Twistor Transform Of Symmetries

by

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Contents

Statement

Chapter 1 consists of known results, stated for use in the rest of the thesis, except Chapter 1, Section (1.5.2) which is my own work.

Chapter 2 consists of necessary background material derived mainly from Merkulov's paper [24].

Chapter 3 is my own work, except where the text indicates otherwise.

[redacted]

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Abstract

The twistor transform introduced by Penrose's fundamental articles ([28],[40],[19],[12]) encodes basic geometric and mathematical physics structures into holomorphic geometry. Differential equations get replaced by complex manifolds and holomorphic bundles over them with well defined properties. Hence direct constructions of such objects lead to constructions of various classes of solutions to basic equations of differential geometry and mathematical physics.

The first successes of the twistor transformation method were associated with the self-dual Einstein and Yang-Mills equations ([29],[36],[3],[1],[2], [10]). Non-Self-dual Yang-Mills equations have been studied by Witten and Manin ([39],[14] and references therein). Deep interconnections between twistor theory and non-linear integrable equations have been unveiled by Mason, Singer and Sparling [23], [21] and [22]. More recently, twistor methods have been successfully used in the study of quaternionic Kahler and hyper-Kahler manifolds [4], [13], [17], [18], [25], [27], [32], [33] and [34].

In 1997, Merkulov [24] has developed the twistor theory for general irreducible G-structures which was applied in 1999 by him and Schwachhofer to solve the long-standing holonomy problem [26].

The main theme of our work is the study of symmetries of G-structures in the twistor theory context. The main result, Theorem (3.14), provides us with a surprisingly simple characterisation of Killing vector fields. This theorem establishes a one-to-one correspondence between Killing symmetry vectors and global sections of the dual contact line bundle on the associated twistor contact manifold (see Theorem (3.14) for a precise statement).

This thesis is organised as follows. In the first chapter we provide a short introduction into the theory of G-structures, and explain our notation. The material is classical except Section (1.5) where we give a new characterisation of Killing vector fields.

In Chapter 2 we explain Merkulov's work which associates to any irreducible G-structure a contact complex manifold (twistor space) and vice versa (via deformation theory). This mathematical set-up is one of the basic requirements of our study.

Chapter 3 is the main part of our thesis where we prove our main Theorems. We start with the special case of conformal structures whose twistor (ambitwistor) spaces were understood long ago. The twistor characterisation of conformal Killing vectors is given by Theorem (3.6). Next we switch to the general case and prove in Section (3.4) the main Theorem (3.14). Finally, we apply this theorem to get a new twistor description of symmetries on quaternionic manifolds (Theorem (3.17)),

Chapter 1

G-Structures.

1.1 Basic notions.

Before we begin our discussion on G-structures, we shall firstly introduce some of the essential background material that is used throughout. Note that, unless stated otherwise, summation from 1 to n takes place over repeated indices.

1.1.1 Fundamental definitions.

In this section we give an introduction to some of the fundamental tools used in differential geometry.

Manifolds.

A Hausdorff topological space M is called a (topological) manifold if every point $x \in M$ has an open neighbourhood U and a homeomorphism $\phi : U \to V \subset \mathbb{R}^n$. The number n is called the dimension of M.

The pair (U, ϕ) is called a chart of M at x. U is called the domain of the chart (U, ϕ).

The coordinates $(x^1, \ldots, x^n)$ of the image $\phi(x) \in \mathbb{R}^n$ are called the coordinates of x in the chart (U, ϕ) or the local coordinates of x. The chart (U, ϕ) itself is often called a local coordinate system at x.

Vector fields and tangent bundles.

Let M be a smooth manifold of dimension n and let x be an arbitrary point in M. Let $\mathcal{O}_x$ be the set of all smooth functions at x, then a tangent vector, X_x, at x is a linear map

$$\mathcal{O}_x \longrightarrow \mathbb{R}$$

$$f(x) \longrightarrow X_x(f)$$

such that

1. X_x is linear,
2. $X_x(fg) = f(x)X_x(g) + g(x)X_x(f)$, where $f(x), g(x) \in \mathbb{R}$ are values of $f, g \in \mathcal{O}_x$ at x respectively. This condition is known as the Leibniz rule.

In a local coordinate chart, any tangent vector, X_x, can be identified with the first order differential operator $X^\alpha \frac{\partial}{\partial x^\alpha}$, where $X^\alpha = X_x(x^\alpha)$

Let T_xM denote the set of all possible tangent vectors of x. T_xM is a linear vector space of dimension n and is called the tangent space at $x \in M$. The dual of T_xM, denoted by Ω^1_xM, is called the cotangent space at $x \in M$.

Define the set $TM = \cup T_xM$. TM is called the tangent bundle of M. It can easily be shown that TM is a smooth manifold of dimension $2n$. Similarly, $\Omega^1 M = \cup\Omega^1_x M$ is called the cotangent bundle of M. $\Omega^1 M$ is also a $2n$-dimensional manifold, however the cotangent bundle is also a symplectic manifold (see Section (2.1)).

A vector field X on M is a smooth map

$$X : M \rightarrow TM$$

such that

$$\pi \circ X = Id.$$

Thus for any $x \in M$, the value of the map at the point x is denoted by $X_x = X(x) \in T_xM = \pi^{-1}(x)$. Since the map $x : M \rightarrow TM$ is smooth, in a local coordinate chart with coordinates (x^α), X is represented as a first-order differential operator

$$X = X^\alpha(x)\frac{\partial}{\partial x^\alpha}$$

where $X^\alpha(x)$ are smooth functions.

The set of all smooth vector fields on M is denoted by $\Gamma(TM)$. $(\Gamma(TM), [,])$ is a Lie algebra under the Lie bracket defined by

$$[X,Y] = XY - YX,$$

where $X, Y \in \Gamma(TM)$.

Note that there is a one-to-one correspondence between $\Gamma(TM)$ and the set of linear operators

$$X : \mathcal{O}_M \to \mathcal{O}_M$$

$$f \mapsto Xf$$

such that

1. X is linear

2. $X(fg) = fX(g) + gX(f)$ for all $f, g \in \mathcal{O}_M$.

Differential of a map.

Let M and N be differentiable manifolds of dimension m and n respectively and let $\mathcal{O}_x$ be the set of all smooth functions at $x \in M$.

Clearly a map $\phi : M \to N$ induces the map

$$\phi^* : \mathcal{O}_{\phi(x)} \to \mathcal{O}_x$$

Indeed, take any $f \in \mathcal{O}_{\phi(x)}$ in a local coordinate chart $(y_1, \ldots, y_m)$ is represented by a smooth function of m variables. Then $\phi^*(f)$ is a smooth function of n variables given by

$$\phi^*(f) = f(y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n)),$$

where $(x_1, \ldots, x_n)$ are any local coordinates at $x \in M$.

The map $\phi : M \to N$ also defines the linear map

$$\phi_* : T_xM \to T_{\phi(x)}M$$

which is called the Jacobian or the differential of ϕ at $x \in M$. The map is defined as follows: Given $X_x \in T_xM$, define $\phi_*(X_x) \in T_{\phi(x)}N$ as $\phi_*(X_x)(f) = X_x(\phi^*(f))$ for some $f \in \mathcal{O}_{\phi(x)}$.

The map $\phi : M \to N$ is called a diffeomorphism at $x \in M$ if $\phi_* : T_xM \to T_{\phi(x)}N$ is a linear isomorphism and the map $\phi : M \to N$ is called a diffeomorphism if it is a diffeomorphism at each point $x \in M$.

1.1.2 Lie Derivatives.

Integral curves of a vector field.

Let M be an n-dimensional manifold and $I = (-r, +r) \subset \mathbb{R}$ be some interval. Then the map

$$\phi : I \to M$$

defines a curve in M (it is assumed that ϕ is a diffeomorphism onto its image).

Clearly any curve $\phi : I \to M$ determines a tangent vector $X_t = \phi_*(\frac{d}{dt}) \in T_{\phi(t)}M$, $(t \in I)$. X_t is called the tangent vector to the curve.

Given a vector field X on M. A smooth curve $\phi : I \to M$ is called an integral curve of X if, for any $t \in I$,

$$\phi_*(\frac{d}{dt}) = X_{\phi(t)}.$$

Theorem 1.1. *Let X be a smooth vector field on M. For any point $x \in M$, there is an $r \in \mathbb{R}^+$ and a smooth curve $\phi : I = (-r, r) \to M$ such that*

1. *$\phi(0) = x$*
2. *$\phi(I)$ is an integral curve of X.*

Proof. The result is just a simple use of the classical theorem of existence and uniqueness of solutions of differential equations. □

Groups of transformations.

Let X be a vector field over a manifold M, then for every $x \in M$, there exists an integral curve $\phi : I = (-r, r) \to M$ of X as defined above $(r \in \mathbb{R}^+)$. Let us fix $t \in I$ and let U be a neighbourhood of x_0 then the map

$$\phi_t : x \mapsto \phi_t(x)$$

is defined on U for $t \in I_{x_0}$. It is a local transformation of M generated by the vector field X. Therefore this map takes a point $x \in U$ and goes to a point $\phi_t(x) \in M$ along the integral curve of X at x and the location of $\phi_t(x)$ along the curve is determined by t.

Theorem 1.2. *A (smooth) vector field X on a compact manifold M generates a one parameter group of transformations.*

Proof. Let I be the intersection of all the intervals I_{x_0} corresponding to a set of neighbourhoods $\{U\}$ covering M. Clearly I is non-empty as M is compact. Therefore ϕ_t with $t \in I$ defines a global transformation of M. We also have the composition law

$$\phi_{t+s} = \phi_t \circ \phi_s$$

and each ϕ_t has an inverse ϕ_{-t}. □

The set $\{\phi_t\}$ is called a one-parameter pseudo group. We may also reverse this process, i.e. let $\{\phi_t\}$ be a one-parameter pseudo group of transformations. Then we can define a vector field X uniquely by the equation

$$X(x) = \frac{d\phi}{dt}|_{t=0}$$

The vector field X is called the generator of $\{\phi_t\}$.

Lie derivatives.

Let X be a vector field on a manifold M and let Y be a p-contravariant tensor field. In local coordinates (x^α), Y will have the form

$$Y = \sum_{i_1,\ldots,i_p} Y^{\alpha_{i_1},\ldots,\alpha_{i_p}}(x)\frac{\partial}{\partial x^{\alpha_{i_1}}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_{i_p}}}$$

Definition 1.3. *The Lie derivative of a contravariant tensor field Y along X is the contravariant tensor field $\mathcal{L}_X Y$ defined by*

$$\mathcal{L}_X Y|_x = \lim_{t\to 0} \frac{1}{t}[(\phi_t^{-1})_*(Y(\phi_t(x))) - Y_x]$$

where $\{\phi_t\}$ is the one parameter group of transformations generated by X.

It is easy to check that $\mathcal{L}$ is an additive operator, i.e

$$\mathcal{L}_X(Y + Z) = \mathcal{L}_X Y + \mathcal{L}_X Z,$$

and that $\mathcal{L}$ satisfies Leibniz rule, i.e

$$\mathcal{L}_X(Y \otimes Z) = \mathcal{L}_X Y \otimes Z + Y \otimes \mathcal{L}_X Z.$$

This allows us to compute Y in a local coordinates, i.e

$$\begin{aligned}\mathcal{L}_X Y &= \mathcal{L}_X(\sum_{i_1,\dots,i_p} Y^{\alpha_{i_1},\dots,\alpha_{i_p}}(x)\frac{\partial}{\partial x^{\alpha_{i_1}}}\otimes\dots\otimes\frac{\partial}{\partial x^{\alpha_{i_p}}})\\ &= \sum_{i_1,\dots,i_p}(\mathcal{L}_X(Y^{\alpha_{i_1},\dots,\alpha_{i_p}}(x))\frac{\partial}{\partial x^{\alpha_{i_1}}}\otimes\dots\otimes\frac{\partial}{\partial x^{\alpha_{i_p}}}\\ &+ Y^{\alpha_{i_1},\dots,\alpha_{i_p}}(x)\mathcal{L}_X(\frac{\partial}{\partial x^{\alpha_{i_1}}})\otimes\dots\otimes\frac{\partial}{\partial x^{\alpha_{i_p}}}\\ &+\dots+Y^{\alpha_{i_1},\dots,\alpha_{i_p}}(x)\frac{\partial}{\partial x^{\alpha_{i_1}}}\otimes\dots\otimes\mathcal{L}_X(\frac{\partial}{\partial x^{\alpha_{i_p}}})).\end{aligned}$$

Clearly,

$$\mathcal{L}_X f|_x = \lim_{t\to 0}\frac{1}{t}[f(\phi_t(x)) - f(x)] = X(f)|_x,$$

so it therefore remains to compute the Lie derivative of the natural basis of vector fields in our coordinate system. Let x, y be the local coordinates in the corresponding points of M, $\phi^\alpha(t,\cdot)$ the coordinate expression of ϕ_t, $(\phi^{-1})^\alpha(t,\cdot)$ the coordinate expression of ϕ_t^{-1} and $\phi^\alpha(0,x) = x^\alpha$, $(\phi^{-1})^\alpha(0,y) = y^\alpha$.

By definition,

$$\mathfrak{L}_X(\frac{\partial}{\partial x^\alpha})|_x = \lim_{t\to 0}\frac{1}{t}[(\phi_t^{-1})_*(\frac{\partial}{\partial x^\alpha}|_{\phi_t(x)}) - \frac{\partial}{\partial x^\alpha}|_x].$$

It is easy to see that

$$(\phi_t^{-1})_*(\frac{\partial}{\partial x^\alpha}|_{\phi_t(x)}) = \frac{\partial(\phi^{-1})^\beta(t,y)}{\partial y^\alpha}\frac{\partial}{\partial x^\beta}|_x.$$

Therefore,

$$\begin{aligned}\mathfrak{L}_X(\frac{\partial}{\partial x^\alpha}) &= \lim_{t\to 0}\frac{1}{t}[\frac{\partial(\phi^{-1})^\beta(t,y)}{\partial y^\alpha}\frac{\partial}{\partial x^\beta}|_x - \frac{\partial}{\partial x^\beta}|_x]\\ &= \frac{d}{dt}\frac{\partial(\phi^{-1})^\beta(t,y)}{\partial y^\alpha}|_{t=0}\frac{\partial}{\partial x^\beta}|_x,\ \text{using } \delta_\alpha^\beta = \frac{\partial(\phi^{-1})^\beta(0,y)}{\partial y^\alpha}\\ &= -\frac{d}{dt}\frac{\partial\phi^\beta}{\partial x^\alpha}|_{t=0}\frac{\partial}{\partial x^\beta}|_x,\ \text{using } \frac{d}{dt}(\frac{\partial(\phi^{-1})^\beta}{\partial y^\gamma}\frac{\partial\phi^\gamma}{\partial x^\alpha}) = \frac{d}{dt}\delta_\alpha^\beta = 0.\end{aligned}$$

Hence,

$$\mathfrak{L}_X\frac{\partial}{\partial x^\alpha} = -\frac{\partial X^\beta}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}$$

since

$$\frac{d\phi}{dt} = X(\phi).$$

In a similar way we can also define the Lie derivative of a p-covariant tensor field Z along X.

Definition 1.4. *The Lie derivative of a covariant tensor field Z along a vector field X on a manifold M is the covariant tensor field $\mathcal{L}_X Z$ defined by*

$$\mathcal{L}_X Z|_x = \lim_{t \to 0} \frac{1}{t}[(\phi_t^*)(Z(\phi_t(x))) - Z_x]$$

where $\{\phi_t\}$ is the one parameter group of transformations generated by X.

As with the contravariant case, $\mathcal{L}$ is an additive operator and it satisfies Leibniz rule. A similar calculation to the one above shows that in a local coordinate system (x^α),

$$\mathcal{L}_X dx^\alpha = \frac{\partial X^\alpha}{\partial x^\beta} dx^\beta.$$

Remark:

If X, Y are both vector fields it is easy to see that

$$\mathcal{L}_X Y = [X, Y].$$

1.1.3 Bundles.

Fibre bundles.

Let E and M be topological spaces and let π be a map

$$\pi : E \to M$$

We call (E, M, π) a bundle if π is a continuous surjective map. M is then called the base and $\pi^{-1}(x)$ is called the fibre at $x \in M$.

Definition 1.5. *Let (E, M, π) be a bundle and let $\pi^{-1}(x)$ be homeomorphic to a space F for all $x \in M$ and let $\{U_\alpha : \alpha \in I\}$ be a covering of M such that*

1. *The bundle is locally trivial, i.e. there exists a homeomorphism*

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F,$$

for all $\alpha \in I$. $\{U_\alpha, \phi_\alpha\}$ are called a family of local trivialisations of the bundle,

2. *Let* $\tilde{\phi}_\alpha = pr \circ \phi_\alpha$*, where* pr *is the projection onto* F*. The homeomorphism*

$$\tilde{\phi}_\beta|_{\pi^{-1}(x)} \circ \tilde{\phi}_\alpha^{-1}|_{\pi^{-1}(x)} : F \to F$$

is an element of G *for all* $x \in U_\alpha \cap U_\beta$ *and all* $\alpha, \beta \in I$ *where* G *is a group of homeomorphisms of* F *onto itself,*

3. *The transition functions are continuous, i.e the induced mappings*

$$M \to G$$
$$x \mapsto g_{\alpha\beta}(x)$$

are continuous, where $g_{\alpha\beta} = \tilde{\phi}_\beta|_{\pi^{-1}(x)} \circ \tilde{\phi}_\alpha^{-1}|_{\pi^{-1}(x)}$,

then (E, M, π, F, G) *is called a fibre bundle. If* E, M *and* F *are differentiable manifolds,* π *and the transition functions are differentiable,* G *is a Lie group and the covering* (U_α) *is an atlas of* M*, then* (E, M, π, F, G) *is called a differentiable fibre bundle. If* F *is a vector space and* G *is the linear group, the fibre bundle is known as a vector bundle.*

Sections.

A section of a fibre bundle (E, M, π, F, G) is a map

$$\sigma : M \to E$$

such that

$$\pi \circ \sigma = \sigma \circ \pi = Id.$$

The space of all sections σ is usually denoted by $H^0(M, E)$

1.2 Connections.

1.2.1 Affine connections.

Definition 1.6. *An affine connection on a smooth manifold* M *is a map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(X, Y) \mapsto \nabla_X Y$$

which satisfies

1. *Linearity in the first argument, i.e* $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$,

2. *Linearity in the second argument, i.e* $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

3. *Leibnitz rule, i.e* $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,

4. *Tensoriality in the first argument, i.e* $\nabla_{fX}Y = f\nabla_X Y$,

for any $X, Y, Z \in \Gamma(TM)$ *and* $f \in \mathcal{O}_M$ *The vector field* $\nabla_X Y$ *is called the covariant derivative of the vector field* Y *along the vector field* X.

Let $(x^1, \ldots, x^n)$ be local coordinates on M, then ∇ is represented by n^3 functions

$$\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\alpha} = \Gamma^\gamma_{\alpha\beta}(x) \frac{\partial}{\partial x^\gamma}$$

The functions $\Gamma^\gamma_{\alpha\beta}(x)$ are called the coordinate symbols (or Christoffel symbols) of the affine connection ∇.

Curvature.

Given a smooth manifold M and an affine connection ∇ on M, for any two vector fields X, Y we define the map $R(X,Y)$.

$$\begin{aligned} R(X,Y) : \Gamma(TM) &\to \Gamma(TM) \\ Z &\mapsto R(X,Y)Z, \end{aligned}$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It can be easily shown that the map, denoted by

$$\begin{aligned} R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X,Y,Z) &\mapsto R(X,Y)Z, \end{aligned}$$

is a tensor for each input i.e

1. $R(fX,Y)Z = fR(X,Y)Z$,

2. $R(X,fY)Z = fR(X,Y)Z$,

3. $R(X,Y)(fZ) = fR(X,Y)Z$,

for any smooth function $f \in \mathcal{O}_M$.

Definition 1.7. *The map $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is called the curvature tensor of the affine connection ∇.*

In a local coordinate chart on M with local coordinates (x^α), it is easy to show that R is represented by the functions $R^\delta_{\alpha\beta\gamma}$

$$R(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\gamma}) = R^\delta_{\alpha\beta\gamma}\frac{\partial}{\partial x^\delta},$$

where

$$R^\delta_{\alpha\beta\gamma} = \frac{\partial\Gamma^\delta_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial\Gamma^\delta_{\alpha\gamma}}{\partial x^\beta} + (\Gamma^\epsilon_{\beta\gamma}\Gamma^\delta_{\alpha\epsilon} - \Gamma^\epsilon_{\alpha\gamma}\Gamma^\delta_{\beta\epsilon}).$$

Torsion.

