# ON THE PRIME SPECTRA OF SOME NOETHERIAN RINGS

by

**Paula Alexandra de Almeida Bastos Carvalho**

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Department of Mathematics

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To

my parents

# Statement

This thesis is submitted in accordance with the degree of Doctor of Philosophy in the University of Glasgow.

Chapter 1 and section 1 of chapter 2 and 3 covers known results.

Chapters 2, 3 and 4 are the author's own work, with the exception of 2.1 and 3.1 as well as other instances indicated within the text.

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# Contents

# Summary


This thesis is devoted to the description of the graph of links of some skew-polynomial rings and skew-Laurent rings; and the characterization of some crossed products which are Azumaya. The characterization of the Azumaya locus and relation with the singular locus is also studied for some crossed products.

In chapter 2 we describe the links between prime ideals in skew-Laurent rings of the form $\mathcal{S} = R[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n]$ and in skew-polynomial rings of the form $\mathcal{T} = R[\theta_1, \ldots, \theta_n; \alpha_1, \ldots, \alpha_n]$ with basis ring $R$, commutative and Noetherian, where $\alpha_1, \ldots, \alpha_n$ are pairwise commuting automorphisms of $R$. In order to do so, we start by studying the strong second layer condition.

**Theorem** *The ring $\mathcal{S}$ is AR-separated.*

**Corollary** *The ring $\mathcal{S}$ satisfies the strong second layer condition.*

**Corollary** *The ring $\mathcal{T}$ satisfies the strong second layer condition.*

We show that, in determining the clique of a prime of $\mathcal{S}$ or $\mathcal{T}$, there is no loss of generality in assuming that $R$ is semilocal and that the primes contract in the basis ring $R$ to an ideal of the form $N = \cap M^g$ where $M$ is a maximal ideal of $R$ and the indicated intersection is finite. We can then describe the links in $\mathcal{S}$ and $\mathcal{T}$.

**Proposition** *Let $P$ and $Q$ be prime ideals of $\mathcal{S}$, with $P \cap R = N$. Suppose that $P \rightsquigarrow Q$. Then $Q \cap R = N$ and one of the following holds:*

1. *$0 \neq N\mathcal{S} = P = Q$;*
2. *$N\mathcal{S} \subsetneq P = Q$;*

*3.* $0 \neq N\mathcal{S} \subsetneq P \neq Q$ *and there exist a prime ideal* $P^\sharp$ *of* $\mathcal{S}_2^\sharp$ *lying over* $\tilde{P}_2/M\mathcal{S}_2$ *and* $i \in \{1, \dots, u\}$ *such that* $\sigma_i(P^\sharp)$ *lies over* $\tilde{Q}_2/M\mathcal{S}_2$ *where* $K^\sharp$ *is the algebraic closure of* $K = R/M$, $\mathcal{S}_2$ *is a skew-Laurent ring,* $\mathcal{S}_2 \subseteq \mathcal{S}$, $\mathcal{S}_2^\sharp = K^\sharp \otimes_K \mathcal{S}_2$, $\tilde{P}_2$ *and* $\tilde{Q}_2$ *are minimal primes over* $P \cap \mathcal{S}_2$ *and* $Q \cap \mathcal{S}_2$, *respectively, such that* $\tilde{P}_2 \cap R = M = \tilde{Q}_2 \cap R$, *and* $\sigma_i$ *are the automorphisms determined by the action of* $\mathcal{S}_2^\sharp$ *in* $K^\sharp \otimes_K M/M^2$.

*Conversely, if one of case 1,2 or 3 holds, then* $P \rightsquigarrow Q$. $\square$

The description of cliques in $\mathcal{T}$ will, in some cases, depend on the description of cliques in $\mathcal{S}$ given before.

**Theorem** *Let* $P, Q \in Spec(\mathcal{T})$ *such that* $\theta_{i+1}, \dots, \theta_n \in P$ *and* $\theta_1, \dots, \theta_i \notin P$. *Then* $P \rightsquigarrow Q$ *if and only if* $\theta_{i+1}, \dots, \theta_n \in Q$ *and either (a)* $P/(\theta_{i+1}\mathcal{T} + \dots + \theta_n\mathcal{T}) \rightsquigarrow Q/(\theta_{i+1}\mathcal{T} + \dots + \theta_n\mathcal{T})$ *in* $R[\theta_1, \dots, \theta_i; \alpha_1, \dots, \alpha_i]$ *or (b) there is* $j \in \{i+1, \dots, n\}$ *such that* $P = \alpha_j(Q)$.

**Corollary** *Let* $P, Q \in Spec(\mathcal{T})$ *such that* $\theta_{i+1}, \dots, \theta_n \in P \cap Q$ *and* $\theta_1, \dots, \theta_i \notin P$. *Let* $Y = \{\theta_1^{j(1)} \dots \theta_i^{j(i)} : j(1), \dots, j(i) \in \mathbb{N}\}$, *an Ore set in* $\mathcal{T}$, $\overline{P} = P/(\theta_{i+1}\mathcal{T} + \dots + \theta_n\mathcal{T})$ *and* $\overline{Q} = Q/(\theta_{i+1}\mathcal{T} + \dots + \theta_n\mathcal{T})$.

*Then* $P \rightsquigarrow Q$ *if and only if* $\overline{P}Y^{-1} \rightsquigarrow \overline{Q}Y^{-1}$ *in* $R[\theta_1, \theta_1^{-1}, \dots, \theta_i, \theta_i^{-1}; \alpha_1, \dots, \alpha_i]$ *or if there is* $j \in \{i+1, \dots, n\}$ *such that* $P = \alpha_j(Q)$.

In chapter 3 we study crossed products which are Azumaya and describe them in terms of H-separability conditions.

**Proposition** *Let* $R$ *be any ring,* $J$ *any group and* $R * J$ *any crossed product. Then the following are equivalent:*

*i)* $R * J$ *is an Azumaya algebra and* $Z(R * J) \subseteq R$;

*ii) $R * J$ is an H-separable extension of $R$ and $R$ is a separable extension of $Z(R)^J$.*

Then we restrict our attention to crossed products $R * G$ where $G$ is a finite group of automorphisms and the action of the crossed product is the one given by $G$.

**Proposition** *Let $R$ be any ring and $G$ a finite $\omega$-outer group of automorphisms of $R$. Let $R * G$ be any crossed product constructed with the given action of $G$ on $R$. If $R * G$ is Azumaya then $Z(R)$ is a $G$-Galois extension of $Z(R)^G$.*

**Theorem** *Let $R$ be any commutative ring, $G$ a finite subgroup of $Aut(R)$ and $R*G$ any crossed product of $G$ over $R$ constructed with the given action of $G$ on $R$. If $R$ is a $G$-Galois extension of $R^G$, then $R * G$ is Azumaya and $Z(R * G) = R^G$.*

**Proposition** *Let $R$ be any ring, $G$ a finite group of automorphisms of $R$ and $R * G$ any crossed product constructed with the given action of $G$ on $R$. Consider the following statements:*

*i) $R * G$ is Azumaya and $Z(R * G) \subseteq R$;*

*ii) a) $R * G$ is an H-separable extension of $R$;*

*b) $R$ is a separable extension of $Z(R)^G$;*

*iii) $R$ is Azumaya and $Z(R)$ is a $G$-Galois extension of $Z(R)^G$.*

*Then $i)$ is equivalent to $ii)$. If $R$ is commutative $iii)$ implies $i)$ and $ii)$.*
*If $G$ is $\omega$-outer on $R$, $i)$ and $ii)$ imply $iii)$.*
*If $R$ is commutative and $G$ is $\omega$-outer on $R$, $i)$, $ii)$ and $iii)$ are equivalent.*

Given a prime Noetherian ring $R$ module-finite over its centre $Z(R)$, the *Azumaya locus* of $R$ is the set

$$\mathcal{A}_R = \{M \in Max(Z(R)) : R_M \text{ is Azumaya}\}$$

and the *singular locus* of $Z(R)$ is the set

$$\mathcal{S}_R = \{M \in Max(Z(R)) : Z(R)_M \text{ is not regular}\}.$$

The rest of chapter 3 is dedicated to the study of the Azumaya locus and the singular locus of some crossed products over some Noetherian domains.

**Proposition** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$ and assume that $D$ is finitely generated over $D^G$. Let $D * G$ be any crossed product constructed with the given action of $G$ on $R$. Then*

$$\mathcal{A}_{D*G} = \{M \in Max(D^G) : I_D(G) \cap D^G \nsubseteq M\}.$$

*Where $I_D(G) = \bigcap_{g \in G \setminus \{1_G\}} I_D(g)$ and $I_D(g)$ is the ideal of $D$ generated by $\{g(d) - d : d \in D\}$.*

The next three results show that the Azumaya locus of some crossed products is, under some conditions, the complement of its singular locus in the set of maximal ideals of its centre.

**Proposition** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$ with $D$ a finitely generated $D^G$-module. Form $D * G$ any crossed product constructed with the given action of $G$ on $R$. If*

*i)* gl.dim$(D * G)$ *is finite,*

*and*

*ii) for all $\overline{P} \in Spec(D)$ of height 1, $I_D(G) \nsubseteq \overline{P}$,* *then* $\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$.

**Corollary** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$. Form $D*G$ any crossed product and assume that $D$ is finitely generated over $D^G$. If*

*i) gl.dim$(D)$ is finite,*

*ii) for every maximal ideal $M$ of $D$ with char$(D/M) = p > 0$, $G_D(M) = \{g \in G : d^g - d \in M$, for all $d \in D\}$ contains no element of order $p$,*

*iii) for every prime ideal $\overline{P}$ of $D$ of height one, $I_D(G) \not\subseteq \overline{P}$,*

*then* $\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$.

**Proposition** *Let $D$ be a commutative domain and an affine algebra over an algebraically closed field $K$ of characteristic zero and $G$ a finite group of $K$-automorphisms of $D$. Form the crossed product $D*G$ constructed with the given action of $G$ on $D$. Assume also that* gl.dim(D) *is finite. We have that $I_D(G) \subseteq \overline{P}$ for all $\overline{P} \in Spec(D)$ of height 1 if and only if $\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$.*

In chapter 4 we study the skew-polynomial and skew-Laurent rings which are fully-bounded Noetherian and the ones which satisfy a polynomial identity. Then, using the results of the previous chapter, chapter 3, we describe the Azumaya locus and singular locus of some skew-Laurent ring $\mathcal{S}$ over a ring $R$. Let $G$ be the group of automorphisms of $R$ such that $\mathcal{S} = RK_e * G$. In particular we prove the following result.

**Proposition** *Suppose $R$ is a commutative Noetherian domain of finite global dimension and $G$ is finite. Assume also that $R$ is finitely generated as a module over $R^G$ and that $I_R(G)$ is not contained in any prime ideal of $R$ of height 1. Then the following sets of maximal ideals of $\mathcal{S}$ are equal* *i)* $\{M \in Max(\mathcal{S}) : \mathcal{S}_{M \cap Z(\mathcal{S})}$ *is Azumaya*$\}$.

*ii)* $\{M \in Max(\mathcal{S}) : Z(\mathcal{S})$ *is regular at* $M \cap Z(\mathcal{S})\}$.

The above sets are contained in

$$\{M \in Max(\mathcal{S}) : I_R(G) \nsubseteq M \cap R\}$$

and they all coincide when $R$ is a Hilbert ring.

# Introduction

It may be said that the theory of Noetherian rings began with Goldie's paper in 1958. Goldie's theorem provides the analogue in the noncommutative case for the usual field of fractions of a domain in the commutative case. Goldie proved that given a prime Noetherian ring and $\mathcal{C} = \mathcal{C}_R(0)$, the set of regular elements of $R$, the set of elements of the form $ac^{-1}$ for $a \in R$ and $c \in \mathcal{C}$ is a ring, the ring of fractions of $R$, isomorphic to a matrix ring over a division ring. The existence of such a ring of fractions is equivalent to the set $\mathcal{C}$ being both right and left Ore ($\mathcal{C}$ is right Ore if for all $a \in R$ and $c \in \mathcal{C}$, there are elements $a' \in R$ and $c' \in \mathcal{C}$ with $ac' = a'c$. The definion of left Ore is symmetric).

In the commutative case, given a prime ideal $P$ of $R$ we can form the ring of fractions of the form $rc^{-1}$ for $r \in R$ and $c \in R \backslash P$ or equivalently, $c \in \mathcal{C}_R(P) = \{r \in R : r + P \in \mathcal{C}_{R/P}(0)\}$, and in this case we say that we localize $R$ at a prime ideal $P$. A natural step after Goldie's theorem and following the case of commutative rings, would be to extend the ideas of localizing at a prime to noncommutative rings; we say that a prime ideal $P$ of a Noetherian ring is localizable if $\mathcal{C}_R(P)$ is a right and left Ore set. If $P$ is localizable we denote the corresponding fraction ring by $R_P$. Prime ideals usually are not localizable; Jategaonkar pointed out that if $P$ and $Q$ are distinct maximal ideals of a Noetherian ring $R$ such that $R/P$ and $R/Q$ are artinian and if $Q \cap P \neq PQ$, then $Q$ cannot be localizable. So prime ideals should be related in some way depending on the existence of some factors of the bimodule $P \cap Q/PQ$. We say that given two prime ideals $P$ and $Q$, $P$ is linked to $Q$, $P \rightsquigarrow Q$, if there exists an ideal $A$ of $R$ such that $PQ \subseteq A \subsetneq P \cap Q$ and $(P \cap Q)/A$ is torsionfree as a left $R/P$-module and as a right $R/Q$-module. The graph of links of $R$ is the directed graph whose vertices are the elements of $Spec(R)$ with an arrow from $P$ to $Q$ whenever $P \rightsquigarrow Q$. The connected components of this graph are called cliques and if $P \in Spec(R)$, the unique clique containing $P$ is denoted by $\mathcal{C}\ell(P)$.

In 1982, Jategaonkar introduced a new condition on a Noetherian ring, the second layer condition. This condition is satisfied by several classes of rings, for instance enveloping algebras of finite dimensional solvable Lie algebras [57, Theorem A.3.9]; group rings of polycyclic by finite groups [18, Proposition 2.2] and [55, Theorem 4.5]. This condition is related with the way one can build some series for a module over a Noetherian ring.

After briefly recalling some definitions and properties of skew-polynomial, skew-Laurent rings and crossed products in chapter 1, we start our research with chapter 2. Chapter 2 is devoted to the description of the graph of links of some skew-polynomial ring, $\mathcal{T}$ and some skew-Laurent ring, $\mathcal{S}$. The work of Chapter 2 is designed to provide analogues for rings such as $\mathcal{S}$ and $\mathcal{T}$ of results obtained by K. R. Goodearl [43] for rings constructed in a similar fashion using derivations rather than automorphisms. As a first step to study the prime links in $\mathcal{S}$ and $\mathcal{T}$, we remark that it is known that $\mathcal{S}$ satisfies the second layer condition. In §2.2 we prove that $\mathcal{S}$ is AR-separated, hence satisfies the strong second layer condition and so does $\mathcal{T}$.

In §2.3 and §2.4 we describe the prime links in $\mathcal{S}$. We will show that the prime links in $\mathcal{S}$ can be reduced to the study of prime links between prime ideals that contract to a maximal ideal in $R$, the coefficient ring of $\mathcal{S}$. In §2.3 we start by describing the links between prime ideals contracting to maximal ideals in the coefficiant ring of $\mathcal{S}$. In §2.4 we show how to reduce the general problem to the one in §2.3 and obtain the description. With the results obtained in §2.4, we can easily describe the prime links in $\mathcal{T}$. Some examples are given in §2.6. These examples are used not only to illustrate the description obtained but also to explain the reason for some conjectures to fail.

So far we have been interested in the way the prime spectrum of skew-Laurent and skew-polynomial rings can be divided into sets of primes, the cliques. The most trivial case happens when there are just trivial links; not only a prime is linked to itself but whenever for some prime $Q$, $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$, then $P = Q$. In this case the cliques will just be singletons. This situation occurs for instance when the rings are Noetherian and Azumaya. If a ring is Azumaya, it is finitely generated over its centre and its prime spectrum is determined by the prime spectrum of its centre. By Müller's Theorem [44, Theorem 11.20], we have that the cliques of a Noetherian Azumaya ring are singletons. Thus it is natural to try to understand those classes of algebras $\mathcal{S}$ and $\mathcal{T}$ for which the graph of links is particularly simple by first identifying those algebras which are Azumaya, and (more generally) to describe the Azumaya locus of algebras of the type considered in this thesis. This is the main objective in Chapter 3 and 4.

Azumaya rings are separable algebras over their centres, or central separable algebras. The study of these algebras led to new notions of separability such as separable extension of ring [49] and H-separability [50].

The main problem is now to describe when are the skew-Laurent rings Azumaya. Similar problem had been studied for skew-group rings of finite groups by Ikehata in [52] and by R. Afaro and G. Szeto in [3]. While studying skew-Laurent rings it became clear that some results obtained to deal with our algebras would apply also to some crossed products of finite groups, and so generalise results in [52] and [3].

In chapter 3, we study separabiliy and H-separability in some crossed products. We start by introducing the definitions and well known results in §3.1. In §3.2 we obtain necessary conditions for a crossed product to be Azumaya. Given a crossed product of the form $R * J$, the condition of $R * J$ being Azumaya is related with an H-separability condition: $R * J$ is an H-separable extension of $R$. In order to be able to describe when $R * J$ is an H-separable extension of $R$, we impose extra conditions on $J$ and on $R * J$; thus we assume $J$ is a finite group of automorphisms of $R$, $\omega$-outer and $R * J$ is defined with the given action of $J$ on $R$. This section is divided into two parts; in the first we study the general crossed product $R * J$ and in the second §3.2.1, we study what happens if $R * J$ is H-separable, assuming the extra conditions on $R$ and on $J$. In §3.3 we get a sufficient condition for a crossed product $R * G$ to be Azumaya, where $R$ is a commutative ring, $G$ is a finite group of automorphisms of $R$ and $R * G$ is any crossed product built with the given action. In §3.4 we study how far some prime Noetherian crossed products are from being Azumaya; we describe their Azumaya locus. Imposing some homological conditions on the commutative Noetherian domain $R$ and some conditions on the group $G$, we will be able to describe the Azumaya locus of $R * G$ in terms of its nonsingular locus, the complement in $Max(Z(R * G))$ of the singular locus.

In chapter 4 we apply the results of chapter 3 to skew-Laurent rings. In a similar way to the work of Damiano and Shapiro in [32], we obtain necessary conditions for the skew-Laurent and skew-polynomial ring with a Noetherian coefficient ring to be fully bounded Noetherian and describe the ones which satisfy a polynomial identity, the PI rings. This study will be done in §4.1 and in §4.2. Section §4.3 is totally dedicated to the study of the Azumaya locus of some skew-Laurent rings.

Throughout this thesis we have tried to quote our references from the original authors. However, sometimes for simplicity we refer to the books [44], [76], [86], [45] and [38] for some well known results.

A short section named Additional remarks, is placed at the end of each chapter to indicate whether a result appearing in the chapter is a well-known or a new one. Most of chapter 2 appeared in [27].

# Notation

| | |
|---|---|
| $\mathbb{N}$ | The set of natural numbers. |
| $\mathbb{N}_0$ | The set of natural numbers and zero. |
| $\mathbb{Z}$ | The set integers. |
| $\mathbb{Q}$ | The field of rational numbers. |
| $\mathbb{R}$ | The field of real numbers. |
| $\mathbb{C}$ | The field of complex numbers. |
| $R$ | Associative ring with unit. |
| $Z(R)$ | The centre of the ring R. |
| $\mathcal{U}(R)$ | The group of units of $R$. |
| $Aut(R)$ | The set of all automorphisms of $R$. |
| $Inn(R)$ | The set of inner automorphisms of $R$. |
| Cl.K.dim($R$) | The classical Krull dimension of $R$. |
| $Spec(R)$ | The prime spectrum of $R$. |
| $Max(R)$ | The set of maximal ideals of $R$. |
| $\mathcal{C}_R(0)$ | The set of regular elements of $R$. |
| $\mathcal{C}_R(P)$ | The set of regular elements of $R$ modulo $P$. |
| $\mathcal{J}(R)$ | The Jacobson radical of $R$. |
| $RX^{-1}$ | The right ring of fractions of $R$ with respect to $X$. |
| $\mathcal{Q}(R)$ | The quotient ring of $R$. |
| $R_P$ | The ring of fractions of $R$ with respect to $C_R(P)$. |
| $R * G$ | The crossed product of $G$ over $R$. |
| $R\#G$ | The skew-group ring of $G$ over $R$. |
| $R^tG$ | The twisted group ring of $G$ over $R$. |
| $Spec^G(R)$ | The set of $G$-prime ideals of $R$. |

| | |
|---|---|
| $P \rightsquigarrow Q$ | Prime link from $P$ to $Q$. |
| $\mathfrak{t}_X(M)$ | The $X$-torsion submodule of $M$. |
| $Ass(M)$ | The set of all associated primes of $M$. |
| $\mathcal{R}_R(I)$ | The Rees ring of the ideal $I$ of $R$. |
| $M_n(K)$ | The ring of square matrices over $K$. |
| $R^G$ | The fixed ring of $G$ on $R$. |
| $\phi_g$ | The set $\{r \in R : rs^g = sr, \forall s \in R\}$. |
| $tr_G$ | The trace map. |
| $\mathcal{A}_R$ | The Azumaya locus of $R$. |
| $\mathcal{S}_R$ | The singular locus of $R$. |
| $G_R(M)$ | The inertia group of $M$ in $R$. |
| $I_R(g)$ | The two-sided ideal of $R$ generated by $\{r^g - r : r \in R\}$. |
| $I_R(G)$ | The ideal $\bigcap_{g \in G \setminus \{1_G\}} I_R(g)$. |

# Chapter 1

# Preliminaries

In this preliminary chapter we will fix some notation and state a few well-known results which will be needed in the following chapters. Other terminology and notation will be introduced either when they appear for the first time in the text or at the beginning of the chapters where they will be used. For more details about this section one can see for instance [44], [76], [85] and [86].

## 1.1 Notation

All our rings are supposed to be associative with identity element. The identity of a ring $R$ will be denoted by $1_R$ or just by 1 if the ring is well understood. A subring of a ring $R$ will always contain the identity of $R$ and ring homomorphisms are supposed to preserve the identity. The centre of $R$ is denoted by $Z(R)$. The group of units of $R$ is denoted by $\mathcal{U}(R)$ and the set of all automorphisms of $R$ is denoted by $Aut(R)$. For $\alpha \in Aut(R)$ the image of $r \in R$ by $\alpha$ will be denoted either by $\alpha(r)$ or by $r^\alpha$. An automorphism $\alpha$ of $R$ is said to be *inner* if there exists a unit $u \in \mathcal{U}(R)$ such that $r^\alpha = u^{-1}ru$, for all $r \in R$; otherwise, $\alpha$ is said to be *outer* (see [80]). The set of all inner automorphisms of $R$ is denoted by $Inn(R)$ and is an invariant subgroup of $Aut(R)$. A Noetherian (resp. Artinian) ring will always mean a right and left Noetherian (resp. Artinian) ring and an ideal a right and left ideal. The classical Krull dimension of a ring $R$ will be denoted by Cl.K.dim($R$); for its definition and properties see [44].

The set of prime ideals of a ring $R$ is denoted, as usual, by $Spec(R)$ and the set of all maximal ideals of $R$ by $Max(R)$.

Given a ring $R$ and $r \in R$, we say that $r$ is *right (resp. left) regular* if whenever $rs = 0$ (resp. $sr = 0$) for some $s \in R$, $s = 0$. An element of a ring is *regular* if it is right and left regular. The set of regular elements of $R$ is denoted by $\mathcal{C}_R(0)$. If $P$ is any ideal of $R$, $\mathcal{C}_R(P)$ denotes the set of regular elements of $R$ modulo $P$.

The intersection of all maximal right ideals (or equivalently, the intersection of all maximal left ideals) of a ring $R$ is denoted by $\mathcal{J}(R)$, the Jacobson radical of $R$. If $R$ is a ring such that $R/\mathcal{J}(R)$ is semisimple Artinian, $R$ is said to be *semilocal*, if $R/\mathcal{J}(R)$ is simple Artinian, $R$ is said to be *local.*

For a ring $R$ and a multiplicatively closed subset $X$ of $R$, we shall denote by $RX^{-1}$ the right ring of fractions of $R$ with respect to $X$ whenever it exists (see [44]). As in [44], we shall abuse notation and write the elements of $RX^{-1}$ in the form $rx^{-1}$, for $r \in R$ and $x \in X$. If $I$ is an ideal of $R$, we denote by $IX^{-1}$ the set $\{ix^{-1} : i \in I, x \in X\}$, the extension of $I$ (see [44]). In the case of $X = \mathcal{C}_R(0)$ instead of $RX^{-1}$ we will write $\mathcal{Q}(R)$, the right quotient ring of $R$. In the case $X = C_R(P)$ and $RX^{-1}$ exists, we denote this ring by $R_P$.

In this thesis all modules will be unitary modules. Given rings $R$ and $S$, we write $M_R$, ${}_SM$, ${}_SM_R$ to denote that $M$ is a right $R$-module, $M$ is a left $S$-module or that $M$ is an $(S, R)$-bimodule, respectively. In the case nothing is said, one should assume that the structure of the module to be considered is the right hand one. If $M$ is a right $R$-module, the ring of $R$-endomorphisms of $M$ will be denoted by $End(M_R)$; similarly, if $M$ is a left $R$-module, the ring of $R$-endomorphisms of the left $R$-module $M$ will be denoted by $End({}_RM)$. If the structure of the module $M$ is well understood, we will just write $End(M)$. For an $(S, R)$-bimodule $M$, l.ann$_S(M)$ and r.ann$_R(M)$ are, respectively, the left annihilator of $M$ in $S$ and the right annihilator of $M$ in $R$. If $N$ is a submodule of $M$, we write $N \leq M$. If $N \leq M$ and for any non-zero $m \in M$, there is $r \in R$ such that $mr \in N\backslash\{0\}$, we say that $N$ is an essential submodule of $M$ or that $M$ is an essential extension of $N$ and write $N \leq_e M$.

## 1.2 Crossed products and skew-polynomial rings

In this section we introduce the definitions of some noncommutative rings which we will be studying throughout this thesis; the skew-Laurent and skew-polynomial rings.

**Definition 1.2.1** Let $R$ be a ring and $\alpha$ an endomorphism of $R$. A *left $\alpha$-derivation* of $R$ is an additive map $\delta : R \to R$ such that $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$, for all $r, s \in R$.

**Proposition 1.2.2** *Let $R$ be a ring, let $\alpha$ be an endomorphism of $R$ and let $\delta$ be a left $\alpha$-derivation of $R$. Then there exists a ring $T$, containing $R$ as a subring, such that $T$ is a free left $R$-module with a basis of the form $1, \theta, \theta^2, \ldots$ and $\theta r = \alpha(r)\theta + \delta(r)$ for all $r \in R$.*

**Proof.** [44, Proposition 1.10] □

**Definition 1.2.3** The ring $T$ determined in Proposition 1.2.2 above is denoted by $R[\theta; \alpha, \delta]$ and called a *skew-polynomial ring* or an *Ore extension* of $R$.

When $\alpha$ is the identity map on $R$, we abbreviate $R[\theta; \alpha, \delta]$ to $R[\theta; \delta]$ and call this ring a *differential operator ring*. In the case $\delta = 0$, we abbreviate $R[\theta; \alpha, \delta]$ to $R[\theta; \alpha]$.

**Remark 1.2.4** We can also define a right $\alpha$-derivation, which is an additive map $\delta$ of $R$ satisfying the rule $\delta(rs) = \delta(r)\alpha(s) + r\delta(s)$, for all $r, s \in R$. Similarly to what was done before, one can construct an Ore extension, which is a free right $R$-module.

If $\alpha$ is an automorphism of $R$ and $\delta$ an additive map of $R$, $\delta$ is a left $\alpha$-derivation of $R$ if and only if $-\delta\alpha^{-1}$ is a right $\alpha^{-1}$-derivation. In this case the rings $R[\theta; \alpha, \delta]$ and $R[\theta; \alpha^{-1}, -\delta\alpha^{-1}]$ coincide.

An obvious example of skew-polynomial ring is the polynomial ring in one indeterminate. For the case of skew-polynomial rings, we also have a version of Hilbert's Basis Theorem.

**Theorem 1.2.5** *Let $R$ be a ring, let $\alpha$ be an automorphism of $R$ and $\delta$ a $\alpha$-derivation of $R$. If $R$ is right (resp. left) Noetherian, then the skew-polynomial ring $T = R[\theta; \alpha, \delta]$ is also right (resp. left) Noetherian.*

**Proof.** [44, Theorem 1.12] □

**Remark 1.2.6** 1) The condition of $\alpha$ to be an automorphism in the Theorem 1.2.5 above, is needed as the example below shows.

2) The above theorem is true for other classes of rings as we will see.

**Example 1.2.7** [76, Example 1.2.11] Let $K$ be a field, $R = K[y]$, $\alpha$ the endomorphism of $R$ defined by $\alpha(f(y)) = f(y^2)$ and $T = R[\theta; \alpha]$. One can see that $\sum T\theta y\theta^i$ is a direct sum. If for any $n \in \mathbb{N}$, we set $I_n = \sum_{i=0}^{n} \theta^i yT$, we get a strictly increasing chain of right ideals. Hence $T$ is neither left nor right Noetherian.

If $\alpha_1$ is an automorphism of $R$ and $\delta_1$ is an $\alpha_1$-derivation of $R$, using Theorem 1.2.2, we can form the ring $R[\theta_1; \alpha_1, \delta]$. Also by Theorem 1.2.2, given $\alpha_2$ an automorphism of $R[\theta_1; \alpha_1, \delta_1]$ and $\delta_2$ an $\alpha_2$-derivation of $R[\theta_1; \alpha_1, \delta_1]$, we can form the ring $R[\theta_1; \alpha_1, \delta_1][\theta_2; \alpha_2, \delta_2]$. Hence applying $n$ times Proposition 1.2.2 and the ideas above, we would get a ring of the form

$$R[\theta_1; \alpha_1, \delta_1][\theta_2; \alpha_2, \delta_2] \dots [\theta_n; \alpha_n, \delta_n].$$

**Definition 1.2.8** The ring $T = R[\theta_1; \alpha_1, \delta_1][\theta_2; \alpha_2, \delta_2] \dots [\theta_n; \alpha_n, \delta_n]$ (assuming the notation above) is called an *iterated skew-polynomial ring*, for short we will just call such a ring $T$ a *skew-polynomial ring.*

There are two cases of iterated skew-polynomial rings of particular interest to us. First, consider the iterated skew-polynomial ring built from a ring $R$ and a finite list $\delta_1, \dots, \delta_n$ of pairwise commuting derivations of $R$

$$R[\theta_1; \delta_1][\theta_2; \delta_2] \dots [\theta_n; \delta_n]$$

which we will denote by $R[\theta_1, \dots, \theta_n; \delta_1, \dots, \delta_n]$. In this case we should note that all the automorphisms in the definition of iterated skew-polynomal ring are assumed to be the identity and that each derivation $\delta_i$ can be extended to a derivation of $R[\theta_1, \dots, \theta_{i-1}; \delta_1, \dots, \delta_{i-1}]$ by setting $\delta_i(\theta_j) = 0$ for any $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, i-1\}$. This is possible because the derivations commute with each other.

Another noteworthy class of iterated skew-polynomial rings can be obtained if we take a family $\alpha_1, \dots, \alpha_n$ of pairwise commuting automorphisms of $R$. For $\alpha_1$ we can form $R[\theta_1; \alpha_1]$ (assume derivation $\delta_1 = 0$) and extend $\alpha_2$ to an automorphism of $R[\theta_1; \alpha_1]$. (We make $\alpha_2(\sum_{i=0}^{m} r_i \theta_1^i) = \sum_{i=0}^{m} \alpha_2(r_i)\theta_1^i$ and then use the fact that $\alpha_1\alpha_2 = \alpha_2\alpha_1$ to prove that $\alpha_2$ is an automorphism of $R[\theta_1; \alpha_1]$). Iterating the above procedure we obtain the ring

$$R[\theta_1; \alpha_1][\theta_2; \alpha_2] \dots [\theta_n; \alpha_n]$$

that we shall denote by

$$R[\theta_1, \dots, \theta_n; \alpha_1, \dots, \alpha_n].$$

If one thinks of the polynomial ring $R[x]$, it is well known that we can form a new ring where the indeterminate becomes a unit, the Laurent ring $R[x, x^{-1}]$. For some skew-polynomial rings the same happens.

**Proposition 1.2.9** *Let $R$ be a ring and $\alpha$ an automorphism of $R$. Then there exists a ring $S$, containing $R$ as a subring, with a unit $\theta \in S$ such that $S$ is a free left $R$-module with a basis of the form $1, \theta, \theta^{-1}, \theta^2, \theta^{-2}, \ldots$ and $\theta r = \alpha(r)\theta$, for all $r \in R$.*

**Proof.** [44, Proposition 1.16]. □

**Definition 1.2.10** Let $R$ be a ring and $\alpha$ an automorphism of $R$. The ring $S$ constructed in Proposition 1.2.9 is denoted $R[\theta, \theta^{-1}; \alpha]$ and called a *skew-Laurent extension of $R$*.

**Remark 1.2.11** 1) Given a ring $R$, if $\alpha = id_R$, then $R[\theta, \theta^{-1}; \alpha]$ is just $R[\theta, \theta^{-1}]$, the ordinary Laurent polynomial ring.

2) As was done for skew-polynomial rings, given $\alpha_1, \ldots, \alpha_n$ pairwise commuting automorphisms of $R$, one can apply Proposition 1.2.9 to form

$$R[\theta_1, \theta_1^{-1}; \alpha_1] \ldots [\theta_n, \theta_n^{-1}; \alpha_n],$$

which we will denote

$$R[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n].$$

Although Definition 1.2.13 below is just a restatement of the definitions given before, we decided to include it as it refers to the rings which we will be studying throughout this thesis, in this way we will also fix some notation.

**Notation 1.2.12** If $R$ is any ring, $\alpha_1, \ldots, \alpha_n \in Aut(R)$ commuting pairwise and $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, we will denote by $\alpha^I$ the automorphism $\alpha_1^{i_1} \ldots \alpha_n^{i_n}$ of $R$. If $\theta_1, \ldots, \theta_n$ are commuting indeterminates, $\theta^I$ denotes the element $\theta_1^{i_1} \ldots \theta_n^{i_n}$.

**Definition 1.2.13** Let $R$ be a ring and $\alpha_1, \dots, \alpha_n$ commuting automorphisms of $R$. We define the skew-polynomial ring $\mathcal{T} = R[\theta_1, \dots, \theta_n; \alpha_1, \dots, \alpha_n]$ and the skew-Laurent ring $\mathcal{S} = R[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1, \dots, \alpha_n]$, whose additive group coincides with the one of $R[\theta_1, \dots, \theta_n]$ and $R[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}]$, respectively, and multiplication is defined by the associative laws and by the rules

$$\theta_i r = \alpha_i(r)\theta_i,$$

$$\theta_i\theta_j = \theta_j\theta_i$$

for all $i, j \in \{1, \dots, n\}$ and $r \in R$.

**Remark 1.2.14** If we form $\mathcal{T}$ and $\mathcal{S}$ as in Definition 1.2.13, the elements of $\mathcal{T}$ and $\mathcal{S}$ are uniquely written in the form $\sum\limits_{I \in \mathbb{N}_0^n} r_I\theta^I$ and $\sum\limits_{I \in \mathbb{Z}^n} r_I\theta^I$, respectively, where $r_I \in R$ and $r_I = 0$ for all but finitely many $I \in \mathbb{Z}^n$.

The skew-Laurent rings defined above in Definition 1.2.13, can be seen as an example of *crossed products.*

**Definition 1.2.15** Let $R$ be any ring and $G$ any multiplicative group. A *crossed product of $G$ over $R$*, denoted by $R * G$, is an associative ring containing for each $g \in G$ an element $\overline{g} \in R * G$. The set $\overline{G} = \{\overline{g} : g \in G\}$, a copy of $G$, is a left $R$-basis for $R * G$ so that every element is uniquely written as a finite sum

$$\sum_{g \in G} r_g\overline{g}$$

with $r_g \in R$. The addition in $R * G$ is the obvious one and the multiplication is defined by the associative laws and by the rules

$$\overline{g}\overline{h} = \varepsilon_{R,G}(g,h)\overline{gh}, \text{ for all } g, h \in G$$

where $\varepsilon_{R,G} : G \times G \to \mathcal{U}(R)$ is a map from $G \times G$ to the group of units of $R$, $\mathcal{U}(R)$, and

$$\overline{g}r = \sigma_{R,G}(g)(r)\overline{g}, \text{ for all } r \in R, g \in G$$

where $\sigma_{R,G} : G \to Aut(R)$.

We say that $\sigma_{R,G}$ is the *action* of $G$ in $R$ and $\varepsilon_{R,G}$ is the twisting.

Whenever $R$ and $G$ are well understood, instead of $\sigma_{R,G}$ and $\varepsilon_{R,G}$ we will write $\sigma$ and $\varepsilon$, respectively.

**Remark 1.2.16** 1) The ring $R * G$ has an identity element, $1 = [\varepsilon(1,1)]^{-1}\overline{1}$, hence without loss of generality we will assume that $\overline{1} = 1$. Moreover each $\overline{g}$ is invertible, for each $g \in G$.

2) One should note that for each ring $R$ and each group $G$, there may be more than one structure of crossed product, depending on the maps, $\sigma$ and $\varepsilon$ defined.

Given a crossed product $R * G$, in general the map $\sigma_{R,G}$ is not a group homomorphism as is shown in the next lemma. The next lemma is well-known but we were unable to find a reference.

**Lemma 1.2.17** *Let $R$ be any ring, $G$ any group and $R * G$ any crossed product. The action of $G$ in $R$, $\sigma$, is a group homomorphism if and only if $\varepsilon(g,h) \in Z(R)$ for all $g, h \in G$.*

**Proof.** Let $g, h \in G$ and $r \in R$. Now

$$\begin{aligned} \overline{g}\overline{h}r &= \overline{g}\sigma(h)(r)\overline{h} \\ &= \sigma(g)(\sigma(h)(r))\varepsilon(g,h)\overline{gh} \end{aligned}$$

and

$$\overline{g}\overline{h}r = \varepsilon(g,h)\overline{gh}r = \varepsilon(g,h)\sigma(gh)(r)\overline{gh}.$$

If $\varepsilon(g,h) \in Z(R)$, for all $g, h \in G$, then since $\varepsilon(g,h)$ and $\overline{gh}$ are units we have that

$$\sigma(g)\sigma(h) = \sigma(gh)$$

for all $g, h$ in $G$. Hence $\sigma$ is a group homomorphism.

Conversely, if we assume that $\sigma$ is a group homomorphism, we have, for any $r \in R$ that

$$\sigma(g)\sigma(h)(r)\varepsilon(g,h) = \varepsilon(g,h)\sigma(g)\sigma(h)(r)$$

and the result follows. $\square$

**Definition 1.2.18** In the definition of crossed products, Definition 1.2.15, if $\varepsilon$ is trivial, that is $\varepsilon(g,h) = 1$, for all $g,h \in G$, the crossed product is called a *skew-group ring* (or *trivial crossed product*); in this case we write $R\#G$. If $\sigma(g)$ is the identity map in $R$ for all $g \in G$, then $R * G$ is called a *twisted group ring* and instead of $R * G$ we write $R^tG$.

**Notation 1.2.19** Given any crossed product $R*G$, in order to simplify notation we will write, for any $r \in R$, $r^g$ instead of $r^{\sigma(g)} = \sigma(g)(r)$.

**Remark 1.2.20** 1) Given a crossed product, if both $\sigma_{R,G}$ and $\varepsilon_{R,G}$ are trivial, the crossed product is just the ordinary group ring.

2) Let $R\#G$ be a skew-group ring. If $G$ is finitely generated, abelian and torsion free, then $R\#G$ is a skew-Laurent ring. Conversely, if $R$ is any ring, $\alpha_1, \dots, \alpha_n$ are pairwise commuting automorphisms of $R$ and $S = R[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1, \dots, \alpha_n]$ the skew-Laurent ring, we can think of $S$ as a crossed product (actually as a skew-group ring) of the form $R * H$ where $H$ is the multiplicative torsionfree group generated by $\theta_1, \dots, \theta_n$, $\sigma$ is the group homomorphism defined by $\sigma(\theta_i) = \alpha_i$ and $\varepsilon$ is trivial.

**Lemma 1.2.21** *Let $R$ be a ring and $G$ a group. Let $R * G$ denote a crossed product. If $G_1$ is a normal subgroup of $G$, then*

$$R * G = (R * G_1) * G/G_1$$

*where the latter is some crossed product of the group $G/G_1$ over the ring $R * G_1$.*

**Proof.** [86, Lemma 1.3] □

The following example not only shows how to apply the lemma but also gives a skew-Laurent ring $S = R\#G$ and $G_1$ a normal subgroup of $G$ such that $S$ is not a skew-group ring over $R\#G_1$.

**Example 1.2.22** Let $S = \mathbb{C}[\theta, \theta^{-1}; \alpha]$ where $\alpha$ is complex conjugation. Hence $S = \mathbb{C}\# < \theta >$. Take $G_1 =< \theta^2 >$. So $< \theta > /G_1 \cong C_2 = \{1, x\}$ and $S = (\mathbb{C}\#G_1) * C_2$ where $\theta^2 = \overline{x}\,\overline{x} = \varepsilon_{\mathbb{C}\#G_1, C_2}(x, x)\overline{x^2} = \varepsilon_{\mathbb{C}\#G_1, C_2}(x, x)$. Also, any other choice of representative $\overline{x}$ for $x$ leads to a non-trivial twisting $\varepsilon$.

**Lemma 1.2.23** *Let $R$ be any ring, $G$ any group and $G_1$ a normal subgroup of $G$. Let $R * G$ be any crossed product. Then any transversal set to $G_1$ in $G$ is a free basis of $R * G$ as a left (or right) $R * G_1$-module.*

**Proof.** [86, §1.1] □

For some groups $G$, there is a version of the Hilbert's Basis Theorem for a crossed product of $G$ over a right Noetherian ring $R$. Before stating this version we need the following definition.

**Definition 1.2.24** A group $G$ is called *polycyclic-by-finite* if $G$ has a finite subnormal series

$$1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$$

with each quotient $G_{i+1}/G_i$ either infinite cyclic or finite for $0 \leq i < n$.

**Proposition 1.2.25** If $R$ is a right (resp. left) Noetherian ring and $G$ is a polycyclic-by-finite group, then the crossed product $R * G$ is also right (resp. left) Noetherian.

**Proof.** [86, Proposition 1.6] □

In [31] E.C. Dade introduced a more general class of ring extensions containing the class of crossed products.

**Definition 1.2.26** Let $R$ be a subring of a ring $S$ and $G$ a group, if there are additive subgroups $S(g)$ of $S$, $g \in G$, such that $S(g)S(h) \subseteq S(gh)$, for any $g, h \in G$ and $R = S(1)$, we say that $S = \oplus_{g \in G} S(g)$ is a *$G$-graded ring over $R$.*

If $S(g)S(h) = S(gh)$ for all $g, h \in G$, we say that $\oplus_{g \in G} S(g)$ is a *strongly $G$-graded ring over $R$.*

## 1.3 Ideals and fraction rings of crossed products

In this section we will be concerned with the ideals of crossed products. A more precise description of the prime ideals will be obtained in the next section for some skew-Laurent rings. We start by describing some relations between ideals of the crossed product and ideals of its basis ring. Also, we will describe the fraction rings of crossed products with respect to some subsets of the basis ring.

**Definition 1.3.1** Let $G$ be a group and $X$ a set. We say that *$G$ acts on $X$* if there is a group homomorphism $\sigma : G \to Sym(X)$, from $G$ to the symmetric group on $X$.

For any $g \in G$, the action of $g$ on $X$ will be denoted by $x \mapsto x^g$, for any $x \in X$.

**Remark 1.3.2** Let $R$ be a ring and $G$ a group. We say that $G$ acts on $R$ if there exists a group homomorphism $\sigma : G \to Aut(R)$.

If $R$ is a ring and $G$ a group, when in Definition 1.2.15 we defined the crossed product $R * G$, we called $\sigma_{R,G}$ the action. One should note that, as we saw in Lemma 1.2.17, in general $G$ doesn't act on $R$ in the sense of Definition 1.3.1.

**Definition 1.3.3** Let $R$ be a ring, $G$ a group acting on the set of ideals of $R$, $I$ an ideal of $R$ and $J$ an ideal of $R$.

We say that $J$ is *$G$-stable* (or *$G$-invariant*) if $J^g = J$, for all $g \in G$.

If $I$ is a proper $G$-stable ideal of $R$ and for all $G$-stable ideals of $R$, $K$ and $J$, whenever $KJ \subseteq I$, either $K \subseteq I$ or $J \subseteq I$, we say that $I$ is a *$G$-prime ideal* of $R$. In particular, we say that $R$ is a *$G$-prime ring* if and only if 0 is a $G$-prime ideal.

The set of all $G$-prime ideals will be denoted by $Spec^G(R)$.

