STABILITY CRITERIA FOR SOME CLASSES OF NONLINEAR MULTIVARIABLE CONTROL SYSTEMS.

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CONTENTS

ABSTRACT

Techniques are well established for using the second method of Liapunov to determine the stability of single loop systems containing one or more nonlinear elements. In this dissertation these techniques are extended to include three separate classes of multivariable systems.

In the first two classes, a linear multivariable process is controlled by many feedback loops, each containing a nonlinear sensing or actuating device. Distinction is made between linear processes which contain only feedforward interactions, and those which contain only feedback interactions. The two classes of systems considered are therefore represented by the equations

$$\underline{y} = -G(p)\underline{f} \quad \text{(feedforward interactions in the process)}$$

and

$$\underline{y} = G_1(p)\,(-\underline{f} + G_2(p)\,\underline{y}) \quad \text{(feedback interactions in the process).}$$

where p is the differential operator, $\underline{y}$ is the system output vector, $\underline{f}$ is the output vector of the nonlinearities and $G(p) = [g_{ij}(p)]$.

Generalisation of the techniques of Lur'e and Letov in applying Liapunov's second method provides sufficient conditions for total stability of such systems. For a given n-variable system, the criteria developed may be applied if the system is stable for all loop gains g_i in the ranges $k_i \leq g_i \leq K_i \leq \infty \quad (i = 1,2,\ldots n)$

provided that the corresponding nonlinear characteristics $f_i(y_i)$ are confined to the same sectors of the input-output plane, namely

$$k_i \leq \frac{f_i(y_i)}{y_i} \leq K_i.$$

For completeness, equivalent criteria for instability of these two classes of systems are included.

The third class of systems considered is one involving multiplication of system variables, and in particular a multinode representation of a nuclear reactor. Such systems, unlike the two classes discussed above, have in most cases only a finite region of stability in the state space. A stability criterion is derived by Liapunov's second method which produces such a finite region provided that the equilibrium point of the system is stable. For the examples considered, this region is entirely adequate for all realistic deviations of the system from equilibrium.

In all criteria developed, the principal objective is to produce systematic stability tests which do not involve prior estimation of, for example, the Liapunov function for any given system. Although computation becomes difficult for high order systems, all criteria involve only one parameter to be chosen such that results are optimum.

The criteria are not applicable to systems which are of a self-oscillatory nature, and a chapter is included on investigation of the oscillatory modes of a particular two variable system containing two functional nonlinearities.

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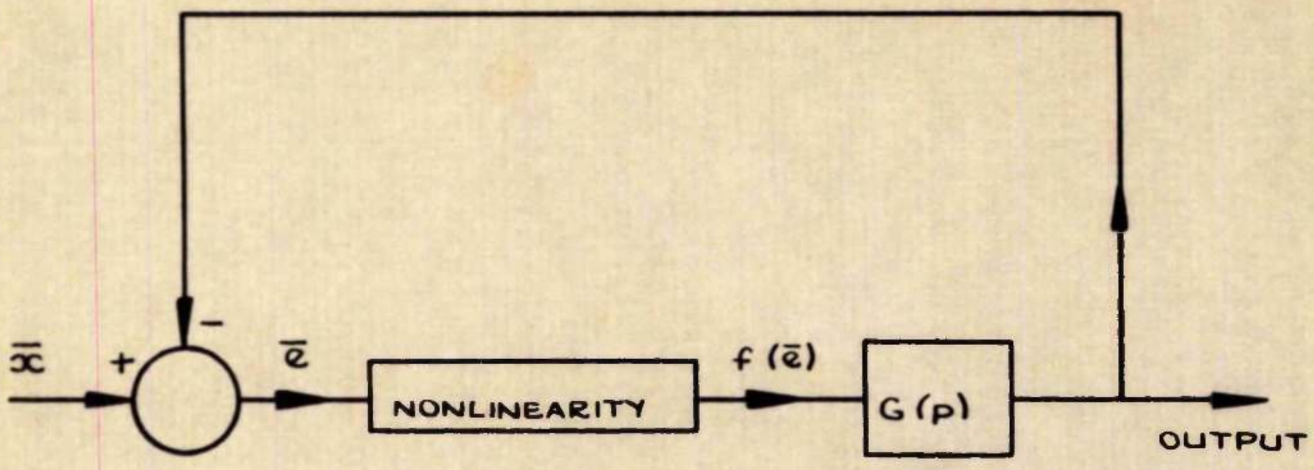

FIG. 1. SINGLE LOOP SYSTEM WITH NONLINEAR ACTUATOR

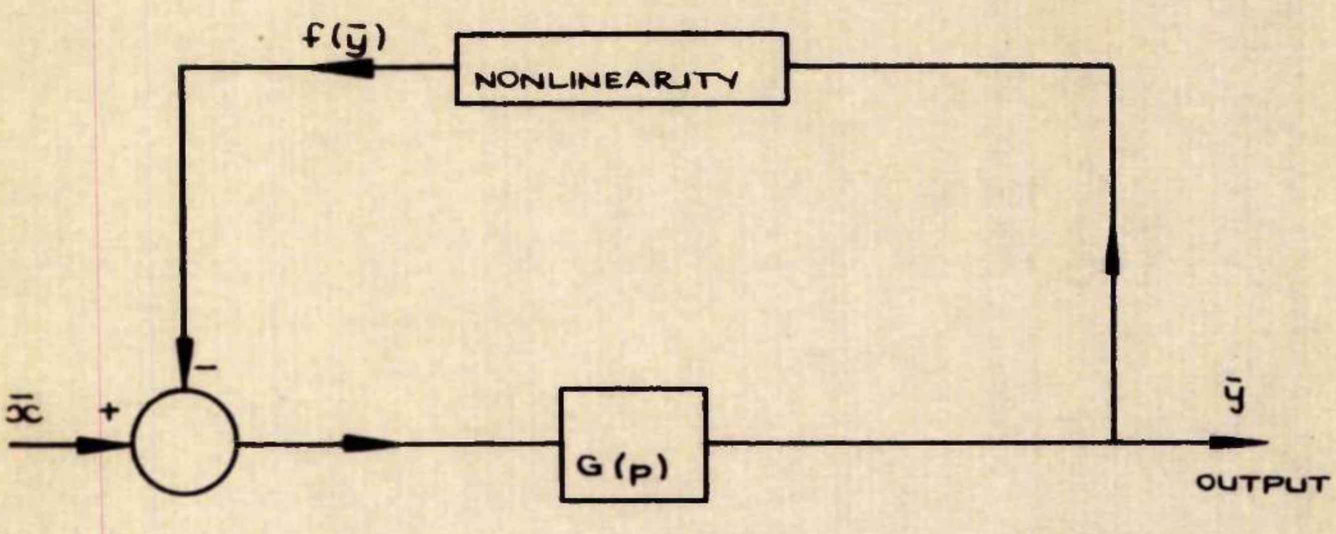

FIG. 2. SINGLE LOOP SYSTEM WITH NONLINEAR SENSOR

CHAPTER I.

SINGLE LOOP SYSTEMS

Introduction

In this chapter, single loop feedback systems containing one nonlinear device are considered. Use of Liapunov techniques has provided many stability criteria for such systems, and in the first section some of these criteria are given in detail to indicate the principles involved.

Whilst these methods are very satisfactory, they do not prove readily extensible to multivariable systems containing many nonlinearities. A more general approach to the stability analysis of single loop systems is established in succeeding sections which proves very useful for extension to multivariable systems.

Criteria are also developed for estimating the bounds of system response to an initial disturbance. These criteria subsume certain of the previously developed stability criteria of this chapter.

In all criteria, the nonlinear functions involved need not be explicitly defined, but must be single - valued and are subject to a general class of restrictions which confines their characteristics to be within certain areas of the input-output plane.

1.1. The stability criteria of Lur'e and Letov.

The class of systems under consideration is that in which a linear process is controlled by a feedback loop containing a nonlinear device. This device may be an actuator, as shown in the block diagram of fig.1, or a sensor, as shown in fig.2.

The operational equations describing such systems are given by

$\overline{y} = G(p)\ (\overline{x} - f(\overline{y})\ )$ (nonlinear sensor)

and

$\overline{e} = \overline{x} - G(p)\ f\ (\overline{e})$ (nonlinear actuator)

where $G(p) = \dfrac{p^m + a_{m-1}\ p^{m-1} + \ldots\ldots + a_o}{p^n + b_{n-1}\ p^{n-1} + \ldots\ldots + b_o}$ $(n \geq m)$

and the nonlinearities are assumed to be single-valued functions. p is the differential operator d/dt.

In the following presentation, the stability of the equilibrium point $\overline{x}\ (t)$ equal to a constant for all $t \geq 0$ will be considered. For this case, both systems may be represented by the equation

$$y = -\ G(p)\ f(y). \qquad (1.1.1)$$

Subject to certain restrictions on G(p), a system defined by eqn. (1.1.1) may be represented by the so-called 'first canonic form' of state variable equations (See Letov[1], Chap. II). Two cases will be considered. Firstly, when $m < n$ in G(p), and secondly when $m = n$ in G(p).

1.1a. $\underline{m < n}$

The first canonic form of state equations is

$$\dot{z}_i = \lambda_i z_i + f(y) \quad (i = 1,2,\ldots n) \qquad (1.1.2)$$

$$y = \sum_{i=1}^{n} \alpha_i z_i \qquad (1.1.3)$$

The λ_i are the n poles, and the α_i the corresponding n residues with sign reversed, of the partial fraction expansion of G(p). The canonic variables z_i will be real corresponding to real poles, and complex conjugate pairs corresponding to complex conjugate pairs of poles.

The canonic form of state variables is obtained by a linear transformation of any initially chosen set of state variables. The transformation will be non-singular only if G(p) contains no multiple poles (see e.g. Rekasius[2] p.51). If m = n, a slightly different form arises which is discussed below.

To investigate the stability of the above system, Lur'e[3] considered the following Liapunov function:

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j z_i z_j}{\lambda_i + \lambda_j} - \int_0^y f(y)\,dy - \frac{1}{2} \sum_{i=1}^{s} A_i z_i^2$$

$$- (A_{s+1} z_{s+1} z_{s+2} + A_{s+3} z_{s+3} z_{s+4} + \ldots + A_{n-1} z_{n-1} z_n) \quad (1.1.4)$$

where there are s real poles, and (n-s)/2 complex conjugate pairs of poles in G(p). A_i are real positive constants, and a_i are as yet undetermined numbers.

Noting that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j z_i z_j}{\lambda_i + \lambda_j} = - \int_0^{\infty} \left(\sum_{i=1}^{n} a_i z_i e^{-\lambda_i \tau} \right)^2 d\tau$$

it is easily proved (see e.g. Letov[1] p.114) that V is a real, negative definite function of the canonic variables and y provided that

(a) The poles λ_i of G(p) are all different

(b) Re $(\lambda_i) < 0$ for all i

(c) $\int_o^y f(y)\,dy \geq 0$ for all $|y| \neq 0$, and $f(o) = 0$

(d) The coefficients a_i are real corresponding to real poles, and form complex conjugate pairs corresponding to complex conjugate pairs of poles. For convenience, this correspondence is described as having coefficients of appropriate algebraic nature.

Differentiating eqn. (1.1.4) gives

$$\dot{V} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{\lambda_i + \lambda_j} (z_i \dot{z}_j + z_j \dot{z}_i) - \dot{y} f(y)$$

$$- \sum_{i=1}^{s} A_i z_i \dot{z}_i - [A_{s+1} (z_{s+1} \dot{z}_{s+2} + z_{s+2} \dot{z}_{s+1}) + \ldots$$

$$\ldots + A_{n-1} (z_{n-1} \dot{z}_n + z_n \dot{z}_{n-1})]. \qquad (1.1.5)$$

From eqns. (1.1.2) and (1.1.3),

$$\dot{y} = \sum_{i=1}^{n} \beta_i z_i - r f(y) \qquad (1.1.6)$$

where $\beta_i = \alpha_i \lambda_i$ and $r = - \sum_{i=1}^{n} \alpha_i$

Substituting from eqns. (1.1.2) and (1.1.6) into eqn. (1.1.5) gives

$$\dot{V} = \left(\sum_{i=1}^{n} a_i z_i\right)^2 + rf^2(y) - \sum_{i=1}^{s} A_i \lambda_i z_i^2$$

$$- [A_{s+1}(\lambda_{s+1} + \lambda_{s+2}) z_{s+1} z_{s+2} \ldots + A_{n-1}(\lambda_{n-1} + \lambda_n) z_{n-1} z_n]$$

$$+ f(y) \sum_{i=1}^{n} z_i \left(2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} - \beta_i - A_i\right) \qquad (1.1.7)$$

where for convenience in writing the last expression one defines

$$A_{s+2} = A_{s+1},\ A_{s+4} = A_{s+3},\ \ldots\ldots A_n = A_{n-1}.$$

Provided that the constant $r \geq 0$, the first four terms in $\dot{V}$ now represent a positive definite function. If the coefficients of the remaining terms $(f(y)).\ z_i$ can all be made zero while satisfying condition (d) above, the system will be proved totally stable (see appendix I). This ca be achieved if the roots of the simultaneous quadratic equations

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = \beta_i + A_i \quad (i = 1,2,\ldots n) \qquad (1.1.8)$$

correspond in algebraic nature to the system poles.

If the numbers A_i are made infinitesimally small, they will not affect the nature of the roots of eqns. (1.1.8) but will still provide a small positive definite term in $\dot{V}$. The remaining terms constitute a positive semidefinite function. The numbers A_i thus play no direct part in the analysis, but are necessary to establish total stability.

An alternative form of the stability criterion is obtained by writing eqn. (1.1.7) in the form (neglecting the small numbers A_i)

$$\dot{V} = \left(\sum_{i=1}^{n} a_i z_i + \sqrt{r}\, f(y)\right)^2 + f(y) \sum_{i=1}^{n} a_i z_i \left(2 \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} - 2\sqrt{r}\, a_i - \beta_i\right)$$

and instead of eqns. (1.1.8) using the equations

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = 2\sqrt{r}\, a_i + \beta_i \quad (i = 1,2,\ldots n) \qquad (1.1.9)$$

to determine the coefficients a_i. The correct algebraic nature of these roots, together with conditions (a) (b) and (c) above constitute a second stability criterion.

1.1b. <u>$m = n$</u>

If $m = n$ in $G(p)$, the corresponding canonic form will be

$$\dot{z}_i = \lambda_i z_i + f(y) \quad (i = 1,2,\ldots n) \qquad (1.1.10)$$

$$y = \sum_{i=1}^{n} \alpha_i z_i - R\, f(y) \qquad (1.1.11)$$

where the numbers α_i and λ_i are as previously defined, and the constant is the remainder in the partial fraction expansion of $G(p)$. Since $\dot{y}$ will now involve $\dot{f}(y)$, the Liapunov function previously considered cannot be used. Take instead the form

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j \; z_i z_j}{\lambda_i + \lambda_j}$$

Differentiating V w.r.t. time using eqns. (1.1.10) and (1.1.11) gives

$$\dot{V} = (\sum_{i=1}^{n} a_i z_i)^2 + 2\, f\,(y) \sum_{i=1}^{n} a_i z_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} \quad .$$

Add to $\dot{V}$ the expression (see eqn. 1.1.11)

$$f(y)\,(y - \sum_{i=1}^{n} \alpha_i z_i + R\, f(y)) = 0.$$

If $R \geq 0$, then

$$\dot{V} = (\sum_{i=1}^{n} a_i z_i + \sqrt{R}\, f(y))^2 + y\, f\,(y)$$

$$+ f(y) \sum_{i=1}^{n} z_i \,(2\, a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} - 2\sqrt{R}\, a_i - \alpha_i).$$

The coefficient equations are therefore

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = 2\sqrt{R}\, a_i + \alpha_i \quad (i = 1,2,\ldots n) \qquad (1.1.12)$$

If eqns. (1.1.12) have roots of the appropriate algebraic nature, the system is proved totally stable (by the addition of an infinitely small functi as before) provided that

$y\, f(y) \geq 0$ for all $|y| \neq 0$, and $f(0) = 0$ (1.1.13)

Since R may be zero, this criterion is also valid when $m < n$. Several other criteria may be established in a similar manner using the first canonic form (Letov[1], pp. 128, 138, 142, 147, 163). Systems which contain poles of G(p) at the origin of the p - plane may also be dealt with (Letov[1], pp. 118, 199, Chap. VI).

1.2. A more general approach

Although the criteria of section 1.1 are very satisfactory, the resulting Liapunov function derivatives contain only sign semidefinite functions of the state variables. While this is of no consequence in single-loop systems, in the analysis of multivariable systems it proves very useful to choose Liapunov functions which have truly sign definite derivatives.

Consider therefore the following form as a Liapunov function for the system of eqns. (1.1.2) and (1.1.3):-

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i\, a_j\, z_i\, z_j}{w_i + w_j} - \eta \int_0^y f(y)\, dy \qquad (1.2.1)$$

where $w_i = c + \lambda_i$, and c and η are real constants. $(i = 1, 2, \ldots n)$

Differentiating V w.r.t. time, using eqns. (1.1.2) and (1.1.3) gives

$$\dot{V} = \Big(\sum_{i=1}^{n} a_i z_i\Big)^2 - 2c\,\overline{V} + \eta r\, f^2(y)$$

$$+ f(y) \sum_{i=1}^{n} z_i \Big(2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} - \eta\, \beta_i\Big) \qquad (1.2.2)$$

where $\overline{V}$ is the quadratic function of eqn. (1.2.1).

If $\eta r \geq 0$, this may be written as

$$\dot{V} = \left(\sum_{i=1}^{n} a_i z_i + \sqrt{\eta r}\, f(y) \right)^2 - 2c\overline{V}$$

$$+ f(y) \sum_{i=1}^{n} z_i \left(2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} - 2\sqrt{\eta r}\, a_i - \eta \beta_i \right)$$

Setting the coefficients of terms $f(y)\, z_i$ to zero gives

$$2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = 2\sqrt{\eta r}\, a_i - \eta \beta_i \quad (i = 1,2,\ldots n) \qquad (1.2.3)$$

Then if there exists some negative value of c such that

(a) Eqns. (1.2.3) have appropriate roots a_i for $\eta = -1$ and $r \leq 0$

(b) Re (w_i) and Re $(\lambda_i) > 0$ for all i

V and $\dot{V}$ are positive definite and the system is unstable.

Conversely, if there exists some positive value of c such that

(c) Eqns. (1.2.3) have appropriate roots a_i for $\eta = +1$ and $r \geq 0$

(d) Re (w_i) and Re$(\lambda_i) < 0$ for all i

V is negative definite, $\dot{V}$ is positive definite and the system is totally stable.

It is sufficient to take $|\eta| = 1$, since the value of $|\eta|$ does not affect the nature of the roots of eqns. (1.2.3).

From the above conditions, stability may only be established when all the poles of G(p) lie in the left-half plane, and instability when the poles lie in the right-half plane.

If one puts $\eta = 0$ in V of eqn. (1.2.1), an alternative criterion similar to the second criterion of section 1.1. may be established.

Add to $\dot{V}$ (eqn.(1.2.2.)) the expression

$$F(y)\ (y - \sum_{i=1}^{n} \alpha_i z_i) \equiv 0.$$

Then

$$\dot{V} = (\sum_{i=1}^{n} a_i z_i)^2 - 2c\overline{V} + yf(y) + f(y) \sum_{i=1}^{n} z_i \ (2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} - \alpha_i)$$

The coefficient equations will therefore be

$$2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = \alpha_i \ (i = 1,2,\ldots n) \qquad (1.2.4)$$

and if the nonlinear characteristic is confined to the first and third quadrants, the system is totally stable, or unstable, depending on the solutions of eqns. (1.2.4), as discussed above for eqns. (1.2.3), but with $\eta = 0$.

When m = n in G(p), the criterion equivalent to eqns. (1.1.12) is that again condition (1.1.13) be satisfied, and the roots of the equations

$$2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = 2\sqrt{R}\, a_i + \alpha_i \ (i = 1,2,\ldots n) \qquad (1.2.5)$$

be of appropriate algebraic nature. Depending on the parameter c, the system is then totally stable or unstable as discussed above, but with

$\eta = 0$, and using eqns. (1.2.5).

Note finally that the above criteria do not represent improvements on those of section 1.1. They are included to illustrate a means of achieving a Liapunov function whose derivative is sign definite, as opposed to the sign-semidefinite forms of section 1.1.

1.3. Assessment of the criteria.

All of the criteria mentioned so far depend on the satisfaction of certain restrictions on the characteristics of the nonlinearities, and the nature of the roots of a set of simultaneous quadratic equations. For sets of four or more such equations, an analytic solution is virtually impossible. This difficulty may be overcome by modifying the Liapunov function used (Letov[1], p.163), but since digital solution of the equations proves readily attainable (See Appendix II) this modification will not be considered here. In any case, the modification produces criteria which are subsets of the results otherwise obtained.

The criterion involving eqns. (1.1.9) and therefore eqns. (1.2.3) apparently produces the most satisfactory results in many cases (Letov[1], pp.131, 156).

