The Outer Automorphism Groups of Three Classes of Groups

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Abstract The theory of outer automorphism groups allows us to better understand groups through their symmetries, and in this thesis we approach outer automorphism groups from two directions. In the first direction we start with a class of groups and then classify their outer automorphism groups. In the other direction we start with a broad class of groups, for example finitely generated groups, and for each group Q in this class we construct a group G_Q such that Q is related, in a suitable sense, to the outer automorphism group of G_Q.

We give a list of 14 groups which precisely classifies the outer automorphism groups of one-ended two-generator, one-relator groups with torsion. We also describe the outer automorphism groups of such groups which have more than one end. Combined with recent algorithmic results of Dahmani–Guirardel, this work yields an algorithm to compute the outer automorphism group of a two-generator, one-relator group with torsion.

We prove a technical theorem which, in a certain sense, writes down a specific subgroup of the outer automorphism group of a particular kind of HNN-extension. We apply this to prove two main results. These results demonstrate a universal property of triangle groups and are as follows. Fix an arbitrary hyperbolic triangle group H. If Q is a finitely generated group then there exists an HNN-extension G_Q of H such that Q embeds with finite index into the outer automorphism group of G_Q. Moreover, if Q is residually finite then G_Q can be taken to be residually finite. Secondly, fix an equilateral triangle group $H = \langle a, b; a^i, b^i, (ab)^i \rangle$ with $i > 9$ arbitrary. If Q is a countable group then there exists an HNN-extension G_Q of H such that Q is isomorphic to the outer automorphism group of G_Q. The proof of this second main result applies a theory of Wise underlying his recent work leading to the resolution of the virtually fibering and virtually Haken conjectures.

We prove a technical theorem which, in a certain sense, writes down a specific subgroup of the outer automorphism group of a semi-direct product $H \rtimes \mathbb{Z}$. We apply this to an open problem of Bumagin–Wise, which asks if every countable group can be realised as the outer automorphism group of a finitely generated, residually finite group. We resolve this question for finitely generated, recursively presented groups. Our resolution is dependent on a positive solution to a question of Osin, and Sapir has stated in a recent paper that he has an (unpublished) proof of a positive solution. If Osin's problem does not admit a positive solution then we obtain the following, slightly weaker result: for every finitely generated, recursively presented group Q there exists a finitely generated, residually finite group G_Q whose outer automorphism group is isomorphic to either Q or $Q \times C_2$.

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Contents

Chapter 1

Introduction

When given a group G it is a natural question to ask what its automorphism and outer automorphism groups are. Finding these groups is often both an interesting and challenging problem, and allows us to better understand the group G through its symmetries. For example, in topology certain surfaces have the property that their mapping class group is isomorphic to the outer automorphism group of the fundamental group of the surface, and so studying the outer automorphism group allows us to better understand the surface, and so its fundamental group. In this thesis we describe, in an appropriate sense, the outer automorphism groups of three classes of groups. The second two classes are related and each very broad, so we give some specific examples. These examples demonstrate that, in a suitable sense, every group can be realised as the outer automorphism group of a group from each of these two classes.

Outline of the thesis. In Chapter 2 we provide background and motivation for this thesis, and include preliminary results on which the later chapters rely. The purpose of Chapter 3 is to prove Theorem A, which completely classifies the outer automorphism groups of two-generator, one-relator groups with torsion. In Chapter 4 we analyse the outer automorphism groups of a certain class of HNN-extensions and a related class of semidirect products, and use this work to prove Theorems B, C and D, each of which constructs groups which all appear well behaved but can have pathological outer automorphism group.

The first class of groups. In Chapter 3 we investigate our first class of groups, which is the class of two-generator, one-relator groups with torsion. These groups are precisely those with a presentation of the form $\langle a, b; R^n\rangle$, $n > 1$. When attempting to find the outer automorphism group of a group it is often fruitful to find something, such as an action on a space, a subgroup or a group element, which is invariant under automorphisms (up to conjugacy) and work backwards to describe the automorphisms. We find two such invariants for two-generator, one-relator groups with torsion, and we use them separately to give two independent proofs of the following theorem.

Theorem (Theorem 3.1.16). *Let $G = \langle a, b; R^n \rangle$, $n > 1$, be a one-ended two-generator, one-relator group with torsion. Then either* $\mathrm{Out}(G)$ *is virtually-cyclic or* $G \cong \langle a, b; [a, b]^n \rangle$.

The restriction to one-ended groups is reasonable because if G has more than one end then $G \cong \mathbb{Z} * C_n$ and the automorphisms of such groups have been much studied [FR40, Gil87]. However, for completeness we state the isomorphism class of the outer automorphism group of $\mathbb{Z} * C_n$, and give a skeleton proof of this fact.

The first of the two invariants which help us to describe the automorphisms of a two-generator, one-relator group with torsion $\langle a, b; R^n \rangle$ is the root R of the relator R^n. More precisely, results of Magnus [MKS04, Theorem N5] and of Pride [Pri77a] imply that every outer automorphism has a natural representative which either freely fixes the word R or sends it freely to R^{-1}. In Section 3.1.3 we apply results of Touikan on equations in free groups [Tou09] to this observation, and so prove the above result, Theorem 3.1.16. A more subtle approach is to note that two-generator, one-relator groups with torsion are hyperbolic, and therefore they have a canonical decomposition as a graph of groups, called a JSJ-decomposition [Bow98]. Because this decomposition is canonical it is invariant under automorphisms. Work of Levitt gives a description of the outer automorphism group of a hyperbolic group using this invariant [Lev05]. Therefore, the purpose of Sections 3.1.1 and 3.1.2 is to prove a structural result regarding the possible JSJ-decomposition of a one-ended two-generator, one-relator group with torsion. This structural result is Theorem 3.1.15, and we apply it to prove the above result, Theorem 3.1.16. We favour the JSJ-decomposition proof over using the root R of the relator R^n as the invariant, and this is because our investigation of the structure of possible JSJ-decompositions in Sections 3.1.1 yields information about the virtually-cyclic splittings of arbitrary one-relator groups with torsion.

The above result, Theorem 3.1.16, is a vast improvement on previous results regarding the outer automorphism groups of two-generator, one-relator groups with torsion. For example, certain specific classes of these groups were known to have residually finite outer automorphism groups [KT09, KT10], but this theorem proves that those groups actually have virtually-cyclic outer automorphism groups or are isomorphic to $\langle a, b; [a, b]^n \rangle$. On the other hand, Theorem 3.1.16 parallels results of Kapovich–Weidmann [KW99], who prove that one-ended two-generator torsion-free hyperbolic groups have virtually-cyclic outer automorphism groups, and indeed the JSJ-decomposition proof of Section 3.1.2 closely follows the path of their proof. Proving that we can follow this path is the purpose of Section 3.1.1.

Now, Theorem 3.1.16 is not the complete classification of 14 groups we mentioned in the abstract. The classification is obtained by viewing how the automorphisms act on the abelianisation. We prove that if an automorphism acts trivially on the abelianisation then it is inner. This, along with a result of Pride, proves the following theorem.

Theorem (Theorem 3.2.1)**.** *If G is a one-ended two-generator, one-relator group with torsion then* $\mathrm{Out}(G)$ *embeds into* $\mathrm{GL}_2(\mathbb{Z})$*.*

We restate Theorem 3.2.1 later, where we explicitly give the embedding. Theorem 3.2.1 implies that if $\mathrm{Out}(G)$ is finite it is a subgroup of either the dihedral group of order twelve or the dihedral group of order eight. There are nine such subgroups, and these all appear in our classification. If G has virtually-$\mathbb{Z}$ outer automorphism group then we can apply a result of Levitt which allows us to write G as $\langle a, b; S^n(a, b^{-1}ab) \rangle$, and thus when we embed the outer automorphism group into $\mathrm{GL}_2(\mathbb{Z})$ the following matrix is in the image.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We prove that, up to isomorphism, there are only four virtually-cyclic subgroups of $\mathrm{GL}_2(\mathbb{Z})$ which contain this matrix. In Section 3.5 we investigate the case of infinitely-ended two-generator, one-relator groups with torsion. This all yields the following theorem.

Theorem A. *Let G be a two-generator, one-relator group with torsion.*

- *If $G \cong \langle a, b; [a, b]^n \rangle$ then* $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$*.*
- *If G is one-ended and $G \not\cong \langle a, b; [a, b]^n \rangle$ then,*
  - *If* $\mathrm{Out}(G)$ *is infinite then it is isomorphic to $D_\infty \times C_2$, D_∞, $\mathbb{Z} \times C_2$ or $\mathbb{Z}$.*
  - *If* $\mathrm{Out}(G)$ *is finite then it is isomorphic to a subgroup of D_6 or of D_4.*
- *If G is infinitely ended, so $G \cong \mathbb{Z} * C_n$, then* $\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$*.*

In Sections 3.3.2 and 3.4 we prove that each of the possibilities from Theorem A occurs.

The second and third class of groups. In Chapter 4 we investigate the outer automorphism groups of HNN-extensions $G = \langle H, t; k^t = k\phi, k \in K\rangle$ where the isomorphism ϕ is induced by an automorphism of the base group H. We call these *automorphism-induced HNN-extensions*. A result of the author with Ateş and Pride allows one to view such an HNN-extension as a Zappa-Szép product of the base group H with the free group on the cosets of H/K, $G \cong H \bowtie F(H/K)$ [ALP14]. If $H = K$ then this is simply the given semidirect product decomposition. We split these HNN-extensions into two cases, which are the second and third classes we investigate. The first case is when $K \lneq H$, so G is a proper HNN-extension, and the second case is when $K = H$, so G is a semi-direct product with $\mathbb{Z}$ (a "mapping torus"). These cases are different because in the semi-direct product case the inducing automorphism ϕ is always associated to an inner automorphism of G while in the former case it may be associated to a non-inner automorphism of G.

We approach these two cases by applying the same technical analysis of (outer) automorphisms, after which the proofs fork. Instead of focusing on the entire outer automorphism group we restrict ourselves to the subgroup $\mathrm{Out}_H(G)$ consisting of outer automorphisms having a representative which setwise fixes the base group H and which sends the stable letter t to an element of t-length one. That is, we artificially take H and the form of the image of t as the invariants and analyse the resulting subgroup of the outer automorphism group. In both cases we produce technical theorems classifying this subgroup of the outer automorphism group. We shall now state a rough form of the technical theorem for the $K \lneq H$ case and state two of our applications. We shall then state the technical theorem for the $K = H$ case and state our application.

Theorem (Theorem 4.2.15)**.** *Let $G = \langle H, t; k^t = k\phi, k \in K\rangle$ with $\phi \in \mathrm{Aut}(H)$ and $K \lneq H$, and let $\mathrm{Out}_H(G)$ be the outer automorphisms of G which fix H and the form of t. Then there exists a short exact sequence*

$$1 \to N \to \mathrm{Out}_H^0(G) \to A \to 1$$

where N is given in terms of subgroups of H, A is a subgroup of $\mathrm{Out}(H)$ and $\mathrm{Out}_H^0(G)$ is a subgroup of $\mathrm{Out}_H(G)$ of index one or two.

We restate Theorem 4.2.15 later, where we both give a more explicit description of the groups N and A and classify when $\mathrm{Out}_H^0(G)$ has index two in $\mathrm{Out}_H(G)$. Theorem 4.2.15

is particularly nice if K has trivial centraliser in G, as then N is isomorphic to $N_H(K)/K$. Our main applications of this theorem both involve fixing a group Q such that there exists $K \lneq H$ which has trivially centraliser and either $N_H(K)/K \cong Q$ or Q embeds with finite index into $N_H(K)/K$. Our first main application of the above theorem, Theorem 4.2.15, is the following result.

Theorem B. *Fix a triangle group $H := \langle a, b; a^i, b^i, (ab)^i \rangle$ with $i > 9$. Then every countable group Q can be realised as the outer automorphism group of an HNN-extension G_Q of H. Moreover,* $\mathrm{Aut}(G_Q) \cong H \rtimes Q$.

Our second main application of Theorem 4.2.15 is the following result, which complements Theorem B.

Theorem C. *Fix a hyperbolic triangle group $H = \langle a, b; a^i, b^j, (ab)^k \rangle$. Then every finitely-generated group Q can be embedded as a finite index subgroup of the outer automorphism group of an HNN-extension G_Q of H, where G_Q is residually finite if Q is.*

In both Theorem B and Theorem C the base group H is a triangle group and so has Serre's property FA, which, along with certain technical conditions on either ϕ or the embedding of K in H (for example, if ϕ is inner), implies that $\mathrm{Out}_H(G) = \mathrm{Out}(G)$. Moreover, the outer automorphism group of a triangle group is finite and so A must be finite. Therefore, if K is chosen to have trivial centraliser and ϕ is chosen to be trivial then $N_H(K)/K$ embeds with finite index into $\mathrm{Out}(G)$. In Theorem C we find for every finitely generated group Q a subgroup $K \leq H$ such that Q has finite index in $N_H(K)/K$, which proves the main statement of the theorem. To obtain the result regarding the residual finiteness of the HNN-extension G_Q, we ensure that K has the property that its normaliser $N_H(K)$ has finite index in H, and the result then follows from a technical theorem, which is Theorem 4.1.3.

The proof of Theorem B is substantial, but we give the general idea now. Take the triangle group $H = \langle a, b; a^i, b^i, (ab)^i \rangle$, $i > 9$, and take ϕ to be the automorphism $a \mapsto b$, $b \mapsto (ab)^{-1}$ of H. In Section 4.3.4 we introduce the notion of a "malcharacteristic" subgroup, which is a generalisation of malnormality, and if K is such that $N_H(K)$ is malcharacteristic and forming $G = \langle H, t; k^t = k\phi, k \in K \rangle$, it follows from the above technical theorem, Theorem 4.2.15, that $\mathrm{Out}(G) \cong N_H(K)/K$. Using rather technical tools, in Section 4.3.4 we prove that H contains a malcharacteristic subgroup which is free of rank two. We shall now explain why doing so proves Theorem B. Suppose Q is an arbitrary countable group, then Q has a presentation $\langle X; \mathbf{r}\rangle$ with $|\mathbf{r}| \geq 1$. Because M is free it contains malnormal subgroups of arbitrary rank, and so specifically of rank $|X|$. We denote such a malnormal subgroup by $M_{|X|}$. The definition of malcharacteristic implies that $M_{|X|}$ is malcharacteristic in H, and so taking K to correspond to the normal closure of $\mathbf{r}$ in $M_{|X|}$ we obtain the group $G_Q = \langle H, t; k^t = k\phi, k \in K\rangle$, and we see that $\mathrm{Out}(G_Q) \cong Q$ as required. Our proof that such equilateral triangle groups have a malcharacteristic subgroup which is free of rank two utilises fibre products of maps of graphs and a small cancellation theory in this setting, which was developed by Wise. This theory of Wise (and its generalisation) is a cornerstone of his recent work which lead to the resolution of Thurston's virtually fibering and Waldhausen's virtually Haken conjectures, as well as his resolution of G. Baumslag's conjecture that all one-relator groups with torsion are residually finite [Wis12].

We shall now state the technical result for the second case, when $G \cong H \rtimes \mathbb{Z}$, and give an application. We shall write $\widehat{\phi}$ for the class of ϕ in $\mathrm{Out}(H)$.

Theorem (Theorem 4.2.17)**.** *Let $G = \langle H, t; h^t = h\phi, h \in H\rangle$ with $\phi \in \mathrm{Aut}(H)$, and let $\mathrm{Out}_H(G)$ be the outer automorphisms of G which fix H. Then we have the following isomorphism, where $\mathrm{Out}^0_H(G)$ has index one or two in $\mathrm{Out}_H(G)$, and has index two precisely when $\widehat{\phi}$ is conjugate to $\widehat{\phi}^{-1}$.*

$$\mathrm{Out}^0_H(G) \cong C_{\mathrm{Out}(H)}(\widehat{\phi})/\langle\widehat{\phi}\rangle$$

Bumagin–Wise have asked whether every countable group can be realised as the outer automorphism group of a finitely generated, residually finite group [BW05]. We apply Theorem D to prove the following theorem, which partially resolves Bumagin–Wise question.

Theorem D. *Every finitely generated, recursively presented group can be realised as the outer automorphism group of a finitely generated, residually finite group.*

To prove Theorem D, if Q is a finitely generated, recursively presented group we use an embedding of Sapir [Sap13] to realise $Q \times C_3$ as a malnormal subgroup of a finitely presented group P. We then use a result of Bumagin–Wise [BW05] to realise P as the outer automorphism group of a finitely presented, residually finite group H. Then taking $\widehat{\phi}$ to correspond to a generator for the cyclic group we have attached to Q, we have that $C_{\mathrm{Out}(H)}(\widehat{\phi})/\langle\widehat{\phi}\rangle \cong Q$. Finally, taking $\phi \in \widehat{\phi}$, we form $G = H \rtimes_\phi \mathbb{Z}$, and certain properties of Bumagin–Wise's result are such that $\mathrm{Out}^0_H(G) = \mathrm{Out}(G)$, and so $\mathrm{Out}(G) \cong Q$.

Chapter 2

Preliminaries

In this chapter we give an overview of a variety of relevant theories and include results from the literature which we apply in subsequent chapters. We begin, in Section 2.1, by defining automorphism and outer automorphism groups. This is in order to motivate the results of Chapter 4, where in Theorem B we prove that every group can be realised as the outer automorphism group of some group, by giving an example of a group which cannot be realised as the automorphism group of a group.

While most of this chapter will be familiar to many readers, note that Section 2.2 and Section 2.10 introduce non-standard concepts which are used in the Chapters 3 and 4 respectively. Section 2.2 introduces the theory of Nielsen equivalence classes, upon which our approach in Chapter 3 is based, while Section 2.10 introduces the theory of maps of graphs, which is a novel way of viewing subgroups of free groups and which we apply in the proof of Theorem B from Chapter 4.

We begin here with a fundamental definition: If a group G has a presentation of the form $\langle X; R^n \rangle$ with $n > 1$ and X contains at least two elements then we say that G is an *one-relator group with torsion.* If X consists of precisely two-elements then G is a *two-generator, one-relator group with torsion.* The reader is referred to Section 2.9 for more information on one-relator groups and a discussion on the "with torsion" label. Note that because X contains at least two elements, a one-relator group is always infinite.

Throughout the thesis ϵ (or some variation such as ϵ', ϵ_0, ϵ_t) will denote an integer of absolute value 1.

All of the results in this chapter can be found in the literature or can be easily derived from known results. The only exception to this is Lemma 2.9.7, which is an original result of the author.

2.1 Automorphism and Outer Automorphism Groups

This thesis deals with outer automorphism groups of certain classes of groups. Therefore, in this section we recall the definition of automorphism groups and outer automorphism groups. We prove that not every (countable) group occurs as the automorphism group of some group, which contrasts with Theorem B from Chapter 4 where we prove that every countable group can be realised as the outer automorphism group of a specific kind of group.

Automorphisms. An *endomorphism* of a group G is a homomorphism from G to itself, $\phi : G \to G$. An *automorphism* of a group G is an endomorphism which is both injective and surjective. We shall write down an endomorphism by prescribing where the generators are sent to. This is sufficient, as endomorphisms are homomorphisms. The automorphisms of a group G form a group in their own right, called the *automorphism group of* G, denoted $\mathrm{Aut}(G)$.

In the following example we write C_2 for the cyclic group of order two. We maintain this notation throughout this thesis, writing C_n for the cyclic group of order n. At different points we shall write C_∞ and $\mathbb{Z}$ for the infinite cyclic group.

Example 2.1.1*:* An easy example is the automorphism group of the non-cyclic group of order four, $G = C_2 \times C_2$. Here, all permutations of the three non-trivial elements preserve the group structure, while every pair of non-trivial elements generates G. Therefore, surjective endomorphisms correspond to permutations of the non-trivial elements. As G is finite, surjective endomorphisms are automorphisms. Thus, $\mathrm{Aut}(G) \cong S_3$.

In this thesis we shall write group homomorphisms on the right, so for $g \in G$ and $\phi \in \mathrm{Aut}(G)$ we write $g\phi$ as opposed to $\phi(g)$. We do this because it makes the working in Chapter 4 clearer. More generally, we shall write arbitrary actions on the right.

Outer automorphism groups. A group acts on itself by conjugation. As we are writing actions on the right, the conjugation action is written $h \cdot g = g^{-1}hg$. The action of an element $g \in G$ corresponds to an automorphism of G, denoted γ_g. This automorphism conjugates every element by g, so $\gamma_g : h \mapsto g^{-1}hg$ for all $h \in G$. Such an automorphism is called an *inner automorphism* of G. Inner automorphisms form a normal subgroup of $\mathrm{Aut}(G)$, denoted $\mathrm{Inn}(G)$, which allows us to form the quotient group. The quotient group is called the *outer automorphism group of* G, denoted $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$. If $\phi \in \mathrm{Aut}(G)$ then we shall write both $\widehat{\phi}$ and $\phi\,\mathrm{Inn}(G)$ for the coset of $\mathrm{Out}(G)$ containing ϕ. This thesis deals with the outer automorphism groups of certain classes of groups.

Theorem B is one of the main results of Chapter 4, and it proves that every countable group can be realised as the outer automorphism group of an HNN-extension of a fixed triangle group. This result is proven in Section 4.3.4. The following lemma on automorphism groups contrasts with Theorem B.

Lemma 2.1.2. *The infinite cyclic group C_∞ cannot occur as the automorphism group of any group.*

Proof. To begin, we observe that $\mathrm{Inn}(G) \cong G/Z(G)$. This is because the automorphism γ_g is trivial if and only if $g \in Z(G)$.

Now, suppose that there exists some group G such that $\mathrm{Aut}(G) \cong C_\infty$. We shall look for a contradiction. Note that every non-trivial subgroup of C_∞ is infinite cyclic, and so $\mathrm{Inn}(G)$ is either infinite cyclic or trivial. Thus, $G/Z(G)$ is either infinite cyclic or trivial. However, $G/Z(G)$ can never be cyclic. Thus, G must be abelian.

As G is an abelian group, the map ϕ which inverts every element of G, so $\phi : h \mapsto h^{-1}$ for all $h \in H$, is an automorphism of G. This map is either trivial or has order two, and as $\mathrm{Aut}(G)$ contains no elements of order two we have that ϕ is trivial. Thus, $h = h^{-1}$ for all $h \in G$, so every element of G has order two. Therefore, G is a direct sum of some groups of order two. However, switching two of the groups in this decomposition is an automorphism which has order two. This is a contradiction, and so the lemma is complete. □

2.2 Nielsen Equivalence Classes

The related notions of "Nielsen equivalence" of generating tuples and of "tame automorphisms" are used in Chapter 3, and so we introduce them here. These notions are fundamental to Chapter 3, where we investigate the outer automorphism groups of two-generator, one-relator groups with torsion. The notions are so fundamental because of Proposition 2.2.2, which states that a (one-ended) two generator, one-relator group with torsion contains a single Nielsen equivalence class of generating pairs, which implies that every automorphism is tame. Chapter 3 contains two important technical theorems, both of which rely on Nielsen equivalence and tame automorphisms. If $G = \langle a, b; R^n \rangle$ is a one-ended two-generator, one-relator group with torsion then the second technical result is

Theorem 3.2.1 from Section 3.2, where we use tame automorphisms and Proposition 2.2.2 to prove that $\mathrm{Out}(G)$ embeds into $\mathrm{GL}_2(\mathbb{Z})$. One of the purposes of this current section is to explain our method of proof used in Theorem 3.2.1. The first technical theorem is Theorem 3.1.16, which says that either $\mathrm{Out}(G)$ is virtually cyclic or $G \cong \langle a, b; [a, b]^n \rangle$. We prove this theorem twice. Our first proof is spread over Sections 3.1.1 and 3.1.2 and is based on JSJ-decompositions, while the second proof is confined to Section 3.1.3 where we give a skeleton proof using equations in free groups.

We favour the JSJ-decomposition proof of Sections 3.1.1 and 3.1.2 as it allows for a more rigorous analysis of the structure of one-relator groups with torsion. However, in this proof much of the Nielsen theory is hidden from view, appearing in statements and proofs of results which we apply. For example, the proofs of Proposition 2.4.2, Proposition 2.4.6 and Proposition 2.9.12 all use Nielsen equivalence classes, and are all applied in Sections 3.1.1 and 3.1.2. The equations in free groups approach of Section 3.1.3 uses Nielsen equivalence classes in a more explicit way. This is because if $G = \langle a, b; R^n \rangle$ is one-ended then every automorphism $a \mapsto A$, $b \mapsto B$ of G must be a Nielsen transformation which solves one of the equations $R(a, b) \equiv R(A, B)$ or $R(a, b) \equiv R(A, B)^{-1}$, by Proposition 3.1.17. However, this second proof is heavily reliant on a paper of Touikan which "solves" equations in free groups which have this form [Tou09], and this approach yields less information about the structure of G than the JSJ-decomposition proof. Section 3.1.3 is therefore included not to give a rigorous proof of Theorem 3.1.16, but rather to demonstrate how Nielsen's theory can be used in the context of two-generator groups to prove results about outer automorphism groups.

Note that the results in this section are only applied in Chapter 3, where the objects of study are two-generator, one-relator groups with torsion. The fact that the groups are two-generated is important. The methods of Chapter 3 do not lift to more generators (although the discussion after Theorem 2.2.6, below, points out that they can be applied to groups with more relators). The reason for the methods' restrictive nature is that they are so reliant on Nielsen's theory, a theory which works particularly well for two-generated groups. We now give the relevant obstacles to three-or-more generation.

For our first technical theorem, Theorem 3.1.16, recall that we give two proofs. For the first proof, from Sections 3.1.1 and 3.1.2, Proposition 2.4.2 gives us information about the generators of HNN-extensions while Proposition 2.4.6 gives us information about the generators of free products with amalgamation and our applications of both of these propositions are reliant on two-generation (and both use Nielsen's theory). For the second proof, from Section 3.1.3, the most obvious obstacle is that we combine Propositions 2.2.2 and Proposition 3.1.17 using Nielsen's theory. They combine to tell us that in a two-generator, one-relator group with torsion every automorphism $a \mapsto A$, $b \mapsto B$ must be a solution to one of the equations $R(a, b) = R(A, B)$ or $R(a, b) = R(A, B)^{-1}$ and the proof only holds in the case of two generators.

For the second technical theorem, we apply Proposition 2.2.2 which gives us information about the Nielsen equivalence classes of two-generator, one-relator groups with torsion. This result is not applicable when there are more generators. Indeed, in Section 3.2 we combine Proposition 2.2.2 with Proposition 2.2.1, which is a classical result of Nielsen saying that $\mathrm{Out}(F_2) = \mathrm{Out}(F_2/F_2')$, but this result of Nielsen only works in the two-generator case. This means that our method of proof for our second technical theorem, Theorem 3.2.1, does not lift to groups with more generators.

Nielsen transformations. Let $X = (x_1, \ldots, x_n)$ be a sequence of letters. An *elementary Nielsen transformation* is one of the following maps. All other x-terms are fixed.

- Swap x_i with x_j, $i \neq j$.
- Replace x_i with x_i^{-1}.
- Replace x_i with $x_i x_j$, $i \neq j$.

A *Nielsen transformation* is a map which is the composition of some elementary Nielsen transformations. Nielsen proved that elementary Nielsen transformations generate the automorphism group of $F(x_1, \ldots, x_n)$, and so a map is an automorphism of $F(x_1, \ldots, x_n)$ if and only if it is a Nielsen transformation of $(x_1, \ldots, x_n)$ [MKS04, Theorem 3.2]. A *primitive element* of $F(x_1, \ldots, x_n)$ is an element which is the image of x_1 under some Nielsen transformation (equivalently, this is an element which is contained in some basis for the free group).

We shall introduce some notation used in this section and in the rest of the thesis. Suppose $G = \langle X; \mathbf{r} \rangle$, $\mathbf{r} \subseteq F(X)$, and let U, V, W be words in $F(X)$, the free group on X. We shall write $U \equiv V$ if U and V represent the same word of $F(X)$. If U and V define the same element of G then it will be said that U *is equal to* V *in* G, written $U =_G V$, or simply $U = V$ if the group G is understood. For a generator $c \in X^{\pm 1}$ of G, an *exponent of* c *in* W is an integer e such that $U \equiv V c^e W$ where neither the last symbol of V nor the first symbol of W are c or c^{-1}. We shall denote the sum of the exponents by $\sigma_c(W)$. If U is a freely reduced word then the sum of the absolute values of the exponents of all the generators of G in U is the length of U, and is denoted $|U|$.

The Nielsen transformations of the free group on two generators have a well-defined structure. This structure can be seen through the equivalence of $\mathrm{Out}(F(a,b))$ with $\mathrm{GL}_2(\mathbb{Z})$, which is given in the following propositions. This is a variant of a classic result of Nielsen [MKS04, Corollary $N4$], and is used in Section 3.2 to prove that the outer automorphism groups of one-ended two-generator, one-relator groups with torsion embed into $\mathrm{GL}_2(\mathbb{Z})$.

Proposition 2.2.1. *(Nielsen, 1924) Taking $\phi : a \mapsto A,\ b \mapsto B$ to be an arbitrary Nielsen transformation of $F(a,b)$, then the following map is an epimorphism.*

$$\begin{aligned} \xi : \mathrm{Aut}(F(a,b)) &\to \mathrm{GL}(2,\mathbb{Z}) \\ \phi &\mapsto \begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ \sigma_a(B) & \sigma_b(B) \end{pmatrix} \end{aligned}$$

Moreover, $\ker(\xi) = \mathrm{Inn}(F(a,b))$.

Nielsen equivalence. Two generating n-tuples $Y = (y_1,\ldots,y_n)$ and $Z = (z_1,\ldots,z_n)$ of a group $G = \langle x_1,\ldots,x_n; \mathbf{r}\rangle$ are *Nielsen equivalent* if there exists some Nielsen transformation ϕ of $(x_1,\ldots,x_n)$ such that if $x_i\phi = w_i(x_1,\ldots,x_n)$ then the following holds.

$$(w_1(Y),\ldots,w_n(Y)) =_G (z_1,\ldots,z_n)$$

The equivalence classes of this equivalence relation are called *Nielsen equivalence classes (of generating n-tuples)*.

The following proposition classifies the Nielsen equivalence classes of generating pairs in a two-generator, one-relator group with torsion. It was proven by Pride in 1977, who used it to solve the isomorphism problem for these groups [Pri77a]. It is fundamental to Chapter 3 where we apply it to investigate the outer automorphism groups of such groups.[1]

[1] It should be noted that a footnote in Pride's paper makes reference to a (at the time) forthcoming joint paper of McCool and Pride [Pri77a] which would have proven that the automorphism groups of two-generator, one-relator groups with torsion are finitely generated. However, according to Professor Pride (who is the author's PhD supervisor) this paper was never written.

Proposition 2.2.2 (Pride, 1977)**.** *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and R is not a proper power. Suppose that R is not a primitive element of $F(a, b)$, or R is a primitive element and $n = 2$. Then G has only one Nielsen Equivalence Class of generating pairs. Suppose, on the other hand, that R is primitive and $n > 2$. Then G has $\frac{1}{2}\varphi(n)$ Nielsen Equivalence Classes (where φ is the Euler totient function).*

Note that the "with torsion" is necessary. To demonstrate this we give the following example.

Example 2.2.3*:* Consider the following group.

$$G = \langle a, b; a^3ba^{-1}b^{-2}a^{-1}b \rangle$$

The map $(a, b) \mapsto (b^{-1}a^{-1}ba, a^{-1}b^{-1}a^{-2}b^{-1}ab)$ defines an automorphism and so is invertible [Rap59]. However, Rapaport proved that the pairs are not Nielsen equivalent [Rap59].

T**-systems.** T-systems shall allow us to connect Nielsen transformations and tame automorphisms. They partition the set of generating n-tuples in a coarser way than Nielsen equivalence classes. Two n-tuples $Y = (y_1, \ldots, y_n)$ and $Z = (z_1, \ldots, z_n)$ of G lie in the same *T-system* if there exists some automorphism ψ of G such that $(y_1\psi, \ldots, y_n\psi)$ is Nielsen equivalent to $(z_1, \ldots, z_n)$. Proposition 2.2.2 implies that a one-ended two-generator, one-relator group with torsion has a single T-system. Again, the "with torsion" label is necessary, as the following example demonstrates.

Example 2.2.4*:* Consider the following group.

$$\mathrm{BS}(2, 3) = \langle a, t; t^{-1}a^2t = a^3 \rangle$$

The map $(a, t) \mapsto (a^2, t)$ defines a surjective endomorphism but it is not invertible [BS62]. Brunner proved that the pairs lie in different T-systems [Bru74].

Tame automorphisms. Let $G = \langle x_1, \ldots, x_n; \mathbf{r} \rangle$, then the following set is the group of *tame automorphisms* of G corresponding to the generating tuple $(x_1, \ldots, x_n)$.

$$\begin{aligned} \mathrm{Tame}_{(x_1,\ldots x_n)}(G) := \{\phi : \phi \in \mathrm{Aut}(G), \\ (x_1\phi, \ldots, x_n\phi) \text{ is Nielsen equivalent to } (x_1, \ldots, x_n)\} \end{aligned}$$

Tame automorphisms are sometimes called *free automorphisms* [MKS04] or *lifting automorphisms* [GS95]. Note that Proposition 2.2.2 implies that if a group is given by a presentation $G = \langle a, b; R^n \rangle$ where R is non-primitive and $n > 1$ then $\mathrm{Tame}_{(a,b)}(G)$ is the whole automorphism group, while for the group $G = \langle a, b; a^3ba^{-1}b^{-2}a^{-1}b \rangle$ from Example 2.2.3 we have that $\mathrm{Tame}_{(a,b)}(G)$ is a proper subgroup of the automorphism group.

If X is a generating n-tuple, the following lemma allows us to connect the number of Nielsen equivalence classes of n-tuples in the T-system of X to the index of $\mathrm{Tame}_X(G)$ in $\mathrm{Aut}(G)$. This lemma is applied to groups of the form $\langle a, b; b^n \rangle$, $n > 1$, in Section 3.5, where we pin together our knowledge of the tame automorphisms with our knowledge of the Nielsen equivalence classes of these groups (which we get from Proposition 2.2.2). This "pinning together" yields a complete description of $\mathrm{Out}(\mathbb{Z} * C_n)$ in terms of n, which is Theorem 3.5.1. An appropriate interpretation of Lemma 2.2.5 is given immediately below the proof.

Lemma 2.2.5. *Let G be an arbitrary, finitely generated group defined by the presentation $\langle x_1, \ldots, x_n; \mathbf{r} \rangle$. Then, for every Nielsen equivalence class $\mathcal{C}$ of G in the T-system of $(x_1, \ldots, x_n)$ there exists some $\psi_c \in \mathrm{Aut}(G)$ with $(x_1\psi_c, \ldots, x_n\psi_c) \in \mathcal{C}$ such that if $\tau \in \mathrm{Aut}(G)$ and $(x_1\tau, \ldots, x_n\tau) \in \mathcal{C}$ then there exists some $\phi \in \mathrm{Tame}_{(x_1,\ldots,x_n)}(G)$ with $\tau = \phi\psi_c$.*

Proof. By the definitions of T-systems and Nielsen equivalence classes, if $\mathcal{C}$ is a Nielsen equivalence class lying in the same T-system of G as $(x_1, \ldots, x_n)$ then there exists some automorphism of G which maps $(x_1, \ldots, x_n)$ into $\mathcal{C}$. So, we can pick some $\psi_c \in \mathrm{Aut}(G)$ arbitrarily, with $\psi_c : x_1 \mapsto y_1, \ldots, x_n \mapsto y_n$, and $(y_1, \ldots, y_n) \in \mathcal{C}$.

Now, let τ be an automorphism of G which takes $(x_1, \ldots, x_n)$ to $(z_1, \ldots, z_n)$ where $(z_1, \ldots, z_n) \in \mathcal{C}$, so $\tau : x_1 \mapsto z_1, \ldots, x_n \mapsto z_n$. Then by the definition of Nielsen equivalence class there exists a Nielsen transformation ϕ of $(x_1, \ldots, x_n)$ with $x_i\phi = w_i(x_1, \ldots, x_n)$, where the following equalities hold.

$$(y_1, \ldots, y_n)\phi = (w_1(y_1, \ldots, y_n), \ldots, w_n(y_1, \ldots, y_n)) =_G (z_1, \ldots, z_n)$$

Then we have the following sequence of equalities.

$$\begin{aligned}
x_i\tau &= z_i \\
&=_G w_i(y_1, \ldots, y_n) \\
&= w_i(x_1\psi_c, \ldots, x_n\psi_c) \\
&= w_i(x_1, \ldots, x_n)\psi_c \\
&= x_i\phi\psi_c
\end{aligned}$$

Noting that ϕ is an automorphism because $\phi = \tau\psi_c^{-1}$ we conclude that $\phi \in \mathrm{Tame}_{(x_1,\ldots,x_n)}(G)$, as required. $\square$

This lemma tells us that if we fix an automorphism ψ_c for every Nielsen equivalence class then, taking $X = \{x_1, \ldots, x_n\}$, they form a set of left coset representatives for the cosets of $\mathrm{Tame}_X(G)$ in $\mathrm{Aut}(G)$. This set is a left transversal, by the definition of $\mathrm{Tame}_X(G)$. Therefore, writing m for the number of Nielsen equivalence classes in the T-system of X, we have the equality $|\mathrm{Aut}(G) : \mathrm{Tame}_X(G)| = m$. This proves the following theorem.

Theorem 2.2.6. *If $G = \langle x_1, \ldots, x_n; \mathbf{r}\rangle$ and there are only finitely many Nielsen equivalence classes in the T-system of $(x_1, \ldots, x_n)$ then $\mathrm{Aut}(G)$ is virtually $\mathrm{Tame}_{(x_1,\ldots,x_n)}(G)$ and $\mathrm{Out}(G)$ is virtually $\mathrm{Tame}_{(x_1,\ldots,x_n)}(G)/\mathrm{Inn}(G)$.*

Theorem 2.2.6 is of interest because it is applicable to two-generated presentations with small-enough cancellation, in the sense of Section 2.3 [HPV84]. Now, the ideas underlying Section 3.2 easily generalise to prove that if $G = \langle a, b\rangle$ and has infinite abelianisation then $\mathrm{Tame}_{(a,b)}(G)/\mathrm{Inn}(G)$ is either virtually cyclic or embeds into $\mathrm{GL}_2(\mathbb{Z})$. Therefore, two-generated groups with small-enough cancellation and infinite abelianisation have linear outer automorphism groups.

We now discuss a difficulty when talking about tame automorphisms. This difficulty is that we must necessarily specify the generating tuple we are dealing with when discussing tame automorphisms, because the set of tame automorphisms can change when we change the generating tuple. This is demonstrated by the following proposition.

Proposition 2.2.7 (Theorem 3.10, [MKS04])**.** *Let $G = \langle x_1, \ldots, x_n; \mathbf{r}\rangle$ be a group which has only finitely many Nielsen equivalence classes in the T-system of $(x_1, \ldots, x_n)$. Then there exists a finite generating set $(y_1, \ldots, y_m)$ for G such that there is only one Nielsen equivalence class in the T-system of $(y_1, \ldots, y_m)$. Moreover, if $\mathbf{r}$ is finite then there exists a finite set $\mathbf{s} \subset F(y_1, \ldots, y_m)$ such that $\langle y_1, \ldots, y_m; \mathbf{s}\rangle$ is a finite presentation for G.*

Proof. Let $G = \langle x_1, \ldots, x_n; \mathbf{r}\rangle$ be a finitely generated group, and write $X = \{x_1, \ldots, x_n\}$. Let $\psi \in \mathrm{Aut}(G)$ where $(x_1\psi, \ldots, x_n\psi)$ is not Nielsen equivalence to $(x_1, \ldots, x_n)$. We shall begin by giving a presentation for G such that ψ is tame. Now, if $W_i(X) =_G x_i\psi$ then the $2n$-generated group with the following presentation is isomorphic to G and is finitely related if $\mathbf{r}$ is a finite set.

$$\langle x_1, \ldots, x_n, z_1, \ldots, z_n; \mathbf{r}, z_1^{-1}W_1(X), \ldots, z_n^{-1}W_n(X)\rangle$$

We shall now prove that we have the following Nielsen equivalence.

$$(x_1\psi, \ldots, x_n\psi, z_1\psi, \ldots, z_n\psi) \sim_{N.e.} (x_1, \ldots, x_n, z_1, \ldots, z_n)$$

To see this, write $Z = \{z_1, \ldots, z_n\}$ and consider the mapping $x_i \mapsto z_i$, $z_i \mapsto x_i V_i^{-1}(Z) W_i(Z)$ where $V_i(X)$ is a word on the X such that $V_i(X)\psi = x_i$. This is a Nielsen transformation, because $V_i^{-1}(Z)W_i(Z)$ are words over Z. Moreover, this defines ψ as $x_i\psi = z_i$ while noting that $x_i = V_i(X)\psi = V_i(Z)$ we have the following sequence of equalities.

$$\begin{aligned} x_i V_i^{-1}(Z) W_i(Z) &= x_i x_i^{-1} W_i(Z) \\ &= W_i(Z) \\ &= W_i(X\psi) \\ &= W_i(X)\psi \\ &= z_i\psi \end{aligned}$$

Therefore, the two generating tuples are Nielsen equivalent, so in the altered presentation ψ is tame. We prove that $|\operatorname{Aut}(G) : \mathrm{Tame}_{(x_1,\ldots,x_n)}| > |\operatorname{Aut}(G) : \mathrm{Tame}_{(x_1,\ldots,x_n,z_1,\ldots,z_n)}|$. This means that we can algorithmically re-write the presentation as above and at every step reduce the index of the Tame automorphisms, and so the number of Nielsen equivalence classes. This proves the result as there are only finitely many Nielsen equivalence classes in the T-system of $(x_1, \ldots, x_n)$.

To prove that the index decreases, we prove that if ϕ is a Nielsen transformation of $(x_1, \ldots, x_n)$ which defines an automorphism of G then there is a Nielsen transformation ϕ' of the $2n$-tuple $(x_1, \ldots, x_n, z_1, \ldots, z_n)$ which defines an automorphism of G and with the following equalities.

$$x_1\phi' = x_1\phi, \ldots, x_n\phi' = x_n\phi$$

So, let ϕ be a Nielsen transformation of $(x_1, \ldots, x_n)$ which defines an automorphism of G, and let $U_i(X) =_G x_i\phi$. Then take ϕ' to be the following Nielsen transformation.

$$\begin{aligned} x_i &\mapsto U_i(X) \\ z_i &\mapsto z_i W_i^{-1}(X) W_i(U_1, \ldots, U_n) \end{aligned}$$

This Nielsen transformation is an automorphism as recalling that $z_i = W_i(X)$ we have the following sequence of equalities.

$$\begin{aligned} z_i W_i^{-1}(X) W_i(U_1, \ldots, U_n) &=_G W_i(U_1, \ldots, U_n) \\ &=_G W_i(X)\phi \end{aligned}$$

Noting that ϕ' and ϕ define the same automorphism of G, the proof is complete. $\square$

Example 2.2.8*:* Consider the group $G = \langle a_1, a_2; a_1^{12} \rangle$. This has $\varphi(12) = 4$ Nielsen equivalence classes by Proposition 2.2.2. However, the following presentation for G has only one Nielsen equivalence class.

$$\begin{aligned} G \cong \langle & a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8; \\ & a_1^{12} = 1, \\ & b_1 = a_1^5, b_2 = a_2, \\ & c_1 = a_1^7, c_2 = a_2, c_3 = b_1^7, c_4 = b_2, \\ & d_1 = a_1^{11}, d_2 = a_2, d_3 = b_1^{11}, d_4 = b_2, d_5 = c_1^{11}, d_6 = c_2, d_7 = c_3^{11}, d_8 = c_4 \rangle \end{aligned}$$

Therefore, $\mathrm{Tame}_{(a_1,\dots,d_8)}(G) = \mathrm{Aut}(G)$.

The method of proof from Section 3.2. In Section 3.2 we prove that if G is a one-ended two-generator, one-relator group with torsion then $\mathrm{Out}(G)$ embeds into $\mathrm{GL}_2(\mathbb{Z})$. We now explain the method of proof we employ to do this. Recall that tame automorphisms are sometimes referred to as lifting automorphisms. The reason for this name is because, taking $X = (x_1, \dots, x_n)$, if we define H_X to be the subgroup of $\mathrm{Aut}(F(X))$ which define automorphisms of G then there exists a surjection from H_X to $\mathrm{Tame}_X(G)$. This means we have Figure 2.1. Now, $\mathrm{Inn}(F(X)) \leq H_X$ and elements of $\mathrm{Inn}(F(X))$ induce inner automorphisms of G. In Section 3.2 we prove that for G given by a presentation of the form $\langle a, b; R^n \rangle$ with R non-primitive and $n > 1$, any two (tame) automorphisms of G are equal mod $\mathrm{Inn}(G)$ if and only if their lifts are equal mod $\mathrm{Inn}(F(X))$. That is, we prove that $\ker(\theta)$ is trivial, where θ is the map from Figure 2.2.

$$\begin{tikzcd} \mathrm{Aut}(F(X)) & \\ H_X \arrow[u, hook] \arrow[r, two heads] & \mathrm{Tame}_X(G) \end{tikzcd}$$

Figure 2.1: Tame automorphisms are precisely those which lift to automorphisms of the ambient free group.

$$\begin{tikzcd} \mathrm{Out}(F(X)) & \\ \frac{H_X}{\mathrm{Inn}(F(X))} \arrow[u, hook] \arrow[r, hook, two heads, "\theta"] & \frac{\mathrm{Tame}_X(G)}{\mathrm{Inn}(G)} \arrow[ul, hook, dotted] \end{tikzcd}$$

Figure 2.2: If the canonical map θ is injective then $\mathrm{Tame}_X(G)/\mathrm{Inn}(G)$ *embeds into* $\mathrm{Out}(F(X))$.

Note that the homomorphism θ, and hence the associated embedding, is canonical in the sense that it is pairing tame automorphisms of G with their associated Nielsen Transformations.

2.3 Small Cancellation Theory

In this section we state certain results in small cancellation theory. This theory is fundamental to the proof of Theorem B, one of the main results of Chapter 4. In Theorem B we take a group $G = \langle a, b; a^i, b^i, (ab)^i \rangle$ with $i > 9$, we take a specific subgroup M of G, and we use small cancellation theory to prove that certain properties of the lift $\overline{M}$ of M to $F(a, b)$ are preserved when we drop down to M. For example, Lemma 4.3.6 is an important step in the proof of Theorem B, and this lemma uses small cancellation theory to prove (in our specific situation) that because $\overline{M}$ is malnormal in $F(a, b)$ then M is malnormal in G. We also introduce this theory here because Section 2.10 gives an introduction to a similar but disjoint small cancellation theory which uses the category of graphs. This novel theory, due to Wise [Wis01], is again used in the proof of Theorem B. The theory using graphs mirrors the classical theory of this current section, and therefore a familiarity with the current section is useful for the understanding and motivation behind Wise's theory in Section 2.10.

(Classical) small cancellation theory deals with presentations of groups which allow for an explicit solution to the word problem and the conjugacy problem for groups. Its study was initiated by Tartakovskii [Tar49] in 1949, who used algebraic methods to solve the word problem for certain groups. The geometric significance of the theory was independently uncovered in 1966 using van Kampen diagrams by Lyndon [Lyn66] and Weinbaum [Wei66]. Note that all the definitions and results in this section can be found Chapter V of Lyndon–Schupp's book [LS77], unless other references are given.

Decision problems for groups. In 1911 Max Dehn posed the three fundamental decision problems in group theory. These problems, along with a desire to generalise associated work of Dehn, provide a motivation for small cancellation theory. The three problems are the *word problem*, the *conjugacy problem*, and the *isomorphism problem*, which are each defined as follows. Note that in a group presentation $\mathcal{P} = \langle X; \mathbf{r} \rangle$, a word w is said to *represent the empty word (in $\mathcal{P}$)* if $w \in \langle\langle \mathbf{r} \rangle\rangle$, where $\langle\langle \mathbf{r} \rangle\rangle$ denotes the normal closure of $\mathbf{r}$.

- Let $\mathcal{P} = \langle X; \mathbf{r} \rangle$ be a presentation of a group. The *word problem* asks if it is possible to determine if a given word over X represents the empty word in $\mathcal{P}$.

- Let $\mathcal{P} = \langle X; \mathbf{r} \rangle$ be a presentation of a group. The *conjugacy problem* asks if it is possible to determine if for two given words u, v over X there is a third word w over X such that $u^{-1}w^{-1}vw$ represents the empty word in $\mathcal{P}$.

- Let $\mathcal{P} = \langle X; \mathbf{r} \rangle$ and $\mathcal{Q} = \langle Y; \mathbf{s} \rangle$ be presentations of groups. The *isomorphism problem* asks if it is possible to determine if $\mathcal{P}$ and $\mathcal{Q}$ define isomorphic groups.

These three problems are insoluble in general [Mil92]. However, when Dehn posed these problems he provided algorithms to solve the word problem and the conjugacy problem for the fundamental groups of closed, orientable two-dimensional manifolds. We shall now point out the key observation which allows the algorithms Dehn provided to be generalised (and which led to the development of small cancellation theory). The fundamental group of a closed orientable two-dimensional manifold of genus greater than two has presentation with a single defining relator R. This relator has the following important property: if S is a cyclic shift of R or R^{-1} then there is very little free cancellation (in proportion to the length of the word R) when we form the product RS. This "small cancellation" is the vital observation, and allows us to generalised Dehn's algorithms to presentations of other groups.

Graphs. We wish to define a geometric way of applying small cancellation theory, called "van Kampen diagrams". In order to do this we must state the notation and conventions which we shall be using when discussing graphs. A *graph* Γ consists of a set of edges E_Γ and a vertex set V_Γ under the following restrictions:

- There exists an *origin* function $\iota : E_\Gamma \to V_\Gamma$ and a *terminal* function $\tau : E_\Gamma \to V_\Gamma$.

- There exists an *inverse* involution $- : E_\Gamma \to E_\Gamma$, such that $\iota(e) = \tau(\overline{e})$ (and so also $\tau(e) = \iota(\overline{e})$).

We shall occasionally write e^{-1} in place of $\overline{e}$, especially in the context of van Kampen diagrams. Note that the inverse involution partitions E_Γ into two disjoint sets E_Γ^+ and E_Γ^-, where $e \in E_\Gamma^+$ if and only if $\overline{e} \in E_\Gamma^-$, and we shall refer to E_Γ^+ and E_Γ^- as the set of *positive* and *negative* edges of Γ respectively. A graph Γ is said to be *trivial* if it consists of a single vertex and no edges, that is, $|V_\Gamma| = 1$ and $|E_\Gamma| = 0$. We shall say that a graph

Γ is *finite* if both $|V_\Gamma|$ and $|E_\Gamma|$ are finite. When we draw graphs we shall often omit the negative edges and draw the remaining (positive) edges, sometimes without direction. The *degree* of a vertex $v \in \Gamma$, denoted $d(v)$, is the number of edges e with $\iota(e) = v$. Note that if Γ consisted of a single vertex v with a single positive edge e then $d(v) = 2$. A *path* p in a graph Γ is a sequence of edges $p = [e_1, \dots, e_n]$ such that $\tau(e_i) = \iota(e_{i+1})$ for $1 \leq i < n$, and we shall define a *trivial path* to be the empty path (that is, it contains no edges). Associated to every path p is an *inverse path* $\overline{p} = [\overline{e_n}, \dots, \overline{e_1}]$, which is the path p in reverse. A graph Γ is said to be *connected* if there exists a path between any two arbitrary vertices. A path is *reduced* if it contains no subpaths $[e_i, \overline{e_i}]$, and a path can be made reduced by removing all such subpaths: this process is called *reduction*.

van Kampen diagrams. We shall now define "(van Kampen) diagrams". These give a geometric view of small cancellation theory. We use a result regarding the structure of certain van Kampen diagrams, called "annular diagrams", in Lemma 4.3.6, which is an important lemma in the proof of Theorem B. Moreover, it is easier to understand the significance of the conditions underlying small cancellation theory using diagrams (as opposed to merely stating the conditions algebraically). Now, consider an embedding $\overline{\Gamma}$ of a graph into a sphere $\mathcal{S}$and note that as it is an embedding no two non-equal edges of the embedded graph intersect. A *region* of $\mathcal{S}$ is a bounded subset homeomorphic to the open unit disk. Note that associated to an embedding $\overline{\Gamma}$ of a connected graph is a set of regions of $\mathcal{S}$, each of which has empty intersection with $\overline{\Gamma}$ but has boundary contained within $\overline{\Gamma}$.

Fixing a presentation $\mathcal{P} = \langle X; \mathbf{r} \rangle$, we can associate to an embedding $\overline{\Gamma}$ of a graph a map $\mathcal{L} : F(X) \to \overline{\Gamma}$ which assigns each oriented edge a *label*. Then, a *van Kampen diagram over the presentation* $\mathcal{P} = \langle X; \mathbf{r} \rangle$, or simply a *diagram over the presentation* $\mathcal{P}$, denoted B, is an embedding $\overline{\Gamma}_B$ of a graph Γ_B into a sphere together with certain specified associated regions and with a labeling map $\mathcal{L} : E_{\overline{\Gamma}_B} \to F(X)$ such that $\mathcal{L}(e^{-1}) = \mathcal{L}(e)^{-1}$, such that $\mathcal{L}(e) \neq 1$ for all edges $e \in \overline{\Gamma}_B$, and such that if a vertex $v \in \overline{\Gamma}_B$ has degree two then the graph Γ_B consists of a single vertex with a single positive loop edge. If the presentation is clear, we shall omit the phrase "over the presentation $\mathcal{P}$". A diagram is called *reduced* if there are no two edges e, f with $\tau(e) = v = \iota(f)$ and $\mathcal{L}(e) = (\mathcal{L}(f))^{-1}$. Note that the union of any the regions of a diagram with their boundaries form subdiagrams of the diagram B with labeling map inherited from B. A *disk diagram* is a contractible diagram, while an *annular diagram* is one whose fundamental group is infinite cyclic. (Note that simply connected does not imply contractible as we are mapping into a sphere.)

Note that a disk diagram has a boundary, $\partial D := \partial(\mathcal{S} \setminus D)$. This boundary is a path, and so fixing a vertex in ∂D and fixing an orientation we obtain an element of $F(X)$. We shall say that this element is a *boundary label* of D. The following lemma allows disk diagrams to be used to give a geometric representation of relators in a presentation.

Lemma 2.3.1 (Lyndon–Schupp, Lemma V.1.2)**.** *Consider the presentation $\mathcal{P} = \langle X; \mathbf{r}\rangle$. A word $W \in F(X)$ is equal to the trivial word in $\mathcal{P}$ if and only if W is a boundary label of a disk diagram D where all regions of D have a boundary label contained in* $\mathbf{r}$.

The small cancellation conditions. We shall now give the conditions underlying small cancellation theory. However, before we can do this we need to define the "symmetrised closure" of a set of definite relators and we need to define a "piece". Let $\langle X; \mathbf{r}\rangle$ be a group presentation such that every element of $\mathbf{r}$ is cyclically reduced. The *symmetrised closure* of $\mathbf{r}$, denoted $\mathbf{r}^*$, is the minimal set of words over X which contains $\mathbf{r}$ and such that if $R \in \mathbf{r}$ then every cyclically reduced conjugate of R and of R^{-1} is in $\mathbf{r}$. If a set $\mathbf{r}$ is its own symmetrised closure, so $\mathbf{r} = \mathbf{r}^*$, then we say that $\mathbf{r}$ is *symmetrised*. A *piece relative to the set* $\mathbf{r}$ is a word p over X such that pu and pv are freely reduced with $u \not\equiv v$, and both $pu, pv \in \mathbf{r}^*$. We shall omit the phrase "relative to the set $\mathbf{r}$" when the set $\mathbf{r}$ is clear. For example, the fundamental group of a closed, orientable two-dimensional manifold of genus g has the following presentation.

$$\mathcal{P}_g = \langle a_1 b_1, \ldots, a_g, b_g; a_1^{-1} b_1^{-1} a_1 b_1 \ldots a_g^{-1} b_g^{-1} a_g b_g\rangle$$

Here, non-trivial pieces are single letters.

We shall now give the small cancellation conditions. There are three such conditions, denoted by $C'(\lambda)$, $C(n)$, and $T(n)$ respectively. In this thesis we only apply the $C'(\lambda)$ condition. This is in Lemma 4.3.6 from Section 4.3.4, which contributes to the proof of Theorem B. However, in Section 2.10 we state conditions which are analogous to each of these three conditions, but using the category of graphs. Therefore, it is pertinent to state all the conditions here. If r is a word over an alphabet X then we use the notation $|r|$ to mean the length of this word.

The $C'(\lambda)$ condition: A presentation $\mathcal{P} = \langle X; \mathbf{r}\rangle$ satisfies the $C'(\lambda)$ condition if whenever $pu \in \mathbf{r}^*$ where p is a piece relative to $\mathbf{r}$ then $|p| < \lambda|pu|$. A group which has a presentation which satisfies $C'(1/6)$ is called a *sixth-group*, while one which satisfies $C'(1/8)$ is an *eighth-group*.

The $C(n)$ condition: A presentation $\mathcal{P} = \langle X; \mathbf{r} \rangle$ satisfies the $C(n)$ condition if no element of $\mathbf{r}^*$ is the product of fewer than n pieces. Equivalently, every interior region of every disc diagram over $\mathcal{P}$ is bounded by at least n edges, where an interior region B is a region where $\partial D \cap \partial B$ is empty.

The $T(n)$ condition: A sequence $R_1, R_2, \ldots, R_i$ is *reduced* if no two successive elements are an inverse pair. Then, a presentation $\mathcal{P} = \langle X; \mathbf{r} \rangle$ satisfies the $T(n)$ condition if for $3 \leq i < n$ every reduced sequence of i elements of $\mathbf{r}^*$, $R_1, R_2, \ldots, R_i$, is such that at least one of the products $R_1R_2, R_2R_3, \ldots, R_iR_1$ is reduced without cancellation. This definition is rather cryptic. However, it makes much more sense from the viewpoint of van Kampen diagrams. In the language of diagrams, this condition holds if and only if for every disc diagram D over $\mathcal{P}$ every interior vertex v of D has degree at least n, where an interior vertex v is a vertex where $v \in D \setminus \partial D$.

Example 2.3.2*:* In $\mathcal{P}_g$ every piece has length one and so this presentation satisfies the $C'(1/(4g-1))$ condition. Further, $\mathcal{P}_g$ satisfies the $C(4g)$ condition.

Example 2.3.3*:* Note that if a presentation satisfies $C'(\lambda)$ then it satisfies $C(n)$ for $\lambda = 1/(n-1)$. In this example we show that the converse is not true. That is, $C(n)$ does not necessarily imply $C'(\lambda)$ for $\lambda = \frac{1}{n-1}$. To see this, take the group with the following presentation.

$$G = \langle a_1, b_1, a_2, b_2, x, y; (x^8y)^{1000}, x^8[a_1, b_1][a_2, b_2] \rangle$$

This presentation satisfies $C(9)$, but x^8 is a piece so this presentation cannot satisfy $C'(\lambda)$ for $\lambda < 8/16 = 1/2$, and so does not satisfy $C'(1/8)$.

Dehn's algorithm and hyperbolic groups. The algorithm which Dehn used to solve the word problem for the fundamental groups of closed, orientable two-dimensional manifolds is related to the specific given presentation $\mathcal{P}_g$. It turns out that the identical algorithm works for $C'(1/6)$ presentations. We shall give this algorithm now and define "hyperbolic groups" using this algorithm. Hyperbolic groups appear sporadically in this thesis. We shall then state the result which proves that Dehn's algorithm solves the word problem for $C'(1/6)$-presentations, which is Proposition 2.3.4.

Let $\mathcal{P} = \langle X; \mathbf{r} \rangle$ be a group presentation. We shall say that a word w contains $> \frac{1}{2}R$ if there exists a word $S \in \mathbf{r}^*$ where $S \equiv S_0 S_1$ such that no free cancellation occurs when forming $S_0 S_1$, where $|S_0| > \frac{1}{2}|S|$, and where $w \equiv u S_0 v$ is freely reduced. A word w over X is *Dehn reduced* if w does not contain $> \frac{1}{2}R$. If the word w contains $> \frac{1}{2}R$, and so is not Dehn reduced, then $w \equiv u S_0 v$ can be replaced with a new word $w_1 \equiv u S_1^{-1} v$ such that $w =_G w_1$ and $|w| > |w_1|$. This replacement process is called *Dehn reduction*, while the process of repeated Dehn reduction on a word is called *Dehn's algorithm.* Suppose that $\mathcal{P}$ has the property that every freely reduced word w which represents the trivial word is *not* Dehn reduced. This supposition implies that w represents the trivial word in $\mathcal{P}$ if and only if Dehn's algorithm terminates at the trivial word. This implies that $\mathcal{P}$ has soluble word problem, and so we shall say that the word problem in $\mathcal{P}$ is *soluble by Dehn's algorithm.* A group is said to be *hyperbolic* if it admits a presentation whose word problem is soluble by Dehn's algorithm [BH99, Theorem III.Γ.2.6].

In his study of the word problem for closed, orientable two-dimensional manifolds, Dehn proved that for the presentations $\mathcal{P}_g$, $g > 1$, if $w =_{\mathcal{P}} 1$ then w is *not* Dehn reduced. Therefore, the word problem in $\mathcal{P}_g$ is soluble by Dehn's algorithm. As the word problem is a group invariant, this means that the word problem is soluble for the fundamental groups of closed, orientable two-manifolds.

The word problem. The observation which Dehn made for the presentations $\mathcal{P}_g$, $g > 1$, holds for $C'(\lambda)$ presentations when $\lambda \leq 1/6$. Therefore, sixth-groups have soluble word problem and are hyperbolic. Indeed, much stronger conditions than the "more than half" required for hyperbolicity hold, which is Greendlinger's Lemma for Sixth-Groups [LS77, Theorem V.4.5]. We apply this in Lemma 4.3.6, which is a key step in the proof of Theorem B from Section 4.3.4. As with $> \frac{1}{2}R$, we shall say that a word w contains $> \frac{a}{b}R$ if there exists a word $S \in \mathbf{r}^*$ where $S = S_0 S_1$ such that no free cancellation occurs when forming $S_0 S_1$, where $|S_0| > \frac{a}{b}|S|$, and where $w \equiv u S_0 v$ is freely reduced.

Proposition 2.3.4 (Greendlinger's Lemma for Sixth-Groups)**.** *Let G be given by a presentation $\mathcal{P} = \langle X; \mathbf{r} \rangle$ which satisfies the $C'(1/6)$ condition. Let w be a non-trivial, cyclically reduced word which represents the trivial word. Then either $w \in \mathbf{r}^*$ or some cyclically reduced conjugate of w contains one of the following.*

- *Two disjoint subwords, each $> \frac{5}{6}R$.*

- *Three disjoint subwords, each* $> \frac{4}{6}R$.
- *Four disjoint subwords, two* $> \frac{4}{6}R$ *and two* $> \frac{3}{6}R$.
- *Five disjoint subwords, four* $> \frac{3}{6}R$ *and one* $> \frac{4}{6}R$.
- *Six disjoint subwords, each* $> \frac{3}{6}R$.

The conjugacy problem. Sixth-groups have soluble conjugacy problem as well as soluble word problem. This is given by the following theorem. Note the strengthened form for eighth-groups (the small cancellation groups we work with in Section 4.3.4 are eighth-groups). If w is a word over X then a *cyclic shift* of w is a word $w^* \equiv vu$ such that $w \equiv uv$ and no free cancellation occurs when forming uv.

Proposition 2.3.5 (Lyndon–Schupp, Theorem V.5.4)**.** *Let G be given by a presentation $\mathcal{P} = \langle X; \mathbf{r} \rangle$ which satisfies the $C'(1/6)$ condition. Let u and v be non-trivial, cyclically reduced words over X which are Dehn reduced and are not conjugate in $F(X)$. Then u and v are conjugate in G if and only if there exists a word $h = r_1 r_2$ over X and cyclic shifts u^* and v^* of u and v respectively such that the following hold.*

- $u^* =_{\mathcal{P}} h^{-1} v^* h$
- *r_1 and r_2 are subwords of relators $R_1, R_2 \in \mathbf{r}^*$. If $\mathcal{P}$ satisfies $C'(1/8)$, r_2 can be taken to be the empty word.*
- $|r_i| < 12 \max(|u|, |v|)$.

Therefore, the conjugacy problem is soluble for finitely generated, recursively presented sixth groups.

Annular diagrams give a geometric view of the conjugacy problem. We use this view in Lemma 4.3.6, which is an important lemma in the proof of Theorem B from Section 4.3.4. This lemma uses the properties of eighth-groups. We shall now explain how annular diagrams encode conjugation. We then state the result which we apply in the proof of Lemma 4.3.6. This result is Proposition 2.3.6, which gives the structure of annular diagrams for $C'(1/8)$ presentations.

An annular diagram A contains precisely two disjoint boundary components which we shall denote by ∂I and ∂E. The component of $\mathcal{S} \setminus A$ with boundary ∂I shall be the *interior*

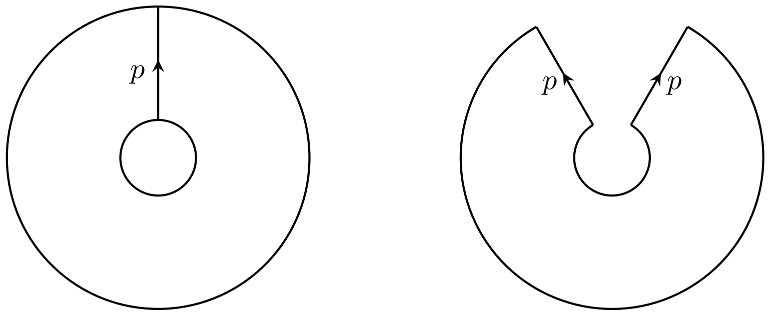

Figure 2.3: An annular diagram is not a disc diagram, but splits along a path to become a disc diagram.

region of A and denoted I, while we obtain the *exterior region* E in an analogous way. A *boundary edge* of A is an edge contained in either ∂I or ∂E. Then there exists a path p in the underlying graph Γ_A of A which connects I to E, that is, $\iota(p) \in \partial I$ and $\tau(p) \in \partial E$. Let W be the label of p, let U be the label of ∂I starting at $\iota(p)$ and reading in a clockwise direction, and let V be the label of ∂E starting at $\tau(p)$ and reading widdershins. Then when A is split along this path p in the sense of Figure 2.3 we obtain a disc diagram D with boundary label $UW^{-1}VW$. Therefore, annular diagrams encode conjugation in the following sense: two words U and V denote conjugate elements of the group given by $\mathcal{P}$ if and only if there exists some annular diagram A such that U is a label for ∂I and V is a label for ∂E, where labels are read in opposite directions. The following proposition gives us the structure of reduced annular diagrams over $C'(1/8)$ presentations.

Proposition 2.3.6 (Lyndon-Schupp, Theorem V.5.3 and Theorem V.5.5)**.** *Let* $\mathcal{P} = \langle X; \mathbf{r}\rangle$ *satisfy* $C'(1/8)$*, and assume the following two hypotheses.*

- *A is a reduced annular diagram.*
- *Every label of ∂I and of ∂E is Dehn reduced.*

Then every region D of A has edges on both ∂I and ∂E, and ∂D contains no more than two pieces which do not intersect with ∂I or ∂E.

If A is an annular diagram described by Proposition 2.3.6 then it is split into *islands*, which are reduced subdiagrams of A whose boundary is not self-intersecting and of the form $\sigma\eta$ with $\sigma \subset \partial I$ and $\eta \subset \partial E$, and *bridges*, which are non-trivial paths in $\partial I \cap \partial E$. An example of an annular diagram described by Proposition 2.3.6 is given by Figure 2.4.

Centralisers. We shall now give a theorem which describes centralisers in sixth-groups.