**Remark 1.3.4** In the proof of Lemma 1.2.17, we have seen that given any ring $R$, any group $G$ and any crossed product $R * G$

$$\sigma_{R,G}(g)\sigma_{R,G}(h)(r)\varepsilon_{R,G}(g,h) = \varepsilon_{R,G}(g,h)\sigma_{R,G}(gh)(r)$$

for any $g, h \in G$ and $r \in R$. It is then easy to see that, although $\sigma_{R,G}$ may not be a group homomorphism, it induces one $\sigma^* : G \to Aut(R)/Inn(R)$, from $G$ to $Aut(R)/Inn(R)$. Since $Inn(R)$ fixes all ideals of $R$, $\sigma^*$ gives $G$ an action on the set of ideals of $R$.

For some groups $G$ and some rings $R$, one can get a good description of $Spec^G(R)$. The following lemma is stated here in a more general setting than we will need as its proof in [86] depends only on the conditions stated below.

**Lemma 1.3.5** *Suppose that $R$ is a right Noetherian ring and $G$ is a group acting on the set of ideals of $R$ with an action that preserves inclusion. Then $Q$ is a $G$-prime ideal of $R$ if and only if $Q = \cap_{g \in G} \overline{Q}^g = \overline{Q}^{g_1} \cap \ldots \cap \overline{Q}^{g_m}$ for any $\overline{Q}$ minimal prime of $R$ over $Q$ and some $g_1, \ldots, g_m \in G$ such that $\{\overline{Q}^{g_1}, \ldots, \overline{Q}^{g_m}\}$ forms a single $G$-orbit of $\overline{Q}$. Thus, every $G$-prime ideal of $R$ is semiprime.*

**Proof.** [86, Lemma 14.2] □

**Proposition 1.3.6** *Let $R$ be any ring and $G$ any group. Form any crossed product $R * G$. We have:*

*i) If $I$ is a $G$-stable ideal of $R$, then $I(R * G)$ is an ideal of $R * G$ with $I(R*G) \cap R = I$. Moreover $(R*G)/I(R*G) \cong (R/I)*G$, where the latter is a suitable crossed product of $G$ over $R/I$.*

*ii) If $J$ is an ideal of $R * G$, then $J \cap R$ is a $G$-stable ideal of $R$ and $(J \cap R)(R * G) \subseteq J$.*

*iii) If $P$ is a prime ideal of $R * G$, then $P \cap R$ is a $G$-prime ideal of $R$.*

**Proof.** *i*) and *ii*) follow from [86, Lemma 1.4] and *iii*) follows from [86, Lemma 14.1] □

It may happen that equality won't hold in *ii*) of Proposition 1.3.6, as we will see in the next example.

**Example 1.3.7** Take $S$ to be the skew-Laurent ring $\mathbb{R}[x][\theta, \theta^{-1}; \alpha]$ where $\alpha$ is the $\mathbb{R}$-algebra homomorphism defined by $\alpha(x) = 2x$. Take $P = xS + (\theta - 1)S$. As $S/P \cong \mathbb{R}[\theta, \theta^{-1}]/(\theta - 1)\mathbb{R}[\theta, \theta^{-1}] \cong \mathbb{R}$, $P$ is a prime ideal of $S$ such that $P \cap \mathbb{R}[x] = x\mathbb{R}[x]$. Hence $(P \cap \mathbb{R}[x])S = xS \subsetneqq P$.

Let $R$ be a ring and $X$ a nonempty multiplicative subset of $R$. We say that $X$ is a *right denominator set* if $X$ is *right reversible* (i.e. for any $r \in R$ and $x \in X$ such that $xr = 0$, there exists $y \in X$ such that $ry = 0$) and a *right Ore set* (i.e. for any $r \in R$ and $x \in X$, $rX \cap xR$ is nonempty). Similarly we can define *left reversible*, *left Ore* and *left denominator*. A nonempty subset $X$ of $R$ is said to be a *denominator*, *reversible* or an *Ore set* if it is a right and left denominator, right and left reversible or right and left Ore set, respectively. If $X$ is a right denominator set we can form $RX^{-1}$, the fraction ring of $R$ with respect to $X$ [44, Theorem 9.7].

The quotient rings of group rings have been studied by P.F. Smith in [101]. Part *i*) of the following lemma is a generalization of [101, Lemma 2.6].

**Lemma 1.3.8** *Let $R$ be a ring, $G$ a group and $R * G$ a crossed product.*

*i) If $\mathcal{C}$ is a right denominator set of $R$ and is $G$-invariant, then $\mathcal{C}$ is a right denominator set of $R * G$, the action and twisting can be extended from $R$ to $R\mathcal{C}^{-1}$, and*

$$(R * G)\mathcal{C}^{-1} \cong R\mathcal{C}^{-1} * G.$$

*ii) If $R$ is a semiprime right Goldie ring and $G$ is finite, then $R * G$ is a right order in an Artinian ring and $Q(R * G) = Q(R) * G$.*

**Proof.** Part $i)$ follows from [108, Lemma 4.2] and part $ii)$ from [71, Lemma 1.5]. □

## 1.4 The Passman correspondence for skew-Laurent rings

In [85] D. S. Passman described the prime ideals in crossed products of polycyclic-by-finite groups over a right Noetherian ring. For the case of skew-Laurent rings of the form $R[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n]$ when $R$ is a commutative Noetherian ring and $\alpha_1, \ldots, \alpha_n$ are automorphisms of $R$ commuting pairwise, his description becomes easier. We shall show how to get it using some of this author's results in [86].

Through this section we will assume the following hypothesis.

**Hypothesis 1.4.1** *Let $R$ be a commutative Noetherian ring, let $\alpha_1, \ldots, \alpha_n$ be automorphisms of $R$ commuting pairwise, $H$ the multiplicative abelian torsionfree group freely generated by $\theta_1, \ldots, \theta_n$ and $G$ the multiplicative abelian subgroup of $Aut(R)$ generated by $\alpha_1, \ldots, \alpha_n$. Let $\Psi : H \to G$ be the group epimorphism defined by $\Psi(\theta_i) = \alpha_i$. Let $\mathcal{S} = R[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n] = R\#H$.*

**Remark 1.4.2** As $H$ acts on $R$ via the group epimorphism $\Psi : H \to G$, we will talk about $G$-prime ideals of $R$ instead of $H$-prime ideals.

**Remark 1.4.3** Given an automorphism $\alpha$ of $R$ that commutes with $\alpha_1, \ldots, \alpha_n$, we can define an automorphism $\alpha'$ of $\mathcal{S}$ given by $\alpha'(\sum_J r_J \theta^J) = \sum_J \alpha(r_J)\theta^J$. Instead of $\alpha'$ we will write $\alpha$. Hence we can think of $G$ acting on $\mathcal{S}$.

**Proposition 1.4.4** *Assume Hypothesis 1.4.1. Let $\overline{H}$ be a subgroup of $H$. Then*

*i)* $\mathcal{S} = (R\#\overline{H}) * H/\overline{H}$.

*ii) If $I$ is a $G$-stable ideal of $R*\overline{H}$, then $I\mathcal{S}$ is an ideal of $\mathcal{S}$ with $I\mathcal{S} \cap (R\#\overline{H}) = I$ and $\mathcal{S}/I\mathcal{S} \cong (R\#\overline{H})/I * H/\overline{H}$.*

*iii) If $J$ is an ideal of $\mathcal{S}$, then $J \cap (R\#\overline{H})$ is a $G$-stable ideal of $R\#\overline{H}$ and $[J \cap (R\#\overline{H})]\mathcal{S} \subseteq J$.*

*iv) If $P$ is a prime ideal of $\mathcal{S}$, then $P \cap (R\#\overline{H})$ is a $G$-prime ideal of $R\#\overline{H}$.*

**Proof.** *i*) follows from Lemma 1.2.21. By Remark 1.4.3, it is easy to see that $G$ acts on $R\#\overline{H}$. The proofs of *ii*), *iii*) and *iv*) are similar to the ones of Proposition 1.3.6. □

**Hypothesis 1.4.5** *Let $N$ be a $G$-prime ideal of $R$.*

Since $N$ is a $G$-prime ideal of $R$, by Lemma 1.3.5, we can write $N = \cap_{\alpha \in G} M^{\alpha} = M \cap M^{x_2} \cap \ldots \cap M^{x_t}$ for some $M$, minimal prime of $R$ over $N$ and $x_2, \ldots, x_t \in G$ such that $\{M, M^{x_2}, \ldots, M^{x_t}\}$ forms a single $G$-orbit of $M$ and $M \neq M^{x_i} \neq M^{x_j}$ for all $i, j \in \{2, \ldots, t\}$ such that $i \neq j$ and $x_i, x_j \neq 1_G$. Assume $x_1 = 1_G$. So $M = M^{x_1}$.

Let $G_{1,M}$ be the subgroup of $G$ defined by

$$G_{1,M} = \{\alpha \in G : M^{\alpha} = M\}$$

and $H_{1,M} = \Psi^{-1}(G_{1,M})$. Thus $H/H_{1,M} \cong G/G_{1,M}$ and this group is finite, being isomorphic to a subgroup of the symmetric group on $t$ symbols. Take $\mathcal{S}_{1,M} = R\#H_{1,M}$, the skew-group ring of $H_{1,M}$ over $R$, so $\mathcal{S}_{1,M}$ is a subring of $\mathcal{S}$.

**Proposition 1.4.6** *Let $P_1$ be a $G$-prime ideal of $\mathcal{S}_{1,M}$ such that $P_1 \cap R = N$. There exists $\tilde{P}_1$, a minimal prime of $\mathcal{S}_{1,M}$ over $P_1$ and $\beta_1, \ldots, \beta_s \in G$, such that $P_1 = \cap_{\alpha \in G} \tilde{P}_1^{\alpha} = \tilde{P}_1^{\beta_1} \cap \ldots \cap \tilde{P}_1^{\beta_s}$, $\tilde{P}_1 \cap R = M$ and $\{\tilde{P}_1^{\beta_1}, \ldots, \tilde{P}_1^{\beta_s}\}$ forms a single $G$-orbit of $\tilde{P}_1$.*

**Proof.** Let $P_1$ be a $G$-prime ideal of $\mathcal{S}_{1,M}$ such that $P_1 \cap R = N$. Therefore $P_1 = \cap_{\alpha \in G} \tilde{P}_1^{\alpha} = \tilde{P}_1^{\beta_1} \cap \ldots \cap \tilde{P}_1^{\beta_s}$, for some $\beta_i \in G$ and some $\tilde{P}_1$, minimal prime of $\mathcal{S}_{1,M}$ over $P_1$.

Since $N$ is a $G$-stable ideal of $R$, $N\mathcal{S}_{1,M}$ is an ideal of $\mathcal{S}_{1,M}$. As $M$ is $G_{1,M}$-stable, $M\mathcal{S}_{1,M}$ is an ideal of $\mathcal{S}_{1,M}$. As $G$ is abelian we have also that for any $i \in \{2, \ldots, t\}$, $M^{x_i}$ is $G_{1,M}$-stable, hence $M^{x_i}\mathcal{S}_{1,M}$ are ideals of $\mathcal{S}_{1,M}$, for each $i \in \{2, \ldots, t\}$. Now $N\mathcal{S}_{1,M} = M\mathcal{S}_{1,M} \cap M^{x_2}\mathcal{S}_{1,M} \cap \ldots \cap M^{x_t}\mathcal{S}_{1,M} \subseteq P_1$. So there is $x_i$ such that $M^{x_i}\mathcal{S}_{1,M} \subseteq \tilde{P}_1$. Without loss of generality, we can assume $M \subseteq \tilde{P}_1 \cap R$. Since $\prod_{j=1}^{s}(\tilde{P}_1^{\beta_j} \cap R) \subseteq P_1 \cap R = N \subseteq M$, there is $j \in \{1, \ldots, t\}$ such that $M^{\beta_j} \subseteq \tilde{P}_1^{\beta_j} \cap R \subseteq M \subseteq \tilde{P}_1 \cap R$. As $M^{\beta_j}$ and $M$ are minimal over the $G$-stable ideal $N$, $M^{\beta_j} = \tilde{P}_1^{\beta_j} \cap R = M$, hence $M = \tilde{P}_1 \cap R$ and the result follows.
$\square$

**Proposition 1.4.7** *Let $P \in Spec(\mathcal{S})$ such that $P \cap R = N$. Then*

$$P = (P \cap \mathcal{S}_{1,M})\mathcal{S}.$$

**Proof.** Let $P$ be a prime ideal of $\mathcal{S}$ such that $P \cap R = N$ and $\tilde{P}_1$ be a minimal prime of $\mathcal{S}_{1,M}$ over $P \cap \mathcal{S}_{1,M}$ as in Proposition 1.4.6. Since the stabilizer of $\tilde{P}_1$ in $H/H_{1,M}$ is $\{H_{1,M}\}$ it follows from [86, Corollary 14.8] that $(P \cap \mathcal{S}_{1,M})\mathcal{S}$ is prime.

It's obvious that $(P \cap \mathcal{S}_{1,M})\mathcal{S} \subseteq P$. Since $H/H_{1,M}$ is finite, by [86, Theorem 16.2], $P$ is a minimal prime over the ideal $(P \cap \mathcal{S}_{1,M})\mathcal{S}$. Then $P = (P \cap \mathcal{S}_{1,M})\mathcal{S}$.
$\square$

For every $\gamma \in G_{1,M}$, we can think of $\overline{\overline{\gamma}}$, the automorphism of $R/M$ induced by $\gamma$, such that $\overline{\overline{\gamma}}(r+M) = \gamma(r)+M$, and define $\overline{\mathcal{S}}_{1,M} = \mathcal{S}_{1,M}/M\mathcal{S}_{1,M} = R/M\#H_{1,M}$ a skew-group ring over the ring $R/M$.

Take $G_{2,M} = \{\gamma \in G_{1,M} : \overline{\overline{\gamma}} = id\}$ and $H_{2,M} = \Psi^{-1}(G_{2,M}) \subseteq H_1$. Note that $G_{2,M} = \{\gamma \in G : \gamma(r) - r \in M, \text{ for all } r \in R\}$. Form $\mathcal{S}_{2,M} = R\#H_{2,M} \subseteq \mathcal{S}_{1,M} \subseteq \mathcal{S}$. So $\overline{\mathcal{S}}_{2,M} = \mathcal{S}_{2,M}/M\mathcal{S}_{2,M} \cong (R/M)H_{2,M}$, the group ring of $H_{2,M}$ over $R/M$.

Let $K$ be the fraction field of the commutative domain $R/M$. For each $\gamma \in G_{1,M}$, we can extend $\overline{\overline{\gamma}}$ to an automorphism $\gamma''$ of $K$ in the obvious way. We will abuse notation and write just $\gamma$ instead of $\overline{\overline{\gamma}}$ or $\gamma''$. Let $\mathcal{S}'_{1,M} = (K \# H_{1,M})$.

**Lemma 1.4.8** *Let $\mathcal{S}'_{1,M}$, $K$ and $H_{2,M}$ be as above. Then $C_{\mathcal{S}'_{1,M}}(K) = KH_{2,M}$ which is a commutative domain.*

**Proof.** Let $x = \sum_{h_1 \in H_{1,M}} k_{h_1} h_1 \in C_{\mathcal{S}'_{1,M}}(K)$. Then, for all $k \in K$, $xk = kx$. That is, $\sum_{h_1 \in H_{1,M}} k_{h_1} \Psi(h_1)(k) h_1 = \sum_{h_1 \in H_{1,M}} k k_{h_1} h_1$. So $k_{h_1}\Psi(h_1)(k) = kk_{h_1}$, for each $h_1 \in H_{1,M}$. Since $K$ is a field, $\Psi(h_1)(k) = k$, for all $k \in K$, whenever $k_{h_1}$ is nonzero. Therefore each $h_1$ with a nonzero coefficient is in $H_{2,M}$ and then $x \in KH_{2,M}$. It's obvious that $KH_{2,M} \subseteq C_{\mathcal{S}'_{1,M}}(K)$. Consequently, $C_{\mathcal{S}'_{1,M}}(K) = KH_{2,M}$. □

Take $\mathfrak{G}'$ a transversal set to $H_{2,M}$ in $H_{1,M}$. Then by Lemma 1.2.23, this set is a basis of $\mathcal{S}_{1,M}$, $\overline{\mathcal{S}}_{1,M}$ or $\mathcal{S}'_{1,M}$ over $\mathcal{S}_{2,M}$, $\overline{\mathcal{S}}_{2,M}$ or $KH_{2,M}$, respectively.

**Proposition 1.4.9** *Let $P'_2$ be a $G_{1,M}$-stable ideal of $KH_{2,M}$. Then*

$$C_{\mathcal{S}'_{1,M}/P'_2\mathcal{S}'_{1,M}}(\{k + P'_2\mathcal{S}'_{1,M} : k \in K\}) = \{s_2 + P'_2\mathcal{S}'_{1,M} : s_2 \in KH_{2,M}\}.$$

**Proof.** If $P'_2$ is a $G_{1,M}$-stable ideal of $C_{\mathcal{S}'_{1,M}}(K) = KH_{2,M}$, $P'_2\mathcal{S}'_{1,M}$ is an ideal of $\mathcal{S}'_{1,M}$. It's obvious that $\{s_2 + P'_2\mathcal{S}'_{1,M} : s_2 \in KH_{2,M}\} \subseteq C_{\mathcal{S}'_{1,M}/P'_2\mathcal{S}'_{1,M}}(\{k + P'_2\mathcal{S}'_{1,M} : k \in K\})$.

We can think of $KH_{2,M}$ as being a $K$-vector space and of $P'_2$ as a $K$-subspace of $KH_{2,M}$. Let $Q'_2$ be a complement of $P'_2$ in $KH_{2,M}$. Let $s_2 + P'_2\mathcal{S}'_{1,M} \in C_{\mathcal{S}'_{1,M}/P'_2\mathcal{S}'_{1,M}}(\{k + P'_2\mathcal{S}'_{1,M} : k \in K\})$ and $s_2 = \sum_{h_2 \in \mathfrak{G}'} s_{h_2} h_2$, for some $s_{h_2} \in KH_{2,M}$. Since $KH_{2,M} = P'_2 \oplus Q'_2$, without loss of generality, we can suppose that $s_2 = \sum_{h_2 \in \mathfrak{G}'} q_{h_2} h_2$, for some $q_{h_2} \in Q'_2$. For all $k \in K$, we have

$$ks_2 - s_2k = \sum_{h_2 \in \mathfrak{G}'} (kq_{h_2} - q_{h_2}\Psi(h_2)(k))h_2.$$

Since $q_{h_2} \in Q'_2 \subseteq KH_{2,M}$, $ks_2 - s_2k = \sum_{h_2 \in \mathfrak{G}'} (k - \Psi(h_2)(k))q_{h_2}h_2$.

By hypothesis, $ks_2 - s_2k \in P'_2\mathcal{S}'_{1,M}$. So, for each $h_2 \in \mathfrak{G}'$,

$$(k - \Psi(h_2)(k))q_{h_2} \in P'_2 \cap Q'_2.$$

So, $ks_2 - s_2k = 0$. Therefore $s_2 \in KH_{2,M}$ and the result follows. □

**Proposition 1.4.10** *Let $P'_1$ be an ideal of $\mathcal{S}'_{1,M}$. Then $(P'_1 \cap KH_{2,M})\mathcal{S}'_{1,M} = P'_1$.*

**Proof.** It's obvious that $P'_2 = P'_1 \cap KH_{2,M}$ is a $G_{1,M}$-stable ideal of $KH_{2,M}$. We have that $P'_2\mathcal{S}'_{1,M} \subseteq P'_1$. Suppose that $P'_2\mathcal{S}'_{1,M} \subsetneq P'_1$ and take $p_1 \in P'_1 \backslash P'_2\mathcal{S}'_{1,M}$. So $p_1 = \sum_{h_1 \in H_{1,M}} k_{h_1}h_1$, for some $k_{h_1} \in K$. We can take $p_1$ with a minimal number of nonzero $k_{h_1}$. Multiplying $p_1$ by an element of $H_{1,M}$, if necessary, we may assume $k_{1_H} \neq 0$, where $1_H$ is the identity of $H$.

Let $k \in K$. Then $kp_1 - p_1k \in P'_1$, but

$$\begin{aligned} kp_1 - p_1k &= \sum_{h_1 \in H_{1,M}} (kk_{h_1}h_1 - \Psi(h_1)(k)k_{h_1}h_1) \\ &= \sum_{h_1 \in H_{1,M}} (kk_{h_1} - \Psi(h_1)(k)k_{h_1})h_1 \end{aligned}$$

has coefficient in $1_H$ equal to 0. So, by the choice of $p_1$, we have

$$kp_1 - p_1k \in P'_2\mathcal{S}'_{1,M}.$$

Since $k$ was arbitrary in $K$, $p_1 + P'_2\mathcal{S}'_{1,M} \in C_{\mathcal{S}'_{1,M}/P'_2\mathcal{S}'_{1,M}}(\{k + P'_2\mathcal{S}'_{1,M} : k \in K\})$. By Proposition 1.4.9, there is $p'_1 \in KH_{2,M}$ such that $p_1 + P'_2\mathcal{S}'_{1,M} = p'_1 + P'_2\mathcal{S}'_{1,M}$. Then $p'_1 \in P'_2$, so $p_1 \in P'_2\mathcal{S}'_{1,M}$, what contradicts the choice of $p_1$. Therefore $(P'_1 \cap KH_{2,M})\mathcal{S}'_{1,M} = P'_1$. □

Let $\oplus_{i=1}^t K$ be the direct sum of $t$ copies of $K$. Given any $\gamma \in G_{1,M}$, then $\gamma$ fixes $M$ and all its $G$-conjugates, so $\gamma$ induces an automorphism of $\oplus_{i=1}^t K$ such that $\gamma(k_1, \ldots, k_t) = (\gamma(k_1), \ldots, \gamma(k_t))$, for any $k_1, \ldots, k_t \in K$. Once again we will abuse notation and write just $\gamma$ for these automorphisms. Let $\mathcal{S}''_{1,M} = (\oplus_{i=1}^t K)\# H_{1,M}$ and $\mathcal{S}''_{2,M} = (\oplus_{i=1}^t K)H_{2,M}$.

**Corollary 1.4.11** *Let $I$ be any ideal of $\mathcal{S}''_{1,M}$. Then $I = (I \cap \mathcal{S}''_{2,M})\mathcal{S}''_{1,M}$.*

**Proof.** Let $e_1, \ldots, e_t$ be the primitive central idempotents of $\oplus_{i=1}^t K$. Also, by definition of $H_{1,M}$, we have that the same elements are primitive central elements of $\mathcal{S}''_{1,M}$ and that for all $i \in \{1, \ldots, t\}$, $\mathcal{S}''_{1,M}e_i = e_i\mathcal{S}''_{1,M} \cong K\#H_{1,M}$ and $\mathcal{S}''_{2,M}e_i = e_i\mathcal{S}''_{2,M} \cong KH_{2,M}$. Then, for all $i \in \{1, \ldots, t\}$, $Ie_i$ is an ideal of $e_i\mathcal{S}''_{1,M} \cong K\#H_{1,M}$ and by Proposition 1.4.10, $Ie_i = (I \cap \mathcal{S}''_{2,M}e_i)\mathcal{S}''_{1,M}e_i$.

So, we have

$$\begin{aligned} I &= \oplus_{i=1}^t Ie_i \\ &= \oplus_{i=1}^t (I \cap \mathcal{S}''_{2,M}e_i)\mathcal{S}''_{1,M}e_i \\ &= [\oplus_{i=1}^t (I \cap \mathcal{S}''_{2,M}e_i)][\oplus_{i=1}^t (I \cap \mathcal{S}''_{1,M}e_i)] \\ &= (I \cap \mathcal{S}''_{2,M})\mathcal{S}''_{1,M} \end{aligned}$$

and the result follows. $\square$

**Proposition 1.4.12** *Let $P$ be a prime ideal of $\mathcal{S}$ such that $P \cap R = N$. Then*

$$P = (P \cap \mathcal{S}_{2,M})\mathcal{S}.$$

**Proof.** By Proposition 1.4.6, $P \cap \mathcal{S}_{1,M} = \cap_{\alpha \in G} \tilde{P}_1^{\alpha} = \tilde{P}_1^{\beta_1} \cap \ldots \cap \tilde{P}_1^{\beta_s}$, for some $\beta_1, \ldots, \beta_s \in G$ and $\tilde{P}_1$ a minimal prime of $\mathcal{S}_{1,M}$ over $P \cap \mathcal{S}_{1,M}$ such that $\tilde{P}_1 \cap R = M$.

Let $\overline{P}_1 = \tilde{P}_1/M\mathcal{S}_{1,M}$, a prime ideal of $\overline{\mathcal{S}}_{1,M}$ and $\mathcal{C} = \mathcal{C}_{R/M}(0)$. Since $R/M$ is a commutative ring, $\mathcal{C}$ is an Ore set in $R/M$. As $M$ is $G_{1,M}$- and $G_{2,M}$-invariant, by Lemma 1.3.8, $\mathcal{C}$ is an Ore set in $\overline{\mathcal{S}}_{1,M}$ and in $\overline{\mathcal{S}}_{2,M}$, $\overline{\mathcal{S}}_{1,M}\mathcal{C}^{-1} \cong \mathcal{S}'_{1,M}$ and $\overline{\mathcal{S}}_2\mathcal{C}^{-1} \cong KH_{2,M}$. By [44, Theorem 9.22], $\overline{P}_1\mathcal{C}^{-1}$ is a prime ideal of $\mathcal{S}'_{1,M}$. By Proposition 1.4.10,

$$\overline{P}_1\mathcal{C}^{-1} = (\overline{P}_1\mathcal{C}^{-1} \cap KH_{2,M})\mathcal{S}'_{1,M}.$$

Since $\overline{P}_1 \cap \mathcal{C} = \emptyset$, $\overline{\mathcal{S}}_{1,M}/\overline{P}_1$ is $\mathcal{C}$-torsionfree [44, Lemma 9.21]. Hence $\overline{\mathcal{S}}_{2,M}/(\overline{P}_1 \cap \overline{\mathcal{S}}_{2,M})$ is $\mathcal{C}$-torsionfree and furthermore

$$\overline{\mathcal{S}}_{1,M}/(\overline{P}_1 \cap \overline{\mathcal{S}}_{2,M})\overline{\mathcal{S}}_{1,M} \qquad \text{is } \mathcal{C}\text{-torsionfree.} \qquad (1)$$

Now we have

$$
\begin{aligned}
\overline{P}_1 &= \overline{P}_1\mathcal{C}^{-1} \cap \overline{\mathcal{S}}_{1,M} \\
&= (\overline{P}_1\mathcal{C}^{-1} \cap KH_2)\mathcal{S}'_{1,M} \cap \overline{\mathcal{S}}_{1,M} && \text{by Proposition 1.4.10} \\
&= (\overline{P}_1 \cap \overline{\mathcal{S}}_{2,M})\mathcal{S}'_{1,M} \cap \overline{\mathcal{S}}_{1,M} \\
&= (\overline{P}_1 \cap \overline{\mathcal{S}}_{2,M})\overline{\mathcal{S}}_{1,M} && \text{by (1) and [44, Theorem 9.17].}
\end{aligned}
$$

So

$$\tilde{P}_1 = (\tilde{P}_1 \cap \mathcal{S}_{2,M})\mathcal{S}_{1,M} \quad (2)$$

and

$$
\begin{aligned}
P \cap \mathcal{S}_{1,M} &= \cap_{j=1}^{s} \tilde{P}_1^{\beta_j} && \text{by Proposition 1.4.6} \\
&= \cap_{j=1}^{s} [(\tilde{P}_1 \cap \mathcal{S}_{2,M})\mathcal{S}_{1,M}]^{\beta_j} && \text{by (2)} \\
&= [(\cap_{j=1}^{s} (\tilde{P}_1 \cap \mathcal{S}_{2,M})^{\beta_j})\mathcal{S}_{1,M}] && \text{since } \mathcal{S}_{1,M} \text{ is free over } \mathcal{S}_{2,M} \\
&= [(\cap_{j=1}^{s} \tilde{P}_1^{\beta_j}) \cap \mathcal{S}_{2,M}]\mathcal{S}_{1,M} \\
&= [P \cap \mathcal{S}_{1,M} \cap \mathcal{S}_{2,M}]\mathcal{S}_{1,M} && \text{by Proposition 1.4.6} \\
&= (P \cap \mathcal{S}_{2,M})\mathcal{S}_{1,M}.
\end{aligned}
$$

Now by Proposition 1.4.7, $P = (P \cap \mathcal{S}_{1,M})\mathcal{S} = (P \cap \mathcal{S}_{2,M})\mathcal{S}_{1,M}\mathcal{S} = (P \cap \mathcal{S}_{2,M})\mathcal{S}$.

□

**Theorem 1.4.13 (Passman)** *There is an one to one correspondence, $\mathcal{P}$, from the set $\{P \in Spec(\mathcal{S}) : P \cap R = N\}$ onto the set $\{P_2 \in Spec^G(\mathcal{S}_{2,M}) : P_2 \cap R = N\}$ such that $\mathcal{P}(P) = P \cap \mathcal{S}_{2,M}$ and $P = (P \cap \mathcal{S}_{2,M})\mathcal{S}$.*

**Proof.** If $P \in Spec(\mathcal{S})$ such that $P \cap R = N$, then $P \cap \mathcal{S}_{2,M}$ is obviously a $G$-stable ideal of $\mathcal{S}_{2,M}$. Moreover $P \cap \mathcal{S}_{2,M}$ is $G$-prime by Proposition 1.4.4.

If $P_2$ is a $G$-prime ideal of $\mathcal{S}_{2,M}$ such that $P_2 \cap R = N$, $P_2\mathcal{S} = \mathcal{S}P_2$ is an ideal of $\mathcal{S}$ and by Proposition 1.4.4, $P_2\mathcal{S} \cap \mathcal{S}_{2,M} = P_2$. Also, we have that $P_2\mathcal{S}_{1,M} = \mathcal{S}_{1,M}P_2$ is an ideal of $\mathcal{S}_{1,M}$. We claim that it is a $G$-prime ideal of $\mathcal{S}_{1,M}$. Let $A_1, B_1$ be $G$-stable ideals of $\mathcal{S}_{1,M}$ such that $A_1B_1 \subseteq P_2\mathcal{S}_{1,M}$, where we may assume without loss of generality that $N \subseteq (A_1 \cap R) \cap (B_1 \cap R)$. Then $(A_1 \cap \mathcal{S}_{2,M})(B_1 \cap \mathcal{S}_{2,M}) \subseteq P_2$

and $A_1 \cap \mathcal{S}_{2,M}$, $B_1 \cap \mathcal{S}_{2,M}$ are $G$-stable ideals of $\mathcal{S}_{2,M}$. Therefore $A_1 \cap \mathcal{S}_{2,M} \subseteq P_2$ or $B_1 \cap \mathcal{S}_{2,M} \subseteq P_2$. If $A_1 \cap \mathcal{S}_{2,M} \subseteq P_2$, then by Corollary 1.4.11, we deduce that $A_1 = (A_1 \cap \mathcal{S}_{2,M})\mathcal{S}_{1,M}$, so that $A_1 \subseteq P_2\mathcal{S}_{1,M}$. Hence, $P_2\mathcal{S}_{1,M}$ is a $G$-prime ideal of $\mathcal{S}_{1,M}$. Then, by Proposition 1.4.6, $P_2\mathcal{S}_{1,M} = \cap_{\alpha \in G} \tilde{P}_1^{\alpha}$, for some minimal prime $\tilde{P}_1$ of $\mathcal{S}_{1,M}$ over $P_2\mathcal{S}_{1,M}$ such that $\tilde{P}_1 \cap R = M$. Now, as $H/H_{1,M}$ is finite and the stabilizer of $\tilde{P}_1/P_2\mathcal{S}_{1,M}$ is $\{H_{1,M}\}$, it follows, by [86, Corollary 14.8], that $\mathcal{S}/P_2\mathcal{S} \cong (\mathcal{S}_{1,M}/P_2\mathcal{S}_{1,M}) * H/H_{1,M}$ is prime. So $P_2\mathcal{S}$ is a prime ideal of $\mathcal{S}$.

By Proposition 1.4.12, we get the desired result. □

## 1.5 Additional remarks

1. All definitions and results of this chapter are well known.

2. The main references for §2 and §3 are [44], [76] and [86].

3. The main references for §4 are [86] and [85].

# Chapter 2

# Prime Links in Skew-Laurent and Skew-Polynomial Rings

Given a commutative domain $R$ and a prime ideal $P$, we can form a ring, containing the first, where the elements of $R\backslash P$ become units - that is, we localize $R$ at $P$. One could try to extend this process to noncommutative rings but, even in Noetherian rings, we have some obstructions to localization. Some of these obstructions are caused by the existence of some relations between prime ideals; given two primes "related" in this way, it is impossible to localize at one without localizing at the other as well. These relations are called *links*, or *prime links*.

In a Noetherian ring $R$, there is a link from $P$ to $Q$, for $P, Q$ prime ideals of $R$, if there is an ideal $A$ of $R$ such that $PQ \subseteq A \subsetneqq P \cap Q$ and $(P \cap Q)/A$ is torsionfree as a left $R/P$-module and as a right $R/Q$-module. In such a case we will write $P \rightsquigarrow Q$.

The *graph of links* of $R$ is the directed graph whose vertices are the elements of $Spec(R)$ with an arrow from $P$ to $Q$ whenever $P \rightsquigarrow Q$. The connected components of this graph are called *cliques*, and if $P \in Spec(R)$ the unique clique containing $P$ will be denoted by $C\ell(P)$.

Links between prime ideals play also an important role in the representation theory of Noetherian rings, see for instance [26]. For some classes of noncommutative Noetherian rings, it was possible to describe the links between prime ideals, for instance, for group rings ( K.A. Brown in [19], [20] ); universal enveloping algebras ( K.A. Brown and F. Du Cloux in [22], [21] ) and certain differential operator rings ( G. Sigurdson in [98] and K.R. Goodearl in [43] ).

In [56], Jategaonkar introduced a condition $(*)$ for Noetherian rings which enables us to localize the ring with respect to the intersection of primes in a finite clique, [44, Theorem 12.21]. This condition became known as the *second layer condition*. A similar condition was introduced by K.A. Brown in [18]. In [55], Jategaonkar introduced another condition $(^{*}_{*})$, later called the *strong second layer condition* by the same author in [57].

In this chapter we describe the prime links in $\mathcal{T} = R[\theta_1, \dots, \theta_n; \alpha_1, \dots, \alpha_n]$ and $\mathcal{S} = R[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1, \dots, \alpha_n]$, the skew-polynomial ring and the skew-Laurent ring, respectively, when $R$ is a commutative Noetherian ring and $\alpha_1, \dots, \alpha_n$ are pairwise commuting automorphisms of $R$. Let $G$ be the group generated by $\alpha_1, \dots, \alpha_n$, $H$ the group generated by $\theta_1, \dots, \theta_n$ and $\Psi$ the group epimorphism from $H$ onto $G$ such that $\Psi(\theta_i) = \alpha_i$, for every $i \in \{1, \dots, n\}$.

It is known that $\mathcal{S}$ satisfies the second layer condition, [13, Corollary 7.4]. In §2 we shall show that $\mathcal{S}$ is AR-separated, Theorem 2.2.11, so it satisfies the strong second layer condition. Using this fact, it is then possible to prove that $\mathcal{T}$ satisfies the strong second layer condition, Corollary 2.2.14.

The study of links between prime ideals in skew-polynomial rings for just one automorphism was carried out by Poole [88]. This chapter extends his results in a similar spirit to the work of Goodearl [43], who described the graph of links of certain differential operator rings over a commutative Noetherian $\mathbb{Q}$-algebra.

Let $N$ be a $G$-prime ideal of $R$, so $N = \bigcap_{\alpha \in G} M^{\alpha}$ for a minimal prime $M$ over $N$ in $R$, Lemma 1.3.5. By a result of Passman, Theorem 1.4.13, there is a one to one correspondence between prime ideals of $\mathcal{S}$ contracting to $N$ and some semiprime (actually $G$-prime) ideals of a new skew-Laurent ring $\mathcal{S}_2$, such that $\mathcal{S}_2/M\mathcal{S}_2 \cong (R/M)[\gamma_1, \gamma_1^{-1}, \dots, \gamma_v, \gamma_v^{-1}]$. Let $\overline{\mathcal{S}}_2 = \mathcal{S}_2/M\mathcal{S}_2$. By a suitable localization we will be able to assume that $M$ is a maximal ideal of $R$ and that $R$ is a semilocal ring with Jacobson radical $N$. Write $K = R/M$, a field.

We start by describing the links between prime ideals contracting in $R$ to $M$ in the special skew-Laurent ring $\mathcal{S}_2$. The links between prime ideals of $\mathcal{S}_2$ contracting to $M$ will be described in §3 with $K^\sharp$-automorphisms, $\sigma_1, \dots, \sigma_u$, of $K^\sharp \otimes \overline{\mathcal{S}}_2 = \mathcal{S}_2^\sharp$, where $K^\sharp$ is the algebraic closure of $K = R/M$. For each $i \in \{1, \dots, u\}$ and $j \in \{1, \dots, v\}$, let $\sigma_i(\gamma_j) = \varepsilon_{ij}\gamma_j$ for some $\varepsilon_{ij} \in K^\sharp \backslash \{0\}$, determined by the action of the subgroup of $G$ generated by $\Psi(\gamma_1), \dots, \Psi(\gamma_v)$ on $K^\sharp \otimes M/M^2$. We will show in Theorem 2.3.9 that if $\tilde{P}_2$ and $\tilde{Q}_2$ are distinct prime ideals of $\mathcal{S}_2$ contracting to $M$, then $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ if and only if there is $i \in \{1, \dots, u\}$ and $P^\sharp$, $Q^\sharp$ are prime ideals of $\mathcal{S}_2^\sharp$ lying over $\tilde{P}_2/M\mathcal{S}_2$ and $\tilde{Q}_2/M\mathcal{S}_2$, respectively, such that $\sigma_i(P^\sharp) = Q^\sharp$.

In §4, we prove that the links between distinct prime ideals $P$ and $Q$ of $\mathcal{S}$ both distinct from $N\mathcal{S}$ and contracting to $N$, arise from links between minimal primes of $\mathcal{S}_2$ over $P \cap \mathcal{S}_2$ and $Q \cap \mathcal{S}_2$ both contracting to the maximal ideal $M$, Theorem 2.4.14.

Ignoring here for the moment technical complications caused by passage to algebraic closures, we can sum up the above results as: *the clique $\mathcal{C}\ell(P)$ of a prime ideal $P$ of $\mathcal{S}$ consists of the set of images of $P$ under the action of a finitely generated abelian group of $R$-automorphisms of $\mathcal{S}$. These automorphisms are determined by the action of $\mathcal{S}_2$ in $M/M^2$.*

In §5, to describe the prime links in $\mathcal{T}$ we reduce this problem to the same one in $\mathcal{S}$, Corollary 2.5.2. In the final section, §6, we give some examples to illustrate the computation of cliques in some skew-Laurent rings and in skew-polynomial rings.

## 2.1 Definitions and background

In this section we introduce the notation, definitions and properties we will need for the rest of this chapter.

Let $R$ be a ring. Given a right $R$-module M and $X$ a right Ore set, we denote by $\mathfrak{t}_X(M) = \{m \in M : mx = 0 \text{ for some } x \in X\}$, the *$X$-torsion submodule* of $M$. The module $M$ is said to be *$X$-torsion* if $\mathfrak{t}_X(M) = M$ and *$X$-torsionfree* if $\mathfrak{t}_X(M) = 0$. In the case $M = R$, $\mathfrak{t}_X(R)$ is an ideal of $R$. If $R$ is a semiprime Goldie ring (in particular if it is a semiprime Noetherian ring), the set of regular elements is an Ore set, by Goldie's Theorem [44, Theorem 5.10]. In this case instead of talking about a $\mathcal{C}_R(0)$-torsion or a $\mathcal{C}_R(0)$-torsionfree module, we will only say *$R$-torsion* (or torsion as an $R$-module) or *$R$-torsionfree* (or torsionfree as an $R$-module).

**Definition 2.1.1** Let $P$ and $Q$ be prime ideals in a Noetherian ring $R$. We say that $P$ is *linked* to $Q$ and write $P \rightsquigarrow Q$, if there is an ideal $A$ of $R$ such that $PQ \subseteq A \subsetneqq P \cap Q$ and $(P \cap Q)/A$ is torsion free as a left $R/P$-module and as a right $R/Q$-module.

There are other definitions of links, the ones just defined are usually called second layer links, although in this thesis we will just call them links. This definition was first introduced by Müller in [83] for some special type of Noetherian rings, the fully bounded Noetherian ones. Another type of link is defined below.

**Definition 2.1.2** Let $P$ and $Q$ be prime ideals in a Noetherian ring. We say that there is an *ideal link* or *internal bond* from $P$ to $Q$, if there are ideals of $R$, $J \subsetneqq I$ such that $PI \subseteq J$, $IQ \subseteq J$ and $I/J$ is a torsionfree right $R/Q$-module and a torsionfree left $R/P$-module.

There is a *bimodule link* or *bond* from $P$ to $Q$, if there exists a nonzero $(R/P, R/Q)$-bimodule which is finitely generated and torsionfree on each side.

**Definition 2.1.3** Let $R$ be a Noetherian ring. The *graph of links* of $R$ is the directed graph whose vertices are the elements of $Spec(R)$ with an arrow from $P$ to $Q$ whenever $P \rightsquigarrow Q$. The connected components of this graph are called *cliques*, and if $P \in Spec(R)$ the unique clique containing $P$ will be denoted by $\mathcal{C}\ell(P)$.

We say that $P \in Spec(R)$ has only a *trivial link* if whenever $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$, we have $P = Q$.

**Remark 2.1.4** 1) In a commutative Noetherian ring the only possible links are the trivial ones. Thus in a commutative Noetherian domain, for every prime $P$, $\mathcal{C}l(P) = \{P\}$.

2) If $P$ and $Q$ are prime ideals in a Noetherian ring and $P \rightsquigarrow Q$, it may happen that $Q$ is not linked to $P$. For instance take $R = \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$, $P = \begin{bmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$ and $Q = \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{bmatrix}$. As $QP = Q \cap P$, $Q$ is not linked to $P$. To prove that $P$ is linked to $Q$, as $P$ and $Q$ are maximal ideals of $R$, it is enough to notice that $(P \cap Q)/PQ$ is nonzero.

Given a Noetherian ring $R$, there is a relation between prime links and some series for some $R$-modules. This relation is given by Jategaonkar's Main Lemma, Theorem 2.1.12. Before stating this theorem we need some more notation.

**Definition 2.1.5** Let $R$ be a ring and $M$ a right $R$-module. A prime ideal $P$ of $R$ is an *associated prime* of $M$ if there exists a submodule $0 \neq N \subseteq M$ such that $P = \text{r.ann}_R(N')$, for all $0 \neq N' \leq N$. The set of all associated primes of $M$ is denoted by $Ass(M)$.

**Lemma 2.1.6** *If $U$ is a uniform right module over a right Noetherian ring $R$, then there exists a unique associated prime of $U$.*

**Proof.** [44, Lemma 4.22] □

**Definition 2.1.7** If $U$ is a uniform right module over a right Noetherian ring, the unique associated prime of $U$ is called the *assassinator* of $U$.

It is easy to see that if $M$ is a right module over a right Noetherian ring that $Ass(M)$ equals the set of assassinators of uniform submodules of $M$. The next proposition follows easily from the definitions given before.

**Proposition 2.1.8** *Let $M$ be a right module over a Noetherian ring. For any submodule $N$ of $M$, $Ass(N) \subseteq Ass(M)$. Moreover, if $N$ is an essential submodule of $M$, then $Ass(N) = Ass(M)$.*

**Proof.** [57, Proposition 4.2.1] □

**Definition 2.1.9** Given a prime ideal $P$ of a ring $R$, a right $R$-module $M$ is called *$P$-primary* if $Ass(M) = \{P\}$.

**Definition 2.1.10** An *affiliated series* of a right $R$-module $M$ is a sequence of submodules of $M$

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_{n-1} \subsetneq M_n = M$$

together with a set of prime ideals of $R$, $\{P_1, \dots, P_n\}$ called affiliated primes such that each $P_i$ is maximal among the annihilators of nonzero submodules of $M/M_{i-1}$ and $M_i/M_{i-1} = \mathrm{l.ann}_{M/M_{i-1}}(P_i)$. In particular each $P_i$ is maximal in $Ass(M/M_{i-1})$.

**Proposition 2.1.11** *Every nonzero finitely generated right module over a right Noetherian ring $R$ has an affiliated series.*

**Proof.** [44, Proposition 2.13] □

**Theorem 2.1.12 (Jategaonkar [56, Lemma 2.2])** *Let $R$ be a Noetherian ring, and let $M$ be a right $R$-module with an affiliated series $0 \subsetneqq U \subsetneqq M$ and corresponding affiliated prime ideals $Q$ and $P$, such that $U \leq_e M$. Let $M'$ be a submodule of $M$, properly containing $U$, such that the ideal $A = \mathrm{r.ann}_R(M')$ is maximal among annihilators of submodules of $M$ properly containing $U$. Then exactly one of the following two alternatives occurs:*

*i) $P \subsetneqq Q$ and $M'P = 0$. In this case, $M'$ and $M'/U$ are faithful torsion $R/P$-modules.*

*ii) $P \rightsquigarrow Q$ and $(P \cap Q)/A$ is a linking bimodule between $P$ and $Q$. In this case, if $U$ is torsionfree as a right $R/Q$-module then $M'/U$ is torsionfree as a right $R/P$-module.*

**Definition 2.1.13** Let $R$ be a Noetherian ring and $Q$ any prime ideal of $R$. We say that $Q$ satisfies the *right strong second layer condition* if given the hypothesis of Theorem 2.1.12, $i$) never occurs.

