A Bivariant Theory for the Cuntz Semigroup and its Rôle for the Classification Programme of C*-algebras

by

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Abstract

A bivariant theory for the Cuntz semigroup is introduced and analysed. This is used to define a Cuntz-analogue of K-homology, which turns out to provide a complete invariant for compact Hausdorff spaces. Furthermore, a classification result for the class of unital and stably finite C*-algebras is proved, which can be considered as a formal analogue of the Kirchberg-Phillips classification result for purely infinite C*-algebras by means of KK-theory, i.e. bivariant K-theory.

An equivariant extension of the bivariant Cuntz semigroup proposed in this thesis is also presented, and some well-known classification results are derived within this new theory, thus showing that it can be applied successfully to the problem of classification of some actions by compact groups over certain C*-algebras of the stably finite type.

Introduction

It is safe to say that the theory of Operator Algebras was founded by the Hungarian mathematician von Neumann, with his pioneering work on *rings of operators*, together with the American mathematician Murray. In the early years of 20th century, the first examples of Hilbert spaces appeared in the work of Hilbert and his pupils at Göttingen about spectral theory, infinite-dimensional quadratic forms, etc.... The modern notion of an ℓ^2 space is due to Schmidt, who introduced it in 1908, while the study of its continuous linear functionals dates to 1912, with the work of Riesz. Not too long after that, the first examples of L^2-spaces also appeared.

Towards the end of 1925, the Austrian physicist Schrödinger, in order to account for the new inexplicable physical phenomena occurring at the atomic level, proposed a new equation for the dynamics at the microscopic scale, known as the *Schrödinger Wave Equation*. Soon after the publication of this work in 1926, von Neumann realised that the solutions to the Schrödinger equation were elements of an L^2-type vector space, and undertook the task of giving a precise mathematical formulation of the newborn theory of *Quantum Mechanics*. Between the years 1927 and 1929, he gave the abstract definition of Hilbert spaces and studied the spectral theory of both bounded and unbounded linear operators defined on them. He then showed that the Quantum Theory developed by Schrödinger could be formulated in the language of Hilbert spaces, in particular of some L^2-spaces, and unbounded operators acting on them, like the position and momentum operators q and p. An account of all these results is provided by [49], which is still a cornerstone of the mathematical foundations of Quantum Mechanics. Unless otherwise stated, we shall assume that all the Hilbert spaces of the present introduction are separable. This should not be viewed as a serious restriction, since almost all the Hilbert spaces that are motivated by physics, which in turn motivated the study of Hilbert spaces, turns out to be separable.

Parallel to the work of Schrödinger was that of the German physicist Heisenberg. Together with Born and Jordan in Göttingen, he worked on a different approach to Quantum Mechanics which was based on the use of *infinite matrices*, that were being investigated by Born at that time. In particular they realised that the so-called Heisenberg relations

$$qp - pq = iI$$

where q, p and I are some linear operators acting on some ℓ^2-type space, q and p being the position and momentum operator respectively, and I being the identity operator, cannot be represented by finite dimensional matrices. The analysis of Born, Heisenberg and Jordan culminated in a paper that appeared in 1926, in which they established what became known as the *matrix mechanics* theory of Quantum Physics. As this new theory also accounted for the new physical phenomena, the question whether the work of Schrödinger and that of Heisenberg were equivalent arose. By the knowledge available at that time, the two theories looked completely different: one was formulated in terms of partial differential equations with solutions in an L^2-type Hilbert space, whereas the other was formulated in terms of exotic and still rather obscure objects like infinite matrices acting on an ℓ^2-type Hilbert space. An answer to this question was provided by von Neumann in [48], where he established what is now known as *von Neumann's uniqueness theorem* for the Scrödinger representation of the Heisenberg relations. Hence, Heisenberg's infinite matrices are unitarily equivalent to a direct sum of copies of the Schrödinger representation.

When von Neumann shifted his attention from the analysis of single operators to families of them, *von Neumann algebras* were born. In the famous works with Murray, he considered sets of linear operators over some Hilbert space that together formed a ring, and called the resulting object a *ring of operators*. In his honour, they are today referred to as *von Neumann algebras*, and provide an extremely rich theory that has a mathematical interest on its own. In the original work of 1929, von Neumann defined them as subalgebras of the set of bounded operators $B(H)$ over some Hilbert space H that are closed under an involution operation $*$ and in the *weak operator topology*, and that contain the unit of $B(H)$, i.e. the identity operator over H. One of the most remarkable discoveries is the celebrated *Double Commutant Theorem* of von Neumann himself, that relates a topological property to a purely algebraic one. This result asserts that every von Neumann algebra $M \subset B(H)$ coincides with its double commutant M'' i.e. the commutant $(M')'$ of its commutant M', where

$$M' := \{S \in B(H) \mid TS = ST \quad \forall T \in M\}.$$

Among all the elements in a von Neumann algebra M, the set of projections $P(M)$, i.e. those elements $p \in M$ that satisfy to $p^*p = p$, play a central rôle in the theory. Not only they generate M as a Banach space, but every spectral projection of a self-adjoint element a, that is $a = a^*$, of M is contained in M itself. Furthermore, $P(M)$ has a natural structure of *orthocomplemented lattice*, and this fact has a great relevance not only from a mathematical point of view, but also in physics, where it provides the right ground for *Quantum Logic*.

With the appearance of this new class of algebras, the problem of the analysis of their structure and that of their classification arose. The structure of commutative von Neumann algebras was determined in [51]. For any such algebra M there exists a locally compact topological space X and a positive measure μ carried by X such that M is isometrically $*$-isomorphic to the set of bounded measurable functions $L^\infty(X, \mu)$. The analysis of the structure of non-commutative von Neumann algebras was carried out in [46], where it emerged that the centre $\mathcal{Z}(M)$ of a von Neumann algebra M, that is the set of elements that are common to both M and M', carries considerable information about the internal structure of M. In particular Murray and von Neumann observed that every non-trivial projection $p \in \mathcal{Z}(M)$ that is not zero or the identity I of M gives a direct sum decomposition of M into pM and $p^\perp M$, where $p^\perp := I - p$ is the complement of p in the lattice $P(M)$. Every von Neumann algebra whose centre is then trivial, in the sense that it consists of only scalar multiples of the identity of the algebra, can be regarded as an elementary building block, which they called a *factor*. Indeed, every von Neumann algebra admits a decomposition into a direct *integral* of factors. As a consequence of this result, the problem of classification of von Neumann algebras can be restricted to the factor case. This problem was tackled by Murray and von Neumann themselves and was based on the introduction of a *dimension function* on the lattice of projections from a factor, with values in $\mathbb{R}_0^+ \cup \{\infty\}$. Roughly speaking, this function gives an idea of how *large* a projection p is when compared to the unit of the algebra, or equivalently how big the subspace pH is with respect to the whole Hilbert space H. From their analysis it turned out that there can only be three cases for the codomain of the dimension functions over factors:

I. $\mathbb{N}_0 \cup \{\infty\}$;

II. $\mathbb{R}_0^+$;

III. $\{0, \infty\}$.

Among the factors of type I one can distinguish the case of finite range for the dimension function, denoted by I_n, where $\{0, 1, \ldots, n\}$ are all the allowed values, and the case I_∞ when the range is infinite. For type II factors one has the finite subtype, for which the dimension function is bounded and can be normalised to have range $[0, 1]$ (type II_1), and the semifinite case (type II_∞). Examples of type I factors are trivial, as they are isomorphic to the von Neumann algebra of bounded operators $B(H)$ for some Hilbert space H. In particular $B(H)$ is of type I_n when H is a finite-dimensional vector space, i.e. $\mathbb{C}^n$, in which case $B(H) = M_n(\mathbb{C})$, i.e. the full $n \times n$ matrix algebra with entries in $\mathbb{C}$. The existence of type II and III factors was not obvious at the time, but Murray and von Neumann themselves provided some examples of type II factors by the group algebra construction. In particular, every group with the ICC (infinite conjugacy classes) property yields a type II factor. An example of a type III factor was only provided in [50]. Almost thirty years later, Powers proved the existence of an uncountable family of non-isomorphic (hyperfinite) type III factors in [58].

The occurrence of type III factors is not confined to the abstract mathematical theory of von Neumann algebras, but extends to the physical literature as well. Despite the belief of von Neumann that type II factors have better properties for a mathematical formulation of Quantum Physics, since he considered type III factors to be rather singular objects, it turns out that most of the algebras that show up in *Algebraic Quantum Field Theory* are indeed of type III. A proof of this fact was provided by Araki in [5], while a physical interpretation of this phenomenon was suggested by Licht in [41], where the notion of *strictly localised states* is introduced. A classification result for type III factors was provided by Connes in the work [14], which earned him the Fields Medal.

The existence of non-isomorphic type II_1 factors was established in [47], where it is also shown that there exists a unique hyperfinite type II_1 factor, up to isomorphism. Many year later, McDuff showed the existence of a countable family of non-isomorphic type II_1 factors in [45]. Furthermore, every factor of type II_∞ turns out to be of the form $M \otimes B(H)$, for a suitable type II_1 factor. Therefore, the problem of classification of type II factors reduces to the problem of classification of *just* type II_1 factors. It was already known to Murray and von Neumann that all the hyperfinite II_1 factors are isomorphic. It was later shown by Connes [15] that injectivity implies hyperfiniteness, thus providing a classification result for injective factors.

In 1943 Gelfand and Naimark started investigating a larger class of operator algebras that became known as C*-algebras. They observed that, for any compact Hausdorff space X, the set of continuous functions $C(X)$ can be equipped with the structure of a commutative $*$-algebra and with a submultiplicative norm, with the involution tied to this norm by the so-called C*-identity

$$\|f^*f\| = \|f\|^2$$

for any $f \in C(X)$, with

$$\|f\| = \sup_{x\in X} |f(x)|.$$

What they showed is that any unital $*$-algebra A that is equipped with a submultiplicative norm that satisfies the C*-identity, and such that A is complete with respect to this norm, is isometrically $*$-isomorphic to $C(X)$ for some compact Hausdorff space X. If A is not unital then X is only locally compact and there is an isometric $*$-isomorphism with $C_0(X)$, i.e. the C*-algebra of continuous functions vanishing at infinity. This result is sometimes considered as an indication of the fact that the general theory of *non-commutative* C*-algebras they have initiated is a generalisation of the classical notion of a topological space. Von Neumann algebras are particular C*-algebras, that can also be characterised in the abstract thanks to results of Sakai. By the analogy with measurable spaces brought in, e.g., by the structure results for the commutative case, they are also considered as non-commutative generalisations of measure theory. However, the connections between C*-algebras and topology goes far beyond this classical result of Gelfand and Naimark, to the extent that theories like *topological* K-theory have generalisations to the operator algebraic setting.

Along with von Neumann algebras, C*-algebras also occur in the algebraic formulation of Quantum Physics. In the Hilbert space formalism of Quantum Mechanics, observable quantities are represented by self-adjoint operators acting on some Hilbert space H. Two such observables that are not compatible yield a non-self-adjoint operator when they are composed, hence something that is physically non-observable. A remedy to this situation was proposed by Jordan with the introduction of a non-associative product that has subsequently led to the notion of Jordan algebras. Today C*-algebras are usually preferred to Jordan algebras for the definition of a quantum mechanical system, and the Jordan structure is recovered through its self-adjoint part.

The structure of a finite dimensional C*-algebra is easy to determine: they are finite direct sums of full matrix algebras $M_n(\mathbb{C})$. Therefore, the study of finite dimensional C*-algebras reduces to the case of the finite matrix algebras. Their classification can be carried out by hand, since it is evident that two matrix algebras $M_m(\mathbb{C})$ and $M_n(\mathbb{C})$ are isomorphic if and only if $m = n$. The next step is then to consider those C*-algebras that are *approximately* finite dimensional, in the sense that every element can be well-approximated by some other element coming from a full matrix algebra. In more precise terms, such algebras arise as inductive limits over finite dimensional C*-algebras, and are known as AF algebras. Special cases of AF algebras are the so called matroid algebras and the UHF algebras. Classification results for the latter were obtained by Glimm in [25] who showed that a complete invariant for this class of C*-algebras is provided by the set of *supernatural* numbers.

A first classification of the larger class of AF algebras was provided by Bratteli in [9], where a Bratteli diagram is associated to each inductive sequence. As there are many equivalent sequences that define the same AF algebras, Bratteli diagrams become a complete invariant up to a certain equivalence relation. Algebraic K-theory makes its appearance in the classification problem of C*-algebras through the celebrated classification result of Elliott [19]. The main theorem asserts that two AF algebras A and B are isomorphic if and only if there exists an ordered group isomorphism between $K_0(A)$ and $K_0(B)$ that also preserves the *scales* and, in the unital case, maps the class of the unit onto the class of the unit. Otherwise stated, Elliott's result asserts that there is a complete invariant for AF algebras which is K-theoretical in nature, i.e. the ordered K_0-group of an AF algebra.

Another class of C*-algebras for which a complete invariant is purely K-theoretic is that of A$\mathbb{T}$-algebras, i.e. inductive limits of circle algebras $C(\mathbb{T})\otimes F$, with F a finite-dimensional C*-algebra, of real rank zero. Their classification was again established by Elliott in [21], where he showed that the invariant must contain all the information coming from the K_0- and the K_1-group, together with a scale constructed out of both these K-groups. This is also the paper where the so-called *Elliott Conjecture*, which is at the heart of the current *Classification Programme* for C*-algebras, first appeared. As more (counter)examples to this conjecture were found during the last few decades, the formulation of this conjecture has undergone many modifications, but the original one was stated in the following terms.

Conjecture 1 (Elliott, 1989)**.** Two simple, nuclear, separable, stably finite, unital C*-algebras A and B are isomorphic if and only if there exist group isomorphisms $\alpha_i : K_i(A) \to$

$K_i(B)$, $i = 1, 2$ such that $\alpha(K_0(A)^+) = K_0(B)^+$ and $\alpha_0([1_A]) = [1_B]$. In this case there exists an isomorphism $\phi : A \to B$ such that $\alpha_i = K_i(\phi)$, $i = 1, 2$.

After the classification result of Elliott for AF algebras, Effros proposed the investigation of a larger class of C*-algebras which arise as inductive limits of direct sums of corners in the matrix algebras $M_n(C(X))$, where X is a compact Hausdorff space. Such algebras were called AH algebras by Blackadar. It became apparent from the work of Goodearl [26], who introduced a special class of AH algebras, known as the *Goodearl Algebras*, that in order to go beyond the real rank zero case, the invariant proposed by Elliott should have been extended to include the trace simplex of the algebras. This follows from the fact that there are Goodearl algebras associated to contractible compact Hausdorff spaces in which the underlying space X can be reconstructed as the extreme boundary of the trace simplex $T(A)$. As $C(X)$ and $\mathbb{C}$ have isomorphic K-theory in this case, it follows that there are uncountably many, up to homeomorphism, Goodearl algebras with isomorphic K-theory that are only distinguished by their trace simplex.

Another contribution to the Elliott invariant came from the considerations of Thomsen on inductive limits of matrix algebras over the closed interval $[0, 1]$, also known as AI algebras. He proposed to include the *pairing* between the traces and the K_0-group of a C*-algebra A, i.e. the map r_A defined as

$$r_A(\tau)([p]) = \tau(p)$$

for any τ in the trace simplex $T(A)$ of A and any projection $p \in A \otimes K$. With the proof that the invariant proposed by Thomsen given by Elliott in [20] classifies simple AI algebras, the conjecture was modified and extended to the unital, finite, simple case, with the Elliott invariant now looking like

$$\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A),$$

which has well studied functoriality properties. Any unital, simple, separable and nuclear C*-algebra is sometimes referred to as an Elliott algebra, although at the moment of writing and to the best of our knowledge, there is no unanimous consensus among operator algebraists. With the above line defining the Elliott invariant, the current Elliott's conjecture can then be stated in the following terms.

Conjecture 2. (Elliott, 1994) The functor Ell gives a complete invariant for Elliott algebras. In particular, every isomorphism $\Phi : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ lifts to a $*$-isomorphism $\phi : A \to B$ between the Elliott algebras A and B.

It must be noted that, by "Elliott Conjecture", one usually refers to a collection of conjectures rather than just a single one, since the invariant involved depends on the class of C*-algebras considered. For C*-algebras A of *infinite type*, i.e. those containing a projection which is Murray-von Neumann equivalent to a proper subprojection, the "correct" invariant is provided by the *graded* Abelian group

$$K_*(A) := K_0(A) \oplus K_1(A).$$

This claim is supported by the classification result for purely infinite C*-algebras obtained independently by Kirchberg and Phillips. A Kirchberg algebra is a simple, nuclear, purely infinite and separable C*-algebra, and any two stable Kirchberg algebras are isomorphic if and only if they have the same K_*-group, up to isomorphism.

As it was recalled earlier, every von Neumann algebra is generated by its lattice of projections as a Banach space. A C*-algebra, however, might not have any projections at all, and therefore there is no useful information that can be extracted from its Murray-von Neumann semigroup. What a C*-algebras always abound of is the set of *positive* elements. Indeed, every element of a C*-algebra is the linear combination of at most four positive elements. It is then reasonable to argue that by *comparing* positive elements within a C*-algebra one might be able to extract useful information about its structure. Among the many notions of comparison between positive elements, one that is now playing an important rôle in the Classification Programme for C*-algebra is the one suggested by Cuntz in [17]. The comparison he proposed in his work is used to define the so-called *Cuntz semigroup* similarly to the Murray-von Neumann semigroup, and which constitutes an invariant for C*-algebras.

Despite its appearance in the early Eighties, the importance of the Cuntz semigroup for the Classification Programme has been fully acknowledged only recently, and most notably with a counterexample to the Elliott Conjecture provided by Toms. In [68] he exhibits two simple AH algebras that agree not only on the Elliott invariant, but also on other topological invariants, like the already cited real rank and the stable rank. One of the invariants for C*-algebras that is capable of telling them apart turned out to be the Cuntz semigroup. It must be said that the scepticism that accompanied the Cuntz semigroup prior to this discovery relied on the fact that, for the purely infinite case, there is no real information coming out of it, and even in the simplest yet interesting cases, like Abelian C*-algebras with contractible spectrum, one ends up with an uncountable object. Furthermore, the original definition that gave rise to the functor W was riddled with some unwanted pathologies, above all the lack of continuity with respect to inductive limits, contrary to the case of K-theory, which proved itself to be quite successful in the Classification Programme.

The problem of continuity was solved by Coward, Elliott and Ivanescu who provided a *Hilbert module picture* for the Cuntz semigroup in [16]. Their new functor Cu preserves inductive limits and is related to the original functor W by the relation

$$\mathsf{Cu}(A) \cong W(A \otimes K).$$

For this reason the Cuntz semigroups arising from Cu are sometimes considered as *stabilised* versions of those produced by W. Another important result of the analysis of [16] is that every Cuntz semigroup Cu, which comes with a natural structure of positively ordered Abelian monoid, is closed under the operation of taking suprema. This has led them to define a new category of partially ordered Abelian monoids, also denoted by Cu, whose objects satisfy certain axioms. With this enriched categorical setting, every Cuntz semigroup produced by the functor Cu on the category of C*-algebras produces an object in the category Cu. Because of the better functoriality properties of Cu, it is this definition of the Cuntz semigroup that is regarded as the new fundamental tool for the Classification Programme for C*-algebras.

In order to account for the situation just discussed, a *seemingly weaker* version of the Elliott Conjecture of 1994 has been proposed in [55], where the Elliott invariant is enriched with the Cuntz semigroup W and the class of the unit $[1_A]_W$ of an Elliott algebra A within $W(A)$. In formal terms this new conjecture can be expressed as follows.

Conjecture 3 (WEC)**.** Let A and B be Elliott algebras. Every isomorphism

$$\Phi : ((W(A), [1_A]_W), \mathrm{Ell}(A)) \to ((W(B), [1_B]_W), \mathrm{Ell}(B))$$

lifts to a $*$-isomorphism $\phi : A \to B$.

At present there are no known counterexamples to the above conjecture. However, the tendency of including the Cuntz semigroup to the Elliott invariant has received some negative critiques because of its fine structure and the difficulties in computing it even in the simplest cases. As a response, [55] bears a result that shows that the Elliott Conjecture of 1994 is equivalent to the *Weak* Elliott Conjecture WEC for the following two classes of C*-algebras:

i. simple, unital AH algebras of slow dimension growth;

ii. simple, unital, nuclear, $\mathcal{Z}$-stable C*-algebras.

Here the equivalence should be understood in the sense that it is possible to reconstruct the Cuntz semigroup $W(A)$ of a C*-algebra A from the knowledge of its Elliott invariant $\mathrm{Ell}(A)$ (see also [2]). In the above, $\mathcal{Z}$ denotes the so-called *Jiang-Su algebra*, a simple, separable, nuclear, unital, infinite dimensional and projectionless C*-algebras firstly discovered in [32] (see [30] for an explicit presentation as a universal C*-algebra). A C*-algebra A is termed $\mathcal{Z}$-stable if there is an isomorphism between A and $A \otimes \mathcal{Z}$. In particular $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$, and in fact $\mathcal{Z}$ is a *minimal* strongly self-absorbing C*-algebras. The above cited result is one of a long series where the Jiang-Su algebra makes its appearance in connection with the Classification Programme. The fact that its Elliott invariant is isomorphic to that of the complex numbers can be viewed as an explanation of this fact, as tensoring by the Jiang-Su algebra does not change the Elliott invariant of a C*-algebra. It is no surprise then that the classification problem in the terms given above can have a positive answer in the $\mathcal{Z}$-stable case.

The way $\mathcal{Z}$-stability interacts with the Cuntz semigroup is via the property of *strict comparison* by traces of positive elements. A C*-algebra A has strict comparison of positive elements if $T(A) \neq \varnothing$ and, for any pair of positive elements $a, b \in (A \otimes K)^+$ and dimension functions d_τ with $d_\tau(b) < \infty$, the condition $d_\tau(a) < d_\tau(b)$ for any $\tau \in T(A)$ implies that $a \precsim b$ in $\mathsf{Cu}(A)$. It was conjectured by Toms and Winter in 2008 that these two properties are equivalent for the unital, separable, simple, nuclear, non-elementary and stably finite case. The conjecture further claims that these two properties are also equivalent to the algebra in said class having *finite* decomposition rank. Despite the attention that it drew for the importance of its implications for the Classification Programme, and the many partial results where this conjecture have been confirmed, to date there is no definitive answer to this question and in fact it represents one of the main and most active areas of research in the field. By results of Winter and Zacharias, the conjecture is nowadays expressed in the following terms.

Conjecture 4 (Toms-Winter)**.** Let A be a simple, separable, unital, nuclear and infinite dimensional C*-algebra. The following are equivalent.

i. A has finite nuclear dimension;

ii. A is $\mathcal{Z}$-stable;

iii. A has strict comparison of positive elements.

What is known at the moment of writing is that i. $\Rightarrow$ ii. in full generality, as shown in [74]. Rørdam [61] has shown that ii. $\Rightarrow$ iii. also in full generality. By results in [69] (also, independently, [37] and [63]) it follows that iii. $\Rightarrow$ ii. when the extreme boundary of the trace simplex $T(A)$ of the C*-algebra A is compact and of finite covering dimension. The implication ii. $\Rightarrow$ i. has been obtained in [44] under the condition of quasidiagonality and unique trace. The latter has been removed in [64], while the former has been replaced by the condition that $T(A)$ is a Bauer simplex in [7]. To summarise, the Toms-Winter conjecture has been verified for all simple, separable, unital, nuclear, non-elementary C*-algebras for which the trace space $T(A)$ is a Bauer simplex with an extreme boundary of finite covering dimension.

To conclude this overview of the Classification Programme for C*-algebras, and the way that structural problems interact with it, we mention the special case of [67, Corollary 6.4] that asserts that the Toms-Winter conjecture holds for C*-algebras of at most one trace in the UCT class. This result is a consequence of a theorem in the same paper which asserts that every faithful trace on a separable, nuclear C*-algebra in the UCT class is *quasidiagonal* ([67, Theorem A]). The far reaching consequence of this theorem is that it settles the Elliott Conjecture for the class of Elliott algebras with finite nuclear dimension that satisfy to the UCT, in the sense that any two such algebras are $*$-isomorphic if and only if they have isomorphic Elliott invariant.

As for a bivariant theory of the Cuntz semigroup, we have already mentioned some facts that allow us to pinpoint its place in the above panorama of the Classification Programme. For the purely infinite case the Cuntz semigroup is trivial, whereas for the stably finite case it exhibits a rich structure that contains abundant information useful for classification purposes. Indeed, the definition of the bivariant Cuntz semigroup that we propose in this thesis provides an object that is capable of classifying all unital and stably finite C*-algebras (Theorem 3.86) thanks to the notion of *strict invertibility*. Like in KK-theory, where one has the notion of KK-equivalence, one can give a notion of Cu-equivalence, together with a scale condition which is a bivariant extension of the scale conditions of the already cited result of Elliott for the classification of AF algebras, and this defines what we have called the *strictly invertible* elements (Definition 3.76). Hence, like in the ordinary theory of the Cuntz semigroup, the *bivariant* extension that we propose here plays a rôle for the unital, finite case of the Classification Programme. As Definition 3.4 shows, the bivariant Cuntz semigroup is defined in terms of a special class of linear maps between C*-algebras known as completely positive maps with the order zero property, or orthogonality preserving (*c.p.c. order zero maps* for short), and the reason is two-fold. One resides in the results obtained in [75], where the structure of such maps is analysed in great details. In particular it emerges that every such map induces a morphism at the level of the Cuntz semigroup and therefore they seem to provide the right framework for the problem of when maps at the level of the Cuntz semigroup lift at the level of the algebras. The other reason is that the class of c.p.c. order zero maps appears in almost all the works that have been cited above relatively to the Toms-Winter conjecture and the last developments for the Elliott Conjecture for Elliott algebras with finite nuclear dimension in the UCT class. Roughly speaking, such maps, when judiciously used, are capable of *lifting* some K-theoretical obstructions associated to $*$-homomorphisms. In particular this last point has led to a new research endeavour towards the so-called *coloured classification*, which is one of the main focuses of [7].

Outline of the Thesis

The present thesis is organised as follows. In **Chapter 1** we give the definition of a C*-algebra and that of a *local* C*-algebra. We then recall the concept of a completely positive map with the order zero property and extend some well-known results concerning these maps to the setting of local C*-algebras as defined in this thesis. We then recall the notion and the main properties of *strongly self-absorbing* C*-algebras as defined by Toms and Winter. Such algebras will occur in one of the following chapters and are used to provide some explicit computations of bivariant Cuntz semigroups. As we also touch upon the open projection picture for the Cuntz semigroup, we recall the main definitions and results from Akemann's theory of Non-commutative Topology. This exposition is then followed by a recollection of the notion of the Murray-von Neumann semigroup, equivariant K-theory and KK-theory. The purpose of many of the sections in this chapter is also that of fixing the notation for the other chapters.

In **Chapter 2** we provide an extensive treatment of the ordinary theory of the Cuntz semigroup. As the previous chapter, the main purpose of many of the sections is to fix the notation for the following chapter. However, we also provide some details about a new proof of the existence of suprema in the *stabilised* Cuntz semigroup in the open projection picture (Theorem 2.24). In the last section we also provide an equivariant theory for the Cuntz semigroup which coincides with the recent work on the same subject of Gardella and Santiago.

Chapter 3 constitutes the core of the thesis. We provide the definition of the bivariant Cuntz semigroup $W(\,\cdot\,,\,\cdot\,)$ by starting from a bivariant extension of the ordinary Cuntz semigroup W (Definition 3.4) and by establishing its main functoriality properties. We then propose a *stabilised* definition of the bivariant Cuntz semigroup $\mathsf{Cu}(\,\cdot\,,\,\cdot\,)$ (Definition 3.32) that extends the ordinary Cuntz semigroup of Coward, Elliott and Ivanescu, and we show how to give an equivalent definition in terms of modules, thus strengthening the analogy with KK-theory. As an analogue of Kasparov product, we introduce a composition product for the bivariant Cuntz semigroup (Proposition 3.36) and we discuss how to use it to introduce a notion of Cu-equivalence between (local) C*-algebras. We then take into consideration the enriched category of positively ordered Abelian monoids W and show that the bivariant functor W has image in this category (Theorem 3.48). A section dedicated to examples of bivariant Cuntz semigroups follows. We show that, even in the bivariant setting proposed in this thesis, the bivariant Cuntz semigroup involving unital Kirchberg algebras in the second argument (and unital and exact C*-algebras in the first) does not yield much useful information. Indeed, it reduces to the ideal lattice of the algebra in the first argument (Theorem 3.58) and therefore we do not expect the bivariant Cuntz semigroup to play an important rôle in the classification for the infinite case. We also show a stability property in the first argument of $W(A, B)$ when both C*-algebras A and B are tensored by a strongly self-absorbing C*-algebra $\mathcal{D}$ (Theorem 3.60). A definition of *Cuntz-homology* is proposed which mimics the way K-homology is recovered from KK-theory and we show that it provides a complete invariant for compact Hausdorff spaces. In the section that follows these series of examples we revisit the notion of Cu-equivalence to include a scale condition and show how to use it to classify all unital and stably finite C*-algebras by an intertwining argument of Elliott (Theorem 3.86). In the last section of the chapter we define an equivariant extension of the bivariant Cuntz semigroup (Definition 3.88) and develop a notion of equivariant Cu-equivalence for the classification of actions over C*-algebras. Particularly, we show that the proposed equivariant version of the bivariant Cuntz semigroup allows recovering the well-known classification result of Handelman and Rossmann (Corollary 3.103), as well as the more recent result of Gardella and Santiago

(Corollary 3.101).

In search for an analogue of the open projection picture of the Cuntz semigroup, **Chapter 4** bears bivariant extensions of other notions of equivalence between positive elements, most notably Pedersen and Blackadar equivalence. By also providing a bivariant extension of open projections, here termed *open ∗-homomorphisms* (Definition 4.1), and of Peligrad-Zsidó equivalence, we prove that the connections among them agree and extend those for the non-bivariant case. We also propose a new bivariant extension of the Cuntz semigroup Cu that is based on a Cuntz-type comparison of open ∗-homomorphisms.

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The time I have spent in Glasgow for my Ph.D. has undoubtedly been one of the best of my life. But as nice as the place itself is, what made it really so were the people I had the pleasure to interact with during the last three years or so. I would like to thank Dr Joachim Zacharias, my supervisor, for his constant support throughout the duration of my Ph.D. and for being an extremely nice person, inside and outside of Academia. The same applies to all the members of the Analysis group of the University of Glasgow that I had the pleasure and the honour to meet during these years. The familiar feeling they managed to bring up is something I will hardly forget. A special thought goes to Jorge, Liam and Joan for their friendship. This thesis also benefited from mathematical

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I believe that, as a Ph.D. student, you can end up either loving or hating your office. In my case I have always considered it a second home because of the amazing people I have had the pleasure to share it with. Office 309 will always remind me of Umar, Xiang, Andrew, and all the others that made it the best office ever!

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$$1 + 2 + 3 + \cdots = -\frac{1}{12}.$$

Chapter 1

Preliminaries

In this chapter we collect some main definitions and results from the theory of operator algebras, and state some more technical results that will be used in the following chapters. Original results that, to the best of our knowledge, are not published elsewhere are also proven. Specifically, we recall the notion of C*-algebra, and introduce that of *local* C*-algebra in the sense of [3] (cf. Definition 1.2). This is justified by the recent developments of the theory of the Cuntz semigroup, even in its unstabilised form, when the target category is enriched with a suitable structure, and the domain category is chosen to be that of local C*-algebras as defined in [3]. We also recall some well-known results about special maps between C*-algebras, introduced in [72] and known as *completely positive order zero maps*, and studied extensively in [75]. Most of the main results of the just cited work are then extended to the setting of local C*-algebras (cf. Proposition 1.8, Corollary 1.9 and Proposition 1.10), in order to apply them to the present work and provide definitions with a larger degree of generality. This digression on the theory of C*-algebras is concluded by a section about a special class of C*-algebras that play an important rôle in the Classification Programme, known as *strongly self-absorbing* (cf. [70]). For the purposes of the present thesis, members of this class can be used to provide some concrete examples of bivariant Cuntz semigroups, as shown in Chapter 3.

Section 1.2 is devoted to the main definitions and results of Akemann's theory of Non-commutative topology. It was shown in [52] that the notion of open projections that arises in Akemann's work can be used to give an equivalent description of the Cuntz semigroup as defined by [16]. Such picture is mentioned in the next chapter, where we also provide an equivariant extension that also agrees with the recent work of Gardella and Santiago [24] on the equivariant theory for the Cuntz semigroup. Furthermore, in Chapter 4 we propose a bivariant extension of the notion of open projections and hint towards a possible bivariant theory for the Cuntz semigroup formulated in terms of these novel bivariant objects, where Cuntz comparison takes a form that closely resembles that in [16].

In Section 1.3 we recall the definition of the Murray-von Neumann semigroup of a C*-algebra. Contrary to von Neumann algebras, which are generated, as a Banach space, by their set of projections, a C*-algebra need not have any projections at all. However, it abounds of positive elements and therefore it makes more sense to compare such elements rather than projections when trying to investigate the structure of a C*-algebra. In this regard, the Cuntz semigroup can be regarded as a generalisation of the Murray-von Neumann semigroup and, in many cases of interested, it can be considered as an extension of it. We do not, however, focus on K-theory, that arises from it, for the following two reasons. One is that the main object of this thesis is a semigroup, whereas in K-theory one deals with groups. The other reason is connected to the previous one by the Grothendieck construction of the enveloping group of a semigroup. When such a construction is performed, (part of) the information contained in the semigroup is lost, and so is any trace of the structure of the associated C*-algebra with it. We do, however, consider equivariant K-theory, which can be formulated without any reference to the specific structure of ordinary K-theory. Besides, there is no particular relevance given to an equivariant extension of the Murray-von Neumann semigroup, though we also define such an object in Definition 1.25.

Since the main idea behind the bivariant theory of the Cuntz semigroup presented in this thesis is to mimic the relationship between the K_0-group with KK-theory, we dedicate Section 1.5 to Kasparov's bivariant K-theory. We collect the most important properties of such theory, as in Chapter 3 we refer back to this section when proving analogous results for the bivariant Cuntz semigroup. Since we also provide an equivariant extension, we briefly mention the main definitions in the equivariant formulation of KK-theory, as well as some main properties and results.

1.1 C*-algebras

The main purpose of this section is to record the definition of local C*-algebras in the sense of [3] and to provide extensions of some well-known results in [75] about completely positive maps with the order zero property (*c.p.c. order zero maps* for short) to them.

1.1.1 Local C*-algebras

We start by recalling the abstract definition of C*-algebra.

Definition 1.1 (C*-algebra)**.** A C*-algebra A is a Banach $*$-algebra that satisfies the C*-identity

$$\|x^*x\| = \|x\|^2$$

for any $x \in A$.

A normed $*$-algebra which satisfies the C*-identity is called a *pre-C*-algebra*, and its completion with respect to the norm is then a C*-algebra.

There are many (possibly inequivalent) definitions of *local C*-algebras* in the literature. For instance, in [6, §3.1], a normed $*$-algebra is a local C*-algebra if every matrix ampliation is closed under holomorphic functional calculus and the algebra itself is equipped with a pre-C*-norm. For the purposes of the present thesis we will borrow the following definition from [3].

Definition 1.2 (Local C*-algebra)**.** A pre-C*-algebra A is a *local C*-algebra* if there exists a family of C*-subalgebras $\{A_i\}_{i\in I}$ of A such that

i. $\forall i, j \in I \quad \exists k \in I \quad | \quad A_i \cup A_j \subset A_k$;

ii. $A = \bigcup_{i\in I} A_i$.

Equivalently, a pre-C*-algebra A is a local C*-algebra if it contains every C*-algebra generated by its finite parts, i.e. if $C^*(F) \subset A$, whenever F is a finite subset of A.

Observe that any C*-algebra is in particular a local C*-algebra. Another typical example of a local C*-algebra, that will be used extensively in this thesis, is provided by the $*$-algebra of all matrices of any finite order with entries in $\mathbb{C}$, $M_\infty(\mathbb{C})$, or simply M_∞, which can be realised as the union of the finite matrix algebras $M_n(\mathbb{C})$. In particular, $M_\infty(A) \cong M_\infty(\mathbb{C}) \odot A$, where $\odot$ denotes the *algebraic* tensor product, is a local C*-algebra for any (local) C*-algebra A. The repeated occurrence of such local C*-algebras in the bivariant theory for the Cuntz semigroup that is developed in this thesis is the main motivation for considering local C*-algebras as defined above.

An element a from a local C*-algebra A is said to be *self-adjoint* if $a = a^*$. A self-adjoint element $b \in A$ is said to be *positive* if there exists $h \in A$ such that $b = h^*h$. Equivalently, b is positive if its spectrum is contained in the set of non-negative real numbers $\mathbb{R}_0^+$. Throughout this thesis we will denote by A^+ the *cone* of all the positive elements of A, i.e. the set

$$A^+ := \{a \in A \mid a = h^*h \text{ for some } h \in A\}.$$

An important feature of local C*-algebras is that they are closed under continuous functional calculus of their normal elements, and as such they are also local in the sense of Blackadar (cf. [6, Definition 3.1.1]).

1.1.2 Order Zero Completely Positive Maps

The theory of the bivariant Cuntz semigroup, as formulated in this thesis, is based on a notion of comparison between completely positive maps with the order zero property (or c.p. order zero maps for short), which we now proceed to introduce. First of all we recall a few basic definitions.

Definition 1.3 (Positive map)**.** A linear map $\phi : A \to B$ between (local) C*-algebras is said to be *positive* if $\phi(A^+) \subset B^+$.

A matrix algebra of order n over a (local) C*-algebra A is the algebra generated by formal $n \times n$ matrices with entries in A. Concretely, this algebra can be identified with the tensor product $M_n(A) := A \otimes M_n(\mathbb{C})$.Throughout this thesis we will use the symbol $\otimes$ to denote the *minimal* tensor product between C*-algebras. Observe that there is no ambiguity with this tensor product, since every matrix algebra $M_n(\mathbb{C})$ is nuclear. Moreover, $M_n(A)$ is a local C*-algebra whenever A is a local C*-algebra, and a C*-algebra whenever A is a C*-algebra.

An n-ampliation of a linear map $\phi : A \to B$ between (local) C*-algebras is the linear map $\phi^{(n)} : M_n(A) \to M_n(B)$ given by $\phi \otimes \mathrm{id}_{M_n(\mathbb{C})}$. That is, $\phi^{(n)}$ acts on a matrix $[a_{ij}] \in M_n(A)$ by mapping every entry $a_{ij} \in A$ to $\phi(a_{ij})$ and giving as a result the matrix $[\phi(a_{ij})] \in M_n(B)$. A linear map $\phi : A \to B$ between (local) C*-algebras is said to be n-positive if its n-ampliation $\phi^{(n)}$ is a positive map between the (local) C*-algebras $M_n(A)$ and $M_n(B)$. If ϕ is n-positive for any $n \in \mathbb{N}$ then ϕ is said to be *completely positive.* For more details on completely positive maps we refer the reader to [53]. Here we limit to mention that completely positive maps are strongly related to the theory of nuclear C*-algebras, as nuclearity has been linked to the completely positive approximation property (CPAP) in the works [12, 35, 38].

Some special completely positive maps have also been used in the attempt to generalise the notion of covering dimension for topological spaces to C*-algebras, which are sometimes regarded as a non-commutative analogue of a topological space, and based on the orthogonality of elements from a C*-algebras. Any two elements $a, b \in A$ from a (local) C*-algebra are said to be orthogonal, and denoted as $a \perp b$, if $ab = ba = ab^* = a^*b = 0$. Observe that, if $a^*a \perp b^*b$, then $a^*ab^*b = 0$ by definition, and so

$$\begin{aligned} 0 &= ba^*ab^*ba^* \\ &= (ab^*)^*(ab^*)(ab^*)^* \\ &= (ab^*)^*(ab^*)(ab^*)^*(ab^*) \\ &= |ab^*|^4, \end{aligned}$$

whence $ab^* = 0$. Therefore, with similar computations, one easily verifies that $a \perp b$ if and only if $a^*a \perp b^*b$, $a^*a \perp bb^*$, $aa^* \perp b^*b$ and $aa^* \perp bb^*$. Furthermore, if a and b are self-adjoint, then $a \perp b$ if and only if $ab = 0$. The following definition stems from [72].

Definition 1.4 (Order n completely positive map)**.** A completely positive map $\phi : A \to B$ between (local) C*-algebras has order n if n is the smallest positive integer for which the following holds: for any finite subset $\{a_1, \ldots, a_{k+2}\}$ of mutually orthogonal elements in A there exist $i, j \in \{1, \ldots, k+2\}$, $i \neq j$, such that $\phi(a_i) \perp \phi(a_j)$.

As argued in [72], completely positive approximations can be regarded as a non-commutative analogue of an open covering for a topological space. The order of the completely positive maps in the approximation can then be defined as the *non-commutative* topological dimension.

Although not strictly necessary in most of the arguments we provide in this thesis, we restrict our attention to completely positive maps that are contractive, i.e. with norm at most one, and we refer to them as c.p.c. maps. Observe that one can always construct a c.p.c. map from a c.p. map by just *normalisation*, that is, if $\phi : A \to B$ is a c.p. map between (local) C*-algebras A and B, then $\frac{1}{\|\phi\|}\phi$ is a c.p.c. map.

Among all the c.p.c. order n maps, a somewhat special rôle is played by c.p.c. *order zero* maps. They are also called *orthogonality preserving*, as the order zero property translates directly into the equivalent property that, if a, b are elements of a (local) C*-algebra such that $a \perp b$, then $\phi(a) \perp \phi(b)$. A stronger version of Wolff's structure theorem for orthogonality preserving linear maps [76] was obtained by Winter and Zacharias in [75].

As this structure theorem is of particular relevance for this thesis, we state it explicitly here.

Theorem 1.5 (Winter-Zacharias)**.** *Let A and B be C*-algebras, $\phi : A \to B$ a completely positive map with the order zero property, and set $C := C^*(\phi(A)) \subset B$. There are a positive element $h \in \mathcal{M}(C) \cap C'$ with $\|h\| = \|\phi\|$ and a $*$-homomorphism $\pi : A \to \mathcal{M}(C) \cap \{h\}'$ such that*

$$\phi(a) = h\pi(a)$$

for any $a \in A$. Furthermore, if A is unital, then $h = \phi(1) \in B$.

In light of this result, the $*$-homomorphism π arising from the above theorem is sometimes referred to as the *support $*$-homomorphism* for the c.p.c. order zero map ϕ.

Observe that the above theorem applies to c.p. order zero maps between C*-algebras. Before showing that such a structure theorem carries over to c.p. order zero maps between *local* C*-algebras we elaborate more on some consequences of the above result. As shown in [75, Corollary 3.2], one can define a functional calculus on c.p. order zero maps. Let $\phi : A \to B$ be any c.p. order zero map between C*-algebras, and let $f \in C_0((0, \|\phi\|])$. Then $f(\phi)$ is the c.p. order zero map defined by

$$f(\phi)(a) := f(h)\pi(a)$$

for any $a \in A$, where h and π are given by the structure theorem 1.5 applied to ϕ. However, such functional calculus can also be defined without making direct use of Theorem 1.5, as shown by the following example.

Example 1.6. Let $\phi : A \to B$ be a c.p.c. order zero map between C*-algebras and consider the function $f \in C_0((0, 1])$ given by $f(t) = t^2$ for any $t \in (0, 1]$. As ϕ is linear and every element of a C*-algebra decomposes as the $\mathbb{C}$-linear combination of at most four positive elements from A^+, it is enough to consider the restriction of the map ϕ to the positive cone A^+. Hence, for any $a \in A^+$ one has

$$\begin{aligned} f(\phi)(a) &= f(h)\pi(a) \\ &= h^2\pi(a) \\ &= h^2\pi(a^{\frac{1}{2}})^2 \\ &= \phi(a^{\frac{1}{2}})^2, \end{aligned}$$

which shows that $f(\phi)(a)$ can be defined in terms of the action of ϕ on some root of a. $\triangle$

By abstracting from the above example one can conclude that for any polynomial p with zero constant term and degree d there exists a polynomial P_p in $m_d + 1$ variables, where $m_d = \lceil \log_2 d \rceil$, such that

$$p(\phi)(a) = P_p(\phi(a), \phi(a^{\frac{1}{2}}), \dots, \phi(a^{\frac{1}{2^{m_d}}})),$$

for any $a \in A^+$. A suitable choice of such polynomials is, for instance,

$$P_p(y_1, \dots, y_{m_n+1}) = \sum_{k=1}^{n} a_k y_{m_k}^{2^{m_k}-k} y_{m_k+1}^{2k-2^{m_k}}, \tag{1.1}$$

for any polynomial

$$p(x) = \sum_{k=1}^{n} a_k x^k, \qquad a_n \neq 0.$$

Since every function $f \in C_0((0, \|\phi\|])$ is a uniform limit of a sequence of polynomials of the type considered above, one sees that $f(\phi)$ can be defined in terms of ϕ alone, without any direct mention of the structure theorem 1.5.

We also record, as another consequence of Theorem 1.5, Corollary 3.1 from [75], which states that there is a one-to-one correspondence between c.p.c. order zero maps $\phi : A \to B$ from and to C*-algebras and $*$-homomorphisms $\rho : C_0((0,1], A) \to B$ from the cone $C_0((0,1], A)$ over A to B. This result can be combined with Loring's projectivity of $*$-homomorphisms from finite dimensional C*-algebras [42] to get to the following result.

Proposition 1.7. *Every c.p.c. order zero map defined on a finite dimensional C*-algebra F to a quotient B/J, where J is a closed two-sided ideal of B, lifts to a c.p.c. order zero map from F to B.*

Hence, c.p.c. order zero maps over finite dimensional domains are also projective in the sense of Loring.

We now focus our attention to c.p. order zero maps between *local* C*-algebras. It turns out that the structure theorem 1.5 carries over to this case, as well as the functional calculus discussed earlier. We start by proving that any c.p. order zero map between local C*-algebras admits a unique extension to a c.p. order zero map between the completions.

