PROPERTIES OF JETS AND WAKES

BY

LAWRENCE JOHN CRANE

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This thesis is a study of the effect of differences in the density of a fluid on the mixing regions of jets, which may be laminar or turbulent. These differences in density are present for three main reasons, namely: when the speed of the fluid is of the same order of magnitude as the local speed of sound; when there are large temperature differences in the fluid; and when the fluid consists of a mixture of components the relative proportions of which vary from point to point.

Three problems are considered. These are: the flow far from the orifice of a plane and of a round jet and the mixing region on the surface of the core of a plane jet near the orifice. This last problem is idealised as the mixing of two semi-infinite streams.

For flows of jet type, the assumption of a coefficient of eddy kinematic viscosity in turbulent flow leads to the possibility of combining in one the equations for laminar and turbulent motion.

The method used is to expand the stream function in a Rayleigh-Jansen series. The first term of this series corresponds to the stream function when the fluid is of constant density. The series is developed in powers of a small parameter whose magnitude depends on the density differences in the fluid. Only the second term of this series is found explicitly. This term gives the first order effect that changes in density have on the flow. The solutions of all examples considered are, with one exception, given in analytical form.

The last appendix to the thesis shows the connection between Stewartson's (1957) approach to the problem of finding uniformly valid approximate solutions to the boundary layer equations and Lighthill's (1948) method. This connection is shown by working out one of the problems considered by Stewartson, namely, the wake past a flat plate, using Lighthill's method.

PROPERTIES OF JETS AND WAKES

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PERSONAL FOREWORD

The work for this thesis was started in April, 1955, and completed in September, 1958, and was carried out in the Mathematics Department, Royal College of Science and Technology, Glasgow, under the supervision of Professor D. C. Pack.

The work of the first four chapters and Chapter VII is mainly of an explanatory character. The substance of Chapter V has been published in the Journal of Fluid Mechanics under the title: The laminar and turbulent mixing of jets of compressible fluid, Part I, Flow far from the orifice, by L. J. Crane and D. C. Pack.

The substance of Chapter VI has been published in the second part of the above paper under the sub-title: "The mixing of two uniform streams," by L. J. Crane.

Chapters VIII and IX consist in the main of original work, which has not yet been published.

I wish to thank Professor Pack for introducing me to the subject and for his constant encouragement and advice.

CONTENTS

D E C L A R A T I O N

I certify that Mr L. J. Crane has fulfilled the conditions of the Ordinance and Regulations for the presentation of the following thesis.

D E C L A R A T I O N

I hereby declare that the following thesis is a record of original work, that it has been composed by me, and that it has not been accepted for any other degree.

CHAPTER I

INTRODUCTION

The motion of a "classical" fluid is described by the Navier-Stokes equations. By a classical fluid is meant a fluid which is continuous, isotropic and whose stress-tensor is related linearly to its strain tensor. Most fluids of practical interest satisfy these requirements to a good degree of approximation. The Navier-Stokes equations are partial differential equations and non-linear in form, so that it is not surprising that few exact solutions of them have been found. Systematic methods of solution of these equations are usually based on two approximations. The first is to neglect the effect of viscosity on the flow. When this approximation is made the Navier-Stokes equations reduce to a set of equations which can be solved by the methods of potential theory. The regions in which this approximation is valid are thus called regions of potential flow.

The approximation does not hold in those regions where the velocity gradients are large. An example of such a region is the thin layer of fluid which surrounds a solid body immersed in a fluid; another example is the "mixing" region of a jet or a wake. In these regions the second type of approximation is used to simplify the Navier-Stokes equations. This simplification was first made by Prandtl in 1904 and is known as the "method of the boundary layer", because its first applications were to the flow of fluid over the boundaries of solid bodies. The flow in the boundary layer is characterised by a non-dimensional constant known as the Reynolds number. This number is defined for a given flow as the product of a typical linear dimension and a typical velocity divided by the kinematic viscosity of the fluid.

When the Reynolds number is increased beyond a certain critical value (which depends on the type of flow) the motion generally becomes unstable and the phenomenon of turbulence sets in. When the turbulence is fully developed the velocity at a point in the fluid is subject to random fluctuations. After the onset of turbulence the shearing stresses in the fluid are very much increased; in some cases they may be several hundred times greater than they were before the motion became turbulent. This is responsible for an important property of turbulent motion, namely the greatly increased rate at which the characteristics of the flow, such as temperature, inside the boundary layer are dissipated or "mixed" into the surroundings when the motion becomes turbulent.

Prior to the last few decades, fluid dynamics was, in the main, the study of the flow of incompressible fluids. In the last twenty years or so, in which the speed of aircraft has approached and finally moved beyond the speed of sound, the study of the motion of compressible fluids has become important. When the speed of the fluid is of the same order of magnitude as the local speed of sound the approximation that the fluid is of constant density is no longer valid. Differences in the density of a fluid in motion may occur for other reasons than sonic or supersonic speeds, for example in a fluid in which there are differences in temperature, or in an inhomogeneous fluid which consists of a mixture of components the relative proportions of which vary from point to point.

The work of this thesis is a study of the effect that differences in the density of the fluid (resulting from the three main causes given above) have on the properties of the mixing regions of jets.

At this point a description of a jet seems appropriate. A jet of fluid, issuing from an orifice, consists of a central "potential" core surrounded by a mixing region. This mixing region separates the core from the surrounding atmosphere. The width of the central core of the jet diminishes with increasing distance (measured along the axis of the jet) from the orifice till finally it vanishes. This decrease in the width of the central core takes place because of the erosive effect of viscosity (or eddy viscosity when the motion is turbulent) on the surface of the core. The mixing region of a jet is conveniently divided into three sections. The first is the mixing region on the surface of the central core; the second starts at the point on the axis of the jet at which the central core vanishes; the third section, which can be considered as a subdivision of the second, is at such a large distance from the orifice that the size of the orifice can be neglected. Typical orders of magnitude of these regions for a turbulent jet are as follows: the central core exists for about five diameters from the orifice; and the third region starts at about ten diameters from the orifice. Three important cases of jet mixing are amenable to mathematical analysis.

These are: the flow far from the orifice of a plane or a round jet and the mixing region on the surface of the core of a plane jet near the orifice. This last problem may be considered, if the distance from the orifice is small compared with the width of the orifice, as the mixing of two semi-infinite streams.

These three problems were first studied in 1926 by Tollmien. Tollmien considered the flow in the mixing regions to be turbulent and the fluid to be incompressible. In 1933 Schlichting published solutions of the problems of the laminar flow of an incompressible fluid in the mixing regions far from the orifice of round and plane jets. (Schlichting used a numerical method in his treatment of the plane jet; it was later shown by Bickley in 1937 that this problem had an analytical solution.) Subsequent to Tollmien's work on turbulent jets in 1926, improved theories of turbulence were developed. It was in the light of one of them that Görtler, in 1924, reconsidered Tollmien's three problems, again for an incompressible fluid. In 1951 Squire published an exact solution of the Navier-Stokes equations for an incompressible fluid. This solution he identified with a round jet issuing from a point source.

The earliest study of the mixing regions of jets of compressible fluid was made by Abramovitch in 1939. Abramovitch considered the case of the turbulent mixing of two semi-infinite streams when (i) there is a large temperature difference between the streams and (ii) when the velocities of the streams are comparable with the local speed of sound. The first significant advance in the theory of the mixing regions of laminar jets of compressible fluid was made by Howarth (1948) and Illingworth (1949). These authors applied a certain transformation of coordinates to solve the problems of the plane jet far from the orifice and the flow in the mixing region between two semi-infinite streams, when the velocity of the fluid is of the same order of magnitude as the local speed of sound. Pai (1949) adopted a different approach to the study of the mixing regions of jets. He reduced the boundary layer equations to a form which resembled that of the equation of heat conduction, for the case in which the exit velocity of the jet is slightly greater than the velocity of the surrounding fluid. Pai applied his simplified equation to problems in the flow of high speed jets and also to the flow of jets of inhomogeneous fluid. This latter type of problem was first attempted by Chou in 1947. Unfortunately Chou's work is spoiled by certain errors in his analysis. Lock in 1951 published a solution of the flow in the boundary layer between two semi-infinite streams of different fluids. He assumed that the streams did not mix at the interface.

In 1954 Pack used the Rayleigh-Jansen method to solve the problem of the mixing of a round laminar jet of high speed fluid at large distances from the orifice. This method is the basis of the work of this thesis. The method is to expand the stream function in an asymptotic series. The first term of this series corresponds to the stream function when the fluid is incompressible. The series is developed in powers of a small parameter whose magnitude depends on the density differences in the fluid. Only the second term of this series is found explicitly. This term gives the first order effect that changes in density have on the flow.

The plan of the thesis will now be given. In Chapters II and III, respectively, the equations of motion of the fluid in laminar and turbulent boundary layers are derived. When Reichardt's theory of turbulence is used the equations of the motion of the fluid in laminar and turbulent boundary layers may be combined in one equation. This result is given in Chapter IV. This combined equation is solved to give the flow far from the orifice in a plane jet of high speed fluid (which may be either laminar or turbulent) in Chapter V. In Chapter VI the mixing of two semi-infinite streams which differ in temperature and whose velocities are comparable with the local speed of sound is considered. The equations of the flow of jets of inhomogeneous fluids are developed in Chapter VII; once again the equations of the (physically distinct) cases of laminar and turbulent motion are reduced to a combined set. Chapters VIII and IX deal with the flow at large distances from the orifice of a plane jet and a round jet respectively when these jets issue into a medium which consists of a different gas. The last part of Chapter IX, namely, Appendix II, deals with the flow in a wake of incompressible fluid. In all the examples considered, with one exception, the solutions are given in analytical form.

CHAPTER II

LAMINAR BOUNDARY LAYER EQUATIONS OF A COMPRESSIBLE FLUID

The starting point of all mathematical investigations in the classical dynamics of a viscous fluid is the set of Navier-Stokes equations. These equations connect the velocities of elements of the fluid with the external forces (if any) acting on the fluid, and take into consideration the macroscopic properties of the fluid, such as pressure and density.

During the nineteenth century there was developed a highly mathematical theory based on an idealisation of the Navier-Stokes equations in which viscosity was neglected. Great as were the successes of this theory, it still left certain phenomena unexplained, and also created certain paradoxes. In the main its great defect was its incapacity to explain drag. The cause of drag was certainly known, at the time, to be the presence of viscosity. The reason why the effect of viscosity was not investigated more fully by mathematical methods during the nineteenth century lies in the intractible nature of the Navier-Stokes equations, which are second order and non-linear.

Then in 1904 Prandtl published his theory of the boundary layer. By the use of this theory it became possible to take into account many of the most important viscous effects. In this theory the flow over surfaces is split up into two regions. The first region, adjacent to the surface, extends only a very small distance into the main stream; it is characterised by a large velocity gradient normal to the surface. In this region it is possible to approximate to the Navier-Stokes equations by the much simpler Boundary-Layer equations. The other region is the space outside the boundary layer. In this region it is a valid approximation to neglect the viscous terms in the Navier-Stokes equations; thus the classical inviscid theory may be applied here. By the use of this combination of boundary layer and classical inviscid theory it became possible to solve many hitherto unexplained problems.

The last twenty years have seen great increases in the speeds of aerodynamic flows occurring in practice. In the mathematical treatment of these high speed flows it is no longer possible to regard the fluid as being incompressible. Compressibility effects in these flows become important when the speeds occurring are comparable with the local speed of sound. The method for solving problems in compressible flow is similar to that employed in incompressible flow; the flow field is split into a thin boundary layer and a potential region (in which the equations of an ideal fluid are a good approximation to the Navier-Stokes equations).

It is to be noted that the term boundary layer may be used to describe certain types of flow in the absence of solid boundaries ("free flows") such as jets and wakes. This is because a boundary layer flow is defined to be a hydrodynamic motion whose velocity [redacted] is mainly in one direction and whose velocity gradient normal to that direction is [redacted] large. Thus a flow may be described as of boundary layer type even if there are no solid boundaries present, provided that the requirements of the above definition are met.

In this chapter the boundary layer equations for steady "free flows" in which compressibility is important are derived. In particular only the equations of flow in two dimensions are derived; the axially symmetric equations are just stated. Since, in the boundary layer, conversion of mechanical energy into heat is comparable with the change in the internal energy of the fluid due to the effect of high speeds it is necessary to form a boundary layer equation of energy. This equation expresses in mathematical terms the law of conservation of energy.

DERIVATION OF THE TWO DIMENSIONAL BOUNDARY LAYER EQUATIONS FOR COMPRESSIBLE FLOW

The Navier-Stokes equations for a two-dimensional viscous compressible fluid are:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} \tag{1}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} \tag{2}$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{3}$$

where

$$\tau_{11} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\tau_{12} = \tau_{21} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

and

$$\tau_{22} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

In these equations x and y are rectangular Cartesian coordinates; u and v are the velocity components with respect to the (x, y) axes; p, ρ, μ are respectively the pressure, density, and coefficient of viscosity of the fluid; and T_{ij} are the components of the stress tensor. Equations (1) and (2) express the consequence of Newton's second law - that the total rate of change of momentum of an element of the fluid is equal to the resultant of the forces acting on it. Equation (3), the continuity equation, puts into mathematical form the fact that matter is conserved.

Let it be assumed that the motion is steady, then $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial \rho}{\partial t} = 0$. The boundary layer postulates are that in a viscous fluid the change in the main velocity of flow U, in the direction normal to the main flow, occurs in a very narrow layer of width δ; and that a variation in velocity of comparable amount in the main direction of flow takes place in a very much longer distance denoted by L. Thus the boundary layer postulate is that $\delta << L$; and the boundary layer equations are obtained by neglecting the terms of order δ/L and higher in the Navier-Stokes equations.

The orders of magnitude of the terms in (1), (2) and (3) are:

$u \sim U$, (The main direction of flow is taken to be in the direction of x increasing.),

$v \sim \frac{U\delta}{L}$ (This follows from equation (3)),

$u_x \sim \frac{U}{L}$, $v_x \sim \frac{U\delta}{L}$, $u_{xy} \sim \frac{U}{\delta L}$, $v_{xy} \sim \frac{U}{L^2}$,

$u_y \sim \frac{U}{\delta}$, $v_y \sim \frac{U}{L}$, $u_{yy} \sim \frac{U}{\delta^2}$, $v_{yy} \sim \frac{U}{\delta L}$,

$u_{xx} \sim \frac{U}{L^2}$ and $v_{xx} \sim \frac{U\delta}{L^3}$

where the suffices denote partial differential coefficients.

Thus the orders of magnitude of the terms on the left of equation (1) are

$$\rho u u_x \sim \rho v u_y \sim \rho U^2/L .$$

The highest order term on the right hand side of equation (1), putting aside $-\frac{\partial p}{\partial x}$ for the moment, is $\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$ which is of order $\frac{\mu U}{\delta^2}$; the remaining terms of this equation are of higher order. Thus neglecting these higher order terms equation (1) becomes, for steady flow,

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right). \tag{4}$$

In order that the motion should be affected by viscosity it is necessary that the term in which it occurs be of magnitude comparable with the other terms in the equation. Thus $\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \sim \rho\frac{U^2}{L}$, hence $\nu = \frac{\mu}{\rho} \sim \frac{U\delta^2}{L}$, or $\frac{\delta}{L} \sim \frac{\nu}{U\delta}$. Now the Reynold's number R_e , of the flow may be defined as $\frac{UL}{\nu}$; thus $\frac{\delta^2}{L^2} \sim \frac{1}{R_e}$.

When the orders of magnitude of the terms occurring in equation (2) are evaluated it is seen that (with the possible exception of $-\frac{\partial p}{\partial y}$) they are all of lower order than the terms on the left side of (4). In fact their order of magnitude may be expressed as $\frac{\delta}{L}\,O\left(\rho u\frac{\partial u}{\partial x}\right)$. Thus $\frac{\partial p}{\partial y} \sim \frac{\delta}{L}\,O\left(\rho u\frac{\partial u}{\partial x}\right)$, and equation (2) becomes $\frac{\partial p}{\partial y} = 0$. The order of magnitude of $\frac{\partial p}{\partial x}$ may now be obtained; for $p \sim f(x) + \frac{\delta^2}{L}\,O\left(\rho u\frac{\partial u}{\partial x}\right)$ when $\frac{\partial p}{\partial y}$ is integrated with respect to y . ($f(x)$ is an arbitrary function.) Now on the boundaries of the flow $\rho u\frac{\partial u}{\partial x}$ vanishes and $p \sim f(x)$ there. In free type flows in which the pressure is constant on the boundaries p = constant, hence $f(x)$ is constant. Thus p = constant + $\frac{\delta^2}{L} O\left(\rho u \frac{\partial u}{\partial x}\right)$ and therefore $\frac{\partial p}{\partial x} \sim \frac{\delta^2}{L^2} O\left(\rho u \frac{\partial u}{\partial x}\right)$. Thus $\frac{\partial p}{\partial x}$ may be neglected in equation (4), to give:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{5}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{6}$$

together with p = constant

as the equations of motion of the boundary layer for steady compressible flow in the absence of solid boundaries.

THE BOUNDARY LAYER EQUATION OF ENERGY

Equations (5) and (6) suffice to determine the motion when the fluid is incompressible; because then μ and ρ are constants. When the fluid is compressible μ and ρ are variables depending upon the pressure p and temperature T. For a complete solution of a problem in compressible flow the variations of these quantities have to be taken into account. Now in free type flows pressure may be taken to be constant; thus μ and ρ are functions of T the absolute temperature alone. μ is related to T by an empirical law which is discussed at the end of this chapter; and ρ is proportional to the reciprocal of the absolute temperature for a perfect gas. Thus the independent variables are u, v and T. Since there are so far only two equations of motion, ((5), (6)) a third equation is necessary to determine the flow completely. This third equation is the equation of energy; this is for steady flow in two dimensions:

$$\rho u \frac{\partial C_p T}{\partial x} + \rho v \frac{\partial C_p T}{\partial y} - \frac{p}{\rho}\left[u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y}\right]$$

$$= \tau_{11}\frac{\partial u}{\partial x} + \tau_{12}\frac{\partial u}{\partial y} + \tau_{21}\frac{\partial v}{\partial x} + \tau_{22}\frac{\partial v}{\partial y} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right). \quad (7)$$

When this equation is multiplied by an element of volume it states the fact that the total energy contained in this element of volume is constant. The four main factors affecting the energy are: the rate of convection of internal energy of the gas into the volume given by the first two terms on the left side of the equation: the rate at which work is being done on the volume by the external pressure, represented by the last two terms on the left side of the equation: the rate of loss of energy through having to overcome the viscous stresses - this corresponds to the terms in τ_{ij} : and lastly the terms $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$ which give the rate at which heat is being conducted out of the volume. (k is the coefficient of heat conduction of the fluid which depends on the absolute temperature.) The energy equation (7) referred to above is too complicated to be useful in any problem of practical interest. For those regions in which the boundary layer assumptions hold it may be simplified by neglecting terms of order δ/L or higher (δ and L are defined earlier in this chapter). It is assumed that in the boundary layer T varies in much the same way as the main velocity of flow. Thus if x is taken to be the main direction of flow:

$$\frac{\partial T}{\partial x} \Big/ \frac{\partial T}{\partial y} \sim \frac{\delta}{L}.$$

When the orders of magnitude of the terms whose coefficients are τ_{ij} are considered it is seen that their dominant part is the element of $\tau_{12}\frac{\partial u}{\partial y}$, $\mu\left(\frac{\partial u}{\partial y}\right)^2$ — the other terms in τ_{ij} being of higher order. Next the term $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$ may be neglected in comparison with $\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$. Thus the right side of (7) may be approximated by

$$\mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right).$$

Turn now to the last two terms on the left side of the energy equation. The order of magnitude of these terms for free flows may be shown to be $\frac{\delta^2}{L^2}\,O\left(\mu\left(\frac{\partial u}{\partial y}\right)^2\right)$; they are therefore neglected in comparison with the right side of the energy equation. Finally, the first two terms, namely $\rho u\frac{\partial c_pT}{\partial x} + \rho v\frac{\partial c_pT}{\partial y}$, may be shown to be of equal magnitude.

Thus the energy equation simplifies to

$$\rho u\frac{\partial c_pT}{\partial x} + \rho v\frac{\partial c_pT}{\partial y} = \mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right). \qquad (8)$$

It is necessary, now, to explain the <u>relative</u> importance of the parts of equation (8). These parts are: the terms due to convection (on the left side of (8)), the term due to viscous stress $\mu\left(\frac{\partial u}{\partial y}\right)^2$ and the term due to the molecular conduction of heat $\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$. The convection terms on the left side of (8) are of order $\frac{\rho u c_p \Delta T}{L}$ where ΔT is the order of magnitude of the variation in T in the flow; the viscous term $\mu\left(\frac{\partial u}{\partial y}\right)^2$ is of order $\mu\frac{U^2}{\delta^2}$; and the heat conduction term $\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$ is of order $\frac{k\,\Delta T}{\delta^2}$. Introduce now the dimensionless quantities:

M the local Mach number defined as the ratio of the velocity at a point in the flow to the local speed of sound; and σ the Prandtl number defined as $\frac{c_p\mu}{k}$. (In the flows to be considered σ is about unity.) Then these three quantities may be shown to have orders of magnitude in the ratios $\Delta T : M^2 T : \Delta T$ respectively; when the multiplicative effect of terms of order unity such as σ and the ratio of the specific heats ~~are~~ is neglected. Thus the term $\mu\left(\frac{\partial u}{\partial y}\right)^2$ is of importance only when M^2 is of the same order as $\Delta T/T$. This will be the case when temperature differences are brought about by changes in velocity. When flows at low Mach numbers are considered in which there are large sources of heat then the viscous term $\mu\left(\frac{\partial u}{\partial y}\right)^2$ may be omitted and the energy equation reduces to one of diffusion type.

SPECIAL RESULTS

EQUATION OF STATE

As mentioned above, only perfect gases are considered. Perfect gases have the simple equations of state $p=\rho RT$, where R is the gas constant. Most gases behave like perfect gases when conditions are far from the liquefaction point of the gas concerned. The gases in ordinary use such as air which require extremely low temperatures before entering the liquid state may therefore be treated as though they were perfect.

VARIATION IN DENSITY

Since the pressure is constant over the whole boundary layer in jet type flow, the result $\rho \propto 1/T$ follows immediately from the equation of state.

VARIATION IN VISCOSITY

Sutherland's empirical formula for the variation of viscosity with temperature is

$$\mu = \mu\left[\text{taken at } 273^{\circ}K\right] \cdot \left(\frac{T}{273}\right)^{1/2} \cdot \frac{1 + T_c/273}{1 + T_c/T}$$

where T is measured in $^{\circ}K$, and T_c is an absolute constant.

This law is too complicated for general use, and it has been modified by Von Karman and Tsien to the form:

$$\mu = \mu\left[\text{taken at } 273^{\circ}K\right] \cdot \left(\frac{T}{273}\right)^{1+\varepsilon}$$

where ε is a constant which is approximately $-0{\cdot}24$ for air. This law will be applied in the analysis of later chapters in the form:

$\mu^{*} = \mu/\mu_0 = (T^{*})^{n}$ where $T^{*} = T/T_0$, T_0 and μ_0 are reference temperatures and viscosity respectively and $n = 1+\varepsilon$.

VARIATION IN THE PRANDTL NUMBER σ

The dimensionless constant σ, which is defined as $C_p\mu/k$, determines the ratio of heat generated by viscosity, to heat conducted by molecular action in an element of volume through which fluid is in motion. Other things being equal, the larger σ is, the smaller the amount of heat conducted and vice versa. Its importance as a parameter lies mainly in heat transfer theory where temperature profiles are of interest. From the standpoint of velocity profiles if σ is taken to be unity, instead of its actual value for air of about 0·7, very little difference is made. This was brought out in the case of boundary layer flow over a flat plate by Emmons and Brainerd (1941). They solved the equations of motion using $\sigma = 0.7$ and $\sigma = 1$, and showed that the difference in velocity profiles was at most of the order of one or two per cent.. It would seem reasonable therefore to assume $\sigma = 1$ in the case of jets, since the velocity profiles are of primary interest. This approximation of taking σ as unity is of some importance because when the Prandtl number is unity it is possible to replace the boundary layer energy equation for jets by a simple algebraic equation which relates velocity to temperature. This result, due to Crocco, will be proved in Chapter IV.

SUMMARY OF THE EQUATIONS OF STEADY MOTION OF COMPRESSIBLE VISCOUS JETS

TWO DIMENSIONAL CASE

Momentum: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

Continuity: $$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

Energy: $$\rho u \frac{\partial}{\partial x}(C_p T) + \rho v \frac{\partial}{\partial y}(C_p T) = \mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)$$

AXIALLY SYMMETRIC CASE

Here the coordinates are cylindrical (x , r), with velocity components (u , v). The direction of increasing gives the main direction of flow.

Momentum: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right)$$

Continuity: $$\frac{\partial}{\partial x}(\rho u r) + \frac{\partial}{\partial r}(\rho v r) = 0$$

Energy: $\rho u \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial r}(c_p T) = \mu\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial}{\partial r}\left(k r \frac{\partial T}{\partial r}\right)$

Also $\rho^* = \rho/\rho_0 = (T/T_0)^{-1} = (T^*)^{-1}$; $\mu^* = \mu/\mu_0 = (T^*)^{1+\varepsilon}$,

and p = constant, where ρ_0, μ_0 and T_0 are constant reference density, viscosity and temperature of the gas.

CHAPTER III

THE EQUATIONS OF MOTION FOR THE FLOW IN A TURBULENT BOUNDARY LAYER

The last chapter dealt with laminar boundary layer flows. This chapter is concerned with the derivation of the equations of motion of a turbulent boundary layer of compressible fluid in the absence of solid boundaries. A turbulent flow is one in which the parameters of the system such as velocity, pressure, etc., fluctuate in a random manner about their mean values. A characteristic of turbulent motions is that the shearing stress and heat transfer developed are very much larger than their counterparts in laminar flows. Because of the extremely complicated nature of turbulence, theoretical treatment of anisotropic flows like jets has depended on certain ad-hoc hypotheses being made about the nature of the shearing stresses. So far no rigorous statistical mechanical treatment, which would seem to be the most logical approach, has been attempted for flows other than isotropic.

Turbulence is a phenomenon that usually occurs at high Reynolds numbers; for example boundary layer flow over a flat plate becomes turbulent for a Reynolds number between $3 - 5 \times 10^5$. (The Reynolds number (R_e) is defined as the product of a typical velocity and a typical length in the flow divided by the kinematic viscosity μ/ρ.) The connection between the onset of turbulence and a critical Reynolds number was first noted by Reynolds in 1883. He carried out experiments on the flow along pipes.

When he introduced a thread of coloured dye into the flow, he observed that this thread became wavy just after a certain critical Reynolds number had been attained, and that on increasing the Reynolds number still further the motion of the thread became no longer discernible - thus indicating turbulence. He also showed that, if a disturbance were introduced into the flow when the Reynolds number was below the critical value, the turbulence decayed and the flow subsequently became laminar.