Definition 1.8. *Given a smooth manifold M and an affine connection ∇ on M, the torsion τ of ∇ is a map*

$$\tau : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$(X,Y) \mapsto T(X,Y)$$

where

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

It is easy to show that T is a tensor.

In a local coordinate chart on M with local coordinates (x^α) we have

$$T(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}) = (\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha})\frac{\partial}{\partial x^\gamma},$$

where $\Gamma^\gamma_{\alpha\beta}(x)$ are the coordinate symbols of ∇.

Definition 1.9. *An affine connection ∇ is called torsion free if $T = 0$.*

Connections in vector bundles.

It is very straightforward to define a connection on a vector bundle in terms of a covariant differentiation operator - you simply take for guidance the rules of for an affine connection on the tangent bundle discussed above, i.e

1. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

2. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,

3. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,

4. $\nabla_{fX} Y = f\nabla_X Y$,

so we have,

Definition 1.10. *Let (E, M, π, V) be a vector bundle, where M is a n-dimensional manifold and V a k-dimensional vector space. A connection in a vector bundle is a map*

$$\nabla : \Gamma(TM) \times H^0(M,E) \to H^0(M,E)$$
$$(X,\sigma) \mapsto \nabla_X \sigma$$

satisfying

1. $\nabla_X(\sigma+\rho) = \nabla_X\sigma + \nabla_X\rho$,

2. $\nabla_{\sigma+\rho} X = \nabla_\sigma X + \nabla_\rho X$,

3. $\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma$,

4. $\nabla_{fX}\sigma = f\nabla_X\sigma$,

for all $\sigma \in H^0(M,E)$, $X \in \Gamma(TM)$ and $f \in \mathcal{O}_M$.

The operator ∇ is called the covariant differentiation along X. Many of the properties of an affine connection discussed above are reproduced in this context (e.g. the curvature R is defined by $R(X,Y)\sigma = \nabla_X\nabla_Y\sigma - \nabla_X\nabla_Y\sigma - \nabla_{[X,Y]}\sigma$).

1.2.2 Parallel transport.

Let

$$\phi : \mathbb{R} \longrightarrow M.$$

be a curve on a manifold M. The tangent vector field to the curve is defined by

$$\phi' = \phi_*(\frac{d}{dt}).$$

Assume that, in addition to the curve ϕ, we have a vector field X on M.

Definition 1.11. *The vector field X is said to be parallel transported along the curve $\phi(t)$, if $\nabla_{\phi'(t)}X = 0$.*

Theorem 1.12. *For any tangent vector $X_0 \in T_{\phi(0)}M$ there is a unique parallel vector field $X(t)$ on the curve $\phi(t)$ such that $X(t) = X_0$.*

Proof. In a local coordinate chart, X satisfies a system of differential equations and the theory of differential equations says that the system has a unique solution. □

1.2.3 Metrics.

Let M be a manifold. A metric g on M is a non-degenerate symmetric tensor field of type $(0, 2)$ on M. In other words, g is a bilinear map

$$g : \Gamma(TM) \times \Gamma(TM) \to \mathcal{O}_M$$
$$(X, Y) \mapsto g(X, Y)$$

such that

$$g(X, Y) = g(Y, X)$$

for all $X, Y \in \Gamma(TM)$. Effectively, the assignment of a metric on M is an assignment of a scalar product in each tangent space of M.

If at each point $x \in M$, $g(X_x, X_x)(x) > 0$ for all non-zero $X \in T_xM$ the g is said to be positive definite. A non-singular, positive definite metric is usually called a Riemannian metric on M (by non-singular we mean that if $g(X, Y) = 0$ for all non-zero $X \in \Gamma(TM)$ then $Y = 0$). A non-singular but not positive definite metric is usually called a pseudo-Riemannian metric.

In a local coordinate chart with coordinates (x^α), the metric has components

$$g_{\alpha\beta}(x) = g(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}),$$

where $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$. It is usual to express the metric in the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta :$$

this should be taken to mean that for any two vector fields $X, Y \in \Gamma(TM)$

$$g(X,Y) = g_{\alpha\beta}dx^\alpha(X)dx^\beta(Y) = g_{\alpha\beta}X^\alpha Y^\beta.$$

Lowering and raising the index.

Let M be a manifold with metric g. Fix $X_x \in T_xM$ for some point $x \in M$. We may then define the map

$$g(X_x) : T_xM \to \mathbb{R}$$
$$Y_x \mapsto g(X_x, Y_x)(x),$$

where $Y \in T_xM$. This map is linear and thus defines an element of Ω^1_xM. In a local coordinate chart with coordinates (x^α) we have

$$g(\frac{\partial}{\partial x^\alpha}) = g_{\alpha\beta}(x)dx^\beta,$$

so if $Y = Y^\alpha \frac{\partial}{\partial x^\alpha}$ then

$$g(Y) = g_{\alpha\beta}(x)Y^\alpha dx^\beta,$$

i.e the components of $g(Y)$ are $g_{\alpha\beta}(x)Y^\alpha$. This process of constructing an element of Ω^1_xM from an element of T_xM, via g, is called lowering the index. In matrix notation, where the components of elements of T_xM are expressed as column vectors and the components of Ω^1_xM are expressed as row vectors, the map is given by

$$X \mapsto X^T G(x)$$

where $G(x) = (g_{\alpha\beta}(x))$. If g is non-singular an inverse map $g^{-1} : \Omega^1_xM \to T_xM$ may be defined such that

$$\omega_x(X_x) = g(g^{-1}(\omega_x), X_x)(x)$$

for any $\omega_x \in \Omega^1_x M$ and $X_x \in T_x M$. In matrix notation, the corresponding element in $T_x M$ of ω_x has components

$$G^{-1}(x)\omega_x^T,$$

where $G^{-1}(x)$ is the matrix inverse to $G(x)$. Denote by $g^{\alpha\beta}(x)$ the entries in $G^{-1}(x)$, then the components ω^α of $g^{-1}(\omega_x)$ are $\omega_\beta g^{\beta\alpha}(x)$. Note that $G^{-1}(x)$ is symmetric, i.e $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, and that the $g^{\alpha\beta}(x)$ and the $g_{\alpha\beta}(x)$ are related by

$$g^{\alpha\gamma}g_{\gamma\beta} = g_{\beta\gamma}g^{\gamma\alpha} = \delta^\alpha_\beta$$

The map g^{-1} is called raising the index.

1.2.4 The Levi-Civita connection.

Let M be a manifold with a non-singular metric g, it is possible to define a torsion-free connection $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ on M which preserves the metric i.e. all parallel transports it defines are isometries. Equivalently, the connection must satisfy

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for any $X, Y, Z \in \Gamma(TM)$. As ∇ is torsion-free it must also satisfy

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for any $X, Y \in \Gamma(TM)$.

From these conditions we have the relation

$$g(\nabla_X Y, Z) = \frac{1}{2}\{X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g(X,[Y,Z])\},$$

for any $X, Y, Z \in \Gamma(TM)$. Using this relation and the initial conditions we may check that ∇ satisfies the axioms for a connection and thus ∇ is known as the Levi-Civita connection for g. In a local coordinate chart with coordinates (x^α), the connection coefficients of a Levi-Civita connection are given by

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(\frac{\partial g_{\beta\delta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\delta}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta}\right).$$

1.3 Principal fibre bundles.

1.3.1 Definition of a principal fibre bundle.

We shall now define Principal fibre bundles and in particular we shall look at two very important examples of principal fibre bundles, the frame and the coframe bundles. These two examples will then directly lead to the definition of a G-structure.

In this section, G is a Lie group and P is a differentiable manifold. Let $\theta : P \times G \rightarrow P$ be a differential map. This induces the map

$$R_g : P \rightarrow P$$
$$p \mapsto \theta(p, g),$$

for some $g \in G$.

Definition 1.13. *G acts differentiably to the right on P if for any $g \in G$, R_g is a diffeomorphism and*

$$R_{gh} = R_h \circ R_g$$

for all $g, h \in G$.

Let us now look at the definition of a principal fibre bundle.

Definition 1.14. *Let M and P be differentiable manifolds and let*

$$\pi : P \longrightarrow M$$

be a differential map. Let G be a Lie group. (P, M, π, G) is then called a (right) principal G-bundle if

1. *G acts on P to the right such that for any $p \in P$, if $R_g p = p$ then $g = e$ (where e is the identity of G),*
2. *$\pi(p_1) = \pi(p_2)$ if and only if there is a $g \in G$ such that $R_g p_1 = p_2$,*
3. *P is locally trivial over M.*

Two very important examples of Principal fibre bundles are the frame bundle and the coframe bundle.

1.3.2 Frame and Coframe bundles.

Let M be an n-dimensional manifold. Let ρ_x and $\hat{\rho}_x$ be any two arbitrary frames in T_xM, this means that ρ_x and $\hat{\rho}_x$ are sets of n linearly independent vectors at $x \in M$. Let $(e^1, \ldots, e^n)$ and $(\hat{e}^1, \ldots, \hat{e}^n)$ be the sets of basis represented by ρ and $\hat{\rho}$ respectively. Obviously each element of ρ can be expressed as a linear combination of $\hat{\rho}$ at x,

$$e^\alpha = g^\alpha_\beta \hat{e}^\beta$$

where $(g^\alpha_\beta) = g \in GL(n, \mathbb{R})$.

This defines a right action of $GL(n, \mathbb{R})$ on the set of frames T_xM. Let $\mathcal{L}(M)$ be the space of all points (x, ρ_x) for all $x \in M$. Then $(\mathcal{L}(M), M, \pi, GL(n, \mathbb{R}))$ is a principal $GL(n, \mathbb{R})$-bundle known as the frame bundle. The coframe bundle has a similar definition: Let ρ_x be an arbitrary frame of the cotangent bundle, $\Omega^1_x M$, and we let $\mathcal{L}^*(M)$ denote the differential coframe bundle whose fibres consist of the set of frames ρ_x for all $x \in M$.

Also notice that a frame ρ_x can be thought of as a nonsingular linear mapping

$$\rho_x : \mathbb{R}^n \longrightarrow T_xM$$
$$(v^1_x, \ldots, v^n_x) \mapsto v^\alpha_x e_\alpha$$

for some $v = (v^\alpha_x) \in \mathbb{R}^n$. The right action of $G = GL(n, \mathbb{R})$ is then just given by

$$R_g(p) = p \circ g,$$

where p is considered as a linear isomorphism $\mathbb{R}^n \longrightarrow T_xM$ as above. The action is similar for the coframe bundle except that a coframe is considered as a linear isomorphism from $\mathbb{R}^n$ to $\Omega^1_x M$.

Let $\rho_x = \{e^\alpha\}$ be a frame and $p = (x, \rho_x)$. Let $\{\theta_\alpha\}$ be the basis dual to $\{e^\alpha\}$. We define the map

$$\Theta_p : T_xM \longrightarrow \mathbb{R}^n$$
$$u \mapsto (\theta_1(u), \ldots, \theta_n(u))$$

where $u \in T_xM$. So Θ_p maps a vector in T_xM into its components with respect to the frame ρ_x. Define a 1-form θ on $\mathcal{L}(M)$ with values in $\mathbb{R}^n$ as

$$\theta_p(v) = \Theta_p \pi_* v,$$

where $v \in T_p\mathcal{L}(M)$. This 1-form is known as the soldering (canonical) form of M.

There is also a canonical 1-form that you can define on the coframe bundle in a similar fashion. This is discussed in Section (2.1), Chapter 2.

1.3.3 Connections on principal fibre bundles.

Definition 1.15. *Let (P, M, π, G) be a principal bundle with Lie group G and let V be a vector space on which G acts to the right via the representation ρ of G on V. Then (E, M, π_1, V, G) is a vector bundle with fibre V and structural group G and is said to be associated to principal bundle (P, M, π, G).*

A connection on a principal fibre bundle will define a connection to any vector bundle associated with it, for example if the principal bundle is the frame bundle then a connection in the frame bundle will define at once the connections in all the tensor bundles.

We shall first look at the notion of horizontal lift on a manifold M with a connection.

Let $X_x \in T_xM$ and let ϕ be a curve through x ($\phi(0) = x$) such that $\phi_*(\frac{d}{dt})|_{t=0} = X_x$. Let Y be parallel along ϕ, then at x

$$\frac{dY^\alpha}{dt}|_{t=0} + \Gamma^\alpha_{\beta\gamma}(x)Y(0)^\beta X^\gamma_x = 0.$$

We can define a curve $\hat{\phi}$ on TM by $\hat{\phi} = (\phi(t), Y(t))$ and the tangent vector to $\hat{\phi}$ at $t = 0$ is

$$\phi^\alpha_*(\frac{d}{dt})|_{t=0}\frac{\partial}{\partial x^\alpha} + \frac{dY^\alpha}{dt}|_{t=0}\frac{\partial}{\partial Y^\alpha} = X^\gamma_x(\frac{\partial}{\partial x^\gamma} - \Gamma^\alpha_{\beta\gamma}(x)Y^\beta(0)\frac{\partial}{\partial Y^\alpha}),$$

where $\phi_* = (\phi^\alpha_*)$. This is independent of the choice of ϕ (apart from the fact that X_x had to be its initial tangent vector) and thus we can define a map

$$\sigma : T_xM \to T_{Y(0)}TM$$
$$X_x \mapsto \sigma(X_x) = X^h_x = X^\gamma_x(\frac{\partial}{\partial x^\gamma} - \Gamma^\alpha_{\beta\gamma}(x)Y^\beta(0)\frac{\partial}{\partial Y^\alpha}),$$

which we call the horizontal lift of X_x to $T_{Y(0)}TM$. This map is linear and injective, its image a subspace of $T_{Y(0)}TM$ isomorphic to $T_{\pi(Y(0))}M$. We call this subspace the horizontal subspace defined by the connection. Thus a connection on M defines a collection of subspaces on TM. In fact, this structure is equivalent to the existence of a connection on M.

We now use a similar process to construct horizontal subspaces in the frame bundle. Let $\mathcal{L}(M)$ be the frame bundle over a manifold M with a affine connection and let ϕ be a curve in M. Let $p \in \mathcal{L}(M)$ be a point such that $\pi(p) = \phi(0)$ then there is a unique horizontal curve $\hat{\phi}$ in $\mathcal{L}(M)$ such that $\hat{\phi}(0) = p$ and $\pi(\hat{\phi}) = \phi$ (this is a well known result). Moreover, for $g \in GL(n, \mathbb{R})$ the horizontal curve through $R_g p$ is just $R_g \hat{\phi}$.

At each point $p \in \mathcal{L}(M)$ there is a set $H_p \subset T_p\mathcal{L}(M)$ consisting of the vectors tangent to horizontal curves through p. This set is a subspace of $T_p\mathcal{L}(M)$ and is known as the horizontal subspace of $T_p\mathcal{L}(M)$. Let $V_p = T_p(\pi^{-1}(x))$ be the subspace of all vectors tangent to the fibre, then $T_p\mathcal{L}(M) = V_p \oplus H_p$. V_p is usually called the vertical subspace of $T_p\mathcal{L}(M)$.

Thus the definition of a connection in a principal bundle is just an adaptation of these ideas:

Definition 1.16. *A connection in the principle bundle (P, M, π, G) is an assignment, to each point $p \in P$, of a subspace $H_p \subset T_pP$ such that*

1. *$H_p \oplus V_p = T_pP$ where V_p is the set of vectors tangent to the fibre,*

2. *H_p defines a smooth distribution on P,*

3. *$H_{R_g p} = (R_g)_* H_p$.*

Connection and curvature forms.

The horizontal subspaces H_p of a connection on a principal bundle may be defined in terms of 1-forms. Let $\mathfrak{g}$ be the Lie algebra of G and let $\hat{X} \in \mathfrak{g}$. For any element $\hat{X} \in \mathfrak{g}$ we can identify an element X_p in V_p. We may then define a 1-form ω as follows: $\omega_p(X)$ is the element $\hat{X} \in \mathfrak{g}$ such that X_p is the vertical component of X. Clearly, ω is well defined, $\mathfrak{g}$-valued, linear and smooth. We call ω the connection 1-form determined by the connection. It can be shown ([8]) that

$$R_g^* \omega = \mathrm{Ad}\ g^{-1} \omega.$$

Definition 1.17. *Let X, Y be vector fields on M and let X^h and Y^h denote the horizontal lifts of X and Y respectively. The curvature form Ω is a $\mathfrak{g}$-valued 2-form satisfying*

1. *$\Omega(X^h, Y^h) = d\omega(X^h, Y^h)$,*

2. *$\Omega(\hat{L}, U) = 0$,*

where $L \in \mathfrak{g}$ and $U \in \Gamma(P)$.

We then have the following fundamental structure equation

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

To verify this, it is sufficient to show that both sides give the same value when applied to all exterior two-vectors $X_p \wedge Y_p$ at every $p \in P$.

1.3.4 Holonomy.

Let ϕ be a closed curve , beginning and ending at x, on a manifold M with a principal fibre bundle (P, M, π, G) with a connection ω. The parallel translation along ϕ of the elements $p \in \pi^{-1}(x)$ maps $\pi^{-1}(x)$ to itself. Thus we have a map, also denoted by ϕ, from $\pi^{-1}(x)$ to itself. The inverse of this map is merely obtained by parallel transport along the same curve ϕ, except that you traverse the curve in the opposite direction. A composition is obtained by defining the product $\phi_1\phi_2$ as the mapping obtained by parallel transporting first along ϕ_2 and then along ϕ_1. Thus the collection of maps of $\pi^{-1}(x)$ has a group structure and it is called the holonomy group, H_p of the connection at x. By fixing a point $p \in \pi^{-1}(x)$ it is possible to consider H_p as a subgroup of G.

1.4 G-Structures.

1.4.1 Definition of G-structures.

Among the class of all fibre bundles there are certain ones which play a central role in differential geometry. Before we define G-structure we must first look at the definition of reduction and then we will look at a few classical examples of G-structures.

Definition 1.18. *Let (P, M, π, G) and (Q, M, π_1, H) be two principal fibre bundles of M such that H is a Lie subgroup of G and Q is a submanifold of P. P is said to be reducible to Q if the injection*

$$\iota : Q \longrightarrow P$$

satisfies

1. $\pi \circ \iota(q) = \pi_1(q)$, *for all* $q \in Q$,

2. $\iota(R_h q) = R_h(\iota(q))$ *for all* $q \in Q$ *and all* $h \in H$.

We now turn to the definition of G-structures.

Definition 1.19. *Let G be a subgroup of $GL(n, \mathbb{R})$. A G-structure on an n-dimensional manifold M, denoted by $\mathcal{G}_G$, is a reduction of the frame bundle, $\mathcal{L}(M)$, to the group G.*

Looking at the definition of reduction we see that a G-structure is a submanifold of the frame bundle such that if $p \in \mathcal{G}_G$ and $g \in GL(n, \mathbb{R})$ then the point $R_g p$ is an element of the G-structure if and only if $g \in G$.

G-structures in a local coordinate chart.

Let M be a complex n-dimensional manifold and let $\mathcal{L}(M)$ be the frame bundle. A G-structure can be understood as follows: Let $\{U_a\}$ be a covering of M, then $TM|_{U_a}$ can be trivialised by a map

$$\phi_a : TM|_{U_a} \to \mathbb{R}^n \times U_a$$

This choice of ϕ_a corresponds to a choice of n-sections, $\{e_i^{(a)}\}$, on TM over U_a which are linearly independent at each $x \in U_a$. Then a G-structure on a manifold M can be understood as a covering $\{U_a, \{e_i^{(a)}\}\}$ such that

$$e_i^{(a)} = g_{i(ab)}^{j}(x) e_j^{(b)}$$

on each $U_a \cap U_b$ where $(g_{i(ab)}^{j}(x))$ is a function on $U_a \cap U_b$ with values in G.

1.4.2 Examples of G-structures.

1. $G = e$. A manifold M with this G-structure is an assignment at each point $x \in M$ a choice of frame p_x. Let $X_x \in T_x M$ be a tangent vector at x. As p_x can be thought of as a non-singular linear map $p_x : \mathbb{R}^n \to T_x M$ (see (1.3.2)), then p_x^{-1} maps X_x to some point $v_x \in \mathbb{R}^n$. Let $y \in M$ and q_y be the choice of frame at y then q_y will map the point v_x to a tangent vector in $T_y M$. Thus we have parallel translated tangent vectors from one point to another on M. Because of this, an $\{e\}$-structure is called a complete parallelism of M.

