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PhD thesis

Noetherian Hopf algebras and their extensions

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Contents

Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 1 covers notation, definitions and known results. Chapters 2–5 contain original work as well as known results; care has been taken to give proper attribution.

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I thank my family for their constant support; it has meant so much. The fact that most of them have no clue what I've been doing over the past four years has certainly helped to keep me grounded! So many people have made my time as a post-graduate student memorable and enjoyable, and many of them will continue to do this well into the future. I like to think they know who they are, so I'll not name names!

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Summary

Chapter 1 covers notation, definitions and basic results that will be used throughout the thesis. It is almost entirely expository.

In Chapter 2, we prove a basic algebraic property that all Hopf algebras over an algebraically closed field of characteristic zero must possess (Lemma 2.13). We then go on to address the existence of particular, but natural, Hopf algebra structures on Ore extensions. In particular we discuss a result of Panov (Theorem 2.19) that gives necessary and sufficient conditions for an Ore extension $R[X; \sigma, \delta]$ to be a Hopf algebra with R a Hopf subalgebra, provided that X is assumed to be skew-primitive. We then answer a related question about skew-Laurent extensions in Theorem 2.22 and Corollary 2.23; we prove that a skew-Laurent extension $R[X^{\pm 1}; \sigma]$, of a Hopf algebra R, has a Hopf algebra structure extending that of R, with X group-like, if and only if the automorphism σ is a morphism of Hopf algebras.

The purpose of the third chapter is to study the character theory of Ore extensions. We state a result, due to Goodearl (Theorem 3.7), describing the relationship between the prime ideals of the Ore extension $T = R[X; \sigma, \delta]$ and those of the coefficient ring R, in the case where R is a commutative noetherian ring. As a corollary of this theorem we obtain a relationship between the sets of characters $\mathrm{Hom}_{k\text{-alg.}}(T, k)$ and $\mathrm{Hom}_{k\text{-alg.}}(R, k)$. The main result of the chapter is Theorem 3.18, where we describe this relationship for a coefficient ring that is not necessarily commutative or noetherian,

thus generalising Goodearl's corollary. We go on to explore the topological properties of these sets of characters. This proves to be particularly fruitful when applied to the study of Hopf algebras.

In Chapter 4 we investigate the circumstances in which a noetherian Hopf algebra H with a Hopf surjection π to a coordinate ring $\mathcal{O}(G)$, for G an affine algebraic group, can be decomposed as a crossed product $H^{\mathrm{co}\,\pi}\#_\sigma\mathcal{O}(G)$. We give examples where this is known to be the case and also counterexamples to show that it is not always possible. Inspired by work of Goodearl and Zhang, we specialise to the case where $G = (k^+)^n$ and explore equivalent conditions to cleftness.

In the fifth and final chapter we expand on the work of Chapter 3 by introducing the class of "iterate Hopf-Ore extension" Hopf algebras and studying some of their ring-theoretic, Hopf-algebraic and homological properties. In particular, we are able to prove a partial converse to Panov's theorem (Theorem 2.19). Theorem 5.26 says that, in a special case, the only Hopf algebra structures that can exist are of the type assumed in Panov's theorem.

Chapter

1 Definitions, notation and background

In this chapter, we collect together the key definitions and terminology used throughout the thesis.

1.1 Noncommutative algebra

1.1.1 Notation

Two good references for standard ideas in noncommutative algebra are [GW04] and [MR88]. Throughout this thesis, we work over a field $\mathbb{k}$; this will, on occasion, be assumed to be algebraically closed or to have characteristic zero (or both) but we do not impose these restrictions from the outset. By an **algebra** we shall mean an associative unital $\mathbb{k}$-algebra, not necessarily commutative or finite-dimensional as a $\mathbb{k}$-vector space. We say that an algebra A is **affine** if it is finitely generated as an algebra. By a **character** of an algebra A we mean an algebra homomorphism $A \to \mathbb{k}$; we shall use the term **character ideal** to refer to the kernel of a character. Unadorned tensor products will denote the tensor product over the field $\mathbb{k}$.

For an algebra A, we denote by $\operatorname{GKdim} A$ the Gelfand-Kirillov dimension of A; see [KL00] for the definition and the properties of this dimension.

1.1.2 Invariant and stable ideals

Definition 1.1. Let Σ be a set of maps from a ring R to itself.

(*i*) An ideal I of R is said to be **Σ-invariant** if $\phi(I) \subseteq I$ for all $\phi \in \Sigma$.

(*ii*) An ideal I of R is said to be **Σ-stable** if $\phi(I) = I$ for all $\phi \in \Sigma$.

(*iii*) A **Σ-prime** ideal is any proper Σ-invariant ideal P such that whenever J, K are Σ-invariant ideals satisfying $JK \subseteq P$, then either $J \subseteq P$ or $K \subseteq P$. $\Diamond$

1.2 Affine algebraic geometry

1.2.1 Notation

We use the notation from [Har97] but favour [Abe80] as a reference for standard theorems because of its more algebraic slant. With the assumption that k is an algebraically closed field, we denote **affine n-space** by $\mathbb{A}^n$. This is the set of all n-tuples of elements of k. As usual, elements of affine n-space are referred to as **points**. An **algebraic set** is the set of common zeros of some finite set of polynomials in n variables with coefficients in k. Affine n-space can be endowed with the Zariski topology (see [Har97, Chapter 1]) where the closed sets are defined to be the algebraic sets. We then call an irreducible algebraic set an **affine variety**.

If an affine variety has a group structure, in which the multiplication and inversion maps are regular functions, then we call it an **affine algebraic group**. The affine line $\mathbb{A}^1$ (which as a set is just the field k) has a group structure with the operation being addition in k. We shall denote this group by k^+ to distinguish it from the algebraic variety $\mathbb{A}^1$.

1.2.2 Algebraic sets and affine commutative algebras

As discussed in [Abe80], when we work over an algebraically closed field k, there is a contravariant equivalence between the category of affine algebraic sets and the category of affine commutative semiprime k-algebras. Concretely, given such an algebra A, the set of maximal ideals of A, denoted $\operatorname{maxspec} A$, is an affine algebraic set. Due to algebraic closure, this can be identified with the set $\mathrm{Hom}_{k\text{-alg.}}(A, k)$ of algebra homomorphisms from A to k. Conversely, given an affine algebraic set X, its **coordinate ring**, that is the set of polynomial functions from X to k, is an affine commutative k-algebra.

1.3 Hopf algebras

1.3.1 Conventions

The book [Mon94] is a standard reference for studying Hopf algebras. For us, Hopf algebras will be over the field k and will *not* be assumed to be finite-dimensional as vector spaces, nor to be commutative or cocommutative. In saying "let H be a Hopf algebra," we mean that we have a tuple of data $(H, m, u, \Delta, \varepsilon, S)$ where

- $m : H \otimes H \to H$ denotes the multiplication in H,
- $u : k \to H$ picks out its unity,
- $\Delta : H \to H \otimes H$ is the coproduct,
- $\varepsilon : H \to k$ is the counit, and
- $S : H \to H$ is the antipode.

In addition, we assume the standard Hopf algebra axioms about various compositions of these maps – see [Mon94, Chapter 1]. We shall often write 1 for both the multiplicative identity of k and the unity of H, suppressing u in doing so. Sometimes we shall use a subscript, for example m_H or Δ_H, to clarify which Hopf algebra's maps we are referring to.

We use Sweedler's sigma notation (see [Mon88, Section 1.4.2]) to work with the coproduct of a coalgebra. For C a coalgebra and $x \in C$, we write

$$\sum x_1 \otimes x_2 \coloneqq \Delta(x).$$

Let H be a Hopf algebra. We say that an element $g \in H$ is **group-like** if $\Delta(g) = g \otimes g$; consequently, if g is group-like then $\varepsilon(g) = 1$ and $S(g) = g^{-1}$. The set of all group-like elements of a Hopf algebra H is denoted by $G(H)$; this set forms a group with operation m, the multiplication in H. Given two group-like elements $g, h \in H$, an element $x \in H$ is said to be **(g, h)-primitive** if $\Delta(x) = x \otimes g + h \otimes x$. We also say that $x \in H$ is **skew-primitive** if there exist group-like elements $g, h \in H$ such that $\Delta(x) = x \otimes g + h \otimes x$. Note that the unity element of a Hopf algebra H is always group-like. We say that $x \in H$ is **primitive** if it is $(1, 1)$-primitive; that is, if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

1.3.2 Morphisms of Hopf algebras

Suppose H and K are two Hopf algebras over a field k. An algebra homomorphism $f : H \to K$ is called a **morphism of Hopf algebras** or a **Hopf morphism** if it is also a coalgebra morphism (see [Mon94, Definition 1.5.1]). In Sweedler notation, f being a coalgebra morphism means that, for all $h \in H$, we have $\sum f(h_1) \otimes f(h_2) = \sum f(h)_1 \otimes f(h)_2$.

1.3.3 Convolution product

Let H be a Hopf algebra and A be an algebra over k. It is a standard fact (see [Mon94, Section 1.2]) that the set of k-linear maps $\mathrm{Hom}_k(H, A)$ has the structure of a k-algebra with the **convolution product**. Given $f, g \in \mathrm{Hom}_k(H, A)$, their convolution $f * g \in \mathrm{Hom}_k(H, A)$ is defined, for each $h \in H$, by

$$(f * g)(h) \coloneqq m \circ (f \otimes g) \circ \Delta(h) = \sum f(h_1)g(h_2).$$

1.3.4 Tensor products

Given two coalgebras H and K, their tensor product $H \otimes K$ is again a coalgebra with coproduct $\Delta_{H\otimes K} \coloneqq (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta_H \otimes \Delta_K)$ where $\tau : H \otimes K \to K \otimes H$ is the **tensor flip**; that is $\tau(h \otimes k) \coloneqq k \otimes h$ for all $h \in H$ and $k \in K$. In Sweedler notation,

$$\Delta_{H\otimes K}(h \otimes k) = \sum h_1 \otimes k_1 \otimes h_2 \otimes k_2.$$

The counit is given by $\varepsilon_{H\otimes K} \coloneqq \varepsilon_H \otimes \varepsilon_K$; that is, $\varepsilon_{H\otimes K}(h \otimes k) = \varepsilon(h)\varepsilon(k)$ for all $h \in H$ and $k \in K$.

1.3.5 The finite dual and winding automorphisms

Let H be a Hopf algebra and consider the **finite dual** H^o whose elements are, by definition, the k-linear maps $H \to k$ whose kernels contain ideals of finite vector space codimension. It is a standard fact that $(H^o, \Delta_H^*, \varepsilon_H^*, m_H^*, u_H^*, S_H^*)$ is a Hopf algebra (see [Mon94, Section 1.2]). Here, the multiplication Δ_H^* is the convolution product. Thus we see that ε_H is the multiplicative identity in H^o because of the counit property of the Hopf algebra H. In the finite dual, the coproduct is $m_H^* : H^o \to$

$H^o \otimes H^o \cong (H \otimes H)^o$ defined, for each $x, y \in H$, by

$$m^* f(x \otimes y) := f(xy).$$

We have the following well-known lemma.

Lemma 1.2. *Let H be a Hopf k-algebra. Then, as groups, $G(H^o) = \mathrm{Hom}_{k\text{-}alg.}(H, k)$, where the operation on both sides is multiplication in H^o.*

Proof. Let $\xi \in G(H^o)$. Then, for all $x, y \in H$,

$$\xi(xy) = (\Delta\xi)(x \otimes y) = (\xi \otimes \xi)(x \otimes y) = \xi(x)\xi(y);$$

hence ξ is an algebra homomorphism. For the converse, let $\chi : H \to k$ be an algebra homomorphism and let $I := \ker \chi$. Then $H/I \cong k$ and so I is an ideal with finite vector space codimension; hence $\chi \in G(H^o)$. Finally, it is a well-known fact that the set of group-like elements of a Hopf algebra forms a group with multiplication taken from the Hopf structure. Thus $\mathrm{Hom}_{k\text{-alg.}}(H, k)$ is a group with convolution as its operation. ■

To each algebra homomorphism $\chi : H \to k$ we can associate a **right winding automorphism** $\tau^r_\chi \in \mathrm{Aut}_{k\text{-alg.}}(H)$ defined, for each $h \in H$, by

$$\tau^r_\chi(h) := m(\mathrm{id} \otimes \chi)\Delta(h) = \sum h_1 \chi(h_2).$$

Similarly, we can also define a **left winding automorphism** $\tau^\ell_\chi \in \mathrm{Aut}_{k\text{-alg.}}(H)$ where, for each $h \in H$,

$$\tau^\ell_\chi(h) := m(\chi \otimes \mathrm{id})\Delta(h) = \sum \chi(h_1)h_2.$$

Lemma 1.3. *Let k be a field and suppose H is a Hopf k-algebra.*

(i) The map $\tau^r : \mathrm{Hom}_{k\text{-}alg.}(H, k) \to \mathrm{Aut}_{k\text{-}alg.}(H)$, mapping $\chi \mapsto \tau^r_\chi$, is an injective group homomorphism.

(ii) The map $\tau^\ell : \mathrm{Hom}_{k\text{-}alg.}(H, k)^{op} \to \mathrm{Aut}_{k\text{-}alg.}(H)$, mapping $\chi \mapsto \tau^\ell_\chi$, is an injective group homomorphism.

(iii) *Let* $\chi, \xi \in \mathrm{Hom}_{k\text{-}alg.}(H, k)$ *be characters. Then* $\tau^r_\chi \tau^\ell_\xi = \tau^\ell_\xi \tau^r_\chi$*; that is, left and right winding automorphisms commute.*

Proof.

(i) Let $\chi, \xi \in \mathrm{Hom}_{\text{k-alg.}}(H, k)$. We check that $\tau^r_{\chi * \xi} = \tau^r_\chi \circ \tau^r_\xi$ so that we have a homomorphism. For all $h \in H$,

$$\begin{aligned}\tau^r_\chi \circ \tau^r_\xi(h) &= \tau^r_\chi\left(\sum h_1 \xi(h_2)\right)\\ &= \xi(h_2) \sum \tau^r_\chi(h_1)\\ &= \sum \xi(h_2) h_{11} \chi(h_{12})\\ &= \sum h_1 \chi(h_2) \xi(h_3)\end{aligned}$$

and

$$\begin{aligned}\tau^r_{\chi * \xi}(h) &= \sum h_1 (\chi * \xi)(h_2)\\ &= \sum h_1 \chi(h_{21}) \xi(h_{22})\\ &= \sum h_1 \chi(h_2) \xi(h_3).\end{aligned}$$

The map is clearly injective because for any τ^r_χ in the image of τ^r we can recover χ by applying the counit ε; $\varepsilon \circ \tau^r_\chi = \chi$, since, for all $h \in H$,

$$\varepsilon\left(\sum h_1 \chi(h_2)\right) = \sum \chi(\varepsilon(h_1) h_2) = \chi(h).$$

So $\mathrm{Hom}_{\text{k-alg.}}(H, k)$ is in bijection with its image under τ^r.

(ii) Exactly similar to the proof of (i).

(iii) Let $\chi, \xi \in G$ be k-algebra maps $H \to k$ and suppose $h \in H$. Then

$$\tau^r_\chi \tau^\ell_\xi(h) = \sum \xi(h_1)(h_2)_1 \chi\big((h_2)_2\big) = \sum \xi(h_1) h_2 \chi(h_3)$$

and

$$\tau^\ell_\xi \tau^r_\chi(h) = \sum \xi\big((h_1)_1\big)(h_1)_2 \chi(h_2) = \sum \xi(h_1) h_2 \chi(h_3). \qquad \blacksquare$$

Thus the set of right winding automorphisms of H (the image of τ^r) forms a group isomorphic to the group of characters $\mathrm{Hom}_{\text{k-alg.}}(H, k)$ and the set of left winding automorphisms forms a group *anti*-isomorphic to $\mathrm{Hom}_{\text{k-alg.}}(H, k)$.

Lemma 1.4. *Let* $G \coloneqq G(H^\circ)$ *be the group of algebra homomorphisms* $H \to k$ *and let* $A \coloneqq \mathrm{Aut}_{k\text{-}alg.}(H)$*. Then the map*

$$\tau : G \times G^{\mathrm{op}} \to A$$
$$(x, y) \mapsto \tau_x^r \tau_y^\ell$$

is a group homomorphism. Moreover, τ *is injective when restricted to* $G \times \{\varepsilon\}$ *or* $\{\varepsilon\} \times G^{\mathrm{op}}$*.*

Proof. Now to see that τ is a group homomorphism we check that, for all $w, x, y, z \in G$, we have $\tau\big((w, x) \cdot (y, z)\big) = \tau(w, x)\tau(y, z)$. The left-hand side is

$$\tau\big((w, x) \cdot (y, z)\big) = \tau(w * y, z * x) = \tau^r_{w*y} \tau^\ell_{z*x} = \tau^r_w \tau^r_y \tau^\ell_x \tau^\ell_z$$

and the right-hand side is

$$\tau(w, x)\tau(y, z) = \tau^r_w \tau^\ell_x \tau^r_y \tau^\ell_z;$$

thus, since $\tau^r_y \tau^\ell_x = \tau^\ell_x \tau^r_y$, τ is a group homomorphism. Now recall that ε is the identity element of G; that is $\tau^\ell_\varepsilon = \tau^r_\varepsilon = \mathrm{id} \in A$. Hence, when restricted to $G \times \{\varepsilon\}$, the image of τ is isomorphic to the group of right winding automorphisms of H which, by Lemma 1.3, is isomorphic to G. Similarly, when τ is restricted to $\{\varepsilon\} \times G^{\mathrm{op}}$, the image is isomorphic to the group of left winding automorphisms, which we know is isomorphic to G^{op} (or, in other words, is anti-isomorphic to G). ∎

Lemma 1.5. *Let* H *be a Hopf algebra. The groups of right and left winding automorphisms of* H *both act transitively on the set of algebra homomorphisms from* H *to* k*.*

Proof. Let G be the group of right winding automorphisms of H. Then G acts on $\mathrm{Hom}_{k\text{-alg.}}(H, k)$ by $\tau^r_\chi \cdot \xi \coloneqq \chi * \xi$ for any characters χ and ξ. Let η be fixed and ζ be any character of H. Then $\tau^r_{\eta * \zeta^{-1}} \cdot \zeta = \eta$; thus G acts transitively on $\mathrm{Hom}_{k\text{-alg.}}(H, k)$. Similarly the group of left winding automorphisms acts on $\mathrm{Hom}_{k\text{-alg.}}(H, k)$ by $\tau^\ell_\chi \cdot \xi \coloneqq \xi * \chi$. ∎

1.3.6 Pointedness and connectedness

We collect together some more important definitions from [Mon94]. The fundamental theorem on coalgebras, as it is called in [Swe69], tells us that coalgebras are locally finite-dimensional; that is, any subset of a coalgebra is contained in a finite-dimensional subcoalgebra [Mon94, Theorem 5.1.1]. A coalgebra is called **simple** if it has no proper subcoalgberas. Thus, by the fundamental theorem, any simple subcoalgebra of a coalgebra is finite-dimensional. Consider the following definition, which appears as [Mon94, Definition 5.1.5].

Definition 1.6. Let C be a coalgebra.

(*i*) The **coradical** of C, denoted C_0, is the sum of all simple subcoalgebras of C.

(*ii*) C is said to be **pointed** if every simple subcoalgebra is one-dimensional.

(*iii*) C is said to be **connected** if C_0 is one-dimensional. ♢

Let H be a Hopf algebra over a field k. Then any one-dimensional subcoalgebra D of H must be the span of a group-like element; that is, $D = \operatorname{span}_k\{g : g \in G(H)\}$, where $G(H)$ is the group of group-like elements of H. Thus we see that H is pointed if and only if $H_0 = \operatorname{span}_k G(H)$. Then, since $k \subseteq H$ is a one-dimensional subcoalgebra, H being connected is equivalent to having $H_0 = k$.

1.3.7 Hopf algebras and algebraic groups

The book [Abe80] deals with affine algebraic groups from the viewpoint of Hopf algebras. This is possible due to a contravariant equivalence between the category of affine commutative semiprime Hopf algebras and the category of affine algebraic groups. Concretely, if G is an affine algebraic group then its coordinate ring $\mathcal{O}(G)$ is an affine commutative semiprime Hopf algebra; if H is an affine commutative semiprime Hopf algebra then $\operatorname{maxspec}(H)$ is an affine algebraic group.

1.3.8 Smash and crossed products

The following definitions are taken from [SS06, Chapter 1].

Definition 1.7 (Measuring)**.** Let A be a k-algebra and T a k-bialgebra. A map $- \cdot - : T \otimes A \to A$ is called a **measuring** if, for all $h \in T$ and all

$a, b \in A$,

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b),$$
$$h \cdot 1 = \varepsilon(h)1.$$

If such a measuring exists then we say that T **measures** A. ◊

Definition 1.8 (Module algebra)**.** Suppose H is a Hopf algebra and A is an algebra and a left (resp. right) H-module. Then A is said to be a **left** (resp. **right**) H**-module algebra** if the action of H on A is a measuring. ◊

Consider the case when $H := kG$ for some group G and let A be a left H-module and a k-algebra. Then the above condition for A to be a left H-module says that, for all $g \in G$ and $a, b \in A$,

$$g \cdot (ab) = (g \cdot a)(g \cdot b)$$
$$g \cdot 1 = 1.$$

Thus A being an H-module algebra says that H acts on A by algebra automorphisms.

Definition 1.9 (Comodule)**.** Let H be a Hopf k-algebra and A be a vector space over k with a k-linear map $\rho : A \to A \otimes H$. Then A is said to be a **right** H**-comodule** with structure map ρ provided that the diagrams

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\rho"'] & A \otimes H \arrow[d, "\mathrm{id} \otimes \rho"] \\ A \otimes H \arrow[r, "\rho \otimes \mathrm{id}"] & A \otimes H \otimes H \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\rho"] \arrow[dr, "- \otimes 1"'] & & A \otimes H \arrow[dl, "\mathrm{id} \otimes \varepsilon"] \\ & A \otimes k & \end{tikzcd}$$

commute. ◊

Definition 1.10 (Morphism of comodules)**.** Let H be a Hopf algebra and suppose A and B are right H-comodules with structure maps ρ_A and ρ_B respectively. Then a linear map $f : A \to B$ is said to be a **morphism of right** H**-comodules** or **right** H**-colinear** if $\rho_B f = (f \otimes \mathrm{id})\rho_A$. ◊

Definition 1.11 (Injective comodule)**.** Let H be a Hopf algebra and A be a right H-comodule. Then A is **injective** if, for every injective right

H-colinear map $i : X \to Y$ and for any right H-colinear map $f : X \to A$, there exists a right H-colinear map $g : Y \to A$ with $gi = f$. $\Diamond$

Note that left H-comodules, morphisms of left H-comodules and injective left H-comodules are defined analogously.

Observe that, given H a Hopf algebra and A a right H-comodule, the Hopf algebra axioms imply that $A \otimes H$ is a right H-comodule with structure map $\mathrm{id} \otimes \Delta$. We shall use the following characterisation of injectivity.

Lemma 1.12. *Let A be a right H-comodule with structure map $\rho : A \to A \otimes H$. Then A is injective if and only if there is a right H-colinear map $\phi : A \otimes H \to A$ such that $\phi\rho = \mathrm{id}_A$.*

Proof.

$\boxed{\text{only if}}$ Suppose A is injective. Observe that the map $\rho : A \to A \otimes H$ is an injective right H-colinear map. Hence take $i := \rho$ and $f := \mathrm{id}$ in the above definition to get that there is some right H-colinear map $\phi : A \otimes H \to A$ such that $\phi\rho = \mathrm{id}$.

$\boxed{\text{if}}$ For the converse, suppose that there is a right H-colinear map $\phi : A \otimes H \to A$ such that $\phi\rho = \mathrm{id}$. Note that $A \otimes H$ is a right H-comodule with structure map $\rho' : A \otimes H \to A \otimes H \otimes H$ defined by

$$\rho' : a \otimes h \mapsto \sum a_0 \otimes h_1 \otimes a_1 h_2.$$

Moreover, $A \otimes H$ is injective by [Gre76, 1.5(a)]. Now let X and Y be two right H-comodules and $i : X \to Y$ be a right H-colinear map. Suppose that $f : X \to A$ is a right H-colinear map. Since $A \otimes H$ is injective, there is a map $\hat{f} : Y \to A \otimes H$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "f"'] & Y \arrow[ddl, "\hat{f}"] \\
A \arrow[d, "\rho"] & \\
A \otimes H \arrow[u, bend left, "\phi"] &
\end{tikzcd}$$

is commutative. Now define $g : Y \to A$ by $g := \phi\hat{f}$, and recall that $\phi\rho = \mathrm{id}$; so $gi = \phi\hat{f}i = \phi\rho f = f$. Hence A is injective. ∎

Definition 1.13 (Comodule algebra)**.** Let H be a Hopf algebra and suppose A is an algebra and a right H-comodule with structure map $\rho : A \to A \otimes H$. Then A is said to be a right H**-comodule algebra** if ρ is an algebra homomorphism. $\lozenge$

Definition 1.14 (Crossed product algebra)**.** Let A be an algebra, H be a Hopf algebra measuring A and $\sigma : H \otimes H \to A$ be a convolution-invertible map. Assume that, for all $a \in A$ and $x, y, z \in H$,

$$\begin{aligned} x \cdot (y \cdot a) &= \sum \sigma(x_1, y_1)\big((x_2 y_2) \cdot a\big)\sigma^{-1}(x_3, y_3), \\ 1 \cdot a &= a; \end{aligned}$$

that is, A is a **twisted module**, and that σ is a 2**-cocycle**; that is

$$\begin{aligned} \sum \big(x_1 \cdot \sigma(y_1, z_1)\big)\sigma(x_2, y_2 z_2) &= \sigma(x_1, y_1)\sigma(x_2 y_2, z), \\ \sigma(x, 1) = \sigma(1, x) &= \varepsilon(x)1. \end{aligned}$$

Then the **crossed product algebra**, written $A\#_\sigma H$, is $A \otimes H$ as a vector space with multiplication

$$(a\#h)(b\#g) \coloneqq \sum a(h_1 \cdot b)\sigma(h_2, g_1)\#h_3 g_2$$

for all $a, b \in A$ and all $h, g \in H$. $\lozenge$

The fact that crossed products exist is confirmed by the following result due to Doi and Takeuchi, and Blattner, Cohen and Montgomery.

Lemma 1.15 ([DT86], [BCM86])**.** *Let A be an algebra, H be a Hopf algebra measuring A and $\sigma : H \otimes H \to A$ be a convolution-invertible map. Then the crossed product $A\#_\sigma H$ is an associative algebra with unity element $1\#1$.*

Examples 1.16. The following examples are taken from [Mon94, pp. 102–103]. The key point to take away from them is that the definition of a crossed product in Definition 1.14 is a generalisation of other notions of crossed products.

1. **Smash products.** Let σ be trivial; that is, $\sigma(h, g) \coloneqq \varepsilon(hg)$ for all $h, g \in H$. Then the condition that A is a twisted module simplifies to the condition that A is an H-module. The 2-cocycle condition is

satisfied trivially and the multiplication in $A\#H \coloneqq A\#_\sigma H$ simplifies: let $a, b \in A$ and $g, h \in H$, then

$$(a\#g)(b\#h) \coloneqq \sum a(g_1 \cdot b)\#g_2 h.$$

The algebra $A\#H$ is called the **smash product algebra**. See [Mon94, 4.1.3] for more details.

2. **Group algebras.** Suppose $H = kG$ is a group algebra and let A be a left H-module algebra. By the discussion above, this means that G acts on A by k-algebra automorphisms. The condition for A to be a twisted module says that, for all $g, h \in G$ and all $a \in A$,

$$g \cdot (h \cdot a) = \sigma(g, h)(gh \cdot a)\sigma^{-1}(g, h).$$

So the conditions needed to form a crossed product $A\#_\sigma kG$ become the conditions for forming an associative crossed product $A * G$ as defined in [Pas89, Chapter 1]. In the special case where σ is trivial, we see that being able to form the smash product $A\#kG$ is equivalent to the map $G \to \mathrm{Aut}_{\text{k-alg.}}(A) : g \mapsto g \cdot -$ being a group homomorphism.

3. **Enveloping algebras.** Let $T = \mathcal{U}(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ and suppose A is a T-module algebra. The condition for A to be a left T-module says that we must have, for all $x \in T$ and all $a, b \in A$,

$$x \cdot (ab) = (x \cdot a)b + a(x \cdot b).$$

Thus T acts on A by derivations and we recover the definition of the crossed product $A * T$ as given in [MR88, 1.7.12]. See [Mon94, 7.1.7] for more details.

Note that, if A and H are both Hopf k-algebras, it is not true in general that the crossed product $A\#_\sigma H$ has a Hopf algebra structure. Indeed, the case where $A\#_\sigma H$ does have a Hopf algebra structure is addressed in [Maj90].

1.3.9 Invariants, coinvariants and cleft extensions

Definition 1.17 (Coinvariants)**.** Let H be a Hopf algebra. Given a right H-comodule algebra A with structure map ρ, the set of **right** H**-coinvariants**

is $A^{\operatorname{co} H} := \{a \in A : \rho(a) = a \otimes 1\}$. Similarly for B a left H-comodule algebra, with structure map $\lambda : B \to H \otimes B$, the set of **left H-coinvariants** is ${}^{\operatorname{co} H}B := \{b \in B : \lambda(b) = 1 \otimes b\}$. The notation $A^{\operatorname{co} \rho}$ (resp. ${}^{\operatorname{co} \lambda}B$) is also used for the set of right (resp. left) H-coinvariants. $\diamondsuit$

Let A be a right (resp. left) H-comodule algebra. Then since, by definition, the map ρ (resp. λ) is an algebra homomorphism, we see that the set of right (resp. left) H-coinvariants in fact forms a subalgebra of A.

Let H be a Hopf algebra and B be a right H-comodule with structure map ρ. For each $a \in A$, let $\rho(a) = \sum a_0 \otimes a_1$. Then, as discussed in [Mon94, Lemma 1.6.4], A has the structure of a left H^*-module where, for each $a \in A$ and each $f \in H^*$, the action is given by

$$f \cdot a := \sum f(a_1)a_0.$$

The set of (left) **invariants** for this action is

$${}^{H^*}A := \{a \in A : f \cdot a = u^*(f)a \text{ for all } f \in H^*\}$$

and, by [Mon94, Lemma 1.7.2(1)], ${}^{H^*}A = A^{\operatorname{co} H}$. Note that, for H a Hopf algebra, it is not in general the case that H^* has a Hopf algebra structure. It is, however, always an augmented algebra with augmentation $u^* : H^* \to k$ given by $u^*(f) := f(1)$; thus the definition of the invariants above makes sense.

Definition 1.18 (Cleft extension)**.** Let H be a Hopf algebra and A a right H-comodule algebra. A right H-colinear and convolution-invertible map $\gamma : H \to A$ is called a **cleaving**. If such a cleaving map exists then $A^{\operatorname{co} H} \subseteq A$ is said to be a **cleft extension** or **H-cleft**. $\diamondsuit$

It turns out that, for H a Hopf algebra and A a right H-comodule algebra, the two notions of crossed product and cleft extension are equivalent.

Theorem 1.19. *Let T be a Hopf algebra and A a right H-comodule algebra.*

(i) If $A = B\#_\sigma H$ is a crossed product then A is H-cleft with cleaving map $\gamma : h \mapsto 1\#h$.

(ii) If A is H-cleft with cleaving γ such that $\gamma(1) = 1$ then

$$\begin{aligned} A^{\operatorname{co} T} \#_\sigma H &\cong A, \\ a \# h &\mapsto a\gamma(h), \end{aligned}$$

as T-comodule algebras. The crossed product is defined by

$$h \cdot a \coloneqq \sum \gamma(h_1) a \gamma^{-1}(h_2), \qquad \sigma(g, h) \coloneqq \sum \gamma(g_1)\gamma(h_1)\gamma^{-1}(g_2 h_2)$$

for all $g, h \in H$ and all $a \in A^{\operatorname{co} H}$.

Proof. Part (*i*) was proved by Blattner and Montgomery [BM89] and part (*ii*) is due to Doi and Takeuchi [DT86]. Proofs of both results can be found in [Mon94]. ■

Our main motivation for studying crossed products can be found in Chapter 4 where we study particular right H-comodule algebras A and ask when we can decompose A as a crossed product $A^{\operatorname{co} H} \#_\sigma H$.

1.3.10 Normality and conormality

We take the following definitions from [SS06, Section 3.2].

Definition 1.20 (Normal and conormal)**.** Let H be a Hopf algebra and $K, I \subseteq H$ be vector subspaces. The subspace K is said to be **left normal** resp. **right normal** in H if, for all $x \in H$ and $y \in K$,

$$\sum x_1 y S(x_2) \in K \quad \text{resp.} \quad \sum S(x_1) y x_2 \in K.$$

The subspace I is called **left conormal** resp. **right conormal** in H if, for all $x \in I$,

$$\sum x_1 S(x_3) \otimes x_2 \in H \otimes I \quad \text{resp.} \quad \sum x_2 \otimes S(x_1) x_3 \in I \otimes H.$$

We use the terms **normal** (resp. **conormal**) to mean both left and right normal (resp. conormal). A Hopf algebra map is said to be **conormal** if its kernel is a conormal Hopf ideal. ◊

1.4 Ore and skew-Laurent extensions

1.4.1 Ore extensions

See [GW04] for the standard definition of an Ore extension. We present an alternative definition here, due to Schneider and Schauenburg [SS06, Section 1.1]. This way of defining Ore extensions has the benefit that, once we know that crossed products exist (which we do by Lemma 1.15), there is no trouble proving the existence of Ore extensions.

Definition 1.21 (σ-derivation)**.** Let A be a k-algebra and $\sigma : A \to A$ be an algebra automorphism. A k-linear map $\delta : A \to A$ is called a **(left)** **σ-derivation** if, for all $a, b \in A$,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b. \qquad \Diamond$$

Let $H \coloneqq k\langle g, x\rangle$ be the free algebra on the generators g and x. Then we can define a coproduct on H by letting g be group-like and x be $(1, g)$-primitive; thus H has the structure of a Hopf algebra. Given any algebra R with an automorphism $\sigma : R \to R$ and a σ-derivation δ, we can construct the **Ore extension** $R[x; \sigma, \delta]$ as follows. Now observe that R becomes a left H-module algebra by setting, for each $r \in R$,

$$g \cdot r \coloneqq \sigma(r) \quad \text{and} \quad x \cdot r \coloneqq \delta(r).$$

Consider the subalgebra $k[x] \subseteq H$ and notice that, because $\Delta(k[x]) \subseteq H \otimes k[x]$, the subspace $R\#k[x]$ is a subalgebra of the smash product $R\#H$; we define the Ore extension $R[x; \sigma, \delta] \coloneqq R\#k[x]$, suppressing the # and writing rx instead of $r\#x$. The multiplication in $R[x; \sigma, \delta]$ is given, for each $r \in R$, by

$$xr = \sum (x_1 \cdot r)x_2 = x \cdot r + (g \cdot r)x = \sigma(r)x + \delta(r).$$

In particular, an extension by derivation $R[x; \delta]$ is nothing more than a smash product with the Hopf algebra $\mathcal{O}(k^+)$ where the action of $\mathcal{O}(k^+)$ on R is by derivations.

Inner σ-derivations

We record another definition and lemma from [GW04].