In appendix II, some necessary (but not sufficient) conditions are given for the roots of some of the quadratic equations derived to be of the appropriate algebraic nature. One significant point arising from these conditions is that equations containing only the residues (α_i) on their right hand sides cannot have the necessary nature of roots if the number $r = 0$. Since $r = 0$ implies that $n - m \geq 2$ in $G(p)$, many practical

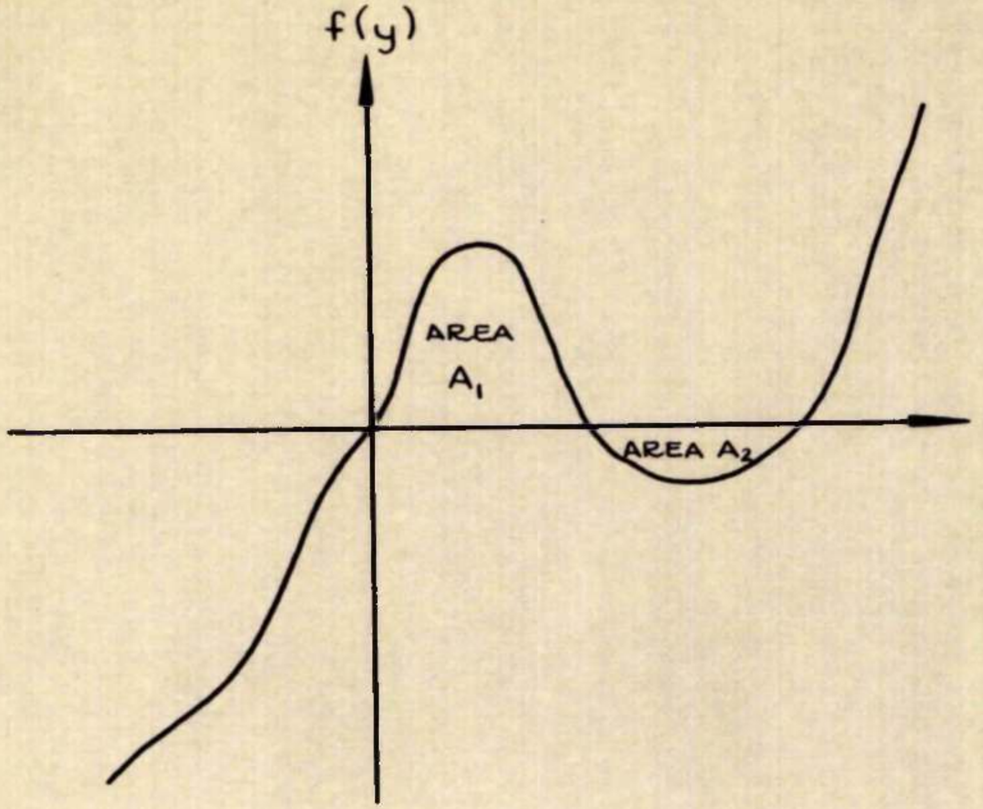

FIG. 3. CONFINEMENT BY A TYPE (2) RESTRICTION

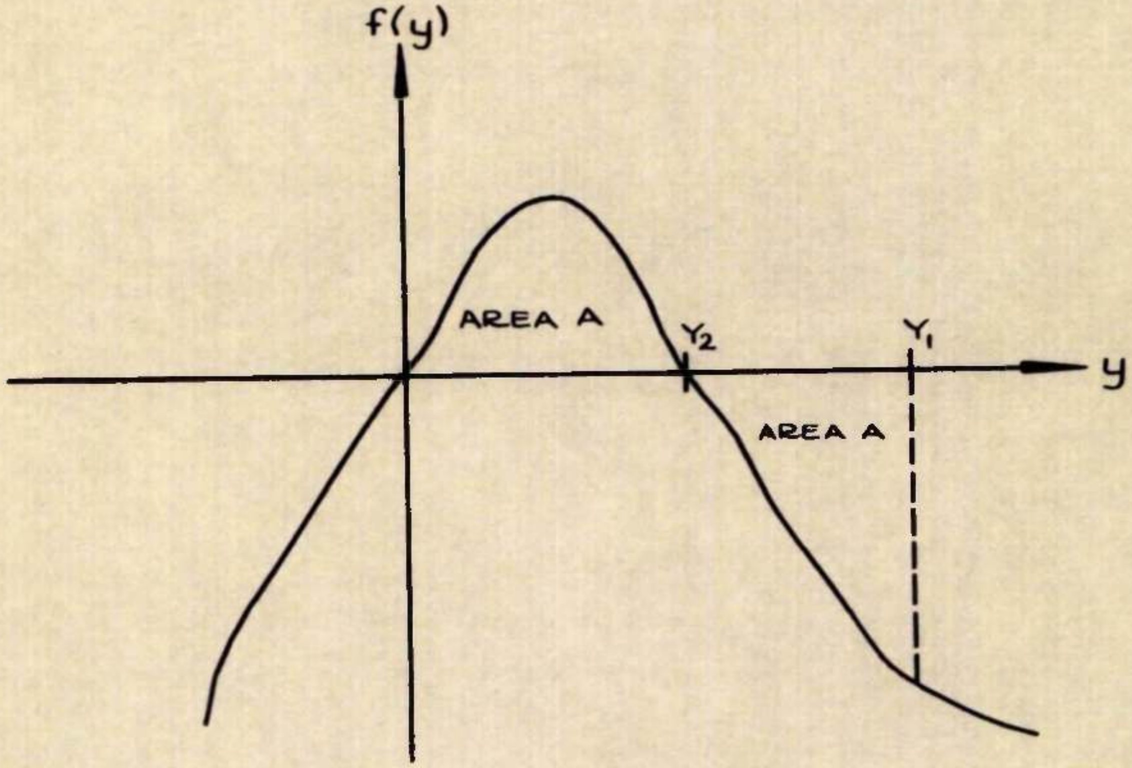

FIG. 4. CHARACTERISTIC WITH LIMITED STABILITY PROPERTIES.

systems will be rejected by such criteria. Eqns. (1.1.9) and (1.2.3) do not have this drawback.

The restriction on the nonlinear characteristics are of two types:

(1) $\int_{0}^{y} f(y)dy \geq 0$ for all $|y| \neq 0$, $f(0) = 0$, and

(2) $yf(y) \geq 0$ for all $|y| \neq 0$, $f(0) = 0$.

For example, in fig.3 the characteristic shown is acceptable by a type (1) restriction provided that area $A_1 \geq$ area A_2. A type (2) restriction merely confines the characteristic to lie within the first and third quadrants. If outwith $y_{min} \leq y \leq y_{max}$ the restriction is violated, then the system is no longer totally stable but merely asymptotically stable provided the initial deviation of the variable y lies within the region $y_{min} \leq y \leq y_{max}$. In fig.4, for example, the system will be stable under a type (1) restriction up to the point Y_1, and the system will be stable under a type (2) restriction up to the point Y_2.

Because of the laxity of both restrictions (1) and (2) above, it is evident that all the tests must fail for many systems which are stable under only slightly more severe restrictions on the nonlinear characteristic. Since neither type (1) or type (2) can distinguish between linear characteristics whose gains vary anywhere between 0 and ∞ the criteria will yield positive results only if the equivalent linear system (where f(y) is replaced by a linear gain Ky) is stable for all values of open-loop gain. That is, a system whose roots-locus is confined entirely to the left-half plane.

If however a method could be derived which would confine the nonlinear characteristic to lie in some finite sector of the first and third quadrants, a great number of otherwise intractable systems could be dealt with. Such a technique has been devised by Gibson[4] and Rekasius[2], in which stability is established provided that the nonlinear characteristic is confined to the sector [k,K]*. This technique is considered in the next section, using the approach of section 1.2.

1.4. Stability in the sector [k,K].

To limit the maximum slope of the nonlinear characteristic at the origin (i.e. the system's small-perturbation feedback gain), a simple linear transformation of the variables y and f(y) may be applied as follows (Gibson[4], p.330).

Rotation of the output axis of the nonlinear characteristic by an angle θ (~~see fig.5~~) is equivalent to defining a new input variable,

$$u = y - \frac{1}{K} f(y) \qquad (1.4.1)$$

where $K = 1/\tan(\theta)$ is the slope of the rotated axis. The output variable will now be defined in terms of the new input as $\phi(u) \equiv f(y)$.

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*The sector [k,K] of a nonlinear characteristic is defined by the area between the lines $f(y) = ky$ and $f(y) = Ky$.

Substituting for y from eqn. (1.4.1) into the canonic eqns. (1.1.2), (1.1.3), (1.1.10) and (1.1.11) gives the modified form

$$\dot{z}_i = \lambda_i z_i + \phi(u) \quad (i = 1,2,\ldots n) \tag{1.4.2}$$

$$u = \sum_{i=1}^{n} \alpha_i z_i - \overline{R}\,\phi(u) \tag{1.4.3}$$

where $\overline{R} \equiv R + 1/K$

Take as a Liapunov function for this system

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j z_i z_j}{w_i + w_j}$$

where w_i are defined as in section 1.2.

Differentiating V w.r.t. time and adding the expression

$$\phi(u)\,(u - \sum_{i=1}^{n} \alpha_i z_i + \overline{R}\;\phi(u)) \equiv 0$$

gives

$$\dot{V} = (\sum_{i=1}^{n} a_i z_i + \sqrt{\overline{R}}\, f(y))^2 - 2cV + u\phi(u)$$

$$+ \phi(u) \sum_{i=1}^{n} z_i \,(2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} - 2\sqrt{\overline{R}}\, a_i - \alpha_i)$$

The coefficient equations are therefore

$$2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = 2\sqrt{\overline{R}}\, a_i + \alpha_i \quad (i = 1,2,\ldots n) \tag{1.4.4.}$$

The requirements for establishing stability or instability are then as follows (cf section 1.2).

The function $\phi(u)$ must be single-valued, and

$$u \phi (u) \geq 0 \text{ for all } |u| \neq 0, \phi (0) = 0 \qquad (1.4.5)$$

Restriction (1.4.5) confines the nonlinear characteristic to the sector [0,K].

If then there exists some negative value of c such that eqns. (1.4.4) have appropriate solutions, and

(a) $Re(w_i)$ and $Re(\lambda_i) > 0$ for all i then V and $\dot{V}$ are positive definite, and the system is unstable.

Conversely, if there exists some positive value of c such that eqns. (1.4.4) have appropriate solutions, and

(b) $Re(w_i)$ and $Re(\lambda_i) < 0$ for all i then V is negative definite, $\dot{V}$ is positive definite and the system is totally stable.

In both cases, the number $\overline{R} = R + 1/K$ must be non-negative. The transformation (1.4.1) used in establishing the above criterion is in fact equivalent to shifting the zeroes of G(p) (Gibson,[4], p.330) and will in future be referred to as the zero-shifting technique.

As in section 1.3, stability and instability may only be established when the poles of G(p) lie respectively in the left and right halves of the p-plane. Since the nonlinear characteristic is now confined to the sector [0,K], the criteria may be applied to systems which are either stable or unstable for all linear feedback gains in the range [0,K].

If a given system is unstable for low values of linear feedback gain (that is has poles with positive real parts) then any applicable criterion must further confine the nonlinear characteristic to some sector [k,K] where $k > 0$.

To restrict the minimum slope of the nonlinear characteristic at the origin, rotation of the input axis through an angle Ω (~~see fig.~~5) is equivalent to defining a new output variable (Gibson[4], p.329).

$$\psi (y) = f(y) - ky \qquad (1.4.6)$$

where $k = \tan (\Omega)$ is the slope of the rotated axis. In this case it is shown that the form of the system equations is unchanged. Substituting for f(y) from eqn. (1.4.6) in eqn. (1.1.1) gives

$$y = - G(p) \; (\psi(y) + ky)$$

or

$$y = - \frac{G(p)}{1 + kG(p)} \; \psi (y) = - G'(p) \; \psi (y) \qquad (1.4.7)$$

Comparing eqns. (1.4.7) and (1.1.1), the criteria of section 1.2. apply unaltered in form, where the parameters λ_i and α_i are now the poles, and residues (with sign reversed) of G'(p).

The corresponding nonlinearity restrictions will now be that $\psi(y)$ is a single-valued function, and that either

(1) $y \, \psi(y) \geq 0$ for all $|y| \neq 0$, $\psi(0) = 0$ (1.4.8)

or

(2) $\int_0^y \psi (y) \, dy \geq 0$ for all $|y| \neq 0$, $\psi(0) = 0$ (1.4.9)

depending on which criterion of section 1.2 is used.

Restriction (1.4.8) confines the nonlinear characteristic to the sector [k, ∞], whereas restriction (1.4.9) is somewhat weaker (cf. section 1.3).

Rotation of the input axis has effectively shifted the poles of G(p), and the above method will in future be referred to as the pole-shifting technique.

By applying first the pole-shifting technique (since the form of the system equation is left unaltered), then the zero-shifting technique, stability or instability criteria may be applied to any system which is stable for all linear feedback gains in the range [k, K].

1.5 Control Quality.

Although stability is an essential property of any control system, some estimate of the quality of response to an initial deviation from equilibrium should be obtained. Liapunov methods can provide estimates of the largest and smallest envelopes of such a response.

Consider the transformation of the state variables of section (1.1) given by

$$x_i = e^{\lambda t} z_i \quad (i = 1,2,\ldots n) \tag{1.5.1}$$

$$v = e^{\lambda t} y \tag{1.5.2}$$

where λ is a real constant.

Substituting in the canonic equations (1.1.10) and (1.1.11) gives

$$\dot{x}_i = (\lambda + \lambda_i)\, x_i + F\,(v,t) \quad (i = 1,2,\ldots n) \tag{1.5.3}$$

$$v = \sum_{i=1}^{n} \alpha_i\, x_i - RF\,(v,t) \tag{1.5.4}$$

where $F(v,t) = e^{\lambda t} f\,(ve^{-\lambda t})$. If some sign definite Liapunov function $V = \underline{x}^T Q\, \underline{x}$ is chosen for the above system, the sign of its derivative will depend in some way upon λ (from eqn. (1.5.3)). Suppose that λ is bounded such that for $\lambda < \lambda_e$, $\dot{V}$ is sign definite of the opposite sign to V, and for $\lambda > \lambda_u$, $\dot{V}$ is sign definite of the same sign as V.

Define $R(t) = \underline{z}^T Q\, \underline{z}$.

Then $V = e^{2\lambda t} R(t)$, and by definition of λ_e and λ_u it follows that

$$R(0)e^{-2\lambda_u t} \leq R(t) \leq R(0)e^{-2\lambda_e t}. \tag{1.5.4}$$

Letov[1] (Chaps. III and VIII) obtains estimates of these constants λ_e and λ_u by use of the 'second canonic form' of state variables. This form however, is valid only when (Rekasius[2]p. 62)

(1) the zeroes of G(p) are all different, and

(2) the order of the numerator of G(p) is one less than the order of the denominator, i.e. $n - m = 1$.

Furthermore, stability may only be established when the zeroes of G(p) are all in the left-half plane (Letov[1], Chap.VI). This means that systems which are unstable for high values of linear feedback gain cannot be dealt with. To apply the zero-shifting technique to state equations of the second canonic form is impossible, since in this form y appears as a state variable, and terms $\frac{df(y)}{dt}$ would appear in the state equations.

Consider instead the first canonic form, of eqns. (1.5.3) and (1.5.4), and take as a Liapunov function the form

$$V = \underline{x}^T Q \underline{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j x_i x_j}{\mu_i + \mu_j} \qquad (1.5.5)$$

where $\mu_i = c + \lambda + \lambda_i$, and c is a real constant.

Differentiate V w.r.t.time and add the expression

$$F(v,t)(v - \sum_{i=1}^{n} \alpha_i x_i + RF(v,t)) \equiv 0.$$

Completing the appropriate squares gives the coefficient equations

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\mu_i + \mu_j} = 2\sqrt{R}\, a_i + \alpha_i \quad (i = 1,2,\ldots n) \qquad (1.5.6)$$

leaving the derivative as

$$\dot{V} = -2cV + vF(v,t) + \left(\sum_{i=1}^{n} a_i x_i + \sqrt{R}\, F(v,t)\right)^2.$$

Since $vF(v,t) = e^{2\lambda t} yf(y)$, if the same nonlinearity restriction applies as in the second criterion of section 1.2, namely

$$y\, f(y) \geq 0 \text{ for all } |y| \neq 0, \; f(0) = 0$$

then the estimates of the bounds on λ, λ_e and λ_u may be obtained as follows.

If there exists some negative value of c such that

(a) eqns. (1.5.6) have appropriate roots for all $\lambda \geq \lambda_u$, say, and

(b) Re $(\mu_i) > 0$ for all i, then V and $\dot{V}$ are positive definite, and λ_u is an estimate of the upper bound of λ.

If there exists some positive value of c such that

(c) eqns. (1.5.6) have appropriate roots for all $\lambda \leq \lambda_e$, say and

(d) Re $(\mu_i) < 0$ for all i

then V is negative definite, $\dot{V}$ is positive definite and λ_e is an estimate of the lower bound of λ.

Since the coefficients of V are dependent on λ, the relation (1.5.4) becomes two:-

$$R_1(0)e^{-2\lambda_u t} \leq R_1(t) \tag{1.5.7}$$

and

$$R_2(t) \leq R_2(0)\, e^{-2\lambda_e t} \tag{1.5.8}$$

where

$$R_1(t) = \underline{z}^T Q_1 \underline{z} \text{ (at } \lambda = \lambda_u)$$

and

$$R_2(t) = \underline{z}^T Q_2 (\underline{z}) \text{ (at } \lambda = \lambda_e).$$

Note that the above criterion subsumes the stability criterion of section 1.2 (when no integral terms are used in the Liapunov function).

Although the above analysis theoretically allows determination of an upper bound of λ, the restriction on the nonlinearity (which allows an infinitely large linear gain) means that eqns. (1.5.7) cannot yield appropriate solutions when Re $(\mu_i) > 0$, if the system is stable. The nonlinearity must be restricted to some sector [k,K] $(k < K < \infty)$ before a finite value of λ_u is obtained.

Use of eqns. (1.4.2) and (1.4.3) then gives the canonic form

$$\dot{x}_i = (\lambda + \lambda_i)\, x_i + L\,(w,t) \qquad (1.5.9)$$

$$\dot{w} = \sum_{i=1}^{n} \alpha_i x_i - \overline{R}\, L\,(w,t) \qquad (1.5.10)$$

where $\overline{R} = R + 1/K$, $w = e^{\lambda t} u$, and $L\,(w,t) = e^{\lambda t} \phi\, (we^{-\lambda t})$

The Liapunov function

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j x_i x_j}{\mu_i + \mu_j} \qquad (1.5.11)$$

yields for the above system the coefficient equations

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\mu_i + \mu_j} = 2\sqrt{R}\, a_i + \alpha_i \quad (i = 1,2,\ldots n) \qquad (1.5.12)$$

provided that $u \phi(u) \geq 0$ for all $|u| \neq 0$, and $\phi(0) = 0$.
The bounds λ_e and λ_u are then obtained as for the sector $[0,\infty]$ above, but using eqns. (1.5.12). This criterion for estimation of λ_e and λ_u subsumes the stability criterion of section 1.4 (See eqns. (1.4.4)).

If the nonlinearity is confined to some sector $[k, \infty]$ the form of the canonic eqns. remains unaltered, and it is sufficient to consider the first criterion of this section, where the poles of G(p) have been shifted as described in section 1.4.

1.6. Examples

Consider the system in which $G(p) = \frac{(p+1)}{(p+2)(p+3)}$

Then $\lambda_1 = -2$, $\lambda_2 = -3$ $\alpha_1 = 1$, $\alpha_2 = -2$.

Using eqns. (1.2.5), the necessary and sufficient conditions for real coefficients a_1 and a_2 are (See appendix II)

$$\alpha_1 w_1 + \alpha_2 w_2 \geq 0 \qquad (1.6.1)$$

and

$$\frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2} \geq 0. \qquad (1.6.2)$$

i.e. $(4-c) \geq 0$

and $\frac{(1-c)}{(c-2)(c-3)} \geq 0.$

The above inequalities are both satisfied provided that $c \leq 1$. Also, for $c \leq 1$ Re $(w_i) < 0$ for $i = 1,2$. It is therefore sufficient to choose $c = 1$. Then $w_1 = -1$, $w_2 = -2$, and the system is totally stable provided that the nonlinear characteristic is confined to the first and third quadrants as indicated in section 1.2. Note that

$$\dot{V} = (a_1 z_1 + a_2 z_2)^2 - 2 \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{a_i a_j z_i z_j}{w_i + w_j}$$

$+ yf(y)$

which is a positive definite function of the state variables.

<u>Example 1.2</u>

Consider the system in which $G(p) = \dfrac{4 - p}{(4+p)(1+p)}$ and the given nonlinear characteristic is confined to the sector [0,2]. Since the linear system is stable for all feedback gains between 0 and 5, one expects this nonlinear system to be stable. G(p) has a zero in the right half plane, and the zero-shifting technique must be applied to confine the nonlinearity to the given sector.

Then

$\lambda_1 = -4$, $\lambda_2 = -1$, $\alpha_1 = 8/3$, $\alpha_2 = -5/3$ and $K = 2$ (see section 1.4)

Using eqns. (1.4.6) the necessary and sufficient conditions for real coefficients a_1 and a2 are (see appendix II)

$$\frac{1}{2} + \frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2} \geq 0$$

and $\frac{1}{4}(w_1^2 - w_2^2)^2 + (w_1^2 - w_2^2)(\alpha_1 w_1 - \alpha_2 w_2) + (\alpha_1 w_1 + \alpha_2 w_2)^2 \geq 0.$

Substituting for α_1, α_2, w_1 and w_2 reduces these inequalities to

$$\frac{c^2 - 3c + 12}{(c-4)(c-1)} \geq 0 \tag{1.6.3}$$

and $(c - 4.325)(c - 0.7425) \geq 0$ (1.6.4)

These inequalities are satisfied provided that either $c \geq 4.325$ or $c \leq 0.7425$.

Since the necessary condition Re $(w_i) < 0$ means that c must be less than 1, it is sufficient to choose any value of c between 0 and 0.7425 to verify stability of the system.

To estimate the bounds λ_e and λ_u, it is sufficient to consider eqns. (1.6.3) and (1.6.4) but with c replaced by $c + \lambda$ (see eqns. 1.5.12); The inequalities are then satisfied provided that

$c + \lambda \geq 4.325$ or $c + \lambda \leq 0.7425$ (1.6.5)

Also, the requirements Re $(\mu_i) > 0$ for $\lambda = \lambda_u$ and Re $(\mu_i) < 0$ for $\lambda = \lambda_e$ mean that

$c + \lambda_u > 4$ and $c + \lambda_e < 1$ (1.6.6)

where c is negative in the first inequality, and positive in the second, of (1.6.6) (see section 1.5). Since $|c|$ may be infinitesimally small in this single-loop case, the inequalities (1.6.5) and (1.6.6) are satisfied provided that

$\lambda_u \geq 4.325$ and $\lambda_e \leq 0.7425.$

Then for $\lambda = \lambda_u$, $\mu_1 = 0.325$ & $\mu_2 = 3.325$, while eqns. (1.4.6) have a pair of roots $a_1 = 0.530$, $a_2 = 6.40$.

For $\lambda = \lambda_e$, $\mu_1 = -3.2575$ and $\mu_2 = -0.7425$, while eqns. (1.4.6) have a pair of roots $a_1 = -0.992$, $a_2 = -0.743$.

From inequalities (1.5.7) and (1.5.8), the transient response of the system is therefore bounded by the inequalities

$$R_1(0)\, e^{-8.65t} \leq R_1(t)$$

and $R_2(t) \leq R_2(0)e^{-1.485t}$

where

$$R_1(t) = 0.433\, z_1^2 + 1.86 z_1 z_2 + 6.16 z_2^2$$

$$R_2(t) = -0.152\, z_1^2 - 0.420 z_1 z_2 - 1.071 z_2^2 .$$

1.7 . Comment on Chapter I

The methods of sections 1.2, 1.4 and 1.5 combine under the same criteria the problems of stability, instability and quality of response. Unlike the criteria of Letov[1], the Liapunov functions chosen in the analysis have truly sign definite derivatives - a fact which proves extremely useful in establishing criteria for multivariable systems.

The prerequisite that any system must be stable for all open-loop linear gain between 0 and ∞ (section 1.2) may be overcome by use of the pole and zero-shifting techniques of section 1.4. In general, the stability criteria may be applied to any system which is stable for all open-loop gains in the sector [k,K], provided that the nonlinear characteristic is confined to the same sector, or in some cases a slightly weaker restriction if integral terms are used in the appropriate Liapunov function. This raises the following question:

If a given system is stable for all open loop linear gains in the sector [k,K], is the corresponding nonlinear system stable in the same sector?

Jury[5] has succeeded in producing a counterexample to this conjecture, although it remains valid in many cases (see e.g. Letov[1], p.156).

The main difficulty in applying the criteria is the solution of simultaneous quadratic equations which may have complex coefficients. The digital method described in Appendix II allows rapid solution of these equations even for high-order systems. For example, a set of six such equations can be solved in some 25 seconds on an English Electric KDF 9 computer. This speed is achieved even when the initial set of values chosen is up to 100 times the initial programme step length from the true set of solutions.

Since all criteria prove total stability, the systems under consideration will also be stable for any bounded inputs applied to them (see e.g. Malkin[6]). Note finally that the criteria represent sufficient conditions only.

