and so is an element of $\widehat{\mathbf{r}}$. Thus $\widehat{\mathbb{P}}$ is an $\widehat{\mathbf{r}}$-picture.

Lemma 5.2.2. *Let $\widehat{\mathbb{P}}$ be the derived picture corresponding to some federal subdivision $\mathbb{F}_{\mathbb{P}}$ of some non-spherical connected simply-connected $\mathbf{r}$-picture $\mathbb{P}$. Then $\widehat{\mathbb{P}}$ satisfies Conditions* (i), (ii) and (iii) *of Lemma 5.1.1.*

Proof. Since $\mathbb{P}$ is connected, $\widehat{\mathbb{P}}$ cannot contain any floating circles. Also, we may assume $\widehat{\mathbb{P}}$ does not contain an arc which has both endpoints on the same disc enclosing a region of degree 1. If such an arc α existed, then (in $\mathbb{P}$) its endpoints would lie on two discs which belong to the same federation $\mathbb{F}$ (see Fig. 5.1). Furthermore, no disc of $\mathbb{P}$ would be contained in the region bounded by α and $\partial\mathbb{F}$. Thus, we could "pull" α into $\mathbb{F}$ as shown in Fig. 5.1. Hence $\widehat{\mathbb{P}}$ satisfies Condition (i) of Lemma 5.1.1.

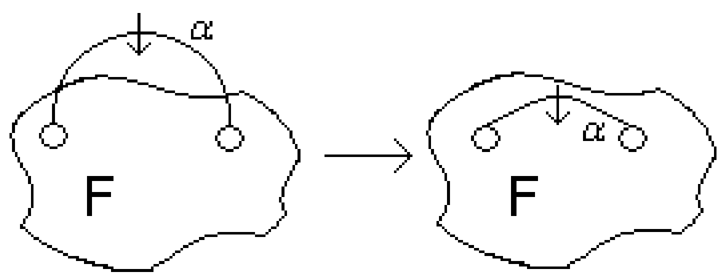


Figure 5.1: Pulling α into $\mathbb{F}$.

Let D be a disc of $\widehat{\mathbb{P}}$ with label W. Then W is an element of $\widehat{\mathbf{r}}_e$ for some $e = \{u, v\} \in E$, so $|W|_s \geqslant m_e$. We deduce that each arc incident with D is coloured u or v and there are at least m_{uv} corners of D joining one arc of each colour. Thus, $\widehat{\mathbb{P}}$ satisfies Condition (ii) of Lemma 5.1.1.

Assume $\widehat{\mathbb{P}}$ contains an interior region F that does not satisfy Condition (iii) of Lemma 5.1.1, i.e. F contains at least two corners κ_1, κ_2 in its boundary that join arcs of the same two distinct colours u and v, say. Suppose κ_1 and κ_2 are corners of two distinct discs D_1 and D_2, respectively. Then there is a simple closed transverse path β in $\widehat{\mathbb{P}}$ enclosing D_1 and D_2 only (see Fig. 5.2). Let

$\mathbb{F}_1$ and $\mathbb{F}_2$ be the federations of $\mathbb{P}$ that correspond to D_1 and D_2, respectively. It follows that in $\mathbb{P}$, there is a simple closed transverse path enclosing $\mathbb{F}_1$ and $\mathbb{F}_2$ only. Therefore the boundary of $\mathbb{F}_1$ can be extended to include all the discs of $\mathbb{F}_2$, contradicting the fact that $\mathbb{F}_1$ is a maximal (u, v)-subpicture.

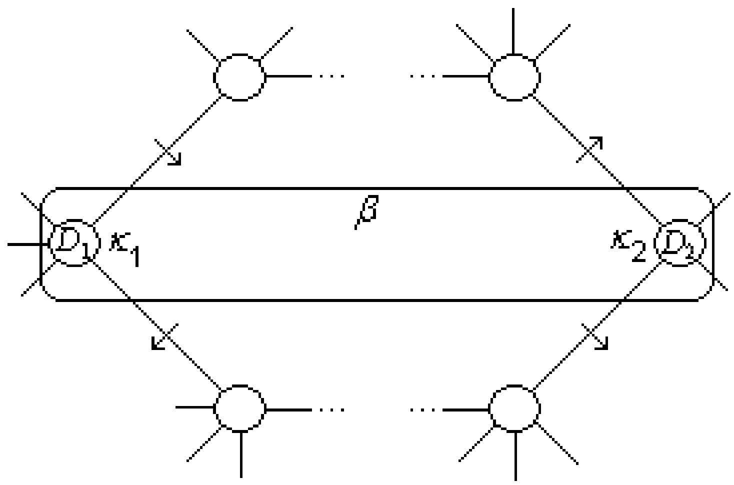


Figure 5.2: A simple closed transverse path enclosing D_1 and D_2.

Now suppose κ_1 and κ_2 are corners of the same disc D (see Fig. 5.3). Draw a simple closed transverse path β enclosing D as shown.

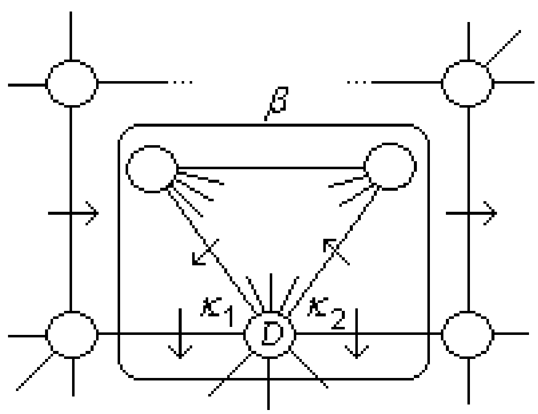


Figure 5.3: A simple closed transverse path enclosing D.

In addition to D, the subpicture bounded by β contains at least one disc so that Condition (i) of Lemma 5.1.1 is not violated. Moreover, by assuming that the subpicture bounded by β is an "innermost" one violating Condition (iii) (meaning that Condition (iii) is satisfied by every region which is enclosed by a simple closed transverse path consisting of discs and arcs of $\widehat{\mathbb{P}}$ that lie in the subpicture bounded by β) we may assume that Lemma 5.1.1 applies to the subpicture bounded by β. Hence each arc in the subpicture is coloured u or v as only arcs of these colours meet its boundary. Now β corresponds to a simple closed transverse path β' in $\mathbb{P}$ that encloses only those federations that correspond to the discs enclosed by β. Each such federation is a maximal (u, v)-subpicture; however, this contradicts the fact that the federation corresponding to D is a maximal (u, v)-subpicture. We deduce that $\widehat{\mathbb{P}}$ must satisfy Condition (iii) of Lemma 5.1.1. This completes the proof of Lemma 5.2.2. $\square$

Let $\mathbb{P}$ be a non-empty connected spherical **r**-picture and let $\mathbb{F}_{\mathbb{P}}$ be a federal subdivision of $\mathbb{P}$. Choose some $\mathbb{F} \in \mathbb{F}_{\mathbb{P}}$ with $\Sigma(\mathbb{F}) = e \in E$, and let $W(\mathbb{F}) \equiv W$. There are three possibilities for W:

(i) W is freely equal to the empty word, or

(ii) W represents the identity element of $\widetilde{G}_e$, or

(iii) W represents a non-identity element of $\ker \psi_e$.

Lemma 5.2.3. *If each federation of $\mathbb{F}_{\mathbb{P}}$ is simply-connected and if $\mathbb{F}_{\mathbb{P}}$ contains at least two federations, then $\mathbb{P}$ contains a federation whose boundary label is either freely equal to the empty word or represents the identity element of $\widetilde{G}_e$ for some $e \in E$.*

Proof. Suppose the boundary label of each federation of $\mathbb{P}$ represents a non-identity element of $\ker \psi_e$ for some $e \in E$. We may then construct the derived picture $\widehat{\mathbb{P}}$ of $\mathbb{P}$ corresponding to $\mathbb{F}_{\mathbb{P}}$. Choose some disc D in $\widehat{\mathbb{P}}$ and let $\mathbb{F}$ be its corresponding federation in $\mathbb{P}$, where $\Sigma(\mathbb{F}) = \{u, v\} \in E$. Identify D as the only disc lying on the northern hemisphere of the two-sphere (see Fig. 5.4). The discs and arcs that lie on the southern hemisphere form a simply-connected $\widehat{\mathbf{r}}$-subpicture $\mathbb{M}$ of $\widehat{\mathbb{P}}$ and so, by Lemma 5.2.2, satisfy the conditions of Lemma 5.1.1. We deduce that each interior arc of $\mathbb{M}$ is coloured u or v, as only arcs of these colours meet its boundary. However, this contradicts the fact that $\mathbb{F}$ is a maximal (u, v)-subpicture. The conclusion of the lemma now follows. $\square$

The following result extends Lemma 5.2.3 to the case where $\mathbb{F}_{\mathbb{P}}$ need not contain only simply-connected federations.

Lemma 5.2.4. *If $|\mathbb{F}_{\mathbb{P}}| \geqslant 2$, then $\mathbb{P}$ contains a simply-connected federation whose boundary label is either freely equal to the empty word or represents the identity element of $\widetilde{G}_e$ for some $e \in E$.*

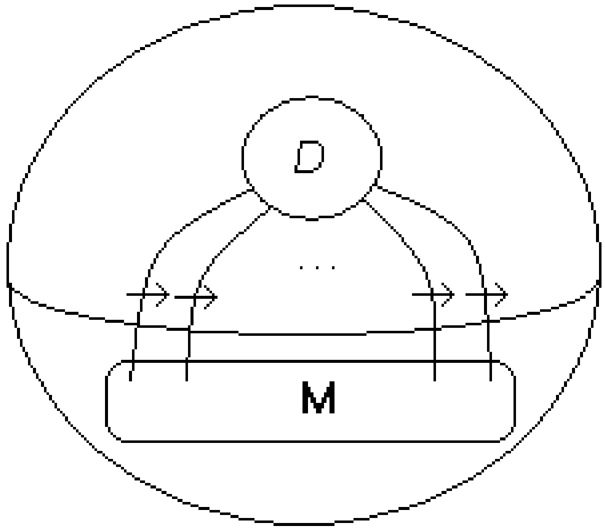


Figure 5.4: Identifying D as lying on the northern hemisphere of the two-sphere.

Proof. In light of Lemma 5.2.3, we may assume $\mathbb{F}_{\mathbb{P}}$ contains a non-simply-connected federation $\mathbb{F}_i$. Let $\Sigma(\mathbb{F}_i) = \{u, v\} \in E$ and let $\mathbb{B}$ be a simply-connected component of $\mathbb{P} - \mathbb{F}_i$. Observe that any arc meeting $\partial\mathbb{B}$ has colour u or v and we may choose $\mathbb{F}_i$ so that each federation contained in $\mathbb{B}$ is simply-connected.

Suppose $\mathbb{B}$ contains only one federation (which we will also denote by $\mathbb{B}$) and let $W(\mathbb{B}) = W$. Then W is either a word on $\mathbf{x}_u^{\pm 1}$ that represents the identity element of G_u, or is a word on $\mathbf{x}_v^{\pm 1}$ that represents the identity element of G_v. In either case, W represents the identity element of $G_u * G_v$. Thus, $\mathbb{B}$ is a simply-connected federation of $\mathbb{P}$ whose boundary label represents the identity element of $\widetilde{G}_e$ for some $e \in E$.

Now suppose $\mathbb{B}$ contains more than one federation and assume that the boundary label of each federation is an element of $\widehat{\mathbf{r}}$. We may then construct the derived picture $\widehat{\mathbb{B}}$ of $\mathbb{B}$. Since $\widehat{\mathbb{B}}$ satisfies the conditions of Lemma 5.1.1, we deduce that each interior arc of $\mathbb{B}$ is coloured u or v, as only arcs of these colours meet its boundary. Thus, $\mathbb{B}$ is a (u, v)-subpicture; however, this contradicts the fact that $\mathbb{F}_i$ is a maximal (u, v)-subpicture. Hence, $\mathbb{B}$ must contain at least one simply-connected federation $\mathbb{F}_j$ whose boundary label is either freely equal to the empty word or represents the identity element of $\widetilde{G}_e$ for some $e \in E$. Thus, $\mathbb{P}$ contains such a federation. $\square$

We are now in a position to prove Theorem 7, which we restate below. Recall that a group is said to be *diagrammatically reducible* if it has a diagrammatically reducible presentation $\mathcal{Q}$, i.e. every non-empty connected spherical picture over $\mathcal{Q}$ contains, after performing bridge moves, a cancelling pair.

Theorem 5.2.1. *Let G be a non-spherical vertex-free Pride group with underlying graph $\Gamma = \{V, E\}$. Then G is diagrammatically reducible if and only if each edge group is diagrammatically reducible.*

Proof. Let $\mathcal{P} = \langle \mathbf{x} \,;\mathbf{r}\rangle$ be the natural presentation of G where each vertex group has the presentation $\langle \mathbf{x}_v \,; -\rangle$ $(v \in V)$. Let $\mathbb{P}$ be a non-empty connected spherical $\mathbf{r}$-picture. If $\mathbb{P}$ is a spherical picture over some edge group presentation, then the diagrammatic reducibility of the edge groups implies that $\mathbb{P}$ must contain, after performing bridge moves, a cancelling pair.

If $\mathbb{P}$ contains more than two federations, then it must contain a simply-connected federation $\mathbb{F}$ whose boundary label W is either freely equal to the empty word or represents the identity element of $\widetilde{G}_e$ for some $e \in E$. Since each vertex group is free, we deduce that W *must* be freely equal to the empty word. By performing bridge moves on the boundary arcs of $\mathbb{F}$ we can split $\mathbb{P}$ into two spherical components, one of which is $\mathbb{F}$. Since each G_e is diagrammatically reducible, $\mathbb{F}$ must contain, after performing bridge moves, a cancelling pair. Thus so does $\mathbb{P}$. This completes the proof of the theorem. □

5.3 A generating set for $\pi_2(\mathcal{P})$

Let G be a non-spherical Pride group with underlying graph $\Gamma = \{V, E\}$, and let $\mathcal{P} = \langle \mathbf{x} \,;\mathbf{r}\rangle$ be the natural presentation of G. Let $E(\mathbb{P})$ to be the set of discs of $\mathbb{P}$ whose labels are elements of $(\mathbf{r}')^s$. We now prove Theorem 6.

Theorem 5.3.1. *For each $e \in E$, let X_e be a generating set for $\pi_2(\mathcal{P}_e)$. Then*

$$X = \bigcup_{e \in E} X_e$$

is a generating set for $\pi_2(\mathcal{P})$. In particular, if each G_e is of type F_3, then G is of type F_3.

Proof. Let $\mathbb{P}$ be a non-empty connected spherical $\mathbf{r}$-picture. Suppose $\mathbb{P}$ does not contain a disc whose label is an element of $(\mathbf{r}')^s$. Then $\mathbb{P}$ is a non-empty connected spherical $\mathbf{r}_v$-picture for some

$v \in V$ and so $\langle \mathbb{P} \rangle \in \pi_2(\mathcal{P}_e)$ where $e = \{u, v\} \in E$. Thus $\mathbb{P}$ is equivalent (modulo X_e) to the empty picture.

Now suppose $\mathbb{P}$ does contain a disc whose label is an element of $(\mathbf{r}')^s$. We proceed by induction on $|E(\mathbb{P})|$.

If $|E(\mathbb{P})| = 1$, then $\mathbb{P}$ is a connected spherical $\mathbf{r}_e$-picture for some $e \in E$ and so $\langle \mathbb{P} \rangle \in \pi_2(\mathcal{P}_e)$. Thus $\mathbb{P}$ is equivalent (modulo X_e) to the empty picture. Assume $|E(\mathbb{P})| = k$ (> 1) and that the result holds for all connected spherical $\mathbf{r}$-pictures $\mathbb{P}'$ with $|E(\mathbb{P}')| < k$. Let $\mathbb{F}_{\mathbb{P}}$ be a federal subdivision of $\mathbb{P}$.

If $\mathbb{F}_{\mathbb{P}}$ contains only one federation, then $\mathbb{P}$ is a non-empty connected spherical $\mathbf{r}_e$-picture for some $e \in E$ and so is equivalent (modulo X_e) to the empty picture.

If $\mathbb{F}_{\mathbb{P}}$ contains at least two federations, then by Lemma 5.2.4, $\mathbb{P}$ contains a simply-connected federation $\mathbb{F}$ whose boundary label W is either

(i) freely equal to the empty word, or

(ii) represents the identity element of $\widetilde{G}_e$ for some $e \in E$.

Suppose (i) holds. By performing bridge moves on the boundary arcs of $\mathbb{F}$ we can split $\mathbb{P}$ into two connected spherical components $\mathbb{C}_1$ and $\mathbb{C}_2$, where $\mathbb{C}_1$ contains $\mathbb{P}' = \mathbb{P} - \mathbb{F}$ and $\mathbb{C}_2$ contains $\mathbb{F}$. It follows by induction that $\mathbb{P}'$ is equivalent (modulo X) to the empty picture. Since $\mathbb{F}$ is a spherical $\mathbf{r}_e$-picture $(e \in E)$, $\mathbb{F}$ is equivalent (modulo X_e) to the empty picture. Thus $\mathbb{P}$ is equivalent (modulo X) to the empty picture.

Now suppose W represents the identity element of $\widetilde{G}_e$, where $e = \{u, v\} \in E$. Then there exists a minimal simply-connected $(\mathbf{r}_u \cup \mathbf{r}_v)$-picture $\mathbb{B}$ with $W(\mathbb{B}) \equiv W$. Let $\mathbb{A}$ be the connected spherical $\mathbf{r}_e$-picture illustrated in Fig. 5.5. By Lemma 1.8.3, we have

$$\langle \mathbb{P} \rangle - \langle \mathbb{P}' \rangle = \overline{U} \cdot \langle \mathbb{A} \rangle \quad (U \in (\mathbf{x}^{\pm 1})^*)$$

where $\mathbb{P}'$ is the non-empty connected spherical $\mathbf{r}$-picture obtained from $\mathbb{P}$ by replacing $\mathbb{F}$ with $\mathbb{B}$. Since $|E(\mathbb{P}')| < |E(\mathbb{P})|$, the induction hypothesis applies to $\mathbb{P}'$. Therefore, $\mathbb{P}'$ is equivalent (modulo X) to the empty picture. It follows that $\mathbb{P}$ is equivalent (modulo $X \cup \mathbb{A}$) to the empty picture and since $\mathbb{A}$ is equivalent (modulo X_e) to the empty picture, we deduce that $\mathbb{P}$ is equivalent (modulo X) to the empty picture. This completes the proof for the case when $\mathbb{P}$ is a connected spherical

r-picture. A simple inductive argument on the number of components proves that the result holds when $\mathbb{P}$ is an arbitrary spherical **r**-picture. $\square$

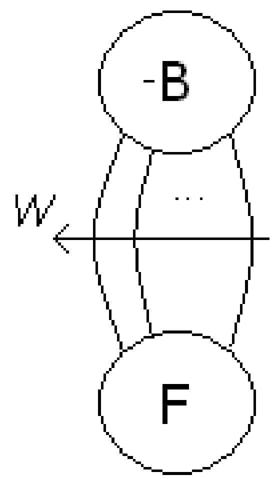


Figure 5.5: The spherical $\mathbf{r}_e$-picture $\mathbb{A}$.