Definition 1.22 (Inner σ-derivation)**.** Suppose A is a k-algebra, $\sigma : A \to A$ is a k-algebra automorphism and δ is a σ-derivation. Then δ is said to be an **inner** σ-derivation if there exists some $d \in R$ such that $\delta(r) = dr - \sigma(r)d$ for all $r \in R$. $\lozenge$

Lemma 1.23. *Let R be a k-algebra and suppose $T = R[X; \sigma, \delta]$ where σ is a k-algebra automorphism of R and δ is an inner σ-derivation with $\delta(r) = dr - \sigma(r)d$ for some $d \in R$ and all $r \in R$. Then $T = R[X - d; \sigma]$.*

1.4.2 Skew-Laurent extensions

Consider the Hopf algebra $\mathcal{O}(k^\times) = k[x^{\pm 1}]$ with x group-like. Given any algebra R with an automorphism $\sigma : R \to R$, we can construct the **skew-Laurent extension** $R[x^{\pm 1}; \sigma]$ as follows. First notice that R becomes a left $\mathcal{O}(k^\times)$-module algebra if we define, for each $r \in R$, $x \cdot r := \sigma(r)$. Then define $R[x^{\pm 1}; \sigma]$ to be the smash product $R\#\mathcal{O}(k^\times)$. Then we see that, again suppressing the #, the multiplication in $R[x^{\pm 1}; \sigma]$ is given, for each $r \in R$, by

$$xr = \sum (x_1 \cdot r)x_2 = (x \cdot r)x = \sigma(r)x.$$

Thus a skew-Laurent extension $R[x^{\pm 1}; \sigma]$ is nothing more than a smash product with the Hopf algebra $\mathcal{O}(k^\times)$ where the action of $\mathcal{O}(k^\times)$ on R is by automorphisms.

1.4.3 Characters, maximal ideals and everything in between

Let k be a field and suppose that A is a k-algebra. Recall from section 1.1.1 that we call an algebra homomorphism $\chi : A \to k$ a **character**, and that $\ker \chi$ is a **character ideal**. Observe that there is a one-to-one correspondence between characters of A and character ideals of A, and that these are precisely the ideals $\mathfrak{m}$ of A such that $A/\mathfrak{m} \cong k$.

There is a hierarchy of sets of ideals, which we should make clear.

$$\begin{aligned}
\mathcal{V} &:= \{\mathfrak{m} \lhd R : R/\mathfrak{m} \cong k\} \\
\mathcal{W} &:= \{\mathfrak{m} \lhd R : R/\mathfrak{m} \cong \text{a field}\} \\
\mathcal{X} &:= \{\mathfrak{m} \lhd R : R/\mathfrak{m} \cong \text{a division ring}\} \\
\mathcal{Y} &:= \{\mathfrak{m} \lhd R : R/\mathfrak{m} \cong \text{a simple Artinian ring}\} \\
\mathcal{Z} &:= \{\mathfrak{m} \lhd R : \mathfrak{m} \text{ is a maximal ideal}\}
\end{aligned}$$

Now $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{Z}$ but, as we demonstrate below, there are examples to show that each of these inclusions can be strict.

$\boxed{\mathcal{V} \subsetneq \mathcal{W}}$ Let $k = \mathbb{R}$ and $R = \mathbb{R}[x]$. Then the ideal $\mathfrak{m} \coloneqq \left\langle x^2 + 1 \right\rangle$ has $R/\mathfrak{m} \cong \mathbb{C} \neq \mathbb{R}$.

$\boxed{\mathcal{W} \subsetneq \mathcal{X}}$ Let R be the division algebra of quaternions, which is an $\mathbb{R}$-algebra but not a field.

$\boxed{\mathcal{X} \subsetneq \mathcal{Y}}$ Let $k = \mathbb{C}$ and $R = \mathbb{C} \oplus \mathbb{C}$. Then define the Ore extension $T = R[x; \sigma]$ where, for all $(a, b) \in R$, $\sigma(a, b) \coloneqq (b, a)$. Let $\mathfrak{m} \coloneqq (x^2 - 1)T$. Then, as can be checked, $T/\mathfrak{m} \cong M_2(\mathbb{C})$, the ring of 2-by-2 matrices over $\mathbb{C}$.

$\boxed{\mathcal{Y} \subsetneq \mathcal{Z}}$ Let $k = \mathbb{C}$ and $R = k[y]$, and consider the Ore extension $T = R[x; d/dy]$, which has relation $xy - yx = 1$, so $T = A_1(\mathbb{C})$, the **first Weyl algebra** over $\mathbb{C}$ (see [GW04, Chapter 2]). Then $\{0\}$ is a maximal ideal but T is not Artinian.

There are situations in which the sets defined above all coincide; most obviously when k is algebraically closed and R is affine commutative.

Chapter

2 Hopf structures on skew group algebras

This chapter is split into two quite different sections. In section 2.1 we explore some ring-theoretic properties that rings must have if they are to support the structure of a Hopf algebra. In section 2.2 we start to study the existence of Hopf algebra structures on extensions of Hopf algebras.

2.1 A Hopf algebra test

In this section, we make an observation that affine or noetherian Hopf algebras over an algebraically closed field of characteristic zero must have a certain property, which is dependent only on the algebra structure. This provides a necessary condition that a given algebra must satisfy in order for it to support a Hopf algebra structure.

2.1.1 Two ideals

First consider the following definitions. The notation introduced in Definitions 2.1, 2.4 and 2.5 will be used throughout the thesis.

Definition 2.1 ($\mathrm{I}(A)$)**.** Let A be an algebra over a field k. Then $\mathrm{I}(A)$ is defined to be the intersection of the kernels of the characters of A; that is,

$$\mathrm{I}(A) \coloneqq \bigcap \left\{ \ker \phi \;\middle|\; \phi : A \to k \text{ is an algebra homomorphism.} \right\}$$

If A has no characters then, by convention, we set $\mathrm{I}(A) = A$. $\Diamond$

Clearly $I(A)$ is an ideal of A.

Lemma 2.2. *Suppose* k *is a field and let* A *be a* k*-algebra.*

(i) *If* $I(A) \neq A$, *then it is semiprime.*

(ii) $I\big(A/I(A)\big) = \{0\}$.

(iii) *Let* $\theta : A \to B$ *be a homomorphism of* k*-algebras. Then* $\theta\big(I(A)\big) \subseteq I(B)$

Proof. (i) $I(A)$ is the intersection of maximal (and hence prime) ideals.

(ii) Suppose, for a contradiction, that $x+I(A) \in I(A/I(A))$ is nonzero; that is, $x+I(A) \in A/I(A)$ is nonzero and, for all characters $\chi : A/I(A) \to k$, $\chi(x + I(A)) = 0$. Then for all characters $\hat{\chi} : A \to k$ we would have $\hat{\chi}(x) = 0$; that is, $x \in I(A)$.

(iii) We show that $\theta(I(A)) \subseteq I(\theta(A)) \subseteq I(B)$. To see the first inclusion, let $\theta(a) \in \theta(A)$ with $\hat{\chi}(\theta(a)) \neq 0$ for some algebra homomorphism $\hat{\chi} : B \to k$. Then we can define an algebra homomorphism $\chi \coloneqq \hat{\chi} \circ \theta : A \to k$. Hence $\chi(a) = \hat{\chi}\theta(a) \neq 0$ and so $a \notin I(A)$; that is, $\theta(a) \notin \theta(I(A))$. For the second inclusion, observe that

$$\begin{aligned} I(\theta(A)) &\subseteq \theta(A) \cap \bigcap\big\{\mathfrak{m} \cap \theta(A) : \mathfrak{m} \lhd B \text{ with } B/\mathfrak{m} \cong k\big\} \\ &= \theta(A) \cap \bigcap\big\{\mathfrak{m} : \mathfrak{m} \lhd B \text{ with } M/\mathfrak{m} \cong k\big\} \\ &= \theta(A) \cap I(B). \end{aligned}$$

■

Remark 2.3. The inclusion in part (iii) can be strict, even if θ is surjective or injective. To see this for θ surjective, let $A \coloneqq \mathbb{C}[x]$ and $B \coloneqq \mathbb{C}[x]/\langle x^2\rangle$ with $\theta : A \to B$ the canonical surjection; then $I(A) = \{0\}$ but $I(B) = \langle x\rangle$. For an example with θ injective, let $A \coloneqq \mathbb{C}[x]$ and $B \coloneqq \mathbb{C}\langle x, y : xy - yx = 1\rangle$ with $\theta : A \to B$ the inclusion map. Then $I(A) = \{0\}$ but $I(B) = B$.

Definition 2.4 (Commutator space)**.** Let A be a k-algebra. The **commutator space**, denoted $[A, A]$, is the vector subspace spanned by commutators of elements of A; that is,

$$[A, A] \coloneqq \operatorname{span}_k\{ab - ba : a, b \in A\}. \qquad \lozenge$$

Definition 2.5 (Commutator ideal)**.** Let A be an algebra. The **commutator ideal**, denoted $\langle[A, A]\rangle$, is the ideal generated by commutators of elements of A; that is,

$$\langle[A, A]\rangle \coloneqq \langle ab - ba : a, b \in A\rangle. \qquad \diamondsuit$$

Lemma 2.6. *Suppose k is a field and let A be a k-algebra.*

(i) $\langle[A, A]\rangle$ is the unique smallest ideal of A such that $A/\langle[A, A]\rangle$ is commutative. In particular, $\langle[A, A]\rangle \subseteq I(A)$.

(ii) Let $\theta : A \to B$ be a homomorphism of k-algebras. Then $\theta\big(\langle[A, A]\rangle\big) \subseteq \langle[B, B]\rangle$ with equality if θ is surjective.

Proof.

(i) The first part is clear from the definition. For the second, observe that any character must vanish on the commutator of two elements.

(ii) Let $a[r, s]b \in \langle[A, A]\rangle$. Then $\theta(a[r, s]b) = \theta(a)[\theta(r), \theta(s)]\theta(b) \in \langle[B, B]\rangle$. ∎

Remark 2.7. The inclusion in part (ii) can be strict: let $A = \mathbb{C}[x]$ and $B = \mathbb{C}\langle x, y : xy - yx = 1\rangle$ with $\theta : A \to B$ the inclusion map; then $\langle[A, A]\rangle = 0$ and $\langle[B, B]\rangle = B$.

In the case where k is algebraically closed and A is affine, the ideals $I(A)$ and $\langle[A, A]\rangle$ are related as follows.

Proposition 2.8. *Let k be an algebraically closed field and A be an affine k-algebra. Then*

(i) there exists a positive integer m such that

$$I(A)^m \subseteq \langle[A, A]\rangle \subseteq I(A);$$

(ii) $\langle[A, A]\rangle$ is semiprime if, and only if, $I(A) = \langle[A, A]\rangle$.

Proof. (i) Observe that $A/\langle[A, A]\rangle$ is affine and commutative and therefore, by Hilbert's basis theorem, it is noetherian.

We see that $\langle[A,A]\rangle \subseteq I(A)$ because, for any algebra homomorphism $\phi : A \to k$, it must be the case that $\phi(uv - vu) = 0$ for all $u, v \in A$.

For the reverse inclusion, note that $A/\langle[A,A]\rangle$ is an affine commutative algebra over k, which we assumed to be algebraically closed. By definition, any character of $A/\langle[A,A]\rangle$ vanishes on $I(A)/\langle[A,A]\rangle$ and hence, by Hilbert's nullstellensatz ([Abe80, Theorem 1.5.5]),

$$\sqrt{\frac{I(A)}{\sqrt{\langle[A,A]\rangle}}} = \mathrm{nil}\left(\frac{A}{\sqrt{\langle[A,A]\rangle}}\right) = 0$$

since $\sqrt{\langle[A,A]\rangle} \triangleleft A$ is a semiprime ideal; thus $I(A)^m \subseteq \langle[A,A]\rangle$ for some m.

(*ii*) The implication from right to left follows from the fact that $I(A)$ is semiprime by Lemma 2.2. Conversely, if $\langle[A,A]\rangle$ is semiprime then, since $I(A)^m \subseteq \langle[A,A]\rangle$ for some $m \geqslant 1$ by part (*i*), we have $I(A) \subseteq \langle[A,A]\rangle$. But we also have $\langle[A,A]\rangle \subseteq I(A)$ from part (*i*). ∎

2.1.2 Properties of these ideals for a Hopf algebra

The following result is mentioned in [BG02, I.9.24] but we give the details of the proof here.

Proposition 2.9. *Let H be a Hopf algebra over k. Then $I(H)$ is a proper Hopf ideal of H.*

Proof. First notice that, since H is a Hopf algebra, we have $I(H) \subseteq \ker \varepsilon \neq H$ and so $I(H)$ is a proper ideal. Now we see that $I(H)$ is a coideal of H as follows. Suppose $a \in I(H)$ so that $\phi(a) = 0$ for all algebra homomorphisms $\phi : H \to k$. In particular $\varepsilon : H \to k$ is an algebra homomorphism and so $\varepsilon(a) = 0$. Hence $\varepsilon(I(H)) = 0$. Next note that the kernels of characters are precisely the annihilators of one-dimensional H-modules. Let V and W be one-dimensional H modules. Then so is $V \otimes W$ and the action of H is given by $h \cdot (v \otimes w) = \sum h_1 \cdot v \otimes h_2 \cdot w$ for each $h \in H$. In particular, for any $x \in I(H)$, $x \cdot (V \otimes W) = 0$ and so, for all one-dimensional H-modules V and

W, $\Delta(x) \in \mathrm{Ann}(V) \otimes H + H \otimes \mathrm{Ann}(W)$; hence

$$\begin{aligned}\Delta(x) &\in \bigcap_{V,W} \big(\mathrm{Ann}(V) \otimes H + H \otimes \mathrm{Ann}(W)\big)\\ &= \Big(\bigcap_{V} \mathrm{Ann}(V)\Big) \otimes H + H \otimes \Big(\bigcap_{W} \mathrm{Ann}(W)\Big)\\ &= I(H) \otimes H + H \otimes I(H).\end{aligned}$$

Thus $\Delta\big(I(H)\big) \subseteq H \otimes I(H) + I(H) \otimes H$. To prove that $S(I) \subseteq I$, note that we have $S : H \to H^{op}$ an algebra homomorphism. Any algebra homomorphism $\phi : H \to k$ is also one $H^{op} \to k$. Then the composition $\phi \circ S : H \to k$ is an algebra homomorphism. Now take $a \in I$ and notice that we must have

$$\phi \circ S(a) = 0;$$

that is, $S(a) \in \ker \phi$ for any algebra homomorphism $\phi : H \to k$. So we have that $S(I) \subseteq I$. ∎

The following result appears in [KM03], but was proven independently.

Proposition 2.10. *Let H be any Hopf algebra. Then the ideal $\langle [H, H] \rangle$ is a Hopf ideal of H.*

Proof. By definition, $H/\langle [H, H] \rangle$ is a commutative algebra. Observe that

$$H/\langle [H, H] \rangle \otimes H/\langle [H, H] \rangle \cong (H \otimes H)/\big(H \otimes \langle [H, H] \rangle + \langle [H, H] \rangle \otimes H\big).$$

is also commutative. Thus, again by the definition of $\langle [H, H] \rangle$,

$$\Big\langle \big[(H \otimes H), (H \otimes H)\big] \Big\rangle \subseteq H \otimes \langle [H, H] \rangle + \langle [H, H] \rangle \otimes H. \tag{2.1}$$

Thus we have that, for any elements $u, v \in H \otimes H$, their commutator $[u, v]$ lies in $H \otimes \langle [H, H] \rangle + \langle [H, H] \rangle \otimes H$. This allows us to prove that $\langle [H, H] \rangle$ is a Hopf ideal as follows.

Firstly, the fact that $\varepsilon(\langle [H, H] \rangle) = 0$ is clear, since any algebra homomorphism to the ground field k sends any commutator to zero. We can check that $\Delta(\langle [H, H] \rangle) \subseteq H \otimes \langle [H, H] \rangle + \langle [H, H] \rangle \otimes H$ by verifying that commutator lies in the right-hand side. Consider $xy - yx \in \langle [H, H] \rangle$ where $x, y \in H$.

Then, since Δ is an algebra homomorphism, and from (2.1),

$$\Delta([x,y]) = [\Delta(x), \Delta(y)] \in H \otimes \langle [H,H] \rangle + \langle [H,H] \rangle \otimes H$$

because $\Delta(x), \Delta(y) \in H \otimes H$.

Finally, we need to check that $\langle [H,H] \rangle$ is closed under the antipode map S. But this is clear since S is an anti-homomorphism of H so that, for any $x, y \in H$,

$$S([x,y]) = [S(y), S(x)].$$

Thus $\langle [H,H] \rangle$ is a Hopf ideal of H. ∎

Corollary 2.11. *Let* H *be an affine or a noetherian Hopf algebra over an algebraically closed field* k*. Then* $H/I(H)$ *is isomorphic to the coordinate ring of some affine algebraic group over* k*.*

Proof. In the case where H is affine, we know that $H/I(H)$ is affine. If H is noetherian then we can reach the same conclusion since we know, from Proposition 2.9, that $I(H)$ is a proper Hopf ideal and so $H/I(H)$ is a commutative noetherian Hopf algebra and, by Molnar's theorem [Mol75], commutative noetherian Hopf algebras over a field are affine. Now $I(H)$ is semiprime, by Proposition 2.8, and so $H/I(H)$ is an affine commutative semiprime Hopf algebra; hence it is the coordinate ring of some affine algebraic group by the discussion in section 1.3.7. ∎

Remarks 2.12.

1. Wu and Zhang have asked whether all noetherian Hopf algebras are affine [WZ03, Question 5.1]; there are no known counter-examples, and this question is still open, but the converse is false. For an easy example, consider F_2, the free group on two generators. Then kF_2 is affine (as is the group algebra of any finitely generated group) but it is not noetherian.

2. The above results tell us that, given a noetherian or an affine Hopf algebra H over an algebraically closed field k, the quotient Hopf algebra $H/I(H)$ is isomorphic, as a Hopf algebra, to the coordinate ring of the group of right winding automorphisms of H. Moreover the group of algebra automorphisms of H contains a copy of G and of G^{op}.

Lemma 2.13. *Suppose* k *is an algebraically closed field of characteristic zero and let* H *be an affine or a noetherian Hopf* k*-algebra. Then* $I(H) = \langle[H, H]\rangle$.

Proof. If H is affine then, since $\langle[H, H]\rangle$ is a Hopf ideal by Proposition 2.10, $H/\langle[H, H]\rangle$ is a commutative affine Hopf algebra. And if H is noetherian then the quotient $H/\langle[H, H]\rangle$ is a commutative noetherian Hopf algebra; hence it is affine by Molnar's theorem [Mol75]. Because k has characteristic zero it follows, by Cartier's theorem [Wat79, Theorem 11.4], that $H/\langle[H, H]\rangle$ is a semiprime ring. Thus $\langle[H, H]\rangle$ is a semiprime ideal of H and so, from Proposition 2.8, it follows that $I(H) = \langle[H, H]\rangle$. ■

This gives us a test to determine when a given affine or noetherian algebra *cannot* support the structure of a Hopf algebra. Note, however, that the converse is false, as demonstrated by Example 1 below. In addition, the assumption that k has characteristic zero is essential; see Example 2.

Remarks 2.14.

1. Consider the cusp curve in $\mathbb{A}^2_{\mathbb{C}}$ given by $y^2 = x^3$ and call its coordinate ring $\mathcal{O}$. Because the cusp is an algebraic set, we know that $\mathcal{O}$ is an affine commutative semiprime algebra. Thus $\langle[\mathcal{O}, \mathcal{O}]\rangle = 0$ by commutativity and, since $\mathcal{O}$ is semiprime, $I(\mathcal{O}) = \langle[\mathcal{O}, \mathcal{O}]\rangle$ by Proposition 2.8. But the cusp is not an algebraic group and hence its coordinate ring $\mathcal{O}$ is not an affine commutative semiprime Hopf algebra. But all of the adjectives can be applied to $\mathcal{O}$ and, therefore, it can't be a Hopf algebra. So $I(H) = \langle[H, H]\rangle$ does not imply that H admits a Hopf algebra structure.

2. Let p be a prime and suppose the field k has characteristic p. Let C_p be the cyclic group of order p generated by x, then $(x - 1)^p = 0$ in kC_p. Thus any character $\chi : kC_p \to k$ must have $\ker \chi = (x - 1)kC_p$ and so $I(kC_p) = (x - 1)kC_p$. On the other hand, since kC_p is commutative, we know that $\left\langle[kC_p, kC_p]\right\rangle = \{0\}$; hence the Hopf algebra kC_p has $I(kC_p) \neq \left\langle[kC_p, kC_p]\right\rangle$. This shows that the assumption in Lemma 2.13 that k has characteristic zero is necessary.

Proposition 2.15. *Suppose* k *is an algebraically closed field of characteristic zero. Let* R *be an affine commutative* k*-algebra and* $\delta \in \mathrm{Der}_k(R)$.

Set $T = R[X; \delta]$ *and suppose that* T *admits a Hopf algebra structure. Then* $\langle \delta(R) \rangle$ *is a semiprime ideal of* T.

Proof. Since $Xr - rX = \delta(r)$, for all $r \in R$, the ideal $\langle \delta(R) \rangle$ of T is the smallest with commutative quotient. Then, since T is a Hopf algebra, $\langle \delta(R) \rangle$ is semiprime by Lemma 2.13 and Proposition 2.8. ∎

Corollary 2.16. *Suppose* k *is an algebraically closed field of characteristic zero. Let* $R = k[y]$ *and* $\delta \in \mathrm{Der}_k(R)$, *so* $\delta = f(d/dy)$ *for some* $f \in k[y]$. *If* $R[X; \delta]$ *admits a Hopf algebra structure, then* f *has no repeated factors in its irreducible factorisation.*

Proof. Write $f = f_1^{m_1} f_2^{m_2} \cdots f_t^{m_t}$ with each $f_i \in k[y]$ irreducible and distinct. By Proposition 2.15, we need $\langle f \rangle$ to be semiprime; that is, we must have $m_1 = m_2 = \cdots = m_t = 1$. ∎

Remarks 2.17.

1. This corollary can also be obtained from stronger (but more difficult) results of Goodearl and Zhang [GZ10].

2. The **Jordan plane** (defined in [Kor91]) is the k-algebra $T \coloneqq k\langle x, y : xy - yx = y^2 \rangle$. We see that $T = k[y][x; y^2(d/dy)]$ and so the corollary tells us that it does not admit the structure of a Hopf algebra.

2.2 Hopf structures on skew group algebras

In this section, we shall state a result, due to Panov [Pan03], which gives necessary and sufficient conditions for an Ore extension of a Hopf algebra to have a particular (but natural) Hopf algebra structure. Panov's result suggests a related question about the existence of Hopf structures on skew-Laurent extensions; we shall discuss this question before providing an answer as a corollary of the main theorem (Theorem 2.22).

Throughout this section, R will be an arbitrary Hopf algebra over an arbitrary field k. In addition, σ will be an algebra automorphism of R and δ will be a σ-derivation. We shall denote by T the Ore extension $R[X; \sigma, \delta]$ and by Q the skew-Laurent extension $R[X^{\pm 1}; \sigma]$.

Note that, in general, we cannot assume that T and Q have Hopf algebra structures even though R has one. For example, if $R = k[y]$ is the polynomial algebra in one indeterminate then the first Weyl algebra $T = R\left[x; \frac{d}{dy}\right]$ cannot have a Hopf algebra structure as follows. In T, we have the relation $xy - yx = 1$ so, if T were to have a Hopf algebra structure with counit ε, we would require $\varepsilon(xy - yx) = \varepsilon(1)$. But, since ε is an algebra homomorphism and k is commutative, the left-hand side of this equation is 0 and the right-hand side is 1, giving a contradiction. On the other hand, the polynomial algebra $k[x, y]$ is trivially an Ore extension of $k[y]$ and clearly has a Hopf algebra structure as it is the coordinate ring of the affine algebraic group $(k^+)^2$.

2.2.1 Skew-primitive Hopf-Ore extensions

In his paper [Pan03], Panov gives necessary and sufficient conditions on the Hopf algebra R so that T is a Hopf algebra extending R in which the new indeterminate X is skew-primitive (see section 1.3.1). To make reference to this notion more straightforward, we introduce the following terminology.

Definition 2.18 (Skew-primitive Hopf-Ore extension)**.** Suppose k is a field. Let R be a Hopf k-algebra and set $T = R[X; \sigma, \delta]$ where σ is an algebra automorphism and δ is a σ-derivation. Then T is said to be a **skew-primitive Hopf-Ore extension** of R if T has a Hopf algebra structure with R a Hopf subalgebra and with X skew-primitive; that is if

$$\Delta(X) = X \otimes g_1 + g_2 \otimes X$$

for some group-like elements $g_1, g_2 \in R$. ◊

Notice that by making the change of variable $X \mapsto X' = g_1^{-1}X$, we have

$$\Delta(X') = (g_1^{-1} \otimes g_1^{-1})(X \otimes g_1 + g_2 \otimes X) = X' \otimes 1 + g_1^{-1}g_2 \otimes X'$$

and so X' is $(1, g)$-primitive where $g \coloneqq g_1^{-1}g_2$ is group-like. We can then modify σ and δ by setting, for all $r \in R$,

$$\sigma'(r) \coloneqq g_1\sigma(r)g_1^{-1}$$

and

$$\delta'(r) \coloneqq g_1^{-1}\delta(r).$$

Then σ' is an algebra automorphism of R, since it is the composition of two such automorphisms. Moreover δ' is a σ'-derivation because, for any $r, s \in R$,

$$\begin{aligned}\delta'(rs) &= g_1^{-1}\delta(rs)\\ &= g_1^{-1}(\delta(r)s + \sigma(r)\delta(s))\\ &= g_1^{-1}\delta(r)s + \sigma'(r)g_1^{-1}\delta(s)\\ &= \delta'(r)s + \sigma'(r)\delta'(s).\end{aligned}$$

Thus, without loss of generality, we can assume that X is $(1, g)$-primitive for some group-like element $g \in R$. Similarly we could change the variable $X \mapsto X'' = g_2^{-1}X$ and then X'' would be $(h, 1)$-primitive, where $h \coloneqq g_2^{-1}g_1$.

Theorem 2.19 ([Pan03, Theorem 1.3]). *Let k be a field. The k-algebra $T = R[X; \sigma, \delta]$ is a skew-primitive Hopf-Ore extension of R, with X being $(1, g)$-primitive for a group-like element $g \in R$, if and only if each of the following conditions holds.*

1. *There is a character $\chi : R \to k$ such that, for any $r \in R$,*

$$\sigma(r) = \sum \chi(r_1)r_2;$$

   that is, σ is a left winding automorphism of R.

2. *For each $r \in R$,*

$$\sum \chi(r_1)r_2 = \sum gr_1g^{-1}\chi(r_2).$$

   That is, $\tau_\chi^\ell = c_g \circ \tau_\chi^r$, where c_g denotes conjugation of R by g.

3. *For each $r \in R$, the σ-derivation δ satisfies the identity*

$$\Delta\delta(r) = \sum \Big(\delta(r_1) \otimes r_2 + gr_1 \otimes \delta(r_2)\Big).$$

Remarks 2.20.

1. Let k be an algebrically closed field of characteristic zero, suppose $n \geqslant 2$ is an integer and let $\nu \in k$ be a primtive nth root of unity. Consider the n^2-dimensional Taft algebra $T(n)$ as defined in [Taf71]. The algebra $T(n)$ can be written as the Ore extension $kC_n[x; \sigma]$ where

$C_n := \langle g : g^n = 1\rangle$ is the cyclic group of order n and $\sigma(g) := \nu g$. The Taft algebra $T(n)$ has a Hopf algebra structure where g is group-like and x is $(1, g^t)$-primtive, for some integer t. Indeed we see from Theorem 2.19 that if $T = kC_n[x; \sigma]$ is any skew-primitive Hopf-Ore extension of kC_n then σ must be a winding automorphism of kC_n and consequently T must be a Taft algebra.

2. The conditions imposed in Definition 2.18 do not allow all Hopf algebras of the form $R[X; \sigma, \delta]$, even when R is a *commutative* Hopf subalgebra. As an example, let k be algebraically closed of characteristic zero and consider the three-dimensional Heisenberg group

$$G := \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} : a_{12}, a_{13}, a_{23} \in k \right\}.$$

Since $i : G \hookrightarrow SL(3, k)$ is an embedding of algebraic groups we get a surjective Hopf morphism

$$\pi := i^* : \mathcal{O}(SL(3, k)) \twoheadrightarrow \mathcal{O}(G)$$

dual to the embedding i. Let us write

$$\mathcal{O}(SL(3, k)) = k\left[\overline{X_{ij}} : 1 \leqslant i, j \leqslant 3\right] \cong k[X_{ij} : 1 \leqslant i, j \leqslant 3]/(\det -1)$$

and so

$$\pi(\overline{X_{ij}}) = \overline{X_{ij}} \circ i = \begin{cases} 0 & i > j \\ 1 & i = j \\ Y_{ij} & i < j \end{cases}$$

where $Y_{ij} \in \mathcal{O}(G)$ picks out the (i, j)th entry. Since π is a Hopf morphism, the coproduct on $\mathcal{O}(G)$ is obtained from the coproduct on $\mathcal{O}(SL(3, k))$. Now

$$\Delta(\overline{X_{ij}}) = \sum_{m=1}^{3} \overline{X_{ik}} \otimes \overline{X_{kj}}$$

and so

$$\begin{aligned}
\Delta(Y_{12}) &= (\pi \otimes \pi)\Delta\big(\overline{X_{12}}\big) = 1 \otimes Y_{12} + Y_{12} \otimes 1 \\
\Delta(Y_{23}) &= (\pi \otimes \pi)\Delta\big(\overline{X_{23}}\big) = 1 \otimes Y_{23} + Y_{23} \otimes 1 \\
\Delta(Y_{13}) &= (\pi \otimes \pi)\Delta\big(\overline{X_{13}}\big) = 1 \otimes Y_{13} + Y_{12} \otimes Y_{23} + Y_{13} \otimes 1.
\end{aligned}$$

We can also calculate how the counit and antipode behave using the fact that $\varepsilon(Y_{ij})$ is the (i,j)th entry of the identity matrix and $S(Y_{ij})$ is the (i,j)th entry of the inverse. Hence $\varepsilon(Y_{12}) = \varepsilon(Y_{23}) = \varepsilon(Y_{13}) = 0$ and

$$\begin{aligned}
S(Y_{12}) &= -Y_{12} \\
S(Y_{23}) &= -Y_{23} \\
S(Y_{13}) &= Y_{12}Y_{23} - Y_{13}
\end{aligned}$$

In particular, note that Y_{12} and Y_{23} are primitive but Y_{13} is not.

Let $T \coloneqq \mathcal{O}(G) \cong k[Y_{12}, Y_{23}, Y_{13}]$ with the Hopf structure just discussed. We shall show that we cannot write T as a skew-primitive Hopf-Ore extension. Suppose that $T = R[X]$ for some Hopf subalgebra $R \subseteq T$ and some primitive $X \in P(T)$. (Note there are no units in R except for nonzero field elements and so the only skew-primitive elements are actually primitive.) Then $R^+T \subseteq T$ would be a Hopf ideal and so $T/R^+T \cong \mathcal{O}(k^+)$. Therefore G must contain a copy of k^+ as a normal subgroup; thus T/R^+T is the coordinate ring of the centre

$$C \coloneqq \left\{ \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a_{13} \in k \right\}.$$

Hence $T/R^+T \cong k[Y_{13}]$; but this is not a Hopf subalgebra of T, contradicting our assumption that T could be written as a skew-primitive Hopf-Ore extension.

3. In Chapter 3, we shall study Hopf algebras of the form $T = R[X; \sigma, \delta]$ without the hypothesis that X is skew-primitive.

2.2.2 Hopf structures on skew group algebras

A related question is suggested by Theorem 2.19; namely if R is a Hopf algebra and $Q = R[X^{\pm 1}; \sigma]$, what are necessary and sufficient conditions on σ so that Q is a Hopf algebra extending R with X group-like? Since the skew-Laurent extension Q is the skew group algebra $R *_\sigma C_\infty$, where C_∞ is the infinite cyclic group, we might hope to find an answer in the more general setting of skew group algebras.

Recall the definition of a left module algebra from Definition 1.8.

Lemma 2.21. *Let* k *be a field,* R *be a Hopf* k*-algebra and* kG *be a group algebra with its usual Hopf algebra structure. Suppose that* R *is a left* kG*-module algebra. Then* $\cdot : kG \otimes R \to R$ *is a coalgebra morphism if and only if, for each* $g \in G$*, the map* $g \cdot - : R \to R$ *is a coalgebra morphism.*

Proof. First observe that, for each $g \in G$, the map $g \cdot - : R \to R$ is always an algebra automorphism as follows. Since, by assumption, kG measures R we know that, for each $a, b \in R$,

$$g \cdot (ab) = \sum (g_1 \cdot a)(g_2 \cdot b) = (g \cdot a)(g \cdot b) \quad \text{and} \quad g \cdot 1 = \varepsilon(g)1 = 1$$

because g is group-like. Note also that $g \cdot -$ is bijective since kG acts on R.

$\boxed{\text{only if}}$ Assume that $\cdot : kG \otimes R \to R$ is a coalgebra morphism with the standard coalgebra structure on $kG \otimes R$. Then we have, for all $g \in G$ and $r \in R$,

$$\Delta(g \cdot r) = \sum (g \cdot r_1) \otimes (g \cdot r_2) = \big((g \cdot -) \otimes (g \cdot -)\big)\Delta(r)$$

and $\varepsilon(g \cdot r) = \varepsilon(g)\varepsilon(r) = \varepsilon(r)$; thus $g \cdot -$ is a coalgebra morphism.

$\boxed{\text{if}}$ Suppose that, for each $g \in G$, the map $g \cdot - : R \to R$ is a coalgebra morphism. Writing down the axioms for a coalgebra morphism gives immediately that $\cdot : kG \otimes R \to R$ is a coalgebra morphism. ∎

Theorem 2.22. *Let* k *be a field,* R *be a Hopf* k*-algebra and* kG *be a group algebra with its usual Hopf algebra structure. The smash product* $R\#kG$ *is a Hopf algebra, with both* R *and* kG *as Hopf subalgebras, if and only if the action* $\cdot : kG \otimes R \to R$ *is a coalgebra morphism.*

Note that the sufficiency of the condition that $\cdot : kG \otimes R \to R$ is a coalgebra morphism was proven independently by Agore and Militaru as a special case of [AM11, Theorem 2.4].

Proof of Theorem 2.22. Recall, from Examples 1.16(2), that $R\#kG \cong R * G$ and the map $G \to \mathrm{Aut}_{\text{k-alg.}}(R) : g \mapsto g \cdot -$ is a group homomorphism. Thus we shall prove the following equivalent statement of the theorem: let R be a Hopf k-algebra and let G be a group with a homomorphism $\rho : G \to \mathrm{Aut}_{\text{k-alg.}}(R)$. Set $U = R *_\rho kG$, the skew group algebra defined using ρ (that is, $gr = \rho(g)(r)g$ for all $r \in R$ and $g \in G$). Then U is a Hopf algebra extending R with the elements of G group-like if, and only if, $\mathrm{im}\,\rho \subseteq \mathrm{Aut}_{\text{Hopf}}(R)$.

