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Coherent Rings and Completions

by

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Abstract

Commutative coherent rings form a standard class of rings which include commutative Noetherian rings. The notion of completion with respect to a maximal ideal is also standard, but it seems not to have been well studied for coherent rings. The aim of this project is to study completion for coherent regular local rings and the derived functors of completion on their module categories. In particular, the theory of L-complete modules due to Greenlees and May should have an analogue and a major goal is to establish which parts of it work for these rings.

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During my little journey in the field of research I realized that at every step the hazards involved in a research work cannot be confronted by a single individual. There are always ideas and persons behind a research that serves as motivating force. These may be the supervisor, colleagues, friends, university, organization, library, books, journals, research papers, experience of the past and vision of the future. It is obligatory upon me to extend my gratitude to all aforementioned with an explicit emphasis on a few.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Master of Science at the University of Glasgow.

Chapter 1 and 2 cover background material and some basic well known characteristics and results.

The results in later sections are the authors original work with the exception of those results which are explicitly referenced.

Contents

Chapter 1

COMMUTATIVE RING THEORY

1.1 Rings, subrings and ring homomorphisms

Most of the results provided in this chapter are very standard and can be found in the books [2], [4] and [25] with the exception of some which are explicitly referenced.

A *ring* is a natural object of study because it crops up in so many varied and important mathematical contexts. The axioms defining a ring are derived from some of the important properties of the set $\mathbb{Z}$ of integers. In fact, the integers may be taken as a prototype for a ring. Like the integers a ring R is a set with two binary operations; these are usually called addition and multiplication. R is then a ring if it forms a commutative group with respect to addition, a semigroup with respect to multiplication and satisfies distributive laws connecting the two operations.

More precisely, a commutative ring with identity element can be defined as:

Definition 1.1. A *commutative ring* R is a set with two binary operations (addition '+' and multiplication '$\times$') such that

(a) R is an abelian group with respect to addition.

(b) Multiplication is associative and distributive over addition.

(c) Multiplication is commutative.

(d) There exists a unique identity element 1.

Throughout this thesis the word ‘ring’ shall mean a commutative ring with an identity element, that is, a ring satisfying above axioms.

Definition 1.2. A subset M of a ring R is a *subring* of R if it is closed under addition and multiplication and contains the identity element of R. The identity mapping of M into R is then a ring homomorphism.

Definition 1.3. A *ring homomorphism* is a mapping φ of a ring R into another ring S such that f respects addition, multiplication and the identity element. In other words, A homomorphism of a ring R into a ring S is a map $\varphi : R \to S$ such that for all x,$y \in R$

(a) $\varphi(x + y) = \varphi(x) + \varphi(y)$,

(b) $\varphi(xy) = \varphi(x)\varphi(y)$,

(c) $\varphi(1) = 1$.

Definition 1.4. Let R and S be rings:

(a) An *epimorphism* $R \to S$ is a surjective homomorphism.

(b) A *monomorphism* $R \to S$ is an injective homomorphism.

(c) An *isomorphism* $R \to S$ is a map which is both an epimorphism and a monomorphism, that is a bijective homomorphism.

(d) An *endomorphism* of a ring R is a homomorphism of R into itself.

(e) An *automorphism* of a ring R is an isomorphism of R into itself.

It is easy to verify that the composition of two homomorphisms is a homomorphism, and that the same is true for any any of the ‘morphisms’ which are defined above. Furthermore, if $\varphi : R \to S$ is an isomorphism of rings, then the inverse map $\varphi^{-1} : S \to R$ which exists since φ is bijective, is also an isomorphism.

Definition 1.5. Let $\varphi : R \to S$ be a ring homomorphism, then

(a) $\operatorname{im}\varphi = \{ y \in S : \exists\, x \in R$ for which $\varphi(x) = y\}$.

(b) $\ker\varphi = \{x \in R : \varphi(x) = 0 \}$.

1.2 Ideals and quotient rings

Definition 1.6. An *ideal* I of a ring R is a subset of R which is an additive subgroup and is such that $RI = IR \subseteq I$. It is written as $I \lhd R$ if $I \neq R$ and $I \unlhd R$ if $I \subseteq R$. The quotient group R/I inherits a uniquely defined multiplication from R which makes it into a ring, called the *quotient ring* R/I. The elements of R/I are the cosets of I in R, and the mapping $\varphi : R \to R/I$ which maps each $x \in R$ to its coset $x + I$, is a surjective ring homomorphism.

Lemma 1.7. *Let R be a ring, Then*

(a) Let $\{S_\lambda : \lambda \in \Lambda\}$ be a family of subrings (respectively ideals) of R. Then,

$$\bigcap_{\lambda \in \bigwedge} S_\lambda$$

is a subring (respectively ideal) of R.

(b) Let $S_1 \subseteq S_2 \subseteq \cdots$ be an ascending chain of subrings(respectively ideals) of R. Then,

$$\bigcup_{i=1}^{\infty} S_i$$

is a subring (respectively ideal) of R.

Definition 1.8. The *subring generated* by a subset X of a ring R is the smallest subring of R containing X. The *ideal generated* by a subset X of a ring R is the smallest ideal of R containing X.

Lemma 1.9. *Let X be a subset of a ring R. Then:*

(a) The subring of R generated by X consists of all finite sums of elements ± 1 and $\pm x_1 x_2 \cdots x_n$ where $x_i \in X$, for $n = 1, 2, \ldots$.

(b) If R is a ring and $X \neq \emptyset$, then the ideal of R generated by X is RX.

Lemma 1.10. *If $J_1, \ldots, J_n$ are ideals of a ring R, then $\sum_{i=1}^{n} J_i$ is an ideal of R.*

Lemma 1.11. *Let R and S be rings and $\varphi : R \to S$ be a ring homomorphism. Then*

(a) $\ker \varphi \lhd R$ and φ is a monomorphism if and only if $\ker \varphi = \{0_R\}$.

(b) $\operatorname{im} \varphi$ is a subring of S.

Theorem 1.12. *Let $J \lhd R$ and let $\alpha : R \to R/J$ be the natural homomorphism. Suppose that $\varphi : R \to S$ is a ring homomorphism whose kernel contains J. Then there exists a unique homomorphism $\psi : R/J \to S$ which makes the diagram*

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & S \\ & R/J \arrow[ur, "\psi"'] & \end{tikzcd}$$

commute and $\ker\psi = \ker\varphi/J$.

Theorem 1.13 (First Isomorphism Theorem)**.** *Let $\varphi : R \to S$ be a ring homomorphism. Then there is a ring isomorphism*

$$R/\ker\varphi \cong \operatorname{im}\varphi.$$

Theorem 1.14 (Second Isomorphism Theorem)**.** *Let R be a ring, S be a subring and J be an ideal of R. Then*

$$S + J = \{s + j : s \in S, j \in J\}$$

is a subring of R, $S \cap J \lhd S$ and

$$(S + J)/J \cong S/(S \cap J).$$

Theorem 1.15 (Third Isomorphism Theorem)**.** *Let R be a ring and let J and K be ideals of R with $J \subseteq K$. Then*

$$K/J \lhd R/J \quad \textit{and} \quad (R/J)/(K/J) \cong R/K.$$

Prime ideals play a central role in the theory of commutative rings. A prime ideal can be defined as:

Definition 1.16. An ideal $\mathfrak{p} \lhd R$ is *prime* if

$$xy \in \mathfrak{p} \quad \text{implies} \quad x \in \mathfrak{p} \quad \text{or} \quad y \in \mathfrak{p}.$$

Definition 1.17. An ideal $\mathfrak{m} \lhd R$ is *maximal* if $\mathfrak{m} \neq R$ and if there is no ideal $I \lhd R$ such that $\mathfrak{m} \subsetneq I$.

Equivalently, $\mathfrak{p}$ is prime if and only if $R/\mathfrak{p}$ is an integral domain and $\mathfrak{m}$ is maximal if and only if $R/\mathfrak{m}$ is a field. Hence a maximal ideal is prime but not conversely, in general. The zero ideal is prime if and only if R is an integral domain.

Theorem 1.18 (see [34]). *Let $\varphi : R \to S$ be a ring epimorphism with $K = \ker \varphi$. If $I \lhd R$ containing K, then I is prime if and only if $\varphi(I)$ is prime.*

Let R be a ring. A multiplicatively closed subset of R is a subset S of R such that $1 \in S$ and it is closed under multiplication.

Theorem 1.19. *Let S be a multiplicatively closed set in a ring R and let $I \lhd R$ maximal with respect to the exclusion of S. Then I is prime.*

Theorem 1.20 ([21, Theorem 7]). *Let I be an ideal in R. Suppose I is not finitely generated, and is maximal among all ideals in R that are not finitely generated. Then I is prime.*

Theorem 1.21 ([21, Theorem 9]). *Let $\{\mathfrak{p}_i\}$ be a chain of prime ideals in a ring R. Then both $\bigcup_i \mathfrak{p}_i$ and $\bigcap_i \mathfrak{p}_i$ are prime ideals in R.*

Theorem 1.22. *Every ring R has at least one maximal ideal.*

Corollary 1.23. *If $I \lhd R$. Then there exists a maximal ideal of R containing I.*

1.3 Zero divisors, nilpotent elements, units and radicals

Definition 1.24. A *zero divisor* in a ring R is a non-zero element x which 'divides 0', i.e., for which there exists $y \neq 0$ in R such that $xy = 0$. A ring with no zero-divisors is called an *integral domain*. For example, $\mathbb{Z}$ and $\mathbb{K}[x_1, \ldots, x_n]$ for $\mathbb{K}$ a field and x_i indeterminates are integral domains.

Definition 1.25. An element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n > 0$. A nilpotent element is a zero-divisor, but not conversely true (in general).

Definition 1.26. A *unit* in R is an element x which 'divides 1', i.e., an element x such that $xy = 1$ for some $y \in R$. The element y is then uniquely determined by x, and is written x^{-1}. The units in R form a multiplicative abelian group.

The multiples $rx = xr$ of an element $x \in R$ form a *principal ideal*, denoted by (x) or Rx. Then x is a unit if and only if $(x) = R$. The *zero ideal* (0) is often denoted by 0. A *field* is a ring R in which every non-zero element is a unit. Every field is an integral domain but not conversely, as $\mathbb{Z}$ is not a field. Every non-unit of R is contained in a maximal ideal.

Let R be a ring then the set of all nilpotent elements of R is an ideal, for if x,y are elements of R such that x^m=y^n=0, then $(x + y)^{m+n} = 0$ by the binomial theorem.

Definition 1.27. Let R be a ring then the ideal of all nilpotent elements in R is called the *nilradical* of R. If $I \lhd R$, the inverse image under the canonical mapping $R \to R/I$, of the nilradical R/I is called the *radical* of I and is denoted by $\sqrt{I}$.

Proposition 1.28. *Every prime ideal of a ring R contains a minimal prime ideal.*

Proposition 1.29. *Let $\mathfrak{p}$ be a minimal prime ideal of a ring R. For all $x \in \mathfrak{p}$, there exists $s \in R - \mathfrak{p}$ and an integer $k > 0$ such that $sx^k = 0$.*

The following proposition gives an alternate interpretation of the nilradical.

Corollary 1.30. *The radical $\sqrt{I}$ of an ideal $I \lhd R$ is the intersection of all the prime ideals of R containing I and it is also the intersection of the minimal elements of this set of prime ideals.*

Proposition 1.31. *Let R be a ring and $I \lhd R$. Let $J \lhd R$ be a finitely generated contained in $\sqrt{I}$. Then there exists an integer $k > 0$ such that $J^k \subseteq I$. In particular, in a Noetherian ring the nilradical is a nilpotent ideal.*

Definition 1.32. A ring R is called *reduced* if $\sqrt{0} = 0$, in other words if no non-zero element of R is nilpotent.

Definition 1.33. The *Jacobson radical* $J(R)$ of R is defined to be the intersection of all the maximal ideals of R.

It can be characterized as follows:

Proposition 1.34. *$x \in J(R)$ if and only if $1 - xy$ is a unit in R for all $y \in R$.*

1.4 Localization, local and semi local rings

Let R be a ring and S be a multiplicatively closed subset of R. Define a relation $\approx$ on $R \times S$ for $a, b \in R$ and $s, t \in S$ as follows:

$$(a, s) \approx (b, t) \quad \text{if and only if} \quad (at - bs)u = 0 \quad \text{for some} \quad u \in S.$$

This relation is reflexive, symmetric and transitive, therefore, is equivalence relation. Let r/s denote the equivalence class of (r, s), and let RS^{-1} denote the set of equivalence classes. We put a ring structure on RS^{-1} by defining addition and multiplication of these fractions in the same way as in elementary algebra for $a, b \in R$ and $s, t \in S$:

$$(a/s) + (b/t) = (at + bs)/st,$$

$$(a/s)(b/t) = ab/st.$$

We also have a ring homomorphism $f : R \to RS^{-1}$ defined by $f(x) = x/1$. This is not in general injective.

Definition 1.35. The ring RS^{-1} is called the *localization* or *ring of fractions* of R with respect to S and is also denoted by R_S or $S^{-1}R$.

It has a universal property.

Proposition 1.36. *Let $g : R \to T$ be a ring homomorphism such that $g(s)$ is a unit in T for all $s \in S$. Then, there exists a unique ring homomorphism $h : R_S \to T$ such that $g = h \circ f$.*

Definition 1.37. If $\mathfrak{p}$ is a prime ideal of R. Then $S = R - \mathfrak{p}$ is multiplicatively closed and we write $R_\mathfrak{p}$ for R_S in this case. The elements a/s with $a \in \mathfrak{p}$ form a maximal ideal $\mathfrak{m}$ in $R_\mathfrak{p}$. If b/t is not in $\mathfrak{m}$, then b is not in $\mathfrak{p}$, hence $b \in S$ and therefore b/t is a unit in $R_\mathfrak{p}$.

If R is Noetherian, integrally closed, factorial, regular, Cohen-Macaulay, Krull, Prufer, Bezout, Pseudo-Bezout, or a valuation domain, then R_S also has the same property.

Definition 1.38 (see [32])**.** A commutative ring R which has exactly one maximal ideal, $\mathfrak{m}$ say, is said to be *local*. In these circumstances, the field $R/\mathfrak{m}$ is called the *residue field* of R.

Proposition 1.39. *Let R be a ring and $\mathfrak{m}$ a maximal ideal of R, such that every element of*

$$1 + \mathfrak{m} = \{1 + x : x \in \mathfrak{m}\}$$

is a unit in R. Then R is a local ring.

Proposition 1.40. *Let R be a ring. The following properties are equivalent:*

(a) The set of maximal ideals of R is finite.

(b) The quotient $R/J(R)$ is the direct product of a finite number of fields.

Definition 1.41. A ring R is called a *semi-local ring* if it satisfies the equivalent conditions of the above Proposition.

Every local ring is semi-local. Every quotient of a semi-local ring is semi-local. Every finite product of semi-local rings is semi-local.

Proposition 1.42. *Let R be a ring and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of R. Write*

$$S = \bigcap_{i=1}^{n} (R - \mathfrak{p}_i) = R - \bigcup_{i=1}^{n} \mathfrak{p}_i.$$

(a) The ring $S^{-1}R$ is semilocal; if $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are the distinct maximal elements (with respect to inclusion) of the set of $\mathfrak{p}_i$, the maximal ideals of $S^{-1}R$ are the $S^{-1}\mathfrak{q}_j$ for $1 \leqslant j \leqslant r$ and these ideals are distinct.

(b) The ring $R_{\mathfrak{p}}$, is canonically isomorphic to $(S^{-1}R)_{S^{-1}\mathfrak{p}_i}$ for $1 \leqslant i \leqslant n$.

(c) If R is an integral domain, then in the field of fractions of R

$$S^{-1}R = \bigcap_{i=1}^{n} R_{\mathfrak{p}_i}.$$

1.5 Modules and modules homomorphisms

Definition 1.43 (see [15])**.** Let R be a ring. An *R-module* is an abelian group M (written additively) on which R acts linearly: more precisely, it is a pair (M, φ), where M is an abelian group and $\varphi : R \times M \to M$, where for $r \in R$ and $x \in M$, we write rx for $\varphi(r, x)$. Then the following axioms are satisfied for $r, s \in R$ and $x, y \in M$:

(a) $r(x + y) = rx + ry$,

(b) $(r + s)x = rx + sx$,

(c) $(rs)x = r(sx)$,

(d) $1x = x$

Definition 1.44. Let M, N be R-modules. A mapping $f : M \to N$ is an *R-module homomorphism* if for all $r \in R$ and all $x, y \in M$

(a) $f(x + y) = f(x) + f(y)$,

(b) $f(rx) = rf(x)$.

Definition 1.45 (see [23, 6]). Let M and N be R-modules. The set $\mathrm{Hom}_R(M, N) = \{f : M \to N\}$ of all R-module homomorphisms f is an abelian group, under the addition defined for $f, g : M \to N$ by

$$(f+g)(m) = f(m) + g(m) \quad \text{for} \quad m \in M.$$

If $M = N$, then $\mathrm{Hom}_R(M, M)$ is a ring under addition and composition of homomorphisms; the ring is called the ring of *R-endomorphisms* of M. If for $r \in R$ and $f \in \mathrm{Hom}_R(M, M)$

$$(rf)(m) = rf(m),$$

the ring $\mathrm{Hom}_R(M, M)$ may be regarded not just as a group but as an R-module.