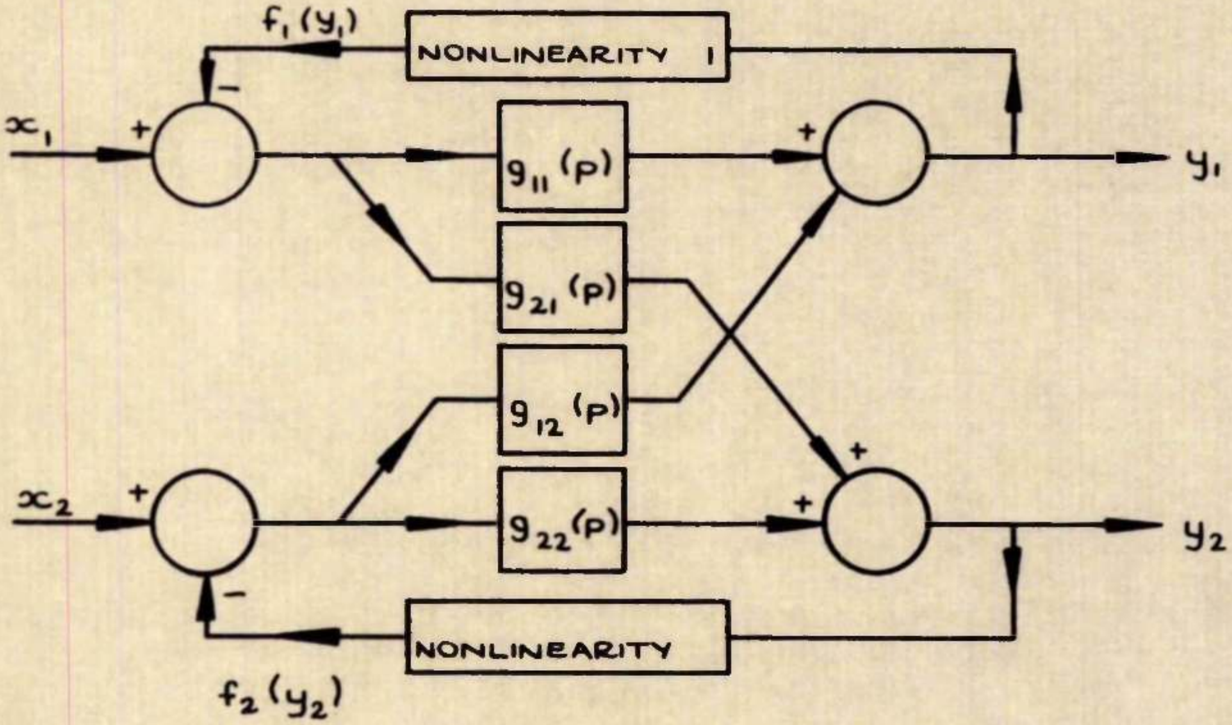

FIG. 6. A TWO-VARIABLE CLASS 1 SYSTEM.

CHAPTER II

Stability of Class 1 Multivariable Systems

Introduction

The natural extension of the single-loop analysis of Chapter I is to systems in which a linear multivariable process is controlled by many feedback loops, each loop containing a single-valued nonlinear element. Class 1 systems are defined to be those in which there are only feedforward interactions in the process (as illustrated in fig.6 for a two-variable process). Experimental determination of the dynamic relationship (transfer functions) between the inputs and outputs of a linear multivariable process always results in mathematical models of a class 1 nature.

Previous analytic work on stability of multiple-nonlinearity systems is discussed in the first section of this chapter, and the disadvantages of the criteria obtained are pointed out.

In extension of the methods of Chapter I to include class 1 systems, construction of suitable canonic forms of state is found to be dependent on the structure of the transfer matrix of the linear process. This leads to two further subclassifications of class 1 systems. Stability and instability criteria are then developed for these two subclasses.

Criteria are also developed for estimating the bounds of a system's response to an initial disturbance. These criteria subsume certain of the previously developed stability criteria.

Three worked examples are included at the end of the chapter.

2.1. The criteria of Letov.

Letov[1] (Chap.IX) considered the stability of a system containing two nonlinearities, described by the following canonic form of state variables:

$$\dot{x}_i = \lambda_i x_i + u_{1i}\, f_1\,(y_1) + u_{2i}\, f_2\,(y_2)\,(i = 1,2,\ldots n) \tag{2.1.1}$$

$$\dot{y}_1 = \sum_{i=1}^{n} \beta_{1i}\, x_i - r_{11}\, f_1\,(y_1) - r_{12}\, f_2\,(y_2) \tag{2.1.2}$$

$$\dot{y}_2 = \sum_{i=1}^{n} \beta_{2i}\, x_i - r_{21}\, f_1\,(y_1) - r_{22}\, f_2\,(y_2) \tag{2.1.3}$$

To reduce the system to block diagram form, substitute eqn. (2.1.1) into eqns. (2.1.2) and (2.1.3), giving

$$y_1 = -\,G_{11}\,(p)\, f_1\,(y_1) - G_{12}\,(p)\, f_2\,(y_2) \tag{2.1.4}$$

$$y_2 = -\,G_{21}\,(p)\, f_1\,(y_1) - G_{22}\,(p)\, f_2\,(y_2) \tag{2.1.5}$$

where

$$G_{ks}\,(p) = -\frac{1}{p}\left(\sum_{i=1}^{n} \frac{\beta_{ki}\ u_{si}}{p - \lambda_i} - r_{ks}\right) \tag{2.1.6}$$

Equations (2.1.4) and (2.1.5) describe a two variable class 1 system as shown in fig.6, but with the inputs x_1 (t) and x_2 (t) removed.

From eqn. (2.1.6), each transfer function G_{ks} (p) has a common denominator

$$p \prod_{i=1}^{n} (p - \lambda_i).$$

Letov states that the poles λ_i must all be different for the canonic form to be valid; but in some cases this need not be true as is seen below (sections 2.2a and 2.2b).

He chooses as a Liapunov function for this system the negative definite form

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i\, a_j\, x_i\, x_j}{\lambda_i + \lambda_j} - \int_0^{y_1} f_1(y_1)\, dy_1 - \int_0^{y_2} f_2(y_2)\, dy_2$$

where the assumptions are made, similar to Chapter I, that

(a) Re $(\lambda_i) < 0$ for all i

(b) $\int_0^{y_1} f_1(y_1)\, dy_1$ and $\int_0^{y_2} f_2(y_2)\, dy_2 \geq 0$ for all

$|y_1|$, $|y_2| \neq 0$, and $f_1(0) = f_2(0) = 0$

(c) the coefficients a_i are of appropriate algebraic nature.

Differentiating V w.r.t time, using eqns. (2.1.1.) to (2.1.3) gives

$$\dot{V} = (\sum_{i=1}^{n} a_i\, x_i)^2 + r_{11} f_1^2 (y_1) + r_{22} f_2^2 (y_2) + (r_{12} + r_{21}) f_1(y_1) f_2(y_2)$$

$$+ f_1(y_1)\left[\sum_{i=1}^{n} x_i \left(2a_i \sum_{j=1}^{n} \frac{a_j u_{1j}}{\lambda_i + \lambda_j} - \beta_{1i}\right)\right.$$

$$+ f_2(y_2)\left[\sum_{j=1}^{n} x_i \left(2a_i \sum_{j=1}^{n} \frac{a_j u_{2j}}{\lambda_i + \lambda_j} - \beta_{2i}\right).\right.$$

$\dot{V}$ is easily constrained to be positive semidefinite by setting the terms $f_1(y_1) z_i$ and $f_2(y_2) z_i = 0$ for all i.

Then

$$2a_i \sum_{j=1}^{n} \frac{a_j u_{1j}}{\lambda_i + \lambda_j} + \beta_{1i} \qquad (2.1.7)$$

$$(i=1,2,\ldots n)$$

$$2a_i \sum_{j=1}^{n} \frac{a_j u_{2j}}{\lambda_i + \lambda_j} = \beta_{2i} \qquad (2.1.8)$$

Applying Sylvester's theorem to the remaining quadratic function of the variables $f_1(y_1)$ and $f_2(y_2)$ gives the following conditions for V to be positive semidefinite in the state variables x_i:

$$r_{11} > 0,\ r_{22} > 0,\ 4r_{11}r_{22} - (r_{12} + r_{21})^2 \geq 0.$$

The 2n eqns. (2.1.7) and (2.1.8), however, contain only n dependent variables a_i; for a satisfactory solution n system parameters must be predetermined. Finding these values such that the resulting roots a_i are also of the appropriate nature is an extremely difficult task (Letov[1], p.264). For a given system in which all parameters are fixed, the possibility of appropriate solutions of these equations is very remote indeed.

Other analytic work on systems with multiple nonlinearities has proceeded along the same lines (e.g. Bedelbaev[7], Sultanov[8]).

Consider as an alternative using only one of the above equations, say (2.1.7). Then the coefficients a_i will be determined for arbitrary system parameters, and

$$\dot{V} = (\sum_{i=1}^{n} a_i x_i)^2 + r_{11} f_1^2 (y_1) + r_{22} f_2^2 (y_2)$$

$$+(r_{12} + r_{21}) f_1(y_1) f_2(y_2) + f_2 \sum_{i=1}^{n} x_i (2a_i \sum_{j=1}^{n} \frac{a_j u_{2j}}{\lambda_i + \lambda_j} - \beta_{2i})$$

Because of the perfect square $(\sum_{i=1}^{n} a_i x_i)^2$, $\dot{V}$ may be positive semidefinite only if certain relationships exist between the system parameters (see Appendix III).

Development of this approach by the author has led to a useful set of stability criteria (Rae and MacLellan[9]). A more general approach, however, is to extend the methods of Chapter I to include multivariable systems. Before doing this, some further classifications must be considered which affect the canonic representation of class 1 systems.

2.2. Canonic representation of class 1 systems.

The operational equation describing a class 1 system is (for all inputs removed)

$$\underline{y} = - G\underline{f} \qquad (2.2.1)$$

where G is the linear transfer matrix of the system. As in Chapter I, the above equation represents systems which contain either actuators or sensors as nonlinear devices. It is assumed in all that follows that each row of G is linearly independent.

To simplify the writing of the mathematics involved, the canonic forms used will be described by vectors and matrices associated with each (i,j)th transfer function of G. For example:

$\underline{z}^{(ij)}$ would be a vector of the Qcanonic variables used in describing the (i,j)th transfer function $g_{ij}(p)$, which contains Q poles.

$\Lambda^{(ij)}$ would be a Q x Q diagonal matrix whose elements are the Q poles of $g_{ij}(p)$

The order of any vector or matrix is thus determined by its superscript (ij), and is equal to the number of poles in the (i,j)th transfer function.

2.2a. No common poles in any row of G.

Consider the canonic form of the state variable equations

$$\dot{\underline{z}}^{(ij)} = \Lambda^{(ij)} \underline{z}^{(ij)} + \underline{f}_j^{(ij)} \quad (i,j = 1,2,\ldots n) \qquad (2.2a.1)$$

$$y_i = \sum_{j=1}^{n} \left(\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} f_j (y_j) \right) \qquad (2.2a.2)$$

where $\underline{z}^{(ij)}$ and $\Lambda^{(ij)}$ are defined above. $\underline{\alpha}^{(ij)}$ is a vector of the residues (with sign reversed) of $g_{ij}(p)$, R_{ij} is the remainder in the partial fraction expansion of $g_{ij}(p)$, and $\underline{f}_j^{(ij)}$ denotes a vector all of whose elements are $f_j\ (y_j)$. n is the number of system outputs (and inputs).

To show that these equations represent a class 1 system, substitution for $\underline{z}^{(ij)}$ into eqn. (2.2a.2) gives

$$y_i = \sum_{j=1}^{n} (\underline{\alpha}^{(ij)T} (p\underline{I}^{(ij)} - \Lambda^{(ij)})^{-1} \underline{f}_j^{(ij)} - R_{ij}\ f_j(y_j))$$

where $\underline{I}$ is the unit matrix.

The linear transformation which is used to obtain the above canonic form from any initial set of state variables is nonsingular only if (see Appendix IV)

(a) there are no multiple poles in any transfer functions

and

(b) the transfer functions of any row of G have no common poles among them.

Since many physical systems will violate condition (b), an alternative form is given below for such systems (section 2.2b).

As an illustration of the above canonic form, consider the system where

$$g_{11}(p) = \frac{2p^2 + 7p + 8}{p^2 + 3p + 2} = \frac{3}{p+1} + \frac{-2}{p+2} + 2,$$

$$g_{12}(p) = \frac{-4}{p+0.1}, \quad g_{21}(p) = \frac{k}{p+5},$$

$$g_{22}(p) = \frac{3p + 15}{p^2 + 9p + 18} = \frac{2}{p+3} + \frac{1}{p+6},$$

Then

$$\Lambda^{(11)} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \underline{\alpha}^{11} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\Lambda^{(12)} = [-0.1] \qquad \underline{\alpha}^{(12)} = [4]$$

$$\Lambda^{(21)} = [-5] \qquad \underline{\alpha}^{21} = [-k]$$

$$\Lambda^{(22)} = \begin{bmatrix} -3 & 0 \\ 0 & -6 \end{bmatrix} \qquad \underline{\alpha}^{(22)} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

and $R_{11} = 2$, $R_{12} = R_{21} = R_{22} = 0$.

2.2b. Common poles in one or more rows of G.

A suitable canonic form in this case is

$$\underline{\dot{z}}^{(ij)} = \Lambda^{(ij)} \underline{z}^{(ij)} + \underline{f}_j^{(ij)} \quad (i,j = 1,2,\ldots n) \qquad (2.2b.1)$$

$$\underline{\dot{z}}^{(i)} = \Lambda^{(i)} \underline{z}^{(i)} + \sum_{j=1}^{n} \underline{\hat{\alpha}}^{(ij)} f_j\,(y_j) \quad (i=1,2,\ldots h) \qquad (2.2b.2)$$

$$y_i = \sum_{j=1}^{n} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij}\, f_j\,(y_j)) + \underline{e}^{(i)T} \underline{z}^{(i)} \qquad (2.2b.3)$$

. h is the number of rows of G which each have one or more common poles among them.

$\Lambda^{(ij)}$ is a matrix of the poles of the (i,j)th transfer function which are not common to any other transfer function in the (i)th row of G.

$\underline{\alpha}^{(ij)}$ is a vector of the residues (with sign reversed) corresponding to the poles of $\Lambda^{(ij)}$.

$\Lambda^{(i)}$ is a matrix of the poles which are common to all transfer functions of the (i)th row of G.

$\underline{\hat{\alpha}}^{(ij)}$ is a vector of the residues (with sign reversed) corresponding to the poles of $\Lambda^{(i)}$ in the (i,j) th transfer function.

$\underline{e}^{(i)}$ is the unit vector.

Substituting for $\underline{z}^{(ij)}$ and $\underline{z}^{(i)}$ from eqns. (2.2b.1) and (2.2b.2) in eqn. (2.2b.3) gives

$$y_i = \sum_{j=1}^{n} \underline{\alpha}^{(ij)T} (p\, \underline{I}^{(ij)} - \Lambda^{(ij)})^{-1} - R_{ij}\, f_j\,(y_j))$$

$$+ \underline{e}^{(i)T} \left(p \, \underline{I}^{(i)} - \Lambda^{(i)}\right)^{-1} \sum_{j=1}^{n} \underline{\hat{a}}^{(ij)} f_j (y_j)$$

The poles of $\Lambda^{(i)}$, being independent of j, are the poles common to the (i)th row.

As is seen in Appendix IV, any linear transformation used to obtain the above canonic form is nonsingular only if there are no multiple poles in any transfer function of G. Systems which have one common characteristic equation can be represented by this canonic form.

As an illustration, consider the system where

$$g_{11}(p) = \frac{4p^3 + 69p^2 + 328\frac{2}{7}p + 469\frac{1}{7}}{p^3 + 17p^3 + 82p + 120}$$

$$= \frac{\frac{2}{7}}{p+4} + \frac{-1}{p+3} + \frac{12/7}{p+10} + 4,$$

$$g_{12}(p) = \frac{K_{12}\,(3p^2 + 15p + 11)}{p^3 + 12p^2 + 47p + 60} = \frac{K_{12}}{p+4} + \frac{-K_{12}}{p+3} + \frac{3K_{12}}{p+5},$$

$$g_{21}(p) = \frac{K_{21}\,(p^2 + 5p + 6)}{p^3 + 10p^2 + 29p + 20} = \frac{\frac{1}{6}K_{21}}{p+1} + \frac{-\frac{2}{3}K_{21}}{p+4} + \frac{\frac{3}{2}K_{21}}{p+5},$$

$$g_{22}(p) = \frac{3p^3 + 45p^2 + 135\ 1/2\ p + 92}{p^3 + 13p^2 + 44p + 32}$$

$$= \frac{\frac{-13}{42}}{p+1} + \frac{\frac{1}{6}}{p+4} + \frac{\frac{8}{7}}{p+8} + 3.$$

Row 1 of G has common poles at -4 and -3.

Row 2 of G has common poles at -1 and -4.

Then

$$\Lambda^{(11)} = [-10] \qquad \underline{\alpha}^{(11)} = [-\tfrac{12}{7}]$$

$$\Lambda^{(12)} = [-5] \qquad \underline{\alpha}^{(12)} = [-3K_{12}]$$

$$\Lambda^{(21)} = [-5] \qquad \underline{\alpha}^{(21)} = [-\tfrac{3}{2} K_{21}]$$

$$\Lambda^{(22)} = [-8] \qquad \underline{\alpha}^{(22)} = [-\tfrac{8}{7}]$$

$$\Lambda^{(1)} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}, \qquad \underline{\hat{\alpha}}^{(11)} = \begin{bmatrix} -\frac{2}{7} \\ 1 \end{bmatrix}, \qquad \underline{\hat{\alpha}}^{(12)} = \begin{bmatrix} -K_{12} \\ K_{12} \end{bmatrix}$$

$$\Lambda^{(2)} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}, \qquad \underline{\hat{\alpha}}^{(21)} = \begin{bmatrix} -\frac{1}{6} K_{21} \\ \frac{2}{3} K_{21} \end{bmatrix}, \qquad \underline{\hat{\alpha}}^{(22)} = \begin{bmatrix} \frac{13}{42} \\ -\frac{1}{6} \end{bmatrix}$$

and $R_{11} = 4$, $R_{22} = 3$, $R_{12} = R_{21} = 0$.

2.3. Stability in the sectors $[0, \infty]$.

The first set of stability criteria derived will be those which involve nonlinearity restrictions of the type

$$y_i \, f_i \, (y_i) \geq 0 \text{ for all } | \, y_i \, | \neq 0 \qquad (2.3.1)$$

or

$$\int_o^{y_i} f_i \, (y_i) \geq 0 \text{ for all } |y_i| \neq 0 \qquad (2.3.2)$$

where the nonlinear functions are single-valued, and are zero for all $y_i = 0$. For convenience, these restrictions will be referred to as confining the characteristics to the sectors $[0,\infty]$, although (2.3.2) is a somewhat weaker restriction (see Chapter I, section 1.3).

2.3a. No common row poles.

Assume that k rows of G have no remainders in the partial fraction expansions of their transfer functions,

i.e. $R_{ij} = 0$ for all $i,j \leq k$.

Any Liapunov function chosen may then involve integrals of the nonlinearities $f_i(y_i)$ only for $i \leq k$, otherwise terms dy_i/dt will arise in its derivative (see eqn. (2.2a.2)). Consider as a Liapunov function for the system of section 2.2a. the following form:

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} \, Q^{(ij)} \, \underline{z}^{(ij)} - n \sum_{i=1}^{k} \int_o^{y_i} f_i \, (y_i) \, dy_i \qquad (2.3a.1)$$

where the symmetric matrices $Q^{(ij)}$ are defined by their (r,s)th element

$$q_{rs}^{(ij)} = \frac{a_r^{(ij)} \; a_s^{(ij)}}{w_r^{(ij)} + w_s^{(ij)}}$$

and $w_r^{(ij)} = c + \lambda_r^{(ij)}$ (cf. section 1.2, Chap.I).

The above Liapunov function consists of groups of quadratic functions of the canonic variables describing each transfer function of G. There are no correlation terms between canonic variables of any two transfer functions, unlike the Liapunov function of section 2.1. As will be seen below, this type of Liapunov function simplifies the choice of the coefficients 'a'. The effects of system interactions appear in the conditions for $\dot{V}$ to be sign definite.

Differentiating V w.r.t. time,

$$\dot{V} = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{\dot{z}}^{(ij)} - n \sum_{i=1}^{k} \dot{y}_i \, f_i \, (y_i). \qquad (2.3a.2)$$

From eqns. (2.2a.1) and (2.2a.2), since $R_{ij} = 0$ for all $i,j \leq k$,

$$\dot{y}_i = \sum_{j=1}^{n} \underline{\alpha}^{(ij)T} \left(\Lambda^{(ij)T} \underline{z}^{(ij)} + \underline{f}_j^{(ij)} \right) \quad (i=1,2,\ldots k)$$

$$\text{or } \dot{y}_i = \sum_{j=1}^{n} \left(\underline{\beta}^{(ij)T} \underline{z}^{(ij)} - r_{ij} \, f_j \, (y_j) \right) \qquad (2.3a.3)$$

where $\underline{\beta}^{(ij)} = \Lambda^{(ij)} \underline{\alpha}^{(ij)}$, and r_{ij} is the sum of the residues of the (i,j)th transfer function.

Substituting from eqns. (2.3a.3) and (2.2a.1) in eqn. (2.3a.2) gives

$$\dot{V} = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} (\Lambda^{(ij)} \underline{z}^{(ij)} + \underline{f}_j^{(ij)})$$

$$- n \sum_{i=1}^{k} \sum_{j=1}^{n} (\underline{\beta}^{(ij)T} \underline{z}^{(ij)} f_i(y_i) - r_{ij} f_i(y_i) f_j(y_j)).$$

By definition of the matrices $Q^{(ij)}$, it follows that

$$2 \underline{z}^{(ij)T} Q^{(ij)} \Lambda^{(ij)} \underline{z}^{(ij)} = (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 - 2c \overline{V}$$

where $\underline{a}^{(ij)}$ is a vector of the coefficients $a_r^{(ij)}$ associated with the matrix $Q^{(ij)}$ and

$$\overline{V} = \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{z}^{(ij)} \qquad \text{(see eqn. (2.3a.1))}.$$

The derivative can therefore be written as

$$\dot{V} = -2c\overline{V} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2$$

$$+ 2 \sum_{i=1}^{n} \sum_{j=1}^{n} f_j (y_j) \underline{z}^{(ij)T} Q^{(ij)} \underline{e}^{(ij)}$$

$$- n \sum_{i=1}^{k} \sum_{j=1}^{n} f_i (y_i) (\underline{\beta}^{(ij)T} \underline{z}^{(ij)} - r_{ij} f_j (y_j)). \qquad (2.3a.4)$$

In constraining $\dot{V}$ to be a possibly sign-definite quadratic function of the state variables and the nonlinear functions $f_i(y_i)$, it should be done in such a manner that if the system interactions were removed (i.e. g_{ij} (p) = 0 for all $i \neq j$), any criteria obtained would be equivalent to n independent criteria similar to section 1.2, Chapter I. Also, some equations must be found which yield specific values for the coefficients $a_r^{(ij)}$ $(i \neq j)$ pertaining to the interaction transfer functions. It is possible to do this in two ways, similar to the criteria of section 1.2, Chapter I.