Theoretical treatments of problems in turbulent flows may be divided into two groups. The first, an empirical method based on analogies with laminar viscous flows was introduced by Boussinesq in 1877. He put forward the hypothesis that the shearing stresses set up by turbulence behave in the same way as viscous stresses, except that the coefficient, which corresponds to μ in his theory, is dependent upon the spatial coordinates. This idea was extended and applied to practical problems by the use of certain 'mixing length' theories. The mixing length theory, introduced by Prandtl in 1914, uses the hypothesis that the properties of the fluid are convected by the random movements of lumps of the fluid which disperse after moving through a certain characteristic distance. This distance Prandtl called the 'mixing length'. The mixing length has some resemblance to the mean free path in the kinetic theory of gases. By the use of Prandtl's theory and its later developments by Tollmien, Von Karman and Taylor many problems in turbulent flow of engineering interest were solved. A recent development of Boussinesq's idea is the work of Reichardt (1941). Reichardt dispensed with mixing length theories and determined the shear stress coefficient (or coefficient of eddy kinematic viscosities) by appeal to experiment, or to laws of similarity when the mean flow was known to be sensibly similar. It is on Reichardt's theory that the work of this thesis is based.

The second group of theories has its basis in statistical mechanics. Although logically more satisfying than the empirical theories mentioned above, it has proved to be extremely difficult to apply to any problems other than the simplest.

The study of turbulent flows in the absence of solid boundaries has one factor which greatly simplifies the work. In turbulent flow over a solid body, there is a thin laminar boundary layer adjacent to the surface of the body, and superposed on this layer is the turbulent flow. This fact greatly complicates turbulent flows over bodies, but does not arise in free flows. In free flows, an example of which is the jet, turbulence is eventually fully developed everywhere except perhaps at the edges of the flow. This partial breakdown of turbulence at the edges, which is called intermittency, does not greatly affect the motion in the region of greatest interest - viz. the central core of the flow.

TWO DIMENSIONAL EQUATIONS OF TURBULENT MOTION

The motion of the turbulent boundary layer is assumed to satisfy the Navier-Stokes equations, together with the equation of energy at any given instant.

That is, in the case of two dimensional flow the motion satisfies the equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \text{terms due to viscous stresses} \qquad (2)$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \text{terms due to viscous stresses} \qquad (3)$$

$$\rho\frac{\partial c_pT}{\partial x} + \rho u\frac{\partial c_pT}{\partial x} + \rho v\frac{\partial c_pT}{\partial y} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \qquad (4)$$

+ terms due to viscous stresses and molecular heat conduction.

On account of the extremely complex nature of turbulent flows it is not possible to deduce their nature directly from these equations. Instead, equations (1) - (4) are replaced by their statistically averaged values. By a statistical average of a quantity such as u , is meant the average value of u taken at a given point and at a given time over all possible turbulent systems satisfying the same boundary conditions. This statistical average is identified in practice with the time average value for turbulence in which the mean motion is constant in time.

The quantities u , ρ etc., are split into two parts, one being the statistical average, in the sense defined above, and denoted by $\bar{u}$, $\bar{\rho}$ etc.; the other is the fluctuation of the quantity from the statistical mean and is denoted by u', ρ' etc..

$$\left.\begin{array}{l} \text{Thus } u=\bar{u}+u' \;, v=\bar{v}+v' \;, \rho=\bar{\rho}+\rho' \;, T=\bar{T}+T' \\ p=\bar{p}+p' \;, \rho u=\overline{\rho u}+(\rho u)' \;\text{ and }\; \rho v=\overline{\rho v}+(\rho v)' \end{array}\right] \quad (5)$$

Prior to the insertion of these expressions into the equations of motion, the equations are simplified by neglecting the terms due to viscous stresses in (2), (3), and (4) and also the terms in (4) due to molecular heat conduction. This is a valid approximation because the shear stresses and the heat transfer developed in turbulent flow are very much greater than the corresponding quantities due solely to molecular motion in the regions where the turbulence is fully developed. Thus, when the expressions (5) for u , ρ etc. are inserted in equation (1):

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial\rho'}{\partial t} + \frac{\partial}{\partial x}\left(\overline{\rho u} + (\rho u)'\right) + \frac{\partial}{\partial y}\left(\overline{\rho v} + (\rho v)'\right) = 0 \; .$$

When the statistical mean of this equation is taken, and the mean flow is stationary

$$\frac{\partial}{\partial x}\left(\overline{\rho u}\right) + \frac{\partial}{\partial y}\left(\overline{\rho v}\right) = 0 \qquad (6)$$

- the quantities $\overline{\frac{\partial\rho'}{\partial t}}$, $\overline{\frac{\partial}{\partial x}(\rho u)'}$ and $\overline{\frac{\partial}{\partial y}(\rho v)'}$ being identically zero by definition. Equation (6) is the turbulent form of the equation of continuity.

When the viscous stresses are neglected, and the instantaneous equation of continuity, i.e. equation (1), is used, equations (2) and (3) may be written

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = -\frac{\partial p}{\partial x}$$

and $$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) = -\frac{\partial p}{\partial y}.$$

When the expressions (5) are used the above equations become:

$$\frac{\partial}{\partial t}(\overline{\rho u}) + \frac{\partial}{\partial t}(\rho u)' + \frac{\partial}{\partial x}\left[(\overline{\rho u} + (\rho u)')(\overline{u} + u')\right]$$
$$+ \frac{\partial}{\partial y}\left[(\overline{\rho v} + (\rho v)')(\overline{u} + u')\right] = -\frac{\partial \overline{p}}{\partial x} - \frac{\partial p'}{\partial x}$$

and

$$\frac{\partial}{\partial t}(\overline{\rho v}) + \frac{\partial}{\partial t}(\rho v)' + \frac{\partial}{\partial x}\left[(\overline{\rho u} + (\rho u)')(\overline{v} + v')\right]$$
$$+ \frac{\partial}{\partial y}\left[(\overline{\rho v} + (\rho v)')(\overline{v} + v')\right] = -\frac{\partial \overline{p}}{\partial y} - \frac{\partial p'}{\partial y}.$$

The statistical mean gives

$$\frac{\partial \overline{\rho u}}{\partial t} + \frac{\partial}{\partial x}(\overline{\rho u}.\overline{u}) + \frac{\partial}{\partial y}(\overline{\rho v}.\overline{u}) = -\frac{\partial \overline{p}}{\partial x} - \frac{\partial}{\partial x}\overline{(\rho u)'u'} - \frac{\partial}{\partial y}\overline{(\rho v)'u'} \qquad (7)$$

and

$$\frac{\partial \overline{\rho v}}{\partial t} + \frac{\partial}{\partial x}(\overline{\rho u}.\overline{v}) + \frac{\partial}{\partial y}(\overline{\rho v}.\overline{v}) = -\frac{\partial \overline{p}}{\partial y} - \frac{\partial}{\partial x}\overline{(\rho u)'v'} + \frac{\partial}{\partial y}\overline{(\rho v)'v'} \qquad (8)$$

Equations (7) and (8) are the turbulent equations of momentum in the x and y directions. The double correlation terms $\overline{(\rho u)'u'}$, etc. on the right side of (7) and (8) are the components of turbulent stress.

If the variation in density is small then the fluctuation in ρ, i.e. ρ', may be neglected in comparison with $\overline{\rho}$. When this approximation is made the equations of continuity and momentum become:

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x}(\bar{\rho}\bar{u}) + \frac{\partial}{\partial y}(\bar{\rho}\bar{v}) = 0 \qquad (9)$$

$$\bar{\rho}\frac{\partial\bar{u}}{\partial t} + \bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial\bar{u}}{\partial y} = -\frac{\partial\bar{p}}{\partial x} - \frac{\partial}{\partial x}\left[\bar{\rho}\,\overline{(u')^2}\right] - \frac{\partial}{\partial y}\left[\bar{\rho}\,\overline{u'v'}\right] \qquad (10)$$

and $$\bar{\rho}\frac{\partial\bar{v}}{\partial t} + \bar{\rho}\bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial\bar{v}}{\partial y} = -\frac{\partial\bar{p}}{\partial y} - \frac{\partial}{\partial x}\left[\bar{\rho}\,\overline{(u'v')}\right] - \frac{\partial}{\partial y}\left[\bar{\rho}\,\overline{(v')^2}\right] \qquad (11)$$

Equations (9), (10) and (11) may be interpreted as the flow of a fluid whose velocity, density and pressure are given by the mean values of these quantities in the turbulent motion. This fictitious fluid is acted upon by the internal shearing stresses

$$\begin{pmatrix} -\bar{\rho}\,\overline{(u')^2}, & -\bar{\rho}\,\overline{u'v'} \\ -\bar{\rho}\,\overline{(u'v')}, & -\bar{\rho}\,\overline{(v')^2} \end{pmatrix}.$$

These components of the shearing stress tensor are assumed to behave in a way analogous to that of viscous stresses, i.e. the shearing stress tensor is written:

$$\begin{pmatrix} -\bar{\rho}\,\overline{(u')^2}, & -\bar{\rho}\,\overline{(u'v')} \\ -\bar{\rho}\,\overline{(u'v')}, & -\bar{\rho}\,\overline{(v')^2} \end{pmatrix} = \bar{\rho}\varepsilon\begin{pmatrix} \frac{\partial\bar{u}}{\partial x}, & \frac{\partial\bar{v}}{\partial x} \\ \frac{\partial\bar{u}}{\partial y}, & \frac{\partial\bar{v}}{\partial y} \end{pmatrix} + \bar{\rho}\varepsilon\begin{pmatrix} \frac{\partial\bar{u}}{\partial x}, & \frac{\partial\bar{u}}{\partial y} \\ \frac{\partial\bar{v}}{\partial x}, & \frac{\partial\bar{v}}{\partial y} \end{pmatrix} \qquad (12)$$

where ε is assumed to be a function of position only.

The main physical assumption used in deriving (12) is that the shearing stress acting as a surface is proportional to the normal velocity gradient across the surface.

When the expression (12) is used equations (10) and (11) become:

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} + \bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + 2\frac{\partial}{\partial x}\left[\bar{\rho}\varepsilon\frac{\partial \bar{u}}{\partial x}\right] + \frac{\partial}{\partial y}\left[\bar{\rho}\varepsilon\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)\right] \tag{13}$$

$$\bar{\rho}\frac{\partial \bar{v}}{\partial t} + \bar{\rho}\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{\partial \bar{p}}{\partial y} + \frac{\partial}{\partial x}\left[\bar{\rho}\varepsilon\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)\right] + 2\frac{\partial}{\partial y}\left[\bar{\rho}\varepsilon\frac{\partial \bar{v}}{\partial y}\right] \tag{14}$$

Now the flows to be considered are boundary layer in type with mean velocities and pressures which are constant in time.

Thus (13) and (14) may be approximated by the equations

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left[\bar{\rho}\varepsilon\frac{\partial \bar{u}}{\partial y}\right]$$

$$\text{and} \quad \frac{\partial \bar{p}}{\partial y} = 0$$

when the boundary layer assumptions (viz: that the main velocity is in the x direction and the main velocity gradient is in the y direction) are applied.

For flows of jet type in which the mean exhaust pressure of the jet is the same as that of the surrounding atmosphere $\frac{\partial \bar{p}}{\partial x}$ may be shown to be very small in the same way as in the case of the laminar jet.

Thus the momentum equation for a two dimensional jet or wake is

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\left(\bar{\rho}\varepsilon\frac{\partial \bar{u}}{\partial y}\right) \tag{15}$$

The mean turbulent equation of energy is derived from equation (4) by a process similar to that used in obtaining equations (9) - (11). The heat transfer terms, which involve terms like $\overline{u'T'}$ etc., in this mean energy equation are then replased by the transformation below-

$$\left(-\bar{\rho}\,C_p\,\overline{u'T'},\ -\bar{\rho}\,C_p\,\overline{v'T'}\right) = \varepsilon_T\left(\frac{\partial \bar{T}}{\partial x},\ \frac{\partial \bar{T}}{\partial y}\right),$$

where ε_T, the eddy coefficient of ~~heat~~ conduction, is a function of the spatial coordinates. Thus a stage has now been reached in which the mean energy equation is in terms of $\bar{u}$, $\bar{v}$, $\bar{T}$, $\bar{\rho}$ and their derivatives only. The boundary layer assumptions are now applied with the additional assumption that $\frac{\partial \bar{T}}{\partial x} << \frac{\partial \bar{T}}{\partial y}$. When these approximations are made the mean equation of energy for the turbulent compressible flow in a boundary layer is found to be:

$$\bar{\rho}\,\bar{u}\,\frac{\partial C_p\bar{T}}{\partial x} + \bar{\rho}\,\bar{v}\,\frac{\partial C_p\bar{T}}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon_T\,\bar{\rho}\,\frac{\partial \bar{T}}{\partial y}\right) + \varepsilon\,\bar{\rho}\left(\frac{\partial \bar{u}}{\partial y}\right)^2. \tag{16}$$

The complete solution of the flow in a turbulent jet in two dimensions therefore satisfies equations (9), (15) and (16). These equations contain, however, two quantities as yet unspecified, namely ε and ε_T. In order to determine these quantities absolutely appeal has to be made to experimental evidence. When certain hypotheses are made this appeal to experiment is reduced to that of fixing a certain scaling factor. Reichardt assumed that ε and ε_T were functions of x alone. Prandtl suggested that $\varepsilon = k\,\frac{U_1 - U_2}{b}$, where U_1 is the maximum and U_2 the minimum velocity at a normal cross-section of the jet and b is the width of the mixing region. When Prandtl's suggestion is used ε may be written $\varepsilon_0\left(\frac{x}{L}\right)^n$ where n is determined from the experimental data on the mean axial velocity, or from a similarity condition on the solution if the flow is known to be similar; L is a constant length, introduced so that the constant ε_0 may have the same dimensions as ε. The constant ε_0 is determined by fitting the theoretical velocity profile to the experimental profile at a given point in the profile.

The constant ε_T is assumed to be related to ε by

$$\varepsilon_T = C_p \, \varepsilon \, .$$

This gives a turbulent Prandtl number of unity. This assumption simplifies the analysis and may be presumed not to affect the mean velocity profile to any great extent. This surmise is based on the fact that in laminar flows, in which the Prandtl number is of order unity, the true velocity profiles do not sensibly deviate from corresponding profiles based on a Prandtl number of unity.

APPENDIX:

The turbulent equations of mean flow for a round compressible jet, whose axis coincides with the x axis, in terms of cylindrical coordinates (x, r) are:

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon \bar{\rho}\, r \frac{\partial \bar{u}}{\partial r}\right],$$

$$\bar{\rho}\bar{u}\frac{\partial C_p\bar{T}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial C_p\bar{T}}{\partial r} = \frac{1}{r}\left[\varepsilon_T \bar{\rho}\, r \frac{\partial \bar{T}}{\partial r}\right] + \bar{\rho}\,\varepsilon\left(\frac{\partial \bar{u}}{\partial r}\right)^2$$

and $\frac{\partial}{\partial x}(\bar{\rho}\bar{u}r) + \frac{\partial}{\partial r}(\bar{\rho}\bar{v}r) = 0 \, .$

SUMMARY

In this chapter the equations of motion of the mean flow of a turbulent jet have been derived. In order to obtain these equations in a suitable mathematical form it has been assumed that turbulent shearing stresses act in an analogous way to molecular (i.e. viscous) shearing stresses. It has further been assumed that the parameters ε (and ε_T) do not change across a section of the jet. These assumptions are justified a posteriori by the agreement of the solutions (which they imply) with experimental evidence. It may be noted that ε (and ε_T) are not in fact found to be constant over a section of the jet - there is some variation at the edges of the jet due to the flow at the edges being only intermittently turbulent. Finally, because fluctuations in ρ have been neglected, equations (9), (15) and (16) are not valid when there are large temperature gradients or when the Mach number is too large.

CHAPTER IV

THE COMBINATION IN ONE OF THE LAMINAR AND TURBULENT BOUNDARY LAYER EQUATIONS

The equations of motion for steady laminar and turbulent compressible jets have been derived in the previous two chapters. In this chapter they are combined in one. Also some useful integrals of the equations are found. The results are obtained in detail in the two dimensional case and merely stated for axially symmetric jets.

TWO DIMENSIONAL MOTION

Subject to the limitation that density changes are small, the laminar equation of continuity and the turbulent equation of continuity for mean flow are the same.

i.e. $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$ (1), in the case of steady flows.

In equation (1) the quantities ρ, u and v are interpreted as the statistical means of the density and velocities if the flow is turbulent; while if the flow is laminar they retain their usual meanings.

Next the momentum equation for both laminar and turbulent jets may be written:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon \rho \frac{\partial u}{\partial y}\right). \tag{2}$$

In laminar flow $\varepsilon = \mu/\rho$, while in turbulent flow ε is the coefficient of eddy kinematic viscosity.

Lastly the energy equation is:

$$\rho u \frac{\partial (C_p T)}{\partial x} + \rho v \frac{\partial (C_p T)}{\partial y} = \frac{\partial}{\partial y}\left(\rho \varepsilon_T \frac{\partial T}{\partial y}\right) + \rho \varepsilon \left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

where $\varepsilon_T = \kappa/\rho$ in the case of laminar flow, κ being the coefficient of heat conduction; when the flow is turbulent ε_T is the eddy coefficient of heat transfer.

THE MOMENTUM INTEGRAL (FOR TWO DIMENSIONAL FLOW)

The boundary conditions on a two dimensional jet where main velocity is in the direction of x are:

on the boundaries of the jet i.e. $y = \pm\infty$: $u = 0$

and on the axis of the jet i.e. $y = 0$: $v = 0$ and $\frac{\partial u}{\partial y} = 0$.

The equation of momentum, i.e. equation (2), may be integrated with respect to y between the limits 0 and y to give

$$\frac{\partial}{\partial x}\int_0^y \rho u^2 \, dy + \left[\rho u v\right]_0^y = \left[\varepsilon \rho \frac{\partial u}{\partial y}\right]_0^y$$

This result is obtained by means of the equation of continuity - equation (1). Let $y \to \infty$ in the above equation, and use the boundary conditions:

then it follows that $\frac{d}{dx}\int_0^\infty \rho u^2 dy = 0$

Thus $\int_0^\infty \rho u^2 dy$ = a constant which does not depend on x. Now $\int_0^\infty \rho u^2 dy$ is just half the momentum flux crossing the section of the jet at x. Thus denoting the total momentum flux by M_0

$$\frac{1}{2}M_0 = \int_0^\infty \rho u^2 dy . \tag{4}$$

THE CROCCO RELATION (FOR TWO DIMENSIONAL FLOW)

This relation is a simple integral of the momentum and energy equations, which is true when the Prandtl number σ has the value unity. It was discovered by Crocco in 1931 for laminar free flows and by Pai in 1949 for turbulent free flows.

The equation of energy may be written

$$\rho u \frac{\partial}{\partial x}\left[C_pT+\tfrac{1}{2}u^2\right] + \rho v\frac{\partial}{\partial y}\left[C_pT+\tfrac{1}{2}u^2\right]$$
$$= \rho\varepsilon\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\rho\varepsilon_T\frac{\partial T}{\partial y}\right) + \rho u^2\frac{\partial u}{\partial x} + \rho uv\frac{\partial u}{\partial y} \qquad (5)$$

In order to obtain this expression the quantity $\rho u^2\frac{\partial u}{\partial x} + \rho uv\frac{\partial u}{\partial y}$ is added to both sides, and the equation of continuity is used to modify the left hand side of (5).

Now the right hand side of (5) is equal to

$$u\frac{\partial}{\partial y}\left[\rho\varepsilon\frac{\partial u}{\partial y}\right] + \rho\varepsilon\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left[\rho\varepsilon_T\frac{\partial T}{\partial y}\right]$$

on using the momentum equation (2).

The above expression may be written:

$$\frac{\partial}{\partial y}\left[\rho\varepsilon\frac{\partial}{\partial y}\left(C_pT+u^2/2\right)\right] + \frac{\partial}{\partial y}\left[\rho\varepsilon_T\frac{\partial T}{\partial y} - \rho\varepsilon\frac{\partial C_pT}{\partial y}\right].$$

Now C_p is a constant and if $\frac{C_p\varepsilon}{\varepsilon_T} = 1$ the second term vanishes and the right side of (5) is just

$$\frac{\partial}{\partial y}\left[\rho\varepsilon\frac{\partial}{\partial y}\left(C_pT+u^2/2\right)\right].$$

Thus, with $P = C_pT + u^2/2$ the equation of energy is now

$$\rho u\frac{\partial P}{\partial x} + \rho v\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\rho\varepsilon\frac{\partial P}{\partial y}\right).$$

When this equation is compared with the momentum equation (2) it is seen that its solution is

$P = A + Bu$ (6), where A and B are arbitrary constants.

This integral satisfies the equations of motion. It may be made to fit the required boundary conditions by taking suitable values of A and B.

Equations (1), (2) and (3) may now be replaced by the set of equations (1), (2) and (6).

THE AXIALLY SYMMETRIC EQUATIONS

It may be shown that the equations of steady motion for laminar and turbulent jets in cylindrical coordinates (,) may be combined in one as:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon \rho r \frac{\partial u}{\partial r}\right],$$

$$\rho u \frac{\partial C_pT}{\partial x} + \rho v \frac{\partial C_pT}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_T \rho r \frac{\partial T}{\partial r}\right] + \rho\varepsilon\left(\frac{\partial u}{\partial r}\right)^2$$

and $\frac{\partial}{\partial x}(\rho u r) + \frac{\partial}{\partial r}(\rho v r) = 0.$

In these equations ε and ε_T are to be interpreted in the same way as in the two dimensional case discussed above.

It may also be shown that the momentum integral is:

$M_0 = \int_0^\infty \rho u^2 r\,dr$, where the constant M_0 is the total momentum flux crossing a section of the jet.

Next when the Prandtl number is unity the Crocco relation takes the same form as in two dimensional flow, i.e.

$$C_pT + \frac{u^2}{2} = A + Bu .$$

CHAPTER V

THE MIXING OF A PLANE JET OF COMPRESSIBLE FLUID FAR FROM THE ORIFICE

Insight into the theory of the flow of a jet of gas issuing from an orifice at full expansion is most conveniently obtained by considering two problems. The first corresponds to the flow at such large distances from the orifice that its finite size may be neglected. This problem is idealised into the flow resulting from a point source (or a line source in two dimensional flow) of momentum whose direction is that of the main flow of the jet. The second problem is the mixing at the interface of two uniform streams which are semi-infinite in extent. The solution of this problem is deferred to Chapter VI. This half-jet problem, as it is called, gives information about the flow in the boundary layer region on the edge of the jet, very near the orifice.

The physically distinct cases of laminar and turbulent flow are treated simultaneously. This is possible because, as is shown in Chapter IV, the equations of laminar and turbulent flow may be combined in one, when a coefficient of eddy kinematic viscosity is introduced into the equations of mean turbulent flow.

The problem in which the fluid is incompressible was first noted by Bickley in 1937 for a laminar jet. In 1942 Görtler solved the corresponding turbulent motion using Reichardt's constant eddy coefficient of kinematic viscosity. In both cases the solutions satisfy the boundary layer equations; these may be shown to be good approximations to the full equations of incompressible motion at large distances from the slit. The method of solution is to assume that the velocity profiles are similar when taken in sections normal to the main direction of flow of the jet. For laminar flow a solution on this basis is completely determined when the total momentum flux and the kinematic viscosity are known. When the flow is turbulent a complete solution is only obtained when in addition to the total momentum flux of the jet, an experimentally determined scaling factor, connecting theoretical and experimental velocity profiles is known.

On the basis of these incompressible solutions an approximate solution of the flow of a jet of compressible fluid at large distances from the orifice may be obtained. To determine this solution in analytical form it is necessary to assume that the Prandtl number of the flow is unity; that the viscosity varies as the n-th power of the absolute temperature and that the stagnation enthalpy of the jet is the same as that of the surrounding gas. The method of solution is to expand the stream function in a power series in squares of a parameter, which is of the same order of magnitude as the local Mach number on the axis of the jet. The method was first used by Pack in 1954 to solve a similar problem in the flow of an axially symmetric jet of compressible fluid. The coefficients of this parameter satisfy certain ordinary linear differential equations. These equations are obtained by substituting the stream function in the equations of motion and equating powers of the parameter.

THE FLOW IN TWO DIMENSIONS OF A JET OF COMPRESSIBLE FLUID AT LARGE DISTANCES FROM THE ORIFICE

EQUATIONS OF MOTION

Let u, v be the velocity components (or mean velocity components in the case of turbulent flow) parallel to Cartesian axes (x, y). Let the origin of coordinates be taken at the orifice (taken to be a source), and take the x axis along the main forward direction of the jet. Let ρ be the density of the gas in the jet and T its absolute temperature.

At large distances from the source in the region of the forward axis of the jet the equation of momentum is the same as for boundary layer flow at constant pressure, i.e.

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\rho\varepsilon\frac{\partial u}{\partial y}\right). \tag{1}$$

In this equation $\varepsilon = \mu/\rho$, the ordinary kinematic coefficient of viscosity when the flow is laminar, while when the flow is turbulent $\varepsilon = \varepsilon(x)$ a coefficient of eddy kinematic viscosity that is assumed, following Reichardt, to be independent of the y coordinate.

The equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0.$$

From this equation it follows that a stream function ψ exists such that

$$\rho u = \frac{\partial \psi}{\partial y} \quad , \quad \rho v = -\frac{\partial \psi}{\partial x} .$$

Let the independent variables be changed from (x, y) to (x, z) where $z = \int_0^y \rho^* \, dy$ and $\rho^* = \rho/\rho_0$, ρ_0 being the density of the fluid on the boundary of the jet. This transformation is due to Howarth, (1948).

Then:

$$\rho u = \left(\frac{\partial \psi}{\partial y}\right)_x = \left(\frac{\partial \psi}{\partial z}\right)_x \left(\frac{\partial z}{\partial y}\right)_x = \rho^* \left(\frac{\partial \psi}{\partial z}\right)_x$$

and it follows that

$$\rho_0 u = \left(\frac{\partial \psi}{\partial z}\right)_x .$$

Now

$$-\rho v = \left(\frac{\partial \psi}{\partial x}\right)_y = \left(\frac{\partial \psi}{\partial x}\right)_z + \left(\frac{\partial \psi}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y .$$

When these expressions are inserted into equation (1), the terms involving $\frac{\partial z}{\partial x}$ cancel out and the momentum equation becomes on simplification

$$\frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial z^2} = \rho_0 \frac{\partial}{\partial z} \left[(\rho^*)^2 \varepsilon \frac{\partial^2 \psi}{\partial z^2} \right]. \tag{2}$$

Now the coefficient of viscosity μ is in general a function of temperature. Its dependence on T is assumed to be of the form

$$\mu^* = \mu/\mu_0 = (T/T_0)^n = (T^*)^n,$$

where μ_0 and T_0 refer to the gas on the boundary of the jet, and n is a constant, which is assumed to be 0·76, according to Karman and Tsien (1938). This law is satisfactory so long as the range of variation in T is not too large. Next, ρ depends in general on temperature and pressure, but since the latter is constant over the boundary layer, the dependence is on T

alone. If the gas is perfect, then it follows that $\rho^* = T^*$ when p is put equal to a constant in the equation of stage of the gas.

The solutions of equation (2) differ in their dependence on x according as they are laminar or turbulent. In order to overcome this lack of symmetry the following device is used.