$\boxed{\text{if}}$ Assume U has a Hopf structure extending R and kG. By the definition of U as the skew group algebra we have that, for all $g \in G$ and all $r \in R$,

$$gr = \rho(g)(r)g. \tag{2.2}$$

First make the observation that, for all $r \in R$ and all $g \in G$,

$$\begin{aligned}
\Delta(g)\Delta(r) &= (g \otimes g)(r_1 \otimes r_2) \\
&= (gr_1 \otimes gr_2) \\
&= (\rho(g)(r_1)g \otimes \rho(g)(r_2)g) \\
&= (\rho(g)(r_1) \otimes \rho(g)(r_2))(g \otimes g) \\
&= (\rho(g) \otimes \rho(g)) \circ \Delta(r)(g \otimes g).
\end{aligned} \tag{2.3}$$

Suppose the structure extends. Then Δ must preserve (2.2), so, for each r and g, applying (2.3) gives

$$\begin{aligned}
\Delta(g)\Delta(r) &= \Delta(\rho(g)(r))\Delta(g) \\
(\rho(g) \otimes \rho(g)) \circ \Delta(r)(g \otimes g) &= \Delta(\rho(g)(r))(g \otimes g).
\end{aligned}$$

But $g \otimes g$ is not a zero divisor in $U \otimes U$, so it follows that $(\rho(g) \otimes \rho(g)) \circ \Delta(r) = \Delta \circ \rho(g)(r)$ for all $r \in R$. We must also have ε preserving (2.2) and so, since $\varepsilon(g) = 1$,

$$\varepsilon(r) = \varepsilon(\rho(g)(r))$$

for all $r \in R$. Thus, for each $g \in G$, $\rho(g)$ is a bialgebra morphism $R \to R$. All that remains is to show that $\rho(g) \circ S = S \circ \rho(g)$ but, since S preserves (2.2),

$$\begin{aligned} S(r)S(g) &= S(g)S(\rho(g)(r)) \\ S(r)g^{-1} &= g^{-1}S(\rho(g)(r)) \\ gS(r)g^{-1} &= S(\rho(g)(r)) \\ \rho(g)(S(r)) &= S(\rho(g)(r)). \end{aligned}$$

Therefore, assuming that the Hopf structure on U extends that of R, it follows that, for every $g \in G$, the automorphism $\rho(g)$ must be a Hopf morphism of R.

only if The converse is a simple verification that if $\rho(g) : R \to R$ is a Hopf morphism for all $g \in G$ then defining each element of G to be grouplike satisfies the axioms of a Hopf algebra. In particular, we must show that the extended coproduct and counit maps are algebra homomorphisms.

Let $u = \sum_{g \in G} r_g g$ and $v = \sum_{h \in G} r_h h$ be two arbitrary elements of U. Then

$$\begin{aligned} \Delta(uv) &= \Delta\left(\sum_{g,h \in G} r_g g s_h h \right) \\ &= \Delta\left(\sum_{g,h \in G} r_g \rho(g)(s_h) h \right) \\ &= \sum_{g,h \in G} \Delta(r_g)\Delta\big(\rho(g)(s_h)\big)(gh \otimes gh) \end{aligned}$$

where we have used the extended definition of Δ and the fact that it is an algebra homomorphism on R. Now, by assumption, $\rho(g)$ is a Hopf morphism on R and so

$$\Delta\big(\rho(g)(s_h)\big) = \big(\rho(g) \otimes \rho(g)\big)\big(\Delta(s_h)\big).$$

Thus we see

$$\begin{aligned}\Delta(uv) &= \sum_{g,h\in G} \Delta(r_g)\Big(\sum \rho(g)(s_{h1}) \otimes \rho(g)(s_{h2})\Big)(h \otimes h)\\ &= \sum_{g,h\in G} \Delta(r_g)\Big(\sum gs_{h1} \otimes gs_{h2}\Big)(h \otimes h)\\ &= \sum_{g,h\in G} \Delta(r_g)(g \otimes g)\Delta(s_h)(h \otimes h)\\ &= \Delta(u)\Delta(v).\end{aligned}$$

Checking that ε is an algebra homomorphism is similar. ∎

Specialising to the case where $G = C_\infty$, the infinite cyclic group, gives the following.

Corollary 2.23. *Suppose* R *is a Hopf algebra. The extension* $Q = R[X^{\pm 1}; \sigma]$ *is a Hopf algebra extending the structure of* R *with* X *group-like if, and only if,* $\sigma : R \to R$ *is a Hopf morphism.*

Proof. Put $G = \{X^n : n \in \mathbb{Z}\} \cong C_\infty$ with $\rho(x) = \sigma$; then $Q = R *_\rho G = R[X^{\pm 1}; \sigma]$. ∎

Remarks 2.24.

1. Since smash products are precisely trivial crossed products [Mon94, 7.1.5], the above theorem suggests that we should study Hopf algebra structures on crossed products.

2. The assumption that R is a Hopf subalgebra of $R[X; \sigma, \delta]$ is rather restrictive. We shall consider the example of the quantised enveloping algebra of the negative Borel subalgebra of $\mathfrak{sl}_2(k)$ in section 3.3.3. We shall see that this has the structure of a Hopf algebra and can be written as an Ore extension. But the coefficient ring is not a Hopf subalgebra.

Chapter

3 Characters of Ore extensions

3.1 Introduction

In this chapter, we analyse the set of characters of an Ore extension. (Recall from section 1.1.1 that by a "character" we mean an algebra homomorphism to the ground field.) Throughout the chapter, suppose R is an algebra over a field k and let $T = R[X; \sigma, \delta]$ be the Ore extension of R defined using the algebra automorphism σ and the σ-derivation δ. (See section 1.4.1 for a reminder of the definitions of these terms.)

In Theorem 3.18, we shall show that there is a correspondence between the characters of the Ore extension T and the characters of the coefficient ring R. Key to this correspondence is the fact that each character of T restricts to a character of R and that the resulting character of R must have one of two types (by Lemma 3.4). We then go on to study the geometric properties of sets of characters of R and of T. Finally, we consider the special case when T is a Hopf algebra and show that, under the additional assumption that $R \subseteq T$ is a Hopf subalgebra, we can deduce stronger results about the character theory of R and of T.

3.2 Characters of Ore extensions

In this section, whenever we say "let $T = R[X; \sigma, \delta]$," we mean that R is an algebra over our arbitrary field k, that σ is an algebra automorphism, and that δ is a σ-derivation of R.

3.2.1 Notation and basic results

Notation 3.1 (Sets of characters and character ideals)**.** Given any algebra A over a field k, let $\hat{\mathbb{X}}(A) \coloneqq \mathrm{Hom}_{\text{k-alg.}}(H, k)$ denote the set of algebra homomorphisms from A to k. Let $\mathbb{X}(A)$ denote the set of character ideals, so that there is a one-to-one correspondence between $\hat{\mathbb{X}}(A)$ and $\mathbb{X}(A)$.

Notation 3.2 (Φ)**.** Let $T = R[X; \sigma, \delta]$. Then we can obtain a character of R by taking a character of T and restricting its domain to R; we call this map $\hat{\Phi} : \hat{\mathbb{X}}(T) \to \hat{\mathbb{X}}(R)$. Equivalently, we have a map $\Phi : \mathbb{X}(T) \to \mathbb{X}(R)$ that intersects a character ideal of T with R.

We now present some basic results about the contraction of ideals from an Ore extension to its coefficient ring. Recall, from Definition 1.1, the meanings of Σ-invariant and Σ-stable ideals.

Lemma 3.3. *Let $T = R[X; \sigma, \delta]$ with R noetherian and suppose the two-sided ideal $\mathfrak{i} \lhd R$ is σ-invariant. Then $\mathfrak{i}$ is σ-stable, that is, $\sigma(\mathfrak{i}) = \mathfrak{i}$.*

Proof. From [Goo92, p. 330]. The chain of ideals

$$\mathfrak{i} \subseteq \sigma^{-1}(\mathfrak{i}) \subseteq \sigma^{-2}(\mathfrak{i}) \subseteq \cdots$$

stabilises and so $\sigma^{-n+1}(\mathfrak{i}) = \sigma^{-n}(\mathfrak{i})$ for some n. Then, since σ^{n} induces an isomorphism of $\sigma^{-n}(\mathfrak{i})/\sigma^{-n+1}(\mathfrak{i})$ onto $\mathfrak{i}/\sigma(\mathfrak{i})$, we have $\mathfrak{i} = \sigma(\mathfrak{i})$. ∎

Lemma 3.4. *Let $T = R[X; \sigma, \delta]$. Suppose I is an ideal of T and define $\mathfrak{i} \coloneqq I \cap R$. If $\mathfrak{i}$ is σ-invariant then it is also δ-invariant; that is, if $\sigma(\mathfrak{i}) \subseteq \mathfrak{i}$ then $\delta(\mathfrak{i}) \subseteq \mathfrak{i}$.*

Proof. Suppose $w \in \mathfrak{i}$ and so, in particular, $w \in I \lhd T$. We have $Xw - \sigma(w)X = \delta(w)$ and $Xw \in I$. Since $\sigma(w) \in \mathfrak{i} \subseteq I$, it follows that $\sigma(w)X \in I$ also; hence $\delta(w) \in I$. But $\delta : R \to R$ and so $\delta(w) \in \mathfrak{i}$. ∎

3.2.2 A theorem of Goodearl

We shall need the following results.

Lemma 3.5. *Suppose k is an arbitrary field and let R be a k-algebra. Let σ be a k-algebra automorphism of R and let $\mathfrak{m}$ be an ideal of R such that $R/\mathfrak{m} \cong k$. If $\mathfrak{m}$ is σ-invariant, then it is σ-stable; that is $\sigma(\mathfrak{m}) = \mathfrak{m}$.*

Proof. By hypothesis $\sigma(\mathfrak{m}) \subseteq \mathfrak{m}$. Hence $\mathfrak{m} \subseteq \sigma^{-1}(\mathfrak{m}) \triangleleft R$ and $\sigma^{-1}(\mathfrak{m}) \neq R$. But $\mathfrak{m}$ is a maximal ideal, so $\mathfrak{m} = \sigma^{-1}(\mathfrak{m})$; thus $\sigma(\mathfrak{m}) = \mathfrak{m}$. ■

The following lemma is well-known and can be found, for example, in [GW04].

Lemma 3.6. *Let* k *be a field and* $T = R[X; \sigma, \delta]$. *If* I *is a* (σ, δ)*-invariant ideal of* R *and* $\sigma(I) = I$*, then* IT *is an ideal of* T *and*

$$T/IT \cong (R/I)[\hat{X}; \hat{\sigma}, \hat{\delta}],$$

where $\hat{\sigma}$ *and* $\hat{\delta}$ *are the maps induced from* σ *and* δ.

Proof. That IT is a right ideal of T is clear, and the commutation relation in T along with the fact that I is (σ, δ)-invariant gives that IT is a two-sided ideal. Let $\hat{\sigma}$ and $\hat{\delta}$ be the maps induced from σ and δ. These are well-defined since I is (σ, δ)-invariant and we can check that $\hat{\delta}$ is $\hat{\sigma}$-derivation. The fact that $\sigma(I) = I$ implies that $\hat{\sigma}$ is an automorphism. Then we can check that, for $\sum_{i=1}^{n} r_i X^i \in T$, the map

$$\left(\sum_{i=1}^{n} r_i X^i\right) + IT \mapsto \sum_{i=1}^{n} (r_i + I)\hat{X}^i$$

is a well-defined algebra automorphism. ■

The following important result about prime ideals in Ore extensions was proved by Goodearl in [Goo92]. We record it here and go on to discuss its consequences for the sets of characters of R and T. Recall the definitions of invariant and stable ideals from Definition 1.1.

Theorem 3.7 ([Goo92, Theorem 3.1])**.** *Let* R *be a commutative noetherian ring and set* $T = R[X; \sigma, \delta]$.

(I) *If* P *is a prime ideal of* T *and* $I := P \cap R$*, then one of the following cases must hold:*

  (a) I *is a* (σ, δ)*-prime ideal of* R. *In this case, either*

    (i) I *is a* σ*-prime* (σ, δ)*-invariant ideal of* R*, or*

    (ii) I *is a* δ*-prime* (σ, δ)*-invariant ideal of* R *and* R/I *has a unique associated prime ideal, which contains* $(1 - \sigma)(R)$.

(b) I *is a prime ideal of* R *and* $\sigma(I) \neq I$.

(II) *Conversely, if* I *is any ideal of* R *satisfying (a) or (b), then* $I = P \cap R$ *for some prime ideal* P *of* T. *More specifically, in case (a),* $IT \in \mathrm{Spec}(T)$, *while in case (b), there exists a unique* $P \in \mathrm{Spec}(T)$ *such that* $P \cap R = I$, *and* T/P *is a commutative domain.*

Recall from section 1.1.1 that, given a k-algebra A, character ideals are just the kernels of algebra homomorphisms from A to the ground field k. By specialising Theorem 3.7, we obtain the following result relating the character ideals of R and T.

Corollary 3.8. *Suppose* k *is an algebraically closed field and* R *is an affine commutative* k*-algebra. Let* $T = R[X; \sigma, \delta]$.

(i) *Let* $\mathfrak{M} \in \mathbb{X}(T)$. *Then* $\mathfrak{M} \cap R \in \mathbb{X}(R)$ *and is either* (σ, δ)*-invariant or it is not* σ*-invariant.*

(ii) *Let* $\mathfrak{m} \in \mathbb{X}(R)$.

(a) *If* $\mathfrak{m}$ *is* (σ, δ)*-invariant then, for each element* λ *of* k, *there is a character ideal* $\mathfrak{M}_\lambda$ *of* T *restricting to* $\mathfrak{m}$, *with* $X - \lambda \in \mathfrak{M}_\lambda$.

(b) *If* $\sigma(\mathfrak{m}) \neq \mathfrak{m}$ *then there is a unique character ideal* $\mathfrak{M} \lhd T$ *such that* $\mathfrak{M} \cap R = \mathfrak{m}$.

Proof. First observe that the intersection of a character ideal $\mathfrak{M} \in \mathbb{X}(T)$ to the subalgebra R gives a character ideal $\mathfrak{m} \in \mathbb{X}(R)$. Then Lemma 3.4 says that either $\mathfrak{m}$ is (σ, δ)-invariant or it is not σ-invariant, proving part (*i*).

Conversely, suppose that $\mathfrak{m}$ is a character ideal of R; hence it is a maximal (therefore prime) ideal of R. We have two cases to consider: either $\mathfrak{m}$ is (σ, δ)-invariant or it is not σ-invariant. If $\mathfrak{m}$ is (σ, δ)-invariant then

$$T/\mathfrak{m}T \cong (R/\mathfrak{m})[\hat{x}] \cong k[\hat{x}].$$

Now $\mathbb{X}(k[\hat{x}]) \cong k$ and for each scalar in k there is a character of T restricting to $\mathfrak{m}$. Now suppose that $\mathfrak{m}$ is not σ-invariant; hence, according to Theorem 3.7, there is a unique prime ideal $P \lhd T$ such that $P \cap R = \mathfrak{m}$. Then P is maximal since, otherwise, this would contradict the uniqueness of P. So T/P is a simple commutative domain containing k; hence $T/P \cong k$. ■

Remark 3.9. The proof of Corollary 3.8(*i*) only uses Lemma 3.4 and so it does not require that k is algebraically closed nor that R is affine or commutative.

In summary, Corollary 3.8 tells us that, for R a commutative noetherian algebra and $T = R[X; \sigma, \delta]$, all characters of T are obtained from characters of R and, moreover, that the character ideals of R that give rise to character ideals of T are precisely those that are either (σ, δ)-invariant or not σ-invariant. We shall later explore the topological and geometrical properties of these sets of character ideals but first we generalise Corollary 3.8 to the case where R is a general (possibly noncommutative) algebra.

3.3 Examples

We give the following examples as applications of Corollary 3.8 and as motivation for trying to extend the result to Ore extensions with a noncommutative coefficient ring.

3.3.1 The two-dimensional solvable non-abelian Lie algebra

Let $\mathfrak{g}$ be the two-dimensional solvable non-abelian complex Lie algebra and consider its universal enveloping algebra $\mathcal{U}(\mathfrak{g}) = \mathbb{C}\langle x, y : yx - xy = x \rangle$. Then we can write $T := \mathcal{U}(\mathfrak{g})$ as an Ore extension in two ways; namely

$$T = \mathbb{C}[x][y; \delta]$$

where $\delta = x(d/dx)$, and

$$T = \mathbb{C}[y][x; \sigma]$$

where $\sigma(y) = y + 1$. Notice that $\mathbb{X}(T) = \left\{ \langle x, y - \lambda \rangle : \lambda \in \mathbb{C} \right\}$ since any character must vanish on $yx - xy = x$ and then there is free choice of where to send y. These two presentations of T demonstrate nicely the two cases appearing in Corollary 3.8.

T **as an extension by derivation.** Let $T_1 := \mathbb{C}[x]$ and consider $T = T_1[y; \delta]$. Observe that $\mathbb{X}(T_1) = \left\{ \langle x - \lambda \rangle : \lambda \in \mathbb{C} \right\}$ and the only δ-invariant member of this set is $\langle x \rangle$. Then, since T_1 is a commutative affine $\mathbb{C}$-algebra, Corollary 3.8 implies that for each scalar $\lambda \in \mathbb{C}$, there is a character ideal of T restricting to $\langle x \rangle$. Indeed, as we observed above, $\mathbb{X}(T) = \{\langle x, y - \lambda \rangle : y \in \mathbb{C}\}$.

T **as an extension by automorphism.** Now let $T_1 := \mathbb{C}[y]$ and consider $T = \mathbb{C}[y][x; \sigma]$. Then $\mathbb{X}(T_1) = \{\langle y - \lambda \rangle : \lambda \in \mathbb{C}\}$ and, for each $\lambda \in \mathbb{C}$, $\sigma(\langle y - \lambda \rangle) = \langle y - (\lambda - 1) \rangle$; thus all character ideals of T_1 are shifted by σ. Then Corollary 3.8 tells us that, for each $\lambda \in \mathbb{C}$, there is a unique character ideal of T restricting to $\langle y - \lambda \rangle$.

3.3.2 The enveloping algebra of $\mathfrak{sl}_2(k)$

Let k be an algebraically closed field of characteristic zero and consider the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2(k)$ This algebra has presentation

$$\mathcal{U}(\mathfrak{sl}_2(k)) = k \left\langle e, f, h : \begin{array}{c} ef - fe = h, \ he - eh = 2e, \\ hf - fh = -2f. \end{array} \right\rangle$$

Let $T := \mathcal{U}(\mathfrak{sl}_2(k))$. Then the only character of T is the algebra map sending e, h and f to zero; thus $\mathbb{X}(T) = \{\langle e, h, f \rangle\}$.

As discussed in [GW04, pp. 40–41], T can be written as an iterated Ore extension as follows. Let $T_1 := k[h]$ be a polynomial ring in one indeterminate. Set $T_2 := T_1[e; \sigma_1]$, where $\sigma_1(h) := h - 2$ and σ_1 is extended to be a k-algebra map; then T_2 is the enveloping algebra of the positive Borel subalgebra of $\mathfrak{sl}_2(k)$. Finally, let $T_3 := T_2[f; \sigma_2, \delta_2]$ with

$$\sigma_2 : \begin{cases} e & \mapsto & e \\ h & \mapsto & h + 2 \end{cases} \qquad \delta_2 : \begin{cases} e & \mapsto & -h \\ h & \mapsto & 0. \end{cases}$$

Then we see that $T = T_3$. We would like to see, at each step of this iterated Ore extension, how the characters of the extension are related to the characters of the coefficient ring.

Step 1. Consider $T_2 := T_1[e; \sigma_1]$. Here, the coefficient ring T_1 is an affine commutative k-algebra and so we are in the setting of Corollary 3.8. Now, after a quick calculation, we see that $\mathbb{X}(T_1) = \{\langle h - \lambda \rangle : \lambda \in k\}$ and we see that each of these character ideals is shifted by σ_1. Hence, by Corollary 3.8, for each $\lambda \in k$, there is a unique character ideal of T_2 restricting to $\langle h - \lambda \rangle$. Indeed by direct calculation we see that $\mathbb{X}(T_2) = \{\langle e, h - \lambda \rangle : \lambda \in k\}$.

Step 2. Now consider $T = T_3 := T_2[f; \sigma_2, \delta_2]$. As remarked above, $\mathbb{X}(T) = \{\langle e, h, f \rangle\}$ and so, after restricting to the coefficient ring, the only character ideal of $\mathbb{X}(T_2)$ we obtain is $\{\langle e, h \rangle\}$. For all nonzero $\lambda \in k$, the character ideal $\langle e, h - \lambda \rangle$ of T_2 is shifted by σ_2 but there is no character ideal of T restricting to it. Thus the hypothesis in Corollary 3.8, that the coefficient ring is commutative, is necessary.

3.3.3 The quantised enveloping algebra of $\mathfrak{sl}_2(k)$

Let k be an algebraically closed field of characteristic zero and suppose $q \in k^\times$ is not a root of unity. Then the quantised enveloping algebra of $\mathfrak{sl}_2(k)$ is the k-algebra with presentation

$$\mathcal{U}_q(\mathfrak{sl}_2(k)) := k \left\langle E, F, K, K^{-1} : \begin{array}{c} KEK^{-1} = q^2 E, \ KFK^{-1} = q^{-2} F, \\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \end{array} \right\rangle$$

Let $T := \mathcal{U}_q(\mathfrak{sl}_2(k))$. As discussed in [Kas95, Section VI.1], T can be written as an iterated Ore extension as follows. Let $T_1 := k[K^{\pm 1}]$ be a Laurent polynomial ring in one indeterminate. Set $T_2 := T_1[F; \sigma_1]$ where σ_1 is the k-algebra automorphism of T_1 defined by $\sigma_1(K) = q^2 K$. (Then T_2 is known as the quantised enveloping algebra of the negative Borel and is sometimes written $\mathcal{U}_q(\mathfrak{b}^-)$.) Now let $T_3 := T_2[E; \sigma_2, \delta_2]$ where

$$\sigma_2 : \begin{cases} F & \mapsto & F \\ K & \mapsto & q^{-2} K \end{cases} \qquad \delta_2 : \begin{cases} F & \mapsto & \frac{K - K^{-1}}{q - q^{-1}} \\ K & \mapsto & 0. \end{cases}$$

The map δ_2 can be extended to be a σ_2-derivation of T_2. Then $T = T_3$. Let us consider the sets of character ideals at each step of this iterated Ore extension.

Step 1. Consider $T_2 := T_1[F; \sigma_1]$. At this step, we are in the setting of Corollary 3.8. We see that $\mathbb{X}(T_1) = \{\langle K - \lambda \rangle : \lambda \in k^\times\}$ and that, for each $\lambda \in k^\times$, the ideal $\langle K - \lambda \rangle$ is shifted by σ_1. Thus, by Corollary 3.8, for each $\lambda \in k^\times$, there is a unique character ideal of T_2 restricting to $\langle K - \lambda \rangle$. By direct calculation we see that $\mathbb{X}(T_2) = \{\langle F, K - \lambda \rangle : \lambda \in k^\times\}$.

Step 2. Consider $T = T_3 \coloneqq T_2[E; \sigma_2, \delta_2]$. By [BG02, Section I.6] there are two characters of T; more precisely, $\mathbb{X}(T) = \big\{\langle E, F, K+1\rangle, \langle E, F, K-1\rangle\big\}$.

In the following section, we generalise Corollary 3.8 to relate the character ideals of T to those of T_2.

3.4 A noncommutative version of Corollary 3.8

For the rest of this section, let k be a field and let R be a (possibly non-commutative and non-noetherian) k-algebra. Suppose that $T = R[X; \sigma, \delta]$, where σ is a k-algebra automorphism and δ is a σ-derivation of R. In this section we prove Theorem 3.16, which identifies those characters of R that are restrictions of characters of T, and gives information about the fibres of the restriction map from $\mathbb{X}(T)$ to $\mathbb{X}(R)$. Our result is a generalisation of Corollary 3.8, but the proof will not use this result or Theorem 3.7, and so it will, in particular, specialise to give a direct proof of Corollary 3.8.

Lemma 3.10. *Let $T = R[X; \sigma, \delta]$ and suppose $\mathfrak{a} \lhd R$ is an ideal such that there exists an ideal $\mathfrak{A} \lhd T$ with $\mathfrak{a} = \mathfrak{A} \cap R$. Then $\delta\big(\mathfrak{a} \cap \sigma^{-1}(\mathfrak{a})\big) \subseteq \mathfrak{a}$.*

Proof. Let $r \in \mathfrak{a} \cap \sigma^{-1}(\mathfrak{a})$. So $\delta(r) = Xr - \sigma(r)X \in \mathfrak{A} \cap R = \mathfrak{a}$, since $\sigma(r) \in \mathfrak{a} \subseteq \mathfrak{A}$. ■

Thus, in particular, if $\mathfrak{M} \lhd T$ with $T/\mathfrak{M} \cong k$ then $\mathfrak{m} \coloneqq \mathfrak{M} \cap R \lhd R$ with $R/\mathfrak{m} \cong k$ and, moreover, $\delta\big(\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})\big) \subseteq \mathfrak{m}$. We now prove, in a number of steps, that the converse also holds. Suppose that $\mathfrak{m} \lhd R$ with $R/\mathfrak{m} \cong k$ and $\sigma(\mathfrak{m}) \neq \mathfrak{m}$. Assume also that

$$\delta\big(\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})\big) \subseteq \mathfrak{m}. \tag{3.1}$$

Given these assumptions, we shall find an ideal $L \lhd T$ such that $T/L \cong k$ and $\mathfrak{m} = L \cap R$.

Lemma 3.11. *With the above hypotheses and notation, there exists an element $r \in \mathfrak{m}$ such that $\sigma(r) \equiv 1 \pmod{\mathfrak{m}}$. Moreover, r is uniquely determined, and nonzero, modulo $\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})$.*

Proof. Observe that, since $\mathfrak{m}$ and $\sigma(\mathfrak{m})$ are maximal ideals, we have $\mathfrak{m} + \sigma(\mathfrak{m}) = R$. Thus there are elements $a, r \in \mathfrak{m}$ such that $1 = a + \sigma(r)$; then $\sigma(r) - 1 \in \mathfrak{m}$. Observe also that $r \notin \sigma^{-1}(\mathfrak{m})$: if this was the case then we would have $\sigma(r) \in \mathfrak{m}$; that is, $\sigma(r) \equiv 0 \pmod{\mathfrak{m}}$. Hence, as k-vector spaces,

$$\frac{\mathfrak{m}}{\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})} \cong \frac{\mathfrak{m} + \sigma^{-1}(\mathfrak{m})}{\sigma^{-1}(\mathfrak{m})} = \frac{R}{\sigma^{-1}(\mathfrak{m})} \cong k.$$

Since $r \in \mathfrak{m} \setminus \sigma^{-1}(\mathfrak{m})$, it is uniquely determined, and nonzero, modulo $\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})$. ∎

Notation 3.12. Let $\mathfrak{m}$ be an ideal of R with $R/\mathfrak{m} \cong k$, for which (3.1) holds, and let r be an element with the properties in Lemma 3.11. Define $\lambda := \delta(r) \pmod{\mathfrak{m}}$, so $\lambda \in k$.

Lemma 3.13. *Retain the notation just introduced. Suppose there exists an ideal $\mathfrak{M}$ of T with $\mathfrak{M} \cap R = \mathfrak{m}$. Then $X + \lambda \in \mathfrak{M}$.*

Proof. Let r be as above. So $Xr \in T\mathfrak{m} \subseteq \mathfrak{M}$, since $\mathfrak{m} \subseteq \mathfrak{M}$. But δ is a σ-derivation and so

$$Xr = \sigma(r)X + \delta(r) \in \mathfrak{M}. \tag{3.2}$$

Now, since $\sigma(r) \equiv 1 \pmod{\mathfrak{m}}$ and $\delta(r) \equiv \lambda \pmod{\mathfrak{m}}$, we can write $\sigma(r) = 1 + t$ and $\delta(r) = \lambda + s$ for some $t, s \in \mathfrak{m} \subseteq \mathfrak{M}$; hence (3.2) becomes

$$X + \underbrace{tX}_{\in\mathfrak{M}} + \lambda + \underbrace{s}_{\in\mathfrak{M}} \in \mathfrak{M}$$

and so $X + \lambda \in \mathfrak{M}$. ∎

This lemma suggests a candidate for the ideal $L \lhd T$ restricting to $\mathfrak{m}$. Observe that, as a k-vector space,

$$T = \bigoplus_{i \geqslant 0} RX^i = \bigoplus_{i \geqslant 0} (k + \mathfrak{m})X^i,$$

and that

$$\bigoplus_{i \geqslant 0} (k + \mathfrak{m})X^i = k[X] \oplus \mathfrak{m}T.$$

So we see that, as left $k[X]$-modules, $T = k[X] \oplus \mathfrak{m}T$. Define $L \subseteq T$ to be the left $k[X]$-submodule

$$L := k[X](X + \lambda) \oplus \mathfrak{m}T.$$

Notice that, as left $k[X]$-modules, $T/L \cong k$.

Lemma 3.14. *Let* $\mathfrak{m}$ *be an ideal of* R *such that* $R/\mathfrak{m} \cong k$*. Assume that* $\sigma(\mathfrak{m}) \neq \mathfrak{m}$ *and that* (3.1) *holds. Retain the notation from Notation 3.12 and from above. Then* L *is an ideal of* T*.*

Proof. Suppose we can show that L is a left ideal of T. Then set

$$L_0 \coloneqq \ell.\operatorname{Ann}(T/L) = \{t \in T : tT \subseteq L\}$$

so $L_0 \lhd T$ and $T/L_0 \hookrightarrow \operatorname{End}_k(T/L) \cong k$, since $T/L \cong k$ as a vector space. But we also know that

$$L_0 \coloneqq \ell.\operatorname{Ann}(T/L) = \bigcap_{v \in T/L} \operatorname{Ann}_T(v) \subseteq L$$

because $L = \operatorname{Ann}_T(1 + L)$. Thus $L_0 = L$ and the lemma is proved.

So it remains to show that L is a left ideal of T. As a k-algebra, T is generated by $\mathfrak{m}$ and X because $R/\mathfrak{m} \cong k$. So it is enough to show that

$$\mathfrak{m}L \subseteq L \tag{3.3}$$

and

$$XL \subseteq L. \tag{3.4}$$

Since $L = k[X](X + \lambda) \oplus \mathfrak{m}T$,

$$\mathfrak{m}L \subseteq \mathfrak{m} \underbrace{k[X](X + \lambda)}_{\in T} \oplus \mathfrak{m}T \subseteq \mathfrak{m}T \subseteq L$$

and so (3.3) holds. For condition (3.4), notice that $Xk[X](X+\lambda) \subseteq k[X](X+\lambda)$ and so it is enough to show that

$$X\mathfrak{m}T \subseteq L. \tag{3.5}$$

As a k-vector space, $R = \mathfrak{m} + k$, and so $T = \mathfrak{m}k[X] + k[X]$. Then $\mathfrak{m}T \subseteq \mathfrak{m}k[X]$ and so $X\mathfrak{m}T \subseteq X\mathfrak{m}k[X]$; hence it suffices to show that $X\mathfrak{m}k[X] \subseteq L$ or, equivalently, that whenever $w \in \mathfrak{m}$ and $f \in k[X]$ we have

$$Xwf \in L. \tag{3.6}$$

First suppose $w \in \mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})$. Then, since $\sigma(w) \in \mathfrak{m}$ and, by assumption (3.1), $\delta(w) \in \mathfrak{m}$,

$$Xwf = \sigma(w)Xf + \delta(w)f \in \mathfrak{m}T \subseteq L.$$

Now observe that, as sets, $\mathfrak{m} = kr + \left(\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})\right)$, where r is the element from Lemma 3.11, and so all that remains to show is that, for any $f \in k[X]$,

$$Xrf \in L. \tag{3.7}$$

Recall that there are elements $t, s \in \mathfrak{m}$ such that $\sigma(r) = 1+t$ and $\delta(r) = \lambda+s$. Thus, using the commutation rule in T,

$$\begin{aligned} Xrf &= \sigma(r)Xf + \delta(r)f \\ &= (1+t)Xf + (\lambda+s)f \\ &= f(X+\lambda) + tXf + sf \\ &\in (X+\lambda)k[X] + \mathfrak{m}T = L. \end{aligned}$$

Thus condition (3.7) holds and so the proof is complete. ■

Lemma 3.15. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two distinct ideals of a ring R and suppose $R/\mathfrak{a} \cong R/\mathfrak{b} \cong k$, where k is a field. Then*

$$\mathfrak{a} \cap \mathfrak{b} = \langle [R, R] \rangle + \mathfrak{a}\mathfrak{b} = \langle [R, R] \rangle + \mathfrak{b}\mathfrak{a}.$$

Proof. We begin by showing that $\mathfrak{a}\mathfrak{b}+\mathfrak{b}\mathfrak{a} = \mathfrak{a}\cap\mathfrak{b}$. The fact that $\mathfrak{a}\mathfrak{b}+\mathfrak{b}\mathfrak{a} \subseteq \mathfrak{a}\cap\mathfrak{b}$ is clear because $\mathfrak{a}$ and $\mathfrak{b}$ are ideals. For the reverse inclusion, first observe that, since $\mathfrak{a}$ and $\mathfrak{b}$ are distinct maximal ideals, it follows that $\mathfrak{a} + \mathfrak{b} = R$. Thus

$$\mathfrak{a} \cap \mathfrak{b} = (\mathfrak{a} \cap \mathfrak{b})R = (\mathfrak{a} \cap \mathfrak{b})(\mathfrak{a} + \mathfrak{b}) \subseteq \mathfrak{b}\mathfrak{a} + \mathfrak{a}\mathfrak{b};$$

hence $\mathfrak{a}\mathfrak{b} + \mathfrak{b}\mathfrak{a} = \mathfrak{a} \cap \mathfrak{b}$. Now notice that $\langle [R, R] \rangle \subseteq \mathfrak{a} \cap \mathfrak{b}$ and so

$$\mathfrak{a} \cap \mathfrak{b} = \langle [R, R] \rangle + \mathfrak{b}\mathfrak{a} + \mathfrak{a}\mathfrak{b} = \langle [R, R] \rangle + \mathfrak{a}\mathfrak{b} = \langle [R, R] \rangle + \mathfrak{b}\mathfrak{a}.$$ ■

We are now ready to state our generalisation of case $(ii)(b)$ of Goodearl's Corollary 3.8. The proof is a straightforward combination of the lemmas proved above.

Theorem 3.16. *Let* R *be a* k*-algebra and suppose* $T = R[X;\sigma,\delta]$*, where* σ *is a* k*-algebra automorphism and* δ *is a* σ*-derivation of* R*. Let* $\mathfrak{m} \lhd R$ *with* $R/\mathfrak{m} \cong k$ *and* $\sigma(\mathfrak{m}) \neq \mathfrak{m}$*. Then the following are equivalent.*

(i) There exists an ideal $\mathfrak{M}$ *of* T *with* $T/\mathfrak{M} \cong k$ *and* $\mathfrak{M} \cap R = \mathfrak{m}$*.*

(ii) There exists an ideal $\mathfrak{M}$ *of* T *with* $\mathfrak{M} \cap R = \mathfrak{m}$*.*

(iii) There exists a unique ideal $\mathfrak{M}$ *of* T *with* $\mathfrak{M} \cap R = \mathfrak{m}$*.*

(iv) There exists a unique ideal $\mathfrak{M}$ *of* T *with* $T/\mathfrak{M} \cong k$ *and* $\mathfrak{M} \cap R = \mathfrak{m}$*.*

(v) $\delta\big(\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})\big) \subseteq \mathfrak{m}$*;*

(vi) $\delta\big([R,R]\big) \subseteq \mathfrak{m}$*.*

Proof.

$\boxed{(i) \Rightarrow (ii)}$ Trivial.

$\boxed{(ii) \Rightarrow (v)}$ Lemma 3.10.

$\boxed{(ii) \Rightarrow (iii)}$ Suppose there exists an ideal $\mathfrak{M}$ of T with $\mathfrak{M} \cap R = \mathfrak{m}$. By Lemma 3.13, $X + \lambda \in \mathfrak{M}$ so that $\mathfrak{M} \supseteq L \coloneqq k[X](X+\lambda) + \mathfrak{m}T$. But $(ii) \Rightarrow (v)$ so Lemma 3.14 applies to show that L is an ideal of T with $T/L \cong k$. In particular L is a maximal ideal of T and so $M = L$, proving (iii).

