Shift invariant preduals of $\ell_1(\mathbb{Z})$, and isomorphisms with $c_0(\mathbb{Z})$

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Abstract

The relation between shift-invariant preduals of the space of summable sequences $\ell_1(\mathbb{Z})$ and the dual Banach algebra $\ell_1(\mathbb{Z})$ equipped with the convolution product have resulted in recent development of research on preduality of this space. According to the survey paper entitled 'Shift Invariant Preduals of $\ell_1(\mathbb{Z})$', written by Matthew Daws, Richard Hadon, Thomas Schlumprecht and Stuart White, we know that there exists an uncountable family $\left\{F^{(\lambda)}\right\}_{\lambda\in\mathbb{C}}$ of shift-invariant preduals of $\ell_1(\mathbb{Z})$ and all these preduals $F^{(\lambda)}$ constructed in the above paper are isomorphic to $c_0(\mathbb{Z})$, the space of sequences converging to zero. This conclusion is based on an abstract theory of the Szlenk index, without stating the explicit form of that isomorphism. This thesis will make an attempt to define this sort of isomorphism. In other words, I will form an isomorphism between $c_0(\mathbb{Z})$ and $F_+^{(\lambda)}$, which is a subspace of $F^{(\lambda)}$.

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During my studies at Glasgow University, I realised that work involved in research in mathematics cannot be confronted by a single individual. There are always ideas and people behind research that serves as a motivating force. Therefore, I would like to take this opportunity to express my gratitude to the people who have been instrumental in my successful completing of this thesis. First of all, I would like to express my sincere gratitude to my advisor, Dr. Stuart White, for his continuous support in my MSc(R) study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me all the time during my research and writing of this thesis. Without his encouragement and guidance this paper would not have materialized. I also offer my regards and appreciation to all those who assisted me morally, intellectually and physically in any capacity during the completion of this project. I am grateful for their constant support and help.

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Statement

The following thesis is submitted in accordance with the regulations for the degree of Master of Science at the University of Glasgow.

Chapter 1, 2, 3 and 4, cover background material and some basic well known facts and results.

The results in chapter 5 are the author's original work with the exception of those results that are explicitly referenced.

Contents

Chapter 1

Introduction

1.1 Overview

The Banach space $\ell_1(\mathbb{Z})$ is a very standard sequence space and is defined as a space of those elements $x = (x_n)_{n\in\mathbb{Z}}$ satisfying $\sum_{n\in\mathbb{Z}} |x_n| < \infty$, where each x_n belongs to the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. The standard norm for which $\ell_1(\mathbb{Z})$ becomes a Banach space is $\|x\| = \sum_{n\in\mathbb{Z}} |x_n|$, $(x \in \ell_1(\mathbb{Z}))$. The initial, result familiar to most of the undergraduate students concerning the study of preduality of $\ell_1(\mathbb{Z})$, says that the sequence space $c_0(\mathbb{Z})$ is an isometric isomorphic predual of $\ell_1(\mathbb{Z})$. In other words, we say that the dual space of $c_0(\mathbb{Z})$ is simply $\ell_1(\mathbb{Z})$. For the sake of completeness, we recall the proof of this fact in the next section. It is important to note that in the study of preduality of $\ell_1(\mathbb{Z})$ mathematicians very often call a space E a predual of $\ell_1(\mathbb{Z})$, if there is an isomorphism between the dual space of E and $\ell_1(\mathbb{Z})$, which does not have to be isometric. This definition also applies to this thesis. There are many examples of predual spaces of $\ell_1(\mathbb{Z})$ and some of them are included in the next section. A short review of other, more sophisticated examples of preduals of $\ell_1(\mathbb{Z})$ is given in the introduction of [7].

As it will be made precise later in Lemma 3.2.4 of this thesis, every predual E of $\ell_1(\mathbb{Z})$ can be canonically viewed as a subspace of $\ell_\infty(\mathbb{Z})$ just as the canonical predual $c_0(\mathbb{Z})$ naturally sits inside $\ell_\infty(\mathbb{Z})$. Therefore, it makes a sense to consider the behaviour of the bilateral shift on $\ell_\infty(\mathbb{Z})$ on E. We say that the predual E of $\ell_1(\mathbb{Z})$ is shift-invariant, if for any $x \in E$ the elements $\sigma(x)$ and $\sigma^{-1}(x)$ also lie in E, where σ is the bilateral shift operator on $\ell_\infty(\mathbb{Z})$. Certainly, as it will be seen in the next section, $c_0(\mathbb{Z})$ is an example of such a predual. Yet, to provide more shift-invariant preduals of $\ell_1(\mathbb{Z})$, a more elaborate construction is required. For example, in [7] the authors construct an uncountable family of shift-invariant preduals and set up systematic study of those preduals.

One of the reasons why we are interested in shift-invariant preduals of $\ell_1(\mathbb{Z})$ is hidden behind the dual Banach algebra theory. Recall that a Banach algebra A, which is also the dual space of some Banach space E, is called a *dual Banach algebra*, if the multiplication product on A is separately weak*-continuous, with respect to E. As explained in section 3.3 of this project if E is a shift-invariant predual of $\ell_1(\mathbb{Z})$, then the convolution product on $\ell_1(\mathbb{Z})$ given by

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k)g(n-k) \qquad f, g \in \ell_1(\mathbb{Z}),\ n \in \mathbb{Z} \tag{1.1.1}$$

is separately weak*-continuous and so E turns $\ell_1(\mathbb{Z})$ into a dual Banach algebra. By the research paper [7] we know that there exists an uncountable family of such shift-invariant preduals. Moreover, each of the preduals constructed in that paper is isomorphic as a Banach space to $c_0(\mathbb{Z})$. As it often happens in Banach space theory, although we now know that two spaces are isomorphic, it can sometimes be difficult to find an explicit form of isomorphism. My thesis will make some attempts in the direction of constructing an isomorphism between these preduals and $c_0(\mathbb{Z})$. Another important result from [7] is that, despite the fact that two shift-invariant preduals may be isomorphic as a Banach spaces, they may induce different weak*-topology. This interesting outcome may also initiate further development of the theory of shift-invariant preduals of $\ell_1(\mathbb{Z})$, but this time, from the Banach space perspective.

Now let us discuss the content of this thesis. Chapter two is preliminary and presents the Čech-Stone compactification of $\mathbb{Z}$. The crucial result here is that the space $\beta\mathbb{Z}$ of ultrafilters on $\mathbb{Z}$ with the Stone topology is the Čech-Stone compactification of $\mathbb{Z}$. At the end of this chapter I describe a very useful approach to examine convergence of sequences in terms of ultrafilters. The theory presented in this part of my thesis will have an application further, in chapters 4 and 5.

An introduction to the study of preduals of $\ell_1(\mathbb{Z})$ is included in chapter 3. I review there the most important tools of functional analysis, which are required for the thesis. This include the Hahn-Banach theorem, the Banach Isomorhphism theorem and the concepts of the weak and weak*-topology. In the next two sections we focus directly on the core knowledge concerning preduals of $\ell_1(\mathbb{Z})$ and we put particular emphasis on the aspect of concrete preduals. In particular, I describe the connection between concrete shift invariant preduals of $\ell_1(\mathbb{Z})$ and dual Banach algebra $\ell_1(\mathbb{Z})$ mentioned before. The research paper [7] is the main reference in this part of my thesis.

In chapter 4, based on chapter 3 of [7], I construct the family of shift-invariant preduals $\{F^{(\lambda)}\}_{|\lambda|>1}$. The shift-invariant predual $F^{(\lambda)}$ is described as follows: fix $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$, and for $n \geq 1$ in $\mathbb{Z}$, $b(n)$ denote the number of ones in the binary expansion of n, for $n < 0$ we set $b(n) = \infty$. Then, define an element $x_0 \in \ell_\infty(\mathbb{Z})$ by $x_0(n) = \lambda^{-b(n)}$ with the convention that $\lambda^{-\infty} = 0$. Hence, x_0 is the element of the form $x_0 = (..., 0, 0, 1, \lambda^{-1}, \lambda^{-1}, \lambda^{-2}, \lambda^{-1}, ...)$ with 1 on the $n = 0$ position of $\mathbb{Z}$. Finally, the space $F^{(\lambda)}$ is the closed shift-invariant subspace of $\ell_\infty(\mathbb{Z})$ generated by x_0. In other words

$$F^{(\lambda)} := \overline{span\,\{\sigma^n(x_0) : n \in \mathbb{Z}\}},$$

In that section I also define a space $F_+^{(\lambda)}$ which is a subspace of $F^{(\lambda)}$. This space is defined as $\overline{span\,\{\sigma^n(x_0) : n \geq 0\}}$ and will be my main object of research.

One of the most important results in [7], also described in this thesis, is Theorem 3.4 which says that the preduals $F^{(\lambda)}$ are G-spaces. Because of that, using the main result of [1] we know that each $F^{(\lambda)}$ is isomorphic as Banach space to some $C(L)$ space for some compact Hausdorff space L. This work was a guideline for my attempts in finding an explicit isomorphism between $F_+^{(\lambda)}$ and $c_0(\mathbb{Z})$, which is the content of the last chapter. First, in section one of this chapter I explain why the space $F^{(\lambda)}$ is isomorphic with $c_0(\mathbb{Z})$. This is caused by the fact that indices of these spaces, called Szlenk index, are the same. I calculate, by definition of the Szlenk index of $c_0(\mathbb{Z})$ and then I rely on Theorem 3.8 of [7], where it was proved that the Szlenk index of $F^{(\lambda)}$ is equal to ω. The last section summarizes all the information gathered and allows us to define a function between $F_+^{(\lambda)}$ and $c_0(\mathbb{N})$ which, by using fundamental theorems of functional analysis, proved to be an isomorphism.

1.2 Basic preduals of $\ell_1(\mathbb{Z})$

Now we look at two important examples of predual spaces of $\ell_1(\mathbb{Z})$. First, we "review the proof" that the space $c_0(\mathbb{Z})$ is a shift-invariant predual of $\ell_1(\mathbb{Z})$ and then show that this predual really makes the convolution product separately continuous. To prove these facts we need to recall some very basic facts of functional analysis.

Definition 1.2.1. Let X and Y be two vector spaces over the field $\mathbb{K}$ of either real numbers or complex numbers. A map $T : X \to Y$ is called a linear operator, if for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$ we have

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y). \tag{1.2.1}$$

Definition 1.2.2. Let X and Y be two normed spaces. A map $T : X \to Y$ is called a bounded linear operator, if it is linear and there exists a constant $M > 0$ such that, for all $x \in X$,

$$\|Tx\| \leq M\|x\|. \tag{1.2.2}$$

The set of those $T : X \to Y$ satisfying (1.2.1) and (1.2.2) with respect to the standard pointwise definitions of additions and scalar multiplications of functions forms a vector space and is denoted by $\mathcal{B}(X, Y)$. Theorem 4.1 in [18] shows that for every $T \in \mathcal{B}(X, Y)$ we can assign a number

$$\|T\| := \sup\{\|Tx\| : x \in X, \|x\| \leq 1\},$$

and this definition of $\|T\|$ turns $\mathcal{B}(X, Y)$ into normed space. In fact, from linearity of T we have

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\} = \sup\{\|Tx\| : x \in X, \|x\| = 1\}. \tag{1.2.3}$$

A very standard fact in the theory of bounded linear operators is that every $T \in \mathcal{B}(X,Y)$ is continuous and each continuous linear operator $T: X \to Y$ belongs to $\mathcal{B}(X,Y)$. The next important fact to recall is that if Y is a Banach space, then $\mathcal{B}(X,Y)$ is also a Banach space (see Theorem 4.3 in [18]). Now, let us assume that Y is the scalar field ($\mathbb{R}$ or $\mathbb{C}$). In this case, the space of bounded linear operators $\mathcal{B}(X,Y)$ is called the *dual space* of X and is simply denoted by X^*. For our convenience we denote the elements of X^* by x^* and for $x \in X$ instead of writing $x^*(x)$ we often write $\langle x^*, x\rangle$. The elements of X^* are called *bounded linear functionals* or, simply, *functionals*.

Lemma 1.2.3. *The dual space of $c_0(\mathbb{Z})$ is canonically isometrically isomorphic with $\ell_1(\mathbb{Z})$.*

Proof. To prove that our statement is true we need to find a bijective, linear and continuous map between $c_0(\mathbb{Z})^*$ and $\ell_1(\mathbb{Z})$ which is also norm-preserving. Let $(e^{(n)})_{n=-\infty}^{n=+\infty}$ be the standard basis for $c_0(\mathbb{Z})$, defined by $e_k^{(n)} = \delta_{n,k}$, where $k \in \mathbb{Z}$ and

$$\delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k; \\ 0 & \text{if} \quad n \neq k. \end{cases} \tag{1.2.4}$$

Let us define the map T in the following way:

$$T : c_0(\mathbb{Z})^* \ni x^* \longmapsto \left(..., \left\langle x^*, e^{(-1)}\right\rangle, \left\langle x^*, e^{(0)}\right\rangle, \left\langle x^*, e^{(1)}\right\rangle, ...\right) \in \ell_\infty(\mathbb{Z}). \tag{1.2.5}$$

Linearity of this map follows directly from the fact that $c_0(\mathbb{Z})^*$ is a vector space. Now, for any $x^* \in c_0(\mathbb{Z})^*$ we define $x^{(l)} = (x_n^{(l)}) \in c_0(\mathbb{Z})$, where $l \in \mathbb{Z}$ by setting

$$x_n^{(l)} := \begin{cases} \frac{\left|\left\langle x^*, e^{(n)}\right\rangle\right|}{\left\langle x^*, e^{(n)}\right\rangle} & \text{if} \quad n \leq |l|, \left\langle x^*, e^{(n)}\right\rangle \neq 0; \\ 0 & \text{if} \quad n > |l| \ \text{ or } \ \left\langle x^*, e^{(n)}\right\rangle = 0. \end{cases}$$

We see that $\|x^{(l)}\| = 1$ for each l and also

$$\begin{aligned} \left|\left\langle x^*, x^{(l)}\right\rangle\right| &= \left| \frac{\left|\left\langle x^*, e^{(-l)}\right\rangle\right|}{\left\langle x^*, e^{(-l)}\right\rangle} \left\langle x^*, e^{(-l)}\right\rangle + ... \frac{\left|\left\langle x^*, e^{(0)}\right\rangle\right|}{\left\langle x^*, e^{(0)}\right\rangle} \left\langle x^*, e^{(0)}\right\rangle + ... + \frac{\left|\left\langle x^*, e^{(l)}\right\rangle\right|}{\left\langle x^*, e^{(l)}\right\rangle} \left\langle x^*, e^{(l)}\right\rangle \right| \\ &= \left|\left\langle x^*, e^{(-l)}\right\rangle\right| + ... + \left|\left\langle x^*, e^{(0)}\right\rangle\right| + ... + \left|\left\langle x^*, e^{(l)}\right\rangle\right|. \end{aligned}$$

Therefore,

$$\sum_{|i|=0}^{\infty} \left|\left\langle x^*, e^{(i)}\right\rangle\right| = \lim_{|l|\to\infty} \left|\left\langle x^*, x^{(l)}\right\rangle\right| \leq \lim_{|l|\to\infty} \left(\|x^*\| \| x^{(l)}\|\right) = \|x^*\| < \infty. \tag{1.2.6}$$

Hence, we see that defined map T is into $\ell_1(\mathbb{Z})$ and, moreover, it is continuous. To show that T is norm preserving, we present $x \in c_0(\mathbb{Z})$ such that $x = \sum_{|i|=0}^{\infty} \alpha_i e^{(i)}$ for some unique sequence of scalars (α_i) and a base $(e^{(n)})$ defined by (1.2.4). Then for $x^* \in c_0(\mathbb{N})^*$ we have

$$|\langle x^*, x\rangle| = \left|\sum_{|i|=0}^{\infty} \alpha_i \left\langle x^*, e^{(i)}\right\rangle\right| \leq \sup_{i\in\mathbb{N}} |\alpha_i| \left|\sum_{|i|=0}^{\infty} \left\langle x^*, e^{(i)}\right\rangle\right| \leq \|x\| \sum_{|i|=0}^{\infty} \left|\left\langle x^*, e^{(i)}\right\rangle\right|. \tag{1.2.7}$$

In particular, considering only these elements $x \in c_0(\mathbb{N})$ such that $\|x\| = 1$ we obtain

$$\|x^*\| \leq 1 \cdot \sum_{i=1}^{\infty} \left|\left\langle x^*, e^{(i)}\right\rangle\right| = \|Tx^*\|.$$

Together, combining (1.2.7) and (1.2.6) gives us $\|Tx^*\| = \|x^*\|$, which also implies that T is injective. Now we show that T maps onto $\ell_1(\mathbb{Z})$. Let $\beta = (..., \beta^{(-2)}, \beta^{(-1)}, \beta^{(0)}, \beta^{(1)}, \beta^{(2)}, ..) \in \ell_1(\mathbb{Z})$. We define a map $g : c_0(\mathbb{Z}) \to \mathbb{Z}$ by setting $g(e^{(n)}) = \beta^{(n)}$ for each $n \in \mathbb{Z}$, where $(e^{(n)})$ is defined by (1.2.4). Then for any $x \in c_0(\mathbb{Z})$ we have $|g(x)| = |\sum_{|n|=0}^{\infty} \alpha_n \beta^{(n)}| \leq (\sup_{n\in\mathbb{Z}} |\alpha_n|) \sum_{|n|=0}^{\infty} |\beta^{(n)}| < +\infty$. Hence, $g \in c_0(\mathbb{Z})^*$ and so $Tg = \beta$. □

In a similar vein, we can show that the dual space of $\ell_1(\mathbb{Z})$ is isometrically isomorphic with $\ell_\infty(\mathbb{Z})$ (See subsection 2.10-6 in [12]). It is easy to see that $c_0(\mathbb{Z})$ is shift-invariant since shifting position of any $x \in c_0(\mathbb{Z})$ one position to the left or one position to the right does not change the limit of x. This fact, as mentioned above, implies that the convolution product on $\ell_1(\mathbb{Z})$ is separately weak*-continuous and so $c_0(\mathbb{Z})$ turns $\ell_1(\mathbb{Z})$ into a dual Banach algebra. Let us make this calculation by hand. Let $(f_i)_{i\in\mathbb{Z}} \subset \ell_1(\mathbb{Z})$ and $f \in \ell_1(\mathbb{Z})$ be such that $f_i \longrightarrow f$ weak*, with respect to $c_0(\mathbb{Z})$ (see section 3.1 for information about weak*-convergence). Fix $g \in \ell_1(\mathbb{Z})$ and take any $a \in c_0(\mathbb{Z})$. We want to show that

$$\langle f_i * g, a\rangle \longrightarrow \langle f * g, a\rangle, \quad (i \longrightarrow \infty), \tag{1.2.8}$$

where $*$ denotes the convolution product on $\ell_1(\mathbb{Z})$ defined by formula (1.1.1). Fix $i \in \mathbb{N}$, then we have

$$\langle f_i * g, a\rangle = \sum_{n\in\mathbb{Z}} (f_i * g)(n)a(n) = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} f_i(k)g(n-k)a(n).$$

Notice, that since the element $(f_i * g)$ is in $\ell_1(\mathbb{Z})$ and $a \in c_0(\mathbb{Z})$ the series $\sum_{n\in\mathbb{Z}}(f_i * g)(n)a(n)$ is absolutely convergent and, therefore, we can interchange the order of summation in the above equation. Hence,

$$\sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} f_i(k)g(n-k)a(n) = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} f_i(k)g(n-k)a(n) = \sum_{k\in\mathbb{Z}} f_i(k) \sum_{n\in\mathbb{Z}} g(n-k)a(n).$$

Since, for every $k \in \mathbb{Z}$, we have

$$|\sum_{n} g(n-k)a(n)| \leq \|g\|_{\ell_1} \|a\|_{c_0},$$

hence, by the Dominated Convergence Theorem for Sums, as $i \longrightarrow \infty$ we obtain

$$\sum_{k\in\mathbb{Z}} f_i(k) \sum_{n\in\mathbb{Z}} g(n-k)a(n) \longrightarrow \sum_{k\in\mathbb{Z}} f(k) \sum_{n\in\mathbb{Z}} g(n-k)a(n).$$

Again, since $f \in \ell_1(\mathbb{Z})$ this must be equal to

$$\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} f(k)g(n-k)a(n) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} f(k)g(n-k)a(n) = \sum_{n\in\mathbb{Z}}(f*g)(n)a(n).$$

Therefore, we see that if $i \to \infty$, then

$$\langle f_i * g, a\rangle = \sum_{n\in\mathbb{Z}}(f_i*g)(n)a(n) \longrightarrow \sum_{n\in\mathbb{Z}}(f*g)(n)a(n) = \langle f*g, a\rangle\,,$$

which, since the convolution product on $\ell_1(\mathbb{Z})$ is commutative, implies separate weak*-continuity of that product with respect to $c_0(\mathbb{Z})$.

Here is another important example of a space which is a predual of $\ell_1(\mathbb{Z})$.

Lemma 1.2.4. *Let $C(K)$ be a space of continuous functions on a countable, infinite, compact Hausdorff space K. Then the dual space of $C(K)$ is isometrically isomorphic with $\ell_1(\mathbb{Z})$.*

Proof. Indeed, the result of the Riesz Theorem (see Theorem 7.4 in [10]) states that the dual space of $C(K)$ is isometrically isomorphic with $\mathcal{M}(K)$ the space of all finite, signed Borel measure on K. As K is countable, write $K = \{x_1, x_2, x_3, ...\}$, where x_n are distinct. Since every $\{x_n\}$ is a Borel set, for every $\mu \in \mathcal{M}(K)$ we can assign the sequence $(a_n)_{n=1}^{\infty}$, where $a_n = \mu(\{x_n\})$. Furthermore, since μ is countably additive and finite we obtain

$$\|\mu\| = |\mu(K)| = \left|\mu(\bigcup_{n=1}^{\infty}\{x_n\})\right| = \sum_{n=1}^{\infty}|a_n| = \|(a_n)_{n=1}^{\infty}\|\,.$$

Hence, we see that $\mathcal{M}(K) \cong \ell_1(K)$. The last thing to notice is that $\ell_1(K) = \ell_1(\mathbb{Z})$, but this simply follows from the fact that both K and $\mathbb{Z}$ are countable and so bijective. $\square$

The example shown above can never be shift-invariant and explanation of that fact will be given later in Example 3.2.7, since more advanced theory will be required. In understanding, which preduals of $\ell_1(\mathbb{Z})$ are and which are not shift-invariant, it would be useful to refer to [8], where the authors were investigating possible preduals of the measure algebra $\mathcal{M}(G)$, where G is a locally compact group.