Definition 1.46. A *submodule* N of M is a subgroup of M which is closed under multiplication by elements of R. The *quotient group* M/N then inherits an R-module structure from M, defined by $r(x+N) = rx + N$. The R-module M/N is the *quotient* of M by N. The natural map of $M \to M/N$ is an R-module homomorphism.

There is a one to one order preserving correspondence between submodules of M which contain N and submodules of M/N.

If $f : M \to N$ is R-module homomorphism, the *kernel* of f is the set

$$\ker(f) = \{x \in M : f(x) = 0\}$$

and is a submodule of M. The *image* of f is the set

$$\mathrm{im}(f) = \{y \in N : \text{there exists} \quad x \in M \quad \text{for which} \quad f(x) = y\}$$

and is a submodule of N. The *cokernel* of f is the quotient module

$$\mathrm{coker}(f) = N/\,\mathrm{im}(f).$$

If N is a submodule of M such that $N \subseteq \ker(f)$, then f gives rise to a homomorphism $\overline{f} : M/N \to N$, defined as follows: if $\overline{x} \in M/N$ is the image of $x \in M$, then $\overline{f}(\overline{x}) = f(x)$. The kernel of $\overline{f}$ is $\ker(f)/N$. The homomorphism $\overline{f}$ is said to be induced by f. In particular, taking $N = \ker(f)$, we have an isomorphism of R-modules

$$M/\ker(f) \cong \mathrm{im}(f).$$

Definition 1.47. If M and N are R-modules, their *direct sum*

$$M \oplus N = \{(x, y) : x \in M, y \in N\}$$

is an R-module if we define addition and scalar multiplication in the obvious way:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$r(x, y) = (rx, ry).$$

More generally, if $(M_i)_{i \in I}$ is any family of R-modules, we can define their direct sum $\oplus_{i \in I} M_i$; its elements are families $(x_i)_{i \in I}$ such that $(x_i) \in M_i$ for each $i \in I$ and almost all x_i are 0. If we drop the restriction on the number of non-zero $x's$ we have the direct product $\prod_{i \in I} M_i$. Direct sum and direct product are therefore the same if the index set I is finite, but not in general.

Definition 1.48. A *free* R-module is one which is isomorphic to an R-module of the form $\bigoplus_{i \in I} R$ for some indexing set I. A finitely generated free R-module is therefore isomorphic to n summands of R i.e. to $R^n = R \oplus \cdots \oplus R$.

M is finitely generated R-module if and only if M is isomorphic to a quotient of R^n for some integer $n > 0$.

Theorem 1.49. *Let M be a finitely generated R-module and $I \lhd R$ such that $IM = M$. Then there exists $a \in I$ such that $(1 + a)M = 0$.*

Proposition 1.50 (Nakayama's lemma)**.** *Let M be a finitely generated R-module, let I be an ideal of R contained in $J(R)$. Then $IM = M$ implies $M = 0$.*

Corollary 1.51. *Let M be a finitely generated R-module, N a submodule of M, $I \subseteq J(R)$ an ideal. Then*

$$M = IM + N \quad \Rightarrow \quad M = N.$$

Definition 1.52. Let R be a ring, $f : F \to G$ and $g : G \to H$ be two R-module homomorphisms. The ordered pair (f, g) is called an *exact sequence* if $\ker g = \operatorname{im} f$. Of course this implies that $gf = 0$.

Consider similarly a diagram consisting of four modules and three homomorphisms:

$$E \xrightarrow{f} F \xrightarrow{g} G \xrightarrow{h} H.$$

This is exact or exact sequence if the diagram $E \xrightarrow{f} F \xrightarrow{g} G$ is exact at F and the diagram $F \xrightarrow{g} G \xrightarrow{h} H$ is exact at G. More generally, a sequence of homomorphisms

$$\cdots \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} \cdots$$

is exact if for each n, the sequence

$$G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1}$$

is exact. An exact sequence of the form

$$0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$$

is called *short exact*. Such a sequence is split if there is a homomorphism $r : G \to F$ (or equivalently $j : H \to G$) so that $rf = \mathrm{id}_F$ (respectively $gj = \mathrm{id}_H$).

$$\begin{tikzcd}
 & & F & & \\
0 \arrow[r] & F \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, "f"'] & G \arrow[u, "r"'] \arrow[r, "g"] & H \arrow[r] & 0 \\
 & & H \arrow[u, "j"] \arrow[ur, dashed, "\mathrm{id}"'] & &
\end{tikzcd}$$

These equivalent conditions imply that $G \cong F \oplus H$. Such homomorphisms r and g are said to be *retractions*, while F and H are said to be *retracts* of G. In particular,

(a) To say that $0 \to E \xrightarrow{f} F$ is an exact sequence is equivalent to saying that f is injective.

(b) To say that $E \xrightarrow{f} F \to 0$ is an exact sequence is equivalent to saying that f is surjective.

(c) To say that $0 \to E \xrightarrow{f} F \to 0$ is an exact sequence is equivalent to saying that f is bijective, that is f is an isomorphism.

(d) If F is a submodule of E and i denotes the canonical injection of F into E and p denotes the canonical surjection of E onto E/F, the diagram

$$0 \to F \xrightarrow{i} E \xrightarrow{p} E/F \to 0$$

is an exact sequence.

(e) If $f : E \to F$ is a homomorphism, the diagram

$$0 \to f^{-1}(0) \xrightarrow{i} E \xrightarrow{f} F \xrightarrow{p} F/f(E) \to 0$$

(where i is the canonical injection of $f^{-1}(0)$ into E and p the canonical projection of $F/f(E)$) is an exact sequence.

Theorem 1.53 (see [19, Thm 5.1]). *In an arbitrary exact sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

of R-module homomorphisms, the following three statements are equivalent:

(a) f is an epimorphism.

(b) g is the trivial homomorphism.

(c) h is a monomorphism.

Corollary 1.54 (see [19, Cor 5.5]). *If the sequence*

$$0 \to C \xrightarrow{g} D \to 0$$

of R-module homomorphisms is exact, then g is an isomorphism.

Proposition 1.55 (The Snake Diagram). *Consider a commutative diagram of R-modules:*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & B \arrow[r, "v"] \arrow[d, "b"'] & C \arrow[d, "c"'] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \end{tikzcd}$$

Suppose that the two rows of the diagram are exact.

(a) If c is injective, we have

$$\operatorname{im}(b) \cap \operatorname{im}(u') = \operatorname{im}(u' \circ a) = \operatorname{im}(b \circ u).$$

(b) If a is surjective, we have

$$\ker(b) + \operatorname{im}(u) = \ker(v' \circ b) = \ker(c \circ v).$$

Corollary 1.56. *Suppose that the above snake diagram is commutative and the two rows are exact. Then:*

(a) *If u', a and c are injective, b is injective.*

(b) *If v, a and c are surjective, b is surjective.*

Corollary 1.57. *Suppose that the above snake diagram is commutative and the two rows are exact. Then:*

(a) *If b is injective and if a and v are surjective, then c is injective.*

(b) *If b is surjective and if c and u' are injective, then a is surjective.*

Proposition 1.58. *Let R be a ring.*

(a) *The sequence of homomorphisms of R-modules*

$$M' \to M \to M'' \to 0$$

is exact if and only if for all R-modules N, the sequence

$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$$

is exact.

(b) *The sequence of homomorphisms of R-modules*

$$0 \to N' \to N \to N''$$

is exact if and only if for all R-modules M, the sequence

$$0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'')$$

is exact.

Proposition 1.59 (The Snake Lemma)**.** *Let*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"'] & M \arrow[r, "v"] \arrow[d, "f"'] & M'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"'] & N \arrow[r, "v'"'] & N'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of R-modules and homomorphisms, with the rows exact. Then there exists an exact sequence

$$0 \to \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \mathrm{coker}(f') \xrightarrow{\overline{u}'} \mathrm{coker}(f) \xrightarrow{\overline{v}'} \mathrm{coker}(f'') \to 0$$

in which $\overline{u}$,$\overline{v}$ are restrictions of u, v, and $\overline{u}'$,$\overline{v}'$ are induced by u', v'.

Definition 1.60. Let M, N, P be three R-modules. A mapping $f : M \times N \to P$ is said to be *R-bilinear* if for each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is R-linear, and for each $y \in N$ the mapping $x \mapsto f(x, y)$ of M into P is R-linear

There is an R-module T, called the *tensor product* of M and N denoted by $T = M \otimes_R N$ with the property that the R-bilinear mappings $M \times N \to P$ are in natural one-to-one correspondence with the R-linear mappings $T \to P$, for all R-modules P.

Definition 1.61 (see [7])**.** Let R be a ring. An R-module P is called a *projective* R-module if for every R-module M and N and every R-homomorphism f and g, where f is surjective, the following diagram can be completed

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "g"] & \\ M \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

Proposition 1.62 (see [23, Prop 5.4])**.** *Every free R-module is projective.*

The converse is not necessarily true. For example, let $R = \mathbb{Z} \oplus \mathbb{Z}$, the direct sum of the ring $\mathbb{Z}$ of integers with itself (with product $(m, n)(m', n') = (mm', nn')$). Then the first summand $\mathbb{Z}$, as submodule of an R-module, is an R-module. It is clearly not free but projective according to

Proposition 1.63 (see [23, Prop 5.5])**.** *An R-module P is projective if and only if it is a direct summand of a free R-module.*

Proposition 1.64 (see [5, Prop 2.4])**.** *In order that P be projective it is necessary and sufficient condition that all exact sequences*

$$0 \to M \to N \to P \to 0$$

split.

Proposition 1.65 (see [16, Prop 4.7])**.** *For an R-module P the following statements are equivalent.*

(a) P is projective.

(b) For every short exact sequence

$$A \xrightarrow{\mu} B \xrightarrow{\varepsilon} C$$

of R-modules the induced sequence

$$0 \to \mathrm{Hom}_R(P, A) \xrightarrow{\mu_*} \mathrm{Hom}_R(P, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_R(P, C) \to 0$$

is exact.

Lemma 1.66 (see [29, Lem 10.2])**.** *Let P be a finitely generated projective R-module.*

(a) Suppose V is a finitely generated R-module and $\mu : V \to P$ is an R-homomorphism. If $\overline{\mu} : V/J(R)V \to P/J(R)P$ is an isomorphism, then μ is also an isomorphism.

(b) If $P/PJ(R)$ is a free $R/J(R)$-module, then P is a free R-module.

Definition 1.67. Let R be a ring and let M be an R-module. An exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

with P_i projective is called a *projective resolution* of M. If P_i are free R-modules, this exact sequence is called a *free resolution* of M. If P_i are finitely generated then this exact sequence is called a *finite projective (respectively free) resolution* of M. If a module M admits a finite projective (respectively free) resolution of type

$$0 \to P_n \to \cdots \to P_0 \to M \to 0,$$

this resolution is called a *finite resolution of length* n, or a resolution of *finite length* if knowledge of n is not important.

Definition 1.68 (see [7])**.** Let R be a ring. An R-module E is called an *injective* R-module if for every R-module M and N and every R-homomorphism f and g,where f is injective, the following diagram can be completed

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "g"] \arrow[r, "f"] & N \arrow[ul, dashed] \end{tikzcd}$$

An important example of an injective module is the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$ where $\mathbb{Z}$ denotes the integers and $\mathbb{Q}$ the rationals.

Proposition 1.69 (see [16, Prop 6.3])**.** *A direct product $\prod_{j\in J} I_j$ is injective if and only if each I_j is injective.*

Proposition 1.70 (see [5, Prop 3.4])**.** *In order that E be injective it is necessary and sufficient condition that every exact sequences*

$$0 \to E \to M \to N \to 0$$

split.

Lemma 1.71 (Baer's Criterion)**.** *The R-module Q is injective if and only if given any ideal $I \subseteq R$ and any R-homomorphism $\sigma : I \to Q$, there exists an R-homomorphism $\sigma^* : R \to Q$ that extends σ.*

Proof. See [29, Lem 21.3] □

Lemma 1.72 (Baer's Theorem)**.** *Every R-module is a submodule of an injective R-module.*

Proof. See [29, Lem 21.6] □

Lemma 1.73 (see [29, Lem 21.7])**.** *Let R be a ring.*

(a) If Q is an injective R-module and if A is a direct summand of Q, then A is injective.

(b) If $Q_i (i \in I)$ is a family of injective R-modules, then $\prod_{i \in I} Q_i$ is injective.

Lemma 1.74 (see [29, Lem 21.8])**.** *An R-module Q is injective if and only if it is a direct summand of every module which contains it.*

Definition 1.75. Let R be a ring and let M be an R-module. An exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots$$

with E_i each an injective R-modules is called an *injective resolution* of M.

Every R-module admits an injective resolution.

Definition 1.76 (see [7])**.** Let R be a ring, an R-module M is called a *flat* R-module if $M \otimes_R -$ is an exact functor, that is if for any exact sequence

$$0 \to N \to N' \to N'' \to 0$$

the sequence

$$0 \to M \otimes_R N \to M \otimes_R N' \to M \otimes_R N'' \to 0$$

is exact.

Proposition 1.77. *Let $\varphi : R \to S$ be a ring homomorphism. We view S as an R-module. If M is a flat R-module, then the S-module $M \otimes_R S$ is S-flat.*

Theorem 1.78. *Let R be a ring and M an R-module. Then M is flat over R if and only if for every finitely generated ideal I of R the canonical map $I \otimes_R M \to R \otimes_R M$ is injective, and therefore $I \otimes M \cong IM$.*

Theorem 1.79. *Let*

$$0 \to E' \xrightarrow{v} E \xrightarrow{w} E'' \to 0$$

be an exact sequence of R-modules; then if E' and E'' are both flat, so is E.

Proposition 1.80. *If $P = \bigoplus_i P_i$, then P is flat if and only if each P_i is flat.*

Proposition 1.81. *Every projective module P is flat.*

The converse is not necessarily true. For example, let $R = \mathbb{Z}$ and $M = \mathbb{Q}$.

Proposition 1.82. *If every finitely generated submodule of a module P is a flat, then P itself is flat.*

Proposition 1.83. *Let E be an R-module. The following four properties are equivalent:*

(a) For a sequence

$$N' \xrightarrow{v} N \xrightarrow{w} N''$$

of R-modules to be exact, it is necessary and sufficient that the sequence

$$E \otimes_R N' \xrightarrow{1 \otimes v} E \otimes_R N \xrightarrow{1 \otimes w} E \otimes_R N''$$

be exact.

(b) E is flat and for every R-module N, the relation $E \otimes_R N = 0$ implies $N = 0$.

(c) E is flat and, for every homomorphism $v : N' \to N$ of R-modules, the relation $1 \otimes_R v = 0$ implies $v = 0$.

(d) E is flat and, for every maximal ideal m of R, $E \neq Em$.

Definition 1.84. An R-module E is called *faithfully flat* if it has the equivalent properties of last proposition.

Free modules are faithfully flat; the converse is not necessarily true. To see this, note that the R-module $\bigoplus R_{\mathfrak{m}}$ as $\mathfrak{m}$ runs over all maximal ideals of R is a faithfully flat R-module for any ring R. Faithfully flat modules are flat; the converse is not necessarily true since for any ring R, any localization of R is a flat R-module.

Theorem 1.85. *Let R be a ring and M an R-module. Then the following conditions are equivalent:*

(a) M is faithfully flat over R;

(b) M is R-flat, and $N \otimes_R M \neq 0$ for any non-zero R-module N;

(c) M is R-flat, and $\mathfrak{m}M \neq M$ for every maximal ideal $\mathfrak{m}$ of R.

1.6 Associated primes

We define $IM \subseteq M$ by

$$IM = \{\sum a_i x_i : a_i \in I, x_i \in M\}.$$

where I is an ideal and M an R-module, and is a submodule of M.

Definition 1.86. For every R-module M, the *annihilator* of M is defined as:

$$\operatorname{Ann}(M) = (0 :_R M) = \{r \in R : rm = 0 \quad \text{for all} \quad m \in M\}.$$

Definition 1.87 ([4, 4.1.1])**.** Let R be a ring and M be an R-module. A prime ideal $\mathfrak{p} \lhd R$ is said to be *associated* with M if there exists $m \in M$ such that $\mathfrak{p}$ is equal to the annihilator of m. The set of all prime ideals associated with M is denoted by $\operatorname{Ass}_R(M)$ or simply $\operatorname{Ass}(M)$.

As the annihilator of 0 is R, an element $m \in M$ whose annihilator is a prime ideal is necessarily non-zero. To say that a prime ideal $\mathfrak{p}$ is associated with M amounts to saying that M contains a submodule isomorphic to $R/\mathfrak{p}$ namely Rm, for some $m \in M$ whose annihilator is $\mathfrak{p}$.

If an R-module M is the union of a family $(M_i)_{i \in I}$ of submodules, then clearly

$$\operatorname{Ass}(M) = \bigcup_{i \in I} \operatorname{Ass}(M_i).$$

Proposition 1.88 ([4, 4.1.1.1])**.** *For every prime ideal $\mathfrak{p} \lhd R$ and every non-zero submodule M of $R/\mathfrak{p}$,*

$$\operatorname{Ass}(M) = \{\mathfrak{p}\}.$$

Proposition 1.89 ([4, 4.1.1.2]). *Let R be a ring and M be an R-module. Every maximal element of the set of ideals $(0 :_R m)$ of R, where m runs through the set of non-zero elements of M, belongs to* $\mathrm{Ass}(M)$.

