Quivers of sections on toric Deligne-Mumford stacks

by

Tarig M H Abdelgadir

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Abstract

Starting from a collection of line bundles on a projective toric DM stack $\mathcal{X}$, we introduce a stacky analogue of the classical linear series. Our first main result extends work of King by building moduli stacks of refined representations of labelled quivers. We associate one such stack to any collection of line bundles on $\mathcal{X}$ to obtain our notion of a stacky linear series; as in the classical case, $\mathcal{X}$ maps to the ambient stack by evaluating sections of line bundles in the collection. As a further application, we describe a finite sequence of GIT wall crossings between $[\mathbb{A}^n/G]$ and $G\text{-Hilb}(\mathbb{A}^n)$ for $G \subset \mathrm{SL}(n, \Bbbk)$ for $n \leq 3$.

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Declaration

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 2 and Section 5.2 cover some background and preliminaries. The rest of this thesis is the author's original work unless explicitly mentioned.

Chapter 1

Introduction

Motivated by Olsson-Starr [OS03] among others, Kresch [Kre09] introduced the notion of a projective Deligne-Mumford (DM) stack. A DM stack $\mathcal{X}$ is said to be projective if it has a projective coarse moduli space X and a generating sheaf. One may think of a generating (or π-very ample) sheaf as a very ample sheaf relative to the morphism $\pi : \mathcal{X} \to X$, or more loosely, as a sheaf that allows one to lift the projectivity of X to $\mathcal{X}$.

The classical linear series construction is a key tool in studying projective varieties. However, one cannot naively extend this construction to stacks. Indeed, requiring a line bundle to be both very ample on the coarse moduli space and π-very ample is too restrictive; the stack must be an algebraic space, forcing all stabilizers to be trivial. One could sidestep this issue by considering sections of more than one line bundle. In the case where $\mathcal{X}$ has cyclic stabilizers, the approach adopted by Abramovich-Hassett [AH] uses sections of tensor powers of a single line bundle L to produce closed immersions into weighted projective stacks $\mathcal{X} \hookrightarrow \mathbb{P}(\bigoplus_{j=n}^{m} \Gamma(\mathcal{X}, L^{\otimes j}))$ for some $n, m \in \mathbb{N}$.

For $\mathcal{X}$ a smooth toric DM stack, this thesis gives an alternative stacky analogue to the linear series construction that generalizes the Abramovich-Hassett construction. We start with a finite collection of line bundles $\mathscr{L}$ and use the quiver of sections as defined by Craw-Smith [CS08] to package efficiently the sections of the line bundles in $\mathscr{L}$. For our ambient space, we introduce moduli stacks $\mathcal{M}_\theta(Q, \mathrm{div})$ of quiver representation-like objects and produce rational maps $\psi_\theta : \mathcal{X} \dashrightarrow \mathcal{M}_\theta(Q, \mathrm{div})$. This construction puts no constraints on the stabilizers. Moreover, the efficiency of the quiver of sections allows for a more streamlined stacky analogue.

Our construction is not limited to projective stacks. In fact for $G \subset \mathrm{GL}(n, \Bbbk)$ a finite abelian group, it recovers the stack quotient $[\mathbb{A}^n/G]$ from the McKay quiver. Under some constraints on G, Craw-Ishii [CI04b] show that the McKay quiver allows

us to move between projective crepant resolutions of $\mathbb{A}^n/G$ by a finite sequence of wall crossings. When $G \subset \mathrm{SL}(n, \Bbbk)$, one may think of the stack $[\mathbb{A}^n/G]$ as a noncommutative crepant resolution of $\mathbb{A}^n/G$. This is because its coordinate ring, as defined by Chan-Ingalls [CI04a], is a noncommutative crepant resolution of $\mathbb{A}^n/G$ in the sense of Van den Bergh [VdB04]. Therefore it is natural to ask whether one can introduce a quiver theoretic construction that allows us to move between a crepant resolution of $\mathbb{A}^n/G$, say Nakamura's G-Hilb$(\mathbb{A}^n)$, and the stack $[\mathbb{A}^n/G]$ by crossing finitely many walls. A slight adaptation of our construction gives an affirmative answer to this question, putting G-Hilb$(\mathbb{A}^n)$ and $[\mathbb{A}^n/G]$ on the same footing.

We now summarize the contents of the thesis in more detail. Motivated by the natural labelling of the quiver of sections on a toric variety by torus-invariant divisors, we define the notion of a labelled quiver. A *labelled quiver* is a quiver Q along with a map of sets $l : Q_1 \to \mathbb{Z}^d$. Naively, one wishes to define a 'representation' of a labelled quiver as a representation of the underlying quiver for which any two paths with the same label are represented by the 'same' linear map. One stumbles when trying to force this on linear maps representing two paths that have the same labels but don't share the same head and tail. The reason for this is that our proposed equations on the representation space are not homogeneous with respect to the change of basis action. We bypass this issue by introducing new 'homogenizing' parameters to the representation space, that homogenize every equation of paths induced by the label. A *refined representation* of a labelled quiver (Q, l) is a representation of the underlying quiver, together with a choice of nonzero homogenizing parameters. Given a refined representation W and weight $\theta \in K_0(\Bbbk Q\text{-mod})^\vee$, we follow King [Kin94] to define a notion of θ-stability on W. For a weight $\theta \in K_0(\Bbbk Q\text{-mod})^\vee$ defined by a character χ_θ of $\mathrm{PGL}(\alpha)$ (the group acting faithfully on the refined representation space), the main result of Chapter 3 relates GIT χ_θ-stability to θ-stability.

Theorem 3.8. *Let χ_θ be a character of $\mathrm{GL}(\alpha)$ and θ the corresponding element of $K_0(mod\text{-}\Bbbk Q)^\vee$. A refined quiver representation W is θ-semistable (resp. θ-stable) if and only if the corresponding point in $\mathcal{R}(Q, l, \alpha)$ is χ_θ-semistable (resp. χ_θ-stable) with respect to action of $\mathrm{GL}(\alpha)$.*

This allows us to introduce families of θ-semistable refined representations, which in turn enables us to define moduli stacks $\mathcal{M}_\theta(Q, l, \alpha)$ of refined representations, given some dimension vector α. The stacks $\mathcal{M}_\theta(Q, l, \alpha)$ form the ambient stacks in our construction.

Now take $\mathcal{X}$ to be a smooth projective toric stack with trivial generic stabilizers. For a given collection of line bundles $\mathscr{L}$ on $\mathcal{X}$, we use techniques very similar to those in [CS08] to define a labelled quiver of sections (Q, div) and give a rational map

$\psi_\theta : \mathcal{X} \dashrightarrow \mathcal{M}_\theta(Q, \mathrm{div}, \alpha)$ where $\alpha = (1, \ldots, 1)$. As in the classical linear series case, when ψ_θ is a morphism the tautological line bundles on $\mathcal{M}_\theta(Q, \mathrm{div}, \alpha)$ pull-back to recover the collection $\mathcal{L}$. Checking whether or not there exists a stability condition θ for which ψ_θ is a morphism can be tedious, hence we introduce a sufficient condition that is straightforward to check. We also explicitly describe the image of ψ_θ and address the question of representability of the morphism ψ_θ. Let $\mathcal{L}_{\mathrm{bpf}}$ denote the collection of line bundles

$$\{L_i^\vee \otimes L_j \mid L_i, L_j \in \mathcal{L} \text{ and } L_i^\vee \otimes L_j \text{ is base-point free}\}$$

we show the following,

Theorem 4.14. *If* $\mathrm{rank}(\mathbb{Z}\mathcal{L}) = \mathrm{rank}(\mathbb{Z}\mathcal{L}_{\mathrm{bpf}})$ *then* $\mathcal{L}$ *is base-point free, i.e. there exists a stability condition* θ *such that* $\psi_\theta : \mathcal{X} \to \mathcal{M}_\theta(Q, \mathrm{div})$ *is a morphism.*

Theorem 4.18. *A morphism* ψ_θ *is representable if and only if* $\bigoplus_{j=1}^r L_j$ *is* π*-ample.*

For $\mathcal{X}$ smooth toric, the Abramovich-Hassett construction may be recovered.

Remark 4.10. Given a polarizing line bundle L on $\mathcal{X}$, the Abramovich-Hassett construction, for $n = 0$, is recovered by applying our machinery to the collection

$$\mathcal{L} = (\mathcal{O}_\mathcal{X}, L, L \otimes L^{\otimes 2}, \ldots, L^{\otimes m(m+1)/2})$$

and if necessary, working with an 'incomplete' quiver of sections. An incomplete quiver of sections is a quiver of sections where not all torus-invariant sections contribute to paths in the quiver, analogous to an incomplete linear series.

We then apply this technology to the McKay quiver associated to a finite abelian group $G \subset \mathrm{GL}(n, \Bbbk)$. After showing that every refined representation of the labelled McKay quiver (Q, div) is θ-stable and deducing ψ_θ is a morphism for any given stability condition θ, we show that $\psi_\theta : \mathbb{A}^n/G \to \mathcal{M}_\theta(Q, \mathrm{div})$ is a closed immersion. We tweak the GIT construction of $\mathcal{M}_\theta(Q, \mathrm{div})$ by allowing the homogenizing parameters to be zero and examine a substack cut-out by an ideal defined naturally from the labels of Q. By studying the GIT chamber decomposition, we observe that certain chambers define semistable loci in which every homogenizing parameter is nonzero, enabling us to recover the stack $[\mathbb{A}^n/G]$. We also show that in the semistable locus of a second chamber the homogenizing variables are completely determined by the variables corresponding to the arrows and are therefore redundant. This recovers the Craw-Maclagan-Thomas [CMT07] construction of the coherent component $\mathrm{Hilb}^G(\mathbb{A}^n)$ of Nakamura's G-Hilbert scheme. Using the results of Ito-Nakamura [IN99] and Nakamura [Nak01] we have the following results.

Theorem 5.10. *For finite abelian $G \subset \mathrm{GL}(n,\Bbbk)$, there exists generic stability conditions $\chi_{\theta_1}, \chi_{\theta_2} \in \mathrm{PGL}(\alpha)^\vee$, such that*

$$[\mathbb{A}^n/G] \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_1}/\mathrm{PGL}(\alpha)] \ \ and \ \ \mathrm{Hilb}^G(\mathbb{A}^n) \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)].$$

Corollary 5.11. *For $n \leq 3$ and finite abelian $G \subset \mathrm{SL}(n,\Bbbk)$, there exists generic stability conditions $\chi_{\theta_1}, \chi_{\theta_2} \in \mathrm{PGL}(\alpha)^\vee$, such that*

$$[\mathbb{A}^n/G] \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_1}/\mathrm{PGL}(\alpha)] \ \ and \ \ G\text{-}\mathrm{Hilb}(\mathbb{A}^n) \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)].$$

The thesis is organized as follows. We assemble some background material in Chapter 2. In Chapter 3, we define the ambient stacks in our construction. We define our stacky analogue of the classical linear series construction in Chapter 4. In Chapter 5, we apply the machinery from Chapter 4 to the McKay quiver. Finally in Chapter 6, we discuss the limitations of our construction and give a possible avenue for generalizations via an example.

Conventions and notation

The symbol $\Bbbk$ will be reserved for an algebraically closed field of characteristic 0. All objects and maps are defined over $\Bbbk$ unless stated. The symbol $\mathbb{N}$ will be reserved for the nonnegative integers. For a finite set C we use $\mathbb{Z}C$ to denote the free abelian group generated by C and $\mathbb{N}C$ to be the free abelian monoid generated by C. For an abelian group G we write $G_\mathbb{Q} := G \otimes_\mathbb{Z} \mathbb{Q}$ and $G^\vee := \mathrm{Hom}(G,\mathbb{Z})$. For vector spaces W and locally free sheaves $\mathcal{W}$ we use $W^\vee$ and $\mathcal{W}^\vee$ to denote the dual vector space and the dual sheaf respectively. For the groups $\mathrm{GL}(\alpha)$ and $\mathrm{PGL}(\alpha)$ we use $\mathrm{GL}(\alpha)^\vee$ and $\mathrm{PGL}(\alpha)^\vee$ to denote their respective groups of characters.

Chapter 2

Background

This chapter establishes the main objects and tools used in this thesis, as well as the context in which the thesis was built.

2.1 Deligne-Mumford stacks

There are many great introductions to algebraic stacks available in print and in video; an in-progress book by Kresch et al. [BCE$^+$] and videos of a lecture series given by Behrend at the Newton Institute [Beh], to name but two. For completeness we include a brief introduction to stacks and stacky ideas relevant to this thesis.

Historically, stacks were motivated by the study of families of algebro-geometric objects (or moduli problems). One starts with some objects that one wishes to parametrize and a notion of a continuous family over a scheme, and one seeks a scheme $\mathcal{M}$ whose S-points, $\mathrm{Hom}(S, \mathcal{M})$, are equivalence classes of families of the aforementioned objects over S. There are some examples where such a scheme $\mathcal{M}$ exists; for example, when the objects of interest are subvarieties of a fixed projective scheme, such schemes $\mathcal{M}$ exist and are called Hilbert schemes. However, this is not always the case.

Example 2.1. Consider families of vector spaces of dimension 1, i.e. line bundles. We will define two such families to be equivalent if the corresponding line bundles are isomorphic. Now assume a parametrizing scheme $\mathcal{M}$ exists. The only line bundle over $\mathrm{Spec}(\Bbbk)$ is the trivial bundle, so $\mathrm{Hom}(\mathrm{Spec}(\Bbbk), \mathcal{M})$ is a singleton and $\mathcal{M}$ has one geometric point. This also implies that $\mathrm{Hom}(\mathbb{P}^n, \mathcal{M})$ is also a singleton. However, there are many line bundles over $\mathbb{P}^n$, so such a scheme $\mathcal{M}$ can not exist.

The problem with the example above lies in the fact that our objects, namely one dimensional vector spaces, have non-trivial groups of automorphisms, i.e. $\mathrm{GL}(1, \Bbbk)$.

After all, to create non-trivial line bundles we use transition functions from GL(1). Stacks are designed with the capacity to remember automorphism groups of objects, bypassing this issue.

We move towards a definition of stacks. The idea is to axiomatize the notion of families of objects over schemes. The first step is the notion of a category fibred in groupoids (CFG). The two features CFGs capture are: every family is parametrized by a base scheme and we can pullback families 'uniquely' via morphisms of base schemes. Let $\mathscr{S}$ denote the category of schemes over $\mathrm{Spec}(\Bbbk)$.

Definition 2.2. A *category fibred in groupoids* (CFG) over our base category $\mathscr{S}$ is a category $\mathcal{X}$ with a functor $p : \mathcal{X} \to \mathscr{S}$ satisfying the following two axioms:

i) (pullbacks exist) for every morphism $f : T \to S$ in $\mathscr{S}$ and object s in $\mathcal{X}$ with $p(s) = S$, there exists an object t in $\mathcal{X}$ such that $p(t) = T$ and a morphism $\varphi : t \to s$ for which $p(\varphi) = f$;

ii) (pullbacks are unique up to unique isomorphism) for every morphism $f : T \to S$ in $\mathscr{S}$ and morphisms $\varphi : t \to s$, $\varphi' : t' \to s$ in $\mathcal{X}$ for which $f = p(\varphi) = p(\varphi')$ there exists a unique morphism $\vartheta : t \to t'$ in $\mathcal{X}$ for which $\varphi = \varphi' \circ \vartheta$ and $p(\vartheta) = \mathrm{id}_T$.

Examples 2.3.   1) The CFG of vector bundles of rank n (families of vector spaces) is the following category:

Ob: vector bundles $E \to S$ over a scheme S;

Mor: pullbacks of vector bundles

$$
\begin{array}{ccc}
F & \longrightarrow & E \\
\downarrow & \square & \downarrow \\
T & \longrightarrow & S.
\end{array}
$$

In this case the functor p takes $E \to S$ to the scheme S.

2) The CFG of trivial vector bundles of rank n is the following category:

Ob: trivial vector bundles $S \times \Bbbk^n \to S$ over a scheme S;

Mor: morphisms of vector bundles

$$
\begin{array}{ccc}
T \times \Bbbk^n & \longrightarrow & S \times \Bbbk^n \\
\downarrow & & \downarrow \\
T & \longrightarrow & S.
\end{array}
$$

The functor p sends $S \times \Bbbk^n \to S$ to S.

3) A scheme X can be viewed as a category fibred in groupoids. The CFG X is the following category:

Ob: morphisms $S \to X$;

Mor: commutative diagrams

$$
\begin{array}{ccc}
T & \longrightarrow & S \\
& \searrow \quad \swarrow & \\
& X &
\end{array} \quad .
$$

The functor p sends $S \to X$ to S.

4) Take X a scheme and G algebraic group acting on X. We define the CFG $[X/G]$ as follows:

Ob: principal G-bundles E over S with a G-equivariant morphism to X;

Mor: commutative diagrams

$$
\begin{array}{ccc}
& X & \\
\nearrow & & \nwarrow \\
F & \longrightarrow & E \\
\downarrow & & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

with F isomorphic to the pullback of E.

Remark 2.4. Given a CFG $\mathcal{X}$ the 'fibre' $\mathcal{X}_S$ over a scheme S, is the subcategory of $\mathcal{X}$ whose objects map to S via p and whose morphisms map to the identity morphism of S. It follows from axiom ii) of Definition 2.2 that every morphism in $\mathcal{X}_S$ is an isomorphism, that is $\mathcal{X}_S$ is a groupoid. This explains the mouthful 'categories fibred in groupoids' but more importantly gives us a mechanism of 'remembering' automorphisms of objects that we wish to parametrize.

Note that morphisms of CFG form a category; the objects of this category are functors between CFGs that commute with the projection functor and the morphisms are invertible natural transformations between functors (sometimes referred to as natural isomorphisms). The following proposition states the close relationship between objects of a CFG $\mathcal{X}$ over S and morphisms of CFGs from S to $\mathcal{X}$.

Proposition 2.5 (Prop. 2.20, [BCE$^+$]). *Let $\mathcal{X}$ be a CFG. Let S be a scheme. Then the functor from $\mathrm{Hom}(S, \mathcal{X})$ to $\mathcal{X}_S$ given by evaluation at the object (S, id_S) is surjective and fully faithful. In particular it is an equivalence of categories.*

As well as having pullbacks, families glue to give other families. The notion of a stack is given by adding extra gluing axioms to CFGs. We endow the category $\mathscr{S}$ with the étale topology.

Definition 2.6. A *stack* is a CFG $\mathcal{X}$ that satisfies the following axioms:

i) given an étale covering $\{S_i \to S\}$ of a scheme S and any E, E' in the fibre over S, for every collection of isomorphisms $\alpha_i : E|_{S_i} \to E'|_{S_i}$ in the fibre over S_i such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$, there exists a unique isomorphism $\alpha : E \to E'$ such that $\alpha|_{S_i} = \alpha_i$;

ii) given an étale covering $\{S_i \to S\}$ of a scheme S and a collection of objects E_i over S_i, with isomorphisms $\alpha_{ij} : E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ for every i, j, satisfying the cocycle condition $\alpha_{ik} = \alpha_{ij} \circ \alpha_{jk}$ over S_{ijk}, there exists a lifting E over S with isomorphisms $\alpha_i : E|_{S_i} \to E_i$ such that $\alpha_{ij} = \alpha_j|_{S_{ij}} \circ (\alpha_i|_{S_{ij}})^{-1}$.

Example 2.7. The CFGs 1), 3) and 4) in Examples 2.3 define stacks. However the CFG of trivial line bundles, CFG 2) in Examples 2.3, does not. It fails axiom ii) of Definition 2.6 due to the existence of nontrivial bundles that are locally trivial.

Before we go on to define Deligne-Mumford stacks, we need one more tool.

Definition 2.8. Let $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of stacks. The fibre product CFG $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is the following category:

Ob: triples (x, φ, y) where x is an object of $\mathcal{X}$ over a base scheme T, y is an object of $\mathcal{Y}$ over T and φ is an isomorphism $\varphi : f(x) \to g(y)$ in $\mathcal{Z}$ over T;

Mor: pairs of morphisms $(\alpha, \beta) : (x, \varphi, y) \to (x', \varphi', y')$ where $\alpha : x \to x'$ and $\beta : y \to y'$ over $T \to T'$ such that the following diagram commutes in $\mathcal{Z}$

$$
\begin{array}{ccc}
f(x) & \xrightarrow{f(\alpha)} & f(x') \\
\varphi \downarrow & & \downarrow \varphi' \\
g(y) & \xrightarrow{g(\beta)} & g(y').
\end{array}
$$

The fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ of two stacks is a stack and comes with projections p_1 and p_2 to $\mathcal{X}$ and $\mathcal{Y}$, respectively, and a natural isomorphism from $f \circ p_1$ to $g \circ p_2$.

It also satisfies the following universal property: given a 2-commutative diagram of stacks (i.e. a diagram where morphisms commute up to a natural isomorphism)

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\;v\;} & \mathcal{Y} \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\
\mathcal{X} & \xrightarrow[\;f\;]{} & \mathcal{Z}
\end{array}
$$

there exists a unique morphism $(u, v) : \mathcal{W} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. with $p_1 \circ (u, v) = u$ and $p_2 \circ (u, v) = v$, so that the natural isomorphism from $f \circ u$ to $g \circ v$ is determined by that from $f \circ p_1$ to $g \circ p_2$ (see page 45, [BCE$^+$]).

We can now define DM stacks.

Definition 2.9. A stack $\mathcal{X}$ is *Deligne-Mumford (DM)* if there exists a family $x : S \to \mathcal{X}$ over S satisfying the following properties:

  i) the stack $R := S \times_{\mathcal{X}} S$ is isomorphic to a scheme and the projection morphisms $R \rightrightarrows S$ are étale;

  ii) for every other family $y : T \to \mathcal{X}$ over T there exists an étale cover $T' \to T$ and a morphism $T' \to S$ such that $y|_{T'} = x|_{T'}$.

Example 2.10. Given $n+1$ natural numbers $w_i \in \mathbb{N}$, the weighted projective stack $\mathbb{P}(w_0, \ldots, w_n)$ is defined by the stack quotient $[\mathbb{A}^{n+1} \setminus \{0\} / \mathbb{k}^\times]$, where $\mathbb{k}^\times$ acts on $\mathbb{A}^{n+1}$ by

$$
\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n).
$$

Weighted projective stacks are DM. In general, stacks $[X/G]$ are DM when G is a smooth separated group scheme acting on X with finite, reduced geometric stabilizers (see Proposition 5.28, [BCE$^+$]).