Firstly, add to $\dot{V}$ the expression (see eqn. (2.2a.2))

$$\sum_{i=k+1}^{n} f_i(y_i)\,(y_i - \sum_{j=1}^{n} \underline{\alpha}^{(ij)T}\,\underline{z}^{(ij)} + \sum_{j=k+1}^{n} R_{ij}\,f_j(y_j)) \equiv 0.$$

Provided that ηr_{ii} and $R_{ii} \geq 0$ for all i, one may then complete the necessary squares in $\dot{V}$ and rewrite it in the following manner (cf. section 1.2, Chapter I):

$$\dot{V} = -2c\overline{V} + \sum_{i=1}^{k}\sum_{\substack{j=1\\(i\neq j)}}^{k} (\underline{a}^{(ii)T}\,\underline{z}^{(ii)} + \underline{a}^{(ji)T}\,\underline{z}^{(ji)} + \sqrt{\eta r_{ii}}\, f_i(y_i))^2$$

$$+ \sum_{\substack{i=k+1\\(i\neq j)}}^{n}\sum_{j=k+1}^{n} (\underline{a}^{(ii)T}\,\underline{z}^{(ii)} + \underline{a}^{(ji)T}\,\underline{z}^{(ji)} + \sqrt{R_{ii}}\, f_i(y_i))^2$$

$$- 2\sum_{i=1}^{n}\sum_{\substack{j=1\\(i\neq j)}}^{n} \underline{a}^{(ii)T}\,\underline{z}^{(ii)}\,\underline{a}^{(ji)T}\,\underline{z}^{(ji)} + \sum_{i=k+1}^{n} y_i\, f_i(y_i)$$

$$+ n \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{n} r_{ij} f_i (y_i) f_j (y_j) + \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} R_{ij} f_i (y_i) f_j (y_j)$$

$$- n \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{n} f_i (y_i) \underline{\beta}^{(ij)T} \underline{z}^{(ij)} - \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=1}^{n} f_i (y_i) \underline{\alpha}^{(ij)T} \underline{z}^{(ij)}$$

$$+ \sum_{i=1}^{k} f_i (y_i) \underline{z}^{(ii)T} (2Q^{(ii)} \underline{e}^{(ii)} - 2 \sqrt{n r_{ii}} \, \underline{a}^{(ii)} - n \, \underline{\beta}^{(ii)})$$

$$+ \sum_{i=k+1}^{n} f_i (y_i) \underline{z}^{(ii)T} (2Q^{(ii)} \underline{e}^{(ii)} - 2 \sqrt{R_{ii}} \, \underline{a}^{(ii)} - \underline{\alpha}^{(ii)})$$

$$+ \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{k} f_j (y_j) \underline{z}^{(ij)T} (2Q^{(ij)} \underline{e}^{(ij)} - 2 \sqrt{n r_{jj}} \, \underline{a}^{(ij)})$$

$$+ \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} f_j (y_j) \underline{z}^{(ij)T} (2Q^{(ij)} \underline{e}^{(ij)} - 2 \sqrt{R_{jj}} \, \underline{a}^{(ij)}) \qquad (2.3a.5)$$

where $\underline{e}$ is the unit vector.

The coefficients $\underline{a}^{(ij)}$ may be determined by setting the last four summations of eqn. (2.3a.5) to zero:

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{\eta r_{ii}}\,\underline{a}^{(ii)} + \eta\,\underline{\beta}^{(ii)} \quad (i = 1,2,\ldots k) \tag{2.3a.6}$$

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{R_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1,\ k+2,\ldots n) \tag{2.3a.7}$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{\eta r_{jj}}\,\underline{a}^{(ij)} \quad (i,j = 1,2,\ldots k,\ i\neq j) \tag{2.3a.8}$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{R_{jj}}\,\underline{a}^{(ij)} \quad (i,j = k+1,\ k+2,\ldots n,\ i\neq j) \tag{2.3a.9}$$

For some given transfer functions each with q poles, say, the above equations written out in full are

$$2a_e^{(ii)} \sum_{m=1}^{q} \frac{a_m^{(ii)}}{w_e^{(ii)} + w_m^{(ii)}} = 2\sqrt{\eta r_{ii}}\, a_e^{(ii)} + \eta\, \beta_e^{(ii)} \tag{2.3a.10}$$

$$2a_e^{(ii)} \sum_{m=1}^{q} \frac{a_m^{(ii)}}{w_e^{(ii)} + w_m^{(ii)}} = 2\sqrt{R_{ii}}\, a_e^{(ii)} + \alpha_e^{(ii)} \tag{2.3a.11}$$

$$\sum_{m=1}^{q} \frac{a_m^{(ij)}}{w_e^{(ij)} + w_m^{(ij)}} = \sqrt{\eta r_{jj}} \tag{2.3a.12}$$

$$\sum_{m=1}^{q} \frac{a_m^{(ij)}}{w_e^{(ij)} + w_m^{(ij)}} = \sqrt{R_{jj}} \tag{2.3a.13}$$

where $e = 1,2,\ldots q$. Eqns. (2.3a.10) and (2.3a.11) are exactly those which would be obtained for the uncoupled system (see eqns. (1.2.3) and (1.2.6)). Eqns. (2.3a.12) and (2.3a.13) are simultaneous linear equations in the coefficients $a_m^{(ij)}$ $(i \neq j)$.

From the above analysis, the following criterion may be formulated.

<u>CRITERION 1.1.</u>

The restrictions on the nonlinearities are

(1) those of eqn. (2.3.1) for $i = 1,2,\ldots k$

(2) those of eqn. (2.3.2) for $i = k+1, k+2,\ldots n$.

Then if there exists some negative value of c such that

(a) Eqns. (2.3a.6) to (2.3a.9) have appropriate roots for $\eta = -1$, $r_{ii} < 0$ and $R_{ii} > 0$ for all i

(b) Re $(\lambda_e^{(ij)})$ and Re $(w_e^{(ij)}) > 0$ for all e,i,j,

then V is positive definite, and if $\dot{V}$ is positive definite the system is unstable.

Conversely, if there exists some positive value of c such that

(c) the eqns. have appropriate roots for $\eta = +1$, $r_{ii} > 0$ and $R_{ii} > 0$ for all i,

(d) Re $(\lambda_e^{(ij)})$ and Re $(w_e^{(ij)}) < 0$ for all e,i,j then V is negative definite, and if $\dot{V}$ is positive definite the system is totally stable.

A prerequisite for establishing either stability or instability is that $\dot{V}$ be positive definite. From eqn. (2.3a.5), taking the nonlinearity restrictions into account, $\dot{V}$ is positive definite if the following quadratic function of the state variables $\underline{z}^{(ij)}$ and the nonlinear functions $f_i(y_i)$ is positive definite:

$$W = -2c\overline{V} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 + \eta \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij} f_i(y_i) f_j(y_j)$$

$$+ 2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\ (i \neq j)}}^{k} \sqrt{\eta r_{ii}}\, f_i(y_i)(\underline{a}^{(ii)T} \underline{z}^{(ii)} + a^{(ji)T} \underline{z}^{(ji)})$$

$$+ 2 \sum_{i=k+1}^{n} \sum_{\substack{j=k+1 \\ (i \neq j)}}^{n} \sqrt{R_{ii}}\, f_i(y_i) (\underline{a}^{(ii)T} \underline{z}^{(ii)} + \underline{a}^{(ji)T} \underline{z}^{(ji)})$$

$$- \eta \sum_{i=1}^{k} \sum_{\substack{j=1 \\ (i \neq j)}}^{n} f_i(y_i)\, \underline{\beta}^{(ij)T} \underline{z}^{(ij)} - \sum_{i=k+1}^{n} \sum_{\substack{j=k+1 \\ (i \neq j)}}^{n} f_i(y_i)\, \underline{\alpha}^{(ij)T} \underline{z}^{(ij)}$$

$$+ \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} R_{ij} f_i(y_i) f_j(y_j) \qquad (2.3a.14)$$

Thanks to the use of the parameter c, the first two expressions in W constitute a positive definite quadratic function of the state variables when the conditions of criterion 1.1 are satisfied.

Since every state vector $\underline{z}^{(ii)}$ in W occurs only in conjunction with the corresponding coefficient vector $\underline{a}^{(ii)}$, definition of the new variables

$$b_e^{(ii)} = a_e^{(ii)} z_e^{(ii)} \qquad (2.3a.15)$$

enables the sign of W to be determined independent of the vectors $\underline{a}^{(ii)}$.

Practical application of the criterion is then as follows.

(1) If all system poles have negative real parts, denote by c_1 the smallest modulus of these real parts. If all system poles have positive real parts, denote by c_2 the smallest of these real parts.

(2) Solve the simultaneous linear eqns. (2.3a.8) and (2.3a.9) to give the coefficient vectors $\underline{a}^{(ij)}$ $(i \neq j)$ in terms of the system parameters and c.

(3) Substitute for these vectors in W of eqn. (2.3a.4). Using the transformation (2.3a.15), determine if W is positive definite for any values of c lying either in the range

$0 < c < c_1$ (stability)

or $-c_2 < c < 0$ (instability)

(4) See if eqns. (2.3a.5) and (2.3a.6) have appropriate roots for any c lying in the pertinent range defined in (3).

As an alternative to the above criterion, one may be established which yields results similar to those of eqns. (1.2.5) in the single-loop analysis.

Add to $\dot{V}$ of eqn. (2.3a.9) the expression

$$\sum_{i=1}^{n} f_i(y_i)\,(y_i - \sum_{j=1}^{n} \underline{\alpha}^{(ij)T}\,\underline{z}^{(ij)} + \sum_{j=k+1}^{n} R_{ij}\,f_j(y_j)) \equiv 0$$

where the summation is over all i, instead of over i = 1,2,...k as in criterion 1.1. Manipulation very similar to that above then yields the following criterion.

CRITERION 1.2.

The nonlinearity restrictions are those of eqn. (2.3.1) for all i.

The coefficient equations are

$$2Q^{(ii)}\,\underline{e}^{(ii)} = \underline{\alpha}^{(ii)} \quad (i = 1,2,\ldots k) \tag{2.3a.16}$$

$$2Q^{(ii)}\,\underline{e}^{(ii)} = \sqrt{R_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1,\ k+2,\ldots n) \tag{2.3a.17}$$

$$Q^{(ij)}\,\underline{e}^{(ij)} = \sqrt{n r_{jj}}\,\underline{a}^{(ij)} \quad (i,j,= 1,2,\ldots k,\ i \neq j) \tag{2.3a.18}$$

$$Q^{(ij)}\,\underline{e}^{(ij)} = \sqrt{R_{jj}}\,\underline{a}^{(ij)} \quad (i,j = k+1,\ k+2,\ldots n,\ i\neq j) \tag{2.3a.19}$$

The function W which must be positive definite is

$$W = -2c\overline{V} + \sum_{i=1}^{n}\sum_{j=1}^{n} (\underline{a}^{(ij)T}\,\underline{z}^{(ij)})^2$$

$$+ \eta \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij} f_i (y_i) f_j (y_j) + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} R_{ij} f_i (y_i) f_j (y_j)$$

$$+ 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{k} \sqrt{\eta r_{ii}}\, f_i (y_i) \underline{a}^{(ji)T} \underline{z}^{(ji)}$$

$$+ 2 \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} \sqrt{R_{ii}}\, f_i (y_i) \underline{a}^{(ji)T} \underline{z}^{(ji)}$$

$$- \eta \sum_{i=1}^{k} \sum_{j=1}^{n} f_i (y_i) \underline{\beta}^{(ij)T} \underline{z}^{(ij)} - \sum_{\substack{i=1 \\ (i \neq j)}}^{n} \sum_{j=1}^{n} f_i (y_i) \underline{\alpha}^{(ij)T} \underline{z}^{(ij)} \qquad (2.3a.20)$$

In this case the sign of W is not independent of the vectors $\underline{a}^{(ii)}$, and the simultaneous quadratic equations (2.3a.16) and (2.3a.17) must be solved before the sign of W may be determined.

Stability or instability are established as outlined for criterion 1.1, but using eqns. (2.3a.16) to (2.3a.19). η is again chosen as -1 (instability) or +1 (stability).

2.3b. Common row poles

Assuming again that k rows of G have no remainders in their partial fraction expansions, take as a Liapunov function the form

$$V = \sum_{i=1}^{h} \underline{z}^{(i)T} Q^{(i)} \underline{z}^{(i)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{z}^{(ij)}$$

$$- \eta \sum_{i=1}^{k} \int_{0}^{y_i} f_i (y_i)\, dy_i \qquad (2.3b.1)$$

where the symmetric matrices $Q^{(i)}$ are defined similarly to the matrices $Q^{(ij)}$, but with reference to the common poles of the ith row of G (see section 2.2b).

As in the previous section, the above Liapunov function contains no interaction terms among the canonic variables of any two transfer functions. Furthermore, the canonic variables pertaining to the common row poles (see eqn. (2.2b.2)) also form separate quadratic groups.

Differentiating V w.r.t.time using eqns. (2.2b.1) to (2.2b.3) gives

$$\dot{V} = \sum_{i=1}^{h} (\underline{a}^{(i)T} \underline{z}^{(i)})^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 - 2c\overline{V}_1$$

$$+ 2 \sum_{i=1}^{h} \sum_{j=1}^{n} \underline{z}^{(i)T} Q^{(i)} \underline{\hat{a}}^{(ij)} f_j (y_j) - \eta \sum_{i=1}^{k} \dot{y}_i f_i (y_i)$$

$$+ 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{e}^{(ij)} f_j (y_j) \qquad (2.3b.2)$$

where $\overline{V}_1$ is the quadratic portion of V, namely

$$\overline{V}_1 = \sum_{i=1}^{h} \underline{z}^{(i)T} Q^{(i)} \underline{z}^{(i)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{z}^{(ij)} .$$

From eqn. (2.2a.2), for $i \leq k$,

$$\dot{y}_i = \sum_{j=1}^{n} \underline{\beta}^{(ij)T} \underline{z}^{(ij)} - \overline{r}_{ij} f_j (y_j) - \hat{r}_{ij} f_j (y_j)) + \underline{\lambda}^{(i)T} \underline{z}^{(i)} \qquad (2.3b.3)$$

where one defines

$$\underline{\lambda}^{(i)T} = \underline{e}^{(i)T} \Lambda^{(i)}$$

$$\hat{r}_{ij} = \underline{e}^{(i)T} \underline{\hat{\alpha}}^{(ij)}$$

$$\overline{r}_{ij} = \underline{e}^{(ij)T} \underline{\alpha}^{(ij)} .$$

$\underline{\lambda}^{(i)}$ is therefore a vector of the poles of $\Lambda^{(i)}$ (common to the ith row of G). $\hat{r}_{ij}$ is the sum of the residues of these poles in the (i,j)th transfer function, and $\overline{r}_{ij}$ is the sum of the residues of all other poles of the (i,j)th transfer function. It therefore follows that $\hat{r}_{ij} + \overline{r}_{ij} = r_{ij}$ is the sum of the residues of the (i,j)th transfer function.

Substituting for $\dot{y}_i$ from eqn. (2.3b.3) in eqn. (2.3b.2,)

$$\dot{V} = \sum_{i=1}^{h} (\underline{a}^{(i)T} \underline{z}^{(i)})^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 - 2c\overline{V}_1$$

$$+ 2 \sum_{i=1}^{h} \sum_{j=1}^{n} \underline{z}^{(i)T} Q^{(i)} \underline{\hat{a}}^{(ij)} f_j (y_j) - \eta \sum_{i=1}^{k} f_i (y_i) \underline{\lambda}^{(i)T} \underline{z}^{(i)}$$

$$+ 2 \sum_{i=1}^{n} \sum_{j=1}^{n} f_j (y_j) \underline{z}^{(ij)T} Q^{(ij)} \underline{e}^{(ij)}$$

$$- \eta \sum_{i=1}^{k} \sum_{j=1}^{n} f_i(y_i) \left(\underline{\beta}^{(ij)T} \underline{z}^{(ij)} - (\hat{r}_{ij} + \bar{r}_{ij}) f_j (y_j)\right) \qquad (2.3b.4)$$

There are now three different groups of coefficients for which equations must be derived. These are $\underline{a}^{(ii)}$, $\underline{a}^{(ij)}$ $(i \neq j)$ and $\underline{a}^{(i)}$. This case is unlike the previous one (section 2.3a) in that the structure of the linear process has a feature (common row poles) which is only of importance for multivariable systems.

Again there are two alternative forms of criterion possible. Firstly, add to $\dot{V}$ of eqn. (2.3b.4) the expression (see eqn. (2.2b.3)):

$$\sum_{i=k+1}^{n} f_i (y_i)(y_i - \sum_{j=1}^{n} \underline{\alpha}^{(ij)T} \underline{z}^{(ij)} + \sum_{j=k+1}^{n} R_{ij} f_j (y_j) - \underline{e}^{(i)T} \underline{z}^{(i)}) \equiv 0 .$$

To obtain equations for the coefficients, and at the same time leave $\dot{V}$ as a possibly sign-definite quadratic form, it is necessary to complete squares as in the previous analysis, but in this case taking the additional vectors $\underline{z}^{(i)}$ and $\underline{a}^{(i)}$ into account. Two cases must be considered, namely

(1) $h \leq k$ and (2) $h > k$.

This natural extension of the analysis of section 2.3a yields the following criterion.

CRITERION 1.3.

The nonlinearity restrictions are

(1) Those of eqn. (2.3.2) for i = 1,2,...k

(2) Those of eqn. (2.3.1) for i = k+1, k+2,...n

The coefficient equations are

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{n r_{ii}} + n\underline{\beta}^{(ii)} \quad (i = 1,2,\ldots k) \qquad (2.3b.5)$$

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{R_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1,k+2,\ldots n) \qquad (2.3b.6)$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{n\bar{r}_{jj}}\,\underline{a}^{(ij)} \quad (i,j = 1,2,\ldots k, i\neq j) \qquad (2.3b.7)$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{R_{jj}}\,\underline{a}^{(ij)} \quad (i,j = k+1,k+2,\ldots n,\ i\neq j) \qquad (2.3b.8)$$

and either (for $h \leq k$)

$$2Q^{(i)}\underline{\hat{a}}^{(ii)} = 2\sqrt{n\hat{r}_{ii}}\,\underline{a}^{(i)} + n\underline{\lambda}^{(i)} \quad (i = 1,2,\ldots h) \qquad (2.3b.9)$$

or(for $h > k$)

$$2Q^{(i)}\underline{\hat{a}}^{(ii)} = 2\sqrt{n\hat{r}_{ii}}\,\underline{a}^{(i)} + n\underline{\lambda}^{(i)} \quad (i = 1,2,\ldots k) \qquad (2.3b.10)$$

$$2Q^{(i)}\underline{\hat{a}}^{(ii)} = 2\sqrt{R_{ii}}\,\underline{a}^{(i)} + \underline{e}^{(i)} \quad (i = k+1,k+2,\ldots h) \qquad (2.3b.11)$$

The quadratic function of the state variables and the functions $f_i\,(y_i)$ which must be positive definite is

$$W = -2c\overline{V}_1 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\underline{a}^{(ij)T}\,\underline{z}^{(ij)}\right)^2 + \sum_{i=1}^{h}\left(\underline{a}^{(i)T}\,\underline{z}^{(i)}\right)^2$$

$$+\ 2\sum_{\substack{i=1\\(i\neq j)}}^{k}\sum_{j=1}^{k}\sqrt{\eta\overline{r}_{ii}}\,f_i\,(y_i)\left(\underline{a}^{(ii)T}\,\underline{z}^{(ii)} + \underline{a}^{(ji)T}\,\underline{z}^{(ji)}\right)$$

$$+\ 2\sum_{\substack{i=k+1\\(i\neq j)}}^{n}\sum_{j=k+1}^{n}\sqrt{R_{ii}}\,f_i\,(y_i)\left(\underline{a}^{(ii)T}\,\underline{z}^{(ii)} + \underline{a}^{(ji)T}\,\underline{z}^{(ji)}\right)$$

$$+\ \eta\sum_{i=1}^{k}\sum_{j=1}^{n} r_{ij}\,f_i\,(y_i)\,f_j\,(y_j) + \sum_{i=k+1}^{n}\sum_{j=k+1}^{n} R_{ij}\,f_i\,(y_i)\,f_j\,(y_j)$$

$$+\ 2\sum_{\substack{i=1\\(i\neq j)}}^{h}\sum_{j=1}^{n} f_j\,(y_j)\,\underline{z}^{(i)T}\,Q^{(i)}\,\underline{\alpha}^{(ij)} + Y$$

$$-\ \eta\sum_{\substack{i=1\\(i\neq j)}}^{k}\sum_{j=1}^{n} f_i\,(y_i)\,\underline{\beta}^{(ij)T}\,\underline{z}^{(ij)} - \sum_{\substack{i=k+1\\(i\neq j)}}^{n}\sum_{j=k+1}^{n} f_i\,(y_i)\,\underline{\alpha}^{(ij)T}\,\underline{z}^{(ij)}$$

(2.3b.12)

where for $h \leq k$,

$$Y = 2 \sum_{i=1}^{h} \sqrt{\eta \hat{r}_{ii}}\, f_i\,(y_i)\, \underline{a}^{(i)T}\, \underline{z}^{(i)} \qquad (2.3b.13)$$

and for $h > k$,

$$Y = 2 \sum_{i=1}^{k} \sqrt{\eta \hat{r}_{ii}}\, f_i\,(y_i)\, \underline{a}^{(i)T}\, \underline{z}^{(i)}$$

$$+ 2 \sum_{i=k+1}^{h} \sqrt{R_{ii}}\, f_i\,(y_i)\, \underline{a}^{(i)T}\, \underline{z}^{(i)} \qquad (2.3b.14)$$

For some given transfer functions each containing q poles, say, eqns. (2.3b.9) to (2.3b.11) are of the form

$$2a_e^{(i)} \sum_{m=1}^{q} \frac{a_m^{(i)}\, \hat{a}_m^{(ii)}}{w_e^{(i)} + \hat{w}_m^{(i)}} = 2\sqrt{\eta \hat{r}_{ii}}\; a_e + \eta \lambda_e$$

and

$$2a_e^{(i)} \sum_{m=1}^{q} \frac{a_m^{(i)}\, \hat{a}_m^{(ii)}}{w_e^{(i)} + w_m^{(i)}} = 2\sqrt{R_{ii}}\, a_e^{(i)} + 1.$$

By defining the new variables

$$A_e^{(i)} = \hat{a}_e^{(ii)}\, a_e^{(i)} \qquad (2.3b.15)$$

the equations may be written in the form

$$2A_e^{(i)} \sum_{m=1}^{q} \frac{A_m^{(i)}}{w_e^{(i)} + w_m^{(i)}} = 2\sqrt{\eta \hat{r}_{ii}}\, A_e^{(i)} + \eta\, \hat{\alpha}_e^{(i)}\, \lambda_e^{(i)}$$

$$2A_e^{(i)} \sum_{m=1}^{q} \frac{A_m^{(i)}}{w_e^{(i)} + w_m^{(i)}} = 2\sqrt{R_{ii}}\, A_e^{(i)} + \hat{\alpha}_e^{(i)}$$

which is identical to that of eqns. (2.3a.10) and (2.3a.11) of section 2.3a. The conditions for appropriate solution of these equations are therefore those discussed in Appendix II, since the variables A_e correspond in algebraic nature to the variables a_e. As in criterion 1.1, the sign of W may be determined independent of the vectors $\underline{a}^{(ii)}$, by definition of the new variables

$$b_e^{(ii)} = a_e^{(ii)}\, z_e^{(ii)} \qquad (2.3b.16)$$

As an alternative to the above criterion, one may add to $\dot{V}$ (eqn. (2.3b.4)) the expression (see eqn. (2.2b.3)

$$\sum_{i=1}^{n} f_i(y_i)\left(y_i - \sum_{j=1}^{n} \alpha^{(ij)T}\, \underline{z}^{(ij)} + \sum_{j=k+1}^{n} R_{ij}\, f_j(y_j) - \underline{e}^{(i)T}\, \underline{z}^{(i)}\right) \equiv 0.$$

Completing the necessary squares in $\dot{V}$, this time to give coefficient equations depending on the vectors $\underline{\alpha}$ rather than the vectors $\underline{\beta}$ (cf. criteria 1.1. and 1.2) yields the following criterion.

CRITERION 1.4

The nonlinearity restrictions are those of eqn. (2.3.1) for all i.