The proof of Corollary 2 (restated below) follows immediately.

Corollary 5.3.1. *Let G be a non-spherical Pride group. If each G_e is combinatorially aspherical, then G is combinatorially aspherical. In particular, if each G_e is aspherical, then G is aspherical.*

5.4 An upper bound for the second order Dehn function of a non-spherical Pride group

We begin this section with the proof of Theorem 8.

Theorem 5.4.1. *Let G be a non-spherical vertex-free Pride group with underlying graph $\Gamma = \{V, E\}$. Assume for each $e \in E$ that G_e is of type F_3 and let $\delta_E^{(2)} = \max\{\bar{\delta}_{G_e}^{(2)} : e \in E\}$. Then*

$$\delta_G^{(2)}(n) \preccurlyeq \delta_E^{(2)}(n)$$

for all $n \in \mathbb{N}$.

Proof. Let $\mathcal{P} = \langle \mathbf{x}\,;\mathbf{r}\rangle$ be the natural presentation of G where each vertex group has the presentation $\langle \mathbf{x}_v\,; - \rangle$ ($v \in V$). Let $\mathbb{P}$ be a non-empty connected spherical **r**-picture of area n and let X be the set of generators of $\pi_2(\mathcal{P})$ described in Theorem 5.3.1. Note, since each G_e ($e \in E$) is of type F_3, X is a finite set.

We proceed by induction on $|E(\mathbb{P})| = k$. If $k = 0$, then $\mathbb{P}$ is a non-empty connected spherical $\mathbf{r}_v$-picture for some $v \in V$. Since G_v is free and hence aspherical, $\mathbb{P}$ is equivalent to the empty picture. Therefore, $V_X(\langle\mathbb{P}\rangle) = 0$.

Now assume $k > 0$ and let $\mathbb{F}_{\mathbb{P}} = \{\mathbb{F}_i\}_{i=1}^{r}$ be a federal subdivision of $\mathbb{P}$. If $r = 1$, then $\mathbb{P}$ is a non-empty connected spherical $\mathbf{r}_e$-picture for some $e \in E$. Therefore, $V_X(\langle\mathbb{P}\rangle) \leqslant \delta_E^{(2)}(n)$.

Now suppose $r \geqslant 2$. It follows from Lemma 5.2.4 that $\mathbb{P}$ must contain a simply-connected federation $\mathbb{F}$ whose boundary label W is either freely equal to the empty word or represents the identity element of $\widetilde{G}_e$ for some $e \in E$. Since each vertex group is free, we deduce that W *must* be freely equal to the empty word. By performing bridge moves on the boundary arcs of $\mathbb{F}$ we can split $\mathbb{P}$ into two spherical components. Thus,

$$\langle\mathbb{P}\rangle = \langle\mathbb{P}_1\rangle + \langle\mathbb{F}\rangle$$

where $\mathbb{P}_1 = \mathbb{P} - \mathbb{F}$. The induction hypothesis applies to $\mathbb{P}_1$, so $V_X(\langle\mathbb{P}_1\rangle) \leqslant \delta_E^{(2)}(n-f)$ where f is the area of $\mathbb{F}$. Also, $V_X(\langle\mathbb{F}\rangle) \leqslant \delta_E^{(2)}(f)$. Therefore,

$$\begin{aligned} V_X(\langle\mathbb{P}\rangle) &= V_X(\langle\mathbb{P}_1\rangle) + V_X(\langle\mathbb{F}\rangle) \\ &\leqslant \delta_E^{(2)}(n-f) + \delta_E^{(2)}(f) \\ &\leqslant \delta_E^{(2)}(n). \end{aligned}$$

The statement of the result now follows. □

We now consider the second order Dehn function of an arbitrary non-spherical Pride group G. If we wish to apply the reduction argument used in the proof of Theorem 5.3.1 to calculate an upper bound for $\delta_G^{(2)}$, then we have to consider the case when a spherical $\mathbf{r}$-picture $\mathbb{P}$ contains a simply-connected federation $\mathbb{F}$ whose boundary label represents the identity element of some $\widetilde{G}_e$ ($e \in E$). (Recall, this case did not arise in the proof of Theorem 5.4.1 as each vertex group was free.) Following the proof of Theorem 5.3.1, we replace $\mathbb{F}$ with some simply-connected picture $\mathbb{B}$ over a presentation $\widetilde{\mathcal{P}}_e$ for $\widetilde{G}_e$. We have to take account of the area of $\mathbb{B}$ as this will affect the volume of $\mathbb{P}$; however, there is no obvious way to calculate Area($\mathbb{B}$). The area distortion function defined below is designed to do precisely this.

Let $\mathcal{Q}_1 = \langle \mathbf{y}_1 ; \mathbf{s}_1 \rangle$ and $\mathcal{Q}_2 = \langle \mathbf{y}_2 ; \mathbf{s}_2 \rangle$ be two finite presentations, and let $\mathbf{s}$ be a finite set of non-empty cyclically reduced words on $(\mathbf{y}_1 \cup \mathbf{y}_2)^{\pm 1}$ where each element of $\mathbf{s}$ involves at least one $\mathbf{y}_1$-letter and at least one $\mathbf{y}_2$-letter. Let $\mathcal{R} = \langle \mathbf{y}_1, \mathbf{y}_2 ; \mathbf{s}_1, \mathbf{s}_2, \mathbf{s} \rangle$. Then $G(\mathcal{R})$ is the quotient group

$$\frac{G(\mathcal{Q}_1) * G(\mathcal{Q}_2)}{<< \mathbf{s} >>}.$$

Suppose the natural maps $G(\mathcal{Q}_1) \to G(\mathcal{R})$ and $G(\mathcal{Q}_2) \to G(\mathcal{R})$ are injective, and let $\mathcal{Q} = \langle \mathbf{y}_1, \mathbf{y}_2 ; \mathbf{s}_1, \mathbf{s}_2 \rangle$ be a finite presentation of $G(\mathcal{Q}_1) * G(\mathcal{Q}_2)$.

Definition 5.4.1. The *area distortion function of $\mathcal{Q}$ relative to $\mathcal{R}$* is the function $\Lambda_{\mathcal{Q}}^{\mathcal{R}} : \mathbb{N} \to \mathbb{N}$ where for all $n \in \mathbb{N}$,

$$\Lambda_{\mathcal{Q}}^{\mathcal{R}}(n) = \max\{\mathrm{Area}_{\mathcal{Q}}(W) : \overline{W} = 1 \text{ in } G(\mathcal{Q}) \text{ and } \mathrm{Area}_{\mathcal{R}}(W) \leqslant n\}.$$

This function is similar to the following area distortion function h defined in [49] (see also [52, 80]). Let $\mathcal{S}$ be a subpresentation of a finite presentation $\mathcal{T}$ so each generator (respectively, relator) of $\mathcal{S}$ is a generator (respectively, relator) of $\mathcal{T}$, and assume that $G(\mathcal{S})$ embeds in $G(\mathcal{T})$. The *area distortion function of $\mathcal{S}$ in $\mathcal{T}$* is the function $h : \mathbb{N} \to \mathbb{N}$ given by

$$h(n) = \max\{\mathrm{Area}_{\mathcal{S}}(W) : \overline{W} = 1 \text{ in } G(\mathcal{S}) \text{ and } \mathrm{Area}_{\mathcal{T}}(W) \leqslant n\}.$$

It is easily shown that h is invariant (up to the standard $\simeq$-equivalence) under change of presentations, thus one may speak of *the* area distortion function of a finitely presented subgroup in a finitely presented group. Unlike h, the area distortion function $\Lambda_{\mathcal{Q}}^{\mathcal{R}}$ is *not* well-defined in general. Arbitrarily long words can have small area with respect to $\mathcal{R}$ but there is no obvious way of bounding their areas with respect to $\mathcal{Q}$. With these comments in mind, we proceed with the proof of Proposition 1.

Proposition 5.4.1. *Let G be a non-spherical Pride group with underlying graph $\Gamma = \{V, E\}$ and let $\mathcal{P} = \langle \mathbf{x} ; \mathbf{r} \rangle$ be the natural presentation of G. For each $e = \{u, v\} \in E$, let $\widetilde{\mathcal{P}}_e = \langle \mathbf{x}_u, \mathbf{x}_v ; \mathbf{r}_u, \mathbf{r}_v \rangle$ be a presentation of $\widetilde{G}_e$ and let Λ_e be the area distortion function of $\widetilde{\mathcal{P}}_e$ relative to $\mathcal{P}_e$. Assume each G_e is of type F_3 and set $\Lambda = \max\{\bar{\Lambda}_e : e \in E\}$. Then for all $n \in \mathbb{N}$,*

$$\delta_{\mathcal{P}}^{(2)}(n) \leqslant \delta_E^{(2)}(n + 2n\Lambda^n(n^2 + n)) + \delta_V^{(2)}(n)$$

where $\delta_V^{(2)} = \max\{\bar{\delta}_{\mathcal{P}_v}^{(2)} : v \in V\}$, $\delta_E^{(2)} = \max\{\bar{\delta}_{\mathcal{P}_e}^{(2)} : e \in E\}$ and Λ^n is the n-th power of Λ.

Proof. Let $\mathbb{P}$ be a non-empty connected spherical $\mathbf{r}$-picture of area n and let X be the set of generators of $\pi_2(\mathcal{P})$ described in Theorem 5.3.1. If $|E(\mathbb{P})| = 0$, then $\mathbb{P}$ is a non-empty connected spherical $\mathbf{r}_v$-picture for some $v \in V$, so

$$V_X(\langle \mathbb{P} \rangle) \leqslant \delta_V^{(2)}(n).$$

Now assume $|E(\mathbb{P})| = k > 0$ and let $\mathbb{F}_{\mathbb{P}} = \{\mathbb{F}_i\}_{i=1}^r$ be a federal subdivision of $\mathbb{P}$. Our aim is to prove that

$$V_X(\langle \mathbb{P} \rangle) \leqslant \delta_E^{(2)}(n + 2\Lambda^k((k+1)n)).$$

We proceed by induction on k. If $k = 1$, then $\mathbb{P}$ is a non-empty connected spherical $\mathbf{r}_e$-picture for some $e \in E$, so

$$V_X(\langle \mathbb{P} \rangle) \leqslant \delta_E^{(2)}(n).$$

Now assume $k \geqslant 2$. If $r = 1$, then we argue as in the previous paragraph. Now suppose $r \geqslant 2$. It follows from Lemma 5.2.4 that $\mathbb{P}$ must contain a simply-connected federation $\mathbb{F}$ whose boundary label is either freely equal to the empty word or represents the identity element of $\widetilde{G}_e$ for some $e \in E$. In the first instance, we can perform bridge moves on the boundary arcs of $\mathbb{F}$ to split $\mathbb{P}$ into two spherical components. Thus,

$$\langle \mathbb{P} \rangle = \langle \mathbb{P}_1 \rangle + \langle \mathbb{F} \rangle$$

where $\mathbb{P}_1 = \mathbb{P} - \mathbb{F}$. The induction hypothesis applies to $\mathbb{P}_1$, so

$$V_X(\langle \mathbb{P}_1 \rangle) \leqslant \delta_E^{(2)}(n - f + 2\Lambda^{k_1}((k_1+1)(n-f)))$$

where $f = \text{Area}(\mathbb{F})$ and $k_1 = |E(\mathbb{P}_1)|$. Hence,

$$\begin{aligned}
V_X(\langle \mathbb{P} \rangle) &= V_X(\langle \mathbb{P}_1 \rangle) + V_X(\langle \mathbb{F} \rangle) \\
&\leqslant \delta_E^{(2)}(n - f + 2\Lambda^{k_1}((k_1+1)(n-f))) + \delta_E^{(2)}(f) \\
&\leqslant \delta_E^{(2)}(n + 2\Lambda^{k_1}((k_1+1)(n-f)) \\
&\leqslant \delta_E^{(2)}(n + 2\Lambda^k((k+1)n)).
\end{aligned}$$

Now suppose the boundary label of $\mathbb{F}$ represents the identity element of $\widetilde{G}_e$ for some $e \in E$. Then there exists a minimal simply-connected picture $\mathbb{B}$ for $W(\mathbb{F})$ over $\widetilde{\mathcal{P}}_e$. As in the proof of Theorem 5.3.1, we replace $\mathbb{F}$ with $\mathbb{B}$ to obtain a connected spherical $\mathbf{r}$-picture $\mathbb{P}_2$ so that

$$\langle \mathbb{P} \rangle = \langle \mathbb{P}_2 \rangle + \overline{U} \cdot \langle \mathbb{A} \rangle \quad (U \in (\mathbf{x}^{\pm 1})^*),$$

where $\mathbb{A}$ is the connected spherical $\mathbf{r}_e$-picture illustrated in Fig. 5.5. Let Area($\mathbb{F}$) $= f$. Then Area($\mathbb{B}$) $\leqslant \Lambda_e(f)$, where Λ_e is the area distortion function of $\widetilde{\mathcal{P}}_e$ relative to $\mathcal{P}_e$. Therefore,

$$\text{Area}(\mathbb{A}) \leqslant f + \Lambda_E(f)$$

and we deduce that

$$V_X(\overline{U} \cdot \langle \mathbb{A} \rangle) = V_X(\langle \mathbb{A} \rangle) \leqslant \delta_E^{(2)}(f + \Lambda(f)).$$

Since $|E(\mathbb{B})| = 0$, $|E(\mathbb{P}_2)| \leqslant k - 1$ and the induction hypothesis applies to $\mathbb{P}_2$. Therefore,

$$V_X(\langle \mathbb{P}_2 \rangle) \leqslant \delta_E^{(2)}(n - f + \Lambda(f) + 2\Lambda^{k-1}(k(n - f + \Lambda(f)))).$$

Thus,

$$\begin{aligned}
V_X(\langle \mathbb{P} \rangle) &= V_X(\langle \mathbb{P}_2 \rangle) + V_X(\overline{U} \cdot \langle \mathbb{A} \rangle) \\
&\leqslant \delta_E^{(2)}(n - f + \Lambda(f) + 2\Lambda^{k-1}(k(n - f + \Lambda(f)))) + \delta_E^{(2)}(f + \Lambda(f)) \\
&\leqslant \delta_E^{(2)}(n + 2\Lambda(f) + 2\Lambda^{k-1}(\Lambda(kn - kf) + \Lambda(kf))) \\
&\leqslant \delta_E^{(2)}(n + 2\Lambda(f) + 2\Lambda^{k-1}(\Lambda(kn))) \\
&\leqslant \delta_E^{(2)}(n + 2\Lambda^k(n) + 2\Lambda^k(kn)) \\
&\leqslant \delta_E^{(2)}(n + 2\Lambda^k((k+1)n))
\end{aligned}$$

as required.

Now let $\mathbb{P}$ be an arbitrary spherical $\mathbf{r}$-picture of area n and let $|E(\mathbb{P})| = k$. If $\mathbb{P}$ has m components, then we claim

$$V_X(\langle \mathbb{P} \rangle) \leqslant \delta_E^{(2)}(n + 2m\Lambda^k((k+1)n)) + \delta_V^{(2)}(n).$$

We proceed by induction on m. If $m = 1$, then the result holds by the first part of the proof. Suppose $m > 1$. Then

$$\langle \mathbb{P} \rangle = \langle \mathbb{M} \rangle + \langle \mathbb{C} \rangle$$

where $\mathbb{C}$ is a non-empty spherical component of $\mathbb{P}$ and $\mathbb{M} = \mathbb{P} - \mathbb{C}$. Let Area($\mathbb{C}$) $= c$ and $|E(\mathbb{C})| = l$.

If $\mathbb{C}$ is a connected spherical $\mathbf{r}_v$-picture for some $v \in V$, then $l = 0$ and

$$\begin{aligned}
V_X(\langle \mathbb{P} \rangle) &= V_X(\langle \mathbb{M} \rangle) + V_X(\langle \mathbb{C} \rangle) \\
&\leqslant \delta_E^{(2)}(n - c + 2(m-1)\Lambda^k((k+1)(n-c))) + \delta_V^{(2)}(n-c) + \delta_V^{(2)}(c) \\
&\leqslant \delta_E^{(2)}(n + 2m\Lambda^k((k+1)n)) + \delta_V^{(2)}(n).
\end{aligned}$$

If $\mathbb{C}$ is a connected spherical $\mathbf{r}_e$-picture for some $e \in E$, then $l \geqslant 1$ and

$$\begin{aligned}
V_X(\langle \mathbb{P} \rangle) &= V_X(\langle \mathbb{M} \rangle) + V_X(\langle \mathbb{C} \rangle) \\
&\leqslant \delta_E^{(2)}(n - c + 2(m-1)\Lambda^{k-l}((k-l+1)(n-c))) + \delta_V^{(2)}(n-c) + \delta_E^{(2)}(c) \\
&\leqslant \delta_E^{(2)}(n + 2m\Lambda^k((k+1)n)) + \delta_V^{(2)}(n).
\end{aligned}$$

Finally, if $\mathbb{C}$ is neither a connected spherical $\mathbf{r}_v$-picture nor a connected spherical $\mathbf{r}_e$-picture for any $v \in V$ or $e \in E$, then

$$\begin{aligned}
V_X(\langle \mathbb{P} \rangle) &= V_X(\langle \mathbb{M} \rangle) + V_X(\langle \mathbb{C} \rangle) \\
&\leqslant \delta_E^{(2)}(n - c + 2(m-1)\Lambda^{k-l}((k-l+1)(n-c))) + \delta_V^{(2)}(n-c) + \delta_E^{(2)}(c + 2\Lambda^l((l+1)c)) \\
&\leqslant \delta_E^{(2)}(n + 2(m-1)\Lambda^k((k+1)n) + 2\Lambda^k((k+1)n)) + \delta_V^{(2)}(n) \\
&\leqslant \delta_E^{(2)}(n + 2m\Lambda^k((k+1)n)) + \delta_V^{(2)}(n).
\end{aligned}$$

This completes the proof of our claim. Since $m \leqslant n$ and $k \leqslant n$, we have

$$V_X(\langle \mathbb{P} \rangle) \leqslant \delta_E^{(2)}(n + 2n\Lambda^n((n+1)n)) + \delta_V^2(n).$$

The statement of the result now follows. $\square$

If one could show that the area distortion function Λ_e is well-defined for each $e \in E$, then the formula given in Proposition 5.4.1 would be an upper bound for the second order Dehn function $\delta_G^{(2)}$ of an arbitrary non-spherical Pride group G. We have obtained results concerning this which will appear elsewhere.

Chapter 6

An introduction to relative presentations

We now turn our attention to the study of relative presentations. Our main interest is in determining when a relative presentation is aspherical. In this chapter we define what it means for a relative presentation to be aspherical and describe various tests that are used to determine whether or not this is the case. The language we use is pictures over relative presentations. In Chapter 7 we give a classification of when relative presentations which belong to a particular family are aspherical.

6.1 Relative presentations

A *relative presentation* consists of the following data: a group H, a set $\mathbf{t}$ that is disjoint from H, and a set $\mathbf{r}$ of cyclically reduced elements of $H * F(\mathbf{t})$. The relative presentation is then $\mathcal{P} = \langle H, \mathbf{t}\,; \mathbf{r}\rangle$ and the *group defined by* $\mathcal{P}$ is

$$G(\mathcal{P}) = (H * F(\mathbf{t}))/ << \mathbf{r} >> .$$

The group H is the *coefficient* group of the relative presentation. If we choose H to be the trivial group, then $\mathcal{P}$ is simply an ordinary presentation as defined in §1.3.