2. Let V be a k-dimensional subspace of $\mathbb{R}^n$ and let G be the group of all linear transformations leaving V invariant. Let $\mathcal{G}_G$ be a G-structure with this group. Let $p_x \in \mathcal{G}_G$ such that $\pi(p_x) = x$. As before, p_x is a non-singular linear map $p_x : \mathbb{R}^n \to T_xM$. Clearly $p_x(V)$ will define a subspace of T_xM which we denote by $\mathcal{D}_x$. Let q_x be another point in $\mathcal{G}_G$ such that $\pi(q_x) = x$, then $q_x = R_g p_x$ for some $g \in G$ (by the definition of a G-structure). Thus,

$$q_x(V) = R_g p_x(V) = p_x \circ g(V) = p_x(gV) = p_x(V).$$

Hence $\mathcal{D}_x$ does not depend on the choice of p_x in $\mathcal{G}_G$. This means that $\mathcal{G}_G$ gives rise to a k-dimensional differential system on M. Conversely, let $\mathcal{D}$ be a differential system on M and let $\mathcal{G}_G$ be a submanifold of the fame bundle such that $p_x \in \mathcal{G}_G$ if and only if $p_x^{-1}(\mathcal{D}_x) = V$. It is easy to see that $\mathcal{G}_G$ is a G-structure. Thus, in this example, a G-structure is the same as a k-dimensional differential system.

3. G is the orthogonal group $O(n)$. Then in local coordinates we may regard the $O(n)$-structure as the sets $(\{U_a\}, \{e_i^{(a)}\})$ as described above. Define $g_{ij}^{(a)}(x)$ in $(U_a, \{e_i^{(a)}\})$ as

$$g_{ij}^{(a)}(x) := \delta_{ij}^{(a)} = g(e_i^{(a)}, e_j^{(a)}).$$

In $U_a \cap U_b$ we have,

$$\begin{aligned} g(e_i^{(b)}, e_j^{(b)}) &= g(B_i^k e_k^{(a)}, B_j^l e_l^{(a)}) \\ &= B_i^k B_j^l g(e_i^{(a)}, e_j^{(a)}) \\ &= g(e_i^{(a)}, e_j^{(a)}), \end{aligned}$$

by the definition of $O(n)$. Thus we may construct a metric as follows: If $e_i = v_i^\alpha \frac{\partial}{\partial x^\alpha}$ then define

$$g_{\alpha\beta} = \delta_{ij} v_\alpha^i v_\beta^j$$

Then the metric $g = g_{\alpha\beta} dx^\alpha dx^\beta$ is a Riemannian metric. Conversely, given a Riemannian metric on a manifold it is then straightforward to construct $(\{U_a\}, \{e_j^{(a)}\})$ and thus we have a $O(n)$-structure on M. Therefore, giving an $O(n)$-structure is the same a giving a Riemannian metric.

4. Let G be the conformal group $CO(n)$. Then a $CO(n)$-structure on M is the same as a conformal structure on M in the sense of an equivalence class of metrics i.e. $g^1_{\alpha\beta}$ is equivalent to $g^2_{\alpha\beta}$ if and only if $g_{\alpha\beta} = g_{\beta\alpha}$ and $g^1_{\alpha\beta} = \Omega^2(x) g^2_{\alpha\beta}$ for some function $\Omega^2(x)$. The calculation used is similar to the one in the previous example.

5. Let $G = Sp(n)$ be the symplectic group. Given a $\mathcal{G}_{Sp(n)}$-structure it is possible to define a non-degenerate antisymmetric bilinear form on the tangent space at each point of M. To do this let ω be a non-degenerate antisymmetric bilinear form on $\mathbb{R}^n$ (so a $g \in GL(n, \mathbb{R})$ belongs to $Sp(n)$ if and only if $\omega(gu, gv) = \omega(u, v)$ for all $u, v \in \mathbb{R}^n$) and let $p_x \in \mathcal{G}_G$ such that $\pi(p_x) = x$, where $x \in M$. Define a 2-form, Θ_{p_x} on the tangent space at x as

$$\Theta_{p_x}(X, Y) = \omega(p_x^{-1}(X), p_x^{-1}(Y))$$

for any $X, Y \in T_x M$. Clearly Θ_p is a non-degenerate antisymmetric bilinear form. Moreover, if $q_x \in \mathcal{G}_G$ is such that $\pi(q_x) = x$ then

$$\Theta_{q_x}(X, Y) = \omega(q_x^{-1}(X), q_x^{-1}(Y)) = \omega((R_g p_x)^{-1}(X), (R_g p_x)^{-1}(Y)) = \omega(p_x^{-1}(X), p_x^{-1}(Y)).$$

Thus Θ_p is independent of the choice of p. Conversely, it is easily shown that a non-degenerate antisymmetric bilinear form gives rise to an $Sp(n)$-structure. In conclusion, giving an $Sp(n)$-structure on a manifold M is the same as giving a 2-form on M. $\mathcal{G}_{Sp(n)}$ is called an almost symplectic structure. A symplectic structure has the extra condition that the 2-form be closed.

6. Let

$$J : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$$

be a linear transformation such that

$$J^2 = -I$$

Let $GL(n, \mathbb{C})$ be the subgroup of $GL(2n, \mathbb{R})$ be such that

$$gJ = Jg$$

for all $g \in GL(n, \mathbb{C})$. A $GL(n, \mathbb{C})$-structure on a manifold M is called an almost complex structure. Let M be a manifold with an almost complex structure and let $p_x \in \mathcal{G}_{GL(n\mathbb{C})}$

be such that $\pi(p_x) = x$ for some $x \in M$. Define the map $J_x^{p_x}$:

$$J_x^{p_x} : T_xM \to T_xM$$
$$X_x \mapsto p_x J p_x^{-1}(X_x)$$

where $X_x \in T_xM$ Let $q_x \in \mathcal{G}_{GL(n,\mathbb{C})}$ be another point in the almost complex structure such that $\pi(q_x) = x$, then $q_x = R_g p_x$ for some $g \in GL(n,\mathbb{C})$. Then

$$\begin{aligned} J_x^{q_x}(X_x) &= q_x J q_x^{-1}(X_x) = R_g p_x J (R_g p_x)^{-1}(X_x) \\ &= (pg)J(pg)^{-1}(X_x) = p(gJg^{-1})p^{-1}(X_x) \\ &= pJp^{-1}(X_x) = J_x^{p_x}(X_x), \end{aligned}$$

for some $X_x \in T_xM$. Thus $J_x^{p_x}$ is independent of the choice of p_x (so the map is instead denoted by J_x) and so J_x is a linear transformation on each tangent space varying differentiably with x. Since $J_x^2 = -I$, an almost complex structure on M is the same as giving a linear transformation J_x on each tangent space T_xM such that $J_x^2 = -I$ and J_x varies differentiably with $x \in M$. Conversely, such a J_x at each $x \in M$ clearly defines a $GL(n,\mathbb{C})$-structure. Let M have an almost complex structure, then each tangent space of M is a vector space over the complex numbers. A complex structure on a manifold M is one which we can introduce complex coordinates such that the transition functions are complex analytic.

7. Let $SL(n,\mathbb{R})$ be the set of all linear transformations of determinant 1. By a similar calculation to example 5 it is easy to see that giving an $SL(n,\mathbb{R})$-structure on a manifold M is the same as giving a non-zero exterior form of degree n (a volume form) on M.

8. Let $GL(n,\mathbb{R})^+$ be the group of matrices with a positive determinant. Giving an $GL(n,\mathbb{R})^+$-structure on a manifold M is the same as giving an orientation on M.

1.4.3 Torsion-free G-structures.

Given a G-structure $\mathcal{G}_G$ on M and a representation $\rho : G \to End(V)$ for some vector space V. We can form an associated vector bundle, $\mathcal{V}$, whose typical fibre is V. Let $\mathfrak{g}$ be the Lie algebra of G. Therefore $\mathfrak{g}$ is a subalgebra of $V \otimes V^*$. Now G acts on $\mathfrak{g}$ via the adjoint representation;

$ad : G \to End(\mathfrak{g})$. Thus we have an associated vector bundle $\mathfrak{G}_M$ whose typical fibre is $\mathfrak{g}$. Thus $\mathfrak{g} \subset V \otimes V^*$ gets transformed into $\mathfrak{G}_M \subset TM \otimes \Omega^1 M$.

A vector space W on which a linear action of G is defined is called a G-module. Therefore V is a G-module, $\mathfrak{g}$ is a G-module and $V \otimes V^*$ is a G-module. Now it can be shown that every G-module gives rise to a vector bundle on M associated with $\mathcal{G}_G$ (see [31]). Now we consider the intersections of two G-modules which are both subsets of $V \otimes V^* \otimes V^*$;

$$\mathfrak{g}^{(1)} := (\mathfrak{g} \otimes V^*) \cap (V \otimes S^2(V^*))$$

This is a G-module, hence there is an associated vector bundle $\mathfrak{g}_M^{(1)}$ on M.

Lemma 1.20. *The set of all torsion-free linear connections in the G-structure $\mathcal{G}_G$ on M is an affine space modelled on the vector space $\Gamma(M, \mathfrak{g}_M^{(1)})$.*

Proof. Given two torsion-free affine connections ∇_1 and ∇_2 on $\mathcal{G}_G$, we have to show that

$$\nabla_1 - \nabla_2 \in \Gamma(M, \mathfrak{g}_M^{(1)}).$$

Since both ∇_1 and ∇_2 are torsion-free then

$$\nabla_1 - \nabla_2 \in \Gamma(M, TM \otimes S^2\Omega^1 M).$$

On the other hand, since they are connections of the G-structure,

$$\nabla_1 - \nabla_2 \in \Gamma(M, \mathfrak{g} \otimes \Omega^1 M).$$

□

Consider a G-module $V \otimes \wedge^2 V^*$ and its associated bundle $TM \otimes \Omega^2 M$ and consider a G-submodule $a(\mathfrak{g} \otimes V^*) \subset V \otimes \wedge^2 V^*$ where a is the antisymmetric map. i.e.,

$$a : \mathfrak{g} \otimes V^* \subset V \otimes V^* \otimes V^* \xrightarrow{antisym} V \otimes \wedge^2 V^*$$

Define a quotient G-module

$$Tor := V \otimes \wedge^2 V^* / a(\mathfrak{g} \otimes V^*),$$

this has an associated vector bundle which we denote by Tor_M. There is a map

$$V \otimes \wedge^2 V^* \xrightarrow{\tilde{a}} V^* \otimes \wedge^2 V^* / a(\mathfrak{g} \otimes V^*).$$

Hence we have a map of vector bundles, which we also denote by $\tilde{a}$;

$$TM \otimes \Omega^2 M \xrightarrow{\tilde{a}} Tor_M$$

Let ∇ be a affine connection on $\mathcal{G}_G$. Compute its torsion tensor, $T_\nabla \in H^0(M, TM \otimes \Omega^2 M)$. In general, T_∇ does depend on the choice of ∇, but

Lemma 1.21. *$\tilde{a}(T_\nabla)$ does not depend on the choice of ∇ on $\mathcal{G}_G$. It is called the invariant torsion, T_G, of the G-structure.*

Proof. Let ∇ be any linear connection on $\mathcal{G}_G$. Let T_∇ be its torsion, i.e, $T_\nabla \in H^0(M, TM \otimes \Omega^2 M)$ Consider the composition

$$\hat{a} : \mathfrak{G}_M \otimes \Omega^1 M \to TM \otimes \Omega^1 M \otimes \Omega^1 M \to TM \otimes \Omega^2 M$$

and define

$$Inv_T = TM \otimes \Omega^2 M / \hat{a}(\mathfrak{G}_M \otimes \Omega^1 M)$$

Then we have the following short exact sequence

$$0 \to \mathfrak{G}_M \otimes \Omega^1 M \xrightarrow{\hat{a}} TM \otimes \Omega^2 M \xrightarrow{\pi} Inv_T \to 0$$

So we claim that $\pi(T_\nabla)$ does not depend on the choice of ∇. Choose ∇_1. Then $\nabla_1 = \nabla + \omega$, where $\omega \in H^0(M, \mathfrak{G}_M \otimes \Omega^1 M)$. Then $T_{\nabla_1} = T_\nabla + \hat{a}(\omega)$. Hence $\pi(T_{\nabla_1}) = \pi(T_\nabla) + \pi \circ \hat{a}a(\omega) = \pi(T_\nabla)$. □

Definition 1.22. *$\mathcal{G}_G$ is called 1-flat (or torsion-free) if $T_G = 0$.*

1.5 Symmetries of G-structures.

Let

$$\varphi : M \longrightarrow N$$

be a diffeomorphism from M to N. This then induces the diffeomorphism

$$\varphi_* : \mathcal{L}(M) \longrightarrow \mathcal{L}(N).$$

Definition 1.23. *let $\mathcal{G}_G^M$ and $\mathcal{G}_G^N$ be G-structures on M and N respectively. Then a diffeomorphism $\varphi : M \to N$ which satisfies*

$$\varphi_* : (\mathcal{G}_G^M) = \mathcal{G}_G^N$$

is said to be an isomorphism of $\mathcal{G}_G^M$ onto $\mathcal{G}_G^N$.

1.5.1 Symmetries of G-structures.

Let $\mathcal{G}_G$ be a G-structure on a manifold M. As explained above, any diffeomorphism $\varphi : M \to M$, gives rise to a new G-structure $\varphi_*(\mathcal{G}_G)$ on M.

Definition 1.24. *A diffeomorphism $\varphi : M \longrightarrow M$ is called a symmetry of the G-structure $\mathcal{G}_G$ if $\varphi_*(\mathcal{G}_G) = \mathcal{G}_G$*

Definition 1.25. *Let $\mathcal{G}_G$ be a G-structure on a manifold M. A vector field $X \subset \Gamma(M, TM)$ is called a Killing vector field if, for any local frame $\{e^a\} \in \mathcal{G}_G$, one has*

$$\mathcal{L}_X e^a = \mathfrak{g}_b^a e^b,$$

where $\mathfrak{g}_b^a$ are some local functions on M with values in the Lie algebra, $\mathfrak{g}$, of the group G.

As explained in Section (1.1.2), vector fields on M generate a 1-parameter family of diffeomorphisms of M. It is well known that such a family of diffeomorphisms associated to a Killing vector field of a G-structure $\mathcal{G}_G$ (if it exists) is a family of symmetries of $\mathcal{G}_G$ in the sense of Definition (1.25). Thus Killing vector fields is a very useful tool in the study of symmetries of a given G-structure.

Example:

Riemannian Geometry. Let $O(n)$ be the orthogonal group in $\mathbb{R}^n$, i.e. the group preserving the scalar product η given by the following matrix,

$$\eta_{ab} = 1, \text{ if } a = b$$

$$0, \text{ otherwise.}$$

If $\mathcal{G}_{O(n)}$ is an $O(n)$-structure on an n-dimensional manifold M, then, as shown in Section (1.4.2), M comes equipped with the Riemannian metric, given locally by,

$$g = \eta_{ab} e^a \otimes e^b,$$

where $\{e^a\}$ is any frame in $\mathcal{G}_{O(n)}$.

Lemma 1.26. *$X \in \Gamma(M, TM)$ is a Killing vector field of $\mathcal{G}_O(n)$ if and only if*

$$\mathcal{L}_X g = 0$$

Proof. We have

$$\begin{aligned}
\mathcal{L}_X g &= \mathcal{L}_X(\eta_{ab} e^a \otimes e^b) \\
&= \eta_{ab}\mathcal{L}_X(e^a) \otimes e^b + \eta_{ab} e^a \otimes \mathcal{L}_X(e^b) \\
&= \eta_{ab} g^a_c e^c \otimes e^b + \eta_{ab} e^a \otimes g^b_c e^c \\
&= g_{bc} e^c \otimes e^b + g_{ac} e^a \otimes e^c, \quad \eta_{ab} g^a_c := g_{bc} \text{ in the Lie algebra of } O(n) \\
&= g_{bc} e^c \otimes e^c + g_{cb} e^c \otimes e^b \\
&= (g_{bc} + g_{cb}) e^c \otimes e^b \\
&= 0
\end{aligned}$$

since $g_{bc} = -g_{cd}$. The latter follows from the well-known fact that the Lie algebra of $O(n)$ consists of skew-symmetric matrices. □

Remark:

If ∇ is the Levi-Civita connection of the metric g, then the "Killing" condition

$$\mathcal{L}_X g = 0$$

can be equivalently rewritten, in a local coordinate chart $\{x^1, \ldots, x^n\}$, as

$$\nabla^\mu X^\nu + \nabla^\nu X^\mu = 0,$$

where $\nabla^\mu = g^{\mu\nu}\nabla_{\frac{\partial}{\partial x^\nu}}$, and X^μ are the components of X in the basis $\frac{\partial}{\partial x^\mu}$.

Example:

Conformal geometry. Let $\mathcal{G}_{CO(n)}$ be a conformal $CO(n)$-structure on an n-dimensional manifold M. As explained in Section (1.4.2), in this the manifold has a canonical class of conformally related metrics,

$$\{\Omega^2(x) g\} = \{\Omega^2 \eta_{ab} e^a \otimes e^b\},$$

where $\{e^a\}$ is an arbitrary frame in $\mathcal{G}_{CO(n)}$.

Lemma 1.27. *$X \in \Gamma(M, TM)$ is a Killing vector field of the conformal structure $\mathcal{G}_{CO(n)}$ if and only if*

$$\mathcal{L}_X g = \Omega(x) g$$

for some function $\Omega(x)$.

Proof. As in the above example we have

$$\mathcal{L}_X g = (g_{ab} + g_{ba}) e^a \otimes e^b,$$

where $g_{ab} = \eta_{ac} g^c_b$, and g^c_b is a function on M with values in the Lie algebra of $CO(n)$. Any such function can be uniquely decomposed as follows,

$$g_{ab} = f_{ab} + \Omega(x)\eta_{ab}$$

where $f_{ab} = -f_{ba}$. Hence

$$\mathcal{L}_X g = \Omega(x)\eta_{ab} e^a \otimes e^b = \Omega(x) g$$

□

Remark:

If ∇ is the Levi-Civita connection of any metric g in the conformal class $\{\Omega^2(x)g\}$, then the condition

$$\mathcal{L}_X g = \Omega(x) g$$

can be rewritten, in a local coordinate chart, as

$$\nabla^\mu X^\nu + \nabla^\nu X^\mu = \Omega(x) g^{\mu\nu},$$

for some function $\Omega(x)$.

1.5.2 A new characterization of Killing vector fields.

Let M be a n-dimensional manifold and $X = X^\alpha \frac{\partial}{\partial x^\alpha}$ be a vector field on M where (x^α) is a local coordinate system for some open set U in M. Let $\mathcal{G}_G$ be a G-structure on M where $G \subset GL(n, \mathbb{C})$ is a closed irreducible Lie subgroup.

Let $e = \{e^a\}_{a=1\ldots n}$ be an element of $\mathcal{G}_G$ over a point $x \in M$, i.e $\{e^a\}$ are sections of $\Omega^1 M$ which form a basis of $\Omega^1 M$ at that point so $e^a = e^a_\alpha dx^\alpha$.

Then for all $v \in \Omega^1 M$,

$$v = v_a e^a = v_a e^a_\alpha dx^\alpha,$$

where $v_a \in \mathbb{R}$.

So the right G-action on the G-structure $\mathcal{G}_G$ is given explicitly by

$$e^a_\alpha \to g^{-1}{}^a_b e^b_\alpha(x)$$

where $(g^a_b) \in G$.