Chapter 2

The Čech-Stone compactification $\beta\mathbb{Z}$

The Čech-Stone compactification is an important tool in examining many mathematical problems. The concepts of duality and preduality of some Banach Spaces are among them. Therefore, to understand the main theme of my thesis we present this compactification at the beginning of my work.

2.1 Filters and ultrafilters

In the first section we introduce the definition of a filter and an ultrafilter on the set of integers $\mathbb{Z}$. We give also some examples and present some basic facts related to these concepts. All of the results provided in this section are very standard and can be found for example in [11], [6] as well as [24].

Definition 2.1.1. A subset $\mathcal{F} \subset \mathcal{P}(\mathbb{Z})$ is said to be a *filter* if:
a) $\mathbb{Z} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;
b) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
c) If $A \in \mathcal{F}$ and the set $B \in \mathcal{P}(\mathbb{Z})$ is such that $B \supseteq A$, then $B \in \mathcal{F}$.

The reader can easily see from this definition that filter on $\mathbb{Z}$ is a non-empty and proper collection of subsets of $\mathbb{Z}$, which is closed under finite intersection and supersets. We proceed now to the two important types of filters that are related.

Definition 2.1.2. A filter $\mathcal{F}$ is said to be *principal* if it is generated by a single, non-empty set. In other words, there exist a single, non-empty set A such that $\mathcal{F} = \{B : B \supseteq A\}$. A filter which is not principal, is called *non-principal.*

An example of a non-principal filter is the family $\mathcal{F}r$ of subsets of $\mathbb{Z}$ such that

$$\mathcal{F}r := \{X \subseteq \mathbb{Z} : \mathbb{Z} \setminus X \text{ is finite } \} .$$

The filter $\mathcal{F}r$ is called Frechet filter.

Now we shall introduce the definition of an ultrafilter which plays a crucial role in this chapter.

Definition 2.1.3. A filter $\mathcal{F}$ on $\mathbb{Z}$ is said to be an *ultrafilter* if for each $A \subseteq \mathbb{Z}$, either $A \in \mathcal{F}$ or $\mathbb{Z} \setminus A \in \mathcal{F}$.

To have better understanding of the idea of filters and ultrafilters we prove some basic facts that refer to them. First, we recall that a family $\mathcal{G} \subset \mathcal{P}(\mathbb{Z})$ has the finite intersection property if each finite subfamily of $\mathcal{G}$ has a non-empty intersection. Now we can start with the following lemma.

Lemma 2.1.4. *Every subset $\mathcal{G} \subset \mathcal{P}(\mathbb{Z})$ with the finite intersection property can be extended to a filter.*

Proof. Let $\mathcal{G} = \{G_i\}_{i \in I}$ be a family with the finite intersection property and let

$$\mathcal{F} := \left\{ A \subseteq \mathbb{Z} \ : \exists \text{ finite E} \subseteq I \text{ such that } A \supseteq \bigcap_{i \in E} G_i \right\}.$$

We prove that $\mathcal{F}$ is a filter. It is easy to see that the whole space $\mathbb{Z}$ does belong to $\mathcal{F}$ and, since, $\mathcal{G}$ has the finite intersection property, the empty set does not belong to $\mathcal{F}$. If A and B belong to $\mathcal{F}$, then there exist finite sets E and F included in I such that $A \supseteq \bigcap_{i \in E} G_i$ and $B \supseteq \bigcap_{i \in F} G_i$. But this implies that $A \cap B \supseteq \bigcap_{i \in E \cup F} G_i$. Now, if $A \in \mathcal{F}$ then $A \supseteq \bigcap_{i \in E} G_i$ for some finite set $E \subseteq I$, then $\bigcap_{i \in E} G_i$ is included in any set $B \supseteq A$ and so $\mathcal{F}$ is a filter. The fact that $\mathcal{G} \subseteq \mathcal{F}$ follows by definition of $\mathcal{G}$. $\square$

Lemma 2.1.5. *A filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ is a maximal filter.*

Proof. ($\Longrightarrow$) If $\mathcal{F}$ is not a maximal filter, then there exists a filter $\mathcal{G}$ such that $\mathcal{G} \supset \mathcal{F}$ and a set A such that $A \in \mathcal{G}$ and $A \notin \mathcal{F}$. If $\mathcal{F}$ were an ultrafilter then $\mathbb{Z} \setminus A \in \mathcal{F}$, also since $A \in \mathcal{G}$ then $\mathbb{Z} \setminus A \notin \mathcal{G}$ which is impossible so $\mathcal{F}$ is not an ultrafilter.

($\Longleftarrow$) Let us assume that $\mathcal{F}$ is not an ultrafilter. Then there exists a set $A \subset \mathbb{Z}$ such that $A \notin \mathcal{F}$ and $\mathbb{Z} \setminus A \notin \mathcal{F}$. Now, we define the family $\mathcal{G} := \mathcal{F} \cup \{A\}$. I show that $\mathcal{G}$ has the finite intersection property. First we notice that if $B \in \mathcal{F}$ then $B \cap A \neq \emptyset$. Otherwise $B \subseteq \mathbb{Z} \setminus A$, but $\mathbb{Z} \setminus A \notin \mathcal{F}$ and we get a contradiction. Therefore, $A \cap B \neq \emptyset$. Now we consider any finite family of sets $B_1, B_1, ..., B_n \in \mathcal{F}$. Then of course $B_1 \cap B_1 \cap ... \cap B_n \in \mathcal{F}$ and so $A \cap (B_1 \cap B_1 \cap ... \cap B_n) \neq \emptyset$. We see now that the family $\mathcal{G}$ has the finite intersection property and so by Lemma 2.1.4 it can be extended to a filter $\mathcal{F}'$. Hence, we obtain that $\mathcal{F} \subset \mathcal{G} \subset \mathcal{F}'$ and so $\mathcal{F}$ is not maximal. $\square$

Before we state the last lemma in this section we recall from the set theory *Zorn's Lemma* which is equivalent to the *axiom of choice.* Zorn's lemma says that if in any non-empty, partially ordered set $\mathcal{X}$, every non-empty, linearly ordered set $\mathcal{C}$ has an upper bound, then in $\mathcal{X}$ there exists at least one maximal element.

Lemma 2.1.6. *Every filter $\mathcal{F}$ can be extended to an ultrafilter.*

Proof. Let us consider the set

$$\mathbf{F} = \left\{ \mathcal{F}^{'} \ : \ \mathcal{F}^{'} \text{ is a filter and } \mathcal{F} \subseteq \mathcal{F}^{'} \right\}.$$

It is easy to observe that the set $\mathbf{F}$ is partially ordered by the inclusion relation $' \subseteq \ '$. Now let $\mathcal{C}$ be a non-empty, lineary ordered set in $\mathbf{F}$ and let $\mathcal{U} = \bigcup \mathcal{C}$. We show that $\mathcal{U}$ must be a filter. By contradiction, if $\mathcal{U}$ were not a filter then there would exist two different sets A and B from $\mathcal{U}$ such that $A \cap B = \emptyset$. Since $A, B \in \mathcal{U}$ there exist filters $\mathcal{F} \in \mathcal{C}$ and $\mathcal{G} \in \mathcal{C}$ such that $A \in \mathcal{F}$ and $B \in \mathcal{G}$. Since $\mathcal{C}$ is a linearly ordered, we obtain that $A \in \mathcal{F}$ and $B \in \mathcal{F}$ or $A \in \mathcal{G}$ and $B \in \mathcal{G}$. But this implies that intersection of the sets A and B cannot be empty. Therefore we see that $\mathcal{U}$ is an filter and, hence, an upper bound for $\mathcal{C}$ in $\mathbf{F}$. Now, applying *Zorn's Lemma*, we obtain the existence of a maximal element in $\mathbf{F}$ which, by Lemma 2.1.5, must be an ultrafilter. $\square$

2.2 The topological space $\beta\mathbb{Z}$

In this section we define the topological space $\beta\mathbb{Z}$. We show that this space is compact and satisfies some separation axioms. At this point we will need certain basic topological definitions and facts, which are not included in my thesis but can easily be found, for example, in any basic undergraduate course of topology, or in [11]. In the beginning we introduce the definition of the space $\beta\mathbb{Z}$ and distinguish some sets in this space.

Definition 2.2.1. The space $\beta\mathbb{Z}$ is defined as the space of all ultrafilters on $\mathbb{Z}$. Thus,

$$\beta\mathbb{Z} := \{ p \subseteq \mathcal{P}(\mathbb{Z}) \ : \ p \text{ is an ultrafilter on } \mathbb{Z} \}.$$

Also for each non-empty set $A \subseteq \mathbb{Z}$ we define the corresponding base set, denoted by $\overline{A}$, as the set of those ultrafilters which contain the set A. Hence,

$$\overline{A} := \{ p \in \beta\mathbb{Z} \ : \ A \in p \}.$$

From this definition we see that $p \in \overline{A}$ if and only if $A \in p$.

Now we prove a lemma that shows us some basic properties of base sets which are very helpful in defining the topology on $\beta\mathbb{Z}$.

Lemma 2.2.2. *Let A and B be any non-empty subsets of $\mathbb{Z}$. Then for the corresponding base sets $\overline{A}$ and $\overline{B}$ the following holds:*
1. $\overline{A} \cap \overline{B} = \overline{A \cap B}$;
2. $\beta\mathbb{Z} \setminus \overline{A} = \overline{(\mathbb{Z} \setminus A)}$;
3. $\overline{\mathbb{Z}} = \beta\mathbb{Z}$.

Proof. (1) Let $p \in \beta\mathbb{Z}$ be such that $p \in \overline{A} \cap \overline{B}$. Using the definition of a base set and a filter we obtain

$$(A \in p) \wedge (B \in p) \Longleftrightarrow A \cap B \in p,$$

which is equivalent with the fact that

$$(p \in \overline{A}) \wedge (p \in \overline{B}) \Longleftrightarrow p \in \overline{A \cap B}.$$

(2) If an ultrafilter $p \in \beta\mathbb{Z}$ is such that $p \notin \overline{A}$, then according to the definition of base set $A \notin p$. Since p is an ultrafilter, we have $\mathbb{Z} \setminus A \in p$, but this says that $p \in \overline{(\mathbb{Z} \setminus A)}$.
(3) Since for every filter p on $\mathbb{Z}$ we have $\mathbb{Z} \in p$ we obtain our equality. $\square$

We shall now introduce the topology on $\beta\mathbb{Z}$ and we do it by using a collection of base sets. First, we recall that if the collection of sets $\mathcal{B}$ of a space $\mathcal{X}$ satisfies the following conditions:

1. $\bigcup \mathcal{B} = \mathcal{X}$;
2. if B_1, $B_2 \in \mathcal{B}$, then there exist $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$,

then there exists a unique topology $\mathcal{O}$ on $\mathcal{X}$ such that $\mathcal{B}$ is a base for $\mathcal{X}$, and so every open set in $\mathcal{X}$ is in fact the union of the sets from $\mathcal{B}$. In connection with this topological fact our situation is as follows. If we put $\mathcal{B} = \{\overline{A} : A \subset \mathbb{Z}\}$ where $\overline{A}$ is a base set introduced above, then, from Lemma 2.2.2 (1), we get that $\mathcal{B}$ is closed under finite intersections. Also, Lemma 2.2.2 (3), implies that $\mathcal{B}$ contains the whole space. Therefore, we obtain that $\mathcal{B}$ is the base of a topology on $\beta\mathbb{Z}$. Now we summarize our discussion in the following definition.

Definition 2.2.3. (topological space $\beta\mathbb{Z}$) The space of ultrafilters $\beta\mathbb{Z}$ is a topological space where the topology on $\beta\mathbb{Z}$ is generated by the base $\mathcal{B} = \{\overline{A} : A \subseteq \mathbb{Z}\}$ where $\overline{A}$ is the set of all ultrafilters containing A.

We finish this section with lemma that says that base sets are also closed. As a consequence, we obtain the characterisation of closed sets in the topological space $\beta\mathbb{Z}$.

Lemma 2.2.4. *The set $\overline{A} \subseteq \beta\mathbb{Z}$ is closed for any $A \subseteq \mathbb{Z}$. In particular, each closed subset of $\beta\mathbb{Z}$ can be written as an intersection of sets from $\mathcal{B} = \{\overline{A} : A \subseteq \mathbb{Z}\}$.*

Proof. The fact that $\overline{A}$ is closed follows directly from point (2) of lemma 2.2.2 which says that

$$\beta\mathbb{Z} \setminus \overline{A} = \overline{(\mathbb{Z} \setminus A)} \in \mathcal{B}.$$

Now, let C denote a closed set in $\beta\mathbb{Z}$. Then, for some open set O in $\beta\mathbb{Z}$, we get

$$C = \beta\mathbb{Z} \setminus O = \beta\mathbb{Z} \setminus \bigcup_{\overline{A} \in \mathcal{A}} \overline{A},$$

where $\mathcal{A}$ is a subset of $\mathcal{B}$. Then

$$C = \bigcap_{\overline{A}\in\mathcal{B}} \beta\mathbb{Z}\setminus\overline{A} = \bigcap_{\overline{A}\in\mathcal{B}} \overline{(\mathbb{Z}\setminus A)}.$$

Thus we see that each closed set must be an intersection of a family of closed (and, at the same time, open) sets from $\mathcal{B}$. □

2.3 Properties of the topological space $\beta\mathbb{Z}$

In this section we present some basic properties of the topological space $\beta\mathbb{Z}$. We start with the presentation of the theorem which tells us that $\beta\mathbb{Z}$ is a compact space and also highlights some separation properties.

Theorem 2.3.1. *The topological space $\beta\mathbb{Z}$ is a compact Hausdorff space.*

Proof. We first show that $\beta\mathbb{Z}$ is a Hausdorff space. If p and q are two different ultrafilters, then there exists a set $A \subseteq \mathbb{Z}$ such that $A \in p$ and $A \notin q$. Using the definition of the base set we see that this is equivalent with the fact that $p \in \overline{A}$ and $q \in \overline{\mathbb{Z}\setminus A}$. Hence, the ultrafilters p and q can be separated by two open sets.

Now we turn to the compactness of $\beta\mathbb{Z}$. Firstly, we recall the condition that helps us to identify the compactness property of topological spaces. This condition says that a topological space X is compact if and only if every collection of closed sets with the finite intersection property has a non-empty intersection. Applying this fact to the basic closed sets $\overline{A}_i$ where $i \in I$, let us assume that $\{\overline{A_i} : i \in I\}$ is a collection with the finite intersection property. According to Lemma 2.2.2 (1) for each finite set $E \subseteq I$ we have:

$$\emptyset \neq \bigcap_{i\in E} \overline{A_i} = \overline{\bigcap_{i\in E} A_i}.$$

However, this implies that $\bigcap_{i\in E} A_i \neq \emptyset$. Hence, the family of sets $\{A_i : i \in I\}$ has the finite intersection property. By Lemma 2.1.4 we know that such a family generates a filter $\mathcal{F}$ which, according to the Lemma 2.1.6, can be extended to an ultrafilter p. Let us notice that each A_i belongs to p, so $p \in \overline{A_i}$ for each $i \in I$, but this implies that $p \in \bigcap_{i\in I} \overline{A_i}$. So now, $\bigcap_{i\in I} \overline{A_i} \neq \emptyset$, since each closed set in $\beta\mathbb{Z}$ can be written as the intersection of basic closed sets (as explained in Lemma 2.2.4), we see that our condition holds also for any collection of closed sets with the finite intersection property. Therefore, we obtain that the space $\beta\mathbb{Z}$ is compact. □

Now, we prove that the space $\mathbb{Z}$ is densely embedded in $\beta\mathbb{Z}$. Let us first define the natural embedding of $\mathbb{Z}$ into $\beta\mathbb{Z}$. Since for each number $n \in \mathbb{Z}$ there is exactly one principal ultrafilter containing the singleton $\{n\}$, we can define the map $\beta : \mathbb{Z} \longrightarrow \beta\mathbb{Z}$ such that

$$\mathbb{Z} \ni n \longmapsto \beta(n) := \{A \subseteq \mathbb{Z} \ : \ n \in A\} \in \beta\mathbb{Z}. \tag{2.3.1}$$

I show that the map β is a homeomorphism onto its range and that the image $\beta(\mathbb{Z}) \subseteq \beta\mathbb{Z}$ of the set $\mathbb{Z}$ is dense in $\beta\mathbb{Z}$.

Theorem 2.3.2. *The topological space $\beta\mathbb{Z}$ contains a homeomorphic copy of $\mathbb{Z}$ as a dense subset.*

Proof. First, we have to show that the map β defined above is a homeomorphism onto its range. It is easy to see that β is injective and also, since $\mathbb{Z}$ carries the discrete topology, the map β must be continuous. To prove that β is a homeomorphism onto its range, it remains to show that β is open. Because $\mathbb{Z}$ is equipped with discrete topology it will be suffice to show that the image of every singleton in $\mathbb{Z}$ is open in $\beta\mathbb{Z}$. Let $n \in \mathbb{Z}$ and we obtain

$$\{\beta(n)\} = \{p \in \beta\mathbb{Z} : \{n\} \in p\} = \overline{\{n\}} \in \mathcal{B}.$$

We see that $\{\beta(n)\}$ is open in $\beta\mathbb{Z}$. Hence, we obtain that $\beta(\mathbb{Z})$ is a homeomorphic copy of $\mathbb{Z}$.

Now we explain why $\beta(\mathbb{Z})$ is dense in $\beta\mathbb{Z}$. I use the definition of density, which says that subset D of a topological space X is dense in X if every non-empty open subset in X contains some element of D. In our case it will be enough to consider open sets from the base $\mathcal{B}$. Let then $\overline{A} \in \mathcal{B}$, we find an element of $\beta(\mathbb{Z})$ in $\overline{A}$. Since $A \neq \emptyset$ we can pick some $n \in A$. By definition, $\{n\} \in \beta(n)$, hence, $\{n\} \subseteq A \in \beta(n)$, but this yields that $\beta(n) \in \overline{A}$. □

2.4 The Čech-Stone compactification $\beta\mathbb{Z}$

Now we present the main essence of this chapter, namely, the fact that the space $\beta\mathbb{Z}$ is a Čech-Stone compatification of $\mathbb{Z}$. We start with the definition of Hausdorff compactification.

Definition 2.4.1. Let X be a topological space. A compact Hausdorff space C is said to be a *Hausdorff compactification* of X if C contains a homeomorphic copy of X as a dense subset.

Before we state the definition of Čech-Stone compactification we recall a few more topological facts.

Definition 2.4.2. A topological space X is called *Tychonoff space* if it is Hausdorff and for any closed set F and any point x that does not belong to F there is a continuous function f from X to the real line $\mathbb{R}$ such that $f(x) = 0$ and, for every y in F, $f(y) = 1$.

Remark 2.4.3. A standard topological fact concearning Tychonoff space says that a topological space has a Hausdorff compactifcation if and only if it is Tychonoff.

Definition 2.4.4. Let X be a Tychonoff space, then the Čech-Stone compactification βX of X is a Hausdorff compactification with an embedding $\beta : X \to \beta X$ and in addition the following universal property is satisfied: for each compact Hausdorff space Y and continuous map $f : X \to Y$, there exist a uniquely determined continuous map $\widetilde{f} : \beta X \to Y$ such that $\widetilde{f}$ restricted to X equals to f.

This definition can be illustrated by the following diagram.

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[dr, "f"'] & \beta X \arrow[d, "\exists! \widetilde{f}"] \\ & Y \end{tikzcd}$$

Remark 2.4.5. A standard topogical argument proves that the Čech-Stone compactification of a Tychonoff space is unique in the sense that every compactification which satisfies the above property is homeomorphic to βX.

Lemma 2.4.6. *Let f be a function from $\mathbb{Z}$ to a compact Hausdorff space Y and let p be an ultrafilter on $\mathbb{Z}$. Let also for $y \in Y$ the set $U(y)$ denote the collection of neighbourhoods of y in Y. Then*

$$f^{-1}(U) \in p \text{ for all } U \in U(y) \Longleftrightarrow y \in \bigcap_{A \in p} \overline{f(A)}.$$

Proof. ($\Longrightarrow$) If we assume that $y \notin \bigcap_{A\in p} \overline{f(A)}$, then we can choose a $U \in U(y)$ such that for some $A \in p$ we get $U \cap f(A) = \emptyset$. But this implies that $f^{-1}(U) \cap A = \emptyset$, so $f^{-1}(U)$ can not be in p.
($\Longleftarrow$) Similarly, if we assume that for some $y \in \bigcap_{A\in p} \overline{f(A)}$ and $U \in U(y)$ we have $f^{-1}(U) \notin p$. Then $\mathbb{Z} \setminus f^{-1}(U) \in p$, but this implies

$$U \cap f(\mathbb{Z} \setminus f^{-1}(U)) \neq \emptyset,$$

which is not possible. □

Before we present the main result of this chapter we recall another topological fact. If Y is a Hausdorff topological space, then the value of a continuous function $f : X \longrightarrow Y$ is completely determined by the value of f on a dense subset D of X. To see this suppose that f and g are two functions that agree on a dense subset D of X, and let $u \in X \setminus D$. If $f(u) \neq g(u)$, then there exist open neighbourhoods U and V of $f(u)$ and $g(u)$, respectively, such that $U \cap V = \emptyset$. Then $f^{-1}(U)$ is an open neighbourhood of u and $g^{-1}(V)$ is an open neighbourhood of u. Their intersection must be an open neighbourhood of u, and, therefore, must contain elements of D; but then any $e \in D$ in the intersection has $f(e) = g(e)$, with $f(e) \in U$ and $g(e) \in V$, contradicting that $U \cap V = \emptyset$. Therefore, $f(u) = g(u)$ for each $u \in X$. Hence $f = g$.

