# On Wave Propagation in Finitely Deformed Magnetoelastic Solids

by

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# Abstract

In this thesis we consider some boundary value problems concerning nonlinear deformations and incremental motions in magnetoelastic solids. Three main problems have been addressed relating to waves propagating on the surface of a finitely deformed half-space and waves propagating along the axis of a thick-walled tube.

First, the equations and boundary conditions governing linearized incremental motions superimposed on an initial motion and underlying electromagnetic field are derived and then specialized to the quasimagnetostatic approximation. The magnetoelastic material properties are characterized in terms of a "total" isotropic energy density function that depends on both the deformation and a Lagrangian measure of the magnetic field.

In the first problem, we analyze the propagation of Rayleigh-type surface waves for different directions of the initial magnetic field and for a simple constitutive model of a magnetoelastic material in order to evaluate the combined effect of the finite deformation and magnetic field on the surface wave speed. Numerical results for a Mooney–Rivlin type magnetoelastic material show that a magnetic field in the considered (sagittal) plane in general destabilizes the material compared with the situation in the absence of a magnetic field. A magnetic field applied in the direction of wave propagation is more destabilizing than that applied perpendicular to it.

In the second problem, the propagation of Love-type waves in a homogeneously and finitely deformed layered half-space is analyzed for a Mooney–Rivlin type and a neo-Hookean type magnetoelastic energy function. The resulting wave speed characteristics in general depend significantly on the initial magnetic field as well as on the initial finite deformation, and the results are illustrated graphically for different combinations of these parameters. In the absence of a layer, shear horizontal surface waves do not exist in a purely elastic material, but the presence of a magnetic field normal to the sagittal plane makes such waves possible,

these being analogous to Bleustein–Gulyaev waves in piezoelectric materials.

Then, we consider nonlinear axisymmetric deformations and incremental motions of a cylindrical magnetoelastic tube. The effects of internal pressure, axial stretch, and magnetic field are studied for two different kinds of energy density functions. It is found that in general an underlying azimuthal magnetic field increases the total internal pressure, affects the axial load, and induces stability in the tube. Dependence of the incremental motion on internal pressure, axial stretch, thickness of tube, and the applied magnetic field is illustrated graphically.

Finally, we consider the general equations of Electrodynamics and Thermodynamics in continua. In particular, we write the equations governing mechanical waves, electromagnetic fields and temperature changes in a magnetoelastic conductor with a motivation to describe the electromagnetic acoustic transduction (EMAT) process. This is a work in progress and an open research problem for the future.

# Acknowledgements

First and foremost, thanks to my parents who are responsible for all of my education and much more, despite many odds.

I am sincerely grateful to my supervisor Prof. Ray W. Ogden FRS for his guidance over the last three years. He has been a constant source of inspiration and a great mentor. I thank him for correcting numerous errors in this thesis.

No words are enough for my wife Swati who has always been very encouraging and supportive throughout this PhD. She smilingly handled all the responsibilities while letting me focus on research. Thanks are also due to my brother Rahul and several other members of family.

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I would like to specially mention and thank MIT's OCW and India's NPTEL for providing many high quality courses free of cost online. These have been instrumental in filling many gaps in my knowledge.

Last but not the least, I gratefully acknowledge the financial support and excellent research facilities of the University of Glasgow.

# Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

Figure 1.2 and the related discussion in Chapter 1 are inspired by author's M.Tech. thesis [Saxena, 2009] submitted to the Indian Institute of Technology Kanpur. Excluding this, no part of this thesis has been submitted in the past for a degree at this or at any other university.

Chapter 2 and parts of Chapter 6 contain some background and preliminaries. The remaining thesis is author's original work with his supervisor Prof. Ray W. Ogden unless otherwise explicitly mentioned with references.

Results from Chapter 3 have been published in the International Journal of Applied Mechanics [Saxena and Ogden, 2011] and were presented at the 2nd International Conference on Material Modelling in Paris, France. Results from Chapter 4 have been published in Zeitschrift für Angewandte Mathematik und Physik [Saxena and Ogden, 2012] while results from Chapter 5 were presented at the 8th European Solid Mechanics Conference in Graz, Austria.

*Dedicated*

*to*

*my parents*

# Contents

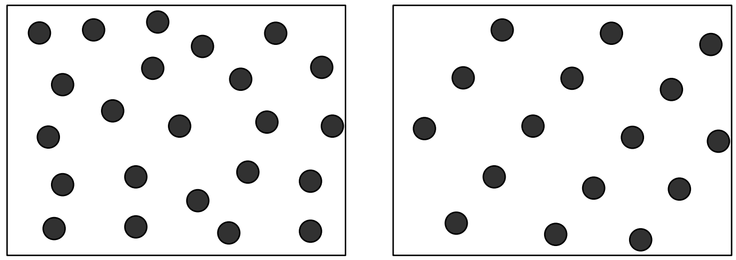
# Chapter 1

# Introduction

Recent times have seen a rapid increase in the engineering devices that exploit and rely on multi-physical couplings. A large subset of these devices work using the electro-magneto-mechanical interactions in solids and liquids. In particular, many synthetic elastomers that are capable of significant changes in their mechanical properties on the application of a magnetic field have been developed, as highlighted in the works of [Jolly et al., 1996], [Ginder et al., 2002], [Lokander and Stenberg, 2003], [Yalcintas and Dai, 2004], [Varga et al., 2006], and [Boczkowska and Awietjan, 2009].

Typically these elastomers consist of a rubber matrix filled with small micron-sized magnetically active particles (see, for example, Figure 1.1). The magnetic particles try to arrange themselves in the direction of applied magnetic field and therefore influence the macroscopic shape and the local elastic modulus of the material. These elastomers exhibit remarkable properties such as tuneable elastic modulus, non-homogeneous deformation, and a quick response to the magnetic field. Hence, they can be used in various engineering applications like vibration dampers and robotics as demonstrated by [Böse et al., 2012].

The above-mentioned developments have motivated a considerable increase in studying the coupling of electromagnetic and mechanical phenomena in the recent literature. Specially, the problem of wave propagation under a state of finite deformation in the presence of an electromagnetic field is very important for various applications. An important application of such analysis is in the experimental determination of the magnetoelastic properties of the materials concerned as done by [Jolly et al., 1996], [Böse and Röder, 2009], and [Johnson et al., 2012]. [Jolly et al., 1996] prepared a form of magnetoelastic elastomers by mixing carbonyl iron particles of 3–4 $\mu$m size in silicone oil and then allowing the mixture to cure in the presence of a magnetic field. They then performed double lap



(a) (b)

Figure 1.1: Schematic of a magnetoelastic material inspired from the results of [Jolly et al., 1996] representing elastomers filled with ferromagnetic particles. **(a)** Randomly distributed particles, isotropic material. **(b)** Particles aligned in a direction by the application of a magnetic field during the curing process. The resulting mechanical response is transversely isotropic.

shear tests for such specimens to determine the effective Young's elastic modulus of the elastomers containing 10%, 20%, and 30% iron by volume. [Johnson et al., 2012] study the behaviour of such elastomers under a dynamic loading in the presence of an underlying magnetic field to observe the effects on shear modulus, and natural frequency of a finite system. They report that initial stress and an underlying magnetic field significantly alter their results. Theoretical analysis of wave propagation under a state of finite deformation in the presence of magnetic field is also important for non-destructive evaluation, such as through electromagnetic acoustic transducers.

The effect of initial stress on the propagation of magnetoelastic waves was addressed as early as 1966 by [Yu and Tang, 1966], who considered the propagation of plane harmonic waves for some special cases of initial stress relevant to seismic wave propagation. [De and Sengupta, 1971, 1972] used the equations of [Yu and Tang, 1966] in order to discuss surface and interfacial waves in magnetoelastic conducting solids.

A paper by [Maugin, 1981] reviewed the major developments in deformable magnetoelastic materials until that time with special emphasis on wave propagation in magnetizable conducting materials. This was followed by a series of works, notably those by [Maugin and Hakmi, 1985] on magnetoelastic surface waves with a bias magnetic field orthogonal to the sagittal plane, by [Abd-Alla and Maugin, 1987] on the general form of the magnetoacoustic equations, by [Abd-Alla and Maugin, 1988] on magnetoelastic waves in anisotropic materials, by [Lee and Its, 1992] on Rayleigh waves in an undeformed magnetoelastic conductor and by [Hefni et al., 1995a, 1995b, 1995c] on surface and bulk magnetoelastic waves in electrical conductors. Similar parallel developments have been there in the field of electroelasticity, but in this thesis we focus solely on the magnetoelastic materials. Most of the work in this field is based on the study of electromagnetic phenomena in continua by [Pao, 1978], [Maugin, 1988], and [Eringen and Maugin, 1990a, 1990b].

Recently, a new constitutive formulation based on a "total" energy density function was developed by [Dorfmann and Ogden, 2004], wherein the solutions of some basic boundary-value problems were obtained using two alternative forms of the energy density with different independent magnetic vectors; see also [Dorfmann and Ogden, 2005] for the discussion of further boundary-value problems. This formulation was based on the equations of nonlinear magnetoelasticity developed by [Dorfmann and Ogden, 2003a], [Brigadnov and Dorfmann, 2003], and [Dorfmann and Ogden, 2003b]. In the paper by [Otténio et al., 2008], which was based on the formulation of [Dorfmann and Ogden, 2004], the equations governing time-independent linearized incremental deformations and magnetic fields superimposed on a static finite deformation and magnetic field were derived. These were then applied to analyze the effect of the presence of a magnetic field normal to the half-space boundary on the stability of a deformed magnetoelastic half-space.

Working with the same formulation, Bustamante and coworkers have worked on various boundary value problems and the variational formulations for obtaining numerical solutions. Static finite deformations of a magnetoelastic tube of finite length was considered in [Bustamante et al., 2007], a variational formulation to solve the governing equations of nonlinear magnetoelasticity using numerical methods was developed in [Bustamante et al., 2008] and [Bustamante, 2009], and [Bustamante and Ogden, 2012] further looked at the second variations of the energy functionals.

In this thesis, we focus on some boundary value problems concerning wave propagation in magnetoelastic materials. In particular, we study waves on the surface of a deformed magnetoelastic half-space and waves propagating in an axisymmetrically deformed magnetoelastic tube. Corresponding problems in the case of pure elasticity have been looked at in the past by various researchers. A general theory for incremental motions in elastic solid can be seen in [Biot, 1965], while a theory of nonlinear elasticity is presented in texts, such as those by [Ogden, 1997] and [Holzapfel, 2000]. Study of mechanical waves on the surface of a half-space goes back more than a hundred years when Lord Rayleigh studied what are now called 'Rayleigh waves'. Waves propagating on the surface of a stratified half-space with an out-of-plane motion of material particles are called 'Love waves'. Such waves, in the context of an incompressible and finitely deformed elastic solid, have been studied by various researchers, such as [Hayes and Rivlin, 1961], [Dowaikh and Ogden, 1990], [Dowaikh, 1999], and several references therein.

Axisymmetric deformations and incremental motions of a finitely deformed thick-walled elastic tube have been studied by various researchers in the past. In particular, we refer to the works of [Wang and Ertepinar, 1972] and [Haughton and Ogden, 1979a, 1979b] with regards to stability and bifurcations of cylindrical tubes; and [Vaughan, 1979], [Dasgupta, 1982], [Haughton, 1982], and [Haughton, 1984] on wave propagation in finitely deformed cylinders and tubes.

Electromagnetic Acoustic Transduction (EMAT) is a technique used to generate mechanical waves in magnetoelastic conductors for the purpose of non-destructive testing. A generic EMAT configuration serves a two-fold purpose, viz., the generation of mechanical waves using electromagnetic body force and sensing of the existence of mechanical waves due to changes in the electromagnetic fields. An EMAT configuration, as shown in Figure 1.2, comprises of an electromagnet that generates a large static magnetic field inside the magnetoelastic material, and an AC current carrying coil that generates a time-varying magnetic field. The time changing magnetic field causes eddy currents and a body force in the magnetoelastic conductor that lead to generation of mechanical waves in the transmitting mode. In the receiving mode, mechanical waves already present in the magnetoelastic conductor cause a time-varying change in the existing magnetic field. This causes the generation of a small AC current in the coil which can be measured and used to determine the existence of such mechanical waves in the bulk material.

EMAT techniques have been greatly used in the recent decades for the purpose of non-destructive testing in magnetoelastic conductors, see, for example, the works of [Ludwig et al., 1993], [Ogi, 1997], [Hirao and Ogi, 2003], [Shapoorabadi et al., 2005], and [Saxena, 2009]. However, most of the existing literature still relies largely on a linear theory of elasticity, and simplistic coupling mechanisms of displacement field with electromagnetic fields. Very often the samples on which testing is performed are finitely deformed or under a pre-stress. Moreover, generation of eddy currents in the material causes a change in temperature due to resistive heating which is rarely taken into account. Such a simplified theory has many limitations, and its use leads to, for example, a reduction in the efficiency of EMATs by reducing the "signal-to-noise ratio".

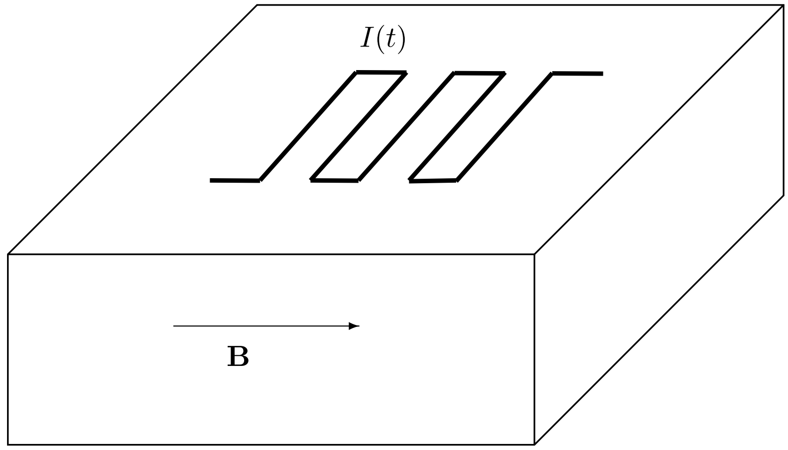



Figure 1.2: Schematic showing an EMAT arrangement. A coil carrying alternating current $I(t)$ placed on a magnetoelastic bulk with an underlying magnetic field $\mathbf{B}$.

Towards the end of this thesis, we have expressed the coupled equations of nonlinear elasticity taking into account the effects due to electromagnetic fields and changes in temperature. This is still a work in progress and will form a basis for future research on theoretical analysis of thermodynamics and electrodynamics in general, and improved mathematical modelling of EMATs in particular.

## 1.1 Thesis outline

In this thesis, we consider different boundary value problems in relation to wave propagation in different geometries. In particular, we focus on Rayleigh type, Love type, and Bleustein-Gulyaev type surface waves on a magnetoelastic half-space, and waves propagating along an infinitely long hollow cylinder. This thesis is divided into seven chapters (including this one) as follows.

In Chapter 2, we detail the basic equations governing nonlinear magnetoelasticity. First, using the equations in [Eringen and Maugin, 1990a] and [Ogden, 2009], we write the equations of electrodynamics in a deformable continua. Then, on the application of a quasimagnetostatic approximation, the equations are simplified to consider only magnetic effects in a finitely deformed static solid continuum. We allow for time-dependent increments on a finite deformation and, using the formulation in [Dorfmann and Ogden, 2004], define magnetoelastic *moduli* tensors. For a simple case of homogeneous plane waves, following [Destrade and Ogden, 2011], a propagation condition is derived that leads to a generalized strong ellipticity condition to be satisfied by the moduli tensors.

In Chapter 3, following [Otténio et al., 2008], we specialize the governing equations and boundary conditions derived in Chapter 2 to two dimensions to consider increments on the surface of a homogeneously finitely deformed magnetoelastic half-space. The combined effect of the underlying magnetic field and the finite deformation is then studied on wave propagation for Rayleigh type waves and numerical solutions obtained for a generalized Mooney–Rivlin magnetoelastic solid. We consider three different cases – when the underlying magnetic field is in the direction of wave propagation, when the magnetic field is perpendicular to the direction of wave propagation but in the plane of deformation, and when the magnetic field is out of plane. For the first two cases, we considered different plane strain problems and observed that in general a magnetic field tends to cause instabilities on the surface of a half-space. The wave speed, in general, reduces on the application of a high magnetic field and its variation has been demonstrated graphically. For the static bifurcations, our results converge to those obtained by [Otténio et al., 2008], while on removing the magnetic field, for the purely elastic case, our results reduce to those obtained by [Dowaikh and Ogden, 1990]. When the magnetic field is out of plane, for the energy function considered, it is observed that the magnetic boundary conditions become incompatible and lead to a trivial solution for the increments in the magnetic field. Hence, the problem is reduced to a purely mechanical problem.

In Chapter 4, we use the two-dimensional specialization of Chapter 3 but allow for out-of-plane mechanical displacements. Thus, we analyze Love type waves on the surface of a layered magnetoelastic half-space and show the existence of Bleustein–Gulyaev type waves on the surface of a half-space. Bleustein–Gulyaev type waves, that have an out of plane motion and exist without a layer on a half-space, require an out-of-plane underlying magnetic field to be present and do not have a counterpart in pure elasticity. The governing equations and boundary conditions for Love-type waves are transformed to a secular equation relating wave speed to various deformation parameters. Multiple modes of wave propagation are observed and we illustrate the wave propagation characteristics for different principal stretches and different directions of the underlying magnetic field. Numerical results are obtained for a Mooney–Rivlin type and a neo-Hookean type magnetoelastic material. In the absence of a magnetic field, our results converge to those obtained by [Dowaikh, 1999] for the purely elastic case.

In Chapter 5, we specialize the governing equations and boundary conditions from Chapter 2 to cylindrical coordinates and consider finite deformations and motions of a cylindrical tube. In the presence of an internal pressure, axial stretch, and an underlying magnetic field (in either the axial or azimuthal direction), the tube undergoes a finite deformation. We obtain numerical solutions for static finite deformations for two energy density functions – a Mooney–Rivlin type magnetoelastic solid, and a generalization of Ogden-type elastic solid to magnetoelasticity. Axially homogeneous magnetic field in the axial direction is not possible for a tube of finite length due to the boundary conditions that need to be satisfied at the ends. A problem concerning non-homogeneous axial magnetic field has been solved numerically by [Bustamante et al., 2007]. Hence to consider an axially homogeneous magnetic field, we take an infinitely long tube for this problem. An azimuthal magnetic field tends to increase the total internal pressure and generates an extensional or compressional axial loading depending on the inflation. An axial magnetic field, on the other hand, has no effect on the internal pressure and generates an extensional axial loading in all the cases.

Superimposed on the finite deformation, we allow for axisymmetric incremental motions while considering two different cases of an axial magnetic field or an azimuthal magnetic field. It is observed that the equations governing incremental motions in the azimuthal direction are decoupled from the equations governing incremental motions in the axial and radial directions, hence we consider these two sub-cases separately. We finally obtain higher order ODEs in each case which are non-dimensionalized and converted to a system of first order ODEs to be solved numerically. An algorithm described by [Haughton and Ogden, 1979b] is used to obtain numerical solutions in Matlab for a Mooney–Rivlin type magnetoelastic solid. For the case of radial and axial displacements, only a purely elastic solution is possible by taking the incremental magnetic fields to be zero. For azimuthal displacements, we obtain multiple modes of wave propagation and the dependence of the wave speed on other deformation parameters is illustrated graphically. On neglecting the magnetic effects and considering only pure elasticity, our equations converge to those obtained by [Haughton and Ogden, 1979b] and [Haughton, 1984].

In Chapter 6, doing away with the quasimagnetostatic approximation used in Chapters 2–5, we use the dipole-current circuit model of [Pao, 1978] to consider the governing equations of electrodynamics in solid continua. We consider a temperature-dependent elastic response of a finitely deformed pre-stressed conductor in the presence of an electric and a magnetic field. The equations derived in this chapter are useful for understanding and mathematically modelling EMATs. Numerical solutions using Finite Element Methods of these equations for a specified geometry are planned to be undertaken as a future work of this research.

Key results and the conclusions of this thesis are summarized in Chapter 7. Some side calculations are detailed in Appendices A–C.

Results from Chapter 3 have been published in the International Journal of Applied Mechanics [Saxena and Ogden, 2011] and presented at the 2nd International Conference on Material Modelling in Paris, France. Results from Chapter 4 have been published in Zeitschrift für Angewandte Mathematik und Physik [Saxena and Ogden, 2012]. Further results from Chapter 5 are in preparation for submission for publication, and have been presented at the 8th European Solid Mechanics Conference in Graz, Austria.

# Chapter 2

# Theory of Nonlinear Magnetoelasticity

In this chapter we first summarize the basic equations governing electrodynamics of a solid continua based on [Eringen and Maugin, 1990a] and [Ogden, 2009]. Then, using the constitutive formulation developed in [Dorfmann and Ogden, 2004], we define the mangetoelastic moduli tensors and obtain the generalized strong ellipticity condition as given in [Destrade and Ogden, 2011].

A body $B$ is a set of points from which we can define a mapping $\theta$ to a Euclidean space $\mathcal{E}$ as shown in Figure 2.1. The region of $\mathcal{E}$ given by $\theta(B)$ is called a configuration of the body. The undeformed stress-free *reference configuration* of a continuous body is denoted by $\mathcal{B}_r$ and its boundary by $\partial\mathcal{B}_r$. Let $\mathcal{B}_t$, the *current configuration*, be the region occupied by the body at time $t$ and $\partial\mathcal{B}_t$ its boundary. Elements of $B$ are called material points and are identified by the position vector $\mathbf{X}$ in $\mathcal{B}_r$ which becomes the position vector $\mathbf{x}$ in $\mathcal{B}_t$.

## 2.1 Basic equations

### 2.1.1 Kinematics

The time-dependent deformation (or motion) of the body is described by an invertible mapping $\boldsymbol{\chi}$ that maps points from $\mathcal{B}_r$ to points in $\mathcal{B}_t$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The function $\boldsymbol{\chi}$ and its inverse are assumed to be sufficiently regular in space and time. The velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of a material particle at $\mathbf{X}$ are defined by

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{x}_{,t} = \frac{\partial}{\partial t}\boldsymbol{\chi}\left(\mathbf{X}, t\right), \quad \mathbf{a}(\mathbf{x}, t) = \mathbf{v}_{,t} = \mathbf{x}_{,tt} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}\left(\mathbf{X}, t\right), \tag{2.1}$$

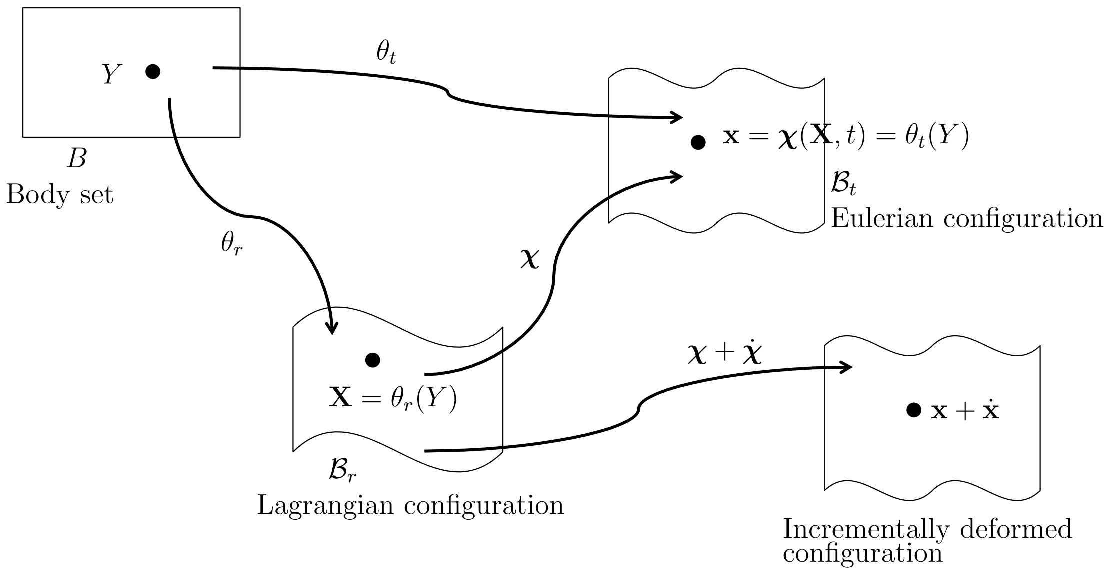



Figure 2.1: Body set $B$, the Lagrangian configuration $\mathcal{B}_r$, the Eulerian configuration $\mathcal{B}_t$, and the incrementally deformed configuration.

where the subscript $t$ following a comma denotes the material time derivative.

Throughout this thesis, grad, div, and curl denote the standard differential operators with respect to $\mathbf{x}$; and Grad, Div, and Curl denote the corresponding operators with respect to $\mathbf{X}$.

The *deformation gradient tensor* is defined as $\mathbf{F} = \text{Grad}\,\boldsymbol{\chi}(\mathbf{X}, t)$ and its determinant is denoted by $J = \det \mathbf{F}$, with $J > 0$. For an incompressible material the constraint

$$J \equiv \det \mathbf{F} = 1, \tag{2.2}$$

has to be satisfied. Incompressibility also imposes the condition $\text{div}\,\mathbf{v} = 0$. Associated with $\mathbf{F}$ are the left and right Cauchy–Green tensors, defined by

$$\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \tag{2.3}$$

respectively.

Let $\boldsymbol{\Gamma} = \text{grad}\,\mathbf{v}$ denote the velocity gradient, tr and $^{\mathrm{T}}$ the trace and transpose of a second-order tensor, respectively, $\mathbf{0}$ the zero vector, and $\mathbf{O}$ the second-order zero tensor. Then the following standard kinematic identities are noted (see, for example, [Ogden, 2009]) which will be useful for switching between Eulerian and Lagrangian descriptions:

$$\mathbf{F}_{,t} = \boldsymbol{\Gamma}\mathbf{F}, \quad \left(\mathbf{F}^{-1}\right)_{,t} = -\mathbf{F}^{-1}\boldsymbol{\Gamma}, \quad J_{,t} = J\,\text{tr}\,\boldsymbol{\Gamma} = J\,\text{div}\,\mathbf{v},$$

$$\text{Div}\left(J\mathbf{F}^{-1}\right) = \mathbf{0}, \quad \text{div}\left(J^{-1}\mathbf{F}\right) = \mathbf{0}, \quad \text{Curl}\left(\mathbf{F}^{\mathrm{T}}\right) = \mathbf{O}, \quad \text{curl}\left(\mathbf{F}^{-\mathrm{T}}\right) = \mathbf{O}. \tag{2.4}$$

In this thesis, we follow the convention that differentiation operates on the first index of the following tensor. In the index notation

$$(\operatorname{div}\mathbf{F})_\alpha = \frac{\partial F_{j\alpha}}{\partial x_j}, \quad (\operatorname{Curl}\mathbf{F}^{\mathrm{T}})_{\alpha i} = \varepsilon_{\alpha\beta\gamma}\frac{\partial F_{i\gamma}}{\partial X_\beta}, \tag{2.5}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the alternating symbol.

Let $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$ be an Eulerian vector with a Lagrangian counterpart $\mathbf{A} = \mathbf{A}(\mathbf{X}, t)$. Then from the above kinematic identities, we obtain

$$\operatorname{Div}(J\mathbf{F}^{-1}\mathbf{a}) = J\operatorname{div}\mathbf{a}, \quad \operatorname{Curl}(\mathbf{F}^{\mathrm{T}}\mathbf{a}) = J\mathbf{F}^{-1}\operatorname{curl}\mathbf{a}, \tag{2.6}$$

$$\operatorname{div}(J^{-1}\mathbf{F}\mathbf{A}) = J^{-1}\operatorname{Div}\mathbf{A}, \quad \operatorname{curl}(\mathbf{F}^{-\mathrm{T}}\mathbf{A}) = J^{-1}\mathbf{F}\operatorname{Curl}\mathbf{A}. \tag{2.7}$$

For the divergence identities to be equivalent, we require $\mathbf{a}$ and $\mathbf{A}$ to be connected by the relation $\mathbf{a} = J^{-1}\mathbf{F}\mathbf{A}$. While, for the curl identities to be equivalent, we require the connection $\mathbf{a} = \mathbf{F}^{-\mathrm{T}}\mathbf{A}$.

Let $\mathbf{a} = J^{-1}\mathbf{F}\mathbf{A}$ and $\mathbf{V} = \mathbf{F}^{-1}\mathbf{v}$, then the material time derivative of $\mathbf{a}$ gives

$$\mathbf{a}_{,t} = \frac{\partial \mathbf{a}}{\partial t} + (\operatorname{grad}\mathbf{a})\mathbf{v}, \tag{2.8}$$

$$\Rightarrow J^{-1}\mathbf{F}\mathbf{A}_{,t} = \frac{\partial \mathbf{a}}{\partial t} + (\operatorname{grad}\mathbf{a})\mathbf{v} - \boldsymbol{\Gamma}\mathbf{a} + (\operatorname{div}\mathbf{v})\mathbf{a}, \tag{2.9}$$

$$\Rightarrow J^{-1}\mathbf{F}\mathbf{A}_{,t} = \frac{\partial \mathbf{a}}{\partial t} - \operatorname{curl}(\mathbf{v}\times\mathbf{a}) + (\operatorname{div}\mathbf{a})\mathbf{v}, \tag{2.10}$$

$$\Rightarrow J\mathbf{F}^{-1}\frac{\partial \mathbf{a}}{\partial t} = \mathbf{A}_{,t} + \operatorname{Curl}(\mathbf{V}\times\mathbf{A}) - (\operatorname{Div}\mathbf{A})\mathbf{V}. \tag{2.11}$$

Here, to obtain the last equation, we have used the relation from $(2.6)_2$

$$\operatorname{curl}(\mathbf{v}\times\mathbf{a}) = J^{-1}\mathbf{F}\operatorname{Curl}[\mathbf{F}^{\mathrm{T}}(\mathbf{v}\times\mathbf{a})], \tag{2.12}$$

along with the relation $\mathbf{F}^{\mathrm{T}}(\mathbf{v}\times\mathbf{a}) = \mathbf{V}\times\mathbf{A}$ which can be obtained by the standard vector identities.

### 2.1.2 Equations of electromagnetism

Let $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{J}$, $\mathbf{K}$, $\rho_{\mathrm{e}}$, and $\sigma_{\mathrm{e}}$ be the electric field, electric displacement, magnetic induction, magnetic field, volume electric current density, surface electric current density, volume electric charge density, and surface electric charge density, respectively. It should be noted that the volume current density $\mathbf{J}$ is different from $J = \det\mathbf{F}$ defined in the above subsection. We work within the non-relativistic framework, with Maxwell's equations of electromagnetism given in Eulerian form by

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{curl}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \quad \operatorname{div}\mathbf{D} = \rho_{\mathrm{e}}, \quad \operatorname{div}\mathbf{B} = 0, \tag{2.13}$$

with the boundary conditions

$$\mathbf{n} \times [\![\mathbf{E} + \mathbf{v} \times \mathbf{B}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{D}]\!] = \sigma_{\mathrm{e}},$$
$$\mathbf{n} \times [\![\mathbf{H} - \mathbf{v} \times \mathbf{D}]\!] = \mathbf{K} - \sigma_{\mathrm{e}}\mathbf{v}_{\mathrm{s}}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0, \tag{2.14}$$

on $\partial\mathcal{B}_t$, where $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}_t$, $\mathbf{v}_{\mathrm{s}}$ is the value of $\mathbf{v}$ on $\partial\mathcal{B}_t$, and surface polarization is not included. Here $[\![\mathbf{a}]\!]$ represents the jump in vector $\mathbf{a}$ across the boundary in the sense $[\![\mathbf{a}]\!] = \mathbf{a}^{\mathrm{o}} - \mathbf{a}^{\mathrm{i}}$, where the superscripts 'o' and 'i' signify 'outside' and 'inside', respectively.

Lagrangian forms of the physical quantities in (2.13) are defined by (see, for example, [Maugin, 1988], [Ogden, 2009])

$$\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}, \quad \mathbf{E}_l = \mathbf{F}^{\mathrm{T}}\mathbf{E}, \quad \mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}, \quad \mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B},$$
$$\mathbf{J}_{\mathrm{E}} = J\mathbf{F}^{-1}\left(\mathbf{J} - \rho_{\mathrm{e}}\mathbf{v}\right), \quad \rho_{\mathrm{E}} = J\rho_{\mathrm{e}}. \tag{2.15}$$

Substituting $(2.15)_2$ in $(2.13)_1$ and using $(2.15)_4$ with (2.11) gives

$$\mathrm{curl}(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l) = -J^{-1}\mathbf{F}\mathbf{B}_{l,t} - J^{-1}\mathbf{F}\,\mathrm{Curl}(\mathbf{V} \times \mathbf{B}_l) + J^{-1}(\mathrm{Div}\,\mathbf{B}_l)\mathbf{F}\mathbf{V}, \tag{2.16}$$

which can be further simplified to obtain

$$\mathrm{Curl}(\mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l) = -\mathbf{B}_{l,t}. \tag{2.17}$$

Using the process described above, we can re-write all the four Maxwell's equations in the Lagrangian form as

$$\mathrm{Curl}\,(\mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l) = -\mathbf{B}_{l,t}, \quad \mathrm{Div}\,\mathbf{D}_l = \rho_{\mathrm{E}}, \tag{2.18}$$
$$\mathrm{Curl}\,(\mathbf{H}_l - \mathbf{V} \times \mathbf{D}_l) = \mathbf{D}_{l,t} + \mathbf{J}_{\mathrm{E}}, \quad \mathrm{Div}\,\mathbf{B}_l = 0, \tag{2.19}$$

along with the boundary conditions

$$\mathbf{N} \times [\![\mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l]\!] = \mathbf{0}, \quad \mathbf{N} \cdot [\![\mathbf{D}_l]\!] = \sigma_{\mathrm{E}}, \tag{2.20}$$
$$\mathbf{N} \times [\![\mathbf{H}_l - \mathbf{V} \times \mathbf{D}_l]\!] = \mathbf{K}_l - \sigma_{\mathrm{E}}\mathbf{V}_{\mathrm{s}}, \quad \mathbf{N} \cdot [\![\mathbf{B}_l]\!] = 0, \tag{2.21}$$

on $\partial\mathcal{B}_r$. The transformation from (2.14) to (2.20) and (2.21) requires use of Nanson's formula $\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}A$ connecting reference and current area elements $\mathrm{d}A$ and $\mathrm{d}a$, where $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_r$. Here each term is evaluated on $\partial\mathcal{B}_r$, $\mathbf{V}_s$ is the value of $\mathbf{V}$ on the boundary, $\mathbf{K}_l$ is the surface current density per unit area of $\partial\mathcal{B}_r$ given by $\mathbf{K}_l = \mathbf{F}^{-1}\mathbf{K}\mathrm{d}a/\mathrm{d}A$ and $\sigma_{\mathrm{E}} = \sigma_{\mathrm{e}}\mathrm{d}a/\mathrm{d}A$ is the surface charge density per unit area of $\partial\mathcal{B}_r$.

### 2.1.3 Continuum electromagnetodynamic equations

In Eulerian form, the linear momentum balance equation may be written as

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \rho\mathbf{a}, \tag{2.22}$$

where $\rho$ is the mass density, $\mathbf{f}$ is the mechanical body force density per unit mass and $\boldsymbol{\tau}$ is the so-called *total Cauchy stress tensor*, which incorporates the electromagnetic body forces. In Lagrangian form, the equation of motion is

$$\operatorname{Div}\mathbf{T} + \rho_r\mathbf{f} = \rho_r\mathbf{a}, \tag{2.23}$$

where $\mathbf{T}$ is the total nominal stress tensor and $\rho_r$ is the reference mass density, and we note the connections

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\mathbf{T}, \quad \rho_r = \rho J. \tag{2.24}$$

The transformation from (2.22) to (2.23) is effected by use of $(2.4)_5$.

If there are no intrinsic mechanical couples, which is assumed to be the case, then, by virtue of the definition of the total stress, the electric and magnetic couples are absorbed in such a way that $\boldsymbol{\tau}$ is *symmetric*. The angular momentum balance equation is then expressed in either of the equivalent forms

$$\boldsymbol{\tau}^{\mathrm{T}} = \boldsymbol{\tau}, \quad (\mathbf{F}\mathbf{T})^{\mathrm{T}} = \mathbf{F}\mathbf{T}. \tag{2.25}$$

On any part of the boundary $\partial\mathcal{B}_r$ where the traction is prescribed, the boundary condition may be given as

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{A}} + \mathbf{t}_{\mathrm{M}}, \tag{2.26}$$

where $\mathbf{t}_{\mathrm{A}}$ and $\mathbf{t}_{\mathrm{M}}$ are the mechanical and magnetic contributions to the traction per unit area on the boundary $\partial\mathcal{B}_r$ in the reference configuration. The above equation in Eulerian form can be written as

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \mathbf{t}_{\mathrm{m}}, \tag{2.27}$$

where $\mathbf{t}_{\mathrm{a}}$ and $\mathbf{t}_{\mathrm{m}}$ are the Eulerian representations of the mechanical and magnetic contributions to the traction per unit area on the boundary $\partial\mathcal{B}_t$.

## 2.2 Incremental equations

On the initial motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, we superimpose an incremental motion given by

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\chi}}(\mathbf{X}, t), \tag{2.28}$$

where here and henceforth incremented quantities are denoted by a superimposed dot. The Eulerian counterpart of $\dot{\mathbf{x}}$ is given by the displacement $\mathbf{u}(\mathbf{x}, t) = \dot{\mathbf{x}}(\mathbf{X}, t)$. Then, an increment in the velocity $\mathbf{v}$ is given by

$$\dot{\mathbf{v}} = \dot{\mathbf{x}}_{,t} = \mathbf{u}_{,t}. \tag{2.29}$$

We define $\mathbf{L} = \operatorname{grad}\mathbf{u}$ and then obtain the useful relations

$$\dot{\mathbf{F}} = \mathbf{LF}, \quad \dot{\mathbf{F}}_{,t} = (\operatorname{grad}\dot{\mathbf{v}})\,\mathbf{F}, \tag{2.30}$$

which supplement those in (2.4).

For an incompressible material, the constraint $J = 1$ leads to $\operatorname{div}\mathbf{v} = 0$ and as a linear approximation, we get

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\dot{\mathbf{v}} = 0. \tag{2.31}$$

The governing equations (2.18) and (2.19) are also satisfied by the incremented quantities $(\mathbf{E}_l + \dot{\mathbf{E}}_l)$, $(\mathbf{B}_l + \dot{\mathbf{B}}_l)$, $(\mathbf{V} + \dot{\mathbf{V}})$, $(\mathbf{D}_l + \dot{\mathbf{D}}_l)$, $(\mathbf{H}_l + \dot{\mathbf{H}}_l)$, $(\mathbf{J}_\mathrm{E} + \dot{\mathbf{J}}_\mathrm{E})$, and $(\rho_\mathrm{E} + \dot{\rho}_\mathrm{E})$. Substituting them in the governing equations gives the incremental forms of the Lagrangian Maxwell's equations as

$$\operatorname{Curl}(\dot{\mathbf{E}}_l + \mathbf{V} \times \dot{\mathbf{B}}_l + \dot{\mathbf{V}} \times \mathbf{B}_l) = -\dot{\mathbf{B}}_{l,t}, \quad \operatorname{Div}\dot{\mathbf{D}}_l = \dot{\rho}_\mathrm{E}, \tag{2.32}$$

$$\operatorname{Curl}(\dot{\mathbf{H}}_l - \mathbf{V} \times \dot{\mathbf{D}}_l - \dot{\mathbf{V}} \times \mathbf{D}_l) = \dot{\mathbf{D}}_{l,t} + \dot{\mathbf{J}}_\mathrm{E}, \quad \operatorname{Div}\dot{\mathbf{B}}_l = 0, \tag{2.33}$$

and similarly from the mechanical balance equations (2.23) and (2.25), we have

$$\operatorname{Div}\dot{\mathbf{T}} + \rho_r\dot{\mathbf{f}} = \rho_r\mathbf{u}_{,tt}, \quad \mathbf{LFT} + \mathbf{F}\dot{\mathbf{T}} = \mathbf{T}^\mathrm{T}\mathbf{F}^\mathrm{T}\mathbf{L}^\mathrm{T} + \dot{\mathbf{T}}^\mathrm{T}\mathbf{F}^\mathrm{T}, \tag{2.34}$$

wherein use has been made of $(2.30)_1$.

Analogously to Equation (2.15), we define updated (i.e. pushed-forward to the Eulerian configuration) forms of the increments $\dot{\mathbf{T}}, \dot{\mathbf{B}}_l, \dot{\mathbf{D}}_l, \dot{\mathbf{E}}_l, \dot{\mathbf{H}}_l$ as

$$\begin{aligned} &\dot{\mathbf{T}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{T}}, \quad \dot{\mathbf{B}}_{l0} = J^{-1}\mathbf{F}\dot{\mathbf{B}}_l, \quad \dot{\mathbf{D}}_{l0} = J^{-1}\mathbf{F}\dot{\mathbf{D}}_l, \\ &\dot{\mathbf{E}}_{l0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_l, \quad \dot{\mathbf{H}}_{l0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{H}}_l, \end{aligned} \tag{2.35}$$

where the subscript 0 is used to indicate the push-forward operation. We use these push-forward forms to update the incremented governing equations to obtain

$$\operatorname{curl}(\dot{\mathbf{E}}_{l0} + \mathbf{v} \times \dot{\mathbf{B}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{l,t0}, \quad \operatorname{div}\dot{\mathbf{D}}_{l0} = \dot{\rho}_{\mathrm{E}0}, \tag{2.36}$$

$$\operatorname{curl}(\dot{\mathbf{H}}_{l0} - \mathbf{v} \times \dot{\mathbf{D}}_{l0} - \dot{\mathbf{v}} \times \mathbf{D}) = \dot{\mathbf{D}}_{l,t0} + \dot{\mathbf{J}}_{\mathrm{E}0}, \quad \operatorname{div}\dot{\mathbf{B}}_{l0} = 0, \tag{2.37}$$

and

$$\operatorname{div}\dot{\mathbf{T}}_0 + \rho_r\dot{\mathbf{f}} = \rho_r\mathbf{u}_{,tt}, \quad \mathbf{L}\boldsymbol{\tau} + \dot{\mathbf{T}}_0 = \boldsymbol{\tau}\mathbf{L}^{\mathrm{T}} + \dot{\mathbf{T}}_0^{\mathrm{T}}. \tag{2.38}$$

It should be noted that the push-forward and material time derivative operations do not commute in general. However, in the special case of $\mathbf{v} = \mathbf{0}$, we have

$$\begin{aligned}\dot{\mathbf{B}}_{l0,t} = (J^{-1}\mathbf{F}\dot{\mathbf{B}}_l)_{,t} &= J^{-1}\mathbf{F}\dot{\mathbf{B}}_{l,t} + J^{-1}\boldsymbol{\Gamma}\mathbf{F}\dot{\mathbf{B}}_{l,t} - J^{-1}(\operatorname{div}\mathbf{v})\mathbf{F}\dot{\mathbf{B}}_{l,t},\\ &= J^{-1}\mathbf{F}\dot{\mathbf{B}}_{l,t},\\ &= \dot{\mathbf{B}}_{l,t0},\end{aligned} \tag{2.39}$$

and similarly $\dot{\mathbf{D}}_{l0,t} = \dot{\mathbf{D}}_{l,t0}$.

### 2.2.1 The quasimagnetostatic approximation

We now consider the initial configuration to be purely static and subject only to magnetic and mechanical effects, i.e. $\mathbf{E} = \mathbf{0}$, $\mathbf{D} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ and no mechanical body forces ($\mathbf{f} = \mathbf{0}$). We assume that there are no volume or surface charges or currents, so that $\rho_{\mathrm{e}} = \sigma_{\mathrm{e}} = 0$ and $\mathbf{J} = \mathbf{0}$, while $\mathbf{H}$ and $\mathbf{B}$ are independent of time. Additionally, we consider a non-conducting material so that $\dot{\mathbf{J}}_{l0} = \mathbf{0}$. The updated incremented governing equations then specialize to

$$\operatorname{curl}(\dot{\mathbf{E}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{l0,t}, \quad \operatorname{div}\dot{\mathbf{D}}_{l0} = 0, \tag{2.40}$$

$$\operatorname{curl}\dot{\mathbf{H}}_{l0} = \dot{\mathbf{D}}_{l,t0}, \quad \operatorname{div}\dot{\mathbf{B}}_{l0} = 0, \tag{2.41}$$

and

$$\operatorname{div}\dot{\mathbf{T}}_0 = \rho_r\mathbf{u}_{,tt}. \tag{2.42}$$

We now focus on the magnetoacoustic (or quasimagnetostatic) approximation of the equations, which allows the incremental electric field and displacement to be neglected. It can be shown that they are of order $v/c\,(\ll 1)$ times the retained terms in the equations, where $c$ is the speed of electromagnetic waves in vacuo and $v$ is a typical magnitude of the acoustic wave speed. After the approximation is applied, the remaining equations, coupling magnetic and mechanical effects, are

$$\operatorname{curl}\dot{\mathbf{H}}_{l0} = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{B}}_{l0} = 0, \quad \operatorname{div}\dot{\mathbf{T}}_0 = \rho_r\mathbf{u}_{,tt}. \tag{2.43}$$

These are the equations we use for the interior of the material in Chapters 3–5.

Outside the material, which may be vacuum (or a non-magnetizable and non-polarizable material) we use a superscript $^*$ to indicate field quantities. Thus, $\mathbf{H}^*$ and $\mathbf{B}^*$, respectively are the magnetic field and magnetic induction, which follow the simple constitutive relation $\mathbf{B}^* = \mu_0 \mathbf{H}^*$, where $\mu_0$ is the magnetic permeability of vacuum. Then the magnetostatic equations are

$$\operatorname{div} \mathbf{B}^* = 0, \quad \operatorname{curl} \mathbf{H}^* = \mathbf{0}, \tag{2.44}$$

and in the quasimagnetostatic approximation their incremental counterparts are

$$\operatorname{div} \dot{\mathbf{B}}^* = 0, \quad \operatorname{curl} \dot{\mathbf{H}}^* = \mathbf{0}, \tag{2.45}$$

with $\dot{\mathbf{B}}^* = \mu_0 \dot{\mathbf{H}}^*$.

Henceforth, we use the notations $\mathcal{B}$ and $\partial\mathcal{B}$ for the (time-independent) initial deformed configuration upon which the infinitesimal motion is superimposed.

### 2.2.2 Incremental boundary conditions

The boundary condition $(2.14)_4$ for the magnetic induction is written $(\mathbf{B} - \mathbf{B}^*) \cdot \mathbf{n} = 0$ on $\partial\mathcal{B}$. Since there is no deformation outside the material (in the case that it is a vacuum, which we assume henceforth) there is no physical meaning attached to a Lagrangian form of the magnetic induction, so when the boundary condition is expressed in Lagrangian form it becomes

$$(\mathbf{B}_l - J\mathbf{F}^{-1}\mathbf{B}^*) \cdot \mathbf{N} = 0 \quad \text{on } \partial\mathcal{B}_r. \tag{2.46}$$

On taking an increment of the above equation, and then updating and using the incompressibility condition $(2.31)_2$ we obtain the incremental boundary condition

$$(\dot{\mathbf{B}}_{l0} - \dot{\mathbf{B}}^* + \mathbf{L}\mathbf{B}^*) \cdot \mathbf{n} = 0 \quad \text{on } \partial\mathcal{B}. \tag{2.47}$$

The boundary condition $(2.14)_3$ for the magnetic field, with $\mathbf{K} = \mathbf{0}$, now becomes $(\mathbf{H} - \mathbf{H}^*) \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{B}$, and in Lagrangian form

$$(\mathbf{H}_l - \mathbf{F}^{\mathrm{T}}\mathbf{H}^*) \times \mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_r. \tag{2.48}$$

On incrementing this and updating we obtain the incremental boundary condition

$$(\dot{\mathbf{H}}_{l0} - \mathbf{L}^{\mathrm{T}}\mathbf{H}^* - \dot{\mathbf{H}}^*) \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\mathcal{B}. \tag{2.49}$$

In order to arrive at the corresponding incremental traction boundary condition we need to define the Maxwell stress outside the material, denoted by $\boldsymbol{\tau}^*$. This is symmetric and given by

$$\boldsymbol{\tau}^* = \mu_0^{-1} \left[ \mathbf{B}^* \otimes \mathbf{B}^* - \tfrac{1}{2} (\mathbf{B}^* \cdot \mathbf{B}^*) \mathbf{I} \right], \tag{2.50}$$

where $\mathbf{I}$ is the identity tensor. The incremental Maxwell stress is then obtained as

$$\dot{\boldsymbol{\tau}}^* = \mu_0^{-1}\left[\dot{\mathbf{B}}^* \otimes \mathbf{B}^* + \mathbf{B}^* \otimes \dot{\mathbf{B}}^* - \left(\dot{\mathbf{B}}^* \cdot \mathbf{B}^*\right)\mathbf{I}\right]. \tag{2.51}$$

The Lagrangian form of the Maxwell stress is $J\mathbf{F}^{-1}\boldsymbol{\tau}^*$, which is defined only on the boundary $\mathcal{B}_r$, and the magnetic contribution $\mathbf{t}_\mathrm{M}$ to the traction on $\mathcal{B}_r$ in (2.26) is given by

$$\mathbf{t}_\mathrm{M} = J\boldsymbol{\tau}^*\mathbf{F}^{-\mathrm{T}}\mathbf{N} \quad \text{on } \partial\mathcal{B}_r. \tag{2.52}$$

On taking an increment of this equation, we obtain

$$\dot{\mathbf{t}}_\mathrm{M} = J\dot{\boldsymbol{\tau}}^*\mathbf{F}^{-\mathrm{T}}\mathbf{N} - J\boldsymbol{\tau}^*\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^\mathrm{T}\mathbf{F}^{-\mathrm{T}}\mathbf{N} + J(\operatorname{div}\mathbf{u})\boldsymbol{\tau}^*\mathbf{F}^{-\mathrm{T}}\mathbf{N}, \tag{2.53}$$

which on pushing forward and using the incompressibility condition $(2.31)_2$, gives

$$\dot{\mathbf{t}}_{\mathrm{M}0} = \dot{\boldsymbol{\tau}}^*\mathbf{n} - \boldsymbol{\tau}^*\mathbf{L}^\mathrm{T}\mathbf{n} \quad \text{on } \partial\mathcal{B}. \tag{2.54}$$

When there is also a mechanical traction $\mathbf{t}_\mathrm{A}$, with increment $\dot{\mathbf{t}}_\mathrm{A}$, the incremental traction boundary condition is written

$$\dot{\mathbf{T}}_0^\mathrm{T}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} + \dot{\mathbf{t}}_{\mathrm{M}0} \tag{2.55}$$

at any point of $\partial\mathcal{B}$ where the traction is prescribed.

## 2.3 Constitutive relations

Following [Dorfmann and Ogden, 2004], we consider a magnetoelastic material for which the constitutive law is given in terms of a total potential energy density function, $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$, defined per unit reference volume. This yields the simple formulas

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial\Omega}{\partial\mathbf{B}_l}, \tag{2.56}$$

for the total nominal stress and the Lagrangian magnetic field. Their Eulerian counterparts are

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\frac{\partial\Omega}{\partial\mathbf{B}_l}. \tag{2.57}$$

In the case of an incompressible material, we have the constraint $J = 1$ and the above equations for the stresses are modified to

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \quad \boldsymbol{\tau} = \mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{I}, \tag{2.58}$$

where $p$ is a Lagrange multiplier associated with the constraint and $\mathbf{I}$ is again the identity tensor.

For an isotropic magnetoelastic material, $\Omega$ can be expressed in terms of six independent scalar invariants of $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{B}_l \otimes \mathbf{B}_l$ (see, for example, [Steigmann, 2004]).

One possible set of invariants, used by [Dorfmann and Ogden, 2004], is

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}(\mathbf{c}^2)], \quad I_3 = \det\mathbf{c} = J^2, \tag{2.59}$$

$$I_4 = \mathbf{B}_l \cdot \mathbf{B}_l, \quad I_5 = (\mathbf{c}\mathbf{B}_l) \cdot \mathbf{B}_l, \quad I_6 = \left(\mathbf{c}^2\mathbf{B}_l\right) \cdot \mathbf{B}_l. \tag{2.60}$$

We adopt these here and confine attention to isotropic magnetoelastic materials.

The total nominal stress and the Lagrangian magnetic field can then be expanded in the forms

$$\mathbf{T} = \sum_{i \in \mathcal{I}} \Omega_i \frac{\partial I_i}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \sum_{i \in \mathcal{J}} \Omega_i \frac{\partial I_i}{\partial \mathbf{B}_l}, \tag{2.61}$$

where $\Omega_i = \partial\Omega/\partial I_i$, $i = 1, ..., 6$, $\mathcal{I}$ is the set $\{1, 2, 3, 5, 6\}$, or $\{1, 2, 5, 6\}$ for an incompressible material, and $\mathcal{J}$ the set $\{4, 5, 6\}$. The derivatives of the $I_i$ with respect to $\mathbf{F}$ and $\mathbf{B}_l$ are given in Appendix A in component form. Explicitly we calculate the expressions for $\boldsymbol{\tau}$ for an incompressible material and $\mathbf{H}$ as

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\Omega_1\mathbf{b} + 2\Omega_2\left(I_1\mathbf{b} - \mathbf{b}^2\right) + 2\Omega_5\mathbf{B} \otimes \mathbf{B} + 2\Omega_6\left(\mathbf{B} \otimes \mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B} \otimes \mathbf{B}\right), \tag{2.62}$$

and

$$\mathbf{H} = 2\left(\Omega_4\mathbf{b}^{-1}\mathbf{B} + \Omega_5\mathbf{B} + \Omega_6\mathbf{b}\mathbf{B}\right), \tag{2.63}$$

where $I_3 \equiv 1$ and we recall that $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy-Green tensor.