Write $\varepsilon = \theta e(x)$. In a laminar jet $\theta = \mu/\rho$ and $e(x)$ has the constant value unity. In the case of turbulent flow θ is Reichardts constant exchange coefficient ε_0, and $e(x)$ is an experimentally determined function of x. Change the independent variables from (x, z) to (ζ, z) where

$$\zeta = \int_0^x e(x)\,dx.$$

Then the equation of motion i.e. equation (2) becomes in terms of the new variables (ζ, z)

$$\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial \zeta \partial z} - \frac{\partial \psi}{\partial \zeta}\frac{\partial^2 \psi}{\partial z^2} = \alpha \frac{\partial}{\partial z}\left[(T^*)^\beta \frac{\partial^2 \psi}{\partial z^2}\right]. \tag{3}$$

In this equation the constants α, β have the values μ_0 and $n-1$ respectively when the flow is laminar; and $\varepsilon_0 \rho_0$ and -2 respectively when the flow is turbulent.

It will be noticed that equation (3) contains two dependent variables viz. ψ, T. It is therefore necessary in order to solve the problem to use some other relation which involves T. This relation is the Energy equation which, for boundary layer flows, is

$$\rho u \frac{\partial c_p T}{\partial x} + \rho v \frac{\partial c_p T}{\partial y} = \frac{\partial}{\partial y}\left[\rho\, \varepsilon_T \frac{\partial T}{\partial y}\right] + \rho \varepsilon \left(\frac{\partial u}{\partial y}\right)^2.$$

It has been shown in Chapter IV that, if the Prandtl number of the flow is unity, then an integral of this equation is

$$\tfrac{1}{2}u^2 + C_p T = A + Bu \qquad \text{— Crocco's relation,}$$

A, B being constants determined by the boundary conditions.

THE SOLUTION FOR INCOMPRESSIBLE FLOW

When the fluid is incompressible (a condition which is satisfied approximately when is small in comparison with the local speed of sound), $T^* \doteq 1$. In this case the equation of motion reduces to

$$\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial\zeta\partial z} - \frac{\partial\psi}{\partial\zeta}\frac{\partial^2\psi}{\partial z^2} = \alpha\frac{\partial^3\psi}{\partial z^3}. \qquad (4)$$

(In this equation z is now equal to y .)

In order to solve equation (4) it is assumed that the velocity profiles taken at sections normal to the axis of the jet are similar. This assumption entails

$$\psi = \zeta^p f(z/\zeta^q) = \zeta^p f(\eta) \qquad \text{where } \eta = z/\zeta^q.$$

In the above form for ψ the indices p and q are partially determined by the fact that the total momentum flux across a normal section of the jet is independent of the x coordinate and therefore of ζ . This result was proved in Chapter IV. Thus $\int_0^\infty \rho u^2 dy$, being equal to $\frac{1}{2}M_0$, is independent of x . (M_0 is the total momentum flux of the jet.)

Now

$$\int_0^\infty \rho u^2 dy = \frac{1}{\rho_0}\int_0^\infty \left(\frac{\partial\psi}{\partial z}\right)^2 dz$$

$$= \frac{1}{\rho_0}\zeta^{2p-q}\int_0^\infty [f'(\eta)]^2 d\eta \qquad \text{on using the}$$

above form for ψ . Thus $2p-q=0$ since the momentum integral is independent of x and therefore of ζ .

When the similarity form for ψ is inserted in equation (4), the left side of (4) is seen to be of the form:

$\zeta^{2p-2q-1} \times$ (functions involving η only), while the right side takes the form

$$\zeta^{p-3q} \times \quad \text{(functions of } \eta \text{ only).}$$

Thus equation (4) has similar solutions of the above form only when

$$2p-2q-1 = p-3q,$$

i.e. when $p+q=1$. Thus using the fact that $2p=q$ it is seen that $p=1/3$ and $q=2/3$. The equation (4) now reduces to an ordinary differential equation in η , viz.

$$(f')^2 + ff'' + 3\alpha f''' = 0. \tag{5}$$

This equation may be put in a non dimensional form by writing

$\dot{z} = \alpha\eta$, and $f(\eta) = 6a\alpha g(z)$, 'a' being a constant which is at present undefined.

Thus (5) becomes

$$2\left[(g')^2 + gg''\right] + g''' = 0. \tag{6}$$

In equation (6) the primes denote differentiation with respect to z . Now the boundary conditions that g satisfies are

$$\text{on } z=0,\ g''=0 \quad \text{and} \quad g=0;$$
$$\text{on } z=0,\ g'=0.$$

Equation (6) may be integrated at once to give

$2gg' + g'' = \text{constant} = 0$, by insertion of the boundary conditions on the axis.

The above equation may be reduced to first order by quadrature to give

$$g^2 + g' = \text{constant} \qquad (7)$$

The constant in (7) may be taken as unity, because there is already one arbitrary constant occurring in the solution, namely 'a'.

The solution of (8) is

$g = \tanh \xi$, using the boundary condition as g at $\xi = 0$.

The value of the constant 'a' may be determined in terms of M_o, ρ_o and α by evaluating the momentum integral. It is in fact given by

$$a = \left(\frac{1}{48} \frac{M_o \rho_o}{\alpha^2}\right)^{1/3}.$$

The x-component of the velocity, i.e. u, is

$$\frac{6a^2\alpha}{\rho_o \xi^{1/3}} \operatorname{sech}^2\left(\frac{az}{\xi^{2/3}}\right).$$

Now a typical term omitted from the Navier-Stokes equations or the general turbulent equations of mean flow is $\varepsilon \rho \frac{\partial^2 u}{\partial x^2}$. This term is neglected in comparison with $\varepsilon \rho \frac{\partial^2 u}{\partial y^2}$. In order that this approximation be valid it is seen from the solution that $\frac{e^2}{a^2 \xi}$ must be small. (e is a function of x defined on page 38.) When the flow is laminar $e = 1$; while Görtler has shown from experimental evidence, based on the fact that the axial velocity is known to very as $x^{-1/2}$, that for the turbulent jet far from the orifice $e(x) \propto x^{1/2}$. Thus if $e^2/a^2\xi$ is to be small then x must be large - i.e. the solution is only true at large distances from the orifice.

HOWARTH'S (1948) AND ILLINGWORTH'S (1949) SOLUTION FOR COMPRESSIBLE FLOW

If the factor $(\rho^*)^2\varepsilon$ occurring in the right side of equation (2) has a constant value, then this equation has the same form as the equation of incompressible flow (4). Thus the solution in this special case is just

$$\psi = 6a\alpha\,\xi^{1/3}\tanh\left(\frac{az}{\xi^{2/3}}\right),$$

where z is now related to the physical coordinate y by

$$z = \int_0^y \rho^*\,dy.$$

Now $\rho^* = (T^*)^{-1}$ and T^* is a known function of u. T^* is in fact given by the Crocco relation

$$\frac{u^2}{2} + C_pT = C_pT_0 \quad \text{i.e.} \quad T^* = 1 - u^2/2C_pT_0.$$

(This relation is obtained from the general form of the Crocco relation (Chapter IV) by using the fact that the stagnation enthalpy is everywhere constant.)

In laminar flow

$$(\rho^*)^2\varepsilon = \mu^*\rho^*\cdot\mu_0/\rho_0 \doteq (T^*)^{n-1}\mu_0/\rho_0.$$

Since $n = 0{\cdot}76$, it is to be expected that the solution in the case of laminar flows is given sensibly by the solution of the corresponding problem when $\mu^*\rho^* = 1$. For turbulent flows, the quantity $(\rho^*)^2\varepsilon$ cannot be considered as a constant. The effect on the velocity profile given by Howarth and Illingworth's method will be called the "change of scale effect" and the rest of this chapter is devoted to finding what corrections have to be made to this solution when $(\rho^*)^2\varepsilon$ is not a constant.

THE APPROXIMATE SOLUTION OF (3) WHEN $(T^*)^\beta$ IS NOT CONSTANT

The equation to be solved is equation (3) viz:

$$\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial \zeta \partial z} - \frac{\partial \psi}{\partial \zeta}\frac{\partial^2 \psi}{\partial z^2} = \alpha \frac{\partial}{\partial z}\left[(T^*)^\beta \frac{\partial^2 \psi}{\partial z^2}\right].$$

T is related to u by Crocco's relation which is here

$$T^* = 1 - u^2/2C_pT_0 .$$

Thus when $u^2/2C_pT_0$ is small the velocity profile is given to a first approximation by the incompressible solution, viz.

$$u \doteqdot \frac{6a^2\alpha}{\rho_0 \zeta^{1/3}} \operatorname{sech}^2 a\eta .$$

The incompressible stream function may be used as a starting point in a Rayleigh-Janzen series expansion for ψ . This expansion, which is assumed to be convergent, takes the form:

$$\psi = 6a\alpha\zeta^{1/3}\left[\tanh a\eta + \varepsilon_1 F_1(a\eta) + \cdots\right].$$

ε_1 is a small parameter. The value of ε_1 is seen from the approximate form of T^* (obtained by inserting the incompressible value of u in Crocco's relation)

$$T^* \doteqdot 1 - \frac{K^2}{\zeta^{2/3}} \operatorname{sech}^4 a\eta \qquad \text{where } K^2 = \frac{18a^4\alpha^2}{C_p \rho_0^2 T_0}.$$

Whence $\varepsilon_1 = K^2/\zeta^{2/3}$.

Thus a suitable form for a Rayleigh-Janzen series expansion of ψ is

$$\psi = 6a\alpha\zeta^{1/3}\left[\tanh a\eta + \frac{K^2}{\zeta^{2/3}} F_1(a\eta) + \cdots\right]$$

and u is therefore

$$\frac{6a^2\alpha}{\rho_0 \zeta^{1/3}}\left[\operatorname{sech}^2 a\eta + k^2/\zeta^{2/3} F_1'(a\eta) + \cdots\right].$$

This parameter $\varepsilon_1 = k^2/\zeta^{2/3}$ is to a first approximation $\frac{1}{5}M_1^2$ — M_1 being the local Mach number of the jet on the axis, (assuming γ, the ratio of the specific heats to be $7/5$.).

When the above expansion for ψ is inserted and the approximate value for T^* used, in equation (3), the terms independent of ε_1 immediately cancel out leaving only terms of order ε_1 and higher in (3). When the terms of order ε_1 are equated to zero the function F_1 is seen to satisfy the equation

$$F_1''' + 2\tanh\xi\, F_1'' + 8\operatorname{sech}^2\xi\, F_1' + 4\tanh\xi\operatorname{sech}^2\xi\, F_1$$
$$= 2\beta\operatorname{sech}^6\xi\,(7\tanh^2\xi - 1),$$

where the primes denote differentiations with respect to ξ .

This equation may be reduced to the simpler form

$$(1-t^2)\frac{d^2H}{dt^2} - 2t\frac{dH}{dt} + 4H = 2\beta(1-t^2)^2(7t^2-1)$$

when the independent variable ξ is changed to $t = \tanh\xi$, and the dependent variable F_1 is changed first to G where G is given by $F_1 = (1-t^2)G$ and then to H where $H(t) = (1-t^2)^2\frac{dG}{dt}$.

The above equation in H is Legendre's equation with a right hand side. A particular integral of the equation may be shown to be

$$f(t) = \frac{1}{38}\beta\,(17 - 72t^2 + 45t^4 - 14t^6).$$

The complementary function consists of the linear combination of Legendre functions

$$H = A P_n(t) + B Q_n(t).$$

Because n is not an integer these Legendre functions are not closed in form but consist in fact of hypergeometric series.

Thus $P_n(t) = F(-n_1, -n_2; 1; \frac{1}{2}(1-t))$

and $Q_n(t) = F(-n_1, -n_2; 1; \frac{1}{2}(1+t))$

where n_1 and n_2 are the roots of the equation $n(n+1) = 4$.

The complete integral of the equation in H given above is:

$$H = AP_n + BQ_n + f(t).$$

A and B are arbitrary constants which are determined by the boundary conditions. These are:

on the axis of the jet $F_1(0) = 0$, $F_1'(0)$ is finite and $F_1''(0) = 0$

and on the boundaries of the jet

$$\lim_{t \to \pm 1} (1-t^2) F_1'(t) = 0.$$

Now $G = \int_0^t \frac{H dt}{(1-t^2)^2}$ and F_1 is therefore $(1-t^2)\int_0^t \frac{H dt}{(1-t^2)^2}$, the lower limit of the integral being zero because $F_1(0) = 0$.

Consider first the boundary condition at $t = \pm 1$,

i.e. $\lim_{t \to +1} (1-t^2) F_1'(t) = 0$

i.e. $\lim_{t \to +1} \left[(1-t^2)(-2t) \int_0^t \frac{H dt}{(1-t^2)^2} + H \right] = 0$;

this becomes

$$\lim_{t \to +1} \left[-(1-t) \int_0^t \frac{A + BK \log(1-t) + f(1)}{(1-t^2)^2} dt + A + BQ_n + f(1) \right]^{*}$$

Thus

$$\lim_{t \to 1} \left[-(A + f(1)) - BK \log(1-t) - BK + A + BQ_n + f(1) \right] = 0.$$

The logarithmic part of Q_n now cancels out and the above expression simplifies to $B = 0$.

Next, on the axis of the jet $F_1''(0) = 0$. This requirement entails that

$$\lim_{t\to 0}\left[-2\int_0^t \frac{H\,dt}{(1-t^2)^2} + \frac{H'}{1-t^2} + \frac{2tH}{(1-t^2)^2}\right] = 0$$

i.e. $H'(0) = 0$.

This implies that

$$2(A+B)\,F'(-n_1,-n_2;\,1,\,\tfrac{1}{2}) = 0 \qquad \text{since } f(0) = 0$$

But $F'(-n_1,-n_2;\,1,\,\frac{1}{2}) \neq 0$ hence $A+B=0$, i.e. $A = B = 0$.

Thus the solution which satisfies three of the boundary conditions is $H = f(t)$. This solution may easily be shown to satisfy the condition on $t = -1$. When the variables H and t are changed back to F_1 and ζ:

t	$\zeta = \tanh^{-1} t$	$57(-2/\beta)F_1(\zeta)$	$57(-2/\beta)F_1'(\zeta)$
0	0	0	-51·000
0·1	0·100	- 5·012	-47·852
0·2	0·203	- 9·490	-38·777
0·3	0·310	-12·919	-24·871
0·4	0·424	-14·794	- 7·888
0·5	0·549	-14·643	9·863
0·6	0·693	-12·019	25·646
0·7	0·867	- 6·482	36·437
0·8	1·099	2·454	39·027
0·9	1·472	15·548	29·720
1·0	∞	36·000	0

TABLE I

$$F_1(\zeta) = -\beta/38\left[12\tanh\zeta - \operatorname{sech}^2\zeta\,(12\zeta + 17\tanh\zeta - 14/3\tanh^3\zeta)\right].$$

---

*Note To obtain this result use is made of the result that if in the hypergeometric series $F(\alpha,\beta;\gamma;x)$, $\gamma-\alpha-\beta = 0$ then

$$\lim_{t\to 1} F(\alpha,\beta;\gamma;x) \div \left[\Gamma(\alpha+\beta)/\Gamma(\alpha)\Gamma(\beta)\ \log\left(\frac{1}{1-x}\right)\right] = 1$$

(Whittaker and Watson - Modern Analysis p. 299.)

Thus near $t = 1$

$Q_n(t) \sim k\log(1-t)$ + a function of t which tends to zero as $t \to 1$. k being a certain constant.

An estimate may now be made of the limits of validity of the solution so far obtained. It may be shown that, assuming $F_2(\zeta)$ to be of the same order of magnitude as $F_1(\zeta)$ the first two terms of the series expansion approximate to ψ within 1% for an axial Mach number of less than $\frac{1}{2}$. When the axial velocity is in the sonic region the maximum error is about 2% for laminar and 14% for turbulent flows respectively.

Thus the second approximation may be expected to give the effect of compressibility with fair accuracy when the axial Mach number is less than sonic. For supersonic flows, there may occur the added complication of shock waves; if however the jet is fully expanded when it leaves the orifice the occurrence of shock waves may be ruled out and the approximate solution given above may yield useful information about laminar flows. In the case of turbulent flows the next term in the series F_2 is clearly needed when the flow is supersonic.

The solution obtained above is given in terms of the variable $\zeta = aZ/\xi^{2/3}$. In order to find the changes made to the velocity profile it is necessary to change from Z to the physical coordinate y. The change in Bickley's incompressible profile resulting from this return to physical coordinates is termed the "change of scale effect". (The effect of the term F_1 on the velocity profile is called the perturbation effect.)

To a first approximation the formula relating y to Z is

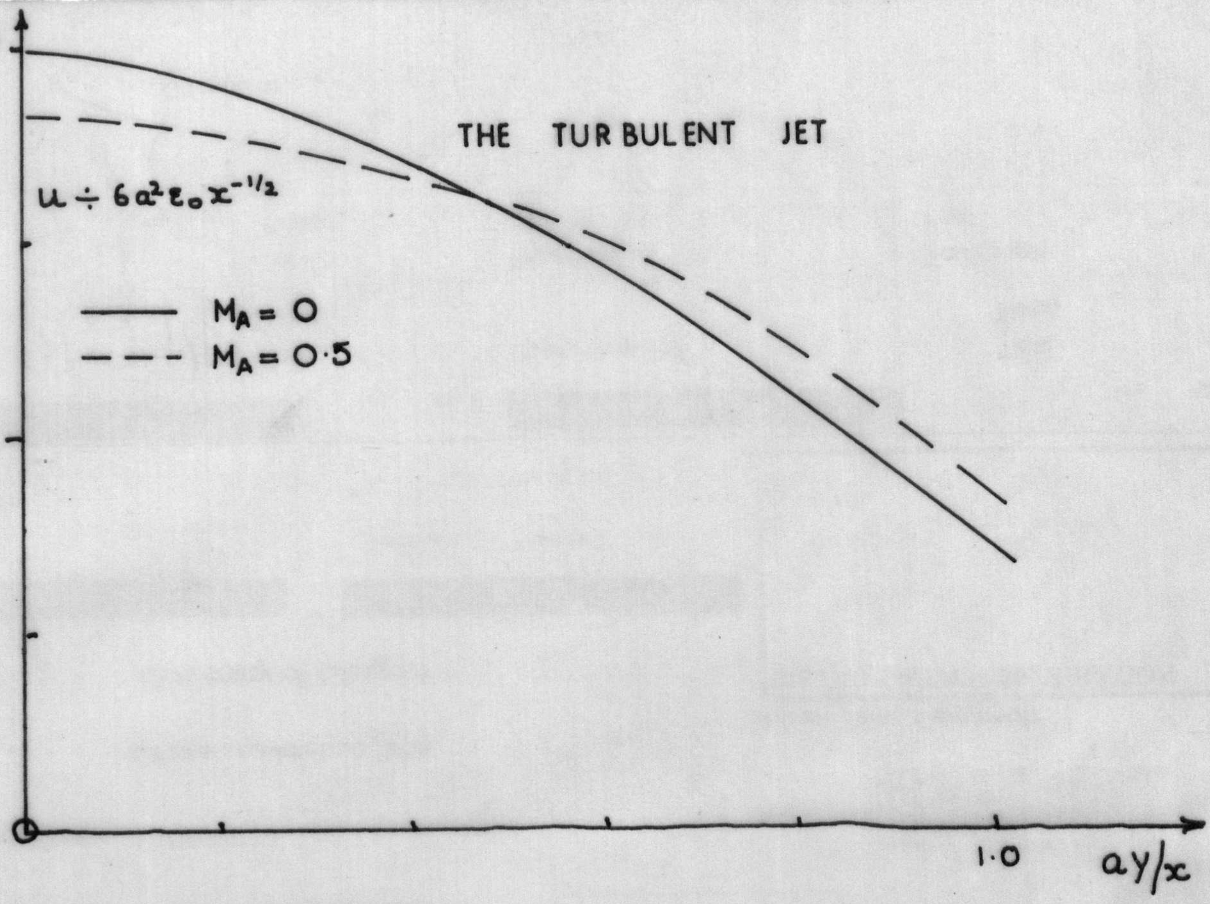
$$ay/\xi^{2/3} = \zeta - k^2/\xi^{2/3} \tanh\zeta \left(1 - \tfrac{1}{3}\tanh^2\zeta\right),$$

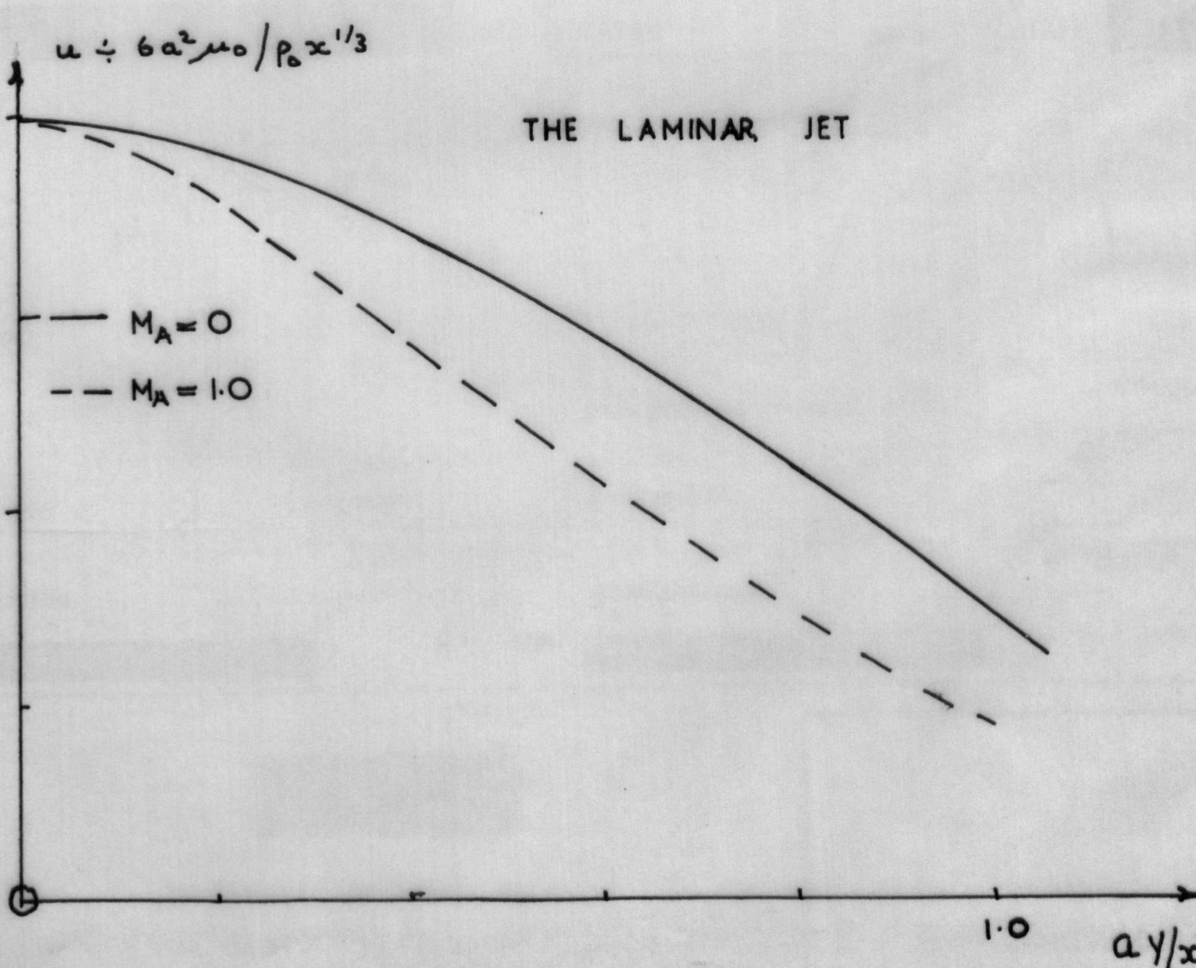
$$\left(\text{as } \zeta \to \infty, \quad ay/\xi^{2/3} \to \zeta - \frac{2}{3}\frac{k^2}{\xi^{2/3}}\right).$$

This relation is obviously the same as was obtained by Illingworth. In the case of laminar flow ξ is just x. In turbulent flow ξ was defined to be $\int_0^x e(x)\,dx$, where $e(x)$ gives the x variation in the coefficient of eddy kinematic viscosity. Now in incompressible flow, the experimental results of Förthmann (1934) show, that at large distances from the slit, the flow is similar and that the axial velocity decays as $x^{-1/2}$. This fixes the ζ as a function of x and gives in fact $\zeta = x^{3/2}$.

RESULTS AND PHYSICAL INTERPRETATION OF THE ABOVE ANALYSIS

The effect of compressibility, for the range of Mach numbers for which the analysis given above is valid, may be conveniently described as the resultant of two effects - the "change of scale effect" and the perturbation effect. The "change of scale effect" gives the true solution if $(T^*)^\beta$ is constant in the equation of motion. The "perturbation effect" gives the additional element due to the variation of $(T^*)^\beta$. Inspection of the approximate relation between y and z shows that the "change of scale" effect tends to draw the streamlines of the jet nearer to the axis, although it does not change the axial velocity. The magnitude of the "change of scale effect" is best expressed in terms of the change in the width of the jet - this being defined as $2\zeta^{2/3}/a$. When the axial Mach number is sonic the width of the mixing region decreases by 20%, due to the change of scale effect. The "change of scale" effect gives the bulk of the change in velocity profile for laminar flows, since the perturbation effect - due to the term F_1 in the expansion of ψ - is negligible. This indicates that Howarth's and Illingworth's assumption that $\mu^* \rho^* \doteq 1$ is a useful approximation to make in laminar flows. When the flow is turbulent the perturbation effect is of the same order of magnitude as the change of scale effect. Now the perturbation term has an additive effect on the velocity profile proportional to $F_1'(\zeta)$, leaving aside the variation due to ζ. The function F_1' is negative on the axis (actually $F_1'(\zeta) = -17/19$), as ζ increases F_1' changes sign and then tends to zero as ζ tends to infinity. The net effect of the perturbation term is therefore to flatten at the profile of the jet i.e. to increase the effective width of the jet. The combination of the two effects is to make the velocity profile more flat in the centre of the jet and to steepen the sides - the effect is rather like the change in velocity profile in pipe flow when the flow goes from laminar to turbulent.


THE TURBULENT JET
$u \div 6a^2\varepsilon_0 x^{-1/2}$
1.0
$M_A = 0$
$M_A = 0.5$
O
1.0
ay/x


$u \div 6a^2\mu_0/\rho_0 x^{1/3}$
1.0
THE LAMINAR JET
$M_A = 0$
$M_A = 1.0$
O
1.0
$ay/x^{2/3}$

VELOCITY PROFILES FOR THE FLOW FAR FROM THE ORIFICE OF A TWO DIMENSIONAL JET
(M_A = AXIAL MACH NUMBER.)

APPENDIX

When the flow is that of an axially symmetric jet, the equations of laminar and turbulent flow may be combined in one as in the two dimensional case. The solution for the laminar problem was given by Pack (1954). This same solution correctly interpreted may also be used for a turbulent jet. The stream function is with respect to cylindrical coordinates (x, r).

$$\psi = ax\left[\frac{b^2\eta^2}{(1+\frac{1}{4}b^2\eta^2)} + \frac{k^2}{x^2}F_1(\eta) + \frac{k^4}{x^4}F_2(\eta) + \cdots\right],$$

where $k^2 = 2b^4/C_p T_0 \rho_0^2$, $\eta = R/xa^{1/2}$, and the relation between b and the momentum flux M_0 across a section of the jet is given by

$b^2 = 3\rho_0 M_0 / 16\pi a$. (The variable R is defined by:

$$R^2 = 2\int_0^r \rho^* r\,dr.)$$

The function $F_1(\eta)$ is defined by means of the new variable $t = \dfrac{1}{1+\frac{1}{4}b^2\eta^2}$.

$$F_1(t) = \frac{6+13n}{494}(135 - 30t^3 - 8t^4) - \left(\frac{325}{76}n + \frac{519}{494}\right)t + \left(\frac{505}{228}n - \frac{917}{1482}\right)t^2 + \frac{28}{57}(1-n)t^5 - \frac{45}{247}(6+13n)(t^2-3t)\log t.$$

For laminar flows n is taken to be 0·76; the function $F_1(t)$ is tabulated for this case in Pack's paper. When the flow is turbulent n is taken as -1.