A relative presentation is said to be *orientable* if no element of $\mathbf{r}$ is a cyclic permutation of its inverse, and it is *injective* if the natural map $H \to G(\mathcal{P})$ is a monomorphism. Injective relative presentations are intimately connected with the study of *equations* over groups in the following way. If $\mathcal{P}$ is an injective relative presentation, then the collection $\{t << \mathbf{r} >>: t \in \mathbf{t}\}$ of elements of $G(\mathcal{P})$ is a solution to the system of equations $R(H, \mathbf{t}) = 1$ ($R \in \mathbf{r}$) in the overgroup $G(\mathcal{P})$ of H. The study of equations over groups has been pursued by numerous authors. We refer to [34, 37, 40, 43, 55, 60, 61, 63] and the references contained therein.

We may obtain from $\mathcal{P}$ an ordinary *lifted* presentation $\widehat{\mathcal{P}}$ in the following way. First, choose an ordinary presentation $\mathcal{Q} = \langle \mathbf{x} \,; \mathbf{s} \rangle$ for H. Select words on $\mathbf{x}^{\pm 1}$ to represent the elements of H and then use these words to rewrite the H-coefficients that appear in the relative relators of $\mathbf{r}$. Denote the resulting set by $\widehat{\mathbf{r}}$. The lifted presentation is then $\widehat{\mathcal{P}} = \langle \mathbf{x}, \mathbf{t} \,; \mathbf{s}, \widehat{\mathbf{r}} \rangle$ and it defines the group

$$\widehat{G} = F(\mathbf{x}, \mathbf{t}) / << \mathbf{s} \cup \widehat{\mathbf{r}} >>,$$

where $<< \mathbf{s} \cup \widehat{\mathbf{r}} >>$ is the normal closure of $\mathbf{s} \cup \widehat{\mathbf{r}}$ in $F(\mathbf{x}, \mathbf{t})$. There is an isomorphism $\nu : \widehat{G} \to G(\mathcal{P})$ induced by the epimorphism

$$\phi * id : F(\mathbf{x}) * F(\mathbf{t}) \to H * F(\mathbf{t}),$$

where $\phi : F(\mathbf{x}) \to H$ is an epimorphism with kernel $<< \mathbf{s} >>$.

Definition 6.1.1. A relative presentation $\mathcal{P} = \langle H, t \,; \mathbf{r} \rangle$ is *aspherical* if for some ordinary presentation $\mathcal{Q} = \langle \mathbf{x} \,; \mathbf{s} \rangle$ of H and for some lifted presentation $\widehat{\mathcal{P}} = \langle \mathbf{x}, \mathbf{t} \,; \mathbf{s}, \widehat{\mathbf{r}} \rangle$, the second homotopy module $\pi_2(\widehat{\mathcal{P}})$ is generated by $\pi_2(\mathcal{Q})$ as a left $\mathbb{Z}\widehat{G}$-module.

In terms of pictures, a relative presentation $\mathcal{P}$ is aspherical if every spherical picture over $\widehat{\mathcal{P}}$ is equivalent (modulo X) to the empty picture, where X is a set of generators of $\pi_2(\mathcal{Q})$. This concept of asphericity was first given in [10] and is more general than that given in [16]. The stronger notion of asphericity according to the theory developed in [16] will be given in §6.3. The main theoretical consequences of asphericity are summarized in the following result.

Theorem 6.1.1. ([10, Theorem 1]) *If $\mathcal{P} = \langle H, \mathbf{t} \,; \mathbf{r} \rangle$ is an injective aspherical relative presentation for a group G, then the following statements are true.*

(1) *The inclusion $H \hookrightarrow G$ induces isomorphisms*

$$H_n(G, -) \cong H_n(H, -) \text{ and } H^n(G, -) \cong H^n(H, -)$$

in all dimensions $n \geqslant 3$ and for all choices of $\mathbb{Z}G$-module coefficients.

(2) *Each finite subgroup of G is contained in a G-conjugate of H.*

For the remainder of this chapter we restrict our study to *one-relator* relative presentations of the form

$$\mathcal{P} = \langle H, t\,; R\rangle, \tag{6.1}$$

where $R = t^{\varepsilon_1}h_1t^{\varepsilon_2}h_2\ldots t^{\varepsilon_m}h_m$ is a cyclically reduced element of $H * \langle t\rangle$. The *exponent sum* of t in the relative relator R is $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_m$ and the integer m is the *t-length* of R. A relative presentation of the form (6.1) is said to be of *t-length* m if the t-length of R is equal to m. The *t-shape* of R is the unreduced word formed by the occurrences of $t^{\pm 1}$, i.e. the unreduced word obtained from R by deleting the H-coefficients. The injectivity and asphericity of relative presentations of the form (6.1) depends very much on the t-shape of R.

6.2 A brief survey of known and original results

Levin [63] proved that the relative presentation $\mathcal{P}_1 = \langle H, t; th_1th_2\ldots th_m\rangle$ is injective for $m \geqslant 2$ and it is easy to analyse the asphericity of $\mathcal{P}_1$ for t-lengths $\leqslant 2$. However, for t-lengths $\geqslant 3$ it is much harder. Bogley and Pride [16] obtained a complete classification of when $\mathcal{P}_1$ is aspherical for the case when it has t-length 3. When $\mathcal{P}_1$ has t-length 4, an almost complete classification was obtained in [10]. More recently, an almost complete classification of when $\mathcal{P}_1$ is aspherical for the case when it has t-length 5 was obtained in [57].

Howie [55] proved that the relative presentation $\mathcal{P}_2 = \langle H, t\,; th_1th_2\ldots th_{m-1}t^{-1}h_m\rangle$ is injective if $m = 3$ and Edjvet [36] obtained a classification of when $\mathcal{P}_2$ is aspherical modulo eight exceptional cases. Injectivity for the case $m = 4$ was shown in [37] and a classification of when $\mathcal{P}_2$ is aspherical was obtained in [2] modulo twelve exceptional cases. Note that if $m = 1$, then $\mathcal{P}_2$ is a relative presentation defining an HNN-extension. The injectivity and asphericity of such presentations is well known [28].

We are interested in the case when $h_1 = h_2 = \ldots = h_{m-2} = 1$ in $\mathcal{P}_2$, i.e. we study relative presentations of the form $\mathcal{P} = \langle H, t\,; t^nat^{-1}b\rangle$. Edjvet [34] has shown that $\mathcal{P}$ is injective for $n \geqslant 2$. The results of [36] and [2] determine when $\mathcal{P}$ is aspherical for the cases $n = 2$ and $n = 3$, respectively. In Chapter 7, we prove the following.

Theorem 9. *Let $\mathcal{P} = \langle H, t\,; t^nat^{-1}a\rangle$ where a is a non-identity element of H and $n \geqslant 4$. Then $\mathcal{P}$ is aspherical if and only if a has infinite order in H.*

Theorem 10. *Let $\mathcal{P} = \langle H, t\,; t^n a t^{-1} b\rangle$ $(n \geqslant 4)$ where a and b are distinct elements of H such that $o(a), o(b) \geqslant 3$. Then $\mathcal{P}$ is aspherical if and only if neither of the following two conditions hold:*

(1) $\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} > 1$ *where* $\frac{1}{\infty} := 0$;

(2) $a = b^{-1}$ *and* $o(a) < \infty$.

There are four exceptional cases (see §7.2) for which asphericity cannot be determined in Theorem 10.

We note that the relative relator $t^n a t^{-1} b$ $(n \geqslant 2)$ has an *amenable* t-shape [43]. Forester and Rourke [45] proved that if H is *torsion-free* and if R has an amenable t-shape, then the relative presentation $\langle H, t\,; R\rangle$ is aspherical in the sense of [16]. The difficulty in proving Theorem 10 comes from the fact that H may have torsion.

Remark 6.2.1. *For the remainder of this chapter, by a relative presentation we will mean a one-relator relative presentation of the form (6.1). Note, such presentations are orientable.*

6.3 Pictures over one-relator relative presentations

Let $\mathcal{P} = \langle H, t\,; R\rangle$ be a relative presentation and let $\mathbb{P}$ be a simply-connected picture. (All pictures in this chapter will be simply-connected.). We say that $\mathbb{P}$ is a *picture over* $\mathcal{P}$ if it admits the following labelling. Each arc of $\mathbb{P}$ has a normal orientation (indicated by an arrow transverse to the arc) and is labelled by t, and each corner of $\mathbb{P}$ is labelled by an element of H. If κ is a corner of a disc D_i, then we denote by $W(\kappa)$ the word obtained by reading in a clockwise order (beginning with the arc at the head of κ) the labels on corners and arcs meeting ∂D_i. The following two conditions must be satisfied:

(i) For each corner κ of $\mathbb{P}$, $W(\kappa)$ is a cyclic permutation of R or R^{-1};

(ii) If $h_1, \ldots, h_r$ is the sequence of corner labels encountered in an anticlockwise traversal of the boundary of an interior region of $\mathbb{P}$, then $h_1 \ldots h_r = 1$ in H.

We say that $h_1 \ldots h_r$ is the *label* of the interior region.

Example 6.3.1. Let $\mathcal{Q} = \langle Z_3, t\,; t^3 a t^{-1} a\rangle$ where a generates the cyclic group Z_3 of order 3. The picture illustrated in Fig. 6.1 is a picture over $\mathcal{Q}$.

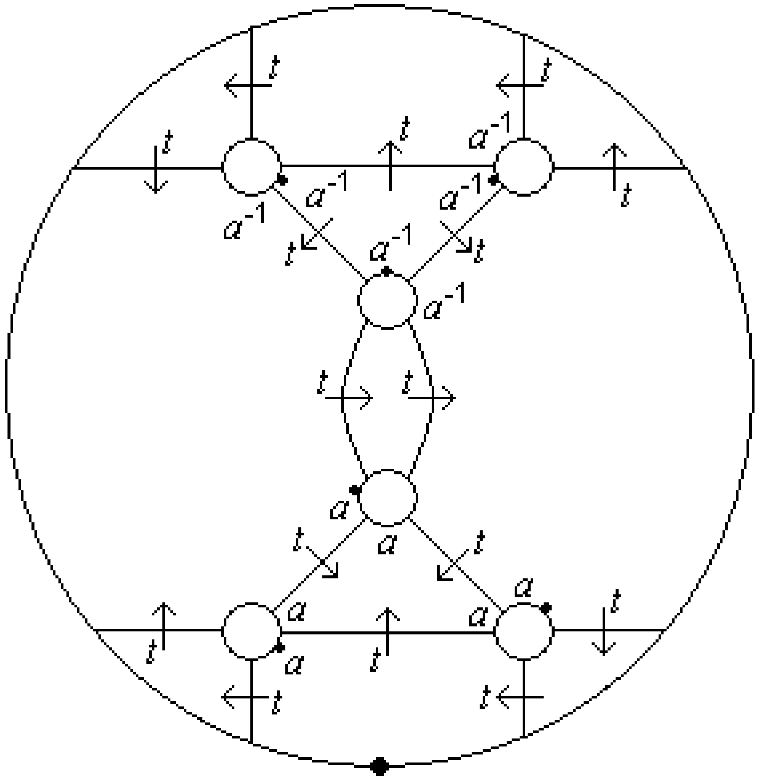


Figure 6.1: A picture over $\mathcal{Q}$.

Recall that we may obtain from $\mathcal{P}$ an ordinary lifted presentation $\widehat{\mathcal{P}}$. Given a connected *spherical* picture $\mathbb{P}$ over $\mathcal{P}$, one may lift $\mathbb{P}$ (though not uniquely) to a simply-connected picture $\widehat{\mathbb{P}}$ over $\widehat{\mathcal{P}}$. Choose a presentation $\mathcal{Q} = \langle \mathbf{x}\,;\mathbf{s}\rangle$ of H. For each corner label h of each interior region F of $\mathbb{P}$, replace h with a succession of $\mathbf{x}$-arcs whose total label W_h is a word on $\mathbf{x}^{\pm 1}$ that represents h. The product of the W_h's (h running over all corner labels of F) represents the identity element of H, so by Theorem 1.8.1 there exists a simply-connected $\mathbf{s}$-picture $\mathbb{B}_F$ whose boundary label is identical to the product of the W_h's. Fill F with $\mathbb{B}_F$ and proceed to fill the remaining interior regions of $\mathbb{P}$ in this way. The one remaining region of $\mathbb{P}$ is an annulus. Replace each corner label in this region by a succession of $\mathbf{x}$-arcs reading the representative word (anticlockwise around the ambient disc). These $\mathbf{x}$-arcs extend radially to the boundary of $\widehat{\mathbb{P}}$. We call $\widehat{\mathbb{P}}$ a *lifted picture* of $\mathbb{P}$.

A connected spherical picture over $\mathcal{P}$ is said to be *strictly* spherical if the product of the corner labels in the unique annular region (taken in anticlockwise order) equals the identity in H. The lifted picture $\widehat{\mathbb{P}}$ of a connected strictly spherical picture over $\mathcal{P}$ is a *spherical* picture over $\widehat{\mathcal{P}}$. (The product of the representative words of the corner labels of the unique annular region of $\mathbb{P}$ represents the identity of H, so we may fill this region with a simply-connected $\mathbf{s}$-picture.)

Example 6.3.2. Let $\mathcal{Q}$ be the relative presentation given in Example 6.3.1. The picture illustrated in Fig. 6.2 is a strictly spherical picture over $\mathcal{Q}$.

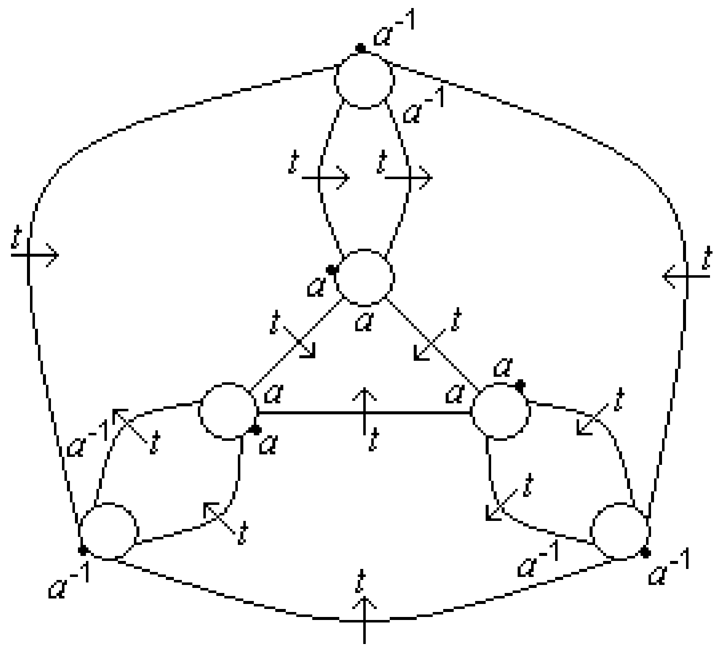


Figure 6.2: A strictly spherical picture over $\mathcal{Q}$.

A *dipole* in a picture $\mathbb{P}$ consists of a pair of corners κ, κ' together with an arc α joining the head of one corner with the tail of the other such that κ and κ' belong to the same region of $\mathbb{P}$, and such that if $W(\kappa) = Sh$ where $h \in H$ and S begins and ends with t or t^{-1}, then $W(\kappa') = S^{-1}h^{-1}$ (see Fig. 6.3). A picture is *reduced* if it does not contain a dipole.

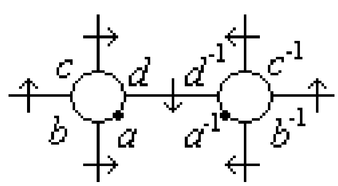


Figure 6.3: A dipole.

Definition 6.3.1. A relative presentation $\mathcal{P}$ is *diagrammatically reducible* if every non-empty connected spherical picture over $\mathcal{P}$ contains a dipole.

In [16] a relative presentation was defined to be aspherical if it was diagrammatically reducible. Suppose $\mathcal{P}$ is an orientable diagrammatically reducible presentation. If R is not a proper power when viewed in the ambient free product, then $\mathcal{P}$ is aspherical as given in Definition 6.1.1. We will make frequent use of the fact that if an orientable injective relative presentation $\mathcal{P}$ is *not* aspherical, then there exists a reduced non-empty connected strictly spherical picture over $\mathcal{P}$.

6.4 Tests for asphericity

In this section we describe five tests that are used to determine, given an orientable relative presentation $\mathcal{P} = \langle H, t\,; R\rangle$, whether or not $\mathcal{P}$ is aspherical.

6.4.1 The weight test

The *star graph* $\mathcal{P}^{st}$ of $\mathcal{P}$ is a graph whose edges are labelled by elements of H. The vertex set of $\mathcal{P}^{st}$ is $t \cup t^{-1}$. The edge set of $\mathcal{P}^{st}$ consists of all cyclic permutations of R and R^{-1} which are of the form Sh, where $h \in H$ and S begins and ends with t or t^{-1}. Let $\widetilde{R} = Sh \in R^*$. The initial and terminal functions, respectively, ι and τ are defined as follows: $\iota(\widetilde{R})$ is the first symbol of S and $\tau(\widetilde{R})$ is the *inverse* of the last symbol of S. The inverse of $\widetilde{R}$ is defined to be $S^{-1}h^{-1}$ and the labelling function is given by $\lambda(\widetilde{R}) = h^{-1}$. The labelling is extended to paths by multiplication in H in the obvious way. A non-empty cyclically reduced closed path in $\mathcal{P}^{st}$ is *admissible* if its label equals the identity in H. Each interior region of a reduced picture over $\mathcal{P}$ supports an admissible cycle in $\mathcal{P}^{st}$.

A *weight function* on $\mathcal{P}^{st}$ is a real valued function ω on the set of edges of $\mathcal{P}^{st}$ such that $\omega(\widetilde{R}^{-1}) = \omega(\widetilde{R})$ for each $\widetilde{R} \in R^*$. The weight of a path is the sum of the weights of its constituent edges. Following [16], we say that a weight function ω is *weakly aspherical* if the following conditions are satisfied:

(1) If $R = t^{\varepsilon_1}h_1 \ldots t^{\varepsilon_m}h_m$, then

$$\sum_{i=1}^{m}(1 - \omega(t^{\varepsilon_i}h_i \ldots t^{\varepsilon_m}h_m t_1^{\varepsilon_1}h_1 \ldots t^{\varepsilon_{i-1}}h_{i-1})) \geq 2;$$

(2) Each admissible cycle in $\mathcal{P}^{st}$ has weight at least 2.