Theorem 2.4.7. *The topological space $\beta\mathbb{Z}$ is the Čech-Stone compactification of $\mathbb{Z}$.*

Proof. Let $f : \mathbb{Z} \to Y$ be a function (which is obviously continuous) from $\mathbb{Z}$ to an arbitrary compact Hausdorff space Y. We are going to define a continuous function $\widetilde{f} : \beta\mathbb{Z} \to Y$ which extends f. Notice that Y is a Hausdorff space and, since $\mathbb{Z}$ is a dense subset of $\beta\mathbb{Z}$, therefore, as explained above, the function $\widetilde{f}$ has prescribed values on $\mathbb{Z}$ so it is uniquely determined.

Now, for each ultrafilter $p \in \beta\mathbb{Z}$ we consider the expression: $\bigcap_{A\in p} \overline{f(A)}$, where $\overline{f(A)}$ denotes the closure in Y of the pointwise image of the set A. We show that for each ultrafilter $p \in \beta\mathbb{Z}$ the set $\bigcap_{A\in p} \overline{f(A)}$ consists of a single point. First, notice that since each filter has the finite intersection property, the family $\left\{\overline{f(A)} : A \in p\right\}$ has also the finite intersection property. Since Y is a compact space the set $\bigcap_{A\in p} \overline{f(A)}$ is non-empty.

Now we prove that for each ultrafilter $p \in \beta\mathbb{Z}$ the expression $\bigcap_{A\in p} \overline{f(A)}$ cannot contain two different points. Let assume that y_1 and y_2 are two distinct points contained in $\bigcap_{A\in p} \overline{f(A)}$. Using the fact that Y is Hausdorff we can find neighbourhoods $U_1 \in U(y_1)$ and $U_2 \in U(y_2)$ such that $U_1 \cap U_2 = \emptyset$. From the last lemma we get that $f^{-1}(U_1) \in p$ and $f^{-1}(U_2) \in p$, which is impossible. Hence, we proved that the set $\bigcap_{A\in p} \overline{f(A)}$ consists of a single point. Therefore, our function $\widetilde{f} : \beta\mathbb{Z} \to Y$ can be defined by

$$p \longmapsto \widetilde{f}(p) \in \bigcap_{A\in p} \overline{f(A)},$$

for each $p \in \beta\mathbb{Z}$.

The remaining thing is to show that the function $\widetilde{f} : \beta\mathbb{Z} \to Y$ is continuous and agrees with f on $\mathbb{Z}$. First, let us show continuity. Let $p \in \beta\mathbb{Z}$ and $V \in U(\widetilde{f}(p))$. Since Y is a compact Hausdorff space we can find a closed set $V^{'} \in U(\widetilde{f}(p))$ such that $V^{'} \subseteq V$. I want to find a neighbourhood of p such that its image under $\widetilde{f}$ is contained in $V^{'} \subseteq V$. If we define $A_0 := f^{-1}(V^{'}) \subseteq \mathbb{Z}$, then by Lemma 2.4.6, this set is in p so the corresponding basic set $\overline{A_0}$ is a neighbourhood of p. Now we can see the following

$$\widetilde{f}(p) \in \bigcap_{A\in p} \overline{f(A)} \subseteq \overline{f(A_0)} = \overline{f(f^{-1}(V'))} \subseteq V^{'}.$$

Hence, we obtain continuity of $\widetilde{f}$.

To show agreement between $\widetilde{f}$ and f on $\mathbb{Z}$ we notice the following. For any $n \in \mathbb{Z}$ and for a principal ultrafilter $\beta(n) = \{\ A \subseteq \mathbb{Z} : n \in A\}$ we have

$$\widetilde{f}(\beta(n)) \in \bigcap_{A\in\beta(n)} \overline{f(A)} \subseteq \overline{f(\{n\})} = \overline{\{f(n)\}} = \{f(n)\},$$

since $\{n\} \in \beta(n)$ and each singleton in Y is closed. Hence we obtain $\widetilde{f}(\beta(n)) = f(n)$ for each $n \in \mathbb{Z}$ which finishes the proof of the theorem. □

2.5 Convergence along an ultrafilter.

In the last short section of this chapter we examine the convergence of sequences in terms of ultrafilters. Given an ultrafilter p defined on $\mathbb{Z}$ and a sequence $\{x_n\}_{n\in\mathbb{Z}}$ on a topological space X, we can consider the limit of $\{x_n\}$ along p. The details are as follows.

Definition 2.5.1. For a topological space X let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence in X and let $x \in X$. Moreover, let p be an ultrafilter defined on $\mathbb{Z}$. We say that a sequence $\{x_n\}_{n\in\mathbb{Z}}$ converges to x along p (denoted by $\lim\limits_{n\to p} x_n = x$) if for each neighbourhood U of x, the set $\{n \in \mathbb{Z} : x_n \in U\}$ is in p.

The most fascinating aspect of this method of convergence is that if the topological space X is compact, then any sequence has limit along p.

Lemma 2.5.2. *For a compact Hausdorff space X and an ultrafilter p on $\mathbb{Z}$ let $\{x_n\}_{n\in\mathbb{Z}}$ be any sequence taking values in X. Then there exists exactly one point $x \in X$ such that $\lim\limits_{n\to p} x_n = x$.*

Proof. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence in X and p be an ultrafilter on $\mathbb{Z}$. First, we prove the existence of $x \in X$ such that $\lim\limits_{n\to p} x_n = x$. By contradiction, suppose that such x does not exists. Then, for each point $y \in X$ there is an open neighbourhood U_y of y such that $\{n \in \mathbb{Z} : x_n \in U_y\} \notin p$. By construction the family $\{U_y\}_{y\in X}$ of subsets of X is a cover for X. Since X is compact there exist a finite collection of points $y_1, ..., y_k$ in X such that the family $\{U_{y_i}\}_{i=1}^k$ is a finite subcover for X. Now we partition $\mathbb{Z}$ into finitely many disjoint pieces K_i $(i \leq k)$ in the following way

$$l \in K_i \Leftrightarrow x_l \in U_i.$$

Let us notice that for some $l \in \mathbb{Z}$ we may have that the element x_l belongs to more than one element of the finite subcover. In this case we choose the element of the subcover where x_l belongs arbitrarily. By construction, no piece in this partition is in p, so we can easily see that this contradicts the fact that p is an ultrafilter.

To prove uniqueness of the limit of a sequence $\{x_n\}_{n\in\mathbb{Z}}$ we assume that there exist two different points x and y in X such that

$$(\lim_{n\to p} x_n = x) \wedge (\lim_{n\to p} x_n = y).$$

Since X is a Hausdorff space we can choose two disjoint neighborhoods U_x and U_y of x and y respectively. The sets $\{n : x_n \in U_x\}$ and $\{n : x_n \in U_y\}$ constructed like above are also disjoint and so they can not both be in p. Hence, we see that points x and y can not be the limit of the same sequence $\{x_n\}$. $\square$

Chapter 3

Duality and Preduality

In this chapter of my thesis we focus on the concepts of duality and preduality. I divide this chapter into three parts. The first part recalls basic tools of functional analysis. In the second part I introduce the idea of concrete preduals and in the last part I reveal the connection between shift-invariant preduals and dual Banach algebras. I assume that reader has basic knowledge about topological, metric and normed spaces which can be found, for example, in [11], [22] and in [18].

3.1 Review of basic tools of functional analysis

Research on duality and preduality demands wide knowledge of functional analysis. This section, mainly based on [18] and [10], presents the core knowledge of this area of mathematics with emphasis on the parts that are extensively used in my thesis. Since these concepts are also very basic they are stated without proofs.

The very standard facts of the theory of bounded linear operators were introduced at the beginning of Section 1.2. Here, we begin with the concept of isomorphic spaces.

Definition 3.1.1. Let X and Y be two normed spaces. We say that X and Y are isomorphic if there exists a linear operator $T : X \to Y$ such that

1. $T \in \mathcal{B}(X, Y)$,
2. T is bijective,
3. T^{-1} is bounded.

The following theorem gives sufficient conditions for a map between Banach spaces to be an isomorphism. In particular, we only have to show that if a bijective map $T : X \to Y$ is continuous, then continuity of the inverse is given for us for free.

Theorem 3.1.2. *Let X and Y be two complete normed spaces and let $T \in \mathcal{B}(X, Y)$ be a bijection. Then T^{-1} is bounded and so T is an isomorphism.*

The next theorem is also helpful in examining the isomorphic structures of two normed spaces.

Theorem 3.1.3. *Let X be a Banach space, Y be a normed space and $T \in \mathcal{B}(X,Y)$. If T is bounded below, in other words there exists $M > 0$ such that*

$$\|Tx\| \geq M\|x\|, \quad x \in X, \tag{3.1.1}$$

then $T(X)$ is complete and so closed in Y.

Now we present one of the most important tool of functional analysis, namely, the Hahn-Banach Theorem. This theorem allows us extend functionals from a subspace to the entire space without increasing the norm. We present here the version for normed spaces. (See Theorem 4.3-2 in [12].)

Theorem 3.1.4. *Let X be a normed space and Y be a subspace of X. Let also X^* denote the dual space of X, and Y^* denote the dual space of Y. Then for any $y^* \in Y^*$ there exists $x^* \in X^*$ such that*

$$\langle x^*, y \rangle = \langle y^*, y \rangle, \quad y \in Y, \tag{3.1.2}$$

and

$$\|x^*\| = \|y^*\|. \tag{3.1.3}$$

There are many important implications of this theorem and among them are the following two corollaries.

Corollary 3.1.5. *Let X be a normed space and X^* its dual space. Then for any $x \in X$*

$$\|x\| = \sup\{|\langle x^*, x \rangle| : x^* \in X^*, \|x^*\| = 1\}. \tag{3.1.4}$$

Corollary 3.1.6. *Let X be a normed space, F be a subspace of X and X^* be the dual of X. Let also $x_0 \in X$ be such that $x_0 \notin \overline{F}$, where $\overline{F}$ denotes the closure of F. Then there exists $x^* \in X^*$ satisfying the following conditions*

$$\langle x^*, x_0 \rangle = 1 \quad and \quad \langle x^*, y \rangle = 0, \quad y \in F. \tag{3.1.5}$$

The proof of the first corollary can be found in Lecture 12 of [10] and Theorem 3.5 in [18] proves the second corollary.

Another important for us space is called the *second dual* space of X. Let X be a Banach space. Since the normed dual space X^* of X is itself a Banach space it has a normed dual space denoted by X^{**}. Then every $x \in X$ defines a unique functional $\kappa x \in X^{**}$ given by the equality

$$\langle \kappa x, x^* \rangle := \langle x^*, x \rangle, \quad x^* \in X^*. \tag{3.1.6}$$

Moreover, by Corollary 3.1.5, we have

$$\|\kappa x\| = \|x\|. \tag{3.1.7}$$

The reader can easily see that by (3.1.6) we obtain a linear map $\kappa : X \to X^{**}$ which by property (3.1.7) is an isometry. The map κ is called *the canonical embedding* of X in X^{**}. For the reason that X is a complete normed space we conclude that κ is an isometric isomorphism from X onto its range $\kappa(X) \subseteq X^{**}$ (see subsection 4.5 in [18] for discussion about the second dual space). In the case when $\kappa(X) = X^{**}$ the space X is called *reflexive*.

Now, let us draw our attention to the following theorem describing the concept of *adjoint operators*. For proof of this theorem check Theorem 4.10 in [18].

Theorem 3.1.7. *Let X and Y be a two normed spaces with dual spaces X^* and Y^*, respectively. For each $T \in \mathcal{B}(X,Y)$ there exist exactly one operator $T^* \in \mathcal{B}(Y^*, X^*)$, called the adjoint of T satisfying*

$$\langle T^* y^*, x\rangle = \langle y^*, Tx\rangle\,, \tag{3.1.8}$$

for each $y^* \in Y^*$ and $x \in X$. Moreover we have $\|T^*\| = \|T\|$.

In the next chapter we will also require some knowledge about *quotient spaces*. Let us now recall this concept.

Definition 3.1.8. Let F be a subspace of a linear space X. For every $x \in X$ we define the *coset* $\pi(x)$ of x in X by

$$\pi(x) := x + F.$$

The set $\{\pi(x) = x + F : x \in X\}$, together with addition and scalar multiplication operations defined respectively as follows

$$\pi(x) + \pi(y) = \pi(x+y) \qquad \text{and} \qquad \alpha\pi(x) = \pi(\alpha x) \qquad x, y \in X,\ \ \alpha \in \mathbb{C}\ (or\ \mathbb{R}),$$

forms a linear space called the *quotient space of X modulo F* and is denoted by X/F. The map π is very often called the *quotient map* from X onto X/F. Note also that if X is a normed space then X/F is as well and the required norm on X/F is defined by

$$\|\pi(x)\| := \inf\{\|x - z\| : z \in F\}\,, \qquad x \in X.$$

Every normed space X is naturally equipped with topology given by a norm. Nevertheless, we often need to consider other topologies on X. These are usually the weak and the weak*-topologies. We now recall these ideas. Let X be a vector space over the field $\mathbb{K}$ of either real numbers or complex numbers. Recall that a function $p : X \to \mathbb{R}$ is called a seminorm if the following condition holds

- $p(x) \geq 0, \quad x \in X;$
- $p(\alpha x) = |\alpha| p(x), \quad x \in X,\ \alpha \in \mathbb{K};$
- $p(x+y) \leq p(x) + p(y), \quad x, y \in X.$

Recall also that we say that a family $\mathcal{P}$ of seminorms on X is separating if $p(x) = 0$ for all $p \in \mathcal{P}$ implies that $x = 0$, $(x \in X)$. Such a family of seminorms $\mathcal{P}$ gives rise to a topology τ_p on X. The details are as follows. For $a \in X$, $p \in \mathcal{P}$ and $\epsilon > 0$ we define a set

$$N(a; p, \epsilon) := \{x \in X : \ p(x-a) < \epsilon\}.$$

A subset G of X is τ_p open if for any $a \in G$ there exist seminorms $p_1, ..., p_n \in \mathcal{P}$ and scalars $\epsilon_1, ...\epsilon_n > 0$ such that

$$\bigcap_{j=1}^{n} N(a; p_j, \epsilon_j) \subseteq G.$$

Definition 3.1.9. Let X be a normed space and X^* its dual. For each $f \in X^*$ we define a seminorm p_f on X by $p_f(x) = |f(x)|$, where $x \in X$. Then the family

$$\mathcal{P}_{X^*} := \{p_f : f \in X^*\}$$

is separating and the corresponding topology, obtained as described above, is called the *weak topology* of X and is usually denoted by $\sigma(X, X^*)$. Similarly, for $x \in X$ let us define a seminorm q_x on X^* by $q_x(f) = |f(x)|$, where $f \in X^*$. Again, the family

$$Q_x := \{q_x : x \in X\}$$

is separating and the corresponding topology on X^*, obtained as described above is called the *weak* − topology* of X^* and is usually denoted by $\sigma(X^*, X)$.

Remark 3.1.10. We can also think about weak and weak*- topologies in the follownig way. Let X be a normed space. Then the weak topology on X is the coarsest topology such that all elements of its dual space X^* are continuous in this topology. Similarly, the weak*-topology on X^* is the coarsest topology such that for each $x \in X$ the functional $x^* \mapsto \langle x^*, x\rangle$ on X^* is continuous in this topology.

With the weak and weak*-topologies, we very often combine the ideas of weak and weak* convergence.

Definition 3.1.11. (Weak and weak* convergence) Let X be a vector space and X^* its dual. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is convergent to the point $x \in X$ in the weak topology $\sigma(X, X^*)$ if for each $x^* \in X^*$

$$\langle x^*, x_n\rangle \longrightarrow \langle x^*, x\rangle \text{ as } (n \longrightarrow \infty). \tag{3.1.9}$$

In this case, we often say that a sequence $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to x. In a similar vein, we define weak*-convergence on X^*. We say that a sequence of functionals $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ is weak*-convergent to x^* if for each $x \in X$

$$\langle x_n^*, x\rangle \longrightarrow \langle x^*, x\rangle \text{ as } (n \longrightarrow \infty). \tag{3.1.10}$$

The last fact in this section is more advanced. (See Item 45.2 in [4] for a proof.)

Lemma 3.1.12. *Let X be a normed space, X^* its dual and X^{**} its second dual. Let also $\kappa : X \to X^{**}$ be the canonical embedding. Then $\kappa(X)$ is dense in X^{**} for the weak*-topology $\sigma(X^{**}, X^*)$.*

3.2 Concrete Preduals

This part of my thesis focuses on informations about concrete preduals. The content of the material included here is almost entirely based on section 2 of [7]. Here we will use the concepts and notations introduced in the previous section of this chapter where we reviewed the basic tools of functional analysis. Nevertheless, before we start learning about concrete preduals let us recall one more fact concerning dual spaces. (See Theorem 4.9 in [18]).

Fact 3.2.1. For a closed subspace F of a Banach space X its dual space F^* is isometrically isomorphic to the quotient space $X^*/F^\perp$, where

$$F^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0, \text{ for all } x \in F\}.$$

Definition 3.2.2. Let F be a closed subspace of $\ell_\infty(\mathbb{Z})$ and r be the restriction map from $\ell_\infty(\mathbb{Z})^*$ to F^*. Let also $\kappa_{\ell_1(\mathbb{Z})} : \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})^{**} = \ell_\infty(\mathbb{Z})^*$ be the canonical embedding. We define a map ι_F as the composition of the map $\kappa_{\ell_1(\mathbb{Z})}$ and r. Thus,

$$\iota_F := r \circ \kappa_{\ell_1(\mathbb{Z})} : \ell_1(\mathbb{Z}) \longrightarrow F^*. \tag{3.2.1}$$

The following diagram illustrates the above definition of the map ι_F combined with Fact 3.2.1.

$$\begin{tikzcd}
\ell_1(\mathbb{Z}) \arrow[r, "\iota_F"] \arrow[dr, "\kappa_{\ell_1(\mathbb{Z})}"'] & \ell_\infty(\mathbb{Z})^*/F^\perp \\
& \ell_\infty(\mathbb{Z})^* \arrow[u, "r"']
\end{tikzcd}$$

Hence, for $x \in F$ and $a \in \ell_1(\mathbb{Z})$ we have

$$\langle x, a\rangle = \left\langle \kappa_{\ell_1(\mathbb{Z})}(a), x\right\rangle = \left(\kappa_{\ell_1(\mathbb{Z})}(a)\right)(x) = \kappa_{\ell_1(\mathbb{Z})}(a)|_F(x) = \langle \iota_F(a), x\rangle. \tag{3.2.2}$$

Now we present the definiton of a *concrete predual* of $\ell_1(\mathbb{Z})$. This is a very useful concept since the work on concrete preduals, as will be explained below, gives identical results as the work on the standard preduals. Moreover, concrete preduals are easier to explore, which gives an advantage over the standard definition of preduality.

Definition 3.2.3. Let F be a closed subspace of $\ell_\infty(\mathbb{Z})$ and let ι_F be the map introduced above. We say that F is a *concrete predual* of $\ell_1(\mathbb{Z})$ if the map ι_F is an isomorphism.

In the next lemma I show how we may obtain a concrete predual having predual $E \cong \ell_1(\mathbb{Z})^*$. This Lemma is based on Lemma 2.1 of [7], where the authors also included facts that weak*-topologies included by preduals and associated with them concrete predual are the same. I added more theoretical details to make this proof more convenient to read. The facts concerning weak*-topologies are explained in the Corollary 3.2.5.

Lemma 3.2.4. *Let E be a Banach space and κ_E be the canonical embedding of E into E^{**}. Let also $\theta : \ell_1(\mathbb{Z}) \to E^*$ be an isomorphism and $\theta^* : E^{**} \to \ell_1(\mathbb{Z})^* = \ell_\infty(\mathbb{Z})$ be its adjoint map. Then the map*

$$T := \theta^* \circ \kappa_E : E \longrightarrow \ell_\infty(\mathbb{Z}), \tag{3.2.3}$$

is an isomorphism onto its range $F := T(E)$. Moreover, the map ι_F introduced in Definition 3.2.2 is an isomorphism and so the subspace F is a concrete predual of $\ell_1(\mathbb{Z})$.

Proof. Let $a \in \ell_1(\mathbb{Z})$, $x \in E$ and $\kappa_{\ell_1(\mathbb{Z})} : \ell_1(\mathbb{Z}) \to \ell_\infty(Z)^*$ be the canonical embedding. Then using formulas (3.1.6) and (3.1.8) from the previous section for the adjoint operator $T^* : \ell_\infty(\mathbb{Z})^* \to E^*$ we have

$$\left\langle T^* \circ \kappa_{\ell_1(\mathbb{Z})}(a), x\right\rangle = \left\langle T^*(\kappa_{\ell_1(\mathbb{Z})}(a)), x\right\rangle = \left\langle \kappa_{\ell_1(\mathbb{Z})}(a), Tx\right\rangle = \langle Tx, a\rangle . \tag{3.2.4}$$

Now, by (3.2.3),

$$\langle Tx, a\rangle = \langle \theta^* \circ \kappa_E(x), a\rangle = \langle \theta^*(\kappa_E(x)), a\rangle = \langle \kappa_E(x), \theta(a)\rangle = \langle \theta(a), x\rangle . \tag{3.2.5}$$

Hence, we see that (3.2.4) and (3.2.5) imply

$$\langle Tx, a\rangle = \langle \theta(a), x\rangle , \qquad a \in \ell_1(\mathbb{Z}),\ x \in E \tag{3.2.6}$$

and, in particular,

$$T^* \circ \kappa_{\ell_1(\mathbb{Z})} = \theta. \tag{3.2.7}$$

Now, let us observe the following. Since θ is an isomorphism, for any $e^* \in E^*$ we have $\theta(\theta^{-1}(e^*)) = e^*$ and $\|\theta^{-1}(e^*)\| \le \|\theta^{-1}\|\|e^*\|$. For $a \in \ell_1(\mathbb{Z})$ this gives us

$$\{\theta(a) : \|a\| \le 1\} = \{e^* \in E^* : \|\theta^{-1}(e^*)\| \le 1\} \supseteq \left\{e^* \in E^* : \|e^*\| \le \frac{1}{\|\theta^{-1}\|}\right\}. \tag{3.2.8}$$

Using this observation we show that T is bounded below.