The ideal $Q$ is said to satisfy the *right second layer condition* if given the hypothesis of Theorem 2.1.12 and the additional hypothesis that $U$ is torsionfree as an $R/Q$-module, $i$) never occurs.

A ring is said to satisfy the *right strong second layer condition* and *right second layer condition* if the corresponding condition holds for every prime $Q$ of $R$.

The *left strong second layer condition* and the *left second layer condition* are defined similarly.

A ring is said to satisfy the *strong second layer condition* or the *second layer condition* if it satisfies these conditions on both the left and the right.

Theorem 2.1.12 shows that links between prime ideals may arise from affiliated series of a module. The next theorem shows that all links between prime ideals arise from affiliated series of some modules.

**Theorem 2.1.14 (Jategaonkar, Brown)** *Let $R$ be a Noetherian ring and $P$ and $Q$ prime ideals of $R$. Then $P \rightsquigarrow Q$ if and only if there exists a finitely generated uniform right $R$-module $M$ with an affiliated series $0 \subsetneqq U \subsetneqq M$ such that $U$ is isomorphic to a (uniform) right ideal of $R/Q$ and $M/U$ is isomorphic to a uniform right ideal of $R/P$.*

**Proof.** [44, Theorem 11.2] □

**Proposition 2.1.15** *Let $Q$ be a prime ideal in a Noetherian ring $R$. The prime ideal $Q$ satisfies the right strong second layer condition if and only if there does not exist a finitely generated uniform right $R$-module $M$ with an affiliated series $0 \subsetneqq U \subsetneqq M$ and corresponding affiliated prime ideals $Q$ and $P$ such that $M/U$ is uniform, $P \subsetneqq Q$ and $MP = 0$.*

**Proof.** [44, Proposition 11.3] □

**Definition 2.1.16** Let $R$ be a Noetherian ring and $P, Q$ prime ideals of $R$. We say that the pair $(P, Q)$ is *strongly undesirable* if $P \subsetneqq Q$ and there is a finitely generated uniform $R$-module $M$ containing a nonzero submodule $U$ whose unique assassinator prime is $Q$ and such that $U = \text{l.ann}_M(Q)$, $\text{r.ann}_R(M) = P$ and $M/U$ has unique associated prime $P$.

The following proposition gives us a useful criterion to check if a ring satisfies the right (or left) strong second layer condition.

**Proposition 2.1.17** *A Noetherian ring $R$ satisfies the right strong second layer condition if and only if there are no pairs of strongly undesirable prime ideals.*

**Proof.** Let $R$ be a Noetherian ring. Suppose that $R$ satisfies the right strong second layer condition and $(P, Q)$ is a pair of strongly undesirable prime ideals of $R$. So $P \subsetneqq Q$ and there is a finitely generated uniform $R$-module $M$ containing a nonzero submodule $U$ whose unique associated prime is $Q$ and such that $U = \text{l.ann}_M(Q)$, $\text{r.ann}_R(M) = P$ and $M/U$ has unique associated prime $P$.

As $M/U$ has a uniform submodule, say $M'/U$, the module $M'$ will satisfy all the properties of $M$ above. Hence, we may assume that $M = M'$. As $U$ and $M/U$ are uniform $R$-modules with assassinator $Q$ and $P$, respectively, the series $0 \subsetneqq U \subsetneqq M$ is an affiliated series with affiliated primes $Q$ and $P$ such that $P \subsetneqq Q$ and $MP = 0$. So, by Proposition 2.1.15, $Q$ does not satisfy the right strong second layer condition, a contradiction. Hence there are no pairs of strongly undesirable primes.

Conversely assume that there are no pairs of strongly undesirable primes of $R$ and that there is $Q$ and a finitely generated uniform right $R$-module $M$ with an affiliated series $0 \subsetneqq U \subsetneqq M$ and corresponding affiliated prime ideals $Q$ and $P$ such that $M/U$ is uniform, $P \subsetneqq Q$ and $MP = 0$. As affiliated primes are in particular associated primes of the corresponding factors, and $M$ and $M/U$ are uniform, the associated primes of each of these modules is unique, hence the assassinator of $M$ is $Q$ and the one of $M/U$ is $P$. Since $M$ is uniform, the assassinator of $M$ and of $U$ is the same. Obviously $P \subseteq \text{r.ann}_R(M)$. If $I = \text{r.ann}_R(M)$, then $I \subseteq \text{r.ann}_R(M/U) \subseteq P$. Hence $\text{r.ann}_R(M) = P$. So $(P, Q)$ is a pair of strongly undesirable primes of $R$, a contradiction. So for each $Q$ there does not exist such a module. The result follows now by Proposition 2.1.15. □

Many classes of rings satisfy the second layer condition. For example, this is the case for enveloping algebras of any solvable Lie algebra (A.V. Jategaonkar [57, Theorem A.3.9]), polycyclic-by-finite group rings over a commutative Noetherian ring (K.A. Brown [18, Proposition 2.2] and A.V. Jategaonkar [55, Theorem 4.5]) strongly graded rings of polycyclic-by-finite groups over commutative Noetherian coefficient rings (A.D. Bell [13, Corollary 7.4]), as well as Noetherian PI-rings or more generally FBN rings. On the other hand, many Noetherian rings do not satisfy the second layer condition, for instance enveloping algebras of semisimple

Lie Algebras (K.A. Brown [18, Theorem 4.4.3]) but no example is known of a ring satisfying the second layer condition and not satisfying the strong second layer condition. The next theorem and lemma describe some properties of bimodules over rings satisfying the second layer condition.

**Theorem 2.1.18 (Jategaonkar [57, 8.2.8])** *Let $R$ and $S$ be Noetherian rings satisfying the second layer condition, and suppose there exists a bimodule ${}_R B_S$ which is finitely generated and faithful on both sides. Then* $\mathrm{Cl.K.dim}(R) = \mathrm{Cl.K.dim}(S)$.

The following lemma is an easy generalization of a similar result for faithful, finitely generated bimodules over Noetherian prime rings satisfying the second layer condition, [42, Lemma 1.3].

**Lemma 2.1.19** *Let $R$ and $S$ be semiprime Noetherian rings satisfying the second layer condition, and let ${}_R B_S$ be a bimodule which is finitely generated and faithful on each side. Suppose also that* $\mathrm{Cl.K.dim}(R) = \mathrm{Cl.K.dim}(R/P)$ *and* $\mathrm{Cl.K.dim}(S) = \mathrm{Cl.K.dim}(S/Q)$, *for all $P$ and $Q$, minimal primes of $R$ and $S$, respectively. Then the torsion submodules of $B$ as a left $R$-module and right $S$-module, are the same and different from $B$. Therefore, there exists a sub-bimodule $B' \subsetneqq B$ such that $B/B'$ is torsionfree on each side.*

**Proof.** Suppose $B$ as above. By Theorem 2.1.18, $\mathrm{Cl.K.dim}(R) = \mathrm{Cl.K.dim}(S)$.

Let $\mathcal{C} = \mathcal{C}_R(0)$ and $\mathcal{D} = \mathcal{C}_S(0)$, Ore sets in $R$ and $S$, respectively. Let $T = \mathfrak{t}_{\mathcal{C}}(B)$, the $\mathcal{C}$-torsion submodule of $B$. In fact, $T$ is an $(R, S)$-bimodule. We claim that $T \subseteq \mathfrak{t}_{\mathcal{D}}(B)$.

As $T$ is an $S$-submodule of a finitely generated module over the Noetherian ring $S$, $T$ is finitely generated as a right $S$-module. Therefore, exists $c \in \mathcal{C}$ such that $cT = 0$. If $T = B$, $c \in \mathrm{l.ann}_R(B) = 0$, a contradiction. Let $I = \mathrm{l.ann}_R(T)$. Since $c \in I$, $I$ is not contained in any minimal prime of $R$, by [44, Proposition

6.3]. Thus setting $J = \mathrm{r.ann}_S(T)$,

$$\mathrm{Cl.K.dim}(S/J) = \mathrm{Cl.K.dim}(R/I) < \mathrm{Cl.K.dim}(R) = \mathrm{Cl.K.dim}(S)$$

by two applications of Theorem 2.1.18. Thus $J \cap \mathcal{D} \neq \emptyset$, proving the claim.

Similarly, $\mathfrak{t}_{\mathcal{D}}(B) \subseteq T$. Take $B' = \mathfrak{t}_{\mathcal{D}}(B)$. $\square$

**Corollary 2.1.20** *Let $R$ be a Noetherian ring satisfying the second layer condition, and let $P$ and $Q$ be prime ideals of $R$. Then $P \rightsquigarrow Q$ if and only if $(P \cap Q)/PQ$ is faithful as a left $R/P$-module and as a right $R/Q$-module.*

**Remark 2.1.21** Lemma 2.1.19 remains true without the assumption that $R$ and $S$ are semiprime: it is enough to assume that $R$ and $S$ have classical quotient rings. The proof is essentially the same, since any minimal prime ideal in a Noetherian ring consists of zero divisors.

## 2.2 The strong second layer condition

In this section we will show that some skew-polynomial rings and skew-Laurent rings over a commutative Noetherian ring satisfy the strong second layer condition. To prove this we show that these rings belong to a larger class, the *AR-separated* rings, which satisfy the strong second layer condition. We start by describing the *AR-property* and by fixing some notation that we will keep throughout the section.

**Definition 2.2.1** An ideal $I$ in a ring $R$ has the *right AR-property* if for every right ideal $K$ of $R$, there is a positive integer $n$ such that $K \cap I^n \subseteq KI$. The *left AR-property* is defined similarly, and $I$ has the *AR-property* if it has both the right and the left AR-properties.

**Definition 2.2.2** If $R$ is a ring and $I$ an ideal of $R$, then the *Rees ring* of $I$ is the subring $\mathcal{R}_R(I)$ of the polynomial ring $R[x]$ defined by

$$\mathcal{R}_R(I) = R + Ix + I^2x^2 + \ldots + I^ix^i + \ldots$$

**Lemma 2.2.3** *If $I$ is an ideal in a ring $R$, and the Rees ring $\mathcal{R}_R(I)$ is right Noetherian, then $I$ has the right AR-property.*

**Proof.** [44, Lemma 11.12] $\square$

**Theorem 2.2.4 (Artin, Rees)** *If $R$ is a Noetherian ring and $I$ is an ideal of $R$ generated by central elements, then $\mathcal{R}_R(I)$ is Noetherian and hence $I$ has the AR-property.*

**Proof.** [44, Theorem 11.13] $\square$

**Definition 2.2.5** A ring $R$ is *right AR-separated* if for every pair of prime ideals $P$ and $Q$ in $R$ such that $P \subsetneqq Q$, there is an ideal $I$ such that $P \subsetneqq I \subseteq Q$ and $I/P$ has the right AR-property in $R/P$. *Left AR-separated* is defined symmetrically. The ring $R$ is said to be *AR-separated* if is both left and right AR-separated.

**Proposition 2.2.6** *If $R$ is a Noetherian ring which is right AR-separated, then $R$ satisfies the right strong second layer condition.*

**Proof.** [44, Lemma 11.14] $\square$

**Proposition 2.2.7** *Let $I$ be an ideal in a Noetherian ring $R$, and let $P$ and $Q$ be prime ideals of $R$ with $P \rightsquigarrow Q$. If $I \subseteq Q$ and $I$ has the right AR-property, then $I \subseteq P$. Similarly, if $I \subseteq P$ and $I$ has the left AR-property, then $I \subseteq Q$.*

**Proof.** [44, Proposition 11.16] $\square$

**Notation 2.2.8** If $R$ is any ring, $\alpha_1, \ldots, \alpha_n \in Aut(R)$, commuting pairwise and $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, we will denote by $\alpha^I$ the automorphism $\alpha_1^{i_1} \ldots \alpha_n^{i_n}$ of $R$ and by $\theta^I$ the element $\theta_1^{i_1} \ldots \theta_n^{i_n}$ of the skew-Laurent ring $R[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n]$.

For the rest of the chapter $R$ will always be a commutative Noetherian ring, $\alpha_1, \ldots, \alpha_n$ pairwise commuting automorphisms of $R$, $\mathcal{T} = R[\theta_1, \ldots, \theta_n; \alpha_1, \ldots, \alpha_n]$ the skew-polynomial ring and $\mathcal{S} = R[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n]$ the corresponding skew-Laurent ring.

We let $H$ be the multiplicative abelian subgroup of the group of units of $\mathcal{S}$ generated by $\theta_1, \ldots, \theta_n$, $G$ the multiplicative abelian subgroup of $Aut(R)$ generated by $\alpha_1, \ldots, \alpha_n$ and $\Psi : H \to G$ the group homomorphism such that $\Psi(\theta_i) = \alpha_i$. Write $\mathcal{S} = R * H$.

The following lemma shows that, when calculating the cliques of $\mathcal{S}$, one can fix an ideal of $R$ as prime ideals of $\mathcal{S}$ in the same clique will contract to the same ideal of $R$.

**Proposition 2.2.9** *If $P$ and $Q$ are prime ideals of $\mathcal{S}$ such that $P \rightsquigarrow Q$, then $P \cap R = Q \cap R$.*

**Proof.** As $R$ is a commutative Noetherian ring, $\mathcal{R}_R(P \cap R)$ is Noetherian and $P \cap R$ has the AR-property, Theorem 2.2.4.

Since $\mathcal{R}_{\mathcal{S}}((P \cap R)\mathcal{S}) = \mathcal{S} + t(P \cap R)\mathcal{S} + t^2(P \cap R)^2\mathcal{S} + \ldots = \mathcal{R}_R(P \cap R)[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n]$, $\mathcal{R}_{\mathcal{S}}((P \cap R)\mathcal{S})$ is Noetherian by [44, Theorem 1.17] and so $(P \cap R)\mathcal{S}$ has the AR-property.

By the two parts of Proposition 2.2.7, $P \cap R = Q \cap R$. $\square$

**Notation 2.2.10** Let $N$ be a $G$-prime ideal of $R$ and let $M$ be a prime ideal of $R$ minimal over $N$. Then by Lemma 1.3.5, $N = \cap_{\alpha \in G} M^{\alpha} = M^{x_1} \cap M^{x_2} \cap \ldots \cap M^{x_t}$, where $x_1, x_2, \ldots, x_t$ is a complete set of coset representatives of the stabilizer

$G_{1,M} = \{\alpha \in G : M^\alpha = M\}$ in $G$. We will write $M_i = M^{x_i}$. For convenience we take $x_1 = 1_G$, whence $M_1 = M$. Note that $M_i \neq M_j$ for all $i \neq j$, $i, j = 1, \ldots, t$ and that $\{M_1, M_2, \ldots, M_t\}$ is the $G$-orbit of $M$ in $Spec(R)$.

We set $H_{1,M} = \Psi^{-1}(G_{1,M})$ and $\mathcal{S}_{1,M} = R * H_{1,M}$, the skew group ring of $H_{1,M}$ over $R$.

For every $\alpha \in G_{1,M}$, we will write $\overline{\alpha}$ for the automorphism of $\overline{R} = R/M$ induced by $\alpha$ and denote by $\overline{G}_{1,M}$, the group of these induced automorphisms of $\overline{R}$. Let $\overline{\mathcal{S}}_{1,M} = \mathcal{S}_{1,M}/M\mathcal{S}_{1,M} \cong (R/M) * H_{1,M}$, a skew group ring over the ring $R/M$.

Take $G_{2,M} = \{\alpha \in G_{1,M} : \overline{\alpha} = id|_{R/M}\}$ and $H_{2,M} = \Psi^{-1}(G_{2,M}) \subseteq H_{1,M}$. Note that $G_{2,M} = \{\alpha \in G : \alpha(r) - r \in M,$ for all $r \in R\}$. Form $\mathcal{S}_{2,M} = R * H_{2,M} \subseteq \mathcal{S}_{1,M} \subseteq \mathcal{S}$. So $\mathcal{S}_{2,M}/M\mathcal{S}_{2,M} \cong (R/M) * H_{2,M}$ and write $\overline{\mathcal{S}}_{2,M} = \mathcal{S}_{2,M}/M\mathcal{S}_{2,M}$. Let $H_{2,M}$ be freely generated by $\gamma_{1,M}, \ldots, \gamma_{v,M}$. As $G_{2,M}$ acts trivially on $R/M$, we have that $\overline{\mathcal{S}}_{2,M} = (R/M)[\gamma_{1,M}, \gamma_{1,M}^{-1}, \ldots, \gamma_{v,M}, \gamma_{v,M}^{-1}]$, a commutative Laurent polynomial ring.

As $N$ is a $G$-stable ideal of $R$, $N\mathcal{S}$ and $N\mathcal{S}_{2,M}$ are ideals of $\mathcal{S}$ and $\mathcal{S}_{2,M}$, respectively. We let $\overline{\overline{\mathcal{S}}} = \mathcal{S}/N\mathcal{S}$ and $\overline{\overline{\mathcal{S}}}_{2,M} = \mathcal{S}_{2,M}/N\mathcal{S}_{2,M} \cong R/N * H_{2,M}$. Since $G$ is an abelian group and because of the way we defined $G_{2,M}$, it is easy to see that this group acts trivially on $R/N$. Hence $\overline{\overline{\mathcal{S}}}_{2,M} \cong (R/N)H_{2,M}$, a commutative Laurent polynomial ring.

Whenever $M$ is well understood, we will just write $\mathcal{S}_1$, $\mathcal{S}_2$, $\overline{\mathcal{S}}_1$, $\overline{\mathcal{S}}_2$, $\overline{\overline{\mathcal{S}}}_2$, $H_1$, $G_1$, $H_2$, $G_2$, $\gamma_1, \ldots, \gamma_v$ instead of $\mathcal{S}_{1,M}$, $\mathcal{S}_{2,M}$, $\overline{\mathcal{S}}_{1,M}$, $\overline{\mathcal{S}}_{2,M}$, $\overline{\overline{\mathcal{S}}}_{2,M}$, $H_{1,M}$, $G_{1,M}$, $H_{2,M}$, $G_{2,M}$, $\gamma_{1,M}, \ldots, \gamma_{v,M}$.

**Theorem 2.2.11** *The ring $\mathcal{S}$ is AR-separated.*

**Proof.** Let $P$ and $Q$ be prime ideals of $\mathcal{S}$ such that $P \subsetneqq Q$.

Suppose first that $P \cap R \subsetneqq Q \cap R$. As $R$ is commutative, $\mathcal{R}_R(Q \cap R)$, the Rees ring of $Q \cap R$ over $R$, is Noetherian, Theorem 2.2.4.

Since $\mathcal{R}_{\mathcal{S}}((Q \cap R)\mathcal{S}) = \mathcal{S} + t(Q \cap R)\mathcal{S} + t^2(Q \cap R)^2\mathcal{S} + \dots = \mathcal{R}_R(Q \cap R)[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1, \dots, \alpha_n]$, $\mathcal{R}_{\mathcal{S}}((Q \cap R)\mathcal{S})$ is Noetherian by [44, Theorem 1.17] and so $(P + (Q \cap R)\mathcal{S})/P$ has the AR-property in $\mathcal{S}/P$.

Suppose now that $Q \cap R = P \cap R = N$. Let $H_2 =< \gamma_1, \dots, \gamma_v >$ be $H_{2,M}$ as before. If $P \cap \mathcal{S}_2 = Q \cap \mathcal{S}_2$, then $Q = P$ by Theorem 1.4.13, a contradiction. Hence $P \subsetneqq P + (Q \cap \mathcal{S}_2)\mathcal{S}$. As $\overline{\overline{\mathcal{S}}}_2 = \mathcal{S}_2/N\mathcal{S}_2$ is commutative, $\mathcal{R}_{\overline{\overline{\mathcal{S}}}_2}((Q \cap \mathcal{S}_2)/N\mathcal{S}_2)$ is Noetherian by Theorem 2.2.4. Since

$$
\begin{aligned}
\mathcal{R}_{\overline{\overline{\mathcal{S}}}}((Q \cap \mathcal{S}_2)\mathcal{S}/N\mathcal{S}) &= \overline{\overline{\mathcal{S}}} + ((Q \cap \mathcal{S}_2)/N\mathcal{S}_2)\overline{\overline{\mathcal{S}}}t + \dots \\
&= < \mathcal{R}_{\overline{\overline{\mathcal{S}}}_2}((Q \cap \mathcal{S}_2)/N\mathcal{S}_2), \theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1} >,
\end{aligned}
$$

and for each $i \in \{1, \dots, n\}$, $\theta_i \mathcal{R}_{\overline{\overline{\mathcal{S}}}_2}((Q \cap \mathcal{S}_2)/N\mathcal{S}_2) = \mathcal{R}_{\overline{\overline{\mathcal{S}}}_2}((Q \cap \mathcal{S}_2)/N\mathcal{S}_2)\theta_i$, $\mathcal{R}_{\overline{\overline{\mathcal{S}}}}((Q \cap \mathcal{S}_2)\mathcal{S}/N\mathcal{S})$ is Noetherian, [75, Theorem 9]. Hence $(P + (Q \cap \mathcal{S}_2)\mathcal{S})/P$ has the AR-property. Thus in all cases we have found a nonzero ideal of $\mathcal{S}/P$ contained in $Q/P$, with the AR-property in $\mathcal{S}/P$, and so $\mathcal{S}$ is AR-separated. □

**Corollary 2.2.12** *The ring $\mathcal{S}$ satisfies the strong second layer condition.*

**Proof.** Theorem 2.2.11 and Proposition 2.2.6. □

The following proposition and Corollary 2.2.12 will allow us to prove that the ring $\mathcal{T}$ satisfies the strong second layer condition.

**Proposition 2.2.13** *Let $W$ be a Noetherian ring, let $Y$ be a set of normal elements in $W$ and let $X$ be the multiplicative submonoid of $W$ generated by $Y$. The ring $W$ satisfies the strong second layer condition if and only if $W/yW$ satisfies the strong second layer condition for all $y \in Y$ and $WX^{-1}$ satisfies the strong second layer condition.*

**Proof.** Assume $Y$ as above. As $Y$ is a set of normal elements of $W$, $X$ is a right and left Ore set.

Assume that $WX^{-1}$ and $W/yW$ satisfy the right strong second layer condition for all $y \in Y$. (The proof for the left case is similar). Suppose that there is a pair $(P, Q)$ of strongly undesirable prime ideals of $W$ as in Definition 2.1.16; that is $P \subsetneqq Q$ and there is a finitely generated uniform $W$-module $V$ containing a nonzero submodule $U$ whose unique assassinator prime is $Q$ and such that $U = \text{l.ann}_V(Q)$, $\text{r.ann}_W(V) = P$ and $V/U$ has unique assassinator prime $P$.

We consider each of the following cases: (1) $P \cap Y = \emptyset = Q \cap Y$; (2) there is $y \in Y$ such that $y \in Q \backslash P$; and (3) $Q \cap Y \neq \emptyset$ and for $y \in Y$, $y \in Q$ if and only if $y \in P$.

If case (1) occurs, since $Y$ is a set of normal elements of $W$, $P \cap X = Q \cap X = \emptyset$. By [44, Theorem 9.22], $PX^{-1}$ and $QX^{-1}$ are distinct prime ideals of $WX^{-1}$. Suppose that $U$ is not $X$-torsionfree. Then, there would be $u \in U \backslash \{0\}$ and $x \in X$ such that $ux = 0$. As $X$ is a set of normal elements, we have $(uW)(xW) = 0$, whence $Q \subsetneqq Q + xW \subseteq \text{r.ann}_W(uW)$, contradicting the assumption that $Q$ is the assassinator prime of $U$. Hence $U$ is $X$-torsionfree. As $U$ is essential in $V$, $V$ is $X$-torsionfree as well.

As $V$ is a finitely generated uniform right $W$-module, $VX^{-1}$ is a finitely generated right $WX^{-1}$-module. The $X$-torsionfreeness of $V$ and $U$ shows that the modules $VX^{-1}$ and $UX^{-1}$ have annihilators $PX^{-1}$ and $QX^{-1}$, respectively. Hence as $PX^{-1}$ and $QX^{-1}$ are distinct, $VX^{-1}$ contains properly the $WX^{-1}$-submodule $UX^{-1}$, whose unique assassinator prime is $QX^{-1}$. Assume that $V/U$ is not $X$-torsionfree. As we may suppose that $V/U$ is uniform, we may as well assume that $V/U$ is $X$-torsion. Hence $VX^{-1} = UX^{-1}$, a contradiction. The $X$-torsionfreeness of $V/U$ implies that $VX^{-1}/UX^{-1}$ has unique assassinator prime $PX^{-1}$. Therefore $(PX^{-1}, QX^{-1})$ will be a strongly undesirable pair of prime ideals of $WX^{-1}$, a contradiction.

If case (2) occurs, let $y \in Y$ be in $Q \backslash P$. As $y$ is a normal element of $W$, $yW$ is an AR-ideal of $W$ [76, Proposition 4.2.6], and so $(P + yW)/P$ is an AR-ideal of $W/P$. Since $V$ is an $W/P$-module and $U(P + yW) = 0$, there is $m$ such that $V(P+yW)^m = 0$, [44, Lemma 11.11]. Since $\text{l.ann}_W(V) = P$, $yW \subseteq P$, a contradiction.

If case (3) occurs, let $y \in P \cap Q \cap Y$. Hence $(P/yW, Q/yW)$ is a strongly undesirable pair of primes of $W/yW$, contradicting the hypothesis.

Hence no strongly undesirable pair of primes exists in $W$ and so $W$ satisfies the right strong second layer condition.

Conversely, assume that $W$ satisfies the right strong second layer condition. Then so does $W/yW$, for all $y \in Y$. We will show that there are no pairs of strongly undesirable primes of $WX^{-1}$.

By [44, Theorem 9.22], every prime ideal of $WX^{-1}$ is of the form $PX^{-1}$ where $P$ is a prime ideal of $W$ such that $P \cap X = \emptyset$. Let $P$ and $Q$ be prime ideals of $W$ such that $P \cap X = \emptyset = Q \cap X$ and $(PX^{-1}, QX^{-1})$ is a pair of strongly undesirable prime ideals of $WX^{-1}$; so $PX^{-1} \subsetneqq QX^{-1}$ and there is a finitely generated uniform $WX^{-1}$-module $V'$ containing a nonzero submodule $U'$ whose unique associated prime is $QX^{-1}$ and such that $U' = \text{l.ann}_{V'}(QX^{-1})$, $\text{r.ann}_{WX^{-1}}(V') = PX^{-1}$ and $V'/U'$ has unique associated prime $PX^{-1}$.

Assume that $V'$ is generated by $v_1, \ldots, v_r$ as a $WX^{-1}$-module. Then $V = \sum_{i=1}^{r} v_i W$ is an $W$-submodule of $V'$, finitely generated and such that $VX^{-1} = V'$. Also $V$ is $X$-torsionfree. If $V_1$ and $V_2$ are $W$-submodules of $V$ such that $V_1 \cap V_2 = 0$, then $V_1X^{-1}$ and $V_2X^{-1}$ are $WX^{-1}$ submodules of $V'$ with intersection 0. As $V'$ is uniform either $V_1X^{-1} = 0$ or $V_2X^{-1} = 0$. As $V$ is $X$-torsionfree, either $V_1 = 0$ or $V_2 = 0$, hence $V$ is uniform as an $W$-module.

Take $U = U' \cap V$. By [44, Proposition 9.17], $U' = UX^{-1}$ and so $U \neq 0$ and $U \neq V$. Since $UQ \subseteq U'QX^{-1} = 0$, $U \subseteq \text{l.ann}_V(Q)$. Since $(\text{l.ann}_V(Q))QX^{-1} = 0$, $\text{l.ann}_V(Q) \subseteq U'$. So $U = \text{l.ann}_V(Q)$. Also we have $VP \subseteq V'PX^{-1} = 0$, so $P \subseteq \text{r.ann}_W(V)$. Let $I = \text{r.ann}_W(V)$, which is an ideal of $W$. As $W$ is Noetherian by [44, Theorem 9.20], $IX^{-1}$ is an ideal of $WX^{-1}$. As $V' = VX^{-1}$, $V'IX^{-1} = VIX^{-1} = 0$, so $IX^{-1} \subseteq PX^{-1}$. Since $P \cap X = \emptyset$, by [44, Lemma 9.21] $X \subseteq \mathcal{C}_W(P)$, so $I \subseteq P$. Hence $\text{r.ann}_W(V) = P$.

As $Ass(U') = \{QX^{-1}\}$, there exists $U''$ an $RX^{-1}$-submodule of $U'$, such that $\text{r.ann}_{WX^{-1}}(U'') = QX^{-1}$ and all non-zero submodules of $U''$ have annihilator $QX^{-1}$. By [44, Theorem 9.17], $U'' = (U'' \cap V)X^{-1}$. As before we can conclude that $\text{r.ann}_W(U'' \cap V) = Q$. Also as the annihilator of any non-zero $W$-submodule of $U'' \cap V$, say $J$, will be such that $JX^{-1}$ will annihilate a $WX^{-1}$-submodule of $U''$, $JX^{-1} \subseteq QX^{-1}$. As $X \subseteq \mathcal{C}_W(Q)$, $J \subseteq Q$. So, as $U$ is uniform, we conclude that $Ass(U) = \{Q\}$.

As $V'/U'$ has unique associated prime $PX^{-1}$, we may assume that $V'/U'$ is uniform with assassinator $PX^{-1}$. As $V$ is $X$-torsionfree so is $U' \cap V$. It is then clear that $V/U' \cap V$ is uniform. By an argument similar to the one before, we can see that $P$ is an associated prime of $V/U' \cap V$, hence is unique. So $(P, Q)$ is a pair of strongly undesirable primes of $W$, contradicting the hypothesis. So $WX^{-1}$ satisfies the right strong second layer condition. The left version is proved in a similar way. $\square$

**Corollary 2.2.14** *The ring $\mathcal{T}$ satisfies the strong second layer condition.*

**Proof.** We will argue by induction on the number $n$ of automorphisms of $R$.

If $n = 0$, then $\mathcal{T} = R$ and by Theorem 2.2.4, $\mathcal{T}$ is an $AR$-separated ring. Hence it satisfies the strong second layer condition by Proposition 2.2.6.

Suppose that for all $l \leq n - 1$ and for all $i_1, \ldots, i_l \in \{1, \ldots, n\}$, $R[\theta_{i_1}, \ldots, \theta_{i_l}; \alpha_{i_1}, \ldots, \alpha_{i_l}]$ satisfies the strong second layer condition. Let $Y = \{\theta_1, \ldots, \theta_n\}$ and $X = \{\theta^J : J \in \mathbb{N}\}$, an Ore subset of $\mathcal{T}$. As $\mathcal{T}X^{-1} \cong \mathcal{S}$, by Corollary 2.2.12, $\mathcal{T}X^{-1}$ satisfies the strong second layer condition. By the induction hypothesis, $\mathcal{T}/\theta_i\mathcal{T} \cong R[\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n; \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n]$ satisfies the strong second layer condition, for any $i \in \{1, \ldots, n\}$. By Proposition 2.2.13, $\mathcal{T}$ satisfies the strong second layer condition. $\square$

## 2.3 Prime links in skew-Laurent rings contracting to a maximal ideal

In this section we describe the links between prime ideals contracting to maximal ideals in some skew-Laurent rings. The general case will be studied in the following sections. Proposition 2.3.2, 2.3.3 and Theorem 2.3.4 can and will be generalised in the next section. We start by fixing some notation that we will keep through the rest of this chapter.

**Notation 2.3.1** Let $R$ be a commutative Noetherian ring, $\beta_1, \ldots, \beta_v$ automorphisms of $R$ commuting pairwise and $\mathcal{S}_2 = R[\gamma_1, \gamma_1^{-1}, \ldots, \gamma_v, \gamma_v^{-1}; \beta_1, \ldots, \beta_v]$ the skew-Laurent ring. Denote by $G_2$ the abelian group generated by $\beta_1, \ldots, \beta_v$ and $H_2$ the torsionfree group generated by $\gamma_1, \ldots, \gamma_v$.

Take $M$ a maximal ideal of $R$, write $K = R/M$ and assume that each $\beta \in G_2$ fixes $M$ and induces the identity in $K$. Write $\overline{\mathcal{S}}_2 = \mathcal{S}_2/M\mathcal{S}_2 \cong K[\gamma_1, \gamma_1^{-1}, \ldots, \gamma_v, \gamma_v^{-1}] = KH_2$. Note that $\mathcal{S}_2$ is $\mathcal{S}_{2,M}$ of Notation 2.2.10 and $\beta_i = \Psi(\gamma_i)$ where $\Psi$ is as in Notation 2.2.8.

**Proposition 2.3.2** *The ideal $M\mathcal{S}_2$ is a prime ideal of $\mathcal{S}_2$, $\mathcal{C}\ell(M\mathcal{S}_2) = \{M\mathcal{S}_2\}$ and $M\mathcal{S}_2 \rightsquigarrow M\mathcal{S}_2$ if and only if $M \neq M^2$.*

**Proof.** As $\mathcal{S}_2/M\mathcal{S}_2 \cong KH_2$ is a commutative Noetherian domain, $M\mathcal{S}_2$ is a prime ideal of $\mathcal{S}_2$.

Suppose that there is a prime $P$ of $\mathcal{S}_2$ such that $P \rightsquigarrow M\mathcal{S}_2$ or $M\mathcal{S}_2 \rightsquigarrow P$. Then by Proposition 2.2.9, $P \cap R = M\mathcal{S}_2 \cap R = M$ and so $M\mathcal{S}_2 \subseteq P$. By Theorem 2.1.18, Cl.K.dim$(\mathcal{S}_2/M\mathcal{S}_2)$ = Cl.K.dim$(\mathcal{S}_2/P)$, so $M\mathcal{S}_2 = P$.

Obviously, if $M\mathcal{S}_2 \rightsquigarrow M\mathcal{S}_2$, then $M\mathcal{S}_2 \neq M^2\mathcal{S}_2$ and so $M \neq M^2$. Conversely, suppose $M \neq M^2$. As $\mathcal{S}_2$ is free as a right and left $R$-module, $M\mathcal{S}_2/M^2\mathcal{S}_2$ is faithful as a right and left $\mathcal{S}_2/M\mathcal{S}_2$-module. Thus $M\mathcal{S}_2 \rightsquigarrow M\mathcal{S}_2$. □

**Proposition 2.3.3** *Every prime ideal of $\mathcal{S}_2$, properly containing $M\mathcal{S}_2$ is linked to itself.*

**Proof.** Let $P$ be a prime ideal of $\mathcal{S}_2$, properly containing $M\mathcal{S}_2$.

As $\overline{\mathcal{S}}_2$ satisfies the second layer condition, by [23, Proposition 2.5], there will be some prime of $\overline{\mathcal{S}}_2$ linked to the non-minimal prime $P/M\mathcal{S}_2$ of $\overline{\mathcal{S}}_2$. Since $\overline{\mathcal{S}}_2$ is commutative, there are no links between distinct primes of $\overline{\mathcal{S}}_2$, so we will have $P/M\mathcal{S}_2 \rightsquigarrow P/M\mathcal{S}_2$ and so $P \rightsquigarrow P$. $\square$

**Theorem 2.3.4** *Let $\tilde{P}_2$ and $\tilde{Q}_2$ be distinct prime ideals of $\mathcal{S}_2$, distinct from $M\mathcal{S}_2$, such that $\tilde{P}_2 \cap R = \tilde{Q}_2 \cap R = M$. Then $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ if and only if $M\mathcal{S}_2/(\tilde{P}_2M + M\tilde{Q}_2)$ is faithful as a left $\mathcal{S}_2/\tilde{P}_2$-module and as a right $\mathcal{S}_2/\tilde{Q}_2$-module*

**Proof.** Let $\tilde{P}_2, \tilde{Q}_2$ be as stated. Suppose $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ via $(\tilde{P}_2 \cap \tilde{Q}_2)/A$, for some ideal $A$ of $\mathcal{S}_2$ such that $\tilde{P}_2\tilde{Q}_2 \subseteq A \subsetneq \tilde{P}_2 \cap \tilde{Q}_2$.

If $M\mathcal{S}_2 \subseteq A$, then $\tilde{P}_2/M\mathcal{S}_2 \rightsquigarrow \tilde{Q}_2/M\mathcal{S}_2$ and since $\overline{\mathcal{S}}_2$ is commutative $\tilde{P}_2 = \tilde{Q}_2$, a contradiction. So $A \subsetneq M\mathcal{S}_2 + A$. Hence $M\mathcal{S}_2/(M\mathcal{S}_2 \cap A)$ is faithful as a left $\mathcal{S}_2/\tilde{P}_2$-module and as a right $\mathcal{S}_2/\tilde{Q}_2$-module. The same happens with $M\mathcal{S}_2/(\tilde{P}_2M + M\tilde{Q}_2)$, since this bimodule has $M\mathcal{S}_2/(M\mathcal{S}_2 \cap A)$ as a factor.

Conversely, suppose that $M\mathcal{S}_2/(\tilde{P}_2M+M\tilde{Q}_2)$ is faithful as a left $\mathcal{S}_2/\tilde{P}_2$-module and as a right $\mathcal{S}_2/\tilde{Q}_2$-module. Since, by Corollary 2.2.12, $\mathcal{S}_2$ satisfies the second layer condition, by Theorem 2.1.18,

$$\text{Cl.K.dim}(\mathcal{S}_2/\tilde{P}_2) = \text{Cl.K.dim}(\mathcal{S}_2/\tilde{Q}_2). \tag{2.1}$$

By Lemma 2.1.19 there is an ideal $A$ of $\mathcal{S}_2$ such that $\tilde{P}_2M + M\tilde{Q}_2 \subseteq A \subsetneq M\mathcal{S}_2$ and $M\mathcal{S}_2/A$ is torsionfree as a left $\mathcal{S}_2/\tilde{P}_2$-module and as a right $\mathcal{S}_2/\tilde{Q}_2$-module. Therefore the bimodule $M\mathcal{S}_2/A$ is a bond from $\tilde{P}_2/A$ to $\tilde{Q}_2/A$. By [42, Theorem 1.1], there are $P_0/A, \ldots, P_r/A$, with $r \geq 1$, distinct prime ideals of $\mathcal{S}_2/A$ such that

$$\tilde{P}_2/A = P_0/A \rightsquigarrow P_1/A \rightsquigarrow \ldots \rightsquigarrow P_r/A = \tilde{Q}_2/A.$$

Suppose $P_0/A \rightsquigarrow P_1/A$ via $(P_0/A \cap P_1/A)/(B/A)$, for some ideal $B$ of $\mathcal{S}_2$ such that $A \subseteq B$ and $P_0P_1 \subseteq B \subsetneqq P_0 \cap P_1$. Then, $P_0 \rightsquigarrow P_1$ via $(P_0 \cap P_1)/B$. As above $M\mathcal{S}_2/(M\mathcal{S}_2 \cap B)$ is faithful as a $(\mathcal{S}_2/\tilde{P}_2, \mathcal{S}_2/P_1)$-bimodule. As $M\tilde{Q}_2 \subseteq M\mathcal{S}_2 \cap A \subseteq M\mathcal{S}_2 \cap B$, it follows that $\tilde{Q}_2 \subseteq P_1$.

Since $\tilde{P}_2/A \rightsquigarrow P_1/A$, Cl.K.dim$(\mathcal{S}_2/\tilde{P}_2)$ = Cl.K.dim$(\mathcal{S}_2/P_1)$ and thus Cl.K.dim$(\mathcal{S}_2/P_1)$ = Cl.K.dim$(\mathcal{S}_2/\tilde{Q}_2)$, by (2.1) above. Hence $P_1 = \tilde{Q}_2$ and $\tilde{P}_2/A \rightsquigarrow \tilde{Q}_2/A$. Therefore $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$. □

Suppose two distinct prime ideals $\tilde{P}_2$ and $\tilde{Q}_2$ of $\mathcal{S}_2$ both contract to $M$. If there is a link between them, then as $\overline{\mathcal{S}}_2 = \mathcal{S}_2/M\mathcal{S}_2$ is a commutative Noetherian ring, $M^2 \subsetneqq M$. Without loss of generality, in studying the relation between $\tilde{P}_2$ and $\tilde{Q}_2$, we can suppose $M^2 = 0$, and we shall do so until Theorem 2.3.11.

By Theorem 2.3.4, any two distinct primes $\tilde{P}_2$ and $\tilde{Q}_2$ of $\mathcal{S}_2$, both distinct from $M\mathcal{S}_2$, such that $\tilde{P}_2 \cap R = \tilde{Q}_2 \cap R = M$, are linked if and only if $B = M\mathcal{S}_2/(\tilde{P}_2M + M\tilde{Q}_2)$ is faithful as a left $\mathcal{S}_2/\tilde{P}_2$-module and as a right $\mathcal{S}_2/\tilde{Q}_2$-module. Let $\overline{\mathcal{S}}_2 = \mathcal{S}_2/M\mathcal{S}_2 \cong KH_2$, $\overline{P}_2 = \tilde{P}_2/M\mathcal{S}_2$ and $\overline{Q}_2 = \tilde{Q}_2/M\mathcal{S}_2$. Then $B$ is a factor bimodule of $M\mathcal{S}_2$, faithful on each side as an $(\overline{\mathcal{S}}_2/\overline{P}_2, \overline{\mathcal{S}}_2/\overline{Q}_2)$-bimodule.

Let $K^\sharp$ be the algebraic closure of $K$. Take $M^\sharp = K^\sharp \otimes_K M$ and $\mathcal{S}_2^\sharp = K^\sharp \otimes_K \overline{\mathcal{S}}_2 \cong K^\sharp[\gamma_1, \gamma_1^{-1}, \dots, \gamma_v, \gamma_v^{-1}]$.

We can view $M^\sharp = K^\sharp \otimes_K M$ as a left $\mathcal{S}_2^\sharp$-module, by defining $\gamma_i.(k \otimes m) = \gamma_i(k \otimes m)\gamma_i^{-1} = k \otimes \gamma_i m \gamma_i^{-1}$, for every $i \in \{1, \dots, v\}$, $m \in M$ and $k \in K^\sharp$. Obviously $\gamma_i m \gamma_i^{-1} = \beta_i(m)$, for each $i \in \{1, \dots, v\}$.

As $M$ is an ideal of a Noetherian ring, it is a finite dimensional vector space over $K$, so the same is true for $M^\sharp$ as a $K^\sharp$-vector space. Hence as a left $\mathcal{S}_2^\sharp$-module, $M^\sharp$ has a composition series whose composition factors are isomorphic to $\mathcal{S}_2^\sharp/E_i$, for some distinct maximal ideals $E_1, \dots, E_u$ (with some multiplicities). Since $K^\sharp$ is algebraically closed, we may write $E_i = \langle \gamma_1 - \varepsilon_{i1}, \dots, \gamma_v - \varepsilon_{iv} \rangle$, where $\varepsilon_{i1}, \dots, \varepsilon_{iv} \in K^\sharp \backslash \{0\}$. As $\mathcal{S}_2^\sharp$ is commutative, we get $(E_u^{\mu_u} \dots E_1^{\mu_1}).M^\sharp = 0$ with each $\mu_i$ chosen minimal. Now, decomposing $U = \mathcal{S}_2^\sharp/E_u^{\mu_u} \dots E_1^{\mu_1}$ into a direct sum of primary rings, we can think of $M^\sharp$ as a left $U$-module and write

$$M^\sharp = D_1^\sharp \oplus \ldots \oplus D_u^\sharp$$

where, for each $i \in \{1, \ldots, u\}$, $D_i^\sharp = \{n^\sharp \in M^\sharp : E_i^{\mu_i} \cdot n^\sharp = 0\}$, an $\mathcal{S}_2^\sharp$-submodule of $M^\sharp$.

For each $i \in \{1, \ldots, u\}$ define the $K^\sharp$-algebra automorphism

$$\begin{aligned} \sigma_i : \mathcal{S}_2^\sharp &\longrightarrow \mathcal{S}_2^\sharp \\ \gamma_j &\longmapsto \varepsilon_{ij}\gamma_j \end{aligned}$$

Taking a composition series of $D_i^\sharp$ as a left $\mathcal{S}_2^\sharp$-module we get a $K^\sharp$-basis for $D_i^\sharp$, say $\{m_{i1}, \ldots, m_{i\eta_i}\}$, such that for all $j \in \{1, \ldots, v\}$ and $k \in \{2, \ldots, \eta_i\}$,

$$(\gamma_j - \varepsilon_{ij}) \cdot m_{ik} \equiv 0 \pmod{\sum_{l=1}^{k-1} K^\sharp m_{il}} \quad \text{and} \quad (\gamma_j - \varepsilon_{ij}).m_{i1} = 0.$$

Then, letting $\gamma = \gamma_1^{j(1)} \ldots \gamma_v^{j(v)}$, for some $j(1), \ldots, j(v) \in \mathbb{Z}$,

$$\gamma_j m_{ik} \gamma_j^{-1} \equiv \varepsilon_{ij} m_{ik}, \quad \gamma m_{ik} \gamma^{-1} \equiv \varepsilon_{i1}^{j(1)} \ldots \varepsilon_{iv}^{j(v)} m_{ik} \pmod{\sum_{l=1}^{k-1} K^\sharp m_{il}},$$

$$\gamma_j m_{i1} \gamma_j^{-1} = \varepsilon_{ij} m_{i1} \quad \text{and} \quad \gamma m_{i1} \gamma^{-1} = \varepsilon_{i1}^{j(1)} \ldots \varepsilon_{iv}^{j(v)} m_{i1}.$$

Hence

$$\gamma_j m_{ik} \equiv m_{ik}\sigma_i(\gamma_j), \quad \gamma m_{ik} \equiv m_{ik}\sigma_i(\gamma) \pmod{\sum_{l=1}^{k-1} \mathcal{S}_2^\sharp m_{il}},$$

$$\gamma_j m_{i1} = m_{i1}\sigma_i(\gamma_j) \quad \text{and} \quad \gamma m_{i1} = m_{i1}\sigma_i(\gamma).$$

**Notation 2.3.5** We will retain the notation introduced before in Notation 2.2.8. In addition, we assume that $M^2 = 0$, and $\mathcal{S}_2^\sharp$, $M^\sharp$, $D_1^\sharp, \ldots, D_u^\sharp$, $\varepsilon_{ij}$, $m_{ik_i}$ for $i \in \{1, \ldots, u\}$, $j \in \{1, \ldots, v\}$, $k_i \in \{1, \ldots, \eta_i\}$ and $\sigma_1, \ldots, \sigma_u$ will be as immediately above.