Corollary 1.90 ([4, 4.1.1.3]). *Let R be a Noetherian ring and M be an R-module. Then $M \neq \{0\}$ if and only if* $\mathrm{Ass}(M) \neq \emptyset$.

Corollary 1.91 ([4, 4.1.1.4]). *Let R be a Noetherian ring. Then the set of zero divisors is the union of the ideals* $\mathfrak{p} \in \mathrm{Ass}(R)$.

Proposition 1.92 ([4, 4.1.1.5]). *Let R be a ring, M an R-module and N a submodule of M. Then*

$$\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N).$$

Theorem 1.93. *Let R be a Noetherian ring. If we denote by $\mathrm{Min}(R)$ the set of all minimal prime ideals of R, then for $M = R$ we have $\mathrm{Min}(R) \subseteq \mathrm{Ass}(R)$ and $\mathrm{Ass}(R)$ is a finite set.*

1.7 Finitely presented modules

Definition 1.94 (see [4]). Let R be a ring. An exact sequence

$$L_1 \to L_0 \to E \to 0$$

of R-modules, where L_0 and L_1 are free, is called a *presentation* of an R-module E.

Every R-module E admits a presentation. We know in fact that there exists a surjective homomorphism $u : L_0 \to E$, where L_0 is free: if K is the kernel of u, there exists similarly a surjective homomorphism $v : L_1 \to K$, where L_1 is free. If v is considered as a homomorphism of L_1 to L_0, the sequence

$$L_1 \xrightarrow{v} L_0 \xrightarrow{u} E \to 0$$

is exact by definition.

Definition 1.95. Let R be a ring. An R-module M is called a *finitely presented* R-module if there exists an exact sequence

$$F_1 \to F_0 \to M \to 0$$

with F_i finitely generated free R-modules.

Lemma 1.96 (see [7, Lem 2.1.1])**.** *Let R be a ring, let M be a finitely presented R-module, and let*

$$0 \to K \to N \to M \to 0$$

be an exact sequence with N a finitely generated R-module, then K is finitely generated.

Proof. Chase the diagram

$$\begin{tikzcd} & F_1 \arrow[r] \arrow[d, "\beta"'] & F_0 \arrow[r] \arrow[d, "\alpha"'] & M \arrow[r] \arrow[d, "1_M"'] & 0 \\ 0 \arrow[r] & K \arrow[r] & N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where the upper row is obtained from the definition of M as a *finitely presented module*, to obtain maps α and β, with β surjective. □

Definition 1.97. Let R be a ring and let M be an R-module. An *n-presentation* of M is an exact sequence

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

with F_i free R-modules. If, in addition, F_i are finitely generated, this presentation is called a *finite n-presentation* of M.

A finite 1-presentation of M is sometimes called a finite presentation of M.

Definition 1.98. If M is a finitely generated R-module, denote by:

$$\lambda_R(M) = \lambda(M) = \sup\{n : \text{there is a finite } n\text{-presentation of } M\}$$

If M is not finitely generated put $\lambda(M) = -1$. It is clear that M is finitely generated iff $\lambda(M) \geqslant 0$, and M is finitely presented iff $\lambda(M) \geqslant 1$.

Theorem 1.99 (see [7, Theorem 2.1.2])**.** *Let R be a ring and let*

$$0 \to P \to N \to M \to 0$$

be an exact sequence of R-modules, then

(a) $\lambda(N) \geqslant \inf\{\lambda(P), \lambda(M)\}$.

(b) $\lambda(M) \geqslant \inf\{\lambda(N), \lambda(P) + 1\}$.

(c) $\lambda(P) \geqslant \inf\{\lambda(N), \lambda(M) - 1\}$.

(d) If $N = M \oplus P$ then $\lambda(N) = \inf\{\lambda(M), \lambda(P)\}$.

In particular, N is finitely presented iff M and P are both finitely presented.

Corollary 1.100. *Let R be a ring and let N_1 and N_2 be two finitely presented submodules of an R-module M. Then $N_1 + N_2$ is finitely presented iff $N_1 \cap N_2$ is finitely generated.*

Proof. We have to show that $\lambda(N_1 + N_2) \geqslant 1$ iff $\lambda(N_1 \cap N_2) \geqslant 0$. Consider the exact sequence

$$0 \to N_1 \cap N_2 \to N_1 \oplus N_2 \to N_1 + N_2 \to 0.$$

By Theorem 1.99(d) $\lambda(N_1 \oplus N_2) \geqslant 1$. Now use Theorem 1.99(b) and (c). □

Theorem 1.101 (see [7, Thm 2.1.4]). *Let R be a ring.*

(a) If R is Noetherian, every finitely generated R-module is finitely presented.

(b) Every finitely generated projective R-module is finitely presented.

Proof.

(a) Any submodule of a finitely generated module over a Noetherian ring is finitely generated; thus, in mapping a finitely generated free module onto a finitely generated module M, we obtain a finitely generated Kernel.

(b) Let P be a finitely generated and projective and let

$$0 \to K \to F \to P \to 0$$

be an exact sequence with F finitely generated and free. Since P is projective the sequence splits and K is isomorphic to a direct summand of F. It follows that K is finitely generated.

□

Theorem 1.102 (see [7, Thm 2.1.7]). *Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism making S a finitely generated R-module. If an S-module M is finitely presented as an R-module, then M is finitely presented as an S-module.*

Proof. Clearly M is a finitely generated S-module. Let

$$0 \to K \to F \to M \to 0$$

be an exact sequence of S-modules with F finitely generated and free as an S-module. We have $\lambda_R(F) \geqslant 0$, $\lambda_R(M) \geqslant 1$ and $\lambda_R(K) \geqslant \inf\{\lambda_R(F), \lambda_R(M) - 1\} \geqslant 0$. Thus, K is finitely generated R-module and, hence, S-module. □

Theorem 1.103 (see [7, Thm 2.1.8]). *Let R be a ring and let $I \lhd R$.*

(a) Let M be a finitely presented R-module, then M/IM is a finitely presented R/I-module.

(b) Assume that I is finitely generated and let M be an R/I-module, then M is a finitely presented R-module iff M is a finitely presented R/I-module.

Proof.

(a) Since $M/IM \cong M \otimes_R R/I$, tensoring a finite presentation of M over R by R/I we obtain a finite presentation of M/IM over R/I.

(b) By Theorem 1.102, we have that if M is a finitely presented R-module, then it is a finitely presented R/I-module. Conversely, let

$$0 \to K \to F \to M \to 0$$

be an exact sequence of R/I-modules with F and K finitely generated and F free. Since $F \simeq (R/I)^n$ and I is a finitely generated ideal, we have that $\lambda_R(F) \geqslant 1$. Since $\lambda_R(K) \geqslant 0$ we have

$$\lambda_R(M) \geqslant \inf(\lambda_R(K)+1, \lambda_R(F)) \geqslant 1.$$

□

Theorem 1.104. *Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism making S a faithfully flat R-module. An R-module M is finitely generated (resp. finitely presented) iff $M \otimes_R S$ is a finitely generated (resp. finitely presented) S-module.*

Proof. see [7, Thm 2.1.9] □

Chapter 2

BASIC HOMOLOGICAL ALGEBRA

2.1 Chain complexes

Definition 2.1. Let R be a ring. A sequence of R-modules and R-homomorphisms

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

is said to be a *chain complex* if each pair of adjacent homomorphisms (d_{n+1}, d_n) satisfies the relation $d_n \circ d_{n+1} = 0$. This is equivalent to saying that $\operatorname{im} d_{n+1} \subseteq \ker d_n$. Such a complex is generally denoted by (C, d) or simply C. We write (C_*, d) for a chain complex and refer to the d_n as boundary homomorphisms. We symbolically write $d^2 = 0$ to indicate that $d_n \circ d_{n+1} = 0$ holds for all n. For clarity we write sometimes (C_*, d^C) to indicate which boundary is being used.

An *exact* or *acyclic* chain complex is one in which each segment

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is exact.

If a chain complex is of finite length we often enlarge it to a doubly infinite complex by adding in trivial modules as homomorphisms. In particular, if M is an R-module we can view it as the chain complex with $M_0 = M$ and $M_n = 0$ whenever $n \neq 0$. It is often useful to consider the null complex $\mathbf{0} = (\{0\}, 0)$.

Definition 2.2. (C_*, d) is called *bounded below* if there is an n_1 such that $C_n = 0$ whenever

$n < n_1$. Similarly, (C_*, d) is called *bounded above* if there is an n_2 such that $C_n = 0$ whenever $n > n_2$. (C_*, d) is called *bounded* if it is bounded both below and above.

The homology of the complex (C_*, d) is defined to be the complex $(H_*(C_*, d), 0)$ where

$$H_n(C_*, d) = \ker d_n / \operatorname{im} d_{n+1}.$$

The homology of a complex measures its deviation from exactness; in particular, (C_*, d) is exact if and only if $H_*(C_*, d) = 0$. Notice that there are exact sequences

$$0 \to \operatorname{im} d_{n+1} \to \ker d_n \to H_n(C_*, d) \to 0,$$

and

$$0 \to \ker d_n \to C_n \to \operatorname{im} d_n \to 0.$$

Example 2.3. Consider the complex of $\mathbb{Z}$-modules where

$$C_n = \mathbb{Z}/4, \quad d : \mathbb{Z}/4 \to \mathbb{Z}/4; \quad d(\bar{t}) = \overline{2t}.$$

Then $\ker d_n = 2\mathbb{Z}/4 = \operatorname{im} d_n$ and $H_n(C_*, d) = 0$, hence (C_*, d) is a acyclic.

Definition 2.4. A *homomorphism* of chain complexes or *chain homomorphism*

$$h : (C_*, d^C) \to (D_*, d^D)$$

is a sequence of homomorphisms $h_n : C_n \to D_n$ for which the following diagram commutes.

$$\begin{tikzcd} C_n \arrow[r, "d_n^C"] \arrow[d, "h_n"'] & C_{n-1} \arrow[d, "h_{n-1}"] \\ D_n \arrow[r, "d_n^D"'] & D_{n-1} \end{tikzcd}$$

We often write $h : C_* \to D_*$ when the boundary homomorphisms are clear from the context. A chain homomorphism for which each $h_n : C_n \to D_n$ is an isomorphism is called a *chain isomorphism* and admits an inverse chain homomorphism $D_* \to C_*$ consisting of the inverse homomorphisms $h_n^{-1} : D_* \to C_*$.

The *category of chain complexes* has chain complexes as its objects and chain homomorphisms as its morphisms. It is also an abelian category.

Definition 2.5. A *cochain complex* is a collection of R-modules C^n together with coboundary homomorphisms $d^n : C^n \to C^{n+1}$ for which $d^{n+1}d^n = 0$; the cohomology of this complex is $(H^*(C^*, d), 0)$ where

$$H^n(C^*, d) = \ker d^n / \operatorname{im} d^{n-1}.$$

Given a chain complex (C_n, d) we can re-index so that $C^n = C_{-n}$ and form the cochain complex (C^n, d); similarly each cochain complex gives rise to a chain complex. We then have

$$H^n(C^*, d) = H_{-n}(C_*, d).$$

We mainly focus on chain complexes, but everything can be reworked for cochain complexes using this correspondence.

Definition 2.6. Given a morphism of chain complexes $h : (C_*, d) \to (D_*, d)$ we may define two new chain complexes

$$\ker h = ((\ker h)_*, d), \quad \operatorname{im} h = ((\operatorname{im} h)_*, d),$$

where

$$(\ker h)_n = \ker h : C_n \to D_n, \quad (\operatorname{im} h)_n = \operatorname{im} h : C_n \to D_n.$$

The boundary homomorphisms are the restrictions of d to each of these.

Let $h : (L_*, d) \to (M_*, d)$ and $k : (M_*, d) \to (N_*, d)$ be chain homomorphisms and suppose that

$$0 \to (L_*, d) \xrightarrow{h} (M_*, d) \xrightarrow{k} (N_*, d) \to 0$$

is *short exact*, i.e.

$$\ker h = 0, \quad \operatorname{im} k = (N_*, d), \quad \ker k = \operatorname{im} h.$$

It is natural to ask about the relationship between the three homology functors $H_*(L_*, d)$, $H_*(M_*, d)$ and $H_*(N_*, d)$.

Theorem 2.7 (see [33, 1.3.1])**.** *Let*

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

be a short exact sequence of chain complexes. Then there are natural maps

$$\partial : H_n(C_*) \to H_{n-1}(A_*),$$

called connecting homomorphisms, such that

$$\cdots \xrightarrow{g} H_{n+1}(C_*) \xrightarrow{\partial} H_n(A_*) \xrightarrow{f} H_n(B_*) \xrightarrow{g} H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \xrightarrow{f} \cdots$$

is an exact sequence. Similarly, if

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

is a short exact sequence of cochain complexes. Then there are natural induced maps

$$\partial : H^n(C_*) \to H^{n+1}(A_*),$$

and a long exact sequence

$$\cdots \xrightarrow{g} H^{n-1}(C_*) \xrightarrow{\partial} H^n(A_*) \xrightarrow{f} H^n(B_*) \xrightarrow{g} H^n(C_*) \xrightarrow{\partial} H^{n+1}(A_*) \xrightarrow{f} \cdots .$$

Definition 2.8. Let (C, d) and (C', d') be chain complexes and $f : C \to C'$, $g : C \to C'$ be chain maps. A *chain homotopy* $\mathcal{D}$ between f and g is a sequence of homomorphisms

$$\{\mathcal{D}_n : C_n \to C'_{n+1}\},$$

so that

$$d'_{n+1} \circ \mathcal{D}_n + \mathcal{D}_{n-1} \circ d_n = f_n - g_n$$

for each n. Thus we have the following diagram

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d] \arrow[dl, "\mathcal{D}_n"] & C_{n-1} \arrow[d, "f_{n-1}-g_{n-1}"] \arrow[dl, "\mathcal{D}_{n-1}"] \\
C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1}
\end{tikzcd}$$

If there exists a chain homotopy between f and g, then f and g are said to be *chain homotopic*. In particular, a chain map $f : C \to D$ is *null homotopic* if there are maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$. The maps $\{s_n\}$ are called a *chain contraction* of f. Similarly, two chain maps f and g from C to D are *chain homotopic* written as $f \simeq g$, if their difference $f - g$ is null homotopic, that is, if

$$f - g = sd + ds.$$

The maps $\{s_n\}$ are called a *chain homotopy* from f to g.

Proposition 2.9. *$\simeq$ is an equivalence relation on the set of chain homomorphisms $C_* \to D_*$.*

The equivalence classes of $\simeq$ are called *chain homotopy classes*.

Proposition 2.10. *Suppose that two chain homomorphisms $f, g : C_* \to D_*$ are chain homotopic. Then the induced homomorphisms $f_*, g_* : H_*(C_*, d) \to H_*(D_*, d)$ are equal.*

Definition 2.11 (see [33, 1.1.2])**.** A morphism $C \to D$ of chain complexes is called *quasi-isomorphism* if the induced maps $H_n(C) \to H_n(D)$ are all isomorphisms.

Definition 2.12 (see [33, 1.4.1]). A complex C is called *split* if there are maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$. The maps s_n are called *the splitting maps*. If in addition C is acyclic (exact as a sequence), we say that C is *split exact*.

2.2 Categories and functors

Definition 2.13 (see [23, 7]). A *category* $\mathfrak{C}$ consists of *objects* and *morphisms* which may sometimes be composed. Formally, a category is a class of objects $A, B, C, \dots$ together with

(a) a family of disjoint sets of morphisms from A to B, $\mathfrak{C}(A, B)$.

(b) for each triple of objects A,B,C a function which assigns to $\alpha \in \mathfrak{C}(A, B)$ and $\beta \in \mathfrak{C}(B, C)$ an element $\beta\alpha \in \mathfrak{C}(A, C)$.

(c) for each object A, a morphism $1_A \in \mathfrak{C}(A, A)$;

subject to the two axioms:

(a) Associativity: If $\alpha \in \mathfrak{C}(A, B)$, $\beta \in \mathfrak{C}(B, C)$, and $\gamma \in \mathfrak{C}(C, D)$, then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$;

(b) Identity: If $\alpha \in \mathfrak{C}(A, B)$, then $\alpha 1_A = \alpha = 1_B\alpha$.

Definition 2.14 (see [23, 7]). A morphism $\theta : A \to B$ is called an *isomorphism* if there is another morphism $\varphi : B \to A$ such that $\varphi\theta = 1_A$ and $\theta\varphi = 1_B$. Then, it can be easily seen that φ is unique.

Example 2.15 (see [23, 7]). A group G is a category with one object G; let $\mathrm{Hom}(G, G)$ be all elements of G. If a set M is closed under an associative multiplication with an identity, it is likewise a category with one object and composition given by multiplication.

Example 2.16 (see [23, 7]). Another example of a category is the category of R-modules over a fixed ring R. The objects of this category are all R-modules A, B, C,.... The set $\mathcal{M}_R = \mathrm{Hom}_R(A, B)$ of all R-module homomorphisms of A to B, while the composite is the usual composite of homomorphisms.

To give other examples of categories it will suffice to specify the objects and the morphisms of the category;

(a) The category of topological spaces, where the objects are all topological spaces and morphisms are all continuous maps of one space to another.

(b) The category of abelian groups, where the objects are all abelian groups and morphisms are all homomorphisms of such.