$\boxed{(iii) \Rightarrow (ii)}$ Trivial.

$\boxed{(iv) \Rightarrow (i)}$ Trivial.

$\boxed{(v) \Rightarrow (iv)}$ Lemmas 3.13 and 3.14.

$\boxed{(v) \Rightarrow (vi)}$ Because $R/\mathfrak{m} \cong R/\sigma^{-1}(\mathfrak{m}) \cong k$, we know that $[R,R] \subseteq \mathfrak{m}$ and $[R,R] \subseteq \sigma^{-1}(\mathfrak{m})$, so that the statement is clear.

$\boxed{(vi) \Rightarrow (v)}$ Assume that $\delta\big([R,R]\big) \subseteq \mathfrak{m}$. First notice that this implies $\delta\big(\big\langle [R,R] \big\rangle\big) \subseteq \mathfrak{m}$ as follows: because δ is a σ-derivation, for any $u, r, s, v \in R$,

$$\delta(u[r,s]v) = \delta(u)[r,s]v + \sigma(u)\delta([r,s])v + \sigma(u)\sigma([r,s])\delta(v). \tag{3.8}$$

Moreover, the right-hand expression in (3.8) is in $\mathfrak{m}$ thanks to hypothesis (vi) and the facts that $[R,R] \subseteq \mathfrak{m}$ and $\sigma([r,s]) = [\sigma(r), \sigma(s)] \in$

$[R, R] \subseteq \mathfrak{m}$. Now applying Lemma 3.15 to $\mathfrak{m}$ and $\sigma^{-1}(\mathfrak{m})$, and using the commutation relation, gives

$$\begin{aligned}\delta\big(\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})\big) &= \delta\big(R[R,R]R + \sigma^{-1}(\mathfrak{m})\mathfrak{m}\big) \\ &\subseteq \underbrace{\delta\big(R[R,R]R\big)}_{\subseteq \mathfrak{m}} + \underbrace{\delta\big(\sigma^{-1}(\mathfrak{m})\big)}_{\subseteq R} \mathfrak{m} + \mathfrak{m}\underbrace{\delta(\mathfrak{m})}_{\subseteq R} \\ &\subseteq \mathfrak{m},\end{aligned}$$

proving the statement. ■

Remarks 3.17.

1. The condition that $\sigma(\mathfrak{m}) \neq \mathfrak{m}$ is necessary in this theorem. To see that this is so, let $R = k[y]$ and define $T := R[x; y(d/dy)]$ to be the enveloping algebra of the two-dimensional solvable non-abelian Lie algebra as in section 3.3.1. Then, since R is commutative, $[R, R] = 0$ and so condition (*vi*) in the theorem is satisfied for all characters of R. But the only one obtained by restriction from T is $\langle y \rangle$.

2. The condition $\delta([R, R]) \subseteq \mathfrak{m}$ which, of course, does not appear when R is commutative, as in Corollary 3.8, is genuinely necessary when R is noncommutative. That is, it does not follow from the assumption that $\sigma(\mathfrak{m}) \neq \mathfrak{m}$. To see this, consider the example of the universal enveloping algebra of $\mathfrak{sl}_2$ as discussed in section 3.3.2. Let $T := \mathcal{U}(\mathfrak{sl}_2(k))$ and $R := k[h][e; \sigma_1]$ be the enveloping algbera of the positive Borel subalgebra; then $T = R[f; \sigma_2, \delta_2]$ with the maps as defined in section 3.3.2. As we saw there, $\mathbb{X}(T) = \big\{\langle e, f, h\rangle\big\}$ and $\mathbb{X}(R) = \big\{\langle e, h - \lambda\rangle : \lambda \in k\big\} \cong \mathbb{A}^1$. By the definition of σ_2, we see that $\sigma_2(h - \lambda) = h + 2 - \lambda$ so that all character ideals $\mathfrak{m} \in \mathbb{X}(R)$ have $\sigma_2(\mathfrak{m}) \neq \mathfrak{m}$. But not all of these are restrictions of character ideals of T – only $\langle e, h\rangle$ is. And indeed we see that this is the only character ideal of R containing $\delta([R, R])$.

3. In fact, having this apparently additional hypothesis in play, which is needed when R is noncommutative, shows that (at least for character ideals) the dichotomy appearing in Theorem 3.7, between the two cases $\sigma(\mathfrak{m}) = \mathfrak{m}$ and $\sigma(\mathfrak{m}) \neq \mathfrak{m}$, is more apparent than real. This is made clear in the following formulation of the complete generalisation

of Goodearl's theorem to an arbitrary (not necessarily commutative, noetherian or affine) coefficient ring, for the case of character ideals.

Theorem 3.18. *Let* k *be an algebraically closed field and* R *be a* k*-algebra. Suppose* $\mathsf{T} = \mathsf{R}[\mathsf{X}; \sigma, \delta]$ *with* σ *a* k*-algebra automorphism of* R *and* δ *a* σ*-derivation. Let* $\Phi : \mathbb{X}(\mathsf{T}) \to \mathbb{X}(\mathsf{R})$ *be the restriction map on character ideals. Let* $\mathfrak{m} \in \mathbb{X}(\mathsf{R})$.

(i) There exists $\mathfrak{M} \in \mathbb{X}(\mathsf{T})$ *with* $\Phi(\mathfrak{M}) = \mathfrak{m}$ *if and only if*

$$\delta\big((\mathfrak{m} \cap \sigma^{-1}(\mathfrak{m})\big) \subseteq \mathfrak{m}. \tag{3.9}$$

(ii) If (3.9) *holds and* $\sigma(\mathfrak{m}) = \mathfrak{m}$*, then* $\mathfrak{m}$ *is* (σ, δ)*-invariant and* $\Phi^{-1}(\mathfrak{m}) \cong \mathbb{A}^1$.

(iii) If (3.9) *holds and* $\sigma(\mathfrak{m}) \neq \mathfrak{m}$*, then there exists a unique* $\mathfrak{M} \in \mathbb{X}(\mathsf{T})$ *with* $\Phi(\mathfrak{M}) = \mathfrak{m}$.

Proof. Let $\mathfrak{m} \in \mathbb{X}(\mathsf{R})$.

(i) Suppose there exists $\mathfrak{M} \in \mathbb{X}(\mathsf{T})$ with $\Phi(\mathfrak{M}) = \mathfrak{m}$. If $\sigma(\mathfrak{m}) \subseteq \mathfrak{m}$ then $\sigma(\mathfrak{m}) = \mathfrak{m}$ by Lemma 3.5, and $\mathfrak{m}$ is (σ, δ)-invariant by Lemma 3.4. In particular, (3.9) holds in this case. Otherwise $\sigma(\mathfrak{m}) \neq \mathfrak{m}$ and (3.9) follows from "(i) $\Rightarrow$ (v)" of Theorem 3.16. The converse implication in (i) is part of (ii) and (iii), which we deal with now.

(ii) Suppose (3.9) holds and $\sigma(\mathfrak{m}) = \mathfrak{m}$. Then clearly $\mathfrak{m}$ is (σ, δ)-invariant, and (ii) now follows from Lemma 3.6.

(iii) Suppose (3.9) holds and $\sigma(\mathfrak{m}) \neq \mathfrak{m}$. Then the result follows from "(v) $\Rightarrow$ (i)" of Theorem 3.16. ■

Question 3.19. More generally, we can ask whether Theorem 3.7 can be generalised to a noncommutative setting; that is, how do the prime ideals of an Ore extension relate to those of the coefficient ring?

3.5 Topological properties of the sets of characters

3.5.1 Definitions and notation

Throughout this section, suppose R is an affine k-algebra with k an algebraically closed field. Let $T = R[X; \sigma, \delta]$, where σ is an automorphism and δ is a σ-derivation of R. We now give some basic results before introducing notation and definitions to describe the sets of characters of R and of T.

Lemma 3.20. *Let k be an algebraically closed field and suppose A is an affine k-algebra. Then there are bijections between the sets $\mathbb{X}(A)$, $\mathbb{X}\big(A/\sqrt{\langle[A,A]\rangle}\big)$ and $\operatorname{maxspec}\big(A/\sqrt{\langle[A,A]\rangle}\big)$. In particular $\mathbb{X}(A)$ has the structure of an algebraic set.*

Proof. Let $\chi \in \mathbb{X}(A)$ be an algebra homomorphism from A to k. Then $\mathfrak{m} := \ker\chi$ is an ideal of A with $A/\mathfrak{m} \cong k$. Now observe that $\sqrt{\langle[A,A]\rangle} \subseteq \mathfrak{m}$ as follows. Suppose $r \in \sqrt{\langle[A,A]\rangle}$; that is, $r^n \in \langle[A,A]\rangle$ for some $n \geqslant 1$. Then $\chi(r^n) = 0$ and so, since $\chi : A \to k$ is an algebra homomorphism and k is a field, we have $\chi(r) = 0$. Thus $\mathfrak{m}$ is an ideal of A containing $\sqrt{\langle[A,A]\rangle}$ and so there is a bijection between $\mathbb{X}(A)$ and $\operatorname{maxspec}\big(A/\sqrt{\langle[A,A]\rangle}\big)$. Finally, since $A/\sqrt{\langle[A,A]\rangle}$ is an affine commutative semiprime k-algebra, its maximal spectrum $\mathbb{X}(A)$ is naturally an algebraic set. ∎

Lemma 3.21. *Let k be an arbitrary field and suppose T is a k-algebra with R a subalgebra. Then the map*

$$\begin{aligned}\Theta : R/\sqrt{\langle[R,R]\rangle} &\to T/\sqrt{\langle[T,T]\rangle}\\ r + \sqrt{\langle[R,R]\rangle} &\mapsto r + \sqrt{\langle[T,T]\rangle}\end{aligned}$$

is a well-defined homomorphism of commutative k-algebras.

Proof. Firstly, to ease notation, let $\overline{R} = R/\sqrt{\langle[R,R]\rangle}$ and $\overline{T} = T/\sqrt{\langle[T,T]\rangle}$. To see that Θ is well-defined, suppose $r + \sqrt{\langle[R,R]\rangle} = s + \sqrt{\langle[R,R]\rangle}$ for $r, s \in R$. Then, for some $n \in \mathbb{N}$, we have $(r-s)^n \in \langle[R,R]\rangle$. But $\langle[R,R]\rangle \subseteq \langle[T,T]\rangle$, since R is a subalgebra of T, and so $(r-s)^n \in \langle[T,T]\rangle$; that is, $r - s \in \sqrt{\langle[T,T]\rangle}$. The proof that Θ is an algebra homomorphism is a simple check. ∎

Corollary 3.22. *Let* k *be an algebraically closed field and suppose* T *is an affine* k*-algebra with* R *an affine subalgebra. Then there is a morphism of algebraic sets*

$$\Phi : \mathbb{X}(T) \to \mathbb{X}(R)$$

where, for each $\chi \in \mathbb{X}(T)$*,* $\Phi(\chi) \coloneqq \chi|_R$ *is the restriction of* χ *to* R*.*

Proof. We translate the result of Lemma 3.21 into equivalent category of algebraic sets. This gives a morphism of algebraic sets $\Phi : \mathbb{X}(T) \to \mathbb{X}(R)$ where

$$\Phi(\mathfrak{M}) \coloneqq \Theta^{-1}(\mathfrak{M} \cap \operatorname{im} \Theta).$$

To see that $\Phi(\mathfrak{M}) = \mathfrak{M} \cap R$, observe that

$$\begin{aligned} r + \sqrt{\langle [R, R] \rangle} \in \Phi(\mathfrak{M}) &\Leftrightarrow \Theta(r + \sqrt{\langle [R, R] \rangle}) \subseteq \mathfrak{M} \cap \operatorname{im} \Theta \\ &\Leftrightarrow r + \sqrt{\langle [T, T] \rangle} \subseteq \mathfrak{M} \\ &\Leftrightarrow r \in \mathfrak{M}. \end{aligned}$$

■

Thus we have that each character ideal of T restricts to a character ideal of R and, moreover, that the restriction map is continuous in the Zariski topology.

Definition 3.23 (Relevant characters)**.** Let k, R and T be as defined above. We call $\mathbb{X}_r(R) \coloneqq \operatorname{im} \Phi$ the set of **relevant characters** of R. These are precisely the characters of R that arise as restrictions of characters of T. ◊

3.5.2 Geometrical descriptions of sets of characters

Given an automorphism σ and a σ-derivation δ of a ring R, it is clear that any ideal of R is either (σ, δ)-invariant, σ-invariant but not δ-invariant, δ-invariant but not σ-invariant, or neither σ- nor δ-invariant. We introduce the following notation to make reference to these different types of ideal more concise.

Definition 3.24 ($\mathbb{X}^{(\sigma,\delta)}$)**.** For any k-algebra A, and maps $\alpha_1, \alpha_2, \dots, \alpha_n$ from A to itself, let $\mathbb{X}^{(\alpha_1,\dots,\alpha_n)}(A)$ be those character ideals $\mathfrak{m}$ of A such that, for all $i = 1, \dots, n$, $\mathfrak{m}$ is α_i-invariant. ◊

Given a k-algebra R, we can write the set $\mathbb{X}(R)$ as the union

$$\mathbb{X}(R) = \mathbb{X}^{(\sigma,\delta)}(R) \cup \mathbb{X}^{\sigma}(R) \cup \mathbb{X}^{\delta}(R) \cup \mathbb{X}^{\varnothing}(R)$$

where $\mathbb{X}^{\varnothing}(R)$ denotes the maximal ideals that are neither σ- nor δ-invariant. Note that this is not a disjoint union since, for example, $\mathbb{X}^{(\sigma,\delta)}(R) \subseteq \mathbb{X}^{\sigma}(R)$ and $\mathbb{X}^{(\sigma,\delta)}(R) \subseteq \mathbb{X}^{\delta}(R)$.

Let k be an algebraically closed field and R be a k-algebra. Suppose $T = R[X; \sigma, \delta]$ with σ an algebra automorphism of R and δ a σ-derivation. We know, from Lemma 3.4, that each character ideal $\mathfrak{M} \in \mathbb{X}(T)$ has one of two types; namely $\mathfrak{m} := \mathfrak{M} \cap R$ is either (σ, δ)-invariant or not σ-invariant. Let

$$\mathcal{A} := \left\{\mathfrak{M} \in \mathbb{X}(T) : \mathfrak{M} \cap R \in \mathbb{X}^{(\sigma,\delta)}(R)\right\}$$

be the set of characters of T whose restriction to R is (σ, δ)-invariant. Similarly, let

$$\mathcal{B} := \left\{\mathfrak{M} \in \mathbb{X}(T) : \mathfrak{M} \cap R \in \mathbb{X}_r(R) \setminus \mathbb{X}^{(\sigma,\delta)}(R)\right\}$$

be the set of characters of T that restrict to a non-σ-invariant character of R. Then we know that $\mathbb{X}(T) = \mathcal{A} \sqcup \mathcal{B}$.

Definition 3.25 (Type $\mathcal{A}$ and type $\mathcal{B}$)**.** We shall say that a character $\mathfrak{M} \in \mathbb{X}(T)$ has **type** $\mathcal{A}$ if $\mathfrak{M} \in \mathcal{A}$; and similarly that $\mathfrak{M} \in \mathbb{X}(T)$ has **type** $\mathcal{B}$ if $\mathfrak{M} \in \mathcal{B}$. $\Diamond$

Thus any character of T is either type $\mathcal{A}$ or type $\mathcal{B}$. We now go on to study the algebraic-geometric properties of the sets $\mathbb{X}^{(\sigma,\delta)}(R)$, $\mathbb{X}^{\sigma}(R)$ and $\mathbb{X}^{\delta}(R)$, and of $\mathbb{X}(T)$.

We shall need the following result in the next few lemmas, so we record it here to prevent repetition. The proof is contained in the proof of Lemma 3.20.

Lemma 3.26. *Let* k *be a field and* A *be a* k*-algebra. Suppose* $\mathfrak{m}$ *is a character ideal of* A*. If* $I \subseteq A$ *is an ideal such that* $I \subseteq \mathfrak{m}$*, then* $\sqrt{I} \subseteq \mathfrak{m}$*.*

Lemma 3.27. *Let* k *be an algebraically closed field and suppose that* R *is an affine* k*-algebra. Then the set* $\mathbb{X}^{\sigma}(R)$ *is a closed subset of* $\mathbb{X}(R)$*. More* *precisely, as algebraic sets,*

$$\mathbb{X}^{\sigma}(R) \cong \mathbb{X}\big(R/W_{\sigma-\mathrm{id}}(R)\big)$$

where

$$W_{\sigma-\mathrm{id}}(R) := \sqrt{R\big\{\sigma(r) - r : r \in R\big\}R + R[R,R]R}.$$

Proof. Assume that $\mathfrak{m} \in \mathbb{X}^{\sigma}(R)$. Since $R/\mathfrak{m} \cong k$, we can write any $r \in R$ as $r = r_0 + \lambda$ where $r_0 \in \mathfrak{m}$ and $\lambda \in k$. Then

$$\sigma(r) - r = \sigma(r_0) + \lambda - r_0 - \lambda = \sigma(r_0) - r_0 \in \mathfrak{m}$$

and so $R\{\sigma(r) - r : r \in R\}R \subseteq \mathfrak{m}$. Since $R/\mathfrak{m}$ is commutative, it is clear that $R[R,R]R \subseteq \mathfrak{m}$; hence $R\{\sigma(r) - r : r \in R\}R + R[R,R]R \subseteq \mathfrak{m}$. Now applying Lemma 3.26 gives $W_{\sigma-\mathrm{id}}(R) \subseteq \mathfrak{m}$. Conversely, if $W_{\sigma-\mathrm{id}}(R) \subseteq \mathfrak{m}$ then, for all $r \in R$, $\sigma(r) - r \in \mathfrak{m}$ and so certainly $\sigma(\mathfrak{m}) \subseteq \mathfrak{m}$. ■

Lemma 3.28. *Let* k *be an algebraically closed field and suppose that* R *is an affine* k*-algebra. Then the set* $\mathbb{X}^{\delta}(R)$ *is a closed subset of* $\mathbb{X}(R)$*. More precisely, as algebraic sets,*

$$\mathbb{X}^{\delta}(R) \cong \mathbb{X}\big(R/W_{\delta}(R)\big)$$

where

$$W_{\delta}(R) := \sqrt{R\big\{\delta(r) : r \in R\big\}R + R[R,R]R}.$$

Proof. Assume that $\mathfrak{m} \in \mathbb{X}^{\delta}(R)$. Since $R/\mathfrak{m} \cong k$, we can write any $r \in R$ as $r = r_0 + \lambda$ where $r_0 \in \mathfrak{m}$ and $\lambda \in k$. Then, since $\lambda \in k$,

$$\delta(r) = \delta(r_0) + \delta(\lambda) = \delta(r_0) \in \mathfrak{m}$$

and so $R\big\{\delta(r) : r \in R\big\}R \subseteq \mathfrak{m}$. Since $R/\mathfrak{m}$ is commutative, $R[R,R]R \subseteq \mathfrak{m}$; hence $R\big\{\delta(r) : r \in R\big\}R + R[R,R]R \subseteq \mathfrak{m}$ and so, by Lemma 3.26, $W_{\delta}(R) \subseteq \mathfrak{m}$. Conversely, if $W_{\delta}(R) \subseteq \mathfrak{m}$ then, for all $r \in R$, $\delta(r) \in \mathfrak{m}$ and so certainly $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$. ■

Recall, from above, that for $T = R[X;\sigma,\delta]$, with σ an automorphism and δ a σ-derivation, $\mathcal{A}$ denotes the set of character ideals $\mathfrak{M}$ of T such that the contraction $\mathfrak{M} \cap R$ is (σ,δ)-invariant.

Lemma 3.29. *Let* k *be an algebraically closed field and suppose that* R *is an affine* k*-algebra. Let* $T = R[X; \sigma, \delta]$ *with* σ *a* k*-algebra automorphism and* δ *a* σ*-derivation.*

(i) $\mathbb{X}^{(\sigma,\delta)}(R)$ *is a closed subset of* $\mathbb{X}(R)$ *with defining ideal*

$$A \coloneqq \sqrt{W_{\sigma-\mathrm{id}}(R) + W_{\delta}(R)} = \Big(\bigcap_{\mathfrak{M}\in\mathcal{A}} \mathfrak{M}\Big) \cap R.$$

(ii) $\mathcal{A}$ *is the affine algebraic set with coordinate ring* T/AT*. Thus* $\mathcal{A}$ *is homeomorphic to* $\mathbb{X}^{(\sigma,\delta)}(R) \times \mathbb{A}^1$*.*

Proof.

(i) Since $\mathbb{X}^{(\sigma,\delta)}(R) = \mathbb{X}^{\sigma}(R) \cap \mathbb{X}^{\delta}(R)$, it is closed by Lemmas 3.27 and 3.28. The first equality follows from the two previous lemmas. To see the second equality, notice that

$$\Big(\bigcap_{\mathfrak{M}\in\mathcal{A}} \mathfrak{M}\Big) \cap R = \bigcap_{\mathfrak{M}\in\mathcal{A}} (\mathfrak{M} \cap R) = \bigcap_{\mathfrak{m}\in\mathbb{X}^{(\sigma,\delta)}(R)} \mathfrak{m}.$$

The fact that the right-hand side is A is part of Hilbert's nullstellensatz.

(ii) It is clear from its definition that A is (σ,δ)-invariant, so AT is an ideal of T. Since the induced maps $\hat{\sigma}$ and $\hat{\delta}$ on R/A are trivial, $T/AT \cong (R/A)[\hat{X}]$. Then

$$\mathcal{A} \cong \mathbb{X}(T/AT) \cong \mathbb{X}\big((R/A)[\hat{X}]\big) \cong \mathbb{X}(R/A) \times \mathbb{A}^1 \cong \mathbb{X}^{(\sigma,\delta)}(R) \times \mathbb{A}^1. \quad \blacksquare$$

Remark 3.30. Observe that A as defined in Lemma 3.29 is the unique smallest (σ,δ)-invariant semiprime ideal of R such that R/A is commutative and the induced maps $\hat{\sigma}$ and $\hat{\delta}$ on R/A are trivial. To see that A is the unique smallest such ideal, suppose $B \subseteq A$ has all the properties mentioned. So R/B is a commutative semiprime affine k-algebra, with k algebraically closed. By the nullstellensatz applied to R/B,

$$B = \bigcap\{\mathfrak{m} : \mathfrak{m} \in \mathbb{X}(R) \text{ with } B \subseteq \mathfrak{m}\}.$$

Since, by hypothesis, B is (σ, δ)-invariant and the induced maps $\overline{\sigma}$ and $\overline{\delta}$ on R/B are trivial, all the ideals $\mathfrak{m}$ in the intersection above are (σ, δ)-invariant. So $A \subseteq B$ and, therefore, $B = A$.

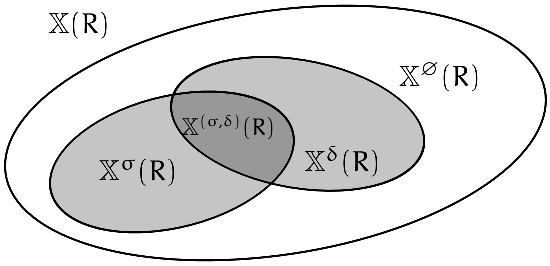
Lemma 3.31. *Let k be an algebraically closed field and suppose that R is an affine k-algebra. Let $T = R[X; \sigma, \delta]$ with σ an automorphism and δ a σ-derivation. Then the map Φ restriced to $\mathcal{B}$ is a continuous bijection onto*

$$\big\{\mathfrak{m} \in \mathbb{X}(R) : \delta([R, R]) \subseteq \mathfrak{m},\ W_{\sigma-\mathrm{id}}(R) \not\subseteq \mathfrak{m}\big\}.$$

If R is commutative, then $\mathcal{B}$ is homeomorphic to an open subset of $\mathbb{X}(R)$.

Proof. From Corollary 3.22 it follows that the restriction of the map $\Phi : \mathbb{X}(T) \to \mathbb{X}(R)$ is a continuous map onto its image, and $\Phi(\mathcal{B})$ is precisely the stated set. If R is commutative, then $[R, R] = \{0\}$ and so $\mathcal{B} \cong \big\{\mathfrak{m} \in \mathbb{X}(R) : W_{\sigma-\mathrm{id}}(R) \not\subseteq \mathfrak{m}\big\}$. ■

It is useful to bear in mind the following picture of $\mathbb{X}(R)$.



We now record the fact that we can describe the set $\mathbb{X}_r(R) \coloneqq \Phi(\mathbb{X}(T))$ in terms of other subsets of $\mathbb{X}(R)$ using Theorem 3.18.

Lemma 3.32. *Let k be an algebraically closed field and suppose that R is a k-algebra. Let $T = R[X; \sigma, \delta]$ with σ a k-algebra automorphism and δ a σ-derivation. Then, as sets,*

$$\mathbb{X}_r(R) = \mathcal{C}(R) \cap \big(\mathbb{X}^{\delta}(R) \sqcup \mathbb{X}^{\varnothing}(R)\big)$$

where

$$\mathcal{C}(R) \coloneqq \{\mathfrak{m} \in \mathbb{X}(R) : \delta\big([R, R]\big) \subseteq \mathfrak{m}\}.$$

Proof. This is just a restatement of Theorem 3.18. Suppose $\mathfrak{m} \in \mathbb{X}_r(R)$, then either $\mathfrak{m}$ is (σ, δ)-invariant (and so contains $\delta\big([R, R]\big)$) or it is not σ-invariant but contains $\delta\big([R,R]\big)$; thus, in either case, $\mathfrak{m} \in \mathcal{C}(R)$. Now if $\mathfrak{m} \in \mathbb{X}_r(R)$ is (σ,δ)-invariant then it is certainly δ-invariant and so $\mathfrak{m} \in \mathbb{X}^{\delta}(R)$. If $\mathfrak{m}$ is not σ-invariant then, from the picture, it must belong to $\mathbb{X}^{\delta}(R) \cup \mathbb{X}^{\varnothing}(R)$.

Conversely, given any $\mathfrak{m} \in \mathcal{C}(R) \cap \big(\mathbb{X}^{\delta}(R) \sqcup \mathbb{X}^{\varnothing}(R)\big)$, we see that $\mathfrak{m}$ is either (σ,δ)-invariant (hence belongs to $\mathbb{X}_r(R)$) or is not σ-invariant and contains $\delta\big([R,R]\big)$ (hence, again, $\mathfrak{m}$ belongs to $\mathbb{X}_r(R)$). ∎

Corollary 3.33. *Let* k *be an algebraically closed field and suppose that* R *is a* k*-algebra. Let* $T = R[X;\sigma,\delta]$ *with* σ *a* k*-algebra automorphism and* δ *a* σ*-derivation. Suppose all characters of* T *have type* $\mathcal{A}$*. Then* $\mathbb{X}_r(R)$ *is a closed subset of* $\mathbb{X}(R)$*.*

Proof. If all characters of T have type $\mathcal{A}$ then $\mathbb{X}_r(R) = \mathcal{C}(R) \cap \mathbb{X}^{(\sigma,\delta)}(R)$. Evidently $\mathcal{C}(R)$ is closed in $\mathbb{X}(R)$, as is $\mathbb{X}^{(\sigma,\delta)}(R)$ from Lemma 3.29. ∎

Remarks 3.34. We demonstrate the results of the previous two sections by looking again at the examples discussed in section 3.3.

1. Let $T \coloneqq \mathcal{U}(\mathfrak{sl}_2(k))$ as discussed in section 3.3.2. With $R = k[h][e;\sigma_1]$, we saw that $T = R[f;\sigma_2,\delta_2]$. We also saw that $\mathbb{X}(R) = \big\{\langle e, h-\lambda\rangle : \lambda \in k\big\}$ and that $\mathbb{X}(T) = \big\{\langle e,f,h\rangle\big\}$. Let $\mathfrak{m} \coloneqq \langle e, h-\lambda\rangle \in \mathbb{X}(R)$ for some $\lambda \in k$. Then $\sigma_2^{-1}(\mathfrak{m}) = \langle e, h-(\lambda+2)\rangle$ and so

   $$\mathfrak{m} \cap \sigma_2^{-1}(\mathfrak{m}) = \big\langle e, (h-\lambda)\big(h-(\lambda+2)\big)\big\rangle = \big\langle e, h^2 - 2(\lambda+1)h + \lambda(\lambda+2)\big\rangle.$$

   Then $\delta_2\big(h^2 - 2(\lambda+1)h + \lambda(\lambda+2)\big) = 0$ and $\delta_2(e) = -h$ and so we see that, by Theorem 3.18, for $\mathfrak{m}$ to be the contraction of a character ideal of T we must have $h \in \mathfrak{m}$; thus $\lambda = 0$. So we see that $\mathcal{A} = \varnothing$ and $\mathcal{B} = \big\{\langle e,h,f\rangle\big\}$.

2. Let $T_1 \coloneqq k[K^{\pm 1}]$, $T_2 \coloneqq T_1[F;\sigma_1] = \mathcal{U}_q(\mathfrak{b}^-)$ and $T_3 \coloneqq T_2[E;\sigma_2,\delta_2] = \mathcal{U}_q(\mathfrak{sl}_2(k))$ be as discussed in section 3.3.3.

   Consider the extension $T_2 \coloneqq T_1[F;\sigma_1]$ and let $\mathfrak{m} \coloneqq \langle K-\lambda\rangle$, for some $\lambda \in k^{\times}$, be a character ideal of T_1. Clearly, since there is no σ_1-derivation, $\mathfrak{m}$ satisfes equation (3.9) of Theorem 3.18; thus there is some $\mathfrak{M} \in \mathbb{X}(T_2)$ such that $\mathfrak{m} = \mathfrak{M} \cap T_1$. Moreover, $\sigma_1\big(\langle K-\lambda\rangle\big) = \langle K - \lambda q^{-2}\rangle$ so $\mathfrak{m}$ is shifted by σ_1. Therefore we see that $\mathcal{A} = \varnothing$ and $\mathcal{B} = \mathbb{X}(T_2)$.

Now consider the extension $T_3 = T_2[E; \sigma_2, \delta_2]$ and let $\mathfrak{m} \coloneqq \langle F, K-\lambda\rangle$, for some $\lambda \in k^\times$, be a character ideal of T_2. Then $\sigma_2^{-1}(\mathfrak{m}) = \langle F, K - \lambda q^{-2}\rangle$ and so $\mathfrak{m} \cap \sigma_2^{-1}(\mathfrak{m}) = \langle F, (K-\lambda)(K-\lambda q^{-2})\rangle$. But $\delta_2\big((K-\lambda)(K-\lambda q^{-2})\big) = 0$ and so to have $\mathfrak{m} \in \mathbb{X}_r(T_2)$ we require $\delta_2(F) = K - K^{-1} \in \mathfrak{m}$. Thus $\mathbb{X}_r(T_2) = \big\{\langle F, K+1\rangle, \langle F, K-1\rangle\big\}$, $\mathcal{A} = \varnothing$ and $\mathcal{B} = \mathbb{X}(T_3) = \big\{\langle E, F, K+1\rangle, \langle E, F, K-1\rangle\big\}$.

3.6 Characters of Ore-extension Hopf algebras

3.6.1 Setting and notation

Throughout this section, let k be an algebraically closed field and let R be an affine k-algebra. As usual, we denote by T the Ore extension $R[X; \sigma, \delta]$ where σ is a k-algebra automorphism and δ a σ-derivation. Additionally, in this section, we shall assume that T is a Hopf algebra (but not always that R is a Hopf subalgebra of T).

3.6.2 Restriction of characters is a group homomorphism

Consider the following theorem, due to Molnar.

Theorem 3.35 ([Mol75])**.** *Let k be an arbitrary field and suppose H is a commutative noetherian Hopf k-algebra. Then H is affine.*

Now assume that k is an algebraically closed field and let R be a Hopf k-algebra. Let $T = R[X; \sigma, \delta]$, where σ is a k-algebra automorphism and δ is a σ-derivation, and suppose that T has a Hopf algebra structure extending R. Since R and T are Hopf algebras, the quotient rings $\overline{R} \coloneqq R/\sqrt{\langle [R, R]\rangle}$ and $\overline{T} \coloneqq T/\sqrt{\langle [T, T]\rangle}$ are commutative Hopf algebras by Proposition 2.10. If we assume that R is an affine k-algebra, then so are $\overline{R}$, T and $\overline{T}$. If we assume that R is noetherian, then so is T by the skew version of Hilbert's basis theorem [GW04, Theorem 2.6]. So $\overline{R}$ and $\overline{T}$ are commutative noetherian Hopf k-algebras; hence they are affine by Molnar's theorem.

If, in addition to the above, we assume that k has characteristic zero, then by Lemma 2.13 we know that $\overline{R}$ and $\overline{T}$ are affine commutative semiprime Hopf algebras. Therefore $\overline{R}$ and $\overline{T}$ are the coordinate rings of affine algebraic groups, and so $\mathbb{X}(R)$ and $\mathbb{X}(T)$ have the structure of affine algebraic groups.

Lemma 3.36. *Let* H *be a noetherian (resp. an affine) Hopf algebra over an algebraically closed field and suppose that* K *is a noetherian (resp. an affine) Hopf subalgebra. Then the map* $\Phi : \mathbb{X}(H) \to \mathbb{X}(K)$ *is a morphism of affine algebraic groups.*

Proof. We already know that $\Phi : \mathbb{X}(H) \to \mathbb{X}(K)$ is a morphism of affine algebraic sets, and that $\mathbb{X}(H)$ and $\mathbb{X}(K)$ are affine algebraic groups, so all that remains is to show that Φ is a group homomorphism; that is, given characters $\chi, \xi \in \mathbb{X}(T)$, we have $\Phi(\chi * \xi) = \Phi(\chi) * \Phi(\xi)$. But after a quick check we see that $\Phi(\chi * \xi) = m(\chi \otimes \xi)\Delta_H|_K$ and $\Phi(\chi) * \Phi(\xi) = m(\Phi(\chi) \otimes \Phi(\xi))\Delta_K$; hence, since $\Delta_K = \Delta_H|_K$, these maps are equal on K, and so Φ is a group homomorphism. ■

As a consequence of this lemma, we have the following result.

Proposition 3.37. *Let* R *be a noetherian or an affine Hopf algebra over an algebraically closed field. Suppose* $T = R[X; \sigma, \delta]$ *is a Hopf algebra with* R *a Hopf subalgebra.*

(i) The set $\mathbb{X}_r(R) \coloneqq \operatorname{im}\Phi$ *is a closed subgroup of* $\mathbb{X}(R)$*.*

(ii) As algebraic groups, $\dim \mathbb{X}(T) = \dim\ker\Phi + \dim \mathbb{X}_r(R)$ *and*

$$\dim\ker\Phi = \begin{cases} 0 & \text{if } \varepsilon|_K \text{ is type } \mathcal{B} \\ 1 & \text{if } \varepsilon|_K \text{ is type } \mathcal{A}. \end{cases}$$

Proof. Given a morphism $f : G \to H$ of affine algebraic groups, $\operatorname{im} f$ is closed and $\dim G = \dim\ker f + \dim\operatorname{im} f$ by [Hum80, Section 7.4, Proposition B]. Moreover

$$\ker\Phi = \Phi^{-1}(\varepsilon|_K) = \begin{cases} \varepsilon & \text{if } \varepsilon|_K \text{ is type } \mathcal{B} \\ \mathbb{A}^1 & \text{if } \varepsilon|_K \text{ is type } \mathcal{A}. \end{cases}$$ ■

3.6.3 Characters are all type $\mathcal{A}$ or all type $\mathcal{B}$

Let k be an algebraically closed field and R be a k-algebra. Suppose $T = R[X; \sigma, \delta]$, where σ is a k-algebra automorphism of R and δ is a σ-derivation. Recall that, by Theorem 3.18, we know that each character of T restricts to a character of R and that the corresponding character ideal of R is either (σ, δ)-invariant (type $\mathcal{A}$) or not σ-invariant (type $\mathcal{B}$), using the notation introduced in Definition 3.25

Theorem 3.38. *Suppose* $T = R[X; \sigma, \delta]$ *is a Hopf algebra with* R *a right or left coideal subalgebra. Then the characters of* T *either all have type* $\mathcal{A}$ *or all have type* $\mathcal{B}$.