Proposition 1.8. *Let A and B be local C*-algebras, $\tilde{A}$ and $\tilde{B}$ their respective completions, and let $\phi : A \to B$ be a c.p.c. order zero map. There exists a unique c.p.c. order zero extension $\tilde{\phi} : \tilde{A} \to \tilde{B}$ of ϕ.*

Proof. Let $\tilde{\phi}$ be the c.p.c. extension of ϕ to the completions. One needs to check that orthogonality of elements in the completion is preserved by $\tilde{\phi}$. For any pair of positive contractions $a, b \in A^+$ with the property $ab = 0$, take sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subset A_1$, where A_1 denotes the unit ball of A, with the property that $a_n \to a$ and $b_n \to b$. Since A is a local C*-algebra, the C*-algebras A_n generated by $\{a_n, b_n\}$ inside $\tilde{A}$ are contained in A and therefore one can consider the restrictions $\phi_n := \phi|_{A_n}$, which are c.p.c. order zero maps over C*-algebras. Therefore, by the structure theorem 1.5 there are positive contractions h_n and $*$-homomorphisms π_n such that $\phi_n(a) = h_n\pi_n(a)$ for any $a \in A_n$ and $n \in \mathbb{N}$. By construction, $\tilde{\phi}$ extends each ϕ_n and therefore one has the identity

$$\tilde{\phi}(a_n)\tilde{\phi}(b_n) = h_n\tilde{\phi}(a_nb_n),$$

for any $n \in \mathbb{N}$. Hence, by the joint continuity of the norm and the boundedness of $\tilde{\phi}$, one has

$$\lim_{n\to\infty} \tilde{\phi}(a_n)\tilde{\phi}(b_n) = \tilde{\phi}(a)\tilde{\phi}(b)$$

and

$$\begin{aligned}\left\|h_n\tilde{\phi}(a_nb_n)\right\| &\leq \left\|\tilde{\phi}(a_nb_n)\right\| \\ &\leq \|a_nb_n\|\end{aligned}$$

for any $n \in \mathbb{N}$, whence

$$\tilde{\phi}(a)\tilde{\phi}(b) = 0.$$

This shows that the extension $\tilde{\phi}$ preserves orthogonality of positive elements and therefore has the order zero property. □

The structure result for c.p.c. order zero maps between local C*-algebras now follows as a corollary to the previous proposition and Theorem 1.5, as it is now shown. We first observe that, for any continuous linear map $\phi : A \to B$, where A and B are local C*-algebras, the extension $\tilde{\phi}$ to the completions $\tilde{A}$ and $\tilde{B}$ is such that $\tilde{\phi}(\tilde{A}) \subset C^*(\phi(A))$, whence $C^*(\tilde{\phi}(\tilde{A})) = C^*(\phi(A))$.

Corollary 1.9. *Let A and B be local C*-algebras, and let $\phi : A \to B$ be a c.p.c. order zero map. Then there is a positive element $h \in \mathcal{M}(C^*(\phi(A))) \cap C^*(\phi(A))'$ and a $*$-homomorphism $\pi : A \to \mathcal{M}(C^*(\phi(A))) \cap \{h\}'$ such that $\|\phi\| = \|h\|$ and $\phi(a) = h\pi(a)$ for any $a \in A$.*

Proof. By Proposition 1.8 there is a unique c.p.c. order zero extension $\tilde{\phi} : \tilde{A} \to \tilde{B}$ of ϕ to a c.p.c. order zero map between the completions of A and B. Hence, by Theorem 1.5 there are a positive element

$$h \in \mathcal{M}(C^*(\tilde{\phi}(\tilde{A}))) \cap C^*(\tilde{\phi}(\tilde{A}))' = \mathcal{M}(C^*(\phi(A))) \cap C^*(\phi(A))'$$

and a $*$-homomorphism

$$\tilde{\pi} : \tilde{A} \to \mathcal{M}(C^*(\tilde{\phi}(\tilde{A}))) \cap \{h\}' = \mathcal{M}(C^*(\phi(A))) \cap \{h\}'$$

such that $\|\phi\| = \left\|\tilde{\phi}\right\| = \|h\|$ and $\tilde{\phi}(a) = h\tilde{\pi}(a)$ for any $a \in \tilde{A}$. It is therefore enough to take π as the restriction of $\tilde{\pi}$ to A to obtain the sought $*$-homomorphism. □

The following proposition shows that continuous functional calculus can be defined on c.p. order zero maps between *local* C*-algebras, thus extending the analogous result in [75].

Proposition 1.10. *Let A, B be local C*-algebras and let $\phi : A \to B$ be a c.p.c. order zero map. For any positive continuous function $f \in C_0((0,1])^+$, the map $f(\phi) : A \to B$ given by*

$$f(\phi)(a) := f(h)\pi(a)$$

for any $a \in A$, where $h \in \mathcal{M}(C^*(\phi(A))) \cap C^*(\phi(A))$ and $\pi : A \to \mathcal{M}(C^*(\phi(A))) \cap \{h\}'$ come from the structure result of Corollary 1.9, is a c.p.c. order zero map between local C-algebras.*

Proof. If A is complete with respect to its C*-norm, i.e. A is a C*-algebra, then one can use Corollary 3.2 of [75] directly. Otherwise, for any positive contraction $a \in A^+$, the C*-algebra $A_a := C^*(a)$ it generates inside the completion of A is σ-unital. Therefore, the image of the restriction of ϕ onto A_a is contained inside a finitely generated C*-subalgebra B_a of the completion of B which is then contained in B itself. For any polynomial p with zero constant term and degree $d > 1$ one can find another polynomial P_p of zero constant term and $m_d + 1$ variables, where $m_d = \lceil \log_2 d \rceil$, such that (cf. Example 1.6 and Equation (1.1))

$$p(\phi)(a) := p(h)\pi(a) = P_p(\phi(a), \phi(a^{\frac{1}{2}}), \ldots, \phi(a^{\frac{1}{2^{m_d}}})) \in B_a.$$

Therefore, by approximating any function $f \in C_0((0,1])$ with suitable polynomials $\{p_n\}$ of zero constant term, one can set

$$f(\phi)(a) := \lim_{n\to\infty} p_n(h)\pi(a) \in B_a,$$

whose extension by linearity to A defines the sought c.p.c. order zero map. □

En passant we observe that, as in the case of c.p.c. order zero maps between C*-algebras, functional calculus on c.p.c. order zero maps can, in principle, be defined without any explicit reference to the structure result of Corollary 1.9.

1.1.3 Strongly Self-absorbing C*-algebras

The class of strongly self-absorbing C*-algebras plays an important rôle in the Classification Programme of C*-algebras. They include the class of UHF algebras of infinite type, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ and the Jiang-Su algebra $\mathcal{Z}$. It emerges from their abstract theory (cf. [70]) that all strongly self-absorbing C*-algebras are simple and nuclear and are either stably finite with unique trace or purely infinite. Their name stems from the fact that if $\mathcal{D}$ is a strongly self-absorbing C*-algebra, then the tensor product $\mathcal{D}\otimes\mathcal{D}$, which is unambiguous since $\mathcal{D}$ is automatically nuclear, is isomorphic to $\mathcal{D}$ in a strong sense, as discussed in [70].

In this section we collect definitions and some well-known facts about strongly self-absorbing C*-algebras, mainly drawn from the already cited [70]. We also prove some further results that are used later on in this thesis, especially to provide a few explicit examples of computation of bivariant Cuntz semigroups (cf. §3.6.2).

We first recall the notion of approximate unitary equivalence of c.p.c. maps as stated in [70], which is the one that we also adopt throughout this thesis.

Definition 1.11. Let A and B be C*-algebras, and let $\phi,\psi : A\to B$ be c.p.c. maps. We say that ϕ is approximately unitarily equivalent to ψ, and we write $\phi\approx_{\text{a.u.}}\psi$ in symbols, if there exists a sequence of unitaries $\{u_n\}_{n\in\mathbb{N}}$ in the multiplier algebra $\mathcal{M}(B)$ of B such that

$$\|\phi(a)-u_n\psi(a)u_n^*\|\to 0$$

for any $a\in A$.

Following [70], all strongly self-absorbing C*-algebras are characterised by the following property.

Definition 1.12 (Strongly self-absorbing C*-algebra)**.** A unital C*-algebra $\mathcal{D}$ which is not the algebra of complex numbers $\mathbb{C}$ is said to be strongly self-absorbing if there is an isomorphism $\phi:\mathcal{D}\to\mathcal{D}\otimes\mathcal{D}$ which is approximately unitarily equivalent to $\mathrm{id}_\mathcal{D}\otimes 1_\mathcal{D}$, i.e. $\phi\approx_{\text{a.u.}}\mathrm{id}_\mathcal{D}\otimes 1_\mathcal{D}$.

Among the properties of strongly self-absorbing C*-algebras we need to refer to the particular one given in part (i) of [70, Proposition 1.10] which, for the purposes of the present thesis, can be stated as follows.

Proposition 1.13. *Let $\mathcal{D}$ be a strongly self-absorbing C*-algebra. There exists a unital $*$-homomorphism $\gamma : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$ such that*

$$\gamma \circ (\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}) \approx_{\mathrm{a.u.}} \mathrm{id}_{\mathcal{D}} .$$

The map γ that appears in this last result can be thought as the inverse map of the isomorphism ϕ that figures in the Definition 1.11 of strongly self-absorbing C*-algebras given above.

Along with the well-known results collected in [70, Proposition 1.2], we also need a further technical lemma about the approximate unitary equivalence of tensor products of maps, which is now given.

Lemma 1.14. *Let A, B, C and D be separable C*-algebras, A nuclear, D unital and nuclear, and let $\phi, \psi : A \to D$, $\eta : B \to C$ be c.p.c. order zero maps, with $\phi \approx_{\mathrm{a.u.}} \psi$. Then $\eta \otimes \phi \approx_{\mathrm{a.u.}} \eta \otimes \psi$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset D$ be a sequence of unitaries such that

$$\|u_n \psi(a) u_n^* - \phi(a)\| \to 0$$

for any $a \in A$. Since one has the inclusion $\mathcal{M}(C) \otimes D \subset \mathcal{M}(C \otimes D)$, the sequence of unitaries $\{1_{\mathcal{M}(C)} \otimes u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(C \otimes D)$ is such that

$$\left\|(1_{\mathcal{M}(C)} \otimes u_n)(\eta \otimes \psi)(x)(1_{\mathcal{M}(C)} \otimes u_n^*) - (\eta \otimes \phi)(x)\right\| \to 0$$

for any $x \in B \otimes A$, whence $\eta \otimes \phi \approx_{\mathrm{a.u.}} \eta \otimes \psi$. □

1.2 Non-commutative Topology

The well-known Gelfand-Naimark Theorem establishes an equivalence between the category of locally compact Hausdorff spaces with that of commutative C*-algebras. For this reason, a general non-commutative C*-algebra is usually regarded as a non-commutative generalisation of topological spaces. This interpretation is substantiated by the fact that many topological properties, like compactness, connectedness, dimension etc. . . , can be reformulated as properties of C*-algebras without any reference to commutativity. Even K-theory, which was originally formulated for topological spaces, can be extended to general non-commutative C*-algebras. In this section we collect some definitions and results at the heart of non-commutative topology in the sense of Akemann [1] that are used later on to give an equivalent picture for the Cuntz semigroup in the sense of [16], in terms of the so-called *open projections* (cf. Section 2.3).

It is well known that open subsets of a compact Hausdorff space X can be characterised by Urysohn's Lemma as follows. A subset $U \subset X$ is open if there exists an increasing net of positive continuous functions $\{f_\alpha\}_{\alpha\in I}$ supported by U such that

$$\chi_U(x) = \lim_{\alpha\in I} f_\alpha(x), \qquad \forall x \in X,$$

i.e. if the characteristic function of U can be realised as the point-wise limit of an increasing net of positive functions supported by U.

In [1], Akemann used this property to generalise the notion of open subsets to non-commutative C*-algebras by naturally replacing sets with projections, and therefore the non-commutative analogue of the above statement leads to the following definition.

Definition 1.15 (Open projection)**.** Let A be any C*-algebra. A projection $p \in A^{**}$ is *open* if it is the strong limit of an increasing net of positive elements $\{a_\alpha\}_{\alpha\in I} \subseteq A^+$.

Equivalently, a projection $p \in A^{**}$ is open if it belongs to the strong closure of the hereditary subalgebra $A_p \subseteq A$ (cf. [1]), where

$$A_p := pA^{**}p \cap A = pAp \cap A. \tag{1.2}$$

Throughout, the set of all the open projections of A in A^{**} will be denoted by $P_o(A^{**})$.

Continuing with the topological analogy, a projection $p \in A^{**}$ is said to be *closed* if its complement $1 - p \in A^{**}$ is an open projection. The supremum of an arbitrary set $P \subset P_o(A^{**})$ of open projections in A^{**} is still an open projection and, likewise, the infimum of an arbitrary family of closed projections is still a closed projection, by results in [1], which extend the commutative analogues. Therefore, the closure of an open projection $p \in A^{**}$ can be defined as

$$\overline{p} := \inf\{q^*q = q \in A^{**} \mid 1 - q \in P_o(A^{**}) \ \wedge \ p \leq q\}.$$

Let B be a C*-subalgebra of A. A closed projection $p \in A^{**}$ is said to be *compact* in B if there exists a positive contraction $a \in B_1^+$ such that $pa = p$.

It is well-known that the association $p \mapsto A_p$ is one-to-one, and surjective when A is separable. Therefore, it is possible to establish a *hereditary* C^**-subalgebra* analogue of the operation of taking suprema of countably many open projections in $P_o(A^{**})$, as argued in [8]. We start by observing that, given two open projections $p, q \in A^{**}$ such that $p \leq q$ (as positive elements), then q obviously acts as a unit on p, and $A_p \subseteq A_q$ (cf. [52, §4.5]). For an increasing sequence of open projections we then have the following result.

Lemma 1.16. *Let A be a separable C^*-algebra and let $\{p_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open projections in A^{**}. Then*

$$A_p = \overline{\bigcup_{k\in\mathbb{N}} A_{p_k}},$$

where $p := \mathrm{SOT}\lim_{n\to\infty} p_n$.

Proof. Let B denote the norm-closure of the union of the hereditary subalgebras $\{A_{p_k}\}_{k\in\mathbb{N}}$. By construction, B is a hereditary subalgebra of A, and therefore, since A is separable, there exists a generator $a \in B$ such that $B = \overline{aAa}$. It is then enough to show that the support projection $q \in A^{**}$ of a coincides with p. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive elements converging to a in norm and such that $a_n \in A_{p_n}$ for any $n \in \mathbb{N}$. Let q be the support projection of a and q_n be the support projection of a_n for any $n \in \mathbb{N}$. It is clear that $q_n \leq p_n \leq q$ for any $n \in \mathbb{N}$ from which it follows that

$$\sup\{q_n\}_{n\in\mathbb{N}} \leq \mathrm{SOT}\lim_{n\to\infty} p_n \leq q.$$

Now suppose that q' is an open projection such that $q_n \leq q'$ for any $n \in \mathbb{N}$. This implies that $a_n q' = q' a_n = a_n$ for any $n \in \mathbb{N}$, whence

$$aq' = q'a = a.$$

Therefore $q \leq q'$, which leads to $q = \sup\{q_n\}_{n\in\mathbb{N}}$, and hence $q \leq p \leq q$, i.e. $p = q$. □

A similar result, which relies on the use of positive elements rather than open projections, can be found in [10, Lemma 4.2]. As an immediate consequence of the above lemma we record the following result, which involves the closure in the strong operator topology of the hereditary subalgebras associated to an increasing sequence of open projections.

Corollary 1.17. *Let A be a separable C^*-algebra and let $\{p_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open projections in A^{**}. Then*

$$\overline{A_p}^{\mathrm{SOT}} = \overline{\bigcup_{k\in\mathbb{N}} \overline{A_{p_k}}^{\mathrm{SOT}}}^{\mathrm{SOT}},$$

where $p := \mathrm{SOT}\lim_{n\to\infty} p_n$.

Proof. From Lemma 1.16 one has

$$\overline{A_p}^{\mathrm{SOT}} = \overline{\bigcup_{n\in\mathbb{N}} A_{p_n}}^{\mathrm{SOT}}.$$

Therefore, by using that

$$\bigcup_{n\in\mathbb{N}} \overline{A_{p_n}}^{\mathrm{SOT}} \subseteq \overline{\bigcup_{n\in\mathbb{N}} A_{p_n}}^{\mathrm{SOT}} \quad \text{and} \qquad A_p \subseteq \overline{\bigcup_{n\in\mathbb{N}} \overline{A_{p_n}}^{\mathrm{SOT}}}^{\mathrm{SOT}},$$

it follows that

$$A_p \subseteq \overline{\bigcup_{n\in\mathbb{N}} \overline{A_{p_n}}^{\mathrm{SOT}}}^{\mathrm{SOT}} \subseteq \overline{\bigcup_{n\in\mathbb{N}} A_{p_n}}^{\mathrm{SOT}} = \overline{A_p}^{\mathrm{SOT}}.$$ □

The result that now follows is an example of an application of Lemma 1.16. The construction of the supremum, i.e. the join of an arbitrary family of projections in the bidual A^{**} of a C*-algebra A can be carried out by relying on the lattice structure on the set of projections in A^{**}. In the case of an *increasing* sequence of projections, Lemma 1.16 shows that the hereditary C*-subalgebra associated to the supremum coincides with the inductive limit of the increasing sequence of hereditary C*-subalgebras associated to each projection in the subset of $P_o(A^{**})$ considered. For the general case we then have the following result.

Proposition 1.18. *Let A be a separable C*-algebra, $\{p_n\}_{n\in\mathbb{N}} \subseteq P_o(A^{**})$ an arbitrary sequence of open projections in A^{**} and $p := \sup\{p_n\}_{n\in\mathbb{N}}$. Then*

$$A_p = \bigvee_{n\in\mathbb{N}} A_{p_n},$$

i.e. A_p coincides with the hereditary C-subalgebra of A generated by the family of hereditary C*-subalgebras $\{A_{p_n}\}_{n\in\mathbb{N}}$.*

Proof. Consider the new sequence of open projections $\{q_n\}_{n\in\mathbb{N}}$ defined by $q_1 := p_1,\ q_{n+1} := q_n \vee p_{n+1},\ \forall n \in \mathbb{N}$. This clearly defines an increasing sequence of open projections, and moreover $p := \sup\{p_n\}_{n\in\mathbb{N}} = \mathrm{SOT}\lim_{n\to\infty} q_n$. Therefore, using Lemma 1.16, one has the identification

$$A_p = \overline{\bigcup_{k\in\mathbb{N}} A_{q_k}}.$$

By definition, A_{p_k} is clearly contained in A_{q_k} for any $k \in \mathbb{N}$, so

$$\bigvee_{k\in\mathbb{N}} A_{p_k} \subseteq \overline{\bigcup_{k\in\mathbb{N}} A_{q_k}}.$$

On the other hand, A_{q_k} is contained in $\bigvee_{n=1}^{k} A_{p_n}$, so

$$\overline{\bigcup_{k\in\mathbb{N}} A_{q_k}} \subseteq \bigvee_{k\in\mathbb{N}} A_{p_k},$$

which shows equality. □

1.3 The Murray-von Neumann Semigroup

In this section we recall the definition and the main properties of the Murray-von Neumann semigroup of a C*-algebra. This object constitutes the foundations for a concrete realisation of topological K-theory which, as argued in the introduction, plays an important rôle in the Classification Programme for C*-algebras. However, we do not need a detailed knowledge of K-theory for the purposes of this thesis, as we are more concerned with objects that have just a semigroup structure. Besides, the enveloping Grothendieck group constructions generally leads to a loss of useful information that is contained within the semigroup the construction is operated upon.

We refer the reader to [40, 71] for more detailed accounts of K-theory for operator algebras.

A self-adjoint element $p \in A$ of a (local) C*-algebra which is also idempotent, i.e. satisfies $p^2 = p$, is called a *projection*. The set of all projections in A will be denoted by $P(A)$ and can be characterised as

$$P(A) := \{p \in A \mid p^*p = p\}.$$

Observe that, contrary to the case of von Neumann algebras, which are generated, as Banach spaces, by their set of projections, a (local) C*-algebra need not have any projections. Indeed, the Abelian C*-algebra $C_0(X)$, where X is a locally compact, non-compact connected Hausdorff space is projectionless. However, they always abound of positive elements, and this is one of the main reasons why one wants to *compare* positive elements from a C*-algebra, like in the theory of the Cuntz semigroup recalled in Chapter 2, in order to gain insight into its internal structure.

Two projections p, q from a (local) C*-algebra A are said to be Murray-von Neumann equivalent, or just equivalent when no confusion over the implied equivalence relation arises, if there exists a partial isometry $v \in A$ such that

$$p = v^*v \qquad \wedge \qquad q = vv^*.$$

Observe that the M_∞-matrix ampliation of a C*-algebra A leads to the following form of stability under direct sum: for any two elements $a, b \in M_\infty(A)$, their direct sum $a \oplus b$ can be identified with an element in $M_\infty(A)$.

Definition 1.19 (Murray-von Neumann semigroup)**.** Let A be a (local) C*-algebra, and let $V(A)$ denote the set of Murray-von Neumann equivalence classes of projections from $M_\infty(A)$. Equipped with the Abelian binary operation

$$[p] + [q] := [p \oplus q],$$

the set $V(A)$ defines the *Murray-von Neumann* semigroup of A.

To any $*$-homomorphism $\phi : A \to B$ between two (local) C*-algebras A and B there corresponds a map ϕ_* between the respective Murray-von Neumann semigroups $V(A)$ and $V(B)$ given by

$$\phi_*([a_{ij}]) := \big[\phi(a_{ij})\big], \qquad \forall [a_{ij}] \in M_\infty(A),$$

and this turns V into a covariant functor from the category of (local) C*-algebras to that of Abelian monoids (cf. [71, Proposition 6.1.3]).

The Murray-von Neumann semigroup is an example of an invariant for (local) C*-algebras. Indeed, if A and B are isomorphic (local) C*-algebras, with isomorphism ϕ, then their Murray-von Neumann semigroups $V(A)$ and $V(B)$ are also isomorphic as monoids through the natural map ϕ_*.

Proposition 1.20. *The functor V is additive, that is, for any pair of C*-algebras A and B there is a natural isomorphism*

$$V(A \oplus B) \cong V(A) \oplus V(B).$$

See [71, Proposition 6.2.1] or [40, Proposition 4.3.4] for a proof of the above property. The same result holds for the Cuntz semigroup (cf. 2.8) and it is shown later on in Chapter 3 that this is also the case for the bivariant Cuntz semigroup in both its arguments (cf. Proposition 3.19 and Proposition 3.23).

Recall that a C*-algebra A is said to be stable if there is an isomorphism between A itself and $A \otimes K$, where K is the C*-algebra of compact operators on an infinite dimensional separable Hilbert space. A functor F whose domain category is that of C*-algebras is said to be stable in its argument if there is a natural isomorphism between $F(A)$ and $F(A \otimes K)$ within the target category.

Proposition 1.21. *The functor V is stable, that is, for every C^*-algebra A there is a natural semigroup isomorphism $V(A) \cong V(A \otimes K)$.*

The proof of the above proposition follows the ideas behind the proof of the analogous result for the K_0 functor and can be found in e.g. [71, Corollary 6.2.11].

We say that a functor F is *continuous* if it preserves inductive limits, that is, if $C = \varinjlim C_i$ in the domain category, then there is a natural isomorphism

$$F(C) \cong \varinjlim F(C_i),$$

where the inductive limit on the right-hand side is taken in the target category, and is assumed to exist.

Proposition 1.22. *The functor V is continuous, that is, if A is the C^*-inductive limit of an inductive sequence of C^*-algebras $\{A_n\}_{n\in\mathbb{N}}$, then*

$$V(A) \cong \varinjlim V(A_n),$$

where the limit of the right-hand side is taken inside the category of partially ordered Abelian monoids.

We refer the reader to [71, Proposition 6.2.9] for a proof of the above result. Observe that Proposition 1.21 follows from Proposition 1.22 if one realises $A \otimes K$ as the inductive limit of the sequence of matrix algebras $M_n(A)$ over A, which induces a constant inductive sequence at the level of the Murray-von Neumann semigroups, since the isomorphism $M_n(M_\infty(A)) \cong M_\infty(A)$ induces a natural ordered semigroup isomorphism $V(M_n(A)) \cong V(A)$ (this latter property is sometimes referred to as *matrix stability*) for any $n \in \mathbb{N}$.

1.4 Equivariant K-theory

In view of the establishment of an equivariant version for the bivariant Cuntz semigroup, which is the subject of Section 3.8, we now give a brief account of equivariant K-theory. This was first formulated by Atiyah for topological spaces acted upon by compact groups (cf. [65]). The theory was later applied to operator algebras with an action of a compact group, and a fundamental result of Julg [33], independently rediscovered by Green and Rosenberg in an unpublished work, established a correspondence between the equivariant K-theory of an action and the ordinary K-theory of the associated crossed product C*-algebra. The main reference for this section is [6, §V.11], which in turn is largely based on [57].

Definition 1.23 (G-algebra)**.** A G-algebra is a triple (A, G, α) consisting of a C*-algebra A, a topological group G and a continuous action $\alpha : G \to \mathrm{Aut}(A)$ of G on A, i.e. a point-norm continuous group homomorphism.

G-algebras are also known in the literature as *C*-dynamical systems* or *C*-covariant systems*. Throughout this section we let G denote a compact topological group, unless otherwise stated. Hence, all the G-algebras we consider are C*-algebras with a compact group action. When the action and the group are clear from the context, or their specification is not necessary, we denote a G-algebra (A, G, α) simply by referring to the underlying C*-algebra A. A homomorphism between G-algebras (A, G, α) and (B, G, β) is a $*$-homomorphism $\phi : A \to B$ that is assumed to intertwine the actions, i.e.

$$\phi(\alpha_g(a)) = \beta_g(\phi(a)), \qquad \forall a \in A, g \in G,$$

or simply by the commutative squares

$$\phi \circ \alpha_g = \beta_g \circ \phi, \qquad \forall g \in G.$$

Like the K_0-group of a C*-algebra A, there are many ways of giving a concrete realisation of the equivariant K_0-group A. The pictures we are interested in are those based on idempotents and finitely generated projective Hilbert right modules, together with their respective equivariant generalisations. We refer the reader to [43] for an introduction to the theory of Hilbert modules. In what follows we will use the notation $B(E)$ to denote the set of all bounded and adjointable operators on the Hilbert right A-module E, where A is any C*-algebra. That is, $T \in B(E)$ if there exists a map $T^* : E \to E$ such that $(x, Ty) = (T^*x, y)$ for any $x, y \in E$, where $(\,\cdot\,,\,\cdot\,)$ denotes the A-valued inner product on E (cf. [43, Lemma 2.1.1]). In this case one also has $T^* \in B(E)$.

Definition 1.24. Let (A, G, α) be a G-algebra. A finitely generated projective (A, G, α)-module is a pair (E, λ) consisting of a finitely generated projective right A-module E and a strongly continuous group homomorphism λ from G to the invertible elements of $B(E)$ with coefficient map α, that is

$$\lambda_g(ea) = \lambda_g(e)\alpha_g(a), \qquad \forall a \in A, g \in G.$$

Let (A, G, α) be a G-algebra, π be a finite-dimensional representation of G over the vector space V and consider the A-module $V \otimes A$[1]. It becomes an (A, G, α)-module when

[1]Unless otherwise specified, tensor products are over the field $\mathbb{C}$.

equipped with the diagonal action $\lambda := \pi \otimes \alpha$, which, in turn, induces an action of G on the C*-algebra of bounded and adjointable operators $B(V \otimes A)$ through

$$gT := \lambda_g \circ T \circ \lambda_g^{-1}, \qquad \forall g \in G.$$

Among all the elements of $B(V \otimes A)$ one can then consider the set of G-invariant projections, i.e.

$$P(V \otimes A)^G := \{P \in B(V \otimes A) \mid P^* P = P \ \wedge GP = \{P\}\}.$$

It is easy to verify that if $p \in P(V \otimes A)^G$ is a G-invariant projection, then $p(V \otimes A)$, that is the range of p, is a finitely generated projective (A, G, α)-module. The converse is also true, namely every finitely generated projective (A, G, α)-module is the range of a projection $p \in B(V \otimes A)$ for some representation π of G over the finite dimensional vector space V (cf. [6, Proposition 11.2.3]). Hence, G-invariant projections and finitely generated projective (A, G, α)-modules are interchangeable objects.

If π and ω are two finite dimensional representations of G over the vector spaces V and W respectively, one can equip the set of bounded and adjointable operators $B(V \otimes A, W \otimes A) \cong B(V, W) \otimes A$, where $B(V, W) \otimes A$ denotes the Banach subspace of the C*-algebra $B(V \oplus W) \otimes A$ given by operators with non-zero entry in the $(2, 1)$ corner, with the G-action given by

$$gT := (\omega_g \otimes \alpha_g) \circ T \circ (\pi_g \otimes \alpha_g)^{-1}.$$

Two G-invariant projections $p \in P(V \otimes A)^G$ and $q \in P(W \otimes A)^G$ are Murray-von Neumann equivalent (in symbols $p \simeq_G q$) if there exist G-invariant elements $v \in B(V \otimes A, W \otimes A)^G$ and $w \in B(W \otimes A, V \otimes A)^G$, i.e. $gv = v$ and $gw = w$ for any $g \in G$, such that $p = w \circ v$ and $q = v \circ w$. Similarly to the standard case, Murray-von Neumann subequivalence is expressed as follows. One says that p is Murray-von Neumann subequivalent to q (in symbols $p \preceq_G q$) if there exist $v \in B(V \otimes A, W \otimes A)^G$ such that $p = v^* \circ v$ and $v \circ v^* \leq q$. The modules $p(V \otimes A)$ and $q(V \otimes A)$ are then isomorphic as (G, A, α)-modules if and only if p and q are Murray-von Neumann equivalent.

Definition 1.25. The equivariant Murray-von Neumann semigroup $V^G(A)$ of a unital G-algebra (A, G, α) is the set of isomorphism classes of finitely generated projective (A, G, α) modules equipped with the operation $+$ derived from the direct sum of modules.

Equivalently, the equivariant Murray-von Neumann semigroup $V^G(A)$ can be defined as the set of classes of Murray-von Neumann equivalent G-invariant projections over all the modules of the form $V \otimes A$, where V is a finite dimensional representation vector space for G. The equivariant K_0 group of the G-algebra A is obtained from $V^G(A)$ through the usual construction of the Grothendieck enveloping group, viz.

$$K_0^G(A) := \Gamma(V^G(A)).$$

However we shall not focus on this construction, as the most relevant object for this thesis is the semigroup V^G. Rather, it is best to reformulate the definition of V^G in a slightly different way, which is more prone to a bivariant generalisation. Let $\hat{G}$ be the set of unitary equivalence classes of irreducible representations of G. Denote by H_G the Hilbert space of the direct sum over all members of $\hat{G}$ of the representation vector spaces of arbitrarily selected representative from each class, viz.

$$H_G := \bigoplus_{\xi \in \hat{G}} V_{\pi_\xi},$$

where V_{π_ξ} is the representation vector space of a representation π_ξ in the class ξ. We then regard each invariant projection $p \in P(V \otimes A)^G$ as a projection from the larger module $H_G \otimes A$. The stabilisation of the Hilbert space H_G, that is $H_G^{\oplus\infty} \cong H_G \otimes \ell^2(\mathbb{N})$, which is needed because of the way the addition in V^G is defined, is then isomorphic to $L^2(G) \otimes \ell^2(\mathbb{N})$ by Peter-Weyl's theorem, and therefore

$$K(H_G \otimes \ell^2(\mathbb{N}) \otimes A) \cong A \otimes K(L^2(G)) \otimes K.$$

By equipping the module $L^2(G) \otimes \ell^2(\mathbb{N}) \otimes A$ with the diagonal action $\lambda \otimes \mathrm{id}_K \otimes \alpha$, where $\lambda : G \to B(L^2(G))$ is the left-regular representation of G, it becomes an (A, G, α)-module and the equivariant Murray-von Neumann semigroup of A can then be identified with the Murray-von Neumann equivalence classes of G-invariant projections in $A \otimes K(L^2(G)) \otimes K$. That is, using K_G as a shorthand notation for $K(L^2(G)) \otimes K$, we have

$$V^G(A) \cong P(A \otimes K_G)^G / \sim . \tag{1.3}$$

Example 1.26. If G is the trivial group $\{e\}$ and A is a G-algebra then $L^2(G) \cong \mathbb{C}$ and therefore $K_{\mathbb{C}} \cong K$. Hence $V^{\mathbb{C}}(A)$ is the ordinary Murray-von Neumann semigroup of the C*-algebra A. △

Example 1.27. Let $G \neq \{e\}$ act trivially on the C*-algebra of complex numbers $\mathbb{C}$. Take any two irreducible representations π_1 and π_2 of G over the vector spaces V_1 and V_2 and consider the direct sum representation $\pi_1 \oplus \pi_2$ over $V_1 \oplus V_2$. The G-invariant elements of $B(V_1 \oplus V_2)$, which can be taken in the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \begin{cases} a_{11} \in B(V_1), a_{22} \in B(V_2), \\ a_{12} \in B(V_2, V_1), a_{21} \in B(V_1, V_2), \end{cases}$$

with respect to the action given by the representations, have to satisfy the conditions

$$[a_{11}, \pi_1] = [a_{22}, \pi_2] = 0, \qquad a_{12}, a_{21}^* \in (\pi_1, \pi_2),$$

where (π_1, π_2) is the set of intertwiners between π_1 and π_2. The irreduciblility of both π_1 and π_2 imply that

$$a_{11} = \alpha_{11} \operatorname{Id}_{V_1} \qquad \text{and} \qquad a_{22} = \alpha_{22} \operatorname{Id}_{V_2}, \qquad \alpha_{11}, \alpha_{22} \in \mathbb{C},$$

as a consequence of Schur's lemma, whereas for the other two conditions one has to distinguish between the cases $\pi_1 \circlearrowleft \pi_2$ and $\pi_1 = \pi_2$, where $\circlearrowleft$ denotes disjointness in the sense of Mackey. The former implies that $(\pi_1, \pi_2) = \{0\}$, whence $a_{12} = a_{21} = 0$, while the latter leads to $[a_{12}, \pi_1] = [a_{21}, \pi_1] = 0$, i.e.

$$a_{12} = \alpha_{12} \operatorname{Id}_{V_1}, \qquad a_{21} = \alpha_{21} \operatorname{Id}_{V_1}, \qquad \alpha_{12}, \alpha_{21} \in \mathbb{C},$$

as a consequence, once more, of Schur's lemma. Hence, the G-invariant elements of $B(V_1 \oplus V_2)$ are either of the form

$$\begin{bmatrix} \alpha_{11} \operatorname{Id}_{V_1} & 0 \\ 0 & \alpha_{22} \operatorname{Id}_{V_2} \end{bmatrix}, \qquad \alpha_{11}, \alpha_{22} \in \mathbb{C}$$

if $\pi_1 \circlearrowleft \pi_2$ or of the form

$$A \otimes \operatorname{Id}_{V_1}, \qquad A \in M_2(\mathbb{C})$$

if $\pi_1 = \pi_2$. It is now immediate to conclude that the Grothendieck enveloping group of $V^G(\mathbb{C})$ coincides with the representation ring $R_{\mathbb{C}}(G)$, or simply $R(G)$, of the group G, i.e.

$$K_0^G(\mathbb{C}) = \Gamma(V^G(\mathbb{C})) \cong R(G),$$

where $R(G)$ is the set of unitary equivalence classes of unitary representations of G, the ring structure coming from direct sums and tensor products. △

About the equivariant K-theory of actions we now recall the already mentioned fundamental theorem of Julg that connects them to the ordinary K-theory of crossed products.

This result has been generalised to the equivariant theory of the Cuntz semigroup in [24]. We also recall that we are here under the assumption that every group G considered is compact.

Theorem 1.28 (Julg)**.** *Let (A, G, α) be a G-algebra. There is a natural isomorphism between $K_0^G(A)$ and $K_0(A \rtimes G)$.*

We refer the reader to [6, Theorem 11.7.1] for a proof of the above result.

1.5 Kasparov's KK-theory

This section is devoted to a brief account of KK-theory, a bivariant formulation of K-theory which is originally due to Kasparov [34]. In this thesis we work under the *ansatz* that the bivariant theory for the Cuntz semigroup that we are after should possess as many of the properties of KK-theory as possible, for this is already the case for the Cuntz semigroup with respect to K-theory. However, because of the abstract characterisation of KK-theory due to Higson [29], not all the properties of the functor $KK(A,\ \cdot\ )$, for any C*-algebra A, like homotopy invariance, stability and split exactness, can be shared by the picture of the bivariant Cuntz semigroup that is presented in this thesis, for otherwise one would end up with yet another picture for KK-theory itself. Indeed, the ordinary Cuntz semigroup is not homotopy invariant, as opposed to what one has in K-theory, where homotopy is employed to actually define the equivalence relations that yields the K-groups. The main references for this section are [31] and [6].

Based on the just recalled result of Higson, it would not be necessary to give any specific picture of KK-theory, but just its abstract properties. However, as we are giving explicit pictures for the bivariant Cuntz semigroup in the present thesis, we shall here recall both Kasparov and Cuntz's approach, as well as the standard simplifications that lead from Kasparov's picture to the Fredholm picture, as analogues of these reductions for the Cuntz semigroup are discussed in both Chapter 3 and Chapter 4.

1.5.1 Kasparov's Picture

As it is standard in KK-theory, we shall assume that all the C*-algebras appearing in this section are σ-unital and $\mathbb{Z}_2$-graded.

Definition 1.29 (Kasparov triple)**.** Let A and B be C*-algebras. A Kasparov A-B triple is a triple (E, ϕ, T) consisting of a countably generated and graded Hilbert right B-module

E, a graded $*$-homomorphism $\phi : A \to L(E)$ and an odd operator $T \in L(E)$ that satisfy the following axioms:

(KT.1) $[\phi(a), T] \in K(E)$ for any $a \in A$, where $[\,\cdot\,,\,\cdot\,]$ is the graded commutator;

(KT.2) $\phi(a)[T^2 - \mathrm{id}_E] \in K(E)$ for any $a \in A$;

(KT.3) $\phi(a)(T - T^*) \in K(E)$ for any $a \in A$.

Observe that, if (E_1, ϕ_1, T_1) and (E_2, ϕ_2, T_2) are Kasparov A-B triples, the direct sum $(E_1, \phi_1, T_1) \oplus (E_2, \phi_2, T_2)$ is defined as the Kasparov A-B triple $(E_1 \oplus E_2, \phi_1 \hat{\oplus} \phi_2, T_1 \oplus T_2)$, where the grading on $E_1 \oplus E_2$ is the *diagonal* one, i.e.

$$S_{E_1 \oplus E_2}(e_1 \oplus e_2) := (S_{E_1} e_1) \oplus (S_{E_2} e_2), \qquad \forall e_1 \oplus e_2 \in E_1 \oplus E_2.$$

This operation between Kasparov triples provides the binary operation on the KK-groups, while the equivalence relation among them is provided by the following notion of homotopy.

Definition 1.30. Let A and B be C*-algebras and let (E_0, ϕ_0, T_0), (E_1, ϕ_1, T_1) be Kasparov A-B triples. A homotopy between such triples is a Kasparov A-$C([0,1], B)$ triple (E, ϕ, T) such that $\mathrm{ev}_{0*}(E, \phi, T) = (E_0, \phi_0, T_0)$ and $\mathrm{ev}_{1*}(E, \phi, T) = (E_1, \phi_1, T_1)$, where $\mathrm{ev}_t : C([0,1], B) \to B$ is given by $\mathrm{ev}_t(\beta) = \beta(t)$ for any $\beta \in C([0,1], B)$.

It is easy to see that homotopy between Kasparov triples defines an equivalence relation (see [31, Lemma 2.1.12]), which we denoted by $\sim$. This can be used to define the KK-groups as follows.

Definition 1.31. Let A and B be C*-algebras. The KK-group of A and B is the set of classes

$$KK(A, B) := \{\text{Kasparov } A\text{-}B \text{ triples}\}/\sim,$$

equipped with the Abelian binary operation given by

$$[(E_1, \phi_1, T_1)] + [(E_2, \phi_2, T_2)] := [(E_1, \phi_1, T_1) \oplus (E_2, \phi_2, T_2)].$$

It is perhaps not evident at first sight that the above definition gives a group. To see this one can observe that to every Kasparov A-B triple (E, ϕ, T) there corresponds another one, namely $(E_-, \phi \circ \beta_A, -T)$, where E_- is the same as E but with opposite grading, and β_A is the grading on A. This new triple is such that $(E, \phi, T) \oplus (E_-, \phi \circ \beta_A, -T) \sim (\{0\}, 0, 0)$, with the right-hand side giving (a representative of) the neutral element for what turns out to then be a group (cf. [31, Theorem 2.1.23]).

1.5.2 Cuntz's Picture

Cuntz's picture of KK-theory is an easier one, and closer in spirit to the main definition of the bivariant Cuntz semigroup that we employ in Chapter 3.

Definition 1.32. Let A and B be C*-algebras. A *quasi-homomorphism* from A to B is a pair of $*$-homomorphisms $\phi_\pm : A \to \mathcal{M}(B \otimes K)$ such that

$$\phi_+(a) - \phi_-(a) \in B \otimes K$$

for any $a \in A$.

Two quasi-homomorphisms (ϕ_+^0, ϕ_-^0) and (ϕ_+^1, ϕ_-^1) are said to be homotopic if there exists a path of quasi-homomorphisms $\{(\eta_+^t, \eta_-^t)\}_{t\in[0,1]}$ such that $t \mapsto \eta_\pm^t(a)$ is strictly continuous, $t \mapsto \eta_+^t(a) - \eta_-^t(a)$ is uniformly continuous, and $(\eta_+^k, \eta_-^k) = (\phi_+^k, \phi_-^k)$ for $k = 0, 1$. The KK-group $KK(A, B)$ of A and B can then be identified as the set of equivalence classes of quasi-homomorphisms with respect to this notion of homotopy, the Abelian binary operation being given by *direct sum* of quasi-homomorphisms, which can formally be expressed as (cf. [66, §2.4])

$$[(\phi_+, \phi_-)] + [(\psi_+, \psi_-)] = [(\phi_+ \hat{\oplus} \psi_+, \phi_- \hat{\oplus} \psi_-)],$$

where by the symbol $\hat{\oplus}$ we denote the direct sum precomposed with the diagonal map, i.e. $(\phi \hat{\oplus} \psi)(a) := (\phi \oplus \psi)(\Delta(a))$, for any $a \in A$, where ϕ and ψ are any two linear maps defined on the same linear space A and taking values in some other linear spaces, and $\Delta : A \to A \oplus A$ is the diagonal map $\Delta(a) := a \oplus a$ for any $a \in A$.

A semigroup $VV(A, B)$ can be constructed using homotopy equivalence as

$$VV(A, B) := \{[(\phi_+, \phi_-)]) \mid \phi_- = 0\}.$$

Observe that any quasi-homomorphism of the form $(\phi_+, 0)$ can be identified with the $*$-homomorphism $\phi_+ : A \to B \otimes K$. Hence, when $A = \mathbb{C}$ one recovers the Murray-von Neumann semigroup of B, viz.

$$VV(\mathbb{C}, B) \cong V(B).$$

One can then think of $VV(A, B)$ as a bivariant extension of the Murray-von Neumann semigroup, and the bivariant Cuntz semigroup introduced in this work is a direct analogue of this object, as it is shown in Chapter 3.

1.5.3 Main Properties

Apart from the characterising properties of the already mentioned Higson's result, one can identify some other properties of the KK-groups. As show in [6, §17.8], $KK(\,\cdot\,,\,\cdot\,)$ provides a bifunctor from the category of C*-algebras to that of Abelian groups which is contravariant in the first argument and covariant in the second. Such a functor is finitely additive and stable in both arguments, i.e.

$$KK(A_1 \oplus A_2, B_1 \oplus B_2) \cong \bigoplus_{i,k=1,2} KK(A_i, B_k)$$

and

$$KK(A \otimes K, B \otimes K) \cong KK(A, B),$$

for any C*-algebras A, A_1, A_2, B, B_1, B_2, but there are no general properties of continuity under inductive limits of C*-algebras. Furthermore, $KK(\,\cdot\,,\,\cdot\,)$ is also countably additive in its second argument. As shown in Chapter 3, all these properties are recovered by the definition of the bivariant Cuntz semigroup given there.

Another important feature of KK-theory is the existence of a bi-additive map

$$\cdot\;: KK(A, B) \times KK(B, C) \to KK(A, C)$$

for any triple of C*-algebras A, B, C, with the following properties:

i. associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad \forall x \in KK(A, B), y \in KK(B, C), z \in KK(C, D)$;

ii. extends composition of $*$-homomorphisms: $[(\phi, 0)] \cdot [(\psi, 0)] = [(\psi \circ \phi, 0)]$ for any pair of $*$-homomorphisms $\phi : A \to B$ and $\psi : B \to C$;

iii. $KK(A, A)$ is a ring with unit $[(\mathrm{id}_A, 0)]$.

Such a map takes the name of *Kasparov product* since it was firstly introduced by Kasparov. However, there is no straightforward description of such a product in both Kasparov and Cuntz's pictures of KK-theory and the construction tends to be rather technical. For the purposes of this thesis it is enough to observe that, on $VV(A, B)$, which comprises classes of $*$-homomorphisms, Kasparov's product can be thought as a mere composition of maps between C*-algebras. Indeed, in Section 3.4 we show that an analogue of Kasparov product can be defined within the theory of the bivariant Cuntz semigroup presented in this thesis.

From the point of view of classification, Kasparov product plays an important rôle since it can be used to define the so-called KK-equivalence by means of *invertible elements*. Throughout we will use the shorthand notation ι_A to denote the unit $[(\mathrm{id}_A, 0)]$ of the ring $KK(A, A)$.

Definition 1.33. Let A and B be C*-algebras. An element $x \in KK(A, B)$ is said to be invertible if there exists $y \in KK(B, A)$ such that $x \cdot y = \iota_A$ and $y \cdot x = \iota_B$.

In Section 3.7 we give an analogous definition of invertibility for the bivariant Cuntz semigroup and exploit it to provide some classification results for unital and stably finite C*-algebras.