The stack of vector bundles (CFG 1) in Examples 2.3) is not DM.

Definition 2.11. A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is *representable* if given any morphism $g : S \to \mathcal{Y}$ where S is isomorphic to a scheme, the fibre product $\mathcal{X} \times_{\mathcal{Y}} S$ is isomorphic to a scheme.

Representable morphisms give us an easy way of associating adjectives to morphisms of stacks. Given a property **P** of morphisms of schemes that it is preserved by arbitrary base change and is local on the étale topology, we say a representable morphism of stacks is **P** if its base change by any scheme is **P**. Examples of such properties **P** include smooth, étale, proper, affine and separated.

We introduce some terminology before stating an alternative criterion for representability of a morphism of stacks.

Definition 2.12. Given a DM stack $\mathcal{X}$, we say x is a *geometric point of $\mathcal{X}$* to mean x is a morphism $x : \mathrm{Spec}(\Bbbk) \to \mathcal{X}$. We will also use $\mathrm{Aut}(x)$ to denote the automorphism group of the geometric point x.

Proposition 2.13 (Lemma 4.4.3, [AV02])**.** *A morphism of DM stacks $f : \mathcal{X} \to \mathcal{Y}$ is representable if and only if for any geometric point $x : \mathrm{Spec}(\Bbbk) \to \mathcal{X}$, the natural homomorphism of group schemes $\mathrm{Aut}(x) \to \mathrm{Aut}(f(x))$ is a monomorphism.*

Definition 2.14. A *coarse moduli space* of a stack $\mathcal{X}$ is a morphism $\pi : \mathcal{X} \to X$ to an algebraic space X that satisfies the following properties:

   i) the morphism π induces a bijection of sets from $\mathcal{X}_{\mathrm{Spec}(\Bbbk)}$ to $X_{\mathrm{Spec}(\Bbbk)}$,

  ii) every morphism $f : \mathcal{X} \to Y$ to an algebraic space Y factors uniquely through π.

Theorem 2.15 (Keel-Mori [KM97])**.** *Every separated DM stack has a coarse moduli space.*

The term *coarse moduli space* is sometimes used in the literature to refer to a morphism $\pi : \mathcal{X} \to X$, to an algebraic space X, that only satisfies condition (ii) of Definition 2.14. The stack $[\mathbb{A}^1/\Bbbk^\times]$, where $\Bbbk^\times$ acts by multiplication, does not have a coarse moduli space in the sense of Definition 2.14, however the morphism $[\mathbb{A}^1/\Bbbk^\times] \to \mathrm{Spec}(\Bbbk)$ satisfies condition (ii) of Definition 2.14.

2.2   Projective DM stacks

A projective scheme is a closed subscheme of projective space. One is naively inclined to define a projective stack to be a closed substack of projective space. This in particular would imply that every 'projective stack' is a projective scheme. To yield a more reasonable notion of a projective stack we have to be a bit creative.

Definition 2.16 (Def. 5.1, [OS03])**.** Let $\mathcal{X}$ be a separated DM stack, with coarse moduli space $\pi : \mathcal{X} \to X$. A locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ is a *generating (or π-very ample) sheaf* if

$$\pi^* \pi_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}$$

is surjective for every quasicoherent sheaf $\mathcal{F}$ on $\mathcal{X}$.

The condition in Definition 2.16 is similar to that in Serre's criterion for relative ampleness, stated below.

Proposition 2.17. *Let Y be a quasicompact scheme and $\pi : X \to Y$ be a proper morphism. An invertible sheaf L on X is ample relative to π if and only if for every coherent sheaf $\mathcal{F}$ there exists an integer n_0 for which the adjunction morphism*

$$\pi^*\pi_*\mathcal{H}om_{\mathcal{O}_X}(L(-n), \mathcal{F}) \otimes L(-n) \to \mathcal{F}$$

is surjective for every integer $n > n_0$.

This suggests that a generating sheaf can be thought of as a π-very ample sheaf and that possession of a generating sheaf can be interpreted as projectivity of the morphism π to the coarse moduli space. This along with the fact that the composition of projective morphisms of schemes is projective gives the following definition of a projective DM stack.

Definition 2.18 (Cor. 5.4, Def. 5.5, [Kre09]). A separated proper DM stack is *projective* if it possesses a generating sheaf and its coarse moduli space is a projective scheme.

There are many equivalent formulations of the definition of a projective DM stack. In fact, Definition 2.18 is not the original definition.

Proposition 2.19 (Cor. 5.4, [Kre09]). *Let $\mathcal{X}$ be a proper separated DM stack. The following are equivalent.*

   i) $\mathcal{X}$ is projective.

   ii) $\mathcal{X}$ has a projective coarse moduli space and is isomorphic to a stack of the form $[P/G]$ for P an algebraic space and G a linear group acting algebraically on P.

   iii) $\mathcal{X}$ admits a closed immersion to a smooth proper DM stack with projective coarse moduli space.

Example 2.20. Weighted projective stacks $\mathbb{P}(w_0, \dots, w_n)$ are examples of projective DM stacks. Moduli stacks of stable maps $\mathcal{M}_{g,n}(X, \beta)$ are also projective DM stacks (see [Kre09]).

We also state an equivalent definition of generating sheaves that will be useful later.

Proposition 2.21 (5.2, [Kre09]). *A locally free sheaf $\mathcal{E}$ on a DM stack $\mathcal{X}$ is generating if and only if for every geometric point of $\mathcal{X}$ the representation of the stabilizer group at that point contains every irreducible representation.*

Definition 2.22 (Def. 2.2, [Nir08])**.** We say a locally free sheaf $\mathcal{E}$ is π-*ample* if for every geometric point the representation of the stabilizer group at that point is faithful.

2.3   The Abramovich-Hassett construction

The first steps towards a stacky analogue of the classical linear series construction were taken by Abramovich-Hassett [AH]. Their primary aim was to describe a suitable notion of a polarized stack in order to study compactifications of moduli of surfaces. The class of stacks for which the Abramovich-Hassett construction is applicable is that of cyclotomic stacks.

Definition 2.23. A separated DM stack $\mathcal{X}$ locally of finite type is *cyclotomic* if it has cyclic stabilizer groups.

Example 2.24. Weighted projective stacks $\mathbb{P}(w_0, \ldots, w_n)$ are examples of cyclotomic stacks. However, products of such stacks need not be cyclotomic.

As in the scheme theoretic setting, Abramovich-Hassett [AH] use a single line bundle L to polarize a cyclotomic stack. For this to work the polarizing line bundle must capture information about the stabilizers as well as the geometric structure of the coarse moduli space. To capture stabilizer information we require L to be π-ample (since π-very ample is too strong by Proposition 2.21). Combining that with a geometric condition yields the notion of a polarizing line bundle.

Definition 2.25. Let $\mathcal{X}$ be a cyclotomic stack and L a line bundle on $\mathcal{X}$. Then L is said to be *polarizing* if L is π-ample and there exists a positive integer N such that $L^{\otimes N} \cong \pi^* M$ for some ample line bundle M on the coarse moduli space X.

Remark 2.26. If a DM stack $\mathcal{X}$ possess a π-ample line bundle L then it is cyclotomic. Indeed, let L be a π-ample line bundle over $\mathcal{X}$ and take x a geometric point of $\mathcal{X}$. Then the finite group $\mathrm{Aut}(x)$ acts faithfully on the fibre over x. Since the fibre over x is a one dimensional vector space it follows that $\mathrm{Aut}(x)$ is cyclic.

The ambient stacks used in this construction (analogues of projective space) are constructed as follows. Given positive integers $n < m$, consider the graded $\Bbbk$-algebra

$$R_{n,m} := \mathrm{Sym}\bigg(\bigoplus_{j=n}^{m} \Gamma(\mathcal{X}, L^{\otimes j}) \bigg).$$

The $\mathbb{Z}$-grading of $R_{n,m}$ induces a $\mathbb{k}^\times$-action on $\mathrm{Spec}(R_{n,m})$. The ambient stack is defined by the stack quotient

$$\underline{\mathrm{Proj}}(R_{n,m}) := \left[\frac{\mathrm{Spec}(R_{n,m}) \setminus \mathbb{V}(R_{n,m}^+)}{\mathbb{k}^\times} \right],$$

where $R_{n,m}^+$ is the ideal generated by positively graded elements of $R_{n,m}$. These stacks, $\underline{\mathrm{Proj}}(R_{n,m})$, are weighted projective stacks. As in the classical case, $\mathcal{X}$ maps to ambient stacks by evaluation on sections.

Proposition 2.27 (Cor. 2.4.4, [AH]). *Let $\mathcal{X}$ be a cyclotomic stack with a polarizing line bundle L. Then there exists positive integers $n < m$ such that*

$$\mathcal{X} \longrightarrow \underline{\mathrm{Proj}}(R_{n,m})$$

is a closed immersion.

Example 2.28. Take $\mathcal{X} = \mathbb{P}(1,1,2)$ with polarizing line bundle $\mathcal{O}(1)$. The vector spaces $\Gamma(\mathcal{X}, \mathcal{O}(1))$ and $\Gamma(\mathcal{X}, \mathcal{O}(2))$ are of dimensions 2 and 4 respectively. Therefore $R_{1,2}$ is isomorphic to the graded ring $\mathbb{k}[y_1, y_2, z_1, z_2, z_3, z_4]$ where the variables y_i and z_i are in degrees 1 and 2 respectively. The construction gives a closed immersion

$$\mathbb{P}(1,1,2) \hookrightarrow \underline{\mathrm{Proj}}(R_{1,2}) \cong \mathbb{P}(1,1,2,2,2,2)$$

that sends (x_1, x_2, x_3) to $(x_1, x_2, x_1^2, x_1 x_2, x_2^2, x_3)$.

2.4 Multilinear series

The Abramovich-Hassett construction uses sections of tensor powers of a *single* line bundle and is therefore limited to cyclotomic stacks (c.f. Remark 2.26). To break free from cyclotomic stacks we consider taking sections of several line bundles. For projective varieties a generalization of linear series to several line bundles was introduced by Craw-Smith [CS08] in the toric case and Craw [Cra11] in general; the resulting ambient spaces are called multilinear series in [CS08] and multigraded linear series in [Cra11]. We will refer to them as multilinear series since the term was imprinted in our consciousness at the start of this journey. In this section we describe this multi-line bundle linear series construction to pave the way for the use of similar techniques later on.

We take a detour to introduce some of the quiver theoretic terminology needed to discuss multilinear series.

2.4.1 Quivers and their representations

Definition 2.29. A *quiver* Q is specified by two finite sets Q_0 and Q_1, whose elements are called vertices and arrows, together with two maps $h, t \colon Q_1 \to Q_0$ indicating the vertices at the head and tail of each arrow.

From now on we will assume our quivers are connected, i.e. the underlying graph is connected.

A nontrivial path in a quiver Q is a sequence of arrows $p = a_1 \cdots a_m$ with $h(a_k) = t(a_{k+1})$ for $1 \le k < m$. Each $i \in Q_0$ gives a trivial path e_i where $t(e_i) = h(e_i) = i$. To a quiver one may associate a $\Bbbk$-algebra, denoted $\Bbbk Q$, whose underlying $\Bbbk$-vector space has a basis consisting of paths in Q and where the product of two basis elements equals the basis element defined by concatenation of the paths if possible or zero otherwise. A cycle is a path p in which $t(p) = h(p)$. We say a quiver is acyclic if it contains no nontrivial cycles. A vertex is a source of the quiver if it is not the head of any arrow and a quiver is rooted if it has a unique source.

The vertex space $\mathbb{Z}^{Q_0}$ is the free abelian group generated by the vertices and the arrow space $\mathbb{Z}^{Q_1}$ is the free abelian group generated by the arrows. We write $\mathbb{N}^{Q_0}$ and $\mathbb{N}^{Q_1}$ for the submonoids generated by the basis elements of $\mathbb{Z}^{Q_0}$ and $\mathbb{Z}^{Q_1}$. The incidence map $\mathrm{inc} \colon \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}$ is defined by $\mathrm{inc}(\mathbf{e}_a) = \mathbf{e}_{h(a)} - \mathbf{e}_{t(a)}$. We define the weight lattice $\mathrm{Wt}(Q)$ to be the image of inc, that is the sublattice give by elements $\theta = \sum_{i \in Q_0} \theta_i \mathbf{e}_i \in \mathbb{Z}^{Q_0}$ for which $\sum_{i \in Q_0} \theta_i = 0$.

Definition 2.30. A *representation* $\overline{W} = (W_i, w_a)$ of a quiver Q consists of a vector space W_i for each $i \in Q_0$ and a linear map $w_a \colon W_{t(a)} \to W_{h(a)}$ for each $a \in Q_1$. The *dimension vector* of $\overline{W}$ is the integer vector $(\dim W_i) \in \mathbb{N}^{Q_0}$.

A map between representations $\overline{W} = (W_i, w_a)$ and $\overline{W}' = (W_i', w_a')$ is a family of linear maps $\xi_i \colon W_i \to W_i'$ for $i \in Q_0$ that are compatible with the structure maps, that is $w_a' \xi_{t(a)} = \xi_{h(a)} w_a$ for all $a \in Q_1$. With composition defined componentwise, we obtain the abelian category of representations of Q denoted $\mathrm{rep}_\Bbbk(Q)$. This category is equivalent to the category $\Bbbk Q$-mod of finitely generated left modules over the path algebra.

Definition 2.31. Given $\theta \in \mathrm{Wt}(Q)$, a representation W is said to be θ-*semistable* if for every proper nonzero subrepresentation $\overline{W}' \subset \overline{W}$, we have $\sum_{i \in Q_0} \theta_i \cdot \dim(W_i') \ge 0$. The notion of θ-*stability* is obtained by replacing $\ge$ with $>$.

For a given dimension vector $\alpha \in \mathbb{N}^{Q_0}$, a family of θ-semistable quiver representations over a connected scheme S is a collection of rank α_i locally free sheaves $\mathcal{W}_i$ together with morphisms $\mathcal{W}_{t(a)} \to \mathcal{W}_{h(a)}$ for every $a \in Q_1$. In general, a fine

moduli space parametrizing families of θ-semistable quiver representations does not exist. However, under certain conditions on θ and α one may alter the definition of a family in order to sidestep this issue (cf. discussion before Definition 3.11). For example, when every θ-semistable representation is θ-stable and an entry α_i of α is 1, one may restrict to families for which $\mathcal{W}_i$ is trivial; such families are representable with fine moduli scheme $\mathcal{M}_\theta(Q, \alpha)$, see Proposition 5.3 in [Kin94].

2.4.2 Quivers of sections

To generalize linear series to several line bundles one must conjure up a way of efficiently packaging sections of all the line bundles in a given collection. One also needs a suitable analogue of projective space. Craw-Smith [CS08] use quivers and moduli of their representations for these purposes.

Definition 2.32. Let X be a projective toric variety with dense torus T_X and $\mathscr{L} = (L_0, \ldots, L_n)$ be a collection of distinct line bundles on X.

1. A T_X-invariant section $s \in \Gamma(X, L_j \otimes L_i^\vee)$ is *reducible* if there exists $L_k \in \mathscr{L}$ and T_X-invariant sections $s' \in \Gamma(X, L_k \otimes L_i^\vee), s'' \in \Gamma(X, L_j \otimes L_k^\vee)$ such that $s = s' \otimes s''$. A T_X-invariant section is *irreducible* if it is not reducible.

2. The *quiver of sections* associated to $\mathscr{L}$ is the quiver $Q_{\mathscr{L}}$ whose vertices correspond to the line bundles in $\mathscr{L}$ with an arrow from i to j for every irreducible T_X-invariant section $s \in \Gamma(X, L_j \otimes L_i^\vee)$.

Example 2.33. Let $X = \mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ be the Hirzebruch surface determined by the fan in Figure 2.1 (a). For $(k, \ell) \in \mathbb{Z}^2$, set $\mathcal{O}_X(k, \ell) := \mathcal{O}_X(kD_1 + \ell D_4)$.

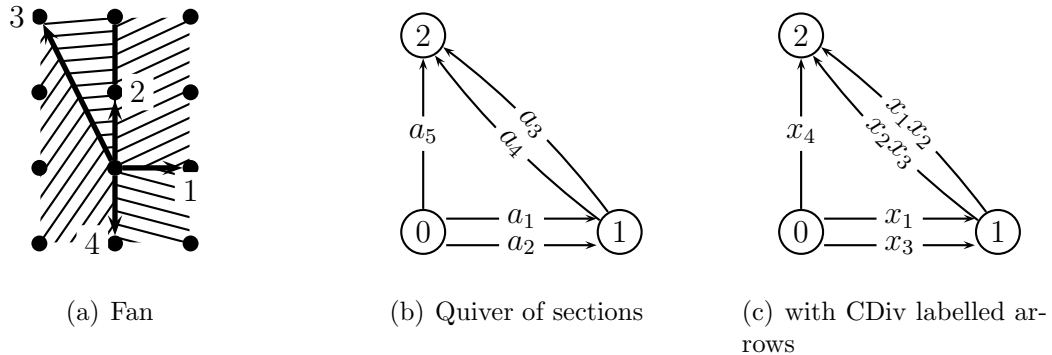(a) Fan   (b) Quiver of sections   (c) with CDiv labelled arrows

Figure 2.1: Hirzebruch surface $\mathbb{F}_2$

The quiver of sections for $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(1,0), \mathcal{O}_X(0,1))$ appears in Figure 2.1 (b). Notice that the section $x_1^2 x_2 \in \Gamma(X, \mathcal{O}_X(0,1))$ is reducible via $\mathcal{O}_X(1,0)$ and so does not give rise to an arrow.

The definition of the quiver of sections depends only on sections of the line bundles $L_j \otimes L_i^\vee$ for $L_i, L_j \in \mathscr{L}$. Consequently, for any line bundle L' on $\mathcal{X}$, we have $Q_\mathscr{L} = Q_{\mathscr{L}'}$ where $\mathscr{L}' = (L_0 \otimes L', \ldots, L_r \otimes L')$. To eliminate this redundancy, Craw-Smith conform to the following conventions.

Conventions 2.34. Let $\mathscr{L} = (L_0, \ldots, L_r) \subset \mathrm{Pic}(X)$ for a toric variety X.

  (a) We will assume that $L_0 = \mathcal{O}_X$.

  (b) We will also assume that $\Gamma(X, L_i) \neq 0$ for $L_i \in \mathscr{L}$, to ensure quivers of sections are connected and rooted at the vertex 0.

The ambient spaces for this construction are moduli spaces of representations of the associated quiver of sections.

Definition 2.35. Take $\mathscr{L} = (\mathcal{O}_X, \ldots, L_r) \subset \mathrm{Pic}(X)$ for a toric variety X and $\vartheta := \sum_{i \in Q_0} (\mathbf{e}_i - \mathbf{e}_0) \in \mathrm{Wt}(Q)$. The *multilinear series* of $\mathscr{L}$ is the moduli space $\mathcal{M}(Q_\mathscr{L})$ of ϑ-semistable representations of the quiver of sections $Q_\mathscr{L}$ of dimension vector $(1, \ldots, 1)$.

One should note that $\vartheta \in \mathrm{Wt}(Q)$ as defined in Definition 2.35 is generic, i.e. ϑ-semistable points are in fact ϑ-stable.

Example 2.36. For a single line bundle L on a projective toric variety X, the quiver of sections of the collection $\mathscr{L} = (\mathcal{O}_X, L)$ has $\dim(\Gamma(X, L))$ arrows and is of the form shown in Figure 2.2. In this case the moduli space of quiver representations $\mathcal{M}(Q)$
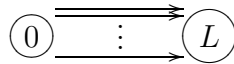
Figure 2.2: Quiver of sections of $\mathscr{L} = (\mathcal{O}_X, L)$

is isomorphic to $\mathbb{P}(\Gamma(X, L))$, that is the moduli space of hyperplanes of the vector space $\Gamma(X, L)$. This isomorphism is stronger than just an isomorphism of schemes; the pullback of the tautological bundle on $\mathcal{M}(Q)$ is $\mathcal{O}_X \oplus \mathcal{O}(1)$, where $\mathcal{O}(1)$ is the tautological bundle on $\mathbb{P}(\Gamma(X, L))$. So for a single line bundle multilinear series reproduce linear series.

2.4.3   Maps to multilinear series

For X and $\mathscr{L}$ as above, Craw-Smith go on to give a rational map $X \dashrightarrow \mathcal{M}(Q_\mathscr{L})$. The map is defined by the 'labelling' of the arrows. Every arrow of a quiver of sections is defined (labelled) by a torus-invariant section of a line bundle giving a

group homomorphism $\mathrm{div} : \mathbb{Z}^{Q_1} \to \mathrm{CDiv}(X)$, where $\mathrm{CDiv}(X)$ is the group of torus-invariant Cartier divisors. Similarly, every vertex of the quiver corresponds to a line bundle and the assignment $\mathbf{e}_i \mapsto L_i$ extends by $\mathbb{Z}$-linearity to a group homomorphism $\mathrm{pic} : \mathrm{Wt}(Q) \hookrightarrow \mathbb{Z}^{Q_0} \to \mathrm{Pic}(X)$. These two maps fit into a commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}^{Q_1} & \xrightarrow{\ \ \mathrm{inc}\ \ } & \mathrm{Wt}(Q) \\
{\scriptstyle \mathrm{div}}\downarrow & & \downarrow{\scriptstyle \mathrm{pic}} \\
\mathrm{CDiv}(X) \subset \mathbb{Z}^{\Sigma(1)} & \xrightarrow{\ \ \mathrm{deg}\ \ } & \mathrm{Pic}(X).
\end{array}
\tag{2.1}
$$

First note that the homomorphisms inc and deg induce $\mathrm{Wt}(Q)$ and $\mathrm{Pic}(X)$ gradings on the monoid algebras $\Bbbk[\mathbb{N}^{Q_1}]$ and $\Bbbk[\mathbb{N}^{\Sigma(1)}]$ respectively. These gradings give the actions of $\mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times)$ on $\mathrm{Spec}(\Bbbk[\mathbb{N}^{Q_1}]) \cong \mathbb{A}^{Q_1}$ and $\mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^\times)$ on $\mathrm{Spec}(\Bbbk[\mathbb{N}^{\Sigma(1)}]) \cong \mathbb{A}^{\Sigma(1)}$. Now X is the quotient

$$
\left(\mathbb{A}^{\Sigma(1)} \setminus \mathbb{V}(B_L) \right) / \left(\mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^\times) \right),
$$

where $\mathbb{V}(B_L)$ is the vanishing locus of the sections an ample line bundle L. Also $\mathcal{M}(Q)$ is

$$
\left(\mathbb{A}^{Q_1} \setminus \mathbb{V}(B_\vartheta) \right) / \left(\mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times) \right),
$$

where $\mathbb{V}(B_\vartheta)$ is the vanishing locus of the ideal $B_\vartheta = \left\langle y^u \in \Bbbk[\mathbb{N}^{Q_1}] \,\middle|\, \mathrm{inc}(u) = \vartheta \right\rangle$, this follows since ϑ is ample in $\mathcal{M}_\vartheta(Q)$.