Proof. We give the proof for R a right coideal subalgebra; the proof for R a left coideal subalgebra is exactly similar. Let $\chi \in \mathbb{X}(T)$ be a character of T. As discussed in section 1.3.5, there is a corresponding right winding automorphism $\tau^r_\chi \in \mathrm{Aut}(T)$. Since $\Delta(R) \subseteq R \otimes T$, because R is a right coideal subalgebra, it follows that $\tau^r_\chi(R) \subseteq R$; similarly $(\tau^r_\chi)^{-1}(R) = \tau^r_{\chi \circ S}(R) \subseteq R$. Hence τ^r_χ restricts to an algebra automorphism of R. Let $\mathfrak{M} \in \mathbb{X}(R)$ have type $\mathcal{A}$ with $\mathfrak{m} := \mathfrak{M} \cap R$. Then $\mathfrak{m}T$ is an ideal of T and $T/\mathfrak{m}T \cong k[\hat{X}]$ as in Theorem 3.18. Now let $\mathfrak{N}$ be any other member of $\mathbb{X}(T)$. By Lemma 1.5 there is a right winding automorphism τ of T such that $\tau(\mathfrak{M}) = \mathfrak{N}$. (If ϕ and ψ are the characters with kernels $\mathfrak{M}$ and $\mathfrak{N}$ respectively, then $\tau = \tau^r_{\psi * \phi^{-1}}$.) Therefore $\tau|_R(\mathfrak{m}) = \tau(\mathfrak{M}) \cap R = \mathfrak{N} \cap R =: \mathfrak{n}$. Thus $\tau(\mathfrak{m}T) = \mathfrak{n}T$, so that $\mathfrak{n}T$ is an ideal of T, and

$$T/\mathfrak{n}T \cong T/\tau(\mathfrak{m}T) \cong T/\mathfrak{m}T \cong k[\hat{X}].$$

Hence $\mathfrak{N}$ has type $\mathcal{A}$ also. ■

Thus we have a dichotomy, and introduce the following terminology.

Definition 3.39. Suppose $T = R[X; \sigma, \delta]$ is a Hopf algebra with R a right or left coideal subalgebra. We shall say that T has **type** $\mathcal{A}$ **over** R if all of the characters of T have type $\mathcal{A}$; similarly we say that T has **type** $\mathcal{B}$ **over** R if all of the characters of T have type $\mathcal{B}$. ♢

Theorem 3.38 provides us with a further algebraic test to determine when an Ore extension supports the structure of a Hopf algebra with the coefficient ring as a right or left coideal subalgebra. Note that the condition that all of the characters of R obtained by restricting characters of T have the same type is not sufficient to conclude that T *does* support a Hopf algebra structure: the Jordan plane (see Remark 2.17(2)) has this property but, as we saw, cannot be made into a Hopf algebra.

To summarise, when $T = R[X; \sigma, \delta]$ is a Hopf algebra with R a Hopf subalgebra we know that $\mathbb{X}_r(R)$ is a closed subgroup of $\mathbb{X}(R)$ and that all of the elements of $\mathbb{X}_r(R)$ have the same type. Namely, if $\ker \varepsilon_R = (\ker \varepsilon_T \cap R)$

is σ-invariant, then T has type $\mathcal{A}$ over R; and if $(\ker \varepsilon_T \cap R)$ is not σ-invariant, then T has type $\mathcal{B}$ over R. By specialising to the type $\mathcal{A}$ and type $\mathcal{B}$ cases we now give an algebraic-geometric description of $\mathbb{X}_r(R)$.

3.7 Extensions of type $\mathcal{A}$

3.7.1 Preliminaries

In this section we begin the study of type $\mathcal{A}$ extensions of a Hopf algebra. Recall the notation $W_{\sigma-\mathrm{id}}(R)$ and $W_\delta(R)$ from Lemmas 3.27 and 3.28.

Theorem 3.40. *Let R be an affine algebra over an algebraically closed field k and suppose $T = R[X;\sigma,\delta]$ is a Hopf algebra. If all characters of T have type $\mathcal{A}$ then, as algebraic sets,*

$$\mathbb{X}_r(R) = \mathbb{X}\big(R/A\big)$$

where $A \coloneqq \sqrt{W_{\sigma-\mathrm{id}}(R) + W_\delta(R)}$. Thus $\mathbb{X}_r(R)$ can be given the structure of an affine algebraic group.

Proof. Since T is an affine Hopf algebra, we know that $\mathbb{X}(T)$ is an affine algebraic group. Then, because all characters of T have type $\mathcal{A}$, Lemma 3.29 tell us that $\operatorname{im}\Phi \eqqcolon \mathbb{X}_r(R) = \mathbb{X}^{(\sigma,\delta)}(R)$ and

$$\mathbb{X}^{(\sigma,\delta)}(R) \cong \mathbb{X}\big(R/A\big)$$

where A is as defined. ∎

Remark 3.41. Retain the hypotheses of Theorem 3.40 and assume, in addition, that k has characteristic zero. Then combining Lemmas 3.29 and 2.13 gives us that $A = I(T) \cap R = \big\langle [T,T] \big\rangle \cap R$.

3.7.2 A type $\mathcal{A}$ extension is an extension by derivation

Let k be an algebraically closed field of characteristic zero. Let R be an affine k-algebra and suppose $T = R[X;\sigma,\delta]$ with σ an automorphism and δ a σ-derivation of R. Assume that T has a Hopf algebra structure with R a Hopf subalgebra and that T has type $\mathcal{A}$ over R.

Since $R \subseteq T$ is a Hopf subalgebra, the counit of R is in the image of the map $\Phi : \mathbb{X}(T) \to \mathbb{X}(R)$; that is, $\varepsilon_R = \varepsilon_T|_R$. Hence, since T has type $\mathcal{A}$ over R, we know that $R^+ := \ker \varepsilon_R$ is (σ, δ)-invariant; therefore R^+T is an ideal of T. Moreover, R^+T is a Hopf ideal of T since

$$\Delta(R^+T) \subseteq (R \otimes R^+ + R^+ \otimes R)(T \otimes T) \subseteq T \otimes R^+T + R^+T \otimes T.$$

Thus, as Hopf algebras, $T/R^+T \cong \mathcal{O}(k^+)$. Now, by [SS06, Lemma 3.2.2(3)], we know that R^+T is conormal in T since $R \subseteq T$ is a Hopf subalgebra (see Definition 1.20 for a reminder of the definition). Let $\psi : T \twoheadrightarrow \mathcal{O}(k^+)$ be the induced surjective Hopf algebra morphism. Then we know that ψ is conormal (by definition, since its kernel is a conormal Hopf ideal) and so, by [Sch93, Lemma 1.3(1)], the left and right coinvariants of ψ are equal; that is, $T^{\operatorname{co}\psi} = {}^{\operatorname{co}\psi}T$. Moreover $T^{\operatorname{co}\psi}$ is a normal Hopf subalgebra of T. Then, applying [GZ10, Theorem 8.3], we see that, as algebras,

$$T \cong T^{\operatorname{co}\psi}[\tilde{X}; \partial]$$

where ∂ is a k-linear derivation of $T^{\operatorname{co}\psi}$. (This is where the hypothesis that k has characteristic zero is used.) We now go on to show that, in fact, $T^{\operatorname{co}\psi} = R$.

Lemma 3.42. *Let S be a ring with $R \subseteq S$ a subring. Suppose that $R[X; \sigma, \delta] = S[\tilde{X}; \partial]$ is a domain, where σ is an automorphism of R, δ is a σ-derivation, and ∂ is a derivation of S. Then $R = S$.*

Proof. We can write $X = \sum_{i=0}^{n} s_i \tilde{X}^i$ and $\tilde{X} = \sum_{j=0}^{m} r_j X^j$ where n and m are non-negative integers and $r_j \in R \subseteq S$ and $s_i \in S$ with r_m and s_n nonzero. Then

$$\tilde{X} = \sum_{j=0}^{m} r_j \left(\sum_{i=0}^{n} s_i \tilde{X}^i \right)^j. \tag{3.10}$$

Observe that, for any $s \in S$, $\tilde{X}s = s\tilde{X} + \partial(s)$; hence after commuting $\tilde{X}^i$ past an element $s \in S$ we see that the highest-degree term is $s\tilde{X}^i$. Thus the term of highest degree in the right-hand side of (3.10) is $r_m s_n \tilde{X}^{mn}$. Now s_n and r_m are not zero-divisors and so, comparing coefficients of degree mn in (3.10), we see that

$$r_m s_n \tilde{X}^{mn} = \begin{cases} \tilde{X} & \text{if } m = n = 1. \\ 0 & \text{otherwise.} \end{cases}$$

The second case gives a contradiction and so we must have $m = n = 1$ and $r_1 s_1 = 1$. A quick calculation also shows that $s_1 r_1 = 1$ and so r_1 and s_1 are units. Now suppose $s \in S \setminus R$. Then, since $s \in R[X; \sigma, \delta]$, we can write

$$s = \sum_{i=0}^{t} u_i X^i = \sum_{i=0}^{t} u_i (s_0 + s_1 \tilde{X})^i$$

where $u_t \neq 0$ for some $t > 0$ (since we are assuming $s \notin R$). Now, just as above, the highest-degree term in the right-hand side is $u_t s_1^t \tilde{X}^t$ and this must be zero. But s_1 and $\tilde{X}$ are not zero-divisors and so we must have $u_t = 0$, contradicting the definition of t. Thus $S \subseteq R$. ∎

Remark 3.43. The assumption in Lemma 3.42 that $R[X; \sigma, \delta] = S[\hat{X}; \partial]$ is a domain is only used to see that s_n and r_m are not zero-divisors. We suspect that the lemma must be true in general and, if so, Lemma 3.44 can then be proved without the assumption that the Ore extensions are domains.

Lemma 3.44. *Let* k *be an algebraically closed field of characteristic zero. Let* R *be an affine* k*-algebra domain and suppose* $T = R[X; \sigma, \delta]$ *with* σ *an automorphism and* δ *a* σ*-derivation of* R*. Assume that* T *has a Hopf algebra structure with* R *a Hopf subalgebra and that* T *has type* $\mathcal{A}$ *over* R*. Retain the notation introduced just before Lemma 3.42. Then* $T^{\mathrm{co}\,\psi} = R$*.*

Proof. First observe that $\psi|_R = \varepsilon|_R$ and so $R \subseteq T^{\mathrm{co}\,\psi}$ as follows. Let $w \in R$. Since $R \subseteq T$ is a Hopf subalgebra, we have $\Delta(w) \in R \otimes R$ and so

$$(\mathrm{id} \otimes \psi)\Delta(w) = (\mathrm{id} \otimes \varepsilon)\Delta(w) = w \otimes \overline{1}$$

by the counit axiom. Now we can apply Lemma 3.42 to complete the proof. ∎

This lemma tells us that whenever we have a type $\mathcal{A}$ Ore-extension Hopf algebra domain then we can change variables so that the automorphism is the identity – we record this result as a theorem.

Theorem 3.45. *Let* k *be an algebraically closed field of characteristic zero. Let* R *be an affine* k*-algebra and a domain, and suppose* $T = R[X; \sigma, \delta]$ *with* σ *a* k*-algebra automorphism and* δ *a* σ*-derivation of* R*. Assume that* T *has a Hopf algebra structure with* R *a Hopf subalgebra and that* T *has type*

$\mathcal{A}$ over R. Then there is a change of variables so that $T = R[\tilde{X}; \partial]$, where ∂ is a derivation of R. Moreover, R is a normal Hopf subalgebra of T.

Proof. This has all been proved above. In particular R is normal since it equals $T^{\mathrm{co}\,\psi}$, which is normal as already noted. ∎

Question 3.46. Can we characterise all the Hopf algebras studied in Theorem 3.45? In other words, which derivations ∂ allow the Hopf algebra structure to be extended from R to $R[X; \partial]$?

3.8 Extensions of type $\mathcal{B}$

3.8.1 Preliminaries

In this section, we continue the study of Ore extensions of Hopf algebras by considering type $\mathcal{B}$ extensions.

Theorem 3.47. *Let R be an affine algebra over an algebraically closed field k and suppose $T = R[X; \sigma, \delta]$ is a Hopf algebra. If all characters of T have type $\mathcal{B}$ then, as algebraic sets,*

$$\mathbb{X}_r(R) = \mathcal{B} = \left\{\mathfrak{m} \in \mathbb{X}(R) : \delta([R, R]) \subseteq \mathfrak{m},\ W_{\sigma-\mathrm{id}}(R) \not\subseteq \mathfrak{m}\right\}$$

where $W_{\sigma-\mathrm{id}}(R) \coloneqq \sqrt{R\left\{\sigma(r) - r : r \in R\right\}R + R[R, R]R}$. Thus $\mathbb{X}_r(R)$ can be given the structure of an affine algebraic group.

Proof. Since T is an affine Hopf algebra, we know that $\mathbb{X}(T)$ is an affine algebraic group. Then, because all characters of T have type $\mathcal{B}$, Lemma 3.31 tell us that $\operatorname{im}\Phi \eqqcolon \mathbb{X}_r(R) \cong \mathcal{B}$. ∎

Remark 3.48. The previous two results apply, in particular, in the case where $T = R[X; \sigma, \delta]$ is a Hopf algebra with R a Hopf subalgebra since then we know from Theorem 3.38 that all characters of T either have type $\mathcal{A}$ or type $\mathcal{B}$.

Lemma 3.49. *Let k be an algebraically closed field and R be an affine commutative semiprime k-algebra so that $V = \operatorname{maxspec}(R)$ is an affine algebraic set over k. Suppose $X \subseteq V$ is both closed and locally closed with*

$$X = \left\{\mathfrak{m} \in \operatorname{maxspec}(R) : I \subseteq \mathfrak{m}\right\} = \left\{\mathfrak{m} \in \operatorname{maxspec}(R) : J \subseteq \mathfrak{m}, L \not\subseteq \mathfrak{m}\right\}$$

for semiprime ideals I, J *and* L *with* $J \subseteq L$. *Then* $R/J \cong (R/I) \oplus (R/L)$. *So* X *is the union of some of the irreducible components of* $V(J)$.

Proof. First note that, by definition, $J \subseteq \mathfrak{m}$ for all $\mathfrak{m} \in X$ and by the nullstellensatz $\bigcap_{\mathfrak{m}\in X} \mathfrak{m} = I$; hence $J \subseteq I$. Define $\hat{R} \coloneqq R/J$, $\hat{V} \coloneqq \text{maxspec}(\hat{R})$, $\hat{I} \coloneqq I/J$ and $\hat{L} \coloneqq L/J$, so that X is homeomorphic to $\hat{X} = V(\hat{I}) = W(\hat{L})$ and we may work in $\hat{R}$ rather than R. Then $\hat{V} = V(\hat{L}) \cup W(\hat{L})$ and $V(\hat{L}) \cap W(\hat{L}) = \varnothing$; hence $\hat{V} = V(\hat{L}) \cup V(\hat{I})$ and $V(\hat{L}) \cap V(\hat{I}) = \varnothing$. Thus $\hat{L} + \hat{I} = \hat{R}$. Moreover, $\hat{L} \cap \hat{I} = \{0\}$; if not, then since $\hat{L} \cap \hat{I} \lhd \hat{R}$ is semiprime there would be a maximal ideal $\mathfrak{m} \lhd \hat{R}$ with $\mathfrak{m} \not\supseteq \hat{L} \cap \hat{I}$. But this contradicts the fact that $\hat{V} = V(\hat{L}) \cup V(\hat{I})$.

Now we have $\hat{R} \cong (\hat{R}/\hat{L}) \oplus (\hat{R}/\hat{L})$ where the isomorphism is the map $\theta : x \mapsto (x + \hat{L}, x + \hat{I})$ for each $x \in \hat{R}$. To see that θ is surjective, write $1 = u + v$ with $u \in \hat{L}$ and $v \in \hat{I}$; then for any $a, b \in \hat{R}$ we have $\theta(au + bv) = (bv + \hat{L}, au + \hat{I}) = (b + \hat{L}, a + \hat{I})$. Thus $R/J \cong (R/I) \oplus (R/L)$. ∎

We now have the following immediate consequence.

Corollary 3.50. *Let* k *be an algebraically closed field and suppose* R *is an affine commutative Hopf* k*-algebra and a domain, so that* $R \cong \mathcal{O}(G)$ *for some connected algebraic group* G. *Let* $T = R[X; \sigma, \delta]$ *be a Hopf algebra with* R *a Hopf subalgebra and suppose that* T *has type* $\mathcal{B}$ *over* R. *Then* $\mathbb{X}(R) = \mathbb{X}_r(R)$ *and* $W_{\sigma-\mathrm{id}}(R) = R$.

Proof. Note first that $\mathbb{X}_r(R)$ is both closed, by Proposition 3.37, and locally closed in $\mathbb{X}(R)$, by Theorem 3.47. Indeed Lemma 3.49 shows that, in this case,

$$\mathbb{X}_r(R) = \left\{\mathfrak{m} \in \mathbb{X}(R) : W_{\sigma-\mathrm{id}}(R) \not\subseteq \mathfrak{m}\right\} = \mathbb{X}(R)$$

using the hypothesis that R is a domain. So $W_{\sigma-\mathrm{id}}(R) = R$. ∎

3.8.2 Type $\mathcal{B}$ extensions of commutative Hopf domains

We now prove a result, analogous to Theorem 3.45, about type $\mathcal{B}$ extensions of a Hopf algebra.

Let k be an algebraically closed field of characteristic zero and suppose R is an affine commutative k-algebra and a domain. Let $T = R[X; \sigma, \delta]$ with σ a k-algebra automorphism and δ a σ-derivation of R. Assume that T has a Hopf algebra structure with R a Hopf subalgebra and that T has type $\mathcal{B}$ over R.

Lemma 3.51. *Retain the above hypotheses. Then* $I(T) \cap R = \{0\}$.

Proof. We know that the map $\Phi : \mathbb{X}(T) \to \mathbb{X}(R)$ is surjective by Corollary 3.50. Thus the canonical map $\Theta = \Phi^* : R \to T/\sqrt{\langle [T,T] \rangle}$, defined by $\Theta : r \mapsto r + \sqrt{\langle [T,T] \rangle}$ is injective. But $\ker \Theta = I(T) \cap R$. ■

Lemma 3.52. *Retain the above hypotheses. Then* $I(T) = (X-r)T = T(X-r)$ *for some* $r \in R$.

Proof. By Lemma 3.36 the canonical map $\Theta : R \to T/\sqrt{\langle [T,T] \rangle}$, defined by $\Theta : r \mapsto r + \sqrt{\langle [T,T] \rangle}$ is a Hopf homomorphism. We claim that it is enough to show that Θ is a bijection. If so then there is some $r \in R$ such that $\Theta(r) = X + I(T)$; that is, $T(X-r) \subseteq I(T)$. But each element of $t \in T$ clearly has a unique expression of the form $t = a(X-r) + b$ with $a \in T$ and $b \in R$. Hence, as left R-modules $T/T(X-r) \cong R$. Since R is noetherian, it cannot be isomorphic to a proper factor of itself and so $I(T) = T(X-r)$. Similarly, working with right R-modules, we see $I(T) = (X-r)T$.

Thus it remains to show that Θ is a bijection. We know that Θ is injective by Lemma 3.51 so R is a Hopf subalgebra of $\overline{T} := T/I(T)$; thus we have to show that $R + I(T) = T$. In what follows, we shall sometimes slightly abuse notation and use the same symbol both for an element of R and its image in $(R + I(T))/I(T) \subseteq \overline{T}$.

We now prove two intermediate claims about the maximal ideals of R. First we claim that, for all maximal ideals $\mathfrak{w}$ of R, we have $\overline{T}/\mathfrak{w}\overline{T} \cong k$. Note that, since $R^+T + I(T)/I(T)$ is a Hopf ideal of $\overline{T}$, we know that $R^+T + I(T)$ is a Hopf ideal of T. Moreoever, there is a unique character ideal $\mathfrak{M}$ of T containing R^+T, by Lemma 3.13. Thus $\hat{T} := T/(R^+T + I(T))$ is an affine commutative Hopf algebra with unique character ideal $\mathfrak{M}$, and the nullstellensatz tells us that this is nilpotent. But, by Cartier's theorem [Wat79, Theorem 11.4], $\hat{T}$ is semiprime and so $\mathfrak{M} = R^+T + I(T)$. Now let $\mathfrak{W}$ by any character of ideal of T and suppose τ is a right winding automorphism of T such that $\tau(\mathfrak{M}) = \mathfrak{W}$. Then

$$\mathfrak{W} = \tau(\mathfrak{M}) = \tau(R^+)T + \tau(I(T)) = \mathfrak{w}T + I(T)$$

for $\mathfrak{w} = \mathfrak{W} \cap R$ a character ideal of R. Thus, for all $\mathfrak{w} \in \mathbb{X}(R)$,

$$\frac{T}{\mathfrak{w}T + I(T)} \cong k.$$

Hence $\overline{T}/\mathfrak{w}\overline{T} \cong k$.

For a maximal ideal $\mathfrak{w}$ of R, let $R_{\mathfrak{w}}$ denote the localisation of R at $\mathfrak{w}$. The second of our claims is that, for all maximal ideals $\mathfrak{w}$ of R, the $R_{\mathfrak{w}}$-module $R_{\mathfrak{w}} \otimes_R \overline{T} = \overline{T}_{\mathfrak{w}}$ is finitely generated. Let Y be a commuting indeterminate. Then there is a k-algebra homomorphism $\psi : R[Y] \twoheadrightarrow \overline{T}$ sending R identically to R and Y to $\overline{X}$. Moreover, $\ker\psi$ is nonzero since, as left R-modules,

$$T = \bigoplus_{i \geqslant 0} RX^i$$

and so if $\ker\psi = 0$ then we would have $T \cong \overline{T}$ as left R-modules; but this is not the case since T is not commutative. Thus there is some nonzero polynomial $f = \sum_{i=0}^{n} r_i Y^n$ inside $\ker\psi$ with $r_n \in R$ nonzero. If r_n is a unit, then Y^n can be written in as a sum of lower degree terms. So $\overline{T}$ is a finitely generated R-module and so, for all maximal ideals $\mathfrak{w}$ of R, we have $\overline{T}_{\mathfrak{w}}$ is a finitely generated $R_{\mathfrak{w}}$-module. Suppose r_n is not a unit. Then, by the nullstellensatz, there is some maximal ideal $\mathfrak{m}$ of R such that $r_n \notin \mathfrak{m}$. Then r_n is a unit in $R_{\mathfrak{m}}$ and so, as above, $R_{\mathfrak{m}} \otimes_R \overline{T} = \overline{T}_{\mathfrak{m}}$ is a finitely generated $R_{\mathfrak{m}}$-module; that is,

$$\overline{T}_{\mathfrak{m}} = \bigoplus_{i=0}^{n-1} R_{\mathfrak{m}} \overline{X}^i \tag{3.11}$$

Now every maximal ideal $\mathfrak{w}$ of R is contained in a unique maximal ideal $\hat{\mathfrak{w}}$ of $\overline{T}$. Take a winding automorphism τ of $\overline{T}$ that maps $\hat{\mathfrak{m}}$ to $\hat{\mathfrak{w}}$. So

$$\tau(\mathfrak{m}) = \tau(\hat{\mathfrak{m}} \cap R) = \hat{\mathfrak{w}} \cap R = \mathfrak{w}.$$

Then apply τ to (3.11) to get that

$$\overline{T}_{\mathfrak{w}} = \sum_{i=0}^{n-1} R_{\mathfrak{w}} \tau(\overline{X}^i);$$

that is, for all maximal ideals $\mathfrak{w}$ of R, $\overline{T}_{\mathfrak{w}}$ is a finitely generated $R_{\mathfrak{w}}$-module, as claimed.

Now suppose $R + I(T) \subsetneq T$. Then $T/(R + I(T))$ is a nonzero R-module. So, by standard commutative algebra, there is some maximal ideal $\mathfrak{n}$ of R such that

$$R_{\mathfrak{n}} \otimes_R \frac{T}{R + I(T)} \neq 0.$$

But this $R_\mathfrak{n}$-module is just $\overline{T}_\mathfrak{n}/R_\mathfrak{n}$, which is finitely generated by above. Then, since $\overline{T}_\mathfrak{n}/R_\mathfrak{n}$ is a nonzero $R_\mathfrak{n}$-module, it has a simple factor. That is

$$\mathfrak{n}(\overline{T}_\mathfrak{n}/R_\mathfrak{n}) \subsetneq \overline{T}_\mathfrak{n}/R_\mathfrak{n}.$$

This is equivalent to

$$\frac{\mathfrak{n}T + (R + I(T))}{R + I(T)} \subsetneq \frac{T}{R + I(T)};$$

that is,

$$\mathfrak{n}\overline{T} + R \subsetneq \overline{T}. \tag{3.12}$$

But, as we saw above, $\overline{T}/\mathfrak{w}\overline{T} \cong k$ for all maximal ideals $\mathfrak{w}$ of R. In particular

$$R + \mathfrak{n}\overline{T} \supset k + \mathfrak{n}\overline{T} = \overline{T},$$

contradicting (3.12). Thus the assumption that $R + I(T) \subsetneq T$ is false, and so Θ is a bijection. ■

We can now prove that, in the above situation, we can change variables to remove the derivation from the Ore extension. We record this as a theorem.

Theorem 3.53. *Let k be an algebraically closed field of characteristic zero and suppose R is an affine commutative k-algebra and a domain. Let $T = R[X; \sigma, \delta]$ with σ a k-algebra automorphism and δ a σ-derivation of R. Assume that T has a Hopf algebra structure with R a Hopf subalgebra and that T has type $\mathcal{B}$ over R. Then there is a change of variables so that $T = R[\tilde{X}; \sigma]$.*

Proof. By the previous two lemmas we know that $X - d \in I(T)$, for some $d \in R$, and that $I(T) \cap R = \{0\}$. Thus, for all $r \in R$,

$$I(T) \ni (X - d)r - \sigma(r)(X - d) = \delta(r) - d\big(r - \sigma(r)\big).$$

But this element also belongs to R and so, for all $r \in R$,

$$\delta(r) = dr - d\sigma(r);$$

that is δ is an inner σ-derivation (see Definition 1.22). Then, by Lemma 1.23, $R[X; \sigma, \delta] = R[X - d; \sigma]$. ■

Remarks 3.54.

1. We saw in section 3.3.1 that the universal enveloping algebra of the two-dimensional solvable non-abelian Lie algebra can be written as $\mathbb{C}[y][x;\sigma]$.

2. We shall see in Chapter 5 that we can characterise all possible Hopf algebras of the type studied in Theorem 3.53. Namely, we prove in Theorem 5.26 that for R an affine commutative Hopf algebra domain and $\sigma \neq \mathrm{id}$, $T = R[X;\sigma]$ if and only if $\sigma : R \to R$ is a winding automorphism and X is primitive.

Chapter

4

Quantum groups with a classical subgroup

4.1 Introduction

Throughout this chapter, let k be an algebraically closed field of characteristic zero. We shall discuss Hopf algebras in the following setting.

(D) Let H be a noetherian Hopf k-algebra with bijective antipode and suppose that

$$\pi : H \twoheadrightarrow \overline{H}$$

is a surjective Hopf morphism with $\overline{H}$ a commutative Hopf domain of GK-dimension at least one (so $\overline{H} \cong \mathcal{O}(G)$ is the coordinate ring of a non-trivial connected affine algebraic group G over k).

Imposing the additional assumption that H is commutative as well as satisfying (D) justifies the title of this chapter, as we shall discuss now. Suppose H satisfies (D) and is commutative. Then, by Molnar's theorem [Mol75], H is an *affine* commutative Hopf algebra over an algebraically closed field of characteristic zero; hence $H \cong \mathcal{O}(K)$ for some affine algebraic group K by the contravariant equivalence of categories discussed in section 1.3.7. Also, because of this equivalence of categories, we know that having a surjective Hopf morphism $\mathcal{O}(K) \twoheadrightarrow \mathcal{O}(G)$ means that there is an embedding of algebraic groups $G \hookrightarrow K$. Thus assuming that a commutative Hopf algebra H satisfies (D) is equivalent to assuming that the group K (of which

H is the coordinate ring) contains G as a closed subgroup. Therefore we can think of assumption (D) as saying that a noncommutative noetherian Hopf algebra H is a "quantum group" with classical subgroup G.

It is a well-known fact, discussed for example in [GZ10, Section 4], that, given any surjective map of Hopf algebras $\pi : H \twoheadrightarrow \overline{H}$, the Hopf algebra H has the structure of a right and left $\overline{H}$-comodule algebra with structure maps

$$\rho \coloneqq (\mathrm{id} \otimes \pi)\Delta : H \to H \otimes \overline{H}$$

and

$$\lambda \coloneqq (\pi \otimes \mathrm{id})\Delta : H \to \overline{H} \otimes H$$

respectively. (Recall the definition of an $\overline{H}$-comodule algebra from Definition 1.13.) We then have, as in Definition 1.17, the subalgebra of right (resp. left) coinvariants given by

$$H^{\mathrm{co}\,\overline{H}} \coloneqq \{h \in H : \rho(h) = h \otimes 1\}$$

and

$${}^{\mathrm{co}\,\overline{H}}H \coloneqq \{h \in H : \lambda(h) = 1 \otimes h\}.$$

Note that $\rho : H \to H \otimes \overline{H}$ is an injective algebra homomorphism as follows. Since H is a right $\overline{H}$-comodule we must have $(\mathrm{id} \otimes \varepsilon)\rho = - \otimes \overline{1}$. Suppose $h, h' \in H$ with $\rho(h) = \rho(h')$. Applying $(\mathrm{id} \otimes \varepsilon)$ to this equation gives that $h \otimes \overline{1} = h' \otimes \overline{1}$; hence $h = h'$. Similarly λ is also an injective algebra homomorphism.

The purpose of this chapter is, given H satisfying (D), to investigate whether the extension $H^{\mathrm{co}\,\overline{H}} \subseteq H$ is $\overline{H}$-cleft (see Definition 1.18 for a reminder of the definition). Recall from Theorem 1.19 that H being $\overline{H}$-cleft is equivalent to saying that, as an algebra, H is isomorphic to a crossed product $H^{\mathrm{co}\,\overline{H}} \#_\sigma \overline{H}$. In section 4.2, we shall see that in general (D) does not imply that the extension is $\overline{H}$-cleft, but some positive cases are known when additional assumptions are added to (D). This will lead us to consider the case where G is a unipotent group and, in particular, where $G = (k^+)^n$. In section 4.4 we explore this latter case and give equivalent conditions to H being $\overline{H}$-cleft.

4.2 Examples and counterexamples

We now motivate the study of Hopf algebras H satisfying (D) by giving some special cases where $\mathrm{H}^{\mathrm{co}\,\overline{\mathrm{H}}} \subseteq \mathrm{H}$ is $\overline{\mathrm{H}}$-cleft and some examples where it is not.

4.2.1 Pointed Hopf algebras

Recall the definition of a pointed Hopf algebra from Definition 1.6. The following result appears in [SS06, Theorem 1.2.6] and was proven by Masuoka.

Theorem 4.1 ([Mas91, 1.3])**.** *Suppose* H *is a pointed Hopf algebra over* $\Bbbk$ *and let* $\pi : \mathrm{H} \to \overline{\mathrm{H}}$ *be a surjective morphism of Hopf algebras. Then* H *is* $\overline{\mathrm{H}}$*-cleft.*

Note that this result requires us to introduce an additional hypothesis on the Hopf algebra structure of H – namely that H is pointed – rather than on the image $\overline{\mathrm{H}}$.

4.2.2 A counterexample when G is not connected

Here we give an example to demonstrate that it is necessary to have the condition that G is connected in (D) if we hope to say that H is $\overline{\mathrm{H}}$-cleft.

Let $\mathrm{K} \coloneqq \mathrm{SL}_2(\mathbb{C})$, the algebraic group of 2×2-matrices with determinant one and let

$$\mathrm{G} \coloneqq \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{K}.$$

Then G is a closed subgroup of K. Define $\mathrm{H} \coloneqq \mathcal{O}(\mathrm{K})$ and $\overline{\mathrm{H}} \coloneqq \mathcal{O}(\mathrm{G})$. Then there is a surjective Hopf morphism $\pi : \mathrm{H} \to \overline{\mathrm{H}}$ dual to the embedding $\mathrm{G} \hookrightarrow \mathrm{K}$. As discussed in [SS06, Ex 1.4.2(5)], H is not free as a $\mathrm{H}^{\mathrm{co}\,\overline{\mathrm{H}}}$-module; in particular it is not $\overline{\mathrm{H}}$-cleft.

4.2.3 A counterexample when G is semisimple

We now give an example of a Hopf algebra H satisfying (D) such that $\mathrm{H}^{\mathrm{co}\,\overline{\mathrm{H}}} \subseteq \mathrm{H}$ is not $\overline{\mathrm{H}}$-cleft.

Let $q \in k^\times$ be a nonzero scalar. Then quantum $SL_2(k)$ is the k-algebra

$$\mathcal{O}_q(SL_2(k)) \coloneqq k\left\langle a, b, c, d : \begin{array}{c} ab = qbc,\ ac = qca,\ bd = qdb, \\ cd = qdc,\ bc = cb, \\ ad - da = (q - q^{-1})bc,\ ad - qbc = 1 \end{array} \right\rangle.$$

Let $H \coloneqq \mathcal{O}_q(SL_2(k))$. Then H has the structure of a Hopf algebra ([Wor87]) with Δ, ε and S as given in [BG02, I.1.9], for example. Let $\pi : \mathcal{O}_q(SL_2(k)) \to k\langle \overline{a}, \overline{d} : \overline{a}\overline{d} = 1\rangle \cong k[X^{\pm 1}]$ be defined by

$$\pi : \begin{cases} a & \mapsto & \overline{a} \\ d & \mapsto & \overline{d} \\ b, c & \mapsto & 0 \end{cases}$$

and extend to an algebra homomorphism. We can check that π is, in fact, a Hopf morphism and $\overline{H} \coloneqq \operatorname{im} \pi \cong \mathcal{O}(k^\times)$; therefore H satisfies (D).

Suppose $H \cong H^{\mathrm{co}\,\overline{H}} \#_\sigma \overline{H}$. Since $\overline{H} \cong \mathcal{O}(k^\times)$, [Mon94, Example 7.1.6] says, in particular, that $\sigma(X, X^{-1})$ is invertible in H. But the only units in H are the nonzero scalars by [Jos95, Lemma 9.1.14] and so, by the definition of multiplication in the crossed product,

$$(1\#X)(1\#X^{-1}) = \lambda(1\#1)$$

for some $\lambda \in k^\times$. This then says that $1\#X$ is an invertible element of H; that is, a scalar, contradicting the definition of H as a crossed product.

Essentially the same calculations also show that $H \coloneqq \mathcal{O}(SL_2(k))$ satisfies (D), also with $G = k^\times$, but that H is not $\overline{H}$-cleft. In this case, the fact that the only units in H are the nonzero scalars follows from [Swe70, Theorem 2.1].

