Solutions to the Reflection Equation:

A bijection between lattice configurations and marked shifted tableaux

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Abstract

This thesis relates Young tableaux and marked shifted tableaux with non-intersecting lattice paths. These lattice paths are generated by certain exactly solvable statistical mechanics models, including the vicious and osculating walkers. These models arise from solutions to the Yang-Baxter and Reflection equations. The Yang-Baxter Equation is a consistency condition in integrable systems; the Reflection Equation is a generalisation of the Yang-Baxter equation to systems which have a boundary. We further establish a bijection between two types of marked shifted tableaux.

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Chapter 1

Introduction

1.1 Combinatorics and statistical physics

In the late 20th century, the connection between combinatorics and statistical physics came to light, wherein many models in statistical physics have been used to prove combinatorial results, and vice versa. One of the first major results relating the two fields was Greg Kuperberg's proof of the alternating sign matrix conjecture, based on the Yang-Baxter equation of the six-vertex model, [14].

Alternating sign matrices were first defined in the 1980s by William Mills, David Robbins, Howard Rumsey ([17]), in the context of the six-vertex model with domain wall boundaries. An alternating sign matrix is a generalisation of the permutation matrix; it is a matrix of 0's, 1's, and -1's such that each row and column sums to 1, and the nonzero entries in each row and column alternate between 1 and -1 and begin and end with 1. One such example is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The alternating sign matrix conjecture postulates that the number of $n \times n$ alternating sign matrices is equal to

$$\prod_{j=1}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

In 1996, Kuperberg gave an alternate proof of this conjecture using the Yang-Baxter equation for the six vertex model in [14].

In more recent years, there have been multiple results relating statistical mechanics models and their partition functions to combinatorial objects, many triggered by Kuperberg's result. For a good overview of Kuperberg's result and its wide reaching effects, see Bressoud's book, e.g. [2]. Some such results include the Razumov-Stroganov conjecture between the O(1) loop model, the fully packaged loop model, and alternating sign matrices, which was proved in 2010 by Cantini and Sportiello in [15] using purely combinatorial methods. In [7], Hamel and King showed that the characters of irreducible representations times the deformed Weyl denominators are equal to the partition functions of certain ice models, while in [3], Bump, Brubaker and Friedberg utilised the Yang-Baxter equation to study these models and their relationships with Schur polynomials. Dmitry Ivanov's 2010 thesis, [8] showed that the partition function of a six vertex model which satisfied the Reflection Equation is equal to product of an irreducible character of the symplectic group $Sp(2n, \mathbb{C})$ and a deformation of the Weyl denominator.

In a similar spirit, this thesis reviews and derives new results: bijections between statistical mechanics configurations, specifically non-intersecting lattice paths, and different types of tableaux.

1.2 Outline

This thesis will begin by introducing relevant combinatorial notions, including partitions, Young diagrams, tableaux, and symmetric functions, as well as the notion of 01-words and their relationship with Young diagrams.

Chapter 3 begins by reviewing existing results from [11], which form the starting point of our discussion regarding lattice models. It focuses on the two specific statistical mechanics models, the vicious walker model and the osculating walker model, and relates both models to solutions of the Yang-Baxter equation, while also proving a result relating lattice configurations to Young tableaux and Schur functions.

Chapter 4 introduces another equation, the Reflection Equation, and its solutions. Further, it presents new generalised solutions to both the Yang-Baxter equation and the Reflection Equation, while introducing another statistical mechanics model, which is a slight generalisation of the vicious walker model from Chapter 3.

Chapter 5 contains the main results of this thesis, as well as introducing the notion of marked shifted tableaux. It proves a bijection between two types of marked shifted tableaux, as well as a bijection between specific lattice configurations and marked shifted tableaux.

Chapter 2

Combinatorics

This chapter covers necessary background material in the combinatorics behind partitions and symmetric functions which will be necessary in later chapters of this thesis. This is not meant to be a complete reference; for such we refer the reader e.g. to Macdonald [16] and Fulton [6]. Most what will be presented will be definitions; however we will also cover some theorems without proof. We will also give a number of examples to help familiarise the reader with the topics.

2.1 Partitions

Definition 2.1. A **partition** λ of a non-negative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers in non-increasing order such that

1. There is an $\ell \geq 0$ such that $\lambda_k = 0$ for all $k > \ell$
2. $\sum_i \lambda_i = n$

We write that $|\lambda| = n$ or $\lambda \vdash n$. The **parts** of λ are the non-zero λ_i in λ. The number of parts is called the **length** of λ, which is denoted by $l(\lambda)$.

For convenience, we say that partitions which only differ by a sequence of zeroes at the end are the same. For example, the partitions $(3, 2, 1)$, $(3, 2, 1, 0)$, and $(3, 2, 1, 0, 0 \ldots)$ are the same.

If $|\lambda| = n$, then we say that λ is a partition of n. We denote by $\mathcal{P}_n$ the set of all partitions of n, and by $\mathcal{P}$ the set of all partitions.

Definition 2.2. The **multiplicity** of i in λ is

$$m_i = m_i(\lambda) = \mathrm{Card}\{j : \lambda_j = i\} \tag{2.1}$$

for $i, j \in \mathbb{N}$

Then the notation

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, r^{m_r}, \ldots)$$

gives the partition with exactly m_i parts equal to i.

We have one final definition regarding partitions which will be of importance, both later in this chapter and through the rest of this thesis.

Definition 2.3. A partition $\lambda \vdash n$ is **strict** if all of its parts are distinct, that is if we have

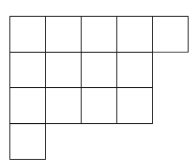
$$\lambda_1 > \lambda_2 > \ldots > \lambda_n.$$

Example 2.4. The partition $\lambda = (5, 4, 2, 1)$ is a strict partition, but $(5, 4, 3, 3, 1)$ is not.

2.2 Diagrams

To each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we associate a **Young diagram**, usually also denoted λ. The Young diagram of a partition λ is obtained by assigning left-justified rows of boxes to each part of λ, with the number of boxes in row i equals λ_i.

Example 2.5. The partition $\lambda = (5, 4, 4, 1)$ has the following Young diagram associated with it:



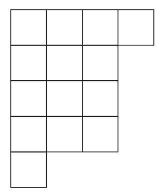
In what follows, when we say 'diagram', we mean 'Young diagram' unless specifically noted otherwise. We will often identify partitions with their diagrams.

Definition 2.6. The **conjugate** of a partition λ is the partition λ' whose diagram is the transpose of the diagram λ; this is the diagram obtained by reflecting across the main diagonal. Hence λ'_i is the number of boxes in the ith column of λ or equivalently

$$\lambda'_i = \text{Card}\{j : \lambda_j \geq i\} \tag{2.2}$$

Thus, $\lambda'_1 = l(\lambda)$ and $\lambda_1 = l(\lambda')$. It is also clear that $(\lambda')' = \lambda$.

Example 2.7. if $\lambda = (5, 4, 4, 1)$, then $\lambda' = (4, 3, 3, 3, 1)$, which has the Young diagram



Combining equations (1.1) and (1.2), we see that

$$m_i(\lambda) = \lambda'_i - \lambda'_{i+1} \tag{2.3}$$

Thus, we see that another way to calculate the multiplicity of i in λ comes from considering its conjugate partition.

2.3 01-words

In this section, we follow [11] and introduce some additional combinatorial concepts.

Consider non-negative integers $N, n, k \in \mathbb{Z}_{\geq 0}$, such that $N = n + k$. We set $I := \{1, \ldots, N\}$. Let $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ be a vector space with inner product $\langle v_i | v_j \rangle = \delta_{ij}, i, j = 0, 1$ which we take to be antilinear in the first term. We may take the tensor product $V^{\otimes N}$, which has the standard basis

$$\mathcal{B} = \{v_{w_1} \otimes \cdots \otimes v_{w_N} : w_i = 0, 1\} \subset V^{\otimes N}.$$

We may identify $\mathcal{B}$ with the set of 01-words of length N,

$$W = \{w = w_1 w_2 \ldots w_N : w_i = 0, 1\}$$

through the following map:

$$w \mapsto b_w := v_{w_1} \otimes \cdots \otimes v_{w_N}. \tag{2.4}$$

Consider the canonical inner product given by

$$\langle b_w | b_{\tilde{w}} \rangle = \prod_{i=1}^{N} \delta_{w_i, \tilde{w}_i}$$

Then we have that the basis $\{b_w\}$ is orthonormal.

Denote by W_n of W the subset which contains all 01-words with n one-letters:

$$W_n = \{w \in W : |w| = \sum_{i=1}^{N} w_i = n\}.$$

Let, $\mathcal{B}_n \subset V^{\otimes N}$ be the image of W_n under the map (2.4) and we denote by $V_n \subset V$ the subspace spanned by the basis elements of $\mathcal{B}_n$. Since the map (2.9) is a bijection it has an inverse map, which we will denote by $w(b)$.

We will now introduce a second description of the elements of $\mathcal{B}_n$, which will be used throughout this thesis. Begin by considering the set of partitions λ whose associated Young diagrams fit into a bounding box which has height n and width k; we will denote such a box by (n, k). Then, define a bijection $(n, k) \to W_n$ via the map

$$\lambda \mapsto w(\lambda) = 0 \cdots \underset{\ell_1}{1} 0 \cdots \underset{\ell_n}{1} 0 \cdots 0, \qquad \ell_i(\lambda) = \lambda_{n+1-i} + 1 \tag{2.5}$$

where $\ell(\lambda) = (\ell_n, \ldots, \ell_n)$ with $1 \le \ell_1 < \ldots < \ell_n \le N$ denote the positions of all the one-letters in the word $w(\lambda)$ from left to right. We assume that $w(\lambda)$ is periodic, i.e we have $\ell_{i+n} = \ell_n + N$. We will further denote the inverse map of (2.5) by $\lambda(w)$, and by b_λ we mean the element $b_{w(\lambda)} \in \mathcal{B}_n$. It is perhaps easier to see the map (2.5) graphically: the Young diagram corresponding to the partition λ traces a path in the $n \times k$ bounding box; this path is encoded in the word w. We see this a follows: starting from the bottom left corner of the bounding box, go one square right for each letter 0 and up one square for each letter 1. Figure 2.1 gives an example of this procedure.

We note also that given the conjugate λ' of $\lambda \in (n, k)$ in the bounding box, we obtain the corresponding 01-word $w(\lambda')$ from $w(\lambda)$ via the map

$$w \mapsto w' = (1 - w_N) \cdots (1 - w_2)(1 - w_1), \tag{2.6}$$

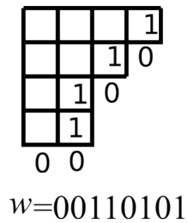


Figure 2.1: For $N = 8$ and $n = k = 4$, we may go from the 01-word 00110101 to the above Young diagram.

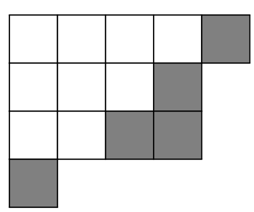
where we have made use of the map (2.10). It is important to note that this map is a bijection $W_n \to W_k$, as all one-letters turn into 0-letters and vice versa; hence we now have k 1-letters. Again we have the corresponding element $b_{w'} \in \mathcal{B}_k$.

2.4 Skew Diagrams and Tableaux

Given two partitions λ, μ, the notation $\mu \subseteq \lambda$ means that the diagram of λ **contains** the diagram of μ, or rather that for all i, we have that $\mu_i \leq \lambda_i$.

Definition 2.8. Given two partitions λ, μ such that $\mu \subseteq \lambda$ then the **skew diagram** $\theta = \lambda/\mu$ (which is also denoted $\theta = \lambda - \mu$) is the set of boxes which are in the Young diagram of λ but not in the Young diagram of μ.

Example 2.9. if we let $\lambda = (5, 4, 4, 1)$ and $\mu = (4, 3, 2)$ then the skew diagram is the region of shaded blocks in the diagram below:



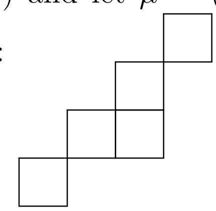
Consider a skew diagram θ containing n boxes. Label the boxes in θ by $x_0, x_1, \ldots x_n$ from right to left starting in the first row, and then the second, and so on. A *path* in θ is a subsequence $x_i, x_i, \ldots, x_{i+j}$ of squares in θ such that x_{i-1} and x_i have a common side, for $i \leq i \leq i+j$. A subset φ of θ is said to be *connected* if any two squares in φ can be connected by a path in φ. The maximal connected subsets of θ are skew diagrams in their own right, and they are called the *connected components* of θ. In the above example, it is clear that there are 3 connected components.

Given a skew diagram $\theta = \lambda/\mu$, let θ' denote its conjugate λ'/μ'. Let $\theta_i = \lambda_i/\mu_i$ and let

$$|\theta| = \sum_i \theta_i = |\lambda|/|\mu|$$

Definition 2.10. A skew diagram θ is a **horizontal m-strip** (resp. **vertical m-strip**) if $|\theta| = m$ and $\theta'_i \leq 1$ (resp. $\theta_i \leq 1|$) for each $i \geq 1$. This means that a horizontal (reps. vertical) strip has at most one square in each column (resp. row).

Example 2.11. Let $\lambda = (4, 3, 3, 1)$ and let $\mu = (3, 2, 1)$. Then the the following shape is the skew diagram $\theta = \lambda/\mu$:



Definition 2.12. We say that a skew diagram θ is called a **border strip** if it is connected and contains no 2×2 block of squares; this means that each successive row or column in θ overlap by no more than 1 square. We say that the **length** of a border strip is $|\theta|$, the total number of boxes it contains, and if a border strip occupies m rows, then its **height** is defined to be $m - 1$.

Given a skew diagram $\theta = \lambda/\mu$ a necessary and sufficient condition for θ to be a horizontal strip is that the two partitions λ and μ are interlaced, i.e.

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \ldots$$

Definition 2.13. Given a partition λ, a **tableau** T is a filling of the squares of the Young diagram of λ with integers $\{1, 2, \ldots\}$ such that the rows and columns are weakly increasing. We say that a tableau T has shape λ. Further, we call a tableau **semistandard** if is weakly increasing across rows, but strictly increasing down columns.

In an identical fashion, we may define **skew tableau** as the filling of the boxes of a skew diagram with the same conditions; a **semistandard skew tableau** is again weakly increasing across rows and strictly down columns.

In other words, we may define T as a map $T : \lambda \to \mathbb{N}$ where $\lambda \subset \mathbb{Z}^2$ is a subset of the integer plane. This is equivalent to considering the tableau to be a sequence of strictly increasing shapes,

$$\emptyset = \lambda_0 \subset \lambda_1 \subset \ldots \lambda_k = \lambda$$

where the skew diagram λ_i/λ_{i-1} is filled with the integer i. By strictly increasing sequence of shapes, we mean a sequence of diagrams, $\lambda_0, \lambda_1, \ldots \lambda_k$ such that the diagram of λ_{i-1} is contained within the diagram of λ_i. Further, we note that the semistandard condition is equivalent to saying that the strictly increasing sequence of shapes above is such that λ_i/λ_{i-1} is a horizontal strip.

Proposition 2.14. Given a sequence of strictly increasing shapes,

$$\emptyset = \lambda_0 \subset \lambda_1 \subset \ldots \lambda_k = \lambda$$

such that λ_i/λ_{i-1} is filled with the integer i, then λ is a semistandard Young tableau.

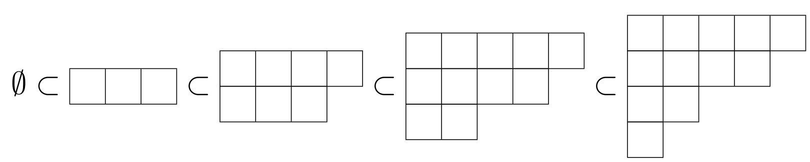
Proof. The skew diagram λ_i/λ_{i-1} is a horizontal strip, filled with the letter i. Because horizontal strips have at most one square in each column, this means that each column may contain at most one i. Further, each skew diagram λ_i/λ_{i-1} may contain only squares in the first i rows. Therefore, there can be no entry i below the i-th row. Thus, columns in λ must be strictly increasing, and so we have a semistandard Young tableau. $\square$

At this juncture, we note for the reader that the definition of tableaux within this thesis differs slightly from the usual definition; in this thesis a tableau is taken to be a sequence of border strips.

Example 2.15. Consider the Young tableau T shown below.