The same general conclusions as were found for the two-dimensional laminar jet for the effects of 'change of scale' and compressibility apply equally to the laminar axially symmetric jet. It is found for axially symmetric turbulent jets, that both of the above effects are of the same

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Note The independent variable x is not changed to ξ because experimental results show for incompressible flow that the axial velocity in turbulent flow varies as x^{-1} as is the case in laminar flow; this leads to the coefficient of eddy kinematic viscosity being constant.

order of magnitude, but just as with the two-dimensional turbulent jet the sign of $\eta^{-1} F_1'(\eta)$ as the axis is negative; as η increases $\eta^{-1} F_1'(\eta)$ changes sign and then tends to zero as $\eta \to \infty$. This behaviour of $\eta^{-1} F_1'(\eta)$ tends to broaden the velocity profile of the jet, but the change of scale effect is more significant and the net result is that the velocity profile is narrowed as the speed rises, in contrast with two-dimensional flow.

CHAPTER VI

THE MIXING OF TWO UNIFORM STREAMS OF COMPRESSIBLE FLUID

In this chapter the problem of the mixing of two semi-infinite streams of compressible fluid is discussed for the physically distinct cases of laminar and turbulent motion. As has been indicated in the introduction to the last chapter, the solution of this problem yields information about the mixing region which lies between the central core of the jet and the surrounding medium, when the flow is two-dimensional.

The solution of this problem, for incompressible flow, has been discussed by Görtler (1942), by Lessen (1949) and by Lock (1951). Görtler's method of analysis, originally used when the streams were turbulent, was adapted by Pai (1954) to cover the case of laminar motion. It is on Görtler's solution that the work of this chapter is based.

Görtler used Reichardt's constant exchange coefficient hypothesis and assumed that the velocity profiles were similar. In this way he was able to reduce the equations of motion to a single ordinary non-linear differential equation. This equation he solved in terms of a series whose coefficients were powers of a certain parameter. This parameter depends on the difference in velocity of the two streams. Since the series does not converge rapidly in the important case when one stream is at rest, it has been necessary to treat this particular problem by a numerical method. This division into two sections, viz. when the two streams have nearly the same velocity, and when one stream is at rest, runs throughout the analysis of this chapter.

As in the previous work on the compressible jet far from the orifice, it has been necessary to make certain assumptions in order to obtain a solution in analytical form. These assumptions have been stated in the introduction to the last chapter. One restriction however has been relaxed - viz. that the stagnation enthalpy is no longer assumed constant over the jet. This means that the analysis includes the case in which there are large differences in temperature between the streams.

Differences in the density of the fluid in this problem are due to two factors. The first of these occurs when the velocities in the jet are comparable with the local speed of sound, the second when there are temperature differences in the streams. The method of solution is to develop the stream function in a double series of powers of two parameters. One of these parameters is proportional to the square of the Mach number; the other depends on the temperature difference of the streams. Analytical expressions are found for the terms up to second order in the series for the stream function when the streams do not differ too greatly either in velocity or in temperature. However, when one of the streams is at rest the analytical method is no longer accurate and for this case numerical solutions are given.

EQUATIONS OF MOTION

Let u, v be the velocity components (or mean velocity components in the case of turbulent flow) parallel to Cartesian (x, y) axes, the origin of coordinates being taken at the point at which mixing begins and the x-axis parallel to the main direction of motion of the streams. As usual ρ and T are the density and absolute temperature of the gas respectively.

If it is assumed that the flow in the mixing regions between the streams is of boundary layer type and that there is no difference in pressure between the streams, then the equation of motion may be written as

$$\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial\xi\partial z} - \frac{\partial\psi}{\partial\xi}\frac{\partial^2\psi}{\partial z^2} = \alpha\left[(T^*)^\beta\frac{\partial^2\psi}{\partial z^2}\right], \tag{1}$$

following Chapter IV. This equation is valid for turbulent flow since, as in Chapter IV, Reichardt's theory of turbulence is used. To recapitulate the meaning of the symbols in equation (1);

$$z = \int_0^y \rho^* \, dy \; ; \; \xi = \int_0^x e(x)\,dx \; ; \; \rho^2 = \rho/\rho_0 \; ; \; T^* = T/T_0,$$

ρ_0 and T_0 referring to a reference state of the gas to be defined later. If the flow is laminar $\xi = x$, $\alpha = \mu_0$ and $\beta = n-1$, n coming from the law of variation of the coefficient of viscosity μ with T (i.e. $\mu \propto T^n$); for turbulent flows $\xi = \int_0^x e(x)dx$, where $e(x)$ is an experimentally determined function of x defined by $\varepsilon = \varepsilon_0 e(x)$, ε_0 is Reichardt's constant coefficient of eddy kinematic viscosity, $\alpha = \varepsilon_0\rho_0$ and $\beta = -2$.

Let the two streams in their uniform states have velocities U_1 and U_2 parallel to the x-axis as they cross the half-lines $y > 0$, $y < 0$ respectively. Let

$$U_0 = \tfrac{1}{2}(U_1 + U_2) \text{ and } \lambda = (U_1 - U_2)/(U_1 + U_2),$$

Since the profiles are assumed to be similar, a suitable form for the stream-function is

$$\psi = (\rho_0 U_0 \alpha \xi)^{1/2} g(\eta), \text{ where } \eta = \left(\frac{U_0\rho_0}{\alpha\xi}\right)^{1/2} z.$$

The power law dependence on ζ in ψ and in η is determined by the requirement that ψ shall satisfy the differential equation and by the fact that the boundary conditions are independent of x.

When the above expression for ψ is inserted in the equation of motion (equation (1)), $g(\eta)$ is seen to satisfy the equation

$$\frac{d}{d\eta}\left[(T^*)^\beta g''\right] + \frac{1}{2} g g'' = 0, \qquad (2),$$

where the primes denote derivatives with respect to η.

If the Prandtl number of the flow is assumed to be unity then (as has been shown in Chapter IV) an integral of the momentum and energy equations is the Crocco relation

$$\frac{1}{2}u^2 + C_p T = A + Bu,$$

A and B being constants which are determined by the boundary conditions.

Now the boundary conditions on T and u are

$u = U_1$, $T = T_1$ in the main upper stream, i.e. at $y = +\infty$

and $u = U_2$, $T = T_2$ in the main lower stream, i.e. at $y = -\infty$.

Thus u and T are related by the equations

$$\frac{1}{2}u^2 + C_p T = A + Bu,$$

$$\frac{1}{2}U_1^2 + C_p T_1 = A + BU_1$$

and $$\frac{1}{2}U_2^2 + C_p T_2 = A + BU_2.$$

When A and B are eliminated from these equations

$$T^* = \frac{h}{\lambda}(u^* - 1) - w_0^2 u^{*2},$$

where $T^* = T/T_0$ and $T_0 = (I_1 + I_2)/2$; $u^* = u/U_0$;

$$h = \frac{I_1 - I_2}{I_1 + I_2} \text{ and } I_{1,2} = \frac{1}{2}U_{1,2}^2 + C_p T_{1,2}.$$

The temperature (T_0) defined above is now the temperature at which μ_0 and ρ_0 are taken. The non-dimensional quantity 'h' is non-zero only when the stagnation point enthalpies of the streams are different - it is, therefore, a measure of that part of the temperature difference between the streams which is not due to their high speeds. The parameter $\omega_0^2 = U_0^2/2C_pT_0$ is of the same order of magnitude as the square of the Mach number of the flow.

The natural parameters in which to express a series solution of (2) are, therefore, 'h' and ω_0^2. Let $g(\eta)$ be expanded in the double series:

$$g(\eta) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{rs} f_{rs}(\xi)\, \omega_0^{2r} h^s , \tag{3}$$

where $\xi = (\eta - b)/2$, $\eta = b$ giving the locus of those points of the flow at which the velocity is U_0 , and all the a_{rs} are equal to $-\beta$ except a_{00} which is equal to 2. Then

$$u^* = u/U_0 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \tfrac{1}{2} a_{rs} f'_{rs}(\xi)\, \omega_0^{2r} h^s \tag{4}$$

and $$(T^*)^\beta = 1 + \beta\left[\frac{h}{\lambda}(u^* - 1) - \omega_0^2 (u^*)^2\right] + \cdots \tag{5}$$

The above expressions, (4) and (5), are valid for sufficiently small h and ω_0^2 - the numerical value of $(u^*-1)/\lambda$ being less than unity.

The boundary conditions on $g'(\eta)$ are

$$g'(+\infty) = 1 + \lambda,\ g'(b) = 1 \text{ and } g'(-\infty) = 1 - \lambda .$$

Lock (1951) has shown for the incompressible case that any given solution of (2) generates an infinity of solutions, all of which satisfy the same boundary conditions at $\eta = \pm\infty$. These solutions may be obtained by replacing η by $\eta + c$ in the given solution, c being an arbitrary constant. This result is true for compressible flow. This is because $(T^*)^\beta$ may be expressed as a function of $g'(\eta)$ alone by virtue of the Crocco relation. Thus equation (2) may be written in the form

$$\frac{d}{d\eta}\left[\phi(g', g'')\right] + \frac{1}{2}gg'' = 0.$$

It may be seen by substitution that if $g(\eta)$ is a solution of this equation so also is $f(\eta) = g(\eta + c)$, c being an arbitrary constant. Each one of this infinity of solutions is an equally valid solution of the boundary layer problem because the solution obtained by replacing η by $\eta + c$ in any given solution leads to a value of the y component of the velocity on the boundary which differs by an amount of order $U_0 c/\sqrt{R_e}$ from that of the given solution, R_e being a large dimensionless constant. (In laminar flow R_e is the usual Reynolds number; when the flow is turbulent R_e is defined as in the laminar case except that the coefficient of kinematic viscosity μ_0/ρ_0 is replaced by the eddy coefficient of kinematic viscosity ε_0 .) To pick out the correct solution it is necessary to take into account those terms of the Navier Stokes equations of order $1/\sqrt{R_e}$ higher than the terms in the boundary layer equations. It follows from the above discussion that the velocity at any finite η is mathematically indeterminate from the boundary layer equations alone. Thus b , which by definition gives the locus of points of the flow whose velocity is U_0 is likewise indeterminate. The value of b can however be found from experiment. When b is known the solution to the problem is uniquely determined.

The equations which the functions f_{rs} satisfy may be found by substituting (4) and (5) into equation (2) and equating the coefficients of $w_0^{2r} h^s$ to zero. The three equations giving the terms of lowest order in the series (3) are

$$f_{00}''' + 2 f_{00} f_{00}'' = 0, \tag{6}$$

$$f_{10}''' + 2 (f_{00} f_{10}'' + f_{00}'' f_{10}) + 2 \frac{d}{d\xi}\left[(f_{00}')^2 f_{00}''\right] = 0, \tag{7}$$

and

$$f_{01}''' + 2 (f_{00} f_{01}'' + f_{00}'' f_{01}) - 2 \frac{d}{d\xi}\left[f_{00}''(f_{00}'-1)/\lambda\right] = 0. \tag{8}$$

The boundary conditions which these functions satisfy are

$$f_{00}'(+\infty) = 1+\lambda \quad , \quad f_{00}'(0) = 1, \quad f_{00}'(-\infty) = 1-\lambda .$$

$$f_{10}'(+\infty) = 0 \quad , \quad f_{10}'(0) = 0, \quad f_{10}'(-\infty) = 0 .$$

$$f_{01}'(+\infty) = 0 \quad , \quad f_{01}'(0) = 0, \quad f_{01}'(-\infty) = 0 .$$

The function f_{00} is just the non-dimensional stream function when the fluid is incompressible, while the functions f_{10} and f_{01} give the first order effects of non-zero Mach number and temperature difference in the streams respectively. The solutions of these equations will be obtained for two distinct cases, namely, when λ is small (say less than 0·3) and when $\lambda = 1$, which corresponds to the problem in which the lower stream is at rest. In this way many examples of practical interest are likely to be covered.

CASE I. SOLUTION FOR SMALL VALUES OF λ

SECTION I. DETERMINATION OF f_{oo}

The functions f_{rs} may be expanded in a series of powers of λ when λ is small. Thus f_{oo} is expanded as

$$f_{oo} = \sum_{n=0}^{\infty} \lambda^n F_n(\xi).$$

The boundary conditions which these functions F_n satisfy are

$$F_o'(0) = 1 \quad , \quad F_o'(\pm\infty) = 1,$$

$$F_1'(0) = 0 \quad , \quad F_1'(\pm\infty) = \pm 1,$$

$$F_n'(0) = 0 \quad \text{and} \quad F_n'(\pm\infty) = 0 \quad , \quad (n = 2, 3 \cdots).$$

The equations for the functions $F_n(\xi)$ are obtained by substituting the above expansion for f_{oo} into the differential equation (6) and equating the coefficients of λ to zero. The functions (F_n) were first evaluated numerically by Görtler (1942). They may also be obtained in analytical form and have been tabulated from these analytical expressions (see table). The numerical values show that the expansion for f_{oo} does not converge very rapidly when λ is greater than about 0·3. This is the reason for treating the case $\lambda = 1$ separately.

The differential equation for F_o is

$$F_o''' + 2F_oF_o'' = 0.$$

A solution of this equation is $F_o = a + b\xi$, where a , b are arbitrary constants. The form of this solution which satisfies the boundary conditions is

$$F_o = \xi + d_o ,$$

where d_0 is a constant which is determined in the process of solving the equation for F_1 .

The equation for F_1 is:

$$F_1''' + 2F_0 F_1'' = 0.$$

Thus

$$F_1' = A\int_0^{\zeta} e^{-\zeta^2 - d_0\zeta} d\zeta$$ (using the boundary condition that $F_1'(0) = 0$).

Next the boundary conditions at $\zeta = \pm\infty$ give:

$$A\int_0^{\infty} e^{-\zeta^2 - d_0\zeta} d\zeta = 1$$

and $$A\int_0^{-\infty} e^{-\zeta^2 - d_0\zeta} d\zeta = -1.$$

Hence $d_0 = 0$ and $A = 2/\sqrt{\pi}$.

Thus $$F_1' = \frac{2}{\sqrt{\pi}}\int_0^{\zeta} e^{-\zeta^2} d\zeta = \Phi(\zeta) = \int_0^{\zeta} \Phi_1(\zeta)\, d\zeta .$$

Hence integrating

$$F_1 = \int_0^{\zeta} \Phi(\zeta)\, d\zeta + d_1$$

where d_1 is to be determined in the same way as d_0 .

In general, since $F_0 = \zeta$, the differential equation for F_n may be written

$$F_n''' + 2\zeta F_n'' = \Phi_1 R_n(\zeta) \quad (9)$$ when $n \geqslant 2$.

(R_n is a known function of F_{n-1} , F_{n-2} , ... ζ.)

The solution of (9) is

$$F_n = \int_0^\zeta \int_0^\zeta \Phi_1 \int_0^\zeta R_n \, d\zeta \, d\zeta \, d\zeta + C_n \int_0^\zeta \Phi \, d\zeta + D_n \zeta + d_n . \tag{10}$$

C_n, D_n and d_n are constants of integration. C_n is found by using the boundary condition at $\zeta = \pm\infty$; D_n is zero because of the boundary condition at $\zeta = 0$; and d_n is determined in the process of evaluating F_{n+1} .

The following is a list of the values of the functions F_0 , F_1 , F_2 , F_3 and their derivatives.

$$F_0 = \zeta \ , \quad F_0' = 1 \ , \quad F_0'' = 0 ,$$

$$F_1 = \zeta \Phi + \tfrac{1}{2}\Phi_1 + \beta_1 , \quad F_1' = \Phi \ , \quad F_1'' = \Phi_1 ,$$

$$F_2 = -\tfrac{1}{2}\zeta\Phi^2 - \tfrac{3}{4}\Phi\Phi_1 + \beta_1\Phi + \sqrt{\tfrac{2}{\pi}}\,\Phi(\sqrt{2}\,\zeta) + \tfrac{1}{2}\zeta ,$$

$$F_2' = -\tfrac{1}{2}\Phi^2 + \tfrac{1}{2}\zeta\Phi\Phi_1 + \tfrac{1}{4}\Phi_1^2 + \beta_1\Phi_1 + \tfrac{1}{2} ,$$

$$F_2'' = -(\zeta^2 + \tfrac{1}{2})\Phi\Phi_1 - \tfrac{1}{2}\zeta\Phi_1^2 - 2\beta_1\zeta\Phi_1 .$$

(F_3 is not evaluated since $F_3(0) \equiv d_3$ is unknown, its value only becoming known when F_4' has been calculated.)

$$F_3' = \tfrac{1}{2}\Phi^3 - (\tfrac{1}{4}\zeta^3 + \tfrac{7}{8}\zeta)\Phi^2\Phi_1 - (\tfrac{1}{4}\zeta^2 + \tfrac{3}{4})\Phi_1^2\Phi - \tfrac{1}{16}\zeta\Phi_1^3$$
$$-\frac{9}{4\pi\sqrt{3}}\Phi(\sqrt{3}\,\zeta) - \beta_1(\zeta^2 + \tfrac{1}{2})\Phi\Phi_1 - \tfrac{1}{2}\beta_1\zeta\Phi_1^2 + \left(\frac{3\sqrt{3}}{4\pi} - \frac{1}{2}\right)\Phi$$
$$+ \zeta(\tfrac{1}{4} - \beta_1^2)\Phi_1 + \sqrt{\tfrac{2}{\pi}}\,\Phi_1\Phi(\sqrt{2}\,\zeta),$$

$$F_3'' = (\tfrac{1}{2}\zeta^4 + \zeta^2 + \tfrac{5}{8})\Phi_1\Phi^2 + (\tfrac{1}{2}\zeta^3 + \tfrac{3}{4}\zeta)\Phi\Phi_1^2 + (\tfrac{1}{8}\zeta^2 - \tfrac{3}{8})\Phi_1^3$$
$$+\beta_1(2\zeta^3 - \zeta)\Phi\Phi_1 + \beta_1(\zeta^2 - 1)\Phi_1^2 + \left(\frac{\pi}{8} + \frac{1}{2\pi}\right)\zeta^2\Phi_1$$
$$-2\sqrt{\tfrac{2}{\pi}}\,\zeta\Phi_1\Phi(\sqrt{2}\,\zeta) + \left[\frac{1}{4\pi}(3\sqrt{3} - 1) - \frac{1}{2} - \frac{\pi}{16}\right]\Phi_1 .$$

where $\beta_1 = -\frac{\sqrt{\pi}}{2}\left(\frac{1}{2}+\frac{1}{\pi}\right) = -0.72521.$

Values of the functions F_1, F_2, F_3 and their derivatives have been tabulated using the above formulae. When they are compared with Görtler's numerical solution, it is seen that there is a systematic error in Görtler's F_2'', and that this error is carried through into F_2, F_2' and F_3''. As can be seen from the numerical values of F_1, F_2 and F_3, when λ is small, the function f_{00}' may be approximated with good accuracy by

$$f_{00}' = 1 + \lambda F_1' + \lambda^2 F_2' + \lambda^3 F_3'.$$

SECTION II

DETERMINATION OF THE ANALYTICAL APPROXIMATION TO f_{10} WHEN λ IS SMALL

The function f_{10} is the coefficient of ω_0^2 in the expansion of the stream function. When λ is small f_{10} may be expressed in the series

$$f_{10} = \sum_{n=0}^{\infty} \lambda^n G_n(\xi).$$

The boundary conditions on the functions G_n are

$$G_n'(0) = G_n'(+\infty) = G_n'(-\infty) = 0, \quad (n = 0, 1, 2 \ldots).$$

The equations that the functions G_n satisfy are found by substituting the expansions for f_{00} and f_{10} into equation (7) and equating the coefficients of λ^n to zero. The equations giving the two terms of lowest order are in fact $G_0''' + 2\xi G_0'' = 0$

and $G_1''' + 2\xi G_1'' + 2G_0 F_1'' + 2F_1''' = 0.$

The solution of the former equation is

$$G_0' = C\int_0^{\xi} e^{-\xi^2}d\xi + D,$$

where C and D are arbitrary constants. These constants are in fact zero because of the boundary conditions. Thus

$$G_0 = a_0$$

a_0 being a constant determined in the evaluation of G_1.

In the equation for G_1 put $G_1'' = \Phi_1 y(\xi)$. Then

$$y' + 2a_0 - 4\xi = 0.$$

Hence, by integration

$$G_1'' = (2\xi^2 - 2a_0\xi + c)\,\Phi_1$$

c being a constant of integration. When the above expression is integrated, and the boundary conditions on G_1' used it is found that

$$c = -1 \qquad \text{and} \qquad a_0 = 0.$$

Subsequent integration gives

$$G_1 = \tfrac{1}{2}\Phi_1 + a_1,$$

where a_1 is a constant whose determination is similar to that of a_0.

When the equation for G_2'' is solved,

$$\begin{aligned} G_2' = {} & -2\Phi^2 + \left(\xi^3 - \tfrac{3}{2}\xi\right)\Phi\Phi_1 + \left(\tfrac{1}{2}\xi^2 - 1\right)\Phi_1^2 \\ & + \left(2\beta_1\xi^2 - \beta_1 + a_1\right)\Phi_1 + 2, \end{aligned}$$

where $a_1 = \sqrt{\pi}\left(\frac{3}{2\pi} - \frac{5}{4}\right)$ and β_1 has the same value as above.

LIST OF VALUES OF THE G FUNCTIONS

$$G_0 = 0,$$

$$G_1 = \tfrac{1}{2}\Phi_1 + a_1, \quad G_1' = -\xi\Phi_1, \quad G_1'' = (2\xi^2 - 1)\Phi_1,$$

$$\begin{aligned} G_2 = {} & -2\xi\Phi^2 - \left(\tfrac{1}{2}\xi^2 + \tfrac{7}{4}\right)\Phi\Phi_1 + \sqrt{\tfrac{2}{\pi}}\,\Phi(\sqrt{2}\xi) - \tfrac{1}{4}\xi\Phi_1^2 \\ & - \beta_1\xi\Phi_1 + a_1\Phi + 2\xi, \end{aligned}$$

$$G_2' = -2\Phi^2 + (\xi^3 - \tfrac{3}{2}\xi)\Phi\Phi_1 + (\tfrac{1}{2}\xi^2 - 1)\Phi_1^2 + (2\beta_1\xi^2 - \beta_1 + a_1)\Phi_1 + 2,$$

$$G_2'' = -(2\xi^4 - 6\xi^2 + \tfrac{11}{2})\Phi\Phi_1 - (\xi^3 - \tfrac{2}{2}\xi)\Phi_1 - (4\beta_1\xi^3 + \overline{2a_1 - 6\beta_1}\,\xi)\Phi_1 .$$

The analytical expressions for the subsequent terms G_3, G_4, etc., could be found in the same way, but the labour involved in evaluating the complicated terms would be very great; for example, G_3 involves about 60 terms. When the above expressions for G_1' and G_2' are used, f_{10}' is obtained to 1% accuracy for values of λ up to about 0·2 if the Mach number is lexx than five.

SECTION III

DETERMINATION OF THE ANALYTICAL APPROXIMATION TO f_{01} WHEN λ IS SMALL

The function f_{01} is the coefficient of 'h' in the expansion of the stream function. When λ is small f_{01} may be expressed (like f_{10}) in the form

$$f_{10} = \sum_{n=0}^{\infty} \lambda^n H_n(\xi).$$

An analysis essentially similar to that of sections I and II above gives the following analytical forms for the H functions:

$$H_0 = c_1 = \tfrac{1}{2}\sqrt{\pi}\left(1 - \tfrac{2}{\pi}\right) = 0.32204,$$

$$H_1 = c_1\Phi + \xi\Phi^2 + \tfrac{1}{2}\Phi\Phi_1 - \xi,$$

$$H_1' = c_1\Phi_1 + \Phi^2 + \xi\Phi\Phi_1 + \tfrac{1}{2}\Phi_1^2 - 1,$$

$$H_1'' = -2c_1\xi\Phi_1 - 2\xi^2\Phi\Phi_1 - \xi\Phi_1^2 + 3\Phi\Phi_1 .$$

(H_2 has not been determined because the constant c_2, being found only after H_3' has been evaluated, is unknown)

$$H_2' = \left\{(3\beta_1 - \frac{c_1}{2}) - \xi^2(c_1 + 2\beta_1)\right\}\Phi\Phi_1 - (\tfrac{1}{2}c_1 + \beta_1)\xi\Phi_1^2 +$$
$$+ \frac{4}{3}\Phi^3 - (\xi^3 - \tfrac{1}{2}\xi)\Phi^2\Phi_1 - \xi^2\Phi_1^2\Phi + \frac{\sqrt{3}}{\pi}\Phi(\sqrt{3}\xi) +$$
$$- \frac{1}{4}\xi\Phi_1^3 + \left(\frac{4}{3} - \frac{\sqrt{3}}{\pi}\right)\Phi - 2\beta_1 c_1 \xi\Phi_1,$$

$$H_2'' = \left\{-(10\beta_1 + c_1)\xi + (2c_1 + 4\beta_1)\xi^3\right\}\Phi\Phi_1 +$$
$$+ \left\{(2\beta_1 - c_1) + (2\beta_1 + c_1)\xi^2\right\}\Phi_1^2 + (2\xi^4 - 4\xi^2 - \tfrac{7}{2})\Phi_1\Phi^2 +$$
$$+ (2\xi^3 - \xi)\Phi\Phi_1^2 + \tfrac{1}{2}(\xi^2 + 1)\Phi_1^3 +$$
$$+ \left\{\left(-\frac{\sqrt{3}}{\pi} - 2\beta_1 c_1 + \frac{4}{3}\right) + 4\beta_1 c_1 \xi^2\right\}\Phi_1 .$$

As before, the expressions for the higher H-functions could be found exactly. However, when only the functions up to H_2', are used, f_{01}' is found quite accurately for the values of λ up to about 0·3. Indeed, on comparing $f_{01}' \doteqdot H_1' + H_2'$, with the exact solution f_{01}' for $\lambda = 1$, it is seen that in the range $|\xi| < 1{\cdot}5$ the agreement is quite good. The maximum error in this region is about 5%.

CASE II

SOLUTION WHEN $\lambda = 1$, i.e. THE LOWER STREAM IS AT REST

SECTION I. DETERMINATION OF f_{00}.

It has been observed that the series representation, in powers of λ, of the function f_{00} is not useful when λ is about unity. Consequently another method must be used for this important case.

Now f_{00} is that solution of equation (6) viz.

$$f_{00}''' + 2f_{00}''f_{00} = 0, \tag{6}$$

which satisfies the boundary conditions

$$f_{00}'(+\infty) = 2, \quad f_{00}'(0) = 1 \text{ and } f_{00}'(-\infty) = 0.$$

This equation has been solved in terms of an asymptotic series by Lock (1951). Lock's work was unknown to the author who developed his own iterative method for dealing with the problem. This method is described below as a matter of interest.