Before concluding this brief overview of KK-theory we also mention how the Kasparov product can be used to define a map $\gamma_0 : KK(A, B) \to \mathrm{Hom}(K_0(A), K_0(B))$[2] for any pair of C*-algebras A and B, which is based on the well-known identification

$$KK(\mathbb{C}, B) \cong K_0(B)$$

for any C*-algebra B. Such map is defined as

$$\gamma(x)(z) = z \cdot x, \qquad \forall x \in KK(A, B), z \in K_0(A).$$

It is shown in Section 3.4 that an analogue of this map for the bivariant Cuntz semigroup agrees with the map that sends a class of a c.p.c. order zero map to the semigroup homomorphism it induces between the corresponding Cuntz semigroups. The surjectivity of this map can then be used, in principle, to introduce a notion of Cuntz-bootstrap class similarly to the standard results of KK-theory of Rosenberg and Schochet [62]. However, this line of research will be pursued elsewhere.

1.5.4 The Equivariant Theory

In Section 3.8 we give an equivariant definition of the bivariant Cuntz semigroup. As a comparison, we include here a brief overview of equivariant KK-theory. As in [6, §20], we also restrict our attention to the second countable case, to which we also add compactness of all the groups, although some of the following results hold in a greater generality.

[2] It is also possible to define a map γ_1 that goes from KK-theory to the K_1 groups, but this is not needed for the purposes of this thesis and therefore we omit it.

Definition 1.34. Let (A, G, α) be a G-algebra. A Hilbert (A, G, α)-module is a Hilbert A-module E with an action of G on E which is continuous in the sense that the map $g \mapsto \|(gx, gx)\|$ is continuous for any $x \in E$, and compatible with the action α on A, i.e.

$$g(xa) = (gx)\alpha_g(a), \qquad \forall g \in G, x \in E, a \in A.$$

The grading can be extended to both G-algebras and equivariant Hilbert modules of the above definition. Then many of the results and operations involving graded Hilbert modules extend to the equivariant setting, including Kasparov's stabilisation theorem (cf. [31, Theorem 1.2.12]). The equivariant analogue of a Kasparov triple is provided by the following definition.

Definition 1.35. Let A and B be graded G-algebras. A Kasparov A-B G-triple is a triple (E, ϕ, T), where E is a countably generated Hilbert (B, G, β)-module, $\phi : A \to B(E)$ is an equivariant graded $*$-homomorphism and $T \in B(E)$ is a G-invariant operator that satisfy

(EKT.1) $[\phi(a), T] \in K(E)$ for any $a \in A$;

(EKT.2) $\phi(a)(T^2 - 1_{B(E)}) \in K(E)$ for any $a \in A$;

(EKT.3) $\phi(a)(T - T^*) \in K(E)$ for any $a \in A$.

It must be noted that the above definition is not the most general one. However, we are here assuming that G is compact and we can therefore make use of Proposition 20.2.4 of [6] to find a compact perturbation of a Kasparov triple where the operator T is G-invariant (and hence G-continuous). The equivariant KK-group of the pair of C*-algebras A and B is then defined as in the non-equivariant case by taking homotopy classes of Kasparov A-B G-triples. One then gets a bivariant functor KK^G which, likewise KK, is contravariant in the first argument and covariant in the second.

With the natural identification $KK^G(\mathbb{C}, B) \cong K_0^G(B)$ one sees immediately that the representation ring of the group G is recovered as $R(G) \cong KK^G(\mathbb{C}, \mathbb{C})$. Furthermore, there is a group homomorphism $j_G : KK^G(A, B) \to KK(A \rtimes G, B \rtimes G)$ that is functorial in A and B and compatible with Kasparov product (cf. [6, Theorem 20.6.2]).

Chapter 2

The Cuntz Semigroup

This chapter is devoted to the theory of the Cuntz semigroup. We give a brief account of its history and of its latest developments, focusing on those aspects that are of particular relevance for the bivariant Cuntz semigroup introduced in Chapter 3, in order to provide an exposition of the topic that is as much self-contained as possible. Perhaps one of the most interesting features of the theory of the Cuntz semigroup is that there are many equivalent pictures that give concrete realisations of it. For one of this, namely the open projection picture, which is discussed in Section 2.3, we give an alternative proof to the result of [16] about the existence of suprema in the Cuntz semigroup. The aim of this approach is to provide a setting of the theory that can be extended easily to a bivariant form, with the goal of proving that the object Cu defined in Chapter 3 belongs to the category Cu, which to date remains an open question. The further bivariant extensions presented in Chapter 4 are meant to provide technical tools for tackling this problem. Our main references for this chapter are [4] for the original definition of the Cuntz semigroup, together with some of its most important developments, [16] for the module picture of the Cuntz semigroup as well as the category Cu and [3] for the latest developments on the structure of the Cuntz semigroup and further categorical aspects.

2.1 Definitions, Properties and Technicalities

The Cuntz semigroup has received a lot of attention in the last decade because of its successes in the endeavours to classify C*-algebras and, as a consequence, it has evolved quite extensively since its original definition. We provide here a brief account of its history, from its original definition as comparison of positive elements from the infinite matrix ampliation $M_\infty(A)$ of a C*-algebra A to its restatement in a stabilised form through the Hilbert module picture of Coward, Elliott and Ivanescu [16]. The reason for following this path relies on the fact that, although rarely used in practice, the original definition of the Cuntz semigroup has important categorical aspects which are intimately related to the notion of local C*-algebras given in the previous chapter, and interesting properties, like e.g. continuity, arise when the target category is chosen wisely, as shown in [3].

The origin of Cuntz comparison, which is the main idea behind the Cuntz semigroup, can be traced back to the work of Cuntz [17] on dimension functions for simple C*-algebras. Here he introduced the group $K_0^*(A)$ of a C*-algebra A, which is just the Grothendieck enveloping group of a semigroup that resembles the Murray-von Neumann semigroup for projections, which became known as the Cuntz semigroup. It was soon realised that, by passing from the Cuntz semigroup to its enveloping Grothendieck group, there is sometimes a loss of information involved in the process, and therefore one usually does not consider this construction in practice, but rather the Cuntz semigroup itself is analysed instead. At the heart of the theory we have the following definition.

Definition 2.1 (Cuntz subequivalence)**.** Let A be a local C*-algebra, and let $a, b \in M_\infty(A)^+$ be two positive elements. We say that a is (Cuntz-)subequivalent to b, in symbols $a \precsim b$, if there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ such that

$$\lim_{n\to\infty} \|x_n b x_n^* - a\| = 0.$$

With the above definition giving Cuntz comparison of positive elements from a C*-algebra, one can define the following semigroup. Let $\sim$ be the antisymmetrisation of the above pre-order relation $\precsim$, that is we say that two positive elements $a, b \in A$ from a local C*-algebra A are (Cuntz)-equivalent if both $a \precsim b$ and $b \precsim a$ hold. With this equivalence relation at hand, the original Cuntz semigroup is defined as follows.

Definition 2.2 (Cuntz semigroup W)**.** Let A be a local C*-algebra. The Cuntz semigroup $W(A)$ of A is the set of equivalence classes

$$W(A) := M_\infty(A)/\sim,$$

endowed with the binary operation $+$ defined as

$$[a] + [b] := [a \oplus b].$$

It is routine to verify that the above operation + is well-defined and that therefore $W(A)$ is indeed a semigroup for every local C*-algebra A[1]. One can also establish a connection between this object and the Murray-von Neumann semigroup, which follows from the fact that any two projections p, q from a local C*-algebra A that are Murray-von Neumann equivalent are also Cuntz equivalent (cf. [3] for the extension of results of Rørdam to local C*-algebras and [4, Lemma 2.18]). This implies that there is a natural map $V(A) \to W(A)$ given by sending the class of a projection p in $V(A)$ to its class inside $W(A)$, which is then well-defined. Such a map becomes an embedding under special circumstances, notably when the local C*-algebra A is stably finite (cf. [4, Lemma 2.20]).

It turns out that every Cuntz semigroup $W(A)$ can be equipped with a structure of positively ordered Abelian monoid through the order relation $\leq$ defined as

$$[a] \leq [b] \qquad \Longleftrightarrow \qquad a \precsim b, \qquad a, b \in M_\infty(A)^+,$$

and that such order extends the algebraic one (cf. [4]). The following result, taken from [4] and stemming from commutative C*-algebras, not only gives a better insight into the meaning of the Cuntz subequivalence described above, but also offers a bridge between classical and non-commutative topology in relation to Cuntz comparison.

Proposition 2.3. *Let X be a compact Hausdorff space, and let f, g be any two positive functions from the commutative C*-algebra $C(X)$ of continuous functions over X. Then $f \precsim g$ if and only if* $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$.

For a proof of the above proposition we refer the reader to [4, Proposition 2.5]. As an immediate corollary to this result we have that $a \sim a^n$ for every positive element a from a C*-algebra A and every $n \in \mathbb{N}$. Furthermore, $a^*a \sim aa^*$ for *any* $a \in A$ (cf. [4, Corollary 2.6]).

We now provide a series of technical results which are of fundamental importance for both the theory of the Cuntz semigroup and the bivariant Cuntz semigroup introduced in Chapter 3. The first one is a special instance of a result of Handelman (cf. [27, Lemma A-1]) that allows relating the natural partial order of the positive cone of a local C*-algebra with Cuntz comparison.

[1]It turns out that every Cuntz semigroup is actually a monoid, the class of the null element being the identity of the monoid. However, it is customary in the literature to refer to it as a semigroup.

Lemma 2.4. *Let A be a local C*-algebra and let $a, b \in A$ be two positive elements of A such that $a \leq b$. Then there exists a sequence of contractions $\{z_n\}_{n\in\mathbb{N}}$, $\|z_n\| \leq 1$, such that*

$$\lim_{n\to\infty} \left\| z_n b^{\frac{1}{2}} - a^{\frac{1}{2}} \right\| = 0.$$

Proof. If A is not unital, consider it as a subalgebra of its minimal unitisation A^+, and let $\{z_n\}_{n\in\mathbb{N}} \subset A$ be the sequence given by

$$z_n := a^{\frac{1}{2}} b^{\frac{1}{2}} (b + \tfrac{1}{n})^{-1}.$$

By the C*-identity we then have the following estimate

$$\begin{aligned}
\left\| z_n b^{\frac{1}{2}} - a^{\frac{1}{2}} \right\|^2 &= \left\| a^{\frac{1}{2}} \left[b(b + \tfrac{1}{n})^{-1} - 1 \right] \right\|^2 \\
&= \left\| \left[(b + \tfrac{1}{n})^{-1} b - 1 \right] a \left[b(b + \tfrac{1}{n})^{-1} - 1 \right] \right\| \\
&\leq \left\| b \left[b(b + \tfrac{1}{n})^{-1} - 1 \right]^2 \right\| \\
&\leq \frac{1}{n^2} \sup_{t\in\mathbb{R}_0^+} \left| \frac{t}{(t + \frac{1}{n})^2} \right| \\
&= \frac{1}{4n},
\end{aligned}$$

where we have used the fact that $x^*ax \leq x^*bx$ for any $x \in A$ whenever $a \leq b$. The same fact can be used to check that all the z_ns are contractive, since

$$\begin{aligned}
\|z_n\|^2 &= \left\| (b + \tfrac{1}{n})^{-1} b^{\frac{1}{2}} a b^{\frac{1}{2}} (b + \tfrac{1}{n})^{-1} \right\| \\
&\leq \left\| b^2 (b + \tfrac{1}{n})^{-2} \right\| \\
&\leq \sup_{t\in\mathbb{R}_0^+} \left| \frac{t^2}{(t + \frac{1}{n})^2} \right| \\
&\leq 1,
\end{aligned}$$

whence $\|z_n\| \leq 1$ for any $n \in \mathbb{N}$. ◻

The above lemma can be used to give a concise proof of the following result which links the natural order of the positive cone of a local C*-algebra and the Cuntz subequivalence relation $\precsim$ introduced earlier (cf. [4, Lemma 2.8]).

Lemma 2.5. *Let A be a C*-algebra and let $a, b \in A$ be any two positive elements such that $a \leq b$. Then $a \precsim b$.*

Proof. By the previous lemma there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset A$ of contractions such that

$$\lim_{n\to\infty} \left\| z_n b^{\frac{1}{2}} - a^{\frac{1}{2}} \right\| = 0,$$

and therefore one has

$$\lim_{n\to\infty} \| z_n b z_n^* - a \| = 0,$$

which shows that $a \precsim b$. □

Other applications of Lemma 2.4 appear in Chapter 3, where we provide a result that can be regarded as a generalisation of the last lemma to the bivariant setting (cf. Proposition 3.8).

Another technical result that we generalise to the bivariant theory revolves around the continuous functional calculus through the continuous function $f_\epsilon \in C_0((0,1])$ given by the description

$$f_\epsilon(x) = \begin{cases} 0 & x \in (0,\epsilon] \\ x - \epsilon & x \in (\epsilon, 1], \end{cases} \tag{2.1}$$

that is, $f_\epsilon(x) = (x-\epsilon)_+$, where $(\,\cdot\,)_+$ denotes the positive part, i.e. $(x)_+ = \max\{0, x\}$. For a positive element a from a local C*-algebra A, the Cuntz relation between $(a-\epsilon)_+$ for any $\epsilon > 0$ and a is a rather predictable one, as shown by the following corollary to the last lemma (cf. [4, Corollary 2.9]).

Corollary 2.6. *Let A be a C*-algebra, $a \in A$ a positive element. Then $(a-\epsilon)_+ \precsim a$ for any $\epsilon > 0$.*

Proof. For any $\epsilon > 0$ one clearly has $(a-\epsilon)_+ \leq a$, whence $(a-\epsilon)_+ \precsim a$ by the previous lemma. □

A generalisation of this result to the bivariant theory is also provided in Chapter 3 (cf. Corollary 3.10).

The functoriality properties of the Cuntz semigroup W are analysed in greater details in Section 2.4, where an enriched category of partially ordered semigroups is taken into consideration. Here we restrict to some basic functorial properties when W is viewed as functor from the category of C*-algebras to that of partially ordered Abelian monoids. In order to fix the most suited domain category, however, we recall the following result of Winter and Zacharias (cf. [75, Corollary 3.5]).

Proposition 2.7. *Let A and B be C*-algebras. Every c.p.c. order zero map $\phi : A \to B$ induces a morphism of partially ordered Abelian monoids*

$$W(\phi) : W(A) \to W(B)$$

which is given by

$$W(\phi)([a]) := [(\phi \otimes \mathrm{id}_{M_\infty})(a)]$$

for any $[a] \in W(A)$.

With the above result at hand one can show that W is a covariant functor from the category whose objects are C*-algebras and whose arrows are given by c.p.c. order zero maps, to the category of partially ordered Abelian monoids. Furthermore, it is clear that the same result holds when A and B are *local* C*-algebras, so that one could consider a larger domain category for W. Already with this rather simple categorical setting, W exhibits many interesting properties, although it fails, for instance, to be continuous, i.e. it does not preserve inductive limits, contrary to the case of the K functor of K-theory. For this reason, a different categorical setting needs to be considered if one wishes to salvage this property, and this aspect is the focus of Section 2.4. For the time being, we limit ourselves to recall a few general properties of the functor W in the functorial setting described in this section, as these turn out to be common to other enriched settings discussed further on in this thesis.

Proposition 2.8. *The functor W is additive, that is, for any pair of local C*-algebras A and B there is a natural isomorphism*

$$W(A \oplus B) \cong W(A) \oplus W(B).$$

The proof of the above proposition is based on the same idea behind the analogous property of the Murray-von Neumann semigroup (cf. Proposition 1.20). However, the following examples show that W is neither stable nor continuous, contrary to the case of the Murray-von Neumann semigroup (cf. propositions 1.21 and 1.22).

Example 2.9. The isomorphism $M_n(M_\infty(A)) \cong M_\infty(A)$ induces a natural isomorphism $W(M_n(A)) \cong W(A)$ at the level of the Cuntz semigroups W for any $n \in \mathbb{N}$, as it is easy to show, and therefore W is a matrix-stable functor. As every positive element in M_k is of finite rank, one has that

$$W(M_n(\mathbb{C})) \cong W(\mathbb{C}) \cong \mathbb{N},$$

for any $n \in \mathbb{N}$. △

Example 2.10. A positive element in the C*-algebra of compact operators K can have infinite rank. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\ell^2(\mathbb{N})$ and let $\theta_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the rank-1 operator given by

$$\theta_n(v) = (e_n, v)e_n$$

for any $n \in \mathbb{N}$, i.e. θ_n is the projection onto the direction identified by e_n. Then

$$a := \sum_{k=1}^{n} \frac{1}{2^k}\theta_k$$

is a compact operator on $\ell^2(\mathbb{N})$, and in fact positive and of infinite rank. Therefore,

$$W(K) \cong \mathbb{N}_0 \cup \{\infty\},$$

where the class of $\infty \in W(K)$ is represented by $a \in K^+$, i.e. $[a] = \infty$. △

Since the C*-algebra of compact operators K can be viewed as the C*-inductive limit of the sequence of matrix algebras $M_n(\mathbb{C})$, the two examples above show that

$$\mathbb{N}_0 \cup \{\infty\} \cong W(K) \not\cong \varinjlim W(M_n(\mathbb{C})) \cong \mathbb{N}_0.$$

Therefore the functor W from the category of C*-algebras to that of partially ordered Abelian monoids cannot be stable and continuous. In Section 2.4 it is shown that, with a different categorical setting, one can recover the property of continuity for W, and hence stability as well.

2.2 The Module Picture

Because of the problems discussed at the end of the previous section, Coward, Elliott and Ivanescu [16] gave a new description of the Cuntz semigroup of a C*-algebra which is based on a suitable notion of comparison between countably generated Hilbert right C*-modules, of which isomorphism is a special instance. They introduced a category, called Cu, to which every Cuntz semigroup in this new picture belongs, and the new functor they introduced, that goes from the category of C*-algebras and $*$-homomorphisms to Cu is shown to be continuous under direct limits. We refer the reader to [31, 39, 43] for the general theory of Hilbert C*-modules.

Cuntz comparison of countably generated Hilbert (right) modules is based on the key notion of *compact containment* introduced in [16, §1].

Definition 2.11 (Compact containment)**.** Let B be a C*-algebra and let X and Y be countably generated Hilbert B-modules such that $X \subset Y$. Then X is compactly contained in Y, $X \Subset Y$ in symbols, if there exists a compact-like self-adjoint operator $a \in K_B(Y)$ such that $a|_X = \mathrm{id}_X$.

Following the original terminology introduced in [16], we shall sometimes call every submodule of a given countably generated Hilbert C*-module a *subobject*. Then Cuntz comparison of countably generated Hilbert C*-modules can be stated in words by saying that a module X is subequivalent to another module Y if every compactly contained subobject is isomorphic, as a B-module, to a compactly contained subobject of Y. To be more definite we give the following formal definition.

Definition 2.12. Let A be a C*-algebra and let X and Y be Hilbert A-modules. We say that X is Cuntz-subequivalent to Y (in symbols $X \precsim Y$) if

$$\forall X' \Subset X \qquad \exists Y' \Subset Y \qquad | \qquad X' \cong Y'.$$

The antisymmetrisation of the above relation $\precsim$ yields an equivalence relation $\sim$, the Cuntz equivalence, between countably generated Hilbert C*-modules, that is $X \sim Y$ if $X \precsim Y$ and $Y \precsim X$. This can then be used to define the Cuntz semigroup Cu of a C*-algebra A as the set of equivalence classes of countably generated Hilbert (right) A-modules. In formal terms we have the following definition.

Definition 2.13 (Cuntz semigroup Cu)**.** Let A be a C*-algebra and let $H(A)$ denote the collection of all the countably generated Hilbert right A-modules. Then the Cuntz semigroup $\mathsf{Cu}(A)$ associated to A is the set of classes

$$\mathsf{Cu}(A) := H(A)/\sim,$$

with the binary operation $+$ given by direct sum of modules, i.e.

$$[X] + [Y] := [X \oplus Y],$$

for any $[X], [Y] \in \mathsf{Cu}(A)$.

Observe that containment of modules is a special instance of subequivalence, that is $X \subset Y$ as modules implies that $X \precsim Y$. Furthermore, isomorphism of modules is a special instance of Cuntz equivalence, so that, in the definition above, one can think of $H(A)$ as denoting either the family of countably generated right Hilbert A-modules or the set of their isomorphism classes.

As in the case of the Cuntz semigroup W introduced earlier, one can equip the Cuntz semigroup Cu with an order relation $\leq$, which is simply given by

$$[X] \leq [Y] \qquad \text{if} \qquad X \precsim Y.$$

This way one can prove that $\mathsf{Cu}(A)$ becomes a partially ordered Abelian monoid for every C*-algebra A (cf. [16, Theorem 1]). Another important result that emerges from the just cited theorem is the existence of suprema for increasing sequences, or more generally for countable upward directed sets. Here the supremum is to be understood as the least upper bound of a given set of elements from a Cuntz semigroup Cu, that is, given a C*-algebra A and a countable subset $S \subset \mathsf{Cu}(A)$, $x \in \mathsf{Cu}(A)$ is the supremum of S, in symbols

$$x = \sup S,$$

if

i. $s \leq x$ for any $s \in S$ and

ii. $s \leq y$ for any $s \in S$ implies $x \leq y$.

The relation between the Cuntz semigroups W and Cu can be inferred from the following two results. The first one asserts that Cu, which turns out to be a functor, is stable (cf. [4, Corollary 4.31]).

Proposition 2.14. *For any C*-algebra A there is an isomorphism*

$$\mathsf{Cu}(A) \cong \mathsf{Cu}(A \otimes K)$$

of partially ordered Abelian monoids.

The next result gives an explicit connection between the Cuntz semigroups W and Cu when the argument is a stable C*-algebra (cf. [4, Theorem 4.33]).

Proposition 2.15. *For any stable C*-algebra A there is an isomorphism*

$$W(A) \cong \mathsf{Cu}(A)$$

of partially ordered Abelian monoids.

By combining the last two cited results together, one reaches the conclusion that the general relation between the Cuntz semigroups W and Cu is given by the isomorphism

$$\mathsf{Cu}(A) \cong W(A \otimes K) \tag{2.2}$$

of partially ordered Abelian monoids, which holds for *any* C*-algebra A. Hence, one can view the Cuntz semigroup Cu as a form of *stabilisation* of the Cuntz semigroup W which leads, among many things, to the existence of suprema for increasing sequences.

Example 2.16. As shown in Example 2.10, The Cuntz semigroup W of the C*-algebra of compact operators K is isomorphic to the extended naturals $\mathbb{N}_0 \cup \{\infty\}$. Therefore, we have the isomorphisms

$$\mathsf{Cu}(\mathbb{C}) \cong \mathsf{Cu}(K) \cong \mathbb{N}_0 \cup \{\infty\}.$$

It is clear that every increasing sequence in $\mathsf{Cu}(\mathbb{C})$ is just an increasing sequence of natural numbers which always has a supremum because of the existence of $\infty \in \mathsf{Cu}(\mathbb{C})$. △

2.3 The Open Projection Picture

There is yet another picture for the Cuntz semigroup that has been studied in [52], and based on a suitable notion of Cuntz comparison of open projections. Roughly speaking it is a restatement of the module picture, where each module is naturally associated to an open projection, and the notion of compact containment and equivalence of modules are suitably redefined. As shown by the main result of [8], one of the advantages of this picture for the Cuntz semigroup is that a proof of the existence of suprema can be given that has more of an algebraic flavour than the one in [16], where this result was first established for the general case.

As with modules, the key notion is that of compact containment between open projections.

Definition 2.17. Let A be a C*-algebra and let $p, q \in A^{**}$ be open projections. We say that p is compactly contained in q (in symbols $p \subset\subset q$) if there exists a positive contraction $e \in A_q$ such that $\bar{p}e = \bar{p}$, i.e. if the closure of p is compact in A_q.

The analogue of isomorphism between countably generated Hilbert modules is provided by the following notion of equivalence between open projections.

Definition 2.18 (Peligrad-Zsidó [54])**.** Let A be a C*-algebra and let $p, q \in A^{**}$ be open projections. We say that p and q are PZ-equivalent (in symbols $p \sim_{\mathrm{PZ}} q$) if there exists a partial isometry $v \in A^{**}$ such that

$$p = v^*v, \qquad q = vv^*$$

and

$$vA_p \subset A, \qquad v^*A_q \subset A.$$

It is clear that PZ equivalence is generally stronger than Murray-von Neumann equivalence, although there are cases where the two are known to coincide (see [52] and references therein). With the two definitions above it is now possible to formulate Cuntz comparison for open projections as in [52].

Definition 2.19. Let A be a C*-algebra and let $p, q \in A^{**}$ be open projections. We say that p is Cuntz-subequivalent to q (in symbols $p \precsim q$) if

$$\forall p' \Subset p \qquad \exists q' \Subset q \qquad | \qquad p' \sim_{\mathrm{PZ}} q'.$$

As remarked in Section 1.2, when a C*-algebra A is separable, there is a bijective correspondence between its hereditary C*-subalgebras, each of which is generated by a positive element of A, and open projections in A^{**}. This suggests the introduction, as in [52], of an equivalence relation $\cong$ between positive elements of the form

$$a \cong b \qquad \Longleftrightarrow \qquad A_a = A_b. \tag{2.3}$$

If a is any positive element of A, and E_a and p_a are the corresponding right Hilbert A-module and the support projection respectively, Proposition 4.13 of [52] shows that the Cuntz comparison of these objects coincide and therefore the Cuntz semigroup $\mathsf{Cu}(A)$ can be identified, up to isomorphism, with a semigroup of open projections, namely

$$\mathsf{Cu}(A) := P_o((A \otimes K)^{**})/\sim,$$

with addition given by direct sum as described in [52, §6.2].

In [8], a slight refinement of the above identification is obtained, where it is shown that for every class in $P_o((A \otimes K)^{**})/\sim$ one can find a representative P which lies in $\mathcal{M}(A \otimes K)$ (cf. [8, Proposition 1.8]). This result is exploited in the proof of existence of suprema in the open projection picture of the Cuntz semigroup. It is firstly established for special cases up to the most general one of arbitrary Cuntz-increasing sequences. For completeness we mention the main logical steps involved. An important technical result is an analogue in the open projection picture of an argument in [16] (cf. [4, Proposition 4.11]).

Lemma 2.20. *Let p be the strong limit of an increasing sequence of open projections $p_1 \leq p_2 \leq \cdots$. Then, for every $q \Subset p$, there is an $n \in \mathbb{N}$ and an open projection $q' \Subset p_n$ such that $q \sim_{PZ} q'$.*

Proof. By the definition of the relation $q \Subset p$ there exists a positive element a in the unit ball of A_p such that $\overline{q}a = \overline{q}$, and by the same argument as in [16] (cf. [4, Proposition 4.11]), one can find $a' \in \mathrm{C}^*(a)$ such that $\overline{q}(a' - \epsilon)_+ = \overline{q}$.

Let $a_n \in A_{p_n}$ be such that $\|a_n - a'\| < \epsilon$, which exists by Lemma 1.16. By [36, Lemma 2.2] there is a contraction $d \in A_p$ such that $da_nd^* = (a' - \epsilon)_+$, and it follows from [54, Theorem 1.4] that

$$\overline{q} \leq p_{x^*x} \sim_{PZ} p_{xx^*} \leq p_n,$$

where $x = a_n^{1/2}d^*$. Since $\leq$ and $\sim_{PZ}$ are special instances of $\precsim_{\mathsf{Cu}}$ and $\sim_{\mathsf{Cu}}$ respectively, using [52, Proposition 4.10] one also has

$$q \Subset p_{x^*x} \sim_{\mathsf{Cu}} p_{xx^*} \precsim_{\mathsf{Cu}} p_n.$$

Therefore there must exist an open projection $q' \Subset p_n$ such that $q \sim_{PZ} q'$. □

The first special instance of increasing sequences in the open projection picture of the Cuntz semigroup is the object of the following result.

Proposition 2.21. *If $p_1 \Subset p_2 \Subset \cdots$ is a rapidly increasing sequence of open projections in $P_o(A^{**})$, then $p_1 \leq p_2 \leq \cdots$ and*

$$\sup[p_n] = [\textsc{sot}\lim_{n\to\infty} p_n].$$

Proof. Let p be the strong limit of the p_ns and suppose that $[q]$ is such that $[p_n] \leq [q]$ for any $n \in \mathbb{N}$. By Lemma 2.20, for every $p' \Subset p$ there is an $n \in \mathbb{N}$ and an open projection q' such that $p' \sim_{PZ} q' \Subset p_n$. But, since $[p_n] \leq [q]$, there exists a $q'' \Subset q$ such that $q'' \sim_{\mathrm{PZ}} q'$. Therefore, $[p] \leq [q]$. Since $[q]$ is arbitrary, it follows that $[p] = \sup[p_n]$. □

The next step is to exploit the above result for more general increasing sequences, like the ones arising from the following partial order relation.

Definition 2.22 (Compact subequivalence)**.** Two open projections $p, q \in P_o(A^{**})$ are said to be compactly subequivalent, $p \Subset\!\!\prec q$ in symbols, if there exists $q' \Subset q$ such that $p \sim_{\mathrm{PZ}} q'$.

Observe that the usual compact containment relation $\Subset$ is a special instance of compact subequivalence $\Subset\!\!\prec$.

Let $\{p_n\}_{n\in\mathbb{N}}$ be any sequence of open projections in $P_o(A^{**})$ with the property that $p_n \Subset\!\!\prec p_{n+1}$ for every $n \in \mathbb{N}$. By assumption there are open projections $\{q_n\}_{n\in\mathbb{N}}$ such that $q_k \Subset p_{k+1}$ and $p_k \sim_{\mathrm{PZ}} q_k$. These determine an inductive sequence $(A_{p_k}, \phi_k)_{k\in\mathbb{N}}$ of hereditary subalgebras of A, where the connecting maps are given by the adjoint action of partial isometries $\{v_n\}_{n\in\mathbb{N}}$ satisfying $p_k = v_k^* v_k$, $q_k = v_k v_k^*$ and $vA_{p_k} \subseteq A$, $v^* A_{q_k} \subseteq A$, i.e. $\phi_k(a) = v_k^* a v_k$ for any $a \in A_{p_k}$. Denoting by $\tilde{A}$ the inductive limit of such a sequence, one gets maps $\{\rho_n\}_{n\in\mathbb{N}}$ that make the following diagram

$$\begin{array}{ccc} A_{p_k} & \xrightarrow{\phi_k} & A_{p_{k+1}} \\ & \searrow^{\rho_k} & \downarrow{\scriptstyle \rho_{k+1}} \\ & & \tilde{A} \end{array}$$

commutative. By considering $A \otimes K$ instead of A, one can extend the above partial isometries to unitaries and conclude that $\tilde{A}$ is a hereditary C*-subalgebra of A.

Lemma 2.23. *Every sequence $\{p_n\}_{n\in\mathbb{N}}$ of open projections in $P_o(A\otimes K)^{**}$ with the property that $p_n \Subset\!\!\prec p_{n+1}$ for every $n \in \mathbb{N}$ has a supremum in $\mathsf{Cu}(A)$.*

Proof. Denote by q_k the element that satisfies $p_{k-1} \sim_{PZ} q_k \Subset p_k$ coming from the definition of the relation $p_k \Subset\!\!\prec p_{k+1}$, and by capital letters (e.g. P_k, Q_k) the Cuntz equivalent projections in $\mathcal{M}(A\otimes K)$ that one can always find by [8, Proposition 1.8]. By [8, Corollary 1.11], there exists a collection of unitaries $\{u_n\}_{n\in\mathbb{N}}$ such that $u_{k-1}P_{k-1}u_{k-1}^* = Q_k$ for all $k \in \mathbb{N}$. Hence, $P_{k-1} = u_{k-1}^* Q_k u_{k-1} \Subset u_{k-1}^* P_k u_{k-1}$ and therefore one has that

$$P_1 \Subset u_1^* P_2 u_1 \Subset u_1^* u_2^* P_3 u_2 u_1 \Subset u_1^* u_2^* u_3^* P_4 u_3 u_2 u_1 \Subset \cdots .$$

Denoting by

$$U_n := \prod_{i=1}^{n-1} u_i$$

and by

$$P_i' := U_n^* P_n U_n,$$

we set $P := \textsc{sot}\lim_{n\to\infty} P_i'$. By the special case of Proposition 2.21 it then follows that $[P] = \sup[P_n']$ which implies that $[P] = \sup[p_i]$ since $[p_i] = [P_i] = [P_i']$. ∎

One then builds up on top of these partial results to get to the proof that every Cuntz-increasing sequence in the Cuntz semigroup admits a supremum, as shown by the following main result of [8].

Theorem 2.24. *Every Cuntz-increasing sequence* $\{p_n\}_{n\in\mathbb{N}}$ *of projections in* $P_o(A\otimes K)^{**}$ *admits a supremum in* $\mathsf{Cu}(A)$*.*

Proof. Without loss of generality we may assume that A is a stable C*-algebra. By assumptions there are positive contractions $\{a_{n,k}\}_{n,k\in\mathbb{N}}\subseteq A_1^+$ such that $p_n = \text{SOT}\lim_{k\to\infty} a_{n,k}$ and $a_{n,k}\leq a_{n,k+1}$ for any $k,n\in\mathbb{N}$.

These elements can be modified to yield rapidly increasing sequences of positive elements by setting

$$a'_{n,k} := \left(a_{n,k}-\tfrac{1}{k}\right)_+ .$$

Denoting by $q_{n,k}$ the support projections associated to these new elements $a'_{n,k}$, one has

$$q_{n,k}\Subset q_{n,k+1},$$

for any $k,n\in\mathbb{N}$ by construction. Now, starting with, e.g., $q_{1,1}$ and applying Lemma 2.20 to $q_{1,1}\Subset q_{1,2}\Subset p_1\precsim p_2$, one gets $m_1\in\mathbb{N}$ and $q_{1,1}\Subset p_{2,m_1}$ such that $q_{1,1}\sim_{\text{PZ}} q'_{1,1}$. By iterating these steps one can construct a sequence of open projections $q_k := q_{k,m_{k-1}}$ that satisfies the relations

$$q_1\sim_{\text{PZ}} q'_{1,1}\Subset q_2\sim_{\text{PZ}} q'_{2,m_1}\Subset q_3\cdots ,$$

which show that

$$q_1 \precsim\!\!\!\!\prec q_2 \precsim\!\!\!\!\prec q_3 \precsim\!\!\!\!\prec q_4 \precsim\!\!\!\!\prec \cdots .$$

Observe that $[q_k]\leq[p_k]$ for any $k\in\mathbb{N}$, and that for any $n,m\in\mathbb{N}$ there exists $l\in\mathbb{N}$ such that $[q_{n,m}]\leq[q_l]$. Therefore

$$[p_n]\leq\sup_k[q_k]\leq\sup_k[p_n],$$

which implies

$$\sup_n[p_n]\leq\sup_k[q_k]\leq\sup_n[p_n],\quad\text{i.e.}\quad \sup_n[p_n]=\sup_k[q_k].$$

The existence of the supremum now follows from Lemma 2.23. □

A summary of the analysis in [8] can then be given in the form of the following corollary to the above results.

Corollary 2.25. *Every element $x \in \mathsf{Cu}(A)$ can be written as the Cuntz class of the strict limit of an increasing sequence of projections in $\mathcal{M}(A \otimes K)$.*

Before concluding this section we record some further results of [52] that link together the comparison of open projections with other notions of comparison of positive elements, notably Blackadar equivalence. We first recall *Pedersen equivalence* between positive elements

Definition 2.26. Two positive elements a, b from a C*-algebra A are said to be *Pedersen equivalent*, in symbols $a \approx b$, if there exists $x \in A$ such that $a = x^*x$ and $b = xx^*$

A combination of both the relation $\cong$ of Equation (2.3) and Pedersen equivalence above leads to the following definition.

Definition 2.27. Two positive elements $a, b \in A$ are said to be *Blackadar equivalent*, in symbols $a \sim_s b$, if there exists $x \in A$ such that $a \cong x^*x$ and $b \cong xx^*$.

Since $A_a = A_b$ if and only if $p_a = p_b$ and $p_{x^*x} \sim_{\mathrm{PZ}} p_{xx^*}$ by Theorem 1.4 of [54], one has that $a \sim_s b$ if and only if $p_a \sim_{\mathrm{PZ}} p_b$ (cf. [52, Proposition 4.3], where it is also shown that this is equivalent to the isomorphism of E_a and E_b). With these results, the Cuntz semigroup of a separable C*-algebra can be realised in the picture of positive elements, countably generated Hilbert C*-modules and open projections, with a definition of the Cuntz comparison that follows that same prototype for each of these objects. One can then abstract from this observation and define a Cuntz comparison of objects in a certain set where a *strong* equivalence relation $\sim_s$ and a compact containment relation $\Subset$ are defined. With this idea in mind we then give the following abstract definition.

Definition 2.28 (Cuntz Comparison)**.** Let $(S, \Subset, \sim_s)$ be a triple consisting of a set S equipped with an equivalence relation $\sim_s$ and some partial order relation $\Subset$. Then $a \in S$ is said to be Cuntz-subequivalent to $b \in S$, in symbols $a \precsim b$, if

$$\forall a' \Subset a \qquad \exists b' \Subset b \qquad | \qquad a' \sim_s b'.$$

By results in [52], some of which have been cited earlier in this section, one sees that all the three pictures for the Cuntz semigroup are based on a Cuntz comparison in the sense above, with obvious meaning of the symbols $\sim_s$ and $\Subset$ for each of the three cases.

In Chapter 4 we give bivariant extensions of the Pedersen, Blackadar and Peligrad-Zsidó equivalence relations. We also give a bivariant extension of open projections, there termed *open* $*$*-homomorphisms*, by means of c.p.c. order zero maps, in order to introduce a Cuntz comparison of the type of Definition 2.28 for such new bivariant objects.

2.4 Categorical Aspects

As already mentioned towards the end of Section 2.1 and in Section 2.2, the main idea behind the work of Coward, Elliott and Ivanescu [16] is to give a new description of the Cuntz semigroup by introducing a suitable functor between suitable categories where one would recover the property of continuity under inductive limits. As this is an important tool for the construction of large classes of C*-algebras, preservation of inductive limits is a very useful property to have for an invariant that is expected to play an important rôle for the Classification Programme. Indeed, one of the main results of the already cited work [16] is enclosed in [16, Theorem 2], where it is shown that the map Cu that sends a C*-algebra A to the partially ordered Abelian monoid $\mathsf{Cu}(A)$ is indeed a functor, which becomes continuous when the target category is suitable chosen.

In this section we recall some of these results, together with the recent analogous analysis carried out in [3] for the functor W stemming from the original definition of the Cuntz semigroup.

2.4.1 The Category Cu

Before we are able to give the definition of the category Cumentioned above, we need to introduce the so-called *way below* relation, which can be stated in abstract form for any partially ordered Abelian monoid in an order-theoretic sense.

Definition 2.29. Let $(M, \leq)$ be a partially ordered Abelian monoid and let $x, y \in M$. We say that $x \ll y$, or that x is way below y, if any increasing sequence $\{y_n\}_{n\in\mathbb{N}}$ whose supremum exists and satisfies $y \leq \sup\{y_n\}_{n\in\mathbb{N}}$ is eventually above x, i.e. there exists $n \in \mathbb{N}$ such that $x \leq y_n$.

It was already observed in [16] that the above relation is equivalent to the following notion of compact containment at the level of elements from a Cuntz semigroup. For $[X], [Y] \in \mathsf{Cu}(A)$, where A is a C*-algebra, we say that $[X] \Subset [Y]$ if there exists a countably generated Hilbert A-module Z such that $[X] \leq [Z]$ and $Z \Subset Y$.

Definition 2.29 above allows to state some of the axioms of the category Cu, which we are about to define according to [16] (see also [4, §4.2] and [3, §3.1]), more concisely.

Definition 2.30 (Category Cu)**.** Let Cu be the category whose objects are positively ordered Abelian monoids, subject to the following extra axioms:

(CuO.1) every increasing sequence has a supremum;

(CuO.2) every element s of an object S in Cu can be represented as the supremum of a sequence $\{x_n\}_{n\in\mathbb{N}}$ with the property that $x_n \ll x_{n+1}$ for any $n \in \mathbb{N}$;

(CuO.3) $\ll$ is compatible with the binary operation $+$, i.e. if $a, b, c, d \in S$ are such that $a \ll c$ and $b \ll d$ then $a + b \ll c + d$;

(CuO.4) suprema are compatible with the binary operation $+$, i.e. if $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are increasing sequences then $\sup\{a_n + b_n\}_{n\in\mathbb{N}} = \sup\{a_n\}_{n\in\mathbb{N}} + \sup\{b_n\}_{n\in\mathbb{N}}$.

The arrows of the category Cu are positively ordered Abelian monoid morphisms that also preserve

(CuM.1) suprema of increasing sequences;

(CuM.2) the way below $\ll$ relation.

It is standard terminology to say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ from an object S in the category Cu is *rapidly increasing* if it satisfies the condition (CuO.2), i.e. if $x_n \ll x_{n+1}$ for any $n \in \mathbb{N}$. Hence, axiom (CuO.2) can also be stated by saying that every element from an object in the category Cu can be represented as the supremum of a rapidly increasing sequence.

As shown in [16] (cf. [3, Proposition 3.2.3]) every Cuntz semigroup Cu(A), where A is a C*-algebra, is an object in the category Cu, and when the target category of the functor Cu is restricted to this new enriched category, it becomes continuous (cf. [16, Theorem 2]; see also [4, §4.9]).

Regarding the way-below relation $\ll$ introduced above, there is a class of elements within the Cuntz semigroup Cu of a C*-algebra that can be singled out by the behaviour under this relation. If p is a projection then for every $\epsilon \in (0, 1)$ there is a positive scalar $\lambda \in (0, 1)$ such that $(p-\epsilon)_+ = \lambda p$. Therefore, $[(p-\epsilon)_+] = [p] \ll [p]$. This property justifies the following definition.

Definition 2.31 (Compact element)**.** Let S be an object in the category Cu. An element $k \in S$ is said to be compact if $k \ll k$.

With this terminology, every projection from a local C*-algebra defines a compact element in the Cuntz semigroup. It is well known that, if A is a simple and stably finite local C*-algebra, then the set of compact elements coincides with the image of the natural inclusion of the Murray-von Neumann semigroup inside the Cuntz semigroup (cf. [23]).

Before moving to the definition of another category that makes the functor W continuous, it is perhaps worth mentioning that every Cuntz semigroup Cu, that is, every object in the category Cu that comes from a C*-algebra A through the functor Cu, satisfies two more properties, that are usually assumed as extra axioms for the objects in the category Cu. These are known as property (O5) and (O6), and can be found stated in, e.g., [4, §4.2] or [3]. We record them here for the reader's convenience.

(CuO.5) Every object S of Cu has *almost algebraic order*, that is, for every $s, s', t, t', r \in S$ that satisfy $s + t \leq r$, $s' \ll s$ and $t' \ll t$ there exists $r' \in S$ such that $a' + r' \leq r \leq s + r'$ and $t' \leq r'$;

(CuO.6) every object S of Cu has *almost Riesz decomposition*, that is, for every $s', s, t, r \in S$ that satisfy $s' \ll s \leq t + r$, there exists $e, f \in S$ such that $s' \leq e + f$, $e \leq s, t$ and $f \leq s, r$.

2.4.2 The Category W

As the Cuntz semigroup $W(A)$ of a C*-algebra A is not an object of Cu in general, one might wonder if there is a suitable category that contains every such object. This study has been undertaken in [3], where a new category, called W, has been introduced. There it is shown that every Cuntz semigroup $W(A)$ is an object of this category for any *local* C*-algebra A. Furthermore, when viewed in this categorical setting, the functor W becomes continuous under arbitrary inductive limits if the domain category is chosen to be that of local C*-algebras, where limits differ from those in the category of C*-algebra by, as it turns out, a crucial norm completion. The purpose of this section is to collect some definitions and results from mainly [3] in order to provide bivariant extensions and analogues in the next chapter.

Definition 2.32 (Auxiliary relation)**.** Let S be a positively ordered monoid. An auxiliary relation $\prec$ on S is a binary relation that satisfies the following requirements.

i. $a \prec b \;\Rightarrow\; a \leq b$ for any $a, b \in S$;

ii. $a \leq b \prec c \leq d \;\Rightarrow\; a \prec d$ for any $a, b, c, d \in S$;

iii. $0 \prec a$ for any $a \in S$.

The way below relation of Definition 2.29 is an example of an auxiliary relation on a Cuntz semigroup $W(A)$, where A is a (local) C*-algebra. As it is customary, we set

$$a^{\prec} := \{x \in S \mid x \prec a\},$$

where a is any element from a positively ordered monoid S which is equipped with an auxiliary relation $\prec$.

Definition 2.33 (Category W)**.** Let W be the category whose objects are positively ordered Abelian monoids subject to the following extra axioms;

(WO.1) $s^{\prec}$ is $\prec$-upward directed and contains a $\prec$-cofinal increasing sequence for any element s of an object S in W.

(WO.2) $s = \sup s^{\prec}$ for any $s \in S$;

(WO.3) $\prec$ is compatible with $+$;

(WO.4) if $r, s, t \in S$ satisfy $r \prec s + t$ then there are $s', t' \in S$ with $s' \prec s$ and $t' \prec t$ such that $r \prec s' + t'$.

The arrows of the category W are positively ordered Abelian monoid morphisms $\phi : S \to T$ with the following properties.

(WM.1) $\forall s \in S, t \in T \mid b \prec \phi(a) \quad \exists s' \in S \mid s' \prec s \;\wedge\; t \leq f(s')$;

(WM.2) ϕ preserves $\prec$.

In [3], property (WM.1) above is referred to as *continuity* of the morphisms in the category W, and a positively ordered Abelian monoid morphism ϕ that satisfies it but not to property (WM.2) is called a *generalised* W*-morphism*.

As shown by Proposition 2.2.5 of [3], one can introduce an auxiliary relation $\prec$ on the Cuntz semigroup $W(A)$ of a local C*-algebra A by requiring that

$$[a] \prec [b] \qquad \Longleftrightarrow \qquad \exists \epsilon > 0 \mid [a] \leq [(b - \epsilon)_+],$$

for $a, b \in A^+$. Then $(W(A), \prec)$ becomes an object in category W. The structure theorem for c.p.c. order zero maps in [75] in the context of category W shows that every such map induces a generalised W-morphism at the level of the Cuntz semigroup W, whereas $*$-homomorphisms induce morphisms in category Cu. With the extension of the correspondence $A \mapsto W(A)$ to local C*-algebras, with target in category W one recovers continuity of the functor W in the terms expressed by Theorem 2.2.9 of [3].