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 2 & 3 & \\ \cline{1-4} 3 & 3 & & & \\ \cline{1-2} 4 & & & & \\ \cline{1-1} \end{array}$$

T corresponds to the following sequence of shapes:



Definition 2.16. The **weight** of a semistandard tableau T is the partition μ such that the part μ_i is equal to the number of times the integer i appears in T. Then, given two partitions, λ and μ, the **Kostka number** $K_{\lambda\mu}$ is the non-negative integer equal to the number of semistandard Young tableaux with shape λ and weight μ.

2.5 Introduction to symmetric functions

Let $\mathbb{Z}[x_1, \ldots, x_n]$ be the ring of polynomials in n independent variables $x_1, \ldots, x_n$. There is a natural action of the symmetric group S_n on elements of this ring, which is to permute the variables. A polynomial is called **symmetric** if it is invariant under this action.

The set of symmetric polynomials forms a subring of $\mathbb{Z}[x_1, \ldots, x_n]$; denoted by

$$\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}.$$

Further, the degree of a polynomial induces a grading of Λ_n which is preserved by the action of S_n:

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$$

where Λ_n^k is the subgroup consisting of all homogeneous symmetric polynomials of degree k, in addition to the zero polynomial.

Definition 2.17. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}$$

Let λ be any partition such that $l(\lambda) \leq n$. Then the **monomial symmetric function of** λ is

$$m_\lambda(x_1, \ldots, x_n) = \sum_\alpha x^\alpha \tag{2.7}$$

where the sum ranges over all distinct permutations α of $\lambda = (\lambda_1, \ldots, \lambda_n)$.

We see that $m_\lambda(x_1, \ldots, x_n) = 0$ if $l(\lambda) > n$, and by definition, the m_λ are symmetric.

Definition 2.18. For each integer $r \geq 0$ the rth **elementary symmetric function** e_r is the sum over square-free monomials in r district variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r} = m_{(1^r)}$$

for $r \geq 1$. The generating function for the e_r is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t).$$

Given a partition λ, we define e_λ to be

$$e_\lambda = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i} \tag{2.8}$$

Similarly, for each $r \geq 0$ the rth **complete symmetric function** h_r is the sum of all monomials of total degree r in the variables $x_1, x_2, \dots$ such that

$$h_r = \sum_{|\lambda|=r} m_\lambda$$

We note that $h_0 = 1$ and $h_1 = e_1$. For simplicity we also define $h_r = e_r = 0$ for $r \leq 0$. The generating function for the h_r is

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

Once again, we say that for a partition λ,

$$h_\lambda = \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i} \tag{2.9}$$

The following theorem is from [16], where its proof may be found.

Theorem 2.19. *The following are $\mathbb{Z}$-bases for the ring Λ_n:*

- $\{m_\lambda(x)\}_{\lambda \in \mathcal{P}_n}$
- $\{e_\lambda(x)\}_{\lambda \in \mathcal{P}_n}$
- $\{h_\lambda(x)\}_{\lambda \in \mathcal{P}_n}$

2.6 Schur polynomials

We will now define another set of symmetric polynomials, which will introduce to us the close relationship between symmetric functions and semistandard tableaux.

Definition 2.20. Given a partition $\lambda \vdash n$, the corresponding **Schur polynomial** s_λ in n variables is given by

$$s_\lambda(x_1, \dots, x_n) = \sum_T x^T, \tag{2.10}$$

where we are summing over all semistandard tableaux T of shape λ with entries in $\{1, \ldots, n\}$. We define x^T to mean

$$x^T = x^{\mathrm{wt}(T)} = x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}.$$

where $\mathrm{wt}(T)$ is the weight of a tableau as in Definition 2.16.

It is not immediately clear that the s_λ are symmetric from this formula.

Proposition 2.21. The Schur polynomials s_λ are symmetric.

Proof. First, note that there are exactly $K_{\lambda\mu}$ tableaux of shape λ such that $\mathrm{wt}(T) = \mu$. Next, since the symmetric group is generated by transpositions, we just need to show that the coefficient of $x^\mu = x_1^\mu \cdots x_i^{\mu_i} x_{i+1}^{\mu_{i+1}} \cdots x_n^{\mu_n}$ is equal to the coefficient of $x_1^{\mu_1} \cdots x_i^{\mu_{i+1}} x_{i+1}^{\mu_i} \cdots x_n^{\mu_n}$. To show this, consider a tableau T of shape λ and weight μ, and look only at the position of i and $i+1$, and set $j = i+1$. If an i and a j are in the same column, ignore them. The remaining i's and j's lie in horizontal strips of the form

i	i	i	j	j

For each such horizontal strip, interchange the number of i's and j's. Then we have

i	i	j	j	j

The total effect on T is that we have switched the total number of i's with the total number of j's. We thus have an involution

$$S_i : T(\lambda) \to T(\lambda)$$

switching the number of i's and $j = (i+1)$'s. Under this involution, a tableau with weight μ and associated monomial $x^\mu = x_1^\mu \cdots x_i^{\mu_i} x_{i+1}^{\mu_{i+1}} \cdots x_n^{\mu_n}$ is mapped to a tableau with weight $\lambda' = (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n)$ and monomial $x^\mu = x_1^\mu \cdots x_i^{\mu_{i+1}} x_{i+1}^{\mu_i} \cdots x_n^{\mu_n}$. This then implies that there are the same number of tableaux T whose weight is a permutation of λ as tableaux with weight λ, and that number is $K_{\lambda\mu}$. Hence we see that we may write

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \tag{2.11}$$

such that $K_{\lambda\mu}$ is the Kostka number defined earlier and m_μ is the monomial symmetric function of the partition μ. In this case we are summing over all possible weights of the Young diagram of shape λ. With this definition, it is clear that the s_λ are symmetric, since they are a sum of the symmetric monomial functions. $\square$

Example 2.22. Consider the partition $\lambda = (2,1)$. We will calculate $s_\lambda(x_1, x_2, x_3)$. The possible Young tableaux are:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

which correspond to the monomials $x_1x_2x_3, x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_3^2, x_2^2x_3$, and $x_2x_3^2$. Summing over these, we get that

$$s_{2,1}(x_1, x_2, x_3) = 2x_1x_2x_3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

We note that $s_k = h_k$, since the Young diagram corresponding $\lambda = k$ is just a row of k boxes. This means that finding all the possible weights of that diagram is the same as finding all partitions of k. Similarly, we see that $s_{1^k} = e_k$, where by 1^k we mean the partition λ which has k parts which are all 1's.
The following two theorems come from [16]; we will omit their proofs.

Theorem 2.23. *The following are equivalent definitions of the Schur polynomials s_λ:*

- $s_\lambda(x_1, \dots, x_n) = \frac{\det(x_j^{\lambda_i + n - i})}{\det(x_j^{n-i})}$
- $s_\lambda(x_1, \dots, x_n) = \det(h_{\lambda_i - i + j})_{1 \le i,j \le \ell(\lambda)}$
- $s_\lambda(x_1, \dots, x_n) = \det(e_{\lambda'_i - i + j})_{1 \le i,j \le \ell(\lambda)}$

Theorem 2.24. *The Schur polynomials $s_\lambda(x_1, \dots, x_n)$ form a $\mathbb{Z}$-basis of the ring Λ_n.*

Chapter 3

Vicious and Osculating Walkers

3.1 Background

In this chapter, we will consider two complementary statistical models, the vicious and osculating walker models of lattice paths. These models were first introduced by Fisher and Brak in [5] and [1], respectively. Korff, in [11], expands upon and generalises these models. Following [11], we will show that each of these models may be associated with a solution of the Yang-Baxter equation, and show that there is a bijection between Young tableaux and lattice paths in both models.

3.2 Vicious and osculating walkers and lattice paths

This section utilises definitions and notions from [11]. We will begin by defining what we mean by a lattice configuration and lattice path, and will define the vicious walker model in terms of its allowed vertices; we will do similarly for the osculating walker model.

Given two integers, $N > 0$ and $0 \leq n \leq N$, consider the square lattice defined by

$$\mathbb{L} = \{\langle i, j\rangle \in \mathbb{Z}^2 | 0 \leq i \leq n+1, 0 \leq j \leq N+1\}.$$

Denote by $\mathbb{E} = \{(p, p') \in \mathbb{L}^2 : p_1 + 1 = p'_1, p_2 = p'_2 \text{ or } p_1 = p'_1, p_2 + 1 = p'_2\}$ the set of horizontal and vertical edges. Then

Definition 3.1. A **lattice configuration** is a map $\Gamma : \mathbb{E} \to \{0, 1\}$ where $\mathbb{E}$ is the set of lattice edges.

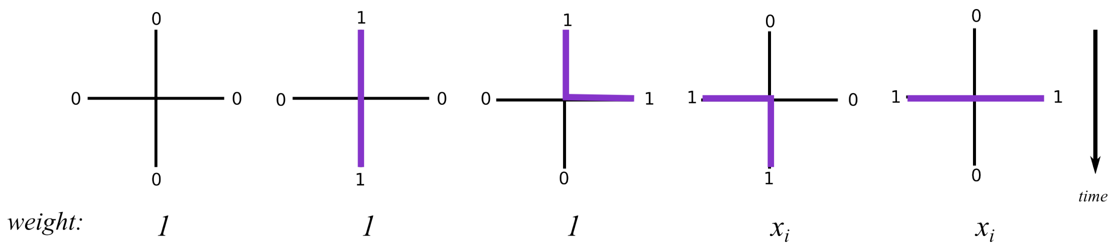


Figure 3.1: The allowed vicious walker vertices and their weights

The weight of a given lattice configuration Γ is given by the product of the weights of its individual vertices,

$$\mathrm{wt}(\Gamma) = \prod_{v \in \Gamma} \mathrm{wt}(v)$$

Each vertex $v \in \Gamma$ can be labeled with indices i, j such that $v_{i,j}$ is the vertex where the ith horizontal lattice line intersects the jth vertical lattice line. We say that a vertex configuration of $v_{i,j} \in \Gamma$ is the 4-tuple $v = (a, b, c, d)$ where $a, b, c, d \in \{0, 1\}$ are the values of the W, N, E, S edges at the lattice point $\langle i, j \rangle$. The five allowed vertex configurations of the vicious walker model are shown in Figure 3.1 along with their weights; all other vertex configurations have weight zero, and thus are disallowed. The weights are given in terms of the commuting variables $(x_1, \ldots, x_n)$, where the weight is x_i in the i-th row of the lattice.

As shown in the above figure, we may draw paths by connecting 1-letters. We see, therefore, that each lattice configuration corresponds to a set of non-intersecting paths. We formally define a path $\gamma = (p_1, \ldots, p_l)$ as a sequence of points $p_r = (i_r, j_r)$ such that either $p_{r+1} = (i_r + 1, j_r)$ or (i_r, j_{r+1}), or rather, it is a connected sequence of horizontal and vertical edges, like those shown in Figure 3.5.

We may define another 5-vertex model, called the osculating walker model, this time on a $k \times N$ lattice with $k = N - n$,

$$\mathbb{L}' := \{\langle i, j \rangle \in \mathbb{Z}^2 : 0 \leq i \leq k+1, 0 \leq j \leq N+1\}. \tag{3.1}$$

In this case, $\mathbb{E}'$ denotes the set of its horizontal and vertical edges. The definitions of lattice configurations are defined analogously to those of the vicious walkers; the five allowed vertices and their weights are shown in Figure 3.2.

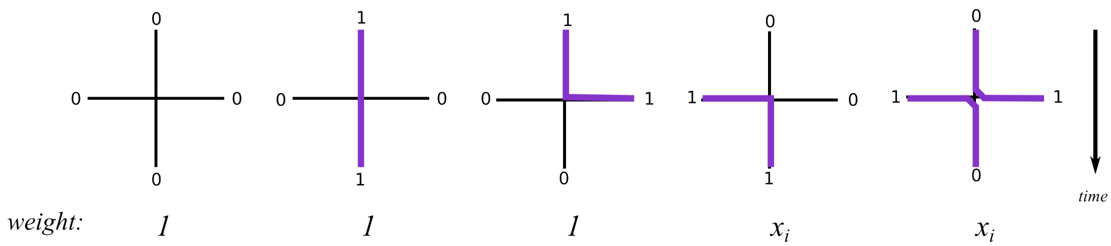


Figure 3.2: The allowed osculating walker vertices and their weights

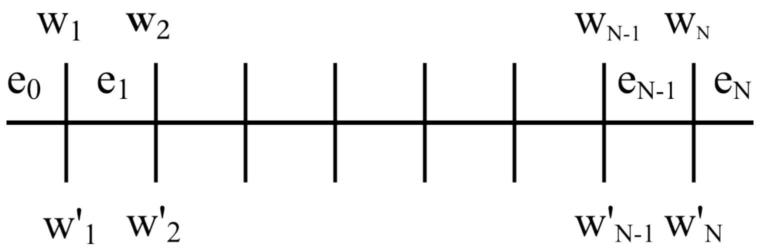


Figure 3.3: The 01-words describing a $1 \times N$ lattice

3.3 Lattice paths and 01-words

We will now take a brief moment to describe how we may consider lattice configurations in terms of 01-words. Again, many of our definitions and notations come from [11]. Let $w = w_1 w_2 \dots w_N$, $w' = w'_1 w'_2 \dots w'_N$, and $e = e_0 e_1 \dots e_N$ be 01-words of length N, N and $N+1$ respectively. Now we will see how these words describe a $1 \times N$ lattice. First, note that such a lattice has exactly $N+1$ horizontal edges, two of which are external (the first and the last) and $N-1$ of which are internal. There are $2N$ vertical edges, N of which are on top and N of which are on the bottom.

As detailed in the previous section, we know that lattice edges through which a path travels are labelled by the letter 1, and edges with no paths are labelled by the letter 0. Let w be the 01-word obtained by considering the labels of all the top edges, and let w' be the 01-word obtained by considering the labels of all the bottom edges. These words are called the top and bottom words, respectively. Similarly, we let e be the 01-word obtained by considering the labels of all the horizontal edges, and we call this the middle word. In case with periodic boundary conditions, note that $e_0 = e_N$. Figure 3.3 depicts this description.

We may extend this description to an $n \times N$ or $(k \times N)$ lattice, as we may consider

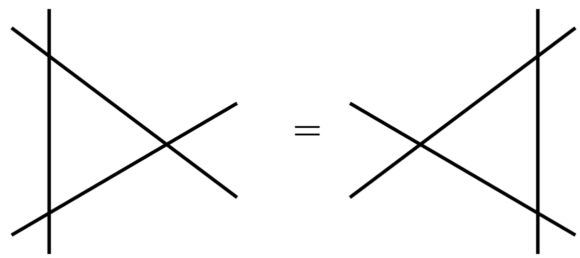


Figure 3.4: The Yang-Baxter equation describes the equivalence of the above situations.

such lattices to be a set of n $1 \times N$ lattice rows. In such cases the top and bottom words describe the top and bottom external lattice edges respectively, and we shall call them the entering and exiting words. We will have n words describing the horizontal edges and $n-1$ words describing the vertical internal edges.

3.4 The Yang-Baxter equation

We now turn our attention in a seemingly unrelated direction, to what is known as the Yang-Baxter equation. We will only briefly introduce it here; a good reference is e.g. [9].

The Yang-Baxter equation, also known as the Star-Triangle equation, first appeared in integrable systems in the 1960s, in [21]. It underpins the integrability, or solvability, of a system of scattering particles. The system described by the Yang-Baxter equation is that of 3-particle scattering with a corresponding 2-particle scattering matrix, R. This is depicted in Figure 3.4. We formally define the Yang-Baxter equation as follows:

Definition 3.2. The **Yang-Baxter equation** is a following matrix equation, involving the scattering matrix R, which describes two different ways of factorising 3-particle scattering:

$$R_{12}(u,v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u,v) \tag{3.2}$$

Mathematically, we describe the Yang-Baxter equation as follows. First, let V be some complex vector space, and let $R(u)$ be a function of the complex variable u. $R(u)$ takes values in $\mathrm{End}_{\mathbb{C}}(V \otimes V)$, and satisfies equation (3.2). By R_{ij} we mean the matrix on $V^{\otimes 3}$ which acts as $R(u)$ on the i-th and j-th position. In other words, we may define three functions, $\phi_{12}, \phi_{13}, \phi_{23} : V \otimes V \to V \otimes V \otimes V$ which act on $a \otimes b$ as follows:

$$\begin{aligned}\phi_{12}(a \otimes b) &= a \otimes b \otimes 1\\ \phi_{13}(a \otimes b) &= a \otimes 1 \otimes b\\ \phi_{23}(a \otimes b) &= 1 \otimes a \otimes b\end{aligned}$$

where 1 is the identity on V. Then we see that $R_{12}(u) = \phi_{12}(R(u)) \in \mathrm{End}(V \otimes V)$ and similarly for R_{13} and R_{23}. We will take $V \cong \mathbb{C}^2 \cong \mathbb{C}v_0 \otimes \mathbb{C}v_1$ for all that follows. This means that we may write that V has a basis consisting of two vectors, v_0 and v_1. Next we will show how solutions of the vicious and osculating walker models arise from solutions of the Yang-Baxter equation.