The method depends on the following relations, which are consequences of equation (6) and the boundary conditions:

$$f_{00}' = 1 + f_{00}''(0)\int_0^{\zeta} \exp\left[\int_0^{\zeta} -2f_{00}\,d\zeta\right] d\zeta, \tag{9}$$

$$f_{00}''(0) = \frac{1}{\int_0^{\infty} \exp\left[\int_0^{\zeta} -2f_{00}\,d\zeta\right] d\zeta}, \tag{10}$$

$$f_{00}''(0) = \frac{-1}{\int_0^{-\infty} \exp\left[\int_0^{\zeta} -2f_{00}\,d\zeta\right] d\zeta}, \tag{11}$$

and

$$f_{00} = -\frac{1}{4}\left\{\frac{d}{d\zeta}\left[(f_{00}'')^2\right]\right\} / (f_{00}'')^2. \tag{12}$$

In order to start the iterative process an approximation to f_{00}, which satisfies the boundary conditions exactly, is required. A suitable approximation is

$$f_{00} \doteq F_0 + F_1 + F_2 + F_3{}^{*} \tag{13}$$

When the values of f_{00} given by equation (13) are substituted in the relations (9), (10) and (11), there is obtained a second approximation to f_{00}'. A difficulty which arises here is that the values of $f_{00}''(0)$ given by (10) and (11) will not necessarily be the same. For the positive range of ξ the value of $f_{00}''(0)$ to be substituted in (9) is given by (10); and for the negative range of ξ the value of $f_{00}''(0)$ is given by (11). This makes the second approximation given by (9) satisfy the boundary conditions.

In order to bring the values of $f_{00}''(0)$ given by (10) and (11) together the following device is used:

$$\text{Put} \quad f_{00} = \phi + a + \varepsilon, \quad \text{where } \phi = \int_0^{\xi} f_{00}'\, d\xi,$$

a is the value of $f_{00}(0)$ given by (13) and ε is a small (constant) correcting factor.

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*Note: F_3' only is known exactly. When F_3' is integrated up F_3 is known except for a constant of integration d_3. This constant is found when F_4' is known. It may, however, be estimated by determining $f_{00}(0)$ from (12) taking the form of f_{00}'' as the right side of (12) as given by

$$f_{00}'' \doteq F_1'' + F_2'' + F_3''.$$

The value of $f_{00}(0)$ calculated in this way is then used on the left side of equation (13), to give an estimate to d_3. This assumes that the contribution of $F_4(0) + F_5(0) + \cdots$ to $f_{00}(0)$ is negligible.

Then:

$$\int_0^{+\infty} \exp\left[-2\int_0^{\xi} f_{00}\, d\xi\right] d\xi = I_+ - 2\varepsilon J_+ + 2\varepsilon^2 K_+ - \cdots$$

and

$$\int_0^{-\infty} \exp\left[-2\int_0^{\xi} f_{00}\, d\xi\right] d\xi = I_- - 2\varepsilon J_- + 2\varepsilon^2 K_- - \cdots$$

where

$$I_\pm = \int_0^{\pm\infty} \exp\left[-2\int_0^{\xi} (\phi + a)\, d\xi\right] d\xi,$$

$$J_\pm = \int_0^{\pm\infty} \xi \exp\left[-2\int_0^{\xi} (\phi + a)\, d\xi\right] d\xi,$$

$$K_\pm = \int_0^{\pm\infty} \xi^2 \exp\left[-2\int_0^{\xi} (\phi + a)\, d\xi\right] d\xi$$ and so on.

When the condition that the values of $f_{00}''(0)$ given by (10) and (11) be equal is used it follows that

$$(I_+ + I_-) - 2\varepsilon(J_+ + J_-) + 2\varepsilon^2(K_+ + K_-)\ldots = 0$$

This gives when ε is small

$$\varepsilon \doteqdot \frac{1}{2}\,\frac{I_+ + I_-}{J_+ + J_-}.$$

This is the required correction to f_{00} at $\xi = 0$. Thus the iterated value of f_{00} is

$$f_{00} = \phi + a + \varepsilon.$$

When this set of values for f_{00} is used in place of those given by (13) the iterative process may be continued till a final result is obtained. It is found that the process converges quite rapidly.

The numerical results are given in table ( ). They may be shown to agree with Lock's solution to the third significant digit. An independent check on the work is given by an application of Meksyn's (1956) method of solution of Blasius' equation to this problem. This is given in the appendix to this chapter.

SECTION II. DETERMINATION OF THE FUNCTIONS f_{10} AND f_{01} WHEN THE LOWER STREAM IS AT REST

Again, as in Section I above, the series method of solution in terms of λ is not useful in determining f_{10} and f_{01}. The equations that f_{10} and f_{01} satisfy are, however, linear and it follows that the standard methods of numerical integration of differential equations apply.

The functions f_{10} and f_{01} when $\lambda = 1$ satisfy respectively the equations:

$$f_{10}''' + 2\left(f_{00} f_{10}'' + f_{00}'' f_{10}\right) + 2\frac{d}{d\zeta}\left[(f_{00}')^2 f_{00}''\right] = 0 \quad (14)$$

and

$$f_{01}''' + 2\left(f_{00} f_{01}'' + f_{00}'' f_{01}\right) - 2\frac{d}{d\zeta}\left[f_{00}''(f_{00}' - 1)\right] = 0. \quad (15)$$

The boundary conditions on f_{10} and f_{01} are:

$$f_{10,01}'(+\infty) = 0, \quad f_{10,01}'(0) = 0 \text{ and } f_{10,01}'(-\infty) = 0.$$

These equations cannot be solved analytically because f_{00} is not given in an analytical form (when $\lambda = 1$). At first sight it might appear to be useful to reduce the order of the equations (14) and (15) by making use of that part of the complementary function given by

$$f_{10,01} = f_{00}'$$

However when the asymptotic behaviour* of the solutions of the reduced equations is examined it is found that they have a complementary function which is badly behaved at $\zeta = \pm\infty$. Thus the reduced form of equation (14) and (15) is not suitable for numerical methods of integration. In fact

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* See Appendix II.

numerical methods of solution have been applied to the equations (14) and (15) as they stand, since they have solutions which are well behaved at $\zeta = \pm\infty$. The method of computation used in evaluating the functions consists in computing two linearly independent parts of the complementary function of (14) which satisfy respectively the initial conditions:

$$f_{10}(0) = 0 \ , \quad f_{10}'(0) = 0 \ , \quad f_{10}''(0) = 1$$

and $f_{10}(0) = 1, \quad f_{10}'(0) = 0 \ , \quad f_{10}''(0) = 0.$

Next a particular integral of the complete equation (14) is calculated with the initial conditions

$$f_{10}(0) = 0 \ , \quad f_{10}'(0) = 0 \ \text{ and } \ f_{10}''(0) = 0.$$

The required solution of equation (14) is obtained by adding a suitable linear combination of the parts of the complementary function to the particular integral, so that the boundary conditions at $\zeta = \pm\infty$ may be satisfied.

The same method may be used to solve equation (15). Values of f_{10} and f_{01} and their first derivatives are given in table (   ).

NOTE ON THE CHANGE OF SCALE EFFECT

It can be shown that, if w_0^2 and h are small,

$$\zeta_1 \doteq \zeta + \frac{h}{\lambda}\int_0^{\zeta}(f_{00}'-1)\,d\zeta - w_0^2\int_0^{\zeta}(f_{00}')^2\,d\zeta,$$

where $\zeta_1 = \frac{1}{2}\left(\frac{U_0\rho_0}{\alpha\xi}\right)^{1/2} \cdot Y.$

When λ is small, the above approximation becomes:

$$\zeta_1 \doteqdot \zeta + h\int_0^{\zeta}\left[F_1' + \lambda F_2' + \cdots\right]d\zeta - w_0^2\int_0^{\zeta}\left[1 + \lambda F_1' + \lambda^2 F_2' \cdots\right]^2 d\zeta$$

$$\doteqdot \zeta + h\left[F_1 - F_1(0) + \lambda F_2 + \lambda^2\left(F_3 - F_3(0)\right) + \cdots\right]$$

$$- w_0^2\left[\zeta + 2\lambda\left(F_1 - F_1(0)\right) + \cdots\right].$$

When $\lambda = 1$, the above approximation becomes

$$\zeta_1 \doteqdot \zeta + h\int_0^{\zeta}\left(f_{00}' - 1\right)d\zeta - w_0^2\int_0^{\zeta}\left(f_{00}'\right)^2 d\zeta .$$

The relations given above show that changes in the width of the mixing region due to the "change of scale effect" depend on the difference in the total energies of the streams (through 'h ') and on the Mach number of the flow (through w_0^2) and that these effects are, to a first approximation independent. It can be shown, by considering $\int_0^{\zeta}\left(f_{00}' - 1\right)d\zeta$ that the <u>net</u> effect on the width of the mixing region of differing total energies is very slight, for $|h|$ less than 0·3 (h 0·3 corresponds roughly to the upper stream having a temperature of 300° C with the lower stream at room temperature). The effect of a non-zero Mach number is to decrease the width of the mixing region as the Mach number increases.

<u>CONCLUSION</u> The effect of compressibility on the non-dimensional velocity profile may be conveniently divided into two parts. One is the "change of scale effect" which depends only on w_0^2 and h . The other is the perturbation effect which enters through the functions f_{01}' and f_{10}' . Each of these effects may be subdivided into the effect of differing total energies (h) and the effect of non-zero Mach number (w_0^2). The non-dimensional velocity profile is

$$\frac{u}{U_0} \doteqdot f_{00}' - \frac{\beta h}{2\lambda} f_{01}' - \frac{\beta}{2} w_0^2 f_{10}' + \cdots .$$

The perturbation effect depends on both compressibility and variation of viscosity with temperature for laminar flows. For laminar flows the 'change of scale' effect is dominant, the perturbation effect being negligible, e.g. for $\lambda = 1$ the result of putting $\mu \propto (T)^{0.76}$ as against $\mu \propto T$ on the velocity profile is at most 4% for Mach numbers up to about 5. Further for $|h| < 0.3$ the effect of 'h' on the "change of scale" is negligible. Thus for compressible laminar flow the basic cause of changes in the velocity profile is the Mach number acting through the "change of scale" effect. Physically the larger the Mach number the smaller is the width of the mixing region, e.g. when the lower stream is at rest and the upper stream has a Mach number of five the width of the mixing region is about three quarters that of an incompressible flow of the same Reynolds number. The bulk of this narrowing of the mixing region is in the higher velocity half of the profile - as is to be expected on physical grounds; since in the lower velocity half of the profile the effects of compressibility are small - due to the low local Mach number there. When the temperature of the upper stream is higher than that of the lower stream (i.e. $h > 0$), the upper half of the profile tends to broaden, and the lower half to shrink. This effect is reversed when $h < 0$. This is to be expected on physical grounds, because molecules in a hot stream will have random velocities which are greater than those in the lower stream and will therefore tend to transfer the momentum of the jet to the surroundings to a greater extent than the colder stream.

For turbulent streams the perturbation terms and the change of scale have effects of the same order of magnitude. It is found on computing the non dimensional velocity profile that they almost cancel each other.

Thus the form of the non-dimensional velocity profile is left sensibly unchanged from that obtained in incompressible flow. In order to test this last result against experiment a scale factor σ must be used[x], where $\xi = \sigma(y-y_0)/x$ if the coefficient of eddy viscosity is taken (following Görtler 1942) to be $\varepsilon_0 x/L$. ($y = y_0$ gives the locus of points in the flow at which $U = U_0$.) From the equations developed above, $\sigma = (U_0 L/2\varepsilon_0)^{1/2}$. The theory has been applied to the case examined experimentally by Laurence (1955), namely to a turbulent jet at Mach number 0·7 entering a medium at rest. The observations chosen for comparison were those taken in the neighbourhood of the orifice where the core of the jet was at constant velocity. This might be expected to give the closest representation to the mixing of two parallel streams. The results have been plotted in the figure with u/U_c as ordinate (U_c is the velocity of the core of the jet) and with $(y_0 - y)/x$ as abscissa. The best fit between theory and experiment was obtained for σ 12·7, when $\xi = \sigma(y_0 - y)/x$. The fit is seen to be good except in the part of the profile where the velocity is lower; in this region the gradient of the theoretical profile is less steep than that of the experimental profile. Now over the whole flow the gradient of the laminar profile corresponding to the laminar Reynolds number of the jet is steeper than the turbulent profile. This suggests that turbulence may not be fully developed in the lower velocity region, that is, the flow may be only intermittently turbulent there.[±]

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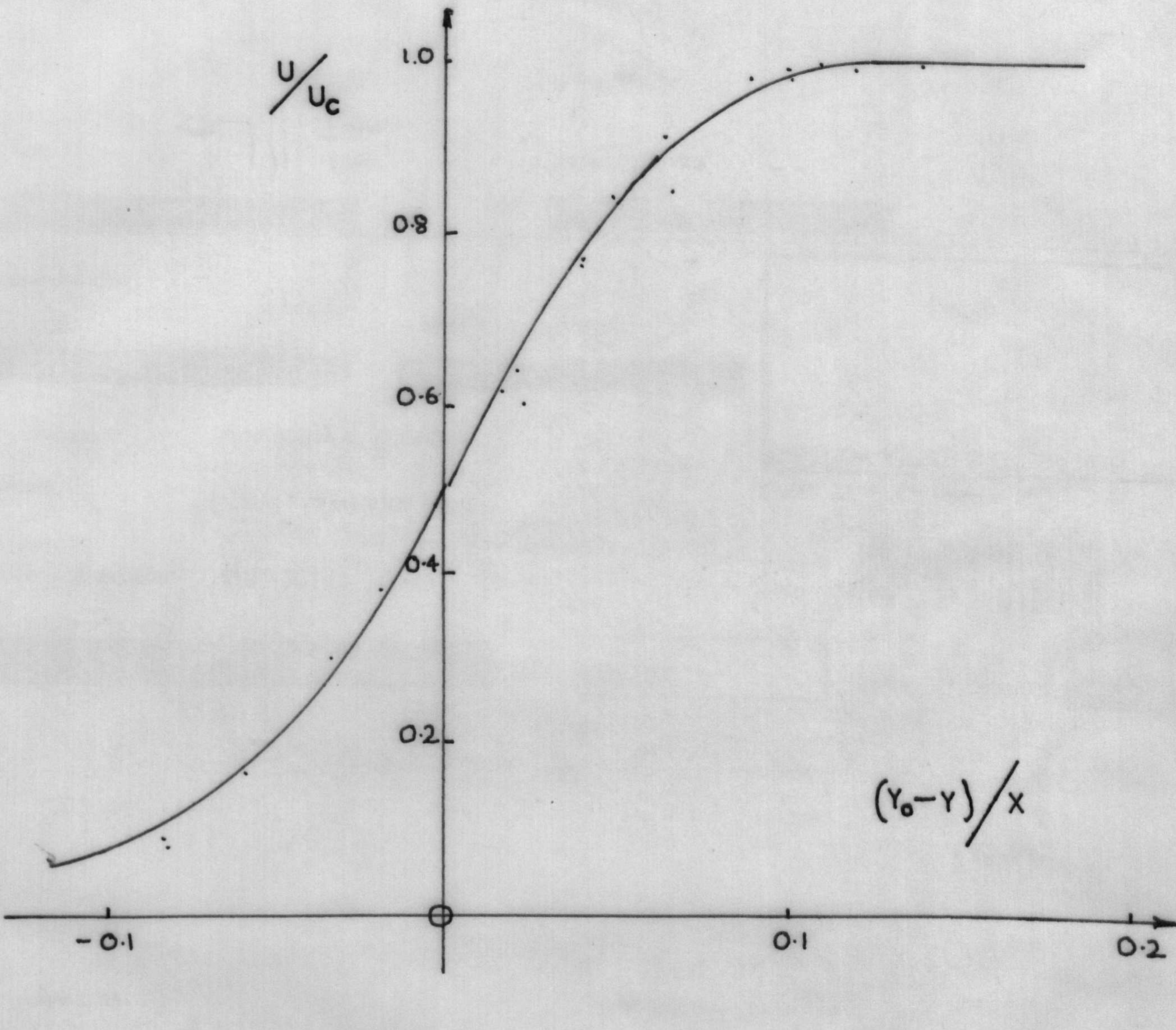
x The scale factor σ is not an absolute constant but may vary with the Mach number of the undisturbed stream, that is with ω_0^2. This does not affect the statement made in the preceding sentence.

± Johannesen has shown that the solution f_{00}' gives an accurate fit, except in the region of lower velocity, to his experimental results for the non-dimensional velocity profile of the mixing region near the orifice when a round turbulent jet issues with Mach number 1·6 into a medium at rest. The value of σ used was 21·9.



COMPARISON BETWEEN LAURENCE'S RESULTS AND THE THEORETICAL PROFILE WITH $\sigma = 12.7$ FOR THE MIXING REGION NEAR THE ORIFICE WHEN A TURBULENT JET OF AIR ISSUES AT MACH NUMBER 0.7 INTO AIR AT REST

APPENDIX I

Discussion of the solutions of the equation

$$y''' + 2 f_{oo} y'' + 2 f_{oo}'' y = 0 \quad \text{when } \lambda = 1. \tag{i}$$

It may be shown[*] that the function f_{oo} near $\zeta = +\infty$ behaves like

$a + 2\zeta$ + an exponentially decaying function of ζ.

and that near $\zeta = -\infty$, f_{oo} behaves like

$-b$ + an exponentially decaying function of ζ.

Since an integral of equation (i) is known to be $y = f_{oo}'$, the equation may be reduced to one of second order. If $y = f_{oo}' q$ and p is put equal to q', then

$$f_{oo}' p'' + (3 f_{oo}'' + 2 f_{oo} f_{oo}') p' + f_{oo}''' p = 0. \tag{ii}$$

Consider first the behaviour of solutions of equation (ii) near $\zeta = +\infty$. Equation (ii) may be written in the simple form

$$v'' - \alpha^2 v = 0 \quad \text{(iii)}, \quad \text{where} \quad \alpha^2 = \frac{2FF''}{F'} + \frac{3}{4}\left(\frac{F''}{F'}\right)^2 + F^2 + F'$$

and v is given by $p = (F')^{-3/2} \exp\left[-\int_0^{\zeta} F\, d\zeta\right]$.

Equation (iii) may be written as

$$\frac{d^2 w}{ds^2} - w(1 + \phi) = 0, \tag{iv}$$

using the changes of variable $w = \sqrt{\alpha}.v$, $s = \int_0^{\zeta} \alpha\, d\zeta$, and where

$$\phi = -\frac{1}{4\alpha^2}\left(\frac{d\alpha}{ds}\right)^2 + \frac{1}{2\alpha}\frac{d^2\alpha}{ds^2}.$$

Now for large positive values of ζ $\alpha^2 \sim f_{oo}^2 + 2$ + an exponentially decaying function.

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[*] Lock (1951)

Thus $$s = \int^{\xi} \alpha \, d\xi \to \infty \text{ as } \xi \to +\infty.$$

Thus the function ϕ may be shown to be $O(s^{-2})$, and $\int^{s} |\phi| ds \sim O(s^{-1})$ when s is very large and positive.

Now a theorem given by Bellman (1953) states that for the equation (iv), if $\int^{s} |\phi| ds \to 0$ as $s \to \infty$ then the two independent integrals of (iv), which may be taken as w_1 and w_2, lie between the limits

$$e^{+s - c_1 \int^{s} |\phi| ds} \le w_1 \le e^{+s + c_1 \int^{s} |\phi| ds},$$

and $$e^{-s - c_1 \int^{s} |\phi| ds} \le w_2 \le e^{-s + c_1 \int^{s} |\phi| ds},$$

respectively.

Hence applying this theorem $w_1 \sim e^{s}$ and $w_2 \sim e^{-s}$ for large s. Thus translating back into terms of v, ξ it may be seen that the two independent solutions of (iii) behave like

$$e^{\eta^2} \text{ and } e^{-\eta^2} \quad \text{respectively, where } \eta = \frac{a}{2} + \xi,$$

when ξ is very large and positive. Finally it may be shown that near $\xi = +\infty$, the integrals of the equation, apart from f_{00}' which is of order 2, behave like ξ and 1.

Consider now the behaviour of solutions of equation (i) near $\xi = -\infty$. For very large negative values of ξ the equation (ii) becomes approximately

$$p'' - 4bp' + 4b^2 p = 0.$$

This equation may be integrated immediately to give two integrals which are $e^{2b\xi}$ and $\xi e^{2b\xi}$.

Thus translating this result back into the variables of equation (i), it may be shown that two of the integrals of equation (i) behave like 1 and ξ near $\xi = -\infty$, the other integral being of course f_{00}'.

From this analysis it is seen that equation (iii), while it is the simplest form of the equation, is not suitable for numerical computation because of its behaviour near $\xi = +\infty$. This leaves the forms (i) and (ii). While both of these are suitable, nevertheless the computation required in calculating the coefficients in equation (ii) as compared with equation (i), makes equation (i) the most useful form of the equation for numerical solution.

APPENDIX II

MEKSYN'S METHOD

In 1956 Meksyn published a method for the solution of the boundary layer equations which depended on the evaluation of a certain integral form of the equations of motion. One of his applications was to Blasius' equation governing the motion of fluid past a flat plate. His method may be adapted to the equation $f_{00}''' + 2 f_{00} f_{00}'' = 0$ (1), with the boundary conditions

$$f_{00}'(-\infty) = 0, \quad f_{00}'(0) = 1, \quad f_{00}'(+\infty) = 2.$$

By means of the solution, the initial values obtained by the earlier method may be checked. The values obtained agree to about 1%.

Let the boundary conditions be stated as

$$f_{00}(0) = \alpha, \quad f_{00}'(0) = 1, \quad f_{00}''(0) = \beta_1,$$

where α and β_1 are constants to be evaluated so that the conditions at $\pm\infty$ may be satisfied.

Make the change of dependent variable from $f_{00}(z)$ to $\phi(z)$ so that

$$f_{00}(z) = \phi(z) + z + \alpha,$$

then

$$f_{00}'(z) = \phi'(z) + 1$$

and

$$f_{00}''(z) = \phi''(z).$$

When equation (1) is integrated and the boundary conditions (at $z = 0$) are used

$$\phi'' = \beta_1 \exp\left[-\left(z^2 + 2\int_0^z \phi \, dz\right)\right] e^{-2\alpha z}.$$

Now it is known that α is negative and is numerically about 0·169. Thus $e^{-2\alpha\zeta}$ is a slowly varying function compared with $e^{-\zeta^2}$ when ζ is large and positive.

Let
$$\tau = \zeta^2 + 2\int_0^\zeta \phi \, d\zeta$$
$$= \zeta^2\left[1 + a_1\zeta + a_2\zeta^2 + \cdots\right]. \qquad \text{(ii)}.$$

((ii) is obtained by assuming a power series expansion for ϕ in terms of ascending powers of ζ .) The constants a_1, etc., in (ii) have the values

$$a_1 = \beta_1/3 \quad , \quad a_2 = -\alpha\beta_1/6 \quad , \quad a_3 = -\beta_1(1-2\alpha^2)/30 ,$$

$$a_4 = -\beta_1(\beta - 6\alpha + 4\alpha^3)/180 \quad \text{etc.}$$

Now reverse the series (ii) to give

$$\zeta = \sum_{m=0}^{\infty} b_m \tau^{m+1/2} . \qquad \text{(iii)}$$

[The method used in reversing the series is due to Watson (1944) and is given in Meksyn's paper.]

Now $\phi'' = \beta_1 e^{-\tau} e^{-2\alpha\zeta}$.

Therefore
$$\phi' = \beta_1 \int_0^\zeta e^{-\tau} e^{-2\alpha\zeta} d\zeta$$
$$= \beta_1 \int_0^\tau e^{-\tau}\left(e^{-2\alpha\zeta}\frac{d\zeta}{d\tau}\right) d\tau . \qquad \text{(iv)}$$

Now the series (ii) and (iii) have only a finite range of convergence. Hence, when they are substituted in (iv) and ζ is allowed to tend to infinity a divergent series is obtained. Since, however, $\phi'(\zeta) \to 1$ as

$\xi \to +\infty$ it is assumed that the divergent series on the right side of (iv) may be summed by Euler's transformation.

Let $e^{-2\alpha\xi}\dfrac{d\xi}{d\tau} = \sum_{m=0}^{\infty} d_m \tau^{\frac{m-1}{2}}$; the coefficients d_m being obtained by Watson's method. In fact

$$d_0 = 1/2, \quad d_1 = -\alpha - \beta_1/6, \quad d_2 = \alpha^2 + 5\alpha\beta_1/8 + 5\beta_1^2/48, \text{ etc.}$$

Thus when $\xi \to \infty$ the right side of (iv) may be written formally as

$$\beta_1 \int_0^{\infty} e^{-\tau}\left(\sum_{m=0}^{\infty} d_m \tau^{\frac{m-1}{2}}\right) d\tau$$

$$= \beta_1 \sum_{m=0}^{\infty} d_m \Gamma\left(\frac{m+1}{2}\right).$$

Using Euler's transformation as the first five terms of this series the following, convergent, finite expansion is obtained for the right side of (iv) in the limit $\xi \to \infty$.

$$\beta_1\left[0{\cdot}8862 - 0{\cdot}0070 + 0{\cdot}0050 + 0{\cdot}0091 + 0{\cdot}0037 - 0{\cdot}0007\right] \qquad \text{(v)},$$

when the values $\alpha = -0{\cdot}1693$, $\beta_1 = 1{\cdot}0996$ are used. (These values of α and β_1 are those obtained by the iterative method.) Thus equating expression (v) to unity (because $\phi'(\infty)=1$) gives a value for β_1 which is

1·1157

When the small number of terms taken in the expansion is taken into account this value for β_1 is quite close to that given by the iterative method. The method shows that the iterative solution is certainly accurate to within one per cent.