Let $x \in E$, then

$$\|Tx\| = \sup\{| \langle Tx, a \rangle | : a \in \ell_1(\mathbb{Z}), \|a\| \leq 1\},$$

which, by equation (3.2.6), is equivalent to

$$\|Tx\| = \sup\{| \langle \theta(a), x \rangle | : a \in \ell_1(\mathbb{Z}), \|a\| \leq 1\}.$$

Applying (3.2.8) and Corollary 3.1.5, we finally obtain

$$\|Tx\| = \sup\{| \langle \theta(a), x \rangle | : a \in l_1(\mathbb{Z}), \|a\| \leq 1\} \geq \sup\left\{ |\langle e^*, x \rangle| : \|e^*\| \leq \frac{1}{\|\theta^{-1}\|} \right\} = \frac{\|x\|}{\|\theta^{-1}\|}.$$

As a result, we obtain that $\|Tx\| \geq \frac{\|x\|}{\|\theta^{-1}\|}$, which implies that T is bounded below and, thus, is an isomorphism onto its range F. Hence, by (3.2.1) and (3.2.6), for $a \in \ell_1(\mathbb{Z})$ and $x \in E$ we have

$$\langle T^* \iota_F(a), x \rangle = \langle T(x), a \rangle = \langle \theta(a), x \rangle ,$$

which gives $T^* \iota_F = \theta$. Since T^* is an isomorphism we obtain that $\iota_F = (T^*)^{-1}\theta$ must also be an isomorphism, but this, by Definition 3.2.3, yields that F is a concrete predual of $\ell_1(\mathbb{Z})$. □

Now let us turn to the weak*-topologies on $\ell_1(\mathbb{Z})$ induced by predual E and the corresponding concrete predual.

Corollary 3.2.5. *In the situation described in Lemma 3.2.4 the weak*-topologies induced by the pairings* $(\ell_1(\mathbb{Z}) \overset{\theta}{\cong} E^*, E)$ *and* $(\ell_1(\mathbb{Z}), F)$ *agree. In other words for any net* $(a_\alpha) \in \ell_1(\mathbb{Z})$ *we have*

$$(\lim_\alpha \langle \theta(a_\alpha), x \rangle = 0 \textit{ for all } x \in E) \Longleftrightarrow (\lim_\alpha \langle y, a_\alpha \rangle = 0 \textit{ for all } y \in F).$$

Proof. Indeed, let us consider (a_α), a null net in $\ell_1(\mathbb{Z})$ for the $\sigma(\ell_1(\mathbb{Z}), F)$ topology. This means that

$$\lim_\alpha \langle Tx, a_\alpha \rangle = 0, \qquad x \in E.$$

By (3.2.6) this is equivalent to

$$\lim_\alpha \langle \theta(a_\alpha), x \rangle = 0, \qquad x \in E.$$

which is equivalent to the fact that the net $(\theta(a_\alpha))$ is weak*-null for the $\sigma(\ell_1(\mathbb{Z}) \cong E^*, E)$ topology. □

The next lemma shows us when two concrete preduals induce the same weak*-topologies on $\ell_1(\mathbb{Z})$. This lemma is based on Lemma 2.2 of [7].

Lemma 3.2.6. *Let* E_1 *and* E_2 *be two preduals of* $\ell_1(\mathbb{Z})$ *and let* F_1 *and* F_2 *be concrete preduals of* $\ell_1(\mathbb{Z})$ *obtained respectively from* E_1 *and* E_2 *as explained in Lemma 3.2.4. Then the weak*-topologies on* $\ell_1(\mathbb{Z})$ *induced by* E_1 *and* E_2 *agree if and only if* $F_1 = F_2$.

Proof. ($\Longleftarrow$) Let F_1 and F_2 be concrete preuals of $\ell_1(\mathbb{Z})$ associated respectively with preduals E_1 and E_2. If $F_1 = F_2$ then the topologies $\sigma(\ell_1(\mathbb{Z}), F_1)$ and $\sigma(\ell_1(\mathbb{Z}), F_2)$ agree, hence by Corollary 3.2.5 the topologies (E_1^*, E_1) and (E_2^*, E_2) must agree as well.
($\Longrightarrow$) Let E_1 and E_2 be preduals of $\ell_1(\mathbb{Z})$ with its dual spaces E_1^* and E_2^*. Let also $\theta_1 : \ell_1(\mathbb{Z}) \to E_1^*$ and $\theta_2 : \ell_1(\mathbb{Z}) \to E_2^*$ be isomorphisms. (We know that such an isomorphisms exist since E_1 and E_2 are preduals of $\ell_1(\mathbb{Z})$). Let us assume that E_1 and E_2 induce the same weak*-topology on $\ell_1(\mathbb{Z})$, but there exists $x \in \ell_\infty(\mathbb{Z})$ such that $x \in F_1$ and $x \notin F_2$. Then by Corollary 3.1.6 we can find $\Lambda \in \ell_\infty(\mathbb{Z})^*$ such that

$$\langle \Lambda, x\rangle = 1 \qquad \text{and} \qquad \langle \Lambda, y\rangle = 0, \qquad y \in F_2. \tag{3.2.9}$$

Now, by an application of Lemma 3.1.12, there exists a bounded net $(a_\alpha) \subset \ell_1(\mathbb{Z})$ converging weak* to Λ in $\ell_\infty(\mathbb{Z})^* = \ell_1(\mathbb{Z})^{**}$. For such a net

$$\lim_\alpha \langle y, a_\alpha\rangle = \langle \Lambda, y\rangle = 0, \qquad y \in F_2.$$

By Corollary 3.2.5 this is equivalent with the fact that

$$\lim_\alpha \langle \theta_2(a_\alpha), z\rangle = 0, \qquad z \in E_2,$$

which, by assumption, implies that

$$\lim_\alpha \langle \theta_1(a_\alpha), z\rangle = 0, \qquad z \in E_1,$$

Now, again by Corollary, 3.2.5 this is equivalent with

$$\lim_\alpha \langle y, a_\alpha\rangle = \langle \Lambda, y\rangle = 0, \qquad y \in F_1,$$

which can not be true since, by (3.2.9), there exists $x \in F_1$ such that $\langle \Lambda, x\rangle = 1$. This implies that we must have $F_1 \subseteq F_2$. The same proof occurs if we suppose that $x \in F_2 \setminus F_1$ and then $F_2 \subseteq F_1$. Hence, ultimately we obtain $F_1 = F_2$. $\square$

Notice that this lemma shows why we care about exactly what subspace of $\ell_\infty(\mathbb{Z})$ we get - not just the Banach space isomorphism class of the predual. The following example shows how the theory presented in this section works if we consider E as the Banach space of continuous functions $C(K)$, where K is a countable, infinite and compact Hausdorff space. In Lemma 1.2.4 we have shown that $C(K)$ is a predual of $\ell_1(\mathbb{Z})$, but here we present $C(K)$ as a concrete predual F, and then we show that concrete predual obtained that way can not be shift-invariant. This is an easier way to prove so and does not require to refer to the more advanced theory included in [8], as we mentioned at the end of section 1.2.

Example 3.2.7. Take a countable infinite compact Hausdorff space K and enumerate K as $\{x_i : i \in \mathbb{N}\}$. Let us define an isomorphism $\theta : \ell_1(\mathbb{Z}) \longrightarrow C(K)^* = \mathcal{M}(K)$ by assigning to every $(a_n) \in \ell_1(\mathbb{Z})$ to the unique measure $\mu \in \mathcal{M}(K)$ such that $\mu(\{x_n\}) = a_n$, where $\{x_n\}$ is a Borel set. According to the above lemma the map

$$T = \theta^* \circ \kappa_{C(K)} : C(K) \longrightarrow T(C(K)) \subseteq \ell_\infty(\mathbb{Z}),$$

where $\kappa_{C(K)}$ is a canonical embedding and θ^* is adjoint to θ, is an isomorphism from $\ell_1(\mathbb{Z})$ onto $F = T(C(K))$ and so F is a concrete predual of $\ell_1(\mathbb{Z})$. Let us write the map T more explicitly. We claim that

$$(Tf)(n) = f(x_n), \quad f \in C(K),\ n \in \mathbb{Z}. \tag{3.2.10}$$

Indeed, let $f \in C(K)$ and $x_n \in K$

$$(Tf)(n) = (\theta^* \kappa_{C(K)} f)(n) = \left\langle \theta^* \kappa_{C(K)} f, \delta_n \right\rangle = \left\langle \kappa_{C(K)} f, \theta(\delta_n) \right\rangle = \langle \theta(\delta_n), f \rangle = f(n). \tag{3.2.11}$$

Hence, we see that the concrete predual of $\ell_1(\mathbb{Z})$, associated with $C(K)$, has a form

$$F = T(C(K)) = \{Tf : (Tf)(n) = f(x_n), f \in C(K)\} \subseteq \ell_\infty(\mathbb{Z}).$$

We now explain why the concrete predual F cannot be shift-invariant. By contradiction let us assume that F is shift-invariant. Since K is a infinite, countable and compact, by the Baire category theorem there exists at least one isolated point in K. Without loss of generality, assume that $x_0 \in K$ is an isolated point and define a function $f : K \longrightarrow \mathbb{C}$ by setting

$$f(x) = \begin{cases} 1 & \text{if} \quad x = x_0; \\ 0 & \text{if} \quad x \neq x_0. \end{cases}$$

Because x_0 is isolated, the function f is continuous on K and so $Tf = \delta_0$. This implies that $\delta_0 \in F$ and by shift-invariance for all n the element δ_n is in F. Hence, we see that $c_0(\mathbb{Z})$ is included in F. On the other hand, if $c_0(\mathbb{Z}) \subseteq F$, then necessarily $c_0(\mathbb{Z}) = F$, but we know that it never happens. The obtained contradiction proves that $C(K)$ is not a shift-invariant predual of $\ell_1(\mathbb{Z})$.

3.3 Relation between shift-invariant preduals and dual Banach algebras

In this section we reveal the connection between shift-invariant preduals of $\ell_1(\mathbb{Z})$ and a dual Banach algebra $\ell_1(\mathbb{Z})$ mentioned in the introduction. First, we recall the definition of Banach algebra which readers should know from a basic course of functional analysis and then we present the definition of dual Banach algebra. Readers interested more in the theory of dual Banach algebras I refer to [16] and [17]. For information about the foundations of Banach algebras I refer to the very beginning of chapter 10 in [18].

Definition 3.3.1. Let X be an algebra and let $*$ denote a multiplication product on X. If X is at the same time a Banach space with norm $\|\cdot\|$ satisfying for each $x, y \in X$ the following condition

$$\|x * y\| \leq \|x\|\|y\|,$$

then X is called Banach algebra.

Definition 3.3.2. A Banach algebra X is said to be a *dual Banach algebra* if there exists a Banach space Y which is a predual of X and moreover the multiplication operation on X is separately weak*-continuous, with respect to Y.

We now present the definition of the bilateral shift.

Definition 3.3.3. Let $\mathbb{K}^{\mathbb{Z}}$ denotes the vector space of all sequences of scalars indexed by $\mathbb{Z}$, where $\mathbb{K}$ is the field either of real or complex numbers. Then the bilateral shift operator $\sigma : \mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{\mathbb{Z}}$ is defined as follows

$$\sigma : \mathbb{K}^{\mathbb{Z}} \ni x = (..., x_{-2}, x_{-1}, \overbrace{x_0}^{(0)}, x_1, x_2, ...) \longrightarrow \sigma(x) = (..., x_{-2}, \overbrace{x_{-1}}^{(0)}, x_0, x_1, x_2, ...) \in \mathbb{K}^{\mathbb{Z}}. \tag{3.3.1}$$

Remark 3.3.4. We see that the bilateral shift operator defined above has the property of shifting elements $x \in \mathbb{K}^{\mathbb{Z}}$ one position to the right. Also notice that the formula (3.3.1) may also be written as $\sigma(x)(n) = x(n-1)$. In a similar, vein we can define bilateral shift which changes position of any $x \in \mathbb{K}^{\mathbb{Z}}$ one position to the left. This operator is σ^{-1} and the formula $\sigma(x)^{-1}(n) = x(n+1)$ describes this operator explicitly. Note that σ restricts to a map on $\ell_1(\mathbb{Z})$, which we also denote by σ.

Fact 3.3.5. Let σ the bilateral shift on $\ell_1(\mathbb{Z})$ introduced in Definition 3.3.3, then the adjoint operator σ^* is the bilateral shift operator on $\ell_\infty(\mathbb{Z})$ going in the opposite direction.

Proof. Indeed, by Theorem 3.1.7 for the bilateral shift operator $\sigma : \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})$ there exists exactly one adjoint operator $\sigma^* : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ such that

$$\langle \sigma^*(y^*), x \rangle = \langle y^*, \sigma(x) \rangle, \qquad y^* \in \ell_\infty(\mathbb{Z}),\ x \in \ell_1(\mathbb{Z}). \tag{3.3.2}$$

Hence, for

$$y^* = (..., y^*_{-2}, y^*_{-1}, \overbrace{y^*_0}^{(0)}, y^*_1, y^*_2, ...) \in \ell_\infty(\mathbb{Z})$$

$$x = (..., x_{-2}, x_{-1}, \overbrace{x_0}^{(0)}, x_1, x_2, ...) \in \ell_1(\mathbb{Z})$$

$$\sigma(x) = (..., x_{-2}, \overbrace{x_{-1}}^{(0)}, x_0, x_1, x_2, ...) \in \ell_1(\mathbb{Z}),$$

we have

$$\langle y^*, \sigma(x)\rangle = ... + y^*_{-2}x_{-3} + y^*_{-1}x_{-2} + y^*_0 x_{-1} + y^*_1 x_0 + y^*_2 x_1 + ... = \sum_{k\in\mathbb{Z}} y^*_k x_{k-1}.$$

On the other hand,

$$\langle \sigma^*(y^*), x\rangle = ... + \sigma^*(y^*)(-1)x_{-1} + \sigma^*(y^*)(0)x_0 + \sigma^*(y^*)(1)x_1 + ... = \sum_{k\in\mathbb{Z}} \sigma^*(y^*)(k)x_k.$$

Hence, we must have

$$\sum_{k\in\mathbb{Z}} y^*_k x_{k-1} = \sum_{k\in\mathbb{Z}} \sigma^*(y^*)(k)x_k, \quad (x_k) \in \ell_1(\mathbb{Z}),$$

and so $\sigma^*(y^*)(k-1) = y^*_k$, which is equivalent to $\sigma^*(y^*)(k) = y^*_{k+1}$. Notice that analogical proof holds for the bilateral shift operator σ^{-1}(see Remark 3.3.4) and in this case we obtain $(\sigma^{-1})^*(y^*)(k) = y^*_{k-1}$. □

The following two lemmas reveal the connection between shift-invariant preduals of $\ell_1(\mathbb{Z})$ and dual Banach algebra $\ell_1(\mathbb{Z})$. These are based on Lemma 2.3 of [7], but we added here more details for better understanding of the material. In particular, the authors in [7] prove these lemmas by working only with the bilateral shift σ considering this operator as invariant in both directions, left and right. Here we make an additional effort and for purpose of this thesis we distinguish left and shift invariance (See definition below).

Definition 3.3.6. Let σ, σ^{-1} be the bilateral shift operators on $\ell_1(\mathbb{Z})$ described by Remark 3.3.4, and let σ^*, $(\sigma^{-1})^*$ be associated with them the bilateral shift operators on $\ell_\infty(\mathbb{Z})$. (See Fact 3.3.5). We say that a concrete predual F of $\ell_1(\mathbb{Z})$ is *left shift-invariant* if $\sigma^*(F) \subseteq F$. Similary, we say that a concrete predual F of $\ell_1(\mathbb{Z})$ is *right shift-invariant* if $(\sigma^{-1})^*(F) \subseteq F$. A predual F it is said to be *shift-invariant* if is both right and left shift-invariant.

Lemma 3.3.7. *Let $F \subseteq \ell_\infty(\mathbb{Z})$ be a concrete predual of $\ell_1(\mathbb{Z})$ and let σ and σ^{-1} be the bilateral shift operators on $\ell_1(\mathbb{Z})$. Then σ is weak*-continuous, with respect to F if and only if the space F is left shift-invariant. Similary, σ^{-1} is weak*-continuous, with respect to F if and only if the space F is right shift-invariant.*

Proof. ($\Longrightarrow$) Let σ be the bilateral shift operator on $\ell_1(\mathbb{Z})$ and let σ^* be the adjoint operator which, by Fact 3.3.5, is the bilateral shift on $\ell_\infty(\mathbb{Z})$ going in the opposite direction. Suppose that the space F is not left shift-invariant. Hence, there exists $x \in F \setminus \sigma^*(F)$ which implies that $(\sigma^*)^{-1}(x) \notin F$. Now, in according to Corollary 3.1.6, we have the existence of $\Lambda \in \ell_\infty(\mathbb{Z})^*$ such that

$$\left\langle \Lambda, (\sigma^*)^{-1}(x)\right\rangle = 1 \qquad \text{and} \qquad \langle \Lambda, y\rangle = 0, \qquad y \in F.$$

Let $(a_\alpha) \subseteq \ell_1(\mathbb{Z})$ be a net convergent weak* to Λ in $\ell_\infty(\mathbb{Z})^*$. Hence,

$$\lim_\alpha \langle y, a_\alpha \rangle = 0, \qquad y \in F.$$

Therefore, we see that (a_α) is weak*-null for the weak*-topology given by F. This implies that the sequence $(\sigma^{-1}(a_\alpha))$ is also weak*-null for the weak*-topology given by F. Hence, for any $x \in F$ we have

$$0 = \lim_\alpha \left\langle x, \sigma^{-1}(a_\alpha) \right\rangle,$$

which is equivalent with

$$0 = \lim_\alpha \left\langle (\sigma^*)^{-1}(x), a_\alpha \right\rangle,$$

and so

$$0 = \left\langle \Lambda, (\sigma^*)^{-1}(x) \right\rangle = 1.$$

This contradiction gives $\sigma^*(F) \subseteq F$ and so F is left shift-invariant. A similar proof holds when we will try to show that $(\sigma^{-1})^*(F) \subseteq F$, whence σ^{-1} is weak*-continuous, with respect to F then the space F is right shift-invariant.

$(\Longleftarrow)$ Let (a_α) be a weak*-null net in $\ell_1(\mathbb{Z})$, σ the bilateral shift on $\ell_1(\mathbb{Z})$ and σ^* the adjoint operator of σ. Let $x \in F$, then by assumption

$$\lim_\alpha \langle \sigma^*(x), a_\alpha \rangle = 0,$$

which implies

$$\lim_\alpha \langle x, \sigma(a_\alpha) \rangle = 0.$$

Hence, we see that $(\sigma(a_\alpha))$ is weak*-null, which automatically gives the weak*-continuity of the bilateral shift operator σ with respect to F. Similary we can prove that, if F is right shift-invariant, then σ^{-1} is weak*-continuous, with respect to F. □

Lemma 3.3.8. *Let $F \subseteq \ell_\infty(\mathbb{Z})$ be a concrete predual of $\ell_1(\mathbb{Z})$. Then F is shift-invariant if and only if $\ell_1(\mathbb{Z})$ is a dual Banach algebra with respect to F.*

Proof. $(\Longleftarrow)$ Let $(f_i) \subset \ell_1(\mathbb{Z})$ be a sequence which converges weak* to $f \in \ell_1(\mathbb{Z})$, with respect to F and let also $\delta_1 \in \ell_1(\mathbb{Z})$ be such that $\delta_1 = (...0, 0, \overbrace{1}^{1}, 0, 0, ..)$. Then by assumption for any $x \in F$ we have

$$\langle f_i * \delta_1, x \rangle \longrightarrow \langle f * \delta_1, x \rangle, \quad (i \to \infty). \tag{3.3.3}$$

Now, notice that for any $g \in \ell_1(\mathbb{Z})$ and $n \in \mathbb{Z}$ we have

$$(g * \delta_1)(n) = \sum_{k \in \mathbb{Z}} g(k)\delta_1(n-k) = g(n-1) = \sigma(g)(n), \tag{3.3.4}$$

where σ is the bilateral shift on $\ell_1(\mathbb{Z})$. Hence, σ is the operator of convolution by δ_1, so by assumption in (3.3.3) we obtain

$$\langle \sigma f_i, x\rangle = \langle f_i * \delta_1, x\rangle \longrightarrow \langle f * \delta_1, x\rangle = \langle \sigma(f), x\rangle \quad (i \longrightarrow \infty). \tag{3.3.5}$$

Now, we see that σ is weak*-continuous what by the previous lemma implies that F is left shift-invariant. Notice that the analogous calculation applied to δ_{-1} in (3.3.4) implies that F is right shift-invariant. Therefore the predual F is both, left and right shift-invariant and so by Definition 3.3.6 is shift-invariant.