The coefficient equations are

$$2Q^{(ii)}\underline{e}^{(ii)} = \underline{\alpha}^{(ii)} \quad (i = 1,2,\ldots k) \tag{2.3b.17}$$

$$2Q^{(ii)}\underline{e}^{(ii)} = \sqrt{R_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1,k+2,\ldots n) \tag{2.3b.18}$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{n\bar{r}_{jj}}\,\underline{a}^{(ij)} \quad (i,j = 1,2,\ldots k,\ i\neq j) \tag{2.3b.19}$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{R_{jj}}\,\underline{a}^{(ij)} \quad (i,j = k+1,\ k+2,\ldots n,\ i\neq j) \tag{2.3b.20}$$

and either (for $h \leq k$)

$$2Q^{(i)}\underline{\hat{\alpha}}^{(ii)} = \underline{e}^{(i)} \quad (i = 1,2,\ldots h) \tag{2.3b.21}$$

or (for $h > k$)

$$2Q^{(i)}\underline{\hat{\alpha}}^{(ii)} = \underline{e}^{(i)} \quad (i = 1,2,\ldots k) \tag{2.3b.22}$$

$$2Q^{(i)}\underline{\hat{\alpha}}^{(ii)} = \sqrt{R_{ii}}\,\underline{a}^{(i)} + \underline{e}^{(i)} \quad (i = k+1,k+2,\ldots h) \tag{2.3b.23}$$

The function W which must be positive definite is

$$W = -2c\overline{V}_1 + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 + \sum_{i=1}^{h} (\underline{a}^{(i)T} \underline{z}^{(i)})^2$$

$$+ 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{k} \sqrt{\eta \overline{r}_{ii}}\; f_i (y_i)\, \underline{a}^{(ji)T} \underline{z}^{(ji)}$$

$$+ 2 \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} \sqrt{R_{ii}}\; f_i (y_i) (\underline{a}^{(ii)T} \underline{z}^{(ii)} + \underline{a}^{(ji)T} \underline{z}^{(ji)})$$

$$+ \eta \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij} f_i (y_i) f_j (y_j) + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} R_{ij} f_i (y_i) f_j (y_j)$$

$$+ 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{h} \sum_{j=1}^{n} f_j (y_j)\, \underline{z}^{(i)T} Q^{(i)} \underline{\hat{\alpha}}^{(ij)} - \sum_{\substack{i=1 \\ (i \neq j)}}^{n} \sum_{j=1}^{n} f_i (y_i) \underline{\alpha}^{(ij)T} \underline{z}^{(ij)}$$

$$- \eta \sum_{i=1}^{k} \sum_{j=1}^{n} \underline{z}^{(ij)} f_i (y_i)\, \underline{\beta}^{(ij)T} - \eta \sum_{i=1}^{h} f_i (y_i)\, \underline{\lambda}^{(i)T} \underline{z}^{(i)}$$

$$+ \; Y \qquad (2.3b.24)$$

where for $h \leq k$, $Y = 0$, and for $h > k$,

$$Y = 2 \sum_{i=k+1}^{k} \sqrt{R_{ii}}\; f_i (y_i)\, \underline{a}^{(i)T} \underline{z}^{(i)} \qquad (2.3b.25)$$

2.4. Stability in the sectors $[0,K_i]$

If a given class 1 system is such that one or more of the uncoupled loops of the system is unstable for high values of linear open-loop gain, the pole shifting technique of section 1.4, Chapter I must be used to establish stability. Conversely, if a given system has one or more loops which are stable for high values of linear gain only, the technique must be applied to establish instability.

Systems which have zeroes in the right-half of the p-plane may then be handled.

Assume that (n - q) loops are unstable (or stable, if instability is being established) for high gains. Then rotation of the output axes of the corresponding (n-q) nonlinear characteristics is equivalent to defining the new input variables (see eqn. (1.4.1), Chapter I).

$$u_i = y_i - \frac{1}{K_i} f_i (y_i) \quad (i = q+1, q+2, \ldots n) \qquad (2.4.1)$$

The outputs will then be defined in terms of the old outputs as

$$\phi_i (u_i) \equiv f_i (y_i) \quad (i = q+1, q+2, \ldots n) \qquad (2.4.2)$$

The restrictions on these (n - q) nonlinearities in the resulting criteria will then be of the form

$$u_i \phi_i (u_i) \geq 0 \text{ for all } | u_i | \neq 0 \qquad (2.4.3)$$

where the functions $\phi_i (u_i)$ are assumed to be single valued and zero at the origin $u_i = 0$. Eqn. (2.4.3) confines the nonlinear characteristics to the sectors $[0,K_i]$.

2.4a. No common row poles

Substituting for y_i and f_i (y_i) from eqns. (2.4.1) and (2.4.2.) in the canonic eqns. of section 2.2a gives

$$\underline{\dot{z}}^{(ij)} = \Lambda^{(ij)} \underline{z}^{(ij)} + \underline{\emptyset}_j^{(ij)} \qquad (2.4a.1)$$

$$u_i = \sum_{j=1}^{n} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - \overline{R}_{ij}\, \emptyset_j\, (u_j)) \qquad (2.4a.2)$$

$$(i,j = 1,2,...n)$$

where for convenience one defines

$$u_i = y_i \text{ and } \overline{R}_{ii} = R_{ii} \text{ for all } i \leq q \qquad (2.4a.3)$$

$$\overline{R}_{ij} = R_{ij} \text{ for all } i \neq j \qquad (2.4a.4)$$

$$\overline{R}_{ii} = R_{ii} + \frac{1}{K_i} \quad \text{for all } i > q \qquad (2.4a.5)$$

Assume the first k rows of G have no remainders in the partial fraction expansions of their transfer functions, that is

$\overline{R}_{ij} = R_{ij} = 0$ for all $i,j \leq k \leq q$.

Since the canonic form of eqns. (2.4a.1) and (2.4a.2) is of identical form to that of section 2.2a, the analysis of section 2.3a may be applied directly to yield the following two criteria (cf. criteria 1.1 and 1.2).

CRITERION 1.5

The nonlinearity restrictions are

(1) those of eqn. (2.3.2) for $i = 1,2,\ldots k$

(2) those of eqn. (2.3.1) for $i = k+1,k+2,\ldots q$

(3) those of eqn. (2.4.2) for $i = q+1, q+2, \ldots n$

The coefficient equations are

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{nr_{ii}}\,\underline{a}^{(ii)} + \eta\underline{\beta} \quad (i = 1,2,\ldots k) \qquad (2.4a.6)$$

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{\bar{R}_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha} \quad (i = k+1,k+2,\ldots n) \qquad (2.4a.7)$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{nr_{jj}}\,\underline{a}^{(ij)} \quad (i,j = 1,2,\ldots k,\ i\neq j) \qquad (2.4a.8)$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{\bar{R}_{jj}}\,\underline{a}^{(ij)} \quad (i,j = k+1,k+2,\ldots n,\ i\neq j) \qquad (2.4a.9)$$

The function W which must be positive definite is

$$W = -2c\bar{V} + \sum_{i=1}^{n}\sum_{j=1}^{n}(\underline{a}^{(ij)T}\underline{z}^{(ij)})^2$$

$$+ 2\sum_{\substack{i=1\\(i\neq j)}}^{k}\sum_{j=1}^{k}\sqrt{nr_{ii}}\,\emptyset_i(u_i)(\underline{a}^{(ii)T}\underline{z}^{(ii)} + \underline{a}^{(ji)T}\underline{z}^{(ji)})$$

$$+ 2\sum_{\substack{i=k+1\\(i\neq j)}}^{n}\sum_{j=k+1}^{n}\sqrt{\bar{R}_{ii}}\,\emptyset_i(u_i)(\underline{a}^{(ii)T}\underline{z}^{(ii)} + \underline{a}^{(ji)T}\underline{z}^{(ji)})$$

$$+ n \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij}\, \emptyset_i (u_i)\, \emptyset_j (u_j) + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \overline{R}_{ij}\, \emptyset_i (u_i)\, \emptyset_j (u_j)$$

$$- n \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{n} \emptyset_i (u_i)\, \underline{\beta}^{(ij)T}\, \underline{z}^{(ij)} - \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} \emptyset_i (u_i)\, \underline{\alpha}^{(ij)T}\, \underline{z}^{(ij)} \tag{2.4a.10}$$

CRITERION 1.6.

The nonlinearity restrictions are

(1) those of eqn. (2.3.1) for i = 1,2,...q

(2) those of eqn. (2.4.2) for i = q+1,q+2,...n

The coefficient equations are

$$2Q^{(ii)}\, \underline{e}^{(ii)} = \underline{\alpha}^{(ii)} \qquad (i = 1,2,\ldots k) \tag{2.4a.11}$$

$$2Q^{(ii)}\, \underline{e}^{(ii)} = \sqrt{\overline{R}_{ii}}\, \underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \qquad (i = k+1,k+2,\ldots n) \tag{2.4a.12}$$

$$Q^{(ij)}\, \underline{e}^{(ij)} = \sqrt{n r_{jj}}\, \underline{a}^{(ij)} \qquad (i,j = 1,2,\ldots k,\ i \neq j) \tag{2.4a.13}$$

$$Q^{(ij)}\, \underline{e}^{(ij)} = \sqrt{\overline{R}_{jj}}\, \underline{a}^{(ij)} \qquad (i,j = k+1,k+2,n,\ i \neq j) \tag{2.4a.14}$$

The function W which must be positive definite is

$$W = -2c\overline{V} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2$$

$$+ 2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\ (i \neq j)}}^{k} \sqrt{\eta r_{ii}}\, \emptyset_i (u_i)\, \underline{a}^{(ji)T} \underline{z}^{(ji)}$$

$$+ 2 \sum_{i=k+1}^{n} \sum_{\substack{j=k+1 \\ (i \neq j)}}^{n} \sqrt{\overline{R}_{ii}}\, \emptyset_i (u_i) (\underline{a}^{(ii)T} \underline{z}^{(ii)} + \underline{a}^{(ji)T} \underline{z}^{(ji)})$$

$$+ \eta \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij}\, \emptyset_i (u_i)\, \emptyset_j (u_j) + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \overline{R}_{ij}\, \emptyset_i (u_i)\, \emptyset_j (u_j)$$

$$- \eta \sum_{i=1}^{k} \sum_{j=1}^{n} \emptyset_i (u_i)\, \underline{\beta}^{(ij)T} \underline{z}^{(ij)} - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ (i \neq j)}}^{n} \emptyset_i(u_i) \underline{\alpha}^{(ij)T} \underline{z}^{(ij)} \tag{2.4a.15}$$

The comments on criteria 1.1 and 1.2 in section 2.3a. apply to criteria 1.5 and 1.6 respectively.

2.4b. Common row poles

Substitution for y_i and $f_i(y_i)$ from eqns. (2.4.1) and (2.4.2) in the appropriate canonic eqns. of section 2.2b gives

$$\underline{\dot{z}}^{(ij)} = \Lambda^{(ij)} \underline{z}^{(ij)} + \underline{\emptyset}_j^{(ij)} \quad (i,j=1,2,\ldots n) \tag{2.4b.1}$$

$$\underline{\dot{z}}^{(i)} = \Lambda^{(i)} \underline{z}^{(i)} + \sum_{j=1}^{n} \underline{\hat{\alpha}}^{(ij)} \emptyset_j (u_j) \quad (i = 1,2,..h) \tag{2.4b.2}$$

$$u_i = \sum_{j=1}^{n} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - \overline{R}_{ij} \emptyset_j (u_j) + \underline{e}^{(i)T} \underline{z}^{(i)}) \tag{2.4b.3}$$

$$(i = 1,2,\ldots n).$$

where for convenience the definitions (2.4a.3) to (2.4a.5) are again made. The above equations are of identical nature to the canonic form of section 2.2b. Assuming again that the first k rows of G have no remainders in their partial fraction expansions, the two following criteria may be written down by direct comparison with section 2.3b, and criteria 1.3 and 1.4.

CRITERION 1.7.

The nonlinearity restrictions are

(1) those of eqn. (2.3.2) for $i = 1,2,\ldots k$

(2) those of eqn. (2.3.1) for $i = k+1, k+2, \ldots q$

(3) those of eqn. (2.4.2) for $i = q+1,q+2,\ldots n$.

The coefficient equations are

$$2Q^{(ii)} \underline{e}^{(ii)} = 2\sqrt{n\overline{r}_{ii}}\, \underline{a}^{(ii)} + n\underline{\beta}^{(ii)} \quad (i = 1,2,\ldots k) \tag{2.4b.4}$$

$$2Q^{(ii)} \underline{e}^{(ii)} = 2\sqrt{\overline{R}_{ii}}\, \underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1,k+2,\ldots n) \tag{2.4b.5}$$

$$Q^{(ij)} \underline{e}^{(ij)} = \sqrt{n\overline{r}_{jj}}\, \underline{a}^{(ij)} \quad (i,j,= 1,2,\ldots k,\ i\neq j) \tag{2.4b.6}$$

$$Q^{(ij)} \underline{e}^{(ij)} = \sqrt{\overline{R}_{jj}}\, \underline{a}^{(ij)} \quad (i,j = k+1,k+2,\ldots n, i \neq j) \qquad (2.4b.7)$$

and either (for $h \leq k$)

$$2Q^{(i)} \underline{\hat{a}}^{(ii)} = 2\sqrt{\eta \hat{r}_{ii}}\; \underline{a}^{(i)} + \eta \underline{\lambda}^{(i)} \quad (i=1,2,\ldots h) \qquad (2.4b.8)$$

or (for $h > k$)

$$2Q^{(i)} \underline{\hat{a}}^{(ii)} = 2\sqrt{\eta \hat{r}_{ii}}\; \underline{a}^{(i)} + \eta \underline{\lambda}^{(i)} \quad (i = 1,2,\ldots k) \qquad (2.4b.9)$$

$$2Q^{(i)} \hat{a}^{(ii)} = 2\sqrt{\overline{R}_{ii}}\; \underline{a}^{(i)} + \underline{e}^{(i)} \quad (i = k+1,k+2,\ldots h) \qquad (2.4b.10)$$

The function W which must be positive definite is

$$W = -2c\overline{V}_1 + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 + \sum_{i=1}^{h} (\underline{a}^{(i)T} \underline{z}^{(i)})^2$$

$$+ 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{k} \sqrt{\eta \overline{r}_{ii}}\; \emptyset_i (u_i) (a^{(ii)T} \underline{z}^{(ii)} + a^{(ji)T} \underline{z}^{(ji)})$$

$$+ 2 \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} \sqrt{\overline{R}_{ii}}\; \emptyset_i (u_i) (\underline{a}^{(ii)T} \underline{z}^{(ii)} + \underline{a}^{(ji)T} \underline{z}^{(ji)})$$

$$+ \; n \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij} \, \phi_i (u_i) \, \phi_j (u_j) \; + \sum_{i=k+1}^{n} \; \sum_{j = k+1}^{n} \bar{R}_{ij} \, \phi_i(u_i) \, \phi_j (u_j)$$

$$+ \; 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{n} \phi_j (u_j) \, \underline{z}^{(i)T} \, Q^{(i)} \, \underline{\hat{\alpha}}^{(ij)} \; - \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \; \sum_{j=k+1}^{n} \phi(u_i) \underline{\alpha}^{(ij)T} \, \underline{z}^{(ij)}$$

$$- \; n \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{n} \phi_i (u_i) \underline{\beta}^{(ij)T} \, \underline{z}^{(ij)} \; + Y \qquad (2.4b.11)$$

where for $h \geq k$,

$$Y = \; 2 \sum_{i=1}^{h} \sqrt{n \hat{r}_{ii}} \, \phi_i (u_i) \, \underline{a}^{(i)T} \, \underline{z}^{(i)} \qquad (2.4b.12)$$

and for $h > k$,

$$Y = \; 2 \sum_{i=1}^{k} \sqrt{n \hat{r}_{ii}} \, \phi_i (u_i) \, \underline{a}^{(i)T} \, \underline{z}^{(i)}$$

$$+ \; 2 \sum_{i=k+1}^{h} \sqrt{\bar{R}_{ii}} \, \phi_i (u_i) \, \underline{a}^{(i)T} \, \underline{z}^{(i)} \qquad (2.4b.13)$$

CRITERION 1.8.

The nonlinearity restrictions are

(1) those of eqn. (2.3.1) for $i = 1,2,\ldots q$

(2) those of eqn. (2.4.2) for $i = q+1,q+2,\ldots n$

The coefficient equations are

$$2Q^{(ii)}\underline{e}^{(ii)} = \underline{\alpha}^{(ii)} \quad (i = 1,2,\ldots k) \qquad (2.4b.14)$$

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{\overline{R}_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1,k+2,\ldots n) \qquad (2.4b.15)$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{n\overline{r}_{jj}}\,\underline{a}^{(ij)} \quad (i,j = 1,2,\ldots k,\ i\neq j) \qquad (2.4b.16)$$

$$Q^{(ij)}\underline{e}^{(ij)} = \sqrt{\overline{R}_{jj}}\,\underline{a}^{(ij)} \quad (i,j = k+1,k+2,\ldots n,\ i\neq j) \qquad (2.4b.17)$$

and either (for $h \leq k$)

$$2Q^{(i)}\underline{\hat{\alpha}}^{(ii)} = \underline{e}^{(i)} \quad (i = 1,2,\ldots h) \qquad (2.4b.18)$$

or (for $h > k$)

$$2Q^{(i)}\underline{\hat{\alpha}}^{(ii)} = \underline{e}^{(i)} \quad (i = 1,2,\ldots k) \qquad (2.4b.19)$$

$$2Q^{(i)}\underline{\hat{\alpha}}^{(ii)} = 2\sqrt{\overline{R}_{ii}}\,\underline{a}^{(i)} + \underline{e}^{(i)} \quad (i = k+1,k+2,\ldots n) \qquad (2.4b.20).$$

The function W which must be sign definite is

$$W = -2c\overline{V}_1 + \sum_{i=1}^{n}\sum_{j=1}^{n}(\underline{a}^{(ij)T}\underline{z}^{(ij)})^2 + \sum_{i=1}^{h}(\underline{a}^{(i)T}\underline{z}^{(i)})^2$$

$$+ 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{k} \sqrt{n \bar{r}_{ii}}\, \emptyset_i (u_i)\, \underline{a}^{(ji)T} \underline{z}^{(ji)}$$

$$+ 2 \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} \sqrt{\bar{R}_{ii}}\, \emptyset_i (u_i) \left(\underline{a}^{(ii)T} \underline{z}^{(ii)} + \underline{a}^{(ji)T} \underline{z}^{(ji)} \right)$$

$$+ n \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij}\, \emptyset_i (u_i)\, \emptyset_j (u_j) + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \bar{R}_{ij}\, \emptyset_i (u_i)\, \emptyset_j (u_j)$$

$$+ 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{n} \dot{\emptyset}_j (u_j)\, \underline{z}^{(i)T} Q^{(i)} \underline{\hat{a}}^{(ij)} - \sum_{\substack{i=1 \\ (i \neq j)}}^{n} \sum_{j=1}^{n} \emptyset_i (u_i)\, \underline{\alpha}^{(ij)T} \underline{z}^{(ij)}$$

$$- n \sum_{i=1}^{k} \sum_{j=1}^{n} \emptyset_i (u_i)\, \underline{\beta}^{(ij)T} \underline{z}^{(ij)} - n \sum_{i=1}^{h} \emptyset_i (u_i)\, \underline{\lambda}^{(i)T} \underline{z}^{(i)} + Y$$

(2.4b.21)

where for $h \leq k$, $Y = 0$ and for $h > k$,

$$Y = 2 \sum_{i=k+1}^{h} \sqrt{\bar{R}_{ii}}\, \underline{a}^{(i)T} \underline{z}^{(i)} \qquad (2.4b.22)$$

2.5. Stability in the sectors $[k_i, \infty]$

If a given class 1 system is such that one or more of the uncoupled loops of the system is unstable (stable) for low values of linear open-loop gain, the pole-shifting technique of section 1.4, Chapter I must be applied to establish stability (instability).

The above statement refers to establishing stability and instability respectively.

Assume that (n - q) of the loops are unstable (stable) for low feedback gains. Then rotation of the input axes of the corresponding (n - q) nonlinear characteristics defines the new output variables (see eqn. (1.4.7)).

$$\psi_i\,(y_i) = f_i(y_i) - k_i y_i \quad (i = q+1,\ q+2, \ldots n) \tag{2.5.1}$$

Substituting for $f_i(y_i)$ from eqn. (2.5.1) in the system equation (2.2.1) gives (for $\underline{x}(t) = \underline{0}$)

$$\underline{y} = -G\,(\underline{\psi} + \kappa\ \underline{y})$$

$$\text{or } \underline{y} = -(I + G\,\kappa)^{-1}\ .\ \ G\ \underline{\psi} = -G'\ \underline{\psi} \tag{2.5.2}$$

where for convenience one defines

$$\psi_i\,(y_i) \equiv f_i\,(y_i) \qquad \text{for all } i \leq q,$$

and κ is a diagonal matrix whose first q elements are zero, and the remaining (n-q) are the constants k_i.

Comparing eqns. (2.5.2) and (2.2.1), the pole-shifting technique has not altered the form of the system equation, and the criteria of section 2.3. (Nos. 1.1 to 1.4) apply unaltered in substance, where the poles and residues of G are now replaced by those of the new transfer matrix,

$$G' = (I + G\kappa)^{-1} G.$$

Inherently unstable systems (containing one or more poles with non-negative real parts) may be dealt with by the above transformation. Since all the elements of G' have a common denominator

$$D = |I + G\kappa|$$

only criteria 1.3 and 1.4 will in fact apply (common row poles).

The nonlinear restrictions (2.3.1) and (2.3.2) will then be replaced by

$$y_i \psi_i (y_i) \geq 0 \text{ for all } |y_i| \neq 0 \qquad (2.5.3)$$

or $$\int_0^{y_i} \psi_i (y_i) dy_i \geq 0 \text{ for all } |y_i| \neq 0 \qquad (2.5.4)$$

(2.5.3) confines the nonlinear characteristics to the sectors $[k_i, \infty]$ whereas (2.5.4) is somewhat weaker (see section 1.3. Chapter I).

Application of both pole and zero-shifting techniques allows a stability or instability test for any nonlinear characteristics confined to the sectors $[k_i, K_i]$. Since pole-shifting does not alter the form of the system equation, this technique should be applied first.

The zero-shifting technique unfortunately requires invertion of the operational matrix of eqn. (2.5.2.), which is by no means a simple task for high-order multivariable systems.

2.6 Control Quality

Estimates of the upper and lower bounds of the envelope of system transient response to an initial disturbance may be obtained by 'λ-transformations' similar to section 1.5, Chapter I. As in establishing stability, it is necessary to choose Liapunov functions which have sign-definite functions of the state variables in their derivatives.

2.6a. No common row poles: sectors [0, ∞].

Define the new variables (cf. section 1.5 Chapter I)

$$\underline{x}^{(ij)} = e^{\lambda t} \underline{z}^{(ij)} \qquad (2.6a.1)$$

$$(i,j = 1,2,\ldots n)$$

$$v_i = e^{\lambda t} y_i \qquad (2.6a.2)$$

Substituting for $\underline{z}^{(ij)}$ and y_i from the above eqns. into eqns. (2.2a.1) and (2.2a.2) gives

$$\dot{\underline{x}}^{(ij)} = (\lambda I^{(ij)} + \Lambda^{(ij)}) \underline{x}^{(ij)} + \underline{F}_j^{(ij)} \qquad (2.6a.3)$$

$$v_i = \sum_{i=1}^{n} (\underline{\alpha}^{(ij)T} \underline{x}^{(ij)} - R_{ij} F_j (v_j,t)) \qquad 2.6a.4)$$

where I is the unit matrix, and

$$F_j (v_j,t) = e^{\lambda t} f_j (v_j . e^{-\lambda t}).$$

Eqns. (2.6a.3) and (2.6a.4) are identical in form to the original canonic eqns. (2.2a.1) and (2.2a.2). Choose therefore as a Liapunov function for the above system the form

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{x}^{(ij)T} \, \overline{Q}^{(ij)} \, \underline{x}^{(ij)} \qquad (2.6a.5)$$

where the symmetric matrices $\bar{Q}^{(ij)}$ are defined by their (r,s)th element

$$\bar{q}^{(ij)}_{rs} = \frac{a_r^{(ij)} \, a_s^{(ij)}}{\mu_r^{(ij)} + \mu_s^{(ij)}}$$

and $\mu_r^{(ij)} = c + \lambda + \lambda_r^{(ij)}$ (cf. eqn. 2.3a.1).