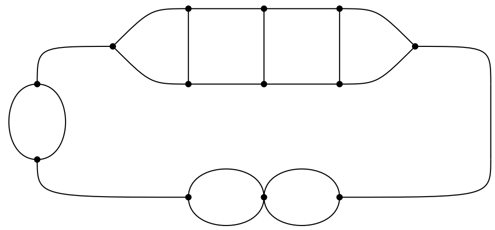

Figure 2.4: An annular diagram with four islands and three bridges. We do not count a single vertex connecting two islands as a bridge.

We apply this result in Section 4.3.4 as a technical step in the proof of Theorem B. We further apply this result in Lemma 2.6.2, which uses Rips' construction to give groups with pathological outer automorphism groups. Greendlinger proved, first for eighth-groups [Gre62] and then for sixth-groups [Gre66], that if two elements in such a group commute then they are powers of a common element. This forms the basis of a proof that elements of sixth groups have cyclic centralisers. Note that the result does not immediately follow, as in an arbitrary group powers of elements can commute even through the elements themselves do not (for example, take the free product with amalgamation $G = \langle a, b; a^2 = b^3 \rangle$, then G centralises a^2 but G is non-cyclic). The key additional property needed for the proof is that elements of sixth groups have unique roots, that is, if $a^n =_G b^n \neq 1$ then $a =_G b$. Lipschutz proved that elements of infinite order in eighth-groups have unique roots [Lip72], while Truffault improved this result to sixth-groups [Tru74a] and extended it to show that elements of finite order in sixth-groups also have unique roots [Tru74b]. Seymour independently proved the same results as Truffault [Sey74]. Therefore, elements of sixth-groups have cyclic centralisers. The result we use in Section 4.3.4 and in Section 2.6 is the following.

Proposition 2.3.7. *If G is a sixth-group then every finitely generated, non-cyclic subgroup has trivial centraliser.*

Proof. Let g and h be two elements of a non-cyclic subgroup H of G, and suppose that they are not contained in a common cyclic subgroup. Such elements exist because H is finitely generated and non-cyclic. Suppose $1 \neq k \in C_G(H)$, then $k \in C_G(g)$ and $k \in C_G(h)$. Therefore, k and g are contained in a common cyclic subgroup, as are k and h. Applying the fact that elements of G have unique roots, this means that g and h are contained in a common cyclic subgroup, a contradiction. $\square$

2.4 Free Constructions

In this section we review *HNN-extensions* and *free products with amalgamation*. These notions are entirely fundamental to this thesis because Chapter 4 investigates the outer automorphism groups of certain HNN-extensions, while Section 3.1 analyses how a two-generator, one-relator group with torsion splits as an HNN-extension or free product with amalgamation over a virtually cyclic subgroup. Almost all of the results stated in this section are applied in Section 3.1, and many of them are also used sporadically throughout this thesis. We refer to HNN-extensions and free products with amalgamation collectively as *free constructions*, in the sense of Kharlampovich–Myasnikov [KM98].

In Section 3.1 we study the *virtually cyclic splittings* of (two-generator) one-relator groups with torsion, that is, the splittings of such groups as HNN-extensions or free products with amalgamation over virtually-$\mathbb{Z}$ subgroups. We undertake this analysis because such splittings encode the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion and such groups are the objects of study in Chapter 3. The JSJ-decomposition is invariant under automorphisms and so proving results on the structure of the JSJ-decomposition of such a group allows us to prove results on the outer automorphism group of the group (the reader is referred to Section 2.8 for more details on JSJ-decompositions). The structural result on the JSJ-decompositions of one-ended two-generator, one-relator groups with torsion is Theorem 3.1.15, while the application to the outer automorphism groups of these groups is Theorem 3.1.16. Most of the results of this section are used in the (rather substantial) build up to these two theorems. Specifically, they are used in Proposition 3.1.13, which analyses how a two-generator, one-relator group with torsion can split as a free product with amalgamation over a virtually cyclic subgroup, and in Proposition 3.1.14, which analyses how such a group can split as an HNN-extension over a virtually cyclic subgroup.

The theories of HNN-extensions and free products with amalgamation parallel one another, and so we begin this section by reviewing certain results on HNN-extensions and then review often analogous results for free products with amalgamation. We finish by stating some results due to Kharlampovich–Myasnikov regarding the hyperbolicity of these constructions. The results at the start of this section can be found in the books of Lyndon–Schupp [LS77, Chapter IV.2] or Magnus–Karrass–Solitar [MKS04, Chapter 4] unless other references are given.

HNN-extensions. If H is a fixed group, A and B isomorphic subgroups with isomorphism $\varphi : A \to B$, then the *HNN-extension of H relative to A, B and φ* is the group G with the following presentation.

$$G = \langle H, t; a^t = a\varphi,\ a \in A\rangle$$

The group H is called the *base group*, the subgroups A and B are called the *associated subgroups*, and t is called the *stable letter*. We shall often write G as $H{*}_{A^t=B}$ or as $\langle H, t; A^t = B\rangle$.

The fundamental result on HNN-extensions is due to Higman–Neumann–Neumann [HNN49], and states that the base group H embeds into the HNN-extension $H{*}_{A^t=B}$ in the natural way.

The elements of an HNN-extension have a relative normal form (that is, there is a canonical way of writing each element of the HNN-extension relative to the base group H) and we use the normal form in the proof of Proposition 3.1.14. To give the relative normal form of an HNN-extension $G = \langle H, t; A^t = B\rangle$ we first define a "reduced sequence" for a word U over H and t. To do this, begin with a sequence $w_0, t^{\epsilon_1}, w_1, \ldots, t^{\epsilon_k}, w_k$. Then this sequence is *reduced* if there is no subsequence of the form t^{-1}, w_i, t with $w_i \in A$, or t, w_j, t^{-1} with $w_j \in B$. This sequence *represents* the word U if U is the concatenation of the elements in the sequence, $U \equiv w_0 t^{\epsilon_1} w_1 \cdots t^{\epsilon_k} w_k$.

Proposition 2.4.1 (Britton's Lemma [LS77])**.** *A reduced sequence with $k \geq 1$ represents a non-trivial element of G.*

Note that this implies a reduced sequence with $k \geq 1$ does not represent an element of H, and to see this implication suppose otherwise. Then there exists $g^{-1} \in H$ which is represented by a reduced sequence $w_0, t^{\epsilon_1}, w_1, \ldots, t^{\epsilon_k}, w_k$ with $k > 1$, and so $g^{-1}g$ has a reduced sequence $g^{-1}w_0, t^{\epsilon_1}, w_1, \ldots, t^{\epsilon_k}, w_k$ which represents the empty word, a contradiction. It can be shown that every element $g \in G$ can be written in a unique way, up to equality of elements of H, as (the concatenation of elements in) a reduced sequence, and the set of reduced sequences yield a relative normal form for the elements of an HNN-extension [LS77, Theorem 2.1]. A word represented by a reduced sequence is called *t-reduced*, and a word is called *t-free* if it does not contain a t-term.

We finish this subsection on HNN-extensions with the following proposition, which is due to Pride [Pri75, Theorem 6]. It gives information on how certain HNN-extensions can be generated. We discuss our motivation below the statement. To state Pride's result we need the notion of a *malnormal* subgroup, which is a subgroup H of a group G such that the following implication holds.

$$H^g \cap H \neq 1 \Rightarrow g \in H$$

Pride's result is as follows.

Proposition 2.4.2 (Pride)**.** *Suppose that $G = \langle H, t; A^t = B\rangle$ can be generated by two elements. Further suppose that A and B are proper, malnormal subgroups of G. If (u, v) is a generating pair for G such that the words u and v define reduced sequences then (u, v) is Nielsen equivalent to a pair (tu', v') such that $u' \in H$ and $v' \in A$.*

We use Proposition 2.4.2 in our proof of Proposition 3.1.14. In the proof, we consider Pride's result in the case when the HNN-extension is additionally a one-relator group with torsion and prove that the analogous result holds if at least one of A and B is malnormal.

Free products with amalgamation. If H and K are fixed groups containing isomorphic subgroups $A \leq H$, $B \leq K$ with isomorphism $\varphi : A \to B$, then the *free product of H and K, amalgamating the subgroups A and B by the isomorphism φ* is the group G with the following presentation.

$$G = \langle H, K; a = a\varphi,\ a \in A\rangle$$

The groups H and K are called the *factors* of the free product with amalgamation, while A and B are called the *amalgamated subgroups*. We shall often write G as $H *_{A=B} K$ or as $H *_C K$, where C is the subgroup of G associated to both A and B. We shall often refer to free products with amalgamation as *amalgams*, in the sense of Serre [SS03]. The fundamental result on free products with amalgamation is that H and K embed into G in the natural way [LS77, Theorem IV.2.6].

The elements of a free product with amalgamation have a relative normal form and a length function L, which measures the "length" of the normal form. We apply these two concepts in the proof of Proposition 3.1.13. Let $G = H *_C K$ be a free product with amalgamation, then to give the relative normal form of an element $g \in G \setminus \{1\}$ begin with a sequence $w_1, w_2, \ldots, w_s$ where each *syllable* w_i is in H or K, where successive syllables w_i, w_{i+1} come from different factors H or K, and if $s > 1$ then each syllable w_i is not contained in C while if $s = 1$ then $w_1 \neq 1$. This is a relative normal form for elements of $H *_C K$ [LS77, Theorem IV.2.6]. The *length function* of $H *_C K$ is the function such that

$L(c) = 0$ if $c \in C$, $L(g) = 1$ if $g \in (H \cup K) \setminus C$, and $L(g) = s$ if $g \notin H \cup K$, where s is the number of syllables in the normal form of g.

The relative normal form addresses the word problem in free products with amalgamation. The following proposition addresses conjugation, and we apply this proposition in the proof of Proposition 3.1.13. We shall say that an element of G is *cyclically reduced* if either $L(g) \leq 1$ or the normal form for g, written $g = w_1 w_2 \ldots w_k$, is such that w_1 and w_k come from different factors.

Proposition 2.4.3 (Theorem 4.6 [MKS04]). *Let $G = H *_C K$ be a free product with amalgamation. Then every element of G is conjugate to a cyclically reduced element of G. Moreover, for g a cyclically reduced element of G then one of the following happens.*

1. *If g is conjugate to an element $h \in C$ then g is in some factor and there is a sequence*

$$h, c_1, c_2, \ldots, c_t, g$$

   where $c_k \in C$ and consecutive terms in the sequence are conjugate in a factor.

2. *If g is conjugate to an element h which is in some factor but not in a conjugate of C then g and h are in the same factor and are conjugate in that factor.*

3. *If g is conjugate to an element h with normal form $w_1 \ldots w_s$ such that $s \geq 2$, then g can be obtained by cyclically permuting the normal form and then conjugating by an element of C.*

We apply Proposition 2.4.3 in Lemma 3.1.10, a technical lemma in the proof of Proposition 3.1.13. The proof of Lemma 3.1.10 involves using Proposition 2.4.3 to prove that if $G = H *_C K$ then under certain conditions an element of one of the factors H or K is centralised by an element of the amalgamated subgroup C. The following proposition gives us information about this centralising element, and so is useful in the proof of Lemma 3.1.10.

Proposition 2.4.4 (Theorem 4.5 [MKS04]). *Let $G = H *_C K$ be a free product with amalgamation. Suppose $g, h \in G$ such that $gh = hg$. Then one of the following happens.*

1. *g is in a conjugate of C.*

2. *h is in a conjugate of C.*

3. *Neither g nor h is in a conjugate of C but g is in a conjugate of a factor. In this case, h is in that same conjugate of a factor.*

4. *Neither h nor g is in a conjugate of C but h is in a conjugate of a factor. In this case, g is in that same conjugate of a factor.*

5. *Neither g nor h is in a conjugate of a factor. In this case, $g = U^{-1}cU \cdot W^i$, $h = U^{-1}c'U \cdot W^j$ where $U, W \in G$, $c, c' \in C$ and $U^{-1}cU$, $U^{-1}c'U$, W pairwise commute.*

In Section 3.1 we are investigating how one-ended one-relator groups with torsion can split over virtually-$\mathbb{Z}$ subgroups. These groups are well-studied, and so we already posses information about their subgroups. For example, if $G = \langle X; R^n \rangle$, $n > 1$, then every two-generator subgroup is either a free product of cyclic groups or a one-relator group with torsion [Pri77b], while the normal closure T of the element R, $T := \langle\langle R \rangle\rangle$, is isomorphic to the free product of infinitely many cyclic groups of order n [FKS72]. We play our knowledge of the subgroup structure of these groups off against the Kurosh Subgroup Theorem [MKS04, Corollary 4.9.1], which gives the form subgroups of free products (without amalgamation) take, and in the built-up to Lemma 3.1.8 we apply this to the subgroup T.

Proposition 2.4.5 (Kurosh Subgroup Theorem)**.** *Every subgroup A of a free product $G = H * K$ is a free product of a free group F with the subgroups of conjugates of H and K which intersect A.*

$$A = F * \prod_{g \in G} (A \cap H^g) * \prod_{g \in G} (A \cap K^g)$$

The following proposition gives us Corollary 2.4.7, below. This corollary is used in the proof of Proposition 3.1.13, which is our analysis of how a two-generator, one-relator group with torsion can split as a free product with amalgamation over virtually-cyclic subgroups, and gives us information regarding generating pairs in a two-generated free product with amalgamation. The proposition is due to Zieschang [Zie70][2].

Proposition 2.4.6 (Zieschang)**.** *Suppose $G = A *_C B$ with $C \neq 1$. Then any generating set X of G is Nielsen equivalent to another generating set X' such that one of the following holds.*

1. *Any element $g \in G$ can be written as a product W_g of elements of X' such that $L(x) \leq L(g)$ for every $x \in X'$ that occurs in W_g.*

2. *There is a subset $\overline{X}$ of X' that lies in a conjugate of either A or B, and there is an element of $\langle \overline{X} \rangle$ that is conjugate to an element of $C \setminus \{1\}$.*

[2]Zieschang's paper is written in German. The result is stated and proved in English in a paper of Collins-Zieschang [CZ88].

Suppose X consists of two elements, and suppose (1) from Proposition 2.4.6 happens. If $g \in A \setminus C$ then $L(g) = 1$, so if $x \in X'$ occurs in W_g then x is in one of A, B or C, and indeed as $g \in A \setminus C$ there must exist some x which occurs in W_g and is in $A \setminus C$. Thus, $X' \cap (A \setminus C)$ is non-empty. By identical logic, $X' \cap (B \setminus C)$ is non-empty. Now, if $g \in C$ then $L(g) = 0$, and so there exists some $x \in X'$ which occurs in W_g such that $x \in C$. Thus, $X' \cap C$ is non-empty. As $A \setminus C$, $B \setminus C$ and C are pairwise disjoint we have that $|X'| \geq 3$, which implies $|X| \geq 3$, a contradiction. Therefore, we have the following corollary due to Kapovich–Weidmann.

Corollary 2.4.7 (Kapovich–Weidmann)**.** *Suppose $G = A *_C B$ with $C \neq 1$. Then any generating set X of G which contains two elements is Nielsen equivalent to another generating set X' such that there is a subset $\overline{X}$ of X' that lies in a conjugate of either A or B, and there is an element of $\langle \overline{X} \rangle$ that is conjugate to an element of $C \setminus \{1\}$.*

The following proposition, due to Karrass–Solitar [KS71, Theorem 6], also relates to two-generated free products with amalgamation. It immediately gives us that the amalgamating subgroup in a one-ended two-generated free product with amalgamation cannot be malnormal in both of the factor groups. This is used alongside the above result, Corollary 2.4.7, in the proof of Proposition 3.1.13.

Proposition 2.4.8 (Karrass–Solitar)**.** *Suppose $G = A *_{C_1 = C_2} B$ where C_1 is a malnormal subgroup of A and C_2 is a malnormal subgroup of B. Then any two-generated subgroup of G is the free products of two cyclic groups or is contained in a conjugate of a factor.*

Hyperbolic groups and free constructions. In Section 3.1 we prove results regarding the structure of the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion. This involves studying how these (hyperbolic) groups can be decomposed as $H *_{A=B} K$ where H and K are hyperbolic and A and B are virtually-$\mathbb{Z}$, or as $H *_{A^t=B}$ where H is hyperbolic and A and B are virtually-$\mathbb{Z}$. Kharlampovich–Myasnikov give concrete algebraic conditions which relate the hyperbolicity of the group with the hyperbolicity of the factor group(s) [KM98]. We state certain algebraic conditions from Kharlampovich–Myasnikov's paper now and we apply them in Section 3.1. Indeed, the following two propositions immediately imply Lemma 3.1.12. Two torsion-free subgroups A and B of H are *separated*[3] if $A \cap B^g$ is trivial for all $g \in H$.

[3] We discuss the wording "separated" before Lemma 3.1.12 in Section 3.1.2.

Proposition 2.4.9 (Corollary 1, Kharlampovich–Myasnikov). *Let H be a hyperbolic group, with A and B infinite cyclic subgroups. Then the HNN-extension $G = H*_{A^t=B}$ is hyperbolic if and only if A and B are separated and one of A or B is malnormal in H.*

Proposition 2.4.10 (Corollary 2, Kharlampovich–Myasnikov). *Let H and K be hyperbolic groups with infinite cyclic subgroups $A \leq H$ and $B \leq K$. Then the free product with amalgamation $G = H *_{A=B} K$ is hyperbolic if and only if either A is malnormal in H or B is malnormal in K.*

The following two propositions are used in the proof of Lemma 3.1.11. Note that they are not precisely the statements given by Kharlampovich–Myasnikov, who give identical results but when A and B are "quasiconvex". Our statements follow immediately because cyclic subgroups of hyperbolic groups are quasiconvex [BH99, Chapter III.Γ.3].

Proposition 2.4.11 (Theorem 4, Kharlampovich–Myasnikov). *Let $G = H*_{A^t=B}$ be hyperbolic with A and B cyclic. Then H is hyperbolic.*

Proposition 2.4.12 (Theorem 6, Kharlampovich–Myasnikov). *Let $G = H *_{A=B} K$ be hyperbolic with A and B cyclic. Then H and K are hyperbolic.*

2.5 Residually Finite Groups

In this section we give an introduction to residually finite groups. We do this for two reasons. Our primary reason is that in Chapter 4 we prove results regarding the outer automorphism groups of residually finite groups. Most notable is Theorem D, from Section 4.3.5, which yields finitely generated, residually finite groups with (almost) arbitrary outer automorphism groups, and thus gives a partial answer to an open problem of Bumagin–Wise. Our secondary reason for introducing residual finiteness is that our motivation for undertaking the work in Chapter 3 was to prove that every two-generator, one-relator group with torsion $G = \langle a, b; R^n\rangle$, $n > 1$, has residually finite outer automorphism group. When the author of this thesis began the work which evolved into Chapter 3 this was a timely problem, as newly published results had proven certain special cases (for example, when R has the form $a^{-1}b^iab^j$ [KT10]). The results obtained in Chapter 3 solve this problem, but this is not why they are interesting. Rather, the results of Chapter 3 are interesting because they completely classify the possible outer automorphism groups of such groups and in doing so obtain much stronger properties than residual finiteness. Note that a recent result of Carette proves that every one-relator group with torsion has a residually finite outer automorphism group, and thus Carette solves our motivating problem in much greater generality [Car13].

A group G is *residually finite* if for all $g \in G$ there exists some finite group K and some homomorphism $\phi : G \to K$ such that $g\phi \neq_K 1$. We shall write $H \leq_f G$ to mean that H is a finite-index subgroup of G. Then g is associated to such a homomorphism if and only if there exists some $N_g \trianglelefteq_f G$ such that $g \notin N_g$ (one simply takes $N_g = \ker \phi$).

In this section we prove that finitely presented residually finite groups have soluble word problem, we give examples of residually finite and non-residually finite groups, we review certain links between the structure of a group and the residual finiteness of the automorphism and outer automorphism groups, and we give a condition concerning the residual finiteness of HNN-extensions, which we apply in the proof of Theorem 4.1.3, an important technical result from Chapter 4. Along the way we prove that the results of Chapter 3 prove that a two-generator, one-relator group with torsion has a residually finite outer automorphism group. Now, Theorem A from Chapter 3 classifies the possible outer automorphism groups, and a rough form of this classification is as follows: a two-generator, one-relator group with torsion has an outer automorphism group which is either finite, virtually-$\mathbb{Z}$, or isomorphic to $\mathrm{GL}_2(\mathbb{Z})$. Then, finite groups are trivially residually finite, while Example 2.5.1 and Example 2.5.6 prove, respectively, that virtually-$\mathbb{Z}$ groups are residually finite and that $\mathrm{GL}_2(\mathbb{Z})$ is residually finite.

The word problem. Residual finiteness is a strong finiteness condition. This strength can be demonstrated by noting that finitely presented residually finite groups have soluble word problem. We prove this now by giving two algorithms, one of which will terminate if a word represents the empty word while the other will terminate if the word does not. Executing the algorithms in parallel therefore yields a solution to the word problem (note that this new algorithm will always terminate). So, suppose $\langle X; \mathbf{r}\rangle$ is a finite presentation of a residually finite group G and let W be a word over the alphabet $X^{\pm 1}$. The first algorithm is the enumeration of all consequences of the defining relations, and this algorithm terminates if it obtains W as a consequence of the relations. As the set $\mathbf{r}$ is finite, this algorithm will always terminate if W represents the empty word. The second algorithm is more complicated. This algorithm enumerates all finite groups, and when it enumerates a finite group K it further enumerates the homomorphisms from $G = \langle X; \mathbf{r}\rangle$ to K, of which there are only finitely many as X is finite, and with each homomorphism ϕ the algorithm checks to see if $W\phi \neq_K 1$ holds. This second algorithm terminates when a finite group K and a homomorphism $\phi : G \to K$ are found such that $W\phi \neq_K 1$, and will always terminate if W does not represent the empty word because G is residually finite.

Examples and non-examples. We shall now give an example of a class of residually finite groups. We then give an example of a non-residually finite group.

Example 2.5.1*:* Virtually-$\mathbb{Z}$ groups are residually finite. Note that these groups occur in the classification from Chapter 3. To see that virtually-$\mathbb{Z}$ groups are residually finite, consider the infinite cyclic group $\mathbb{Z}$. Suppose $n \in \mathbb{Z} \setminus \{0\}$, then $n \notin \langle 2n \rangle \trianglelefteq_f \mathbb{Z}$ and so $\mathbb{Z}$ is residually finite. The following lemma then proves the result.

Lemma 2.5.2. *Subgroups of residually finite groups are residually finite. On the other hand, suppose that H is a finite-index subgroup of a group G and suppose that H is residually finite, then G is residually finite.*

Proof. Suppose H is a subgroup of a residually finite group G and take some $h \in H$. Consider h as an element of G, so we can associate a homomorphism $\phi : G \to K$ where K is finite and $g \neq_K 1$. Then the restriction of ϕ to H is a homomorphism, $\phi|_H : H \to K$, and moreover $h\phi|_H \neq_K 1$. This proves the first point.

Suppose H is residually finite and is a finite-index subgroup of G. As H has finite-index in G it has finitely many conjugate subgroups. Intersecting these conjugates gives a finite-index, normal subgroup N of G. Note that N is a subgroup of H so N is residually finite. Consider an element $g \in G$. If $g \notin N$ we are done, so suppose $g \in N$. As N is residually finite there exists some subgroup $N_g \trianglelefteq_f N$ with $g \notin N_g$. Taking N'_g to be the intersection of the (finitely many) conjugates of N_g in G, we are done. □

Example 2.5.3*:* A group G is called *Hopfian* if every surjective endomorphism $\phi : G \to G$ is injective. We shall give an example of a non-Hopfian group, and then prove that this implies that the group is also not residually finite. Consider the following Baumslag–Solitar group.

$$\mathrm{BS}(2,3) = \langle a, t; t^{-1}a^2t = a^3 \rangle$$

This group became famous because it is the non-Hopfian group with the simplest possible presentation. Hopfian groups first arose in a topological context, with Hopf asking whether a finitely generated group can be isomorphic to a proper factor of itself and drawing parallels with the problem of listing all classes of maps of the closed orientable surface of genus g onto the closed orientable surface of genus g [Hop31][4]. The first answer was due to B. H. Neumann who gave a two-generator group with infinitely many defining relators [Neu50]. Higman gave a three-generator group with two defining relations, which was the first finitely presented example [Hig51]. Finally, G.Baumslag–Solitar proved that their group $\mathrm{BS}(2,3)$ is non-Hopfian, which is the simplest possible example in terms of presentations [BS62]. To see that $\mathrm{BS}(2,3)$ is non-Hopfian, consider the following map.

$$\begin{aligned} \phi : a &\mapsto a^2 \\ t &\mapsto t \end{aligned}$$

Noting that $t^{-1}a^2ta^{-3}$ is mapped to $t^{-1}a^4ta^{-6} = 1$, we have that ϕ is a homomorphism. Further, ϕ is surjective because $a^3 = a^{-2}(t^{-1}a^2t)$. Thus, to prove that $\mathrm{BS}(2,3)$ is non-Hopfian it suffices to prove that ϕ has non-trivial kernel. To do this, consider the following word.

$$t^{-1}a^{-1}tat^{-1}a^{-1}ta^2$$

This is t-reduced, and so by Britton's Lemma it does not define the trivial word. However, this word is mapped to the identity because of the following sequence of equalities.

$$\begin{aligned} (t^{-1}a^{-1}tat^{-1}a^{-1}ta^2)\phi &= t^{-1}a^{-2}ta^2t^{-1}a^{-2}ta^4 \\ &= a^{-3}a^2a^{-3}a^4 \\ &= 1 \end{aligned}$$

Therefore, $\mathrm{BS}(2,3)$ is non-Hopfian, as required. Then, $\mathrm{BS}(2,3)$ is not residually finite by the following proposition due to Mal'cev [Mal40].

Proposition 2.5.4 (Mal'cev)**.** *A finitely generated, residually finite group is Hopfian.*

Proof. Suppose G is residually finite and non-Hopfian, and we shall find a contradiction. As G is non-Hopfian, there exists a map $\phi : G \to G$ such that $\ker\phi \neq 1$. Consider

[4]Hopf [Hop31] wrote "I believe the problem of listing all classes of maps of the closed orientable surface of genus g onto the closed orientable surface of genus g is interesting, both on account of the connection with function-theoretical [analytic?] questions...as well as from a purely topological viewpoint. This question is only solved for special cases...Apart from these, it is easy to show that the sought-after list is identical to that of all homomorphisms of the fundamental group of the manifold into the fundamental group of the same manifold. However, this group-theoretic problem is most likely no easier to solve than the original geometric one." *Translated from the original German.*

$g \in \ker\phi \setminus \{1\}$. Then as G is residually finite there exists a subgroup $N_g \trianglelefteq_f G$ such that $g \notin N_g$, with associated map $\psi : G \to K = G/N_g$. As G is finitely generated there are only finitely many, n say, homomorphisms $G \to K = G/N_g$. We shall denote these homomorphisms by $\psi_1, \ldots, \psi_n$. Now, each of the maps $\phi\psi_i$ are distinct and so they constitute all the homomorphisms from G to K. However, $g\phi\psi_i =_K 1$ for all i but $g\psi \neq 1$. This is a contradiction, as required. □

Note that the finitely generated assumption in Proposition 2.5.4 is necessary. For example, the free group on countably many generators is residually finite but is non-Hopfian.

Automorphism and outer automorphism groups. We shall now discuss certain connection between residually finite groups and automorphism and outer automorphism groups. We mention these results as the first, Proposition 2.5.5, allows us to prove, in Example 2.5.6, that $\mathrm{GL}_2(\mathbb{Z})$ is residually finite (which is the final, remaining group from the classification of Chapter 3) and because the second result, Proposition 2.5.7, was employed by Kim–Tang to prove that certain specific two-generator, one-relator groups with torsion have residually finite outer automorphism groups and so motivated Chapter 3. The first result we mention is the following remarkable result of G. Baumslag [Bau63]. It is remarkable for its implications, for the elegance of the statement, and for the brevity of the proof.

Proposition 2.5.5 (G. Baumslag)**.** *Suppose G is finitely generated and residually finite. Then G has residually finite automorphism group.*

The proof uses the following definition: a subgroup N of a group G is called *characteristic* if $N\phi = N$ for all $\phi \in \mathrm{Aut}(G)$.

Proof. Begin by noting that if $H \leq_f G$ then there exists a characteristic subgroup N of G with $N \leq_f H$, which is obtained by intersecting the subgroups of index $|G : H| =: n$. The resulting group is characteristic because automorphisms preserve index, and it has finite index because there are only finitely many subgroups of index n as G is finitely generated.

Now, suppose $\phi \in \mathrm{Aut}(G)$ is non-trivial. We shall find a finite group K such that $\mathrm{Aut}(G) \to \mathrm{Aut}(K)$ and ϕ is not mapped to the identity under this homomorphism. To do this, note that as ϕ is non-trivial there exists some $g \in G$ such that $g^{-1}(g\phi) \neq 1$. Choose $x \neq g^{-1}(g\phi)$. Then there exists some characteristic subgroup $N_x \trianglelefteq_f G$ with $x \notin N_x$. Set

$K = G/N_x$. Then ϕ acts on G/N_x because N_x is characteristic, and as $g^{-1}(g\phi) \notin N_x$ we have that $gN_x \neq (gN_x)\phi$. Therefore, the action of ϕ on K is non-trivial, as required. $\square$

Example 2.5.6*:* The group $\mathrm{GL}_2(\mathbb{Z})$ is residually finite. This is because $\mathbb{Z} \times \mathbb{Z}$ is finitely generated and residually finite and so we can apply Proposition 2.5.5 to get that its automorphism group is residually finite. As its automorphism group is $\mathrm{GL}_2(\mathbb{Z})$, the result follows.

Note that Proposition 2.5.5, using Example 2.5.6, gives a remarkably simple proof that free groups are residually finite. The proof is as follows: By Lemma 2.5.2 it is sufficient to prove that the free group on two generators F_2 is residually finite, because every countable free group embeds into F_2. The result then holds because the following two matrices from $\mathrm{GL}_2(\mathbb{Z})$ generate a free group of rank two.

$$F_2 \cong \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

It is natural to wish to see if the proof of Proposition 2.5.5 extends to the outer automorphism group. We shall, however, need additional assumptions, as Wise has proven that there are finitely generated, residually finite groups whose outer automorphism groups are not residually finite [Wis03]. The most obvious assumption to add would be *conjugacy separability*, that is, for every pair of elements $g, h \in G$ such that g is not conjugate to h there exists a homomorphism ϕ onto a finite group K, $\phi : G \to K$, such that $g\phi$ is not conjugate to $h\phi$. Taking $h = 1$, we see that if G is conjugacy separable then it is also residually finite, so this is the conjugation-analogue of residual finiteness. Now, we wish to extend G. Baumslag's proof of Proposition 2.5.5 to the outer automorphism group situation. It turns out that simply replacing "residually finite" by "conjugacy separable" is not enough. We need a further assumption, as the following proposition shows. A group G has *Grossman's Property A* if the only automorphisms $\alpha \in \mathrm{Aut}(G)$ which are such that $g\alpha$ is conjugate to g for all $g \in G$ are the inner automorphisms. Grossman proved the following result and applied it to prove that mapping class groups of compact, two-dimensional manifolds are residually finite [Gro74].

Proposition 2.5.7 (Grossman)**.** *Suppose G is finitely generated, conjugacy separable and has Grossman's Property A. Then* $\mathrm{Out}(G)$ *is residually finite.*

This proposition has found recent applications in numerous situations. For example, it has been applied to Fuchsian groups [AKT05, MS06], to certain Seifert 3-manifold groups [AKT09], to certain HNN-extensions [WW11], and to hyperbolic groups [MO10, LM13]. We mention this proposition here, however, because its application was the method favoured by Kim–Tang in their proofs that the outer automorphism groups of certain, specific two-generator, one-relator groups with torsion are residually finite [KT10], and this result motivates our Chapter 3. We shall now say a few words on their proof. When Kim–Tang's paper was published, proving the residual finiteness of all one-relator groups with torsion was a famous unsolved conjecture of G. Baumslag [Bau67]. Therefore, Kim–Tang could not extend their method of proof to all (two-generator) one-relator groups with torsion. However, between the publication of their paper and the writing of this thesis, Wise has proven that all one-relator groups with torsion are residually finite [Wis12] and this has been applied by Minasyan–Zalesskii to prove that they are conjugacy separable [MZ13]. Therefore, it is conceivable that Proposition 2.5.7 could be applicable to all one-relator groups with torsion. Note, however, that it is already known that all one-relator groups with torsion have residually finite outer automorphism group. This is a result of Carette, who proves that if G is a one-relator group with torsion then $\mathrm{Out}(G)$ contains a finite index subgroup which embeds into $\mathrm{Aut}(G)$ [Car13]. To see that this is sufficient, note that by Wise's result G is residually finite and so Proposition 2.5.5 implies that $\mathrm{Aut}(G)$ is residually finite. Lemma 2.5.2 then completes the proof. Note that our results in Chapter 3 do not use Wise's (very deep) results.

Residually finite HNN-extensions. In Example 2.5.3 we gave an example of a non-residually finite HNN-extension. However, HNN-extensions can be residually finite, and the following lemma, due to B.Baumslag–Tretkoff [BT78, Lemma 4.4], gives conditions which imply certain HNN-extensions are residually finite. In Theorem 4.1.3 we use this lemma to give a necessary and sufficient condition for certain HNN-extensions to be residually finite.

Proposition 2.5.8 (B.Baumslag–Tretkoff)**.** *Let H be a finitely generated group, let K be a subgroup of H and let ϕ be an automorphism of H. Let G be the following HNN-extension of H induced by the automorphism ϕ.*

$$G = \langle H, t; t^{-1}kt = k\phi,\ k \in K\rangle$$

Then G is residually finite if the following conditions hold.

- *H is residually finite.*

- *The subgroup K is such that for an arbitrary finite set of elements not from H, $\{g_1, g_1, \ldots, g_n\}$ with each g_i not an element of H, then there exists a normal subgroup N of finite index in H such that $g_iK \cap N$ is empty.*

In Chapter 4 we study such "automorphism-induced" HNN-extensions, and in Theorem 4.1.3 we combine Proposition 2.5.8 with Proposition 2.5.5, along with an observation about the automorphism group of G, to prove that if the base group H is finitely generated and residually finite and if K is such that $N_H(K)$ has finite index in H, then G is residually finite if and only if $N_H(K)/K$ is residually finite.

2.6 Rips' Construction

Rips' construction is a way of using small cancellation theory to produce hyperbolic groups whose subgroups have certain pathological properties. It does this by constructing a short exact sequence, and Wise observed that these pathological properties can often be lifted to the outer automorphism group of the kernel of the sequence [Wis03]. Bumagin–Wise refined this observation and gave an alternative version of Rips' construction [BW05]. For every countable group Q, their construction obtains a finitely generated, residually finite groups N with $\mathrm{Out}(N) \cong Q$. This result of Bumagin–Wise motivates Chapter 4, and one of the two key steps in the proof of Theorem D from Section 4.3.5 is an application of this result. Section 4.3.2 also includes a minor application of Rips' construction.

In this current section we give Rips' original construction and then prove Wise's observation regarding outer automorphism groups. We then outline the variation of Rips' construction due to Bumagin–Wise, which proves Propositions 2.6.4 and 2.6.5. Theorem B gives an alternative proof of Proposition 2.6.4. We end this section by stating an open problem of Bumagin–Wise, and Theorems C and D from Chapter 4 each provide partial answers to this problem.

Rips' construction. Rips' original version of this construction is as follows [Rip82].

Theorem 2.6.1 (Rips)**.** *Let $\lambda > 0$ be given and let Q be a finitely presented group. Then there exists a short exact sequence of groups*

$$1 \to N \to H_Q \to Q \to 1$$

such that the following hold.

1. *H_Q is a finitely presented group which has a presentation satisfying the $C'(\lambda)$ small cancellation condition.*

2. *N is finitely generated.*

Proof. Let Q be given by the following presentation.

$$Q = \langle x_1, x_2, \ldots, x_m; R_1, R_2, \ldots, R_n \rangle$$

Consider the following infinite word.

$$W = abab^2ab^3 \ldots ab^iab^{i+1}a \ldots$$

Note that for any number k there is a set of k disjoint partitions of W which satisfies the $C'(\lambda')$ small cancellation condition for arbitrary $\lambda' > 0$. We shall take H_Q to be the following group, where the finite set $\{U_i, V_{j,\epsilon}, W_{j,\epsilon} : 0 < i \leq n, 0 < j \leq m\}$ consists of disjoint partitions of W and satisfies $C'(\lambda')$, where $\lambda' << \lambda$ is such that the following presentation satisfies $C'(\lambda)$ (for example, take λ' such that $M\lambda' < \lambda$, where M is the maximum length of a relator $R \in \mathbf{r}$).

$$\begin{aligned} H_Q = \langle a, b, x_1, x_2, \ldots, x_m; & \\ & R_1U_1, R_2U_2, \ldots, R_nU_n \\ & x_1^\epsilon a x_1^{-\epsilon} V_{(1,\epsilon)},\ x_2^\epsilon a x_2^{-\epsilon} V_{(2,\epsilon)},\ \ldots,\ x_m^\epsilon a x_m^{-\epsilon} V_{(m,\epsilon)} \\ & x_1^\epsilon b x_1^{-\epsilon} W_{(1,\epsilon)},\ x_2^\epsilon b x_2^{-\epsilon} W_{(2,\epsilon)},\ \ldots,\ x_m^\epsilon b x_m^{-\epsilon} W_{(m,\epsilon)} \rangle \end{aligned}$$

Define the map $\phi : H_Q \to Q$ given by $\phi : a \mapsto 1, b \mapsto 1, x_i \mapsto x_i$ for $0 < i \leq m$. Writing $N = \langle a, b \rangle$, then as $x_i^\epsilon a x_i^{-\epsilon}, x_i^\epsilon b x_i^{-\epsilon} \in N$ for all $0 < i \leq m$ we have that $N \lhd G$, and so $N = \ker \phi$. This proves the theorem. $\square$

Rips' construction is of general interest because it implies finitely generated subgroups of hyperbolic groups can be badly behaved. For example, if Q has insoluble word problem then N has insoluble membership problem. A more complicated implication is as follows. If Q contains a finitely generated subgroup which is not finitely presentable then so does H_Q, and to see this suppose that K is a finitely generated but not finitely presentable subgroup of the finitely presented group Q. Then the full preimage $K\phi^{-1}$ is finitely generated but cannot be finitely presented as K has relative presentation $\langle K\phi^{-1}, a, b; a, b \rangle$, as required[5].

[5]Note that it follows from a result of Bieri that N is finitely presented if and only if Q is finite [Bie76]. However, we have been unable to obtain this reference. This application of Bieri's result was pointed out by Bridson [Bri06, Section 5.1]. See also Bridson–Haefliger [BH99, Exercise 5.47].

Note that such a group Q exists, for example, Grigorchuk's group is finitely generated but not finitely presentable [dLH00, Chapter VIII] while it is recursively presentable and so embeds into a finitely presented group by Higman's Embedding Theorem [LS77, Theorem 7.1]. The above two consequences of the construction are both due to Rips' [Rip82].

Numerous variations of Rips' construction exist, where the groups H_Q and N are bestowed with additional properties. For example, Wise gave a version of Rips' construction where H_Q, and so N, is residually finite [Wis03], and Cotton-Barratt–Wilton [CBW12] proved that here N is conjugacy separable, while Bridson–Grunewald used this construction to settle a problem of Grothendieck [BG04] (note that recent results of Wise show that the groups H_Q and N in Rips' original construction are residually finite [Wis12]). Olliver–Wise gave a version where N has Kazhdan's Property T, which allows them to give Property T groups with infinite outer automorphism group. Belegradek–Osin gave a version where N is the quotient group of a fixed (but arbitrary) non-elementary hyperbolic group, which allows them to prove that certain properties, such as Property T and Serre's Property FA, are not recursively recognisable within the class of hyperbolic groups [BO08]. Baumslag–Bridson–Miller–Short give an enhanced version of Rips' construction where conditions (relating to homotopy) are placed on the complex associated to the given presentation of H_Q, and this allows them to prove that there exists a torsion-free hyperbolic group H and a finitely presented subgroup $P \leq H \times H$ such that there is no algorithm to decide membership of P, and the conjugacy problem for P is insoluble [BBMS00].

Rips' construction and outer automorphism groups. Wise and Bumagin–Wise have altered Rips' construction to produce finitely generated groups with pathological outer automorphism groups. Consider the subgroup N from the short exact sequence. Wise proved that Q embeds into $\mathrm{Out}(N)$ [Wis03] while Bumagin–Wise altered Rips' construction in such a way that Q can be an arbitrary countable group and that $\mathrm{Out}(N) \cong Q$ [BW05]. Both Wise and Bumagin–Wise are motivated by residual finiteness, that is, if H_Q is residually finite then so is N and so they produce finitely generated, residually finite groups whose outer automorphism groups have pathological properties. Note that Bumagin–Wise prove that H_Q is residually finite only when Q is finitely presented (Wise's construction is only applicable when Q is finitely presented).

We shall first explain Wise's result and then Bumagin–Wise's result. Wise gives an altered version of Rips' construction which is designed to allow certain previous results to be applied, and these results mean that H_Q is residually finite. Cotton-Barratt–Wilton extended the proof to show that H_Q (and therefore N) is conjugacy separable [CBW12]. Wise combined his alternative construction with the following result on outer automorphism groups.

Lemma 2.6.2 (Wise)**.** *Suppose we have a short exact sequence*

$$1 \to N \to H_Q \to Q \to 1$$

such that H_Q satisfies $C'(1/6)$. Then Q embeds into $\mathrm{Out}(N)$.

Proof. Note that H_Q acts by conjugation on N, so there is a homomorphism $\psi : H_Q \to \mathrm{Out}(N)$. Then, $N \leq \ker(\psi)$. We shall prove that $N = \ker(\psi)$, which is clearly sufficient. Suppose $h \in \ker(\psi)$ but $h \notin N$. Then there exists $g \in N$ such that $k^h = k^g$ for all $k \in N$. Thus, $gh^{-1} \in C_{H_Q}(N)$. By Proposition 2.3.7, $C_{H_Q}(N)$ is trivial so $h = g \in N$, a contradiction. Therefore, $\ker(\psi) = N$, and the result follows. □

Wise used his residually finite version of Rips' construction and Lemma 2.6.2 to essentially obtain the following corollary. It is enhanced here by Cotton-Barratt–Wilton's result that N is conjugacy separable. Note that this implies that, in general, the additional assumption of Grossman's Property A in Proposition 2.5.7 is necessary.

Corollary 2.6.3 (Wise, Cotton-Barratt–Wilton)**.** *For each finitely presented group Q there exists a finitely generated, conjugacy separable group N such that Q embeds into* $\mathrm{Out}(N)$.

We shall now give Bumagin–Wise's version of Rips' construction. We shall outline the proof that the group H_Q in this construction is $C'(1/6)$ for Q an arbitrary countable group, which means that we can apply Lemma 2.6.2. However, we shall not prove that the groups N and Q from their construction are such that $\mathrm{Out}(N) \cong Q$.

Bumagin–Wise's version of Rips' construction uses a variation of the infinite word used in Rips' construction. Specifically, they use partitions of the following infinite word.

$$W = ab(ab^2)ab(ab^2)^2ab(ab^2)^3\ldots$$

They prove that for each infinite sequence of natural numbers $\{n_1, n_2, \ldots\}$, one can find an infinite family of finite-length partitions of W

$$\mathcal{W}_\infty = \{w_1, w_2, \ldots\}$$

which satisfies the $C'(1/20)$ small-cancellation condition and such that for each i we have $|w_i| \geq n_i$ [BW05, Lemma 6]. Bumagin–Wise's construction is then as follows. Suppose $Q = \langle X; \mathbf{r} \rangle$ and $\mathbf{r} = \{R_1, R_2, \ldots\}$. Take the sequence of numbers

$$\{1, 1, 1, 1, 30|R_1|, 1, 1, 1, 1, 30|R_2|, \ldots\}$$

which consists of quintuples of the form $\{1, 1, 1, 1, 30|R_i|\}$. This is associated to a family of non-empty finite-length partitions of W

$$\begin{aligned} \mathcal{W}_\infty = \{ & W_{(1,1)}, W_{(1,-1)}, V_{(1,1)}, V_{(1,-1)}, U_1, \\ & W_{(2,1)}, W_{(2,-1)}, V_{(2,1)}, V_{(2,-1)}, U_2, \\ & \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \} \end{aligned}$$

which satisfies the $C'(1/20)$ small-cancellation condition and so that $|U_i| \geq 30|R_i|$ for each i. Pick $p > 97$ and $q = 45$. We then form the following presentation.

$$\begin{aligned} H_Q = \langle a, b,\ & x_1, x_2, \ldots, x_m; \\ & R_1U_1, R_2U_2, \ldots, R_nU_n \\ & x_1^\epsilon a x_1^{-\epsilon} V_{(1,\epsilon)},\ x_2^\epsilon a x_2^{-\epsilon} V_{(2,\epsilon)},\ \ldots,\ x_m^\epsilon a x_m^{-\epsilon} V_{(m,\epsilon)} \\ & x_1^\epsilon b x_1^{-\epsilon} W_{(1,\epsilon)},\ x_2^\epsilon b x_2^{-\epsilon} W_{(2,\epsilon)},\ \ldots,\ x_m^\epsilon b x_m^{-\epsilon} W_{(m,\epsilon)} \\ & a^p, b^p, abab^2ab^3ab^4 \ldots ab^q \rangle \end{aligned}$$

This presentation is a $C'(1/11)$ presentation, and hence the map $Q \to \mathrm{Out}(N)$, where $N = \langle a, b \rangle$, is injective by Lemma 2.6.2. Bumagin–Wise prove that this map is also surjective [BW05, Lemma 9], and therefore they prove the following result.

Proposition 2.6.4 (Bumagin–Wise)**.** *Every countable group can be realised as the outer automorphism group of a finitely generated group.*

Theorem B of Chapter 4 gives a very different proof of this theorem, and indeed Theorem B is interesting in its own right. Now, if the group Q is finitely presented then so is the group H_Q, and this allows Bumagin–Wise to apply a previous result of Wise to obtain that H_Q is residually finite, and so N is residually finite. This proves the following result.

Proposition 2.6.5 (Bumagin–Wise)**.** *Every finitely presented group can be realised as the outer automorphism group of a finitely generated, residually finite group.*

They then ask the following question.

Question 1. *Is every countable group Q isomorphic to* $\mathrm{Out}(N)$ *where N is finitely generated and residually finite?*

Theorem D of Chapter 4 (almost) resolves this question for finitely generated, recursively presented groups, and the proof utilised Bumagin–Wise's construction. Theorem C of Chapter 4 gives a partial answer to this question for finitely generated residually finite groups.

2.7 Bass–Serre Theory

This section serves as an introduction to Bass–Serre theory, which is the theory of groups acting on trees, and is split into two parts. A graph of groups encodes the action of a group G on a tree T, and in the first part, Section 2.7.1, we describe, without proof, how to recover the group G, the tree T and the action of G on T from this description. In the second part, Section 2.7.2, we define Serre's property FA and prove that triangle groups posses this property. All the information in this current section can be found in Serre's book on Bass–Serre theory [SS03]. An alternative reference is the introductory book of Meier [Mei08]. We give this introduction to Bass–Serre theory for three reasons.

The first reason is that Section 3.1.2 from Chapter 3 investigates how a one-ended two-generator, one-relator group with torsion splits as a specific kind of graph of groups, called a JSJ-decomposition. The relevant structural result is Theorem 3.1.15.

The second reason is that if a group H has Serre's property FA then, in Section 4.2, Theorem 4.2.15 combined with Lemma 4.2.2 and Theorem 4.2.3 gives descriptions of the outer automorphism groups of certain HNN-extensions $G = H*_{A^t=B}$ of H, called "automorphism induced" HNN-extensions. Triangle groups posses Serre's property FA, and Theorems B and C each use the description of the outer automorphism groups of these HNN-extensions to construct for every group Q in a specific class HNN-extensions of triangle groups whose outer automorphism groups are, in an appropriate sense, related to Q.

The third reason is that in Section 2.11 we outline a bridge between the work of Chapter 3 and the work of Chapter 4, and this bridge is an analysis, due to Levitt, of the outer automorphism groups of groups acting on trees. Moreover, Section 2.11 includes an important result, Proposition 2.11.6, which relates how the automorphisms of a vertex group in certain types of graphs of groups can lift to automorphisms of the fundamental groups of the graph of groups, and we apply this proposition to the JSJ-decompositions of outer automorphism groups of two-generator, one-relator groups with torsion in Theorem 3.1.16.

Trees. Bass–Serre theory is the theory of groups acting on trees, and trees are a specific kind of graph (recall the definition of a graph from Section 2.3). A *cycle* is a path $(e_1, \ldots, e_n)$ in a graph which begins and ends at the same vertex (that is, $\iota(e_1) = \tau(e_n)$) but after reduction is non-trivial. A *tree* is a connected graph which contains no cycles. A *forest* is a graph where all the connected components are trees. If u and v are vertices of a tree we shall write $[u, v]$ for the unique geodesic connecting them.

2.7.1 Graphs of groups

A *graph of groups* $\mathbf{\Gamma}$ is a concise way of describing an action of a group G on a tree T, and consists of a connected graph Γ with an associated set of vertex groups $\{G_v; v \in V_\Gamma\}$ and edge groups $\{G_e; e \in E_\Gamma, G_e = G_{\overline{e}}\}$, and a set of injections $\theta_e : G_e \to G_{\iota(e)}$. The groups G_v are called *vertex stabilisers* or *vertex groups*, while the groups G_e are called *edge stabilisers* or *edge groups*. Throughout this thesis we assume that a group element $g \in G$ acts *without inversion* on a tree T, that is, there is no edge $e \in E_T$ such that $e \cdot g = \overline{e}$.

In this section we explain how obtain the action of a group on a tree encoded by a graph of groups, with a specific emphasis on HNN-extensions. This emphasis is because in Chapter 4 we analyse the outer automorphism groups of a specific class of HNN-extension, and this is related to an analysis of Levitt on the outer automorphism groups of the fundamental groups of graphs of groups. The relationship is based on the fact that HNN-extensions can be given as the fundamental group of a (non-trivial) graph of groups, and we discuss these connections in Section 2.11.

The fundamental group of a graph of groups. In order to define the fundamental group G of a graph of groups $\mathbf{\Gamma}$, take a maximal subtree T_Γ of the underlying graph Γ and define N to be the normal closure of the following set in $\prod_{v \in V_\Gamma} G_v * F(E_\Gamma)$.

$$\{e\overline{e}; e \in E_\Gamma\} \cup \{e; e \in E_{T_\Gamma}\} \cup \{e^{-1}(g\theta_e)e = (g\theta_{\overline{e}}); e \in E_\Gamma, g \in G_e\}$$

Then the *fundamental group of the graph of groups* $\mathbf{\Gamma}$ *with respect to the maximal subtree*

T_Γ is defined to be the following group.

$$\pi_1(\mathbf{\Gamma}, T_\Gamma) = \prod_{v \in V_\Gamma} G_v * F(E_\Gamma)/N$$

It should be noted that picking a different maximal subtree T'_Γ yields an isomorphic group, $\pi_1(\mathbf{\Gamma}, T_\Gamma) \cong \pi_1(\mathbf{\Gamma}, T'_\Gamma)$. The graph of groups $\mathbf{\Gamma}$ encodes an action of $G \cong \pi_1(\mathbf{\Gamma}, T_\Gamma)$ on a tree T, called the "Bass–Serre tree" T of $\mathbf{\Gamma}$.

Example 2.7.1*:* Consider the graph of groups $\mathbf{\Gamma}$ with underlying graph from Figure 2.5, with $G_v = \langle a \rangle$, $G_e = \langle x \rangle = G_{\overline{e}}$ and $\langle a \rangle \cong \mathbb{Z} \cong \langle x \rangle$. The injections are $\theta_e : x \mapsto a^2$, $\theta_{\overline{e}} : x \mapsto a^3$. Then, necessarily, $T_\Gamma = \{v\}$ so we have the group $\mathrm{BS}(2,3)$, which is an HNN-extension of the infinite cyclic group.

$$\begin{aligned}\pi_1(\mathbf{\Gamma}, v) &= \langle a, e, \overline{e}; e\overline{e} = 1, e^{-1}a^2e = a^3 \rangle \\ &\cong \langle a, t; t^{-1}a^2t = a^3 \rangle = \mathrm{BS}(2,3)\end{aligned}$$

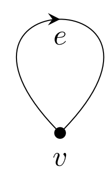

Figure 2.5: The graph Γ *underlying the natural graph of groups* $\mathbf{\Gamma}$ *associated to an HNN-extension.*

HNN-extensions. An HNN-extension can be viewed as a graph of groups $\mathbf{\Gamma}$ where Γ is a graph with a single vertex v and a single positive (loop) edge e, as in Figure 2.5. The base group is the vertex stabiliser, $G_v = H$, and the associated subgroups are the embeddings of the edge stabiliser in G_v, $\theta_e(G_e) = A$ and $\theta_{\overline{e}}(G_{\overline{e}}) = B$. For example, the group $\mathrm{BS}(2,3) = \langle a, t; t^{-1}a^2t = a^3 \rangle$ from Example 2.7.1 is an HNN-extension of the infinite cyclic group where $H = \langle a \rangle$, $A = \langle a^2 \rangle$ and $B = \langle a^3 \rangle$.

Bass–Serre trees. Given a graph of groups $\mathbf{\Gamma}$, we have just discussed how to obtain a presentation for the group G described by this decomposition, but a graph of groups encodes an action of this group G on some tree T called the *Bass–Serre tree*. Therefore, we now briefly explain how to obtain this tree and action before giving an instructive example in Example 2.7.2, below. Define a *hanging tree* to be a tree but possibly with certain degree one vertices removed. To obtain the tree Bass–Serre tree T, begin by taking the maximal subtree T_Γ associated to $G = \pi_1(\mathbf{\Gamma}, T_\Gamma)$, and attach only one end of each of the remaining edges (in such a way that the inverse involution is preserved) to obtain a hanging tree T_H. Then tiling together copies of the hanging tree T_H using the information given by G yields the tree T.

Example 2.7.2*:* We shall now give an example of a Bass–Serre tree with the associated action of the fundamental group G of a graph of groups $\mathbf{\Gamma}$. The example is illustrated in Figure 2.6. Let $\mathbf{\Gamma}$ be the graph of groups from Example 2.7.1, and this has underlying graph consisting of a single vertex with a single positive loop edge e. There are then two choices for the hanging tree T_H, which correspond to taking $v = \tau(e)$ or taking $v = \iota(e)$ where e is the edge in T_H, and we shall choose the hanging tree to be such that $\iota(e) = v$. Recall that $\pi_1(\mathbf{\Gamma}, v) \cong \langle a, t; t^{-1}a^2t = a^3\rangle$. Now, no power of the element $t \in G$ is equal to a power of a, and so no power of t fixes an edge or a vertex. Therefore, we begin constructing T by tiling infinitely many copies of T_H to obtain a linear tree T_L, that is, an infinite tree where every vertex has degree two. Fix a copy of T_H in T_L with vertex $v \in V_{T_L}$ and positive edge $e_1 \in E_{T_L}$, so $\iota(e_1) = v$, and write f_1 for the edge with $\tau(f_1) = v$. We shall take the action of t on T_L to be such that $f_1 \cdot t = e_1$. The first injection $\theta_e : x \mapsto a^2$ given by the graph of groups $\mathbf{\Gamma}$ implies that the edge e_1 is fixed by a^2 but not by a, so in T the vertex v will have precisely two edges e_1 and e_2 with initial vertex v, $\iota(e_1) = v = \iota(e_2)$. This corresponds to attaching a copy of the infinite linear tree beginning at v and extending along e_1, and tells us how G acts on the edges with initial vertex v. The second injection $\theta_{\overline{e}} : x \mapsto a^3$ given by the graph of groups implies that the edge f_1 is fixed by a^3 but not by a or by a^2. We therefore attach two additional infinite linear trees corresponding to the orbit of f_1 under G. As every vertex is in the orbit of the vertex v under G, every vertex has the same local structure as v. Therefore, we have the tree T illustrated in Figure 2.6, which is is the (infinite) tree where every vertex has degree five, with precisely three positive incoming edges and precisely two positive outgoing edges, that is, for every vertex v there are positive edges f_1^v, f_2^v, f_3^v, e_1^v, e_2^v such that $\tau(f_i^v) = v$ and $\iota(e_i^v) = v$. The group G acts on T by letting the generator a cycle the (trees extending from the) two positive incoming edges and cycle the (trees extending from the) three positive outgoing edges of the fixed vertex v, so $f_i^v \cdot a = f_{i+1}^v$ (subscripts computed modulo three) and $e_i^v \cdot a = e_{i+1}^v$ (subscripts computed modulo two), while t acts by translating the graph along the edge f_1^v, that is, $v \cdot t = \iota(f_1^v)$ while $f_i^v \cdot t = f_i^{\iota(f_1)}$ and $e_i^v \cdot t = e_i^{\iota(f_1)}$, $f_1^v \cdot t = e_1^v$. Both generators act on the appropriate inverse edges in the analogous way. Note that because $e_1 = f_1 \cdot t^{-1}$, we have that $f_1 \cdot t^{-1}a^2t = f_1 \cdot a^3$, and so the group $G = \mathrm{BS}(2,3)$ acts on T in the way described. This tree T is the Bass–Serre tree.

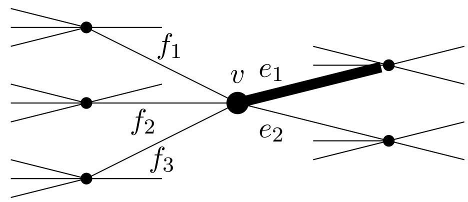

Figure 2.6: The tree represents the infinite tree T and where the bold vertex and bold edge represent the hanging tree T_H which tiles to give T.

2.7.2 Serre's property FA

A finitely generated group G has a decomposition as a non-trivial graph of groups only if G acts without a global fixed point on a tree T [SS03, Theorem 15]. A group G is said to have *Serre's property FA*, or simply *property FA*, if every action of G on a tree has a global fixed point. Examples of groups with property FA are finite groups and *triangle groups*, which are groups with presentation of the form $\langle a, b; a^i, b^j, (ab)^k \rangle$, $i, j, k > 0$. In this section we prove that finite groups have property FA, and we apply this to prove that triangle groups have property FA. We do this because property FA allows a technical theorem from Chapter 4, Theorem 4.2.15, to be applied to the outer automorphism groups of a certain kind of HNN-extensions of triangle groups. This application is proven in Lemma 4.2.2 and Theorem 4.2.3, and is applied in Theorems B and C from Chapter 4 which are two of the three main results from Chapter 4. Without property FA, Theorem 4.2.15 is only applicable to a specific subgroup of the outer automorphism group.

We begin this section by proving that whenever a finite group acts on a tree then there is a global fixed point. Note that this means that if $G = \pi_1(\mathbf{\Gamma}, T_\Gamma)$ is the fundamental group of a non-trivial graph of groups and $g \in G$ is an element of finite order then g fixes some vertex $v \in V_T$ and thus is contained in a conjugate of some vertex group G_w of $\mathbf{\Gamma}$.

Proposition 2.7.3. *Finite groups have Serre's property FA.*

Proof. Suppose G is a finite group acting on a tree T. Let $v \in V_T$ be a vertex of T and consider the orbit $\mathcal{O}_G(v) = \{v \cdot g; g \in G\}$. Note that if the orbit consists only of v then we are done, thus we can assume $|\mathcal{O}_G(v)| \geq 2$. Form the tree T^0 which consists of $\mathcal{O}_G(v)$

along with, for each $g \in G$, the geodesics $[v, v \cdot g]$ connecting v to $v \cdot g$. Note that for $g, h \in G$ arbitrary then the geodesic $[v \cdot g, v \cdot h]$ connecting $v \cdot g$ and $v \cdot h$ is in T^0. Now, G acts on T^0 as it acts on $\mathcal{O}_G(v)$. Then, because $|\mathcal{O}_G(v)| \geq 2$, T^0 contains some degree one vertices which are permuted by the action of G. Removing these vertices yields a tree T^1 upon which G acts. Note that T^0 was finite, and so repeating this pruning process for finitely many, m say, steps yields a single vertex or two vertices connected by a positive edge, and G acts upon this subtree T^m. As G acts without inversion, the vertices of T^m are fixed by G, as required. □

We shall now prove that triangle groups have Serre's property FA. The proof uses the above result, Proposition 2.7.3, along with the following fact: if G acts on a tree T and H is a subgroup of G, $H \leq G$, then the fixed points of the action of H on T form a tree, denoted $\mathrm{Fix}_T(H)$. Note that $\mathrm{Fix}_T(H)$ is a tree because if $v, w \in V_T$ are vertices of T which are fixed by H then the geodesic $[v, w]$ connecting them must also be fixed by H. If H is cyclic and generated by an element g then we shall write $\mathrm{Fix}_T(g)$. Figure 2.7 serves as a companion to the proof of the following proposition, Proposition 2.7.4.

Proposition 2.7.4. *Triangle groups have Serre's property FA.*

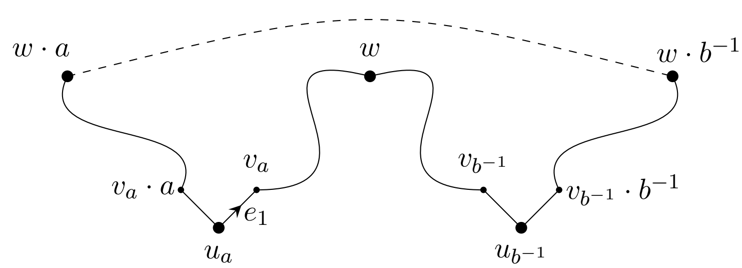

Figure 2.7: The line $p_a = [w, u_a]$ is the geodesic connecting $\mathrm{Fix}_T(a)$ to $\mathrm{Fix}_T(ab)$, and have $u_a \in \mathrm{Fix}_T(a)$ and $w \in \mathrm{Fix}_T(ab)$, while $[w, u_{b^{-1}}]$ is the geodesic connecting $\mathrm{Fix}_T(b)$ to w. The dashed line indicates that $w \cdot a = w \cdot b^{-1}$. Thus, $[w, w \cdot a] = [w, w \cdot b^{-1}]$ and so $u_a = u_{b^{-1}}$.

Proof. Suppose $G = \langle a, b; a^i, b^j, (ab)^k \rangle$ acts on a tree T. Note that because a and b generate G, the fixed points of the action of G on T correspond precisely to the points fixed by both a and b, that is, $\mathrm{Fix}_T(G) = \mathrm{Fix}_T(a) \cap \mathrm{Fix}_T(b)$. Now, a, b and ab each have a fixed point, by Proposition 2.7.3. Consider the geodesic $p_a = [e_1, \ldots, e_n]$ connecting $\mathrm{Fix}_T(a)$ to $\mathrm{Fix}_T(ab)$, with $u_a = \iota(e_1) \in \mathrm{Fix}_T(a)$ and $w = \tau(e_n) \in \mathrm{Fix}_T(ab)$. Then $v_a = \tau(e_1)$ is not fixed by a but $\iota(e_1)$ is fixed. Therefore, the path $[\overline{e_1} \cdot a, e_1, \ldots, e_n]$ is the geodesic from $\iota(\overline{e_1} \cdot a)$ to $\tau(e_n)$. We can repeat this extending process and obtain the following path, which, because $\mathrm{Fix}_T(a)$ is a tree, is the geodesic connecting $w \cdot a$ to w.

$$p'_a = [\overline{e_n} \cdot a, \overline{e_{n-1}} \cdot a, \ldots, \overline{e_1} \cdot a, e_1, \ldots, e_n]$$

Note that the midpoint of this geodesic is $u_a \in \mathrm{Fix}_T(a)$. Now, consider the geodesic $p_{b^{-1}}$ connecting w to $\mathrm{Fix}_T(b)(= \mathrm{Fix}_T(b^{-1}))$. One can repeat the above working with b^{-1} in place of a to obtain the geodesic $p'_{b^{-1}}$ from w to $w \cdot b^{-1}$, and the midpoint of this geodesic is $u_{b^{-1}} \in \mathrm{Fix}_T(b)$. Then because $w \cdot ab = w$ we have that $w \cdot b^{-1} = w \cdot a$, and so $p'_a = p'_{b^{-1}}$. Therefore, $u_a = u_{b^{-1}}$, and so $\mathrm{Fix}_T(a) \cap \mathrm{Fix}_T(b) = \mathrm{Fix}_T(G)$ is non-trivial, as required. □

2.8 JSJ-decompositions

In this section we give the definition of the JSJ-decomposition of a one-ended hyperbolic group. In Chapter 3 we apply the theory JSJ-decompositions of one-ended hyperbolic groups to the outer automorphism groups of two-generator, one-relator groups with torsion. This allows us to prove Theorem 3.1.16, the first of two keystone theorems in Chapter 3.

Hyperbolic groups. Recall that in Section 2.3 we defined a hyperbolic group to be a group which admits a presentation whose word problem is soluble by Dehn's algorithm. This definition is most relevant to this thesis as it allows us to see that sixth-groups are hyperbolic by Greendlinger's lemma (this is Proposition 2.3.4), and that one-relator groups with torsion are hyperbolic by the Newman–Gurevich spelling theorem (this is Proposition 2.9.5). However, many other, equivalent, definitions exist, and these are often based on geometric notions (for example, the word metric satisfying certain conditions relating to hyperbolic geometry). Indeed, Gromov's seminal work on hyperbolic groups greatly developed the idea of adopting geometric notions, tools and results to obtain algebraic results on the structure of hyperbolic groups and their subgroups [Gro87]. We mention the general, geometric approach to hyperbolic groups because the purpose of this section is to explain how a certain geometric notion, the notion of the JSJ-decomposition of a 3-manifold, can be adapted to the context of hyperbolic groups.

Classical JSJ-decompositions. A 3-manifold can be split along spheres in a unique way [Jac80] [Hat00, Theorem 1.5]. This is the *prime decomposition.* One can then split the components of the prime decomposition along tori [Hat00, Theorem 1.9]. In general this splitting is not unique, as Seifert manifolds decompose along tori in a non-unique way. Jaco–Shalen [JS79] and Johannson [Joh79] independently proved that decomposing along tori gives a unique decomposition if Seifert manifolds are treated as pieces to be left intact and not decomposed.

Group-theoretic JSJ-decompositions. Sela proved that one-ended, torsion-free hyperbolic groups decompose in a similar way to 3-manifolds, that is, they decompose as a graph of groups where edge groups are infinite cyclic and certain subgroups, which correspond to *quadratically hanging vertices* in the graph of groups, are treated as pieces to be left intact and not decomposed [RS97]. Such a decomposition is referred to as a *JSJ-decomposition (of a one-ended, torsion-free hyperbolic group).* However, these JSJ-decompositions are not unique, but rather are unique up to certain operations on the graph of groups. We give an example of one such operation, a "slide move", in Example 2.8.1. Sela used his JSJ-decompositions to obtain results regarding the outer automorphism groups of one-ended, torsion-free hyperbolic groups [RS97] and regarding the isomorphism problem for one-ended torsion-free hyperbolic groups [Sel95].

Bowditch generalised Sela's JSJ-decomposition to one-ended hyperbolic groups with torsion [Bow98]. The addition of torsion means that edge groups are virtually-$\mathbb{Z}$ as opposed to infinite cyclic. The notion of JSJ-decomposition due to Bowditch also differ from Sela's notion in that Bowditch's JSJ-decompositions are unique. Bowditch gains uniqueness by adding vertices whose corresponding groups are virtually-$\mathbb{Z}$ (called *elementary vertices*). We shall give an instructive example in Example 2.8.1. Levitt used Bowditch's JSJ-decompositions to analyse the outer automorphism groups of one-ended hyperbolic groups [Lev05]. We review parts of Levitt's proof in Section 2.11, but his main result is Propisition 2.8.2, below.