($\Longrightarrow$) Let $(f_i) \subset \ell_1(\mathbb{Z})$, $f \in \ell_1(\mathbb{Z})$ and suppose that for any $x \in F$ we have $\langle f_i, x\rangle \longrightarrow \langle f, x\rangle$ as $i \to \infty$. Now fix $g \in \ell_1(\mathbb{Z})$. We want to show that $\langle f_i * g, x\rangle \longrightarrow \langle f * g, x\rangle$ as $i \to \infty$. First, notice that g can presented as $g = \sum_{n\in\mathbb{Z}} \alpha_n \delta_n$, where $\delta_n = (..., 0, 0, \overbrace{1}^{n}, 0, 0, ...)$ for some sequence of scalars $(\alpha_n)_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} |\alpha_n| < \infty$. Moreover, since the convolution product on $\ell_1(\mathbb{Z})$ is commutative, by applying δ_n in (3.3.4) in the previous lemma we obtain

$$f * g = g * f = \sum_{n\in\mathbb{Z}} \alpha_n \delta_n * f = \sum_{n\in\mathbb{Z}} \alpha_n \sigma^n(f), \tag{3.3.6}$$

where σ is the bilateral-shift operator on $\ell_1(\mathbb{Z})$. Notice, that since there exists a constant $M > 0$ such that for each $i \in \mathbb{N}$ we have $\|\sigma(f_i)\| < M$ the expressions $\sum_{n\in\mathbb{Z}} \alpha_n \langle \sigma^n(f_i), x\rangle$ are uniformly bouned. Thus, since F is shift invariant, by the previous lemma and by the Dominate Convergence Theorem we obtain, as $i \to \infty$ that

$$\langle f_i * g, x\rangle = \sum_{n\in\mathbb{Z}} \alpha_n \langle \sigma^n(f_i), x\rangle \longrightarrow \sum_{n\in\mathbb{Z}} \alpha_n \langle \sigma^n(f), x\rangle = \langle f * g, x\rangle, \qquad x \in F. \tag{3.3.7}$$

Hence, the convolution product on $\ell_1(\mathbb{Z})$ is separately weak*-continuous what implies that $\ell_1(\mathbb{Z})$ is a dual Banach algebra. $\square$

Chapter 4

The Predual $F^{(\lambda)}$

This chapter of my thesis focuses on the definition and basic properties of the space $F^{(\lambda)}$. I follow the construction of this space from chapter 3 of [7]. The space $F^{(\lambda)}$ constructed in that paper proved to be a shift-invariant predual of $\ell_1(\mathbb{Z})$. In my work I add some new elements to that paper, such as a space $F_+^{(\lambda)}$ and examine some properties of this space. In this chapter we also use the theory of Čech-Stone compactification introduced in chapter 2 to show that $F^{(\lambda)}$ is a $G - space$ - a fact needed in [1] to obtain non-constructive isomorphism between $F^{(\lambda)}$ and $c_0(\mathbb{Z})$.

4.1 Necessary instruments for defining the space $F^{(\lambda)}$

In this section I introduce some notation and make some provisions which will be used throughout this chapter and then in the rest of this thesis. We start with the following two definitions, which can be found at the very beginning of chapter 3 in [7].

Definition 4.1.1. Let $n \in \mathbb{Z}$, then the function b defined on $\mathbb{Z}$ takes the following values

$$b(n) := \begin{cases} k & \text{if} \quad n > 0; \\ 0 & \text{if} \quad n = 0; \\ -\infty & \text{if} \quad n < 0. \end{cases} \tag{4.1.1}$$

where k denotes the number of ones in the binary expansion of n.

Having defined the value $b(n)$, we may now introduce an important for us an element $x_0 \in \ell_\infty(\mathbb{Z})$.

Definition 4.1.2. Let λ denotes a complex scalar with absolute value $|\lambda| > 1$. Then, the element $x_0 \in \ell_\infty(\mathbb{Z})$ is defined by the formula $x_0(n) = \lambda^{-b(n)}$, where $n \in \mathbb{Z}$ and $b(n)$ is defined above. Writing more explicitly, we obtain

$$x_0(n) = \begin{cases} \lambda^{-b(n)} & \text{if} \quad n > 0; \\ 1 & \text{if} \quad n = 0; \\ 0 & \text{if} \quad n < 0. \end{cases} \tag{4.1.2}$$

Hence,

$$x_0 = (..., 0, 0, \overbrace{1}^{0}, \lambda^{-1}, \lambda^{-1}, \lambda^{-2}, \lambda^{-1}, \lambda^{-2}, \lambda^{-2}, \lambda^{-3}, \lambda^{-1}, ...).$$

The definition of the element is crucial for us since as it will be clear in the next section we use this element to define the space $F^{(\lambda)}$. Now, we introduce a bounded linear operator acting between $\ell_\infty(\mathbb{Z})$ space. This operator and the bilateral shift operator by Definition 3.3.3 satisfy interesting equation given in Lemma 4.1.5. The material which will be presented here is taken form the beginning of chapter 3 of [7].

Definition 4.1.3. A bounded linear operator $\tau : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ is defined by the formula

$$\tau(x)(n) = \begin{cases} x(n/2) & n \text{ even}; \\ 0 & n \text{ odd}, \end{cases} \tag{4.1.3}$$

where $n \in \mathbb{Z}$ and $x \in \ell_\infty(\mathbb{Z})$.

It is easy to observe that the operator τ has the property of spreading out elements of $x \in \ell_\infty(\mathbb{Z})$. For example, application of this operator to the element x_0 gives:

$$x_0 = (..., 0, 0, \overbrace{1}^{0}, 0, \lambda^{-1}, 0, \lambda^{-1}, 0, \lambda^{-2}, 0, \lambda^{-1}, 0, \lambda^{-2}, 0, \lambda^{-2}, 0, \lambda^{-3}, 0, \lambda^{-1}, 0, ...).$$

Recall also that the bilateral shift operator σ, introduced in Definition 3.3.3, which has the formula

$$\sigma(x)(n) = x(n-1), \tag{4.1.4}$$

where $x \in \ell_\infty(\mathbb{Z})$ and $n \in \mathbb{Z}$. The next short lemma shows a relation between the operators τ and σ.

Lemma 4.1.4. *For the operators σ and τ defined above the following equation holds*

$$\tau\sigma = \sigma^2\tau. \tag{4.1.5}$$

Proof. Let $n \in \mathbb{Z}$ be even. Starting from the lefthand side for any element $x \in \ell_\infty(\mathbb{Z})$, applying formulas (4.1.4) and (4.1.3) we obtain

$$\tau\sigma(x) = \sigma(x)(n/2) = x(n/2-1) = \tau(x)(n-2) = \sigma^2\tau(x)(n).$$

In the case when n is odd it is easy to see that both sides are equal to zero. Hence, our equation is satisfied for all $n \in \mathbb{Z}$. □

Using this easy result, we now prove a more complicated formula, which is based on Lemma 3.1 in [7]. We add here more theoretical details to make this proof more convenient for understanding. This lemma will be very valuable in proving the fact that $\tau(x_0)$ is the element of $F_+^{(\lambda)}$ (see Lemma 4.2.5).

Lemma 4.1.5. *Let $id : \ell_\infty(\mathbb{Z}) \longrightarrow \ell_\infty(\mathbb{Z})$ be the identity operator. Then for the element x_0, scalar λ and operators σ, τ introduced above, the following equation holds:*

$$(id - \lambda^{-1}\sigma)(x_0)(n) = (\lambda - 1)\sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0)(n) \qquad n \in \mathbb{Z}. \tag{4.1.6}$$

Proof. To obtain the above equation for any $n \in \mathbb{Z}$ we subdivide its proof into three cases, namely: a) $n < 0$, b) $n = 0$ and c) $n > 0$.

a) If $n < 0$, then on the left-hand side of the equation (4.1.6) we have

$$(id - \lambda^{-1}\sigma)(x_0)(n) = x_0(n) - \lambda^{-1}\sigma(x_0)(n) = x_0(n) - \lambda^{-1}x_0(n-1) = 0 - 0 = 0.$$

On the right-hand side of (4.1.6), since $\tau^j(x_0)(n) = 0$ for any $j \in \mathbb{N}$ we obtain

$$(\lambda - 1)\sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0)(n) = \sum_{j=1}^{\infty} \lambda^{-j+1}\tau^j(x_0)(n) - \sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0)(n) = 0 - 0 = 0.$$

Therefore, we see that the equation (4.1.6) holds for $n < 0$.

b) Very similarly to the previous case we prove the case with $n = 0$. Here, on the lefthand side of (4.1.6) we get

$$(id - \lambda^{-1}\sigma)(x_0)(0) = x_0(0) - \lambda^{-1}\sigma(x_0)(0) = x_0(0) - \lambda^{-1}x_0(-1) = 1 - 0 = 1.$$

Since for all $j \in \mathbb{N}$ we have $\tau^j(x_0)(0) = 1$ we see that

$$(\lambda - 1)\sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0)(n) = \sum_{j=1}^{\infty} \lambda^{-j+1}\tau^j(x_0)(n) - \sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0)(n) = \sum_{j=2}^{\infty} \lambda^{-j} - \sum_{j=1}^{\infty} \lambda^{-j} = 1.$$

And so the equation (4.1.6) also holds for $n = 1$.

c) The last case is a bit more complicated. Let us first write a given natural number $n > 0$ in the binary expansion form. Therefore, there exists a natural number l such that

$$n = \epsilon_0 2^0 + \epsilon_1 2^1 + ... + \epsilon_j 2^j + ... + \epsilon_l 2^l, \qquad \text{where } \epsilon_j \in \{0, 1\} \text{ and } j \in \{0, 1, ..., l\}.$$

Now, let k be the smallest number from the set $\{0, 1, ..., l\}$ such that $\epsilon_k = 1$. Since $2^k - 1 =$

$\sum_{j=0}^{k-1} 2^j$, we obtain that the binary expansion of $n-1$ is

$$n - 1 = 2^0 + 2^1 + ... + 2^{k-1} + \epsilon_{k+1}2^{k+1} + ... + \epsilon_l 2^l.$$

In other words,

$$n - 1 = \sum_{j=0}^{k-1} 2^j + \sum_{j=k+1}^{l} \epsilon_j 2^j.$$

Which immediately implies that

$$b(n-1) = b(n) - 1 + k. \tag{4.1.7}$$

Now let us make the following observation. For a natural number n, the number k mentioned above can be also viewed as the largest one such that $2^k|n$. This fact yields that $2^j|n$ for any natural number $j \leq k$. As an easy consequence of this fact, we obtain that $b(n) = b(\frac{n}{2^k})$ and so

$$\tau^j(x_0)(n) = \begin{cases} x_0(n) & \text{if} \quad j \leq k; \\ 0 & \text{if} \quad j > k. \end{cases} \tag{4.1.8}$$

Having all this information presented we check the equation (4.1.6) starting from the left-hand side. Firstly using the equation (4.1.7) we obtain:

$$(id - \lambda^{-1}\sigma)(x_0)(n) = x_0(n) - \lambda^{-1}\sigma(x_0)(n) = \lambda^{-b(n)} - \lambda^{-b(n)-k} = (1 - \lambda^{-k})x_0(n).$$

For the reason that $(1 - \lambda^{-k}) = (\lambda - 1)\sum_{j=1}^{k} \lambda^{-j}$ we have that

$$(id - \lambda^{-1}\sigma)(x_0)(n) = (\lambda - 1)\sum_{j=1}^{k} \lambda^{-j} x_0(n).$$

Finally, applying (4.1.8), we receive

$$(1 - \lambda^{-k})(x_0)(n) = (\lambda - 1)\sum_{j=1}^{k} \lambda^{-j}\tau^j(x_0)(n) = (\lambda - 1)\sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0)(n).$$

Hence, we see that the equation (4.1.6) is satisfied for $n > 0$. □

4.2 The Space $F^{(\lambda)}$

Now we introduce the main object of our study, namely, the space $F^{(\lambda)}$. At this point, we will need to recall the definition of *linear span* introduced, for example, during the course of linear algebra. We also mention here an important concept of *shift-invariant* space.

Definition 4.2.1. For a *vector space* X and subset $A \subseteq X$ the space $span(A)$ is defined as the set of all finite *linear combinations* of the elements of A. Hence,

$$span(A) := \left\{ \sum_{i=1}^{k} \lambda_i v_i \mid k \geq 1, v_i \in A, \lambda_i \in \mathbb{K} \right\}.$$

Definition 4.2.2. Let σ be the bilateral shift operator introduced in Definition 3.3.3. A linear subspace A of a vector space $\ell_\infty(\mathbb{Z})$ is said to be *shift-invariant* if for each element $a \in A$, the element $\sigma^n(a) \in A$, for any $n \in \mathbb{Z}$.

Now we present the definitions of the space $F^{(\lambda)}$ and $F_+^{(\lambda)}$.

Definition 4.2.3. Let us define the sets $A := \{\sigma^n(x_0) : n \in \mathbb{Z}\}$ and $A_+ := \{\sigma^n(x_0) : n \geq 1\}$, then the spaces $F^{(\lambda)}$ and $F_+^{(\lambda)}$ are defined as the closure of $span(A)$ and $span(A_+)$, respectively, with the norm inherited from $\ell_\infty(\mathbb{Z})$. Hence,

$$F^{(\lambda)} := \overline{span(A)} = \overline{span\, \{\sigma^n(x_0) : n \in \mathbb{Z}\}}, \tag{4.2.1}$$

$$F_+^{(\lambda)} := \overline{span(A_+)} = \overline{span\, \{\sigma^n(x_0) : n \geq 0\}}. \tag{4.2.2}$$

Fact 4.2.4. From the definition of a subspace, spaces $F^{(\lambda)}$ and $F_+^{(\lambda)}$ are subspaces of $\ell_\infty(\mathbb{Z})$. Also, since $F_+^{(\lambda)} \subset F^{(\lambda)}$, we obtain that $F_+^{(\lambda)}$ is a subspace of $F^{(\lambda)}$. The reader can easily see from the fact that the operator σ is bounded and linear, both $F^{(\lambda)}$ and $F_+^{(\lambda)}$ are shift-invariant.

The next two lemmas show from what elements we can expect to be in $F_+^{(\lambda)}$ and so in $F^{(\lambda)}$ space. These are based on the second part of Lamma 3.1 in [7], where was shown that $\tau(x_0) \in F^{(\lambda)}$. We change the proof of this lemma a little showing, in the first Lemma that $\tau(x_0)$ is in $F_+^{(\lambda)}$ and, in the second Lemma, that $\tau^k(x_0)$ is in $F_+^{(\lambda)}$, for any $k \geq 1$.

Lemma 4.2.5. *With the notation introduced above,* $\tau(x_0) \in F_+^{(\lambda)}$.

Proof. To prove that our statement is true let σ and τ be the operators defined by the formulas (4.1.4) and (4.1.3) respectively. Let also id be the identity operator on $\ell_\infty(\mathbb{Z})$ and let $\lambda \in \mathbb{C}$ be such that $|\lambda| > 1$. Multiplying $(id - \lambda^{-1}\tau)$ by $(id - \lambda^{-1}\sigma)$ we receive

$$(id - \lambda^{-1}\tau)(id - \lambda^{-1}\sigma)(x_0) = (id - \lambda^{-1}\sigma)(x_0) - \lambda^{-1}\tau(id - \lambda^{-1}\sigma)(x_0). \tag{4.2.3}$$

By the result of Lemma 4.1.5 we obtain

$$(id - \lambda^{-1}\sigma)(x_0) - \lambda^{-1}\tau(id - \lambda^{-1}\sigma)(x_0) = (\lambda - 1)\sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0) - \lambda^{-1}\tau(\lambda - 1)\sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0).$$

But this implies

$$(id - \lambda^{-1}\tau)(id - \lambda^{-1}\sigma)(x_0) = (\lambda - 1)(\sum_{j=1}^{\infty} \lambda^{-j}\tau^j(x_0) - \sum_{j=2}^{\infty} \lambda^{-j}\tau^j(x_0)).$$

Hence, we obtain that

$$(id - \lambda^{-1}\tau)(id - \lambda^{-1}\sigma)(x_0) = \frac{\lambda - 1}{\lambda}\tau(x_0).$$

Now we solve this equation for $(id - \lambda^{-1}\sigma)(x_0)$. We see that by (4.2.3)

$$(id - \lambda^{-1}\sigma)(x_0) = \frac{\lambda - 1}{\lambda}\tau(x_0) + \lambda^{-1}\tau(id - \lambda^{-1}\sigma)(x_0). \tag{4.2.4}$$

Since

$$\frac{\lambda - 1}{\lambda}\tau(x_0) + \lambda^{-1}\tau(id - \lambda^{-1}\sigma)(x_0) = \frac{\lambda - 1}{\lambda}\tau(x_0) + \lambda^{-1}\tau(x_0) - \lambda^{-2}\tau\sigma(x_0) = \tau(x_0) - \lambda^{-2}\tau\sigma(x_0),$$

by the result of Lemma 4.1.4, we ultimately obtain

$$(id - \lambda^{-1}\sigma)(x_0) = (id - \lambda^{-2}\sigma^2)\tau(x_0). \tag{4.2.5}$$

My intention is to find the formula for $\tau(x_0)$ from the equation (4.2.5). Notice, that this will be possible if the operator $(id - \lambda^{-2}\sigma^2)$ will be invertible. The well known fact from the theory of operators, which can be found for example in Theorem 10.7 of [18], explains that this is the case if $\left\|\lambda^{-2}\sigma^2\right\| < 1$, which is true since $\|\sigma\| = 1$ and $|\lambda| > 1$. Also, from the same theorem, we get $(id - \lambda^{-2}\sigma^2)^{-1} = \sum_{j=0}^{\infty} \lambda^{-2j}\sigma^{2j}$. This immediately implies that

$$\tau(x_0) = (id - \lambda^{-2}\sigma^2)^{-1}(id - \lambda^{-1}\sigma)(x_0) = \sum_{j=0}^{\infty} \lambda^{-2j}\sigma^{2j}(id - \lambda^{-1}\sigma)(x_0). \tag{4.2.6}$$

Since the space $F_+^{(\lambda)}$ is shift-invariant for the shift operator σ the element $(id - \lambda^{-1}\sigma)(x_0)$ is in $F_+^{(\lambda)}$. Moreover, since by definition $F_+^{(\lambda)}$ is closed, we obtain that the series in (4.2.6) is convergent in $F_+^{(\lambda)}$, which implies that $\tau(x_0) \in F_+^{(\lambda)}$. □

As an easy consequence of this lemma we get the following important result.

Lemma 4.2.6. *With the notation introduced above, the element $\tau^k(x_0) \in F_+^{(\lambda)}$, for any $k \geq 1$.*

Proof. We conduct the proof of this lemma by induction. The case with $k = 1$ is the result of the last lemma, therefore, let us assume that for $k \geq 2$ the element $\tau^{k-1}(x_0)$ is in $F_+^{(\lambda)}$. I show that $\tau^k(x_0)$ is also in $F_+^{(\lambda)}$. By the equation (4.2.6) we obtain

$$\tau^k(x_0) = \tau^{k-1}\tau(x_0) = \tau^{k-1}\left(\sum_{j=0}^{\infty} \lambda^{-2j}\sigma^{2j}(id - \lambda^{-1}\sigma)(x_0)\right),$$

which implies

$$\tau^k(x_0) = \sum_{j=0}^{\infty} \left(\lambda^{-2j}\tau^{k-1}\sigma^{2j}(x_0) - \lambda^{-2j-1}\tau^{k-1}\sigma^{2j+1}(x_0)\right). \tag{4.2.7}$$

Now notice the following. The formula proved in Lemma 4.1.4 implies that for any $k, l \geq 1$ we have $\tau^k \sigma^l = \sigma^{2^k l} \tau^k$. Indeed, let $k = 1$ and $l \geq 1$, then by Lemma 4.1.4 we have

$$\tau \circ \sigma^l = (\tau \circ \sigma) \circ \overbrace{\sigma \circ \ldots \circ \sigma}^{l-1} = \sigma^2 \circ (\tau \circ \sigma) \circ \overbrace{\sigma \circ \ldots \circ \sigma}^{l-2} = \sigma^{2+2} \circ (\tau \circ \sigma) \circ \overbrace{\sigma \circ \ldots \circ \sigma}^{l-3} = \ldots = \sigma^{2(l-1)} \circ (\tau \circ \sigma),$$

which implies that

$$\tau \circ \sigma^l = \sigma^{2l} \circ \tau. \tag{4.2.8}$$

Using formula (4.2.8), let $k \geq 1$ and $l \geq 1$, then

$$\tau^k \sigma^l = \overbrace{\tau \circ \ldots \circ \tau}^{k-1} \circ \tau \circ \sigma^l = \overbrace{\tau \circ \ldots \circ \tau}^{k-1} \circ \sigma^{2l} \circ \tau = \overbrace{\tau \circ \ldots \circ \tau}^{k-2} \circ \sigma^{2 \cdot 2l} \circ \tau^2 = \ldots = \tau \circ \sigma^{2^{k-1} l} \circ \tau^{k-1},$$

which ultimately implies that

$$\tau^k \sigma^l = \sigma^{2^k l} \tau^k. \tag{4.2.9}$$

Now applying (4.2.9) for (4.2.7), we obtain

$$\tau^k(x_0) = \sum_{j=0}^{\infty} \left(\lambda^{-2j} \sigma^{2^{k-1} 2j} \tau^{k-1}(x_0) - \lambda^{-2j-1} \sigma^{2^{k-1}(2j+1)} \tau^{k-1}(x_0) \right).$$

By the assumption the element $\tau^{k-1}(x_0)$ belongs to $F_+^{(\lambda)}$, also since $F_+^{(\lambda)}$ is shift-invariant the elements $\sigma^{2^{k-1} 2j} \tau^{k-1}(x_0)$ and $\sigma^{2^{k-1}(2j+1)} \tau^{k-1}(x_0)$ are in $F_+^{(\lambda)}$. Finally, since $F_+^{(\lambda)}$ is closed we obtain required result. □

Remark 4.2.7. Note now that, since the operator τ is bounded and linear, we obtain $\tau(F_+^{(\lambda)}) \subseteq F_+^{(\lambda)}$.

4.3 $F^{(\lambda)}$ as a predual of $\ell_1(\mathbb{Z})$

In chapter 2 we introduced the concept of Čech-Stone compactification, mentioning the usefulness of this theory in examining of preduality of $\ell_1(\mathbb{Z})$ space. Now we utilise this theory to show that the space $F^{(\lambda)}$ from Definition 4.2.3 is a predual of $\ell_1(\mathbb{Z})$. In other words, we show that the map $\iota_{F^{(\lambda)}} : \ell_1(\mathbb{Z}) \longrightarrow (F^{(\lambda)})^*$, introduced in Definition 3.2.2, is an isomorphism. Just as in the last two sections, this part of my thesis is based on chapter 3 of [7]. Throughout this section we use the notation concerning the theory of Čech-Stone compactification from chapter 2.