2.5 The Equivariant Theory

An equivariant theory for the Cuntz semigroup has been established recently, with the work of Gardella and Santiago [24]. Independently from them, we have obtained an equivalent formulation as a special case of the equivariant theory for the bivariant Cuntz semigroup that we present in Section 3.8. There it is shown that the equivariant Cuntz semigroup of a G-algebra B is recovered when the first argument of the equivariant bivariant Cuntz semigroup is chosen as the C*-algebra $\mathbb{C}$ with the trivial action of G. Throughout this section only we will use G again to denote a compact group.

2.5.1 Definitions and Properties

Our starting observation is the concrete realisation of the equivariant Murray-von Neumann semigroup, as described by Equation (1.3). We recall also that the formal way of obtaining the Cuntz semigroup of a (local) C*-algebra is by looking at positive elements from the stabilisation $A \otimes K$ instead of projections, and then consider their Cuntz-equivalence classes. Similarly, we can define the *equivariant* Cuntz semigroup as Cuntz-equivalence classes of G-invariant positive elements from $A \otimes K_G$.

Definition 2.34 (Equivariant Cuntz semigroup)**.** Let (A, G, α) be a G-algebra. Its equivariant Cuntz semigroup is the set of classes

$$\mathsf{Cu}^G(A) := (A \otimes K_G)^G_+ / \sim,$$

where Cuntz comparison is now witnessed by G-invariant sequences, that is, if A is a G-algebra and $a, b \in A^G_+$, then

$$a \precsim_G b \qquad \text{if} \qquad \exists\, \{x_n\}_{n\in\mathbb{N}} \subset A^G \mid \|x_n b x_n^* - a\| \to 0,$$

where A^G denotes the fixed point algebra of A with respect to the action of G. The binary operation is still derived from direct sum of positive elements, that is

$$[a] + [b] := [a \oplus b],$$

for any $[a], [b] \in \mathsf{Cu}^G(A)$.

The approach of [24] is different, closer in spirit to the original definition (see Definition 1.25) of equivariant K-theory (cf [24, Definition 2.4]). Finite dimensional representations of G are replaced by separable ones, that is those representations μ of G over a separable Hilbert space H_μ, and classes of G-invariant positive elements from the C*-algebras $K(H_\mu \otimes A)$ are now considered, with Cuntz comparison implemented by G-invariant elements from $K(H_\mu \otimes A, H_\nu \otimes A)$, where ν is any other separable representation of G (cf. [24, Definition 2.6]).

With arguments similar to those of Section 1.4 one can see that indeed these two different approaches lead to the same equivariant Cuntz semigroup for a continuous action of a compact group G over a C*-algebra A. The map Cu^G turns out to be a sequentially continuous functor from the category of C*-algebras to that of Coward, Elliott and Ivanescu, namely Cu. This means that, for every G-algebra (A, G, α), the equivariant Cuntz semigroup $\mathsf{Cu}^G(A)$ is an object in Cu and, if (B, G, β) is another G-algebra, then every equivariant $*$-homomorphism $\pi : A \to B$ induces a morphism $\mathsf{Cu}^G(\pi) : \mathsf{Cu}^G(A) \to \mathsf{Cu}^G(B)$ in the category Cu.

As with the ordinary Murray-von Neumann and Cuntz semigroups, there are similar connections between the equivariant versions of these objects. Let (A, G, α) be a G-algebra and $p \in (A \otimes K_G)^G$ a projection. The map that sends the class of p in $V^G(A)$ to the class of p in $\mathsf{Cu}^G(A)$ is a well defined semigroup homomorphism, as a consequence of the following results, which generalises [4, Lemma 2.18] to the equivariant setting.

Lemma 2.35. *Let (A, G, α) be a G-algebra and let $p, q \in A^G$ be G-invariant projections. Then $p \preceq_G q$ if and only if $p \precsim_G q$.*

Proof. Thanks to [24, Proposition 2.5], the same proof of [4, Lemma 2.18] applies almost verbatim by taking all the elements to be G-invariant. ☐

As in the ordinary theory, there are special cases where the equivariant Murray-von Neumann semigroup embeds into the equivariant Cuntz semigroup. A stably finite G-algebra (A, G, α) is a G-algebra whose underlying C*-algebra A is stably finite. The following result is an equivariant generalisation of [4, Lemma 2.20].

Lemma 2.36. *Let (A, G, α) be a stably finite G-algebra. Then the natural map $V^G(A) \to \mathsf{Cu}^G(A)$ is injective.*

Proof. The same proof of [4, Lemma 2.20] applies almost verbatim by taking all the elements to be G-invariant. ☐

2.5.2 The Completed Representation Semiring

As shown in Example 1.27, the representation ring $R(G)$ of a group is the Grothendieck enveloping group of the equivariant Murray-von Neumann semigroup $V^G(\mathbb{C})$. When equipped with the multiplication operation that corresponds to taking the class of the tensor product of representations, $V^G(\mathbb{C})$ becomes a semiring. The *completed* representation semiring $\mathsf{Cu}(G)$, or simply the representation semiring, as defined in [24, Definition 3.1], is the semiring arising by considering separable representations G rather than just the finite dimensional ones. We choose to include the word *complete* here because $\mathsf{Cu}(G)$ can be regarded as a sup-completion of the semiring $V^G(\mathbb{C})$, however we sometimes refrain from specifying this explicitly, since the name *representation semiring*, to the best of our knowledge, was not associated to any particular object before the work of Gardella and Santiago appeared. As in the case of K-theory, where $R(G) \cong K_0^G(\mathbb{C})$, it turns out that there is a ordered semigroup isomorphism between $\mathsf{Cu}(G)$ and $\mathsf{Cu}^G(\mathbb{C})$ [24, Theorem 3.4], which is then an object in the category Cu.

Let (A, G, α) be a G-algebra. Definition 3.10 and Theorem 3.11 of [24] show that the equivariant Cuntz semigroup $\mathsf{Cu}^G(A)$ has a natural $\mathsf{Cu}(G)$-semimodule structure and, as such, $\mathsf{Cu}^G(A)$ belongs to a subcategory of Cu, denoted Cu^G (cf. [24, Definition 3.7]). As we are not particularly interested in this category, we refer the reader to the already cited work of Gardella and Santiago for more details. Here we limit ourself to observing that, thanks to [24, Theorem 3.11], by equipping every equivariant Cuntz semigroup with this $\mathsf{Cu}(G)$-semimodule structure, Cu^G becomes a functor from the category of G-algebras to the category Cu^G.

2.5.3 The Module Picture

A module picture for the equivariant Cuntz semigroup is introduced in Section 4 of [24]. Some of the definitions we give here differ slightly from those given in the cited work, but nonetheless lead to the same objects and results.

Definition 2.37 (Equivariant Hilbert C*-module)**.** Let (A, G, α) be a G-algebra. An equivariant Hilbert A-module is a pair (E, ρ) consisting of a Hilbert A-module E and a strongly continuous group homomorphism $\rho : G \to U(E)$ such that

i. $\rho_g(a\xi) = \alpha_g(a)\rho_g(\xi), \quad \forall g, \in G, a \in A, \xi \in E;$

ii. $(\rho_g(\xi), \rho_g(\eta)) = \alpha_g((\xi, \eta)), \quad \forall g \in G, \xi, \eta \in E.$

When the actions are understood, or there is no risk of confusion, we denote an equivariant Hilbert A-module (E, ρ) by its underlying Hilbert A-module E alone. Two equivariant Hilbert A-modules (E, ρ) and (F, σ) are said to be equivariantly isomorphic (in symbols $E \cong_G F$) if there exists a G-invariant unitary u, that is $u \circ \rho_g = \sigma_g \circ u$ for any $g \in G$, in $B(E, F)^G$. An equivariant A-module (Y, η) is said to be an *equivariant A-submodule* of (E, ρ) if Y is an A-submodule of E which is stable under the action ρ on E, i.e. $\rho_g(Y) \subset Y$ for any $g \in G$, and η coincides with the restriction of ρ onto Y, that is $\eta_g = \rho_g|_Y$ for any $g \in G$.

In order to define a Cuntz comparison of equivariant Hilbert C*-modules that resembles the ordinary definition of [16] we need a notion of *equivariant compact containment*. This is done in [24, Definition 4.10], of which we give a slightly different version, that already incorporates the comment that follows it in [24], namely that the contraction a below can always be chosen to be G-invariant by a simple averaging with respect to the normalised Haar measure on G.

Definition 2.38. Let (A, G, α) be a G-algebra, (E, ρ) an equivariant A-module and (F, σ) an equivariant A-submodule of E. We say that (F, σ) is equivariantly compactly contained in (E, ρ) (in symbols $F \subset\subset_G E$) if there exists a G-invariant positive contraction $a \in K(E)_+^G$ such that $a|_F = \mathrm{id}_F$.

With the above definition, together with the notion of isomorphism of equivariant Hilbert C*-modules we can now give a notion of Cuntz comparison for these objects, that resembles that of Definition 2.12.

Definition 2.39. Let (A, G, α) be a G-algebra and let E and F be equivariant Hilbert A-modules. We say that E is equivariantly Cuntz-subequivalent to F (in symbols $E \precsim_G F$) if

$$\forall E' \subset\subset_G E \qquad \exists F' \subset\subset_G F \qquad | \qquad E' \cong_G F'.$$

An equivariant Hilbert A-module E is said to be countably generated if its underlying Hilbert A-module is. Denoting the antisymmetrisation of $\precsim_G$ by $\sim_G$ we can then define the module picture of the Cuntz semigroup of the G-algebra A as the set of $\sim_G$-equivalence classes of countably generated equivariant Hilbert A-modules, equipped with the binary operation arising from the direct sum of modules, viz.

$$\mathsf{Cu}_H^G(A) := \{E \mid E \text{ is a countably generated equivariant Hilbert } A\text{-module}\}/\sim_G,$$

where the subscript H indicates that we are dealing with the module picture. A relation between the functors Cu^G and Cu_H^G can be established when G is second countable. In this case it turns out that there is a natural isomorphism between $\mathsf{Cu}^G(A)$ and $\mathsf{Cu}_H^G(A)$ for any G-algebra A that can be taken not only in the category Cu, but even in the category Cu^G of partially ordered $\mathsf{Cu}(G)$-semimodules briefly mentioned above (cf. [24, Proposition 4.13]).

2.5.4 The Open Projection Picture

In this section we extend the work of [24] by providing an open projection picture for the equivariant Cuntz semigroup, which can be regarded as an equivariant generalisation of the results of [52] on the comparison of open projections and its relation to the Cuntz semigroup, as already discussed in Section 2.3.

Definition 2.40. Let A be a G-algebra. A G-invariant open projection is an open projection in $(A^G)^{**}$.

The above definition entails that every G-invariant open projection is the strong limit of an increasing sequence of positive elements from the fixed point algebra.

Lemma 2.41. *If (E, ρ) is an equivariant Hilbert A-module of the form $\overline{aA}$ for some $a \in A^+$, then there exists $\bar{a} \in A^G$ such that $E \cong_G \overline{\bar{a}A}$.*

Proof. Clearly $a \in E$. Since the map $g \mapsto \rho_g(a)$ is uniformly continuous, for every $\epsilon > 0$ there exists a neighbourhood N of the identity e of the group G such that $\|\rho_g(a) - a\| < \epsilon$, for any $g \in N$. Hence,

$$\begin{aligned}\int_G \rho_g(a)\mathrm{d}\mu(g) &\geq \int_N \rho_g(a)\mathrm{d}\mu(g)\\ &\geq \int_N (a-\epsilon)_+\mathrm{d}\mu(g)\\ &= \mu(N)(a-\epsilon)_+,\end{aligned}$$

with $\mu(N) > 0$ by the regularity of the Haar measure μ on G. By setting

$$\bar{a} := \int_G \rho_g(a)\mathrm{d}\mu(g)$$

one has $p_{\bar{a}} \geq p_{(a-\epsilon)_+}$ for any $\epsilon > 0$, so that $E \cong \overline{\bar{a}A}$, and $\rho_g(\bar{a}) = \bar{a}$ for any $g \in G$. For the inner product one has

$$\begin{aligned}(\rho_g(\bar{a}b), \rho_g(\bar{a}c)) &= \rho_g(\bar{a}b)^* \rho_g(\bar{a}c) \\ &= \alpha_g(b)^* \bar{a}^2 \alpha_g(c) \\ &= \alpha_g(b^* \bar{a}^2 c), \qquad \forall g \in G\end{aligned}$$

and by taking approximate units for b and c one then finds $\bar{a}^2 = \alpha_g(\bar{a}^2)$ for any $g \in G$, whence $\bar{a} \in A^G$. □

Let $a \in A^G$ and, like in the non-equivariant case, denote by E_a the equivariant Hilbert A-module generated by $(\overline{aA}, \rho)$, where the strongly continuous action ρ is given by

$$\rho_g(ab) := a\alpha_g(b)$$

for any $g \in G$. We give the following equivariant version of Blackadar equivalence as defined in Definition 2.27.

Definition 2.42. Let A be a G-algebra. Two positive elements $a, b \in A^G$ are said to be *equivariantly Blackadar equivalent*, in symbols $a \sim_{G,s} b$, if there exists $x \in A^G$ such that $A_a = A_{x^*x}$ and $A_b = A_{xx^*}$.

For open projections $p, q \in (A^G)^{**}$ we give the following equivariant version of Peligrad-Zsidó equivalence.

Definition 2.43. Let A be a G-algebra. Two G-invariant open projections $p, q \in (A^G)^{**}$ are said to be *equivariantly PZ equivalent*, in symbols $p \sim_{G,\mathrm{PZ}} q$, if they are PZ equivalent with respect to A^G, i.e. if there exists a partial isometry $v \in (A^G)^{**}$ such that

$$p = v^*v, \qquad q = vv^*,$$

and

$$v(A^G)_p \subset A, \qquad v^*(A^G)_q \subset A.$$

A direct application of Kaplanski's density theorem shows that one might as well use the notation A_p^G to denote either $(A^G)_p$ or $(A_p)^G$, since both these hereditary subalgebras coincide.

Proposition 2.44. *Let A be a G-algebra and let p be a G-invariant open projection. Then $(A^G)_p = (A_p)^G$.*

Proof. The inclusion $(A^G)_p \subset (A_p)^G$ is obvious. By Kaplanski's density theorem, every element $a \in (A_p)^G$ is a strong limit of a sequence of elements $\{a_n\}_{n\in\mathbb{N}} \subset (A_p)_{\|a\|}$. For any vectors $\xi, \eta \in pH_u$, where H_u denotes the universal Hilbert space of A, one has the estimate

$$|(\xi, \alpha_g(a_n)\eta)| \leq \|\xi\| \, \|\eta\| \, \|a\|, \qquad \forall n \in \mathbb{N}, g \in G.$$

Therefore, by Lebesgue's dominated convergence theorem one can interchange the order of limit and integral in

$$a = \int_G \text{SOT} \lim_{n\to\infty} p\alpha_g(a_n)p \, \mathrm{d}\mu(g)$$

to get

$$a = \text{SOT} \lim_{n\to\infty} p \left(\int_G \alpha_g(a_n) \mathrm{d}\mu(g) \right) p$$

with the average $\int_G \alpha_g(a_n)\mathrm{d}\mu(g)$ belonging to A^G for any $n \in \mathbb{N}$. Hence $a \in (A^G)_p$. ☐

The result that follows can be regarded as an equivariant extension of Proposition 4.3 of [52].

Proposition 2.45. *Let A be a G-algebra and let a and b be G-invariant positive elements of A. The following are equivalent:*

i. $a \sim_{G,s} b$;

ii. E_a *and* E_b *are equivariantly isomorphic;*

iii. $\exists x \in A^G$ *such that* $E_a = E_{x^*x}$ *and* $E_b = E_{xx^*}$;

iv. $p_a \sim_{G,PZ} p_b$.

Proof. *i.* $\Rightarrow$ *iv.* As a direct consequence of [54, Theorem 1.4] one has that $p_{x^*x} \sim_{G,PZ} p_{xx^*}$, since this is true for $p_{x^*x} \sim_{PZ} p_{xx^*}$ in A^G. Furthermore, $A_a = A_b$, with $a, b \in A^G$, implies that $p_a = p_b$, with p_a and p_b in $(A^G)^{**}$.

iv. $\Rightarrow$ *i.* By the arguments of [52, Proposition 4.3], one sees that, if v denotes the partial isometry that witnesses the PZ equivalence of p_a and p_b, then $vav^* \in A^G$ has the same support projection of b, i.e. p_b, in A^G.

ii. $\Rightarrow$ *iii.* Let u be the map that implements the equivariant isomorphism and set $x := ua$. Then $E_{xx^*} = \overline{xA} = \overline{uaA} = E_b$ and

$$\sigma_g(x) = (\sigma_g \circ u \circ \rho_g^{-1} \circ \rho_g)(a) = u\rho_g(a) = ua = x, \qquad \forall g \in G,$$

therefore $x \in A^G$. Furthermore, $x^*x = a^2$ since u is isometric, so that $E_a = E_{x^*x}$.

iii. ⇒ *ii.* Let $x = v|x|$ be the polar decomposition of x, with $v \in (A^G)^{**}$. Then

$$\begin{aligned} v\rho_g(|x|b) &= v|x|\alpha_g(b) \\ &= |x^*|v\alpha_g(b) \\ &= \sigma_g(|x^*|v)\alpha_g(xb) \\ &= \sigma_g(|x^*|vb) \\ &= \sigma_g(v|x|b) \end{aligned}$$

for any $b \in A$, whence $v \in B(E_{x^*x}, E_{xx^*})^G$ is the sought equivariant isomorphism.

i. ⇔ *iii.* This is a restatement of the definitions involved and based on the one-to-one correspondence between hereditary subalgebras and right ideals. □

The following is an equivariant version of the compact containment relation for open projections.

Definition 2.46. Let A be a G-algebra. Define $q \Subset_G p$ by requiring the existence of $e \in A_p^G$ such that $\bar{q}e = \bar{q}$.

The proposition below can be regarded as an equivariant extension of part of the results established in [52, Proposition 4.10].

Proposition 2.47. *Let A be a G-algebra and let a, b be G-invariant positive elements. Then $E_a \Subset_G E_b$ if and only if $p_a \Subset_G p_b$.*

Proof. Identify $K(E_b)$ with A_b and observe that a rank-1 operator $\theta_{ab,ac}$ is sent to the element abc^*a. Hence the action $\rho_g \circ T \circ \rho_g^{-1}$ on $K(E_b)$ coincides with the action of α_g on A_b. Therefore, if $e \in K(E_b)^G$ is such that $e|_{E_a} = \mathrm{id}_{E_a}$, then $e \in A_b^G$ is such that $\overline{p_a}e = \overline{p_a}$. For the converse, observe that all the above implications can be reversed. □

Propositions 2.45 and 2.47 can now be used to *translate* the module picture of the previous section into the open projection picture for the equivariant Cuntz semigroup.

Theorem 2.48. *Let G be a second countable compact group. Then $\mathsf{Cu}^G(A) \cong P_o(((A \otimes K_G)^G)^{**})$.*

Proof. By Proposition 2.45, equivariant isomorphism of modules coincides with equivariant PZ equivalence of the corresponding G-invariant open projection, whereas by 2.47 compact containment of equivariant modules corresponds to compact containment of G-invariant open projections. Hence it is enough to show that there exists a bijection between

$$E_a^{\Subset_G} := \{X \mid X \Subset_G E_a\}$$

and

$$p_a^{\Subset_G} := \{p \mid p \Subset p_a\}$$

for any $a \in (A \otimes K_G)^G$. To this end, suppose that $X \Subset_G E_a$. Since $A \otimes K_G$ is a stable C*-algebra, there exists $a' \in A \otimes K_G$ such that $X = \overline{a' A \otimes K_G}$, and by Lemma 2.41 one can assume that a' is G-invariant. By Proposition 2.47, $E_{a'} \Subset_G E_b$ is equivalent to $p_{a'} \Subset_G p_a$, so that one can associate the G-invariant projection $p_{a'}$ to the equivariant module X. To see that this correspondence is well-defined and independent from the choice of a', observe that, if $a'' \in A \otimes K_G$ is another G-invariant positive element such that $X = \overline{a'' A \otimes K_G}$, then the hereditary subalgebra generated by a'' is the same as that generated by a' and therefore $p_{a''} = p_{a'}$. Conversely, for every $p \Subset p_a$ there exists $a' \in A \otimes K_G$ such that $p = p_{a'}$, and by Proposition 2.47 again this implies that $E_{a'} \Subset E_a$. Any other choice of a positive element that gives the same open projection leads to the same hereditary subalgabra and hence to the same module, whence it follows that the correspondence $p \mapsto E_{a'}$ is well-defined and independent from the choice of a'. It is now immediate to verify that this correspondence is the inverse of the one above and therefore it provides a bijection between $p_a^{\Subset_G}$ and $E_a^{\Subset_G}$. □

2.5.5 Relation with Crossed Products

The original Julg's Theorem 1.28 establishes a connection between the equivariant K-theory of G-algebras with the ordinary K-theory of the corresponding crossed products. As shown by [24, Theorem 5.3], an analogue of this result generalises to the equivariant theory of the Cuntz semigroup. We state such an important result in its entirety for the sake of completeness.

Theorem 2.49 (Gardella-Santiago)**.** *Let (A, G, α) be a G-algebra. There exists a natural isomorphism between $\mathsf{Cu}^G(A)$ and $\mathsf{Cu}(A \rtimes G)$ which lies in the category Cu.*

An isomorphism in the category Cu^G of $\mathsf{Cu}(G)$-semimodules can be obtained in the case of a second countable compact group G, when the Cuntz semigroup of the crossed product is equipped with the only $\mathsf{Cu}(G)$-semimodule structure that makes the existence of such an isomorphism possible. For more details we refer the reader to [24, Theorem 5.14].

2.5.6 Classification of Actions

One of the main application of the equivariant theory of the Cuntz semigroup developed in [24] is to the problem of classification of certain actions (see Definition 2.50 below) by finite Abelian groups over a certain class of C*-algebras, namely those that can be classified by the information contained in the Cuntz semigroup alone (cf. [60]).

Definition 2.50. Let (A, G, α) be a G-algebra. The action α on A is said to be *representable* if there exists a group homomorphism $u : G \to U(\mathcal{M}(A))$ such that $\alpha_g = \mathrm{Ad}(u_g)$ for any $g \in G$. The action α is said to be *locally representable* if there exists an increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of α-invariant C*-subalgebras of A such that $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A and $\alpha|_{A_n}$ is representable for every $n \in \mathbb{N}$.

Actions of this kind for compact groups over AF algebras have been considered in the classical work of Handelman and Rossmann [28], which has led to the conclusion that such actions are classified by the K_0-group of the crossed product. Hence, as Julg's theorem shows, such actions are classified by equivariant K-theory, which is then a complete invariant for this case.

Following [24], let $\mathbf{R}$ denote the class of all the C*-algebras that are isomorphic to inductive limits of either Razak building blocks [59], or Robert's one-dimensional NCCW complexes [60] with trivial K_1-group and that have a countable approximate identity consisting of projections. The classification result we are interested in is that enclosed in part (2) of [24, Theorem 8.4], to which we include the extra assumption of unitality, viz.

Theorem 2.51 (Gardella-Santiago)**.** *Let G be a finite Abelian group, (A, G, α), (B, G, β) unital G-algebras such that $A, B \in \mathbf{R}$ and α, β are locally representable in $\mathbf{R}$. The G-algebras A and B are equivariantly isomorphic if and only if there exists a $\mathsf{Cu}(G)$-semimodule isomorphism $\rho : \mathsf{Cu}^G(A) \to \mathsf{Cu}^G(B)$ such that $\rho([1_A]) = [1_B]$ and $\rho([1_A \otimes e_G]) = [1_B \otimes e_G]$.*

In order to have a genuine equivalence of actions rather than just a cocycle equivalence, the extra condition on the units of the crossed products is required. To this end, observe that, if A is a unital G-algebra, then $1_A \otimes e_G \in A \otimes K_G$ projects onto the constant A-valued functions over G.

Chapter 3

The Bivariant Cuntz Semigroup

In this chapter we introduce the main object of this thesis along with all the properties that have been discovered so far. As already stated in the Introduction, the concrete realisation we propose here is based on a notion of comparison between c.p.c. order zero maps, for reasons that should become clearer as the reader progresses through this chapter.

The material is organised as follows. In Section 3.1 we introduced the already mentioned notion of comparison of c.p.c. order zero maps between local C*-algebras, and use it to define the bivariant Cuntz semigroup $W(A, B)$ of the pair of local C*-algebras A and B. The main properties are investigated and it is shown that one can equip the bivariant Cuntz semigroup with an order structure. Furthermore, the properties of functoriality, additivity and stability in both the arguments are studied, and the exactness in the second argument and the results obtained in these directions strengthen the analogy with KK-theory.

In Section 3.2 we introduced a *stabilised* version of the bifunctor W of the previous section, which we call Cu. To justify this choice of notation we show that, with this new definition, $\mathsf{Cu}(\mathbb{C}, B)$ can be identified with $\mathsf{Cu}(B)$ for any C*-algebra B.

In Section 3.3 we provide a module picture for the bivariant Cuntz semigroup by introducing some further terminology, like order zero pairs and triples, the latter being an analogue of Kasparov triples of KK-theory.

In Section 3.4 we define an analogue of Kasparov product in KK-theory, which takes here the form of a composition between c.p.c. order zero maps. The importance of such map resides in the fact that it can be used to define invertible elements in the bivariant Cuntz semigroup, and hence a notion of Cu-equivalence. This will be revisited in Section 3.7 to provide some classification results.

Section 3.5 takes into consideration further categorical aspects for the bivariant Cuntz semigroup. In particular we show that every bivariant Cuntz semigroup $W(A, B)$ belongs to the category W for any pair of local C*-algebras A and B. We also introduce a notion of compact elements that agrees with the one given for the ordinary Cuntz semigroup when the first argument is chosen to be $\mathbb{C}$. By means of counterexamples we also show that the bivariant Cuntz semigroup is not continuous in both arguments in general. This leads to the notion of Cu-semiprojectivity, as an analogue of KK-semiprojectivity as presented in [18].

In Section 3.6 we give a series of explicit examples of computations of bivariant Cuntz semigroups. We show that $\mathsf{Cu}(A, B)$ reduces to the closed two-sided ideal lattice of A whenever A is unital and exact and B is a unital Kirchberg algebra, thus providing a bivariant extension of the ordinary result that asserts that the Cuntz semigroup of any Kirchberg algebra is trivial. We then use the class of strongly self-absorbing C*-algebras in the sense of [70] to derive a "stability" result (cf. Theorem 3.60) that allows us to compute some bivariant Cuntz semigroups explicitly, by relating them to some ordinary Cuntz semigroups up to isomorphism. We also consider the special case of commutative C*-algebras as the first argument and the field $\mathbb{C}$ for the second and regard the resulting bivariant Cuntz semigroup as defining the *Cuntz-homology* semigroups for compact Hausdorff spaces.

In Section 3.7 we state and prove a classification result of unital and stably finite C*-algebras that involves the bivariant Cuntz semigroup. In order to obtain such a result we need to introduce a notion of scale for the bivariant Cuntz semigroup, and a stricter notion of invertibility that makes use of this. Hence, we obtain that two unital and stably finite C*-algebras A and B are isomorphic if and only if there exists such a *strictly* invertible element in the bivariant Cuntz semigroup $\mathsf{Cu}(A, B)$.

Section 3.8 concludes the chapter with the introduction of an equivariant extension of the bivariant Cuntz semigroup. Its explicit construction is based on an equivariant notion of c.p.c. order zero maps. As a by-product we recover the theory of the equivariant Cuntz semigroup, as recently developed in [24]. We also show how to use this new object for the purposes of classification of actions by recovering the results of [28] and [24] of certain locally representable actions.

3.1 Main Definitions and Properties

In this section we introduce a notion of comparison among c.p.c. order zero maps between local C*-algebras that will be used to define a bivariant Cuntz semigroup that belongs to the category W introduced in 2.4.2 (cf. Section 3.5). In turn this is used to define the bivariant Cuntz semigroup W that maps a pair of local C*-algebras to a monoid. We also give a *stabilised* definition that allows recovering the ordinary Cuntz semigroup Cu of Section 2.2.

Proposition 3.1. *Let A and B be separable local C*-algebras, and let $\phi, \psi : A \to B$ be two c.p.c. order zero maps. The following are equivalent.*

i. $\exists \{b_n\}_{n\in\mathbb{N}} \subset B \quad | \quad \|b_n^* \psi(a) b_n - \phi(a)\| \to 0$ *for any* $a \in A$;

ii. $\forall F \Subset A, \epsilon > 0 \quad \exists b \in B \quad | \quad \|b^* \psi(a) b - \phi(a)\| < \epsilon$ *for any* $a \in F$.

The proof of the above proposition is routine and therefore we omit it. We will sometimes refer to point *ii.* above as the *local form* of Cuntz comparison of c.p.c. order zero maps.

Definition 3.2. Let A and B be local C*-algebras. If $\phi, \psi : A \to B$ are c.p.c. order zero maps that satisfy one of the two equivalent conditions of Proposition 3.1 then ϕ is said to be Cuntz-subequivalent, or simply subequivalent, to ψ. We will denote this relation by the symbol $\phi \precsim \psi$.

It is left to the reader to check that the above relation defines a pre-order among c.p.c. order zero maps between local C*-algebras.

The antisymmetrisation of the Cuntz subequivalence relation $\precsim$ introduced above yields the Cuntz-equivalence relation $\sim$ between c.p.c. order zero maps, that is $\phi \sim \psi$ if $\phi \precsim \psi$ and $\psi \precsim \phi$. We shall sometimes simply say that ϕ is equivalent to ψ when no confusion is likely to arise.

Let A and B be local C*-algebras, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. The C*-algebra C_ψ generated by the image of ψ, that is $C_\psi := C^*(\psi(A))$ is contained in the completion $\tilde{B}$ of B. In some of the results that are derived in this chapter, one shows the existence of a sequence $\{x_n\}_{n\in\mathbb{N}} \subset C_\psi \subset \tilde{B}$ such that

$$\lim_{n\to\infty} \|x_n \psi(a) x_n^* - \phi(a)\| = 0$$

for any A. This is enough to conclude that $\phi \precsim \psi$, as shown by the following result.

Lemma 3.3. *Let A and B be local C^*-algebras, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. If for every finite subset F of A and $\epsilon > 0$ there is $x \in \tilde{B}$, the completion of B, such that*

$$\|x\psi(a)x^* - \phi(a)\| < \epsilon$$

for any $a \in F$, then $\phi \precsim \psi$.

Proof. Fix a finite subset F of A, define the number

$$M := \max_{a \in F} \|\psi(a)\|,$$

and fix $0 < \epsilon < M$. There is $x \in \tilde{B}$ such that

$$\|x\psi(a)x^* - \phi(a)\| < \epsilon$$

for any $a \in F$. By the density of B in $\tilde{B}$ there exists $y \in B$ such that

$$\|x - y\| < \frac{\epsilon}{M(1 + 2\|x\|)}.$$

Furthermore, we have the following estimate

$$\begin{aligned}
\|y\psi(a)y^* - \phi(a)\| &\leq \|y\psi(a)y^* - x\psi(a)x^*\| + \|x\psi(a)x^* - \phi(a)\| \\
&< \|x - y\|^2 \|\psi(a)\| + 2\|x - y\| \|\psi(a)\| \|x\| + \epsilon \\
&< \frac{\epsilon^2}{M} + \epsilon + \epsilon \\
&< 3\epsilon,
\end{aligned}$$

for any $a \in F$, where we have used that $\frac{\epsilon}{M(1+2\|x\|)} \leq \frac{\epsilon}{M} < 1$. Hence $\phi \precsim \psi$ by Proposition 3.1. ☐

We recall that by the symbol $\hat{\oplus}$ we mean the ordinary direct sum $\oplus$ precomposed with the diagonal map Δ, as discussed in Section 1.5.2.

Definition 3.4. Let A and B be two local C*-algebras. The bivariant Cuntz semigroup $W(A, B)$ of A and B is the set of equivalence classes

$$W(A, B) = \{\phi : A \to M_\infty(B) \mid \phi \text{ is c.p.c. order zero}\}/\sim$$

endowed with the binary operation $+ : W(A, B) \times W(A, B) \to W(A, B)$ given by

$$[\phi] + [\psi] = [\phi \,\hat{\oplus}\, \psi],$$

where $\phi \mathbin{\hat{\oplus}} \psi$ denotes the map from A to $M_2(M_\infty(B)) \cong M_\infty(B)$ given by

$$(\phi \mathbin{\hat{\oplus}} \psi)(a) := \begin{bmatrix} \phi(a) & 0 \\ 0 & \psi(a) \end{bmatrix}.$$

One can also introduce an order structure on the set $W(A, B)$, where A and B are any local C*-algebras. Indeed, one can set $[\phi] \leq [\psi]$ whenever the two c.p.c. order zero maps $\phi, \psi : A \to B$ are such that $\phi \precsim \psi$. Thus, we also define the *ordered* bivariant Cuntz semigroup $(W(A, B), +, \leq)$ as the semigroup $W(A, B)$ equipped with such order relation $\leq$.

We say that a local C*-algebra B is σ-unital if it admits a countable approximate unit.

Lemma 3.5. *Let A and B be local C*-algebras, B σ-unital, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. Then $\phi \mathbin{\hat{\oplus}} \psi \precsim \psi \mathbin{\hat{\oplus}} \phi$ in $M_2(B)$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset B$ be a countable approximate unit for B and set

$$x_n := u_n \otimes (e_{12} + e_{21}) \in B \otimes M_2, \qquad \forall n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} \left\| x_n^*(\psi \mathbin{\hat{\oplus}} \phi)(a)x_n - (\phi \mathbin{\hat{\oplus}} \psi)(a) \right\| = 0$$

for any $a \in A$, i.e. $\phi \mathbin{\hat{\oplus}} \psi \precsim \psi \mathbin{\hat{\oplus}} \phi$. □

We shall make the blanket assumption that all the local C*-algebras are σ-unital, unless otherwise stated. As the above lemma shows, it is enough, in general, to assume σ-unitality for the second argument of the bivariant functor W. The following result justifies the use of the word "semigroup" in Definition 3.4.

Proposition 3.6. *For any pair of local C*-algebras A and B, $(W(A, B), +, \leq)$ is a positively ordered Abelian monoid.*

Proof. It is clear that the binary operation $+$ is well-defined and, since $\phi \mathbin{\hat{\oplus}} \psi \sim \psi \mathbin{\hat{\oplus}} \phi$ by Lemma 3.5, it follows that such operation on $W(A, B)$ is Abelian. The class of the zero map is clearly giving the neutral element with respect to $+$. Moreover, $0 \precsim \phi$ by means of the zero constant sequence in $M_\infty(B)$, so $[0] \leq [\phi]$ for any c.p.c. order zero map $\phi : A \to M_\infty(B)$. □

The order $\leq$ defined above extends the algebraic one, for if $x, y \in W(A, B)$ are such that there exists $z \in W(A, B)$ with $x + z = y$, then any transversal $\{\alpha, \beta, \gamma\}$ of $\{x, y, z\}$ is obviously such that $[\alpha]+[\gamma]=[\beta]$ by definition, and this implies that

$$\exists\,\{b_n\}_{n\in\mathbb{N}}\subset M_\infty(B)\quad|\quad\lim_{n\to\infty}\left\|b_n^*\beta(a)b_n-(\alpha\,\hat{\oplus}\,\gamma)(a)\right\|=0\quad\forall a\in A.$$

Taking the sequence $(u_n\otimes e_{11})b_n$, where $\{u_n\}_{n\in\mathbb{N}}$ is an approximate unit for $M_\infty(B)$, one then has

$$\lim_{n\to\infty}\left\|{b_n'}^*\beta(a)b_n'-\alpha(a)\otimes e_{11}\right\|=0$$

for any $a\in A$, whence $x\leq y$.

The following example shows that the above definition contains the ordinary Cuntz semigroup for local C*-algebras as a special instance, and can then be regarded as a bivariant extension of it.

Example 3.7. Let B be a local C*-algebra and let $\phi:\mathbb{C}\to M_\infty(B)$ be a c.p.c. order zero map. By the structure result of Corollary 1.9 there exists a positive element $h\in M_\infty(B)^+$ such that

$$\phi(z)=zh$$

for any $z\in\mathbb{C}$. Therefore, we can identify the set of c.p.c. order zero maps from $\mathbb{C}$ to $M_\infty(B)$ with the positive cone of $M_\infty(B)$. If $\phi,\psi:\mathbb{C}\to M_\infty(B)$ are c.p.c. order zero maps associated to the positive elements $h_\phi,h_\psi\in M_\infty(B)^+$ respectively, then the condition $\phi\precsim\psi$ implies that

$$\exists b_n\subset M_\infty(B)\qquad|\qquad\lim_{n\to\infty}\|b_n\psi(z)b_n^*-\phi(z)\|=0$$

for any $z\in\mathbb{C}$ and, in particular, for $z=1$ one has that

$$\lim_{n\to\infty}\|b_nh_\psi b_n^*-h_\phi\|=0,$$

whence $h_\phi\precsim h_\psi$ in the ordinary Cuntz comparison of positive elements from $M_\infty(B)$. Therefore, one can identify $W(\mathbb{C},B)$ with the Cuntz semigroup $W(B)$ naturally through the structure result of Corollary 1.9. △

We now record some technical results concerning the Cuntz comparison of c.p.c. order zero maps that are used extensively in the proof of the main properties of the bivariant Cuntz semigroup of Definition 3.4.

Proposition 3.8. *Let A and B be local C*-algebras and let $\phi,\psi:A\to B$ be two c.p.c. order zero maps with the same support $*$-homomorphism $\pi:A\to\mathcal{M}(C)$, $C\subset B$, but such that $h_\phi\leq h_\psi$ in $\mathcal{M}(C)$. Then $\phi\precsim\psi$.*

Proof. The result of Handelman 2.4 for C*-algebras can be extended to local C*-algebras since they are closed under functional calculus. Therefore, there exists a sequence of contractions $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(C)$ such that $\zeta_n h_\psi^{\frac{1}{2}} \to h_\phi^{\frac{1}{2}}$. By using an approximate unit $\{e_n\}_{n\in\mathbb{N}}$ of C one can introduce the sequence of contractions $\{z_n\}_{n\in\mathbb{N}} \subset C$ given by

$$z_n := \zeta_n e_n, \qquad \forall n \in \mathbb{N},$$

which has the property that

$$\lim_{n\to\infty} \|z_n \psi(a) z_n^* - \phi(a)\| = 0$$

for any $a \in A$. Hence, $\phi \precsim \psi$. □

The following result, which makes use of the functional calculus on c.p.c. order zero maps between local C*-algebras, as described by Proposition 1.10, is an immediate corollary to the above proposition.

Corollary 3.9. *Let A and B be local C*-algebras, $\phi : A \to B$ a c.p.c. order zero map and $f \in C_0((0,1])^+$ any positive function such that $x - f(x) \geq 0$ for all $x \in (0,1]$. Then $f(\phi) \precsim \phi$.*

Proof. The maps ϕ and $f(\phi)$ share the same support $*$-homomorphism, in the sense that

$$f(\phi) = f(h_\phi)\pi_\phi,$$

where π_ϕ is the support $*$-homomorphism of ϕ. Moreover, $f(h) \leq h$ in $\mathcal{M}(C)$, where $C := C^*(\phi(A))$, and therefore one can make use of Proposition 3.8. □

Other important technical results are based on functional calculus of c.p.c. order zero maps through the continuous function $f_\epsilon \in C_0((0,1])$ given by Equation 2.1. Here we use it to introduce the following shorthand notation

$$\phi_\epsilon := f_\epsilon(\phi),$$

where $\phi : A \to B$ is any c.p.c. order zero map between the local C*-algebras A and B. The following result can be regarded as the *bivariant* analogue of the relation $(a-\epsilon)_+ \precsim a$ between positive elements with respect to the ordinary Cuntz comparison of Definition 2.1.

Corollary 3.10. *Let A and B be local C*-algebras and let $\phi : A \to B$ be a c.p.c. order zero map. Then $\phi_\epsilon \precsim \phi$ for any $\epsilon > 0$.*

Proof. Since ϕ is contractive, it follows from Corollary 1.9 that $\|h\| = \|\phi\| \leq 1$. Furthermore, by the properties of functional calculus, $f_\epsilon(h) \leq h$ for any $\epsilon > 0$ and any contractive positive element h of a C*-algebra, and by Corollary 3.9 this implies that $\phi_\epsilon \precsim \phi$ for any $\epsilon > 0$. □

More generally, we have the following result that applies to the functional calculus of c.p.c. order zero maps.

Proposition 3.11. *Let A and B be local C*-algebras and let $\phi : A \to B$ be c.p.c. order zero. For any pair of positive continuous functions $f, g \in C_0((0,1])^+$ such that* $\operatorname{supp} f \subseteq \operatorname{supp} g$ *we have that $f(\phi) \precsim g(\phi)$.*

Proof. Fix a finite subset F of A and an $\epsilon > 0$. For a given pair of positive continuous functions $f, g \in C_0((0,1])$ such that $\operatorname{supp} f \subseteq \operatorname{supp} g$, find $k \in C_0((0,1])^+$ with the property that $\|gk - f\| < \frac{\epsilon}{M}$, where $M := \max_{a\in F} \|a\|$, e.g. like in the proof of [4, Proposition 2.5]. By the existence of approximate units in C*-algebras, take $e \in C^*(\phi(A))$ such that

$$\|e(gk)(\phi)(a)e^* - (gk)(\phi)(a))\| < \epsilon$$

for any $a \in F$. The element $ek(h_\phi)^{\frac{1}{2}} \in C^*(\phi(A))$, where $h_\phi \in \mathcal{M}(C^*(\phi(A)))$ is the positive element coming from Theorem 1.9 applied to ϕ, leads to the estimate

$$\begin{aligned}
\|e(gk)(\phi)(a)e^* - f(\phi)(a)\| &\leq \|e(gk)(\phi)(a)e^* - (gk)(\phi)(a)\| + \|(gk)(\phi)(a) - f(\phi)(a)\| \\
&< \epsilon + \frac{\epsilon \|a\|}{M} \\
&\leq 2\epsilon,
\end{aligned}$$

for any $a \in F$, where we are using that $e(gk)(\phi)(a)e^* = ek(h_\phi)^{\frac{1}{2}} g(\phi)(a)k(h_\phi)^{\frac{1}{2}} e^*$ by the functional calculus on c.p.c. order zero maps, Proposition 1.10. Hence the result follows by Lemma 3.3. □

If A and B are C*-algebras and $\pi : A \to B$ is a $*$-homomorphism, then for every positive element $a \in A$ one has the identity

$$\pi((a - \epsilon)_+) = (\pi(a) - \epsilon)_+$$

for any $\epsilon > 0$, as one can easily verify by making use of the properties of functional calculus. In this respect, c.p.c. order zero maps behave differently in general, as shown by the following result.

Lemma 3.12. *Let A and B be local C^*-algebras and let $\phi : A \to B$ be a c.p.c. order zero map. Then $\phi((a-\epsilon)_+) \geq (\phi(a)-\epsilon)_+$ for any $\epsilon > 0$ and $a \in A^+$.*

Proof. By Proposition 1.9 one can assume that ϕ has the form $\phi = h\pi$ for some positive element $h \in \mathcal{M}(C^*(\phi(A))) \cap C^*(\phi(A))'$ and a $*$-homomorphism $\pi : \mathcal{M}(C^*(\phi(A))) \cap \{h\}'$. By the details contained in the proof of [75, Theorem 2.3], the positive element h comes from the image of the unit of the minimal unitisation of A through the unique c.p.c. order zero extension $\phi^{(+)} : A^+ \to B^{**}$ of ϕ. Since $\|\phi\| = \|h\| \leq 1$ one has

$$\begin{aligned} \phi^{(+)}(a - \epsilon 1_{A^+}) &= \phi(a) - \epsilon h \\ &\geq \phi(a) - \epsilon 1_{\mathcal{M}(C^*(\phi(A)))}, \end{aligned}$$

with both sides generating a commutative C*-algebra. Therefore, by identifying the unit $1_{\mathcal{M}(C^*(\phi(A)))}$ with that of the minimal unitisation of $C^*(\phi(A))$, one has $(\phi(a)-\epsilon)_+ \leq \phi^{(+)}(a-\epsilon 1_{A^+})_+$. Moreover, since the map ϕ is positive, it follows that $\phi^{(+)}(a-\epsilon 1_{A^+})_+ = \phi^{(+)}((a-\epsilon)_+) = \phi((a-\epsilon)_+)$, whence the result. ☐

As a straightforward consequence of the above lemma and the already cited result of Handelman [27] we have the following corollary.

Corollary 3.13. *Let A and B be local C^*-algebras and let $\phi : A \to B$ be a c.p.c. order zero map. Then $(\phi(a)-\epsilon)_+ \precsim \phi((a-\epsilon)_+)$ for any $\epsilon > 0$ and $a \in A^+$.*

Proof. By the above lemma we have that $(\phi(a)-\epsilon)_+ \leq \phi((a-\epsilon)_+)$, which by the result of Handelman [27] implies that $(\phi(a)-\epsilon)_+ \precsim \phi((a-\epsilon)_+)$, which holds for any choice of $\epsilon > 0$ and a in the positive cone of A. ☐

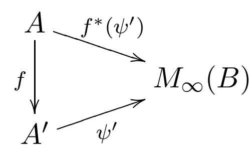
3.1.1 Functoriality

We now show that the map $W(\,\cdot\,,\,\cdot\,)$ of Definition 3.4 yields a functor from the bicategory ${\mathsf{C}^*_{\mathsf{loc}}}^{\mathrm{op}} \times \mathsf{C}^*_{\mathsf{loc}}$ to $\mathsf{OrdAMon}$, where $\mathsf{C}^*_{\mathsf{loc}}$ denotes the category of local C*-algebras, and $\mathsf{OrdAMon}$ that of ordered Abelian monoids. Proposition 2.7, together with the comments that follow it, allows including c.p.c. order zero maps in the set of arrows of $\mathsf{C}^*_{\mathsf{loc}}$.

A deeper analysis of these categorical aspects is postponed until Section 3.5, where we show that the target category can be taken to be W, which was described in Section 2.4.2. For the moment we focus on the functoriality properties of W in the categorical setting discussed thus far.