The morphism div induces a morphism of monoid algebras $\Phi : \Bbbk[\mathbb{N}^{Q_1}] \to \Bbbk[\mathbb{N}^{\Sigma(1)}]$ or in other words a morphism $\Phi^* : \mathbb{A}^{\Sigma(1)} \to \mathbb{A}^{Q_1}$. Since diagram (2.1) commutes, this morphism is equivariant with respect to the actions of $\mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times)$ on $\mathbb{A}^{Q_1}$ and $\mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^\times)$ on $\mathbb{A}^{\Sigma(1)}$ giving a rational map $\varphi_{|\mathscr{L}|} : X \dashrightarrow \mathcal{M}(Q)$.

Proposition 2.37 (Cor. 4.2, [CS08]). *The rational map $\varphi_{|\mathscr{L}|} : X \dashrightarrow \mathcal{M}(Q)$ is a morphism if and only if every line bundle in the collection $\mathscr{L}$ is base-point free.*

Proposition 2.38 (Prop. 4.3, [CS08]). *Given a collection of base-point free line bundles $\mathscr{L}$ on X the image of the morphism $\varphi_{|\mathscr{L}|} : X \longrightarrow \mathcal{M}(Q)$ is cut out by the following $\mathrm{Wt}(Q)$-homogeneous ideal*

$$
I := \left\langle y^{u_1} - y^{u_2} \in \Bbbk[\mathbb{N}^{Q_1}] \,\middle|\, \mathrm{div}(u_1 - u_2) = 0, \mathrm{inc}(u_1 - u_2) = 0 \right\rangle \subset \mathrm{Spec}(\Bbbk[\mathbb{N}^{Q_1}]).
$$

Example 2.39. In the case where $\mathscr{L} = (\mathcal{O}, L)$ (Example 2.36) the morphism described by Craw-Smith reproduces the morphism to the linear series.

The ambient spaces used in linear series and multilinear series are fine moduli spaces, they therefore come with a collection of tautological objects. In linear series, the pullback of the tautological line bundle recovers the line bundle whose sections define the morphism. A similar thing happens with multilinear series.

Proposition 2.40 (Thm. 4.15, [CS08])**.** *Let $\mathscr{L}$ be a collection of base-point free line bundles on a projective toric variety X and Q be the quiver of sections of $\mathscr{L}$. For every $i \in Q_0$, the pullback, via $\varphi_{|\mathscr{L}|}$, of the tautological line bundle $\mathscr{W}_i$ of $\mathcal{M}(Q)$ is L_i.*

2.5   Smooth toric DM stacks

The aim is to use the techniques of multilinear series to define a multi-line bundle analogue for projective stacks freeing us from the cyclotomic restriction. However, the definition of the morphism to multilinear series makes use of some techniques from toric geometry. We will therefore limit ourselves to stacks that are suitably toric, that is, to smooth toric DM stacks.

Toric DM stacks were introduced by Borisov-Chen-Smith [BCS05] using stack quotients. Later, Fantechi-Mann-Nironi [FMN07] gave an equivalent definition analogous to the classical definition of a toric variety. In this thesis we are concerned only with toric orbifolds, that is toric DM stacks whose generic stabilizer is trivial or equivalently stacks whose dense DM torus is just an algebraic torus T. We will begin by introducing the Fantechi-Mann-Nironi approach then discuss the Borisov-Chen-Smith approach.

Definition 2.41. A *smooth toric DM stack* is a smooth separated DM stack $\mathcal{X}$ together with an open immersion $\iota : T \hookrightarrow \mathcal{X}$ with dense image such that the action of T on itself extends to an action of T on $\mathcal{X}$.

Example 2.42. Again all weighted projective stacks $\mathbb{P}(w_0, \ldots, w_n)$ are toric stacks, however some have nontrivial generic stabilizers. A weighted projective stack is a toric orbifold if and only if $\gcd(w_0, \ldots, w_n) = 1$. The moduli stack of stable elliptic curves $\mathcal{M}_{1,1}$ is also toric (in fact, it is a weighted projective space). Products of toric stacks are toric and so toric stacks need not be cyclotomic.

In the same spirit as toric varieties, toric stacks also have a combinatorial description. In fact, the combinatorial definition historically preceded the more geometric definition above and will be the one we refer to most frequently.

Definition 2.43. A *stacky fan* is a triple $\boldsymbol{\Sigma} := (N, \Sigma, \beta)$, where N is a finitely generated free abelian group, Σ is a rational simplicial fan in $N_{\mathbb{Q}}$ with d rays that span $N_{\mathbb{Q}}$, denoted $\rho_1, \ldots, \rho_d \in \Sigma(1)$, and $\beta : \mathbb{Z}^d \to N$ is a morphism of groups for which $\beta(\mathbf{e}_i) \otimes 1$ lies on the ray $\rho_i \in N_{\mathbb{Q}}$.

Remark 2.44. In the original definition of a stacky fan N is not required to be free. We add this extra restriction since we are only concerned with toric orbifolds (see Lemma 7.14, [FMN07]).

Using a construction very similar to the Cox construction one associates a toric stack to a stacky fan. Let $\mathbb{Z}^{\Sigma(1)} := (\mathbb{Z}^d)^{\vee}$ and consider the exact sequence

$$N^{\vee} \xrightarrow{\beta^{\vee}} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Coker}(\beta^{\vee}) \longrightarrow 0. \tag{2.2}$$

After applying the functor $\mathrm{Hom}(-, \Bbbk^{\times})$ to the extact sequence (2.2) we get an inclusion $G := \mathrm{Hom}(\mathrm{Coker}(\beta^{\vee}), \Bbbk^{\times}) \subset (\Bbbk^{\times})^{\Sigma(1)}$ and therefore have a natural action of G on $\mathbb{A}^{\Sigma(1)}$. For a cone $\sigma \in \Sigma$, $\widehat{\sigma}$ is the set of one-dimentional cones in Σ not contained in σ. The Cox unstable locus is then defined

$$B_{\mathcal{X}} := \left\langle \prod_{\rho \in \widehat{\sigma}} x_{\rho} \in \Bbbk[x_{\rho} \mid \rho \in \Sigma(1)] \;\middle|\; \sigma \in \Sigma \right\rangle. \tag{2.3}$$

The stack $\mathcal{X}_{\boldsymbol{\Sigma}}$ associated to the stacky fan is defined by

$$\left[\frac{\mathbb{A}^{\Sigma(1)} \setminus \mathbb{V}(B_{\mathcal{X}})}{G} \right].$$

The group of line bundles $\mathrm{Pic}(\mathcal{X}_{\boldsymbol{\Sigma}})$ is given by $\mathrm{Hom}(G, \Bbbk^*) \cong \mathrm{Coker}(\beta^{\vee})$ and the group of torus-invariant divisors of $\mathcal{X}_{\boldsymbol{\Sigma}}$ is given by $\mathbb{Z}^{\Sigma(1)}$.

Proposition 2.45 (Thm 7.23, [FMN07])**.** *Given any toric stack $\mathcal{X}$ there exists a stacky fan $\boldsymbol{\Sigma}$ such that $\mathcal{X}_{\boldsymbol{\Sigma}} \cong \mathcal{X}$. In particular, $\mathcal{X}$ is a global quotient.*

One of the equivalent characterizations of projective stacks (Proposition 2.19) then gives,

Corollary 2.46. *A smooth toric DM stack is projective if and only if its coarse moduli space is projective.*

The coarse moduli space of a toric stack also admits an easy description.

Proposition 2.47 (Thm 3.7, [BCS05])**.** *The toric variety $X(\Sigma)$ is the coarse moduli space of $\mathcal{X}(\Sigma)$.*

Chapter 3

Moduli of refined quiver representations

This chapter is devoted to defining a suitable ambient stack for our construction, that is an analogue of $\mathbb{P}(\Gamma(X, L))$ in the linear series construction.

Multilinear series are fine moduli spaces of quiver representations. In particular, they are schemes and hence do not admit closed immersions from non-representable stacks; making them unsuitable for our construction. The next example motivates the approach we adopt in defining our ambient stacks.

3.1 Motivating example

Example 3.1. Let $\mathcal{X}$ be the weighted projective stack $\mathbb{P}(1, 1, 2)$ with homogeneous coordinates x_1, x_2, x_3. Note that $\mathrm{Pic}(\mathcal{X}) = \mathbb{Z}$. The $\mathrm{Pic}(\mathcal{X})$-grading of $\mathbb{k}[x_1, x_2, x_3]$ giving rise to the quotient construction of $\mathcal{X}$ is generated in degrees 1 and 2, so morally the collection $\mathscr{L} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ should be sufficient to 'reproduce' $\mathcal{X}$. More precisely, the construction we seek should yield a closed immersion into some ambient stack. The corresponding quiver of sections Q is shown in Figure 3.1 below.

As in classical linear series, $\mathbb{P}(1, 1, 2)$ should map to ambient stacks by evaluating sections. Therefore our desired morphism of stacks descends from the morphism

$$\Psi^* : \mathrm{Spec}(\mathbb{k}[x_1, x_2, x_3]) \cong \mathbb{A}^3 \longrightarrow \mathrm{Spec}(\mathbb{k}[y_a \mid a \in Q_1])$$

mapping (x_1, x_2, x_3) to $(x_1, x_2, x_1, x_2, x_3)$.

First note that Ψ^* is a closed immersion with the ideal $I := \langle y_1 - y_3, y_2 - y_4 \rangle$ cutting out its image. The multilinear series corresponding to Q is the quotient of

(a) with numbered arrows   (b) with labelled arrows
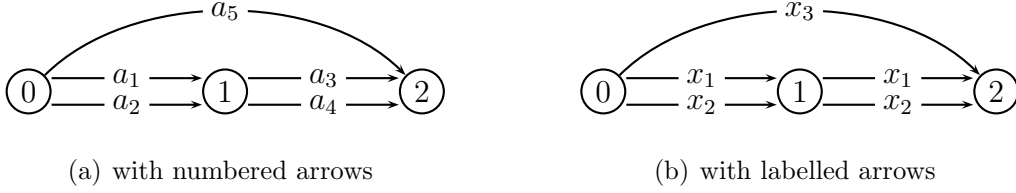
Figure 3.1: Quiver of sections of $\mathbb{P}(1, 1, 2)$

$\mathrm{Spec}(\mathbb{k}[y_a \,|\, a \in Q_1])$ by the following action of $\mathrm{Spec}(\mathbb{k}[s, t, s^{-1}, t^{-1}])$,

$$(s, t) \cdot (y_1, y_2, y_3, y_4, y_5) = (sy_1, sy_2, s^{-1}y_3t, s^{-1}y_4t, y_5t).$$

Observe that I is not homogeneous with respect to this action. To engineer a closed immersion from Ψ^*, we act on $\mathbb{A}^5$ with the largest subgroup of $\mathrm{Spec}(\mathbb{k}[s, t, s^{-1}, t^{-1}])$ for which I is homogeneous, that is $G := \mathbb{V}(s^2 - t) \subset (\mathbb{k}^\times)^2$.

The morphism Ψ^* is equivariant with respect to the action of $\mathbb{k}^\times$ on $\mathbb{A}^3$ and the action of G on $\mathbb{A}^5$, and hence descends to a morphism

$$\psi : \mathbb{P}(1, 1, 2) \cong \left[\frac{\mathbb{A}^3 \setminus \{0\}}{\mathbb{k}^\times} \right] \longrightarrow \mathbb{P}(1, 1, 1, 1, 2) \cong \left[\frac{\mathbb{A}^5 \setminus \{0\}}{G} \right]$$

which is a closed immersion.

To define suitable ambient stacks we generalize the strategy detailed in Example 3.1 as outlined below:

i) **The ideal I.** As in multilinear series, the morphism of monoid algebras $\Psi : \mathbb{k}[\mathbb{N}^{Q_1}] \to \mathbb{k}[\mathbb{N}^{\Sigma(1)}]$ is induced by the labelling map $\mathrm{div} : \mathbb{Z}^{Q_1} \to \mathbb{Z}^{\Sigma(1)}$, which is generated by the map $Q_1 \to \mathbb{Z}^{\Sigma(1)}$ sending an arrow to its label. The image of Ψ^* is then cut out by the toric ideal corresponding to div, i.e.

$$I := \left\langle y^{u_1} - y^{u_2} \in \mathbb{k}[\mathbb{N}^{Q_1}] \,\middle|\, u_1 - u_2 \in \ker(\mathrm{div}) \right\rangle.$$

ii) **Making I homogeneous.** The $\mathrm{Hom}(\mathrm{Wt}(Q), \mathbb{k}^\times)$-action on $\mathrm{Spec}(\mathbb{k}[\mathbb{N}^{Q_1}])$ is induced by the $\mathrm{Wt}(Q)$-grading of $\mathbb{k}[\mathbb{N}^{Q_1}]$ which is given by the incidence map $\mathrm{inc} : \mathbb{Z}^{Q_1} \to \mathrm{Wt}(Q)$. The ideal I is homogeneous if for every $y^{u_1} - y^{u_2} \in I$ the monomial y^{u_1} lies in the same grade as y^{u_2}, that is if $\mathrm{inc}(u_1 - u_2) = 0$. Therefore the free abelian group $R := \mathrm{inc}(\ker(\mathrm{div}))$ is the obstruction to the homogeneity of I.

Restricting the action to a subgroup of $\mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^{\times})$ feels a bit artificial. Instead we add a nonzero 'homogenizing' parameter z_i to $\mathrm{Spec}(\Bbbk[\mathbb{N}^{Q_1}])$ for every dimension of R, such that for every generator $y^{u_1} - y^{u_2} \in I$ there exists a product of nonzero parameters z_i for which $y^{u_1} - z_1 \cdots z_m y^{u_2}$ is homogeneous. This approach produces the same ambient stack as restricting the action to the homogenizing subgroup of $\mathrm{Spec}(\Bbbk[\mathbb{N}^{Q_1}])$.

iii) **Moduli description.** We then mimic the construction of moduli of quiver representations for our quiver representations with additional homogenizing parameters, to yield our desired ambient stacks.

3.2   Labelled quivers

The labels of the quiver play a crucial role in adopting the strategy detailed above. We begin by formalizing the notion of a labelled quiver.

Definition 3.2. A *labelled quiver* (Q, l) is a connected finite quiver Q along with a free abelian group $\mathbb{Z}^d$ for some $d \in \mathbb{N}$ and a map of sets $l : Q_1 \to \mathbb{Z}^d$.

Abusing notation we use l to denote the *labelling map* $\mathbb{Z}^{Q_1} \to \mathbb{Z}^d$ generated by l. Let R denote the image of $\ker(l)$ under the incidence map and consider the following commutative diagram,

$$\begin{array}{ccc}
\ker(l) & \longrightarrow & R := \mathrm{inc}(\ker(l)) \\
\downarrow & & \downarrow \\
\mathbb{Z}^{Q_1} & \xrightarrow{\;\mathrm{inc}\;} & \mathbb{Z}^{Q_0} \\
{\scriptstyle l}\downarrow & & \\
\mathbb{Z}^d. & &
\end{array} \tag{3.1}$$

Let Q be a quiver and let (W_i, w_a) be a representation of Q. We introduce useful bit of notation.

Definition 3.3. Given an element $b = \sum_{i \in Q_0} b_i \mathbf{e}_i \in \mathbb{Z}^{Q_0}$ define

$$\mathrm{det}_b W := \bigotimes_{i \in Q_0} (\det W_i)^{\otimes b_i}.$$

Here we use the conventions $W_i^{\otimes -1} := W_i^{\vee}$ and $\det W_i := \bigwedge^{(\dim W_i)} W_i$.

Pick a basis $\mathfrak{B}$ of R.

Definition 3.4. An *refined representation W of a labelled quiver* (Q, l) consists of a finite dimensional representation $\overline{W} := (W_i, w_a)$ of Q together with an isomorphism $f_b : \Bbbk \to \det_b W$ for every $b \in \mathfrak{B}$. The *dimension vector* of an R-refined representation $W = (W_i, w_a, f_b)$ is the integer vector $(\dim(W_i))_{i \in Q_0}$.

We say that two refined representations $W = (W_i, w_a, f_b)$, $W' = (W_i', w_a', f_b')$ are *isomorphic* if there exist isomorphisms of vector spaces $\gamma_i : W_i \to W_i'$ for every vertex $i \in Q_0$ such that $\gamma_{t(a)}^{-1} \circ w_a \circ \gamma_{h(a)} = w_a'$ for all $a \in Q_1$ and $f_b \circ \gamma_b = f_b'$ for all $b \in \mathfrak{B}$ where $\gamma_b : \det_b W \to \det_b W'$ is the isomorphism induced by the isomorphisms γ_i.

Remark 3.5.   i) The independence of the choice of basis $\mathfrak{B}$ will be addressed in Remark 3.10.

  ii) Refined representations and their moduli maybe defined without appealing to a labelling map l; the crucial ingredient is the subgroup $R \subset \mathrm{Wt}(Q)$. Given a quiver Q and an arbitrary subgroup $K \subset \mathrm{Wt}(Q)$ with a choice of basis $\mathfrak{B}_K$, one may define *K-refined representations of Q* to be finite dimensional representation $\overline{W}$ of Q together isomorphisms $f_b : \Bbbk \to \det_b W$ for every $b \in \mathfrak{B}_K$. All the definitions and results in this chapter may be lifted to this setting. With the immediate applications in mind, we restrict ourselves to subgroups R arising from a labelling map l.

For $i \in Q_0$, let W_i be a vector space of dimension α_i and $\alpha := (\alpha_i) \in \mathbb{N}^{Q_0}$. Let $\Bbbk[z_b \,|\, b \in \mathfrak{B}]$ denote the coordinate ring of the vector space $\bigoplus_{b \in \mathfrak{B}} \det_b W$. The isomorphism classes of refined representations of (Q, l) are in one-to-one correspondence with the orbits in the refined representation space

$$\mathcal{R}(Q, l, \alpha) := \left(\bigoplus_{a \in Q_1} \mathrm{Hom}(W_{t(a)}, W_{h(a)}) \oplus \bigoplus_{b \in \mathfrak{B}} \det_b W \right) \setminus \mathbb{V}\left(\prod_{b \in \mathfrak{B}} z_b \right)$$

of the symmetry group

$$\mathrm{GL}(\alpha) := \prod_{i \in Q_0} \mathrm{GL}(W_i)$$

under the change of basis action. Note that $\mathrm{GL}(\alpha)$ contains the diagonal one-parameter subgroup $\Delta = \{(\lambda \cdot 1, \ldots, \lambda \cdot 1) : \lambda \in \Bbbk^\times\}$ acting trivially and define $\mathrm{PGL}(\alpha) := \mathrm{GL}(\alpha)/\Delta$.

We note that the characters of $\mathrm{GL}(\alpha)$ are given by

$$\chi_\theta(g) = \prod_{i \in Q_0} \det(g_i)^{\theta_i}$$

for $\theta = \sum_i \theta_i \mathbf{e}_i \in \mathbb{Z}^{Q_0}$ and that every character of $\mathrm{GL}(\alpha)$ is of the form χ_θ for some $\theta \in \mathbb{Z}^{Q_0}$. As the diagonal $\Delta \subset \mathrm{GL}(\alpha)$ acts trivially on $\mathcal{R}(Q, \mathrm{div}, \alpha)$ we are interested in characters χ_θ that satisfy $\sum_i \theta_i \alpha_i = 0$.

It is convenient to identify $\mathbb{Z}^{Q_0}$, and hence the character group $\mathrm{GL}(\alpha)^\vee$, with a subgroup of $K_0(\mathrm{mod}\text{-}\mathbb{k}Q)^\vee := \mathrm{Hom}(K_0(\mathrm{mod}\text{-}\mathbb{k}Q), \mathbb{Z})$ as follows. Let $\overline{W} = (W_i, w_a)$ be a representation of Q. Implicitly using the equivalence of categories $\mathrm{mod}\text{-}\mathbb{k}Q \cong \mathrm{rep}_\mathbb{k}(Q)$, define $\theta(\overline{W}) := \sum_i \theta_i \dim W_i$, and observe that this is additive on short exact sequences.

3.3 θ-stability

Now we move towards building moduli spaces of refined representation. The idea is to study quotients of $\mathcal{R}(Q, l, \alpha)$ under the $\mathrm{GL}(\alpha)$ action. To guarantee well be-haved quotients, we will only consider quotients of GIT-semistable points for a given character of $\mathrm{GL}(\alpha)$. If we still wish to use the term moduli stack we must express GIT-stability in terms intrinsic to refined representation.

The standard trick (as in Section 3 of [Kin94]) starts by characterizing GIT-semistability using Mumford's Numerical Criterion. For a one-parameter subgroup, $\lambda : \mathrm{Spec}(\mathbb{k}[t, t^{-1}]) \to \mathrm{GL}(\alpha)$ and a character $\chi_\theta : \mathrm{GL}(\alpha) \to \mathrm{Spec}(\mathbb{k}[t, t^{-1}])$, define $\langle \chi_\theta, \lambda \rangle = m$ when $\chi_\theta(\lambda(t)) = t^m$.