First, we recall the definition of spaces $F^{(\lambda)}$ and $F^{(\lambda)}$ and then we apply this definitions to the context of ι_F map introduced in Definition 3.2.2. Spaces $F^{(\lambda)}$ and $F^{(\lambda)}$ are defined as

$$F^{(\lambda)} = \overline{span(A)} = \overline{span\,\{\sigma^n(x_0) : n \in \mathbb{Z}\}},$$

$$F_+^{(\lambda)} = \overline{span(A_+)} = \overline{span\,\{\sigma^n(x_0) : n \geq 0\}}.$$

Hence, the map ι_F has a form

$$\iota_{F^{(\lambda)}} = r \circ \kappa_{\ell_1(\mathbb{Z})} : \ell_1(\mathbb{Z}) \longrightarrow (F^{(\lambda)})^*,$$

for the space $F^{(\lambda)}$, and

$$\iota_{F_+^{(\lambda)}} = r \circ \kappa_{\ell_1(\mathbb{Z})} : \ell_1(\mathbb{Z}) \longrightarrow (F_+^{(\lambda)})^*,$$

for the space $F_+^{(\lambda)}$, where r is the restriction map from $\ell_\infty(\mathbb{Z})^*$ to $(F^{(\lambda)})^*$ and $(F_+^{(\lambda)})^*$, respectively, and $\kappa_{\ell_1(\mathbb{Z})} : \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})^{**} = \ell_\infty(\mathbb{Z})^*$ is the canonical embedding.

Now we show that the map $\iota_{F^{(\lambda)}}$ is injective (see Lemma 3.2 in [7]).

Lemma 4.3.1. *Let $F^{(\lambda)}$ and $\iota_{F^{(\lambda)}}$ be like above, then the map $\iota_{F^{(\lambda)}}$ is injective.*

Proof. Let τ be the operator introduced in Definition 4.1.3 and let $a = (a_n) \in \ell_1(\mathbb{Z})$. According to the definition the map $\iota_{F^{(\lambda)}}$ will be injective if $\iota_{F^{(\lambda)}}(a) = 0$ implies $a = 0$.
First, fix $k \in \mathbb{N}$ and notice that

$$\tau^k(x_0)(n) = \begin{cases} 0 & \text{if} \quad |n| < 2^k \text{ and } n \neq 0, \\ 1 & \text{if} \quad n = 0, \end{cases} \tag{4.3.1}$$

where x_0 is the element defined by the formula (4.1.2). This implies that

$$\lim_{k\to\infty} \tau^k(x_0)(n) = \begin{cases} 0 & \text{if} \quad n \neq 0, \\ 1 & \text{if} \quad n = 0. \end{cases} \tag{4.3.2}$$

Hence, by Lemma 4.2.6, since $\tau^k(x_0)$ and $a \in \ell_1(\mathbb{Z})$ are in duality we have

$$\lim_{k\to\infty} \left\langle \tau^k(x_0), a \right\rangle = \lim_{k\to\infty} \sum_{n=-\infty}^{n=+\infty} \tau^k(x_0)(n)a(n) = \begin{cases} 0 & \text{if} \quad n \neq 0, \\ a_0 & \text{if} \quad n = 0. \end{cases} \tag{4.3.3}$$

which implies that if $\lim_{k\to\infty} \left\langle \iota_{F^{(\lambda)}}(a), \tau^k(x_0) \right\rangle = 0$ then $a_0 = 0$.
Similarly, by shift-invariance of $F^{(\lambda)}$, for any $l \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \sigma^l(\tau^k)(x_0)(n) = \begin{cases} 0 & \text{if} \quad n \neq l, \\ 1 & \text{if} \quad n = l. \end{cases} \tag{4.3.4}$$

where σ is the bilateral shift operator defined by the formula (4.1.4). Hence

$$\lim_{k\to\infty} \left\langle \sigma^l(\tau^k)(x_0), a \right\rangle = \lim_{k\to\infty} \sum_{n=-\infty}^{n=+\infty} \sigma^l \tau^k x_0(n)a(n) = \begin{cases} 0 & \text{if} \quad n \neq l, \\ a_l & \text{if} \quad n = l. \end{cases} \tag{4.3.5}$$

Therefore, if $\lim_{k\to\infty} \left\langle \iota_{F^{(\lambda)}}(a), \sigma^l(\tau^k)(x_0) \right\rangle = 0$, then $a_l = 0$ for any $l \in \mathbb{N} \cup 0$. Hence, we see that $\iota_{F^{(\lambda)}}(a) = 0$ implies $a = 0$. $\square$

Let us turn our attention towards surjectivity of $\iota_{F^{(\lambda)}}$. Before we do that, we add some informations about ultrafilters. The next definition and following the definition remark can be found before Lemma 3.3 in [7].

Definition 4.3.2. Let $t \in \mathbb{Z}$ and $k \geq 1$. We define subsets $X_t^{(k)}$ and $X^{(\infty)}$ of the space of non-principal ultrafilters $\mathbb{Z}^*$ as follows

$$X_t^{(k)} := \{\mathcal{U} \in \mathbb{Z}^* : \forall_{m>0} \{2^{n_1} + 2^{n_2} + ... + 2^{n_k} + t : m < n_1 < n_2 < ... < n_k\} \in \mathcal{U}\}, \tag{4.3.6}$$

$$X^{(\infty)} := \mathbb{Z}^* \setminus \bigcup_{t \in \mathbb{Z}, k \geq 1} X_t^{(k)}. \tag{4.3.7}$$

Remark 4.3.3. Let us notice that if $\mathcal{U}$ is a *non − principal* ultrafilter containing the set $\{2^n + t : n > 0\}$, then $\mathcal{U}$ must be in $X_t^{(1)}$. To see it we suppose by contradiction that for some $m > 0$ the set $\{2^n + t : n > m\} \notin \mathcal{U}$ then we get that $\mathbb{Z} \setminus \{2^n + t : n > m\} \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. But the set

$$(\{2^n + t : n > 0\}) \cap (\mathbb{Z} \setminus \{2^n + t : n > m\})$$

is finite and must lie in $\mathcal{U}$. As the ultrafilter $\mathcal{U}$ is *non − principal* this can not be true.

The next lemma shows a relation between the sets $X_t^{(k)}$ defined above (See Lemma 3.3 in [7]).

Lemma 4.3.4. *Let $s, t \in \mathbb{Z}$ and let $k, l \geq 1$. In addition, let $s \neq t$ or $k \neq l$. Then $X_s^{(k)} \cap X_t^{(l)} = \emptyset$.*

Proof. Let $s, t \in \mathbb{Z}$ and $k, l \geq 1$. We prove the contradictive statement which says that

$$(X_s^{(k)} \cap X_t^{(l)} \neq \emptyset) \Longrightarrow (k = l \wedge s = t).$$

If $X_s^{(k)} \cap X_t^{(l)} \neq \emptyset$, then there exist a non-principal ultrafilter $\mathcal{U} \in \mathbb{Z}^*$ such that $\mathcal{U} \in X_s^{(k)}$ and $\mathcal{U} \in X_t^{(l)}$. This implies that for all natural numbers $n, m > 0$ the intersection

$$\{2^{n_1} + ... + 2^{n_k} + s : n < n_1 < ... < n_k\} \cap \{2^{m_1} + ... + 2^{m_l} + t : m < m_1 < ... < m_l\}$$

belongs to $\mathcal{U}$. In particular, we can choose $n = m$ such that $2^n > |s - t|$. Now, suppose that for such n and natural numbers $n_1, ..., n_k, m_1, ..., m_l$ such that $n < n_1 < ... < n_k, m < m_1 < ... < m_l$, we have

$$2^{n_1} + ... + 2^{n_k} + (s - t) = 2^{m_1} + ... + 2^{m_l}.$$

Since $\sum_{j=1}^{k-1} 2^{m_j} \geq 2^{m_1} > 2^m$ we get

$$2^{m_l} = \sum_{i=1}^{k} 2^{n_i} + (s - t) - \sum_{j=1}^{k-1} 2^{m_j} < 2^{n_k+1} + |s - t| - 2^m < 2^{n_k+1},$$

which gives $m_l < n_k + 1$ and so $m_l \leq n_k$. By symmetry, we also obtain that $2^{n_k} < 2^{m_l+1}$, which yields $n_k \leq m_l$ and so $n_k = m_l$. The reader can now see that the same process can be made for numbers $2^{n_1} + ... + 2^{n_{k-1}} + s$ and $2^{m_1} + ... + 2^{m_{l-1}} + t$ to obtain that $n_{k-1} = m_{l-1}$. Consequently, we may proceed in this way to get $n_1 = m_1$. Therefore, we must have $k = l$ and so $n_i = m_i$ for all $i \leq k$. As a result, we also obtain $s = t$. □

Fact 4.3.5. As an easy consequence of this lemma and Definition 4.3.2 we see that the set of non-principal ultrafilters $\mathbb{Z}^*$ is a disjoint union of $X^{(\infty)}$ and the sets $X_t^{(k)}$.

Remark 4.3.6. In section 3 of chapter 2 we showed that the topological space $\beta\mathbb{Z}$ is the Čech-Stone compactification of $\mathbb{Z}$. In the proof of Theorem 2.4.7 we explained why every bounded function $f : \mathbb{Z} \to \mathbb{R}$ can be uniquely extended to a continuous function $\widetilde{f} : \beta\mathbb{Z} \to \mathbb{R}$. Let us define the following map I

$$I : \ell_\infty(\mathbb{Z}) \ni f \longrightarrow \widetilde{f} \in C(\beta\mathbb{Z}), \tag{4.3.8}$$

where $\widetilde{f}$ is the unique extension of f such that $\widetilde{f}(\mathcal{U}) = \lim_{n\to\mathcal{U}} f(n)$. Since $\|I(f)\| = \|f\|$ we see that I is a isometry between $\ell_\infty(\mathbb{Z})$ and $C(\beta\mathbb{Z})$ and so I is injective. Now, let $\widetilde{f} \in C(\beta\mathbb{Z})$. Taking $f = \widetilde{f}_{|(\beta\mathbb{Z}\setminus\mathbb{Z}^*)}$, we obtain $I(f) = \widetilde{f}$, hence I must be also surjective. Therefore, I is an isometric isomorphism between $\ell_\infty(\mathbb{Z})$ and $C(\beta\mathbb{Z})$ spaces and so we can consider these spaces as the same.

The following theorem proves surjectivity of the map $\iota_{F^{(\lambda)}}$ and also indicates this elements of $\ell_\infty(\mathbb{Z})$ which also lie in $F^{(\lambda)}$. This theorem is entirely based on Theorem 3.4 of [7], but in addition gives more detailed explanantion of the facts presented there.

Theorem 4.3.7. *Let us identify the space $\ell_\infty(\mathbb{Z})$ with $C(\beta\mathbb{Z})$ as described above and let G be the closed subspace of those elements $f \in \ell_\infty(\mathbb{Z}) \cong C(\beta\mathbb{Z})$ which satisfy the following condition*

$$f(\mathcal{U}) = \begin{cases} \lambda^{-k} f(t) & \text{if} \quad \mathcal{U} \in X_t^{(k)}; \\ 0 & \text{if} \quad \mathcal{U} \in X^{(\infty)}. \end{cases} \tag{4.3.9}$$

Then $G = F^{(\lambda)}$, where $F^{(\lambda)}$ is the space introduced in Definition 4.2.3.

Proof. We subdivide the proof of this theorem into three steps.

a) We show that the space G is shift-invariant,

b) We show the inclusion that $F^{(\lambda)} \subseteq G$,

c) We show that $F^{(\lambda)} = G$.

a) Let $\mathcal{U} \in \mathbb{Z}^*$ and $s \in \mathbb{Z}$. For $A \subset \mathbb{Z}$, define $A + s := \{a + s : a \in A\} \subseteq \mathbb{Z}$. Then define the expression $\mathcal{U} + s$ such that

$$\mathcal{U} + s := \{A + s : A \in \mathcal{U}\}.$$

For the reason that

$$\bigcap_{A\in\mathcal{U}} A = \emptyset \Longleftrightarrow \bigcap_{A\in\mathcal{U}} A + s = \emptyset$$

we see that $\mathcal{U}$ is a non-principal ultrafilter if and only if $\mathcal{U}+s$ is a non-principal ultrafilter. Moreover, for $t,s \in \mathbb{Z}$ and $k>0$ we have $\mathcal{U} \in X_t^{(k)}$ if and only if $\mathcal{U}+s \in X_{t+s}^{(k)}$. To see this let A be any subset of $\mathbb{Z}$ such that $A \in \mathcal{U}$. Since every $n \in A$ is represented as

$$n = 2^{n_1} + ... + 2^{n_k} + t \qquad \text{for some } n_1 < n_2 < ... < n_k,$$

we can observe that

$$n + s = 2^{n_1} + ... + 2^{n_k} + t + s$$

belongs to $A+s$. But since $A+s$ is an element of $\mathcal{U}+s$ we see that if $\mathcal{U} \in X_t^{(k)}$, then $\mathcal{U}+s \in X_{t+s}^{(k)}$.
Now, let us assume that $\mathcal{U} \in X^{(\infty)}$. According to the Definition 4.3.2 for each $k \geq 1$ and $t \in \mathbb{Z}$ there exist $n > 0$ such that

$$\{2^{n_1} + 2^{n_2} + ... + 2^{n_k} + t : n < n_1 < n_2 < ... < n_k\} \notin \mathcal{U}.$$

But this implies that for any $s \in \mathbb{Z}$ we have

$$\{2^{n_1} + 2^{n_2} + ... + 2^{n_k} + t + s : n < n_1 < n_2 < ... < n_k\} \notin \mathcal{U} + s.$$

Hence, we see that $\mathcal{U}+s$ can not belong to any of the sets $X_{t+s}^{(k)}$ and so, by the Definition 4.3.2, the set $\mathcal{U}+s \in X^{(\infty)}$.
Therefore, if $f \in G$ then for any $s \in \mathbb{Z}$ we have

$$\sigma^s f(\mathcal{U}) = f(\mathcal{U}+s) = \begin{cases} \lambda^{-k} f(t+s) & \text{if} \quad \mathcal{U}+s \in X_t^{(k)}; \\ 0 & \text{if} \quad \mathcal{U}+s \in X^{(\infty)}, \end{cases} \tag{4.3.10}$$

which implies that $\sigma^s(f)$ is in G and so G is shift-invariant.

b) To prove the inclusion $F^{(\lambda)} \subseteq G$, we notice that $F^{(\lambda)}$ is the smallest shift-invariant subspace containing the set $A = \{\sigma^n x_0 : n \in \mathbb{Z}\}$. Since we know that G is shift-invariant it will be enough to show that the element x_0 defined by (4.1.2) is in G. For this purpose we need to prove that x_0 satisfies the equation (4.3.9).
First, we prove that for any $t \in \mathbb{Z}$ and $k > 0$ if $\mathcal{U} \in X_t^{(k)}$, then we have $x_0(\mathcal{U}) = \lambda^{-k} x_0(t)$. Let us notice that if $k = 1$ and $t \geq 0$ then for n sufficiently large we have $b(2^n + t) = b(t) + 1$. This fact implies that $\lambda^{-b(2^n+t)} = \lambda^{-b(t)-1}$ and so by the Remark 4.3.6 and the Definition 2.5.1

$$x_0(\mathcal{U}) = \lim_{n\to\infty} x_0(2^n + t) = \lim_{n\to\infty} \lambda^{-b(2^n+t)} = \lim_{n\to\infty} \lambda^{-b(t)-1} = \lambda^{-1} x_0(t).$$

Hence, $x_0(\mathcal{U}) = \lambda^{-k} x_0(t)$ for $t \geq 0$ and $k = 1$.
Now, let $k = 1$ and $t < 0$. Writing t in the binary expansion form ,we obtain $-t = \sum_{j=0}^{p} \epsilon_j 2^j$

where $\epsilon_j \in \{0,1\}$. Consequently, if we define $\epsilon_j^{'} := 1-\epsilon_j$, we have

$$\sum_{j=0}^{p} \epsilon_j 2^j + \sum_{j=0}^{p} \epsilon_j^{'} 2^j = 2^{p+1}-1.$$

For the reason that not every $\epsilon_j = 0$, we also obtain the following inequality

$$1+\sum_{j=0}^{p} \epsilon_j^{'} 2^j \leq 2^{p+1}-1. \tag{4.3.11}$$

Now we notice that for $n > p+1$ we obtain

$$2^n + t = 2^n - 2^{p+1} + (2^{p+1} - \sum_{j=0}^{p} \epsilon_j 2^j) = 1 + \sum_{j=p+1}^{n-1} 2^j + \sum_{j=0}^{p} \epsilon_j^{'} 2^j.$$

Using this expression and inequality (4.3.11) we obtain $b(2^n+t) \geq n-k$, which gives

$$\lim_{n\to\infty} x_0(2^n+t) = \lim_{n\to\infty} \lambda^{-b(2^n+t)} = 0 = x_0(t).$$

Now, using the case when $k=1$ we show that equation (4.3.9) holds for any $\mathcal{U} \in X_t^{(k)}$, where $k \geq 2,\ t \in \mathbb{Z}$. Notice, that if $\mathcal{U} \in X_t^{(k)}$, then according to Definition 4.3.2 and Definition 2.5.1 we have

$$\lim_{n\to\mathcal{U}} x_0(n) = \lim_{n_1\to\infty} \lim_{n_2\to\infty} \dots \lim_{n_k\to\infty} x_0(t+2^{n_1}+2^{n_2}+\dots+2^{n_k}), \tag{4.3.12}$$

which implies that

$$\lim_{n\to\mathcal{U}} x_0(n) = \lim_{n_1\to\infty} \left(\lim_{n_2\to\infty} \dots \left(\lim_{n_k\to\infty} x_0(t+2^{n_1}+2^{n_2}+\dots+2^{n_k}) \right) \dots \right). \tag{4.3.13}$$

Since, by the case with $k=1$, we already know that

$$\lim_{n_k\to\infty} x_0(t+2^{n_1}+2^{n_2}+\dots+2^{n_k}) = \lambda^{-1} x_0(t+2^{n_1}+2^{n_2}+\dots+2^{n_{k-1}}), \tag{4.3.14}$$

we obtain

$$\lim_{n\to\mathcal{U}} x_0(n) = \lim_{n_1\to\infty} \left(\lim_{n_2\to\infty} \dots \left(\lim_{n_{k-1}\to\infty} \lambda^{-1} x_0(t+2^{n_1}+2^{n_2}+\dots+2^{n_{k-1}}) \right) \dots \right). \tag{4.3.15}$$

We now see that we can repeat this way of calculation for $n_{k-1}, n_{k-1}, \dots$ and finally for n_1, which will imply

$$\lim_{n\to\mathcal{U}} x_0(n) = \lim_{n_1\to\infty} \lambda^{k-1} x_0(n)(t+2^{n_1}) = \lambda^{-k} x_0(t). \tag{4.3.16}$$

Hence, the equation (4.3.9) is satisfied for any $\mathcal{U} \in X_t^{(k)}$, $k \geq 1,\ t \in \mathbb{Z}$.