3.5 Solutions of the Yang-Baxter equation

Begin by defining $\sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to be the Pauli matrices which act on V via the maps $v_0 = \sigma^- v_1$, $v_1 = \sigma^+ v_0$ and $\sigma^z = (-1)^\alpha v_\alpha$, $\alpha = 0, 1$. Now, given a vertex configuration in the i-th row and j-th column, we may interpret it as a map

$$L(x_i) : V_i(x_i) \otimes V_j \to V_i(x_i) \otimes V_j$$

where we have set $V_i(x_i) = V_i \otimes \mathbb{C}(x_i)$ and $V_i \cong V_j \cong V$ for all $\langle i, j\rangle \in \mathbb{L}$. Therefore, the values of the vertical edges label the basis vectors in V_j, while the values of the horizontal edges label the basis vectors in V_i. The mapping which gives us this labelling is from the NW to the SE direction through the vertex. By this, we mean label the values of the edges with $a, b, c, d = 0, 1$ in the clockwise direction starting from the W edge, for vertices such as those in Figures 3.1 and 3.2. Then, let $\mathrm{wt}(v_{ij}) = L_{cd}^{ab}$ be the matrix element of the map L. We set $L_{cd}^{ab} = 0$ whenever a vertex configuration has weight zero, i.e. it is not allowed. Then, for the vicious walkers model, we obtain

$$L(x_i)v_a \otimes v_b = \sum_{c,d=0,1} L_{cd}^{ab} v_c \otimes v_d = x_i^a \left[v_0 \otimes (\sigma^+)v_b + v_1 \otimes \sigma^-(\sigma^+)^a v_b\right] \tag{3.3}$$

Then, given the basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_0\}$ of V^2 we can rewrite this map as

$$L(x_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x_i & 0 \\ 0 & 1 & x_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

In the same fashion, we define a map $L'(x_i)$ for the osculating walkers model:

$$L'(x_i) v_a \otimes v_b = \sum_{c,d=0,1} L'^{ab}_{cd} v_c \otimes v_d = x_i^a \left[v_0 \otimes (\sigma^+)^a v_b + v_1 \otimes (\sigma^+)^a \sigma^- v_b \right] \tag{3.5}$$

which can again be written in the basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ as

$$L'(x_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x_i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_i \end{pmatrix} \tag{3.6}$$

Proposition 3.3. The 5-vertex L-matrix of the vicious walkers model satisfies the Yang-Baxter equation,

$$R_{12}(x,y) L_{13}(x) L_{23}(y) = L_{23}(y) L_{13}(x) R_{12}(x,y), \tag{3.7}$$

where the matrix R is given by

$$R(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y/x & 1 - y/x & 0 \\ 0 & 0 & 0 & y/x \end{pmatrix} \tag{3.8}$$

Similarly for the osculating L-matrix: The 5-vertex L-matrix of the osculating walker model solves the Yang-Baxter equation:

$$R'_{12}(x/y) L'_{13}(x) L'_{23}(y) = L'_{23}(y) L'_{13}(x) R'_{12}(x/y), \tag{3.9}$$

where the matrix R' is given by

$$R'(x/y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - x/y & x/y & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x/y \end{pmatrix}. \tag{3.10}$$

Proof. These are straightforward computations that we leave to the reader. □

3.6 Transfer matrices and partition functions

Now, given that the vicious and osculating models are statistical mechanics models, we would like to calculate the partition functions of each, assuming periodic boundary conditions. For each model, the partition function will be the sum of weighted paths over all possible lattice configurations Γ which have some fixed entering and exiting words. For the vicious walkers, we consider an $n \times N$ lattice, with entering word μ and exiting word ν, which will have the following partition function:

$$\mathcal{Z}_n = \sum_{\Gamma} \prod_{v \in \Gamma} \mathrm{wt}(v).$$

For the osculating walkers, we consider an $k \times N$ lattice, again with entering word μ and exiting word ν, which has the partition function

$$\mathcal{Z}'_k = \sum_{\Gamma} \prod_{v \in \Gamma} \mathrm{wt}'(v).$$

Now, let us focus on the case of vicious walkers. Given an $n \times N$ lattice, we may think of it as n $1 \times N$ lattices, i.e. it is made of of n lattices with 1 row. Therefore, let us define $\mathcal{Z}_{n=1}$ to be the partition function of the $1 \times N$ lattice, i.e. a lattice row with N vertices. Then, under periodic boundary conditions, we have that $\mathcal{Z}_n = (\mathcal{Z}_{n=1})^n$.

Now, for each model, we have the associated L-operators, $L : V \otimes V \to V \otimes V$. The matrix elements of the L-operators are the Boltzmann weights of the vertex configurations. Consider now the space End(V) $\otimes$ End$(V^N) \cong$ End$(V_0 \otimes V_1 \otimes \cdots \otimes V_n)$ where $V_i \cong V$. Let L_{0i} denote the L-matrix acting on the 0-th and i-th subspace. We define the *monodromy matrix* T as follows:

$$T(x) = L_{0N}(x) \cdots L_{02}(x) L_{01}(x) \tag{3.11}$$

This monodromy matrix encodes all the possible weights of N vertices in a row. As such, we may obtain from the matrix elements of $T(x)$ the partition functions for a single lattice row of length N with different boundary conditions; the diagonal elements yield the partition functions of rows with periodic boundary conditions, given entering word μ and exiting word ν. Thus, we may define the transfer matrix $t(x)$ as

$$t(x) = \mathrm{Tr}_0 T(x), \tag{3.12}$$

where we are taking the partial trace over V_0. This implies that $t(x)$ is an element of $\mathrm{End}(V_1 \otimes \cdots \otimes V_N)$, and $t(x)$ is the sum of the diagonal entries of $T(x)$. Hence we may write the partition function as

$$\mathcal{Z}_{n=1} = \langle \nu | t(x) | \mu \rangle$$

where λ and μ are row lattice configurations, and we are summing over every such configuration. Then, we see that the partition function of a lattice with n rows can be written as

$$\mathcal{Z}_n = \langle \nu | t(x_1) t(x_2) \cdots t(x_n) | \mu \rangle. \tag{3.13}$$

We will further explore the monodromy matrix in the following section.

3.7 Monodromy matrix and the Yang-Baxter algebra

Consider once again the L-matrices of the vicious and osculating models, (3.4) and (3.6). We note from equations (3.5) and (3.7) that both matrices may be written in terms of the Pauli matrices. In this case, the matrix elements are polynomials in the variable x_i, which have coefficients in $\mathrm{End}(V)$. Now, we can define a co-product algebra homomorphism $\Delta : \mathrm{End}(V \otimes V) \to \mathrm{End}(V) \otimes \mathrm{End}(V \otimes V)$ by $\Delta(L) = L_{13} L_{12}$.

Proposition 3.4. $\Delta(L)$ also solves the Yang-Baxter equation.

Proof. Consider the left-hand side of the Yang-Baxter equation with $L \mapsto \Delta(L)$.

$$\begin{aligned}
&R_{00'} \Delta(L_0) \Delta(L_{0'}) \\
&= R_{00'} L_{02} L_{01} L_{0'2} L_{0'1} = R_{00'} L_{02} L_{0'2} L_{01} L_{0'1} \\
&= L_{0'2} L_{02} R_{00'} L_{01} L_{0'1} = L_{0'2} L_{02} L_{0'1} L_{01} R_{00'} \\
&= L_{0'2} L_{0'1} L_{02} L_{01} R_{00'} = \Delta(L_{0'}) \Delta(L_0) R_{00'}
\end{aligned}$$

where we have utilised the Yang-Baxter equation for L and the fact that the L-matrices commute if they have no common indices. □

Now consider applying Δ to $L(x)$ N times. This results in the monodromy matrix defined earlier. Clearly, $T(x)$ will also solve the Yang-Baxter equation. Thinking of $T(x)$ as a 2×2 matrix whose entries are themselves $2^N \times 2^N$ matrices, we may rewrite it as follows:

$$T(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \tag{3.14}$$

Similarly for the osculating case, we define

$$T'(x) = L'_{0N}(x) \cdots L'_{02}(x) L'_{01}(x) = \begin{pmatrix} A'(x) & B'(x) \\ C'(x) & D'(x) \end{pmatrix} \tag{3.15}$$

The elements $A(x), B(x), C(x)$ and $D(x)$ in fact generate an algebra, called the Yang-Baxter algebra. The commutation relations between these elements may be computed from the Yang-Baxter equation. We find the following relations for the vicious walker model:

$$\begin{aligned} A(x)A(y) = A(y)A(x), \quad D(x)D(y) = D(y)D(x), \\ B(x)B(x) = \frac{y}{x}B(y)B(x), \quad C(x)C(y) = \frac{x}{y}C(y)C(y), \\ xB(x)A(y) = xB(y)A(x) + (x-y)A(y)B(x), \\ yB(x)D(y) + (x-y)D(x)B(y) = yB(y)D(x) \end{aligned} \tag{3.16}$$

and similar such relations for the osculating walker model.

As was discussed in the previous section, the monodromy matrix encodes all the possible configurations of a $1 \times N$ lattice. Then, considering its entries, we see that $A(x)$ corresponds to all possible lattice configurations in which both the left and right boundary edge are labelled with a 0. Similarly, $B(x)$ corresponds to boundary conditions in which the left boundary edge is a 1, and the right boundary edge is a 0; $C(x)$ corresponds to boundary conditions in which the left boundary edge is a 0 and the right is a 1, and $D(x)$ corresponds to boundary conditions in both the left and right edge are 1. Thus, the transfer matrix $t(x) = A(x) + D(x)$ encodes the partition function of all lattice configurations with periodic boundary conditions.

Given these definitions, we realise that we may think of the matrix entries as operators which add rows to a lattice by acting on 01-words. For instance, if we have an $n-1 \times N$ lattice with exiting word w', and act on it with an A-operator, it is the same as adding a row to the lattice with 0's at both the left and right boundary edge and creating several new $n \times N$ lattices, each with a new exiting word, w''. These new exiting words are created because acting with an A-operator allows paths to propagate through one more row of the lattice. Hence, given n A-operators acting on the 01-word w, we mean all possible $n \times N$ lattices with top word w and all left and right boundary edges labelled 0. We may similarly describe lattices with other boundary conditions as series of B, C or D operators, or a mix of any of the above.

To understand concretely the actions of these operators on 01-words, we first introduce some terminology. For the following calculations we will work with the vicious walker case; we will consider the osculating walker case at the end.

Recall the Pauli matrices $\sigma^+, \sigma^-, \sigma^z$. Let us now denote by σ_i the Pauli matrix acting on the i-th component of $\operatorname{End}(V) \otimes \operatorname{End}(V^{\otimes N})$. Then, define the hopping operators f_i by

$$f_i = \sigma_{i+1}^+ \sigma_i^-, \quad i = 1, \ldots, N-1 \tag{3.17}$$

and

$$f_N = \sigma_1^+ \sigma_N^- \tag{3.18}$$

Now, consider the action of f_i acting on b_w, where w is some 01-word with an 0 in the $(i+1)$-th position, and a 1 in the i-th position. Clearly, f_i acts by "hopping" the 1 from the i-th position to the $(i+1)$-th position, leaving a zero at the i-th spot.

Proposition 3.5. We may express the elements of the Yang-Baxter algebra in terms of the f_i's as follows:

$$\begin{gathered} A(x) = (1 + x f_{N-1}) \cdots (1 + x f_1), \\ B(x) = x A(x) \sigma_1^+, \ C(x) = \sigma_N^- A(x), \quad D(x) = x \sigma_N^- A(x) \sigma_1^+ \end{gathered} \tag{3.19}$$

Proof. We will proceed via induction. The case $N = 1$ is clear; we need only to look at (1.12) to see that these definitions hold. Now consider the case $N = 2$. We have that

$$\begin{aligned} L_{02}(x) L_{01}(x) &= \begin{pmatrix} 1 & x\sigma_2^+ \\ \sigma_2^- & x\sigma_2^- \sigma_2^+ \end{pmatrix} \begin{pmatrix} 1 & x\sigma_1^+ \\ \sigma_1^- & x\sigma_1^- \sigma_1^+ \end{pmatrix} \\ &= \begin{pmatrix} 1 + x\sigma_2^+ \sigma_1^- & x\sigma_1^+ + x^2 \sigma_2 + \sigma_1^- \sigma_1^+ \\ \sigma_2^- + x\sigma_2^- \sigma_1^+ \sigma_1^- & x\sigma_2^- \sigma_1^+ + x^2 \sigma_2^- \sigma_2^+ \sigma_1^- \sigma_1^+ \end{pmatrix} \end{aligned}$$

Once again, the relations (1.19) hold. So considering them true for N, we consider the case $N+1$:

$$\begin{aligned}
T(x) &= L_{0(N+1)}L_{0N}\cdots L_{01} \\
&= \begin{pmatrix} 1 & x\sigma^+_{N+1} \\ \sigma^-_{N+1} & x\sigma^-_{N+1}\sigma^+_{N+1} \end{pmatrix} \begin{pmatrix} (1+xf_{N-1})\cdots(1+xf_1) & x(1+xf_{N-1})\cdots(1+xf_1)\sigma^+_1 \\ \sigma^-_N(1+xf_{N-1})\cdots(1+xf_1) & x\sigma^-_N(1+xf_{N-1})\cdots(1+xf_1)\sigma^+_1 \end{pmatrix}
\end{aligned}$$

Consider the entry for $A(x)$. We have

$$\begin{aligned}
A(x) &= (1+xf_{N-1})\cdots(1+xf_1) + x\sigma^+_{N+1}\sigma^-_N(1+xf_{N-1})\cdots(1+xf_1) \\
&= (1+x\sigma^+_{N+1}\sigma^-_N)(1+xf_{N-1})\cdots(1+xf_1) \\
&= (1+xf_N)(1+xf_{N-1})\cdots(1+xf_1)
\end{aligned}$$

which is just as we wanted. We find find similarly that

$$\begin{aligned}
B(x) &= x(1+x\sigma^+_{N+1}\sigma^-_N)(1+xf_{N-1})\cdots(1+xf_1)\sigma^+_1 \\
C(x) &= \sigma^-_{N+1}(1+x\sigma^+_{N+1}\sigma^-_N)(1+xf_{N-1})\cdots(1+xf_1) \\
D(x) &= \sigma^-_{N+1}(1+x\sigma^+_{N+1}\sigma^-_N)(1+xf_{N-1})\cdots(1+xf_1)\sigma^+_1
\end{aligned}$$

□

Recall from section 2.3 that there is a bijection between 01-words, the basis $\mathcal{B}$ of $V^{\otimes N}$ and Young diagrams. Consider the entering and exiting words associated with a given lattice configuration Γ which is a series of A-operators; let us call them μ and λ. Due to the periodic boundary conditions, the same amount of paths leave the lattice as enter it and because all paths must propagate to the right, it must be true that $\mu \subseteq \lambda$. We see that the action of the A-operators must add boxes to the diagram associated with μ to give us the diagram associated with λ. We explore this idea further with the following lemma and its proof, from [11].

Lemma 3.6. *Let $\mu \in (n,k)$ be a partition, and A_r-operators such that $A(x) = \sum_r x^r A_r$. Then we may write A_r as a polynomial in the f_i:*

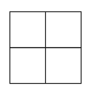
$$A_r = \sum_{\alpha \vdash r} f_{N-1}^{\alpha_{N-1}} \cdots f_1^{\alpha_1} \tag{3.20}$$

where we are summing over all compositions $\alpha = (\alpha_1, \ldots, \alpha_{N-1})$ with $\alpha_i = 0, 1$. Then A_r acts on the basis vector b_μ by adding all possible horizontal r-strips to the Young diagram of μ such that the result λ lies within the $n \times k$ bounding box, $A_r b_\mu = \sum_{\lambda/\mu = r} b_\lambda$.