(c) The category of groups, where the objects are all groups (not necessarily abelian) and morphisms are all homomorphisms of groups.

(d) The category of sets, where the objects are all sets and morphisms are all functions of one set to another.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories and suppose that to each object C of $\mathfrak{C}$ there is associated an object $T(C)$ of $\mathfrak{D}$, and also with each morphism f of $\mathfrak{C}$ there is associated a morphism $T(f)$ of $\mathfrak{D}$. $T : \mathfrak{C} \to \mathfrak{D}$ is called a *covariant functor* when the following conditions are satisfied

(a) If $f : C_1 \to C_2$ is a morphism, then so is $T(f) : T(C_1) \to T(C_2)$,

(b) $T(1_C) = 1_{T(C)}$ for every object $C \in \mathfrak{C}$,

(c) If gf is defined, then $T(gf) = T(g)T(f)$.

Contravariant functors are defined similarly except that the above conditions are replaced by

(a) If $f : C_1 \to C_2$ is a morphism, then so is $T(f) : T(C_2) \to T(C_1)$,

(b) $T(1_C) = 1_{T(C)}$ for every object $C \in \mathfrak{C}$,

(c) If gf is defined, then $T(gf) = T(f)T(g)$.

Example 2.17 (see [5, 2.1])**.** Let R and S be any two rings. Suppose that for each R-module A, an S-module $T(A)$ is given and that to each R-homomorphism $\varphi : A \to A'$, a S-homomorphism $T(\varphi) : T(A) \to T(A')$ is given such that

(a) $T(1_A) = 1_{T(A)}$,

(b) $T(\varphi'\varphi) = T(\varphi')T(\varphi)$ for homomorphisms $\varphi : A \to A'$, $\varphi' : A' \to A''$.

We then say that the pair of functions $T(A)$, $T(\varphi)$ forms a *covariant* functor T on the category of R-modules with values in the category of S-modules. In the case of a *contravariant* functor we have $T(\varphi) : T(A') \to T(A)$ and $T(\varphi'\varphi) = T(\varphi)T(\varphi')$.

Definition 2.18 (see [19, 2.4])**.** Let F and G be any two covariant functors from a category $\mathfrak{C}$ to a category $\mathfrak{D}$. By a *natural transformation* of the functor F into the functor G, we mean

(a) For each $X \in \mathfrak{C}$, we have

$$\varphi(X) : F(X) \to G(X).$$

(b) For every morphism $\alpha : X \to Y$ of $\mathfrak{C}$, we have

$$F(\alpha)\varphi(Y) = \varphi(X)G(\alpha).$$

The first condition is equivalent to the condition that the products in second condition are always defined. The second condition asserts that the following square is commutative:

$$\begin{tikzcd} F(X) \arrow[r, "F(\alpha)"] \arrow[d, "\varphi(X)"'] & F(Y) \arrow[d, "\varphi(Y)"] \\ G(X) \arrow[r, "G(\alpha)"'] & G(Y) \end{tikzcd}$$

In case F and G are contravariant functors, then the places of X and Y are replaced with each other in the second condition.

Definition 2.19 (see [26, 1.3])**.** A functor $T : \mathcal{M}_R \to \mathcal{M}_S$ between the categories of modules over the commutative rings R and S respectively, is said to be *additive* if whenever $f_1 : A \to A'$ and $f_2 : A \to A'$ are two R-homomorphisms, then

$$T(f_1 + f_2) = T(f_1) + T(f_2).$$

Definition 2.20. A covariant functor T is said to be:

(a) *left exact* if whenever

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is an exact sequence, then

$$0 \to TA \xrightarrow{T\alpha} TB \xrightarrow{T\beta} TC$$

is also an exact sequence.

(b) *right exact* if whenever

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is an exact sequence, then

$$TA \xrightarrow{T\alpha} TB \xrightarrow{T\beta} TC \to 0$$

is also an exact sequence.

A contravariant functor T is said to be:

(a) *left exact* if whenever

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is an exact sequence, then

$$0 \to TC \xrightarrow{T\beta} TB \xrightarrow{T\alpha} TA$$

is also an exact sequence.

(b) *right exact* if whenever

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is an exact sequence, then

$$TC \xrightarrow{T\beta} TB \xrightarrow{T\alpha} TA \to 0$$

is also an exact sequence.

The functors which will concern us most are those of the category $\mathcal{M}_R$ of R-modules.

Definition 2.21 (see [31, 2.11.2])**.** A functor F is *exact* if whenever

$$K \xrightarrow{f} M \xrightarrow{g} N$$

is exact then

$$FK \xrightarrow{Ff} FM \xrightarrow{Fg} FN$$

is exact for R-modules K, M and N.

Proposition 2.22 (see [27, 3.11.5])**.** *A functor T is exact if and only if it is both left exact and right exact.*

Proposition 2.23 (see [31, 2.11.3])**.** *A functor F is exact if and only if whenever*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is exact then

$$0 \to FK \xrightarrow{Ff} FM \xrightarrow{Fg} FN \to 0$$

is exact for R-modules K, M and N.

2.3 Derived functors

It was noted in various quite different settings that a short exact sequence often gives rise to a long exact sequence. The concept of derived functor explains and clarifies many of these observations. Suppose we are given a covariant left exact functor $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories $\mathcal{A}$ and $\mathcal{B}$. If

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{A}$, then applying F yields the exact sequence

$$0 \to F(A) \to F(B) \to F(C)$$

and one could ask how to continue this sequence to the right to form a long exact sequence. Strictly speaking, this question is ill-posed, since there are always numerous different ways to continue a given exact sequence to the right. But it turns out that if $\mathcal{A}$ is 'nice' enough, then there is one canonical way of doing so, given by the right derived functors of F. For every $i \geqslant 1$, there is a functor $R^iF : \mathcal{A} \to \mathcal{B}$, and the above sequence continues like so:

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \cdots$$

From this we see that F is an exact functor if and only if $R^iF = 0$; so in a sense the right derived functors of F measure how far F is from being exact.

If the object A in the above short exact sequence is injective, then the sequence splits. Applying any additive functor to a split sequence results in a split sequence, so in particular $R^1F(A) = 0$. Right derived functors are zero on injectives: this is the motivation for the construction given below.

The crucial assumption we need to make about our abelian category $\mathcal{A}$ is that it has enough injectives, meaning that for every object A in $\mathcal{A}$ there exists a monomorphism $A \to I$ where I is an injective object in $\mathcal{A}$.

The right derived functors of the covariant left-exact functor $F : \mathfrak{A} \to \mathfrak{B}$ are then defined as follows. Start with an object X of $\mathcal{A}$. Because there are enough injectives, we can construct a long exact sequence of the form

$$0 \to X \to I^0 \to I^1 \to I^2 \to \dots$$

where the I^i are all injective, an injective resolution of X. Applying the functor F to this sequence, and omitting the first term, we obtain the chain complex

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots$$

Note that this is in general not an exact sequence anymore. But we can compute its homology at the i-th spot, we call the result $R^iF(X)$. Of course, various things have to be checked: the end result does not depend on the given injective resolution of X, and any morphism $X \to Y$ naturally yields a morphism $R^iF(X) \to R^iF(Y)$, so that we indeed obtain a functor.

Note that left exactness means that

$$F(X) \to F(I^0) \to F(I^1)$$

is exact, so $R^0F(X) = F(X)$, so we only get something interesting for $i > 0$.

To produce well-defined derived functors of F, we would have to fix an injective resolution for every object of $\mathcal{A}$. This choice of injective resolutions then yields functors R^iF. Different choices of resolutions yield naturally isomorphic functors, so in the end the choice doesn't really matter.

If X is itself injective, then we can choose the injective resolution $0 \to X \to X \to 0$, and we obtain that $R^iF(X) = 0$ for all $i \geqslant 1$. In practice, this fact, together with the long exact sequence property, is often used to compute the values of right derived functors.

An equivalent way to compute $R^iF(X)$ is the following: take an injective resolution of X as above, and let K^i be the image of the map $I^{i-1} \to I^i$ (for $i = 0$, define $I^{i-1} = 0$), which is the same as the kernel of $I^i \to I^{i+1}$. Let $\varphi_i : I^{i-1} \to K^i$ be the corresponding surjective map. Then $R^iF(X)$ is the cokernel of $F(\varphi_i)$.

If one starts with a covariant right-exact functor G, and the category $\mathcal{A}$ has enough projectives (i.e. for every object A of $\mathcal{A}$ there exists an epimorphism $P \to A$ where P is a projective object), then one can define analogously the left-derived functors L_iG. For an object X of A we first construct a projective resolution of the form

$$P_2 \to P_1 \to P_0 \to X \to 0$$

where the P_i are projective. We apply G to this sequence, chop off the last term, and compute homology to get $L_iG(X)$. As before, $L_0G(X) = G(X)$.

In this case, the long exact sequence will grow to the left rather than to the right:

$$0 \to A \to B \to C \to 0$$

is turned into

$$\cdots \to L_2G(C) \to L_1G(A) \to L_1G(B) \to L_1G(C) \to G(A) \to G(B) \to G(C) \to 0.$$

Left derived functors are zero on all projective objects.

One may start with a contravariant left exact functor F, the resulting right-derived functors are then also contravariant. In this case the short exact sequence

$$0 \to A \to B \to C \to 0$$

is turned into the long exact sequence

$$0 \to F(C) \to F(B) \to F(A) \to R^1F(C) \to R^1F(B) \to R^1F(A) \to R^2F(C) \to \cdots .$$

Example 2.24. Some common examples are:

(a) Ext functors. If R is a ring, then the category of all left R-modules is an abelian category with enough injectives. If A is a fixed left R-module, then the functor $\mathrm{Hom}_R(A, -)$ is left exact, and its right derived functors are the Ext functors $\mathrm{Ext}^i_R(A, B)$.

(b) Tor functors. The category of left R-modules also has enough projectives. If A is a fixed right R-module, then the tensor product with A gives a right exact covariant functor on the category of left R-modules; its left derived functors are the Tor functors $\mathrm{Tor}^R_i(A, B)$.

(c) Sheaf cohomology. If X is a topological space, then the category of all sheaves of abelian groups on X is an abelian category with enough injectives. The functor which assigns to each such sheaf L the group $L(X)$ of global sections is left exact, and the right derived functors are the sheaf cohomology functors, usually written as $H^i(X, L)$. Slightly more generally: if $(X, \mathcal{O}_X)$ is a ringed space, then the category of all sheaves of $\mathcal{O}_X$-modules is an abelian category with enough injectives, and we can again construct sheaf cohomology as the right derived functors of the global section functor.

Derived functors and the long exact sequences are 'natural' in several technical senses. First, given a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B_1 \arrow[r, "g_1"] \arrow[d, "\beta"'] & C_1 \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "f_2"] & B_2 \arrow[r, "g_2"] & C_2 \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact, the two resulting long exact sequences are related by commuting squares:

$$\begin{tikzcd}
0 \arrow[r] & F(A_1) \arrow[r, "F(f_1)"] \arrow[d, "F(\alpha)"'] & F(B_1) \arrow[r, "F(g_1)"] \arrow[d, "F(\beta)"'] & F(C_1) \arrow[r] \arrow[d, "F(\gamma)"'] & R^1(F(A_1)) \arrow[r, "R^1(F(f_1))"] \arrow[d, "R^1(F(\alpha))"'] & R^1(F(B_1)) \arrow[r] \arrow[d, "R^1(F(\beta))"'] & \cdots \\
0 \arrow[r] & F(A_2) \arrow[r, "F(f_2)"] & F(B_2) \arrow[r, "F(g_2)"] & F(C_2) \arrow[r] & R^1(F(A_2)) \arrow[r, "R^1(F(f_2))"] & R^1(F(B_2)) \arrow[r] & \cdots
\end{tikzcd}$$

Second, suppose $\eta : F \to G$ is a natural transformation from the left exact functor F to the left exact functor G. Then natural transformations $R^i\eta : R^iF \to R^iG$ are induced, and indeed R^i becomes a functor from the functor category of all left exact functors from $\mathcal{A}$ to $\mathcal{B}$ to the full functor category of all functors from $\mathcal{A}$ to $\mathcal{B}$. Furthermore, this functor is compatible with the long exact sequences in the following sense: if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence, then a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[r, "F(g)"] \arrow[d, "\eta_B"'] & F(C) \arrow[r] \arrow[d, "\eta_C"'] & R^1(F(A)) \arrow[r, "R^1(F(f))"] \arrow[d, "R^1(\eta_A)"'] & R^1(F(B)) \arrow[r] \arrow[d, "R^1(\eta_B)"'] & \cdots \\
0 \arrow[r] & G(A) \arrow[r, "G(f)"] & G(B) \arrow[r, "G(g)"] & G(C) \arrow[r] & R^1(G(A)) \arrow[r, "R^1(G(f))"] & R^1(G(B)) \arrow[r] & \cdots
\end{tikzcd}$$

is induced.

Both of these naturalities follow from the naturality of the sequence provided by the snake lemma.

Conversely, the following characterization of derived functors holds: given a family of functors $R^i : \mathcal{A} \to \mathcal{B}$, satisfying the above, i.e. mapping short exact sequences to long exact sequences, such that for every injective object I of $\mathcal{A}$, $R^i(I) = 0$ for every positive i, then these functors are the right derived functors of R^0.

2.4 Hom and Ext functors

Definition 2.25. Let M, N be two R-modules. Recall that we define $\mathrm{Hom}_R(M, N)$ to be the set of all R-homomorphisms $M \to N$. Since R is commutative so $\mathrm{Hom}_R(M, N)$ is an R-module.

Proposition 2.26. *Let M, N be two R-modules.*

(a) Given a short exact sequence of R-modules

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M_3, N) \xrightarrow{f_2^*} \mathrm{Hom}_R(M_2, N) \xrightarrow{f_1^*} \mathrm{Hom}_R(M_1, N)$$

is exact.

(b) *Given a short exact sequence of R-modules*

$$0 \to N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M, N_1) \xrightarrow{g_{1*}} \mathrm{Hom}_R(M, N_2) \xrightarrow{g_{2*}} \mathrm{Hom}_R(M, N_3)$$

is exact.

Therefore, $\mathrm{Hom}_R(M, -)$ is a left exact functor and $\mathrm{Hom}(-, M)$ is a left exact contravariant functor for any R-module M.

Proposition 2.27 (see [31, 2.11.6])**.** $\mathrm{Hom}_R(P, -)$ *is an exact functor if and only if P is a projective R-module.* $\mathrm{Hom}_R(-, E)$ *is an exact contravariant functor if and only if E is an injective R-module.*

In homological algebra, the Ext functors are the derived functors of Hom functors.

Let R be a ring and $\mathcal{M}_R$ the category of R-modules. Let $N \in \mathcal{M}_R$ and set $T(N) = \mathrm{Hom}_R(M, N)$, for fixed $M \in \mathcal{M}$. Then T is a left exact functor from $\mathcal{M}_R$ to $\mathcal{M}_R$ and thus has right derived functors R^nT. The Ext functor is defined by

$$\mathrm{Ext}_R^n(M, N) = (R^nT)N.$$

This can be calculated by taking any injective resolution

$$0 \to N \to I^0 \to I^1 \to \cdots,$$

and computing

$$0 \to \mathrm{Hom}_R(M, I^0) \to \mathrm{Hom}_R(M, I^1) \to \cdots.$$

The $(R^nT)N$ is the homology of this complex. Note that $\mathrm{Hom}_R(M, N)$ is excluded from the complex.

An alternative definition is given using the functor $G(M) = \mathrm{Hom}_R(M, N)$. For a fixed module N, this is contravariant left exact functor, and thus we also have right derived functors R^nG, and can define

$$\mathrm{Ext}_R^n(M, N) = (R^nG)M.$$

This can be calculated by taking any projective resolution

$$\cdots \to P^1 \to P^0 \to M \to 0,$$

and proceeding dually by computing

$$0 \to \mathrm{Hom}_R(P^0, N) \to \mathrm{Hom}_R(P^1, N) \to \cdots .$$

Then $(R^nG)M$ is the homology of this complex. Again note that $\mathrm{Hom}_R(M, N)$ is excluded from the complex.

These two constructions turn out to yield isomorphic results, and so both may be used to calculate the Ext functor.

Ext^n_R has the following properties:

(a) $\mathrm{Ext}^i_R(M, N) = 0$ for $i > 0$ if either N is injective or M is projective.

(b) $\mathrm{Ext}^n_R(\bigoplus_\alpha M_\alpha, N) \cong \prod_\alpha \mathrm{Ext}^n_{\mathcal{M}_R}(M_\alpha, N)$.

(c) $\mathrm{Ext}^n_R(M, \prod_\beta N_\beta) \cong \prod_\beta \mathrm{Ext}^n_{\mathcal{M}_R}(M, N_\beta)$.

2.5 Tensor products and Tor functors

In homological algebra, the Tor functors are the derived functors of the tensor product functors.

Let R be a commutative ring and $\mathcal{M}_R$ the category of R-modules. Pick a fixed module N in $\mathcal{M}_R$. For $M \in \mathcal{M}_R$, set $T(M) = M \otimes_R N$. Then T is a right exact functor from $\mathcal{M}_R$ to $\mathcal{M}_R$ and its left derived functors L_nT are defined. We set

$$\mathrm{Tor}^R_n(M, N) = (L_nT)M$$

i.e., we take a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

the chop off the last term M and tensor it with N to get the complex

$$\cdots \to P_2 \otimes N \to P_1 \otimes N \to P_0 \otimes N \to 0$$

and take the homology of this complex.