Let $p \in \Omega^1 M$ then $p = p_\beta dx^\beta$ for some $p_\beta \in \mathbb{C}^n$. In the basis $\{e^a\}$, one has $p = \tilde{p}_a e^a$. Thus,

$$\begin{aligned} p = \tilde{p}_a e^a &= \tilde{p}_a e^a_\beta dx^\beta \\ \Rightarrow p_\beta &= e^a_\beta \tilde{p}_a \\ \Rightarrow \tilde{p}_a &= (e^{-1})^\beta_a p_\beta \end{aligned}$$

Let X be a vector field on M and let $\tilde{X}$ be its unique lift onto $\Omega^1 M$, satisfying the conditions $\pi_*(\tilde{X}) = X$ and $\mathcal{L}_{\tilde{X}}\theta = 0$ where $\theta = p_\beta dx^\beta$. Then $\tilde{X}$ will have the form $\tilde{X} = X^\alpha(x)\frac{\partial}{\partial x^\alpha} + Y^\beta(x,p)\frac{\partial}{\partial p_\beta}$. Then the equation in general,

$$\mathcal{L}_{\tilde{X}}\theta = \mathcal{L}_{\tilde{X}}(p_\alpha dx^\alpha) = \tilde{X}(p_\alpha)dx^\alpha + p_\alpha \mathcal{L}_{\tilde{X}} dx^\alpha = Y^\beta(x,p)dx^\beta + p_\alpha \frac{\partial X^\alpha}{\partial x^\beta}dx^\beta = 0$$

implies

$$Y^\beta = -p_\alpha \frac{\partial X^\alpha}{\partial x^\beta}$$

and so finally this unique lift of X is given by

$$\tilde{X} = X^\alpha(x)\frac{\partial}{\partial x^\alpha} - p_\alpha \frac{\partial X^\alpha}{\partial x^\beta}\frac{\partial}{\partial p_\beta}. \tag{1.1}$$

Let us find the coordinates of $\tilde{X}$ in the $(x^\alpha, \tilde{p}_a)$ coordinate system. We have

$$\begin{aligned} \frac{\partial}{\partial x^\alpha} &\mapsto \frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} + \frac{\partial \tilde{p}_a}{\partial x^\alpha}\frac{\partial}{\partial \tilde{p}_a} \\ &= \frac{\partial}{\partial x^\alpha} + \frac{\partial (e^{-1})^\beta_a}{\partial x^\alpha} p_\beta \frac{\partial}{\partial \tilde{p}_a} \\ &= \frac{\partial}{\partial x^\alpha} + \frac{\partial (e^{-1})^\beta_a}{\partial x^\alpha} e^b_\beta \tilde{p}_b \frac{\partial}{\partial \tilde{p}_a}, \end{aligned}$$

or

$$\frac{\partial}{\partial x^\alpha} \mapsto \frac{\partial}{\partial x^\alpha} - (e^{-1})^\beta_a \frac{\partial e^b_\beta}{\partial x^\alpha} \tilde{p}_b \frac{\partial}{\partial \tilde{p}_a}.$$

Since $(e^{-1})^\beta_a e^b_\beta = \delta^b_a$ we have $\frac{\partial (e^{-1})^\beta_a}{\partial x^\alpha} e^b_\beta + (e^{-1})^\beta_a \frac{\partial e^b_\beta}{\partial x^\alpha} = 0$, and

$$\frac{\partial}{\partial p_\alpha} \mapsto \frac{\partial x^\beta}{\partial p_\alpha}\frac{\partial}{\partial x^\beta} + \frac{\partial \tilde{p}_a}{\partial p_\alpha}\frac{\partial}{\partial \tilde{p}_a} = 0 + (e^{-1})^\alpha_a \frac{\partial}{\partial \tilde{p}_a}.$$

Thus our lift is given by,

$$\tilde{X} = X^\alpha \frac{\partial}{\partial x^\alpha} - X^\alpha (e^{-1})^\beta_a \frac{\partial e^b_\beta}{\partial x^\alpha} \tilde{p}_b \frac{\partial}{\partial \tilde{p}_a} - e^b_\alpha \tilde{p}_b \frac{\partial X^\alpha}{\partial x^\beta}(e^{-1})^\beta_a \frac{\partial}{\partial \tilde{p}_a}. \tag{1.2}$$

Let m_G be the right action of G on the fibre bundle $\Omega^1 M$, where G a Lie group. The the right action of G on the $(x^a, \tilde{p}_a)$ coordinate system to is given explicitly by

$$m_G : x^\alpha \to x^\alpha,$$

$$m_G : \tilde{p}_a \to g^b_a(x)\hat{p}_b,$$

$$m_G : e^a_\beta \to (g^{-1})^a_b e^b_\beta,$$

where $(x^a, \hat{p}_a)$ is the local coordinate system following this transformation.

The following result seems to be new.

Theorem 1.28. *Suppose there exists a nontrivial class of invariant gauge transformations such that $m_{G*}(\tilde{X}) = \tilde{X}$ (i.e the Lie lift of a vector is preserved) then X is a killing vector field (and hence a symmetry).*

Proof. Let us find the coordinates of $m_{G*}(\tilde{X})$ in the $(x^\alpha, \hat{p})$ coordinate system. We have

$$\begin{aligned} m_{G*}\left(\frac{\partial}{\partial x^\alpha}\right) &= \frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} + \frac{\partial \hat{p}_a}{\partial x^\alpha}\frac{\partial}{\partial \hat{p}_a} \\ &= \frac{\partial}{\partial x^\alpha} + \frac{\partial (g^{-1})^c_a}{\partial x^\alpha}\tilde{p}_c \frac{\partial}{\partial \hat{p}_a} \\ &= \frac{\partial}{\partial x^\alpha} + \frac{\partial (g^{-1})^d_c}{\partial x^\alpha} g^d_c \hat{p}_d \frac{\partial}{\partial \hat{p}_a} \\ &= \frac{\partial}{\partial x^\alpha} - (g^{-1})^c_a \frac{\partial g^d_c}{\partial x^\alpha}\hat{p}_d \frac{\partial}{\partial \hat{p}_a} \end{aligned}$$

since $g^d_c (g^{-1})^c_a = \delta^d_a$ we have $\frac{\partial g^d_c}{\partial x^\alpha}(g^{-1})^c_a + g^d_c \frac{\partial (g^{-1})^c_a}{\partial x^\alpha} = 0$. Also,

$$m_{G*}\left(\frac{\partial}{\partial \tilde{p}_a}\right) = \frac{\partial x^\beta}{\partial \tilde{p}_a}\frac{\partial}{\partial x^\beta} + \frac{\partial \hat{p}_b}{\partial \tilde{p}_a}\frac{\partial}{\partial \hat{p}_b} = 0 + (g^{-1})^a_b \frac{\partial}{\partial \hat{p}_b}$$

So under this action $\tilde{X}$ is transformed into

$$\begin{aligned} m_{G*}(\tilde{X}) = X^\alpha \frac{\partial}{\partial x^\alpha} &- X^\alpha (g^{-1})^c_a \frac{\partial g^d_c}{\partial x^\alpha}\hat{p}_d \frac{\partial}{\partial \hat{p}_a} \\ &+ X^\alpha g^h_a (e^{-1})^\beta_h \frac{\partial ((g^{-1})^b_i e^i_\beta)}{\partial x^\alpha} g^d_b \hat{p}_d (g^{-1})^a_f \frac{\partial}{\partial \hat{p}_f} \\ &+ (g^{-1})^b_i e^i_\alpha g^d_b \hat{p}_d \frac{\partial X^\alpha}{\partial x^\beta} g^h_a (e^{-1})^\beta_h (g^{-1})^a_f \frac{\partial}{\partial \hat{p}_f} \end{aligned}$$

Thus, $m_{G*}(\tilde{X}) = \tilde{X}$ if and only if

$$\begin{aligned} X^\alpha (e^{-1})^\beta_a \frac{\partial e^b_\beta}{\partial x^\alpha} p_b \frac{\partial}{\partial p_a} + e^b_\alpha p_b \frac{\partial X^\alpha}{\partial x^\beta} (e^{-1})^\beta_a \frac{\partial}{\partial p_a} &= X^\alpha (g^{-1})^c_a \frac{\partial g^d_c}{\partial x^\alpha} p_d \frac{\partial}{\partial p_a} \\ &- X^\alpha g^h_a (e^{-1})^\beta_h e^i_\beta \frac{\partial (g^{-1})^h_i}{\partial x^\alpha} g^d_b p_d (g^{-1})^a_f \frac{\partial}{\partial p_f} \\ &- X^\alpha g^h_a (e^{-1})^\beta_h \frac{\partial e^i_\beta}{\partial x^\alpha} (g^{-1})^b_i g^d_b p_d (g^{-1})^a_f \frac{\partial}{\partial p_f} \\ &- (g^{-1})^b_i e^i_\alpha g^d_b p_d \frac{\partial X^\alpha}{\partial x^\beta} g^h_a (e^{-1})^\beta_h (g^{-1})^a_f \frac{\partial}{\partial p_f} \end{aligned}$$

Rearranging this we have;

$$(e^{-1})^\beta_a (X^\alpha \frac{\partial e^b_\beta}{\partial x^\alpha} + e^b_\alpha \frac{\partial X^\alpha}{\partial x^\beta}) p_b \frac{\partial}{\partial p_a} = (X^\alpha (g^{-1})^c_a \frac{\partial g^b_c}{\partial x^\alpha}) p_b \frac{\partial}{\partial p_a}$$

and from this we can see that

$$(e^{-1})^\beta_a (X^\alpha \frac{\partial e^b_\beta}{\partial x^\alpha} + e^b_\alpha \frac{\partial X^\alpha}{\partial x^\beta}) = (A_\alpha)^b_a \tag{1.3}$$

Where $(A_\alpha)^a_b = X^\alpha (g^{-1})^c_b \frac{\partial g^a_c}{\partial x^\alpha}$ and hence takes values in the Lie algebra of G.

As the gauge transformation is invariant, it is easy to show that the equality (1.3) does, in fact imply, that X is a Killing vector field.

□

Chapter 2

Twistor Theory of G-structures.

2.1 Symplectic Manifolds.

Definition 2.1. *Let M be a smooth n-dimensional manifold and let ω be a non-degenerate closed 2-form on M. The pair (M, ω) is then called a symplectic manifold and we say that ω is a symplectic structure on M.*

As noted in Section (1.4.2), Example 5, a manifold with a $Sp(n)$-structure gives an exterior 2-form on M and M is then a symplectic manifold if this 2-form is closed. Thus an isomorphism defined by definition (1.23) is usually called a symplectic map and a symmetry defined by definition (1.24) is usually called a symplectomorphism.

Let us look at a few examples

2.1.1 Examples.

1. Let M be an orientable surface. Thus there exists on M a non-degenerate closed 2-form on M (the volume form) so M is a symplectic manifold.

2. Let $M = \mathbb{R}^{2n}$. Let ω be the following 2-form

$$\omega = dx^1 \wedge dx^{n+1} + \ldots + dx^n \wedge dx^{2n}.$$

This is known as the "standard" symplectic structure on $\mathbb{R}^{2n}$.

3. Let (M, ω) be a symplectic manifold and let N be an even dimensional submanifold of M such that the form ω pulls back to a non-degenerate 2-form ω_N on N. Then (N, ω_N) is a symplectic manifold known as a symplectic submanifold of M.

4. Let M be a manifold and let $\Omega^1 M$ be the cotangent bundle of M. Define a 1-form θ on $\Omega^1 M$ by the equation

$$\theta(v) = \mu(\pi_*(v)),$$

for any $v \in T_\alpha(\Omega^1 M)$, where $\mu \in \Omega^1 M$ and $\pi : \Omega^1 M \to M$ is the basepoint projection. Let

$$\omega = d\theta,$$

then ω is a symplectic form on $\Omega^1 M$.

To see this, let us compute θ in a local coordinate chart with coordinates (x^α). Let $\mu \in \Omega^1_x M$ for $x \in M$. Then locally μ can be written as

$$\mu = p_1(\mu)dx^1|_x + \ldots + p_n(\mu)dx^n|_x$$

Thus the functions $x^1, \ldots, x^n, p_1, \ldots, p_n$ form a local coordinate system on $\Omega^1 M$. It is then straightforward to compute that, in this coordinate system,

$$\theta = p_\alpha dx^\alpha.$$

Hence,

$$\omega = d\theta = dp_\alpha \wedge dx^\alpha$$

and so is obviously a non-degenerate closed 2-form.

2.1.2 Symplectic reduction.

Before we discuss the theory of symplectic reduction we state fundamental theorem due to Darboux

Theorem 2.2. *(Darboux' Theorem). Let M be a manifold and ω be a closed 2-form on M such that ω^n is nowhere vanishing, then for every $x \in M$, there is a local coordinate chart $x^1, \ldots, x^n, p_1, \ldots, p_n$ on some neighbourhood U of x such that*

$$\omega|_U = dp_\alpha \wedge dx^\alpha.$$

Proof. See [31] □

We shall also discuss the notion of a distribution on a manifold M.

Definition 2.3. *A distribution $\mathcal{D}$ on a manifold M is a subset of the tangent bundle TM such that the fibre $\mathcal{D}_x = \mathcal{D} \cap T_xM$ is a vector subspace of T_xM for all $x \in M$. The dimension of $\mathcal{D}_x$ is called the rank of the distribution $\mathcal{D}$.*

An integral manifold of a distribution $\mathcal{D}$ on a manifold M is a submanifold N of M which satisfies

$$\iota_*(T_xN) = \mathcal{D}_{\iota(x)}$$

for all $x \in N$, where $\iota : N \to M$ denotes the canonical injection.

Definition 2.4. *A distribution $\mathcal{D}$ on a manifold M is integrable if, for every point $x \in M$ there exists an integral manifold of $\mathcal{D}$ which contains x.*

Let (M, ω) be a symplectic manifold of dimension $2n$ and let $F \hookrightarrow M$ be a submanifold of M of dimension k, $0 < k < 2n$. We restrict ω from M to F. We get a 2-form on F denoted by $\omega|_F$ or ω_F. Since

$$\omega_F = \iota^*(\omega)$$

where $\iota : F \to M$ is the inclusion map, then

$$d\omega_F = \iota^*(d\omega) = 0, \text{ using } d\iota^* = \iota^* d.$$

Thus ω_F is closed.

However, in general, ω_F is degenerate. For example, ω_F is always degenerate when the dimension of F is odd. On a local coordinate chart with coordinates (y^α) then

$$\omega_F = \hat{\omega}_{\alpha\beta} dy^\alpha \wedge dy^\beta$$

Define a distribution $\mathcal{D} \subset TF$ (a subbundle of TF) as follows:

$$\mathcal{D}_y := \{Y_y \in T_yF : Y_y \lrcorner \omega_F = 0\}.$$

If ω_F is non-degenerate (i.e if ω_F is a symplectic form) then $\mathcal{D}_y \equiv 0$. But, in general, $\mathcal{D}_y$ is non-zero and moreover, $\mathcal{D}$ is an integrable distribution.

Lemma 2.5. *Let F be a submanifold of a manifold M and let $\mathcal{D}$ be a distribution of F ($\mathcal{D} \subset TF$). The following statements are equivalent:*

1. *$\mathcal{D}$ is integrable,*

2. *for all local sections X, Y of $\mathcal{D}$, $[X, Y]$ is a local section of $\mathcal{D}$,*

3. *$\mathcal{D}$ is a sheaf of Lie algebras.*

Proof. See [31] □

We also have the following theorem:

Theorem 2.6. *(Frobenius Theorem). Let F be a submanifold of a manifold M with a distribution $\mathcal{D} \subset TF$ such that $\mathcal{D}$ is integrable, then through any $y \in F$, there passes a local submanifold N of dimension p such that*

$$TN = \mathcal{D}|_N$$

with rank $\mathcal{D} = p$.

Proof. See [31] □

Thus, if ω_F is degenerate, then F is foliated by p-dimensional submanifolds (leaves). Let F' be an open subset of F, then there is a local submersion

$$\nu : F' \longrightarrow Z$$

such that $\nu^{-1}(z)$ is a leaf from the above foliation for all $z \in Z$. Z is called the quotient manifold (or the parameter space) of the foliation.

Theorem 2.7. *Z comes equipped with a symplectic form ω_Z such that*

$$\omega_F = \nu^*(\omega_Z)$$

Thus (Z, ω_Z) is a symplectic manifold.

This process, beginning with a submanifold F of a symplectic manifold (M, ω) and ending in the symplectic manifold (Z, ω_Z), is known as symplectic reduction.

Hamiltonian vector fields.

Let (M, ω) be a symplectic manifold and let H be a function on $\Omega^1 M$. Define the vector field Y_H as follows:

$$Y_H \lrcorner \omega = dH$$

The pair $(H, \Omega^1 M)$ is known as a Hamiltonian system and the integral curves of Y_H are solutions of equations called Hamilton equations which are given locally as

$$\frac{dx^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha},$$
$$\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha},$$

where $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ are local coordinates in $\Omega^1 M$. Or alternatively we have

$$Y_H = \frac{\partial H}{\partial p_\alpha}\frac{\partial}{\partial x^\alpha} - \frac{\partial H}{\partial x^\alpha}\frac{\partial}{\partial p_\alpha}.$$

2.1.3 Contact manifolds.

Let M be a complex $(2n+1)$-dimensional manifold. Let $D \subset TM$ be a subbundle in TM of rank $2n$. Then define the quotient bundle

$$L = TM/D$$

It is a line bundle so there is an exact sequence

$$0 \to D \to TM \to L \to 0.$$

With any such $D \subset TM$ one can associate a Frobenius form

$$\phi : \wedge^2 D \to L$$

$$(X, Y) \mapsto [X, Y] \bmod D$$

Definition 2.8. *A contact structure on a $(2n+1)$-dimensional manifold M is a rank $2n$ subbundle D of the tangent bundle such that the associated Frobenius form ϕ is non-degenerate. L is called a contact line bundle.*

The non-degeneracy condition in the above definition is as follows; Let $\{e_\alpha\}$ be any local basis of D and $\{e_0\}$ be any local basis of L. Then

$$\phi(e_\alpha, e_\beta) = \phi_{\alpha\beta}(x).e_0$$

where $\phi_{\alpha\beta} = -\phi_{\beta\alpha}$, $\alpha, \beta = 1, \ldots, 2n$. ϕ is called non-degenerate if $\det \phi_{\alpha\beta} \neq 0$. It can be easily shown that this is well defined.

One can also easily verify that the above non-degeneracy of D is equivalent to the fact that the 1-form $\theta \in H^0(M, L \otimes \Omega^1 M)$ satisfies the condition

$$\theta \wedge (d\theta)^n \neq 0$$

where ϕ is the "twisted" 1-form defined by the exact sequence

$$0 \to D \to MY \xrightarrow{\theta} L \to 0.$$

We also have the following important definition:

Definition 2.9. *Let Z be a complex submanifold of a $(2n+1)$-dimensional contact complex manifold Y such that $TZ \subset D$ (Z is said to be isotropic). Then if Z has dimension n then Z is said to be a Legendre submanifold of Y.*

2.2 Jet bundles.

Let M and N be manifolds and let $x \in M$ and $y \in N$. Let f be a differentiable map

$$f : U \longrightarrow N,$$

where U is some open neighbourhood of x, such that $f(x) = y$. Let $\mathcal{O}_{x,y}(M,N)$ denote the set of all such maps. In a local coordinate chart in neighbourhoods of x and y respectively, with coordinates (x^α) on M and (y^α) on N, we can express an element f of $\mathcal{O}_{x,y}(M,N)$ in terms of a Taylor expansion. This leads to the following definition:

Definition 2.10. *Let f and g be elements of $\mathcal{O}_{x,y}(M,N)$ and let (x^α) and (y^α) be local coordinates of local coordinate charts defined in a neighbourhood of $x \in M$ and $y \in N$ respectively. Then we say that f and g are tangent to the k-th order at x if they have the same Taylor expansion up to the order k.*

This definition allows us to then define an equivalence relation on $\mathcal{O}_{x,y}(M,N)$.

Definition 2.11. *Let f and g be elements of $\mathcal{O}_{x,y}(M,N)$. We say that f and g are equivalent to order k if they are tangent to the k-th order at x. It can be shown that this relation is an* *equivalence relation in $\mathcal{O}_{x,y}(M,N)$ and, moreover, that this relation does not depend on the choice of local coordinates. The equivalence classes for this relation are called the k-th jets from x to y thus an equivalence class for an element $f \in \mathcal{O}_{x,y}(M,N)$ is called the k-th jet of f at the point x and is denoted by $j^k_x f$. The set of k-jets from x to y is denoted by $J^k_{x,y}(M,N)$ and the set of all the k-jets from M to N are denoted by $J^k(M,N)$ and it can be shown that $J^k(M,N)$ has a manifold structure.*

Let $f \in \mathcal{O}_{x,y}(M,N)$. Define the map $j^k f$ as follows:

$$M \to J^k_{x,y}(M,N)$$

$$x \mapsto j^k_x f$$

We call $j^k f$ the k-jet extension of f.

J^1E in a local coordinate chart.