4.2.4 The case when G is unipotent

We saw in the previous section that a Hopf algebra H satifying (D) need not be $\overline{H}$-cleft when G is a semisimple group. Thus we turn our attention to the case when G is a unipotent group. The following theorem appears in [Mon94, Theorem 8.4.8] but was proven by Schneider.

Theorem 4.2 ([Sch92, Theorem 4.2])**.** *Let $\pi : H \to \overline{H}$ be a surjective map of Hopf algebras. Assume* *(a)* H *is injective as a right* $\overline{H}$*-comodule, and*

(b) the coradical of $\overline{H}$ *is liftable along* π*.*

If $B = A\#_\sigma H$ *is a crossed product then* $B \cong B^{\mathrm{co}\,\pi}\#_\tau\overline{H}$*.*

The condition in (b) means that there is a coalgebra map $g : \overline{H}_0 \to H$ such that πg is the embedding $\overline{H}_0 \hookrightarrow \overline{H}$.

Let $A \coloneqq k$ and $B \coloneqq H$ in the above theroem. Then we obtain the following.

Corollary 4.3. *Let* H *satisfy (D) with* $\overline{H}$ *being the coordinate ring of a connected unipotent algebraic group. Then* H *is* $\overline{H}$*-cleft if and only if* H *is an injective right* $\overline{H}$*-comodule.*

Proof. Condition (b) of Theorem 4.2 always holds for such H; we show that $\overline{H}_0 = k$ and so the required map g is the unit map u. To see that $\overline{H}_0 = k$ note that, since $\overline{H} \cong \mathcal{O}(U)$ is the coordinate ring of a unipotent group, we have that $\overline{H}$ is *pointed* commutative by [Abe80, Section 5.1]; thus $\overline{H}_0 = kG(\overline{H})$. But, since U is unipotent, the only group-like element is 1. (We can see this because, as an algebra, $\overline{H}$ is isomorphic to a polynomial ring and so the only units are scalars.) Hence $\overline{H}_0 = k$. Thus we can apply the theorem with $B = H$ and $A = k$ to get the statement of the corollary. ∎

We know no example of a Hopf algebra H satisfying (D), with G a unipotent group, that is not $\overline{H}$-cleft. Thus we introduce the following notation and question.

(DU) Let H satisfy (D) with G a unipotent affine algebraic group.

Question 4.4. Let H satisfy (DU). Does it follow that H is $\overline{H}$-cleft?

By Corollary 4.3, an equivalent question is: if H satisfies (DU), does it follow that H is an injective $\overline{H}$-comodule? We know of no examples to show the answer is negative.

4.2.5 Hopf algebra domains of GK-dimension two

Here we show that if H is a domain satisfying (DU) and having GK-dimension two then H is $\overline{H}$-cleft. We shall refer to results from later in this chapter, but there is no circularity. With this set-up [KL00, Lemma 3.1] implies that, since $\overline{H}$ is a homomorphic image of H, we have $\mathrm{GKdim}\,\overline{H} \leqslant 2$. Thus $\mathrm{GKdim}\,\overline{H} \in \{0,1,2\}$ by [KL00, Proposition 1.4] and Bergman's gap theorem [KL00, Theorem 2.5].

Lemma 4.5. *Let A and B be k-algebras of GK-dimension $n < \infty$. Suppose A is a domain and that $\phi : A \twoheadrightarrow B$ is a surjective homomorphism. Then* $\ker\phi = \{0\}$.

Proof. We have $A/\ker\phi \cong B$. Suppose, for a contradiction, that $\ker\phi \neq \{0\}$. Then, since A is a domain, applying [KL00, Proposition 3.15] says that $\mathrm{GKdim}\,A/\ker\phi \leqslant \mathrm{GKdim}\,A - 1 = n - 1$. But this would mean $n = \mathrm{GKdim}\,B \leqslant n - 1$. ■

If it were the case that $\mathrm{GKdim}\,\overline{H} = 2$, then we would have $\ker\pi = \{0\}$, by Lemma 4.5, and so $H \cong \overline{H}$ which is trivially $\overline{H}$-cleft. If $\mathrm{GKdim}\,\overline{H} = 0$ then, since k is algebraically closed and $\overline{H}$ is a domain, we have $\overline{H} = k$ because the only domain which is a finite-dimensional k-algebra is k itself, by the Artin-Wedderburn theorem, as k is algebraically closed. Hence $G = \mathrm{maxspec}\,\overline{H}$ is the trivial group, contradicting the assumption that H satisfies (DU).

Therefore the only non-trivial case to consider is when $\mathrm{GKdim}\,\overline{H} = 1$ but then $\overline{H} = \mathcal{O}(k^+) = k[t]$ by [Hum75, Theorem 20.5], which says that the only one-dimensional connected affine algebraic groups are k^+ and $k^\times$. By [GZ10, Lemma 9.3] we know that $\mathrm{GKdim}\,H^{\mathrm{co}\,\overline{H}} = \mathrm{GKdim}^{\mathrm{co}\,\overline{H}}\,H = 1$ and consequently, by [GZ10, Proposition 9.4], $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$. The second paragraph of the proof of [GZ10, Theorem 8.3] tells us that $H^{\mathrm{co}\,\overline{H}} \subsetneq H$, since any $x \in H$ such that $\pi(x) = t$ cannot belong to $H^{\mathrm{co}\,\overline{H}}$. We can then apply Lemma 4.22 to show that any domain H of GK-dimension two satisfying (DU) is $\overline{H}$-cleft.

In their paper [GZ10], Goodearl and Zhang classify (among other things) all Hopf algebra domains of GK-dimension two that have a surjective Hopf morphism to $\mathcal{O}(k^+) = k[t]$ with t primitive; that is, they list all Hopf domains of GK-dimension two satisfying (DU). Their list is as follows:

- $k[x,y]$, the coordinate ring of $(k^+)^2$;
- $k[y][x; y\frac{d}{dy}]$, the universal enveloping algebra of the two-dimensional solvable non-abelian Lie algebra;

- $k[y^{\pm 1}][x; (y^n - y)\frac{d}{dy}]$, for n a positive integer, where y is group-like and x is skew-primitive with $\Delta(x) = x \otimes y^{n-1} + 1 \otimes x$.

Observe that all of these Hopf algebras are smash products with the image $\mathcal{O}(k^+)$, since we observed in section 1.4.1 that such a smash product, where $\mathcal{O}(k^+)$ acts by derivations, is nothing more than an extension by derivation. This motivates the study of the link between a Hopf algebra H satisfying (D) and being an $\overline{H}$-cleft extension.

4.3 Equivalent conditions to cleftness

Let H satisfy (D). In this section, we shall give some equivalent conditions to H being $\overline{H}$-cleft.

Corollary 4.6 ([SS06, Examples 1.2.5(1)])**.** *Let H satisfy (D). If there is a coalgebra map $\gamma : \overline{H} \to H$ with $\pi\gamma = \mathrm{id}_{\overline{H}}$ then H is cleft with cleaving map γ.*

We also record the following result, which combines results of Schneider and Schauenburg, and Takeuchi. The theorem applies to any Hopf algebra H satisfying (D) since if $\overline{H}$ is commutative then it has a bijective antipode [Mon94, 1.5.12].

Theorem 4.7 ([SS06], [Tak77])**.** *Let H be a Hopf algebra, $K \subseteq H$ be a Hopf ideal and assume that the antipode of the quotient Hopf algebra $\overline{H} \coloneqq H/K$ is bijective. Then the following are equivalent:*

(i) H *is injective as a left* $\overline{H}$*-comodule.*

(ii) H *is injective as a right* $\overline{H}$*-comodule.*

(iii) H *is coflat as a left* $\overline{H}$*-comodule.*

(iv) H *is coflat as a right* $\overline{H}$*-comodule.*

(v) H *is faithfully coflat as a left* $\overline{H}$*-comodule.*

(vi) H *is faithfully coflat as a right* $\overline{H}$*-comodule.*

(vii) H *is faithfully flat as a left* $H^{\mathrm{co}\,\overline{H}}$*-module and* $K = (H^{\mathrm{co}\,\overline{H}})^+H$.

(viii) H *is faithfully flat as a right* $H^{\mathrm{co}\,\overline{H}}$*-module and* $K = (H^{\mathrm{co}\,\overline{H}})^+H$.

(ix) H *is faithfully flat as a right* ${}^{\mathrm{co}\,\overline{H}}H$*-module and* $K = H({}^{\mathrm{co}\,\overline{H}}H)^+$.

(x) H *is faithfully flat as a left* $H^{\mathrm{co}\,\overline{H}}$*-module and* $K = H({}^{\mathrm{co}\,\overline{H}}H)^+$.

Proof. Statements (*iii*)–(*x*) are equivalent by [SS06, Theorem 3.1.10]. Then, since k is a field, (*i*) $\Leftrightarrow$ (*iii*) and (*ii*) $\Leftrightarrow$ (*iv*) by [Tak77]. ∎

Let H satisfy (DU). Corollary 4.3 tells us that H is $\overline{H}$-cleft if and only if H is an injective $\overline{H}$-comodule. Thus H is $\overline{H}$-cleft if and only if one of the equivalent conditions of Theorem 4.7 holds.

4.4 The case when G is $(k^+)^n$

In this section, we shall investigate Question 4.4 in the special case where $G = (k^+)^n$, so that $\overline{H} = \mathcal{O}(G)$ is a polynomial ring in n variables with the standard cocommutative Hopf algebra structure. We introduce the labels (A) and (B) below to make statements of results more concise.

(A) H is an affine Hopf algebra domain over an algebraically closed field of characteristic zero with a Hopf epimorphism

$$\pi : H \twoheadrightarrow \overline{H} := \mathcal{O}((k^+)^n) = k[t_1, t_2, \ldots, t_n]$$

for some $n \in \mathbb{N}$.

(B) $\operatorname{GKdim} H = n + 1$.

Under assumption (D), the map $\pi : H \twoheadrightarrow \overline{H}$ may or may not be conormal, even for the the same H and $\overline{H}$, as illustrated by the following example.

4.4.1 Conormality of π

Recall from Definition 1.20 the definition of a conormal map $\pi : H \to \overline{H}$. We have the following lemma, due to Schneider.

Lemma 4.8 ([Sch93, Lemma 1.3(1)])**.** *Let* H *satisfy (D) with* $\pi : H \to \overline{H}$ *conormal. Then* $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$ *is a normal Hopf subalgebra of* H*.*

So in the setting of (D) with π conormal we have $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$ and $H^{\mathrm{co}\,\overline{H}}$ is a normal Hopf subalgebra of H. Intuitively, having $H \to \overline{H} = \mathcal{O}(G)$ with π conormal tells us that "the group G is a normal subgroup of the quantum group H" and so we should be able to use some group theory results as inspiration for results about (noncommutative) affine Hopf algebras.

Example 4.9 (The coordinate ring of the Heisenberg group)**.** Let $H = \mathcal{O}(G)$ be the coordinate ring of the three-dimensional Heisenberg group as discussed in Remarks 2.20(2). Then

Let

$$A := \left\{ \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a_{13} \in k \right\} \leqslant G$$

be a subgroup isomorphic to k^+. Dual to the canonical embedding $A \hookrightarrow G$ we have a Hopf surjection $\pi_1 : H \twoheadrightarrow \mathcal{O}(A) := k[\overline{Y}_{13}]$ Since A is central in G it is a normal subgroup and so π_1 is a conormal Hopf surjection.

On the other hand, let

$$B := \left\{ \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a_{12} \in k \right\} \leqslant G$$

be another subgroup isomorphic to k^+. Again, we obtain a Hopf surjection $\pi_2 : H \twoheadrightarrow \mathcal{O}(B) := k[\overline{Y}_{12}]$. The map π_2 is not cornormal: if it were then, by definition, $\ker \pi_2 = \langle Y_{13}, Y_{23} \rangle$ would be a normal Hopf ideal. This requires, in particular, that $\rho_r(h) := \sum h_2 \otimes (Sh_1)h_3$ lies in $\ker \pi_2 \otimes H$ for all $h \in \ker \pi_2$. However a quick calculation shows that

$$\rho_r(Y_{13}) = Y_{12} \otimes Y_{23} - \underbrace{1 \otimes Y_{12}Y_{23}}_{\notin \ker \pi_2 \otimes \mathcal{O}(G)} + Y_{13} \otimes 1 - Y_{23} \otimes Y_{12} \notin \ker \pi_2 \otimes H.$$

We can also see that π_2 is not cormormal because the subgroup $B \subseteq H$ is not normal, by the discussion in [Mon94, p. 36].

Consider again the classical case where H satisfies (D) and is commutative, so that $H = \mathcal{O}(K)$ for some affine algebraic group K. Then, as noted in the previous example, π is conormal if and only if $G \subseteq K$ is a closed normal subgroup. If this is the case then $\mathcal{O}(K/G) \cong H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$ is a Hopf subalgebra of H. All this goes through when H is not commutative, as we record now.

Lemma 4.10. *Let* H *satisfy (D) and suppose that the antipode of* H *is bijective. Then the following are equivalent:*

(i) $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$;

(ii) $H^{\mathrm{co}\,\overline{H}}$ *is a Hopf subalgebra of* H;

(iii) $H^{\mathrm{co}\,\overline{H}}$ *is a normal Hopf subalgebra of* H.

Proof.

$\boxed{(i)\Rightarrow(iii)}$ This does not require the antipode of H to be bijective. By [GZ10, Lemma 4.3(c)] we know that if $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$ then $H^{\mathrm{co}\,\overline{H}}$ is a Hopf subalgebra. It remains to check that it is normal. Let $h \in H$, $b \in H^{\mathrm{co}\,\overline{H}}$ and define $c := (\mathrm{ad}_l\, h)(b) = \sum h_1 b S(h_2)$ and $d := (\mathrm{ad}_r\, h)(b) = \sum S(h_1) b h_2$; so, since $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$, saying $H^{\mathrm{co}\,\overline{H}}$ is normal is equivalent to saying

$$(\mathrm{id} \otimes \pi)\Delta(c) = c \otimes \overline{1} \tag{4.1}$$

and

$$(\pi \otimes \mathrm{id})\Delta(d) = \overline{1} \otimes d. \tag{4.2}$$

First we prove (4.1). Applying $(\mathrm{id} \otimes \pi)\Delta$ to c, and using the fact that $(\mathrm{id} \otimes \pi)\Delta(b) = b \otimes 1$, gives

$$\begin{aligned}
\sum c_1 \otimes \pi(c_2) &= \sum h_1 b S(h_4) \otimes \pi\big(h_2 S(h_3)\big) \\
&= \sum h_1 b S(h_3) \otimes \pi\big(\varepsilon(h_2)\big) \\
&= \sum h_1 b S\big(\varepsilon(h_2) h_3\big) \otimes \pi(1) \\
&= \sum h_1 b S(h_2) \otimes \overline{1} \\
&= c \otimes \overline{1}.
\end{aligned}$$

We can prove (4.2) in a similar way using the fact that $(\pi \otimes \mathrm{id})\Delta(b) = \overline{1} \otimes b$.

$\boxed{(iii)\Rightarrow(ii)}$ This is clear.

$\boxed{(ii)\Rightarrow(i)}$ First of all, since π is a Hopf morphism, a simple check using Sweedler notation shows that

$$(\mathrm{id} \otimes \pi)(S \otimes S)\tau\Delta = (S \otimes S)\tau(\pi \otimes \mathrm{id})\Delta. \tag{4.3}$$

Let $b \in H^{\operatorname{co}\overline{H}}$. Then, since $H^{\operatorname{co}\overline{H}}$ is a Hopf subalgebra, we have $S(b) \in H^{\operatorname{co}\overline{H}}$. We show that $b \in {}^{\operatorname{co}\overline{H}}H$; that is, $(\pi \otimes \mathrm{id})\Delta(b) = 1 \otimes b$. Since $S(b) \in H^{\operatorname{co}\overline{H}}$,

$$(\mathrm{id} \otimes \pi)\Delta\big(S(b)\big) = S(b) \otimes 1.$$

Applying $(S^{-1} \otimes S^{-1})$ gives

$$\begin{aligned}(S^{-1} \otimes S^{-1})(\mathrm{id} \otimes \pi)\Delta\big(S(b)\big) &= b \otimes 1\\ (S^{-1} \otimes S^{-1})(\mathrm{id} \otimes \pi)(S \otimes S)\tau\Delta(b) &= b \otimes 1\\ (S^{-1} \otimes S^{-1})(S \otimes S)\tau(\pi \otimes \mathrm{id})\Delta(b) &= b \otimes 1,\end{aligned}$$

where the last two lines follow from a standard property of the antipode and (4.3). Hence we obtain

$$\tau(\pi \otimes \mathrm{id})\Delta(b) = b \otimes 1,$$

and applying τ gives us that $b \in {}^{\operatorname{co}\overline{H}}H$.

We have $S(H^{\operatorname{co}\overline{H}}) \subseteq {}^{\operatorname{co}\overline{H}}H$ and $S(S^{-1}({}^{\operatorname{co}\overline{H}}H)) = {}^{\operatorname{co}\overline{H}}H$. To show that $S^{-1}({}^{\operatorname{co}\overline{H}}H) \subseteq H^{\operatorname{co}\overline{H}}$, suppose for a contradiction that $x \in H^{\operatorname{co}\overline{H}} \setminus S^{-1}({}^{\operatorname{co}\overline{H}}H)$. Because $x \notin S^{-1}({}^{\operatorname{co}\overline{H}}H)$, it follows that $x \notin {}^{\operatorname{co}\overline{H}}H$; but $S(x) \in H^{\operatorname{co}\overline{H}} \subseteq {}^{\operatorname{co}\overline{H}}H$, giving a contradiction. Therefore $S^{-1}({}^{\operatorname{co}\overline{H}}H) \subseteq H^{\operatorname{co}\overline{H}}$ and so ${}^{\operatorname{co}\overline{H}}H \subseteq S(H^{\operatorname{co}\overline{H}}) \subseteq H^{\operatorname{co}\overline{H}}$, since $H^{\operatorname{co}\overline{H}}$ is a Hopf subalgebra; hence $H^{\operatorname{co}\overline{H}} = {}^{\operatorname{co}\overline{H}}H$. ■

4.4.2 The $\overline{H}$-comodule structures on H

We describe the right and left $\overline{H}$-comodule structures on H satisfying (A). The results is this sub-section are based on the ideas for $n = 1$ from [GZ10, Section 8]. First we introduce some notation. For any $\underline{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$, define $\underline{m}! \coloneqq m_1!m_2! \cdots m_n!$ and $\underline{t}^{\underline{m}} \coloneqq t_1^{m_1}t_2^{m_2} \cdots t_n^{m_n} \in k[t_1, \dots, t_n]$.

Lemma 4.11. *Let* H *satisfy (A). Then there are* n *locally nilpotent commuting* k*-linear derivations* $\delta_1, \delta_2, \dots, \delta_n : H \to H$ *such that, for all* $h \in H$*,*

$$\rho(h) = \sum_{\underline{m}} \frac{1}{\underline{m}!}\underline{\delta}^{\underline{m}}(h) \otimes \underline{t}^{\underline{m}},$$

where $\underline{\delta}^{\underline{m}} := \delta_1^{m_1}\delta_2^{m_2}\cdots\delta_n^{m_n}$ *and* $\delta_i^0 := \mathrm{id}_H$.

Similarly, there are n *locally nilpotent commuting* k*-linear derivations* $\nu_1, \nu_2, \ldots, \nu_n : H \to H$ *such that the left* $\overline{H}$*-comodule structure* $\lambda : H \to \overline{H} \otimes H$ *is described by the* ν_i*. In this case, for all* $h \in H$*,*

$$\lambda(h) = \sum_{\underline{m}} \frac{1}{\underline{m}!} \underline{t}^{\underline{m}} \otimes \underline{\nu}^{\underline{m}}(h).$$

Proof. We know, from the fact that ρ is k-linear, that there exist k-linear maps $d_{\underline{m}} : H \to H$ such that

$$\rho(h) = \sum_{\underline{m}} d_{\underline{m}}(h) \otimes \underline{t}^{\underline{m}}. \tag{4.4}$$

Apply the map $(\mathrm{id} \otimes \varepsilon)$. The counit axiom for ρ tells us that $(\mathrm{id} \otimes \varepsilon)\rho = \mathrm{id} \otimes 1$ and so, using the fact that $\varepsilon(\underline{t}^{\underline{m}}) = 0$ for all $\underline{m} \neq \underline{0}$, we have $h \otimes 1 = d_{\underline{0}}(h) \otimes \underline{t}^{\underline{0}}$. Hence $d_{\underline{0}} = \mathrm{id} : H \to H$.

Now fix $i \in \{1, 2, \ldots, n\}$ and suppose that $\underline{u} = (\delta_{j,i})_{j=1}^n$; that is, that $\underline{u}$ is a vector with 1 in the ith entry and zeros everywhere else. We show that $d_{\underline{u}}$ is a derivation. Since ρ is an algebra homomorphism we have, for all $h, h' \in H$,

$$\begin{aligned} \rho(hh') &= \rho(h)\rho(h') \\ \sum_{\underline{m}} d_{\underline{m}}(hh') \otimes \underline{t}^{\underline{m}} &= \left(\sum_{\underline{p}} d_{\underline{p}}(h) \otimes \underline{t}^{\underline{p}}\right)\left(\sum_{\underline{q}} d_{\underline{q}}(h) \otimes \underline{t}^{\underline{q}}\right). \end{aligned}$$

Comparing terms $- \otimes t_i$ we obtain

$$d_{\underline{u}}(hh') = d_{\underline{0}}(h) d_{\underline{u}}(h') + d_{\underline{u}}(h) d_{\underline{0}}(h').$$

But we know that $d_{\underline{0}} = \mathrm{id}$ and so we have

$$d_{\underline{u}}(hh') = h d_{\underline{u}}(h') + d_{\underline{u}}(h) h';$$

that is, $d_{\underline{u}}$ is a k-linear derivation. To ease notation, let δ_i be the derivation with index $(\delta_{j,i})_{j=1}^n$.

Next we prove a formula for the composition of two maps $d_{\underline{i}}$ and $d_{\underline{p}}$.

Using (4.4) and the coassociativity axiom for ρ gives

$$\begin{aligned}(\rho \otimes \mathrm{id})\rho(h) &= (\mathrm{id} \otimes \overline{\Delta})\rho(h)\\ \sum_{\underline{i},\underline{p}} d_{\underline{i}} d_{\underline{p}}(h) \otimes \underline{t}^{\underline{i}} \otimes \underline{t}^{\underline{p}} &= \sum_{\underline{m}} d_{\underline{m}}(h) \otimes \overline{\Delta}(\underline{t}^{\underline{m}}),\end{aligned} \tag{4.5}$$

where $\overline{\Delta} : \overline{H} \to \overline{H} \otimes \overline{H}$ is the map induced from Δ. Notice that

$$\begin{aligned}\overline{\Delta}(\underline{t}^{\underline{m}}) &= \overline{\Delta}(t_1)^{m_1}\overline{\Delta}(t_2)^{m_2} \cdots \overline{\Delta}(t_n)^{m_n}\\ &= (t_1 \otimes 1 + 1 \otimes t_1)^{m_1}(t_2 \otimes 1 + 1 \otimes t_2)^{m_2} \cdots (t_n \otimes 1 + 1 \otimes t_n)^{m_n}\\ &= \left(\sum_{j_1=0}^{m_1} \binom{m_1}{j_1} t_1^{j_1} \otimes t_1^{m_1-j_1}\right) \cdots \left(\sum_{j_n=0}^{m_n} \binom{m_n}{j_n} t_n^{j_n} \otimes t_n^{m_n-j_n}\right)\\ &= \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \underline{t}^{\underline{j}} \otimes \underline{t}^{\underline{m}-\underline{j}}.\end{aligned}$$

Comparing terms $- \otimes \underline{t}^{\underline{i}} \otimes \underline{t}^{\underline{p}}$ in (4.5) gives

$$d_{\underline{i}} d_{\underline{p}}(h) = \binom{\underline{p}+\underline{i}}{\underline{i}} d_{\underline{p}+\underline{i}} \tag{4.6}$$

where we define

$$\binom{\underline{p}+\underline{i}}{\underline{i}} \coloneqq \binom{p_1+i_1}{i_1} \cdots \binom{p_n+i_n}{i_n}.$$

As a consequence, we see that any two maps $d_{\underline{i}}$ and $d_{\underline{p}}$ commute and, in particular, the derivations δ_i commute. Fix i, let $\underline{u} = (\delta_{j,i})_{j=1}^n$ be a unit vector and a a non-negative integer. Notice that applying the above formula in this case yields $d_{a\underline{u}} = (1/a!)\delta_i^a$.

By repeatedly applying formula (4.6) we see that

$$\begin{aligned}d_{\underline{m}} &= d_{(m_1,0,\ldots,0)} d_{(0,m_2,\ldots,0)} \cdots d_{(0,\ldots,0,m_n)}\\ &= \frac{1}{m_1!}\frac{1}{m_2!} \cdots \frac{1}{m_n!} \delta_1^{m_1}\delta_2^{m_2} \cdots \delta_n^{m_n}\\ &= \frac{1}{\underline{m}!}\underline{\delta}^{\underline{m}}.\end{aligned}$$

Hence we see that

$$\rho(h) = \sum_{\underline{m}} \frac{1}{\underline{m}!} \underline{\delta}^{\underline{m}}(h) \otimes \underline{t}^{\underline{m}}.$$

The proof of the result for the left $\overline{H}$-comodule structure is exactly similar. ■

Assuming that H satisfies (A), we have the following picture.

$$\begin{tikzcd}
 & & k[t_1] \\
 & & k[t_2] \\
H \arrow[r, "\pi"] & \overline{H} \arrow[uur, "\theta_1"] \arrow[ur, "\theta_2"'] \arrow[dr, "\theta_n"'] & \vdots \\
 & & k[t_n]
\end{tikzcd}$$

Here each map $\theta_i : \overline{H} \to k[t_i]$ is the obvious projection of $\overline{H} \cong k[t_1, \ldots, t_n]$ to $k[t_i]$. Defining $\pi_i \coloneqq \theta_i \circ \pi : H \to k[t_i]$ puts us in the situation of [GZ10, Section 8]. Let $\rho_i \coloneqq (\mathrm{id} \otimes \pi_i)\Delta$ and $\lambda_i \coloneqq (\pi_i \otimes \mathrm{id})\Delta$ for each $i \in \{1, \ldots, n\}$ and let $H^{\mathrm{co}\,\rho_i}$ and ${}^{\mathrm{co}\,\lambda_i}H$ be, respectively, the subalgebras of right and left coinvariants.

Observe that

$$\rho_i \coloneqq (\mathrm{id} \otimes \pi_i)\Delta = (\mathrm{id} \otimes (\theta_i \circ \pi))\Delta = (\mathrm{id} \otimes \theta_i)(\mathrm{id} \otimes \pi)\Delta = (\mathrm{id} \otimes \theta_i)\rho$$

and similarly $\lambda_i = (\theta_i \otimes \mathrm{id})\lambda$.

Lemma 4.12. *For each $i \in \{1, \ldots, n\}$, we have $H^{\mathrm{co}\,\rho_i} = \ker \delta_i$. (Similarly, ${}^{\mathrm{co}\,\lambda_i}H = \ker \nu_i$.)*

Proof.

$\boxed{\subseteq}$ Let $h \in H^{\mathrm{co}\,\rho_i}$. By definition, $\rho_i(h) = h \otimes \hat{1}$ and so, by the above observation, $(\mathrm{id} \otimes \theta_i)\rho(h) = h \otimes \hat{1}$. Using the formula from Lemma 4.11 gives

$$(\mathrm{id} \otimes \theta_i) \sum_{\underline{m}} \frac{1}{\underline{m}!} \underline{\delta}^{\underline{m}}(h) \otimes \underline{t}^{\underline{m}} = h \otimes \hat{1}.$$

But, because $\theta_i(t_j) = 0$ for $j \neq i$, this implies

$$\sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \delta_i^{\alpha}(h) \otimes t_i^{\alpha} = h \otimes \overline{1},$$

and comparing terms $- \otimes t_i$ we see that $\delta_i(h) = 0$.

$\boxed{\supseteq}$ For the reverse inclusion, suppose that $h \in \ker \delta_i$. Then, using the formula from Lemma 4.11 once more, we see that $\rho(h)$ has no terms of the form $- \otimes \underline{t}^{\underline{m}}$ with $m_i \neq 0$. Applying $(\mathrm{id} \otimes \theta_i)$ to $\rho(h)$ then leaves only $h \otimes \overline{1}$. ■

Corollary 4.13. *Let* H *satisfy (A) and retain the notation from above. Then* $H^{\mathrm{co}\,\overline{H}} = \bigcap_{i=1}^n H^{\mathrm{co}\,\rho_i}$ *and* ${}^{\mathrm{co}\,\overline{H}}H = \bigcap_{i=1}^n {}^{\mathrm{co}\,\lambda_i}H$.

Proof. This follows immediately from Lemmas 4.11 and 4.12 since $h \in H^{\mathrm{co}\,\overline{H}}$ if and only if $\rho(h) = h \otimes 1$. ■

Lemma 4.14. *For each* $i \in \{1, \ldots, n\}$, *the derivation* $\delta_i : H \to H$ *is an* $H^{\mathrm{co}\,\overline{H}}$*-bimodule homomorphism and* ν_i *is a* ${}^{\mathrm{co}\,\overline{H}}H$*-bimodule homomorphism. Consequently, for any* $\underline{m} \in \mathbb{Z}^n_{\geqslant 0}$, $\ker \underline{\delta}^{\underline{m}}$ *is an* $H^{\mathrm{co}\,\overline{H}}$*-bimodule and* $\ker \underline{\nu}^{\underline{m}}$ *is an* ${}^{\mathrm{co}\,\overline{H}}H$*-bimodule.*

Proof. First, recall that $H^{\mathrm{co}\,\overline{H}}$ and ${}^{\mathrm{co}\,\overline{H}}H$ are subalgebras of H and so H is naturally an $H^{\mathrm{co}\,\overline{H}}$- and ${}^{\mathrm{co}\,\overline{H}}H$-bimodule. For each $i \in \{1, \ldots, n\}$, and for any $x \in H$ and $h \in H^{\mathrm{co}\,\overline{H}}$,

$$\delta_i(hx) = \delta_i(h)x + h\delta_i(x) = h\delta_i(x)$$

since δ_i is a derivation and $h \in H^{\mathrm{co}\,\overline{H}} = \bigcap_{i=1}^n \ker \delta_i$ by Corollary 4.13. Similarly, $\delta_i(xh) = \delta_i(x)h$ and so δ_i is an $H^{\mathrm{co}\,\overline{H}}$-bimodule homomorphism. The proof that ν_i is a ${}^{\mathrm{co}\,\overline{H}}H$-bimodule homomorphism is exactly similar.

Now for any $\underline{m} \in \mathbb{Z}^n_{\geqslant 0}$, and any $x \in H$ and $h \in H^{\mathrm{co}\,\overline{H}}$, because $\delta_i(hx) = h\delta_i(x)$ for each i, we also have $\underline{\delta}^{\underline{m}}(hx) = h\underline{\delta}^{\underline{m}}(x)$; that is, $\underline{\delta}^{\underline{m}} : H \to H$ is left $H^{\mathrm{co}\,\overline{H}}$-module map. Similarly $\underline{\delta}^{\underline{m}}$ is, in fact, an $H^{\mathrm{co}\,\overline{H}}$-bimodule homomorphism. Again, the fact that $\underline{\nu}^{\underline{m}} : H \to H$ is a ${}^{\mathrm{co}\,\overline{H}}H$-bimodule homomorphism follows by the same argument. ■

Lemma 4.15. *Let* H *satisfy (A) and* $\delta_1, \delta_2, \ldots, \delta_n : H \to H$ *be the derivations describing the* $\overline{H}$*-module structure on* H. *Then, for all* $\underline{m} \in \mathbb{Z}^n_{\geqslant 0}$,

(a) $\Delta\underline{\delta}^{\underline{m}} = (\mathrm{id} \otimes \underline{\delta}^{\underline{m}})\Delta$;

(b) $\Delta(\ker \underline{\delta}^{\underline{m}}) \subseteq H \otimes \ker \underline{\delta}^{\underline{m}}$.

Proof.

(*a*) Using [GZ10, Lemma 4.2] we have, for all $h \in H$,

$$(\Delta \otimes \mathrm{id})\rho(h) = (\mathrm{id} \otimes \rho)\Delta(h)$$
$$\sum_{\underline{m}} \frac{1}{\underline{m}!}\Delta(\underline{\delta}^{\underline{m}}(h)) \otimes \underline{t}^{\underline{m}} = \sum_{(h)} \sum_{\underline{m}} \frac{1}{\underline{m}!} h_1 \otimes \underline{\delta}^{\underline{m}}(h_2) \otimes \underline{t}^{\underline{m}};$$

hence comparing terms $- \otimes - \otimes \underline{t}^{\underline{m}}$ gives $\Delta\underline{\delta}^{\underline{m}} = (\mathrm{id} \otimes \underline{\delta}^{\underline{m}})\Delta$.

(*b*) Suppose $\underline{\delta}^{\underline{m}}(h) = 0$ so that $\Delta(\underline{\delta}^{\underline{m}}(h)) = 0$. From part (*a*) it follows that $(\mathrm{id} \otimes \underline{\delta}^{\underline{m}})\Delta(h) = 0$; that is $\Delta(h) \in H \otimes \ker \underline{\delta}^{\underline{m}}$. ■

4.4.3 Injectivity under (A)

The classification in [GZ10] of H satisfying (A) and (B) in the case GKdim $H = 2$ shows that, in this case, $H^{\mathrm{co}\,\overline{H}} \subseteq H$ is a Hopf subalgebra and that, as an algebra, $H \cong H^{\mathrm{co}\,\overline{H}}[x; \partial]$ for some $\partial \in \mathrm{Der}_k H^{\mathrm{co}\,\overline{H}}$. In particular we have that $H \cong H^{\mathrm{co}\,\overline{H}} \#_\sigma \overline{H}$ since any Ore extension is a smash product.

Recall that since $\overline{H} = \mathcal{O}((k^+)^n)$ is pointed we know, by Corollary 4.3, that H is $\overline{H}$-cleft if, and only if, H is an injective $\overline{H}$-comodule. We shall, in effect, reprove this result here but, in doing so, will prove an equivalent condition on H. This condition will later allow us to prove, in Proposition 4.25, that any cleft extension by $\mathcal{O}((k^+)^n)$ is an iterated extension by derivation.

Lemma 4.16. *Suppose* H *satisfies (A) and let* $\delta_1, \dots, \delta_n$ *be the derivations describing the right* $\overline{H}$*-comodule structure on* H*. Then* H *is an injective right* $\overline{H}$*-comodule if and only if, for each* $i \in \{1, \dots, n\}$*,*

$$1 \in \delta_i\left(\ker \delta_i^2 \cap \bigcap_{j \neq i} \ker \delta_j\right). \tag{4.7}$$

Proof.