Proof. As was discussed earlier, $f_i b_\mu = b_\lambda$ if $w_i(\mu) = 1$ and $w_{i+1}(\mu) = 0$; otherwise we have that $f_i b_\mu = 0$. This corresponds to adding a box in the $(i-n)$-th diagonal of the Young diagram of μ. Now suppose there is a consecutive string $f_{i+r'} \cdots f_{i+1} f_i b_\mu$ with $r' \leq r$ and suppose $w_i(\mu) = 1, w_j(\mu) = 0$ for $i < j \leq i + r'$; otherwise the action is trivial. Then the 1-letter at position i in $w(\mu)$ is moved past r' 0-letter whose position each decreases by one. Since $\mu'_{k+1-j} = \ell_j(\mu') + j$ where $N + 1 - \ell_{k+1-j}(\mu')$ are the position of the 0-letters in the word $w(\mu)$, we see that $\lambda'_{k+1-j} - \mu'_{k+1-j} = 1$. Thus, adding a horizontal strip of length r' to μ results in λ. $\square$

Using Lemma 3.6, we may realise the actions of the other elements of the Yang-Baxter algebra. Observe that $\sigma_1^+ b_\lambda = b_{(\lambda_1 - 1, \ldots, \lambda_{n-1})}$ if $\lambda_n \neq 0$ and 0 otherwise. This results in removing the first column from the diagram of λ. Similarly, $\sigma_N^- b_\lambda = b_\mu$ where μ is the diagram obtained from the Young diagram λ by adding a column of maximal height and then subtracting a border strip of length N which starts in the first row.

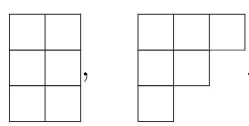
Example 3.7. Consider the case $N = 6, n = 3, k = 3$. Let $\mu = 100110$. Then μ has the following Young diagram:



which lies within a 3×3 bounding box. We can now examine the action of A_2 on b_μ. From (3.20), we know that A_2 is the coefficient of x^2 in the expansion of $A(x)$, and we find that

$$A_2 = f_5 f_4 + f_5 f_3 + f_5 f_2 + f_5 f_1 + f_4 f_3 + f_4 f_2 + f_4 f_1 + f_3 f_2 + f_3 f_1 + f_2 f_1$$

We see that $f_2 f_1 b_\mu = 001110$ and $f_5 f_1 b_\mu = 010101$, which yields the diagrams



All the other terms in A_2 result in 0 when they act upon b_μ, so we are left with only the above diagrams, both of which remain in the 3×3 bounding box,.

Now let us return to the case of osculating walkers.

Proposition 3.8. The matrix elements of (3.16) may be expressed in terms of the f_i of (3.18) and (3.19) as:

$$A'(x) = (1 + x f_1) \cdots (1 + x f_{N-1}), \tag{3.21}$$
$$B'(x) = x \sigma_1^+ A'(x), \;\; C'(x) = A'(x) \sigma_N^-, \;\;\; D(x) = x \sigma_1^+ A'(x) \sigma_N^-$$

Proof. Proceed identically as in Proposition 3.5. In this case make use of the fact that the $\sigma_i\sigma_j = \sigma_j\sigma_i$ as long as $i \neq j$. $\square$

Now, as before, we want to understand the action of $A'(x)$ and the other elements of the Yang-Baxter algebra. We have the following lemma and its proof, again from [11].

Lemma 3.9. *Let $A'(x) = \sum_{r\geq 0} x^r A'_r$. Then the polynomials:*

$$A'_r = \sum_{\alpha\vdash r} f_1^{\alpha_1} \cdots f_{N-1}^{\alpha_{N-1}} \tag{3.22}$$

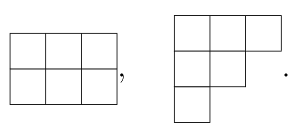
act on b_μ (for a partition $\mu \in (n,k)$) by adding all possible vertical r-strips to the Young diagram of μ such that the resulting diagram λ still lies within the $n \times k$ bounding box, $A'_r b_\mu = \sum_{\lambda/\mu} = (1^r) b_\lambda$.

Proof. We proceed similarly to the case of vicious walkers. Consider $f_i f_{i+1} \cdots f_{i+r} b_\mu = b_\lambda$; this is nontrivial if and only if we have that $w_{i+r'+1}(\mu) = 0$ and $w_j(\mu) = 1$ with $i \leq j \leq i+r', r' \leq r$. Using again the bijection (2.13), we see that $\lambda_{n+1-j} - \mu_{n+1-j} = 1$. Therefore, we obtain λ from μ by adding a vertical strip of height r. $\square$

Example 3.10. Let μ be as in Example 3.9. We have from (3.22) that A_2^{prime} is the coefficient of x^2 in the expansion of $A'(x)$, which yields:

$$A'_2 = f_1f_2 + f_1f_3 + f_1f_4 + f_1f_5 + f_2f_3 + f_2f_4 + f_2f_5 + f_3f_4 + f_3f_5 + f_4f_5.$$

We see that only the terms f_4f_5 and f_1f_5 have non-zero actions on b_μ, and yields the following diagrams:



3.8 A bijection between Young tableaux and lattice configurations

Once again, let us consider an $n \times N$ vicious walker lattice configuration Γ in which all left and right external edges have value zero and which has entering word μ and exiting word λ. With these boundary conditions, we see that it is a product of A-operators. From Lemma 3.6, we see that we may associate with Γ a skew diagram

$\theta = \lambda/\mu$. Note that there are many different lattice configurations which will have the same entering and exiting word, corresponding to different tableaux.

Let $\mathcal{C}$ be the set of $n \times N$ lattice configurations Γ described above. Then the following theorem describes a bijection between these lattice configurations and the set of semistandard skew tableaux θ of shape λ/μ.

Theorem 3.11. *There is a bijection between $\mathcal{C}$ and all semistandard skew tableaux θ of shape λ/μ which fit in an $n \times k$ bounding box.*

Proof. Since the left and right external edges of any $\Gamma \in \mathcal{C}$ are zero, we may consider Γ to be a series of n A-operators acting on the entering word μ. Let us denote by λ_i the exiting word obtained after considering the action of i of the A-operators. Then consider the sequence of diagrams

$$\mu \subset \lambda_1 \subset \cdots \subset \lambda_n = \lambda$$

From Lemma 3.6, we see that λ_i/λ_{i-1} is a horizontal strip. Fill the horizontal strip λ_i/λ_{i-1} with the integer i. From Chapter 2, we know that this results in a semistandard skew tableau, which has shape λ/μ. $\square$

We will now explain how one may actually go from a lattice configurations Γ to a skew tableaux with shape λ/μ. Take the lattice configuration Γ with entering and exiting words λ and μ. Draw the Young diagram associated with μ. Then, consider the lattice configuration row by row, and let λ_i denote the exiting word after the i-th row. Start with the first row and consider λ_1. Add the extra boxes associated with the Young diagram of λ_1 to the diagram of μ and let their entries be 1. Continue in this fashion through all the rows, labelling the boxes added by λ_i with the integer i. Proceeding in this fashion, we label all the boxes in $\theta = \lambda/\mu$.

Example 3.12. Let Γ be the lattice configuration with entering word $\mu = 1100$ and exiting word $\lambda = 0101$ shown in Figure 3.5. Then the sequence of adding labelled boxes to result in a final skew shape is shown as follows:

$$\emptyset \to \begin{array}{|c|}\hline 1 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

We see as an immediate consequence of Theorem 3.11 that if we consider the subset $\mathcal{C}_0$ of $\mathcal{C}$ which consists of those lattice configuration whose entering word μ corresponds to the empty partition, then there is a bijection between $\mathcal{C}_0$ and semistandard Young tableaux of shape λ. Therefore, we may associate with $\mathcal{C}_0$ the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$.

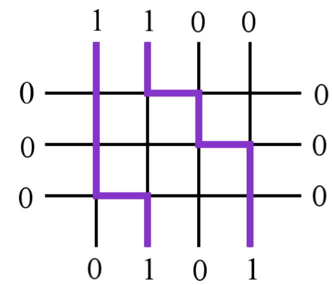


Figure 3.5: The lattice configuration Γ in Example 3.12

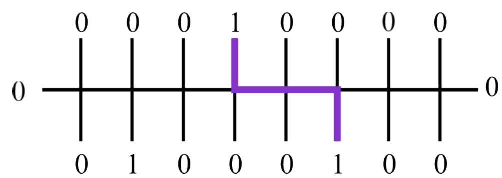


Figure 3.6: A lattice configuration Γ with one path

Proposition 3.13. Given a lattice configuration $\Gamma \in \mathcal{C}_0$, the weight of Γ is x^α where α is the weight of the associated semistandard Young tableau and $x = (x_1, \ldots, x_n)$.

Proof. Recall from proof of Lemma 3.6 that $f_i b_\mu = b_\lambda$ if $w_i(\mu) = 1$ and $w_{i+1}(\mu) = 0$ for 01-words μ and λ. This action corresponds to adding a box in the $(i-n)$-th diagonal of the Young diagram of μ. Let us specify that μ is the 01-word with a 1 in the i-th position and zeros elsewhere. Then λ is the 01-word with a 1 in the $(i+1)$-th position and zeroes elsewhere.

Similarly, acting with $f_{i+s} \ldots f_i$ on b_μ gives an 01-word λ with a 1 in the $(i+s)$-th position and zeroes elsewhere. A $1 \times N$ lattice configuration with these entering an exiting words would have a path come in, travel s edges to the right, and then exit. An example of such a lattice configuration is shown in Figure 3.6. From the proof of Lemma 3.6 we know that such an action corresponds to adding a horizontal strip of length s to the Young diagram of μ. We thus see a correspondence: the length of the horizontal strip added to μ corresponds to the number of horizontal path edges in the $1 \times N$ lattice configuration with entering word μ and exiting word λ.

Now recall from Figure 3.1 the weight of the vicious walker vertices. It is clear that the weight of a $1 \times N$ lattice configuration Γ is x^s where s is the number of horizontal path edges in Γ. Since λ/μ is a horizontal strip with s boxes, the corresponding skew tableau λ/μ will also have weight x^s as each box will be filled with the same letter. This proves the assertion. □

Now, we will consider the case in which the only allowed vertex configurations are those of the osculating walkers. Let Γ be a $k \times N$ lattice configuration in which all left and right external edges are zero and which has entering word μ and exiting word λ. With these boundary conditions, we see that it is a product of A'-operators. Since from Lemma 3.9, we know the action of A' is to add a vertical strip, we see that we may associate with Γ a skew diagram $\theta = \lambda'/\mu'$. Let $\mathcal{C}'$ be the set of lattice configurations Γ with the conditions just described. Then we have the following bijection.

Theorem 3.14. *There is a bijection between $\mathcal{C}'$ and all semistandard skew tableaux θ of shape λ'/μ' which fit in an $n \times k$ bounding box.*

Proof. The proof is virtually identical to that of Theorem 3.11. Note that we may consider Γ to be a product of k A'-operators, and then use the fact that the vertical strip λ_i/λ_{i-1} becomes a horizontal strip upon reflection across the main diagonal. □

We will now demonstrate how one may actually go from the lattice configuration Γ to the skew tableau with shape λ'/μ'.

Take the lattice configuration Γ with entering and exiting words λ and μ. Draw the Young diagram associated with μ and shade in the boxes. Then, consider the lattice configuration row by row, and let λ_i denote the exiting word after the i-th row. Start with the first row and consider λ_1. Add the extra boxes associated with the Young diagram of λ_1 to the diagram of μ and let their entries be 1. Continue in this fashion through all the rows, labelling the boxes added by λ_i with the integer i. Proceeding in this fashion, we label all the boxes in the diagram of λ/μ. Then, take the diagram and reflect across its main diagonal. In this way we get a semistandard skew tableau $\theta = \lambda'/\mu'$.

Example 3.15. Let Γ be the lattice configuration with entering word $\mu = 1100$ and exiting word $\lambda = 0101$ shown in Figure 3.7. Then the sequence of adding labelled boxes and then rotating across the main diagonal is shown below:

$$\emptyset \to \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Corollary 3.16. *If we consider the subset $\mathcal{C}'_0$ of $\mathcal{C}'$ which is those lattice configuration whose entering word μ corresponds to the empty partition, then there is a bijection between $\mathcal{C}'_0$ and semistandard Young tableaux of shape λ'.*

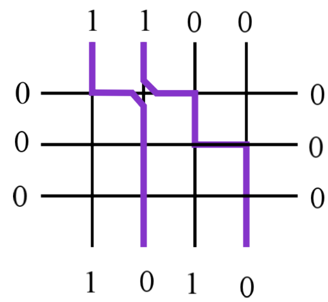


Figure 3.7: The lattice configuration associated with Example 3.15

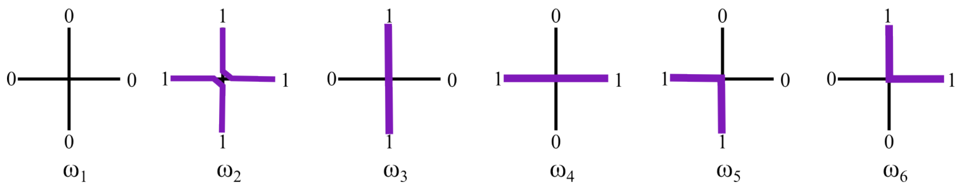


Figure 3.8: The allowed vertices of the six vertex model

3.9 The Six Vertex Model

This section is based upon unpublished results, [10]. The osculating and vicious walkers are actually special cases of another model, called the asymmetric six vertex model. The allowed vertices and their weights are shown in Figure 3.8. The six vertex model has an associated L-matrix,

$$L^{6\mathrm{V}} = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_3 & \omega_5 & 0 \\ 0 & \omega_6 & \omega_4 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix} \tag{3.23}$$

Let us define the weights $\omega_1, \ldots, \omega_6$ subject to complex parameters α and β as follows:

$$\omega_1 = 1, \quad \omega_2 = \alpha x, \quad \omega_3 = 1, \quad \omega_4 = \beta x, \quad \omega_5(\alpha+\beta)x, \quad \omega_6 = 1.$$

Then clearly, when $\alpha = 0$ and $\beta = 1$ we have the vicious walkers, and when $\beta = 0$ and $\alpha = 1$ we have the osculating walkers. The L-matrix associated with this configuration of weights is

$$L''(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (\alpha+\beta)x & 0 \\ 0 & 1 & \beta x & 0 \\ 0 & 0 & 0 & \alpha x \end{pmatrix} \tag{3.24}$$

Proposition 3.17. $L''(x)$ solves the Yang-Baxter equation

$$R''_{12}(x,y)L''_{13}(x)L''_{23}(y) = L''_{23}(y)L''_{13}R''_{12}(x,y), \tag{3.25}$$

where the matrix R'' is given by

$$R''(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha(y-x)}{\alpha y+\beta x} & \frac{(\alpha+\beta)x}{\alpha y+\beta x} & 0 \\ 0 & \frac{(\alpha+\beta)y}{\alpha y+\beta x} & \frac{\beta(x-y)}{\alpha y+\beta x} & 0 \\ 0 & 0 & 0 & \frac{\alpha x+\beta y}{\alpha y+\beta x} \end{pmatrix}. \tag{3.26}$$

Chapter 4

The Reflection Equation

4.1 Introduction to the Reflection Equation

In the case of periodic boundary conditions, the Yang-Baxter Equation (YBE) suffices to establish the integrability of a given model. However, if we consider situations with more general boundary conditions, this ceases to be true; we need an extra condition to ensure integrability. We will now discuss one model which has more general boundary conditions. In the picture of scattering particles which we used to motivate the YBE, this corresponds to introducing an impenetrable boundary. This effectively is equivalent to inserting a wall at position $x = 0$, which results in restricting particles to the half line between negative infinity and $x = 0$. Particles approaching the wall reflect off it. To describe this new process, we must introduce a new 2×2 matrix, called the reflection matrix, which describes the interaction of the particle with the boundary, just as the $R-$matrix describes the interaction of two particles during a scattering event in the bulk.

4.2 The Reflection Equation

As stated in the introductory analogy, when we have a particle undergoing a reflection off a wall, we must introduce a matrix to describe this process of the particle interacting with the wall. We call this reflection matrix $K(x)$. To maintain factorisability of the scattering at the boundary, we must introduce a new condition, in addition to the Yang-Baxter equation. If we want factorisability, the reflection matrix K must satisfy an equation known as the Reflection Equation (RE).