Let $(E, M, \pi, \mathbb{R}^a)$ be a rank a vector bundle E over a manifold M of dimension n. We define the set J^kE as the space of k-jets of local sections of E. Thus J^kE is contained in $J^k(M,E)$. Since E is locally trivial we can cover M by open neighbourhoods $\{(U_i, \phi_i)\}$ such that the map

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^a$$

is a homeomorphism. Let $x \in \pi^{-1}(U_i) := E|_{U_i}$ and let $(x^p_{(i)})$ be coordinates in U_i, then

$$\phi_i(p) = (x^p_{(i)}, v^\alpha_{(i)})$$

where $\alpha = 1, \dots, a$ and $p = 1, \dots, n$. For $x \in E|_{U_i \cap U_j}$ we have

$$x^p_{(i)} = f^p_{(ij)}(x^q_{(j)}) \text{ for } p, q = 1, \dots, n$$

$$v^\alpha_{(i)} = g^\alpha_{(ij)\beta}(x^q_{(j)})v^\beta_{(j)} \text{ for } \alpha, \beta = 1, \dots, a$$

Let $v \in J^1E$ then in $U_i \cap U_j$ we have

$$v_{(ij)}(x_{(i)}) = (x^p_{(i)}, v^\alpha_{(i)}(x^p_{(i)}))$$

where each $v^\alpha_{(i)}$ is given by

$$v^\alpha_{(i)} = v^\alpha_{(i)} + w^\alpha_{(i)p}x^p_{(i)}.$$

Hence $v_{(i)}$ is given by the map

$$x^p_{(i)} \mapsto (x^p_{(i)}, v^\alpha_{(i)}, w^\alpha_{(i)p})$$

such that on $U_i \cap U_j$,

$$\begin{aligned} x^p_{(i)} &= f^p_{(ij)}(x^q_{(j)}) \\ v^\alpha_{(i)} &= g^\alpha_{(ij)\beta}(x^q_{(j)})v^\beta_{(j)} \\ w^\alpha_{s(i)} &= \frac{\partial x^{\tilde{s}}_{(j)}}{\partial x^s_{(i)}}\left(\frac{\partial g^\alpha_{(ij)\beta}}{\partial x^s_{(j)}}v^\beta_{(j)} + g^\alpha_{(ij)\beta}w^\beta_{\tilde{s}(j)}\right), \end{aligned}$$

where $p, q, s, \tilde{s} = 1, \ldots, n$ and $\alpha, \beta = 1, \ldots, a$.

Consider a subspace of vectors of the form $(0, w^\alpha_{s(j)})$, then

$$w^\alpha_{s(i)} = g^\alpha_{(ij)\beta}\frac{\partial x^{\tilde{s}}_{(j)}}{\partial x^s_{(i)}}w^\alpha_{\tilde{s}(j)},$$

thus $E\otimes\Omega^1 M$ is a subbundle of J^1E consisting of all vectors of the form $(0, w^\alpha_{s(i)})$. Since $E\otimes\Omega^1 M$ has rank na and J^1E has rank $na + a$ then we have the following short exact sequence:

$$0 \to E \otimes \Omega^1 M \to J^1 E \to E \to 0$$

$$0 \to (v^\alpha_s, 0) \to (v^\alpha, v^\alpha_s) \to (v^\alpha) \to 0$$

Locally, a connection ∇ on E is a set of functions $\Gamma^\gamma_{\alpha\beta} \in E \otimes E^* \otimes \Omega^1 M$ and so ∇ defines a map (also denoted by ∇):

$$\nabla : J^1 E \to E \otimes \Omega^1 M$$

$$(v^\alpha, v^\alpha_s) \mapsto (v^\alpha_s + \Gamma^\alpha_{s\beta}v^\beta)$$

and this in turn defines a map (also denoted by ∇):

$$\nabla : E \to J^1 E$$

$$(v^\alpha) \mapsto (v^\alpha, -\Gamma^\alpha_{s\beta}v^\beta)$$

2.3 The Normal bundle.

Let Y be a manifold and let Z be a submanifold of Y. Then we define the space

$$N_z = T_z Y / T_z Z$$

for each $z \in Z$. N_z is called the Normal space of $Z \hookrightarrow Y$ at the point $z \in Z$. The set $N_{Z|Y} = \cup N_z$ for all $z \in Z$ is called the Normal bundle of $Z \hookrightarrow Y$ and is a vector bundle. Denote $TY|_Z$ to be the vector bundle TY restricted to Z, then by definition, we have

$$0 \to TZ \to TY|_Z \to N_{Z|Y} \to 0,$$

which gives,

$$0 \to N^*_{Z|Y} \to \Omega^1 Y|_Z \to \Omega^1 Z \to 0.$$

The normal bundle in a local coordinate chart.

Let Z be a r-dimensional submanifold of an $n+r$-dimensional manifold Y. Obviously, Z is covered by coordinate neighbourhoods Y_i in Y. We choose a local coordinate

$$(y_i, z_i) = (y_i^1, \ldots, y_i^n, z_i^1, \ldots, z_i^r)$$

on each neighbourhood Y_i such that $Z \cap Y_i$ coincides with the subspace of Y_i determined by

$$y_i^1 = \ldots = y_i^n = 0.$$

A general 1-form on $\Omega^1 Y|_{Y_i}$ is given by

$$\omega = \omega_A(y, z) dy_i^A + \omega_a(y, z) dz_i^a,$$

and a typical vector of TZ is given by $\frac{\partial}{\partial z_i^a}$, where $A = 1, \ldots, n$ and $a = 1, \ldots, r$. Thus,

$$\begin{aligned} \omega \in N^* &\Leftrightarrow \omega \lrcorner \frac{\partial}{\partial z_i^a} = 0 \\ &\Leftrightarrow \omega_A(0, z) dy_i^A \lrcorner \frac{\partial}{\partial z_i^b} + \omega_a(0, z) dz_i^a \lrcorner \frac{\partial}{\partial z_i^b} = 0, \end{aligned}$$

where $A = 0, \ldots, n$ and $a, b = 0, \ldots, r$.

But

$$\begin{aligned} d\omega_i^A \lrcorner \frac{\partial}{\partial z_i^b} &= \frac{\partial \omega_i^A}{\partial z_i^b} = 0 \\ dz^a \lrcorner \frac{\partial}{\partial z_i^b} &= \delta_b^a. \end{aligned}$$

Hence $\omega \in N^*$ if and only if $\omega_b(0, z) = 0$ and so

$$\omega|_Z = \omega_A(0, z) dy_i^A|_Z,$$

for any $\omega \in N^*$, where $A = 0, \ldots, n$.

Similarly, a tangent vector v in N at the point $z \in Z$ will have the form

$$v_z = \nu_A(0, z)\frac{\partial}{\partial y_i^A}|_z.$$

The Normal bundle of a Legendre submanifold.

Let Z be a n-dimensional Legendre submanifold of a $(2n+1)$-dimensional complex contact manifold with contact structure D and line bundle L. Then the normal bundle $N_{Z|Y}$ of Z is isomorphic to $J^1 L_Z$, where $L_Z = L|_Z$.

From Section (2.2) we have the exact sequence

$$0 \to E \otimes \Omega^1 M \to J^1 E \to E \to 0$$

where E was any vector bundle over a manifold M. Therefore, in this case, with Z a Legendre submanifold of a contact manifold Y, because $J^1 L_Z = N_{Z|Y}$, $N_{Z|Y}$ fits into the exact sequence

$$0 \to \Omega^1 M \otimes L_Z \to N_{Z|Y} \to L_Z \to 0.$$

2.4 Kodaira relative deformation theory.

We shall now give an overview of a theorem by Kodaira from 1962. For further details see [15] and [16].

2.4.1 Analytic families.

Let Y be a complex manifold of dimension $n + r$. Let M be a complex manifold. We can form the product space $Y \times M$ and let π_1 be the canonical projection of $Y \times M$ onto M.

Definition 2.12. *Let $\mathcal{Z}$ be a complex analytic submanifold of $Y \times M$ of codimension r such that*

1. *for each $x \in M$ the set $\mathcal{Z} \cap (Y \times x)$ is a connected, compact submanifold of $Y \times x$ of dimension n,*

2. *for each $z \in \mathcal{Z}$, there exists r holomorphic functions*

$$f_1(y, x), \ldots, f_r(y, x)$$

defined on a neighbourhood U_z of z in $Y \times M$ with coordinates $(y^1, \ldots, y^{n+r}, x)$ such that

$$rank \frac{\partial(f_1, \ldots, f_r)}{\partial(y^1, \ldots, y^{n+r})} = r$$

and $\mathcal{Z}$ is defined by the simultaneous equations

$$f_1(y, x) = \ldots = f_r(y, x) = 0.$$

then the pair $(\mathcal{Z}, M)$ is said to be a complex analytic family of Y. We call M the Kodaira moduli space of the family. For each point $x \in M$ we set

$$Z_x \times x = \mathcal{Z} \cap (Y \times x).$$

The submanifold Z_x of Y is called the fibre of $\mathcal{Z}$ over x.

So an analytic family is a complex submanifold $\mathcal{Z} \hookrightarrow Y \times M$ such that the restriction of the projection π_1 of $\mathcal{Z}$ is a proper regular map. Let π_2 be the canonical projection from $Y \times M$ to Y and let $\nu := \pi_1|_{\mathcal{Z}}$ and $\mu := \pi_2|_{\mathcal{Z}}$, the family $\mathcal{Z}$ then has a double fibration structure

$$Y \overset{\mu}{\leftarrow} \mathcal{Z} \overset{\nu}{\rightarrow} M$$

For each $x \in M$ we say that the compact complex submanifold

$$Z_x = \mu \circ \nu^{-1}(x) \hookrightarrow Y$$

belongs to the family $\mathcal{Z}$.

Definition 2.13. *Let x_0 be a point in M and let $(\mathcal{Z}, M)$ be an analytic family of compact submanifolds Z_x of Y (where $x \in M$) such that for any analytic family $(\mathcal{W}, N)$ of compact submanifolds W_y, $y \in N$, of Y with the property*

$$W_{y_0} = Z_{x_0},$$

for a point y_0 of N, there exists a neighbourhood U_{y_0} of N and a holomorphic map

$$h : y \rightarrow x = h(y)$$

of U_{y_0} into M sending y_0 into x_0 such that

$$W_y = Z_{h(y)}$$

for $y \in U_{y_0}$. Then we say that $(\mathcal{Z}, M)$ is maximal at the point x_0 of M.

Let $(\mathcal{Z}, M)$ be an analytic family of Y. There is a linear map j_x,

$$j_x : T_xM \longrightarrow H^0(\nu^{-1}(x), N_{\nu^{-1}|\mathcal{Z}}),$$

which is the natural lift of a tangent vector $v \in T_xM$ to a global section of the normal bundle of the submanifold $\nu^{-1}(x) = Z_z \times x \hookrightarrow \mathcal{Z}$. We then define the composition

$$k_x := \mu_* \circ j_x,$$

where

$$\mu_* : H^0(\nu^{-1}(x), N_{\nu^{-1}(x)|\mathcal{Z}}) \longrightarrow H^0(Z_x, N_{Z_x|Y}),$$

is the differential of μ.

Definition 2.14. *An analytic family $(\mathcal{Z}, M)$ of Y is called complete if the map k_x is an isomorphism for all $x \in M$. Thus dim $M = h^0(Z_x, N_{Z_x|Y})$.*

In 1962 Kodaira proved the following theorem:

Theorem 2.15. *Let Y be a complex manifold and let Z be a compact complex submanifold of Y with normal bundle N. If $H^1(Z, N) = 0$ then Z belongs to a complete maximal analytic family $(\mathcal{Z}, M)$ of compact submanifolds of Y.*

Proof. See Kodaira ([15]) □

2.5 Deformations of compact Legendre submanifolds of complex contact manifolds.

We shall now be interested in the specialisation (which will actually turn out to be a generalisation) of the Kodaira relative deformation problem where the initial manifold Y is a complex contact manifold with a compact Legedre submanifold Z and we are concerned with the set of all holomorphic deformations of Z inside Y which remain Legendre.

Definition 2.16. *Let $(\mathcal{Z}, M)$ be an analytic family of a complex contact manifold Y be such that for any point $x \in M$, the corresponding subset $Z_x = \mu \circ \nu^{-1}(x)$ is a Legendre submanifold of Y. Then we say that $(\mathcal{Z}, M)$ is an analytic family of compact Legendre submanifolds. The parameter space M is called a Legendre moduli space.*

Let $(\mathcal{Z}, M)$ be a analytic family of compact Legendre submanifolds of a complex manifold Y. According to Kodaira there is a natural linear map $k_x : T_x M \to H^0(Z_x, N_{Z_x|Y})$. We say that the analytic family $(\mathcal{Z}, M)$ is complete at a point $x \in M$ if the composition

$$s_x : T_x M \xrightarrow{k_x} H^0(Z_x, N_{Z_x|Y}) \xrightarrow{pr} H^0(Z_x, L_{Z_x})$$

provides an isomorphism between the tangent space of M at the point x and the vector space of global sections of the contact line bundle $L_{Z_x} := L|_{Z_x}$ over Z_x, where L is the line bundle defined by the contact structure on Y.

In 1995 Merkulov ([24]) proved a simple condition for the existence of complete Legendre moduli spaces.

Theorem 2.17. *Let Z be a compact complex Legendre submanifold of a complex contact manifold (Y, L). If $H^1(Z, L_Z) = 0$, then there exists a complete analytic family of compact Legendre submanifolds $(\mathcal{Z}, M)$ containing Z. This family is maximal and dim $M = h^0(Z, L_Z)$.*

Proof. See Merkulov [24] □

Let Z be a complex manifold and L_Z a line bundle on Z. There is a natural "evaluation" map $H^0(Z, L_Z) \otimes \mathcal{O}_Z \to J^1 L_Z$ whose dualization gives rise to the canonical map

$$L_Z \otimes S^{k+1}(J^1 L_Z)^* \to L_Z \otimes S^k(J^1 L_Z)^* \otimes [H^0(Z, L_Z)]^*$$

which in turn gives rise to the map of cohomology groups

$$H^1(Z, L_Z \otimes S^{k+1}(J^1 L_Z)^*) \xrightarrow{\phi} H^1(Z, L_Z \otimes S^k(J^1 L_Z)^*) \otimes [H^0(Z, L_Z)]^*.$$

For future reference, we define a vector subspace

$$\tilde{H}^1(Z, L_Z \otimes S^{k+1}(J^1 L_Z)^*) := \text{ker } \phi \subset H^1(Z, L_Z \otimes S^{k+1}(J^1 L_Z)^*).$$

2.6 G-structures induced on Legendre moduli spaces of generalised flag varieties.

Recall that a generalised flag variety Z is a compact simply connected homogeneous Kahler manifold. Any such a manifold is of the form $Z = G/P$, where G is a complex semisimple

Lie group and $P \subset G$ a fixed parabolic subgroup. Assume that such an Z is embedded as a Legendre submanifold into a complex contact manifold (Y, L) with contact line bundle L such that $L_Z := L|_Z$ is very ample. Then the Bott-Borel-Weil theorem and the fact that any holomorphic line bundle on Z is homogeneous imply that $H^1(Z, L_Z) = 0$. Therefore, the above theorem, there exists a complete analytic family of compact Legendre submanifolds $\{Z_x \hookrightarrow Y | x \in M\}$, i.e. the initial data "$Z \hookrightarrow Y$" give rise to a new complex manifold M which, as the following result shows, comes equipped with a rich geometric structure.

Theorem 2.18. *Let Z be a generalised flag variety embedded as a Legendre submanifold into a complex contact manifold Y with contact line bundle L such that L_Z is very ample on Z. Then*

1. *There exists a complete analytic family $(\mathcal{Z}, M)$ of compact Legendre submanifolds with moduli space M being an $h^0(Z, L_Z)$-dimensional complex manifold. For each $x \in M$, the associated Legendre submanifold Z_x is isomorphic to Z.*

2. *The Legendre moduli space M comes equipped with an induced irreducible G-structure, $\mathcal{G}_{ind} \to M$, with G isomorphic to the connected component of the identity of the group of all global biholomorphisms $\phi : L_Z \to L_Z$ which commute with the projection $\pi : L_Z \to Z$. The Lie algebra of G is isomorphic to $H^0(Z, L_Z \otimes (J^1 L_Z)^*)$.*

3. *If $\mathcal{G}_{ind}$ is k-flat, $k \geq 0$, then the obstruction for $\mathcal{G}_{ind}$ to be $(k+1)$-flat is given by a tensor field on M whose value at each $x \in M$ is represented by a cohomology class $\rho_x^{[k+1]} \in \tilde{H}^1(Z_x, L_{Z_x} \otimes S^{k+2}(J^1 L_{Z_x})^*)$.*

4. *If $\mathcal{G}_{ind}$ is 1-flat, then the bundle of all torsion-free connections in $\mathcal{G}_{ind}$ has the typical fibre of an affine space modelled on $H^0(Z, L_Z \otimes S^2(J^1 L_Z)^*)$.*

Proof. See Merkulov ([24]) □

Remark:

The above theorem is actually valid for a larger class of compact complex manifolds Z than the class of generalised flag varieties - the only vital assumptions are that Z is rigid and the cohomology groups $H^1(Z, \mathcal{O}_Z)$ and $H^1(Z, L_Z)$ vanish.

The geometric meaning of cohomology classes $\rho_x^{[k+1]} \in \tilde{H}^1(Z_x, L_{Z_x} \otimes S^{k+2}(J^1L_{Z_x})^*)$ of Theorem (2.18 (3)) is very simple - they compare to $(k+2)$th order the germ of the Legendre embedding $Z_x \hookrightarrow Y$ with the "flat" model, $Z_x \hookrightarrow J^1L_{Z_x}$, where the ambient contact manifold is just the total space of the vector bundle $J^1L_{Z_x}$ together with its canonical contact structure and the Legendre submanifold Z_x is realised as a zero section of $J^1L_{Z_x} \to Z_x$. Therefore, the cohomology class $\rho_x^{[k]}$ can be called the kth Legendre jet of Z_x in Y. Then it is natural to call a Legendre submanifold $Z_x \hookrightarrow Y$ k-flat is $\rho_x^{[k]} = 0$. With this terminology, the item (3) of Theorem (2.18) acquires a rather symmetric form: *the induced G-structure on the moduli space M of a complete analytic family of compact Legendre submanifolds is k-flat if and only if the family consists of k-flat Legendre submanifolds.*

2.6.1 Examples

This general construction can be illustrated by two well known examples.

1. The first example is a "generic" $GL(n,\mathbb{C})$-structure on an n-dimensional manifold M. The associated twistorial data $Z \hookrightarrow Y$ is easy to describe: the complex contact manifold Y is the projectivized cotangent bundle $\mathbb{P}(\Omega^1 M)$ with its natural contact structure while $Z = \mathbb{CP}^{n-1}$ is just a fiber of the projection $\mathbb{P}(\Omega^1 M) \to M$. The corresponding complete family $\{Z_x \hookrightarrow Y | x \in M\}$ is the set of all fibres of this fibration. Since $L_Z = \mathcal{O}(1)$ and $J^1L_Z = \mathbb{C}^n \otimes \mathcal{O}_Z$, we have $H^1(Z, L_Z \otimes S^{k+2}(J^1L_Z)^*) = 0$ for all $k \geq 0$ which confirms the well-known fact that any $GL(n,\mathbb{C})$-structures on an n-dimensional manifold are locally flat.

2. The second example is a pair $Z \hookrightarrow Y$ consisting of an n-quadric Q_n embedded into a $(2n+1)$-dimensional contact manifold (Y, L) with $L_z \simeq i^*\mathcal{O}_{\mathbb{CP}^{n+1}}(1)$, $i : Q_n \to \mathbb{CP}^{n+1}$ being a standard projective realisation of Q_n. It is easy to check that in this case $H^0(Z, L_Z \otimes (J^1L_Z)^*)$ is precisely the conformal algebra implying that the associated $(n+2)$-dimensional Legendre moduli space M comes equipped canonically with a conformal structure. Since $H^1(Z, L_Z \otimes S^2(J^1L_Z)^*) = 0$, the induced conformal structure must be torsion-free in agreement with the classical result of differential geometry [6]. Easy calculations show that the vector space $H^1(Z, L_Z \otimes S^3(J^1L_Z)^*)$ is exactly the subspace

of $TM \otimes \Omega^1 M \otimes \Omega^2 M$ consisting of tensors with Weyl curvature symmetries [6]. Thus Theorem (2.18(3)) implies the well-known Schouten conformal flatness criterion [6]. Since $H^0(Z, L_Z \otimes S^2(J^1 L_Z)^*)$ is isomorphic to the typical fibre of $\Omega^1 M$, the set of all torsion-free affine connections preserving the induced conformal structure is the affine space modelled on $H^0(M, \Omega^1 M)$, again in agreement with the classical result [6].

How large is the family of G-structures which can be constructed by twistor methods of Theorem (2.18)? As the following result shows, in the category of irreducible 1-flat G-structures this class is a large as one could wish.

Theorem 2.19. *1. Let H be one of the following representations:*

(a) Spin$(2n+1, \mathbb{C})$ acting on $\mathbb{C}^{2^n}$, $n \geq 3$;

(b) Sp$(2n, \mathbb{C})$ acting on $\mathbb{C}^{2n}$, $n \geq 2$;

(c) G_2 acting on $\mathbb{C}^7$.