**Lemma 2.3.6** *Let $B$ be a finitely generated faithful module over a commutative Noetherian ring $R$. Let $Q$ be a minimal prime of $R$. Then $B/BQ$ is faithful as an $R/Q$-module, so $B$ has a nonzero factor which is torsionfree as an $R/Q$-module.*

**Proof.** By Proposition 2.1.11, we can build an affiliated series for $B$,

$$0 = B_0 \subsetneqq B_1 \subsetneqq \dots \subsetneqq B_n = B$$

with corresponding affiliated primes $P_1, \dots, P_n$. Hence $BP_n \dots P_1 = 0$ and so $P_n \dots P_1 = 0 \subseteq Q$. Then, there is $i \in \{1, \dots, n\}$ such that $P_i = Q$. Without loss of generality, we can suppose that $P_i$ is the last occurence of $Q$ in the list $\{P_1, \dots, P_n\}$.

Let $I = \text{r.ann}(B/BQ)$. Then $BIP_n \dots P_{i+1} \subseteq B_{i-1}$. Hence $IP_n \dots P_{i+1} \subseteq \text{r.ann}(B_i/B_{i-1}) = Q$. As $Q \notin \{P_{i+1}, \dots, P_n\}$ and $Q$ is a minimal prime of $R$, $I \subseteq Q$. Hence $I = Q$ and $B/BQ$ is faithful as a right $R/Q$-module. □

**Theorem 2.3.7** *Retain Notation 2.3.5. Let $\tilde{P}_2$ and $\tilde{Q}_2$ be distinct primes of $\mathcal{S}_2$, both distinct from $M\mathcal{S}_2$, such that $\tilde{P}_2 \cap R = \tilde{Q}_2 \cap R = M$. Take $\overline{P}_2 = \tilde{P}_2/M\mathcal{S}_2$, $\overline{Q}_2 = \tilde{Q}_2/M\mathcal{S}_2$ and $Q^\sharp$ a prime ideal of $\mathcal{S}_2^\sharp$ lying over $\overline{Q}_2$. If $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ then there is $i \in \{1, \dots, u\}$ such that $\sigma_i^{-1}(Q^\sharp)$ lies over $\overline{P}_2$.*

**Proof.** Let $\tilde{P}_2$, $\tilde{Q}_2$ and accompanying notation be as in the statement of the theorem. As $\mathcal{S}_2^\sharp$ is Noetherian and $\overline{P}_2\mathcal{S}_2^\sharp$, $\overline{Q}_2\mathcal{S}_2^\sharp$ are ideals of $\mathcal{S}_2^\sharp$, there is just a finite number of minimal primes over $\overline{P}_2\mathcal{S}_2^\sharp$ and $\overline{Q}_2\mathcal{S}_2^\sharp$. These are exactly the primes of $\mathcal{S}_2^\sharp$ lying over $\overline{P}_2$ and $\overline{Q}_2$, respectively, by GU and INC, [63, Theorem 44].

By Theorem 2.3.4, $M\mathcal{S}_2/(\tilde{P}_2M + M\tilde{Q}_2)$ is faithful as an $(\overline{\mathcal{S}}_2/\overline{P}_2, \overline{\mathcal{S}}_2/\overline{Q}_2)$-bimodule. As $\mathcal{S}_2^\sharp$ is commutative and free over $\mathcal{S}_2$, $\mathcal{S}_2^\sharp \otimes_{\overline{\mathcal{S}}_2} (M\mathcal{S}_2/\tilde{P}_2M + M\tilde{Q}_2)$ is faithful as a left $\mathcal{S}_2^\sharp/\overline{P}_2\mathcal{S}_2^\sharp$-module and as a right $\mathcal{S}_2^\sharp/\overline{Q}_2\mathcal{S}_2^\sharp$-module. Hence the $(\mathcal{S}_2^\sharp/\overline{P}_2\mathcal{S}_2^\sharp, \mathcal{S}_2^\sharp/\overline{Q}_2\mathcal{S}_2^\sharp)$-bimodule, $M^\sharp\mathcal{S}_2^\sharp/(\overline{P}_2\mathcal{S}_2^\sharp M^\sharp + M^\sharp\overline{Q}_2\mathcal{S}_2^\sharp)$ is faithful. Using the notation introduced in Notation 2.3.5, each $D_i^\sharp$ is a left $\mathcal{S}_2^\sharp$-module under the conjugation action. So it is easy to see that each $D_i^\sharp\mathcal{S}_2^\sharp$ is an $(\mathcal{S}_2^\sharp, \mathcal{S}_2^\sharp)$-bimodule left and right under the multiplication actions and we can write, as $(\mathcal{S}_2^\sharp, \mathcal{S}_2^\sharp)$-bimodule,

$$\frac{M^\sharp\mathcal{S}_2^\sharp}{\overline{P}_2\mathcal{S}_2^\sharp M^\sharp + M^\sharp\overline{Q}_2\mathcal{S}_2^\sharp} \cong \frac{D_1^\sharp\mathcal{S}_2^\sharp}{\overline{P}_2\mathcal{S}_2^\sharp D_1^\sharp + D_1^\sharp\overline{Q}_2\mathcal{S}_2^\sharp} \oplus \dots \oplus \frac{D_u^\sharp\mathcal{S}_2^\sharp}{\overline{P}_2\mathcal{S}_2^\sharp D_u^\sharp + D_u^\sharp\overline{Q}_2\mathcal{S}_2^\sharp}.$$

By Lemma 2.3.6, the $(\mathcal{S}_2^\sharp/\overline{P}_2\mathcal{S}_2^\sharp, \mathcal{S}_2^\sharp/Q^\sharp)$-bimodule

$$\frac{M^\sharp \mathcal{S}_2^\sharp}{\overline{P}_2\mathcal{S}_2^\sharp M^\sharp + M^\sharp Q^\sharp} \cong \frac{D_1^\sharp \mathcal{S}_2^\sharp}{\overline{P}_2\mathcal{S}_2^\sharp D_1^\sharp + D_1^\sharp Q^\sharp} \oplus \ldots \oplus \frac{D_u^\sharp \mathcal{S}_2^\sharp}{\overline{P}_2\mathcal{S}_2^\sharp D_u^\sharp + D_u^\sharp Q^\sharp}$$

is faithful on the right.

So $\cap_{i=1}^{u} \text{r.ann}_{\mathcal{S}_2^\sharp}(D_i^\sharp \mathcal{S}_2^\sharp/(\overline{P}_2\mathcal{S}_2^\sharp D_i^\sharp + D_i^\sharp Q^\sharp)) \subseteq Q^\sharp$. As $Q^\sharp$ is a minimal prime of $\mathcal{S}_2^\sharp$, there will be $i \in \{1, \ldots, u\}$ such that $D_i^\sharp \mathcal{S}_2^\sharp/(\overline{P}_2\mathcal{S}_2^\sharp D_i^\sharp + D_i^\sharp Q^\sharp)$ is faithful on the right as a $\mathcal{S}_2^\sharp/Q^\sharp$-module. Without loss of generality suppose $i = 1$. Factoring this bimodule by its $\mathcal{S}_2^\sharp/Q^\sharp$-torsion bimodule, we get a $(\mathcal{S}_2^\sharp/\overline{P}_2\mathcal{S}_2^\sharp, \mathcal{S}_2^\sharp/Q^\sharp)$-bimodule, say $D_1^\sharp \mathcal{S}_2^\sharp/A$, faithful and torsionfree on the right.

By [44, Proposition 7.7], there is a left affiliated series for $D_1^\sharp \mathcal{S}_2^\sharp/A$, say

$$0 = A_0/A \subsetneq A_1/A \subsetneq \ldots \subsetneq A_w/A = D_1^\sharp \mathcal{S}_2^\sharp/A$$

such that each factor $D_1^\sharp \mathcal{S}_2^\sharp/A_{l-1}$, $l \in \{1, \ldots, w\}$, is a torsionfree right $\mathcal{S}_2^\sharp/Q^\sharp$-module. Let $P_1/\overline{P}_2\mathcal{S}_2^\sharp, \ldots, P_w/\overline{P}_2\mathcal{S}_2^\sharp$ be the left affiliated primes of such a series. Then $D_1^\sharp \mathcal{S}_2^\sharp/A_{w-1}$ is faithful as a left $\mathcal{S}_2^\sharp/P_w$-module. As $D_1^\sharp \mathcal{S}_2^\sharp/A_{w-1}$ is finitely generated and torsionfree on the right over the prime ring $\mathcal{S}_2^\sharp/Q^\sharp$, $D_1^\sharp \mathcal{S}_2^\sharp/A_{w-1}$ is faithful as a $(\mathcal{S}_2^\sharp/P_w, \mathcal{S}_2^\sharp/Q^\sharp)$-bimodule. Then,

$$\text{Cl.K.dim}(\mathcal{S}_2^\sharp/P_w) = \text{Cl.K.dim}(\mathcal{S}_2^\sharp/Q^\sharp),$$

by Theorem 2.1.18. As $\overline{P}_2\mathcal{S}_2^\sharp \subseteq P_w$, $P_w$ contains one of the minimal primes over $\overline{P}_2\mathcal{S}_2^\sharp$, say $P^\sharp$. By [63, Theorem 44, Theorem 47],

$$\text{Cl.K.dim}(\mathcal{S}_2^\sharp/P^\sharp) = \text{Cl.K.dim}(\overline{\mathcal{S}}_2/\overline{P}_2) \text{ and } \text{Cl.K.dim}(\mathcal{S}_2^\sharp/Q^\sharp) = \text{Cl.K.dim}(\overline{\mathcal{S}}_2/\overline{Q}_2).$$

As $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$, we have $\text{Cl.K.dim}(\overline{\mathcal{S}}_2/\overline{P}_2) = \text{Cl.K.dim}(\overline{\mathcal{S}}_2/\overline{Q}_2)$. But then

$$\text{Cl.K.dim}(\mathcal{S}_2^\sharp/P_w) = \text{Cl.K.dim}(\mathcal{S}_2^\sharp/P^\sharp).$$

Therefore $P_w = P^\sharp$ and $D_1^\sharp \mathcal{S}_2^\sharp/A_{w-1}$ is a faithful $(\mathcal{S}_2^\sharp/P^\sharp, \mathcal{S}_2^\sharp/Q^\sharp)$-bimodule torsionfree on the right. By Lemma 2.1.19, $D_1^\sharp \mathcal{S}_2^\sharp/A_{w-1}$ is also torsionfree as a left $\mathcal{S}_2^\sharp/P^\sharp$-module.

Take $\{m_{11}, \ldots, m_{1\eta_1}\}$ the $K^\sharp$-basis of $D_1^\sharp$ built in the beginning of the section. Take the first $l \in \{1, \ldots, \eta_1\}$ such that $m_{1l} \notin A_{w-1}$. We have

$$0 = P^\sharp(m_{1l} + A_{w-1}) = m_{1l}\sigma_1(P^\sharp) + A_{w-1}.$$

As $D_1^\sharp \mathcal{S}_2^\sharp / A_{w-1}$ is torsionfree as a right $\mathcal{S}_2^\sharp/Q^\sharp$-module, $\sigma_1(P^\sharp) \subseteq Q^\sharp$. We have seen that $\mathrm{Cl.K.dim}(\mathcal{S}_2^\sharp/P^\sharp) = \mathrm{Cl.K.dim}(\mathcal{S}_2^\sharp/Q^\sharp)$. Hence

$$\mathrm{Cl.K.dim}(\mathcal{S}_2^\sharp/\sigma_1(P^\sharp)) = \mathrm{Cl.K.dim}(\mathcal{S}_2^\sharp/Q^\sharp).$$

Therefore $\sigma_1(P^\sharp) = Q^\sharp$. $\square$

**Theorem 2.3.8** *Retain Notation 2.3.5. Let $\tilde{P}_2$ and $\tilde{Q}_2$ be distinct primes of $\mathcal{S}_2$, both distinct from $M\mathcal{S}_2$, such that $\tilde{P}_2 \cap R = \tilde{Q}_2 \cap R = M$. Take $\overline{P}_2 = \tilde{P}_2/M\mathcal{S}_2$ and $\overline{Q}_2 = \tilde{Q}_2/M\mathcal{S}_2$. If there is $i \in \{1, \ldots, u\}$ such that $\sigma_i(P^\sharp) = Q^\sharp$, for some prime ideals, $P^\sharp, Q^\sharp$, of $\mathcal{S}_2^\sharp$, lying over $\overline{P}_2$ and $\overline{Q}_2$, respectively, then $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$.*

**Proof.** Suppose that $P^\sharp, Q^\sharp$ are primes of $\mathcal{S}_2^\sharp$ lying over $\overline{P}_2$ and $\overline{Q}_2$, respectively, such that $\sigma_i(P^\sharp) = Q^\sharp$. Take the basis of $D_i^\sharp$ chosen in the beginning of the section and $V_i = \sum_{j=1}^{\eta_i - 1} K^\sharp m_{ij}$. Hence $V_i$ is a left $\mathcal{S}_2^\sharp$-module under the conjugation action and $V_i\mathcal{S}_2^\sharp$ is a right and left $\mathcal{S}_2^\sharp$-module under the multiplication action. Take the left $\mathcal{S}_2^\sharp/P^\sharp$-module,

$$B = \frac{D_i^\sharp \mathcal{S}_2^\sharp}{P^\sharp m_{i\eta_i} + \mathcal{S}_2^\sharp V_i}.$$

As $\mathcal{S}_2^\sharp D_i^\sharp / \mathcal{S}_2^\sharp V_i \cong \mathcal{S}_2^\sharp \otimes_{K^\sharp} (D_i^\sharp/V_i)$ is a free left $\mathcal{S}_2^\sharp$-module of rank one with basis $m_{i\eta_i} + \mathcal{S}_2^\sharp V_i$, $B$ is a free left $\mathcal{S}_2^\sharp/P^\sharp$-module of rank one with basis element $m_{i\eta_i} + (P^\sharp m_{i\eta_i} + \mathcal{S}_2^\sharp V_i)$. As $m_{i\eta_i} Q^\sharp \equiv \sigma_i^{-1}(Q^\sharp) m_{i\eta_i} \equiv P^\sharp m_{i\eta_i} \pmod{\mathcal{S}_2^\sharp V_i}$, $B$ is a right $\mathcal{S}_2^\sharp/Q^\sharp$-module. Let $I = \mathrm{r.ann}_{\mathcal{S}_2^\sharp}(B)$. Then

$$\mathrm{Cl.K.dim}(\mathcal{S}_2^\sharp/I) = \mathrm{Cl.K.dim}(\mathcal{S}_2^\sharp/P^\sharp) = \mathrm{Cl.K.dim}(\mathcal{S}_2^\sharp/Q^\sharp).$$

As $Q^\sharp \subseteq I$, we have $Q^\sharp = I$. Hence $B$ is faithful as an $(\mathcal{S}_2^\sharp/P^\sharp, \mathcal{S}_2^\sharp/Q^\sharp)$-bimodule. Thus, so too is the $(\mathcal{S}_2^\sharp/P^\sharp, \mathcal{S}_2^\sharp/Q^\sharp)$-bimodule $D_i^\sharp \mathcal{S}_2^\sharp/(P^\sharp D_i^\sharp + D_i^\sharp Q^\sharp)$.

As we have the following isomorphism of $\mathcal{S}_2^\sharp$-$\mathcal{S}_2^\sharp$-bimodules,

$$\frac{M^\sharp \mathcal{S}_2^\sharp}{P^\sharp M^\sharp + M^\sharp Q^\sharp} \cong \frac{D_1^\sharp \mathcal{S}_2^\sharp}{P^\sharp D_1^\sharp + D_1^\sharp Q^\sharp} \oplus \cdots \oplus \frac{D_u^\sharp \mathcal{S}_2^\sharp}{P^\sharp D_u^\sharp + D_u^\sharp Q^\sharp},$$

$M^\sharp \mathcal{S}_2^\sharp/(P^\sharp M^\sharp + M^\sharp Q^\sharp)$ is faithful as an $(\mathcal{S}_2^\sharp/P^\sharp, \mathcal{S}_2^\sharp/Q^\sharp)$-bimodule.

Let $\overline{s} \in \overline{\mathcal{S}}_2$ be such that $\overline{s}M \subseteq \overline{P}_2 M + M\overline{Q}_2$. Then

$$\overline{s}M^\sharp \subseteq P^\sharp M^\sharp + M^\sharp Q^\sharp.$$

Hence $\overline{s} \in P^\sharp \cap \overline{\mathcal{S}}_2 = \overline{P}_2$. Therefore $M\overline{\mathcal{S}}_2/(\overline{P}_2 M + M\overline{Q}_2)$ is faithful as a left $\overline{\mathcal{S}}_2/\overline{P}_2$-module and, similarly, as a right $\overline{\mathcal{S}}_2/\overline{Q}_2$-module. Hence by Theorem 2.3.4, $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$. $\square$

Combining the previous two results yields:

**Theorem 2.3.9** *Retain Notation 2.3.5. Let $\tilde{P}_2$ and $\tilde{Q}_2$ be distinct primes of $\mathcal{S}_2$, both distinct from $M\mathcal{S}_2$, such that $\tilde{P}_2 \cap R = \tilde{Q}_2 \cap R = M$. Take $\overline{P}_2 = \tilde{P}_2/M\mathcal{S}_2$, $\overline{Q}_2 = \tilde{Q}_2/M\mathcal{S}_2$ and $Q^\sharp$ a prime ideal of $\mathcal{S}_2^\sharp$ lying over $\overline{Q}_2$. Then $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ if and only if there is $i \in \{1, \ldots, u\}$ such that $\sigma_i^{-1}(Q^\sharp)$ lies over $\overline{P}_2$.* $\square$

**Theorem 2.3.10** *Retain Notation 2.3.5. Let $\tilde{P}_2$ and $\tilde{Q}_2$ be distinct primes of $\mathcal{S}_2$, both distinct from $M\mathcal{S}_2$, such that $\tilde{P}_2 \cap R = \tilde{Q}_2 \cap R = M$. Take $\overline{P}_2 = \tilde{P}_2/M\mathcal{S}_2$, $\overline{Q}_2 = \tilde{Q}_2/M\mathcal{S}_2$ and $P^\sharp$ a prime ideal of $\mathcal{S}_2^\sharp$ lying over $\overline{P}_2$. Then $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ if and only if there is $i \in \{1, \ldots, u\}$ such that $\sigma_i(P^\sharp)$ lies over $\overline{Q}_2$.*

**Proof.** Let $\Theta$ be the natural ring isomorphism from $\mathcal{S}_2$ onto $\mathcal{S}_2^{op} \cong R[\gamma_1, \gamma_1^{-1}, \ldots, \gamma_v, \gamma_v^{-1}; \beta_1^{-1}, \ldots, \beta_v^{-1}]$, defined by $\Theta(\gamma_i) = \gamma_i^{-1}$ and $\Theta(\gamma_i^{-1}) = \gamma_i$, for every $i \in \{1, \ldots, v\}$ with $\Theta(r) = r$ for all $r \in R$. As $\Theta(M\mathcal{S}_2) = M\mathcal{S}_2$, $\Theta$ induces an automorphism of $\overline{\mathcal{S}}_2$. In this case, the $K^\sharp$-algebra automorphism of $\mathcal{S}_2^\sharp$, determined by the action of the subgroup generated by $\beta_1^{-1}, \ldots, \beta_v^{-1}$ on $M$ will be $\sigma_1^{-1}, \ldots, \sigma_u^{-1}$. Now $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ in $\mathcal{S}_2$ if and only if $\tilde{Q}_2 \rightsquigarrow \tilde{P}_2$ in $\mathcal{S}_2^{op}$. By Theorem 2.3.9 and what was said before, this happens if and if there is $i \in \{1, \ldots, u\}$ such that $\sigma_i(P^\sharp) = (\sigma_i^{-1})^{-1}(P^\sharp)$ lies over $\overline{Q}_2$. $\square$

It is now possible to describe the cliques of primes of $\mathcal{S}_2$ contracting to the maximal ideal $M$ and distinct from $M\mathcal{S}_2$.

**Theorem 2.3.11** *Retain Notation 2.3.5, let $\tilde{P}_2$ be a prime ideal of $\mathcal{S}_2$, distinct from $M\mathcal{S}_2$ such that $\tilde{P}_2 \cap R = M$. Take $\overline{P}_2 = \tilde{P}_2/M\mathcal{S}_2$ and $P^\sharp$ a prime ideal of $\mathcal{S}_2^\sharp$ lying over $\overline{P}_2$. Then*

$\mathcal{C}\ell(\tilde{P}_2) = \{\tilde{Q}_2 \in Spec(\mathcal{S}_2) : \tilde{Q}_2/M\mathcal{S}_2 = \sigma_1^{i(1)} \dots \sigma_u^{i(u)}(P^\sharp) \cap \overline{\mathcal{S}}_2 \; for \; i(1), \dots, i(u) \in \mathbb{Z}\}.$

**Proof.** Let $\tilde{Q}_2$ be a prime ideal in the clique of $\tilde{P}_2$. Let $\overline{Q}_2 = \tilde{Q}_2/M\mathcal{S}_2$. If $\tilde{Q}_2 = \tilde{P}_2$, then clearly $\tilde{Q}_2$ belongs to the set on the right. If $\tilde{P}_2 \neq \tilde{Q}_2$, there is a sequence of primes in $\mathcal{S}_2$, $\tilde{P}_2, \tilde{P}_3, \dots, \tilde{P}_n = \tilde{Q}_2$, such that for all $i \in \{2, \dots, n-1\}$, either $\tilde{P}_i \rightsquigarrow \tilde{P}_{i+1}$ or $\tilde{P}_{i+1} \rightsquigarrow \tilde{P}_i$. Write $P^\sharp = P_2^\sharp$. By Theorems 2.3.9 and 2.3.10, for each $j \in \{2, \dots, n-1\}$, there are $i_j \in \{1, \dots, u\}$ and primes $P_3^\sharp, \dots, P_n^\sharp$ in $\mathcal{S}_2^\sharp$ such that $P_{j+1}^\sharp = \sigma_{i_j}(P_j^\sharp)$ or $P_{j+1}^\sharp = \sigma_{i_j}^{-1}(P_j^\sharp)$ and $P_{j+1}^\sharp$ lies over $\overline{P}_{j+1} = \tilde{P}_{j+1}/M\mathcal{S}_2$. Hence, as the $K^\sharp$-automorphisms $\sigma_1, \dots, \sigma_u$ commute pairwise, the result follows.

Conversely, let $\tilde{Q}_2$ be a prime ideal of $\mathcal{S}_2$ such that $\tilde{Q}_2/M\mathcal{S}_2 = \sigma_1^{i(1)} \dots \sigma_u^{i(u)}(P^\sharp) \cap \overline{\mathcal{S}}_2$ for some $i(1), \dots, i(u) \in \mathbb{Z}$. If $i(1) = \dots = i(u) = 0$, then $\tilde{Q}_2 = \tilde{P}_2$ and by Proposition 2.3.3 it follows that $\tilde{Q}_2$ is linked to itself. If some of the $i(1), \dots, i(u)$ are nonzero, as $\sigma_1, \dots, \sigma_u$ commute pairwise, we can write $\tilde{Q}_2/M\mathcal{S}_2 = \sigma_{i_1}^{\delta_1} \dots \sigma_{i_v}^{\delta_v}(P^\sharp) \cap \overline{\mathcal{S}}_2$, for some $i_1, \dots, i_v \in \{1, \dots, u\}$ and $\delta_1, \dots, \delta_v \in \{-1, 1\}$. For $l \in \{1, \dots, v\}$, write $P_{2,l}^\sharp = \sigma_{i_l}^{\delta_l}\sigma_{i_{l+1}}^{\delta_{l+1}} \dots \sigma_{i_v}^{\delta_v}(P^\sharp)$ and take $\tilde{P}_{2,l}$ a prime ideal of $\mathcal{S}_2$ containing $M\mathcal{S}_2$ such that $P_{2,l}^\sharp$ lies over $\overline{P}_{2,l} = \tilde{P}_{2,l}/M\mathcal{S}_2$. Obviously $\tilde{P}_{2,1} = \tilde{Q}_2$. By Theorem 2.3.9 and Theorem 2.3.10, either $\tilde{P}_{2,l} \rightsquigarrow \tilde{P}_{2,l+1}$ or $\tilde{P}_{2,l+1} \rightsquigarrow \tilde{P}_{2,l}$ for $l \in \{1, \dots, v-1\}$ and one of the cases happens, $\tilde{P}_2 \rightsquigarrow \tilde{P}_{2,v}$ or $\tilde{P}_{2,v} \rightsquigarrow \tilde{P}_2$. So $\tilde{Q}_2$ belongs to the clique of $\tilde{P}_2$. $\Box$

## 2.4 Prime links in skew-Laurent rings

In this section we generalize the results obtained in the last. By Proposition 2.2.9, linked prime ideals contract to the same ideal of $R$, say $N$. We will see that it is possible to assume that $R$ is semilocal and that $N$ is its Jacobson radical.

**Theorem 2.4.1** *Let $N$ be a $G$-prime ideal of $R$. Then*

1. *The sets $\{P \in Spec(\mathcal{S}) : P \cap R = N\}$ and $\{P' \in Spec(\mathcal{S}\mathcal{C}_R^{-1}(N)) : P' \cap R\mathcal{C}_R^{-1}(N) = N\mathcal{C}_R^{-1}(N)\}$ are link-closed;*

2. *There is an isomorphism of directed graphs between the graph of links of the two above sets of primes given by the rule $P \longmapsto P\mathcal{S}\mathcal{C}_R^{-1}(N)$.*

**Proof.** By Proposition 2.2.9 the above two sets are link closed. By [44, Theorem 9.22] contraction and extension provide inverse bijections between the set of prime ideals of $\mathcal{S}\mathcal{C}_R^{-1}(N)$ and those prime ideals of $\mathcal{S}$ that are disjoint from $\mathcal{C}_R^{-1}(N)$ and by [98, Lemma 2.11] two prime ideals of $\mathcal{S}$ are linked if and only if their extension are linked in $\mathcal{S}\mathcal{C}_R^{-1}(N)$, so we have *2*. □

Theorem 2.4.1 reduces our study of links in $\mathcal{S} = R[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1, \dots, \alpha_n]$ to the special case where $R$ is semilocal and the primes intersect $R$ in its Jacobson radical.

**Notation 2.4.2** We retain Notation 2.2.8 and 2.2.10. In addition, we suppose that $R$ is semilocal with Jacobson Radical $N$, a $G$-prime ideal of $R$. Hence $R/N = \overline{\overline{R}}$, is a semisimple Artinian ring. Let $K = R/M$, a field.

**Theorem 2.4.3** *Retain Notation 2.4.2. Every prime ideal $\overline{\overline{P}}$ of $\overline{\overline{\mathcal{S}}} = S/N\mathcal{S}$ is such that $\mathcal{R}_{\overline{\overline{\mathcal{S}}}}(\overline{\overline{P}})$ is Noetherian.*

**Proof.** Let $P$ be a prime ideal of $\mathcal{S}$ such that $N\mathcal{S} \subseteq P$ and $\overline{\overline{P}} = P/N\mathcal{S}$. As $P$ contracts in $R$ to a $G$-prime ideal containing $N$, the Jacobson radical of the semilocal commutative Noetherian ring $R$, $P \cap R = N$. By Passman's Theorem, Theorem 1.4.13, $P = (P \cap \mathcal{S}_2)\mathcal{S}$ and $P_2 = P \cap \mathcal{S}_2$ is a $G$-prime ideal of $\mathcal{S}_2$.

As $\overline{\overline{\mathcal{S}}}_2 = \mathcal{S}_2/N\mathcal{S}_2$ is a commutative ring and $P/N\mathcal{S} = ((P \cap \mathcal{S}_2)/N\mathcal{S}_2)\mathcal{S}/N\mathcal{S}$, by Theorem 2.2.4, $\mathcal{R}_{\overline{\overline{\mathcal{S}}}_2}((P \cap \mathcal{S}_2)/N\mathcal{S}_2)$ is Noetherian and by [13, Lemma 7.1], $\mathcal{R}_{\overline{\overline{\mathcal{S}}}}(\overline{\overline{P}})$ is Noetherian. □

**Proposition 2.4.4** *Retain Notation 2.4.2. There are no links between distinct primes of* $\overline{\overline{\mathcal{S}}} = \mathcal{S}/N\mathcal{S}$.

**Proof.** Let $P$ and $Q$ be prime ideals of $\mathcal{S}$ containing $N\mathcal{S}$ such that $P/N\mathcal{S} \rightsquigarrow Q/N\mathcal{S}$. By Theorem 2.4.3 and Lemma 2.2.3, $P/N\mathcal{S}$ and $Q/N\mathcal{S}$ have the AR-property. By Proposition 2.2.7, $P/N\mathcal{S} = Q/N\mathcal{S}$. $\square$

The previous notations, Notation 2.2.8, 2.2.10 and 2.4.2, will remain in effect throughout this section.

**Lemma 2.4.5** *Let $W$ be any commutative Noetherian ring and $I$ an ideal of $W$ which is the intersection of distinct maximal ideals, $I_1, \dots, I_p$. Then*

$$I/I^2 \cong I_1/I_1^2 \oplus \dots \oplus I_p/I_p^2.$$

*as $W$-modules.*

**Proof.** By [63, Theorem 166], $I^2 = I_1^2 \cap \dots \cap I_p^2$. It follows from [110, Theorem 31], that the ideals $I_i^2$ and $I_j^2$ are comaximal (that is, their sum is $W$), for $i \neq j$. Now by [110, Theorem 32], there is a ring isomorphism

$$\frac{W}{I^2} \cong \frac{W}{I_1^2} \oplus \dots \oplus \frac{W}{I_p^2}.$$

Comparing the Jacobson radical of each of the sides of the isomorphism, we have $\mathcal{J}(W/I^2) = I/I^2$ and $\mathcal{J}(\oplus_{i=1}^p W/I_i^2) = \oplus_{i=1}^p \mathcal{J}(W/I_i^2) = \oplus_{i=1}^p (I_i/I_i^2)$, hence we have the result. $\square$

**Proposition 2.4.6** *Retain Notation 2.4.2. The ideal $N\mathcal{S}$ is a prime ideal of $\mathcal{S}$, $\mathcal{C}\ell(N\mathcal{S}) = \{N\mathcal{S}\}$ and $N\mathcal{S} \rightsquigarrow N\mathcal{S}$ if and only if $N \neq 0$.*

**Proof.** By [86, Corollary 14.8], $\mathcal{S}/N\mathcal{S} \cong R/N * H$ is prime if and only if $(R/N * H_1)/(M/N * H_1) \cong K * H_1$ is prime. As $K$ is a field, $K * H_1$ is a Noetherian domain, and so $N\mathcal{S}$ is a prime ideal of $\mathcal{S}$.

Suppose that there is a prime $P$ of $\mathcal{S}$ such that $P \rightsquigarrow N\mathcal{S}$ or $N\mathcal{S} \rightsquigarrow P$. Then by Proposition 2.2.9, $P \cap R = N\mathcal{S} \cap R = N$ and so $N\mathcal{S} \subseteq P$. By Theorem 2.1.18, Cl.K.dim$(\mathcal{S}/N\mathcal{S}) =$ Cl.K.dim$(\mathcal{S}/P)$, so $N\mathcal{S} = P$.

As $N$ is the Jacobson radical of $R$, $N \neq 0$ if and only if $N^2 \neq N$.

By [63, Theorem 166], $N^2 = M_1^2 \cap \ldots \cap M_t^2$. One checks easily that $N \neq N^2$ if and only if $M \neq M^2$. Obviously, if $N\mathcal{S} \rightsquigarrow N\mathcal{S}$, then $N\mathcal{S} \neq N^2\mathcal{S}$ and so $N \neq 0$. Conversely, suppose $N \neq 0$. Then $M \neq M^2$. By Lemma 2.4.5 and the $G$-conjugacy of $M_j$, $N/N^2$ is a faithful $R/N$-module. As $\mathcal{S}$ is free as a right and left $R$-module, $N\mathcal{S}/N^2\mathcal{S}$ is faithful as a right and left $\mathcal{S}/N\mathcal{S}$-module. Thus $N\mathcal{S} \rightsquigarrow N\mathcal{S}$. $\square$

**Proposition 2.4.7** *Retain Notation 2.4.2. Every prime ideal of $\mathcal{S}$ properly containing $N\mathcal{S}$ is linked to itself.*

**Proof.** Let $P$ be a prime ideal of $\mathcal{S}$, properly containing $N\mathcal{S}$. Write $\overline{\overline{P}} = P/N\mathcal{S}$.

By Proposition 2.4.6, $\overline{\overline{\mathcal{S}}} = \mathcal{S}/N\mathcal{S}$ is a prime ring and by [23, Proposition 2.5], as $\overline{\overline{\mathcal{S}}}$ satisfies the second layer condition, there will be some prime of $\overline{\overline{\mathcal{S}}}$ linked to the non-minimal prime $\overline{\overline{P}}$ of $\overline{\overline{\mathcal{S}}}$. Since by Proposition 2.4.4 there are no links between distinct primes of $\overline{\overline{\mathcal{S}}}$, we will have $P/N\mathcal{S} \rightsquigarrow P/N\mathcal{S}$ and so $P \rightsquigarrow P$. $\square$

**Theorem 2.4.8** *Retain Notation 2.4.2. Let $P$ and $Q$ be distinct prime ideals of $\mathcal{S}$, distinct from $N\mathcal{S}$, such that $P \cap R = Q \cap R = N$. Then $P \rightsquigarrow Q$ if and only if $N\mathcal{S}/(PN + NQ)$ is faithful as a left $\mathcal{S}/P$-module and as a right $\mathcal{S}/Q$-module.*

**Proof.** Let $P, Q$ be as stated. Suppose $P \rightsquigarrow Q$ via $(P \cap Q)/A$, for some ideal $A$ of $\mathcal{S}$ such that $PQ \subseteq A \subsetneq P \cap Q$.

If $N\mathcal{S} \subseteq A$, then $P/N\mathcal{S} \rightsquigarrow Q/N\mathcal{S}$ and by Proposition 2.4.4, $P = Q$, a contradiction. So $A \subsetneq N\mathcal{S} + A$. Hence $N\mathcal{S}/(N\mathcal{S} \cap A)$ is faithful as a left $\mathcal{S}/P$-module and as a right $\mathcal{S}/Q$-module. The same happens with $N\mathcal{S}/(PN + NQ)$, since this bimodule has $N\mathcal{S}/(N\mathcal{S} \cap A)$ as a factor.

Conversely, suppose that $\mathcal{NS}/(PN + NQ)$ is faithful as a left $\mathcal{S}/P$-module and as a right $\mathcal{S}/Q$-module. Since, by Corollary 2.2.12, $\mathcal{S}$ satisfies the second layer condition, by Theorem 2.1.18,

$$\text{Cl.K.dim}(\mathcal{S}/P) = \text{Cl.K.dim}(\mathcal{S}/Q). \tag{2.2}$$

By Lemma 2.1.19 there is an ideal $A$ of $\mathcal{S}$ such that $PN + NQ \subseteq A \subsetneqq N\mathcal{S}$ and $N\mathcal{S}/A$ is torsionfree as a left $\mathcal{S}/P$-module and as a right $\mathcal{S}/Q$-module. Therefore the bimodule $N\mathcal{S}/A$ is a bond from $P/A$ to $Q/A$. By [42, Theorem 1.1], there are $P_0/A, \ldots, P_r/A$, with $r \geq 1$, distinct prime ideals of $\mathcal{S}/A$ such that

$$P/A = P_0/A \rightsquigarrow P_1/A \rightsquigarrow \ldots \rightsquigarrow P_r/A = Q/A.$$

Suppose $P_0/A \rightsquigarrow P_1/A$ via $(P_0/A \cap P_1/A)/(B/A)$, for some ideal $B$ of $\mathcal{S}$ such that $A \subseteq B$ and $P_0P_1 \subseteq B \subsetneqq P_0 \cap P_1$. Then, $P \rightsquigarrow P_1$ via $(P \cap P_1)/B$. As above $N\mathcal{S}/(N\mathcal{S} \cap B)$ is faithful as an $(\mathcal{S}/P, \mathcal{S}/P_1)$-bimodule. As $NQ \subseteq N\mathcal{S} \cap A \subseteq N\mathcal{S} \cap B$, $Q \subseteq P_1$.

Since $P/A \rightsquigarrow P_1/A$, $\text{Cl.K.dim}(\mathcal{S}/P) = \text{Cl.K.dim}(\mathcal{S}/P_1)$ and thus $\text{Cl.K.dim}(\mathcal{S}/P_1) = \text{Cl.K.dim}(\mathcal{S}/Q)$, by (2.2) above. Hence $P_1 = Q$ and $P/A \rightsquigarrow Q/A$. Therefore $P \rightsquigarrow Q$. □

**Theorem 2.4.9** *Retain Notation 2.4.2. Let $P, Q$ be distinct prime ideals of $\mathcal{S}$, distinct from $N\mathcal{S}$, such that $P \cap R = Q \cap R = N$. Let $P \cap \mathcal{S}_2 = P_2$ and $Q \cap \mathcal{S}_2 = Q_2$. Then $P \rightsquigarrow Q$ if and only if $N\mathcal{S}_2/(P_2N + NQ_2)$ is faithful as a left $\mathcal{S}_2/P_2$-module and as a right $\mathcal{S}_2/Q_2$-module.*

**Proof.** Suppose $P, Q$ are distinct prime ideals of $\mathcal{S}$, different from $N\mathcal{S}$, such that $P \cap R = N = Q \cap R$ and $P \rightsquigarrow Q$. Then, by Theorem 2.4.8, $N\mathcal{S}/(PN + NQ)$ is faithful as a left $\mathcal{S}/P$-module and as a right $\mathcal{S}/Q$-module.

By Theorem 1.4.13, $P = P_2\mathcal{S}$, $Q = Q_2\mathcal{S}$ and $P_2, Q_2$ are $G$-prime ideals of $\mathcal{S}_2$.

Let $s_2 \in \mathcal{S}_2$ be such that $s_2 N\mathcal{S}_2 \subseteq P_2 N + NQ_2$. As $\mathcal{S}N$ is an ideal of $\mathcal{S}$, we will have

$$s_2 N\mathcal{S} = s_2 N\mathcal{S}_2\mathcal{S} \subseteq P_2 N\mathcal{S} + NQ_2\mathcal{S} = PN + NQ.$$

Therefore, $s_2 \in P \cap \mathcal{S}_2$ and $N\mathcal{S}_2/(P_2N + NQ_2)$ is faithful as a left $\mathcal{S}_2/P_2$-module. Similarly, $N\mathcal{S}_2/(P_2N + NQ_2)$ is faithful as a right $\mathcal{S}_2/Q_2$-module.

Now, suppose that $N\mathcal{S}_2/(P_2N + NQ_2)$ is faithful as a left $\mathcal{S}_2/P_2$-module and as a right $\mathcal{S}_2/Q_2$-module. As $\mathcal{S}$ is free as a right or left $\mathcal{S}_2$-module, it follows easily from Theorem 2.4.8 that $P \rightsquigarrow Q$. $\square$

**Lemma 2.4.10** *Retain Notation 2.4.2. Let $P$ be a prime ideal of $\mathcal{S}$ such that $P \cap R = N$. Then there is a minimal prime $\widetilde{P}_2$ over $P \cap \mathcal{S}_2$ in $\mathcal{S}_2$ such that $\widetilde{P}_2 \cap R = M$. It is then possible to write $P \cap \mathcal{S}_2$ in the form*

$$P \cap \mathcal{S}_2 = \widetilde{P}_2^{\beta_{1,1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{1,w}} \cap \ldots \cap \widetilde{P}_2^{\beta_{t,1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{t,w}}$$

*for some $\beta_{l,k} \in G$, $l \in \{1, \ldots, t\}$, $k \in \{1, \ldots, w\}$ and $\beta_{1,1} = 1_G$, such that*

*i) $M^{\beta_{l,k}} = \widetilde{P}_2^{\beta_{l,k}} \cap R = M^{x_l}$, for any $l \in \{1, \ldots, t\}$ and $k \in \{1, \ldots, w\}$;*

*ii) $\{\widetilde{P}_2^{\beta_{1,1}}, \ldots, \widetilde{P}_2^{\beta_{t,w}}\}$ forms a single $G$-orbit of $\widetilde{P}_2$ in $\mathcal{S}_2$;*

*iii) for any $l \in \{1, \ldots, t\}$, $\{\widetilde{P}_2^{\beta_{l,1}}, \ldots, \widetilde{P}_2^{\beta_{l,w}}\}$ forms a single $G_1$-orbit of $\widetilde{P}_2^{x_l}$ in $\mathcal{S}_2$ and*

$$\widetilde{P}_2^{\beta_{l,1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{l,w}} = \cap_{g \in G_1} (\widetilde{P}_2^{x_l})^g.$$

**Proof.** Let $P$ be a prime ideal of $\mathcal{S}$ such that $P \cap R = N$. By Theorem 1.4.13, $P \cap \mathcal{S}_2$ is a $G$-prime ideal of $\mathcal{S}_2$. By Lemma 1.3.5

$$P \cap \mathcal{S}_2 = \cap_{\alpha \in G} \widetilde{P}_2^{\alpha} = \widetilde{P}_2^{y_1} \cap \ldots \cap \widetilde{P}_2^{y_{t'}}$$

for any minimal prime $\widetilde{P}_2$ over $P \cap \mathcal{S}_2$ in $\mathcal{S}_2$ and some $y_1, y_2, \ldots, y_{t'} \in G$. We can and shall assume that $y_1 = 1_G$. Thus $\{\widetilde{P}_2, \ldots, \widetilde{P}_2^{y_{t'}}\}$ form a single $G$-orbit of $\widetilde{P}_2$ and we can assume that $\widetilde{P}_2^{y_i} \neq \widetilde{P}_2^{y_j}$, whenever $i \neq j$, for all $i, j \in \{1, \ldots, t'\}$.

Since $M$ is a $G_2$-stable ideal of $R$, $M^{x_i}\mathcal{S}_2$ are ideals of $\mathcal{S}_2$. Since $N\mathcal{S}_2 = M^{x_1}\mathcal{S}_2 \cap \ldots \cap M^{x_t}\mathcal{S}_2 \subseteq \widetilde{P}_2$, there will be $i \in \{1, \ldots, t\}$ such that $M^{x_i}\mathcal{S}_2 \subseteq \widetilde{P}_2$. As $M^{x_i}$ is maximal, $M^{x_i} = \widetilde{P}_2 \cap R$. Without loss of generality, we can suppose $\widetilde{P}_2 \cap R = M$.

As $\cap_{i=1}^{t} M^{x_i} = N = P \cap R = P \cap \mathcal{S}_2 \cap R = \cap_{j=1}^{t'}(\widetilde{P}_2^{y_j} \cap R)$ for each $i \in \{1, \ldots, t\}$, there is $j_i \in \{1, \ldots, t'\}$ such that

$$\widetilde{P}_2^{y_{j_i}} \cap R \subseteq M^{x_i}.$$

Also $\widetilde{P}_2^{y_{j_i}} \cap R = (\widetilde{P}_2 \cap R)^{y_{j_i}} = M^{y_{j_i}}$. As $M$ is maximal, $M^{x_i} = M^{y_{j_i}}$ and $\widetilde{P}_2^{y_{j_i}} \cap R = M^{x_i}$. Since $M^{x_i} \neq M^{x_j}$ for $i, j \in \{1, \ldots, t\}$ and $i \neq j$, it is then possible to conclude that each $M^{x_i}$ is the intersection with $R$ of one of the primes $\widetilde{P}_2^{y_1}, \ldots, \widetilde{P}_2^{y_{t'}}$ and that each of these primes contract in $R$ to one and only one $M^{x_j}$, for $j \in \{1, \ldots, t\}$. Hence $t \leq t'$ and we can write

$$P_2 = \widetilde{P}_2^{\beta_{1,1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{1,i_1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{t,1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{t,i_t}}$$

where $\{\beta_{1,1} = 1_G, \beta_{1,2}, \ldots, \beta_{t,i_t}\} = \{1_G, y_2, \ldots, y_{t'}\}$, and $\widetilde{P}_2^{\beta_{j,1}}, \ldots, \widetilde{P}_2^{\beta_{j,i_j}}$ are distinct and contract to $M^{x_j}$ in $R$, for every $j \in \{1, \ldots, t\}$.