Theorem 6.4.1. ([16, Theorem 2.1]) *If $\mathcal{P}^{st}$ admits a weakly aspherical weight function, then $\mathcal{P}$ is diagrammatically reducible.*

6.4.2 The curvature test

Let $\mathbb{P}$ be a spherical picture over a relative presentation $\mathcal{P}$. An *angle function* on $\mathbb{P}$ is a real valued function θ on the set of corners of $\mathbb{P}$. Associated to θ is a *curvature function* γ defined on discs of $\mathbb{P}$ by

$$\gamma(D) = 2\pi - \sum_{\kappa \subseteq \partial D} \theta(\kappa),$$

and on regions of $\mathbb{P}$ by

$$\gamma(F) = 2\pi - \sum_{\kappa \subseteq \partial F} (\pi - \theta(\kappa)).$$

If $\mathbb{P}$ is connected, then there is the fundamental *curvature formula*

$$\sum_{D} \gamma(D) + \sum_{F} \gamma(F) = 2\pi\chi(S^2) = 4\pi,$$

where the sum is taken over all discs and all regions of $\mathbb{P}$ including the unique annular region. The following lemma is an obvious consequence of the curvature formula.

Lemma 6.4.1. *If θ is any angle function on a connected spherical picture $\mathbb{P}$, then some disc or region of $\mathbb{P}$ has positive curvature.*

A spherical picture $\mathbb{P}$ is *flat* at a disc D if $\gamma(D) = 0$, and θ is a *flat* angle function if $\mathbb{P}$ is flat at every disc. If θ is a flat angle function on $\mathbb{P}$, then it follows from Lemma 6.4.1 that there exists a region F such that $\gamma(F) > 0$. Let d be the degree of such a region. Then

$$\sum_{\kappa \subseteq \partial F} \theta(\kappa) > (d-2)\pi$$

and we say that F is an *exceptional* region. Let $h_1 \dots h_d$ be the label of an exceptional region. If $\mathbb{P}$ is strictly spherical, then $h_1 \dots h_d = 1$ for any exceptional region. We may then obtain some restrictions on the elements $h_1, \dots, h_d$.

6.4.3 The distribution test

The distribution test introduced by Edjvet [36] is a more flexible application of the curvature formula. Assign a flat angle function θ on a connected strictly spherical picture $\mathbb{P}$ so that by Lemma 6.4.1 there exists at least one exceptional region F. The idea of the distribution test is to "flatten" F by distributing curvature to its neighbouring regions. If we can flatten every exceptional region, then we obtain a contradiction to the fundamental curvature formula, thus proving that $\mathbb{P}$ cannot exist.

Let F and F' be two neighbouring regions of $\mathbb{P}$ that share an arc α which is directed from F to F', and let ζ be any real number. Subtract ζ from the angle of one of the corners in F that touches α and add ζ to the angle of the adjacent corner in F'. There results a new angle function θ^* on $\mathbb{P}$ with associated curvature function γ^*. It is clear that $\gamma^*(D) = \gamma(D)$ for each disc D of $\mathbb{P}$, that $\gamma^*(F) = \gamma(F) - \zeta$, and that $\gamma^*(F') = \gamma(F') + \zeta$. Other regions are unaffected. We may think of this process as assigning a real number ζ to the pair of regions (F, F'). More generally, let $\mathcal{F}$ denote the set of regions of $\mathbb{P}$. A *distribution scheme* on $\mathbb{P}$ is a function

$$\eta : \mathcal{F} \times \mathcal{F} \to \mathbb{R}.$$

For example, in the situation described above the distribution function would be defined as:

$$\eta(F, F') = \begin{cases} \zeta & \text{if } F \text{ and } F' \text{ have an arc in common which is directed from } F \text{ to } F'; \\ 0 & \text{otherwise.} \end{cases}$$

If we are given $\mathbb{P}, \theta, \gamma$ as above and if η is any distribution scheme on $\mathbb{P}$, then there is an angle function θ^* with associated curvature function γ^* such that $\gamma^*(D) = \gamma(D)$ for each disc D of $\mathbb{P}$, and

$$\gamma^*(F) = \gamma(F) + \sum_{F'} (\eta(F', F) - \eta(F, F'))$$

for each region F of $\mathbb{P}$. Here the sum is taken over all regions F' of $\mathbb{P}$. The function γ^* is called the *distributed* curvature function. A distribution scheme η on $\mathbb{P}$ is a γ-*flattening* of $\mathbb{P}$ if the following two conditions are satisfied:

(1) For all regions F and F', if $\eta(F, F') > 0$, then $\gamma(F') \leq 0$;

(2) For all regions F, if $\gamma(F) > 0$, then $\gamma(F) \leq \sum_{F'} \eta(F, F')$.

Lemma 6.4.2. ([10, Lemma 2]) *Suppose that γ is a curvature function on a connected strictly spherical picture $\mathbb{P}$ in which each disc is non-positively curved. If η is a γ-flattening of $\mathbb{P}$ with distributed curvature function γ^*, then there exists a region K of $\mathbb{P}$ such that $\gamma^*(K) > \gamma(K)$ and $\gamma^*(K) > 0$.*

6.4.4 Degenerate pictures

Let $\mathbb{P}$ be a connected strictly spherical picture over $\mathcal{P} = \langle H, t\,; R\rangle$. If one can show that $\mathbb{P}$ is not *degenerate*, then $\mathcal{P}$ is not aspherical in the most general sense. The following account of degenerate pictures is taken from [2]. See also [10, §8].

Recall that $\mathbb{P}$ can be lifted (though not uniquely) to a spherical picture $\widehat{\mathbb{P}}$ over $\widehat{\mathcal{P}}$ for some appropriate choice of presentation $\mathcal{Q} = \langle \mathbf{x}\,; \mathbf{s}\rangle$ for H (see §6.3). Consider the image of $\widehat{\mathbb{P}}$ under the standard embedding [84, p. 692]

$$\mu \,:\, \pi_2(\widehat{\mathcal{P}}) \to (\bigoplus_{S\in\mathbf{s}} \mathbb{Z}\widehat{G}e_S) \oplus \mathbb{Z}\widehat{G}e_{\widehat{R}},$$

where $\widehat{G}$ is the group defined by $\widehat{\mathcal{P}}$. Our aim is to calculate the coefficient $\lambda_{\widehat{\mathbb{P}}}$ of $e_{\widehat{R}}$.

Let $\{D_j : j = 1, \dots, l\}$ be the discs of $\widehat{\mathbb{P}}$ whose labels are $\widehat{R}^{\pm 1}$. Let F be any region in $\mathbb{P}$ and let $\mathbb{B}_F$ be the $\mathbf{s}$-picture that we filled F with to obtain $\widehat{\mathbb{P}}$. Corresponding to F, there is a subpicture $\widehat{F}$ in $\widehat{\mathbb{P}}$ enclosed by t-arcs as illustrated in Fig. 6.4.

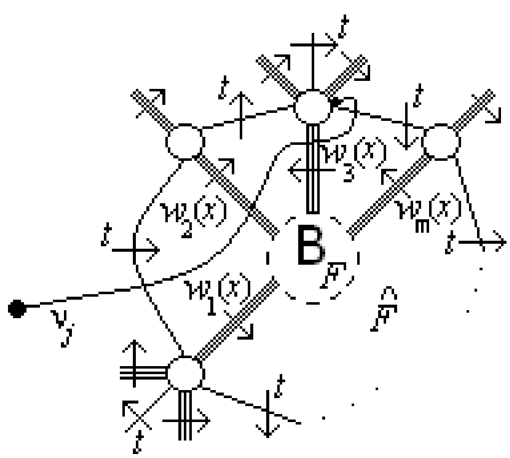


Figure 6.4: A subpicture of the lifted picture $\widehat{\mathbb{P}}$ corresponding to F.

Let v_j be a transverse path from the basepoint of $\widehat{\mathbb{P}}$ to the basepoint of D_j such that v_j is always "close" to t-arcs and $\widehat{R}$-discs, in the sense that if $\widehat{F'}$ is any subpicture of $\widehat{\mathbb{P}}$ corresponding to a region F', then v_j does not cut through $\mathbb{B}_{F'}$. Note, it does not matter for v_j to be on the left or right of $\mathbb{B}_F$. Let V_j be the label on v_j and note that V_j is a word on $\mathbf{x}^{\pm 1} \cup \{t^{\pm 1}\}$. Then

$$\lambda_{\widehat{\mathbb{P}}} = \sum_{j=1}^{l} \delta_j g_j$$

where g_j is the element of $\widehat{G}$ represented by V_j and $\delta_j = \pm 1$. The group isomorphism $\nu : \widehat{G} \to G(\mathcal{P})$ induces a ring isomorphism $\nu^* : \mathbb{Z}\widehat{G} \to \mathbb{Z}G(\mathcal{P})$, so let $\nu^*(\lambda_{\widehat{\mathbb{P}}}) = \lambda_{\mathbb{P}}$.

Suppose we choose another presentation $\mathcal{Q}_1 = \langle \mathbf{x}_1 \,; \mathbf{s}_1 \rangle$ for H that determines a lifted presentation $\widehat{\mathcal{P}}_1 = \langle \mathbf{x}_1, t \,; \mathbf{s}_1, \widehat{R}_1 \rangle$ where $\widehat{G}_1 = G(\widehat{\mathcal{P}}_1)$. We then obtain an ordinary lifted picture $\widehat{\mathbb{P}}_1$ over $\widehat{\mathcal{P}}_1$, which may be viewed as a copy of $\widehat{\mathbb{P}}$ by replacing:

- $\widehat{R}$-discs by $\widehat{R}_1$-discs;
- successions of $\mathbf{x}$-arcs labelling corners of $\mathbb{P}$ by successions of $\mathbf{x}_1$-arcs labelling corners of $\mathbb{P}$;
- $\mathbf{s}$-pictures $\mathbb{B}_F$ by $\mathbf{s}_1$-pictures $\mathbb{E}_F$.

For each transverse path v_j in $\widehat{\mathbb{P}}$, the copy of v_j in $\widehat{\mathbb{P}}_1$ is a transverse path from the basepoint of $\widehat{\mathbb{P}}_1$ to the basepoint of E_j, where E_j is the $\widehat{R}_1$-disc which replaced the $\widehat{R}$-disc D_j. Denote this path by u_j and let U_j be the label of u_j. Note that U_j is a word on $\mathbf{x}_1^{\pm 1} \cup \{t^{\pm 1}\}$. Since the presentations $\mathcal{Q}$ and $\mathcal{Q}_1$ define isomorphic groups, we have $\overline{V_j} = \overline{U_j}$ in H. Therefore, if $\lambda_{\widehat{\mathbb{P}}_1}$ is the image of the coefficient of $e_{\widehat{R}_1}$ in $\widehat{\mathbb{P}}_1$ under the embedding

$$\mu_1 \,:\, \pi_2(\widehat{\mathcal{P}}_1) \to (\bigoplus_{S_1 \in \mathbf{s}_1} \mathbb{Z}\widehat{G}_1 e_{S_1}) \oplus \mathbb{Z}\widehat{G}_1 e_{\widehat{R}_1},$$

then $\lambda_{\widehat{\mathbb{P}}_1} = \lambda_{\widehat{\mathbb{P}}}$. Thus, $\lambda_{\mathbb{P}}$ is independent of the choice of presentation for H, the choice of $\widehat{R}$ and the choice of lift. We say that $\mathbb{P}$ is *degenerate* if $\lambda_{\mathbb{P}} = 0$.

Lemma 6.4.3. *If there exists a non-empty connected reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$ such that $\mathbb{P}$ is not degenerate, then $\mathcal{P}$ is not aspherical.*

Proof. Assume $\mathcal{P}$ is aspherical and let $\mathbb{P}$ be such a picture. Then there exists a presentation $\mathcal{Q} = \langle \mathbf{x} \,; \mathbf{s} \rangle$ of H such that $\widehat{\mathbb{P}}$ is equivalent (modulo a generating set for $\pi_2(\mathcal{Q})$) to the empty picture. It follows that the image of $\widehat{\mathbb{P}}$ under the standard embedding μ lies entirely in $\bigoplus_{S \in \mathbf{s}} \mathbb{Z}\widehat{G} e_S$. We deduce that the coefficient of $e_{\widehat{R}}$ must be zero. Thus, $\lambda_{\mathbb{P}} = 0$. $\square$

Let $\mathbb{P}$ be a reduced strictly spherical picture over $\mathcal{P} = \langle H, t\,; R\rangle$ where $R = t^{\varepsilon_1}h_1 \ldots t^{\varepsilon_m}h_m$, and let $k = \varepsilon_1 + \ldots + \varepsilon_m$. Let Z_k be the cyclic group of order k given by the presentation $\langle x\,; x^k\rangle$. There exists a group homomorphism $\psi : G(\mathcal{P}) \to Z_k : H \mapsto 1, t \mapsto x$, and ψ induces a ring homomorphism $\psi^* : \mathbb{Z}G(\mathcal{P}) \to \mathbb{Z}Z_k$. The point is, if $\psi^*(\lambda_{\mathbb{P}}) \neq 0$, then $\lambda_{\mathbb{P}} \neq 0$. Let $\mathcal{P}^o = \langle t, R^o\rangle$ where $R^o = t^{\varepsilon_1} \ldots t^{\varepsilon_m}$, and let $\mathbb{P}^o$ be the picture over $\mathcal{P}^o$ which is obtained from $\mathbb{P}$ by deleting all the corner labels. Then $\psi^*(\lambda_{\mathbb{P}})$ is just the coefficient of e_{R^o} under the embedding

$$\mu^o : \pi_2(\mathcal{P}^o) \to \mathbb{Z}Z_k e_{R^o}.$$

This gives us an efficient way to determine whether $\lambda_{\mathbb{P}} \neq 0$.

Example 6.4.1. Let $\mathcal{P} = \langle H, t\,; R\rangle$ where $R = t^n a t^{-1} a$ $(n \geqslant 2)$ and where $o(a) = p < \infty$. Consider the reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$ illustrated in Fig. 6.5. Discs that have a corner in the boundary region of $\mathbb{P}$ contribute px^{-1} to the coefficient of e_{R^o}, while the remaining discs of $\mathbb{P}$ contribute $-px^{-2}$. It follows that the coefficient of e_{R^o} is $px^{-1}(1 - x^{-1})$. Thus, $\psi^*(\lambda_{\mathbb{P}}) \neq 0$ and so $\mathcal{P}$ is not aspherical.

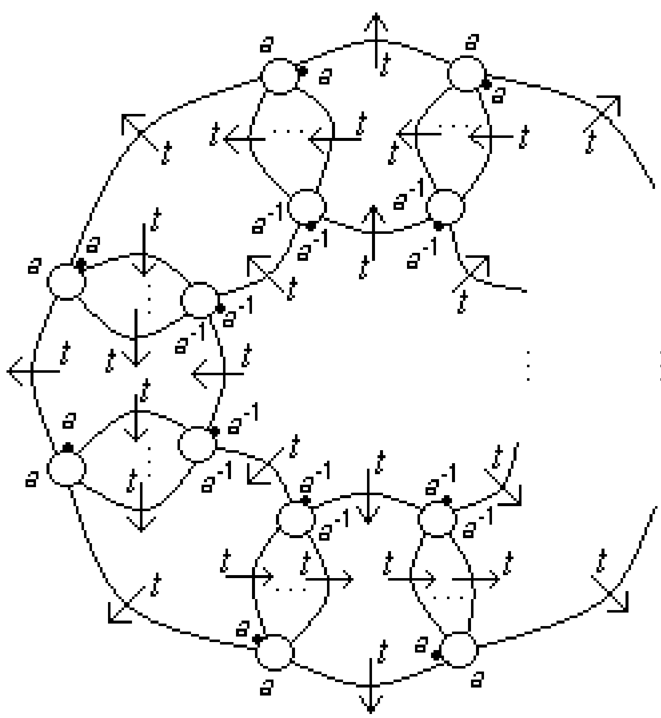


Figure 6.5: A reduced strictly spherical picture over $\mathcal{P}$.

6.4.5 Calculating the order of t

Let $\mathcal{P} = \langle H, t\,; R\rangle$ be an injective presentation where $R = t^{\varepsilon_1}h_1 \ldots t^{\varepsilon_m}h_m$, and let $k = \varepsilon_1 + \ldots + \varepsilon_m$. The factor group $G(\mathcal{P})/N$, where N is the normal closure of H in $G(\mathcal{P})$, is cyclically generated by tN and is of order k (it has infinite order if $k = 0$). If $k \neq \pm 1$, then $t \notin N$. Hence if we can show that t has finite order in $G(\mathcal{P})$, then $\mathcal{P}$ is *not* aspherical by Theorem 6.1.1 (the finite subgroup generated by t is not contained in a $G(\mathcal{P})$-conjugate of H). Thus, we have the following lemma.

Lemma 6.4.4. *If the exponent sum of t is not equal to ± 1 and if t has finite order in $G(\mathcal{P})$, then $\mathcal{P}$ is not aspherical.*

Chapter 7

The asphericity of relative presentations of the form $\langle H, t\,; t^n a t^{-1} b\rangle$

Let $\mathcal{P} = \langle H, t\,; t^n a t^{-1} b\rangle$ where $n \geqslant 4$ and a, b are non-identity elements of H. Note that $\mathcal{P}$ is orientable and injective [34].

7.1 The subcase $a = b$

Theorem 7.1.1. *Let $\mathcal{P} = \langle H, t\,; t^n a t^{-1} a\rangle$ where a is a non-identity element of H and $n \geqslant 4$. Then $\mathcal{P}$ is aspherical if and only if a has infinite order in H.*

Proof. Suppose a has infinite order in H and consider the star graph $\mathcal{P}^{st}$ in Fig. 7.1. Edges α_1, α_2 have label a^{-1} and edges α_i $(i = 3, \ldots, n+1)$ have label 1. Define a weight function ω on $\mathcal{P}^{st}$ as follows: $\omega(\alpha_1) = \omega(\alpha_2) = 0$ and $\omega(\alpha_i) = 1$ for $i = 3, \ldots, n+1$. Since a has infinite order any admissible cycle must include at least two edges from the set $\{\alpha_3, \ldots, \alpha_{n+1}\}$ and so will have weight at least 2. Thus, ω satisfies Condition (2) of §6.4.1. Condition (1) is also satisfied, so ω is a weakly aspherical weight function. Hence, by Theorem 6.4.1, $\mathcal{P}$ is diagrammatically reducible.

In Example 6.4.1 we proved that $\mathcal{P}$ is not aspherical when a has finite order in H. This completes the proof of Theorem 7.1.1. $\square$

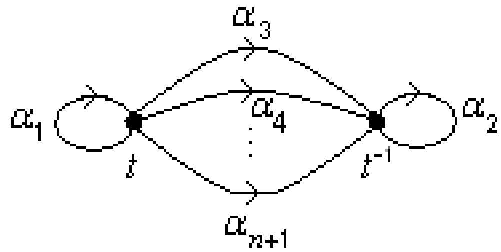


Figure 7.1: The star graph $\mathcal{P}^{st}$.

7.2 The subcase $a \neq b$

Theorem 7.2.1. *Suppose* $\mathcal{P} = \langle H, t\,; t^n a t^{-1} b\rangle$ $(n \geqslant 4)$ *is not one of the exceptional cases* (E1), (E2), (E3), (E4) *listed below and that* a, b *are distinct elements of* H *such that* $o(a), o(b) \geqslant 3$. *Then* $\mathcal{P}$ *is aspherical if and only if neither of the following two conditions hold:*

(1) $\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} > 1$ *where* $\frac{1}{\infty} := 0$;

(2) $a = b^{-1}$ *and* $o(a) < \infty$.