Proposition 3.6 (Mumford's Numerical Criterion). *A point $x \in \mathcal{R}(Q, l, \alpha)$ is χ_θ-semistable if and only if $\chi_\theta(\Delta) = \{1\}$ and for every one-parameter subgroup λ of $\mathrm{GL}(\alpha)$, for which $\lim_{t \to 0} \lambda(t) \cdot x$ exists satisfies $\langle \chi_\theta, \lambda \rangle \geq 0$. Such a point is χ_θ-stable if and only if the only one-parameter subgroups λ of $\mathrm{GL}(\alpha)$ for which $\lim_{t \to 0} \lambda(t) \cdot x$ exists and $\langle \chi_\theta, \lambda \rangle = 0$ are in Δ.*

Let $\mathcal{R}(Q, \alpha)$ denote the quiver representations space $\bigoplus_{a \in Q_1} \mathrm{Hom}(W_{t(a)}, W_{h(a)})$. For a quiver representation $\overline{W} \in \mathcal{R}(Q, \alpha)$, King [Kin94] shows that one-parameter subgroups for which $\lim_{t \to 0} \lambda(t) \cdot \overline{W}$ exists define filtrations $W_\bullet$,

$$0 \subsetneq W_1 \subset \ldots \subset W_{n-1} \subsetneq W_n = \overline{W}$$

of the associated quiver representation $\overline{W}$. Moreover, given a filtration $W_\bullet$ of $\overline{W}$ there exists a one-parameter subgroup for which the associated filtration coincides with $W_\bullet$. Under this correspondence the pairing $\langle \chi, \lambda \rangle$ can be expressed as follows,

$$\langle \chi_\theta, \lambda \rangle = \theta(W_\bullet) := \sum_{j=1}^{n-1} \theta(W_j)$$

for a given character χ_θ of $\mathrm{GL}(\alpha)$. Translating Mumford's Numerical Criterion in terms of filtrations gives the definition of θ-stability for quiver representations (Definition 2.31). We adopt an almost identical strategy to define θ-stability for refined quiver representations of labelled quivers.

Definition 3.7. Let $\theta \in K_0(\text{mod-}\Bbbk Q)^\vee$. A refined representation W is θ-*semistable* if $\theta(W) = 0$ and $\theta(W_\bullet) \geq 0$ for every proper filtration $W_\bullet$ of the $\Bbbk Q$-module $\overline{W}$ that satisfies $b(W_\bullet) = 0$ for every $b \in \mathfrak{B}$. The notion of θ-*stability* is obtained by replacing $\geq$ with $>$.

Notice that the definition of θ-semistability makes sense for any $\theta \in K_0(\text{mod-}\Bbbk Q)^\vee$, not necessarily ones coming from characters of $\mathrm{GL}(\alpha)$. However, we introduced the notion of θ-semistability to be able to make sense of families of refined representations and use the term 'moduli stack'. So in practice, we are interested primarily in functions $\theta \in K_0(\Bbbk Q\text{-mod})^\vee$ coming from characters of $\mathrm{GL}(\alpha)$.

Theorem 3.8. *Let χ_θ be a character of $\mathrm{GL}(\alpha)$ and θ the corresponding element of $K_0(\text{mod-}\Bbbk Q)^\vee$. A refined quiver representation W is θ-semistable (resp. θ-stable) if and only if the corresponding point in $\mathcal{R}(Q, l, \alpha)$ is χ_θ-semistable (resp. χ_θ-stable) with respect to action of $\mathrm{GL}(\alpha)$.*

Proof. We begin by pinning down the one-parameter subgroups λ of $\mathrm{GL}(\alpha)$ for which $\lim_{t \to 0}(\lambda(t) \cdot W)$ exists. Write $\mathcal{R}(Q, l, \alpha) \cong \mathcal{R}(Q, \alpha) \times (\Bbbk^\times)^{\mathfrak{B}}$ and π_1, π_2 for the first and second projection respectively. The limit $\lim_{t \to 0}(\lambda(t) \cdot W)$ exists if and only if $\lim_{t \to 0}(\lambda(t) \cdot \pi_1(W))$ and $\lim_{t \to 0}(\lambda(t) \cdot \pi_2(W))$ exist. By the discussion preceding Definition 3.7, $\lim_{t \to 0}(\lambda(t) \cdot \pi_1(W))$ exists if and only if λ defines a $\mathbb{Z}$-filtration, $W_\bullet$, of the $\Bbbk Q$-module $\pi_1(W) = \overline{W}$,

$$\ldots \subset W_{n-1} \subset W_n \subset W_{n+1} \subset \ldots$$

for which $W_n = 0$ for $n \ll 0$ and $W_n = \overline{W}$ for $n \gg 0$. Now consider $\lim_{t \to 0}(\lambda(t) \cdot \pi_2(W))$. The one-parameter subgroup λ defines a $\mathbb{Z}$-grading on the coordinate ring $\Bbbk[z_b, z_b^{-1} \mid b \in \mathfrak{B}]$ of $(\Bbbk^\times)^{\mathfrak{B}}$. The limit $\lim_{t \to 0}(\lambda(t) \cdot \pi_2(W))$ exists if and only if the variables z_b and z_b^{-1} are simultaneously non-negatively graded. Notice that this holds precisely when they are zero graded, that is when $\langle \chi_b, \lambda \rangle = 0$ for every $b \in \mathfrak{B}$. Therefore, for λ and W as above, $\lim_{t \to 0}(\lambda(t) \cdot W)$ exists if and only if λ gives a $\mathbb{Z}$-filtration $(W_n)_{n \in \mathbb{Z}}$ of the quiver representation $\pi_1(W) = \overline{W}$ and $\langle \chi_b, \lambda \rangle = 0$, for every $b \in \mathfrak{B}$.

Now assume W is θ-semistable. Take λ to be a one-parameter subgroup for which the limit $\lim_{t \to 0}(\lambda(t) \cdot W)$ exists and let $W_\bullet$ be the associated filtration as

discussed before Definition 3.7. Then $\langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} \theta(W_n)$ which is equal to $\theta(W_\bullet)$ since $\theta(\overline{W}) = 0$. In particular, this implies $\langle \chi_b, \lambda \rangle = b(W_\bullet) = 0$ for all $b \in \mathfrak{B}$. Seeing that W is θ-semistable we get $\langle \chi_\theta, \lambda \rangle = \theta(W_\bullet) \geq 0$. GIT semistability of W then follows from Mumford's Numerical Criterion.

Next assume $W \in \mathcal{R}(Q, l, \alpha)$ is χ_θ-semistable. By the fact that Δ acts trivially we have $\langle \chi_\theta, \Delta \rangle = \theta(W) = 0$. Let $W_\bullet$ be a proper filtration satisfying the conditions of Definition 3.7. By the discussion preceding Definition 3.7 there exists a one-parameter subgroup λ for which the associated filtration is $W_\bullet$. By assumption we have that $b(W_\bullet) = \langle \chi_b, \lambda \rangle = 0$ for every $b \in \mathfrak{B}$, so $\lim_{t \to 0}(\lambda(t) \cdot W)$ exists. Mumford's Numerical Criterion gives $\theta(W_\bullet) = \langle \chi_\theta, \lambda \rangle \geq 0$, as required. $\qquad\square$

3.4 Moduli of refined representations

We now have all the ingredients to define moduli stacks of refined representations.

Definition 3.9. For $\chi_\theta \in \mathrm{PGL}(\alpha)^\vee \subset \mathrm{GL}(\alpha)^\vee$, let $\mathcal{R}(Q, l, \alpha)_\theta^{ss}$ denote the open subscheme of $\mathcal{R}(Q, l, \alpha)$ parametrizing the θ-semistable refined representation. The *moduli stack of θ-semistable refined representations* is the stack quotient

$$\mathcal{M}_\theta(Q, l, \alpha) := [\mathcal{R}(Q, l, \alpha)_\theta^{ss} / \mathrm{PGL}(\alpha)].$$

Remark 3.10. The definition of $\mathcal{M}_\theta(Q, l, \alpha)$ depends a priori on a choice of basis $\mathfrak{B}$ of R. However, any alternative basis $\mathfrak{B}'$ gives an isomorphic stack. Indeed, given $W = (W_i, w_a, f_b) \in \mathcal{R}(Q, l, \alpha)$ write $b' = n_1 b_1 + \cdots + n_m b_m$ and

$$f_{b'} : \mathbb{k} \cong \mathbb{k}^{\otimes n_1} \otimes \cdots \otimes \mathbb{k}^{\otimes n_m} \to (\det_{b_1} W)^{\otimes n_1} \otimes \cdots \otimes (\det_{b_m} W)^{\otimes n_m} \cong \det_{b'} W$$

for every $b' \in \mathfrak{B}'$. The assignment $(W_i, w_a, f_b) \mapsto (W_i, w_a, f_{b'})$ gives an equivariant isomorphism from $\mathcal{R}(Q, l, \alpha)$ to $\mathcal{R}(Q, l, \alpha)$, under which semistable points are sent to semistable points. This follows from the fact semistability depends only on the subgroup $R \subset \mathrm{Wt}(Q)$ and the factor (W_i, w_a) of W. The factor (W_i, w_a) is not altered by the proposed isomorphism; checking $b(W) = 0$ for basis elements $b \in \mathfrak{B}$ is equivalent to checking $r(W) = 0$ on every element $r \in R$. Hence the equivariant isomorphism above defines an isomorphism of the resulting stacks $\mathcal{M}_\theta(Q, l, \alpha)$.

This is not to say that the choice of basis is unimportant. It only becomes unimportant when we insist that the linear maps $f_b : \mathbb{k} \to \det_b W$ are isomorphisms. Indeed, let $b' = -b$. Then given linear map $f_b : \mathbb{k} \to \det_b W$ there exists a natural linear map $(f_b)^\vee : \det_{b'} W \cong (\det_b W)^\vee \to \mathbb{k}^\vee \cong \mathbb{k}$. If f_b is an isomorphism then we

define $f_{b'} := (f_b^\vee)^{-1}$, otherwise there is no natural definition for $f_{b'}$. In Chapter 5, we will allow the maps f_b to be zero and will have to choose a basis carefully.

To justify the term 'moduli stack' in the definition above we must give a suitable notion of families over schemes for which the moduli stack is $\mathcal{M}(Q, l, \alpha)$. One could define a family of refined representations over a scheme S to be a refined representation of (Q, l) in the category of locally free $\mathcal{O}_S$-modules of S, that is, Definition 3.11 without the isomorphism of line bundles $\mathcal{O}_S \to \det_{\theta_\Delta} \mathcal{W}$. However, this would imply that Δ is a subgroup of the automorphism group of any given object. This gives stacks that are unsuitable for our applications as they don't admit closed immersions from DM stacks.

If α is primitive, that is the greatest common factor of its components is 1, we can alter the definition of a family to sidestep this issue. We do this by adding an extra nonzero parameter to $\mathcal{R}(Q, l, \alpha)$ on which Δ acts with weight 1. This amounts to finding a character $\theta_\Delta \in \mathbb{Z}^{Q_0}$ for which $\langle \theta_\Delta, \Delta \rangle = \sum_i \theta_i \alpha_i = 1$, one may find such θ_Δ precisely when α is primitive. For the rest of the section fix a primitive dimension vector α and pick $\theta_\Delta \in \mathbb{Z}^{Q_0}$ such that $\langle \theta_\Delta, \Delta \rangle = 1$.

Definition 3.11. A *flat family of refined representations of* (Q, l) over a connected scheme S is a collection of rank α_i locally free sheaves $\mathcal{W}_i$ for $i \in Q_0$, together with a choice of morphisms $\mathcal{W}_{t(a)} \to \mathcal{W}_{h(a)}$ for $a \in Q_1$, isomorphisms of line bundles $\mathcal{O}_S \to \det_b \mathcal{W}$ for $b \in \mathfrak{B}$ and an isomorphism of line bundles $\mathcal{O}_S \to \det_{\theta_\Delta} \mathcal{W}$.

Proposition 3.12. *The stack* $\mathcal{M}_\theta(Q, l, \alpha)$ *is the moduli stack of families of* θ-*semistable refined representations of* (Q, l).

Proof. First we identify the nonzero elements of $\det_{\theta_\Delta} W$ with $\mathbb{k}^\times$, $\mathrm{GL}(\alpha)$ acts on $\det_{\theta_\Delta} W$ by change of basis. Consider the stack quotient $[\mathcal{R}(Q, l, \alpha)_\theta^{ss} \times \mathbb{k}^\times / \mathrm{GL}(\alpha)]$. We claim that this represents the moduli problem defined by Definition 3.11. An object in $[\mathcal{R}(Q, l, \alpha)_\theta^{ss} \times \mathbb{k}^\times / \mathrm{GL}(\alpha)](S)$ is a principal $\mathrm{GL}(\alpha)$-bundle $\mathcal{P} := \bigoplus_{i \in Q_0} \mathcal{P}_i$ over S with a $\mathrm{GL}(\alpha)$-equivariant morphism $\mathcal{P} \to \mathcal{R}(Q, l, \alpha)_\theta^{ss} \times \mathbb{k}^\times$. Define $\mathcal{W}_i$ to be the W_i-bundles corresponding to $\mathcal{P}_i$. Let $(U_j)_{j \in J}$ be an open cover of S that trivializes $\mathcal{P}_i$. For $j \in J$ an equivariant morphism $U_j \times \mathrm{GL}(\alpha) \to \mathcal{R}(Q, l, \alpha)_\theta^{ss} \times \mathbb{k}^\times$ is determined by the image of the identity fibre and so is determined by a morphism $U_j \to \mathcal{R}(Q, l, \alpha)_\theta^{ss} \times \mathbb{k}^\times$. This morphism in turn defines a section of the vector bundle $U_j \times \mathrm{Hom}(W_{t(a)}, W_{h(a)})$ for every $a \in Q_1$, a nonzero section of $U_j \times \det_b W$ for every $b \in \mathfrak{B}$ and a nonzero section of $U_j \times \det_{\theta_\Delta} W$. Since these sections come from a globally defined map $\mathcal{P} \to \mathcal{R}(Q, l, \alpha)_\theta^{ss} \times \mathbb{k}^\times$, they glue to give the required family over S. Similarly a family over S defines an object of $[\mathcal{R}(Q, l, \alpha)_\theta^{ss} \times \mathbb{k}^\times / \mathrm{GL}(\alpha)](S)$.

Morphisms of families over S correspond naturally to morphisms of objects of $[\mathcal{R}(Q,l,\alpha)_\theta^{ss} \times \Bbbk^\times/\mathrm{GL}(\alpha)](S)$.

The choice of θ_Δ implies that Δ acts with weight one on the space of isomorphisms from $\Bbbk$ to $\det_{\theta_\Delta} W$. For every element $(W,t) \in \mathcal{R}(Q,l,\alpha)_\theta^{ss} \times \Bbbk^\times$ there exists a unique element of Δ that acts on (W,t) to give $(W,1)$. The subgroup of $\mathrm{GL}(\alpha)$ that fixes the $\Bbbk^\times$ component is isomorphic to $\mathrm{PGL}(\alpha)$, so we have a stack isomorphism

$$[\mathcal{R}(Q,l,\alpha)_\theta^{ss} \times \Bbbk^\times/\mathrm{GL}(\alpha)] \cong [\mathcal{R}(Q,l,\alpha)_\theta^{ss}/\mathrm{PGL}(\alpha)] = \mathcal{M}_\theta(Q,l,\alpha)$$

as required. $\qquad\square$

The choice θ_Δ might seem ad hoc at the moment, but a natural choice presents itself in our applications, see Remark 4.3.

Chapter 4

Quivers of sections

In this chapter we introduce our generalization of the classical linear series construction to smooth projective toric DM stacks $\mathcal{X}$ with trivial generic stabilizers. Starting with a collection of distinct line bundles $\mathcal{L} = (L_0, L_1, \ldots, L_r)$ on $\mathcal{X}$ we produce a labelled quiver (Q, div) and give a rational map $\mathcal{X} \dashrightarrow \mathcal{M}_\theta(Q, \mathrm{div}, \alpha)$ with $\alpha = (1, \ldots, 1)$. We then go on to study certain properties of this rational map. Throughout this section we assume that all our stacks $\mathcal{X}$ are smooth projective toric DM stacks with trivial generic stabilizers; we will also fix the dimension vector to be $\alpha := (1, \ldots, 1)$ and drop it from the notation.

4.1 The map to $\mathcal{M}(Q, \mathbf{div})$

The main hurdle in defining a sensible stacky multilinear series construction was the definition of the ambient stack. Now that we have a suitable ambient stack, the methods used to define the map $\varphi_{|\mathcal{L}|} : X \dashrightarrow \mathcal{M}(Q)$, in Subsections 2.4.2 and 2.4.3, go through to the stacks case almost word for word.

We begin by extending the definition of a quiver of sections, as defined in Definition 2.32, to toric DM stacks.

Definition 4.1. Let $\mathcal{X}$ be a projective toric stack with dense torus $T_\mathcal{X}$ and let $\mathcal{L} = (L_0, \ldots, L_n) \subset \mathrm{Pic}(\mathcal{X})$ be a collection of distinct line bundles.

1. A $T_\mathcal{X}$-invariant section $s \in \Gamma(\mathcal{X}, L_j \otimes L_i^\vee)$ is *reducible* if there exists $L_k \in \mathcal{L}$ and $T_\mathcal{X}$-invariant sections $s' \in \Gamma(\mathcal{X}, L_k \otimes L_i^\vee), s'' \in \Gamma(\mathcal{X}, L_j \otimes L_k^\vee)$ such that $s = s' \otimes s''$. A $T_\mathcal{X}$-invariant section is *irreducible* if it is not reducible.

2. The *labelled quiver of sections* associated to $\mathcal{L}$ is the labelled quiver $(Q_\mathcal{L}, \mathrm{div})$ where the vertices of $Q_\mathcal{L}$ correspond to the line bundles in $\mathcal{L}$ with an arrow from i to j for every irreducible $T_\mathcal{X}$-invariant section $s \in \Gamma(\mathcal{X}, L_j \otimes L_i^\vee)$. The

labelling map div : $Q_1 \rightarrow \mathbb{Z}^{\Sigma(1)}$ sends a to the corresponding $T_{\mathcal{X}}$-invariant divisor in $\mathbb{Z}^{\Sigma(1)}$.

We will adopt similar conventions to Conventions 2.34.

Conventions 4.2. Let (Q, div) be the labelled quiver of sections corresponding to the collection $\mathscr{L} = (L_0, \ldots, L_r)$.

(a) We will assume that $L_0 = \mathcal{O}_{\mathcal{X}}$.

(b) We will also assume that $\Gamma(\mathcal{X}, L_i) \neq 0$ for $L_i \in \mathscr{L}$.

Remark 4.3. For labelled quivers of sections of line bundles, Convention 4.2 a) fixes $\theta_\Delta = (1, 0, \ldots, 0)$.

Keeping the notation of Section 2.4.1, define pic : $\mathrm{Wt}(Q) \rightarrow \mathrm{Pic}(\mathcal{X})$ by $\theta = \sum_{i \in Q_0} \theta_i \mathbf{e}_i \mapsto \bigotimes_{i \in Q_0} L_i^{\otimes \theta_i}$ and let deg : $\mathbb{Z}^{\Sigma(1)} \rightarrow \mathrm{Pic}(\mathcal{X})$ be the homomorphism in short exact sequence (2.2). We then have the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}^{Q_1} & \xrightarrow{\ \mathrm{inc}\ } & \mathrm{Wt}(Q) \\
{\scriptstyle \mathrm{div}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{pic}} \\
\mathbb{Z}^{\Sigma(1)} & \xrightarrow{\ \mathrm{deg}\ } & \mathrm{Pic}(\mathcal{X}).
\end{array}
\tag{4.1}
$$

The subgroup R is by definition the image under inc of the kernel of div, so diagram (4.1) restricts to the following commutative diagram

$$
\begin{array}{ccc}
R & \xhookrightarrow{\ \iota\ } & \mathrm{Wt}(Q) \\
{\scriptstyle 0}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{pic}} \\
\mathbb{Z}^{\Sigma(1)} & \xrightarrow{\ \mathrm{deg}\ } & \mathrm{Pic}(\mathcal{X}).
\end{array}
\tag{4.2}
$$

Define a $\mathrm{Wt}(Q)$-grading of the monoid algebra $\Bbbk[\mathbb{N}^{Q_1} \oplus R]$ by assigning the monomial $y^u z^v \in \Bbbk[\mathbb{N}^{Q_1} \oplus R]$ degree $\mathrm{inc}(u) + \iota(v)$. After noting that the characters of $\mathrm{PGL}(\alpha)$ are given by elements of $\mathrm{Wt}(Q) \subset \mathbb{Z}^{Q_0}$, we observe that this $\mathrm{Wt}(Q)$-grading of $\Bbbk[\mathbb{N}^{Q_1} \oplus R]$ induces the change of basis action of $\mathrm{PGL}(\alpha)$ on $\mathcal{R}(Q, \mathrm{div}) \cong \mathrm{Spec}(\Bbbk[\mathbb{N}^{Q_1} \oplus R])$. On the other hand, the map deg gives the $\mathrm{Pic}(\mathcal{X})$-grading of $\Bbbk[\mathbb{N}^{\Sigma(1)}]$ that arises from the short exact sequence (2.2).

By construction $\mathrm{div}(\mathbb{N}^{Q_1}) \subset \mathbb{N}^{\Sigma(1)}$, so the map $\mathrm{div} \oplus 0 : \mathbb{N}^{Q_1} \oplus R \rightarrow \mathbb{N}^{\Sigma(1)}$ induces a map of monoid algebras $\Psi : \Bbbk[\mathbb{N}^{Q_1} \oplus R] \rightarrow \Bbbk[\mathbb{N}^{\Sigma(1)}]$, which in turn defines a morphism Ψ^* from $\mathbb{A}^{\Sigma(1)}$ to $\mathcal{R}(Q, \mathrm{div})$. This morphism is equivariant with respect

to the actions of the groups $\mathrm{Hom}(\mathrm{Pic}(\mathcal{X}), \Bbbk^\times)$ and $\mathrm{PGL}(\alpha) \cong \mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times)$ on $\mathbb{A}^{\Sigma(1)}$ and $\mathcal{R}(Q, \mathrm{div})$ because the diagrams (4.1) and (4.2) commute. Thus for any $\theta \in \mathrm{Wt}(Q)$, Ψ induces a rational map

$$\psi_\theta : \mathcal{X} \dashrightarrow \mathcal{M}_\theta(Q, \mathrm{div}).$$

4.2 Base-point free collections

In this section we discuss the conditions under which there exists a sufficiently nice $\theta \in \mathrm{Wt}(Q)$ for which ψ_θ is a morphism.

We say a character $\chi_\theta \in \mathrm{PGL}(\alpha)^\vee$ is *generic* if every χ_θ-semistable point is χ_θ-stable.