### 2.3.1 Magnetoelastic moduli tensors

By taking the increments of (2.56) we obtain the linearized equations

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \boldsymbol{\mathcal{C}}\dot{\mathbf{B}}_l, \quad \dot{\mathbf{H}}_l = \boldsymbol{\mathcal{C}}^{\mathrm{T}}\dot{\mathbf{F}} + \mathsf{K}\dot{\mathbf{B}}_l, \tag{2.64}$$

where the magnetoelastic 'moduli' tensors are defined by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \boldsymbol{\mathcal{C}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{B}_l}, \quad \boldsymbol{\mathcal{C}}^{\mathrm{T}} = \frac{\partial^2\Omega}{\partial\mathbf{B}_l\partial\mathbf{F}}, \quad \mathsf{K} = \frac{\partial^2\Omega}{\partial\mathbf{B}_l\partial\mathbf{B}_l}, \tag{2.65}$$

the products in (2.64) are defined by

$$(\boldsymbol{\mathcal{A}}\dot{\mathbf{F}})_{\alpha i} = \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, \quad (\boldsymbol{\mathcal{C}}\dot{\mathbf{B}}_l)_{\alpha i} = \mathcal{C}_{\alpha i|\beta}\dot{B}_{l\beta},$$
$$(\boldsymbol{\mathcal{C}}^{\mathrm{T}}\dot{\mathbf{F}})_{\beta} = \mathcal{C}_{\beta|\alpha i}\dot{F}_{i\alpha}, \quad (\mathsf{K}\dot{\mathbf{B}}_l)_{\alpha} = \mathsf{K}_{\alpha\beta}\dot{B}_{l\beta}, \tag{2.66}$$

and we note the symmetries

$$\mathcal{A}_{\alpha i\beta j} = \mathcal{A}_{\beta j\alpha i}, \quad \mathcal{C}_{\alpha i|\beta} = \mathcal{C}_{\beta|\alpha i}, \quad \mathsf{K}_{\alpha\beta} = \mathsf{K}_{\beta\alpha}, \tag{2.67}$$

which reflect the commutativity of the partial derivatives. The vertical bar between the indices on $\mathcal{C}$ is a separator used to distinguish the single subscript from the pair of subscripts that always go together.

For an incompressible material $(2.64)_1$ is replaced by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \boldsymbol{\mathcal{C}}\dot{\mathbf{B}}_l - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{2.68}$$

and, subject to $\det\mathbf{F} = 1$, $(2.64)_2$ is unchanged.

On updating the incremented constitutive equations (2.64) and (2.68), we obtain

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0}, \quad \dot{\mathbf{H}}_{l0} = \boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathsf{K}}_0\dot{\mathbf{B}}_{l0}. \tag{2.69}$$

and

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} - \dot{p}\mathbf{I} + p\mathbf{L}, \tag{2.70}$$

respectively, where $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and $\boldsymbol{\mathsf{K}}_0$ are defined in component form by

$$\mathcal{A}_{0piqj} = \mathcal{A}_{0qjpi} = J^{-1}F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j} = J^{-1}F_{p\alpha}F_{q\beta}\mathcal{A}_{\beta j\alpha i}, \tag{2.71}$$

$$\mathcal{C}_{0ij|k} = \mathcal{C}_{0k|ij} = F_{i\alpha}F^{-1}_{\beta k}\mathcal{C}_{\alpha j|\beta} = F_{i\alpha}F^{-1}_{\beta k}\mathcal{C}_{\beta|\alpha j}, \tag{2.72}$$

$$\mathsf{K}_{0ij} = \mathsf{K}_{0ji} = JF^{-1}_{\alpha i}F^{-1}_{\beta j}\mathsf{K}_{\alpha\beta}, \tag{2.73}$$

which, for an incompressible material, apply with $J = 1$. Explicit formulas for these components for an isotropic magnetoelastic material referred to the principal axes of the left Cauchy–Green tensor $\mathbf{b}$ are given in Appendix B.

On substituting $(2.69)_1$ and (2.70) into (2.38) in turn, we obtain

$$\boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbf{L}\boldsymbol{\tau} = (\boldsymbol{\mathcal{A}}_0\mathbf{L})^{\mathrm{T}} + \boldsymbol{\tau}\mathbf{L}^{\mathrm{T}}, \quad \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} = (\boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0})^{\mathrm{T}} \tag{2.74}$$

and

$$\boldsymbol{\mathcal{A}}_0\mathbf{L} + p\mathbf{L} + \mathbf{L}\boldsymbol{\tau} = (\boldsymbol{\mathcal{A}}_0\mathbf{L})^{\mathrm{T}} + p\mathbf{L}^{\mathrm{T}} + \boldsymbol{\tau}\mathbf{L}^{\mathrm{T}}, \quad \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} = (\boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0})^{\mathrm{T}}, \tag{2.75}$$

respectively.

Writing Equation $(2.75)_2$ in component form gives

$$\mathcal{C}_{0ij|k}\dot{B}_{l0k} = \mathcal{C}_{0ji|k}\dot{B}_{l0k}. \tag{2.76}$$

The above equality holds for all the values of $\dot{B}_{l0}$ while $\boldsymbol{\mathcal{C}}_0$ is a material constant. Hence, we obtain the symmetry

$$\mathcal{C}_{0ij|k} = \mathcal{C}_{0ji|k}. \tag{2.77}$$

Equation $(2.75)_1$, when written in component form, gives

$$\mathcal{A}_{0ijkl}L_{lk} + pL_{ij} + L_{ik}\tau_{kj} = \mathcal{A}_{0jikl}L_{lk} + pL_{ji} + \tau_{ik}L_{jk}, \tag{2.78}$$

$$\Rightarrow \mathcal{A}_{0ijkl}L_{lk} + p\delta_{il}\delta_{jk}L_{lk} + \delta_{il}L_{lk}\tau_{kj} = \mathcal{A}_{0jikl}L_{lk} + p\delta_{jl}\delta_{ki}L_{lk} + \delta_{jl}\tau_{ik}L_{lk}, \tag{2.79}$$

where $\delta_{ij}$ is the Kronecker delta. The above equality holds for all the values of the displacement gradient $L_{lk} = u_{l,k}$. Hence, we deduce the symmetries

$$\mathcal{A}_{0ijkl} + \delta_{il}(\tau_{kj} + p\delta_{jk}) = \mathcal{A}_{0jikl} + \delta_{jl}(\tau_{ik} + p\delta_{ki}), \quad \mathcal{C}_{0ij|k} = \mathcal{C}_{0ji|k}, \tag{2.80}$$

which are additional to (2.67). Here $p = 0$ in the case of an unconstrained material.

Henceforth we restrict attention to incompressible materials. We now use the constitutive equations $(2.69)_2$ and (2.70) together with (2.43) to arrive at the governing equations

$$\mathrm{curl}\,(\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \mathsf{K}_0\dot{\mathbf{B}}_{l0}) = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{B}}_{l0} = 0, \quad \mathrm{div}\,\mathbf{u} = 0, \tag{2.81}$$

$$\mathrm{div}\,(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0}) - \mathrm{grad}\,\dot{p} + \mathbf{L}^{\mathrm{T}}\mathrm{grad}\,p = \rho\mathbf{u}_{,tt}. \tag{2.82}$$

If the underlying configuration is homogeneous so that $p$, $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{C}}_0$, and $\mathsf{K}_0$ are uniform, then in Cartesian component form, equations (2.81) and (2.82) become

$$\varepsilon_{ijk}(\mathcal{C}_{0pq|k}u_{p,qj} + \mathsf{K}_{0kp}\dot{B}_{l0p,j}) = 0, \quad \dot{B}_{l0i,i} = 0, \quad u_{i,i} = 0, \tag{2.83}$$

$$\mathcal{A}_{0piqj}u_{j,pq} + \mathcal{C}_{0pi|q}\dot{B}_{l0q,p} - \dot{p}_{,i} = \rho u_{i,tt}. \tag{2.84}$$

We note here that alternatively we can define the energy density function in terms of $\mathbf{F}$ and $\mathbf{H}_l$ that gives a different set of magnetoelastic moduli tensors. This is detailed in Chapter 5 to work on a boundary value problem in cylindrical geometry.

## 2.4 Homogeneous plane waves

We now consider infinitesimal homogenous plane waves propagating with speed $v$ in the direction of unit vector $\mathbf{n}$ in the form

$$\mathbf{u} = \mathbf{m}f\,(\mathbf{n}\cdot\mathbf{x} - vt)\,, \quad \dot{\mathbf{B}}_{l0} = \mathbf{q}g\,(\mathbf{n}\cdot\mathbf{x} - vt)\,, \quad \dot{p} = P\,(\mathbf{n}\cdot\mathbf{x} - vt)\,, \tag{2.85}$$

where $\mathbf{m}$ and $\mathbf{q}$ are constant (polarization) unit vectors in the directions of the incremental displacement and magnetic induction, respectively, and $f$, $g$ and $P$ are appropriately regular functions of the argument $\mathbf{n}\cdot\mathbf{x}-vt$. Substituting these expressions into Eq. (2.83) and Eq. (2.84), we obtain

$$\mathbf{n}\times\{\mathbf{R}(\mathbf{n})^{\mathrm{T}}\mathbf{m}f''+\boldsymbol{\mathsf{K}}_0\mathbf{q}g'\}=\mathbf{0},\quad \mathbf{q}\cdot\mathbf{n}=0,\quad \mathbf{m}\cdot\mathbf{n}=0, \tag{2.86}$$

$$\mathbf{Q}(\mathbf{n})\mathbf{m}f''+\mathbf{R}(\mathbf{n})\mathbf{q}g'-P'\mathbf{n}=\rho v^2\mathbf{m}f'', \tag{2.87}$$

where $\mathbf{Q}(\mathbf{n})$, the *acoustic tensor*, and $\mathbf{R}(\mathbf{n})$, the *magneto-acoustic tensor*, are given by

$$[\mathbf{Q}(\mathbf{n})]_{ij}=\mathcal{A}_{0piqj}n_pn_q,\quad [\mathbf{R}(\mathbf{n})]_{ij}=\mathcal{C}_{0ip|j}n_p, \tag{2.88}$$

and a prime signifies differentiation with respect to the argument $\mathbf{n}\cdot\mathbf{x}-vt$. Note that $\mathbf{Q}(\mathbf{n})$ is symmetric but in general $\mathbf{R}(\mathbf{n})$ is not.

Let $\hat{\mathbf{I}}(\mathbf{n})=\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$ denote the symmetric projection tensor onto the plane with normal $\mathbf{n}$. Then, following [Destrade and Ogden, 2011], we define the notations

$$\hat{\mathbf{Q}}(\mathbf{n})=\hat{\mathbf{I}}(\mathbf{n})\mathbf{Q}(\mathbf{n})\hat{\mathbf{I}}(\mathbf{n}),\quad \hat{\mathbf{R}}(\mathbf{n})=\hat{\mathbf{I}}(\mathbf{n})\mathbf{R}(\mathbf{n})\hat{\mathbf{I}}(\mathbf{n}),\quad \hat{\boldsymbol{\mathsf{K}}}_0(\mathbf{n})=\hat{\mathbf{I}}(\mathbf{n})\boldsymbol{\mathsf{K}}_0(\mathbf{n})\hat{\mathbf{I}}(\mathbf{n}), \tag{2.89}$$

which are the projections of $\mathbf{Q}(\mathbf{n})$, $\mathbf{R}(\mathbf{n})$, and $\boldsymbol{\mathsf{K}}_0(\mathbf{n})$, respectively, onto the plane normal to $\mathbf{n}$.

Using $(2.86)_3$ we obtain from (2.87)

$$P'=[\mathbf{Q}(\mathbf{n})\mathbf{m}]\cdot\mathbf{n}f''+[\mathbf{R}(\mathbf{n})\mathbf{q}]\cdot\mathbf{n}g', \tag{2.90}$$

and substitution of this back into (2.87) enables the latter to be written as

$$\hat{\mathbf{Q}}(\mathbf{n})\mathbf{m}f''+\hat{\mathbf{R}}(\mathbf{n})\mathbf{q}g'=\rho v^2\mathbf{m}f''. \tag{2.91}$$

Similarly, from $(2.86)_1$ we deduce that

$$\mathbf{R}(\mathbf{n})^{\mathrm{T}}\mathbf{m}f''+\boldsymbol{\mathsf{K}}_0\mathbf{q}g'=\{[\mathbf{R}(\mathbf{n})^{\mathrm{T}}\mathbf{m}]\cdot\mathbf{n}f''+[\boldsymbol{\mathsf{K}}_0\mathbf{q}]\cdot\mathbf{n}g'\}\mathbf{n}, \tag{2.92}$$

which can be written more compactly as

$$\hat{\mathbf{R}}(\mathbf{n})^{\mathrm{T}}\mathbf{m}f''+\hat{\boldsymbol{\mathsf{K}}}_0\mathbf{q}g'=\mathbf{0}. \tag{2.93}$$

As in [Destrade and Ogden, 2011] we assume that $\hat{\boldsymbol{\mathsf{K}}}_0$ is non-singular as an operator restricted to the plane normal to $\mathbf{n}$ and also positive definite in view of its interpretation as the inverse of the incremental permeability tensor. We then obtain $\mathbf{q}g'=$

$-\hat{\mathsf{K}}_0^{-1}\hat{\mathbf{R}}(\mathbf{n})^{\mathrm{T}}\mathbf{m}f''$, and substitution into (2.91) and elimination of $f'' \neq 0$ yields the propagation condition for acoustic waves under the influence of a magnetic field, explicitly

$$\hat{\mathbf{P}}(\mathbf{n})\mathbf{m} \equiv \hat{\mathbf{Q}}(\mathbf{n})\mathbf{m} - \hat{\mathbf{R}}(\mathbf{n})\hat{\mathsf{K}}_0^{-1}\hat{\mathbf{R}}(\mathbf{n})^{\mathrm{T}}\mathbf{m} = \rho v^2\mathbf{m}, \tag{2.94}$$

wherein the generalized acoustic (or Christoffel) tensor $\hat{\mathbf{P}}$ is defined as $\hat{\mathbf{Q}}(\mathbf{n})-\hat{\mathbf{R}}(\mathbf{n})\hat{\mathsf{K}}_0^{-1}\hat{\mathbf{R}}(\mathbf{n})^{\mathrm{T}}$, which is symmetric. This is a generalization of the propagation condition for homogeneous plane waves in an incompressible elastic solid in the absence of a magnetic field. This prompts a corresponding generalization of the *strong ellipticity condition* in the form

$$\mathbf{m} \cdot [\hat{\mathbf{P}}(\mathbf{n})\mathbf{m}] > 0, \tag{2.95}$$

for all unit vectors $\mathbf{m}$ and $\mathbf{n}$ such that $\mathbf{m} \cdot \mathbf{n} = 0$, as given in [Destrade and Ogden, 2011]. This guarantees that homogeneous plane waves have real wave speeds. In component form, which will be useful later, the generalized strong ellipticity inequality (2.95) can be written as

$$\left(\mathcal{A}_{0piqj} - \mathcal{C}_{0ip|k}\hat{\mathsf{K}}_{0kl}^{-1}\mathcal{C}_{0jq|l}\right) m_i m_j n_p n_q > 0. \tag{2.96}$$

# Chapter 3

# Surface Waves on a Half-Space: In-Plane Motion

In this chapter, Rayleigh type surface waves propagating in an incompressible isotropic half-space of nonconducting magnetoelastic material are studied for a half-space subjected to a finite pure homogeneous strain and a uniform magnetic field. We first specialize the equations obtained in the previous chapter to two dimensions corresponding to a homogeneously deformed half-space. The combined effect of the finite deformation and the underlying magnetic field is then studied on wave propagation for different directions of the magnetic field.

Analysis and results in this chapter have been published in [Saxena and Ogden, 2011] and are being reproduced here in greater detail.

## 3.1 Two-dimensional specialization

Let the initial deformation of the material be given by the pure homogeneous strain

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{3.1}$$

where the principal stretches $\lambda_1$, $\lambda_2$, $\lambda_3$ are uniform. The component matrix $[\mathbf{F}]$ of the deformation gradient is then $[\mathbf{F}] = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. We also assume that the initial (uniform) magnetic induction has components $(B_1, B_2, 0)$ in the material and $(B_1^*, B_2^*, 0)$ outside.

In this chapter, we study two-dimensional motions in the $(1, 2)$ plane and seek solutions depending only on the in-plane variables $x_1$ and $x_2$ such that $u_3 = \dot{B}_{l03} = \dot{B}^*_{l03} = 0$. The third component of the equation of motion (2.84) and the first two components of $(2.83)_1$

are then satisfied trivially, and the remaining equations are

$$\begin{aligned}\mathcal{A}_{01111}u_{1,11} &+ 2\mathcal{A}_{01121}u_{1,12} + \mathcal{A}_{02121}u_{1,22} + \mathcal{A}_{01112}u_{2,11}\\ &+ (\mathcal{A}_{01122} + \mathcal{A}_{01221})\,u_{2,12} + \mathcal{A}_{02122}u_{2,22} + \mathcal{C}_{011|1}\dot{B}_{l01,1}\\ &+ \mathcal{C}_{021|1}\dot{B}_{l01,2} + \mathcal{C}_{011|2}\dot{B}_{l02,1} + \mathcal{C}_{021|2}\dot{B}_{l02,2} - \dot{p}_{,1} = \rho u_{1,tt},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathcal{A}_{01211}u_{1,11} &+ (\mathcal{A}_{01221} + \mathcal{A}_{01122})\,u_{1,12} + \mathcal{A}_{02221}u_{1,22}\\ &+ \mathcal{A}_{01212}u_{2,11} + 2\mathcal{A}_{01222}u_{2,12} + \mathcal{A}_{02222}u_{2,22} + \mathcal{C}_{012|1}\dot{B}_{l01,1}\\ &+ \mathcal{C}_{022|1}\dot{B}_{l01,2} + \mathcal{C}_{012|2}\dot{B}_{l02,1} + \mathcal{C}_{022|2}\dot{B}_{l02,2} - \dot{p}_{,2} = \rho u_{2,tt},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathcal{C}_{011|2}u_{1,11} &+ (\mathcal{C}_{021|2} - \mathcal{C}_{011|1})u_{1,12} - \mathcal{C}_{021|1}u_{1,22}\\ &+ \mathcal{C}_{012|2}u_{2,11} + (\mathcal{C}_{022|2} - \mathcal{C}_{012|1})u_{2,12} - \mathcal{C}_{022|1}u_{2,22}\\ &+ \mathsf{K}_{012}\dot{B}_{l01,1} - \mathsf{K}_{011}\dot{B}_{l01,2} + \mathsf{K}_{022}\dot{B}_{l02,1} - \mathsf{K}_{012}\dot{B}_{l02,2} = 0.\end{aligned} \tag{3.4}$$

Elimination of $\dot{p}$ from (3.2) and (3.3) by cross differentiation and subtraction yields

$$\begin{aligned}\mathcal{A}_{01211}u_{1,111} &+ (\mathcal{A}_{01221} + \mathcal{A}_{01122} - \mathcal{A}_{01111})u_{1,112} + (\mathcal{A}_{02221} - 2\mathcal{A}_{01121})u_{1,122}\\ &- \mathcal{A}_{02121}u_{1,222} + \mathcal{A}_{01212}u_{2,111} + (2\mathcal{A}_{01222} - \mathcal{A}_{01112})u_{2,112}\\ &- (\mathcal{A}_{01122} + \mathcal{A}_{01221} - \mathcal{A}_{02222})u_{2,122} - \mathcal{A}_{02122}u_{2,222} + \mathcal{C}_{012|1}\dot{B}_{l01,11}\\ &+ (\mathcal{C}_{022|1} - \mathcal{C}_{011|1})\dot{B}_{l01,12} - \mathcal{C}_{021|1}\dot{B}_{l01,22} + \mathcal{C}_{012|2}\dot{B}_{l02,11}\\ &+ (\mathcal{C}_{022|2} - \mathcal{C}_{011|2})\dot{B}_{l02,12} - \mathcal{C}_{021|2}\dot{B}_{l02,22} = \rho(u_{2,1} - u_{1,2})_{,tt}.\end{aligned} \tag{3.5}$$

The corresponding equations in (2.45) outside the material may be written as

$$\dot{B}^*_{1,1} + \dot{B}^*_{2,2} = 0, \quad \dot{B}^*_{2,1} - \dot{B}^*_{1,2} = 0. \tag{3.6}$$

Since $\dot{\mathbf{B}}_{l0}$ and $\mathbf{u}$ satisfy the equations $(2.83)_{2,3}$ and $\dot{\mathbf{B}}^*$ satisfies Equation $(3.6)_1$, we can define potentials $\psi$, $\phi$, and $\psi^*$ such that

$$\begin{aligned}\dot{B}_{l01} = \psi_{,2}, \quad \dot{B}_{l02} = -\psi_{,1}, \quad u_1 = \phi_{,2}, \quad u_2 = -\phi_{,1},\\ \dot{B}^*_1 = \psi^*_{,2}, \quad \dot{B}^*_2 = -\psi^*_{,1}.\end{aligned} \tag{3.7}$$

Substituting these expressions into the governing equations (3.5), (3.4), and $(3.6)_2$, we obtain the two coupled equations

$$\alpha\phi_{,1111} + 2\delta\phi_{,1112} + 2\beta\phi_{,1122} + 2\varepsilon\phi_{,1222} + \gamma\phi_{,2222}$$
$$+ a\psi_{,111} + b\psi_{,112} + c\psi_{,122} + d\psi_{,222} = \rho\left(\phi_{,11} + \phi_{,22}\right)_{,tt}, \tag{3.8}$$

$$a\phi_{,111} + b\phi_{,112} + c\phi_{,122} + d\phi_{,222} + \mathsf{K}_{011}\psi_{,22} + \mathsf{K}_{022}\psi_{,11} - 2\mathsf{K}_{012}\psi_{,12} = 0, \tag{3.9}$$

for $\phi$ and $\psi$ in the material, where, for compactness of representation, we have introduced the notations

$$\alpha = \mathcal{A}_{01212}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{01221}, \quad \gamma = \mathcal{A}_{02121}, \tag{3.10}$$

$$\delta = \mathcal{A}_{01222} - \mathcal{A}_{01211}, \quad \varepsilon = \mathcal{A}_{01121} - \mathcal{A}_{02221}, \quad a = \mathcal{C}_{012|2}, \tag{3.11}$$

$$b = \mathcal{C}_{022|2} - \mathcal{C}_{011|2} - \mathcal{C}_{012|1}, \quad c = \mathcal{C}_{011|1} - \mathcal{C}_{022|1} - \mathcal{C}_{021|2}, \quad d = \mathcal{C}_{021|1}. \tag{3.12}$$

Outside the material we have the single equation

$$\psi^*_{,11} + \psi^*_{,22} = 0. \tag{3.13}$$

When there is no time dependence and $B_1 = 0$, equations (3.8) and (3.9) reduce to equations given in Section 5.2 of [Otténio et al., 2008], but partly in different notation.

## 3.2 Surface waves

In this section we consider two separate cases: first, $B_1 = 0$ with $B_2 \neq 0$; and second, $B_1 \neq 0$ with $B_2 = 0$. The material forms a half-space $X_2 < 0$ in the reference configuration, with unit outward normal $\mathbf{N}$ to its boundary $X_2 = 0$ having components $(0, 1, 0)$. Under the deformation (3.1), the material occupies the half space $x_2 < 0$ in the deformed configuration and the unit outward normal $\mathbf{n}$ to its boundary $x_2 = 0$ has components $(0, 1, 0)$.

### 3.2.1 Magnetic induction components $(0, B_2, 0)$

In this first example we take the initial magnetic induction to be perpendicular to the surface of the half-space so that the components of $\mathbf{B}$ are $(0, B_2, 0)$. The boundary condition $\mathbf{B} \cdot \mathbf{n} = \mathbf{B}^* \cdot \mathbf{n}$ applied to $x_2 = 0$ then gives $B^*_2 = B_2$. It follows from (2.50) and (2.51) that the matrix representations of $\boldsymbol{\tau}^*$ and $\dot{\boldsymbol{\tau}}^*$ are, respectively,

$$[\boldsymbol{\tau}^*] = \frac{B_2^2}{2\mu_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B_2}{\mu_0}\begin{bmatrix} -\dot{B}_2^* & \dot{B}_1^* & 0 \\ \dot{B}_1^* & \dot{B}_2^* & 0 \\ 0 & 0 & -\dot{B}_2^* \end{bmatrix}. \tag{3.14}$$

**Incremental equations and boundary conditions**

For the given values of $\mathbf{F}$ and $\mathbf{B}$, many of the components of the moduli listed in Appendix B vanish, and equations (3.8) and (3.9) simplify to

$$\alpha\phi_{,1111} + 2\beta\phi_{,1122} + \gamma\phi_{,2222} + b\psi_{,112} + d\psi_{,222} = \rho\left(\phi_{,11} + \phi_{,22}\right)_{,tt}, \tag{3.15}$$

$$b\phi_{,112} + d\phi_{,222} + \mathsf{K}_{011}\psi_{,22} + \mathsf{K}_{022}\psi_{,11} = 0. \tag{3.16}$$

Using the values of $\boldsymbol{\tau}^*$ and $\dot{\boldsymbol{\tau}}^*$ from (3.14) and assuming there is no incremental mechanical traction on $x_2 = 0$ the components of the incremental traction are obtained from (2.54) with $\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{N} = \dot{\mathbf{t}}_{\mathrm{M}0}$ as

$$\dot{T}_{021} - \frac{B_2}{\mu_0}\dot{B}_1^* - \frac{B_2^2}{2\mu_0}u_{2,1} = 0, \quad \dot{T}_{022} - \frac{B_2}{\mu_0}\dot{B}_2^* + \frac{B_2^2}{2\mu_0}u_{2,2} = 0 \quad \text{on } x_2 = 0, \tag{3.17}$$

with $\dot{T}_{023} = 0$ satisfied identically. From the boundary conditions (2.47) and (2.49) we obtain

$$\dot{B}_{l02} - \dot{B}_2^* + B_2 u_{2,2} = 0, \quad \dot{H}_{l01} - \frac{B_2}{\mu_0}u_{2,1} - \dot{H}_1^* = 0 \quad \text{on } x_2 = 0. \tag{3.18}$$

By substituting the updated incremented constitutive equations (2.70) and $(2.69)_2$, appropriately specialized, into the incremental boundary conditions (3.17) and $(3.18)_1$ and making use of the connection

$$\mathcal{A}_{01221} + \tau_{22} + p = \mathcal{A}_{02121}, \tag{3.19}$$

which comes from $(2.80)_1$, we obtain

$$(\mathcal{A}_{02121} - \tau_{22} - \frac{B_2^2}{2\mu_0})u_{2,1} + \mathcal{A}_{02121}u_{1,2} + \mathcal{C}_{021|1}\dot{B}_{l01} - \frac{B_2}{\mu_0}\dot{B}_1^* = 0, \tag{3.20}$$

$$\mathcal{A}_{01122}u_{1,1} + (\mathcal{A}_{02222} + p + \frac{B_2^2}{2\mu_0})u_{2,2} + \mathcal{C}_{022|2}\dot{B}_{l02} - \dot{p} - \frac{B_2}{\mu_0}\dot{B}_2^* = 0, \tag{3.21}$$

$$\mathcal{C}_{012|1}u_{2,1} + \mathcal{C}_{021|1}u_{1,2} + \mathsf{K}_{011}\dot{B}_{l01} - \frac{B_2}{\mu_0}u_{2,1} - \frac{1}{\mu_0}\dot{B}_1^* = 0, \tag{3.22}$$

each holding on $x_2 = 0$.

Next, we differentiate (3.21) with respect to $x_1$ and make use of (3.2) to eliminate $\dot{p}_{,1}$. We then introduce the potentials $\phi, \psi$, and $\psi^*$ into the resulting equation and equations

(3.18), (3.20), and (3.22) and use the notations (3.10)–(3.12). We also note that if there is no mechanical traction applied on the boundary $x_2 = 0$ in the underlying configuration then the normal stress $\tau_{22}$ in the material must balance the Maxwell stress $\tau_{22}^*$ on $x_2 = 0$, which gives

$$\tau_{22} = \tau_{22}^* = \frac{B_2^2}{2\mu_0}. \tag{3.23}$$

The boundary conditions can then be written as

$$(\gamma - 2\tau_{22}^*)\phi_{,11} - \gamma\phi_{,22} - d\psi_{,2} + \frac{B_2}{\mu_0}\psi_{,2}^* = 0, \tag{3.24}$$

$$(2\beta + \gamma)\phi_{,112} + \gamma\phi_{,222} + (b + d)\psi_{,11} + d\psi_{,22} - \frac{B_2}{\mu_0}\psi_{,11}^* - \rho\phi_{,2tt} = 0, \tag{3.25}$$

$$B_2\phi_{,12} + \psi_{,1} - \psi_{,1}^* = 0, \tag{3.26}$$

$$d(\phi_{,11} - \phi_{,22}) - \mathsf{K}_{011}\psi_{,2} - \frac{B_2}{\mu_0}\phi_{,11} + \frac{1}{\mu_0}\psi_{,2}^* = 0, \tag{3.27}$$

which apply on $x_2 = 0$.

Hence, the problem is reduced to solving the governing equations (3.15), (3.16) in $x_2 < 0$ and (3.13) in $x_2 > 0$, and applying the boundary conditions (3.24)–(3.27) on $x_2 = 0$ and appropriate decay behaviour as $x_2 \to \pm\infty$.

### Surface wave propagation

We now study two-dimensional surface waves propagating in the $x_1$ direction with the increments having non-zero components lying in the $(1, 2)$ plane. We consider harmonic solutions of the form

$$\phi = P\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \quad \psi = kQ\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t) \quad \text{in} \ \ x_2 < 0, \tag{3.28}$$

$$\psi^* = kR\exp(s^*kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t) \quad \text{in} \ \ x_2 > 0, \tag{3.29}$$

where $P$, $Q$, $R$ are constants, $k$ is the wave number, $\omega$ is the angular frequency, and $s$ and $s^*$ are to be determined subject to the requirements $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s^*) < 0$ needed for decay of the surface wave amplitude away from the boundary. Substituting these solutions into the governing equations (3.13), (3.15), and (3.16) we obtain

$$[\alpha - 2\beta s^2 + \gamma s^4 + \rho v^2(s^2 - 1)]P + (ds^2 - b)sQ = 0, \tag{3.30}$$

$$(ds^2 - b)sP + (\mathsf{K}_{011}s^2 - \mathsf{K}_{022})Q = 0, \tag{3.31}$$

and $s^{*2} = 1$, where the wave speed is $v = \omega/k$. For the solution $\psi^*$ to decay as $x_2 \to \infty$, we necessarily take $s^* = -1$. For non-trivial solutions for $P$ and $Q$ from (3.30) and (3.31), we set the determinant of coefficients to be zero and obtain a cubic equation for $s^2$, namely

$$(\gamma \mathsf{K}_{011} - d^2)s^6 + [\mathsf{K}_{011}(\rho v^2 - 2\beta) - \gamma \mathsf{K}_{022} + 2bd]s^4$$
$$+ [\mathsf{K}_{022}(2\beta - \rho v^2) + \mathsf{K}_{011}(\alpha - \rho v^2) - b^2]s^2 + (\rho v^2 - \alpha)\mathsf{K}_{022} = 0. \quad (3.32)$$

We denote by $s_1, s_2, s_3$ the three solutions satisfying the requirement $\mathrm{Re}(s) > 0$. The general solutions of the equations that satisfy the decay conditions are now given by

$$\phi = (P_1 \mathrm{e}^{s_1 k x_2} + P_2 \mathrm{e}^{s_2 k x_2} + P_3 \mathrm{e}^{s_3 k x_2})\mathrm{e}^{\mathrm{i}(kx_1 - \omega t)}, \quad (3.33)$$

$$\psi = k(Q_1 \mathrm{e}^{s_1 k x_2} + Q_2 \mathrm{e}^{s_2 k x_2} + Q_3 \mathrm{e}^{s_3 k x_2})\mathrm{e}^{\mathrm{i}(kx_1 - \omega t)}, \quad (3.34)$$

$$\psi^* = kR\mathrm{e}^{-kx_2 + \mathrm{i}(kx_1 - \omega t)}. \quad (3.35)$$

For each $i$, $Q_i$ is related to $P_i$ by equation (3.31), which we re-write here as

$$Q_i = \frac{(b - ds_i^2)s_i}{\mathsf{K}_{011}s_i^2 - \mathsf{K}_{022}} P_i, \quad i = 1, 2, 3. \quad (3.36)$$

Next, we substitute the general solutions (3.33)–(3.35) into the boundary conditions (3.24)–(3.27) to obtain

$$(\gamma - 2\tau_{22}^*)\Sigma_j P_j + \gamma \Sigma_j s_j^2 P_j + d\Sigma_j s_j Q_j + \frac{B_2}{\mu_0} R = 0, \quad (3.37)$$

$$(2\beta + \gamma - \rho v^2)\Sigma_j s_j P_j - \gamma \Sigma_j s_j^3 P_j + (b + d)\Sigma_j Q_j - d\Sigma_j s_j^2 Q_j - \frac{B_2}{\mu_0} R = 0, \quad (3.38)$$

$$B_2 \Sigma_j s_j P_j + \Sigma_j Q_j - R = 0, \quad (3.39)$$

$$d\Sigma_j (s_j^2 + 1)P_j + \mathsf{K}_{011}\Sigma_j s_j Q_j - \frac{B_2}{\mu_0}\Sigma_j P_j + \frac{1}{\mu_0} R = 0, \quad (3.40)$$

where $\Sigma_j$ indicates summation over $j$ from 1 to 3.

We now have seven linear equations in $P_1, P_2, P_3, Q_1, Q_2, Q_3$, and $R$, and for a non-trivial solution the determinant of coefficients must vanish. The result is the secular equation relating the wave speed $v$ to the initial deformation, the material properties and the initial magnetic induction $B_2$, and we note that, by (3.23), the stress $\tau_{22}^*$ depends on $B_2$.

### Pure elastic case

Here we take the magnetic field to vanish in order to reduce our results to known results in the purely elastic case. For this purpose we set $\boldsymbol{\mathcal{C}} = \mathbf{0}$, $Q_i = 0$, $i = 1, 2, 3$, and $R = 0$. Equation (3.32) reduces to a quadratic for $s^2$, namely

$$\gamma s^4 - (2\beta - \rho v^2)s^2 + \alpha - \rho v^2 = 0, \tag{3.41}$$

from which we deduce that the solutions $s_1^2$ and $s_2^2$ satisfy

$$\gamma(s_1^2 + s_2^2) = 2\beta - \rho v^2, \quad \gamma s_1^2 s_2^2 = \alpha - \rho v^2. \tag{3.42}$$

For a surface wave we take $s_1$ and $s_2$ to be the solutions satisfying $\mathrm{Re}(s) > 0$, and, as discussed in [Dowaikh and Ogden, 1990], we require $\gamma > 0$ and $\rho v^2 \leq \alpha$.

The boundary conditions (3.24)–(3.27) reduce to the two equations

$$(\gamma - \tau_{22})\phi_{,11} - \gamma\phi_{,22} = 0, \quad (2\beta + \gamma - \tau_{22})\phi_{,112} + \gamma\phi_{,222} - \rho\phi_{,2tt} = 0 \quad \text{on } x_2 = 0, \tag{3.43}$$

where, for comparison with the results of [Dowaikh and Ogden, 1990], we have assumed that there is a normal mechanical traction $\tau_{22}$ on $x_2 = 0$ in the underlying configuration. The general solution for $\phi$ can be rewritten as

$$\phi = (P_1 \mathrm{e}^{s_1 k x_2} + P_2 \mathrm{e}^{s_2 k x_2})\mathrm{e}^{\mathrm{i}(kx_1 - \omega t)}. \tag{3.44}$$

Substitution into the boundary conditions then yields

$$(\gamma - \tau_{22} + \gamma s_1^2)P_1 + (\gamma - \tau_{22} + \gamma s_2^2)P_2 = 0, \tag{3.45}$$

$$(2\beta + \gamma - \tau_{22} - \rho v^2 - \gamma s_1^2)s_1 P_1 + (2\beta + \gamma - \tau_{22} - \rho v^2 - \gamma s_2^2)s_2 P_2 = 0, \tag{3.46}$$

from which, on use of (3.42), the explicit secular equation is obtained as

$$\gamma(\alpha - \rho v^2) + (2\beta + 2\gamma - 2\tau_{22} - \rho v^2)\sqrt{\gamma(\alpha - \rho v^2)} = (\gamma - \tau_{22})^2. \tag{3.47}$$

Apart from some minor differences of notation, this agrees with the formula (5.17) obtained by [Dowaikh and Ogden, 1990].

### Application to a Mooney–Rivlin magnetoelastic material

For purposes of illustration we now consider the energy function of a Mooney–Rivlin magnetoelastic material as used by [Otténio et al., 2008]. This has the form

$$\Omega = \frac{1}{4}\mu(0)[(1+\nu)(I_1 - 3) + (1-\nu)(I_2 - 3)] + lI_4 + mI_5, \tag{3.48}$$

where $\mu(0)$ is the shear modulus of the material in the absence of magnetic fields. According to the convention in literature, the shear modulus $\mu$ is a function of the invariants $I_4, I_5, I_6$ in general (as has been used in Chapter 4). For this particular energy function, we consider it to be a constant $\mu(0)$. To avoid a conflict of notation, we use $l$, $m$, $\nu$, respectively, in place of the $\alpha/\mu_0$, $\beta/\mu_0$, $\gamma$ used by [Otténio et al., 2008]. Note that $l\mu_0$, $m\mu_0$ and $\nu$ are dimensionless, with $\nu$ restricted to the range $-1 \leq \nu \leq 1$, as for the classical Mooney–Rivlin model.

For this model, from Equation (2.62), the total stress $\boldsymbol{\tau}$ reduces to

$$\boldsymbol{\tau} = -p\mathbf{I} + \frac{1}{2}\mu(0)(1+\gamma)\mathbf{b} + \frac{1}{2}\mu(0)(1-\gamma)\left(I_1\mathbf{b} - \mathbf{b}^2\right) + 2m\mathbf{B}\otimes\mathbf{B}, \tag{3.49}$$

while from Equation (2.63), $\mathbf{H}$ is given as

$$\mathbf{H} = 2l\mathbf{b}^{-1}\mathbf{B} + 2m\mathbf{B}. \tag{3.50}$$

If $l = 0$ then the magnetic constitutive equation is unaffected by deformation while if $m = 0$ then the the total stress is unaffected by the magnetic field. Thus, a two-way coupling require inclusion of both the constants.

It can be seen from the above equations that the parameter $l$ has no effect on the total stress. On the other hand, $m$, if positive, stiffens the material in the direction of the magnetic field which is consistent with the experimental results obtained by [Jolly et al., 1996]. Hence we require $m$ to be positive. If $m = 0$ and there is no initial deformation, i.e. $\mathbf{b} = \mathbf{I}$, then from the magnetic constitutive equation, for $\mathbf{H}$ and $\mathbf{B}$ to be in the same direction we require $l$ to be positive.

The relevant non-zero components of the magnetoelastic tensors are easily calculated from the formulas in Appendix B as

$$\mathcal{A}_{01111} = \frac{1}{2}\mu(0)\lambda_1^2[1+\nu+(1-\nu)(\lambda_2^2+\lambda_3^2)], \tag{3.51}$$

$$\mathcal{A}_{02222} = \frac{1}{2}\mu(0)\lambda_2^2[1+\nu+(1-\nu)(\lambda_1^2+\lambda_3^2)] + 2mB_2^2, \tag{3.52}$$

$$\mathcal{A}_{01212} = \frac{1}{2}\mu(0)\lambda_1^2[1+\nu+(1-\nu)\lambda_3^2], \quad \mathcal{A}_{02121} = \lambda_2^2\lambda_1^{-2}\mathcal{A}_{01212} + 2mB_2^2, \tag{3.53}$$

$$\mathcal{A}_{01122} = -2\mathcal{A}_{01221} = \mu(0)(1-\nu)\lambda_1^2\lambda_2^2, \tag{3.54}$$

$$\mathcal{C}_{022|2} = 2\mathcal{C}_{012|1} = 4mB_2, \tag{3.55}$$

$$\mathsf{K}_{011} = 2(m+\lambda_1^{-2}l), \quad \mathsf{K}_{022} = 2(m+\lambda_2^{-2}l), \tag{3.56}$$

from which we deduce, using the notation defined in (3.10)–(3.12), that

$$2\beta = \alpha + \gamma, \quad b = d. \tag{3.57}$$

With these values, Equation (3.32) factorizes in the form

$$(s^2 - 1)\left\{(\gamma \mathsf{K}_{011} - d^2)s^4 - [\gamma \mathsf{K}_{022} + (\alpha - \rho v^2)\mathsf{K}_{011} - d^2]s^2 + (\alpha - \rho v^2)\mathsf{K}_{022}\right\} = 0. \tag{3.58}$$

Let the solutions with positive real part be denoted $s_1$ $(= 1)$, $s_2$ and $s_3$. Then,

$$s_2^2 + s_3^2 = \frac{\gamma \mathsf{K}_{022} + (\alpha - \rho v^2)\mathsf{K}_{011} - d^2}{\gamma \mathsf{K}_{011} - d^2}, \quad s_2^2 s_3^2 = \frac{(\alpha - \rho v^2)\mathsf{K}_{022}}{\gamma \mathsf{K}_{011} - d^2}. \tag{3.59}$$

Note that when $v = 0$ the bi-quadratic in (3.58) factorizes easily to give the equation

$$(s^2 - \lambda^4)[(\mu_0 \mathsf{K}_{011} + 4lmB_2^2)s^2 - \mu_0 \mathsf{K}_{022}] = 0, \tag{3.60}$$

as shown by [Otténio et al., 2008], although there is a slight error in their equation (112), wherein their $\alpha$ and $\beta$ should be replaced by $2\alpha$ and $2\beta$, respectively. This has only minor repercussions for their subsequent results. We also note in passing that for $v \neq 0$, in the special case $\lambda = 1$, the bi-quadratic factorizes as $(s^2 - 1)[(\gamma \mathsf{K}_{011} - d^2)s^2 - \mathsf{K}_{011}(\alpha - \rho v^2)]$.

Now, by specializing the generalized strong-ellipticity condition (2.96) to the present constitutive model and setting $n_1 = 1, n_2 = 0, m_1 = 0, m_2 = 1$ we obtain $\gamma \mathsf{K}_{011} - d^2 > 0$. Then, following the same argument as used in the purely elastic case, we require $s_2^2 s_3^2 \geq 0$ and we therefore conclude that

$$\rho v^2 \leq \alpha. \tag{3.61}$$

For the considered model, this upper bound is identical to that in the purely elastic case and hence independent of the magnetic field.

We now use $s_1 = 1$ and the expressions (3.59) in the boundary conditions (3.37)–(3.40) and set the determinant of coefficients to zero to obtain the secular equation. The resulting equation is too lengthy to reproduce here, and we obtain the solutions numerically. For this purpose, we use the standard value $4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}}$ of $\mu_0$ together with the value $2.6 \times 10^5\,\mathrm{N\,m^{-2}}$ of $\mu(0)$ that was adopted by [Otténio et al., 2008] based on data for an elastomer filled with 10% by volume of iron particles from [Jolly et al., 1996]. We also use a series of values of $l$ and $m$ consistent with the values of the magnetoelastic coupling constants used in [Otténio et al., 2008].

First, we consider the underlying deformation to be one of plane strain in the $(1, 2)$ plane, and we take $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$, $\lambda_3 = 1$. In this case the results are independent of

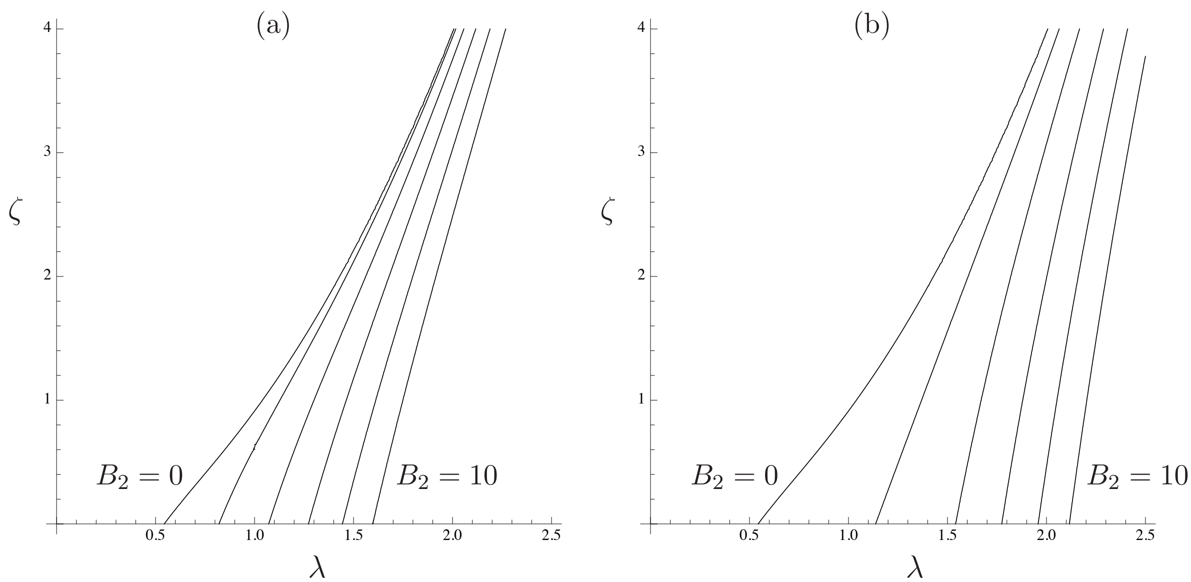



Figure 3.1: Plot of $\zeta = \rho v^2/\mu(0)$ vs $\lambda_1 = \lambda$ with $\lambda_3 = 1$ for $B_2 = 0, 2, 4, 6, 8, 10\,\mathrm{T}$ (curves reading from left to right): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 0.1, \mu_0 m = 1$.

the parameter $n$ in the Mooney–Rivlin model and the upper bound (3.61) is $\mu(0)\lambda^2$. Let $\zeta = \rho v^2/\mu(0)$. Then we plot the variation of $\zeta$ with $\lambda$ for a selection of values of $l$ and $m$ and a range of values of $B_2$ in Figures 3.1 and 3.2. We also consider a deformation for which $\lambda_1 = 1$, $\lambda_2 = \lambda$, $\lambda_3 = \lambda^{-1}$ and we use the value $n = 0.3$ in the Mooney–Rivlin model. Then, the upper bound (3.61) is $\mu(0)(0.65 + 0.35\lambda^{-2})$. Results for this case are plotted in Figure 3.3 for two representative pairs of values of $l$ and $m$ and a range of values of $B_2$.

Figures 3.1 and 3.2 relate to a plane strain deformation in which the half-space is subject to compression or extension parallel to its boundary. The result for $B_2 = 0$ corresponds to the purely elastic case and provides a point of reference. The $B_2 = 0$ curve cuts the $\lambda$ axis at $\lambda = \lambda_c \simeq 0.5437$, which agrees with the classical result for the critical value of $\lambda$ corresponding to loss of stability of the half-space under compression for the neo-Hookean model (for which $n = 1$); see [Biot, 1965] and [Dowaikh and Ogden, 1990] for details. By referring to the $\zeta = 0$ axis in Figure 3.1 and Figure 3.2(b) it can be seen that the magnetic field destabilizes the material, i.e. instability occurs at a compression closer to the undeformed configuration where $\lambda = 1$. For each value of $B_2$ there is a critical value of $\lambda$ beyond which a surface wave exists, and the wave speed increases with $\lambda$ consistently with the upper bound (3.61). Note, in particular, that the undeformed configuration $\lambda = 1$

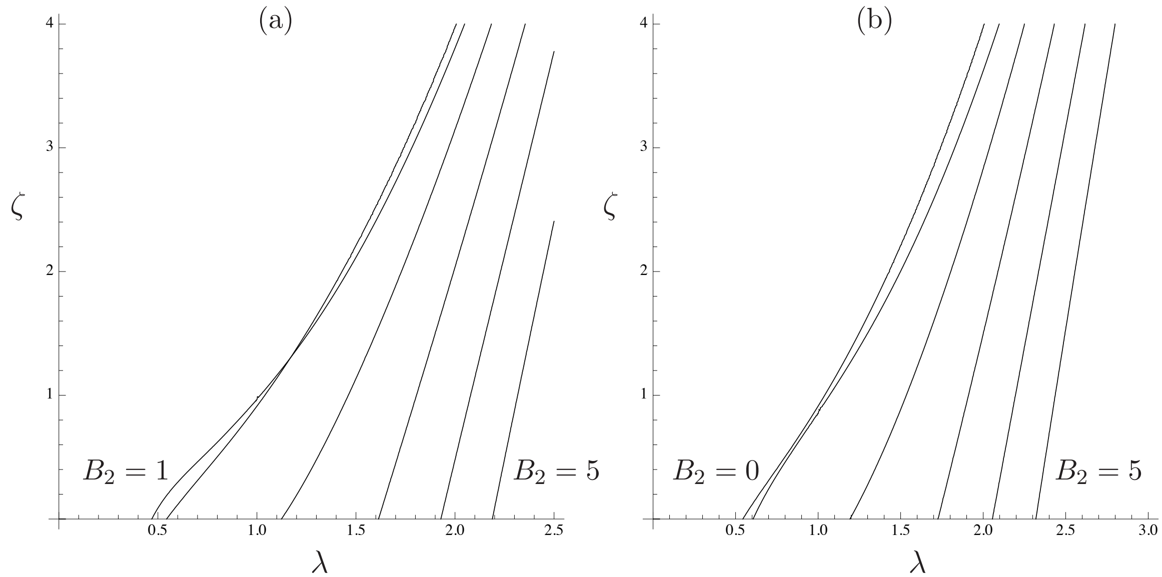



Figure 3.2: Plot of $\zeta = \rho v^2/\mu(0)$ vs $\lambda_1 = \lambda$ with $\lambda_3 = 1$: (a) $\mu_0 l = 2, \mu_0 m = 0.2$ with $B_2 = 1, 0, 2, 3, 4, 5\,\mathrm{T}$; (b) $\mu_0 l = 0.2, \mu_0 m = 0.2$ with $B_2 = 0, 1, 2, 3, 4, 5\,\mathrm{T}$ (curves reading from left to right in each case).

becomes unstable as $B_2$ increases. In Figure 3.2(a) the situation is slightly different since for small values of $B_2$ the half-space is initially stabilized as $B_2$ increases (i.e. the critical value of $\lambda$ decreases below the classical value $\lambda_c$), but then as $B_2$ is increased further stability is lost again. Note that the $B_2 = 0$ and $B_2 = 1$ curves cross over in this case. These results are consistent with the stability analysis of [Otténio et al., 2008].

When there is no compression or extension parallel to $x_2 = 0$ in the sagittal plane but there is extension (or compression) normal to the boundary and a corresponding compression (or extension) normal to the sagittal plane the effect of the magnetic field is different. Figure 3.3 illustrates this case.

Now there is instability for $\lambda > 1$, at $\lambda \simeq 3.4$ for $B_2 = 0$, and the critical value of $\lambda$ decreases with increasing $B_2$, i.e. the magnetic field again has a destabilizing effect. The wave speed increases as $\lambda$ decreases, again consistently with the upper bound (3.61). Figure 3.4 shows plots of the dimensionless squared wave speed as a function of $B_2$ for the undeformed configuration $\lambda = 1$ for (a) a fixed value of $m$ and a series of values of $l$, and (b) a fixed value of $l$ and a series of values of $m$. For $B_2 = 0$ the curves cut the $\zeta$ axis at the classical Rayleigh value ($\simeq 0.9126$). As $B_2$ increases then, depending on the values of the parameters $l$ and $m$, the wave speed either increases or decreases initially but

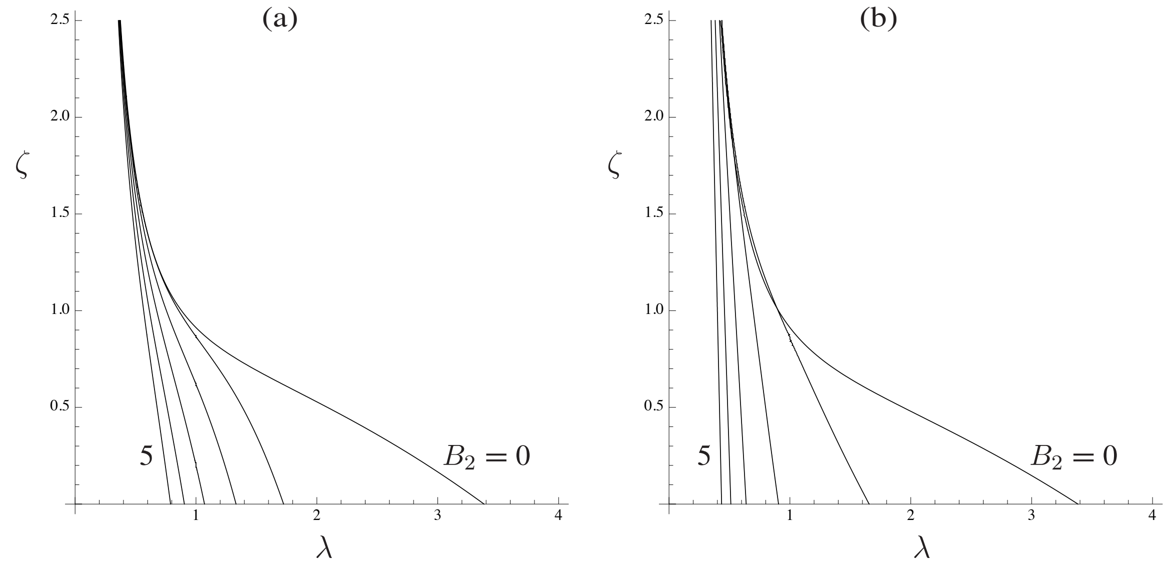



Figure 3.3: Plot of $\zeta = \rho v^2/\mu(0)$ vs $\lambda_2 = \lambda$ with $\lambda_1 = 1$ for $B_2 = 0, 1, 2, 3, 4, 5\,\mathrm{T}$ (curves reading from right to left): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 0.2, \mu_0 m = 0.2$.