A key observation regarding Bowditch's JSJ-decompositions is that an edge group "contributes" to the outer automorphism group if and only if it has infinite center [Lev05, Theorem 1.4] [MNS99]. This observation led to Dahmani–Guirardel's notion of the $\mathcal{Z}$-max JSJ-decomposition of a hyperbolic group, where edge groups are virtually-$\mathbb{Z}$ but have infinite center [DG11]. As with Sela's notion, this decomposition is, in a certain sense, unique, and is used by Dahmani–Guirardel to resolve the isomorphism problem for all hyperbolic groups and to give an algorithm which obtains the generators of the outer automorphism group of a hyperbolic group. We use this latter algorithm in Section 3.6.2 of Chapter 3 to prove that given a presentation $\langle a, b; R^n \rangle$, $n > 1$, there exists an algorithm to determine the outer automorphism group of the group defined by this presentation.

The notion of a group-theoretic JSJ-decomposition is not restricted to the theory of hyperbolic groups. For example, Kropholler developed JSJ-decompositions for some Poincaré duality groups [Kro90], and many authors have developed theories applicable to various classes of finitely presented groups [DS99, DS00, SS02, FP06].

Orbifold vertices. Recall that in torsion-free hyperbolic groups, quadratically hanging vertices of the JSJ-decomposition are the analogue of Seifert manifolds. If we add torsion then orbifold vertices form the appropriate analogue: *orbifold vertices* correspond to vertex groups which can decompose in a non-unique way whilst maintaining the graph of groups structure, and, as with Seifert manifolds in the 3-manifold setting, the JSJ-decomposition of a one-ended hyperbolic group treats these vertices as pieces to be left intact and not decomposed. We shall formally define orbifold vertices now. In order to do this we (loosely) define the related notions of a "bounded Fuchsian group", which is a specific kind of Fuchsian group, and a "(maximal) hanging Fuchsian subgroup" of a hyperbolic group. A *bounded Fuchsian group* F is a finitely-generated, non-virtually-cyclic group which acts on the hyperbolic plane $\mathbb{H}$ such that the corresponding quotient 2-orbifold $\mathbb{H}/F$ is canonical and such that if K is the "convex core" of $\mathbb{H}/F$ then K has a non-empty set of boundary components, and these have empty intersection [Bow98]. The *peripheral subgroups* of a bounded Fuchsian group F are the maximal virtually-cyclic subgroups which project onto the fundamental groups of the boundary components of the convex core K. A subgroup $H \leq G$ is a *hanging Fuchsian subgroup* if there is a finite splitting of G as a graph of groups such that H is a vertex stabiliser and such that H admits an isomorphism with a bounded Fuchsian group F where the incident edge groups of H in the graph of groups are precisely the peripheral subgroups of F. A subgroup is *maximal hanging Fuchsian* if it is not contained in any other hanging Fuchsian subgroup. It is a theorem that every hanging Fuchsian subgroup of a one-ended hyperbolic group is contained in a maximal hanging Fuchsian subgroup [Bow98]. An *orbifold vertex* is one whose associated subgroup is maximal hanging Fuchsian.

The above definition of orbifold vertices is rather technical. In Section 3.1.2 we prove that the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion cannot have an orbifold vertex. However, our proof of this fact side-steps the above definition. Rather, orbifold vertices contribute infinitely many outer automorphisms to the hyperbolic group, and in Theorem 3.1.16 we prove that every vertex in the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion contributes only finitely many outer automorphisms. The proof of this part of Theorem 3.1.16 uses Proposition 2.11.6, which is a technical result of Levitt [Lev05, Proposition 2.3.].

The JSJ-decomposition of a one-ended hyperbolic group. We shall now define the canonical JSJ-decomposition of a one-ended hyperbolic group. This definition is due to Bowditch, who proved that it is canonical [Bow98]. First, however, we define certain terms used in the definition. An *elementary subgroup* of a hyperbolic group is a virtually-$\mathbb{Z}$ subgroup, and a *maximal elementary subgroup* is an elementary subgroup not contained in any other elementary subgroup. A *refinement* of a graph of groups is a splitting of a vertex group G_v which respects the graph of groups structure. More formally, a refinement is a splitting of a vertex group G_v in one of the following two ways.

- $G_v = A \ast_C B$ such that if $\iota(e) = v$ then $\theta_e(G_e) \leq A$ or $\theta_e(G_e) \leq B$
- $G_v = A \ast_C$ such that if $\iota(e) = v$ then $\theta_e(G_e) \leq A$.

If such a splitting exists then we say that G_v can be *refined over* C.

Let G be a one-ended hyperbolic group which is not a Fuchsian group. Then a *JSJ-decomposition* of G is a splitting of G as a graph of groups $\mathbf{\Gamma}$ with three types of vertices.

1. *Elementary vertices*, whose group is a maximal elementary subgroup.
2. *Orbifold vertices*, whose group is a maximal hanging Fuchsian subgroup.
3. *Rigid vertices*, whose group cannot be refined over an elementary subgroup.

Every edge connects an elementary vertex to either a rigid or an orbifold vertex. Edge groups of orbifold vertices correspond to the peripheral subgroups of the hanging Fuchsian group. Finally, edge groups are maximal elementary in the corresponding rigid or orbifold vertex group.

A JSJ-decomposition of a one-ended hyperbolic group is unique and so we can talk about *the* JSJ-decomposition of a one-ended hyperbolic group. However, Sela's definition (for one-ended, torsion-free hyperbolic groups) yielded non-unique splittings. The key difference is that Sela did not include elementary vertices. To illustrate the issue which elementary vertices resolve, we give the following example.

Example 2.8.1*:* Let $G_t = \langle a_t, b_t; a_t^{i_t}, b_t^{j_t}, (a_t b_t)^{k_t}\rangle$ for $t = 1, 2, 3$ be three triangle groups with $i_t, j_t, k_t >> 1$, write $x_t := a_t^2 b_t$ and take the free product with amalgamation $G_1 *_{\langle x_1 = x_2\rangle} G_2 *_{\langle x_2 = x_3\rangle} G_3$. Note that each of the groups G_t have Serre's property FA, and so this splitting cannot be refined any further. However, it is not unique as permuting the G_t result in different splittings, for example, $G_2 *_{\langle x_2 = x_1\rangle} G_1 *_{\langle x_2 = x_3\rangle} G_3$. The operation of moving between these splittings is called a *slide move*. Inserting an elementary vertex, as in Figure 2.8, results in a unique splitting, as it encodes all of the permutations of the G_t. Note that the decomposition described by Figure 2.8 is the JSJ-decomposition of the group (the group is hyperbolic by Proposition 2.4.10). Sela proved that in his definition,

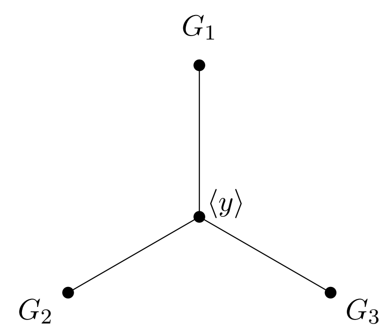

Figure 2.8: Edge groups are $\langle x_t = y\rangle$ for $t = 1, 2, 3$.

the splittings are unique up to conjugation, slide moves, and "modifying the boundary monomorphisms by conjugation".

JSJ-decompositions and outer automorphism groups. The JSJ-decomposition of a one-ended hyperbolic group is entirely canonical. Therefore, it is invariant under automorphisms and Levitt used this invariance to study the outer automorphism group of a one-ended hyperbolic group. The purpose of Sections 3.1.1 and 3.1.2 is to apply Proposition 2.8.2, which is a theorem of Levitt on the outer automorphism groups of hyperbolic groups, in the case of one-ended two-generator, one-relator groups with torsion. We state Proposition 2.8.2 below.

We shall write $\mathcal{V}_1$ for the number of elementary vertices, $\mathcal{V}_2$ for the number of orbifold vertices and $\mathcal{V}_3$ for the number of rigid vertices, while $\mathcal{E}_2$ shall denote the number of edges whose initial vertex is an orbifold vertex, and $\mathcal{E}_3$ the number of edges whose initial vertex is a rigid vertex. Then $\mathcal{E}^\infty$ and V_1^∞ are used to respectively denote the set of edges and the set of elementary vertices with infinite center.

Proposition 2.8.2. *Let G be a one-ended hyperbolic group. Then there is an exact sequence*

$$1 \to \mathcal{T} \to \mathrm{Out}_2(G) \to \prod_{v \in \mathcal{V}_2} \mathrm{PMCG}(G_v) \to 1$$

where $\mathrm{Out}_2(G)$ has finite index in $\mathrm{Out}(G)$. The kernel $\mathcal{T}$ is virtually $\mathbb{Z}^n$ where $n = |\mathcal{E}^\infty| - |\mathcal{V}_1^\infty|$. The group $\mathrm{PMCG}(G_v)$ is the mapping class group of the punctured surface corresponding to the hanging Fuchsian group G_v.

Proposition 2.8.2 connects the JSJ-decomposition of a one-ended hyperbolic group with its outer automorphism group, and we apply it in Theorem 3.1.16 at the culmination of Sections 3.1.1 and 3.1.2. However, Sections 3.1.1 and 3.1.2 deal with virtually-cyclic splittings of one-ended two-generator, one-relator groups with torsion, not with JSJ-decompositions. The following lemma allows us to apply Proposition 2.8.2 using our work on virtually-cyclic splittings. The lemma is a naive statement of the fact that the JSJ-decomposition encodes all virtually-$\mathbb{Z}$ splittings of a one-ended hyperbolic group, and is easily proven by sinking each elementary vertex group into some adjacent vertex group.

Lemma 2.8.3. *Suppose $\mathbf{\Gamma}$ is the JSJ-decomposition of a one-ended hyperbolic group G. Then there exists a virtually-$\mathbb{Z}$ splitting of G as a graph of groups $\mathbf{\Gamma}'$ such that no vertex groups are virtually-$\mathbb{Z}$ and there are precisely $\mathcal{E}_2 + \mathcal{E}_3 - \mathcal{V}_1$ positive edges and $\mathcal{V}_2 + \mathcal{V}_3$ vertices.*

In Chapter 3 we prove that if a two-generator, one-relator group with torsion splits over a virtually-$\mathbb{Z}$ subgroup such that no vertex group is elementary then the underlying graph of the decomposition is either a single vertex or a single vertex with a single positive loop edge. This, combined with Lemma 2.8.3, means that the graph underlying the "modified" JSJ-decomposition of such a group is either a single vertex or two vertices joined by two positive edges (we define modified JSJ-decompositions before Theorem 3.1.16).

Fuchsian groups. Note that JSJ-decompositions are only valid if the group is one-ended and not Fuchsian. In Chapter 3 we prove results regarding the possible JSJ-decompositions of certain two-generator, one-relator groups, but for this chapter to be exhaustive we must also study the Fuchsian groups of this form. Fuchsian groups which occur in this chapter are two-generated, and two-generator Fuchsian groups have been classified [FR93, Theorem A] [Kat92]. We use this classification in Theorem 3.1.16.

Proposition 2.8.4. *A two-generator Fuchsian group has one of the following presentations.*

- $\langle a, b; a^2, a^{-1}ba = b^{-1}\rangle$.
- $G = \langle a, b; -\rangle$.
- $G = \langle a, b; a^i\rangle$ *for* $i > 1$.
- $G = \langle a, b; a^i, b^j\rangle$ *for* $i \geq j \geq 2$ *such that* $i + j \geq 5$.
- $G = \langle a, b; a^i, b^j, (ab)^k\rangle$ *for* $i \geq j \geq k \geq 2$ *such that* $i^{-1} + j^{-1} + k^{-1} < 1$.
- $G = \langle a, b; [a, b]^i\rangle$ *for* $i > 1$.
- $G = \langle a, b, c; a^2, b^2, c^2, (abc)^i\rangle$ *for* $i \geq 3$ *odd.*

2.9 One-Relator Groups

Chapter 3 classifies the outer automorphism groups of certain one-relator groups with torsion. In this section we give some background on these groups and we state certain results used in Chapter 3. A group G is called a *one-relator group* if it has a presentation of the form $G = \langle X; S\rangle$ where S is a non-empty word over X and X contains at least two elements. We assume that $|X| \geq 2$ as otherwise G is cyclic (and possibly trivial). The general study of one relator groups was instigated by Magnus. He proved the "Freiheitssatz" [Mag30] and resolved the word problem for one-relator groups [Mag32]. In his paper resolving the word problem, Magnus introduced a general method for proving results about one-relator groups. This method involves inducting on the length of the relator S and is called *Magnus' method.* Proofs which utilise the original version of this method involve passing to a larger class of groups, groups with "staggered" presentations [MKS04]. Moldovanskii modified Magnus' method using HNN-extensions [Mol67], and this more modern adaptation allows for neater proofs without the need for staggered presentations [MS73].

The Freiheitssatz. Underlying Magnus' method is the famous *Freiheitssatz* (that is, *Freeness Theorem*). One can prove the Freiheitssatz using Magnus' method [MKS04, MS73, FR94]. The proof inducts on the length of the relator, so the argument is not circular.

Proposition 2.9.1 (Freiheitssatz). *Let $G = \langle x_1, x_2, \ldots, x_n; S\rangle$ be a one-relator group. Suppose that the relator S is cyclically reduced in the free group $F(x_1, \ldots, x_n)$ and contains all the generators. Then the subgroup of G generated by $x_1, \ldots, x_{n-1}$ is free on the generators $x_1, \ldots, x_{n-1}$.*

Note that this implies that, under the conditions of Proposition 2.9.1, every proper subset of the generators generates a free group on those generators, and also note that, by taking free products, this result can be extended to arbitrary one-relator groups. That is, a subgroup which is generated by a subset of the generators while omitting a generator which occurs in the (cyclically reduced) relator is free. Such a subgroup is called a *Magnus subgroup*. The following result is due to B. B. Newman [New73], which we apply in Section 3.2 to prove that the outer automorphism groups of certain two-generator, one-relator groups with torsion embed into $\mathrm{GL}_2(\mathbb{Z})$.

Proposition 2.9.2. *Magnus subgroups of a one-relator group with torsion are malnormal.*

A similar result is due to Wise, who proved that Magnus subgroups of a one-relator group with torsion are quasiconvex [Wis12] (this is a key step in Wise's proof that one-relator groups with torsion are residually finite).

Moldovanskii rewriting. Moldovanskii adapted Magnus' method so that proofs use HNN-extensions as opposed to staggered presentations. In order to do this, he gave a method for re-writing a relator S of a group $\langle x_1, \ldots, x_m; S\rangle$ to get another word S' such that the following hold.

- $\langle x_1, \ldots, x_m; S\rangle \cong \langle x_1, \ldots, x_m; S'\rangle$.
- S' is cyclically reduced.
- x_1 occurs in S'.
- $\sigma_{x_1}(S) = 0$.

This rewriting process is called *Moldovanskii rewriting*, and given a word S there is an algorithm which will put it in this form. In Chapter 3, we often assume that if $G =$

$\langle a, b; R^n \rangle$ then $\sigma_a(R) = 0$, and we can only assume this due to Moldovanskii rewriting. For example, in Lemma 3.2.4 this assumption gives us a useful view of the abelianisation of G. We shall now give the rewriting algorithm, followed by an example. Rewrite S such that it is cyclically reduced, such that $0 \leq \sigma_{x_1}(S) \leq \sigma_{x_2}(S) \leq \ldots \leq \sigma_{x_k}(S)$ and such that x_j does not occur in S for $j > k$. Suppose that $\sigma_{x_1}(S) \neq 0$ (otherwise we are done). Take i to be the greatest integer such that $i\sigma_{x_1}(S) < \sigma_{x_k}(S)$. Then, taking S_1 to be the cyclic reduction of the word $S(x_1 x_k^{-i}, x_2, \ldots, x_m)$ we see that $\langle x_1, \ldots, x_m; S\rangle \cong \langle x_1, \ldots, x_m; S_1\rangle$ and $\sigma_{x_k}(S_1) < \sigma_{x_k}(S)$ while all other exponent sums are unchanged. Therefore, repeating we see that this algorithm will terminate at a word S_n such that some generator has exponent sum zero, and then relabelling so that x_1 has exponent sum zero we obtain a word S' with the required properties.

Example 2.9.3*:* Consider the group $G = \langle a, b; a^2b^{-1}a^4b^5\rangle$, so the relator S is the word $S := a^2b^{-1}a^4b^5$. Then $\sigma_a(S) = 6$ while $\sigma_b(S) = 4$. Our first step is to rewrite S so that $\sigma_a(S) \leq \sigma_b(S)$, so S become $b^2a^{-1}b^4a^5$. Then we have $i := 1$, so we obtain the following word.

$$\begin{aligned} S_1' &:= S(ab^{-1}, b) \\ &\equiv b^2(ab^{-1})^{-1}b^4(ab^{-1})^5 \\ &\equiv b^3a^{-1}b^4(ab^{-1})^5 \\ \Rightarrow S_1 &:= b^2a^{-1}b^4a(b^{-1}a)^4 \end{aligned}$$

Repeating these steps, we re-write S_1 as $a^2b^{-1}a^4b(a^{-1}b)^4$ and $i := 2$ so we obtain the following word.

$$\begin{aligned} S_2' &:= S_1(ab^{-2}, b) \\ &\equiv (ab^{-2})^2b^{-1}(ab^{-2})^4b((ab^{-2})^{-1}b)^4 \\ &\equiv ab^{-2}ab^{-3}(ab^{-2})^4(b^3a^{-1})^4b \\ \Rightarrow S_2 &:= ab^{-2}ab^{-3}(ab^{-2})^4(b^3a^{-1})^4b \end{aligned}$$

Then $S' := ba^{-2}ba^{-3}(ba^{-2})^4(a^3b^{-1})^4a$ is the required word, so after rewriting the group G becomes $G \cong \langle a, b; ba^{-2}ba^{-3}(ba^{-2})^4(a^3b^{-1})^4a\rangle$.

2.9.1 Torsion

In this thesis we shall write the relator of a one-relator group G either using the letter S, so $G = \langle X; S\rangle$, or using R^n where the letter R denotes a word which is not a proper power of an element of $F(X)$, so $G = \langle X; R^n \rangle$. This latter convention is because in Chapter 3 we study the case when $n > 1$. Therefore, a one-relator group has a presentation $G = \langle X; R^n \rangle$ for some $n \geq 1$, and also $G = \langle X; S \rangle$ if we do not want to state whether the relator is a proper power or not.

If G has a presentation $\langle X; R^n \rangle$ where $n > 1$ then we shall say that G is a *one-relator group with torsion*. In the theory of one-relator groups there is a divide between those "with torsion" and those "without torsion". The remainder of this section illustrates some of these differences. First, however, we shall note that the "with torsion" label makes sense. That is, it is conceivable that there exists a one-relator group $G = \langle X; R \rangle$ where R is not a proper power but such that G has torsion. However, this cannot happen by the following result [MKS04, Theorem 4.12].

Proposition 2.9.4. *Let $G = \langle x_1, x_2, \ldots, x_m; R^n \rangle$ be a one-relator group. Suppose R is cyclically reduced and not a proper power of any element from $F(x_1, \ldots, x_m)$. If $n = 1$ then G is torsion free. If $n > 1$ then G has an element of order n and all elements of finite order are conjugates of powers of R.*

The word problem. Magnus used his Freiheitssatz and his method to prove that one-relator groups have soluble word problem. The following spelling theorem gives a stronger result in the case of one-relator groups with torsion, proving that the word problem for the standard presentation $\langle X; R^n \rangle$ of a one-relator groups with torsion is soluble by Dehn's algorithm. As with Greendlinger's Lemma for small cancellation groups, if $n >> 1$ then this theorem gives a much quicker solution to the word problem than the "more than half" needed for Dehn's algorithm. We use this spelling theorem extensively in Chapter 3. Suppose $G = \langle X; R^n \rangle$ with $n \geq 1$, then a *Gurevich subword for R^n* of a word W is a subword of W which has the form $S^{n-1}S_0$ where $S = S_0S_1$ is a cyclic shift of R or R^{-1}, and every generator which appears in R appears in S_0

Proposition 2.9.5 (Newman–Gurevich Spelling Theorem [HP84])**.** *Let $G = \langle X; R^n \rangle$, $n \geq 1$. Suppose $W =_G 1$ but W is freely reduced and not the empty word. Then W contains a Gurevich subword for R^n. If, further, W is cyclically reduced, then either W is a cyclic shift of R^n or R^{-n}, or some cyclic shift of W contains two disjoint subwords, each of which is a Gurevich subword for R^n.*

Note that the Newman–Gurevich Spelling Theorem implies that one-relator groups with torsion are hyperbolic. It is not true that all one-relator groups are hyperbolic. Indeed, no Baumslag–Solitar group $\mathrm{BS}(m,n) = \langle a,t; t^{-1}a^m t = a^n\rangle$ is hyperbolic, or even isomorphic to a subgroup of a hyperbolic group [GS91].

Note also that if G is the fundamental group of a closed, orientable surface of genus $g \geq 2$ then the Newman–Gurevich Spelling Theorem can be used in place of Greendlinger's Lemma to resolve the word problem for G. Therefore, the Newman–Gurevich Spelling Theorem generalises Dehn's original result but in a different direction to Greendlinger's Lemma.

The ends of a one-relator group. We shall prove a result which relates the number of ends of a one-relator group with torsion $G = \langle X; R^n\rangle$ to the structure of the related one-relator group *without* torsion $\widehat{G} = \langle X; R\rangle$. This result is Lemma 2.9.7, which is used in Lemma 3.1.4 and Lemma 3.1.8 from Section 3.1 to prove that if a one-relator group with torsion splits as an HNN-extension or free product with amalgamation over a virtually cyclic subgroup C then C is a subgroup of a malnormal infinite cyclic subgroup.

If we write $\Gamma(G,X)$ to be the Cayley graph of G with finite generating set X, and write $B(n)$ for the set of words over X of length less than or equal to n, then the number of *ends* of $\Gamma(G,X)$ is the limit as n tends to infinity of the number of disjoint connected components in $\Gamma(G,X) \setminus B(n)$. The ends of the Cayley graph are a group invariant, and so we can talk about *the ends of a group* as opposed to just of the Cayley graph [Mei08]. Stallings' proved the following important theorem which classifies the number of ends in a finitely generated group [Sta68, Sta71].

Proposition 2.9.6 (Stallings' Theorem on Ends of Groups). *Let G be a finitely generated group.*

- *G has zero ends if and only if G is finite.*
- *G has two ends if and only if G is virtually infinite cyclic.*
- *G has infinitely many ends if and only if either G splits as a free product with amalgamation where the amalgamating subgroup is finite and does not have index two in both of the factor groups, or G splits as an HNN-extension such that the associated subgroups are finite and are not both of index two in the base group.*
- *G has one end otherwise.*

We shall now apply Stallings' theorem to one-relator groups with torsion. Recall that a *primitive element* of a free group $F(X)$ is an element which is contained in some basis of $F(X)$. We shall generalise the following observation: a two-generator, one-relator group with torsion $G = \langle a, b; R^n \rangle$ is infinitely ended if and only if the relator R is a primitive element of $F(a, b)$, and G is one-ended otherwise. To generalise this, note that if G is an (arbitrarily-generated) one-relator group with torsion then G surjects onto a one-relator group without torsion $\widehat{G}$ in the obvious way:

$$G = \langle x_1, x_2, \ldots; R^n \rangle \twoheadrightarrow \langle x_1, x_2, \ldots; R \rangle = \widehat{G}$$

The kernel of this map is $T := \langle\langle R \rangle\rangle$. The following lemma links the structure of $\widehat{G}$ to the number of ends of G, and is essential to Section 3.1, where we prove that if a one-relator group with torsion splits as an HNN-extension or free product with amalgamation over a virtually cyclic subgroup C then C is infinite cyclic and a subgroup of a malnormal infinite cyclic subgroup which intersects T trivially.

Lemma 2.9.7. *Let $G = \langle X; R^n \rangle$ with $n > 1$ be a one-relator group with torsion. Then G is infinitely ended if and only if $\widehat{G}$ is either infinitely ended or infinite cyclic. Otherwise both G and $\widehat{G}$ are one-ended.*

The proof of Lemma 2.9.7 is based on the following three results.

Proposition 2.9.8 (Proposition II.5.10 [LS77])**.** *Let $G = \langle x_1, x_2, \ldots, x_m; S \rangle$ be a one-relator group. If G is a free group then S is a primitive element of F_m.*

Proposition 2.9.9 (Proposition II.5.13 [LS77])**.** *Let $G = \langle x_1, x_2, \ldots, x_m; S \rangle$ be a one-relator group. Suppose that S is of minimal length under $\mathrm{Aut}(F_m)$ and contains precisely the generators $x_1, x_2, \ldots, x_k$ for some $1 \leq k \leq m$. Then $G \cong G_1 * G_2$ where $G_1 = \langle x_1, \ldots, x_k; S \rangle$ is freely indecomposable and G_2 is free with basis $x_{k+1}, \ldots, x_m$.*

Proposition 2.9.10 (Fischer–Karrass–Solitar [FKS72])**.** *Suppose that $G = \langle x_1, x_2, \ldots, x_m; R^n \rangle$ with $n > 1$ is finitely generated and has more than one end. Then it has infinitely many ends and is a free product of a nontrivial free group and an indecomposable one-relator group.*

Proof of Lemma 2.9.7. Let $G = \langle x_1, x_2, \ldots, x_m; S \rangle$ be a one-relator group with torsion. Without loss of generality, we can assume that S has minimal length in $\mathrm{Aut}(F_m)$, because we can re-write S to get that $G \cong G^* = \langle x_1, x_2, \ldots, x_m; S^* \rangle$ where S^* has minimal length in $\mathrm{Aut}(F_m)$ and $\widehat{G} \cong \widehat{G}^*$.

Now, G and $\widehat{G}$ are infinite (as we always assume $m > 1$ so the Freiheitssatz yields an element of infinite order in each case). Thus, G and $\widehat{G}$ are each either one-ended, two-ended or infinitely ended, by Stallings' Theorem, while Proposition 2.9.10 tells us that G is either one-ended or infinitely ended.

Suppose G has infinitely many ends. Then by Proposition 2.9.10, G is a free product of a nontrivial free group and an indecomposable one-relator group or a finite cyclic group. Then Proposition 2.9.9 tells us that there exists some $1 \leq k < m$ such that $G \cong G_1 * G_2$ where $G_1 = \langle x_1, \ldots, x_k; S\rangle$ is freely indecomposable and G_2 is free with basis $x_{k+1}, \ldots, x_m$. Thus, $\widehat{G} = \widehat{G}_1 * G_2$. If $\widehat{G}_1$ is non-trivial then $\widehat{G}$ is infinitely ended. If $\widehat{G}_1$ is trivial then $\widehat{G}$ is free, and so is either infinite cyclic or has infinitely many ends, as required.

In order to prove the lemma, it is now sufficient to prove that if $\widehat{G}$ is infinitely ended or infinite cyclic then G is infinitely ended. Firstly, suppose that $G = \langle x_1, x_2, \ldots, x_m; R^n\rangle$, $n > 1$, is one-ended but $\widehat{G}$ is infinitely ended. Again, we can assume that $S = R^n$ has minimal length in its $\mathrm{Aut}(F_m)$ orbit, and so R also has minimal length. As $\widehat{G}$ is a one-relator group (note that it is possibly isomorphic to a free group) it can be decomposed as $\widehat{G}_1 * G_2$ where $\widehat{G}_1 = \langle x_1, \ldots, x_k; R\rangle$ is freely indecomposable (possibly trivial) and where G_2 is free, by Proposition 2.9.9. Note that as $\widehat{G} = \widehat{G}_1 * G_2$ we have that $G = G_1 * G_2$. We can then apply the fact that G is one-ended to get that G_2 is trivial, and so $\widehat{G} = \widehat{G}_1$ is freely indecomposable. As $\widehat{G}$ is torsion-free and freely indecomposable it is not infinitely ended, a contradiction. Secondly, suppose that $G = \langle x_1, \ldots, x_m; R^n\rangle$, $n > 1$, is one-ended but $\widehat{G}$ is two-ended, and again assume that R^n, and so R, has minimal length in its $\mathrm{Aut}(F_m)$ orbit. Then $\widehat{G}$ is free of rank one, because the only two-ended torsion-free group is the infinite cyclic group. Therefore, R is primitive by Proposition 2.9.8. As R has minimal length in $\mathrm{Aut}(F_m)$, G has presentation $\langle x_1, \ldots, x_m; x_1^n\rangle$ and so cannot be one-ended, a contradiction. This completes the proof of the proposition. □

Recall that our application of Lemma 2.9.7 uses the subgroup $T := \langle\langle R\rangle\rangle$. For this application we need a description of the subgroup T, and this description is given by the following result.

Proposition 2.9.11 (Fischer–Karrass–Solitar [FKS72])**.** *Suppose that $G = \langle x_1, x_2, \ldots, x_m; R^n\rangle$ with $n > 1$. Then the subgroup $T = \langle\langle R\rangle\rangle$ is isomorphic to the free product of infinitely many copies of the cyclic group of order n.*

We finish this section with the following result of Pride, which classifies the two-generator subgroups of a one-relator group with torsion. This is used in Lemma 3.1.1 and Lemma 3.1.3 of Section 3.1 to classify the virtually cyclic subgroups of one-relator groups with torsion.

Proposition 2.9.12 (Pride [Pri77b]). *Suppose that $G = \langle x_1, x_2, \ldots, x_m; R^n \rangle$ with $n > 1$. If H is subgroup of G which can be generated by two elements then H is either cyclic, a free product of cyclic groups, or is a one-ended one-relator group with torsion.*

2.10 Maps of Graphs

A *map of graphs* is a map α between graphs Γ_A and Γ, $\alpha : \Gamma_A \to \Gamma$, such that vertices are mapped to vertices, edges are mapped to edges, and such that the graph structure is preserved, that is, $\iota(e) \mapsto \iota(\alpha(e))$ and $\overline{e} \mapsto \overline{\alpha(e)}$. In this section, we shall define Wise's small cancellation theory of maps of graphs [Wis01]. This theory allows one to determine if a finitely generated subgroup of a free group is malnormal and we apply it in Lemma 4.3.6, a lemma in the proof of Theorem B from Chapter 4. The underlying idea of this theory is the analysis of fibre products of these maps, and we use this underlying idea in Lemma 4.3.11, another lemma in the proof of Theorem B from Chapter 4, to give conditions under which it is decidable to determine if a subgroup of a two-generated free group is "malcharacteristic", which is a generalisation of malnormality.

A one-relator group $\langle X; S \rangle$ is *positive* if no more than one of x and x^{-1} appears in S for all $x \in X$. For example, $\langle a, b; ab^{-2}a^3b^{-1} \rangle$ is positive. The small cancellation theory of maps of graphs was introduced by Wise [Wis01], who used it to prove that every positive $C'(1/6)$ one-relator group is residually finite. Wise further developed the theory to obtain a "graded" small cancellation theory of maps of graphs [Wis02], which he used to prove that if a one-relator group $G = \langle X; S \rangle$ is "sufficiently small cancellation" and S is "sufficiently positive" then G is residually finite. More recently, Wise has faithfully generalised this theory to the setting of cubical presentations, and cubical small cancellation theory also faithfully generalises the classical small cancellation theory. The small cancellation theory of cubical presentations is a key tool in Wise's work on groups with a quasi-convex hierarchy [Wis12]. This work resolves Waldhausen's Virtually Haken Conjecture and G. Baumslag's conjecture that all one-relator groups with torsion are residually finite. Agol has applied Wise's results on groups with a quasi-convex hierarchy to resolve Thurston's Virtually Fibering Conjecture [AGM13].

We begin this section by describing a canonical way of viewing a subgroup of a free group as a map of graphs. We then discuss fibre products in preparation for their use in the proof of Lemma 4.3.11. We then state Wise' small cancellation conditions for maps of graphs and use these to prove Proposition 2.10.5, which we apply in Lemma 4.3.6. Both of Lemma 4.3.6 and Lemma 4.3.11 are important steps in the proof of Theorem B from Chapter 3. These lemmata are each applied in a similar way. Consider the following subgroup of $G = \langle x, y; x^i, y^i, (xy)^i \rangle$, $i > 9$, where $\rho >> i$.

$$M = \langle x^3(xy^{-1})^3x^3(xy^{-1})^4 \dots x^3(xy^{-1})^{\rho+2}, x^3(xy^{-1})^{\rho+3} \dots x^3(xy^{-1})^{2\rho+2} \rangle$$

Then Lemma 4.3.6 uses Wise's small cancellation theory to prove that the lift $\overline{M}$ of M to $F(a, b)$ is malnormal, while Lemma 4.3.11 uses fibre products of maps of graphs to prove that the same subgroup $\overline{M}$ is malcharacteristic. Both of these lemmata then use classical small cancellation theory to prove that the relevant property falls down to the subgroup M of G.

Conventions. Unless otherwise stated, we shall assume the graphs Γ_A and Γ involved in a map of graphs $\alpha : \Gamma_A \to \Gamma$ are connected. We shall assume Γ is labelled, and we shall label the edges of Γ_A with their image under α. Similarly, we shall call an edge of Γ_A positive (respectively negative) if it is mapped under α to a positive (respectively, negative) edge of Γ.

Subgroups of free groups as maps of graphs. For A and B finitely generated subgroups of the free group F_k on k generators, in Chapter 4 we use maps of graphs to investigate the space consisting of subgroups $A \cap B^w$ for $w \in F_k$. Doing so allows us to determine if a subgroup is malcharacteristic in F_k. This investigation is based around computing the fibre products of maps of graphs. In order to think of a subgroup A as a map of graphs we shall view the free group on k generators as a graph Γ_F with a single vertex and k positive edges. For example, if $k = 2$ then label one positive edge by x and the other by y, and then $\pi_1(\Gamma_F) = F(x, y)$. Now, let $A = \langle a_1, \dots, a_n \rangle$ where the a_i are elements of F_k, and assume $k = 2$ (if $k > 2$ then it is analogous). Let Γ_A be the graph with a single root vertex v, where positive edges have labels from $\{x, y\}$ while negative edges have labels from $\{x^{-1}, y^{-1}\}$, such that there are n paths each beginning and ending at v, the labels of each path spells out one of the words a_i, and the path labels are in one-to-one correspondence with the generators a_i of A. We associate A with the map of graphs $\alpha : \Gamma_A \to \Gamma_F$ given by wrapping the edges of each path around the corresponding edge of Γ_F, as in Figure 2.9. We shall call the map α the *canonical map associated to the subgroup A of F_k*, or simply the *canonical map associated to A* if the free group F_k is understood. We call the loops corresponding to each generator a_i of A the *arcs*, and call the central vertex *distinguished*.

Note that every loop of Γ_A is subdivided into edges corresponding to $x^{\pm 1}$ and $y^{\pm 1}$. This ensures that the map of graphs is entirely combinatorial (so edges map to single edges and vertices map to vertices). We adhere to this convention, but it is pertinent to point out that Wise does not [Wis01].

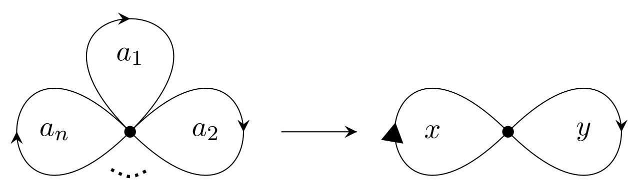

Figure 2.9: The map $\alpha : \Gamma_A \to \Gamma_F$ acts by wrapping each letter of every word a_i around the corresponding edge of Γ_F.

2.10.1 Fibre products

If A and B are subgroups of a free group F_k then fibre products of maps of graphs allow one to study the space consisting of all subgroups of the form $A^w \cap B$, $w \in F_k$. In Lemma 4.3.6 we are interested in the space when $B = A$, and this space is related to "malnormal" maps of graphs. In Lemma 4.3.11 we are interested in each of the spaces when $B = A\phi$ and ϕ is an automorphism of F_k (note that if ϕ is trivial then we get $B = A$). We begin by defining certain motivating terms.

A map of graphs $\alpha : \Gamma_A \to \Gamma$ is said to be π_1-*injective* if every essential closed path in Γ_A maps to an essential closed path in Γ. This corresponds to the map α defining an injection on the fundamental groups. A map of graphs is said to be an *immersion* if it is locally injective. Note that immersions are not necessarily embeddings. An example of a non-embedded immersion is given in Figure 2.10. Note that immersions are π_1-injective.

Recall that a subgroup H of a group G is called *malnormal* if the following implication

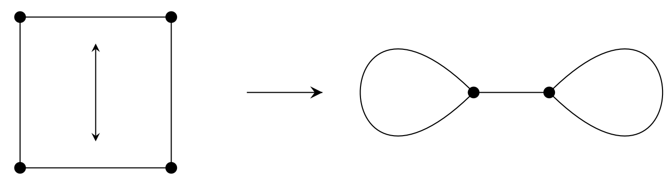

Figure 2.10: Gluing together the top and bottom edges gives an immersion.

holds.

$$H^g \cap H \neq 1 \Rightarrow g \in H$$

Denote by α_* the map on fundamental groups induced by a map of graphs $\alpha : \Gamma_A \to \Gamma$, so $\alpha_* : \pi_1(\Gamma_A, v) \to \pi_1(\Gamma, \alpha(v))$. A map of graphs $\alpha : \Gamma_A \to \Gamma$ is *malnormal* if it is an immersion and if for any two distinct vertices v_1 and v_2 in Γ_A such that $\alpha(v_1) = w = \alpha(v_2)$, the following intersection is trivial in $\pi_1(\Gamma, w)$.

$$\alpha_*(\pi_1(\Gamma_A, v_1)) \cap \alpha_*(\pi_1(\Gamma_A, v_2))$$

As Γ_A and Γ are connected, this is equivalent to $\alpha_*(\pi_1(\Gamma_A, v_1))$ being a malnormal subgroup of $\pi_1(\Gamma, w)$.

Fibre products. Let $\alpha : \Gamma_A \to \Gamma$ and $\beta : \Gamma_B \to \Gamma$ be arbitrary maps of graphs and write A and B for the respective subgroups of F_k associated to the images of Γ_A and Γ_B in Γ. We define the *fibre product* of these maps, denoted $\Gamma_A \otimes_{\Gamma_F} \Gamma_B$, to be the graph whose vertices are pairs of vertices (v_A, v_B) where $v_A \in \Gamma_A$ and $v_B \in \Gamma_B$ and whose edges are pairs of edges (e_A, e_B) where $e_A \in \Gamma_A$ and $e_B \in \Gamma_B$ are such that $\alpha(e_A) = \beta(e_B)$. Then, $A \cap B^w = 1$ for all $w \in F_k$ if and only if $\Gamma_A \otimes_{\Gamma_F} \Gamma_B$ is a forest, while A is a malnormal subgroup of F_k if and only if the non-diagonal components of the fibre product $\Gamma_A \otimes_{\Gamma_F} \Gamma_A$ are trees. Note that the fibre product is such that the diagram in Figure 2.11 commutes.

We give an example of a fibre product in Example 2.10.1, below. This example also includes “folding”.

Stallings’ foldings. Stallings’ Folding Algorithm [Sta83] allows for a simplification when computing fibre products. We use this algorithm in the remainder of this section, so we define it now. Let $\alpha : \Gamma_A \to \Gamma$ be a map of graphs. A *fold* is a map $\Gamma_A \to \Gamma_A^1$ obtained by

$$\begin{tikzcd}
\Gamma_A \otimes_{\Gamma_F} \Gamma_B \arrow[r] \arrow[d] & \Gamma_B \arrow[d, "\beta"] \\
\Gamma_A \arrow[r, "\alpha"] & \Gamma_F
\end{tikzcd}$$

Figure 2.11: $\Gamma_A \otimes_{\Gamma_F} \Gamma_A$ is the fibre product *of the maps α and β. Arrows represent maps of graphs.*

identifying two edges with a common vertex and which are mapped to the same edge of Γ. If no such edges exist then Γ_A is said to be *folded*. A fold yields a factorisation of α as $\Gamma_A \to \Gamma_A^1 \to \Gamma$. If Γ_A^1 is not folded then we can perform a folding on the map $\Gamma_A^1 \to \Gamma$. The process of repeated folding is *Stallings' Folding Algorithm*, and it terminates at a folded graph $\Gamma_A^n =: \widehat{\Gamma}_A$. The algorithm yields a composition of maps where each map $\Gamma_A^i \to \Gamma_A^{i+1}$ is π_1-surjective and the map $\widehat{\Gamma}_A \to \Gamma$ is an immersion and so π_1-injective.

$$\Gamma_A \to \Gamma_A^1 \to \Gamma_A^2 \to \cdots \to \Gamma_A^n = \widehat{\Gamma}_A \to \Gamma$$

Stallings proved that the decomposition $\Gamma_A \to \widehat{\Gamma}_A \to \Gamma$ is unique. However, the individual steps of the algorithm are not necessarily unique.

The map $\alpha : \Gamma_A \to \Gamma$ is π_1-injective if and only if the folding map $\Gamma_A \to \widehat{\Gamma}_A$ is π_1-injective, because $\widehat{\Gamma}_A \to \Gamma$ is π_1-injective. If α is π_1-injective then performing Stallings' foldings on Γ_A does not change the image of the map. Thus, one can fold Γ_A and Γ_B then take the fibre product of the these new graphs $\widehat{\Gamma}_A$ and $\widehat{\Gamma}_B$, and the fibre product $\widehat{\Gamma}_A \otimes_{\Gamma_F} \widehat{\Gamma}_B$ has the same fundamental groupoid as $\Gamma_A \otimes_{\Gamma_F} \Gamma_B$. This can be seen in Figure 2.12. Thus, $\Gamma_A \otimes_{\Gamma_F} \Gamma_B$ is a forest if and only if $\widehat{\Gamma}_A \otimes_{\Gamma_F} \widehat{\Gamma}_B$ is a forest, while the non-diagonal components of $\Gamma_A \otimes_{\Gamma_F} \Gamma_A$ are trees if and only if the non-diagonal components of $\widehat{\Gamma}_A \otimes_{\Gamma_F} \widehat{\Gamma}_A$ are trees. This means that if a pair of maps of graphs are π_1-injective, we can fold each one and then take the fibre product as opposed to simply taking the fibre product, and this yields much neater proofs.

Example 2.10.1*:* Let A be the subgroup $A = \langle x^{-1}y^{-1}xyx \rangle$ of $G = F(x, y)$. Then A is malnormal in G as maximal cyclic subgroups of free groups are malnormal. We shall verify malnormality using maps of graphs. Consider the canonical map $\alpha : \Gamma_A \to \Gamma$ associated to A. This folds as in Figure 2.13. The fibre product $\Gamma_A \otimes_\Gamma \Gamma_A$ has 25 vertices, while $\widehat{\Gamma}_A \otimes_\Gamma \widehat{\Gamma}_A$ has 9 vertices. We compute $\widehat{\Gamma}_A \otimes_\Gamma \widehat{\Gamma}_A$ in Figure 2.14. Note that the non-diagonal components of the fibre product $\widehat{\Gamma}_A \otimes_{\Gamma_F} \widehat{\Gamma}_A$ are trees. This means that the map α is malnormal, and so A is a malnormal subgroup of G.

$$
\begin{tikzcd}
\Gamma_A \otimes_{\Gamma_F} \Gamma_B \arrow[rr] \arrow[dr, "\delta"] \arrow[dd] & & \Gamma_B \arrow[d] \\
& \widehat{\Gamma}_A \otimes_{\Gamma_F} \widehat{\Gamma}_B \arrow[r] \arrow[d] & \widehat{\Gamma}_B \arrow[d] \\
\Gamma_A \arrow[r] & \widehat{\Gamma}_A \arrow[r] & \Gamma_F
\end{tikzcd}
$$

Figure 2.12: Taking the fibre product without first folding yields a space with the same fundamental groupoid as that produced by first performing Stallings' foldings and then taking the fibre product. This is because the map δ folds the fibres of $\Gamma_A \otimes_{\Gamma_F} \Gamma_B$.

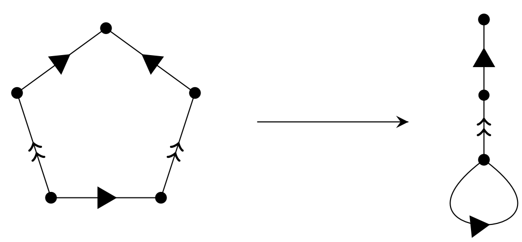

Figure 2.13: An example of folding.

2.10.2 Small cancellation theory of maps of graphs

We shall now define Wise's small cancellation theory of maps of graphs. This theory allow us to determine if a subgroup of a free group is malnormal. It does this by giving conditions which imply that the non-diagonal components of the fibre product $\widehat{\Gamma}_A \otimes_{\Gamma_F} \widehat{\Gamma}_A$ of a map of graphs $\alpha : \Gamma_A \to \Gamma$ with itself are trees, and therefore the map α is malnormal. That is, it allows one to view the structure of the fibre product $\widehat{\Gamma}_A \otimes_{\Gamma_F} \widehat{\Gamma}_A$ without actually computing the product. Note that the intuition behind the classical small cancellation theory is of fitting "circles" (disk diagrams) together to tile a disk (word problem) or an annulus (conjugacy problem). Here, the intuition is lying "lines" (paths) on top of one another. In order to present this theory we need some definitions.

A set $D \subset V_{\Gamma_A}$ is a set of *distinguished vertices* if each component of $\Gamma_A \setminus D$ is homeomorphic to an open interval and each end of this interval is connected to a vertex from D. That is, each interval consists of a reduced path with initial and terminal vertices from D. We shall refer to these intervals as *arcs*. Note that every cycle in Γ_A is the concatenation of arcs.

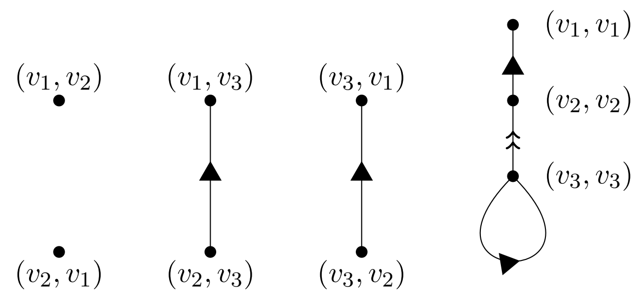

Figure 2.14: The fibre product $\widehat{\Gamma}_A \otimes_\Gamma \widehat{\Gamma}_A$ of the map α from Example 2.10.1 after performing Stallings' foldings. There are 9 vertices while $\Gamma_A \otimes_\Gamma \Gamma_A$ has 25 vertices. Note that the non-diagonal fibres are trees, so α is malnormal.

Consider a map of graphs $\alpha : \Gamma_A \to \Gamma$ with a set of distinguished vertices D such that each arc of Γ_A is immersed into Γ. A *piece* is a reduced subpath p of an arc of Γ_A such that there exists a non-equal reduced subpath q of an arc of Γ_A such that p and q are sent by α to the same path in Γ, that is, $p \neq q$ but $\alpha(p) = \alpha(q)$. Such a pair p and q is called a *piece pair*. In what follows, we shall always assume that maps are such that no arc is a piece.

Example 2.10.2*:* Consider the the canonical map associated to the subgroup $\langle a^{10} \rangle$ of $F(a)$, illustrated in Figure 2.15. Here, Γ_A is a loop with a single distinguished vertex and pieces correspond to subpaths of the loop which do not pass through (but can contain) the distinguished vertex, so to words a^i for $0 < i < 10$. However, in classical small cancellation theory the presentation $\langle a; a^{10} \rangle$ does not have any pieces.

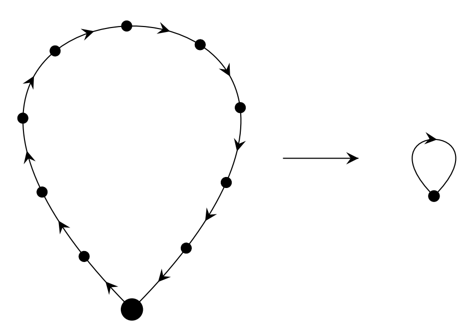

Figure 2.15: The map $\alpha : \Gamma_A \to \Gamma$ given by Example 2.10.2. The graph Γ_A consists of a single arc, beginning and ending at the distinguished (bold) vertex. Every reduced subpath of this arc is a piece.

We shall now give the small cancellation conditions.

The $c'(\lambda)$ condition: A map $\alpha : \Gamma_A \to \Gamma$ satisfies the $c'(\lambda)$ condition if every piece p which is a subpath of an arc r of Γ_A is such that $|\alpha(p)| < \lambda|\alpha(r)|$.

The $c(n)$ condition: A map $\alpha : \Gamma_A \to \Gamma$ satisfies the $c(n)$ condition if no arc r in Γ_A is the concatenation of fewer than n pieces. Note that if a map satisfies $c'(\lambda)$ then it satisfies $c(n)$ for $\lambda = 1/(n-1)$ (as with the classical theory, the inverse implication is not necessarily true). Our assumption that no arc is a piece is therefore the assumption that all our maps satisfy $c(2)$.

The $t(n)$ condition: A *cycle of m pieces* is a set of pairs of pieces $\{(p_i, q_i); 1 \leq i \leq m\}$ such that the following three conditions hold.

1. p_i and q_i are subpaths of the same arc of Γ_A.
2. p_i and q_i have the same terminal vertex, $\tau(p_i) = \tau(q_i)$.
3. p_{i+1} and q_i form a piece pair, where i is computed modulo m.

Then the map α satisfies the $t(n)$ condition if there does not exist a cycle of m pieces for $2 < m < n$. An illustration of a cycle of four pieces is given in Figure 2.16.

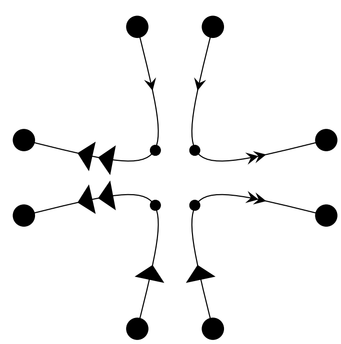

Figure 2.16: A cycle of four pieces.

Small cancellation and canonical maps. Note that if $\alpha : \Gamma_A \to \Gamma_F$ is the canonical map associated to the subgroup $A = \langle a_1, \ldots, a_k \rangle$ of F, then if the set of words $\{a_1, \ldots, a_k\}$ satisfies the classical $C(n)$ condition (respectively the $C'(\lambda)$ condition) then the map α satisfies the $c(n)$ condition (respectively the $c'(\lambda)$ condition), where D is taken to consist of only the central vertex of Γ_A. Note that this implies that the subgroup $\overline{M}$ of $F(x,y)$, defined at the beginning of Section 2.10 and used in the proof of Theorem B from Chapter 3, has associated map which satisfies $c'(\lambda)$ for some $\lambda << 1$.

The $t(n)$ condition. In the current section we focus on the $c(n)$ condition, because all the results we need follow from an analysis of the $c(5)$ case. However, we shall now briefly examine the $t(n)$ condition, specifically the $t(4)$ condition, as the $c(4) - t(4)$ conditions give the same results which we need and are applicable in our case. Reviewing the proofs of these results for the $c(4) - t(4)$ condition would be superfluous (as the proofs of the $c(5)$ case are much shorter and less involved). Therefore, we shall state these results here without proof. As with the classical theory, the $t(n)$ condition is rather cryptic. However, Wise points out that one can verify the $t(n)$ condition by constructing and analysing the "cycle graph" of the map, which is analogous to the star graph from the classical theory [EH88]. A map of graphs $\alpha : \Gamma_A \to \Gamma$ with distinguished vertices D is said to be *orientation preserving* if every arc of Γ_A is mapped to only positive or only negative edges of Γ. Cycle graphs can be use to show that an orientation preserving map satisfies $t(4)$. We shall now give an example of an orientation preserving map. Suppose a subgroup A of $F(x_1, x_2, \ldots)$ is given by a *positive* generating set, that is, $A = \langle a_1, \ldots, a_k \rangle$ and no x_i^{-1} is a subword of any a_j. Then the canonical associated map is orientation preserving, and so $t(4)$. Wise then proves that a $c(4) - t(4)$ map is malnormal. Thus, we have the following result.

Proposition 2.10.3. *If a subgroup A of a free group F is generated by a positive set of words which satisfies the $C(4)$ condition then A is a malnormal subgroup of F.*

Note that Proposition 2.10.3 can be applied to prove that the subgroup $\overline{M}$ of $F(x,y)$ is malnormal.

The $c(5)$ condition. The goal of the remainder of this section is to prove that if a subgroup A of a free group F is generated by a set of words which satisfies the $C(5)$ condition then A is a malnormal subgroup of F. We shall do this by proving that if $\alpha : \Gamma_A \to \Gamma$ is a map of graphs which satisfies the $c(5)$ condition then it is a malnormal map of graphs. This result implies that the subgroup $\overline{M}$ defined at the beginning of Section 2.10 is a malnormal subgroup of $F(x,y)$, and we use this fact in the proof of Lemma 4.3.6, which is an important step in the proof of Theorem B from Chapter 4. In the remainder of this section we prove that $c(5)$ maps are π_1-injective and malnormal, and therefore explicitly prove this property of the subgroup $\overline{M}$.

Crowns. A *circle* is a reduced path which forms a cycle and which is homeomorphic to a topological circle, that is, a reduced cycle which contains no proper subgraph which is a cycle. After folding a circle $c \subset \Gamma_A$ the image $\widehat{c}$ is either null-homotopic or after reduction is a circle. If the image is not null-homotopic we call the image a *crown*. If $\alpha : \Gamma_A \to \Gamma$ is a map of graphs then as the map $\widehat{\alpha} : \widehat{\Gamma}_A \to \Gamma$ is π_1-injective, α is π_1-injective if and only if every circle c folds to give a crown. Therefore, crowns are useful tools for determining if a map is π_1-injective. It turns out that they are also useful for showing a map is malnormal. Note that a graph C which embeds to give a crown is a cycle which reduces to a circle such that every vertex in C has degree no more than three as a vertex of C. Moreover, all valency one vertices of a crown are distinguished while all other distinguished vertices lie in the circle of the crown. An example of a crown is given in Figure 2.17. If v is a distinguished vertex contained in a crown $\mathcal{C}$ with underlying circle c then the *tail of* $\mathcal{C}$ *containing* v is the subgraph of $(\mathcal{C} \setminus c) \cup v$ containing v. A tail is *trivial* if it consists entirely of v, and note that the tail of w in Figure 2.17 is trivial. Note that tails correspond to piece-pairs. The *segments* of the underlying circle c are the paths of c which join consecutive distinguished vertices or tails of distinguished vertices. A segment and a tail have been pointed out in Figure 2.17.

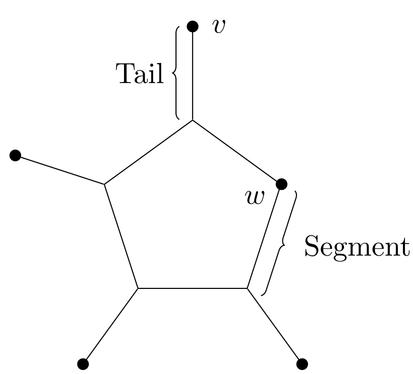

Figure 2.17: A crown with five segments and four tails. All drawn vertices are distinguished. The tail of the vertex w is trivial.

We shall now apply crowns to the injectivity of fundamental groups in the following proposition. We then apply them to show that $c(5)$ maps are malnormal. This is Proposition 2.10.5.

Proposition 2.10.4. *If a map of graphs $\alpha : \Gamma_A \to \Gamma$ satisfies $c(3)$ then α is π_1-injective.*

Proof. Begin by noting that a circle in Γ_A is the concatenation of at least two arcs, by the definition of distinguished vertices. Suppose pqr is a path in Γ_A where p, q and r are arcs and $q = stu$ where s is the maximal initial subpath of q which forms a piece pair with an initial path of $\overline{p}$ and where u is the maximal terminal subpath of q which forms a piece pair with a terminal path of $\overline{r}$. That is, after folding pqr we have Figure 2.18. By the $c(3)$ condition, t is non-trivial.

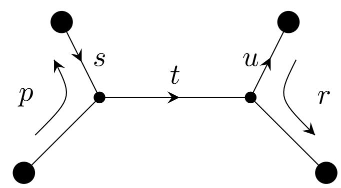

Figure 2.18: Bold vertices are distinguished. The map satisfies the $c(3)$ condition and so t is non-trivial.

Now, suppose $c = p_1 p_2 \cdots p_n$ is a circle in Γ_A, $n \geq 2$. Then each p_i is an arc and so has the form $a_i p_i' \overline{a_{i+1}}$ where each p_i' is non-trivial (and subscripts are computed modulo n). After folding we have a crown with underlying circle $\widehat{c} = p_1' p_2' \cdots p_n'$. Therefore, every essential closed path in Γ_A maps to an essential closed path in $\widehat{\Gamma}_A$ and so folding is π_1-injective. As $\widehat{\alpha} : \widehat{\Gamma}_A \to \Gamma$ is π_1-injective, we can conclude that α is π_1-injective, as required. $\square$

The following proposition implies that the subgroup $\overline{M}$ defined at the beginning of Section 2.10 is a malnormal subgroup of $F(x, y)$, which is the result we apply in the proof of Lemma 4.3.6.

Proposition 2.10.5. *If $\alpha : \Gamma_A \to \Gamma$ satisfies $c(5)$ then it is malnormal.*

Proof. Note that α is π_1-injective, by Proposition 2.10.4. Therefore, it is sufficient to prove that there does not exist circles $c_1, c_2 \subset \Gamma_A$ which are inequivalent but where $\alpha(c_1)$ and $\alpha(c_2)$ are equivalent. Suppose otherwise, and as in the proof of Theorem 2.10.4 note that $\alpha(c_1)$ folds to give a crown with underlying circle $p_1' p_2' \cdots p_n'$ contained in $\widehat{\Gamma}$. By the $c(5)$ condition, each p_i' is the concatenation of no fewer than three pieces. Similarly, $\alpha(c_2)$ folds to give a crown with underlying circle $q_1' q_2' \cdots q_n'$ and each q_i' is the concatenation of no fewer than three pieces. Consider the path $p_1' p_2'$, and write $p_1' = p_1^* z_1$ where z_1 is a piece.

Similarly, write $p_2' = x_2p_2^*$ where x_2 is a piece. As $\alpha(c_1)$ and $\alpha(c_2)$ are equivalent, the terminal end of p_1' overlaps with some path q_i'.

Note that the overlap of the terminal end of p_1' with some path q_i' must be a piece but it cannot be an internal piece of q_i', as otherwise by the definition of pieces we have that one of p_1' or q_i' is a subword of the other and so is a piece. This contradicts the $c(3)$ assumption. Therefore, the overlap must be an initial or a terminal piece of q_i', and without loss of generality it is an initial piece a_i, so $q_i' = a_iq_i^*$. However, this overlap cannot contain more than z_1, as z_1 is a piece. Therefore, the terminal end of q_i' overlaps with the initial end of p_2', and again this overlap cannot contain more that x_2. Therefore, q_i' is a subword of z_1x_2 where z_1 and x_2 are pieces. This is illustrated in Figure 2.19. This means that q_i' is the concatenation of fewer than three pieces and so the arc q_i is the concatenation of fewer than five pieces, a contradiction. This completes the proof. □

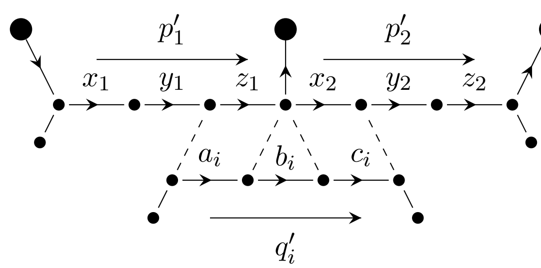

Figure 2.19: The paths a_i and z_1 form a piece-pair, as do c_i and x_2. Therefore, b_i is trivial.

2.11 Automorphisms of Graphs of Groups

In Chapter 3 we apply a result of Levitt (Propositions 2.8.2) on the outer automorphism groups of one-ended hyperbolic groups. This proposition is related to the JSJ-decomposition of these groups, that is, a canonical action of these groups on a canonically determined tree. In Chapter 4 we analyse the outer automorphism groups of certain HNN-extensions, and our analysis closely parallels certain aspects of the aforementioned work of Levitt. Levitt uses the theory of groups acting on trees while our work on HNN-extensions is entirely algebraic. In this section we review Levitt's work, so that the reader can understand the connections between Chapter 4 and the work of Levitt. We also state a technical result of Levitt, Proposition 2.11.6, which we apply in the proof of Theorem 3.1.16 from Chapter 3 to coarsely classify the outer automorphism groups of two-generator, one-relator groups using a result on the structure of their JSJ-decompositions.

It should be noted that the results we obtain regarding certain HNN-extensions in Chapter 4 do not follow from Levitt's work, and this is for the following reasons. Firstly, we have a different starting point from Levitt: we analyse "Pettet's subgroup" of the outer automorphism group (of an HNN-extension) while Levitt analyses "Levitt's subgroup". Secondly, we look at a specific class of HNN-extensions which allows us to gain more detailed results than Levitt, who works in a more general setting. Finally, our algebraic approach in Chapter 4 is applicable to mapping tori $G = H \rtimes \mathbb{Z}$, while Levitt's results are not. The application of our methods to mapping Tori is Section 4.2.3, which leads to the proof of Theorem D, and this section is similar to our approach to proper HNN-extensions in Section 4.2.2.

Levitt studies the outer automorphism group of the fundamental group of a "minimal" graph of groups which is not a "mapping torus". We define these terms now. Let G be the fundamental group of a graph of groups $\mathbf{\Gamma}$ with associated Bass–Serre tree T. We shall say that the graph of groups $\mathbf{\Gamma}$ is *minimal* if $\pi_1(\mathbf{\Gamma}')$ is a proper subgroup of $\pi_1(\mathbf{\Gamma})$ for every proper, connected subgraph Γ' of Γ. Two conditions which are equivalent to minimality are that T contains no G-invariant subtree, and that if v is a leaf of Γ then $G_e \lneq G_v$. We shall say that G is a *mapping torus* if G is isomorphic to a semidirect product $G \cong H \rtimes \mathbb{Z}$.

2.11.1 Levitt's and Pettet's subgroups

If G is given by a minimal graph of groups $\mathbf{\Gamma}$ and G is not a mapping torus then we can define and study two subgroups of $\mathrm{Out}(G)$, "Levitt's subgroup" and "Pettet's subgroup". In this section we shall define these subgroups and prove a certain connection between them. Levitt's starting point for his work on the outer automorphism groups of hyperbolic groups is Levitt's subgroup, while the starting point for our applications in Chapter 4 is Pettet's subgroup. We first define the analogues of Levitt's subgroup and of Pettet's subgroup in $\mathrm{Aut}(G)$. These both contain $\mathrm{Inn}(G)$, and so in each case we quotient out to obtain the respective subgroup of $\mathrm{Out}(G)$.

Before we define Levitt's and Pettet's subgroups we shall prove the following rather useful lemma, Lemma 2.11.1. This is used to prove that Levitt's subgroup is actually a subgroup. For G given by a graph of groups $\mathbf{\Gamma}$ with Bass-Serre tree T, we shall write g for the automorphism of T associated to $g \in G$, that is, we shall write g for the image of g in the map $G \to \mathrm{Aut}(T)$. We say that $c \in \mathrm{Aut}(T)$ *centralises* the action of G on T if

$x \cdot ch = x \cdot hc$ for all $h \in G$ and $x \in T$. By a *linear tree* we mean the infinite tree where every vertex has degree two, that is, a tree isomorphic to the Cayley graph of $\langle a; - \rangle$.

Lemma 2.11.1. *Suppose G acts minimally on a tree T. Then this action has trivial centraliser in* $\mathrm{Aut}(T)$*, or T is a linear tree and* $\mathrm{Aut}(T) \cong \mathbb{Z}$*.*

Proof. Suppose $c \in \mathrm{Aut}(T)$ and define V_c to be the vertices x of T such that the geodesics $[x, x \cdot c]$ are of minimal length. Define T_c to be the minimal subtree of T whose vertex set contains V_c. If c fixes some point then $T_c = \mathrm{Fix}_T(c)$. Otherwise, T_c is a linear subtree, and to see that T_c is a linear subtree begin by supposing otherwise. Then there exists two vertices u and v of T_c such that u, v and $u \cdot c$ do not lie in a linear subtree. Consider the geodesic $[u, u \cdot c]$. Write p_v for the geodesic connecting v to $[u, u \cdot c]$ and $p_{v \cdot c}$ for the geodesic connecting $v \cdot c$ to $[u \cdot c, u \cdot c^2]$. Note that we can replace u with $u \cdot c^i$ for some i such that p_v does not contain an edge of $[u \cdot c^{-1}, u]$ or of $[u \cdot c, u \cdot c^2]$. Then $p_{v \cdot c}$ does not contain an edge of $[u, u \cdot c]$ or of $[u \cdot c^2, u \cdot c^3]$. Because v is not contained in $[u, u \cdot c]$, we have that p_v and $p_{v \cdot c}$ have length at least one. Now, write w for the vertex where p_v meets $[u, u \cdot c]$. Then $p_{v \cdot c}$ meets $[u \cdot c, u \cdot c^2]$ at $w \cdot c$ and the geodesic $[w, w \cdot c]$ has minimal length. Thus, $[v, v \cdot c]$ is the concatenation of $p_v \cdot [w, w \cdot c] \cdot \overline{p_{v \cdot c}}$. Thus, $[v, v \cdot c]$ cannot have minimal length, a contradiction. Therefore, either $T_c = \mathrm{Fix}_T(c)$ or T_c is a linear subtree of T

Suppose that the action of G has non-trivial centraliser in $\mathrm{Aut}(T)$, and take $1 \neq c \in C_{\mathrm{Aut}(T)}(G)$. Then T_c is G-invariant, as if $[v, v \cdot c]$ is minimal then $[v \cdot g, v \cdot gc]$ is minimal by the following.

$$[v \cdot g, v \cdot cg] = [v \cdot g, v \cdot gc] = [v', v' \cdot c]$$

Therefore, because T contains no G invariant subtree (by minimality) we have that $T_c = T$ and so T is a linear tree, as required. $\square$

Levitt's subgroup. We define *Levitt's subgroup of* $\mathrm{Aut}(G)$, denoted $\mathrm{Aut}^{\mathbf{\Gamma}}(G)$, to be the set of automorphisms $\alpha \in \mathrm{Aut}(G)$ such that there exists $H_\alpha \in \mathrm{Aut}(T)$ which induces the action of α on the elements of $g \in \mathrm{Aut}(T)$ in the following sense.

$$H_\alpha g = (g\alpha) H_\alpha$$

We shall now prove that H_α is unique, and so the map $\alpha \mapsto H_\alpha$ defines an action of $\mathrm{Aut}^{\mathbf{\Gamma}}(G)$ on T. To see that H_α is unique, suppose otherwise. Then there exists $K_\alpha \in$

$\mathrm{Aut}(T)$ such that $H_\alpha \neq K_\alpha$ and $H_\alpha g H_\alpha^{-1} = K_\alpha g K_\alpha^{-1}$ for all $g \in G$. Therefore, $K_\alpha^{-1}H_\alpha \in C_{\mathrm{Aut}(T)}(G)$. However, $C_{\mathrm{Aut}(T)}(G)$ is trivial by Lemma 2.11.1 and because G is not infinite cyclic (as it is not a mapping torus). Therefore, $K_\alpha = H_\alpha$, a contradiction. Thus, $\alpha \mapsto H_\alpha$ is an action of $\mathrm{Aut}^{\mathbf{\Gamma}}(G)$ on T, as required.

Noting that $\mathrm{Inn}(G) \leq \mathrm{Aut}^{\mathbf{\Gamma}}(G)$ by the map $\gamma_g \mapsto g^{-1}$, we can define *Levitt's subgroup of* $\mathrm{Out}(G)$ to be $\mathrm{Out}^{\mathbf{\Gamma}}(G) := \mathrm{Aut}^{\mathbf{\Gamma}}(G)/\mathrm{Inn}(G)$. Levitt subjected $\mathrm{Out}^{\mathbf{\Gamma}}(G)$ to a detailed analysis and applied his results to the case where G is a one-ended hyperbolic group [Lev05], while Gilbert–Howie–Metaftsis–Raptis proved that if G is a generalised Baumslag–Solitar group, that is, G_x is infinite cyclic for all $x \in T$, then under certain conditions $\mathrm{Out}^{\mathbf{\Gamma}}(G) = \mathrm{Out}(G)$ [GHMR00].