Tor^R_n has the following properties:

(a) For every $n \geqslant 1$, Tor_n^R is an additive functor from $\mathcal{M}_R \times \mathcal{M}_R$ to $\mathcal{M}_R$.

(b) If $r \in R$ is not a zero-divisor then

$$\mathrm{Tor}_1^R(R/(r), N) = \{n \in N : rn = 0\}.$$

(c) $\mathrm{Tor}_n^R(M, N)$ can be computed by using projective resolutions of either variable and the answers agree up to isomorphisms.

(d) Given R-module homomorphisms $f : M_1 \to M_2$ and $g : N_1 \to N_2$, there are homomorphisms

$$(f \otimes g)_* = f_* \otimes g_* : \mathrm{Tor}_n^R(M_1, N_1) \to \mathrm{Tor}_n^R(M_2, N_2)$$

generalizing $f_* \otimes g_* : M_1 \otimes_R N_1 \to M_2 \otimes_R N_2$.

(e) For a projective R-module P (resp. Q) and $n > 0$, we have

$$\mathrm{Tor}_n^R(P, N) = 0 = \mathrm{Tor}_n^R(M, Q).$$

(f) Associated to a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_{n+1}^R(M_3, N) \arrow[dll] & \\
\mathrm{Tor}_n^R(M_1, N) \arrow[r] & \mathrm{Tor}_n^R(M_2, N) \arrow[r] & \mathrm{Tor}_n^R(M_3, N) \arrow[dl] & \\
 & \mathrm{Tor}_{n-1}^R(M_1, N) \arrow[r] & \cdots &
\end{tikzcd}$$

$$\cdots \to M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0$$

and associated to a short exact sequence of R-modules

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_{n+1}^R(M, N_3) \arrow[dll] & \\
\mathrm{Tor}_n^R(M, N_1) \arrow[r] & \mathrm{Tor}_n^R(M, N_2) \arrow[r] & \mathrm{Tor}_n^R(M, N_3) \arrow[dl] & \\
 & \mathrm{Tor}_{n-1}^R(M, N_1) \arrow[r] & \cdots &
\end{tikzcd}$$

$$\cdots \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0.$$

(g) In the case of abelian groups (i.e. if $R = \mathbb{Z}$), then $\operatorname{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for all $n \geqslant 2$. The reason is that every abelian group A has a free resolution of length 2, since subgroups of free abelian groups are free abelian.

(h) The Tor functors commute with arbitrary direct sums, i.e. there is a natural isomorphism

$$\operatorname{Tor}_n^R(\bigoplus_i A_i, \bigoplus_j B_j) \cong \bigoplus_i \bigoplus_j \operatorname{Tor}_n^R(A_i, B_j).$$

(j) A module $M \in \mathcal{M}_R$ is flat if and only if $\operatorname{Tor}_1^R(M, -) = 0$. In this case, we even have $\operatorname{Tor}_n^R(M, -) = 0$ for all $n \geqslant 1$. In fact, to show $\operatorname{Tor}_n^R(M, N) = 0$, one may use a *flat resolution* of M or N, instead of a projective resolution because every projective resolution is a flat resolution, but the converse is not true, so allowing flat resolution is more flexible.

Corollary 2.28. *Let Q be an R-module for which $\operatorname{Tor}_n^R(M, Q) = 0$ for all $n > 0$ and M. Then for any exact complex (C_*, d), the complex $(C_* \otimes_R Q, d \otimes 1)$ is exact, and*

$$H_n(C_* \otimes_R Q, d \otimes 1) \cong H_n(C_*, d) \otimes_R Q.$$

Proposition 2.29. *If $F_* \to M \to 0$ is a flat resolution, then*

$$\operatorname{Tor}_n^R(M, N) = H_n(F_* \otimes_R N, d \otimes 1).$$

2.6 Homological dimensions over rings

Definition 2.30 (see [7, 1.3])**.** Let R be a ring and let M be an R-module. The *projective dimension of* M over R, denoted by $\operatorname{proj.dim}_R M$ is equal to the least nonnegative integer n, for which there is an exact sequence

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

with P_i projective R-modules. If no such n exists, set $\operatorname{proj.dim}_R M = \infty$.

Definition 2.31 (see [7, 1.3])**.** Let R be a ring and let M be an R-module. The *injective dimension of* M over R, denoted by $\operatorname{inj.dim}_R M$ is equal to the least nonnegative integer n, for which there is an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$$

with E_i injective R-modules. If no such n exists, set $\operatorname{inj.dim}_R M = \infty$.

Definition 2.32 (see [7, 1.3]). Let R be a ring. The *global dimension of* R denoted by $\operatorname{gl.dim} R$ is defined by

$$\operatorname{gl.dim} R = \sup\{\operatorname{proj.dim}_R M : \text{M an } R\text{-module}\}.$$

Definition 2.33 (see [7, 1.3]). Let R be a ring and M be an R-module. The *weak dimension of* M over R denoted by $\operatorname{w.dim}_R M$ is is equal to the least nonnegative integer n, for which there is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

with F_i flat R-modules. If no such n exists, set $\operatorname{w.dim}_R M = \infty$.

Definition 2.34 (see [7, 1.3]). Let R be a ring. The *weak dimension of* R denoted by $\operatorname{w.dim} R$ is defined by

$$\operatorname{w.dim} R = \sup\{\operatorname{w.dim}_R M : \text{M an } R\text{-module}\}.$$

Theorem 2.35 ([7, 1.3.10]). *Let R be a ring, then:*

(a) $\operatorname{w.dim} R \leqslant \operatorname{gl.dim} R$.

(b) If R is Noetherian then $\operatorname{w.dim} R = \operatorname{gl.dim} R$.

Chapter 3

COHERENT RINGS AND COMPLETION

3.1 Elementary properties of coherent modules

Definition 3.1 (see [7]). Let R be a ring. An R-module, M is called a *coherent* R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

Every finitely generated submodule of a coherent module is a coherent module. Over a Noetherian ring every finitely generated module is a coherent module.

Theorem 3.2 (see [7, Thm 2.2.1]). *Let R be a ring and let*

$$0 \to P \xrightarrow{\alpha} N \xrightarrow{\beta} M \to 0$$

be an exact sequence of R-modules.

(a) *If N is a coherent module and P is a finitely generated module then M is a coherent module.*

(b) *If M and P are coherent modules then so is N.*

(c) *If N and M are coherent modules then so is P.*

In particular, if any two of the modules are coherent, so is the third.

Proof.

(a) Since N is finitely generated, so is M. Let M_1 be a finitely generated submodule of M. Since N is a coherent module and P is a finitely generated module, P is finitely presented. Consider the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & K_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^n \arrow[r] \arrow[d] & R^{n+s} \arrow[r] \arrow[d] & R^s \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & \beta^{-1}(M_1) \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where, since $0 \in M_1$ we have that $P \subseteq \beta^{-1}(M_1)$, the left column is derived from a finite presentation of P, the right column is a result of the finite generation of M_1. Now $\beta^{-1}(M_1)$ is a finitely generated submodule of the coherent module N; hence, K_2, and therefore K_3, is finitely generated.

(b) Since by Theorem 1.99(a)

$$\lambda(N) \geqslant \inf\{\lambda(P), \lambda(M)\} \geqslant 1,$$

N is finitely presented. Let N_1 be a finitely generated submodule of N. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\beta/N_1) \arrow[r, "\alpha"] \arrow[d] & N_1 \arrow[r, "\beta"] \arrow[d] & \beta(N_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r, "\alpha"] & N \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

$\beta(N_1)$ is a finitely generated submodule of M and, hence, finitely presented. Since N_1 is finitely generated it follows that $\ker(\beta/N_1)$ is finitely generated. P is coherent; therefore, $\ker(\beta/N_1)$ is finitely presented. We conclude that

$$\lambda(N_1) \geqslant \inf\{\lambda(\ker(\beta/N_1)), \lambda(\beta(N_1))\} \geqslant 1.$$

(c) Since

$$0 \to P \xrightarrow{\alpha} N \xrightarrow{\beta} M \to 0$$

is an exact sequence of R-modules and N is coherent. Therefore $P \subseteq N$ is finitely generated. Now, $P \subseteq N$ is a finitely generated submodule of a coherent module, therefore, coherent. $\square$

Corollary 3.3 (see [7, Cor 2.2.2]). *Let R be a ring, let M and N be coherent R-modules and let $\varphi : M \to N$ be a homomorphism. Then $\ker \varphi$, $\operatorname{im} \varphi$ and $\operatorname{coker} \varphi$ are coherent R-modules.*

Proof. Use Theorem 3.2, and the exact sequences:

$$0 \to \ker \varphi \to M \to \operatorname{im} \varphi \to 0$$

and

$$0 \to \operatorname{im} \varphi \to N \to \operatorname{coker} \varphi \to 0.$$ $\square$

Corollary 3.4 (see [7, Cor 2.2.3]). *Every finite direct sum of coherent modules is a coherent module.*

Proof. Let $\{M_i\}_{i=1}^n$ be a family of coherent modules. Use Theorem 3.2, and the exact sequence

$$0 \to M_1 \to M_1 \oplus \cdots \oplus M_n \to M_2 \oplus \cdots \oplus M_n \to 0$$

to prove the statement by induction on n. $\square$

Corollary 3.5 (see [7, Cor 2.2.4]). *Let R be a ring and let M and N be coherent submodules of a coherent module E, Then $M + N$ and $M \cap N$ are coherent modules.*

Proof. Since $M + N$ is a finitely generated submodule of the coherent module E, we have that $M + N$ is a coherent module. $M \oplus N$ is a coherent module by Corollary 3.4. Now use Theorem 3.2 and the exact sequence

$$0 \to N \cap M \to N \oplus M \to N + M \to 0.$$ $\square$

Corollary 3.6 (see [7, Cor 2.2.5]). *Let R be a ring and let M and N be coherent modules, then $M \otimes_R N$ and $\operatorname{Hom}_R(M, N)$ are coherent modules.*

Proof. Let

$$F_1 \to F_0 \to M \to 0$$

be a finite free presentation of M, then:

$$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(F_0, N) \to \operatorname{Hom}_R(F_1, N)$$

is an exact sequence. $\mathrm{Hom}_R(F_1, N)$ is isomorphic to a finite direct sum of copies of N, and hence, a coherent module. It follows from Corollary 3.3 that $\mathrm{Hom}_R(M, N)$ is coherent. A similar argument yields $M \otimes_R N$ as the cokernel of a map between coherent modules. $\square$

Theorem 3.7 (see [7, Thm 2.2.7])**.** *Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism making S a finitely generated R-module. Let M be an S-module which is coherent as an R-module, then M is coherent as an S-module.*

Proof. Observe that every finitely generated S submodule of M is finitely generated as an R-module and apply Theorem 1.102. $\square$

3.2 Coherent rings

Definition 3.8 (see [7])**.** A ring R is called a *coherent ring* if it is coherent module over itself, that is, if every finitely generated ideal of R is finitely presented.

Theorem 3.9 (see [7, Lem 2.3.2])**.** *Let R be a ring. The following conditions are equivalent.*

(a) R is a coherent ring.

(b) Every finitely presented R-module is a coherent module.

(c) Every finitely generated submodule of a free R-module is finitely presented.

(d) Every R-module R^S with S an arbitrary set, is a flat R-module.

(e) Every direct product of flat R-modules is a flat R-module.

(f) $(I : a) = \{\lambda \in R : \lambda a \in I\}$ is a finitely generated ideal of R for every finitely generated ideal I of R and any element $a \in R$.

(g) $(0 : a)$ is a finitely generated ideal for every element $a \in R$, and the intersection of two finitely generated ideals of R is a finitely generated ideal of R.

Theorem 3.10 (see [7, Thm 2.3.3])**.** *Let $\{R_\alpha : \alpha \in S\}$ be a directed system of rings and let $R = \mathrm{colim}\, R_\alpha$. Suppose that for $\alpha \leqslant \beta$, R_β is a flat R_α-module and that R_α is a coherent ring for every α, then R is a coherent ring.*

Theorem 3.11 (see [7, Thm 2.3.4]). *Let R be a ring and let $x_1, x_2, \ldots$ be indeterminates over R. Set $S = R[x_1, x_2, \ldots]$ be the polynomial ring in $x_1, x_2, \ldots$ over R. Assume that R is Noetherian, then S is a coherent ring.*

Theorem 3.12 (see [7, Thm 2.4.1]). *Let R be a ring and let I be a finitely generated ideal of R. Then an R/I-module M is R/I-coherent iff it is R-coherent. In particular, for a ring R and an ideal I of R, we have*

(a) If R is a coherent ring and I is a finitely generated ideal, then R/I is a coherent ring.

(b) If R/I is a coherent ring and I is a coherent R-module, then R is a coherent ring.

Proof. Note that an R/I module is finitely generated iff it is finitely generated as an R-module. Now apply Theorem 1.103. □

Theorem 3.13 (see [7, Thm 2.2.6]). *Let R be a ring and let U be a multiplicatively closed subset of R. Let M be a coherent R-module, then the localization of M, i.e. M_U is a coherent R_U-module.*

Proof. Clearly, M_U is a finitely generated R_U module. A finitely generated R_U submodule of M_U is of the form N_U, where N is a finitely generated submodule of M. Since R_U is a flat R-module, N_U is finitely presented along with N. □

Theorem 3.14 (see [7, Thm 2.4.2]). *Let R be a ring and let U be a multiplicatively closed subset of R. If R is a coherent ring, then the localization R_U is a coherent ring.*

Theorem 3.15 (see [7, Thm 2.4.3]). *Let R_i with $1 \leqslant i \leqslant n$ be a family of coherent rings, then $R = \prod_{i=1}^{n} R_i$ is a coherent ring.*

Proof. Using induction on n, it suffices to prove the assertion for $n = 2$. In the exact sequence

$$0 \to R_1 \to R \to R/R_1 \to 0$$

the quotient R/R_1 is the coherent ring R_2. Since R_1 is finitely generated as an R-ideal, it follows from Theorem 3.12 that R/R_1 is R-coherent; that is to say, R_2 is a coherent R-module. It now follows from Theorem 3.12(b) that R is a coherent ring, because R_2 is a finitely generated R-ideal and a coherent R-module, and because R/R_2 is the coherent ring R_1. □

Theorem 3.16. *Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism making S a faithfully flat R-module. Let M be an R-module. If $M \otimes_R S$ is a coherent S-module then M is a coherent R-module.*

Proof. see [7, Thm 2.4.4] $\square$

Corollary 3.17 (see [7, Cor 2.4.5])**.** *Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism making S a faithfully flat R-module. If S is a coherent ring, then R is a coherent ring.*

Theorem 3.18 (see [7, Thm 2.4.6])**.** *Let R be a semilocal ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ and such that $R_{\mathfrak{m}_i}$ is a coherent ring for each $1 \leqslant i \leqslant n$, then R is a coherent ring.*

Proof. By Theorem 3.15, we have that $S = \prod_{i=1}^{n} R_{\mathfrak{m}_i}$ is a coherent ring, and S is a faithfully flat extension of R; therefore, R is a coherent ring. $\square$

3.3 Homological dimensions over coherent rings

Theorem 3.19 ([7, 2.5.1])**.** *Let R be a coherent ring and let*

$$0 \to P \to N \to M \to 0$$

be an exact sequence of R-modules. If any two of the modules P, N and M are finitely presented, so is the third.

Theorem 3.20 ([7, 2.5.2])**.** *Let R be a coherent ring and let M be a finitely presented R-module, then M admits a finite free resolution, that is, there exists an exact sequence*

$$\cdots \to F_1 \to F_0 \to M \to 0$$

with F_i finitely generated free R-modules.

Theorem 3.21 ([7, 2.5.3])**.** *Let R be a coherent ring and let M and N be two coherent R-modules, then for each $n \geqslant 0$, $\operatorname{Tor}_n^R(M, N)$ and $\operatorname{Ext}_R^n(M, N)$ are coherent modules.*

Theorem 3.22 ([7, 2.5.4])**.** *Let R be a coherent ring and let M be a finitely presented R-module. The following conditions are equivalent*

(a) $\operatorname{proj.dim}_R M \leqslant n$.

(b) $\operatorname{Tor}_{n+1}^{R}(M, R/I) = 0$ *for all finitely generated ideals* $I \lhd R$.

(b) $\operatorname{Ext}_{R}^{n+1}(M, R/I) = 0$ *for all finitely generated ideals* $I \lhd R$.

Theorem 3.23 ([7, 2.5.5]). *Let R be a ring and let M be an R-module admitting a finite free resolution, then* $\operatorname{w.dim}_R M = \operatorname{proj.dim}_R M$. *In particular, this equality holds for any finitely presented module M over a coherent ring R.*

Corollary 3.24 ([7, 2.5.10]). *Let R be a coherent ring and let M be a finitely presented R-module, then* $\operatorname{proj.dim}_R M \leqslant n$ *if and only if* $\operatorname{Tor}_{n+1}^{R}(M, R/\mathfrak{m}) = 0$ *for every maximal ideal* $\mathfrak{m} \lhd R$.

3.4 Weakly associated prime ideals and Euler characteristic

Recall Section 1.6, that for a ring R and for an R-module M, the *associated primes of M*, denoted by $\operatorname{Ass}(M)$ is defined as the set of all prime ideals associated with M.