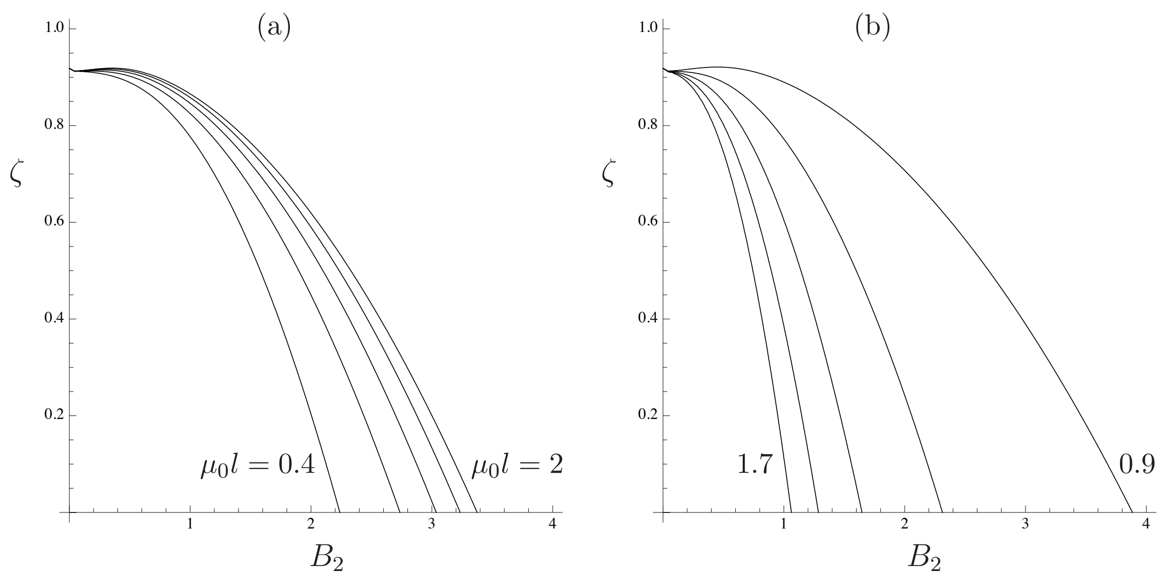



Figure 3.4: Plot of $\zeta = \rho v^2/\mu(0)$ vs $B_2$ with $\lambda_1 = \lambda_2 = \lambda_3 = 1$: (a) $\mu_0 l = 0.4, 0.8, 1.2, 1.6, 2$ and $\mu_0 m = 1$ (curves reading from left to right); (b) $\mu_0 l = 1$ and $\mu_0 m = 0.9, 1.1, 1.3, 1.5, 1.7$ (curves reading from right to left).

in each case subsequently decreases to zero with further increase in $B_2$. This emphasizes that the undeformed configuration is destabilized at a critical value of $B_2$ dependent on the material parameters. From Figure 3.4(a), for the selected value of $m$, it can be seen that increasing the value of $l$ has a stabilizing effect, while from Figure 3.4(b) the reverse is true for increasing $m$ at a fixed value of $l$.

### 3.2.2 Magnetic induction components $(B_1, 0, 0)$

The initial deformed configuration is considered to be the same as in Section 3.2.1, but now we take the magnetic induction $\mathbf{B}$ to have components $(B_1, 0, 0)$. The corresponding magnetic field $\mathbf{H}$ is given by (2.63) and has components $(H_1, 0, 0)$, with

$$H_1 = 2(\Omega_4\lambda_1^{-2} + \Omega_5 + \Omega_6\lambda_1^2)B_1, \tag{3.62}$$

which, for the model (3.48), reduces to $H_1 = 2(l\lambda_1^{-2} + m)B_1$. The magnetic boundary conditions on $x_2 = 0$ require that $H_1^* = H_1$, so that $B_1^* = \mu_0 H_1^* = 2\mu_0(l\lambda_1^{-2} + m)B_1$.

From (2.50) and (2.51), the components of Maxwell stress and its increment in $x_2 > 0$ are given by

$$[\boldsymbol{\tau}^*] = \frac{B_1^{*2}}{2\mu_0}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B_1^*}{\mu_0}\begin{bmatrix} \dot{B}_1^* & \dot{B}_2^* & 0 \\ \dot{B}_2^* & -\dot{B}_1^* & 0 \\ 0 & 0 & -\dot{B}_1^* \end{bmatrix}. \tag{3.63}$$

#### Incremental equations and boundary conditions

For the present situation, equations (3.8) and (3.9) reduce to

$$\alpha\phi_{,1111} + 2\beta\phi_{,1122} + \gamma\phi_{,2222} + a\psi_{,111} + c\psi_{,122} = \rho(\phi_{,11} + \phi_{,22})_{,tt}, \tag{3.64}$$

$$a\phi_{,111} + c\phi_{,122} + \mathsf{K}_{022}\psi_{,11} + \mathsf{K}_{011}\psi_{,22} = 0, \tag{3.65}$$

for $x_2 < 0$, while again (3.13) holds for $x_2 > 0$.

Using the values of $\boldsymbol{\tau}^*$ and $\dot{\boldsymbol{\tau}}^*$ from (3.63) and assuming there is no incremental mechanical traction on $x_2 = 0$ the components of the incremental traction are obtained from (2.54) with $\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{N} = \dot{\mathbf{t}}_{\mathrm{M}0}$ as

$$\dot{T}_{021} - \frac{B_1^*}{\mu_0}\dot{B}_2^* + \frac{B_1^{*2}}{2\mu_0}u_{2,1} = 0, \quad \dot{T}_{022} + \frac{B_1^*}{\mu_0}\dot{B}_1^* - \frac{B_1^{*2}}{2\mu_0}u_{2,2} = 0 \quad \text{on } x_2 = 0, \tag{3.66}$$

with $\dot{T}_{023} = 0$ satisfied identically. From (2.47) and (2.49) we obtain

$$\dot{B}_{l02} - \dot{B}_2^* + B_1^* u_{2,1} = 0, \quad \dot{H}_{l01} - H_1^* u_{1,1} - \dot{H}_1^* = 0 \quad \text{on } x_2 = 0. \tag{3.67}$$

Next, we substitute the updated incremented constitutive equations (2.69) and (2.70) into equations (3.66) and (3.67) and use (3.19) and the boundary condition $\tau_{22} = \tau^*_{22}$, where $\tau^*_{22} = -B_1^{*2}/2\mu_0$, and follow the same procedure as in the previous section to eliminate $\dot{p}$. This yields

$$(\gamma - 2\tau^*_{22})\phi_{,11} - \gamma\phi_{,22} + a\psi_{,1} - \frac{B_1^*}{\mu_0}\psi^*_{,1} = 0, \tag{3.68}$$

$$(2\beta + \gamma)\phi_{,112} + \gamma\phi_{,222} - \rho\phi_{,2tt} + c\psi_{,12} - \frac{B_1^*}{\mu_0}\psi^*_{,12} = 0, \tag{3.69}$$

$$B_1^*\phi_{,11} + \psi_{,1} - \psi^*_{,1} = 0, \tag{3.70}$$

$$(c + a - \frac{B_1^*}{\mu_0})\phi_{,12} - \frac{1}{\mu_0}\psi^*_{,2} = 0, \tag{3.71}$$

each of which holds on $x_2 = 0$.

**Surface waves in a Mooney–Rivlin magnetoelastic half-space**

We again study surface waves as in Section 3.2.1, with solutions of the form (3.28) and (3.29). Substituting these solutions into equations (3.64), (3.65) and (3.13), we obtain

$$[\gamma s^4 - (2\beta - \rho v^2)s^2 + \alpha - \rho v^2]P + \mathrm{i}(cs^2 - a)Q = 0, \tag{3.72}$$

$$\mathrm{i}(cs^2 - a)P + (\mathsf{K}_{011}s^2 - \mathsf{K}_{022})Q = 0, \tag{3.73}$$

and $s^{*2} = 1$, where the wave speed is again given by $v = \omega/k$.

For the solution $\psi^*$ to decay as $x_2 \to \infty$, we take $s^* = -1$. For non-trivial solutions for $P$ and $Q$, we set the determinant of coefficients to zero and obtain a cubic equation in $s^2$:

$$\begin{aligned}&\gamma\mathsf{K}_{011}s^6 - [\mathsf{K}_{011}(2\beta - \rho v^2) + \gamma\mathsf{K}_{022} - c^2]s^4 \\ &\quad + [\mathsf{K}_{011}(\alpha - \rho v^2) + \mathsf{K}_{022}(2\beta - \rho v^2) - 2ac]s^2 - \mathsf{K}_{022}(\alpha - \rho v^2) + a^2 = 0.\end{aligned} \tag{3.74}$$

For the Mooney–Rivlin magnetoelastic material given by (3.48), the non-zero components of the magnetoelastic tensors are obtained from the general formulas in Appendix B as

$$\mathcal{A}_{01111} = \frac{1}{2}\mu(0)\lambda_1^2[1 + \nu + (1 - \nu)(\lambda_2^2 + \lambda_3^2)] + 2mB_1^2, \tag{3.75}$$

$$\mathcal{A}_{02222} = \frac{1}{2}\mu(0)\lambda_2^2[1 + \nu + (1 - \nu)(\lambda_1^2 + \lambda_3^2)], \tag{3.76}$$

$$\mathcal{A}_{02121} = \frac{1}{2}\mu(0)\lambda_2^2[1 + \nu + (1 - \nu)\lambda_3^2], \quad \mathcal{A}_{01212} = \lambda_1^2\lambda_2^{-2}\mathcal{A}_{02121} + 2mB_1^2, \tag{3.77}$$

$$\mathcal{A}_{01122} = -2\mathcal{A}_{01221} = \mu(0)(1 - \nu)\lambda_1^2\lambda_2^2, \tag{3.78}$$

$$\mathcal{C}_{011|1} = 2\mathcal{C}_{012|2} = 4mB_1, \tag{3.79}$$

$$\mathsf{K}_{011} = 2(m + \lambda_1^{-2}l), \quad \mathsf{K}_{022} = 2(m + \lambda_2^{-2}l), \tag{3.80}$$

from which, using the notation defined in (3.10)–(3.12), we obtain

$$2\beta = \alpha + \gamma, \quad c = a. \tag{3.81}$$

Substitution of these values in (3.74) yields the factorization

$$(s^2 - 1)\{\gamma\mathsf{K}_{011}s^4 - [\gamma\mathsf{K}_{022} + (\alpha - \rho v^2)\mathsf{K}_{011} - a^2]s^2 + (\alpha - \rho v^2)\mathsf{K}_{022} - a^2\} = 0. \tag{3.82}$$

We note in passing that the second factor in the above equation can be factorized in simple form in two cases: for $v = 0$ we obtain $(s^2 - \lambda^4)(\gamma\mathsf{K}_{011}s^2 - \gamma\mathsf{K}_{022} - 4lm\lambda^{-2}B_1^2)$; for $\lambda = 1$, the result is $(s^2 - 1)[\gamma\mathsf{K}_{011}s^2 + a^2 - \mathsf{K}_{011}(\alpha - \rho v^2)]$.

Let $s_1 = 1$, and let $s_2$ and $s_3$ be the solutions of the second factor with positive real part. As in the previous section we require $s_2^2 s_3^2 \geq 0$, which, after noting that $\gamma > 0$, $\mathsf{K}_{011} > 0$ and $\mathsf{K}_{022} > 0$, and specializing the generalized strong ellipticity condition as in Section 3.2.1, gives

$$\rho v^2 \leq \alpha - a^2/\mathsf{K}_{022}, \tag{3.83}$$

the right-hand side of which is positive. As distinct from (3.61) the upper bound in (3.83) *does* in general depend on the magnetic field.

We again take the solutions for $\phi$, $\psi$ and $\psi^*$ as (3.28) and (3.29). Substituting these into the boundary conditions (3.68)–(3.71), we obtain

$$(\gamma - 2\tau_{22}^*)\Sigma_j P_j + \gamma\Sigma_j s_j^2 P_j - \mathrm{i}a\Sigma_j Q_j + \mathrm{i}\frac{B_1^*}{\mu_0}R = 0, \tag{3.84}$$

$$(2\beta + \gamma - \rho v^2)\Sigma_j s_j P_j - \gamma\Sigma_j s_j^3 P_j - \mathrm{i}c\Sigma_j s_j Q_j - \mathrm{i}\frac{B_1^*}{\mu_0}R = 0, \tag{3.85}$$

$$B_1^*\Sigma_j P_j - \mathrm{i}\Sigma_j Q_j + \mathrm{i}R = 0, \tag{3.86}$$

$$(c + a - \frac{B_1^*}{\mu_0})\Sigma_j s_j P_j - \mathrm{i}\frac{1}{\mu_0}R = 0, \tag{3.87}$$

along with the connection between $Q_i$ and $P_i$ from (3.73):

$$Q_i = \frac{\mathrm{i}(a - cs_i^2)}{\mathsf{K}_{011}s_i^2 - \mathsf{K}_{022}}P_i, \quad i = 1, 2, 3. \tag{3.88}$$

Again, $\Sigma_j$ signifies summation over $j$ from 1 to 3.

As in the previous section, we have seven linear equations in $P_1$, $P_2$, $P_3$, $Q_1$, $Q_2$, $Q_3$ and $R$, and the solution follows the pattern therein. The results for $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$,

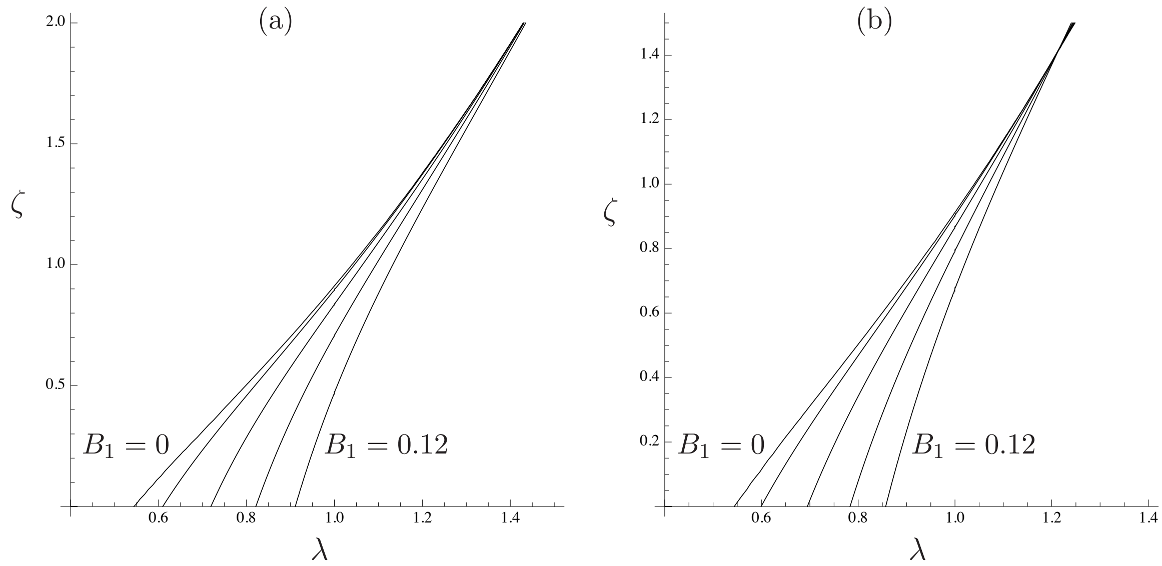



Figure 3.5: Plot of $\zeta = \rho v^2/\mu(0)$ vs $\lambda_1 = \lambda$ with $\lambda_3 = 1$ for $B_1 = 0, 0.03, 0.06, 0.09, 0.12\,\mathrm{T}$ (curves reading from left to right): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 1, \mu_0 m = 1$.

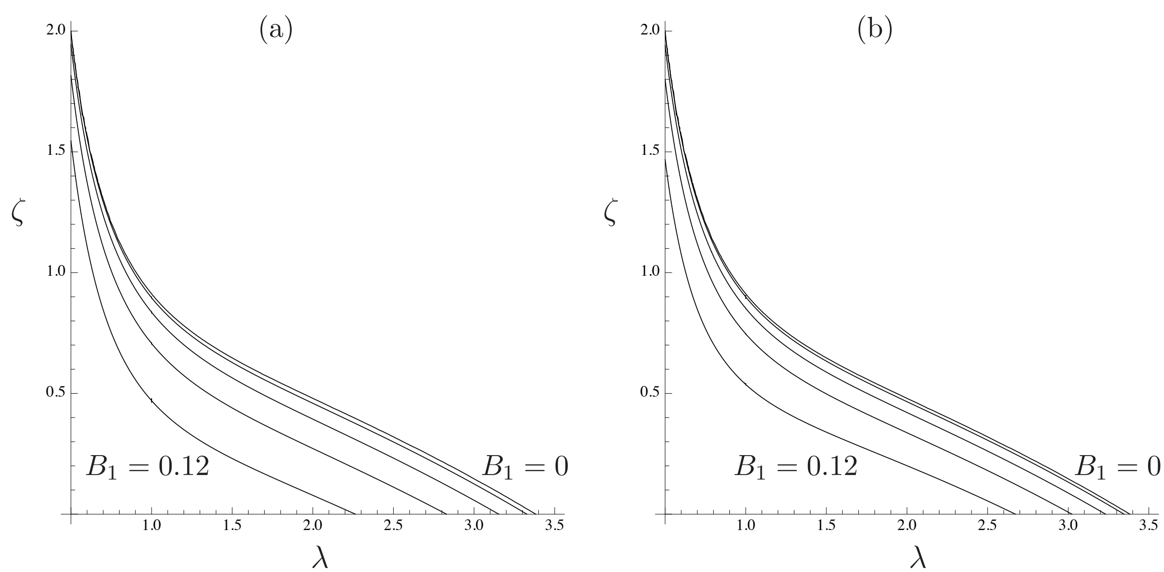



Figure 3.6: Plot of $\zeta = \rho v^2/\mu(0)$ vs $\lambda_2 = \lambda$ with $\lambda_1 = 1$ for $B_1 = 0, 0.03, 0.06, 0.09, 0.12\,\mathrm{T}$ (curves reading from right to left): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 2, \mu_0 m = 0.2$.

$\lambda_3 = 1$ and $\lambda_1 = 1$, $\lambda_2 = \lambda$, $\lambda_3 = \lambda^{-1}$ are shown in Figures 3.5 and 3.6, respectively. The results are broadly similar to those shown in Figures 3.1 and 3.3 except that the effect of $B_1$ is significantly stronger than that for $B_2$. Indeed, much smaller values of $B_1$ than $B_2$ are required to produce comparable effects. The upper bound (3.83) depends on the magnitude $B_1$ but the values of $\zeta$ shown do not reflect this because of the relatively small values $B_1$ used.

## 3.3 Out-of-plane considerations

### 3.3.1 Magnetic induction components $(0, 0, B_3)$

The initial and deformed configurations are considered to be the same as in Section 3.2.2 except that the magnetic induction is taken to have components $(0, 0, B_3)$. The incremental quantities are as in the previous sections, i.e. we consider only incremental motions and magnetic induction components within the $(1, 2)$ plane. In fact, the full three-dimensional equations decouple in this case and the out-of-plane motion can be considered separately, as discussed in [Maugin and Hakmi, 1985].

From equations (3.8) and (3.9), with the components of the moduli tensors appropriately specialized, we obtain

$$\alpha\phi_{,1111} + 2\beta\phi_{,1122} + \gamma\phi_{,2222} = \rho\left(\phi_{,11} + \phi_{,22}\right)_{,tt}, \tag{3.89}$$

$$\mathsf{K}_{022}\psi_{,11} + \mathsf{K}_{011}\psi_{,22} = 0, \tag{3.90}$$

which apply in $x_2 < 0$, and again (3.13) holds in $x_2 > 0$.

The boundary conditions for the underlying configuration require that $H_3^* = H_3$. Thus, $B_3^* = \mu_0 H_3 = 2\mu_0(l\lambda_3^{-2} + m)$. If we assume there are no mechanical tractions, then $\tau_{22} = \tau_{22}^*$. The normal components of the Maxwell stress are $\tau_{22}^*(1, 1, -1)$, where $\tau_{22}^* = -B_3^{*2}/2\mu_0$. The incremental boundary conditions reduce to $\dot{T}_{021} = -\tau_{22}^* u_{2,1}$, $\dot{T}_{022} = -\tau_{22}^* u_{2,2}$, $\dot{T}_{023} = \mu_0^{-1} B_3^* \dot{B}_2^*$, $\dot{B}_{l02} = \dot{B}_2^*$ and $\dot{H}_{l01} = \dot{H}_1^*$. Note, in particular, the appearance of the out-of-plane shear traction term. After differentiating the $\dot{T}_{022}$ condition with respect to $x_1$, substituting for $\dot{p}_{,1}$ from an appropriately specialized form of (3.2) and then substituting for the potentials $\phi$, $\psi$ and $\psi^*$, we obtain (on dropping the factor $\gamma \neq 0$ from the first equation)

$$\phi_{,11} - \phi_{,22} = 0, \quad (2\beta + \gamma)\phi_{,112} + \gamma\phi_{,222} - \rho\phi_{,2tt} = 0 \quad \text{on } x_2 = 0, \tag{3.91}$$

$$m\psi_{,1} = (l + m\lambda_3^{-2})\psi^*_{,1}, \quad \psi_{,1} = \psi^*_{,1}, \quad \mathsf{K}_{011}\psi_{,2} = \mu_0^{-1}\psi^*_{,2} \quad \text{on } x_2 = 0. \tag{3.92}$$

Except in the very special case for which $l = 0$ and $\lambda_3 = 1$ the latter equations are incompatible unless there is no incremental magnetic field. Thus, the problem reduces to a purely mechanical problem for the potential $\phi$. For the considered model none of the moduli components depend on $B_3$, so the magnetic field has no effect on the propagation of elastic surface waves. More generally, however, for the considered underlying deformation and magnetic field, equation (3.89) and the boundary conditions (3.91) apply for an arbitrary form of isotropic energy function $\Omega$ and therefore the coefficients then do involve $B_3$.

# Chapter 4

# Surface Waves on a Half-Space: Out of Plane Motion

In this chapter, the propagation of Love-type waves in a homogeneously and finitely deformed layered half-space of an incompressible non-conducting magnetoelastic material in the presence of an initial magnetic field is analyzed. The equations and the boundary conditions obtained in Chapter 2 are used to study the problem for different directions of the initial magnetic field for two different magnetoelastic energy functions. Bleustein–Gulyaev type waves, which can exist in a half-space without a layer in the presence of a magnetic field, are discussed briefly at the end of the chapter.

Analysis and results in this chapter have been published in [Saxena and Ogden, 2012] and are being reproduced here in further detail.

## 4.1 Two-dimensional specialization

Consider a magnetoelastic material that is deformed homogeneously and the deformation is given by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{4.1}$$

where the principal stretches $\lambda_1, \lambda_2, \lambda_3$ are uniform. The component matrix of the deformation gradient is then given by $[\mathbf{F}] = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. The initial uniform magnetic induction vector is taken to have either components $(B_1, B_2, 0)$ in the material with $(B_1^*, B_2^*, 0)$ outside the material, or $(0, 0, B_3)$ in the material with $(0, 0, B_3^*)$ outside the material. Note that the boundary condition $(2.14)_4$ requires that $B_2^* = B_2$.

For such a configuration, the in-plane displacement components $u_1$ and $u_2$ are coupled with each other in the governing equations, and are independent of the out-of-plane component $u_3$. In this chapter we seek solutions depending on the in-plane variables $x_1$ and $x_2$ such that $u_1 = u_2 = 0$ and $u_3$ depends on $(x_1, x_2, t)$. The problem concerning the displacement components $u_1$ and $u_2$ is discussed in the previous chapter. The incremental incompressibility condition $\operatorname{div}\mathbf{u} = 0$ is then automatically satisfied, and with all incremental quantities independent of $x_3$, $\dot{p}_{,3} = 0$ and, from $(2.81)_2$ we obtain

$$\dot{B}_{l01,1} + \dot{B}_{l02,2} = 0. \tag{4.2}$$

On expanding the governing equations $(2.81)_1$ and (2.82) in component form, we obtain

$$\mathcal{A}_{01113}u_{3,11} + (\mathcal{A}_{01123} + \mathcal{A}_{02113})\,u_{3,12} + \mathcal{A}_{02123}u_{3,22} + \mathcal{C}_{011|1}\dot{B}_{l01,1} + \mathcal{C}_{011|2}\dot{B}_{l02,1} + \mathcal{C}_{011|3}\dot{B}_{l03,1} + \mathcal{C}_{021|1}\dot{B}_{l01,2} + \mathcal{C}_{021|2}\dot{B}_{l02,2} + \mathcal{C}_{021|3}\dot{B}_{l03,2} - \dot{p}_{,1} = 0, \tag{4.3}$$

$$\mathcal{A}_{01213}u_{3,11} + (\mathcal{A}_{01223} + \mathcal{A}_{02213})\,u_{3,12} + \mathcal{A}_{02223}u_{3,22} + \mathcal{C}_{012|1}\dot{B}_{l01,1} + \mathcal{C}_{012|2}\dot{B}_{l02,1} + \mathcal{C}_{012|3}\dot{B}_{l03,1} + \mathcal{C}_{022|1}\dot{B}_{l01,2} + \mathcal{C}_{022|2}\dot{B}_{l02,2} + \mathcal{C}_{022|3}\dot{B}_{l03,2} - \dot{p}_{,2} = 0, \tag{4.4}$$

$$\mathcal{A}_{01313}u_{3,11} + 2\mathcal{A}_{01323}u_{3,12} + \mathcal{A}_{02323}u_{3,22} + \mathcal{C}_{013|1}\dot{B}_{l01,1} + \mathcal{C}_{013|2}\dot{B}_{l02,1} + \mathcal{C}_{013|3}\dot{B}_{l03,1} + \mathcal{C}_{023|1}\dot{B}_{l01,2} + \mathcal{C}_{023|2}\dot{B}_{l02,2} + \mathcal{C}_{023|3}\dot{B}_{l03,2} = \rho u_{3,tt}, \tag{4.5}$$

$$(\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{013}\dot{B}_{l01} + \mathsf{K}_{023}\dot{B}_{l02} + \mathsf{K}_{033}\dot{B}_{l03})_{,2} = 0, \tag{4.6}$$

$$(\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{013}\dot{B}_{l01} + \mathsf{K}_{023}\dot{B}_{l02} + \mathsf{K}_{033}\dot{B}_{l03})_{,1} = 0, \tag{4.7}$$

$$\mathcal{C}_{013|2}u_{3,11} + \left(\mathcal{C}_{023|2} - \mathcal{C}_{013|1}\right)u_{3,12} - \mathcal{C}_{023|1}u_{3,22} + \mathsf{K}_{012}\dot{B}_{l01,1} + \mathsf{K}_{022}\dot{B}_{l02,1} + \mathsf{K}_{023}\dot{B}_{l03,1} - \mathsf{K}_{011}\dot{B}_{l01,2} - \mathsf{K}_{012}\dot{B}_{l02,2} - \mathsf{K}_{013}\dot{B}_{l03,2} = 0. \tag{4.8}$$

Associated boundary conditions will be considered in the specializations that follow.

From here on, we consider two separate cases, for which the underlying magnetic field is, first, parallel to the $(1, 2)$ plane and, second, normal to the plane.

We consider a half-space of magnetoelastic material for which $X_2 < 0$ in the undeformed configuration ($x_2 < 0$ in the deformed configuration). A layer of a different magnetoelastic material of thickness $h$ in the deformed configuration is attached on top of the half-space as shown in Figure 4.1. The layer occupies the region $0 < x_2 < h$. Quantities in the half-space are distinguished by a prime ($'$); those in the layer are unprimed.

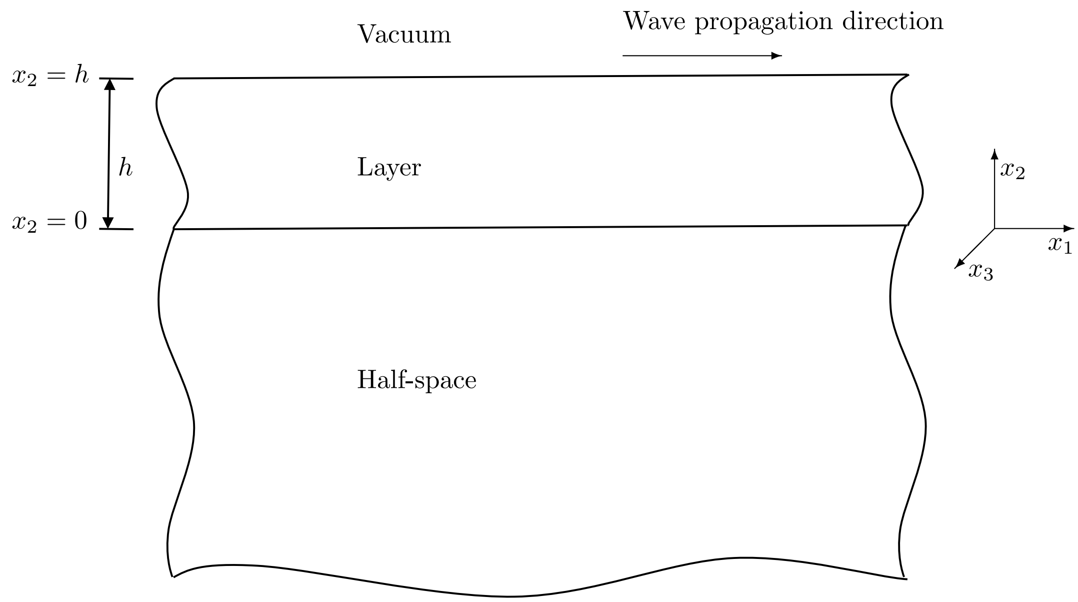



Figure 4.1: Layer–Half-space configuration for Love-type wave propagation.

## 4.2 In-plane magnetic field: $\mathbf{B} = (B_1, B_2, 0)$

Here we take $B_3 = B_3^* = 0$ so that the Maxwell stress and its increment are obtained in component form from equations (2.50) and (2.51) as

$$[\boldsymbol{\tau}^*] = \frac{1}{\mu_0}\begin{bmatrix} \frac{1}{2}(B_1^{*2} - B_2^{*2}) & B_1^* B_2^* & 0 \\ B_1^* B_2^* & \frac{1}{2}\left(-B_1^{*2} + B_2^{*2}\right) & 0 \\ 0 & 0 & \frac{1}{2}(-B_1^{*2} - B_2^{*2}) \end{bmatrix}, \tag{4.9}$$

and

$$[\dot{\boldsymbol{\tau}}^*] = \frac{1}{\mu_0}\begin{bmatrix} \dot{B}_1^* B_1^* - \dot{B}_2^* B_2^* & \dot{B}_1^* B_2^* + \dot{B}_2^* B_1^* & \dot{B}_3^* B_1^* \\ \dot{B}_2^* B_1^* + \dot{B}_1^* B_2^* & \dot{B}_2^* B_2^* - \dot{B}_1^* B_1^* & \dot{B}_3^* B_2^* \\ \dot{B}_3^* B_1^* & \dot{B}_3^* B_2^* & -\left(\dot{B}_1^* B_1^* + \dot{B}_2^* B_2^*\right) \end{bmatrix}, \tag{4.10}$$

respectively.

Equations (4.3)–(4.8) simplify to

$$\mathcal{C}_{011|1}\dot{B}_{l01,1} + \mathcal{C}_{021|1}\dot{B}_{l01,2} + \mathcal{C}_{021|2}\dot{B}_{l02,2} + \mathcal{C}_{011|2}\dot{B}_{l02,1} - \dot{p}_{,1} = 0, \tag{4.11}$$

$$\mathcal{C}_{012|1}\dot{B}_{l01,1} + \mathcal{C}_{012|2}\dot{B}_{l02,1} + \mathcal{C}_{022|2}\dot{B}_{l02,2} + \mathcal{C}_{022|1}\dot{B}_{l01,2} - \dot{p}_{,2} = 0, \tag{4.12}$$

$$\mathcal{A}_{01313}u_{3,11} + 2\mathcal{A}_{01323}u_{3,12} + \mathcal{A}_{02323}u_{3,22} + \mathcal{C}_{013|3}\dot{B}_{l03,1} + \mathcal{C}_{023|3}\dot{B}_{l03,2} = \rho u_{3,tt}, \tag{4.13}$$

$$\left(\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{033}\dot{B}_{l03}\right)_{,2} = 0, \tag{4.14}$$

$$\Big(\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{033}\dot{B}_{l03}\Big)_{,1} = 0, \tag{4.15}$$

$$\mathsf{K}_{022}\dot{B}_{l02,1} + \mathsf{K}_{012}\dot{B}_{l01,1} - \mathsf{K}_{012}\dot{B}_{l02,2} - \mathsf{K}_{011}\dot{B}_{l01,2} = 0. \tag{4.16}$$

Equations (4.14), (4.15) and the assumption of independence of $x_3$ imply that $\dot{H}_{l03}$ depends only on $t$ and hence we take $\dot{H}_{l03} = f(t)$. We also observe that $\dot{B}_{l01}$ and $\dot{B}_{l02}$ are coupled through equations (4.2), (4.11), (4.12), and (4.16) and are independent of $u_3$, while $u_3$ is coupled with $\dot{B}_{l03}$ through equations (4.13), (4.14), and (4.15). Since we are only interested here in $u_3$ it suffices to take $\dot{B}_{l01} = \dot{B}_{l02} = 0$ in both half-space and layer. Indeed, in general $\dot{B}_{l01}$ and $\dot{B}_{l02}$ are overdetermined by equations (4.2), (4.11), (4.12), and (4.16). It follows from $(2.69)_2$ and the components of $\boldsymbol{\mathcal{C}}_0$ and $\boldsymbol{\mathsf{K}}_0$ given in Appendix B that $\dot{H}_{l01} = \dot{H}_{l02} = 0$.

The governing equations now reduce to

$$\mathcal{A}_{01313}u_{3,11} + 2\mathcal{A}_{01323}u_{3,12} + \mathcal{A}_{02323}u_{3,22} + \mathcal{C}_{013|3}\dot{B}_{l03,1} + \mathcal{C}_{023|3}\dot{B}_{l03,2} = \rho u_{3,tt}, \tag{4.17}$$

$$\mathcal{C}_{013|3}u_{3,1} + \mathcal{C}_{023|3}u_{3,2} + \mathsf{K}_{033}\dot{B}_{l03} = f(t), \tag{4.18}$$

in the layer, while in the half-space they are

$$\mathcal{A}'_{01313}u'_{3,11} + 2\mathcal{A}'_{01323}u'_{3,12} + \mathcal{A}'_{02323}u'_{3,22} + \mathcal{C}'_{013|3}\dot{B}'_{l03,1} + \mathcal{C}'_{023|3}\dot{B}'_{l03,2} = \rho' u'_{3,tt}, \tag{4.19}$$

$$\mathcal{C}'_{013|3}u'_{3,1} + \mathcal{C}'_{023|3}u'_{3,2} + \mathsf{K}'_{033}\dot{B}'_{l03} = f'(t), \tag{4.20}$$

where $f'(t)$ is the counterpart of $f(t)$ for the half-space.

The boundary conditions (2.47) and (2.49) reduce to

$$\dot{B}^*_2 = \dot{B}_{l02} = 0, \tag{4.21}$$

$$\dot{B}^*_1 = \mu_0\dot{H}^*_1 = \mu_0\dot{H}_{l01} = 0, \tag{4.22}$$

$$\dot{H}^*_3 = \dot{H}_{l03} = f(t), \tag{4.23}$$

on $x_2 = 0$, and hence we may take $\dot{B}^*_1 = \dot{B}^*_2 = 0$ outside the material.

From the boundary condition (2.55) applied at the layer–vacuum boundary the only non-trivial component is $\dot{T}_{023} = \dot{\tau}^*_{23}$, which yields

$$\mathcal{A}_{02313}u_{3,1} + \mathcal{A}_{02323}u_{3,2} + \mathcal{C}_{023|3}\dot{B}_{l03} = 0 \quad \text{on } x_2 = h, \tag{4.24}$$

and at the layer–half-space interface $\dot{T}_{023} = \dot{T}'_{023}$, which leads to

$$\mathcal{A}_{02313}u_{3,1} + \mathcal{A}_{02323}u_{3,2} + \mathcal{C}_{023|3}\dot{B}_{l03} = \mathcal{A}'_{02313}u'_{3,1} + \mathcal{A}'_{02323}u'_{3,2} + \mathcal{C}'_{023|3}\dot{B}'_{l03}, \tag{4.25}$$

on $x_2 = 0$. Additionally, the displacement must be continuous at the interface, i.e.

$$u_3 = u_3' \quad \text{on } x_2 = 0. \tag{4.26}$$

The problem is therefore reduced to solving equations (4.17) and (4.18) in $0 < x_2 < h$ and equations (4.19) and (4.20) in $x_2 < 0$ using the boundary conditions (4.24), (4.25), and (4.26).

### 4.2.1 Wave propagation

On the basis of the above equations and boundary conditions we now study Love-type waves propagating in the $x_1$ direction. We consider harmonic solutions of the form

$$u_3 = P \exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{4.27}$$

$$\dot{B}_{l03} = Q \exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{4.28}$$

$$u_3' = P' \exp(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{4.29}$$

$$\dot{B}_{l03}' = Q' \exp(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{4.30}$$

with the condition $\mathrm{Re}\,(s') > 0$ for the wave to decay away from the surface of the half-space. As defined earlier, $\mathrm{i} = \sqrt{-1}$, $k$ is the wave number, and $\omega$ is the angular frequency.

Substitution of (4.27) and (4.28) into the governing equation (4.18) yields

$$[\mathrm{i}(\mathcal{C}_{013|3} + s\mathcal{C}_{023|3})Pk + \mathsf{K}_{033}Q]\mathrm{e}^{\mathrm{i}(skx_2+kx_1-\omega t)} = f(t), \tag{4.31}$$

which is satisfied non-trivially only when $f(t) = 0$. Similarly, we obtain $f'(t) = 0$. Now using Equation (4.17), and defining the wave speed $v = \omega/k$ we have the two equations

$$\left(\rho v^2 - \mathcal{A}_{01313} - 2s\mathcal{A}_{01323} - \mathcal{A}_{02323}s^2\right) Pk + \mathrm{i}\left(\mathcal{C}_{013|3} + s\mathcal{C}_{023|3}\right) Q = 0, \tag{4.32}$$

$$\mathrm{i}\left(\mathcal{C}_{013|3} + s\mathcal{C}_{023|3}\right) Pk + \mathsf{K}_{033}Q = 0. \tag{4.33}$$

For non-trivial solutions for $P$ and $Q$, the determinant of coefficients must vanish, which yields a quadratic equation for $s$, which we write compactly as

$$As^2 + 2Bs + C - \rho v^2 = 0, \tag{4.34}$$

where we have introduced the notations

$$A = \mathcal{A}_{02323} - \frac{\mathcal{C}_{023|3}^2}{\mathsf{K}_{033}}, \quad B = \mathcal{A}_{02313} - \frac{\mathcal{C}_{013|3}\mathcal{C}_{023|3}}{\mathsf{K}_{033}}, \quad C = \mathcal{A}_{01313} - \frac{\mathcal{C}_{013|3}^2}{\mathsf{K}_{033}}. \tag{4.35}$$

Let $s_1$ and $s_2$ be the two solutions of this quadratic. Then the general solution of the considered form is

$$u_3 = \left(P_1 \mathrm{e}^{\mathrm{i}s_1 k x_2} + P_2 \mathrm{e}^{\mathrm{i}s_2 k x_2}\right) \exp[\mathrm{i}(kx_1 - \omega t)], \tag{4.36}$$

$$\dot{B}_{l03} = \left(Q_1 \mathrm{e}^{\mathrm{i}s_1 k x_2} + Q_2 \mathrm{e}^{\mathrm{i}s_2 k x_2}\right) \exp[\mathrm{i}\,(kx_1 - \omega t)]. \tag{4.37}$$

The coefficients $P_j$ and $Q_j$, $j = 1, 2$, are related by either one of the equations (4.32) or (4.33) as

$$Q_j = -\frac{\mathrm{i}k(\mathcal{C}_{013|3} + s_j \mathcal{C}_{023|3})}{\mathsf{K}_{033}} P_j, \quad j = 1, 2. \tag{4.38}$$

Substituting the solutions (4.29) and (4.30) into equations (4.19) and (4.20), we obtain a similar quadratic for $s'$, namely

$$A' s'^2 + 2\mathrm{i}B' s' + \rho' v^2 - C' = 0, \tag{4.39}$$

where the coefficients are defined by

$$A' = \mathcal{A}'_{02323} - \frac{\mathcal{C}'^2_{023|3}}{\mathsf{K}'_{033}}, \quad B' = \mathcal{A}'_{02313} - \frac{\mathcal{C}'_{013|3}\mathcal{C}'_{023|3}}{\mathsf{K}'_{033}}, \quad C' = \mathcal{A}'_{01313} - \frac{\mathcal{C}'^2_{013|3}}{\mathsf{K}'_{033}}. \tag{4.40}$$

This has at most one solution satisfying the requirement $\mathrm{Re}(s') > 0$. Equation (4.30) also yields the connection

$$k(\mathrm{i}\mathcal{C}'_{013|3} + s'\mathcal{C}'_{023|3})P' + \mathsf{K}'_{033} Q' = 0. \tag{4.41}$$

From the generalized strong ellipticity condition (2.95), we deduce that

$$A > 0, \quad C > 0, \quad A' > 0, \quad C' > 0, \tag{4.42}$$

and hence that there is a solution for $s'$ with positive real part provided

$$A'(C' - \rho' v^2) - B'^2 > 0. \tag{4.43}$$

Substituting the solutions (4.29), (4.30), (4.36), and (4.37) into the boundary conditions (4.24), (4.25), and (4.26), we obtain

$$\mathrm{i}k(\mathcal{A}_{02313} + s_1\mathcal{A}_{02323})P_1 \mathrm{e}^{\mathrm{i}s_1 kh} + \mathrm{i}k(\mathcal{A}_{02313} + s_2\mathcal{A}_{02323})P_2 \mathrm{e}^{\mathrm{i}s_2 kh} + \mathcal{C}_{023|3}\left(Q_1 \mathrm{e}^{\mathrm{i}s_1 kh} + Q_2 \mathrm{e}^{\mathrm{i}s_2 kh}\right) = 0, \tag{4.44}$$

$$\mathrm{i}k(\mathcal{A}_{02313} + s_1\mathcal{A}_{02323})P_1 + \mathrm{i}k(\mathcal{A}_{02313} + s_2\mathcal{A}_{02323})P_2 + \mathcal{C}_{023|3}\,(Q_1 + Q_2) = k(\mathrm{i}\mathcal{A}'_{02313} + s'\mathcal{A}'_{02323})P' + \mathcal{C}'_{023|3}Q', \tag{4.45}$$

$$P_1 + P_2 = P'. \tag{4.46}$$

We may then use the relations (4.38) and (4.41) to eliminate $Q_1, Q_2$, and $Q'$ to obtain

$$(s_1 A + B)\,\mathrm{e}^{\mathrm{i}s_1 kh} P_1 + (s_2 A + B)\,\mathrm{e}^{\mathrm{i}s_2 kh} P_2 = 0, \tag{4.47}$$

$$(s_1 A + B)\,P_1 + (s_2 A + B)\,P_2 + \left(\mathrm{i}s' A' - B'\right) P' = 0, \tag{4.48}$$

$$P_1 + P_2 - P' = 0. \tag{4.49}$$

The three linear equations for $P_1, P_2$, and $P'$ have non-trivial solutions provided the determinant of their coefficients vanishes. This gives rise to the secular equation

$$[(s_1 A + B)(s_2 A + B) + (\mathrm{i}s' A' - B')B](\mathrm{e}^{\mathrm{i}s_2 kh} - \mathrm{e}^{\mathrm{i}s_1 kh}) + (\mathrm{i}s' A' - B')A(s_2 \mathrm{e}^{\mathrm{i}s_2 kh} - s_1 \mathrm{e}^{\mathrm{i}s_1 kh}) = 0, \tag{4.50}$$

where $s_1$ and $s_2$ are the solutions of equation (4.34) and $s'$ is the solution of (4.39) with positive real part.

### 4.2.2 Pure elastic case

We now take the magnetic field to vanish in order to reduce our results to the purely elastic case. For this purpose we take $\mathbf{B} = \mathbf{0}, \boldsymbol{\mathcal{C}} = \mathbf{O}$, and $Q_1 = Q_2 = Q' = 0$. Under this specialization, the governing equations (4.17) and (4.19) reduce to

$$\mathcal{A}_{01313} u_{3,11} + \mathcal{A}_{02323} u_{3,22} = \rho u_{3,tt}, \quad \mathcal{A}'_{01313} u'_{3,11} + \mathcal{A}'_{02323} u'_{3,22} = \rho' u'_{3,tt}, \tag{4.51}$$

in the layer and half-space, respectively. The relations (4.34) and (4.39) become

$$s^2 = \frac{\rho v^2 - \mathcal{A}_{01313}}{\mathcal{A}_{02323}}, \quad s'^2 = \frac{\mathcal{A}'_{01313} - \rho' v^2}{\mathcal{A}'_{02323}}. \tag{4.52}$$

For these simplifications, the secular equation (4.50) becomes

$$\tan(skh) = \frac{s' \mathcal{A}'_{02323}}{s \mathcal{A}_{02323}}, \quad \rho v^2 > \mathcal{A}_{01313}, \tag{4.53}$$

where $s > 0$ and we note that to qualify for a surface wave the inequality $\rho' v^2 < \mathcal{A}'_{01313}$ must be satisfied and that there are no real solutions for the wave speed if $s^2 < 0$. The above equation is equivalent to Equation (3.12) in [Dowaikh, 1999]. Note, however, the result (3.17) in [Dowaikh, 1999] corresponding to $s^2 < 0$ is incorrect. Thus,

$$\mathcal{A}_{01313}/\rho < v^2 < \mathcal{A}'_{01313}/\rho'. \tag{4.54}$$

For the isotropic linear elastic case, $\mathcal{A}_{01313} = \mathcal{A}_{02323} = \mu(0)$, $\mathcal{A}'_{01313} = \mathcal{A}'_{02323} = \mu'(0)$, where $\mu(0)$ and $\mu'(0)$ are the shear moduli of the layer and the half-space, respectively. If the transverse wave speed is denoted by $v_\mathrm{T} = (\mu(0)/\rho)^{1/2}$ in the layer and

$v'_{\rm T} = (\mu'(0)/\rho')^{1/2}$ in the bulk, then the above secular equation reduces to

$$\tan\left[\left(\frac{v^2}{v_{\rm T}^2} - 1\right)^{\frac{1}{2}} kh\right] = \frac{\mu'}{\mu}\frac{\left[1-(v/v'_{\rm T})^2\right]^{\frac{1}{2}}}{\left[(v/v_{\rm T})^2 - 1\right]^{\frac{1}{2}}}, \quad v_{\rm T} < v < v'_{\rm T}, \tag{4.55}$$

thus recovering the well-known dispersion relation for Love waves in linear elasticity (see, for example, [Achenbach, 1975]).

### 4.2.3 Application to a Mooney–Rivlin magnetoelastic material

To illustrate the results, we now consider the energy function of Mooney–Rivlin type magnetoelastic material as defined in Equation (3.48). We note that if the underlying magnetic induction is either parallel or perpendicular to the boundary, i.e. either $\mathbf{B} = (B_1, 0, 0)$ or $\mathbf{B} = (0, B_2, 0)$, then $\mathcal{A}_{01323} = 0 = \mathcal{C}_{013|3}\mathcal{C}_{023|3}$ ($\mathcal{C}_{013|3} = 0$ if $B_1 = 0$ and $\mathcal{C}_{023|3} = 0$ if $B_2 = 0$). Hence $B = 0$ and similarly $B' = 0$. Equations (4.34) and (4.39) then simplify to

$$s^2 = (\rho v^2 - C)/A, \quad s'^2 = (C' - \rho' v^2)/A'. \tag{4.56}$$

We require $s'^2 > 0$ for a surface wave to exist. By taking account of the strong ellipticity condition (2.95) this requirement imposes the conditions

$$C < \rho v^2 < \rho C'/\rho' \tag{4.57}$$

on the wave speed, and these inequalities also impose certain restrictions on the energy functions used and the deformations in the layer and the half-space for the existence of Love-type waves.

The secular equation (4.50) reduces to

$$\tan(skh) = \frac{s'A'}{sA}. \tag{4.58}$$

We now analyze this equation numerically by plotting the non-dimensionalized squared wave speed $\zeta = \rho v^2/\mu(0)$ against the dimensionless wave number $kh$. We take the following values of the material constants in order to obtain some representative solutions:

$$l\mu_0 = 2, \quad l'\mu_0 = 1.7, \quad m\mu_0 = 2, \quad m'\mu_0 = 0.7, \quad \nu = 0.3, \quad \nu' = 0.8,$$

$$\mu_0 = 4\pi\times 10^{-7}\,\mathrm{N/A^2}, \quad \mu(0) = 2.6\times 10^5\,\mathrm{N/m^2}, \quad \mu'(0)/\mu(0) = 2, \quad \rho'/\rho = 1/3. \tag{4.59}$$

We assume the initial deformations in the layer and the half-space to be the same, i.e. $\lambda_1 = \lambda'_1, \lambda_2 = \lambda'_2, \lambda_3 = \lambda'_3$. An infinite number of propagation modes are obtained

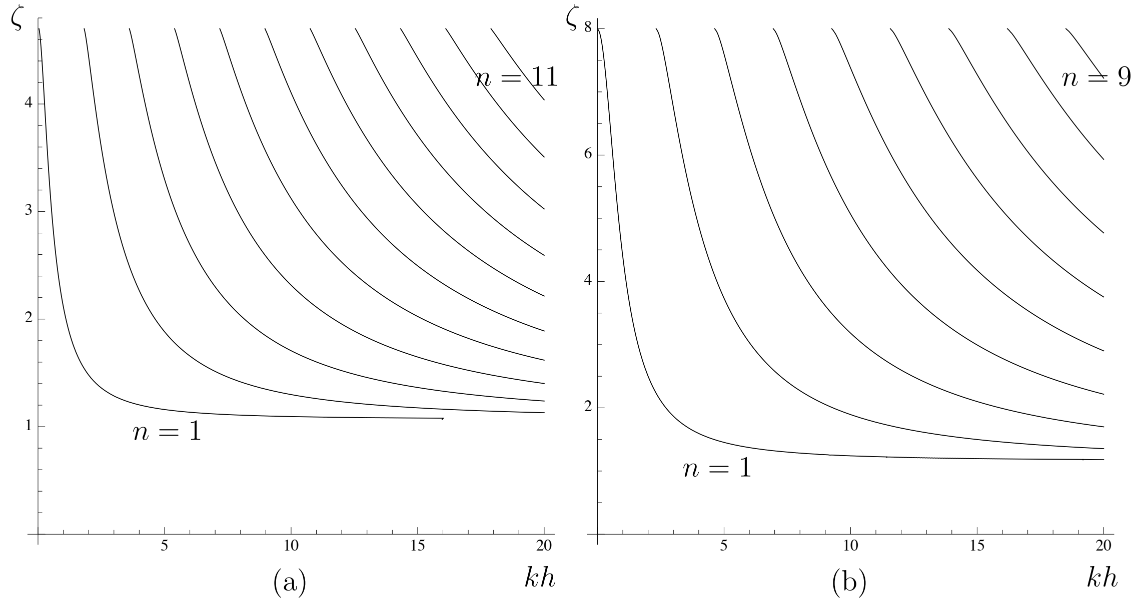
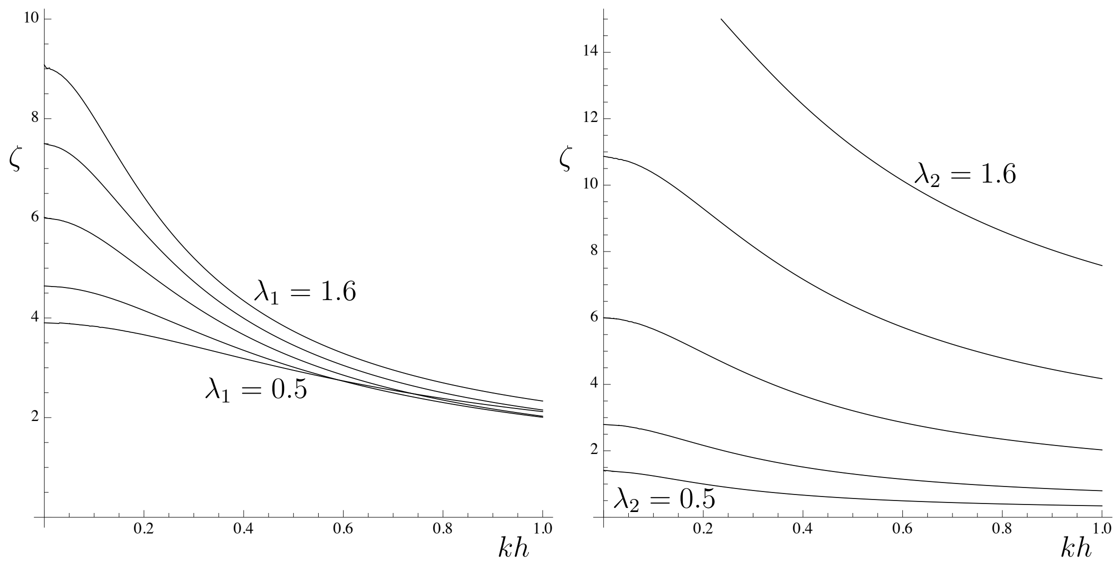



Figure 4.2: Dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. $kh$ for various mode numbers $n$ satisfying equation (4.58), illustrated for **(a)** $B_2 = 0 = B_3, B_1 = 0.1$ T, $\lambda_1 = 0.7 = \lambda_2^{-1}, \lambda_3 = 1$, $n = 1$ to $n = 11$; **(b)** $B_1 = 0 = B_3, B_2 = 0.5$ T, $\lambda_1 = 1.4 = \lambda_2^{-1}, \lambda_3 = 1$, $n = 1$ to $n = 9$.

due to the dispersive nature of Equation (4.58). Multiple modes of wave propagation corresponding to Equation (4.58) are illustrated in Figure 4.2 for two sets of representative values ($B_1 = 0.1$ T, $\lambda_1 = 0.7 = \lambda_2^{-1}$) and ($B_2 = 0.5$ T, $\lambda_1 = 1.4 = \lambda_2^{-1}$). For other values of the parameters the pattern of the higher-order modes is similar and we therefore show only the lowest mode henceforth from (4.58) for each of a selection of values of the magnetic induction and deformation.