Definition 4.1. The **Reflection Equation**, also known as the boundary Yang-

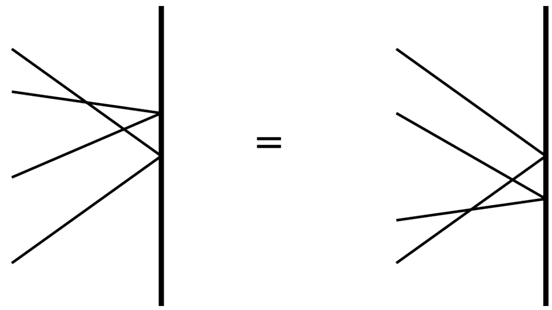


Figure 4.1: The Reflection Equation describes the equivalence of these two processes

Baxter equation is the following matrix equation

$$R_{12}(x/y)K_1(x)R_{21}(xy)K_2(y) = K_2(y)R_{12}(xy)K_1(x)R_{21}(x/y) \tag{4.1}$$

where

$$R : \mathrm{End}(V \otimes V) \to \mathrm{End}(V \otimes V)$$

is also a solution of the Yang-Baxter equation and

$$K(x) : \mathrm{End}(V) \to \mathrm{End}(V)$$

is a matrix describing the process of reflection.

The Reflection Equation describes the factorisability of the diagram in Figure 5.1. It was described by Cherednik in [4] and Skylanin in [19]. See also e.g. [13], [12],.

4.3 Solutions of the Reflection Equation

Now, given an R-matrix which is a solution of the Yang-Baxter equation, we would like to find solutions to the Reflection Equation, i.e. find the matrix $K(x)$. Consider the $R-$matrix given by

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha(1-x)}{\alpha+\beta x} & \frac{(\alpha+\beta)x}{\alpha+\beta x} & 0 \\ 0 & \frac{(\alpha+\beta)}{\alpha+\beta x} & \frac{\beta(x-1)}{\alpha+\beta x} & 0 \\ 0 & 0 & 0 & \frac{\alpha x+\beta}{\alpha+\beta x} \end{pmatrix} \tag{4.2}$$

which is a solution of the Yang-Baxter equation

$$R_{12}(x/y)L_{13}(x)L_{23}(y) = L_{23}(y)L_{13}(x)R_{12}(x/y)$$

for the L-matrix given by

$$L(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (\alpha+\beta)x & 0 \\ 0 & 1 & \beta x & 0 \\ 0 & 0 & 0 & \alpha x \end{pmatrix}. \tag{4.3}$$

We have $R(x) = R_{12}(x)$ since R takes values in $\mathrm{End}(V \otimes V)$. By $R_{21}(x)$ we mean $\mathcal{P}R(x)\mathcal{P}$ where $\mathcal{P}$ is the permutation operator.

$K(x)$ is a function of x which takes values in $\mathrm{End}(V)$, and $K_1(x)$ and $K_2(x)$ take values in $\mathrm{End}(V \otimes V)$, and act as $K(x)$ in the first and second factor, respectively, and trivially in the other. We will only consider diagonal solutions, so let us write $K(x)$ as follows:

$$K(x) = \begin{pmatrix} k(x) & 0 \\ 0 & 1 \end{pmatrix}$$

$K_1(x)$ and $K_2(x)$ are easily calculated, and we will not include them here.

Proposition 4.2. There are four independent solutions for the diagonal reflection matrix.

Proof. Substituting in for the explicit forms of $K_1(x)$, $K_2(x)$, $R_{12}(x)$, and $R_{21}(x)$ into the Reflection equation, we obtain two 4×4 matrices. On the left-hand side, we have the following matrix:

$$\begin{pmatrix} k(x)k(y) & 0 & 0 & 0 \\ 0 & \begin{array}{c} \frac{(-x+y)(-1+xy)\alpha\beta k(x)}{(y\alpha+x\beta)(\alpha+xy\beta)} + \\ \frac{x^2 y(\alpha+\beta)^2}{(y\alpha+x\beta)(\alpha+xy\beta)} \end{array} & \begin{array}{c} k(y)\left(\frac{(-x+y)\alpha(\alpha+\beta)k(x)}{(y\alpha+x\beta)(\alpha+xy\beta)}\right) + \\ k(y)\left(\frac{x(1-\mathrm{xy})\alpha(\alpha+\beta)}{(y\alpha+x\beta)(\alpha+\mathrm{xy}\beta)}\right) \end{array} & 0 \\ 0 & \begin{array}{c} \frac{y(-1+xy)\beta(\alpha+\beta)k(x)}{(y\alpha+x\beta)(\alpha+xy\beta)} + \\ \frac{x(x-y)y\beta(\alpha+\beta)}{(y\alpha+x\beta)(\alpha+xy\beta)} \end{array} & \begin{array}{c} k(y)\left(\frac{y(\alpha+\beta)^2 k(x)}{(y\alpha+x\beta)(\alpha+xy\beta)}\right) + \\ k(y)\left(\frac{(1-\mathrm{xy})(x-y)\alpha\beta}{(y\alpha+x\beta)(\alpha+\mathrm{xy}\beta)}\right) \end{array} & 0 \\ 0 & 0 & 0 & \frac{(\mathrm{xy}\alpha+\beta)(x\alpha+y\beta)}{(y\alpha+x\beta)(\alpha+xy\beta)} \end{pmatrix}$$

On the right-hand side, we have the matrix:

$$\begin{pmatrix} k(x)k(y) & 0 & 0 & 0 \\ 0 & \begin{array}{c}\frac{(x-y)(1-xy)\alpha\beta k(x)}{(y\alpha+x\beta)(\alpha+xy\beta)}+ \\ \frac{x^2 y(\alpha+\beta)^2}{(y\alpha+x\beta)(\alpha+xy\beta)}\end{array} & \begin{array}{c}\frac{y(1-xy)\alpha(\alpha+\beta)k(x)}{(y\alpha+x\beta)(\alpha+xy\beta)}+ \\ \frac{\mathrm{xy}(-x+y)\alpha(\alpha+\beta)}{(y\alpha+x\beta)(\alpha+xy\beta)}\end{array} & 0 \\ 0 & \begin{array}{c}\frac{(x-y)\beta(\alpha+\beta)k(x)k(y)}{(y\alpha+x\beta)(\alpha+xy\beta)}+ \\ \frac{x(-1+\mathrm{xy})\beta(\alpha+\beta)k(y)}{(y\alpha+x\beta)(\alpha+xy\beta)}\end{array} & \begin{array}{c}\frac{y(\alpha+\beta)^2 k(x)k(y)}{(y\alpha+x\beta)(\alpha+xy\beta)}+ \\ \frac{(-1+\mathrm{xy})(-x+y)\alpha\beta k(y)}{(y\alpha+x\beta)(\alpha+xy\beta)}\end{array} & 0 \\ 0 & 0 & 0 & \frac{(y\alpha\mathrm{x}+\beta)(x\alpha+y\beta)}{(y\alpha+x\beta)(\alpha+xy\beta)} \end{pmatrix}$$

Consider the second entry on the third row for both matrices. Equating them and cancelling the common α and β factors, we have the following functional equation:

$$y(xy-1)k(x) + xy(x-y) = (x-y)k(x)k(y) + x(xy-1)k(y)$$

Factorising this we get

$$k(x)k(y) - xy + \left(\frac{xy-1}{x-y}\right)(xk(y) - yk(x)) = 0$$

If we let $y = 0$, we see that

$$k(x)k(0) - \frac{1}{x}xk(0) = 0 \Rightarrow k(x)k(0) = k(0)$$

Thus we have that either $k(0) = 0$ or $k(x) = 1$. If we let $y = 1$, we get

$$k(x)k(1) - x + xk(1) - k(x) = 0 \Rightarrow k(1)(k(x) + x) = k(x) + x$$

And so we see that $k(-1) = 1$ or $k(x) = -x$. Lastly, if we let $y = -1$ we get

$$k(x)k(-1) - xk(-1) - k(x) + x = 0 \Rightarrow k(-1)(k(x) - x) = k(x) - x$$

so we have that $k(x) = x$ or $k(-1) = 1$. The three points $k(1) = 0, k(1) = 1$, and $k(-1) = 1$ define the curve $k(x) = x^2$. Therefore we see that we have four solutions to the functional equation:

$$k_{\mathrm{I}}(x) = 1, \quad k_{\mathrm{II}}(x) = x, \quad k_{\mathrm{III}}(x) = -x, \quad k_{\mathrm{IV}}(x) = x^2 \tag{4.4}$$

These solutions then correspond to four solutions of the reflection equation:

$$K_{\mathrm{I}} = I, \quad K_{\mathrm{II}} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{\mathrm{III}} = \begin{pmatrix} -x & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{\mathrm{IV}} = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}$$

□

4.4 Generalised solutions of the RE and the YBE

Now, we will show how we can generalise solutions of the Reflection Equation. Just as we did in the case of the Yang-Baxter Equation, we will create a monodromy matrix which will also satisfy the Reflection Equation. We do this in several steps, as follows.

In similar spirit as for the monodromy matrix for $L(x)$, we construct a generalised K-matrix that is also a solution of the Reflection equation. Let us define the generalised K-matrix $\mathcal{K}$ as follows:

$$\mathcal{K}(x) = L_N(x) \cdots L_1(x) K(x) L_1^{-1}(x^{-1}) \cdots L_N^{-1}(x^{-1}) \tag{4.5}$$

Note that this may also be written $\mathcal{K}(x) = T(x)K(x)T^{-1}(x^{-1})$ where $T(x)$ is the monodromy matrix associated with $L(x)$.

Proposition 4.3. $\mathcal{K}(x)$ satisfies the Reflection Equation.

Proof. This proof follows [10]. It suffices to consider the case $N = 1$. The general case then follows by repeatedly applying the YBE.

First, we note that by rearranging the YBE, we obtain the following identities:

$$L_{23}^{-1}(y) R_{12}(x/y) L_{13}(x) = L_{12}(x) R_{12}(x/y) L_{23}^{-1}(y) \tag{4.6}$$

and

$$L_{23}(y) R_{21}(xy) L_{13}^{-1}(x^{-1}) = L_{13}^{-1}(x^{-1}) R_{21}(xy) L_{23}(y) \tag{4.7}$$

The second identity is obtained by letting the indices $1 \leftrightarrow 2$ and letting first $x \leftrightarrow y$ and then $x \leftrightarrow x^{-1}$.

Let us consider the left-hand side of the Reflection Equation with $\mathcal{K}(x)$. We begin with:

$$R_{00'}(x/y) L_{01} K_0(x) L_{01}^{-1}(x^{-1}) R_{0'0}(xy) L_{0'1}(y) K_{0'}(y) L_{0'1}^{-1}(y^{-1})$$

We show this is equal to the right-hand side of the Reflection Equation as follows:

$$
\begin{aligned}
&R_{00'}(x/y)L_0K_0(x)L_0^{-1}(x^{-1})R_{0'0}(xy)L_{0'}(y)K_{0'}(y)L_{0'}^{-1}(y^{-1})\\
=&L_{0'}(y)L_{0'}^{-1}(y)R_{00'}(x/y)L_0K_0(x)L_0^{-1}(x^{-1})R_{0'0}(xy)L_{0'}(y)K_{0'}(y)L_{0'}^{-1}(y^{-1})\\
=&L_{0'}(y)L_0(x)R_{00'}(x/y)K_0(x)L_{0'}^{-1}(y)L_0^{-1}(x^{-1})R_{0'0}(xy)L_{0'}K_{0'}L_{0'}^{-1}(y^{-1})\\
=&L_{0'}(y)L_0(x)R_{00'}(x/y)K_0(x)L_{0'}^{-1}(y)L_{0'}(y)R_{0'0}(xy)L_0^{-1}(x^{-1})K_{0'}(y)L_{0'}^{-1}(y^{-1})\\
=&L_{0'}(y)L_0(x)K_{0'}(y)R_{00'}(xy)K_0(x)R_{0'0}(x/y)L_0(x^{-1})L_{0'}(y^{-1})\\
=&L_{0'}(y)K_{0'}(y)L_{0'}^{-1}(y^{-1})L_{0'}(y^{-1})L_0(x)R_{00'}(xy)K_0(x)R_{0'0}(x/y)L_0^{-1}(x^{-1})L_0^{-1}(y^{-1})\\
=&\mathcal{K}_{0'}(y)L_{0'}(y^{-1})L_0(x)R_{00'}(xy)L_{0'}^{-1}(y^{-1})L_{0'}(y^{-1})K_0(x)R_{0'0}(x/y)L_0^{-1}(x^{-1})L_0^{-1}(y^{-1})\\
=&\mathcal{K}_{0'}(y)L_{0'}(y^{-1})L_{0'}^{-1}(y^{-1})R_{00'}(xy)L_0(x)K_0(x)L_{0'}(y^{-1})R_{0'0}(x/y)L_0^{-1}(x^{-1})L_{0'}^{-1}(y^{-1})\\
=&\mathcal{K}_{0'}(y)R_{00'}(xy)L_0(x)K_0(x)L_0(x^{-1})R_{0'0}(x/y)L_{0'}(y^{-1})L_{0'}^{-1}(y^{-1})\\
=&\mathcal{K}_{0'}(y)R_{00'}(xy)\mathcal{K}_0(x)R_{0'0}(x/y)
\end{aligned}
$$

The first and second equalities follow from the insertion of $L_{0'}^{-1}L_{0'}$ and then the use of identity (3.6). The other lines follow similarly. □

So, $\mathcal{K}(x)$ satisfies the reflection equation. As we did with the monodromy matrix $T(x)$, we may write $\mathcal{K}(x)$ as a 2×2 matrix whose entries are operators:

$$\mathcal{K}(x) = \begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{C}(x) & \mathcal{D}(x) \end{pmatrix} \tag{4.8}$$

We would like to express these operators in terms of $T(x)$. Consider $\mathcal{K}(x)$, written in terms of the monodromy matrix $T(x)$ of $L(x)$, $\mathcal{K}(x) = T(x)K(x)T^{-1}(x)$. We know that we may write

$$T(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$$

However we do not have a simple expression for $T^{-1}(x)$. To obtain such an expression, choose the following set of Boltzmann weights:

$$L(x) : \omega_1 = \omega_3 = \omega_6 = 1, \quad \omega_2 = \alpha x, \quad \omega_4 = \beta x, \quad \omega_5 = (\alpha+\beta)x \tag{4.9}$$

which corresponds to the L-matrix (4.3). Then $L(x)$ is a solution to the Yang-Baxter equation

$$R_{12}(x/y)L_{12}(x)L_{23}(y) = L_{23}(y)L_{13}(x)R_{12}(x/y)$$

where $R(x)$ is the matrix (3.2). Now we ask the question: given $L(x)$, can we write $L^{-1}(x)$ in terms of another of $L(x)$ or some other matrix $\tilde{L}$ which corresponds to a different choice of Boltzmann weights?