An alternative description of $\mathrm{Aut}^{\mathbf{\Gamma}}(G)$ is that it is the maximal subgroup of $\mathrm{Aut}(G)$ such that the diagram in Figure 2.20 commutes, where θ is the canonical map from G to $\mathrm{Aut}(G)$ whose image is the inner automorphisms.

$$\begin{tikzcd} G \arrow[r, "\theta"] \arrow[d] & \mathrm{Aut}^{\mathbf{\Gamma}}(G) \arrow[dl, dotted] \\ \mathrm{Aut}(T) & \end{tikzcd}$$

Figure 2.20: The map θ is the canonical homomorphism whose image is the inner automorphisms $\mathrm{Inn}(G)$*. Levitt's subgroup* $\mathrm{Aut}^{\mathbf{\Gamma}}(G)$ *is the maximal subgroup of* $\mathrm{Aut}(G)$ *such that this diagram commutes.*

In the case of HNN-extensions, an alternative description of $\mathrm{Out}^{\mathbf{\Gamma}}(G)$ is that it is the maximal subgroup of $\mathrm{Out}(G)$ consisting of elements $\widehat{\alpha}$ with a representative α where $H\alpha = H$ and $t\alpha$ has t-length one, so $t\alpha = h_1 t^\epsilon h_2$ for some $h_1, h_2 \in H$.

Pettet's subgroup. We define *Pettet's subgroup of* $\mathrm{Aut}(G)$, denoted $\mathrm{Aut}_{\mathbf{\Gamma}}(G)$, to be the subgroup consisting of automorphisms which send each vertex group of $\mathbf{\Gamma}$ to a G-conjugate of some vertex group. As every inner automorphism sends each vertex subgroup to a conjugate of itself, we can define *Pettet's subgroup of* $\mathrm{Out}(G)$ to be $\mathrm{Out}_{\mathbf{\Gamma}}(G) := \mathrm{Aut}_{\mathbf{\Gamma}}(G)/\mathrm{Inn}(G)$.

Relating Levitt's and Pettet's subgroups. Levitt's and Pettet's subgroups are related through a common subgroup, which we shall now define. The action of Levitt's subgroup $\mathrm{Aut}^{\mathbf{\Gamma}}(G)$ on T induces an action of $\mathrm{Out}^{\mathbf{\Gamma}}(G)$ on $\Gamma = T/G$. As Γ is finite, this means that there exists a finite index subgroup of $\mathrm{Out}^{\mathbf{\Gamma}}(G)$ which acts trivially on Γ. We shall denote this subgroup by $\mathrm{Out}_0^{\mathbf{\Gamma}}(G)$, and we shall denote its pre-image in $\mathrm{Aut}(G)$ by $\mathrm{Aut}_0^{\mathbf{\Gamma}}(G)$. Karrass–Pietrowski–Solitar studied the relationship between Pettet's and Levitt's subgroups in the case of free products with amalgamation [KPS84] while M. Pettet extended their analysis to graphs of groups, proving that under certain conditions $\mathrm{Out}^{\mathbf{\Gamma}}(G) = \mathrm{Out}_{\mathbf{\Gamma}}(G)$ [Pet99]. We shall now prove that $\mathrm{Out}_0^{\mathbf{\Gamma}}(G)$ is a subgroup of Pettet's subgroup, $\mathrm{Out}_0^{\mathbf{\Gamma}} \leq \mathrm{Out}_{\mathbf{\Gamma}}(G)$, and so Levitt's subgroup virtually embeds into Pettet's subgroup. The proof begins with Lemma 2.11.2, which is an observation on the actions of G and of Levitt's subgroup $\mathrm{Out}^{\mathbf{\Gamma}}(G)$ on the edge and vertex stabilisers of T.

Lemma 2.11.2. *Let $x \in T$. If $\alpha \in \mathrm{Aut}^{\mathbf{\Gamma}}(G)$ then $G_{xH_\alpha} = G_x\alpha^{-1}$, while if $g \in G$ then $G_{x\cdot g} = g^{-1}G_xg$.*

Proof. Suppose $\alpha \in \mathrm{Aut}^{\mathbf{\Gamma}}(G)$. Then $G_{xH_\alpha} = G_x\alpha^{-1}$ because, taking $g \in G$, we have the following sequence of equivalences.

$$\begin{aligned} x \cdot H_\alpha &= x \cdot H_\alpha g \\ \Leftrightarrow x &= x \cdot H_\alpha g H_\alpha^{-1} \\ \Leftrightarrow x &= x \cdot (g\alpha) \end{aligned}$$

Suppose $g \in G$ and let $h \in G_{x\cdot g}$. Then $x \cdot gh = x \cdot g$ so $ghg^{-1} \in G_x$, and so $G_{x\cdot g} \leq g^{-1}G_xg$. Suppose $k \in G_x$ and consider $g^{-1}kg$. Then $x \cdot g \cdot g^{-1}kg = x \cdot g$, so $g^{-1}kg \in G_{x\cdot g}$, and so $g^{-1}G_xg \leq G_{x\cdot g}$. Therefore, $g^{-1}G_xg = G_{x\cdot g}$, as required. □

The following lemma immediately implies that $\mathrm{Out}_0^{\mathbf{\Gamma}}(G)$ is a subgroup of Pettet's subgroup, $\mathrm{Out}_0^{\mathbf{\Gamma}}(G) \leq \mathrm{Out}_{\mathbf{\Gamma}}(G)$, and so Levitt's subgroup virtually embeds into Pettet's subgroup.

Lemma 2.11.3. *For all $x \in T$ and all $\alpha \in \mathrm{Aut}_0^{\mathbf{\Gamma}}(G)$, G_x is conjugate in G to $G_x\alpha$.*

Proof. Take H_α such that $\alpha \in \mathrm{Aut}_0^{\mathbf{\Gamma}}(G)$, and take $x \in T$. Note that as H_α acts trivially on $\Gamma = T/G$, for all vertices $x \in T$ there exists some $g \in G$ such that $xH_\alpha = x \cdot g$. The result then holds as, applying Lemma 2.11.2, we have the following sequence of equalities.

$$\begin{aligned} G_x\alpha^{-1} &= G_{xH_\alpha} \\ &= G_{x\cdot g} \\ &= g^{-1}G_xg \end{aligned}$$

□

Note that if we are viewing $G = H*_{K^t=K'}$ as an HNN-extension then Levitt's subgroup is a subgroup of Pettet's subgroup. This is because if $\alpha \in \mathrm{Aut}^{\mathbf{\Gamma}}(G)$ then $H\alpha$ is conjugate to H. Therefore, we can view $\mathrm{Out}^{\mathbf{\Gamma}}(G)$ as the subgroup of $\mathrm{Out}_{\mathbf{\Gamma}}(G)$ consisting of elements $\widehat{\alpha}$ with a representative α where $H\alpha = H$ and $t\alpha$ has t-length one, so $t\alpha = h_1t^{\epsilon}h_2$ for some $h_1, h_2 \in H$.

In Chapter 4 we prove a technical theorem, Theorem 4.2.15, about the finite index subgroup $\mathrm{Out}_0^{\mathbf{\Gamma}}$ of Levitt's subgroup. Our main results of Chapter 4 apply this theorem, but these applications require us to prove that, in the groups we are interested in, Pettet's and Levitt's subgroups are equal.

2.11.2 Levitt's Analysis

Suppose that G is given by a minimal graph of groups $\mathbf{\Gamma}$ and that G is not a mapping torus. In this section we state a technical result of Levitt, Proposition 2.11.6, which we use in the proof of Theorem 3.1.16 from Chapter 3. The statement of Proposition 2.11.6 requires a substantial introduction, which is what this section provides. Theorem 3.1.16 is one of the two keystone theorems of Chapter 3, and it uses an analysis of the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion $G = \langle a, b; R^n \rangle$, given by Theorem 3.1.15, to prove that such a group either has virtually-cyclic outer automorphism group or is isomorphic to $\langle a, b; [a, b]^n \rangle$. Note that the results of this section are from the build-up to Levitt's result which connects the structure of the JSJ-decomposition of a one-ended hyperbolic group to the outer automorphism group of the hyperbolic group [Lev05].

We mentioned at the end of Section 2.11.1 that in our analysis of Levitt's subgroup of certain groups we study $\mathrm{Out}_0^{\mathbf{\Gamma}}(G)$. It turns out that our method of approaching this subgroup $\mathrm{Out}_0^{\mathbf{\Gamma}}(G)$ is very similar to Levitt's approach, in the sense that both investigations begin by splitting the subgroup using essentially identical short exact sequences: Levitt calls this method the *extension construction.*

The extension construction. Levitt's investigation of $\mathrm{Out}^{\mathbf{\Gamma}}(G)$ is based around a decomposition of $\mathrm{Out}_0^{\mathbf{\Gamma}}(G)$ as a short exact sequence, using a homomorphism ρ.

$$1 \longrightarrow N \longrightarrow \mathrm{Out}_0^{\mathbf{\Gamma}}(G) \xrightarrow{\rho} \mathcal{I} \longrightarrow 1$$

The groups N and $\mathcal{I}$ are then analysed, and this is what we do now. Levitt refers to this decomposition as the *extension construction*. In Theorem 4.2.15, we give a detailed analysis of this short exact sequence for the specific groups we are interested in, and classify when $\mathrm{Out}_0^{\mathbf{\Gamma}}(G) = \mathrm{Out}^{\mathbf{\Gamma}}(G)$.

The map ρ is defined by its restriction to the vertex stabilisers. That is, fix a vertex $v \in V_\Gamma$ in the graph of groups $\mathbf{\Gamma}$, then by Lemma 2.11.3 every coset $\widehat{\alpha}$ has a representative which fixes G_v, while one can prove that the class of $\beta \in \widehat{\alpha}$ in $\mathrm{Out}(G_v)$ depends only on $\widehat{\alpha}$. Therefore, for each vertex $v \in V_\Gamma$ there exists a homomorphism $\rho_v : \mathrm{Out}_0^{\mathbf{\Gamma}}(G) \to \mathrm{Out}(G_v)$. We can then extend these homomorphisms to define $\rho = \prod \rho_v : \mathrm{Out}_0^{\mathbf{\Gamma}}(G) \to \prod_{v \in V} \mathrm{Out}(G_v)$.

The image of ρ, which we shall denote by $\mathcal{I}$, preserves the peripheral structure of G_v. This is more formally given by the following lemma of Levitt (which we do not prove).

Lemma 2.11.4. *For $\widehat{\alpha} \in \mathrm{Out}_0^{\mathbf{\Gamma}}(G)$, if $\overline{\beta}$ is in the coset $\widehat{\alpha}\rho_v \in \mathrm{Out}(G_v)$ then there exists $g_e \in G_v$ such that $G_e\overline{\beta} = g_e^{-1}G_eg_e$.*

Define the (pure) mapping class group $\mathrm{PMCG}(G_v)$ to be the subgroup of $\mathrm{Out}(G_v)$ consisting of those cosets containing automorphisms which act on each edge group G_e as conjugation by some $g_e \in G_v$. The technical result of Levitt which we use in Chapter 3, which is Proposition 2.11.6, involves the relationship of the subgroup $\prod \mathrm{PMCG}(G_v)$ to the image $\mathcal{I}$. Levitt proves the following lemma, which is the first step towards proving this relationship.

Lemma 2.11.5. *For every vertex v,* $\mathrm{PMCG}(G_v)$ *is a subgroup of $\mathcal{I}$.*

Note that this does not prove that $\prod \mathrm{PMCG}(G_v) = \mathcal{I}$. This is because although $\mathcal{I}$ preserves the peripheral structure it is not necessarily acting by conjugation. That is, if $G_v\beta = G_v$ and $\iota(e) = v$ then there exists $g_e \in G_v$ such that for all $g \in G_e$ there exists $h \in G_e$ such that $g\beta = g_e^{-1}hg_e$. In $\mathrm{PMCG}(G_v)$, we require $h = g$. An example of a graph of groups $\mathbf{\Gamma}$ where $\prod \mathrm{PMCG}(G_v)$ is a proper subgroup of the image $\mathcal{I}$ is the usual graph of groups decomposition of the following Baumslag–Solitar group.

$$G = \langle a, t; t^{-1}a^2t = a^{-2} \rangle$$

Here, the underlying graph is a single vertex v with a positive loop edge e, while the stabilisers are $G_v = \langle a \rangle$ and $G_e = \langle a^2 \rangle = G_{\overline{e}}$. Consider the following map.

$$\alpha : a \mapsto a^{-1}$$
$$t \mapsto t$$

Now, α inverts the relator $t^{-1}a^2ta^2$ and so α ia a surjecvtive homomorphism of G, and then because G is Hopfian α is an automorphism [Mes72]. Clearly, $\widehat{\alpha} \in \mathrm{Out}_0^{\mathbf{\Gamma}}(G)$. Now, we shall prove that if $g \in G$ is such that $G_v\alpha\gamma_g = G_v$ then $g \in G_v = \langle a \rangle$, and we then explain why this is sufficient. So, taking such a $g \in G$ we have that $g^{-1}ag = a^i$ for some $i \in \mathbb{Z}$ and also $g^{-1}a^jg = a$ for some $j \in \mathbb{Z}$, but then $g^{-1}a^jg = a^{ij}$ and so $ij = 1$. This means that $|i| = 1 = |j|$, and so $g^{-1}ag = a^{\epsilon}$. Now, taking g to be t-reduced we have that $g^{-1}aga^{-\epsilon}$ is t-reduced so g must be t-free, so $g \in G_v = \langle a \rangle$. Therefore, α does not act on G_e by conjugation although clearly $G_e\alpha = G_e$. Thus, the coset containing $\overline{\alpha} : a \mapsto a^{-1}$ is not in $\mathrm{PMCG}(G_v)$ and so $\mathrm{PMCG}(G_v) \lneq \mathcal{I}$.

We shall now state the proposition of Levitt which we apply in Theorem 3.1.16. In Theorem 3.1.16 we wish to prove that no two-generator, one-relator group with torsion can be a bounded Fuchsian group. The following proposition, Proposition 2.11.6, allows us to side-step the definition of a bounded Fuchsian group from Section 2.8 by letting us instead prove that no two-generator, one-relator group with torsion can be the fundamental group of a vertex v of a JSJ-decomposition of a two-generator, one-relator group with torsion such that $\mathrm{PMCG}(G_v)$ is infinite.

Proposition 2.11.6. *Suppose that* $\mathbf{\Gamma}$ *is the JSJ-decomposition of a hyperbolic group* G*. Then* $\prod_{v\in V} \mathrm{PMCG}(G_v)$ *has finite index in the image* $\mathcal{I}$ *while the kernel of* ρ *is virtually-*$\mathbb{Z}^n$*, where* $n = \mathcal{E}^{\infty} - \mathcal{V}^{\infty}$*.*

Proving results about the kernel of ρ is beyond the scope of this section. However, we shall prove that $\prod \mathrm{PMCG}(G_v)$ has finite index in $\mathcal{I}$. This is based on the fact that virtually-$\mathbb{Z}$ groups have finite outer automorphism groups [Pet95, Theorem 3.4].

Proof. To prove that $\prod_{v\in V} \mathrm{PMCG}(G_v)$ has finite index in the image $\mathcal{I}$ it suffices to prove that $\mathrm{PMCG}(G_v)$ has finite index in the image of ρ_v. To do this, consider $\widehat{\phi} \in \mathrm{Im}\,\rho_v \leq \mathrm{Out}(G_v)$. Then ϕ is such that $G_e\phi = g_{(e,\phi)}^{-1}G_eg_{(e,\phi)}$ for e with $\iota(e) = v$, that is, there exists an automorphism ψ of G_e such that the following holds.

$$g\phi = g_{(e,\phi)}^{-1}(g\psi)g_{(e,\phi)} \forall\, g \in G_e$$

Consider the subgroup $N = \langle \gamma_h ; h \in N_{G_v}(G_e) \rangle$. Now, every class $\widehat{\phi} \in \mathrm{Im}\,\rho_v$ induces a fixed class $[\widehat{\psi}]$ of $\mathrm{Out}(G_e)/N$ Therefore, we have a map $\mathrm{Im}\,\rho_v \to \mathrm{Out}(G_e)/N$ whose kernel consists of the $\widehat{\phi}$ which induce the trivial outer automorphism of G_e. As $\mathrm{Out}(G_e)$ is finite, the kernel has finite index in $\mathrm{Im}\,\rho_v$. Intersecting these (finitely many) kernels yields a finite index subgroup K of $\operatorname{Im}\rho_v$ such that if $\widehat{\phi} \in K$ and e is such that $\iota(e) = v$ then $g\phi = g_{(e,\phi)}^{-1} g g_{(e,\phi)}$. Thus, $K \leq \mathrm{PMCG}(G_v)$, as required. $\square$

To prove Levitt's main theorem, Proposition 2.8.2, it would suffice to prove that if G is a hyperbolic group with JSJ-decomposition $\mathbf{\Gamma}$ then firstly the "group of twists" $\mathcal{T}$ (this is the same $\mathcal{T}$ as in Proposition 2.8.2) is virtually-$\mathbb{Z}^n$, where $n = |\mathcal{E}^\infty| - |V^\infty|$, and secondly $\mathrm{PMCG}(G_v)$ is infinite only if v is an orbifold vertex. The latter point holds as if v is an elementary vertex then G_v is virtually-cyclic so $\mathrm{Out}(G_v)$ is finite, while Levitt points out that if v is a rigid vertex then $\mathrm{PMCG}(G_v)$ is finite by Paulin's Theorem [Pau91] and by certain results of Bestvina–Feighn on Rips' theory of $\mathbb{R}$-trees [BF95] (these are substantial black-boxes). To prove that $\mathcal{T}$ is virtually-$\mathbb{Z}^n$, Levitt changes the exact sequence we are working with. We shall now give this exact sequence. Now, $\prod_{v \in V_2} \mathrm{PMCG}(G_v)$ has finite index in $\mathcal{I}$ and so its pre-image, denoted $\mathrm{Out}_1(G)$, has finite index in $\mathrm{Out}_0(G)$, and so has finite index in $\mathrm{Out}(G)$. Consider the following homomorphism.

$$\rho_1 : \mathrm{Out}_1(G) \twoheadrightarrow \prod_{v \notin V_2} \mathrm{PMCG}(G_v)$$

Take $\mathrm{Out}_2(G)$ to be the kernel of ρ_1. Then once again $\mathrm{Out}_2(G)$ has finite index in $\mathrm{Out}(G)$ and we have the following short exact sequence.

$$1 \to N_2 \to \mathrm{Out}_2(G) \to \prod_{v \in V_2} \mathrm{PMCG}(G_v) \to 1$$

Levitt proves that $\mathcal{T} = N_2$, and then that N_2 is virtually-$\mathbb{Z}^n$, and so he recovers the main result of his paper, Proposition 2.8.2.

Chapter 3

Two-Generator, One Relator Groups with Torsion

A *two-generator, one-relator group with torsion* is a group of the form $G = \langle a, b; R^n \rangle$, where R is not a true power of any element of $F(a, b)$ and where $n > 1$. The "with torsion" label is because G has torsion if and only if $n > 1$, by Proposition 2.9.4.

Two-generator, one-relator groups with torsion are interesting in their own right, but they are hyperbolic and as such they serve as important testbeds for this larger class of groups. For example, the isomorphism problem for two-generator, one-relator groups with torsion was shown to be soluble [Pri77a] long before Dahmani–Guirardel's recent resolution of the isomorphism problem for all hyperbolic groups [DG11]. Another example of one-relator groups with torsion being used in this capacity is in the residual finiteness of hyperbolic groups. Wise recently resolved the classical conjecture of G. Baumslag that all one-relator groups with torsion (equivalently, all two-generator, one-relator groups with torsion) are residually finite [Wis12], while it is still an open question as to whether all hyperbolic groups are residually finite.

Writing D_n for the dihedral group of order $2n$, the main result of this chapter is as follows.

Theorem A. *Let G be a two-generator, one-relator group with torsion.*

- *If $G \cong \langle a, b; [a, b]^n \rangle$ then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$.*
- *If G is one-ended and $G \ncong \langle a, b; [a, b]^n \rangle$ then,*
  - *If $\mathrm{Out}(G)$ is infinite then it is isomorphic to $D_\infty \times C_2$, D_∞, $\mathbb{Z} \times C_2$ or $\mathbb{Z}$.*
  - *If $\mathrm{Out}(G)$ is finite then it is isomorphic to a subgroup of D_6 or of D_4.*

- *If G is infinitely ended, so $G \cong \mathbb{Z} * C_n$, then* $\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$.

We give examples to demonstrate that every possibility occurs. Combining Theorem A with recent work of Dahmani–Guirardel on algorithms in hyperbolic groups [DG11], we give an algorithm to compute the outer automorphism group of a two-generator, one-relator group with torsion. We also explain how to write down the full automorphism group of such a group.

A general theory exists for the outer automorphism groups of hyperbolic groups, based on JSJ-decompositions. However, this theory is limited as it describes only a finite-index subgroup of the outer automorphism group. Our characterisation in Theorem A improves upon the finite-index description by combining JSJ-decompositions with faithful linear representations over $\mathbb{Z}$ of the outer automorphism groups. It is a recent result of Carette [Car13] that the outer automorphism group of an arbitrarily-generated one-relator group with torsion has a faithful linear representation over $\mathbb{Z}$, so it may be possible to generalise our method of combining the JSJ-decomposition with a faithful linear representation of the outer automorphism group to give complete descriptions of the possible outer automorphism groups of all one-relator groups with torsion.

JSJ-decompositions. The standard approach to analysing the outer automorphism groups of classes of (one-ended, non-Fuchsian) hyperbolic groups is to prove structural results for their JSJ-decompositions. Section 3.1 proves results on the structure of the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion G.

Kapovich–Weidmann proved that a one-ended two-generated torsion-free hyperbolic group has virtually-cyclic outer automorphism group [KW99]. Clearly this case is disjoint from, but similar to, the case of one-ended two-generator, one-relator groups with torsion. Indeed, the analysis of the outer automorphism groups in each of these two cases yields similar results, with the outer automorphism group of a two-generator, one-relator group with torsion being virtually-cyclic unless the group has the form $\langle a, b; [a, b]^n \rangle$. The results are similar because the possibilities for the graph underlying the JSJ-decompositions in each case are identical, and the exceptional groups are precisely the Fuchsian ones. Indeed, in Section 3.1.1 we prove that the treatment by Kapovich–Weidmann [KW99] of the structure of the JSJ-decompositions for the torsion-free case applies to the present setting with minimal alterations, while in Section 3.1.2 we give a version of Kapovich–Weidmann's proof altered to the current setting.

Previous results. Much of the previous work on the outer automorphism groups of one-relator groups with torsion is based around residual finiteness. For example, Kim–Tang proved certain specific two-generator, one-relator groups with torsion have residually finite outer automorphism groups [KT09, KT10], while it is a recent result of Carette that an (arbitrarily-generated) one-relator group with torsion has residually finite outer automorphism group [Car13]. Note that Carette uses the fact that one-relator groups with torsion are residually finite [Wis12]. Theorem A gives a complete description of the outer automorphism groups of two-generator, one-relator groups with torsion, and so this is a much stronger theorem than the previous results. Note that Theorem A implies that such outer automorphism groups are residually finite.

It is worth mentioning that our results show that the outer automorphism groups of two-generator, one-relator groups with torsion are very similar to the outer automorphism groups of one-relator groups with non-trivial center [GHMR00] and to the outer automorphism groups of Baumslag–Solitar groups $\mathrm{BS}(m,n) = \langle b, s; s^{-1}b^m s = b^n\rangle$ with $|m| \geq |n|$ and n is not a proper divisor of m [Cla06]. In each of these two cases, the groups involved are non-hyperbolic two-generator, one-relator groups without torsion, but their outer automorphism groups are found in the list given in Theorem A.

Overview of the chapter. Let G be a two-generator, one-relator group with torsion, then G is either one-ended or infinitely ended. Note that G is infinitely ended if and only if $G \cong \mathbb{Z} * C_n$, and the outer automorphism groups of such free products have been studied before [FR40, Gil87]. Therefore, in this chapter we focus on the one-ended case. Our proof of the main result, Theorem A, is built around two keystone theorems: Theorem 3.1.16, which uses JSJ-decompositions to prove that if G is one-ended and $G \not\cong \langle a, b; [a, b]^n\rangle$ then $\mathrm{Out}(G)$ is virtually-cyclic, and Theorem 3.2.1, which gives a faithful linear representation for $\mathrm{Out}(G)$ when G is one-ended.

In Section 3.1 we prove our first keystone theorem, Theorem 3.1.16. In Section 3.2 we prove our second keystone theorem, Theorem 3.2.1, and we prove that if $G \cong \langle a, b; [a, b]^n\rangle$ then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$. In Section 3.3 we determine the possibilities for $\mathrm{Out}(G)$ if G is one-ended, $\mathrm{Out}(G)$ is infinite, and $G \not\cong \langle a, b; [a, b]^n\rangle$. In Section 3.4 we determine the possibilities for $\mathrm{Out}(G)$ if G is one-ended and $\mathrm{Out}(G)$ is finite. In Section 3.5 we sketch a proof that if G is infinitely-ended then $\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$. In Section 3.6 we assemble the proof of Theorem A from the previous sections, we give an algorithm to compute $\mathrm{Out}(G)$, and we explain how to obtain a presentation for $\mathrm{Aut}(G)$.

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3.1 The JSJ-decompositions of one-relator groups with torsion

In this section we determine the possible structure of the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion. The JSJ-decomposition of a group G is a specific type of decomposition of G as a graph of groups where every edge group is virtually-cyclic. If G is a hyperbolic group then this decomposition is canonical [Bow98] and so yields information about the outer automorphism group of G, by Proposition 2.8.2. The reader is referred to Section 2.8 for more details on JSJ-decompositions. We use our main structural result on JSJ-decompositions, Theorem 3.1.15, to prove that if G is a one-ended two-generator, one-relator group with torsion then either $\mathrm{Out}(G)$ is virtually-cyclic or G is Fuchsian (and so $G \cong \langle a, b; [a, b]^n \rangle$).

JSJ-decompositions encode the virtually-cyclic splittings of a group, so we begin this section by classifying, in Lemma 3.1.4 and Lemma 3.1.8, the virtually-cyclic subgroups which a one-relator group with torsion can split over. In Proposition 3.1.13 and Proposition 3.1.14 we apply these two lemmata to prove results regarding the structure of a virtually-cyclic splitting of a two-generator, one-relator group with torsion. We then combine these two propositions to prove Theorem 3.1.15, which is our structural result on the JSJ-decompositions of one-ended two-generator, one-relator groups with torsion. We in turn apply Theorem 3.1.15 to prove Theorem 3.1.16, which proves that a one-ended non-Fuchsian two-generator, one-relator group with torsion has virtually-cyclic outer automorphism group. The purpose of this current section, Section 3.1, is to prove Theorem 3.1.16, and in Section 3.1.3 we outline an additional proof of this theorem using Nielsen transformations and equations in free groups. This second approach is in-line with the ideas of the later sections but it yields less structural information about arbitrary one-relator groups with torsion than the proof based on JSJ-decompositions of Sections 3.1.1 and 3.1.2.

The subgroup T. If $G = \langle X; R^n \rangle$ is a one-relator group with torsion, then throughout this section we shall use the fact that the subgroup $T := \langle\langle R \rangle\rangle$ is isomorphic to the free product of infinitely many cyclic groups of order n, by Proposition 2.9.11.

Two-generator subgroups. Throughout this section we shall also use the fact that if H is a two generated subgroup of an arbitrary one-relator group with torsion then H is either a one-ended one-relator group with torsion or a free product of cyclic groups, by Proposition 2.9.12.

3.1.1 Virtually cyclic splittings of one-ended one-relator groups with torsion

In this section we prove that if a one-ended one-relator group with torsion G splits as an HNN-extension or free product with amalgamation over a virtually-cyclic group C then either C is a subgroup of a malnormal infinite cyclic subgroup of G or $G \cong A*_C B \cong \langle X; R^2 \rangle$ where A and C are infinite dihedral. We do not know if this latter case, when A and C are infinite dihedral, ever occurs. However, even if this case does occur it has no impact on our analysis of the outer automorphism groups of two-generator, one-relator groups with torsion.

Therefore, the results of this section show that, with some possible exceptions, the possible JSJ-decompositions of one-ended one-relator groups with torsion are similar to the JSJ-decompositions of one-ended torsion-free hyperbolic groups, in the sense that in both cases all edge groups in a JSJ-decomposition are subgroups of malnormal infinite cyclic groups (note that all maximal virtually-cyclic subgroups of a torsion-free hyperbolic group are malnormal and infinite cyclic). This similarity allows us, in Section 3.1.2, to transfer proofs relating to the JSJ-decompositions of one-ended two-generator torsion-free hyperbolic groups [KW99] to the setting of one-ended two-generator one-relator groups with torsion.

In this current section, G is not necessarily two-generated. We begin with the following observation which we apply at numerous points in this section.

Lemma 3.1.1. *If G is a one-relator group with torsion and $x^{-1}y^i x = y^j$ with $y^i \neq 1 \neq y^j$ then $H = \langle x, y \rangle$ is either cyclic or a free product of cyclic groups.*

Proof. Note that H is either cyclic, a free product of cyclic groups, or a one-ended two-generator, one-relator group with torsion, by Proposition 2.9.12. We wish to prove that this latter case cannot happen.

Suppose $H = \langle x, y\rangle$ is a one-ended one-relator group with torsion. Then as such groups contain a single Nielsen equivalence class of generating pairs, by Proposition 2.2.2, the pair (x, y) yields a presentation $\langle x, y; S^m\rangle$. However, this is a contradiction as in such a presentation $\langle x\rangle$ and $\langle y\rangle$ are malnormal, by Proposition 2.9.2. $\square$

Virtually cyclic subgroups. In order to analyse the virtually-cyclic splittings of G we first classify, in the following lemma, the isomorphism classes of the virtually-cyclic subgroups of a one-relator group with torsion. We then classify, in a certain sense, those virtually-$\mathbb{Z}$ subgroups which are not contained in an infinite dihedral subgroup of G. Recall that for $G = \langle X; R^n\rangle$, $T := \langle\langle R\rangle\rangle$ denotes the normal closure of the element R. Thus, every element of finite order of G is contained in the subgroup T.

Lemma 3.1.2. *If C is a virtually-$\mathbb{Z}$ subgroup of a one-relator group with torsion then C is either infinite cyclic or infinite dihedral.*

Proof. Suppose C is virtually-$\mathbb{Z}$ but not infinite cyclic. We shall prove that C is a subgroup of T, $C \leq T$. As T is the free product of infinitely many cyclic groups of order n, this proves that C is infinite dihedral by the Kurosh Subgroup Theorem.

Begin by taking a subgroup C' generated by two elements g and h which are not powers of a common element, so C' is non-cyclic. However, C' is either a free product of cyclic groups or a one-ended one-relator group with torsion, and as it is two-ended it is infinite dihedral, $C' \cong C_2 * C_2$. Therefore, the subgroup C', and so C, must contain an element of order two, x say. Now, take an arbitrary element $y \in C$ of infinite order. Then $\langle x, y\rangle$ is again either a free product of cyclic groups or a one-ended one-relator group with torsion, and again it is two-ended it must be infinite dihedral. As the infinite dihedral group can be generated by two elements of finite order we have that $y \in T$. Thus, every element of C is in T, as required. This proves the lemma. $\square$

Our current goal is to prove that, apart from possibly certain specific cases, every edge group of the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion G is a subgroup of a malnormal infinite-cyclic group. We shall prove that if such a group G splits as an amalgam or HNN-extension over a virtually-cyclic subgroup C with non-virtually-cyclic edge group(s) then C cannot be contained in the normal closure $T :=$

$\langle\langle R\rangle\rangle$ of the element R, and so, by Lemma 3.1.3, below, we achieve our aim. Lemma 3.1.4 proves the result for HNN-extensions while Lemma 3.1.8 proves it for free products with amalgamation.

We now prove the following lemma, the proof of which relies on the fact that in hyperbolic groups every cyclic subgroup is contained in a unique maximal virtually-cyclic subgroup [Bow98].

Lemma 3.1.3. *Let C be a virtually-$\mathbb{Z}$ subgroup of $G = \langle X; R^n\rangle$, $n > 1$. If C is not a subgroup of an infinite dihedral group then C a subgroup of a malnormal infinite-cyclic subgroup of G.*

Proof. It suffices to prove that a maximal virtually-$\mathbb{Z}$ subgroup C of G which is not a subgroup of an infinite dihedral subgroup of G is malnormal in G. Note that such a subgroup C is in fact infinite cyclic, by Lemma 3.1.2. Suppose that our result is false: there exists some maximal virtually-$\mathbb{Z}$ subgroup C which is not malnormal but is not a subgroup of an infinite dihedral group. Then, writing $C = \langle x\rangle$, there exists $y \in G \setminus C$ and integers i, $j \in \mathbb{Z} \setminus \{0\}$ such that $y^{-1}x^iy = x^j$.

Consider the subgroup $H = \langle x, y\rangle$ which is either cyclic, a free product of cyclic groups, or a one-ended two-generator, one-relator group with torsion. Note, however, that H cannot be cyclic as C is maximal, and H cannot be a one-ended one-relator group by Lemma 3.1.1. Therefore, H must be a free product of cyclic groups, $H = K_1 * K_2$. We shall now use the conjugacy theorem for free products, which is Proposition 2.4.3, to prove that $|i| = |j|$. To see this, conjugate the generator x to some word z which is a cyclically reduced word in the free product $H = K_1 * K_2$, $z = u_1u_2 \dots u_m$. Then $(u_1u_2 \dots u_m)^i$ is conjugate to $(u_1u_2 \dots u_m)^j$, so $|m \cdot i| = |m \cdot j|$. Thus, $|i| = |j|$.

Now, consider $H_i := \langle x^i, y\rangle$. Again, this cannot be a one-ended one-relator group with torsion and it cannot be infinite cyclic as $C = \langle x\rangle$ is the unique maximal virtually-cyclic subgroup containing x^i and $y \notin C$. Thus, H_i must be a free product of cyclic groups, $H_i = K_3 * K_4$. However, H_i is the homomorphic image either of $\langle z, y; y^{-1}zy = z\rangle$ or of $\langle z, y; y^{-1}zy = z^{-1}\rangle$. Both these groups do not contain non-abelian free subgroups (they are soluble) and so H_i cannot contain non-abelian free subgroups. Thus, $H_i = K_3 * K_4$ is infinite dihedral which contradicts the fact that x^i is contained in a unique maximal virtually-$\mathbb{Z}$ subgroup which is infinite cyclic. Therefore, C is malnormal in G, as required. ∎

Virtually cyclic splittings. We shall now prove that if G is a one-ended one-relator group with torsion which splits over a virtually-cyclic subgroup C then either C is contained in a malnormal infinite cyclic subgroup of G or $G \cong A *_C B$ where A and C are infinite dihedral. Recall that if $G = \langle X; R^n \rangle$ then $T := \langle\langle R \rangle\rangle$ denotes the normal closure of the element R while $\widehat{G} = G/T = \langle X; R \rangle$ denotes the associated torsion-free one-relator group.

Lemma 3.1.4. *Assume that G is a one-ended one-relator group with torsion which splits as an HNN-extension $G \cong H *_{A^t=B}$. If A and B are virtually-cyclic groups then $A, B \not\leq T$ and so A and B are subgroups of malnormal, infinite cyclic subgroups of G.*

Proof. Suppose, without loss of generality, that $A \leq T$. As $G = \langle H, t; A^t = B \rangle$, we further have that $B \leq T$ as B is contained in the normal closure of A. Therefore, $\widehat{G}$ is the free product $\widehat{H} * \langle t \rangle$ where $\widehat{H}$ is obtained from H by quotienting out the normal closure of the torsion elements of H. Thus, by Lemma 2.9.7, G is infinitely ended, a contradiction. So $A, B \lneq T$ and the result follows from Lemma 3.1.3. $\square$

We shall prove the corresponding result, Lemma 3.1.8, for free products with amalgamation. This states that if G is a one-ended one-relator group with torsion and if $G = A *_C B$ where C is virtually-cyclic then either $C \not\leq T$, as in Lemma 3.1.4, or C and one of A or B is infinite dihedral. The proof of Lemma 3.1.8 is much longer than the HNN-case and comprises of Lemma **??**, which gives a form for one of the factor groups A or B, Lemma 3.1.6, which proves that if C is infinite cyclic then $C \not\leq T$, and Lemma 3.1.7, which proves that if G is one-ended and if $C \leq T$ then C is infinite dihedral and one of A or B is infinite dihedral.

Note that Lemma 3.1.5 and Lemma 3.1.6 include the case of $\mathbb{Z} * C_n$. This inclusion allows us to prove Proposition 3.1.13 for two-generator, one-relator groups with no restriction on the number of ends, and we wish to do this because we apply Proposition 3.1.13 in the proof of Proposition 3.1.14 and this application requires no restriction on the number of ends.

We begin our proof of Lemma 3.1.8 by giving a form for one of the factor groups A or B of $G = A *_C B$ when the the amalgamating subgroup C is subgroup of $T := \langle\langle R \rangle\rangle$. A splitting $A *_C B$ is called *non-trivial* if $C \lneq A, B$.

Lemma 3.1.5. *Assume that G is a one-ended one-relator group with torsion or $G \cong \mathbb{Z} * C_n$, for $n > 1$. Suppose that G splits non-trivially as a free product with amalgamation*

$G \cong A *_C B$ where C is a subgroup of T. Then either $A \leq T$ or $B \leq T$. If $A \leq T$ (the case of $B \leq T$ is analogous) then $A \cong F_m * A_1 * A_2 * \dots$ with each $A_i = \langle a_i \rangle$ non-trivial cyclic of order n_i dividing n and F_m is free of rank $m \geq 0$. There may be only finitely many A_i, and indeed there may be none.

Proof. Let $\widehat{A}$ and $\widehat{B}$ be the images of A and B in $\widehat{G} = G/T$. Note that the image $\widehat{C}$ of C in $\widehat{G}$ is trivial, $\widehat{C} = 1$, and so $\widehat{G} = \widehat{A} * \widehat{B}$. If G is one-ended then one of the factors $\widehat{A}$ or $\widehat{B}$ must be trivial, by Lemma 2.9.7. If $G \cong \mathbb{Z} * C_n$ then $\widehat{G}$ is infinite cyclic so again one of the factors $\widehat{A}$ or $\widehat{B}$ must be trivial. Thus, either $A \leq T$ or $B \leq T$.

Suppose that $A \leq T$. As T is the free product of infinitely many cyclic subgroups of order n, we can apply the Kurosh Subgroup Theorem to see that A is the free product of a free group with some (non-trivial) cyclic subgroups of order dividing n, $A \cong F_m * A_1 * A_2 * \dots$ with each A_i non-trivial cyclic of order n_i dividing n. This proves the lemma. □

Now, if $G = A *_C B$ is a one-relator group with torsion and C is virtually-$\mathbb{Z}$ then either C is infinite cyclic or infinite dihedral, by Lemma 3.1.2. We shall now, in Lemma 3.1.6, investigate the case when C is infinite cyclic, while in Lemma 3.1.7, below, we investigate the case when C is infinite dihedral. We shall write C_A (respectively C_B) for the copy of C in A (respectively B), and so $G = A *_{C_A = C_B} B$.

If A is a (not necessarily proper) subgroup of a group G we define the A-*normal closure* of a set $S \subset A$, denoted $\langle\langle S \rangle\rangle_A$, to be the normal closure of the set S in the abstract group A, as opposed to the normal closure of S in G. Recall that a splitting $A *_C B$ is called non-trivial if $C \lneq A, B$.

Lemma 3.1.6. *Assume that G is a one-ended one-relator group with torsion or $G \cong \mathbb{Z} * C_n$, for $n > 1$. Suppose that G splits non-trivially as a free product with amalgamation $G \cong A *_C B$ where C is infinite cyclic. Then C is not a subgroup of T.*

Proof. Suppose that C is infinite cyclic and is contained in T, and we shall find a contradiction. As $C \leq T$, we can apply Lemma 3.1.5 to get that, without loss of generality, $A \cong F_m * A_1 * A_2 * \dots$ with each $A_i = \langle a_i \rangle$ non-trivial cyclic of order n_i dividing n and F_m is free of rank m. Now, we have two cases: either the root R of the relator R^n is contained in a conjugate of A or is contained in a conjugate of B.

- Suppose $R \in g^{-1}Bg$. If A contains torsion, so $A_1 = \langle a_1 \rangle$ is non-trivial, then a_1 is conjugate in G to a power of $gRg^{-1} \in B$, and by the conjugacy theorem for free products with amalgamation, which is Proposition 2.4.3, we have that a_1 is conjugate to an element of C_A, a contradiction as C_A is torsion-free.

If A is torsion-free then $A \cong F_m$ is the A-normal closure of $C_A \cong \mathbb{Z}$, as $A \leq T$, and so the abstract group A is the normal closure of a single element. As a group with more generators than relators is infinite, we have that $m = 1$, and indeed $C_A = A$, a contradiction. Therefore, if $C_A \cong C$ is infinite cyclic then R cannot be contained in a conjugate of B.

- Suppose that $R \in g^{-1}Ag$, and by re-writing R we can assume that $R \in A$ and indeed that $R = a_1$, where $A_1 = \langle a_1 \rangle$. If A_2 is non-trivial then the generator a_2 of A_2 is conjugate to a power of a_1, and so there exists some $k \in \mathbb{Z}$ such that a_1^k and a_2 are conjugate in G but not in A. Therefore, by the conjugacy theorem for free products with amalgamation, a_1^k and a_2 are both contained in conjugates of the amalgamating subgroup C_A, a contradiction as C_A is torsion-free. Thus, $A \cong F_m * A_1$ where F_m is free and A_1 is finite cyclic.

  Now, as the subgroup T is the G-normal closure of $R = a_1$, and because A is a subgroup of T we have that C_A intersects the A-normal closure of a_1 non-trivially, $\langle\langle a_1 \rangle\rangle_A \cap C_A \neq 1$. Then, as $A/\langle\langle a_1 \rangle\rangle \cong F_m$ is torsion-free and because C_A is infinite cyclic, we have that C_A is completely contained in the A-normal closure of a_1. On the other hand, as A is contained in the G-normal closure of a_1 we have that the A-normal closure of a_1 and C_A must be the whole of A, $\langle\langle a_1, C_A \rangle\rangle_A = A$. However, as $C_A \leq \langle\langle a_1 \rangle\rangle_A$ this means that the group $A \cong F_m * A_1$ is the normal closure of a single element, and so $m = 0$. Thus, $A = A_1$ is finite cyclic, a contradiction. Therefore, if $C_A \cong C$ is infinite cyclic then R cannot be contained in a conjugate of A.

We conclude that if C is infinite cyclic then R cannot be contained in a conjugate of A or of B. This is a contradiction, as required. $\square$

We shall now analyse how a one-ended one-relator group with torsion can split as a free-product with amalgamation over an infinite dihedral group. This, combined with Lemma 3.1.6, shall complete our proof of Lemma 3.1.8. We shall again use C_A to denote the copy of the amalgamating subgroup C contained in the factor group A.

If A is a (not necessarily proper) subgroup of a group G we say two elements $g, h \in A \leq G$ are A-*conjugate* if there exists an element $k \in A$ such that $k^{-1}gk = h$. Recall that a splitting $A *_C B$ is called non-trivial if $A \lneq C$ and $B \lneq C$.

Lemma 3.1.7. *Assume that $G = \langle X; R^n \rangle$ is a one-ended one-relator group with torsion. Suppose that G splits non-trivially as a free product with amalgamation $G \cong A *_C B$ where C is infinite dihedral. Then $n = 2$ and either A or B is infinite dihedral.*

Proof. As C is infinite dihedral it is generated by two elements of order two, $C \cong C_2 * C_2$, and so $C \leq T$. We can then apply Lemma 3.1.5 to get that, without loss of generality, $A \leq T$ and $A \cong F_m * A_1 * A_2 * \ldots$ with each $A_i = \langle a_i \rangle$ non-trivial cyclic of order n_i dividing n and F_m is free of rank m. Then $m = 0$, because A is equal to the A-normal closure of C_A along with the finite cyclic factors, $A = \langle\langle C_A, A_1, A_2, \ldots \rangle\rangle_A$. Note that $A_1 = \langle a_1 \rangle$ and $A_2 = \langle a_2 \rangle$ must be non-trivial as otherwise A is finite cyclic. Suppose, without loss of generality, that A_1 is of maximal order in the free-factor groups $A_i = \langle a_i \rangle$.

In order to prove the lemma it is sufficient to prove that A is infinite dihedral. To prove this we shall use the fact that, for all $i > 1$, a_1 and a_i are both conjugates of powers of the element R. There are two cases: either R is contained in a conjugate of A or R is contained in a conjugate of B.

- Suppose $R \in g^{-1}Bg$. Then by the conjugacy theorem for free products with amalgamation we have that conjugates of a_1 and a_i are contained in the free-factor group C_A, and so a_1 and each a_i for $i > 1$ have order two. Suppose A is not infinite dihedral, then $A_3 = \langle a_3 \rangle$ is non-trivial. As A is a free product with a_1, a_2 and a_3 in different free factors we have that these three elements are pairwise non-conjugate in A. However, they are each A-conjugate to an element of C_A and C_A is infinite dihedral. As infinite dihedral groups have precisely two conjugacy classes of elements of finite order, we have that two of a_1, a_2 and a_3 are A-conjugate, which is a contradiction. Thus, A is infinite dihedral, as required.

- Suppose $R \in g^{-1}Ag$. Then we can re-write R to get that $R \in A$, and indeed that $R = a_1$ (because A_1 has maximal order in the subgroups A_i). Therefore, a_i is a G-conjugate of a power of a_1, a_1^k say, but not an A-conjugate of a_1^k. Thus, applying the conjugacy theorem for free products with amalgamation, we have that an A-conjugate of $a_1^{n/2}$ is contained in C_A and an A-conjugate of a_i is contained in C_A. As with the previous case, applying the fact that the infinite dihedral group has two conjugacy classes of elements of order two yields that $A = A_1 * A_2$ with $A_2 \cong C_2$, and here $A_1 \cong C_n$.

Note that we can rewrite G using some inner automorphism so that we can assume $a_1^{n/2}$ is contained in C_A, and we shall write $\hat{a}_2 := h^{-1}a_2h$ for the conjugate of a_2 contained in C_A, so $a_1^{n/2}, \hat{a}_2 \in C_A$. We shall now prove that we can assume that $C_A = \langle a_1^{n/2}, \hat{a}_2\rangle$. To see this, note that $a_1^{n/2}$ and $\hat{a}_2$ are both contained in C_A but are not conjugate in $C_A \cong C_2 * C_2$. As the infinite dihedral group has two conjugacy classes of elements of order two, we have that C_A is generated by a C_A-conjugate of $a_1^{n/2}$ and a C_A-conjugate of $\hat{a}_2$. Therefore, $C_A = \langle h_1^{-1}a_1^{n/2}h_1, h_2^{-1}\hat{a}_2h_2\rangle$ where $h_1, h_2 \in C_A$, and we can rewrite G such that $C_A = \langle a_1^{n/2}, h_3^{-1}\hat{a}_2h_3\rangle$ where $h_3, \hat{a}_2 \in C_A$. Now, $C_A = \langle a_1^{n/2}\rangle * \langle h_3^{-1}\hat{a}_2h_3\rangle$ because both $a_1^{n/2}$ and $h_3^{-1}\hat{a}_2h_3$ have order two and have no relations between them (otherwise C_A would not be infinite dihedral), and so as $\hat{a}_2 \in C_A = \langle a_1^{n/2}\rangle * \langle h_3^{-1}\hat{a}_2h_3\rangle$, we have that $h_3 \in \langle a_1^{n/2}\rangle$. Therefore, $C_A = \langle a_1^{n/2}, \hat{a}_2\rangle$, as required.

To complete the proof of this case it is sufficient to prove that $n = 2$, as $A \cong C_n * C_2$. Now, $G = \langle X; R^n\rangle$ and consider $G' = \langle X; R^{n/2}\rangle$. Then the image of C in G' is trivial and so $G' \cong A' * B'$ where $A' \cong C_{n/2}$ and B' is non-trivial (as $\widehat{G'} \cong \langle X; R\rangle$ is non-trivial and B' surjects onto $\widehat{G'}$). Thus, G' is infinitely ended. Then, because $\widehat{G'} \cong \widehat{G}$ we can apply Lemma 2.9.7 to get that G is infinitely ended, a contradiction. Thus, $n = 2$ and A is infinite dihedral, as required.

We therefore conclude that if $G = \langle X; R^n\rangle$ is one-ended and splits as a free product with amalgamation $G \cong A *_C B$ where C is infinite dihedral then $n = 2$ and either A or B is infinite dihedral, as required. $\square$

We now give our classification of the ways in which a one-ended one-relator group with torsion and the group $\mathbb{Z} * C_n$ can split as a free product with amalgamation over a virtually-cyclic subgroup. Recall that for $G = \langle X; R^n\rangle$, $T := \langle\langle R\rangle\rangle$ denotes the normal closure of the element R and that a splitting $A *_C B$ is called non-trivial if $C \lneq A, B$.

Lemma 3.1.8. *Let $G = \langle X; R^n\rangle$ be a one-ended one-relator group with torsion. If G splits non-trivially as a free product with amalgamation $G \cong A *_C B$ over a virtually-$\mathbb{Z}$ subgroup C then one of the following occurs.*

- *$C \not\leq T$, and so C is a subgroup of a malnormal, infinite cyclic subgroup of G.*
- *$n = 2$ and both C and A are infinite dihedral.*
- *$n = 2$ and both C and B are infinite dihedral.*

Proof. Recall that by Lemma 3.1.2, C is either infinite cyclic or infinite dihedral. By Lemma 3.1.3, if $C \not\leq T$ then C is a subgroup of an infinite cyclic, malnormal subgroup of G. Thus, it suffices to prove that if $C \leq T$ then C is infinite dihedral, $n = 2$, and one of A or B is infinite dihedral. So, suppose $C \leq T$. Then, by Lemma 3.1.6, C is infinite dihedral, and then by Lemma 3.1.7 $n = 2$ and one of A or B is infinite dihedral, as required. □

We conclude Section 3.1.1 with the following theorem, which gives a description of the JSJ-decompositions of one-ended one-relator groups with torsion.

Theorem 3.1.9. *Suppose $G = \langle X; R^n \rangle$ is a one-ended one-relator group with torsion. Suppose $n > 2$. If v is an elementary vertex in the JSJ-decomposition of G then $G_v \not\leq T$. Therefore, every elementary vertex group is a subgroup of malnormal, infinite cyclic subgroup of G. Suppose $n = 2$. Then there is the additional possibility of elementary vertices of degree one whose vertex groups and adjacent edge groups are infinite-dihedral.*

Proof. Let Γ be the graph underlying the JSJ-decomposition of G. Let v be an arbitrary elementary vertex of Γ and let e be an edge incident to v. We shall prove that either $G_e \not\leq T$ (which implies $G_v \not\leq T$) or $n = 2$ and v is an elementary vertex of degree one whose vertex group and adjacent edge group are each infinite dihedral. This proves the theorem. Recall that G_e is virtually-$\mathbb{Z}$.

Suppose e is *not* a separating edge of Γ. Then G splits as an HNN-extension over G_e, so $G = A*_{G_e^t = G_e'}$. Thus, by Lemma 3.1.4, we have that $G_e \not\leq T$, as required.

Suppose e is a separating edge of Γ. Then G splits as a free product with amalgamation over G_e, so $G = A *_{G_e} B$ where $G_e \lneq A, B$. Thus, by Lemma 3.1.8, we have that either $G_e \not\leq T$ or $n = 2$ and one of A or B is infinite dihedral. Suppose, without loss of generality, that A is infinite dihedral, then A corresponds to an elementary vertex v of degree one in the JSJ-decomposition, where $G_v = A$ is infinite dihedral and G_e is infinite dihedral, by Lemma 3.1.8, as required. □

3.1.2 JSJ-decompositions in the two-generator case

In this section we give, in a certain sense, the possibilities for the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion. We begin with three technical lemmata, which we then combine with Lemma 3.1.4 and Lemma 3.1.8 to see how a two-generator, one-relator group with torsion can split over a virtually-cyclic subgroup. We do this for splittings as free products with amalgamation in Proposition 3.1.13 and for splittings as HNN-extensions in Proposition 3.1.14, and both of these propositions have substantial proofs. We then apply this analysis in Theorem 3.1.15 to determine the possible graphs underlying the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion, and in Theorem 3.1.16 we apply this structural theorem to obtain a finite-index description of the outer automorphism groups of these groups. The purpose of the later sections of this chapter, beginning at Section 3.2, is to use this finite-index description to give a classification of these outer automorphism groups up to isomorphism.

Applying torsion-free results. One-ended two-generator torsion-free hyperbolic groups have a very strict structure, in the sense that the graph underlying the JSJ-decomposition (in the sense of Sela) of such a group is either a single vertex or a single vertex with a loop edge. This was proven in a paper of Kapovich–Weidmann [KW99]. Now, Lemma 3.1.4 and Lemma 3.1.8 prove that the virtually-$\mathbb{Z}$ subgroups which a one-ended two-generator, one-relator group with torsion can split over are similarly placed in the groups as the virtually-$\mathbb{Z}$ subgroups of torsion-free hyperbolic groups, in the sense that they are subgroups of malnormal infinite cyclic subgroups (apart from when $G \cong A \ast_C B$ with C and one of A or B infinite dihedral, but, it turns out, we can ignore this case). This observation allows us to apply the proofs from Kapovich–Weidmann's paper with very little modification to the setting of two-generator, one-relator groups with torsion. This section, especially Propositions 3.1.13 and 3.1.14, parallels their argument. Theorem 3.1.16, which is our finite-index description of the outer automorphism groups of two-generator, one-relator groups with torsion, then follows.

We begin with the following technical lemma, Lemma 3.1.10, on malnormality. It is applied in Proposition 3.1.13 and is proven using Propositions 2.4.3 and 2.4.4, which deal with, respectively, conjugation and commutativity in free products with amalgamation. Lemma 3.1.10 appears in Kapovich–Weidmann's paper without proof, but we include the proof for completeness.

Lemma 3.1.10. *Suppose $G = A \ast_C B$ where $A = \langle h \rangle$ is infinite cyclic and where C is malnormal in B. Then A is malnormal in G.*

Proof. Suppose that A is not malnormal in G. Then there exists some integers $i, j \in \mathbb{Z}$ and some $g \in G$ such that $g^{-1}h^i g = h^j$. Note that both $L(h^i) \leq 1$ and $L(h^j) \leq 1$, which means that one of (1) or (2) from Proposition 2.4.3 is applicable here (but (3) is never applicable). There are two cases: Either $L(h^i) = 0$ or $L(h^i) = 1$. The proof of the former case is substantially longer then the latter case, so we begin with the shorter case.

Firstly, suppose that $L(h^i) = 1$, and so $h^i \in A \setminus C$. Then, h^i and h^j are conjugate in A, so $i = j$. Thus, g centralises h^i, and applying Proposition 2.4.4 we have that $g \in A$, as required.

Secondly, suppose that $L(h^i) = 0$, and so $h^i \in C$. Then h^j is conjugate to h^i and so, applying (1) from Proposition 2.4.3, there is a sequence

$$h^i, c_1, c_2, \ldots, c_t, h^j$$

where $c_k \in C\setminus\{1\}$ and consecutive terms in the sequence are conjugate in a factor. Assume that $t \geq 1$ is minimal for such a sequence connecting h^i and h_j. Then h^i is conjugate to c_1 in a factor. As C is malnormal in B, we have that h^i is conjugate to c_1 in A. However, A is abelian and so $h^i = c_1$. Thus, we can remove c_1 from the sequence, which contradicts the minimality of t. Therefore, we have that h^i is conjugate to h^j in a factor, and again using the malnormality of C in B this factor must be A, so $h^i = h^j$ which implies that $i = j$.

Write g using the normal form for free products with amalgamation, so $g = w_1 \ldots w_t$ and if $t > 1$ then $w_i \notin C$. We shall prove that $t = 1$ which proves that $g \in A$, as required, because C is malnormal in B. Now, $L(w_t^{-1} \ldots w_1^{-1} h^i w_1 \ldots w_t) = 0$ because $g^{-1}h^i g = h^i \in C$. The only place where reduction can happen is over the subword $w_1^{-1}h^i w_1$. There are three cases: $w_1^{-1}h^i w_1 \in C$, $w_1^{-1}h^i \in C$, or $h^i w_1 \in C$. If either of the latter two cases happen then $w_1 \in C$ and so $t = 1$, as required. So, suppose $w_1^{-1}h^i w_1 \in C$ and suppose $t \geq 2$, then as C is malnormal in B we have that $w_1 \in A$ and so g is equal to the following word, where $h^{i_0} \in C$.

$$w_t^{-1} \ldots w_2^{-1} h^{i_0} w_2 \ldots w_t$$

Again, reduction must happen in this word, and the only place where this can happen is over the subword $w_2^{-1}h^{i_0}w_2$, and again because $w_2 \notin C$ we require $w_2^{-1}h^{i_0}w_2 \in C$, which, again using the malnormality of C in B, implies that $w_2 \in A$. This means that $w_1, w_2 \in A$, and therefore $w_1 w_2 \ldots w_t$ is not a normal form for g, a contradiction. Thus, $t = 1$ and so $g \in A$, which lets us conclude that A is malnormal in G, as required. □

The following lemma tells us how virtually-cyclic subgroups not contained in T are embedded into $G = \langle X; R^n \rangle$, and is the analogue of Kapovich–Weidmann's Proposition 3.4. Recall that if $G = \langle X; R^n \rangle$ then $T := \langle\langle R \rangle\rangle$ denotes the normal closure of the word R.

Lemma 3.1.11 (Proposition 3.4, Kapovich–Weidmann). *The following properties hold for $G = \langle X; R^n \rangle$ an arbitrary one relator group with torsion.*

- *Every cyclic subgroup of G which is not a subgroup of T is contained in a unique maximal cyclic subgroup of G.*
- *Every cyclic subgroup of G which is not a subgroup of T is maximal cyclic if and only if it is malnormal.*
- *If a virtually-cyclic subgroup has non-trivial intersection with T then it is wholly contained in T.*

Proof. Recall that the only virtually-cyclic subgroups of G are either cyclic or infinite dihedral, by Lemma 3.1.3.

To prove the first point, suppose $\langle g \rangle \not\leq T$ and let C be the unique maximal virtually-cyclic group containing g (recall that this subgroup C exists because G is hyperbolic). If C is not cyclic it must be infinite dihedral, and thus can be generated by two elements of order two. Thus, $C \leq T$, so $g \in T$, a contradiction.

We shall now prove the second point. Suppose $C \not\leq T$ is malnormal but not maximal cyclic. Then $C \lneq C_0$ with C_0 cyclic. Let $g \in C_0 \setminus C$, then g commutes with every element of C and so $C^g = C$. This contradicts the malnormality of C. Therefore, C is maximal cyclic, as required. On the other hand, suppose $C \not\leq T$ is maximal cyclic in G. Then by Lemma 3.1.3, C is contained in a malnormal cyclic subgroup C_0 of G. As C is maximal cyclic we have that $C = C_0$, so C is malnormal in G.

To prove the third point, note that $G/T = \langle X; R \rangle$ where R is not a proper power. Thus, G/T is torsion-free. Now, suppose C is cyclic and that $C \cap T \neq 1$, then G/T is torsion free so $C \leq T$. As infinite dihedral groups are wholly contained in T, this proves the third point. $\square$

Separated subgroups. Let G be a group and let A and B be torsion-free subgroups of G. Then A and B are said to be *separated*[1] if for every $g \in G$ the following holds.

$$A^g \cap B = 1$$

[1] Kapovich–Weidmann call this concept "conjugacy separated". However, Kharlampovich–Myasnikov reserve the term "conjugacy separated" for a different notion, and this notion underlies the results from which Lemma 3.1.12 follows [KM98]. Our definition of "separated" is related to Kharlampovich–Myasnikov's definition of a "separated HNN-extension" [KM98].

The following lemma is an immediate consequence of Propositions 2.4.9 2.4.10 2.4.12 and 2.4.11 because G is hyperbolic. It allows us to apply malnormality in our analysis of virtually-cyclic splittings.

Lemma 3.1.12 (Lemma 3.6, Kapovich–Weidmann)**.** *Let G be an arbitrary one-relator group with torsion.*

- *Suppose that G is an amalgamated free product of the form $A *_{\langle a=b \rangle} B$ where a and b are non-trivial elements of infinite order. Then at least one of the subgroups $C_A = \langle a \rangle$ and $C_B = \langle b \rangle$ is malnormal in A and B respectively.*
- *Suppose that G is an HNN-extension of the form $\langle H, t; a^t = b \rangle$ where a and b are non-trivial elements of H. Then $A = \langle a \rangle$ and $B = \langle b \rangle$ are separated in H. Moreover, at least one of the subgroups A, B is malnormal in H.*
- *If $G = A *_{\langle a=b \rangle} B$ then A and B are hyperbolic.*
- *If $G = \langle H, t; a^t = b \rangle$ then H is hyperbolic.*

We shall now prove Proposition 3.1.13, which is our first main technical result of this section. This result classifies how a two-generator, one-relator group with torsion G can split as a free product with amalgamation over an infinite cyclic subgroup. Note that we make no assumption about the number of ends of G, while in the analogous result for HNN-extensions, which is Proposition 3.1.14, we assume that the group G is one-ended. We cannot assume one-ended in the following proposition as in the HNN-case we apply Proposition 3.1.13 to an arbitrary two-generator, one-relator group with torsion, making no assumptions about the number of ends. At certain points in the proof of Proposition 3.1.13 we split our analysis into the one-ended and the infinitely ended case, so the case when G is freely indecomposable and the case when $G \cong \mathbb{Z} * C_n$, although whenever we do this we obtain identical intermediate results for both cases.

Proposition 3.1.13 (Proposition 3.7, Kapovich–Weidmann)**.** *Let $G = A *_C B$ be a two-generator, one-relator group with torsion and let $C = \langle c \rangle$ be infinite-cyclic. Assume also that the splitting is non-trivial, that is $A \neq C$ and $B \neq C$. Then either A or B is cyclic. In the case that A is cyclic (the case that B is cyclic is analogous) we further get the following.*

1. *The group G is an amalgam of the form $G = \langle a \rangle *_{\langle a^m = c \rangle} B$ where a generates $A = \langle a \rangle$ and $C = \langle a^m \rangle = \langle c \rangle$ is malnormal in B.*

2. *There exists an element $b \in B$ such that $G = \langle a, b \rangle$ and $B = \langle a^m = c, b \rangle$. In particular B is also two-generated.*

Proof. As C is cyclic it is not a subgroup of T, by Lemma 3.1.8, and has trivial intersection with the subgroup T, by Lemma 3.1.11. We shall begin by proving that, simultaneously, either A or B is cyclic and that (1) holds. By Lemma 3.1.12, C is malnormal in A or B, so suppose, without loss of generality, that C is malnormal in B. Now, C is contained in a unique maximal virtually-cyclic subgroup A_0 of G, and because C intersects T trivially A_0 is infinite cyclic while by Lemma 3.1.11 we have that A_0 is malnormal in G. Note that $A_0 \leq A$ because C is malnormal in B. Now, by Lemma 3.1.12, A is hyperbolic and so C is contained in a unique maximal, virtually-cyclic subgroup of A, and this must be A_0. We shall now prove that $A_0 = A$, which will complete the proof of (1). Assume otherwise, so $A_0 \lneq A$, and we shall look for a contradiction. Then G can be written as an amalgam in the following way.

$$\begin{aligned} G &= A *_C B \\ &= A *_{A_0} A_0 *_C B \\ &= A *_{A_0} B_0 \end{aligned}$$

Suppose G is one-ended. Then we know that A_0 is malnormal in A while, by Lemma 3.1.10, its image is malnormal in $B_0 = A_0 *_C B$. However, one-ended two-generated groups cannot have the form $P *_Q R$ where Q is malnormal in both P and R, by Proposition 2.4.8. Thus, we have our required contradiction, and so $A = A_0$. Therefore, (1) holds for G one-ended.

Now, suppose G is infinitely ended. Then $G \cong \mathbb{Z} * C_n = \langle x, y; x^n \rangle$, and recall that $G = A *_{A_0} B_0$ where $A_0 \cong \mathbb{Z}$. Consider the generator $x \in G$ of finite order. As it has finite order it must be contained in one of the factor groups A or B_0. However, $x \in T$ while A_0 intersects T trivially and so A_0 is not contained in the normal closure of x. This means that when we quotient out the subgroup T to obtain $\widehat{G}$ we yield an amalgam where one of A or B_0 is unaffected, so we have that $\widehat{G} = A *_{A_0} \widehat{B_0}$ or $\widehat{G} = \widehat{A} *_{A_0} B_0$. However, $\widehat{G}$ is infinite cyclic and so the first case implies that $A = A_0 = \widehat{B_0}$ while the second case implies that $\widehat{A} = A_0 = B_0$. However, by assumption we have that $A_0 \neq A$, and so $A_0 = B_0$. This means that $B = C$, which contradicts the non-triviality of the splitting $G = A *_C B$. Thus, we have our required contradiction, and so $A = A_0$. This completes the proof of (1).

In order to prove (2) we consider a generating pair $X = (p, q)$ of G. By Corollary 2.4.7, this pair is Nielsen equivalent to another generating pair X' such that there is a subset

$\overline{X}$ of X' that lies in a conjugate of either A or B, and there is an element of $\langle \overline{X} \rangle$ that is conjugate to an element of $C \setminus \{1\}$. Now, if $|\overline{X}| = 2$ then $\overline{X} = X'$. This means that X' lies in a conjugate of A or of B, and so cannot generate G, a contradiction. Thus, $\overline{X}$ consists of a single element x such that x^k is conjugate to an element of C for some integer k. Without loss of generality, we can assume that $x^k \in C$, as we can conjugate X' and conjugation is a Nielsen transformation. Therefore, x is contained in the maximal infinite cyclic subgroup of G containing C, which is the subgroup $A = \langle a \rangle$. Thus, $G = \langle g, a \rangle$ for some $g \in G$. We shall prove that $g \in B$ and then that $B = \langle a^m = c, g \rangle$, which will complete the proof of (2).

Pick g such that the length of g, $L(g)$, is minimal such that g generates G with a. Then the normal form of g begins and ends with elements of $B \setminus C$, and so g is of the following form where $u_i \in B \setminus C$ and $v_i \in A \setminus C$.

$$g = u_1 v_1 \dots u_{l-1} v_{l-1} u_l$$

We shall prove that $l = 1$, and so $g \in B$. It is then sufficient to prove that $B = \langle c = a^n, g \rangle$. Assume $l \geq 2$, so $L(g) \geq 3$.

We begin by showing that we can assume $u_l c^k u_1 \notin C$ for all non-zero integers $k \in \mathbb{Z} \setminus \{0\}$. To do this, suppose $u_l c^k u_1 \in C$ for some non-zero integer $k \in \mathbb{Z} \setminus \{0\}$. Then replace g with gc^k (so replace u_l with $u_l c^k$) where $\langle c \rangle = C$. As C is a subgroup of both A and B, this does not change the length of g, and because $c \in \langle a \rangle$ we have that G is still generated by a and the new g. It then holds that in the new g, $u_l c^p u_1 \notin C$ for all non-zero integers $p \in \mathbb{Z} \setminus \{0\}$, and this is because $u_l u_1 \in C$ so $u_l = c^q u_1^{-1}$ for some integer $q \in \mathbb{Z}$, and so $u_l c^p u_1 = c^q u_1^{-1} c^p u_1 \notin C$ by the malnormality of C in B (recall that $u_1 \in B \setminus C$). Therefore, we can assume that $u_l c^k u_1 \notin C$ for all non-zero integers $k \in \mathbb{Z} \setminus \{0\}$.