Let $l \in \{1, \ldots, t\}$. For any $g \in G_1$ we have

$$(\widetilde{P}_2^{x_l})^g \cap R = (\widetilde{P}_2^{x_l} \cap R)^g = (M^{x_l})^g = (M^g)^{x_l} = M^{x_l}.$$

As $\{\widetilde{P}_2^{\beta_{1,1}}, \ldots, \widetilde{P}_2^{\beta_{1,i_1}}, \ldots, \widetilde{P}_2^{\beta_{t,1}}, \ldots, \widetilde{P}_2^{\beta_{t,i_t}}\}$ forms a single $G$-orbit of $\widetilde{P}_2$ and by the way we chose the $\widetilde{P}_2^{\beta_{l,j}}$, we have

$$(\widetilde{P}_2^{x_l})^g \in \{\widetilde{P}_2^{\beta_{l,1}}, \ldots, \widetilde{P}_2^{\beta_{l,i_l}}\}.$$

Hence $\{\widetilde{P}_2^{\beta_{l,1}}, \ldots, \widetilde{P}_2^{\beta_{l,i_l}}\}$ forms a single $G_1$-orbit of $\widetilde{P}_2^{x_l}$ in $\mathcal{S}_2$ and

$$\widetilde{P}_2^{\beta_{l,1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{l,i_l}} \subseteq \cap_{g \in G_1}(\widetilde{P}_2^{x_l})^g.$$

Also, as for each $k \in \{1, \ldots, i_l\}$, we have $M^{\beta_{l,k}} = M^{x_l} = M_l$, $\beta_{l,k}x_l^{-1} = x_l^{-1}\beta_{l,k} \in G_1$ and so $\widetilde{P}_2^{\beta_{l,k}} \in \{(\widetilde{P}_2^{x_l})^g : g \in G_1\}$. Hence

$$\widetilde{P}_2^{\beta_{l,1}} \cap \ldots \cap \widetilde{P}_2^{\beta_{l,i_l}} = \cap_{g \in G_1}(\widetilde{P}_2^{x_l})^g.$$

It remains to prove that $i_1 = i_2 = \ldots = i_t$. As for each $l \in \{1, \ldots, t\}$, $\{(\tilde{P}_2^{\beta_{1,1}})^{x_l}, \ldots, (\tilde{P}_2^{\beta_{1,i_1}})^{x_l}\} \subseteq \{\tilde{P}_2^{\beta_{l,1}}, \ldots, \tilde{P}_2^{\beta_{l,i_l}}\}$ and also $\{(\tilde{P}_2^{\beta_{l,1}})^{x_l^{-1}}, \ldots, (\tilde{P}_2^{\beta_{l,i_l}})^{x_l^{-1}}\} \subseteq \{\tilde{P}_2^{\beta_{1,1}}, \ldots, \tilde{P}_2^{\beta_{1,i_1}}\}$, $i_1 = i_l$. Let $w = i_1 = \ldots = i_t$.
□

**Remark 2.4.11** 1) If $\mathcal{S}_1 = \mathcal{S}_2$, then for $P$ satisfying the same conditions as in Lemma 2.4.10, we would have

$$P \cap \mathcal{S}_2 = \tilde{P}_2^{x_1} \cap \ldots \cap \tilde{P}_2^{x_t}$$

and $\{\tilde{P}_2^{x_1}, \ldots, \tilde{P}_2^{x_t}\}$ forms a single $G$-orbit.

Assuming the same notation as in the statement of Lemma 2.4.10, and in addition $\mathcal{S}_1 = \mathcal{S}_2$, $\tilde{P}_2^{\beta_{i,l}} \cap R = \tilde{P}_2^{\beta_{i,k}} \cap R$ for any $i \in \{1, \ldots, t\}$ and $l, k \in \{1, \ldots, i\}$. So

$$M^{\beta_{i,l}} = M^{x_i} = M^{\beta_{i,k}}$$

hence $\beta_{i,l}\beta_{i,k}^{-1} \in G_1 = G_2$. As $\tilde{P}_2$ is an ideal of $\mathcal{S}_2 = \mathcal{S}_1$, it is $G_1$-stable, hence $\tilde{P}_2^{\beta_{i,l}} = \tilde{P}_2^{\beta_{i,k}}$. So $w = 1$ and $\{\tilde{P}_2^{\beta_{1,1}}, \ldots, \tilde{P}_2^{\beta_{t,1}}\}$ is a single $G$-orbit of $\tilde{P}_2$ in $\mathcal{S}_2$. As $\tilde{P}_2^{x_1}, \ldots, \tilde{P}_2^{x_t}$ are all distinct (as the same happens with their intersections in $R$) and belong to $\{\tilde{P}_2^{\beta_{1,1}}, \ldots, \tilde{P}_2^{\beta_{t,1}}\}$, we will have

$$P \cap \mathcal{S}_2 = \tilde{P}_2^{x_1} \cap \ldots \cap \tilde{P}_2^{x_t}.$$

2) Assuming the same notation as in Lemma 2.4.10, it is possible that $w \neq 1$. For instance take $\mathcal{S} = \mathbb{C}[\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}; id_{\mathbb{C}}, \alpha_2]$ where $\alpha_2$ is complex conjugation. Take $M = N = 0$, so $\mathcal{S}_2 = \mathbb{C}[\theta_1, \theta_1^{-1}, \theta_2^2, \theta_2^{-2}]$. Let $\tilde{P}_2 = (\theta_1 - i)$, a prime ideal of $\mathcal{S}_2$. Take $P = [(\theta_1 - i) \cap (\theta_1 + i)]\mathcal{S}$. As $\tilde{P}_2 \cap \tilde{P}_2^{\alpha_2}$ is an $< \alpha_2 >$-stable ideal of $\mathbb{C}[\theta_1, \theta_1^{-1}, \theta_2^2, \theta_2^{-2}]$, by Theorem 1.4.13, $P$ is a prime ideal of $\mathcal{S}$. In this case $P \cap \mathbb{C} = 0$ but $P \cap \mathcal{S}_2 = (\theta_1 - i)\mathcal{S}_2 \cap (\theta_1 + i)\mathcal{S}_2$.

**Proposition 2.4.12** *Retain Notation 2.4.2. Let $P$ be a prime ideal of $\mathcal{S}$, distinct from $N\mathcal{S}$, such that $P \cap R = N$. Put $P_2 = P \cap \mathcal{S}_2$ and let $\tilde{P}_2$ be a minimal prime over $P \cap \mathcal{S}_2$ in $\mathcal{S}_2$ such that $\tilde{P}_2 \cap R = M$.*

*If there is a prime ideal $\widetilde{Q}_2$ of $\mathcal{S}_2$, distinct from $\widetilde{P}_2$, such that $\widetilde{P}_2 \rightsquigarrow \widetilde{Q}_2$ in $\mathcal{S}_2$, then $P \rightsquigarrow (\cap_{\alpha\in G}\widetilde{Q}_2^{\alpha})\mathcal{S}$ in $\mathcal{S}$.*

**Proof.** Let $P$ be as stated and suppose there is a prime ideal $\widetilde{Q}_2$ of $\mathcal{S}_2$, distinct from $\widetilde{P}_2$, such that $\widetilde{P}_2 \rightsquigarrow \widetilde{Q}_2$ in $\mathcal{S}_2$. Then $M = \widetilde{P}_2 \cap R = \widetilde{Q}_2 \cap R$, by Proposition 2.2.9.

As $Q_2 = \cap_{\alpha\in G}\widetilde{Q}_2^{\alpha}$ is a $G$-prime ideal of $\mathcal{S}_2$ such that $Q_2 \cap R = N$, $(\cap_{\alpha\in G}\widetilde{Q}_2^{\alpha})\mathcal{S}$ is a prime ideal of $\mathcal{S}$, by Theorem 1.4.13.

If $\widetilde{Q}_2 = M\mathcal{S}_2$ then, since Cl.K.dim$(\mathcal{S}_2/\widetilde{P}_2)$ = Cl.K.dim$(\mathcal{S}_2/\widetilde{Q}_2)$ and $M\mathcal{S}_2 \subseteq \widetilde{P}_2$, $\widetilde{P}_2 = \widetilde{Q}_2$, a contradiction. By Theorem 2.3.4 the bimodule $M\mathcal{S}_2/(\widetilde{P}_2M + M\widetilde{Q}_2)$ is faithful as a left $\mathcal{S}_2/\widetilde{P}_2$-module and as a right $\mathcal{S}_2/\widetilde{Q}_2$-module.

If we think about the $(\mathcal{S}_2, \mathcal{S}_2)$-bimodule

$$C_1 = \frac{M\mathcal{S}_2}{(\cap_{g\in G_1}\widetilde{P}_2^{g})M + M(\cap_{g\in G_1}\widetilde{Q}_2^{g})}$$

the left and right annihilators of $C_1$ in $\mathcal{S}_2$ are $G_1$-stable ideals of $\mathcal{S}_2$. Also, as $M\mathcal{S}_2/(\widetilde{P}_2M + M\widetilde{Q}_2)$ is a factor bimodule of $C_1$, we have $\text{l.ann}_{\mathcal{S}_2}(C_1) \subseteq \widetilde{P}_2$ and $\text{r.ann}_{\mathcal{S}_2}(C_1) \subseteq \widetilde{Q}_2$. Hence $\text{l.ann}_{\mathcal{S}_2}(C_1) \subseteq \cap_{g\in G_1}\widetilde{P}_2^{g}$ and $\text{r.ann}_{\mathcal{S}_2}(C) \subseteq \cap_{g\in G_1}\widetilde{Q}_2^{g}$. So $\text{l.ann}_{\mathcal{S}_2}(C_1) = \cap_{g\in G_1}\widetilde{P}_2^{g}$ and $\text{r.ann}_{\mathcal{S}_2}(C_1) = \cap_{g\in G_1}\widetilde{Q}_2^{g}$.

Similarly, for all $i \in \{1, \ldots, t\}$, $M_i = M^{x_i}$ and

$$C_i = \frac{M_i\mathcal{S}_2}{(\cap_{g\in G_1}(\widetilde{P}_2^{x_i})^{g})M_i + M_i(\cap_{g\in G_1}(\widetilde{Q}_2^{x_i})^{g})}$$

is a left $\mathcal{S}_2/\cap_{g\in G_1}(\widetilde{P}_2^{x_i})^{g}$-module and a right $\mathcal{S}_2/\cap_{g\in G_1}(\widetilde{Q}_2^{x_i})^{g}$-module, faithful on both sides.

By Lemma 2.4.5, we have

$$\frac{N\mathcal{S}_2}{N^2\mathcal{S}_2} \cong \frac{M_1\mathcal{S}_2}{M_1^2\mathcal{S}_2} \oplus \ldots \oplus \frac{M_t\mathcal{S}_2}{M_t^2\mathcal{S}_2}.$$

Hence,

$$\frac{N\mathcal{S}_2}{P_2N + NQ_2} \cong \frac{\mathcal{S}_2M_1}{P_2M_1 + M_1Q_2 + M_1^2\mathcal{S}_2} \oplus \ldots \oplus \frac{\mathcal{S}_2M_t}{P_2M_t + M_tQ_2 + M_t^2\mathcal{S}_2}.$$

Thus $\text{l.ann}_{\mathcal{S}_2}(N\mathcal{S}_2/(P_2N + NQ_2)) \subseteq \bigcap_{i=1}^t \text{l.ann}_{\mathcal{S}_2}(\mathcal{S}_2M_i/(\bigcap_{g\in G_1}(\tilde{P}_2^{x_i})^g)M_i + M_i(\bigcap_{g\in G_1}(\tilde{Q}_2^{x_i})^g) = \bigcap_{i=1}^t(\bigcap_{g\in G_1}(\tilde{P}_2^{x_i})^g)$. By Lemma 2.4.10, $\bigcap_{i=1}^t(\bigcap_{g\in G_1}(\tilde{P}_2^{x_i})^g) = P_2$, and the left $\mathcal{S}_2/P_2$-module $N\mathcal{S}_2/(P_2N + NQ_2)$ is faithful. Similarly, $N\mathcal{S}_2/(P_2N + NQ_2)$ is faithful as a right $\mathcal{S}_2/Q_2$-module. Since $Q_2\mathcal{S} \cap \mathcal{S}_2 = Q_2$, by Theorem 2.4.9, we have that $P \rightsquigarrow (\bigcap_{\alpha\in G}\tilde{Q}_2^{\alpha})\mathcal{S}$ in $\mathcal{S}$. □

**Theorem 2.4.13** *Retain Notation 2.4.2. Let $P$ and $Q$ be distinct prime ideals of $\mathcal{S}$, distinct from $N\mathcal{S}$, such that $P \rightsquigarrow Q$ and $P \cap R = N = Q \cap R$. Let $P_2 = P \cap \mathcal{S}_2$ and $Q_2 = Q \cap \mathcal{S}_2$. There are minimal primes, $\tilde{P}_2$ and $\tilde{Q}_2$, over $P_2$ and $Q_2$ in $\mathcal{S}_2$, respectively, such that $\tilde{P}_2 \cap R = M = \tilde{Q}_2 \cap R$ and $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$.*

**Proof.** Let $P$ and $Q$ be as stated. Then, by Theorem 2.4.9, $N\mathcal{S}_2/(P_2N+NQ_2)$ is faithful as a left $\mathcal{S}_2/P_2$-module and as a right $\mathcal{S}_2/Q_2$-module. By [57, Proposition 8.2.6],

$$\begin{aligned}
\text{Cl.K.dim}(\mathcal{S}_2/P_2) &= \max\{\text{Cl.K.dim}(\mathcal{S}_2/\tilde{P}_2^{\beta_{i,j}}) : i \in \{1,\dots,t\}, j \in \{1,\dots,w\}\} \\
&= \text{Cl.K.dim}(\mathcal{S}_2/\tilde{P}_2^{x_i}),
\end{aligned}$$

for any $i \in \{1,\dots,t\}$. Therefore $\text{Cl.K.dim}(\mathcal{S}_2/P_2) = \text{Cl.K.dim}(\mathcal{S}_2/P_2')$, for any $P_2'$, minimal prime over $P_2$ in $\mathcal{S}_2$. The same happens to $\mathcal{S}_2/Q_2$. As $\mathcal{S}_2$ is a Noetherian ring that satisfies the second layer condition, if $\mathcal{C} = \mathcal{C}_{\mathcal{S}_2/P_2}(0)$ and $\mathcal{D} = \mathcal{C}_{\mathcal{S}_2/Q_2}(0)$, by Lemma 2.1.19, the left $\mathcal{C}$-torsion submodule and the right $\mathcal{D}$-torsion submodule of $N\mathcal{S}_2/(P_2N + NQ_2)$ are the same and different from $N\mathcal{S}_2/(P_2N + NQ_2)$. Since

$$\frac{N\mathcal{S}_2}{P_2N + NQ_2} \cong \frac{\mathcal{S}_2M_1}{P_2M_1 + M_1Q_2 + M_1^2\mathcal{S}_2} \oplus \dots \oplus \frac{\mathcal{S}_2M_t}{P_2M_t + M_tQ_2 + M_t^2\mathcal{S}_2},$$

as an $\mathcal{S}_2$-module (right or left), it follows that, for all $j \in \{1,\dots,t\}$,

$$\mathrm{t}_{\mathcal{C}}(M_j\mathcal{S}_2/(P_2M_j + M_jQ_2 + M_j^2\mathcal{S}_2)) = \mathrm{t}_{\mathcal{D}}(M_j\mathcal{S}_2/(P_2M_j + M_jQ_2 + M_j^2\mathcal{S}_2)).$$

Also, there is $i \in \{1,\dots,t\}$, such that $\mathrm{t}_{\mathcal{C}}(M_i\mathcal{S}_2/(P_2M_i + M_iQ_2 + M_i^2\mathcal{S}_2)) \neq M_i\mathcal{S}_2/(P_2M_i + M_iQ_2 + M_i^2\mathcal{S}_2)$. Let $A_i/(P_2M_i + M_iQ_2 + M_i^2\mathcal{S}_2)$ be this torsion

$(\mathcal{S}_2/P_2, \mathcal{S}_2/Q_2)$-bimodule. Therefore $\mathcal{S}_2 M_i/A_i$ is nonzero and torsionfree as a left $\mathcal{S}_2/P_2$-module and as a right $\mathcal{S}_2/Q_2$-module.

Take a $(\mathcal{S}_2/P_2, \mathcal{S}_2/Q_2)$-subbimodule $B_1$ of $\mathcal{S}_2 M_i$ such that $A_i \leq B_1 \leq M_i\mathcal{S}_2$ and the bimodule $B_1/A_i$ has prime annihilators $P'/P_2$, $Q'/Q_2$ as a left $\mathcal{S}_2/P_2$-module and as a right $\mathcal{S}_2/Q_2$-module and is torsionfree as a left $\mathcal{S}_2/P'$-module and as a right $\mathcal{S}_2/Q'$-module. Such a bimodule exists by [44, Corollary 7.6]. Since $M_i\mathcal{S}_2/A_i$ is torsionfree as a left $\mathcal{S}_2/P_2$-module and as a right $\mathcal{S}_2/Q_2$-module, $P'$ and $Q'$ are minimal primes over $P_2$ and $Q_2$, respectively, [44, Proposition 6.3].

As $\mathcal{S}_2 M_i \subseteq \mathrm{l.ann}_{\mathcal{S}_2}(B_1/A_i) = P'$, $P' \cap R = M_i$. Also $Q' \cap R = M_i$. Therefore $B_1/A_i$ is a bimodule subfactor of $M_i\mathcal{S}_2/M_i^2\mathcal{S}_2$ which is a bond from $P'$ to $Q'$. As $P'$, $Q'$ are prime ideals of $\mathcal{S}_2$ containing $M_i\mathcal{S}_2$ and $\mathcal{S}_2/M_i\mathcal{S}_2$ is a commutative Noetherian domain, $P' \rightsquigarrow Q'$ by [23, Lemma 2.9]. If we let $\tilde{P}_2 = (P')^{x_i^{-1}}$ and $\tilde{Q}_2 = (Q')^{x_i^{-1}}$, the result follows. □

The next result summarises the conclusions of Theorem 2.4.1, Propositions 2.4.6, 2.4.7, 2.4.12 and Theorem 2.4.13.

**Theorem 2.4.14** *Retain Notation 2.2.8, 2.2.10 where $M$ is not necessarily maximal. Let $P$ and $Q$ be prime ideals of $\mathcal{S}$, with $P \cap R = N$. Suppose that $P \rightsquigarrow Q$. Then $Q \cap R = N$ and one of the following holds:*

1. $0 \neq N\mathcal{S} = P = Q$;
2. $N\mathcal{S} \subsetneq P = Q$;
3. $0 \neq N\mathcal{S} \subsetneq P \neq Q$ *and* $\tilde{P}_2 \rightsquigarrow \tilde{Q}_2$ *where* $\tilde{P}_2$ *and* $\tilde{Q}_2$ *are minimal primes over* $P \cap \mathcal{S}_2$ *and* $Q \cap \mathcal{S}_2$ *such that* $\tilde{P}_2 \cap R = \tilde{Q}_2 \cap R$.

*Conversely, if one of case 1,2 or 3 holds, then $P \rightsquigarrow Q$.* □

The description of links in a skew-Laurent ring is now obtained if we combine Theorem 2.4.14 and the results of section 3.

**Proposition 2.4.15** *Retain Notation 2.2.8, 2.2.10 where $M$ is not necessarily maximal. Let $P$ and $Q$ be prime ideals of $\mathcal{S}$, with $P \cap R = N$. Suppose that $P \rightsquigarrow Q$. Then $Q \cap R = N$ and one of the following holds:*

1. $0 \neq N\mathcal{S} = P = Q$;

2. $N\mathcal{S} \subsetneq P = Q$;

3. $0 \neq N\mathcal{S} \subsetneq P \neq Q$ *and there exist a prime ideal $P^\sharp$ of $\mathcal{S}_2^\sharp$ lying over $\widetilde{P}_2/M\mathcal{S}_2\mathcal{C}_R^{-1}(N)$ and $i \in \{1, \ldots, u\}$ such that $\sigma_i(P^\sharp)$ lies over $\widetilde{Q}_2/M\mathcal{S}_2\mathcal{C}_R^{-1}(N)$ where, $\widetilde{P}_2$ and $\widetilde{Q}_2$ are minimal primes over $P\mathcal{C}_R^{-1}(N) \cap \mathcal{S}_2\mathcal{C}_R^{-1}(N)$ and $Q\mathcal{C}_R^{-1}(N) \cap \mathcal{S}_2\mathcal{C}_R^{-1}(N)$, respectively, such that $\widetilde{P}_2 \cap R\mathcal{C}_R^{-1}(N) = M\mathcal{C}_R^{-1}(N) = \widetilde{Q}_2 \cap R\mathcal{C}_R^{-1}(N)$, $\sigma_i$ are the automorphisms defined as in section 3 and if $K = R\mathcal{C}_R^{-1}(N)/M\mathcal{C}_R^{-1}(N)$ and $\mathcal{S}_2^\sharp = K^\sharp \otimes_K \mathcal{S}_2\mathcal{C}_R^{-1}(N)$.*

*Conversely, if one of case 1,2 or 3 holds, then $P \rightsquigarrow Q$.* □

As the statement of the following result depends on some conditions on the elements $\varepsilon_{ij}$ for $i \in \{1, \ldots, u\}$ and $j \in \{1, \ldots, v\}$ determined in section 3, in order to simplify we will assume once again that the ideal $M$ of Notation 2.2.8 and 2.2.10 is maximal.

**Proposition 2.4.16** *Retain Notation 2.4.2 and 2.3.5. All the cliques of prime ideals of $\mathcal{S}$ contracting to the ideal $N$ are finite if and only if the multiplicative subgroup of $K^\sharp$ generated by $\varepsilon_{ij}$ for $i \in \{1, \ldots, u\}$ and $j \in \{1, \ldots, v\}$ is finite.*

**Proof.** If the order of the group generated by $\varepsilon_{ij}$ is finite, then each $\sigma_i$ has finite order and by Theorem 2.3.11 and Proposition 2.4.15, the clique of any prime of $\mathcal{S}$ will be finite.

Conversely, if the cliques of prime ideals of $\mathcal{S}$ contracting to the ideal $N$ and different from $N\mathcal{S}$ are finite, by Proposition 2.4.15, the cliques of primes of $\mathcal{S}_2$ contracting to $M$ and different from $M\mathcal{S}_2$ will be finite as well. Take the prime of $\mathcal{S}_2$, $\widetilde{P}_2 = M\mathcal{S}_2 + (\gamma_1 - 1, \ldots, \gamma_v - 1)\mathcal{S}_2$. By Theorem 2.3.10, for every $i \in \{1, \ldots, u\}$ there are $j, k \in \mathbb{Z}$ such that $j \neq k$ and $(\gamma_1 - \varepsilon_{i1}^{-j}, \ldots, \gamma_v - \varepsilon_{iv}^{-j}) = \sigma_i^j(\gamma_1 - 1, \ldots, \gamma_v - 1) = \sigma_i^k(\gamma_1 - 1, \ldots, \gamma_v - 1) = (\gamma_1 - \varepsilon_{i1}^{-k}, \ldots, \gamma_v - \varepsilon_{iv}^{-k})$.

But then $\varepsilon_{il}^{-j} = \varepsilon_{il}^{-k}$, for every $l \in \{1, \ldots, v\}$. So, for every $i \in \{1, \ldots, u\}$ and $l \in \{1, \ldots, v\}$, $\varepsilon_{il}$ has finite order. □

**Remark 2.4.17** Suppose that every $\Psi(\gamma_j)|_M$ has finite order, $p_j$ say, for $j \in \{1, \ldots, n\}$. Then for each $i \in \{1, \ldots, u\}$, if we take the basis $D_i^\sharp$ chosen before in section 3, then $(\gamma_j - \varepsilon_{ij}).m_{i1} = 0$. So $\gamma_j.m_{i1} = \varepsilon_{ij}m_{i1}$. By induction, we can see that $\gamma_j^{p_j} \cdot m_{i1} = \varepsilon_{ij}^{p_j}m_{i1}$. Hence each $\varepsilon_{ij}$ has finite order and by Proposition 2.4.16, it follows that the clique of each prime $P$ of $\mathcal{S}$ is finite. However the converse is not true as example 4 in §2.6 shows.

## 2.5 Prime links in skew-polynomial rings

In this section we reduce the study of links between prime ideals in skew-polynomial rings to the similar problem in skew-Laurent rings. We will keep the notation introduced before in Notation 2.2.8 and 2.2.10. The description of cliques in $\mathcal{T}$ will, in some cases, depend on the description of cliques in $\mathcal{S}$ given before in section 4.

**Theorem 2.5.1** *Retain Notation 2.2.8 and 2.2.10. Let $P, Q \in Spec(\mathcal{T})$ such that $\theta_{i+1}, \ldots, \theta_n \in P$ and $\theta_1, \ldots, \theta_i \notin P$. Then $P \rightsquigarrow Q$ if and only if $\theta_{i+1}, \ldots, \theta_n \in Q$ and either (a) $P/(\theta_{i+1}\mathcal{T} + \ldots + \theta_n\mathcal{T}) \rightsquigarrow Q/(\theta_{i+1}\mathcal{T} + \ldots + \theta_n\mathcal{T})$ in $R[\theta_1, \ldots, \theta_i; \alpha_1, \ldots, \alpha_i]$ or (b) there is $j \in \{i+1, \ldots, n\}$ such that $P = \alpha_j(Q)$.*

**Proof.** Suppose $P \rightsquigarrow Q$ in $\mathcal{T}$ via $(P \cap Q)/A$.

As $\theta_1, \ldots, \theta_n$ are normal elements of $\mathcal{T}$, $\theta_{i+1}, \ldots, \theta_n \in Q$ and $\theta_1, \ldots, \theta_i \notin Q$, by Proposition 2.2.7.

If there is $j \in \{i+1, \dots, n\}$ such that $\theta_j \notin A$, then $(A + \theta_j \mathcal{T})/A$ is a nonzero subbimodule of $(P \cap Q)/A$. As $(P \cap Q)/A$ is torsionfree as an $(R/P, R/Q)$-bimodule, so is $(A + \theta_j \mathcal{T})/A$, and hence this is a faithful bimodule. Since $\theta_j \alpha_j^{-1}(P) = P\theta_j \subseteq A$ and $\alpha_j(Q)\theta_j = \theta_j Q \subseteq A$, we have $\alpha_j(Q) = P$ and so $(b)$ happens.

If $\theta_{i+1}, \dots, \theta_n \in A$, obviously case $(a)$ occurs.

Conversely, if we have $(a)$, obviously $P \rightsquigarrow Q$ in $\mathcal{T}$. Suppose $(b)$ and take the skew-polynomial ring,

$$R_j = R[\theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \dots, \theta_n; \alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_n].$$

Let $t \in \mathcal{T}$ be such that $t(\alpha_j(Q) \cap Q) \subseteq \alpha_j(Q)Q$ and write $Q = \theta_j \mathcal{T} + Q_j$, where $Q_j = Q \cap R_j$. Hence

$$t\theta_j \in (\theta_j \mathcal{T} + \alpha_j(Q_j))(\theta_j \mathcal{T} + Q_j) \subseteq \mathcal{T}\theta_j^2 + \alpha_j(Q_j)\theta_j + \alpha_j(Q_j)Q_j.$$

Since $\mathcal{T} = \oplus_{u \geq 0} R_j \theta_j^u$ and $\alpha_j(Q_j)Q_j \subseteq R_j$, the above inclusion shows that $t \in \mathcal{T}\theta_j + \alpha_j(Q_j)$. That is, $t \in \alpha_j(Q)$. So, $(\alpha_j(Q) \cap Q)/\alpha_j(Q)Q$ is faithful as a left $\mathcal{T}/\alpha_j(Q)$-module. Similarly, one can prove that the module is faithful as a right $\mathcal{T}/Q$-module. Hence $P = \alpha_j(Q) \rightsquigarrow Q$, since $\mathcal{T}$ satisfies the second layer condition by Corollary 2.2.14. □

**Corollary 2.5.2** *Retain Notation 2.2.8 and 2.2.10. Let $P, Q \in Spec(\mathcal{T})$ such that $\theta_{i+1}, \dots, \theta_n \in P \cap Q$ and $\theta_1, \dots, \theta_i \notin P$. Let $Y = \{\theta_1^{j(1)} \dots \theta_i^{j(i)} : j(1), \dots, j(i) \in \mathbb{N}\}$, an Ore set in $\mathcal{T}$, $\overline{P} = P/(\theta_{i+1}\mathcal{T} + \dots + \theta_n \mathcal{T})$ and $\overline{Q} = Q/(\theta_{i+1}\mathcal{T} + \dots + \theta_n \mathcal{T})$.*

*Then $P \rightsquigarrow Q$ if and only if $\overline{P}Y^{-1} \rightsquigarrow \overline{Q}Y^{-1}$ in $R[\theta_1, \theta_1^{-1}, \dots, \theta_i, \theta_i^{-1}; \alpha_1, \dots, \alpha_i]$ or if there is $j \in \{i+1, \dots, n\}$ such that $P = \alpha_j(Q)$.*

**Proof.** As $R[\theta_1, \dots, \theta_i; \alpha_1, \dots, \alpha_i]Y^{-1} \cong R[\theta_1, \theta_1^{-1}, \dots, \theta_i, \theta_i^{-1}; \alpha_1, \dots, \alpha_i]$ the result follows from Theorem 2.5.1 and from the fact that two prime ideals are linked if and only if their extensions are, [98, Lemma 2.11]. □

The complete description of the cliques in $\mathcal{T}$ follows now from Corollary 2.5.2 and Proposition 2.4.15.

## 2.6 Examples

The following examples illustrate how to apply the theorem proved before. In particular with examples 4 and 5 we discuss the problem of finite cliques in relation with what has been done in Proposition 2.4.16.

1. Let $\mathcal{S} = \mathbb{R}[x][\theta, \theta^{-1}; \alpha]$, where $\alpha$ is the $\mathbb{R}$-automorphism of $\mathbb{R}[x]$ such that $\alpha(x) = ax$ for some $a \in \mathbb{R}\backslash\{0\}$. Let $N = x\mathbb{R}[x]$. Then $\mathcal{S}_2 = \mathcal{S}$ and any prime of $\mathcal{S}$ strictly containing $N$ is of the form $P = x\mathcal{S} + p(\theta)\mathcal{S}$ where $p(\theta)$ is an irreducible polynomial of $\mathbb{R}[\theta]$.

In this case $\overline{\mathcal{S}}_2 = \mathbb{R}[\theta, \theta^{-1}]$, $\mathcal{S}_2^\sharp = \mathbb{C}[\theta, \theta^{-1}]$ and $N^\sharp = x\mathbb{C}[x]$. The composition series of $N^\sharp/(N^\sharp)^2$ has length one with a factor isomorphic to $\mathcal{S}_2^\sharp/(\theta - a)\mathcal{S}_2^\sharp$. Hence the automorphism of $\mathbb{C}$-algebras $\sigma_1$ is such that $\sigma_1(\theta) = a\theta$.

If we take $P = x\mathcal{S} + (\theta^2+1)\mathcal{S}$, then $\overline{P}_2 = (\theta^2+1)\overline{\mathcal{S}}_2$ and $(\theta - i)\mathcal{S}_2^\sharp$ is a prime of $\mathbb{C}[\theta, \theta^{-1}]$ lying over $\overline{P}_2$. Take $Q$ a prime ideal of $\mathcal{S}$, distinct from $P$. Hence $P \rightsquigarrow Q$ if and only if $\overline{Q}_2 = (\theta - a^{-1}i)\mathcal{S}_2^\sharp$ lies over $Q/N\mathcal{S}_2$. That is

$$P \rightsquigarrow Q \quad \text{if and only if} \quad Q = x\mathcal{S} + (\theta^2 + a^{-2})\mathcal{S}.$$

Thus $\mathcal{C}\ell(P) = \{x\mathcal{S} + (\theta^2 + a^{2l})\mathcal{S} : l \in \mathbb{Z}\}$.

2. Let $\mathcal{S} = \mathbb{C}[x,y][\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}; \alpha_1, \alpha_2]$, where $\alpha_1, \alpha_2 \in Aut(\mathbb{C}[x,y])$ are $\mathbb{C}$-algebra automorphisms such that $\alpha_1(x) = 3x, \alpha_1(y) = 2y, \alpha_2(x) = 2x$ and $\alpha_2(y) = 4y$. Take $N = x\mathbb{C}[x,y] + y\mathbb{C}[x,y]$. In this case $\mathcal{S}_1 = \mathcal{S} = \mathcal{S}_2 = \mathbb{C}[x,y][\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}; \alpha_1, \alpha_2]$ and $\overline{\mathcal{S}}_2 = \mathcal{S}_2^\sharp = \mathbb{C}[\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}]$.

The composition series for $N/N^2$ has length 2 and the factors are isomorphic to $\mathcal{S}_2^\sharp/(\theta_1 - 2, \theta_2 - 4)$ and $\mathcal{S}_2^\sharp/(\theta_1 - 3, \theta_2 - 2)$. So the link generating $\mathbb{C}$-algebra automorphisms $\sigma_1, \sigma_2$ are given by $\sigma_1(\theta_1) = 2\theta_1, \sigma_1(\theta_2) = 4\theta_2, \sigma_2(\theta_1) = 3\theta_1$ and

$\sigma_2(\theta_2) = 2\theta_2$.

Let $P = N\mathcal{S} + (\theta_1 - a, \theta_2 - b)\mathcal{S}$ for some $a, b \in \mathbb{C}\backslash\{0\}$. Then

$$\mathcal{C}\ell(P) = \{N\mathcal{S} + (\theta_1 - 3^j 2^l a, \theta_2 - 2^j 4^l b)\mathcal{S} : l, j \in \mathbb{Z}\}.$$

3. Let $\mathcal{T} = \mathbb{C}[x,y][\theta_1, \theta_2, \theta_3; \alpha_1, \alpha_2, \alpha_3]$ where $\alpha_1, \alpha_2, \alpha_3 \in Aut(\mathbb{C}[x,y])$ are such that $\alpha_1, \alpha_2$ are $\mathbb{C}$-algebra automorphisms defined by $\alpha_1(x) = 3x$, $\alpha_1(y) = 2y$, $\alpha_2(x) = 2x$, $\alpha_2(y) = 4y$, and $\alpha_3$ is an $\mathbb{R}$-algebra automorphism such that $\alpha_3(i) = -i$, $\alpha_3(x) = x$ and $\alpha_3(y) = y$.

Take $N = x\mathbb{C}[x,y] + y\mathbb{C}[x,y]$ and $P = N\mathcal{T} + \theta_3\mathcal{T} + (\theta_1 + i, \theta_2 - i)\mathcal{T}$, a prime ideal of $\mathcal{T}$. As $\theta_3 \in P$, we have by Theorem 2.5.1 that

$$P \rightsquigarrow \alpha_3^{-1}(P) = N\mathcal{T} + \theta_3\mathcal{T} + (\theta_1 - i, \theta_2 + i)\mathcal{T} \rightsquigarrow P.$$

Write $X = \{\theta_1^j \theta_2^l : j, l \in \mathbb{N}\}$ and $\mathcal{S} = \mathbb{C}[x,y][\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}; \alpha_1, \alpha_2]$, so that $(P/\theta_3\mathcal{T})X^{-1} = N\mathcal{S} + (\theta_1 + i, \theta_2 - i)\mathcal{S}$. By example 2 we can calculate the cliques of $(P/\theta_3\mathcal{T})X^{-1}$ and of $(\alpha_3^{-1}(P)/\theta_3\mathcal{T})X^{-1}$. Putting these together and noting that $\alpha_3^2 = id|_{\mathbb{C}[x,y]}$, we find by Corollary 2.5.2 that

$$\mathcal{C}\ell(P) = \{N\mathcal{T}+\theta_3\mathcal{T}+(\theta_1+3^j 2^l i, \theta_2-2^j 4^l i)\mathcal{T}, N\mathcal{T}+\theta_3\mathcal{T}+(\theta_1-3^j 2^l i, \theta_2+2^j 4^l i)\mathcal{T} : l, j \in \mathbb{Z}\}.$$

4. Let $\alpha$ be the $\mathbb{C}$-automorphism of $\mathbb{C}[x,y]$ such that $\alpha(x) = x + 2y$ and $\alpha(y) = y$. Define $\mathcal{S} = \mathbb{C}[x,y][\theta, \theta^{-1}; \alpha]$ and take $N = x\mathbb{C}[x,y] + y\mathbb{C}[x,y]$. In this case $\mathcal{S}_2 = \mathcal{S}_1 = \mathcal{S}$, $\overline{\mathcal{S}}_2 = \mathbb{C}[\theta, \theta^{-1}]$ and $\sigma_1 : \mathbb{C}[\theta, \theta^{-1}] \to \mathbb{C}[\theta, \theta^{-1}]$ is just the identity.

In example 4 the cliques are obviously finite but the order of $\Psi(\theta) = \alpha$ is not (in this case $\gamma_1 = \theta$). The next example, example 5, shows that it may well happen that the clique of a fixed prime in a skew-Laurent ring is finite but the order of the multiplicative subgroup of $K^\sharp$ generated by $\varepsilon_{ij}$, for $i \in \{1, \ldots, u\}$ and $j \in \{1, \ldots, v\}$ is not.

5. Let $\alpha_1$ be a $\mathbb{C}$-automorphism of $\mathbb{C}[x,y]$ such that $\alpha_1(x) = ix$, $\alpha_1(y) = y$ and $\alpha_2$ the $\mathbb{C}$-automorphism defined by $\alpha_2(x) = 2x$ and $\alpha_2(y) = 4y$. Take $\mathcal{S} = \mathbb{C}[x,y][\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}; \alpha_1, \alpha_2]$, $N = x\mathbb{C}[x,y] + y\mathbb{C}[x,y]$ and the prime $P = N\mathcal{S} + (\theta_1 - 1)\mathcal{S}$. In this case $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$, $\overline{\mathcal{S}}_2 = \mathbb{C}[\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}]$ and the link generating $\mathbb{C}$-algebra automorphisms $\sigma_1, \sigma_2$ are given by $\sigma_1(\theta_1) = i\theta_1, \sigma_1(\theta_2) = 2\theta_2, \sigma_2(\theta_1) = \theta_1$ and $\sigma_2(\theta_2) = 4\theta_2$. Hence

$$\mathcal{C}\ell(P) = \{N\mathcal{S} + (\theta_1 - 1)\mathcal{S}, N\mathcal{S} + (\theta_1 + 1)\mathcal{S}, N\mathcal{S} + (\theta_1 - i)\mathcal{S}, N\mathcal{S} + (\theta_1 + i)\mathcal{S}\}.$$

## 2.7 Additional remarks

1. Most of this chapter is part of the paper [27].

2. All the results in §1 are well known and can be found in [44] or in [57]. The only exception is Lemma 2.1.19 which is our generalization of [43, Lemma 1.3].

3. Proposition 2.2.13 was obtained with a suggestion of the referee of the paper [27].

# Chapter 3

# Azumaya Algebras and Crossed Products

The concept of Azumaya algebra is related with several other notions such as separability and H-separability. There have been some studies for some classes of rings to decide whether they are Azumaya or not. For instance, given a ring $R$ and a finite group $G$, DeMeyer and Janusz in [36] studied when the group ring $RG$ is an Azumaya algebra.

In [52], Ikehata proves that given a commutative ring $R$ and $G$ a finite group of automorphisms of $R$, the skew-group ring $R\#G$ is an Azumaya $R$-algebra if and only if $R$ is a $G$-Galois extension of $R^G$. ( For the definition of $G$-Galois extension and other terms introduced in these introductory paragraphs, see §3.1 and §3.2.1.)

Ricardo Alfaro and George Szeto in [3] generalize Ikehata's result proving that given a ring $R$ and $G$ a finite group of automorphisms of the ring, the following are equivalent:

i) $R\#G$ is Azumaya and $Z(R\#G) \subseteq R$;

ii) $R\#G$ is an H-separable extension of $R$ and $R$ is a separable extension of $Z(R)^G$;

iii) (a) $R^G$ is Azumaya;

(b) $R$ is finitely generated and projective as an $R^G$-module;

(c) $R\#G \cong End_{R^G}(R)$ as rings.

Ricardo Alfaro and George Szeto's result doesn't apply for instance to the skew-Laurent ring $\mathbb{C}[\theta, \theta^{-1}; \alpha]$, where $\alpha$ is complex conjugation, since this ring can be expressed as a crossed product of a finite group over a commutative ring, but not as a skew-group ring of a finite group. In this chapter we try to obtain a similar description to the one in [3] for some crossed products. Such description will allow us to conclude that rings such as $\mathbb{C}[\theta, \theta^{-1}; \alpha]$, are Azumaya.

In [48] R.B. Howlett and I.M.Isaacs, for the proof of their main result, Theorem 8.2, built a non-abelian finite group of central type. There is then a finite nontrivial group, $J$, and a twisted group ring $\mathbb{C}^t J$ of $J$ over $\mathbb{C}$, such that $\mathbb{C}^t J$ is simple, Artinian and $Z(\mathbb{C}^t J) = \mathbb{C}$. Hence $\mathbb{C}^t J$ is a central simple algebra and by [76, Proposition 7.7], an Azumaya algebra. Hence $i)$ as above, holds but as $\mathbb{C}^t J$ or $\mathbb{C}J$ are not isomorphic to $\mathbb{C}$, it follows that $iii)(c)$ of the same result does not. Therefore, we will not be able to generalize the above to crossed products in general.

Throughout this chapter, whenever we have a ring $R$, $G$ a group of automorphisms and form a crossed product, we will assume that the action $\sigma$, as defined in Definition 1.2.15, will be such that $\sigma(g) = g$, for any $g \in G$, and the twisting $\varepsilon$ is arbitrary, unless stated otherwise. We will show that given a commutative ring $R$, a finite subgroup of automorphisms of $R$ and a crossed product of $G$ over $R$, if $R$ and $G$ satisfy the same conditions as $iii)b)$ and $iii)c)$ above, then $R * G$ is Azumaya and $Z(R * G) = R^G$, Theorem 3.3.6. The converse is true provided we impose an extra condition on $G$.

## 3.1 Azumaya algebras and separability

Let $A$ be an algebra over a commutative ring $R$. The opposite algebra of $A$ is the $R$-algebra $A^{op}$ which coincides with $A$ as an $R$-module and has multiplication $o$ defined by $yox = xy$ for each $x$ and $y$ in $A$. The $R$-algebra $A \otimes_R A^{op} = A^e$ we call the *enveloping algebra* of $A$.

We can define a left $A^e$-module structure on $A$ induced by $(a \otimes a')b = aba'$, for all $a, a', b \in A$.

**Definition 3.1.1** An $R$-algebra $A$ is *separable* if $A$ is projective as a left $A^e$-module.

We define a left $A^e$-module homomorphism $\mu$, from $A^e$ onto $A$ given by

$$\begin{array}{rccc} \mu : & A \otimes_R A^{op} & \longrightarrow & A \\ & \sum_i a_i \otimes a'_i & \longmapsto & \sum_i a_i a'_i. \end{array}$$

Let $J = \text{Ker}(\mu)$. Then $J$ is the left ideal of $A^e$ generated by all elements of the form $a \otimes 1 - 1 \otimes a$, for any $a \in A$.

**Theorem 3.1.2** *Let $A$ be an $R$-algebra. Then the following are equivalent;*

*i) $A$ is separable;*

*ii) $0 \to J \to A^e \xrightarrow{\mu} A \to 0$ splits as a sequence of left $A^e$-modules;*

*iii) There is $e \in A^e$ such that $\mu(e) = 1$ and $Je = 0$.*

**Proof.** [34, Theorem II.1.1]. $\square$

**Remark 3.1.3** The element $e \in A \otimes_R A^{op}$ taken in Theorem 3.1.2 *iii)* is an idempotent. To see this, note that $\mu(e - 1 \otimes 1) = 0$ and write $e^2 - e = (e - 1 \otimes 1)e \in Je = 0$.

**Definition 3.1.4** If $A$ is a faithful $R$-algebra such that $R.1$ coincides with the centre of $A$, we say that $A$ is a *central* $R$-algebra. A separable central $R$-algebra is said to be an *Azumaya $R$-algebra.*

**Proposition 3.1.5** *Let $A$ be an $R$-algebra. Then the following are equivalent:*

*i) $A$ is an Azumaya $R$-algebra;*

*ii) $A$ is finitely generated and projective as an $R$-module and $A \otimes_R A^{op} \cong End_R(A)$ via the map $\theta : a \otimes b \mapsto \lambda_a \rho_b$, where $\lambda_a \rho_b(x) = axb$, for all $x \in A$.*

**Proof.** [34, Theorem II.3.4 and Corollary I.1.10] □

**Definition 3.1.6** A ring is said to be *Azumaya* if it is an Azumaya algebra over the centre.

In [49], Hirata and Sugano, generalized the notion of separable algebras defining "*separable extensions of a ring*".

**Definition 3.1.7** Let $S$ be a ring and $T$ a subring of $S$. We say that $S$ is a *separable extension* of $T$ if there exists an element $\sum s_i \otimes s_i'$ in $S \otimes_T S$ such that

i) $\sum s_i s_i' = 1$;

ii) $\sum x s_i \otimes s_i' = \sum s_i \otimes s_i' x$, for all $x \in S$.