Then the analysis of criterion 1.2, section 2.3a, may be applied directly to this system by replacing $\lambda_r^{(ij)}$ by $\lambda + \lambda_r^{(ij)}$ throughout.

The coefficient equations, nonlinear restrictions and the function W of criterion 1.2 (for k = 0, since no integral terms have been used in this case) may be used to determine bounds on λ by replacing $\lambda_r^{(ij)}$ by $\lambda + \lambda_r^{(ij)}$ for all r,i,j.

Since system stability may only be established for positive values of c, it follows that Re $(\lambda + \lambda_r^{(ij)})$ must be negative for all r,i,j (see criterion 1.2). λ must therefore be less than the smallest modulus of the real parts of the system poles. No upper bound on λ is obtainable using the above Liapunov function. In the single loop case, it was sufficient to take c = 0, when the relevant Liapunov function derivative is positive definite regardless of the sign of the function itself (see section 1.5, chapter I, and example 1.2).

It is sufficient therefore to use criterion 1.2 to estimate the lower bound of λ, λ_e, by putting k = 0 and replacing $\lambda_r^{(ij)}$ by $\lambda + \lambda_r^{(ij)}$ for all r,i,j.

Note that k = 0 requires that at least every transfer function $g_{ii}(p)$ has a remainder in its partial fraction expansion, and many systems which may be established as stable by criterion 1.2 will reject the above method for estimating transient response.

It is easily shown that the remaining criteria can be used in a similar manner to determine a lower bound of λ for a given system which has been established as stable. In section 2.6b below the appropriate Liapunov functions for the λ-transformed systems are given for the various criteria. Having determined the lower bounds of λ, λ_e, then $e^{-2\lambda_e t}$ is guaranteed to contain, for all t, the modulus of these functions (see eqns. (1.5.7) and (1.5.8), section 1.5, Chapter I).

2.6b. Liapunov functions parallel to the stability criteria.

Criterion 1.4

$$V = \sum_{i=1}^{h} \underline{x}^{(i)T} \overline{Q}^{(i)} \underline{x}^{(i)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{x}^{(ij)T} \overline{Q}^{(ij)} \underline{x}^{(ij)}$$

Criterion 1.6

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{x}^{(i)T} \overline{Q}^{(i)} \underline{x}^{(i)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{x}^{(ij)T} \overline{Q}^{(ij)} \underline{x}^{(ij)}$$

For all three cases, k is effectively zero in the stability criteria, and as for criterion 1.2 (see section 2.6a) not all systems which may be established as stable can be given an estimate of transient response by the above methods.

For systems in which the nonlinearities are confined to sectors $[k_i, \infty]$ the argument of section 2.5 is valid.

2.7. Examples

Example 2.1.

Consider the system where

$$G = \begin{bmatrix} \dfrac{p-1}{p^2 + 13p + 36} & 0.26 \\ 0.4 & \dfrac{2(p-1)}{p^2 + 6p + 13} \end{bmatrix}.$$

Since $g_{11}(p)$ and $g_{22}(p)$ both have a zero in the right half of the p-plane, stability may only be established for nonlinear characteristics confined to some finite sectors $[0, K_i]$ $(i = 1,2)$ such that the zero-shifting technique moves these zeroes into the left half plane. Assume that the nonlinear characteristics are confined to the sectors [0, 1/3] and [0, 1/10], i.e. $K_1 = 1/3$ and $K_2 = 1/10$. Criterion 1.5 may then be used to establish stability. Note that since $k = 0$ (see section 2.4a) criteria 1.5 and 1.6 will yield identical results.

The partial fraction expansions of $g_{11}(p)$ and $g_{22}(p)$ are

$$g_{11}(p) = \frac{2}{p+9} + \frac{-1}{p+4}$$

$$g_{22}(p) = \frac{1-j2}{p+(3+j2)} + \frac{1+j2}{p+(3-j2)}\ .$$

From section 2.4a, the relevant system parameters are

$$\lambda_1^{(11)} = -9 \qquad \alpha_1^{(11)} = -2$$

$$\overline{R}_{11} = \frac{1}{K_1} = 3$$

$$\lambda_2^{(11)} = -4 \qquad \alpha_2^{(11)} = +1$$

$$\lambda_1^{(22)} = -3-j2 \qquad \alpha_1^{(22)} = -1+j2$$

$$\overline{R}_{22} = \frac{1}{K_2} = 10$$

$$\lambda_2^{(22)} = -3+j2 \qquad \alpha_2^{(22)} = -1-j2$$

$$\overline{R}_{12} = R_{12} = 0.26 \qquad \overline{R}_{21} = R_{21} = 0.4.$$

Since there are no interaction dynamics, the variables $\underline{z}^{(ij)}$ $(i \neq j)$ vanish, and the effects of system interactions appear only as the constants R_{12} and R_{21}.

To determine the sign of the function W (eqn. (2.4a.10) define the new variables (see eqn. (2.3a.15))

$$b_1^{(11)} = a_1^{(11)} z_1^{(11)} \qquad b_2^{(11)} = a_2^{(11)} z_2^{(11)}$$

$$b_1^{(22)} = a_1^{(22)} z_1^{(22)} \qquad b_2^{(22)} = a_2^{(22)} z_2^{(22)}\ .$$

Substituting for the above variables and the system parameters in eqn. (2.4a.10) gives

$$W = \frac{9}{9-c} b_1^{(11)2} + \frac{26}{13-2c} b_1^{(11)} b_2^{(11)} + \frac{4}{4-c} b_2^{(11)2}$$

$$+ \frac{3+j2}{3+j2-c} b_1^{(22)2} + \frac{12}{6-2c} b_2^{(22)} b_1^{(22)} + \frac{3-j2}{3-j2-c} b_2^{(22)2}$$

$$+ 2\sqrt{3}\, \varnothing_1(u_1)\, (b_1^{(11)} + b_2^{(11)}) + 2\sqrt{10}\, \varnothing_2(u_2)(b_1^{(22)} + b_2^{(22)})$$

$$+ 3\, \varnothing_1^2 (u_1) + 10\, \varnothing_2^2 (u_2) + 0.66\, \varnothing_1 (u_1)\, \varnothing_2 (u_2)$$

W may be written in the quadratic form $\underline{x}^T A\, \underline{x}$, where

$$\underline{x}^T = \left[b_1^{(11)}, b_2^{(11)}, b_1^{(22)}, b_2^{(22)}, \varnothing_1 (u_1), \varnothing_2 (u_2) \right].$$

Since the variables $b_r^{(ii)}$ $(r, i = 1,2)$ constitute a positive definite function in W provided that $c < 3$ (see detailed explanation in criterion 1.1: V is negative definite for $c < 3$) it is sufficient, by Sylvester's theorem, that the fifth and sixth leading minors of A be positive. These minors will of course be real, since A is the matrix of a real quadratic function. The minors are readily found to be

$$\Delta_5 = \frac{-9000c^3 \ (c - 6)}{(c-9)(c-4)(2c-13)^2 \ (2c-6)^2}$$

$$\Delta_6 = \frac{2.66.10^5 \ (c + 1.39)(c - 0.942)c^2 .}{(c-9)(c-4)(2c-13)^2(2c-6)^2}$$

Provided that $c < 3$, Δ_5 is positive for all positive c, and Δ_6 is positive for $c > 0.942$. So far, therefore, the restriction imposed upon c is that

$$0.942 < c < 3 \qquad (2.7.1)$$

From appendix II, the necessary and sufficient conditions for appropriate solution of the coefficient equations (2.4a.7) are

(1) $$A = \frac{1}{K_i} + \frac{a_1^{(ii)}}{w_1^{(ii)}} + \frac{a_2^{(ii)}}{w_2^{(ii)}} \geq 0 \quad (i = 1,2)$$

(2) $$\alpha_1^{(ii)} w_1^{(ii)} + \alpha_2 w_2^{(ii)} + \frac{1}{K_i} (w_i^{(ii)2} + w_2^{(ii)2})$$

$$\pm \frac{2}{\sqrt{K_i}} w_1^{(ii)} w_2^{(ii)} \sqrt{A} \geq 0 \ (i = 1,2).$$

Substituting for the system parameters in inequalities (1),

$$3 + \frac{-2}{c-9} + \frac{1}{c-4} = \frac{3c^2 - 40c + 77}{(c-9)(c-4)} \geq 0 \qquad (2.7.2)$$

$$10 + \frac{-2(c+1)}{(c-3)^2 + 4} = \frac{10c^2 - 62c + 128}{(c-3)^2 + 4} \geq 0 \qquad (2.7.3)$$

The numerators of the expressions on the left of inequalities (2.7.2) and (2.7.3) are positive for all c: these inequalities are therefore satisfied for all $0 < c < 3$.

Noting that

$$\alpha_1^{(11)} w_1^{(11)} + \alpha_2^{(11)} w_2^{(11)} = 14-c$$

and

$$\alpha_1^{(22)} w_1^{(22)} + \alpha_2^{(22)} w_2^{(22)} = 7-c$$

then inequalities (2) are also satisfied for all $0 < c < 3$, by choosing the positive square root in each case.

Eqn. (2.7.1) therefore represents necessary and sufficient conditions for V to be negative definite, $\dot{V}$ to be positive definite. The system is totally stable.

To obtain an estimate of transient response, replacing $\lambda_r^{(ii)}$ by $\lambda + \lambda_r^{(ii)}$ $(r,i = 1,2)$ in the above criterion yields as the corresponding minors of W

$$\Delta_5(\lambda) = \frac{-9000c^3 (c + 2\lambda - 6)}{(c + \lambda - 9)(c+\lambda-4)(2c+2\lambda-13)^2(2c+2\lambda-6)^2}$$

$$\Delta_6(\lambda) = \frac{-15696c^2 (\lambda^2 + (c - 9.5)\lambda - 16.93 (c + 1.39)(c - 0.942))}{(c + \lambda - 9)(c + \lambda - 4)(2c + 2\lambda - 13)^2 (2c \pm 2\lambda - 6)^2}$$

where λ must be less than 3 (see section 2.6a).

Also, $\mathrm{Re}\,(\mu_r^{(ii)}) < 0$ for all r,i.

$\therefore \quad c + \lambda - 3 < 0$

or $\quad \lambda < 3 - c \qquad (2.7.4.)$

Choose c = 0.942 (or as near as desired to 0.942, to satisfy inequality (2.7.1)) since this gives the largest value of λ possible, and is therefore the best estimate obtainable. Then inequality (2.7.4) is satisfied provided that $\lambda < 2.058$. It now remains to verify that for c = 0.942, λ = 2.058, Δ_5 and Δ_6 are positive and the coefficient equations have appropriate solutions. Since c is effectively replaced by $c + \lambda$ in the coefficient equations, the latter condition is inherently satisfied since $(c+\lambda) = 3$. For Δ_5 and Δ_6 to be positive it is sufficient that

$$c + 2\lambda - 6 < 0 \qquad (2.7.5)$$

and

$$\lambda^2 + (c - 9.5)\lambda - 16.93\,(c + 1.39)(c - 0.942) < 0 \qquad (2.7.6)$$

The left hand sides of inequalities (2.7.5) and (2.7.6) are then -0.942 and -4.328 respectively. $\lambda = \lambda_e = 2.508$ is therefore a lower bound to the envelope of the modulus of the function

$$V = \frac{a_1^{(ii)2}}{2\mu_1^{(ii)}}\, z_1^{(ii)2} + \frac{2a_1^{(ii)}\, a_2^{(ii)}\, z_1^{(ii)}\, z_2^{(ii)}}{\mu_1^{(ii)} + \mu_2^{(ii)}} + \frac{a_2^{(ii)2}}{2\mu_2^{(ii)}}\, z_2^{(ii)2}$$

$$+ \frac{a_1^{(22)2}}{2\mu_1^{(22)}} z_1^{(22)2} + \frac{2a_1^{(22)} a_2^{(22)} z_1^{(22)} z_2^{(22)}}{\mu_1^{(22)} + \mu_2^{(22)}} + \frac{a_2^{(22)2}}{2\mu_2^{(22)}} z_2^{(22)}$$

(see criteria 1.5 and 1.1).

Example 2.2

Consider now the system where

$$G = \begin{bmatrix} \frac{K_{11}}{p+1} & \frac{K_{12}}{p+1} \\ \\ \frac{K_{21}}{p+2} & \frac{K_{22}}{p+2} \end{bmatrix}.$$

G has a common pole of -1 in the first row, and a common pole of -2 in the second row. Provided that K_{11} and K_{22} are positive, criterion 1.4 may be used to investigate stability of this system in the sectors [0, ∞].

From section 2.2b, the relevant system parameters are

$$\lambda_1^{(1)} = -1 \qquad \hat{a}_1^{(11)} = -K_{11} \qquad \hat{a}_1^{(12)} = -K_{12}$$

$$\lambda_1^{(2)} = -2 \qquad \hat{a}_1^{(21)} = -K_{21} \qquad \hat{a}_1^{(22)} = -K_{22}$$

$$\hat{r}_{11} = r_{11} = K_{11} \qquad \hat{r}_{22} = r_{22} = K_{22}$$

$$r_{12} = K_{12} \qquad r_{21} = K_{21}.$$

The necessary condition Re $(w_r^{(i)}) < 0$ for all r,i is satisfied provided that $c < 1$.

From the coefficient equations (2.3b.24),

$$a_1^{(1)2} = \frac{1-c}{K_{11}} \qquad a_1^{(2)2} = \frac{2-c}{K_{22}} \tag{2.7.6}$$

These solutions are real as required if again $c < 1$. The function W of eqn. (2.3b.23) is then

$$W = \frac{1}{K_{11}} z_1^{(1)2} + \frac{2}{K_{22}} z_1^{(2)2} + K_{11} f_1^2 (y_i) + K_{22} f_2^2 (y_2)$$

$$+ (K_{12} + K_{21}) f_1 (y_1) f_2 (y_2) + f_1(y_1) z_1^{(1)} + f_2 (y_2) z_1^{(2)}$$

$$+ \frac{K_{12}}{K_{11}} f_2 (y_2) z_1^{(1)} + \frac{K_{21}}{K_{22}} f_1 z_1^{(2)} .$$

Because each transfor function of G is first order, and hence from eqns. (2.7.6), the sign of W is independent of the parameter c.

W may be written in the form $\underline{x}^T A \underline{x}$, where

$$\underline{x}^T = [z_1^{(1)}, z_1^{(2)}, f_1 (y_1), f_2 (y_2)].$$

By the argument of example 2.1, W is positive definite if the third and fourth leading minors of A are positive.These minors are

$$\Delta_3 = \frac{1}{2K_{22}} (3 - \frac{K_{21}^2}{2K_{11}K_{22}})$$

$$\Delta_4 = \frac{3}{4} - \frac{1}{2K_{11}K_{22}} (K_{12}^2 + \frac{1}{2}K_{21}^2 + \frac{1}{2} K_{12} K_{21} - \frac{(K_{12}K_{21})^2}{8K_{11}K_{22}})$$

Since K_{11} and K_{22} are positive, it is therefore sufficient that

$$6K_{11}K_{22} - K_{21}^{2} > 0 \qquad (2.7.7)$$

and

$$3(K_{11}K_{22})^2 - 2K_{11}K_{22}\,(K_{12}^{2} + \tfrac{1}{2}K_{21}^{2} + \tfrac{1}{2}K_{12}K_{21})$$

$$+ \tfrac{1}{4}(K_{12}K_{21})^2 > 0. \qquad (2.7.8.)$$

The equivalent linear system on the other hand is stable in the sectors [0, ∞], provided that

$$K_{11}K_{22} - K_{12}K_{21} > 0 \qquad (2.7.9)$$

To compare the results obtained for nonlinear and linear systems, put $K_{11} = K_{22} = K$ and $K_{12} = K_{21} = k$.

Inequalities (2.7.7) and (2.7.8) then become

$$6K^2 - k^2 > 0 \qquad (2.7.10)$$

and

$$3K^4 - 4k^2 + \tfrac{1}{4}k^4 > 0$$

or $$3(K^2 - (\tfrac{2}{3} + \sqrt{\tfrac{13}{6}})\,k^2)(K^2 - (\tfrac{2}{3} - \sqrt{\tfrac{13}{6}})\,k^2) > 0. \qquad (2.7.11)$$

Inequality (2.7.9) becomes

$$K^2 - k^2 > 0 \qquad (2.7.12)$$

The nonlinear stability conditions of inequalities (2.7.10) and (2.7.11) are satisfied provided that

$$\frac{\frac{k^2}{2}}{K} < \frac{1}{\frac{2}{3} + \frac{\sqrt{13}}{6}} \doteq 0.788$$

whereas the linear system, from inequality (2.7.12), is stable provided that

$$\frac{\frac{k^2}{2}}{K} < 1.$$

Since stability is established for any choice of $c < 1$, and the lower bound of λ, λ_e, must be less than the smallest modulus of system poles, it is sufficient to choose $c = 0$, then in the limit

$$\lambda_e = 1.$$

2.8. Comment on Chapter II

In deriving the criteria of this chapter, Liapunov functions have been chosen which contain independent quadratic groups of state variables, each relevant to one system transfer function. When the transfer matrix contains common row poles, these groups are further subdivided. Thanks to this choice, the initially unknown coefficient vectors $\underline{a}^{(ij)}$ are determined by the solution of n^2 separate sets of simultaneous equations, one for each system transfer function. Only the n sets of vectors $\underline{a}^{(ii)}$ are determined by quadratic equations; the remaining n(n-1) sets $\underline{a}^{(ij)}$ $(i \neq j)$ are obtained by solution of linear equations. The determination of these coefficients is therefore very little more difficult than if the system loops were uncoupled, since solution of simultaneous linear equations is trivial compared to the quadratic case.

A further calculation which must be performed in all criteria is the evaluation of the sign of the quadratic functions 'W'. The determination of the sign of these functions may be reduced to the evaluation of a number of determinants (equal to the number of system variables, n, and not the total order of the system plus n) by arranging them in the appropriate form as shown in examples 2.1 and 2.2. This operation should also select the value of the parameter c which gives optimum results.

Although the relatively simple examples in section 2.7 were worked out using only a slide rule, for high order systems use of a digital computer would be necessary.

The criteria for estimation of system transient response are very simply obtained from the stability criteria, by adding the parameter λ to each system pole and performing the same calculations. These criteria cannot be applied to as wide a range of systems as the stability criteria.

As for single-loop systems, a stability (instability) test may be applied for given nonlinearities confined to the sectors $[k_i, K_i]$ only if the equivalent linear system is stable (unstable) in the same sectors.

In principle, the criteria could be embodied in a digital computer programme which could produce stability or instability tests from the given system data, and would require no analysis or estimation by the user.

Finally note that the criteria represent sufficient but not necessary conditions for stability or instability.

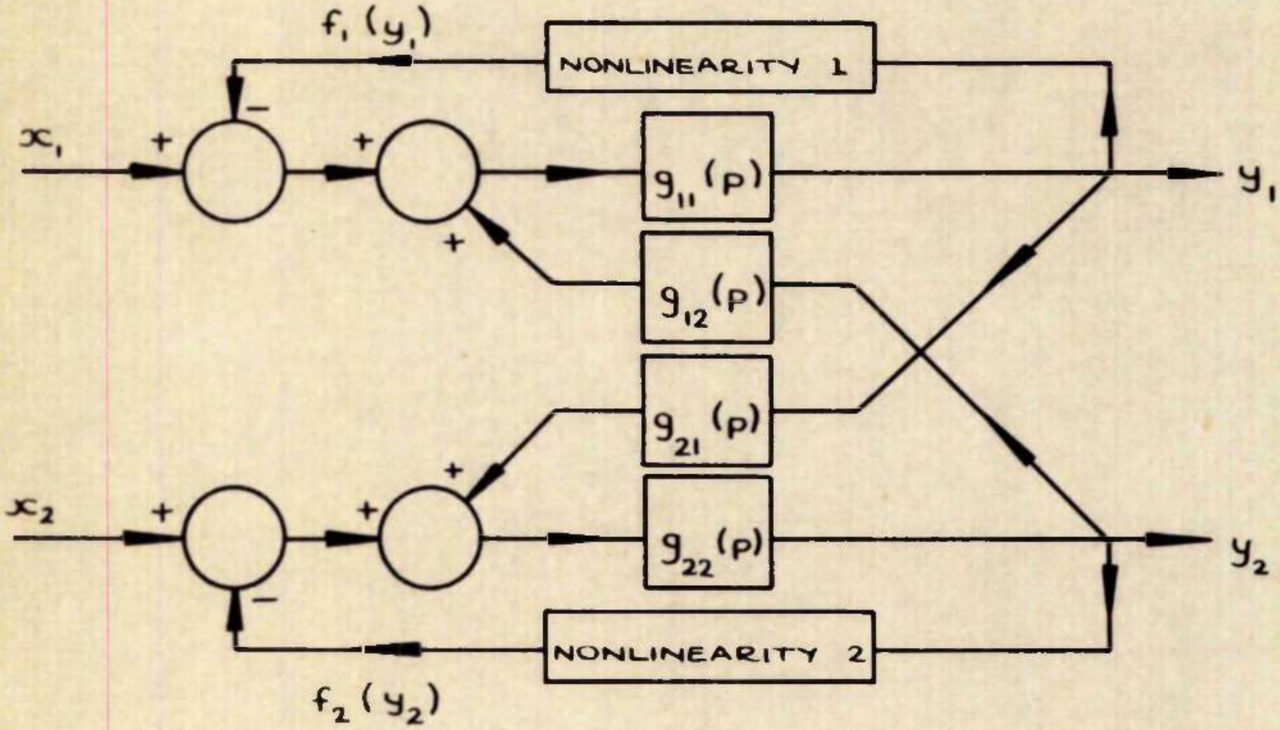

FIG. 7. A TWO-VARIABLE CLASS 2 SYSTEM.

CHAPTER III

STABILITY OF CLASS 2 MULTIVARIABLE SYSTEMS

Introduction

Class 2 systems are defined to be those in which the linear multivariable process to be controlled contains only feed~~forward~~back interactions (see fig.7). The process is assumed to be controlled by many feedback loops each containing a single-valued nonlinear device, as for class 1 systems. In general, class 2 systems may be transformed into equivalent class 1 systems, but due to the difficulty of inverting matrices of transfer functions it is convenient to consider the two classes separately.

Examples of processes described by a class 2 transfer matrix are linear nuclear reactor models (Harvey[10]) and some types of electrical power systems (e.g. parallel connected synchronous machines).

3.1. Canonic representation of class 2 systems

The general equation describing a class 2 system is

$$\underline{y} = G_1 \, (\underline{x} - \underline{f} + G_2 \, \underline{y}) \qquad (3.1.1.)$$

where G_1 is an n x n diagonal matrix of the transfer functions $g_{ii}(p)$, and G_2 is an n x n matrix of the remaining transfer functions $g_{ij}(p)(i \neq j)$. A two-variable class 2 system is represented in block-diagram form in fig.7.