Our next step is to show that any positive power g^k has a normal form of type $u_1 \dots u_l$ and $L(g^k) \geq 3$. This is clear unless the normal form of g is of one of the following two forms.

(i) $g = w a^i w^{-1}$ with $w = u_1 v_1 \dots u_t$ and $a^i \in A \setminus C$.

(ii) $g = w b w^{-1}$ with $w = u_1 v_1 \dots u_t$ and $b \in B \setminus C$.

Case (i) cannot happen as then w generates G with a, but as $L(w) < L(g)$ this is a contradiction. In (ii), the normal form for g^k has the following form and so is of the required type.

$$U_1 V_1 \dots V_{t-1} (U_t b^k U_t^{-1}) V_{t-1}^{-1} \dots V_1^{-1} U_1^{-1}$$

We shall write $g^k = u_1 V_k u_l$ where V_k has a normal form which begins and ends with elements of A.

We shall now combine the assumption that $u_l c^k u_1 \notin C$ for all non-zero integers $k \in \mathbb{Z} \setminus \{0\}$ with the above fact regarding the form of g^k to prove that the number l in the normal form of g cannot be greater than one, so $l \ngeq 2$, and so $g \in B$ (because g generates G with $a \in A$). To do this, pick some element $h \in B \setminus C$. As $L(h) = 1$ and $h \notin A$, h is not a power of a nor of g (all powers of g have length greater than one). Combining this with the fact that $G = \langle a, g \rangle$, we have that $h = a^{i_0} g^{j_0} \dots a^{i_p} g^{j_p}$ where $i_q \neq 0$ for $0 < q \leq p$ and $j_q \neq 0$ for $0 \leq q < p$. This means that h has the following form, where $w_0 = a^{i_0} u_1$ and $w_t = u_l a^{i_t} u_1$ for $0 < t \leq p$.

$$\begin{aligned} h &= (a^{i_0} u_1 V_{j_0} u_l)(a^{i_1} u_1 V_{j_1} u_l) \dots (a^{i_{p-1}} u_1 V_{j_{p-1}} u_l)(a^{i_p} u_1 V_{j_p} u_l) \\ &= w_0 V_{j_0} w_1 V_{j_1} w_2 \dots w_{p-1} V_{j_{p-1}} w_p V_{j_p} u_l \end{aligned}$$

Note that $w_0 = a^{i_0} u_1 \notin A$ as $u_1 \in B \setminus C$. Also note that $w_t \notin A$ for $0 < t \leq p$ and to see this suppose otherwise. Then $u_l a^{i_t} u_1$ is not a normal form so $a^{i_t} \in C$ (because $u_l, u_1 \in B \setminus C$). This means that $u_l a^{i_t} u_1 \in B$, but that we have that $u_l a^{i_t} u_1 \in A \cap B = C$, which contradicts the assumption that $u_l c^k u_1 \notin C$ for all non-zero integers k. We now observe that $L(h) \geq L(w_0) + L(V_{j_0}) + 1 \geq 3$, where the final "+1" comes from w_1 if $p > 1$ and from u_l otherwise. This means that the case $l \geq 2$ is impossible, and so $g \in B$.

Therefore, there exists a generating pair (a, b) of G where $\langle a \rangle = A$ and $b \in B$. To complete the proof of (2), it suffices to prove that B is generated by b and $c = a^m$ where $C = \langle c \rangle$. To do this, let $g \in B$ be arbitrary and view g as a word of minimal length over a and b as follows, where $i_t \neq 0$ for $0 < t \leq p$ and $j_t \neq 0$ for $0 \leq t < p$.

$$g = a^{i_0} b^{j_0} \dots a^{i_p} b^{j_p}$$

Suppose there exists some i_t which does not divide m (and so $a^{i_t} \notin B$) and look for a contradiction. We can isolate those a-terms a^{i_t} with m not dividing i_t and write w_t for the words partitioning them as follows, where m does not divide k_t and $k_t \neq 0$ for $0 < t \leq r$.

$$g = a^{k_0} W_0 a^{k_1} W_1 \dots a^{k_r} W_r$$

Note that $W_t \in B \setminus C$ for $0 \leq t < r$, because $W_t \in B$ by construction while if $W_t \in C$ then $W_t \in \langle a \rangle$ which contradicts the minimal length of g. Therefore, this expression is a normal form for g. By assumption, there exists some i_t which does not divide m and so

$L(g) > 1$. Therefore, $g \notin B$, a contradiction. Thus, g must be a word in a^m and in b, and so $B = \langle c = a^m, b \rangle$ which completes the proof of (2). □

We shall now prove Proposition 3.1.14, which is our second main technical result of this section. This result classifies how a one-ended two-generator, one-relator group with torsion G can split as a HNN-extension over a virtually-cyclic subgroup. Note that we assume G is one-ended, while in the analogous result for free products with amalgamation, which is Proposition 3.1.13, we make no assumptions about the ends of G.

Proposition 3.1.14 (Proposition 3.8, Kapovich–Weidmann)**.** *Let G be a one-ended two-generator, one-relator group with torsion which splits as an HNN-extension over a virtually-cyclic subgroup $G = H*_{C_A^t = C_B}$. Then either C_A or C_B is malnormal in G. In the case that C_B is malnormal (the case that C_A is malnormal is analogous) we get that G has the following HNN-presentation where a, b are non-trivial elements of H, the subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$ are malnormal in H, and $C_A = \langle a^m \rangle$ while $C_B = \langle b \rangle$.*

$$G = \langle H, t; t^{-1}a^m t = b \rangle$$

Moreover, the following hold.

1. *The group G has a generating pair (th, a), where $h \in H$.*

2. *The group H is not cyclic.*

3. *The group H has a generating pair $(a, h^{-1}bh)$, where h is as in (1).*

4. *The group H is a two-generator, one-relator group with torsion.*

Proof. Suppose $G = H*_{C_A^t = C_B}$ where $C_A \cong C_B$ are isomorphic virtually-cyclic groups. Then, by Lemma 3.1.4, C_A (respectively C_B) is a subgroup of a malnormal infinite cyclic subgroup $A = \langle a \rangle$ (respectively $B = \langle b \rangle$). Moreover, by Lemma 3.1.12, either $A = C_A$ or $B = C_B$, so we can assume that $B = C_B$. Thus, G has the prescribed HNN-presentation.

The group H cannot be cyclic because Lemma 3.1.12 gives us that $\langle a^m \rangle$ and B are separated, which is impossible if H is cyclic. Thus, (2) holds.

We shall prove (3) under the assumption that (1) holds. Write $d = h^{-1}bh$ and $s = th$, so $\langle s, a \rangle = G$ and we shall prove that $\langle a, d \rangle = H$. Note that $\langle a, d \rangle \leq H$, so we assume that it is a proper subgroup and look for a contradiction. So, suppose there exists $g \in H \setminus \langle a, d \rangle$. Then g can be written in terms of a and s, as these elements generate G, and, moreover, g can be written in terms of a, s and d. Write g as a word W in a, s and d such that the number of occurrences of s is minimal. That is, write g in the following way, where $h_j \in \langle a, d\rangle$ and k is minimal (note that h_i can be trivial).

$$g =_G W(a, s, d) = h_0 s^{\epsilon_1} h_1 s^{\epsilon_2} h_2 \dots s^{\epsilon_k} h_k$$

Note that $k > 0$ as $g \notin \langle a, d\rangle$. If W has a subword of the form $s^{-1}a^{lm}s (= (h^{-1}t^{-1}a^m th)^l)$ then this can be replaced by d^l to gain a word with fewer s-terms. Similarly, if W has a subword of the form $sd^l s^{-1} (= (tbt^{-1})^l)$ then this can be replaced by a^{lm} to gain a word with fewer s-terms. Therefore, as k is minimal W contains no subwords of the form $s^{-1}a^{ln}s$ or of the form $sd^l s^{-1}$. It follows that this is a reduced sequence for g which contains k t-terms. Applying Britton's Lemma (Proposition 2.4.1), we have that $g \notin H$, a contradiction. Therefore, $H = \langle a, d\rangle$, as required.

We shall now prove (1). Note that if $m = 1$ then there exists some integer $i \in \mathbb{Z}$ such that $G = \langle th, a^i\rangle$ where $h \in H$, by Proposition 2.4.2. Thus, $G = \langle th, a\rangle$ and so the result holds when $m = 1$. Therefore, (3) holds when $m = 1$.

We use the case of $m = 1$ to prove the result for $m > 1$. To do this, assume $m > 1$ and begin by re-writing G in the following way, where H_0 is the amalgam $H *_{\langle b=b_0^m\rangle} \langle b_0\rangle$.

$$\begin{aligned} G &= \langle H, t; t^{-1}a^m t = b\rangle \\ &\cong \langle H, t; (t^{-1}at)^m = b\rangle \\ &\cong \langle H, t_0; t_0^{-1}at_0 = b_0, b_0^m = b\rangle \\ &\cong \langle H_0, t_0; t_0^{-1}at_0 = b_0\rangle \end{aligned}$$

Note that $\langle a\rangle$ is malnormal in H_0 as $\langle a\rangle$ is malnormal in H and $\langle a\rangle$ and $\langle b_0\rangle$ are separated, while $\langle b_0\rangle$ is malnormal because it is conjugate to $\langle a\rangle$. The case of $m = 1$ can now be applied. Therefore, G is generated by a pair (t_0h_0, a) with $h_0 \in H_0$, and we can apply (3) to get that H_0 is generated by the pair $(h_0 a h_0^{-1}, b_0)$ (note that we have conjugated by h_0). We now prove that $h_0 \in H$.

Now, because (t_0h_0, a) is a generating pair for G then so is $(h_0t_0, t_0^{-1}at_0) = (h_0t_0, b_0)$. Thus, $(b_0^i h_0 t_0, b_0)$ is a generating pair of G for all integers i which means that we can assume that the word $h_0 \in H_0$ is not contained in $\langle b_0^m = b\rangle$. Therefore, h_0 has the following form in the amalgam $H_0 = H *_{\langle b=b_0^m\rangle} \langle b_0\rangle$, where $h_j \in H \setminus \langle b\rangle$ and $b_j \in \langle b_0\rangle \setminus \langle b_0^m\rangle$ but h_1 and h_{l+1} are possible trivial.

$$h_1 b_1 h_2 b_2 \dots h_l b_l h_{l+1}$$

As $A = \langle a \rangle$ is separated from $B = \langle b \rangle$, it is also separated from $\langle b_0 \rangle$, as b_0 is a root of b. This means that $h_{l+1} a^k h_{l+1}^{-1} \in H \setminus \langle b \rangle$ for all non-zero integers $k \in \mathbb{Z} \setminus \{0\}$, and so $h_0 a^k h_0^{-1}$ has a reduced sequence, with respect to the decomposition $H_0 = H *_{b_0^m = b} \langle b_0 \rangle$, of the following form, where h_j and b_j are as above and $h_{l+1} a h_{l+1}^{-1} \in H \setminus \langle b \rangle$.

$$h_0 a^k h_0^{-1} = h_1 b_1 \ldots h_l b_l (h_{l+1} a^k h_{l+1}^{-1}) b_l^{-1} h_l^{-1} \ldots b_1^{-1} h_1^{-1}$$

Assume that $h_0 \notin H$, which implies that $l > 0$ and $L(h_0 a^k h_0^{-1}) \geq 3$. Note that $h_1^{-1} b_0^j h_1 \notin B$ for $j \neq 0$ as $B = \langle b \rangle$ is malnormal in H and b_0 is a root of b. This means that an arbitrary word over b_0 and $h_0 a h_0^{-1}$ is either a proper power of b_0 or has length greater than or equal to 3. Now choose an element $g \in H \setminus \langle b \rangle$. Then $L(g) \leq 1$ but g is not a power of b_0, and so $g \notin \langle b_0, h_0 a h_0^{-1} \rangle$, a contradiction. Thus, $h_0 \in H$.

As $h_0 \in H$ and taking $h := h_0^{-1}$, we have that $H_0 = H *_{\langle b = b_0^m \rangle} \langle b_0 \rangle = \langle b_0, hah^{-1} \rangle$ with $h \in H$, and by conjugating by b we can further assume that either h is trivial or $h \in H \setminus \langle b \rangle$. Now, H_0 is a two-generator subgroup of a one-relator group with torsion, and so is either a two-generator, one-relator group with torsion or a free product of cyclic groups. If $H_0 \cong C_p * C_q$ with p and q both finite then $H_0 \leq T := \langle\langle R \rangle\rangle$ so $G/T = \widehat{G} \cong \mathbb{Z}$ which means that G is infinitely ended by Lemma 2.9.7, a contradiction. If H_0 is free of rank two then G is torsion-free as amalgams of torsion-free groups are torsion free, a contradiction. Therefore, H_0 is a one-relator group with torsion and so we can apply (2) from Proposition 3.1.13 to get that (b, hah^{-1}) is a generating pair for H. Thus, as $h \in H$ we can conjugate this generating pair by h to get that $(h^{-1} b h, a) = (h^{-1} t^{-1} a^m t h, a)$ is a generating pair for H. Thus, (th, a) is a generating pair for H where $h \in H$, as required.

We conclude by proving (4), that H is a two-generator, one-relator group with torsion. Suppose otherwise. Note that H is two-generated, by (3), so applying the fact that G is a one-relator group with torsion this means that H is either a free product of two finite cyclic groups or is free of rank two. If H is a free product of two finite cyclic groups then $H \leq T := \langle\langle R \rangle\rangle$ so $G/T = \widehat{G} \cong \mathbb{Z}$ and so G is infinitely ended, by Lemma 2.9.7, a contradiction. Thus, H must be free of rank two. However, then H is torsion free and as an HNN-extension of a torsion-free group is torsion-free we have that G is torsion free, a contradiction. Therefore, H is a two-generator, one-relator group with torsion, as required. $\square$

The JSJ-decompositions of one-ended two-generator, one-relator groups with

torsion. We shall now apply Propositions 3.1.13 and 3.1.14 to prove the following results on the structure of the JSJ-decomposition of a one-ended two-generator, one-relator group with torsion. The first is the analogue of Theorem A from Kapovich–Weidmann's paper on the JSJ-decompositions of two-generator hyperbolic groups without torsion, and the second is the analogue of Theorem B from their paper [KW99].

Theorem 3.1.15. *Let G be a one-ended two-generator, one-relator group with torsion. Suppose that G is the fundamental group of a graph of groups $\mathbf{\Gamma}$ where all edge stabilisers are virtually-cyclic and no vertex-stabiliser is virtually-cyclic. Then the graph Γ underlying $\mathbf{\Gamma}$ is one of the following.*

- *A single vertex with no edges.*
- *A single vertex with a single positive loop edge.*

Proof. Consider a collapsing tree T for the graph of groups $\mathbf{\Gamma}$, and write k for the number of positive edges in $\Gamma \setminus T$. Now, recall that G is isomorphic to the fundamental group of $\mathbf{\Gamma}$ with respect to this collapsing tree, and so G can be described in the following way, where the subgroup N is defined below.

$$\pi_1(\mathbf{\Gamma}, T_\Gamma) = \prod_{v \in V_\Gamma} G_v * F(E_\Gamma)/N$$

Here, N is the normal closure of the following set.

$$\{e\bar{e}; e \in E_\Gamma\} \cup \{e; e \in E_{T_\Gamma}\} \cup \{e^{-1}(g\theta_e)e = (g\theta_{\bar{e}}); e \in E_\Gamma, g \in G_e\}$$

Therefore, G maps onto the free group on k generators, by collapsing the vertex groups G_v. As G is a non-free two-generated group we have that $k = 0$ or $k = 1$. This means that in order to prove the theorem it is sufficient to prove that T consists of a single vertex, which is what we shall do.

Now, T inherits a graph of groups structure $\boldsymbol{T}$ from $\mathbf{\Gamma}$, and the fundamental group H of $\boldsymbol{T}$ is a subgroup of G. Indeed, if $k = 0$ then $G = H$ while if $k = 1$ then $G = H*_{A^t=B}$ where A is the stabiliser of the positive edge from $\Gamma \setminus T$. We shall begin by proving that H is a two-generator, one-relator group with torsion. If $k = 0$ this clearly holds, as $H = G$. Suppose $k = 1$. Then $G = H*_{A^t=B}$ and as A is an edge stabiliser in $\mathbf{\Gamma}$ we have that A is virtually-cyclic and so by Proposition 3.1.14, H is a two-generator, one-relator group with torsion.

We shall now prove that T consists of a single vertex, which completes the proof. Suppose otherwise, then T contains a separating edge and so H splits as a free product with amalgamation over this edge. As the edge stabilisers of T are virtually-cyclic, this means that $H = A *_C B$ where C is virtually-$\mathbb{Z}$. By Theorem 3.1.9, C is infinite cyclic. Then, by Proposition 3.1.13, one of A or B is virtually-cyclic and so $\mathbf{\Gamma}$ contains a vertex stabiliser which is virtually-cyclic, a contradiction. This completes the proof. $\square$

Modified JSJ-decompositions. Note that the JSJ-decomposition of a group (as defined by Bowditch) may have elementary vertices of degree one, but that sinking each of these into their unique adjacent vertex yields a canonical decomposition of the group as a graph of groups. We shall call this new decomposition the *modified JSJ-decomposition*. We maintain the notation *rigid* and *orbifold* from Bowditch's definition, that is, a vertex in the modified JSJ-decomposition of a group is a rigid (respectively, orbifold/elementary) vertex if it corresponds to a rigid (respectively, orbifold/elementary) vertex in the JSJ-decomposition.

We shall now use the above result, Theorem 3.1.15, to prove the following theorem, Theorem 3.1.16, on the outer automorphism groups of two-generator, one-relator groups with torsion. The proof shows that if $G = \langle a, b; R^n \rangle$ is one-ended then either G is Fuchsian or G has modified JSJ-decomposition consisting of a single rigid vertex or a single rigid vertex and a single elementary vertex connected by two positive edges, as in Figure 3.1, and in each case the rigid vertex group is a two-generator, one-relator group with torsion. In the proof we use the fact that if $G \cong \mathbb{Z} * C_n$ then $\mathrm{Out}(G)$ is finite, which is a well-known result and is easily proven using the normal form for free products. However, for the sake of completeness, we prove this fact on the outer automorphism groups of free products in Section 3.5, and Section 3.5 is proven entirely independently of this current section.

Theorem 3.1.16. *Let G be a one-ended two-generator, one-relator group with torsion. Then either* $\mathrm{Out}(G)$ *is virtually-cyclic or $G \cong \langle a, b; [a, b]^n \rangle$ for some $n > 1$.*

Proof. Note that G is a Fuchsian group if and only if $G \cong \langle a, b; [a, b]^n \rangle$ for some $n > 1$, by Proposition 2.8.4. Therefore, we shall prove that if G is not Fuchsian then $\mathrm{Out}(G)$ is virtually-cyclic. So, assume that G is not Fuchsian. We shall analyse the modified JSJ-decomposition of G. Write $\mathcal{V}_1$ for the number of elementary vertices of the modified JSJ-decomposition of G, $\mathcal{V}_2$ for the number of orbifold vertices, and $\mathcal{V}_3$ for the number of rigid vertices. Write $\mathcal{E}_2$ for the number of edges with origin an orbifold vertex, and $\mathcal{E}_3$ for the number of edges with origin a rigid vertex.

We begin by proving that the modified JSJ-decomposition of G has an underlying graph Γ which is either a single vertex, or two vertices connected by two positive edges. To see that Γ has this form, push every elementary vertex into some adjacent vertex v (note that this operation may change the group G_v of the adjacent vertex v). In this new graph of groups $\mathbf{\Gamma}_0$ there are precisely $\mathcal{V}_2 + \mathcal{V}_3$ vertices, none of which are elementary, and there are $\mathcal{E}_2 + \mathcal{E}_3 - \mathcal{V}_1$ positive edges. Now, all edge stabilisers are virtually-cyclic and no vertex stabilisers are virtually-cyclic, so we can apply Theorem 3.1.15 to get that the underlying graph Γ_0 of $\mathbf{\Gamma}_0$ consists of a single vertex or a single vertex with a single positive loop edge. Therefore, $\mathcal{V}_2 + \mathcal{V}_3 = 1$ and so Γ necessarily consists of a central either orbifold or rigid vertex and k elementary vertices which are connected to the central vertex with two positive edges. As each of these elementary vertices would result in a positive edge of Γ_0, $k \leq 1$ as required.

If $|V_\Gamma| = 1$ then $\mathrm{Out}(G)$ is finite by Proposition 2.8.2, as G is non-Fuchsian. So, suppose that $|V_\Gamma| = 2$, and we shall prove that neither vertex is an orbifold vertex by proving that neither vertex contributes, in a certain sense, infinitely many automorphisms to the outer automorphism group. Now, because $|V_\Gamma| = 2$ we have that G is isomorphic to an HNN-extension $G = H*_{A^t=B}$ where A and B are virtually-cyclic, by sinking the elementary vertex into the non-elementary one. Therefore, by Proposition 3.1.14, H is a two-generator, one-relator group with torsion and G has a presentation of the following form for some $m \neq 0$ (note that we have replaced t with th^{-1} in the results given by Proposition 3.1.14).

$$G \cong \langle a, b, t; R(a,b)^n, t^{-1}a^m t = b\rangle$$

This HNN-decomposition implies that G splits as a graph of groups as in Figure 3.1, with vertices v and w such that $G_v = \langle a, b; R(a,b)^n\rangle$, $G_w = \langle c\rangle$, positive edges e and f with $\iota(e) = v = \iota(f)$ such that $G_e = \langle x\rangle$ and $G_f = \langle y\rangle$, and with injections as follows.

$$\theta_e : x \mapsto a \qquad\qquad \theta_{\overline{e}} : x \mapsto c$$
$$\theta_f : y \mapsto b^m \qquad\qquad \theta_{\overline{f}} : y \mapsto c$$

This graph of groups satisfies all the properties of a modified JSJ-decomposition, and by uniqueness we conclude that this is, indeed, the modified JSJ-decomposition of G.

Therefore, by Proposition 2.11.6, to prove the theorem it is sufficient to prove that

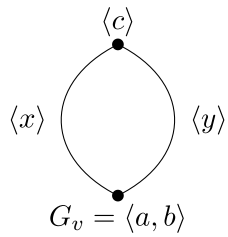

Figure 3.1: If a non-Fuchsian two-generator, one-relator group with torsion has a non-trivial modified JSJ-decomposition then it must look like the graph of groups pictured. The group G_v is generated by the elements a and b and is a two-generator, one-relator group with torsion while x, y and c all generate infinite cyclic groups. Edge maps are given by $x \mapsto c$, by $y \mapsto c$, by $x \mapsto a$ and by $y \mapsto b^m$.

$\mathrm{PMCG}(G_v)$ and $\mathrm{PMCG}(G_w)$ are finite. Note that $\mathrm{PMCG}(G_w)$ is finite as $\mathrm{Out}(G_w) \cong C_2$ is finite. We shall now prove that $\mathrm{PMCG}(G_v)$ is finite, that is, we shall prove that there are only finitely many outer automorphisms $\widehat{\beta}$ of G_v with a representative $\beta \in \mathrm{Aut}(G_v)$ such that $a \mapsto g^{-1}ag$ and $b^m \mapsto h^{-1}b^m h$ for some $g, h \in G_v$. To do this, begin by noting that taking $\beta\gamma_h^{-1}$ we can assume that h is trivial. Recall that G_v is a two-generator, one-relator group with torsion, and therefore is either a free product $\mathbb{Z} * C_n$ or G_v has only one Nielsen equivalence class of generating pairs, by Proposition 2.2.2. If $G_v \cong \mathbb{Z} * C_n$, then $\mathrm{Out}(G_v)$ is finite which means that $\mathrm{PMCG}(G_v)$ is finite, as required. So, suppose that G_v has a single Nielsen equivalence class of generating pairs. Then we can assume that β is a Nielsen transformation and so $a \mapsto g^{-1}ag$, $b \mapsto b$. By Proposition 2.2.1 we therefore have that $\widehat{\beta}$ is trivial and so $\mathrm{PMCG}(G_v)$ is trivial, as required. This completes the proof of the theorem. □

3.1.3 Equations in free groups

An alternative approach to proving Theorem 3.1.16 (which is the purpose of Section 3.1) would be to observe, as we do below using Proposition 3.1.17, that $\mathrm{Out}(G)$ is related to equations in free groups, and then one can use Touikan's treatment of these [Tou09] to get the required results. We do this here. The solutions to equations in free groups are relevant to these outer automorphism groups because finding automorphisms up to conjugacy corresponds to solving the equations $R(x, y) = R^{\pm 1}(a, b)$ in free groups (x and y are the variables), which follows from the following proposition due to Magnus [MKS04, Theorem N5] and from the fact that the proposition's converse is also true.

Proposition 3.1.17. *Let G be a group on generators x_ν ($\nu = 1, 2, \ldots, n$) with a single defining relator $R(x_\nu)$. If there exists a set of words $W_\nu(x_\mu)$ such that the mapping*

$$x_\nu \mapsto W_\nu(x_\mu)\,(\nu = 1, 2, \ldots, n)$$

is a Nielsen transformation acting on the x_ν which defines an automorphism of G, then $R(W_\nu)$ is freely equal (as a word in the x_ν) to a transform

$$T(x_\nu) \cdot R(x_\nu)^{\pm 1} \cdot T^{-1}(x_\nu)$$

of $R^{\pm 1}$.

The converse of this Proposition is also true: If there exists a set of words $W_\nu(x_\mu)$ such that the mapping

$$x_\nu \mapsto W_\nu(x_\mu)\,(\nu = 1, 2, \ldots, n)$$

is a Nielsen transformation ϕ acting on the x_ν and $R(W_\nu)$ is freely equal (as a word in the x_ν) to a transform

$$T(x_\nu) \cdot R(x_\nu)^{\pm 1} \cdot T^{-1}(x_\nu)$$

of $R^{\pm 1}$ then the Nielsen transformation ϕ defines an automorphism of G. This is true because the mapping ϕ is a homomorphism, and because ϕ is a Nielsen transformation it is surjective, and as two-generator, one-relator groups with torsion are Hopfian [Pri77a], the map ϕ is in fact an automorphism.

We now call on the work of Touikan [Tou09]. Specifically, we need the following proposition which outlines the forms a rank 2 solution to the equation $w(x, y) = u$ can take, $u \in F(a, b)$. In the proposition, a *solution* is a map $\phi : x \mapsto x'$, $y \mapsto y'$ with $x', y' \in F(a, b)$, such that $w(x', y') \equiv u$, or equivalently a pair (x', y') under the same conditions. A *rank 2 solution* is a solution such that x', y' are not contained in some cyclic subgroup of $F(a, b)$. A *primitive solution* is a solution (x', y') such that (x', y') is a primitive pair of $F(a, b)$. Two equations $w(x, y) = u$ and $w'(x, y) = u'$ are *rationally equivalent* if there is a Nielsen Transformation of (x, y), φ say, such that $w\varphi = w'$.

If (t, p) is a primitive pair of $F(x, y)$, write

$$\begin{aligned} \bar{\delta}_t : t &\mapsto pt \\ p &\mapsto p \end{aligned}$$

while γ_v denotes the (inner) automorphism of $F(a,b)$ corresponding to conjugation by v, $\gamma_v : a \mapsto v^{-1}av$, $b \mapsto v^{-1}bv$.

Proposition 3.1.18. *Suppose that $w(x,y) = u$ has rank 2 solutions and that $w(x,y)$ is neither primitive nor a proper power. Then there exists a finite set of solutions $\{\phi_i : i \in I\}$ such that the rank 2 solutions are given by one of the following.*

1. *All solutions are of the form $\phi_i\gamma_u^j$, $j \in \mathbb{Z}$.*

2. *We have $\langle x,y\rangle = \langle H, t; t^{-1}pt = q\rangle$, with $H = \langle p,q\rangle$, $w \in H$, and we can write the elements x,y as words $x = X(p,t)$, $y = Y(p,t)$. All solutions are are of the following form, where $j,k \in \mathbb{Z}$.*
$$\bar{\delta}_t^k\phi_i\gamma_u^j$$

3. *Up to rational equivalence, $w(x,y) \equiv [x,y]$ and all solutions are of the following form where $\sigma \in \langle\bar{\delta}_x, \bar{\delta}_y, \gamma_w\rangle$.*
$$\sigma\phi_i$$

Touikan specifies the finite set of solutions $\{\phi_i : i \in I\}$, and these are precisely the “Δ-minimal” solutions. The map $\phi : x \mapsto a, y \mapsto b$ is a Δ-minimal solution to the equation $R(x,y) \equiv R(a,b)$, and so we can abuse notation to equate x with a and y with b, so we can write $\bar{\tau} \circ \phi = \bar{\tau}$, $\phi \circ \tau = \tau$, and $w\phi = w$ for $w \in F(a,b)$, and working mod $\mathrm{Inn}(G)$ we see that if $G = \langle a,b; R^n\rangle$ then one of only precisely three things happen.

1. There are only finitely many solutions to $R(x,y) \equiv R^{\pm1}(a,b)$,

2. $G \cong \langle a,b; S^n\rangle$, $S \in \langle a^{-1}ba, b\rangle$,

3. $G \cong \langle a,b; S^n\rangle$ with $S = [a,b]$.

We now use Touikan's solutions to prove the following result about the structure of $\mathrm{Out}(G)$.

Theorem 3.1.16 (Alternative proof)**.** *Let G be a one-ended two-generator, one-relator group with torsion. Then either $\mathrm{Out}(G)$ is virtually-cyclic or $G \cong \langle a,b; [a,b]^n\rangle$ for some $n > 1$.*

Proof. Firstly, note that if $\mathrm{Out}(G)$ is finite then it is trivially virtually-cyclic. So we restrict ourselves to the case where $\mathrm{Out}(G)$ is infinite; to the second two cases of Touikan's solution to $R(x,y) \equiv R(a,b)$. If the third case of Touikan's solution holds then $G \cong \langle a,b; [a,b]^n\rangle$.

We prove that if the second case of Touikan's solution holds then $\mathrm{Out}(G)$ is virtually-cyclic. Re-write R in terms of p and t, and then $\bar{\delta}_t \in \mathrm{Aut}(G)$. We essentially prove that the subgroup $\langle \bar{\delta}_t \,\mathrm{Inn}(G)\rangle$ has finite index in $\mathrm{Out}(G)$[2].

Assume the second case of Touikan's solution holds, and define $\mathcal{S}_p$ to be the set of automorphisms of G which (freely) fix R or send it to R^{-1}. One can view the set $\mathcal{S}_p$ as the set of primitive solutions to $R(x,y) \equiv R(a,b)$ unioned with the set of primitive solutions to $R(x,y) \equiv R(a,b)^{-1}$. Clearly, $\mathcal{S}_p$ is closed under products and inverses, and contains the identity automorphism, so $\mathcal{S}_p \leq \mathrm{Aut}(G)$, and noting that $\mathcal{S}_p \,\mathrm{Inn}(G) = \mathrm{Aut}(G)$, this means that we have the following isomorphism.

$$\mathrm{Out}(G) \cong \frac{\mathcal{S}_p}{\mathcal{S}_p \cap \mathrm{Inn}(G)}$$

We prove that $\mathcal{S}_p/\mathcal{S}_p \cap \mathrm{Inn}(G)$ is virtually-cyclic, which proves the result. The subgroup $\mathcal{S}_p$ contains as a normal subgroup $\mathrm{Stab}_p(R) := \mathcal{S}_p \cap \mathrm{Stab}(R)$, the *primitive stabiliser* of R. We prove that $\mathrm{Stab}_p(R)/\mathcal{S}_p \cap \mathrm{Inn}(G)$ is virtually-cyclic, which is sufficient by certain results from Touikan's paper which we outline now. Proposition 2.21 of [Tou09] tells us that if $\phi_0, \phi_1 \in \mathcal{S}_p$ and ϕ_0, ϕ_1 have the same *terminal pair* and the same *terminal word* then there exists some element $\beta \in \mathrm{Stab}_p(R)$ such that $\phi_0\beta = \phi_1$, while Proposition 2.19 of [Tou09] gives us that there are only finitely many possible terminal pairs and terminal words. That is, $\mathrm{Stab}_p(R)$ is of finite index in $\mathcal{S}_p$, as required.

Write $N := \langle \gamma_R \rangle \leq \mathrm{Inn}(G)$, and clearly $N \cap \mathrm{Inn}(G) = N$ because γ_R is inner, while $N \trianglelefteq \mathrm{Stab}_p(R)$ because if $R\phi \equiv R^{\epsilon}$ then $\gamma_R\phi = \phi\gamma_{R^{\epsilon}}$. We therefore have the following factorisation of the map $\mathrm{Stab}_p(R) \twoheadrightarrow \mathrm{Stab}_p(R)/(\mathcal{S}_p \cap \mathrm{Inn}(G))$.

$$\mathrm{Stab}_p(R) \twoheadrightarrow \frac{\mathrm{Stab}_p(R)}{N} \twoheadrightarrow \frac{\mathrm{Stab}_p(R)}{\mathcal{S}_p \cap \mathrm{Inn}(G)}$$

We shall now use results from Touikan's paper to prove that $\mathrm{Stab}_p(R)/N$ is virtually-cyclic, which proves the theorem. Looking at Touikan's paper, Corollary 2.12 and Section 2.4.1 combine to give us that $\Delta = \langle \gamma_R, \bar{\delta}_t \rangle$ is of finite index in $\mathrm{Stab}_p(R)$. Thus, as Δ/N is virtually-cyclic so is $\mathrm{Stab}_p(R)/N$, as required. □

3.2 $\mathrm{Out}(G)$ embeds into $\mathrm{Out}(F(a,b))$

In this section we assume that G is one ended and prove that $\mathrm{Out}(G)$ embeds into $\mathrm{GL}_2(\mathbb{Z})$. This embedding gives us a particularly nice way of viewing $\mathrm{Out}(G)$. We use this view,

[2] Later, in Theorem 3.2.1 from Section 3.2, we prove that no power of $\bar{\delta}_t$ is inner. Therefore, what we are actually proving here is that $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$.

along with our knowledge of the JSJ-structure of G, in Sections 3.3 and 3.4 to determine the possibilities for $\mathrm{Out}(G)$.

As G is one-ended there is a single Nielsen equivalence class of generating pairs of G, by Proposition 2.2.2, and so every automorphism is tame, that is, every automorphism of $G = \langle a, b; R^n \rangle$ lifts to an automorphism of the ambient free group $F(a, b)$. As we explained in Section 2.2, this implies that there exists a homomorphism θ from some subgroup H of $\mathrm{Out}(F(a, b))$ onto $\mathrm{Out}(G)$, $\theta : H \twoheadrightarrow \mathrm{Out}(G)$. The purpose of this section is to prove Theorem 3.2.1, which states that the homomorphism θ is an isomorphism. Therefore, using Proposition 2.2.1 we can view the elements of $\mathrm{Out}(G)$ as elements of $\mathrm{GL}_2(\mathbb{Z})$. This representation turns out to be a very powerful tool in our analysis of $\mathrm{Out}(G)$.

Recall that such a group G has a single Nielsen equivalence class by Proposition 2.2.2, and so every automorphism has the form $\phi : a \mapsto A$, $b \mapsto B$ and this lifts to an automorphism ϕ_0 of the ambient free group.

Theorem 3.2.1. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and R non-primitive. Then* $\mathrm{Out}(G)$ *embeds in* $\mathrm{Out}(F(a, b))$*. Moreover, the embedding is as follows. First realise an automorphism ϕ as a Nielsen transformation $\phi_0 : a \mapsto A$, $b \mapsto B$. Then the following map gives the embedding.*

$$\begin{aligned} \mathrm{Out}(G) &\to \mathrm{GL}(2, \mathbb{Z}) \\ \widehat{\phi} &\mapsto \begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ \sigma_a(B) & \sigma_b(B) \end{pmatrix} \end{aligned}$$

Note that the embedding is the composition of the map $\widehat{\phi} \mapsto \widehat{\phi_0}$ with the isomorphism $\overline{\xi} : \mathrm{Out}(F(a, b)) \to \mathrm{GL}_2(\mathbb{Z})$ induced by the map ξ from Proposition 2.2.1. Therefore, to prove the theorem it is sufficient to prove that if ϕ is an inner automorphism of G which can be realised by the Nielsen transformation ϕ_0 then ϕ_0 defines an inner automorphism of $F(a, b)$.

Theorem 3.2.1 gives us the following corollary, which is proven by combining the theorem with the fact that every automorphism of $F(a, b)$ maps $[a, b]$ to a conjugate of $[a, b]^{\pm 1}$ [MKS04, Theorem 3.9].

Corollary 3.2.2. *If $G = \langle a, b; [a, b]^n \rangle$ with $n > 1$ then* $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$.

Recall that $\mathrm{GL}_2(\mathbb{Z}) \cong \mathrm{Out}(F(a, b))$. Then note that one can interpret this corollary as saying $\mathrm{Out}(G) = \mathrm{Out}(F(a, b))$, rather than just $\mathrm{Out}(G) \cong \mathrm{Out}(F(a, b))$. This is because the embedding given by Theorem 3.2.1 is the obvious embedding given by lifting the automorphisms to the ambient free group.

The proof of Theorem 3.2.1 is split between two lemmata, corresponding to the cases $R \in F(a,b)'$ and $R \notin F(a,b)'$, where $F(a,b)'$ denotes the derived subgroup of $F(a,b)$. The first case, when $R \in F(a,b)'$, is proven easily and we do this in Lemma 3.2.3. The proof of the second case, when $R \notin F(a,b)'$, is more substantial, and is proven in Lemma 3.2.5 using Lemma 3.2.4.

Lemma 3.2.3. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and $R \in F(a,b)'$. Then* $\mathrm{Out}(G)$ *embeds in* $\mathrm{Out}(F(a,b))$ *by the map given in Theorem 3.2.1.*

Proof. It is sufficient to prove that if ϕ is a Nielsen transformation and $\phi \in \mathrm{Inn}(G)$ then $\phi \in \mathrm{Inn}(F(a,b))$. So, let ϕ be some Nielsen transformation of (a,b) with $a\phi := A$ and $b\phi := B$ and such that there exists $W \in F(a,b)$ with $a^W =_G A$ and $b^W =_G B$, and we shall prove that $\phi \in \mathrm{Inn}(F(a,b))$. As $a^W = A$ and $b^W = B$ in G it must hold that $a^W = A \mod G'$ and $b^W = B \mod G'$. However, $G^{ab} = \langle a, b; [a,b] \rangle \cong \mathbb{Z} \times \mathbb{Z}$, as $R \in F(a,b)'$. Therefore, it must hold that $\sigma_a(A) = 1$ and $\sigma_b(A) = 0$, and that $\sigma_a(B) = 0$ and $\sigma_b(B) = 1$. Then Proposition 2.2.1 implies that under the homomorphism $\xi : \mathrm{Aut}(G) \to \mathrm{GL}_2(\mathbb{Z})$, ϕ is mapped to the identity matrix. This means that $\phi \in \mathrm{Inn}(F(a,b))$, as required. □

We shall now prove Theorem 3.2.1 in the case when $G = \langle a, b; R^n \rangle$ with $R \notin F(a,b)'$. We begin by proving Lemma 3.2.4, which gives a description of the automorphisms of G by looking at their action on the abelianisation. We use this to prove Lemma 3.2.5, which completes the proof of Theorem 3.2.1. The description of automorphisms given by Lemma 3.2.4 is used at a number of points in the remaining sections of this chapter.

Now, by applying Moldovanskii rewriting to the relator R we can assume that $\sigma_a(R) = 0$. After rewriting, $\sigma_b(R) \neq 0$ while G being primitive corresponds to $R = b^\epsilon$. We shall now give the form which (outer) automorphisms of $G = \langle a, b; R^n \rangle$ must have when $R \notin F(a,b)'$. Recall that if $\psi \in \mathrm{Aut}(G)$ then $\widehat{\psi}$ denotes the element of $\mathrm{Out}(G)$ with representative ψ. The proof uses the fact that every automorphism of G is tame by Proposition 2.2.2.

Lemma 3.2.4. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$, $\sigma_a(R) = 0$, $\sigma_b(R) \neq 0$, and $R \neq b^\epsilon$ cyclically reduced. Let ψ be an arbitrary automorphism of G. Then $\widehat{\psi} \in \mathrm{Out}(G)$ has a representative $\phi \in \mathrm{Aut}(G)$ of the following form.*

$$\phi : a \mapsto a^{\epsilon_0} b^k$$
$$b \mapsto b^{\epsilon_1}$$

Proof. Note that if $\phi : a \mapsto a^{\epsilon_0}b^k, b \mapsto b^{\epsilon_1}$ is a homomorphism then it is also an automorphism. Now, begin by assuming that ψ is a Nielsen transformation of (a, b) (we can do this by Proposition 2.2.2). We shall write $a\psi := A$ and $b\psi := B$, and so (A, B) is a primitive pair of $F(a, b)$.

Let $\pi : G \to G^{ab}$ be the abelianisation map. The abelianisation has presentation $G^{ab} = \langle x, y; y^m, [x, y]\rangle$ because $\sigma_a(R) = 0$ but $\sigma_b(R) \neq 0$ (here, $x := a\pi$ and $y := b\pi$ while $m := \sigma_b(R)$). Let $x^i y^\alpha := A\pi$ and let $x^j y^\beta := B\pi$. Then as G' is characteristic in G, automorphisms of G define automorphisms of $G^{ab} = G/G'$, so $B\pi$ has order $m \neq 0$. We therefore have the following implications.

$$(x^j y^\beta)^m = 1 \Rightarrow x^{mj} y^{m\beta} = 1 \Rightarrow x^{mj} = 1$$

This means that $mj = 0$ as x has infinite order in G^{ab}. Thus, $j = 0$ and so $B\pi = b^\beta$. Therefore, $\sigma_a(B) = 0$.

By Proposition 2.2.1, the Nielsen transformation ψ corresponds to the following matrix of $GL_2(\mathbb{Z})$.

$$\begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ 0 & \sigma_b(B) \end{pmatrix}$$

Therefore, $|\sigma_a(A)| = 1 = |\sigma_b(B)|$. Taking $k := \sigma_b(A)$, $\epsilon_0 := \sigma_a(A)$ and $\epsilon_1 := \sigma_b(B)$, the Nielsen transformation, $\phi : a \mapsto a^{\epsilon_0}b^k, b \mapsto b^{\epsilon_1}$ also corresponds to this matrix. Now, if two Nielsen transformations are equal $\mod \mathrm{Inn}(F(a, b))$ they must also be equal $\mod \mathrm{Inn}(G)$, and so we are done. □

We now apply the above lemma, Lemma 3.2.4, to prove the following lemma which completes the proof of Theorem 3.2.1. The fact that $\langle b\rangle$ is a malnormal subgroup of $G = \langle a, b; R^n\rangle$, $n > 1$, is used throughout the proof of the lemma (this fact is Proposition 2.9.2).

Lemma 3.2.5. *Let $G = \langle a, b; R^n\rangle$ with $n > 1$, $\sigma_a(R) = 0$, $\sigma_b(R) \neq 0$ and $R \neq b^\epsilon$. Then* $\mathrm{Out}(G)$ *embeds in* $\mathrm{Out}(F(a, b))$.

Proof. It is sufficient to prove that if ϕ is an inner automorphism of G, $\phi \in \mathrm{Inn}(G)$, such that $\phi : a \mapsto a^{\epsilon_0}b^k$, $b \mapsto b^{\epsilon_1}$ then $a \equiv a^{\epsilon_0}b^k$ and $b \equiv b^{\epsilon_1}$, by Lemma 3.2.4. So, let $\phi : a \mapsto a^{\epsilon_0}b^k$, $b \mapsto b^{\epsilon_1}$ with either $\epsilon_0 \neq 1$, or $\epsilon_1 \neq 1$, or $k \neq 0$, and assume that ϕ is inner, $\phi \in \mathrm{Inn}(G)$. Therefore, there exists some word $W(a, b)$ such that $a^W =_G a^{\epsilon_0}b^k$ and $b^W =_G b^{\epsilon_1}$. Now, $W =_G b^i$ for some $i \in \mathbb{Z}$ and $\epsilon_1 = 1$, as $\langle b\rangle$ is malnormal in G, and so we can assume $W \equiv b^i$. We shall now prove that $i \neq 0$, or equivalently, that $W \neq_G 1$.

To do this, suppose that $i = 0$, so $a =_G a^{\epsilon_0} b^k$ and $b =_G b^{\epsilon_1}$. If $\epsilon_0 = -1$ then $a^2 = b^k$, but a has infinite order in the abelianisation while b has finite order, a contradiction. Therefore, $\epsilon_0 = 1$ and so $b^{\epsilon_1} =_G b$ and $b^k =_G 1$. Now, the assumptions of the lemma tells us that b has infinite order, and so $\epsilon_1 = 1$ and $k = 0$, a contradiction. Thus, we have that $i \neq 0$. Therefore, we have that $b^{-i} a b^i = a^{\epsilon_0} b^k$, so $a^{\epsilon_0} b^{k-i} a^{-1} b^i =_G 1$. We shall prove that $a^{\epsilon_0} b^{k-i} a^{-1} b^i$ cannot represent the trivial word, which is a contradiction and so proves the lemma.

To begin our proof that $a^{\epsilon_0} b^{k-i} a^{-1} b^i \neq_G 1$, we shall prove that $\epsilon_0 = 1$ and $i \neq k$. To do this, note that if $U =_G 1$ then $\sigma_a(U) = 0$. This is because the order of a under the abelianisation map is infinite. Thus, $\sigma_a(a^{\epsilon_0} b^{k-i} a^{-1} b^i) = 0$ and so $\epsilon_0 = 1$. We shall now prove that $i \neq k$. If $i = k$ then we have that $b^i = 1$, and because $i \neq 0$ this means that b has finite order. However, by the assumptions of the lemma b has infinite order. Thus, $i \neq k$.

We now analyse words of the form $ab^p a^{-1} b^q$, $p, q \neq 0$, and prove that they cannot represent the trivial word in $G = \langle a, b; R^n \rangle$. As $a^{\pm 1} \leq R$, we can apply the Newman-Gurevich spelling theorem, which is Proposition 2.9.5, to get that either $ab^p a^{-1} b^q$ is a cyclic shift of R^n or R^{-n}, or there exists two disjoint subwords $S^{n-1} S_0$ and $T^{n-1} T_0$, where $a^{\pm 1} \leq S, S_0, T, T_0$. We shall now prove that neither case can happen. Suppose the latter case occurs. The four words S, S_0, T, and T_0 are disjoint and each contain an $a^{\pm 1}$-term, but there are only 2 occurrences of $a^{\pm 1}$ in $ab^p a^{-1} b^q$ so we have a contradiction. On the other hand, the former case cannot happen as this word is not a proper power in $F(a, b)$, which we shall now prove. To see that $ab^p a^{-1} b^q$ is not a proper power in $F(a, b)$, suppose otherwise. Then $ab^p a^{-1} b^q \equiv S^n$, $n > 1$, and S must begin with an a and end in a b^ϵ. This means that no free cancellation happens when forming the word S^n, and so there exist two positive a-terms in S^n which is a contradiction. □

3.3 The possibilities for $\mathrm{Out}(G)$ when it is infinite

In this section we work under the assumption that G is a one-ended two-generator, one-relator group with torsion which has infinite outer automorphism group and determine the possible isomorphism classes for $\mathrm{Out}(G)$. We prove that every possibility occurs. Note that $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$ when $G \cong \langle a, b; [a, b]^n \rangle$, by Corollary 3.2.2. Therefore, in this section we assume that $G \ncong \langle a, b; [a, b]^n \rangle$. We maintain our assumptions from Section 3.2 that G is one-ended, that is, we assume that that R is not primitive.

Theorem 3.2.1 tells us that there is a faithful linear representation for $\mathrm{Out}(G)$. In this Section we use this representation to determine the possible forms which (outer) automorphisms of $G = \langle a, b; R^n \rangle$ can take when $\mathrm{Out}(G)$ is infinite (but G is one-ended and not isomorphic to $\langle a, b; [a, b]^n \rangle$). Note that we know the possible forms when $R \notin F(a, b)'$, by Lemma 3.2.4. The possible forms for R arbitrary are still very restrictive, and under the assumptions of this section the forms are precisely those given in Lemma 3.2.4. However, the proofs in this section are very different from the proof of Lemma 3.2.4. We use these forms to prove that $\mathrm{Out}(G)$ must be one of $\mathbb{Z}$, $\mathbb{Z} \times C_2$, D_∞, or $D_\infty \times C_2$.

3.3.1 The form of (outer) automorphisms when $\mathrm{Out}(G)$ is infinite

Note that under the assumptions of this section $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$, by Theorem 3.1.16.

Lemma 3.3.1. *Suppose $G \ncong \langle a, b; [a, b]^n \rangle$. Then the following equivalence holds.*

$$\mathrm{Out}(G) \text{ is virtually-}\mathbb{Z} \Longleftrightarrow G \cong \langle a, b; S^n(a, b^{-1}ab) \rangle$$

Proof. Suppose $G \cong \langle a, b; S(a, b^{-1}ab)^n \rangle$. Then the map $a \mapsto ab$, $b \mapsto b$ is an automorphism of G. It has infinite order by Theorem 3.2.1. As $G \ncong \langle a, b; [a, b]^n \rangle$, Theorem 3.1.16 implies that $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$.

Suppose $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$. Then G splits as an HNN-extension or free product with amalgamation with vertex group(s) having finite center and edge group virtually cyclic with infinite center [Lev05, Theorem 1.4]. Thus, G must split as an HNN-extension with infinite cyclic edge group, then by Proposition 3.1.14 the vertex group is a two-generator, one-relator group with torsion. This implies that G is isomorphic to a group of the required form. □

We shall prove, under the assumptions that $R \in \langle a, b^{-1}ab \rangle$ and $G \ncong \langle a, b; [a, b]^n \rangle$, that if ψ is an automorphism of G, $\psi \in \mathrm{Aut}(G)$, then there exists some automorphism $\phi \in \mathrm{Aut}(G)$ such that $\psi = \phi \mod \mathrm{Inn}(G)$ and $\phi : a \mapsto a^{\epsilon_0} b^k, b \mapsto b^{\epsilon_1}$. Thus, as in Lemma 3.2.4, the automorphisms can be assumed to take one of the following four forms.

$$\begin{array}{cccc} \alpha_i : a \mapsto a^{-1}b^i & \beta_i : a \mapsto ab^i & \zeta_i : a \mapsto a^{-1}b^i & \delta_i : a \mapsto ab^i \\ b \mapsto b & b \mapsto b^{-1} & b \mapsto b^{-1} & b \mapsto b \end{array}$$

We shall use the labels α_i, β_i, ζ_i, and δ_i in the rest of the chapterto refer to these forms.

Virtually cyclic subgroups. Note that we can assume that δ_1 is an automorphism of

G by Lemma 3.3.1. We shall work out all possible virtually cyclic subgroups of $\mathrm{GL}_2(\mathbb{Z})$ which contain the following matrix.

$$\Delta_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

This approach shall give a description of $\mathrm{Out}(G)$ because it embeds into $\mathrm{GL}_2(\mathbb{Z})$ by Theorem 3.2.1, and the embedding maps δ_1 to Δ_1. To work out these subgroups of $GL_2(\mathbb{Z}$ we begin with the following lemma, Lemma 3.3.2, which allows one to compute roots of the matrix Δ_i, $i \neq 0$. This lemma is easily proven by induction on m, and so the proof is left to the reader.

Lemma 3.3.2. *Let A be a matrix from* $\mathrm{GL}_2(\mathbb{Z})$*.*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then A^m has the following form where x_m, y_m and z_m are such that $x_m - y_m = z_m(a-d)$.

$$A^m = \begin{pmatrix} x_m & bz_m \\ cz_m & y_m \end{pmatrix}$$

We apply this lemma to the following result, which tells us about certain virtually cyclic subgroups of $\mathrm{GL}_2(\mathbb{Z})$.

Lemma 3.3.3. *If $\psi \in \mathrm{Aut}(F(a,b))$ is such that $\langle \widehat{\psi}, \widehat{\delta_1} \rangle$ is a virtually cyclic subgroup of $\mathrm{Out}(F(a,b))$ then, modulo the inner automorphisms, ψ corresponds to one of α_i, β_i, ζ_i, or δ_i.*

Proof. We prove this by using the equivalence of $\mathrm{Out}(F(a,b))$ and $\mathrm{GL}_2(\mathbb{Z})$. So, write $\Delta := \Delta_1$ for the matrix corresponding to δ_1 and Ψ for the matrix corresponding to ψ. Take $A := \Psi$ in Lemma 3.3.2, and we shall use Lemma 3.3.2 to prove that $c = 0$. This is sufficient as then $|x_m| = 1 = |y_m|$, by looking at the determinant of Ψ, which implies that ψ is of the required form.

Suppose that Ψ has infinite order, which implies that $\Psi^j = \Delta^k$ for some $j, k \neq 0$. Then, and applying the fact that $\Psi^j = \Delta^k$ we have that $x_j = 1 = y_j$, $cz_j = 0$ and $bz_j \neq 0$. Thus, $z_j \neq 0$ and so $c = 0$, as required.

Suppose Ψ has finite order. Then, $\Psi\Delta\Psi^{-1}$ has infinite order and so because we are in an infinite cyclic group there must exist integers $j, k \neq 0$ such that $\Psi\Delta^j\Psi^{-1} = \Delta^k$.

Writing $\epsilon := \det(\Psi) = \pm 1$, we have the following equivalences.

$$\begin{aligned}
\Psi\Delta^j\Psi^{-1} &= \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \epsilon \begin{pmatrix} ad - bc - jac & ja^2 \\ -jc^2 & jac + ad - bc \end{pmatrix} \\
&= \Delta^k
\end{aligned}$$

Then because $j \neq 0$ we have that $c = 0$, as required. □

The following lemma, Lemma 3.3.4, gives a form which the elements of $\mathrm{Out}(G)$ take if $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$. We use this form in Section 3.3.2 to determine the possible isomorphism classes for $\mathrm{Out}(G)$. The above lemma, Lemma 3.3.3, immediately implies Lemma 3.3.4 as subgroups of virtually cyclic groups are themselves virtually cyclic.

Lemma 3.3.4. *Suppose* $G \cong \langle a, b; S(a^{-1}ba, b)^n\rangle$. *If* $\psi \in \mathrm{Aut}(G)$ *then there exists some* $\phi \in \mathrm{Aut}(G)$ *such that* $\psi = \phi \mod \mathrm{Inn}(G)$ *and* $\phi \in \{\alpha_i, \beta_i, \zeta_i, \delta_i\}$.

3.3.2 The possibilities

We wish to give the possible isomorphism classes for $\mathrm{Out}(G)$ when it is virtually-$\mathbb{Z}$. So, in this section we take $G = \langle a, b; R^n\rangle$ and assume that $R \in \langle aba^{-1}, b\rangle$ but $G \ncong \langle a, b; [a, b]^n\rangle$, and we write down the possible isomorphism classes for $\mathrm{Out}(G)$. As $R \in \langle aba^{-1}, b\rangle$, so $\delta \in \mathrm{Aut}(G)$, one can view ζ_i as $\delta^{-i}\zeta$ where $\zeta := \zeta_0$, and so if $\zeta_i \in \mathrm{Aut}(G)$ then so is ζ_j for all $j \in \mathbb{Z}$. Similarly, $\alpha_i = \delta^i\alpha$ and $\beta_i = \delta^{-i}\beta$, where $\alpha := \alpha_0$ and $\beta := \beta_0$, and so if $\alpha_i \in \mathrm{Aut}(G)$ (respectively $\beta_i \in \mathrm{Aut}(G)$) then so is α_j (respectively β_j) for all $j \in \mathbb{Z}$. Now, as $\alpha\beta = \zeta$, if α and β are in $\mathrm{Aut}(G)$ then so is ζ. Similarly, if ζ and α are then so is β and if ζ and β are then so is α. What this means is that we have five choices of generating set for $\mathrm{Out}(G)$ if $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$. We always have $\delta \in \mathrm{Aut}(G)$, and we either have none of, one of or all three of α, β and ζ. We shall prove that the following isomorphisms hold.

1. If $\alpha, \beta, \zeta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong \mathbb{Z}$.

2. If $\alpha \in \mathrm{Aut}(G)$ but $\beta, \zeta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong D_\infty$.

3. If $\beta \in \mathrm{Aut}(G)$ but $\alpha, \zeta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong D_\infty$.

4. If $\zeta \in \mathrm{Aut}(G)$ but $\alpha, \beta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong \mathbb{Z} \times C_2$.

5. If $\alpha, \beta, \zeta \in \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong D_\infty \times C_2$.

Each of these possibilities occurs, and the following examples can be verified by checking if $R\phi$ is freely conjugate to $R^{\pm 1}$ or not for each $\phi \in \{\alpha, \beta\zeta\}$. This works because none of these maps change the length of the relator R so we can apply the Newman–Gurevich Spelling Theorem, which is Proposition 2.9.5.

1. If $R = aba^{-1}b^2ab^3a^{-1}b^4$ then $\alpha, \beta, \zeta \notin \mathrm{Aut}(G)$.
2. If $R = aba^{-1}b^2ab^3a^{-1}bab^2a^{-1}b^3$ then $\alpha \in \mathrm{Aut}(G)$ but $\beta, \zeta \notin \mathrm{Aut}(G)$.
3. If $R = aba^{-1}b^2$ then $\beta \in \mathrm{Aut}(G)$ but $\alpha, \zeta \notin \mathrm{Aut}(G)$.
4. If $R = aba^{-1}b^2ab^2a^{-1}bab^3a^{-1}b^3$ then $\zeta \in \mathrm{Aut}(G)$ but $\alpha, \beta \notin \mathrm{Aut}(G)$.
5. If $R = aba^{-1}b$ then $\alpha, \beta, \zeta \in \mathrm{Aut}(G)$.

The possibile isomorphism classes. We wish to verify the isomorphisms for $\mathrm{Out}(G)$ in each of these five cases. If $\mathrm{Out}(G) = \langle \delta \rangle$ then clearly $\mathrm{Out}(G) \cong \mathbb{Z}$. Otherwise, the presentations are easily acquired as there is a normal form; every element is of the form $\delta^i\widehat{\sigma}$ with $\sigma \in \{\alpha, \beta, \zeta, e\}$, where e denotes the trivial automorphism. By Theorem 3.2.1, an element of this normal form is trivial modulo the inner automorphisms if and only if $i = 1$ and $\sigma = e$. This means that once we have added the relators to the group which get elements into this normal form (which we can work out as we have a representation for $\mathrm{Out}(G)$ in terms of Nielsen transformations) we need add no more relators. The groups are as follows. Note that for the sake of clarity we have abused notation in the following presentations and written ϕ in place of $\widehat{\phi}$ for each $\phi \in \{\alpha, \beta, \zeta, \delta\}$.

1. $\alpha, \beta, \zeta \notin \mathrm{Aut}(G)$, and so $\mathrm{Out}(G) \cong \mathbb{Z}$.
2. $\alpha \in \mathrm{Aut}(G)$ but $\beta, \zeta \notin \mathrm{Aut}(G)$, and so we have the following isomorphism.
$$\mathrm{Out}(G) \cong \langle \delta, \alpha; \alpha^2, \alpha\delta = \delta^{-1}\alpha \rangle \cong D_\infty$$
3. $\beta \in \mathrm{Aut}(G)$ but $\alpha, \zeta \notin \mathrm{Aut}(G)$, and so we have the following isomorphism.
$$\mathrm{Out}(G) \cong \langle \delta, \beta; \beta^2, \beta\delta = \delta^{-1}\beta \rangle \cong D_\infty$$
4. $\zeta \in \mathrm{Aut}(G)$ but $\alpha, \beta \notin \mathrm{Aut}(G)$, and so we have the following isomorphism.
$$\mathrm{Out}(G) \cong \langle \delta, \zeta; \zeta^2, [\delta, \zeta] \rangle \cong \mathbb{Z} \times C_2$$

5. $\alpha, \beta, \zeta \in \mathrm{Aut}(G)$, and so we have the following relations, modulo the inner automorphisms.

$$\alpha^2 = e, \beta^2 = e, \zeta^2 = e, \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \delta\zeta = \zeta\delta$$

$$\alpha\beta = \zeta, \alpha\zeta = \beta, \beta\alpha = \zeta, \beta\zeta = \alpha, \zeta\alpha = \beta, \zeta\beta = \alpha$$

All of the relations in the second line are consequences of the following relations.

$$\alpha^2 = e, \beta^2 = e, \zeta^2 = e, \alpha\beta\zeta = e$$

We therefore have the following isomorphism.

$$\mathrm{Out}(G) \cong \langle \alpha, \beta, \delta, \zeta; \alpha^2, \beta^2, \zeta^2, [\delta, \zeta], \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \alpha\beta\zeta \rangle$$

Replacing β with $\alpha\zeta$, and following the Tietze transformations through we obtain the required isomorphism:

$$\begin{aligned}\mathrm{Out}(G) &\cong \langle \alpha, \delta, \zeta; \alpha^2, \zeta^2, \alpha\delta\alpha = \delta^{-1}, [\alpha, \zeta], [\delta, \zeta] \rangle \\ &\cong D_\infty \times C_2\end{aligned}$$

This section yields the following lemma.

Lemma 3.3.5. *Suppose G is a one-relator group with torsion and $G \not\cong \langle a, b; [a, b]^n \rangle$. Then $\mathrm{Out}(G)$ is one of $\mathbb{Z}$, $\mathbb{Z} \times C_2$, D_∞ or $D_\infty \times C_2$. Moreover, each of these possibilities occurs.*

Note that this proves that there exists an algorithm to find $\mathrm{Out}(G)$ if $R \in \langle a^{-1}ba, b \rangle$. One determines if $\phi \in \mathrm{Aut}(G)$ for each $\sigma \in \{\alpha, \beta, \zeta\}$. The isomorphism class of $\mathrm{Out}(G)$ is obtained by comparing which of these maps ϕ are in $\mathrm{Aut}(G)$ with the above list.

3.4 The possibilities for $\mathrm{Out}(G)$ when it is finite

In this section we work under the assumption that G is a one-ended two-generator, one-relator group with torsion which has finite outer automorphism group and we determine the possible isomorphism classes for $\mathrm{Out}(G)$. We prove that every possibility occurs. Note that in Section 3.3.2 we did this for when $\mathrm{Out}(G)$ is infinite.

Suppose $\mathrm{Out}(G)$ is finite, then $\mathrm{Out}(G)$ must be isomorphic to a finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$, by Theorem 3.2.1. Every finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$ is a subgroup of D_6 or of D_4 [Zim96], where D_n denotes the dihedral group of order $2n$. We shall now show that the groups D_6, D_4, D_3, C_6, C_4 and C_3 can be realised as the outer automorphism group of a two-generator, one-relator group with torsion. The remaining three groups which occur as subgroups of D_4 or D_6, which are $C_2 \times C_2$, C_2 and the trivial group, also each occur as the outer automorphism group of a one-ended two-generator, one-relator group with torsion. However, these cases require more working to prove and so are treated later, in Section 3.4.1.

Lemma 3.4.1. *For $Q \in \{D_6, D_4, D_3, C_6, C_4, C_3\}$ there exists a group $G = \langle a, b; R^n \rangle$, $n > 1$, such that $\mathrm{Out}(G) \cong Q$. The following groups give explicit examples.*

- *If $G = \langle a, b; (a^2bab^2a^{-2}b^{-1}a^{-1}b^{-2})^n \rangle$ then $\mathrm{Out}(G) \cong D_6$.*
- *If $G = \langle a, b; [a^2, b^2]^n \rangle$ then $\mathrm{Out}(G) \cong D_4$.*
- *If $G = \langle a, b; (a^2(ab)^{-2}b^2)^n \rangle$ then $\mathrm{Out}(G) \cong D_3$.*
- *If $G = \langle a, b; R^n \rangle$ where R is the word*

$$a^2b^3aba^{-1}b^{-2}ababa^2ba^{-1}b^{-1}a^{-1}b^{-1}a^2b^{-1}ab$$
$$a^{-2}b^{-3}a^{-1}b^{-1}ab^2a^{-1}b^{-1}a^{-1}b^{-1}a^{-2}b^{-1}ababa^{-2}ba^{-1}b^{-1}$$

  then $\mathrm{Out}(G) \cong C_6$.
- *If $G = \langle a, b; (ab^2aba^{-2}ba^{-1}b^{-2}a^{-1}b^{-1}a^2b^{-1})^n \rangle$ then $\mathrm{Out}(G) \cong C_4$.*
- *If $G = \langle a, b; (ab^{-1}a^2b^{-1}a^{-2}b^{-1}a^{-1}b^{-1}a^{-1}bab^3)^n \rangle$ then $\mathrm{Out}(G) \cong C_3$.*

Proof. We shall use the fact that if $\mathrm{Out}(G)$ is infinite and $G \not\cong \langle a, b; [a, b]^n \rangle$ then any finite order elements of $\mathrm{Out}(G)$ have order two, which follows from Lemma 3.3.5. We also use the fact that none of the groups G in the statement of the lemma are isomorphic to $\langle a, b; [a, b]^n \rangle$, as $\langle a, b; R^n \rangle \cong \langle a, b; [a, b]^n \rangle$ if and only if R is freely conjugate to $[a, b]^{\pm 1}$, which follows from the solution to the isomorphism problem for two-generator, one-relator groups with torsion [Pri77a]. Note that by Theorem 3.2.1, all interactions between outer automorphisms can be verified by viewing them as elements of $\mathrm{GL}_2(\mathbb{Z})$.

Proof of D_6: Note that the maps $\phi : a \mapsto b^{-1}, b \mapsto ab$ and $\psi : a \mapsto ab,\ b \mapsto b^{-1}$ define automorphisms of $G = \langle a, b; (a^2bab^2a^{-2}b^{-1}a^{-1}b^{-2})^n \rangle$. Now, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order six but $G \not\cong \langle a, b; [a, b]^n \rangle$. Therefore, $\mathrm{Out}(G)$ is isomorphic to a subgroup of D_4 or D_6 which contains an element of order six, and so it is isomorphic to either D_6 or C_6. Then, as $\widehat{\phi}^3 \neq \widehat{\psi}$ but $\widehat{\psi}$ has order two, we conclude that $\mathrm{Out}(G) \cong D_6$, as required.

Proof of D_4: Note that the maps $\phi : a \mapsto b, b \mapsto a^{-1}$ and $\psi : a \mapsto b,\ b \mapsto a$ define automorphisms of $G = \langle a, b; [a^2, b^2]^n \rangle$. Now, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order four but $G \ncong \langle a, b; [a, b]^n \rangle$. Therefore, $\mathrm{Out}(G)$ is isomorphic to a subgroup of D_4 or D_6 which contains an element of order four, and so it is isomorphic to either D_4 or C_4. Then, as $\widehat{\phi}^2 \neq \widehat{\psi}$ but $\widehat{\psi}$ has order two, we conclude that $\mathrm{Out}(G) \cong D_2$, as required.

Proof of D_3: Note that the maps $\phi : a \mapsto a^{-1}b^{-1}, b \mapsto a$ and $(\beta_1 =)\psi : a \mapsto ab,\ b \mapsto b^{-1}$ define automorphisms of $G = \langle a, b; (a^2(ab)^{-2}b^2)^n \rangle$. As in the D_6 case, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order three but $G \ncong \langle a, b; [a, b]^n \rangle$. Noting that $\widehat{\psi}$ has order two, $\mathrm{Out}(G)$ is isomorphic to a subgroup of D_4 or D_6 which contains an element of order three and an element of order two, and so it is isomorphic to one of D_6, C_6 or D_3. We shall prove that $\mathrm{Out}(G)$ does not contain an element of order six, which is sufficient. To do this, we look at the embedding of $\mathrm{Out}(G)$ in $\mathrm{GL}_2(\mathbb{Z})$. There are two matrices, Ω_1 and Ω_2, which satisfy the relation $\Omega^2 = \Phi$, where the matrix Φ is the image of ϕ in $\mathrm{GL}_2(\mathbb{Z})$, and so there are only two possible Nielsen transformations which would have order six in $\mathrm{Out}(G)$. The corresponding matrices are as follows:

$$\Omega_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \Omega_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

We can ignore the matrix $\Omega_2 = \Phi^{-1}$. Now, any Nielsen transformation which corresponds to Ω_1 does not preserve the relation of the group, $(a^2(ab)^{-2}b^2)^n$, and so does not correspond to an automorphism of G. Thus, $\mathrm{Out}(G)$ contains no element of order six and so $\mathrm{Out}(G) \cong D_3$, as required.