Suppose that $G \subset GL(m, \mathbb{C})$ is a connected semisimple Lie subgroup whose decomposition into a locally direct product of simple groups contains H. If $\mathcal{G}$ is any irreducible 1-flat $G \cdot \mathbb{C}^*$-structure on an m-dimensional manifold M, then there exists a complex contact manifold (Y, L) and a generalised flag variety Z embedded into Y as a Legendre submanifold with L_Z being very ample, such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and $\mathcal{G} \subset \mathcal{G}_{ind}$. In particular, when $G = H$ one has

(a) $Z = SO(2n+2, \mathbb{C})/U(n+1)$ and $\mathcal{G}_{ind}$ is a Spin$(2n+2, \mathbb{C}) \cdot \mathbb{C}^*$-structure;

(b) $Z = \mathbb{CP}^{2n+1}$ and $\mathcal{G}_{ind}$ is a $GL(2n, \mathbb{C})$-structure;

(c) $Z = Q_5$ and $\mathcal{G}_{ind}$ is a $CO(7, \mathbb{C})$-structure.

2. Let $G \subset GL(m, \mathbb{C})$ be an arbitrary connected semisimple Lie subgroup whose decomposition into a locally direct product of simple groups does not contain any of the groups H considered in (1). If $\mathcal{G}$ is any irreducible 1-flat $G \cdot \mathbb{C}^*$-structure on an m-dimensional manifold M, then there exists a complex contact manifold (Y, L) and a Legendre submanifold $Z \hookrightarrow Y$ with $Z = G/P$ for some parabolic subgroup $P \subset G$ and with L_Z being very ample, *such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and $\mathcal{G} = \mathcal{G}_{ind}$.*

Proof. See Merkulov ([24]) □

The conclusion is that there are very few irreducible G-structures which can *not* be constructed by the twistor methods discussed. It is also worth pointing out that Theorem (2.18) gives rise to a new and rather effective machinery to search for exotic holonomies.

2.7 From Kodaira to Legendre moduli spaces and back.

In this subsection we first show that any complete Kodaira moduli space can be interpreted as a complete Legendre moduli space and then use this fact to prove a proposition about canonically induced geometric structures on Kodaira moduli spaces.

If $Z \hookrightarrow Y$ is a complex submanifold, there is an exact sequence of vector bundles

$$0 \to N^*_{Z|Y} \to \Omega^1 Y|_Z \to \Omega^1 Z \to 0,$$

which induces a natural embedding, $\mathbb{P}(N^*_{Z|Y}) \hookrightarrow \mathbb{P}(\Omega^1 Y)$, of total spaces of the associated projectivised bundles. The manifold $\hat{Y} = \mathbb{P}(\Omega^1 Y)$ carries a natural contact structure such that the constructed embedding $\hat{Z} = \mathbb{P}(N^*_{Z|Y}) \hookrightarrow \hat{Y}$ is a Legendre one. Indeed, the contact distribution $D \subset T\hat{Y}$ at each point $\hat{y} \in \hat{Y}$ consists of those tangent vectors $X_{\hat{y}} \in T_{\hat{y}}\hat{Y}$ which satisfy the equation $< \hat{y}, \tau_*(X_{\hat{y}}) >= 0$, where $\tau : \hat{Y} \to Y$ is a natural projection and the angular brackets denote the pairing of 1-forms and vectors at $\tau(\hat{y}) \in Y$. Since the submanifold $\hat{Z} \subset \hat{Y}$ consists precisely of those projective classes of 1-forms in $\Omega^1 Y|Z$ which vanish when restricted on TZ, we conclude that $T\hat{Z} \subset D|_{\hat{Z}}$. One may check that this association

$$\text{Kodaira moduli space} \to \text{Legendre moduli space}$$

$$\{Z_x \hookrightarrow Y | x \in M\} \to \{\hat{Z}_x := \mathbb{P}(N^*_{X_x|Y}) \hookrightarrow \hat{Y} := \mathbb{P}(\Omega^1 Y) | x \in M\}$$

preserves completeness while changing its meaning, i.e a complete Kodaira family of compact submanifolds is mapped into a complete family of compact complex Legendre submanifolds (which is usually not complete in the Kodaira sense).

Chapter 3

Twistor Transform Of Symmetries.

3.1 Basic Notions.

Let M be any complex manifold and let $\Omega^1 M$ be its cotangent bundle. There is a canonical holomorphic 2-form on $\Omega^1 M$ which can be defined as follows. If $\pi : \Omega^1 M \to M$ is the base point projection, then, for every $v \in T_\alpha(\Omega^1 M)$, define a 1-form θ on the total space $\Omega^1 M$ by the equation

$$\theta(v) = \alpha(\pi_*(v)).$$

Thus θ is a holomorphic 1-form on $\Omega^1 M$ and $\omega = d\theta$ is the canonical symplectic form on $\Omega^1 M$.

To see this, let us compute θ in local coordinates. Let $z : U \to \mathbb{R}^n$ be a local holomorphic coordinate chart. Since the 1-forms $dz^1, \ldots, dz^n$ are linearly independent at every point of U, it follows that there are unique functions p_i on $\Omega^1 M$ so that, for $\alpha \in \Omega^1_a U$,,

$$\alpha = p_1(\alpha) dz^1|_a + \ldots + p_n(\alpha) dz^n|_a.$$

The functions $z^1, \ldots, z^n, p_1, \ldots, p_n$ then form a holomorphic coordinate system on $\Omega^1 M$ in which the projection mapping π is given by

$$\pi(z, p) = z.$$

It is then straightforward to compute that, in this coordinate system,

$$\theta = p_\alpha dz^\alpha$$

Hence, $\omega = dp_\alpha \wedge dz^\alpha$ and so is obviously non-degenerate.

Thus the total space of the cotangent bundle $\Omega^1 M$ has a canonical holomorphic symplectic structure represented in a natural local coordinate system $\{z^\alpha, p_\alpha\}$ by the 2-form $\omega = dp_\alpha \wedge dz^\alpha$. Then the sheaf of holomorphic functions on $\Omega^1 M$ is a sheaf of Lie algebras relative to the Poisson bracket $\{f, g\} = \omega^{-1}(df, dg)$, or, in local coordinates,

$$\{f, g\} = \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial z^\alpha} - \frac{\partial f}{\partial z^\alpha}\frac{\partial g}{\partial p^\alpha}.$$

Suppose $\mathfrak{G}$ is a $CO(n, \mathbb{C})$-structure on an n-dimensional manifold M. With such a $\mathfrak{G}$ there is naturally associated a subbundle $\tilde{\mathfrak{F}} \subset \Omega^1 M$

$$\tilde{\mathfrak{F}} = \{\mu \in \Omega^1 M \backslash 0 : g(\mu, \mu) = 0\}$$

where g is any metric in the conformal class $[\Omega^2 g_{ab}]$ defined by $\mathfrak{G}$. This subspace $\tilde{\mathfrak{F}}$, called the subspace of "null" or "isotropic" 1-forms, is a hypersurface in $\Omega^1 M$. There is a natural $\mathbb{C}^*$-action on $\tilde{\mathfrak{F}}$:

$$\mathbb{C}^* : \tilde{\mathfrak{F}} \rightarrow \tilde{\mathfrak{F}},$$

$$\mu \rightarrow \lambda\mu$$

where $\lambda \in \mathbb{C}^*$. Note that if $g(\mu, \mu) = 0$, then $g(\lambda\mu, \lambda\mu) = \lambda^2 g(\mu, \mu) = 0$. Let $\mathbb{P}(\Omega^1 M) = \Omega^1 M \backslash 0 / \mathbb{C}^*$ and let $F = \tilde{\mathfrak{F}}/\mathbb{C}^* \subset \mathbb{P}(\Omega^1 M)$ to be the quadric lying in the projectivized cotangent bundle $\mathbb{P}(\Omega^1 M)$.

3.1.1 Rank 1 Distribution On $\tilde{\mathfrak{F}}$.

Consider the symplectic form ω restricted to $\tilde{\mathfrak{F}} \subset \Omega^1 M$. We shall denote it by $\omega|_{\tilde{\mathfrak{F}}}$. Now

$$d(\omega|_{\tilde{\mathfrak{F}}}) = (d\omega)|_{\tilde{\mathfrak{F}}} = 0|_{\tilde{\mathfrak{F}}} = 0$$

so that $\omega|_{\tilde{\mathfrak{F}}}$ is a closed 2-form on $\tilde{\mathfrak{F}}$. Clearly, $\omega|_{\tilde{\mathfrak{F}}}$ is degenerate (as the dimension of $\tilde{\mathfrak{F}}$ is odd). Hence, there is a distribution $\mathfrak{D}_{\tilde{\mathfrak{F}}} \subset T\tilde{\mathfrak{F}}$ defined by

$$\mathfrak{D}_{\tilde{\mathfrak{F}}} := \{v \in T\tilde{\mathfrak{F}} : \omega(v, .)|_{\tilde{\mathfrak{F}}} = 0\}$$

which is non-empty. Actually, rank $\mathfrak{D} = 1$ and it is locally spanned by one vector field

$$\mathcal{D} = \frac{\partial(g^{ab} p_a p_b)}{\partial p^c}[\frac{\partial}{\partial z^c} - \Gamma^b_{ca} p_b \frac{\partial}{\partial p_a}]$$

where g^{ab} is inverse to a metric g_{ab} and Γ^b_{ca} is the Levi-Civita connection of g_{ab}, g_{ab} being any metric in the conformal class G

3.1.2 Killing Vector Fields Versus $\mathfrak{D}_{\tilde{\mathfrak{F}}}$.

Take any vector field $X = X^a(z)\frac{\partial}{\partial z^c}$ on M. Define an associated function $h_X = X^a p_a$ on $\Omega^1 M$.

Theorem 3.1. *$h_X|_{\tilde{\mathfrak{F}}}$ is constant along the distribution $\mathfrak{D}_{\tilde{\mathfrak{F}}}$ if and only if X is a conformal Killing vector field, in other words $\mathfrak{D}h_X|_{\tilde{\mathfrak{F}}} = 0$ if and only if X satisfies $\nabla^\mu X^\nu + \nabla^\nu X^\mu = \Omega^2 g^{\mu\nu}$.*

Proof. $\tilde{\mathfrak{F}}$ is a quadratic hypersurface in $\tilde{\mathfrak{F}} \hookrightarrow \Omega^1 M\backslash 0$, given by one equation

$$f = g^{\alpha\beta} p_\alpha p_\beta = 0$$

Indeed,

$$\begin{aligned}\tilde{\mathfrak{F}} &= \{\mu \in \Omega^1 M\backslash 0 : g(\mu,\mu) = 0\} = \{p_\alpha dz^\alpha : g(p_\alpha dz^\alpha, p_\beta dz^\beta) = 0\} \\ &= \{p_\alpha dz^\alpha : p_\alpha p_\beta g(dz^\alpha, dz^\beta) = 0\} = \{p_\alpha dz^\alpha : p_\alpha p_\beta g^{\alpha\beta} = 0\}\end{aligned}$$

Let X be a vector field then

$$X = X^\alpha(z)\frac{\partial}{\partial z^\alpha} \Rightarrow h_X = X^\alpha(z) p_\alpha$$

on $\Omega^1 M$. Clearly $h_X|_{\tilde{\mathfrak{F}}}$ is constant along the leaves of the distribution $\mathfrak{D}_{\tilde{\mathfrak{F}}}$ if and only if $\mathcal{D}h_X|_{\tilde{\mathfrak{F}}} = 0$, where

$$\mathcal{D} = \frac{\partial f}{\partial p_\alpha}\left(\frac{\partial}{\partial z^\alpha} + \Gamma^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\beta}\right)|_{\tilde{\mathfrak{F}}}$$

Let us study the equation $\mathfrak{D}h_X|_{\tilde{\mathfrak{F}}} = 0$. In local coordinates (z^α, p_α), it takes the form

$$\begin{aligned}&\frac{\partial(g^{\theta\phi} p_\theta p_\phi)}{\partial p_\alpha}\left(\frac{\partial(X^\delta p_\delta)}{\partial z^\alpha} + \Gamma^\gamma_{\alpha\beta} p_\gamma \frac{\partial(X^\delta p_\delta)}{\partial p_\beta}\right)|_{\tilde{\mathfrak{F}}} = 0 \\ &\Rightarrow (2g^{\theta\alpha} p_\theta)\left(\frac{\partial X^\delta}{\partial z^\alpha} p_\delta + \Gamma^\gamma_{\alpha\beta} p_\gamma X^\beta\right)|_{\tilde{\mathfrak{F}}} = 0 \\ &\Rightarrow \left(2g^{\theta\alpha}\frac{\partial X^\delta}{\partial z^\alpha} p_\delta p_\theta + 2g^{\theta\alpha}\Gamma^\gamma_{\alpha\beta} p_\gamma p_\theta X^\beta\right)|_{\tilde{\mathfrak{F}}} = 0 \\ &\Rightarrow \left(2g^{\theta\alpha}\frac{\partial X^\delta}{\partial z^\alpha} p_\delta p_\theta + 2g^{\theta\alpha}\Gamma^\gamma_{\alpha\beta} p_\gamma p_\theta X^\beta\right) = \Omega^2 g^{\alpha\beta} p_\alpha p\beta\end{aligned}$$

for some function Ω. Changing the notation slightly we have,

$$g^{\mu\alpha}\frac{\partial X^\nu}{\partial z^\alpha} p_\mu p_\nu + g^{\mu\alpha}\Gamma^\nu_{\alpha\beta} X^\beta p_\mu p_\nu = \Omega^2 g^{\mu\nu} p_\mu p_\nu$$

or

$$g^{\mu\alpha}\frac{\partial X^\nu}{\partial z^\alpha} + g^{\nu\alpha}\frac{\partial X^\mu}{\partial z^\alpha} + g^{\mu\alpha}\Gamma^\nu_{\alpha\beta} X^\beta + g^{\nu\alpha}\Gamma^\mu_{\alpha\beta} X^\beta = \Omega^2 g^{\mu\nu}$$

which is precisely the equation

$$\nabla^\mu X^\nu + \nabla^\nu X^\mu = \Omega^2 g^{\mu\nu}.$$

We have already shown that this equation is equivalent to saying that X is a conformal Killing vector field (see Section (1.5.1)). □

3.2 Twistor Transform Of A Conformal Killing Vector Field.

The integral curves of the rank 1 distribution $\mathfrak{D}|_{\tilde{\mathfrak{F}}}$ foliate $\tilde{\mathfrak{F}}$. Let $\tilde{Y}$ be the parameter space of these integral curves. If M is sufficiently "small" $\tilde{Y}$ is a holomorphic complex manifold of dimension $2n-2$. Moreover, $\tilde{Y}$ has a symplectic form $\tilde{\omega}$ such that $\mu^*(\tilde{\omega}) = \omega|_{\tilde{\mathfrak{F}}}$. There is a $\mathbb{C}^*$-action on $\tilde{Y}$. The quotient $\tilde{Y}/\mathbb{C}^* = Y^{2n-3}$, is a contact manifold of dimension $2n-3$. When the dimension of Y is one, then this space is called the ambitwistor space. We shall also call Y an ambitwistor space for any dimension. The contact line bundle L on Y is just the quotient $L = \tilde{\mathfrak{F}} \times \mathbb{C}/\mathbb{C}^*$ relative to the natural multiplication map

$$\tilde{\mathfrak{F}} \times \mathbb{C} \to \tilde{\mathfrak{F}} \times \mathbb{C},$$

$$(p, c) \to (\lambda p, \lambda, c)$$

where $\lambda \in \mathbb{C}^*$.

Given a conformal Killing vector field X, we produce the function $h_X = X^a p_a$. We know that this function is constant along the integral curves of $\mathfrak{D}|_{\tilde{\mathfrak{F}}}$. Hence $h_X = \mu^{-1}(f_X)$, for some function f_X on $\tilde{Y}$ which is homogenous of degree 1 with respect to the $\mathbb{C}^*$ action.

Theorem 3.2. *Any conformal Killing vector field X on (M, G) gives rise to a global section $s_X \in H^0(Y, L^*)$ on the associated ambitwistor space.*

To show this we must first introduce the notion of Euler vector fields.

3.2.1 Euler Vector Fields.

Let $E \to Y$ be a holomorphic vector bundle of rank k over a n-dimensional manifold Y. Let $\{\mathfrak{U}_{(k)}, (z^a_{(k)}, v^i_{(k)})\}$ be a coordinate covering of E. Then on $\mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)} \neq \emptyset$ we have

$$z^a_{(k)} = g^a(z^b_{(l)}), v^i_{(k)} = g^i_j(z^b_{(l)}) v^j_{(l)}$$

for some holomorphic functions $g^a(z), g^i_j(z)$, where $z^a_{(k)}$ are coordinates in the base and $v^i_{(k)}$ are coordinates in the fibre.

Define in $\mathfrak{U}_{(k)}$ the vector field

$$\epsilon_{(k)} = \sum_{i=1}^{n} v^i_{(k)} \frac{\partial}{\partial v_{(k)}{}^i}$$

It is easy to show that $\epsilon_{(k)} = \epsilon_{(l)}$ on each $\mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)} \neq \emptyset$. Hence, we have a globally defined vector field on the total space E which we call the Euler vector field.

Now let be L be a line bundle and $(z^a_{(k)}, s_{(k)})$ a local coordinate system of L. In this case $\epsilon = s_{(k)} \frac{\partial}{\partial s_{(k)}}$ is the Euler vector field of L.

Lemma 3.3. *Let $L \to M$ be a complex line bundle over M. Let $\mathcal{O}_L$ be the sheaf of holomorphic functions on L. Then*

$$\{f \in \Gamma(L, \mathcal{O}_L); \epsilon f = kf\} = \Gamma(M, (L^*)^{\otimes^k})$$

Proof. Let $\mathfrak{U}_{(k)}$ be a covering of L such that $\pi^{-1}(\mathfrak{U}_{(k)}) = \mathfrak{U}_{(k)} \times \mathbb{C}$ Then $(z_{(k)}, s_{(k)}) \in \mathfrak{U}_{(k)} \times \mathbb{C}$ and $(z_{(l)}, s_{(l)}) \in \mathfrak{U}_{(l)} \times \mathbb{C}, \mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)} \neq \emptyset$ are the same point on L if $s_{(k)} = c_{(kl)}(z_{(k)})s_{(l)}$ where the transition function $c_{(kl)}(z_{(k)})$ is a non-vanishing holomorphic function on $\mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)}$. Let $f \in \Gamma(L, \mathcal{O}_L)$. Then locally on $\mathfrak{U}_{(k)}$ we have $f = f_{(k)}(z^a_{(k)}, s_{(k)})$ and

$$\epsilon_{(k)} f_{(k)} = k f_{(k)} \Leftrightarrow s_{(k)} \frac{\partial f_{(k)}}{\partial s_{(k)}} = k f_{(k)} \Leftrightarrow f_{(k)} = f^0_{(k)}(z_{(k)}) s^k_{(k)}$$

Thus, on $\mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)} \neq \emptyset$, $f_{(l)} = f^0_{(l)}(z_{(l)}) s^k_{(l)} = f^0_{(l)}(z_{(l)}) c_{(kl)}(z_{(k)})^k s^k_{(k)} = f^0_{(k)}(z_{(k)}) s^k_{(k)} = f_{(k)}$. Thus,

$$f^0_{(k)}(z_{(k)}) = c_{(kl)}(z_{(k)})^k f^0_{(l)}(z_{(l)})$$

$f^0_{(k)}((k))$ defines a global section of $(L^*)^{\otimes^k}$ over M since the transformation law of $f^0_{(k)}(z_{(k)})$ is the same as the transformation law of a section $\omega \in \Gamma(M, (L^*)^{\otimes^k})$. To see this, locally on $\mathfrak{U}_{(k)}$, $\omega_{(k)} \in \Gamma(\mathfrak{U}_{(k)}, (L^*)^{\otimes^k})$ must have the form $\omega_{(k)} = r_{(k)}(z_{(k)}) e \otimes \ldots \otimes e$ where $e \in \Gamma(\mathfrak{U}_{(k)}, L^*)$ which vanishes nowhere. On $\mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)} \neq \emptyset$, $e_{(k)} = h_{(kl)}(z_{(k)}) e_{(l)}$, where $h_{(kl)}(z_{(k)})$ is a non-vanishing holomorphic function on $\mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)}$. Thus we have on $\mathfrak{U}_{(k)} \cap \mathfrak{U}_{(l)} \neq \emptyset$

$$\begin{aligned} \omega_{(l)} &= r_{(l)}(z_{(l)}) e_{(l)} \otimes \ldots \otimes e_{(l)} = r_{(l)}(z_{(l)})(h_{(kl)}(z_{(k)}) e_{(k)}) \otimes \ldots \otimes h_{(kl)}((z_{(k)}) e_{(k)}) \\ &= r_{(l)}(z_{(l)})(h_{(kl)}(z_{(k)}))^k e_{(k)} \otimes \ldots \otimes e_{(k)} = r_{(k)}(z_{(k)}) e_{(k)} k \otimes \ldots \otimes e_{(k)} = \omega_{(k)} \end{aligned}$$

Thus,

$$r_{(k)}(z_{(k)}) = (h_{(kl)}(z_{(k)}))^k r_{(l)}(z_{(l)})$$

as required. □

Proof Of Theorem: From the above Lemma, when $k = 1$ we have

$$\{f \in \Gamma(L, \mathcal{O}_L); \epsilon f = f\} = \Gamma(M, L^*)$$

Then since s_X, given by $h_X|_{\tilde{\mathfrak{F}}} = \mu^*(s_X)$, is a global function on $\tilde{Y}$ homogenous of degree one and is an element of $\{f \in \Gamma(L, \mathcal{O}_L); \epsilon f = f\}$, s_X is an element of $H^0(Y, L^*)$. QED.