ξ	F_1 (e)	F_1' (o)	F_2 (o)	F_2' (e)	F_3' (o)	G_1 (e)	G_1' (o)	G_2 (o)	G_2' (e)	H_1 (o)	H_1' (e)	H_2' (o)
0	−0·1610	0	0	0	0	−0·8051	0	0	0	0	0	0
0·1	−0·1554	0·1125	0	0·0018	−0·0024	−0·8107	−0·1117	−0·0007	−0·0215	0·0003	0·0090	−0·0124
0·2	−0·1386	0·2227	0·0004	0·0070	−0·0047	−0·8272	−0·2168	−0·0059	−0·0890	0·0024	0·0347	−0·0220
0·3	−0·1110	0·3286	0·0015	0·0148	−0·0067	−0·8536	−0·3094	−0·0171	−0·1738	0·0077	0·0735	−0·0269
0·4	−0·0731	0·4284	0·0035	0·0244	−0·0084	−0·8885	−0·3846	−0·0398	−0·2808	0·0173	0·1202	−0·0270
0·5	−0·0256	0·5205	0·0064	0·0347	−0·0099	−0·9299	−0·4394	−0·0733	−0·3881	0·0318	0·1688	−0·0236
0·6	0·0307	0·6039	0·0104	0·0443	−0·0114	−0·9757	−0·4723	−0·1169	−0·4807	0·0509	0·2133	−0·0192
0·7	0·0949	0·6778	0·0153	0·0524	−0·0129	−1·0236	−0·4839	−0·1685	−0·5474	0·0741	0·2489	−0·0168
0·8	0·1660	0·7421	0·0208	0·0583	−0·0145	−1·0718	−0·4760	−0·2253	−0·5814	0·1003	0·2726	−0·0186
0·9	0·2430	0·7969	0·0268	0·0614	−0·0162	−1·1183	−0·4518	−0·2837	−0·5815	0·1282	0·2827	−0·0256
1·0	0·3250	0·8427	0·0330	0·0619	−0·0180	−1·1617	−0·4151	−0·3405	−0·5508	0·1564	0·2798	−0·0375
1·1	0·4112	0·8802	0·0391	0·0598	−0·0196	−1·2010	−0·3701	−0·3930	−0·4960	0·1838	0·2655	−0·0527
1·2	0·5008	0·9103	0·0449	0·0557	−0·0209	−1·2356	−0·3208	−0·4392	−0·4254	0·2092	0·2425	−0·0687
1·3	0·5931	0·9340	0·0502	0·0501	−0·0217	−1·2652	−0·2707	−0·4779	−0·3479	0·2321	0·2139	−0·0832
1·4	0·6875	0·9523	0·0549	0·0436	−0·0219	−1·2898	−0·2225	−0·5088	−0·2710	0·2519	0·1826	−0·0940
1·5	0·7834	0·9661	0·0589	0·0368	−0·0213	−1·3098	−0·1784	−0·5323	−0·2004	0·2686	0·1511	−0·0998
1·6	0·8806	0·9763	0·0622	0·0301	−0·0201	−1·3257	−0·1396	−0·5492	−0·1400	0·2822	0·1214	−0·1004
1·7	0·9786	0·9838	0·0650	0·0240	−0·0184	−1·3379	−0·1066	−0·5607	−0·0914	0·2930	0·0949	−0·0962
1·8	1·0777	0·9891	0·0670	0·0186	−0·0163	−1·3472	−0·0796	−0·5670	−0·0546	0·3013	0·0722	−0·0881
1·9	1·1764	0·9928	0·0687	0·0141	−0·0139	−1·3540	−0·0580	−0·5720	−0·0285	0·3076	0·0535	−0·0774
2·0	1·2758	0·9953	0·0699	0·0104	−0·0116	−1·3589	−0·0413	−0·5739	−0·0117	0·3122	0·0387	−0·0655

Table 1. Values of F_1, etc., for small values of λ. (o denotes an odd function of ξ; e denotes an even function of ξ.)

ξ	f_{00} $(+\xi)$	f'_{00} $(+\xi)$	f_{00} $(-\xi)$	f'_{00} $(-\xi)$	f_{01} $(+\xi)$	f'_{01} $(+\xi)$	f_{01} $(-\xi)$	f'_{01} $(-\xi)$	f_{10} $(+\xi)$	f'_{10} $(+\xi)$	f_{10} $(-\xi)$	f'_{10} $(-\xi)$
0	−0·169	1·000	−0·169	1·000	0·286	0	0·286	0	−0·863	0	−0·863	0
0·1	−0·064	1·112	−0·264	0·892	0·285	−0·006	0·284	0·023	−0·866	−0·084	−0·866	0·045
0·2	0·053	1·224	−0·348	0·790	0·285	0·006	0·280	0·060	−0·880	−0·209	−0·870	0·058
0·3	0·181	1·336	−0·422	0·695	0·287	0·037	0·272	0·105	−0·908	−0·370	−0·875	0·047
0·4	0·320	1·444	−0·487	0·607	0·293	0·081	0·259	0·153	−0·953	−0·555	−0·878	0·020
0·5	0·470	1·545	−0·544	0·527	0·304	0·132	0·242	0·200	−1·007	−0·741	−0·877	−0·011
0·6	0·629	1·637	−0·593	0·456	0·319	0·180	0·219	0·242	−1·098	−0·905	−0·873	−0·043
0·7	0·797	1·718	−0·635	0·392	0·339	0·219	0·193	0·277	−1·194	−1·023	−0·865	−0·071
0·8	0·972	1·787	−0·672	0·337	0·362	0·242	0·164	0·303	−1·298	−1·077	−0·855	−0·093
0·9	1·154	1·844	−0·703	0·288	0·387	0·247	0·133	0·321	−1·404	−1·062	−0·842	−0·110
1·0	1·340	1·889	−0·729	0·245	0·411	0·235	0·101	0·331	−1·505	−0·983	−0·828	−0·120
1·1	1·531	1·924	−0·752	0·209	0·444	0·209	0·067	0·333	−1·595	−0·857	−0·814	−0·124
1·2	1·725	1·949	−0·771	0·177	0·453	0·174	0·034	0·330	−1·671	−0·705	−0·798	−0·123
1·3	1·921	1·968	−0·788	0·150	0·468	0·138	0·002	0·321	−1·732	−0·548	−0·783	−0·119
1·4	2·118	1·980	−0·801	0·127	0·480	0·102	−0·030	0·308	−1·757	−0·401	−0·769	−0·111
1·5	2·316	1·988	−0·813	0·107	0·489	0·072	−0·060	0·293	−1·774	−0·276	−0·754	−0·102
1·6	2·516	1·993	−0·823	0·090	0·495	0·048	−0·088	0·278	−1·794	−0·178	−0·742	−0·091
1·7	2·715	1·996	−0·831	0·076	0·499	0·030	−0·115	0·257	−1·805	−0·100	−0·730	−0·080
1·8	2·915	1·998	−0·838	0·064	0·501	0·018	−0·140	0·239	−1·811	−0·055	−0·719	−0·068
1·9	3·115	1·999	−0·844	0·054	0·502	0·009	−0·162	0·220	−1·813	−0·021	−0·708	−0·057
2·0	3·315	2·000	−0·849	0·045	0·503	0·004	−0·184	0·202	−1·813	−0·008	−0·698	−0·046

Table 2. Values of f_{00}, f'_{00}, f_{01}, f'_{01}, f_{10}, f'_{10}, when $\lambda = 1$ (i.e. the lower stream is at rest).

CHAPTER VII

THE EQUATION OF DIFFUSION FOR LAMINAR AND TURBULENT FLOWS

This chapter together with the following two chapters is concerned with the diffusion of a jet of gas into an atmosphere composed of a different gas. To simplify this problem, the temperature is assumed to be constant throughout the flow field; a necessary condition for this to be true is that the Mach number of the motion be small. Next the flow is taken to be of boundary layer type. It follows, therefore, that the appropriate equations of momentum and continuity are in the case of the laminar jet equations II (5), (6) and for the turbulent jet equations III (6), (15) respectively (when the Reichardt theory of turbulence is used). The boundary layer assumptions when applied to free flows give the result that pressure is constant to a first approximation. Thus, since the temperature is also constant, the variation in density at a point in the field of flow depends only on the relative proportions of the two gases at that point. In order to find what these proportions are at points in the flow, it is necessary to set up a diffusion equation, in a simplified boundary layer form, and then to solve this equation simultaneously with the momentum and continuity equations. This approach was first tried by Chou in 1947 for laminar jets.

In this chapter the boundary layer equations of diffusion for both laminar and turbulent flows are derived and are combined in one. The last section of this chapter deals with the variation of quantities such as viscosity, density, etc., with varying concentrations of one gas in another.

DERIVATION OF THE BOUNDARY LAYER

EQUATION OF DIFFUSION FOR STEADY LAMINAR MOTION

Ficks law states that in a gas mixture containing N_1 molecules of gas '1' and N_2 molecules of gas '2' per unit volume, the flux of molecules of gas 1 is given by $-D_{12}\,\mathrm{grad}\,N_1$, where D_{12} depends (slightly) on the proportions of the gases 1, 2 present.

Consider now a control volume τ bounded by a surface S.
The number of molecules of gas '1' entering this volume by convection per unit time is

$$-\int_S N_1\,\underset{\sim}{u}\cdot d\underset{\sim}{S}$$

where $\underset{\sim}{u}$ is the velocity vector of the flow.

The number of molecules entering by molecular diffusion in unit time is, using Fichs law:

$$-\int_S \left(-D_{12}\,\mathrm{grad}\,N_1\right)\cdot d\underset{\sim}{S}.$$

When the law of conservation of matter is used it follows that:

$$\frac{\partial}{\partial t}\int_\tau N_1\,d\tau = -\int_S N_1\,\underset{\sim}{u}\cdot d\underset{\sim}{S} + \int_S D_{12}\,\mathrm{grad}\,N_1\cdot d\underset{\sim}{S}.$$

When Gauss' theorem is applied to the right hand side of this equation it follows that, since the volume τ is arbitrary,

$$\frac{\partial N_1}{\partial t} = -\,\mathrm{div}\,(N_1\,\underset{\sim}{u}) + \mathrm{div}\,(D_{12}\,\mathrm{grad}\,N_1)$$

at all points in the flow.

In the case of two dimensional flow with respect to axes (x, y) whose velocity components relative to these axes are (u, v) respectively, this equation becomes:

$$\frac{\partial N_1}{\partial t} + \frac{\partial (N_1 u)}{\partial x} + \frac{\partial (N_1 V)}{\partial y} = \frac{\partial}{\partial x}\left(D_{12}\frac{\partial N_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{12}\frac{\partial N_1}{\partial y}\right). \tag{1}$$

For steady boundary layer flows whose main direction of motion is that of x increasing this equation reduces to:

$$\frac{\partial (N_1 u)}{\partial x} + \frac{\partial (N_1 V)}{\partial y} = \frac{\partial}{\partial y}\left(D_{12}\frac{\partial N_1}{\partial y}\right). \tag{2}$$

It is assumed, in the derivation of equation (2) that $\frac{\partial N_1}{\partial x} << \frac{\partial N_1}{\partial y}$ - the usual boundary layer postulate.

DERIVATION OF THE TURBULENT BOUNDARY LAYER

EQUATION OF DIFFUSION FOR STEADY MEAN MOTION

When the flow is turbulent the transport of molecules of gas '$_1$' by molecular diffusion is unimportant compared with the transport by turbulent convection. Thus the turbulent flow sensibly satisfies the left side of (1) equated to zero. Write now the fluctuating quantities N_1 as $\overline{N_1} + N_1'$, where $\overline{N_1}$ is the statistical mean* of N_1 and N_1' is the turbulent fluctuation of N_1 about the mean.

Thus the turbulent diffusion equation may be written:

$$\frac{\partial \overline{N_1}}{\partial t} + \frac{\partial N_1'}{\partial t} + \frac{\partial}{\partial x}(\overline{N_1}\,\overline{u}) + \frac{\partial}{\partial y}(\overline{N_1}\,\overline{v}) + \frac{\partial}{\partial x}\left[N_1'\overline{u} + \overline{N_1}u'\right] +$$
$$+ \frac{\partial}{\partial y}\left[N_1'\overline{v} + \overline{N_1}v'\right] + \frac{\partial}{\partial x}(N_1'u') + \frac{\partial}{\partial y}(N_1'v') = 0.$$

When the statistical mean of this equation is taken, the equation of steady mean motion is:

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* See p. (22)

$$\frac{\partial}{\partial x}(\bar{N}_1 \bar{u}) + \frac{\partial}{\partial y}(\bar{N}_1 \bar{v}) = -\frac{\partial}{\partial x}\overline{(N_1' u')} - \frac{\partial}{\partial y}\overline{(N_1' v')}.$$

This equation involves two quantities which have to be determined experimentally - namely $\overline{N_1' u'}$ and $\overline{N_1' v'}$. A hypothesis similar to that of Boussinesq in Chapter III, is now used, to give:

$$\left(-\overline{N_1' u'}, -\overline{N_1' v'}\right) = \varepsilon_D \left(\frac{\partial \bar{N}_1}{\partial x}, \frac{\partial \bar{N}_1}{\partial y}\right).$$

ε_D the eddy diffusion coefficient is a function of the partial coordinates. Thus the diffusion equation may now be written:

$$\frac{\partial}{\partial x}(\bar{N}_1 \bar{u}) + \frac{\partial}{\partial y}(\bar{N}_1 \bar{v}) = \frac{\partial}{\partial x}\left(\varepsilon_D \frac{\partial \bar{N}_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_D \frac{\partial \bar{N}_1}{\partial y}\right).$$

When the boundary layer postulates are used, viz. that $\frac{\partial \bar{N}_1}{\partial x} << \frac{\partial \bar{N}_1}{\partial y}$ the above equation reduces to

$$\frac{\partial}{\partial x}(\bar{N}_1 \bar{u}) + \frac{\partial}{\partial y}(\bar{N}_1 \bar{v}) = \frac{\partial}{\partial y}\left(\varepsilon_D \frac{\partial \bar{N}_1}{\partial y}\right). \qquad (3)$$

Equation (3) is the turbulent boundary layer equation of diffusion for steady mean motion. Now Reichardt's hypothesis is that ε_D is a function of x alone; i.e. that ε_D does not vary across a section of the jet. The exact form of the variation of ε_D is found as in the case of ε by a similarity hypothesis or by appeal to experiment. ε_D is found to be an approximately constant multiple of ε ; it may be written $\varepsilon_D = E\varepsilon$ where ε is the eddy coefficient of kinematic viscosity.

Equations (2) and (3) may be combined in the equation:

$$\frac{\partial}{\partial x}(N_1 u) + \frac{\partial}{\partial y}(N_1 v) = \frac{\partial}{\partial y}\left(\varepsilon_D \frac{\partial N_1}{\partial y}\right), \qquad (4)$$

where $\varepsilon_D = D_{12}$ in laminar motion, and is the eddy coefficient of diffusion in turbulent flow; when the flow is turbulent the physical dependent variables N_1, u, etc., refer to statistical means of these quantities.

THE MASS FLUX EQUATION

The boundary conditions on a two dimensional jet issuing from an orifice into an atmosphere which is of gas '2' are:

on the axis of the jet, i.e. $y = 0$, $v = 0$, $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial N_1}{\partial y} = 0$,

on the boundaries of the jet, i.e. on $y = \pm\infty$; $u = 0$ and $N_1 = 0$.

Integrating equation (4) with respect to y between the limits 0 and ∞, and using the boundary conditions:

$$\frac{d}{dx}\int_0^\infty N_1 u \, dy = 0,$$

i.e. $\int_0^\infty N_1 u \, dy$ is a constant which is independent of x. This integral is in fact half the flux of the gas and of the orifice.

EXPERIMENTAL AND THEORETICAL LAWS FOR THE VARIATION OF DENSITY, VISCOSITY, ETC., OF GAS MIXTURES

This section is concerned with the variation of the physical quantities like density and viscosity for gas mixtures. It is convenient first to define a new variable namely the concentration, which will be denoted by c and used in place of N_1. c is defined as the number of molecules of gas 1 in unit volume divided by the total number of molecules in that volume. Since both pressure and temperature are constant it follows from Dalton's law of partial pressures that the total number of molecules per unit volume is a constant which is denoted by N. Thus $c = N_1/N$ and $N_1 \propto c$. Equation (4) in terms of the new variable c is:

$$\frac{\partial}{\partial x}(cu) + \frac{\partial}{\partial y}(cv) = \frac{\partial}{\partial y}\left(\varepsilon_D \frac{\partial c}{\partial y}\right). \tag{5}$$

Let m_1 and m_2 be the molecular weights of gas 1 and gas 2 respectively. Then the density ρ of the gas mixture is given by

$$\begin{aligned} \rho &= N_1 m_1 + (N - N_1) m_2 \\ &= (1 + \beta c)\rho_2 , \end{aligned} \tag{6}$$

where ρ_2 is the density of the second gas.

The variation of the coefficient of viscosity of a binary gas mixture is in general a very complicated function of their relative properties, and of the physical properties of the molecules of the gases.

Thiesen (1902), using as a basis the kinetic theory of gases, has derived an expression for the viscosity μ_{12} of a mixture of two gases denoted by the subscripts 1, 2 respectively. This expression is

$$\mu_{12} = \frac{\mu_1}{1 + A_1 N_2/N_1} + \frac{\mu_2}{1 + A_2 N_1/N_2},$$

where N_1 and N_2 are the number of molecules of gases 1, 2 per unit volume respectively, and A_1, A_2 are constants relating to the two gases, which depend on certain properties of the gas molecules.

Schroer (1936) gives numerical values for these constants for various gas mixtures and compares them with experimental results.

When the concentration of the first gas is small, the law of variation simplifies to:

$$\mu^* = \mu_{12}/\mu_2 \doteq 1 + c\left[\frac{\mu_1}{\mu_2}\cdot\frac{1}{A_1} - A_2\right],$$

where c is the concentration of gas 1 in the mixture and μ_1, μ_2 are the viscosities of the two gases respectively.

This relation may be written:

$$\mu^* \doteq 1 + \gamma c, \qquad \text{where} \qquad \gamma = \frac{\mu_1}{\mu_2}\cdot\frac{1}{A_1} - A_2 .$$

<u>Coefficient of Diffusion D_{12}</u> This coefficient does not vary very much with the varying mixture of gases - (Jeans 1948). Accordingly it has here been taken to be constant.

CHAPTER VIII

TWO DIMENSIONAL MIXING OF A JET OF ONE GAS INTO AN ATMOSPHERE OF A SECOND GAS FAR FROM THE ORIFICE

This chapter is concerned with the two-dimensional flow of gas out of a narrow slit into a medium consisting of another gas. The profiles of velocity and concentration in the jet are determined for both laminar and turbulent flows. The solution found is valid only at large distances from the slit.

In order to solve the problem analytically four basic assumptions have been made. These are:

(i) the gases are incompressible, i.e. the velocities are small in comparison with the local speed of sound;

(ii) the temperature is constant throughout;

(iii) the boundary layer approximations are valid;

and (iv) the pressure is constant over the mixing region.

In turbulent flow the Reichardt constant exchange coefficient hypothesis is used.

In this chapter a method similar to that used in studying the effects of compressibility on jet mixing is employed. This method consists in expanding both the stream function and the concentration in a Rayleigh-Jansen series of powers of a parameter which is in this case proportional to the concentration of one gas in the mixture. As in the case of compressible flows in previous chapters the laminar and turbulent states are treated simultaneously.

EQUATIONS OF MOTION

Let u and v be the velocity components (or mean velocity components in the case of turbulent flow) parallel to rectangular Cartesian axes (x, y). Let the origin of coordinates be taken at the point at which mixing begins, i.e. at the slit. It was shown in the previous chapter that, assuming pressure and temperature to be constant throughout the mixing region, the density ρ of the gas mixture in the jet downstream is given by:

$$\rho^* = \rho/\rho_2 = 1 + \beta c,$$

where ρ_2 is the density of the gas in the surrounding medium (ρ_1 is the density of the gas '1' coming out of the slit); $\beta = \frac{m_1}{m_2} - 1$, m_1 and m_2 being the molecular weights of the two gases respectively; and c is the molecular concentration of gas '1' in the mixture. When c is small the viscosity μ of the mixture of gases in the jet is given by

$$\mu^* = \mu/\mu_2 \doteqdot 1 + \gamma c,$$

to a first approximation. (μ_1 and μ_2 refer to the viscosities of gases 1, 2 respectively and γ is a constant defined in Chapter VIII.) Then is to a first approximation

$$\mu^* \rho^* \doteqdot 1 + (\beta + \gamma) c .$$

The equation of motion for both laminar and turbulent motion of a steady two dimensional jet whose main direction of motion is that of x increasing may be written (IV (2))

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon \rho \frac{\partial u}{\partial y}\right) \quad \text{— momentum} \qquad (1)$$

and

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad \text{— continuity} \qquad (2)$$

In laminar flow $\varepsilon = \mu / \rho$, and in turbulent flow ε is Reichardt's coefficient of eddy kinematic viscosity.

In addition to the above equations of momentum and continuity it is also necessary to take into account the equation which governs the diffusion of the gas in the jet into the surrounding atmosphere. This equation may be written for both laminar and turbulent flows as (VII (5))

$$\frac{\partial}{\partial x}(cu) + \frac{\partial}{\partial y}(cv) = \frac{\partial}{\partial y}\left(\varepsilon_D \frac{\partial c}{\partial y}\right), \tag{3}$$

where $\varepsilon_D = D_{12}$ in the case of laminar flow and ε_D is the eddy coefficient of diffusion when the motion is turbulent. ε_D may be shown by appeal to conditions of similarity and to experimental evidence to be a constant multiple of ε ; this constant E , which is of order unity, is defined by $\varepsilon_D = E\varepsilon$.

The equation of continuity is replaced by the suitably defined stream function ψ given by

$$\rho u = \psi_y \,, \quad \rho v = -\psi_x \,.$$

As in the treatment of the compressible jet, the change of variables from (x, y) to (x, z) defined by

$$z = \int_0^y \rho^* dy$$

is introduced. In terms of the new variables equation (1) becomes

$$\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial \zeta \partial z} - \frac{\partial \psi}{\partial \zeta}\frac{\partial^2 \psi}{\partial z^2} = \alpha \frac{\partial}{\partial z}\left[\phi(c)\frac{\partial^2 \psi}{\partial z^2}\right]. \tag{4}$$

In this equation ζ is a variable which replaces x defined in Chapter V. In laminar flow $\zeta = x$, while in the case of turbulent motion

$$\zeta = \int_0^x \frac{\varepsilon(x)}{\varepsilon_0}\, dx \,,$$

where ε_0 is Reichardt's constant exchange coefficient. In laminar flow $\alpha = \mu_2$ and $\phi(c) = \mu^* \rho^* \doteq 1 + (\beta + \gamma) c$; when the motion is turbulent $\alpha = \varepsilon_0 \rho_2$ and $\phi(c) = (\rho^*)^2 \doteq 1 + 2\beta c$. Thus for both laminar and turbulent flow $\phi(c) \doteq 1 + nc$ where n takes the values $\beta + \gamma$ and 2β in laminar and turbulent flows respectively.

The equation of diffusion may be written in terms of the new variables as

$$\frac{\partial \psi}{\partial z} \frac{\partial c}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \frac{\partial c}{\partial z} = \alpha \sigma (\rho^*)^2 \frac{\partial}{\partial z} \left[\rho^* \frac{\partial c}{\partial z} \right], \tag{5}$$

where σ is a dimensionless constant of order unity which is equal to $D_{12} \rho_2 / \mu_2$ in laminar flow and E when the flow is turbulent.

BOUNDARY CONDITIONS

The boundary conditions on the velocity are exactly those of Chapter V, i.e. on the axis of the jet, $y = 0$; $v = 0$ and $\frac{\partial u}{\partial y} = 0$.
and on the boundaries of the jet $y = \pm\infty$; $u = 0$.
In terms of the stream function ψ , these conditions imply:

on $z = 0$; $\psi = 0$ and $\frac{\partial^2 \psi}{\partial z^2} = 0$

and on $z = \pm\infty$; $\frac{\partial \psi}{\partial z} = 0$.

The boundary conditions that the concentration c satisfies are:

on the axis of the jet, i.e. $y = 0$ or $z = 0$

$$\frac{\partial c}{\partial y} = 0 \quad \text{or} \quad \frac{\partial c}{\partial z} = 0$$

and on the boundaries of the jet, i.e. $y = \pm\infty$ or $z = \pm\infty$

$$c = 0 .$$

SOLUTIONS OF EQUATIONS (4), (5)

A first approximation to the solution of equation (4) is obtained by putting $\rho^* = 1$, i.e. by neglecting any variation in density. When this approximation is made this equation has as its solution

$$\psi = 6 a \alpha \xi^{1/3} \tanh \eta , \tag{6}$$

a result due to Bickley, which has been proved in Chapter V. In (6) the similarity variable η is equal to $\alpha z / \xi^{2/3}$, and 'α' is a constant of the motion which depends on the initial momentum flux.

A first approximation to c may now be obtained by inserting the approximate stream function given by (6) in equation (5) and neglecting second order terms in c in that equation.

Equation (5) then becomes

$$\frac{\partial \psi}{\partial z} \frac{\partial c}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \frac{\partial c}{\partial z} = \alpha \sigma \frac{\partial^2 c}{\partial z^2} . \tag{7}$$

A solution is now sought for c in the form:

$$c = \frac{1}{\xi^m} g(\eta).$$

The value of m is found from the condition, found in Chapter VII, that the flux of molecules of gas '1' crossing any section of the jet is a constant which is independent of x . This requirement leads to m having the value 1/3,

i.e. to $c = \xi^{-1/3} g(\eta)$

When a concentration function of this type is inserted in equation (7) the required solution which satisfies the boundary conditions is

$$c = \frac{b}{\xi^{1/3}} (\operatorname{sech}^2 \eta)^{1/\sigma} ,$$

where b is a constant whose magnitude depends on the flux of molecules of gas '1' issuing from the slit. Its value is calculated using the result that the total mass flus (M) of molecules of gas '1' from the slit is given by:

$$M = 2\,(\beta+1)\,\rho_2 \int_0^{\infty} c u \, dy,$$

which when evaluated gives

$$M = 6 a \alpha b\,(\beta+1)\,B\left(1+\frac{1}{\sigma}, \frac{1}{2}\right)$$

where $B(\ell, m)$ is the Beta function.

It now remains to see what effect the diffusion of the first gas into the surrounding atmosphere has on the velocity profile. The method used is to expand the stream function and the concentration in a Rayleigh-Janzen series in terms of a parameter which should be related to the concentration in some way. In fact this parameter is found to be the first approximation to the concentration on the axis of the jet, viz $b\xi^{-1/3}$. Thus the appropriate expansions for ψ and c are:

$$\psi = 6 a \alpha \xi^{1/3}\left[\tanh \zeta + \frac{b}{\xi^{1/3}} F_1(\zeta) + \cdots \right] \tag{8}$$

and $$c = \frac{b}{\xi^{1/3}}\left[\operatorname{sech}^{2/\sigma}\zeta + \frac{b}{\xi^{1/3}} g_1(\zeta) + \cdots \right], \tag{9}$$

making use of the first order approximations. When these expansions (8) and (9) are inserted into the full equation for ψ i.e. equation (4) and the terms of order $b/\xi^{1/3}$ equated to zero, $F_1'(\zeta)$ is found to satisfy the equation:

$$F_1''' + 2\tanh\zeta\, F_1'' + 6\operatorname{sech}^2\zeta\, F_1' = 2n\frac{d}{d\zeta}\left[\tanh\zeta\,(\operatorname{sech}^2\zeta)^{\frac{1}{\sigma}+1}\right]. \tag{10}$$

Equation (10) may be readily integrated to give

$$F_1'(\zeta) = \frac{nt(1-t^2)}{\frac{1}{\sigma}+2}\left[\left(\frac{1}{\sigma}+2\right)B\left(t^2;\frac{1}{\sigma},\frac{1}{2}\right)^* + \int_0^t \frac{(1-t^2)^{1/\sigma}-1}{t^2}\,dt - 1\right]$$

as the solution which satisfies the boundary conditions: $F_1'(\pm\infty)=0$ and $F_1''(0)=0$.

In the important case when $\sigma = 1$ the solution has the simple form

$$F_1'(\zeta) = \frac{n}{3}(1-t^2)(5t^2-1)$$

TABLE OF VALUES OF $(1-t^2)(5t^2-1)$

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ζ	0	0.100	0.203	0.310	0.424	0.549	0.693	0.867	1.099	1.472	∞
$(1-t^2)(5t^2-1)$	-1.000	-0.941	-0.768	-0.501	-0.168	+0.188	+0.512	+0.740	+0.792	+0.580	0

The value of $F_1'(\zeta)$ on the axis of the jet is

$-n/(\frac{1}{\sigma}+2)$, a result which holds for all values of σ and n.