Theorem 3.14. *Let B be a local C^*-algebra. $W(\,\cdot\,, B)$ is a contravariant functor from the category of local C^*-algebras to that of ordered Abelian monoids.*

Proof. Let A, A' be any local C*-algebras. Consider a $*$-homomorphism $f \in \mathrm{Hom}(A, A')$ and a completely positive map of order zero $\psi' : A' \to M_\infty(B)$. Define $f^*(\psi')$ in such a way that the diagram



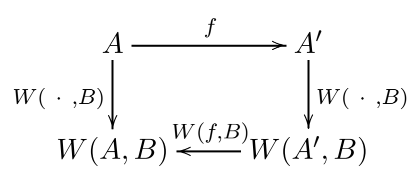
commutes, i.e. set

$$f^*(\psi') := \psi' \circ f.$$

Then $f^*(\psi)$ is a completely positive map of order zero between A and $M_\infty(B)$, and therefore f^* defines a pull-back between c.p.c. order zero maps which can be projected onto equivalence classes from the corresponding bivariant Cuntz semigroups by setting

$$W(f, B)([\psi']) = [f^*(\psi')], \qquad \forall [\psi'] \in W(A', B).$$

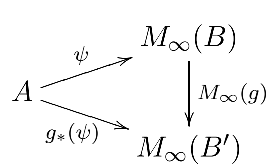
It is easy to check that this yields a well-defined map. This implies that for every $*$-homomorphism f there exists a semigroup homomorphism $W(f, B)$ such that the following diagram



commutes. To see that such map preserves the order consider another c.p.c. order zero map $\phi' : A' \to M_\infty(B)$ with $\phi' \precsim \psi'$. Then there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \subset M_\infty(B)$ such that $\|b_n^* \psi'(a) b_n - \phi'(a)\| \to 0$ for any $a \in A'$. In particular this is true if a is restricted to $f(A) \subset A'$, whence $f^*\phi \precsim f^*\psi$. □

Theorem 3.15. *Let A be a local C^*-algebra. $W(A,\,\cdot\,)$ is a covariant functor from the category of local C^*-algebras to that of ordered Abelian monoids.*

Proof. Let B and B' be any local C*-algebras, and a $*$-homomorphism $g \in \mathrm{Hom}(B, B')$ and a completely positive map of order zero ψ between A and $M_\infty(B)$. Define $g_*(\psi)$ such that the diagram



commutes, i.e. set

$$g_*(\psi) := g^{(\infty)} \circ \psi.$$

Such map is clearly completely positive with the order zero property and well-defined, and therefore it defines a push-forward between c.p.c. order zero maps that gives rise to the semigroup homomorphism

$$W(A,g)([\psi]) = [g_*(\psi)], \qquad \forall [\psi] \in W(A,B).$$

Hence the diagram

$$\begin{array}{ccc} B & \xrightarrow{\;g\;} & B' \\ {\scriptstyle W(A,\,\cdot\,)}\downarrow & & \downarrow{\scriptstyle W(A,\,\cdot\,)} \\ W(A,B) & \xrightarrow{W(A,g)} & W(A,B') \end{array}$$

commutes. To see that such map preserves the order consider another c.p.c. order zero map $\phi : A \to M_\infty(B)$ such that $\phi \precsim \psi$. Then there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \subset M_\infty(B)$ such that $\|b_n^*\psi(a)b_n - \phi(a)\| \to 0$ for any $a \in A'$. Since g is necessarily contractive, the sequence $\{g^{(\infty)}(b_n)\}_{n\in\mathbb{N}} \subset M_\infty(B')$ is easily seen to witness the relation $g_*(\phi) \precsim g_*(\psi)$. □

3.1.2 Additivity

We now analyse the properties of additivity of the bifunctor W in both its arguments. It turns out that, like KK-theory, W is finitely additive in both its arguments.

Let A_1, A_2 and B be local C*-algebras. We shall say that two c.p.c. order zero maps $\phi : A_1 \to B$ and $\psi : A_2 \to B$ are orthogonal, and we shall indicate this by $\phi \perp \psi$, if $\phi(A_1)\psi(A_2) = \{0\}$. This implies, in particular, that the positive elements $h_\phi, h_\psi \in B^{**}$ coming from Theorem 1.5 applied to ϕ and ψ respectively are orthogonal, i.e. $h_\phi h_\psi = 0$ in B^{**}. Furthermore, we have the following result.

Proposition 3.16. *Let A_1, A_2 and B be C*-algebras and let $\phi : A_1 \to B$ and $\psi : A_2 \to B$ be c.p.c. order zero maps such that $\phi \perp \psi$. Then $\phi(A_1) \cap \psi(A_2) = \{0\}$.*

Proof. Assume that there are $a_1 \in A_1, a_2 \in A_2$ such that $b = \phi(a_1) = \psi(a_2)$. Then

$$\begin{aligned} \|b\|^2 &= \|b^*b\| \\ &= \|\phi(a_1)^*\psi(a_2)\| \\ &= \|\phi(a_1^*)\psi(a_2)\| \\ &= 0 \end{aligned}$$

by orthogonality of ϕ and ψ. Hence $b = 0$. □

Let A_1 and A_2 be local C*-algebras. We observe that, given two c.p.c. order zero maps $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$, their direct sum $\phi_1 \oplus \phi_2$ is easily seen to be a c.p.c. order zero map. For the converse of this property we provide the following results.

Lemma 3.17. *Let A_1, A_2, B be local C*-algebras. A map $\phi : A_1 \oplus A_2 \to B$ is c.p.c. order zero if and only if there are c.p.c. order zero maps $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ such that*

i. $\phi_1(a_1) + \phi_2(a_2) = \phi(a_1 \oplus a_2)$*, for any* $a_1 \in A_1$ *and* $a_2 \in A_2$*;*

ii. $\phi_1 \perp \phi_2$.

Proof. Let $\phi : A_1 \oplus A_2 \to B$ be a c.p.c. order zero map. Define the maps $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ as

$$\phi_1(a_1) := \phi(a_1 \oplus 0) \qquad \text{and} \qquad \phi_2(a_2) := \phi(0 \oplus a_2)$$

respectively. Then clearly one has that $\phi(a_1 \oplus a_2) = \phi_1(a_1) + \phi_2(a_2)$. Furthermore, the order zero property implies that

$$\phi_1(a_1)\phi_2(a_2) = \phi(a_1 \oplus 0)\phi(0 \oplus a_2) = 0,$$

whence $\phi_1 \perp \phi_2$.

Conversely, assume that there are c.p.c. order zero maps ϕ_1 and ϕ_2 with the desired properties. Their sum $\phi_1 + \phi_2$ is clearly a c.p. map. Contractivity follows from the orthogonality $\phi_1 \perp \phi$. Let h_1 and h_2 be the positive elements coming from Theorem 1.5 applied to ϕ_1 and ϕ_2 respectively. Then

$$\|\phi\| = \|h_1 + h_2\| \leq 1,$$

since $h_1 \perp h_2$. The order zero property follows immediately from the fact that both ϕ_1 and ϕ_2 are assumed to be orthogonality preserving. □

Lemma 3.18. *Let A_1, A_2 and B be local C*-algebras, and let $\phi : A_1 \to B$ and $\psi : A_2 \to B$ be c.p.c. order zero maps such that $\phi \perp \psi$. Then*

$$\begin{bmatrix} \phi + \psi & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \phi & 0 \\ 0 & \psi \end{bmatrix}$$

in $M_2(B)$, where $\phi + \psi$ is the c.p.c. order zero map from $A_1 \oplus A_2$ to B given by $(\phi + \psi)(a_1 \oplus a_2) := \phi(a_1) + \psi(a_2)$.

Proof. By the previous lemma, the matrix on the left is a well-defined c.p.c. order zero map from $A_1 \oplus A_2$ to $M_2(B)$. Let $h_\phi \in \mathcal{M}(C_\phi)$, $h_\psi \in \mathcal{M}(C_\psi)$, π_ϕ and π_ψ be the positive elements and the support $*$-homomorphisms coming from Corollary 1.9 applied to ϕ and ψ respectively, $\{e_n\}_{n\in\mathbb{N}} \subset C_\phi$ and $\{f_n\}_{n\in\mathbb{N}} \subset C_\psi$ approximate units, and set

$$x_n := \begin{bmatrix} e_n h_\phi^{\frac{1}{4}} & 0 \\ f_n h_\psi^{\frac{1}{4}} & 0 \end{bmatrix}, \quad y_n := \begin{bmatrix} e_n h_\phi^{\frac{1}{4}} & f_n h_\psi^{\frac{1}{4}} \\ 0 & 0 \end{bmatrix},$$

for any $n \in \mathbb{N}$, which define sequences in $M_2(\tilde{B})$, where $\tilde{B}$ is the completion of B. One easily sees that

$$\lim_{n\to\infty} \left\| x_n \begin{bmatrix} \phi+\psi & 0 \\ 0 & 0 \end{bmatrix} (a) x_n^* - \begin{bmatrix} \phi^2 & 0 \\ 0 & \psi^2 \end{bmatrix} (a) \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| y_n \begin{bmatrix} \phi^{\frac{1}{2}} & 0 \\ 0 & \psi^{\frac{1}{2}} \end{bmatrix} (a) y_n^* - \begin{bmatrix} \phi+\psi & 0 \\ 0 & 0 \end{bmatrix} (a) \right\| = 0$$

for any $a \in A_1 \oplus A_2$, whence

$$\begin{bmatrix} \phi^2 & 0 \\ 0 & \psi^2 \end{bmatrix} \precsim \begin{bmatrix} \phi+\psi & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \phi+\psi & 0 \\ 0 & 0 \end{bmatrix} \precsim \begin{bmatrix} \phi^{\frac{1}{2}} & 0 \\ 0 & \psi^{\frac{1}{2}} \end{bmatrix}.$$

by Lemma 3.3. By Proposition 3.11 we have

$$\begin{bmatrix} \phi^2 & 0 \\ 0 & \psi^2 \end{bmatrix} \sim \begin{bmatrix} \phi^{\frac{1}{2}} & 0 \\ 0 & \psi^{\frac{1}{2}} \end{bmatrix} \sim \begin{bmatrix} \phi & 0 \\ 0 & \psi \end{bmatrix},$$

which concludes the proof. □

Proposition 3.19. *For any triple of local C^*-algebras A_1, A_2 and B the semigroup isomorphism*

$$W(A_1 \oplus A_2, B) \cong W(A_1, B) \oplus W(A_2, B)$$

holds.

Proof. Let $\sigma : W(A_1, B) \oplus W(A_2, B) \to W(A_1 \oplus A_2, B)$ be the map given by

$$\sigma([\phi_1] \oplus [\phi_2]) = [\phi_1 \oplus \phi_2].$$

By the above two lemmas it is clear that this map is surjective. To prove injectivity we show that $\phi_1 \oplus \phi_2 \precsim \psi_1 \oplus \psi_2$ implies $\phi_k \precsim \psi_k$, $k = 1, 2$. By hypothesis there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \subset M_\infty(B)$ such that

$$b_n^*(\psi_1(a_1) \oplus \psi_2(a_2)) b_n \to \phi_1(a_1) \oplus \phi_2(a_2)$$

in norm for every $a_1 \in A_1, a_2 \in A_2$. As $M_2(M_\infty(B)) \cong M_\infty(B)$, the sequence b_n has the structure

$$b_n = \sum_{i,j=1}^{2} b_{n,ij} \otimes e_{ij},$$

where $b_{n,ij} \in M_\infty(B)$ for any $i,j = 1,2$ and $\{e_{ij}\}_{i,j=1,2}$ form the standard basis of matrix units for M_2. Thus, for $a_2 = 0$, one finds that

$$\lim_{n\to\infty} \left\| b_{n,11}^* \psi_1(a_1) b_{n,11} - \phi_1(a_1) \right\| = 0$$

for any $a_1 \in A_1$, i.e. $\phi_1 \precsim \psi_1$. A similar argument with $a_1 = 0$ leads to the conclusion that $\phi_2 \precsim \psi_2$ as well.

To check that σ preserves the semigroup operations it suffices to show that

$$(\phi_1 \hat{\oplus} \psi_1) \oplus (\phi_2 \hat{\oplus} \psi_2) \sim (\phi_1 \oplus \phi_2) \hat{\oplus} (\psi_1 \oplus \psi_2).$$

A direct computation reveals that such equivalence is witnessed by the sequence $\{b_n\}_{n\in\mathbb{N}} \subset M_4(M_\infty(B))$ given by

$$b_n := u_n \otimes (e_{11} + e_{44} + e_{23} + e_{32}),$$

where $\{u_n\}_{n\in\mathbb{N}} \subset M_\infty(B)$ is an approximate unit for $M_\infty(B)$. □

As mentioned in Section 1.5.3, the KK bifunctor is actually countable additive in the first argument. The same holds true for the bivariant Cuntz semigroup provided that one uses a stable C*-algebra as second argument, for the countable direct sum of maps ending up in $M_\infty(B)$ may lie in $B \otimes K$ instead.

Lemma 3.20. *Let A and B be local C*-algebras, $\phi : A \to B$ be a countable sum of pair-wise orthogonal c.p.c. order zero maps, that is*

$$\phi(a) = \sum_{k=1}^{\infty} \phi_k(a), \qquad \forall a \in A,$$

where each ϕ_k is a c.p.c. order zero map and $\phi_k \perp \phi_i$ for any $i \neq k$ in $\mathbb{N}$, and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ a summable sequence. Then

$$\phi \sim \sum_{k=1}^{\infty} a_k \phi_k.$$

Proof. Fix a finite subset F of A and $\epsilon > 0$. Find an $n \in \mathbb{N}$ such that

$$\left\| \sum_{n+1}^{\infty} \phi_k(a) \right\| < \epsilon$$

for any $a \in F$. Define the C*-subalgebras $B_k := C^*(\phi_k(A)) \subset \tilde{B}$ of the completion of B for any $k \in \mathbb{N}$, and by the existence of approximate units find elements $e_k \in B_k$, $k = 1, \ldots, n$ such that

$$\|e_k\phi_k(a)e_k^* - \phi_k(a)\| < \frac{\epsilon}{n}$$

for any $a \in F$ and $k = 1, \ldots, n$. Observe that the orthogonality of the maps ϕ_k implies that $e_i \perp e_k$ for any $i \neq k$. With the element $x \in \tilde{B}$ defined as

$$x := \sum_{k=1}^{n} \frac{e_k}{\sqrt{a_k}}$$

one has the estimate

$$\begin{aligned}
\left\| x\left(\sum_{k=1}^{\infty} a_k\phi_k(a)\right)x^* - \phi(a)\right\| &\leq \sum_{k=1}^{n} \|e_k\phi_k(a)e_k^* - \phi_k(a)\| + \left\|\sum_{n+1}^{\infty}\phi_k(a)\right\| \\
&< \sum_{k=1}^{n}\frac{\epsilon}{n} + \epsilon \\
&\leq 2\epsilon
\end{aligned}$$

for any $a \in F$. Hence $\phi \precsim \sum_{k=1}^{\infty} a_k\phi_k(a)$ by Lemma 3.3. For the converse subequivalence, let S be the sum of the sequence $\{a_n\}_{n\in\mathbb{N}}$ and assume, without loss of generality, that $S = 1$. Find, if necessary, a new n for which

$$\left\|\sum_{k=n+1}^{\infty} a_k\phi_k(a)\right\| < \epsilon$$

for any $a \in F$, and new elements $e_k \in B_k$, $k = 1, \ldots, n$ such that

$$\|e_k\phi_k(a)e_k^* - \phi_k(a)\| < \frac{\epsilon}{n}.$$

With the element $y \in \tilde{B}$ defined as

$$y := \sum_{k=1}^{n} \sqrt{a_k}e_k$$

one has the estimate

$$\begin{aligned}
\left\| y\phi(a)y^* - \sum_{k=1}^{\infty} a_k\phi_k(a)\right\| &\leq \sum_{k=1}^{n} \|e_k\phi_k(a)e_k^* - \phi_k(a)\| + \left\|\sum_{n+1}^{\infty}a_k\phi_k(a)\right\| \\
&< \sum_{k=1}^{n}\frac{\epsilon}{n} + \epsilon \\
&\leq 2\epsilon,
\end{aligned}$$

for any $a \in F$, which by Lemma 3.3 again implies that $\sum_{k=1}^{\infty} a_k\phi_k(a) \precsim \phi$. □

Lemma 3.21. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of local C^*-algebras and let B be a local C^*-algebras. Then any c.p.c. order zero map $\phi : \bigoplus_{k=1}^{\infty} A_k \to B$ is such that*

$$\phi \otimes e \sim \bigoplus_{k=1}^{\infty} \frac{\phi|_{A_k}}{2^k}$$

in $B \odot K$, where $e \in K$ is a minimal projection and $\phi|_{A_k}$ is defined as

$$\phi|_{A_k}(a_k) := \phi(a_k \otimes e_{kk}), \qquad \forall k \in \mathbb{N}, a_k \in A_k.$$

Proof. Assume, without loss of generality, that $e = e_{11}$, and denote by A and ψ the direct sum of the A_ks and of the $\frac{\phi|_{A_k}}{2^k}$s respectively. Let $B_k := C^*(\phi|_{A_k}(A_k))$ and let $\{e^k{}_n\}_{n\in\mathbb{N}}$ be an approximate unit for B_k, for any $k \in \mathbb{N}$. For any $\phi|_{A_k}$, let h_k be the positive element coming from Corollary 1.9 applied to it. With the sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \tilde{B} \otimes K$ given by

$$x_n := \begin{bmatrix} e_n^1 \frac{h_1^{\frac14}}{2} & 0 & \cdots \\ e_n^2 \frac{h_2^{\frac14}}{4} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad y_n := \begin{bmatrix} e_n^1 \left(\frac{h_1}{2}\right)^{\frac14} & e_n^2 \left(\frac{h_2}{4}\right)^{\frac14} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

one has

$$\lim_{n\to\infty} \left\| x_n(\phi \otimes e)(a)x_n^* - \psi^2(a) \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| y_n \psi^{\frac12}(a) y_n^* - \left(\sum_{k=1}^{\infty} \frac{\phi|_{A_k}}{2^k} \otimes e \right)(a) \right\| = 0$$

for any $a \in A$. Since $\psi \sim \psi^2$ by Proposition 3.11 and $\phi \sim \sum_{k=1}^{\infty} \frac{\phi|_{A_k}}{2^k}$ by the previous lemma, the result now follows from Lemma 3.3. □

Proposition 3.22. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of local C^*-algebras and let B be a stable C^*-algebra. Then the semigroups $\prod_{n\in\mathbb{N}} W(A_n, B)$ and $W(\bigoplus_{n\in\mathbb{N}} A_n, B)$ are isomorphic.*

Proof. Let $\sigma : \prod_{n\in\mathbb{N}} W(A_n, B) \to W(\bigoplus_{n\in\mathbb{N}} A_n, B)$ be the semigroup homomorphism defined by

$$\sigma([\phi_1], [\phi_2], \ldots) := \left[\gamma \circ \bigoplus_{n\in\mathbb{N}} \frac{1}{2^n} \phi_n \right],$$

where $\gamma : M_\infty \odot K \to M_\infty$ is any isomorphism. For any fixed projection $e \in K$ such isomorphism satisfies $\gamma \circ (\mathrm{id}_{M_\infty} \otimes e) \sim \mathrm{id}_{M_\infty}$. An inverse is provided by the semigroup homomorphism $\rho : W(\bigoplus_{n\in\mathbb{N}} A_n, B) \to \prod_{n\in\mathbb{N}} W(A_n, B)$ given by

$$\rho([\phi]) := ([\phi|_{A_1}], [\phi|_{A_2}], \ldots),$$

where $\phi|_{A_k}(a_k) := \phi(a_k \otimes e_{kk})$ for any $k \in \mathbb{N}$ and $a_k \in A_k$. Indeed, by the previous lemma

$$\gamma \circ \bigoplus_{n\in\mathbb{N}} \frac{\phi|_{A_n}}{2^n} \sim \gamma \circ (\phi \otimes e) \sim \phi$$

and

$$\gamma \circ \left(\frac{\phi_k}{2^k} \otimes e_{kk}\right) \sim \gamma \circ (\phi_k \otimes e) \sim \phi_k, \qquad \forall k \in \mathbb{N}$$

since every minimal projection e_{kk} is Cuntz-equivalent to e, and $a\phi_k \sim \phi_k$ for any $a \in (0,1)$. □

Proposition 3.23. *For any triple of local C*-algebras A, B_1 and B_2 the semigroup isomorphism*

$$W(A, B_1 \oplus B_2) \cong W(A, B_1) \oplus W(A, B_2)$$

holds.

Proof. Since $M_\infty(B_1 \oplus B_2)$ is isomorphic to $M_\infty(B_1) \oplus M_\infty(B_2)$ one has that for every c.p.c. order zero map $\phi : A \to M_\infty(B_1 \oplus B_2)$ there are c.p.c. order zero maps $\phi_k : A \to M_\infty(B_k)$, $k = 1, 2$ such that ϕ can be identified, up to isomorphism, with $\phi_1 \hat{\oplus} \phi_2$[1]. This shows that the map $\rho : W(A, B_1) \oplus W(A, B_2) \to W(A, B_1 \oplus B_2)$ given by

$$\rho([\phi_1] \oplus [\phi_2]) := [\phi_1 \hat{\oplus} \phi_2]$$

is surjective. Injectivity comes from the fact that $\phi_1 \hat{\oplus} \phi_2 \precsim \psi_1 \hat{\oplus} \psi_2$ implies $\phi_1 \precsim \psi_1$ and $\phi_2 \precsim \psi_2$, which is obvious. □

3.1.3 Stability

We elaborate now on the property of stability of the bifunctor W in both its arguments. Recall that, for the purposes of this thesis, a functor F on the category of local C*-algebras is said to be stable if $F(A) \cong F(A \otimes K)$ naturally for any local C*-algebra A, K being the C*-algebra of compact operators on an separable infinite-dimensional Hilbert space. F is said to be matrix-stable if $F(M_n(A)) \cong F(A)$ naturally for any $n \in \mathbb{N}$. Observe that stability implies matrix stability, but the converse is not true in general.

Matrix stability on the second argument of the bifunctor W can be inferred immediately from the definition of the bivariant Cuntz semigroup, Definition 3.4, since $M_n(M_\infty(B)) \cong$

[1]such maps are given by $\phi_k := \pi_k^{(\infty)} \circ \phi$, where $\pi_k^{(\infty)}$ is the ∞-ampliation of the natural projection $\pi_k : B_1 \oplus B_2 \to B_k$, that is, $\pi_k^{(\infty)} := \pi_k \otimes \mathrm{id}_{M_\infty}$.

$M_\infty(B)$ for any local C*-algebra B and $n \in \mathbb{N}$, and therefore

$$W(A, M_n(B)) \cong W(A, B), \qquad \forall n \in \mathbb{N},$$

naturally, for any pair of local C*-algebras A and B. On the other hand, matrix stability on the first argument is not as immediate, but it turns out to hold. Firstly we recall that, for every $*$-isomorphism $\gamma : M_\infty \odot M_\infty \to M_\infty \subset B(\ell^2(\mathbb{N}))$, there exists a partial isometry $v \in B(\ell^2(\mathbb{N}))$ such that

$$\mathrm{Ad}_v \circ \gamma \circ (\mathrm{id}_{M_\infty} \otimes e) = \mathrm{id}_{M_\infty},$$

for some minimal projection $e \in M_\infty$, and that the flip

$$\mathrm{id}_{M_\infty} \otimes e \mapsto e \otimes \mathrm{id}_{M_\infty}$$

from $M_\infty \odot M_\infty$ to itself is unitarily implemented, in the sense that the unitary lies in $B(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$. The same holds true when M_∞ is replaced by its norm completion K. We then record the following result.

Proposition 3.24. *Let A, B and C be separable local C*-algebras and let $\phi, \psi : A \to B$, $\eta, \theta : B \to C$ be c.p.c. order zero maps such that $\phi \precsim \psi$ and $\eta \precsim \theta$. Then $\eta \circ \phi \precsim \eta \circ \psi$ and $\eta \circ \phi \precsim \theta \circ \phi$.*

Proof. The implication $\eta \precsim \theta \Rightarrow \eta \circ \phi \precsim \theta \circ \phi$ is trivial. For the other implication, let $\{e_n\}_{n\in\mathbb{N}} \subset C^*(\eta(B))$ be an approximate unit, and let π_η be the support $*$-homomorphism of η. If $\{b_n\}_{n\in\mathbb{N}} \subset B$ is any sequence that witnesses $\phi \precsim \psi$, then the sequence $\{d_n\}_{n\in\mathbb{N}} \subset C^*(\eta(B))$ given by $d_n := e_n \pi_\eta(b_n)$ leads to the estimate

$$\begin{aligned}
\|d_n(\eta \circ \psi)(a)d_n^* - (\eta \circ \phi)(a)\| &= \|e_n \eta(b_n \psi(a) b_n^*) e_n^* - (\eta \circ \phi)(a)\| \\
&\leq \|e_n \eta(b_n \psi(a) b_n^*) e_n^* - e_n(\eta \circ \phi)(a) e_n^*\| \\
&\qquad + \|e_n(\eta \circ \phi)(a) e_n^* - (\eta \circ \phi)(a)\| \\
&\leq \|b_n \psi(a) b_n^* - \phi(a)\| + \|e_n(\eta(\phi(a))) e_n^* - \eta(\phi(a))\|,
\end{aligned}$$

which tends to 0 as $n \to \infty$, for every $a \in A$. Therefore $\eta \circ \phi \precsim \eta \circ \psi$ by Lemma 3.3. ☐

Lemma 3.25. *Let A, B, C and D be local C*-algebras, with the completions of C and D giving nuclear C*-algebras, and let $\phi, \psi : A \to B$, $\eta : C \to D$ be c.p.c. order zero maps such that $\phi \precsim \psi$ in B. Then $\phi \otimes \eta \precsim \psi \otimes \eta$ in $B \odot D$.*

Proof. If $\{b_n\}_{n\in\mathbb{N}} \subset B$ is the sequence that witnesses the Cuntz subequivalence between ϕ and ψ then $\{b_n \otimes e_n\}_{n\in\mathbb{N}}$, where $\{e_n\}_{n\in\mathbb{N}} \subset D$ is an approximate unit, witnesses the sought Cuntz subequivalence between $\phi \otimes \eta$ and $\psi \otimes \eta$. □

Corollary 3.26. *Let A and B be local C^*-algebras and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. Then $\phi \precsim \psi$ in B if and only if $\phi \otimes \mathrm{id}_{M_\infty} \precsim \psi \otimes \mathrm{id}_{M_\infty}$ in $B \odot M_\infty$. The same holds true with K in place of M_∞.*

Proof. The implication $\phi \precsim \psi \Rightarrow \phi \otimes \mathrm{id}_{M_\infty} \precsim \psi \otimes \mathrm{id}_{M_\infty}$ follows from the previous lemma. For the other implication observe that B embeds into $B \odot M_\infty$ by means of the injective map $b \overset{\iota}{\mapsto} b \otimes e$, where $e \in M_\infty$ is any minimal projection. If $\{b_n\}_{n\in\mathbb{N}} \subset B \odot M_\infty$ is the sequence that witnesses the relation $\phi \otimes \mathrm{id}_{M_\infty} \precsim \psi \otimes \mathrm{id}_{M_\infty}$ then, with $x_n := (1_{B^+} \otimes e) b_n (1_{B^+} \otimes e) \in B \otimes \{e\}$, where 1_{B^+} is either the unit of B or that of its minimal unitisation B^+, we have

$$\|x_n^*(\psi(a) \otimes e)x_n - \phi(a) \otimes e\| \to 0, \qquad \forall a \in \ A$$

which can be pulled back to B through ι to give

$$\left\|\iota^{-1}(x_n)^*\psi(a)\iota^{-1}(x_n) - \phi(a)\right\| \to 0, \qquad \forall a \in A,$$

whence $\phi \precsim \psi$. The same argument works with K in place of M_∞. □

Proposition 3.27. *Let A and B be local C^*-algebras. Then $W(M_\infty(A), B) \cong W(A, B)$ and $W(A \otimes K, B \otimes K) \cong W(A, B \otimes K)$.*

Proof. Since any isomorphism $\gamma : M_\infty \odot M_\infty \to M_\infty$ induces a semigroup isomorphism $W(A, B) \cong W(A, M_\infty(B))$ it is enough to show that $W(A, B) \cong W(M_\infty(A), M_\infty(B))$. Mutual inverses are then given by the maps

$$[\phi] \mapsto [\phi \otimes \mathrm{id}_{M_\infty}], \qquad [\phi] \in W(A, B)$$

and

$$[\Phi] \mapsto [(\mathrm{id}_B \otimes \gamma) \circ \Phi \circ (\mathrm{id}_A \otimes e)],$$

where e is a minimal projection in M_∞. Indeed, by making use of Proposition 3.24 and

Lemma 3.25 above one has

$$\begin{aligned}(\mathrm{id}_B \otimes \gamma) \circ (\phi \otimes \mathrm{id}_{M_\infty}) \circ (\mathrm{id}_A \otimes e) &= (\mathrm{id}_B \otimes \gamma) \circ (\phi \otimes e) \\ &= (\mathrm{id}_B \otimes \gamma) \circ (\mathrm{id}_B \otimes \mathrm{id}_{M_\infty} \otimes e) \circ \phi \\ &\sim (\mathrm{id}_B \otimes \mathrm{id}_{M_\infty}) \circ \phi \\ &= \phi\end{aligned}$$

and

$$\begin{aligned}((\mathrm{id}_B \otimes \gamma) \circ \Phi \circ (\mathrm{id}_A \otimes e)) \otimes \mathrm{id}_{M_\infty} = \\ &= (\mathrm{id}_B \otimes \gamma \otimes \mathrm{id}_{M_\infty}) \circ (\Phi \otimes \mathrm{id}_{M_\infty}) \circ (\mathrm{id}_A \otimes e \otimes \mathrm{id}_{M_\infty}) \\ &\sim (\mathrm{id}_B \otimes \gamma \otimes \mathrm{id}_{M_\infty}) \circ (\Phi \otimes \mathrm{id}_{M_\infty}) \circ (\mathrm{id}_A \otimes \mathrm{id}_{M_\infty} \otimes e) \\ &= (\mathrm{id}_B \otimes \gamma \otimes \mathrm{id}_{M_\infty}) \circ (\Phi \otimes e) \\ &= (\mathrm{id}_B \otimes \gamma \otimes \mathrm{id}_{M_\infty}) \circ (\mathrm{id}_B \otimes \mathrm{id}_{M_\infty} \otimes \mathrm{id}_{M_\infty} \otimes e) \circ \Phi \\ &\sim (\mathrm{id}_B \otimes \gamma \otimes \mathrm{id}_{M_\infty}) \circ (\mathrm{id}_B \otimes \mathrm{id}_{M_\infty} \otimes e \otimes \mathrm{id}_{M_\infty}) \circ \Phi \\ &\sim (\mathrm{id}_B \otimes \mathrm{id}_{M_\infty} \otimes \mathrm{id}_{M_\infty}) \circ \Phi \\ &= \Phi,\end{aligned}$$

which become equalities at the level of the Cuntz classes. The result involving K follows from analogous steps. □

As a corollary of this result one has that every c.p.c. order zero map $\Phi : A \otimes K \to B \otimes K$ is Cuntz-equivalent to a K-ampliation of a c.p.c. order zero map $\phi : A \to B \otimes K$, that is, $\Phi \sim \phi \otimes \mathrm{id}_K$. Hence, every c.p.c. order zero map $\Phi : M_\infty(A)$ to $M_\infty(B)$ is Cuntz-equivalent to an ∞-ampliation of a c.p.c. order zero map $\phi : A \to B \otimes M_\infty$. That is to say that, for every c.p.c. order zero map $\Phi : M_\infty(A) \to M_\infty(B)$, where A and B are local C*-algebras, there exists a c.p.c. order zero map $\phi : A \to M_\infty(B)$ such that $\Phi \sim \phi \otimes \mathrm{id}_{M_\infty}$.

It must be noted that one cannot expect a semigroup isomorphism $W(A, B \otimes K) \cong W(A, B)$ in general, unless B is a stable C*-algebra. A counterexample is provided by the special case $A, B = \mathbb{C}$, where

$$\begin{aligned} W(\mathbb{C}, \mathbb{C} \otimes K) &\cong W(K) \\ &\cong \mathbb{N}_0 \cup \{\infty\} \\ &\neq \mathbb{N}_0 \\ &\cong W(\mathbb{C}) \\ &\cong W(\mathbb{C}, \mathbb{C}). \end{aligned}$$

3.1.4 Exactness

We now investigate the behaviour of the bivariant Cuntz semigroup Cu under extensions in the second argument. As a corollary of our analysis we obtain a new proof of the exactness of the functor Cu in the ordinary sense which relies on Loring's semiprojectivity of c.p.c. order zero maps on finite dimensional domains (Proposition 1.7).

We start by recording the following lemma, which is used to prove the injectivity of the induced map from $W(A, J)$ to $W(A, B)$, where A and B are C*-algebras and J is a closed two-sided $*$-ideal of B.

Lemma 3.28. *Let A and B be C*-algebras, J a closed two-sided $*$-ideal of B, and $\phi, \psi : A \to J$ two c.p.c. order zero maps such that $\phi \precsim \psi$ in B (i.e. the sequence $\{b_n\}_{n\in\mathbb{N}}$ that witnesses the subequivalence lies in B). Then $\phi \precsim \psi$ in J.*

Proof. By the local description of Cuntz equivalence of c.p.c. order zero maps we have that for every finite subset F of A and $\epsilon > 0$ there exists an element $b \in B$ such that

$$\|b\psi(a)b^* - \phi(a)\| < \epsilon$$

for any $a \in F$. By the existence of approximate units in C*-algebras, there is $e \in J$ such that

$$\|eb\psi(a)b^*e^* - b\psi(a)b^*\| < \epsilon$$

for any $a \in F$. Hence

$$\begin{aligned} \|eb\psi(a)b^*e^* - \phi(a)\| &\leq \|eb\psi(a)b^*e^* - b\psi(a)b^*\| + \|b\psi(a)b^* - \phi(a)\| \\ &< 2\epsilon, \end{aligned}$$

for any $a \in F$, with $eb \in J$. Therefore $\phi \precsim \psi$ in J. ∎

For any homomorphism $\phi : S \to T$ between the partially ordered Abelian monoids S and T, we define its kernel and image as the sets

$$\ker \phi := \{s \in S \mid \phi(s) = 0\}$$

and

$$\operatorname{im} \phi := \phi(S).$$

We shall say that a sequence $S \xrightarrow{f} T \xrightarrow{g} R$ of semigroup is exact if $\operatorname{im}(f) = \ker(g)$. We then have the following result about the exactness of the bifunctor Cu with respect to the second argument.

Proposition 3.29. *Let A and B be C*-algebras and let J be a two-sided closed $*$-ideal of B. The c.p.c. order zero split short exact sequence of C*-algebras*

$$0 \longrightarrow J \longrightarrow B \leftrightarrows B/J \longrightarrow 0$$

induces the split short exact sequence of monoids

$$0 \longrightarrow W(A, J) \longrightarrow W(A, B) \leftrightarrows W(A, B/J) \longrightarrow 0.$$

Proof. Let $\iota : J \to B$ be the inclusion map, $\pi : B \to B/J$ the quotient map and $\sigma : B/J \to B$ c.p.c. order zero such that $\pi \circ \sigma = \mathrm{id}_{B/J}$. The functoriality of W yields the induced maps $W(A, \iota) : W(A, J) \to W(A, B)$ and $W(A, \pi) : W(A, B) \to W(A, B/J)$, which are explicitly given by

$$W(A, \iota)([\phi]) = [(\iota \otimes \mathrm{id}_{M_\infty}) \circ \phi]$$

for any $[\phi]$ in $W(A, J)$, and

$$W(A, \pi)([\psi]) = [(\pi \otimes \mathrm{id}_{M_\infty}) \circ \psi],$$

for any $[\psi]$ in $W(A, B)$, respectively. The injectivity of $W(A, \iota)$ is a consequence of the following fact. If $\phi, \psi : A \to M_\infty(J)$ are c.p.c. order zero maps such that $\phi \precsim \psi$ by a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $M_\infty(B)$, then by an adaptation of Lemma 3.28 to the local C*-algebras $M_\infty(J)$ and $M_\infty(B)$ there is a sequence $\{b'_n\}_{n\in\mathbb{N}}$ in $M_\infty(J) \lhd M_\infty(B)$ such that $\phi \precsim \psi$ in $M_\infty(J)$. Therefore, $[\phi] \leq [\psi]$ in $W(A, B)$ implies $[\phi] \leq [\psi]$ in $W(A, J)$. It remains to show that $W(A, \pi) \circ W(A, \iota)$ is the zero map, but this is evident, and it follows that the map induced by σ is also injective. To see that $W(A, \pi)$ is surjective, observe that, again by functoriality, $W(A,\pi)\circ W(A,\sigma) = \mathrm{id}_{W(A,B/J)}$, so that any c.p.c. order zero map $\phi : A \to M_\infty(B/J)$ lifts to a c.p.c. order zero map $\sigma_*(\phi) : A \to M_\infty(B)$ given by

$$\sigma_*(\phi) = (\sigma \otimes \mathrm{id}_{M_\infty}) \circ \phi,$$

with the property that $W(A,\pi)([\sigma_*(\phi)]) = [\phi]$. Hence $W(A,\pi)$ is also surjective. Exactness in the middle is immediate, since any map from A to B that is sent to the zero map by π_* must have its range in J. □

The above result shows that $W(A,\ \cdot\ )$ is a split-exact functor for any C*-algebra A. Observe that an analogous result holds for the bifunctor Cu, defined in the next section, in place of W, as one can easily see by replacing any occurrence of the local C*-algebra M_∞ by that of the compact operators K in the proof of the above proposition.

When A is a finite dimensional C*-algebra we can make use of Proposition 1.7 to show that the map $\mathsf{Cu}(A,\pi)$ is surjective. Thus, we get to the following conclusion.

Corollary 3.30. *Let F and B be C*-algebras, F finite dimensional, and let J be a two-sided closed $*$-ideal of B. Then the short exact sequence of C*-algebras*

$$0 \longrightarrow J \longrightarrow B \longrightarrow B/J \longrightarrow 0$$

induces the short exact sequence of monoids

$$0 \longrightarrow \mathsf{Cu}(F,J) \longrightarrow \mathsf{Cu}(F,B) \longrightarrow \mathsf{Cu}(F,B/J) \longrightarrow 0.$$

Proof. If $\phi : F \to (B/J)\otimes K$ is a c.p.c. order zero map, use Proposition 1.7 to find a lift $\tilde{\phi} : F \to B \otimes K$. Hence the map $\mathsf{Cu}(F,\pi) : \mathsf{Cu}(F,B) \to \mathsf{Cu}(F,B/J)$ is surjective. □

Observe that, as a consequence of stability, the above result is true when F is any elementary C*-algebra. If we take $F = \mathbb{C}$ in the above corollary we can make use of the identification $\mathsf{Cu}(\mathbb{C},B) = \mathsf{Cu}(B)$, which holds for any C*-algebra B, to prove that the ordinary functor Cu is exact.

Corollary 3.31. *The functor Cu is exact. That is, for any short exact sequence of C*-algebras*

$$0 \longrightarrow J \longrightarrow B \longrightarrow B/J \longrightarrow 0$$

there is a short exact sequence of Cuntz semigroups

$$0 \longrightarrow \mathsf{Cu}(J) \longrightarrow \mathsf{Cu}(B) \longrightarrow \mathsf{Cu}(B/J) \longrightarrow 0.$$

Proof. This is a special case of the previous corollary where the finite dimensional C*-algebra F is just the algebra of complex numbers $\mathbb{C}$. □

3.2 The Bivariant Functor Cu

With Example 3.7 we have argued that the bivariant Cuntz semigroup of Definition 3.4 provides a bivariant extension of the ordinary Cuntz semigroup W. In this section we provide a *stabilised* definition for the bivariant Cuntz semigroup that allows recovering the Cuntz semigroup Cu of Section 2.2. We do so by relying on the isomorphism of Equation (2.2), and only after we restate this new definition in terms of modules, in order to strengthen the analogy with Kasparov's picture of KK-theory.

Definition 3.32. Let A and B be C*-algebras. The bivariant Cuntz semigroup $\mathsf{Cu}(A,B)$ is the set of equivalence classes

$$\mathsf{Cu}(A,B) = \{\phi : A\otimes K \to B\otimes K \mid \phi \text{ is c.p.c. order zero}\}/\sim,$$

endowed with the binary operation given by the direct sum $\hat{\oplus}$.

Observe that, equivalently, one can set

$$\mathsf{Cu}(A,B) := W(A\otimes K, B\otimes K),$$

for any pair of C*-algebras A and B. It is easily seen through Proposition 3.27 that stabilisation is only necessary in the second argument, since we have the isomorphism

$$W(A\otimes K, B\otimes K) \cong W(A, B\otimes K)$$

for any pair of C*-algebras A and B. Hence, up to isomorphism, we have the identification

$$\mathsf{Cu}(A,B) \cong W(A, B\otimes K), \tag{3.1}$$

for any pair of C*-algebras A and B.

It is easy to verify that all the properties of the bifunctor W that have been established in the previous section carry over to Cu. Therefore Cu is a bifunctor from the category ${\mathsf{C}^*_{\mathsf{loc}}}^{\mathrm{op}} \times \mathsf{C}^*_{\mathsf{loc}}$ with arrows given by c.p.c. order zero maps, to the category $\mathsf{OrdAMon}$. In this case, stability in both argument is easier to check, since it is now a direct consequence of the definition. In greater generality we then have the identification

$$\mathsf{Cu}(A\otimes K, B\otimes K) \cong \mathsf{Cu}(A,B),$$

which holds for any pair of C*-algebras A and B.

3.3 The Module Picture

In this Section we show how to *translate* the definition of the bivariant Cuntz semigroup Cu of the previous section in terms of countably generated Hilbert right modules. It is shown that one can define suitable triples that are somewhat reminiscent of Kasparov triples of KK-theory. Some of the ideas and terminology employed here are inspired by notes on the subject by Winter [73].

Definition 3.33 (Order zero pair)**.** Let A and B be C*-algebras. An A-B order zero pair is a pair (X, ϕ) consisting of a countably generated Hilbert right B-module X and a non-degenerate c.p.c. order zero map $\phi : A \to K(X)$.

If (X, ϕ) and (Y, ψ) are A-B order zero pairs, we say that (X, ϕ) is Cuntz subequivalent to (Y, ψ), $(X, \phi) \precsim (Y, \psi)$ in symbols, if there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \in K(X, Y)$ such that

$$\lim_{n\to\infty} \|s_n^*\psi(a)s_n - \phi(a)\| = 0$$

for all $a \in A$. The antisymmetrisation of such a subequivalence relation gives an equivalence relation, namely $(X, \phi) \sim (Y, \psi)$ if $(X, \phi) \precsim (Y, \psi)$ and $(Y, \psi) \precsim (X, \phi)$.

Definition 3.34. For separable C*-algebras A and B, we define

$$\mathcal{C}u(A, B) := \{A \otimes K\text{-}B \otimes K \text{ order zero pairs}\}/\sim,$$

endowed with the binary operation arising from the direct sum of pairs, i.e.

$$[(X, \phi)] + [(Y, \psi)] := [(X \oplus Y, \phi \,\hat{\oplus}\, \psi)]$$

The above definition gives a module picture for the bivariant Cuntz semigroup Cu introduced in the previous subsection. Indeed, this claim is supported by the following

Theorem 3.35. *For any pair of separable C*-algebras A and B, there is a natural isomorphism*

$$\mathsf{Cu}(A, B) \cong \mathcal{C}u(A, B).$$

Proof. Observe that, by Kasparov stabilisation theorem, one has the relations

$$K(X) \subset K(X \oplus H_B) \cong K(H_B) \cong B \otimes K.$$

For convenience we set

$$E_\phi := E_\phi$$

for the Hilbert B-module generated by a c.p.c. order zero map $\phi : A \otimes K \to B \otimes K$, where we identify the codomain with $B \otimes K$ with $K(H_B)$. To every $A \otimes K$-$B \otimes K$ order zero pair (X, ϕ) we associate the c.p.c. order zero map

$$\phi : A \otimes K \to K(X) \subset B \otimes K.$$

An inverse to this correspondence is provided by the map that sends a c.p.c. order zero map $\phi : A \otimes K \to B \otimes K$ to the pair

$$(E_\phi, \phi)$$

after the identification $\phi(A \otimes K) \cong K(E_\phi)$. To see this, consider two order zero pairs (X, ϕ) and (Y, ψ) such that $(X, \phi) \precsim (Y, \psi)$. Then there exists $\{s_n\}_{n\in\mathbb{N}} \in K(X, Y)$ such that

$$\lim_{n\to\infty} \|s_n^* \psi(a) s_n - \phi(a)\| = 0$$

for all $a \in A \otimes K$. Now

$$K(X, Y) \subset K(X \oplus H_B, Y \oplus H_B) \subset K(H_B) \cong B \otimes K$$

so that $\{s_n\}_{n\in\mathbb{N}}$ can be identified as a sequence of $B \otimes K$. Hence, up to this identification

$$\lim_{n\to\infty} \|s_n^* \psi(a) s_n - \phi(a)\| = 0$$

i.e. $\phi \precsim \psi$. Conversely, let $\phi, \psi : A \otimes K \to B \otimes K$ be c.p.c. order zero maps such that $\phi \precsim \psi$. Hence, there exists $\{z_n\}_{n\in\mathbb{N}} \subset B \otimes K \cong K(H_B)$ such that

$$\lim_{n\to\infty} \|z_n^* \phi(a) z_n - \psi(a)\| = 0$$

for all $a \in A \otimes K$. Since E_ϕ and E_ψ are countably generated Hilbert modules, there are projections $p, q \in B(H_B)$ such that $pH_B = E_\phi$ and $qH_B = E_\psi$. Therefore, the sequence $\{w_n\}_{n\in\mathbb{N}} \subset K(E_\phi, E_\psi)$ given by

$$w_n := p z_n q \in K(E_\phi, E_\psi), \qquad \forall n \in \mathbb{N},$$

is such that

$$\lim_{n\to\infty} \|w_n^* \psi(a) w_n - \phi(a)\| = 0,$$

for any $a \in A$, and this shows precisely that $(E_\phi, \phi) \precsim (E_\psi, \psi)$. □

3.4 The Composition Product

In order to strengthen the analogy with KK-theory we now introduce a product between elements of the bivariant Cuntz semigroup that resembles the analogous operation that is defined among classes of KK-groups. Before giving the details of such a product we state some technical results that are used in what follows.