For the pure elastic problem (no magnetic field) with a finite initial deformation the results are shown in Figure 4.3 for different values of initial stretch for the first mode. It is noted from Figures 4.3(a) and 4.3(b) that for the linear elastic case ($\lambda_1 = \lambda_2 = \lambda_3 = 1$), the curves intersect the $\zeta$ axis at $\mu'(0)\rho/\mu(0)\rho'$, which agrees with the classical solution (obtained by taking the limit $kh \to 0$ in Equation (4.55)) and is equal to 6 for the values adopted here.

The effect of the magnetic field without a finite deformation on the wave propagation characteristics is illustrated in Figure 4.4. It is noted that as $kh \to 0$, a magnetic (induction) field $B_2$ perpendicular to the boundary has no effect while that parallel to the boundary ($B_1$) changes the wave speed significantly. Either $B_1$ or $B_2$ tends to increase




(a) (b)

Figure 4.3: First mode dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. $kh$ under finite deformation in the absence of a magnetic field: **(a)** $\lambda_1 = 0.5, 0.7, 1, 1.3, 1.6,\ \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$; **(b)** $\lambda_1 = 1, \lambda_2 = 0.5, 0.7, 1, 1.3, 1.6,\ \lambda_3 = \lambda_2^{-1}$.

the wave speed.

The effect of the magnetic field when there is an initial finite deformation is illustrated in Figure 4.5 for two different values of $\lambda_1$: 0.7 and 1.4. The character of the results is similar qualitatively to the situation when there is no initial stretch.

## 4.3 Out-of-plane magnetic field: $\mathbf{B} = (0, 0, B_3)$

We now consider the case when the magnetic field is out of the plane, i.e. in the same direction as the mechanical displacement. The initial and deformed configurations are considered to be the same as in the previous section. For this value of the underlying magnetic induction, using equations (2.50) and (2.51), the components of the Maxwell stress and its increment are given by

$$[\boldsymbol{\tau}^*] = \frac{B_3^{*2}}{2\mu_0} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.60}$$

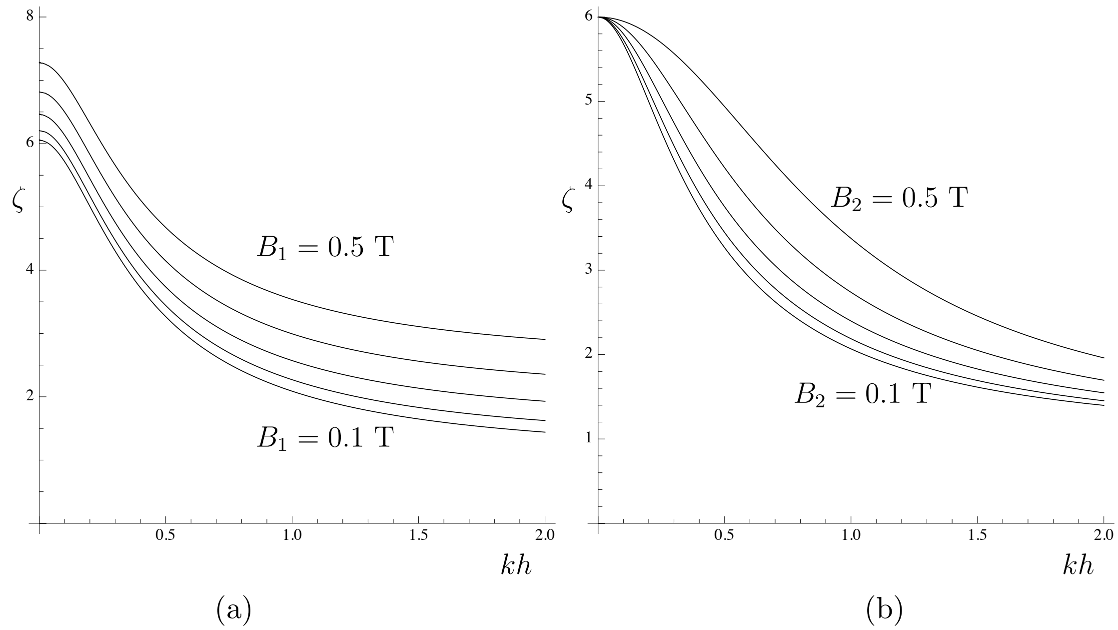



Figure 4.4: First mode dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. $kh$ for the linear elastic case in the presence of a magnetic field: **(a)** $B_2 = 0$, $B_1 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; **(b)** $B_1 = 0$, $B_2 = 0.1, 0.2, 0.3, 0.4, 0.5$ T.

$$[\dot{\boldsymbol{\tau}}^*] = \frac{1}{\mu_0}\begin{bmatrix} -\dot{B}_3^* B_3^* & 0 & \dot{B}_1^* B_3^* \\ 0 & -\dot{B}_3^* B_3^* & \dot{B}_2^* B_3^* \\ \dot{B}_1^* B_3^* & \dot{B}_2^* B_3^* & \dot{B}_3^* B_3^* \end{bmatrix}, \tag{4.61}$$

respectively.

The governing equations (4.3)–(4.8) reduce to $\dot{p}_{,1} = 0$, $\dot{p}_{,2} = 0$, $\dot{B}_{l03,1} = 0$, and $\dot{B}_{l03,2} = 0$ along with (4.2) and

$$\mathcal{A}_{01313} u_{3,11} + \mathcal{A}_{02323} u_{3,22} + \mathcal{C}_{013|1} \dot{B}_{l01,1} + \mathcal{C}_{023|2} \dot{B}_{l02,2} = \rho u_{3,tt}, \tag{4.62}$$

$$\left(\mathcal{C}_{023|2} - \mathcal{C}_{013|1}\right) u_{3,12} + \mathsf{K}_{022} \dot{B}_{l02,1} - \mathsf{K}_{011} \dot{B}_{l01,2} = 0, \tag{4.63}$$

for the layer, and similarly for the half-space. In this case $u_3$, $\dot{B}_{l01}$ and $\dot{B}_{l02}$ are coupled with each other through equations (4.62) and (4.63). Clearly, since there is no dependence on $x_3$, we may infer that $\dot{B}_{l03}$ is a function of $t$ which may be taken to be zero as for $f(t)$ in Section 4.1.1.

Let $u_3 = \phi$. Since the pairs $\{\dot{B}_{l01}, \dot{B}_{l02}\}$, $\{\dot{B}'_{l01}, \dot{B}'_{l02}\}$ and $\{\dot{B}^*_{l01}, \dot{B}^*_{l02}\}$ satisfy equation (4.2), we may define potentials $\psi, \psi'$, and $\psi^*$ such that

$$\dot{B}_{l01} = \psi_{,2}, \quad \dot{B}_{l02} = -\psi_{,1}, \quad \dot{B}'_{l01} = \psi'_{,2}, \quad \dot{B}'_{l02} = -\psi'_{,1}, \quad \dot{B}^*_1 = \psi^*_{,2}, \quad \dot{B}^*_2 = -\psi^*_{,1}. \tag{4.64}$$

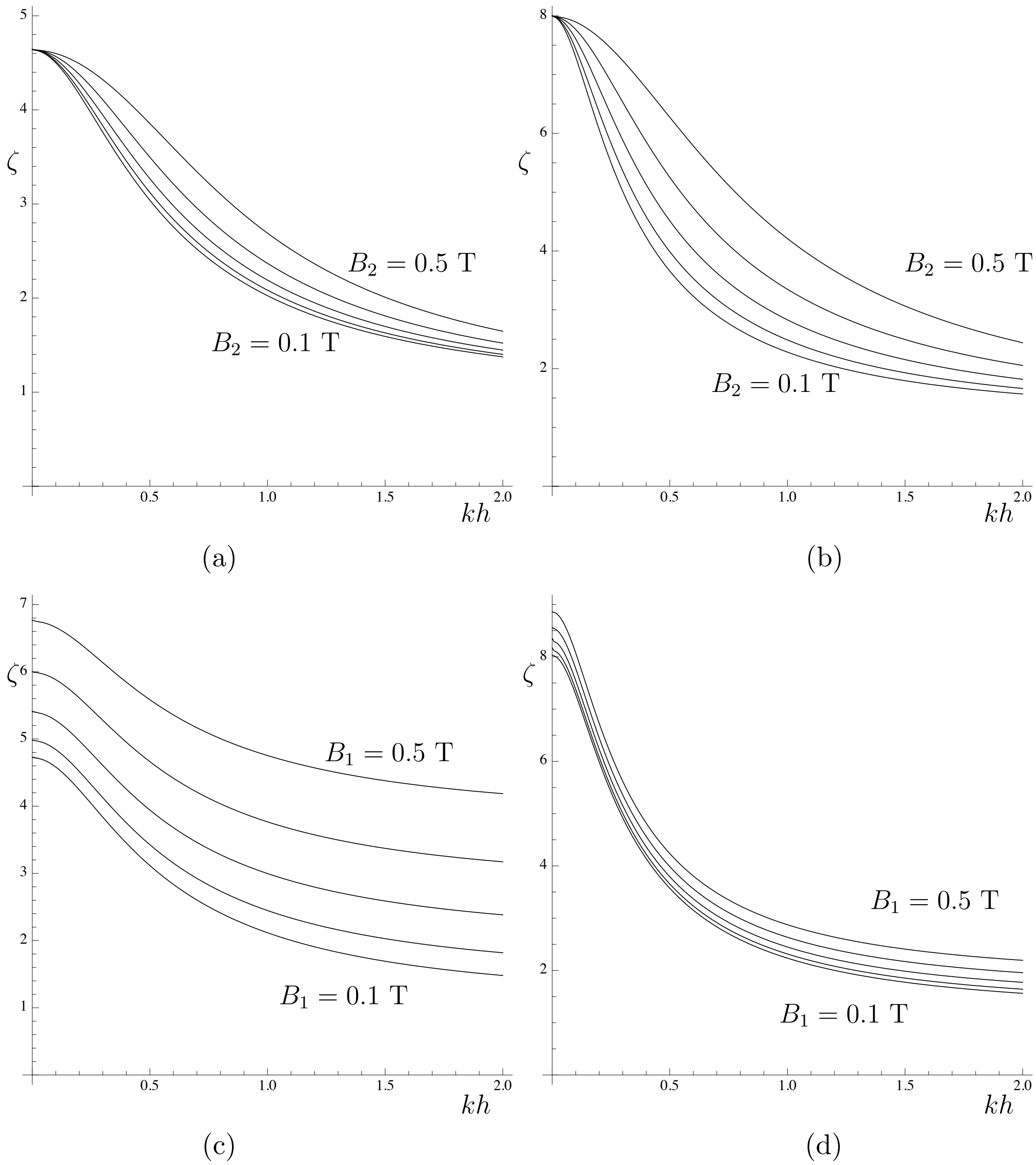



Figure 4.5: First mode dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. $kh$ for the material under finite deformation and magnetic field satisfying equation (4.58): (a) $\lambda_1 = 0.7, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_1 = 0$, $B_2 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; (b) $\lambda_1 = 1.4, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_1 = 0$, $B_2 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; (c) $\lambda_1 = 0.7, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_2 = 0$, $B_1 = 0.1, 0.2, 0.3, 0.4, 0.5$ T; (d) $\lambda_1 = 1.4, \lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$, $B_2 = 0$, $B_1 = 0.1, 0.2, 0.3, 0.4, 0.5$ T.

Substituting these potentials in the governing equations, we obtain

$$\mathcal{A}_{01313}\phi_{,11} + \mathcal{A}_{02323}\phi_{,22} + \mathcal{C}_{013|1}\psi_{,12} - \mathcal{C}_{023|2}\psi_{,12} = \rho\phi_{,tt}, \tag{4.65}$$

$$\left(\mathcal{C}_{023|2} - \mathcal{C}_{013|1}\right)\phi_{,12} - \mathsf{K}_{011}\psi_{,22} - \mathsf{K}_{022}\psi_{,11} = 0, \tag{4.66}$$

in the layer, while for the half-space we obtain

$$\mathcal{A}'_{01313}\phi'_{,11} + \mathcal{A}'_{02323}\phi'_{,22} + \mathcal{C}'_{013|1}\psi'_{,12} - \mathcal{C}'_{023|2}\psi'_{,12} = \rho'\phi'_{,tt}, \tag{4.67}$$

$$\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)\phi'_{,12} - \mathsf{K}'_{011}\psi'_{,22} - \mathsf{K}'_{022}\psi'_{,11} = 0, \tag{4.68}$$

and outside the material

$$\psi^*_{,11} + \psi^*_{,22} = 0. \tag{4.69}$$

### 4.3.1 Incremental boundary conditions

From the boundary conditions (2.47), (2.49), and (2.55) the only non-trivial remaining components are

$$\dot{T}_{023} = \dot{\tau}^*_{23}, \quad \dot{B}_{l02} = \dot{B}^*_2, \quad \dot{H}_{l01} - u_{3,1}H^*_3 - \dot{H}^*_1 = 0, \tag{4.70}$$

which, in terms of $\phi$ and the potential functions, yield, at the layer–vacuum interface $x_2 = h$,

$$\mathcal{A}_{02323}\phi_{,2} - \mathcal{C}_{023|2}\psi_{,1} + \psi^*_{,1}H^*_3 = 0, \tag{4.71}$$

$$\psi_{,1} - \psi^*_{,1} = 0, \tag{4.72}$$

$$\left(\mathcal{C}_{013|1} - H^*_3\right)\phi_{,1} + \mathsf{K}_{011}\psi_{,2} - \frac{1}{\mu_0}\psi^*_{,2} = 0, \tag{4.73}$$

and at the layer–half-space interface $x_2 = 0$,

$$\phi = \phi', \tag{4.74}$$

$$\mathcal{A}_{02323}\phi_{,2} - \mathcal{C}_{023|2}\psi_{,1} = \mathcal{A}'_{02323}\phi'_{,2} - \mathcal{C}'_{023|2}\psi'_{,1}, \tag{4.75}$$

$$\mathcal{C}_{013|1}\phi_{,1} + \mathsf{K}_{011}\psi_{,2} = \mathcal{C}'_{013|1}\phi'_{,1} + \mathsf{K}'_{011}\psi'_{,2}, \tag{4.76}$$

$$\psi_{,1} = \psi'_{,1}, \tag{4.77}$$

the first of which corresponds to continuity of displacement.

Hence the problem is reduced to solving equations (4.65) and (4.66) in $0 < x_2 < h$, equations (4.67) and (4.68) in $x_2 < 0$, and equation (4.69) in $x_2 > h$ using the boundary conditions (4.71), (4.72), and (4.73) at $x_2 = h$, and (4.74), (4.75), (4.76), and (4.77) at $x_2 = 0$.

### 4.3.2 Wave propagation

We again study Love-type waves in the same form as in the previous section and consider harmonic solutions of the form

$$\phi = P\exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{4.78}$$

$$\psi = Q\exp[\mathrm{i}\,(skx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{4.79}$$

$$\phi' = P'\exp\left(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{4.80}$$

$$\psi' = Q'\exp\left(s'kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{4.81}$$

$$\psi^* = R\exp\left(s^*kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 > h, \tag{4.82}$$

with the conditions $\mathrm{Re}(s') > 0$ and $\mathrm{Re}(s^*) < 0$ for the solutions to decay as $x_2 \to -\infty$ and $x_2 \to \infty$, respectively.

Substituting the solutions (4.80) and (4.81) in equations (4.67) and (4.68), we obtain

$$\left(-\mathcal{A}'_{01313} + s'^2\mathcal{A}'_{02323} + \rho' v^2\right)P' + \mathrm{i}s'\left(\mathcal{C}'_{013|1} - \mathcal{C}'_{023|2}\right)Q' = 0, \tag{4.83}$$

$$\mathrm{i}s'\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)P' + \left(\mathsf{K}'_{022} - s'^2\mathsf{K}'_{011}\right)Q' = 0. \tag{4.84}$$

For non-trivial solutions of $P'$ and $Q'$, the determinant of the coefficients of the above equations should be zero which gives

$$\mathcal{A}'_{02323}\mathsf{K}'_{011}s'^4 + \{\mathsf{K}'_{011}\left(\rho' v^2 - \mathcal{A}'_{01313}\right) - \mathsf{K}'_{022}\mathcal{A}'_{02323} + \left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)^2\}s'^2 \\ -\mathsf{K}'_{022}\left(\rho' v^2 - \mathcal{A}'_{01313}\right) = 0. \tag{4.85}$$

Let $s'_1$ and $s'_2$ be the two solutions satisfying the condition $\mathrm{Re}(s') > 0$, then we note that the condition $s'^2_1 s'^2_2 \geq 0$ gives an upper bound on the wave speed, which we express in the form

$$\rho' v^2 \leq \mathcal{A}'_{01313}. \tag{4.86}$$

With the two possible values of $s'$, the relevant general solutions for $\phi'$ and $\psi'$ are

$$\phi' = \left(P'_1\mathrm{e}^{s'_1kx_2} + P'_2\mathrm{e}^{s'_2kx_2}\right)\exp[\mathrm{i}\,(kx_1 - \omega t)], \tag{4.87}$$

$$\psi' = \left(Q'_1\mathrm{e}^{s'_1kx_2} + Q'_2\mathrm{e}^{s'_2kx_2}\right)\exp[\mathrm{i}\,(kx_1 - \omega t)], \tag{4.88}$$

where $P'_j$ and $Q'_j$ are related by (4.84) as

$$Q'_j = \frac{-\mathrm{i}s'_j\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)}{\left(\mathsf{K}'_{022} - s'^2_j\mathsf{K}'_{011}\right)}P'_j, \quad j = 1, 2. \tag{4.89}$$

Substituting the solutions (4.78) and (4.79) into equations (4.65) and (4.66), we obtain

$$\left(-\mathcal{A}_{01313} - s^2\mathcal{A}_{02323} + \rho v^2\right) P - s\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right) Q = 0, \tag{4.90}$$

$$s\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right) P + \left(\mathsf{K}_{011}s^2 + \mathsf{K}_{022}\right) Q = 0. \tag{4.91}$$

For non-trivial solutions for $P$ and $Q$, the determinant of the coefficients should be zero, which gives

$$\mathcal{A}_{02323}\mathsf{K}_{011}s^4 + \{\mathsf{K}_{011}\left(\mathcal{A}_{01313} - \rho v^2\right) + \mathcal{A}_{02323}\mathsf{K}_{022} - \left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)^2\}s^2 + \mathsf{K}_{022}\left(\mathcal{A}_{01313} - \rho v^2\right) = 0. \tag{4.92}$$

Let the solutions of this equation be $s_1, s_2, s_3$, and $s_4$. Then the general solutions for $\phi$ and $\psi$ may be written in the form

$$\phi = \left(P_1 e^{is_1kx_2} + P_2 e^{is_2kx_2} + P_3 e^{is_3kx_2} + P_4 e^{is_4kx_2}\right)\exp[i\left(kx_1 - \omega t\right)], \tag{4.93}$$

$$\psi = \left(Q_1 e^{is_1kx_2} + Q_2 e^{is_2kx_2} + Q_3 e^{is_3kx_2} + Q_4 e^{is_4kx_2}\right)\exp[i\left(kx_1 - \omega t\right)], \tag{4.94}$$

where $P_j$ and $Q_j$ are related by (4.91) as

$$Q_j = \frac{-s_j\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)}{\left(\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}\right)}P_j, \quad j = 1, 2, 3, 4. \tag{4.95}$$

Substituting the solution (4.82) into equation (4.69), we obtain $s^{*2} = 1$, and to satisfy the condition $\mathrm{Re}(s^*) < 0$, we take $s^* = -1$. Hence

$$\psi^* = R\exp\left(-kx_2 + ikx_1 - i\omega t\right). \tag{4.96}$$

Substituting the modified solutions (4.87), (4.88), (4.93), (4.94), and (4.96) into the boundary conditions (4.71), (4.72), and (4.73) at $x_2 = h$, and (4.74), (4.75), (4.76), and (4.77) at $x_2 = 0$, we obtain

$$\mathcal{A}_{02323}\sum_{j=1}^{4} P_j s_j e^{is_jkh} - \mathcal{C}_{023|2}\sum_{j=1}^{4} Q_j e^{is_jkh} + H_3^* Re^{-kh} = 0, \tag{4.97}$$

$$\sum_{j=1}^{4} Q_j e^{is_jkh} - Re^{-kh} = 0, \tag{4.98}$$

$$i\left(\mathcal{C}_{013|1} - H_3^*\right)\sum_{j=1}^{4} P_j e^{is_jkh} + i\mathsf{K}_{011}\sum_{j=1}^{4} Q_j s_j e^{is_jkh} + \frac{1}{\mu_0}Re^{-kh} = 0, \tag{4.99}$$

$$\sum_{j=1}^{4} P_j - \sum_{j=1}^{2} P_j' = 0, \tag{4.100}$$

$$\mathrm{i}\mathcal{A}_{02323}\sum_{j=1}^{4}P_j s_j - \mathrm{i}\mathcal{C}_{023|2}\sum_{j=1}^{4}Q_j - \mathcal{A}'_{02323}\sum_{j=1}^{2}P'_j s'_j + \mathrm{i}\mathcal{C}'_{023|2}\sum_{j=1}^{2}Q'_j = 0, \tag{4.101}$$

$$\mathrm{i}\mathcal{C}_{013|1}\sum_{j=1}^{4}P_j + \mathrm{i}\mathsf{K}_{011}\sum_{j=1}^{4}Q_j s_j - \mathrm{i}\mathcal{C}'_{013|1}\sum_{j=1}^{2}P'_j - \mathsf{K}'_{011}\sum_{j=1}^{2}s'_j Q'_j = 0, \tag{4.102}$$

$$\sum_{j=1}^{4}Q_j - \sum_{j=1}^{2}Q'_j = 0. \tag{4.103}$$

Using the relations (4.89) and (4.95) between $P_j$–$Q_j$ and $P'_j$–$Q'_j$, we can modify the above equations to

$$\sum_{j=1}^{4}\left[\mathcal{A}_{02323} + \frac{\mathcal{C}_{023|2}\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}}\right] s_j \mathrm{e}^{\mathrm{i}s_j kh} P_j + H_3^* R\,\mathrm{e}^{-kh} = 0, \tag{4.104}$$

$$\sum_{j=1}^{4} s_j \frac{\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}} \mathrm{e}^{\mathrm{i}s_j kh} P_j + R\,\mathrm{e}^{-kh} = 0, \tag{4.105}$$

$$\mathrm{i}\sum_{j=1}^{4}\left[\mathcal{C}_{013|1} - H_3^* - s_j^2 \frac{\mathsf{K}_{011}\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}}\right] \mathrm{e}^{\mathrm{i}s_j kh} P_j + \frac{1}{\mu_0} R\,\mathrm{e}^{-kh} = 0, \tag{4.106}$$

$$\sum_{j=1}^{4}P_j - \sum_{j=1}^{2}P'_j = 0, \tag{4.107}$$

$$\sum_{j=1}^{4}\left[\mathcal{A}_{02323} + \frac{\mathcal{C}_{023|2}\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}}\right] s_j P_j + \mathrm{i}\sum_{j=1}^{2}\left[\mathcal{A}'_{02323} - \frac{\mathcal{C}'_{023|2}\left(\mathcal{C}'_{013|1} - \mathcal{C}'_{023|2}\right)}{\mathsf{K}'_{011}s_j'^2 - \mathsf{K}'_{022}}\right] s'_j P'_j = 0, \tag{4.108}$$

$$\sum_{j=1}^{4}\frac{\mathsf{K}_{022}\mathcal{C}_{013|1} + s_j^2\mathsf{K}_{011}\mathcal{C}_{023|2}}{\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}} P_j + \sum_{j=1}^{2}\frac{\mathsf{K}'_{022}\mathcal{C}'_{013|1} - s_j'^2\mathsf{K}'_{011}\mathcal{C}'_{023|2}}{\mathsf{K}'_{011}s_j'^2 - \mathsf{K}'_{022}} P'_j = 0, \tag{4.109}$$

$$\sum_{j=1}^{4}\frac{\left(\mathcal{C}_{013|1} - \mathcal{C}_{023|2}\right)}{\mathsf{K}_{011}s_j^2 + \mathsf{K}_{022}} s_j P_j - \mathrm{i}\sum_{j=1}^{2}\frac{\left(\mathcal{C}'_{013|1} - \mathcal{C}'_{023|2}\right)}{\mathsf{K}'_{011}s_j'^2 - \mathsf{K}'_{022}} s'_j P'_j = 0. \tag{4.110}$$

These are seven equations for the seven constants $P_1, P_2, P_3, P_4, P'_1, P'_2$, and $R$. For non-trivial solutions, the determinant of the matrix formed by their coefficients should be zero. This condition gives the secular equation for the problem. We now illustrate the results for particular constitutive laws.

### 4.3.3 Application to a Mooney–Rivlin magnetoelastic material

In the underlying configuration, the boundary conditions require that $H_3^* = H_3 = H_3'$. Thus $B_3^* = \mu_0 H_3, B_3 = 0.5\lambda_3^2 H_3/(l + m\lambda_3^2), B_3' = 0.5\lambda_3'^2 H_3/(l' + m'\lambda_3'^2)$. Also, we have $\mathcal{C}_{023|2} = \mathcal{C}_{013|1} = 2mB_3$. Hence the governing equations (4.62) and (4.63) reduce to

$$\mathcal{A}_{01313}u_{3,11} + \mathcal{A}_{02323}u_{3,22} = \rho u_{3,tt}, \tag{4.111}$$

$$\mathsf{K}_{022}\psi_{,11} + \mathsf{K}_{011}\psi_{,22} = 0. \tag{4.112}$$

On substituting the harmonic solutions (4.78) and (4.79) in the above equations we get one value of $s^2$ for each of the mechanical and magnetic equations, say $s_1^2$ and $s_2^2$, respectively, i.e.

$$s_1^2 = \frac{\rho v^2 - \mathcal{A}_{01313}}{\mathcal{A}_{02323}}, \quad s_2^2 = -\frac{\mathsf{K}_{022}}{\mathsf{K}_{011}}. \tag{4.113}$$

Since the equations are decoupled these need not be the same, although in general there will be a coupling of the mechanical and magnetic effects through the boundary conditions.

When the mechanical and magnetic fields are combined the general solution may be written

$$\phi = \left(P^+ \mathrm{e}^{\mathrm{i}s_1 kx_2} + P^- \mathrm{e}^{-\mathrm{i}s_1 kx_2}\right) \exp[\mathrm{i}\,(kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{4.114}$$

$$\psi = \left(Q^+ \mathrm{e}^{\mathrm{i}s_2 kx_2} + Q^- \mathrm{e}^{-\mathrm{i}s_2 kx_2}\right) \exp[\mathrm{i}\,(kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{4.115}$$

$$\phi' = P' \exp\left(s_1' kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{4.116}$$

$$\psi' = Q' \exp\left(s_2' kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{4.117}$$

$$\psi^* = R \exp\left(-kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t\right), \quad x_2 > h, \tag{4.118}$$

where

$$s_1'^2 = \frac{\mathcal{A}_{01313}' - \rho' v^2}{\mathcal{A}_{02323}'}, \quad s_2'^2 = \frac{\mathsf{K}_{022}'}{\mathsf{K}_{011}'}. \tag{4.119}$$

After substituting these into the seven boundary conditions (4.71)–(4.77) we find that

$$P^+ + P^- = P', \quad Q^+ + Q^- = Q', \quad Q^+ \mathrm{e}^{\mathrm{i}s_2 kh} + Q^- \mathrm{e}^{-\mathrm{i}s_2 kh} = Re^{-kh}, \tag{4.120}$$

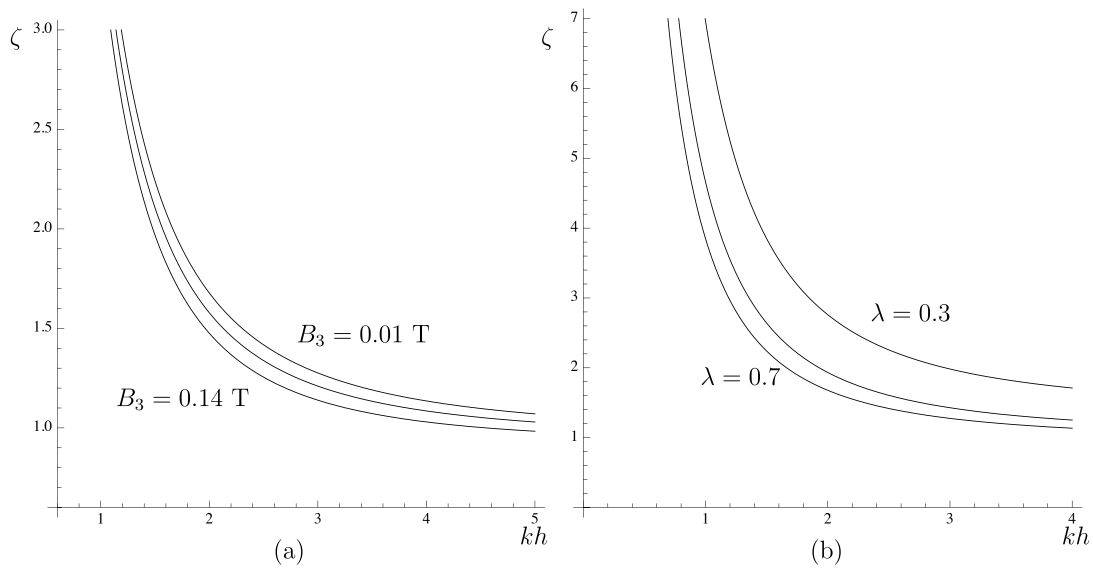



Figure 4.6: First mode of the dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. $kh$ for a Mooney–Rivlin type material in the presence of an out-of-plane magnetic field $B_3$. $\lambda = \lambda_1 = 1/\lambda_2, \lambda_3 = 1$. (a) $\lambda_1 = 0.7$, $B_3 = 0.01, 0.1, 0.14$ T; (b) $B_3 = 0.01$ T, $\lambda_1 = 0.3, 0.5, 0.7$.

and the remaining four boundary conditions expressed in terms of $P^+, P^-, Q^+, Q^-$ are

$$s_1\mathcal{A}_{02323}(P^+ - P^-) + \mathrm{i}\mathcal{A}'_{02323}s'_1(P^+ + P^-) + (\mathcal{C}'_{023|2} - \mathcal{C}_{023|2})(Q^+ + Q^-) = 0, \quad (4.121)$$

$$(\mathcal{C}_{013|1} - \mathcal{C}'_{013|1})(P^+ + P^-) + \mathsf{K}_{011}s_2(Q^+ - Q^-) + \mathrm{i}\mathsf{K}'_{011}s'_2(Q^+ + Q^-) = 0, \quad (4.122)$$

$$s_1\mathcal{A}_{02323}(P^+\mathrm{e}^{\mathrm{i}s_1kh} - P^-\mathrm{e}^{-\mathrm{i}s_1kh}) - \mathcal{C}_{023|2}(Q^+\mathrm{e}^{\mathrm{i}s_2kh} + Q^-\mathrm{e}^{-\mathrm{i}s_2kh}) + H_3^*(Q^+\mathrm{e}^{\mathrm{i}s_2kh} + Q^-\mathrm{e}^{-\mathrm{i}s_2kh}) = 0, \quad (4.123)$$

$$(\mathcal{C}_{013|1} - H_3^*)(P^+\mathrm{e}^{\mathrm{i}s_1kh} + P^-\mathrm{e}^{-\mathrm{i}s_1kh}) + \mathsf{K}_{011}s_2(Q^+\mathrm{e}^{\mathrm{i}s_2kh} - Q^-\mathrm{e}^{-\mathrm{i}s_2kh}) - \mathrm{i}\mu_0^{-1}(Q^+\mathrm{e}^{\mathrm{i}s_2kh} + Q^-\mathrm{e}^{-\mathrm{i}s_2kh}) = 0. \quad (4.124)$$

We plot the variation of the non-dimensionalized wave speed $\zeta = \rho v^2/\mu(0)$ against the non-dimensionalized wave number $kh$ in Figure 4.6 to study the effects of magnetic field and deformation. Values of the material constants listed in (4.59) are used for the numerical calculations. In general, the wave speed decreases with an increase in the wave number and in the magnetic field $B_3$. Considering a plane strain deformation ($\lambda_3 = 1$), a compression represented by the stretch $\lambda_1$ parallel to the surface in the direction of wave propagation tends to increase the wave speed.

Equation (4.111) is the same as that obtained for the pure elastic case in Section 4.2.2, and if the incremental magnetic field vanishes the problem reduces to a purely mechanical problem to solve for $u_3$. However, in the presence of a magnetic field vanishing of the incremental magnetic field (so that $Q^+ = Q^- = 0$) in general forces $u_3 = 0$. There is an exception to this if both coefficients $\mathcal{C}_{013|1} - H_3^*$ and $\mathcal{C}_{013|1} - \mathcal{C}'_{013|1}$ vanish. For the considered material we have

$$\mathcal{C}_{013|1} - H_3^* = -2l\lambda_3^{-2}B_3, \quad \mathcal{C}_{013|1} - \mathcal{C}'_{013|1} = (lm'{\lambda'_3}^2 - l'm\lambda_3^2)H_3. \tag{4.125}$$

Thus, for a purely mechanical wave to propagate in the presence of a magnetic field we must have $l = 0$ and either $l' = 0$ or $m = 0$. If both $l$ and $m$ vanish then the layer is not a magnetic material. In either case it is easy to show that the wave speed does not depend on the value of the magnetic field since, for the considered model, $\mathcal{A}_{01313}$ and $\mathcal{A}_{02323}$ are independent of $B_3$.

Similarly, if $u_3 = 0$, i.e. $P^+ = P^- = 0$, then in general a purely magnetic wave cannot exist except when both $\mathcal{C}_{023|2} - \mathcal{C}'_{023|2}$ and $\mathcal{C}_{023|2} - H_3^*$ are zero. For the Mooney–Rivlin model we have

$$\mathcal{C}_{023|2} - H_3^* = 2l\lambda_3^{-2}B_3, \quad \mathcal{C}_{023|2} - \mathcal{C}'_{023|2} = \left(\frac{m\lambda_3^2}{l + m\lambda_3^2} - \frac{m'\lambda_3'^2}{l' + m'\lambda_3'^2}\right)H_3. \tag{4.126}$$

If we take the deformation in the layer and the bulk half space to be the same, i.e. $\lambda_3 = \lambda'_3$, then for a purely magnetic wave to propagate we must have $l = 0$ and either $l' = 0$ or $m = m' = 0$. Vanishing of both $l$ and $m$ will make the layer non-magnetic.

In order to consider the case in which there is coupling through the equations we specialize the constitutive law to a version of the neo-Hookean solid.

### 4.3.4 Application to a neo-Hookean type magnetoelastic material

We consider a generalization of the neo-Hookean energy function for the magnetoelastic case which is a slight modification of the one used in [Dorfmann and Ogden, 2005] and given by

$$\Omega = \frac{\mu(0)}{2}(1 + \alpha I_4)(I_1 - 3) + lI_4 + mI_5 + qI_6, \tag{4.127}$$

where $\mu(I_4) = \mu(0) \times (1 + \alpha I_4)$ is a shear modulus that varies with the magnetic field, and $\alpha, l, m$, and $q$ are magnetoelastic coupling parameters. For this function, the relevant

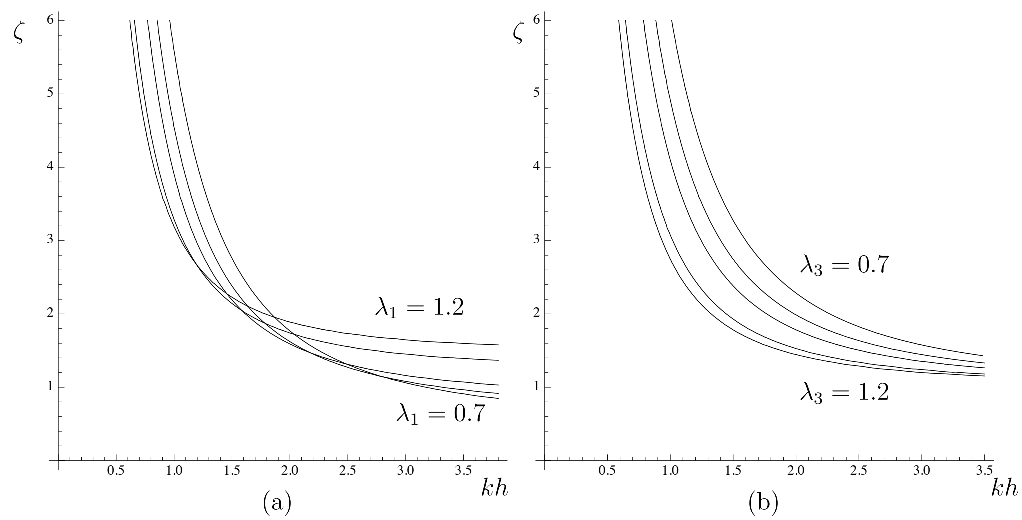



Figure 4.7: First mode dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. $kh$ for a neo-Hookean type material in the presence of an out-of-plane magnetic field $B_3 = 0.03$ T: (a) $\lambda_3 = 1, \lambda_2^{-1} = \lambda_1 = 0.7, 0.8, 0.9, 1.1, 1.2$; (b) $\lambda_1 = 1, \lambda_2^{-1} = \lambda_3 = 0.7, 0.8, 0.9, 1.1, 1.2$.

components of the moduli tensors are

$$\mathcal{A}_{01313} = \mu(0)\lambda_1^2(1+\alpha I_4) + 2\lambda_1^2 B_3^2 q, \quad \mathcal{A}_{02323} = \mu(0)\lambda_2^2(1+\alpha I_4) + 2\lambda_2^2 B_3^2 q,$$
$$\mathcal{C}_{013|1} = 2B_3[m + (\lambda_1^2+\lambda_3^2)q], \quad \mathcal{C}_{023|2} = 2B_3[m + (\lambda_2^2+\lambda_3^2)q],$$
$$\mathsf{K}_{011} = \lambda_1^{-2}[\mu\alpha(I_1-3)+2l] + 2m + 2q\lambda_1^2,$$
$$\mathsf{K}_{022} = \lambda_2^{-2}[\mu\alpha(I_1-3)+2l] + 2m + 2q\lambda_2^2. \quad (4.128)$$

For this model, we use equations (4.104)–(4.110) to study the variation of the non-dimensionalized wave speed $\zeta = \rho v^2/\mu(0)$ with the underlying magnetic field and deformation. We use the following values of the material parameters for the numerical calculations:

$$\mu(0) = 2.6\times 10^5\,\mathrm{N/m}^2, \quad \mu'(0) = 2\mu(0), \quad \rho' = 2\rho, \quad \alpha = 2, \quad \alpha' = 0.7,$$
$$l\mu_0 = 2, \quad l'\mu_0 = 1.7, \quad m\mu_0 = 2, \quad m'\mu_0 = 0.7, \quad q\mu_0 = 2, \quad q'\mu_0 = 0.1. (4.129)$$

The equations are dispersive and we obtain an infinite number of wave modes. We plot $\zeta$ against the non-dimensionalized wave number $kh$ for the first modes in Figures 4.7 and 4.8.

In general the wave speeds decrease with increasing wave number and a higher magnetic field tends to increase the wave speed for the material described by the generalized neo-

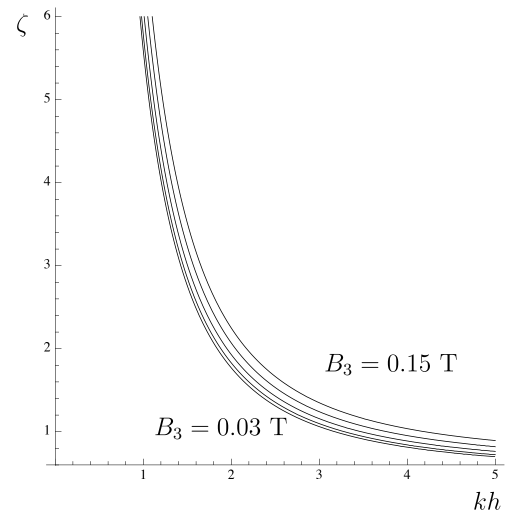



Figure 4.8: First mode dispersion curves $\zeta = \rho v^2/\mu(0)$ vs. $kh$ for a neo-Hookean type material in the presence of an out-of-plane magnetic field $B_3$. $\lambda_1 = 0.7 = \lambda_2^{-1}, \lambda_3 = 1$; $B_3 = 0.03, 0.06, 0.09, 0.12, 0.15$ T.

Hookean model. Considering an underlying deformation of plane strain ($\lambda_3 = 1$), a larger stretch $\lambda_1$ parallel to the surface in the direction of wave propagation tends to increase the wave speed. When a plane strain in the plane perpendicular to the wave propagation direction is considered ($\lambda_1 = 1$), a larger principal stretch $\lambda_3$ in the out-of-plane direction tends to decrease the wave speed.

## 4.4 Shear horizontal surface waves without a layer

We now consider a magnetoelastic half-space without a layer and seek the possibility of waves with an out-of-plane displacement component. Waves of this type, first described in [Parekh, 1969a, 1969b], are similar to the Bleustein–Gulyaev waves in electroelasticity (see, for example, [Bleustein, 1968]) and do not have a counterpart in pure elasticity. We consider the two cases of in-plane and out-of-plane directions of the underlying magnetic induction.

### 4.4.1 $\mathbf{B} = (B_1, B_2, 0)$

The relevant governing equations are (4.13), (4.14), and (4.15) in $x_2 < 0$ with the boundary condition (4.24) at $x_2 = 0$. We consider solutions of the type (4.29) and (4.30) and substitute into the boundary conditions to obtain

$$(\mathrm{i}s'A' - B')P' = 0. \tag{4.130}$$

This cannot be satisfied since for a non-trivial wave we must have $P' \neq 0$, but also, since $A'$ and $B'$ are real, and, by strong ellipticity $A' > 0$ the real part of $s'$ must vanish. Therefore such a mode of wave propagation does not exist when the underlying magnetic field is in-plane.

### 4.4.2 $\mathbf{B} = (0, 0, B_3)$

In this case, we consider the governing equations (4.67) and (4.68) in $x_2 < 0$, and equation (4.69) in $x_2 > 0$ to solve with the boundary conditions (4.71), (4.72), and (4.73) at $x_2 = 0$. We consider solutions similar to (4.87), (4.88), and (4.96). Substituting into the boundary conditions we obtain

$$\mathcal{A}'_{02323}(s'_1P'_1 + s'_2P'_2) - \mathrm{i}\mathcal{C}_{023|2}(Q'_1 + Q'_2) + \mathrm{i}H_3^*R = 0, \tag{4.131}$$

$$Q'_1 + Q'_2 - R = 0, \tag{4.132}$$

$$\mathrm{i}(\mathcal{C}_{013|1} - H_3^*)(P'_1 + P'_2) + \mathsf{K}_{011}(s'_1Q'_1 + s'_2Q'_2) + \mu_0^{-1}R = 0, \tag{4.133}$$

while Equation (4.84) gives the relations

$$\mathrm{i}s'_1\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)P'_1 + \left(\mathsf{K}'_{022} - s_1'^2\mathsf{K}'_{011}\right)Q'_1 = 0, \tag{4.134}$$

$$\mathrm{i}s'_2\left(\mathcal{C}'_{023|2} - \mathcal{C}'_{013|1}\right)P'_2 + \left(\mathsf{K}'_{022} - s_2'^2\mathsf{K}'_{011}\right)Q'_2 = 0, \tag{4.135}$$

Here $s'_1$ and $s'_2$ are the solutions of Equation (4.85) satisfying the criterion $\mathrm{Re}(s') > 0$.

For non-trivial solutions for $P'_1, P'_2, Q'_1, Q'_2$, and $R$, the determinant of their coefficients should be zero which gives an equation to solve for the wave speed. We therefore illustrate the results in Figure 4.9 by considering again the modified neo-Hookean energy function defined in (4.127). The non-dimensionalized wave speed $\zeta = \rho' v^2/\mu'(0)$ is plotted against the underlying axial stretch for different values of the underlying magnetic field.

For a plane strain deformation ($\lambda_3 = 1$) illustrated in Figure 4.9(a), it is observed that a stretch parallel to the direction of wave propagation $\lambda_1$ tends to increase the wave speed. A higher underlying magnetic field also increases the wave speed.

For the plane strain deformation when there is no compression or extension parallel to the wave propagation direction ($\lambda_1 = 1$) as shown in Figure 4.9(b), a critical value

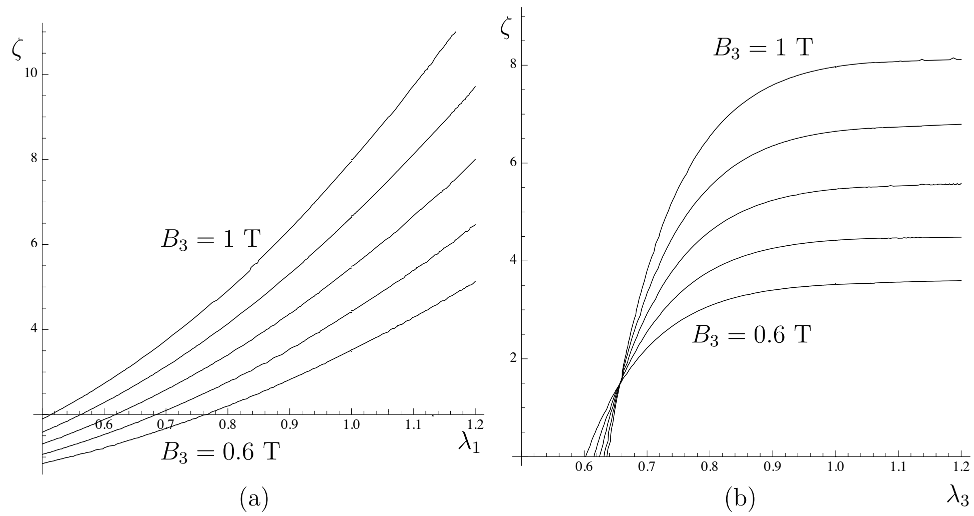



Figure 4.9: Variation of $\zeta = \rho' v^2/\mu'(0)$ with the underlying deformation and the underlying magnetic field for a Bleustein–Gulyaev type wave in a neo-Hookean type solid. $B_3 = 0.6, 0.7, 0.8, 0.9, 1$ T; (a) $\lambda = \lambda_1 = \lambda_2^{-1}, \lambda_3 = 1$; (b) $\lambda_1 = 1, \lambda_3 = \lambda_2^{-1} = \lambda$.

of $\lambda_3 = \lambda_c$ is observed at which the wave speed becomes independent of the underlying magnetic field $B_3$. The critical stretch $\lambda_c$ depends on the parameters of the energy function used. When $\lambda_3 < \lambda_c$ the wave speed decreases with an increase in $B_3$ while in the region $\lambda_3 > \lambda_c$ the wave speed increases with an increase in $B_3$. For large values of compression (small $\lambda_3$) $\zeta$ goes to zero which coincides with the onset of instability in the material. The wave speed increases with an increase in $\lambda_3$ and reaches an asymptotic value dependent on the underlying magnetic field $B_3$.

When there is no underlying deformation, for the considered model we have $\mathcal{C}'_{013|1} = \mathcal{C}'_{023|2}$ and $\mathsf{K}'_{011} = \mathsf{K}'_{022}$. Equation (4.85) can be factorized to obtain the roots $s_1'^2 = 1 - \rho' v^2/\mathcal{A}_{01313}$ and $s_2'^2 = 1$. This results in (4.135) becoming identically zero and hence the above procedure yields no solution for the wave speed. So in this case we consider the solutions

$$\phi' = P' \exp\left(s_1' k x_2 + \mathrm{i} k x_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{4.136}$$

$$\psi' = Q' \exp\left(s_2' k x_2 + \mathrm{i} k x_1 - \mathrm{i}\omega t\right), \quad x_2 < 0, \tag{4.137}$$

$$\psi^* = R \exp\left(-k x_2 + \mathrm{i} k x_1 - \mathrm{i}\omega t\right), \quad x_2 > 0, \tag{4.138}$$

where the boundary conditions are (4.71)–(4.73), applied on $x_2 = 0$. These yield $R = Q'$

and

$$\mathcal{A}'_{02323} s'_1 P' + \mathrm{i}(H_3^* - \mathcal{C}'_{023|2})Q' = 0 \tag{4.139}$$

$$-\mathrm{i}(H_3^* - \mathcal{C}'_{013|1})P' + (\mu_0^{-1} + s'_2 \mathsf{K}'_{011})Q' = 0, \tag{4.140}$$

Requiring a non-trivial solution yields the following explicit formula for the wave speed:

$$\rho' v^2 = \mathcal{A}'_{01313} - \frac{(H_3^* - \mathcal{C}'_{013|1})^4}{(\mu_0^{-1} + \mathsf{K}'_{011})^2 \mathcal{A}'_{01313}}. \tag{4.141}$$

The value of the wave speed thus obtained for the linear elastic case is consistent with those illustrated in Figure 4.9.

The Mooney–Rivlin type energy function requires special treatment, and we follow the procedure as above for the linear elastic case and obtain an explicit formula of the wave speed

$$\rho' v^2 = \mathcal{A}'_{01313} - \frac{(H_3^* - \mathcal{C}'_{013|1})^4}{(\mu_0^{-1} + \sqrt{\mathsf{K}'_{011}\mathsf{K}'_{022}})^2 \mathcal{A}'_{02323}}, \tag{4.142}$$

When the specific forms of the Mooney–Rivlin constants are substituted (for the case $\nu = 1$ for illustration), we get

$$\rho' v^2/\mu'(0) = \lambda_1^2 - \frac{16 l^4 \lambda_3^{-8} B_3^4}{[\mu_0^{-1} + 2\sqrt{(m + l\lambda_1^{-2})(m + l\lambda_2^{-2})}]^2 \lambda_2^2 \mu'(0)^2}. \tag{4.143}$$

The above formula suggests that there is an upper bound on the underlying magnetic field for the wave speed to be real. When evaluated for no underlying deformation this reduces to

$$\rho' v^2/\mu'(0) = 1 - \frac{16 l^4 B_3^4}{[\mu_0^{-1} + 2(l + m)]^2 \mu'(0)^2}. \tag{4.144}$$

# Chapter 5

# Finite Deformation and Axisymmetric Motions of a Cylindrical Tube

In this chapter, we specialize the equations of nonlinear magnetoelasticity to cylindrical coordinates to consider deformations and motions of a thick-walled tube. In the presence of an internal pressure, axial force, and an underlying magnetic field in the azimuthal or axial directions, the tube undergoes a finite deformation. Nonlinear static deformation of the tube and its dependence on the intensity of the applied magnetic field are analyzed for two different kinds of energy density functions. Thereafter we study the axisymmetric motions of the said finitely deformed tube and their dependence on the applied magnetic field, internal pressure, and axial stretch.