**Remark 3.1.8** In the situation of Definition 3.1.7, we note the following points:

i) $S \otimes_T S$ is just an $(S, S)$-bimodule and not, in general, a ring.

ii) If $T$ is in the centre of $S$, by Theorem 3.1.2 *iii)*, $S$ is separable as an $T$-algebra. If $T$ is the centre of $S$, then $S$ is Azumaya.

Given $T$ a subring of a ring $S$, we can define an $(S,S)$-homomorphism $\varphi$ from $S \otimes_T S$ onto $S$ such that

$$\begin{array}{rccc} \varphi : & S \otimes_T S & \longrightarrow & S \\ & \sum_i s_i \otimes s'_i & \longmapsto & \sum_i s_i s'_i. \end{array}$$

The following proposition is well-known, but we were unable to find a reference.

**Proposition 3.1.9** *Let $T$ be a subring of a ring $S$. Then $S$ is a separable extension of $T$ if and only if $\varphi$ splits as an $(S,S)$-homomorphism.*

**Proof.** Suppose $S$ a separable extension of $T$ and take $\sum_i s_i \otimes s'_i \in S \otimes_T S$ as in Definition 3.1.7. Then we can define a $(S,S)$-homomorphism $\psi : S \to S \otimes_T S$ such that $\psi(1) = \sum s_i \otimes s'_i$. It is easy to verify that $\varphi\psi = id_S$.

Conversely, suppose that $\varphi$ splits. Let $\psi$ be an $(S,S)$-homomorphism such that $\varphi\psi = id_S$. Let $\psi(1) = \sum_i s_i \otimes s'_i$. Then $\varphi(\psi(1)) = 1$ and for all $x \in S$, $\sum_i x s_i \otimes s'_i = x\psi(1) = \psi(x) = \psi(1)x = \sum_i s_i \otimes s'_i x$. □

**Proposition 3.1.10** *Let $S$ be a ring, $T$ and $U$ subrings of $S$ such that $U \subseteq T$.*

*i) If $S$ is a separable extension of $U$, then $S$ is a separable extension of $T$;*

*ii) If $S$ is a separable extension of $T$ and $T$ is a separable extension of $U$, then $S$ is a separable extension of $U$.*

**Proof.** [49, Proposition 2.5]. □

In [50], Hirata gave new insight into the notion of separable extensions. He proved

**Theorem 3.1.11** *Let $S$ be a ring with centre $C$, $T$ a subring of $S$. If $S \otimes_T S$ is isomorphic to a two-sided $S$-direct summand of a finite direct sum of copies of $S$, then $C_S(T)$ is $C$-finitely generated projective and $S$ is a separable extension of $T$.*

**Proof.** [50, Theorem 2.2] □

**Definition 3.1.12** Let $S$ and $T$ satisfy the hypothesis of Theorem 3.1.11. We say that $S$ is an *H-separable extension* of $T$.

By Theorem 3.1.11, every H-separable extension is a separable extension. This theorem was an attempt to introduce the notion of central separable algebras to separable extensions; as we shall see in the next proposition, every Azumaya algebra is an H-separable extension of its centre.

**Proposition 3.1.13** *A ring $R$ is an H-separable extension of its centre if and only if $R$ is Azumaya.*

**Proof.** [103, Proposition 1.1] □

**Proposition 3.1.14** *Let $S$ be a ring and $T$ a subring of $S$. If $S$ is an H-separable extension of $T$ and $T$ is a direct summand of $S$ as a $(T,T)$-bimodule, then $C_S(T)$ is a separable extension of $Z(S)$.*

**Proof.** [104, Proposition 1.3] □

## 3.2 A necessary condition

One could try to state a similar result to the one of Ricardo Alfaro and George Szeto in [3], for some crossed product. Their proof (to prove that *ii*) implies *iii*)), and indeed the very statement of *iii*)*c*) depends on the fact that given any ring $R$, any group of automorphisms $G$ of $R$ and any skew-group ring $R\#G$, one can think of $R$ as an $R\#G$-module. In general this is not the case for crossed products.

We will split this section into two parts. In the first one we will work with any crossed product $R * J$ and describe the ones which are Azumaya in terms of $H$-separability and separability conditions. This description will follow the same ideas as Ricardo Alfaro and George Szeto in [3]. In this part we will also describe the centre of $R * J$ and centralizers of $R$ in $R * J$. In the second part the idea is to relate, like Ricardo Alfaro in [1] did for skew-group rings, the H-separability condition with the concept of Galois extensions. For this to be possible we will impose some restrictions on the crossed product; we will take any ring $R$ and any group $G$ of automorphisms of $R$ and a crossed product $R * G$. As it was already mentioned in the introduction to this chapter, whenever we will form a crossed product of a group of automorphisms $G$ of a ring $R$ over $R$, the action $\sigma$ as defined in Definition 1.2.15, is such that $\sigma(g) = g$ for all $g \in G$, and the twisting will be arbitrary.

**Definition 3.2.1** Let $R$ be any ring and $G$ a subgroup of $Aut(R)$. The *fixed subring of* $G$ *on* $R$ is $R^G = \{r \in R : g(r) = r, \text{ for all } g \in G\}$. If $r \in R^G$, we say that $r$ is a *fixed point*.

If $R$ is any ring, $J$ is any group and $R * J$ any crossed product, by $R^J$ we denote the set of elements of $r \in R$ such that $r^j = r$, for all $j \in J$.

**Lemma 3.2.2** *Let $R$ be any ring, $J$ any group and $R * J$ any crossed product, then $Z(R * J) \subseteq R$ if and only if $Z(R * J) = Z(R)^J$.*

**Proof.** It's obvious that $Z(R)^J \subseteq Z(R * J)$.

If $Z(R * J) \subseteq R$, as any element $r \in Z(R * J)$ will commute with the elements of $R$ and with each $\overline{g}$, for any $g \in J$, we have $r \in Z(R)^J$, and equality holds. □

As we shall see later, there are natural circumstances where the condition $Z(R * J) \subseteq R$ holds, see for instance Lemma 3.2.11. The proof of the following result is similar to that of the analogous result for skew-group rings, [3, Theorem

1], and shows that the first two of the three equivalent statements of Alfaro and Szeto in the introduction of the chapter remain equivalent for crossed products.

**Proposition 3.2.3** *Let $R$ be any ring, $J$ any group and $R*J$ any crossed product. Then the following are equivalent:*

*i) $R*J$ is an Azumaya algebra and $Z(R*J) \subseteq R$;*

*ii) $R*J$ is an H-separable extension of $R$ and $R$ is a separable extension of $Z(R)^J$.*

**Proof.** Assume that $R*J$ is Azumaya with $Z(R*J)$ contained in $R$. As $R*J$ is free as an $R$-module, $R*J$ is projective as an $R$-module. As $Z(R*J) = Z(R)^J \subseteq R$ by Lemma 3.2.2, $R*J$ is Azumaya and is projective over a ring containing its centre, it follows from [52, Theorem 1] that $R*J$ is an H-separable extension of $R$. Since $R$ is a two-sided $R$-direct summand of $R*J$, $C_{R*J}(R)$ is a separable extension of $Z(R*J) = Z(R)^J$, Proposition 3.1.14. Hence, by [34, Theorem II.4.3], $C_{R*J}(C_{R*J}(R))$ is a separable extension of $Z(R)^J$.

As $R*J$ is an H-separable extension of $R$ and $R$ is a two-sided $R$-direct summand of $R*J$, by [103, Proposition 1.2], $C_{R*J}(C_{R*J}(R)) = R$. Hence $R$ is a separable extension of $Z(R)^J$.

Conversely, assume that $R*J$ is an H-separable extension of $R$ and $R$ is a separable extension of $Z(R)^J$. Hence, by Theorem 3.1.11, $R*J$ is a separable extension of $R$ and by Proposition 3.1.10, $R*J$ is a separable extension of $Z(R)^J$.

By [103, Proposition 1.2], $R = C_{R*J}(C_{R*J}(R))$. As $Z(R*J) \subseteq C_{R*J}(C_{R*J}(R))$, $Z(R*J) = Z(R)^J$ by Lemma 3.2.2. Hence $R*J$ is Azumaya and $Z(R*J) \subseteq R$.
□

Given a ring $R$ and $G$ a group of automorphisms of $R$, by Proposition 3.2.3, if a crossed product $R*G$ is Azumaya and $Z(R*G) \subseteq R$, $R*G$ is an $H$-separable extension of $R$. In order to study when this happens, we will need to describe the centre of such rings and the centralizer of $R$ in $R * G$, $C_{R*G}(R)$. We start by introducing the definition of an $\omega$-outer group of automorphisms. This definition will play an important role in the description of some crossed products that are Azumaya.

**Definition 3.2.4** Let $R$ be any ring and $g$ an automorphism of $R$. Let

$$\phi_g = \{r \in R : rs^g = sr, \forall s \in R\}.$$

If $J$ is any group and $R * J$ is any crossed product, for each $j \in J$, $\phi_j = \phi_{\sigma(j)}$ is defined exactly in the same way as before.

**Definition 3.2.5** Let $R$ be any ring and $G$ a nontrivial group of automorphisms of $R$. If for all $g \in G \backslash \{id\}$, $\phi_g = 0$, we say that $G$ is *$\omega$-outer*.

Obviously, if $R$ is a commutative domain and $G$ is any nontrivial group of automorphisms of $R$, $G$ is $\omega$-outer. One should note that, as the next example shows, $R$ commutative is not sufficient for a nontrivial group of automorphisms of $R$ to be $\omega$-outer.

**Example 3.2.6** Let $\mathbb{C}$ be the complex field, $R = \mathbb{C}^2$ and $g$ an automorphism of $\mathbb{C}^2$ such that $g(a, b) = (a, \overline{b})$, where $\overline{b}$ denotes the complex conjugation of $\mathbb{C}$. Let $G$ be the group of order two generated by $g$. We have that, for any $r \in R$

$$(g(r) - r)(1, 0) = 0.$$

In this case we can easily see that $\phi_g = \mathbb{C} \oplus 0$ and $\phi_1 = \mathbb{C} \oplus \mathbb{C}$.

We would like to remark that there are other concepts of outer, for instance the definition of outer automorphism introduced in §1.1 and

**Definition 3.2.7** *(V.K. Kharchenko in [65], [82])* Let $R$ be a semiprime ring and $\mathcal{M}(R)$ the left Martindale ring of quotients. If $G$ is a group of automorphisms of $R$, for each $g \in G$, let

$$\overline{\phi}_g = \{x \in \mathcal{M}(R) : xy^g = yx, \text{ for all } y \in R\}.$$

Let $G_{inn} = \{g \in G : \overline{\phi}_g \neq 0\}$. If $G_{inn}$ is a subgroup of $G$, let $G_{out} = G/G_{inn}$. We say that $G$ is $X$-outer if $G_{inn} = \{id\}$.

**Remark 3.2.8** *i)* In definition 3.2.7, if $G_{inn}$ is a subgroup, it will be a normal one. To see this, let $g \in G_{inn}$ and $h \in G$. Then there is $x \neq 0$ such that $xz^g = zx$, for all $z \in R$. Let $h \in G$ and $w = x^{h^{-1}} \neq 0$. Then, for all $y \in R$, $wy^{hgh^{-1}} = (xy^{hg})^{h^{-1}} = (y^h w^h)^{h^{-1}} = yw$.

*ii)* If $R$ is a semiprime ring and $\mathcal{F}$ the set of all essential two-sided ideals of $R$, then $\mathcal{M}(R) = R_{\mathcal{F}} = \varinjlim_{I\in\mathcal{F}}\mathrm{Hom}({}_RI,\ {}_RR)$, the ring of left quotients of $R$ with respect to $\mathcal{F}$.

*iii)* The ring $\mathcal{M}(R)$ was first defined for prime rings by W.S.Martindale in [73] [see also [82]]. S.A.Amitsur in [4] [see also [82]], extended the definition to semiprime rings.

*iv)* $X$-outer automorphisms are sometimes called $\mathcal{F}$-outer (see for instance [82]) to indicate that this definition is related to the filter $\mathcal{F}$ = {essential ideals of $R$}.

*v)* There is at least one more definition of outer automorphisms, the one of completely outer automorphisms given by Y. Miyashita in [78] [see also [82]].

Obviously, if a group $G$ of automorphisms of a semiprime ring $R$ is $\omega$-outer when extended to $\mathcal{M}(R)$, then $G$ is $X$-outer. If $R$ is a simple ring with identity then $\mathcal{M}(R) = R$ and the notions of $X$-outer and $\omega$-outer are the same. The following theorem gives a clarification of the concept of $X$-outer.

**Theorem 3.2.9** *Let $R$ be a semiprime right and left Goldie ring and $G$ a group of automorphisms of $R$. Then, $G$ is $X$-outer on $R$ if and only if $G$ is $X$-outer when extended to $\mathcal{Q}(R)$, the classical ring of quotients of $R$. In particular, when $R$ is a prime right and left Goldie ring, $G$ is $X$-outer on $R$ if and only if the identity is the only inner automorphism of $\mathcal{Q}(R)$ in $G$.*

**Proof.** [82, Theorem 1.4]. □

**Remark 3.2.10** If $R$ is a semiprime Goldie PI ring, one can form $Q(R)$ by inverting central elements. From Theorem 3.2.9 it follows that the two definitions 3.2.5 and 3.2.7 coincide in this case.

**Lemma 3.2.11** *Let $R$ be any ring, $J$ any multiplicative group and $R * J$ be any crossed product of $J$ over $R$.*

*Then*

$$C_{R*J}(R) = \sum_{g\in J} \phi_g \overline{g}.$$

*If $J$ is a group of automorphisms of $R$ which is $\omega$-outer, then*

$$C_{R*J}(R) = Z(R) \qquad \textit{and} \qquad Z(R * J) = Z(R)^J.$$

*If $R\#J$ is any skew-group ring and $J$ is abelian then $\phi_g$ is $J$-invariant for all $g \in J$ and*

$$Z(R\#J) = \sum_{g\in J} \phi_g^J g.$$

**Proof.** Let $R$ be any ring and $s = \sum_g r_g \overline{g} \in C_{R*J}(R)$. Then, for all $r \in R$,

$$\sum_g r r_g \overline{g} = \sum_g r_g \overline{g} r = \sum_g r_g r^{\sigma(g)} \overline{g}.$$

So, for each $g$, $r_g r^{\sigma(g)} = r r_g$, for all $r \in R$ and so $s \in \sum_{g\in J} \phi_g \overline{g}$. The other inclusion is obvious.

If $J$ is an $\omega$-outer group of automorphisms, then $\phi_g = Z(R)$ for $g = id$, otherwise $\phi_g = 0$. So, $C_{R*J}(R) \subseteq Z(R)$. Obviously, $Z(R) \subseteq C_{R*J}(R)$. Hence $Z(R) = C_{R*J}(R)$. Let $s \in Z(R * J) \subseteq C_{R*J}(R) = Z(R)$. Then, for all $g \in J$, $s^g \overline{g} = \overline{g} s = s\overline{g}$, and so $s \in Z(R)^J$. Then $Z(R * J) = Z(R)^J$.

Suppose that $R\#J$ is a skew-group ring and $J$ is abelian. Let $\sigma$ be the action of the crossed product as in Definition 1.2.15 and take $g, h \in J$, $r \in \phi_g$ and $s \in R$. Then $r^h s^g = (r s^{gh^{-1}})^h = (r s^{h^{-1}g})^h = (s^{h^{-1}} r)^h = s r^h$. Hence $r^h \in \phi_g$ and $\phi_g$ is $J$-invariant. We already have $Z(R\#J) \subseteq C_{R\#J}(R) \subseteq \sum_{g\in J} \phi_g g$. Let

$s \in Z(R\#J)$. Then $s = \sum_{g \in J} r_g g$ for some $r_g \in \phi_g$. Now, for all $h \in J$, $hs = sh$. So

$$\sum_{g \in J} r_g^{\sigma(h)} hg = \sum_{g \in J} r_g gh$$

and so $r_g^{\sigma(h)} = r_g$ for all $h \in J$. Then $Z(R\#J) \subseteq \sum_{g \in J} \phi_g^J g$ and the equality holds. $\square$

**Example 3.2.12** Let $R$ and $G$ be as in Example 3.2.6 and form $S = R\#G$ the skew-group ring of $G$ on $R$ constructed with the given action of $G$ on $R$. Then by Lemma 3.2.11, $Z(S) = (\mathbb{C} \oplus 0)^G g + (\mathbb{C} \oplus \mathbb{C})^G = (\mathbb{C} \oplus 0)g + (\mathbb{C} \oplus \mathbb{R})$.

**Remark 3.2.13** Example 3.2.12 gives a counterexample to what has been claimed in Remark 1 of [3]. In particular it shows that Ikehata's result [52, Theorem 2] cannot be deduced as a corollary of [3, Theorem 1].

### 3.2.1 Galois extensions and H-separability

In the first part of section 2, we described some crossed products that are Azumaya in terms of H-separability and separability. In this subsection we will describe the H-separability condition obtained in terms of Galois extensions in a similar spirit to the work of Alfaro in [1]. We will have to make some restrictions on the crossed products we consider.

In [9], M.Auslander and O. Goldman introduced the notion of a Galois extension of commutative rings. In [61], T.Kanzaki generalized the notion of Galois extensions to noncommutative rings, as follows.

**Definition 3.2.1.1** Let $T$ be a ring, $U$ a subring of $T$ and $G$ a finite subgroup of $Aut(T)$. We say that $T$ is a *G-Galois extension* of $U$ if

i) $U = T^G$;

ii) $T$ is a finitely generated projective right $U$-module;

iii) the natural map $\phi : T\#G \to End(T_U)$ such that $\phi(tg)(s) = tg(s)$, for any $t, s \in T$ and $g \in G$, is an isomorphism of rings.

**Remark 3.2.1.2** *i)* To be precise, in Definition 3.2.1.1, we should have called such an extension a *right G-Galois extension* of $U$. We will omit the word "right", so that we will use the same term as the one found in the literature about the subject.

*ii)* We regard $T$ as a left $T\#G$-module by setting $s.t = \phi(s)(t)$, for all $s \in T\#G$ and $t \in T$.

*iii)* For a commutative ring $T$, *ii)* and *iii)* together imply that $T$ is a projective $T\#G$-generator. (See [8, Proposition A.3] and [76, Definition 3.5.3] or [34] for the definition of generator.)

If we assume $T$ to be a $T\#G$-generator (in this case we won't need the commutative hypothesis) it is easy to verify that $\mathrm{Hom}_{T\#G}(T, T)$ may be identified with $T^G$. Then by [8, Theorem A.2, c) and f)] we have that $T$ is a finitely generated projective $T^G$-module and the natural map $\phi : T\#G \to End(T_U)$ is an isomorphism.

*iv)* In [10], M. Auslander, I. Reiten and S. O. Smalø gave a different definition of Galois extension for noncommutative rings: *Given a ring $T$, $U$ a subring of $T$ and $G$ a finite group of automorphisms of $T$, $T$ is a pregalois extension of $U$ with group $G$ if $U = T^G$, $T$ is a finitely generated $U$-module and $T$ is a projective $T\#G$-generator.*

For these authors, a Galois extension would be a pregalois extension $T$ of $U$ with a group $G$ such that for every simple left or right $T$-module $S$, $U/ann_U(S)$ is a semisimple artinian ring.

It is a consequence of [8, Theorem A.2] or [10, Proposition 1.6] that if $T$ is a pregalois extension of $U$ in the Auslander-Reiten-Smalø sense then $T$ is a $G$-Galois extension of $U$ in the sense of Definition 3.2.1.1. Moreover the reverse implication is valid when $T$ is commutative, by note *iii)*. Whether the two definitions are equivalent in general appears unclear. (It's worth noting, however, that a finitely generated projective $R$-module is not in general projective over its endomorphism ring; see Remark in [8] after Proposition A.3.)

*v*) Throughout this chapter we will use the definition of Galois extension given in Definition 3.2.1.1.

The following proposition is a well-known characterization of Galois extensions for commutative rings. For more details see Chapter III of [34].

**Proposition 3.2.1.3** *Let $T$ be a commutative ring, $U$ a subring of $T$ and $G$ a finite subgroup of $Aut(T)$. The following are equivalent:*

1. *$T$ is a $G$-Galois extension of $U$.*
2. *(a) $T^G = U$;*

   *(b) For each non-zero idempotent $e \in T$ and each pair $g \neq h$ in $G$, there is an element $x \in T$ such that $g(x)e \neq h(x)e$;*

   *(c) $T$ is a separable $U$-algebra.*
3. *(a) $T^G = U$;*

   *(b) There exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n; y_1, \ldots, y_n$ in $T$ such that $\sum\limits_{j=1}^{n} x_j g(y_j) = \delta_{g,1}$.*
4. *(a) $T^G = U$;*

   *(b) For every maximal ideal $M$ of $T$ and each $g \in G \backslash \{1\}$ there is $t \in T$ such that $g(t) - t \notin M$.*

**Proof.** [34, Proposition III.1.2] □

For the case of Galois extensions for noncommutative rings, at least condition 3 of the above proposition holds, as we record below. For other characterizations of Galois extensions for noncommutative rings, see [1], [39], [61] and [62].

**Proposition 3.2.1.4** *Let $T$ be any ring, $U$ a subring of $T$ and $G$ a finite subgroup of $Aut(T)$. Then $T$ is a $G$-Galois extension of $U$ if and only if*

1. $T^G = U$;

2. *There exist $n \in \mathbb{N}$ and $x_1, \dots, x_n; y_1, \dots, y_n$ in $T$ such that, for $g \in G$,* $\sum_{j=1}^{n} x_j g(y_j) = \delta_{g,1}$.

**Proof.** [62, Proposition 2.4]. □

**Lemma 3.2.1.5** *Let $T$ be a ring, $H$ any group and $G$ a finite subgroup of $Aut(T)$. Let $TH$ be the group ring and extend each automorphism of $T$ to one of $TH$ in the usual way. Then $T$ is a $G$-Galois extension of $T^G$ if and only if $TH$ is a $G$-Galois extension of $T^G H$.*

**Proof.** Suppose first that $T$ is a $G$-Galois extension of $T^G$. Obviously, $(TH)^G = T^G H$. By Proposition 3.2.1.4, we have that $TH$ is a $G$-Galois extension of $T^G H$.

Conversely suppose that $TH$ is a $G$-Galois extension of $T^G H$. Again by Proposition 3.2.1.4, there are $x_1, \dots, x_n, y_1, \dots, y_n \in TH$ such that, for $g \in G$ $\sum_{j=1}^{n} x_j g(y_j) = \delta_{g,1}$.

For each $j \in \{1, \dots, n\}$, write $x_j = \sum_{h \in H} a_{j,h} h$ and $y_j = \sum_{l \in H} b_{j,l} l$, for some $a_{j,h}, b_{j,l} \in T$. Then

$$\sum_{j=1}^{n} \sum_{h,l} a_{j,h} g(b_{j,l}) hl = \delta_{g,1}.$$

Hence

$$\sum_{j=1}^{n} \sum_{h \in H} a_{j,h} g(b_{j,h^{-1}}) = \delta_{g,1}.$$

So, $\{a_{j,h}; b_{j,h^{-1}} : 1 \leq j \leq n, h \in H\}$ is a family of elements of $T$ satisfying condition 2 of Proposition 3.2.1.4. Hence $T$ is a $G$-Galois extension of $T^G$. □

**Definition 3.2.1.6** If $G$ is a finite group of automorphisms of a ring $R$, we define the *trace map*, an $R^G$-module homomorphism, by

$$\begin{array}{rrcl} tr_G: & R & \longrightarrow & R^G \\ & r & \longmapsto & \sum_{g\in G} g(r). \end{array}$$

Whenever the group $G$ is well understood, we will just write $tr$.

**Remark 3.2.1.7** In the situation of Definition 3.2.1.6, obviously $tr_G(r) \in R^G$ and if $r$ is a fixed point then $tr_G(r) = |G|r$.

The following lemma follows easily from the remark and definitions above.

**Lemma 3.2.1.8** *If $G$ is a finite group of automorphisms of the ring $R$, then $tr_G$ is an $R^G$-bimodule homomorphism from $R$ to $R^G$ and $|G|R^G \subseteq tr_G(R) \lhd R^G$.* □

**Lemma 3.2.1.9** *Let $R$ be a commutative ring and $G$ a finite group of automorphisms of $R$. If $|G|^{-1} \in R$ or $R$ is a $G$-Galois extension of $R^G$ then $1 \in Im(tr)$ and $tr$ is onto as a map from $R$ to $R^G$. If $tr$ is onto as a map from $R$ to $R^G$, $R^G$ is isomorphic to a direct summand of $R$ as an $R^G$-module.*

**Proof.** If $|G|^{-1} \in R$, then $1 = tr(|G|^{-1}.1)$ and by Lemma 3.2.1.8 $tr$ is onto.

If $R$ is a $G$-Galois extension of $R^G$, then $1 \in Im(tr)$ by [34, Corollary III.1.3], and hence $tr$ is onto as a map from $R$ to $R^G$. Since $R \twoheadrightarrow R^G \rightarrow 0$ is an exact sequence of $R^G$-modules, the sequence splits and the result follows. □

**Remark 3.2.1.10** Given any ring $R$, $G$ any subgroup of $Aut(R)$, and $M$ a $G$-stable ideal of $R$, for every $g \in G$ we can define an automorphism $g'$ of $R/M$ by $g'(r + M) = g(r) + M$. We shall abuse notation and denote $g'$ by $g$ and the subgroup of $Aut(R/M)$ generated by these elements by $G$ (even if $G$ is not isomorphic to a subgroup of $Aut(R/M)$).

**Lemma 3.2.1.11** *Let $R$ be a ring and $G$ a finite subgroup of $Aut(R)$ such that $tr$ is onto as a map from $R$ to $R^G$. If $M$ is a proper $G$-stable ideal of $R$ then:*

*i)* $(R/M)^G = (R^G + M)/M \cong R^G/(M \cap R^G)$;

*ii)* *If $R$ is a commutative ring and a $G$-Galois extension of $R^G$ then $R/M$ is a $G$-Galois extension of $(R^G + M)/M$.*

**Proof.** Suppose $R$ and $G$ as above. Assume that $tr$ is onto as a map from $R$ to $R^G$. Take $d \in R$ such that $tr(d) = 1$.

Let $r \in R$ be such that $g(r) - r \in M$ for all $g \in G$. Then, as $M$ is an ideal of $R$, $g(r)g(d) - rg(d) \in M$, for all $g \in G$. Hence

$$\sum_g g(rd) - \sum_g rg(d) \in M.$$

So $tr(rd) - r \in M$ and $r + M = tr(rd) + M \in (R^G + M)/M$. Obviously, $(R^G + M)/M \subseteq (R/M)^G$ and the equality holds. So we have $i)$.

Assume that $R$ is a commutative ring and a $G$-Galois extension of $R^G$. Let $\overline{M}/M$ be a maximal ideal of $R/M$ and $g \in G \backslash \{id\}$. As $R$ is a $G$-Galois extension of $R^G$, there is $r \in R$, such that $g(r) - r \notin \overline{M}$. But then $g(r + M) - (r + M) \notin \overline{M}/M$. So, by $i)$ and Proposition 3.2.1.3 (4), we have that $R/M$ is a $G$-Galois extension of $(R^G + M)/M$. □

**Lemma 3.2.1.12** *Let $R$ be any ring and $G$ a finite subgroup of $Aut(R)$ such that $R$ is a $G$-Galois extension of $R^G$ and $R^G$ is a field. Then $R$ is semiprime.*

**Proof.** Since $R$ is a $G$-Galois extension of $R^G$, by definition $R\#G \cong End_{R^G}(R)$ and $R$ is finitely generated as a right module over $R^G$. As $R^G$ is a field, $R$ has a finite basis over $R^G$, say of cardinality $t$. Hence $End_{R^G}(R) \cong M_t(R^G)$ is a simple artinian ring.

Let $\mathcal{J}(R)$ and $\mathcal{J}(R\#G)$ be the Jacobson radical of $R$ and $R\#G$, respectively. As $\mathcal{J}(R)$ is $G$-invariant and is nilpotent (Hopkins Levitzki's Theorem [44, Theorem 3.15]), so is $\mathcal{J}(R)\#G$. As $R\#G$ is prime, $\mathcal{J}(R)\#G = 0$ and so $\mathcal{J}(R) = 0$. So $R$ is semisimple, hence semiprime. □

In [1] Ricardo Alfaro gives a necessary condition for a skew-group ring $R\#G$ to be $H$-separable over $R$ where $G$ is a finite group acting faithfully as automorphisms of $R$. He proves that:

*Let $R$ be any ring, $G$ a finite group acting faithfully as automorphisms of $R$ and $R\#G$ the skew-group ring. If $R\#G$ is an H-separable extension of $R$, then:*

*i) $G$ is $\omega$-outer;*

*ii) $C_{R\#G}(R) = Z(R)$;*

*iii) $Z(R)$ is a $G$-Galois extension of $Z(R)^G$.*

His proof depends essentially on the fact that, assuming $R$ and $G$ as above, $C_{R\#G}(R) = Z(R)$ and if $R\#G$ is an H-separable extension of $R$, then $G$ is $\omega$-outer. The next proposition and its proof is the analogue to [1, Theorem 3.4], although his result is just for skew-group rings and his proof is rather obscure. In this thesis we clarify the proof using some of Alfaro's ideas and especially his remark that precedes [1, Theorem 3.4] and generalize to some crossed products of the form $R * G$ where $G$ is a finite group acting faithfully on $R$ as automorphisms of $R$.

**Remark 3.2.1.13** We should note that if $R$ is any ring, $G$ any group and $R * G$ any crossed product of $G$ over $R$, we are assuming that an action $\sigma : G \to Aut(R)$ as in Definition 1.2.15, is defined such that $r^g = r^{\sigma(g)}$ . Hence it is obvious that, under these conditions, $R^G$ is a subring of $R$.

As already discussed in Lemma 1.2.17, the action $\sigma$ does not have to be a homomorphism. Although, by the same lemma, $\sigma$ is a homomorphism if $R$ is a commutative ring. For some results in this section and in the following one, we will have to assume that $\sigma$ is actually a homomorphism or even that it is a monomorphism; this is the real meaning of phrases like "$G$ a group of automorphisms of $R$".

**Proposition 3.2.1.14** *Let $R$ be any ring and $G$ a finite group acting faithfully on $R$ as automorphisms of $R$. Suppose that $G$ is $\omega$-outer on $R$. Suppose that there exists a crossed product $R * G$ whose action of $G$ on $R$ is the given one, such that $R * G$ is an H-separable extension of $R$. Then $Z(R)$ is a $G$-Galois extension of $Z(R)^G$.*

**Proof.** As $R*G$ is an H-separable extension of $R$ and since $R$ is a direct summand of $R*G$ as an $(R, R)$-bimodule, $C_{R*G}(R)$ is a separable extension of $Z(R*G)$ by Proposition 3.1.14.

Since $G$ is $\omega$-outer and acts faithfully on $R$ as a group of automorphisms of $R$, Proposition 3.2.11 implies that $Z(R*G) = Z(R)^G$ and $C_{R*G}(R) = Z(R)$. Hence $Z(R)$ is a separable extension of $Z(R)^G$.

Assume there is a nonzero idempotent $e$ in $Z(R)$ and $h \neq g$ in $G$ such that $r^h e = r^g e$, for all $r \in Z(R)$. Then for all $x \in Z(R)$,

$$xe^{g^{-1}} = x^{hg^{-1}}e^{g^{-1}}.$$

Hence, for all $x \in Z(R)$,

$$\begin{aligned} e^{g^{-1}}\overline{hg^{-1}}x &= e^{g^{-1}}x^{hg^{-1}}\overline{hg^{-1}} \\ &= x^{hg^{-1}}e^{g^{-1}}\overline{hg^{-1}} \\ &= xe^{g^{-1}}\overline{hg^{-1}}. \end{aligned}$$

So, $e^{g^{-1}}\overline{hg^{-1}} \in C_{R*G}(Z(R)) = C_{R*G}(C_{R*G}(R))$. As $R*G$ is an H-separable extension of $R$ and $R$ is a direct summand of $R*G$, we know that $C_{R*G}(C_{R*G}(R)) = R$, by [103, Proposition 1.2]. Therefore $hg^{-1} = 1$ and so $g = h$, a contradiction. Hence there is $r \in Z(R)$ such that $r^h e \neq r^g e$. By Proposition 3.2.1.3, 2, $Z(R)$ is a $G$-Galois extension of $Z(R)^G$. □

**Proposition 3.2.1.15** *Let $R$ be any ring and $G$ a finite $\omega$-outer group of automorphisms of $R$. Let $R * G$ be any crossed product constructed with the given* *action of $G$ on $R$. If $R * G$ is Azumaya then $Z(R)$ is a $G$-Galois extension of $Z(R)^G$.*

**Proof.** Assume $R$ and $G$ as above. By Lemma 3.2.11, $Z(R * G) \subseteq Z(R)$. If $R * G$ is Azumaya, then by Proposition 3.2.3, $R * G$ is an H-separable extension of $R$ and by Proposition 3.2.1.14, $Z(R)$ is a $G$-Galois extension of $Z(R)^G$. □

**Corollary 3.2.1.16** *Let $D$ be a commutative domain and $G$ a finite group acting faithfully as automorphisms of $D$ and $D * G$ any crossed product constructed with the given action of $G$ on $R$. If $D * G$ is Azumaya, then $D$ is a $G$-Galois extension of $D^G$.*

**Proof.** Suppose $D * G$ as stated above. Then, $G$ is $\omega$-outer and the result follows from Proposition 3.2.1.15. □

**Remark 3.2.1.17** We should note that in the statement of Corollary 3.2.1.16 we could have started with a commutative domain $D$ and any crossed product $D * G$ and assumed that the action is faithful. By Remark 3.2.1.13, this would mean that $G$ is actually isomorphic to a group of automorphisms of $D$.

## 3.3 A sufficient condition

In this section we would like to give a sufficient condition for a crossed product of a finite group $G$ over a commutative domain $D$ such that $G$ acts faithfully on $D$ to be Azumaya. For that, we start by taking any ring $R$ and any finite group of automorphisms of $R$, then we try to describe the maximal ideals of $R^G$ in terms of $G$-prime ideals of $R$. To do so, we need some general results of the theory of finite group actions.

**Proposition 3.3.1** *If $R$ is any commutative ring, $G$ a finite group of automorphisms of $R$ and $P, Q \in Spec(R)$ are such that $P \cap R^G = Q \cap R^G$, then $P$ and $Q$ are in the same $G$-orbit of $Spec(R)$.*

**Proof.** The proof follows the proof of [81, Proposition 1.1]. □

**Proposition 3.3.2** *Let $R$ and $G$ be as stated in Proposition 3.3.1. Then the pair $R^G, R$ of rings satisfy:*

*i) Lying over (LO): for any prime $P$ of $R^G$ there exists a prime $Q$ in $R$ with $Q \cap R^G = P$;*

*ii) Going up (GU): given primes $P \subsetneqq P_0$ in $R^G$ and $Q$ in $R$ with $Q \cap R^G = P$, there exists a prime $Q_0$ in $R$ satisfying $Q \subsetneqq Q_0$ and $Q_0 \cap R^G = P_0$;*

*iii) Going down (GD): given primes $P \subsetneqq P_0$ in $R^G$ and $Q_0$ in $R$ with $Q_0 \cap R^G = P_0$, there exists a prime $Q$ in $R$ satisfying $Q \subsetneqq Q_0$ and $Q \cap R^G = P$;*

*iv) Incomparability (INC): if $Q$ and $Q_0$ are distinct primes in $R$ with $Q \cap R^G = Q_0 \cap R^G$, then $Q \not\subseteq Q_0$ and $Q_0 \not\subseteq Q$.*

**Proof.** As every element $r \in R$ is a root of the polynomial of $R^G[x]$, $\prod_{g \in G}(x - g(r))$, $R$ is integral over $R^G$. So $R^G \subseteq R$ satisfies LO, GU and INC, [63, Theorem 42 and Theorem 44]. Let $P \subsetneqq P_0$ be prime ideals of $R^G$ and $Q_0$ a prime ideal of $R$ such that $Q_0 \cap R^G = P_0$. By LO, there is a prime ideal of $R$, $\overline{Q}$, such that $\overline{Q} \cap R^G = P \subsetneqq Q_0 \cap R^G$. By GU there is a prime ideal $\overline{Q}_1$ of $R$ such that $\overline{Q} \subsetneqq \overline{Q}_1$ and $\overline{Q}_1 \cap R^G = P_0 = Q_0 \cap R^G$. By Lemma 3.3.1, there is $\beta \in G$ such that $\overline{Q}_1^\beta = Q_0$. Hence $\overline{Q}^\beta \subsetneqq \overline{Q}_1^\beta = Q_0$ and $\overline{Q}^\beta \cap R^G = \overline{Q} \cap R^G = P$. If we take $Q = \overline{Q}^\beta$, $Q$ will satisfy *iii*). □

**Proposition 3.3.3** *Let $R$ be a commutative ring and $G$ a finite group of automorphisms of $R$. If $M$ is a maximal ideal of $R^G$ then $M = \cap_{g \in G} \overline{M}^g \cap R^G$, for some maximal ideal $\overline{M}$ of $R$.*

**Proof.** By LO, there is a prime ideal $\overline{M}$ of $R$ such that $M = \overline{M} \cap R^G$. Obviously, $\cap_{g\in G}\overline{M}^g \cap R^G = \overline{M} \cap R^G$. Since $M$ is maximal in $R^G$ by INC we have that $\overline{M}$ is maximal in $R$. □

**Proposition 3.3.4** *Let $R$ be a commutative ring, $G$ a finite group of automorphisms of $R$ and $M$ a maximal ideal of $R^G$. If $MR$ is a semiprime ideal of $R$, then $MR = \cap_{g\in G}\overline{M}^g$ for some maximal ideal $\overline{M}$ of $R$.*

**Proof.** By INC and LO the set $\mathcal{M}$ of primes of $R$ lying over $M$ is non-empty and consists of maximal ideals. By Proposition 3.3.1, $\mathcal{M}$ consists of a single $G$-orbit. In particular, $\mathcal{M}$ is finite. Finally, $MR = \cap\{\overline{M} : \overline{M} \in \mathcal{M}\}$, because $MR$, being semiprime is certainly an intersection of prime ideals of $R$, but all such are by definition in $\mathcal{M}$. □

**Proposition 3.3.5** *Let $R$ be a commutative ring, $G$ a finite group of automorphisms of $R$ such that $R$ is a $G$-Galois extension of $R^G$. Then for all maximal ideals $M$ of $R^G$, $M = MR \cap R^G$ and $MR = \cap_{g\in G}\overline{M}^g$ for some maximal ideal $\overline{M}$ of $R$.*

**Proof.** Let $R$ and $G$ be as stated above and $M$ a maximal ideal of $R^G$. Obviously, $MR$ is a $G$-stable ideal of $R$. By Lemma 3.2.1.9, $R^G$ is a direct summand of $R$ as an $R^G$-module, hence $M = MR \cap R^G$. So $MR \neq R$. By Lemma 3.2.1.11, $(R/MR)^G \cong R^G/M$, where this ring is a field and $R/MR$ is a $G$-Galois extension of $(R/MR)^G$. Now, by Lemma 3.2.1.9 and Lemma 3.2.1.12, $R/MR$ is a semiprime ring and the result follows from Proposition 3.3.4. □

**Theorem 3.3.6** *Let $R$ be any commutative ring, $G$ a finite subgroup of $Aut(R)$ and $R * G$ any crossed product of $G$ over $R$ constructed with the given action of $G$ on $R$. If $R$ is a $G$-Galois extension of $R^G$, then $R * G$ is Azumaya and $Z(R * G) = R^G$.*

In order to prove the Theorem we need the following Lemma:

**Lemma 3.3.7** *Let $R$ be a commutative ring, $G$ a finite subgroup of $Aut(R)$ such that $R$ is a $G$-Galois extension of $R^G$ and $R*G$ any crossed product of $G$ over $R$ constructed with the given action of $G$ on $R$. Then $G$ is $\omega$-outer and $Z(R*G) = R^G$.*

**Proof.** Let $g \in G\backslash\{id\}$ and $r \in R$ be such that $rg(x) - xr = 0$ for all $x \in R$. That is, $r(g(x)-x) = 0$ for all $x \in R$. Let $\text{r.ann}_R(r) = I$ and suppose $r \neq 0$. Then $I$ is a proper ideal of $R$ and there is a maximal ideal $M$ of $R$, such that $I \subseteq M$. Since $R$ is a $G$-Galois extension of $R^G$, there is $x \in R$ such that $g(x) - x \notin M$. So $g(x) - x \notin I$ and $r(g(x) - x) \neq 0$, a contradiction. So $G$ is $\omega$-outer and by Lemma 3.2.11 we have $Z(R*G) = R^G$. □

**Proof of Theorem 3.3.6:**

Let $R$ and $G$ be as stated above. Assume that $R$ is a $G$-Galois extension of $R^G$. So $R$ is finitely generated over $R^G$. Since $G$ is finite, $R*G$ is finitely generated over $R$ and also over $R^G$, which equals $Z(R*G)$ by Lemma 3.3.7.

By [34, Theorem II.7.1], $R*G$ is Azumaya if and only if for all maximal ideals $M$ of $Z(R*G)$, $(R*G)/M(R*G)$ is a separable $R^G/M$-algebra.

Let $M$ be a maximal ideal of $R^G$. Obviously $MR$ is a $G$-stable ideal of $R$ and by Lemma 3.3.5, $M = MR \cap R^G$. So $MR \neq R$. As $(R*G)/M(R*G) \cong (R/MR)*G$, it follows from Lemma 3.3.7 and 3.2.1.11 that $Z((R*G)/M(R*G)) \cong R^G/M$. So $(R*G)/M(R*G)$ is a separable $R^G/M$-algebra if and only if $(R*G)/M(R*G)$ is Azumaya.

We claim that for all maximal ideals $M$ of $R^G$, $M(R*G)$ is a maximal ideal of $R*G$. We show first that this claim will prove the result. Since $G$ is finite $R*G$ is finitely generated as a module over a commutative ring so, by [76, Corollary 13.1.13 iii)] $R*G$ is PI. Hence so is $(R*G)/M(R*G)$. Using the claim $(R*G)/M(R*G)$ is a simple PI ring and by Kaplansky's Theorem [76, Theorem 13.3.8], a central simple algebra, hence Azumaya [76, Proposition 13.7.7].

We will now prove the claim. Let $M$ be a maximal ideal of $R^G$. By Proposition 3.3.5, there is a maximal ideal $\overline{M}$ of $R$ such that

$$M = \cap_{\alpha \in G} \overline{M}^{\alpha} \cap R^G = \overline{M} \cap R^G,$$

and

$$MR = \cap_{\alpha \in G} \overline{M}^{\alpha}.$$

As $R/MR$ is artinian, $\overline{M}$ is a minimal prime over $MR$.

Let $\overline{M}^{\alpha_1}, \ldots, \overline{M}^{\alpha_t}$ be the distinct maximal ideals in $\{\overline{M}^{\alpha} : \alpha \in G\}$. Hence

$$\frac{R * G}{M(R * G)} \cong \left(\frac{R}{\overline{M}^{\alpha_1}} \oplus \ldots \oplus \frac{R}{\overline{M}^{\alpha_t}}\right) * G.$$

By [86, Corollary 14.8], $(R * G)/M(R * G)$ is prime if and only if $(R/\overline{M}) * G_1$ is prime for $G_1 = \{g \in G : \overline{M}^g = \overline{M}\}$. Set $G_2 = \{g \in G : \overline{M}^g = \overline{M}$ and $g$ induces the identity on $R/\overline{M}\}$. Then $R/\overline{M}$ is a field and by [86, Corollary 15.9], $(R/\overline{M}) * G_1$ is prime if and only if the twisted group ring $[R/\overline{M}]^t[G_2]$ is $G_1$-prime. Let $g \in G_2$, so that $g(x) - x \in \overline{M}$, for all $x \in R$. So $g = id$ since $R$ is a $G$-Galois extension of $R^G$. Hence $G_2 = \{id\}$ and $(R * G)/M(R * G)$ is prime.

As $(R*G)/M(R*G)$ is finite dimensional over the field $R^G/\overline{M}$, $(R*G)/M(R*G)$ is Artinian, hence $M(R * G)$ is maximal. This proves the claim. □

**Example 3.3.8** Take $S = \mathbb{C}[\theta, \theta^{-1}; \alpha]$, where $\alpha$ is complex conjugation on $\mathbb{C}$. We can think of $S$ as $\mathbb{C}[\theta^2, \theta^{-2}] * G$ where $G$ is the group of order 2 generated by $\alpha$. It is obvious that $(\mathbb{C}[\theta^2, \theta^{-2}])^G = \mathbb{R}[\theta^2, \theta^{-2}]$. Let $M$ be a maximal ideal of $\mathbb{C}[\theta^2, \theta^{-2}]$. If $\alpha(i) - i = -2i \in M$, then $1 \in M$, a contradiction. Hence we can deduce that $\mathbb{C}[\theta^2, \theta^{-2}]$ is a $G$-Galois extension of $\mathbb{R}[\theta^2, \theta^{-2}]$. By Theorem 3.3.6, $S$ is Azumaya.

Combining several results obtained so far, we get

**Proposition 3.3.9** *Let $R$ be any ring, $G$ a finite group of automorphisms of $R$ and $R * G$ any crossed product constructed with the given action of $G$ on $R$. Consider the following statements:*

*i)* $R * G$ *is Azumaya and* $Z(R * G) \subseteq R$*;*

*ii)* *a)* $R * G$ *is an H-separable extension of* $R$*;*

*b)* $R$ *is a separable extension of* $Z(R)^G$*;*

*iii)* $R$ *is Azumaya and* $Z(R)$ *is a G-Galois extension of* $Z(R)^G$.