Proposition 4.4. We may write $L^{-1}(x)$ as follows:

$$L^{-1}(x) = \frac{1}{\alpha x}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{L}(x)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\tilde{L}$ is another L-matrix which corresponds to the Boltzmann weights

$$\tilde{L} : \omega_1 = \omega_6 = 1, \quad \omega_3 = -1, \quad \omega_2 = \beta x, \quad \omega_4 = \alpha x, \quad \omega_5 = (\alpha+\beta)x \tag{4.10}$$

Proof.

$$\begin{aligned} L^{-1}(x) &= \frac{1}{\alpha x}\begin{pmatrix} \alpha x & 0 & 0 & 0 \\ 0 & -\beta x & (\alpha+\beta x) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{\alpha x}\begin{pmatrix} x(\alpha\sigma^-\sigma^+ - \beta\sigma^+\sigma^-) & x(\alpha+\beta)\sigma^+ \\ \sigma^- & -\sigma^z \end{pmatrix} \\ &= \frac{1}{\alpha x}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \sigma^z & x(\alpha+\beta)\sigma^+ \\ \sigma^- & x(\beta\sigma^+\sigma^- - \alpha\sigma^-\sigma^+) \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \frac{1}{\alpha x}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{L}(x)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

It is a straightforward computation to show that $\tilde{L}(x)$ also satisfies the YBE

$$\tilde{R}_{12}(x)\tilde{L}_1(x)\tilde{L}_2(x) = \tilde{L}_2(x)\tilde{L}_1(x)\tilde{R}_{12}(x)$$

where $\tilde{R}(x)$ is given by

$$\tilde{R}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(x-1)\beta}{\alpha x+\beta} & \frac{(\alpha+\beta)x}{\alpha x+\beta} & 0 \\ 0 & \frac{\alpha+\beta}{\alpha x+\beta} & \frac{(1-x)\alpha}{\alpha x+\beta} & 0 \\ 0 & 0 & 0 & \frac{\alpha+\beta x}{\alpha x+\beta} \end{pmatrix}$$

□

Now, we may rewrite $T^{-1}(x)$ in terms of $\tilde{L}(x)$, as follows:

$$\begin{aligned}
T^{-1}(x) &= L_1^{-1}(x)\dots L_N^{-1}(x) \\
&= \frac{1}{\alpha x}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\tilde{L}_1^t(x)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdots\frac{1}{\alpha x}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\tilde{L}_N^t(x)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \\
&= \left(\frac{1}{\alpha x}\right)^N\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\tilde{L}_1^t(x)\cdots\tilde{L}_N^t(x)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \\
&= \left(\frac{1}{\alpha x}\right)^N\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\left(\tilde{L}_N(x)\cdots\tilde{L}_1(x)\right)^t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \\
&= \left(\frac{1}{\alpha x}\right)^N\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\tilde{T}(x)^t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}
\end{aligned}$$

where $\tilde{T}(x)$ is the monodromy matrix associated with $\tilde{L}(x)$. Therefore, we may rewrite $\mathcal{K}(x)$ as follows:

$$\begin{aligned}
\mathcal{K}(x) &= L_N(x)\cdots L_1(x)K(x)L_1^{-1}(x^{-1})\cdots L_n^{-1}(x^{-1}) \\
&= (x/\alpha)^N\begin{pmatrix}A(x) & B(x)\\ C(x) & D(x)\end{pmatrix}K(x)\begin{pmatrix}\tilde{D}(x^{-1}) & -\tilde{B}(x^{-1})\\ -\tilde{C}(x^{-1}) & \tilde{A}(x^{-1})\end{pmatrix} \\
&= \alpha^{-N}\begin{pmatrix}\mathcal{A}(x) & \mathcal{B}(x)\\ \mathcal{C}(x) & \mathcal{D}(x)\end{pmatrix},
\end{aligned} \tag{4.11}$$

where we have used the definitions

$$L_N(x)\cdots L_2(x)L_1(x) = \begin{pmatrix}A(x) & B(x)\\ C(x) & D(x)\end{pmatrix}$$

and

$$\tilde{L}_N(x)\cdots\tilde{L}_2(x)\tilde{L}_1(x) = \begin{pmatrix}\tilde{A}(x) & \tilde{B}(x)\\ \tilde{C}(x) & \tilde{D}(x)\end{pmatrix}$$

To obtain concrete expressions for $\mathcal{A}(x)$, $\mathcal{B}(x)$, $\mathcal{C}(x)$, and $\mathcal{D}(x)$, we return to the second line of (4.11) and substitute in for $K(x)$ with

$$K(x) = \begin{pmatrix}k(x) & 0\\ 0 & 1\end{pmatrix}$$

This yields the following equations for the entries of $\mathcal{K}(x)$:

$$\mathcal{A}(x) = x^N \left(A(x)k(x)\tilde{D}(x^{-1}) - B(x)\tilde{C}(x^{-1}) \right) \tag{4.12}$$

$$\mathcal{B}(x) = x^N \left(B(x)\tilde{A}(x^{-1}) - k(x)A(x)\tilde{B}(x^{-1}) \right) \tag{4.13}$$

$$\mathcal{C}(x) = x^N \left(A(x)C(x)\tilde{D}(x^{-1}) - D(x)\tilde{C}(x^{-1}) \right) \tag{4.14}$$

$$\mathcal{D}(x) = x^N \left(D(x)\tilde{A}(x^{-1}) - k(x)C(x)\tilde{B}(x^{-1}) \right) \tag{4.15}$$

Equations (4.12-15) define the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. $\mathcal{K}(x)$ plays the role in the reflection equation that the monodromy matrix played in the YBE. Then in the same vein as the Yang-Baxter algebra, we see that we have another algebra in $\mathrm{End}(V)^{\otimes N}$ whose elements are the entries of $\mathcal{K}(x)$. From the reflection equation, we find the following commutation relations between these entries

$$\begin{aligned} \mathcal{A}(x)\mathcal{A}(y) &= \mathcal{A}(y)\mathcal{A}(x) \\ \mathcal{B}(x)\mathcal{B}(y) &= \frac{\alpha x + \beta y}{\alpha y + \beta x}\mathcal{B}(y)\mathcal{B}(x) \end{aligned} \tag{4.16}$$

among other relations.

For the rest of this thesis, we will focus on the case where $\alpha = 1$ and $\beta = 0$. This reduces (4.9) and (4.10) to the following Boltzmann weights:

$$L(x) : \omega_1 = \omega_3 = \omega_6 = 1, \quad \omega_2 = \omega_5 = x, \quad \omega_4 = 0$$

and

$$\tilde{L}(x) : \omega_1 = \omega_6 = 1, \quad \omega_3 = -1, \quad \omega_2 = 0, \quad \omega_4 = \omega_5 = x.$$

$L(x)$ is now identical to the osculating walker model, while $\tilde{L}(x)$ is a slightly modified version of the vicious walkers model; it differs only in changing the weight $\omega_3 = 1$ to $\omega_3 = -1$. The allowed vertex configurations are the same; it is only the one weight which differs. We call this new model the generalised vicious walker model. Therefore we may conclude that $\mathcal{A}(x), \mathcal{B}(x), \mathcal{C}(x)$ and $\mathcal{D}(x)$ are products of the operators in the Yang-Baxter algebras of the generalised vicious and osculating walker models.

4.5 The generalised vicious walker model

In this section we will examine in more detail the generalised vicious walker model.

Given the vertex weights of the generalised vicious walker model, we have the following L-matrix:

$$\tilde{L}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & x & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.17}$$

We may rewrite this in terms of the Pauli matrices as

$$\tilde{L}(x) = \begin{pmatrix} \sigma^z & x\sigma^+ \\ \sigma^- & x\sigma^-\sigma^z \end{pmatrix} \tag{4.18}$$

Then the monodromy matrix of this model is $\tilde{T}(x)$, and as before we may write it in terms of a 2×2 matrix whose entries are operators:

$$\tilde{T}(x) = \begin{pmatrix} \tilde{A}(x) & \tilde{B}(x) \\ \tilde{C}(x) & \tilde{D}(x) \end{pmatrix} \tag{4.19}$$

Proposition 4.5. Given an 01-word μ, then the operators $\tilde{A}_r$, where $\tilde{A}(x) = \sum_r x^r A_r$, act on b_μ in the same way as the operators A_r from Chapter 3, up to a sign.

Proof. From the definition of σ^z, we see that $\sigma^z = \sigma^-\sigma^+ - \sigma^+\sigma^-$. Therefore, we see that $\sigma_i^z b_\mu$ is equal to either b_μ or $-b_\mu$. In both cases, the 01-word μ is unchanged. From (4.18) we see that we must have

$$A_r = \sum_{\theta \vdash r} g_\theta(\sigma_r^z, \ldots, \sigma_1^z) f_{N-1}^{\theta_{N-1}} \cdots f_1^{\theta_1} h_\theta(\sigma_r^z, \ldots, \sigma_1^z) \tag{4.20}$$

where g and h are θ-dependent monomials in the σ_i^z. But, we know that $g_\theta b_\mu = \pm b_\mu$ and $h_\theta b_\mu = \pm b_\mu$ and so both functions leave the word μ unchanged. Therefore the only component of $\tilde{A}_r$ which affects the word μ is identical to that of A_r. Hence we see that $\tilde{A}_r b_\mu = \sum_{\lambda/\mu} \pm b_\lambda$, or rather that $\tilde{A}_r$ acts on b_μ by adding all possible horizontal r-strips to the Young diagram of μ such that the resulting diagram λ still lies within the $n \times k$ bounding box. $\square$

In a similar way we may argue that $\tilde{B}(x)$ acts on 01-words in the same way as $B(x)$, up to a sign.

Chapter 5

Bijections on marked shifted tableaux

In this chapter, we will first review the concept of marked shifted tableaux, and then prove some results concerning them. We will follow this with the two main proofs of this thesis, both of which are on properties of generalised versions of marked shifted tableaux.

5.1 Marked shifted tableaux and Schur's Q-functions

In this first section, we introduce the notion of marked shifted tableaux, and an associated set of symmetric functions. Both of these concepts will play an important role in the main proofs of this chapter.

In his 1911 paper [18], Schur introduced a new type of symmetric function, in conjunction with construction of irreducible spin characters of the symmetric group S_n. The functions are now called **Schur's Q-functions**. They are defined as follows. Let $x = (x_1, \ldots, x_n)$ be a set of variables. Then

$$Q(t) = \prod_i \frac{1 + x_i t}{1 - x_i t} = E(t)H(t) = \sum_{r \geq 0} q_r t^r \tag{5.1}$$

for q_r functions of x and where $E(t)$ and $H(t)$ are the generating functions of the elementary and complete symmetric functions. It thus follows that

$$Q(t)Q(-t) = 1$$

or in other words that

$$\sum_{r+s=n} (-1)^r q_r q_s = 0 \tag{5.2}$$

for all integers $n \geq 1$. Now, given integers $r, s \geq 0$, further define

$$Q(r,s) = q_r q_s + 2\sum_{i=1}^{s} (-1)^i q_{r+i} q_{s-i}.$$

It is clear that $Q(r,s) = -Q(s,r)$ from (2.10).

Now let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2m})$ be a strict partition of n. Further, by M_λ denote the skew-symmetric matrix,

$$M_\lambda = (Q(\lambda_i, \lambda_j))_{1 \leq i,j \leq 2m}$$

Then the Schur Q-function Q_λ is given by

$$Q_\lambda = \mathrm{Pf}(M_\lambda)$$

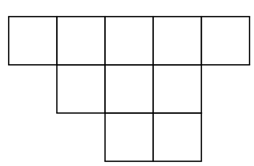
where Pf means the Pfaffian of a matrix, defined by the equality $\mathrm{Pf}(A) = \mathrm{set}(A)^2$ for some skew-symmetric matrix A.

Like the Schur polynomials, it turns out that we may find an alternative presentation of Schur's Q-function in terms of certain types of tableaux, in this case marked shifted tableaux. This definition was put forward by Stembridge in [20].

Definition 5.1. Given a strict partition λ and its associated Young diagram, let $S(\lambda)$ denote the shifted diagram obtained by shifting the i-th row $(i-1)$ squares to the right for $i > 1$.

For example, let $\lambda = (5, 3, 2)$. Then we have that

$S(\lambda) =$



Let P'_n denote the ordered alphabet $\{1' < 1 < 2' < 2 < \ldots < n\}$. The symbols $1', 2', \ldots$ are called *marked*. Given $a \in P'_n$, let $|a|$ denote the unmarked character.

Definition 5.2. A **marked shifted tableau** T of shape $S(\lambda)$ is a map $T : S(\lambda) \to P'_n$ (where $S(\lambda) \subset \mathbb{Z} \times \mathbb{Z}$) such that

- **MST1** The labels increase weakly along each row and down each column, i.e. $T(i,j) \geq T(i+1,j)$ and $T(i,j+1) \geq T(i,j)$
- **MST2** Each column contains at most one k, for each $k \geq 1$
- **MST3** Each row contains at most one k', for each $k \geq 1$

The last two conditions, MST2 and MST3 mean that the set of squares labelled by k form a horizontal strip, and the set of squares labelled by k' form a vertical strip.

For a marked shifted tableau T, its weight is the partition μ such that μ_i is the number of times an i or an i' appears in T. An example of a marked shifted tableau is the following tableau T:

$$T = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 2 & 2 & 3' \\ \hline \multicolumn{1}{c|}{} & 2' & 3' & 3 & \multicolumn{1}{|c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3' & \multicolumn{2}{|c}{} \\ \cline{3-3} \end{array}$$

Theorem 5.3. *Schur's Q-functions can also be expressed in terms of marked shifted tableaux. Let $x = (x_1, \ldots, x_n)$ and let λ be a strict partition, then:*

$$Q_\lambda = \sum_T x^T \tag{5.3}$$

where we are summing over all marked shifted tableaux T of shape λ with entries in P'_n and $x_{i'} = x_i$.

5.2 Some results on marked shifted tableaux

In this section, we prove some results on marked shifted tableaux. All of the results on standard marked shifted tableaux come from [16]; the results on generalised marked shifted tableaux are new.

Consider a marked shifted tableau T. Let $|T|$ denote the tableau in which every marked element in T is replaced by its unmarked version $|a|$.

Lemma 5.4. *No 2×2 block of squares in $|T|$ can bear the same label.*

Proof. Consider a 2×2 block of squares in T, as shown below.

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

Let us assume the squares are labelled by either i or i', i.e. $\{a, b, c, d\} = k, k'$. By MST2, we see that we cannot have both a and c equal to k. Hence we must have one of two situations: either both a and c are k' or $a = k'$ and $c = k$. In both cases, by MST1 and MST3, we must have that $b = k$. Then, since columns are weakly increasing, we must have $d = k$, which contradicts MST2. This proves the assertion. $\square$

Lemma 5.4 means that $|T|$ is composed of a series of strict partitions with entries in P_n

$$\emptyset = \lambda_0 \subset \lambda_1 \subset \cdots \subset \lambda_n = |T|$$

such that λ_k/λ_{k-1} is a disjoint union of border strips whose entries are k. We let $b(T)$ denote the number of border strips in $|T|$. It is clear that for each border strip in $|T|$, there is a corresponding border strip in T whose entries are labelled by k and k', and hence there is a sequence of strict partitions which have entries in P'_n

$$\emptyset = \lambda'_0 \subset \lambda'_1 \subset \cdots \subset \lambda'_n = T$$

where the partition λ'_k is of the same shape as the partition λ_k. Then, we have that $\lambda'_k/\lambda'_{k-1}$ is a disjoint union of border strips whose entries are k and k'.

Lemma 5.5. *Given a border strip $\beta \in \lambda'_k/\lambda'_{k-1}$, then β is a union of horizontal strips whose entries are k, and vertical strips whose entries are k'.*

Proof. By MST2, each column in β contains at most one k, and by MST3, each row in in β contains at most one k'. Hence, given a row in β of length r, it must consist of either a horizontal strip of k's of length r, or the first entry is a k', and the rest of the row is a horizontal strip of k's of length $r - 1$. Similarly, given a column in β of height s, it must consist of either a vertical strip of k' of height s, or the bottom entry is a k, and the rest of the column is a vertical strip of k' of height $s - 1$. The diagram below gives an example of such a border strip.

$$\begin{array}{ccccc} & & & & \boxed{k'} \\ & & & & \boxed{k'} \\ & & & & \boxed{k'} \\ & & & & \boxed{k} \\ \boxed{k'} & \boxed{k} & \boxed{k} & \boxed{k} & \boxed{k'} \end{array}$$

□

The following lemma and its proof follow [16], III.8, page 256.

Lemma 5.6. *Given a border strip $\beta \in T$, the conditions MST1-MST3 uniquely determine the filling of β by k and k', save for the square $(i,j) \in \beta$ nearest the diagonal, or rather the square (i,j) such that $(i+1,j) \notin \beta$ and $(i,j-1) \notin \beta$. This square can either be marked or unmarked, and we call it the* ***free square****.*

Proof. From its definition, the free square must be the square in β that is left-most and bottom-most. Consider the boxes in β which are also in row i, and suppose there are m of them. By MST1 and MST3, all of these boxes must be labelled by k. Now consider row $(i-1)$ Recall from the definition of a border strip that each successive row or column in β overlap by no more than one square. Therefore, there is only one box in beta in row $i-1$ and column $j+m$. This box must be labelled by k', by MST1 and MST2. The rest of the boxes in row $m-1$ must be labelled by k for the same reasons as before. Continuing inductively, we see that every box in β is determined by MST1- MST3. However, the free square may be either k or k', since the box $(i-1,j) \notin \beta$ by the definition of a border strip. □

We see that the number of free squares in T is equal to $b(T)$. Since there are two possible entries for each free square, we see that the number of marked shifted tableaux T which arise from a (non marked) shifted tableau given by $|T|$ is equal to $2^{b(T)}$.

5.3 Generalised marked shifted tableaux

Now, we will define a new type of marked shifted tableaux, which arises when we change the ordering of the primed and unprimed characters. This differs from [16].