3.3 Inverse Twistor Transform.

In the previous section we have shown that any conformal Killing vector field X on the conformal manifold M gives rise to a global section $s_X \in H^0(Y, L^*)$ on the associated ambitwistor space. Now we want to show that this association;

$$\{Killing\ \ Vectors\ \ on\ \ M\} \to \{global\ \ sections\ \ of\ \ L^* \to Y\}$$

is one-to-one.

Let $s \in H^0(Y, L^*)$ by Lemma (3.3), s can be viewed as a global holomorphic function f_s on the total space of the contact line bundle L on Y of homogenity one with respect to the Euler field. Then $\mu^*(f_s)$ is a global holomorphic function on F which is homogenous of degree 1 with respect to the $\mathbb{C}^*$ action $p_\alpha \to \lambda p_\alpha$. By Lemma (3.3) again, $\mu^*(f_s)$ is a global section of the line bundle $\mathcal{O}(1)|_F$, where $\mathcal{O}(1)$ is the line bundle on the total space $\mathbb{P}(\Omega^1 M)$ dual to the tautological line bundle $\mathcal{O}(-1)$. We denote this element of $H^0(F, \mathcal{O}(1)|_F)$ by $\hat{s}$.

Lemma 3.4.

$$H^0(F, \mathcal{O}(1)|_F) = H^0(\mathbb{P}(\Omega^1 M), \mathcal{O}(1)).$$

Proof. The space F is a relative irreducible hypersurface in the projectivised cotangent bundle $\mathbb{P}(\Omega^1 M)$. Since it is also irreducible any function $g \in \mathcal{O}|_{\mathbb{P}(\Omega^1 M)}$ vanishing on F must be of the form $g' g^{\alpha\beta} p_\alpha p_\beta$ for some g'. Since $g^{\alpha\beta} p_\alpha p_\beta$ is homogenous of degree two in p_α, it can be viewed of an element of $\mathcal{O}(2)$. Hence $g' \in \mathcal{O}(-2)$. Thus an the ideal sheaf of $F \subset \mathbb{P}(\Omega^1 M)$ is isomorphic to $\mathcal{O}(-2)$ and we have an exact sequence of sheafs,

$$0 \to \mathcal{O}_{\mathbb{P}(\Omega^1 M)}(-2) \to \mathcal{O}_{\mathbb{P}(\Omega^1 M)} \to \mathcal{O}_F \to 0$$

Tensoring this extension with $\mathcal{O}(1)$, we get

$$0 \to \mathcal{O}_{\mathbb{P}(\Omega^1 M)}(-1) \to \mathcal{O}_{\mathbb{P}(\Omega^1 M)}(1) \to \mathcal{O}_F(1) \to 0$$

This in turn implies the associated long exact sequence

$$0 \to H^0(\mathbb{P}(\Omega^1 M), \mathcal{O}(-1)) \to H^0(\mathbb{P}(\Omega^1 M), \mathcal{O}(1)) \to H^0(F, \mathcal{O}_F(1)) \to H^1(\mathbb{P}(\Omega^1 M), \mathcal{O}(-1)) \to \dots$$

However, by Leray spectral sequence (see [5]), we have for M "small",

$$H^0(\mathbb{P}(\Omega^1 M), \mathcal{O}(-1)) = H^0(M, \pi_*^0(\mathcal{O}(-1)))$$

and

$$H^1(\mathbb{P}(\Omega^1 M), \mathcal{O}(-1)) = H^1(M, \pi_*^1(\mathcal{O}(-1)))$$

In our case, $\pi^{-1}(x) = \mathbb{P}^{n-1}$. Hence,

$$\pi_*^0(\mathcal{O}(-1)) = H^0(\mathbb{P}^{n-1}, \mathcal{O}(-1)) = 0$$

and

$$\pi_*^1(\mathcal{O}(-1)) = H^1(\mathbb{P}^{n-1}, \mathcal{O}(-1)) = 0$$

and the result is proved. □

Corollary 3.5. *$H^0(Y, L^*)$ is isomorphic to the subspace* $\{f \in \mathcal{O}_{\Omega^1 M \setminus 0}, Ef = f, \mathfrak{D}f|_{\mathfrak{F}} = 0\}$

Proof. We have shown already that $s \in H^0(Y, L^*)$, defines an element $\hat{s}$ of $H^0(F, \mathcal{O}_F)$ and hence, by Lemma (3.4), an element $\hat{s}$ of $H^0(\mathbb{P}(\Omega^1 M), \mathcal{O}(1))$. By Lemma (3.3), $H^0(\mathbb{P}(M), \mathcal{O}(1)) = \{f \in \mathcal{O}_{\Omega^1 M \setminus 0}, Ef = f\}$, i.e. $\hat{s}$ gives rise to a global function on $\Omega^1 M$ of homogeneity of degree 1 with respect to the transformation $p_\alpha \to \lambda p_\alpha$, i.e. f is linear in p_α,

$$\hat{s} = X^\alpha(x) p_\alpha \tag{3.1}$$

for some smooth functions X^α on M.

By construction,

$$\hat{s}|_{\tilde{\mathfrak{F}}} = \mu^*(f_s)$$

implying

$$\mathfrak{D}\hat{s}|_{\tilde{\mathfrak{F}}} = 0$$

Thus we have produced an injection

$$H^0(Y, L^*) \to \{f \in \mathcal{O}_{\Omega^1 M \setminus 0}, Ef = f, \mathfrak{D}f|_{\tilde{\mathfrak{F}}} = 0\}$$

$$s \longrightarrow \hat{s}$$

□

This is actually one-to-one according to Section (3.2), thus we have the following theorem:

Theorem 3.6. *There is a one-to-one correspondence between the vector space of Killing vector fields X on a conformal manifold (M, G) and the vector space $H^0(Y, L^*)$, where Y is the associated ambitwistor space and L is the contact line bundle on Y.*

3.3.1 Grassmannian Manifolds.

We shall begin this section with a review of Grassmannian manifolds. Let V be a n-dimensional $\mathbb{C}$-vector space and let $Gr(k, V) :=$ {the set of k-dimensional subspaces of V} for $k < n$. Such a $Gr(k, V)$ is called a *Grassmannian Manifold.* The Grassmannian manifolds are clearly generalisations of the projective spaces (in fact, $\mathbb{P}(V) = Gr(1, V)$) and can be given a manifold structure in a fashion analogous to that used for projective spaces.

Consider, for example, $Gr(k, \mathbb{C}^n)$. We can define the map

$$\pi : M_{k\times n}(\mathbb{C}) \to Gr(k, \mathbb{C}^n)$$

where $M_{k\times n}(\mathbb{C})$ denotes the set of complex-valued $k \times n$ matrices and

$$\pi(A) = \pi \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{pmatrix} = \{k\text{-dimensional subspace of } \mathbb{C}^n \text{ spanned by the row vectors } \{a_j\} \text{ of } A\}$$

We notice that for $g \in GL(k, \mathbb{C})$ we have $\pi(gA) = \pi(A)$ (where gA is matrix multiplication), since the action of g merely changes the basis of $\pi(A)$. The mapping π is surjective. Let $Gr(k, \mathbb{C}^n)$ have the quotient topology induced by the map π, i.e $U \subset Gr(k, \mathbb{C}^n)$ is open if and only if $\pi^{-1}(U)$ is open in $M_{k\times n}(\mathbb{C})$. Hence π is continuous and $Gr(k, \mathbb{C}^n)$ is a Hausdorff space with a countable basis.

We can also make $Gr(k, \mathbb{C}^n)$ into a complex manifold. Consider $A \in M_{k\times n}(\mathbb{C})$ and let $\{A_1, \ldots, A_l\}$ be the collection of $k \times k$ minors of A. Since A has rank k, A_α is nonsingular for some α and there is a permutation matrix P_α such that

$$AP_\alpha = [A_\alpha \tilde{A}_\alpha]$$

where $\tilde{A}_\alpha$ is a $k \times (n-k)$ matrix. Note that is $g \in GL(k, \mathbb{C})$, then gA_α is a nonsingular minor of gA and $gA_\alpha = (gA)_\alpha$. Let $U_\alpha = \{S \in Gr(k, \mathbb{C}^n); S = \pi(A), \text{where } A_\alpha \text{ is nonsingular}\}$. This is well defined. The set U_α is defined by the condition det $A_\alpha \neq 0$; hence it is an open set in $Gr(k, \mathbb{C}^n)$, and $\{U_\alpha\}_{\alpha=1}^{l}$ covers $Gr(k, \mathbb{C}^n)$.

We define a map

$$h_\alpha : U_\alpha \to \mathbb{C}^{k(n-k)}$$

by setting

$$h_\alpha(\pi(A)) = A_\alpha^{-1} \tilde{A}_\alpha \in \mathbb{C}^{k(n-k)}$$

where $AP_\alpha = [A_\alpha \tilde{A}_\alpha]$. Again this is well defined and it is easy to see that this defines a holomorphic structure on $Gr(k, \mathbb{C}^n)$. Hence, $Gr(k, \mathbb{C}^n)$ is a complex manifold.

Let $x \in Gr(k, V)$ where V is a n-dimensional complex vector space, and let $\mathcal{S}_x$ be the k-dimensional subspace corresponding to x. We denote

$$\mathcal{S} = \{(x, v); x \in Gr(k, V), v \in \mathcal{S}_x\} \subset Gr(k, V) \times V,$$

We then let $\pi : \mathcal{S} \to Gr(k, V)$ be the projection onto the first component. We call $\mathcal{S}$ the tautological vector bundle on $Gr(k, V)$ and the fibre of $\mathcal{S}$ at $x \in Gr(k, V)$ is just the k-dimensional subspace $\mathcal{S}_x$.

We can represent the tangent bundle of $Gr(k, V)$, denoted by $\mathcal{T}Gr(k, V)$, in terms of the tautological vector bundle as follows. Let $Gr(k, V) \times V$ be trivial vector bundle with fibre V.

Then we define a rank $n-k$ vector bundle $\tilde{S}$ by the exact sequence

$$0 \to \mathcal{S} \to Gr(k,V) \times V \to \tilde{\mathcal{S}} \to 0.$$

The tangent bundle of the Grassmannian factorises as follows.

Theorem 3.7.

$$\mathcal{T}Gr(k,V) = \mathcal{S} \otimes \tilde{\mathcal{S}}^*.$$

Proof. See Manin [20]. □

We shall now be interested in $Gr(2,\mathbb{C}^4)$ to illustrate the use of Theorem (3.6).

Proposition 3.8. *$Gr(2,\mathbb{C}^4)$ has a canonical conformal structure.*

Proof. We know that a metric g_{ab} on a manifold M is a section of $\Gamma(M, \Omega^1 M \odot \Omega^1 M)$. Therefore, a conformal class of metrics $\{\Omega^2(x)g_{ab}\}$ must determine a line subbundle, say L, in $\Omega^1 M \odot \Omega^1 M$, where the fibre of L at $x \in M$ is $\{\mathbb{C}g_{ab}\}$. In other words, a manifold M will have a canonical conformal structure if and only if there exists a line subbundle in $\Omega^1 M \odot \Omega^1 M$ admitting a non-degenerate section. We shall show that in the case $M = Gr(2,\mathbb{C}^4)$ the bundle $\Omega^1 Gr(2,\mathbb{C}^4) \odot \Omega^1 Gr(2,\mathbb{C}^4)$ does indeed have a canonical line subbundle.

We know that $\Omega^1 Gr(2,\mathbb{C}^4) = \mathcal{S}^* \otimes \tilde{\mathcal{S}}$, therefore,

$$\begin{aligned} \Omega^1 Gr(2,\mathbb{C}^4) \odot \Omega^1 Gr(2,\mathbb{C}^4) &= (\mathcal{S}^* \otimes \tilde{\mathcal{S}}) \odot (\mathcal{S}^* \otimes \tilde{\mathcal{S}}) \\ &= \mathcal{S}^* \odot \mathcal{S}^* \otimes \tilde{\mathcal{S}} \odot \tilde{\mathcal{S}} + \wedge^2 \mathcal{S}^* \otimes \wedge^2 \tilde{\mathcal{S}} \end{aligned}$$

Since, $\wedge^2 \mathcal{S}^* \otimes \wedge^2 \tilde{\mathcal{S}}$ has a rank of one and locally has non-degenerate sections, this is the canonical line bundle that represents the canonical conformal structure on $Gr(2,\mathbb{C}^4)$. □

According to Penrose [30], the pair $(Gr(2,\mathbb{C}^4), \wedge^2 \mathcal{S}^* \otimes \wedge^2 \tilde{\mathcal{S}})$ represents the compactified conformal Minkowski space.

Now let us return to the notation used in Theorem (3.6) with $M = Gr(2,\mathbb{C}^4)$. Then with this notation $F = F(1,2,3 : \mathbb{C}^4)$ and $Y = F(1,3;\mathbb{C}^4)$, where $F(1,2,3;\mathbb{C}^4) =$ {The set of all pairs (L_x, S_x, D_x) in $\mathbb{C}^4$ where L_x is a line in $\mathbb{C}^4$, S_x is a 2-surface in $\mathbb{C}^4$ and D_x is a 3-surface in $\mathbb{C}^4$}.

Since $Gr(1,\mathbb{C}^4) = \mathbb{CP}^3$ with coordinates $z^a, a = 1,\ldots,n$ and $Gr(3,\mathbb{C}^4) = \mathbb{CP}^3$ with coordinates $w_a, a = 1,\ldots,n$, then clearly Y is a hypersurface in $\mathbb{CP}^3 \times \mathbb{CP}^3$ given by $z^a w_a = 0$. Hence, Y has a contact structure with contact line bundle $L^* = \mathcal{O}(1,1)|_Y$. We shall now use Theorem (3.6) to prove the following result.

Theorem 3.9. *There are 15 linearly independent Killing vector fields on $Gr(2,\mathbb{C}^4)$.*

Proof. Let n be the number of linearly independent Killing vector fields. By Theorem (3.6),

$$n = H^0(Y, L^*)$$

Let us compute n. We have the short exact sequence

$$0 \to \mathcal{O}(0,0) \to \mathcal{O}(1,1) \to \mathcal{O}_F(1,1) \to 0$$

which gives the long exact sequence

$$\begin{aligned} 0 \to H^0(\mathbb{CP}^3 \times \mathbb{CP}^3, \mathcal{O}(0,0)) &\to H^0(\mathbb{CP}^3 \times \mathbb{CP}^3, \mathcal{O}(1,1)) \\ &\to H^0(Y, L^*) \to H^1(\mathbb{CP}^3 \times \mathbb{CP}^3, \mathcal{O}(0,0)) \to \ldots \end{aligned}$$

However,

$$H^0(\mathbb{CP}^3 \times \mathbb{CP}^3, \mathcal{O}(0,0)) = H^0(\mathbb{CP}^3, \mathcal{O}(0)) \otimes H^0(\mathbb{CP}^3, \mathcal{O}(0)) = \mathbb{C},$$

$$H^0(\mathbb{CP}^3 \times \mathbb{CP}^3, \mathcal{O}(1,1)) = H^0(\mathbb{CP}^3, \mathcal{O}(1)) \otimes H^0(\mathbb{CP}^3, \mathcal{O}(1)) = \mathbb{C}^4 \otimes \mathbb{C}^4,$$

and

$$H^1(\mathbb{CP}^3 \times \mathbb{CP}^3, \mathcal{O}(0,0)) = H^1(\mathbb{CP}^3, \mathcal{O}(0)) \otimes H^0(\mathbb{CP}^3, \mathcal{O}(0)) \oplus H^0(\mathbb{CP}^3, \mathcal{O}(0)) \otimes H^1(\mathbb{CP}^1, \mathbb{O}(0)) = 0$$

Thus, the long exact sequence becomes,

$$0 \to \mathbb{C} \to \mathbb{C}^4 \otimes \mathbb{C}^4 \to H^0(Y, L^*) \to 0$$

and so

$$H^0(Y, L^*) = \mathbb{C}^{16}/\mathbb{C} = \mathbb{C}^{15}$$

Hence, $n = 15$ as required. □

This is a new proof of the classical result in differential geometry (see Novikov [11]).

3.4 The General Case.

The next natural question that arises from this result is whether or not that this result can be generalised, in other words a Killing vector field X on a complex manifold M with an arbitrary G-structure corresponds to a global section of $\Gamma(Y, L^*)$.

3.4.1 The Twistor Construction Of All "Elementary" Geometries.

Let M be an n-dimensional complex manifold and $\mathcal{L}(M)$ the holomorphic coframe bundle $\pi : L^*M \to M$. If G is a closed subgroup of $GL(n, \mathbb{C})$, then the set of holomorphic G-structures $\mathcal{G}$ on M, i.e the set of principal subbundles $\mathcal{G} \subset \mathcal{L}(M)$ with the group G, can be identified with the set of holomorphic sections σ of the quotient bundle $\tilde{\pi} : \mathcal{L}(M)/G \to M$ whose typical fibre is isomorphic to $GL(n, \mathbb{C})/G$.

Suppose that $G \subset GL(n, \mathbb{C})$ is a irreducible Lie subgroup,and $\mathcal{G}$ is a G-structure on M. With any such $\mathcal{G}$ there is naturally associated a subbundle $\tilde{F} \subset \Omega^1 M$ whose typical fibre is isomorphic to the cone in $\mathbb{C}^n$ defined as the G-orbit of the line spanned by a highest weight vector. Denote $\tilde{\mathfrak{F}} = \tilde{F} \backslash 0_{\tilde{F}}$, where $0_{\tilde{F}}$ is the "zero" section of $\tilde{p} : \tilde{F} \to M$ whose value at each $z \in M$ is the vertex of the cone $\tilde{p}^{-1}(z)$. The quotient bundle $\nu : \mathfrak{F} := \tilde{\mathfrak{F}}/\mathbb{C}^* \to M$ is then a subbundle of the projectivized cotangent bundle $\mathbb{P}(\Omega^1 M)$. Let the dimension of the fibres of $\mathbb{P}(\Omega^1 M)$ be equal to m.

The total space of the cotangent bundle $\Omega^1 M$ has a canonical holomorphic symplectic structure represented in a natural local coordinate system (z^α, p_α) by the 2-form $\omega = dp_\alpha \wedge dz^\alpha$.

The pullback, $i^*\omega$, of the symplectic form ω from $\Omega^1 M \backslash 0_{\Omega^1 M}$ to its submanifold $i : \tilde{\mathfrak{F}} \to \Omega^1 M \backslash 0_{\Omega^1 M}$ defines a distribution $\mathcal{D} \subset T\tilde{\mathfrak{F}}$ as the kernel of the natural "lowering of indices" map $T\tilde{\mathfrak{F}} \xrightarrow{\lrcorner\omega} \Omega^1 \tilde{\mathfrak{F}}$, i.e.

$$\mathcal{D}_e := \{V \in T_e\tilde{\mathfrak{F}} : V \lrcorner i^*\omega = 0\}$$

at each point $e \in \tilde{\mathfrak{F}}$.

Using the fact that $d(i^*\omega) = i^*d\omega = 0$, one can show that this distribution is integrable and thus defines a foliation of $\tilde{\mathfrak{F}}$ by holomorphic leaves. We shall assume that this space of leaves, $\tilde{Y}$, is a complex manifold. This assumption imposes no restriction on the local structure on M.

Since the Lie derivative vanishes

$$\mathcal{L}_V(i^*\omega) = V \lrcorner i^* d\omega + d(V \lrcorner i^*\omega) = 0$$

for any vector field V tangent to the leaves, the 2-form $i^*\omega$ is the pullback relative to the canonical projection $\tilde{\mu} : \tilde{\mathfrak{F}} \to \tilde{Y}$ of a closed 2-form $\tilde{\omega}$ on $\tilde{Y}$. It is easy to check that $\tilde{\omega}$ is non-degenerate which means that the quotient $(\tilde{Y}, \tilde{\omega})$ is a symplectic manifold. The pair $(\tilde{Y}, \tilde{\omega})$ is what is usually called the symplectic reduction of a symplectic manifold $(\Omega^1 M \backslash 0_{\Omega^1 M}, \omega)$ via its submanifold $\tilde{\mathfrak{F}}$.