We cannot determine whether or not $\mathcal{P}$ is aspherical in the following four exceptional cases:

(E1) $a = b^2$ where $3 < o(b) < \infty$;

(E2) $a = b^{-2}$ where $3 < o(b) < \infty$;

(E3) $a^2 = b$ where $3 < o(a) = o(b) < \infty$;

(E4) $a = b^3$ where $o(b) = 9$.

7.2.1 Preliminary comments and an outline of the proof of Theorem 7.2.1

We may assume, without loss of generality, that $o(a) \leqslant o(b)$. For if $o(a) > o(b)$, then we may use the substitution $s = t^{-1}$ to transform the relator $t^n a t^{-1} b$ into the equivalent relator $s^n b^{-1} s^{-1} a^{-1}$ in which $o(b^{-1}) < o(a^{-1})$. *This assumption will not be repeated.*

The proof of Theorem 7.2.1 is split into two parts. In Part A we determine when $\mathcal{P}$ is diagrammatically reducible and hence aspherical. Lemma 7.2.1 states that $\mathcal{P}$ is diagrammatically reducible if a and b both have infinite order in H. Next, we suppose $a \neq b^{-1}$ and $\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} \leqslant 1$

where $o(a) < \infty$. To prove $\mathcal{P}$ is diagrammatically reducible in this case we assume there exists a reduced non-empty connected strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$ and argue for a contradiction. We define a flat angle function θ on $\mathbb{P}$ with associated curvature function γ. Since θ is flat, $\mathbb{P}$ must contain at least one exceptional region and we deduce that an exceptional region has degree $d < 4$. The exceptional regions of degree 2 are the $\mathcal{D}_i$-regions ($i = 1, 2, 3$) (see Fig. 7.3) and the exceptional regions of degree 3 are the $\mathcal{T}_j$-regions ($j = 1, 2$) (see Fig. 7.4). Next, we define a γ-flattening distribution scheme η on $\mathbb{P}$ and conclude from Lemma 6.4.2 that there exists a region K such that $\gamma^*(K) > \gamma(K)$ and $\gamma^*(K) > 0$. Let K have degree m. If curvature is not distributed from any $\mathcal{D}_i$-region ($i = 1, 2, 3$) to K, then $m < 5$ and it is easy to show that K cannot exist in this case. If curvature is distributed from at least one $\mathcal{D}_i$-region ($i = 1, 2, 3$) to K, then $m < 12$. In Lemmas 7.2.2, 7.2.3, 7.2.5, 7.2.6 and 7.2.8 - 7.2.10 we prove that m cannot equal 3, 4, 6, 7, 9, 10, or 11. If m equals 5 or 8, then in Lemmas 7.2.4 and 7.2.7 we show that K must be a P_51-region (see Fig. 7.8), a P_8-region (see Fig. 7.13) or a Q_5-region (see Fig. 7.14). In Lemma 7.2.12 we prove that some of the neighbouring regions of a P_51-region are sufficiently negatively curved to allows us to flatten this region, thus obtaining a contradiction to the fundamental curvature formula. Regions Q_5 and P_8 are dealt with in a similar way in Lemmas 7.2.13 and 7.2.14, respectively. This will complete Part A of the proof of Theorem 7.2.1.

In Part B we determine when $\mathcal{P}$ is not aspherical. First, we suppose $\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} > 1$ which holds if and only if $(o(a), o(b), o(ab^{-1})) = (3, 3, 2), (3, 4, 2), (3, 5, 2)$. For each solution we exhibit a reduced non-degenerate strictly spherical picture over $\mathcal{P}$ (see Lemmas 7.2.15 - 7.2.17 and Figs. 7.19 - 7.21). Next, we suppose $a = b^{-1}$ and $o(a) < \infty$. In Lemma 7.2.18 we prove that t must then have finite order in $G(\mathcal{P})$. This will complete Part B of the proof of Theorem 7.2.1.

7.2.2 The proof of Theorem 7.2.1

Part A

Lemma 7.2.1. *If a and b both have infinite order in H, then $\mathcal{P}$ is diagrammatically reducible.*

Proof. Consider the star graph $\mathcal{P}^{st}$ in Fig. 7.1 where edges α_1 and α_2 have labels b^{-1} and a^{-1}, respectively, and edges α_i ($i = 3, \ldots, n+1$) have label 1. Define a weight function ω on $\mathcal{P}^{st}$ as follows: $\omega(\alpha_1) = \omega(\alpha_2) = 0$ and $\omega(\alpha_i) = 1$ for $i = 3, \ldots, n+1$. It is easy to see that ω is a weakly aspherical weight function. Hence, $\mathcal{P}$ is diagrammatically reducible by Theorem 6.4.1. $\square$

Suppose $a \neq b^{-1}$ and $\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} \leqslant 1$ where $o(a) < \infty$. We assume $\mathcal{P}$ is not aspherical and argue for a contradiction. It follows that there exists a reduced non-empty connected strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. The discs of $\mathbb{P}$ are illustrated in Fig. 7.2. For clarity, we leave blank those corners that have label 1 and omit the label t on arrows.

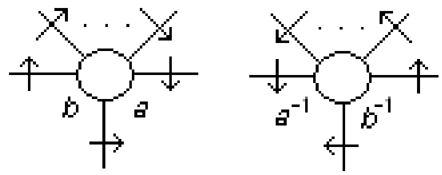


Figure 7.2: The discs of $\mathbb{P}$.

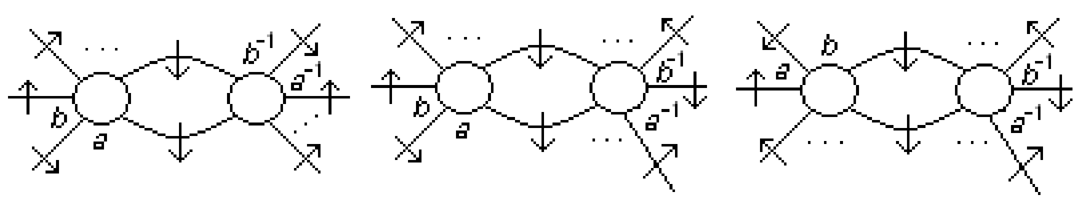
Define an angle function θ on $\mathbb{P}$ with associated curvature function γ as follows: corners labelled $a^{\pm 1}$ or $b^{\pm 1}$ have angle $\frac{\pi}{2}$; corners that are adjacent to corners labelled $a^{\pm 1}$ or $b^{\pm 1}$ have angle $\frac{\pi}{2}$; all other corners have angle 0. Then θ is a flat angle function on $\mathbb{P}$ and it follows from Lemma 6.4.1 that $\mathbb{P}$ contains an exceptional region F of degree d, say. Since

$$\sum_{\kappa \subseteq \partial F} \theta(\kappa) > (d-2)\pi,$$

and since

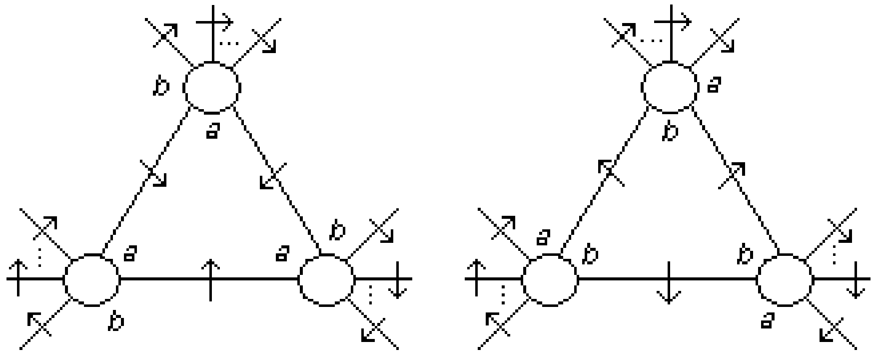
$$\sum_{\kappa \subseteq \partial F} \theta(\kappa) \leqslant d\frac{\pi}{2}$$

we deduce that $d < 4$. In Figs. 7.3 and 7.4 we exhibit (up to mirror image) the positively curved regions of degree 2 and degree 3. A $\mathcal{D}_1$-region has curvature π, and a $\mathcal{D}_2$- or a $\mathcal{D}_3$-region has curvature $\frac{\pi}{2}$. Regions $\mathcal{T}_1$ and $\mathcal{T}_2$ also have curvature $\frac{\pi}{2}$. By a $\mathcal{D}$-region we shall mean a $\mathcal{D}_i$-region for some $i = 1, 2, 3$ and by a $\mathcal{T}$-region we shall mean a $\mathcal{T}_j$-region for some $j = 1, 2$.



(a) $\mathcal{D}_1$ (b) $\mathcal{D}_2$ (c) $\mathcal{D}_3$

Figure 7.3: Positively curved regions of degree 2.



(a) $\mathcal{T}_1$ (b) $\mathcal{T}_2$

Figure 7.4: Positively curved regions of degree 3.

Define a distribution scheme η on $\mathbb{P}$ as follows:

$$\eta(F,F') = \begin{cases} \frac{\pi}{2} & F \text{ is a } \mathcal{D}_1\text{-region and } F \text{ is separated from } F' \text{ by an arc which is oriented from } F \text{ to } F'; \\ \frac{\pi}{2} & F \text{ is a } \mathcal{D}_1\text{-region and } F \text{ is separated from } F' \text{ by an arc which is oriented from } F' \text{ to } F; \\ \frac{\pi}{2} & F \text{ is a } \mathcal{D}_2\text{-region and } F \text{ is separated from } F' \text{ by an arc which is oriented from } F \text{ to } F'; \\ \frac{\pi}{2} & F \text{ is a } \mathcal{D}_3\text{-region and } F \text{ is separated from } F' \text{ by an arc which is oriented from } F' \text{ to } F; \\ \frac{\pi}{6} & F \text{ is a } \mathcal{T}_1\text{-region and } F \text{ is separated from } F' \text{ by an arc which is oriented from } F' \text{ to } F; \\ \frac{\pi}{6} & F \text{ is a } \mathcal{T}_2\text{-region and } F \text{ is separated from } F' \text{ by an arc which is oriented from } F \text{ to } F'; \\ 0 & \text{otherwise.} \end{cases}$$

If $\eta(F,F') > 0$, then there exist at least two corners κ_1 and κ_2 of F' such that κ_1 has label $a^{\pm 1}$ or $b^{\pm 1}$ and κ_2 has label 1. It follows easily that η is a γ-flattening of $\mathbb{P}$ so by Lemma 6.4.2 there exists a region K satisfying $\gamma^*(K) > \gamma(K)$ and $\gamma^*(K) > 0$, where γ^* is the distributed curvature function. Let K have degree m. Our aim is to prove that K cannot exist.

Suppose K does not receive curvature from any $\mathcal{D}$-region. Then K can receive curvature only from a $\mathcal{T}$-region. Since at most every second arc of ∂K can be on the boundary of a $\mathcal{T}$-region, we have

$$\sum_{\kappa \subseteq \partial K} \theta^*(\kappa) \leqslant m\frac{\pi}{2} + \frac{m}{2}\frac{\pi}{6} = m\frac{7\pi}{12},$$

and since $\gamma^*(K) > 0$, we have

$$\sum_{\kappa \subseteq \partial K} \theta^*(\kappa) > (m-2)\pi.$$

Thus, $m < \frac{24}{5} < 5$. By considering all possible arrangements of corner labels we find that the label of K is either a, b, a^2, b^2 or $ab^{\pm 1}$. Each of these labels is a non-identity element of H, so K cannot exist if $m < 5$.

Now suppose at least one arc of ∂K is on the boundary of a $\mathcal{D}$-region and that curvature is distributed from this region to K. To estimate an upper bound for the degree of K we first calculate the maximum value of $\sum_{\kappa \subseteq \partial K} \theta^*(\kappa)$. Before we apply η, each corner of K either has angle 0 or $\frac{\pi}{2}$. Suppose each corner has angle $\frac{\pi}{2}$ and let κ be a corner of K. Then the label of κ is either $a^{\pm 1}$ or $b^{\pm 1}$, or κ has label 1 and precisely one adjacent corner to κ has label $a^{\pm 1}$ or $b^{\pm 1}$ (see Fig. 7.5). Therefore, if each corner of K has angle $\frac{\pi}{2}$, then at most every second arc of ∂K can be on the boundary of a $\mathcal{D}$-region. Each of the remaining arcs of ∂K may lie on the boundary of a $\mathcal{T}$-region, so after applying η, we have

$$\sum_{\kappa \subseteq \partial K} \theta^*(\kappa) \leqslant m\frac{\pi}{2} + \frac{m}{2}\frac{\pi}{2} + \frac{m}{2}\frac{\pi}{6} = m\frac{5\pi}{6}. \tag{7.1}$$

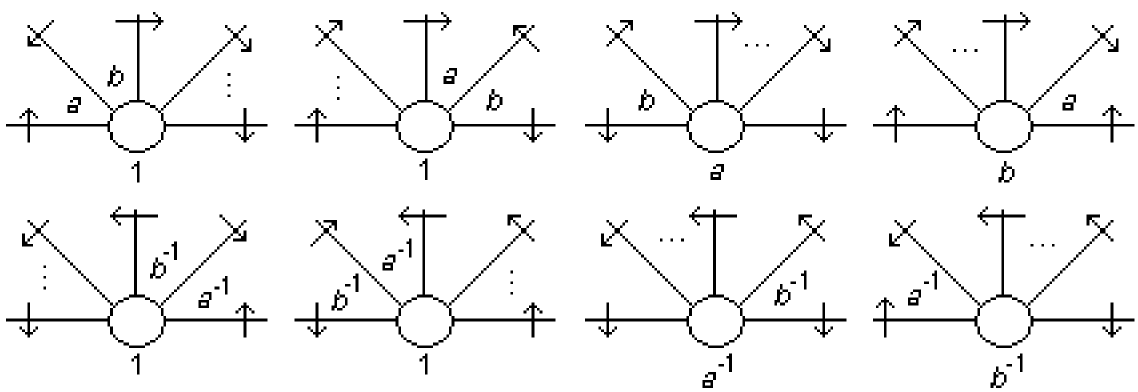


Figure 7.5: The possible labels of κ and its adjacent corners.

If two consecutive arcs of ∂K are on the boundaries of $\mathcal{D}$-regions $\mathcal{D}'$ and $\mathcal{D}''$ which both contribute $\frac{\pi}{2}$ to $\gamma^*(K)$, then the corner κ of K that is on the boundary of the disc which is common to both $\mathcal{D}'$ and $\mathcal{D}''$ must have angle 0 (see Fig. 7.6). After applying η, we have $\theta^*(\kappa) = \pi$. If $\theta(\kappa) = \frac{\pi}{2}$ and only one adjacent arc is on the boundary of a $\mathcal{D}$-region which contributes $\frac{\pi}{2}$, then $\theta^*(\kappa) = \pi$. Thus, both situations are equivalent in terms of the contribution κ makes to the sum $\sum_{\kappa' \subseteq \partial K} \theta^*(\kappa')$. Therefore, we may assume that each corner of K has angle $\frac{\pi}{2}$ and that at most every second arc of ∂K is on the boundary of a $\mathcal{D}$-region. Since

$$\sum_{\kappa \subseteq \partial K} \theta^*(\kappa) > (m-2)\pi,$$

we deduce from (7.1) that $m < 12$.

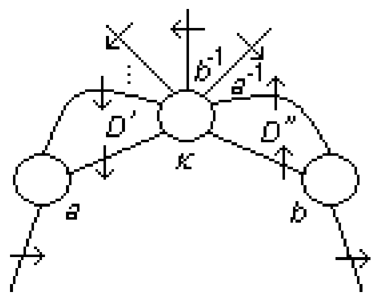
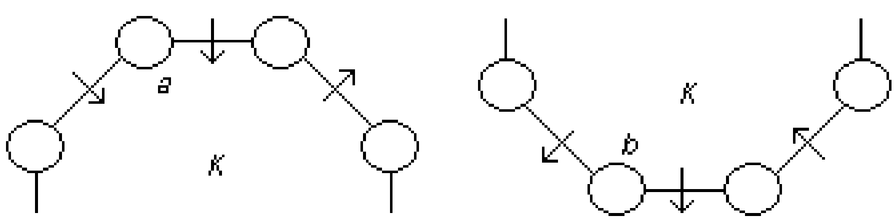


Figure 7.6: Consecutive $\mathcal{D}$-regions.

At least one arc of ∂K is on the boundary of a $\mathcal{D}$-region and curvature is distributed from this region to K. It follows that part of ∂K must have Configuration A or Configuration B as illustrated in Fig. 7.7.



(a) Configuration A

(b) Configuration B

Figure 7.7: Configurations of ∂K.

Beginning from Configuration A, we now consider all possible arrangements of corner labels which can exist in K. Our aim is to prove that the label of K is a non-identity element of H or that $\gamma^*(K) \leqslant 0$. Note, when estimating an upper bound for $\gamma^*(K)$ we assume each corner of K has angle $\frac{\pi}{2}$ and that at most every second arc of ∂K is on the boundary of a $\mathcal{D}$-region.

Lemma 7.2.2. *K cannot have degree 3.*

Proof. If $m = 3$, then K must have label $a \neq 1$. $\square$

Lemma 7.2.3. *K cannot have degree 4.*

Proof. If $m = 4$, then K either has label a^2 or $ab^{\pm 1}$ and it is clear that both of these elements are non-identity elements of H. $\square$

Lemma 7.2.4. *If $m = 5$, then $\gamma^*(K) > 0$ only if K is a $P_5 1$-region.*

Proof. Suppose K is a $P_5 1$-region (see Fig. 7.8). At most one $\mathcal{D}$-region and at most one $\mathcal{T}_2$-region can contribute, respectively, $\frac{\pi}{2}$ and $\frac{\pi}{6}$ to $\gamma^*(P_5 1)$. Since $\gamma(P_5 1) \leqslant -\frac{\pi}{2}$, we have $\gamma^*(P_5 1) > 0$ if and only if curvature is distributed from both of these regions to $P_5 1$, in which case $\gamma^*(P_5 1) = \frac{\pi}{6}$.

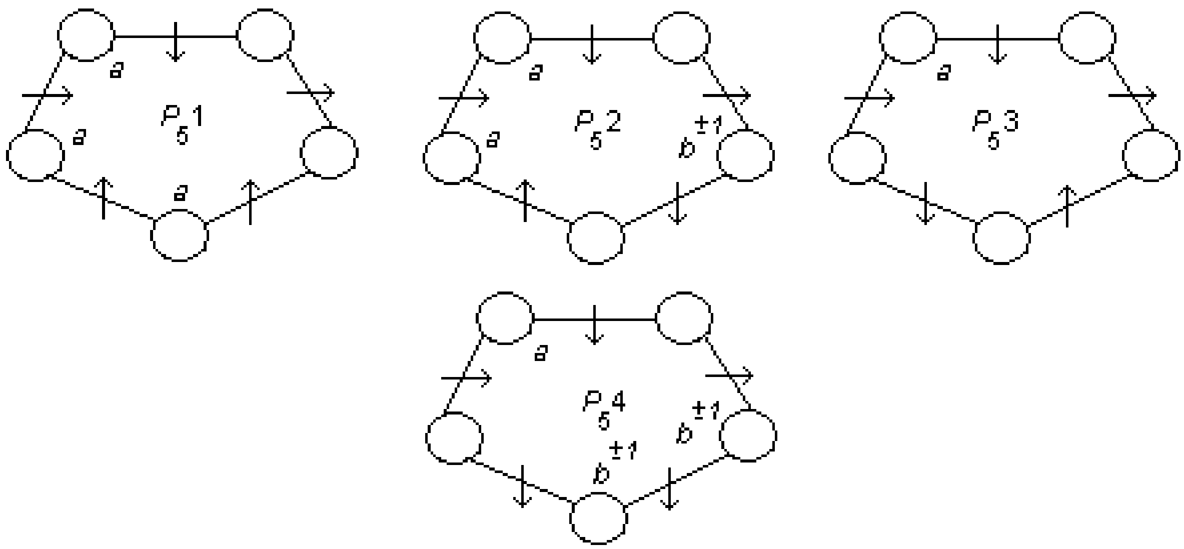


Figure 7.8: Possible regions of degree 5.