## 5.1 Constitutive relations

For specialization to cylindrical geometry, it is useful to consider $\mathbf{H}_l$ as an independent variable of the energy density function rather than $\mathbf{B}_l$ (see [Dorfmann and Ogden, 2005] for details). The two different energy functions are related by the Legendre transformation as

$$\Omega(\mathbf{F}, \mathbf{H}_l) = \Omega^*(\mathbf{F}, \mathbf{B}_l) - \mathbf{B}_l \cdot \mathbf{H}_l. \tag{5.1}$$

The above equation when combined with the constitutive relations (2.56) yields

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega}{\partial \mathbf{H}_l}, \tag{5.2}$$

for the total nominal stress and the Lagrangian magnetic induction vector. If the material is incompressible, then the first of above equations becomes

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{5.3}$$

where $p$ is a Lagrange multiplier associated with the constraint of incompressibility. The above equations when written in Eulerian form, become

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{B} = -J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{H}_l}, \tag{5.4}$$

for compressible materials, while for incompressible materials the first of the above equations becomes

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}. \tag{5.5}$$

For an incompressible isotropic magnetoelastic material, the energy function can be expressed in terms of five scalar invariants which we choose to be

$$\begin{gathered} I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}\left(I_1^2 - \operatorname{tr}\left(\mathbf{c}^2\right)\right), \\ K_4 = \mathbf{H}_l \cdot \mathbf{H}_l, \quad K_5 = (\mathbf{c}\mathbf{H}_l) \cdot \mathbf{H}_l, \quad K_6 = \left(\mathbf{c}^2\mathbf{H}_l\right) \cdot \mathbf{H}_l, \end{gathered} \tag{5.6}$$

where $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is the right Cauchy-Green tensor. We use $K_4, K_5, K_6$ above instead of $I_4, I_5, I_6$ to maintain consistency as the latter are used in Chapter 2 to define invariants in terms of $\mathbf{B}_l$, and $I_3 = \det \mathbf{c}$ is unity in the present case of incompressibility. Hence, the constitutive relations $(5.2)_2$ and (5.3) can be expanded to be written in the form

$$\begin{aligned} \mathbf{T} = &- p\mathbf{F}^{-1} + 2\Omega_1\mathbf{F}^{\mathrm{T}} + 2\Omega_2\left(I_1\mathbf{F}^{\mathrm{T}} - \mathbf{c}\mathbf{F}^{\mathrm{T}}\right) + 2\Omega_5\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l \\ &+ 2\Omega_6\left(\mathbf{H}_l \otimes \mathbf{F}\mathbf{c}\mathbf{H}_l + \mathbf{c}\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l\right), \end{aligned} \tag{5.7}$$

and

$$\mathbf{B}_l = -2\left(\Omega_4\mathbf{H}_l + \Omega_5\mathbf{c}\mathbf{H}_l + \Omega_6\mathbf{c}^2\mathbf{H}_l\right), \tag{5.8}$$

where $\Omega_k = \partial\Omega/\partial I_k$ for $k = 1, 2$ and $\Omega_k = \partial\Omega/\partial K_k$ for $k = 4, 5, 6$. In the Eulerian form the above equations are

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\Omega_1\mathbf{b} + 2\Omega_2\left(I_1\mathbf{b} - \mathbf{b}^2\right) + 2\Omega_5\mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H} + 2\Omega_6\left(\mathbf{b}\mathbf{H} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{b}\mathbf{H}\right), \tag{5.9}$$

and

$$\mathbf{B} = -2\left(\Omega_4\mathbf{b}\mathbf{H} + \Omega_5\mathbf{b}^2\mathbf{H} + \Omega_6\mathbf{b}^3\mathbf{H}\right), \tag{5.10}$$

where $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy–Green tensor.

On incrementing the equations $(5.2)_2$ and (5.3), we get

$$\dot{\mathbf{T}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}\dot{\mathbf{F}} + \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{H}_l}\dot{\mathbf{H}}_l - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{5.11}$$

$$\dot{\mathbf{B}}_l = -\frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{F}}\dot{\mathbf{F}} - \frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{H}_l}\dot{\mathbf{H}}_l. \tag{5.12}$$

Using the relations in Equation (2.35) with $J = 1$, we update the above equations to get

$$\dot{\mathbf{T}}_0 = \mathbf{F}\boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbf{F}\boldsymbol{\mathcal{C}}\dot{\mathbf{H}}_l - \dot{p}\mathbf{I} + p\mathbf{L}, \quad \dot{\mathbf{B}}_{l0} = -\mathbf{F}\boldsymbol{\mathcal{C}}^{\mathrm{T}}\dot{\mathbf{F}} - \mathbf{F}\mathsf{K}\dot{\mathbf{H}}_l, \tag{5.13}$$

where we have used the notations

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \boldsymbol{\mathcal{C}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{H}_l}, \quad \boldsymbol{\mathcal{C}}^{\mathrm{T}} = \frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{F}}, \quad \mathsf{K} = \frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{H}_l}, \tag{5.14}$$

which now redefine the magnetoelastic *moduli* tensors. The updated magnetoelastic tensors, $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and $\mathsf{K}_0$ can be defined in component form as

$$\begin{aligned} \mathcal{A}_{0ipjq} &= \mathcal{A}_{0jqip} = F_{i\alpha}F_{j\beta}\mathcal{A}_{\alpha p\beta q}, \\ \mathcal{C}_{0ij|k} &= \mathcal{C}_{0ji|k} = F_{i\alpha}F^{-1}_{\beta k}\mathcal{C}_{\alpha j|\beta}, \\ \mathsf{K}_{0ij} &= \mathsf{K}_{0ji} = F^{-1}_{\alpha i}F^{-1}_{\beta j}\mathsf{K}_{\alpha\beta}. \end{aligned} \tag{5.15}$$

It is worth noting that the magnetoelastic moduli tensors above are different from those used in Chapter 2 and elaborated in Appendix B. Here they are defined in terms of the magnetic field vector $\mathbf{H}_l$ while earlier they were defined in terms of the magnetic induction vector $\mathbf{B}_l$. These are substituted in the updated incremented constitutive equations above to give

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{H}}_{l0} - \dot{p}\mathbf{I} + p\mathbf{L}, \quad \dot{\mathbf{B}}_{l0} = -\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} - \mathsf{K}_\mathbf{0}\dot{\mathbf{H}}_{l0}. \tag{5.16}$$

On substituting the above forms of constitutive equations in the governing equations $(2.43)_{2,3}$ (assuming no mechanical body forces) we get

$$\operatorname{div}\left(\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \mathsf{K}_\mathbf{0}\dot{\mathbf{H}}_{l0}\right) = 0, \tag{5.17}$$

$$\operatorname{div}\left(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{H}}_{l0} + p\mathbf{L}\right) - \operatorname{grad}\dot{p} = \rho\mathbf{u}_{,tt}. \tag{5.18}$$

Using $(5.16)_1$, and from the symmetry of the total stress tensor in incremental form $(2.38)_2$, we obtain the identities

$$\mathcal{A}_{0ipjq} + \delta_{iq}\left(\tau_{jp} + p\delta_{jp}\right) = \mathcal{A}_{0pijq} + \delta_{pq}\left(\tau_{ij} + p\delta_{ij}\right), \quad \mathcal{C}_{0ij|k} = \mathcal{C}_{0ji|k}, \tag{5.19}$$

the first of which can be used to obtain the useful relation

$$p = \mathcal{A}_{01313} - \mathcal{A}_{01331} - \tau_{11} = \mathcal{A}_{01212} - \mathcal{A}_{01221} - \tau_{11}. \tag{5.20}$$

## 5.2 Specialization to a cylindrical geometry

We consider an infinite circular cylindrical tube made of an incompressible non-conducting magnetoelastic material. We work in terms of cylindrical polar coordinates, which in the reference configuration $\mathcal{B}_\mathrm{r}$ are denoted by $(R, \Theta, Z)$ and in the deformed configuration $\mathcal{B}$ by $(r, \theta, z)$. In the reference configuration, let the internal and external radii of the tube be given by $A$ and $B$, respectively.

The tube is deformed by inflating and stretching in the radial and axial directions, respectively, and then by the application of a magnetic field in the azimuthal and the axial direction to maintain axisymmetry. After the deformation, the new inner and outer radii are $a$ and $b$ such that $a \le r \le b$. The deformation assumes the form

$$r = \left[a^2 + \frac{1}{\lambda_z}\left(R^2 - A^2\right)\right]^{\frac{1}{2}}, \qquad z = \lambda_z Z, \qquad \theta = \Theta, \tag{5.21}$$

where the first relation is due to incompressibility and $\lambda_z$ is the (uniform) axial stretch.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denote the basis vectors corresponding to the $r, \theta, z$ coordinates and $\lambda_1, \lambda_2, \lambda_3$ be the corresponding principal stretches. From here onwards, we will take $(1, 2, 3)$ to correspond to $(r, \theta, z)$. Hence the underlying magnetic field is given as $\mathbf{H} = (0, H_2, H_3)$. Using the constraint of incompressibility $\lambda_1\lambda_2\lambda_3 = 1$, the principal stretches in the azimuthal, axial, and radial directions are given by

$$\lambda_2 = \lambda = \frac{r}{R}, \qquad \lambda_3 = \lambda_z, \qquad \lambda_1 = \lambda^{-1}\lambda_z^{-1}, \tag{5.22}$$

respectively, wherein the notation $\lambda$ is introduced.

From Equation (5.9), we obtain

$$\begin{aligned}
\tau_{11} &= -p + 2\Omega_1\lambda_1^2 + 2\Omega_2\lambda_1^2\left(\lambda_2^2 + \lambda_3^2\right), \\
\tau_{22} &= -p + 2\Omega_1\lambda_2^2 + 2\Omega_2\lambda_2^2\left(\lambda_1^2 + \lambda_3^2\right) + 2\Omega_5\lambda_2^4 H_2^2 + 4\Omega_6\lambda_2^6 H_2^2, \\
\tau_{33} &= -p + 2\Omega_1\lambda_3^2 + 2\Omega_2\lambda_3^2\left(\lambda_1^2 + \lambda_2^2\right) + 2\Omega_5\lambda_3^4 H_3^2 + 4\Omega_6\lambda_3^6 H_3^2.
\end{aligned} \tag{5.23}$$

The equilibrium equation $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$ gives

$$\frac{d\tau_{11}}{dr} = \frac{1}{r}\left(\tau_{22} - \tau_{11}\right), \tag{5.24}$$

which on substituting the values of $\tau_{11}$ and $\tau_{22}$ becomes

$$\frac{d\tau_{11}}{dr} = \frac{1}{r}\left\{2\Omega_1\left(\lambda_2^2 - \lambda_1^2\right) + 2\Omega_2\lambda_3^2\left(\lambda_2^2 - \lambda_1^2\right) + 2\Omega_5\lambda_2^4 H_2^2 + 4\Omega_6\lambda_2^6 H_2^2\right\}. \tag{5.25}$$

Boundary conditions on the lateral surfaces of the cylinder ($r = a, b$) are given by the balance of traction (2.27) as

$$\tau_{11} = \tau_{11}^* - P_{\text{in}} \quad \text{at } r = a, \quad \text{and} \quad \tau_{11} = \tau_{11}^* - P_{\text{out}} \quad \text{at } r = b. \tag{5.26}$$

Here, $P_{\text{in}}$ and $P_{\text{out}}$ are the mechanically applied internal and external pressures, respectively, while $\tau_{11}^*$ obtains the value $-\mu_0(H_2^2 + H_3^2)/2$ from Equation (2.50).

We note here that in the case of a tube of finite length, the magnetic boundary conditions at the two ends of the tube are easily satisfied if the magnetic field is in azimuthal direction. For an axial magnetic field, a detailed analysis for a tube of finite length has been done by [Bustamante et al., 2007].

Since the independent parameters of the deformation process are $\lambda, \lambda_z, H_{l2}$, and $H_{l3}$, we can write the energy function as

$$\Omega(\mathbf{F}, \mathbf{H}_l) = \hat{\Omega}(\lambda, \lambda_z, H_{l2}, H_{l3}). \tag{5.27}$$

The scalar invariants can be then written in the form

$$I_1 = \lambda^2 + \lambda_z^2 + \lambda^{-2}\lambda_z^{-2}, \quad I_2 = \lambda^{-2} + \lambda_z^{-2} + \lambda^2\lambda_z^{-2},$$
$$K_4 = H_{l2}^2 + H_{l3}^2, \quad K_5 = \lambda^2 H_{l2}^2 + \lambda_z^2 H_{l3}^2, \quad K_6 = \lambda^4 H_{l2}^2 + \lambda_z^4 H_{l3}^2, \tag{5.28}$$

using which we write

$$\frac{\partial \hat{\Omega}}{\partial \lambda} = \Omega_1 \frac{\partial I_1}{\partial \lambda} + \Omega_2 \frac{\partial I_2}{\partial \lambda} + \Omega_5 \frac{\partial K_5}{\partial \lambda} + \Omega_6 \frac{\partial K_6}{\partial \lambda}, \tag{5.29}$$
$$= \frac{2}{\lambda}\left(\Omega_1 + \Omega_2 \lambda_z^2\right)(\lambda^2 - \lambda^{-2}\lambda_z^{-2}) + 2\Omega_5 \lambda H_{l2}^2 + 4\Omega_6 \lambda^3 H_{l2}^2, \tag{5.30}$$

which gives

$$\frac{d\tau_{11}}{dr} = \frac{\lambda}{r}\frac{\partial \hat{\Omega}}{\partial \lambda}, \tag{5.31}$$

where use has been made of (5.25).

We also mention the following useful relations for a finitely deformed tube, which are derived in Appendix C and can be seen in, for example, [Haughton and Ogden, 1979b]

$$r\frac{d\lambda}{dr} = \lambda(1 - \lambda^2\lambda_z), \tag{5.32}$$

$$A^{-2}B^2\left(\lambda_b^2\lambda_z - 1\right) = \lambda_a^2\lambda_z - 1 = R^2 A^{-2}(\lambda^2\lambda_z - 1), \qquad \frac{\partial \lambda_b}{\partial \lambda_a} = \frac{\lambda_a A^2}{\lambda_b B^2}. \tag{5.33}$$

Here $\lambda_a = \lambda|_{r=a}$ and $\lambda_b = \lambda|_{r=b}$.

On integrating Equation (5.25) using the boundary conditions (5.26), we obtain

$$\int_a^b \frac{1}{r}\left\{2\Omega_1\left(\lambda_2^2-\lambda_1^2\right)+2\Omega_2\lambda_3^2\left(\lambda_2^2-\lambda_1^2\right)+2\Omega_5\lambda_2^4H_2^2+4\Omega_6\lambda_2^6H_2^2\right\}dr$$
$$= P_{\text{in}}-P_{\text{out}}+\frac{\mu_0}{2}\left(H_2^2|_a-H_2^2|_b\right), \tag{5.34}$$

using which along with $(5.21)_1$, we can evaluate the inner and outer radii $(a, b)$ of the tube after deformation for a given pressure difference and magnetic field. (The contributions due to $H_3$ in the above formula cancel out.) We now use the above calculated value of the inner radius $a$ to evaluate the value of $\tau_{11}$ as a function of $r$ by integrating Equation (5.25) as

$$\tau_{11} = -\frac{\mu_0}{2}\left(H_2^2+H_3^2\right)|_a - P_{\text{in}}$$
$$+\int_a^r \frac{1}{r}\left\{2\Omega_1\left(\lambda_2^2-\lambda_1^2\right)+2\Omega_2\lambda_3^2\left(\lambda_2^2-\lambda_1^2\right)+2\Omega_5\lambda_2^4H_2^2+4\Omega_6\lambda_2^6H_2^2\right\}dr. \tag{5.35}$$

The above process can be equivalently repeated by using Equation (5.31) instead of (5.25) depending on the requirements of the energy density function used.

In the following subsections we study the total pressure and the axial force generated in the tube due to static nonlinear axisymmetric deformations in the presence of an underlying magnetic field.

### 5.2.1 Total internal pressure in the tube

We define the net total internal pressure $P_{\text{T}}$ as the difference between the surface traction per unit area on the inside and on the outside of the tube.

$$P_{\text{T}} = (P_{\text{in}} - \tau_{11}^*|_a) - (P_{\text{out}} - \tau_{11}^*|_b) \tag{5.36}$$
$$= \left(P_{\text{in}} + \frac{\mu_0}{2}(H_2^2+H_3^2)_a\right) - \left(P_{\text{out}} + \frac{\mu_0}{2}(H_2^2+H_3^2)|_b\right). \tag{5.37}$$

On integrating Equation (5.31) using the boundary conditions (5.26), we get

$$P_{\text{T}} = \int_a^b \frac{\lambda}{r}\frac{\partial\hat{\Omega}}{\partial\lambda}dr. \tag{5.38}$$

which is slightly more general than the formula (127) given in [Dorfmann and Ogden, 2005]. We use Equation (5.32) to change the variable of integration from $r$ to $\lambda$

$$P_{\text{T}} = \int_{\lambda_b}^{\lambda_a} \frac{1}{(\lambda^2\lambda_z-1)}\frac{\partial\hat{\Omega}}{\partial\lambda}d\lambda. \tag{5.39}$$

On differentiating this with respect to $\lambda_a$ and using Equation $(5.33)_1$, we get

$$\frac{(\lambda_a^2\lambda_z - 1)}{\lambda_a}\frac{\partial P_T}{\partial \lambda_a} = \frac{1}{\lambda_a}\frac{\partial}{\partial \lambda}\hat{\Omega}(\lambda, \lambda_z, H_{l2}, H_{l3})|_{\lambda=\lambda_a} - \frac{1}{\lambda_b}\frac{\partial}{\partial \lambda}\hat{\Omega}(\lambda, \lambda_z, H_{l2}, H_{l3})|_{\lambda=\lambda_b}. \quad (5.40)$$

This is a generalization of the formula (15) obtained by [Haughton and Ogden, 1979b] in the context of pure elasticity. It is evident from the above equation that a necessary condition for the pressure turning points to exist is

$$\frac{\partial}{\partial \lambda}\left(\frac{1}{\lambda}\frac{\partial}{\partial \lambda}\hat{\Omega}(\lambda, \lambda_z, H_{l2}, H_{l3})\right) = 0, \qquad \text{for at least one} \quad \lambda \in (\lambda_b, \lambda_a)\,. \quad (5.41)$$

For rubber-like solids it is observed experimentally (at least for thin-walled tubes) that as $\lambda$ increases, the internal pressure increases up to a maximum, then decreases until it attains a minimum and then again increases monotonically until rupture. We can predict a similar behaviour for the total pressure $P_{\mathrm{T}}$ if the above condition is satisfied. To show this we use a generalization of the energy function used in [Haughton and Ogden, 1979b] to the magnetoelastic context.

$$\Omega = \hat{\Omega}(\lambda, \lambda_z, H_{l2}, H_{l3}) = \sum_{r=1}^{3}\frac{\mu_r}{\alpha_r}\left(\lambda^{\alpha_r} + \lambda_z^{\alpha_r} + \lambda^{-\alpha_r}\lambda_z^{-\alpha_r} - 3\right) + qK_5, \quad (5.42)$$

where the last term is $K_5 = (\lambda^2 H_{l2}^2 + \lambda_z^2 H_{l3}^2)$. Here, $\mu_r$'s are material constants with the dimensions of stress, $\alpha_r$'s are dimensionless constants while $q$ is a magnetoelastic coupling parameter with $q/\mu_0$ being dimensionless. Following the terminology in [Haughton and Ogden, 1979b], we call it a three-term magnetoelastic energy function.

The non-dimensionalized total internal pressure $\hat{P}_{\mathrm{T}} = P_{\mathrm{T}}/\mu$ calculated for this energy function is plotted in Figure 5.1 for different values of the underlying magnetic field and the following values of the energy function parameters (as used by [Haughton and Ogden, 1979b]):

$$\alpha_1 = 1.3, \quad \alpha_2 = 5, \quad \alpha_3 = -2, \quad \mu_1 = 1.491\mu, \quad \mu_2 = 0.003\mu,$$
$$\mu_3 = -0.023\mu, \quad \mu = 2.6 \times 10^5\,\mathrm{N/m^2}, \quad q = \mu_0/2 = 2\pi \times 10^{-7}\,\mathrm{N/A^2}. \quad (5.43)$$

The ratio of the internal radius to the external radius is taken as $A/B = 0.6$, while we plot for two values of the axial stretch, viz. $\lambda_z = 2$ and $\lambda_z = 0.7$. The axial magnetic field $H_3$ has no effect on the internal pressure for this energy function due to the nature of the expressions in (5.30) and (5.39). A reference value $H_0$ is taken for the azimuthal magnetic field so that at a radius $r$, $H_2$ is given by

$$H_2(r) = H_0 B/r. \quad (5.44)$$

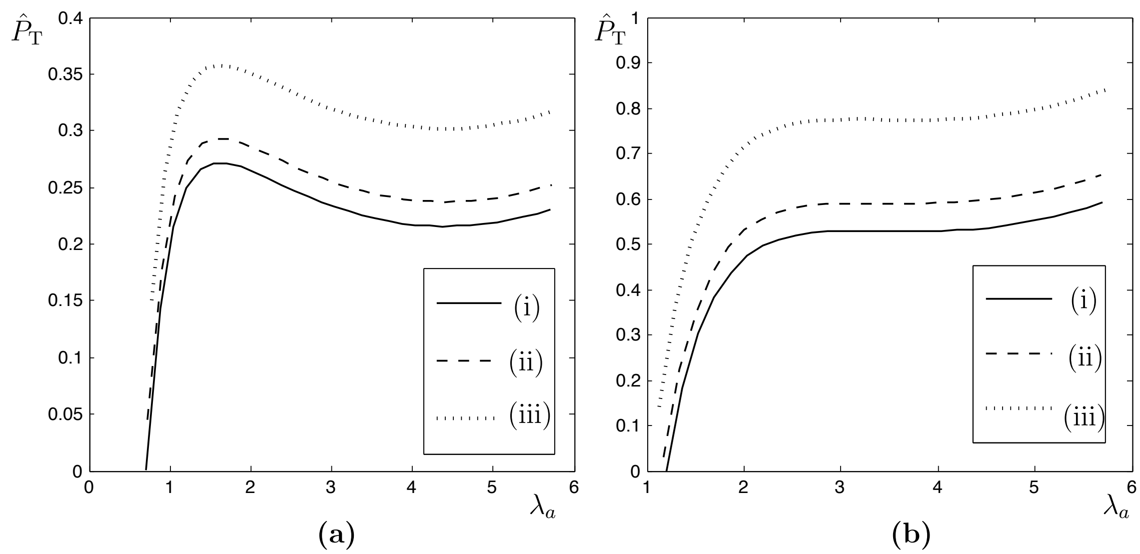



Figure 5.1: Plot of the non-dimensionalized total internal pressure $\hat{P}_{\mathrm{T}}$ against the stretch $\lambda_a$ for different values of an underlying azimuthal magnetic field for the three-term magnetoelastic energy function; $A/B = 0.6$. **(a)** $\lambda_z = 2$; **(b)** $\lambda_z = 0.7$. (i) $H_0 = 0$; (ii) $H_0 = 1 \times 10^5$ A/m; (iii) $H_0 = 2 \times 10^5$ A/m.

The plot starts from the value of $\lambda_a$ that corresponds to zero internal mechanical pressure and $\lambda_a$ is then increased quasi-statically.

At the starting point when there is no internal pressure, an underlying magnetic field tends to inflate the tube when $\lambda_z = 2$, and it tends to deflate the tube when $\lambda_z = 0.7$. A larger underlying magnetic field creates higher total internal pressure while stretching (increasing $\lambda_z$) the tube in the axial direction decreases $\hat{P}_T$. The behaviour of a rubber-like material is captured from the graphs since it is observed that as $\lambda_a$ increases, the internal pressure first rises, then falls and then increases monotonically. This agrees with the results in [Haughton and Ogden, 1979b] for the purely elastic case.

We now analyze this problem for a different form of magnetoelastic material defined by a Mooney–Rivlin magnetoelastic energy function given as

$$\Omega = \frac{\mu}{4}[(1+\gamma)(I_1 - 3) + (1-\gamma)(I_2 - 3)] + mK_5, \tag{5.45}$$

similar to the one defined in Equation (3.48) but with a different term corresponding to energy of the magnetic field. Here $\mu$ is the shear modulus of the material in the absence of a magnetic field, $\gamma$ is a dimensionless parameter in the range $-1 \leq \gamma \leq 1$, and $m$ is a magnetoelastic coupling constant such that $m/\mu_0$ is dimensionless. The total internal

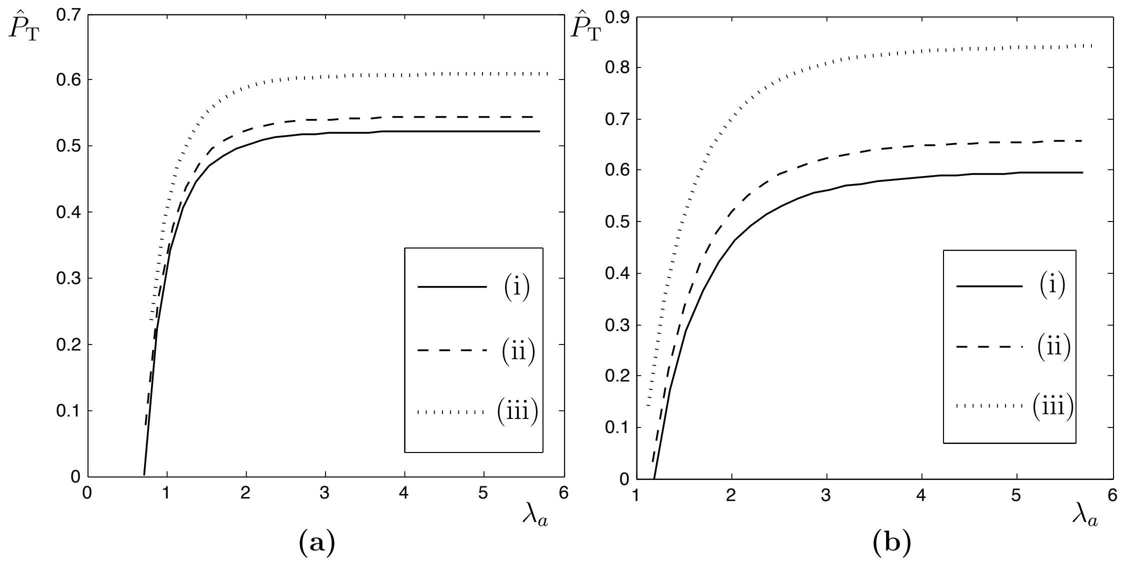



Figure 5.2: Plot of the non-dimensionalized total internal pressure $\hat{P}_{\mathrm{T}}$ against the stretch $\lambda_a$ for different values of an underlying azimuthal magnetic field for a Mooney–Rivlin magnetoelastic model; $A/B = 0.6$. **(a)** $\lambda_z = 2$; **(b)** $\lambda_z = 0.7$. (i) $H_0 = 0$; (ii) $H_0 = 1 \times 10^5$ A/m; (iii) $H_0 = 2 \times 10^5$ A/m.

pressure calculated using this energy function is plotted in Figure 5.2 for different magnetic fields and the following values of the material parameters:

$$\mu = 2.6 \times 10^5 \,\mathrm{N/m}^2, \quad \gamma = 0.3, \quad m = \mu_0/2. \tag{5.46}$$

The axial magnetic field $H_3$ has no effect on the internal pressure for this energy function and therefore we consider dependence only on $H_2$. For this function, unlike for the previous one, the total internal pressure increases monotonically as a function of $\lambda_a$. However, the effect of the magnetic field and the axial stretch is qualitatively the same as before, i.e. a larger magnetic field tends to increase while a larger axial stretch tends to decrease the total internal pressure.

### 5.2.2 Total axial load on the cylinder

The principal stress in the axial direction is given as

$$\tau_{33} = \lambda_3 \frac{\partial \Omega}{\partial \lambda_3} - p, \tag{5.47}$$

which on using Equation (C.18) from Appendix C can be rewritten as

$$\tau_{33} = \frac{1}{2}\left(2\lambda_z \frac{\partial \hat{\Omega}}{\partial \lambda_z} - \lambda \frac{\partial \hat{\Omega}}{\partial \lambda}\right) + \frac{1}{2r}\frac{d}{dr}\left(r^2 \tau_{11}\right). \tag{5.48}$$

This can also be expressed in terms of the scalar invariants defined in Equation (5.6) and their derivatives using Equation (C.19) as

$$\begin{aligned}\tau_{33} =& \Omega_1(3\lambda_3^2 - I_1) + \Omega_2(I_2 - 3\lambda_1^2\lambda_2^2) \\ &+ \Omega_5\left(2\lambda_3^4 H_3^2 - \lambda_2^4 H_2^2\right) + 2\Omega_6\left(2\lambda_3^6 H_3^2 - \lambda_2^6 H_2^2\right) + \frac{1}{2r}\frac{d}{dr}\left(r^2\tau_{11}\right). \end{aligned} \tag{5.49}$$

The total axial force on the cylinder is given as

$$N = \int_0^{2\pi}\int_a^b \tau_{33}\, r\, dr\, d\theta, \tag{5.50}$$

which on using the value of $\tau_{33}$ from Equation (5.48) can be rewritten as

$$N = \pi\int_a^b \left(2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda}\right) r dr + \pi\left(a^2 P_{\text{in}} - b^2 P_{\text{out}}\right) - \frac{\pi\mu_0}{2}H_3^2\left(b^2 - a^2\right), \tag{5.51}$$

which is similar to the formula (128) obtained by [Dorfmann and Ogden, 2005]. Using Equation (5.32), we can change the variable of integration in the first term from $r$ to $\lambda$ to get

$$\begin{aligned} N = \pi A^2\left(\lambda_a^2\lambda_z - 1\right)\int_{\lambda_b}^{\lambda_a} \frac{\lambda}{(\lambda^2\lambda_z - 1)^2}\left(2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda}\right) d\lambda \\ +\pi\left(a^2 P_{\text{in}} - b^2 P_{\text{out}}\right) - \frac{\pi\mu_0}{2}H_3^2\left(b^2 - a^2\right). \end{aligned} \tag{5.52}$$

Alternatively, we can use Equation (5.49) to write the expression for $\tau_{33}$ in terms of the invariants and their derivatives

$$\begin{aligned} N =& \pi\int_a^b [\Omega_1(3\lambda_3^2 - I_1) + \Omega_2(I_2 - 3\lambda_1^2\lambda_2^2) + \Omega_5\left(2\lambda_3^4 H_3^2 - \lambda_2^4 H_2^2\right) \\ &+ 2\Omega_6\left(2\lambda_3^6 H_3^2 - \lambda_2^6 H_2^2\right)] r\, dr + \pi\left(a^2 P_{\text{in}} - b^2 P_{\text{out}}\right) - \frac{\pi\mu_0}{2}H_3^2\left(b^2 - a^2\right), \end{aligned} \tag{5.53}$$

which on changing the variable of integration from $r$ to $\lambda$, gives

$$\begin{aligned} N = \pi A^2\left(\lambda_a^2\lambda_z - 1\right)\int_{\lambda_b}^{\lambda_a} \frac{\lambda}{(\lambda^2\lambda_z - 1)^2}[\Omega_1(3\lambda_3^2 - I_1) + \Omega_2(I_2 - 3\lambda_1^2\lambda_2^2) \\ +\Omega_5\left(2\lambda_3^4 H_3^2 - \lambda_2^4 H_2^2\right) + 2\Omega_6\left(2\lambda_3^6 H_3^2 - \lambda_2^6 H_2^2\right)] d\lambda \\ +\pi\left(a^2 P_{\text{in}} - b^2 P_{\text{out}}\right) - \frac{\pi\mu_0}{2}H_3^2\left(b^2 - a^2\right). \end{aligned} \tag{5.54}$$

We now study the dependence of the axial load $N$ on the underlying magnetic fields $H_2$ and $H_3$, the axial stretch $\lambda_z$, and the inflation given by the stretch $\lambda_a$. The numerical

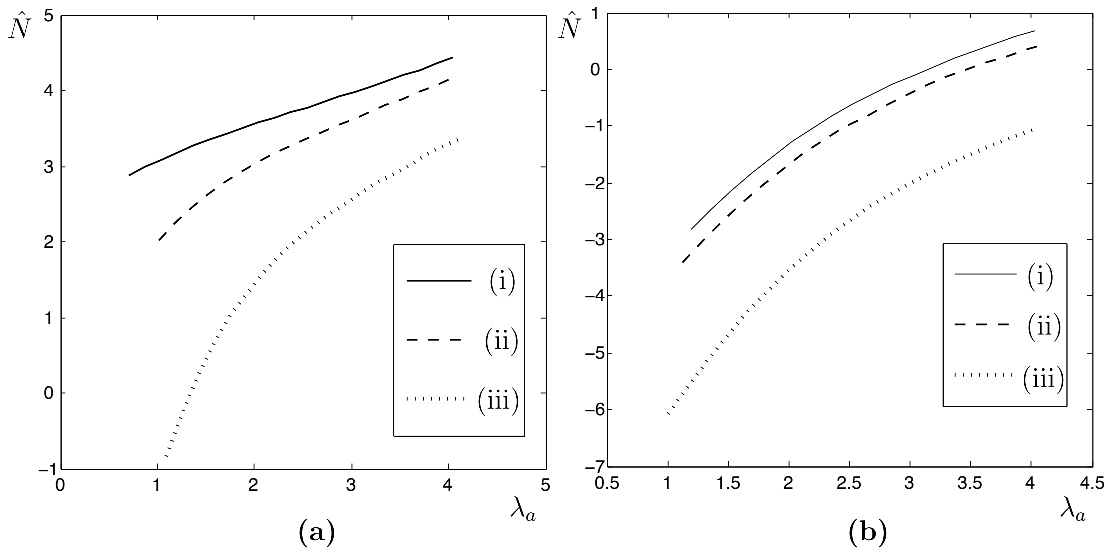



Figure 5.3: Plot of the non-dimensionalized axial force $\hat{N}$ against the stretch $\lambda_a$ for the three-term magnetoelastic energy function. **(a)** $\lambda_z = 2$; (i) $H_0 = 0$, (ii) $H_0 = 5 \times 10^5$ A/m, (iii) $H_0 = 1 \times 10^6$ A/m. **(b)** $\lambda_z = 0.7$; (i) $H_0 = 0$, (ii) $H_0 = 2 \times 10^5$ A/m, (iii) $H_0 = 5 \times 10^5$ A/m.

calculations are done for the values $A/B = 0.6$, $P_{\text{in}} = 0.2\mu$, and $P_{\text{out}} = 0$. We consider a tube of finite length in the presence of azimuthal magnetic field and take two values of axial stretch $\lambda_z = 0.7$ and $\lambda_z = 2$. In the presence of an axial magnetic field, we consider a tube of infinite length and consider only an extensional axial stretch $\lambda_z = 2$.

We plot the variation of the non-dimensionalized axial load $\hat{N} = N/\mu$ with $\lambda_a$ for the three-term magnetoelastic energy function in Figure 5.3. Increasing the radius of the tube, keeping $\lambda_z$ fixed, causes an increase in the axial load $\hat{N}$. In the absence of magnetic field, as expected, there is a positive (extensional) $\hat{N}$ for $\lambda_z > 1$ and a negative (compressional) $\hat{N}$ for $\lambda_z < 1$. An underlying magnetic field tends to create a compressional loading in the axial direction and hence reduces $\hat{N}$.

Results for the Mooney–Rivlin type magnetoelastic energy function are plotted in Figure 5.4. Similar to the previous case, $\hat{N}$ increases with an increase in $\lambda_a$ keeping $\lambda_z$ constant. However, there exists a value of $\lambda_a$ (say $\lambda^c$) depending on $\lambda_z$ such that when $\lambda_a < \lambda^c$, an underlying magnetic field decreases $\hat{N}$ while it increases $\hat{N}$ for $\lambda_a > \lambda^c$.

An underlying magnetic field in the axial direction tends to cause an extensional loading and hence a higher magnetic field increases $\hat{N}$. Similar behaviour is observed for both the kinds of materials as shown in Figure 5.5.

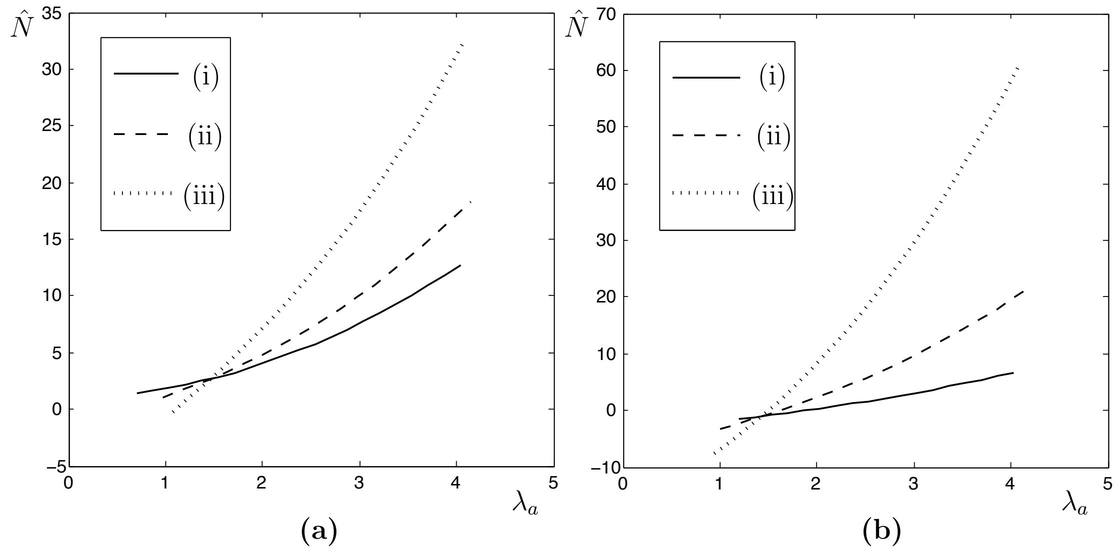



Figure 5.4: Plot of the non-dimensionalized axial force $\hat{N}$ against the stretch $\lambda_a$ for the Mooney–Rivlin type magnetoelastic material. **(a)** $\lambda_z = 2$; **(b)** $\lambda_z = 0.7$. (i) $H_0 = 0$, (ii) $H_0 = 5 \times 10^5$ A/m, (iii) $H_0 = 1 \times 10^6$ A/m.

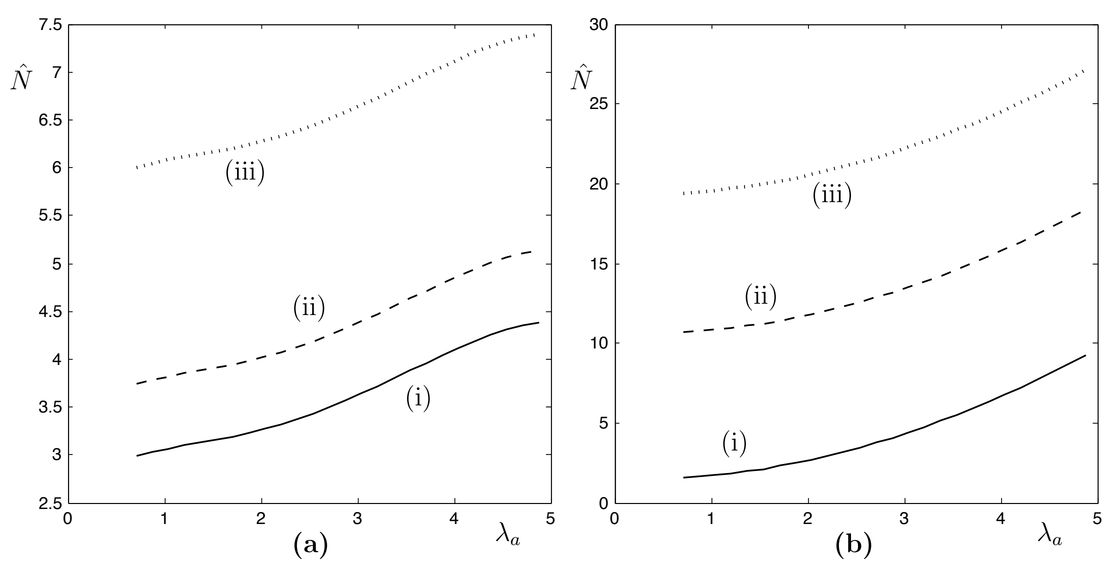



Figure 5.5: Plot of the non-dimensionalized axial force $\hat{N}$ against the stretch $\lambda_a$ in the presence of an axial magnetic field $H_3$ (in A/m). $\lambda_z = 2$. **(a)** Three-term magnetoelastic material; (i) $H_3 = 0$, (ii) $H_3 = 1 \times 10^5$ A/m, (iii) $H_3 = 2 \times 10^5$ A/m. **(b)** Mooney-Rivlin type magnetoelastic material; (i) $H_3 = 0$, (ii) $H_3 = 5 \times 10^5$ A/m, (iii) $H_3 = 7 \times 10^5$ A/m.

## 5.3 Incremental motions

We now consider time-dependent increments in the displacement and the magnetic field on top of the underlying finite deformation. Consider a small increment $\mathbf{u}$ in the deformation such that $\mathbf{u} = \{u_1, u_2, u_3\}$. The constraint of incompressibility requires $\mathbf{u}$ to satisfy the condition $\operatorname{div} \mathbf{u} = 0$. We consider only axisymmetric motions so that there is no dependence on $\theta$ and the components of the displacement gradient and the increment in the deformation gradient are given in matrix form by

$$[\mathbf{L}] = [\operatorname{grad} \mathbf{u}] = \begin{bmatrix} u_{1,1} & -u_2/r & u_{1,3} \\ u_{2,1} & u_1/r & u_{2,3} \\ u_{3,1} & 0 & u_{3,3} \end{bmatrix}, \tag{5.55}$$

$$[\dot{\mathbf{F}}] = [\text{Grad } \mathbf{u}] = [\mathbf{LF}] = \begin{bmatrix} \lambda^{-1}\lambda_z^{-1}u_{1,1} & -\lambda u_2/r & \lambda_z u_{1,3} \\ \lambda^{-1}\lambda_z^{-1}u_{2,1} & \lambda u_1/r & \lambda_z u_{2,3} \\ \lambda^{-1}\lambda_z^{-1}u_{3,1} & 0 & \lambda_z u_{3,3} \end{bmatrix}, \tag{5.56}$$

where here and henceforth we use the subscript $i$ followed by a comma to denote a derivative with respect to the $i$th coordinate, $i \in \{1, 3\}$.

In vacuum, the governing equations (2.45) can be written in component form as

$$\dot{H}^*_{1,1} + \frac{\dot{H}^*_1}{r} + \dot{H}^*_{3,3} = 0, \tag{5.57}$$

$$\dot{H}^*_{1,3} - \dot{H}^*_{3,1} = 0, \tag{5.58}$$

$$\dot{H}^*_{2,3} = 0, \tag{5.59}$$

$$\dot{H}^*_{2,1} + \frac{\dot{H}^*_2}{r} = 0, \tag{5.60}$$

while the incremental incompressibility constraint ($\operatorname{div} \mathbf{u} = 0$) is given as

$$u_{1,1} + \frac{u_1}{r} + u_{3,3} = 0. \tag{5.61}$$

Boundary conditions on the curved faces ($\mathbf{n} = \mathbf{e}_1$ for outer surface and $\mathbf{n} = -\mathbf{e}_1$ for inner surface) of the cylindrical tube are $H_2 = H^*_2, H_3 = H^*_3$, and given by equations

(2.47), (2.49), and (2.55) as

$$\begin{Bmatrix} \dot{T}_{011} \\ \dot{T}_{012} \\ \dot{T}_{013} \end{Bmatrix} = \begin{Bmatrix} \dot{\tau}^*_{11} \\ \dot{\tau}^*_{21} \\ \dot{\tau}^*_{31} \end{Bmatrix} - \begin{Bmatrix} \tau^*_{11}L_{11} + \tau^*_{12}L_{12} + \tau^*_{13}L_{13} \\ \tau^*_{21}L_{11} + \tau^*_{22}L_{12} + \tau^*_{23}L_{13} \\ \tau^*_{31}L_{11} + \tau^*_{32}L_{12} + \tau^*_{33}L_{13} \end{Bmatrix}, \tag{5.62}$$

$$\dot{B}_{l01} - \dot{B}^*_1 + B^*_2 L_{12} + B^*_3 L_{13} = 0, \tag{5.63}$$

$$\dot{H}_{l02} - \dot{H}^*_2 - H_2 L_{22} - H_3 L_{32} = 0, \tag{5.64}$$

$$\dot{H}_{l03} - \dot{H}^*_3 - H_2 L_{23} - H_3 L_{33} = 0. \tag{5.65}$$

In the case of a tube of finite length $L$ (and in the absence of $H_3$), boundary conditions at the ends ($z = 0, \lambda_z L$ and $\mathbf{n} = \pm\mathbf{e}_3$) for the incremental magnetic fields are given as

$$\dot{H}_{l01} - \dot{H}^*_1 - H_2 u_{2,1} = 0, \tag{5.66}$$

$$\dot{H}_{l02} - \dot{H}^*_2 - H_2 u_1/r = 0, \tag{5.67}$$

$$\dot{B}_{l03} - \dot{B}^*_3 = 0. \tag{5.68}$$

Boundary conditions for increments in the deformation and traction at the ends of the cylinder are given later in the sections that follow. We now consider the two cases of the underlying magnetic field being in the axial and in the azimuthal directions separately.

## 5.4 Axial magnetic field: $\mathbf{H} = (0, 0, H_3)$

In this first case, we consider an infinite tube with a uniform initial magnetic field in the axial direction. From the boundary condition $(2.14)_3$, we note that $H_3 = H^*_3$ at the lateral surfaces $r = a, b$. The Maxwell stress and its increment are given in component form by

$$[\boldsymbol{\tau}^*] = \frac{B^{*2}_3}{2\mu_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B^*_3}{\mu_0}\begin{bmatrix} -\dot{B}^*_3 & 0 & \dot{B}^*_1 \\ 0 & -\dot{B}^*_3 & \dot{B}^*_2 \\ \dot{B}^*_1 & \dot{B}^*_2 & \dot{B}^*_3 \end{bmatrix}. \tag{5.69}$$

In the presence of an axial magnetic field, the non-zero components of the magnetoelastic tensors are $\mathcal{A}_{0iiii}$, $\mathcal{A}_{0iijj}$, $\mathcal{A}_{0ijij}$, $\mathcal{A}_{0ijji}$, $\mathcal{C}_{0ii|3}$, $\mathcal{C}_{0i3|i}$, $\mathsf{K}_{0ii}$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Expanding the incremental governing equations $(2.43)_1$, (5.17), and (5.18) in component form, we obtain

$$\dot{H}_{l02,3} = 0, \quad \dot{H}_{l02,1} + \frac{\dot{H}_{l02}}{r} = 0, \tag{5.70}$$

$$\dot{H}_{l01,3} - \dot{H}_{l03,1} = 0, \tag{5.71}$$

$$\frac{1}{r}\left\{r\left(\mathcal{C}_{013|1}(u_{1,3}+u_{3,1})+\mathsf{K}_{011}\dot{H}_{l01}\right)\right\}_{,1} + \left(\mathcal{C}_{011|3}u_{1,1}+\mathcal{C}_{022|3}u_1/r+\mathcal{C}_{033|3}u_{3,3}+\mathsf{K}_{033}\dot{H}_{l03}\right)_{,3}=0, \tag{5.72}$$

$$\frac{1}{r}\left\{r\left((\mathcal{A}_{01111}+p)u_{1,1}+\mathcal{A}_{01122}u_1/r+\mathcal{A}_{01133}u_{3,3}+\mathcal{C}_{011|3}\dot{H}_{l03}\right)\right\}_{,1} -\frac{1}{r}\left(\mathcal{A}_{01122}u_{1,1}+(\mathcal{A}_{02222}+p)u_1/r+\mathcal{A}_{02233}u_{3,3}+\mathcal{C}_{022|3}\dot{H}_{l03}\right) + \left(\mathcal{A}_{03131}u_{1,3}+(\mathcal{A}_{03113}+p)u_{3,1}+\mathcal{C}_{013|1}\dot{H}_{l01}\right)_{,3}-\dot{p}_{,1}=\rho u_{1,tt}, \tag{5.73}$$

$$\frac{1}{r}\left[\left\{r\left(\mathcal{A}_{01212}u_{2,1}-(\mathcal{A}_{01221}+p)\frac{u_2}{r}\right)\right\}_{,1}-\mathcal{A}_{02121}u_2/r+(\mathcal{A}_{01221}+p)u_{2,1}\right] +\left\{\mathcal{A}_{03232}u_{2,3}+\mathcal{C}_{032|2}\dot{H}_{l02}\right\}_{,3}=\rho u_{2,tt}, \tag{5.74}$$

$$\frac{1}{r}\left\{r\left(\mathcal{A}_{01313}u_{3,1}+(\mathcal{A}_{01331}+p)u_{1,3}+\mathcal{C}_{013|1}\dot{H}_{l01}\right)\right\}_{,1} +\left\{\mathcal{A}_{01133}u_{1,1}+\mathcal{A}_{02233}u_1/r+(\mathcal{A}_{03333}+p)u_{3,3}+\mathcal{C}_{033|3}\dot{H}_{l03}\right\}_{,3}-\dot{p}_{,3}=\rho u_{3,tt}. \tag{5.75}$$

If we consider the purely elastic case (neglecting $\dot{\mathbf{H}}_{l0}$ and $\boldsymbol{\mathcal{C}}$) and only quasi-static bifurcations (no dependence on time), then the equations (5.73) and (5.75) reduce to equations (47) and (48) of [Haughton and Ogden, 1979b] after taking into account the differences in notation. We can eliminate $\dot{p}$ from equations (5.73) and (5.75) to get

$$-\frac{1}{r}\left\{r\left(\mathcal{A}_{01313}u_{3,1}+(\mathcal{A}_{01331}+p)u_{1,3}+\mathcal{C}_{013|1}\dot{H}_{l01}\right)\right\}_{,11}+\frac{1}{r^2}\{r(\mathcal{A}_{01313}u_{3,1} +(\mathcal{A}_{01331}+p)u_{1,3}+\mathcal{C}_{013|1}\dot{H}_{l01}\Big)\Big\}_{,1}+\frac{1}{r}\{r((\mathcal{A}_{01111}+p)u_{1,1}+\mathcal{A}_{01122}u_1/r +\mathcal{A}_{01133}u_{3,3}+\mathcal{C}_{011|3}\dot{H}_{l03}\Big)\Big\}_{,13}-(\mathcal{A}_{01133}u_{1,1}+\mathcal{A}_{02233}u_1/r+(\mathcal{A}_{03333}+p)u_{3,3} +\mathcal{C}_{033|3}\dot{H}_{l03}\Big)_{,13}+\left\{\mathcal{A}_{03131}u_{1,3}+(\mathcal{A}_{01331}+p)\,u_{3,1}+\mathcal{C}_{031|1}\dot{H}_{l01}\right\}_{,33} -\frac{1}{r}\left\{\mathcal{A}_{01122}u_{1,1}+(\mathcal{A}_{02222}+p)\,\frac{u_1}{r}+\mathcal{A}_{02233}u_{3,3}+\mathcal{C}_{022|3}\dot{H}_{l03}\right\}_{,3}=\rho(u_{1,3}-u_{3,1})_{,tt}. \tag{5.76}$$

It can be seen from the equations above that $u_2$ and $\dot{H}_{l02}$ are coupled with each other and are independent of $u_1, u_3, \dot{H}_{l01}$, and $\dot{H}_{l03}$ which are related to each other. We now consider both these cases separately.

### 5.4.1 Displacement in the $(r, z)$ plane

We now work only with the equations that have incremental motion in the radial and the axial directions. Considering that the magnetoelastic moduli tensors are uniform along the $z$ direction, we can rewrite Equation (5.76) as

$$A_1 u_{1,3} + A_2 u_{1,13} + A_3 u_{1,113} + A_4 u_{3,1} + A_5 u_{3,11} + A_6 u_{3,111} + A_7 u_{3,133} + A_8 u_{3,33}$$
$$A_9 u_{1,333} + A_{10} \dot{H}_{l01} + A_{11} \dot{H}_{l01,1} + A_{12} \dot{H}_{l01,11} + A_{13} \dot{H}_{l03,3} + A_{14} \dot{H}_{l03,13}$$
$$-A_{12} \dot{H}_{l01,33} = \rho \left(u_{1,3} - u_{3,1}\right)_{,tt}, \quad (5.77)$$

where

$$A_1 = -(\mathcal{A}_{01331} + p)_{,11} + \frac{1}{r}(\mathcal{A}_{01122} - \mathcal{A}_{02233} - \mathcal{A}_{01331} - p)_{,1} + \frac{1}{r^2}(\mathcal{A}_{01331} + \mathcal{A}_{02233}$$
$$-\mathcal{A}_{02222}), A_2 = (\mathcal{A}_{01111} - 2\mathcal{A}_{01331} - \mathcal{A}_{01133} - p)_{,1} + \frac{1}{r}(\mathcal{A}_{01111} - \mathcal{A}_{01331} - \mathcal{A}_{02233}),$$
$$A_3 = \mathcal{A}_{01111} - \mathcal{A}_{01313} - \mathcal{A}_{01133}, \quad A_4 = -\mathcal{A}_{01313,11} - \frac{\mathcal{A}_{01313,1}}{r} + \frac{\mathcal{A}_{01313}}{r^2},$$
$$A_5 = -\frac{\mathcal{A}_{01313}}{r} - 2\mathcal{A}_{01313,1}, \quad A_6 = -\mathcal{A}_{01313}, \quad A_7 = \mathcal{A}_{01133} - \mathcal{A}_{03333} + \mathcal{A}_{01331},$$
$$A_8 = (\mathcal{A}_{01133} - \mathcal{A}_{03333} - p)_{,1} + \frac{\mathcal{A}_{01133} - \mathcal{A}_{02233}}{r}, \quad A_9 = \mathcal{A}_{03131},$$
$$A_{10} = \frac{\mathcal{C}_{013|1}}{r^2} - \frac{\mathcal{C}_{013|1,1}}{r} - \mathcal{C}_{013|1,11}, \quad A_{11} = -\frac{\mathcal{C}_{013|1}}{r} - 2\mathcal{C}_{013|1,1}, \quad A_{12} = -\mathcal{C}_{013|1},$$
$$A_{13} = \left(\mathcal{C}_{011|3} - \mathcal{C}_{033|3}\right)_{,1} + \frac{\mathcal{C}_{011|3} - \mathcal{C}_{022|3}}{r}, \quad A_{14} = \mathcal{C}_{011|3} - \mathcal{C}_{033|3}. \quad (5.78)$$

Equation (5.72) gives

$$\left(\mathcal{C}_{013|1,1} + \frac{\mathcal{C}_{013|1}}{r}\right) u_{3,1} + \left(\mathcal{C}_{013|1,1} + \frac{\mathcal{C}_{013|1} + \mathcal{C}_{022|3}}{r}\right) u_{1,3} + \left(\mathcal{C}_{013|1} + \mathcal{C}_{011|3}\right) u_{1,13}$$
$$+\mathcal{C}_{013|1} u_{3,11} + \mathcal{C}_{033|3} u_{3,33} + \left(\mathsf{K}_{011,1} + \frac{\mathsf{K}_{011}}{r}\right) \dot{H}_{l01} + \mathsf{K}_{011} \dot{H}_{l01,1} + \mathsf{K}_{033} \dot{H}_{l03,3} = 0, \quad (5.79)$$

while from the boundary conditions (5.62)–(5.65), we get

$$\mathcal{A}_{01111} u_{1,1} + \mathcal{A}_{01122} \frac{u_1}{r} + \mathcal{A}_{01133} u_{3,3} + \mathcal{C}_{011|3} \dot{H}_{l03} - \dot{p} + p u_{1,1}$$
$$= -\mu_0 H_3 \dot{H}_3^* + \frac{\mu_0 H_3^2}{2} u_{1,1}, \quad (5.80)$$

$$\mathcal{A}_{01313} u_{3,1} + \mathcal{A}_{01331} u_{1,3} + \mathcal{C}_{013|1} \dot{H}_{l01} + p u_{1,3} = \mu_0 H_3 \dot{H}_1^* - \frac{\mu_0 H_3^2}{2} u_{1,3}, \quad (5.81)$$

$$-\mathcal{C}_{013|1} \left(u_{1,3} + u_{3,1}\right) - \mathsf{K}_{011} \dot{H}_{l01} - \mu_0 \dot{H}_1^* + \mu_0 H_3 u_{1,3} = 0, \quad (5.82)$$

$$\dot{H}_{l03} - H_3 u_{3,3} - \dot{H}_3^* = 0, \quad (5.83)$$

at $r = a$ and $r = b$.