The momentum flux of the jet at any point in the flow is given by

$$\rho u^2 \doteq \frac{36a^4\alpha^2}{\rho_2 \xi^{2/3}}\,\mathrm{sech}^4\zeta\left[1 + \frac{b}{\xi^{1/3}}\left\{\beta\,(1-t^2)^{1/\sigma} + \frac{2n}{3}(5t^2-1)\right\}\right].$$

<u>'Change of scale' effect</u> The change of scale effect has already been defined above. It arises when the change is made from the independent variable z or ζ back to the physical variable y or $ay/\xi^{2/3}$

It can be shown that

$$ay/\xi^{2/3} = \zeta_1 \doteq \zeta - \frac{b\beta}{\xi^{1/3}}\tanh\zeta.$$

Thus to obtain the profile in terms of the physical variable ζ_1, the profile in terms of the variable ζ is distorted by a translation of amount

$$-\frac{b\beta}{\xi^{1/3}}\tanh\zeta.$$

* Note $B(x;\alpha,\beta)$ denotes the incomplete Beta function $\int_0^x x^{\alpha-1}(1-x)^{\beta-1}dx$.
Also note $t = \tanh\zeta$.

CONCLUSION

The x component of the velocity may be shown to be given by

$$u = \frac{6a^2\alpha}{\rho_2 \xi^{1/3}}\left[\operatorname{sech}^2 \zeta + \frac{b}{\xi^{1/3}} F_1'(2) + \cdots\right].$$

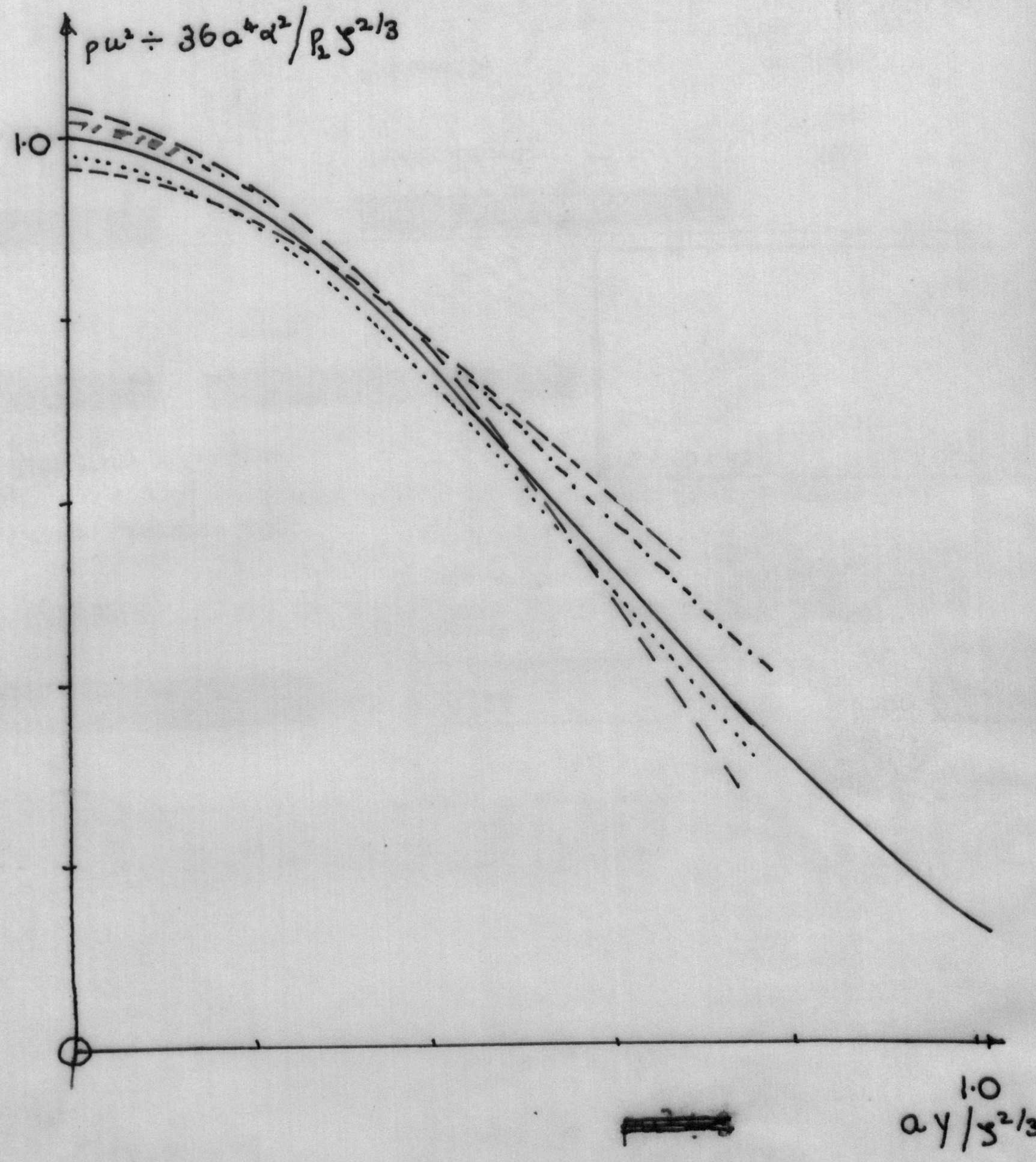
A first glance at this result shows that the fractional change in the axial velocity is given by

$$\frac{b}{\xi^{1/3}}\left(\frac{-n}{\frac{1}{\sigma}+2}\right).$$

In laminar flow γ is in general only about a third or a quarter of β, and so the order of magnitude of the proportional change in velocity on the axis is about $-\frac{b\beta}{3}$ (assuming $\sigma \sim 1$). This might at first sight be thought to be in disagreement with what is to be expected from physical arguments. For this result means that the heavier the molecules of gas, issuing from the slit, are in comparison with the molecules of the surrounding medium, the greater the <u>decrease</u> in the axial velocity, and vice versa. It might have been expected that a jet of dense gas would be much sharper than that of a less dense gas, because the more massive the molecule, the smaller is its deflection in any random collision with a lighter molecule. This apparent paradox is resolved by remembering that the total mass flux, and the total momentum flux have been kept constant. This means that when the gas issuing from the slit is made up of massive molecules its velocity is from the start correspondingly reduced so that the total momentum flux may be kept constant. Thus the velocity profile is not in this instance very significant.

A more significant characteristic of the motion is the momentum flux, i.e. ρu^2. It is found, when the change of scale effect is taken into account, that the momentum flux profile is only slightly affected by the differing densities. In the case of a density difference between the gas mixture on the axis and the surrounding atmosphere of $\frac{1}{10}$, i.e. $\frac{b\beta}{\zeta^{1/3}} = \frac{1}{10}$, it is found that, for the typical examples of a jet of hydrogen going into an atmosphere of oxygen and vice versa, the variation in the momentum profile over the central core of the jet is only a few per cent. This variation is not susceptible to experimental observation. This result is true for both laminar and turbulent flows. When the flow is turbulent it is to be noted that the above solution predicts quite appreciable changes in the profile near the boundaries of the jet. These however would be masked in any experimental observations because the flow is only intermittently turbulent there.

Finally it is to be remembered that these results only hold when ζ is large, i.e. at large distances from the orifice, because it is only in this region that the original solution due to Bickley is valid.



MOMENTUM PROFILES

THE FLOW FAR FROM THE ORIFICE OF A TWO DIMENSIONAL JET

—·—·— LAMINAR JET OF OXYGEN IN ATMOSPHERE OF HYDROGEN ($\beta = -0\cdot93, \gamma = -0\cdot19$)
(EXCESS DENSITY ON AXIS = + 0·1)

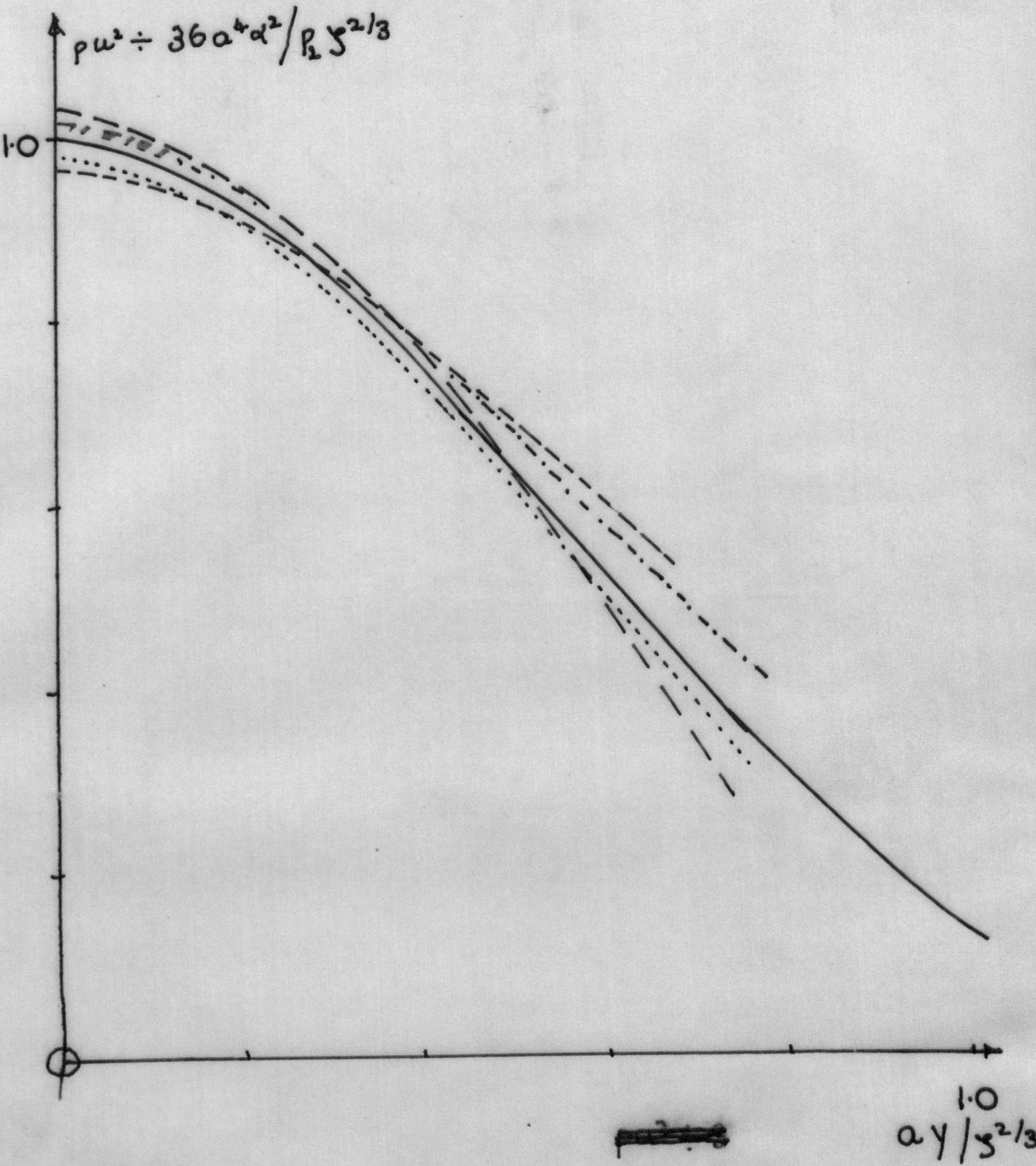
············ LAMINAR JET OF HYDROGEN IN ATMOSPHERE OF OXYGEN ($\beta = 15, \gamma = 4\cdot79$)
(EXCESS DENSITY ON AXIS = + 0·1)

— — — — TURBULENT JET EXCESS DENSITY ON AXIS = -0·1

-------- TURBULENT JET EXCESS DENSITY ON AXIS = + 0·1

(EXCESS DENSITY ON AXIS = $\beta b / \zeta^{1/3}$)

——— HOMOGENEOUS JET



MOMENTUM PROFILES

THE FLOW FAR FROM THE ORIFICE OF A TWO DIMENSIONAL JET

LAMINAR JET OF OXYGEN IN ATMOSPHERE OF HYDROGEN (β = −0·93, γ = [illegible]9)
(EXCESS DENSITY ON AXIS = + 0·1)

LAMINAR JET OF HYDROGEN IN ATMOSPHERE OF OXYGEN (β = 15, γ = [illegible])
(EXCESS DENSITY ON AXIS = + 0·1)

TURBULENT JET EXCESS DENSITY ON AXIS = −0·1

TURBULENT JET EXCESS DENSITY ON AXIS = + 0·1

(EXCESS DENSITY ON AXIS = $\beta b / \zeta^{1/3}$)

HOMOGENEOUS JET

CHAPTER IX

AXIALLY SYMMETRIC JET MIXING OF TWO DIFFERENT GASES FAR FROM THE ORIFICE

This chapter is concerned with the analogue, in axially symmetric flow, of the problem considered in the last chapter. That is to say the problem considered is that of an axially symmetric jet of one gas which diffuses into an atmosphere made up of a different gas. As in the previous chapter certain basic assumptions have been made so that the solution may be found in an analytical form. The solution is developed in an asymptotic series whose coefficients depend on the axial concentration, in the gas mixture, of the gas which comes out of the orifice.

EQUATIONS OF MOTION

The flow being assumed to be of boundary layer type at constant pressure, the equations of motion are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\varepsilon \rho r \frac{\partial u}{\partial r}\right) \qquad \text{— momentum} \quad (1),$$

and

$$\frac{\partial}{\partial x}(\rho u r) + \frac{\partial}{\partial y}(\rho v r) = 0. \qquad \text{— continuity} \quad (2).$$

A system of cylindrical coordinates (x, r) is chosen so that the x axis lies along the forward axis of the jet. u and v are the components* of velocity along and perpendicular to the axis of the jet respectively. The quantity, ε in equation (1), is equal to μ/ρ in laminar flow and is the coefficient of eddy kinematic viscosity when the flow is turbulent.

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* or mean components in the case of turbulent flow.

In addition to equations (1) and (2) it is necessary to take into account the diffusion of the first gas (1) (which issues from the orifice) into the atmosphere (which consists of gas (2)). The boundary layer equation of diffusion is in this case

$$\frac{\partial}{\partial x}(cur) + \frac{\partial}{\partial r}(cvr) = \frac{\partial}{\partial r}\left(\varepsilon_D\, r\frac{\partial c}{\partial r}\right) \qquad (3),$$

where c is the concentration* of gas (1) in the mixture. The quantity ε_D is the coefficient of diffusion in laminar flow and is the eddy coefficient of diffusion when the flow is turbulent.

It may be shown that, for a turbulent jet at large distances from the orifice, experimental data on the similarity of velocity profiles and variation of axial velocity lead, in this case, to the eddy coefficients of viscosity and diffusion having constant values, ε_0 and $E\varepsilon_0$ respectively. (E is an experimentally determined constant whose value may be shown on theoretical grounds to be close to unity.)

The continuity equation, i.e. equation (2), is automatically satisfied by introducing a stream function ψ defined by

$$\frac{\partial\psi}{\partial r} = (\rho ur) \qquad \text{and} \qquad \frac{\partial\psi}{\partial x} = -(\rho vr).$$

For convenience a new variable R is introduced to replace the coordinate r. This variable R has already been defined, in Chapter V, by the relation

$$R^2 = 2\int_0^r \rho^* r\, dr,$$

with ρ^* now equal to $1 + \beta c$,

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* or mean concentration in the case of turbulent flow.

In terms of the new variables (x, R) the momentum equation i.e. equation (1) becomes

$$\frac{1}{R}\frac{\partial\psi}{\partial R}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial R}\frac{\partial^2\psi}{\partial R\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial R^2} = \alpha R\frac{\partial}{\partial R}\left[\phi(c) R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\right]. \quad (4).$$

In this equation $\phi(c) = \mu^* \rho^* r^2/R^2$ and $\alpha = \mu_2$ when the flow is laminar, while for turbulent flow $\phi(c) = (\rho^*)^2 r^2/R^2$ and $\alpha = \varepsilon_0 \rho_2$.

Next the equation of diffusion is

$$\frac{\partial\psi}{\partial R}\frac{\partial c}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial c}{\partial R} = \alpha\sigma(\rho^*)^2\frac{\partial}{\partial R}\left[\left(\rho^*\frac{r^2}{R^2}\right)R\frac{\partial c}{\partial R}\right], \quad (5),$$

when the variables (x, R) are used instead of (x, r). (σ in equation (5) is equal to $D_{12}\rho_2/\mu_2$ and E in laminar and turbulent flows respectively.)

BOUNDARY CONDITIONS

The boundary conditions that the velocity components satisfy are:

on the boundary of the jet i.e. $r = \infty$; $u = 0$,

and on the axis of the jet i.e. $r = 0$; $\frac{\partial u}{\partial r} = 0$ and $v = 0$.

In terms of the stream function ψ and the new variable R, these become:

as $R \to \infty$; $\frac{1}{R}\frac{\partial\psi}{\partial R} \to 0$,

and at $R = 0$; $\psi = 0$ and $\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) = 0$.

The boundary conditions on the concentration are:

on the boundary of the jet i.e. $r = \infty$ or $R = \infty$, $c = 0$,

and on the axis of the jet i.e. $r = R = 0$; $\frac{\partial c}{\partial r} = 0$ or $\frac{\partial c}{\partial R} = 0$.

SOLUTIONS OF EQUATIONS (4) and (5)

A first approximation to the stream function is obtained by neglecting the variations in density. An approximation valid at large distances from the orifice is that due to Schlichting (1933), i.e.

$$\psi = \alpha x \left(\frac{4\zeta}{1+\zeta} \right), \tag{6}$$

where $\zeta = \dfrac{a^2 R^2}{x^2}$, 'a' being a constant which depends on the total momentum flux of the jet.

Turn now to the concentration equation, equation (5). The first approximation to c is found by inserting the approximate stream function, (6) in equation (5) and neglecting terms of second and higher order in c . Then the approximate concentration function satisfies the equation

$$\frac{\partial \psi}{\partial R}\frac{\partial c}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial c}{\partial R} = \alpha \sigma \frac{\partial}{\partial R}\left(R \frac{\partial c}{\partial R} \right). \tag{7}$$

The appropriate solution, valid far from the orifice, may be obtained in the form

$$\frac{1}{x}\, g(\zeta).$$

(The reason for the index -1 lies in the fact that the flux of molecules of gas '1' crossing a plane normal to the axis of the jet is a constant which is independent of x . When this condition, which follows immediately from the diffusion equation and the continuity equation, is used the index -1 follows immediately.)

The function $g(\zeta)$ then satisfies the ordinary differential equation

$$\frac{g}{(1+\zeta)^2} + \frac{\zeta g'}{(1+\zeta)} + \frac{\sigma}{2}\frac{d}{d\zeta}(\zeta g') = 0.$$

The solution of this equation which satisfies the boundary conditions is

$$g(\zeta) = \frac{b}{(1+\zeta)^{2/\sigma}}, \tag{8}$$

where b is a constant which depends on the initial flux of molecules of gas (1) out of the orifice. In fact b is given by

$$H_1 = \frac{2(1+\beta)}{\left(1+\frac{2}{\sigma}\right)}\, b\mu_2,$$

where H_1 is the total mass flux of gas (1) out of the orifice.

DETERMINATION OF THE SECOND APPROXIMATION TO THE STREAM FUNCTION

Since the first approximation to the variation in density depends mainly on the axial concentration $\frac{b}{x}$, it would appear that the most natural form in which to expand the stream function is:

$$\psi = \alpha x\left[\frac{4\zeta}{1+\zeta} + \frac{b}{x}F_1(\zeta) + \cdots\right]. \tag{9}$$

This expansion (9) is then substituted into equation (5) and the coefficients of $\frac{b}{x}$ equated to zero. (The terms in c are replaced by the approximate value $\frac{1}{x}\, g(\zeta)$.) This gives the equation for $F_1(\zeta)$ as

$$\zeta F_1''' + \left(\frac{3\zeta+1}{1+\zeta}\right)F_1'' + \frac{6}{(1+\zeta)^2}F_1' = -\frac{d}{d\zeta}\left[\chi\zeta F_0''\right]. \tag{10}$$

In equation (10) primes denote derivatives with respect to ζ, $F_0 = \frac{4\zeta}{1+\zeta}$, and χ is a function of ζ defined by the relation:

$$\phi(c) \doteq 1 + \frac{b}{x}\chi + \cdots, \qquad \text{when } x \text{ is large.}$$

Write $P = \frac{dF_1}{d\zeta}$ and change the independent variable from ζ to t where $t = \frac{1}{1+\zeta}$.

Then (10) becomes

$$t(1-t)\frac{d^2P}{dt^2} - \frac{dP}{dt} + 6P = R$$

where (11)

$$R = \frac{d}{dt}\left[\chi\zeta F_0''\right] = -8\frac{d}{dt}\left[\left(nt^{2/\sigma} - \beta t\right)(1-t)t^2\right].$$

(The constant n occurring in the expression for R is $\beta+\gamma$ in the case of laminar flow and 2β when the flow is turbulent. The constants β and γ are defined in Chapter VII pp.87-88.)

Equation (11) is of hypergeometric type. Its complementary function is a linear combination of two integrals P_1 and P_2, where

$$P_1 = t^2(3-4t)$$

and $$P_2 = P_1 \log\left(\frac{t}{1-t}\right) - \frac{1}{6} - t + \frac{16}{3}t^3.$$

The required solution $\left(P = \frac{dF_1}{d\xi}\right)$ of equation (11) satisfies the following boundary conditions:

at $t=0$, i.e. $\xi = \infty$; $P = 0$

and at $t=1$, i.e. $\xi = 0$; P is finite.

The complete integral of equation (11) is

$$P = -P_1 \int^t \frac{R}{t^2} P_2\, dt + P_2 \int^t \frac{R}{t^2} P_1\, dt, \qquad (13),$$

the lower limits of the integrals being in the general case arbitrary. Now the first boundary condition, i.e. that $P=0$ at $t=0$, gives the lower limit of the second integral in (13) as zero. Next, it will be noticed that P_2 has a logarithmic singularity at $t=1$, thus unless $\int_0^1 \frac{R}{t^2} P_1\, dt = 0$ the solution P will also have a logarithmic singularity at $t=1$. This condition does not hold for $\int_0^1 \frac{R}{t^2} P_1\, dt$ is in fact $32\left[\frac{\beta}{20} - \frac{n\sigma^2}{(2+3\sigma)(2+4\sigma)}\right]$ which is in general not zero. Thus equation (10) does not yield a satisfactory solution. (At this stage it might be thought that the boundary conditions given above were not true. There seems, however, to be no good physical reason for supposing that the velocity on the axis of the jet becomes infinite when a small quantity of another gas is introduced into the system.) The flaw in the above analysis would appear to be that the wrong form of expansion for the stream function has been used.

The problem is now to find the right kind of expansion for the stream function. Recently other problems in boundary layer theory in which [redacted] this type of difficulty arises have been met and solved, notably by Kuo (1957) and by Stewartson (1957). The method used by Kuo is a modification of Lighthill's (1949) technique for rendering approximate solutions of differential equations uniformly valid. Kuo has used this method to obtain uniformly valid approximate solutions of the boundary layer equations.

The method consists in expanding the stream function in the series

$$\psi = \alpha X \left[\frac{4\zeta}{1+\zeta} + \frac{b}{X} F_1(\zeta) + \cdots \right], \tag{9a}$$

X is a function of x defined by the series:

$$x = X + \frac{b}{X} x_1(x) + \cdots ,$$

ζ has changed its meaning; it now stands for $\frac{a^2R^2}{X^2}$ instead of $\frac{a^2R^2}{x^2}$ as earlier. Since the first order approximation to the concentration now becomes $\frac{1}{X} g_1(\zeta)$, and thus the concentration on the axis of the jet is $\frac{b}{X}$ it will be seen that the series for ψ and x will be useful when the axial concentration $\frac{b}{X}$ is small.

The crux of Lighthill's and Kuo's method consists in the determination of $x_1(X)$. The value of $x_1(X)$ is chosen so as to nullify the effect of those terms in the differential equation which give rise to the singularity in the solution $F_1'(\zeta)$. It will be observed that additional terms arise in the differential equation when the derivative with respect to x is changed to a derivative with respect to X.

Now

$$\left(\frac{\partial}{\partial x}\right)_R = \frac{dX}{dx}\cdot\frac{\partial}{\partial X} = \left[\frac{1}{1+\frac{b}{X}\left(x_1' - \frac{x_1}{X}\right)+\cdots}\right]\frac{\partial}{\partial X}$$

$$\doteq \left[1 - \frac{b}{X}\left(x_1' - \frac{x_1}{X}\right)+\cdots\right]\frac{\partial}{\partial X} \qquad \text{when } X \text{ is large.}$$

Thus the differential equation (4) becomes:

$$\left[\frac{1}{R}\frac{\partial\psi}{\partial R}\frac{\partial\psi}{\partial X} + \frac{\partial\psi}{\partial R}\frac{\partial^2\psi}{\partial R\partial X} - \frac{\partial\psi}{\partial X}\frac{\partial^2\psi}{\partial R^2}\right]\left[1 - \frac{b}{X}\left(x_1' - \frac{x_1}{X}\right)+\cdots\right]$$

$$= \alpha R\frac{\partial}{\partial R}\left[\phi(c)R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\right], \tag{14}$$

when the independent variables are changed from (x, R) to (X, R). The change of variable thus introduces new terms into the equation. These terms are of order $\frac{b}{X}\left(x_1' - \frac{x_1}{X}\right)$ times as large as the lowest order terms in equation (14). Now the terms which give rise to the singularity in the solution of $F_1'(\xi)$ are of order $\frac{b}{X}$ times as large as the terms of lowest order in equation (14). The function $x_1(X)$ is to be chosen so as to nullify the effect of the terms giving rise to the singularity. A necessary condition for this to take place is that

$$\frac{b}{X}\left(x_1' - \frac{x_1}{X}\right) \sim \frac{b}{X}$$

or $\quad x_1' - \frac{x_1}{X} = A \qquad$ where A is a constant.

Thus $x_1 = AX\log X + B$, where B is an arbitrary constant.

The transformation of coordinates is thus defined by:

$$x = X\left[1 + \frac{b}{X}\log X + \frac{bB}{X} + \cdots\right].$$

It is to be noted that the constant B has the effect of changing the origin of coordinates when the transformation is made.

The equation for $F_1'(\xi)$ may now be found by substituting the expansion

(9a) in equation (14) and then equating the coefficients of $\frac{b}{X}$ to zero. (The concentration c is replaced by its first approximation $\frac{1}{X} g(\zeta)$.)

The equation for $F_1'(\zeta)$ has the same left side as equation (10). Its right side, however, contains the term

$$\frac{8A}{(1+\zeta)^4}(1-2\zeta) \qquad \text{in addition to} \quad -\frac{d}{d\zeta}\left[X \zeta F_0''\right].$$

The equation for $F_1(\zeta)$ may now be simplified as before by the changes of variable $P = F_1'(\zeta)$ and $t = \frac{1}{1+\zeta}$.