Proof of C_6: Note that the map $\phi : a \mapsto b^{-1}, b \mapsto ab$ defines an automorphism of $G = \langle a, b; R^n \rangle$, where R is as in the statement of the lemma. As in the D_6 case, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order six but $G \ncong \langle a, b; [a, b]^n \rangle$. Therefore, $\mathrm{Out}(G)$ is isomorphic to a subgroup of D_4 or D_6 which contains an element of order six, and so it is isomorphic to either D_6 or C_6. Suppose $\mathrm{Out}(G) \cong D_6$, and we shall find a contradiction. There are six matrices, $\Psi_1, \ldots, \Psi_6$, which are of order two and which satisfy the relator $(\Phi\Psi_i)^2$, where Φ is the image of ϕ in $\mathrm{GL}_2(\mathbb{Z})$, and so there are only six possible Nielsen transformations which will generate $\mathrm{Out}(G) \cong D_6$ with $\widehat{\phi}$. The matrices, up to multiplication by $-Id$, are as follows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

Note that $-Id = \widehat{\phi}^3$, and so we only need to verify that the above three matrices do not correspond to automorphisms of G. However, any Nielsen transformation which corresponds to one of these six matrices do not preserve the relator of the group, and so do not define automorphisms of G. This is our required contradiction, and so $\mathrm{Out}(G) \cong C_6$.

Proof of C_4: Note that the map $\phi : a \mapsto b, b \mapsto a^{-1}$ defines an automorphism of $G = \langle a, b; (ab^2aba^{-2}ba^{-1}b^{-2}a^{-1}b^{-1}a^2b^{-1})^n \rangle$. As in the D_4 case, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order four but $G \ncong \langle a, b; [a, b]^n \rangle$. Therefore, $\mathrm{Out}(G)$ is isomorphic to a subgroup of D_4 or D_6 which contains an element of order four, and so it is isomorphic to either D_4 or C_4. Suppose $\mathrm{Out}(G) \cong D_4$, and we shall find a contradiction. There are two matrices, Ψ_1 and Ψ_2, which are of order two and which satisfy the relator $(\Phi\Psi_i)^2$, where Φ is the image of ϕ in $\mathrm{GL}_2(\mathbb{Z})$, and so there are only two possible Nielsen transformation which will generate $\mathrm{Out}(G) \cong D_4$ with $\widehat{\phi}$. The matrices are as follows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

However, any Nielsen transformation which corresponds to one of these two matrices do not preserve the relator of the group and so do not define automorphisms of G. This is our required contradiction, and so $\mathrm{Out}(G) \cong C_4$.

Proof of C_3: Note that the map $\phi : a \mapsto a^{-1}b^{-1}, b \mapsto a$ defines an automorphism of $G = \langle a, b; (ab^{-1}a^2b^{-1}a^{-2}b^{-1}a^{-1}b^{-1}a^{-1}bab^3)^n \rangle$. As in the D_6 case, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order three but $G \ncong \langle a, b; [a, b]^n \rangle$. Therefore, $\mathrm{Out}(G)$ is isomorphic to a subgroup of D_4 or D_6 which contains an element of order three, and so it is isomorphic to one of D_6, C_6, D_3 or C_3. We shall prove that the former three cases cannot happen. Applying identical logic to the D_3 case (the same matrices appear), we get that $\mathrm{Out}(G)$ cannot contain an element of order six and so must be isomorphic to either D_3 or C_3. Finally, applying identical logic to the C_6 case (again, the same matrices appear) we get that $\mathrm{Out}(G)$ cannot be isomorphic to D_3 and so is isomorphic to C_3. □

The remaining three finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$, which are $C_2 \times C_2$, C_2 and the trivial group, do each occur as the outer automorphism group of a one-ended two-generator, one-relator group with torsion. The purpose of the following section is to prove this.

3.4.1 The relator is not in the derived subgroup of $F(a, b)$

In this section we assume $R \notin F(a, b)'$. We begin with Lemma 3.4.2, which proves that if $G \cong \langle a, b; R^n \rangle$ is a one-ended two-generator, one-relator group such that $R \notin F(a, b)'$ that either $\delta_k \in \mathrm{Aut}(G)$ for some $k \neq 0$ or $\mathrm{Out}(G)$ is isomorphic to one of $C_2 \times C_2$, C_2 or the trivial group. We analyse the implications of the presence of an automorphism δ_k, which allows us to give an algorithm which determines $\mathrm{Out}(G)$ if it is finite with $R \notin F(a,b)$. This allows us, in Lemma 3.4.6, to construct two-generator, one-relator groups G such that $\mathrm{Out}(G) \cong C_2$, $\mathrm{Out}(G) \cong C_2 \times C_2$ and $\mathrm{Out}(G)$ is trivial. This completes the proof that each of the possibilities given in Theorem A occur for the case of one-ended two-generator, one-relator groups with torsion.

Lemma 3.4.2. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$, R not primitive, $\sigma_a(R) = 0$, and $\sigma_b(R) \neq 0$. Then $\mathrm{Out}(G)$ is either infinite, $C_2 \times C_2$, C_2 or trivial. If $\mathrm{Out}(G)$ is infinite then there exists $k \in \mathbb{Z}$ such that $\delta_k \in \mathrm{Aut}(G)$.*

The proof of this result uses certain equalities between Nielsen transformations. These can be quickly and easily verified by using the corresponding matrices in $\mathrm{GL}_2(\mathbb{Z})$, given by Theorem 3.2.1.

Proof. Firstly, note that Lemma 3.2.4 tells us that every element of $\mathrm{Out}(G)$ can be viewed as one of $\widehat{\alpha_i}$, $\widehat{\beta_i}$, $\widehat{\zeta_i}$ or $\widehat{\delta_i}$, $i \in \mathbb{Z}$.

Now, by Theorem 3.2.1, if δ_i is an automorphism of G then it has infinite order modulo the inner automorphisms, for $i \neq 0$. Then, $\widehat{\zeta_i}^2 = \widehat{\delta}_{-2i}$ and so if $\zeta_i \in \mathrm{Aut}(G)$ with $i \neq 0$ then $\mathrm{Out}(G)$ is infinite. Next, $\widehat{\alpha_i}\widehat{\alpha_j} = \widehat{\delta}_{i-j}$ and so if α_i and $\alpha_j \in \mathrm{Aut}(G)$ with $i \neq j$ then $\mathrm{Out}(G)$ is infinite. Now, $\widehat{\beta_i}\widehat{\beta_j} = \widehat{\delta}_{j-i}$ and so if β_i and $\beta_j \in \mathrm{Aut}(G)$ with $i \neq j$ then $\mathrm{Out}(G)$ is infinite. Finally, $\widehat{\beta_j}\widehat{\alpha_i} = \widehat{\zeta}_{i+j}$ and so if α_i and $\beta_j \in \mathrm{Aut}(G)$ with $i + j \neq 0$ then $\mathrm{Out}(G)$ is infinite. Therefore, if $\mathrm{Out}(G)$ is finite we only have the following possibilities. Note that the isomorphisms can be obtained using the corresponding matrices in $\mathrm{GL}_2(\mathbb{Z})$.

$$
\begin{aligned}
&\mathrm{Out}(G) \text{ is trivial.}\\
\text{or}\ \ &\mathrm{Out}(G) = \langle \widehat{\alpha_i} \rangle \cong C_2.\\
\text{or}\ \ &\mathrm{Out}(G) = \langle \widehat{\beta_i} \rangle \cong C_2.\\
\text{or}\ \ &\mathrm{Out}(G) = \langle \widehat{\zeta_0} \rangle \cong C_2.\\
\text{or}\ \ &\mathrm{Out}(G) = \langle \widehat{\alpha_i}, \widehat{\beta}_{-i} \rangle \cong C_2 \times C_2.
\end{aligned}
$$

Otherwise, $\delta_k \in \mathrm{Aut}(G)$ for some $k \in \mathbb{Z}$, as required. □

Note that the above lemma, Lemma 3.4.2, says that $\mathrm{Out}(G)$ is infinite if and only if there exists some $k \in \mathbb{Z}$ such that $\delta_k \in \mathrm{Aut}(G)$. This is similar to Lemma 3.3.1. The difference is that if $R \notin F(a,b)'$ then the Lemma 3.4.2 can be interpreted (using

Lemma 3.4.4, below) as saying that, after Moldovanskii rewriting, if G is infinite then it has a presentation of the form given by Lemma 3.3.1, while Lemma 3.3.1 merely states that such a presentation exists. Note that it is not obvious that if δ_k defines an automorphism of G for $|k| > 1$ then δ_1 defines an automorphism of G, but we prove this in Lemma 3.4.4, below.

We wish to find out when the Nielsen transformations α_k, β_k and δ_k define automorphisms of G for a given $k \in \mathbb{Z}$. This will allows us to use Lemma 3.4.2 to verify that each of $C_2 \times C_2$, C_2 and the trivial group occur as the outer automorphism group of a one-ended two-generator, one-relator group with torsion. Lemma 3.4.4, below, proves that $\delta_1 \in \mathrm{Aut}(G)$ if and only if $\mathrm{Out}(G)$ is infinite, and, further, that δ_1 fixes R or a cyclic shift of R if and only if $\mathrm{Out}(G)$ is infinite, where $\sigma_a(R) = 0$ and $\sigma_b(R) \neq 0$. The lemma also proves that if $\mathrm{Out}(G)$ is finite then one can find a finite set of integers A_R (respectively B_R) obtained from the relator R such that α_k (respectively β_k) can be in $\mathrm{Aut}(G)$ only if $k \in A_R$ (respectively $k \in B_R$). Note that this yields an algorithm to compute $\mathrm{Out}(G)$ (for G satisfying the assumptions of this section): rewrite $G = \langle a, b; R^n \rangle$ such that $\sigma_a(R) = 0$, then $\mathrm{Out}(G)$ is infinite if δ_1 defines an automorphism of G while if δ_1 does not define an automorphism of G then one obtains the sets A_R and B_R and checks if α_i and β_i define automorphisms of G, and finally one checks if ζ_0 defines an automorphism of G. One can then apply this knowledge to Lemma 3.4.2 or to Section 3.3.2 to obtain the isomorphism class of $\mathrm{Out}(G)$.

Recall that the purpose of this current section is to realise each of $C_2 \times C_2$, C_2 and the trivial group as the outer automorphism group of a two-generator, one-relator group with torsion. In Lemma 3.4.6, the algorithm given by Lemma 3.4.4 allows us to realise these groups in this way. To prove Lemma 3.4.4 we need the following technical result.

Lemma 3.4.3. *Let ϕ_k be the following Nielsen transformation.*

$$\begin{aligned} \phi_k : a &\mapsto a^{\epsilon_0} b^k \\ b &\mapsto b^{\epsilon_1} \end{aligned}$$

Let W be an arbitrary, freely reduced word in $F(a, b)$. Then we have the following.

1. *If W begins in a, $W\phi_k$ begins in a^{ϵ_0}.*

2. *If W begins in a^{-1}, $W\phi_k$ begins in $b^{-k}a^{-\epsilon_0}$.*

3. *If W ends in a^{-1}, $W\phi_k$ ends in $a^{-\epsilon_0}$.*

4. *If W ends in a, $W\phi_k$ ends in $a^{\epsilon_0}b^k$.*

Proof. Note that once we have proven (1) and (2) then (3) and (4) follow immediately, by looking at W^{-1}. To prove (1) and (2) we assume that $W \equiv a^{\epsilon}\overline{W}$ is a word starting with an a-term and induct on the number of a-terms in the word W.

If W contains one a-term then $W = a^{\epsilon}b^i$ so $W\phi_k = a^{\epsilon_0}b^{\epsilon_1 i+k}$ if $\epsilon = 1$ while $W\phi_k = b^{-k}a^{-\epsilon_0}b^{\epsilon_1 i}$ if $\epsilon = -1$, as required.

Assume the result holds for all words beginning with an a-term and containing n a-terms, and let W be a word containing $n+1$ a-terms and beginning with an a-term. Then $W = a^{\epsilon}b^i a^{\epsilon'}\overline{W}$ where $a^{\epsilon'}\overline{W}$ satisfies the induction hypothesis and $i \neq 0$ if $\epsilon + \epsilon' = 0$. We thus have four cases to consider, which are as follows. These prove the lemma.

- $\epsilon = 1$, $\epsilon' = 1$: $W\phi_k = (ab^i)\phi_k(a\overline{W})\phi_k = a^{\epsilon_0}(b^{\epsilon_1 i+k}a^{\epsilon_0}\widehat{W})$.
- $\epsilon = 1$, $\epsilon' = -1$: $W\phi_k = (ab^i)\phi_k(a^{-1}\overline{W})\phi_k = a^{\epsilon_0}(b^{\epsilon_1 i}a^{-\epsilon_0}\widehat{W})$.
- $\epsilon = -1$, $\epsilon' = 1$: $W\phi_k = (a^{-1}b^i)\phi_k(a\overline{W})\phi_k = b^{-k}a^{-\epsilon_0}(b^{\epsilon_1 i}a^{\epsilon_0}\widehat{W})$.
- $\epsilon = -1$, $\epsilon' = -1$: $W\phi_k = (a^{-1}b^i)\phi_k(a^{-1}\overline{W})\phi_k = b^{-k}a^{-\epsilon_0}(b^{\epsilon_1 i-k}a^{-\epsilon_0}\widehat{W})$.

□

We shall now prove the following lemma, Lemma 3.4.4, which gives us an algorithm to calculate $\mathrm{Out}(G)$ if $\mathrm{Out}(G)$ is finite and $R \notin F(a,b)'$. The lemma also proves that if $R \notin F(a,b)'$ then performing Moldovanskii rewriting on R will show you if $\mathrm{Out}(G)$ is infinite or not. We use this algorithm in the proof of Lemma 3.4.6 to verify that $C_2 \times C_2$, C_2 and the trivial group each occur as the outer automorphism group of a one-ended two-generator, one-relator group with torsion. Define $\min_+$ to be the least integer such that ab^ia is a subword of some cyclic shift of R^n and define $\max_+$ to be the greatest such integer. Further, define $\min_-$ to be the least integer such that $a^{-1}b^ia^{-1}$ is a subword of some cyclic shift of R^n and define $\max_-$ to be the greatest such integer.

Lemma 3.4.4. *Let $G = \langle a, b; R^n\rangle$ with $n > 1$, R not primitive, and $\sigma_a(R) = 0$ but $\sigma_b(R) \neq 0$. If $a^{\epsilon'}b^ia^{\epsilon'}$ is a subword of some cyclic shift of R^n for $i \in \mathbb{Z}$ then the following points hold.*

- $\alpha_k \in \mathrm{Aut}(G)$ *only if either $k = -(\max_+ + \min_+)$ or k is in the following range.*

$$\min_- - \min_+ \leq k \leq \max_- - \max_+$$

- $\beta_k \in \mathrm{Aut}(G)$ *only if either* $k = \min_+ + \max_+$ *or* k *is in the following range.*

$$\max_+ - \max_- \leq k \leq \min_+ - \min_-$$

- $\delta_k \notin \mathrm{Aut}(G)$ *for* $k \neq 0$.

Proof. Note that if $a^{\epsilon'}b^i a^{\epsilon'}$ is a subword of some cyclic shift of R^n then there exists some $j \in \mathbb{Z}$ such that $a^{-\epsilon'}b^j a^{-\epsilon'}$ is a subword of some cyclic shift of R^n. This is because $\sigma_a(R^n) = 0$. Therefore, we assume that ab^ia is a subword of some cyclic shift of R^n and prove that the points hold.

We prove the following points which prove the lemma. We explain, below, why they prove the lemma, and we then prove that they hold.

- If $\alpha_k \in \mathrm{Aut}(G)$ and ab^ia is a subword of some cyclic shift of R^n then $a^{-\epsilon}b^{(i+k)\epsilon}a^{-\epsilon}$ is a subword of some cyclic shift of R^n.

- If $\beta_k \in \mathrm{Aut}(G)$ and ab^ia is a subword of some cyclic shift of R^n then $a^{\epsilon}b^{(k-i)\epsilon}a^{\epsilon}$ is a subword of some cyclic shift of R^n.

- If $\delta_k \in \mathrm{Aut}(G)$ and ab^ia is a subword of some cyclic shift of R^n then either $ab^{i-k}a$ or $ab^{i-2k}a$ is a subword of some cyclic shift of R^n.

Too see that these points prove the lemma, note that i can take any value between $\min_+$ and $\max_+$, $\min_+ \leq i \leq \max_+$, so:

Consider α_k and suppose $\epsilon = -1$. Then $\min_+ \leq -i - k \leq \max_+$, and so substituting in $i := \min_+$ and separately $i := \max_+$ we get two inequalities which combine to give $-(\min_+ + \max_+) \leq k \leq -(\min_+ + \max_+)$ which yields the required equality. Suppose $\epsilon = -1$, then we have $\min_- \leq i + k \leq \max_-$, and so substituting in $i := \min_+$ and separately $i := \max_+$ we obtain the following two inequalities.

$$\min_- - \max_+ \leq k \leq \max_- - \max_+$$

$$\min_- - \min_+ \leq k \leq \max_- - \min_+$$

These combine to give $\min_- - \min_+ \leq k \leq \max_- - \max_+$ as required.

Consider β_k, and suppose $\epsilon = 1$. Then $\min_+ \leq k - i \leq \max_+$, and so substituting in $i := \min_+$ and separately $i := \max_+$ we get two inequalities which combine to give $\min_+ + \max_+ \leq k \leq \min_+ + \max_+$ which yields the required equality. Suppose $\epsilon = -1$, then we have $\min_- \leq i-k \leq \max_-$, and so substituting in $i := \min_+$ and separately $i := \max_+$ we obtain the following two inequalities.

$$\min_+ - \min_- \geq k \geq \min_+ - \max_-$$

$$\max_+ - \min_- \geq k \geq \max_+ - \max_-$$

These combine to give $\max_+ - \max_- \leq k \leq \min_+ - \min_-$ as required.

Consider δ_k and note that $\delta_k^{-1} = \delta_{-k}$ so we can assume $k > 0$. Then, taking i to be the least integer such that ab^ia is a subword of some cyclic shift of R^n (so, $i = \min_+$) we have that $ab^{i-k}a$ is a subword of a cyclic shift of R^n, a contradiction.

We now prove the three statements. note that in each statement we can cyclically shift the word R^n to obtain a new word S^n which contains the subword ab^ia. Therefore, we shall assume that ab^ia is a subword of R^n

To prove the statements, we start by using the Newman–Gurevich Spelling Theorem, which is Proposition 2.9.5. We begin by proving that for all $\phi \in \mathrm{Aut}(G)$ there exists $\epsilon \in \{1, -1\}$ such that $a^{\epsilon}b^{\epsilon i}a^{\epsilon} \leq R^n\phi$. This is because we have that $S^{n-1}S_0 \leq R^n\phi$, S a cyclic shift of R or R^{-1}, and either $a^{\epsilon}b^{\epsilon i}a^{\epsilon} \leq S$ or $S \equiv b^{\epsilon j}a^{\epsilon}\bar{S}a^{\epsilon}b^{\epsilon k}$, $i = j+k$, and in each case $a^{\epsilon}b^{\epsilon i}a^{\epsilon} \leq SS_0$, as required. [3] Note that we have that $a^{\epsilon}b^{\epsilon i}a^{\epsilon}$ is a subword of $R^n\phi$, not just a subword of a cyclic shift.

Writing γ_g for the automorphism inducing conjugation by g, so $a\gamma_g = g^{-1}ag$ and $b\gamma_g = g^{-1}bg$, we investigate the three cases:

Let α_k define an automorphism of G: As ab^ia is a subword of R^n we have that $a^{\epsilon}b^{\epsilon i}a^{\epsilon}$ is a subword of $R^n\alpha_k$, so $R^n\alpha_k \equiv Ua^{\epsilon}b^{\epsilon i}a^{\epsilon}V$. Then as $\alpha_k^{-1} = \alpha_k\gamma_{b^{-k}}$ we can apply Lemma 3.4.3 to $R^n\alpha_k$ to obtain R^n as follows.

$$\begin{aligned} R^n &\equiv (Ua^{\epsilon})\alpha_k\gamma_{b^{-k}}(b^{\epsilon i})\alpha_k\gamma_{b^{-k}}(a^{\epsilon}V)\alpha_k\gamma_{b^{-k}} \\ &\equiv (U'a^{-\epsilon}b^{(i+k)\epsilon}a^{-\epsilon}V')\gamma_{b^{-k}} \\ &\equiv b^kU'a^{-\epsilon}b^{(i+k)\epsilon}a^{-\epsilon}V'b^{-k} \end{aligned}$$

This yields the required result.

Let β_k define an automorphism of G: As ab^ia is a subword of R^n we have that $a^{\epsilon}b^{\epsilon i}a^{\epsilon}$ is a subword of $R^n\beta_k$, so $R^n\beta_k \equiv Ua^{\epsilon}b^{\epsilon i}a^{\epsilon}V$. Then as $\beta_k^{-1} = \beta_k$ we can apply Lemma 3.4.3

[3] Newman's original spelling theorem merely stipulated that $|S_0| \geq 1$ []. Note that this is insufficient here, and instead we need enhancement due to Gurevich. The enhanced result is the version stated in Proposition 2.9.5.

to $R^n\beta_k$ to obtain R^n as follows.

$$\begin{aligned} R^n &\equiv (Ua^\epsilon)\beta_k(b^{\epsilon i})\beta_k(a^\epsilon V)\beta_k \\ &\equiv U'a^\epsilon b^{(k-i)\epsilon}a^\epsilon V' \end{aligned}$$

This yields the required result.

Let δ_k define an automorphism of G: As ab^ia is a subword of R^n we have that $a^\epsilon b^{\epsilon i}a^\epsilon$ is a subword of $R^n\delta_k$. We have two cases, $\epsilon = 1$ and $\epsilon = -1$.

1. Assume $\epsilon = 1$. That is, ab^ia is a subword of $R^n\delta_k$, so $R^n\delta_k \equiv Uab^iaV$. Then as $\delta_k^{-1} = \delta_{-k}$ we can apply Lemma 3.4.3 to $R^n\delta_k$ to obtain R^n as follows.

$$\begin{aligned} R^n &\equiv (Ua)\delta_{-k}(b^i)\delta_{-k}(aV)\delta_{-k} \\ &\equiv U'ab^{i-k}aV' \end{aligned}$$

   This yields the required result.

2. Assume $\epsilon = -1$. That is, $a^{-1}b^{-i}a^{-1}$ is a subword of $R^n\delta_k$ but ab^ia is not. We thus have $R^n\delta_k \equiv Ua^{-1}b^{-i}a^{-1}V$. Then as $\delta_k^{-1} = \delta_{-k}$ we can apply Lemma 3.4.3 to $R^n\delta_k$ to obtain R^n as follows.

$$\begin{aligned} R^n &\equiv (Ua^{-1})\delta_{-k}(b^{-i}\delta_{-k})(a^{-1}V)\delta_{-k} \\ &\equiv U'a^{-1}b^{k-i}a^{-1}V' \end{aligned}$$

   Therefore, $a^{-1}b^{k-i}a^{-1}$ is a subword of a cyclic shift of R^n.

   Now, by the Newman-Gurevich Spelling Theorem there exists a cyclic shift S of R or R^{-1} such that $S^{n-1}S_0$ is a subword of $R^n\delta_k$. Note that ab^ia is a subword of $S^{n-1}S_0$, and as ab^ia is not a subword of $R^n\delta_k$ but is a subword of R^n we must have that S is a cyclic shift of R^{-1}. Then as $a^{-1}b^{k-i}a^{-1}$ is a subword of R^n we have that $a^{-1}b^{k-i}a^{-1}$ is a subword of $S^{n-1}S_0$ and so $ab^{i-k}a$ is a subword of $R^n\delta_k$. Therefore, $R^n\delta_k \equiv U_0ab^{i-k}aV_0$. Then as $\delta_k^{-1} = \delta_{-k}$ we can apply Lemma 3.4.3 to $R^n\delta_k$ to obtain R^n as follows.

$$\begin{aligned} R^n &\equiv (U_0a)\delta_{-k}(b^{i-k}\delta_{-k})(aV_0)\delta_{-k} \\ &\equiv U_0'ab^{i-2k}aV_0' \end{aligned}$$

   This yields the required result.

Therefore, the lemma holds. $\square$

The above result, Lemma 3.4.4, leads to the following theorem, which can be summarised as saying "if $R \notin F(a,b)'$ then Moldovanskii rewriting determines if $\mathrm{Out}(G)$ is infinite or not".

Theorem 3.4.5. *Let $G = \langle a, b; R^n\rangle$ with $n > 1$, $R \notin F(a,b)$ and R not primitive. After re-writing R such that $\sigma_a(R) = 0$, the following are equivalent.*

1. $\mathrm{Out}(G)$ *is infinite.*

2. $R \in \langle aba^{-1}, b\rangle \cup \langle a^{-1}ba, b\rangle$.

The proof of Theorem 3.4.5 is an easy application the following three facts. Firstly, R has a subword of the form $a^{\epsilon}b^{i}a^{\epsilon}$ if and only if $R \notin \langle aba^{-1}, b\rangle \cup \langle a^{-1}ba, b\rangle$. Secondly, G is Hopfian [Pri77a]. Thirdly, if $\delta : a \mapsto ab$, $b \mapsto b$ is a homomorphism of $G = \langle a, b; R^n\rangle$ then $\mathrm{Out}(G)$ is infinite, by Theorem 3.2.1. We therefore leave the proof to the reader.

We end the section by giving three one-ended two-generator, one-relator groups whose outer automorphism groups are respectively isomorphic to $C_2 \times C_2$, to C_2 and to the trivial group. This means that if $G = \langle a, b; R^n\rangle$, $n > 1$, is one-ended with $\mathrm{Out}(G)$ finite then the possibilities for $\mathrm{Out}(G)$ are D_6, D_4, D_3, $C_2 \times C_2$, D_6, C_4, C_3, C_2 or the trivial group, and every possibility occurs.

Lemma 3.4.6. *For Q one of $C_2 \times C_2$, C_2 or the trivial group there exists a group $G = \langle a, b; R^n\rangle$, $n > 1$, such that $\mathrm{Out}(G) \cong Q$. The following groups give explicit examples.*

- *If $G = \langle a, b; (a^2ba^{-2}b)^n\rangle$ then $\mathrm{Out}(G) \cong C_2 \times C_2$.*

- *If $G = \langle a, b; (a^2ba^{-3}b)^n\rangle$ then $\mathrm{Out}(G) \cong C_2$.*

- *If $G = \langle a, b; (a^{-2}ba^4ba^{-3}ba^5b)^n\rangle$ then $\mathrm{Out}(G)$ is trivial.*

Proof. Note that a^2 is a subword of each of the three words and $\sigma_a(R) = 0$, and so each group has finite outer automorphism group by Theorem 3.4.5. In each case we apply Lemma 3.4.4 to determine the possible α_i and β_j which define automorphisms of G. We then apply ζ_0 and each of the α_i and β_j given by Lemma 3.4.4 to work out which of these finitely many maps define automorphisms of G. Finally, applying Lemma 3.4.2, we obtain the isomorphism class of $\mathrm{Out}(G)$.

The maps α_0 and β_0 define automorphisms of $G = \langle a, b; (a^2ba^{-2}b)^n\rangle$. Hence, $\mathrm{Out}(G) \cong C_2 \times C_2$, as required.

If $R \equiv a^2ba^{-1}ba^{-1}b$, we have that $\min_+ = 0 = \max_+$ and that $\min_- = 0 = \max_-$. Thus, the only possible elements of $\mathrm{Out}(G)$ are (the cosets of) α_0, β_0 and ζ_0. The only one of these maps which defines an automorphism of G is ζ_0 and so $\mathrm{Out}(G) \cong C_2$, as required.

If $R \equiv a^2ba^{-1}baba^{-1}ba^{-1}b$, we have that $\min_+ = 0 = \max_+$ and that $\min_- = 1 = \max_-$. Thus, the only possible elements of $\mathrm{Out}(G)$ are (the cosets of) α_0, β_0, and ζ_0. None of these maps define automorphisms of G and so $\mathrm{Out}(G)$ is trivial, as required. □

3.5 $\mathrm{Out}(G)$ for G infinitely-ended

Let $G = \langle a, b; R^n \rangle$ with R primitive. Equivalently, G has more than one end. We shall now give a skeleton proof of the following theorem.

Theorem 3.5.1. *If $G = \langle a, b; R^n \rangle$ R is a primitive element of $F(a, b)$ then the following isomorphism holds, where $\mathrm{Aut}(C_n)$ commutes with the flip generator of D_n and acts on the rotation generator in the natural way as automorphisms of C_n.*

$$\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$$

To see this theorem, recall that as R is primitive then Moldovanskii rewriting yields $G \cong \langle a, b; b^n \rangle$. That is, $G \cong \mathbb{Z} * C_n$. There are a number of ways of approaching the outer automorphism group of such a group, and indeed the automorphism groups of free products have been much studied [FR40, Gil87]. We have chosen to give a proof using ideas surrounding Nielsen equivalence classes, ideas which are in line with the the rest of this chapter. We begin by stating the following two lemmata, both of which generalise lemmata from earlier sections to the setting of tame automorphisms. Both lemmata can be proven using easy modifications of the results they generalise, and so the proofs are left to the reader. The first lemma is an adaption of Lemma 3.2.4, and is as follows.

Lemma 3.5.2. *Let $G = \langle a, b; \mathbf{r} \rangle$ such that $\sigma_a(R) = 0$ for all $R \in \mathbf{r}$ and $\mathbf{r} \not\subset F(a, b)'$. Then, every tame automorphism $\psi \in \mathrm{Tame}_{(a,b)}(G)$ has a representative $\phi \in \widehat{\psi}$ such that $\phi \in \mathrm{Tame}_{(a,b)}(G)$ and has the following form.*

$$\begin{aligned} \phi : a &\mapsto a^{\epsilon_0} b^k \\ b &\mapsto b^{\epsilon_1} \end{aligned}$$

The second lemma, Lemma 3.5.3, is a generalisation of Lemma 3.2.5, in the sense that it is the statement of what we prove in Lemma 3.2.5. However, Lemma 3.2.5 only holds when G is one-ended but Lemma 3.5.3 makes no assumptions about ends. Lemma 3.5.3 implies Lemma 3.2.5 if G is one-ended, and is as follows. Recall that if $\phi \in \mathrm{Aut}(G)$ then $\widehat{\phi}$ denotes the coset of $\mathrm{Out}(G)$ containing ϕ.

Lemma 3.5.3. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and $\sigma_a(R) = 0$. Then if*

$$\begin{aligned}\phi_1 : a &\mapsto a^{\epsilon_0} b^i\\ b &\mapsto b^{\epsilon_1}\end{aligned}$$

and

$$\begin{aligned}\phi_2 : a &\mapsto a^{\epsilon'_0} b^j\\ b &\mapsto b^{\epsilon'_1}\end{aligned}$$

are two automorphisms of G ($\epsilon_0 \neq \epsilon'_0$, or $\epsilon_1 \neq \epsilon'_1$, or $i \neq j$) which are non-equal in $\mathrm{Aut}(G)$ then they lie in different cosets of $\mathrm{Aut}(G)/\mathrm{Inn}(G)$, so $\widehat{\phi_1} \neq \widehat{\phi_2}$.

Proof of Theorem 3.5.1. Recall that $G \cong \mathbb{Z} * C_n$, so we can assume that $G = \langle a, b; b^n \rangle$ with $n > 1$.

It is clear that every function of the form $\phi : a \mapsto a^{\epsilon_0} b^i$, $b \mapsto b^{\epsilon_1}$ is in an automorphism of G. Now, Lemma 3.5.3 gives us that these are all non-equal modulo $\mathrm{Inn}(G)$ for $0 \leq i < n$, while by Lemma 3.5.2 these are the only automorphisms, modulo the inner automorphisms, which keep (a, b) in the same Nielsen equivalence class. Therefore, keeping the same notation as Section 3.3.2, α, β, δ and ζ are all in $\mathrm{Aut}(G)$, are all non-equal modulo $\mathrm{Inn}(G)$ (unless $n = 2$), and every tame automorphism has a representative in $\mathrm{Out}(G)$ of the form $\delta^i \sigma$ where $\sigma \in \{\alpha, \beta, \zeta, e\}$. Note that $\delta \in \mathrm{Aut}(G)$ and $\widehat{\delta}$ both have order n. Thus, by Lemma 2.2.5, every outer automorphism is of the form $\delta^i \sigma \psi_c$ where $\sigma \in \{\alpha, \beta, \zeta, e\}$, and where ψ_c is the distinguished automorphism such that $(a\psi_c, b\psi_c)$ is contained in the Nielsen equivalence class $\mathcal{C}$. Note that there are only finitely many choices for ψ_c, by Proposition 2.2.2, and so we have that $\mathrm{Out}(G)$ is finite.

If $n = 2$ there is only one Nielsen equivalence class, so the ψ_c can be ignored and we get that $\mathrm{Out}(G) = \langle \widehat{\alpha}, \widehat{\delta} \rangle$ and indeed $\mathrm{Out}(G) \cong C_2 \times C_2 \cong D_2$, as required.

If $n > 2$ we want to find out what the ψ_c are; what are the maps which take (a, b) to the other Nielsen equivalence classes. Now, the Grushko–Neumann Theorem implies that if $G = H * K$ and $G = \langle g_1, g_2 \rangle$ then the pair (g_1, g_2) can be obtained by a Nielsen transformation from a generating pair (h, k) such that $h \in H$ and $k \in K$. Therefore, one can take the automorphisms ψ_c to be (a certain subset of) the automorphisms $\psi_k : a \mapsto a$, $b \mapsto b^k$ where $\gcd(k, n) = 1$ and $0 < k < n$. Now, the generator (a, b^{n-k}) is in the same Nielsen equivalence class as (a, b^k) for all i, as (a, b^k) is mapped to (a, b^{n-k}) via the automorphism $\alpha : a \mapsto a$, $b \mapsto b^{-1}$, which is a Nielsen Transformation. Thus we restrict the range for k to $0 < k < \frac{n}{2}$.

We shall now prove that no two maps $\delta^i\sigma\psi_k$ are equal modulo the inner automorphisms, for $\sigma \in \{\alpha, \beta, \zeta, \delta\}$, $\gcd(k, n) = 1$ and $0 < k < \frac{n}{2}$. As inner automorphisms will keep the generating pair (a, b) in the same Nielsen equivalence class, none of the automorphisms ψ_k are equal modulo the inner automorphisms.

So, we have that the maps $\phi : a \mapsto a^{\epsilon_0}b^i$, $b \mapsto b^{\epsilon_1}$ are pairwise non-equal modulo the inner automorphisms, and that the ψ_k are also pairwise non-equal modulo the inner automorphisms. It now suffices to prove that no two automorphisms of the form $\phi_1\psi_j$ and $\phi_2\psi_k$, with $0 < j, k < \frac{n}{2}$, are equal modulo the inner automorphisms. Now, note that if ϕ is some Nielsen transformation, $\phi : a \mapsto U_a(a, b)$, $b \mapsto U_b(a, b)$ say, then $(a\phi\psi_k, b\phi\psi_k) = (a\psi_k, b\psi_k)\phi$ because of the following equivalencies.

$$\begin{aligned}
(a\phi\psi_k, b\phi\psi_k) &= (U_a(a, b)\psi_k, U_b(a, b)\psi_k) \\
&= (U_a(a\psi_k, b\psi_k), U_b(a\psi_k, b\psi_k)) \\
&= (a\psi_k, b\psi_k)\phi
\end{aligned}$$

So, suppose $\phi_1\psi_j = \phi_2\psi_k\gamma_g$ for some $g \in G$ but $\phi_1\psi_j \neq \phi_2\psi_k$. Now, because $\phi_1\psi_j = \phi_2\psi_k\gamma_g$, we have that $(a\phi_1\psi_j, b\phi_1\psi_j) = (a\psi_j, b\psi_j)\phi_1$ (as ϕ_1 is a Nielsen transformation) is in the same Nielsen equivalence class as $(a\phi_2\psi_k, b\phi_2\psi_k) = (a\psi_k, b\psi_k)\phi_2$, and so $(a\psi_j, b\psi_j)$ and $(a\psi_k, b\psi_k)$ are Nielsen equivalent. Thus, $j = k$. This means that $\widehat{\psi_k}\widehat{\phi_1} = \widehat{\psi_k}\widehat{\phi_2}$, and so $\widehat{\phi_1} = \widehat{\phi_2}$. Thus, $\phi_1 = \phi_2$, by Lemma 3.5.3, and so $\phi_1\psi_j = \phi_2\psi_k$, a contradiction. Thus, the maps $\delta^i\sigma\psi_k$, where $\sigma \in \{\alpha, \beta, \zeta, \delta\}$, $\gcd(k, n) = 1$ and $0 < k < \frac{n}{2}$, form a transversal for $\mathrm{Out}(G)$.

Now, we have that the elements $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\delta}$, $\widehat{\zeta}$ and $\widehat{\psi_k}$ for $0 < k < n$, $\gcd(k, n) = 1$, generate $\mathrm{Out}(G)$ (note that we have included ψ_k for $\frac{n}{2} < k < n$). The generators $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\delta}$ and $\widehat{\zeta}$ give a homomorphic image of the group with the following presentation by Section 3.3.2 and because δ has order n (as b has order n). As in Section 3.3.2, we abuse notation and write ϕ in place of $\widehat{\phi}$ for each $\phi \in \{\alpha, \beta, \zeta, \delta\}$.

$$\langle \alpha, \beta, \delta, \zeta; \alpha^2, \beta^2, \zeta^2, [\delta, \zeta], \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \alpha\beta\zeta, \delta^n \rangle$$

Further, we have the following relations modulo the inner automorphisms.

$$\delta\psi_i = \psi_i\delta^i$$

$$\alpha\psi_i = \psi_i\alpha$$

$$\beta\psi_i = \psi_i\beta$$

$$\zeta\psi_i = \psi_i\zeta$$

$$\psi_i\psi_j = \psi_{ij \mod n}$$

Therefore, $\mathrm{Out}(G)$ is a homomorphic image of the following group.

$$\langle \alpha, \beta, \delta, \zeta, \psi_i; \beta = \psi_{n-1}, \alpha^2, \beta^2, \zeta^2, \delta^n, [\delta, \zeta], \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \alpha\beta\zeta,$$
$$\psi_i^{-1}\delta\psi_i = \delta^i, [\alpha, \psi_i], [\beta, \psi_i], [\zeta, \psi_i], \psi_i\psi_j = \psi_{ij \mod n}\rangle$$

However, every element in the group given by this presentation has the form $\delta^i\sigma\psi_k$ for $\sigma \in \{\alpha, \beta, \zeta, \delta\}$, $\gcd(k, n) = 1$ and $0 < k < \frac{n}{2}$ (as $\beta = \psi_{n-1}$), but there is no element $\delta^i\sigma\psi_k$ in the kernel of this homomorphism. Therefore, $\mathrm{Out}(G)$ is isomorphic to this group. Replacing ζ with $\alpha\beta$ and β with ψ_{n-1}, and following the Tietze transformations through yields the following isomorphism.

$$\mathrm{Out}(G) \cong \langle \alpha, \delta, \psi_i; \alpha^2, \delta^n, \alpha\delta\alpha = \delta^{-1},$$
$$\psi_i^{-1}\delta\psi_i = \delta^i, [\alpha, \psi_i], \psi_i\psi_j = \psi_{ij \mod n}\rangle$$

Writing $H = \langle \alpha, \delta \rangle$ and $K = \langle \psi_k\ (0 \leq k < n, \gcd(k, n) = 1); \psi_i\psi_j = \psi_{ij \mod n}\rangle$, clearly $G = HK$, $H \cap K = \langle 1 \rangle$ and $H \lhd G$. Thus, $G = H \rtimes K$. Finally, $H \cong D_n$ while the group K is the automorphisms group of C_n, $\mathrm{Aut}(C_n)$. This completes the proof. $\square$

3.6 Conclusion

We conclude this chapter by completing the proof of Theorem A, by providing an algorithm to compute the outer automorphism group of a two-generator, one-relator group with torsion, and by using the knowledge we have obtained so far in this chapter to describe how to give the presentation of the automorphism group of a two-generator, one-relator group with torsion.

3.6.1 The proof of Theorem A

We shall now assemble the proof of Theorem A, which was stated in the introduction to this chapter.

Proof of Theorem A. Let $G = \langle a, b; R^n \rangle$, $n > 1$, be a two-generator, one-relator group with torsion. Note that G is either one-ended or $G \cong \mathbb{Z} * C_n$.

If $G \cong \langle a, b; [a, b]^n \rangle$ then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$ by Corollary 3.2.2.

If G is one-ended, not isomorphic to $\langle a, b; [a, b]^n \rangle$ and $\mathrm{Out}(G)$ is infinite then $\mathrm{Out}(G)$ is one of $D_\infty \times C_2$, D_∞, $\mathbb{Z} \times C_2$ or $\mathbb{Z}$, by Lemma 3.3.5.

If G is one-ended, not isomorphic to $\langle a, b; [a, b]^n \rangle$ and $\mathrm{Out}(G)$ is finite then $\mathrm{Out}(G)$ is isomorphic to a finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Then every finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$ is isomorphic to a subgroup of D_4 or D_6 [Zim96], as required.

If G has more than one end then $G \cong \mathbb{Z} * C_n$ [Pri77a], and then $\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$ by Theorem 3.5.1. □

Note that in Section 3.3.2 we give examples of one-ended two-generator, one-relator groups with torsion which realise each of the groups $D_\infty \times C_2$, D_∞, $\mathbb{Z} \times C_2$ or $\mathbb{Z}$ as their outer automorphism group. In Lemma 3.4.1 we give examples of one-ended two-generator, one-relator groups with torsion which realise each of the groups D_6, D_4, D_3, C_6, C_4 and C_3 as their outer automorphism group. In Lemma 3.4.6 we give examples of one-ended two-generator, one-relator groups with torsion which realise each of the groups $C_2 \times C_2$, C_2 and the trivial group as their outer automorphism group.

3.6.2 An algorithm to compute $\mathrm{Out}(G)$

In this section we give an algorithm to compute $\mathrm{Out}(G)$, where G is given by a presentation $\langle a, b; R^n \rangle$ with $n > 1$ and R not a proper power. The algorithm uses certain results of Dahmani–Guirardel, specifically that it is decidable if a hyperbolic group splits over a virtually-cyclic group with infinite center [DG11] and that there exists an algorithm to determine the generators of the outer automorphism group of a hyperbolic group [DG11, Theorem 3]. Note that if it is known that the JSJ-decomposition of G splits then $G \cong \langle a, b; S^n(a^{-1}ba, b) \rangle$ for some word S and the word S can be found by enumerating the Nielsen transformations of $F(a, b)$ and applying them to R [Pri77a]. The algorithm to compute $\mathrm{Out}(G)$ where $G = \langle a, b; R^n \rangle$ with $n > 1$ is as follows.

- Rewrite R such that $\sigma_a(R) = 0$ and R is cyclically reduced.
  - If $R = b^\epsilon$ then $\mathrm{Out}(G) \cong D_{2n} \rtimes \mathrm{Aut}(C_n)$.
  - If R is a cyclic shift of $[a, b]^{\pm 1}$ then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$.

- Does the JSJ-decomposition of G split?
  - If *yes* then rewrite R as $S(a^{-1}ba, b)$ and apply Section 3.3.2 to find $\mathrm{Out}(G)$.
  - If *no* then obtain the generators for $\mathrm{Out}(G)$ and apply Section 3.4 to find $\mathrm{Out}(G)$.

3.6.3 What does $\mathrm{Aut}(G)$ look like?

In this section we describe how to write down a presentation for $\mathrm{Aut}(G)$ using $\mathrm{Out}(G)$, where G is an arbitrary one-ended two-generator, one-relator group with torsion (the infinitely-ended case has appeared in print before [FR40, Gil87]) and we give examples. Begin by noting that G is centerless [BT67]. Then, take a transversal T for $\mathrm{Out}(G)$ which consists of Nielsen transformations and denote by $\mathcal{O}$ a subset of this transversal which will generate $\mathrm{Out}(G)$. Note that this transversal T exists by Theorem 3.2.1.

To obtain a presentation for $\mathrm{Aut}(G)$, note that the inner automorphisms are isomorphic to G in the canonical way (as G has trivial center), so we immediately have the following relation.

$$\gamma_{R^n} = 1$$

Next, we have that $\gamma_w^{\psi} = \gamma_{w\psi}$ for all $\psi \in \mathrm{Aut}(G)$. Therefore, we have the following relations for all $\psi \in \mathcal{O}$.

$$\gamma_a^{\psi} = \gamma_{a\psi}$$
$$\gamma_b^{\psi} = \gamma_{b\psi}$$

We now have to ascertain how the elements of $\mathcal{O}$ multiply together. However, this is easily computed: let $\langle X; \mathbf{r}\rangle$ be a presentation for $\mathrm{Out}(G)$ which corresponds to the generators $\mathcal{O}$, then if $S \in \mathbf{r}$ is a relator we have that $S = \gamma_w$ is a relation in $\mathrm{Aut}(G)$ where γ_w is the appropriate inner automorphism. These three kinds of relations are all the relations, as any other non-trivial relation would have one of the two following forms, where $U(a, b) \neq_G 1$ and where $W(\mathcal{O})$ is a word over $\mathcal{O}$.

$$U(\gamma_a, \gamma_b) = 1 \tag{3.1}$$
$$V(\gamma_a, \gamma_b) = W(\mathcal{O}) \tag{3.2}$$

However, (3.1) cannot happen as $G \cong \mathrm{Inn}(G)$ under the isomorphism $a \mapsto \gamma_a$, $b \mapsto \gamma_b$, because G is centerless, while (3.2) cannot happen as it corresponds to a relator in $\mathrm{Out}(G)$ and we have captured all of these.

For example, writing w for γ_w (so w represents the automorphism corresponding to conjugation by w), if $\mathrm{Out}(G) = \langle \widehat{\alpha}_i \rangle$ then $\mathrm{Aut}(G)$ is the following group.

$$\mathrm{Aut}(G) = \langle \alpha_i, a, b; R^n(a, b), \alpha_i^2 = b^i, a^{\alpha_i} = a^{-1}b^i, b^{\alpha_i} = b \rangle$$

If $\mathrm{Out}(G) = \langle \widehat{\beta}_i \rangle$ then $\mathrm{Aut}(G)$ is the following group.

$$\begin{aligned}\mathrm{Aut}(G) &= \langle \beta_i, a, b; R^n(a, b), \beta_i^2 = 1, a^{\beta_i} = ab^i, b^{\beta_i} = b^{-1} \rangle \\ &\cong G \rtimes C_2\end{aligned}$$

Our final example corresponds to G being Fuchsian: if $G \cong \langle a, b; [a, b]^n \rangle$ and writing $\mathrm{Aut}(F(a, b)) = \langle a, b, X; \mathbf{r} \rangle$ then we have the following group.

$$\mathrm{Aut}(G) = \langle a, b, X; \mathbf{r}, [a, b]^n \rangle$$

Chapter 4

Automorphism-Induced HNN-Extensions

Every group can be realised as the outer automorphism group of some group [Mat89]. One can ask what restrictions can be placed on the groups involved. Notably, Bumagin–Wise proved that every countable group Q can be realised as the outer automorphism group of a finitely generated group G_Q [BW05]. Several other authors have achieved results in a similar vein (see, for example, [Koj88], [GP00], [DGG01], [BG03], [FM05]).

To prove their results, Bumagin–Wise construct G_Q as the kernel of a short exact sequence using a version of a construction due to Rips' [Rip82]. Their proof also shows that if Q is finitely presented then G_Q can be taken to be residually finite. They then pose the question: can every countable group be realised as the outer automorphism group of a finitely generated, residually finite group?

In this chapter we give partial answers this question of Bumagin–Wise, and we give a new proof of their result that every countable group Q can be realised as the outer automorphism group of a finitely generated group G_Q. Our construction in this new proof is explicit and more elementary than that of Bumagin–Wise, in the sense that we construct the group G_Q as an HNN-extension obtained from any countable presentation of Q.

A *triangle group* is a group with a presentation of the following form.

$$T_{i,j,k} := \langle a, b; a^i, b^j, (ab)^k \rangle$$

A *hyperbolic triangle group* is one where $i^{-1} + j^{-1} + k^{-1} < 1$. If $i = j = k$ we shall write $T_i := T_{i,i,i}$ for the corresponding *equilateral triangle group*. Our first two theorems reveal a certain universal property possessed by hyperbolic triangle groups. Our first main theorem of this section, Theorem B, implies Bumagin–Wise's result, that every group can be realised as the outer automorphism group of a finitely generated group, as the group G_Q in the statement of the Theorem B is finitely generated.

Theorem B. *Fix an equilateral triangle group $H := T_i$ with $i > 9$. Then every countable group Q can be realised as the outer automorphism group of an HNN-extension G_Q of H. Moreover,* $\mathrm{Aut}(G_Q) \cong H \rtimes Q$.

Our second main theorem, Theorem C, gives a partial answer to the question asked by Bumagin–Wise, as again the group G_Q in the statement of the theorem is finitely generated. As in Theorem B, the construction is both explicit and elementary, with G_Q being an HNN-extension obtained from a presentation of Q.

Theorem C. *Fix a hyperbolic triangle group $H := T_{i,j,k}$. Then every finitely-generated group Q can be embedded as a finite index subgroup of the outer automorphism group of an HNN-extension G_Q of H, where G_Q is residually finite if Q is.*

Our third main theorem, Theorem D, gives a stronger result than Theorem C, in the sense that Q being "residually finite" is replaced by being "recursively presented" (or equivalently, "embeds in a finitely presented group"), and because it gives a much more precise description of the outer automorphism group. The proof is based upon the construction of Bumagin–Wise and utilises an embedding theorem of Sapir [Sap13].

Theorem D. *If Q is a finitely generated, recursively presented group then either Q or $Q \times C_2$ can be realised as the outer automorphism group of a finitely-generated, residually finite group G_Q.*

This theorem admits a possible improvement: a positive answer to a question of Osin would allow us to dispense of the $Q \times C_2$ possibility, and thus implies that every finitely generated, recursively presented group can be realised as the outer automorphism group of a finitely generated, residually finite group. We discuss this in more detail in Section 4.3.5.

Automorphism-induced HNN-extensions. Our main tool in the proofs of the above theorems is the notion of an *automorphism-induced* HNN-extension. Such HNN-extensions have been studied before [BT78,ALP14], but their outer automorphism groups have not yet been closely analysed. We do this in Section 4.2. The second class of groups from the title of this thesis is the class of automorphism-induced HNN-extensions $G = \langle H, t; K^t = K' \rangle$

where the associated subgroups are proper subgroups of the base group H, $K, K' \lneq K$. The third class of groups is the class of automorphism-induced HNN-extensions $G = \langle H, t; K^t = K' \rangle$ where the base group H has trivial center and where the associated subgroups are both H, and so the group G is a mapping torus $G = H \rtimes \mathbb{Z}$.

The class of HNN-extensions exhibits a variety of pathological properties, and the standard examples of badly-behaved HNN-extensions are Baumslag–Solitar groups, that is, HNN-extensions of the infinite cyclic group. However, if a Baumslag–Solitar group is *automorphism-induced* then it is "nice": it is residually finite and has virtually cyclic outer automorphism group. The constructions in Sections 4.3 are surprising because they show that automorphism-induced HNN-extensions can still be "wild", even when the base group is well-behaved. In particular, we prove that an automorphism-induced HNN-extension is not necessarily residually finite, and can have arbitrary outer automorphism group.

We show that an underlying reason for these wild properties is that if H is a hyperbolic triangle group then for every countable group Q there exists some subgroup K of H such that $Q \cong N_H(K)/K$, where $N_H(K)$ is the normaliser of K in H. This is proved in Section 4.3.4. In an automorphism-induced HNN-extension $G = \langle H, t; K^t = K' \rangle$, the quotient $N_H(K)/K$ embeds into $\mathrm{Aut}(G)$ and, under certain conditions, into $\mathrm{Out}(G)$. Thus the properties of $N_H(K)/K$ are in a certain sense bestowed upon G. However, if H is cyclic (and hence G is a Baumslag–Solitar group) then $N_H(K)/K$ is necessarily cyclic and so G inherits no pathological properties from this subgroup quotient.

Outline of the chapter. In Section 4.3 we prove our three main theorems, Theorems B, C and D, and we give two other constructions of groups and classes of groups with pathological properties. These results rely on three technical results which we prove in Sections 4.1 and 4.2. In Section 4.1 we define automorphism-induced HNN-extensions, introduce a way of viewing them as Zappa-Szép products, and prove the first technical result, Theorem 4.1.3, which gives a criterion under which these groups are residually finite. In Section 4.2 we use the Zappa-Szép product viewpoint to prove the second and third technical results of the chapter, Theorems 4.2.15 and 4.2.17. Theorem 4.2.15 deals with the second class of groups from the title of the thesis, and gives a decomposition of a certain subgroup of the outer automorphism groups of a proper automorphism-induced HNN-extensions, while Theorem 4.2.17 does this for the second class of groups from the title of the thesis, specifically, when the base group has trivial center and the associated subgroups are equal to the base group, so the HNN-extension is a semi-direct product.

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4.1 Automorphism-induced HNN-extensions as Zappa-Szép products

In this section we define automorphism-induced HNN-extensions and explain how to view them as Zappa–Szép products. We use this approach to prove results regarding their residual finiteness. This view allows us to think of such HNN-extensions as "generalised Mapping Tori", and it, along with most of the results contained in this section, were first observed by Ateş–Logan–Pride [ALP14].

Automorphism-induced HNN-extensions. Let H be a group and let $K, K' \leq H$ be non-trivial, isomorphic subgroups of H. Then we say that an HNN-extension $G = \langle H, t; k^t = k',\ k \in K,\ k' \in K'\rangle$ is *automorphism-induced* if the isomorphism $K \to K'$, $k \mapsto k'$ is induced by an automorphism of H, that is, there exists $\phi \in \mathrm{Aut}(H)$ such that $k\phi = k'$ for all $k \in K$. We shall write $G = \langle H, t; K^t = K\phi\rangle$, and here $K^t = K\phi$ means $k^t = k\phi$ for all $k \in K$.

Zappa–Szép products. A group Z is an *(internal) Zappa–Szép product* of $A, B \leq Z$ if $AB = Z$ and $A \cap B = 1$. We write $Z = A \bowtie B$. This is also known as a *general product* or *knit product*, and is the "next obvious thing" after one has defined direct and semi-direct products.

Denote by F_k the free group on the cosets of H/K, $F_k := F(H/K)$, let T be a transversal for $H/K = \{Kg : g \in H\}$, and for $h \in H$ let $\bar{h}$ denote the unique element of T such that $\bar{h}h^{-1} \in K$. We have the following proposition.

Proposition 4.1.1 (Ateş–Logan–Pride)**.** *Let H be a group, $K \leq H$ a subgroup and $\phi \in \mathrm{Aut}(H)$ an automorphism. Then the associated automorphism-induced HNN-extension can be viewed as a Zappa–Szép product in the following way.*

$$\begin{aligned} G &= \langle H, t; K^t = K\phi \rangle \\ &\cong \langle H, x_a, a \in T; x_a h = (h\phi) x_{\overline{ah}} \ (h \in H, a \in T) \rangle \\ &= H \bowtie F_k \end{aligned}$$

Proof. We use Tietze transformations to prove that G can be presented in this way, beginning at the proposed Zappa-Szép product Z.

$$\begin{aligned} Z &= \langle H, x_a, a \in T; x_a h = (h\phi) x_{\overline{ah}} \ (h \in H, a \in T) \rangle \\ &\cong \langle H, x_a, a \in T; x_1 a = (a\phi) x_a, x_a h = (h\phi) x_{\overline{ah}} \ (h \in H, a \in T) \rangle \\ &\cong \langle H, x_a, a \in T; x_a = (a\phi)^{-1} x_1 a, x_a h = (h\phi) x_{\overline{ah}} \ (h \in H, a \in T) \rangle \\ &\cong \langle H, x_1; (a\phi)^{-1} x_1 a h = (h\phi)(\overline{ah}\phi)^{-1} x_1 \overline{ah} \ (h \in H, a \in T) \rangle \\ &\cong \langle H, x_1; x_1 a h \overline{ah}^{-1} x_1^{-1} = (a\phi)(h\phi)(\overline{ah}\phi)^{-1} \ (h \in H, a \in T) \rangle \\ &\cong \langle H, x_1; x_1 a h \overline{ah}^{-1} x_1^{-1} = (a h \overline{ah}^{-1})\phi \ (h \in H, a \in T) \rangle \\ &\cong \langle H, x_1; x_1 k x_1^{-1} = k\phi \ (k \in K) \rangle \\ &\cong G \end{aligned}$$

To complete the proof, we need to prove that Z is, indeed, a Zappa-Szép product, which reduces to proving that $H \cap \langle x_a, a \in T \rangle = 1$. So, suppose $g =_Z W(x_a, a \in T)$ for some $g \in H$ and W a freely reduced word on the x_a. We shall prove that W is the empty word. Now, the above working shows that the following map is an isomorphism.

$$\begin{aligned} Z &\to G \\ h &\mapsto h && \forall h \in H \\ x_a &\mapsto (a\phi)^{-1} t a && \forall a \in T \end{aligned}$$

Thus, to prove that Z is a Zappa-Szép product it suffices to prove that if the following holds then W is the empty word.

$$W((a\phi)^{-1} t a, a \in T) =_G g \in H$$

Suppose $g^{-1} W((a\phi)^{-1} t a, a \in T) =_G 1$, then this cannot be t-reduced by Britton's Lemma and thus there exists some subword of the form tkt^{-1} with $k \in K$ or some subword of the form $t^{-1}(k\phi)t$ with $k \in K$. However, this corresponds to $W(x_a, a \in T)$ having a subword of the form $x_a x_a^{-1}$ or of the form $x_a^{-1} x_a$, which contradicts the assumption that it was freely reduced. Therefore, Z is a Zappa-Szép product and so the proposition holds. □

Therefore, G is isomorphic to a Zappa-Szép product of H with the free group on the set H/K. This is the way we shall view G for the remainder of the Chapter.

Residual finiteness. We now apply the above viewpoint in an elementary way to the residual finiteness of an automorphism-induced HNN-extension. Let $g \in N_H(K)$, and so $g \cdot K = Kg$. Then $g \cdot Ka = K\overline{ga}$ so we have that the following function is an automorphism of G.

$$\begin{aligned} \psi_g : h &\mapsto h \\ x_a &\mapsto x_{\overline{ga}} \end{aligned}$$

Noting that the maps ψ_{g_1} and ψ_{g_2} define the same automorphism of G if and only if $g_1 g_2^{-1} \in K$ we have the following proposition.

Proposition 4.1.2 (Ateş–Logan–Pride)**.** *If $G = \langle H, t; k^t = k\phi, k \in K\rangle$ with K non-trivial and $\phi \in \mathrm{Aut}(H)$ then $N_H(K)/K \leq \mathrm{Aut}(G)$.*

This allows us to prove the following theorem, which is our first technical theorem of the chapter. This result is applied in the proof of Theorem C.

Theorem 4.1.3. *Suppose that $G = \langle H, t; k^t = k\phi, k \in K\rangle$ with K non-trivial and $\phi \in \mathrm{Aut}(H)$. Suppose H is finitely generated and residually finite, and suppose that $N_H(K)$ has finite index in H. Then G is residually finite if and only if $N_H(K)/K$ is residually finite.*

Proof. As H finitely generated, G is finitely generated. This means that if G is residually finite then so is $\mathrm{Aut}(G)$, by Proposition 2.5.5, and so $N_H(K)/K$ is residually finite by Proposition 4.1.2.

Now, G is residually finite if for all finite sets $\{g_1, \ldots, g_n\}$ with $g_i \in H \setminus K$ there exists some finite index normal subgroup N of H, $N \trianglelefteq_f H$, such that $g_iK \cap N$ is empty for all $i \in \{1, \ldots, n\}$, by Proposition 2.5.8. This condition holds if $N_H(K)/K$ is residually finite. To see this, we shall find for each g_i a normal subgroup N_i of finite index in H such that $g_i \notin N_i$. Then, intersecting the finitely many subgroups N_i, each of which has finite index in H, yields a finite-index subgroup $N := \cap N_i$ with the required properties. So, begin by noting that if $g_i \notin N_H(K)$ we can take the normal subgroup N_i to be contained in $N_H(K)$ (for example, take N_i to be the intersection of the (finitely many) conjugates of $N_H(K)$. We shall now consider the case when $g_i \in N_H(K)$. Then $g_iK \neq K$ and because $N_H(K)/K$ is residually finite there exists a map ψ_i from $N_H(K)/K$ onto a finite group Q_i, $\psi_i : N_H(K)/K \to Q_i$, such that g_iK is not contained in the kernel of ψ_i. Therefore, these exists a map $\widetilde{\psi}_i : N_H(K) \to N_H(K)/K \xrightarrow{\psi_i} Q_i$ such that g_i is not contained in the kernel of $\widetilde{\psi}_i$, and take N_i to be the kernel of the map $\widetilde{\psi}_i$. Therefore, the condition holds and so G is residually finite. □

4.2 The outer automorphism groups of automorphism-induced HNN-extensions

In this section we prove Theorems 4.2.15 and 4.2.17, which are the two main technical results of the chapter. Theorem 4.2.15 gives a short exact sequence for an index-one or -two subgroup of Levitt's subgroup of $\mathrm{Out}(G)$ for G a proper automorphism-induced HNN-extension, while Theorem 4.2.17 gives an explicit isomorphism for the analogous subgroup for G a mapping torus of a group H with trivial center, $G = H \rtimes \mathbb{Z}$ and $Z(H) = 1$. Theorem 4.2.15 forms the basis of the proofs of Theorem B and C from Section 4.3 while Theorem 4.2.17 forms the basis of the proof of Theorem D, also from Section 4.3.

The layout of this current section is as follows. We begin by recalling Levitt's and Pettet's subgroups and providing certain conditions, in Lemma 4.2.1 and Lemma 4.2.2, which imply that Pettet's subgroup is the whole outer automorphism group, and we then provide, in Theorem 4.2.3, conditions which imply that Levitt's subgroup is equal to Pettet's subgroup. We then prove that the elements of Levitt's subgroup have a specific form, which we do in Lemma 4.2.6 from Section 4.2.1. This lemma forms the basis of this current section, Section 4.2, and we prove this lemma by viewing G as a Zappa–Szép product, as in Section 4.1. In Section 4.2.2 we prove Theorem 4.2.15, which gives a description of Levitt's subgroup of proper automorphism-induced HNN-extensions, and in Section 4.2.3 we prove Theorem 4.2.17, which is the analogous result Levitt's subgroup of mapping tori of groups with trivial centre.

Levitt's subgroup. The main results of this section look at Levitt's subgroup of $\mathrm{Aut}(G)$

and of $\mathrm{Out}(G)$, denoted $\mathrm{Aut}_H(G)$ and $\mathrm{Out}_H(G)$ respectively, in the specific case where G is an automorphism-induced HNN-extension. We now recall the definition of Levitt's subgroup from Section 2.11. Let $G = \langle H, t; K^t = K' \rangle$ be a (not-necessarily automorphism-induced) HNN-extension where $K \lneq H$, then Levitt's subgroup of $\mathrm{Aut}(G)$, denoted $\mathrm{Aut}_H(G)$, is the maximal subgroup of $\mathrm{Aut}(G)$ such that the diagram in Figure 4.1 commutes, where θ is the canonical map from G to $\mathrm{Aut}(G)$ whose image is the inner automorphisms. As inner automorphisms are contained in this subgroup, $\mathrm{Inn}(G) \leq \mathrm{Aut}_H(G)$, we

$$\begin{tikzcd} G \arrow[r, "\theta"] \arrow[d] & \mathrm{Aut}_H(G) \arrow[dl, dotted] \\ \mathrm{Aut}(T) & \end{tikzcd}$$

Figure 4.1: The map θ is the canonical homomorphism whose image is the inner automorphisms $\mathrm{Inn}(G)$*. Levitt's subgroup* $\mathrm{Aut}_H(G)$ *is the maximal subgroup of* $\mathrm{Aut}(G)$ *such that this diagram commutes.*

can define *Levitt's subgroup of* $\mathrm{Out}(G)$ in the obvious way, $\mathrm{Out}_H(G) := \mathrm{Aut}_H(G)/\mathrm{Inn}(G)$.

If G is a mapping torus, so $G = \langle H, t; H^t = H\phi \rangle \cong H \rtimes_\phi \mathbb{Z}$, then we define Levitt's subgroup $\mathrm{Aut}_H(G)$ to be the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms of G which send H to a conjugate of H, and again this contains the inner automorphisms so we define $\mathrm{Out}_H(G)$ in the obvious way. Note that this is the same definition as that of Pettet's subgroup, below. This is because Levitt's subgroup can be thought of as the subgroup of Pettet's subgroup consisting of elements $\widehat{\alpha}$ with a representative α where $H\alpha = H$ and $t\alpha$ has t-length one, so $t\alpha = h_1 t^\epsilon h_2$ for some $h_1, h_2 \in H$.

Conditions implying $\mathrm{Out}_H(G) = \mathrm{Out}^H(G)$. Levitt's subgroup is most of interest when $\mathrm{Out}_H(G) = \mathrm{Out}(G)$. We shall now give certain conditions which imply that this is so. Our first condition is for mapping tori, so the case when $H = K$. The observation we prove, below, appears in a paper of Arzhantseva–Lafont–Minasyan [ALM11], although it is somewhat hidden in the proof of their Proposition 2.1.

Lemma 4.2.1 (Arzhantseva–Lafont–Minasyan)**.** *Suppose H has no epimorphisms onto $\mathbb{Z}$. If $G = H \rtimes \mathbb{Z}$ then every automorphism of G maps H to itself. Thus, Pettet's subgroup* $\mathrm{Out}_H(G)$ *of the outer automorphism group is the whole outer automorphism group.*

Proof. Consider the following composition of maps, where the first embedding is the natural one of H into G, where the map $\phi : G \to G$ is an automorphism of G, and where the final surjection is the natural one of G onto $\mathbb{Z}$ by quotienting out H.

$$H \hookrightarrow G \xrightarrow{\phi} G \twoheadrightarrow \mathbb{Z}$$

As H does not map onto $\mathbb{Z}$, these maps compose to give the trivial map. Therefore, $H\phi \leq H$. Using the same argument with ϕ^{-1}, we see that $H\phi^{-1} \leq H$ and so $H\phi = H$ as required. □

We now wish to prove an analogous result for proper automorphism-induced HNN-extensions, for the case when $K \lneq H$. The following results use an additional subgroup of $\mathrm{Out}(G)$, "Pettet's subgroup". The first result gives conditions when Pettet's subgroup is the whole of the outer automorphism group, and the second result gives conditions when Pettet's subgroup is equal to Levitt's subgroup. Therefore, we begin by defining this subgroup: *Pettet's subgroup of* $\mathrm{Aut}(G)$ is the subgroup consisting of automorphisms of G which send H to a conjugate of H.

$$\mathrm{Aut}^H(G) := \{\psi \in \mathrm{Aut}(G) : H\psi = H^g\}$$

As inner automorphisms are contained in this subgroup, $\mathrm{Inn}(G) \leq \mathrm{Aut}^H(G)$, we can define *Pettet's subgroup of* $\mathrm{Out}(G)$ in the obvious way, $\mathrm{Out}_H(G) := \mathrm{Aut}^H(G)/\mathrm{Inn}(G)$. M. Pettet has studied $\mathrm{Aut}^H(G)$, proving that if K is conjugacy-maximal in H (that is, no conjugate of K properly contains K) then Pettet's subgroup is equal to Levitt's subgroup [Pet99].