We differentiate Equation (5.80) with respect to $z$ and replace $\dot{p}_{,3}$ using Equation (5.75) to get

$$\xi_1 u_{1,3} + \xi_2 u_{1,13} + \xi_3 u_{3,1} + \xi_4 u_{3,33} + A_6 u_{3,11} + \xi_5 \dot{H}_{l01} + A_{12} \dot{H}_{l01,1} + A_{14} \dot{H}_{l03,3}$$
$$+\mu_0 H_3 \dot{H}_{3,3}^* + \rho u_{3,tt} = 0, \quad (5.84)$$

where

$$\xi_1 = \frac{1}{r}\left(\mathcal{A}_{01122} - \mathcal{A}_{01331} - \mathcal{A}_{02233} - p\right) - \left(\mathcal{A}_{01331} + p\right)_{,1},$$
$$\xi_2 = \mathcal{A}_{01111} - \mathcal{A}_{01331} - \mathcal{A}_{01133} - \frac{\mu_0 H_3^2}{2}, \quad \xi_3 = -\left(\mathcal{A}_{01313,1} + \frac{\mathcal{A}_{01313}}{r}\right),$$
$$\xi_4 = \mathcal{A}_{01133} - \mathcal{A}_{03333} - p, \quad \xi_5 = -\left(\mathcal{C}_{013|1,1} + \frac{\mathcal{C}_{013|1}}{r}\right). \tag{5.85}$$

Since $u_1$ and $u_3$ satisfy Equation (5.61), $\dot{H}_{l01}$ and $\dot{H}_{l03}$ satisfy Equation (5.71), and $\dot{H}_1^*$ and $\dot{H}_3^*$ satisfy Equation (5.58), we can define the potentials $\phi(r,z,t)$, $\psi(r,z,t)$, and $\psi^*(r,z,t)$ such that

$$u_1 = \frac{\phi_{,3}}{r}, \quad u_3 = \frac{-\phi_{,1}}{r}, \quad \dot{H}_{l01} = \psi_{,1}, \quad \dot{H}_{l03} = \psi_{,3}, \quad \dot{H}_1^* = \psi^*_{,1}, \quad \dot{H}_3^* = \psi^*_{,3}. \tag{5.86}$$

Substituting the potentials and their derivatives in the governing equations, we get

$$\begin{aligned}
&\phi_{,1}\left(\frac{A_4}{r^2} - 2\frac{A_5}{r^3} + 6\frac{A_6}{r^4}\right) + \phi_{,11}\left(-\frac{A_4}{r} + 2\frac{A_5}{r^2} - 6\frac{A_6}{r^3}\right) + \phi_{,111}\left(-\frac{A_5}{r} + 3\frac{A_6}{r^2}\right) \\
&\quad - \frac{A_6}{r}\phi_{,1111} + \phi_{,33}\left(\frac{A_1}{r} - \frac{A_2}{r^2} + 2\frac{A_3}{r^3}\right) + \phi_{,133}\left(\frac{A_2}{r} - 2\frac{A_3}{r^2} - \frac{A_8}{r} + \frac{A_7}{r^2}\right) \\
&\quad + \phi_{,1133}\left(\frac{A_3}{r} - \frac{A_7}{r}\right) + \frac{A_9}{r}\phi_{,3333} + A_{10}\psi_{,1} + A_{11}\psi_{,11} + A_{12}\psi_{,111} + A_{13}\psi_{,33} \\
&\quad + \left(A_{14} - A_{12}\right)\psi_{,133} = \rho\left(\frac{\phi_{,33}}{r} + \frac{\phi_{,11}}{r} - \frac{\phi_{,1}}{r^2}\right)_{,tt},
\end{aligned} \tag{5.87}$$

$$\begin{aligned}
&\phi_{,1}\left(\frac{\mathcal{C}_{013|1,1}}{r^2} - \frac{\mathcal{C}_{013|1}}{r^3}\right) + \phi_{,11}\left(-\frac{\mathcal{C}_{013|1,1}}{r} + \frac{\mathcal{C}_{013|1}}{r^2}\right) - \frac{\mathcal{C}_{013|1}}{r}\phi_{,111} \\
&\quad + \left(\frac{\mathcal{C}_{013|1,1}}{r} + \frac{\mathcal{C}_{022|3} - \mathcal{C}_{011|3}}{r^2}\right)\phi_{,33} + \frac{\phi_{,133}}{r}\left(\mathcal{C}_{013|1} + \mathcal{C}_{011|3} - \mathcal{C}_{033|3}\right) \\
&\quad + \left(\mathsf{K}_{011,1} + \frac{\mathsf{K}_{011}}{r}\right)\psi_{,1} + \mathsf{K}_{011}\psi_{,11} + \mathsf{K}_{033}\psi_{,33} = 0,
\end{aligned} \tag{5.88}$$

for $a < r < b$ and

$$\psi^*_{,11} + \frac{1}{r}\psi^*_{,1} + \psi^*_{,33} = 0, \tag{5.89}$$

for $r < a$ and $r > b$.

The boundary conditions become

$$\frac{\mathcal{A}_{01313}}{r^2}\phi_{,1} - \frac{\mathcal{A}_{01313}}{r}\phi_{,11} + \phi_{,33}\left(\frac{\mathcal{A}_{01331} + p}{r} + \frac{\mu_0 H_3^2}{2r}\right) + \mathcal{C}_{013|1}\psi_{,1} - \mu_0 H_3 \psi^*_{,1} = 0, \tag{5.90}$$

$$\frac{\mathcal{C}_{013|1}}{r^2}\phi_{,1} - \frac{\mathcal{C}_{013|1}}{r}\phi_{,11} + \left(\frac{\mathcal{C}_{013|1} - \mu_0 H_3}{r}\right)\phi_{,33} + \mathsf{K}_{011}\psi_{,1} + \mu_0\psi^*_{,1} = 0, \tag{5.91}$$

$$\psi_{,3} + \frac{H_3^*}{r}\phi_{,13} - \psi^*_{,3} = 0, \tag{5.92}$$

$$\phi_{,1}\left(\frac{\xi_3}{r^2}-2\frac{A_6}{r^3}\right)+\phi_{,11}\left(\frac{-\xi_3}{r}+2\frac{A_6}{r^2}\right)+\left(-\frac{\xi_2}{r^2}+\frac{\xi_1}{r}\right)\phi_{,33}+\phi_{,133}\left(\frac{\xi_2-\xi_4}{r}\right)$$
$$-\frac{A_6}{r}\phi_{,111}+\xi_5\psi_{,1}+A_{12}\psi_{,11}+A_{14}\psi_{,33}+\mu_0H_3\psi^*_{,33}-\frac{\rho}{r}\phi_{,1tt}=0, \qquad (5.93)$$

at $r=a$ and $r=b$.

### Wave propagation solutions

For the above partial differential equations, by separation of variables we consider wave type solutions of the form

$$\phi = F(r)\exp\left(ikz-i\omega t\right), \quad a<r<b, \qquad (5.94)$$

$$\psi = G(r)\exp\left(ikz-i\omega t\right), \quad a<r<b, \qquad (5.95)$$

$$\psi^* = M_1(r)\exp\left(ikz-i\omega t\right), \quad r<a, \qquad (5.96)$$

$$\psi^* = M_2(r)\exp\left(ikz-i\omega t\right), \quad r>b, \qquad (5.97)$$

which convert the equations to a system of coupled ODEs as follows

$$\left\{\frac{k^4}{r}A_9-k^2\left(\frac{A_1}{r}-\frac{A_2}{r^2}+2\frac{A_3}{r^3}+\frac{\rho\omega^2}{r}\right)\right\}F$$
$$+\left\{\frac{A_4}{r^2}-2\frac{A_5}{r^3}+6\frac{A_6}{r^4}-k^2\left(\frac{A_2-A_8}{r}+\frac{A_7-2A_3}{r^2}\right)-\frac{\rho\omega^2}{r}\right\}F'$$
$$+\left\{-\frac{A_4}{r}-2\frac{A_5}{r^2}-6\frac{A_6}{r^3}-k^2\frac{A_3-A_7}{r}+\frac{\rho\omega^2}{r}\right\}F''+\left(3\frac{A_6}{r^2}-\frac{A_5}{r}\right)F'''$$
$$-\frac{A_6}{r}F''''-k^2A_{13}G+\left\{A_{10}+k^2\left(A_{12}-A_{14}\right)\right\}G'+A_{11}G''+A_{12}G'''=0, \qquad (5.98)$$

$$-k^2\left(\frac{\mathcal{C}_{013|1,1}}{r}+\frac{\mathcal{C}_{022|3}-\mathcal{C}_{011|3}}{r^2}\right)F+\left(-\frac{\mathcal{C}_{013|1,1}}{r}+\frac{\mathcal{C}_{013|1}}{r^2}\right)F''$$
$$+\left\{-\frac{\mathcal{C}_{013|1}}{r^3}+\frac{\mathcal{C}_{013|1,1}}{r^2}-\frac{k^2}{r}\left(\mathcal{C}_{013|1}+\mathcal{C}_{011|3}-\mathcal{C}_{033|3}\right)\right\}F'-\frac{\mathcal{C}_{013|1}}{r}F'''-k^2\mathsf{K}_{033}G$$
$$+\left(\mathsf{K}_{011,1}+\frac{\mathsf{K}_{011}}{r}\right)G'+\mathsf{K}_{011}G''=0, \qquad (5.99)$$

for $a<r<b$, and

$$M_1''+\frac{1}{r}M_1'-k^2M_1=0, \quad r<a, \quad M_2''+\frac{1}{r}M_2'-k^2M_2=0, \quad r>b. \qquad (5.100)$$

Here and henceforth, a prime denotes a derivative with respect to $r$. The boundary conditions reduce to

$$-k^2\left(\frac{\mathcal{A}_{01331}+p}{a}+\frac{\mu_0H_3^2}{2a}\right)F+\frac{\mathcal{A}_{01313}}{a^2}F'-\frac{\mathcal{A}_{01313}}{a}F''$$
$$+\mathcal{C}_{013|1}G'-\mu_0H_3M_1'=0, \qquad (5.101)$$

$$-\frac{k^2}{a}\left(\mathcal{C}_{013|1}-\mu_0 H_3\right)F+\frac{\mathcal{C}_{013|1}}{a^2}F'-\frac{\mathcal{C}_{013|1}}{a}F''+\mathsf{K}_{011}G'+\mu_0 M_1'=0, \tag{5.102}$$

$$G+\frac{H_3}{a}F'-M_1=0, \tag{5.103}$$

$$-k^2\left(-\frac{\xi_2}{a^2}+\frac{\xi_1}{a}\right)F+\left\{\frac{\xi_3}{a^2}-2\frac{A_6}{a^3}-\frac{k^2}{a}(\xi_2-\xi_4)+\frac{\rho c^2k^2}{a}\right\}F'$$
$$+\left(-\frac{\xi_3}{a}+2\frac{A_6}{a^2}\right)F''-\frac{A_6}{a}F'''-k^2A_{14}G+\xi_5G'+A_{12}G''-k^2\mu_0H_3M_1=0, \tag{5.104}$$

at $r=a$ and

$$-k^2\left(\frac{\mathcal{A}_{01331}+p}{b}+\frac{\mu_0H_3^2}{2b}\right)F+\frac{\mathcal{A}_{01313}}{b^2}F'-\frac{\mathcal{A}_{01313}}{b}F''$$
$$+\mathcal{C}_{013|1}G'-\mu_0H_3M_2'=0, \tag{5.105}$$

$$-\frac{k^2}{b}\left(\mathcal{C}_{013|1}-\mu_0 H_3\right)F+\frac{\mathcal{C}_{013|1}}{b^2}F'-\frac{\mathcal{C}_{013|1}}{b}F''+\mathsf{K}_{011}G'+\mu_0 M_2'=0, \tag{5.106}$$

$$G+\frac{H_3}{b}F'-M_2=0, \tag{5.107}$$

$$-k^2\left(-\frac{\xi_2}{b^2}+\frac{\xi_1}{b}\right)F+\left\{\frac{\xi_3}{b^2}-2\frac{A_6}{b^3}-\frac{k^2}{b}(\xi_2-\xi_4)+\frac{\rho c^2k^2}{b}\right\}F'$$
$$+\left(-\frac{\xi_3}{b}+2\frac{A_6}{b^2}\right)F''-\frac{A_6}{b}F'''-k^2A_{14}G+\xi_5G'+A_{12}G''-k^2\mu_0H_3M_2=0, \tag{5.108}$$

at $r=b$.

Let the governing equations (5.98) and (5.99) be written in the form

$$p_1F+p_2F'+p_3F''+p_4F'''+p_5F''''+p_6G+p_7G'+p_8G''+p_9G'''=0, \tag{5.109}$$
$$q_1F+q_2F'+q_3F''+q_4F'''+q_5G+q_6G'+q_7G''=0, \tag{5.110}$$

where $p_i$'s and $q_i$'s are the coefficients in (5.98) and (5.99), and let

$$y_1=F,\quad y_2=F',\quad y_3=F'',\quad y_4=F''',\quad y_5=G,\quad y_6=G',\quad y_7=G'', \tag{5.111}$$

then the above equations can be written as a system of first order ODEs of the form

$$\Pi\mathbf{y}'=\mathbf{g}. \tag{5.112}$$

Here $\Pi, \mathbf{y}'$, and $\mathbf{g}$ are matrices of size $7 \times 7$, $7 \times 1$ and $7 \times 1$, respectively, and are given by

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p_5 & 0 & 0 & p_9 \\ 0 & 0 & q_4 & 0 & 0 & q_7 & 0 \end{bmatrix}, \quad \mathbf{y}' = \begin{Bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \\ y_6' \\ y_7' \end{Bmatrix}, \tag{5.113}$$

$$\mathbf{g} = \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_6 \\ y_7 \\ -p_1y_1 - p_2y_2 - p_3y_3 - p_4y_4 - p_6y_5 - p_7y_6 - p_8y_7 \\ -q_1y_1 - q_2y_2 - q_3y_3 - q_5y_5 - q_6y_6 \end{Bmatrix}. \tag{5.114}$$

Here we have eight boundary conditions but have to solve for nine variables, viz. $y_1, ..., y_7, M_1$, and $M_2$. Hence we have infinitely many solutions to this problem and a unique solution is only possible when $H_2 = 0$. Vanishing of the underlying magnetic field would cause the increments in magnetic field to be identically zero ($G = M_1 = M_2 \equiv 0$) and only the increments in mechanical displacement $F$ remain. Such purely elastic waves have already been studied in papers, such as those by, [Vaughan, 1979] and [Haughton, 1984].

### 5.4.2 Displacement in the azimuthal direction

Now considering the set of equations that contain only $u_2$ and $\dot{H}_{l02}$, the governing equations (5.70) and (5.74) are written in component form as

$$-\left(\frac{\mathcal{A}_{01212,1} - \tau_{11,1}}{r} + \frac{\mathcal{A}_{02121}}{r^2}\right)u_2 + \left(\mathcal{A}_{01212,1} + \frac{\mathcal{A}_{01212}}{r}\right)u_{2,1} + \mathcal{A}_{01212}u_{2,11}$$
$$+\mathcal{A}_{03232}u_{2,33} = \rho u_{2,tt}, \tag{5.115}$$

$$\dot{H}_{l02,3} = 0, \quad \dot{H}_{l02,1} + \frac{\dot{H}_{l02}}{r} = 0, \tag{5.116}$$

in $a < r < b$, along with (5.59) and (5.60) in vacuum. The boundary conditions (5.62) and (5.64) give

$$\mathcal{A}_{01212} u_{2,1} - \left( \mathcal{A}_{01221} + p - \frac{\mu_0 H_3^2}{2} \right) \frac{u_2}{r} = 0, \tag{5.117}$$

$$\dot{H}_{l02} - \dot{H}_2^* = 0, \tag{5.118}$$

at $r = a$ and $r = b$.

Due to $(5.70)_1$, the governing equations for $u_2$ and $\dot{H}_{l02}$ are decoupled. So, equation (5.115) is of the form what one would normally obtain for a purely mechanical problem except that the coefficients still depend on $H_3$. The governing equations for $\dot{H}_{l02}$ and $\dot{H}_2^*$ can be integrated analytically to give $\dot{H}_{l02} = c_1/r$ in $a < r < b$, $\dot{H}_2^* = c_2/r$ in $r < a$, and $\dot{H}_2^* = c_3/r$ in $r > b$. The boundary conditions (5.118) at $r = a, b$ require that $c_1 = c_2 = c_3$.

For the mechanical displacement, if we consider propagating wave type solution of the form

$$u_2 = F(r) \exp\left(\mathrm{i}kz - \mathrm{i}\omega t\right), \tag{5.119}$$

the governing equations and boundary conditions are transformed to

$$\left( \frac{-\mathcal{A}_{01212,1} + \tau_{11,1}}{r} - \frac{\mathcal{A}_{02121}}{r^2} - k^2 \mathcal{A}_{03232} + \rho\omega^2 \right) F + \left( \mathcal{A}_{01212,1} + \frac{\mathcal{A}_{01212}}{r} \right) F' + \mathcal{A}_{01212} F'' = 0, \tag{5.120}$$

for $a < r < b$, and

$$\mathcal{A}_{01212} F' - \left( \mathcal{A}_{01212} - \tau_{11} - \frac{\mu_0 H_3^2}{2} \right) \frac{F}{r} = 0, \tag{5.121}$$

at $r = a, b$.

The above set of equations can be non-dimensionalized by defining

$$\zeta = \frac{\rho\omega^2}{k^2\mu}, \quad \hat{r} = \frac{r}{A}, \quad \hat{k} = Ak, \quad \hat{F}(\hat{r}) = \frac{F(r)}{A} \quad \hat{\boldsymbol{\mathcal{A}}} = \frac{\boldsymbol{\mathcal{A}}}{\mu}, \quad \hat{\boldsymbol{\tau}} = \frac{\boldsymbol{\tau}}{\mu}, \tag{5.122}$$

and are rewritten as

$$\left\{ \frac{1}{\hat{r}\hat{k}^2} \left( \hat{\tau}'_{11} - \hat{\mathcal{A}}'_{01212} \right) - \frac{\hat{\mathcal{A}}_{02121}}{\hat{r}^2 \hat{k}^2} - \hat{\mathcal{A}}_{03232} + \zeta \right\} \hat{F} + \left( \hat{\mathcal{A}}'_{01212} + \frac{\hat{\mathcal{A}}_{01212}}{\hat{r}} \right) \frac{\hat{F}'}{\hat{k}^2} + \frac{\hat{\mathcal{A}}_{01212}}{\hat{k}^2} \hat{F}'' = 0, \tag{5.123}$$

for $\hat{a} < \hat{r} < \hat{b}$, and

$$\hat{\mathcal{A}}_{01212} \hat{F}' - \left( \hat{\mathcal{A}}_{01212} - \hat{\tau}_{11} - \frac{\mu_0 H_3^2}{2\mu} \right) \frac{\hat{F}}{\hat{r}} = 0, \tag{5.124}$$

at $\hat{r} = \hat{a}, \hat{b}$.

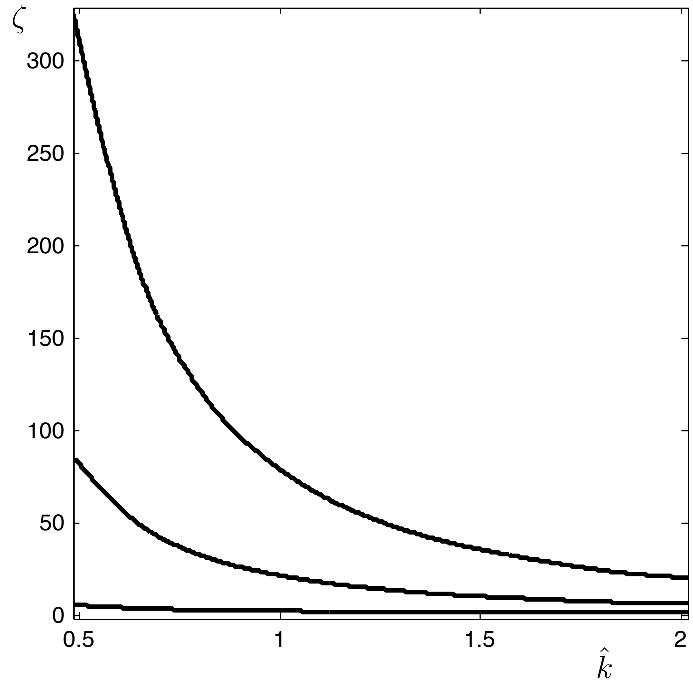



Figure 5.6: Plot of the non-dimensionalized squared wave speed $\zeta$ against the non-dimensionalized wave number $\hat{k}$ for the first three modes of wave propagation for $\lambda_z = 1.5, P_{\text{in}} = 0.2\mu, H_3 = 1 \times 10^5$ A/m.

The above equations are converted to a system of two first order ODEs and solved numerically using the algorithm described in Section 5.5.2. Variation of wave speed with various deformation parameters is illustrated in the following plots.

We observe existence of more than one mode of wave propagation due to presence of a finite length scale $(B - A)$ in the problem. These are illustrated in Figure 5.6 for the Mooney–Rivlin type magnetoelastic material of Equation (5.45) and the following material and deformation parameters

$$\mu = 2.6 \times 10^5\,\text{N/m}^2, \quad \gamma = 0.3, \quad m = \mu_0, \quad A/B = 0.6,$$
$$\lambda_z = 1.5, \quad P_{\text{in}} = 0.2\mu, \quad H_3 = 1 \times 10^5\,\text{A/m}. \tag{5.125}$$

Dispersion relations for different magnetic fields are plotted in Figure 5.7 and it is observed that a large magnetic field tends to increase the speed of wave propagation. In general, the wave speed decreases with an increasing wave number. Figure 5.8 illustrates the variation of wave speed with the underlying axial stretch for the deformation parameters in Equation (5.125) and $\hat{k} = 1$. Stretching the tube in the axial direction causes an

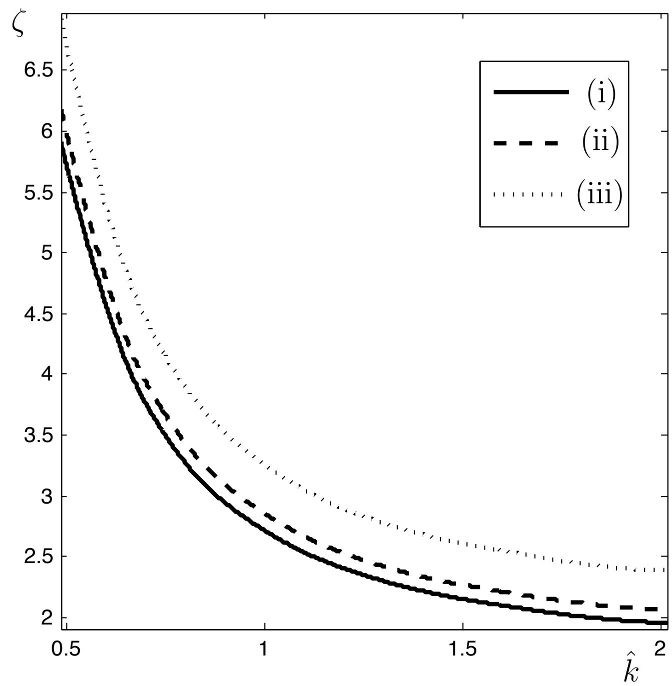



Figure 5.7: Plot of the non-dimensionalized squared wave speed $\zeta$ against the non-dimensionalized wave number $\hat{k}$ for an underlying magnetic field in axial direction. (i) $H_3 = 0$, (ii) $H_3 = 1 \times 10^5$ A/m, (iii) $H_3 = 2 \times 10^5$ A/m.

increase in the speed of wave propagation.

In Figure 5.9, we consider the effect of the magnetoelastic coupling parameter $m$ through the underlying magnetic field on the wave speed. Wave speed increases with an increase in $H_3$ and is also higher for a material with a larger value of $m$. As $H_3$ tends to zero, the problem tends to a purely elastic case and $m$ has no effect on $\zeta$. Hence the three curves in Figure 5.9 converge in that region.

## 5.5 Azimuthal magnetic field: $\mathbf{H} = (0, H_2, 0)$

We now consider an initial magnetic field in the azimuthal direction. Such a field can be generated by a long current carrying wire placed on the axis of the hollow tube so that $H_2$ has dependence only on $r$. From the boundary condition $(2.14)_3$, we have $H_2 = H_2^*$ at the lateral surfaces $r = a, r = b$. We shall consider the tube to be of infinite length for the current problem.

For this specialization, the Maxwell stress and its increment are given in the component

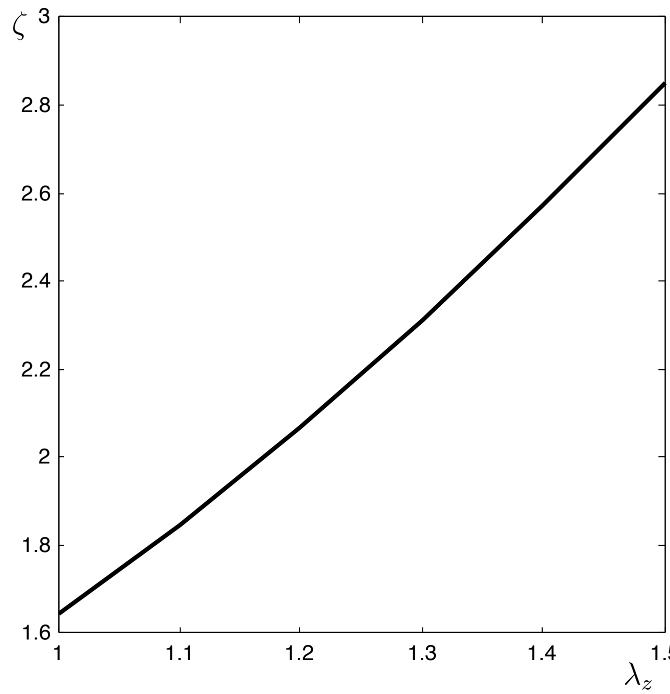



Figure 5.8: Variation of wave speed with axial stretch for $P_{\text{in}} = 0.2\mu, H_3 = 1 \times 10^5$ A/m.

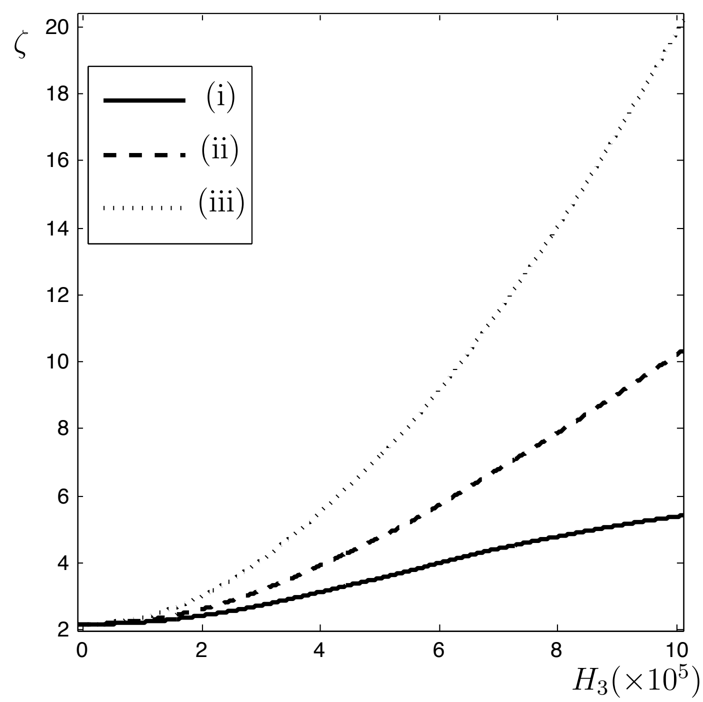



Figure 5.9: Variation of wave speed with the underlying magnetic field for different values of the magnetoelastic coupling parameter $m$ and $\lambda_z = 1.5, P_{\text{in}} = 0.2\mu$. $H_3$ is in $10^5$ A/m, (i) $m = \mu_0/2$, (ii) $m = \mu_0$, (iii) $m = 2\mu_0$.

form by

$$[\boldsymbol{\tau}^*] = \frac{B_2^{*2}}{2\mu_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B_2^*}{\mu_0}\begin{bmatrix} -\dot{B}_2^* & \dot{B}_1^* & 0 \\ \dot{B}_1^* & \dot{B}_2^* & \dot{B}_3^* \\ 0 & \dot{B}_3^* & -\dot{B}_2^* \end{bmatrix}. \tag{5.126}$$

To work with the governing equations (5.17) and (5.18) in the presence of an azimuthal magnetic field, the non-zero components of the magnetoelastic tensors are $\mathcal{A}_{0iiii}$, $\mathcal{A}_{0iijj}$, $\mathcal{A}_{0ijij}$, $\mathcal{A}_{0ijji}$, $\mathcal{C}_{0ii|2}$, $\mathcal{C}_{0i2|i}$, $\mathsf{K}_{0ii}$ for $i,j \in \{1,2,3\}$ and $i \neq j$. Explicit formulas for these components for the generalized Mooney–Rivlin magnetoelastic material given by Equation (5.45) are given below as

$$\mathcal{A}_{01111} = 2\lambda_1^2\left\{\Omega_1 + (\lambda_2^2+\lambda_3^2)\Omega_2\right\}, \tag{5.127}$$

$$\mathcal{A}_{02222} = 2\lambda_2^2\left\{\Omega_1 + (\lambda_1^2+\lambda_3^2)\Omega_2 + \lambda_2^2 H_2^2\Omega_5\right\}, \tag{5.128}$$

$$\mathcal{A}_{03333} = 2\lambda_3^2\left\{\Omega_1 + (\lambda_1^2+\lambda_2^2)\Omega_2\right\}, \tag{5.129}$$

$$\mathcal{A}_{01122} = 4\lambda_1^2\lambda_2^2\Omega_2, \quad \mathcal{A}_{01133} = 4\lambda_1^2\lambda_3^2\Omega_2, \quad \mathcal{A}_{02233} = 4\lambda_2^2\lambda_3^2\Omega_2, \tag{5.130}$$

$$\mathcal{A}_{01212} = 2\lambda_1^2\left(\Omega_1 + \lambda_3^2\Omega_2\right), \quad \mathcal{A}_{02121} = 2\lambda_2^2\left(\Omega_1 + \lambda_3^2\Omega_2\right) + 2\lambda_2^4 H_2^2\Omega_5, \tag{5.131}$$

$$\mathcal{A}_{01313} = 2\lambda_1^2\left(\Omega_1 + \lambda_2^2\Omega_2\right), \quad \mathcal{A}_{03131} = 2\lambda_3^2\left(\Omega_1 + \lambda_1^2\Omega_2\right), \tag{5.132}$$

$$\mathcal{A}_{02323} = 2\lambda_2^2\left(\Omega_1 + \lambda_1^2\Omega_2\right) + 2\lambda_2^4 H_2^2\Omega_5, \quad \mathcal{A}_{03232} = 2\lambda_3^2\left(\Omega_1 + \lambda_1^2\Omega_2\right), \tag{5.133}$$

$$\mathcal{A}_{01221} = -2\lambda_1^2\lambda_2^2\Omega_2, \quad \mathcal{A}_{01331} = -2\lambda_1^2\lambda_3^2\Omega_2, \quad \mathcal{A}_{02332} = -2\lambda_2^2\lambda_3^2\Omega_2, \tag{5.134}$$

$$\mathcal{C}_{011|2} = 0 = \mathcal{C}_{033|2}, \quad \mathcal{C}_{022|2} = 4\lambda_2^2 H_2\Omega_5 = 2\mathcal{C}_{012|1} = 2\mathcal{C}_{032|3}, \tag{5.135}$$

$$\mathsf{K}_{011} = 2\lambda_1^{-2}\Omega_4 + 2\Omega_5, \quad \mathsf{K}_{022} = 2\lambda_2^{-2}\Omega_4 + 2\Omega_5, \quad \mathsf{K}_{033} = 2\lambda_3^{-2}\Omega_4 + 2\Omega_5. \tag{5.136}$$

Expanding the incremental governing equations $(2.43)_1$, (5.17), and (5.18) in component form, we obtain

$$\dot{H}_{l02,3} = 0, \quad \dot{H}_{l02,1} + \frac{\dot{H}_{l02}}{r} = 0, \tag{5.137}$$

$$\dot{H}_{l01,3} - \dot{H}_{l03,1} = 0, \tag{5.138}$$

$$\frac{1}{r}\left[r\left\{\mathcal{C}_{012|1}\left(u_{2,1} - \frac{u_2}{r}\right) + \mathsf{K}_{011}\dot{H}_{l01}\right\}\right]_{,1} + \left(\mathcal{C}_{032|3}u_{2,3} + \mathsf{K}_{033}\dot{H}_{l03}\right)_{,3} = 0, \tag{5.139}$$

$$\begin{aligned} \frac{1}{r}\Bigg[&\left\{r\left((\mathcal{A}_{01111}+p)u_{1,1} + \mathcal{A}_{01122}\frac{u_1}{r} + \mathcal{A}_{01133}u_{3,3} + \mathcal{C}_{011|2}\dot{H}_{l02}\right)\right\}_{,1} \\ &- \left\{(\mathcal{A}_{02222}+p)\frac{u_1}{r} + \mathcal{A}_{01122}u_{1,1} + \mathcal{A}_{02233}u_{3,3} + \mathcal{C}_{022|2}\dot{H}_{l02}\right\}\Bigg] \\ &+ \left\{\mathcal{A}_{03131}u_{1,3} + (\mathcal{A}_{01331}+p)u_{3,1}\right\}_{,3} - \dot{p}_{,1} = \rho u_{1,tt}, \end{aligned} \tag{5.140}$$

$$\frac{1}{r}\left[\left\{r\left(\mathcal{A}_{01212}u_{2,1}-(\mathcal{A}_{01221}+p)\frac{u_2}{r}+\mathcal{C}_{012|1}\dot{H}_{l01}\right)\right\}_{,1}+(\mathcal{A}_{01221}+p)\,u_{2,1}\right.$$
$$\left.-\mathcal{A}_{02121}\frac{u_2}{r}+\mathcal{C}_{012|1}\dot{H}_{l01}\right]+\left(\mathcal{A}_{03232}u_{2,3}+\mathcal{C}_{032|3}\dot{H}_{l03}\right)_{,3}=\rho u_{2,tt}, \quad (5.141)$$

$$\left(\mathcal{A}_{01133}u_{1,1}+\mathcal{A}_{02233}\frac{u_1}{r}+(\mathcal{A}_{03333}+p)u_{3,3}+\mathcal{C}_{033|2}\dot{H}_{l02}\right)_{,3}$$
$$\frac{1}{r}\left[r\left\{\mathcal{A}_{01313}u_{3,1}+(\mathcal{A}_{01331}+p)\,u_{1,3}\right\}\right]_{,1}-\dot{p}_{,3}=\rho u_{3,tt}, \quad (5.142)$$

in the material along with the equations (5.57)–(5.60) in vacuum and the constraint of incompressibility (5.61). We can eliminate $\dot{p}$ from equations (5.140) and (5.142) and use $(5.137)_1$ to get

$$A_1u_{1,3}+A_2u_{1,13}+A_3u_{1,113}+A_4u_{3,1}+A_5u_{3,11}+A_6u_{3,111}$$
$$+A_7u_{3,133}+A_8u_{3,33}+A_9u_{1,333}=\rho(u_{1,3}-u_{3,1})_{,tt}, \quad (5.143)$$

where we have assumed that the magnetoelastic moduli tensors are uniform along the axial $z$ direction and $A_1, ..., A_9$ are defined in (5.78).

From the above governing equations, we observe that $u_2, \dot{H}_{l01}$, and $\dot{H}_{l03}$ are related to each other and independent of $u_1, u_3$, and $\dot{H}_{l02}$. Hence we analyze these two cases separately.

### 5.5.1 Displacement in the azimuthal direction

We now consider the set of equations with $u_2, \dot{H}_{l01}, \dot{H}_{l03}, \dot{H}_1^*$, and $\dot{H}_3^*$. Since $\dot{H}_{l01}$ and $\dot{H}_{l03}$ satisfy Equation (5.138) while $\dot{H}_1^*$ and $\dot{H}_3^*$ satisfy Equation (5.58), we can define the potential functions $\psi$ and $\psi^*$ that satisfy equations $(5.86)_{3,4,5,6}$. On substituting them in to the governing equations (5.139) and (5.141), we obtain

$$\mathcal{C}_{012|1,1}\left(u_{2,1}-\frac{u_2}{r}\right)+\mathcal{C}_{012|1}u_{2,11}+\mathcal{C}_{032|3}u_{2,33}+\left(\frac{\mathsf{K}_{011}}{r}+\mathsf{K}_{011,1}\right)\psi_{,1}$$
$$+\mathsf{K}_{011}\psi_{,11}+\mathsf{K}_{033}\psi_{,33}=0, \quad (5.144)$$

$$-\left(\frac{\mathcal{A}_{02121}}{r^2}+\frac{(\mathcal{A}_{01221}+p)_{,1}}{r}\right)u_2+\left(\frac{\mathcal{A}_{01212}}{r}+\mathcal{A}_{01212,1}\right)u_{2,1}+\mathcal{A}_{01212}u_{2,11}$$
$$+\mathcal{A}_{03232}u_{2,33}+\left(2\frac{\mathcal{C}_{012|1}}{r}+\mathcal{C}_{012|1,1}\right)\psi_{,1}+\mathcal{C}_{012|1}\psi_{,11}+\mathcal{C}_{032|3}\psi_{,33}=\rho u_{2,tt}, \quad (5.145)$$

for $a<r<b$ along with Equation (5.57) in vacuum. Boundary conditions are given by the equations (5.62)–(5.65) as

$$-\left(\mathcal{A}_{01221}+p+\frac{\mu_0H_2^2}{2}\right)\frac{u_2}{r}+\mathcal{A}_{01212}u_{2,1}+\mathcal{C}_{012|1}\psi_{,1}-\mu_0H_2^*\psi_{,1}^*=0, \quad (5.146)$$

$$\left(\mathcal{C}_{012|1}-\mu_0 H_2\right)\frac{u_2}{r}-\mathcal{C}_{012|1}u_{2,1}-\mathsf{K}_{011}\psi_{,1}-\mu_0\psi^*_{,1}=0, \tag{5.147}$$

$$\psi_{,3}-\psi^*_{,3}-H_2 u_{2,3}=0, \tag{5.148}$$

at $r=a$ and $r=b$.

### 5.5.2 Wave propagation solutions

Using separation of variables we assume solutions of the form

$$u_2=F(r)\exp\left(\mathrm{i}kz-\mathrm{i}\omega t\right)\quad \text{for}\quad a<r<b, \tag{5.149}$$

$$\psi=G(r)\exp\left(\mathrm{i}kz-\mathrm{i}\omega t\right)\quad \text{for}\quad a<r<b, \tag{5.150}$$

$$\psi^*=M_1(r)\exp\left(\mathrm{i}kz-\mathrm{i}\omega t\right)\quad \text{for}\quad r<a, \tag{5.151}$$

$$\psi^*=M_2(r)\exp\left(\mathrm{i}kz-\mathrm{i}\omega t\right)\quad \text{for}\quad r>b, \tag{5.152}$$

with $\mathrm{i}=\sqrt{-1}$, $k$ being the wave number, and $\omega$ being the frequency.

On substituting these solutions in the governing equations we obtain

$$-\left(\frac{\mathcal{C}_{012|1,1}}{r}+k^2\mathcal{C}_{032|3}\right)F+\mathcal{C}_{012|1,1}F'+\mathcal{C}_{012|1}F''-k^2\mathsf{K}_{033}G \\ +\left(\mathsf{K}_{011,1}+\frac{\mathsf{K}_{011}}{r}\right)G'+\mathsf{K}_{011}G''=0, \tag{5.153}$$

$$\left(-\frac{\mathcal{A}_{02121}}{k^2r^2}-\frac{(\mathcal{A}_{01221}+p)_{,1}}{k^2r}-\mathcal{A}_{03232}+\rho v^2\right)F+\frac{1}{k^2}\left(\frac{\mathcal{A}_{01212}}{r}+\mathcal{A}_{01212,1}\right)F' \\ +\frac{1}{k^2}\mathcal{A}_{01212}F''-\mathcal{C}_{032|3}G+\frac{1}{k^2}\left(2\frac{\mathcal{C}_{012|1}}{r}+\frac{\mathcal{C}_{012|1,1}}{k^2}\right)G'+\mathcal{C}_{012|1}G''=0, \tag{5.154}$$

for $a<r<b$, and

$$M_1''+\frac{M_1'}{r}-k^2M_1=0\quad \text{for}\ \ r<a,\qquad M_2''+\frac{M_2'}{r}-k^2M_2=0\quad \text{for}\ \ r>b, \tag{5.155}$$

where we have taken a prime to denote a derivative with respect to $r$ and $v=\omega/k$ is the wave speed. The boundary conditions are

$$-\left(\mathcal{A}_{01221}+p+\frac{\mu_0 H_2^2}{2}\right)\frac{F}{r}+\mathcal{A}_{01212}F'+\mathcal{C}_{012|1}G'-\mu_0 H_2^* M'=0, \tag{5.156}$$

$$\left(\mathcal{C}_{012|1}-\mu_0 H_2\right)\frac{F}{r}-\mathcal{C}_{012|1}F'-\mathsf{K}_{011}G'-\mu_0 M'=0, \tag{5.157}$$

$$G-H_2F-M_1=0, \tag{5.158}$$

at $r=a$, and

$$-\left(\mathcal{A}_{01221}+p+\frac{\mu_0 H_2^2}{2}\right)\frac{F}{r}+\mathcal{A}_{01212}F'+\mathcal{C}_{012|1}G'-\mu_0 H_2^* M'=0, \tag{5.159}$$

$$\left(\mathcal{C}_{012|1} - \mu_0 H_2\right) \frac{F}{r} - \mathcal{C}_{012|1} F' - \mathsf{K}_{011} G' - \mu_0 M' = 0, \tag{5.160}$$

$$G - H_2 F - M_2 = 0, \tag{5.161}$$

at $r = b$.

To obtain numerical solutions, we non-dimensionalize the above governing equations and boundary conditions. For this purpose we define $H_{2a} = H_2|_{r=a}$ and define the following non-dimensional quantities (with a superposed hat) in addition to those in (5.122)

$$\hat{\mathcal{C}} = \frac{\mathcal{C}}{H_{2a}\mu_0}, \quad \hat{\mathsf{K}} = \frac{\mathsf{K}}{\mu_0}, \quad \hat{G}(\hat{r}) = \frac{G(r)}{H_{2a}A}, \quad \hat{M}(\hat{r}) = \frac{M(r)}{H_{2a}A},$$
$$\hat{M}_1(\hat{r}) = \frac{M_1(r)}{H_{2a}A}, \quad \hat{M}_2(\hat{r}) = \frac{M_2(r)}{H_{2a}A}. \tag{5.162}$$

On non-dimensionalization, the governing equations become

$$-\left(\frac{\hat{\mathcal{C}}'_{012|1}}{\hat{r}} + \hat{k}^2 \hat{\mathcal{C}}_{032|3}\right)\hat{F} + \hat{\mathcal{C}}'_{012|1}\hat{F}' + \hat{\mathcal{C}}_{012|1}\hat{F}'' - \hat{k}^2 \hat{\mathsf{K}}_{033}\hat{G}$$
$$+\left(\hat{\mathsf{K}}'_{011} + \frac{\hat{\mathsf{K}}_{011}}{\hat{r}}\right)\hat{G}' + \hat{\mathsf{K}}_{011}\hat{G}'' = 0, \tag{5.163}$$

$$-\left(\frac{\hat{\mathcal{A}}_{02121}}{\hat{k}^2\hat{r}^2} + \frac{\hat{\mathcal{A}}'_{01221} + \hat{p}'}{\hat{k}^2\hat{r}} + \hat{\mathcal{A}}_{03232} - \zeta\right)\hat{F} + \frac{1}{\hat{k}^2}\left(\frac{\hat{\mathcal{A}}_{01212}}{\hat{r}} + \hat{\mathcal{A}}'_{01212}\right)\hat{F}'$$
$$+\frac{\hat{\mathcal{A}}_{01212}}{\hat{k}^2}\hat{F}'' + \frac{\mu_0 H_{2b}^2}{\hat{k}^2\mu}\left\{-\hat{k}^2\hat{\mathcal{C}}_{032|3}\hat{G} + \left(2\frac{\hat{\mathcal{C}}_{012|1}}{\hat{r}} + \hat{\mathcal{C}}'_{012|1}\right)\hat{G}' + \hat{\mathcal{C}}_{012|1}\hat{G}''\right\} = 0, \tag{5.164}$$

for $\hat{a} \le \hat{r} \le \hat{b}$ and

$$\hat{M}_1'' + \frac{\hat{M}_1'}{\hat{r}} - \hat{k}^2\hat{M}_1 = 0 \quad \text{for } \hat{r} < \hat{a}, \qquad \hat{M}_2'' + \frac{\hat{M}_2'}{\hat{r}} - \hat{k}^2\hat{M}_2 = 0 \quad \text{for } \hat{r} > \hat{b}, \tag{5.165}$$

where a prime now denotes a derivative with respect to $\hat{r}$. The boundary conditions are

$$-\left(\hat{\mathcal{A}}_{01221} + \hat{p} + \frac{\mu_0 H_2^2}{2\mu}\right)\frac{\hat{F}}{\hat{r}} + \hat{\mathcal{A}}_{01212}\hat{F}' + \frac{\mu_0 H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{012|1}\hat{G}' - \hat{M}_1'\right) = 0, \tag{5.166}$$

$$\left(\hat{\mathcal{C}}_{012|1} - 1\right)\frac{\hat{F}}{\hat{r}} - \hat{\mathcal{C}}_{012|1}\hat{F}' - \hat{\mathsf{K}}_{011}\hat{G}' - \hat{M}_1' = 0, \tag{5.167}$$

$$\hat{G} - \hat{F} - \hat{M}_1 = 0, \tag{5.168}$$

at $\hat{r} = \hat{a}$, and

$$-\left(\hat{\mathcal{A}}_{01221} + \hat{p} + \frac{\mu_0 H_2^2}{2\mu}\right)\frac{\hat{F}}{\hat{r}} + \hat{\mathcal{A}}_{01212}\hat{F}' + \frac{\mu_0 H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{012|1}\hat{G}' - \frac{H_2}{H_{2a}}\hat{M}_2'\right) = 0, \tag{5.169}$$

$$\left(\hat{\mathcal{C}}_{012|1} - \frac{H_2}{H_{2a}}\right)\frac{\hat{F}}{\hat{r}} - \hat{\mathcal{C}}_{012|1}\hat{F}' - \hat{\mathsf{K}}_{011}\hat{G}' - \hat{M}_2' = 0, \tag{5.170}$$

$$\hat{G} - \frac{H_2}{H_{2a}}\hat{F} - \hat{M}_2 = 0, \tag{5.171}$$

at $\hat{r} = \hat{b}$.

Equations (5.165) are modified Bessel's equations and the solution not diverging at $\hat{r} = 0$ and $\hat{r} = \infty$ are $\hat{M}_1 = e_5 J_0(\mathrm{i}\hat{r}/\hat{k})$, $\hat{M}_2 = e_6 J_0(\mathrm{i}\hat{r}/\hat{k})$, where $J_0$ is the Bessel's function of first kind and order zero, and $e_5$ and $e_6$ are scaling parameters. To obtain a numerical solution of the system of coupled ODEs, we convert them into a system of first order ODEs by defining

$$y_1 = \hat{F}, \;\; y_2 = \hat{F}', \;\; y_3 = \hat{G}, \;\; y_4 = \hat{G}'. \tag{5.172}$$

Let the ODEs be then given by

$$p_1 y_1 + p_2 y_2 + p_3 y_2' + p_4 y_3 + p_5 y_4 + p_6 y_4' = 0, \tag{5.173}$$

$$q_1 y_1 + q_2 y_2 + q_3 y_2' + q_4 y_3 + q_5 y_4 + q_6 y_4' = 0, \tag{5.174}$$

where $p_i$s and $q_i$s $(i = 1, ..., 6)$ correspond to the coefficients in the equations (5.163) and (5.164) respectively.

Hence, we obtain the following system of first order ODEs

$$\Pi \mathbf{y}' = \mathbf{g}, \tag{5.175}$$

to be solved for $\hat{a} < \hat{r} < \hat{b}$ where the matrices $\Pi, \mathbf{y}'$, and $\mathbf{g}$ are given by

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & p_3 & 0 & p_6 \\ 0 & q_3 & 0 & q_6 \end{pmatrix}, \quad \mathbf{y}' = \begin{Bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{Bmatrix}, \tag{5.176}$$

$$\mathbf{g} = \begin{Bmatrix} y_2 \\ y_4 \\ -p_1 y_1 - p_2 y_2 - p_4 y_3 - p_5 y_4 \\ -q_1 y_1 - q_2 y_2 - q_4 y_3 - q_5 y_4 \end{Bmatrix}. \tag{5.177}$$

Given the internal pressure, the underlying magnetic field, and the axial stretch, we first evaluate $a$ using Equation (5.34) and then consider the initial value problem defined by

$$y_i(a) = \delta_{ik}, \quad i = 1, ..., 4 \tag{5.178}$$

for each of $k = 1, ..., 4$, $\delta_{ik}$ being the Kronecker delta. Subject to these initial conditions, we solve the differential equation described by Equation (5.175) using the '`ode15s`' solver in MATLAB. The four solutions thus generated are denoted by $\mathbf{y}^k\,(k=1,...,4)$ and a general solution to the problem is expressed as

$$\mathbf{y}=\sum_{k=1}^{4}e_k\mathbf{y}^k,\quad \hat{M}_1=e_5J_0(\mathrm{i}\hat{r}/\hat{k}),\quad \hat{M}_2=e_6J_0(\mathrm{i}\hat{r}/\hat{k}), \tag{5.179}$$

where $e_k\,(k=1,...,6)$ are constants. For the solutions to exist, there should be a set of non-trivial constants $\{e_k\}_{k=1}^{6}$ such that the general solution (5.179) satisfies the following boundary conditions.

$$-\left(\hat{\mathcal{A}}_{01221}+\hat{p}+\frac{\mu_0H_2^2}{2\mu}\right)\frac{y_1}{\hat{r}}+\hat{\mathcal{A}}_{01212}y_2+\frac{\mu_0H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{012|1}y_4-\hat{M}_1'\right)=0, \tag{5.180}$$

$$\left(\hat{\mathcal{C}}_{012|1}-1\right)\frac{y_1}{\hat{r}}-\hat{\mathcal{C}}_{012|1}y_2-\hat{\mathsf{K}}_{011}y_4-\hat{M}_1'=0, \tag{5.181}$$

$$y_3-y_1-\hat{M}_1=0, \tag{5.182}$$

at $\hat{r}=\hat{a}$, and

$$-\left(\hat{\mathcal{A}}_{01221}+\hat{p}+\frac{\mu_0H_2^2}{2\mu}\right)\frac{y_1}{\hat{r}}+\hat{\mathcal{A}}_{01212}y_2+\frac{\mu_0H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{012|1}y_4-\frac{H_2}{H_{2a}}\hat{M}_2'\right)=0, \tag{5.183}$$

$$\left(\hat{\mathcal{C}}_{012|1}-\frac{H_2}{H_{2a}}\right)\frac{y_1}{\hat{r}}-\hat{\mathcal{C}}_{012|1}y_2-\hat{\mathsf{K}}_{011}y_4-\hat{M}_2'=0, \tag{5.184}$$

$$y_3-\frac{H_2}{H_{2a}}y_1-\hat{M}_2=0, \tag{5.185}$$

at $\hat{r}=\hat{b}$.

This yields a $6\times 6$ determinant of the coefficients of $e_k$, vanishing of which gives the relationship between $\zeta$ and other parameters. This solution process is similar to the numerical routine described by [Haughton and Ogden, 1979b].

### 5.5.3 Numerical results

We solve the above equations numerically for the Mooney–Rivlin type magnetoelastic material defined in Equation (5.45). Internal and external radii of the tube are taken to have the ratio $A/B=0.6$, the external pressure is taken to be zero while the material parameters are taken to have the values as in (5.125). Multiple modes of wave propagation are obtained as in Section 5.4.2 and we plot only the first modes here.