*Then i) is equivalent to ii). If* $R$ *is commutative iii) implies i) and ii).*
*If* $G$ *is* $\omega$*-outer on* $R$*, i) and ii) imply iii).*
*If* $R$ *is commutative and* $G$ *is* $\omega$*-outer on* $R$*, i), ii) and iii) are equivalent.*

**Proof.** Assume $R$ and $G$ as above. By Proposition 3.2.3, *i)* is equivalent to *ii)*. If $R$ is commutative then *iii)* implies *i)* by Theorem 3.3.6.

If $G$ is $\omega$-outer on $R$ then $Z(R*G) = Z(R)^G$ by Lemma 3.2.11. By Proposition 3.2.1.14 and Proposition 3.1.10, *ii)* implies *iii)*.

If $R$ is commutative and $G$ is $\omega$-outer on $R$, by the above, *i)*, *ii)* and *iii)* are equivalent. □

**Corollary 3.3.10** *Let* $D$ *be a commutative domain,* $G$ *a finite subgroup of* $Aut(D)$*. Let* $D * G$ *be any crossed product of* $D$ *by* $G$ *constructed with the given action of* $G$ *on* $D$*. Then* $D * G$ *is Azumaya if and only if* $D$ *is a G-Galois extension of* $D^G$.

**Proof.** As $D$ is a commutative domain, $G$ is $\omega$-outer and the corollary follows from Proposition 3.3.9 □

We will state an improved version of Corollary 3.3.10 as Proposition 3.4.9 in §3.4, after introducing the notion of *inertia group.*

One could try to generalize the results obtained so far to any crossed product. In the general case, given a crossed product $R * J$, we could try to replace *iii)* of Proposition 3.3.9 by *iii')* *a)* $R$ *is Azumaya;*

*b)* $R$ *is finitely generated projective as an* $R^J$*-module;*

*c)* $R\#J \cong End(R_{R^J})$.

The following example shows that such a generalization will not be possible.

**Example 3.3.11** In [48], R.B. Howlett and I.M. Isaacs, for the proof of their main result, Theorem 8.2, built a non-abelian finite group $H$ of central type (given any field $K$, there exists a faithful irrreducible $K$-representation of degree $|H : Z(H)|^{1/2}$). By [35, Theorem 1] and its proof, if $H$ is of central type then we can form a certain twisted group algebra $\mathbb{C}^t[H/Z(H)]$ with centre $\mathbb{C}$ and isomorphic to $M_n(\mathbb{C})$ where $n^2 = [H : Z(H)]$. Hence $\mathbb{C}^t[H/Z(H)]$ is simple. So if $J = H/Z(H)$, $J$ is a nontrivial finite group such that $\mathbb{C}^t J$ is simple. As it is finitely generated as a $\mathbb{C}$-module, it is artinian. Hence $\mathbb{C}^t J$ is a central simple algebra and by [76, Proposition 13.7.7], Azumaya. Hence $i)$ of Proposition 3.3.9 holds but as $\mathbb{C}^J = \mathbb{C}$ ( $J$ acts trivially on $\mathbb{C}$ by definition of the twisted-group ring ), $\mathbb{C}J = \mathbb{C}\#J$ is not isomorphic to $\mathbb{C} \cong End(\mathbb{C}_{\mathbb{C}})$.

## 3.4 The Azumaya locus

Let $R$ be any ring and $M$ any maximal ideal of $Z(R)$. It is obvious that one can form $Z(R)_M$. Also the set $\mathcal{C}_{Z(R)}(M)$ is a right denominator set in $R$, hence we can form $R\mathcal{C}_{Z(R)}^{-1}(M)$ which we shall denote by $R_M$.

Given a ring $R$ which is finitely generated as a module over its centre $Z(R)$, $R$ is Azumaya if and only if $R_M$ is separable over $Z(R)_M$ for all maximal ideals $M$ of $Z(R)$, [34, Theorem II.7.1].

When $R$ is prime, it is trivial to confirm that $Z(R)_M$ is $Z(R_M)$. If $R$ is finitely generated as a module over its centre, $R$ is Azumaya if and only if

$$\{M \in Max(Z(R)) : R_M \text{ is Azumaya}\} = Max(Z(R)),$$

[34, Theorem II.7.1].

**Notation and Definition 3.4.1** For a ring $R$ and a (right or left) $R$-module $X$, the projective dimension and injective dimension of $X$ are denoted by $\text{pr.dim}_R(X)$ and $\text{inj.dim}_R(X)$, respectively. If there is no ambiguity we may omit the ring $R$ and simple denote them by pr.dim($X$) and inj.dim($X$), respectively. The right (respectively left) global dimension of $R$ is denoted by r.gl.dim($R$) (respectively l.gl.dim($R$)). If the right global dimension and the left global dimension are equal, we simple denote the common value by gl.dim($R$). The injective dimension of a ring $R$ as a right (respectively left) $R$-module is denoted by r.inj.dim($R$) (respectively l.inj.dim($R$)), and simply by inj.dim($R$) if the two values are the same. A ring $R$ is said to be *regular* if has finite global dimension. For the definitions of these concepts see [87] and [91].

**Remark 3.4.2** *i*) If $R$ is a Noetherian ring, r.gl.dim($R$) = l.gl.dim($R$), by [91, Corollary 9.20].

*ii*) One should note that the definition of regular rings given in Definition 3.4.1 is not the same as the one given in [76]. In [76, Definition 7.7.1], a ring is right regular if every right cyclic module has finite projective dimension. In [76, Example 7.7.2.], an example is given of a commutative Noetherian integral domain of infinite global dimension but regular in the sense of [76]. However, if $R$ is a commutative Noetherian local ring, the two concepts coincide [63, Theorem 121 and Theorem 173].

The definition of regular ring that we will use throughout this thesis, is the one given in Definition 3.4.1.

**Definition 3.4.3** Let $R$ be a prime Noetherian ring which is module-finite over its centre $Z(R)$. The *Azumaya locus* of $R$ is the set

$$\mathcal{A}_R = \{M \in Max(Z(R)) : R_M \text{ is Azumaya}\},$$

and the *singular locus* of $Z(R)$ is the set

$$\mathcal{S}_R = \{M \in Max(Z(R)) : Z(R)_M \text{ is not regular}\}.$$

**Remark 3.4.4** *i*) Let $R$ be a prime ring and $M$ a maximal ideal of $Z(R)$ such that $R_M$ is Azumaya. By [76, Proposition 13.7.11] $R_M/MR_M$ is a central simple (simple artinian) algebra with centre $Z(R)_M/MZ(R)_M$. If $R$ is a prime Noetherian ring module finite over its centre, $R_M/MR_M \cong R/MR$ and $\mathcal{A}_R$ is the set of maximal ideals $M$ of $Z(R)$ such that $R/MR$ is a central simple algebra over its centre, $Z(R)/M$.

*ii*) Under suitable hypothesis on the ring $R$, the Azumaya locus of $R$ contains a non-empty open set of $Max(Z(R))$. (In this case we are considering the Zariski topology defined on $Max(Z(R))$ in which open sets are the ones of the form $W(I) = \{P \in Max(Z(R)) : I \nsubseteq P\}$, for any ideal $I$ of $Z(R)$.)

Let $R$ be a prime PI ring of degree $n$. By [93, Corollary 6.1.36] and definition of PI ring of degree $n$ [92, Definition 1.4.30], there is $s \in Z(R)\backslash\{0\}$ such that $R[s^{-1}]$ is Azumaya. Hence, for any maximal ideal of $Z(R)$ such that $s \notin M$, $R_M$ is Azumaya. So $W(sR) := \{M \in Max(Z(R)) : sR \nsubseteq M\} \subseteq \mathcal{A}_R$, and $W(sR)$ is open and nonempty.

*iii*) If we impose some conditions on the centre of the prime Noetherian ring $R$, the singular locus of $Z(R)$ is a proper closed subset of $Max(Z(R))$.

Assume for instance that $K$ is an algebraically closed field of characteristic zero and $Z(R)$ a domain which is affine over $K$. Then $Z(R) = K[x_1, \dots, x_n]/P$ which is the affine coordinate ring of some affine variety $Y$ ($Y = Z(P) = \{r \in K^n : f(r) = 0, \text{for all } f \in P\}$.) Let $c$ be the height of $P$ in $K[x_1, \dots, x_n]$. In this case by [38, Corollary 16.20 and Theorem 19.12] the singular locus of $R$ (and $Z(R)$) is defined by the set of maximal ideals $M$ of $Z(R)$ containing the $c \times c$ minors of the Jacobian matrix $[\partial f_i/\partial x_j]$ (built with $f_1, \dots, f_n$ generators of $P$). Hence $\mathcal{S}_R$ is closed. By the Nullstellensatz, [38, Corollary 1.9], there is a one to one correspondence between the maximal ideals of $Z(R)$ and the points of $Y$, by

[45, Theorem 1.5.3 and Theorem 1.3.2] we have that $\mathcal{S}_R$ is a proper closed subset of $Max(Z(R))$.

Moreover, if $Q$ is any prime ideal of $Z(R)$ and if $J$ is the ideal of $Z(R)$ generated by the minors of the Jacobian matrix $[\partial f_i/\partial x_j]$, by [38, Corollary 16.20], $Z(R)_Q$ is regular if and only if $Q$ doesn't contain $J$.

Let $D$ be any commutative Noetherian domain and $G$ a finite group acting faithfully on $D$ as automorphisms of $D$. Form $D * G$, any crossed product of $G$ over $D$, and assume that $D*G$ is finitely generated over its centre $Z(D*G) = D^G$, Lemma 3.2.11. In this section we will describe the Azumaya locus of $D * G$ and try to relate it with the singular locus of $D^G$. To achieve this we will impose some basic homological conditions on $D$ (for instance we will assume that $D$ has finite global dimension) and on $D * G$ (we will assume that $D * G$ is height 1 Azumaya, see below for definition). We will start by describing when $(D * G)_M$ is Azumaya for $M$ a maximal ideal of $D^G$. In order to do so, we need to introduce some definitions.

**Definition 3.4.5** Let $R$ be any ring, $G$ a group acting on $R$ and $M$ an ideal of $R$. Set

$$G_R(M) = \{g \in G : r^g - r \in M, \text{ for all } r \in R\}.$$

$G_R(M)$ is usually called the *inertia group* of $M$.

For each $g \in G$, let

$$I_R(g) =< r^g - r : r \in R >$$

the two-sided ideal of $R$ generated by $\{r^g - r : r \in R\}$. Define the two-sided ideal of $R$

$$I_R(G) = \bigcap_{g \in G \setminus \{1_G\}} I_R(g).$$

The following lemma follows easily from the definitions.

**Lemma 3.4.6** *Suppose that $R$ is a ring, $G$ a group acting on $R$ and $M$ an ideal of $R$. Then*

*i) $G_R(M)$ is the unique largest subgroup $H$ of $G$ such that $M$ is $H$-invariant and $H$ acts trivially on $R/M$;*

*ii) if $N$ is an ideal of $R$ such that $N \subseteq M$, then $G_R(N) \subseteq G_R(M)$;*

*iii) if $N$ is a $G$-invariant ideal of $R$ such that $N \subseteq M$, then $G_R(M) = G_{R/N}(M/N)$;*

*iv) for $1_G \neq g \in G$ and an ideal $M$ of $R$, $I_R(g) \subseteq M$ if and only if $g \in G_R(M)$.*

**Corollary 3.4.7** *Let $R$ be any ring and $G$ a finite group acting on $R$ and $M$ a prime ideal of $R$. Then $G_R(M) = \{1_G\}$ if and only if $I_R(G) \not\subseteq M$.*

**Proof.** Assume $G_R(M) = \{1_G\}$. By Lemma 3.4.6 ($iv$), $I_R(g) \not\subseteq M$, for any $g \neq 1_G$. So, as $M$ is prime, $I_R(G) \not\subseteq M$. Conversely assume $I_R(G) \not\subseteq M$. Then $I_R(g) \not\subseteq M$ for any $g \in G\backslash\{1_G\}$. Hence by Lemma 3.4.6 ($iv$), $G_R(M) = \{1_G\}$. □

**Lemma 3.4.8** *Let $R$ be a commutative ring and $G$ a finite group acting on $R$. The following are equivalent:*

*i) $R$ is a $G$-Galois extension of $R^G$;*

*ii) For all maximal ideals $M$ of $R$, $G_R(M) = \{1_G\}$;*

*iii) $I_R(G) = R$.*

**Proof.** By Proposition 3.2.1.3, $R$ is a $G$-Galois extension of $R^G$ if and only if for every maximal ideal $M$ of $R$ and each $g \in G\backslash\{1_G\}$, there is $t \in R$ such that $g(t) - t \notin M$. But this is equivalent to $ii$). As $I_R(G)$ is an ideal of $R$, by Corollary 3.4.7, $ii$) is equivalent to $iii$). □

**Proposition 3.4.9** *Let $G$ be a finite group of automorphisms of a commutative domain $D$ and $D*G$ be any crossed product of $D$ by $G$ constructed with the given action of $G$ on $D$. The following are equivalent:*

*i)* $D * G$ *is Azumaya;*

*ii)* $I_D(G) = D$;

*iii) For all* $\overline{M} \in Max(D)$, $G_D(\overline{M}) = \{1_G\}$.

**Proof.** By Corollary 3.3.10, $D * G$ is Azumaya if and only if $D$ is a $G$-Galois extension of $D^G$. The result follows now from Lemma 3.4.8. □

**Lemma 3.4.10** *Let $R$ be any ring and $G$ a subgroup of $Aut(R)$. Then the following holds:*

*i) For any ideal $M$ of $R$ and any $g \in G$,* $G_R(M^g) = (G_R(M))^{g^{-1}}$;

*ii) For any subgroup $H$ of $G$ and any $g \in G$,* $I_R(H^g) = (I_R(H))^{g^{-1}}$;

*iii) Let $H$ be any subgroup of $G$ and $N = N_G(H) = \{g \in G : gHg^{-1} = H\}$. Then $I_R(H)$ is $N$-stable. In particular if $g \in G$ and $< g >$ is a normal subgroup of $G$, then $I_R(< g >)$ is $G$-stable.*

**Proof.** Let $M$ be any ideal of $R$ and $g \in G$. If $h \in G_R(M^g)$, then for all $r \in R$, $h(r) - r \in M^g$. So, for all $r \in R$, $g^{-1}hg(r) - r = g^{-1}hg(r) - g^{-1}g(r) \in M$. Then $g^{-1}hg \in G_R(M)$ and so $G_R(M^g) \subseteq G_R(M)^{g^{-1}}$. Conversely, if $h \in G_R(M)^{g^{-1}}$, $g^{-1}hg \in G_R(M)$ and so, for all $r \in R$, $g^{-1}hg(r) - r \in M$. Then $hg(r) - g(r) \in M^g$, for all $r \in R$. Therefore $h \in G_R(M^g)$, whence $i)$ holds.

Let $H$ be any subgroup of $G$ and $g \in G$. As for any $h \in G$, we have

$$\begin{aligned} I_R(g^{-1}hg) &= < g^{-1}hg(r) - r : r \in R > \\ &= < g^{-1}h(r) - g^{-1}(r) : r \in R > \\ &= < h(r) - r : r \in R >^{g^{-1}} \end{aligned}$$

Hence $I_R(H^g) = \cap_{h\in H} I_R(g^{-1}hg) = \cap_{h\in H} I_R(h)^{g^{-1}} = (I_R(H))^{g^{-1}}$ and we have $ii)$.

Let $H$ be any subgroup of $G$ and $N = N_G(H)$. Take $g \in N$. By $ii)$ $(I_R(H))^g = I_R(H^{g^{-1}}) = I_R(H)$. Hence $I_R(H)$ is $N$-stable. If $g \in G$ and $< g >$ is a normal subgroup of $G$, $N_G(< g >) = G$ and by the above $I_R(< g >)$ is $G$-stable. □

**Remark 3.4.11** If $G$ is not abelian $I_R(g)$ doesn't have to be $G$-invariant. For instance take $G =< a, b : a^2 = b^3 = 1, aba = b^{-1} >$ and $R = \mathbb{C}[x, y]$. One can define an action of $G$ on $V = \mathbb{C}x + \mathbb{C}y$ and then extend it to an action on $R$. We think of $G$ acting on $V$ as a group of homomorphisms of a $\mathbb{C}$-vector space in the following way

$$a(x) = y$$

$$a(y) = x$$

and for any $\gamma \in \mathbb{C}\backslash\{1\}$ such that $\gamma^3 = 1$

$$b(x) = \gamma x$$

$$b(y) = \gamma^2 y.$$

If $I_R(a)$ were $G$-stable, in particular we would have $(I_R(a))^b = I_R(a)$. By Lemma 3.4.10 $ii)$, we would have

$$I_R(b^{-1}ab) = I_R(a).$$

As $b^{-1}ab = ba$, in particular, we would have

$$I_R(ba) \cap V = I_R(a) \cap V.$$

Now $I_R(a) \cap V = \{cx - cy : c \in \mathbb{C}\}$ and $ba(y) - y = b(x) - y = \gamma x - y$. So

$$\gamma x - y \in I_R(ba)\backslash I_R(a).$$

Hence $I_R(a)$ is not $G$-invariant.

**Lemma 3.4.12** *Let $R$ be a commutative Noetherian ring, $G$ a finite subgroup of $Aut(R)$, $M$ a maximal ideal of $R^G$ and $\overline{M}$ any maximal ideal of $R$ such that $\overline{M} \cap R^G = M$. We can form the ring $R_M = RC_{R^G}^{-1}(M)$ and extend each $g \in G$ to an automorphism of $R_M$. Then, the following are equivalent*

*i) $R_M$ is a G-Galois extension of $(R_M)^G$;*

*ii) $G_R(\overline{M}) = \{1_G\}$;*

*iii) $I_R(G) \nsubseteq \overline{M}$.*

**Proof.** Assume $R$, $G$ and $M$ as stated above. If $g \in G$, then the map $rx^{-1} \to g(r)x^{-1}$, for all $r \in R$, $x \in C_{R^G}(M)$ is an automorphism of $R_M$ (One should note that, as was said in the first chapter, we are abusing notation when we write $rx^{-1}$ for an arbitrary element of $RX^{-1}$).

Since $M$ is a maximal ideal of $R^G$, by Proposition 3.3.3, there is a maximal ideal $\overline{M}$ of $R$ such that $M = \overline{M} \cap R^G$.

By [44, Theorem 9.22], every prime ideal of $R_M$ is of the form $IR_M$ for $I$ a prime ideal of $R$ such that $I \cap R^G \subseteq M$.

Let $\overline{\overline{M}}R_M$ be a maximal ideal of $R_M$, where $\overline{\overline{M}}$ is an ideal of $R$ such that $\overline{\overline{M}} \cap R^G \subseteq M = \overline{M} \cap R^G$. So $\overline{\overline{M}}$ is a prime ideal of $R$ maximal among the primes $M'$ such that $M' \cap R^G \subseteq M = \overline{M} \cap R^G$. Then by GU, Proposition 3.3.2, we have that $\overline{\overline{M}} \cap R^G = M = \overline{M} \cap R^G$ and by Proposition 3.3.1, there is $\alpha \in G$ such that $\overline{\overline{M}} = \overline{M}^\alpha$. Obviously, for any $\beta \in G$, $\overline{M}^\beta R_M$ is a maximal ideal of $R_M$. So

$$Max(R_M) = \{\overline{M}^\alpha R_M : \alpha \in G\}$$

By Proposition 3.2.1.3, $R_M$ is a $G$-Galois extension of $(R_M)^G$ if and only if for all $\beta \in G$ and for each $\alpha \in G \backslash \{1_G\}$, there is $x \in R_M$ such that

$$\alpha(x) - x \notin \overline{M}^\beta R_M.$$

As $R_M = RC_{R^G}^{-1}(M)$, it is then easy to conclude that $R_M$ is a $G$-Galois extension of $(R_M)^G$ if and only if for all $\beta \in G$ and for each $\alpha \in G\backslash\{1_G\}$, there is $r \in R$ such that

$$\alpha(r) - r \notin \overline{M}^{\beta}.$$

This holds, if and only if, for all $\beta \in G$ and each $\alpha \in G\backslash\{1_G\}$, $I_R(\alpha) \nsubseteq \overline{M}^{\beta}$ i.e. if and only if $I_R(G) \nsubseteq \overline{M}^{\beta}$, for any $\beta \in G$. By Lemma 3.4.10 *ii*), $I_R(G) \nsubseteq \overline{M}^{\beta}$ if and only if $I_R(G) = I_R(G^{\beta}) = (I_R(G))^{\beta^{-1}} \nsubseteq \overline{M}$. So we have that *i*) is equivalent to *iii*). By Corollary 3.4.7, *ii*) is equivalent to *iii*). □

**Proposition 3.4.13** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$ and assume that $D$ is finitely generated over $D^G$. Let $D*G$ be any crossed product constructed with the given action of $G$ on $D$. Then*

$$\begin{aligned}
\mathcal{A}_{D*G} &= \{M \in Max(D^G) : \forall \overline{M} \in Max(D) \text{ with } \overline{M} \cap D^G = M, I_D(G) \nsubseteq \overline{M}\} \\
&= \{M \in Max(D^G) : \exists \overline{M} \in Max(D) \text{ with } \overline{M} \cap D^G = M, I_D(G) \nsubseteq \overline{M}\} \\
&= \{M \in Max(D^G) : I_D(G) \cap D^G \nsubseteq M\}.
\end{aligned}$$

**Proof.** By [86, Proposition 1.6 and Corollary 12.6], $D * G$ is prime and Noetherian. Let $M$ be a maximal ideal of $D^G$. By Corollary 3.3.10, $(D*G)_M \cong D_M * G$ is Azumaya if and only if $D_M$ is a $G$-Galois extension of $(D_M)^G$. The first two equalities follow now from Lemma 3.4.12.

Let $\mathcal{A}' = \{M \in Max(D^G) : I_D(G) \cap D^G \nsubseteq M\}$. It is obvious that $\mathcal{A}' \subseteq \mathcal{A}_{D*G}$. Conversely let $M \in \mathcal{A}_{D*G}$. Then, by the first two equalities above, $I_D(G) \nsubseteq \overline{M}$ for all maximal ideals $\overline{M}$ of $D$ such that $\overline{M} \cap D^G = M$. It is then easy to see (for instance by induction on the number of maximal ideals contracting in $D^G$ to $M$), that $I_D(G) \nsubseteq \cup\{\overline{M} \in Max(D)$ with $\overline{M} \cap D^G = M\}$. Take $d \in I_D(G)\backslash \cup\{\overline{M} \in Max(D)$ with $\overline{M} \cap D^G = M\}$ and $\overline{d} = \prod_{g \in G} d^g \in I_D(G) \cap D^G$. If $\overline{d} \in \overline{M}$ for some $\overline{M}$ maximal ideal of $D$ such that $\overline{M} \cap D^G = M$, then $d^g \in \overline{M}$, for some $g \in G$. Hence $d \in \overline{M}^{g^{-1}}$ but $\overline{M}^{g^{-1}} \in Max(D)$ and $\overline{M}^{g^{-1}} \cap D^G = M$, contradicting the choice of $d$. Hence $\overline{d} \in (I_D(G) \cap D^G) \setminus \cup\{\overline{M} \in Max(D)$ with $\overline{M} \cap D^G = M\}$. So $I_D(G) \cap D^G \not\subseteq \overline{M}$ and $I_D(G) \cap D^G \not\subseteq \overline{M} \cap D^G = M$. □

The following results relate the Azumaya locus of some prime Noetherian rings module-finite over their centres, with the singular locus of their centres.

**Lemma 3.4.14** *Let $R$ be a prime Noetherian ring, module-finite over its centre. If $gl.dim(R)$ is finite, then $\mathcal{A}_R \subseteq Max(Z(R)) \setminus \mathcal{S}_R$.*

**Proof.** See [24, Lemma 3.3] □

**Corollary 3.4.15** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$ of finite global dimension. Form $D * G$ any crossed product of $G$ over $D$ constructed with the given action of $G$ on $D$ and assume that $D * G$ is Azumaya. Then $\mathcal{S}_{D*G} = \emptyset$.*

**Proof.** Let $G$ and $D$ be as above. By Lemma 3.2.11, $Z(D * G) = D^G$. As $D * G$ is Azumaya, $\mathcal{A}_{D*G} = Max(D^G)$. By Corollary 3.4.9, we have that for each maximal ideal $M$ of $D$, $G_D(M) = \{1_G\}$. Then by [108, Corollary 5.7], $D * G$ has finite global dimension. The result follows from Lemma 3.4.14. □

**Definition 3.4.16** Let $R$ be a prime Noetherian ring, module finite over its centre. We say that $R$ is *height 1 Azumaya* if $R_P$ is Azumaya over $Z(R)_P = Z(R_P)$, for all primes $P$ of $Z(R)$ of height 1.

**Lemma 3.4.17** *Let $D$ be a commutative Noetherian domain, $G$ a finite subgroup of $Aut(D)$ and $P$ a prime ideal of $D^G$ of height one. Let $\overline{P}$ be any prime ideal of $D$ such that $P = \overline{P} \cap D^G$. Then $ht(\overline{P}) = 1$ and the following are equivalent:*

*i) $D_P$ is a $G$-Galois extension of $(D_P)^G$;*

*ii) $G_D(\overline{P}) = \{1_G\}$;*

*iii)* $I_D(G) \not\subseteq \overline{P}$.

**Proof.** Let $D$, $G$ and $P$ be as above. If $\tilde{P}$ is a prime ideal of $D$ such that $P = \tilde{P} \cap D^G$, then by INC, Proposition 3.3.2, $\tilde{P}$ is a prime of height less than or equal to 1. As $P \neq 0$ ($P$ is of height 1), $\tilde{P} \neq 0$ and so $\tilde{P}$ is of height 1.

Let $\mathcal{C} = \mathcal{C}_{D^G}(P) = D^G \backslash P$ and $M\mathcal{C}^{-1} \in Max(D_P)$. Hence $M$ is maximal among the prime ideals $N$ of $D$ such that $N \cap D^G \subseteq P$. As $P$ has height 1 and $D$ is a domain, $M \cap D^G = 0$ or $M \cap D^G = P$. If $M \cap D^G = 0$, by INC, $M = 0$, a contradiction. Hence $M \cap D^G = P = \tilde{P} \cap D^G$. So $M = \tilde{P}^\alpha$, for some $\alpha \in G$, by Lemma 3.3.1. Hence the set of maximal ideals of $M\mathcal{C}^{-1}$ is just $\{\tilde{P}^\alpha \mathcal{C}^{-1} : \alpha \in G\}$. Now the proof follows as the proof of Lemma 3.4.12. □

**Proposition 3.4.18** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$ with $D$ a finitely generated $D^G$-module. Form $D * G$ any crossed product constructed with the given action of $G$ on $D$. Then the following are equivalent*

*i)* $D * G$ *is height 1 Azumaya;*

*ii) for all* $\overline{P} \in Spec(D)$ *of height 1,* $I_D(G) \not\subseteq \overline{P}$*;*

*iii) for all* $\overline{P} \in Spec(D)$ *of height 1,* $G_D(\overline{P}) = \{1_G\}$*;*

*iv) for all* $\overline{P} \in Spec(D)$ *of height 1 and for all* $1_G \neq g \in Stab_G(\overline{P}) = \{g \in G : \overline{P}^g = \overline{P}\}$*,* $g$ *acts non-trivially on* $D/\overline{P}$*.*

**Proof.** Let $D$ and $G$ be as above. In this case, $D * G$ is finitely generated as a $D^G$-module. The crossed product $D * G$ is height 1 Azumaya if and only if for all $P \in Spec(D^G)$ such that $ht(P) = 1$, $D_P * G$ is Azumaya. By Corollary 3.3.10 $D * G$ is height 1 Azumaya if and only if for all $P \in Spec(D^G)$ such that $ht(P) = 1$, $D_P$ is a $G$-Galois extension of $(D_P)^G$.

As prime ideals of $D$ of height 1 contract to prime ideals of $D^G$ of height 1 (by GD), by Lemma 3.4.17 *i)* and *ii)* are equivalent. By Corollary 3.4.7, *ii)* is equivalent to *iii)* and it is obvious that *iii)* is equivalent to *iv)* □

To prove the following proposition, we need to introduce some more definitions which will only be used in the proof of Proposition 3.4.21 and Lemma 3.4.22.

**Definition 3.4.19** Let $R$ be a ring and $M$ a finitely generated right or left $R$-module. The *grade* of $M$ is defined to be

$$j(M) = inf\{n | Ext^n_R(M, R) \neq 0\} \in \mathbb{N}_0 \cup \{+\infty\}$$

see [59] and [91].

**Definition 3.4.20** Let $R$ be a Noetherian ring and let $M$ be a finitely generated right or left $R$-module. We say that $M$ *satisfies the Auslander condition* if the following holds: for every non-negative integer $i$ and every non-zero submodule $N$ of $Ext^i_R(M, R)$, $j(N) \geq i$. (Note that $Ext^i_R(M, R)$ is an $R$-module on the opposite side to $M$ in view of the fact that $R$ is an $R$-module.) If every finitely generated right and left $R$-module satisfies the Auslander condition, then we say that $R$ *satisfies the Auslander condition.* A Noetherian ring is called *Auslander-Gorenstein* (respectively *Auslander-regular*), if it satisfies the Auslander condition and has finite right and left injective dimension (respectively global dimension).

An *Auslander-Gorenstein* ring $R$ is called *Macaulay* if $j(M) + \text{K.dim}(M) = \text{K.dim}(R)$ holds for every finitely generated right or left $R$-module $M$, where K.dim( ) denotes the (Gabriel-Rentschler) Krull dimension (see [44] for the definition and properties of K.dim).

**Proposition 3.4.21** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$ with $D$ a finitely generated $D^G$-module. Form any crossed product $D * G$ constructed with the given action of $G$ on $D$. If* *i)* $\text{gl.dim}(D * G)$ *is finite,*

*and*

*ii) for all* $\overline{P} \in Spec(D)$ *of height 1,* $I_D(G) \not\subseteq \overline{P}$,

*then* $\mathcal{A}_{D*G} = Max(D^G) \backslash \mathcal{S}_{D*G}$.

In order to prove Proposition 3.4.21, we need the following lemma.

**Lemma 3.4.22** *Let* $D$ *be a commutative Noetherian domain of finite injective dimension. Assume that* $D$ *is a semilocal ring with maximal ideals* $X_1, \dots, X_t$ *and that* $\text{K.dim}(D) = \text{K.dim}(D_{X_i})$*, for all* $i \in \{1, \dots, t\}$*. Then* $D$ *is Auslander-Gorenstein and Macaulay.*

**Proof.** Let $D$ be as above. By [12, Corollary 3.4 and §1], every commutative Noetherian ring of finite injective dimension is Auslander-Gorenstein. So $D$ is Auslander-Gorenstein. Also $D$ is such that $D_X$ is Auslander Gorenstein-Macaulay for every $X$ maximal ideal of $D$, [109, Proposition 3.6]. Hence, if $M$ is a finitely generated $D$-module, $MD_{X_i}$ is a finitely generated $D_{X_i}$-module and

$$j(MD_{X_i}) + \text{K.dim}(MD_{X_i}) = \text{K.dim}(D_{X_i}) = \text{K.dim}(D) \qquad (1)$$

for all $i \in \{1, \dots, t\}$.

As for all $i \in \{1, \dots, t\}$, $(Ext^i_D(M, D))D_{X_i} \cong Ext^i_{D_{X_i}}(MD_{X_i}, D_{X_i})$, [25], $j(M) \le j(MD_{X_i})$. Also as $\text{ann}_D(Ext^i_D(M, D))$ is an ideal of $D$, it will be contained in one of the maximal ideals, $X_k$ say. Hence $Ext^i_D(M, D)$ is not $(D \backslash X_k)$-torsion and $j(M) = j(MD_{X_k})$. So

$$j(M) = inf\{j(MD_{X_i}) : i \in \{1, \dots, t\}\} \qquad (2).$$

Let $M$ be a finitely generated $D$-module. Then if $ht(\ )$ denotes the height of a prime ideal we have $\text{K.dim}(M) = \text{K.dim}(D/\text{r.ann}_D(M)) =$

$max\{ht(X_i/\text{r.ann}_D(M)) : \text{r.ann}_D(M) \subseteq X_i\}$. For each $i = 1, \ldots, t$,

$$\text{K.dim}(MD_{X_i}) = \begin{cases} -1 & \text{if}\, \text{ann}_D(M) \not\subseteq X_i \\ ht(X_i/\text{r.ann}_D(M)) & \text{if}\, \text{r.ann}_D(M) \subseteq X_i \end{cases},$$

since $ht(X_iD_{X_i}/\text{r.ann}_D(M)D_{X_i}) = ht(X_i/\text{r.ann}_D(M))$ in the second case. Hence

$$\text{K.dim}(M) = max\{\text{K.dim}(MD_{X_i}), i \in \{1, \ldots, t\}\} \qquad (3).$$

Without loss of generality we will assume $\text{K.dim}(MD_{X_u}) = \text{K.dim}(M)$ and $j(M_{D_{X_v}}) = j(M)$, for some $u, v \in \{1, \ldots, t\}$. Then by (1) and (2)

$$\begin{aligned} \text{K.dim}(D) &= \text{K.dim}(MD_{X_u}) + j(MD_{X_u}) = \text{K.dim}(M) + j(M_{D_{X_u}}) \\ &\geq \text{K.dim}(M) + j(M); \end{aligned}$$

also, by (1) and (3)

$$\begin{aligned} \text{K.dim}(D) &= \text{K.dim}(MD_{X_v}) + j(MD_{X_v}) = \text{K.dim}(MD_{X_v}) + j(M) \\ &\leq \text{K.dim}(M) + j(M). \end{aligned}$$

So $D$ is Auslander-Gorenstein Macaulay □

**Remark 3.4.23** We should note that Lemma 3.4.22 is false without the unmixedness condition. For instance take $R = \mathbb{C}[x, y]$, $P =< x, y >$ the ideal of $R$ generated by $x$ and $y$ and $Q =< x + 1 >$ the ideal of $R$ generated by $x + 1$. Let $D$ be the localization of $R$ at $P \cap Q$. The commutative Noetherian domain $D$ has finite injective dimension. In this case the maximal ideals are $QD$ and $PD$. As $Q$ has height 1, $\text{K.dim}(D_{QD}) = 1$ but $\text{K.dim}(D) = 2$. If we take the $D$-module $W = D/QD$, $\text{K.dim}(W) = 0$. As $\text{pr.dim}(W) = 1$, by [91, Theorem 9.6], $j(W) \leq 1$ and the ring $D$ is not Auslander-Gorenstein Macaulay.

**Proof of Theorem 3.4.21:** As $D$ is a commutative domain $Z(D * G) = D^G$, by Lemma 3.2.11.

Let $M \in Max(D^G)$. Then $MD_M^G$ is a maximal ideal of $D_M^G$. Also $M \in \mathcal{A}_{D*G}$ if and only if $D_M * G$ is Azumaya. As $Z(D_M * G) = D_M^G$ and $(D_M)_{MD_M^G}$ is just $D_M$, $M \in \mathcal{A}_{D*G}$ if and only if $MD_M^G \in \mathcal{A}_{D_M*G}$. Moreover $M \in \mathcal{S}_{D*G}$ if and only if $MD_M^G \in \mathcal{S}_{D_M^G*G}$.

As $D$ is finitely generated over $D^G$, $D_M/MD_M$ is finitely generated over $D^G/M$, a field, hence is Artinian. As in the proof of Lemma 3.4.12, we have that $Max(D_M) = \{\overline{M}^g D_M : g \in G\}$, where $\overline{M}$ is a maximal ideal of $D$ such that $\overline{M} \cap D^G = M$. So $MD_M \subseteq \mathcal{J}(D_M)$ and $D_M$ is semilocal.

As $D$ is a subring of $D*G$ and a direct summand of $D*G$ as a $D$-bimodule, by [76, Theorem 7.2.8], gl.dim$(D) \leq$ gl.dim$(D*G)+$pr.dim$_D(D*G)$. As $D*G$ is free as a module over $D$ and of finite global dimension, we have that $D$ also has finite global dimension. Hence $D_M$ has finite global dimension as well, [76, Proposition 15.2.8]. By [91, Theorem 9.12], the supremum of the injective dimensions of $D_M$-modules is equal to the supremum of the projective dimensions of $D_M$-modules, hence the injective dimension of $D_M$ (as a $D_M$-module) is finite.

By Lemma 3.4.22, we have that $D_M$ is an Auslander-Gorenstein Macaulay ring. By [109, Proposition 3.9] any ring strongly graded by a finite group and with Noetherian coefficient ring is Auslander-Gorenstein and Macaulay if and only if the same happens for the basis ring, so $D_M * G$ is Auslander-Gorenstein and Macaulay. As $D * G$ is regular, $D_M * G$ is regular as well [76, Corollary 7.4.3]. So $D_M * G$ is Auslander-regular Macaulay.

By *ii*) and Proposition 3.4.18, $D_M * G$ is height 1 Azumaya. By [86, Corollary 12.6] $D_M*G$ is prime. Hence $D_M*G$ is a prime Noetherian ring, module finite over its centre, Auslander-regular and Macaulay, and $D_M*G$ is height 1 Azumaya. So, by [24, Theorem 3.8], we have $\mathcal{A}_{D_M*G} = Max(D_M^G)\backslash\mathcal{S}_{D_M*G}$. By the discussion at the start of the proof, $\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$. $\square$

**Remark 3.4.24** Let $D = K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ and identify the multiplicative abelian group generated by $x_1, \ldots, x_n$ with $\mathbb{Z}^n$. Take $G$ a subgroup of the group of linear isomorphisms of $\mathbb{Z}^n$, $GL(\mathbb{Z}^n) \cong GL_n(\mathbb{Z})$, the group of invertible $n \times n$ matrices with integer entries. We say that $g \in G$ is a reflection if and only if it is a conjugate in $GL_n(\mathbb{Z})$ of one the matrices

$$d = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{bmatrix} \text{ or } e = \begin{bmatrix} -1 & 1 & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{bmatrix}.$$

In this case, by [60, Lemma 2.3], condition *ii)* of Proposition 3.4.21 is equivalent to

*ii') There are no reflections in* $G \backslash \{1\}$.

We would like to establish when, given a finite group $G$ of automorphisms of a commutative Noetherian domain $D$ satisfying condition *ii)* of Proposition 3.4.21, the crossed product will have finite global dimension.

In [108], Zhong Yi, gave a sufficient condition for a crossed product of a commutative Noetherian ring with finite global dimension by a polycyclic-by-finite group to have finite global dimension. In [108, Corollary 5.7] he proved: *Let $R$ be a commutative Noetherian ring with finite global dimension. Let $G$ be a polycyclic-by-finite group and $S = R * G$ a crossed product. If for every maximal ideal of $M$ of $R$ with characteristic char$(R/M)$ of $R/M$ equal to $p$, a positive (prime) integer, $G_R(M)$ contains no element of order $p$, then* gl.dim$(R * G)$ *is finite.* This condition has already been used in the proof of Corollary 3.4.15, where $G_R(M)$ was $\{1_G\}$.

**Corollary 3.4.25** *Let $G$ be a finite group of automorphisms of a commutative Noetherian domain $D$. Form $D * G$ any crossed product constructed with the given action of $G$ on $D$ and assume that $D$ is finitely generated over $D^G$. If*

*i) gl.dim$(D)$ is finite,*

*ii) for every maximal ideal $M$ of $D$ with char$(D/M) = p > 0$, $G_D(M)$ contains no element of order $p$,* *iii) for every prime ideal $\overline{P}$ of $D$ of height one, $I_D(G) \not\subseteq \overline{P}$,*

*then $\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$. Moreover the closed subset of singular (or, equivalently, non-Azumaya) points of $Max(D)$ is given by*

$$\mathcal{S}_{D*G} = \{M \in Max(D^G) : I_D(G) \cap D^G \subseteq M\}.$$

**Proof.** Since $G$ is finite, $D * G$ is finitely generated as a module over $D$. As $D$ is finitely generated over $D^G$, $D * G$ is finitely generated over $D^G$. By [108, Corollary 5.7] gl.dim($D * G$) is finite and the Corollary follows from Proposition 3.4.21, with the last sentence being a consequence of Proposition 3.4.13. □

In the case when $D$ is a commutative domain and an affine algebra over an algebraically closed field of characteristic zero, every maximal ideal $M$ of $D$ is such that $D/M \cong K$ and the characteristic of $D/M$ is zero, so if $D$ has finite global dimension then so too does $D * G$, [108, Corollary 5.7].

In the next proposition we show that condition *iii)* of Corollary 3.4.25 is necessary as well as sufficient provided we assume that $D$ has finite global dimension and is an affine commutative Noetherian algebra over an algebraically closed field $K$ of characteristic zero.

**Proposition 3.4.26** *Let $D$ be a commutative domain and an affine algebra over an algebraically closed field $K$ of characteristic zero and $G$ a finite group of $K$-automorphisms of $D$. Form the crossed product $D * G$ constructed with the given action of $G$ on $D$. Assume also that* gl.dim(D) *is finite. Then $I_D(G) \not\subseteq \overline{P}$ for all $\overline{P} \in Spec(D)$ of height 1 if and only if $\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$.*

**Proof.** Assume $D$ and $G$ as above. By Noether's Theorem [99, Theorem 2.3.1], $D^G$ is affine and $D$ is finitely generated as a module over $D^G$. By Lemma 3.2.11, $Z(D * G) = D^G$.

If $\overline{M}$ is a maximal ideal of $D$, $D/\overline{M} \cong K$ of characteristic zero, hence $D * G$ has finite global dimension, [108, Corollary 5.7].

By Corollary 3.4.25 and Proposition 3.4.18, we may assume that

$$\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$$

and aim to prove that $D * G$ is height 1 Azumaya.

As $D$ is a Noetherian domain of finite global dimension, $D$ is integrally closed [74, Theorem 19.4 and Theorem 11.5]. Hence so is $D^G$. By Serre's Theorem [45, Theorem II.8.22A], as $D^G$ is Noetherian, for every prime ideal $P$ of $D^G$ of height 1, $D^G_P$ is regular. (See also [38, Theorem 19.12] for the relation of the different concepts of regular rings used).

Let $P$ be a prime of $D^G$ of height 1. As $D^G$ is a domain and an affine $K$-algebra, for $K$ an algebraically closed field of characteristic zero, as in Remark 3.4.4 *iii*), $\mathcal{S}_{D*G} = \{M \in Max(D^G) : J \subseteq M\}$, for a certain ideal $J$ of $D^G$. Suppose that for all maximal ideals $M$ of $D^G$ such that $P \subseteq M$, $M \in \mathcal{S}_{D*G}$. As $D^G$ is a Hilbert ring every prime ideal is the intersection of maximal ideals [58, Corollary 5.4]. So $\cap\{M \in Max(D^G) : P \subseteq M\} = P$. Hence $J \subseteq P$ and so $D^G_P$ is not regular, a contradiction. So there is a maximal ideal $M$ of $D^G$ such that $P \subseteq M$ and $M \notin \mathcal{S}_{D*G}$. As by hypothesis $\mathcal{A}_{D*G} = Max(D^G)\backslash\mathcal{S}_{D*G}$, we have $D_M * G$ Azumaya. As $P \subseteq M$, $D_P * G$ is Azumaya, [34, Corollary II.1.7]. So $D * G$ is height 1 Azumaya as required. □

The next three examples show that there are cases of commutative Noetherian domains $D$ of finite global dimension which are affine algebras over algebraically closed fields of characteristic zero and such that $\mathcal{A}_{D*G} \neq Max(D^G)\backslash\mathcal{S}_{D*G}$ or equivalently, condition *iii*) of Corollary 3.4.25 is not satisfied. In both of the first two examples the singular locus is empty. The last example gives a case when all the conditions of Proposition 3.4.26 are verified and the singular locus is not empty.

**Example 3.4.27** 1) Take the group ring $\mathbb{C}G$ where $G = < a, b : b^{-1}ab = a^{-1} >$. We can think of $\mathbb{C}G$ as $\mathbb{C} < a, b^2 > *\overline{G}$ where $\overline{G} = G/ < b^2 > \cong C_2$. As $b^2ab^{-2} =$

$b(bab^{-1})b^{-1} = ba^{-1}b^{-1} = a$, $b^2a = ab^2$ and $\mathbb{C}G = \mathbb{C}[a, a^{-1}, b^2, b^{-2}] * \overline{G}$ where the action $\sigma$ is a homomorphism such that $\sigma(b)(c) = c$, for all $c \in \mathbb{C}$, $\sigma(b)(b^2) = b^2$, $\sigma(b)(a) = bab^{-1} = (ba^{-1}b^{-1})^{-1} = a^{-1}$ and $\sigma(b)(a^{-1}) = ba^{-1}b^{-1} = (bab^{-1})^{-1} = a$. So we can think of $\overline{G}$ as a finite group of $\mathbb{C}$-automorphisms of the commutative Noetherian domain and affine $\mathbb{C}$-algebra, $\mathbb{C}[a, a^{-1}, b^2, b^{-2}]$.