Definition 3.25 (see [4, IV-1, Ex. 17]). Let R be a ring and M an R-module. A prime ideal $\mathfrak{p}$ of R is said to be *weakly associated* with M if there exists $x \in M$ such that $\mathfrak{p}$ is a minimal element of the set of prime ideals containing $\operatorname{Ann}(x)$; we denote by $\operatorname{Ass}_f(M)$ the set of ideals weakly associated with M. Then

$$\operatorname{Ass}(M) \subseteq \operatorname{Ass}_f(M).$$

Definition 3.26 (see [7, 3.3]). Let $(R, \mathfrak{m})$ be a local ring. R is called a *self-associated* if $\mathfrak{m} \in \operatorname{Ass}_f(R)$.

A self-associated ring R satisfies the following property:

(P) : Every proper finitely generated ideal of R has a nonzero annihilator.

Corollary 3.27 (see [7, 3.3.19]). *Let $(R, \mathfrak{m})$ be a local ring satisfying* (P), *and let M be an R-module admitting a finite free resolution*

$$0 \to F_n \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} F_0 \to M \to 0.$$

Then M is free.

Corollary 3.28 (see [7, 3.3.20]). *Let $(R, \mathfrak{m})$ be a local coherent ring satisfying* (P), *and let M be a finitely presented R-module. If* $\operatorname{w.dim} M < \infty$, *then M is free.*

Definition 3.29 (see [7, 3.2]). Let R be a domain with field of fractions K, and let M be an R-module, the rank *of M over R* denoted by $\operatorname{rank}_R M$ or $\operatorname{rank} M$, if there is no ambiguity, is defined as $\operatorname{rank}_R M = \operatorname{rank}_K M \otimes_R K$, that is, the cardinality of the basis of the vector space $M \otimes_R K$ over K.

Definition 3.30 (see [7, 3.4]). Let R be a domain and let M be a finitely generated R-module admitting a finite free resolution of finite length

$$0 \to F_n \to \cdots \to F_0 \to M \to 0.$$

The *Euler characteristic of M*, denoted by $\chi_R(M)$ or $\chi(M)$ when there is no ambiguity, is defined as

$$\chi_R(M) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} F_i.$$

As a direct consequence of Schanuel's lemma, the Euler characteristic of a module is independent of the free resolution.

Theorem 3.31 (see [7, 3.4.6]). *Let R be a ring and let M be a finitely generated R-module admitting a finite free resolution of finite length, then:*

(a) $\chi(M) \geqslant 0$.

(b) $\chi(M) = 0$ *if and only if* $(0 : M) \neq 0$.

Corollary 3.32 (see [7, 3.4.7]). *Let R be a ring and let I be an ideal of R admitting a finite free resolution of finite length*

$$0 \to F_n \to \cdots \to F_0 \to I \to 0$$

then $(0 : I) = 0$.

Lemma 3.33 (see [7, 4.2.3]). *Let R be a coherent local ring in which every principal ideal has finite projective dimension. Then R is a domain.*

3.5 Regular and super regular coherent rings

Definition 3.34 ([7, 6.2]). A ring R is called a *regular ring* if every finitely generated ideal of R has finite projective dimension.

Theorem 3.35 ([7, 6.2.1]). *Let R be a coherent ring, then R is a regular ring iff every finitely presented R-module has finite projective dimension.*

Definition 3.36 ([7, 6.2]). Let $(R, \mathfrak{m})$ be a coherent local ring. R is called a *super regular ring* if $\operatorname{gl.dim} R = \operatorname{w.dim} R < \infty$.

Theorem 3.37. *If R is a coherent super regular local ring, then it is a domain.*

Proof. Let R be a coherent super regular local ring and M be a finitely presented R-module. Then M has a finite projective dimension. Since R is coherent and local, there exists finitely generated free modules F_i for $0 \leqslant i \leqslant n$ such that

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

is exact. By Lemma 3.33 we get, R is a domain. □

Theorem 3.38. *Let $(R, \mathfrak{m})$ be a coherent super regular local ring. Then $\mathfrak{m}$ is finitely generated ideal of R.*

Proof. See [7, 6.2.15]. □

3.6 Filtration and completion

Definition 3.39 (see [1, 4.1.1]). A *graded ring* is a ring that is expressible as $\bigoplus_{n \geqslant 0} R_n$ where the R_n are additive subgroups such that

$$R_m R_n \subseteq R_{m+n}.$$

Sometimes R_n is referred to as the n^{th} graded piece and elements of R_n are said to be homogeneous of degree n. The prototype is a polynomial ring in several variables, with R_d consisting of all homogeneous polynomials of degree d (along with the zero polynomial).

Note that the identity element of a graded ring R must belong to R_0. For if 1 has a component a of maximum degree $n > 0$, then $1a = a$ forces the degree of a to exceed n, a contradiction.

Definition 3.40. A *graded module* over a graded ring R is a module M expressible as $\bigoplus_{n \geqslant 0} M_n$, where

$$R_m M_n \subseteq M_{m+n}.$$

Now suppose that $\{R^n\}$ is a *filtration* of the ring R, in other words, the R^n are additive subgroups such that

$$R = R^0 \supseteq R^1 \supseteq \cdots \supseteq R^n \supseteq \cdots$$

with

$$R^m R^n \subseteq R^{m+n}.$$

We call R a filtered ring. A *filtered module*

$$M = M^0 \supseteq M^1 \supseteq \cdots \supseteq M^n \supseteq \cdots$$

over the filtered ring R may be defined similarly. In this case, each M^n is a submodule and we require that

$$R^m M^n \subseteq M^{m+n}.$$

If $I \lhd R$ and M is an R-module, we will be interested in the *I-adic filtration* of R and of M, given respectively by $R^n = I^n$ and $M^n = I^n M$.

Definition 3.41 ([25, 3.8])**.** Let R be a ring and M an R-module. Suppose that $S = \{M_\lambda\}_{\lambda\in\Lambda}$ for a directed set Λ, is a family of submodules of M indexed by Λ and such that $\lambda < \mu \Rightarrow M_\lambda \supseteq M_\mu$. Then taking S as a system of neighborhoods of 0 makes M in to a topological group under addition. In this topology, for any $x \in M$ a system of neighborhoods of x is given by $\{x+M_\lambda\}_{\lambda\in\Lambda}$. In M addition and subtraction are continuous, as is scalar multiplication $x \mapsto rx$ for any $r \in R$. When $M = R$ each M_λ is an Ideal, so that multiplication is also continuous:

$$(r_1 + M_\lambda)(r_2 + M_\lambda) \subseteq r_1 r_2 + M_\lambda$$

This type of topology is called a *linear topology* on M; it is separated (that is, Hausdorff) if and only if $\bigcap_\lambda M_\lambda = 0$. Each $M_\lambda \subseteq M$ is an open set, each coset $x + M_\lambda$ is again open, and the complement $M - M_\lambda$ of M_λ is a union of cosets, so is also open. Hence M_λ is an open and closed subset; the quotient module M/M_λ is the discrete in the quotient topology. $M/\bigcap_\lambda M_\lambda$ is called the separated module associated with M. Moreover, since for $\lambda < \mu$ there is a natural linear map $\varphi_{\lambda\mu} : M/M_\mu \to M/M_\lambda$, we can construct the inverse system $\{M/M_\lambda; \varphi_{\lambda\mu}\}$ of R-modules; its inverse limit

$$\lim_{\lambda} M/M_\lambda$$

is called the *completion* of M, and is written $\widehat{M}$. We give each M/M_λ the discrete topology, the direct product $\prod_\lambda M/M_\lambda$ the product topology, and $\widehat{M}$ the subspace topology in $\prod_\lambda M/M_\lambda$. Let $\psi : M \to \widehat{M}$ be the natural R-linear map; then ψ is continuous, and $\psi(M)$ is dense in $\widehat{M}$. Write $\pi_\lambda : \widehat{M} \to M/M_\lambda$ for the projection, and set $M^*_\lambda = \ker \pi_\lambda$; it is easy to see that topology of $\widehat{M}$ coincides with the linear topology defined by $S = \{M_\lambda^*\}_{\lambda\in\wedge}$. The map π_λ is surjective, in fact

$$\pi_\lambda(\psi(M)) = M/M_\lambda,$$

so that $\widehat{M}/M_\lambda^* \cong M/M_\lambda$, and the completion of $\widehat{M}$ coincides with $\widehat{M}$ itself. If $\psi : M \to \widehat{M}$ is an isomorphism, we say that M is complete. If $S' = \{M'_\gamma\}_{\gamma\in\Gamma}$ is another family of submodules of M indexed by a directed set Γ,then S and S' give the same topology on M if and only if for each M_λ there is a $\gamma \in \Gamma$ such that $M'_\gamma \subseteq M_\lambda$, and for every M'_γ there is a $\mu \in \Lambda$ such that $M_\mu \subseteq M'_\gamma$. It is then easy to see that there is an isomorphism of topological modules

$$\varprojlim_\lambda M/M_\lambda \cong \varprojlim_\gamma M/M'_\gamma.$$

Thus $\widehat{M}$ depends only on the topology of M, as does the question of whether M is complete. When $M = R$, $\{M/M_\lambda; \varphi_{\lambda\mu}\}$ becomes an inverse system of rings, $\widehat{M} = \widehat{R}$ is a ring, and $\psi : R \to \widehat{R}$ a ring homomorphism. $M_\lambda^* \subseteq \widehat{A}$ is not just an R-submodule, but an ideal of $\widehat{R}$; this is clear from the fact that $\pi_\lambda : \widehat{R} \to R/M_\lambda$ is a ring homomorphism. If $N \subseteq M$ is a submodule, then closure $\overline{N}$ of N in M is given by the following formula:

$$\overline{N} = \bigcap_\lambda (N + M_\lambda).$$

Indeed,

$$x \in \overline{N} \iff (x + M_\lambda) \cap N \neq \emptyset \iff x \in N + M_\lambda \quad \text{for all} \quad \lambda.$$

If we write M'_λ for the image of M_λ in the quotient module M/N, the quotient topology of M/N is just the linear topology defined by $\{M'_\lambda\}_\lambda \in \Lambda$. In fact, let $G \subseteq M$ be the inverse image of $G' \subseteq M/N$; then G' is open in the quotient topology of M/N if and only if G is open in M if and only if for every $x \in G$ there is an M_λ such that $x + M_\lambda \subseteq G$ if and only if for every $x' \in G'$ there is an M'_λ such that $x' + M'_\lambda \subseteq G'$. Hence the condition for M/N to be separated is that $\bigcap_\lambda M'_\lambda = 0$, that is $\bigcap_\lambda (N + M_\lambda) = N$, or in other words, that N is closed in M. Moreover, the subspace topology of N is clearly the same thing as the linear topology defined by $\{N \cap M_\lambda\}_{\lambda\in\Lambda}$. Set $M/N = M'$; then

$$0 \to N/(N \cap M_\lambda) \to M/M_\lambda \to M'/M'_\lambda = M/(N + M_\lambda) \to 0$$

is an exact sequence, so that taking the inverse limit, we see that

$$0 \to \widehat{N} \to \widehat{M} \to \widehat{(M/N)}$$

is exact. If we view $\widehat{N}$ as a submodule of $\widehat{M}$, the condition that $\xi = (\xi_\lambda)_{\lambda\in\Lambda} \in \widehat{M}$ belongs to $\widehat{N}$ is that each ξ_λ can be represented by an element of N, or in other words that $\xi \in \psi(N) + M_\lambda^*$ for each λ. Hence $\widehat{N}$ is the same thing as the closure of $\psi(N)$ in M. In general it is not clear whether $\widehat{M} \to \widehat{M/N}$ is surjective, but this holds in the case $\mathbb{N}$. In fact then

$$\widehat{M/N} = \varprojlim_{\lambda} M/(N + M_\lambda);$$

given an element $\xi' = (\xi'_1, \xi'_2, \ldots) \in \widehat{M/N}$, with $\xi'_n \in M/(N + M_n)$, let $x_1 \in M$ be an inverse image of ξ'_1, and $y_2 \in M$ an inverse image of ξ'_2; then $y_2 - x_1 \in N + M_1$, so that we can write

$$y_2 - x_1 = t + m_1$$

with $t \in N$ and $m_1 \in M_1$. If we set $x_2 = y_2 - t$ then $x_2 \in M$ is also an inverse image of ξ'_2, and satisfies $x_2 - x_1 \in M_1$. Similarly we can successively choose inverse images $x_n \in M$ of the ξ'_n in such a way that for $n = 1, 2, \ldots$, we have $x_{n+1} - x_n \in M_\lambda$. If we write $\xi_n \in M/M_n$ for the image of x_n, then by construction $\xi = (\xi_1, \xi_2, \ldots)$ is an element of

$$\varprojlim_{\lambda} M/M_\lambda = \widehat{M}$$

which maps to $\xi' \in \widehat{M/N}$. This proves the following theorem.

Theorem 3.42 (see [25, 3.8.1])**.** *Let R be a ring, M an R-module with a linear topology, and $N \subseteq M$ a submodule. We give N the subspace topology, and M/N the quotient topology.*

(a) These are linear topologies on N and M/N.

(b) The sequence

$$0 \to \widehat{N} \to \widehat{M} \to (\widehat{M/N})$$

is exact and $\widehat{N}$ is the closure of $\psi(N)$ in $\widehat{M}$, where $\psi : M \to \widehat{M}$ is the natural map.

(c) If the topology of M is defined by a decreasing chain of submodules

$$M_1 \supseteq M_2 \supseteq \cdots,$$

then

$$0 \to \widehat{N} \to \widehat{M} \to (\widehat{M/N}) \to 0$$

is exact. In other words,

$$\widehat{M/N} \cong \widehat{M}/\widehat{N}.$$

Definition 3.43. The *I-adic topology* of an R-module M is defined by taking the submodules I^nM as basic neighborhoods of 0. A filtration $\{M_n\}$ of M is an *I-filtration* if

$$IM_n \subseteq M_{n+1},$$

for all n and an *I-good filtration* if

$$IM_n = M_{n+1}$$

for all sufficiently large n. Thus $\{I^nM\}$ is an I-good filtration.

Lemma 3.44. *If $\{M_n\}_{n\in\mathbb{N}}$ and $\{M'_n\}_{n\in\mathbb{N}}$ are I-good filtrations of M, then they have bounded difference: that is, there exists an integer n_0 such that*

$$M_{n+n_0} \subseteq M'_n \quad \textit{and} \quad M'_{n+n_0} \subseteq M_n \quad \textit{for sufficiently large } n.$$

Hence all I-good filtrations determine the same topology on M, namely the I-adic topology.

Now, let M be an R-module with filtration $\{M_n\}$. The filtration determines a topology on M as discussed above, with the M_n forming a base for the neighbourhoods of 0. We have the following result.

Proposition 3.45 (see [2]). *If N is a submodule of M, then the closure of N is given by*

$$\overline{N} = \bigcap_{n=0}^{\infty}(N + M_n).$$

Proof. Let x be an element of M. Then x fails to belong to $\overline{N}$ iff some neighbourhood of x is disjoint from N, in other words, $(x + M_n) \cap N = \emptyset$ for some n. To justify the last step, note that if x is in $N + M_n$, then $x = y + z$, $y \in N$, $z \in M_n$. Thus

$$y = x - z \in (x + M_n) \cap N.$$

Conversely,if $y \in (x + M_n) \cap N$, then for some $z \in M_n$ we have $y = x - z$, so

$$x = y + z \in N + M_n. \qquad \square$$

Let $\{M_n\}$ be a filtration of the R-module M. Recalling the construction of the reals from the rationals, or the process of completing an arbitrary metric space, let us try to come up with something similar in this case. If we go far out in a Cauchy sequence, the difference between terms becomes small. Thus we can define a Cauchy sequence $\{x_n\}$ in

M by the requirement that for every positive integer r there is a positive integer N such that $x_n - x_m \in M_r$ for $n, m \geqslant N$. We identify the Cauchy sequence $\{x_n\}$ and $\{y_n\}$ if they get close to each other for large n. More precisely, given a positive integer r there exists a positive integer N such that $x_n - y_n \in M_r$ for all $n \geqslant N$. Notice that the condition $x_n - x_m \in M_r$ is equivalent to $x_n + M_r = x_m + M_r$. This suggests that the essential feature of the Cauchy condition is that the sequence is coherent with respect to the maps $\theta_n : M/M_n \to M/M_{n-1}$.

Definition 3.46 (see [2])**.** Let $\{M_n\}$ be the filtration of an R-module M. We define the *completion* of M by

$$\widehat{M} = \lim_{n}(M/M_n).$$

The functor that assigns the inverse limit to an inverse system of modules is left exact, and becomes exact under certain conditions.

Theorem 3.47. *Let $\{M'_n, \theta'_n\}, \{M_n, \theta_n\}$, and $\{M''_n, \theta''_n\}$ be inverse systems of modules, and assume that the diagram below is commutative with exact rows.*

$$\begin{tikzcd}
0 \arrow[r] & M'_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\theta'_{n+1}"'] & M_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\theta_{n+1}"'] & M''_{n+1} \arrow[r] \arrow[d, "\theta''_{n+1}"'] & 0 \\
0 \arrow[r] & M'_n \arrow[r, "f_n"] & M_n \arrow[r, "g_n"] & M''_n \arrow[r] & 0
\end{tikzcd}$$

Then the sequence

$$0 \to \lim_{n} M'_n \to \lim_{n} M_n \to \lim_{n} M''_n$$

is exact. Moreover, if θ'_n is surjective for all n, then

$$0 \to \lim_{n} M'_n \to \lim_{n} M_n \to \lim_{n} M''_n \to 0$$

is exact.