When $\underline{x}(t) \equiv \underline{0}$ for all t, eqn. (3.1.1) also represents systems in which the nonlinear devices are actuators.

A suitable canonic form of state variable equations for $\underline{x}(t) \equiv 0$ is given by

$$\underline{\dot{z}}^{(ii)} = \Lambda^{(ii)} \underline{z}^{(ii)} + \underline{s}^{(ii)} + \underline{f}_i^{(ii)} \qquad (3.1.2)$$

$$\underline{\dot{z}}^{(ij)} = \Lambda^{(ij)} \underline{z}^{(ij)} + \underline{y}_j^{(ij)} \quad (i \neq j) \qquad (3.1.3)$$

$$y_i = \underline{\alpha}^{(ii)T} \underline{z}^{(ii)} - R_{ii} \, [f_i \, (y_i) + s] \; (i,j = 1,2,\ldots n) \qquad (3.1.4)$$

where the notation is as in Chapter II. $\underline{s}^{(ii)}$ is a column vector all of whose elements are equal to the scalar quantity

$$s = \sum_{j=1}^{n} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} \, y_j) \quad (j \neq i) \qquad (3.1.5)$$

To show that this form represents a class 2 system, substituting from eqns. (3.1.2) and (3.1.3) into eqn. (3.1.4) gives

$$y_i = \left[\alpha^{(ii)T} (PI^{(ii)} - \Lambda^{(ii)})^{-1} \underline{e}^{(ii)} - R_{ii} \right] X$$

$$X \left[f_i \, (y_i) + \sum_{\substack{j=1 \\ (j \neq i)}}^{n} \left(\underline{\alpha}^{(ij)T} (pI^{(ij)} - \Lambda^{(ij)})^{-1} \underline{e}^{(ij)} - R_{ij} \right) y_j \right] .$$

The linear transformation which is used to obtain the above canonic form from any initial set of state variables must be nonsingular. One necessary condition for nonsingularity is that no transfer function contains multiple poles (see Appendix V). The necessary and sufficient conditions are difficult to obtain in general, but a simple test for nonsingularity is that the determinant of a certain matrix (See Appendix V) of order equal to that of the system should be finite and not equal to zero.

3.2. Stability in the sectors [0, ∞].

Assume that the first k transfer functions of G_1 have no remainders in their partial fraction expansions, i.e.

$R_{ii} = 0$ for all $i = 1,2,\ldots k$.

Any Liapunov function chosen for the system may then involve integrals of the nonlinear functions $f_i(y_i)$ only for $i = 1,2,\ldots k$.

Consider the following form as a Liapunov function for the system of section 3.1:-

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{z}^{(ij)} - \eta \sum_{i=1}^{k} \int_{o}^{y_i} f_i(y_i)\, dy_i \qquad (3.2.1)$$

where the matrices $Q^{(ij)}$ are defined similarly to Chapter II, namely by their (r,s)th element

$$q_{rs}^{(ij)} = \frac{a_r^{(ij)} \; a_s^{(ij)}}{w_r^{(ij)} + w_s^{(ij)}} \qquad (w_r^{(ij)} = c + \lambda_r^{(ij)}).$$

Differentiating V w.r.t. time, using eqns. (3.1.2)-(3.1.5) gives

$$\dot{V} = -2c\overline{V} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 + \eta \sum_{i=1}^{k} r_{ii} f_i^2 (y_i)$$

$$+ \eta \sum_{i=1}^{k} \sum_{\substack{j=1 \\ (i \neq j)}}^{n} r_{ii} f_i (y_i) (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} y_j)$$

$$- n \sum_{i=1}^{k} f_i(y_i)\, \underline{\beta}^{(ii)T}\, \underline{z}^{(ii)} + 2\sum_{i=1}^{n} f_i(y_i)\, \underline{z}^{(ii)T} Q^{(ii)} \underline{e}^{(ii)}$$

$$+ 2 \sum_{\substack{i=1 \\ (i\neq j)}}^{n} \sum_{j=1}^{n} \underline{z}^{(ii)T} Q^{(ii)} \underline{e}^{(ii)} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} f_j(y_j))$$

$$+ 2 \sum_{\substack{i=1 \\ (i\neq j)}}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{e}^{(ij)} y_j \qquad (3.2.2)$$

As for class 1 systems (section 2.3a, Chap.II), at this stage $\dot{V}$ must be constrained to be a possibly sign-definite quadratic function of all the state variables and the nonlinear functions $f_i(y_i)$. This should be done in such a manner that if the system interactions were removed, any criteria obtained would be equivalent to n independent single-loop criteria (see Chap.I). Equations must also be found for the coefficients $a_r^{(ij)}$ pertaining to the interacting transfer functions $(i\neq j)$.

It is possible to achieve this in two alternative ways (cf. Chap.II, section 2.3a.)

Firstly, add to $\dot{V}$ the expression (see eqn. (1.3.4)

$$\left(\sum_{i=k+1}^{n} f_i(y_i) + \sum_{\substack{i=1 \\ (i\neq j)}}^{n} \sum_{j=1}^{n} \underline{\alpha}^{2(ji)T} \underline{z}^{(ji)} \right) X$$

$$X\, (y_i - \underline{\alpha}^{(ii)T} \underline{z}^{(ii)} + R_{ii}[f_i(y_i) + s]) \equiv 0 \qquad (3.2.3)$$

where $\underline{\alpha}^{2(ji)}$ denotes a vector whose elements are the squares of the elements of the vector $\underline{\alpha}^{(ji)}$.

The reason for introducing these squares is to allow for systems which have interacting transfer function gains of either sign, as will be seen in the examples at the end of this chapter (see also Rae and MacLellan[9]).

Completing the appropriate squares in $\dot{V}$ and arranging it in the necessary form (similar to all previous criteria) then yields the following criterion.

CRITERION 2.1

The nonlinearity restrictions are

(1) $\int_0^{y_i} f_i(y_i)\, dy_i \geq 0$ for all $|y_i| \neq 0$, and

$f_i(0) = 0$ for all $i = 1,2,\ldots k$

(2) $y_i f_i(y_i) \geq 0$ for all $|y_i| \neq 0$, and

$f_i(0) = 0$ for all $i = k+1,k+2,\ldots n$.

The coefficient equations are

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{n\, r_{ii}}\ \underline{a}^{(ii)} + \underline{\beta}^{(ii)} \quad (i = 1,2,\ldots k) \tag{3.2.4}$$

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{R_{ii}}\ \underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1, k+2,\ldots n) \tag{3.2.5}$$

$$2Q^{(ij)}\underline{e}^{(ij)} = -\underline{\alpha}^{2(ij)} \quad (i,j = 1,2,\ldots n,\ i\neq j) \tag{3.2.6}$$

The nature of eqns. (3.2.6) is such that three of the necessary conditions for their appropriate solution are satisfied (see Appendix II).

If there exists some negative value of c such that

(a) the above eqns. have appropriate roots $\underline{a}^{(ij)}$ for

$$\eta = -1, \quad r_{ii} \leq 0, \quad \text{and} \quad R_{ii} \geq 0 \text{ for all } i,$$

(b) $\mathrm{Re}\,(\lambda_e^{(ij)})$ and $\mathrm{Re}\,(w_e^{(ij)}) > 0$ for all e,i,j, then V is positive definite, and if $\dot{V}$ is positive definite the system is unstable.

Conversely, if there exists some positive value of c such that

(c) the eqns. have appropriate roots for $\eta = +1$ and r_{ii}, $R_{ii} \geq 0$ for all i,

(d) $\mathrm{Re}\,(\lambda_e^{(ij)})$ and $\mathrm{Re}\,(w_e^{(ij)}) < 0$ for all i

then V is negative definite, and if $\dot{V}$ is positive definite the system is totally stable.

The prerequisite that $\dot{V}$ be positive definite is satisfied if the following quadratic function is positive definite:

$$W = -2c\overline{V} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 + \eta \sum_{i=1}^{k} r_{ii} f_i^2 (y_i)$$

$$+ 2 \sum_{i=1}^{k} \sqrt{\eta\, r_{ii}}\, f_i(y_i)\, \underline{a}^{(ii)T} \underline{z}^{(ii)}$$

$$+ 2 \sum_{i=k+1}^{n} \sqrt{R_{ii}}\, f_i\, (y_i)\, \underline{a}^{(ii)T} \underline{z}^{(ii)}$$

$$+ \; n \sum_{\substack{i=1 \\ (i\neq j)}}^{k} \sum_{j=1}^{n} [r_{ii} \, f_i(y_i) \; \underline{\alpha}^{(ij)T} \, \underline{z}^{(ij)} \; - R_{ij} \, (\underline{\alpha}^{(jj)T} \, \underline{z}^{(jj)} \; - R_{jj} [f_j \, (y_j) + s])$$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{k} \; (2 \sqrt{n \, r_{ii}} \; \underline{a}^{(ii)} + \underline{\beta}^{(ii)T}) \; \underline{z}^{(ii)} \; (\underline{\alpha}^{(ij)T} \, \underline{z}^{(ij)} - R_{ij} \, y_j)$$

$$+ \sum_{\substack{i=k+1 \\ (i\neq j)}}^{n} \sum_{j=k+1}^{n} \; (2 \sqrt{R_{ii}} \; \underline{a}^{(ii)} + \underline{\alpha}^{(ii)\,T}) \; \underline{z}^{(ii)} \; (\underline{\alpha}^{(ij)T} \, \underline{z}^{(ij)} - R_{ij} \, y_j)$$

$$- \sum_{\substack{i=1 \\ (i\neq j)}}^{n} \sum_{j=1}^{n} \underline{\alpha}^{2(ij)T} \, \underline{z}^{(ij)} \; (\underline{\alpha}^{(jj)T} \, \underline{z}^{(jj)} \; - R_{jj} [f_j \, (y_j) + s]) \qquad (3.2.7)$$

For brevity in writing, the substitution for y_j in terms of the state variables (see eqn. (3.1.4)) has not been made in the above equation.

Unlike class 1 systems, the criterion is valid when some or all of the constants r_{ii} are zero.

The above criterion is similar to criterion 1.1 in that it subsumes the same stability properties of the uncoupled system (see eqns. (2.3a.4) and (2.3a.5)).

Alternatively, one may add to equn. (3.2.2), instead of the expression (3.2.3), the expression

$$(\sum_{i=1}^{n} f_i \, (y_i) + \sum_{\substack{i=1 \\ (i\neq j)}}^{n} \sum_{j=1}^{n} \underline{\alpha}^{2(ji)T} \, \underline{z}^{(ji)}) \; X$$

$$X \; (y_i - \underline{\alpha}^{(ii)T} \, \underline{z}^{(ii)} + R_{ii} \, f_i \, (y_i) \; + s]) \equiv 0$$

Similar manipulation then yields the following criterion (compare with criterion 1.2, Chap.II).

CRITERION 2.2

The nonlinearity restrictions are

$y_i f_i(y_i) \geq 0$ for all $|y_i| \neq 0$, $f_i(0) = 0$

for all i.

The coefficient eqns. are

$$2Q^{(ii)} \underline{e}^{(ii)} = \underline{\alpha}^{(ii)} \quad (i = 1,2,\ldots k) \tag{3.2.8}$$

$$2Q^{(ii)} \underline{e}^{(ii)} = \sqrt{R_{ii}}\, \underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \quad (i = k+1, k+2, \ldots n) \tag{3.2.9}$$

$$2Q^{(ij)} \underline{e}^{(ij)} = -\underline{\alpha}^{2(ij)} \quad (i,j = 1,2,\ldots n,\ i \neq j) \tag{3.2.10}$$

Since the constant η does not enter into the coefficient equations, and the resulting Liapunov function derivative has all coefficients of terms involving $f_i(y_i)$ multiplied by η for $i \leq k$, it proves sufficient in this case to put $\eta = 0$. Then the function W which must be positive definite is

$$W = -2c\overline{V} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^2 + \sum_{i=k+1}^{n} R_{ii} f_i^2(y_i)$$

$$+ 2 \sum_{i=k+1}^{n} \sqrt{R_{ii}}\, f_i(y_i)\, \underline{a}^{(ii)T} \underline{z}^{(ii)}$$

$$+ \sum_{\substack{i=1\\(i\neq j)}}^{k} \sum_{j=1}^{k} \underline{\alpha}^{(ii)T} \underline{z}^{(ii)} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij}\, y_j)$$

$$+ \sum_{\substack{i=k+1\\(i\neq j)}}^{n} \sum_{j=k+1}^{n} (2\sqrt{R_{ii}}\, \underline{a}^{(ii)} + \underline{\alpha}^{(ii)})^{T} \underline{z}^{(ii)} (\underline{\alpha}^{(ij)} \underline{z}^{(ij)} - R_{ij}\, y_j)$$

$$- \sum_{\substack{i=1\\(i\neq j)}}^{n} \sum_{j=1}^{n} \underline{\alpha}^{2(ij)T} \underline{z}^{(ij)} (\underline{\alpha}^{(jj)T} \underline{z}^{(jj)} - R_{jj} [f_j (y_j) + s] \qquad (3.2.11)$$

If there are no remainders in the partial fraction expansions of $g_{ii}(p)$ $(i = 1,2,\ldots n)$, W reduces to a quadratic function of the state variables only.

3.3 Stability in the sectors $[0,K_i]$

If a given system is such that one or more of the uncoupled loops of the system is unstable for high values of open loop linear gain, the zero-shifting technique must be applied before stability can be established.

Assume that (n-q) loops are unstable for high gain. Then rotation of the output axes of the corresponding (n-q) nonlinear characteristics defines the new input variables (section 1.4, Chap.I).

$$u_i = y_i - \frac{1}{K_i} f_i (y_i) \quad (i=q+1,q+2,\ldots n) \qquad (3.3.1)$$

The outputs will then be defined in terms of the new inputs as

$$\emptyset_i (u_i) \equiv f_i (y_i) \qquad (i = q+1,q+2,\ldots n) \qquad (3.3.2)$$

Substituting for y_i and $f_i(y_i)$ from eqns. (3.3.1) and (3.3.2) into eqns. (3.1.2) to (3.1.4) gives the modified canonic form

$$\underline{\dot{z}}^{(ii)} = \Lambda^{(ii)} \underline{z}^{(ii)} + \underline{s}^{(ii)} + \underline{\emptyset}_i^{(ii)} \qquad (3.3.3)$$

$$\underline{\dot{z}}^{(ij)} = \Lambda^{(ij)} \underline{z}^{(ij)} + \underline{u}_j^{(ij)} + \frac{1}{K_j} \underline{\emptyset}_j^{(ij)} \quad (i \neq j) \qquad (3.3.4)$$

$$u_i = \underline{\alpha}^{(ii)T} \underline{z}^{(ii)} - \overline{R}_{ii} \emptyset_i (u_i) + s \; (i,j = 1,2,\ldots n) \qquad (3.3.5)$$

where all the elements of the vector $\underline{s}^{(ii)}$ are equal to the scalar quantity

$$s = \sum_{j=1}^{n} \left(\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} \left(u_j + \frac{1}{K_j} \emptyset_j (u_j)\right)\right) \; (i \neq j)$$

and for convenience one defines

$$u_i = y_i \text{ for all } i \leq q \qquad (3.3.6)$$

$$\overline{R}_{ii} = R_{ii} + \frac{1}{K_i} \text{ for all } i > q \qquad (3.3.7)$$

$$\overline{R}_{ii} = R_{ii} \text{ for all } i \leq q \qquad (3.3.8)$$

Assuming again that the first k transfer functions of G_1 $(k \leq q)$ have no remainders in their partial fraction expansions, take the following form as a Liapunov function for the above system:-

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{z}^{(ij)T} Q^{(ij)} \underline{z}^{(ij)} - n \sum_{i=1}^{k} \int_{0}^{y_i} f_i (y_i)\, dy_i$$

Differentiating V w.r.t. time, and adding the expression

$$\left(\sum_{i=k+1}^{n} \emptyset_i (u_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ (j \neq 1)}}^{n} \underline{\alpha}^{2(ji)T} \underline{z}^{(ji)} \right) X$$

$$X \left(u_i - \underline{\alpha}^{(ii)T} \underline{z}^{(ii)} + \overline{R}_{ii} [\emptyset_i (u_i) + s]\right) = 0$$

yields the following criterion (compare with criterion 1.5).

CRITERION 2.3

The nonlinearity restrictions are

(1) $\int_0^{y_i} f_i (y_i) \geq 0$ for all $| y_i| \neq 0$, and $f_i (0) = 0$

for all $i = 1,2,\ldots k$

(2) $y_i f_i (y_i) \geq 0$ for all $| y_i | \neq 0$, and $f_i (0) = 0$

for all $i = k+1, k+2,\ldots q$.

(3) The functions $\emptyset_i (u_i)$ must be single-valued, and $u_i \emptyset_i (u_i) \geq 0$ for all $| u_i | \neq 0$,

$\emptyset_i(0) = 0$ for all $i = q+1, q+2,\ldots n$.

The coefficient equations are

$$2Q^{(ii)} \underline{e}^{(ii)} = 2\sqrt{n\, r_{ii}}\, \underline{a}^{(ii)} + \underline{\beta}^{(ii)} \qquad (i = 1,2,\ldots k) \tag{3.3.9}$$

$$2Q^{(ii)} \underline{e}^{(ii)} = 2\sqrt{\overline{R}_{ii}}\, \underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \qquad (i = k+1, k+2, \ldots n) \qquad (3.3.10)$$

$$2Q^{(ij)} \underline{e}^{(ij)} = -\underline{\alpha}^{2(ij)} \qquad (i,j = 1,2,\ldots n,\ i \neq j) \qquad (3.3.11)$$

The function W which must be positive definite is

$$W = -2c\overline{V} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{a}^{(ij)T} \underline{z}^{(ij)})^{2} + n \sum_{i=1}^{k} r_{ii}\, \emptyset_i^{2}(u_i)$$

$$+ 2 \sum_{i=1}^{k} \sqrt{n r_{ii}}\, \emptyset_i (u_i)\, \underline{a}^{(ii)T} \underline{z}^{(ii)}$$

$$+ 2 \sum_{i=k+1}^{n} \sqrt{\overline{R}_{ii}}\, \emptyset_i (u_i)\, \underline{a}^{(ii)T} \underline{z}^{(ii)}$$

$$+ n \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{k} r_{ii}\, \emptyset_i (u_i)\, (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} u_j)$$

$$+ \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \sum_{j=1}^{k} (2\sqrt{n r_{ii}}\, \underline{a}^{(ii)} + \underline{\beta}^{(ii)})^{T} \underline{z}^{(ii)} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} u_j)$$

$$+ \sum_{\substack{i=k+1 \\ (i \neq j)}}^{n} \sum_{j=k+1}^{n} (2\sqrt{\overline{R}_{ii}}\, \underline{a}^{(ii)} + \underline{\alpha}^{(ii)})^{T} \underline{z}^{(ii)} (\underline{\alpha}^{(ij)T} \underline{z}^{(ij)} - R_{ij} u_j)$$

$$+ \sum_{i=k+1}^{n} \overline{R}_{ii}\, \emptyset_1^{2} (u_i) - \sum_{\substack{i=1 \\ (i \neq j)}}^{n} \sum_{j=1}^{n} \underline{\alpha}^{2(ij)T} \underline{z}^{(ij)} (\underline{\alpha}^{(jj)T} \underline{z}^{(jj)}$$

$$-R_{jj}[\emptyset_j (u_j) + s] \qquad (3.3.12)$$

The alternative to the above criterion is obtained by adding to V the expression

$$\left(\sum_{i=1}^{n} \emptyset_i(u_i) + \sum_{\substack{i=1 \\ (i\neq j)}}^{n} \sum_{j=1}^{n} \underline{\alpha}^{2(ji)T} \underline{z}^{(ji)}\right) X$$

$$X\left(u_i - \underline{\alpha}^{(ii)T} \underline{z}^{(ii)}\right) + \overline{R}_{ii}\left[\emptyset_i(u_i) + s\right] \equiv 0$$

which gives (compare with criterion 1.6)

<u>CRITERION 2.4</u>

The nonlinearity restrictions are

(1) $y_i f_i(y_i) \geq 0$ for all $|y_i| \neq 0$, and $f_i(0) = 0$ for all $i = 1,2,\ldots q$

(2) The functions $\emptyset_i(u_i)$ must be single-valued, and $u_i\,\emptyset_i(u_i) \geq 0$ for all $|u_i| \neq 0$, $\emptyset_i(0) = 0$ for all $i = q+1, q+2,\ldots n$.

The coefficient equations are

$$2Q^{(ii)}\underline{e}^{(ii)} = \underline{\alpha}^{(ii)} \qquad (i = 1,2,\ldots k) \tag{3.3.13}$$

$$2Q^{(ii)}\underline{e}^{(ii)} = 2\sqrt{\overline{R}_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha}^{(ii)} \qquad (i = k+1, k+2,\ldots n) \tag{3.3.14}$$

$$2Q^{(ij)}\underline{e}^{(ij)} = -\underline{\alpha}^{2(ij)} \qquad (i,j = 1,2\ldots n,\ i\neq j) \tag{3.3.15}$$

The function W which must be positive definite is

$$W = -2c\overline{V} + \sum_{i=1}^{n}\sum_{j=1}^{n} (\underline{a}^{(ij)T}\underline{z}^{(ij)})^2 + \sum_{i=k+1}^{n} \overline{R}_{ii}\,\phi^2(u_i)$$

$$+ 2\sum_{i=k+1}^{n} \sqrt{\overline{R}_{ii}}\,\phi_i(u_i)\,\underline{a}^{(ii)T}\underline{z}^{(ii)}$$

$$+ \sum_{\substack{i=1\\(i\neq j)}}^{k}\sum_{j=1}^{k} \underline{\alpha}^{(ii)T}\underline{z}^{(ii)}(\underline{\alpha}^{(ij)T}\underline{z}^{(ij)} - R_{ij}u_j)$$

$$+ \sum_{\substack{i=k+1\\(i\neq j)}}^{n}\sum_{j=k+1}^{n} (2\sqrt{\overline{R}_{ii}}\,\underline{a}^{(ii)} + \underline{\alpha}^{(ii)})^T\underline{z}^{(ii)}(\underline{\alpha}^{(ij)T}\underline{z}^{(ij)} - R_{ij}u_j)$$

$$+ \sum_{\substack{i=1\\(i\neq j)}}^{n}\sum_{j=1}^{n} \underline{\alpha}^{2(ij)T}\underline{z}^{(ij)}(\underline{\alpha}^{(jj)T}\underline{z}^{(jj)} - \overline{R}_{jj}[\phi_j(u_j) + s]) \qquad (3.3.16)$$

3.4. Stability in the sectors $[k_i, \infty]$.

If a given system is such that one or more of the uncoupled loops of the system is unstable for low values of open-loop linear gain, the pole-shifting technique must be used to establish stability.

Assume that (n-q) loops are unstable for low values of gain. Then rotation of the input axes of the corresponding (n-q) nonlinear characteristics defines the new output variables (section 1.4, Chap.I)

$$\psi_i\ (y_i) = f_i\ (y_i) - k_i\ y_i\ (i = q+1,\ q+2,\ldots n) \qquad (3.4.1)$$

Substituting for $f_i\ (y_i)$ from eqn. (3.4.1) in the system operational eqn. (3.1.1) gives (for $\underline{x}(t) = \underline{0}$)

$$\underline{y} = G_1\ (-\ \psi + (G_2 - K)\ \underline{y}) \qquad (3.4.2)$$

where for convenience one defines

$$\psi_i\ (y_i) \equiv\ f_i\ (y_i) \text{ for all } i \leq q \qquad (3.4.3)$$

and K is a diagonal matrix whose first q elements are zero, and the remaining (n-q) are the constants k_i. Rearranging eqn. (3.4.2),

$$\underline{y} = (I + G_1 K)^{-1}\ G_1\ (\underline{\psi} + G_2\ \underline{y})$$

$$= G_1'\ (\underline{\psi} + G_2\ \underline{y}) \qquad (3.4.4)$$

Since G_1 and K are diagonal, G_1' is also diagonal.