Let P_n'' denote the ordered alphabet $\{1 < 1' < 2 \ldots < n < n'\}$. Then, we define a new type of marked shifted tableaux:

Definition 5.7. A **generalised marked shifted tableau** T of shape $S(\lambda)$ is a map $T : S(\lambda) \to P_n''$ (where $S(\lambda) \subset \mathbb{Z} \times \mathbb{Z}$) such that

- **GST1** The labels increase weakly along each row and down each column, i.e. $T(i,j) \geq T(i+1,j)$ and $T(i,j+1) \geq T(i,j)$
- **GST2** Each column contains at most one k, for each $k \geq 1$
- **GST3** Each row contains at most one k', for each $k \geq 1$

GST4 All entries in row i are greater than or equal to i and if $T(i,1) = k$ or k' then $T(i+1,1) \geq (k+1)$

The necessity of the condition GST4 arises due to the fact if we only have conditions GST1-3, then tableaux such as

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$$
$$\begin{array}{cc} & \boxed{1'} \end{array}$$

would be allowed.

Now consider a generalised marked shifted tableau (GST) S'. As before $|S'|$ denotes the shifted tableau obtained from S' when we replace all marked characters a with their unmarked analogue, $|a|$.

Lemma 5.8. *There exists a sequence of partitions*

$$\emptyset = \mu_0 \subset \mu_1 \subset \cdots \subset \mu_n = |S'| \tag{5.4}$$

such that μ_k/μ_{k-1} is a disjoint union of border strips whose entries are k.

Proof. The lemma follows by noting that no 2×2 block of squares in S' may be labelled solely with k and k'. The proof is nearly identical to that of Lemma 5.4. □

Then, as in the case of the standard marked shifted tableaux, we have a sequence of partitions $\emptyset = \mu'_0 \subset \mu'_1 \cdots \subset \mu'_n = \mu'$ such that μ'_k/μ'_{k-1} is a disjoint union of border strips whose entries are k or k', and thus each border strip $\beta \in \mu'_k/\mu'_{k-1}$ is a union of vertical strips whose entries are k' and horizontal strips whose entries are k.

Lemma 5.9. *Given a border strip $\beta' \in S'$, the conditions GST1-GST4 uniquely determine the filling of β' by k and k', other than the square $(i,j) \in S'$ which is furthest from the diagonal, i.e. the square (i,j) such that $(i,j+1) \notin \beta'$ and $(i-1,j) \notin \beta'$. We say that such a square is the free square of S'.*

Proof. Again, this proof is virtually identical to that of Lemma 5.6 and thus will be omitted. □

We now introduce the main theorem of this section.

Theorem 5.10. *There is a bijection between marked shifted tableaux of shape λ whose fillings are in P'_n and generalised marked shifted tableaux of shape λ whose fillings are in P''_n.*

We prove this theorem by establishing a number of smaller claims.

Claim 5.3.1. Consider a standard marked shifted tableau, T. The conditions MST1-3 automatically imply that all entries in the k-th row of T are greater than or equal to k or k'.

Proof. We will show this by induction in k. First, let us assume there is a 1 in the second row. This means that the first box of the second row must contain a 1, since rows are weakly increasing. Since columns must also be weakly increasing, we see that the first and second squares of the first row must contain $1'$. However, this means that the first row would have two instances of $1'$, which violates MST3. Similarly, if we now assume there is a $1'$ in the second row, it again must be in the first position, and we again must have a $1'$ in the second box of the first row, which violates MST3. Hence, there can be no entry in the second row which is less than a $2'$. This proves the case $k = 1$.

Now let us assume that there is an k or an k' in the $(k+1)th$ row. Since rows are weakly increasing, this implies that the k (or k') must be the first entry in the row. Consider the first two boxes in the ith row. We may assume that the first entry is either an k or an k'. If it is otherwise we will have violated our inductive hypothesis by having an entry less than k in the k-th row. The diagram below shows this situation.

$$\begin{array}{|c|c|}\hline x & y \\ \hline \multicolumn{1}{c|}{} & z \\ \cline{2-2}\end{array}$$

We have that $x = k, k'$ and $z = k, k'$. By MST1, we must have that $y \leq z$. However, we must also have that $y \geq x$. This is impossible without violating either MST2 or MST3, and hence we see that there cannot be an i or an k' in the $(k+1)th$ row. □

Claim 5.3.2. Again considering a standard marked shifted tableau, T, the conditions MST1-MST3 imply that if $T(i,1) = k$ or k' then $T(i+1,1) \geq (k+1)'$.

Proof. Let us assume that there is a k or a k' in the first box of the ith row, and the same in the first box of the $(i+1)$th row. In the notation of the diagram above, we have that $x = k, k'$ and $z = k, k'$. Then, because rows must be weakly increasing, and the fact that we cannot have a second k' in the ith row, we see that we must have $y \geq k$. If $y = k$ and $z = k'$ we violate MST1, and if $y = k$ and $z = k$ we violate MST2. If $y > k$ then we violate MST1, and so we see that there cannot be a k or a k' in the first box of the $(i+1)$th row, and so we must have $S(i+1,1) \geq (k+1)'$ □

Now, we describe the bijection.

Let MST(λ) denote all the standard marked shifted tableaux of shape λ. Then let GST(λ) denote all the generalised marked shifted tableaux of shape λ. Furthermore, given $T \in MST(\lambda)$, let β_i be the border strips in $|T|$. For each β_i, let ϵ_i be the label of free square of T. Let $T' \in S(\lambda)$ be a shifted diagram of shape λ. For each of the $\beta_i \in |T|$, consider the same set of squares in T', call them β_i'. In the square furthest from the diagonal, place the label ϵ_i. Then, using conditions GST1-GST4, label the rest of the squares in β_i'. In this way, we label all the squares of T' such that we have $T' \in$ GST(λ). The following example demonstrates this process.

Example 5.11. Let T be a member of MST(3,1):

$$T = \begin{array}{|c|c|c|c|c|} \hline 1' & 2' & 2 & 3' & 3 \\ \hline & 3' & 3 & 3 & \\ \hline & & 4 & & \\ \hline \end{array}$$

Clearly there are four border strips in T. The free square of β_1 is $\epsilon_1 = (1,1)$, containing $1'$, the free square of β_2 is $\epsilon_2 = (1,2)$, containing the $2'$. The free square of β_3 is $(2,1)$, and the free square of β_4 is $(3,1)$. Now, given a blank shifted diagram, consider the squares corresponding to β_1 in T. This is only one square in our example, so we again label $(1,1)$ with ϵ_1. Now consider the squares corresponding to β_2. These are squares $(1,2)$ and $(1,3)$. The square furthest from the diagonal is $(1,3)$ so we label that square with ϵ_2. In the squares corresponding to β_3, the square furthest from the diagonal is $(1,5)$, so we label that square with ϵ_3. Lastly, there is only one square in β_4, $(3,1)$, so we label that square in the blank diagram with ϵ_4. Our blank diagram now looks like the following:

$$\begin{array}{ccccc} 1' & \; & 2' & \; & 3' \\ & \; & \; & \; & \\ & & 4 & & \end{array}$$

Now, using GST1-4, we fill in the empty squares to get

$$T' = \begin{array}{ccccc} 1' & 2 & 2' & 3 & 3' \\ & 3 & 3 & 3' & \\ & & 4 & & \end{array}$$

We see that T' is indeed a GST.

Finally, we come to the proof of Theorem 5.10.

Proof. Let λ be a strict partition. Define $\psi : MST(\lambda) \to GST(\lambda), T \mapsto T'$ to be the map described above. We must show that this is a bijection.

From the description of the map, we see that ψ is a surjection. Since this map generates every possible allocation of the free squares of T', it must also generate every possible associated GST, otherwise we would contradict Lemma 5.9. So we must show that it is injective. To see this, assume that there exist $S, T \in MST(\lambda)$ such that under ψ, both S and T map to $T' \in GST(\lambda)$. Let β_i^T be the border strips in $|T$ and let β_i^S be the border strips in $|S|$. Since $\psi(S) = \psi(T) = T'$, we that we must have $\epsilon_i^T = \epsilon_i^S$, and thus rules MST1-MST3 force us to have $S = T$, and so we see that ψ is indeed injective. Therefore, since ψ is surjective and injective, it is a bijection. $\square$

5.4 A bijection between marked shifted tableaux and lattice configurations

We shall now connect the GST to lattice configurations. In this section, we will introduce and prove one final theorem, which is analogous to the bijection in Chapter 3 between certain lattice configurations with periodic boundary conditions and semi-standard Young tableaux. We will state the theorem, and then prove some relevant claims before proceeding with the proof of the theorem.

In Chapter 3, we saw how we may associate the entries in the monodromy matrix with $1 \times N$ lattice configurations which have specific boundary conditions.

Let us consider the entries of the generalised K-matrix, $\mathcal{K}$. We will focus specifically on the operator $\mathcal{B}(x)$. Recall from Equation (4.12) of Chapter 4 that we have

$$\mathcal{B}(x) = x^N \left(B(x)\tilde{A}(x^{-1}) - k(x)A(x)\tilde{B}(x^{-1}) \right) \tag{5.5}$$

where $\tilde{A}$ and $\tilde{B}$ are entries in the monodromy matrix $\tilde{T}(x)$ from the generalised vicious walkers model, (4.19), and A and B are entries in the monodromy matrix $T'(x)$ of the osculating walkers model, (3.17). Denote by $\mathcal{B}_1(x)$ the operator product $B(x)\tilde{A}(x^{-1})$ and by $\mathcal{B}_2(x)$ the product $A(x)\tilde{B}(x^{-1})$. Then we can write

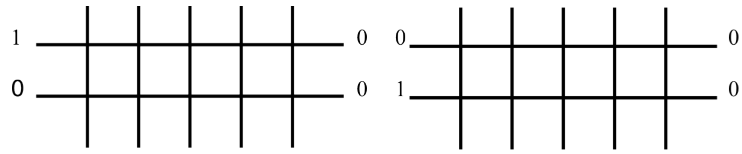


Figure 5.1: The left-hand lattice corresponds to a $\mathcal{B}_2$ term, and the right-hand lattice corresponds to a $\mathcal{B}_1$ term.

$\mathcal{B}(x) = x^N(\mathcal{B}_1(x) - k(x)\mathcal{B}_2(x))$.

Since each term in $\mathcal{B}(x)$ is a product of two A- and B-operators, we see that we can associate with $\mathcal{B}(x)$ a $2 \times N$ lattice which has specific boundary conditions. From the $\mathcal{B}_1$ term, we may associate with $\mathcal{B}(x)$ a $2 \times N$ lattice in which both right external edges are 0, and the first left external edge is a 0, while the second left external edge is a 1. From the $\mathcal{B}_2$ term, we may associate with $\mathcal{B}(x)$ a $2 \times N$ lattice in which the first left external edge is a 1, and the second is a 0, and both right external edges are 0. Figure 5.1 depicts such lattices.

Now consider an $n - 2 \times N$ lattice configuration with exiting word λ, and act on it with a $\mathcal{B}$-operator. This results in two distinct sets of $n \times N$ lattice configurations: those in which the left external edges of the two new rows are a 0 and then a 1, corresponding to the $\mathcal{B}_1$ operator, or a 1 and then a 0, corresponding to the $\mathcal{B}_2$ operator.

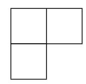
Therefore, to understand the possible exiting words of the new $n \times N$ lattice configurations, we would like to understand how a $\mathcal{B}$-operator acts when applied to an 01-word b_μ of length N with n 1-letters. Remember that the operators $\tilde{A}$ and $\tilde{B}$ come from the generalised vicious walkers model, which differs from the vicious walkers model only in that $\omega_3 = -1$ rather than 1, and that $\tilde{A}(x)$ and $\tilde{B}(x)$ act on the 01-word μ in the same way as the A- and B-operators of the vicious walker models. We would like to examine in more depth how $\tilde{B}(x)$ and $B(x)$ from the osculating walkers actually act on μ.

Lemma 5.12. *$\tilde{B}(x)$ acts on b_μ to give b_λ by adding all possible horizontal strips to the diagram of $(\mu_1 - 1, \ldots, \mu_n - 1)$ such that λ is still in the $(n+1) \times (k-1)$ bounding box.*

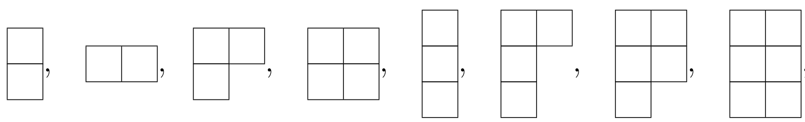
Proof. Recall from Equation (3.21) that we have $B(x) = xA(x)\sigma_1^+$ for the vicious walkers, or if we write $B(x) = \sum_r x^r B_r$ we have that $B_r = xA_r\sigma_1^+$. Further we know that $\sigma_1^+ b_\mu = b_{(\mu_1-1,\ldots,\mu_n-1)}$ or 0 if $\lambda_n = 0$. This is the same as removing the first column from μ, so that we are now in a $(n+1)\times(k-1)$ bounding box. This happens because the σ_1^+ operator adds a 1-letter to the 01-word μ. Then we act with A_r to add all possible horizontal strips of length r to the diagram of $(\mu_1 - 1, \ldots, \mu_n - 1)$ such that the resulting diagram λ is still with the $(n+1)\times(k-1)$ bounding box. $\square$

The following example demonstrates the action of $\tilde{B}(x)$.

Example 5.13. Let $\mu = 01010$. This has the following Young diagram



which lies within a 2×3 bounding box. Acting with $\tilde{B}(x)$ on b_μ yields the following Young diagrams in a 3×2 bounding box:



Now, we look at how $B(x)$ acts, where $B(x)$ is the operator from the osculating walker model.

Lemma 5.14. *$B(x)$ acts on b_μ to give b_λ by adding all possible vertical strips such that b_λ is still in the $(n+1)\times(k-1)$ bounding box.*

This lemma is proved in the same fashion as Lemma 5.12.

Given a $1\times N$ lattice configuration, which has entering word μ and exiting word λ, we know that we may associate Young diagrams with both μ and λ, and we know that λ is obtained from μ by adding a series of boxes, which correspond to the horizontal edges of the lattice configuration. However now we must understand how to draw λ when the left horizontal edge is a 1 and we introduce a path, since μ is a diagram in a $n \times k$ bounding box, and λ is a diagram in the $(n+1)\times(k-1)$ bounding box. We will start by looking at a basic case.

Let us consider a $1 \times N$ lattice Γ, with only vicious vertices allowed. Γ has entering 01-word μ and exiting 01-word λ and is such that the left external edge is a 1 and the right external edge is a 0. This corresponds to one vicious B-operator acting on the word μ to produce the word λ, i.e. $B(x)b_\mu = b_\lambda$. We will now demonstrate how to draw the resulting diagram. First, draw the Young diagram which corresponds

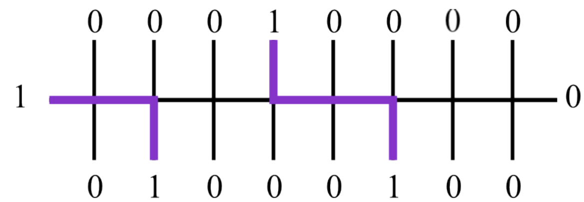
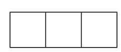


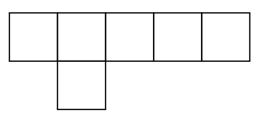
Figure 5.2: The lattice diagram corresponding to Example 1

to μ. Now consider the exiting word λ. Since we are now in an $(n+1)\times(k-1)$ bounding box, we must start one row below the last row of μ, and one box to the right. Starting from this position, go one box right for each letter 0 in λ, and go one box up for each letter 1. In this way we trace the outline of the diagram of λ. Thus in the resulting diagram, ignoring the first column gives the Young diagram of λ. The following example will illustrate this procedure.

Example 5.15. Let $\mu = 00010000$ and $\lambda = 01000100$. This corresponds to the following 1×8 lattice depicted in Figure 5.1. The Young diagram corresponding to μ is the following:



Then, drawing the diagram of λ on top of this diagram gives the following:



where we have added two boxes in the first row, and one to the shifted second row.

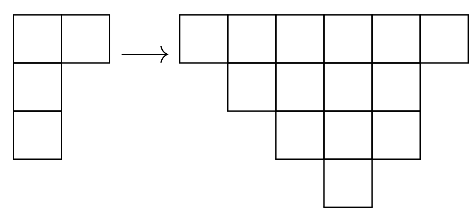
We see therefore that every time we introduce a path to a lattice, or equivalently, add a 1-letter to an 01-word, we must shift the resulting diagram one row down, and one row to the left.