Let A and B be local C*-algebras, and let $\phi : A \to M_\infty(B)$ be a c.p.c. order zero map. Then the ∞-ampliation $\phi^{(\infty)}$ is easily seen to be a completely positive map of order zero from $M_\infty(A)$ to $M_\infty(B)$, since $\phi^{(\infty)} = \phi \otimes \mathrm{id}_{M_\infty}$. Furthermore, if $\phi, \psi : A \to B$ are c.p.c. order zero maps such that $\phi \precsim \psi$, then the same subequivalence relation holds between their m-ampliations, namely $\phi^{(m)} \precsim \psi^{(m)}$, for any $m \in \mathbb{N} \cup \{\infty\}$. This follows from the fact that $\phi^{(m)} = \phi \otimes \mathrm{id}_{M_n}$ and by Lemma 3.25.

Let A, B, C be local C*-algebras, and take any c.p.c. order zero maps $\phi : A \to M_\infty(B)$ and $\psi : B \to M_\infty(C)$. Since $\psi^{(\infty)} : M_\infty(B) \to M_\infty(C)$ is a c.p.c. order zero map, the composition

$$\phi \cdot \psi := \psi^{(\infty)} \circ \phi$$

defines a c.p.c. order zero map from A to $M_\infty(C)$. One can then define a composition product among elements of *composable* bivariant Cuntz semigroups by just pushing the above composition product towards the corresponding classes. What follows is a proposition and definition.

Proposition 3.36 (Composition product)**.** *Let A, B and C be separable local C*-algebras. The binary map $W(A,B) \to W(B,C) \to W(A,C)$ given by*

$$[\phi] \cdot [\psi] := [\psi \cdot \phi]$$

is well-defined. We call such map the composition product *for the bivariant Cuntz semigroup.*

Proof. Let $\phi, \phi' : A \to M_\infty(B)$, $\psi, \psi' : B \to M_\infty(C)$ be c.p.c. order zero maps such that $\phi \precsim \phi'$ and $\psi \precsim \psi'$. Since the latter condition implies $\psi^{(\infty)} \precsim \psi'^{(\infty)}$, it follows from Proposition 3.24 that $\phi \cdot \psi \precsim \phi \cdot \psi'$ and $\phi \cdot \psi \precsim \phi' \cdot \psi$. ◻

As a consequence of the above result, it follows that the product preserves the order structure in the following sense.

Corollary 3.37. *The composition product on the bivariant Cuntz semigroup and its order structure are compatible, in the sense that, if ϕ, ϕ', ψ, ψ' are as in the above proposition, then $\phi \cdot \psi \precsim \phi' \cdot \psi'$.*

It is clear that, for any separable local C*-algebra A, $W(A, A)$ has a natural semiring structure, and it is easy to check that the class of the embedding $\iota_A : A \to M_\infty(A)$ in $W(A, A)$ provides a unit $[\iota_A]$. The following example gives a feeling of the behaviour of the composition product for the bivariant Cuntz semigroup.

Example 3.38. With $A = \mathbb{C}$ we obtain the semigroup $W(\mathbb{C}, \mathbb{C}) = W(\mathbb{C}) = \mathbb{N}$. It is an easy exercise to verify that, if $[\phi], [\psi] \in W(\mathbb{C}, \mathbb{C})$, the product for the corresponding positive elements $[h_\phi], [h_\psi] \in W(\mathbb{C})$ is given by the tensor product $[h_\phi \otimes h_\psi]$. Therefore, the composition product corresponds to the ordinary product between natural numbers in $\mathbb{N}$. △

The composition product that we have just introduced is of particular importance for the theory of classification of C*-algebras. With such an object at our disposal we can then introduce a notion of invertibility of elements of the bivariant Cuntz semigroup and link them to isomorphy of the associated C*-algebras. We give the following series of definitions with the goal of classification in mind.

Definition 3.39 (Invertible element in W)**.** Let A and B be separable C*-algebras. An element $\Phi \in W(A, B)$ is said to be invertible if there exists a $\Psi \in W(B, A)$ such that $\Phi \cdot \Psi = [\iota_A]$ and $\Psi \cdot \Phi = [\iota_B]$.

Definition 3.40 (W-equivalence)**.** Two separable C*-algebras A and B are W-equivalent if there exists an invertible element in $W(A, B)$.

The semigroup $\mathsf{Cu}(A, B)$ inherits the composition product directly from $W(A \otimes K, B \otimes K)$. However, one can give an equivalent definition where it takes the form of a genuine composition, since there is no need of considering matrix ampliations in this case. Thus, if A, B and C are separable C*-algebras, and $\phi : A \to B$ and $\psi : B \to C$ are c.p.c. order zero maps, then one can set

$$\phi \cdot \psi := \psi \circ \phi,$$

and from this introduce the composition product on $\mathsf{Cu}(A, B)$ in the same way that it has already been done on the semigroup $W(A, B)$. Again, $\mathsf{Cu}(A, A)$ has a natural semiring structure, and the unit is seen to be represented by the identity map on $A \otimes K$, that is

$[\mathrm{id}_{A\otimes K}]$. The notion of invertibility also generalises to this case, with the due modifications. We then give the following additional definitions for the bifunctor Cu.

Definition 3.41 (Invertible element in Cu)**.** Let A and B be separable C*-algebras. An element $\Phi \in \mathsf{Cu}(A,B)$ is said to be invertible if there exists a $\Psi \in \mathsf{Cu}(B,A)$ such that $\Phi \cdot \Psi = [\mathrm{id}_{A\otimes K}]$ and $\Psi \cdot \Phi = [\mathrm{id}_{B\otimes K}]$.

Definition 3.42 (Cu-equivalence)**.** Two separable C*-algebras A and B are Cu-equivalent if there exists an invertible element in $\mathsf{Cu}(A,B)$.

3.5 Further Categorical Aspects

We now move to the question whether the bivariant Cuntz semigroup $W(A,B)$ belongs to the category W introduced in Section 2.4.2. To this end we define the following auxiliary relation on $W(A,B)$.

Definition 3.43 (Auxiliary relation)**.** Let A and B be C*-algebras, and let $\Phi,\Psi \in W(A,B)$. Define the auxiliary relation $\prec$ on $W(A,B)$ by setting $\Phi \prec \Psi$ if there exists $\epsilon > 0$ such that $\Phi \leq [\psi_\epsilon]$, where ψ is any representative of Ψ.

Lemma 3.44. *Let A and B be local C*-algebras, and let $\Phi \in W(A,B)$. For any representative $\phi \in \Phi$, the sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ given by $\Phi_n := [\phi_{\frac{1}{n}}]$ for any $n \in \mathbb{N}$ is increasing in $W(A,B)$ and is such that $\sup \Phi_n = \Phi$.*

Proof. Given $\epsilon_1,\epsilon_2 > 0$ one has $\phi_{\epsilon_1+\epsilon_2} = (\phi_{\epsilon_1})_{\epsilon_2}$ and therefore the sequence Φ_n described above is increasing by Corollary 3.10. Moreover, from the same corollary we have that $\Phi_n \leq \Phi$ for any $n \in \mathbb{N}$, whence $\sup \Phi_n \leq \Phi$. Suppose $\Psi \in W(A,B)$ is such that $\Phi_n \leq \Psi$ for any $n \in \mathbb{N}$, and let ψ be any representative of Ψ. From the local description of Cuntz comparison of c.p.c. order zero maps we have that

$$\forall n \in \mathbb{N}, F \Subset A, \epsilon > 0\ \exists b_{n,F,\epsilon} \in M_\infty(B) \mid \left\| b^*_{n,F,\epsilon}\psi(a)b_{n,F,\epsilon} - \phi_{\frac{1}{n}}(a)\right\| \to 0,\ \forall a \in A.$$

Since the continuous functional calculus is norm-continuous, we have that

$$\lim_{n\to\infty} \left\|\phi_{\frac{1}{n}}(a) - \phi(a)\right\| = 0, \qquad \forall a \in A,$$

i.e.

$$\forall a \in A, \epsilon > 0 \quad \exists n_{a,\epsilon} \in \mathbb{N} \quad \mid \quad n > n_{a,\epsilon} \quad \Rightarrow \quad \left\|\phi_{\frac{1}{n}}(a) - \phi(a)\right\| < \epsilon.$$

For any finite subset $F \Subset A$ and $\epsilon > 0$ one can take

$$N_\epsilon := \max_{a \in F} n_{a,\epsilon},$$

so that there exists $b_{N_\epsilon,F,\epsilon} \in M_\infty(B)$ with the property that

$$\left\| b^*_{N_\epsilon,F,\epsilon}\psi(a)b_{N_\epsilon,F,\epsilon} - \phi_{\frac{1}{N_\epsilon}}(a) \right\| < \epsilon, \qquad \forall a \in F.$$

Setting $b := b^*_{N_\epsilon,F,\epsilon}$ one has that

$$\begin{aligned} \|b^*\psi(a)b - \phi(a)\| &= \left\| b^*\psi(a)b_n - \phi_{\frac{1}{N_a}}(a) + \phi_{\frac{1}{N_a}}(a) - \phi(a) \right\| \\ &\leq \left\| b^*\psi(a)b_n - \phi_{\frac{1}{N_a}}(a) \right\| + \left\| \phi_{\frac{1}{N_a}}(a) - \phi(a) \right\| \\ &< 2\epsilon \end{aligned}$$

for any $a \in F$. Therefore $\Phi \leq \Psi$, and by the arbitrariness of Ψ we conclude that $\Phi = \sup \Phi_n$. □

Proposition 3.45. *Let A and B be local C^*-algebras. Then $W(A,B)$ belongs to* W.

Proof. One has to verify that $W(A,B)$ has all the properties stated in Definition 2.33. By the above definition of the auxiliary relation $\prec$ on $W(A,B)$, it follows that, for $0 < \epsilon_1 < \epsilon_2$, one has

$$\phi_{\epsilon_2} \prec \phi_{\epsilon_1} \prec \phi$$

for $[\phi] \in W(A,B)$; therefore $\{\phi_{\frac{1}{n}}\}_{n\in\mathbb{N}}$ is easily seen to be a cofinal $\prec$-increasing sequence in $[\phi]^{\prec} \subset W(A,B)$, which shows that $W(A,B)$ has property (WO.1), and consequently property (WO.2) by the previous lemma. Property (WO.3) follows from the fact that

$$(\phi \mathbin{\hat{\oplus}} \psi)_\epsilon = \phi_\epsilon \mathbin{\hat{\oplus}} \psi_\epsilon$$

for any c.p.c. order zero maps $\phi, \psi : A \to M_\infty(B)$, whereas property (WO.4) comes immediately from the fact that $[\phi_\epsilon] \prec [\phi]$ for any c.p.c. order zero map $\phi : A \to M_\infty(B)$. □

We now enclose, in the following two lemmas, the technical results needed to prove that, if $f : A \to A'$ and $g : B \to B'$ are $*$-homomorphisms between C*-algebras, then the induced maps $W(f,B) : W(A',B) \to W(A,B)$ and $W(A,g) : W(A,B) \to W(A,B')$ are morphisms in the category W.

Lemma 3.46. *Let A, B, B' be local C^*-algebras, $g : B \to B'$ a $*$-homomorphism and $\phi : A \to B$ a c.p.c. order zero map. Then $(g \circ \phi)_\epsilon = g \circ \phi_\epsilon$ for any $\epsilon > 0$.*

Proof. Let h_ϕ and π_ϕ be such that $\phi = h_\phi \pi_\phi$ as described by the structure result of Corollary 1.9. Let $g^{**} : B^{**} \to B'^{**}$ be the bitranspose of g. Then it is easily checked that the decomposition

$$g \circ \phi = g^{**}(h_\phi)(g^{**} \circ \pi_\phi).$$

satisfies again Corollary 1.9. Therefore the result follows from the definition of functional calculus on c.p.c. order zero maps. □

Lemma 3.47. *Let A, A', B be local C^*-algebras, $f : A \to A'$ a $*$-homomorphism and $\phi : A' \to B$ a c.p.c. order zero map. Then $\phi_\epsilon \circ f = (\phi \circ f)_\epsilon$ for any $\epsilon > 0$.*

Proof. Observe that, by applying the structure result of Corollary 1.9 at different stages, the c.p.c. order zero map $\phi \circ f$ can be expressed in the equivalent forms

$$\phi \circ f = h_{\phi \circ f} \pi_{\phi \circ f} = h_\phi (\pi_\phi \circ f).$$

Set $C_\phi := C^*(\phi(A))$ and let $\{u_n\}_{n \in \mathbb{N}} \subset A$ be an increasing approximate unit for A. Define the projection $p \in C_\phi^{**}$ by the strong limit

$$p := \text{SOT} \lim_{n \to \infty} \pi_\phi(f(u_n)),$$

which clearly commutes with h_ϕ and $\pi_\phi \circ f$, and $p\pi_\phi(f(a)) = \pi_\phi(f(a))$ for any $a \in A$. A direct computation shows that $h_{\phi \circ f} = p h_\phi$, and therefore $\pi_{\phi \circ f} = p(\pi_\phi \circ f) = \pi_\phi \circ f$. Since p is a projection, $F(h_{\phi \circ f}) = pF(h_\phi)$ for any $F \in C_0((0, 1])$ whence

$$\begin{aligned}
(\phi \circ f)_\epsilon &= (h_{\phi \circ f} \pi_{\phi \circ f})_\epsilon \\
&= f_\epsilon(h_{\phi \circ f})(\pi_\phi \circ f) \\
&= f_\epsilon(p h_\phi)(\pi_\phi \circ f) \\
&= f_\epsilon(h_\phi) p (\pi_\phi \circ f) \\
&= \phi_\epsilon \circ f,
\end{aligned}$$

for any $\epsilon > 0$. □

Theorem 3.48. *W is a bifunctor from the category of local C^*-algebras to the category* W*, contravariant in the first argument and covariant in the second.*

Proof. It has already been shown that $W(A,B)$ is in the category W for any choice of local C*-algebras A and B. It is left to check that any $*$-homomorphisms $f : A \to A'$ and $g : B \to B'$ between local C*-algebras induce maps $W(f,B)$ and $W(A,g)$ respectively which are morphisms in $\mathcal{W}$.

The continuity of $W(A,g)$ follows from the fact that, if $[\psi] \prec W(A,g)([\phi])$, then there exists $\epsilon > 0$ such that $\psi \precsim (g^{(\infty)} \circ \phi)_\epsilon$, which by Lemma 3.46 is seen to coincide with $\psi \precsim g^{(\infty)} \circ \phi_\epsilon$. Therefore, it is enough to take ϕ_ϵ to witness the continuity, since $[\phi_\epsilon] \prec [\phi]$ and $[\psi] \leq W(A,g)([\phi_\epsilon])$, as it was just shown. Suppose now that $[\phi] \prec [\psi]$ in $W(A,B)$. This is equivalent to the statement that

$$\exists \epsilon > 0 \qquad | \qquad \phi \precsim \psi_\epsilon.$$

Since $W(A,g)$ is order preserving, we must have

$$W(A,g)([\phi]) \leq W(A,g)([\psi_\epsilon]),$$

whereas by Lemma 3.46 we can conclude that the right-hand side coincides with $[(g^{(\infty)} \circ \psi)_\epsilon]$. Hence

$$W(A,g)([\phi]) \prec W(A,g)([\psi]).$$

By exchanging post-composition by $g^{(\infty)}$ with pre-composition by f, and using Lemma 3.47 in place of Lemma 3.46, the same argument shows that $W(f,B)$ has both properties (WM.1) and (WM.2) as well. □

3.5.1 Compact Elements

In the ordinary theory of the Cuntz semigroup there is the notion of compactness (cf. Definition 2.31). It has been observed that, according to this definition, every projection of a C*-algebra A defines a compact element in the Cuntz semigroup $\mathsf{Cu}(A)$. As shown by the structure theorem for c.p.c. order zero maps of [75], the natural bivariant extension of projections are $*$-homomorphisms. Hence, we look at a definition of compact elements for the bivariant Cuntz semigroup for which the class of every $*$-homomorphism between C*-algebras A and B turns out to be compact in $\mathsf{Cu}(A,B)$. The following result constitutes our starting point.

Proposition 3.49. *Let A and B be separable local C*-algebras. Then every c.p.c. order-zero map $\phi : A \to B$ naturally induces a generalised W-morphism $W(\phi) : W(A) \to W(B)$.*

If ϕ is a $*$-homomorphism, then $W(\phi)$ preserves the auxiliary relation and thus is a W-*morphism.*

Proof. It follows from [75, Corollary 4.5] that $W(\phi)$ is a well-defined morphism between semigroups (i.e. $W(\phi)$ preserves addition, order and the zero element).

To check that $W(\phi)$ is continuous, let $t \in W(B)$ and $s \in W(A)$ be such that $t \prec W(\phi)(s)$. We need to show the existence of $s' \in W(A)$ such that $s' \prec s$ and $t \leq W(\phi(s'))$. To this end, let $[x] = s$. Since $t \prec W(\phi)(s) = [\phi(x)]$, there exists $\epsilon > 0$ such that $t \leq [(\phi(x) - \epsilon)_+]$. Moreover, from Corollary 3.13 we have that $(\phi(x) - \epsilon)_+ \precsim \phi((x - \epsilon)_+)$ and therefore, by setting $s' := [(x - \epsilon)_+]$, we have $s' \prec s$ in $W(A)$ and $t \leq W(\phi)(s')$. ☐

Observe that, if ϕ and ψ are Cuntz-equivalent c.p.c. order zero maps, then the induced maps at the level of the Cuntz semigroups coincide. If the map ϕ in the proposition above is a $*$-homomorphism, then one has $\phi((a - \epsilon)_+) = (\phi(a) - \epsilon)_+$, which implies that $W(\phi)$ preserves the *way-below* relation $\ll$ (equivalently the compact containment relation $\subset\subset$) of Definition 2.29. As a consequence of this result, by viewing every Cuntz semigroup $W(A)$ of a local C*-algebra A as a subsemigroup of the corresponding completion, namely $\mathsf{Cu}(\tilde{A})$, one gets that the induced map $\mathsf{Cu}(\phi)$ is in the category Cu. These considerations, together with the fact that $*$-homomorphisms over $\mathbb{C}$ correspond to a projection in the target algebra, lead to the following definition.

Definition 3.50. Let A and B be C*-algebras. An element $\Phi \in \mathsf{Cu}(A, B)$ is compact if there exists a c.p.c. order zero map $\phi \in \Phi$ such that $\mathsf{Cu}(\phi)$ is an arrow in the category Cu.

When the first argument of the bivariant functor Cu is set to $\mathbb{C}$ one recovers the usual definition for compact elements of the ordinary Cuntz semigroup. To see this, consider a C*-algebra B and a c.p.c. order zero map $\phi : \mathbb{C} \to B \otimes K$. By the structure theorem of [75] this is of the form $\phi(z) = zb$ for any $z \in \mathbb{C}$, with $b := \phi(1)$. Furthermore, we have that $\mathsf{Cu}(\phi)$ is an arrow in the category Cu by definition of compactness for the bivariant Cuntz semigroup given above. The induced map $\mathsf{Cu}(\phi) : \tilde{\mathbb{N}}_0 \to \mathsf{Cu}(\tilde{B})$ maps n to $n[b]$, with $[b] \in \mathsf{Cu}(B)$. Since $n = 1$ arises from any minimal projection in K, one has $1 \ll 1$ inside $\mathsf{Cu}(\mathbb{C})$. Since $\mathsf{Cu}(\phi)$ preserves the way-below relation by hypothesis, it follows that $\mathsf{Cu}(\phi)(1) \ll \mathsf{Cu}(\phi)(1)$, i.e. $[b] \ll [b]$ in $\mathsf{Cu}(B)$. Other examples of compact elements in the bivariant Cuntz semigroup are given by the classes of c.p.c. order zero maps that have a $*$-homomorphism as a representative. This follows from the fact that c.p.c. order zero maps in the same class induce the same map at the level of the Cuntz semigroups and that every $*$-homomorphism preserves the relation $\ll$.

When stable and finite C*-algebras are considered, one has a natural embedding of the Murray-von Neumann semigroups $V(A)$ and $V(B)$ inside $W(A)$ and $W(B)$, and hence in $\mathsf{Cu}(A)$ and $\mathsf{Cu}(B)$, respectively. If $\Phi \in \mathsf{Cu}(A,B)$ is compact then there is a representative $\phi \in \Phi$ such that $\mathsf{Cu}(\phi) \in \mathsf{Cu}$. This implies, by definition, that $\mathsf{Cu}(\phi)$ preserves the way-below relation, and therefore the class of a projection is sent to the class of a projection. As a consequence of this fact one is then allowed to restrict $\mathsf{Cu}(\phi)$ to $V(A)$ to obtain a semigroup homomorphism in $\mathrm{Hom}(V(A), V(B))$.

Inside $W(A,B) \subset \mathsf{Cu}(\tilde{A}, \tilde{B})$, where A and B are local C*-algebras, one can identify a special subsemigroup that is given by all those elements that have a $*$-homomorphism as a representative. We shall call this subsemigroup $V(A,B)$, i.e. we set

$$V(A,B) := \{\Phi \in W(A,B) \mid \Phi = [\pi] \text{ for some } *\text{-homomorphism } \pi\}.$$

Thanks to [4, Lemma 2.20], which asserts that $V(A)$ order-embeds in $W(A)$ for any stably finite C*-algebra, one can see that $V(\mathbb{C}, B) \cong V(B)$, i.e. the Murray-von Neumann semigroup of B, whenever B is a stably finite local C*-algebra. In analogy with the contents of Section 2.4.2 of [4], we denote by $W(A,B)_+$ all the other elements of $W(A,B)$ that do not belong to $V(A,B)$. Hence we shall call *purely c.p.c. order zero* any c.p.c. order zero map $\phi : A \to M_\infty(B)$ such that $[\phi] \in W(A,B)_+$. We then have a decomposition of the bivariant Cuntz semigroup as the disjoint union

$$W(A,B) = V(A,B) \sqcup W(A,B)_+$$

for any pair of local C*-algebras A and B. A similar decomposition can be obtained when considering compact and non-compact elements in $W(A,B)$. However, there are notable cases where these two decompositions coincide.

Theorem 3.51. *Let A and B be C*-algebras, with A unital and B stable and finite. Then $V(A,B)$ is the subsemigroup of all the compact elements of $\mathsf{Cu}(A,B)$.*

Proof. Clearly every element in $V(A,B)$ is compact in $\mathsf{Cu}(A,B)$. Conversely, assume that $\Phi \in \mathsf{Cu}(A,B)$ is compact. Then there exists a representative of the form $\phi \otimes \mathrm{id}_K \in \Phi$ such that $\mathsf{Cu}(\phi)$ preserves the way-below relation. Since $1_A \otimes e \in A \otimes K$ is a projection, its class in $W(A)$ is a compact element and therefore $\mathsf{Cu}(\phi)([1_A \otimes e]) \ll \mathsf{Cu}(\phi)([1_A \otimes e])$, which implies that $[\phi(1_A)] \ll [\phi(1_A)]$ in $\mathsf{Cu}(B)$. Since B is stably finite, the positive element

$h := \phi(1_B)$ is then Cuntz-equivalent to its support projection p_h by [11, Theorem 3.5]. Hence $h^{\frac{1}{2}}p = h^{\frac{1}{2}}$ and there exists $\{x_n\}_{n\in\mathbb{N}} \subset B \otimes K$ such that

$$\lim_{n\to\infty} \left\| x_n h^{\frac{1}{2}} - p \right\| = 0,$$

where the sequence $\{x_n\}_{n\in\mathbb{N}}$ is explicitly given by

$$x_n := (h + \tfrac{1}{n})^{-1} h^{\frac{1}{2}}.$$

Thus one has that $h^{\frac{1}{2}}\pi_\phi(a)h^{\frac{1}{2}} = \phi(a)$ and

$$\lim_{n\to\infty} \|x_n^* \phi(a) x_n - \pi_\phi(a)\| = 0$$

for any $a \in A$, which shows that ϕ is Cuntz-equivalent to its support $*$-homomorphism. □

The above theorem can be regarded as the bivariant version of the analogous result for the Cuntz semigroup of [11, Theorem 3.5].

3.5.2 Continuity

Since the bivariant Cuntz semigroups W can be taken in the enriched category W, as shown in Section 3.5, we now address the properties of continuity in both the arguments. Like in KK-theory we do not expect to have continuity under the most general circumstances. The results that follow indeed show that the situation with the bivariant Cuntz semigroup is no different, with continuity recovered in some special cases. This opens up for a definition of Cu-semiprojectivity along the lines of the notion of KK-semiprojectivity of [18]. However, we will not touch upon this topic in this thesis.

It is known that the category W has inductive limits and that the functor W is sequentially continuous (cf [3]). In fact W turns out to be continuous under arbitrary inductive limits. Therefore the bivariant functor W is continuous in the second variable whenever the first argument is a finite dimensional C*-algebra, for in this case it just reduces to the ordinary Cuntz semigroup W. In more general cases, however, this property fails, as shown by the following (counter)examples.

Example 3.52. Let A be the algebraic direct limit of the sequence

$$\mathbb{C} \xrightarrow{\phi_0} M_2 \xrightarrow{\phi_1} M_4 \xrightarrow{\phi_2} \cdots,$$

where the connecting maps are given by

$$\phi_k(a) := a \oplus a, \qquad \forall a \in M_{2^k}, k \in \mathbb{N}_0.$$

Since there are no $*$-homomorphisms from the CAR algebra to matrix algebras apart from the trivial one, one has

$$W(A, M_{2^n}) = \{0\},$$

however

$$W(A, A) = W(A) = \mathbb{N}_0[\tfrac{1}{2}] \sqcup (0, \infty],$$

as an immediate consequence of the simplicity of A. Continuity in this case is recovered if one takes the completion $\tilde{A}$ of A, i.e. the CAR algebra, since, again by simplicity, one has $W(\tilde{A}, A) = \{0\}$. △

A similar computation shows that the functor is not continuous in the first argument as well in the most general case.

Example 3.53. Let A be the CAR algebra, i.e. the C*-inductive limit of the sequence of matrix algebras of the previous example. Then

$$W(M_{2^n}, \mathbb{C}) \cong \mathbb{N}_0$$

for any $n \in \mathbb{N}$, and the connecting maps are just multiplication by 2 at each step. Therefore

$$\varprojlim W(M_{2^n}, \mathbb{C}) = \{0\},$$

which coincides with $W(A, \mathbb{C}) = \{0\}$. However

$$W(M_{2^n}, K) = \mathbb{N}_0 \cup \{\infty\},$$

so that

$$\varprojlim W(M_{2^n}, K) = \{0, \infty\} \neq \{0\} = W(A, K).$$

Hence $W(\,\cdot\,, B)$ does not turn C*-inductive limits into projective limits for a general local C*-algebra B. △

There are however cases where the algebra in the first argument is not finite dimensional, but the bifunctor W is nevertheless continuous in the first argument, as shown by the following example.

Example 3.54. Let A be the CAR algebra. Then

$$W(M_{2^n}, A) \cong W(A) \cong \mathbb{N}_0[\tfrac{1}{2}] \sqcup (0, \infty],$$

with the connecting maps that are now automorphisms of $\mathbb{N}_0[\frac{1}{2}] \sqcup (0,\infty]$. Hence

$$\varinjlim W(M_{2^n}, A) \cong \mathbb{N}_0[\tfrac{1}{2}] \sqcup (0,\infty],$$

which coincides with $W(A,A)$. The same result is obtained if A is stabilised, i.e. replaced by $A \otimes K$, so that, according to Equation 3.1, this gives an analogue of this example for the bifunctor Cu. △

3.6 Examples

In this section we provide some computations of bivariant Cuntz semigroups. It is well-known that the task of computing ordinary Cuntz semigroups is not an easy one for general C*-algebras, and the bivariant Cuntz semigroup, being an extension of it, is no exception. Therefore, we use special classes of C*-algebras in order to give some concrete examples of computations.

3.6.1 Purely Infinite C*-algebras

An interesting class of C*-algebras that plays an important rôle in the Classification Programme is that of Kirchberg algebras, that is separable, nuclear, simple and purely infinite C*-algebras. The ordinary Cuntz semigroup of a Kirchberg algebra turns out to be trivial because of the following fact. If A is a Kirchberg algebras, then for any pair of positive elements $a, b \in A^+$ there exists $s \in A$ such that $sas^* = b$. As a consequence of this property, and of the fact that $M_\infty(A)$ as well as $A \otimes K$ are Kirchberg algebras, one has that $\mathsf{Cu}(A) = \{0, \infty\}$, where 0 is simply the class of the zero element of A, and ∞ is the class of any other positive element of $A \otimes K$.

When Kirchberg algebras are considered within the bivariant theory of the Cuntz semigroup introduced in this chapter, one observes a similar behaviour. Our results make use of the following fundamental approximation result for unital completely positive (u.c.p. for short) maps on unital Kirchberg algebras (cf. [66, Corollary 6.3.5]).

Lemma 3.55. *Let B be a unital Kirchberg algebra, $\rho : B \to B$ a u.c.p. map, $F \subset B$ a finite subset and $\epsilon > 0$. There exists an isometry $s \in B$ such that $\|s^*bs - \rho(b)\| \leq \epsilon \|b\|$ for all $b \in F$.*

We also make use of the following extension result for u.c.p. maps, which follows from Arveson's extension theorem (cf. [66, Theorem 6.1.5]). Recall that an operator system in a unital C*-algebra A is a closed self-adjoint subspace of A containing the unit of A.

Lemma 3.56. *Let B be a nuclear C*-algebra, $E \subset B$ a finite dimensional operator system, $\eta : E \to B$ a u.c.p. map, and $\epsilon > 0$. There exists a u.c.p. map $\tilde{\eta} : B \to B$ such that $\|\tilde{\eta}|_E - \eta\| \leq \epsilon$.*

Proof. The nuclearity of B makes the inclusion $E \hookrightarrow B$ a nuclear map. Therefore, for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ and u.c.p. maps $\rho : E \to M_n$ and $\phi : M_n \to B$ such that

$$\|\phi \circ \rho - \eta\| \leq \epsilon.$$

By Arveson's extension theorem, the map ρ admits a u.c.p. extension to B, i.e. there exists $\tilde{\rho} : B \to M_n$ u.c.p. such that $\tilde{\rho}|_E = \rho$. The situation is depicted in the following diagram

$$\begin{array}{ccccc} E & & \xrightarrow{\quad\eta\quad} & & B \\ \big\downarrow & \searrow^{\rho} & & \nearrow^{\phi} & \\ B & \xrightarrow{\tilde{\rho}} & M_n & & \end{array}$$

which commutes up to ϵ. By setting $\tilde{\eta} := \phi \circ \tilde{\rho}$ we then have $\|\tilde{\eta}|_E - \eta\| \leq \epsilon$. □

Let A and B be C*-algebras, A unital. By the structure theorem 1.5, any c.p.c. order zero map $\phi : A \to B$ has a decomposition of the form $h\pi$, where $\pi : A \to \mathcal{M}(C^*(\phi(A)))$ and $h \in \mathcal{M}(C^*(\phi(A)))$. In fact the range of π lies in $\mathcal{M}(B_\phi)$, where B_ϕ is the hereditary C*-subalgebra $\overline{\phi(A)B\phi(A)}$ of B. Let g_ϵ be the continuous function on $[0, \infty)$ which vanishes on $[0, \epsilon/2)$, it is 1 on $[\epsilon, \infty)$ and linear otherwise, and let $h_\epsilon = g_\epsilon(h)$ and $\phi_\epsilon = h_\epsilon \pi$, i.e. $\phi_\epsilon = g_\epsilon(\phi)$ according to the definition of functional calculus on c.p.c. order zero maps. There exists a continuous positive function k_ϵ vanishing on $[0, \epsilon/2]$ and such that $tk_\epsilon(t) = g_\epsilon(t)$, e.g.

$$k_\epsilon(t) = \begin{cases} 0 & t \in [0, \epsilon/2) \\ \frac{g_\epsilon(t)}{t} & t \geq \epsilon/2. \end{cases}$$

With $\bar{h}_\epsilon := k_\epsilon(h)$ one has the identity $\bar{h}_\epsilon h = h_\epsilon$ and $\|\bar{h}_\epsilon\| \leq \epsilon^{-1}$. By letting p_ϵ denote the support projection of h_ϵ in $P_o(B^{**})$, one has that $\pi_\epsilon := p_\epsilon \pi$ can be regarded as a $*$-homomorphism from A to the multiplier algebra $\mathcal{M}(B_\epsilon)$, where $B_\epsilon := h_\epsilon B h_\epsilon$. Moreover, $h_\epsilon = \phi_\epsilon(1_A) \in B_\epsilon$, so that $\phi_\epsilon = h_\epsilon \pi_\epsilon$. Since $p_{2\epsilon} h_\epsilon = p_{2\epsilon}$ by the definitions above, there is a completely positive linear map from $\phi_\epsilon(A)$ to $\pi_{2\epsilon}(A)$ which is given by the mapping

$$x \mapsto p_{2\epsilon} x p_{2\epsilon}.$$

Furthermore

$$p_{2\epsilon}\phi_\epsilon(a) = \phi_\epsilon(a)p_{2\epsilon} = \pi_{2\epsilon}(a)$$

for all $a \in A$. We also observe that ϕ is injective if and only if π is injective, and in this case

$$\lim_{\epsilon\to 0^+} \|\pi_\epsilon(a)\| = \|a\|$$

for every $a \in A$. Moreover, $\pi_\epsilon(A)$ may be identified with $A/\ker\pi_\epsilon$. If A is nuclear then this quotient map admits a completely positive lift, and if A is only exact then the quotient map $A \to A/\ker\pi_\epsilon$ has the local lifting property, i.e. given any finite dimensional operator system $E \subset A/\ker\pi_\epsilon$ there exists a u.c.p. map $\lambda_\epsilon : E \to A$ with $\pi_\epsilon \circ \lambda_\epsilon = \mathrm{id}_E$. If A is simple then $\ker\pi_\epsilon = \{0\}$ for sufficiently small ϵ, so that the existence of such lift is obvious under this extra hypothesis.

Lemma 3.57. *Let A be a unital, simple, separable and exact C*-algebra, B a unital Kirchberg algebra and $\phi_1, \phi_2 : A \to B$ be c.p.c. order zero maps, with ϕ_1 injective. Then $\phi_2 \precsim \phi_1$.*

Proof. It is enough to show that, given a finite dimensional operator system $E \subset A$ and $\epsilon > 0$, there exists $b \in B$ such that $\|b^*\phi_1(e)b - \phi_2(e)\| \leq \epsilon$ for all e in the unit ball of E. With the same notation as above, consider the decompositions $\phi_k = h_k\pi_k$ and the elements $h_{k,\epsilon}$, $\bar{h}_{k,\epsilon}$, $\pi_{k,\epsilon}$, for $k = 1, 2$. Assume that, for every $\delta > 0$, we have $h_{k,\delta} \neq 1$ for $k = 1, 2$, otherwise the following argument works but with minor changes. Fix a δ small enough so that $\pi_{1,\delta}$ is non-zero, hence injective, with inverse $\lambda_\delta = \pi_{1,\delta}^{-1} : \pi_{1,\delta}(A) \to A$. We define a u.c.p. map $\rho_{1,\delta}$ from the operator system

$$E_{1,\delta} := \phi_{1,\delta}(E) + \mathbb{C}1 = \phi_{1,\delta}(E) + \mathbb{C}(1 - h_{1,\delta})$$

to A as follows,

$$\rho_{1,\delta}(\phi_{1,\delta}(e) + \lambda(1 - h_{1,\delta})) = \lambda_\delta(p_{1,2\delta}(\phi_{1,\delta}(e) + \lambda(1 - h_{1,\delta}))p_{1,2\delta}).$$

Since $1 - h_{1,\delta}$ and $p_{1,2\delta}$ are orthogonal, this is equal to e for any $e \in E$. Now fix any state ω on A and consider the *unitisation* of $\phi_{2,\delta}$ given by

$$a \mapsto \phi_{2,\delta}(a) + \omega(a)(1 - h_{2,\delta}).$$

The composition of $\rho_{1,\delta}$ followed by this map gives a u.c.p. map $\eta : E_{1,\delta} \to B$. By Lemma 3.56 one can find $\tilde{\eta} : B \to B$ u.c.p. such that

$$\left\|\tilde{\eta}|_{E_{1,\delta}} - \eta\right\| \leq \tfrac{\epsilon}{6}.$$

In particular, since $\eta(1-h_{1,\delta})=0$, we have the estimate

$$\|\tilde{\eta}(1-h_{1,\delta})\| \leq \tfrac{\epsilon}{6}.$$

By Lemma 3.55 we can find an isometry $s\in B$ such that

$$\|s^*xs-\tilde{\eta}(x)\| \leq \tfrac{\epsilon}{6}\,\|x\|$$

for any $x\in E_{1,\delta}$. Therefore, one has that

$$\|s^*[\phi_{1,\delta}(e)+\lambda(1-h_{1,\delta})]s-s^*\phi_{1,\delta}(e)s\| \leq \tfrac{2\epsilon}{6}|\lambda| \leq \tfrac{\epsilon}{3},$$

for any $e\in E$ and $\lambda\in\mathbb{C}$ with $\|e\|\leq 1$ and $|\lambda|\leq 1$. It follows that

$$\left\|h_2^{1/2}s^*\phi_{1,\delta}(e)sh_2^{1/2}-h_2^{1/2}\phi_{2,\delta}(e)h_2^{1/2}\right\| \leq \tfrac{2\epsilon}{3}$$

whenever e is in the unit ball of E. Furthermore, we have

$$\|h_2(1-h_{2,\delta})\| = \|h_2-h_2h_{2,\delta}\| \leq \delta,$$

so that, choosing $\delta\leq\frac{\epsilon}{6}$, we find that

$$\left\|h_2^{1/2}\phi_{2,\delta}(e)h_2^{1/2}-\phi_2(e)\right\| \leq \tfrac{\epsilon}{6}\,\|e\|$$

for all $e\in E$. By identifying $\phi_{1,\delta}$ with $\bar{h}_{1,\delta}\phi_1$ we finally get

$$\left\|h_2^{1/2}s^*\bar{h}_{1,\delta}^{1/2}\phi_1(e)\bar{h}_{1,\delta}^{1/2}sh_2^{1/2}-\phi_{2,\delta}(e)\right\| \leq \epsilon,$$

for any e in the unit ball of E, so that $b=\bar{h}_{1,\delta}^{1/2}sh_2^{1/2}$ is as required. □

If A is not simple in the lemma above, the result still holds but the proof needs to be modified as follows. One replaces the lift $\pi_{1,\delta}^{-1}$ with a local lift $\lambda_\delta : E/\ker\pi_{1,\delta}\to A$, depending on E and δ. Hence $\lambda_\delta\circ\pi_{1,\delta}(e)=e+j_\delta(e)$, for some $j_\delta(e)\in J_\delta:=\ker\pi_{1,\delta}$. Note that $\|j_\delta(e)\|\leq 2\,\|e\|$ and that for any bounded linear functional ω we have $\|\omega|_{J_\delta}\|\to 0$ as $\delta\to 0^+$. This follows from the fact that $\bigcap_\delta J_\delta=\{0\}$. The same is true for every u.c.p. map that goes into a finite dimensional C*-algebra. Using nuclearity of the map $\phi_2 : A\to B$ we can conclude the proof similarly to the remaining part of the proof above.

For any C*-algebra A, let $\mathcal{J}(A)$ denote the lattice of its two-sided closed ideals. This set can be turned into an Abelian monoid by introducing the binary operation $+$ defined as

$$I+J := I\cap J,$$

i.e. as the intersection of ideals. The neutral element of $\mathcal{J}(A)$ is clearly provided by the improper ideal A. Furthermore, $\mathcal{J}(A)$ can be turned into an partially ordered Abelian monoid by the partial order $\leq$ defined as

$$I \leq J \qquad \Longleftrightarrow \qquad J \subset I.$$

It is easy to see that $(\mathcal{J}(A), \leq)$ is closed under suprema, i.e. every increasing sequence in $\mathcal{J}(A)$ admits a least upper bound according to the partial order given by $\leq$.

Theorem 3.58. *Let A be a unital, exact C*-algebra and let B be a unital Kirchberg algebra. Then there is a partially ordered Abelian monoid isomorphism $\mathsf{Cu}(A, B) \cong \mathcal{J}(A)$, which is explicitly given by*

$$\Phi \mapsto \ker \phi,$$

where $\phi : A \to B \otimes K$ is any c.p.c. order zero map in the class Φ.

Proof. Firstly we show that any c.p.c. order zero map $\phi : A \to B \otimes K$ is Cuntz-equivalent to a c.p.c. order zero map with range in $B \otimes e$, where $e \in K$ is any minimal projection. To this end, choose a sequence of pairwise orthogonal projections $p_0, p_1, p_2, \ldots$ each of which are Murray-von Neumann equivalent to the unit of B. Such a sequence exists by, e.g., an induction argument. Let

$$B_0 = \lim_{n\to\infty}(p_0 + \cdots + p_n)B(p_0 + \cdots + p_n) \subset B \otimes e.$$

Then there exists a sequence of partial isometries $v_n \in B \otimes K$ such that $v_n v_n^* = (p_0 + \cdots + p_n) \otimes e$ and $v_n^* v_n = 1 \otimes q_n$, with $e \leq q_n$, $q_n < q_{n+1}$, and v_{n+1} extends v_n, for any $n \in \mathbb{N}$. It follows that $v_n \phi v_n^*$ converges point-wise to an order zero map $\phi_0 : A \to B_0 \otimes e \subset B \otimes e$ and therefore $\{v_n\}_{n\in\mathbb{N}}$ implements the sought equivalence between ϕ and ϕ_0. This shows that every class in $\mathsf{Cu}(A, B)$ has a representative of the form $\phi \otimes e$, where $\phi : A \to B$ is c.p.c. order zero. From Lemma 3.57 one sees that two order zero maps $\phi, \psi : A \to B$ are equivalent if and only if they have the same kernel. Hence, the Cuntz-equivalence classes are in a one-to-one correspondence with the elements of $\mathcal{J}(A)$. To see that this correspondence is also onto, one can apply the same argument as in the simple case to obtain an embedding of the quotient A/J, with $J \in \mathcal{J}(A)$, which is an exact C*-algebra, into B. Furthermore, it is clear that the class of the direct sum of two c.p.c. order zero maps corresponds to the intersection of the corresponding ideals. □

3.6.2 Strongly Self-absorbing C*-algebras

The bivariant Cuntz semigroup exhibits an interesting behaviour of stability when its two arguments are tensored by a strongly self-absorbing C*-algebra. This result, of general interest on its own, can be exploited to determine some bivariant Cuntz semigroups.

As recalled with Proposition 1.13, for every strongly self-absorbing C*-algebra $\mathcal{D}$ there is a unital $*$-homomorphism $\gamma : \mathcal{D}\otimes\mathcal{D} \to \mathcal{D}$ which is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}}$. Hence, in what follows, we shall assume that every strongly self-absorbing C*-algebra $\mathcal{D}$ comes equipped with such a map γ. Our notion of approximate unitary equivalence between c.p.c. maps is the same as that used in [70, Definition 1.1], which we have recalled with Definition 1.12 for the reader's convenience. We also refer to some results collected in [70, Proposition 1.2] that are used in this section, along with Lemma 1.14. Here we add the following result to the list, which links approximate unitary equivalence and the Cuntz comparison of c.p.c. order zero maps.

Proposition 3.59. *Let A and B be separable C*-algebras, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps such that $\phi \approx_{\text{a.u.}} \psi$. Then $\phi \sim \psi$.*

Proof. The sequence of unitaries $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(B)$ used to witness the approximate unitary equivalence $\phi \approx_{\text{a.u.}} \psi$ can be cut down to a sequence in B by an approximate unit $\{e_n\}_{n\in\mathbb{N}} \subset B$, which then witnesses Cuntz (sub)equivalence. □

As a consequence of the above result and Proposition 3.24 we have that, if A, B and C are separable local C*-algebras, and $\phi, \psi : A \to B$, $\eta : B \to C$ are c.p.c. order zero maps such that $\phi \approx_{\text{a.u.}} \psi$, then $\eta \circ \phi \sim \eta \circ \psi$.

The following theorem can be found stated without proof in the notes [73] and involves the bivariant Cuntz semigroup Cu as defined in this chapter, there denoted by W. We first give a proof that involves our bifunctor W, and retrieve the actual result of [73] as a corollary to our more general statement.