Suppose K is a $P_5 2$-region. The label of such a region is either $a^2 b$ or $a^2 b^{-1}$, so we must have $o(a) > 3$. Since $\mathcal{P}$ is not the exceptional case (E3), $a^2 b^{-1} \neq 1$. Therefore, the label must be $a^2 b$. Since $o(a) > 3$, $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed to $P_5 1$ from at most one $\mathcal{D}$-region, we have

$$\gamma^*(P_5 2) \leqslant \gamma(P_5 2) + \frac{\pi}{2} \leqslant -\frac{\pi}{2} + \frac{\pi}{2} = 0.$$

Clearly K cannot be a $P_5 3$-region so suppose K is a $P_5 4$-region. In this case we must have $o(b) > 3$. Since $\mathcal{P}$ is neither the exceptional case (E1) or (E2), we have $ab^{\pm 2} \neq 1$. $\square$

Lemma 7.2.5. *K cannot have degree* 6.

Proof. Suppose K is a $P_6 1$-region (see Fig. 7.9). In this case we must have $o(a) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed from at most one $\mathcal{D}$-region to $P_6 1$, we have

$$\gamma^*(P_6 1) \leqslant \gamma(P_6 1) + \frac{\pi}{2} \leqslant -\pi + \frac{\pi}{2} = -\frac{\pi}{2}.$$

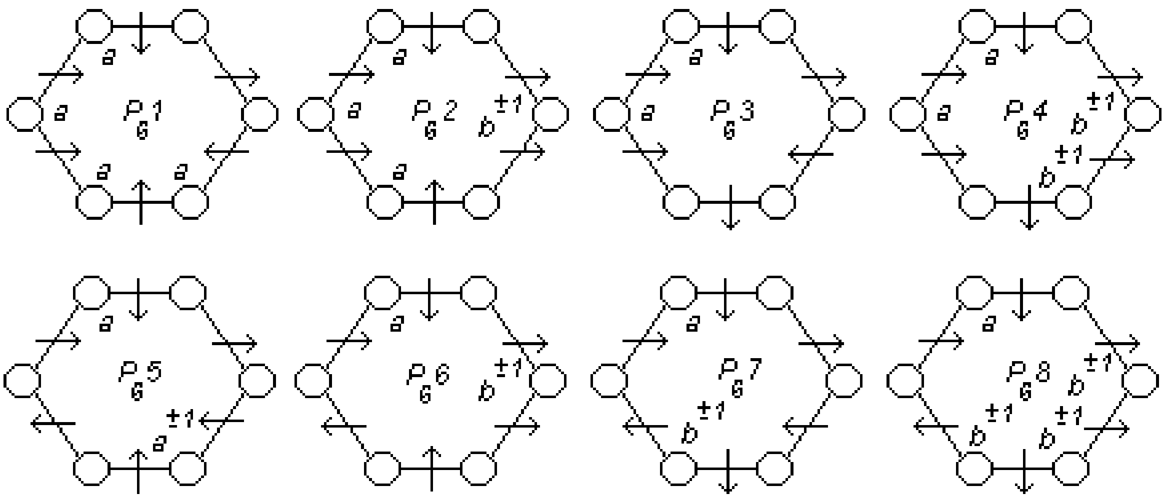


Figure 7.9: Possible regions of degree 6.

Suppose K is a $P_6 2$-region. In this case we must have $o(a) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed from at most two $\mathcal{D}$-regions to $P_6 2$, we have

$$\gamma^*(P_6 2) \leqslant \gamma(P_6 2) + 2\frac{\pi}{2} \leqslant -\pi + \pi = 0.$$

Clearly K cannot be a $P_6 3$-region, a $P_6 6$-region, or a $P_6 7$-region.

Suppose K is a $P_6 4$-region. In this case we cannot have $o(a) = o(b) = 3$. Since $\mathcal{P}$ is neither the exceptional case (E1) or (E2), we cannot have $o(a) = 3$ and $o(b) > 3$. If $o(a) > 3$, then $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed from at most two $\mathcal{D}$-regions to $P_6 4$, we have

$$\gamma^*(P_6 4) \leqslant \gamma(P_6 4) + 2\frac{\pi}{2} \leqslant -\pi + \pi = 0.$$

Suppose K is a $P_6 5$-region. Since $o(a) \geqslant 3$, the label of this region must be aa^{-1}. In this case $\frac{\pi}{2}$ is distributed from at most two $\mathcal{D}$-regions to $P_6 5$. Suppose curvature is distributed from exactly two such regions. This is only possible if $\mathbb{P}$ contains a dipole (see Fig. 7.10). Therefore, curvature is distributed from at most one $\mathcal{D}$-region. Since $\frac{\pi}{6}$ is distributed from at most two $\mathcal{T}$-regions to P_65, we have

$$\gamma^*(P_65) \leqslant \gamma(P_65) + \frac{\pi}{2} + 2\frac{\pi}{6} = -\frac{\pi}{6}.$$

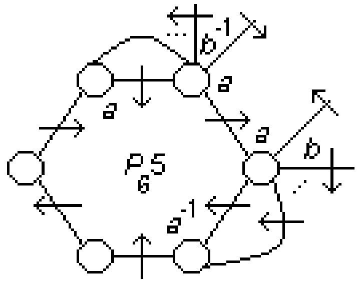


Figure 7.10: A P_65-region with a dipole.

Suppose K is a P_68-region with label ab^{-3}. In this case we must have $o(b) > 3$. Since $\mathcal{P}$ is not the exceptional case (E4), $o(a) > 3$ and so $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed from at most two $\mathcal{D}$-regions to P_68, we have

$$\gamma^*(P_68) \leqslant \gamma(P_68) + 2\frac{\pi}{2} \leqslant -\pi + \pi = 0.$$

Now suppose the label is ab^3. Then as above $o(b) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}_2$-region. In this case $\frac{\pi}{2}$ and $\frac{\pi}{6}$ is distributed from, respectively, at most one $\mathcal{D}$-region and at most one $\mathcal{T}_1$-region to P_68. Therefore,

$$\gamma^*(P_68) \leqslant \gamma(P_68) + \frac{\pi}{2} + \frac{\pi}{6} \leqslant -\pi + \frac{2\pi}{3} = -\frac{\pi}{3}.$$

□

Lemma 7.2.6. *K cannot have degree 7.*

Proof. Suppose K is a P_71-region (see Fig. 7.11). In this case we must have $o(a) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed from at most one $\mathcal{D}$-region to P_71, we have

$$\gamma^*(P_71) \leqslant \gamma(P_71) + \frac{\pi}{2} \leqslant -\frac{3\pi}{2} + \frac{\pi}{2} = -\pi.$$

Suppose K is a P_72-region. In this case we must have $o(a) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed from at most two $\mathcal{D}$-regions to P_72, we have

$$\gamma^*(P_72) \leqslant \gamma(P_72) + 2\frac{\pi}{2} \leqslant -\frac{3\pi}{2} + \pi = -\frac{\pi}{2}.$$

Suppose K is a P_73-region. In this case $\frac{\pi}{2}$ and $\frac{\pi}{6}$ is distributed from, respectively, at most one $\mathcal{D}$-region and at most one $\mathcal{T}_2$-region to P_73. Therefore,

$$\gamma^*(P_73) \leqslant \gamma(P_73) + \frac{\pi}{2} + \frac{\pi}{6} \leqslant -\frac{3\pi}{2} + \frac{2\pi}{3} = -\frac{\pi}{2}.$$

Suppose K is a P_74-region. In this case we must have $o(a) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\frac{\pi}{2}$ is distributed from at most two $\mathcal{D}$-regions to P_74, we have

$$\gamma^*(P_74) \leqslant \gamma(P_74) + 2\frac{\pi}{2} \leqslant -\frac{3\pi}{2} + \pi = -\frac{\pi}{2}.$$

Suppose K is a P_75-region. Clearly, the label must be a^3. In this case $\frac{\pi}{2}$ and $\frac{\pi}{6}$ is distributed from, respectively, at most two $\mathcal{D}$-regions and at most two $\mathcal{T}$-regions to P_75. Therefore,

$$\gamma^*(P_75) \leqslant \gamma(P_75) + 2\frac{\pi}{2} + 2\frac{\pi}{6} \leqslant -\frac{3\pi}{2} + \pi + \frac{\pi}{3} = -\frac{\pi}{6}.$$

Suppose K is a P_76-region. In this case we must have $o(a) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}$-region. Since $\mathcal{P}$ is not the exceptional case (E3), $a^2b^{-1} \neq 1$ and so the label must be a^2b. In this case $\frac{\pi}{2}$ is distributed from at most one $\mathcal{D}$-region to P_76. Therefore,

$$\gamma^*(P_76) \leqslant \gamma(P_76) + \frac{\pi}{2} \leqslant -\frac{3\pi}{2} + \frac{\pi}{2} = -\pi.$$

If K is a P_77-region, then, arguing as in the case when K is a P_76-region, we find that $\frac{\pi}{2}$ is distributed from at most two $\mathcal{D}$-regions to P_77. Therefore,

$$\gamma^*(P_77) \leqslant \gamma(P_77) + 2\frac{\pi}{2} \leqslant -\frac{3\pi}{2} + \pi = -\frac{\pi}{2}.$$

Suppose K is a P_78-region with label a^2b^3. In this case we must have $o(b) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}_2$-region. Since $\frac{\pi}{2}$ and $\frac{\pi}{6}$ is distributed from, respectively, at most one $\mathcal{D}$-region and at most one $\mathcal{T}_1$-region to P_78, we have

$$\gamma^*(P_78) \leqslant \gamma(P_78) + \frac{\pi}{2} + \frac{\pi}{6} \leqslant -\frac{3\pi}{2} + \frac{2\pi}{3} = -\frac{5\pi}{6}.$$

Now suppose the label is a^2b^{-3}. Then as above $o(b) > 3$, so $\mathbb{P}$ cannot contain a $\mathcal{T}_2$-region. In this case $\frac{\pi}{2}$ and $\frac{\pi}{6}$ is distributed from, respectively, at most two $\mathcal{D}$-regions and at most one $\mathcal{T}_1$-region to P_78. Therefore,

$$\gamma^*(P_78) \leqslant \gamma(P_78) + 2\frac{\pi}{2} + \frac{\pi}{6} \leqslant -\frac{3\pi}{2} + \frac{2\pi}{3} = -\frac{\pi}{3}.$$

The arguments used to discount regions P_79 - P_716 are very similar to those used above. This completes the proof Lemma 7.2.6. $\square$

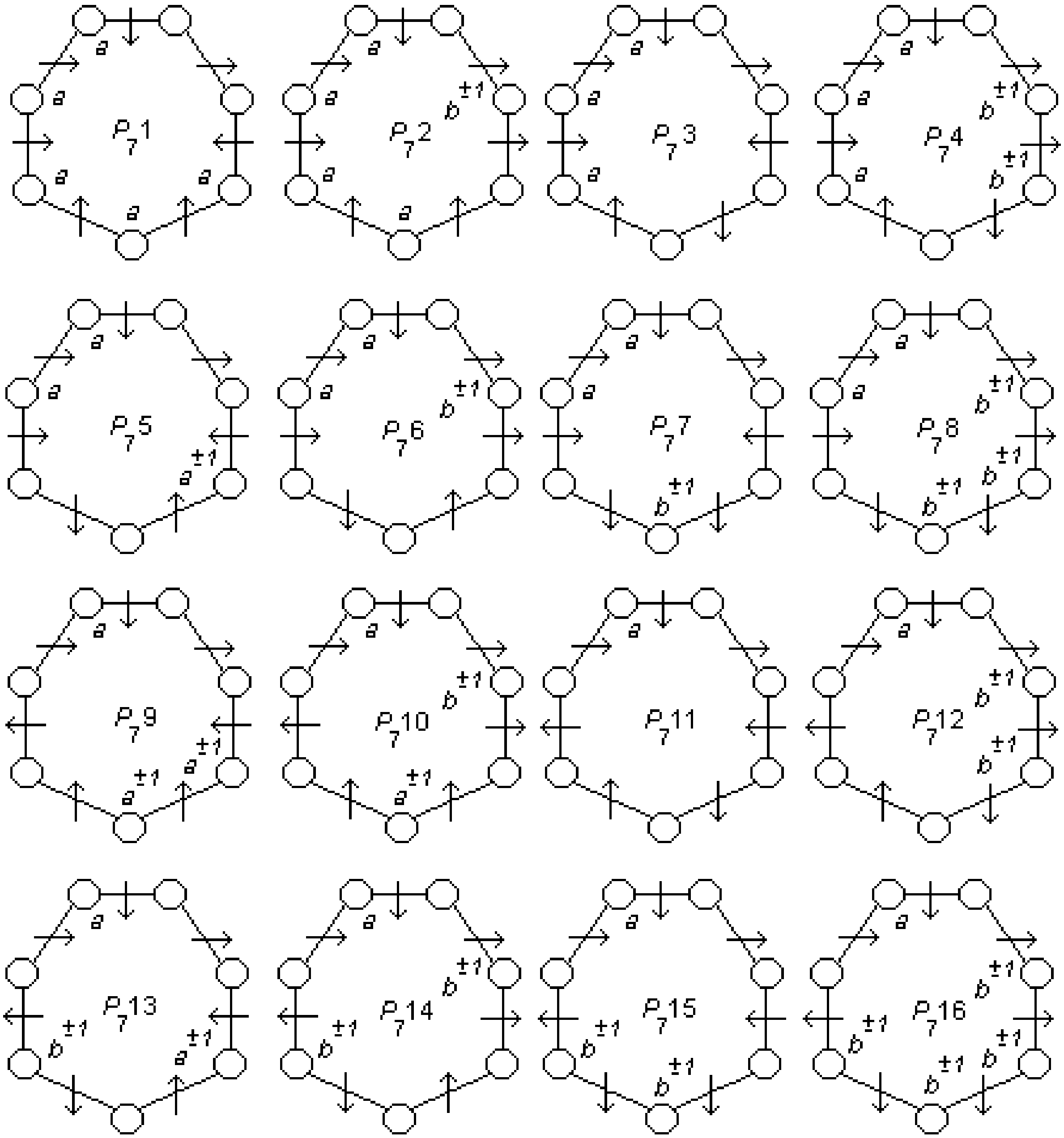


Figure 7.11: Possible regions of degree 7.

Lemma 7.2.7. *If $m = 8$, then $\gamma^*(K) > 0$ only if K is a P_8-region.*

Proof. At most four arcs of ∂K can each be on the boundary of a $\mathcal{T}$-region. Suppose that K receives $\frac{\pi}{6}$ from precisely four such regions. Up to mirror image, there is only one possible configuration for K and its neighbouring regions (see Fig. 7.12). In this case the relations $a^3 = b^3 = ab^{-1}ab^{-1} = 1$

hold in H. Therefore

$$\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} > 1,$$

which contradicts one of our main assumptions.

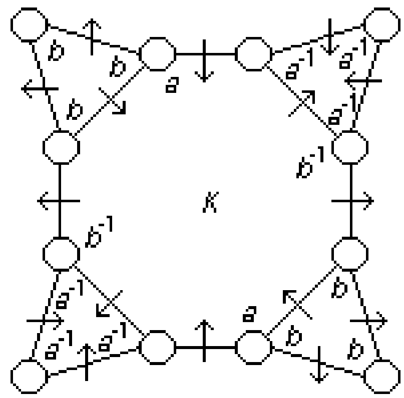


Figure 7.12: Four neighbouring $\mathcal{T}$-regions.

It follows that at most three $\mathcal{T}$-regions each contribute $\frac{\pi}{6}$. In this case at least four neighbouring $\mathcal{D}$-regions must each contribute $\frac{\pi}{2}$. For otherwise,

$$\gamma^*(K) \leqslant \gamma(K) + 3\frac{\pi}{2} + 3\frac{\pi}{6} \leqslant -2\pi + 2\pi = 0.$$

We deduce that K and its neighbours must have the configuration shown in Fig. 7.13, where at least one of F, F' is a $\mathcal{T}_1$-region. If only one of F, F' is a $\mathcal{T}_1$-region, then $\gamma^*(P_8) = \frac{\pi}{6}$. If both F and F' are $\mathcal{T}_1$-regions, then $\gamma^*(P_8) = \frac{\pi}{3}$. Note that $\mathbb{P}$ cannot contain a $\mathcal{T}_2$-region. For otherwise, the relations $a^3 = b^3 = ab^{-1}ab^{-1} = 1$ hold in H. □

Lemma 7.2.8. *K cannot have degree 9.*

Proof. If K receives $\frac{\pi}{6}$ from at most four neighbouring $\mathcal{T}$-regions and $\frac{\pi}{2}$ from at most three neighbouring $\mathcal{D}$-regions, then

$$\gamma^*(K) \leqslant \gamma(K) + 3\frac{\pi}{2} + 4\frac{\pi}{6} \leqslant -\frac{5\pi}{2} + \frac{13\pi}{6} = -\frac{\pi}{3}.$$

If K receives $\frac{\pi}{6}$ from at most three $\mathcal{T}$-regions and $\frac{\pi}{2}$ from at most four $\mathcal{D}$-regions, then

$$\gamma^*(K) \leqslant \gamma(K) + 4\frac{\pi}{2} + 3\frac{\pi}{6} \leqslant -\frac{5\pi}{2} + \frac{5\pi}{2} = 0.$$

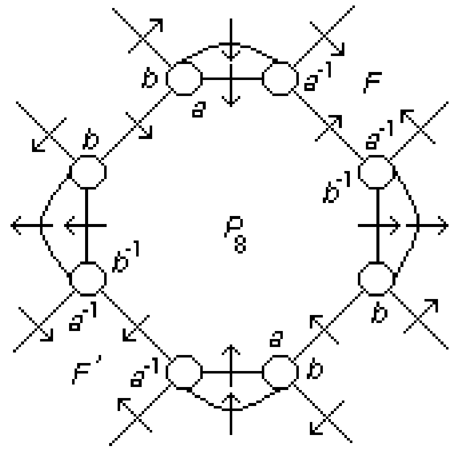


Figure 7.13: The possible region of degree 8.