Lemma 5.16. *If we act on the empty word $b_{0\cdots 0}$ of length N with a series of n $\mathcal{B}(x)$-operators to get the word b_λ, then each resulting diagram is a shifted Young diagram.*

Proof. Consider $\mathcal{B}(x_n)\cdots\mathcal{B}(x_1)b_{0\cdots 0}$. Ignoring signs and coefficients, this may be rewritten as $(\mathcal{B}_1(x_n)+\mathcal{B}_2(x_n))\cdots(\mathcal{B}_1(x_1)+\mathcal{B}_2(x_1))\,b_{0\cdots 0}$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are as defined after equation (5.5). Without loss of generality, let us consider the case $\mathcal{B}_2(x_n)\cdots\mathcal{B}_2(x_1)b_{0\cdots 0}$. Acting with $\mathcal{B}_2(x_1)$on $b_{0\cdots 0}$ introduces one 1-letter, so each resulting word b_{λ_1} must have a diagram in a $1\times N-1$ bounding box. Then acting with $\mathcal{B}_2(x_1)$ adds another 1-letter, so the resulting diagram b_{λ_2} must be in a $2 \times N-2$ bounding box. So when drawing λ_2, we must start one row down and column to the left. Continuing in this fashion, we see that the diagram of λ is in the bounding box of dimensions $n \times k$, and so the overall diagram must be a shifted Young diagram. $\square$

First, we will demonstrate how to encode a strict partition in terms of 01-words. Let λ be an 01-word of length N with n 1-letters. Draw the Young diagram associated with λ in the standard fashion. Then, add n boxes to the left of the first row, $n-1$ boxes to the second row, and so on. If there are n rows in the standard Young diagram this process finishes by adding one box to the last row. If the diagram has only m rows, with $m < n$, create the $(m+1)$-th row by adding $n-m$ boxes, starting one box below and one box to the right of the start of the m-th row. Continue in this way until one box is added to create the n-th row. Then denote by λ^s the new strict partition obtained by counting the number of boxes in each row. Clearly $S(\lambda^s)$ gives the shifted diagram drawn. Example 5.13 demonstrates this process.

Example 5.17. Let $\lambda = 101101 = (2, 1, 1)$.



Then in the above diagram, we have the standard Young diagram on the left, and the shifted Young diagram, $S(\lambda)$ on the right. Then $\lambda^s = (6, 4, 3, 1)$.

Let $\mathcal{C}$ be the set of all $2n \times N$ lattice configurations in which the entering word is the empty word and the exiting word is λ, and such that all right external edges are 0, and the left external edges are either 0 or 1, and are such that there are no more than two 1's or two 0's in a row. Further, in the odd rows, allow only the vertices of the generalised vicious walker model, and in the even rows only allow the vertices of the osculating walker model.

Theorem 5.18. *There is a bijection between $\mathcal{C}$ and the set of generalised marked shifted tableaux of shape $S(\lambda^s)$ with entries in P''_n and which have n rows.*

We will prove Theorem 5.18 via a series of smaller claims.

Claim 5.4.1. The set $\mathcal{C}$ is the set of lattice configurations generated by acting with n $\mathcal{B}(x)$-operators on the empty word $b_{0\cdots0}$, $\mathcal{B}(x_n)\mathcal{B}(x_{n-1})\cdots\mathcal{B}(x_1)b_{0\cdots0}$.

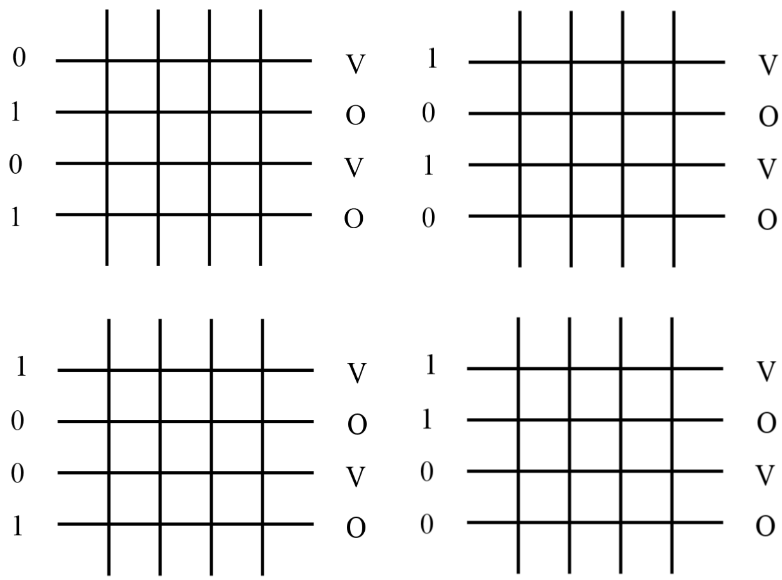


Figure 5.3: These four lattices show the possible combinations of 0's and 1's for any subset of four rows in a given lattice.

Proof. This follows naturally from the definition of $\mathcal{B}(x)$. Writing $\mathcal{B}(x) = \mathcal{B}_1(x) + \mathcal{B}_2(x)$, when we multiply n different $\mathcal{B}(x)$-operators, there are 2^n resulting terms, in conjunction with all the ways of ordering the products of all the $\mathcal{B}_1(x_i)$ and $\mathcal{B}_2(x_i)$. So, any given term in the product $\mathcal{B}(x_n)\cdots\mathcal{B}(x_1)$, when expanding with respect to $x_1, x_2, \ldots, x_n$ is a monomial of the form $\mathcal{B}_{\alpha_n}\mathcal{B}_{\alpha_{n_1}}\cdots\mathcal{B}_{\alpha_1}$ where $\alpha_i = 1, 2$. The n-tuple α determines the left external edges of the lattice configurations generated by the product. Since both $\mathcal{B}_1$ and $\mathcal{B}_2$ are made up of a product of A- and B- operators, the two left outer lattice edges corresponding to either operator must contain both a 0 and a 1. Hence, considering any two adjacent operators in the expansion of $\mathcal{B}(x_n)\cdots\mathcal{B}(x_1)$, the four associated external lattice edges must contain two 0's and two 1's, and so it is never possible to have more than two 0's or 1's in a row on the left outer lattice edges. Further, examining $\mathcal{B}_1$ and $\mathcal{B}_2$, we see in both operators that the $\tilde{B}$ and $\tilde{A}$ always act first, which means that the first row, the third row, and so on can only have allowed vertices from the generalised vicious walker model. Similarly, the second operator in both $\mathcal{B}_1$ and $\mathcal{B}_2$ is either A or B, meaning that the second row, and every even row after that must only allow vertices of the osculating walker model. Figure 5.3 shows the four possible combinations of 0's and 1's for any set of four left external lattice edges, along with their allowed vertices. $\square$

Now we will describe the bijection, mapping a lattice configuration $\Gamma \in \mathcal{C}$ to a gen-

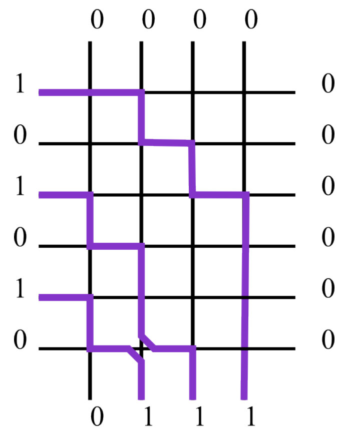


Figure 5.4: The four allowed configurations for left external lattice edges.

eralised marked shifted tableau.

First, label the rows of Γ as follows. The first two rows are labelled 1 and $1'$. The next two rows are labelled 2 and then $2'$. Each subsequent pair of rows, is labelled by k and k' so that all the labels from P_n'' are used in order. Then consider the lattice configuration row by row, and let λ_i denote the exiting word after the row labelled by i, and let λ_i' denote the exiting word after the row labelled by i'. Start with the first row and consider λ_1. Draw the diagram associated with λ_1, and let the entries in those boxes be 1. Then, draw the diagram associated with λ_1' and label the boxes added with $1'$. Continue in this vein through all the rows, labelling the boxes added by λ_i with an i and boxes added by λ_i' with i', remembering to start one box down and one box to the left when a 1-letter is added. Then, add one box to the front of each row. Label the extra box in the first with a 1 if the left external edge in the first row is a 1, otherwise add a $1'$. Similarly, for the box added to the i-th row, consider the i-th pair of rows in the lattice diagram; if the left external edge in the first row of the pair is a 1, label the box with an i, otherwise label it with i'. Example 5.19 shows an example of this bijection.

Example 5.19. Consider the lattice configuration in Figure 5.4. The sequence of adding boxes goes as follows:

$$\emptyset \to \begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 1' \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 1' & 2 \\ \hline\end{array} \to \begin{array}{ccc} 1 & 1' & 2 \\ & 2' & \end{array} \to \begin{array}{ccc} 1 & 1' & 2 \\ & 2' & 3' \\ & & 3' \end{array}$$

Then, adding boxes to the beginning of each row, and labelling them appropriately gives

$$\begin{array}{cccc} 1 & 1 & 1' & 2 \\ & 2 & 2' & 3' \\ & & 3 & 3' \end{array}$$

since every left external edge labelled by 1 is in an odd row.

To understand why we must add a box to each row of the diagram, recall from Chapter 3 how the movement of 1-letters in an 01-word corresponds to the addition of boxes in a Young diagram. In the vicious walker model, when a 1-letter moves through r zeroes, we add a horizontal strip of length r. In the osculating walker model, when a string of r 1-letters, all next to each other, move to the right one box, we add a vertical strip of height r. Further, we have from Chapter 2 that moving the i-th 1-letter one position to the right adds 1 to $\mu_{(n+1-i)}$, where μ is the corresponding partition. This is the same as adding one box to the $(n+1-i)$-th row in the Young diagram of μ. This is the reason for adding one box in each row: when we introduce a new 1-letter into an 01-word, as we do with each $\mathcal{B}$-operator, effectively we have moved the 1-letter from outside the word into the word, or moved it one place to the right, and so we add a box for this action.

Claim 5.4.2. Let $\Gamma \in \mathcal{C}$ be a lattice configuration with exiting word λ and T the GST associated with it. Now, let λ'_i denote the exiting word after the row labelled by i'. Then $S(\lambda'^{s}_i)$, where λ'^{s} is as described above, is exactly the subdiagram of T which contains all letters in P''_n less than or equal to i'.

Proof. Let Γ_i denote the diagram which is identical to Γ, but stops after the row labelled by i'. Then Γ_i has λ'_i as its exiting word, and has the empty word as its entering word, and corresponds to the action of i $\mathcal{B}$-operators on the empty word.

Now, consider the 01-word λ'_i, which has i 1-letters, and let $\ell_m(\lambda'_i)$ denote the position of the m-th 1-letter in λ'_i, for $1 \leq m \leq i$. If γ is the partition associated with λ'_i, then from Equation (2.13) of Chapter 2, we have that $\gamma_{i+1-m} = \ell_m(\lambda'_i) - m$, which counts the number of 0-and 1- letters in front of the m-th 1-letter. If we add m to γ_{i+1-m} to get γ^s_{i+1-m}, then the number of boxes in the $(i+1-m)$-th row of the Young diagram of γ^s is equal to the position of the m-th 1-letter, or $\ell_m(\lambda'_i)$.

Let μ_i denote the diagram drawn after the action of i $\mathcal{B}$-operators. Examine the action of the first $\mathcal{B}$-operator: it acts either as $\mathcal{B}_1(x_1)$ or as $\mathcal{B}_2(x_1)$, and corresponds to the first two rows of Γ_i. In either case, a path is introduced to the lattice, which corresponds to the introduction of a 1-letter to the empty word, so that 1 is the first letter of the word. After the first row, the 1-letter is moved r positions to the right, and after the second row, it has moved at most one position further to the right. When we draw the diagram corresponding to λ'_1 we draw r boxes labelled with 1's, corresponding to the horizontal stripped added by the right row, and then add a vertical strip of height at most one box, labelled by $1'$. Overall we have added either r or $r+1$ boxes to the first row, and the number of boxes corresponds to the number of positions the 1-letter has moved.

Then the action of the second $\mathcal{B}$-operator is to introduce a second 1-letter, and move both letters to the right. If the first 1-letter moves s positions to the right, we add s boxes to the first row, labelled by 2 and $2'$, and if the second 1-letter moves t positions to the right, we start a second row with t boxes one box down and one box to the left of the right row, which will be labelled by 2 and $2'$. After this, the first 1-letter has moved either $s+r$ or $s+r+1$ positions to the right, and we have added $s+r$ or $s+r+1$ boxes to the first row, and the second letter has moved t positions to the right, and there are t boxes in the second row. If we continue in this way, we see that the number of boxes in each row of μ_i corresponds to the total number of spaces each 1-letter has moved to the right. Since each 1-letter initially started in the first position, the number of spaces it has moved to the right is related to its final position. For example, consider the m-th 1-letter; its end position is $\ell_m(\lambda'_i)$, and since it started in the first position, it moved $\ell_m(\lambda'_i) - 1$ spaces to the right, and so the $(i+1-m)$-th row of μ_i has $\ell_m(\lambda'_i)$ boxes in it. Finally, we add one box to each row to account for entering each 1 into the 01-word, and thus the $(i+1-m)$-th row of μ_i has $\ell_m(\lambda'_i)$ boxes.
Therefore, since the length of each row inγ^s is equal to the length of each row in μ_i, we see that the diagram $S(\gamma^s)$ is the same as the diagram μ_i. $\square$

This claim then implies that the shape of the GST T associated with Γ is in fact $S(\lambda^s)$.

Given $\Gamma \in \mathcal{C}$, consider the i-th pair of rows as a lattice diagram in its own right; it will have entering word λ'_{i-1} and exiting word λ'_i and corresponds to the action of either $\mathcal{B}_1(x_i)$ or $\mathcal{B}_2(x_i)$, depending on whether the left external edges are 0 and then 1, or 1 and then 0. Let μ'_{i-1} denote the shifted diagram associated with λ'_{i-1}, and let μ'_i denote the shifted diagram associated with λ'_i.

Claim 5.4.3. The boxes in the diagram of μ'_i/μ'_{i-1} are a disjoint union of border strips whose entries are i and i'.

Proof. Let us first consider the case in which we are acting with $\mathcal{B}_1(x_i)$. Then the lattice diagram from the i-th pair of rows of Γ have left external edges 0 and then 1. The first row corresponds to the action of the operator $\tilde{A}(x_i^{-1})$. We know that this corresponds to the addition of a set of horizontal strips to the diagram of μ'_{i-1} such that the resulting diagram μ_i is still in the $(i-1) \times (N+1-i)$ bounding box. These horizontal strips will be labelled by i. The second row corresponds to the action of the operator $B(x_i)$, which adds a set of vertical strips to μ_i such that the resulting diagram, μ'_i is in a $i \times (N-i)$ bounding box. These vertical strips will be labelled by i'.

We claim that the union of these strips is a disjoint union of border strips. To see this, first note that each horizontal and vertical strip is a border strip in its own right. Second, note that if the top box in a vertical strip is directly next to the left-most box of a horizontal strip, then the union of these two is one border strip. Similarly, if the left-most box of a horizontal strip is next to the bottom-most box in a vertical strip, then the union of the two is one border strip. Lastly we see that there can be no 2×2 set of boxes in the union of these strips, as this would imply that either two horizontal strips overlap, which is not allowed, or two vertical strips overlap, which is not allowed.

Similarly if we consider the case where we are acting with $\mathcal{B}_2(x_i)$, we see that first we act with $\tilde{B}(x_i^{-1})$. The result is adding all possible horizontal strips to the diagram of μ'_{i-1} with the first column removed such that μ_i is in the $i \times (N-i)$ bounding box. Label these horizontal strips by i. Then we act with $A(x)$ to add every possible vertical strip to μ_i such that μ'_i is in the $i \times (N-i)$ bounding box, and we label these strips by i'. Then the union of these horizontal and vertical strips is a union of border strips, as argued above. $\square$

Lastly, when we add a box to each row, we are either extending an existing border strip, or creating a new border strip which contains only one box.

We are now in the position to prove Theorem 5.18.

Proof. Let $\Gamma \in \mathcal{C}$ be a lattice configuration with exiting word λ, and let λ_i denote the exiting word after the action of i $\mathcal{B}$-operators. Then let μ_i be the marked shifted tableau associated with the first $2i$ rows of Γ. Then, by Claims 5.4.1 and 5.4.2 we have a sequence of strict partitions

$$\emptyset \subset \mu_1 \subset \mu_2 \subset \cdots \subset \mu_n = \mu$$

such that μ_i/μ_{i-1} is a disjoint union of border strips labelled by i and i', by Claim 5.4.3. Therefore, μ is a marked shifted tableau of shape $S(\lambda^s)$.

$\square$

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