Theorem 3.60. *Let A, B and $\mathcal{D}$ be separable C*-algebras, $\mathcal{D}$ strongly self-absorbing. The following isomorphism holds,*

$$W(A \otimes \mathcal{D}, B \otimes \mathcal{D}) \cong W(A, B \otimes \mathcal{D}).$$

Proof. Since $\mathcal{D}$ is strongly self-absorbing, any isomorphism $\phi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ induces an isomorphism between $W(A, B \otimes \mathcal{D})$ and $W(A, B \otimes \mathcal{D} \otimes \mathcal{D})$ by functoriality. Therefore, it is enough to show that $W(A, B\otimes\mathcal{D})$ is isomorphic to $W(A\otimes\mathcal{D}, B\otimes\mathcal{D}\otimes\mathcal{D})$. We claim that the maps[2]

$$\begin{aligned} W(A, B\otimes\mathcal{D}) &\longrightarrow W(A\otimes\mathcal{D}, B\otimes\mathcal{D}\otimes\mathcal{D}) \\ [\phi] &\longmapsto [\phi\otimes\mathrm{id}_{\mathcal{D}}] \end{aligned}$$

and

$$\begin{aligned} W(A\otimes\mathcal{D}, B\otimes\mathcal{D}\otimes\mathcal{D}) &\longrightarrow W(A, B\otimes\mathcal{D}) \\ [\psi] &\longmapsto [(\mathrm{id}_{B\otimes K}\otimes\gamma)\circ\psi\circ(\mathrm{id}_A\otimes 1_{\mathcal{D}})] \end{aligned}$$

are mutual inverses, where $\gamma:\mathcal{D}\otimes\mathcal{D}\to\mathcal{D}$ is the unital $*$-homomorphism that comes with $\mathcal{D}$. Indeed, by a repeated use of Lemma 1.14, we have

$$\begin{aligned} (\mathrm{id}_{B\otimes K}\otimes\gamma)\circ(\phi\otimes\mathrm{id}_{\mathcal{D}})\circ(\mathrm{id}_A\otimes 1_{\mathcal{D}}) &= (\mathrm{id}_{B\otimes K}\otimes\gamma)\circ(\mathrm{id}_{B\otimes K}\otimes\mathrm{id}_{\mathcal{D}}\otimes 1_{\mathcal{D}})\circ\phi \\ &\approx_{\text{a.u.}} (\mathrm{id}_{B\otimes K}\otimes\mathrm{id}_{\mathcal{D}})\circ\phi \\ &= \phi, \end{aligned}$$

and

$$\begin{aligned} ((\mathrm{id}_{B\otimes K}\otimes\gamma)\circ\psi\circ(\mathrm{id}_A\otimes 1_{\mathcal{D}}))\otimes\mathrm{id}_{\mathcal{D}} &= (\mathrm{id}_{B\otimes K}\otimes\gamma\otimes\mathrm{id}_{\mathcal{D}})\circ(\psi\otimes\mathrm{id}_{\mathcal{D}})\circ(\mathrm{id}_A\otimes 1_{\mathcal{D}}\otimes\mathrm{id}_{\mathcal{D}}) \\ &\sim (\mathrm{id}_{B\otimes K}\otimes\gamma\otimes\mathrm{id}_{\mathcal{D}})\circ(\psi\otimes\mathrm{id}_{\mathcal{D}})\circ(\mathrm{id}_A\otimes\mathrm{id}_{\mathcal{D}}\otimes 1_{\mathcal{D}}) \\ &= (\mathrm{id}_{B\otimes K}\otimes\gamma\otimes\mathrm{id}_{\mathcal{D}})\circ(\psi\otimes 1_{\mathcal{D}}) \\ &= (\mathrm{id}_{B\otimes K}\otimes\gamma\otimes\mathrm{id}_{\mathcal{D}})\circ(\mathrm{id}_K\otimes\mathrm{id}_{\mathcal{D}}\otimes\mathrm{id}_{\mathcal{D}}\otimes 1_{\mathcal{D}})\circ\psi \\ &\approx_{\text{a.u.}} (\mathrm{id}_{B\otimes K}\otimes\gamma\otimes\mathrm{id}_{\mathcal{D}})\circ(\mathrm{id}_K\otimes\mathrm{id}_{\mathcal{D}}\otimes 1_{\mathcal{D}}\otimes\mathrm{id}_{\mathcal{D}})\circ\psi \\ &\approx_{\text{a.u.}} (\mathrm{id}_{B\otimes K}\otimes\mathrm{id}_{\mathcal{D}}\otimes\mathrm{id}_{\mathcal{D}})\circ\psi \\ &= \psi, \end{aligned}$$

which reduce to equalities at the level of the Cuntz semigroups, since $\approx_{\text{a.u.}}$ implies Cuntz equivalence $\sim$ by Proposition 3.59. □

The result stated in [73] now follows as a corollary to the above proposition and, with the notation introduced in this thesis, takes the following form.

Corollary 3.61. *Let A, B and $\mathcal{D}$ be separable C^*-algebras, $\mathcal{D}$ strongly self-absorbing. The following isomorphism holds,*

$$\mathsf{Cu}(A\otimes\mathcal{D}, B\otimes\mathcal{D})\cong\mathsf{Cu}(A, B\otimes\mathcal{D}).$$

[2]here $\mathrm{id}_{B\otimes K}$ is used instead of the identity map on $M_\infty(B)\subset B\otimes K$.

Proof. Thanks to Equation 3.1 we have the following chain of isomorphisms

$$\begin{aligned}\mathsf{Cu}(A\otimes\mathcal{D},B\otimes\mathcal{D}) &\cong W(A\otimes\mathcal{D},B\otimes\mathcal{D}\otimes K)\\ &\cong W(A,B\otimes\mathcal{D}\otimes K)\\ &\cong \mathsf{Cu}(A,B\otimes\mathcal{D}).\end{aligned}$$ □

The above theorem can be used to explicitly compute some bivariant Cuntz semigroups when strongly self-absorbing C*-algebras of the stably finite type are considered. For any UHF algebra of the infinite type we have the following result.

Example 3.62. Let U be any UHF algebra of infinite type. Then $W(U,U)\cong W(U)$. In particular, if $\mathcal{Q}$ is the universal UHF algebra, then $W(U,\mathcal{Q})\cong W(\mathcal{Q})$. △

A rather important class of C*-algebras which play a central rôle in the current theory of classification is that of the so-called $\mathcal{Z}$-stable C*-algebras, i.e. C*-algebras A that satisfy the isomorphism $A\otimes\mathcal{Z}\cong\mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra. For this class of C*-algebras we have the following result.

Example 3.63. Let A be any separable C*-algebra, and let $\mathcal{Z}$ be Jiang-Su algebra. Then $W(\mathcal{Z},A\otimes\mathcal{Z})\cong W(A\otimes\mathcal{Z})$. In particular, if A is $\mathcal{Z}$-stable, then $W(\mathcal{Z},A\otimes\mathcal{Z})\cong W(A)$. △

The following example uses the Jiang-Su algebra $\mathcal{Z}$ again to show that, if $\mathcal{D}$ is a strongly self-absorbing C*-algebra, one cannot expect the isomorphism

$$W(A\otimes\mathcal{D},B\otimes\mathcal{D})\cong W(A\otimes\mathcal{D},B)$$

to hold in general.

Example 3.64. Consider the bivariant Cuntz semigroup $W(\mathcal{Z},\mathbb{C})$. As there are no c.p.c. order zero maps from $\mathcal{Z}\to M_\infty(\mathbb{C})$, which would imply the existence of finite-dimensional representations of $\mathcal{Z}$ (the same holds if $M_\infty(\mathbb{C})$ is replaced by the C*-algebra of compact operators K, hence for Cu in place of W), one has

$$W(\mathcal{Z},\mathbb{C})=\{0\}.$$

On the other hand

$$\begin{aligned}W(\mathcal{Z},\mathcal{Z}) &\cong W(\mathbb{C},\mathcal{Z})\\ &\cong W(\mathcal{Z})\\ &\cong \mathbb{N}\sqcup\mathbb{R}^+\end{aligned}$$

whence $W(\mathcal{Z},\mathbb{C})\ncong W(\mathcal{Z},\mathcal{Z})$. △

3.6.3 Cuntz Homology for Compact Hausdorff Spaces

In this section we give an explicit computation of the special bivariant Cuntz semigroup $\mathsf{Cu}(C(X), \mathbb{C})$, which can be regarded as a first step towards a Cuntz-analogue of K-homology for compact Hausdorff spaces in the setting of the Cuntz theory. Throughout this section we let X denote a compact and metrisable Hausdorff space, unless otherwise stated.

We observe that, if $\phi : A \to B$ is a c.p.c. order zero map between C*-algebras, its kernel coincides with that of its support $*$-homomorphism π_ϕ. When $A = C(X)$ then $\ker\phi$ can be identified with a closed subspace C_ϕ of X, on which every function in $\ker\phi$ vanishes exactly.

Definition 3.65. The spectrum $\sigma(\phi)$ of a c.p.c. order zero map $\phi : C(X) \to K$ is the closed subset $C_\phi \subset X$ associated to the kernel of ϕ, i.e.

$$\sigma(\phi) := \{x \in X \mid f(x) = 0\ \forall f \in \ker\phi\}.$$

It is convenient to split the set of isolated points of the spectrum of a c.p.c. order zero map from $C(X)$ to K from the set of accumulation points. The former is denoted by $\sigma_i(\phi)$, while the latter is defined as $\sigma_{\text{ess}}(\phi) := \sigma(\phi) \setminus \sigma_i(\phi)$. Our notation follows the usual definition of the essential spectrum of a normal operator, with the only difference that here we do not include isolated points with infinite multiplicity in it. We could have aligned our terminology to the standard one, but our choice simplifies the notation in some of the proofs that follow. If x is an isolated point from a subset C of X, then there exists a neighbourhood U of x that does not contain other points of C. By Urysohn's Lemma one can then find a continuous function $\tilde{\chi}_{\{x\}} \in C(X)$ that vanishes on the outside of U and such that $\tilde{\chi}_{\{x\}}(x) = 1$. We will use this fact to provide continuous indicator functions $\tilde{\chi}$ for isolated points of subsets in the relative topology.

Definition 3.66 (Multiplicity function)**.** Let $\phi : C(X) \to K$ be a c.p.c. order zero map. The multiplicity function ν_ϕ of ϕ is the map from X to $\mathbb{N}_0 \cup \{\infty\}$ given by

$$\nu_\phi(x) = \begin{cases} 0 & x \notin \sigma(\phi) \\ \infty & x \in \sigma_{\text{ess}}(\phi) \\ \operatorname{rk}\pi_\phi(\tilde{\chi}_{\{x\}}) & x \in \sigma_i(\phi). \end{cases}$$

Let $\phi : C(X) \to K$ be a c.p.c. order zero map. Then by Theorem 1.5 there exists a compact operator $h = \phi(1_{C(X)})$ that commutes with a representation $\pi_\phi : C(X) \to B(\ell^2(\mathbb{N}))$ and such that $\phi(f) = h\pi(f)$. From the theory of representations of commutative C*-algebras it follows that there exists a dense sequence $\{x_n\}_{n\in\mathbb{N}} \subset \sigma(\phi)$ such that

$$\pi_\phi(f) = \bigoplus_{i\in\mathbb{N}} f(x_i) \qquad \forall f \in C(X).$$

The multiplicity function ν_ϕ associated to ϕ gives the number of occurrence of every x_i in the sequence, with the assumption that every accumulation point has, by definition, infinite multiplicity. A reordering of the elements in $\{x_n\}_{n\in\mathbb{N}}$ gives a new representation of $C(X)$ which is clearly unitarily equivalent to π_ϕ. Therefore, up to unitary equivalence, one can split π_ϕ into

$$\pi_\phi = \pi_{\phi,i} \,\hat{\oplus}\, \pi_{\phi,\mathrm{ess}}, \tag{3.2}$$

where

$$\pi_{\phi,i}(f) := \bigoplus_{x\in\sigma_i(\phi)} f(x)\,\mathrm{id}_{\nu_\phi(x)}$$

and

$$\pi_{\phi,\mathrm{ess}}(f) := \bigoplus_{x\in\sigma_{\mathrm{ess}}(\phi)} f(x)\,\mathrm{id}_{n_x},$$

with id_n denoting the unit of M_n for every $n \in \mathbb{N}$, $\mathrm{id}_0 = 0$ and $\mathrm{id}_\infty := \mathrm{id}_{B(\ell^2)}$ by definition, and with $n_x \in \mathbb{N} \cup \{\infty\}$ denoting the number of occurrences of x inside $\{x_n\}_{n\in\mathbb{N}}$. To this decomposition of representations corresponds a decomposition of the associated order zero map of the analogous form

$$\phi = \phi_i \,\hat{\oplus}\, \phi_{\mathrm{ess}}, \tag{3.3}$$

with obvious meaning of the symbols.

Lemma 3.67. *Let $\phi, \psi : C(X) \to K$ be c.p.c. order zero maps such that $\sigma_{\mathrm{ess}}(\phi) = \sigma_{\mathrm{ess}}(\psi) = \varnothing$. Then $\phi \precsim \psi$ if and only if $\nu_\phi \leq \nu_\psi$.*

Proof. If $\phi \precsim \psi$ and $\eta_\phi > \eta_\psi$ then there exists at least one more $x \in \sigma_i(\phi)$ than the ones appearing in the decomposition of ψ_i and it is immediate to conclude that there cannot be a sequence that witnesses $\phi \precsim \psi$. Conversely, if $\nu_\phi \leq \nu_\psi$ then there clearly is a unitary that conjugates π_ϕ to a subrepresentation of π_ψ. To witness $\phi \precsim \psi$ it is enough to rescale every basis vector by the appropriate factors coming from the eigenvalues of h_ϕ and h_ψ and combine this with the unitary, together with an approximate unit for $K \subset B(H)$. ☐

The following result shows that the multiplicity of every point in the essential spectrum of a c.p.c. order zero map $\phi : C(X) \to K$ is irrelevant since, by accumulation and the continuity of the functions in $C(X)$, they can be replaced by nearby points in $\sigma_{\mathrm{ess}}(\phi)$.

Lemma 3.68. *Let $\phi : C(X) \to K$ be a c.p.c. order zero map with $\sigma_i(\phi) = \varnothing$ and let $\{y_n\}_{n\in\mathbb{N}}$ be a dense sequence of $\sigma_{\mathrm{ess}}(\phi)$. Then there exists $\{v_n\}_{n\in\mathbb{N}} \subset B(H)$ such that $\mathrm{Ad}_{v_n} \circ \bigoplus_{i\in\mathbb{N}} \mathrm{ev}_{y_i}$ converges to π_ϕ in the point-norm topology.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be the dense sequence in $\sigma_{\mathrm{ess}}(\phi)$ such that $\pi_\phi = \bigoplus_{i\in\mathbb{N}} \mathrm{ev}_{x_i}$ and fix a countable neighbourhood basis $B(x_i) = \{U_{n,i}\}_{n\in\mathbb{N}}$ for every $i \in \mathbb{N}$. For every $n, i \in \mathbb{N}$ choose a point $y_{k_i^{(n)}} \in U_{n,i}$ which is not one of the points of $\{y_n\}_{n\in\mathbb{N}}$ chosen previously. This determines a subsequence $\{y_{k_i^{(n)}}\}$ of $\{y_n\}_{n\in\mathbb{N}}$ for every n such that $y_{k_i^{(n)}} \to x_i$ as $n \to \infty$ for every $i \in \mathbb{N}$. Therefore there is an isometry v_n that conjugates $\bigoplus_{i\in\mathbb{N}} \mathrm{ev}_{y_i}$ to $\bigoplus_{i\in\mathbb{N}} \mathrm{ev}_{y_{k_i^{(n)}}}$ for every $n \in \mathbb{N}$. Hence, on every continuous function one has $f(y_{k_i^{(n)}}) \to f(x_i)$ as $n \to \infty$, and therefore

$$\left\| \pi_\phi(f) - \bigoplus_{i\in\mathbb{N}} f(y_{k_i^{(n)}}) \right\| = \sup_{i\in\mathbb{N}} |f(x_i) - f(y_{k_i^{(n)}})| \to 0,$$

which proves the assertion. □

By the symmetric rôle of the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in the above lemma, it follows that there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subset B(H)$ such that $\mathrm{Ad}_{w_n} \circ \pi_\phi$ converges to $\bigoplus_{i\in\mathbb{N}} \mathrm{ev}_{y_i}$ in the point-norm topology. Hence, with abuse of terminology, we can say that π_ϕ and $\bigoplus_{i\in\mathbb{N}} \mathrm{ev}_{y_i}$ are "Cuntz equivalent", so that the class of ϕ depends only on the closure of the sequences. The above lemma can be used to prove a *complement* of Lemma 3.67.

Lemma 3.69. *Let $\phi, \psi : C(X) \to K$ be c.p.c. order zero maps such that $\sigma_i(\phi) = \sigma_i(\psi) = \varnothing$. Then $\phi \precsim \psi$ if and only if $\nu_\phi \leq \nu_\psi$.*

Proof. If $\phi \precsim \psi$ and $\eta_\phi > \eta_\psi$ then there exists a point $x \in \sigma_{\mathrm{ess}}(\phi)$ which is not in $\sigma_{\mathrm{ess}}(\psi)$. Since the space X is Hausdorff there exists a neighbourhood U of x which has empty intersection with $\sigma_{\mathrm{ess}}(\phi)$ and by an argument similar to the proof of Lemma 3.67 one can conclude that there cannot be a sequence that witnesses $\phi \precsim \psi$. Conversely, if $\nu_\phi \leq \nu_\psi$ then there exists a projection E such that $E\pi_\psi E$ is "Cuntz equivalent" to π_ϕ as a consequence of the previous lemma. To witness $\phi \precsim \psi$ it is enough to rescale every basis vector by the appropriate factors coming from the eigenvalues of h_ϕ and h_ψ and combine the sequences of said "Cuntz equivalence", together with an approximate unit for $K \subset B(H)$ in order to cut these sequences down to K. ☐

If $\{x_n\}_{n\in\mathbb{N}} \subset \sigma_{\text{ess}}(\phi)$ is a dense sequence in the essential spectrum of a c.p.c. order zero map ϕ, then $\{x_1, x_1, x_2, x_2, \ldots\}$ is also dense and therefore $\phi \cong \phi_{\text{ess}} \hat{\oplus} \phi$. Furthermore, the proof of the above lemma, which relies on Lemma 3.68, easily adapts to the case of c.p.c. order zero maps ϕ and ψ for which $\sigma_i(\psi) = \sigma_{\text{ess}}(\phi) = \varnothing$ and $\sigma_i(\phi) \subset \sigma_{\text{ess}}(\psi)$, at the price of having a sequence of partial isometries rather than isometries in general.

Theorem 3.70. *Let $\phi, \psi : C(X) \to K$ be c.p.c. order zero maps. Then $\phi \precsim \psi$ if and only if $\nu_\phi \leq \nu_\psi$.*

Proof. With the above remarks in mind, and the decomposition of Equation (3.3), it is enough to decompose ϕ and ψ in

$$\phi = \phi_i \hat{\oplus} \phi_{\text{ess}} \qquad \text{and} \qquad \psi = \psi_i \oplus \hat{\oplus} \psi_{\text{ess}}.$$

Now ϕ_i can be decomposed further according to $\sigma(\psi)$ into

$$\phi_i = \phi_{i,\text{ess}} \hat{\oplus} \phi_{i,i},$$

where $\sigma(\phi_{i,\text{ess}}) \subset \sigma_{\text{ess}}(\psi)$ and $\sigma(\phi_{i,i}) \subset \sigma_i(\psi)$. By replacing ψ with the equivalent map $\psi_{\text{ess}} \hat{\oplus} \psi$ we then have the decompositions

$$\phi = \phi_{\text{ess}} \hat{\oplus} \phi_{i,\text{ess}} \hat{\oplus} \phi_{i,i} \qquad \text{and} \qquad \psi = \psi_{\text{ess}} \hat{\oplus} \psi_{\text{ess}} \hat{\oplus} \psi_i.$$

It is now enough to apply Lemma 3.67 to $\phi_{i,i}$ and ψ_i, and Lemma 3.69 to the remaining direct summands, to get to the sought conclusion. ☐

Before proceeding with the proof that Cuntz homology, as defined in this section, is a complete invariant for compact Hausdorff spaces, we make the observation that any function $\nu \in \tilde{\mathbb{N}}_0^X$, where $\tilde{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$ with compact support can be split into the sum of two functions, ν_i and ν_{ess}, with disjoint compact supports, such that the former is supported by the isolated points of $\operatorname{supp}\nu$ and the latter on the rest of $\operatorname{supp}\nu$.

Definition 3.71. The set of multiplicity functions $\operatorname{Mf}(X)$ over X is the subset of $\tilde{\mathbb{N}}_0^X$ given by

$$\operatorname{Mf}(X) := \{\nu \in \tilde{\mathbb{N}}_0^X \mid \operatorname{supp}\nu = \overline{\operatorname{supp}\nu} \ \wedge \nu_{\text{ess}}(X) = \{\infty\}\},$$

Observe that $\mathrm{Mf}(X)$ has a natural structure of partially ordered Abelian monoid when equipped with the point-wise operation of addition and partial order inherited from $\tilde{\mathbb{N}}_0$, and that every multiplicity function ν_ϕ of a c.p.c. order zero map $\phi : C(X) \to K$ is an element of $\mathrm{Mf}(X)$. It is clear from the results thus obtained that every multiplicity function in $\mathrm{Mf}(X)$ is associated to a representation of $C(X)$ onto K and hence to a c.p.c. order zero map, which is unique up to Cuntz equivalence. If we insist on calling $\mathsf{Cu}(C(X), \mathbb{C})$ the Cuntz homology of X we then have the following result.

Corollary 3.72. *The Cuntz homology of X is order isomorphic to the partially ordered Abelian monoid* $\mathrm{Mf}(X)$.

For the bivariant Cuntz semigroup $W(C(X), \mathbb{C})$ one sees that the only representations of $C(X)$ involved are finite dimensional, for the positive element h_ϕ of a c.p.c. order zero map $\phi : C(X) \to M_\infty$ is actually a matrix in M_n for some $n \in \mathbb{N}$. Therefore, if we denote by $\mathrm{Mf}_i(X)$ the subsemigroup of $\mathrm{Mf}(X)$ given by

$$\mathrm{Mf}_i(X) := \{\nu \in \mathbb{N}_0^X \mid |\operatorname{supp}\nu| < \infty\},$$

i.e. all the finitely supported multiplicity functions over X with values in $\mathbb{N}_0$, then $W(C(X), \mathbb{C}) \cong \mathrm{Mf}_i(X)$ as semigroups. As the following result shows, the monoid $\mathrm{Mf}(X)$ can be regarded as the sup-completion of $\mathrm{Mf}_i(X)$, for any compact Hausdorff space X.

Proposition 3.73. *For every $\nu \in \mathrm{Mf}(X)$ there exists an increasing sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset \mathrm{Mf}_i(X)$ such that $\nu = \sup \nu_n$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset \operatorname{supp}\nu_{\mathrm{ess}}$ be a dense sequence, $Y \subset \operatorname{supp}\nu_i$ be such that $\nu_i(y) = \infty$ for any $y \in Y$, $Z := \operatorname{supp}\nu_i \smallsetminus Y$ and set

$$\nu_n(x) := \begin{cases} \nu_i(x) & x \in Z \\ n & x \in \{x_1, \dots, x_n\} \cup Y \\ 0 & \text{otherwise} \end{cases}, \qquad \forall x \in X.$$

Then $\nu_n \in \mathrm{Mf}_i(X)$ and $\nu_n \leq \nu$ for every $n \in \mathbb{N}$. Suppose that $\mu \in \mathrm{Mf}(X)$ is such that $\nu_n \leq \mu$ for any $n \in \mathbb{N}$, then $\operatorname{supp}\nu_n \subset \operatorname{supp}\mu$ for all $n \in \mathbb{N}$ and by the closedness of the supports and the density of $\{x_n\}_{n\in\mathbb{N}}$ in $\operatorname{supp}\nu_{\mathrm{ess}}$ it follows that $\operatorname{supp}\nu \subset \operatorname{supp}\mu$. This inclusion implies that $\nu_{\mathrm{ess}} \leq \mu_{\mathrm{ess}}$, while $\nu_i \leq \mu$ by construction of the sequence $\{\nu_n\}_{n\in\mathbb{N}}$, whence $\nu = \nu_i + \nu_{\mathrm{ess}} \leq \mu$. ∎

Remark. The notation in the above proof would have been a bit simpler if, in our definition of essential spectrum of a c.p.c. order zero map, we included the isolated points with infinite multiplicity.

We now proceed to prove that Cuntz homology, as defined in this section, provides a complete invariant for compact Hausdorff spaces, that is, $\mathrm{Mf}(X) \cong \mathrm{Mf}(Y)$ if and only if X and Y are homeomorphic. To this end we first observe that any semigroup $\mathrm{Mf}(X)$ can be described as a quotient of countably supported functions on X taking values in $\tilde{\mathbb{N}}_0$. The equivalence relation is provided by checking that two functions agree in value on the isolated points and have the same closure of the accumulation points, where the functions take the value ∞. If $\mathrm{Lf}(X)$ denotes the set of all such countably supported functions over X and $\sim$ is said equivalence relation, then it is immediate to see that

$$\mathrm{Mf}(X) \cong \mathrm{Lf}(X)/\sim .$$

An element f of $\mathrm{Lf}(X)$ can be represented as a formal sum

$$f = \sum_{k=1}^{\infty} n_k \delta_{x_k},$$

where $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a sequence of points from X, $\{n_n\}_{n\in\mathbb{N}} \subset \tilde{\mathbb{N}}_0$ is such that each $n_k = \infty$ whenever x_k is an accumulation point and δ_{x_k} is the function that takes value 1 on x_k and 0 everywhere else. The quotient map $\pi : \mathrm{Lf}(X) \to \mathrm{Mf}(X)$ can then be represented as the formal sum

$$\pi(f) = \sum_{i=1}^{\infty} n_{k_i} \delta_{x_{k_i}} + C,$$

where C is the closure of all the accumulation points of the sequence $\{x_n\}_{n\in\mathbb{N}}$, and the sum is on the isolated points only.

The topology τ_X of the space X can be recovered from the knowledge of $\mathrm{Mf}(X)$ as follows. For every two multiplicity functions $f, g \in \mathrm{Mf}(X)$ we say that f is τ-equivalent to g, in symbols $f \sim_\tau g$, if $\operatorname{supp} f = \operatorname{supp} g$. It is easy to see that the quotient

$$\mathrm{T}(X) := \mathrm{Mf}(X)/\sim_\tau$$

is in a bijective correspondence with all the closed subsets of X and hence with the topology τ_X on X. The set $\mathrm{T}(X)$ can also be identified with the family of those functions ω in $\mathrm{Mf}(X)$ whose range is in the set $\{0, \infty\}$. Such functions have the absorption property

$$\omega + f = \omega$$

for any $f \in \mathrm{Mf}(X)$ such that $\operatorname{supp} f \subset \operatorname{supp} \omega$, and the stability property

$$n\omega = \omega, \qquad \forall n \in \tilde{\mathbb{N}},$$

that characterise them uniquely, that is, if $\omega \in \mathrm{Mf}(X)$ has at least one of these two properties, then the range of ω is in $\{0, \infty\}$. The correspondence is explicitly realised by mapping such functions to their support, and an inverse is provided by sending a closed subset $C \subset X$ to the map

$$\omega_C(x) := \begin{cases} \infty & x \in C \\ 0 & x \notin C. \end{cases}$$

With this notation it is then clear that

$$f + f + \cdots = \omega_{\operatorname{supp} f},$$

for any $f \in \mathrm{Mf}(X)$.

Let now X and Y be compact Hausdorff spaces and assume that there exists a semigroup isomorphism $\eta : \mathrm{Mf}(X) \to \mathrm{Mf}(Y)$. For a fixed point $x \in X$, the map δ_x is sent to some function $\eta(\delta_x) \in \mathrm{Mf}(Y)$ that can be represented as

$$\eta(\delta_x) = \sum_{k=1}^{\infty} m_k \delta_{y_k} + C_Y.$$

By applying η^{-1} one gets

$$\delta_x = \sum_{i,k=1}^{\infty} m_k n_i^{(k)} \delta_{x_i^{(k)}} + \eta^{-1}(C_Y).$$

which means that the only contribution to the left-hand side is either coming from the double sum or $\eta^{-1}(C_Y)$. For the latter case, observe that $nC_y = C_y$ for any $n \in \tilde{\mathbb{N}}$ and therefore $\eta^{-1}(C_Y) = 0$, which implies that $C_y = \varnothing$. In the former case one has that $x_{i^*}^{(k^*)} = x$ for some $k^*, i^* \in \mathbb{N}$, and since η is bijective, there exists at least one $i \in \mathbb{N}$ for each $k \in \mathbb{N}$ such that $n_i^{(k)} \neq 0$. Hence $m_k = 0$ for all k but k^* and $\delta_x = \eta^{-1}(\delta_{y_{k^*}})$, which defines a bijective map $f : X \to Y$ with $f(x) = y_{k^*}$. It remains to show that X and Y have the same topology, and this can be done by checking that η gives a well-defined map between $\mathrm{T}(X)$ and $\mathrm{T}(Y)$. We observe that, for any closed subset $C \subset X$ we have $\eta(\omega_C) = \omega_{\eta_* C}$ for some closed subset $\eta_* C \subset Y$. This follows from the stability property of ω_C, whereby $n\omega_C = \omega_C$ for any $n \in \tilde{\mathbb{N}}$ implies $n\eta(\omega_C) = \eta(\omega_C)$ for any $n \in \tilde{\mathbb{N}}$. From the absorption property of ω_C we get that $\omega_C + \delta_x = \omega_C$ for any $x \in C$ implies that

$$\omega_{\eta_* C} + \delta_f(x) = \omega_{\eta_* C}$$

for any $x \in C$, whence $f(C) \subset \eta_* C$. But $\omega_{\eta_* C} + \delta_y = \omega_{\eta_* C}$ for any $y \in \eta_* C$, which implies that

$$\omega_C + \delta_f^{-1}(y) = \omega_C,$$

i.e. $y \in f(C)$, and so $\eta_* C = f(C)$. If $\phi, \psi \in \mathrm{Mf}(X)$ are such that $\operatorname{supp}\phi = \operatorname{supp}\psi$, then $\phi + \phi + \cdots = \psi + \psi + \cdots$, which implies that

$$\eta(\phi) + \eta(\phi) + \cdots = \eta(\psi) + \eta(\psi) + \cdots ,$$

and therefore $\operatorname{supp}(\eta(\phi)) = \operatorname{supp}(\eta(\psi)) = f(\operatorname{supp}\psi)$. This shows that η induces a well-defined bijection between $\mathrm{T}(X)$ and $\mathrm{T}(Y)$ and hence that f is a homeomorphism.

3.7 Classification Results

In this section we provide some classification results within the theory of the bivariant Cuntz semigroup developed in this thesis. Our main result is the classification of unital and stably finite C*-algebras by the existence of special invertible elements in the bivariant Cuntz semigroup. We start by arguing that the notion of invertibility and equivalence given in Section 3.4 is not strong enough to capture isomorphism in general. Following the classical result of Elliott [19] on the classification of AF algebras we argue that a notion of scale is somehow needed and we give a stronger notion of invertibility that gives invertible elements in the scale. Hence, two unital and stably finite C*-algebras A and B turn out to be isomorphic if and only if there is at least one of such elements in the bivariant Cuntz semigroup $\mathsf{Cu}(A,B)$. This result contains, as special cases, the already cited result of Elliott [19], as well as the more recent one on AI algebras [13] and on inductive limits of one-dimensional non-commutative CW complexes with trivial K_1-group of [60].

We start by recalling that, for every element $\Phi \in \mathsf{Cu}(A,B)$, there exists a c.p.c. order zero map $\phi : A \to B \otimes K$ such that $[\phi \otimes \mathrm{id}_K] = \Phi$ and that, by definition of invertible elements in Cu, $\Phi \in \mathsf{Cu}(A,B)$ is invertible if there exists a c.p.c. order zero map $\psi : B \otimes K \to A \otimes K$ such that $\psi \circ \phi \sim \mathrm{id}_{A\otimes K}$ and $\phi \circ \psi \sim \mathrm{id}_{B\otimes K}$, for any representative $\phi \in \Phi$. As in the case of K-theory, where the unordered K_0-group is only capable of capturing stable isomorphisms between AF algebras, this notion of invertibility presents the same sort of limitations, as shown by the following example.

Example 3.74. Let $n, m > 0$ be any natural numbers. Then $\mathsf{Cu}(M_n, M_m) \cong \tilde{\mathbb{N}}_0$ has an invertible element, namely 1, for any $n, m \in \mathbb{N}$. However M_m and M_n are isomorphic only when $m = n$; otherwise they are stably isomorphic for any n, m in $\mathbb{N}$. △

The example above shows that, in order to capture isomorphism, a stricter notion of invertibility is required. As a guiding principle we have Elliott's classification theory of AF algebras through their *dimension groups*, that is the collection of the *ordered* K_0-groups, its scale, and the class of the unit of the algebra in the unital case.

We observe that, in the stably finite and unital case, pairs of c.p.c. order zero maps which are invertible up to Cuntz equivalence are Cuntz equivalent to their support $*$-homomorphisms.

Proposition 3.75. *Let A, B be separable, unital and stably finite C^*-algebras. If $\phi : A \to B$ and $\psi : B \to A$ are two c.p.c. order zero maps such that $\psi \circ \phi \sim \mathrm{id}_A$ and $\phi \circ \psi \sim \mathrm{id}_B$ then there are unital $*$-homomorphisms $\pi_\phi : A \to B$ and $\pi_\psi : B \to A$ such that*

i. $[\pi_\phi] = [\phi]$ *and* $[\pi_\psi] = [\psi]$*;*

ii. $\pi_\psi \circ \pi_\phi \sim \mathrm{id}_A$ *and* $\pi_\phi \circ \pi_\psi \sim \mathrm{id}_B$*.*

Proof. By Theorem 1.5 we can find positive elements h_ϕ, h_ψ and $*$-homomorphisms π_ϕ, π_ψ such that $\phi = h_\phi \pi_\phi$ and $\psi = h_\psi \pi_\psi$. Evaluating on the unit of A and B respectively we get

$$h_\psi^{\frac{1}{2}} \pi_\psi(h_\phi) h_\psi^{\frac{1}{2}} \sim_{\mathsf{Cu}} 1_A \qquad \text{and} \qquad h_\phi^{\frac{1}{2}} \pi_\phi(h_\psi) h_\phi^{\frac{1}{2}} \sim_{\mathsf{Cu}} 1_B.$$

From the first relation on the left we get, by definition of $\sim_{\mathrm{Cu}}$, the existence of a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $x_n h_\psi^{\frac{1}{2}} \pi_\psi(h_\phi) h_\psi^{\frac{1}{2}} x_n^*$ converges to 1_A, and therefore $x_n h_\psi^{\frac{1}{2}} \pi_\psi(h_\phi) h_\psi^{\frac{1}{2}} x_n^*$ is eventually invertible. Hence, there exists $c \in A$ such that

$$x_n h_\psi^{\frac{1}{2}} \pi_\psi(h_\phi) h_\psi^{\frac{1}{2}} x_n^* c = 1_A$$

for sufficiently large values of n, which shows that x_n is right invertible. Since A is stably finite, it follows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually invertible, and therefore

$$h_\psi^{\frac{1}{2}} \pi_\psi(h_\phi) h_\psi^{\frac{1}{2}} x_n^* c x_n = 1_A,$$

which shows that h_ψ is also right invertible, hence invertible. By symmetry of the problem one also deduces the invertibility of h_ϕ. Since $\pi_\phi(a) = h_\phi^{-1}\phi(a) \in B$ for any $a \in A$, and analogously for π_ψ, one sees that π_ϕ and π_ψ satisfy (i) and (ii). Now set $p = \pi_\phi(1_A)$ and $q = \pi_\psi(1_B)$. Since $\pi_\psi(p) \sim_{\mathsf{Cu}} 1_A$ and $\pi_\phi(q) \sim_{\mathsf{Cu}} 1_B$, stably finiteness of A and B implies $\pi_\phi(q) = 1_B$ and $\pi_\psi(p) = 1_A$. Now $1_A - q$ is a positive element in A, but

$$\pi_\phi(1_A - q) = p - 1_B \leq 0,$$

which is possible only if $p = 1_B$. Similarly, one finds that $q = 1_A$, and therefore π_ϕ and π_ψ are unital. □

In order to lift an invertible element in a bivariant Cuntz semigroup it suffices to show the existence of representatives which are $*$-homomorphisms, but in a strict sense. These considerations motivate the following definition.

Definition 3.76 (Strictly invertible element)**.** An element $\Phi \in \mathsf{Cu}(A, B)$ is strictly invertible if there exist c.p.c. order zero maps $\phi : A \to B$ and $\psi : B \to A$ such that

i. $[\phi \otimes \mathrm{id}_K] = \Phi$;

ii. $\psi \circ \phi \sim \mathrm{id}_A$ and $\phi \circ \psi \sim \mathrm{id}_B$.

Observe that every $*$-isomorphism between two C*-algebras A and B induces a strictly invertible element in $\mathsf{Cu}(A, B)$. Hence, if there are no strictly invertible elements in $\mathsf{Cu}(A, B)$ then A and B cannot be isomorphic.

Definition 3.77 (Strict Cu-equivalence)**.** Two C*-algebras A and B are strictly Cu-equivalent if there exists a strictly invertible element in $\mathsf{Cu}(A, B)$.

Remark. The notions of strictly invertible elements and strict Cu-equivalence can be reformulated for the bivariant semigroup W as well. It is then safe to replace any occurrence of Cu with W to get to the same classification results that appear in this section.

Observe that any c.p.c. order zero map $\phi : A \to B$ induces an element in $\mathsf{Cu}(A, B)$ through the class of its ampliation, viz. $[\phi \otimes \mathrm{id}_K]$. To make tangible contact with the current theory of classification we also give the following definition.

Definition 3.78 (Scale of the bivariant Cuntz semigroup Cu)**.** The scale $\Sigma(\mathsf{Cu}(A, B))$ of the Bivariant Cuntz Semigroup $\mathsf{Cu}(A, B)$ is the set of all classes of c.p.c. order zero maps that arise from c.p.c. order zero maps from A to B through ∞-ampliation, i.e. the set

$$\Sigma(\mathsf{Cu}(A, B)) = \{[\phi \otimes \mathrm{id}_K] \in \mathsf{Cu}(A, B) \mid \phi : A \to B \text{ c.p.c. order zero}\}.$$

Example 3.79. Let B be a C*-algebra. Since any c.p.c. order zero map $\phi : \mathbb{C} \to B$ is generated by a positive element of B, i.e.

$$\phi(z) = zh_\phi, \qquad \forall z \in \mathbb{C},$$

for some positive element $h_\phi \in B^+$, one can identify the scale of $\mathsf{Cu}(\mathbb{C}, B)$ with the Cuntz equivalence classes of the elements of B embedded in $B \otimes K$ through a minimal projection

e of K. Apart from a sup-completion, $\Sigma(\mathsf{Cu}(\mathbb{C}, B))$ coincides with the notion of scale for the ordinary Cuntz semigroup introduced in [56]. △

The reader might have noticed already that there are clear connections among the last few definitions. Indeed, strictly invertible elements in $\mathsf{Cu}(A, B)$ are the invertible elements of $\mathsf{Cu}(A, B)$ that are contained in the scale $\Sigma(\mathsf{Cu}(A, B))$, with inverse in the scale of $\mathsf{Cu}(B, A)$.

To see how the above notions of strict invertibility is tied to classification we now give a proof by examples for the classification of UHF algebras, starting by revisiting the matrix example given above first.

Example 3.80. Let $0 < n \leq m$ be natural numbers, and consider the full matrix algebras M_n and M_m. We claim that there is a strictly invertible element in $\mathsf{Cu}(M_n, M_m)$ if and only if $n = m$. One direction is clearly obvious, so suppose that $\Phi \in \mathsf{Cu}(M_n, M_m)$ is a strictly invertible element. Then there are c.p.c. order zero maps $\phi : M_n \to M_m$ and $\psi : M_m \to M_n$ such that $\Phi = [\phi \otimes \mathrm{id}_K]$ and $\psi \circ \phi \sim 1_{M_n}$, $\phi \circ \psi \sim 1_{M_m}$. By the above proposition we can find unital $*$-homomorphisms $\pi_\phi : M_n \to M_m$ and $\pi_\psi : M_m \to M_n$, but such a π_ψ can only exist if $m = n$. *En passant* we observe that, under these circumstances, both π_ϕ and π_ψ would be surjective and hence $*$-isomorphisms. △

Example 3.81. With UHF algebras A and B in place of the matrix algebras M_m and M_n in the above example one gets unital injective $*$-homomorphisms $\pi_\phi : A \to B$ and $\pi_\psi : B \to A$, which can exist only if A and B factor tensorially through B and A respectively, i.e. only if A and B have the same supernatural number. By the standard classification result of Glimm it follows that there is a strictly invertible element in $\mathsf{Cu}(A, B)$ if and only if A and B are isomorphic. △

With the above example for UHF algebras we have established the following classification result.

Theorem 3.82. *Two UHF algebras A and B are isomorphic if and only if there is a strictly invertible element in* $\mathsf{Cu}(A, B)$.

Before turning our attention to the more general case of unital AF algebras we would like to recall that, for any simple and stably finite C*-algebra A or with stable rank one, one has an embedding of the Murray-von Neumann semigroup $V(A)$ into $\mathsf{Cu}(A)$ (cf. [4]). Therefore, the ordinary Cuntz semigroup contains enough information to classify all AF algebras. Indeed, as shown in [13], there is enough information to even classify the larger class of AI algebras. The result that follows is a confirmation of the fact that the bivariant Cuntz semigroup Cu described in this paper contains at least the same amount of information to recover this result, at least in the unital case.

Theorem 3.83. *Two unital AI algebras A and B are isomorphic if and only if there is a strictly invertible element in* $\mathsf{Cu}(A, B)$*.*

Proof. By the results in [13] it is enough to show that there exists a semigroup isomorphism $\alpha : \mathsf{Cu}(A) \to \mathsf{Cu}(B)$ such that $\alpha([1_A]) = [1_B]$, for this implies that there exists a lift of α, i.e. a $*$-isomorphism $\phi : A \to B$ such that $\alpha = \mathsf{Cu}(\phi)$. Since $\mathsf{Cu}(A, B)$ has a strictly invertible element, we can find unital $*$-homomorphisms $\pi_1 : A \to B$ and $\pi_2 : B \to A$ such that $\pi_2 \circ \pi_1 \sim \mathrm{id}_A$ and $\pi_1 \circ \pi_2 \sim \mathrm{id}_B$. By functoriality one can check that the induced maps at the level of the Cuntz semigroups Cu satisfy

$$\mathsf{Cu}(\pi_2) \circ \mathsf{Cu}(\pi_1) = \mathrm{id}_{\mathsf{Cu}(A)} \quad \text{and} \quad \mathsf{Cu}(\pi_1) \circ \mathsf{Cu}(\pi_2) = \mathrm{id}_{\mathsf{Cu}(B)} .$$

Moreover, since π_1 is unital, one also has $\mathsf{Cu}(\pi_1)([1_A]) = [1_B]$, and therefore one can take $\alpha = \mathsf{Cu}(\pi_1)$. □

Observe that AF algebras are contained in the class of AI algebras and therefore the above classification result applies to the former as well. A direct application of the classical result of Elliott [19] can then be made on the restriction of the semigroup isomorphism $\mathsf{Cu}(\pi_1)$ of the above proof to the Murray-von Neumann semigroup $V(A)$. Furthermore, we observe that the strictly invertible elements between unital stably finite C*-algebras are always in the set of compact elements, for every class has a $*$-homomorphism as a representative.

The classification results given above are all special instances of a more general result, which is a consequence of Elliott's intertwining argument [22] and Proposition 3.75. Before recalling Elliott's argument and stating this more general classification result we record the following lemma.

Lemma 3.84. *Let A and B be unital C*-algebras, B stably finite. Two unital $*$-homomorphisms $\pi_1, \pi_2 : A \to B$ are Cuntz equivalent if and only if they are approximately unitarily equivalent.*

Proof. One implication is obvious, so let us show the converse. Assuming $\pi_1 \sim \pi_2$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ such that

$$\lim_{n\to\infty} \|x_n^*\pi_1(a)x_n - \pi_2(a)\| \to 0$$

for any $a \in A$. Since π_1 and π_2 are unital, in particular one has

$$\lim_{n\to\infty} x_n^* x_n \to 1.$$

Now, by stably finiteness of B, one obtains that x_n is eventually left invertible and hence invertible. Therefore, by removing the first few non-invertible elements, without loss of generality one can replace x_n by the unitaries u_n coming from the polar decomposition $x_n = u_n|x_n|$. It is the immediate to check that this new sequence witnesses the desired approximate unitary equivalence. □

We now recall the already mentioned intertwining argument of Elliott. For the purposes of this thesis, such result can be stated in the following form.

Theorem 3.85 (Elliott [22])**.** *Let A and B be separable, unital C*-algebras. If there are unital $*$-homomorphisms $\pi_1 : A \to B$ and $\pi_2 : B \to A$ such that $\pi_2 \circ \pi_1 \approx_{\text{a.u.}} \mathrm{id}_A$ and $\pi_1 \circ \pi_2 \approx_{\text{a.u.}} \mathrm{id}_B$ then A and B are isomorphic.*

By combining Proposition 3.75 with Lemma 3.84 and Theorem 3.85 we get to the following classification result for unital and stably finite C*-algebras.

Theorem 3.86. *Let A and B be unital, stably finite C*-algebras. Then A and B are isomorphic if and only if there exists a strictly invertible element in $\mathsf{Cu}(A, B)$.*

Proof. It is clear that any isomorphism between A and B gives a strictly invertible element. In order to prove the converse assume that $\Phi \in \mathsf{Cu}(A, B)$ is a strictly invertible element and that ϕ is a representative of Φ. By Proposition 3.75, one finds unital $*$-homomorphisms $\pi_1 : A \to B$ and $\pi_2 : B \to A$ such that $\pi_2 \circ \pi_1 \sim \mathrm{id}_A$ and $\pi_1 \circ \pi_2 \sim \mathrm{id}_B$. By Lemma 3.84, Cuntz equivalence implies approximately unitary equivalence and, hence, one gets that A is isomorphic to B by a direct application of Theorem 3.85. □

Observe that, under the assumptions of the above theorem, every strictly invertible element has a $*$-homomorphism as a representative and therefore it is compact in the bivariant Cuntz semigroup.

3.8 The Equivariant Theory

In order to obtain an equivariant extension of the bivariant theory for the Cuntz semigroup developed in this thesis we work under the principle that this should be based on a suitable notion of Cuntz comparison of *equivariant* c.p.c. order zero maps. This tenet justifies our choice of presenting the equivariant Cuntz semigroup in the form given by Definition 2.34. As the main goal is to provide a tool for the classification of actions, we also give equivariant extension of the notions of strict invertibility and show how to use it to recover some well-known as well as more recent classification results for locally representable actions.

A c.p.c. order zero map ϕ between two G-algebras (A, G, α) and (B, G, β) is said to be *equivariant* if it is an intertwiner for the actions α and β, that is

$$\phi \circ \alpha_g = \beta_g \circ \phi, \qquad \forall g \in G.$$

Unless otherwise stated, it will be assumed that a c.p.c. order zero map $\phi : A \to B$ between G-algebras A and B is always equivariant. The Cuntz comparison of equivariant c.p.c. order zero maps then takes the following form.

Definition 3.87. Let (A, G, α) and (B, G, β) be G-algebras, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. We say that ϕ is equivariantly Cuntz-subequivalent to ψ (in symbols $\phi \precsim_G \psi$) if there exists a G-invariant sequence $\{b_n\}_{n\in\mathbb{N}} \subset B^G$ such that

$$\|b_n\psi(a)b_n^* - \phi(a)\| \to 0$$

for any $a \in A$.