We shall now prove that Pettet's subgroup of $G = \langle H, t; K^t = K\phi\rangle$, $K \lneq H$, is equal to the whole outer automorphism group when H has Serre's property FA. Note that our proof shows that the base group H is always conjugacy maximal in the HNN-extension G, and is an extension of an argument of Pettet [Pet99]. We shall write γ_g to mean the inner automorphism corresponding to conjugation by g, that is, $h\gamma_g = g^{-1}hg$ for all $h \in G$.

The following proofs use the fact that in an automorphism-induced HNN-extension $G = \langle H, t; K^t = K\phi\rangle$, if $g, h \in H$ are conjugate in G then $g = h\phi^i\gamma_p$ for some $p \in H$. This is because when we view $uWgW^{-1}u^{-1}(= h)$ in the Zappa-Szép product ($u \in H$, $W \in F(H/K)$) we have that $(g\phi^i)\overline{W}W^{-1} = u^{-1}hu$ and $\overline{W}W^{-1}$ must be trivial by the properties of a Zappa-Szép product.

Lemma 4.2.2. *Suppose H has Serre's property FA. If G is an automorphism-induced HNN-extension of H then every automorphism of G maps H to a conjugate of itself.*

Thus, Pettet's subgroup $\mathrm{Out}^H(G)$ *of the outer automorphism group is the whole outer automorphism group.*

Proof. Let ψ be an automorphism of G. We shall begin by proving that $H\psi$ is a subgroup of a conjugate of H. Note that G is an HNN-extension and so acts on the associated Bass–Serre tree T. Consider the action of $H\psi$ on T. Now, $H\psi$ has Serre's property FA and so acts with a global fixed point on T. Thus, $H\psi$ stabilises some vertex. Every vertex stabiliser of T is a conjugate of H. Therefore, $H\psi$ is a subgroup of a conjugate of H, $H\psi \leq H^{g_1}$, as required.

Suppose that $H\psi \lneq H^g$ and we shall look for a contradiction. Using the above argument with ψ^{-1}, we see that $H\psi^{-1} \leq H^{g_2}$ for some $g_2 \in G$, and so $H^{g_3} \leq H\psi$ for some $g_3 \in G$. Thus, we have that $H^{g_3} \lneq H^{g_1}$, and so $H^g \lneq H$. It is therefore sufficient to prove that, for all $g \in G$, if $H^g \leq H$ then $H^g = H$ (that is, H is conjugacy maximal in G). So, let $h \in H$ be arbitrary, and let $g = uW \in G$ be an arbitrary element of g with $u \in H$ and $W \in F_k$, and assume $h^g \in H$. Then $W^{-1}u^{-1}huW = (u^{-1}hu)\phi^i = h\gamma_u\phi^i$ for some $i \in \mathbb{Z}$. Noting that i depends on W not on h, conjugation by an element of G induces an automorphism of H, and so $H^g = H$ as required. $\square$

We now give conditions which imply that Levitt's subgroup is equal to Pettet's subgroup. Combining this with the above result, Lemma 4.2.2, gives conditions which imply that Levitt's subgroup is the whole of the outer automorphism group.

Theorem 4.2.3. *Suppose that* $G = \langle H, t; K^t = K\phi\rangle$ *is an automorphism-induced HNN-extension, and further suppose that one of the following holds.*

1. K *is conjugacy maximal in* H.

2. *The automorphism* ϕ *sends* K *to a conjugate of* K, *so* $K\phi = K\gamma_h$ *for some* $h \in H$.

3. *There does not exist any* $h \in H$ *such that* $K\phi \lneq K\gamma_h$ *and there does not exist any* $g \in H$ *such that* $K\gamma_g \lneq K\phi$.

Then Pettet's subgroup $\mathrm{Out}^H(G)$ *is equal to Levitt's subgroup* $\mathrm{Out}_H(G)$ *of* $\mathrm{Out}(G)$.

What we prove is that every element $\widehat{\psi} \in \mathrm{Out}^H(G)$ has a representative $\psi \in \mathrm{Aut}(G)$ such that $H\psi = H$ and $t\phi = g_1t^{\epsilon}g_2$ for some $g_1, g_2 \in H$. This implies that $\mathrm{Aut}^H(G)$ acts on the Bass–Serre tree in the appropriate way as by definition Pettet's subgroup acts on the vertices and this result implies that the action on the vertices extends to the corresponding action on the edges.

Proof. If either (1) or (2) hold then the theorem holds by results of Pettet [Pet99, Lemma 2.6 and Theorem 1]. We shall suppose that there does not exist any $g \in H$ and $\epsilon_g = \pm 1$ such that $K\gamma_g \lneq K\phi^{\epsilon_g}$ and use this to conclude that $t\phi = g_1 t^{\epsilon} g_2$ for some $g_1, g_2 \in H$.

Note that as $\widehat{\psi} \in \mathrm{Out}_H(G)$ there exists a representative $\psi \in \mathrm{Aut}(G)$ such that $H\psi = H$. We begin by picking such an automorphism ψ. Now, let v be the vertex such that $G_v = H$ and let p be the vertex such that $G_p = tHt^{-1}$. Note that v and p are connected by a single edge, e', which is stabilised by the subgroup K. Let w be the vertex stabilised by $(t\psi)H(t\psi)^{-1}$, and consider the geodesic $[v, w]$. Note that the first edge in this geodesic is stabilised by either an H-conjugate of K or of $K\phi$, and as we are working modulo the inner automorphisms can assume that the first edge of the geodesic $[v, w]$ is fixed by either K or $K\phi$, and we shall denote this subgroup K_1.

There exists a positive edge e in the geodesic $[v, w]$ such that $G_e = K\psi$ [Pet99, Lemma 2.2]. Note that $K\psi$ is G-conjugate to K as it is an edge stabiliser, and so we can re-write $K\psi$ as $Z^{-1}g^{-1}KgZ$ for some $g \in H$, $Z \in F(H/K)$. Now, $K\psi$ stabilises $[v, w]$ and so $K\psi = Z^{-1}g^{-1}KgZ =\leq K_1$, which means that $G_e = K\phi^r\gamma_{h_0}$ for some $r \in \mathbb{Z}$, $h_0 \in H$.

We shall now shift to an algebraic viewpoint. Write $t\psi = g_1 t^{\epsilon_1} g_2 t^{\epsilon_2} \dots g_n t^{\epsilon_n} g_{n+1}$ in its t-reduced form, and more generally write $X_i = g_i t^{\epsilon_i} \dots g_n t^{\epsilon_n} g_{n+1}$. Algebraically, we have shown that there exists some $j \in \{1, \dots, n\}$ such that $X_j K_1 X_j^{-1} = K\phi^r\gamma_{h_j}$ for some $h_j \in H$, and that for each $i \in \{1, \dots, n\}$ there exists some $h_i \in H$ such that $K\phi^r\gamma_{h_i} \leq X_i K_1 X_i^{-1}$. Consider $X_{j-1}K_1X_{j-1}^{-1}$. Then we have the following.

$$\begin{aligned} K\phi^r\gamma_{h_{j-1}} &\leq X_{j-1}K_1X_{j-1}^{-1} \\ &= g_{j-1}t^{\epsilon_{j-1}}(K\phi^r\gamma_{h_j})t^{-\epsilon_{j-1}}g_{j-1}^{-1} \end{aligned}$$

This means that every element of $K\phi^r\gamma_{h_{j-1}}$ is equal to an element of the following form.

$$g_{j-1}t^{\epsilon_{j-1}}(k\phi^r\gamma_{h_j})t^{-\epsilon_{j-1}}g_{j-1}^{-1}$$

Such an element is equal to an element of the form $k\phi^{r+\epsilon_{j-1}}\gamma_{h'_j}$. Hence, $K\phi^r\gamma_{h_{j-1}} \leq K\phi^{r+\epsilon_{j-1}}\gamma_{h'_j}$, and so there exists some $g \in H$ such that $K\gamma_g \leq K^{\epsilon_{j-1}}$. This is a contradiction, and so we conclude that $j = 1$. However, repeating the above argument with X_{j+1} in place of X_{j-1} proves that there exists some $g \in H$ such that $K\gamma_g \leq K^{\epsilon_{j+1}}$ and so we obtain another contradiction. Hence, $j = 1$ and $n = 1$, so $t\phi = g_1 t^{\epsilon} g_2$ for some $g_1, g_2 \in H$ as required. $\square$

4.2.1 The form of (outer) automorphisms

Let G denote an automorphism-induced induced HNN-extension, $G = \langle H, t; K^t = K\phi\rangle$, and recall that such a group G can be viewed as a Zappa–Szép product with the following presentation.

$$\langle H, x_a, a \in T; x_a h = (h\phi)x_{\overline{ah}} \ (h \in H, a \in T)\rangle$$

Our two main technical theorems, Theorems 4.2.15 and 4.2.17, follow from a classification of the elements of Levitt's subgroup $\mathrm{Out}_H(G)$. That is, to prove the two main technical theorems we begin by finding coset representatives for $\mathrm{Out}_H(G)$. The purpose of this section is to prove Lemma 4.2.6, which gives this classification. The proof of this lemma views G as a Zappa–Szép product as above.

We begin by proving that certain maps, which are used as representatives for elements of $\mathrm{Out}(G)$ in Lemma 4.2.6, define automorphisms of G. There are two forms these representatives take, and Lemma 4.2.4 considers the first form while Lemma 4.2.5 considers the second form.

Lemma 4.2.4. *If $\delta \in \mathrm{Aut}(H)$ and $g \in H$ such that $K\delta = K$ and $(k\delta\phi) = g^{-1}(k\phi\delta)g$ for all $k \in K$ then the pair (δ, g) induces an automorphism of G in the following way, where $v_1 := g$, $v_a = (a^{-1}\phi\delta)v_1(a\delta\phi)$, and $\tau_a = \overline{a\delta}$.*

$$\begin{aligned} \alpha_{(\delta,g)} : h &\mapsto h\delta & \forall h \in H \\ x_a &\mapsto v_a x_{\tau_a} \end{aligned}$$

Proof. We begin by proving that each $\alpha_{(\delta,g)}$ is a homomorphism. We then prove that they are each surjective and then that they are right-invertible (that is, injective).

To see that $\alpha_{(\delta,g)}$ is a homomorphism note that it satisfies all the relators of H, as $\alpha_{(\delta,g)}|_H = \delta \in \mathrm{Aut}(H)$, so it is sufficient to prove that $(x_a h)\alpha_{(\delta,g)} = (h\phi x_{\overline{ah}})\alpha_{(\delta,g)}$ for all $a \in T$, $h \in H$. So, the left hand side is evaluated as follows.

$$\begin{aligned} (x_a h)\alpha_{(\delta,g)} &= (a^{-1}\phi\delta)g(a\delta\phi)x_{\overline{a\delta}}(h\delta) \\ &= (a^{-1}\phi\delta)g\left((ah)\delta\phi\right)x_{\overline{(ah)\delta}} \end{aligned}$$

We now evaluate the right hand side as follows. Note that (4.1), below, is obtained because $\overline{\overline{ah}\delta} = \overline{(kah)\delta} = \overline{ah\delta}$ as $K\delta = K$, while (4.2), below, is obtained because g is such that

$(k\delta\phi) = g^{-1}(k\phi\delta)g$ for all $k \in K$, which is rearranged to give the substitution used.

$$\begin{aligned}
(h\phi x_{\overline{ah}})\alpha_{(\delta,g)} &= (h\phi\delta)\left((\overline{ah})^{-1}\phi\delta\right) g(\overline{ah}\delta\phi)x_{\overline{(ah)\delta}} &(4.1)\\
&= (h\phi\delta)\left((h^{-1}a^{-1}k^{-1})\phi\delta\right) g\left((kah)\delta\phi\right) x_{\overline{(ah)\delta}}\\
&= (a^{-1}\phi\delta)(k^{-1}\phi\delta)g(k\delta\phi)\left((ah)\delta\phi\right) x_{\overline{(ah)\delta}}\\
&= (a^{-1}\phi\delta)g\left((ah)\delta\phi\right) x_{\overline{(ah)\delta}} &(4.2)
\end{aligned}$$

Thus, the left and right hand sides are equal and so $\alpha_{(\delta,g)}$ is a homomorphism.

To see that $\alpha_{(\delta,g)}$ is surjective, note that its restriction to H is surjective, and further note that $x_1 \mapsto u_1x_1^{\epsilon}$ for some $u_1 \in H$ so x_1 is in the image. Then because $(a^{-1}\phi)x_1a = x_a$, each x_a for $a \in T$ is contained in the image. As G is generated by H and the x_a we are done.

To see that $\alpha_{(\delta,g)}$ is right-invertible, and so injective, we shall prove that $\alpha_{(\delta^{-1},g^{-1}\delta^{-1})}$ is also a homomorphism and that $\alpha_{(\delta,g)}\alpha_{(\delta^{-1},g^{-1}\delta^{-1})}$ is trivial. To prove that $\alpha_{(\delta^{-1},g^{-1}\delta^{-1})}$ is a homomorphism, it suffices to prove that $K\delta^{-1} = K$ and that $(k\delta^{-1}\phi) = (g^{-1}\delta^{-1})^{-1}(k\phi\delta^{-1})(g^{-1}\delta^{-1})$. Now, $K\delta = K$ and so $K\delta^{-1} = K$, while for all $k \in K$ we have the following implications (note that we replace k with $k\delta^{-1}$, which is a valid step because $K = K\delta^{-1}$).

$$\begin{aligned}
(k\delta\phi) &= g^{-1}(k\phi\delta)g\\
\Rightarrow (k\phi) &= g^{-1}(k\delta^{-1}\phi\delta)g\\
\Rightarrow (k\phi\delta^{-1}) &= (g^{-1}\delta^{-1})(k\delta^{-1}\phi)(g\delta^{-1})\\
\Rightarrow (k\delta^{-1}\phi) &= (g^{-1}\delta^{-1})^{-1}(k\phi\delta^{-1})(g^{-1}\delta^{-1})
\end{aligned}$$

Therefore, the two required properties hold and so $\alpha_{(\delta^{-1},g^{-1}\delta^{-1})}$ is a homomorphism. This map is the right inverse of $\alpha_{(\delta,g)}$ as clearly $h\alpha_{(\delta,g)}\alpha_{(\delta^{-1},g^{-1}\delta^{-1})} = h$ while we have the following working.

$$\begin{aligned}
x_1\alpha_{(\delta,g)}\alpha_{(\delta^{-1},g^{-1}\delta^{-1})} &= (gx_1)\alpha_{(\delta^{-1},g^{-1}\delta^{-1})}\\
&= (g\delta^{-1})(g^{-1}\delta^{-1})x_1\\
&= x_a
\end{aligned}$$

Then, because $\alpha_{(\delta,g)}$ is a homomorphism and because G is generated by x_1 and H, we have that $\alpha_{(\delta,g)}$ is injective, as required. $\square$

The second form which automorphism can take is given by the following lemma.

Lemma 4.2.5. *If $\delta \in \mathrm{Aut}(H)$ and $g \in H$ such that $K\delta = K\phi$, $K\delta^2\gamma_g = K$, and $g^{-1}(k\phi\delta)g = k\delta\phi^{-1}$ for all $k \in K$ then the pair (δ, g) induces an automorphism of G in the following way, where $v_1 := g$, $v_a = (a^{-1}\phi\delta)v_1(a\delta\phi^{-1})$, and $\tau_a = \overline{a\delta\phi^{-1}}$.*

$$\begin{aligned} \zeta_{(\delta,g)} : h &\mapsto h\delta && \forall h \in H \\ x_a &\mapsto v_a x_{\tau_a}^{-1} \end{aligned}$$

Proof. We begin by proving that each $\zeta_{(\delta,g)}$ is a homomorphism. We then prove that they are each surjective and then that they are right-invertible (that is, injective).

To see that $\zeta_{(\delta,g)}$ is a homomorphism note that it satisfies all the relators of H, as $\zeta_{(\delta,g)}|_H \in \mathrm{Aut}(H)$, so it is sufficient to prove that $(x_a h)\zeta_{(\delta,g)} = (h\phi x_{\overline{ah}})\zeta_{(\delta,g)}$ for all $a \in T$, $h \in H$. So, the left hand side is evaluated as follows.

$$\begin{aligned} (x_a h)\zeta_{(\delta,g)} &= (a^{-1}\phi\delta)g(a\delta\phi^{-1})x_{\overline{a\delta\phi^{-1}}}^{-1}(h\delta) \\ &= x_{\overline{g^{-1}(a\phi\delta)}}^{-1}(a^{-1}\phi\delta\phi)(g\phi)\left((ah)\delta\right) \end{aligned}$$

We now evaluate the right hand side as follows. Note that (4.3), below, is obtained because $\overline{\overline{ah}\delta\phi^{-1}} = \overline{(kah)\delta\phi^{-1}} = \overline{(ah)\delta\phi^{-1}}$ as $K\delta = K\phi$ so $K\delta\phi^{-1} = K$, while (4.4), below, is obtained because g is such that $k\delta\phi^{-1} = g^{-1}(k\phi\delta)g$ for all $k \in K$, which is rearranged to give the substitution used.

$$\begin{aligned} (h\phi x_{\overline{ah}})\zeta_{(\delta,g)} &= (h\phi\delta)\left((\overline{ah})^{-1}\phi\delta\right) g(\overline{ah}\delta\phi^{-1})x_{\overline{(ah)\delta\phi^{-1}}}^{-1} && (4.3) \\ &= (h\phi\delta)\left((h^{-1}a^{-1})\phi\delta\right)(k^{-1}\phi\delta)g(k\delta\phi^{-1})\left((ah)\delta\phi^{-1}\right) x_{\overline{(ah)\delta\phi^{-1}}}^{-1} \\ &= (a^{-1}\phi\delta)g\left((ah)\delta\phi^{-1}\right) x_{\overline{(ah)\delta\phi^{-1}}}^{-1} && (4.4) \\ &= x_{\overline{g^{-1}(a\phi\delta)}}^{-1}(a^{-1}\phi\delta\phi)(g\phi)\left((ah)\delta\right) \end{aligned}$$

Thus, the left and right hand sides are equal and so $\zeta_{(\delta,g)}$ is a homomorphism.

To see that $\zeta_{(\delta,g)}$ is surjective, note that its restriction to H is surjective, and further note that $x_1 \mapsto u_1 x_1^{\epsilon}$ for some $u_1 \in H$ so x_1 is in the image. Then because $(a^{-1}\phi)x_1 a = x_a$, each x_a for $a \in T$ is contained in the image. As G is generated by H and the x_a we are done.

In order to prove that $\zeta_{(\delta,g)}$ is right-invertible, and so injective, we shall prove that $\alpha_{(\delta^2\gamma_g, g^{-1}(g\delta))}$ is an automorphism and that $\zeta_{(\delta,g)}^2\gamma_g = \alpha_{(\delta^2\gamma_g, g^{-1}(g\delta))}$. This is sufficient, because we have already proven that $\zeta_{(\delta,g)}$ is a homomorphism. To prove that $\alpha_{(\delta^2\gamma_g, g^{-1}(g\delta))}$ is an automorphism, by Lemma 4.2.4 we are required to prove that $K\delta^2\gamma_g = K$ and that we have the following equality.

$$k\delta^2\gamma_g\phi = (g\delta)^{-1}g(k\phi\delta^2\gamma_g)g^{-1}(g\delta) \ \left(= k\phi\delta^2\gamma_{g\delta}\right)$$

The first point, that $K\delta^2\gamma_g = K$, holds by the definition of $\zeta_{(\delta,g)}$ (that is, this equality is explicitly assumed). To prove the second point, we verify that $k\delta^2\gamma_g\phi\delta^{-1} = k\phi\delta\gamma_g$ for all $k \in K$, which is sufficient. To do this, recall that $K\delta = K\phi$, so we can replace k with $k\phi\delta^{-1}$ throughout, or with $k\delta\phi^{-1}$ throughout. The following working then proves that $\alpha_{(\delta^2\gamma_g, g^{-1}(g\delta))}$ is an automorphism of G, as required. Our starting point is $k\delta\phi^{-1} = k\phi\delta\gamma_g$, which holds by the definition of $\zeta_{(\delta,g)}$.

$$\begin{aligned}
k\delta\phi^{-1} &= k\phi\delta\gamma_g \\
\Leftrightarrow k &= k\phi\delta^{-1}\phi\delta\gamma_g && (k \mapsto k\phi\delta^{-1} \text{ throughout}) \\
\Leftrightarrow k\delta\phi^{-1}\phi\delta^{-1} &= k\phi\delta^{-1}\phi\delta\gamma_g \\
\Leftrightarrow k\phi\delta\gamma_g\phi\delta^{-1} &= k\phi\delta^{-1}\phi\delta\gamma_g && (k\delta\phi^{-1} \mapsto k\phi\delta\gamma_g) \\
\Leftrightarrow k\delta^2\gamma_g\phi\delta^{-1} &= k\phi\delta\gamma_g && (k \mapsto k\delta\phi^{-1} \text{ throughout})
\end{aligned}$$

To complete the result, it is now sufficient to prove that $\zeta^2_{(\delta,g)}\gamma_g = \alpha_{(\delta^2\gamma_g, g^{-1}(g\delta))}$. To see this, note that as as their restriction to H is identical and because $\zeta^2_{(\delta,g)}$ is a homomorphism, it is sufficient to prove that $x_1\zeta^2_{(\delta,g)}\gamma_g = g^{-1}(g\delta)x_1 \left(= x_1\alpha_{(\delta^2\gamma_g, g^{-1}(g\delta))}\right)$. This holds by the following working.

$$\begin{aligned}
x_1\zeta^2_{(\delta,g)}\gamma_g &= (gx_1^{-1})\zeta_{(\delta,g)}\gamma_g \\
&= \left((g\delta)(gx_1^{-1})^{-1}\right)\gamma_g \\
&= \left((g\delta)x_1g^{-1}\right)\gamma_g \\
&= g^{-1}(g\delta)x_1
\end{aligned}$$

Therefore, $\zeta_{(\delta,g)}$ is an automorphism of G, as required. □

It should be noted that it is sometimes possible to replace certain restrictions on the maps from Lemma 4.2.4 and Lemma 4.2.5 with others to obtain surjective endomorphisms with non-trivial kernel, and so automorphism-induced HNN-extensions are not necessarily Hopfian. For example, in the definition of the map $\zeta_{(\delta,g)}$ from Lemma 4.2.5, replacing the condition $K\delta^2\gamma_g = K$ with the condition $K\delta^2 = K$ yields a surjective endomorphism, but this is an injection only if the element g from the pair (δ, g) is contained in the normaliser of K, $g \in N_H(K)$.

Classifying the elements of $\mathrm{Out}_H(G)$. Our classification of the coset representatives for $\mathrm{Out}_H(G)$ is as follows.

Lemma 4.2.6 (Main technical lemma). *Every element $\widehat{\psi}$ of $\mathrm{Out}_H(G)$ has a representative in $\mathrm{Aut}_H(G)$ of the form $\alpha_{(\delta,g)}$ or of the form $\zeta_{(\delta,g)}$. Moreover every map $\alpha_{(\delta,g)}$ and $\zeta_{(\delta,g)}$ defines an automorphism of G.*

The proof of the first part of this lemma, that every element of $\mathrm{Aut}_H(G)$ has a representative in $\mathrm{Out}_H(G)$ of one of the stipulated forms, is substantial, but note that we have already proven, in Lemma 4.2.4 and Lemma 4.2.5, that the prospective representatives define automorphisms of G. We shall now prove the following lemma, Lemma 4.2.7, which gives a rough form of the representatives of $\mathrm{Out}_H(G)$ and is the first step in the proof of the first part of Lemma 4.2.6, on the form of the representatives.

Lemma 4.2.7. *If $\psi \in \mathrm{Aut}_H(G)$ then, modulo the inner automorphisms, ψ is of the form,*

$$\begin{aligned}\psi : h &\mapsto u_h \\ x_a &\mapsto v_a x_{\tau_a}^{\epsilon}\end{aligned}$$

where $u_h, v_a \in H$, $\tau_1 = 1$ and ϵ is fixed for ψ. Further, $\psi|_H \in \mathrm{Aut}(H)$.

Proof. Note that as we are working modulo the inner automorphisms we can assume $H\psi = H$. Thus, the restriction of ψ to H must be an automorphism of H, $\psi|_H \in \mathrm{Aut}(H)$. Therefore, as G is a Zappa–Szép product of the form $H \bowtie F_k$ every automorphism must be of the following form, modulo $\mathrm{Inn}(G)$, where $W_a \in F_k$ and $u_h, v_a \in H$ (recall that $F_k := \langle x_a; a \in T\rangle$).

$$\begin{aligned}\psi : h &\mapsto u_h \\ x_a &\mapsto v_a W_a\end{aligned}$$

Now, for notational convenience, we shall let W_* and v_* be (non-injective) functions from H to respectively F_k and H, so $W_* : H \to F_k$ and $v_* : H \to H$. This allows us to write W_h and v_h for any $h \in H$ meaningfully.

We shall now prove that W_a is of the form $x_{\tau_a}^{\epsilon_a}$, where $\tau_a \in T$. to see this, begin by noting that as ψ is a homomorphism we must have the following equalities.

$$\begin{aligned}(x_a\psi)(h\psi) &= (h\phi\psi)(x_{\overline{ah}}\psi) \\ \Rightarrow v_a W_a u_h &= u_{h\phi} v_{ah} W_{ah} \qquad (4.5)\end{aligned}$$

Note that as $G = H \bowtie F_k$ is a Zappa–Szép product, and be looking at the Zappa–Szép presentation, we have that $W_a g$ can be written as $\hat{g}\widehat{W_a}$ where $\hat{g} \in H$ and $|\widehat{W_a}| = |W_a|$, that is, $W_a g = \hat{g}\widehat{W_a}$. Then, as $v_a W_a u_h = u_{h\phi} v_{ah} W_{ah}$ we have that $|W_a| = |W_{ah}|$ and so the length of W_a is constant for all $a \in T$. Now, x_1 corresponds to the generator t of the HNN-extension, and so $|x_1\psi| = 1$ as we are in Levitt's subgroup. Therefore, $|W_a| = 1$ for all $a \in T$, and so we can write $W_a = x_{\tau_a}^{\epsilon_a}$.

Writing $W_a = x_{\tau_a}^{\epsilon_a}$ in (4.5), we have that the equality $v_1 x_{\tau_1}^{\epsilon_1} u_a = u_{a\phi} v_a x_{\tau_a}^{\epsilon_a}$ holds for all $a \in T$, and so we must have that $\epsilon_1 = \epsilon_a$ for all $a \in T$. Thus, $W_a = x_{\tau_a}^{\epsilon}$ and ϵ is fixed for all $a \in T$.

To complete the proof of the lemma it suffices to prove that we can assume that $\tau_1 = 1$. To see this, begin by recalling that we are working modulo $\mathrm{Inn}(G)$. Let $\psi' : h \mapsto u_h, x_a \mapsto v_a x_{\tau_a}^{\epsilon}$ be an automorphism of G, so $x_1\psi' = hx_a^{\epsilon}$. Then we can take $\psi := \psi'\gamma_{a^{-1}}$ if $\epsilon = 1$ or take $\psi := \psi'\gamma_{a^{-1}\phi}$ if $\epsilon = -1$ to get that $x_1\psi = gx_1^{\epsilon}$ for some $g \in H$. This completes the proof of the lemma. □

In a similar way to W_* and v_*, we can define τ_* to be a non-injective function from H to the transversal T of H/K, so $\tau_* : H \to T$, and so we can write τ_h meaningfully for all $h \in H$. We shall prove that for ψ to be a homomorphism we need the functions v_* and τ_* to be well-defined, that is, we need $v_h = v_{\bar{h}}$ and $\tau_h = \tau_{\bar{h}}$ all to hold for all $h \in H$. Doing so and replacing $h \in H$ with $a \in T$ yields certain of the conditions in Lemma 4.2.6.

Proof of Lemma 4.2.6. We use Lemma 4.2.7 to prove Lemma 4.2.6, and proving this lemma is the purpose of this current section, Section 4.2.1.

Proof of Lemma 4.2.6. We shall write $\delta \in \mathrm{Aut}(H)$ for $\psi|_H$ from now on. So, by Lemma 4.2.7, modulo the inner automorphisms of G, if $\psi \in \mathrm{Aut}_H(G)$ then there exists some $\delta \in \mathrm{Aut}(H)$ such that ψ has the following form, where $v_a \in H$, $\tau_1 = 1$ and ϵ is fixed for ψ.

$$\begin{aligned} \psi : h &\mapsto h\delta \\ x_a &\mapsto v_a x_{\tau_a}^{\epsilon} \end{aligned}$$

We have two cases: $\epsilon = 1$ and $\epsilon = -1$.

The case of $\epsilon = 1$*:* Suppose $\epsilon = 1$. We must establish the following facts, and doing so proves the lemma for this case (by replacing $h \in H$ with $a \in T$ in (i) and (ii)).

(i) $\tau_h = \overline{h\delta}$ holds for all $h \in H$ (that is, τ_* is a well-defined function).

(ii) $v_h = (h^{-1}\phi\delta)v_1(h\delta\phi)$ holds for all $h \in H$ (that is, v_* is a well-defined function).

(iii) $K = K\delta$.

(iv) $k\delta\phi = v_1^{-1}(k\phi\delta)v_1$ holds for all $k \in K$.

We begin by proving that (i) holds. That is, $(h\delta)\tau_h^{-1} \in K$ for all $h \in H$. This holds because of the following sequence of implications.

$$\begin{aligned} x_a h &= (h\phi)x_{\overline{ah}} \\ v_a x_{\tau_a}(h\delta) &= (h\phi\delta)v_{ah}x_{\tau_{ah}} \\ v_a(h\delta\phi)x_{\overline{\tau_a(h\delta)}} &= (h\phi\delta)v_{ah}x_{\tau_{ah}} \\ \tau_a(h\delta)\tau_{ah}^{-1} &\in K \end{aligned} \tag{4.6}$$

Then, $\tau_1 = 1$ so $(h\delta)\tau_h^{-1} \in K$, as required.

We shall now prove that (ii) holds, that is, $v_h = (h^{-1}\phi\delta)v_1(h\delta\phi)$ for all $h \in H$. To see this, by (4.6) we have that $v_a(h\delta\phi) = (h\phi\delta)v_{ah}$ which means, taking $a = 1$, we have that $v_h = (h^{-1}\phi\delta)v_1(h\delta\phi)$ for all $h \in H$ as required.

We shall now prove that (iii) holds, that is, $K\delta = K$ (recall that $\psi|_H = \delta$). Note that because (i) holds we have that $\tau_k = 1$ for all $k \in K$, and so $K\delta \leq K$. On the other hand, suppose $g\delta \in K$. Then $\tau_g = 1$, which means that $\psi : (v_g^{-1}\delta^{-1})x_{\overline{g}} \mapsto x_1$. As $(v_1^{-1}\delta^{-1})x_1 \mapsto x_1$ we have that $(v_g^{-1}\delta^{-1})x_{\overline{g}} = (v_1^{-1}\delta^{-1})x_1$ which means that $\overline{g} = 1$ so $g \in K$. Thus, $K\delta = K$, as required.

Finally, we prove that (iv) holds, that is, $k\delta\phi = v_1^{-1}(k\phi\delta)v_1$ for all $k \in K$. To do this, note that because (ii) holds we have that $v_h = v_a$ for all $h \in H$ such that $ah^{-1} \in K$. So, for $ah^{-1} \in K$ we have the following.

$$\begin{aligned} v_h &= v_a \\ (h^{-1}\phi\delta)v_1(h\delta\phi) &= (a^{-1}\phi\delta)v_1(a\delta\phi) \\ v_1 &= (ah^{-1})\phi\delta \cdot v_1 \cdot (ha^{-1})\delta\phi \end{aligned}$$

Then, because $ah^{-1} \in K$, and because we could take $a = 1$ and $h = k \in K$, we have that $v_1 = (k^{-1}\phi\delta)v_1(k\delta\phi)$ for all $k \in K$. This completes the proof of the $\epsilon = 1$ case.

The case of $\epsilon = -1$*:* Suppose $\epsilon = -1$. We must establish the following facts, and doing so proves the lemma for this case (by replacing $h \in H$ with $a \in T$ in (i) and (ii)).

(i) $\tau_h = \overline{h\delta\phi^{-1}}$ holds for all $h \in H$ (that is, τ_* is a well-defined function).

(ii) $v_h = (h^{-1}\phi\delta)v_1(h\delta\phi^{-1})$ holds for all $h \in H$ (that is, v_* is a well-defined function).

(iii) $K\delta = K\phi$.

(iv) $K\delta^2\gamma_{v_1} = K$.

(v) $v_1^{-1}(k\phi\delta)v_1 = k\delta\phi^{-1}$ holds for all $k \in K$.

To establish these facts we begin with the following implications.

$$\begin{aligned} x_a h &= (h\phi)x_{\overline{ah}} \\ \Rightarrow v_a x_{\tau_a}^{-1}(h\delta) &= (h\phi\delta)v_{ah}x_{\tau_{ah}}^{-1} \\ \Rightarrow x_{\overline{\tau_a v_a^{-1}}}^{-1}(v_a\phi)(h\delta) &= x_{\overline{\tau_{ah}v_{ah}^{-1}(h^{-1}\phi\delta)}}^{-1}(h\phi\delta\phi)(v_{ah}\phi) \end{aligned}$$

These give us the following facts.

$$\tau_{ah}v_{ah}^{-1}(h^{-1}\phi\delta)v_a\tau_a^{-1} \in K$$

$$(v_a\phi)(h\delta) = (h\phi\delta\phi)(v_{ah}\phi)$$

Picking $a = 1$ in both equations, and recalling that $\tau_1 = 1$, we get the following facts, with (4.9) following from (4.8).

$$\tau_h v_h^{-1}(h^{-1}\phi\delta)v_1 \in K \tag{4.7}$$

$$(v_1\phi)(h\delta) = (h\phi\delta\phi)(v_h\phi) \tag{4.8}$$

$$v_1 = (h\phi\delta)v_h(h^{-1}\delta\phi^{-1}) \tag{4.9}$$

By replacing the v_1 in (4.7) using (4.9), we see that $\tau_h(h\delta\phi^{-1})^{-1} \in K$. Thus, (i) holds, while (ii) follows from (4.9).

We now prove (iii) holds. To begin, we prove that if there exists an automorphism with $\epsilon = -1$ then $K\delta\phi^{-1} = K$ (and so $K\delta = K\phi$). Now, $K\delta\phi^{-1} \leq K$ as, by (i), $\overline{k\delta\phi^{-1}} = \tau_k = \tau_1 = 1$ for all $k \in K$. Now, suppose $K\delta\phi^{-1} \neq K$. This means that $K\delta\phi^{-1} \lneq K$, and so there exists $h \in H \setminus K$ such that $h\delta\phi^{-1} \in K$. Then, $\psi : (v_h^{-1}\psi^{-1})x_{\overline{h}} \mapsto x_1^{-1}$ and $\psi : (v_1^{-1}\psi^{-1})x_1 \mapsto x_1$, so $x_1 = x_{\overline{h}}$, and so $h \in K$, our required contradiction. This establishes (iii).

To establish (iv), note that $x_1\psi^2\gamma_{v_1} = ((v_1\psi)x_1v_1^{-1})\gamma_{v_1}$ and so $\psi^2\delta_{v_1}$ is of the form investigated in the previous case (the case of $\epsilon = 1$). All automorphisms with $\epsilon = 1$ fix K (not necessarily pointwise), and so $K\psi^2\gamma_{v_1} = K$. Therefore, (iv) holds.

We now establish (v). We need to prove that $v_h = v_{\bar{h}}$ for all $h \in H$, that is, we need v_* to be a well-defined function. Thus, by (ii), we require the following equality to hold.

$$\left(v_{\bar{h}} =\right)\ (\bar{h}^{-1}\phi\delta)v_1(\bar{h}\delta\phi^{-1}) = (h^{-1}\phi\delta)v_1(h\delta\phi^{-1})\ \ (= v_h)$$

This means we need that $(h\bar{h}^{-1})\phi\delta \cdot v_1 \cdot (\bar{h}h^{-1})\delta\phi^{-1} = v_1$. As $\bar{h}h^{-1} \in K$ we get that $v_1 = (k^{-1}\phi\delta)v_1(k\delta\phi^{-1})$ for all $k \in K$, and (v) holds.

Therefore, every automorphism is equal, modulo the inner automorphisms, to an automorphism of the form $\alpha_{(\delta,g)}$ or an automorphism of the form $\zeta_{(\delta,g)}$. Then, as every map $\alpha_{(\delta,g)}$ and $\zeta_{(\delta,g)}$ defines an automorphism of G by, respectively, Lemma 4.2.4 and Lemma 4.2.5, the proof of the lemma is complete. □

The subgroup $\mathbf{Out}^0_H(G)$. Having proven Lemma 4.2.6, we know, in a certain sense, what the elements of $\mathrm{Out}_H(G)$ are. We shall write $\widehat{\alpha}_{(\delta,g)}$ to denote the coset of $\mathrm{Aut}(G)/\operatorname{Inn}(G)$ containing $\alpha_{(\delta,g)}$. In the remainder of Section 4.2 we describe the group formed by the elements of the form $\widehat{\alpha}_{(\delta,g)}$, denoted $\mathrm{Out}^0_H(G)$, in the two disjoint cases of when $H \lneq K$ (Theorem 4.2.15) and when $H = K$ and $Z(H) = 1$ (Theorem 4.2.17). Note that the purpose of Section 4.2 is to prove Theorems 4.2.15 and 4.2.17, and these two results form the basis of the proofs of the main theorems, Theorems B, C and D.

We shall now explain why we do not consider the automorphisms $\zeta_{(\delta,g)}$, but instead restrict our investigations to the subgroup $\mathrm{Out}^0_H(G)$ of $\mathrm{Out}_H(G)$. If there does not exist any automorphisms of the form $\zeta_{(\delta,g)}$ then $\mathrm{Out}^0_H(G) = \mathrm{Out}_H(G)$. Otherwise, noting that the $\alpha_{(\delta,g)}$ maps x_1 to gx_1 while $\zeta_{(\delta',g')}$ maps x_1 to $g'x_1^{-1}$, we see that $\mathrm{Out}^0_H(G)$ is an index two subgroup of $\mathrm{Out}_H(G)$. Therefore, we have the following lemma.

Lemma 4.2.8. *The subgroup* $\mathrm{Out}^0_H(G)$ *consisting of the outer automorphisms of the form* $\widehat{\alpha}_{(\delta,g)}$ *has index two in Levitt's subgroup* $\mathrm{Out}_H(G)$ *if there exists an automorphism* $\zeta_{(\delta,g)}$ *of* G*. Otherwise,* $\mathrm{Out}^0_H(G)$ *is equal to Levitt's subgroup* $\mathrm{Out}_H(G)$ *itself.*

This lemma is why in Theorems 4.2.15 and 4.2.17 we restrict our analysis to $\mathrm{Out}^0_H(G)$. Note that Lemma 4.2.6 classifies when $\mathrm{Out}^0_H(G)$ has index one or two in $\mathrm{Out}_H(G)$ as it classifies when an automorphism $\zeta_{(\delta,g)}$ exists.

Note that the automorphisms of the form $\alpha_{(\delta,g)}$ are such that the following hold. We use these equalities throughout the remainder of Section 4.2.

$$\alpha_{(\delta,g)}\alpha_{(\xi,h)} = \alpha_{(\delta\xi,(g\xi)h)}$$
$$\alpha_{(\delta,g)}^{-1} = \alpha_{(\delta^{-1},g^{-1}\delta^{-1})}$$

4.2.2 Proper automorphism-induced HNN-extensions

Take G to be a proper automorphism-induced HNN-extension, that is, $G = \langle H, t; K^t = K\phi\rangle$ where ϕ is an automorphism of H and where K is a proper subgroup of H, $K \lneq H$. These are the second class of groups from the title of this thesis. In this section we prove Theorem 4.2.15, which gives a description of $\mathrm{Out}_H(G)$ for G in this class of groups.

In many of the proofs in this section the calculations are written in terms of cosets, for example we write $\widehat{\alpha}_{(\gamma_{h_1},g_1)}\widehat{\alpha}_{(\gamma_{h_2},g_2)} = \widehat{\alpha}_{(\gamma_{h_1h_2},h_2^{-1}g_1h_2g_2)}$ in the proof of Lemma 4.2.9. However, without exception, all of the calculations hold for the written automorphisms. For example, it holds that $\alpha_{(\gamma_{h_1},g_1)}\alpha_{(\gamma_{h_2},g_2)} = \alpha_{(\gamma_{h_1h_2},h_2^{-1}g_1h_2g_2)}$.

Automorphisms acting on the base group as inner automorphisms. We shall now analyse the subgroup $\overline{\mathrm{Inn}(H)}$ of $\mathrm{Out}(G)$, which consists of the outer automorphisms $\widehat{\alpha}_{(\delta,g)}$ such that δ is an inner automorphism of H, $\delta \in \mathrm{Inn}(H)$, that is, those outer automorphisms who have a representative which acts on the base group as an inner automorphism. Recall that the purpose of the current section, Section 4.2.2, is to prove Theorem 4.2.15, which gives a description of $\mathrm{Out}_H^0(G)$ by splitting this subgroup using a short exact sequence. The subgroup $\overline{\mathrm{Inn}(H)}$ forms the kernel of this short exact sequence.

By definition, the maps $\alpha_{(\delta,g)}$ form a set of coset representatives for $\mathrm{Out}_H^0(G)$ (although not necessarily a transversal). We wish to break down this set into parts we can understand, and to do this we use the subgroup $\overline{\mathrm{Inn}(H)}$, which is formally defined as follows.

$$\overline{\mathrm{Inn}(H)} = \{\widehat{\alpha}_{(\gamma_h,g)} : \gamma_h \in \mathrm{Inn}(H), hg(h\phi)^{-1} \in C_H(K\phi), h \in N_H(K)\}$$

In Lemma 4.2.9 we prove that $\overline{\mathrm{Inn}(H)}$ is normal while in Lemma 4.2.11 we prove that $\overline{\mathrm{Inn}(H)}$ splits as a semidirect product. In Lemma 4.2.12 and in Lemma 4.2.13 we give a description of each of the factor groups in this semidirect product decomposition. In Lemma 4.2.14 we consider the quotient $\mathrm{Out}_H^0(G)/\overline{\mathrm{Inn}(H)}$. We use the subgroup $\overline{\mathrm{Inn}(H)}$ and the resulting short exact sequence in the proof of Theorem 4.2.15. We begin by proving that $\overline{\mathrm{Inn}(H)}$ is normal in $\mathrm{Out}_H^0(G)$.

Lemma 4.2.9. *Taking* $\overline{\mathrm{Inn}(H)}$ *as above, we have* $\overline{\mathrm{Inn}(H)} \trianglelefteq \mathrm{Out}_H^0(G)$.

Proof. Note that, by definition, $\overline{\mathrm{Inn}(H)}$ consists of those cosets of $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ containing some $\alpha_{(\delta,g)}$ with $\delta \in \mathrm{Inn}(H)$. Let $\widehat{\alpha}_{(\gamma_{h_1},g_1)}, \widehat{\alpha}_{(\gamma_{h_2},g_2)} \in \overline{\mathrm{Inn}(H)}$, then the following equalities hold.

$$\begin{aligned} \widehat{\alpha}^{-1}_{(\gamma_{h_1},g_1)} &= \widehat{\alpha}_{(\gamma_{{h_1}^{-1}},h_1 g_1^{-1} {h_1}^{-1})} && \in \overline{\mathrm{Inn}(H)} \\ \widehat{\alpha}_{(\gamma_{h_1},g_1)}\widehat{\alpha}_{(\gamma_{h_2},g_2)} &= \widehat{\alpha}_{(\gamma_{h_1 h_2},h_2^{-1} g_1 h_2 g_2)} && \in \overline{\mathrm{Inn}(H)} \end{aligned}$$

Therefore, noting that $\overline{\mathrm{Inn}(H)}$ is non-empty, we have that $\overline{\mathrm{Inn}(H)}$ is a subgroup of $\mathrm{Out}_H^0(G)$. Then, $\overline{\mathrm{Inn}(H)}$ is closed under conjugation, as the following working shows.

$$\begin{aligned} \widehat{\alpha}^{-1}_{(\delta,g_2)}\widehat{\alpha}_{(\gamma_h,g_1)}\widehat{\alpha}_{(\delta,g_2)} &= \widehat{\alpha}_{(\delta^{-1},g_2^{-1}\delta^{-1})}\widehat{\alpha}_{(\gamma_h\delta,(g_1\delta)g_2)} \\ &= \widehat{\alpha}_{(\delta^{-1}\gamma_h\delta,(g_2^{-1}\delta^{-1}\gamma_h\delta)(g_1\delta)g_2)} \\ &= \widehat{\alpha}_{(\gamma_{h\delta},(h^{-1}\delta)g_2^{-1}(h\delta)(g_1\delta)g_2)} && \in \overline{\mathrm{Inn}(H)} \end{aligned}$$

Thus, we conclude that $\overline{\mathrm{Inn}(H)} \trianglelefteq \mathrm{Out}_H^0(G)$, as required. □

So, the subgroup $\overline{\mathrm{Inn}(H)}$ is a normal subgroup of $\mathrm{Out}_H^0(G)$, and in Lemma 4.2.14 we gave a description of the quotient group $\mathrm{Out}_H^0(G)/\overline{\mathrm{Inn}(H)}$. Comparing this to the statement of Theorem 4.2.15, we see that in order to prove this theorem we are required to write $\overline{\mathrm{Inn}(H)}$ as a semidirect product using subgroups of H.

Splitting $\overline{\mathrm{Inn}(H)}$. We wish to break down the representatives $\widehat{\alpha}_{(\gamma_h,g)}$ for $\overline{\mathrm{Inn}(H)}$ into parts we can understand, and to do this we use the subgroups B_K and C_K, defined below. In the definition of B_K the symbol 1 denotes the trivial automorphism of H.

$$\begin{aligned} B_K &= \{\widehat{\alpha}_{(1,g)} : g \in C_H(K\phi)\} \\ C_K &= \{\widehat{\alpha}_{(\gamma_h,h^{-1}(h\phi))} : \gamma_h \in \mathrm{Inn}(H), h \in N_H(K)\} \end{aligned}$$

In Lemma 4.2.11 we prove that $\overline{\mathrm{Inn}(H)}$ is a semidirect product of the subgroups B_K and C_K, $\overline{\mathrm{Inn}(H)} = B_K \rtimes C_K$. In Lemma 4.2.12 we give the isomorphism class of B_K while in Lemma 4.2.13 we give the isomorphism class of C_K. However, we begin with the following technical lemma, Lemma 4.2.10, which is used in the proofs of Lemma 4.2.11 and Lemma 4.2.12.

Lemma 4.2.10. *Assume $K \lneq H$. Let $W \in F_k$, and let $u, g_1, g_2 \in H$. Suppose that both of the following hold.*

$$\begin{aligned} u^{-1}W^{-1}g_2x_1Wu &= g_1^{-1}(g_1\phi)x_1 \\ u^{-1}W^{-1}hWu &= g_1^{-1}hg_1 && \forall\, h \in H \end{aligned}$$

Then W is the empty word and $u \in K$.

Proof. Rearranging $u^{-1}W^{-1}hWu = g_1^{-1}hg_1$ gives us that $W^{-1}h = ug_1^{-1}hg_1u^{-1}W^{-1}$ for all $h \in H$. Then, taking $h := g_2$ and post-multiplying by x_1W, we have the following.

$$W^{-1}g_2x_1W = ug_1^{-1}g_2g_1u^{-1}W^{-1}x_1W \tag{4.10}$$

Now, we can conjugate the first equality in the statement of the lemma by u to get that $W^{-1}g_2x_1W = ug_1^{-1}(g_1u^{-1})\phi x_{\overline{u^{-1}}}$, and we can use this to replace $W^{-1}g_2x_1W$ in (4.10). This yields the following.

$$g_2g_1u^{-1}W^{-1}x_1W = (g_1u^{-1})\phi x_{\overline{u^{-1}}}$$

This means that $x_{\overline{u^{-1}}} = x_1$, and so $u \in K$, as required. It also means that $W = x_1^i$, $i \in \mathbb{Z}$. We therefore have the the following holds for all $h \in H$.

$$x_1^{-i}hx_1^i = ug_1^{-1}hg_1u^{-1}\forall h \in H$$

We wish to prove that $i = 0$. If $i \neq 0$ then we either have $x_1hx_1^{-1} \in H$ for all $h \in H$ or that $x_1^{-1}hx_1 \in H$ for all $h \in H$. However, if $x_1hx_1^{-1} \in H$ then $h \in K$, while if $x_1^{-1}hx_1 \in H$ then $h \in K\phi$. Thus, $H = K$, a contradiction. Therefore, $i = 0$ and so W is the empty word, as required. □

In the following lemma we prove that $\overline{\mathrm{Inn}(H)}$ is a semidirect product of B_K and C_K. It is an easy observations that the action of C_K on B_K is $\alpha_{(1,h)}^{\alpha_{(\gamma_g, g^{-1}(g\phi))}} = \alpha_{(1,h\gamma_{g\phi})}$.

Lemma 4.2.11. *Taking B_K, C_K and $\overline{\mathrm{Inn}(H)}$ as above, we have $\overline{\mathrm{Inn}(H)} = B_K \rtimes C_K$.*

Proof. We begin by proving that $C_K \leq \overline{\mathrm{Inn}(H)}$ and then that $B_K \trianglelefteq \overline{\mathrm{Inn}(H)}$. We use these to prove that $\overline{\mathrm{Inn}(H)} = B_K \rtimes C_K$. To see that C_K is a subgroup of $\overline{\mathrm{Inn}(H)}$, let $\widehat{\alpha}_{(\gamma_g, g^{-1}(g\phi))}, \widehat{\alpha}_{(\gamma_h, h^{-1}(h\phi))} \in C_K$ and note that the following equalities hold.

$$\begin{aligned} \widehat{\alpha}_{(\gamma_g, g^{-1}(g\phi))}\widehat{\alpha}_{(\gamma_h, h^{-1}(h\phi))} &= \widehat{\alpha}_{(\gamma_{gh}, h^{-1}g^{-1}(gh)\phi)} && \in C_K \\ \widehat{\alpha}^{-1}_{(\gamma_h, h^{-1}(h\phi))} &= \widehat{\alpha}_{(\gamma_{h^{-1}}, h(h^{-1}\phi))} && \in C_K \end{aligned}$$

Then, each product is contained in C_K as $N_H(K)$ is a subgroup of H. Therefore, noting that C_K is non-empty, we conclude that C_K is a subgroup of $\overline{\mathrm{Inn}(H)}$.

We shall now prove that B_K is a normal subgroup of $\overline{\mathrm{Inn}(H)}$. Let $\widehat{\alpha}_{(1,g_1)}, \widehat{\alpha}_{(1,g_2)} \in B_K$. Then the following equalities hold.

$$\begin{aligned} \widehat{\alpha}_{(1,g_1)}\widehat{\alpha}_{(1,g_2)} &= \widehat{\alpha}_{(1,g_1g_2)} && \in B_K \\ \widehat{\alpha}^{-1}_{(1,g_1)} &= \widehat{\alpha}_{(1,g_1^{-1})} && \in B_K \end{aligned}$$

Note that each product is contained in B_K as $C_H(K\phi)$ is a subgroup of H. Therefore, noting that B_K is non-empty, we conclude that B_K is a subgroup of $\overline{\mathrm{Inn}(H)}$. Now, to see that B_K is closed under conjugation consider the following working, where $\widehat{\alpha}_{(\gamma_h,g_2)}$ is an arbitrary element of $\overline{\mathrm{Inn}(H)}$, so $h \in N_H(K)$ and $g_3 := hg_2(h^{-1}\phi) \in C_H(K\phi)$.

$$\begin{aligned} \widehat{\alpha}^{-1}_{(\gamma_h,g_2)}\widehat{\alpha}_{(1,g_1)}\widehat{\alpha}_{(\gamma_h,g_2)} &= \widehat{\alpha}_{(\gamma_{h^{-1}},hg_2^{-1}h^{-1})}\widehat{\alpha}_{(\gamma_h,h^{-1}g_1hg_2)} \\ &= \widehat{\alpha}_{(1,g_2^{-1}h^{-1}g_1hg_2)} \\ &= \widehat{\alpha}_{(1,(h^{-1}g_3(h\phi))^{-1}h^{-1}g_1h(h^{-1}g_3(h\phi)))} \\ &= \widehat{\alpha}_{(1,(h\phi)^{-1}g_3^{-1}g_1g_3(h\phi))} \end{aligned}$$

Then, as $h \in N_H(K)$ we have that $h\phi \in N_H(K\phi)$, and so $C_H(K\phi)$ is normal in $N_H(K\phi)$ we have that $(h\phi)^{-1}g_3^{-1}g_1g_3(h\phi) \in C_H(K\phi)$. Therefore, $\widehat{\alpha}_{(1,(h\phi)^{-1}g_3^{-1}g_1g_3(h\phi))} \in B_K$, as required.

Next, we prove that $\overline{\mathrm{Inn}(H)} = B_K \rtimes C_K$. We begin by proving that $B_K \cap C_K = \mathrm{Inn}(G)$. Assume that $\alpha_{(\gamma_{g_1},g_1^{-1}(g_1\phi))} = \alpha_{(1,g_2)} \mod \mathrm{Inn}(G)$, and we shall prove that $\alpha_{(1,g_2)}$ is an inner automorphism of G, $\alpha_{(1,g_2)} \in \mathrm{Inn}(G)$. Now, as these two automorphisms are equal modulo the inner automorphisms of G, there exists some $Wu \in G$, with $W \in F_k$ and $u \in H$, such that $u^{-1}W^{-1}hWu = g_1^{-1}hg_1$ for all $h \in H$ and $u^{-1}W^{-1}g_2x_1Wu = g_1^{-1}(g_1\phi)x_1$. Then by Lemma 4.2.10 we have that $W = 1$ and $u \in K$. We thus have that $u^{-1}hu = g_1^{-1}hg_1$ for all $h \in H$ and so $ug_1^{-1} \in Z(H)$. Note that this means that $h\alpha_{(1,g_2)} = (g_1u^{-1})^{-1}h(g_1u^{-1})$. Further, because $u \in K$ we have the following equalities.

$$g_2x_1 = ug_1^{-1}(g_1\phi)x_1u^{-1} = ug_1^{-1}(g_1u^{-1})\phi x_1$$

This implies that $g_2 = ug_1^{-1}(g_1u^{-1})\phi$, and so we have the following equalities

$$x_1\alpha_{(1,g_2)} = ug_1^{-1}(g_1u^{-1})\phi x_1 = (g_1u^{-1})^{-1}x_1(g_1u^{-1})$$

Therefore, $\alpha_{(1,g_2)}$ acts as conjugation by g_1u^{-1} on H and on x_a, and so $\alpha_{(1,g_2)} = \gamma_{g_1u^{-1}}$ is an inner automorphism of G, as required.

Finally, note that if $\widehat{\alpha}_{(\gamma_g,v_1)} \in \overline{\mathrm{Inn}(H)}$ then the following holds.

$$\widehat{\alpha}_{(\gamma_g,v_1)} = \widehat{\alpha}_{(\gamma_g,g^{-1}(g\phi))}\widehat{\alpha}_{(1,(g\phi)^{-1}gv_1)}$$

Therefore, $\overline{\mathrm{Inn}(H)} \leq C_K B_K$, and so as $C_K, B_K \leq \overline{\mathrm{Inn}(H)}$, we have that $C_K B_K = \overline{\mathrm{Inn}(H)}$. We thus conclude that $\overline{\mathrm{Inn}(G)} = B_K \rtimes C_K$, as required. □

Describing the subgroup $\overline{\mathbf{Inn}(H)} = B_K \rtimes C_K$. We shall now give descriptions of the subgroups B_K and C_K of $\mathrm{Out}_H(G)$ in terms of subgroups of H. Doing this gives a description of the normal subgroup $\overline{\mathrm{Inn}(H)} = B_K \rtimes C_K$ in terms of subgroups of H, and recall that $\overline{\mathrm{Inn}(H)}$ is to be the kernel of the short exact sequence in Theorem 4.2.15. We begin, in the following lemma, by giving the isomorphism class of B_K.

Lemma 4.2.12. *If $L = \{k^{-1}(k\phi) : k \in K \cap Z(H)\}$ then the following holds.*

$$B_K \cong \frac{C_H(K\phi)}{L}$$

Proof. Note that because $B_K = \{\widehat{\alpha}_{(1,g)} : g \in C_H(K\phi)\}$ we have that B_K is a homomorphic image of $C_H(K\phi)$. It is therefore sufficient to prove that $\alpha_{(1,g)} \in \mathrm{Inn}(G)$ if and only if $g \in L$. Suppose that $g \in L$ and write $g = k^{-1}(k\phi)$ for some $k \in K \cap Z(H)$. Then we have the following two equalities, the first of which is obtained using the fact that $k, k\phi \in Z(H)$ while the second uses the fact that $x_{\overline{ak^{-1}}} = x_a$ as $k \in K \cap Z(H)$.

$$\begin{aligned} h\alpha_{(1,g)} &= (k\phi)^{-1}khk^{-1}(k\phi) = h && \forall h \in H \\ x_a\alpha_{(1,g)} &= k^{-1}(k\phi)x_a = k^{-1}x_a k \end{aligned}$$

Therefore, we conclude that $\alpha_{(1,g)}$ is the inner automorphism corresponding to conjugation by k, so $\alpha_{(1,g)} = \gamma_k$.

We now prove the other direction of the equivalence, that is, suppose $\alpha_{(1,g)} \in \mathrm{Inn}(G)$ and we shall prove that $g \in L$. As $\alpha_{(1,g)} \in \mathrm{Inn}(G)$, it acts as conjugation by uW for some $u \in H$ and some $W \in F_k$. Thus, we have $W^{-1}u^{-1}huW = h$ for all $h \in H$ and $W^{-1}u^{-1}x_a uW = gx_a$. Then, taking g_1 to be trivial in Lemma 4.2.10, we have that W is the empty word and $u \in K$. Therefore, we have $u^{-1}hu = h$ for all $h \in H$ and $u^{-1}x_a u = gx_a$, where $u \in K$. The first equality implies $u \in Z(H)$ (and so $u \in Z(H) \cap K$), while the second implies $g = u^{-1}(u\phi)$. Thus, if $\alpha_{(1,g)} \in \mathrm{Inn}(G)$ then $g \in L$, as required. □

Having dealt with the subgroup B_K, we now wish to give the isomorphism class of the subgroup C_K. Indeed, we wish to prove that taking $J = Z(H) \cap \mathrm{Fix}(\phi)$ we have $C_K \cong N_H(K)/JK$. To ease the proof of this we take the coset representative used in the proof of Proposition 4.1.2, that is, if we take $g \in N_H(K)$ then because $g \cdot Ka = K\overline{ga}$ we have that the following function is an automorphism of G.

$$\psi_g : h \mapsto h$$

$$x_a \mapsto x_{\overline{ga}}$$

Moreover, the maps ψ_{g_1} and ψ_{g_2} define the same automorphism if and only if $g_1 g_2^{-1} \in K$. Now, notice that for all $g \in N_H(K)$ we have that $\psi_{g^{-1}} = \alpha_{(\gamma_g, g^{-1}(g\phi))}\gamma_g^{-1}$ and so $\psi_{g^{-1}}$ and $\alpha_{(\gamma_g, g^{-1}(g\phi))}$ are equal modulo the inner automorphisms of G and so we can take the automorphisms ψ_g as representatives for the elements of C_K. We now use these representatives to prove that $C_K \cong N_H(K)/JK$.

Lemma 4.2.13. *If $J = Z(H) \cap \mathrm{Fix}(\phi)$ then the following holds.*

$$C_K \cong \frac{N_H(K)}{JK}$$

Proof. Taking the coset representatives ψ_g, we see that C_K is a homomorphic image of $N_H(K)$. We prove that $\psi_g \in \mathrm{Inn}(G)$ if and only if $g \in JK$, which is therefore sufficient.

If $g \in JK$ then $g = uk$, $k \in K$ and $u \in Z(H) \cap \mathrm{Fix}(\phi) = J$, and $h\psi_g = h = uhu^{-1}$ for all $h \in H$ while we obtain the following for $x_a\psi_g$. Note that as $u \in \mathrm{Fix}(\phi)$ we have that $u(u\phi)^{-1}$.

$$x_a\psi_g = x_{\overline{kua}} = x_{\overline{ua}} = u(u\phi)^{-1}x_{\overline{au}} = ux_a u^{-1}$$

Thus, $\psi_g = \gamma_u^{-1} \in \mathrm{Inn}(G)$, as required.

On the other hand, let $Wu \in G$ with $u \in H$ and $W \in F_k$, and assume that ψ_g is inner an acts as conjugation by Wu. Then we have the following.

$$h = u^{-1}W^{-1}hWu \qquad \forall\, h \in H \tag{4.11}$$

$$x_{\overline{ga}} = u^{-1}W^{-1}x_a Wu \qquad \forall\, a \in T$$

The second equality gives us that $u(u^{-1}\phi)x_{\overline{gau^{-1}}} = W^{-1}x_a W$, which implies that $u = u\phi$, that $x_{\overline{gau^{-1}}} = x_a$ and that $W = x_a^i$ for some $i \in \mathbb{Z}$. However, as $a \in T$ is arbitrary but W is fixed we have that W is the empty word (that is, $i = 0$). As W is empty, (4.11) gives us that $u \in Z(H)$. Thus, we have that $u \in Z(H)$, that $u = u\phi$ and that $x_{\overline{gau^{-1}}} = x_a$ for all $a \in T$. This last fact means that $Kgau^{-1} = Ka$, and as $u \in Z(H)$ this means that $gu^{-1} \in K$, so $g = ku$ for some $k \in K$ and $u \in Z(H) \cap \mathrm{Fix}(\phi) = J$, as required. $\square$

The proof of Theorem 4.2.15. We shall now prove our first main technical theorem, Theorem 4.2.15. We begin by considering the quotient of $\mathrm{Out}^0_H(G)$ by the normal subgroup $\overline{\mathrm{Inn}(H)}$. The image is given in terms of a subgroup A_K of $\mathrm{Aut}(H)$, which is defined as follows.

$$A_K := \{\delta \in \mathrm{Aut}(H) : K\delta = K, \exists\, g \in H \text{ s.t. } (k\delta\phi) = g^{-1}(k\phi\delta)g \;\forall\, k \in K\}$$

Note that this image is precisely the image of the short exact sequence in Theorem 4.2.15, below, while putting $H = K$ into the definition of A_K the quotient is $C_{\mathrm{Out}(H)}(\widehat{\phi})$ which is related to the statement of Theorem 4.2.17. Recall that if ψ is an automorphism of a group Q then $\widehat{\psi}$ denotes the element of $\mathrm{Out}(Q)$ with representative ψ.

Lemma 4.2.14. *Let $G = \langle H, t; K^t = K\phi\rangle$ be an automorphism-induced HNN-extensions, and assume $K \lneq H$. Then the following map has kernel $\overline{\mathrm{Inn}(H)}$ and image $A_K \mathrm{Inn}(H)/\mathrm{Inn}(H)$.*

$$\begin{aligned}\chi : \mathrm{Out}^0_H(G) &\to \mathrm{Out}(H)\\ \widehat{\alpha}_{(\delta,g)} &\mapsto \widehat{\delta}\end{aligned}$$

Proof. Note that χ is a homomorphism as $\widehat{\alpha}_{(\delta_1,g_1)}\widehat{\alpha}_{(\delta_2,g_2)} = \widehat{\alpha}_{(\delta_1\delta_2,(g_1\delta_2)g_2)}$. Clearly the kernel consists of all cosets containing those $\alpha_{(\delta,g)}$ with $\delta = \gamma_h$. Moreover, if $\delta = \gamma_h$ then, by the definition of $\alpha_{(\delta,g)}$ in Lemma 4.2.4, $K\gamma_h = K$ and $(h^{-1}kh)\phi = g^{-1}h^{-1}(k\phi)hg$ for all $k \in K$, so h and g are such that $h \in N_H(K)$ and $hg(h^{-1}\phi) \in C_H(K\phi)$. Therefore, the kernel is $\overline{\mathrm{Inn}(H)}$. That is, we have the following working.

$$\begin{aligned}&\psi\,\mathrm{Inn}(G) \mapsto \mathrm{Inn}(H)\\ \Leftrightarrow\;& \widehat{\psi} = \widehat{\alpha}_{(\gamma_h,g)} \text{ for some } \gamma_h \in \mathrm{Inn}(H)\\ \Leftrightarrow\;& \widehat{\psi} = \widehat{\alpha}_{(\gamma_h,g)} \text{ with } h \in N_H(K) \text{ and } hg(h^{-1}\phi) \in C_H(K\phi)\\ \Leftrightarrow\;& \widehat{\psi}\,\mathrm{Inn}(G) \in \overline{\mathrm{Inn}(H)}\end{aligned}$$

Thus, the map χ is also well-defined. Its image is $A_K \mathrm{Inn}(H)/\mathrm{Inn}(H)$ as A_K consists precisely of those automorphisms $\delta \in \mathrm{Aut}(H)$ of H forming a pair (δ, g) which defines an automorphism $\alpha_{(\delta,g)}$ of G, by Lemma 4.2.6. $\square$

Our first main technical theorem is now as follows, where A_K is defined as above (before Lemma 4.2.14).

Theorem 4.2.15. *Let G be a proper automorphism-induced HNN-extension of H with associated subgroup $K \lneq H$ and associated automorphism ϕ.*

$$G \cong \langle H, t; k^t = k\phi, k \in K\rangle$$

Let $L = \{k^{-1}(k\phi) : k \in K \cap Z(H)\}$ and let $J = Z(H) \cap \mathrm{Fix}(\phi)$. Then we have the following short exact sequence,

$$1 \to \frac{C_H(K\phi)}{L} \rtimes \frac{N_H(K)}{JK} \to \mathrm{Out}_H^0(G) \to \frac{A_K \operatorname{Inn}(H)}{\operatorname{Inn}(H)} \to 1$$

where either $\mathrm{Out}_H^0(G) = \mathrm{Out}_H(G)$ or there exists some $\delta \in \mathrm{Aut}(H)$ and some $g \in H$ such that $K\delta = K\phi$, $K\delta^2\gamma_g = K$ and $g^{-1}(k\phi\delta)g = k\delta\phi^{-1}$ for all $k \in K$, whence $\mathrm{Out}_H^0(G)$ has index two in $\mathrm{Out}_H(G)$.

Proof. By Lemma 4.2.8, $\mathrm{Out}_H^0(G)$ has index one or two in $\mathrm{Out}_H(G)$, and further has index two precisely when there exists a pair (δ, g) such that $\zeta_{(\delta,g)}$ is an automorphism of G. Lemma 4.2.5 thus gives the conditions stipulated by the theorem which imply $\mathrm{Out}_H^0(G)$ has index two.

We shall now prove that $\mathrm{Out}_H^0(G)$ splits as in the statement of the theorem, which completes the proof of the result. By Lemma 4.2.14, we have that $\mathrm{Out}_G^0(G)/\overline{\operatorname{Inn}(H)} = A_K \operatorname{Inn}(H)/\operatorname{Inn}(H)$ which yields a short exact sequence. The description of $\overline{\operatorname{Inn}(H)}$ given by combining Lemma 4.2.11 with Lemma 4.2.12 and with Lemma 4.2.13 then completes the proof. □

An alternative view. It is interesting to note that the following isomorphism holds, where $\operatorname{Inn}(N_H(K)) = \{\gamma_g : g \in N_H(K)\}$.

$$\frac{A_K}{\operatorname{Inn}(N_H(K))} \cong \frac{A_K \operatorname{Inn}(H)}{\operatorname{Inn}(H)}$$

This is because $\operatorname{Inn}(H) \cap A_K = \operatorname{Inn}(N_H(K)) \trianglelefteq A_K$, and to see this note that $K\gamma_g = K$ if and only if $g \in N_H(K)$, while if $K\gamma_g = K$ then $\gamma_g \in A_K$ because of the following equality.

$$(k\gamma_g\phi) = (g^{-1}(g\phi))^{-1}(k\phi\gamma_g)(g^{-1}(g\phi))$$

Describing $\mathbf{Aut}_H^0(G)$ completely. We finish this section with the following lemma, Lemma 4.2.16, which is used in Corollary 4.3.3 (and thus in Theorem B) to give a complete description of $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ under certain conditions. We write $\mathrm{Aut}_H^0(G)$ for the pre-image of $\mathrm{Out}_H^0(G)$ in $\mathrm{Aut}(G)$.