By Lemma 3.2.11, $Z(\mathbb{C}G) = (\mathbb{C}[a, a^{-1}, b^2, b^{-2}])^{\overline{G}} = \mathbb{C}[b^2, b^{-2}, a + a^{-1}]$ and $\mathbb{C} < a, b^2 >$ is finitely generated over $(\mathbb{C}[a, a^{-1}, b^2, b^{-2}])^{\overline{G}}$.

By [108, Corollary 5.7] $\mathbb{C}[a, a^{-1}, b^2, b^{-2}]$ and $Z(\mathbb{C}G)$ have finite global dimension. So, by [76, Proposition 15.2.8], $\mathcal{S}_{\mathbb{C}G} = \emptyset$ but $\mathbb{C}G$ is not Azumaya as the augmentation ideal $\{\sum_{g \in G} c_g g : \sum_{g \in G} c_g = 0\}$ and $< a - 1, b + 1 >$, the ideal generated by $a - 1$ and $b + 1$, are both maximal but contract to the same maximal ideal of $Z(\mathbb{C}G)$, $< b^2 - 1, a + a^{-1} - 2 >$, the ideal generated by $b^2 - 1$ and $a + a^{-1} - 2$. So the non-Azumaya locus must be non-empty.

2) Take $D = \mathbb{C}[x]$ and $G = \{1, g\}$ where $g$ is the $\mathbb{C}$-automorphism of $D$ defined by $g(x) = -x$. Hence $D^G = \mathbb{C}[x^2]$ and $D$ is finitely generated over $D^G$. Take the crossed product $D * G$ constructed with the given action. As $Z(D * G) = \mathbb{C}[x^2]$, a regular ring [76, Theorem 7.5.3], $\mathcal{S}_{D*G} = \emptyset$, [76, Proposition 15.2.8]. As $I_D(G) =< x >$, the ideal generated by $x$ which is a prime ideal of height 1, $\mathcal{A}_{D*G} \neq Max(Z(D * G)) \backslash \mathcal{S}_{D*G}$.

3) Let $D = \mathbb{C}[x, y]$ and $G = \{1, g\}$ where $g$ is the $\mathbb{C}$-algebra automorphism of $D$ such that $g(x) = -x$ and $g(y) = -y$. Form the crossed product $D * G$ with the given action. In this case $D^G = \mathbb{C}[x^2, xy, y^2]$ and does not have finite global dimension, see [76, Example 7.8.10]. So $\mathcal{S}_{D*G} \neq \emptyset$ by [76, Proposition 15.2.8]. Since $x$ and $y$ are in $I_D(G)$ and $G$ fixes the ideal of $D$ generated by $x$ and $y$, $< x, y >$, and acts trivially on the factor, we have $I_D(G) =< x, y >$. So $\mathcal{A}_{D*G} = Max(D^G) \backslash \mathcal{S}_{D*G} = Max(D^G) \backslash \{< x^2, xy, y^2 >\}$, by Corollary 3.4.25.

**Remark 3.4.28** It is not true in general that given a ring $R$ and a finite group of automorphisms $G$, that $R$ will be finitely generated over $R^G$.

For instance take $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$, and the automorphism $g$ of $R$ such that $g : \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \to \begin{bmatrix} a & 0 \\ -b & c \end{bmatrix}$. Then if $G = \{id, g\}$, we have $R^G = \begin{bmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$ but $R$ is not finitely generated over $R^G$.

However, if $R$ is a commutative affine algebra over a field $K$ and $G$ is a finite subgroup of $K$-automorphisms of the algebra, then $R^G$ is affine and $R$ is finitely generated as an $R^G$-module [99, Noether's Theorem, Theorem 2.3.1]. The problem of when is $R$ finitely generated over $R^G$ is discussed in [41], §10 and in [80], §2.

## 3.5 Additional remarks

1. All definitions and results of §1 are well known. The main references are [34], [49], [50], [103] and [104].

2. Section §2 follows the ideas of [3] and [1]. Proposition 3.2.2 and Proposition 3.2.1.14, are respectively, our generalization for crossed products of [3, Theorem 1] and [1, Theorem 3.4], using the same type of arguments as in [3] and [1].

3. We consider Theorem 3.3.6 to be our main result of section 3. The other results of section 3, unless stated otherwise and with the exception of Proposition 3.3.2, are new.

4. Lemma 3.4.6, Corollary 3.4.7 and Lemma 3.4.10 are easy and known, their proof was included as we were not able to find any good reference. Proposition 3.4.13, Proposition 3.4.21 and Proposition 3.4.26 are our main results of this section.

# Chapter 4

# FBN and PI Skew-Laurent Rings

In this chapter we study when some skew-polynomial and skew-Laurent rings over Noetherian rings are fully bounded Noetherian rings (or FBN rings for short). (See below for the definitions). So far we have been unable to completely determine which of these rings are FBN, although one can describe completely the ones which satisfy polynomial identities, a subfamily of FBN rings.

The results in §1 and §2 are an easy generalization of the ones of R. F. Damiano and J. Shapiro in [32], who studied the above properties for the case of skew-polynomial rings with one indeterminate and one automorphism of the Noetherian basis ring.

As every Azumaya ring is a PI ring, in section 3, we combine the results of section 2 and chapter 3 to describe the Azumaya Locus of some skew-Laurent rings.

## 4.1 Fully bounded rings

In this section we introduce the basic definitions and some descriptions of FBN rings. For more details see [29] and [44]. We show also that the necessary condition obtained for our rings to be FBN is not sufficient.

**Definition 4.1.1** A ring $R$ is said to be *right bounded* if every essential right ideal of $R$ contains an ideal which is essential as a right ideal.

**Remark 4.1.2** A prime ring is right bounded if and only if every essential right ideal contains a non-zero ideal.

**Definition 4.1.3** A ring $R$ is said to be *right fully bounded* if every prime factor ring of $R$ is right bounded.

A *right (left) FBN ring* is any right (left) fully bounded right (left) Noetherian ring. An *FBN ring* is any right and left FBN ring.

**Proposition 4.1.4** *Let $R$ be a right Noetherian ring. Then $R$ is right FBN if and only if for every right ideal $I$ of $R$, there are $r_1, \ldots, r_n \in R$ such that*

$$r.ann_R(R/I) = \cap_{i=1}^{n}\{r \in R : r_i r \in I\} \qquad (*)$$

**Proof.** [28, Corollaire II 9] and [67, Theorem 3.5]. □

**Definition 4.1.5** Condition $(*)$ is called *Gabriël's condition.*

**Proposition 4.1.6** *Let $S$ be a right FBN ring and $R$ a subring of $S$. If as a left $R$-module, $S$ is free of basis $\{1, s_\alpha : \alpha \in \Lambda, s_\alpha \in S\}$, then $R$ is right FBN.*

**Proof.** [29, Proposition 2.1] □

**Theorem 4.1.7 (Letzter)** *If $R$ is a right FBN ring and $R$ is a subring of a ring $S$ such that $S$ is finitely generated as a right $R$-module, then $S$ is right FBN.*

**Proof.** [44, Theorem 10.7] □

**Proposition 4.1.8** *Let $R$ be a ring and $X \subseteq C_R(0)$ a right Ore set. If $R$ is right FBN so is $RX^{-1}$.*

**Proof.** [29, Proposition 1.5] and [67, Theorem 3.5] □

The following notation will be kept through the rest of this chapter.

**Notation 4.1.9** Let $R$ be any Noetherian ring, $\alpha_1, \ldots, \alpha_n$ automorphisms of $R$ commuting pairwise, $\mathcal{T} = R[\theta_1, \ldots, \theta_n; \alpha_1, \ldots, \alpha_n]$ and $\mathcal{S} = R[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n]$ the skew-polynomial ring and the skew-Laurent ring, respectively.

We will denote by $G$ the abelian group generated by $\alpha_1, \ldots, \alpha_n$ and by $H$ the abelian torsionfree group generated by $\theta_1, \ldots, \theta_n$.

Define $\Psi$ the group homomorphism from $H$ onto $G$ such that $\Psi(\theta_i) = \alpha_i$, for all $i \in \{1, \ldots, n\}$. Let $K_e = \text{Ker}(\Psi)$. Hence $G \cong H/K_e$.

We can think of $\mathcal{S}$ as being a crossed product of $R$ over $H$, $R * H$.

Each $s \in \mathcal{S}$ can be written as

$$s = \sum_{h \in H} r_h h$$

for some $r_h \in R$ or

$$s = \sum_{J} r_J \theta^J$$

where $J = (j(1), \ldots, j(n)) \in \mathbb{Z}^n$, $r_J \in R$ and $\theta^J = \theta_1^{j(1)} \ldots \theta_n^{j(n)}$.

**Remark 4.1.10** One should note that although Notation 4.1.9 is almost the same as Notation 2.2.8 in the second chapter, in this chapter we won't assume that $R$ is commutative as we did in the previous one.

**Notation 4.1.11** If $R$ is any ring and $\mathcal{P}$ the prime radical of $R$, for every automorphism $\alpha$ of $R$, we can define an induced automorphism of $R/\mathcal{P}$, $\overline{\alpha}$, such that $\overline{\alpha}(r + \mathcal{P}) = \overline{\alpha}(r) + \mathcal{P}$, for any $r \in R$.

As was said in the introduction of the chapter, in [32] Damiano and Shapiro gave a necessary condition for a twisted polynomial ring of the form $R[\theta; \alpha]$, where $\alpha$ is an automorphism of the Noetherian ring $R$, to be an FBN ring. The following proposition is a generalization of their results.

**Proposition 4.1.12** *Let $\mathcal{P}$ be the prime radical of the Noetherian ring $R$. If $\mathcal{S}$ is right (left) FBN, then $R$ is right (left) FBN and for all $i \in \{1, \ldots, n\}$, $\overline{\alpha}_i|_{Z(R/P)}$ has finite order.*

**Proof.** Suppose that $\mathcal{S}$ is right FBN. Then by Proposition 4.1.6 $R$ is right FBN.

We will assume first that $R$ is a prime ring. So $\mathcal{S}$ is a prime ring as well [76, Theorem 1.2.9]. Fix $i \in \{1, \ldots, n\}$. Since $\theta_i - 1$ is a regular element of $\mathcal{S}$, $(\theta_i - 1)\mathcal{S}$ is an essential right ideal of $\mathcal{S}$. By hypothesis, $\mathcal{S}$ is right FBN, so there exists a nonzero ideal $I$ of $\mathcal{S}$ contained in $(\theta_i - 1)\mathcal{S}$.

Take a nonzero element of $I$, $p = \sum_{J\in\Lambda} r_J\theta^J \in I$ say, with a minimal number of nonzero coefficients. Multiplying $p$ by suitable powers of the $\theta_i$, if necessary, we can assume that $\Lambda \subseteq \mathbb{N}_0^n$. We will assume $j(l) \in \{0, \ldots, m\}$ for any $J \in \Lambda$ and $1 \leq l \leq n$ such that $r_J \neq 0$.

Since $p \in (\theta_i - 1)\mathcal{S}$, there will be $r'_J \in R$ such that

$$\sum_{J\in\Lambda} r_J\theta_1^{j(1)} \ldots \theta_n^{j(n)} = (\theta_i - 1) \sum_{J'\in\Lambda'} r'_{J'}\theta_1^{j'(1)} \ldots \theta_n^{j'(n)}. \tag{4.1}$$

Comparing both sides of 4.1, as $\Lambda \subseteq \mathbb{N}_0^n$ those $J'$ in $\Lambda'$ such that $r'_{J'} \neq 0$ are in $\mathbb{N}_0^n$. So we can assume that $\Lambda' \subseteq \mathbb{N}_0^n$. Also by adding zero in either side of the equation if necessary, we can and will assume that $\Lambda = \Lambda'$.

Then, for each $j(1), \ldots, j(i-1), j(i+1), \ldots, j(n)$,

$$\begin{aligned}
r_{(j(1),\ldots,j(i-1),0,j(i+1),\ldots,j(n))} &= -r'_{(j(1),\ldots,j(i-1),0,j(i+1),\ldots,j(n))}, \\
r_{(j(1),\ldots,j(i-1),l,j(i+1),\ldots,j(n))} &= \alpha_i(r'_{(j(1),\ldots,j(i-1),l-1,j(i+1),\ldots,j(n))}) - r'_{(j(1),\ldots,j(i-1),l,j(i+1),\ldots,j(n))} \\
&\qquad 0 < l \leq m, \\
0 &= \alpha_i(r'_{(j(1),\ldots,j(i-1),m,j(i+1),\ldots,j(n))}).
\end{aligned}$$

From (4.1) and the equalities above it is obvious that there is $A = (a(1), \ldots, a(n)) \in \Lambda$ such that $a(i) \neq 0$ and $r_A \neq 0$. Otherwise, for every

$A \in \Lambda$ we would have

$$
\begin{aligned}
r_{(a(1),\dots,a(i-1),0,a(i+1),\dots,a(n))} &= -r'_{(a(1),\dots,a(i-1),0,a(i+1),\dots,a(n))},\\
0 &= \alpha_i(r'_{(a(1),\dots,a(i-1),l-1,a(i+1),\dots,a(n))}) - r'_{(a(1),\dots,a(i-1),l,a(i+1),\dots,a(n))}\\
&\quad 0 < l \leq m,\\
0 &= \alpha_i(r'_{(a(1),\dots,a(i-1),m,a(i+1),\dots,a(n))}),
\end{aligned}
$$

but then $r_A = 0$ for any $A$, a contradiction.

We claim that there is an element $B = (b(1),\dots,b(n))$ of $\Lambda$ such that $r_B \neq 0$, $b(i) \neq a(i)$ and $b(l) = a(l)$ for all $l \neq i$. Suppose not. Then we would have

$$
\begin{aligned}
0 &= -r'_{(a(1),\dots,a(i-1),0,a(i+1),\dots,a(n))},\\
0 &= \alpha_i(r'_{(a(1),\dots,a(i-1),0,a(i+1),\dots,a(n))}) - r'_{(a(1),\dots,a(i-1),1,a(i+1),\dots,a(n))}\\
&\ \vdots\\
0 &= \alpha_i(r'_{(a(1),\dots,a(i-1),a(i)-2,a(i+1),\dots,a(n))}) - r'_{(a(1),\dots,a(i-1),a(i)-1,a(i+1),\dots,a(n))}\\
r_A &= \alpha_i(r'_{(a(1),\dots,a(i-1),a(i)-1,a(i+1),\dots,a(n))}) - r'_{(a(1),\dots,a(i-1),a(i),a(i+1),\dots,a(n))}\\
0 &= \alpha_i(r'_{(a(1),\dots,a(i-1),a(i),a(i+1),\dots,a(n))}) - r'_{(a(1),\dots,a(i-1),a(i)+1,a(i+1),\dots,a(n))}\\
&\ \vdots\\
0 &= \alpha_i(r'_{(a(1),\dots,a(i-1),m-1,a(i+1),\dots,a(n))}) - r'_{(a(1),\dots,a(i-1),m,a(i+1),\dots,a(n))}\\
0 &= \alpha_i(r'_{(a(1),\dots,a(i-1),m,a(i+1),\dots,a(n))}),
\end{aligned}
$$

so $r_A = 0$, a contradiction.

Then, there are $A, B \in \Lambda$ such that $r_B, r_A \neq 0$, $a(i) \neq b(i)$ and $a(l) = b(l)$ for all $l \in \{1,\dots,m\}\setminus\{i\}$.

Let $z \in Z(R)$. Hence, as $\alpha_1,\dots,\alpha_n$ commute pairwise,

$$
\begin{aligned}
\overline{p} &= zp - p\alpha_1^{-a(1)}\cdots\alpha_n^{-a(n)}(z)\\
&= \sum_{J=(j(1),\dots,j(n))\in\Lambda} (zr_J - r_J\alpha_1^{j(1)-a(1)}\cdots\alpha_n^{j(n)-a(n)}(z))\theta^J \in I
\end{aligned}
$$

has a number of nonzero coefficients strictly less than that of $p$ (because $r_A \neq 0$ but the coefficient of $\theta^A$ in $\overline{p}$ is zero). So $\overline{p} = 0$. In particular the coefficient of $\theta^B$

is zero. As $a(l) = b(l)$ for all $l \in \{1, \dots, m\} \backslash \{i\}$, we have that, for any $z \in Z(R)$,

$$zr_B - r_B \alpha_i^{b(i)-a(i)}(z) = 0. \tag{4.2}$$

Since $z \in Z(R)$ also $\alpha_i^{b(i)-a(i)}(z) \in Z(R)$. As $R$ is prime and $r_B \neq 0$ from (4.2), we have $\alpha_i^{b(i)-a(i)}(z) = z$. Hence $\alpha_i|_{Z(R)}$ has finite order.

Now assume that $R$ is any right Noetherian ring and let $\mathcal{P}$ be its prime radical. Since $\mathcal{P}$ is a $G$-stable ideal of $R$

$$\mathcal{S}/\mathcal{P}\mathcal{S} \cong R/\mathcal{P}[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \overline{\alpha}_1, \dots, \overline{\alpha}_n].$$

Moreover, if $\mathcal{S}$ is right FBN then so is $\mathcal{S}/\mathcal{P}\mathcal{S}$. Thus, without loss of generality, in deriving the conclusions of the proposition, we can assume $\mathcal{P} = (0)$. As $R$ is Noetherian $(0) = P_1 \cap \dots \cap P_\zeta$, a finite intersection of distinct minimal primes of $R$.

Since, for each $i \in \{1, \dots, n\}$, $\alpha_i$ permutes the minimal primes $P_1, \dots, P_\zeta$, there exists $\nu_i \in \mathbb{N}$, such that $\alpha_i^{\nu_i}(P_j) = P_j$, for all $j \in \{1, \dots, \zeta\}$.

As $\mathcal{S}$ is right FBN so is $\mathcal{S}' = R[\theta_1^{\nu_1}, \theta_1^{-\nu_1}, \dots, \theta_n^{\nu_n}, \theta_n^{-\nu_n}; \alpha_1^{\nu_1}, \dots, \alpha_n^{\nu_n}]$, by Proposition 4.1.6. Hence, for each $j \in \{1, \dots, \zeta\}$, $P_j\mathcal{S}'$ is an ideal of $\mathcal{S}'$ and

$$\mathcal{S}'/P_j\mathcal{S}' \cong R/P_j[\theta_1^{\nu_1}, \theta_1^{-\nu_1}, \dots, \theta_n^{\nu_n}, \theta_n^{-\nu_n}; \alpha_1^{\nu_1}, \dots, \alpha_n^{\nu_n}]$$

is bounded. From what was said above, there are $k_{i,j}$ such that

$$\alpha_i|_{Z(R/P_j)}^{\nu_i k_{i,j}} = id_{Z(R/P_j)}.$$

Take $k_i = l.c.m.\{\nu_i k_{i,j} : j \in \{1, \dots, \zeta\}\}$. Hence, for any $z \in Z(R)$, we have

$$\alpha_i^{k_i}(z) - z \in P_j, \text{ for all } j \in \{1, \dots, \zeta\}.$$

So,

$$\alpha_i^{k_i}(z) - z = 0$$

and $\alpha_i|_{Z(R)}$ has finite order. The proof for the left case is similar. □

**Proposition 4.1.13** *Let $\mathcal{P}$ be the prime radical of the Noetherian ring $R$. If $\mathcal{T}$ is right (left) FBN, then $R$ is right (left) FBN and for all $i \in \{1, \dots, n\}$, $\overline{\alpha}_i|_{Z(R/\mathcal{P})}$ has finite order.*

**Proof.** By Proposition 4.1.6, we have that $R$ is right FBN. Then as $\mathcal{S} \cong \mathcal{T}X^{-1} \cong X^{-1}\mathcal{T}$ for $X = \{\theta^J : J = (j(1), \dots, j(n)) \in \mathbb{Z}^n\}$, the result follows from Proposition 4.1.8 and Proposition 4.1.12. □

In the converse direction to the above results, we have:

**Proposition 4.1.14** *Let $\mathcal{P}$ be the prime radical of the Noetherian ring $R$. Suppose that $R/\mathcal{P}$ is finitely generated as a $Z(R/\mathcal{P})$-module and for all $i \in \{1, \dots, n\}$, $\overline{\alpha}_i|_{Z(R/\mathcal{P})}$ has finite order. Then $\mathcal{S}$ and $\mathcal{T}$ are FBN.*

**Proof.** Since $R$ is Noetherian, $\mathcal{P}$ is nilpotent, [44, Theorem 2.11]. Hence $\mathcal{PS}$, [resp. $\mathcal{PT}$] is contained in every prime ideal of $\mathcal{S}$, [resp. $\mathcal{T}$]. It suffices to prove that

$$\mathcal{S}/\mathcal{PS} \cong R/\mathcal{P}[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \overline{\alpha}_1, \dots, \overline{\alpha}_n]$$

and

$$\mathcal{T}/\mathcal{PT} \cong R/\mathcal{P}[\theta_1, \dots, \theta_n; \overline{\alpha}_1, \dots, \overline{\alpha}_n]$$

are FBN. Hence, without loss of generality suppose $\mathcal{P} = (0)$.

Let $n_i$ be the order of $\alpha_i|_{Z(R)}$. Then $Z(R)[\theta_1, \dots, \theta_n; \alpha_1, \dots, \alpha_n]$ and $Z(R)[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1, \dots, \alpha_n]$ are finitely generated as modules over the commutative subrings $Z(R)[\theta_1^{n_1}, \dots, \theta_n^{n_n}]$, $Z(R)[\theta_1^{n_1}, \theta_1^{-n_1}, \dots, \theta_n^{n_n}, \theta_n^{-n_n}]$ respectively. As $R$ is a Noetherian ring finitely generated over the centre $Z(R)$, by Einsenbud's Theorem [37, Theorem 1], $Z(R)$ is a Noetherian ring and so $Z(R)[\theta_1^{n_1}, \dots, \theta_n^{n_n}]$ and $Z(R)[\theta_1^{n_1}, \theta_1^{-n_1}, \dots, \theta_n^{n_n}, \theta_n^{-n_n}]$ are FBN rings. By Letzter's Theorem, Theorem 4.1.7, we have that $Z(R)[\theta_1, \dots, \theta_n; \alpha_1, \dots, \alpha_n]$ and $Z(R)[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1, \dots, \alpha_n]$ are FBN.

Since $\mathcal{S}$ and $\mathcal{T}$ are finitely generated over $Z(R)[\theta_1, \ldots, \theta_n; \alpha_1, \ldots, \alpha_n]$ and $Z(R)[\theta_1, \theta_1^{-1}, \ldots, \theta_n, \theta_n^{-1}; \alpha_1, \ldots, \alpha_n]$, respectively, $\mathcal{S}$ and $\mathcal{T}$ are FBN, by Letzter's Theorem, Theorem 4.1.7. □

One might naturally ask whether the converses of Propositions 4.1.12 and 4.1.13 are true. Proposition 4.1.17 provides a negative answer to this question.

In [90], Guy Renault proved:

**Proposition 4.1.15** *Let $D$ be any division ring. Then the following are equivalent:*

1. *$D[x]$ is (left, right) FBN;*

2. *for all $n \geq 1$ $M_n(D)$ is algebraic over $Z(D)$.*

**Proof.** The result follows from [90, Proposition 8 and Theorem 3]. □

In [29], Cauchon generalized this result, proving the following:

**Theorem 4.1.16** *Let $U$ be any ring. Then the following are equivalent:*

1. *$U[x]$ is (left, right) FBN;*

2. *(a) $U$ is (left, right) FBN;*

   *(b) for every prime ideal $P$ of $U$, the (left, right) quotient ring of $U/P$ is isomorphic to the ring $M_n(K)$ for some division ring $K$ satisfying: For every integer $m > 0$, $M_m(K)$ is algebraic over its centre.*

Following the same ideas as [90] and [29], we have

**Proposition 4.1.17** *Let $U$ be any ring. Then the following are equivalent*

1. *$U[x]$ is (left, right) FBN;*

2. *$U[x, x^{-1}]$ is (left, right) FBN;*

*3.* *i)* *$U$ is (left, right) FBN;*

*ii)* *For every prime ideal $P$ of $U$, the (left, right) quotient ring of $U/P$ is isomorphic to the ring $M_n(K)$ for some division ring $K$ satisfying: For every integer $m > 0$, $M_m(K)$ is algebraic over its centre.*

**Proof.** If we assume *1* then, by Proposition 4.1.8, $U[x, x^{-1}]$ is (left, right) FBN.

Now assume that $U[x, x^{-1}]$ is left FBN. As $U[x, x^{-1}]$ is a free $U$-module of basis $\{x^i : i \in \mathbb{Z}\}$, by Proposition 4.1.6 $U$ is left FBN and we have 3.*i*).

Let $P$ be a prime ideal of $U$. Obviously, $\overline{P} = PU[x, x^{-1}]$ is an ideal of $U[x, x^{-1}]$ and as

$$U[x, x^{-1}]/\overline{P} \cong U/P[x, x^{-1}]$$

$\overline{P}$ is a prime ideal of $U[x, x^{-1}]$. Let $S = \mathcal{C}_{U/P}(0)$ and $U_1 = U/P$. Then $S^{-1}U_1[x, x^{-1}]$ is left FBN and $S^{-1}U_1$ is a left FBN simple artinian ring. Hence $S^{-1}U_1 \cong M_n(D)$ for some division ring $D$. As $S^{-1}U_1[x, x^{-1}] \cong M_n(D)[x, x^{-1}] \cong M_n(D[x, x^{-1}])$, by Proposition 4.1.6 and Letzer's Theorem, Theorem 4.1.7, $S^{-1}U_1[x, x^{-1}]$ is left FBN if and only if $D[x, x^{-1}]$ is left FBN.

Let $p(x)$ be a nonzero element of $D[x]$. Then $p(x)$ is a regular element of $D[x, x^{-1}]$ and $D[x, x^{-1}]p(x)$ is essential as a left ideal of $D[x, x^{-1}]$. As $D[x, x^{-1}]$ is left FBN, there is a non-zero two-sided ideal $I$ of $D[x, x^{-1}]$ such that $I \subseteq D[x, x^{-1}]p(x)$. Whence $I$ is generated by an element of $Z(D)[x, x^{-1}]$, $q(x, x^{-1})$ say. Hence, there is $a(x, x^{-1}) \in D[x, x^{-1}]$ such that

$$q(x, x^{-1}) = a(x, x^{-1})p(x). \tag{4.3}$$

Multiplying (4.3) by a suitable power of $x$ it follows that every polynomial of $D[x]$ is a factor of a polynomial of $Z(D)[x]$. Hence by [90, Theorem 3], $M_m(D)$ is algebraic over $Z(D)$ for every $m > 0$ and 3.*ii*) follows.

If we had assumed $U[x, x^{-1}]$ right FBN, in a similar way, we would get condition 3.

If we assume 3, condition 1 follows from Theorem 4.1.16. □

## 4.2 Polynomial identities in skew-polynomial and skew-Laurent rings

**Definition 4.2.1** Let $R$ be any ring and $f(x_1, \dots, x_n)$ an element of the free algebra $\mathbb{Z} < x_1, \dots, x_n >$. We say that $R$ *satisfies* $f$ or $f$ is a *polynomial identity* of $R$ if $f(r_1, \dots, r_n) = 0$, for all $r_i \in R$.

If at least one of the monomials of $f$ of highest degree has coefficient 1, we say that $f$ is monic.

If $R$ satisfies some monic polynomial in $\mathbb{Z} < x_1, \dots, x_n >$, for some $n \in \mathbb{N}$, we say that $R$ is a *polynomial identity ring* or a *PI ring*.

**Proposition 4.2.2** *Let $R$ be a Noetherian ring with prime radical $\mathcal{P}$. The following are equivalent:*

*i) $\mathcal{S}$ is a PI ring;*

*ii) $\mathcal{T}$ is a PI ring;*

*iii) $R$ is a PI ring and for all $i \in \{1, \dots, n\}$, $\overline{\alpha}_i|_{Z(R/\mathcal{P})}$ has finite order;*

*iv) The ring $R$ is PI and the image of the canonical homomorphism from $G$ to $Aut(Z(R/\mathcal{P}))$ is finite.*

**Proof.** We shall prove that $i) \Rightarrow ii)$, $ii) \Rightarrow iii)$ and $iii) \Rightarrow i)$. It is clear that $iii) \Leftrightarrow iv)$.

As $R$ is a Noetherian ring, $\mathcal{S}$ and $\mathcal{T}$ are Noetherian as well.

If $\mathcal{S}$ is a PI ring, as $\mathcal{T}$ is a subring of $\mathcal{S}$, $\mathcal{T}$ is a PI ring. Hence $i) \Rightarrow ii)$.

Suppose that $\mathcal{T}$ is a PI ring. As $R$ is a subring of $\mathcal{T}$, $R$ will be PI as well. By [76, Corollary 13.6.6], $\mathcal{T}$ is FBN and by Proposition 4.1.13, for any $i \in \{1, \dots, n\}$, there is $m_i \in \mathbb{N}$ such that $(\overline{\alpha}_i|_{Z(R/\mathcal{P})})^{m_i} = id_{Z(R/\mathcal{P})}$ and we have $iii)$.

Suppose $iii)$. As the prime radical $\mathcal{P}$ of $R$ is obviously $G$-stable, $\mathcal{PS}$ is an ideal of $\mathcal{S}$ and

$$\mathcal{S}/\mathcal{PS} \cong (R/\mathcal{P}) * H.$$

As $R$ is Noetherian, by [44, Theorem 2.11], $\mathcal{P}$ is a nilpotent ideal and so is $\mathcal{PS}$. By [76, Lemma 13.1.7], $\mathcal{S}$ is PI if and only if $(R/\mathcal{P}) * H$ is PI. So, without loss of generality we can suppose that $\mathcal{P} = (0)$ and so $R$ is semiprime.

Let $X$ be the set of all regular elements of $Z = Z(R)$. Since $R$ is a semiprime PI ring, $Z(RX^{-1}) = ZX^{-1}$ by [92, Proposition 1.7.18]. By [92, Proposition 1.7.22], $RX^{-1}$ is semisimple Artinian. As $ZX^{-1} = Z(RX^{-1})$, $ZX^{-1}$ is semisimple Artinian as well.

We can thus write $ZX^{-1} = \oplus_{i=1}^{t} F_i$, for $F_i$ a field, and then, by [92, Theorem 1.7.20], $RX^{-1} = \oplus_{i=1}^{t} R_i$, for $R_i$ a simple ring of centre $F_i$. As $R$ is a PI-ring, by Kaplansky's Theorem, [76, Theorem 13.3.8], each $R_i$ is a central simple algebra over $F_i$, hence $RX^{-1}$ is finitely generated over $ZX^{-1}$.

For every $i \in \{1, \dots, n\}$, since $\alpha_i(X) \subseteq X$, we can define $\alpha_i' \in Aut(RX^{-1})$, by setting $\alpha_i'(rx^{-1}) = \alpha_i(r)\alpha_i(x)^{-1}$, for each $r \in R$ and $x \in X$. Then, we can form the skew-Laurent ring, $\mathcal{S}' = RX^{-1}[\theta_1, \theta_1^{-1}, \dots, \theta_n, \theta_n^{-1}; \alpha_1', \dots, \alpha_n']$. As $R$ is a semiprime PI ring, by [92, Lemma 1.7.17], $X \subseteq \mathcal{C}_R(0)$, hence we can think of $R$ as a subring of $RX^{-1}$ and of $\mathcal{S}$ as a subring of $\mathcal{S}'$. So it will be enough to prove that $\mathcal{S}'$ is a PI ring.

Also, for every $i \in \{1, \dots, n\}$, $(\alpha_i'|_{Z(R)})^{m_i} = id_{Z(R)}$.

Set $\mathcal{S}'' = RX^{-1}[\theta_1^{m_1}, \theta_1^{-m_1}, \dots, \theta_n^{m_n}, \theta_n^{-m_n}; (\alpha_1')^{m_1}, \dots, (\alpha_n')^{m_n}]$. The ring $\mathcal{S}'$ is obviously finitely generated as an $\mathcal{S}''$-module. As $RX^{-1}$ is finite dimensional over $ZX^{-1}$, $\mathcal{S}''$ is finitely generated as a $ZX^{-1}[\theta_1^{m_1}, \theta_1^{-m_1}, \dots, \theta_n^{m_n}, \theta_n^{-m_n}]$ module. Hence $\mathcal{S}'$ is finitely generated over the commutative subring $ZX^{-1}[\theta_1^{m_1}, \theta_1^{-m_1}, \dots, \theta_n^{m_n}, \theta_n^{-m_n}]$. So $\mathcal{S}'$ is a PI ring [76, Corollary 13.1.13], and we have $i$). □

**Remark 4.2.3** There are examples of semiprime PI rings $R$ and $\alpha \in Aut(R)$, such that $\alpha|_{Z(R)}$ has finite order but $\alpha$ doesn't. For instance take, as in [32, Example 2], $R$ to be the ring of $2 \times 2$ matrices over the rationals and let $u =$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Take $\alpha$ to be the conjugation by $u$. Then $\alpha|_{Z(R)} = 1_{Z(R)}$ but no power of $\alpha$ is the identity.

## 4.3 The Azumaya locus of some skew-Laurent rings

In this section we apply the results of the preceding sections and chapter 3 to some skew-Laurent rings. Throughout this section we retain Notation 4.1.9.

**Proposition 4.3.1** *Assume that $R$ is a commutative Noetherian domain. The ring $\mathcal{S}$ is Azumaya if and only if*

*i) $G$ is finite;*

*ii) $R$ is a $G$-Galois extension of $R^G$.*

**Proof.** Suppose $\mathcal{S}$ as stated above is Azumaya. Then by [76, Proposition 13.7.7] $\mathcal{S}$ is PI and by Proposition 4.2.2, $G$ is finite.

We can write $\mathcal{S}$ as $\mathcal{S} \cong RK_e * H/K_e$ where $H/K_e$ is isomorphic to $G$, a subgroup of $Aut(R)$. By Lemma 3.2.1.5, $RK_e$ is a $G$-Galois extension of $R^G K_e$ if and only if $R$ is a $G$-Galois extension of $R^G$. Now the result follows from Corollary 3.3.10. □

**Example 4.3.2** Let $\mathcal{S} = \mathbb{C}[\theta, \theta^{-1}; \alpha]$ where $\alpha$ is the automorphism of $\mathbb{C}$ defined by $\alpha(z) = \overline{z}$, for all $z \in \mathbb{C}$ as in Example 3.3.8. As $G =< \alpha >$ is of finite order and $\mathbb{C}$ is a $G$-Galois extension of $\mathbb{C}^\alpha = \mathbb{R}$ (by Proposition 3.2.1.3), it follows from Proposition 4.3.1 that $\mathcal{S}$ is Azumaya.

The following proposition describes the Azumaya locus of skew-Laurent rings over commutative Noetherian domains. One can obtain a better description if we assume $R$ not only to be a commutative Noetherian domain but also affine over an algebraically closed field $K$ and $G$ a group of $K$-automorphisms, Corollary 4.3.4.

**Proposition 4.3.3** *Suppose that $R$ is a commutative Noetherian domain, module finite over $R^G$, and that $G$ is finite. Then*

$$\mathcal{A}_S = \{M \in Max(R^G K_e) : I_R(G) \cap R^G \not\subseteq M \cap R^G\}.$$

**Proof.** Let $R$ and $G$ be as above. We can write $\mathcal{S} = RK_e * H/K_e$ where $H/K_e \cong G$ and this group acts faithfully on the commutative Noetherian domain $RK_e$. By Lemma 3.2.11, $Z(S) = R^G K_e$. As $R$ is finitely generated over $R^G$, $RK_e$ is finitely generated over $R^G K_e$. Let $M \in Max(R^G K_e)$. By Proposition 3.4.13, $M \in \mathcal{A}_S$ if and only if $I_{RK_e}(G) \cap R^G K_e \not\subseteq M$. By definition it is easy to see that $I_{RK_e}(G) = I_R(G)RK_e$. Hence $I_{RK_e}(G) \cap R^G K_e \not\subseteq M$ if and only if $I_R(G) \cap R^G \not\subseteq M \cap R^G$ and the equality follows. □

**Corollary 4.3.4** *Suppose that $R$ is a commutative Noetherian domain which is affine over an algebraically closed field $K$, and that $G$ is a finite group of $K$-automorphisms of $R$. Then*

$$\mathcal{A}_S = \{M \in Max(R^G K_e) : M \cap R^G \textit{ is contained in } |G| \textit{ distinct maximal ideals of } R\}.$$

**Proof.** By Noether's Theorem [99, Theorem 2.3.1], $R^G$ is an affine $K$-algebra and $R$ is finitely generated over $R^G$. So $S$ is finitely generated as a module over its center $Z(S) = R^G K_e$, Lemma 3.2.11.

Let $M \in \mathcal{A}_S$. By Proposition 4.3.3, $I_R(G) \cap R^G \not\subseteq M \cap R^G$. As $M$ is a maximal ideal of $R^G K_e$, a commutative affine algebra over an algebraically closed field, by the Nullstellensatz $R^G K_e/M \cong K$. Then $R^G/(M \cap R^G) \cong K$ and $M \cap R^G$ is a maximal ideal of $R^G$. By Proposition 3.3.3, there are maximal ideals of $R$ lying over $M \cap R^G$. Take $\overline{M} \in Max(R)$ such that $\overline{M} \cap R^G = M \cap R^G$. So $I_R(G) \not\subseteq \overline{M}$.

Hence, by Corollary 3.4.7, $G(\overline{M}) = \{1_G\}$. Take $\beta \in G\backslash\{1_G\}$. Then, there is $r \in R$ such that $\beta(r) - r \notin \overline{M}$.

Since $R$ is a commutative affine algebra over an algebraically closed field $K$, by the Nullstellensatz $R/\overline{M} \cong K$. Now write $r = m + r_0$, for some $m \in \overline{M}$ and $r_0 \in K$. Hence $\beta(m) - m \notin \overline{M}$ and $\overline{M}^\beta \neq \overline{M}$ for any $\beta \in G\backslash\{1_G\}$. Hence $\#\{\overline{M}^\alpha : \alpha \in G\} = |G|$.

Conversely assume that $M \in Max(R^G K_e)$ is such that $M \cap R^G$ is contained in $|G|$ distinct maximal ideals of $R$. Let $\overline{M} \in Max(R)$ be such that $\overline{M} \cap R^G = M \cap R^G$. So $\overline{M}^\alpha \neq \overline{M}$ for any $\alpha \in G\backslash\{1_G\}$. So, for $\beta \in G\backslash\{1_G\}$, there is $\overline{m} \in \overline{M}$ such that $\beta(\overline{m}) - \overline{m} \notin \overline{M}$. Hence, for all $\beta \in G\backslash\{1_G\}$, $I_R(\beta) \nsubseteq \overline{M}$. Therefore $I_R(G) \nsubseteq \overline{M}$. Let $N$ be any maximal ideal of $RK_e$ such that $N \cap R^G K_e = M$. As $I_{RK_e}(G) = I_R(G)K_e$ and $N \cap R$ is a maximal ideal of $R$ lying over $M \cap R^G$, we have $I_{RK_e}(G) \nsubseteq N$ and by Proposition 3.4.13 we have that $M \in \mathcal{A}_S$. □

**Corollary 4.3.5** *Suppose that $R$ is a commutative Noetherian domain, affine over an algebraically closed field $K$ and $G$ a finite group of $K$-automorphisms. Then the ring $\mathcal{S}$ is Azumaya if and only if for all $\overline{M} \in Max(R)$, $\#\{\overline{M}^\alpha : \alpha \in G\} = |G|$.*

**Proof.** As in the proof of Corollary 4.3.4, $\mathcal{S}$ is finitely generated as a module over its centre. Hence $\mathcal{S}$ is Azumaya if and only if $\mathcal{A}_S = Max(Z(\mathcal{S})) = Max(R^G K_e)$. So the result is an immediate consequence of Corollary 4.3.4. □

**Proposition 4.3.6** *Suppose that $R$ is a commutative Noetherian domain, $G$ is finite and $R$ is finitely generated over $R^G$. Then $\mathcal{S}$ is height 1 Azumaya if and only if $I_R(G)$ is not contained in any height 1 prime of $R$.*

**Proof.** If $Q$ is a prime ideal of $R$, $QRK_e$ is a prime ideal of $RK_e$. Hence prime ideals of height 1 of $RK_e$ contract in $R$ to prime ideals of height less or equal than 1.

Let $P$ be a prime ideal of $RK_e$ of height 1. Then either the height of $P \cap R$ is zero and in this case $I_R(G) \not\subseteq P$ (by definition we have that $I_R(G) \neq 0$ when $R$ is a domain) or the height of $P \cap R$ is 1 and in this case $P = (P \cap R)RK_e$. So we have that $I_{RK_e}(G) \not\subseteq P$ if and only if $I_R(G) \not\subseteq P \cap R$. The result follows now from Proposition 3.4.18. □

**Proposition 4.3.7** *Suppose $R$ is a commutative Noetherian domain of finite global dimension and $G$ is finite. Assume also that $R$ is finitely generated as a module over $R^G$ and that $I_R(G)$ is not contained in any prime ideal of $R$ of height 1. Then the following sets of maximal ideals of $\mathcal{S}$ are equal*

*i)* $\{M \in Max(\mathcal{S}) : \mathcal{S}_{M \cap Z(\mathcal{S})}$ *is Azumaya*$\}$.

*ii)* $\{M \in Max(\mathcal{S}) : Z(\mathcal{S})$ *is regular at* $M \cap Z(\mathcal{S})\}$.

*The above sets are contained in*

$$\{M \in Max(\mathcal{S}) : I_R(G) \not\subseteq M \cap R\}$$

*and they all coincide when $R$ is a Hilbert ring.*

**Proof.** Suppose $R$, $G$ and $\mathcal{S}$ as above. As $\mathcal{S} \cong RK_e * G$, $Z(\mathcal{S}) = R^G K_e$. As $R$ is finitely generated as a module over $R^G$ and $G$ is finite, $\mathcal{S}$ is finitely generated over its centre. Hence by [37, Theorem 1], $Z(\mathcal{S})$ is Noetherian. So $\mathcal{S}$ is a PI ring integral over $Z(\mathcal{S})$, [76, Lemma 13.8.4], and by [76, Theorem 13.8.14] $Z(\mathcal{S}) \subseteq \mathcal{S}$ satisfy going up and lying over. Whence, if $M \in Max(\mathcal{S})$, $M \cap Z(\mathcal{S})$ is a maximal ideal of $Z(\mathcal{S})$ (by GU).

Since $R$ has finite global dimension, so does the Laurent ring $RK_e$, [76, Theorem 7.5.3]. By Proposition 4.3.6 and Proposition 3.4.21, we have $\mathcal{A}_{\mathcal{S}}$ is the complement in $Max(R^G K_e)$ of the singular locus of $\mathcal{S}$.

Let $M \in Max(\mathcal{S})$. The ring $\mathcal{S}_{M \cap Z(\mathcal{S})}$ is Azumaya if and only if $M \cap Z(\mathcal{S}) \in \mathcal{A}_{\mathcal{S}}$. Hence $\mathcal{S}_{M \cap Z(\mathcal{S})}$ is Azumaya if and only if $Z(\mathcal{S})_{M \cap Z(\mathcal{S})}$ is regular. So the first two sets are equal.

Also, if $M \cap Z(\mathcal{S}) \in \mathcal{A}_{\mathcal{S}}$, by Proposition 4.3.3, $I_R(G) \cap R^G \nsubseteq M \cap R^G$. Hence $I_R(G) \nsubseteq M \cap R$ and

$$\{M \in Max(\mathcal{S}) : \mathcal{S}_{M \cap Z(\mathcal{S})} \text{ is Azumaya}\} \subseteq \{M \in Max(\mathcal{S}) : I_R(G) \nsubseteq M \cap R\}.$$

Assume now that $R$ is a Hilbert ring. Take $M$ a maximal ideal of $\mathcal{S}$ such that $I_R(G) \nsubseteq M \cap R$. As $M$ is a maximal ideal of $\mathcal{S}$, by Proposition 1.3.6 and Lemma 1.3.5, $M \cap R = \cap_{g \in G} P^g$, for some prime ideal $P$ of $R$. So $I_R(G) \nsubseteq P^g$ for some $g \in G$. By Lemma 3.4.10, $I_R(G) \nsubseteq P$. Take $\overline{M}$ any maximal ideal of $RK_e$ such that $\overline{M} \cap R^G K_e = M \cap Z(\mathcal{S})$, so $\overline{M} \cap R^G = M \cap R^G$. As $R$ is a Hilbert ring $\overline{M} \cap R$ is a maximal ideal of $R$, by [63, Theorem 27 and Theorem 30], and $(\overline{M} \cap R) \cap R^G = P \cap R^G$. By Proposition 3.3.1, $\overline{M} \cap R = P^g$, for some $g \in G$. If $I_{RK_e}(G) \subseteq \overline{M}$, then $I_R(G) \subseteq \overline{M} \cap R = P^g$. By Lemma 3.4.10, $I_R(G) \subseteq P$, a contradiction. So $I_{RK_e}(G) \nsubseteq \overline{M}$ for all $\overline{M} \in Max(RK_e)$ such that $\overline{M} \cap R^G K_e = M \cap Z(\mathcal{S})$ and by Proposition 3.4.13, $M \cap Z(\mathcal{S}) \in \mathcal{A}_{\mathcal{S}}$. So the above three sets are all equal. □

## 4.4 Additional remarks

1. The main Proposition of section 1 is Propostion 4.1.12, which extends the results of [32, Proposition 4]. The main references for this section are [29] and [44].

2. Proposition 4.2.2 is the main result of this section and follows the ideas of [32]. The main reference for this section is [92].

3. All the results of section 3 are new.

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