The equation for P is:

$$t(1-t)\frac{d^2P}{dt^2} - \frac{dP}{dt} + 6P = R_1, \tag{15}$$

where $R_1 = 8At(3t-2) - 8\frac{d}{dt}\left[t^2(1-t)(nt^{2/\sigma} - \beta t)\right].$

The value of A is chosen so that the singularity in the solution of equation (15) is eliminated. Since the boundary conditions imposed on the stream function do not depend on the x coordinate they are unaltered by the change of variable from x to X. Thus the boundary conditions on P do not differ from those stated on page 103. The complete integral of equation (15) is:

$$P = -P_1\int^t \frac{R_1}{t^2}P_2\,dt + P_2\int^t \frac{R_1}{t^2}P_1\,dt, \tag{16}$$

the lower limits of the integrals being in the general case arbitrary. It will be remembered that if the boundary conditions on P are to be satisfied then the lower limit of the second integral in (16) must be zero and also

$$\int_0^1 \frac{R_1}{t^2}P_1\,dt = 0. \tag{17}$$

This latter condition gives the value of the constant A which is in fact

$$12\left(\frac{\beta}{4} - \frac{n+\beta}{5} + \frac{n}{6}\right)$$

when the Schmidt number ($\sigma = D_{12}\rho_2/\mu_2$) is unity. In the case of turbulent flow when $n = 2\beta$ or the special case of laminar motion in which $\beta = \gamma$ (which gives the same value of n),

$$A = -\tfrac{1}{5}\beta .$$

When this value of n is used

$$F_1'(\xi) = \frac{2}{15}\beta\left[4t^2(3-4t)\log t + 61t^2 - 100t^4\right] + a_1 t^2(3-4t).$$

The undetermined constant a_1 arises because the lower limit of the first integral in (16) is undetermined by the boundary conditions, or in other words because P_1 satisfies both boundary conditions. When the above expression for $F_1'(\xi)$ is integrated and the condition that the stream function is zero on the axis of the jet is imposed

$$F_1 = \frac{2}{15}\beta\left[-4(3t-2t^2)\log t + \frac{59}{3} - 49t - 4t^2 + \frac{100}{3}t^3\right] + \\ + a_1(1-3t+2t^2).$$

The stream function of the jet is

$$\psi = \alpha X\left[\frac{4\xi}{1+\xi} + \frac{b}{X}F_1(\xi) + \cdots\right],$$

where

$$x = X\left(1 - \frac{1}{5}\frac{b\beta}{X}\log X - \frac{bB}{X} + \cdots\right) \qquad (18)$$

when X is large.

The series giving x in terms of X may be reversed to give

$$X = x\left(1 + \frac{1}{5}\frac{b\beta}{x}\log x + \frac{bB}{x} + \cdots\right).$$

This result may be used to give the stream function in terms of the coordinates (x, R) as

$$\psi = \alpha x \left[4t^2 - \frac{4}{5}\beta(1-3t+2t^2)\frac{b}{x}\log\frac{b}{x} + \right.$$

$$+ \frac{b}{x}\left\{ \frac{2}{15}\beta\left[-4(3t-2t^2)\log t + \frac{59}{3} - 49t - 4t^2 + \frac{100}{3}t^3\right] + \right.$$

$$\left. \left. + a_2(1-3t+2t^2) \right\} + O\left(\left(\frac{b}{x}\right)^2\left(\log\frac{b}{x}\right)^2\right) \right]. \qquad (19).$$

In this expression for ψ, $t = \frac{1}{1+\zeta}$, ζ has its original value of $\frac{a^2R^2}{x^2}$ and a_2 is an arbitrary constant which is different from a, in that it contains the arbitrary constant β (arising out of expression (18)) and also because a term $-\frac{4}{5}\beta(1-3t+2t^2)b\log b$ has been introduced into (19) for dimensional reasons. The arbitrary constant β can be absorbed in a_2 because β introduces, in the first approximation, a multiple of $\frac{b}{x}(1-3t+2t^2)$ when the change of variable from X to x, defined by (18), is made.

CONCLUSION

The approximate stream function has been found, for an axially symmetric jet of one gas which issues into an atmosphere made up of a different gas. This stream function is valid at large distances from the orifice, where the concentration of the gas, which comes from the orifice, is small. Since the stream function is not analytic about $x = \infty$ it was necessary to use Lighthill's technique, as modified by Kuo, in solving the problem. The stream function contains an arbitrary constant which is not fixed by the boundary conditions. This indeterminacy arises in other cases of approximate solutions of the boundary layer equations when Kuo's method is used. Kuo (1957) found an undetermined constant in his treatment of the flow of a compressible fluid over a flat plate. He suggested that the constant might, in this case, be fixed when full Navier-Stokes equations were considered. This is not the case in the problem considered here. It is shown in the Appendix I that the arbitrary constant is still present in the approximate solution of the full Navier-Stokes equations. Thus conditions outside the boundary layer do not fix the constant in this case. Now Stewartson (1957) has also studied approximate solutions of the boundary layer equations in which there is an undetermined constant. Stewartson has shown, in the case of the wake past a flat plate, that the indeterminacy may be removed by taking the flow at the source of the motion into consideration. It seems reasonable, in view of the fact that the constant is not determined by conditions outside the boundary layer to suppose that Stewartson's explanation holds in the case of the axially symmetric jet.

Indeed these arbitrary constants must inevitably occur in series solution, about the point $x = \infty$, of the jet flow. This is because as more terms of the series become known so also does the flow of the jet at distances nearer and nearer to the orifice become known. As the orifice is approached the specific properties of the jet as it leaves the orifice become of increasing importance. The first term of the series, in the present case, only takes into account the most general properties of the jet at the orifice namely the momentum flux and the mass flux. Now as the orifice is approached from $x = \infty$ a point must be reached at which properties of the jet at the orifice, other than momentum and mass flux, have a significant effect on the motion. These additional properties of the jet at the orifice enter into the series solution through the undetermined constants.

The existence of the particular constant, a_2, may be inferred from the form of the boundary layer equations. These equations are linear in their derivatives with respect to x. Thus once a solution of the boundary layer equations, say $f(x, r)$, has been found it is possible to deduce an infinity of solutions. $f(x+c, r)$ which satisfy the same boundary conditions. (c is an arbitrary constant.) Furthermore each one of this infinity of solutions is equally valid because no direct reference has been made to conditions near the orifice. Now the original stream function is of the form

$$\psi = \alpha x \left[F_0(\xi) + \frac{b}{x} \log \frac{b}{x} H_1(\xi) + \frac{b}{x} H_2(\xi) + \cdots \right], \tag{20}$$

where $\xi = \dfrac{a^2 R^2}{x^2}$.

Thus the set of solutions which may be derived from (20) is

$$\psi = \alpha(x+c)\left[F_0(\xi_1) + \frac{b}{x+c}\log\left(\frac{b}{x+c}\right)H_1(\xi_1) + \frac{b}{x+c}H_2(\xi_1) + \cdot\right] \quad (21),$$

where $\xi_1 = \left(\frac{a^2R^2}{x^2}\right)_{x=x+c}$

When x is large the terms of (21) may be developed in powers of $\frac{c}{x}$ to give

$$\psi = \alpha x\left[F_0(\xi) + \frac{b}{x}\log\frac{b}{x}H_1(\xi) + \frac{b}{x}\left(H_2(\xi) + \frac{c}{b}h(\xi)\right) + \cdots\right] \quad (22)$$

The function $h(\xi)$ is in fact equal to

$\lim\limits_{x\to\infty}\frac{\partial}{\partial x}\left[x\ F_0(\xi)\right]$ which is a multiple of the function

$(1 - 3t + 2t^2)$ given in (19).

Equation (22) shows that the solution of the boundary layer equations, in this case contains an arbitrary constant in the term whose coefficient is $\alpha x \cdot \frac{b}{x}$

Since the expansion (19) for the stream function is not valid near the orifice, it is not possible to find the value of a_2 in terms of the properties of the jet at the orifice, directly from (19). The constant a_2 could be found by matching (19) with a stream function which was valid near the orifice. Unfortunately such a stream function has yet to be found.

APPENDIX I

Squire (1951) has found an exact solution of the Navier-Stokes equations for the case of an axially symmetric flow of incompressible fluid. This solution gives the flow resulting from a point source of momentum immersed in a fluid of infinite extent.

The stream function of this flow is

$$2\mu_2 r\left[\frac{1-\alpha^2}{\alpha-\mu}+2\alpha-(\alpha-\mu)\right]=2\mu_2 r\,\frac{(1-\mu^2)}{\alpha-\mu}=\mu_2 r f_0(\mu)$$

and the radial component of velocity is

$$-2\frac{\mu_2}{r}\left[\frac{1-\alpha^2}{(\alpha-\mu)^2}+1\right]$$

when spherical coordinates (r,θ) are used. (In the expressions given above $\mu=\cos\theta$ and $\alpha=d+1$ where d is a constant.) The spherical coordinates (r,θ) are chosen with the axis, $\theta=0$, along the main forward direction of flow and with the origin, $r=0$, at the source of momentum. As tends to zero this solution tends to that given by Schlichting (1933) for the round jet, when a change of variable of boundary layer type is used.

This may be seen by introducing the change of variable

$$t=\frac{\alpha}{\alpha-\mu}.$$

Then near the axis of the jet, i.e. for $1-\mu\sim O(d)$, the stream function is

$$4\mu_2 r\left[(1-t)+O(d)\right]$$

and the radial velocity is

$$\frac{4\mu_2}{rd}\left[t^2+O(d)\right].$$

In the region $1-\mu \sim O(d)$, which is the boundary layer region of the jet, t lies between 1 (which corresponds to the forward axis of the jet) and $o(d)$ (which corresponds to the edge of the boundary layer of the jet). Let d tend to zero, then inside the boundary layer the stream function becomes

$$4\mu_2 x(1-t),$$

the radial velocity becomes

$$\frac{4\mu_2 t^2}{xd}$$

and the variable t ranges from 1 to 0. In these expressions the approximation $x' \doteq r$, which holds inside the boundary layer, is used. The value of t in terms of cylindrical coordinates (x, r_1) (whose origin and axis are the same as those of the spherical coordinates (r, θ)) is given by

$$\frac{1-t}{t} \doteq \frac{r_1^2}{2dx^2},$$

inside the boundary layer. Write

$$\xi = \frac{r_1^2}{2x^2 d} \qquad \text{then} \qquad t = \frac{1}{1+\xi}.$$

Thus putting $a^2 = \frac{1}{2d}$ the stream function is $\dfrac{4\mu_2 x\xi}{1+\xi}$

and the x-component of the velocity is $\dfrac{8a^2\mu_2 x}{(1+\xi)^2}$.

These expressions are identical with Schlichting's (1933) solution. Outside the boundary layer region of the jet the stream function is $\mu_2 r\,O(1)$ and the radial component of velocity is $\frac{\mu_2}{r}O(1)$. It is to be noted that the ratio of a typical velocity inside the boundary layer to a typical velocity outside the boundary layer is of order d^{-1}.

Since a first approximation to the stream function, valid both inside and outside the boundary layer, is known, it is possible to find a second approximation to the stream function (which is also valid inside and outside the boundary layer), for the case when the fluid is inhomogeneous. The form of this stream function is found by generalising the stream function found in Chapter IX. Thus ψ is expanded in the series

$$\mu_2 r\left[\frac{2(1-\mu^2)}{\alpha-\mu} + \frac{b}{r}\log\frac{b}{r}\, f_1(\mu) + \frac{b}{r} f_2(\mu) + \cdots\right]$$

where $\frac{b}{r}$ is the concentration on the axis of the jet. This expansion for $f_1(\mu)$ is substituted in the Navier-Stokes equations of a compressible fluid. The equation for $f_1(\mu)$ is found by equating the coefficients of $\frac{b}{r}\log\frac{b}{r}$ to zero. This gives

$$(1-\mu^2)f_1''' - (f_0 + 2\mu)f_1'' + (6 - 3f_0')f_1' = 0$$

as the equation for f. (In this equation and in the equation for f_2 primes denote derivatives with respect to μ.) Now the boundary conditions on the stream function are as follows: the stream function is finite and has a finite derivative with respect to μ in the range $-1 \leq \mu \leq 1$,

and the stream function is zero on the back and forward axes of the jet i.e. on $\mu = \pm 1$

These conditions imply that

$$f_1 = f_2 = 0 \qquad \text{on} \quad \mu = \pm 1$$

and that f_1' and f_2' are finite in the range $-1 \leq \mu \leq 1$. Now the equation with f_1' as dependent variable has two independent solutions Q_1 and Q_2.

where

$$Q_1 = 1 + 3\frac{(1-\alpha^2)}{(\alpha-\mu)^2} + \frac{2(1-\alpha^2)^2}{\alpha(\alpha-\mu)^3}$$

and

$$Q_2 = \frac{-3\alpha^2}{(\alpha-\mu)^3}\log\left(\frac{1+\mu}{1-\mu}\right)Q_1 + \frac{8-20\alpha^2+18\alpha^3\mu-6\alpha^2\mu^2}{(\alpha-\mu)^3}$$

The second of these solutions is clearly inadmissible on account of its singularities at $\mu = \pm 1$ equal to Q_1 multiplied by a constant. This constant is determined by the condition that the function $f_2'(\mu)$ be finite in $-1 \leq \mu \leq 1$.

The equation for $f_2(\mu)$ is found by equating the coefficients of $\frac{b}{r}$ to zero in the Navier-Stokes equations. It is in fact

$$(1-\mu^2)f_2''' - (f_0+2\mu)f_2'' + (6-3f_0')f_2' = R_1.$$

The known function R_1 contains terms involving f_1 as well as f_0. The solution of this equation in f_2' which is bounded in the region $-1 \leq \mu \leq 1$ is $f_2' = CQ_1$ together with a particular integral of the equation*. The constant C is not determined by the boundary conditions because Q_1 is bounded in $-1 \leq \mu \leq 1$. Neither is it determined by consideration of orders of magnitude because the order of magnitude of Q_1 behaves in the same way as f_0'. (Inside the boundary layer Q_1 is $-\frac{3t^2}{d}(3-4t)$ and Q_1 is thus of order d^{-1}; outside the boundary layer Q_1 is of order unity.)

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* Particular integrals of this equation (in f_2') are not in general bounded in the region $-1 \leq \mu \leq 1$. A bounded particular integral only exists for a certain value of the constant which multiplies f_1. This in fact determines the constant multiplying f_1.

To summarise, an approximate solution of the full Navier-Stokes equations of a compressible fluid has been outlined for the problem of Chapter IX. This solution tends inside the boundary layer to that given in Chapter IX. It will be seen that the indeterminacy, present in the solution given in Chapter IX of the boundary layer equations, remains even when the full Navier-Stokes equations are considered. In other words the undetermined constant a_2 occurring in (18) is not fixed by conditions outside the boundary layer.

APPENDIX II

THE TWO-DIMENSIONAL WAKE

The problem considered in Chapter IX was basically that of determining an approximate, uniformly valid, solution of the boundary layer equations. The method used was that of Kuo. The form of the solution for the stream function (19), containing as it does a term in $\frac{b}{x} \log \frac{b}{x}$ as well as a term in the small parameter $\frac{b}{x}$, resembles the type of expansion used by Stewartson (1957) in his paper "On asymptotic expansions in the theory of boundary layers". The stream function (19) was in fact first obtained by use of an expansion of the type used by Stewartson, namely

$$\alpha x \left[F_0(\xi) + \frac{b}{x} \log \frac{b}{x} G_1(\xi) + \frac{b}{x} F_1(\xi) + \cdots \right]$$

Thus one particular problem, arising out of the boundary layer equations, can be solved by both Kuo's and Stewartson's approaches. This fact leads one to expect that other problems in Stewartson's paper can be solved by Kuo's method. Certainly, as the following analysis shows, the first of Stewartson's problems is soluble in this way.

The problem is to find the flow in the wake of a flat plate, (symmetrically placed in a uniform stream) at large distances from the plate. In 1931 Tollmien found the first approximation to the velocity profile. This work of Tollmien was extended by Goldstein in 1933. Goldstein found the second approximation, but was not able to find a third approximation, which was uniformly valid on the boundaries of the wake. A valid third approximation was found by Stewartson in 1957.

TOLLMIEN'S AND GOLDSTEIN'S METHOD

When Goldstein's notation is used the equations of motion of the two-dimensional wake of incompressible fluid take the form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \qquad \text{and} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

x, y, u, v are non dimensional quantities defined by:

$$u = \frac{U}{U_0}, \quad v = \left(\frac{U_0 d}{\nu}\right)^{1/2}\frac{V}{U_0}, \quad x = \frac{X}{d}, \quad y = \left(\frac{U_0 d}{\nu}\right)^{1/2}\frac{Y}{d},$$

where (X, Y) are rectangular Cartesian coordinates, U and V are the components of velocity parallel to the axes, $U = U_0$ gives the free stream velocity, d is a constant length and ν is the kinematic viscosity of the fluid.

Let u and v be expanded in the series:

$$\left.\begin{aligned} u &= 1 + \frac{1}{\sqrt{x}}u_1(\eta) + \frac{1}{x}u_2(\eta) + \frac{1}{x^{3/2}}u_3(\eta) + \cdots \\ v &= \qquad\qquad\qquad\quad \frac{1}{x}v_1(\eta) + \frac{1}{x^{3/2}}v_2(\eta) + \cdots \end{aligned}\right] \tag{2}$$

where η is a variable defined by

$$\eta = y/\sqrt{2x} = \left(\frac{U_0}{2\nu X}\right)^{1/2} Y.$$

The equations which determine the functions u_1, u_2, etc., are found by inserting the expansions (2) in equation (1) and equating successive powers of x.

Tollmien found the function u_1 which is in fact $-\frac{A}{\sqrt{2}}\eta e^{-\frac{1}{2}\eta^2}$, where A is a constant whose value depends on the drag of the plate.

The term $u_2 = -\frac{A^2}{2}\left[e^{-\eta^2} + \sqrt{\frac{\pi}{2}}\,\eta\, e^{-\frac{1}{2}\eta^2}\,\mathrm{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right]$

was found by Goldstein. Goldstein, after finding u_2 went on to find u_3. He showed that it was not possible to obtain a function u_3 which tended exponentially to zero on the edges of the wake.

STEWARTSON'S APPROACH

Stewartson, at this stage, wrote the equation of motion as:

$$\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = F(x, y), \tag{3}$$

where $F(x, y) = (1-u)\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}$.

He then set up an iterative process by means of which a known approximate solution may be improved. The process consists in solving an approximate form of equation (3). The left hand side of this approximate equation is the same as the left hand side of (3); the right hand side is obtained by substituting the known approximate values of u and v in $F(x, y)$. The resulting approximate form of equation (3) is then solved for u.

The solution of this equation is

$$u = 1 - \int_{-\infty}^{\infty} dy' \, \frac{(1 - f(y'))}{2\sqrt{\pi x}} \exp\left[-\frac{(y-y')^2}{4x}\right] + \int_{-\infty}^{\infty} dy' \int_{0}^{x} \frac{F(x', y')}{2\sqrt{\pi(x-x')}} \exp\left[-\frac{(y-y')^2}{4(x-x')}\right] dx' \tag{4}$$

This solution satisfies the following boundary conditions

at $x = 0$; $u = f(y)$ the initial velocity profile,

at $y = \pm\infty$; $u = 0$

and at $y = 0$; $\frac{\partial u}{\partial y} = 0$.

To obtain the third approximations (valid for large x) the approximations

$$u \doteq 1 + \frac{1}{\sqrt{x}} u_1 + \frac{1}{x} u_2$$

and $$V \doteq \frac{1}{x} V_1 + \frac{1}{x^{3/2}} V_2$$

are used in the determination of an approximation to $F(x, y)$. It may be shown that $F(x, y)$ is in this case $\sim x^{-5/2} F_3(\eta)$ when x is large. When this approximation to $F(x, y)$ is inserted in (4) it is found that part of the second integral is of order $x^{-3/2} \log x$ when x is large

This shows that the asymptotic expansion of u is

$$u = 1 + \frac{1}{\sqrt{x}} u_1 + \frac{1}{x} u_2 + \frac{1}{x^{3/2}} \log x \; g_3(\eta) + \frac{1}{x^{3/2}} u_3(\eta) + \cdots.$$

When this expansion is substituted in equation (1) it is found that g_3 and u_3 satisfy the equations:

$$g_3'' + \eta g_3' + 3 g_3 = 0 \qquad (5)$$

and $$u_3'' + \eta u_3' + 3 u_3 = -2 g_3 + F_3 . \qquad (6)^*$$

The solution of equation (5) is

$$g_3 = a(1-\eta^2) e^{-\frac{1}{2}\eta^2} ;$$

the solution of equation (6) is

$$u_3 = b(1-\eta^2) e^{-\frac{1}{2}\eta^2} + I(\eta),$$

where a and b are constants and $I(\eta)$ is that particular integral of (6) which tends exponentially to zero on the boundaries of the wake. A particular

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* Note: There is a slip in Stewartson's analysis which occurs in his form of equation (6). In this equation Stewartson has $-g_3$ instead of $-2g_3$ This leads to a term $-\frac{A^3}{4\sqrt{3}} \frac{\log x}{x^{3/2}}$ in his result for the velocity on the axis.

integral with this behaviour only exists when the condition

$$\int_0^\infty (1-\eta^2)(F_3 - 2g_3)\,d\eta = 0$$

is satisfied. This condition leads to a unique value of a. The value of b, however, is not determined by the boundary conditions since $(1-\eta^2)e^{-\frac{1}{2}\eta^2}$ satisfies the conditions on the axis and on the boundary of the wake. Stewartson's solution gives the following value of u on the axis of symmetry of the wake

$$1 - \frac{A}{\sqrt{x}} + \frac{A^2}{2x} - \frac{A^3}{8\sqrt{3}}\frac{\log x}{x^{3/2}} + \frac{\beta}{x^{3/2}} + O\left(\frac{\log x}{x^2}\right).$$

APPLICATION OF KUO'S METHOD TO FIND THE THIRD APPROXIMATION TO

Kuo's method has already been used in Chapter IX to solve a problem whose basic difficulties are the same as occur in the present problem. Kuo's method, it will be remembered, is to expand the stream function defined by $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$ and the x coordinate in series.

Thus

$$\psi = \eta\sqrt{2\xi} + F_1(\eta) + \frac{1}{\sqrt{\xi}}F_2(\eta) + \frac{1}{\xi}F_3(\eta) + \cdots$$

and

$$x = \xi + x_1(\xi) + \cdots,$$

The variable η in the above expression is $y/\sqrt{2\xi}$ and not $y/\sqrt{2x}$ as earlier.

This leads to the following expressions for u and v:

$$u = 1 + \frac{u_1(\eta)}{\sqrt{\xi}} + \frac{u_2(\eta)}{\xi} + \frac{u_3(\eta)}{\xi^{3/2}} + \cdots$$

and

$$v = \left(\frac{v_1(\eta)}{\xi} + \frac{v_2(\eta)}{\xi^{3/2}} + \cdots\right)\cdot\frac{d\xi}{dx}.$$

In these expressions u_1, u_2, v_1, v_2 are the same functions as given by Goldstein. Now

$$\frac{d\xi}{dx} = \left(\frac{dx}{d\xi}\right)^{-1} = \left[1 + x_1'(\xi) + \cdots\right]^{-1} \doteq 1 - x_1'(\xi) + \cdots ,$$

assuming that x_1' is small compared to unity when x is large.

Thus $$v \doteq \left[\frac{v_1}{\xi} + \frac{v_2}{\xi^{3/2}} + \cdots\right]\left[1 - x_1'(\xi) + \cdots\right].$$

The equation of motion (1) becomes in terms of ξ, η :

$$\left[u\frac{\partial u}{\partial \xi} + \left(\frac{v_1}{\xi} + \frac{v_2}{\xi^{3/2}} + \cdots\right)\frac{\partial u}{\partial \eta}\cdot\frac{1}{\sqrt{2\xi}}\right](1 - x_1' + \cdots) = \frac{1}{2\xi}\frac{\partial^2 u}{\partial \eta^2} . \qquad (4)$$

New terms have been introduced by the change of variable into the equation of motion. These terms are needed to nullify the effect of certain terms in the differential equation. The latter terms give rise to that part of the solution u_3 which does not tend exponentially to zero on the boundaries of the wake. Now the terms, whose effect is to be nullified, are of order $\xi^{-5/2}$ in equation (4). It may be seen by examining equation (4) that a necessary condition for the terms, introduced by the change of variable, to nullify terms of order $\xi^{-5/2}$ is that x_1' should be of order $\xi^{-1/2}$ or ξ^{-1}. The former value, however, introduces terms of lower order which affect u_1 and u_2; it is therefore to be rejected. Since the latter value does not give rise to such complications it is accepted and leads to

$$x_1 = B \log \xi + C,$$

where B and C are constants.

The equation for u_3 is

$$\left.\begin{aligned} u_3'' + \eta u_3' + 3u_3 &= -AB e^{-1/2\eta^2}(1-\eta^2) + F_3(\eta) \\ &= \Phi e^{-\frac{1}{2}\eta^2} \end{aligned}\right] \qquad (5)$$

where F_3 is the same function as used by Stewartson. F_3 has been shown by Goldstein to be

$$\frac{1}{2}A^3\left[3e^{-\eta^2} + (2\pi)^{1/2}\eta e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right) + \sqrt{\pi}\,\eta \operatorname{erf}\eta\right]e^{-\frac{1}{2}\eta^2}$$

Write now $u_3 = e^{-\frac{1}{2}\eta^2}(1-\eta^2)\,h_3$ Then equation (5) becomes

$$\frac{d}{d\eta}\left[e^{-\frac{1}{2}\eta^2}(1-\eta^2)^2 h_3'\right] = \Phi e^{-\frac{1}{2}\eta^2}(1-\eta^2).$$

Hence

$$h_3' e^{-\frac{1}{2}\eta^2}(1-\eta^2)^2 = \int_0^\eta \Phi e^{-\frac{1}{2}\eta^2}(1-\eta^2)\,d\eta.$$

The lower limit of the integral is zero because the boundary condition $\frac{\partial u_3}{\partial y} = 0$ implies $h_3' = 0$ on $\eta = 0$

In order that u_3 should tend exponentially to zero on the boundaries of the jet it is necessary that

$$\int_0^\infty \Phi e^{-\frac{1}{2}\eta^2}(1-\eta^2)\,d\eta = 0.$$

This condition determines the value of the constant B which is in fact $\frac{A^2}{4\sqrt{3}}$.

A further quadrature gives h_3 and u_3 which is

$$e^{-\frac{1}{2}\eta^2}(1-\eta^2)\int_0^\eta \frac{e^{\frac{1}{2}\eta^2}}{(1-\eta^2)^2}\,d\eta \int_0^\eta \Phi e^{-\frac{1}{2}\eta^2}(1-\eta^2)\,d\eta.$$

The lower limit of the outer part of this integral is not fixed by the boundary conditions because $e^{-\frac{1}{2}\eta^2}(1-\eta^2)$ satisfies the boundary conditions both on the axis and on the edges of the wake.

The approximate value of u on the axis is

$$u(0) \doteq 1 - \frac{A}{\sqrt{\xi}} - \frac{A^2}{2\xi} + \frac{\beta_1}{\xi^{3/2}},$$

where β_1 is an arbitrary constant and

$$\begin{aligned} \xi &\doteq x - x_1(x) \\ &= x\left[1 - \frac{A^2}{4\sqrt{3}}\frac{\log x}{x} - \frac{c}{x}\right]. \end{aligned}$$

In terms of the variable x the approximate value of u on the axis is

$$1 - \frac{A}{\sqrt{x}} - \frac{A^2}{2x} - \frac{A^3}{8\sqrt{3}}\frac{\log x}{x^{3/2}} + \frac{\gamma}{x^{3/2}} + \cdots$$

(γ is a constant which differs from β_1 in that it absorbs the constant C arising out of the change of variable.)

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