If we restrict our attention to a coordinate domain $U \times \mathbb{C}^n$ in $\Omega^1 M$ with coordinate functions (z^α, p_α) such that $\omega|_{U\times(\mathbb{C}^n\backslash 0)} = dp_\alpha \wedge dz^\alpha$. Shrinking $U \subset M$ as necessary, we may assume that $\tilde{\mathfrak{F}}|_U$ is locally realised in $U \times \mathbb{C}^n$ by a system of equations

$$f_i(z,p) = 0, i = 1, \ldots, k$$

where $f_i(z,p)$ are irreducible homogeneous polynomials in p_α and $k \leq n - m - 1$. Let ∇ be any torsion-free connection on the G-structure then it can be shown (see [24]) that the distribution $\mathcal{D}$ defined above is locally spanned by the vector fields

$$d_i = \frac{\partial f_i}{\partial p_\alpha}\left(\frac{\partial}{\partial z^\alpha} + \Gamma^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\beta}\right).$$

A G-structure on an n-dimensional manifold M is locally represented by an equivalence class of frames,

$$\{e^\alpha_a(z)\}$$

where $\{e^\alpha_a\} \sim \{\hat{e}^\beta_b\}$ if $e^\alpha_a = G^b_a(z) e^\alpha_b(z)$, where $G^b_a(z)$ is a local function on M with values in the group G. The index α above is associated with a local coordinate system $\{x^\alpha\}$ on M, and index a enumerates basis vectors in TM.

We have already shown that a Killing vector field X on M is, by definition, a vector field X satisfying, in any local coordinate system $\{x^\alpha\}$, where $X = X^\alpha(z)\frac{\partial}{\partial z^\alpha}$ and any frame $\{e^\alpha_a(z)\}$ from the G-structure, the equation

$$\mathcal{L}_X e_a = g^b_a(z) e_b,$$

where $e_b = e^\alpha_b(z)\frac{\partial}{\partial z^\alpha}$, and $g^b_a(z)$ is a function on M with values in a Lie algebra, $\mathfrak{g}$, of G.

Lemma 3.10. *If X is a Killing vector field of a given G-structure $\{e_a^\alpha(z)\}$ on M, then*

$$\nabla_{e_a} X = \hat{g}_a^b e_b,$$

where $\hat{g}_a^b(z)$ is a function on M with values in $\mathfrak{g} = Lie(G)$ and $\nabla = "\frac{\partial}{\partial z^\alpha} + \Gamma_{\alpha\beta}^\gamma"$ is any torsion-free connection on the G-structure.

Proof. A linear connection $\nabla : TM \to TM \otimes \Omega^1 M$ on M is tangent to the given G-structure $\{e_a = e_a^\alpha(z)\frac{\partial}{\partial z^\alpha}\}$ if and only if

$$\nabla_{\frac{\partial}{\partial z^\alpha}} e_a = \omega_{\alpha a}^b(z) e_b$$

where $\omega_a^b := \omega_{\alpha a}^b(z) dz^\alpha$ is a 1-form with values in $\mathfrak{g}$.

Then

$$\mathcal{L}_X e_a = g_a^b(z) e_b$$

$$\Leftrightarrow [X, e_a] = g_a^b(z) e_b$$

$$\Leftrightarrow \nabla_X e_a - \nabla_{e_a} X = g_a^b(z) e_b, \text{ because } \nabla \text{ is torsion free}$$

$$\Leftrightarrow X^\alpha \omega_{\alpha a}^b e_b - \nabla_{e_a} X = g_a^b(z) e_b$$

$$\Leftrightarrow \nabla_{e_a} X = \hat{g}_a^b(z) e_b,$$

where $\hat{g}_a^b(z) := -g_a^b(z) + X^\alpha \omega_{\alpha a}^b$ takes values in $\mathfrak{g}$. □

3.4.2 Homogeneous manifolds.

Let M be a complex manifold and let G be a matrix $(n \times n)$ group acting in $\mathbb{C}^n = \{p_a, a = 1, \ldots, n\}$, by

$$G : \mathbb{C}^n \to \mathbb{C}^n$$

$$p_a \mapsto G_b^a p_b,$$

where $(G_b^a) \in G$.

Hence G acts on the associated projective space $\mathbb{P}(\mathbb{C}^n) = (\mathbb{C}^n \backslash 0)/\mathbb{C}^*$. If a point $p \in \mathbb{P}(\mathbb{C}^n)$ is represented by homogeneous coordinates $[p_a]$ then the homogeneous coordinates of $G.p$ are $[G_a^b p_b]$. A submanifold $N \subset \mathbb{P}(\mathbb{C}^n)$ is called G-homogeneous if, for any $x \in N$ one has $G.x \in N$.

Near any point $x_0 \in N \subset \mathbb{P}(\mathbb{C}^n)$, the submanifold N can be described by algebraic equations:

$$f_i(p_a) = 0, i = 1, \ldots, k,$$

where $f_i = \sum_{a_1 \ldots a_l} f_i^{a_1 \ldots a_l} p_{a_1} \ldots p_{a_l}$, are homogeneous polynomials in p_a of degree $l \in \mathbb{N}$, (for some l).

If N is G-homogeneous, then, for any $\mathfrak{g}_a^b \in \mathfrak{g}$ (the Lie algebra of G) one must have

$$\sum_{a_1 \ldots a_l b_1} f_i^{a_1 \ldots a_l} \mathfrak{g}_{a_1}^{b_1} p_{b_1} p_{a_2} \ldots p_{a_l}|_N = 0 \tag{3.2}$$

This follows from the condition that, for any $x \in N$, one has $G.x$ again in N for any element G in the group.

3.4.3 The General Result.

We have the diagram,

$$\tilde{Y} \stackrel{\tilde{\mu}}{\leftarrow} \tilde{\mathfrak{F}} \subset \Omega^1 M \backslash 0 \stackrel{\tilde{\nu}}{\rightarrow} M$$

and we have the projective version,

$$Y \stackrel{\mu}{\leftarrow} \mathfrak{F} \stackrel{\nu}{\rightarrow} M.$$

Locally, the embedding $\mathfrak{F} \subset \Omega^1 M \backslash 0$ is given by homogeneous functions,

$$f_i(p_a) = \sum_{a_1 \ldots a_l} f_i^{a_1 \ldots a_l}(z) p_{a_1} \ldots p_{a_l} = 0, i = 1, \ldots k$$

where p_a are the coordinates in the fibre of $\Omega^1 M \to M$ given by

$$p_a = e_a^\alpha p_\alpha,$$

p_α being coordinates associated to z^α.

Thus the functions f_i are given by

$$f_i(z,p) = \sum_{a_1 \ldots a_l} f_i^{a_1 \ldots a_l}(z) e_{a_1}^{\alpha_1} \ldots e_{a_l}^{\alpha_l} p_{\alpha_1} \ldots p_{\alpha_l}.$$

As already mentioned, the fibres of the projection μ are generated by vector fields

$$d_i = \frac{\partial f}{\partial p_\alpha}\left(\frac{\partial}{\partial z^\alpha} + \Gamma_{\alpha\beta}^\gamma(z) p_\gamma \frac{\partial}{\partial p_\beta}\right).$$

Assume $X = X^\alpha(z)\frac{\partial}{\partial z^\alpha}$ is a Killing vector field. We associate to it a homogeneous (degree 1) function

$$h_X := X^\alpha(z) p_\alpha.$$

Theorem 3.11. *If X is a Killing vector field of an arbitrary G-structure on an arbitrary complex manifold M, then*

$$d_i h_X|_{\tilde{\mathfrak{F}}} = 0, i = 1, \ldots, k.$$

Proof. We have

$$\begin{aligned} d_i h_X|_{\tilde{\mathfrak{F}}} &= \frac{\partial f_i}{\partial p_\alpha}(\frac{\partial X^\beta}{\partial z^\alpha} + \Gamma^\beta_{\alpha\gamma} X^\gamma) p_\beta|_{\tilde{\mathfrak{F}}} \\ &= \frac{\partial f_i}{\partial p_\alpha}(\nabla_{\frac{\partial}{\partial z^\alpha}} X)^\beta p_\beta|_{\tilde{\mathfrak{F}}} \\ &= l \sum_{a_1 \ldots a_l} f_i^{a_1 \ldots a_l} e^{\alpha_1}_{a_1} (\nabla_{\frac{\partial}{\partial z^{\alpha_1}}} X)^\beta p_\beta p_{a_2} \cdots p_{a_l}|_{\tilde{\mathfrak{F}}} \\ &= l \sum_{a_1 \ldots a_l} f_i^{a_1 \ldots a_l} (\nabla_{e_{a_1}} X)^\beta p_\beta p_{a_2} \cdots p_{a_l}|_{\tilde{\mathfrak{F}}} \\ &= l \sum_{a_1 \ldots a_l} f_i^{a_1 \ldots a_l} (\mathfrak{g}^{b_1}_{a_1} e^\beta_{b_1} p_\beta) p_{a_2} \cdots p_{a_l}|_{\tilde{\mathfrak{F}}} \text{ by Lemma (3.10)} \\ &= l \sum_{a_1 \ldots a_l} f_i^{a_1 \ldots a_l} \mathfrak{g}^{b_1}_{a_1} p_{b_1} p_{a_2} \cdots p_{a_l}|_{\tilde{\mathfrak{F}}} \\ &= 0 \end{aligned}$$

since the fibres of $\tilde{\mathfrak{F}} \to M$ are G-homogeneous manifolds, and $g^{b_1}_{a_1}$ takes values in $\mathfrak{G}$. □

Corollary 3.12.

$$h_X = \tilde{\mu}^*(\tilde{h}_X),$$

for some function $\tilde{h}_X$ on $\tilde{Y}$ which is homogeneous of degree 1. This is the same as a section of $\tilde{h}_X \in \Gamma(Y, L)$.

Thus we have now proved that any Killing vector field of M with a G-structure gives rise to a global section of L over Y.

Theorem 3.13. *To any global section, $s \in \Gamma(Y, L)$, there corresponds a Killing vector field on M.*

Proof. s gives rise to a function, $\tilde{s}$, on $\tilde{Y}$ homogeneous of degree 1; the latter gives rise to a function, $\tilde{s}_{\tilde{\mathfrak{F}}}$, on $\tilde{\mathfrak{F}}$, homogeneous of degree 1; the latter is just a global section of the bundle $\mathcal{O}(+1)|_{\mathfrak{F}}$, where $\mathcal{O}(+1)$ is the tautological line bundle on $\mathbb{P}(\Omega^1 M)$, i.e.

$$s \to \tilde{s} \to \tilde{s}_{\tilde{\mathfrak{F}}} = H^0(\mathfrak{F}, \mathcal{O}(+1)_{\mathfrak{F}}).$$

It is well-known, that for generalised flag varieties (see [5]), $N \subset \mathbb{P}(\mathbb{C}^n)$, one has

$$H^0(N, \mathcal{O}(+1)_N) = H^0(\mathbb{P}(\mathbb{C}^n), \mathcal{O}(+1))$$

Thus,

$$H^0(\mathfrak{F}, \mathcal{O}(+1)_{\mathcal{F}}) = H^0(\mathbb{P}(\Omega^1 M), \mathcal{O}(+1))$$

since the fibres of $\mathfrak{F} \to M$ are generalized flag varieties.

Thus, in a local coordinate chart (z^α, p_β), $\tilde{s}_{\mathfrak{F}}$ is of the form

$$\tilde{s}_{\mathfrak{F}} = X^\alpha p_\alpha$$

i.e $\tilde{s}_{\mathfrak{F}} = h_X$, for some vector field $X = X^\alpha(z)\frac{\partial}{\partial z^\alpha}$ on M.

It is easy to see that the equation

$$d_i h_X|_{\tilde{\mathfrak{F}}} = 0$$

imply $\nabla_{e_a} X = \mathfrak{g}_a^b e_b$ so that X is a Killing vector field. □

Thus we have proved the following theorem:

Theorem 3.14. Main Theorem. *Let G be a semisimple Lie group whose decomposition into a locally direct product of simple groups does not contain any of the groups*

1. *$Spin(2n+1, \mathbb{C})$ acting on $\mathbb{C}^{2^n}$, $n \geq 3$,*
2. *$Sp(2n, \mathbb{C})$ acting on $\mathbb{C}^{2n}$, $n \geq 2$,*
3. *G_2 acting on $\mathbb{C}^7$.*

Then, given any irreducible $G \circ \mathbb{C}^*$-structure, B_G, on a manifold M, there is a one-to-one correspondence,

$$\{\textit{Killing vectors on } B_G\} \leftrightarrow H^0(Y, L^*),$$

between the vector space of Killing vectors on (M, B_G) and the cohomology group $H^0(Y, L^*)$ where (Y, L) is Merkulov's contact manifold associated to B_G.

Remark:

We have excluded

1. $Spin(2n+1,\mathbb{C})$ acting on $\mathbb{C}^{2^n}$, $n \geq 3$,
2. $Sp(2n,\mathbb{C})$ acting on $\mathbb{C}^{2n}$, $n \geq 2$,
3. G_2 acting on $\mathbb{C}^7$.

because in these cases the associated twistor spaces do not encode full information about G-structures. This phenomenon was first noticed by Merkulov (see Theorem (2.19)).

3.5 Quaternionic Manifolds.

3.5.1 Basic Definitions.

Let $G = GL(n,\mathbb{H})GL(1,\mathbb{H}) \subset GL(4n,\mathbb{R})$ be a natural subgroup. A quaternionic manifold is, by definition, a pair (M, B_G) consisting of an $4n$-dimensional manifold M and an irreducible torsion-free G-structure B_G on M.

If B_G reduces to the subgroup $G' = Sp(p,q)Sp(1)$, then the pair $(M, B_{G'})$ is called a quaternionic Kahler manifold. If it further reduces to $G'' = Sp(p,q)$, the pair $(M, B_{G''})$ is called a hyper-Kahler manifold.

3.5.2 Salamon's twistor space.

In order to talk about a quaternionic structure without reference to a Riemannian metric, one must replace $Sp(n)$ by the full group $GL(n,\mathbb{H})$ of non-singular quaternionic $n \times n$ matrices. Indeed consider $GL(n,\mathbb{H})$ and $GL(1,\mathbb{H})$ as subgroups of $GL(4n,\mathbb{R})$ by letting them act on $\mathbb{H}^n$ by left and right multiplication respectively. These actions commute, so the product

$$G = GL(n,\mathbb{H})GL(1,\mathbb{H}) \subset GL(4n,\mathbb{R})$$

is well defined and has maximal compact subgroup $Sp(n)Sp(1)$. Because $GL(n,\mathbb{H})$ and $GL(1,\mathbb{H})$ share a 1-dimensional centre, one can economise by using $Sp(1)$ instead of $GL(1,\mathbb{H})$ as in the Riemmanian case, and G is double covered by $GL(n,\mathbb{H}) \times Sp(1)$. We then have the following definition.

Definition 3.15. *A quaternionic manifold is a 4n-dimensional manifold ($n > 1$) with a G-structure admitting a torsion-free connection*

Let us consider the 2-sphere

$$S^2 == \{ai + bj + ck \in \mathbb{H} : a^2 + b^2 + c^2 = 1\}$$

of unit imaginary quaternions. Let M be a quaternionic manifold with principle G-bundle Q, and fix a frame $f \in Q$ corresponding to an isomorphism

$$f : \mathbb{R}^{4n} \to T_x M$$

Any $u \in S^2$ acts on $\mathbb{R}^{4n} \cong \mathbb{H}^n$ by right multiplication, and so determines an endomorphism

$$\phi(f, u) = f \circ u \circ f^{-1}$$

of $T_x M$ with square -1, i.e. an almost complex structure. If $g = Aq \in G$ with $A \in GL(n, \mathbb{H})$ and $q \in Sp(1)$, then

$$\phi(fg, u) = \phi(f, q^{-1}uq) \tag{3.3}$$

Consequently the set

$$Z_x = \{\phi(f, u) : u \in S^2\}$$

of almost complex structures does not depend upon f, and is therefore the fibre of a bundle Z over M. From (3.3), Z is none other that the bundle

$$Q \times_G S^2$$

associated to Q by means of of the adjoint action of $Sp(1)$ on $S^2 \subset \operatorname{Im} \mathbb{H} \cong sp(1)$.

Any local section $s \in \Gamma(U, Q)$ converts the basis i, j, k of Im $\mathbb{H}$ into a triple of almost complex structures $I, J, K \in \Gamma(U, Z)$ satisfying

$$IJ = -JI = K.$$

In general, it will be impossible to define I, J, K globally; for example $\mathbb{H}P^n$ cannot admit even one almost complex structure for topological reasons.

Now regard $\mathbb{H}$ as a right vector space over the complex numbers, and let the group $Sp(1)$ of unit quaternions act on $\mathbb{H}$ by left multiplication. Then any $u \in S^2$ determines a complex linear transformation of $\mathbb{H}$ with $u^2 = -1$. Thus $\mathbb{H} = L_+ \oplus L_-$, where $L_\pm$ are the $\pm$-eigenspaces of u. We will use the correspondence

$$u \to L_-$$

to identify S^2 with the complex projective line $\mathbb{P}_\mathbb{C}(\mathbb{H})$; this identification is equivariant with the respective action of $Sp(1)$. Now suppose that $\epsilon = 0$ (this is always true on a sufficiently small open set), and choose a lifting $\tilde{Q} \in H^1(M, GL(n, \mathbb{H}) \times Sp(1))$. Then

$$Z \cong \tilde{Q} \times_{Sp(1)|} \mathbb{P}_\mathbb{C}\mathbb{H} \cong \mathbb{P}_\mathbb{C}(H)$$

is the complex projective bundle formed from the vector bundle $H = \tilde{Q} \times_{Sp(1)} \mathbb{H}$.

Consider $\mathbb{H}^n$ also as a right vector space over $\mathbb{C}$, and let $GL(n, \mathbb{H})$ act by matrix multiplication on the left. Then the tensor product $\mathbb{H}^n \otimes_\mathbb{C} \mathbb{H}$ is naturally the complexification of a real vector space $\mathbb{R}^{4n}$ because it admits the complex conjugation

$$\xi \bar{\otimes} \eta = \xi j \otimes \eta j$$

$\xi \in \mathbb{H}^n, \eta \in \mathbb{H}$.

The action of $GL(n, \mathbb{H}) \times Sp(1)$ on $\mathbb{H}^n \otimes_\mathbb{C} \mathbb{H}$ induces the representation of G on $\mathbb{R}^{4n}$ already determined. Therefore defining an associated vector bundle

$$E = \tilde{Q} \times_{GL(n,\mathbb{H})} H,$$

the complexified tangent bundle has the form

$$T_c M \cong E \otimes_\mathbb{C} H$$

Since any representation of $Sp(1)$ is self-dual, we can identify $\Omega^1_c M = E^* \otimes_\mathbb{C} H$. Moreover, it follows from the above that if h belongs to the fibre $H_x \backslash \{0\}$ over $x \in M$, the space of (1,0)-forms associated to the almost complex structure $z = \mathbb{C}h \in \mathbb{P}_\mathbb{C}(H) \cong Z$ is equal to

$$\wedge^{1,0}_z M = E^*_x \otimes_\mathbb{C} \mathbb{C}h.$$

In the case $\mathbb{H}P^n, \epsilon = 0$ and H is uniquely defined as the tautologous quaternionic line bundle. The total space of H minus its zero section can be identified with $\mathbb{H}^{n+1}/\{0\}$, and Z is the complex projective space $\mathbb{C}P^{2n+1}$.

Theorem 3.16. *If M is a quaternionic manifold, the total space of the associated bundle Z is a complex manifold.*

Proof. See Salamon [35]. □

Theorem 3.17. *There is a one-to-one correspondence between the vector space of Killing vector fields on $(M, B_{GL(n,\mathbb{H})GL(1,\mathbb{H})})$ and the vector space, $H^0(Z, TZ)$ of global holomorphic vector fields on Salamon's twistor space.*

Proof. The Merkulov's twistor space (Y, L) and the Salomon's twistor spaces are related as follows,

$$Y = \mathbb{P}(\Omega^1 Z), L = \mathcal{O}_{\Omega^1 Z}(-1)$$

By the Main Theorem, the vector space of Killing vectors is in a 1-1 correspondence with $H^0(Y, L^*)$.

Let us compute the latter. It is well-known that for a projective space $\mathbb{P}(V)$, V being some vector space, one has

$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) = V^*.$$

Hence, in our case

$$H^0(Y, L^*) = H^0(\mathbb{P}(\Omega^1 Z), \mathcal{O}(1)) = H^0(Z, (\Omega^1 Z)^*) = H^0(Z, TZ)$$

This completes the proof. □

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