Definition 4.4. Let $\mathcal{X}$ be as above. A collection of line bundles $\mathscr{L} = (\mathcal{O}_\mathcal{X}, L_1, \ldots, L_r)$ is *base-point free* if there exists a generic $\chi_\theta \in \mathrm{PGL}(\alpha)^\vee$ for which the rational map $\psi_\theta : \mathcal{X} \dashrightarrow \mathcal{M}_\theta(Q, \mathrm{div})$ is a morphism.

Remark 4.5. i) For a base-point free collection $\mathscr{L}$, the dependence of the morphism ψ_θ on θ is addressed in Remark 4.16.

ii) In the case where χ_θ is not generic $\mathcal{M}_\theta(Q, \mathrm{div})$ is not a Deligne-Mumford stack. Furthermore, it rarely has a coarse moduli space (as in Definition 2.14). We insist χ_θ is generic to avoid such ambient stacks.

We follow the traditional definition of base-point freeness of a line bundle on a variety to give the following definition of a base-point free line bundle on a stack.

Definition 4.6. Let L be an effective line bundle over $\mathcal{X}$ and $(s_0, \ldots, s_n)$ be a basis of $\Gamma(\mathcal{X}, L)$. Then L is *base-point free* if the rational map $\varphi_{|L|} : \mathcal{X} \dashrightarrow \mathbb{P}(\Gamma(\mathcal{X}, L))$, which takes $x \in \mathcal{X}$ to $(s_0(x), \ldots, s_n(x)) \in \mathbb{P}(\Gamma(\mathcal{X}, L))$, is a morphism.

Lemma 4.7. *Let $\mathcal{X}$ be a projective toric DM stack. A nontrivial line bundle L on $\mathcal{X}$ is base-point free if and only if the collection $\mathscr{L} = (\mathcal{O}_\mathcal{X}, L)$ is base-point free.*

Proof. The map pic takes the basis vector $\mathbf{e}_1 - \mathbf{e}_0$ of $\mathrm{Wt}(Q) \cong \mathbb{Z}$ to L. This implies that $\ker(\mathrm{pic})$ is trivial, otherwise $L^{\otimes n} \cong \mathcal{O}_\mathcal{X}$ for some $n > 0$ contradicting projectivity of $\mathcal{X}$. Now R is a subgroup of $\ker(\mathrm{pic})$ and is therefore trivial. Therefore $\mathcal{M}_\theta(Q, \mathrm{div})$ is $\mathcal{M}_\theta(Q)$. Since Q is acyclic, the only chamber in $\mathrm{Wt}(Q)_\mathbb{Q} \cong \mathbb{Q}$ is $\mathbb{Q}_{>0}$; take θ in this chamber, then $\mathcal{M}_\theta(Q) \cong \mathbb{P}(\Gamma(\mathcal{X}, L))$ from which the claim follows. $\square$

Given a base-point free line bundle L on $\mathcal{X}$, the morphism $\varphi_{|L|}$ factors through the coarse moduli space. This follows by the universal property of coarse moduli spaces and the fact that $\mathbb{P}(\Gamma(\mathcal{X}, L))$ is a scheme.

$$\mathcal{X} \xrightarrow{\varphi_{|L|}} \mathbb{P}(\Gamma(\mathcal{X}, L))$$
$$\pi \searrow \qquad \nearrow$$
$$X$$

Now L is the pullback of the tautological bundle on $\mathbb{P}(\Gamma(\mathcal{X}, L))$; pulling back through the composite $\mathcal{X} \to X \to \mathbb{P}(\Gamma(\mathcal{X}, L))$ we see that every base-point free line bundle can be pulled back from the coarse moduli space. This further emphasizes the need for sections of several line bundles.

In general, the rational map ψ_θ is a morphism of stacks if and only if the inverse image under Ψ^* of the θ-unstable locus of $\mathcal{R}(Q, \mathrm{div})$ is contained in the Cox unstable locus $\mathbb{V}(B_\mathcal{X})$, as defined in (2.3) (see Perroni's work on morphisms of toric stacks [Per08]).

Example 4.8. Let $\mathcal{X} = \mathbb{P}(1, 2, 3)$ and $\mathscr{L} = (\mathcal{O}_\mathcal{X}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3))$. The labelled quiver of sections of $\mathscr{L}$ is shown in Figure 4.1,
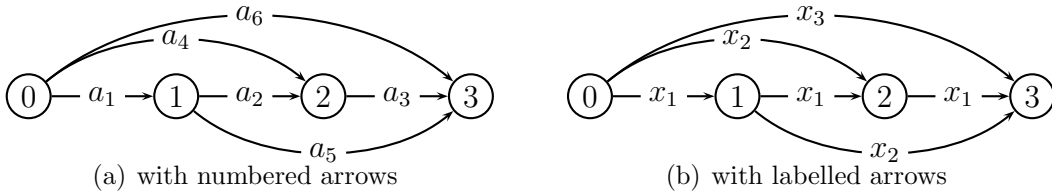
Figure 4.1: A labelled quiver of sections of $\mathbb{P}(1, 2, 3)$.

with labelling map $\mathrm{div} : \mathbb{Z}^6 \to \mathbb{Z}^{\Sigma(1)} \cong \mathbb{Z}^3$ defined by the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The subgroup $\ker(\mathrm{div})$ is then given by

$$\ker(\mathrm{div}) = \langle \mathbf{e}_{a_1} - \mathbf{e}_{a_2}, \mathbf{e}_{a_1} - \mathbf{e}_{a_3}, \mathbf{e}_{a_4} - \mathbf{e}_{a_5} \rangle \subset \mathbb{Z}^{Q_1}.$$

Mapping $\ker(\mathrm{div})$ under the incidence map gives

$$R = \langle \mathbf{e}_0 - 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 \rangle \subset \mathrm{Wt}(Q).$$

The free group R is of rank 2, we therefore have 2 homogenizing variables z_1, z_2. So $\mathcal{R}(Q, \mathrm{div}) \cong \mathrm{Spec}(\Bbbk[y_1, \ldots, y_6, z_1^{\pm}, z_2^{\pm}])$. The map $\Psi^* : \mathbb{A}^3 \longrightarrow \mathcal{R}(Q, \mathrm{div})$ sends (x_1, x_2, x_3) to $(x_1, x_1, x_1, x_2, x_2, x_3, 1, 1)$. For $\theta = -3\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \in \mathrm{Wt}(Q)$ the θ-unstable locus is given by the vanishing locus of the ideal

$$B_\theta := \left\langle y^u z^v \in \Bbbk[\mathbb{N}^{Q_1} \oplus \mathbb{Z}^2] \,\middle|\, \mathrm{inc}(u) + \iota(v) = \theta \right\rangle.$$

The ideal B_θ contains the monomials

$$y_1{}^6 z_1{}^2 z_2, \quad y_2{}^6 z_1{}^{-4} z_2, \quad y_3{}^6 z_1^2 z_2^{-5}, \quad y_4{}^3 z_1{}^{-1} z_2, \quad y_5{}^3 z_1{}^{-1} z_2{}^{-2}, \quad y_6{}^2 z_2{}^{-1}.$$

Since the parameters z_i are nonzero this implies $\mathbb{V}(B_\theta) = \mathbb{V}(y_1, \ldots, y_6)$ which in turn implies that

$$\mathcal{M}_\theta(Q, \mathrm{div}) \cong \left[\frac{\mathbb{A}^6 \times (\Bbbk^\times)^2 \setminus \{0\} \times (\Bbbk^\times)^2}{(\Bbbk^\times)^3} \right] \cong \mathbb{P}(1, 1, 1, 2, 2, 3).$$

Now the Cox unstable locus of $\mathbb{P}(1, 2, 3)$ is $\mathbb{V}(x_1, x_2, x_3) = \{0\} \subset \mathbb{A}^3$ and the morphism Ψ^* maps $(x_1, x_2, x_3) \in \mathbb{A}^3 \setminus \{0\}$ to $(x_1, x_1, x_1, x_2, x_2, x_3, 1, 1) \in \mathcal{R}(Q, \mathrm{div}) \setminus \mathbb{V}(B_\theta)$. Therefore Ψ^* descends to a morphism

$$\psi_\theta : \mathbb{P}(1, 2, 3) \longrightarrow \mathbb{P}(1, 1, 1, 2, 2, 3)$$

that sends (x_1, x_2, x_3) to $(x_1, x_1, x_1, x_2, x_2, x_3)$.

Example 4.9 (cf. Example 2.28). Let $\mathcal{X} = \mathbb{P}(1, 1, 2)$, $\mathscr{L} = (\mathcal{O}_\mathcal{X}, \mathcal{O}(1), \mathcal{O}(3))$. The associated quiver of sections is displayed in Figure 4.2.

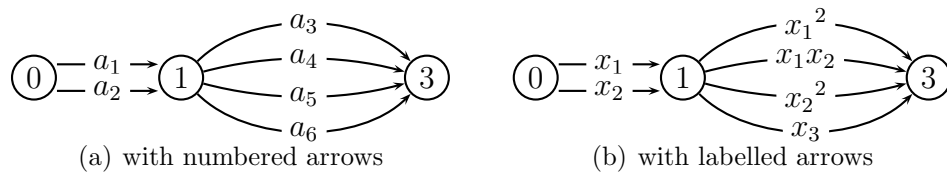Figure 4.2: Abramovich-Hassett construction for $\mathbb{P}(1, 1, 2)$ and $n = 0, m = 2$.

The group R is given by $\langle 2\mathbf{e}_0 - 3\mathbf{e}_1 + \mathbf{e}_3 \rangle \subset \mathrm{Wt}(Q)$, so

$$\mathcal{R}(Q, \mathrm{div}) \cong \mathrm{Spec}(\Bbbk[y_1, \ldots, y_6, z^{\pm 1}])$$

. For $\theta = -2\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_3$ the θ-unstable locus is cut out by the vanishing locus of

the ideal

$$B_\theta := \Big\langle y^u z^v \in \Bbbk[\mathbb{N}^6 \oplus \mathbb{Z}] \,\Big|\, \mathrm{inc}(u) + \iota(v) = \theta \Big\rangle.$$

The ideal B_θ the contains monomials $y_1^4 z$, $y_2^4 z$, $y_3^2 z^{-1}$, $y_4^2 z^{-1}$, $y_5^2 z^{-1}$, $y_6^2 z^{-1}$, therefore $\mathbb{V}(B_\theta) = \mathbb{V}(y_1, \ldots, y_6)$. The moduli stack $\mathcal{M}_\theta(Q, \mathrm{div})$ is then given by

$$\mathcal{M}_\theta(Q, \mathrm{div}) = \left[\frac{\mathbb{A}^6 \times \Bbbk^\times \setminus \{0\} \times \Bbbk^\times}{(\Bbbk^\times)^2} \right] \cong \mathbb{P}(1, 1, 2, 2, 2, 2).$$

The morphism $\psi_\theta : \mathbb{P}(1, 1, 2) \longrightarrow \mathbb{P}(1, 1, 2, 2, 2, 2)$ is given by

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_1^2, x_1 x_2, x_2^2, x_3).$$

Remark 4.10. Example 4.9 recovers the Abramovich-Hassett [AH] construction for $\mathcal{X} = \mathbb{P}(1, 1, 2)$ with polarizing line bundle $L = \mathcal{O}(1)$ and natural numbers $n = 0$ and $m = 2$. More generally, given a polarizing line bundle L on $\mathcal{X}$, one may recover the Abramovich-Hassett construction when $n = 0$ by applying our machinery to the collection

$$\mathscr{L} = (\mathcal{O}_\mathcal{X}, L, L \otimes L^{\otimes 2}, \ldots, L^{\otimes m(m+1)/2})$$

and if necessary, working with an 'incomplete' quiver of sections. An incomplete quiver of sections is a quiver of sections where not all torus-invariant sections contribute to paths in the quiver, analogous to an incomplete linear series.

We now seek a condition on a collection $\mathscr{L}$ under which $\mathscr{L}$ is base-point free. Unlike Examples 4.8 and 4.9, in general, the θ-unstable locus in $\mathcal{R}(Q, \mathrm{div})$ does not coincide with the vanishing locus of the ideal

$$B_\theta := \Big\langle y^u z^v \in \Bbbk[\mathbb{N}^{Q_1} \oplus R] \,\Big|\, \mathrm{inc}(u) + \iota(v) = \theta \Big\rangle,$$

for $\theta \in \mathrm{Wt}(Q)$. This further complicates the pursuit of the sought after condition on $\mathscr{L}$. However, for any $\theta \in \mathrm{Wt}(Q)$ there is a positive integer m for which the θ-unstable locus coincides with the vanishing locus of the ideal $B_{m\theta}$.

Lemma 4.11. *Let $\mathcal{X}$ be a toric DM stack and $L_1, \ldots, L_n \in \mathrm{Pic}(\mathcal{X})$ be such that every L_i, for $1 \le i \le n$, is base-point free. Given a section s of $L := L_1 \otimes \cdots \otimes L_n$, there exists $m \in \mathbb{N}$ such that s^m is in the image of the multiplication map*

$$\mu_m : \Gamma(\mathcal{X}, L_1)^{\otimes m} \otimes_\Bbbk \cdots \otimes_\Bbbk \Gamma(\mathcal{X}, L_n)^{\otimes m} \to \Gamma(\mathcal{X}, L^{\otimes m}).$$

Proof. Let $\Bbbk[x_0, \ldots, x_n]$ be the Cox ring of $\mathcal{X}$ and $\mu : \Gamma(\mathcal{X}, L_1) \otimes \cdots \otimes \Gamma(\mathcal{X}, L_n) \to \Gamma(\mathcal{X}, L)$ be the multiplication map. The line bundles L_i are base-point free and

so correspond to polytopes P_{L_i}. The polytope P_L corresponding to L is given by $P_{L_1} + \ldots + P_{L_n}$ (see page 69 of Fulton [Ful93]). While the lattice points of P_L are not, in general, a sum of lattice points of the polytopes P_{L_i}, the vertices of P_L are given by sums of vertices of P_{L_i} (see Theorem 3.1.2 of Weibel's Ph.D. thesis [Wei07]). Therefore the sections corresponding to the vertices of P_L lie in $\operatorname{im}(\mu)$. Since the vanishing locus of the sections corresponding to the vertices of P_L is equal to that of the sections of L, we have that the vanishing locus of the sections in $\operatorname{im}(\mu)$ is equal to that of the sections of L. Now let $s \in \Gamma(\mathcal{X}, L)$, then by Hilbert's Nullstellensatz there exists a natural number $m \in \mathbb{Z}$ such that s^m is in the ideal generated by $\operatorname{im}(\mu)$ as required. $\qquad\square$

For a collection of line bundles $\mathscr{L} = (\mathcal{O}_\mathcal{X}, L_1, \ldots, L_r)$ on $\mathcal{X}$, define

$$\mathscr{L}_{\mathrm{bpf}} := \{ L_i^\vee \otimes L_j \mid L_i, L_j \in \mathscr{L} \text{ and } L_i^\vee \otimes L_j \text{ is base-point free} \}$$

and let $\operatorname{pic}_\mathbb{Q} : \operatorname{Wt}(Q)_\mathbb{Q} \to \operatorname{Pic}(\mathcal{X})_\mathbb{Q}$ be $\operatorname{pic} \otimes \operatorname{id}$.

Lemma 4.12. *If* $\operatorname{rank}(\mathbb{Z}\mathscr{L}) = \operatorname{rank}(\mathbb{Z}\mathscr{L}_{\mathrm{bpf}})$ *then* $\ker(\operatorname{pic} \otimes \mathbb{Q}) \subset R \otimes \mathbb{Q}$.

Proof. In this proof we use additive notion for the binary operation on the Picard group to avoid confusion with $- \otimes \mathbb{Q}$.

Let $\omega = \sum_{i \in Q_0} \omega_i \otimes q_i \in \ker(\operatorname{pic}_\mathbb{Q})$ and pick $n \in \mathbb{N}$ sufficiently large so that $n\omega = \sum_{i \in Q_0} n_i \omega_i \otimes 1$ for $n_i \in \mathbb{Z}$ and set $\lambda := \sum_{i \in Q_0} n_i \omega_i$. Since $R_\mathbb{Q}$ is a vector subspace of $\operatorname{Wt}(Q)_\mathbb{Q}$ we have $\omega \in R_\mathbb{Q}$ if and only if $n\omega \in R_\mathbb{Q}$ for any $n \in \mathbb{Q} \setminus \{0\}$. Therefore it suffices to show $n\omega = \lambda \otimes 1 \in R_\mathbb{Q}$, that is there exists an element $\tau \in \mathbb{Z}^{Q_1} \otimes 1$ such that $\operatorname{inc}_\mathbb{Q}(\tau) = \lambda \otimes 1$ and $\operatorname{div}_\mathbb{Q}(\tau) = 0$.

Take the basis $E := \{\mathbf{e}_1 - \mathbf{e}_0, \ldots, \mathbf{e}_r - \mathbf{e}_0\}$ of $\operatorname{Wt}(Q)$ and write λ as a difference of positive and negative parts, that is, write $\lambda = \lambda_+ - \lambda_-$ for $\lambda_+, \lambda_- \in \mathbb{N}E$ without cancellation. Let $L_\pm = \operatorname{pic}(\lambda_\pm)$. The fact that $\lambda \otimes 1 \in \ker(\operatorname{pic}_\mathbb{Q})$ implies $L_+ \otimes 1 = L_- \otimes 1$ and the rank assumption gives us

$$L_+ \otimes 1 = L_- \otimes 1 = \sum L_{b_i} \otimes q_i \quad \text{for } L_{b_i} \in \mathscr{L}_{\mathrm{bpf}} \text{ and } q_i \in \mathbb{Q}. \tag{4.3}$$

We may take n big enough to ensure that each $q_i \in \mathbb{Z}$. Rearrange equations (4.3) to get

$$(L_+ \otimes 1) + \left(\sum_{q_i < 0} -q_i L_{b_i} \otimes 1 \right) = \sum_{q_i > 0} q_i L_{b_i} \otimes 1 \tag{4.4}$$

$$(L_- \otimes 1) + \left(\sum_{q_i < 0} -q_i L_{b_i} \otimes 1 \right) = \sum_{q_i > 0} q_i L_{b_i} \otimes 1. \tag{4.5}$$

Fix a section of each of the following line bundles: L_+, L_- and L_{b_i} for which $q_i < 0$ (in turn fixing a section of $\sum_{q_i<0} -q_i L_{b_i}$). Using equations (4.4) and (4.5), this fixes sections $s_\pm$ of $\sum_{q_i>0} q_i L_{b_i} + L_{t\pm}$ for some torsion line bundles $L_{t\pm}$. Without loss of generality we assume $L_{t\pm} = 0$, otherwise multiply n in the beginning of the proof by the orders of $L_{t\pm}$.

Since the incidence map is onto $\mathrm{Wt}(Q)$ there exists elements $\tau_1 \in \mathbb{Z}^{Q_1}$ and $\tau_2 \in \mathbb{Z}^{Q_1}$ such that $\mathrm{inc}(\tau_1) = \lambda_+$ and $\mathrm{inc}(\tau_2) = \lambda_-$. By Lemma 4.11 there exists $m_\pm \in \mathbb{N}$ such that $s_\pm^{m_\pm}$ are a product of sections of the line bundles L_{b_i} for which $q_i > 0$. By definition of the quiver of sections Q, every section of a line bundle in $\mathscr{L}_{\mathrm{bpf}}$ gives rise to a path in the quiver, so there exists $\tau_\pm \in \mathbb{Z}^{Q_1}$ such that $\mathrm{div}(\tau_\pm) = s_\pm^{m_\pm}$. Define

$$\tau := (\tau_1 \otimes 1) - \left(\tau_+ \otimes \frac{1}{m_+}\right) - (\tau_2 \otimes 1) + \left(\tau_- \otimes \frac{1}{m_-}\right).$$

We have that $\mathrm{inc}_\mathbb{Q}(\tau) = \lambda \otimes 1$ because $\mathrm{inc}_\mathbb{Q}(\tau_+ \otimes \frac{1}{m_+}) = \mathrm{inc}_\mathbb{Q}(\tau_- \otimes \frac{1}{m_-})$. We also have that $(\tau_1 \otimes 1) - (\tau_+ \otimes \frac{1}{m_+})$ and $(\tau_2 \otimes 1) - (\tau_- \otimes \frac{1}{m_-})$ map via div to the section of the line bundle $\sum_{q_i<0} -q_i L_{b_i}$ fixed above, and hence $\mathrm{div}_\mathbb{Q}(\tau) = 0$, as required. $\square$

The following example highlights that tensoring with $\mathbb{Q}$ in the statement of Lemma 4.12 is necessary.

Example 4.13. Consider the $(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$-grading of $\mathbb{k}[x_1, x_2, x_3, x_4]$ given by:

$$\deg(x_1) = (1,0,0); \ \deg(x_2) = (1,1,0); \ \deg(x_3) = (1,0,1); \ \deg(x_4) = (1,1,1)$$

and let $(\mathbb{k}^\times \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \curvearrowright \mathbb{A}^4$ be the corresponding action. Take

$$\mathcal{X} = [(\mathbb{A}^4 \setminus \{0\})/(\mathbb{k}^\times \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})]$$
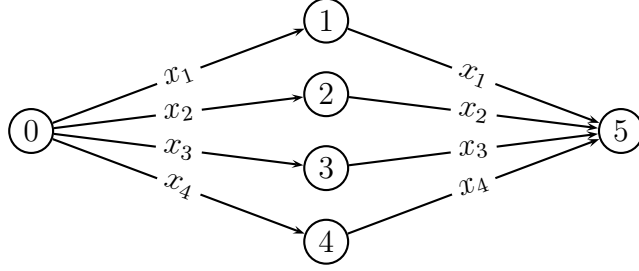
and

$$\mathscr{L} = (\mathcal{O}, \mathcal{O}(1,0,0), \mathcal{O}(1,1,0), \mathcal{O}(1,0,1), \mathcal{O}(1,1,1), \mathcal{O}(2,0,0)).$$

The quiver of sections of $\mathscr{L}$ is given in Figure 4.3. We have that $\mathcal{O}(2,0,0)$ is base-point free and so is an element of $\mathscr{L}_{\mathrm{bpf}}$ therefore $\mathrm{rank}(\mathbb{Z}\mathscr{L}) = \mathrm{rank}(\mathbb{Z}\mathscr{L}_{bpf})$. The free group R is given by

$$R = \langle \mathbf{e}_5 - 2\mathbf{e}_1 + \mathbf{e}_0, \ \mathbf{e}_5 - 2\mathbf{e}_2 + \mathbf{e}_0, \ \mathbf{e}_5 - 2\mathbf{e}_3 + \mathbf{e}_0, \ \mathbf{e}_5 - 2\mathbf{e}_4 + \mathbf{e}_0 \rangle \subset \mathrm{Wt}(Q).$$

Now $\ker(\mathrm{pic})$ contains $\mathbf{e}_1 + \mathbf{e}_4 - \mathbf{e}_2 - \mathbf{e}_3 \notin R$, hence $\ker(\mathrm{pic}) \not\subset R$. Note that $2(\mathbf{e}_1 + \mathbf{e}_4 - \mathbf{e}_2 - \mathbf{e}_3) \in R$.