Dispersion curves are plotted in Figure 5.10 for different values of the underlying magnetic fields, $P_{\text{in}}=0.2\mu,\lambda_z=1.5$. Here $H_0$ is the reference value of the magnetic field as defined in Equation (5.44). It is observed that the non-dimensionalized wave speed $\zeta$

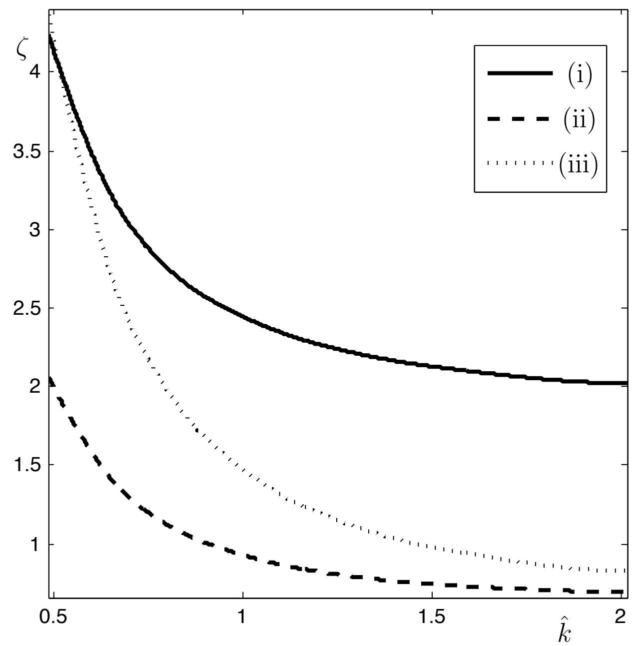



Figure 5.10: Variation of $\zeta$ with $\hat{k}$ for $P_{\text{in}} = 0.2\mu$, $\lambda_z = 1.5, A/B = 0.6$, (i) $H_0 = 0$; (ii) $H_0 = 1 \times 10^5$ A/m; (iii) $H_0 = 2 \times 10^5$ A/m.

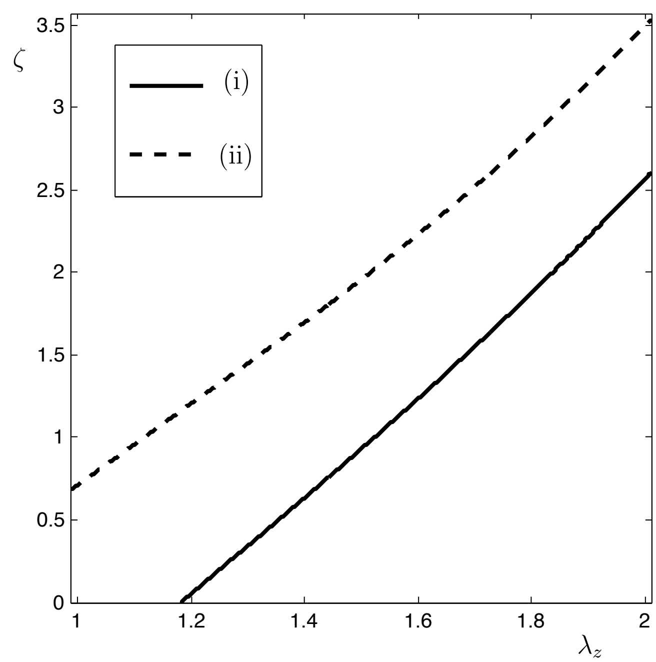



Figure 5.11: Variation of $\zeta$ with $\lambda_z$ for $P_{\text{in}} = 0.2\mu$, $\hat{k} = 1, H_0 = 1\times 10^5$ A/m, (i) $A/B = 0.6$; (ii) $A/B = 0.8$.

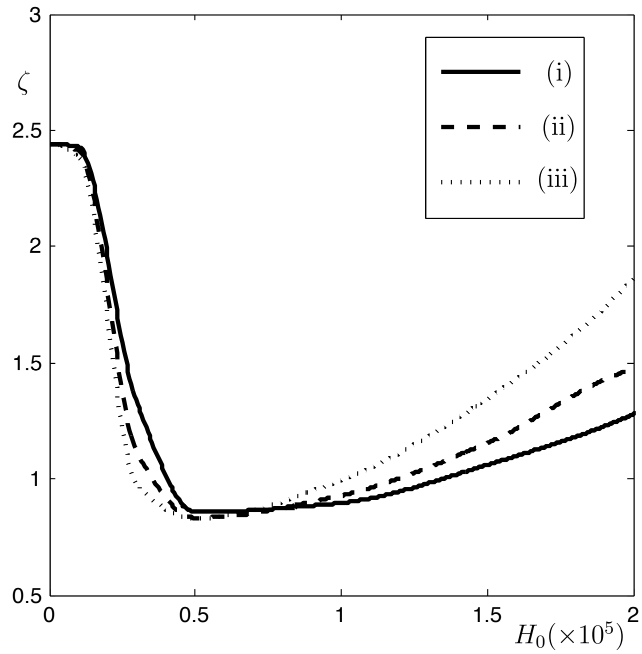



Figure 5.12: Dependence of $\zeta$ on the underlying magnetic field $H_0$ (in A/m) for different values of the magnetoelastic coupling parameter $m$. Here, $\lambda_z = 1.5, P_{\text{in}} = 0.2\mu, \hat{k} = 1$, (i) $m = 0.5\mu_0$, (ii) $m = \mu_0$, (iii) $m = 2\mu_0$.

decreases with an increasing non-dimensionalized wave number $\hat{k}$. An underlying magnetic field may increase or decrease $\zeta$ depending on $\hat{k}$ and other parameters.

Influence of the tube thickness $(A/B)$ and the underlying axial stretch $\lambda_z$ on the wave speed is illustrated in Figure 5.11. As $\lambda_z$ is reduced, the wave speed decreases eventually becoming zero which corresponds to an onset of buckling/instability in the tube. The nearly linear variation of $\zeta$ with $\lambda_z$ is similar to what is shown by [Haughton, 1984] for the problem with purely elastic waves. Tube with thinner walls $(A/B = 0.8)$ has waves with a higher speed than a tube with thicker walls $(A/B = 0.6)$.

We plot the variation of $\zeta$ with the magnetic field for different values of the magnetoelastic coupling parameter $m$ in Figure 5.12. The wave speed $\zeta$ first decreases up to a minimum and then increases with an increasing magnetic field. A small $m$ increases $\zeta$ for small magnetic fields while it decreases $\zeta$ for large magnetic fields.

### 5.5.4 Displacement in the $(r, z)$ plane

We now consider the incremental displacements in the radial and axial directions and hence deal with the equations involving $u_1, u_3$, and $\dot{H}_{l02}$. Since $u_1$ and $u_3$ satisfy Equation

(5.61), we can define a potential $\phi$ that satisfy equations $(5.86)_{1,2}$ and substitute in the governing equation (5.143) to get

$$\phi_{,1}\left(\frac{A_4}{r^2}-2\frac{A_5}{r^3}+6\frac{A_6}{r^4}\right)+\phi_{,11}\left(-\frac{A_4}{r}+2\frac{A_5}{r^2}-6\frac{A_6}{r^3}\right)+\phi_{,111}\left(-\frac{A_5}{r}+3\frac{A_6}{r^2}\right)$$
$$-\frac{A_6}{r}\phi_{,1111}+\phi_{,33}\left(\frac{A_1}{r}-\frac{A_2}{r^2}+2\frac{A_3}{r^3}\right)+\phi_{,133}\left(\frac{A_2-A_8}{r}+\frac{A_7-2A_3}{r^2}\right)$$
$$\phi_{,1133}\frac{A_3-A_7}{r}+\frac{A_9}{r}\phi_{,3333}=\rho\left(\frac{\phi_{,11}+\phi_{,33}}{r}-\frac{\phi_{,1}}{r^2}\right)_{,tt}. \quad (5.186)$$

Equations (5.137) govern $\dot{H}_{l02}$ while $\dot{H}_2^*$ satisfies equations (5.59) and (5.60). The boundary conditions (5.62)–(5.65) give

$$\left(\mathcal{A}_{01111}+p-\frac{\mu_0 H_2^{*2}}{2}\right)u_{1,1}+\mathcal{A}_{01122}\frac{u_1}{r}+\mathcal{A}_{01133}u_{3,3}$$
$$+\mathcal{C}_{011|2}\dot{H}_{l02}+\mu_0 H_2\dot{H}_2^*-\dot{p}=0, \quad (5.187)$$
$$\left(\mathcal{A}_{01331}+p-\frac{\mu_0 H_2^2}{2}\right)u_{1,3}+\mathcal{A}_{01331}u_{3,1}=0, \quad (5.188)$$
$$\dot{H}_{l02}-H_2\frac{u_1}{r}-\dot{H}_2^*=0, \quad (5.189)$$

at $r=a$ and $r=b$. We differentiate Equation (5.187) with respect to $z$ and substitute $\dot{p}_{,3}$ from Equation (5.142) to get

$$u_{1,3}\left(\frac{1}{r}\left(\mathcal{A}_{01122}-\mathcal{A}_{02233}-\mathcal{A}_{01331}-p\right)-\left(\mathcal{A}_{01331}+p\right)_{,1}\right)-\mathcal{A}_{01313}u_{3,11}$$
$$-u_{3,1}\left(\frac{\mathcal{A}_{01313}}{r}+\mathcal{A}_{01313,1}\right)+u_{1,13}\left(\mathcal{A}_{01111}-\mathcal{A}_{01133}-\mathcal{A}_{01331}-\frac{\mu_0 H_2^{*2}}{2}\right)$$
$$+\left(\mathcal{A}_{01133}-\mathcal{A}_{03333}-p\right)u_{3,33}+\rho u_{3,tt}=0. \quad (5.190)$$

Using the definition of $\phi$ from Equation $(5.86)_{1,2}$ in the above boundary conditions, we get

$$\phi_{,1}\left(\frac{\mathcal{A}_{01313}}{r^3}-\frac{\mathcal{A}_{01313,1}}{r^2}\right)+\phi_{,11}\left(\frac{\mathcal{A}_{01313,1}}{r}-\frac{\mathcal{A}_{01313}}{r^2}\right)+\frac{\mathcal{A}_{01313}}{r}\phi_{,111}$$
$$+\phi_{,33}\left\{-\frac{\left(\mathcal{A}_{01331}+p\right)_{,1}}{r}+\frac{1}{r^2}\left(\frac{\mu_0 H_2^2}{2}+\mathcal{A}_{01122}-\mathcal{A}_{02233}-\mathcal{A}_{01111}+\mathcal{A}_{01133}\right.\right.$$
$$\left.\left.-p\right)\right\}+\frac{\phi_{,133}}{r}\left(\mathcal{A}_{01111}+\mathcal{A}_{03333}-2\mathcal{A}_{01133}-\mathcal{A}_{01331}+p-\frac{\mu_0 H_2^2}{2}\right)$$
$$-\frac{\rho}{r}\phi_{,1tt}=0, \quad (5.191)$$
$$\mathcal{A}_{01313}\left(-\frac{\phi_{,11}}{r}+\frac{\phi_{,1}}{r^2}\right)+\left(\mathcal{A}_{01331}+p-\frac{\mu_0 H_2^2}{2}\right)\frac{\phi_{,33}}{r}=0, \quad (5.192)$$
$$\dot{H}_{l02}-H_2\frac{\phi_{,3}}{r^2}-\dot{H}_2^*=0, \quad (5.193)$$

at $r=a$ and $r=b$.

### 5.5.5 Wave propagation solutions

We consider the solutions of the above mentioned differential equations of the form

$$\phi = F(r)\exp(\mathrm{i}kz - \mathrm{i}\omega t), \quad a < r < b, \tag{5.194}$$

$$\dot{H}_{l02} = G(r)\,\mathrm{e}^{-\mathrm{i}\omega t}, \quad a < r < b, \tag{5.195}$$

$$\dot{H}_2^* = M_1(r)\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{for } r < a, \qquad \dot{H}_2^* = M_2(r)\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{for } r > b. \tag{5.196}$$

Substituting these solutions in the governing equations (5.60), (5.137), and (5.186) gives

$$\left(-k^2 b_4 + b_7 k^4 - \rho\omega^2 \frac{k^2}{r}\right) F + \left(b_1 - k^2 b_5 - \frac{\rho\omega^2}{r^2}\right) F' - \left(r b_1 + k^2 b_6 - \frac{\rho\omega^2}{r}\right) F'' + b_2 F''' + b_3 F'''' = 0, \quad a < r < b, \tag{5.197}$$

$$G' + \frac{G}{r} = 0, \quad a < r < b, \tag{5.198}$$

$$M_1' + \frac{M_1}{r} = 0, \quad r < a, \qquad M_2' + \frac{M_2}{r} = 0, \quad r > b, \tag{5.199}$$

where prime denotes a derivative with respect to $r$ and we have defined

$$b_1 = \frac{A_4}{r^2} - 2\frac{A_5}{r^3} + 6\frac{A_6}{r^4}, \quad b_2 = \frac{-A_5}{r} + 3\frac{A_6}{r^2}, \quad b_3 = -\frac{A_6}{r}, \quad b_7 = \frac{A_9}{r},$$
$$b_4 = \frac{A_1}{r} - \frac{A_2}{r^2} + 2\frac{A_3}{r^3}, \quad b_5 = \frac{A_2 - A_8}{r} + \frac{A_7 - 2A_3}{r^2}, \quad b_6 = \frac{A_3 - A_7}{r},$$
$$C_1 = \{\frac{1}{r^2}\left(-\mathcal{A}_{01111} + \mathcal{A}_{01122} + \mathcal{A}_{01133} - \mathcal{A}_{02233} - p + \frac{\mu_0 H_2^2}{2}\right) - \frac{1}{r}\left(\mathcal{A}_{01331,1} + p_{,1}\right)\}, \quad C_2 = \left(\frac{\mathcal{A}_{01313}}{r^3} - \frac{\mathcal{A}_{01313,1}}{r^2}\right), \quad C_4 = \frac{\mathcal{A}_{01313}}{r},$$
$$C_3 = \frac{1}{r}\left(\mathcal{A}_{01111} - \mathcal{A}_{01331} - 2\mathcal{A}_{01133} + \mathcal{A}_{03333} + p - \frac{\mu_0 H_2^2}{2}\right). \tag{5.200}$$

The boundary conditions (5.191)–(5.193) become

$$-k^2 C_1 F + \left(C_2 - k^2 C_3 + \frac{\rho\omega^2}{r}\right) F' - r C_2 F'' + C_4 F''' = 0 \quad \text{at}\, r = a, r = b, \tag{5.201}$$

$$-k^2\left(C_4 + \frac{p}{r} - \frac{\mu_0 H_2^2}{2r}\right) F + \frac{C_4}{r} F' - C_4 F'' = 0, \quad \text{at}\, r = a, r = b, \tag{5.202}$$

$$G - \frac{1}{r^2}\mathrm{i}k H_2 F \mathrm{e}^{\mathrm{i}kz} - M_1 = 0 \quad \text{at}\, r = a, \tag{5.203}$$

$$G - \frac{1}{r^2}\mathrm{i}k H_2 F \mathrm{e}^{\mathrm{i}kz} - M_2 = 0 \quad \text{at}\, r = b. \tag{5.204}$$

Since the last two boundary conditions apply for all $z$ and considering the fact that $G, M_1$, and $M_2$ do not depend on $z$, they can be split into

$$G = M_1 \quad \text{at } r = a, \quad G = M_2 \quad \text{at } r = b, \tag{5.205}$$

$$H_2 F = 0 \quad \text{at } r = a, b. \tag{5.206}$$

The governing equations for $G, M_1$ and $M_2$ can be integrated analytically to get $G = c_1/r$ in $a < r < b$, $M_1 = c_2/r$ in $r < a$, and $M_2 = c_3/r$ in $r > b$. The boundary conditions (5.205), however, require that $c_1 = c_2 = c_3$.

The fourth order ODE (5.197), with the boundary conditions (5.201), (5.202), and (5.206) is overdetermined for non-zero $H_2$, a solution is possible only for $H_2 = 0$ which reduces the problem to the purely elastic case. A non-trivial solution (for incremental magnetic field) can be obtained only in a very special case when the parameters $C_1, ..., C_4$ obtain values such that two of the boundary conditions become linearly dependent.

# Chapter 6

# Wave Propagation in a Finitely-Deformed Pre-Stressed Conductor

In this chapter, we discard the quasimagnetostatic approximation as used in Chapters 2–5 and work with complete equations of electrodynamics in continua. Specifically, we consider the equations required to study wave propagation in a magnetized electric conductor with residual stress. This analysis is useful for an accurate description of the electromagnetic acoustic transduction process which is an important experimental tool for non-destructive evaluation techniques (see, for example, the works of [Ludwig et al., 1993] and [Hirao and Ogi, 2003]). Sections 6.1 and 6.2 are based on the existing literature by, for example, [Pao, 1978] and [Eringen and Maugin, 1990a] and our calculations and analysis is presented in Section 6.3 onwards.

## 6.1 Basic equations

The governing equations of Electrodynamics in Continua are given by the Dipole-current Circuit Model [Pao, 1978] as

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad \varepsilon_0 \operatorname{div}\mathbf{E} = \rho_{\mathrm{e}} - \operatorname{div}\mathbf{P}, \tag{6.1}$$

$$\frac{1}{\mu_0}\operatorname{curl}\mathbf{B} - \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{P}}{\partial t} + \operatorname{curl}\mathbf{M} + \mathbf{J}, \tag{6.2}$$

where we have used the field relations

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{6.3}$$

$\mathbf{P}$ being the electric polarization and $\mathbf{M}$ being the magnetization. Here all other terms have their meanings as defined in Chapter 2.

The first four equations can be used to derive the law of balance of electric charge as

$$\operatorname{div}\mathbf{J} + \frac{\partial \rho_{\mathrm{e}}}{\partial t} = 0. \tag{6.4}$$

Defining the pushback version of the field variables $\mathbf{P}$ and $\mathbf{M}$ as

$$\mathbf{P}_l = J\mathbf{F}^{-1}\mathbf{P}, \quad \mathbf{M}_l = \mathbf{F}^{\mathrm{T}}\mathbf{M}, \tag{6.5}$$

and using Equation (2.15), we rewrite the above Maxwell's equations in Lagrangian form as

$$\operatorname{Div}\mathbf{B}_l = 0, \tag{6.6}$$

$$\operatorname{Curl}(\mathbf{E}_l + \mathbf{V}\times\mathbf{B}_l) = -\mathbf{B}_{l,t}, \tag{6.7}$$

$$\varepsilon_0 \operatorname{Div}(J\mathbf{c}^{-1}\mathbf{E}_l) = \rho_{\mathrm{E}} + \operatorname{Div}\mathbf{P}_l, \tag{6.8}$$

$$\begin{aligned}\operatorname{Curl}\left(\frac{J^{-1}}{\mu_0}\mathbf{c}\mathbf{B}_l - \varepsilon_0\mathbf{V}\times(J\mathbf{c}^{-1}\mathbf{E}_l)\right) &- \varepsilon_0(J\mathbf{c}^{-1}\mathbf{E}_l)_{,t} \\ &= \mathbf{P}_{l,t} + \operatorname{Curl}(\mathbf{M}_l + \mathbf{V}\times\mathbf{P}_l) + \mathbf{J}_{\mathrm{E}},\end{aligned} \tag{6.9}$$

where as defined previously $\rho_{\mathrm{E}} = J\rho_{\mathrm{e}}$ and $\mathbf{J}_{\mathrm{E}} = J\mathbf{F}^{-1}(\mathbf{J} - \rho_{\mathrm{e}}\mathbf{v})$.

At the boundary, the following conditions need to be satisfied

$$\mathbf{N}\times[\![\mathbf{E}_l + \mathbf{V}\times\mathbf{B}_l]\!] = \mathbf{0}, \tag{6.10}$$

$$\mathbf{N}\cdot[\![\mathbf{B}_l]\!] = 0, \tag{6.11}$$

$$\mathbf{N}\cdot[\![\mathbf{D}_l]\!] = \sigma_{\mathrm{E}}, \tag{6.12}$$

$$\mathbf{N}\times[\![J^{-1}\mu_0^{-1}\mathbf{c}\mathbf{B}_l - \mathbf{M}_l - \mathbf{V}\times(\varepsilon_0 J\mathbf{c}^{-1}\mathbf{E}_l + \mathbf{P}_l)]\!] = \mathbf{K}_l - \sigma_{\mathrm{E}}\mathbf{V}_{\mathrm{s}}. \tag{6.13}$$

### 6.1.1 Mechanical balance laws

We consider an elastic body which, in the reference configuration, has a residual stress $\mathbf{S}$. Hence, the equation of equilibrium is

$$\operatorname{Div}\mathbf{S} = \mathbf{0}. \tag{6.14}$$

After a finite deformation, in the current configuration, the balance of linear momentum is given by

$$\operatorname{div}\boldsymbol{\tau} + \mathbf{f}_e = \rho\mathbf{a}, \tag{6.15}$$

where $\boldsymbol{\tau}$ is the Cauchy stress tensor and the electromagnetic body force is

$$\mathbf{f}_e = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B} + (\operatorname{grad} \mathbf{E})^{\mathrm{T}} \mathbf{P} + (\operatorname{grad} \mathbf{B})^{\mathrm{T}} \mathbf{M} + \frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B}) + \operatorname{div}[\mathbf{v} \otimes (\mathbf{P} \times \mathbf{B})]. \quad (6.16)$$

Balance of angular momentum gives

$$\boldsymbol{\varepsilon\tau} + \mathbf{L}_e = \mathbf{0}, \quad (6.17)$$

where $\boldsymbol{\varepsilon}$ is the third order permutation tensor with components $\varepsilon_{ijk}$ and $(\boldsymbol{\varepsilon\tau})_i = \varepsilon_{ijk}\tau_{jk}$. $\mathbf{L}_e$ is the electromagnetic body couple vector given by

$$\mathbf{L}_e = \mathbf{P} \times \mathbf{E} + (\mathbf{M} + \mathbf{v} \times \mathbf{P}) \times \mathbf{B}. \quad (6.18)$$

The total stress tensor of [Dorfmann and Ogden, 2004] that incorporates the magnetic body force and has been utilized in Chapters 2–5 cannot be used here since the above expressions for electromagnetic body force and electromagnetic body couple have terms including both electric and magnetic quantities. The above balance equations can be written in Lagrangian form using the nominal stress tensor $\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}$ as

$$\operatorname{Div} \mathbf{T} + J\, \mathbf{f}_E = \rho_r \mathbf{a}, \quad (6.19)$$

and

$$\boldsymbol{\varepsilon}(\mathbf{FT}) + J\mathbf{L}_E = \mathbf{0}, \quad (6.20)$$

where $\mathbf{f}_E$ and $\mathbf{L}_E$ are Lagrangian counterparts of the corresponding vectors and are given by

$$\begin{aligned}\mathbf{f}_E = J^{-1}\rho_{\mathrm{E}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l + J^{-2}(\mathbf{F}\mathbf{J}_l) \times (\mathbf{F}\mathbf{B}_l) + \mathbf{F}^{-\mathrm{T}}\left[\operatorname{Grad}(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l)\right]^{\mathrm{T}}(J^{-1}\mathbf{F}\mathbf{P}_l) \\ +\mathbf{F}^{-\mathrm{T}}\left[\operatorname{Grad}(J^{-1}\mathbf{F}\mathbf{B}_l)\right]^{\mathrm{T}}(\mathbf{F}^{-\mathrm{T}}\mathbf{M}_l) + \frac{\partial}{\partial t}\left[J^{-2}\left(\mathbf{F}\mathbf{P}_l\right) \times (\mathbf{F}\mathbf{B}_l)\right] \\ +J^{-1}\operatorname{Div}\left[J^{-1}\mathbf{V} \otimes \{(\mathbf{F}\mathbf{P}_l) \times (\mathbf{F}\mathbf{B}_l)\}\right], \quad (6.21)\end{aligned}$$

$$\mathbf{L}_E = J^{-1}\left(\mathbf{F}\mathbf{P}_l\right) \times \left(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l\right) + J^{-1}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el}\right) \times \left(\mathbf{F}\mathbf{B}_l\right). \quad (6.22)$$

On any part of the boundary where the traction is prescribed, the boundary condition may be given as

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{A}}, \quad (6.23)$$

where $\mathbf{t}_{\mathrm{A}}$ is the Lagrangian representation of the traction force.

### 6.1.2 Energy balance laws

First law of Thermodynamics gives the balance of energy as

$$\rho \frac{dU}{dt} = \boldsymbol{\tau} \colon \operatorname{grad} \mathbf{v} - \operatorname{div} \mathbf{q} + q + w_e, \tag{6.24}$$

where $U$ is the internal energy, $\mathbf{q}$ is the heat flux at the surface, $q$ is the volumetric heat generation, the symbol : denotes a scalar product between two second order tensors given in component form as $\boldsymbol{\tau} \colon \boldsymbol{\Gamma} = \tau_{ij}\Gamma_{ji}$, and $w_e$ is the electromagnetic power given by

$$w_e = \mathbf{J}_e \cdot \mathbf{E}_e + \rho \frac{d}{dt}\left(\frac{\mathbf{P}}{\rho}\right) \cdot \mathbf{E}_e - \mathbf{M}_e \cdot \frac{d\mathbf{B}}{dt}, \tag{6.25}$$

where for a dynamic problem we have defined the effective field variables as

$$\mathbf{J}_e = \mathbf{J} - \rho_e \mathbf{v}, \quad \mathbf{E}_e = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad \mathbf{M}_e = \mathbf{M} + \mathbf{v} \times \mathbf{P}. \tag{6.26}$$

Let $\vartheta$ be the absolute temperature, then we can write the above first law of Thermodynamics as

$$\rho\, c_p \frac{\partial \vartheta}{\partial t} = q + w_e + \boldsymbol{\tau} \colon \operatorname{grad} \mathbf{v} - \operatorname{div} \mathbf{q}. \tag{6.27}$$

Here $c_p$ is the specific heat capacity and $\rho$ is the mass density. On defining the pushback versions of the physical quantities

$$\mathbf{q}_l = J\mathbf{F}^{-1}\mathbf{q}, \quad q_l = Jq, \quad w_E = Jw_e, \quad \vartheta_l = J\vartheta, \tag{6.28}$$

the above equation can be written in Lagrangian form as

$$\rho_r c_p \frac{\partial}{\partial t}\left(J^{-1}\vartheta_l\right) = \mathbf{T} \colon \operatorname{Grad}(\mathbf{FV}) + q_l + w_E - \operatorname{Div} \mathbf{q}_l. \tag{6.29}$$

Here, the Lagrangian form of electromagnetic power is given as

$$\begin{aligned} w_E = (\mathbf{F}\mathbf{J}_{el}) \cdot (\mathbf{F}^{-\mathrm{T}}\mathbf{E}_{el}) + \rho_r \left[\frac{\partial}{\partial t}\left(\frac{\mathbf{F}\mathbf{P}_l}{\rho_r}\right) + \operatorname{Grad}\left(\frac{\mathbf{F}\mathbf{P}_l}{\rho_r}\right)\mathbf{V}\right] \cdot \left(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_{el}\right) \\ -J\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el} \cdot \left[\frac{\partial}{\partial t}\left(J^{-1}\mathbf{F}\mathbf{B}_l\right) + \operatorname{Grad}\left(J^{-1}\mathbf{F}\mathbf{B}_l\right)\mathbf{V}\right]. \end{aligned} \tag{6.30}$$

If $S$ is the entropy density, then the second law of Thermodynamics gives

$$\rho \frac{dS}{dt} + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) - \frac{q}{\vartheta} \geq 0. \tag{6.31}$$

## 6.2 Constitutive relations

Substituting equation (6.24) to (6.31), we get

$$\rho\left(\frac{dU}{dt}-\vartheta\frac{dS}{dt}\right)+\operatorname{div}\mathbf{q}-\vartheta\operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right)-\boldsymbol{\tau}:\operatorname{grad}\mathbf{v}-w_e\leq 0. \tag{6.32}$$

We consider a free energy $\Psi$ obtained by $U$ through a Legendre transformation of the form

$$\Psi=U-\vartheta S-\mathbf{E}_e\cdot\frac{\mathbf{P}}{\rho}, \tag{6.33}$$

using which we can rewrite the entropy inequality as

$$\rho\left(\frac{d\Psi}{dt}+S\frac{d\vartheta}{dt}\right)+\mathbf{P}\cdot\frac{d\mathbf{E}_e}{dt}+\operatorname{div}\mathbf{q}-\vartheta\operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right)-\boldsymbol{\tau}:\operatorname{grad}\mathbf{v}+\mathbf{M}_e\cdot\frac{d\mathbf{B}}{dt}-\mathbf{J}_e\cdot\mathbf{E}_e\leq 0. \tag{6.34}$$

We consider the free energy to be dependent on the deformation gradient $\mathbf{F}$, the electric field, the magnetic field, temperature, and the push-forward residual stress $\boldsymbol{\sigma}=J^{-1}\mathbf{F}\mathbf{S}$; and is given as $\Psi=\Psi(\mathbf{F},\mathbf{E}_e,\mathbf{B},\vartheta,\boldsymbol{\sigma})$. Considering the pre-stress to be independent of time, we have

$$\frac{d\Psi}{dt}=\frac{\partial\Psi}{\partial\mathbf{F}}:\frac{d\mathbf{F}}{dt}+\frac{\partial\Psi}{\partial\mathbf{E}_e}\cdot\frac{d\mathbf{E}_e}{dt}+\frac{\partial\Psi}{\partial\mathbf{B}}\cdot\frac{d\mathbf{B}}{dt}+\frac{\partial\Psi}{\partial\vartheta}\frac{d\vartheta}{dt}, \tag{6.35}$$

substituting which in the above entropy inequality gives

$$\begin{aligned}\rho\left(\frac{\partial\Psi}{\partial\vartheta}+S\right)\frac{d\vartheta}{dt}+\left(\rho\frac{\partial\Psi}{\partial\mathbf{E}_e}+\mathbf{P}\right)\cdot\frac{d\mathbf{E}_e}{dt}+\left(\rho\frac{\partial\Psi}{\partial\mathbf{B}}+\mathbf{M}_e\right)\cdot\frac{d\mathbf{B}}{dt}\\+\left(\rho\frac{\partial\Psi}{\partial\mathbf{F}}-\mathbf{F}^{-1}\boldsymbol{\tau}\right):\frac{d\mathbf{F}}{dt}+\frac{\mathbf{q}}{\vartheta}\cdot\operatorname{grad}\vartheta-\mathbf{J}_e\cdot\mathbf{E}_e\leq 0.\end{aligned} \tag{6.36}$$

For the above inequality to be satisfied for all admissible processes, the following constitutive laws should be satisfied

$$S=-\frac{\partial\Psi}{\partial\vartheta}, \tag{6.37}$$

$$\mathbf{P}=-\rho\frac{\partial\Psi}{\partial\mathbf{E}_e}, \tag{6.38}$$

$$\mathbf{M}_e=-\rho\frac{\partial\Psi}{\partial\mathbf{B}}, \tag{6.39}$$

$$\boldsymbol{\tau}=\rho\mathbf{F}\frac{\partial\Psi}{\partial\mathbf{F}}, \tag{6.40}$$

and the entropy inequality reduces to

$$\mathbf{J}_e\cdot\mathbf{E}_e-\frac{\mathbf{q}}{\vartheta}\cdot\operatorname{grad}\vartheta\geq 0. \tag{6.41}$$

We take the constitutive law relating the heat flow to temperature gradient to be given by the Fourier's law of heat conduction as

$$\mathbf{q}=-\boldsymbol{\kappa}\operatorname{grad}\vartheta, \tag{6.42}$$

where $\boldsymbol{\kappa}$ is a positive definite symmetric tensor quantifying thermal conductivity that, in general, varies with the underlying electromagnetic fields, deformation and pre-stress. The above equation is given in Lagrangian form as

$$\mathbf{q}_l = -J\mathbf{F}^{-1}\boldsymbol{\kappa}\mathbf{F}^{-\mathrm{T}}\mathrm{Grad}(J^{-1}\vartheta_l). \tag{6.43}$$

The constitutive law relating electric current density to electric field is assumed to be given by Ohm's law as

$$\mathbf{J} = \boldsymbol{\xi}\,\mathbf{E}, \tag{6.44}$$

where $\boldsymbol{\xi}$ is a positive definite symmetric tensor quantifying electrical conductivity that varies with the underlying electromagnetic fields, deformation, and initial stress. The above equation is given in Lagrangian form as

$$\mathbf{J}_l = J\mathbf{F}^{-1}\boldsymbol{\xi}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l. \tag{6.45}$$

Substituting these constitutive equations in the entropy inequality gives

$$\frac{1}{\vartheta}\mathrm{grad}\,\vartheta\cdot(\boldsymbol{\kappa}\mathrm{grad}\,\vartheta) + \mathbf{E}\cdot(\boldsymbol{\xi}\mathbf{E}) + (\boldsymbol{\xi})\,\mathbf{E}\cdot(\mathbf{v}\times\mathbf{B}) - \rho_{\mathrm{e}}\mathbf{v}\cdot\mathbf{E} \geq 0. \tag{6.46}$$

### 6.2.1 Alternative constitutive formulation

We may rewrite the Equations (6.3) in Lagrangian form using the pushback relations (2.15) as

$$\mathbf{D}_l = \varepsilon_0 J\mathbf{c}^{-1}\mathbf{E}_l + \mathbf{P}_l, \quad \frac{1}{\mu_0 J}\mathbf{c}\mathbf{B}_l = \mathbf{H}_l + \mathbf{M}_l, \tag{6.47}$$

where we have defined new pushback relations for $\mathbf{P}_l$ and $\mathbf{M}_l$ as

$$\mathbf{P}_l = J\mathbf{F}^{-1}\mathbf{P}, \quad \mathbf{M}_l = \mathbf{F}^{\mathrm{T}}\mathbf{M}. \tag{6.48}$$

We may now equivalently consider an energy function $\Phi$ that depends on the Lagrangian variables rather than the Eulerian variables. Let $\mathbf{E}_{el} = \mathbf{F}^{\mathrm{T}}\mathbf{E}_e, \mathbf{M}_{el} = \mathbf{F}^{\mathrm{T}}\mathbf{M}_e$, and

$$\Phi(\mathbf{F}, \mathbf{E}_{el}, \mathbf{B}_l, \vartheta_l, \mathbf{S}) = \rho_r\Psi(\mathbf{F}, \mathbf{E}_e, \mathbf{B}, \vartheta, \boldsymbol{\sigma}), \tag{6.49}$$

such that $\Phi$ is energy per unit volume rather than per unit mass. This gives

$$\frac{\partial\Phi}{\partial\mathbf{E}_{el}} = \rho_r\frac{\partial\mathbf{E}_e}{\partial\mathbf{E}_{el}}\frac{\partial\Psi}{\partial\mathbf{E}_e} = -J\mathbf{F}^{-1}\mathbf{P}, \quad \frac{\partial\Phi}{\partial\mathbf{B}_l} = \rho_r\frac{\partial\mathbf{B}}{\partial\mathbf{B}_l}\frac{\partial\Psi}{\partial\mathbf{B}} = -\mathbf{F}^{\mathrm{T}}\mathbf{M}_e. \tag{6.50}$$

Thus, we have the constitutive relations

$$\mathbf{T} = \frac{\partial\Phi}{\partial\mathbf{F}}, \quad \mathbf{P}_l = -\frac{\partial\Phi}{\partial\mathbf{E}_{el}}, \quad \mathbf{M}_{el} = -\frac{\partial\Phi}{\partial\mathbf{B}_l}. \tag{6.51}$$

## 6.3 Incremental equations

On the initial motion and underlying electromagnetic fields, we consider an incremental mechanical motion $\mathbf{u}(\mathbf{x},t)$, and increments in electromagnetic fields which are denoted by a superposed dot.

The incremented forms of the Lagrangian Maxwell's equations (6.6)–(6.9) are given as

$$\mathrm{Div}\,\dot{\mathbf{B}}_l = 0, \tag{6.52}$$

$$\mathrm{Curl}\,\dot{\mathbf{E}}_{el} = -\dot{\mathbf{B}}_{l,t}, \tag{6.53}$$

$$\varepsilon_0\,\mathrm{Div}\left[J(\mathrm{div}\,\mathbf{u})\mathbf{c}^{-1}\mathbf{E}_l + J\mathbf{c}^{-1}\dot{\mathbf{E}}_l - J\mathbf{F}^{-1}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l\right] = \dot{\rho}_{\mathrm{E}} + \mathrm{Div}\,\dot{\mathbf{P}}_l, \tag{6.54}$$

$$\begin{aligned}\mu_0^{-1}\mathrm{Curl}\left[J^{-1}\left\{(\mathrm{div}\,\mathbf{u})\mathbf{c}\mathbf{B}_l + 2\mathbf{F}^{\mathrm{T}}\mathbf{L}\mathbf{F}\mathbf{B}_l + \mathbf{c}\dot{\mathbf{B}}_l\right\}\right] - \varepsilon_0\mathrm{Curl}\Big[\dot{\mathbf{V}}\times(J\mathbf{c}^{-1}\mathbf{E}_l)\\ +\mathbf{V}\times\left\{J(\mathrm{div}\,\mathbf{u})\mathbf{c}^{-1}\mathbf{E}_l + J\mathbf{c}^{-1}\dot{\mathbf{E}}_l - J\mathbf{F}^{-1}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l\right\}\Big]\\ -\varepsilon_0\left[J\left\{J(\mathrm{div}\,\mathbf{u})\mathbf{c}^{-1}\mathbf{E}_l + J\mathbf{c}^{-1}\dot{\mathbf{E}}_l - J\mathbf{F}^{-1}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l\right\}\right]_{,t}\\ = \dot{\mathbf{P}}_{l,t} + \mathrm{Curl}\,\dot{\mathbf{M}}_{el} + \dot{\mathbf{J}}_{\mathrm{E}},\end{aligned} \tag{6.55}$$

which can be updated to Eulerian form using the relations (2.35) to get

$$\mathrm{div}\,\dot{\mathbf{B}}_{l0} = 0, \tag{6.56}$$

$$\mathrm{curl}\,\dot{\mathbf{E}}_{el0} = [\mathbf{\Gamma} - (\mathrm{div}\,\mathbf{v})\mathbf{I}]\,\dot{\mathbf{B}}_{l0} - \dot{\mathbf{B}}_{l0,t}, \tag{6.57}$$

$$\varepsilon_0\mathrm{div}\,\hat{\mathbf{E}} = \dot{\rho}_{\mathrm{E}0} + \mathrm{div}\,\dot{\mathbf{P}}_{l0}, \tag{6.58}$$

$$\begin{aligned}\mu_0^{-1}\mathrm{curl}\left[\{(1+\mathrm{div}\,\mathbf{u})\mathbf{I} + 2\mathbf{L}\}\,\dot{\mathbf{B}}_{l0}\right] - \varepsilon_0\mathrm{curl}\left(\mathbf{u}_{,t}\times\mathbf{E} + \mathbf{v}\times\hat{\mathbf{E}}\right) - \varepsilon_0\hat{\mathbf{E}}_{,t}\\ = \mathrm{curl}\,\dot{\mathbf{M}}_{el0} + \dot{\mathbf{P}}_{l0,t} + [(\mathrm{div}\,\mathbf{v})\mathbf{I} - \mathbf{\Gamma}]\,\dot{\mathbf{P}}_{l0} + \dot{\mathbf{J}}_{\mathrm{E}0},\end{aligned} \tag{6.59}$$

where

$$\hat{\mathbf{E}} = \dot{\mathbf{E}}_{l0} + (\mathrm{div}\,\mathbf{u})\mathbf{E} - (\mathbf{L}+\mathbf{L}^{\mathrm{T}})\mathbf{E}, \tag{6.60}$$

$$\dot{\mathbf{E}}_{el0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_{el} = \dot{\mathbf{E}}_{l0} + \mathbf{v}\times\dot{\mathbf{B}}_{l0} + (\mathbf{u}_{,t} - \mathbf{L}\mathbf{v})\times\mathbf{B}, \tag{6.61}$$

$$\dot{\mathbf{J}}_{E0} = J^{-1}\mathbf{F}\dot{\mathbf{J}}_E = \boldsymbol{\xi}\hat{\mathbf{E}} - \dot{\rho}_{\mathrm{E}0}\mathbf{v} - \rho_{\mathrm{e}}\left(\mathbf{u}_{,t} - \mathbf{L}\mathbf{v}\right). \tag{6.62}$$

The incremented momentum and angular momentum balance equations are given as

$$\mathrm{Div}\,\dot{\mathbf{T}} + J(\mathrm{div}\,\mathbf{u})\mathbf{f}_E + J\dot{\mathbf{f}}_E = \rho_r\dot{\mathbf{a}}, \tag{6.63}$$

$$\boldsymbol{\varepsilon}(\mathbf{LFT} + \mathbf{F}\dot{\mathbf{T}}) + J(\operatorname{div}\mathbf{u})\mathbf{L}_E + J\dot{\mathbf{L}}_E = \mathbf{0}. \tag{6.64}$$

where the increments in electromagnetic body force and moment are given by

$$\begin{aligned}
\dot{\mathbf{f}}_E = J^{-1}\Big\{&-(\operatorname{div}\mathbf{u})\rho_{\mathrm{E}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l + \dot{\rho}_{\mathrm{E}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l - \rho_{\mathrm{E}}\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l + \rho_{\mathrm{E}}\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_l\Big\} \\
&-J^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\left[\operatorname{Grad}(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l)\right]^{\mathrm{T}}(\mathbf{FP}_l) \\
&+J^{-1}\mathbf{F}^{-\mathrm{T}}\Big[\operatorname{Grad}(-\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l + \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_l)\Big]^{\mathrm{T}}(\mathbf{FP}_l) \\
&+J^{-1}\mathbf{F}^{-\mathrm{T}}\left[\operatorname{Grad}(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l)\right]^{\mathrm{T}}\Big[-(\operatorname{div}\mathbf{u})\mathbf{FP}_l + \mathbf{LFP}_l + \mathbf{F}\dot{\mathbf{P}}_l\Big] \\
&+2J^{-2}(\operatorname{div}\mathbf{u})(\mathbf{FJ}_l)\times(\mathbf{FB}_l) + J^{-2}\left(\mathbf{LFJ}_l + \mathbf{F}\dot{\mathbf{J}}_l\right)\times(\mathbf{FB}_l) \\
&+J^{-2}(\mathbf{FJ}_l)\times(\mathbf{LFB}_l + \mathbf{F}\dot{\mathbf{B}}_l) - \mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\left[\operatorname{Grad}\left(J^{-1}\mathbf{FB}_l\right)\right]^{\mathrm{T}}(\mathbf{F}^{-\mathrm{T}}\mathbf{M}_l) \\
&+\mathbf{F}^{-\mathrm{T}}\Big[\operatorname{Grad}\Big(-J^{-1}(\operatorname{div}\mathbf{u})\mathbf{FB}_l + J^{-1}\mathbf{LFB}_l + J^{-1}\mathbf{F}\dot{\mathbf{B}}_l\Big)\Big]^{\mathrm{T}}(\mathbf{F}^{-\mathrm{T}}\mathbf{M}_l) \\
&+\mathbf{F}^{-\mathrm{T}}\left[\operatorname{Grad}(J^{-1}\mathbf{FB}_l)\right]^{\mathrm{T}}\Big(-\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{M}_l + \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{M}}_l\Big) \\
&+\frac{\partial}{\partial t}\Big[2J^{-2}(\operatorname{div}\mathbf{u})(\mathbf{FP}_l)\times(\mathbf{FB}_l) + J^{-2}\left(\mathbf{LFP}_l + \mathbf{F}\dot{\mathbf{P}}_l\right)\times(\mathbf{FB}_l) \\
&+J^{-2}(\mathbf{FP}_l)\times\Big(\mathbf{LFB}_l + \mathbf{F}\dot{\mathbf{B}}_l\Big)\Big] - J^{-1}(\operatorname{div}\mathbf{u})\operatorname{Div}\left[J^{-1}\mathbf{V}\otimes\{(\mathbf{FP}_l\times(\mathbf{FB}_l)\}\right] \\
&+J^{-1}\operatorname{Div}\Big[J^{-1}\dot{\mathbf{V}}\otimes\{(\mathbf{FP}_l)\times(\mathbf{FB}_l)\} - J^{-1}(\operatorname{div}\mathbf{u})\mathbf{V}\otimes\{(\mathbf{FP}_l)\times(\mathbf{FB}_l)\} \\
&+J^{-1}\mathbf{V}\otimes\Big\{\Big(\mathbf{LFP}_l + \mathbf{F}\dot{\mathbf{P}}_l\Big)\times(\mathbf{FB}_l) + (\mathbf{FP}_l)\times\Big(\mathbf{LFB}_l + \mathbf{F}\dot{\mathbf{B}}_l\Big)\Big\}\Big]
\end{aligned} \tag{6.65}$$

$$\begin{aligned}
\dot{\mathbf{L}}_E = &-J^{-1}(\operatorname{div}\mathbf{u})\left\{(\mathbf{FP}_l)\times\left(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l\right) + \left(\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el}\right)\times(\mathbf{FB}_l)\right\} \\
&+J^{-1}\left(\mathbf{LFP}_l + \mathbf{F}\dot{\mathbf{P}}_l\right)\times\left(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l\right) + J^{-1}(\mathbf{FP}_l)\times\left(-\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_l + \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_l\right) \\
&+J^{-1}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el}\right)\times\left(\mathbf{LFB}_l + \mathbf{F}\dot{\mathbf{B}}_l\right) + J^{-1}\left(-\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el} + \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{M}}_{el}\right)\times(\mathbf{FB}_l)\,.
\end{aligned} \tag{6.66}$$

When updated to Eulerian form, the balance equations become

$$\operatorname{div}\dot{\mathbf{T}}_0 + (\operatorname{div}\mathbf{u})\mathbf{f}_e + \dot{\mathbf{f}}_{E0} = \rho\mathbf{u}_{,tt}, \tag{6.67}$$

$$\boldsymbol{\varepsilon}(\mathbf{L}\boldsymbol{\tau} + \dot{\mathbf{T}}_0) + (\operatorname{div}\mathbf{u})\mathbf{L}_e + \dot{\mathbf{L}}_{E0} = \mathbf{0}, \tag{6.68}$$

where $\dot{\mathbf{f}}_{E0}$ and $\dot{\mathbf{L}}_{E0}$ are the push-forward forms of the incremental body force and moment, respectively, and are given by

$$\begin{aligned}
\dot{\mathbf{f}}_{E0} = &-(\operatorname{div}\mathbf{u})\rho_{\mathrm{e}}\mathbf{E} + \dot{\rho}_{E0}\mathbf{E} - \rho_{\mathrm{e}}\mathbf{L}^{\mathrm{T}}\mathbf{E} + \rho_{\mathrm{e}}\dot{\mathbf{E}}_{l0} - \mathbf{L}^{\mathrm{T}}\left(\operatorname{grad}\mathbf{E}\right)^{\mathrm{T}}\mathbf{P} \\
&+ \left[\operatorname{grad}\left(-\mathbf{L}^{\mathrm{T}}\mathbf{E} + \dot{\mathbf{E}}_{l0}\right)\right]^{\mathrm{T}}\mathbf{P} + (\operatorname{grad}\mathbf{E})^{\mathrm{T}}\left[-(\operatorname{div}\mathbf{u})\mathbf{P} + \mathbf{L}\mathbf{P} + \dot{\mathbf{P}}_{l0}\right] \\
&-2(\operatorname{div}\mathbf{u})\,\mathbf{J}\times\mathbf{B} + \left(\mathbf{L}\mathbf{J} + \dot{\mathbf{J}}_{l0}\right)\times\mathbf{B} + \mathbf{J}\times\left(\mathbf{L}\mathbf{B} + \dot{\mathbf{B}}_{l0}\right) - \mathbf{L}^{\mathrm{T}}\left(\operatorname{grad}\mathbf{B}\right)^{\mathrm{T}}\mathbf{M} \\
&+ \left[\operatorname{grad}\left(-(\operatorname{div}\mathbf{u})\mathbf{B} + \mathbf{L}\mathbf{B} + \dot{\mathbf{B}}_{l0}\right)\right]^{\mathrm{T}}\mathbf{M} + (\operatorname{grad}\mathbf{B})^{\mathrm{T}}\left(-\mathbf{L}^{\mathrm{T}}\mathbf{M} + \dot{\mathbf{M}}_{l0}\right) \\
&+\frac{\partial}{\partial t}\left[2(\operatorname{div}\mathbf{u})\mathbf{P}\times\mathbf{B} + \left(\mathbf{L}\mathbf{P} + \dot{\mathbf{P}}_{l0}\right)\times\mathbf{B} + \mathbf{P}\times\left(\mathbf{L}\mathbf{B} + \dot{\mathbf{B}}_{l0}\right)\right] \\
&-(\operatorname{div}\mathbf{u})\operatorname{div}\left[\mathbf{v}\otimes(\mathbf{P}\times\mathbf{B})\right] + \operatorname{div}\left[(\dot{\mathbf{v}} - \mathbf{L}\mathbf{v})\otimes(\mathbf{P}\times\mathbf{B}) - (\operatorname{div}\mathbf{u})\mathbf{v}\otimes(\mathbf{P}\times\mathbf{B})\right. \\
&\left. +\mathbf{v}\otimes\left\{\left(\mathbf{L}\mathbf{P} + \dot{\mathbf{P}}_{l0}\right)\times\mathbf{B}\right\} + \mathbf{P}\times\left(\mathbf{L}\mathbf{B} + \dot{\mathbf{B}}_{l0}\right)\right],
\end{aligned} \tag{6.69}$$

$$\begin{aligned}
\dot{\mathbf{L}}_{E0} = &-(\operatorname{div}\mathbf{u})\left(\mathbf{P}\times\mathbf{E} + \mathbf{M}_e\times\mathbf{B}\right) + \left(\mathbf{L}\mathbf{P} + \dot{\mathbf{P}}_{l0}\right)\times\mathbf{E} + \mathbf{P}\times\left(-\mathbf{L}^{\mathrm{T}}\mathbf{E} + \dot{\mathbf{E}}_{l0}\right) \\
&+\mathbf{M}_e\times\left(\mathbf{L}\mathbf{B} + \dot{\mathbf{B}}_{l0}\right) + \left(-\mathbf{L}^{\mathrm{T}}\mathbf{M}_e + \dot{\mathbf{M}}_{el0}\right)\times\mathbf{B}.
\end{aligned} \tag{6.70}$$

The heat equation (6.29) can be incremented to give

$$\begin{aligned}
\operatorname{Div}\dot{\mathbf{q}}_l + \rho_r c_p\frac{\partial}{\partial t}\left[J^{-1}\dot{\vartheta}_l - J^{-1}(\operatorname{div}\mathbf{u})\vartheta_l\right] = &\ \dot{q}_l + \dot{w}_E + \dot{\mathbf{T}}\colon \operatorname{Grad}(\mathbf{F}\mathbf{V}) \\
&+\mathbf{T}\colon \operatorname{Grad}(\mathbf{L}\mathbf{F}\mathbf{V} + \mathbf{F}\dot{\mathbf{V}}),
\end{aligned} \tag{6.71}$$

which when updated to the Eulerian configuration becomes

$$\operatorname{div}\dot{\mathbf{q}}_{l0} + \rho c_p\frac{\partial}{\partial t}\left[\dot{\vartheta}_{l0} - (\operatorname{div}\mathbf{u})\vartheta\right] = \dot{q}_{l0} + \dot{w}_{E0} + \dot{\mathbf{T}}_0\colon \operatorname{grad}\mathbf{v} + \boldsymbol{\tau}\colon \operatorname{grad}\dot{\mathbf{v}}. \tag{6.72}$$

We have used the push-forward relations $\dot{\mathbf{q}}_{l0} = J^{-1}\mathbf{F}\dot{\mathbf{q}}_l, \dot{\vartheta}_{l0} = J^{-1}\dot{\vartheta}_l, \dot{w}_{E0} = J^{-1}\dot{w}_E$, and $\dot{q}_{l0} = J^{-1}\dot{q}_l$ to effect the above transformation, and the increments in the electromagnetic power is given by

$$\begin{aligned}
\dot{w}_E = &\left(\mathbf{L}\mathbf{F}\mathbf{J}_{el} + \mathbf{F}\dot{\mathbf{J}}_{el}\right)\cdot\left(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_{el}\right) + \left(\mathbf{F}\mathbf{J}_{el}\right)\cdot\left(-\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_{el} + \mathbf{F}^{-\mathrm{T}}\cdot\dot{\mathbf{E}}_{el}\right) \\
&+\rho_r\left[\frac{\partial}{\partial t}\left(\frac{\mathbf{L}\mathbf{F}\mathbf{P}_l + \mathbf{F}\dot{\mathbf{P}}_l}{\rho_r}\right) + \operatorname{Grad}\left(\frac{\mathbf{L}\mathbf{F}\mathbf{P}_l + \mathbf{F}\dot{\mathbf{P}}_l}{\rho_r}\right)\mathbf{V} + \operatorname{Grad}\left(\frac{\mathbf{F}\mathbf{P}_l}{\rho_r}\right)\dot{\mathbf{V}}\right] \\
&\cdot\left(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_{el}\right) + \rho_r\left[\frac{\partial}{\partial t}\left(\frac{\mathbf{F}\mathbf{P}_l}{\rho_r}\right) + \operatorname{Grad}\left(\frac{\mathbf{F}\mathbf{P}_l}{\rho_r}\right)\mathbf{V}\right]\cdot\left(-\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{E}_{el} + \mathbf{F}^{-\mathrm{T}}\cdot\dot{\mathbf{E}}_{el}\right) \\
&-J\left((\operatorname{div}\mathbf{u})\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el} - \mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el} + \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{M}}_{el}\right)\cdot\left[\frac{\partial}{\partial t}\left(J^{-1}\mathbf{F}\mathbf{B}_l\right)\right. \\
&\left.+\operatorname{Grad}\left(J^{-1}\mathbf{F}\mathbf{B}_l\right)\mathbf{V}\right] - J\mathbf{F}^{-\mathrm{T}}\mathbf{M}_{el}\cdot\left[\frac{\partial}{\partial t}J^{-1}\left(-(\operatorname{div}\mathbf{u})\mathbf{F}\mathbf{B}_l + \mathbf{L}\mathbf{F}\mathbf{B}_l + \mathbf{F}\dot{\mathbf{B}}_l\right)\right. \\
&\left.+\operatorname{Grad}J^{-1}\left\{-(\operatorname{div}\mathbf{u})\mathbf{F}\mathbf{B}_l + \mathbf{L}\mathbf{F}\mathbf{B}_l + \mathbf{F}\dot{\mathbf{B}}_l\right\}\mathbf{V} + \operatorname{Grad}\left(J^{-1}\mathbf{F}\mathbf{B}_l\right)\dot{\mathbf{V}}\right].
\end{aligned} \tag{6.73}$$