Therefore, $\gamma^*(K) > 0$ if and only if K receives $\frac{\pi}{2}$ and $\frac{\pi}{6}$ from four neighbouring $\mathcal{D}$-regions and four neighbouring $\mathcal{T}$-regions, respectively. In this case K must have label $ab^{-2}ab^{-2}$ or $a^2b^{-1}ab^{-1}$, and the relations

(i) $a^3 = b^3 = ab^{-2}ab^{-1} = 1$, or

(ii) $a^3 = b^3 = a^2b^{-1}ab^{-1} = 1$

must hold in H. Suppose (i) hold. Then $1 = ab^{-2}ab^{-1} = a^{-2}bab^{-1}$ and so $bab^{-1} = a^2$. From this we deduce that $b^2ab^{-2} = a^4$. Therefore,

$$a = b^3ab^{-3} = b(b^2ab^{-2})b^{-1} = ba^4b^{-1} = (bab^{-1})^4 = a^8$$

from which we deduce $a = 1$. By a similar argument we can show that if (ii) hold, then $b = 1$. Thus, K cannot have degree 9. $\square$

Lemma 7.2.9. *K cannot have degree* 10.

Proof. Suppose $\frac{\pi}{2}$ is distributed from at most four neighbouring $\mathcal{D}$-regions to K. In this case $\frac{\pi}{6}$ can be distributed from at most five $\mathcal{T}$-regions to K. Therefore,

$$\gamma^*(K) \leqslant \gamma(K) + 4\frac{\pi}{2} + 5\frac{\pi}{6} \leqslant -3\pi + \frac{17\pi}{6} = -\frac{\pi}{6}.$$

Now suppose $\frac{\pi}{2}$ and $\frac{\pi}{6}$ is distributed from, respectively, five $\mathcal{D}$-regions and at most three $\mathcal{T}$-regions to K. Then,

$$\gamma^*(K) \leqslant \gamma(K) + 5\frac{\pi}{2} + 3\frac{\pi}{6} \leqslant -3\pi + 3\pi = 0.$$

Therefore, $\gamma^*(K) > 0$ if and only if five $\mathcal{D}$-regions and four $\mathcal{T}$-regions each contribute $\frac{\pi}{2}$ and $\frac{\pi}{6}$, respectively. However, it is impossible to construct a region of degree 10 that has five neighbouring $\mathcal{D}$-regions and four neighbouring $\mathcal{T}$-regions. □

Lemma 7.2.10. *K cannot have degree 11.*

Proof. If $m = 11$, then $\frac{\pi}{2}$ and $\frac{\pi}{6}$ is distributed from, respectively, at most five $\mathcal{D}$-regions and at most five $\mathcal{T}$-regions. Therefore,

$$\gamma^*(K) \leqslant \gamma(K) + 5\frac{\pi}{2} + 5\frac{\pi}{6} \leqslant -\frac{7\pi}{2} + \frac{10\pi}{3} = -\frac{\pi}{6}.$$

□

We now have to consider all possible arrangements of corner labels that arise from Configuration B (see Fig. 7.7). The analysis of such arrangements is very similar to those that arise from Configuration A. We quickly find that $\gamma^*(K) > 0$ if and only if K is a mirror image of a P_8-region or is a Q_5-region as illustrated in Fig. 7.14.

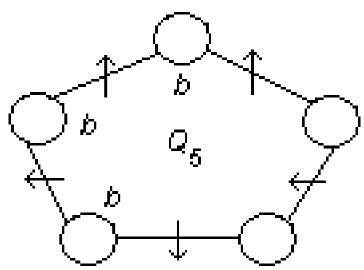


Figure 7.14: A Q_5-region.

Lemma 7.2.11. *If $o(b) = 3, 4, 5$, then $\mathbb{P}$ cannot contain a P_8-region such that $\gamma^*(P_8) > 0$.*

Proof. Suppose $o(b) = 3$ and that $\mathbb{P}$ contains a P_8-region of positive curvature. Now $\gamma^*(P_8) > 0$ only if $\mathbb{P}$ contains a $\mathcal{T}_1$-region. In this case the relations $a^3 = b^3 = (ab^{-1})^2 = 1$ must hold in H, so

$$\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} > 1$$

which contradicts one of our main assumptions. We obtain the same contradiction if $o(b) = 4, 5$. □

We conclude from Lemma 7.2.11 that $\mathbb{P}$ cannot contain a $P_5 1$-region with $\gamma^*(P_5 1) > 0$, or a Q_5-region, together with a P_8-region where $\gamma^*(P_8) > 0$.

Lemma 7.2.12. *After suitably distributing curvature, $\mathbb{P}$ cannot contain a $P_5 1$-region with $\gamma^*(P_5 1) > 0$.*

Proof. Suppose $\mathbb{P}$ contains a $P_5 1$-region (see Fig. 7.8). Before we apply the distribution scheme η to $\mathbb{P}$, distribute $\frac{\pi}{6}$ from all $P_5 1$-regions as shown in Fig. 7.15 below.

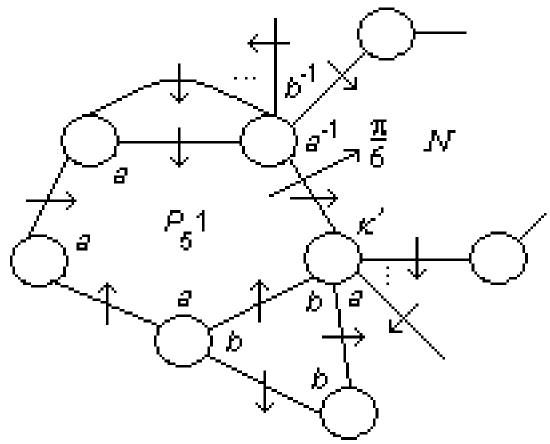


Figure 7.15: Distributing curvature from a $P_5 1$-region.

Note that $\theta(\kappa') = 0$, so after distributing this additional curvature we have $\theta^*(\kappa') = \frac{\pi}{6}$ and $\gamma(P_5 1) \leqslant -\frac{\pi}{6}$. Now apply η to $\mathbb{P}$. Then $\gamma^*(P_5 1) \leqslant 0$. We now prove that $\gamma^*(N) \leqslant 0$, thus obtaining a contradiction to the fundamental curvature formula.

From Lemmas 7.2.2 - 7.2.10, we have $\gamma^*(N) > 0$ if and only if N is a $P_5 1$-region, a Q_5-region, or a P_8-region. Since $\mathbb{P}$ contains a $\mathcal{T}_2$-region, it follows from Lemma 7.2.11 that N cannot be a P_8-region. Moreover, at least one corner of N has label a^{-1}, which prevents N from being a Q_5-region. We are left with the possibility that N is a $P_5 1$-region (see Fig. 7.16). We observe that no arc of ∂N can be on the boundary of a $\mathcal{D}$-region, so N can only receive curvature from at most one $\mathcal{T}_2$-region. Thus,

$$\gamma^*(N) \leqslant \gamma(N) + \frac{\pi}{6} \leqslant 2\pi - 4(\pi - \frac{\pi}{2}) - (\pi - \frac{\pi}{6}) + \frac{\pi}{6} = -\frac{2\pi}{3}.$$

$\square$

The proof of the following lemma is very similar to that of Lemma 7.2.12.

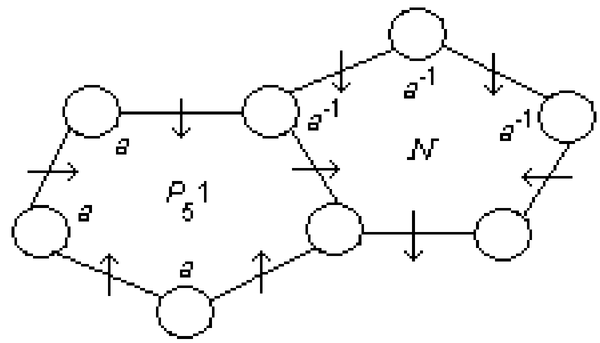


Figure 7.16: The configuration for when N is a $P_5 1$-region.

Lemma 7.2.13. *After suitably distributing curvature, $\mathbb{P}$ cannot contain a Q_5-region with $\gamma^*(Q_5) > 0$.*

We require a final lemma to complete Part A of the proof of Theorem 7.2.1.

Lemma 7.2.14. *After suitably distributing curvature, $\mathbb{P}$ cannot contain a P_8-region with $\gamma^*(P_8) > 0$.*

Proof. Suppose $\mathbb{P}$ contains a P_8-region (see Fig. 7.13). Recall, if neither of F or F' is a $\mathcal{T}_1$-region, then $\gamma^*(P_8) \leqslant 0$. So at least one of F, F' is a $\mathcal{T}_1$-region. Before applying the distribution scheme η to $\mathbb{P}$, distribute $\frac{\pi}{6}$ from all P_8-regions as shown in Fig. 7.17. (If only one of F, F' is a $\mathcal{T}_1$-region, then we distribute $\frac{\pi}{6}$ to N only. Otherwise, we distribute $\frac{\pi}{6}$ to N and $\frac{\pi}{6}$ to N' as shown.) After applying η to $\mathbb{P}$, we have $\gamma^*(P_8) \leqslant 0$. Our aim now is to show prove that $\gamma^*(N) \leqslant 0$ and $\gamma^*(N') \leqslant 0$.

Suppose N does not receive $\frac{\pi}{2}$ from a $\mathcal{D}$-region. Then each arc of ∂N can be on the boundary of a P_8-region which contributes $\frac{\pi}{6}$ to N. If r is the degree of N, then

$$\sum_{\kappa \subseteq N} \theta^*(\kappa) \leqslant r\frac{\pi}{2} + r\frac{\pi}{6} = r\frac{2\pi}{3}.$$

If $\gamma^*(N) > 0$, then

$$\sum_{\kappa \subseteq N} \theta^*(\kappa) > (r-2)\pi$$

and we deduce that $r < 6$. We find that if N has degree at most 5, then its label is either b^2, b^3, b^4, b^5, ab^2 or $a^{-1}b^2$. Since $o(b) > 2$, N cannot have label b^2. Also, N cannot have label b^3, b^4, b^5 by Lemma 7.2.11. Since $\mathcal{P}$ is neither the exceptional case (E1) or (E2), N cannot have label ab^2 or $a^{-1}b^2$. Thus, $\gamma^*(N) \leqslant 0$.

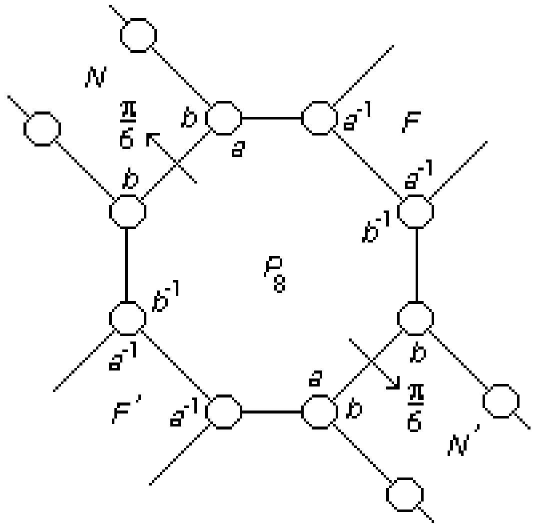


Figure 7.17: Distributing curvature from a P_8-region.

Now suppose N does receive $\frac{\pi}{2}$ from at least one $\mathcal{D}$-region. Since $\gamma^*(N)$ is maximal when each corner of N has angle $\frac{\pi}{2}$; when every second arc of ∂N is on the boundary of a $\mathcal{D}$-region which contributes $\frac{\pi}{2}$ to N; and when every other arc of ∂N is on the boundary of a $\mathcal{T}$-region or a P_8-region which contributes $\frac{\pi}{6}$, we have

$$\sum_{\kappa \subseteq N} \theta^*(N) \leqslant r\frac{5\pi}{6}.$$

If $\gamma^*(N) > 0$, then $r < 12$. We have already shown that $r \neq 3, 4, 5$ and by Lemma 7.2.10, $r \neq 11$. In Appendix A we prove that $r \neq 6, 7, 8, 9, 10$. Thus, $\gamma^*(N) \leqslant 0$. Similarly, $\gamma^*(N') \leqslant 0$. □

From Lemmas 7.2.1 - 7.2.14 we conclude that if neither Condition (1) nor Condition (2) of Theorem 7.2.1 holds, then $\mathcal{P}$ is diagrammatically reducible. This completes Part A of the proof of Theorem 7.2.1.

Part B

Let $\mathcal{P} = \langle H, t\,; t^n a t^{-1} b\rangle$. To prove $\mathcal{P}$ is not aspherical when Condition (1) of Theorem 7.2.1 holds, we exhibit three reduced non-degenerate strictly spherical pictures over $\mathcal{P}$.

The presentation $\mathcal{P}^o = \langle t\,; t^n t^{-1}\rangle$ is a presentation of the cyclic group Z_{n-1} of order $n-1$, and there exists a ring homomorphism $\psi^* : \mathbb{Z}G(\mathcal{P}) \to \mathbb{Z}Z_{n-1}$ induced from the obvious group homomorphism. Let $\mathbb{P}$ be a reduced non-empty connected strictly spherical picture over $\mathcal{P}$. To prove $\lambda_{\mathbb{P}} \neq 0$ (i.e. to prove that $\mathbb{P}$ is not degenerate) recall (§6.4.4) that it is enough to show that

$\psi^*(\lambda_{\mathbb{P}}) \neq 0$, where $\psi^*(\lambda_{\mathbb{P}})$ is the coefficient of e_{R^o} under the embedding

$$\mu^o : \pi_2(\mathcal{P}^o) \to \mathbb{Z}Z_{n-1}e_{R^o},$$

where $R^o = t^n t^{-1}$.

Lemma 7.2.15. *If $o(a) = 3$, $o(b) = 3$ and $o(ab^{-1}) = 2$, then $\mathcal{P}$ is not aspherical.*

Proof. Consider the reduced non-empty connected strictly spherical picture $\mathbb{P}_1$ over $\mathcal{P}$ illustrated in Fig. 7.19. We find that $\psi^*(\lambda_{\mathbb{P}_1}) = 12(1 - x^{-1}) \neq 0$, so $\mathbb{P}_1$ is not degenerate. $\square$

Lemma 7.2.16. *If $o(a) = 3$, $o(b) = 4$ and $o(ab^{-1}) = 2$, then $\mathcal{P}$ is not aspherical.*

Proof. Consider the reduced non-empty connected strictly spherical picture $\mathbb{P}_2$ over $\mathcal{P}$ illustrated in Fig. 7.20. We find that $\psi^*(\lambda_{\mathbb{P}_2}) = 24(1 - x^{-1}) \neq 0$, so $\mathbb{P}_2$ is not degenerate. $\square$

Lemma 7.2.17. *If $o(a) = 3$, $o(b) = 5$ and $o(ab^{-1}) = 2$, then $\mathcal{P}$ is not aspherical.*

Proof. Consider the reduced non-empty connected strictly spherical picture $\mathbb{P}_3$ over $\mathcal{P}$ illustrated in Fig. 7.21. We find that $\psi^*(\lambda_{\mathbb{P}_3}) = 60(1 - x^{-1}) \neq 0$, so $\mathbb{P}_3$ is not degenerate. $\square$

Each double bond in Figs. 7.19 - 7.21 represents the configuration in Fig. 7.18. Also, regions of degree 3 in Fig. 7.21 have label a^3, regions of degree 5 have label b^{-5} and regions of degree 8 (including the boundary region) have label $a^{-1}ba^{-1}b$.

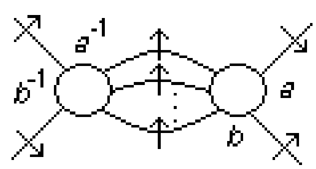


Figure 7.18: The double bond configuration.

Lemma 7.2.18. *If $a = b^{-1}$ and $o(a) < \infty$, then $\mathcal{P}$ is not aspherical.*

Proof. Suppose $a = b^{-1}$ is of finite order. Then in $G(\mathcal{P})$, $ata^{-1} = t^n$ and by induction on m, $a^m ta^{-m} = t^{n^m}$ for any integer m. Let $o(a) = k < \infty$. Then $t^{n^k} = a^k ta^{-k} = t$ and so $t^{n^k - 1} = 1$ in $G(\mathcal{P})$. It follows from Lemma 6.4.4 that $\mathcal{P}$ is not aspherical. $\square$

It follows from Lemmas 7.2.15 - 7.2.17 that $\mathcal{P}$ is not aspherical if

$$\frac{1}{o(a)} + \frac{1}{o(b)} + \frac{1}{o(ab^{-1})} > 1,$$

and it follows from Lemma 7.2.18 that $\mathcal{P}$ is not aspherical if $a = b^{-1}$ and $o(a) < \infty$. This completes Part B of the proof of Theorem 7.2.1. The proof of Theorem 7.2.1 is now complete.

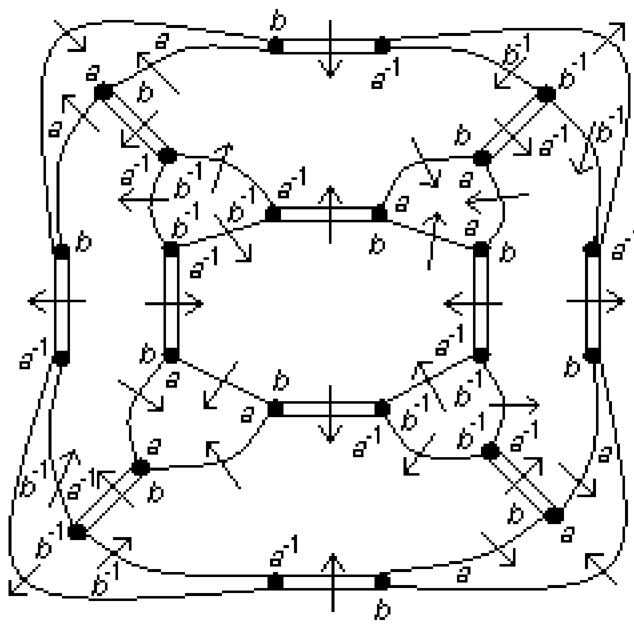


Figure 7.19: $\mathbb{P}_1$.