Figure 4.3: Quiver of sections for $\mathscr{L}$ on $\mathcal{X}$.

Theorem 4.14. *If* $\operatorname{rank}(\mathbb{Z}\mathscr{L}) = \operatorname{rank}(\mathbb{Z}\mathscr{L}_{\mathrm{bpf}})$ *then* $\mathscr{L}$ *is base-point free.*

Proof. Let $\theta \in \operatorname{Wt}(Q)$. The associated character $\chi_\theta \in \operatorname{PGL}(\alpha)^\vee$ gives a morphism of stacks $\mathcal{X} \to \mathcal{M}_\theta(Q, \operatorname{div})$ if and only if the inverse image of the θ-unstable points of $\mathcal{R}(Q, \operatorname{div})$ is contained in $\mathbb{V}(B_\mathcal{X})$. After picking a higher multiple if necessary, we may assume that the θ-unstable locus in $\mathcal{R}(Q, \operatorname{div})$ is precisely the vanishing locus of the monomial ideal

$$B_\theta := \Big\langle\, y^u z^v \in \Bbbk[\mathbb{N}^{Q_1} \oplus R] \,\Big|\, \operatorname{inc}(u) + \iota(v) = \theta \,\Big\rangle$$

and its inverse image $\psi^{-1}(\mathbb{V}(B_\theta)) \subset \mathbb{A}^{\Sigma(1)}$ is the vanishing locus of the monomial ideal

$$\operatorname{div} B_\theta := \Big\langle\, x^{\operatorname{div}((u,v))} \in \Bbbk[\mathbb{N}^{\Sigma(1)}] \,\Big|\, \operatorname{inc}(u) + \iota(v) = \theta \,\Big\rangle.$$

Now let $L := \operatorname{pic}(\theta)$ and define

$$B_L := \Big\langle\, x^\nu \in \Bbbk[\mathbb{N}^{\Sigma(1)}] \,\Big|\, \deg(\nu) = L \,\Big\rangle.$$

Now given χ_θ for which $L = \operatorname{pic}(\theta) \in \mathscr{L}_{\mathrm{bpf}}$, pick m big enough such that the $m\theta$-unstable locus is cut out by $B_{m\theta}$. The line bundle $L \in \mathscr{L}_{\mathrm{bpf}}$ so for every $\nu \in \mathbb{N}^{\Sigma(1)}$ for which $\deg(\nu) = L$ there exists an element $\rho_s \in \mathbb{Z}^{Q_1}$ such that $\operatorname{div}(\rho_s, 0) = \nu$, so $B_L = \operatorname{div} B_\theta$. The ideal $\operatorname{div} B_{m\theta}$ is contained in $B_{L^{\otimes m}}$ and the vanishing locus of $\operatorname{div} B_{m\theta}$ is contained in that of $\operatorname{div} B_\theta$. Therefore

$$\mathbb{V}(B_{L^{\otimes m}}) \subset \mathbb{V}(\operatorname{div} B_{m\theta}) \subset \mathbb{V}(\operatorname{div} B_\theta) = \mathbb{V}(B_L).$$

Since $\operatorname{pic}(m\theta) = L^{\otimes m}$ and L is base-point free, Lemma 4.11 implies $\mathbb{V}(B_L) = \mathbb{V}(B_{L^{\otimes m}})$, therefore $\mathbb{V}(\operatorname{div} B_{m\theta}) = \mathbb{V}(\operatorname{div} B_\theta)$. The line bundle L base-point free so $\mathbb{V}(B_L) \subset \mathbb{V}(B_\mathcal{X})$ and hence $\psi^{-1}(\mathbb{V}(B_{m\theta})) \subset \mathbb{V}(B_\mathcal{X})$. Hence, the rational map $\psi_\theta : \mathcal{X} \dashrightarrow \mathcal{M}_\theta(Q, \operatorname{div})$ is in fact a morphism of stacks.

It remains to show we may pick a generic character for which the map ψ_θ is a morphism. Let $S := \{\mathbf{e}_j - \mathbf{e}_i \in \operatorname{Wt}(Q) \,|\, \operatorname{pic}(\mathbf{e}_j - \mathbf{e}_i) \text{ is base-point free}\}$ and let

$\sigma \subset \mathrm{Wt}(Q)_{\mathbb{Q}}$ be the cone generated by elements of S and R. Since the generators of the cone map, under pic, to line bundles in $\mathscr{L}_{\mathrm{bpf}}$, we have that any θ in σ gives a morphism $\psi_\theta : \mathcal{X} \to \mathcal{M}_\theta(Q, \mathrm{div})$. We claim that $\sigma \subset \mathrm{Wt}(Q)_{\mathbb{Q}}$ is top dimensional. The vector space $\mathrm{Wt}(Q)_{\mathbb{Q}}$ is isomorphic to $(\ker(\mathrm{pic}_{\mathbb{Q}})) \oplus (\mathrm{im}(\mathrm{pic}_{\mathbb{Q}}))$. The image of $\mathrm{pic}_{\mathbb{Q}}$ is generated by $\mathscr{L}$, the rank assumption then implies that the elements of $\mathscr{L}_{\mathrm{bpf}}$ also generate $\mathrm{im}(\mathrm{pic}_{\mathbb{Q}})$. We have $\mathrm{pic}(S) = \mathscr{L}_{\mathrm{bpf}}$. In addition Lemma 4.12 give us that $\ker(\mathrm{pic}_{\mathbb{Q}}) \subset R_{\mathbb{Q}}$, therefore elements of σ span $\mathrm{Wt}(Q)_{\mathbb{Q}}$ which proves the claim. So one may pick a generic θ in the interior of σ that gives a well defined morphism, hence $\mathscr{L}$ is base-point free.   □

The collection of line bundles $\mathscr{L}_{\mathrm{bpf}}$ for Example 4.8 is empty giving an example of a base-point free collection that does not satisfy the rank condition $\mathrm{rank}(\mathbb{Z}\mathscr{L}) = \mathrm{rank}(\mathbb{Z}\mathscr{L}_{\mathrm{bpf}})$.

Given a base-point free collection of line bundles $\mathscr{L}$, the next proposition explicitly describes the image of ψ_θ. Let $I_{\mathscr{L}} \subset \Bbbk[\mathbb{N}^{Q_1} \oplus R]$ be the ideal given by the following

$$I_{\mathscr{L}} := \Big\langle y^{u_1} z^{v_1} - y^{u_2} z^{v_2} \mid \mathrm{div}(u_1 - u_2) = 0, \mathrm{inc}(u_1 - u_2) + \iota(v_1 - v_2) = 0 \Big\rangle. \quad (4.6)$$

Proposition 4.15. *Let $\mathscr{L}$ be a base-point free collection of line bundles on $\mathcal{X}$ and $\theta \in \mathrm{Wt}(Q)$ be such that ψ_θ is a morphism. Then the image of ψ_θ is given by $[(\mathbb{V}(I_{\mathscr{L}}) \setminus \mathbb{V}(B_\theta))/\mathrm{PGL}(\alpha)] \subset \mathcal{M}_\theta(Q, \mathrm{div})$.*

Proof. The image of the map from $\mathbb{A}^{\Sigma(1)}$ to $\mathbb{A}^{Q_1} \times (\Bbbk^\times)^R$ induced by the monoid homomorphism $\mathrm{div} \oplus 0 : \mathbb{N}^{Q_1} \oplus R \to \mathbb{N}^{\Sigma(1)}$ is given by the vanishing locus of the toric ideal

$$I := \Big\langle y^{u_1} z^{v_1} - y^{u_2} z^{v_2} \in \Bbbk[\mathbb{N}^{Q_1} \oplus R] \,\Big|\, \mathrm{div}(u_1 - u_2) = 0 \Big\rangle. \quad (4.7)$$

For any element $y^{u_1} z^{v_1} - y^{u_2} z^{v_2} \in \Bbbk[\mathbb{N}^{Q_1} \oplus R]$, its $\mathrm{Wt}(Q)$-grade is defined to be $\mathrm{inc}(u_1 - u_2) + \iota(v_1 - v_2)$. Therefore, $I_{\mathscr{L}} \subset \Bbbk[\mathbb{N}^{Q_1} \oplus R]$ is defined to give the $\mathrm{Wt}(Q)$-homogeneous part of I. We conclude that the image ψ_θ is given by

$$[(\mathbb{V}(I_{\mathscr{L}}) \setminus \mathbb{V}(B_\theta))/\mathrm{PGL}(\alpha)].$$

□

Remark 4.16. Let $\theta_1, \theta_2 \in \mathrm{Wt}(Q)$ be generic and such that the maps $\psi_{\theta_1}, \psi_{\theta_2}$ are morphisms. The fact that $\psi_{\theta_1}, \psi_{\theta_2}$ are morphisms implies that their images do not intersect the unstable loci $\mathbb{V}(B_{\theta_1})$ and $\mathbb{V}(B_{\theta_2})$ and so are independent of the unstable

loci. Since the only difference between the two morphisms is the unstable loci of the target this implies that the image of ψ_{θ_1} is isomorphic to that of ψ_{θ_2}.

Let L be a base-point free line bundle on a variety X. The linear series construction gives a morphism $\varphi_{|L|} : X \to \mathbb{P}(\Gamma(X, L))$, under which the pullback of the tautological line bundle on $\mathbb{P}(\Gamma(X, L))$ is L. We now reap the rewards of the moduli description of our ambient stacks.

Proposition 4.17. *Let* $\mathscr{L} = (\mathcal{O}_\mathcal{X}, L_1, \ldots, L_r)$ *be base-point free collection of line bundles on* $\mathcal{X}$. *The pullback of the tautological bundles on* $\mathcal{M}(Q, \mathrm{div})$ *via* ψ_θ *is the collection* $\mathscr{L}$.

Proof. The group $\mathrm{GL}(\alpha) \cong (\Bbbk^\times)^{r+1}$ acts on $\det_{\theta_\Delta} W \cong \mathrm{Spec}(\Bbbk[y_{\theta_\Delta}])$ (cf. Remark 4.3) by $(t_0, \ldots, t_r) \cdot y_{\theta_\Delta} = t_0 \cdot y_{\theta_\Delta}$. So the subgroup fixing nonzero y_{θ_Δ} is given by $G_{\theta_\Delta} := \{(t_0, \ldots, t_r) \in \mathrm{GL}(\alpha) \,|\, t_0 = 1\}$. Restricting the quotient $\mathrm{GL}(\alpha) \twoheadrightarrow \mathrm{PGL}(\alpha)$ to G_{θ_Δ} we get an isomorphism $\mathrm{PGL}(\alpha) \cong G_{\theta_\Delta}$.

The tautological line bundles of $\mathcal{M}_\theta(Q, \mathrm{div}) \cong [(\mathcal{R}(Q, l)_\theta^{ss} \times \Bbbk^\times)/\mathrm{GL}(\alpha)]$ are given by the standard basis elements of $\mathbb{Z}^{Q_0} \cong \mathrm{GL}(\alpha)^\vee$. Under the isomorphism of stacks $[\mathcal{R}(Q, l)_\theta^{ss} \times \Bbbk^\times)/\mathrm{GL}(\alpha)] \cong [(\mathcal{R}(Q, l)_\theta^{ss}/G_{\theta_\Delta}]$ the pullbacks of the tautological line bundles is given by the image of the basis elements of $\mathbb{Z}^{Q_0}$ under the map dual to the inclusion $G_{\theta_\Delta} \hookrightarrow \mathrm{GL}(\alpha)$; now under the isomorphism $\mathrm{PGL}(\alpha) \cong G_{\theta_\Delta}$ these are mapped to the elements $0, \mathbf{e}_1 - \mathbf{e}_0, \ldots, \mathbf{e}_r - \mathbf{e}_0 \in \mathrm{Wt}(Q)$.

For $\eta \in \mathrm{Wt}(Q)$ the pullback of the associated line bundle of $[\mathcal{R}(Q, l)_\theta^{ss}/\mathrm{PGL}(\alpha)]$ to $\mathcal{X}$ is given by $\mathrm{pic}(\eta)$. Therefore the pullbacks of the tautological line bundles $0, \mathbf{e}_1 - \mathbf{e}_0, \ldots, \mathbf{e}_r - \mathbf{e}_0 \in \mathrm{Wt}(Q)$ to $\mathcal{X}$ are the line bundles $\mathcal{O}_\mathcal{X}, L_1, \ldots, L_r$ as required. $\qquad\square$

4.3 Representability of ψ_θ

Now we investigate representability of the morphism ψ_θ. Let $\pi : \mathcal{X} \to X$ be the map to the coarse moduli space X of $\mathcal{X}$. Recall, from Definition 2.22, the definition of a π-ample vector bundle.

Theorem 4.18. *Let* $\mathscr{L} = (\mathcal{O}_\mathcal{X}, L_1, \ldots, L_r)$ *be a base-point free collection of line bundles. Then* ψ_θ *is representable if and only if* $\bigoplus_{j=1}^r L_j$ *is* π-*ample.*

Proof. By Lemma 2.3.9 of [AH], ψ_θ is representable if and only if the group homomorphism $g : \mathrm{Aut}(x) \to \mathrm{Aut}(\psi_\theta(x))$ is injective for every geometric point $x \in \mathcal{X}$.

The map g fits into the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{Aut}(x) & \xrightarrow{\ \ g\ \ } & \mathrm{Aut}(\psi_\theta(x)) \\[2pt]
\downarrow & & \downarrow \\[2pt]
\mathrm{Hom}(\mathrm{Pic}(\mathcal{X}),\Bbbk^\times) & \xrightarrow{\ \mathrm{pic}^\vee\ } & \mathrm{Hom}(\mathrm{Wt}(Q),\Bbbk^\times).
\end{array}
\tag{4.8}
$$

Here $\mathrm{pic}^\vee$ denotes the map given by applying the functor $\mathrm{Hom}(-,\Bbbk^\times)$ to pic. We claim that the representation of $\mathrm{Aut}(x)$ given by $\bigoplus_{j=1}^r L_j$ is the composite

$$
\mathrm{Aut}(x) \hookrightarrow \mathrm{Hom}(\mathrm{Pic}(\mathcal{X}),\Bbbk^\times) \xrightarrow{\ \mathrm{pic}\ } \mathrm{Hom}(\mathrm{Wt}(Q),\Bbbk^\times).
\tag{4.9}
$$

Indeed, take the basis $\{\mathbf{e}_i - \mathbf{e}_0 \in \mathrm{Wt}(Q)\,|\,i = 1,\ldots r\}$ of $\mathrm{Wt}(Q)$ giving an isomorphism $\mathrm{Hom}(\mathrm{Wt}(Q),\Bbbk^\times) \cong (\Bbbk^\times)^r$. Evaluating at $\mathbf{e}_i - \mathbf{e}_0 \in \mathrm{Wt}(Q)$ gives a map $\mathrm{Hom}(\mathrm{Wt}(Q),\Bbbk^\times) \to \Bbbk^\times$. By definition of the Hom-functor the composite

$$
\mathrm{Hom}(\mathrm{Pic}(\mathcal{X}),\Bbbk^\times) \xrightarrow{\ \mathrm{pic}\ } \mathrm{Hom}(\mathrm{Wt}(Q),\Bbbk^\times) \to \Bbbk^\times
$$

is given by evaluating at $\mathrm{pic}(\mathbf{e}_i - \mathbf{e}_0) = L_i$ and is therefore the representation induced by the line bundle L_i. This proves the claim. So we have that the composite (4.9) is injective for every $x \in \mathcal{X}$ precisely when $\bigoplus_{j=1}^r L_j$ is π-ample. Commutativity of (4.8) gives that (4.9) is injective if and only if g is injective, as required. $\qquad\square$

Define $\mathrm{Pic}(X) := \{L \in \mathrm{Pic}(\mathcal{X})\,|\,L \cong \pi^*(M)$ for $M \in \mathrm{Pic}(X)\}$ then we have the following corollary.

Corollary 4.19. *Let $\mathscr{L}$ be a base-point free collection of line bundles on $\mathcal{X}$. If $\mathscr{L}$ generates $\mathrm{Pic}(\mathcal{X})/\mathrm{Pic}(X)$ then ψ_θ is representable.*

Proof. This follows from the fact that elements of $\mathrm{Pic}(X)$ give trivial representations of $\mathrm{Aut}(x)$ for every geometric point $x \in \mathcal{X}$. $\qquad\square$

The following example shows that for a given base-point free collection $\mathscr{L}$, representability of ψ_θ is weaker than $\mathscr{L}$ generating $\mathrm{Pic}(\mathcal{X})/\mathrm{Pic}(X)$.

Example 4.20. Take $N = \mathbb{Z}$ and let Σ be the fan associated to the toric variety $\mathbb{P}^1$ with rays $\rho_+ := \mathbb{Q}_{\geq 0}$ and $\rho_- := \mathbb{Q}_{\leq 0}$. Let $\beta : \mathbb{Z}^{\Sigma(1)} \to \mathbb{Z}$ take $\mathbf{e}_{\rho_\pm}$ to ± 2. Let $\boldsymbol{\Sigma}$ be the stacky fan (N,Σ,β) and take $\mathcal{X} = \mathcal{X}_{\boldsymbol{\Sigma}}$. Note that $\mathrm{Pic}(\mathcal{X}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Pic}(\mathcal{X})/\mathrm{Pic}(X) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let $\mathscr{L} = (\mathcal{O}_\mathcal{X}, \mathcal{O}(2,1), \mathcal{O}(4,0))$. The collection $\mathscr{L}$ does not generate $\mathrm{Pic}(\mathcal{X})/\mathrm{Pic}(X)$ since $\mathcal{O}(4,0) \in \mathrm{Pic}(X)$. For $\theta = -3\mathbf{e}_0 + 2\mathbf{e}_1 + \mathbf{e}_2 \in \mathrm{Wt}(Q)$, $\mathscr{L}$ gives a morphism $\psi_\theta : \mathcal{X} \to \mathbb{P}(1,1,2,2)$ that takes a point

$(x, y) \in \mathbb{A}^{\Sigma(1)}$ to (xy, xy, x^4, y^4). The morphism ψ_θ induces injections on stabilizer groups on geometric points and so is representable.

Example 4.21. Each example in this section defines a representable morphism. For an example that does not, take $\mathcal{X} = \mathbb{P}(1, 1, 2)$ and $\mathscr{L} = (\mathcal{O}, \mathcal{O}(2))$.

4.4 What if $\mathcal{X}$ is a toric variety?

We conclude the chapter by comparing our construction to the multilinear series of Craw-Smith [CS08] in the case where $\mathcal{X} = X$ is a toric variety. Let $\mathscr{L} = (\mathcal{O}_X, L_1, \ldots, L_r)$ be a collection of base-point free line bundles on a toric variety X and $\vartheta = -r\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_r$, then Craw-Smith use the commutative diagram (2.1) to produce a morphism $\varphi_{|\mathscr{L}|} : X \to \mathcal{M}_\vartheta(Q)$.

Proposition 4.22. *Let $\mathcal{X} = X$ be a toric variety and $\mathscr{L}$ be a collection of base-point free line bundles on X. Then the image of ψ_ϑ is isomorphic to that of $\varphi_{|\mathscr{L}|}$.*

Proof. We have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Z}^{Q_1} & \xrightarrow{\text{inc}} & \text{Wt}(Q) \\
{\scriptstyle (\text{id},0)}\downarrow & & \downarrow{\scriptstyle \text{id}} \\
\mathbb{Z}^{Q_1} \oplus R & \xrightarrow{\text{inc}\oplus\iota} & \text{Wt}(Q) \\
{\scriptstyle \text{div}\oplus 0}\downarrow & & \downarrow{\scriptstyle \text{pic}} \\
\mathbb{Z}^{\Sigma(1)} & \xrightarrow{\text{deg}} & \text{Pic}(X).
\end{array}
\qquad (4.10)
$$

The maps of monoids $\mathbb{N}^{Q_1} \xrightarrow{(\text{id},0)} \mathbb{N}^{Q_1} \oplus R \xrightarrow{\text{div}\oplus 0} \mathbb{N}^{\Sigma(1)}$ give maps of monoid algebras $\Bbbk[\mathbb{N}^{Q_1}] \to \Bbbk[\mathbb{N}^{Q_1} \oplus R] \to \Bbbk[\mathbb{N}^{\Sigma(1)}]$. After applying the functor Spec these give morphisms

$$
\mathbb{A}^{\Sigma(1)} \xrightarrow{\Psi^*} \mathbb{A}^{Q_1} \times (\Bbbk^\times)^P \xrightarrow{\pi} \mathbb{A}^{Q_1}.
$$

The morphism Ψ^* is induced by the monoid map $\mathbb{N}^{Q_1} \oplus R \xrightarrow{\text{div}\oplus 0} \mathbb{N}^{\Sigma(1)}$, therefore its image lies in the subvariety $\mathbb{A}^{Q_1} \times (1, \ldots, 1) \subset \mathbb{A}^{Q_1} \times (\Bbbk^\times)^P$. On the other hand, the morphism π is just the projection to the first factor, therefore the image of Ψ^* is isomorphic to the image of $\pi \circ \Psi^*$. The commutativity of diagram (4.10) implies that Ψ^* is equivariant with respect to the actions of the groups $\text{Hom}(\text{Pic}(X), \Bbbk^\times)$ and $\text{Hom}(\text{Wt}(Q), \Bbbk^\times)$ induced by deg and inc, similarly π is equivariant. So the

maps Ψ^* and π give rise to rational maps:

$$X - \!\!\xrightarrow{\psi_\vartheta}\!\!\!\!> \mathcal{M}_\vartheta(Q, \mathrm{div}) - \!\!\xrightarrow{\pi}\!\!\!\!> \mathcal{M}_\vartheta(Q) \,.$$

The composite $\pi \circ \psi_\theta$ is equal to the rational map $\varphi_{|L|}$ and since $\mathscr{L}$ is a collection base-point free line bundles Proposition 2.37 implies it is a morphism. By virtue of Theorem 4.14 we have that ψ_ϑ is a morphism. Now, by definition, ψ_ϑ and $\varphi_{|L|}$ descend from Ψ^* and $\pi \circ \Psi^*$ respectively and since Ψ^* and $\pi \circ \Psi^*$ have isomorphic images the images of ψ_ϑ and $\varphi_{|L|}$ are isomorphic. □

Example 4.23. Let $X = \mathbb{P}^1$ and $\mathscr{L} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$. The corresponding quiver of sections is shown in Figure 4.4. The morphism φ_ϑ is given by the diagonal

Figure 4.4: A quiver of sections of $\mathbb{P}^1$.

morphism $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, while ψ_ϑ is given by $\psi_\vartheta : \mathbb{P}^1 \to \mathbb{P}^3$ sending (x_1, x_2) to (x_1, x_2, x_1, x_2). Although these two morphisms share the same image they are very different.