In the Eulerian form, this gives

$$\begin{aligned}\dot{w}_{E0} &= \left(\mathbf{L}\mathbf{J}_e + \dot{\mathbf{J}}_{el0}\right)\cdot\mathbf{E}_e + \mathbf{J}_e\cdot\left(-\mathbf{L}^{\mathrm{T}}\mathbf{E}_e + \dot{\mathbf{E}}_{el0}\right)\\ \rho\left[\frac{d}{dt}\left(\frac{\mathbf{L}\mathbf{P}+\dot{\mathbf{P}}_{l0}}{\rho}\right) + \operatorname{grad}\left(\frac{\mathbf{P}}{\rho}\right)(\dot{\mathbf{v}}-\mathbf{L}\mathbf{v})\right]\cdot\mathbf{E}_e &+ \rho\frac{d}{dt}\left(\frac{\mathbf{P}}{\rho}\right)\cdot\left(-\mathbf{L}^{\mathrm{T}}\mathbf{E}_e + \dot{\mathbf{E}}_{el0}\right)\\ -\left\{(\operatorname{div}\mathbf{u})\mathbf{M}_e - \mathbf{L}^{\mathrm{T}}\mathbf{M}_e + \dot{\mathbf{M}}_{el0}\right\}\cdot\frac{d\mathbf{B}}{dt} &- \mathbf{M}_e\cdot\left[\frac{d}{dt}\left\{-(\operatorname{div}\mathbf{u})\mathbf{B} + \mathbf{L}\mathbf{B} + \dot{\mathbf{B}}_{l0}\right\}\right.\\ &\left. + (\operatorname{grad}\mathbf{B})(\dot{\mathbf{v}}-\mathbf{L}\mathbf{v})\right]. \quad (6.74)\end{aligned}$$

On incrementing the constitutive equations (6.51), we obtain

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \boldsymbol{\mathcal{B}}\dot{\mathbf{E}}_{el} + \boldsymbol{\mathcal{C}}\dot{\mathbf{B}}_l + \boldsymbol{\mathcal{D}}\dot{\vartheta}_l, \quad (6.75)$$

$$\dot{\mathbf{P}}_l = -\left(\boldsymbol{\mathcal{F}}\dot{\mathbf{F}} + \boldsymbol{\mathcal{G}}\dot{\mathbf{E}}_{el} + \boldsymbol{\mathcal{H}}\dot{\mathbf{B}}_l + \boldsymbol{\mathcal{I}}\dot{\vartheta}_l\right), \quad (6.76)$$

and

$$\dot{\mathbf{M}}_{el} = -\left(\boldsymbol{\mathcal{K}}\dot{\mathbf{F}} + \boldsymbol{\mathcal{L}}\dot{\mathbf{E}}_{el} + \boldsymbol{\mathcal{M}}\dot{\mathbf{B}}_l + \boldsymbol{\mathcal{N}}\dot{\vartheta}_l\right), \quad (6.77)$$

where the moduli tensors are defined as

$$\begin{aligned}\boldsymbol{\mathcal{A}} &= \frac{\partial^2\Phi}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \boldsymbol{\mathcal{B}} = \frac{\partial^2\Phi}{\partial\mathbf{E}_{el}\partial\mathbf{F}}, \quad \boldsymbol{\mathcal{C}} = \frac{\partial^2\Phi}{\partial\mathbf{B}_l\partial\mathbf{F}}, \quad \boldsymbol{\mathcal{D}} = \frac{\partial^2\Phi}{\partial\vartheta_l\partial\mathbf{F}},\\ \boldsymbol{\mathcal{F}} &= \frac{\partial^2\Phi}{\partial\mathbf{F}\partial\mathbf{E}_{el}}, \quad \boldsymbol{\mathcal{G}} = \frac{\partial^2\Phi}{\partial\mathbf{E}_{el}\partial\mathbf{E}_{el}}, \quad \boldsymbol{\mathcal{H}} = \frac{\partial^2\Phi}{\partial\mathbf{B}_l\partial\mathbf{E}_{el}}, \quad \boldsymbol{\mathcal{I}} = \frac{\partial^2\Phi}{\partial\vartheta_l\partial\mathbf{E}_{el}},\\ \boldsymbol{\mathcal{K}} &= \frac{\partial^2\Phi}{\partial\mathbf{F}\partial\mathbf{B}_l}, \quad \boldsymbol{\mathcal{L}} = \frac{\partial^2\Phi}{\partial\mathbf{E}_{el}\partial\mathbf{B}_l}, \quad \boldsymbol{\mathcal{M}} = \frac{\partial^2\Phi}{\partial\mathbf{B}_l\partial\mathbf{B}_l}, \quad \boldsymbol{\mathcal{N}} = \frac{\partial^2\Phi}{\partial\vartheta_l\partial\mathbf{B}_l}. \quad (6.78)\end{aligned}$$

We note that $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$ defined here are different from those in Chapters 2–5 and in Appendices A and B. Products in (6.75), (6.76), and (6.77) are defined in component form as

$$\begin{aligned}&(\boldsymbol{\mathcal{A}}\dot{\mathbf{F}})_{\alpha i} = \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, \quad (\boldsymbol{\mathcal{B}}\dot{\mathbf{E}}_{el})_{\alpha i} = \mathcal{B}_{\alpha i|\beta}\dot{E}_{el\beta}, \quad (\boldsymbol{\mathcal{C}}\dot{\mathbf{B}}_l)_{\alpha i} = \mathcal{C}_{\alpha i|\beta}\dot{B}_{l\beta},\\ &(\boldsymbol{\mathcal{F}}\dot{\mathbf{F}})_i = \mathcal{F}_{i|\alpha j}\dot{F}_{j\alpha}, \quad (\boldsymbol{\mathcal{G}}\dot{\mathbf{E}}_{el})_\alpha = \mathcal{G}_{\alpha\beta}\dot{E}_{el\beta}, \quad (\boldsymbol{\mathcal{H}}\dot{\mathbf{B}}_l)_\alpha = \mathcal{H}_{\alpha\beta}\dot{B}_{l\beta},\\ &(\boldsymbol{\mathcal{K}}\dot{\mathbf{F}})_i = \mathcal{K}_{i|\alpha j}\dot{F}_{j\alpha}, \quad (\boldsymbol{\mathcal{L}}\dot{\mathbf{E}}_{el})_\alpha = \mathcal{L}_{\alpha\beta}\dot{E}_{el\beta}, \quad (\boldsymbol{\mathcal{M}}\dot{\mathbf{B}}_l)_\alpha = \mathcal{M}_{\alpha\beta}\dot{B}_{l\beta}, \quad (6.79)\end{aligned}$$

and

$$\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{C}}^{\mathrm{T}}, \quad \boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{B}}^{\mathrm{T}}, \quad \boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{H}}^{\mathrm{T}}. \quad (6.80)$$

On updating the incremented constitutive equations (6.75), (6.76), and (6.77), we obtain

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{B}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{D}}_0\dot{\vartheta}_{l0}, \quad (6.81)$$

$$\dot{\mathbf{P}}_{l0} = -\left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{G}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{H}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{I}}_0\dot{\vartheta}_{l0}\right), \tag{6.82}$$

and

$$\dot{\mathbf{M}}_{el0} = -\left(\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{M}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{N}}_0\dot{\vartheta}_{l0}\right), \tag{6.83}$$

where the updated moduli tensors are defined as

$$\begin{aligned} \mathcal{A}_{0piqj} = J^{-1}F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j}, \quad \mathcal{B}_{0ij|k} = J^{-1}F_{i\alpha}F_{k\beta}\mathcal{B}_{\alpha j|\beta}, \quad \mathcal{C}_{0ij|k} = F_{i\alpha}F^{-1}_{\beta k}\mathcal{C}_{\alpha j|\beta}, \\ \mathcal{D}_{0ij} = F_{ik}\mathcal{D}_{kj}, \quad \mathcal{G}_{0ij} = J^{-1}F_{i\alpha}F_{j\beta}\mathcal{G}_{\alpha\beta}, \quad \mathcal{H}_{0ij} = F_{i\alpha}F^{-1}_{\beta j}\mathcal{H}_{\alpha\beta}, \\ \mathcal{I}_{0i} = F_{ik}\mathcal{I}_{k}, \quad \mathcal{M}_{0ij} = JF^{-1}_{\alpha i}F^{-1}_{\beta j}\mathcal{M}_{\alpha\beta}, \quad \mathcal{N}_{0i} = J^{-1}F^{-1}_{ki}\mathcal{N}_{k}, \end{aligned} \tag{6.84}$$

and $\dot{\vartheta}_{l0} = J\dot{\vartheta}_l$.

Incrementing and updating the constitutive equation (6.43), we get

$$\dot{\mathbf{q}}_{l0} = -(\operatorname{div}\mathbf{u})\boldsymbol{\kappa}\operatorname{grad}\vartheta + 2\mathbf{L}\boldsymbol{\kappa}\operatorname{grad}\vartheta - \boldsymbol{\kappa}\operatorname{grad}\left[\dot{\vartheta}_{l0} - (\operatorname{div}\mathbf{u})\vartheta\right], \tag{6.85}$$

while the updated incremented form of (6.45) is

$$\dot{\mathbf{J}}_{l0} = \boldsymbol{xi}\left[\left\{(\operatorname{div}\mathbf{u})\mathbf{I} - \mathbf{L} - \mathbf{L}^{\mathrm{T}}\right\}\mathbf{E} + \dot{\mathbf{E}}_{l0}\right]. \tag{6.86}$$

On substituting the incremented updated constitutive equations into the incremented updated balance equations (6.58), (6.59), (6.67), (6.68), and (6.72), we obtain

$$\begin{aligned} \varepsilon_0\operatorname{div}\left[\dot{\mathbf{E}}_{l0} + \left\{(\operatorname{div}\mathbf{u})\mathbf{I} - (\mathbf{L}+\mathbf{L}^{\mathrm{T}})\right\}\mathbf{E}\right] = \dot{\rho}_{E0} \\ -\operatorname{div}\left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{G}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{H}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{I}}_0\dot{\vartheta}_{l0}\right), \end{aligned} \tag{6.87}$$

$$\begin{aligned} \mu_0^{-1}\operatorname{curl}\left[\{(1+\operatorname{div}\mathbf{u})\mathbf{I} + 2\mathbf{L}\}\dot{\mathbf{B}}_{l0}\right] - \varepsilon_0\operatorname{curl}\left(\mathbf{u}_{,t}\times\mathbf{E} + \mathbf{v}\times\hat{\mathbf{E}}\right) - \varepsilon_0\hat{\mathbf{E}}_{,t} \\ = -\operatorname{curl}\left(\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{M}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{N}}_0\dot{\vartheta}_{l0}\right) + \dot{\mathbf{J}}_{\mathrm{E}0} \\ - [(\operatorname{div}\mathbf{u})\mathbf{I} - \boldsymbol{\Gamma}]\left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{G}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{H}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{I}}_0\dot{\vartheta}_{l0}\right) \\ - \left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{G}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{H}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{I}}_0\dot{\vartheta}_{l0}\right)_{,t}, \end{aligned} \tag{6.88}$$

$$\operatorname{div}\left(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{B}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{D}}_0\dot{\vartheta}_{l0}\right) + (\operatorname{div}\mathbf{u})\mathbf{f}_e + \dot{\mathbf{f}}_{E0} = \rho\mathbf{u}_{,tt}, \tag{6.89}$$

$$\boldsymbol{\varepsilon}\left(\mathbf{L}\boldsymbol{\tau} + \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{B}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{D}}_0\dot{\vartheta}_{l0}\right) + (\operatorname{div}\mathbf{u})\mathbf{L}_e + \dot{\mathbf{L}}_{E0} = \mathbf{0}, \tag{6.90}$$

$$\begin{aligned} \operatorname{div}\left[-(\operatorname{div}\mathbf{u})\boldsymbol{\kappa}\operatorname{grad}\vartheta + 2\mathbf{L}\boldsymbol{\kappa}\operatorname{grad}\vartheta - \boldsymbol{\kappa}\operatorname{grad}\left(\dot{\vartheta}_{l0} - (\operatorname{div}\mathbf{u})\vartheta\right)\right] \\ = -\rho c_p\frac{\partial}{\partial t}\left[\dot{\vartheta}_{l0} - (\operatorname{div}\mathbf{u})\vartheta\right] + \dot{q}_{l0} + \dot{w}_{E0} + \boldsymbol{\tau}:\operatorname{grad}\mathbf{u}_{,t} \\ + \left(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{B}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{D}}_0\dot{\vartheta}_{l0}\right):\operatorname{grad}\mathbf{v}, \end{aligned} \tag{6.91}$$

along with (6.56) and (6.57).

## 6.4 Application to Electromagnetic Acoustic Transduction (EMAT) process

We now specialize the equations derived until now for application to EMATs. Let the magnetoelastic conductor occupy a region $\mathcal{B}$ while the current carrying coil occupy the region $\mathcal{P}$. Let their respective boundaries be denoted by $\partial\mathcal{B}$ and $\partial\mathcal{P}$. In the presence of a bias magnetic field (which is a uniform field $\mathbf{B}$ at infinity), the current coil carries an alternating current $I(t)$ as shown in Figure 1.2.

If the relative electric permittivity and relative magnetic permeability of the current carrying coil be given by constants $\varepsilon_r$ and $\mu_r$ respectively, then the governing Maxwell's equations of the incremental fields to be satisfied in $\mathcal{P}$ are

$$\operatorname{div}\dot{\mathbf{B}} = 0, \quad \varepsilon_r \operatorname{div}\dot{\mathbf{E}} = 0, \tag{6.92}$$

$$\operatorname{curl}\dot{\mathbf{E}} = -\frac{\partial\dot{\mathbf{B}}}{\partial t}, \quad \frac{1}{\mu_r}\operatorname{curl}\dot{\mathbf{B}} = \dot{\mathbf{J}} + \varepsilon_r\frac{\partial\dot{\mathbf{E}}}{\partial t}. \tag{6.93}$$

The electric current density $\mathbf{J}$ is integrated along the cross-section of the current-carrying wire to obtain $I(t)$. At the boundary $\partial\mathcal{P}$, the following conditions need to be satisfied

$$\left(\dot{\mathbf{B}} - \dot{\mathbf{B}}^*\right)\cdot\mathbf{n} = 0, \quad \left(\dot{\mathbf{H}} - \dot{\mathbf{H}}^*\right)\times\mathbf{n} = \mathbf{0}, \tag{6.94}$$

where $\mathbf{n}$ is the normal to the boundary and a superscript $^*$ represents a quantity in vacuum.

In the magnetoelastic conducting bulk, we consider the material to be incompressible. This imposes the constraint

$$\operatorname{div}\mathbf{u} = 0, \tag{6.95}$$

and the constitutive law for stress is modified to

$$\mathbf{T} = \frac{\partial\Phi}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \tag{6.96}$$

to include the Lagrange multiplier $p$ which is associated with the incompressibility constraint. This changes the incremental nominal stress to

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{B}}_0\dot{\mathbf{E}}_{l0} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{D}}_0\dot{\vartheta}_{l0} - \dot{p}\mathbf{I} + p\mathbf{L}. \tag{6.97}$$

Normally, in the context of EMATs, the material is magnetized with a bias field. Hence, we assume that there is no underlying finite electric field or charges. There will, however, be incremental electric field generated due to secondary effects. For further simplicity we take the underlying underlying configuration to be static ($\mathbf{v} \equiv \mathbf{0}$). Then the incremental governing equations (6.56), (6.57), (6.87)–(6.90) are simplified to

$$\operatorname{div} \dot{\mathbf{B}}_{l0} = 0, \tag{6.98}$$

$$\operatorname{curl} \dot{\mathbf{E}}_{el0} = -\dot{\mathbf{B}}_{l0,t}, \tag{6.99}$$

$$\varepsilon_0 \operatorname{div}\dot{\mathbf{E}}_{l0} = -\operatorname{div}\left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{G}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{H}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{I}}_0\dot{\vartheta}_{l0}\right), \tag{6.100}$$

$$\begin{aligned}\mu_0^{-1}\operatorname{curl}\left[(\mathbf{I}+2\mathbf{L})\,\dot{\mathbf{B}}_{l0}\right] - \varepsilon_0\dot{\mathbf{E}}_{l0,t} = &-\operatorname{curl}\left(\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{M}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{N}}_0\dot{\vartheta}_{l0}\right)\\ &+\dot{\mathbf{J}}_{\mathrm{E}0} - \left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{G}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{H}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{I}}_0\dot{\vartheta}_{l0}\right)_{,t},\end{aligned} \tag{6.101}$$

$$\operatorname{div}\left(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{B}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{D}}_0\dot{\vartheta}_{l0}\right) - \operatorname{grad}\dot{p} + \dot{\mathbf{f}}_{E0} = \rho\mathbf{u}_{,tt}, \tag{6.102}$$

$$\boldsymbol{\varepsilon}\left(\mathbf{L}\boldsymbol{\tau} + \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{B}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{D}}_0\dot{\vartheta}_{l0} - \dot{p}\mathbf{I} + p\mathbf{L}\right) + \dot{\mathbf{L}}_{E0} = \mathbf{0}. \tag{6.103}$$

$$\operatorname{div}\left[\kappa\operatorname{grad}\dot{\vartheta}_{l0}\right] = \rho c_p\frac{\partial}{\partial t}\dot{\vartheta}_{l0} - \dot{w}_{E0} - \boldsymbol{\tau}\colon\operatorname{grad}\mathbf{u}_{,t}. \tag{6.104}$$

For the above-stated simplifications, $\dot{\mathbf{f}}_{E0}$, $\dot{\mathbf{L}}_{E0}$, and $\dot{w}_{E0}$ are reduced to

$$\begin{aligned}\dot{\mathbf{f}}_{E0} =& \dot{\mathbf{J}}_{l0}\times\mathbf{B} - \mathbf{L}^{\mathrm{T}}(\operatorname{grad}\mathbf{B})^{\mathrm{T}}\mathbf{M} + \left[\operatorname{grad}(\mathbf{L}\mathbf{B}+\dot{\mathbf{B}}_{l0})\right]^{\mathrm{T}}\mathbf{M}\\ &+(\operatorname{grad}\mathbf{B})^{\mathrm{T}}\left(-\mathbf{L}^{\mathrm{T}}\mathbf{M}+\dot{\mathbf{M}}_{l0}\right) + \frac{\partial}{\partial t}\left(\dot{\mathbf{P}}_{l0}\times\mathbf{B}\right),\\ =& \dot{\mathbf{J}}_{l0}\times\mathbf{B} - \mathbf{L}^{\mathrm{T}}(\operatorname{grad}\mathbf{B})^{\mathrm{T}}\mathbf{M} + \left[\operatorname{grad}(\mathbf{L}\mathbf{B}+\dot{\mathbf{B}}_{l0})\right]^{\mathrm{T}}\mathbf{M}\\ &-(\operatorname{grad}\mathbf{B})^{\mathrm{T}}\left(\mathbf{L}^{\mathrm{T}}\mathbf{M} + \boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{M}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{N}}_0\dot{\vartheta}_{l0}\right)\\ &+\frac{\partial}{\partial t}\left[\mathbf{B}\times\left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{G}}_0\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{H}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{I}}_0\dot{\vartheta}_{l0}\right)\right],\end{aligned} \tag{6.105}$$

$$\begin{aligned}\dot{\mathbf{L}}_{E0} =& \mathbf{M}_e\times\left(\mathbf{L}\mathbf{B}+\dot{\mathbf{B}}_{l0}\right) + \left(-\mathbf{L}^{\mathrm{T}}\mathbf{M}_e + \dot{\mathbf{M}}_{el0}\right)\times\mathbf{B},\\ =& \mathbf{M}\times\left(\mathbf{L}\mathbf{B}+\dot{\mathbf{B}}_{l0}\right) - \left(\mathbf{L}^{\mathrm{T}}\mathbf{M} + \boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\dot{\mathbf{E}}_{el0}\right.\\ &\left.+\boldsymbol{\mathcal{M}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{N}}_0\dot{\vartheta}_{l0}\right)\times\mathbf{B},\end{aligned} \tag{6.106}$$

$$\begin{aligned}\dot{w}_{E0} =& -\left\{-\mathbf{L}^{\mathrm{T}}\mathbf{M}_e + \dot{\mathbf{M}}_{el0}\right\}\cdot\frac{d\mathbf{B}}{dt} - \mathbf{M}_e\cdot\left[\frac{d}{dt}\left\{\mathbf{L}\mathbf{B}+\dot{\mathbf{B}}_{l0}\right\} + (\operatorname{grad}\mathbf{B})\,\mathbf{u}_{,t}\right],\\ =& \left[\mathbf{L}^{\mathrm{T}}\mathbf{M} + \boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\dot{\mathbf{E}}_{el0} + \boldsymbol{\mathcal{M}}_0\dot{\mathbf{B}}_{l0} + \boldsymbol{\mathcal{N}}_0\dot{\vartheta}_{l0}\right]\cdot\frac{d\mathbf{B}}{dt}\\ &-\mathbf{M}\cdot\left[\frac{d}{dt}\left\{\mathbf{L}\mathbf{B}+\dot{\mathbf{B}}_{l0}\right\} + (\operatorname{grad}\mathbf{B})\,\mathbf{u}_{,t}\right].\end{aligned} \tag{6.107}$$

As a simplification, we take $\boldsymbol{\kappa} = \kappa\mathbf{I}, \boldsymbol{\xi} = \xi\mathbf{I}$, $\mathbf{I}$ being the identity tensor, and consider the finite deformation and the initial (residual) stress to be homogeneous. This causes the moduli tensors to be uniform and hence we can now consider the propagation of bulk homogeneous waves whose direction of propagation is given by the unit vector $\mathbf{n}$ and the wave speed is given by $v$. Thus, we seek the solutions of the above incremental equations of the form

$$\begin{aligned} \mathbf{u} = \mathbf{m}\, f(\mathbf{n}\cdot\mathbf{x} - vt), \quad \dot{\mathbf{B}}_{l0} = \mathbf{q}\, g(\mathbf{n}\cdot\mathbf{x} - vt), \quad \dot{\mathbf{E}}_{el0} = \mathbf{r}\, h(\mathbf{n}\cdot\mathbf{x} - vt), \\ \dot{p} = P(\mathbf{n}\cdot\mathbf{x} - vt), \quad \dot{\vartheta} = T(\mathbf{n}\cdot\mathbf{x} - vt), \end{aligned} \tag{6.108}$$

where $\mathbf{m}, \mathbf{q}$, and $\mathbf{r}$ are unit vectors specifying the polarizations of $\mathbf{u}, \dot{\mathbf{B}}_{l0}$, and $\dot{\mathbf{E}}_{l0}$, respectively. From equations (6.95) and (6.98), we get

$$\mathbf{m}\cdot\mathbf{n} = 0, \quad \mathbf{q}\cdot\mathbf{n} = 0, \tag{6.109}$$

while equations (6.99)–(6.104) give

$$\mathbf{n}\times\mathbf{r}h' = vg'\mathbf{q}, \tag{6.110}$$

$$\begin{aligned} &-\varepsilon_0 h'\mathbf{r}\cdot\mathbf{n} - \varepsilon_0 f'' v\mathbf{n}\cdot(\mathbf{m}\times\mathbf{B}) \\ &\quad = (\boldsymbol{\mathcal{B}}_0\mathbf{n})\colon(\mathbf{m}\otimes\mathbf{n})f'' + \boldsymbol{\mathcal{G}}_0\colon(\mathbf{r}\otimes\mathbf{n})h' + \boldsymbol{\mathcal{H}}_0\colon(\mathbf{q}\otimes\mathbf{n})g' + \boldsymbol{\mathcal{I}}_0\cdot\mathbf{n}T', \end{aligned} \tag{6.111}$$

$$\begin{aligned} \mu_0^{-1}\mathbf{n}\times\mathbf{q}\,g' - \varepsilon_0 vh'\mathbf{r} = -\mathbf{n}\times\left[\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}(\mathbf{m}\otimes\mathbf{n})f'' + \boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\mathbf{r}h' + \boldsymbol{\mathcal{M}}_0\mathbf{q}g' + \boldsymbol{\mathcal{N}}_0 T'\right] \\ -\xi\mathbf{r}h + \rho_e v f'\mathbf{m} + v\left(f''\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}(\mathbf{m}\otimes\mathbf{n}) + h'\boldsymbol{\mathcal{G}}_0\mathbf{r} + g'\boldsymbol{\mathcal{H}}_0\mathbf{q} + T'\boldsymbol{\mathcal{I}}_0\right), \end{aligned} \tag{6.112}$$

$$\begin{aligned} &\left(\boldsymbol{\mathcal{A}}_0(\mathbf{m}\otimes\mathbf{n})\right)^{\mathrm{T}}\mathbf{n}f'' + (\boldsymbol{\mathcal{B}}_0\mathbf{r})^{\mathrm{T}}\mathbf{n}h' + (\boldsymbol{\mathcal{C}}_0\mathbf{q})^{\mathrm{T}}\mathbf{n}g' + \boldsymbol{\mathcal{D}}_0\mathbf{n}T' - \mathbf{n}P' \\ &\quad +\xi h\mathbf{r}\times\mathbf{B} + f''(\mathbf{n}\otimes\mathbf{n})(\mathbf{B}\otimes\mathbf{m})\mathbf{M} + g'(\mathbf{n}\otimes\mathbf{q})\mathbf{M} \\ &\quad -v\mathbf{B}\times\left(\boldsymbol{\mathcal{B}}_0^{\mathrm{T}}(\mathbf{m}\otimes\mathbf{n})f'' + h'\boldsymbol{\mathcal{G}}_0\mathbf{r} + g'\boldsymbol{\mathcal{H}}_0\mathbf{q} + \boldsymbol{\mathcal{I}}_0 T'\right) = \rho v^2\mathbf{m}f'', \end{aligned} \tag{6.113}$$

$$\begin{aligned} &\left\{\mathbf{m}\times\left(\boldsymbol{\tau}^{\mathrm{T}}\mathbf{n}\right) + \boldsymbol{\varepsilon}\left(\boldsymbol{\mathcal{A}}_0(\mathbf{m}\otimes\mathbf{n})\right) + p\mathbf{m}\times\mathbf{n}\right\}f' + +\boldsymbol{\varepsilon}\left(\boldsymbol{\mathcal{B}}_0\mathbf{r}h + \boldsymbol{\mathcal{C}}_0\mathbf{q}g + \boldsymbol{\mathcal{D}}_0 T\right) \\ &\quad +\mathbf{M}\times\left(f'(\mathbf{m}\otimes\mathbf{n})\mathbf{B} + g\mathbf{q}\right) + \mathbf{B}\times\left(f'\mathbf{n}\otimes\mathbf{m} + f'\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}(\mathbf{m}\otimes\mathbf{n}) + h\boldsymbol{\mathcal{H}}_0^{\mathrm{T}}\mathbf{r}\right. \\ &\quad \left. + g\boldsymbol{\mathcal{M}}_0\mathbf{q} + \boldsymbol{\mathcal{N}}_0 T\right) = \mathbf{0}, \end{aligned} \tag{6.114}$$

$$\rho v c_p T' + \kappa T''\mathbf{n}\cdot\mathbf{n} - v\boldsymbol{\tau}\colon(\mathbf{m}\otimes\mathbf{n}) + vg'\mathbf{M}\cdot\mathbf{q} = 0. \tag{6.115}$$

The above equations should, in theory, yield a propagation condition for bulk homogeneous waves similar to that in Equation (2.95). The governing equations (6.98)–(6.104) can be solved numerically to analyze wave propagation characteristics in the context of EMATs. Such analysis has been done, albeit for simpler cases of linear theory, in papers by [Ludwig et al., 1993], [Ogi, 1997], [Shapoorabadi et al., 2005], [Ribichini et al., 2010] and many references therein. For the more general problem presented here, our work is still in progress and we plan to deploy 3-D Finite Element Method for numerical simulations.

# Chapter 7

# Conclusions

## 7.1 Summary

Problems concerning electromagnetic and mechanical interactions in a solid continuum are both mathematically interesting and useful for engineering purposes. In particular, with the development of various synthetic magneto-sensitive elastomers, a need has arisen for better mathematical models explaining this phenomena. In this thesis, following the mathematical models of magnetoelasticity developed by [Dorfmann and Ogden, 2004], we have studied three boundary value problems concerning nonlinear deformations and wave propagation in finitely deformed magnetoelastic materials.

The basic governing equations of nonlinear deformations and incremental motions in magneoelastic solids were detailed in Chapter 2 where we also defined the magnetoelastic moduli tensors.

Rayleigh type waves on the surface of a finitely deformed magnetoelastic half-space were considered in Chapter 3. It was shown that magnetic field can have a significant effect on the speed of surface waves propagating in a half-space of magnetoelastic material and on the mechanical stability of the half-space. For each of the in-plane directions of the magnetic field an upper limit on the wave speed is obtained, similar to that obtained in the purely elastic case but with, in general, dependence on the magnetic field. In the absence of a magnetic field, the equations reduced to those of the purely elastic case given by [Dowaikh and Ogden, 1990], and for the purely static problem results on the stability of a magnetoelastic half-space due to [Otténio et al., 2008] were recovered.

For a Mooney–Rivlin type magnetoelastic material an initial magnetic induction in the sagittal plane in general destabilizes the material and surface waves exist only for values of the stretch beyond a certain critical value (which depends on the chosen material parameters). If the magnetic induction is in the direction of wave propagation, it has a significantly stronger effect than in the case when it is perpendicular to the direction of wave propagation within the sagittal plane. For configurations in which the half-space is stable the dependence of the surface wave speed on both the underlying finite deformation and the magnitude of the magnetic induction was illustrated graphically.

Love type waves on the surface of a finitely deformed layer magnetoelastic half-space were considered in Chapter 4. Similar to the previous case of Rayleigh type waves, the magnetic field can have a significant effect on the wave speed. A secular equation is obtained for the wave speed which is dispersive and multiple modes of wave propagation are obtained. In the absence of a magnetic field, the equation reduces to that of a purely elastic case given by [Achenbach, 1975] and [Dowaikh, 1999]. For a Mooney–Rivlin type magnetoelastic material, upper and lower bounds for wave speed were obtained which depend on the underlying magnetic field and material parameters in general. This imposes certain restrictions on the deformation and the admissible parameters of the energy functions of the layer and half-space for the existence of Love-type waves.

Dependence of wave speed on the finite deformation and in-plane magnetic field for a Mooney–Rivlin type material is illustrated graphically. For this material the problem reduces to a purely-elastic one when the underlying magnetic field is out-of-plane and we then consider a neo-Hookean magnetoelastic material to obtain solutions in this case. Wave speeds, in general, decrease with an increasing wave number. An in-plane magnetic field tends to increase the wave speed while an out-of-plane field decreases the wave speed in general. It is also shown that waves with an out-of-plane displacement can exist in the presence of an out-of-plane magnetic field without a layer. Such waves, analogous to Bleustein–Gulyaev waves in piezoelectric materials, do not exist in pure elasticity.

In Chapter 5, we considered finite axisymmetric deformations and motions of a thick-walled magnetoelastic tube in the presence of an axial and an azimuthal magnetic field. Variation of the total internal pressure and the axial load with magnetic field was studied for a Mooney–Rivlin type and a three-term (Ogden type) magnetoelastic energy function. An azimuthal magnetic field tends to increase the total internal pressure while an axial magnetic field has no effect for the type of materials considered. For these materials, an azimuthal magnetic field causes a compressional loading in general; while an axial magnetic field tends to create an extensional loading.

Thereafter, we considered axisymmetric waves propagating along the axis of the tube. When the underlying magnetic field is either in the axial or in the azimuthal direction, it is observed that the equations governing displacements in the azimuthal direction are decoupled from the equations governing displacements in the axial and radial directions. In the latter case, a unique solution is possible only when the underlying magnetic field vanishes and the problem is reduced to a pure elastic case as considered by [Vaughan, 1979] and [Haughton, 1984]. For azimuthal motions, multiple modes of wave propagation are obtained and the dependence of wave speed on axial stretch, underlying magnetic field, and material parameters is illustrated graphically. Numerical solutions obtained for a Mooney–Rivlin type magnetoelastic material show that, in general, increasing the axial magnetic field increases the wave speed monotonically, while increasing the azimuthal magnetic field first decreases and then increases the wave speed.

Finally, using the equations given by [Pao, 1978], we write the general equations of electrodynamics and thermodynamics in a finitely deformed, electrically conducting continuum. The equations are then linearized to consider wave propagation with a motivation to mathematically model EMATs.

## 7.2 Future work

Chapter 6 of this thesis is written with a motivation to form a basis for future research towards a general theoretical development of Thermodynamics and Electrodynamics in continuum solids. In particular, this could lead to development of better models and a proper understanding of EMATs.

The synthetically developed polymers as discussed in Figure 1.1 in Chapter 1 are, in general, not just elastic, but viscoelastic. Hence, to get a proper understanding of such materials and for a proper explanation of the experimental results, a theory of magneto-(visco)-elasticity that takes dissipation into account is required.

# Appendix A

# Derivatives of the Invariants

The first and second derivatives of the invariants (2.59) and (2.60) with respect to **F** and $\mathbf{B}_l$ were given in [Otténio *et al.*, 2008]. We repeat the non-zero ones here for ease of reference.

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(c_{\gamma\gamma}F_{i\alpha} - c_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3F^{-1}_{\alpha i},$$

$$\frac{\partial I_5}{\partial F_{i\alpha}} = 2B_{l\alpha}(F_{i\gamma}B_{l\gamma}), \quad \frac{\partial I_6}{\partial F_{i\alpha}} = 2(F_{i\gamma}B_{l\gamma}c_{\alpha\beta}B_{l\beta} + F_{i\gamma}c_{\gamma\beta}B_{l\beta}B_{l\alpha}),$$

$$\frac{\partial I_4}{\partial B_{l\alpha}} = 2B_{l\alpha}, \quad \frac{\partial I_5}{\partial B_{l\alpha}} = 2c_{\alpha\beta}B_{l\beta}, \quad \frac{\partial I_6}{\partial B_{l\alpha}} = 2c_{\alpha\gamma}c_{\gamma\beta}B_{l\beta}, \quad \frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta},$$

$$\frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\left(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + c_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - b_{ij}\delta_{\alpha\beta} - c_{\alpha\beta}\delta_{ij}\right),$$

$$\frac{\partial^2 I_3}{\partial F_{i\alpha}\partial F_{j\beta}} = 4I_3F^{-1}_{\alpha i}F^{-1}_{\beta j} - 2I_3F^{-1}_{\alpha j}F^{-1}_{\beta i}, \quad \frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}B_{l\alpha}B_{l\beta},$$

$$\begin{aligned}\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2[\delta_{ij}(c_{\alpha\gamma}B_{l\gamma}B_{l\beta} + c_{\beta\gamma}B_{l\gamma}B_{l\alpha}) + \delta_{\alpha\beta}F_{i\gamma}B_{l\gamma}F_{j\delta}B_{l\delta} \\ &+ F_{i\gamma}B_{l\gamma}F_{j\alpha}B_{l\beta} + F_{j\gamma}B_{l\gamma}F_{i\beta}B_{l\alpha} + b_{ij}B_{l\alpha}B_{l\beta}],\end{aligned}$$

$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial B_{l\beta}} = 2\delta_{\alpha\beta}F_{i\gamma}B_{l\gamma} + 2B_{l\alpha}F_{i\beta},$$

$$\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial B_{l\beta}} = 2F_{i\beta}c_{\alpha\gamma}B_{l\gamma} + 2F_{i\gamma}B_{l\gamma}c_{\alpha\beta} + 2F_{i\gamma}c_{\gamma\beta}B_{l\alpha} + 2\delta_{\alpha\beta}F_{i\gamma}c_{\gamma\delta}B_{l\delta},$$

$$\frac{\partial^2 I_4}{\partial B_{l\alpha}\partial B_{l\beta}} = 2\delta_{\alpha\beta}, \quad \frac{\partial^2 I_5}{\partial B_{l\alpha}\partial B_{l\beta}} = 2c_{\alpha\beta}, \quad \frac{\partial^2 I_6}{\partial B_{l\alpha}\partial B_{l\beta}} = 2c_{\alpha\gamma}c_{\gamma\beta}.$$

# Appendix B

# Magnetoelastic Tensors

For an isotropic material, $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and $\mathsf{K}_0$ can be expanded in terms of the derivatives of the invariants as follows, with $\Omega_n = \partial\Omega/\partial I_n$ and $\Omega_{mn} = \partial^2\Omega/\partial I_m \partial I_n$:

$$\mathcal{A}_{0piqj} = J^{-1}\sum_{m\in\mathcal{I}}\sum_{n\in\mathcal{I}}\Omega_{mn}F_{p\alpha}F_{q\beta}\frac{\partial I_n}{\partial F_{i\alpha}}\frac{\partial I_m}{\partial F_{j\beta}} + J^{-1}\sum_{n\in\mathcal{I}}\Omega_n F_{p\alpha}F_{q\beta}\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial F_{j\beta}},$$

$$\mathcal{C}_{0ji|k} = \sum_{m\in\mathcal{J}}\sum_{n\in\mathcal{I}}\Omega_{mn}F_{j\alpha}F^{-1}_{\beta k}\frac{\partial I_m}{\partial B_{l\beta}}\frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6}\Omega_n F_{j\alpha}F^{-1}_{\beta k}\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial B_{l\beta}},$$

$$\mathsf{K}_{0ij} = J\sum_{m\in\mathcal{J}}\sum_{n\in\mathcal{J}}\Omega_{mn}F^{-1}_{\alpha i}F^{-1}_{\beta j}\frac{\partial I_m}{\partial B_{l\alpha}}\frac{\partial I_n}{\partial B_{l\beta}} + J\sum_{n\in\mathcal{J}}\Omega_n F^{-1}_{\alpha i}F^{-1}_{\beta j}\frac{\partial^2 I_n}{\partial B_{l\alpha}\partial B_{l\beta}}.$$

We recall from Section 2.3 that $\mathcal{I} = \{1,2,3,5,6\}$ and $\mathcal{J} = \{4,5,6\}$. For an incompressible material $\mathcal{I} = \{1,2,5,6\}$ and $J = 1$.

When referred to the principal axes of the left Cauchy–Green tensor $\mathbf{b}$ with principal stretches $\lambda_1, \lambda_2, \lambda_3$ and components $(B_1, B_2, B_3)$ of the magnetic induction $\mathbf{B}$ the components of $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and $\mathsf{K}_0$ are given explicitly for a compressible material as, for $i \neq j \neq k \neq i$,

$$\begin{aligned}
\mathcal{A}_{0iiii} &= 2J^{-1}\lambda_i^2[\Omega_1 + (\lambda_j^2+\lambda_k^2)\Omega_2 + \lambda_j^2\lambda_k^2\Omega_3 + \lambda_j^2\lambda_k^2 B_i^2(\Omega_5 + 6\lambda_i^2\Omega_6)] \\
&+ 4J^{-1}\lambda_i^4\{\Omega_{11} + 2(\lambda_j^2+\lambda_k^2)\Omega_{12} + (\lambda_j^2+\lambda_k^2)^2\Omega_{22} \\
&+ \lambda_j^2\lambda_k^2[2\Omega_{13} + 2(\lambda_j^2+\lambda_k^2)\Omega_{23} + \lambda_j^2\lambda_k^2\Omega_{33}] + 2\lambda_j^2\lambda_k^2 B_i^2[\Omega_{15} + 2\lambda_i^2\Omega_{16} \\
&+ (\lambda_j^2+\lambda_k^2)\Omega_{25} + 2\lambda_i^2(\lambda_j^2+\lambda_k^2)\Omega_{26} + \lambda_j^2\lambda_k^2\Omega_{35} + 2I_3\Omega_{36}] \\
&+ \lambda_j^4\lambda_k^4 B_i^4(\Omega_{55} + 4\lambda_i^2\Omega_{56} + 4\lambda_i^4\Omega_{66})\},
\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0iiij} &= 4B_iB_jJ\lambda_i^2\{\Omega_6 + \Omega_{15} + (\lambda_j^2+\lambda_k^2)\Omega_{25} + \lambda_j^2\lambda_k^2\Omega_{35} \\ &+ (\lambda_i^2+\lambda_j^2)[\Omega_{16} + (\lambda_j^2+\lambda_k^2)\Omega_{26} + \lambda_j^2\lambda_k^2\Omega_{36}] \\ &+ \lambda_j^2\lambda_k^2B_i^2[\Omega_{55} + (3\lambda_i^2+\lambda_j^2)\Omega_{56} + 2\lambda_i^2(\lambda_i^2+\lambda_j^2)\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0iiji} &= 2B_iB_jJ\{\Omega_5 + (\lambda_j^2+3\lambda_i^2)\Omega_6 + 2\lambda_i^2[\Omega_{15} + (\lambda_j^2+\lambda_k^2)\Omega_{25} + \lambda_j^2\lambda_k^2\Omega_{35}] \\ &+ 2\lambda_i^2(\lambda_i^2+\lambda_j^2)[\Omega_{16} + (\lambda_j^2+\lambda_k^2)\Omega_{26} + \lambda_j^2\lambda_k^2\Omega_{36}] \\ &+ 2J^2B_i^2[\Omega_{55} + (3\lambda_i^2+\lambda_j^2)\Omega_{56} + 2\lambda_i^2(\lambda_i^2+\lambda_j^2)\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0iijj} &= 4J^{-1}\lambda_i^2\lambda_j^2\{\Omega_2 + \lambda_k^2\Omega_3 + \Omega_{11} + (I_1+\lambda_k^2)\Omega_{12} + (I_2+\lambda_k^4)\Omega_{22} \\ &+ \lambda_k^2[(\lambda_i^2+\lambda_j^2)\Omega_{13} + (I_2+\lambda_i^2\lambda_j^2)\Omega_{23} + I_3\Omega_{33}] \\ &+ \lambda_k^2(\lambda_j^2B_i^2+\lambda_i^2B_j^2)(\Omega_{15}+\lambda_k^2\Omega_{25}) + 2I_3(\lambda_i^2B_i^2+\lambda_j^2B_j^2)(\Omega_{26}+\lambda_k^2\Omega_{36}) \\ &+ I_3(B_i^2+B_j^2)(2\Omega_{16}+\Omega_{25}+2\lambda_k^2\Omega_{26}+\lambda_k^2\Omega_{35}) \\ &+ I_3\lambda_k^2B_i^2B_j^2[\Omega_{55} + 2(\lambda_i^2+\lambda_j^2)\Omega_{56} + 4\lambda_i^2\lambda_j^2\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0ijij} &= 2J^{-1}\lambda_i^2\{\Omega_1 + \lambda_k^2\Omega_2 + B_i^2\lambda_j^2\lambda_k^2\Omega_5 + \lambda_j^2\lambda_k^2(2B_i^2\lambda_i^2 + B_i^2\lambda_j^2 + B_j^2\lambda_i^2)\Omega_6 \\ &+ 2B_i^2B_j^2J^2\lambda_j^2\lambda_k^2[\Omega_{55} + 2(\lambda_i^2+\lambda_j^2)\Omega_{56} + (\lambda_i^2+\lambda_j^2)^2\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0ijji} &= 2J^{-1}\lambda_i^2\lambda_j^2\{-\Omega_2 - \lambda_k^2\Omega_3 + \lambda_k^2(\lambda_j^2B_i^2+\lambda_i^2B_j^2)\Omega_6 \\ &+ 2B_i^2B_j^2J^2\lambda_k^2[\Omega_{55} + 2(\lambda_i^2+\lambda_j^2)\Omega_{56} + (\lambda_i^2+\lambda_j^2)^2\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{0iijk} &= 4B_jB_kJ\lambda_i^2\{\Omega_{15} + (\lambda_j^2+\lambda_k^2)(\Omega_{25}+\Omega_{16}) + (\lambda_j^2+\lambda_k^2)^2\Omega_{26} + \lambda_j^2\lambda_k^2\Omega_{35} \\ &+ \lambda_j^2\lambda_k^2(\lambda_j^2+\lambda_k^2)\Omega_{36} + B_i^2\lambda_j^2\lambda_k^2[\Omega_{55} + (I_1+\lambda_i^2)\Omega_{56} + 2\lambda_i^2(\lambda_j^2+\lambda_k^2)\Omega_{66}]\},\end{aligned}$$

$$\mathcal{A}_{0ijki} = \mathcal{A}_{0ijik} = 2B_jB_kJ\{\lambda_i^2\Omega_6 + 2B_i^2J^2[\Omega_{55} + (I_1+\lambda_i^2)\Omega_{56} + (I_2+\lambda_i^4)\Omega_{66}]\},$$

$$\mathcal{A}_{0jiki} = 2B_jB_kJ\{\Omega_5 + I_1\Omega_6 + 2B_i^2J^2[\Omega_{55} + (I_1+\lambda_i^2)\Omega_{56} + (I_2+\lambda_i^4)\Omega_{66}]\},$$

$$\begin{aligned}\mathcal{C}_{0ii|i} &= 4B_iJ\{\Omega_5 + 2\lambda_i^2\Omega_6 + \Omega_{14} + \lambda_i^2\Omega_{15} + \lambda_i^4\Omega_{16} \\ &+ (\lambda_j^2+\lambda_k^2)(\Omega_{24} + \lambda_i^2\Omega_{25} + \lambda_i^4\Omega_{26}) + \lambda_j^2\lambda_k^2(\Omega_{34} + \lambda_i^2\Omega_{35} + \lambda_i^4\Omega_{36}) \\ &+ B_i^2\lambda_j^2\lambda_k^2[\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56} + 2\lambda_i^2(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})]\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{0ii|j} &= 4B_jJ\lambda_i^2\lambda_j^{-2}\{\Omega_{14}+\lambda_j^2\Omega_{15}+\lambda_j^4\Omega_{16}+(\lambda_j^2+\lambda_k^2)(\Omega_{24}+\lambda_j^2\Omega_{25}+\lambda_j^4\Omega_{26})\\ &+ \lambda_j^2\lambda_k^2(\Omega_{34}+\lambda_j^2\Omega_{35}+\lambda_j^4\Omega_{36})+B_i^2\lambda_j^2\lambda_k^2[\Omega_{45}+\lambda_j^2\Omega_{55}+\lambda_j^4\Omega_{56}\\ &+ 2\lambda_i^2(\Omega_{46}+\lambda_j^2\Omega_{56}+\lambda_j^4\Omega_{66})]\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{0ij|i} &= 2B_jJ\{\Omega_5+(\lambda_i^2+\lambda_j^2)\Omega_6+2B_i^2\lambda_j^2\lambda_k^2[\Omega_{45}+\lambda_i^2\Omega_{55}+\lambda_i^4\Omega_{56}\\ &+ (\lambda_i^2+\lambda_j^2)(\Omega_{46}+\lambda_i^2\Omega_{56}+\lambda_i^4\Omega_{66})]\},\end{aligned}$$

$$\mathcal{C}_{0ij|k}=4B_iB_jB_kJ\lambda_i^2\lambda_j^2[\Omega_{45}+\lambda_k^2\Omega_{55}+\lambda_k^4\Omega_{56}+(\lambda_i^2+\lambda_j^2)(\Omega_{46}+\lambda_k^2\Omega_{56}+\lambda_k^4\Omega_{66})],$$

$$\begin{aligned}\mathsf{K}_{0ii} &= 2J\lambda_i^{-2}\{\Omega_4+\lambda_i^2\Omega_5+\lambda_i^4\Omega_6+2B_i^2\lambda_j^2\lambda_k^2[\Omega_{44}+\lambda_i^2\Omega_{45}+\lambda_i^4\Omega_{46}\\ &+ \lambda_i^2(\Omega_{45}+\lambda_i^2\Omega_{55}+\lambda_i^4\Omega_{56})+\lambda_i^4(\Omega_{46}+\lambda_i^2\Omega_{56}+\lambda_i^4\Omega_{66}]\},\end{aligned}$$

$$\begin{aligned}\mathsf{K}_{0ij} &= 4B_iB_jJ\lambda_k^2[\Omega_{44}+\lambda_i^2\Omega_{45}+\lambda_i^4\Omega_{46}+\lambda_j^2(\Omega_{45}+\lambda_i^2\Omega_{55}+\lambda_i^4\Omega_{56})\\ &+ \lambda_j^4(\Omega_{46}+\lambda_i^2\Omega_{56}+\lambda_i^4\Omega_{66})].\end{aligned}$$

For an incompressible material the above formulas apply with $J=1, I_3=1$ and with all terms in $\Omega$ carrying a subscript 3 omitted.

# Appendix C

# Some Calculations

Here we list some of the calculations used in Chapter 5 for reference. The derivations can be seen in, for example, [Haughton and Ogden, 1979b] and [Shams, 2010].

### • Change of the variable of integration from $r$ to $\lambda$

As given in Equation $(5.21)_1$, due to incompressibility the deformation of the tube is of the form

$$r^2 = a^2 + \frac{1}{\lambda_z}\left(R^2 - A^2\right). \tag{C.1}$$

On differentiating both sides with respect to $r$, we get

$$\frac{dR}{dr} = \frac{r\lambda_z}{R}, \tag{C.2}$$

while on differentiating $r = \lambda R$ with respect to $r$, we obtain

$$1 = \lambda\frac{dR}{dr} + R\frac{d\lambda}{dr}. \tag{C.3}$$

Together, the above two equations give

$$r\frac{d\lambda}{dr} = \lambda(1 - \lambda^2\lambda_z). \tag{C.4}$$

### • Stretch relations

We define $\lambda_a = a/A$ and $\lambda_b = b/B$. On substituting $r = b, R = B$ in Equation $(5.21)_1$ we get

$$\frac{b^2}{B^2} = \frac{a^2}{B^2} + \frac{1}{\lambda_z}\left(1 - \frac{A^2}{B^2}\right). \tag{C.5}$$

This can be rewritten as

$$A^{-2}B^2(\lambda_b^2\lambda_z - 1) = \lambda_a^2\lambda_z - 1. \tag{C.6}$$

On dividing $(5.21)_1$ throughout by $R$, we obtain

$$\frac{r^2}{R^2} = \frac{a^2}{R^2} + \frac{1}{\lambda_z}\left(1 - \frac{A^2}{R^2}\right), \tag{C.7}$$

$$\Rightarrow \lambda^2\lambda_z - 1 = \frac{A^2}{R^2}\left(\lambda_a^2\lambda_z - 1\right). \tag{C.8}$$

Hence we obtain the relationship

$$A^{-2}B^2\left(\lambda_b^2\lambda_z - 1\right) = \lambda_a^2\lambda_z - 1 = R^2A^{-2}\left(\lambda^2\lambda_z - 1\right). \tag{C.9}$$

which gives

$$\frac{\partial\lambda_b}{\partial\lambda_a} = \frac{\lambda_a A^2}{\lambda_b B^2}. \tag{C.10}$$

Also, Equation (C.9) gives the condition

$$\lambda_b < \lambda < \lambda_a. \tag{C.11}$$

## • Expression of the stress $\tau_{33}$

Equation (5.5) gives the axial component of the principal Cauchy stress as

$$\tau_{33} = \lambda_3\frac{\partial\Omega}{\partial\lambda_3} - p, \tag{C.12}$$

$$= \lambda_3\frac{\partial\Omega}{\partial\lambda_3} - p + 2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda} - \left(2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda}\right), \tag{C.13}$$

$$= \lambda_3\frac{\partial\Omega}{\partial\lambda_3} - p + 2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda} - 2\lambda_z\frac{\partial\Omega}{\partial\lambda_3} + 2\lambda_1\frac{\partial\Omega}{\partial\lambda_1} + \lambda\frac{\partial\Omega}{\partial\lambda_2} - \lambda_1\frac{\partial\Omega}{\partial\lambda_1}, \tag{C.14}$$

$$= 2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda} + \tau_{11} + \tau_{22} - \tau_{33}, \tag{C.15}$$

$$= \frac{1}{2}\left(2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda} + 2\tau_{11} + \tau_{22} - \tau_{11}\right) \tag{C.16}$$

$$= \frac{1}{2}\left(2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda} + 2\tau_{11} + r\frac{d\tau_{11}}{dr}\right) \tag{C.17}$$

$$= \frac{1}{2}\left(2\lambda_z\frac{\partial\hat{\Omega}}{\partial\lambda_z} - \lambda\frac{\partial\hat{\Omega}}{\partial\lambda}\right) + \frac{1}{2r}\frac{d}{dr}\left(r^2\tau_{11}\right), \tag{C.18}$$

where we have used Equation (5.27).

Writing the above expression in terms of the invariants defined in Equation (5.6) and their derivatives, we get

$$\begin{aligned}\tau_{33} =& \Omega_1(3\lambda_3^2 - I_1) + \Omega_2(I_2 - 3\lambda_1^2\lambda_2^2) \\ &+ \Omega_5\left(2\lambda_3^2 H_{l3}^2 - \lambda_2^2 H_{l2}^2\right) + 2\Omega_6\left(2\lambda_3^4 H_{l3}^2 - \lambda_2^4 H_{l2}^2\right) + \frac{1}{2r}\frac{d}{dr}\left(r^2\tau_{11}\right). \end{aligned} \tag{C.19}$$

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