Now, we take into account ultrafilters $\mathcal{U} \in X^{(\infty)}$ and we check that for them we have $x_0(\mathcal{U}) = 0$. By contradiction, suppose that for some $\mathcal{U} \in X^{(\infty)}$ we get $x_0(\mathcal{U}) \neq 0$. As on the set $\mathbb{Z}$ the element x_0 takes value from the set $\{0\} \cup \left\{\lambda^{-k} : k \geq 0\right\}$ there must exist $k \geq 0$ such that $x_0(\mathcal{U}) = \lambda^{-k}$. Since, by Definition 2.5.1, the set

$$\{n > 0 : b(n) = k\} = \left\{n \in \mathbb{Z} : x_0(n) = \lambda^{-k}\right\} \in \mathcal{U}, \tag{4.3.17}$$

we obtain that the set

$$\{2^{n_1} + 2^{n_2} + ... + 2^{n_k} : n_1 < n_2 < ... < n_k\} \in \mathcal{U} \tag{4.3.18}$$

For the reason that $\mathcal{U} \notin X_0^{(k)}$, there exists $m_1 > 0$ such that

$$\{2^{n_1} + 2^{n_2} + ... + 2^{n_k} : m_1 < n_1 < n_2 < ... < n_k\} \notin \mathcal{U}. \tag{4.3.19}$$

Hence, the set

$$\mathbb{Z} \setminus \{2^{n_1} + 2^{n_2} + ... + 2^{n_k} : m_1 < n_1 < n_2 < ... < n_k\} \in \mathcal{U}. \tag{4.3.20}$$

Now the intersection of the sets (4.3.18) and (4.3.20) gives the set

$$\{2^{n_1} + 2^{n_2} + ... + 2^{n_k} : n_1 < n_2 < ... < n_k, n_1 \leq m_1\}, \tag{4.3.21}$$

which also must be in $\mathcal{U}$. Now, setting any $\ell_1 \in \{1, ..., m_1\}$, we obtain that for $t_1 = 2^{\ell_1}$ the set

$$\{t_1 + 2^{n_2} + ... + 2^{n_k} : \ell_1 < n_2 < ... < n_k\} \in \mathcal{U}. \tag{4.3.22}$$

Similarly, since $\mathcal{U} \notin X_{t_1}^{(k-1)}$, there exists $m_2 > 0$ such that

$$\{t_1 + 2^{n_2} + ... + 2^{n_k} : l_1 < n_2 < n_3 < ... < n_k, m_2 < n_2\} \notin \mathcal{U}, \tag{4.3.23}$$

which implies that

$$\mathbb{Z} \setminus \{t_1 + 2^{n_2} + ... + 2^{n_k} : l_1 < n_2 < n_3 < ... < n_k, m_2 < n_2\} \in \mathcal{U}. \tag{4.3.24}$$

The intersection of (4.3.22) and (4.3.24) implies that

$$\{t_1 + 2^{n_2} + ... + 2^{n_k} : \ell_1 < n_2 < ... < n_k, n_2 \leq m_2\} \tag{4.3.25}$$

belongs to $\mathcal{U}$. We see that, setting $l_2 \in \{\ell_1 + 1, ..., m_2\}$, we obtain that for $t_2 = 2^{\ell_1} + 2^{l_2}$

$$\{t_2 + 2^{n_3} ... + 2^{n_k} : l_2 < n_3 < ... < n_k, n_2 \leq m_2\}$$

is in $\mathcal{U}$. As, again $\mathcal{U} \notin X_{t_2}^{(k-2)}$, we proceed consequently in the same way to obtain finally that for $t_{k-1} = 2^{l_1} + 2^{l_2} + ... + 2^{l_{k-1}}$ the set

$$\{t_{k-1} + 2^{n_k} : l_{k-1} < n_k\}$$

must be in $\mathcal{U}$. But this shows that an ultrafilter $\mathcal{U}$ must be in $X_t^{(1)}$, contradicting with the fact that $\mathcal{U} \in X^{(\infty)}$.

c)Let us notice that by Hahn-Banach Theorem (see Theorem 3.1.4) and by Riesz Theorem (see Theorem 7.4 in [10]) every $\mu \in G^*$ can be extended to an element of $\mathcal{M}(\beta\mathbb{Z})$, where G^* is the dual space of G and $\mathcal{M}(\beta\mathbb{Z})$ denotes the family of signed Borel measures on $\beta\mathbb{Z}$. Since we know, by Definition 4.3.2 and Lemma 4.3.4, that the sets $X^{(\infty)}$, $(X_t^{(k)})_{t\in\mathbb{Z},\, k>0}$ are pairwise disjoint, by countable additivity for every $x \in G$ we obtain

$$\langle \mu, x\rangle = \int_{\beta\mathbb{Z}} x d\mu = \int_{X^{(\infty)}} x d\mu + \sum_{t\in\mathbb{Z}} \left(x(t)\mu(\{t\}) + \sum_{k=1}^{\infty} \int_{X_t^{(k)}} x d\mu \right) =$$

$$= \sum_{t\in\mathbb{Z}} x(t) \left(\mu(\{t\}) + \sum_{k=1}^{\infty} \lambda^{-k} \mu(X_t^{(k)}) \right).$$

If we put $a_t = \mu(\{t\}) + \sum_{k=1}^{\infty} \lambda^{-k}\mu(X_t^{(k)})$, where $t \in \mathbb{Z}$ we obtain that the element $a = (a_t)$ is in $\ell_1(\mathbb{Z})$. To see this we evaluate $\|a\|$. The triangle inequality guarantees that

$$\|a\| = \sum_{t\in\mathbb{Z}} |a_t| = \sum_{t\in\mathbb{Z}} |\mu(\{t\}) + \sum_{k=1}^{\infty} \lambda^{-k}\mu(X_t^{(k)})| \leq \sum_{t\in\mathbb{Z}} |\mu(\{t\})| + \sum_{t\in\mathbb{Z}} |\sum_{k=1}^{\infty} \lambda^{-k}\mu(X_t^{(k)})|.$$

Since μ is finite and countably additive we have

$$\sum_{t\in\mathbb{Z}} |\mu(\{t\})| = |\mu(\mathbb{Z})| < \infty. \tag{4.3.26}$$

Also by countable additivity and by the facts that $\sum_{k=1}^{\infty} |\lambda|^{-k}$ is a geometric series and $\mu(\{t\}) = 0$ for any $t \in \mathbb{Z}$, we obtain

$$\sum_{t\in\mathbb{Z}} |\sum_{k=1}^{\infty} \lambda^{-k}\mu(X_t^{(k)})| = \sum_{t\in\mathbb{Z}} (\lambda-1)^{-1} |\mu(\bigcup_{k>0} X_t^{(k)})| = (\lambda-1)^{-1}|\mu(\bigcup_{k>0, t\in\mathbb{Z}} X_t^{(k)})| < \infty. \tag{4.3.27}$$

Hence, combining (4.3.26) and (4.3.27) we obtain that $\|a\| < \infty$ and so $a \in \ell_1(\mathbb{Z})$. This fact implies that $\langle \mu, x\rangle = \langle x, a\rangle$ for each $x \in G$ and, as a result of that, we obtain that ι_G must be surjective. Now, since we know by part b) of this proof that $F \subseteq G$ we obtain that ι_F is simply the composition of ι_G and the restriction map form G^* onto F^* and so must be surjective as well. Moreover, for the same reason and by Lemma 4.3.1, we obtain injectivity of ι_G. Finally, by the Banach Isomorphism Theorem (see Theorem 3.1.2) we obtain that both ι_F and ι_G are isomorphisms and so both F and G are preduals of $\ell_1(\mathbb{Z})$.

Now we show that $F = G$. Suppose that there exists $x \in G \setminus F$. Then, by Corollary 3.1.6 there exists $a \in \ell_1(\mathbb{Z})$ such that

$$\langle x, a \rangle \neq 0 \qquad \text{and} \qquad \langle y, a \rangle = 0 \quad \text{for each } y \in F.$$

This implies that $a = 0$ and so $\ell_1(\mathbb{Z})$ separates points of F. Since $a = 0$ this automatically contradicts with the fact that $\langle x, a \rangle \neq 0$. Hence, by the part b), we obtain that $F = G$. $\square$

Chapter 5

Isomorphism

Since the predual $F^{(\lambda)}$ of $\ell_1(\mathbb{Z})$ from the last chapter can be regarded as subspace of $\ell_\infty(\mathbb{Z})$ and the space $\ell_1(\mathbb{Z})$ has a canonical predual $c_0(\mathbb{Z})$ we may suspect that $F^{(\lambda)}$ could be isomorphic with the $c_0(\mathbb{Z})$. The first section of this chapter shows that this is really the case although without stating the explicit form of that isomorphism. This sections relies mostly on the discussion following Remark 3.7 and Theorem 3.8 in [7]. In the second section we present an explicit isomorphism between $F_+^{(\lambda)}$ and $c_0(\mathbb{N})$ with the hope for further development of this theory. In this chapter we use all the notations introduced in the previous chapters of this thesis.

5.1 Isomorphic spaces $F^{(\lambda)}$ and $c_0(\mathbb{Z})$

In the first section of this chapter we present the main motivation for writing the thesis. As explained in the discussion in section 3 of [7], by using results of [1] and [19], the predual $F^{(\lambda)}$ described in the previous chapter has an interesting property which enables us to classify this predual as isomorphic to $c_0(\mathbb{Z})$. We start describing this outcome by recalling the definition of a $G-space$ (see for example very beginning of [1]).

Definition 5.1.1. Let H be a compact Hausdorff space. Then a closed subspace X of $C(H)$ is said to be a $G-space$ if there exists an index set A and for each $\alpha \in A$, there exist $x_\alpha, y_\alpha \in H$ and $\lambda_\alpha \in \mathbb{C}$ such that

$$X = \{f \in C(H) : f(x_\alpha) = \lambda_\alpha f(y_\alpha)\}. \tag{5.1.1}$$

Remark 5.1.2. Theorem 4.3.7 describing the predual $F^{(\lambda)}$ as a closed subspace of $C(\beta\mathbb{Z})$ space shows in particular by condition (4.3.9) that $F^{(\lambda)}$ is a G-space. Indeed, by Theorem 2.3.1 we know that $\beta\mathbb{Z}$ is a compact Hausdorff space, moreover, by Definition 3.2.2, $F^{(\lambda)}$ is closed. Therefore, for an index set $A = \mathbb{N} \times \mathbb{Z} \cup \{\infty\}$, if $\alpha = (k, t) \in A$, then we can choose $x_\alpha = \mathcal{U} \in X_t^{(k)} \subset \beta\mathbb{Z}$, $y_\alpha = \{t\} \in \beta\mathbb{Z}$ and scalar $\lambda_\alpha = \lambda^{-k}$ and then, by Theorem 4.3.7, the equation (5.1.1) is satisfied. Also, if $\alpha = \infty \in A$, then for $x_\alpha = \mathcal{U} \in X^{(\infty)} \subset \beta\mathbb{Z}$, scalar $\lambda_\alpha = 0$ and any element $y_\alpha \in \beta\mathbb{Z}$ the equation (5.1.1) holds as well. Hence, we obtain that $F^{(\lambda)}$ is a $G-space$.

Now, let us refer to [1] where Benyamini proved that every separable G-space is isomorphic to a $C(L)$ space for some compact Hausdorff space L. Clearly, by the above remark, the space $F^{(\lambda)}$ is a G-space. It is easy to see that it is also separable. To find the space L for which the spaces $F^{(\lambda)}$ and $C(L)$ will be isomorphic we need to use a useful tool called the *Szlenk index.* Here we present an equivalent definition of the Szlenk index which can be found, for example, in [14]. In this definition, a basic notation about ordinals is used and so an ordinal α denotes the set of all ordinals β such that $\beta < \alpha$. By $\alpha+$ we denote the *successor ordinal* of α and an non-zero ordinal number, which is not a successor, is called a *limit ordinal.* Moreover ω denotes the first countable limit ordinal and ω_1 denotes the first uncountable limit ordinal. More very useful information about ordinal numbers can be found in [11] or in [20].

Definition 5.1.3. Let X be a separable Banach space and let us fix $\epsilon > 0$. For an ordinal number $\alpha < \omega_1$ we inductively define sets $P_\alpha(X, \epsilon)$ as follows

- If $\alpha = 0$, then
$$P_0(X, \epsilon) = B_{X^*} := \{x^* \in X^* : \|x^*\| \leq 1\}.$$
- If $\alpha \geq 0$ is a non-limit ordinal, then
$$P_{\alpha+}(X, \epsilon) := \{x^* \in P_\alpha(X, \epsilon) : \text{for all weak*neighbourhoods } \mathfrak{U} \text{ of } x^*,\ diam(\mathfrak{U} \cap P_\alpha(X, \epsilon)) > \epsilon\}.$$
- If α is a limit ordinal, then
$$P_\alpha(X, \epsilon) := \bigcap_{\beta<\alpha} P_\beta(X, \epsilon).$$

We next set
$$\eta(\epsilon, X) = \sup\{\alpha : P_\alpha(X, \epsilon) \neq 0\}.$$
The Szlenk index of X is defined as
$$\eta(X) = \sup_{\epsilon>0}\{\eta(\epsilon, X)\}.$$

Remark 5.1.4. One of the most important properties of the Szlenk index is that for Banach space X its value $\eta(X) < \omega_1$ if and only if X^* is separable. In fact, the possible countable values of the Szlenk index are ordinals of the form ω^{ω^β} for $\beta < \omega_1$. (See section 3 in [14] for more details)

From our perspective, it would be useful to calculate the Szlenk index for $c_0(\mathbb{N})$ space. The following lemma and corollary are a modified version of the example given at the very beginning of section 3 in [14] where the author explained how to calculate the Szlenk index for the space $\ell_1(\mathbb{N})$.

Lemma 5.1.5. *Let $X = c_0(\mathbb{N})$ then, with the notation introduced above, for $\epsilon > 0$ and any*

$n \in \mathbb{N} \cup \{0\}$, *we have*

$$P_{n+1}(c_0(\mathbb{N}), \epsilon) \subseteq (1 - n\frac{\epsilon}{2})B_{\ell_1(\mathbb{N})}. \tag{5.1.2}$$

Proof. Notice that if $n = 0$, then the above formula takes the form

$$P_1(c_0(\mathbb{N}), \epsilon) \subseteq B_{\ell_1(\mathbb{N})}, \tag{5.1.3}$$

which, by Definition 5.1.3, must be true. Now, by induction, assume that

$$P_n(c_0(\mathbb{N}), \epsilon) \subseteq (1 - (n-1))\frac{\epsilon}{2})B_{\ell_1(\mathbb{N})}. \tag{5.1.4}$$

Fix $\epsilon > 0$ and let

$$\begin{aligned} &x^* \in P_{n+1}(c_0(\mathbb{N}), \epsilon) = \\ &= \{y^* \in P_n(c_0(\mathbb{N}), \epsilon) : \text{ for all weak*-neighbourhoods } \mathcal{U} \text{ of } y^*,\ diam(\mathcal{U} \cap P_n(c_0(\mathbb{N}), \epsilon) > \epsilon\}\,. \end{aligned} \tag{5.1.5}$$

For δ such that $0 < \delta < \|x^*\|$, where $x^* = (a_i)_{i\in\mathbb{N}}$ we can choose $n_0 \in \mathbb{N}$ so that

$$\sum_{i=1}^{n_0} |a_i| > \|x^*\| - \frac{\delta}{2}.$$

Then, for $\delta_1 := \frac{\delta}{2n_0}$, we define $\mathcal{U}$, a weak*-neighbourhood of x^*, by

$$\mathcal{U} := \{y^* = (b_i)_{i\in\mathbb{N}} \in \ell_1(\mathbb{N}) : |a_i - b_i| < \delta_1 \text{ for } i \leq n_0\}.$$

In particular, for any $z^* = (k_i)_{i\in\mathbb{N}} \in \mathcal{U} \cap P_n(c_0(\mathbb{N}), \epsilon)$, we have

$$\sum_{i=1}^{n_0} |k_i| > \sum_{i=1}^{n_0}(|a_i| - \delta_1) = \sum_{i=1}^{n_0} |a_i| - n_0\delta_1 > \sum_{i=1}^{n_0} |a_i| - \frac{\delta}{2} > \|x^*\| - \delta.$$

Hence, by the inductive hypothesis

$$(1 - (n-1)\frac{\epsilon}{2}) \geq \|z^*\| = \sum_{i=0}^{n_0} |k_i| + \sum_{|i|=n_0+1}^{\infty} |k_i| > (\|x^*\| - \delta) + \sum_{|i|=n_0+1}^{\infty} |k_i|.$$

Which implies

$$\sum_{i=n_0+1}^{\infty} |k_i| < (1 - (n-1)\frac{\epsilon}{2}) - (\|x^*\| - \delta). \tag{5.1.6}$$

In particular, for any two elements $z_1^* = (k_i)_{i\in\mathbb{N}}$, $z_2^* = (l_i)_{i\in\mathbb{N}}$ of $\mathcal{U} \cap P_n(c_0(\mathbb{N}), \epsilon)$ we have

$$\|z_1^* - z_2^*\| = \sum_{i=1}^{n_0} |k_i - l_i| + \sum_{i=n_0+1}^{\infty} |k_i - l_i| \leq \sum_{i=1}^{n_0} |k_i - a_i| + \sum_{i=1}^{n_0} |a_i - l_i| + \sum_{i=n_0+1}^{\infty} |k_i| + \sum_{i=n_0+1}^{\infty} |l_i|.$$

Hence,

$$\|z_1^* - z_2^*\| \leq 2n_0\delta_1 + \sum_{i=n_0+1}^{\infty} |k_i| + \sum_{i=n_0+1}^{\infty} |l_i|. \tag{5.1.7}$$

By (5.1.6) we obtain

$$\|z_1^* - z_2^*\| \leq \delta + 2\left(1 - (n-1)\frac{\epsilon}{2} - (\|x^*\| - \delta)\right) = 3\delta + 2(1 - \|x^*\|) - (n-1)\epsilon. \tag{5.1.8}$$

Now, since $0 < \delta < x^*$ in (5.1.8) is arbitrary, we obtain

$$\|z_1^* - z_2^*\| \leq 2(1 - \|x^*\|) - (n-1)\epsilon. \tag{5.1.9}$$

On the other hand, since for any weak*-neighbourhood $\mathcal{U}$ of x^* we have $diam(\mathcal{U} \cap P_n(c_0, \epsilon)) > \epsilon$ then there exist $z_1^*, z_2^* \in \mathcal{U} \cap P_n(c_0, \epsilon))$ such that $\epsilon < \|z_1^* - z_2^*\|$ and then applying (5.1.9) we obtain

$$\epsilon < \|z_1^* - z_2^*\| \leq 2(1 - \|x^*\|) - (n-1)\epsilon,$$

which ultimately yields

$$\|x^*\| < 1 - n\frac{\epsilon}{2}.$$

Hence, we see that

$$P_{n+1}(c_0(\mathbb{N}), \epsilon) \subseteq (1 - n\frac{\epsilon}{2})B_{\ell_1(\mathbb{N})}.$$

which finishes the proof for any $n \in \mathbb{N}$. Therefore, by induction the above formula is true for any $n \in \mathbb{N} \cup \{0\}$. ☐

Corollary 5.1.6. *The Szlenk index of $c_0(\mathbb{N})$ is equal to ω.*

Proof. The formula which was proved in the previous lemma says that for $\epsilon > 0$ and any $n \in \mathbb{N} \cup \{0\}$ we have

$$P_{n+1}(c_0(\mathbb{N}), \epsilon) \subseteq \left(1 - n\frac{\epsilon}{2}\right) B_{\ell_1(\mathbb{Z})}. \tag{5.1.10}$$

Hence, we can see that for some $n' \in \mathbb{N}$ we have $n'\frac{\epsilon}{2} > 1$ and, therefore, the set $P_{n'}(c_0(\mathbb{N}), \epsilon)$ must be empty. Now, the fact implying that the Szlenk index of $c_0(\mathbb{N})$ equals to ω is a consequence of the result of Remark 5.1.4 which says that the Szlenk index of separable Banach space must be a limit ordinal. ☐

The result saying that the Szlenk index of the space $c_0(\mathbb{Z})$ equals to ω may also be obtained indirectly by result of Samuel's work [19], where he proved that the Szlenk index of $C(\omega^{\omega^\alpha} + 1)$ space is $\omega^{\alpha+1}$. The identification of $c_0(\mathbb{Z})$ with $C(\omega^1 + 1)$, where $\omega^1 + 1$ is equipped with order topology yields this result immediately. On the other hand, the earliest result proved by Bessaga and Pelczynski in [3] shows that for an infinite countable compact metric space K the space $C(K)$ is isomorphic to $C(\omega^{\omega^\alpha} + 1)$ for some α and moreover two spaces $C(\omega^{\omega^\alpha} + 1)$ and $C(\omega^{\omega^\beta} + 1)$ are isomorphic only if $\alpha = \beta$. This implies that two $C(K)$ spaces are isomorphic if they have the same Szlenk index.

Remark 5.1.7. By Theorem 3.8 in [7] the Szlenk index for the predual $F^{(\lambda)}$ is ω. This fact combined with the above discussion implies that spaces $F^{(\lambda)}$ and $c_0(\mathbb{Z})$ are isomorphic. Moreover, since by Fact 4.2.4 the space $F_+^{(\lambda)}$ is the subspace of $F^{(\lambda)}$, by Remark 5.1.4 we obtain that the Szlenk index of $F_+^{(\lambda)}$ also must be equal to ω.

5.2 Explicit isomorphism between $F_+^{(\lambda)}$ and $c_0(\mathbb{N})$

The previous section was ended with the remark that predual $F^{(\lambda)}$ of $\ell_1(\mathbb{Z})$ is isomorphic to $c_0(\mathbb{Z})$ unfortunately without showing an explicit form of that isomorphism. In this section I make a movement in direction of defining such a map, namely, I present an explicit form of an isomorphism between the spaces $F_+^{(\lambda)}$ and $c_0(\mathbb{N})$.

Definition 5.2.1. Let us define a map $r : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ in the following way. If $n \in \mathbb{N}$ then the number $r(n)$ that is obtained by deleting the largest one from the binary expansion of n and if $n = 0$, then we put $r(n) = 0$.

Fact 5.2.2. In the last definition we can observe that every number $n \in \mathbb{N}$ can be uniquely presented as $n = 2^p + r(n)$ for some $p \in \mathbb{N}$. On the other hand for $k \in \mathbb{N}$, let $p \in \mathbb{N}$ be such that $2^p > k$. If we put $n = 2^p + k$, then we obtain $r(n) = k$ and so r is surjective.

Definition 5.2.3. Let r be the map introduced above. We define a map S in the following way

$$S : F_+^{(\lambda)} \ni f \longmapsto f - \lambda^{-1}(f \circ r) \in \ell_\infty(\mathbb{Z}). \tag{5.2.1}$$

In other words,

$$(Sf)(n) := f(n) - \lambda^{-1} f(r(n)), \qquad (n \in \mathbb{N}). \tag{5.2.2}$$

In the next few lemmas we show a couple of properties of the map S which will contribute to the final theorem proving that S is an isomorphism between $F_+^{(\lambda)}$ and $c_0(\mathbb{N})$. First, we show that S maps *into* $c_0(\mathbb{Z})$. To prove this we need to add some theory concerning filters and ultrafilters.

Lemma 5.2.4. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{Z}$. Then the family*

$$\mathcal{F} := \{B \in \mathcal{P}(\mathbb{Z}) : \exists A \in \mathcal{U} \ \ \textit{such that}\ r(A) \subseteq B\} \tag{5.2.3}$$

is a filter.

Proof. Let $\mathcal{U} \in \mathbb{Z}^*$. To prove that the family $\mathcal{F}$ is a filter we need to check three conditions of Definition 2.1.1. Firstly, we notice $\mathbb{Z} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. This is because for any $A \in \mathcal{U}$ we have $r(A) \subseteq \mathbb{Z}$ and $A \neq \emptyset$ respectively. Now, assume that B_1 and B_2 are in $\mathcal{F}$. Then there exist corresponding sets A_1 and A_2 belonging to the ultrafilter $\mathcal{U}$ such that $r(A_1) \subseteq B_1$ and $r(A_2) \subseteq B_2$. Since $r(A_1 \cap A_2) \subseteq r(A_1) \cap r(A_2) \subseteq B_1 \cap B_2$ we obtain that $B_1 \cap B_2 \in \mathcal{F}$. The fact that the family of subsets $\mathcal{F}$ on $\mathbb{Z}$ is closed under supersets follows from the fact that if $B_1 \in \mathcal{F}$ and $r(A) \subseteq B_1$ for some set $A \in \mathcal{U}$ than $r(A) \subseteq B_2$ for any subset $B_2 \supseteq B_1$ and so $B_2 \in \mathcal{F}$. $\square$

Remark 5.2.5. Notice that for the element $x_0 \in F_+^{(\lambda)}$ and an ultrafilter $\mathcal{U} \in X^{(\infty)}$ the number of ones $b(n)$ in the binary expansion of n tends to infinity if n goes to infinity through the ultrafilter $\mathcal{U}$. This follows directly from Theorem 4.3.7 as according to that theorem we have

$$\lim_{n \to \mathcal{U}} x_0(n) = \lim_{n \to \mathcal{U}} \lambda^{-b(n)} = 0.$$

Because $|\lambda| > 1$ this can happen only if $b(n) \to \infty$. Now, since $b(r(n)) = b(n) - 1$, we obtain that the number of ones in the binary expansion of $r(n)$ tends to infinity as n goes to infinity through the ultrafilter $\mathcal{U}$.