Chapter 5

Application to the McKay correspondence

In this section, we apply the construction from the previous section to toric quotient singularities. For $G \subset \mathrm{GL}(n, \Bbbk)$ a finite abelian group and (Q, div) the labelled McKay quiver, we construct a closed immersion $[\mathbb{A}^n/G] \hookrightarrow \mathcal{M}_\theta(Q, \mathrm{div})$ for any $\theta \in \mathrm{Wt}(Q)$. In the case where $G \subset \mathrm{SL}(n, \Bbbk)$ and $n \leq 3$, we proceed to alter the construction of $\mathcal{M}_\theta(Q, \mathrm{div})$ and yield a GIT problem for which one generic stability condition gives $[\mathbb{A}^n/G]$ and another gives $G\text{-Hilb}(\mathbb{A}^n)$. We will assume $\alpha = (1, \ldots, 1)$ throughout this section and drop it from the notation.

5.1 Motivating example

Observe that the quiver of sections corresponding to $\mathscr{L}_{\mathbb{P}(1,1,2)} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ on $\mathbb{P}(1,1,2)$ is almost identical to that corresponding to $\mathscr{L}_{\mathbb{F}_2} = (\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1))$ on $\mathbb{F}_2$, see Figure 5.1. The only difference lies in that the labels of the arrows going
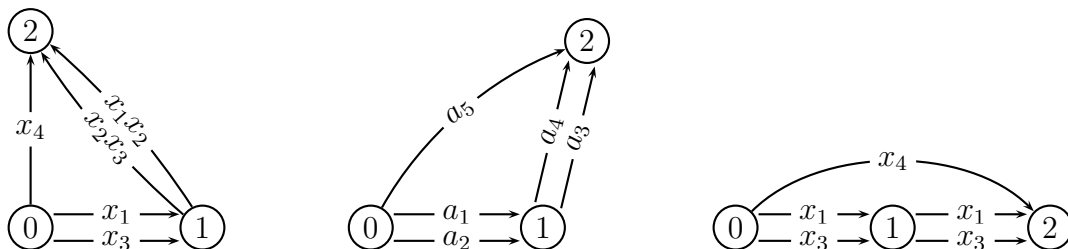
(a) Quiver of sections of $\mathscr{L}_{\mathbb{F}_2}$.   (b) Q with labelled arrows   (c) Quiver of sections of $\mathscr{L}_{\mathbb{P}(1,1,2)}$.

Figure 5.1: A quiver of sections on $\mathbb{P}(1,1,2)$ versus another on $\mathbb{F}_2$.

from vertex 1 to vertex 2 differ by a factor of x_2.

The group $R_{\mathbb{P}(1,1,2)}$ corresponding to the labelled quiver in Figure 5.1 a) is generated by a single element $\mathbf{e}_0 - 2\mathbf{e}_1 + \mathbf{e}_2 \in \mathrm{Wt}(Q)$. Let $\mathcal{R}(Q_{\mathbb{P}(1,1,2)}, \mathrm{div}) = \mathrm{Spec}(\Bbbk[y_1, \ldots, y_5, z^{\pm 1}])$ For $\theta = -3\mathbf{e}_0 + 2\mathbf{e}_1 + \mathbf{e}_2 \in \mathrm{Wt}(Q)$ one gets a morphism

$$\psi_\theta : \mathbb{P}(1,1,2) \longrightarrow \mathcal{M}_\theta(Q, \mathrm{div}) \cong \left[\frac{\mathbb{A}^5 \times \Bbbk^\times \setminus \{0\} \times \Bbbk^\times}{(\Bbbk^\times)^2} \right].$$

The image of ψ_θ is cut out by the ideal

$$I_{\mathscr{L}_{\mathbb{P}(1,1,2)}} := \left\langle y^{u_1} z^{v_1} - y^{u_2} z^{v_2} \,\middle|\, \mathrm{div}(u_1 - u_2) = 0, \mathrm{inc}(u_1 - u_2) + \iota(v_1 - v_2) = 0 \right\rangle.$$

Note that $I_{\mathscr{L}_{\mathbb{P}(1,1,2)}}$ contains the polynomials $y_3 - zy_1$, $y_4 - zy_2$, or, in words, the linear map on arrow 1 differs that of arrow 3 by a factor of z. So the homogenizing variable z plays the role of the label x_2 in the $Q_{\mathbb{F}_2}$, except for the fact that z is always nonzero.

It is important to note that the variety $\mathbb{F}_2$ and the stack $\mathbb{P}(1,1,2)$ are closely related. The variety $\mathbb{F}_2$ is a crepant resolution of the singularity of the coarse moduli space of $\mathbb{P}(1,1,2)$. The strategy for this chapter is to allow our homogenizing variables to be zero to explore the relation between toric DM stacks whose coarse moduli spaces have finite quotient singularities and the crepant resolutions of such singularities, when they exist. Since smoothness is a local property, we work with quotient stacks $[\mathbb{A}^n/G]$ for finite abelian groups $G \subset \mathrm{GL}(n, \Bbbk)$. We begin by setting the scene with a brief journey through the wonders of the McKay correspondence and then extending the ideas of Chapter 4 to quotient stacks $[\mathbb{A}^n/G]$.

5.2   Background: the McKay correspondence and G-Hilb

The general principle behind the McKay correspondence is the following:

> *For $n \in \mathbb{N}$ let $G \subset \mathrm{SL}(n, \Bbbk)$ be a finite group. Given a crepant resolution $\tau : Y \to \mathbb{A}^n/G$, the geometry of Y is equivalent to the geometry of $[\mathbb{A}^n/G]$.*

One manifestation of the above principle, suggested by Reid [Rei97] is the following:

Conjecture 5.1. *For $n \in \mathbb{N}$ let $G \subset \mathrm{SL}(n, \Bbbk)$ be a finite subgroup. Given a crepant resolution $\tau : Y \to \mathbb{A}^n/G$ the bounded derived category of coherent sheaves on Y, $\mathrm{D}^b(Y)$, is equivalent to the bounded derived category of coherent sheaves on $[\mathbb{A}^n/G]$, $\mathrm{D}^b([\mathbb{A}^n/G])$.*

Theorem 5.2 (Bridgeland-King-Reid [BKR01])**.** *Conjecture 5.1 holds for $n \leq 3$.*

For n and $G \subset \mathrm{SL}(n, \Bbbk)$ as above, crepant resolutions of $\mathbb{A}^n/G$ do not always exist. However, for $n \leq 3$ they do exist.

Definition 5.3 (Reid [Rei97])**.** The G-*Hilbert scheme* of $\mathbb{A}^n$ denoted G-Hilb$(\mathbb{A}^n)$ is the fine moduli space of G-invariant subschemes of $\mathbb{A}^n$ whose coordinate ring is isomorphic to $\Bbbk[G]$ as a $\Bbbk[G]$-module.

Although the scheme G-Hilb$(\mathbb{A}^n)$ is reducible in general, it has a distinguished irreducible component, denoted $\mathrm{Hilb}^G(\mathbb{A}^n)$, birational to $\mathbb{A}^n/G$. The following proposition was proved at various levels of generality by several authors. Ito-Nakamura [IN99] proved it for $G \subset \mathrm{SL}(2, \Bbbk)$, Nakamura [Nak01] for abelian $G \subset \mathrm{SL}(3, \Bbbk)$ and Bridgeland-King-Reid [BKR01] for general finite $G \subset \mathrm{SL}(3, \Bbbk)$.

Proposition 5.4. *If $n \leq 3$ and $G \subset \mathrm{SL}(n, \Bbbk)$, the scheme G-Hilb$(\mathbb{A}^n)$ is smooth and isomorphic to $\mathrm{Hilb}^G(\mathbb{A}^n)$. Furthermore the map $\tau : G\text{-Hilb}(\mathbb{A}^n) \to \mathbb{A}^n/G$ sending a subscheme to the orbit supporting it, is a crepant resolution of $\mathbb{A}^n/G$.*

For G abelian, Craw-Maclagan-Thomas [CMT07] show that the distinguished component $\mathrm{Hilb}^G(\mathbb{A}^n)$ of G-Hilb$(\mathbb{A}^n)$ can be recovered from the labelled McKay quiver. From now on we assume G is abelian.

Definition 5.5. The *labelled McKay quiver* of $G \subset \mathrm{SL}(n, \Bbbk)$ is the quiver whose vertices are given by irreducible representations of G with an arrow a_i^ρ from $\rho\rho_i$ to ρ for every ρ and $1 \leq i \leq n$. The labelling map div is given by assigning an arrow a_i^ρ a label x_i.

Proposition 5.6 (Proposition 5.2, [CMT07])**.** *The coherent component $\mathrm{Hilb}^G(\mathbb{A}^n)$ of G-Hilb$(\mathbb{A}^n)$ is the subvariety of*

$$\mathcal{M}_\vartheta(Q) = \left(\mathbb{A}^{Q_1}\right)_\vartheta^{ss} / \mathrm{PGL}(\alpha)$$

cut out by the ideal

$$I_Q := \left\langle y^{u_1} - y^{u_2} \,\middle|\, \mathrm{div}(u_1 - u_2) = 0, \mathrm{inc}(u_1 - u_2) = 0 \right\rangle.$$

5.3 $[\mathbb{A}^n/G]$ from labelled McKay quiver

Take $n \in \mathbb{N}$ and G a finite abelian subgroup of $\mathrm{GL}(n, \Bbbk)$ with no quasireflections. We may assume that G is contained in the subgroup $(\Bbbk^\times)^n$ of diagonal matrices with

nonzero entries in $\mathrm{GL}(n, \Bbbk)$. Line bundles on $[\mathbb{A}^n/G]$ are given by G-equivariant line bundles on $\mathbb{A}^n$, which in turn are determined by G-equivariant isomorphisms $\mathcal{O}_{\mathbb{A}^n \times G} \to \mathcal{O}_{\mathbb{A}^n \times G}$. From this it follows that the Picard group of $[\mathbb{A}^n/G]$ is naturally isomorphic to the group characters $G^\vee$. With these preparations, take

$$\mathscr{L} = (\mathcal{O}_{\mathbb{A}^n} \otimes \rho \,|\, \rho \in G^\vee).$$

Then the labelled quiver of sections (Q, div) of $\mathscr{L}$ coincides with the McKay quiver, see the beginning of Section 4.1 of [CV10].

From now on we will use the isomorphism $\mathrm{Pic}([\mathbb{A}^n/G]) \cong G^\vee$ tacitly. In much the same way as we have commutative diagrams (4.1) and (4.2) we have

$$
\begin{array}{ccc}
\mathbb{Z}^{Q_1} \xrightarrow{\text{inc}} \mathrm{Wt}(Q) & \qquad & R \xhookrightarrow{\iota} \mathrm{Wt}(Q) \\
\downarrow{\scriptstyle \mathrm{div}} \qquad \downarrow{\scriptstyle \mathrm{pic}} & & \downarrow{\scriptstyle 0} \qquad \downarrow{\scriptstyle \mathrm{pic}} \\
\mathbb{Z}^n \xrightarrow{\text{deg}} G^\vee & & \mathbb{Z}^{Q_1} \xrightarrow{\text{deg}} G^\vee.
\end{array}
\qquad (5.1)
$$

As in Section 4, the monoid morphism $\mathrm{div} \oplus 0 : \mathbb{N}^{Q_1} \oplus R \to \mathbb{N}^n$ gives a morphism $\Psi^* : \mathbb{A}^n \to \mathbb{A}^{Q_1} \times (\Bbbk^\times)^R$. The commutativity of (5.1) gives that Ψ^* is equivariant with respect to the actions of G on $\mathbb{A}^n$ and $\mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times)$ on $\mathbb{A}^{Q_1} \times (\Bbbk^\times)^R$. Given $\theta \in \mathrm{Wt}(Q)$ this gives a rational map

$$\psi_\theta : [\mathbb{A}^n/G] \dashrightarrow \mathcal{M}_\theta(Q, \mathrm{div}).$$

From now on we identify the lattice $\mathrm{Wt}(Q)$ with the lattice $\{\theta \in \mathbb{Z}^{Q_0} \,|\, \theta_0 = 0\}$ whose basis is $\{\mathbf{e}_\rho \,|\, \rho \in G^\vee \setminus \{0\}\}$.

Proposition 5.7. *For any $\chi_\theta \in \mathrm{PGL}(\alpha)^\vee$,*

$$\psi_\theta : [\mathbb{A}^n/G] \dashrightarrow \mathcal{M}_\theta(Q, \mathrm{div})$$

is a closed immersion.

Proof. We begin by studying the θ-semistable points. By definition of the quiver of sections, a path p from ρ_0 to ρ corresponds to a section $s \in \mathrm{Hom}(\rho_0, \rho)$. Then there exists a path p' from ρ' to $\rho \otimes \rho'$ given rise to by the same section $s \in \mathrm{Hom}(\rho', \rho' \otimes \rho)$, that is $\mathrm{div}(p) = \mathrm{div}(p')$. Note that $\mathrm{inc}(p) = \mathbf{e}_\rho$ and $\mathrm{inc}(p') = \mathbf{e}_{\rho' \otimes \rho} - \mathbf{e}_{\rho'}$. Since $\mathrm{div}(p') - \mathrm{div}(p) = 0$ we have that $-\mathbf{e}_\rho - \mathbf{e}_{\rho'} + \mathbf{e}_{\rho \otimes \rho'} \in R$. Given that $\ker(\mathrm{pic})$ is generated by elements of the form $-\mathbf{e}_\rho - \mathbf{e}_{\rho'} + \mathbf{e}_{\rho \otimes \rho'}$ this shows $\ker(\mathrm{pic}) \subset R$. The commutativity of the diagrams (5.1) gives $R \subset \ker(\mathrm{pic})$ and therefore $R = \ker(\mathrm{pic})$.

The image of pic is a torsion $\mathbb{Z}$-module, so $R = \ker(\text{pic})$ $\mathbb{Q}$-spans $\text{Wt}(Q)$ and hence any basis $\mathfrak{B}$ of R $\mathbb{Q}$-spans $\text{Wt}(Q)$. Now let W be a refined representation and let $\theta \in \text{Wt}(Q)$. We claim that W is θ-semistable. Indeed, let $W_\bullet$ be $\Bbbk Q$-module filtration satisfying the conditions in Definition 3.7 and write θ as a $\mathbb{Q}$-linear combination of $b \in \mathfrak{B}$. Then since $b(W_\bullet) = 0$, we have $\theta(W_\bullet) = 0$. This in particular implies that the θ-unstable locus is empty. It follows at once that the rational map ψ_θ is a morphism

$$\psi_\theta : [\mathbb{A}^n/G] \longrightarrow \mathcal{M}_\theta(Q, \text{div})$$

for any $\theta \in \text{Wt}(Q)$. Let $I_{\mathscr{L}}$ be the $\Bbbk[\mathbb{N}^{Q_1} \oplus R]$ ideal defined in (4.6). Again, after noting the θ-unstable locus is empty, an argument similar to that of Proposition 4.15 gives that the image of ψ_θ is $[\mathbb{V}(I_{\mathscr{L}})/\text{PGL}(\alpha)]$. It remains to show that $[\mathbb{A}^n/G]$ is isomorphic to $[\mathbb{V}(I_{\mathscr{L}})/\text{PGL}(\alpha)]$. Consider $\Bbbk[\mathbb{N}^{Q_1} \oplus R]/I_{\mathscr{L}}$ and multiply the generators $y^{u_1} z^{v_1} - y^{u_2} z^{v_2}$ of $I_{\mathscr{L}}$ by the units z^{-v_1} to get an alternative set of generators given by elements of the form $y^{u_1} - y^{u_2} z^{v_2 - v_1}$. Then $I_{\mathscr{L}}$ is given by

$$\left\langle y^{u_1} - y^{u_2} z^v \in \Bbbk[\mathbb{N}^{Q_1} \oplus R] \,\middle|\, \text{div}(u_1 - u_2) = 0, \text{inc}(u_1 - u_2) - \iota(v) = 0 \right\rangle.$$

Pick $a_1, \ldots, a_n \in Q_1$ such that $\text{div}(a_i)$ is the ith basis element of $\mathbb{Z}^n$. Since every arrow in the McKay quiver is labelled by a basis element of $\mathbb{Z}^n$ the kernel of div is generated by differences $\mathbf{e}_{a_i'} - \mathbf{e}_{a_i}$ with $\text{div}(a_i') = \text{div}(a_i)$. By definition of R, for every generator $\mathbf{e}_{a_i'} - \mathbf{e}_{a_i}$ of $\ker(\text{div})$ there exists $v' \in R$ such that $\text{inc}(\mathbf{e}_{a_i} - \mathbf{e}_{a_i'}) = v'$. Therefore $I_{\mathscr{L}}$ is generated by elements of the form $y_{a_i'} - y_{a_i} z^{v'}$. This implies that for every $a \in Q_1$ not in the list $a_1, \ldots, a_n$, the monomial y_a is equivalent in the quotient $\Bbbk[\mathbb{N}^{Q_1} \oplus R]/I_{\mathscr{L}}$ to a product of elements in $\Bbbk[y_{a_1}, \ldots, y_{a_n}] \otimes \Bbbk[R]$. Our choice of $a_1, \ldots, a_n$ implies that $\mathbb{Z}\mathbf{e}_{a_1} \oplus \cdots \oplus \mathbb{Z}\mathbf{e}_{a_n}$ maps injectively into $\mathbb{Z}^n$ and so $\Bbbk[\mathbb{N}^{Q_1} \oplus R]/I_{\mathscr{L}} \cong \Bbbk[y_{a_1}, \ldots, y_{a_n}] \otimes \Bbbk[R]$. Therefore $[\mathbb{V}(I_{\mathscr{L}})/\text{PGL}(\alpha)] \cong [\mathbb{A}^n \times (\Bbbk^\times)^R/\text{PGL}(\alpha)]$.

We note that we may always fix the $(\Bbbk^\times)^R$ component to 1. Now, the characters of the subgroup of $\text{PGL}(\alpha)$ fixing the $(\Bbbk^\times)^R$ component are given by $\text{Wt}(Q)/R$. The map pic is surjective onto $G^\vee$ and its kernel is given by R, so that $\text{Wt}(Q)/R \cong G^\vee$. Hence the aforementioned subgroup is naturally isomorphic to G. Consequently, we have stack isomorphisms

$$[\mathbb{V}(I_{\mathscr{L}})/\text{PGL}(\alpha)] \cong [\mathbb{A}^n \times (\Bbbk^\times)^R/\text{PGL}(\alpha)] \cong [\mathbb{A}^n \times \{1\}/G] \cong [\mathbb{A}^n/G].$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5.4   From $[\mathbb{A}^n/G]$ to G-**Hilb**$(\mathbb{A}^n)$

We proceed to relate our construction to that of $\mathrm{Hilb}^G(\mathbb{A}^n)$ by defining a GIT problem in which $[\mathbb{A}^n/G]$ and $\mathrm{Hilb}^G(\mathbb{A}^n)$ are separated by a finite series of wall-crossings. We begin by carefully picking a basis $\mathfrak{B}$ of R.

Write $G^\vee$ as a direct sum of cyclic groups $\bigoplus_{j=1}^m H_j$ and take ρ_j a generator of H_j. Define

$$\overline{\mathfrak{B}} := \left\{ -\mathbf{e}_{\rho_j} - \mathbf{e}_{\rho'\otimes\rho_j^{-1}} + \mathbf{e}_{\rho'} \in \mathrm{Wt}(Q) \,|\, \forall\, 1 \leq j \leq m,\ \rho' \in G^\vee \setminus \{\rho_1,\ldots,\rho_m\} \right\}.$$

Lemma 5.8. *The set $\overline{\mathfrak{B}}$ generates the lattice $R \subset \mathrm{Wt}(Q)$.*

Proof. For notational purposes, we use $+$ for the binary operation on $G^\vee$ in this proof. First we show that

$$\widetilde{\mathfrak{B}} := \left\{ -\mathbf{e}_{\rho_j} - \mathbf{e}_{\rho'-\rho_j} + \mathbf{e}_{\rho'} \in \mathrm{Wt}(Q) \,|\, \forall\, 1 \leq j \leq m,\ \rho' \in G^\vee \right\}$$

generates R. Let $\rho = \sum_j \gamma_j \rho_j$ and without loss of generality assume $\gamma_j > 0$. Since

$$\sum_{1 \leq \kappa_j \leq \gamma_j} -\mathbf{e}_{\rho_j} - \mathbf{e}_{\rho'-(\kappa_j-1)\rho_j-\rho_j} + \mathbf{e}_{\rho'-(\kappa_j-1)\rho_j} = -\gamma_j \mathbf{e}_{\rho_j} - \mathbf{e}_{(\rho'-\gamma_j\rho_j)} + \mathbf{e}_{\rho'}$$

we deduce $(\sum_j -\gamma_j \mathbf{e}_{\rho_j}) + \mathbf{e}_{\rho'}$ is an element of $\mathbb{N}\widetilde{\mathfrak{B}}$. Moreover, for $\rho' = \sum_j \gamma_j \rho_j$ and $\rho'' = \sum_j \gamma'_j \rho_j$, we have that $-\mathbf{e}_{\rho'} - \mathbf{e}_{\rho''} + \mathbf{e}_{\rho'+\rho''}$ is equal to

$$\left(\left(\sum_j \gamma_j \mathbf{e}_{\rho_j}\right) - \mathbf{e}_{\rho'}\right) + \left(\left(\sum_j \gamma'_j \mathbf{e}_{\rho_j}\right) - \mathbf{e}_{\rho''}\right) + \left(\left(\sum_j -(\gamma_j + \gamma'_j)\mathbf{e}_{\rho_j}\right) + \mathbf{e}_{\rho'+\rho''}\right)$$

showing that $-\mathbf{e}'_\rho - \mathbf{e}_{\rho''} + \mathbf{e}_{\rho'+\rho''} \in \mathbb{Z}\widetilde{\mathfrak{B}}$. Therefore $\widetilde{\mathfrak{B}}$ generates $\ker(\mathrm{div}) = R$.