$\boxed{\text{only if}}$ Suppose that H is right $\overline{H}$-injective. Then, by [Doi85, 1.6] (see [SS06, Theorem 2.3.2]) there is a right $\overline{H}$-comodule homomorphism $\gamma : \overline{H} \to H$ with $\gamma(1) = 1$; thus

$$(\gamma \otimes \mathrm{id})\Delta = \rho\gamma$$

as maps from H to $H \otimes \overline{H}$, since γ is a right $\overline{H}$-comodule homomorphism. In particular, for each $i \in \{1, \dots, n\}$, by Lemma 4.11,

$$(\gamma \otimes \mathrm{id})\Delta(t_i) = \rho(\gamma(t_i));$$

that is,

$$\gamma(t_i) \otimes 1 + 1 \otimes t_i = \sum_{\underline{m}} \frac{1}{\underline{m}!} \underline{\delta}^{\underline{m}}(\gamma(t_i)) \otimes \underline{t}^{\underline{m}}.$$

Comparing terms of the form $- \otimes t_i$, we see that $\delta_i(\gamma(t_i)) = 1$; and comparing terms $- \otimes t_i^2$, we see that $\delta_i^2(\gamma(t_i)) = 0$. Finally, comparing terms $- \otimes t_j$ we see that, for $j \neq i$, $\delta_j(\gamma(t_i)) = 0$. Hence $\gamma(t_i) \in \ker \delta_i^2 \cap \bigcap_{j \neq i} \ker \delta_j$ with $\delta_i(\gamma(t_i)) = 1$ as required.

$\boxed{\text{if}}$ Suppose that (4.7) holds. We shall show that there is a right $\overline{H}$-comodule homomorphism $\gamma : \overline{H} \to H$ with $\gamma(1) = 1$.

By hypothesis, for each i, there is some element $x_i \in \ker \delta_i^2 \cap \bigcap_{j \neq i} \ker \delta_j$ such that $\delta_i(x_i) = 1$ and consequently

$$\rho(x_i) = x_i \otimes 1 + 1 \otimes t_i.$$

For each $\underline{m} \in \mathbb{Z}^n_{\geqslant 0}$, define $\gamma : \overline{H} \to H$ to be the k-linear map given by

$$\gamma(\underline{t}^{\underline{m}}) := \underline{x}^{\underline{m}}.$$

The $\overline{H}$-comodule structure on $\overline{H}$ is given by $\Delta_{\overline{H}}$ and so γ is a right $\overline{H}$-comodule homomorphism $\overline{H} \to H$ as we check here. For each

$\underline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$,

$$\begin{aligned}
(\gamma \otimes \mathrm{id})\Delta(\underline{t}^{\underline{m}}) &= (\gamma \otimes \mathrm{id})\left(\prod_{i=1}^{n}(t_i \otimes 1 + 1 \otimes t_i)^{m_i}\right) \\
&= (\gamma \otimes \mathrm{id})\left(\sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \underline{t}^{\underline{j}} \otimes \underline{t}^{\underline{m}-\underline{j}}\right) \\
&= \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \underline{x}^{\underline{j}} \otimes \underline{t}^{\underline{m}-\underline{j}} \\
&= \prod_{i=1}^{n}(x_i \otimes 1 + 1 \otimes t_i)^{m_i} \\
&= \rho(\underline{x})^{\underline{m}} = \rho(\underline{x}^{\underline{m}}) = \rho(\gamma(\underline{t}^{\underline{m}})),
\end{aligned}$$

and so $\rho\gamma = (\gamma \otimes \mathrm{id})\Delta$ as required for γ to be an $\overline{H}$-comodule homomorphism. Hence, since γ is an $\overline{H}$-comodule homomorphism and $\gamma(1) = 1$, we know that H is an injective $\overline{H}$-comodule by [SS06, Theorem 2.3.2]. ■

In general, H being $\overline{H}$-cleft is a stronger condition than H being an injective $\overline{H}$-comdule. We can see this from the examples in sections 4.2.3 and 4.2.2 since, in both, $\overline{H}$ is cosemisimple and so any right $\overline{H}$-comodule is injective by [SS06, Theorem 2.3.1]. Indeed, H is an injective $\overline{H}$-comodule if and only if there is a right $\overline{H}$-comodule homomorphism $\gamma : \overline{H} \to H$ with $\gamma(1) = 1$ [SS06, Theorem 2.3.2]; whereas H is $\overline{H}$-cleft, by definition, if there is such a γ that is *convolution-invertible*. We know, by Corollary 4.3, that under assumption (A), these two properties are equivalent, but we can prove it directly using Lemma 4.16.

Proposition 4.17. *Suppose* H *satisfies (A). Then* H *is an injective* $\overline{H}$-*comodule if, and only if,* H *is* $\overline{H}$-*cleft.*

Proof. Suppose H is $\overline{H}$-cleft. By definition, there is a (convolution-invertible) right $\overline{H}$-comodule homomorphism $\gamma : \overline{H} \to H$. We can ensure that $\gamma(1) = 1$ by replacing it with $\gamma(1)^{-1}\gamma$ if necessary. Then [Doi85, 1.6] says that H is an injective $\overline{H}$-comodule; so if H is $\overline{H}$-cleft it is certainly $\overline{H}$-injective.

For the converse, suppose that H is an injective right $\overline{H}$-comodule. Let $\gamma : \overline{H} \to H$ be defined as in the proof of Lemma 4.16; we now check that γ is convolution-invertible by finding a convolution-inverse $\gamma' : \overline{H} \to H$. Define, for each $\underline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$,

$$\gamma'(\underline{t}^{\underline{m}}) \coloneqq (-1)^{\underline{m}} \underline{x}^{\underline{m}},$$

where $(-1)^{\underline{m}} \coloneqq (-1)^{m_1}(-1)^{m_2} \cdots (-1)^{m_n}$, and extend γ' to a linear map $\overline{H} \to H$. We show that γ' is the convolution-inverse of γ by checking that $\gamma * \gamma' = \varepsilon$. First,

$$(\gamma * \gamma')(1) = \mu(\gamma \otimes \gamma')(1 \otimes 1) = 1 = \varepsilon(1).$$

For each nonzero $\underline{m} \in \mathbb{Z}^n_{\geqslant 0}$,

$$\begin{aligned}
(\gamma * \gamma')(\underline{t}^{\underline{m}}) &= \mu(\gamma \otimes \gamma')\left(\sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \underline{t}^{\underline{j}} \otimes \underline{t}^{\underline{m}-\underline{j}}\right) \\
&= \underline{x}^{\underline{m}} \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} (-1)^{m_1-j_1} \cdots (-1)^{m_n-j_n} \\
&= \underline{x}^{\underline{m}} \left(\sum_{j_1=0}^{m_1} \binom{m_1}{j_1} (-1)^{j_1}\right) \cdots \left(\sum_{j_n=0}^{m_n} \binom{m_n}{j_n} (-1)^{j_n}\right) \\
&= \underline{x}^{\underline{m}} \big(1 + (-1)\big)^{m_1} \big(1 + (-1)\big)^{m_2} \cdots \big(1 + (-1)\big)^{m_n}
\end{aligned}$$

by the binomial theorem. Hence, since at least one of the components of $\underline{m}$ is positive,

$$(\gamma * \gamma')(\underline{t}^{\underline{m}}) = 0 = \varepsilon(\underline{t}^{\underline{m}})$$

and so γ' is the convolution-inverse of γ. Therefore γ is a cleaving and H is cleft. ■

Remarks 4.18.

1. The definition of γ^{-1} in the above proof is for monomials of the form $t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$ but this is sufficient, since $\overline{H}$ is commutative.

2. In general, injective comodules are not necessarily cleft. Consider the example where $H \coloneqq \mathcal{O}_q(SL_2(k))$ and $\overline{H} = \mathcal{O}(k^\times)$ as described in section 4.2.3. Since $\overline{H}$ is cosemisimple, all right $\overline{H}$-comodules are injective (see [Swe69, 14.0.3]). In particular H is an injective right $\overline{H}$-comodule but, as we saw in section 4.2.3, H is not $\overline{H}$-cleft.

The following question will turn out to be crucial.

Question 4.19. Is it the case that, for each $i \in \{1, \ldots, n\}$,

$$H^{\operatorname{co}\overline{H}} \subseteq H \cap \bigcap_{j=1, j\neq i}^{n} H^{\operatorname{co}\rho_j}$$

is a proper subset? In other words, are all of the δ_i "necessary" to define $H^{\operatorname{co}\overline{H}}$?

Answering this question positively means proving that for each $i = 1, \ldots, n$, there exists an $x \in \ker \delta_i$ such that, for some $j \neq i$, $x \notin \ker \delta_j$. The answer is positive in the case where $H^{\operatorname{co}\overline{H}} \subseteq H$ is cleft since, by Lemma 4.16, for each i there is some $x \in \bigcap_{j\neq i} \ker \delta_j$ with $\delta_i(x) = 1 \neq 0$; hence $x \notin H^{\operatorname{co}\overline{H}}$. This question is important because of the following results.

Remarks 4.20.

1. The answer to this question is positive when $n = 1$ by part of the proof of [GZ10, Theorem 8.3]: having $H = H^{\operatorname{co}\overline{H}}$ is impossible since there must be some $u \in H$ with $\pi(u) = t_1$.

2. Note that, as mentioned on page 13, $H^{\operatorname{co}\overline{H}} = {}^{\overline{H}^*}H$. Now clearly

$$H^{\operatorname{co}\overline{H}} = {}^{\overline{H}^*}H \subseteq {}^{\overline{H}^{\circ}}H \subseteq {}^{G(\overline{H}^{\circ})}H$$

since $G(\overline{H}^{\circ}) \subseteq \overline{H}^{\circ} \subseteq \overline{H}^*$. But in fact ${}^{G(\overline{H}^{\circ})}H \subseteq H^{\operatorname{co}\overline{H}}$ too; that is if $h \in H$ is such that $f \cdot h = h$ for all $f \in \operatorname{Hom}_{k\text{-alg.}}(\overline{H}, k)$, then $h \in \ker \delta_i$ for all $i \in \{1, \ldots n\}$. To see this, let $h \in {}^{G(\overline{H}^{\circ})}H$; thus, for all $f \in \operatorname{Hom}_{k\text{-alg.}}(\overline{H}, k)$,

$$h = \sum_{\underline{m}} \frac{1}{\underline{m}!} f(\underline{t}^{\underline{m}}) \underline{\delta}^{\underline{m}}(h).$$

For each $i \in \{1, \ldots, n\}$ define $f_i \in \operatorname{Hom}_{k\text{-alg.}}(\overline{H}, k)$ by $f_i(t_i) = 1$ and $f_i(t_j) = 0$ for all $j \neq i$, and extend to an algebra homomorphism. Then, for all $h \in {}^{G(\overline{H}^{\circ})}H$ and for all $i \in \{1, \ldots, n\}$, we must have

$$h = \sum_{\underline{m}} \frac{1}{\underline{m}!} f_i(t_1)^{m_1} \cdots f_i(t_n)^{m_n} \underline{\delta}^{\underline{m}}(h) = \sum_{m=0}^{\infty} \frac{1}{m!} \delta_i^m(h);$$

hence $\sum_{m=1}^{\infty} \frac{1}{m!} \delta_i^m(h) = 0$ as $\delta_i^0(h) = h$. Suppose $h \in {}^{G(\overline{H}^{\circ})}H$ and fix $i \in \{1, \ldots, n\}$. Define $M \in \mathbb{Z}_{\geqslant 0}$ by the property that $\delta_i^M(h) \neq 0$ and

$\delta_i^{M+1}(h) = 0$ and suppose, for a contradiction, that $M \in \mathbb{N}$. Since $h \in {}^{G(\overline{H}^\circ)}H$, we have $\sum_{m=1}^{\infty} \frac{1}{m!}\delta_i^m(h) = 0$. Applying the linear map δ^{M-1} gives $\delta_i^M(h) = 0$ since, by assumption, $\delta_i^{M+m}(h) = 0$ for all $m > 0$. This contradicts our definition of M and so $M \notin \mathbb{N}$; hence $M = 0$.

Thus one possible strategy for answering Question 4.19 positively would be to work with invariants rather than coinvariants.

Lemma 4.21. *Suppose that, for each* $i \in \{1, \dots, n\}$,

$$H^{\operatorname{co}\overline{H}} \subsetneq H \cap \bigcap_{j=1, j\neq i}^{n} H^{\operatorname{co}\rho_j}$$

and define $J_i \coloneqq \ker \delta_i^2 \cap \bigcap_{j\neq i} \ker \delta_j$. *Then* $\delta_i : J_i \to H^{\operatorname{co}\overline{H}}$ *is a nonzero* $H^{\operatorname{co}\overline{H}}$*-bimodule homomorphism. Consequently* $\delta_i(J_i)$ *is a nonzero ideal of* $H^{\operatorname{co}\overline{H}}$.

Proof. We already know by Lemma 4.14 that, for any $\underline{m} \in \mathbb{Z}^n_{\geqslant 0}$, $\ker \underline{\delta}^{\underline{m}}$ is an $H^{\operatorname{co}\overline{H}}$-bimodule; hence so is J_i for each $i \in \{1, \dots, n\}$. It remains to show it is nonzero.

By Lemma 4.12 we know that, for all $i \in \{1, \dots, n\}$, $H^{\operatorname{co}\rho_i} = \ker \delta_i$. Thus, for any $i \in \{1, \dots, n\}$, there is some $x \in \left(\bigcap_{j\neq i} \ker \delta_j\right) \setminus \ker \delta_i$. Since δ_i is locally nilpotent, there is some minimal $k \geqslant 2$ such that $\delta_i^k(x) = 0$ and so $\delta_i^2\left(\delta_i^{k-2}(x)\right) = 0$; that is, $\delta_i^{k-2}(x) \in \ker \delta_i^2 \cap \bigcap_{j\neq i} \ker \delta_j$. (The fact that $\delta_i^{k-2}(x) \in \ker \delta_j$ for each $j \neq i$ follows because δ_i and δ_j commute so that $\delta_j\left(\delta_i^{k-2}(x)\right) = \delta_i^{k-2}\left(\delta_j(x)\right)$, but $\delta_j(x) = 0$ by assumption.) The minimality of k implies that $\delta_i\left(\delta_i^{k-2}(x)\right) \neq 0$; that is, that $\delta_i^{k-2}(x) \notin \ker \delta_i$. Hence $\delta_i^{k-2}(x) \notin H^{\operatorname{co}\overline{H}}$ as $H^{\operatorname{co}\overline{H}} \subseteq \ker \delta_i$. Therefore $J_i \neq H^{\operatorname{co}\overline{H}}$ for each $i \in \{1, \dots, n\}$.

To see that $\delta_i : J_i \to H^{\operatorname{co}\overline{H}}$ is nonzero, suppose, for a contradiction, that $\delta_i(J_i) = 0$; that is $J_i \subseteq \ker \delta_i$. Then, since $H^{\operatorname{co}\overline{H}} \subseteq J_i$, it would follow that $J_i = H^{\operatorname{co}\overline{H}}$, contradicting the conclusion of the previous paragraph. ■

We can now generalise [GZ10, Theorem 8.3] as follows; the arguments in the proof are based heavily on those of Goodearl and Zhang in their proof.

Lemma 4.22. *Let* H *satisfy (A) and suppose that, for each* $i \in \{1, \dots, n\}$, $H^{\operatorname{co}\overline{H}} \subsetneq \bigcap_{j\neq i} H^{\operatorname{co}\rho_j}$. *If* $H^{\operatorname{co}\overline{H}} = {}^{\operatorname{co}\overline{H}}H$ *then* H *is* $\overline{H}$*-cleft.*

Proof. For each $i \in \{1, \dots, n\}$, define $J_i := \ker \delta_i^2 \cap \bigcap_{j \neq i} \ker \delta_j$ and $I_i := \delta_i(J_i)$; so we want to show that $1 \in I_i$ for all i. Then we can apply Lemma 4.16, to get that H is injective and hence cleft by Proposition 4.17.

By Lemma 4.21 we know that, for each i, I_i is a nonzero ideal of $H^{\mathrm{co}\,\overline{H}}$. Moreover I_i is a left coideal of $H^{\mathrm{co}\,\overline{H}}$ as follows. By Lemma 4.15,

$$\begin{aligned}
\Delta(I_i) &= \Delta\delta_i\left(\ker \delta_i^2 \cap \bigcap_{j\neq i} \ker \delta_j\right)\\
&= (\mathrm{id} \otimes \delta_i)\Delta\left(\ker \delta_i^2 \cap \bigcap_{j\neq i} \ker \delta_j\right)\\
&\subseteq (\mathrm{id} \otimes \delta_i)\left(\left(H \otimes \ker \delta_i^2\right) \cap \bigcap_{j\neq i} \left(H \otimes \ker \delta_j\right)\right)\\
&= (\mathrm{id} \otimes \delta_i)\left(H \otimes \left(\ker \delta_i^2 \cap \bigcap_{j\neq i} \ker \delta_j\right)\right)\\
&= H \otimes I_i.
\end{aligned}$$

But, since $I_i = \delta_i(J_i)$, we also have $I_i \subseteq H^{\mathrm{co}\,\overline{H}}$ and so $\Delta(I_i) \subseteq H^{\mathrm{co}\,\overline{H}} \otimes H^{\mathrm{co}\,\overline{H}}$, since $H^{\mathrm{co}\,\overline{H}}$ is a Hopf subalgebra by Lemma 4.10. Hence $\Delta(I_i) \subseteq (H \otimes I_i) \cap (H^{\mathrm{co}\,\overline{H}} \otimes H^{\mathrm{co}\,\overline{H}}) = H^{\mathrm{co}\,\overline{H}} \otimes I_i$.

Fix $i \in \{1, \dots, n\}$ and choose $v \in J_i \setminus H^{\mathrm{co}\,\overline{H}}$. Write $\Delta(v) = \sum_j a_j \otimes b_j \in H \otimes J_i$. Then, by Lemma 4.15, we have

$$\Delta\delta_i(v) = (\mathrm{id} \otimes \delta_i)\Delta(v) = \sum a_j \otimes \delta_i(b_j).$$

Applying $(\mathrm{id} \otimes \varepsilon)$ and using the counit axiom gives $\delta_i(v) = \sum_j a_j \varepsilon(\delta_i(b_j)) \neq 0$ since $v \notin H^{\mathrm{co}\,\overline{H}}$. Hence there is some b_k such that $\varepsilon(\delta_i(b_k)) \neq 0$ and $b_k \in \ker \delta_i^2 \setminus H^{\mathrm{co}\,\overline{H}}$. By replacing v by $\varepsilon(b_k)^{-1} b_k$ we can assume, without loss of generality, that $\varepsilon(\delta_i(v)) = 1$.

Now write $\Delta(v) = \sum_{j=1}^{p} a_j \otimes b_j \in H \otimes J_i$ with b_j linearly independent and order the b_j so that $b_1, \dots, b_{m-1} \in H^{\mathrm{co}\,\overline{H}}$ and $b_m, \dots, b_p \in J_i \setminus H^{\mathrm{co}\,\overline{H}}$ are linearly independent modulo $H^{\mathrm{co}\,\overline{H}}$. Then $\delta_i(b_m), \dots, \delta_i(b_p)$ are linearly independent as follows. Suppose $\sum_{j=m}^{p} \lambda_j \delta_i(b_j) = 0$ for some $\lambda_m, \dots, \lambda_p \in k$. Then $\delta_i\left(\sum_{j=m}^{p} \lambda_j b_j\right) = 0$; hence $\sum_{j=m}^{p} \lambda_j b_j \in \ker \delta_i \cap J_i = H^{\mathrm{co}\,\overline{H}}$. But $b_m, \dots b_p$ are linearly independent modulo $H^{\mathrm{co}\,\overline{H}}$ and so $\lambda_m = \cdots = \lambda_p = 0$;

therefore $\delta_i(b_m), \dots, \delta_i(b_p)$ are linearly independent. Now

$$\Delta(\delta_i(v)) = (\mathrm{id} \otimes \delta_i)\Delta(v) = \sum_{j=m}^{p} a_j \otimes \delta_i(b_j) \in H^{\mathrm{co}\,\overline{H}} \otimes I_i,$$

where the second equality holds because $b_j \in \ker \delta_i$ for $1 \leqslant j \leqslant m-1$. Because $\delta_i(b_m), \dots, \delta_i(b_p)$ are linearly independent, it must be the case that $a_m, \dots, a_p \in H^{\mathrm{co}\,\overline{H}}$; hence $S(a_m), \dots, S(a_p) \in H^{\mathrm{co}\,\overline{H}}$ because $H^{\mathrm{co}\,\overline{H}}$ is a Hopf subalgebra by Lemma 4.10. The antipode axiom for H says

$$\mu(S \otimes \mathrm{id})\Delta(\delta_i(v)) = \varepsilon(\delta_i(v))$$
$$\sum_{j=m}^{p} S(a_j)\delta_i(b_j) = 1 \in I_i$$

since $S(a_j) \in H^{\mathrm{co}\,\overline{H}}$ and $\delta_i(b_j) \in I_i$, which is an ideal of $H^{\mathrm{co}\,\overline{H}}$.

Thus, for each $i \in \{1, \dots, n\}$, $1 \in I_i$; hence $1 \in \bigcap_{i=1}^{n} I_i$ and so, by Lemma 4.16, H is an injective right $\overline{H}$-comodule and, therefore, cleft by Corollary 4.3. ■

In summary, we have the following.

Theorem 4.23. *Let* H *satisfy (A) and consider the following statements.*

(i) As an algebra $H \cong H^{\mathrm{co}\,\overline{H}} \#_\sigma \overline{H}$*, a crossed product.*

(ii) $H^{\mathrm{co}\,\overline{H}} \subseteq H$ *is a cleft extension.*

(iii) H *is an injective right* $\overline{H}$*-comodule.*

(iv) For each $i \in \{1, \dots, n\}$, $H^{\mathrm{co}\,\overline{H}} \subsetneq \bigcap_{j \neq i} H^{\mathrm{co}\,\rho_j}$.

Then (i) $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* $\Rightarrow$ *(iv). If* π *is conormal, then statements (i)–(iv) are equivalent.*

Proof.

$\boxed{(i) \Leftrightarrow (ii)}$ Theorem 1.19.

$\boxed{(ii) \Leftrightarrow (iii)}$ Corollary 4.3.

$\boxed{(iii) \Rightarrow (iv)}$ Suppose that H is an injective right $\overline{H}$-comodule. Hence, by Lemma 4.16, we know that, for each $i \in \{1, \ldots, n\}$, there is some $x_i \in \ker \delta_i^2 \cap \bigcap_{j \neq i} \ker \delta_j$ with $\delta_i(x_i) = 1$. Therefore $x_i \in \bigcap_{j \neq i} \ker \delta_j$ but $x_i \notin \ker \delta_i$; that is $H^{\mathrm{co}\,\overline{H}} \subsetneq \bigcap_{j \neq i} H^{\mathrm{co}\,\rho_j}$.

$\boxed{(iv) \Rightarrow (ii)}$ If π is conormal then $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$ by Lemma 4.8 and so we can apply Lemma 4.22 to get that H is $\overline{H}$-cleft. ∎

Remarks 4.24.

1. We have $(i) \Leftrightarrow (ii) \Rightarrow (iii)$ for any H satisfying (D).

2. The proof of $(iv) \Rightarrow (ii)$ is based heavily on the proof in the $n = 1$ case from [GZ10].

4.4.4 Cleft extensions are iterated extensions by derivation

Now we use the above results to show that, in the case $\overline{H} = \mathcal{O}((k^+)^n)$, cleft extensions are all iterated extensions by derivation.

Proposition 4.25. *Let* $\overline{H} = \mathcal{O}((k^+)^n)$ *and suppose that* $H = H^{\mathrm{co}\,\overline{H}} \#_\sigma \overline{H}$ *is a crossed product. Then, as an algebra,*

$$H \cong H^{\mathrm{co}\,\overline{H}}[x_1; \partial_1] \cdots [x_n; \partial_n]$$

where, for each $i \in \{1, 2, \ldots, n\}$*, the derivation* $\partial_i = [x_i, -]$ *maps the subalgebra* $H[x_1; \partial_1] \cdots [x_{i-1}; \partial_{i-1}]$ *into* $H^{\mathrm{co}\,\overline{H}}$.

Proof. Since H is a crossed product, Theorem 1.19 tells us that H is $\overline{H}$-cleft and, therefore, H is an injective $\overline{H}$-comodule by Proposition 4.17. Then Lemma 4.16 says that, for each $i \in \{1, \ldots, n\}$, there exists an element $x_i \in H$ with

$$\rho(x_i) = x_i \otimes 1 + 1 \otimes t_i.$$

Notice that, for each $a \in H^{\mathrm{co}\,\overline{H}}$ and each $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} \rho(x_i a - a x_i) &= (x_i \otimes 1 + 1 \otimes t_i)(a \otimes 1) - (a \otimes 1)(x_i \otimes 1 + 1 \otimes t_i) \\ &= (x_i a - a x_i) \otimes 1. \end{aligned}$$

and so $(x_i a - a x_i) \in H^{\mathrm{co}\,\overline{H}}$. In addition, for each $i, j \in \{1, \dots, n\}$,

$$\begin{aligned} &\rho(x_i x_j - x_j x_i) \\ &= (x_i \otimes 1 + 1 \otimes t_i)(x_j \otimes 1 + 1 \otimes t_j) - (x_i \otimes 1 + 1 \otimes t_i)(x_j \otimes 1 + 1 \otimes t_j) \\ &= (x_i x_j - x_j x_i) \otimes 1; \end{aligned}$$

therefore $(x_i x_j - x_j x_i) \in H^{\mathrm{co}\,\overline{H}}$ too.

Thus we can form $\hat{H} := H^{\mathrm{co}\,\overline{H}}[x_1; \partial_1][x_2; \partial_2] \cdots [x_n; \partial_n]$ where, for each $i \in \{1, 2, \cdots, n\}$, $\partial_i := [x_i, -]$. It remains to show that $H \cong \hat{H}$. But observe that, since $H = H^{\mathrm{co}\,\overline{H}} \#_\sigma \overline{H}$, we know that $H = \left\langle (H^{\mathrm{co}\,\overline{H}} \# 1), (1 \# t_1), \cdots (1 \# t_n) \right\rangle$. Thus define a linear map $\theta : \hat{H} \to H$ by

$$\theta : \begin{cases} h & \mapsto & (h \# 1) \quad \text{if } h \in H^{\mathrm{co}\,\overline{H}} \\ x_i & \mapsto & (1 \# t_i). \end{cases}$$

It is clear that θ is a bijection. We have to see that it is an algebra homomorphism; that is, that $\theta(x_i x_j) = (1 \# t_i)(1 \# t_j)$ for all $1 \leqslant i, j \leqslant n$. But using the definition of the multiplication in the crossed product, we see that

$$(1 \# t_i)(1 \# t_j) = 1 \# t_i t_j + \sigma(t_i, t_j) \# 1 \tag{4.8}$$

and so we need $\sigma(t_i, t_j) = 0$. By the definition of the crossed product (see Definition 1.14) we have that $\sigma : \overline{H} \otimes \overline{H} \to H$ is a convolution-invertible map; that is, there is some map $\sigma' : \overline{H} \otimes \overline{H} \to H$ such that $\sigma * \sigma' = \varepsilon$. Now, recalling the Hopf algebra structure on $\overline{H} \otimes \overline{H}$ as discussed in section 1.3.4, we have $\varepsilon(t_i \otimes t_j) = 0$ and so

$$m(\sigma \otimes \sigma')\Delta(t_i \otimes t_j) = \sigma(t_i, t_j)\sigma'(1, 1) = 0.$$

But $\varepsilon(1 \otimes 1) = 1$ and so

$$m(\sigma \otimes \sigma')((1, 1) \otimes (1, 1)) = \sigma(1, 1)\sigma'(1, 1) = 1.$$

In addition, since σ is a 2-cocycle, $\sigma(1, 1) = 1$; thus $\sigma'(1, 1) = 1$ and hence $\sigma(t_i, t_j) = 0$. Therefore (4.8) now says that the map θ is an algebra automorphism. ■

Proposition 4.26. *Let* H *satisfy (A) and suppose* $H = H^{\mathrm{co}\,\overline{H}} \#_\sigma \overline{H}$ *is a crossed product. If* H *is right (resp. left) noetherian then* $H^{\mathrm{co}\,\overline{H}}$ *is right* *(resp. left) noetherian.*

Proof. Suppose H is right noetherian and let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of right ideals of $A := H^{\mathrm{co}\,\overline{H}}$; thus $I_1H \subseteq I_2H \subseteq \cdots$ is an ascending chain of right ideals of H. Since H is right noetherian, this chain stabilises and so $I_nH = I_{n+1}H = \cdots$ for some n. Now, for any m, we have a short exact sequence

$$I_m \hookrightarrow I_{m+1} \twoheadrightarrow I_{m+1}/I_m$$

where the maps are the inclusion and canonical surjection. Since H is $\overline{H}$-cleft, it is a faithfully flat left A-module by Theorem 4.7. Flatness means that

$$I_m \otimes_A H \hookrightarrow I_{m+1} \otimes_A H \twoheadrightarrow (I_{m+1}/I_m) \otimes_A H$$

is a short exact sequence of right A-modules. Since $A \subseteq H$ is a subring, we have $I_m \otimes_A H \cong I_mH$ and $I_{m+1} \otimes_A H \cong I_{m+1}H$ as right A-modules; thus $I_mH = I_{m+1}H$ is equivalent to having $(I_{m+1}/I_m) \otimes_A H = 0$.

Now put $m = n$ in the above and we see that we must have $(I_{n+1}/I_n) \otimes_A H = 0$. But *faithful* flatness of H as a left A-module then says that $I_{n+1}/I_n = 0$; hence $I_{n+1} = I_n$ and so the ascending chain of right ideals of A must stabilise. ∎

Corollary 4.27. *Let* H *satisfy (A) and (B) with* π *conormal and suppose that one of the equivalent conditions in the statement of Theorem 4.23 hold. Then* $\operatorname{GKdim} H^{\mathrm{co}\,\overline{H}} = 1$ *and, as an algebra,*

$$H \cong H^{\mathrm{co}\,\overline{H}}[x_1; \partial_1][x_2; \partial_2] \cdots [x_n; \partial_n].$$

Moreover, $H^{\mathrm{co}\,\overline{H}}$ *is isomorphic, as a Hopf algebra, to* $\mathcal{O}(k^+)$ *or* $\mathcal{O}(k^\times)$*.*

Proof. By [KL00, Lemma 3.4], $\operatorname{GKdim} H \geqslant \operatorname{GKdim} H^{\mathrm{co}\,\overline{H}} + n$ and, since we assumed that $\operatorname{GKdim} H = n + 1$, it follows that $\operatorname{GKdim} H^{\mathrm{co}\,\overline{H}} \leqslant 1$. If $\operatorname{GKdim} H^{\mathrm{co}\,\overline{H}} = 0$ then, by [KL00, p. 14], $H^{\mathrm{co}\,\overline{H}}$ is locally finite-dimensional. It follows that $H^{\mathrm{co}\,\overline{H}}$ is, in fact, finite dimensional: for any $h \in H^{\mathrm{co}\,\overline{H}}$, the finite-dimensional vector space $k\langle h\rangle$ is a finite-dimensional division algebra over k and, since k is algebraically closed, this must be k itself. Thus $H^{\mathrm{co}\,\overline{H}} = k$. But this would mean, by [KL00, Lemma 3.5], that $\operatorname{GKdim} H = n$, since k is certainly a finitely generated k-algebra.

Since $H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$, $H^{\mathrm{co}\,\overline{H}}$ is a Hopf subalgebra of H and so it is a Hopf algebra domain of GK-dimension one. In addition, since H is $\overline{H}$-cleft, and H is noetherian, Proposition 4.26 says that $H^{\mathrm{co}\,\overline{H}}$ is noetherian. Finally, the fact that $H^{\mathrm{co}\,\overline{H}}$ is commutative follows because it is a domain, and because k is algebraically closed, by [GZ10, Lemma 4.5].

Therefore $H^{\mathrm{co}\,\overline{H}}$ is a commutative noetherian Hopf algebra hence affine by Molnar's Theorem [Mol75]. Consequently $H^{\mathrm{co}\,\overline{H}}$ must be either $\mathcal{O}(k^+)$ or $\mathcal{O}(k^\times)$. ■

Chapter

5

Iterated Hopf-Ore extensions

5.1 Introduction

Throughout this chapter, suppose that k is an algebraically closed field of characteristic zero. Not all results require these hypotheses but we impose them nonetheless.

Definition 5.1 (Iterated Ore extension)**.** Let $T_0 \coloneqq R$ be a k-algebra. For each $i \in \{0, 1, \dots n-1\}$, set $T_{i+1} \coloneqq T_i[X_{i+1}; \sigma_i, \delta_i]$, where each σ_i is an algebra automorphism of T_i and each δ_i is a σ_i-derivation of T_i. Then we call T_n an **iterated Ore extension** of R $\lozenge$

When $R = k$ in the above definition, we call such an extension an **iterated Ore extension of polynomial type**.

Definition 5.2 (Iterated Hopf-Ore extension)**.** Let $T_0 \coloneqq R$ be a Hopf k-algebra. For each $i \in \{0, 1, \dots n-1\}$, set $T_{i+1} \coloneqq T_i[X_{i+1}; \sigma_i, \delta_i]$, where each σ_i is an algebra automorphism of T_i and each δ_i is a σ_i-derivation of T_i. Suppose that at each step T_{i+1} is a Hopf k-algebra, with T_i a Hopf subalgebra of T_{i+1}. Then we call T_n an **iterated Hopf-Ore extension** of R. $\lozenge$

In the special case where $R = k$ in the previous definition, we call such a T_n an **iterated Hopf-Ore extension of polynomial type**. Observe that a skew-primitive Hopf-Ore extension, as defined in Definition 2.18, is an example of an iterated Hopf-Ore extension (with one step).

The aim of this chapter is to study the class of iterated Hopf-Ore extensions of a Hopf algebra and the properties they share. We start by showing that many well-known Hopf algebras are of this type. We then go on to explore general properties that iterated Hopf-Ore extensions have. Finally, in section 5.7, we begin the process of classifying iterated Hopf-Ore extensions of polynomial type.

5.2 Examples

5.2.1 Enveloping algebras of solvable Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over an algebraically closed field k. Then, by [Dix77, 1.3.14], there is a chain of Lie subalgebras

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}$$

where each $\mathfrak{g}_i \subseteq \mathfrak{g}$ is an ideal and $\dim \mathfrak{g}_i = i$. Let $\mathcal{U}(\mathfrak{g})$ denote the enveloping algebra of the Lie algebra $\mathfrak{g}$, as defined, for example, in [Hum80, Section 17.2]. Then $\mathcal{U}(\mathfrak{g})$ is a cocommutative Hopf algebra [Mon94, Example 1.3.3], with $\mathfrak{g}$ as the subspace of primitive elements of $\mathcal{U}(\mathfrak{g})$. We shall show that $T_n \coloneqq \mathcal{U}(\mathfrak{g})$ can be written as an iterated Hopf-Ore extension of polynomial type.

Let $\{x_1, x_2, \ldots, x_n\}$ be a vector space basis for $\mathfrak{g}$ such that $\{x_1, x_2, \ldots, x_i\}$ is a basis for $\mathfrak{g}_i$. Then the Poincaré-Birkhoff-Witt theorem [Dix77, Theorem 2.1.11] tells us, in particular, that the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is generated, as an algebra, by $x_1, x_2, \ldots, x_n$. For each $i \in \{1, 2, \ldots, n-1\}$, define $T_i \coloneqq k\langle x_1, x_2, \ldots, x_i\rangle$ so that each $T_i \subseteq T_n$ is a Hopf subalgebra. Now consider the subalgebra $T_{n-1} \subseteq T_n$. Since $\mathfrak{g}_{n-1}$ is an ideal of $\mathfrak{g}$, we know that $[\mathfrak{g}, \mathfrak{g}_{n-1}] \subseteq \mathfrak{g}_{n-1}$ and so, in particular, $[x_n, \mathfrak{g}_{n-1}] \subseteq \mathfrak{g}_{n-1}$. Hence

$$\partial_{n-1} \coloneqq [x_n, -] : T_{n-1} \to T_{n-1}$$

is a derivation. It follows from this that $T_n = \langle T_{n-1}, x_n\rangle$ is isomorphic to a factor algebra of the Ore extension $T_{n-1}[X; \partial_{n-1}]$ (where we map T_{n-1} to itself with the identity map and X to x_n). The Poincaré-Birkhoff-Witt theorem then shows that this map is an isomorphism, so we can write

$$T_n \cong T_{n-1}[x_n; \partial_{n-1}].$$

Now we can repeat the argument for T_{n-1} since, as T_n is solvable, we have $\mathfrak{g}_{n-2}$ an ideal of $\mathfrak{g}_{n-1}$ and so $[x_{n-1}, \mathfrak{g}_{n-2}] \subseteq \mathfrak{g}_{n-2}$; hence $T_{n-1} \cong T_{n-2}[x_{n-1}; \partial_{n-2}]$ and $T_n \cong T_{n-2}[x_{n-1}; \partial_{n-2}][x_n; \partial_{n-1}]$. Continuing in this manner, we see that $T_n \cong k[x_1][x_2; \partial_1] \cdots [x_n; \partial_{n-1}]$ and so T_n is an iterated Hopf-Ore extension of polynomial type.