Lemma 4.2.16. *If* $A_k \leq \mathrm{Inn}(H)$ *and* $C_H(K) = 1$ *then* $\mathrm{Out}^0_H(G) \cong N_H(K)/K$ *and* $\mathrm{Aut}^0_H(G)$ *can be described as follows.*

$$\begin{aligned}\mathrm{Aut}^0_H(G) &= \mathrm{Inn}(G) \rtimes \mathrm{Out}^0_H(G)\\ &\cong G \rtimes \frac{N_H(K)}{K}\end{aligned}$$

Proof. We have $\mathrm{Out}(G) \cong N_H(K)/K$ immediately by looking at the short exact sequence from Theorem 4.2.15. The imposed conditions further give us that, modulo the inner automorphisms, every element of $\mathrm{Aut}^0_H(G)$ is an element of the subgroup C_K and so has the following form (by using the coset representatives described before, and used in, Lemma 4.2.13).

$$\begin{aligned}\psi_g : h &\mapsto h\\ x_a &\mapsto x_{\overline{ga}}\end{aligned}$$

We shall call the set of maps of this form Ψ, $\Psi = \{\psi_g; g \in N_H(K)\}$. Then, a map $\psi_g \in \Psi$ is inner if and only if $g \in K$, and so is inner if and only if it is the trivial automorphism. Thus, $\Psi \cong N_H(K)/K$ and $\Psi \cap \mathrm{Inn}(G)$ is trivial, and so $\mathrm{Aut}^0_H(G) = \mathrm{Inn}(G) \rtimes \Psi \cong G \rtimes N_H(K)/K$ as $\mathrm{Inn}(G) \cong G$ because G has trivial center. Applying the natural equivalence of Ψ and $\mathrm{Out}^0_H(G)$, the proof is complete. □

4.2.3 Mapping tori

Take G to be a mapping torus with base group H, that is, $G = \langle H, t; H^t = H\phi\rangle = H \rtimes_\phi \mathbb{Z}$ where ϕ is an automorphism of H, and also assume that H has trivial center, $Z(H) = 1$. These are the third class of groups from the title of this thesis. In this section we prove Theorem 4.2.17, which gives a description of $\mathrm{Out}_H(G)$ for G in this class of groups.

Our proof of the theorem considers a map similar to the homomorphism χ from Lemma 4.2.14, but the map "goes the other way", in the sense that it has image $\mathrm{Out}_H(G)$ as opposed to pre-image $\mathrm{Out}_H(G)$.

Theorem 4.2.17. *Let* $G = H \rtimes_\phi \mathbb{Z}$ *be a mapping torus with base group* H *and associated automorphism* ϕ*. Assume* H *has trivial center and has no epimorphisms onto* $\mathbb{Z}$*. Then we have the following isomorphism,*

$$\mathrm{Out}^0(H_\phi) \cong \frac{C_{\mathrm{Out}(H)}(\widehat{\phi})}{\langle\widehat{\phi}\rangle}$$

where either $\mathrm{Out}^0(G) = \mathrm{Out}(G)$ *or* $\widehat{\phi}$ *is conjugate to* $\widehat{\phi}^{-1}$ *in* $\mathrm{Out}(H)$*, whence* $\mathrm{Out}^0(G)$ *has index two in* $\mathrm{Out}(G)$*.*

Proof. By Lemma 4.2.8, $\mathrm{Out}^0_H(G)$ has index one or two in $\mathrm{Out}_H(G)$, and further has index two precisely when there exists a pair (δ, g) such that $\zeta_{(\delta,g)}$ is an automorphism of G. Lemma 4.2.5 thus gives the conditions stipulated by the theorem which imply $\mathrm{Out}^0_H(G)$ has index two.

Consider the following map. We shall prove that it is a well-defined surjective homomorphism with kernel $\langle\widehat{\phi}\rangle$, which proves the theorem.

$$\begin{aligned} \eta : C_{\mathrm{Out}(H)}(\widehat{\phi}) &\to \mathrm{Out}^0_H(G) \\ \widehat{\delta} &\mapsto \widehat{\alpha}_{(\delta,g)} \qquad\qquad g \text{ is such that } [\delta, \phi] = \gamma_g \end{aligned}$$

Note that the map η is a homomorphism by the following working.

$$(\widehat{\delta}_1\eta)(\widehat{\delta}_2\eta) = \widehat{\alpha}_{(\delta_1,g_1)}\widehat{\alpha}_{(\delta_2,g_2)} = \widehat{\alpha}_{(\delta_1\delta_2,(g_1\delta_2)g_2)}$$

To see that η is well-defined, suppose that $\delta_2 = \delta_1\gamma_h$. Note that $[\delta_1, \phi] = \gamma_{hg_2(h^{-1}\phi)})$.

$$\begin{aligned} \alpha_{(\delta_2,g_2)} &= \alpha_{(\delta_1\gamma_h,g_2)} \\ &= \alpha_{(\delta_1\gamma_h,hg_2(h^{-1}\phi))}\gamma_h \end{aligned}$$

Writing $g_1 := hg_2(h^{-1}\phi)$ we have that $\widehat{\alpha}_{(\delta_2,g_2)} = \widehat{\alpha}_{(\delta_1,g_1)}$, as required.

Finally, to prove that the map η has kernel $\langle\widehat{\phi}\rangle$ begin by supposing that $\widehat{\alpha}_{(\delta,g)}$ is inner, and so $\widehat{\alpha}_{(\delta,g)} = \gamma_{kt^i}$ for some $k \in H$ and $i \in \mathbb{Z}$. This means that $h = t^{-i}k^{-1} \cdot (h\delta^{-1}) \cdot kt^i$ for all $h \in H$, so $h\phi^i = h\delta^{-1}\gamma_k$ for all $h \in H$, and so $\widehat{\delta} = \widehat{\phi}^j$ in $\mathrm{Out}(H)$ for some $j \in \mathbb{Z}$. Therefore, $\ker\eta \leq \langle\widehat{\phi}\rangle$. On the other hand, the image of $\widehat{\phi}$ is $\widehat{\alpha}_{(\phi,1)}$, and $\widehat{\alpha}_{(\phi,1)}$ is inner because $h\widehat{\alpha}_{(\phi,1)} = h\phi = tht^{-1}$ while $t\widehat{\alpha}_{(\phi,1)} = t$. Therefore, $\langle\widehat{\phi}\rangle \leq \ker\eta$. Thus, we conclude that $\widehat{\alpha}_{(\delta,g)} \in \mathrm{Inn}(H_\phi)$ if and only if $\widehat{\delta} \in \langle\widehat{\phi}\rangle$, as required. □

4.3 Main theorems

In this section we give four applications of Theorem 4.2.15 and one application of Theorem 4.2.17. The first two applications, Theorems 4.3.1 and 4.3.2, are novel by seem to be of no real significance. The second two applications, Theorems B and C, demonstrate a universal property of triangle groups. Note that we prove Theorem C before Theorem B, and this is because its proof is the shorter of the two and serves as a good introduction to the proof of Theorem B. The application of Theorem 4.2.17 is Theorem D, which gives a partial answer to an open problem of Bumagin–Wise.

4.3.1 A triply-universal finitely-presented group

Our first theorem proves the existence of a, in a certain sense, triply universal finitely presented group.

Theorem 4.3.1. *There exists a finitely presented group G such that each of G,* $\mathrm{Out}(G)$ *and* $\mathrm{Aut}(G)$ *contain every finitely presented group as a subgroup.*

Proof. Let us begin by noting that there exists a finitely presented group P which contains every finitely presented group as a subgroup. To see this, one first enumerates all finite presentations on two generators then takes their free product. Applying Higman's embedding theorem to this we obtain P [Mil92].

To construct the group G, rewrite $P := P * \mathbb{Z}$, so P is centerless. Take $K := \langle k; k^2\rangle$ to be cyclic of order two and form $H := P \times K = \langle P, k\rangle$. Then, we obtain the following group.

$$G = \langle H, t; k^t = k\rangle$$

Now, $N_H(K) = H$ while $KJ = K(Z(H) \cap \mathrm{Fix}(\phi)) = K$ so we have that $P \cong H/K = N_H(K)/K$ is a subgroup of G, of $\mathrm{Aut}(G)$ (by Proposition 4.1.2) and of $\mathrm{Out}(G)$ (by Theorem 4.2.15). This proves the theorem as P contains every finitely presented group. □

4.3.2 Using Rips' construction

Let Q be a finitely presented group. Recall that Rips' construction allows us to construct a short exact sequence

$$1 \to N \to H \to Q \to 1$$

such that H is a torsion-free finitely-presented $C'(1/6)$ group and N is finitely generated with trivial centraliser. Variations of this construction give H and N additional properties. Using the groups H and N from Rips' construction, we can form the following HNN-extension which is finitely presented because N is finitely generated.

$$G = \langle H, t; g^t = g\ \forall g \in N\rangle$$

This is a very natural place for Rips' construction to live. Now, the group H in Rips' construction can be taken to be residually finite [Wis03], and so G is residually finite if and only if Q is, by Theorem 4.1.3, while because $C_H(N)$ is trivial we have that $Q \cong H/N \leq \mathrm{Out}(G)$, by Theorem 4.2.15. We thus have the following theorem.

Theorem 4.3.2. *For all finitely presented groups Q there exists a finitely presented group G such that Q embeds into $\mathrm{Out}(G)$. If Q is residually finite then G can be taken to be residually finite.*

We state this construction because it is in a certain sense more natural than the similar constructions of Wise [Wis03] and Bumagin–Wise [BW05], where they take G to be the kernel N. Indeed, the result presented here is similar to Wise's Corollary 3.3 [Wis03]. The difference is that in the constructions of Wise and Bumagin–Wise the group G is always residually finite and it is never finitely presented for Q infinite [Bri06, Section 5.1].

4.3.3 Proof of Theorem C

Recall from the introduction that a *triangle group* is a group with a presentation of the following form.

$$T_{i,j,k} := \langle a, b; a^i, b^j, (ab)^k \rangle.$$

A *hyperbolic triangle group* is one where $i^{-1} + j^{-1} + k^{-1} < 1$. If $i = j = k$ we shall write $T_i := T_{i,i,i}$ for the corresponding *equilateral triangle group*. Triangle groups have Serre's property FA, by Proposition 2.7.4, and so $\mathrm{Out}^H(G) = \mathrm{Out}(G)$, by Lemma 4.2.2. In this section we prove Theorem C, which we now recall.

Theorem C. *Fix a hyperbolic triangle group $H := T_{i,j,k}$. Then every finitely-generated group Q can be embedded as a finite index subgroup of the outer automorphism group of an HNN-extension G_Q of H, where G_Q is residually finite if Q is.*

Note that this theorem does not imply Theorem 4.3.2, as in Theorem C the group G_Q is finitely generated but not necessarily finitely presented. We use F_n to denote the free group of rank n.

Proof. We give the construction, and then we prove that the required properties hold. The group G_Q is an automorphism-induced HNN-extension, and we shall begin by specifying the associated subgroup K. The inducing automorphism ϕ shall be the trivial automorphism.

As $1/i + 1/j + 1/k < 1$, H is a *large* group [BMS87], that is, H has a finite-index subgroup V which maps onto F_2. Let N be the subgroup of H such that $V/N \cong F_2$. Note that we can assume V is torsion-free, as H contains a torsion-free subgroup of finite index U [Feu71], then noting that the image of $V \cap U$ under the map induced by N is free and non-abelian we can rewrite $V := V \cap U$. Then, for every natural number n it holds that H contains a torsion-free finite-index subgroup V_n which maps onto F_n, because the free group on two-generators contain finite-index free subgroups of arbitrary rank and applying the correspondence theorem.

Let Q be a finitely-generated group. Then take a presentation $\langle X; \mathbf{r}\rangle$ of Q with $|X| < \infty$ and $\mathbf{r}$ non-empty, and so V_n maps onto Q with $n := |X|$. Take K to be the subgroup of V_n (and so of H) associated with the kernel of this map, so $V_n/K \cong Q$. Note that because V_n has finite index in H, $N_H(K)$ has finite index in H and V_n/K has finite index in $N_H(K)/K$.

We shall take G_Q to be the group $G_Q = \langle H, t; K^t = K\rangle$, where the inducing automorphism is trivial.

We shall now prove that the required properties hold. As $N_H(K)$ has finite index in H we can apply Theorem 4.1.3 to get that $G_Q = \langle H, t; K^t = K\rangle$ is residually finite if and only if Q is residually finite.

We now prove that Q can be embedded as a finite index subgroup into $\mathrm{Out}(G_Q)$, which completes the proof. We begin by proving that the kernel of the short exact sequence given in Theorem 4.2.15 has finite index in $\mathrm{Out}(G_Q)$, and to see this begin by recalling that the base group H is a triangle group. Thus, H possesses Serre's property FA and has finite outer automorphism group. As H has Serre's property FA we have that $\mathrm{Out}^H(G_Q) = \mathrm{Out}(G_Q)$, by Lemma 4.2.2, while as $\mathrm{Out}(H)$ is finite we have that the kernel of the short exact sequence embeds with finite index in $\mathrm{Out}_H(G_Q)$. Finally, noting that the inducing automorphism is trivial, so $\mathrm{Out}_H(G_Q) = \mathrm{Out}^H(G_Q)$ by Theorem 4.2.3, we have that the kernel of the short exact sequence has finite index in $\mathrm{Out}(G_Q)$, as required.

It is therefore sufficient to prove that $V_n/K \cong Q$ has finite index in the kernel of the short exact sequence. To do this we prove that $C_H(K)$ is trivial. This implies that the kernel of the short exact sequence is isomorphic to $N_H(K)/K$, because $C_H(K\phi)$ and $Z(H)$ are trivial, and as $V_n/K \cong Q$ has finite index in $N_H(K)/K$ this completes the proof of the theorem. So, suppose $C_H(K)$ is non-trivial, so there exists $1 \neq g \in C_H(K)$, and look for a contradiction. As $g \in C_H(K)$ we have $K \leq C_H(g)$. As H is hyperbolic, $|C_H(g) : \langle g\rangle| < \infty$ [BH99, Corollary 3.10]. Thus, K is virtually cyclic, and so cyclic (as it is torsion-free). However, K is non-cyclic because the map $V_n \to V_n/K$ factors through a non-cyclic free group (because we assumed that the set of relators $\mathbf{r}$ in the presentation for Q was non-empty).

This means that $V_n/K \cong Q$ embeds with finite index into $\mathrm{Out}(G_Q)$, as required. $\square$

4.3.4 Proof of Theorem B

In this section we prove Theorem B, which we now recall.

Theorem B. *Fix an equilateral triangle group $H := T_i$ with $i > 9$. Then every countable group Q can be realised as the outer automorphism group of an HNN-extension G_Q of H. Moreover,* $\mathrm{Aut}(G_Q) \cong H \rtimes Q$.

The construction of the group G_Q is given in the proof of Lemma 4.3.4.

We begin by giving, in Corollary 4.3.3, conditions which make the short exact sequence in Theorem 4.2.15 "collapse". We then define malcharacteristic subgroups, and in Lemma 4.3.4 we use Corollary 4.3.3 to prove that if T_i contains a malcharacteristic subgroup which is free of rank two then Theorem B holds. The remaining lemmata prove that T_i contains such a subgroup, by first proving that the ambient free group $F(a,b)$ contains such a subgroup $\overline{M}$, and then by using this, along with some small cancellation theory, to prove that the image M of $\overline{M}$ in T_i is free of rank two and malcharacteristic.

Applying Theorem 4.2.15. To prove Theorem B, we require the following corollary of Theorem 4.2.15 giving conditions on H, K and ϕ which "collapse" the short exact sequence. The corollary is applied by proving that if $H := T_i$ is an equilateral triangle group with $i > 9$ then for all countable groups Q there exists a subgroup K and an automorphism ϕ such that the triple (H, K, ϕ) satisfies the conditions of Corollary 4.3.3 and such that $N_H(K)/K \cong Q$. Recall that $\widehat{\phi}$ denote the image of $\phi \in \mathrm{Aut}(H)$ in $\mathrm{Out}(H)$.

Corollary 4.3.3. *Suppose H, K and ϕ are such that the following properties hold.*

1. *H has Serre's property FA.*

2. *$C_H(K)$ is trivial.*

3. *$K\psi \cap K = 1$ for all automorphisms $\psi \notin \mathrm{Inn}(H)$.*

4. *$\phi \notin \mathrm{Inn}(H)$.*

5. *$\widehat{\phi}$ has odd order in* $\mathrm{Out}(H)$.

Then $\mathrm{Out}(G) \cong N_H(K)/K$ *and* $\mathrm{Aut}(G) \cong G \rtimes N_H(K)/K$.

Proof. First, note that if $K\psi \cap K = 1$ for all automorphisms $\psi \notin \mathrm{Inn}(H)$ then $\mathrm{Out}_H(G) = \mathrm{Out}^H(G)$, by Theorem 4.2.3 and (4), while if H also has Serre's property FA then $\mathrm{Out}_H(G) = \mathrm{Out}(G)$, by Lemma 4.2.2, so we establish that $\mathrm{Out}_H(G) \cong N_H(K)/K$ and the result on $\mathrm{Out}(G)$ immediately follows. Now, by (2), $Z(H)$ and $C_H(K\phi)$ are both trivial so we have the following short exact sequence.

$$1 \to \frac{N_H(K)}{K} \to \mathrm{Out}^0_H(G) \to \frac{A_K \,\mathrm{Inn}(H)}{\mathrm{Inn}(H)} \to 1$$

By (3), $K\psi \neq K$ for all $\psi \notin \mathrm{Inn}(H)$, and this means that $A_K \leq \mathrm{Inn}(G)$, and so $\mathrm{Out}^0_H(G) \cong N_H(K)/K$. Then, by Lemma 4.2.16, $\mathrm{Aut}^0_H(G) = G \rtimes N_H(K)/K$.

To complete the proof we establish that $\mathrm{Out}^0_H(G) = \mathrm{Out}_H(G)$. To do this it is sufficient to prove that there does not exist a pair (δ, g) with $\delta \in \mathrm{Aut}(H)$ and $g \in H$ such that $K\delta = K\phi$, $K\delta^2\gamma_g = K$ and $g^{-1}(k\phi\delta)g = k\delta\phi^{-1}$ for all $k \in K$. Suppose otherwise, then there exists some pair (δ, g) such that $K\delta\phi^{-1} = K = K\delta^2\gamma_g$. Now, $K\delta^2\gamma_g = K$ implies that $\delta^2 \in \mathrm{Inn}(H)$, by (3), and so either $\delta \in \mathrm{Inn}(H)$ or $\widehat{\delta}$ has order two in $\mathrm{Out}(H)$. On the other hand, $K\delta\phi^{-1} = K$ implies that $\widehat{\delta} = \widehat{\phi}$, again by (3), and so $\widehat{\delta}$ has odd order in $\mathrm{Out}(H)$ by (5). Therefore, $\delta \in \mathrm{Inn}(H)$, and so as $\widehat{\delta} = \widehat{\phi}$ we have that $\phi \in \mathrm{Inn}(H)$, which contradicts (4). Therefore, such a pair (δ, g) cannot exist, and so $\mathrm{Out}^0_H(G) = \mathrm{Out}_H(G)$, which proves the result. □

In Lemma 4.3.4, below, we prove that Corollary 4.3.3 combined with the existence of a subgroup M of T_i which is free of rank two and "malcharacteristic" in T_i proves Theorem B.

Malcharacteristic subgroups. A subgroup $M \leq H$ is *malnormal* in H if $M^g \cap M = 1$ for all $g \notin M$. We define a subgroup $M \leq H$ to be *malcharacteristic* in H if M is malnormal in H and for all $\delta \in \mathrm{Aut}(H)$ the following implication holds.

$$M\delta \cap M \neq 1 \Rightarrow \delta \in \mathrm{Inn}(H)$$

We now make two observations which are central to the construction given in Lemma 4.3.4, which is the construction underlying Theorem B. Firstly, note that if M is malcharacteristic in H and M' is malnormal in M then M' is malcharacteristic in H. This means that if H contains a malcharacteristic subgroup M which is free of rank at least two then H contains a malcharacteristic subgroup M_n which is free of rank n where n is arbitrary (possibly countably infinite). Secondly, note that if M is malcharacteristic in H and K is a normal subgroup of M then $N_H(K) = M$.

We shall now prove the following lemma, which reduces the proof of Theorem B to proving the existence of a malcharacteristic subgroup of T_i which is free of rank two.

Lemma 4.3.4. *Let $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$ be a equilateral triangle group. If for all $i > 9$ the group T_i contains a malcharacteristic subgroup which is free of rank two then Theorem B holds.*

Proof. Fix $i > 9$ and take $H := T_i$. Suppose M is a malcharacteristic subgroup of H which is free of rank two. We shall prove that for every countable group Q there exists an automorphism-induced HNN-extension of H, $G_Q = \langle H, t; K^t = K\phi \rangle$ say, such that $\mathrm{Out}(G_Q) \cong Q$ and $\mathrm{Aut(G_Q)} \cong G_Q \rtimes Q$.

We shall take the inducing automorphism ϕ to be the following automorphism of H of order three, $\phi : a \mapsto b$, $b \mapsto b^{-1}a^{-1}$. We choose K as follows: The group Q has a presentation $\langle X; \mathbf{r} \rangle$ where $\mathbf{r}$ is non-empty and $|X| > 1$ (possibly countably infinite). Choose $M_{|X|}$ to be malnormal in M of rank $|X|$ and take K to be the normal subgroup of $M_{|X|}$ associated with the normal closure of $\mathbf{r}$. Note that as $N_H(K) = M_{|X|}$ we have that $N_H(K)/K \cong Q$.

To see that the construction works we just need to prove that H, K and ϕ satisfy the conditions of Corollary 4.3.3, as $N_H(K)/K \cong Q$. So:

1. H has Serre's property FA because it is a triangle group.

2. $C_H(K)$ is trivial. To see this, recall that K is a normal subgroup of $M_{|X|}$, the malnormal free subgroup of rank $|X| > 1$ of H. Now, if $[k, g] = 1$ for all $k \in K$ and $1 \neq g \in H$ then $g \in M_{|X|}$, by malnormality, and so $C_H(K)$ is a subgroup of $M_{|X|}$. As $M_{|X|}$ is free we have that $g \in K$ and K is cyclic, but normal subgroups of non-cyclic free groups can never be cyclic. Thus, $C_H(K)$ is trivial, as required.

3. $K\psi \cap K = 1$ for all automorphisms $\psi \notin \mathrm{Inn}(H)$ because K is a subgroup of the malcharacteristic subgroup M.

4. $\phi \notin \mathrm{Inn}(H)$ by our choice of ϕ.

5. $\widehat{\phi}$ has order three in $\mathrm{Out}(H)$.

Therefore, we can apply Corollary 4.3.3 to get that Theorem B holds, as required. $\square$

Note that because the subgroup K is free the presentation in the construction is aspherical, and so minimal [CCH81] (we thank Jim Howie for this observation). Thus, the group G_Q in the construction is finitely presented if and only if $Q \cong \mathrm{Out}(G_Q)$ is finite.

The malcharacteristic free subgroup. We now work towards proving Lemma 4.3.13, which proves that $T_i = \langle a, b; a^i, b^i, (ab)^i\rangle$, $i > 9$, contains a malcharacteristic subgroup which is free of rank two. Applying Lemma 4.3.4, this proves Theorem B. Indeed, let M be the subgroup of $T_i = \langle a, b; a^i, b^i, (ab)^i\rangle$ which is generated by the following elements, with $\rho >> i$.

$$\begin{aligned} x &:= a^3(ab^{-1})^3a^3(ab^{-1})^4 \dots a^3(ab^{-1})^{\rho+2} \\ y &:= a^3(ab^{-1})^{\rho+3} \dots a^3(ab^{-1})^{2\rho+2} \end{aligned}$$

We shall prove that when $i > 9$ the subgroup $M = \langle x, y\rangle$ of T_i is free of rank two and malcharacteristic. Recall that Malcharacteristic subgroups are malnormal, and in order to prove that M is malcharacteristic we shall first prove that it is malnormal. To do this we need to understand how a word $U(a, b)$ can be conjugate to a word $V(x, y)$, and the following lemma aids our understanding of this. A word $U(a, b)$ is *Dehn reduced* in $T_i = \langle a, b; a^i, b^i, (ab)^i\rangle$ if it contains no more than half of $a^{\pm i}$, $b^{\pm i}$, $(ab)^{\pm i}$ or $(ba)^{\pm i}$ as a subword. Recall that x and y are specific words over a and b, defined above.

Lemma 4.3.5. *Take $T_i = \langle a, b; a^i, b^i, (ab)^i\rangle$ with $i > 9$. Suppose that $U(a, b)$ and $W(a, b)$ are Dehn reduced words such that there exists a word $V(x, y)$ such that the following holds.*

$$1 \not\equiv U(a, b)W(a, b)V^{-1}(x, y)W^{-1}(a, b) =_{T_i} 1$$

Then $U(a, b)$ contains one of $a^{\pm 6}$, $b^{\pm 6}$, $(ab)^{\pm 5}$ or $(ba)^{\pm 5}$ as a subword.

Figure 4.2 serves as a companion to the proof of this lemma.

Proof. Note that no cyclic shift of $V(x, y)$ in $F(a, b)$ contains the subwords $a^{\pm 5}$, $b^{\pm 5}$ or $(abab)^{\pm 1}$, that is, when $V(x, y)$ is considered as a word over a and b and written on a circle then the boundary label of the circle does not contain these subwords.

Suppose that $U(a, b)W(a, b)V^{-1}(x, y)W^{-1}(a, b) =_{T_i} 1$ but that this word is not the empty word. This means that there exists some annular diagram A where $U(a, b)$ is a label of the interior boundary ∂I, $V(x, y)$ is a label of the exterior boundary ∂E, and $W(a, b)$ connects the two labels. Now, recall that T_i is given by a $C'(1/8)$ presentation,

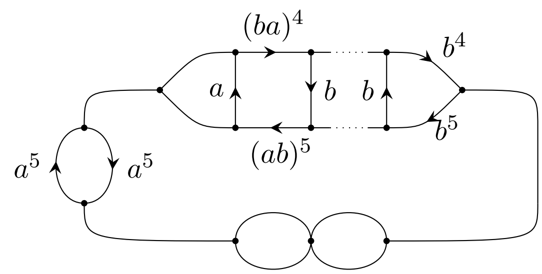

Figure 4.2: Suppose, for example, $H = \langle a, b; a^{10}, b^{10}, (ab)^{10}\rangle$. Any annular diagram has the above form. This example has four islands. Note that no label of either the interior or exterior boundaries can be a word over x and y.

and so by the structural theorem for annular diagrams (Proposition 2.3.6) the diagram A is split into *islands*, which are reduced subdiagrams of A whose boundary is not self-intersecting and of the form $\sigma\eta$ with $\sigma \subset \partial I$ and $\eta \subset \partial E$, and *bridges*, which are non-trivial paths in $\partial I \cap \partial E$. We shall analyse the islands of A and this analysis shall yield the required subwords.

Suppose A contains a diagram D with boundary label $(ab)^i$. Now, $V(x, y)$ does not contain $(abab)^{\pm 1}$ and so neither does $\partial D \cap \partial E$. This means that $|D \cap \partial E| \leq 3$. Now, the pieces of D have length one and D borders no more than two diagrams. Therefore, $\partial D \cap \partial I$ has length at least $|ab| \cdot i - 3 - 2 = 2i - 5 > i$, so we conclude that either $(ab)^{\pm 5}$ or $(ba)^{\pm 5}$ is a subword of $\partial D \cap \partial I$, and so is contained in $U(a, b)$.

Suppose A contains an island A' such that A' has more than one subdiagram. Then one of these subdiagrams D has boundary $\partial D = (ab)^i$ as otherwise A is not reduced. Therefore, by the above working, we have that either $(ab)^{\pm 5}$ or $(ba)^{\pm 5}$ is a subword of $U(a, b)$.

The only cases left to consider are islands consisting of a single diagram with boundary $a^{\pm i}$ or $b^{\pm i}$. So, suppose D is an island of A with boundary label c^i where $c \in \{a^{\pm 1}, b^{\pm 1}\}$. Now, the highest power of c contained in ∂E is $c^{\pm 4}$, and as $i > 9$ we conclude that ∂I contains $c^{\pm 6}$. This completes the proof of the lemma. $\square$

To prove that $M = \langle x, y\rangle$ is free, and to begin the proof that M is malcharacteristic in T_i, we use the following lemma, Lemma 4.3.6. The proof uses the above lemma, Lemma 4.3.5, to prove that M is malnormal in T_i. Taking U and V to be words over the alphabet $X^{\pm 1}$, we shall write $\langle U, V\rangle_F$ to mean the subgroup of $F(X)$ generated by the words U and V, and $\langle U, V\rangle_H$ to mean the corresponding subgroup of $H = \langle X; \mathbf{r}\rangle$. Recall that a word $U(a,b)$ is Dehn reduced in $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$ if it contains no more than half of $a^{\pm i}$, $b^{\pm i}$, $(ab)^{\pm i}$ or $(ba)^{\pm i}$ as a subword.

Lemma 4.3.6. *Let M be the subgroup of $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$, $i > 9$, which is generated by the following elements, with $\rho >> i$.*

$$x := a^3(ab^{-1})^3a^3(ab^{-1})^4 \dots a^3(ab^{-1})^{\rho+2}$$
$$y := a^3(ab^{-1})^{\rho+3} \dots a^3(ab^{-1})^{2\rho+2}$$

Then M is free of rank two. Moreover, M is a malnormal subgroup of T_i.

Proof. We shall write $H := T_i$. Note that, because $i > 9$, H is given by a small cancellation $C'(1/8)$ presentation. Suppose that $M = \langle x, y \rangle_H$ is not free of rank two, then there exists some freely reduced, non-empty word $U(x,y)$ such that $U(x,y) =_H 1$. However, every word over x and y is Dehn reduced, and so no such word U exists by Greendlinger's lemma (Proposition 2.3.4). Therefore, M is free of rank two.

To prove malnormality, begin by noting that the lift of M to $F(a,b)$, denoted $\overline{M} := \langle x, y \rangle_F$, is a malnormal subgroup of $F(a,b)$. This holds because as $\rho >> i$ the generators x, y satisfy Wise's $c(5)$ small cancellation condition and so $\overline{M}$ is malnormal, by Proposition 2.10.5. Now, recall that x and y are words over a and b, and suppose $U(x,y)$, $V(x,y)$ and $W(a,b)$ are such that $U(x,y)W(a,b)V^{-1}(x,y)W^{-1}(a,b) =_H 1$ and $W(a,b)$ is Dehn reduced. Then because neither $U(x,y)$ nor $V(x,y)$ contains $a^{\pm 6}$, $b^{\pm 6}$, $(ab)^{\pm 5}$ or $(ba)^{\pm 5}$, we can apply Lemma 4.3.5 to get that $U(x,y)W(a,b)V^{-1}(x,y)W^{-1}(a,b) \equiv 1$. This then proves the lemma because as $\overline{M} = \langle x, y \rangle_F$ is malnormal in $F(a,b)$ we have that $W \in \langle x, y \rangle_F$, so $W \in \langle x, y \rangle_H$, as required. □

The automorphic orbit of M. Recall that, by Lemma 4.3.4, to prove Theorem B it is sufficient to prove that the candidate subgroup $M = \langle x, y \rangle$ is free of rank two and is malcharacteristic in $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$ for $i > 9$. Now, Lemma 4.3.6 proves that it is free of rank two and malnormal in T_i, and so to prove Theorem B it is sufficient to prove that if $\phi \in \mathrm{Aut}(T_i)$ such that $M \cap M\phi \neq 1$ then $\phi \in \mathrm{Inn}(T_i)$. Our approach is based on the following transversal for $\mathrm{Out}(T_i)$, denoted Ψ [Zie76].

$$\phi_{(1,\epsilon)}: \begin{array}{l} a \mapsto a^{\epsilon} \\ b \mapsto b^{\epsilon} \end{array} \qquad \phi_{(2,\epsilon)}: \begin{array}{l} a \mapsto b^{\epsilon} \\ b \mapsto a^{\epsilon} \end{array} \qquad \phi_{(3,\epsilon)}: \begin{array}{l} a \mapsto (ab)^{\epsilon} \\ b \mapsto a^{-\epsilon} \end{array}$$

$$\phi_{(4,\epsilon)}: \begin{array}{l} a \mapsto a^{\epsilon} \\ b \mapsto (ab)^{-\epsilon} \end{array} \qquad \phi_{(5,\epsilon)}: \begin{array}{l} a \mapsto b^{\epsilon} \\ b \mapsto (ab)^{-\epsilon} \end{array} \qquad \phi_{(6,\epsilon)}: \begin{array}{l} a \mapsto (ab)^{\epsilon} \\ b \mapsto b^{-\epsilon} \end{array}$$

Now, to prove that if $M \cap M\phi \neq 1$ then $\phi \in \mathrm{Inn}(T_i)$ we prove, in Lemma 4.3.9, that for $\phi = \phi_{(l,\epsilon)} \in \Psi$, if $U(x,y)W(a,b)V^{-1}(x\phi, y\phi)W^{-1}(a,b)$ represents the identity in T_i then this word is the empty word. Then, because every automorphism of T_i lifts to an automorphism of the ambient free group $F(a,b)$, in order to prove that M is malcharacteristic in T_i it is sufficient to prove that the lift $\overline{M}$ of M to $F(a,b)$ is malcharacteristic in $F(a,b)$, and we do this in Lemma 4.3.12.

In order to prove that if $U(x,y)W(a,b)V^{-1}(x\phi, y\phi)W^{-1}(a,b)$ represents the identity in T_i for $\phi \in \Psi$ then this word is the empty word, we wish to understand the image $U(x\phi, y\phi)$ of $U(x,y)$ under such an automorphism $\phi = \phi_{(l,\epsilon)} \in \Psi$ from the transversal. Lemma 4.3.7 and Lemma 4.3.8, below, gives us the information we require.

Lemma 4.3.7. *Let $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$, with $i > 9$. Suppose $\phi := \phi_{(l,\epsilon)}$ is contained in the transversal Ψ for $\mathrm{Out}(T_i)$, and we shall write $A := a\phi$, $B := (a\phi)(b^{-1}\phi)$. Then for all $j \in \mathbb{Z}$ with $|j| \geq 3$ the word $B^j A^3 B^{-j}$ does not contain $a^{\pm 4}$, $b^{\pm 4}$, $(ab)^{\pm 4}$ or $(ba)^{\pm 4}$, and so is Dehn reduced.*

Proof. The proof is simply by inspection of the appropriate words. Indeed the following words are all Dehn reduced for $j \in \mathbb{Z}$ with $|j| \geq 3$ and the longest subwords of relators occurring are a^3, b^3, $(ab)^3$, and $(ba)^3$. Each word represents $B^j A^3 B^{-j}$ for the indicated $\phi \in \Psi$.

$$\begin{array}{llll}
\phi_{(1,1)}: & (ab^{-1})^j a^3 (ab^{-1})^{-j} & = \ldots ab^{-1}ab^{-1}a^3ba^{-1}ba^{-1}\ldots & j>0 \\
& \text{or} & = \ldots ba^{-1}ba^3b^{-1}ab^{-1}\ldots & j<0 \\
\phi_{(2,1)}: & (ba^{-1})^j b^3 (ba^{-1})^{-j} & = \ldots ba^{-1}ba^{-1}b^3ab^{-1}ab^{-1}\ldots & j>0 \\
& \text{or} & = \ldots ab^{-1}ab^3a^{-1}ba^{-1}\ldots & j<0 \\
\phi_{(3,1)}: & (aba)^j (ab)^3 (aba)^{-j} & = \ldots aba^2ba(ab)^3a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}\ldots & j>0 \\
& \text{or} & = \ldots a^{-1}b^{-1}a^{-1}(ba)^3aba\ldots & j<0 \\
\phi_{(4,1)}: & (aba)^j a^3 (aba)^{-j} & = \ldots aba^2ba^3b^{-1}a^{-2}b^{-1}a^{-1}\ldots & j>0 \\
& \text{or} & = \ldots a^{-1}b^{-1}a^{-2}b^{-1}a^3ba^2ba\ldots & j<0 \\
\phi_{(5,1)}: & (b^2a)^j b^3 (b^2a)^{-j} & = \ldots b^2ab^2ab^3a^{-1}b^{-2}a^{-1}b^{-2}\ldots & j>0 \\
& \text{or} & = \ldots a^{-1}b^{-2}a^{-1}b^3ab^2a\ldots & j<0 \\
\phi_{(6,1)}: & (ab^2)^j (ab)^3 (ab^2)^{-j} & = \ldots ab^2ab(ba)^3b^{-1}a^{-1}b^{-2}a^{-1}\ldots & j>0 \\
& \text{or} & = \ldots b^{-2}a^{-1}b^{-1}(ab)^3bab^2\ldots & j<0
\end{array}$$

Noting that if the result holds for $\phi_{(l,1)}$ then it holds for $\phi_{(l,-1)}$, the proof is complete. $\square$

Lemma 4.3.8. *Let $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$, with $i > 9$. Suppose $\phi := \phi_{(l,\epsilon)}$ is contained in the transversal Ψ for $\mathrm{Out}(T_i)$, and we shall write $A := a\phi$, $B := (a\phi)(b^{-1}\phi)$. Then for all $j, k \geq 3$ the word $A^3B^jA^3B^k$ is freely reduced and does not contain $a^{\pm 6}$, $b^{\pm 6}$, $(ab)^{\pm 5}$ or $(ba)^{\pm 5}$, and so is Dehn reduced.*

Proof. As with Lemma 4.3.7, the proof is simply by inspection of the appropriate words. Indeed the following words are all freely reduced for $j, k \geq 3$ and the longest subwords of relators occurring are a^5, b^5, $(ab)^4a$, and $b(ab)^4$. Each word represents $A^3B^jA^3B^k$ for the indicated $\phi \in \Psi$.

$$\begin{array}{lr}
\phi_{(1,1)}: & a^3(ab^{-1})^ja^3(ab^{-1})^k \\
\phi_{(2,1)}: & b^3(ba^{-1})^jb^3(ba^{-1})^k \\
\phi_{(3,1)}: & (ab)^3(aba)^j(ab)^3(aba)^k \\
\phi_{(4,1)}: & a^3(aba)^ja^3(aba)^k \\
\phi_{(5,1)}: & b^3(b^2a)^jb^3(b^2a)^k \\
\phi_{(6,1)}: & (ab)^3(ab^2)^j(ab)^3(ab^2)^k
\end{array}$$

Noting that if the result holds for $\phi_{(l,1)}$ then it holds for $\phi_{(l,-1)}$, so the proof is complete. $\square$

We shall now prove the implication which reduces the proof that $M = \langle x, y\rangle$ is malcharacteristic in $T_i = \langle a, b; a^i, b^i, (ab)^i\rangle$ to proving that the lift $\overline{M}$ of M to $F(a, b)$ is malcharacteristic in $F(a, b)$. the proof is based on the above two lemmata, Lemma 4.3.7 and Lemma 4.3.8, and on Lemma 4.3.5, which deals with conjugacy. Recall that a word $U(a, b)$ is Dehn reduced in $T_i = \langle a, b; a^i, b^i, (ab)^i\rangle$ if it contains no more than half of $a^{\pm i}$, $b^{\pm i}$, $(ab)^{\pm i}$ or $(ba)^{\pm i}$ as a subword.

Lemma 4.3.9. *Let $\phi := \phi_{(l,\epsilon)}$ be an element of the transversal Ψ for* $\mathrm{Out}(T_i)$*. Suppose that the following holds, where $W(a, b)$ is Dehn reduced.*

$$U(x, y)W(a, b)V^{-1}(x\phi, y\phi)W^{-1}(a, b) =_{T_i} 1$$

Then $U(x, y)W(a, b)V^{-1}(x\phi, y\phi)W^{-1}(a, b)$ is the empty word in $F(a, b)$.

Proof. Recall that x and y are specific words over a and b, defined before Lemma 4.3.5, and so $x = X(a, b)$ and $y = Y(a, b)$. Then by $x\phi$ and $y\phi$ we mean, respectively, the words $X(a\phi, b\phi)$ and $Y(a\phi, b\phi)$.

We first prove that if $\phi := \phi_{(l,\epsilon)}$ is one of the elements of the transversal Ψ for $\mathrm{Out}(T_i)$ then after free reduction $V(x\phi, y\phi)$ does not contain any of the words $a^{\pm 6}$, $b^{\pm 6}$, $(ab)^{\pm 5}$ or $(ba)^{\pm 5}$, and so is Dehn reduced (note that $U(x, y)$ is Dehn reduced). We then use this along with Lemma 4.3.5 to prove the implication in the statement of the theorem. We shall call the potential subwords $a^{\pm 6}$, $b^{\pm 6}$, $(ab)^{\pm 5}$ and $(ba)^{\pm 5}$ *illegal subwords.*

To prove that if $\phi \in \Psi$ then $V(x\phi, y\phi)$ does not contain any illegal subwords for all freely reduced words V, begin by noting that if $A := a\phi$ and $B := (a\phi)(b^{-1}\phi)$ then $A^3B^jA^3B^{j+1}$ is freely reduced and does not contain any illegal subwords for all $j > 3$, by Lemma 4.3.8. This implies that both $X(a\phi, b\phi) = x\phi$ and $Y(a\phi, b\phi) = y\phi$ are freely reduced and do not contain any illegal subwords, as they are both built up from overlapping words of the form $A^3B^jA^3B^{j+1}$. Now, because $X(a\phi, b\phi) = x\phi$ and $Y(a\phi, b\phi) = y\phi$ are both freely reduced, when we form the word $(x\phi)(y\phi)$ any free cancellation corresponds to free cancellation in the product xy. Then, as $\rho >> i$ the words x, y are small cancellation $C'(1/6)$-words, and so when we form any product PQ where $P, Q \in \{x^{\pm 1}\phi, y^{\pm 1}\phi\}$ we obtain a word P_0Q_0 such that P_0 is a subword of P with $|P_0| > \frac{5}{6}|P|$, and Q_0 is a subword of Q with $|Q_0| > \frac{5}{6}|Q|$. Therefore, the word $V(x\phi, y\phi)$ has the following form, where each P_j is of length $|P_j| > \frac{2}{3}\min\{|x\phi|, |y\phi|\}$ and does not contain any illegal subwords.

$$V(x\phi, y\phi) = P_1P_2 \ldots P_n$$

Now, because each P_j does not contain any illegal subwords, and because each P_j is sufficiently long (as $\rho >> i$), any illegal subwords must bridge two of these P_j terms, that is, if $V(x\phi, y\phi)$ contains an illegal subword then there exists some j such that P_jP_{j+1} contains an illegal subword, but both P_j and P_{j+1} do not contain any illegal subwords. We shall now prove that P_jP_{j+1} cannot contain any illegal subword, and so $V(x, y)$ contains no illegal subwords. Now, again writing $A := a\phi$ and $B := (a\phi)(b^{-1}\phi)$ we see that if P_jP_{j+1} contains an illegal subword then one of the following products contains an illegal subword.

$$A^3B^{\rho+2} \cdot A^3B^{\rho+3} \tag{4.12}$$

$$B^{\rho+1}A^3B^{\rho+2} \cdot B^{-2\rho-2} = B^{\rho+1}A^3B^{-\rho+2} \tag{4.13}$$

$$B^{-4}A^{-3}B^{-3}A^{-3} \cdot A^3B^{\rho+2} = B^{-4}A^{-3}B^{\rho-1} \tag{4.14}$$

$$B^{-3}A^{-3} \cdot B^{-2\rho-2}A^{-3} \tag{4.15}$$

Now, (4.12) and (4.15) do not contain any illegal subwords by Lemma 4.3.8, while (4.13) and (4.14) do not contain any illegal subwords by Lemma 4.3.7. Thus, we conclude that $V(x\phi, y\phi)$ does not contain any illegal subwords for $\phi := \phi_{(l,\epsilon)}$ an arbitrary element of the transversal Ψ for $\mathrm{Out}(T_i)$, as required.

Finally, suppose that $U(x, y)W(a, b)V^{-1}(x\phi, y\phi)W^{-1}(a, b) =_{T_i} 1$. If this word is not the empty word then, by Lemma 4.3.5, we have that $V(x\phi, y\phi)$ contains an illegal subword. However, we have already established that no such subword exists, and so we conclude that $U(x, y)W(a, b)V^{-1}(x\phi, y\phi)W^{-1}(a, b)$ is the empty word in $F(a, b)$, as required. $\square$

Malcharacteristic subgroups of free groups. In Lemma 4.3.13 we use Lemma 4.3.9 to reduce the proof that $M = \langle x, y\rangle$ is malcharacteristic in $T_i = \langle a, b; a^i, b^i, (ab)^i\rangle$ to proving that the lift $\overline{M}$ of M to $F(a, b)$ is malcharacteristic. Now, in Lemma 4.3.11 we use fibre products of maps of graphs to obtain an algorithm which determines whether certain subgroups of $F(a, b)$ are malcharacteristic. This allows us to prove, in Lemma 4.3.12, that the lift $\overline{M}$ of M to $F(a, b)$ is malcharacteristic in $F(a, b)$. We then use this in Lemma 4.3.13 to prove that M is malcharacteristic in T_i, which proves Theorem B. The following lemma allows us to obtain the algorithm of Lemma 4.3.11. We shall call an automorphism $\psi \in \mathrm{Aut}(F(a, b))$ *length-preserving* if $|a\psi| = 1 = |b\psi|$. We shall call a word W *fully cyclically reduced* if either $|W| = 1$ or W is both freely and cyclically reduced and begins and ends with different letters. Note that a cyclically reduced word W' is always a cyclic shift of a fully cyclically reduced word.

Lemma 4.3.10. *Suppose that ϕ is an automorphism of $F(a,b)$ which is not inner and not length-preserving. Then there exists an inner automorphism γ such that both of the words $a\phi\gamma$ and $b\phi\gamma$ are fully cyclically reduced and either both do not contain $a^{\pm 3}$ or both do not contain $b^{\pm 3}$.*

Proof. As ϕ is not length preserving, we have that either $|a\phi| > 1$ or $|b\phi| > 1$. Begin by noting that if A and B are a primitive pair (so either $a\phi = A$ and $b\phi = B$ or $a\phi = B$ and $b\phi = A$ for some Nielsen transformation ϕ) then either there exists an inner automorphism γ and some length-preserving automorphism ψ such that $A\gamma\psi = a$ and $B\gamma\psi = ba^k$ for some $k \in \mathbb{Z}$, or there exists inner automorphisms γ_1 and γ_2 and some length-preserving automorphism ψ and $\epsilon \in \{1, -1\}$ such that $A^\epsilon\gamma_1\psi$ and $B^\epsilon\gamma_2\psi$ have the following forms, where $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in \{k, k+1\}$ for some fixed k [CMZ81].

$$A^\epsilon\gamma_1\psi = a^{\alpha_1}ba^{\alpha_2}b\dots a^{\alpha_m}b$$
$$B^\epsilon\gamma_2\psi = a^{\beta_1}b^{\epsilon'}a^{\beta_2}b^{\epsilon'}\dots a^{\beta_n}b^{\epsilon'}$$

Therefore, to prove the theorem it is sufficient to prove that there is an inner automorphism γ such that $A\gamma$ and $B\gamma$ are fully cyclically reduced, as then $A^\epsilon\gamma\psi$ and $B^\epsilon\gamma\psi$ will have the above forms so $A\gamma$ and $B\gamma$ do not contain $a^{\pm 3}$ or do not contain $b^{\pm 3}$. To prove that such an inner automorphism exists, consider the following three automorphisms of $F(a,b)$.

$$\psi_0 : a \mapsto ab \qquad \psi_1 : a \mapsto b \qquad \psi_2 : a \mapsto a^{-1}$$
$$b \mapsto b \qquad b \mapsto a \qquad b \mapsto b$$

These three automorphisms generate the automorphism group of $F(a,b)$, because $\mathrm{Aut}(F(a,b))$ is generated by the elementary Nielsen transformations [MKS04, Theorem 3.2] and every elementary Nielsen transformation can be written in terms of these three generators (see Section 2.2 for the definition of an elementary Nielsen transformation). We shall write ϕ as a word over these three generators, $\phi = \phi_N \dots \phi_2\phi_1$ where $\phi_j^\epsilon \in \{\psi_0, \psi_1, \psi_2\}$, and induct on the length of the word, so induct on the number N.

If $N = 0$ the result holds. Suppose $a\phi\gamma$ and $b\phi\gamma$ are fully cyclically reduced and consider $\phi_{N+1}\phi$ where ϕ_{N+1} is an elementary Nielsen transformation. If $\phi_{N+1} = \psi_1$ then $a\phi_N\phi = b\phi$ and $b\phi_N\phi = a\phi$, while if $\phi_{N+1} = \psi_2$ then $a\phi_N\phi = (a\phi)^{-1}$ and $a\phi_N\phi = b\phi$, and in both cases $a\phi_{N+1}\phi\gamma$ and $b\phi_{N+1}\phi\gamma$ are fully cyclically reduced, as required. So it suffices to prove the result for $\phi_{N+1} = \psi_0$. So, writing $a\phi = A$, $b\phi = B$ and $a\psi_0\phi = AB$, $b\psi_0\phi = B$, and so to prove the result it suffices to prove that if $W^{-1}AW$ and $W^{-1}BW$ are fully cyclically reduced then there exists some word W_0 such that $W_0^{-1}ABW_0$ and $W_0^{-1}BW_0$ are fully cyclically reduced, and indeed we can assume W is trivial (so A and B are fully cyclically reduced). If AB is not fully cyclically reduced then there exists a word C such that $A = CA_1$ and $B = B_1C^{-1}$ are freely reduced words such that A_1B_1 is cyclically reduced. If $C^{-1}B_1$ is not fully cyclically reduced then consider $C^{-1}B_2c^{-j}$, where $|c| = 1$, $j \in \mathbb{Z}$ and $c^{-j}C^{-1}B_2$ is fully cyclically reduced. Then take $W_0 = Cc^j$. Now, $W_0^{-1}BW_0 = c^{-j}C^{-1}B_2$ is fully cyclically reduced while $W_0^{-1}ABW_0 = c^{-j}A_1B_2$ is fully cyclically reduced as $c^{-j}B_2$ is fully cyclically reduced and because $A_1B_2c^{-j}$ is cyclically reduced (so this cyclic shift induces no free reduction), as required. □

We now give conditions which allow for an algorithm to determine if a subgroup of the free group on two generators is a malcharacteristic subgroup or not. Note that the conditions imply the existence of an algorithm because there are only finitely many length-preserving automorphisms, and because malnormality is decidable in free groups. In Lemma 4.3.12, we apply this result to prove that the subgroup $\overline{M}$ is a malcharacteristic subgroup of $F(a, b)$. By a positive word in $F(a, b)$ we mean a word over $\{a, b\}$ not containing a^{-1} or b^{-1}.

Lemma 4.3.11. *Let $W_1, \ldots, W_n$ be a finite collection of freely reduced positive words in the free group $F(a, b)$, and write $C := \langle W_1, \ldots, W_n\rangle$. Suppose the following hold:*

- *Every circuit in the folded Stalling's graph $\widehat{\Gamma}_C$ contains some a-term and some b-term.*
- *If $z \in \{a, b\}$ then every instance of z in any word W_i is part of a z^3 term.*

Then C is malcharacteristic in $F(a, b)$ if and only if C is malnormal and for all non-trivial $\psi \in \mathrm{Aut}(F(a, b))$ such that $|a\psi| = 1 = |b\psi|$ the fibre product $\Gamma_C \otimes_{\Gamma_{F(x,y)}} \Gamma_{C\psi}$ is a forest.

Proof. We begin by noting that these conditions imply that every instance of some $z \in \{a, b\}$ in any circuit of $\widehat{\Gamma}_C$ is part of a z^3.

Now, if $\psi \in \mathrm{Aut}(F(a, b))$ then by Lemma 4.3.10 we have that, modulo the inner automorphisms, either $|a\psi| = 1 = |b\psi|$ or no positive word over $a\psi$ and $b\psi$ contains $a^{\pm 3}$, or no positive word over $a\psi$ and $b\psi$ contains $b^{\pm 3}$. Note that inner automorphisms leave the Stallings graph unchanged after folding. Thus, if we assume that either $|a\psi| > 1$ or $|b\psi| > 1$ then either no $a^{\pm 3}$ or no $b^{\pm 3}$ can appear in $\widehat{\Gamma}_{C\psi}$ and so $\widehat{\Gamma}_C \otimes \widehat{\Gamma}_{C\psi}$ is a forest.

Therefore, to verify that C is malcharacteristic we only need to verify that C is malnormal and that $\widehat{\Gamma}_C \otimes \widehat{\Gamma}_{C\psi}$ is a forest for the non-trivial automorphisms ψ such that $|a\psi| = 1 = |b\psi|$. Thus, the proof is complete. □

The following lemma, Lemma 4.3.12, proves that the subgroup $\overline{M}$ is malcharacteristic in $F(a, b)$, which is used in Lemma 4.3.13 to prove that M is malcharacteristic in T_i. Theorem B then follows.

Lemma 4.3.12. *Let $\overline{M}$ be the subgroup of $F(a, b)$ which is generated by the following elements, with $\rho >> 1$.*

$$x := a^3(ab^{-1})^3 a^3(ab^{-1})^4 \dots a^3(ab^{-1})^{\rho+2}$$
$$y := a^3(ab^{-1})^{\rho+3} \dots a^3(ab^{-1})^{2\rho+2}$$

Then $\overline{M}$ is a malcharacteristic subgroup of $F(a, b)$.

Proof. We shall write $\overline{M}_0$ for the subgroup of $F(a, b)$ which is generated by the following elements.

$$x_0 := a^3b^3a^3b^4 \dots a^3b^{\rho+2}$$
$$y_0 := a^3b^{\rho+3} \dots a^3b^{2\rho+2}$$

As $\overline{M}_0 = \langle x_0, y_0\rangle$ is in the automorphic orbit of $\overline{M} = \langle x, y\rangle$, we have that $\overline{M}$ is malcharacteristic in $F(a, b)$ if and only if $\overline{M}_0$ is malcharacteristic in $F(a, b)$. We shall prove that $\overline{M}_0$ is malcharacteristic in $F(a, b)$, which therefore proves the lemma.

Now, $\overline{M}_0$ is malnormal in $F(a, b)$ because these words satisfy Wise's $c(5)$ small cancellation condition for $\rho >> 1$ (by Proposition 2.10.5). To prove that $\overline{M}_0 = \langle x_0, y_0\rangle$ is malcharacteristic suppose ψ is non-trivial such that $|a\psi| = 1 = |b\psi|$. Then any word over $x_0\psi$ and $y_0\psi$ either does not include a b^5-term or does not include a subword of the form $(a^3b^pa^{\epsilon 3}b^q)^{\pm 1}$ where $p, q > 0$. This means that the fibre product $\Gamma_{\overline{M}_0} \otimes_{\Gamma_{F(x,y)}} \Gamma_{\overline{M}_0\psi}$ is a forest for each non-trivial length-preserving automorphism ψ. We can then apply Lemma 4.3.11 to get that $\overline{M}_0$ is malcharacteristic. □

Proof of Theorem B. We shall now prove Theorem B. Indeed, by Lemma 4.3.4, it is sufficient to prove that $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$ contains a malcharacteristic subgroup which is free of rank two. The following lemma proves the existence of such a subgroup. Recall that $\widehat{\phi}$ denote the image of $\phi \in \mathrm{Aut}(H)$ in $\mathrm{Out}(H)$.

Lemma 4.3.13. *Let M be the subgroup of $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$, $i > 9$, which is generated by the following elements, with $\rho >> i$.*

$$x := a^3(ab^{-1})^3 a^3(ab^{-1})^4 \ldots a^3(ab^{-1})^{\rho+2}$$
$$y := a^3(ab^{-1})^{\rho+3} \ldots a^3(ab^{-1})^{2\rho+2}$$

Then M is free of rank two. Moreover, M is a malcharacteristic subgroup of T_i.

Proof. By Lemma 4.3.6, M is free of rank two and is a malnormal subgroup of T_i. Therefore, to prove the lemma it is sufficient to prove that if $\psi \in \mathrm{Aut}(T_i)$ such that $M\psi \cap M \neq 1$ then $\psi \in \mathrm{Inn}(T_i)$. So, suppose that there exists an automorphism ψ of T_i, $\psi \in \mathrm{Aut}(T_i)$, and two words U and V such that $U(x, y) = V(x\psi, y\psi)$, and we wish to prove that ψ is inner. Now, $\widehat{\psi} \in \mathrm{Out}(T_i)$ has representative $\phi := \phi_{(l,\epsilon)}$ from the transversal Ψ for $\mathrm{Out}(T_i)$, that is, $\psi = \phi\gamma_W$ where $\gamma_W \in \mathrm{Inn}(T_i)$ and $W(a, b)$ is Dehn reduced. Therefore, we can re-write $V(x\psi, y\psi)$ as $V(x\phi\gamma_W, y\phi\gamma_W) = V(x\phi, y\phi)\gamma_W$. Thus, $U(x, y)$ and $V(x\phi, y\phi)$ are conjugate in T_i, and by Lemma 4.3.9 the word $U(x, y)W^{-1}V^{-1}(x\phi, y\phi)W$ is the empty word and so $U(x, y) \equiv V(x\psi, y\psi)$. Now, because ψ lifts to an automorphism of the ambient free group $F(a, b)$ we can apply the fact that the lift $\overline{M}$ of M to the ambient free group $F(a, b)$ is malcharacteristic in $F(a, b)$, by Lemma 4.3.12, to get that the lift of ψ to $F(a, b)$ is inner. Thus, ψ is an inner automorphism of T_i, as required. □

We shall now formally prove Theorem B.

Proof of Theorem B. By Lemma 4.3.4, Theorem B holds if $T_i = \langle a, b; a^i, b^i, (ab)^i \rangle$ contains a malcharacteristic subgroup which is free of rank two, where $i > 9$ is arbitrary. By Lemma 4.3.13, T_i contains such a subgroup for all $i > 9$. Thus, Theorem B holds. □

4.3.5 The Bumagin–Wise question for recursively presented groups

In this section we prove Theorem D. We do this by analysing the outer automorphism groups of certain mapping tori, $G = H_\phi = \langle H, t; h^t = \phi(h), h \in H \rangle$, using Theorem 4.2.17. We shall write $\widehat{\phi}$ for the coset of $\mathrm{Out}(H)$ containing ϕ.

Sapir's embedding. To use Theorem 4.2.17 we need to have some knowledge or control over the centralisers of elements in $\mathrm{Out}(H)$. To do this, we use an embedding theorem of Sapir [Sap13]. If A is a finitely generated, recursively presented group and $x \in A$ then Sapir's embedding gives a finitely presented group P such that $C_A(x) = C_P(x)$. It is an open problem of Osin that every recursively presented group can be embedded as a malnormal subgroup of a finitely presented group [Sap13], and Sapir remarks that in his embedding A is malnormal in P, and so the open problem of Osin has a positive solution. However, he does not give a proof, but rather states that this will be proven in his next paper. Our following construction has two possible outcomes, with a stronger result occurring if Osin's problem has a positive solution.

The Bumagin–Wise question. We now prove two theorems, which combine to prove Theorem D. The first theorem, Theorem 4.3.14, gives a partial answer to Bumagin–Wise's question for certain groups, while the second theorem, Theorem 4.3.15, gives a complete answer to Bumagin–Wise's question for certain groups. Theorem 4.3.15 is of most interest if Osin's problem has a positive solution.

Theorem 4.3.14. *Let Q be a finitely generated, recursively presented group. Then there exists a finitely-generated, residually finite group G such that* $\mathrm{Out}(G) \cong Q \times C_2$*.*

Proof. Form $Q_2 = Q \times C_2$. As Q is finitely generated and recursively presented, we can use Sapir's embedding to construct a finitely presented group P which contains Q_2 and such that $C_P(g) = Q_2$ where g is the element of order two we added to Q (we equate Q and g with their images in P). As P is finitely presented, there exists a finitely generated, residually finite group H such that $\mathrm{Out}(H) \cong P$ [BW05]. Note that this group H is generated by elements of finite order, and so does not map onto $\mathbb{Z}$, and also note that H is a (possibly infinitely presented) $C'(1/6)$ group and therefore has trivial center, by Proposition 2.3.7.

Now, as H does not map onto $\mathbb{Z}$, we can apply Lemma 4.2.1 to get that $\mathrm{Out}_H(G) = \mathrm{Out}(G)$ for any $G = H \rtimes \mathbb{Z}$. We shall write $\mathrm{Out}^0(G)$ for the subgroup corresponding to $\mathrm{Out}^0_H(G)$ in the case that $\mathrm{Out}_H(G) = \mathrm{Out}(G)$. Thus, $\mathrm{Out}^0(G)$ has index one or two in $\mathrm{Out}(G)$.

Let $\widehat{\phi}$ by the element of $\mathrm{Out}(H)$ associated with $g \in Q_2$. Thus, $C_{\mathrm{Out}(H)}(\widehat{\phi}) \cong Q_2$. Form $G = \langle H, t; h^t = \phi(h)\rangle$ for some $\phi \in \widehat{\phi}$. Then $\mathrm{Out}^0(G) \cong Q$ by Theorem 4.2.17. Note that G is finitely generated and residually finite.

To complete the theorem, it is sufficient to prove that $\mathrm{Out}(G) = \mathrm{Out}^0(G) \times C_2$. To see this, note that $g = g^{-1}$. Thus, the following the automorphism can be taken as the coset representative for $\mathrm{Out}(G)/\mathrm{Out}^0(G)$.

$$\begin{aligned} \psi : h &\mapsto h \\ t &\mapsto t^{-1} \end{aligned}$$

This automorphism has order two and generates a normal subgroup of $\mathrm{Out}(G)$. Therefore, $\mathrm{Out}(G) = \mathrm{Out}^0(G) \times \langle \widehat{\psi} \rangle \cong Q \times C_2$, as required. $\square$

The following theorem will allow us to apply a positive solution of Osin's problem to get a positive solution to Bumagin–Wise's question for finitely generated, recursively presented groups. This is because if Q is finitely generated and recursively presented then the conditions of the theorem hold if, for example, $Q \times C_3$ embeds malnormally into a finitely presented group, and a positive solution to Osin's question gives us this embedding.

Theorem 4.3.15. *Let $Q' = Q \times C$ where $C = \langle g \rangle$ is cyclic of order greater than two (possibly infinite.)Suppose that Q' can be embedded into a finitely presented group P where g is not conjugate to g^{-1} in P. Then there exists a finitely generated, residually finite group G such that* $\mathrm{Out}(G) \cong Q$.

Proof. Write H for the finitely generated, residually finite group such that $\mathrm{Out}(H) \cong P$, and, as in the proof of Theorem 4.3.14, form the finitely generated, residually finite group $G \cong H \rtimes \mathbb{Z}$ such that $\mathrm{Out}^0(G) \cong Q$. Finally, because g is not conjugate to g^{-1} in P we conclude that $\mathrm{Out}(G) = \mathrm{Out}^0(G) \cong Q$, as required. $\square$

Then Theorem D follows immediately from Theorems 4.3.14 and 4.3.15.

Theorem D. *If Q is a finitely generated, recursively presented group then either Q or $Q \times C_2$ can be realised as the outer automorphism group of a finitely-generated, residually finite group G_Q.*

Note that if Osin's open problem has a positive solution, so every finitely generated, recursively presented group is a malnormal subgroup of a finitely presented group, then we can use Theorem 4.3.15 and disregard Theorem 4.3.14 to get the following improvement.

Theorem D (Dependent on Osin's problem)**.** *Every finitely generated, recursively presented group can be realised as the outer automorphism group of a finitely-generated, residually finite group.*

Recursive presentability. It is natural to ask if Theorem D is the complete solution to Bumagin–Wise's question. This is not so, and the reason is that there exist a finitely generated, residually finite group Q which is not recursively presentable (Bridson–Wilton point out that this follows from work of Slobodskoi [BW13]). Applying this group Q to Theorem C yields the following corollary of Theorem C.

Corollary 4.3.16. *There exists a finitely generated, residually finite group G such that* $\mathrm{Out}(G)$ *is finitely generated but not recursively presentable.*

Proof. Let Q be a finitely generated, residually finite group which is not recursively presentable. Then by Theorem C, Q embeds with finite index into the outer automorphism group $\mathrm{Out}(G_Q)$ of a finitely generated residually finite group G_Q. As $\mathrm{Out}(G_Q)$ contains a finitely generated group which is not recursively presentable, namely Q, $\mathrm{Out}(G_Q)$ cannot be recursively presentable. As Q is finitely generated and has finite index in $\mathrm{Out}(G_Q)$ we have that $\mathrm{Out}(G_Q)$ is finitely generated. Taking $G := G_Q$, this proves the result. □

Note, however, that the group G_Q is not recursively presentable. We end this chapter by suggesting the following question, which asks if (the stronger version of) Theorem B gives a complete solution to Bumagin–Wise's original problem in the restricted case of finitely generated, residually finite groups which are recursively presentable.

Question 2. *Does there exist a finitely generated group Q which is not recursively presentable but which occurs as the outer automorphism group* $\mathrm{Out}(G)$ *of a recursively presented finitely generated, residually finite group G?*

Notation

$\epsilon, \epsilon', \epsilon_0, \epsilon_i, \ldots$	Integers of absolute value 1.
$\mathrm{Aut}(G), \mathrm{Inn}(G)$	Automorphism group, inner automorphism group of G.
$\mathrm{Out}(G)$	Outer automorphism group of G, $\mathrm{Out}(G) := \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$.
ϕ, ψ	Greek symbols denote homomorphisms (usually automorphisms) of a group.
$g\phi$, $K\phi$	The image of the element g, subgroup K under the homomorphism ϕ.
γ_g	The inner automorphism corresponding to g. So $h\gamma_g = h^{-1}gh$.
$\widehat{\phi}$	The element of $\mathrm{Out}(G)$ with representative $\phi \in \mathrm{Aut}(G)$.
g^h	$g^h := h^{-1}gh$.
$[g, h]$	$[g, h] := g^{-1}h^{-1}gh$.
$U \equiv V$	Equality of words.
$U =_G V$	The words $U(X)$ and $V(X)$ represent the same element of the group $G = \langle X \rangle$.
$\sigma_a(U)$	The exponent sum of the generator a in the word U.
$\lvert U \rvert$	The length of the word U.
G^{ab}, G'	Abelianisation of G, derived subgroup of G.
$C_H(G), N_H(G)$	Centraliser, normaliser of H in G.
$\langle S \rangle_H$	Subgroup generated by set S in a subgroup $H \leq G$.
$\langle\langle S \rangle\rangle$	Normal closure of the set S in a previously-specified group G.
$\langle\langle S \rangle\rangle_H$	Normal closure of the set S in a subgroup $H \leq G$, the *H-normal closure of S*.
$H \rtimes K, H \bowtie K$	Semidirect product, Zappa–Szép product of H with K.
$\langle X; \mathbf{r} \rangle$	Group given by generators X and relators $\mathbf{r}$.
$\langle X; S \rangle, \langle X; R^n \rangle$	A one-relator group, a one-relator group where the word R has order n.
F_n	The free group of rank $n \in \mathbb{Z} \cup \{\mathbb{N}\}$.
$\mathrm{GL}_2(\mathbb{Z})$	Group of two-by-two matrices with integer entries and non-zero determinant.
$T_{i,j,k}$	The triangle group $T_{i,j,k} := \langle a, b; a^i, b^j, (ab)^k \rangle$.
T_i	The equilateral triangle group $T_i := \langle a, b; a^i, b^i, (ab)^i \rangle$.

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