Proof. Let $M = \prod_n M_n$ and define an R-homomorphism $d_M : M \to M$ by $d_M(x_n) = (x_n - \theta_{n+1}(x_{n+1}))$. The kernel of d_M is the inverse limit of the M_n. Now the maps f_n and g_n induce $f = \prod f_n : M' = \prod M'_n \to M$ and $g = \prod g_n : M \to M'' = \prod M''_n$. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "d_{M'}"'] & M \arrow[r, "g"] \arrow[d, "d_M"'] & M'' \arrow[r] \arrow[d, "d_{M''}"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

We now apply the Snake lemma, which results in an exact sequence

$$0 \to \ker d_{M'} \to \ker d_M \to \ker d_{M''} \to \operatorname{coker} d_{M'},$$

proving the first assertion. Now, if θ'_n is surjective for all n, then $d_{M'}$ is surjective, and consequently the cokernel of $d_{M'}$ is 0. The second assertion follows immediately. □

Corollary 3.48 (see [2]). *Suppose that the sequence*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact. Let $\{M_n\}$ be a filtration of M, so that $\{M_n\}$ induces filtrations $\{M' \cap f^{-1}(M_n)\}$ and $\{g(M_n)\}$ on M' and M'' respectively. Then the sequence

$$0 \to \widehat{(M')} \to \widehat{M} \to \widehat{(M'')} \to 0$$

is exact.

Proof. Exactness of the given sequence implies that the diagram below is commutative with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & M'/(M' \cap f^{-1}(M_{n+1})) \arrow[r] \arrow[d, "\theta'_{n+1}"'] & M/M_{n+1} \arrow[r] \arrow[d, "\theta_{n+1}"'] & M''/g(M_{n+1}) \arrow[r] \arrow[d, "\theta''_{n+1}"'] & 0 \\
0 \arrow[r] & M'/(M' \cap f^{-1}(M_n)) \arrow[r] & M/M_n \arrow[r] & M''/g(M_n) \arrow[r] & 0
\end{tikzcd}$$

Since θ'_n is surjective for all n, therefore, Theorem 3.47 allows us to pass to the inverse limit. □

Remark 3.49. A filtration $\{M_n\}$ of an R-module M induces in a natural way a filtration $\{N \cap M_n\}$ on a given submodule N, and a filtration $\{(N \cap M_n)/N\}$ on the quotient module M/N. We have already noted this in Corollary 3.48 with f the inclusion map and g the canonical epimorphism.

Corollary 3.50 (see [2]). *Let $\{M_n\}$ be the filtration of an R-module M. Let $\widehat{M_n}$ be the completion of M_n with respect to the induced filtration on M_n. Then $\widehat{M_n}$ is a submodule of $\widehat{M}$ and*

$$\widehat{M}/\widehat{M_n} \cong M/M_n$$

for all n.

Proof. Apply Corollary 3.48 with $M' = M_n$ and $M'' = M/M_n$, to obtain the exact sequence

$$0 \to \widehat{M_n} \to \widehat{M} \to \widehat{M/M_n} \to 0.$$

Thus we may identify $\widehat{M_n}$ with a submodule of $\widehat{M}$, and

$$\widehat{M}/\widehat{M_n} \cong \widehat{M/M_n} = \widehat{M''}.$$

Now the m^{th} term of the induced filtration on M'' is

$$M''_m = (M_n + M_m)/M_n = M_n/M_n = 0$$

for $m \geqslant n$. Thus M'' has the discrete topology, so cauchy sequences (and coherent sequences) can be identified with single points. Therefore M'' is isomorphic to its completion, and we have

$$\widehat{M}/\widehat{M_n} \cong M/M_n$$

for all n. □

Definition 3.51. Two filtrations $\{M_n\}$ and $\{M'_n\}$ of a given R-module are said to be *equivalent* if they induce the same topology. For example, the filtrations $\{I^n N\}$ and $\{N \cap I^n M\}$ of the submodule N are equivalent. Since equivalent filtrations give rise to the same set of Cauchy sequences, it follows that completions of a given module with respect to equivalent filtrations are isomorphic.

3.7 Completion of coherent rings

Proposition 3.52. *Let R be a commutative coherent ring, I a finitely generated ideal and M a finitely generated R-module. Then all the I-good filtrations on M define the same I-adic topology.*

Proof. Let $\{M_n\}$ be an I-good filtration on M. As this filtration is exhaustive, every element of M belongs to one of the M_n and, M is finitely generated and the M_n are R-modules, there exists an integer n_1 such that $M_{n_1} = M$. On the other hand, let n_0 be such that $IM_n = M_{n+1}$ for $n \geqslant n_0$. Then for $n > n_0 - n_1$,

$$I^n M \subseteq M_{n+n_1} = I^{n+n_1-n_0} M_{n_0} \subseteq I^{n+n_1-n_0} M.$$ □

Let R be a ring, I a finitely generated ideal of R. Then we can form a graded ring

$$R^* = \bigoplus_{n=0}^{\infty} I^n.$$

Similarly, if M is an R-module and M_n is an I-filtration of M, then

$$M^* = \bigoplus_n M_n$$

is a graded R^*-module, since $I^m M_n \subseteq M_{m+n}$.

Lemma 3.53. *Let M be a coherent module over a coherent ring R, $I \lhd R$ finitely generated and $\{M_n\}$ an I-filtration of M. Suppose that each M_n is coherent. Then the following conditions are equivalent:*

(a) M^* is a finitely generated R^*-module.

(b) $\{M_n\}$ is an I-good filtration.

Proof. Each M_n is finitely generated, hence so is each $N_n = \bigoplus_{i=0}^n M_i$. This is a subgroup of M^* but not in general an R^*-submodule. However, it generates one, namely

$$M_n^* = M_0 \oplus \ldots \oplus M_n \oplus IM_n \oplus I^2 M_n \oplus \ldots$$

Since, N_n is finitely generated as an R-module, M_n^* is finitely generated as an R^*-module. By definition, M^* is the union of the M_n^* over all $n \geqslant 0$. Therefore, M^* is finitely generated over R^* if and only if $M^* = M_m^*$ for some m, In other words,

$$M_{m+k} = I^k M_m$$

for all $k \geqslant 1$. Hence, $\{M_n\}$ is an I-good filtration. $\square$

Proposition 3.54. *Let M be a coherent module over a coherent ring R, $I \lhd R$ finitely generated and assume that $\{M_n\}$ is an I-good filtration of M such that each M_n is coherent. If N is a coherent submodule of M, then the filtration $\{N_n = N \cap M_n\}$ induced by M on N is also an I-good filtration.*

Proof. We have

$$I(N \cap M_n) \subseteq IN \cap IM_n \subseteq N \cap M_{n+1}.$$

Hence, $\{N \cap M_n\}$ is an I-filtration. Therefore, it defines a graded R^*-module N^* which is a submodule of M^* and therefore finitely generated. Now, using Lemma 3.53, it follows that $N \cap M_n$ is an I-good filtration of N. $\square$

Taking $M_n = I^n M$, we obtain a version of the Artin-Rees Lemma.

Corollary 3.55. *There exists an integer c such that for all $n \geqslant c$*

$$(I^n M) \cap N = I^{n-c}(I^c M \cap N).$$

Theorem 3.56. *Let M be a coherent module over a coherent ring R, $I \lhd R$ finitely generated and N a coherent submodule of M. Then the I-adic topology of N coincides with the topology induced by the I-adic topology of M on N.*

Proof. By Corollary 3.55, for $n > c$ we have,

$$I^n N \subseteq I^n M \cap N \subseteq I^{n-c} N.$$

The topology of N as a subspace of M is the linear topology defined by $\{I^n M \cap N\}_{n \geqslant 1}$ and the above formula says that this defines the same topology as $\{I^n N\}_{n \geqslant 1}$. $\square$

Now let $\widehat{M} = \lim_n M/I^n M$ be the *I-adic completion* of an R-module M. In particular $\widehat{R} = \lim_n R/I^n R$.

Theorem 3.57. *Let R be a coherent ring, $I \lhd R$ a finitely generated ideal and M a coherent R-module. Then*

$$M \otimes_R \widehat{R} \cong \widehat{M}.$$

Proof. Since, the I-adic completion of an exact sequence of finitely generated R-modules is again exact. Now, given M, let

$$R^p \to R^q \to M \to 0$$

be an exact sequence; since completion commutes with direct sums, the commutative diagram

$$\begin{tikzcd}
\widehat{R}^p \arrow[r] & \widehat{R}^q \arrow[r] & \widehat{M} \arrow[r] & 0 \\
R^p \otimes \widehat{R} \arrow[r] \arrow[u] & R^q \otimes \widehat{R} \arrow[r] \arrow[u] & M \otimes \widehat{R} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

has exact rows. The vertical rows are the natural maps; the two left hand arrows are obviously isomorphisms, and hence, the right hand arrow is an isomorphism proving

$$M \otimes_R \widehat{R} \cong \widehat{M}. \qquad \square$$

Theorem 3.58. *Let R be a coherent ring, $I \lhd R$ be finitely generated. Then $\widehat{R}$ is flat over R.*

Proof. By Theorem 3.57, we know

$$M \otimes_R \widehat{R} \cong \widehat{M}.$$

So, to prove $\widehat{R}$ is flat over R, it is enough to show that for any finitely generated ideal J of R, $J \otimes \widehat{R} \to \widehat{R}$ is injective. But

$$J \otimes \widehat{R} \cong \widehat{J}$$

and so, $\widehat{J} \to \widehat{R}$ is injective, proving $J \otimes \widehat{R} \to \widehat{R}$ is injective. i.e, $\widehat{R}$ is flat over R. □

Proposition 3.59. *Let $I \lhd R$ be finitely generated where R is a coherent ring. Then*

(a) $\widehat{I} = \widehat{R}I \cong \widehat{R} \otimes_R I$

(b) $\widehat{(I^n)} \cong (\widehat{I})^n$

(c) $I^n/I^{n+1} \cong \widehat{I}^n/\widehat{I}^{n+1}$

(d) $\widehat{I}$ *is contained in the Jacobson radical of* $\widehat{R}$.

Proof.

(a) By Theorem 3.57, we know

$$\widehat{R} \otimes_R I \cong \widehat{I}$$

It follows that the map $\widehat{R} \otimes_R I \to \widehat{I}$, whose image is $\widehat{R}I$, is an isomorphism.

(b) Apply (a) to I^n, we get

$$\widehat{(I^n)} = \widehat{R}I^n \cong (\widehat{R}I)^n = (\widehat{I})^n$$

(c) Since $R/I^n \simeq \widehat{R}/\widehat{I}^n$ proving that $I^n/I^{n+1} \cong \widehat{I}^n/\widehat{I}^{n+1}$.

(d) Since $\widehat{R}$ is complete for its $\widehat{I}$-adic topology, hence for any $x \in \widehat{I}$

$$(1-x)^{-1} = 1 + x + x^2 + \cdots$$

converges in $\widehat{R}$, so $(1-x)$ is a unit. This shows that $\widehat{I}$ is contained in $J(R)$ of $\widehat{R}$.

□

Theorem 3.60. *Let $(R, \mathfrak{m})$ be a coherent local ring and $\mathfrak{m}$ be finitely generated. If $\hat{M_{\mathfrak{m}}}$ is the $\mathfrak{m}$-adic completion of an R-module M, then* *(a)* $(M^{\hat{}}_{\mathfrak{m}})^{\hat{}}_{\mathfrak{m}} = M^{\hat{}}_{\mathfrak{m}}$.

(b) If M is finitely generated then $M^{\hat{}}_{\mathfrak{m}} = R^{\hat{}}_{\mathfrak{m}} \otimes_R M$.

(c) $R^{\hat{}}_{\mathfrak{m}}$ *is flat over* R.

(d) If $f : M \to N$ is an epimorphism, then so is the induced map $M^{\hat{}}_{\mathfrak{m}} \to N^{\hat{}}_{\mathfrak{m}}$.

Proof.

(a) Because $M^{\hat{}}_{\mathfrak{m}}$ is defined as the inverse limit of $\{M/\mathfrak{m}^k M\}$, there is a map

$$\pi_k : M^{\hat{}}_{\mathfrak{m}} \to M/\mathfrak{m}^k M.$$

Because the maps in the inverse system are epimorphisms, so the map π_k is also an epimorphism. It is clear that π_k factors through an epimorphism

$$M^{\hat{}}_{\mathfrak{m}}/\mathfrak{m}^k M^{\hat{}}_{\mathfrak{m}} \to M/\mathfrak{m}^k M.$$

On the other hand, the obvious map $M \to M^{\hat{}}_{\mathfrak{m}}$ induces a map

$$M/\mathfrak{m}^k M \to M^{\hat{}}_{\mathfrak{m}}/\mathfrak{m}^k M^{\hat{}}_{\mathfrak{m}}.$$

One can check that these maps are mutually inverse, so that

$$(M^{\hat{}}_{\mathfrak{m}})^{\hat{}}_{\mathfrak{m}} = \lim_k M^{\hat{}}_{\mathfrak{m}}/\mathfrak{m}^k M^{\hat{}}_{\mathfrak{m}} = \lim_k M/\mathfrak{m}^k M = M^{\hat{}}_{\mathfrak{m}}.$$

(b) and (c) are clear from Theorem 3.57 and Theorem 3.58 respectively.

(d) It is easy to see that the induced maps $M/\mathfrak{m}^n M \to N/\mathfrak{m}^n N$ and $\mathfrak{m}^n M \to \mathfrak{m}^n N$ are epimorphisms, and thus that $\mathfrak{m}^n M/\mathfrak{m}^{n+1} M \to \mathfrak{m}^n N/\mathfrak{m}^{n+1} N$ is also an epimorphism. Let K_n be the kernel of the map $M/\mathfrak{m}^n M \to N/\mathfrak{m}^n N$. We get a diagram as follows, in which the columns and the last two rows are exact

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & \mathfrak{m}^n M/\mathfrak{m}^{n+1} M \arrow[r] \arrow[d] & \mathfrak{m}^n N/\mathfrak{m}^{n+1} N \arrow[d] \\
K_{n+1} \arrow[r] \arrow[d] & M/\mathfrak{m}^{n+1} M \arrow[r] \arrow[d] & N/\mathfrak{m}^{n+1} N \arrow[d] \\
K_n \arrow[r] & M/\mathfrak{m}^n M \arrow[r] & N/\mathfrak{m}^n N
\end{tikzcd}$$

It follows from the snake lemma that the map $K_{n+1} \to K_n$ is epimorphism, and thus that $\lim^1_n K_n = 0$. We therefore have a short exact sequence

$$\lim_n K_n \to \lim_n M/\mathfrak{m}^n M \to \lim_n N/\mathfrak{m}^n N$$

so $M^{\hat{}}_{\mathfrak{m}} \to N^{\hat{}}_{\mathfrak{m}}$ is an epimorphism as claimed.

□

Note that completion does not preserve monomorphisms and is not right exact.

Proposition 3.61. *Let $(R, \mathfrak{m})$ be a coherent local ring with $\mathfrak{m}$ a finitely generated ideal. Then the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a local ring with maximal ideal $\widehat{\mathfrak{m}}$.*

Proof. Since $R/\mathfrak{m} \cong \widehat{R}/\widehat{\mathfrak{m}}$, the quotient $\widehat{R}/\widehat{\mathfrak{m}}$ is a field, hence $\widehat{\mathfrak{m}}$ is maximal. But $\widehat{\mathfrak{m}}$ is contained in $J(\widehat{R})$ of $\widehat{R}$, so $\widehat{\mathfrak{m}}$ is the unique maximal ideal. Therefore $\widehat{R}$ is a local ring. □

Chapter 4

DERIVED FUNCTORS AND COMPLETIONS

4.1 Derived functors of I-adic completion

Recall the theory of local (co)homology and the derived functors of the completion functor. One can define the left derived functors of any additive functor F. One only needs F to be right exact in order to prove that the zeroth derived functor of F is F but this is false in the present context. J.P.C. Greenlees and J.P. May in [8] studied the left derived functors $L_k^I M$ of completion in the following sense.

Let R be a Noetherian regular local ring with a unique maximal homogeneous ideal $\mathfrak{m}$, which is generated by a regular sequence of homogeneous elements $(x_0, \cdots, x_{n-1})$.