Take $-\mathbf{e}_{\rho_j} - \mathbf{e}_{\rho'_j-\rho_j} + \mathbf{e}_{\rho'_j}$ for $1 \leq j, j' \leq m$ and $|\rho_j|$ to be the order of ρ_j. Then we have

$$\sum_{0 \leq \kappa \leq |\rho_j|-2} -\mathbf{e}_{\rho_j} - \mathbf{e}_{\kappa\rho_j+\rho_{j'}} + \mathbf{e}_{(\kappa+1)\rho_j+\rho_{j'}} = (1 - |\rho_j|)\mathbf{e}_{\rho_j} - \mathbf{e}_{\rho_{j'}} + \mathbf{e}_{\rho_{j'}-\rho_j}$$

which along with the fact that $|\rho_j|\,\mathbf{e}_{\rho_j} \in \mathbb{Z}\overline{\mathfrak{B}}$ shows that $\overline{\mathfrak{B}}$ generates R. $\square$

Remark 5.9. Note that for any j as above, the $\mathbf{e}_{\rho_j}$ coefficient of elements of $\mathbb{N}\overline{\mathfrak{B}}$ is non-positive. This will prove crucial in the proof of the theorem below.

Fix a basis $\mathfrak{B} \subset \overline{\mathfrak{B}}$ of R. We have the following diagram

$$
\begin{array}{ccc}
\mathbb{N}^{Q_1} \oplus \mathbb{N}\mathfrak{B} & \xrightarrow{\;\mathrm{inc}\oplus\iota\;} & \mathrm{Wt}(Q) \\
{\scriptstyle \mathrm{div}}\downarrow & & \\
\mathbb{N}^n & &
\end{array}
\tag{5.2}
$$

The monoid homomorphism induces a $\mathrm{Wt}(Q)$-grading on $\Bbbk[\mathbb{N}^n \oplus \mathbb{N}\mathfrak{B}]$ and hence an action of $\mathrm{PGL}(\alpha)$ on $\mathbb{A}^{Q_1} \times \mathbb{A}^{\mathfrak{B}}$. Define $I_{\mathscr{L},\mathfrak{B}}$ to be the $\mathrm{Wt}(Q)$-homogeneous ideal of $\Bbbk[\mathbb{N}^n \oplus \mathbb{N}\mathfrak{B}]$ given by

$$
I_{\mathscr{L},\mathfrak{B}} := \Big\langle y^{u_1} z^{v_1} - y^{u_2} z^{v_2} \;\Big|\; \mathrm{div}(u_1 - u_2) = 0, \mathrm{inc}(u_1 - u_2) + \iota(v_1 - v_2) = 0 \Big\rangle.
$$

For $\theta \in \mathrm{Wt}(Q)$ we consider the stack quotient $[\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_\theta / \mathrm{PGL}(\alpha)]$.

Theorem 5.10. *There exists generic stability conditions* $\chi_{\theta_1}, \chi_{\theta_2} \in \mathrm{PGL}(\alpha)^\vee$, *such that*

$$
[\mathbb{A}^n/G] \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_1} / \mathrm{PGL}(\alpha)] \;\; and \;\; \mathrm{Hilb}^G(\mathbb{A}^n) \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_2} / \mathrm{PGL}(\alpha)].
$$

Proof. Because of Proposition 5.7, to establish the first isomorphism it suffices to find θ_1 for which $\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_1} = \mathbb{V}(I_{\mathscr{L}})^{ss}_{\theta_1}$. The cone $\mathbb{Q}_{\geq 0}\mathfrak{B} \subset \mathrm{Wt}(Q)_{\mathbb{Q}}$ is top dimensional, so we may pick a generic $\theta_1 \in \mathbb{N}\mathfrak{B}$. After picking a higher multiple if necessary, we may assume that the χ_θ-unstable locus in $\mathbb{A}^{Q_1} \times \mathbb{A}^{\mathfrak{B}}$ is given by the vanishing locus of the ideal

$$
B_{\theta_1} := \Big\langle y^u z^v \in \Bbbk[\mathbb{N}^{Q_1} \oplus \mathbb{N}\mathfrak{B}] \;\Big|\; \mathrm{inc}(u) + \iota(v) = \theta_1 \Big\rangle
$$

We claim that for any monomial $y^u z^v \in B_{\theta_1}$ there exists $u' \in \mathbb{N}^{Q_1}$ such that $y^{u'} z^{\theta_1} - y^u z^v \in I_{\mathscr{L},\mathfrak{B}}$. Since $\theta_1 \in \mathbb{N}\mathfrak{B}$, $\mathrm{inc}(u) = \theta_1 - \iota(v) \in R$. The commutative diagrams (5.1) imply $\mathrm{div}(u)$ is a torus-invariant section of the trivial line bundle. Take $u' \in \mathbb{N}^{Q_1}$ to be a cycle in the quiver of sections corresponding to $\mathrm{div}(u)$ and note that $\mathrm{inc}(u') = 0$. We then have that $\mathrm{div}(u - u') = 0$ and $\mathrm{inc}(u - u') + \iota(v - \theta_1) = 0$, as claimed. From this it follows that any point for which $z^{\theta_1} = 0$ is unstable. Now, θ_1 is in the interior of $\mathbb{Q}_{\geq 0}\mathfrak{B}$ therefore any point for which $z_b = 0$ is unstable. That is $\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_1} := \mathbb{V}(I_{\mathscr{L},\mathfrak{B}}) \setminus \mathbb{V}(B_{\theta_1}) \subset \mathbb{A}^{Q_1} \times (\Bbbk^\times)^{\mathfrak{B}}$ and hence

$$
\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_1} = \mathbb{V}(I_{\mathscr{L}})^{ss}_{\theta_1}.
$$

Now take a generic $\theta_2 \in \mathrm{Wt}(Q)$ in the interior top-dimensional cone $\Theta := \mathbb{Q}_{\geq 0}\{\mathbf{e}_\rho \mid \rho \in G^\vee \setminus \{0\}\}$. Once again, taking a higher multiple if necessary we may assume that the χ_θ-unstable locus in $\mathbb{A}^{Q_1} \times \mathbb{A}^P$ is given by the vanishing locus of

the ideal

$$B_{\theta_2} := \Big\langle\, y^u z^v \in \Bbbk[\mathbb{N}^{Q_1} \oplus \mathbb{N}P] \mid \mathrm{inc}(u) + \iota(v) = \theta_2 \,\Big\rangle.$$

If we set

$$B'_{\theta_2} := \big\langle\, y^u \in \Bbbk[\mathbb{N}^{Q_1}] \mid \mathrm{inc}(u) = \theta_2 \,\big\rangle$$

then the vanishing locus of B'_{θ_2} is equal to that of B_ϑ, since θ_2 and ϑ lie in the same chamber. Let $y^u z^v \in B_{\theta_2}$ and take $y^{u'}$ to be the unique monomial in $\Bbbk[\mathbb{N}^{Q_1}]/I_Q$ for which $\mathrm{div}(u) = \mathrm{div}(u')$ and $\mathrm{inc}(u') = \theta_2$. We next show that

$$\left(\frac{\Bbbk[\mathbb{N}^{Q_1} \oplus \mathbb{N}B]}{I_{\mathscr{L},B}} \right)_{y^u z^v} \cong \left(\frac{\Bbbk[\mathbb{N}^{Q_1}]}{I_Q} \right)_{y^{u'}}. \qquad (5.3)$$

Remark 5.9 gives that v has a non-positive coefficient for each basis element $\mathbf{e}_{\rho_j}$. Since $\mathrm{inc}(u) = \theta_2 - \iota(v)$ and θ_2 is in the interior of Θ, $\mathrm{inc}(u)$ has a strictly positive coefficients for each basis element $\mathbf{e}_{\rho_j}$. Write $u = u_1 + \cdots u_m + u''$ for $u_j, u'' \in \mathbb{N}^{Q_1}$ satisfying $\mathrm{inc}(u_j) = \mathbf{e}_{\rho_j}$. We have that $y^u = y^{u_1} \cdots y^{u_m} y^{u''}$ and therefore in the localization above the monomials y^{u_j} are invertible. Take an arbitrary element $b := -\mathbf{e}_{\rho_j} - \mathbf{e}_{\rho'} + \mathbf{e}_{\rho_j \otimes \rho'}$ of $\mathfrak{B}$. Then there exists a path p_j from ρ' to $\rho' \otimes \rho_j$ with label $\mathrm{div}(u_j) \in \mathrm{Hom}(\rho', \rho' \otimes \rho_j)$. Let $u'_j \in \mathbb{N}^{Q_1}$ be the element determined by p_j. We then have $\mathrm{div}(u_j - u'_j) = 0$ and $\mathrm{inc}(u_j - u'_j) + \iota(b) = 0$ which implies that $z_b y^{u_j} - y^{u'_j} \in I_{\mathscr{L},\mathfrak{B}}$. Since y^{u_j} is invertible in the localization we may replace $z_b y^{u_j} - y^{u'_j}$ by $z_b - y^{u'_j - u_j}$, thereby eliminating z_b for every $b \in \mathfrak{B}$. Next, consider the general generator $y^{u_1} z^{v_1} - y^{u_2} z^{v_2}$ of $I_{L,\mathfrak{B}}$, eliminate the monomials z^{v_1}, z^{v_2} and multiply by the invertible elements y^{u_j} to get a polynomial. Since every z_b is replaced by some $y^{u'_j - u_j}$ for which $\mathrm{div}(u_j - u'_j) = 0$ the resulting polynomial will be in I_Q. After noting that the construction above enables us to write $y^u z^v$ as a monomial in $\Bbbk[\mathbb{N}^{Q_1}]$ we have the isomorphisms (5.3).

The isomorphisms (5.3) allow us to conclude that $\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_2} \cong \mathbb{V}(I_Q)^{\theta_2}_{ss}$ and consequently $[\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)] \cong [\mathbb{V}(I_Q)^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)]$. Now the stack $[\mathbb{V}(I_Q)^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)]$ is a substack of the representable stack $\mathcal{M}_{\theta_2}(Q)$ cut out by the homogeneous ideal I_Q and is therefore the variety $\mathbb{V}(I_Q)^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)$. Proposition 5.6 then gives $\mathrm{Hilb}^G \cong [\mathbb{V}(I_Q)^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)]$, completing the proof. $\qquad\square$

Corollary 5.11. *For $n \leq 3$ and abelian $G \subset \mathrm{SL}(n, \Bbbk)$, there exists generic stability conditions $\chi_{\theta_1}, \chi_{\theta_2} \in \mathrm{PGL}(\alpha)^\vee$, such that*

$$[\mathbb{A}^n/G] \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_1}/\mathrm{PGL}(\alpha)] \;\; \text{and} \;\; G\text{-}\mathrm{Hilb}(\mathbb{A}^n) \cong [\mathbb{V}(I_{\mathscr{L},\mathfrak{B}})^{ss}_{\theta_2}/\mathrm{PGL}(\alpha)].$$

Proof. By Proposition 5.4 we have $G\text{-}\mathrm{Hilb}(\mathbb{A}^n) \cong \mathrm{Hilb}^G(\mathbb{A}^n)$ for $n \leq 3$. $\qquad\square$

Remark 5.12. The careful choice of basis $\mathfrak{B}$ was made with G-Hilb($\mathbb{A}^n$) in mind. Moving between $[\mathbb{A}^n/G]$ and another crepant resolution via wall-crossings may require a different choice of basis.

Chapter 6

What if $\mathcal{X}$ is not toric?

Through out this thesis we have restricted ourselves to the case of $\mathcal{X}$ a toric DM stack. In particular, we restricted ourselves to stacks whose stabilizers are abelian. We briefly discuss the limitations of our construction in the non-toric case.

We state the following analogue to Remark 2.26.

Remark 6.1. If a stack $\mathcal{X}$ possesses a collection of line bundles $(L_1, \ldots, L_n)$ for which $\bigoplus_{i=1}^{n} L_i$ is π-ample then the automorphisms of its geometric points are abelian. Indeed, take x a geometric point of $\mathcal{X}$. If $\bigoplus_{i=0}^{n} L_i$ is π-ample, we have a faithful representation of $\mathrm{Aut}(x)$ in the group of diagonal matrices $(\Bbbk^\times)^n$, therefore $\mathrm{Aut}(x)$ is abelian.

To generalize to non-abelian stabilizers one needs to incorporate vector bundles in collections $\mathscr{L}$. This adds two levels of difficulty. The first is, we have to abandon the dimension vector $\alpha = (1, \ldots, 1)$ and work with general dimension vectors of quiver representations. The second and more fundamental hurdle, is that arrows no longer correspond to 1-by-1 matrices and are therefore not labelled by elements of free abelian groups; so the labelled quivers technology does not lift directly.

The finite groups $G \subset \mathrm{SL}(2, \Bbbk)$ have been comprehensively studied and admit an A-D-E classification. The A family corresponds to abelian G; the D and E families correspond to non-abelian groups. Therefore, the McKay quivers for the subgroups of type D and E form a good testing ground for any generalization of our construction to non-abelian stabilizers. Note that our construction gives a suitable ambient stack $\mathcal{M}(Q, \mathrm{div})$ and a closed immersion $[\mathbb{A}^2/G]$ to $\mathcal{M}(Q, \mathrm{div})$ for the A family, see Proposition 5.7.

Example 6.2. We consider the simplest finite non-abelian subgroup $\mathrm{SL}(2, \Bbbk)$, that

is the binary dihedral group $\mathrm{BD}(3)_4$ defined as follows

$$G := \mathrm{BD}(3)_4 := \left\langle \alpha := \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^3 \end{pmatrix}, \beta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ \middle| \ \epsilon^4 = 1 \right\rangle.$$

This is the finite non-abelian group corresponding to the D_4 Dynkin diagram.

The group G has four 1-dimensional irreducible representations and one 2 dimensional irreducible representation. The McKay quiver of G is given in Figure 6.1 below. The vertex 0 corresponds to the trivial representation and the vertex 4 corresponds to the only two dimensional representation so we take representations of dimension vector $\alpha = (1, 1, 1, 1, 2)$. We seek an analogue to Proposition 5.7 for $G = \mathrm{BD}(3)_4$, that is we seek a suitable ambient stack $\mathcal{M}$ and a closed immersion from $[\mathbb{A}^2/G]$ into that stack.

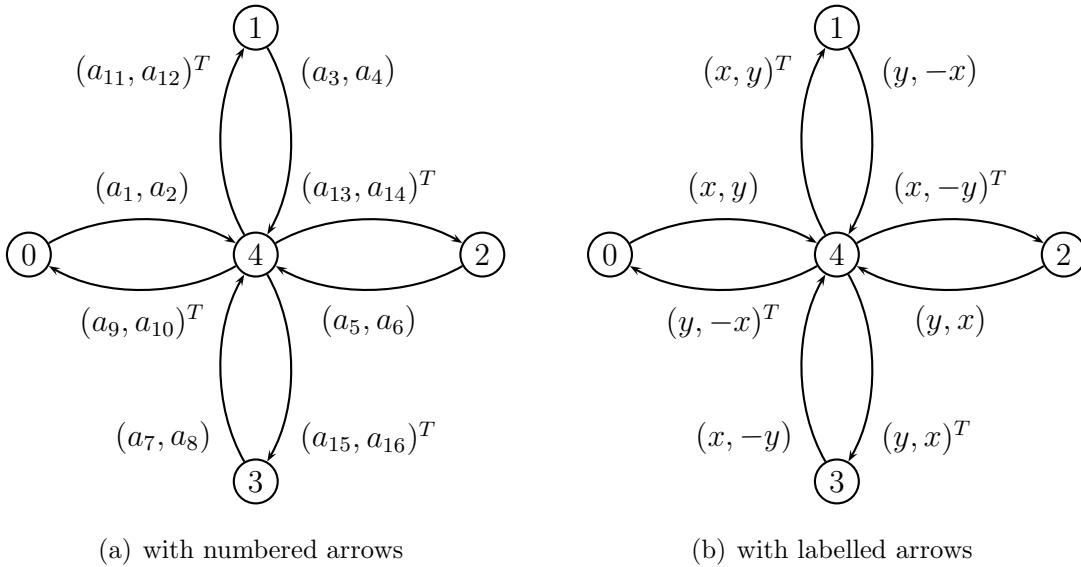
(a) with numbered arrows

(b) with labelled arrows

Figure 6.1: Labelled McKay quiver of $\mathrm{BD}(3)_4$.

The labels in Figure 6.1 (b) are due to Michael Wemyss and can be found on page 87 of Álvaro Nolla de Celis's thesis [dC09]. We will take the approach detailed in Example 3.1 to tackle this example. That is, we will restrict the action of $\mathrm{PGL}(\alpha)$ to the biggest subgroup that 'respects' the labels. More precisely, the labels on Figure 6.1 (b) define a morphism

$$\Psi^* : \mathbb{A}^2 \longrightarrow \mathrm{Spec}(\mathbb{k}[y_1, \ldots, y_{16}]),$$

taking (x, y) to $(x, y, y, -x, y, x, x, -y, y, -x, x, y, x, -y, y, x)$. The image of this morphism is cut out by an ideal I. Note that the labels are designed such that the

homogenous part of I cuts out G-Hilb in $\mathcal{M}_{\vartheta}(Q, \alpha)$, where ϑ is a stability condition for which $\vartheta_i > 0$ for $i \neq 0$, and $\vartheta_0 < 0$. The plan is to restrict the action of $\mathrm{PGL}(\alpha)$ to the biggest subgroup for which I is homogeneous.

We choose the following coordinates for $\mathrm{PGL}(\alpha) \subset \mathbb{A}^8$,

$$\mathrm{PGL}(\alpha) \cong \left\{ u := \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \, t_0, \, t_1, \, t_2, \, t_3 \, \middle| \, \det u \neq 0, \, t_0 = 1, \, t_1 \neq 0, \, t_2 \neq 0, \, t_3 \neq 0 \right\}.$$

A general element of $g \in \mathrm{PGL}(\alpha)$ acts on an arrow $(a, b)^T$ from vertex 4 to vertex i as follows

$$g \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} t_i$$

and on arrow (a, b) from vertex i to vertex 4 by

$$g \cdot \begin{pmatrix} a & b \end{pmatrix} = t_i^{-1} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.$$

Now g acts on (a_1, a_2) and $(a_9, a_{10})^T$ as follows

$$g \cdot (a_1, a_2) = (u_1 a_1 + u_3 a_2, u_2 a_1 + u_4 a_2) \tag{6.1}$$

$$g \cdot (a_9, a_{10})^T = (\det u)^{-1} (u_4 a_9 - u_2 a_{10}, -u_3 a_9 + u_1 a_{10})^T. \tag{6.2}$$

The equations $a_1 = -a_{10}$ and $a_2 = a_9$ belong to I. After acting by g we have, $u_1 a_1 + u_3 a_2 = (\det u)^{-1}(u_3 a_9 - u_1 a_{10})$ and $u_2 a_1 + u_4 a_2 = (\det u)^{-1}(u_4 a_9 - u_2 a_{10})$. Forcing these equations to be homogenous restricts us to elements of $\mathrm{PGL}(\alpha)$ for which $u_2 = (\det)^{-1} u_2$ and $u_4 = (\det^{-1}) u_4$. Now $\det u \neq 0$ so u_2 and u_4 can not be zero simultaneously therefore $\det u = 1$. We will use this fact implicitly through out the calculation. We also have

$$g \cdot (a_5, a_6) = (t_2)^{-1} (u_1 a_5 + u_3 a_6, u_2 a_5 + u_4 a_6) \tag{6.3}$$

$$g \cdot (a_{11}, a_{12})^T = t_1 (\det u)^{-1} (u_4 a_{11} - u_2 a_{12}, -u_3 a_{11} + u_1 a_{12})^T \tag{6.4}$$

$$g \cdot (a_{15}, a_{16})^T = t_3 (\det u)^{-1} (u_4 a_{15} - u_2 a_{16}, -u_3 a_{15} + u_1 a_{16})^T. \tag{6.5}$$

By (6.1) and (6.3), the equations $a_1 = a_6$ and $a_2 = a_5$ are homogeneous if

$$t_1 u_1 = u_4 \qquad\qquad -t_1 u_2 = u_3$$
$$-t_1 u_3 = u_2 \qquad\qquad t_1 u_4 = u_1.$$

This implies $t_1^2 = 1$. Similarly, by (6.1) and (6.4) the equations $a_1 = a_{11}$ and $a_2 = a_{12}$

are homogeneous if

$$t_2 u_1 = u_4 \qquad\qquad\qquad t_2 u_2 = u_3$$
$$t_2 u_3 = u_2 \qquad\qquad\qquad t_2 u_4 = u_1.$$

Therefore $t_2^2 = 1$. Also, by (6.1) and (6.5) the equations $a_1 = a_{11}$ and $a_2 = a_{12}$ are homogeneous if

$$t_3 u_1 = u_1 \qquad\qquad\qquad -t_3 u_2 = u_2$$
$$-t_3 u_3 = u_3 \qquad\qquad\qquad t_3 u_4 = u_4.$$

This in particular implies u_1 and u_2 can not be non-zero simultaneously and that $t_3^2 = 1$. Now

$$(\det u)^2 = (u_1 u_4 - u_2 u_3)^2 = (t_2(u_1)^2 - t_2(u_2)^2)^2 = u_1^4 - 2u_1^2 u_2^2 + u_2^4 = 1.$$

Since u_1 and u_2 can not be non-zero simultaneously this implies that $u_1^4 = 1$ and $u_2^4 = 1$.

This set of equations cuts out the subgroup G in $\mathrm{PGL}(\alpha)$. Restricting to the action of $\mathrm{PGL}(\alpha)$ on $\mathbb{A}^{16}$ to the subgroup G we get a closed immersion

$$\psi : [\mathbb{A}^2/G] \longrightarrow \mathcal{M} := [\mathbb{A}^{16}/G].$$

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