5.2.2 Coordinate rings of unipotent groups

Let U_n be the group of $n \times n$ upper triangular matrices over k (with ones on the diagonal). Since U_n is a closed subgroup of $SL(n, k)$, its coordinate ring $T := \mathcal{O}(U_n)$ is a quotient Hopf algebra of $\mathcal{O}(SL(n, k))$. Following through the calculation, we see that as an algebra $T \cong k[X_{ij} : i < j \leqslant n]$ and

$$\Delta(X_{ij}) = X_{ij} \otimes 1 + 1 \otimes X_{ij} + \sum_{i<k<j} X_{ik} \otimes X_{kj}.$$

For each $m \in \{1, 2, \dots, n\}$, define $S_m := k[X_{ij} : 0 < j - i \leqslant m - 1] \subseteq T$. Then we see that $S_n = T$ and there is a chain of Hopf subalgebras

$$T = S_n \supseteq S_{n-1} \supseteq \cdots \supseteq S_2 \supseteq S_1 = k.$$

So we can write $T = S_{n-1}[X_{1n}]$ with $S_{n-1} \subseteq T$ a Hopf subalgebra. Next we can write

$$S_{n-1} = S_{n-2}[X_{ij} : 0 < j - i = n - 2] = S_{n-2}[X_{1,n-1}][X_{2,n}]$$

and, at each step of the extension, we have that the coefficient ring is a Hopf subalgebra. Continuing in this manner, we see that T can be written as an iterated Hopf-Ore extension of polynomial type.

Now let U be a general unipotent group. Then we know, by [Bor91, Corollary 4.8], that U is isomorphic to a closed subgroup of U_n for some n. Then we can find a chain of subgroups

$$1 = N_0 \subseteq \cdots \subseteq N_t = U \subseteq N_{t+1} \subseteq \cdots \subseteq N_m = U_n$$

with, for each i, $N_i \triangleleft N_{i+1}$ and $N_{i+1}/N_i \cong k^+$. Then, by induction on t, we see that $\mathcal{O}(U)$ is an iterated Hopf-Ore extension of polynomial type.

5.2.3 The enveloping algebra of $\mathfrak{sl}_2(k)$

We saw in section 3.3.2 that $T := \mathcal{U}(\mathfrak{sl}_2(k))$ can be written as an iterated Ore extension $k[h][e;\sigma_1][f;\sigma_2,\delta_2]$ for certain maps σ_1, σ_2 and δ_2. Since each of h, e and f are primitive, we see that

$$k[h] \subseteq k[h][e;\sigma_1] \subseteq k[h][e;\sigma_1][f;\sigma_2,\delta_2]$$

is a chain of Hopf subalgebras. Thus T is an iterated Hopf-Ore extension of polynomial type.

5.2.4 The quantised enveloping algebra of $\mathfrak{sl}_2(k)$

Recall the definition of $T_3 := \mathcal{U}_q(\mathfrak{sl}_2(k))$ from section 3.3.3. We saw that T_3 can be written as an Ore extension $T_2[E;\sigma_2,\delta_2]$ of the quantised enveloping algebra of the negative Borel subalgebra $T_2 := k[K^{\pm 1}][F;\sigma_1]$. As discussed in [Kas95, section VII.1], T_3 has a Hopf algebra structure where K is group-like, E is $(K,1)$-primitive and F is $(1,K^{-1})$-primitive. Thus $k[K^{\pm 1}] \subseteq T_2 \subseteq T_3$ is a chain of Hopf subalgebras and, therefore, $T_3 = k[K^{\pm 1}][F;\sigma_1][E;\sigma_2,\delta_2]$ is an iterated Hopf-Ore extension.

5.2.5 Taft algebras

Given integers n, m and t, and q a primitive nth root of unity, let $H(n,m,t)$ be the k-algebra with presentation

$$H = H(n,m,t) := k\left\langle x, g : g^n = 1,\ xg = q^m gx \right\rangle.$$

Then H is a Hopf algebra if we define g to be group-like and x to be $(1,g^t)$-primitive [Taf71]. Observe that we can write $H = kC_n[x;\sigma]$, where $C_n = \langle g \rangle$ is the cyclic group of order n and $\sigma(g) = q^m g$, and $kC_n \subseteq H$ is a Hopf subalgebra; thus H is a Hopf-Ore extension of kC_n.

5.2.6 Zhuang's examples

The following Hopf algebras were defined in the preprint [Zhu12]. Let k be an algebraically closed field of characteristic zero.

The family A

Given $\lambda_1, \lambda_2, \alpha \in k$ and suppose $\alpha = 0$ if $\lambda_1 \neq \lambda_2$, and $\alpha = 0$ or 1 if $\lambda_1 = \lambda_2$. Define

$$A = A(\lambda_1, \lambda_2, \alpha) \coloneqq k\left\langle x, y, z : \begin{array}{c} xy = yx,\ zy - yz = \lambda_2 y, \\ zx - xz = \lambda_1 x + \alpha y \end{array} \right\rangle.$$

Then A is a Hopf algebra if we define x and y to be primitive and $\Delta(z) \coloneqq 1 \otimes z + z \otimes 1 + x \otimes y$, $\varepsilon(y) \coloneqq 0$ and $S(z) \coloneqq -z + xy$. We can write A as an Ore extension

$$A = k[x, y][z; \delta] \quad \text{with} \quad \delta = \lambda_2 y \frac{\partial}{\partial y} + (\lambda_1 x + \alpha y)\frac{\partial}{\partial x}.$$

Then we see that $k[x, y] \subseteq A$ is a Hopf subalgebra and A is a Hopf-Ore extension of polynomial type.

The family B

Let k be an algebraically closed field of characteristic zero and fix $\lambda \in k$. The k-algebra $B(\lambda)$ is defined, by generators and relations, as

$$B(\lambda) \coloneqq k\left\langle x, y, z : \begin{array}{c} xy - yx = y,\ zy - yz = \frac{1}{2}y^2, \\ zx - xz = -z + \lambda y \end{array} \right\rangle.$$

Zhuang then proves that $B(\lambda)$ has a Hopf algebra structure where x and y are primitive, and

$$\begin{aligned} \Delta(z) &= z \otimes 1 + 1 \otimes z + x \otimes y \\ \varepsilon(z) &= 0 \\ S(z) &= -z + xy. \end{aligned}$$

Lemma 5.3. *Let* k *be an algebraically closed field of characteristic zero and fix* $\lambda \in k$. *Then the Hopf algebra* $B(\lambda)$ *is an iterated Hopf-Ore extension* *of polynomial type. More precisely,*

$$B(\lambda) = k[y]\left[x; y\tfrac{d}{dy}\right][z; \sigma, \delta],$$

where

$$\sigma : \begin{cases} x & \mapsto & x-1 \\ y & \mapsto & y \end{cases} \quad \textit{and} \quad \delta : \begin{cases} x & \mapsto & \lambda y \\ y & \mapsto & \frac{1}{2}y^2 \end{cases}.$$

Proof. Note that, for a k-algebra R and a k-algebra automorphism σ of R, a k-linear map $\delta : R \to R$ is a σ-derivation if and only if the map $f : R \to M_2(R)$ defined, for each $a \in R$, by

$$f : a \mapsto \begin{pmatrix} \sigma(a) & \delta(a) \\ 0 & a \end{pmatrix}$$

is an algebra homomorphism [GW04, p. 44]. Let $R := k[y]\left[x; y\frac{d}{dy}\right]$. Then the map f, as defined above, can be extended to an algebra homomorphism from the free algebra $k\langle x, y\rangle$ to $M_2(R)$. To see that this extended f induces an algebra homomorphism $R \to M_2(R)$, we just have to check that f preserves the relation in R; this is an easy calculation. ∎

Thus we see that $B(\lambda)$ is a type $\mathcal{B}$ Hopf-Ore extension of the universal enveloping algebra of the two-dimensional nonabelian Lie algebra over k.

5.3 Pointedness and connectedness

Recall the notions of pointed and connected Hopf algebras from Definition 1.6. In his preprint [Zhu12], Zhuang classifies all connected Hopf algebras of GK-dimension three. He also proves that the GK-dimension of a connected Hopf algebra over an algebraically closed field of characteristic zero is infinite or an integer [Zhu12, Theorem 6.10]. Consequently we have a classification of all connected Hopf algberas of GK-dimension at most three. Moreover, [Zhu12, Proposition 7.14] says that all of the Hopf algebras A and B are isomorphic, as k-algebras, to the universal enveloping algebra of a solvable Lie algebra. Observe that, as a coalgebra, the Hopf algebras A and B are all isomorphic to the coordinate ring of the Heisenberg group (see Remarks 2.20(2)). Indeed, by Zhuang's classification, we see that all connected Hopf algebras of GK-dimension three are isomorphic, as coalgebras, to the coordinate ring of a three-dimensional unipotent affine algebraic group and, as algebras, to the universal enveloping algebra of a solvable Lie algebra. This suggests we should think of connected Hopf algebras of finite GK-dimension simultaneously as deformations of coordinate rings of unipotent groups and of universal enveloping algebras of solvable Lie algebras.

Interestingly, all connected Hopf algebras of GK-dimension three are iterated Hopf-Ore extensions of polynomial type, suggesting that this is a class worth studying. An obvious first question to ask is as follows.

Question 5.4. Is every iterated Hopf-Ore extension of polynomial type connected?

We know of no examples to demonstrate that the answer to this question is negative. The converse is not true however; there are connected Hopf algebras that are not iterated Hopf-Ore extensions of polynomial type. Let $\mathfrak{g}$ be a semisimple Lie algebra not isomorphic to a direct sum of copies of $\mathfrak{sl}_2(k)$. Then $\mathcal{U}(\mathfrak{g})$ is connected. But $\mathfrak{g}$ does not have a chain of Lie subalgebras

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}.$$

Hopf subalgebras of $\mathcal{U}(\mathfrak{g})$ are all of the form $\mathcal{U}(\mathfrak{h})$ for some Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, hence $\mathcal{U}(\mathfrak{g})$ cannot be an iterated Hopf-Ore extension of polynomial type.

Some evidence of a positive answer to Question 5.4 is provided by the following result.

Proposition 5.5. *Let $T = R[X; \sigma, \delta]$ be a Hopf algebra with R a Hopf subalgebra and suppose that $\Delta(X) \in RX \otimes R + R \otimes RX + R \otimes R$. Then the coradical of R is equal to the coradical of T; that is, $T_0 = R_0$.*

Proof. Let $A_0 := R$ and $A_1 := R + RX$ be vector subspaces of T with $A_0 \subseteq A_1$. Then A_0 is a subalgebra, T is generated, as an algebra, by A_1, and $A_0A_1 + A_1A_0 \subseteq A_1$. We also have $\Delta(A_0) \subseteq A_0 \otimes A_0$, since A_0 is a Hopf subalgebra. Now we need to see that $\Delta(A_1) \subseteq A_1 \otimes A_0 + A_0 \otimes A_1$; let

$r, s \in R$, then

$$\begin{aligned} \Delta(r+sX) &= \Delta(r)+\Delta(s)\Delta(X) \\ &\in R\otimes R+(R\otimes R)\big(RX\otimes R+R\otimes RX+R\otimes R\big) \\ &= RX\otimes R+R\otimes RX+R\otimes R. \end{aligned}$$

But

$$A_1\otimes A_0+A_0\otimes A_1=(R+RX)\otimes R+R\otimes(R+RX)=RX\otimes R+R\otimes RX+R\otimes R;$$

hence $\Delta(A_1)\subseteq A_1\otimes A_0+A_0\otimes A_1$. Thus all of the hypotheses of [Mon94, Lemma 5.5.1] hold and we can conclude that $T_0\subseteq A_0=R$. On the other hand, since R is a subcoalgebra of T, we have $R_0=R\cap T_0$ by [Mon94, Lemma 5.1.9]; hence $T_0=T_0\cap R=R_0$. ■

Remark 5.6. This result applies, in particular, to the case when X is skew-primitive.

Corollary 5.7. *Let $T=R[X;\sigma,\delta]$ be a Hopf algebra with R a Hopf subalgebra and suppose that $\Delta(X)\in RX\otimes R+R\otimes RX+R\otimes R$. If R is pointed (resp. connected) then T is pointed (resp. connected).*

Proof. By Proposition 5.5 we know that $T_0=R_0$; hence $T_0=k$ if $R_0=k$. If R is pointed, then $T_0=R_0=kG(R)$. Since the group-likes of T are just the group-likes of R, we have that T is pointed too. ■

This corollary can be applied to many of the examples discussed above to see that they are connected. Indeed we see that universal enveloping algebras of solvable Lie algebras, coordinate rings of unipotent groups and Zhuang's families A and B are all connected, since they are all iterated Hopf-Ore extension of polynomial type and, at each step, the coproduct satisfies the property required to apply Corollary 5.7.

Question 5.8. Does Proposition 5.5 hold without the hypothesis about the coproduct of X?

Again, we know of no examples to show that the answer is negative.

5.4 Gelfand-Kirillov dimension

We now turn our attention to the GK-dimension of iterated Hopf-Ore extensions. In particular, we ask the following.

Question 5.9. Suppose R is a noetherian Hopf $\Bbbk$-algebra of finite GK-dimension. Let $T = R[X; \sigma, \delta]$ be a Hopf-Ore extension of R. Does it follow that $\operatorname{GKdim} T = \operatorname{GKdim} R + 1$?

Note that the answer to this question is negative if we drop the hypothesis that R and T have the structures of Hopf algebras; the example provided by [Lor82], as discussed in [KL00, Proposition 3.9], shows that it is possible for R to have GK-dimension zero but T to have infinite GK-dimension.

Lemma 5.10 ([HK96, Lemma 2.2])**.** *Let $\Bbbk$ be a field and let A be a $\Bbbk$-algebra. Let σ be a $\Bbbk$-algebra automorphism of A and δ be a σ-derivation. If each finite-dimensional subspace of A is contained in a finite-dimensional subspace $V \subseteq A$ such that $\sigma(V) \subseteq V$ and $\delta(V) \subseteq V^m$ for some $m \geqslant 1$, then* $\operatorname{GKdim}\big(A[x; \sigma, \delta]\big) = \operatorname{GKdim} A + 1$.

A natural question to ask, therefore, is as follows.

Question 5.11. Suppose R is a noetherian Hopf algebra over a field $\Bbbk$ and let $T = R[X; \sigma, \delta]$ be a Hopf-Ore extension of R. Does it follow that σ and δ must satisfy the hypotheses of Lemma 5.10?

A positive answer to this question implies a positive answer to Question 5.9. Note that we can provide a positive answer in the following special cases.

Lemma 5.12. *Let $\Bbbk$ be a field. Suppose R is a Hopf $\Bbbk$-algebra and σ is a left or right winding automorphism of R. Set $T = R[X; \sigma]$. Then* $\operatorname{GKdim} T = \operatorname{GKdim} R + 1$.

Proof. By [Mon94, Theorem 5.1.1] each finite-dimensional subspace of R is contained in a finite-dimensional subcoalgebra $C \subseteq R$. Suppose σ is a left or right winding automorphism. Then, since $\Delta(C) \subseteq C \otimes C$, we have $\sigma(C) \subseteq C$. Thus, since in this case $\delta = 0$, we see that the hypotheses of Lemma 5.10 are satisfied. ■

Lemma 5.13. *Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let R be an affine Hopf $\Bbbk$-algebra domain, and suppose $T = R[X; \sigma, \delta]$ is a Hopf-Ore extension of R of type $\mathcal{A}$. Then* $\operatorname{GKdim} T = \operatorname{GKdim} R + 1$.

Proof. By Theorem 3.45 we know that $T = R[\tilde{X}; \partial]$ for some derivation ∂ of R. Then, since R is affine, [KL00, Proposition 3.5] gives that $\operatorname{GKdim} T = \operatorname{GKdim} R + 1$. ■

5.5 Characters form a unipotent algebraic group

In this section, we show that the set of characters of an iterated Hopf-Ore extension of polynomial type forms a unipotent affine algebraic group.

Lemma 5.14. *A closed subgroup of a unipotent affine algebraic group is unipotent.*

Proof. Let G be a unipotent affine algebraic group. By [Bor91, Corollary 4.8], unipotent affine algebraic groups are precisely closed subgroups of upper triangular matrices with ones on the diagonal. Now any closed subgroup of a group of upper triangular matrices with ones on the diagonal is again unipotent. ■

We now show that each $\mathbb{X}(T_i)$ is a unipotent affine algebraic group.

Lemma 5.15. *Let* k *be an algebraically closed field of characteristic zero and suppose* R *is a noetherian Hopf* k*-algebra with* $\mathbb{X}(R)$ *a unipotent affine algebraic group. Let* $T = R[X; \sigma, \delta]$ *be a Hopf-Ore extension of type* $\mathcal{A}$*. Then, as algebraic groups,* $\mathbb{X}(T)$ *is an extension of* k^+ *by* $\mathbb{X}_r(R)$*. Consequently* $\mathbb{X}(T)$ *is also unipotent.*

Proof. First observe that we have a surjective morphism of algebraic groups $\Phi : \mathbb{X}(T_{i+1}) \twoheadrightarrow \mathbb{X}_r(T_i)$ by Lemma 3.36. Moreover,

$$\ker \Phi = \left\{ \mathfrak{m} \in \mathbb{X}(T_{i+1}) : \mathfrak{m} \cap T_i = \ker \varepsilon|_{T_i} \right\}.$$

Since T_{i+1} has type $\mathcal{A}$ over T_i, $(\ker \varepsilon|_{T_i})T_{i+1} \triangleleft T_{i+1}$, and this is a Hopf ideal with

$$T_{i+1}/(\ker \varepsilon|_{T_i})T_{i+1} \cong k[x] \cong \mathcal{O}(k^+)$$

as Hopf algebras. Expressed dually, this says that $\ker \Phi \cong k^+$, and we have a short exact sequence of algebraic groups

$$0 \to k^+ \to \mathbb{X}(T_{i+1}) \to \mathbb{X}_r(T_i) \to 0. \tag{5.1}$$

By induction on i and Lemma 5.14, $\mathbb{X}_r(T_i)$ is unipotent of dimension at most i. Hence, by (5.1), noting that the class of unipotent groups is closed under extensions and that the dimension of algebraic groups is additive on short exact sequences ([Hum75, Theorem 19.3]), $\mathbb{X}(T_{i+1})$ is unipotent of dimension at most $i+1$. ■

Note that this lemma applies, in particular, to the case when R is the coordinate ring of a unipotent group. The lemma provides the bulk of the proof of the following result.

Theorem 5.16. *Let T_n be an iterated Hopf-Ore extension and suppose that $\mathbb{X}(T_0)$ is a unipotent affine algebraic group. Then the set of characters $\mathbb{X}(T_n)$ is a unipotent affine algebraic group.*

Proof. By hypothesis, $\mathbb{X}(T_0)$ is a unipotent affine algebraic group. Now suppose, for $i \in \{0, 1, \ldots, n-1\}$, that $\mathbb{X}(T_i)$ is unipotent. Since $T_i \subseteq T_{i+1}$ is a Hopf subalgebra, we know from Theorem 3.38 that $\mathbb{X}(T_{i+1})$ has type $\mathcal{A}$ or type $\mathcal{B}$; we consider these two cases in turn. Suppose $\mathbb{X}(T_{i+1})$ has type $\mathcal{B}$ so that, as algebraic varieties, $\mathbb{X}(T_{i+1}) \cong \mathbb{X}_r(T_i)$. By Lemma 3.36 we know that $\mathbb{X}_r(T_i)$ is a closed subgroup of $\mathbb{X}(T_i)$ and so, by Lemma 5.14, it is unipotent. Now suppose that $\mathbb{X}(T_{i+1})$ has type $\mathcal{A}$. Then Lemma 5.15 tells us that $\mathbb{X}(T_{i+1})$ is unipotent. ■

Remark 5.17. Let $T_0 = k$ and suppose that, at each step, the automorphism and derivation are trivial; then T_n is a polynomial ring in n variables and is the coordinate ring of some unipotent affine algebraic group (because in this case $T_n = \mathcal{O}(\mathbb{X}(T_n))$).

5.6 Homological properties

We now turn our attention to various homological properties of Hopf-Ore extensions. We will not work much with the following definitions but they are collected here for completeness. For a ring A, we denote its injective, projective and global dimensions by $\operatorname{inj.dim} A$, $\operatorname{pr.dim}$ and $\operatorname{gl.dim}$, respectively (see [Rot07] for the definitions).

Definition 5.18 (AS-Gorenstein)**.** Let A be a noetherian k-algebra with fixed augmentation $\varepsilon : A \to k$. Then A is said to be **AS-Gorenstein** if (*i*) $d = \operatorname{inj.dim}(A)$ is finite,

(*ii*) for the left A-modules A and k,

$$\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d \\ 0 & \text{otherwise,} \end{cases}$$

(*iii*) the condition in (*ii*) also holds for the right A-modules A and k. ◊

Definition 5.19 (AS-regular)**.** Let A be a noetherian k-algebra with fixed augmentation $\varepsilon : A \to k$. Then A is said to be **AS-regular** if it is AS-Gorenstein and has finite global dimension. ◊

Definition 5.20 (Auslander-Gorenstein)**.** Suppose A is a noetherian ring. Then A is said to be **Auslander-Gorenstein** if

(*i*) A has finite injective dimension, and

(*ii*) each finitely generated left or right A-module M satisfies that **Auslander condition**; that is, for every integer ν and every submodule N of $\operatorname{Ext}^\nu_A(M, A)$, we have $\operatorname{Ext}^i_A(N, A) = 0$ for all $i < \nu$. ◊

Definition 5.21 (Auslander regular)**.** Suppose A is a noetherian ring. Then A is said to be **Auslander regular** if it is Auslander-Gorenstein and has finite global dimension. ◊

Brown and Goodearl asked, in [BG97, 1.15], whether all noetherian Hopf algebras are AS-Gorenstein. Thus we see, by part (*iii*) of the following result, that this question has a positive answer when restricted to the class of iterated Hopf-Ore extensions. Indeed, other homological properties also pass from the coefficient ring to the Hopf-Ore extension.

Theorem 5.22. *Let* k *be a field and suppose* R *is a noetherian Hopf* k*-algebra. Let* $T = R[X; \sigma, \delta]$ *be a Hopf-Ore extension of* R*.*

(i) $\operatorname{inj.dim}(T) < \infty$ *if and only if* $\operatorname{inj.dim}(R) < \infty$*. Moreover,* $\operatorname{inj.dim} R \leqslant \operatorname{inj.dim} T \leqslant \operatorname{inj.dim} R + 1$.

(ii) $\operatorname{gl.dim}(T) < \infty$ *if and only if* $\operatorname{gl.dim}(R) < \infty$*. Moreover,* $\operatorname{gl.dim}(T) = \operatorname{gl.dim}(R) + 1$.

(iii) *T is AS-Gorenstein if and only if R is AS-Gorenstein. Moreover, in this case,* $\operatorname{inj.dim}(T) = \operatorname{inj.dim}(R) + 1$.

(iv) *T is AS-regular if and only if R is AS-regular.*

(v) *T is Auslander-Gorenstein if R is Auslander-Gorenstein.*

(vi) *T is Auslander-regular if R is Auslander-regular.*

Proof.

(i) This is [Yi97, Proposition 1.9].

(ii) If $\operatorname{gl.dim} R$ is finite, then [MR88, Theorem 5.3(i)] says that $\operatorname{gl.dim} T$ is at most $\operatorname{gl.dim} R + 1$. If $\operatorname{gl.dim} R$ is infinite then it is a standard fact that $\operatorname{gl.dim} T$ is too. For the second part of the statement, note that for a Hopf algebra H, $\operatorname{gl.dim} H = \operatorname{pr.dim}_H k$ by [LL95, Corollary 2.4]. Now [Rot07, Proposition 8.6] tells us that $\operatorname{pr.dim}_H k = \max\{i : \operatorname{Ext}^i_H(k, H) \neq 0\}$ and so

$$\operatorname{gl.dim} H = \max\{i : \operatorname{Ext}^i_H(k, H) \neq 0\}.$$

Now suppose $\operatorname{gl.dim} R = d$; so, for all $j > d$, $\operatorname{Ext}^j_R(k, R) = 0$. Then, by [Sch86, Theorem 8], we have

$$\operatorname{Ext}^{j+1}_T(k, T) \cong \operatorname{Ext}^j_R(k, R).$$

Since $\operatorname{Ext}^d_R(k, R) \neq 0$ then $\operatorname{Ext}^{d+1}_T \neq 0$ and $d + 1$ is maximal with this property; hence we obtain the result.

(iii) This follows from part (i) and [Sch86, Theorem 8].

(iv) This follows from parts (i) and (iii).

(v)-(vi) This is [Eks89, Theorem 4.2]. ∎

5.7 Almost commutative iterated Hopf-Ore extensions of type $\mathcal{B}$

Ideally, we would like to describe all iterated Hopf-Ore extensions of polynomial type. A natural first step towards doing this would be to describe the "almost commutative" ones, which we do below in the special case of a type $\mathcal{B}$ extension.

The proof of Theorem 5.26 will depend on the following definition, which we reproduce here for completeness. Suppose A is a k-algebra. Let A^e denote the k-algebra $A \otimes A^{op}$; then we call A-A-bimodules A^e-modules. Let M be an A^e-module and suppose α and β are algebra maps $A \to A$. Then the vector space M becomes an A^e-module with left and right actions given, for each $m \in M$ and $r, s \in A$, by $a \cdot m \cdot b = \alpha(r)m\beta(s)$, where the actions on the right-hand side of the equation are the original left and right actions of A^e on M. We denote this new A^e-module by ${}^{\alpha}M^{\beta}$.

Definition 5.23 (Twisted Calabi-Yau algebra)**.** A k-algebra A is said to be ν-**twisted Calabi-Yau** of dimension d, where ν is a k-algebra automorphism of A and $d \geqslant 0$ is an integer, if

(*i*) as an A^e-module, A has a finitely generated projective resolution of finite length, and

(*ii*) as A^e-modules,

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} A^{\nu} & \text{if } i = d \\ 0 & \text{otherwise.} \end{cases} \qquad \Diamond$$

The key point, for our purposes, is the following result, due to Brown and Zhang, and Liu, Wang and Wu.

Lemma 5.24 ([BZ08b, Lemma 5.2 and Proposition 4.5], [LWW12, Lemma 1.3])**.** *Suppose that A is a noetherian Hopf k-algebra. Then A is twisted Calabi-Yau if and only if A is AS-regular.*

We shall need the following technical lemma.

Lemma 5.25. *Let k be a field, and R and T be Hopf k-algebras with $\pi : T \to R$ a surjective Hopf algebra morphism. Then T has the structure of a left R^*-module. Let $G \subseteq R^*$ be a dense subset; that is, given distinct elements $r, s \in R$, then there is some $f \in G$ such that $f(r) \neq f(s)$. Then ${}^{G}T = {}^{R^*}T$.*

Proof. Recall, from the discussion in section 4.1, that T is a right R-comodule with structure map $\rho \coloneqq (\mathrm{id} \otimes \pi)\Delta$ and so, as discussed in section 1.3.9, we know that T is also a left R^*-module and that $T^{\mathrm{co}\,R} = {}^{R^*}T$. Now ${}^{R^*}T \subseteq {}^{G}T$, since $G \subseteq R^*$. So it remains to see that the reverse inclusion holds. Let $x \in {}^{G}T$; that is, $f \cdot x = \varepsilon(f)x$ for all $f \in G$. Let $\{e_i\}_{i \in I}$ be a basis of T with $e_0 = x$ and write $\rho(x) = \sum_{i \in I} e_i \otimes r_i$ with $r_i \in R$. Then, since the action of $f \in G$ on T is given by $m(\mathrm{id} \otimes f)(\mathrm{id} \otimes \pi)\Delta$, we see that for all $f \in G$

$$f \cdot x = \sum_{i \in I} f(r_i)e_i = \varepsilon(f)x$$

Therefore

$$f(r_i) = \begin{cases} 0 & \text{if } i \neq 0, \\ \varepsilon(f) & \text{if } i = 0. \end{cases}$$

Hence, using the fact that $G \subseteq R^*$ is dense, we see that $r_i = 0$ for all $i \neq 0$ and $r_0 = 1$, since $\varepsilon(f) = f(1)$. So we have $\rho(x) = x \otimes 1$. ■

Now recall that, if k is an algebraically closed field of characteristic zero and R is a commutative Hopf k-algebra, then Molnar's theorem [Mol75] tells us that R is affine if and only if R is noetherian. We can now prove the following result.

Theorem 5.26. *Let k be an algebraically closed field of characteristic zero and let $R = \mathcal{O}(U)$ be the coordinate ring of a unipotent affine algebraic group U over k. Suppose $T = R[X; \sigma, \delta]$ is a Hopf-Ore extension of type $\mathcal{B}$ over R. Then there is a change of variables so that $T = R[\hat{X}; \sigma]$ and*

(i) σ is a winding automorphism of R, and

(ii) $\hat{X}$ is primitive.

(iii) Consequently, $\sigma = \tau^r_\chi = \tau^\ell_\chi$ for some character $\chi : R \to k$.

Proof. First, we can apply Theorem 3.53 to get that $T = R[\hat{X}; \sigma]$. In the proof we shall also need to use the fact that the antipode of T is bijective. This follows because, since R is a domain, then T is a domain by [MR88, Theorem 2.9]. Then [Skr06, Corollary 1] says that the antipode of a semiprime noetherian Hopf algebra is bijective.

(*i*) Since R is a commutative polynomial ring, it has finite global dimension by Hilbert's syzygy theorem [Rot07, Theorem 8.37]. Then, since R is an affine commutative noetherian Hopf k-algebra, it is AS-regular by [BZ08a, (6.2)]. So T is noetherian (by the skew Hilbert basis theorem [GW04, Theorem 2.6]) and AS-regular (by Theorem 5.22). Now, by the discussion in [LWW12], a noetherian Hopf algebra is AS-regular if and only if it is twisted Calabi-Yau. Thus we can apply [LWW12, Theorem 0.2] to get that T is ν-twisted Calabi-Yau where $\nu|_R = \sigma^{-1}$. On the other hand, since T is AS-Gorenstein, [BZ08a, Theorem 0.3] implies that $\nu = S^2\tau^{\ell}_{\chi}$ for some character $\chi : T \to k$. Thus we see that $\nu|_R = S^2\tau^{\ell}_{\chi}\big|_R = S^2_R\tau^{\ell}_{\chi|_R}$. Now, since R is commutative, $S^2_R = \mathrm{id}$ by [Mon94, Corollary 1.5.12] and so we have $\nu|_R = \tau^{\ell}_{\chi|_R}$. Equating the two expressions for $\nu|_R$ we see that $\sigma^{-1} = \tau^{\ell}_{\chi|_R}$.

(*ii*) By Lemma 3.52 we know that $I(T) = \hat{X}T$ and so, since $I(T)$ is a Hopf ideal, there is a Hopf surjection $\pi : T \to T/XT \cong R$. Hence T has the structure of a right R-comodule. Thus, by the discussion in [Abe80, Section 1.2], T is a rational left R^*-module and hence a rational left U-module, since $U = \mathbb{X}(R) = G(R^o) \subseteq R^*$. Now the only simple rational module of a unipotent algebraic group is trivial [GTT07, Example 1.5]. Let $\chi \in U$. Then, since $\hat{X}$ generates $I(T)$, $\chi \cdot \hat{X} = \tau^r_{\chi}(\hat{X})$ also generates $I(T)$ and so $\chi \cdot \hat{X} = u\hat{X}$ for some unit $u \in T$. But the only units in T are those in R, and the nonzero scalars are the only units in R. Thus, for all $\chi \in U$, we have $\chi \cdot \hat{X} = \lambda\hat{X}$ for some nonzero $\lambda \in k$. Thus the vector space $k\hat{X}$, being one-dimensional and U-invariant, is such a simple rational module and so $\hat{X}$ must be fixed by the action of U.

Next we claim that

$$T^{\mathrm{co}\,R} = k[\hat{X}]. \tag{5.2}$$

Once we have this then, by repeating the above argument with the left R-comodule structure of T, we get that $T^{\mathrm{co}\,R} = {}^{\mathrm{co}\,R}T = k[\hat{X}]$ and so $k[\hat{X}]$ is a Hopf subalgebra of T by Lemma 4.10. But $k[\hat{X}]$ has a unique Hopf algebra structure, and $\hat{X} \in I(T) \subseteq \ker \varepsilon_T$; so $\hat{X}$ must be primitive.

To prove (5.2), note that $\mathbb{X}(R)$ is dense in R^*; that is, given distinct elements $r, s \in R$, then there is some $\chi \in \mathbb{X}(R)$ such that $\chi(r) \neq \chi(s)$. We can see this is true since, if $r \neq s$ but $\chi(r) = \chi(s)$ for all $\chi \in \mathbb{X}(R)$, then $r - s \in I(R) = 0$. Hence ${}^{\mathbb{X}(R)}T = {}^{R^*}T$ by Lemma 5.25. We know from above that $k[\hat{X}] \subseteq {}^{U}T$ since $\hat{X}$ is fixed by the action of U. Now suppose $t = \sum_{i \geqslant 2} r_i X^i \in {}^{U}T \setminus k[\hat{X}]$ with the nonzero $r_i \in R \setminus k$. Then, for all $\chi \in U$,

$$\tau^r_\chi(t) = \sum_{i \geqslant 2} \tau^r_\chi(r_i) X^i = \sum_{i \geqslant 2} r_i X^i = t$$

because τ^r_χ is an algebra automorphism and $\hat{X}$ is fixed by the above. Thus each r_i is fixed by all winding automorphisms of R. Now, since the only units in R are the nonzero scalars, we must have $r := r_j \in R$ a non-unit for some j. Then $rR \triangleleft R$ is a proper ideal and so $rR \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \mathbb{X}(R)$. But then, since r is fixed by all winding automorphisms of R, we get that $r \subseteq I(R) = \{0\}$, a contradiction. Hence ${}^{U}T = k[\hat{X}]$ and the proof is complete.

(*iii*) Now that we know $\hat{X}$ is primitive, this follows from Theorem 2.19. ■

Remark 5.27. We expect more to follow from this result, as the requirement that σ is the left and right winding automorphism of the same character is quite strong.

Corollary 5.28. *Retain the hypotheses of Theorem 5.26. Then* $\operatorname{GKdim} T = \operatorname{GKdim} R + 1$.

Proof. Just apply Lemma 5.12. ■

Corollary 5.29. *Retain the hypotheses of Theorem 5.26. If* R *is pointed (resp. connected) then so is* T.

Proof. This follows from Corollary 5.7. ■

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