For any $x \in R$ we let $K^\bullet(x)$ denote the complex $R \to R[1/x]$, with R in degree 0 and $R[1/x]$ in degree 1. We also write

$$K^\bullet(\mathfrak{m}) = K^\bullet(x_0) \otimes \cdots \otimes K^\bullet(x_{n-1});$$

there is then a natural map $K^\bullet(\mathfrak{m}) \to R$. Note that $K^\bullet(\mathfrak{m})$ is complex of flat modules. In fact, $K^\bullet(\mathfrak{m})$ is the finite acyclisation of R determined by $R/\mathfrak{m}$ in the derived category of R, so it is determined by the ideal $\mathfrak{m}$ up to quasi-isomorphism over R. To see this, write $K^\bullet(x)$ as the colimit of the complexes

$$R \xrightarrow{x^k} R$$

and tensor these complexes together. We also write $PK^\bullet(x)$ for the complex of projectives

$$R \oplus R[t] \xrightarrow{(1,tx-1)} R[t],$$

which is quasi-isomorphic to $K^\bullet(x)$. Here the map

$$tx - 1 : R[t] \to R[t]$$

is the $R[t]$-module map which takes 1 to $tx - 1$. We write

$$PK^\bullet(\mathfrak{m}) = PK^\bullet(x_0) \otimes \cdots \otimes PK^\bullet(x_{n-1}),$$

which is quasi-isomorphic to $K^\bullet(\mathfrak{m})$ by flatness. The local homology and cohomology groups of a module M are

$$L_* M = H_*^{\mathfrak{m}}(M) = H_*(\mathrm{Hom}(PK^\bullet(\mathfrak{m}), M))$$

$$H^*_{\mathfrak{m}}(M) = H^*(PK^\bullet(I) \otimes M) = H^*(K^\bullet(I) \otimes M).$$

The last equality again uses the flatness of $K^\bullet(\mathfrak{m})$. Because R is a regular local ring, we have

$$H^k_{\mathfrak{m}}(R) = \begin{cases} 0 & k < n \\ R/(x_0^\infty, \cdots, x_{n-1}^\infty) & k = n \end{cases}$$

Let R be a commutative ring (it is not assumed here that R is Noetherian) and M be an R-module. Let $I \lhd R$ and $\hat{M_I}$ be the I-adic completion of M. The I-adic completion is an additive covariant functor from the category of $\mathcal{M}_R$. Let $L_i^I(M)$ be the i-th left derived module of $\hat{M_I}$.

Remark 4.1.

(a) Since the tensor functor is not left exact and the inverse limit is not right exact on the category of R-modules, the functor of the I-adic completion is neither left nor right exact. Therefore, in general, $L_0^I(M) \neq \hat{M_I}$. However, L_0^I is a right exact functor and its left derived functor for $i > 0$ are the same as those of the functor of I-adic completion. Let

$$0 \to N \xrightarrow{f} P \xrightarrow{g} M \to 0$$

be a short exact sequence of R-modules with P projective. Then we get a sequence

$$\hat{N_I} \xrightarrow{\hat{f_I}} \hat{P_I} \xrightarrow{\hat{g_I}} \hat{M_I} \to 0,$$

which is not necessarily exact. But $\mathrm{im}(\hat{f_I}) \subseteq \ker(\hat{g_I})$ and $\hat{g_I}$ is surjective, so

$$L_0^I(M) \cong \hat{P_I}/\operatorname{im}(\hat{f_I}) \quad \text{and} \quad \hat{M_I} \cong \hat{P_I}/\ker(\hat{g_I}).$$

Therefore, there is a natural epimorphism $\varphi_M : L_0^I(M) \to \hat{M_I}$.

(b) Two ideals I and J of the ring R are called *radically equivalent* if there are positive integer n and m such that $I^n \subseteq J$ and $J^m \subseteq I$. If the ideals I and J are radically equivalent then $\hat{M_I} \cong \hat{M_J}$ for all R-modules M. Therefore we have

$$L_i^I(M) \cong L_i^J(M) \quad \text{for all} \quad i \geqslant 0.$$

(c) Suppose that R is a Noetherian ring and M a finitely generated R-module. Since the functor is exact on finitely generated R-modules, we get $L_i^I(M) = 0$ for all $i > 0$ and $L_0^I(M) \cong \hat{M_I}$.

In general, the natural epimorphism φ_M defined in Remark 4.1(a) is not isomorphism. The following theorem gives us a condition for φ_M to be an isomorphism.

Theorem 4.2 (see [6, Theorem 2.3])**.** *Let M be an R-module and $I \lhd R$. Suppose that the system $\{I^t M\}$ is stationary, i.e. there exists a positive integer n such that $I^t M = I^n M$ for all $t \geqslant n : n \in \mathbb{N}$. Then the natural epimorphism $\varphi_M : L_0^I(M) \to \hat{M_I}$ is an isomorphism.*

Let $(R, \mathfrak{m})$ be a commutative Noetherian regular local ring, and let $n = \dim R$. We consider its left derived functors $L_s = L_s^{\mathfrak{m}}$ $(s \geqslant 0)$.

Theorem 4.3. *Let R be a commutative Noetherian regular local ring with a maximal ideal $\mathfrak{m}$, and let $n = \dim R$. Let M be an R-module. Then*

(a) There are natural maps

$$M \xrightarrow{\eta_M} L_0 M \xrightarrow{\varepsilon_M} \hat{M_{\mathfrak{m}}}.$$

Moreover, ε is an epimorphism, and the composite $M \to \hat{M_{\mathfrak{m}}}$ is the obvious map.

(b) There is a short exact sequence

$$0 \to \lim_k{}^1 \operatorname{Tor}_{s+1}^R(R/\mathfrak{m}^k, M) \to L_s M \to \lim_k \operatorname{Tor}_s^R(R/\mathfrak{m}^k, M) \to 0.$$

In particular, there is a short exact sequence

$$0 \to \lim_k{}^1 \operatorname{Tor}_1^R(R/\mathfrak{m}^k, M) \to L_0 M \to \hat{M_{\mathfrak{m}}} \to 0.$$

(c) For any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

there is a long exact sequence

$$L_{k+1} M'' \to L_k M' \to L_k M \to L_k M'' \to L_{k-1} M'.$$

(d) *There is a natural isomorphism*

$$L_sM = \mathrm{Ext}_R^{n-s}(H^n_{\mathfrak{m}}(R), M).$$

Moreover, both sides vanish if $s < 0$ or $s > n$. Thus L_0 is right exact and L_n is left exact.

(e) *If M is projective then ε_M is an isomorphism.*

Proof. See [18, Theorem A.2]. □

4.2 L-complete modules

Let $(R, \mathfrak{m})$ be a commutative Noetherian regular local ring, and let $n = \dim R$.

Definition 4.4 (see [18, A.5])**.** An R-module M is *L-complete* if $\eta_M : M \to L_0M$ is an isomorphism. We write $\widehat{\mathcal{M}}_R$ for the full subcategory of $\mathcal{M}_R$ consisting of L-complete modules.

Proposition 4.5. *Let R be a commutative Noetherian regular local ring and let M, N be R-modules with M finitely generated. Then there is a natural isomorphism*

$$M \otimes_R L_0N \to L_0(M \otimes_R N).$$

In particular,

$$L_0M \cong M^{\hat{}}_{\mathfrak{m}} \cong \widehat{R} \otimes_R M,$$

$$R/\mathfrak{m}^k \otimes_R L_0N \cong R/\mathfrak{m}^k \otimes_R N = N/\mathfrak{m}^kN.$$

Hence, if N is a bounded $\mathfrak{m}$-torsion module then it is L-complete.

Proof. See [3, Prop 1.1]. □

Another analogue of Nakayama's Lemma provided by [3, Proposition 1.2] is.

Proposition 4.6 ([3, Proposition 1.2])**.** *For $M \in \widehat{\mathcal{M}}_R$,*

$$M = \mathfrak{m}M \quad \Longrightarrow \quad M = 0.$$

This can be used to prove many analogues of standard results in the theory of finitely generated modules over commutative rings.

Corollary 4.7 ([3, Corollary 1.3]). *Let $M \in \widehat{\mathcal{M}}_R$ and suppose that $N \subseteq M$ is the image of a morphism $N' \to M$ in $\widehat{\mathcal{M}}_R$. Then*

$$M = N + \mathfrak{m}M \quad \Longrightarrow \quad N = M.$$

Corollary 4.8 ([3, Corollary 1.4]). *Let $M \in \widehat{\mathcal{M}}_R$ and suppose that F is a free module for which there is an isomorphism $F/\mathfrak{m}F \cong M/\mathfrak{m}M$. Then there is an epimorphism $L_0F \to M$.*

Proposition 4.9 ([3, Proposition 1.6]). *Let M, N be L-complete R-modules, when N is a finitely generated $\mathfrak{m}$-torsion module. Then*

$$\widehat{\mathrm{Tor}}^R_*(M,N) \cong \mathrm{Tor}^R_*(M,N) \cong \widehat{\mathrm{Tor}}^R_*(N,M).$$

When N is a finitely generated $\mathfrak{m}$-torsion module, we may also consider the composite functor $\mathcal{M}_R \to \widehat{\mathcal{M}}_R$ for which

$$M \mapsto L_0(N \otimes_R M).$$

Since

$$L_0(N \otimes_R M) = N \otimes_R L_0M = N \otimes_R M,$$

this functor has for its left derived functors $\mathrm{Tor}^R_*(N, -)$ and there is an associated composite functor spectral sequence.

Proposition 4.10 ([3, Proposition 1.7]). *Let N be a finitely generated $\mathfrak{m}$-torsion module. Then for each R-module M, there is a natural first quadrant spectral sequence*

$$E^2_{s,t} = \widehat{\mathrm{Tor}}^R_s(N, L_tM) = \mathrm{Tor}^R_s(N, L_tM) \Longrightarrow \mathrm{Tor}^R_{s+t}(N,M).$$

Lemma 4.11 ([3, Lemma 1.8]). *Let M be a flat R-module. Then*

$$L_sM = \begin{cases} M\hat{_{\mathfrak{m}}} & \text{if } s=0 \\ 0 & \text{otherwise,} \end{cases}$$

and L_0M is pro-free.

It is also important that if M is a finitely generated R-module then it has bounded $\mathfrak{m}$-torsion, hence by [8, Theorem 1.9], $L_0M = M\hat{_{\mathfrak{m}}}$ and $L_sM = 0$ for $s > 0$. More generally, If F is a free module, then $F \otimes M$ has bounded $\mathfrak{m}$-torsion, so

$$L_s(F \otimes_R M) = F\hat{_{\mathfrak{m}}} \otimes_R M \quad if \quad s = 0,$$

Proposition 4.12 ([3, Proposition 1.9]). *Let $P \in \widehat{\mathcal{M}}_R$ be L-flat, then P is pro-free.*

In general, tensoring with a pro-free module need not be left exact on $\widehat{\mathcal{M}}_R$ as is shown by an example in [3, Appendix B].

4.3 L-complete modules for super regular coherent local ring

Now, let R be a super regular coherent local ring as in Definition 3.36 and $n = \mathrm{gl.dim}\, R$. We write $\mathcal{M}'_R$ for the category of coherent R-modules, and $\widehat{\mathcal{M}'}_R$ for the subcategory of L-complete modules.

Theorem 4.13.

(a) *For any $M \in \mathcal{M}'_R$, the modules $M^{\hat{}}_{\mathfrak{m}}$ and $L_k M$ lie in $\widehat{\mathcal{M}'}_R$. In particular, $L_0^2 M = L_0 M$.*

(b) *If $M \in \widehat{\mathcal{M}'}_R$, then $L_k M = 0$ for $k > 0$.*

(c) *$L_0 M = 0$ if and only if $M^{\hat{}}_{\mathfrak{m}} = 0$ if and only if $M = \mathfrak{m}M$.*

(d) *If $M \in \widehat{\mathcal{M}'}_R$ and $M = \mathfrak{m}M$, then $M = 0$.*

(e) *$\widehat{\mathcal{M}'}_R$ is an abelian subcategory of $\mathcal{M}'$, which is closed under extensions.*

(f) *$L_0 : \mathcal{M}' \to \widehat{\mathcal{M}'}_R$ is left adjoint to the inclusion $\widehat{\mathcal{M}'}_R \to \mathcal{M}'$.*

(g) *If $\{M_k\}$ is a collection of L-complete modules, then $\prod_k M_k$ is L-complete. If they form an inverse system then $\lim_k M_k$ is L-complete. If they form a tower then ${\lim_k}^1 M_k$ is also L-complete.*

Proof.

(a) Suppose $N = M^{\hat{}}_{\mathfrak{m}}$ or $N = L_k M$ then $L_0 N = N$ and $L_k N = 0$ for $k > 0$ by [8, Theorem 4.1]. It follows that the modules $M^{\hat{}}_{\mathfrak{m}}$ and $L_k M$ are L-complete.

(b) It also follows that if M is L-complete and $k > 0$ then $L_k M = L_k L_0 M = 0$.

(c) We have epimorphisms

$$L_0 M \to M^{\hat{}}_{\mathfrak{m}} \to M/\mathfrak{m}M,$$

so

$$L_0M = 0 \quad \Longrightarrow \quad M^{\hat{}}_{\mathfrak{m}} = 0 \quad \Longrightarrow \quad M = \mathfrak{m}M.$$

Suppose that $M = \mathfrak{m}M$. Then $M = \mathfrak{m}^k M$ for all k by induction on k, so $M^{\hat{}}_{\mathfrak{m}} = 0$. We will also prove that $L_0M = 0$. Since $M = \mathfrak{m}^k M$, we have $R/\mathfrak{m}^k \otimes M = 0$ for all k. Using the short exact sequences

$$\mathfrak{m}^k/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^k$$

and the fact that $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a free module over $R/\mathfrak{m}$, we get an exact sequence

$$\mathrm{Tor}_1^R(\mathfrak{m}^k/\mathfrak{m}^{k+1}, M) \to \mathrm{Tor}_1^R(R/\mathfrak{m}^{k+1}, M) \to \mathrm{Tor}_1^R(R/\mathfrak{m}^k, M) \to 0.$$

From this we see that the maps in the tower

$$\cdots \leftarrow \mathrm{Tor}_1^R(R/\mathfrak{m}^k, M) \leftarrow \mathrm{Tor}_1^R(R/\mathfrak{m}^{k+1}, M) \leftarrow \cdots$$

are surjective, so that

$$\lim_k{}^1 \mathrm{Tor}_1^R(R/\mathfrak{m}^k, M) = 0.$$

Now, using the fact that $M^{\hat{}}_{\mathfrak{m}} = 0$ and that

$$0 \to \lim_k{}^1 \mathrm{Tor}_1^R(R/\mathfrak{m}^k, M) \to L_0M \to M^{\hat{}}_{\mathfrak{m}} \to 0,$$

is exact, we get that $L_0M = 0$ as claimed.

(d) If $M \in \widehat{\mathcal{M}'}_R$ and $M = \mathfrak{m}M$, then $M = L_0M = 0$ by (c).

(e) First, we claim that the image of any map $f : M' \to M''$ of L-complete modules is L-complete. To see this, factor f as a composite $M' \xrightarrow{q} M \xrightarrow{j} M''$, with q epimorphism and j monomorphism, so that M is the image of f. We have a diagram as follows:

$$\begin{tikzcd}
M' \arrow[r, "q"] \arrow[d, "\eta'"', "\cong"] & M \arrow[r, "j"] \arrow[d, "\eta"'] & M'' \arrow[d, "\eta''"', "\cong"] \\
L_0M' \arrow[r, "L_0q"'] & L_0M \arrow[r, "L_0j"'] & L_0M''
\end{tikzcd}$$

Note that L_0q is epimorphism because L_0 is right exact. Because the left square commutes, we see that η is epimorphism; because the right square commutes, we see that it is monomorphism. Thus η is an isomorphism, and M is L-complete.

Next suppose that

$$0 \to N' \to N \to N'' \to 0$$

is a short exact sequence, and that any two of the terms are L-complete. We can easily see that the third of these is also L-complete because of the long exact sequence relating the L_k-groups of N',N and N''. Thus $\mathcal{M}'_R$ is closed under extensions, and under kernels and cokernels of epimorphisms and monomorphisms. For any map $f : M' \to M''$, we have $\ker f = \ker q$ and $\operatorname{coker} f = \operatorname{coker} j$. It follows that $\mathcal{M}'_R$ is closed under kernels and cokernels, and thus that it is Abelian.

(f) Suppose that N is L-complete and M is arbitrary. We need to show that $\eta : M \to L_0M$ induces an isomorphism $\eta_M^* : \mathrm{Hom}_R(L_0M, N) \to \mathrm{Hom}_R(M, N)$. There is a map $\lambda : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(L_0M, N)$, defined by the commutativity of the following diagram:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\eta_M"'] & N \arrow[d, "\cong"', "\eta_N"] \\ L_0M \arrow[r, "L_0 f"'] \arrow[ur, "\lambda(f)"] & L_0N \end{tikzcd}$$

It is clear that $\eta_M^* \lambda(f) = f$, so that η_M^* is epimorphism. Suppose that $f \in \ker(\eta_M^*)$, so that f can be factored as $L_0M \xrightarrow{q} M' \xrightarrow{g} N$, where M' is the cokernel of η_M. By the above argument, we have an epimorphism $\eta_M'^* : \mathrm{Hom}(L_0M', N) \to \mathrm{Hom}(M', N)$. However, because L_0 is right exact and idempotent, we see that $L_0M' = 0$. It follows that $g = 0$ and thus $f = 0$. Thus $\eta_M^* : \mathrm{Hom}(L_0M, N) \to \mathrm{Hom}(M, N)$ is an isomorphism, as required.

(g) It is easy to see from the definitions that $H_s^{\mathfrak{m}}(\prod_k M_k) = \prod_k H_s^{\mathfrak{m}}(M_k)$. It follows that a product of L-complete modules is L-complete. If the modules $\{M_k\}$ form an inverse system involving various maps $u : M_k \to M_l$ then $\lim_k M_k$ is the kernel of a map $\prod_k M_k \to \prod_l M_l$, so it is L-complete by (d). If the inverse system is a tower, then $\lim_k^1 M_k$ is the cokernel of a map $\prod_k M_k \to \prod_k M_k$, and thus is L-complete.

□

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