Let $\sim_G$ denote the equivalence relation arising from the antisymmetrisation of the relation $\precsim_G$ just defined, that is $\phi \sim_G \psi$ if $\phi \precsim_G \psi$ and $\psi \precsim_G \phi$. If (A, G, α) is a G-algebra, we shall always assume that the tensor product $A \otimes K_G$ is equipped with the diagonal action $\alpha \otimes \mathrm{id}_K \otimes \lambda_G$, where λ_G is the left-regular representation of G on $L^2(G)$, unless otherwise stated. As an equivariant generalisation of the bivariant Cuntz semigroup Cu of Definition 3.32 we then give the following definition.

Definition 3.88. Let (A, G, α) and (B, G, β) be G-algebras. The equivariant bivariant Cuntz semigroup $\mathsf{Cu}^G(A, B)$ of A and B is the set of equivalence classes

$$\mathsf{Cu}^G(A, B) := \{\phi : A \otimes K_G \to B \otimes K_G \mid \phi \text{ equivariant c.p.c. order zero map}\}/\sim_G .$$

The following result concerning equivariant c.p.c. order zero maps and their Cuntz comparison is an equivariant generalisation of the isomorphism γ of Section 3.1.3, which is mainly due to Fell's absorption principle. First of all we observe that, inside the algebra K_G there is a minimal G-invariant projection e_G. This is given by $e \otimes e_0$, where e is any minimal projection of K and e_0 is the minimal projection in $K(L^2(G))$ that projects onto the representation space of the trivial representation of G. Furthermore, the flip $a \otimes e_G \mapsto e_G \otimes a$ is also implemented by a G-invariant unitary.

Proposition 3.89. *There is an equivariant isomorphism* $\gamma_G : K_G \otimes K_G \to K_G$ *for which there exists a* G*-invariant isometry* $v \in B(L^2(G) \otimes \ell^2(\mathbb{N}))^G$ *such that* $\mathrm{Ad}_v \circ \gamma_G \circ (\mathrm{id}_{K_G} \otimes e_G) = \mathrm{id}_{K_G}$.

Proof. By Fell's absorption principle the G-algebra $(K_G \otimes K_G, \mathrm{id}_K \otimes \lambda_G \otimes \mathrm{id}_K \otimes \lambda_G)$ is conjugate to $(K_G \otimes K_G, \mathrm{id}_K \otimes \lambda_G \otimes \mathrm{id}_{K_G})$ through a map ϕ that is such that $\phi \circ (\mathrm{id}_{K_G} \otimes e_G) = \mathrm{id}_{K_G} \otimes e_G$. Since for every isomorphism $\gamma : K \otimes (K_G, \mathrm{id}_{K_G}) \to K$ there exists an isometry $w \in B(\ell^2(\mathbb{N}))$ with the property that $\mathrm{Ad}_w \circ \gamma \circ (\mathrm{id}_K \otimes e_G) = \mathrm{id}_K$, the sought map is then given by $\gamma_G := \mathrm{id}_{K(L^2(G))} \otimes \gamma$, with the G-invariant isometry given by $v := 1_{B(L^2(G))} \otimes w$. □

It is clear that the composition of two equivariant c.p.c. order zero maps yields another equivariant c.p.c. order zero map. We continue our generalisation to the equivariant case of results in this chapter by presenting an extension of Theorem 1.5.

Theorem 3.90. *Let* (A, G, α) *and* (B, G, β) *be* G*-algebras and let* $\phi : A \to B$ *be an equivariant c.p.c. order zero map. Set* $C_\phi := C^*(\phi(A))$ *and introduce an action of* G *on* $\mathcal{M}(C_\phi)$ *by restricting the bidual maps* β_g^{**} *onto it. Then there exists a* G*-invariant positive element* $h_\phi \in \mathcal{M}(C_\phi)_+^G \cap C_\phi'$*, with* $\|h_\phi\| = \|\phi\|$*, and a non-degenerate equivariant* $*$*-homomorphism* $\pi_\phi : A \to \mathcal{M}(C_\phi) \cap \{h_\phi\}'$*, that is,* $\beta_g^{**} \circ \pi_\phi = \pi_\phi \circ \alpha_g$ *for any* $g \in G$*, such that*

$$\phi(a) = h_\phi \pi_\phi(a), \qquad \forall a \in A.$$

Proof. By Theorem 1.5 there are a positive element $h_\phi \in \mathcal{M}(C_\phi)_+ \cap C_\phi'$ with $\|h_\phi\| = \|\phi\|$ and a non-degenerate $*$-homomorphism $\pi_\phi : A \to \mathcal{M}(C_\phi) \cap \{h_\phi\}'$ such that $\phi(a) = h_\phi \pi_\phi(a)$ for any $a \in A$. If $\{e_n\}_{n \in \mathbb{N}} \subset A$ is an approximate unit, the equivariance of ϕ implies

$h_\phi \pi_\phi(\alpha_g(e_n)) = \beta_g(h_\phi \pi_\phi(e_n))$ for any $n \in \mathbb{N}$, whence

$$\begin{aligned} 0 &= \textsc{sot} \lim_{n\to\infty} [\phi(\alpha_g(e_n)) - \beta_g(\phi(e_n))] \\ &= h_\phi - \beta_g^{**}(h_\phi), \qquad \forall g \in G, \end{aligned}$$

which shows that h_ϕ is G-invariant in $\mathcal{M}(C_\phi)$, with the action given by the restriction of β^{**} to this multiplier algebra. Since $h_\phi^{\frac{1}{n}}$ is also G-invariant, equivariance also implies $h_\phi^{\frac{1}{n}}[\pi_\phi(\alpha_g(a)) - \beta_g^{**}(\pi_\phi(a))] = 0$ for any $n \in \mathbb{N}$ and $a \in A$, whence

$$\begin{aligned} 0 &= \textsc{sot} \lim_{n\to\infty} h_\phi^{\frac{1}{n}}[\pi_\phi(\alpha_g(a)) - \beta_g^{**}(\pi_\phi(a))] \\ &= \pi_\phi(\alpha_g(a)) - \beta_g^{**}(\pi_\phi(a)), \qquad \forall a \in A \end{aligned}$$

i.e $\pi_\phi \circ \alpha_g = \beta_g^{**} \circ \pi_\phi$. □

The fact that such a result holds for the equivariant case allows to give equivariant generalisations of some of the previous results, like the ones mentioned in the following statement.

Proposition 3.91. *The results of Proposition 3.24, Lemma 3.25 and Corollary 3.26 with K_G in place of M_∞ all extend to the equivariant setting.*

Proof. In the proof of Proposition 3.24 one can take an approximate unit $\{e_n\}_{n\in\mathbb{N}}$ in the fixed point algebra, where the image of π_η belongs. Hence, the sequence $\{d_n\}_{n\in\mathbb{N}}$ is G-invariant and can be perturbed into a G-invariant sequence $\{c_n\}_{n\in\mathbb{N}}$ in C by density as described there.

In the proof of Lemma 3.25 one can, again, take a G-invariant approximate unit $\{e_n\}_{n\in\mathbb{N}}$. By equipping the tensor product with the diagonal action, the sequence $\{e_n \otimes b_n\}_{n\in\mathbb{N}}$ is then G-invariant as required.

For the proof of Corollary 3.26 with K_G in place of M_∞ it is enough to take e_G as the minimal projection and observe that $B^G \otimes \{e_G\} = B \otimes \{e_G\} \cap (B \otimes K_G)^G$. □

Thanks to the above proposition and the map γ_G of Proposition 3.89, the stability of Cu^G holds in the rather general form of the following result.

Theorem 3.92. *Let (A, G, α) and (B, G, β) be G-algebras. Then $\mathsf{Cu}^G(A, B) \cong \mathsf{Cu}^G(A \otimes K_G, B \otimes K_G)$.*

Proof. In the proof of Proposition 3.27 one only needs to replace e, γ and M_∞ by e_G and γ_G and K_G respectively, and consider the equivariant version of the results mentioned thereof. □

The following example shows that Definition 3.88 gives an equivariant extension of Definition 3.32.

Example 3.93. Let G be the trivial group $\{e\}$. Then $K_G \cong \mathbb{C}$ with the trivial action and therefore $\mathsf{Cu}^G(A, B) \cong \mathsf{Cu}(A, B)$. △

The example that follows shows that Definition 3.88 can be regarded as a bivariant extension of Definition 2.34.

Example 3.94. Let (A, G, α) and (B, G, β) be G-algebras. Theorem 3.92 implies that for every class $\Phi \in \mathsf{Cu}^G(A, B)$ there exists a representative of the form $\phi \otimes \mathrm{id}_{K_G}$, where $\phi : A \to B \otimes K_G$ is an equivariant c.p.c. order zero map. When $A = \mathbb{C}$ with the trivial action of G then

$$\phi(z) = zh_\phi, \qquad \forall z \in C,$$

where h_ϕ is a G-invariant positive element in $B \otimes K_G$ by Theorem 3.90. Hence, $\mathsf{Cu}^G(\mathbb{C}, B)$ can be identified with Cuntz-equivalence classes of G-invariant positive elements from $B \otimes K_G$, i.e.

$$\mathsf{Cu}^G(\mathbb{C}, B) \cong \mathsf{Cu}^G(B),$$

which shows that the equivariant definition of the bivariant Cuntz semigroup of this section indeed is a bivariant extension of the equivariant Cuntz semigroup of Section 2.5. △

Let $G\mathsf{C}^*$ denote the category of G-algebras, with morphisms given by equivariant c.p.c. order zero maps. The results of Section 3.1.1 generalise to the equivariant setting in the following sense.

Proposition 3.95. *The map* $\mathsf{Cu}^G : G\mathsf{C}^{*\mathrm{op}} \times G\mathsf{C}^* \to \mathsf{OrdAMon}$ *is a functor.*

Proof. Since the composition of equivariant c.p.c. order zero maps yields equivariant c.p.c. order zero maps, it is enough to check that all the sequences required to witness Cuntz comparison of equivariant c.p.c. order zero maps in the results of Section 3.1.1 can be taken and remain in the fixed point algebras, and this is obvious from the definitions. □

3.8.1 Relation with Crossed Products

With Julg's Theorem in the form of Theorem 2.49 it is possible to establish a connection between the equivariant Cuntz semigroup of a G-algebra and the ordinary Cuntz semigroup of the crossed product. In KK-theory one can provide a group homomorphism between the equivariant KK-group and the KK-group of the crossed product (cf. [6, §2.6]). It is now shown that an analogue of this last result holds for the bivariant theory of the Cuntz semigroup.

Proposition 3.96. *Let (A, G, α) and (B, G, β) be G-algebras. Every equivariant c.p.c. order zero map $\phi : A \to B$ induces a c.p.c. order zero map $\phi_\rtimes : A \rtimes G \to B \rtimes G$ between the crossed products.*

Proof. Consider the map $\phi_* : L^1(G, A) \to L^1(G, B)$ given by post-composition, that is,

$$\phi_*(f)(g) := \phi(f(g)), \qquad \forall f \in L^1(G, A), g \in G.$$

If $a, b \in L^1(G, A)$ are such that $a * b = 0$ and μ is the Haar measure on G, then one has

$$\begin{aligned}
(\phi_*(a) * \phi_*(b))(g) &= \int_G \phi\big(a(h)\big)\beta_h\left(\phi(b(h^{-1}g))\right) \mathrm{d}\mu(h) \\
&= \int_G h_\phi \phi\big(a(h)\alpha_h(b(h^{-1}g))\big) \mathrm{d}\mu(h) \\
&= h_\phi \phi((a * b)(g)), \qquad \forall g \in G
\end{aligned}$$

whence $\phi_*(a) * \phi_*(b) = 0$. Therefore ϕ_* extends to a c.p.c. order zero map $\phi_\rtimes : A \rtimes G \to B \rtimes G$. ☐

Proposition 3.97. *Let (A, G, α) and (B, G, β) be G-algebras and let $\phi, \psi : A \to B$ be equivariant c.p.c. order zero maps such that $\phi \precsim_G \psi$. Then $\phi_\rtimes \precsim \psi_\rtimes$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^1(G, B)$ be an approximate unit. If $\{b_n\}_{n\in\mathbb{N}} \subset B^G$ is the sequence witnessing the subequivalence $\phi \precsim_G \psi$, then a direct computation shows that the sequence $\{b_{\rtimes n}\}_{n\in\mathbb{N}} \subset L^1(G, B)$ given by

$$b_{\rtimes n} := b_n \otimes f_n$$

is such that

$$\|b_{\rtimes n} \psi_\rtimes(a \otimes f) {b_{\rtimes n}}^*\| \to \phi_\rtimes(a \otimes f), \qquad \forall a \otimes f \in L^1(G, A),$$

whence $\phi_\rtimes \precsim \psi_\rtimes$. ☐

This last result shows that the assignment $\phi \mapsto \phi_{\rtimes}$ becomes well-defined when considered at the level of classes. Therefore, one has the following result.

Theorem 3.98. *Let (A, G, α) and (B, G, β) be G-algebras. There is a natural semigroup homomorphism*

$$j^G : \mathsf{Cu}^G(A, B) \to \mathsf{Cu}(A \rtimes G, B \rtimes G)$$

which is functorial in A and B and compatible with the composition product.

Proof. The sought map j^G is defined as

$$j^G([\phi]) := [\phi_{\rtimes}]$$

which is well-defined as a consequence of the above proposition. □

We observe that, if A is a G-algebra with the trivial G action on it, then $\mathsf{Cu}^G(A, B)$ comprises of equivalence classes of equivariant c.p.c. order zero maps from A to the fixed point algebra of $B \otimes K_G$, i.e. $(B \rtimes G) \otimes K$, so that there is a natural isomorphism

$$\mathsf{Cu}^G(A, B) \cong \mathsf{Cu}(A, B \rtimes G).$$

This generalises the result of Example 3.94, since in the case $A = \mathbb{C}$ with the trivial action of G one has

$$\mathsf{Cu}^G(\mathbb{C}, B) \cong \mathsf{Cu}(\mathbb{C}, B \rtimes G) \cong \mathsf{Cu}(B \rtimes G)$$

which is isomorphic to $\mathsf{Cu}^G(B)$ by Theorem 2.49.

3.8.2 Classification of Actions

The classification result of Gardella and Santiago, namely Theorem 2.51, which is an equivariant version of the classification result of Robert [60], can be recovered from the equivariant theory of the bivariant Cuntz semigroup discussed in this section. In order to capture the right notion of equivalence, which includes part of the scale conditions of part (2) of [24, Theorem 8.4] we give the following definition as the equivariant analogue of Definition 3.76.

Definition 3.99. Let (A, G, α) and (B, G, β) be G-algebras. An element $\Phi \in \mathsf{Cu}^G(A, B)$ is said to be *strictly invertible* if there exist equivariant c.p.c. order zero maps $\phi : A \to B$ and $\psi : B \to A$ such that

i. $[\phi \otimes \mathrm{id}_{K_G}] = \Phi$;

ii. $\psi \circ \phi \sim_G \mathrm{id}_A$ and $\phi \circ \psi \sim_G \mathrm{id}_B$.

As in the standard theory of the bivariant Cuntz semigroup we have the following result, which generalises Proposition 3.75 to the equivariant setting.

Theorem 3.100. *Let (A, G, α) and (B, G, β) be unital, stably finite G-algebras. Every strictly invertible element $\Phi \in \mathsf{Cu}^G(A, B)$ is compact and induces a $\mathsf{Cu}(G)$-semimodule isomorphism $\rho : \mathsf{Cu}^G(A) \to \mathsf{Cu}^G(B)$ such that $\rho([1_A]) = [1_B]$ and $\rho([1_A \otimes e_G]) = [1_B \otimes e_G]$.*

Proof. It is easily seen that Proposition 3.75 generalises to the equivariant case. Hence, if $\Phi \in \mathsf{Cu}^G(A, B)$ is a strictly invertible element, there are equivariant c.p.c. order zero maps $\phi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \phi \sim_G \mathrm{id}_A$ and $\phi \circ \psi \sim_G \mathrm{id}_B$, which can then be replaced by their support $*$-homomorphisms π_ϕ and π_ψ respectively. Then $\rho := \mathsf{Cu}^G(\pi_\phi)$ is a $\mathsf{Cu}(G)$-semimodule isomorphism that satisfies $\rho([1_A]) = [1_B]$ and clearly sends the constant function $1_A(g) = 1_A$ to $1_B(g) = 1_B$, whence $\rho([1_A \otimes e_G]) = [1_B \otimes e_G]$. □

Corollary 3.101. *Let G be a finite Abelian group and let (A, G, α) and (B, G, β) be unital G-algebras in the class **R** with locally representable actions α and β (cf. Section 2.5.6). Then (A, G, α) and (B, G, β) are equivariantly isomorphic if and only if there is a strictly invertible element in $\mathsf{Cu}^G(A, B)$.*

Proof. It follows directly from the above theorem, together with Theorem 2.51. □

Locally representable actions for the larger class of compact groups have been considered by Handelman and Rossmann. Their definition of local representability is restricted to AF algebras, and it is assumed that an action α over an AF algebra A is locally representable if it is representable along a given inductive sequence of C*-algebras whose limit is A. We shall say that a G-algebra (A, G, α) is AF if the underlying C*-algebra A is. Their main classification result [28, Theorem III.1], for the purposes of this thesis, can be stated in the following way.

Theorem 3.102 (Handelman-Rossmann)**.** *Let G be a compact group and let (A, G, α) and (B, G, β) be unital AF G-algebras, with α and β locally representable actions along given inductive sequences for A and B respectively. Then A and B are equivariantly isomorphic if and only if there exists a $V^G(\mathbb{C})$-semimodule isomorphism $\rho : V^G(A) \to V^G(B)$ such that $\rho([1_A]) = [1_B]$.*

This classical classification result can be recovered from the equivariant bivariant Cuntz semigroup as a corollary to Theorem 3.100.

Corollary 3.103. *Let G be a compact group and let (A, G, α) and (B, G, β) be unital AF-G-algebras, with α and β locally representable actions along given inductive sequences for A and B respectively. Then A and B are equivariantly isomorphic if and only if there exists a strictly invertible element in $\mathsf{Cu}^G(A, B)$.*

Proof. By Theorem 3.100, every strictly invertible element is compact and therefore induces a $V^G(\mathbb{C})$-semimodule homomorphism between $V^G(A)$ and $V^G(B)$ that satisfies all the hypotheses of Theorem 3.102. □

Chapter 4

Further Bivariant Extensions

In this chapter we take into considerations the other notions of comparison of positive elements that have been briefly recalled towards the end of Section 2.3. The aim is to reach a bivariant extension of the notion of open projections as presented in Section 2.3, and thus provide a bivariant analogue of the open projection picture of the ordinary Cuntz semigroup. If one regards c.p.c. order zero maps as a bivariant extension of positive elements, it is natural to conclude that the objects we are after are some special $*$-homomorphisms, since their image on the unit is a projection. To further justify the investigation carried out in this chapter we make the observation that, in general, projections have better algebraic properties than positive elements, and the same can be said for $*$-homomorphisms as opposed to c.p.c. order zero maps. What we propose here is similar in spirit to the theory of standard simplifications of KK-theory where, by *moving* the information contained in the Fredholm-type operator to the $*$-homomorphism that implements the left action on a Kasparov's module one gets the Cuntz's picture of KK-theory (cf. Section 1.5.2).

We start by defining *open $*$-homomorphisms* and by showing with an example that they can be regarded as a bivariant extension of the open projections of Section 2.3. We then propose suitable bivariant extensions of the Pedersen and Blackadar equivalence relations among c.p.c. order zero map. We introduce a comparison of open $*$-homomorphisms in the abstract sense of Definition 2.28 by introducing a bivariant extension of the Peligrad-Zsidó equivalence relation among open $*$-homomorphisms, and show that some of the results of [52] extend to this bivariant setting. We conclude by defining a new bivariant extension of the Cuntz semigroup, which is based on this new notion of Cuntz comparison among open $*$-homomorphism, and by showing how it relates to the bivariant Cuntz semigroup of Chapter 3.

4.1 Open $*$-homomorphisms

In this section we introduce the notion of open $*$-homomorphism as a bivariant extension of the open projections of Section 1.2.

Definition 4.1 (Open $*$-homomorphism)**.** A $*$-homomorphism $\pi : A \to B^{**}$ is of order zero if there exists an increasing sequence of c.p.c. order zero maps $\{\phi_n\}_{n\in\mathbb{N}}$ such that $\phi_n \to \pi$ in the point-SOT topology, i.e.

$$\phi_n(a) \leq \phi_{n+1}(a), \qquad \forall n \in \mathbb{N}, a \in A$$

and

$$\pi(a) = \text{SOT} \lim_{n\to\infty} \phi_n(a), \qquad \forall a \in A.$$

The following example can regarded as a justification of the claim that open $*$-homomorphisms as defined above are bivariant extensions of open projections, as defined in Section 1.2.

Example 4.2. A $*$-homomorphism $\pi : \mathbb{C} \to B^{**}$ is of the form $\pi(z) = zp$ for some projection $p \in B^{**}$. If π is open, then there exists an increasing sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of c.p.c. order zero maps from $\mathbb{C}$ to B such that

$$\pi(z) = \text{SOT} \lim_{n\to\infty} \phi_n(z), \qquad \forall z \in \mathbb{C}.$$

A c.p.c. order zero map $\phi_n : \mathbb{C} \to B$ is of the form $\phi_n(z) = h_n z$ for some positive element $h_n \in B$ and for any $z \in \mathbb{C}$, as a consequence of the structure theorem 1.5. From the evaluation of π on the unit of $\mathbb{C}$ one gets

$$p = \text{SOT} \lim_{n\to\infty} h_n, \tag{4.1}$$

with $h_n \leq h_{n+1}$ for any $n \in \mathbb{N}$. Hence, by Definition 1.15, the projection p is open.

Conversely, every open projection is of the form (4.1) above, for some increasing sequence of positive elements $\{h_n\}_{n\in\mathbb{N}} \subset B$. By setting

$$\pi(z) := pz, \qquad \forall z \in \mathbb{C}$$

and

$$\phi_n(z) := h_n z, \qquad \forall z \in \mathbb{C},$$

one sees that π is an open $*$-homomorphism, since it clearly is the point-SOT limit of an increasing sequence of c.p.c. order zero maps $\{\phi_n\}_{n\in\mathbb{N}}$. This shows that open projections arise as special cases of open $*$-homomorphisms. △

Other examples of open $*$-homomorphisms, which come once again from the structure theorem 1.5, are provided by the support $*$-homomorphisms of c.p.c. order zero maps, as shown by the example below.

Example 4.3. Every support $*$-homomorphism $\pi_\phi : A \to \mathcal{M}(C^*(\phi(A))) \subset B^{**}$ of a c.p.c. order zero map $\phi : A \to B$ is open. This can be seen by considering the functional calculus $\phi^{\frac{1}{n}}$ in the sense of Corollary 3.2 of [75] (cf. Proposition 1.10). Indeed $h_\phi^{\frac{1}{n}}$ converges strongly to the support projection p_{h_ϕ} of h_ϕ, and $\phi^{\frac{1}{n}}(a) \leq \phi^{\frac{1}{n+1}}(a)$ for any $n \in \mathbb{N}$ and $a \in A$. Therefore the point-SOT limit of the sequence $\phi^{\frac{1}{n}}$ gives π_ϕ, i.e. the support $*$-homomorphism of ϕ, which is then an open $*$-homomorphism. △

Definition 4.4. Let $\pi : A \to B^{**}$ be a $*$-homomorphism, and let $\{u_n\}_{n\in\mathbb{N}} \subset A$ be an approximate unit for A. The support projection p_π of π is the element of B^{**} defined by

$$p_\pi := \textsc{sot}\lim_{n\to\infty} \pi(u_n).$$

Lemma 4.5. *Let A and B be C^*-algebras, A unital. If $\phi, \psi : A \to B$ are c.p.c. order zero maps such that $\phi \leq \psi$, i.e. $\phi(a) \leq \psi(a)$ for any $a \in A^+$, then $\pi_\phi \subset \pi_\psi$, where $\subset$ denotes the inclusion in the sense of subrepresentations.*

Proof. Since $\pi_\phi(a) = \textsc{sot}\lim_{n\to\infty} \phi^{1/n}(a)$ for any $a \in A$, and similarly for π_ψ, one also has $\pi_\phi(a) \leq \pi_\psi(a)$ for any $a \in A^+$. Taking the biduals of these open $*$-homomorphisms one has

$$\begin{aligned}
\pi_\phi(1_A)\pi_\psi(p) &= [\pi_\phi(p) + \pi_\phi(p^\perp)]\pi_\psi(p)\\
&= \pi_\psi(p)\pi_\phi(p) + \pi_\phi(p^\perp)[\pi_\psi(1_A) - \pi_\psi(p^\perp)]\\
&= \pi_\psi(p)\pi_\phi(p) + [\pi_\psi(1_A) - \pi_\psi(p^\perp)]\pi_\phi(p^\perp)\\
&= \pi_\psi(p)\pi_\phi(1_A),
\end{aligned}$$

for any $p \in A^{**}$, which shows that the projection $\pi_\phi(1_A)$ is in the commutant of the image of π_ψ. □

From the above lemma one can conclude that an increasing sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of c.p.c. order zero maps is of the form

$$\phi_k = h_k\pi,$$

for a common $*$-homomorphism π and an increasing sequence $\{h_n\}_{n\in\mathbb{N}}$ of positive elements that converges strongly to p_π. Observe that, in general, π will be larger than any support

$*$-homomorphism π_{ϕ_k}, which is contained in π in the sense of the lemma above. Hence the map

$$\phi := \sum_{k\in\mathbb{N}} \frac{1}{2^k}\phi_k = \left(\sum_{k=1}^{\infty}\frac{h_k}{2^k}\right)\pi$$

is c.p.c. order zero and such that $\pi_\phi = \pi$. This shows the following structure result for open $*$-homomorphisms, which can be regarded as the bivariant analogue of the fact that, when A is separable, an open projection in A^{**} is the support projection of a positive element from A (cf. [52, §2]).

Theorem 4.6. *Let A and B be C*-algebras, A unital. Every open $*$-homomorphism from A to B^{**} is the support $*$-homomorphism of a c.p.c. order zero map.*

4.2 Bivariant Pedersen and Blackadar Equivalence

We recall that the equivalence defined in Equation 2.3 can be defined in terms of open projections rather than hereditary subalgebras, since for any positive elements a, b from a C*-algebra A, one has that $A_a = A_b$ if and only if $p_a = p_b$, by results in [52]. As a bivariant extension of this relation we propose the following. Let A and B be C*-algebras and $\phi, \psi : A \to B$ c.p.c. order zero map. We say that ϕ and ψ are equivalent, in symbols $\phi \cong \psi$, if $\pi_\phi = \pi_\psi$, i.e. if they have the same support $*$-homomorphism. As a bivariant extension of the Pedersen equivalence (cf. Definition 2.26), we give the following definition.

Definition 4.7 (Bivariant Pedersen Equivalence)**.** Let A, B be C*-algebras, A unital, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. We say that ϕ is Pedersen equivalent to ψ, in symbols $\phi \sim \psi$, if there exists $x \in B$ such that $\phi(a) = x^*\pi_\psi(a)x$ and $\psi(a) = x\pi_\phi(a)x^*$ for any $a \in A$, with $\phi(1_A) = x^*x$ and $\psi(1_A) = xx^*$.

Example 4.8. Let B be a C*-algebra and let $\phi, \psi : \mathbb{C} \to B$ be c.p.c. order zero maps such that $\phi \sim \psi$. Then there are two positive elements $h_\phi, h_\psi \in B$ such that $\phi(z) = zh_\phi$ and $\psi(z) = zh_\psi$ for any $z \in \mathbb{C}$, and an $x \in B$ such that $h_\phi = x^*x$ and $h_\psi = xx^*$. Hence h_ϕ and h_ψ are Pedersen equivalent in the sense of Definition 2.26. △

By combining the above equivalence $\cong$ among c.p.c. order zero maps with the equivariant Pedersen equivalence we get to the following bivariant extension of Blackadar equivalence (cf. Definition 2.27).

Definition 4.9 (Bivariant Blackadar Equivalence)**.** Let A, B be C*-algebras, A unital, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. We say that ϕ is Blackadar equivalent to

ψ, in symbols $\phi \sim_s \psi$, if there exists $x \in B$ such that $\phi \cong x^*\pi_\psi(\,\cdot\,)x$ and $\psi \cong x\pi_\phi(\,\cdot\,)x^*$, with $x^*\pi_\psi(1_A)x = x^*x$ and $x\pi_\phi(1_A)x^* = xx^*$.

Example 4.10. Let B be a C*-algebra and let $\phi, \psi : \mathbb{C} \to B$ be c.p.c. order zero maps such that $\phi \sim_s \psi$. Then there are two positive elements $h_\phi, h_\psi \in B$ such that $\phi(z) = zh_\phi$ and $\psi(z) = zh_\psi$ for any $z \in \mathbb{C}$, and an $x \in B$ such that $h_\phi \cong x^*x$ and $h_\psi \cong xx^*$. Hence h_ϕ and h_ψ are Blackadar equivalent in the sense of Definition 2.27. △

4.3 Cuntz Comparison of Open ∗-homomorphisms

In this section we introduce a notion of comparison between open ∗-homomorphisms. By regarding them as bivariant extensions of open projections, we model our definitions on those of [52]. Therefore we need to define a Cuntz comparison in the abstract sense of Definition 2.28, where the set S is that of open ∗-homomorphisms. To this end we need to introduce the notion of compact containment and of strong equivalence for open ∗-homomorphisms that also provide a bivariant extension of Definition 2.17 and of the Peligrad-Zsidó equivalence of Definition 2.18 respectively.

Let $\pi : A \to B^{**}$ be an open ∗-homomorphism. We define the C*-hereditary subalgebra B_π of B generated by π as

$$B_\pi := \pi(A)B\pi(A) \cap B.$$

Definition 4.11. Let A and B be C*-algebras and $\omega, \pi : A \to B^{**}$ open ∗-homomorphisms. We say that ω is compactly contained in π, in symbols $\omega \subset\subset \pi$, if $\omega \subset \pi$ and $\overline{p_\omega}$ is compact in B_π in the usual sense of open projections.

Example 4.12. Let B be a C*-algebra and let $\pi, \omega : \mathbb{C} \to B^{**}$ be open ∗-homomorphisms such that $\pi \subset\subset \omega$. Then there are two open projections $p_\pi, p_\omega \in B^{**}$ such that $\pi(z) = zp_\pi$ and $\omega(z) = xp_\omega$ for any $z \in \mathbb{C}$, and moreover $\overline{p_\pi}$ is compact in $B_\omega = B_{p_\omega}$. Hence $p_\pi \subset\subset p_\omega$ in the sense of ordinary open projections (cf. Definition 2.17). △

Let A be a C*-algebra and let π, ω be any two representations of A on the Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ respectively. By (ω, π) we denote the set of all the intertwiners from π to ω, i.e. the set of all the bounded linear operators $T : \mathscr{H}_1 \to \mathscr{H}_2$ such that $\omega(a) \circ T = T \circ \pi(a)$ for any $a \in A$.

Definition 4.13. Let A and B be C*-algebras. Two open ∗-homomorphisms $\omega, \pi : A \to B^{**}$ are said to be PZ-equivalent, $\omega \sim_{\mathrm{PZ}} \pi$ in symbols, if there exists an isometric intertwiner $v \in (\omega, \pi) \subset B^{**}$, that is a partial isometry v with the property that $v \circ \pi(a) = \omega(a) \circ v$ for any $a \in A$, such that $vB_\pi \subset B$ and $v^* B_\omega \subset B$, together with $\pi(1) = v^* v$ and $\omega(1) = vv^*$.

The following example shows that the above definition can be regarded as a bivariant extension of Definition 2.18.

Example 4.14. Let B be a C*-algebra and let $\pi, \omega : \mathbb{C} \to B^{**}$ be open $*$-homomorphisms such that $\pi \sim_{\text{PZ}} \omega$. Then there are two open projections $p_\pi, p_\omega \in B^{**}$ such that $\pi(z) = zp_\pi$ and $\omega(z) = xp_\omega$ for any $z \in \mathbb{C}$, and moreover there exists a partial isometry $v \in B^{**}$ such that

$$v^* v = p_\pi \qquad \text{and} \qquad vv^* = p_\omega$$

together with

$$vB_\pi = vB_{p_\pi} \subset B \qquad \text{and} \qquad v^* B_\omega = v^* B_{p_\omega} \subset B.$$

Hence $p_\pi \sim_{\text{PZ}} p_\omega$ in the sense of Definition 2.18. △

With the two definitions above one can now define Cuntz comparison of open $*$-homomorphisms in the form of abstract Cuntz comparison of Definition 2.28.

Definition 4.15. Let A, B be C*-algebras and $\pi, \omega : A \to B^{**}$ open $*$-homomorphisms. We say that π is subequivalent to ω if

$$\forall \omega' \Subset \omega \qquad \exists \pi' \Subset \pi \qquad | \qquad \omega' \sim_{\text{PZ}} \pi'.$$

We now state and prove some results that can be considered as bivariant extensions of analogous results in [52], linking the above definitions of equivalences, namely the bivariant Pedersen and Blackadar equivalences, between c.p.c. order zero maps, with the Peligrad-Zsidó equivalence of open $*$-homomorphisms of Definition 4.13. The following proposition can be regarded as a bivariant extension of [54, Theorem 1.4].

Proposition 4.16. *Let A and B be C*-algebras, A unital, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps such that $\phi \sim \psi$. Then $\pi_\phi \sim_{PZ} \pi_\psi$.*

Proof. Since $\phi \sim \psi$ there exists $x \in B$ such that $\phi(a) = x^* \pi_\psi(a) x$ and $\psi(a) = x \pi_\phi(a) x^*$ for any $a \in A$. With $x = v|x|$ be the polar decomposition of x, $\pi_\phi(\,\cdot\,) = v^* \pi_\psi(\,\cdot\,) v$, and since the relation $\sim$ is symmetric this implies that v is an isometric intertwiner in (π_ϕ, π_ψ) such that $vB_\phi, v^* B_\psi \subset B$. Hence $\pi_\phi \sim_{\text{PZ}} \pi_\psi$. □

The following result can be regarded as a converse of the above proposition, as well as a bivariant extension of [52, Proposition 3.3].

Proposition 4.17. *Let A and B be C^*-algebras, A unital, and let $\pi, \omega : A \to B^{**}$ be open $*$-homomorphisms such that $\pi \sim_{PZ} \omega$. If π is the support $*$-homomorphism of a c.p.c. order zero map, then so is ω. In this case there exists $x \in B$ such that the expressions $\phi(a) := x^*\omega(a)x$ and $\psi(a) := x\pi(a)x^*$, with $a \in A$, define c.p.c. order zero maps with $\pi = \pi_\phi$ and $\omega = \pi_\psi$.*

Proof. Let $\phi : A \to B^{**}$ be the c.p.c. order zero map for which $\pi = \pi_\phi$. Since $\pi \sim_{\mathrm{PZ}} \omega$, there exists a partial isometry $v \in (\omega, \pi)$ with $p_\pi = v^*v$ and $p_\omega = vv^*$. The map on A defined by $\psi(a) := v\phi(a)v^*$ for any $a \in A$ is c.p.c. order zero, since

$$\begin{aligned}\psi(a)\psi(b) &= v\phi(a)v^*v\phi(b)v^* \\ &= v\phi(a)p_\phi\phi(b)v^* \\ &= v\phi(a)\phi(b)v^*.\end{aligned}$$

By setting $x := vh_\phi^{\frac{1}{2}}$, where $h_\phi = \phi(1_A)$, we then have $\psi(a) = x\pi(a)x^*$, with $\psi(1_A) = |x^*|^2$, whence[1]

$$\begin{aligned}\pi_\psi(a) &= v\pi(a)v^* \\ &= \omega(a)\end{aligned}$$

for any $a \in A$, i.e. $\pi_\psi = \omega$. ☐

Corollary 4.18. *Let A and B be C^*-algebras, A unital, and let $\pi, \omega : A \to B^{**}$ be open $*$-homomorphisms such that $\pi \sim_{PZ} \omega$. Then there are Pedersen equivalent c.p.c. order zero maps $\phi, \psi : A \to B$ such that $\pi = \pi_\phi$, $\omega = \pi_\psi$.*

Proof. Since in the separable case every open $*$-homomorphism is the support $*$-homomorphism of a c.p.c. order zero map, there exists $\phi : A \to B$ c.p.c. order zero such that $\pi_\phi = \pi$. The result then follows from the above proposition. ☐

The following proposition can be regarded as a bivariant extension of (part of) [52, Proposition 4.3].

Proposition 4.19. *Let A and B be C^*-algebras, A unital, and $\phi, \psi : A \to B$ c.p.c. order zero maps. Then $\phi \sim_s \psi$ if and only if $\pi_\phi \sim_{PZ} \pi_\psi$.*

[1]this follows by using the identity $x = v|x| = |x^*|v$ from the polar decomposition of x.

Proof. If $\phi \sim_s \psi$ then there is $x \in B$ such that $\phi \cong \phi'$ and $\psi \cong \psi'$, where $\phi'(a) := x\pi_\psi(a)x^*$ and $\psi'(a) := x^*\pi_\phi(a)x$ for any $a \in A$. Hence $\phi' \sim \psi'$, which by Proposition 4.16 implies that $\pi_\phi = \pi_{\phi'} \sim_{\mathrm{PZ}} \pi_{\psi'} = \pi_\psi$, i.e. $\pi_\phi \sim_{\mathrm{PZ}} \pi_\psi$.

Conversely, assume that $\pi_\phi \sim_{\mathrm{PZ}} \pi_\psi$ and set $x := vh_\phi^{\frac{1}{2}}$. Then $\phi(a) = x^*\pi_\psi(a)x$ for any $a \in A$, and $a \mapsto x\pi_\phi(a)x^*$ is a c.p.c. order zero map which has π_ψ as support $*$-homomorphism for the same argument contained in the proof of Proposition 4.17. Hence $\phi \sim_s \psi$. □

Lemma 4.20. *Let A be a C*-algebra and let $a \in A^+$ be a positive contraction. Then $\{a^{\frac{1}{n}}\}_{n\in\mathbb{N}}$ is an approximate unit for $A_a := \overline{aAa}$.*

Proof. For any positive element $b \in A_a$ and $\epsilon > 0$ there exists a positive element $e \in A$ such that $\|b - aea\| < \epsilon$. Hence

$$\begin{aligned}
\left\|a^{\frac{1}{n}}ba^{\frac{1}{n}} - b\right\| &= \left\|(a^{\frac{1}{n}} - 1)b(a^{\frac{1}{m}} - 1)\right\| \\
&\leq \left\|(a^{\frac{1}{n}} - 1)aea(a^{\frac{1}{n}} - 1)\right\| + \epsilon \\
&\leq \|e\| \left\|a^2(a^{\frac{1}{n}} - 1)^2\right\| + \epsilon \\
&\leq \|e\| \sup_{t\in\mathbb{R}} |t(t^{\frac{1}{n}} - 1)|^2 + \epsilon \\
&= \left(1 + \frac{1}{n}\right)^{-2n} \frac{1}{n^2} + \epsilon \\
&\leq \frac{1}{4n^2} + \epsilon.
\end{aligned}$$

Taking the limit over $n \to \infty$ and remembering that $\epsilon > 0$ is arbitrary one has that $\left\|a^{\frac{1}{n}}ba^{\frac{1}{n}} - b\right\| \to 0$. □

Lemma 4.21. *Let A be a C*-algebra and let $a, b \in A^+$ be such that $p_a \Subset p_b$. Then there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset A$ such that*

$$\lim_{n\to\infty} \left\|z_n b^{\frac{1}{2}} - a^{\frac{1}{2}}\right\| = 0.$$

Proof. One can use the same sequence that comes from Handelman's lemma. To see that such a sequence works in this case as well observe that, with $e \in A^+$ such that $\overline{p}_a e = \overline{p}_a$, one has

$$\lim_{n\to\infty} \left\|b^{\frac{1}{n}}eb^{\frac{1}{n}} - e\right\| = 0$$

because of the previous lemma. Hence for any $\epsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\left\|b^{\frac{1}{k}}eb^{\frac{1}{k}} - e\right\| < \epsilon$$

for any $k > m$. Thus, with

$$z_n := a^{\frac{1}{2}} b^{\frac{1}{2}} (b + \tfrac{1}{n})^{-1}.$$

one has the estimate

$$\begin{aligned}
\left\| z_n b^{\frac{1}{2}} - a^{\frac{1}{2}} \right\|^2 &= \left\| a^{\frac{1}{2}} \left[b(b + \tfrac{1}{n})^{-1} - 1 \right] \right\|^2 \\
&= \left\| \left[(b + \tfrac{1}{n})^{-1} b - 1 \right] a \left[b(b + \tfrac{1}{n})^{-1} - 1 \right] \right\| \\
&< \|a\| \left\| \left[(b + \tfrac{1}{n})^{-1} b - 1 \right] b^{\frac{1}{m}} e b^{\frac{1}{m}} \left[b(b + \tfrac{1}{n})^{-1} - 1 \right] \right\| + \|a\| \epsilon \\
&< \|a\| \left\| \left[(b + \tfrac{1}{n})^{-1} b - 1 \right] b^{\frac{2}{m}} \left[b(b + \tfrac{1}{n})^{-1} - 1 \right] \right\| + \|a\| \epsilon \\
&< \|a\| \sup_{t \in \mathbb{R}_0^+} \left| \frac{t^{\frac{1}{m}}}{nt + 1} \right|^2 + \|a\| \epsilon \\
&= \|a\| \frac{1}{m^2 (m-1)^{\frac{2}{m} + 2}} \frac{1}{n^{\frac{2}{m}}} + \|a\| \epsilon \\
&\leq \|a\| \left(\frac{1}{4n} + \epsilon \right),
\end{aligned}$$

where we have used that e is a contraction and hence $e \leq p_b$ in this case. By taking the limit over n we have

$$\lim_{n \to \infty} \left\| z_n b^{\frac{1}{2}} - a^{\frac{1}{2}} \right\|^2 < \epsilon$$

for any $\epsilon > 0$, whence the sought convergence. □

The following result establishes a (one-sided) connection between the Cuntz comparison between open $*$-homomorphisms and that between c.p.c. order zero map of the previous chapter.

Proposition 4.22. *Let A and B be C^*-algebras and let $\phi, \psi : A \to B$ be c.p.c. order zero maps such that $\pi_\phi \precsim \pi_\psi$. Then $\phi \precsim \psi$.*

Proof. Suppose that $\pi_\phi \precsim \pi_\psi$. Since $\pi_{\phi_\epsilon} \Subset \pi_\phi$ there exists $\omega \Subset \pi_\psi$ such that $\pi_{\phi_\epsilon} \sim_{\mathrm{PZ}} \omega$ through some isometric intertwiner v. Now $v p_{h_{\phi_\epsilon}} v^* \leq p_{h_\psi}$ and by the previous lemma there exists $\{z_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(C^*(\psi(A)))$ such that $z_n h_\psi^{\frac{1}{2}} \to v h_{\phi_\epsilon}^{\frac{1}{2}} v^*$. Therefore, by redefining z_n as

$$z_n \mapsto e_n z_n f_n \in B, \qquad \forall n \in \mathbb{N},$$

where $\{e_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ are approximate units of B_{ϕ_ϵ} and B_ψ respectively, one has

$$\begin{aligned} h_{\phi_\epsilon}^{\frac{1}{n}} v^* z_n \psi(a) z_n^* v h_{\phi_\epsilon}^{\frac{1}{n}} &= h_{\phi_\epsilon}^{\frac{1}{n}} v^* z_n h_\psi^{\frac{1}{2}} \pi_\psi(a) h_\psi^{\frac{1}{2}} z_n^* v h_{\phi_\epsilon}^{\frac{1}{n}} \\ &\to h_{\phi_\epsilon}^{\frac{1}{2}} v^* \pi_\psi(a) v h_{\phi_\epsilon}^{\frac{1}{2}} \\ &= h_{\phi_\epsilon}^{\frac{1}{2}} \pi_{\phi_\epsilon}(a) h_{\phi_\epsilon}^{\frac{1}{2}} \\ &= \phi_\epsilon(a) \end{aligned}$$

whence $\phi_\epsilon \precsim \psi$. Since ϵ is arbitrary and $\sup_{n\in\mathbb{N}}[\phi_{\frac{1}{n}}] = [\phi]$ one has that $[\phi] \leq [\psi]$, i.e. $\phi \precsim \psi$. □

Let $\mathrm{Hom}_o(A, B^{**})$ denote the set of open $*$-homomorphisms from A to B^{**} and define

$$\mathsf{Cu}_o(A, B) := \mathrm{Hom}_o(A, (B \otimes K)^{**})/ \sim_{\mathrm{Cu}},$$

where here $\sim_{\mathrm{Cu}}$ is the antisymmetrisation of the relation $\precsim$ among open $*$-homomorphisms. A semigroup structure on $\mathsf{Cu}_o(A, B)$ is introduced by the binary operation $+ : \mathsf{Cu}_o(A, B) \times \mathsf{Cu}_o(A, B) \to \mathsf{Cu}_o(A, B)$ defined as

$$[\pi] + [\omega] := [\pi \mathbin{\hat{\oplus}} \omega],$$

where $\pi, \omega : A \to (B \otimes K)^{**}$ are open $*$-homomorphisms. It is easy to see that the above operation is well defined on classes, since $M_2((B \otimes K)^{**}) \cong (B \otimes K)^{**}$. The usual order structure can also be introduced by setting

$$[\pi] \leq [\omega]$$

whenever $\pi \precsim \omega$ in the sense of Cuntz comparison of open $*$-homomorphisms.

The following example shows that $\mathsf{Cu}_o(A, B)$ is also a bivariant extension of the ordinary Cuntz semigroup Cu.

Example 4.23. Let B be any C*-algebra. An open $*$-homomorphism $\pi : \mathbb{C} \to (B \otimes K)^{**}$ is of the form

$$\pi(z) = zp$$

for some open projection $p \in (B \otimes K)^{**}$. As shown in this section, both $\Subset$ and $\sim_{\mathrm{PZ}}$ for open $*$-homomorphisms reduce to $\Subset$ and $\sim_{\mathrm{PZ}}$ between open projections and therefore one has a natural identification between $\mathsf{Cu}_o(\mathbb{C}, B)$ and $\mathsf{Cu}(B)$ in the open projection picture. △

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