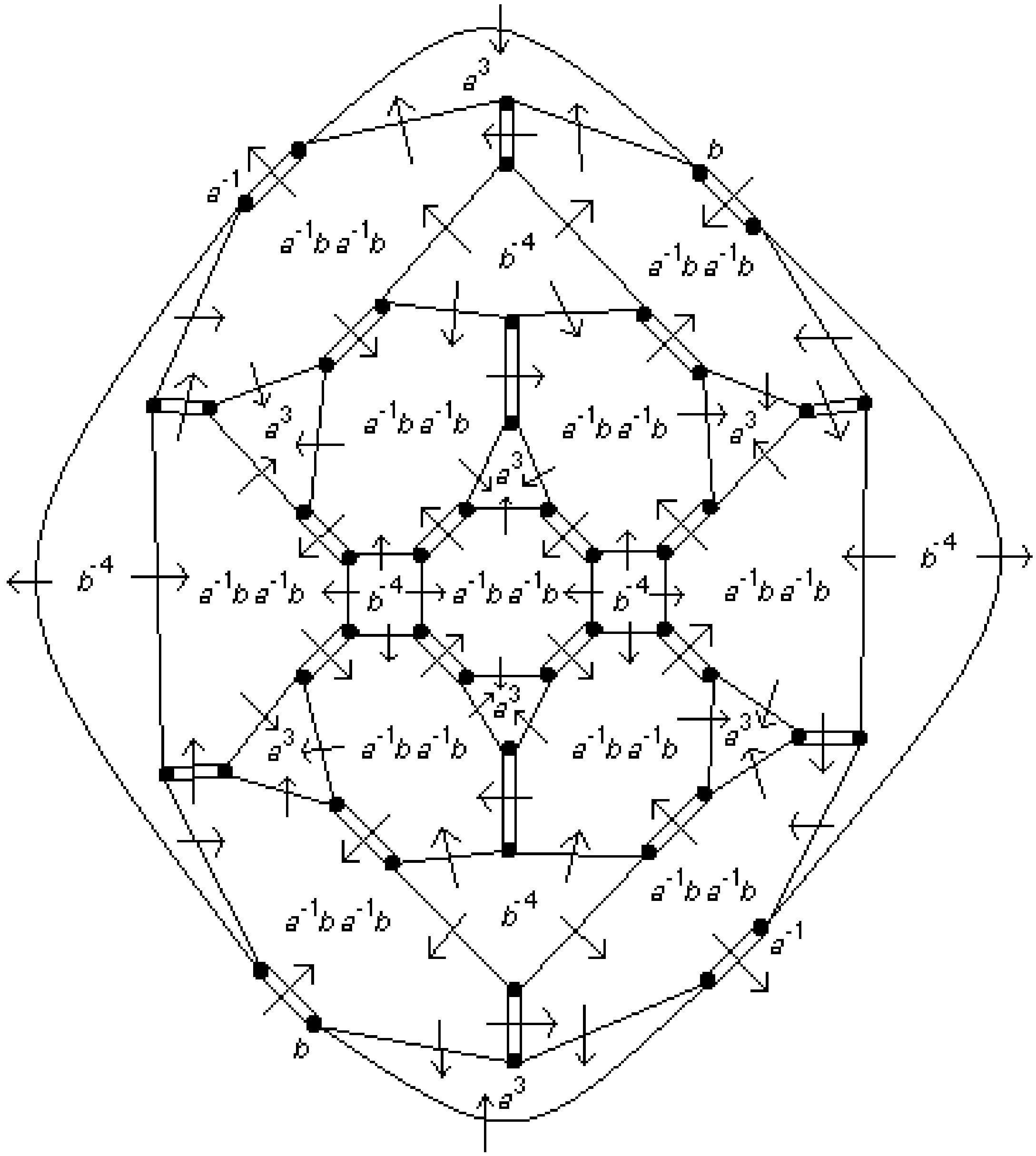


Figure 7.20: $\mathbb{P}_2$.

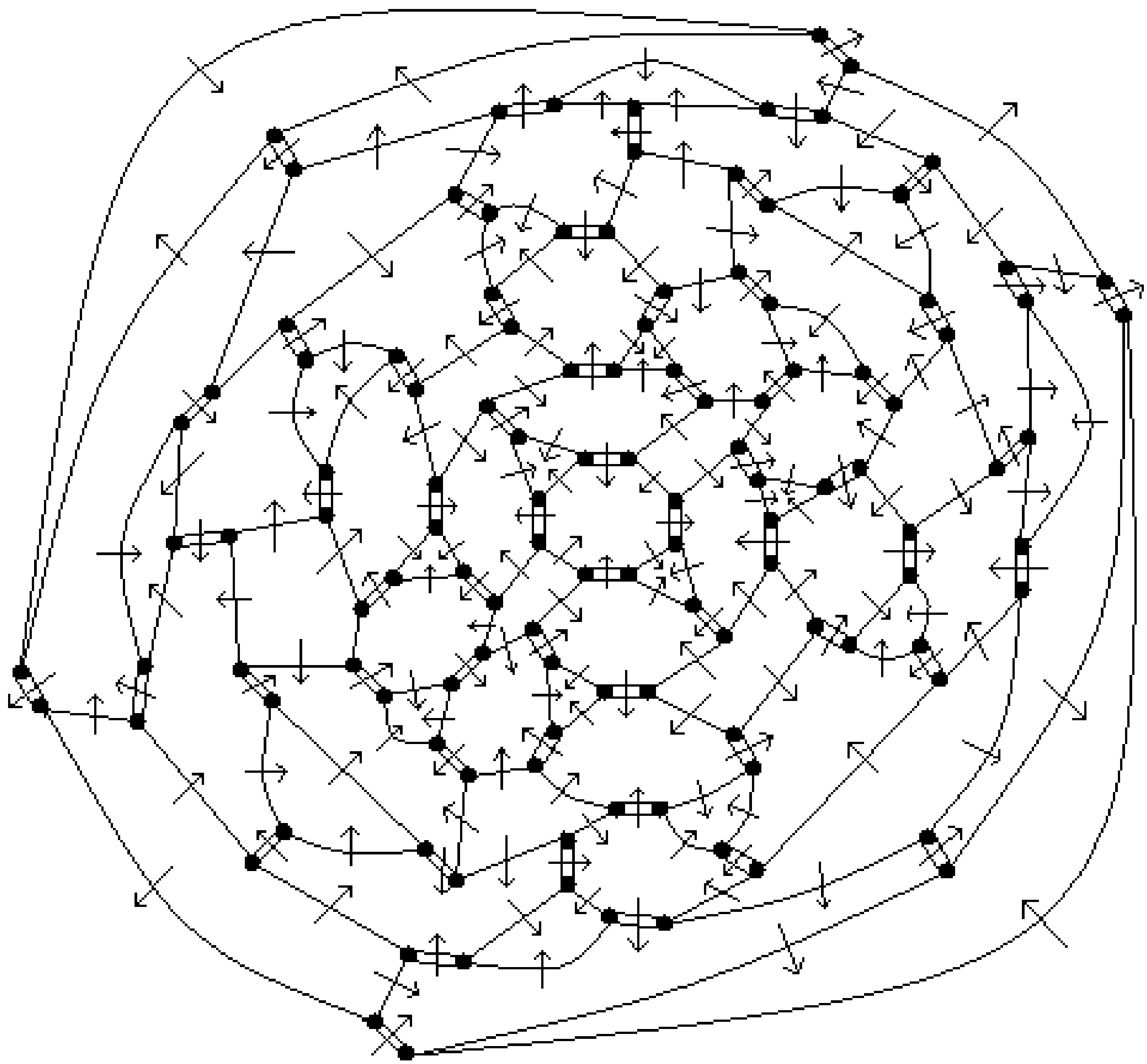


Figure 7.21: $\mathbb{P}_3$.

Appendix A

References for Lemma 7.2.14

In this appendix we prove that $r \neq 6, 7, 8, 9, 10$, where r is the degree of N in Lemma 7.2.14. Recall, the relations $a^3 = ab^{-1}ab^{-1} = 1$ hold in H and $o(b) \geqslant 6$ by Lemma 7.2.11.

Lemma A.0.19. *If $r = 6$, then $\gamma^*(N) \leqslant 0$.*

Proof. Suppose N is an $N_6 1$-region (see Fig. A.1). Then $\frac{\pi}{2}$ cannot be distributed from any $\mathcal{D}$-region to N, which is a contradiction.

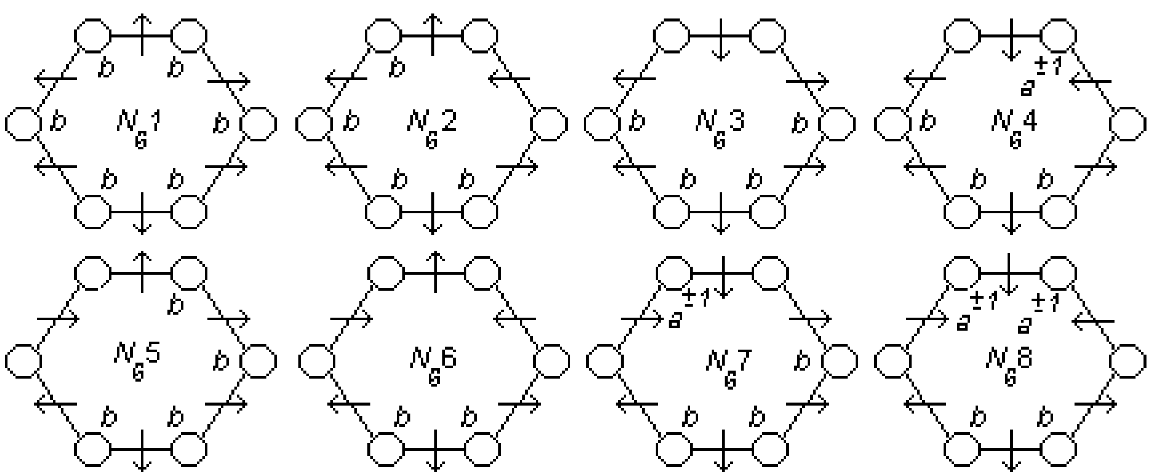


Figure A.1: Possible regions of degree 6.

Since $o(b) \geqslant 6$, N cannot be an $N_6 2$-, $N_6 3$-, $N_6 5$- or an $N_6 6$-region.

Suppose N is an $N_6 4$-region or an $N_6 7$-region. If $b^3a^{-1} = 1$, then $1 = (ab^{-1})^2 = b^4$, contradicting the fact that $o(b) \geqslant 6$. If $b^3a = 1$, then $1 = (ab^{-1})^2 = b^8$. However, $b^9 = a^{-3} = 1$ and we deduce that $b = 1$, which is a contradiction.

Finally, suppose N is an $N_6 8$-region. The label of such a region is either $a^{-2}b^2$ or a^2b^2. Since $a^3 = 1$, the labels become ab^2 and $a^{-1}b^2$, respectively. Now $o(b) > 3$ and $\mathcal{P}$ is neither the exceptional case (E1) or (E2). Therefore, $ab^2 \neq 1$ and $a^{-1}b^2 \neq 1$. Thus, N cannot be an $N_6 8$-region. $\square$

Lemma A.0.20. *If $r = 7$, then $\gamma^*(N) \leqslant 0$.*

Proof. Suppose N is an $N_7 1$-region (see Fig. A.2). Then $\frac{\pi}{2}$ cannot be distributed from any $\mathcal{D}$-region to N, which is a contradiction.

Since $o(b) \geqslant 6$, we see that N cannot be an $N_7 2$-, $N_7 3$-, $N_7 5$-, $N_7 6$-, $N_7 9$-, $N_7 10$- or an $N_7 11$-region.

Suppose N is an $N_7 4$-region with label $a^{-1}b^4$. Then $1 = (ab^{-1})^2 = b^6$. The label then becomes $a^{-1}b^{-2}$. However, $a^{-1}b^{-2} \neq 1$ since $\mathcal{P}$ is not the exceptional case (E2). Thus, $a^{-1}b^4 \neq 1$. If the label is ab^4, then $1 = (ab^{-1})^2 = b^{-10}$ and so $b^{10} = 1$. However, $b^{12} = a^{-3} = 1$ and we deduce that $b^2 = 1$, which contradicts $o(b) \geqslant 6$. Similarly, N cannot be an $N_7 13$-region.

Suppose N is an $N_7 8$-region. The label of such a region is either a^2b^3 or $a^{-2}b^3$ and since $a^3 = 1$, the labels become $a^{-1}b^3$ and ab^3, respectively. Arguing as in Lemma A.0.19, we deduce that $a^{-1}b^3 \neq 1$ and $ab^3 \neq 1$. Thus, N cannot be an $N_7 8$-region. Similarly, N cannot be an $N_7 15$-region.

Suppose N is an $N_7 12$-region or an $N_7 14$-region. Since $\mathcal{P}$ is not the exceptional case (E1), $b^2a^{-1} \neq 1$. Similarly, since $\mathcal{P}$ is not the exceptional case (E2), $b^2a \neq 1$.

Finally, suppose N is an $N_7 16$-region. The label of such a region is either a^3b^2 or $a^{-3}b^2$. Since $a^3 = 1$, we have $1 = a^{\pm 3}b^2 = b^2$, which contradicts $o(b) \geqslant 6$. Thus, N cannot be an $N_7 16$-region. $\square$

Lemma A.0.21. *If $r = 8$, then $\gamma^*(N) \leqslant 0$.*

Proof. If $r = 8$, then $\gamma(N) \leqslant -2\pi$. At most three arcs of ∂N can each be on the boundary of a $\mathcal{D}$-region which contributes $\frac{\pi}{2}$ to $\gamma^*(N)$. Suppose (at most) only two arcs are on the boundaries of such regions. Then each of the remaining six arcs of ∂N can be on the boundary of a $\mathcal{T}_1$-region or a P_8-region which contributes $\frac{\pi}{6}$ to $\gamma^*(N)$. Therefore,

$$\gamma^*(N) \leqslant \gamma(N) + 2\frac{\pi}{2} + 6\frac{\pi}{6} \leqslant -2\pi + 2\pi = 0.$$

It follows that exactly three $\mathcal{D}$-regions must each contribute $\frac{\pi}{2}$ to $\gamma^*(N)$. We find that there are three possible configurations in which this can happen. In each case, at most three arcs of ∂N

can each be on the boundary of a $\mathcal{T}_1$-region or a P_8-region which contributes $\frac{\pi}{6}$ to $\gamma^*(N)$. Thus,

$$\gamma^*(N) \leqslant \gamma(N) + 3\frac{\pi}{2} + 3\frac{\pi}{6} \leqslant -2\pi + 2\pi = 0.$$

□

Lemma A.0.22. *If $r = 9$, then $\gamma^*(N) \leqslant 0$.*

Proof. If $r = 9$, then $\gamma(N) \leqslant -\frac{5\pi}{2}$. At most four arcs of ∂N can each be on the boundary of a $\mathcal{D}$-region which contributes $\frac{\pi}{2}$ to $\gamma^*(N)$. Suppose (at most) only three arcs are on the boundaries of such regions. Then each of the remaining six arcs of ∂N can be on the boundary of a $\mathcal{T}_1$-region or a P_8-region which contributes $\frac{\pi}{6}$ to $\gamma^*(N)$. Therefore,

$$\gamma^*(N) \leqslant \gamma(N) + 3\frac{\pi}{2} + 6\frac{\pi}{6} \leqslant -\frac{5\pi}{2} + \frac{5\pi}{2} = 0.$$

It follows that exactly four $\mathcal{D}$-regions must each contribute $\frac{\pi}{2}$ to $\gamma^*(N)$. We find that there is only one possible configuration in which this can happen. In this case, at most three arcs of ∂N can each be on the boundary of a $\mathcal{T}_1$-region or a P_8-region which contributes $\frac{\pi}{6}$ to $\gamma^*(N)$. Thus,

$$\gamma^*(N) \leqslant \gamma(N) + 4\frac{\pi}{2} + 3\frac{\pi}{6} \leqslant -\frac{5\pi}{2} + \frac{5\pi}{2} = 0.$$

□

Lemma A.0.23. *If $r = 10$, then $\gamma^*(N) \leqslant 0$.*

Proof. If $r = 10$, then $\gamma(N) \leqslant -3\pi$. At most four arcs of ∂N can each be on the boundary of a $\mathcal{D}$-region which contributes $\frac{\pi}{2}$ to $\gamma^*(N)$. If exactly four arcs are on the boundaries of such regions, then each of the remaining six arcs of ∂N can be on the boundary of a $\mathcal{T}_1$-region or a P_8-region which contributes $\frac{\pi}{6}$ to $\gamma^*(N)$. Thus,

$$\gamma^*(N) \leqslant \gamma(N) + 4\frac{\pi}{2} + 6\frac{\pi}{6} \leqslant -3\pi + 3\pi = 0.$$

□

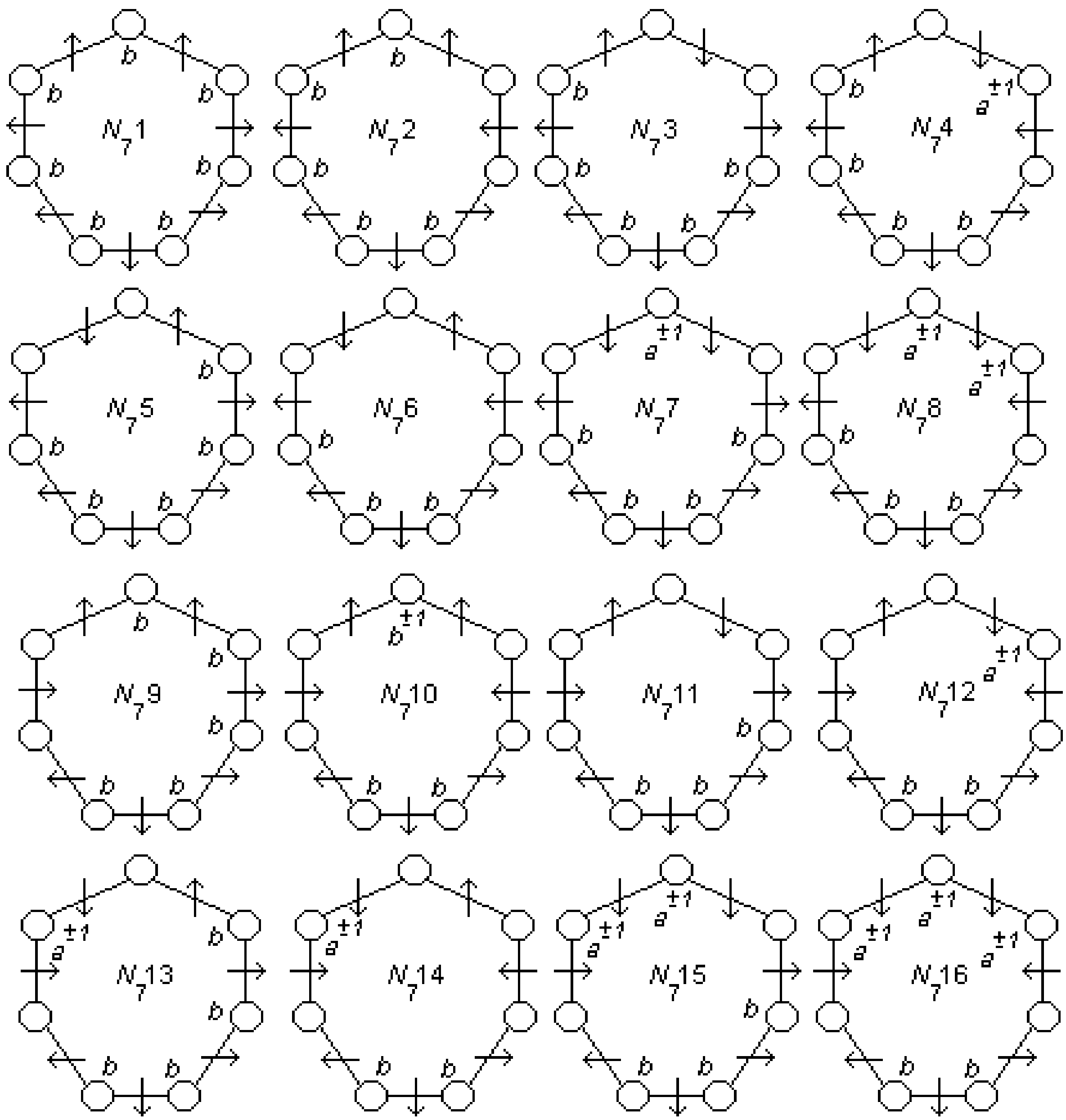


Figure A.2: Possible regions of degree 7.

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