Now we refer to Definition 4.3.2 where we have defined subsets $X_t^{(k)}$ $(k \geq 1, t \in \mathbb{Z})$ of the space of non-principal ultrafilters $\mathbb{Z}^*$. If in this definition we consider $k = 0$, then as a result, for fixed t we obtain a subset $X_t^{(0)} \subseteq \beta\mathbb{Z}$, which is in fact a set of the principle ultrafilers. In other words,

$$X_t^{(0)} := \{\mathcal{U} \in \beta\mathbb{Z} : \ \{t\} \in \mathcal{U}\}. \tag{5.2.4}$$

We apply this result to the following important lemma where we use the above definition of $X_t^{(0)}$ and also the definitions of sets $X_t^{(k)}$ and $X^{(\infty)}$ introduced in Definition 4.3.2.

Lemma 5.2.6. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{Z}$ and let $\mathcal{V}$ be an ultrafilter obtained from $\mathcal{U}$ by some extension of the filter $\mathcal{F}$ from the previous lemma (See Lemma 2.1.6). Then if $\mathcal{U}$ is in $X^{(\infty)}$, then $\mathcal{V}$ is in $X^{(\infty)}$ and if $\mathcal{U}$ is in $X_t^{(k)}$ then $\mathcal{V}$ is in $X_t^{(k-1)}$, where $k \geq 1$ and $t \in \mathbb{Z}$.*

Proof. Let $\mathcal{U} \in \mathbb{Z}^*$, r be the map introduced in Definition 5.2.1, $\mathcal{F}$ be the filter obtained from $\mathcal{U}$ by Lemma 5.2.4 and $\mathcal{V}$ an ultrafilter extending the filter $\mathcal{F}$. First suppose that $\mathcal{U} \in X^{(\infty)}$. We show that $\mathcal{V} \in X^{(\infty)}$. Since if $\mathcal{U} \in X^{(\infty)}$ then, by the above remark, we have that $b(n) \to \infty$ as $n \to \mathcal{U}$. But according to Definition 2.5.1 this implies that

$$\forall_{K>0} \, \{n : b(n) > K\} \in \mathcal{U}.$$

Fix $K > 0$ and notice the following

$$r(\{n : b(n) > K + 1\}) \subseteq \{r(n) : b(r(n)) > K\} = \{n : b(n) > K\}.$$

Hence, we see that $\{n : b(n) > K\} \in \mathcal{F}$. Therefore, extending $\mathcal{F}$ to an ultrafilter $\mathcal{V}$, we obtain that $\mathcal{V} \in X^{(\infty)}$.

Now suppose that $\mathcal{U} \in X_t^{(k)}$ for some $k \geq 1$ and $t \in \mathbb{Z}$. According to Definition 2.5.1 for any $m > 0$ the set

$$\{n : n = 2^{n_1} + 2^{n_2} + ... + 2^{n_k} + t : m < n_1 < n_2 < ... < n_k\} \in \mathcal{U}.$$

Applying the map r to this set we see that, if $k > 1$ then

$$\{r(n) : r(n) = 2^{n_1} + 2^{n_2} + ... + 2^{n_{k-1}} + t : m < n_1 < n_2 < ... < n_{k-1}\} \in \mathcal{F},$$

which shows that $\mathcal{F} \subseteq \mathcal{V} \in X_t^{(k-1)}$. If $k = 1$, then application of the map r to the set

$$\{n : n = 2^n + t : m < n\} \in \mathcal{U},$$

implies that $\{t\} \in \mathcal{F}$ and the only ultrafilter extending $\mathcal{F}$ is the principal ultrafilter $\mathcal{V} = \{t\} \in X_t^{(0)}$. $\square$

Lemma 5.2.7. *Let S be the map introduced in Definition 5.2.3 then S maps $F_+^{(\lambda)}$ into $c_0(\mathbb{N})$.*

Proof. Let f be any element of $F_+^{(\lambda)}$. We need to show that $Sf \in c_0(\mathbb{N})$. In other words, according to the definition of convergence introduced in Definition 2.5.1, we must show that for each non-principal ultrafilter $\mathcal{U} \in \mathbb{Z}^*$ we have $\lim_{n\to\mathcal{U}}(Sf)(n) = 0$. Indeed, if this is not the case, then there exists $\epsilon > 0$ and a sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that $n_1 < n_2 < n_3 < ...$ and $|Sf(n_i)| > \epsilon$. We choose a non-principal ultrafilter $\mathcal{U}$ containing $\{n : |Sf(n)| > \epsilon\}$ and then $\lim_{n\to\mathcal{U}}(Sf)(n) \neq 0$. As if $\lim_{n\to\mathcal{U}} Sf(n) = 0$, then $\{n : |Sf(n)| < \epsilon\} \in \mathcal{U}$ we obtain contradiction since the intersection $\{n : |Sf(n)| > \epsilon\} \cap \{n : |Sf(n)| < \epsilon\}$ is empty and can not belong to $\mathcal{U}$.
Let then $\mathcal{U} \in \mathbb{Z}^*$. From Definition 5.2.3 we have

$$\lim_{n\to\mathcal{U}}(Sf)(n) = \lim_{n\to\mathcal{U}} f(n) - \lambda^{-1} \lim_{n\to\mathcal{U}} f(r(n)).$$

First, I show that $\lim_{n\to\mathcal{U}} f(r(n)) = \lim_{n\to\mathcal{V}} f(n)$, where $\mathcal{V}$ is an ultrafilter obtained from $\mathcal{U}$ by extending the filter $\mathcal{F}$ as in Lemma 5.2.4. Suppose that $L = \lim_{n\to\mathcal{U}} f(r(n))$. This means that for each $\epsilon > 0$ we have

$$\{n : |f(r(n)) - L| < \epsilon\} \in \mathcal{U}.$$

Fix $\epsilon > 0$ and then we get

$$r(\{n : |f(r(n)) - L| < \epsilon\}) = \{r(n) : |f(r(n)) - L| < \epsilon\} \in \mathcal{F} \subseteq \mathcal{V}.$$

Since the map r is surjective we obtain that

$$\{n : |f(n) - L| < \epsilon\} \in \mathcal{F} \subseteq \mathcal{V}.$$

In particular, $L = \lim_{n\to\mathcal{V}} f(n)$. By Fact 4.3.4, the set of *non − principal* ultrafilters $\mathbb{Z}^*$ is a disjoint union of $X^{(\infty)}$ and sets $X_t^{(k)}$ where $k \geq 1$ and $t \in \mathbb{Z}$. Suppose that a *non − principal* ultrafilter $\mathcal{U}$ is in $X^{(\infty)}$. Since f is in $F_+^{(\lambda)}$ we have $\lim_{n\to\mathcal{U}} f(n) = 0$. Also, since $\lim_{n\to\mathcal{U}} f(r(n)) = \lim_{n\to\mathcal{V}} f(n)$ and by Lemma 5.2.6 the ultrafilter $\mathcal{V} \in X^{(\infty)}$ we obtain that $\lim_{n\to\mathcal{V}} f(n) = 0$. It

follows that $\lim_{n\to\mathcal{U}}(Sf)(n) = 0$.
Now, suppose that $\mathcal{U} \in X_t^{(k)}$, where k and t are any numbers such that $k \geq 1$ and $t \in \mathbb{Z}$. By Lemma 5.2.6 we have $\lim_{n\to\mathcal{U}} f(r(n)) = \lim_{n\to\mathcal{V}} f(n) = \lambda^{-k+1}f(t)$. Hence, we have $\lim_{n\to\mathcal{U}}(Sf)(n) = \lambda^{-k}f(t) - \lambda^{-1}\lambda^{-k+1}f(t) = 0$.
Therefore, for any non-principal ultrafilter $\mathcal{U}$ we have

$$\lim_{n\to\mathcal{U}}(Sf)(n) = \lim_{n\to\mathcal{U}} f(n) - \lambda^{-1}\lim_{n\to\mathcal{V}} f(n) = \lambda^{-k}f(t) - \lambda^{-1}\lambda^{-k+1}f(t) = 0,$$

which implies that

$$\lim_{n\to\infty}(Sf)(n) = 0,$$

i.e. $Sf \in c_0(\mathbb{N})$. □

Lemma 5.2.8. *Let S be the map introduced in Definition 5.2.3 then S is continuous and bounded below, so injective.*

Proof. According to the definition of the map S for each $f \in F_+^{(\lambda)}$ and $n \in \mathbb{N}$ we have:

$$|(Sf)(n)| = \left|f(n) - \lambda^{-1}f(r(n))\right| \leq \|f\| + \lambda^{-1}\,|\,f(r(n))| \leq \|f\| + \lambda^{-1}\|f\| = (1+\lambda^{-1})\|f\|\,.$$

Hence, S is bounded and so continuous.
We show that S is bounded from below. Fix $\epsilon > 0$ and find $n \in \mathbb{N}$ such that $|f(n)| \geq \|f\| - \epsilon$. Now, we obtain

$$|(Sf)(n)| = \left|f(n) - \lambda^{-1}f(r(n))\right| \geq \left|f(n) - \lambda^{-1}\|f\|\right| \geq |f(n)| - |\lambda|^{-1}\|f\|\,.$$

Hence

$$|(Sf)(n)| \geq \|f\| - \epsilon - |\lambda|^{-1}\|f\| \geq (1 - |\lambda|^{-1})\|f\| - \epsilon.$$

Since ϵ is arbitrary we get

$$\|Sf\| \geq (1 - |\lambda|^{-1})\|f\|\,.$$

The fact that S is bounded from below tells us also that S is an injective operator with closed range. □

Before we present the next lemma let us observe the following fact.

Fact 5.2.9. Write a given natural number m in the form $m = 2^k + l$, $(2^k \leq l)$. Then for $n \leq k$ and $2^n \leq l$ we have $2^n|(2^k+l) \Leftrightarrow 2^n|l$, which is equivalent to the statement that $2^n \nmid (2^k+l) \Leftrightarrow 2^n \nmid l$.

Let us now recall the definitions of the operators σ and τ introduced in the last chapter.

These operators for $x \in \ell_\infty(\mathbb{Z})$ and $n \in \mathbb{Z}$ were defined respectively by the formulas

$$\sigma(x)(n) = x(n-1) \qquad \textit{and} \qquad \tau(x)(n) = \begin{cases} x(n/2) & \text{n even;} \\ 0 & \text{n odd.} \end{cases}$$

Now we can state the following lemma.

Lemma 5.2.10. *For operators σ, τ and $n \in \mathbb{N}$ the element $f_n = \sigma^n \tau^n x_0$ belongs to $F_+^{(\lambda)}$. Also for this element and for the function S introduced in Definition 5.2.3 the following condition holds:*

$$S(f_n)(m) = \begin{cases} 1 & \textit{if } m = n, \\ 0 & \textit{if } m \neq n. \end{cases} \tag{5.2.5}$$

Proof. The fact that f_n is an element of $F_+^{(\lambda)}$ for $n \in \mathbb{N}$ is a consequence of Lemma 4.2.6 and the fact that $F_+^{(\lambda)}$ is a one-sided shift-invariant subspace of $F^{(\lambda)}$.

Now we check that for the element f_n the above condition is satisfied for any $n \in \mathbb{N}$. The proof of this will be divided into two parts a) and b). In part a) I show that equation (5.2.5) is satisfied for $n = 1$. This give us the idea of the proof for the general case when $n \in \mathbb{N}$ and this will be the content of part b).

a) Let $n = 1$. We need to show that

$$S(f_1)(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m > 1. \end{cases} \tag{5.2.6}$$

Let $m = 1$, then we have

$$S(f_1)(1) = S(\sigma\tau x_0)(1) = \sigma\tau x_0(1) - \lambda^{-1}\sigma\tau x_0(0) = \tau x_0(0) - \lambda^{-1}\tau x_0(-1) = 1 - 0 = 1.$$

Hence, we see that (5.2.6) is satisfied for $m = 1$. Now we check another case.

Let $m > 1$ and let m be odd, then writing m in the form $m = 2^p + r(m)$ we have

$$S(f_1)(m) = S(\sigma\tau x_0)(2^p + r(m)) = \sigma\tau x_0(2^p + r(m)) - \lambda^{-1}\sigma\tau x_0(r(m)),$$

which consequently gives

$$S(f_1)(m) = \tau x_0(2^p + r(m) - 1) - \lambda^{-1}\tau x_0(r(m) - 1).$$

Since $2^p + r(m) - 1$ is even by Fact 5.2.9 we obtain that $2|(r(m) - 1)$. As a result, continuing our calculations we have

$$S(f_1)(m) = x_0\left[\frac{2^p + r(m) - 1}{2}\right] - \lambda^{-1}x_0\left[\frac{r(m) - 1}{2}\right] = \lambda^{-b(2^{p-1} + \frac{r(m)-1}{2})} - \lambda^{-1}\lambda^{-b(\frac{r(m)-1}{2})}.$$

The fact that $2^p + r(m) - 1$ is even yields in particular that $b(2^p + r(m) - 1) = b(\frac{2^p + r(m) - 1}{2})$

and so we get $b(2^{p-1}+\frac{r(m)-1}{2}) = 1 + b(\frac{r(m)-1}{2})$. Hence we finally obtain

$$S(f_1)(m) = \lambda^{-b(2^{p-1}+\frac{r(m)-1}{2})} - \lambda^{-1}\lambda^{-b(\frac{r(m)-1}{2})} = \lambda^{-1}\lambda^{-b(\frac{r(m)-1}{2})} - \lambda^{-1}\lambda^{-b(\frac{r(m)-1}{2})} = 0.$$

Therefore condition (5.2.6) holds for odd natural numbers $n > 1$. Now we check the last case. Let $m > 1$ and let m be even, then writing m in the form $m = 2^p + r(m)$ we have

$$S(f_1)(m) = S(\sigma\tau x_0)(2^p + r(m)) = \sigma\tau x_0(2^p + r(m)) - \lambda^{-1}\sigma\tau x_0(r(m)).$$

Similarly like in the previous case, since $2^p + r(m) - 1$ is odd then by Fact 5.2.9, because $2 \nmid (2^p + r(m) - 1)$ then $2 \nmid (r(m) - 1)$ and so from Definition 4.1.3 we obtain

$$S(f_1)(m) = \tau x_0(2^p + r(m) - 1) - \lambda^{-1}\tau x_0(r(m) - 1) = 0 - 0 = 0.$$

Now we see that (5.2.6) is satisfied when $n = 1$. Let us then move on to the general case.
b) Let $n \in \mathbb{N}$ such that $n > 1$. We show that the element $f_n = \sigma^n\tau^n x_0$ satisfies condition (5.2.5). We first consider the case when $m = n$. As $\sigma^n\tau^n(x_0(n)) = \tau^n x_0(0) = 1$ and $\sigma^n(\tau^n(r(n)) = \tau^n(r(n) - n) = 0$, as $r(n) < n$, we obtain

$$(Sf_n)(n) = \sigma^n\tau^n x_0(n) - \lambda\sigma^n\tau^n x_0(r(n)) = 1 - 0 = 1.$$

Therefore, we see that condition (5.2.5) is satisfied when $m = n$. Let us consider another case. Let $m > n$ be such that $2^n|(m - n)$ and write m in the form $m = 2^p + r(m)$ then we have

$$S(f_n)(m) = \sigma^n\tau^n x_0(2^p+r(m))-\lambda^{-1}\sigma^n\tau^n x_0(r(m)) = \tau^n x_0(2^p+r(m)-n)-\lambda^{-1}\tau^n x_0(r(m)-n).$$

This gives us by Fact 5.2.9

$$S(f_n)(m) = x_0\left[\frac{2^p + r(m) - n}{2^n}\right] - \lambda^{-1}x_0\left[\frac{r(m) - n}{2^n}\right] = \lambda^{-b(2^{p-n}+\frac{r(m)-n}{2^n})} - \lambda^{-1}\lambda^{-b(\frac{r(m)-n}{2^n})}.$$

Notice that the binary expansion of 2^{p-n} consists of 1 on the $(p-n)-th$ position and 0 elsewhere. This implies, in according to operation on binary numbers that $b(2^{p-n}+\frac{r(m)-n}{2^n}) = 1+b(\frac{r(m)-n}{2^n})$. Hence, we obtain

$$S(f_n)(m) = \lambda^{-b(2^{p-n}+\frac{r(m)-n}{2^n})} - \lambda^{-1}\lambda^{-b(\frac{r(m)-n}{2^n})} = \lambda^{-1}\lambda^{-b(\frac{r(m)-n}{2^n})} - \lambda^{-1}\lambda^{-b(\frac{r(m)-n}{2^n})} = 0.$$

And so condition (5.2.5) holds for all $m > n$ such that $2^n|(m - n)$.
Let $m > n$ be such that $2^n \nmid (m - n)$ and let $m = 2^p + r(m)$ then likewise in the previous case we have

$$S(f_n)(m) = \sigma^n\tau^n x_0(2^p+r(m))-\lambda^{-1}\sigma^n\tau^n x_0(r(m)) = \tau^n x_0(2^p+r(m)-n)-\lambda^{-1}\tau^n x_0(r(m)-n).$$

By Fact 5.2.9, since $2^n \nmid (2^p + r(m) - n)$, we obtain that that $2^n \nmid (r(m) - n)$ and so we get

$$S(f_n)(m) = \tau^n x_0(2^p + r(m) - n) - \lambda^{-1}\tau^n x_0(r(m) - n) = 0 - 0 = 0.$$

Hence, we see that the condition $S(f_n)(m) = 0$ is also satisfied for $m > n$ such that $2^n \nmid (m-n)$. Now let us have a look on the case when $m < n$. In this case we obtain

$$S(f_n)(m) = \sigma^n\tau^n x_0(m) - \lambda^{-1}\sigma^n\tau^n x_0(r(m)) = \tau^n x_0(m - n) - \lambda^{-1}\tau^n x_0(r(m) - n).$$

We can observe that since for all negative arguments $n \in \mathbb{Z}$ the element $x_0(n) = 0$ and since the operator τ spreads out the elements of x_0 the value $S(f_n)(m) = 0$ for all $m < n$. Thus, we have proved the last case which finishes the proof of our lemma. $\square$

Now we sum up the discussion in this section in the following theorem.

Theorem 5.2.11. *Let S be the map introduced in Definition 5.2.3. Then S is a linear isomorphism between $F_+^{(\lambda)}$ and $c_0(\mathbb{N})$.*

Proof. Linearity of the map S is very straightforward. The fact that S maps into $c_0(\mathbb{Z})$ is the result of Lemma 5.2.7. On the other hand, by Lemma 5.2.8 we have that S is continuous and bounded from below. This implies that S is an isomorphism between $F_+^{(\lambda)}$ and its closed image $S(F_+^{(\lambda)}) \subseteq c_0(\mathbb{N})$. The remaining thing is to show that $S(F_+^{(\lambda)}) = c_0(\mathbb{N})$. Let $(e_i)_{i\in\mathbb{N}}$ be the canonical base for $c_0(\mathbb{N})$, hence, for $y = (a_i)_{i\in\mathbb{N}} \in c_0(\mathbb{N})$ we have $y = \sum_{i=1}^{\infty} a_i e_i$. By Lemma 5.2.10 for each $i \in \mathbb{N}$ there exists $f_i = a_i\sigma^i\tau^i(x_0) \in F_+^{(\lambda)}$ such that $S(f_i) = (..., 0, a_i, 0, ...)$ where a_i appears on the $i-th$ position. Hence, by linearity the image of S contains the dense subspace of finitely supported sequences. Since $S(F_+^{(\lambda)})$ is a closed we see that S is onto $c_0(\mathbb{N})$ and so describes isomorphism between $F_+^{(\lambda)}$ and $c_0(\mathbb{N})$. $\square$

5.3 Outlook

In the previous section we presented an explicit form of an isomorphism between the space $F_+^{(\lambda)}$ and $c_0(\mathbb{Z})$. The map S defined as

$$S : F_+^{(\lambda)} \ni f \longmapsto f - \lambda^{-1}(f \circ r) \in \ell_\infty(\mathbb{Z}), \tag{5.3.1}$$

where r is defined by Definition 5.2.1, proved to be that isomorphism. As the predual $F^{(\lambda)}$ is two-sided shift-invariant space, finding an isomorphism between $F^{(\lambda)}$ and $c_0(\mathbb{Z})$ proves to be much more difficult. Nevertheless, this work does show the potential for achieving this aim. It should rather be possible to use the explicit form of the presentation of $F^{(\lambda)}$ as a $G - space$ in the proof in [1] to shed light on an isomorphism. Analogical strategy, but applied to the simpler space $F_+^{(\lambda)}$, led to the isomorphism I constructed in this thesis. However, many of the spaces produced in the shift-invariant preduals papers are not presented as $G-spaces$, therefore despite the fact that the Szlenk index of most of these spaces is ω (in [7], considerable effort is expended to produce an example of a shift-invariant predual whose Szlenk index exceeds ω, and hence is not isomorphic to $c_0(\mathbb{Z})$) this doesn't help us determine the Banach-space isomorphism type of these preduals. New methods will be needed to understand these problems, and a better understanding of the isomorphisms between the original spaces $F^{(\lambda)}$ and $c_0(\mathbb{Z})$, will surely be of help.

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