Eqn. (3.4.4) is therefore identical in form to the original eqn. (3.1.1) and the criteria of section 3.2 now apply to this transformed system (compare with section 2.5, Chap.II).

The general element of the matrix G_1' is

$$g_{ii}'\ (p) = \frac{g_{ii}(p)}{1 + k_i\ g_{ii}(p)} \qquad (3.4.5.)$$

The transformation (3.4.1) has therefore effectively shifted the poles of only the transfer functions $g_{ii}(p)$.

The nonlinearity restrictions will be either

(1) $y_i \psi_i (y_i) \geq 0$ for all $| y_i | \neq 0$, $\psi_i (0) = 0$

or (2) $\int_0^{y_i} \psi_i (y_i) dy_i \geq 0$ for all $| y_i | \neq 0$, $\psi_i (0) = 0$

depending on whether criterion 2.1 or 2.2 is used.

Application of first the pole-shifting technique then the zero-shifting technique, will allow stability or instability tests for any system corresponding stable or unstable for all linear gains in the sectors $[k_i, K_i]$.

3.5. Control Quality

Similar to class 1 systems, it may readily be shown that the stability criteria of the preceeding sections may be used to determine a lower bound of the envelope of transient response. Replacing $\lambda_r^{(ij)}$ by $\lambda + \lambda_r^{(ij)}$ in criteria 2.2 and 2.4 enables a lower bound of λ, λ_e to be determined. No integral terms may be used in the Liapunov functions chosen for the λ-transformed class 2 systems (cf. section 2.6, Chapter II) and the method is only valid for $k = 0$ in criteria 2.2 and 2.4.

Many systems which may be established as stable cannot therefore be given an estimate of transient response by this method.

3.6 Examples

Example 3.1

Consider the system where

$$g_{11}(p) = \frac{1}{p + 1} \qquad g_{12}(p) = \frac{k}{p+2}$$

$g_{21}(p) = \frac{k}{p+4}$ $\qquad$ $g_{22}(p) = \frac{1}{p+3}$

and the nonlinear characteristics are both assumed to be confined to the sectors $[0, \infty]$.

The canonic equations of section 3.1 are then

$$\dot{z}_1^{(11)} = -z_1^{(11)} - kz_1^{(12)} + f_1(y_1)$$

$$\dot{z}_1^{(22)} = -3z_1^{(22)} - kz_1^{(21)} + f_2(y_2)$$

$$\dot{z}_1^{(12)} = -2z_1^{(12)} + y_2$$

$$\dot{z}_1^{(21)} = -4z_1^{(21)} + y_1$$

$$y_1 = -z_1^{(11)}$$

$$y_2 = -z_1^{(22)}$$

From appendix V, or directly from the above equations, the canonic variables $\underline{z}^{(ij)}$ may be expressed in terms of the physical variables y_1 and y_2 as

$$z_1^{(11)} = -y_1$$

$$z_1^{(12)} = \frac{1}{k}(y_1 + \dot{y}_1 + f_1(y_1))$$

$$z_1^{(21)} = \frac{1}{k}(3y_2 + \dot{y}_2 + f_2(y_2))$$

$$z_1^{(22)} = -y_2$$

Using criterion 2.2, the coefficient eqns. (3.2.8) and (3.2.10) give

$$\frac{a_1^{(11)2}}{w_1^{(11)}} = -1 \qquad \frac{a_1^{(22)2}}{w_1^{(22)}} = -1 \tag{3.6.1}$$

$$\frac{a_1^{(12)2}}{w_1^{(12)}} = -k^2 \qquad \frac{a_1^{(21)2}}{w_1^{(21)}} = -k^2 \tag{3.6.2}$$

Substituting for these variables and the system parameters in eqn. (3.2.11) gives

$$W = z_1^{(11)2} + kz_1^{(11)} z_1^{(12)} + 2k^2 z_1^{(12)2} + 4k^2 z_1^{(21)2}$$

$$+ kz_1^{(21)} z_1^{(22)} + 3z_1^{(22)2} + k^2 z_1^{(12)} z_1^{(22)} + k^2 z_1^{(11)} z_1^{(21)} .$$

Writing W in the form $\underline{x}^T A\underline{x}$, where

$\underline{x}^T = [z_1^{(11)}, z_1^{(12)}, z_1^{(21)}, z_1^{(22)}]$.

the leading minors of A are easily found to be

$\Delta_1 = -1$

$\Delta_2 = \frac{7}{4} k^2$

$\Delta_3 = \frac{k^4}{2} (14 - k^2)$

$\Delta_4 = \frac{k^4}{16} (k - 31.6)(k^2 - 10.4)$

The inequalities $\Delta_i > 0$ $(i = 1,2,3,4)$ are satisfied provided that

$$k^2 < 10.4.$$

The above example was previously considered by the author using the method mentioned in section 1, Chapter II (see Rae and MacLellan[9], example 1). To obtain a result then required selection of two arbitrary constants, and total stability was established for $k^2 < 10.1$. Since each transfer function is first order, the parameter c does not enter into the sign of W (cf. example 2.2. Chapter II). For real coefficients, however it is necessary that $c < 1$ (see eqns. (3.6.1) and (3.6.2)). Since $c < 1$ is also the necessary and sufficient condition for real, negative definite V the lower bound λ_e of the transient response is also 1.

Example 3.2

Consider the system where

$$g_{11}(p) = \frac{K_{11}}{p-6} \qquad g_{12}(p) = \frac{K_{12}}{p+8}$$

$$g_{21}(p) = \frac{K_{21}}{p+10} \qquad g_{22}(p) = \frac{K_{22}}{p+4}$$

Since $g_{11}(p)$ has a pole in the right half of the p-plane, the pole-shifting technique must be applied to confine the nonlinear characteristic $f_1(y_1)$ to some finite sector $[k, \infty]$ such that the pole at +6 is effectively moved into the left half plane (see section 3.4).

From eqn. (3.4.5) this defines the new transfer function

$$g'_{11}(p) = \frac{K_{11}}{p+(k_1K_{11}-6)}$$

where $f_1(y_1)$ is now confined to the sector $[k_1, \infty]$.

The relevant system parameters are then

$$\lambda_1^{(11)} = 6 - k_1K_{11} \qquad \alpha_1^{(11)} = -K_{11}$$

$$\lambda_1^{(12)} = -8 \qquad \alpha_1^{(12)} = -K_{12}$$

$$\lambda_1^{(21)} = -10 \qquad \alpha_1^{(21)} = -K_{21}$$

$$\lambda_1^{(22)} = -4 \qquad \alpha_1^{(22)} = -K_{22}.$$

The necessary conditions Re $(\lambda_r^{(ij)})$ and Re $(w_r^{(ij)}) < 0$ for establishing stability are satisfied provided that

$$6 - k_1K_{11} < 0, \quad c < 4, \quad c+6 - k_{11}K_{11} < 0 \tag{3.6.3}$$

Using criterion 2.2, the coefficient equations are

$$\frac{a_1^{(11)2}}{w_1^{(11)}} = -K_{11} \qquad \frac{a_1^{(22)2}}{w_1^{(22)}} = -K_{22}$$

$$\frac{a_1^{(12)2}}{w_1^{(12)}} = -K_{12}^2 \qquad \frac{a_1^{(21)2}}{w_1^{(21)}} = -K_{21}^2$$

Provided that K_{11} and $K_{22} > 0$, the coefficients are real when inequalities (3.6.3) are satisfied. The function W of eqn. (3.2.11) is then

$$W = (k_1K_{11} - 6)\, K_{11}\, z_1^{(11)2} + K_{11}K_{12}\, z_1^{(11)}\, z_1^{(12)} + 8K_{12}^{2}\, z_1^{(12)2}$$

$$+ \; 10K_{21}^{2}\, z_1^{(21)2} + K_{21}\, K_{22}\, z_1^{(21)}\, z_1^{(22)} + 4K_{22}\, z_1^{(22)2}$$

$$+ \, K_{12}^{2}\, K_{22}\, z_1^{(12)}\, z_1^{(22)} + K_{11}\, K_{21}^{2}\, z_1^{(11)}\, z_1^{(21)}.$$

Writing W in the appropriate quadratic form, the relevant leading minors are found to be

$$\Delta_1 = K_{11}\, (k_1K_{11} - 6)$$

$$\Delta_2 = 8K_{11}K_{12}^{2}\, (k_1K_{11} - 6 - \frac{k}{32}\, K_{11})$$

$$\Delta_3 = 80K_{11}\, K_{12}^{2}\, K_{21}^{2}\, (k_1\, K_{11} - 6 - \frac{1}{32}\, K_{11} - \frac{1}{40}\, K_{11}\, K_{21}^{2})$$

$$\Delta_4 = K_{11}\, K_{22}\, K_{12}^{2}\, K_{21}^{2} \Big[(k_1\, K_{11} - 6 - \frac{1}{32}\, K_{11} - \frac{1}{40}\, K_{11}\, K_{21}^{2}$$

$$+ \; \frac{K_{11}\, K_{21}\, K_{22}\, (\frac{1}{20}\, K_{21} + \frac{1}{16}\, K_{12})}{\frac{5}{2}\, K_{12}\, K_{22} + 2K_{22} - 320}) \Big] \times (320 - 2K_{22} - \frac{5}{2}\, K_{12}\, K_{22})$$

The satisfaction of the inequalities $\Delta_i > 0$ $(i = 1,2,3,4)$ represents sufficient conditions for total stability. Since c may be taken as zero, the lower bound of λ is obtained directly from inequalities (3.6.3):

$$\lambda_e = 4 \text{ or } k_1 K_{11} - 6,$$

whichever is smaller.

Example 3.3

Consider the three variable system where

$$g_{11}(p) = \frac{6}{3p + 19}, \quad g_{12}(p) = \frac{1}{2p + 8}, \quad g_{13}(p) = \frac{1}{p + 9}$$

$$g_{21}(p) = \frac{1}{2p + 14}, \quad g_{22}(p) = \frac{8}{2p + 11}, \quad g_{23}(p) = \frac{3}{2p + 12}$$

$$g_{31}(p) = \frac{1}{4p + 40}, \quad g_{32}(p) = \frac{1}{3p + 45}, \quad g_{33}(p) = \frac{4}{p+1}$$

and the three nonlinear characteristics are confined to the sectors $[0, \infty]$. Using criterion 2.2, the function W of eqn. (3.2.11) is

$$W = \frac{76}{3} z_1^{(11)2} + \frac{11}{3} z_1^{(22)2} + 4z_1^{(33)2} + z_1^{(12)2}$$

$$+ 18z_1^{(13)2} + \frac{7}{4} z_1^{(21)2} + 27z_1^{(23)2} + \frac{5}{4} z_1^{(31)2}$$

$$+ \frac{30}{9} z_1^{(32)2} + z_1^{(11)} z_1^{(12)} + 2z_1^{(11)} z_1^{(13)} + 2z_1^{(22)} z_1^{(21)}$$

$$+ 6z_1^{(22)} z_1^{(23)} + z_1^{(33)} z_1^{(31)} + \frac{4}{3} z_1^{(33)} z_1^{(32)}$$

$$+ z_1^{(12)} z_1^{(22)} + 4z_1^{(13)} z_1^{(33)} + \frac{1}{2} z_1^{(21)} z_1^{(11)}$$

$$+ \frac{1}{8} z_1^{(31)} z_1^{(11)} + 18 z_1^{(23)} z_1^{(33)} + \frac{4}{9} z_1^{(32)} z_1^{(22)} .$$

The leading minors of W, evaluated on a digital computer, are found to be

$\Delta_1 = +2.533.10^1$ $\Delta_4 = +7.872.10^2$ $\Delta_7 = +6.657.10^4$

$\Delta_2 = +2.508.10^1$ $\Delta_5 = +2.235.10^3$ $\Delta_8 = +2.206.10^5$

$\Delta_3 = +4.505.10^2$ $\Delta_6 = +5.326.10^4$ $\Delta_9 = +8.832.10^5$

The system is therefore totally stable. The time required to evaluate these minors on an English Electric KDF9 computer was 13 seconds. As in all systems with first order transfer functions (cf. previous examples) it is sufficient to take

$$\lambda_e = | \text{ smallest real part of poles } | = 1.$$

3.7 Comment on Chapter III

Liapunov functions have again been chosen which involve quadratic groups of the state variables pertaining to each transfer function of the system, which again reduces the complexity of the resulting coefficient equations.

In some ways, the criteria are more difficult to apply than the class 1 criteria. The signs of the relevant functions W cannot be determined independently of the coefficient vectors $\underline{a}^{(ii)}$, and the coefficient vectors $\underline{a}^{(ij)}$ $(i \neq j)$ are determined from quadratic and not linear equations.

In criterion 2.2, however, W is a function of the state variables only. Whereas a class 1 system would require evaluation of the last (n-m) minors of an (m+n)-square matrix, a class 2 system would require evaluation of all the minors of an m-square matrix (m is the system order, and n the number of system variables). In some cases the latter calculation is easier to perform.

As mentioned at the beginning of this chapter, Class 2 systems are in general transformable into equivalent class 1 systems. This requires invertion of the system transfer matrices, and the resulting system is more complex in the sense that each individual transfer function is of higher order.

As an illustration, for the system of example 3.1 the equivalent class 1 system is

$$y_1 = \frac{-1}{(p+1)(p+2)(p+3)(p+4)-k^2} \left[(p+2)(p+3)(p+4)f_1(y_1) + k(p+4)f_2(y_2) \right]$$

$$y_2 = \frac{-1}{(p+1)(p+2)(p+3)(p+4)-k^2} \left[k(p+2)f_1(y_1) + (p+1)(p+2)(p+4)f_2(y_2) \right]$$

Each transfer function has 4 common poles, and if criterion 1.4 were used, two sets of 4 simultaneous quadratic equations would have to be solved to obtain the coefficient vectors $\underline{a}^{(11)}$ and $\underline{a}^{(22)}$. This calculation is unavoidable, since the sign of the function W of criterion 1.4 depends upon these coefficient vectors.

The application of the pole shifting technique to class 2 systems is very simple (see section 3.4) since it does not require inversion of the system transfer matrices (cf. section 2.5, Chapter II).

Estimation of the lower bound of transient response λ_e, is carried out in a manner exactly analagous to class 1 systems.

As in all previous cases, the criteria represent only sufficient conditions for stability or instability.

CHAPTER IV

THE NUCLEAR REACTOR

Introduction

In the first three chapters, criteria have been developed for systems which contain purely functional nonlinearities, i.e. of the form $y = f(x)$. While many physical processes can be adequately described in terms of linear and functional devices, there are obviously very many other different types of nonlinear systems.

One commonly occuring nonlinearity is multiplication of several variables together. Examples of systems with inherent multiplying media are nuclear fission reactors, and some metabolic processes (for example the relationships between vagus inhibition changes and heart beat rate are of a multiplicative nature).

Since the method of reducing systems to a special 'canonic form' has proved fairly successful for functional nonlinearities, in this chapter the method is modified to handle systems involving multiplying media, and in particular the nuclear reactor.

It is found in general, that only a finite region of stability exists in the state space for the nuclear reactor, unlike the total stability which could be established for systems containing functional nonlinearities.

The success of the method, if any, will therefore depend upon how large a region of stability Liapunov techniques will provide, compared to both the true region, and the region of physically realistic deviations from normal operation.

Two models of a controlled, unmoderated nuclear reactor are considered, firstly, a one-node, 'lumped parameter' model of the reactor core in which spatial distribution of neutron flux is neglected. Secondly, a multinode model of the reactor core is examined, in which the core is represented by a set of neutron sources, each interacting with every other source.

The ensuing stability analysis provides criteria for estimating the region of stability to initial deviations, which give, in the examples considered, quite adequate results for all practical deviations from normal system operation.

4.1. A one node model

In this section the reactor kinetics are assumed to be representable by a single point source, and spatial distribution of neutron flux is neglected. The basic equations of neutron evolution are then (see for example Schultz[11] p.18).

$$\frac{dn}{dt} = \frac{\delta k - \beta}{\ell} n + \sum_{i=1}^{m} \rho_i C_i \qquad (4.1.1)$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{\ell} n - \rho_i C_i \qquad (4.1.2)$$

where

n is the total neutron flux at any instant.

δk is the reactivity

C_i is the concentration of delayed neutrons in the i th group of delayed neutrons

ρ_i is the decay constant of the i th group of delayed neutrons

β_i is the fraction of delayed neutrons in the i th group

$\beta = \sum_{i=1}^{m} \beta_i$ is the fraction of total neutrons which are delayed neutrons.

ℓ is the average neutron lifetime.

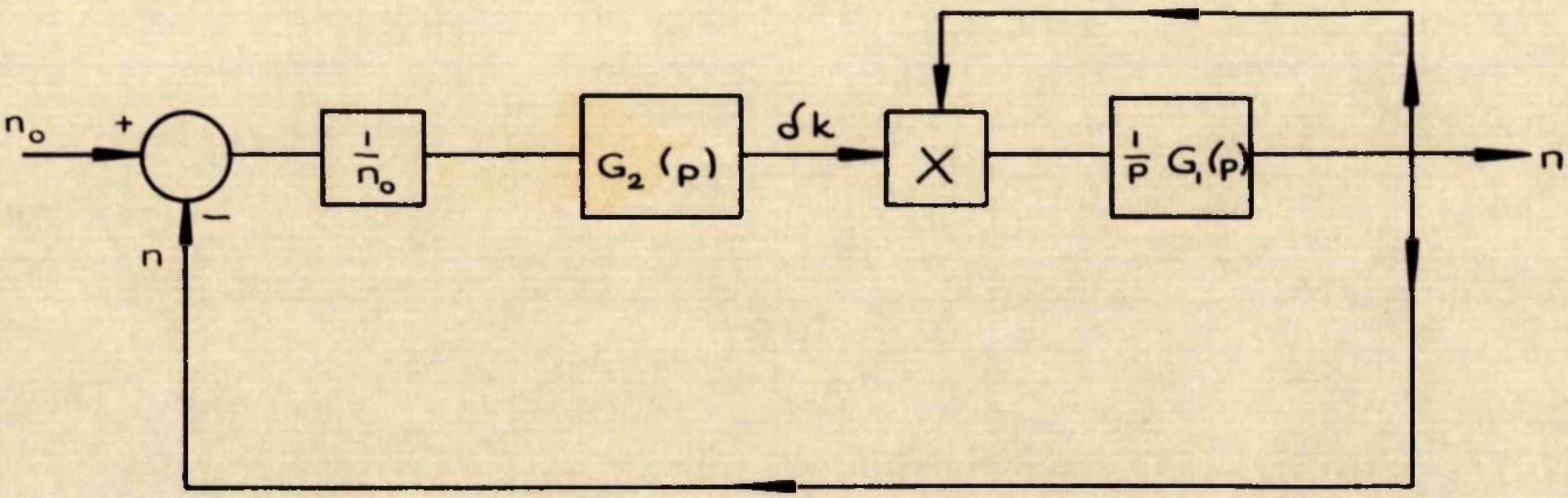

FIG.8. A CONTROL LOOP FOR AN UNMODERATED NUCLEAR REACTOR.

Substituting for C_i from eqn. (4.1.2) into eqn. (4.1.1) gives

$$n = \frac{1}{p} G_1(p). \delta kn \qquad (4.1.3)$$

where

$$G_1(p) = \frac{1}{\ell + \sum_{i=1}^{m} \frac{\beta_i}{p + \rho_i}} \qquad (4.1.4)$$

and p is the differential operator $\frac{d}{dt}$. From eqn. (4.1.3), the response of the open-loop reactor to a change in reactivity is unbounded.

In controlling the unmoderated reactor, the system shown in block diagram form in fig.8 is considered as one of the many possible control configurations (Schultz[11] p.198). The neutron flux is measured, compared with a desired neutron level, and the resulting error is normalised and used to provide a change in reactivity, which in turn alters the current neutron level. The assumptions made in this representation are that (1) the device used for measuring the neutron flux is linear, and independent of frequency. For most forms of ionisation chamber detectors, the output current is proportional to the number of neutrons per second entering the chamber, and response can be considered instantaneous.

(2) that the relationship between control rod position, or some such actuating variable, and reactivity is sensibly linear. Justification of this assumption will depend upon the nature of the actuating system used, and upon the geometry of the reactor core.

The equation of the control loop will then be (see fig.8)

$$\delta k = G_2 (p) . \frac{n_o - n}{n_o} \qquad (4.1.5)$$

By defining the normalised variable $y = \dfrac{n - n_o}{n_o}$, eqns. (4.1.3)

and (4.1.5) may be written as

$$y = \frac{1}{p} G_1(p) \quad \delta k \, (y + 1) \qquad (4.1.6)$$

$$\delta k = -G_2 (p) \, y \qquad (4.1.7)$$

Assuming that $G_2(p)$ has no poles at the origin of the p-plane, then

the equilibrium states of the system are

(1) $\delta k = 0$, $y = 0$. This is the desired operating state.

(2) $y = -1$, $\delta k = G_2(0)$. This state corresponds to zero neutron flux and a finite value of reactivity, and is obviously not a stable mode of operation.

To reduce eqns. (4.1.6) and (4.1.7) to a canonic form of state equations, re-arrange them in the form

$$y = \frac{G_1(p)}{p + G_1(p)\, G_2(p)} \quad \delta ky = R(p) \quad \delta ky \qquad (4.1.8)$$

$$\delta k = - \frac{G_1(p)\, G_2(p)}{p + G_1(p)\, G_2(p)} \quad \delta ky = - H(p) \quad \delta ky \qquad (4.1.9)$$

It may easily be shown that the common denominator of R(p) and H(p) is in fact the characteristic equation of the closed-loop system for small perturbations.

A suitable canonic form is then

$$\dot{z}_i = \lambda_i z_i + \delta ky \quad (i = 1,2,\ldots n) \qquad (4.1.10)$$

$$y = - \sum_{i=1}^{n} \alpha_i z_i \qquad (4.1.11)$$

$$\delta k = \sum_{i=1}^{n} \gamma_i z_i \qquad (4.1.12)$$

where λ_i are the poles of R(p) (and therefore H(p)) and α_i and γ_i are the residues (with sign reversed) of R(p) and H(p) respectively. The conditions for the above canonic form to be valid are given in Appendix VI. One necessary condition, as in all previous types of system considered, is that there are no multiple poles present. For convenience in determining the constants γ_i without finding the residues of H(p), it is easily shown (see appendix VI) that

$\gamma_i = G_2 (\lambda_i). \ \alpha_i.$

The reason for reducing the system to this canonic form becomes apparent when the following Liapunov function is chosen (similar to all other systems):

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j z_i z_j}{w_i + w_j} \qquad (4.1.13)$$

where $w_i = c + \lambda_i$. The conditions for V to be real and negative definite are that (cf. section 1.1a, Chapter I)

(a) the coefficients a_i should be of appropriate algebraic nature, and

(b) Re (λ_i) and Re (w_i) < 0 for all i.

If the system is small-perturbation stable, then Re $(\lambda_i) < 0$ for all i, and c can always be chosen such that condition (b) is satisfied.

Since the system has two singularities in the state space, one of which is unstable from physical considerations, only a finite region of stability is to be expected. To determine this region, the following theorem due to La Lalle and Lefshetz[12] may be applied.

If

(1) there exists a sign definite function $V(\underline{z})$ which is continuous, together with its first partial derivatives, in a region Ω of the state space $\underline{z}$ which includes the origin, and

(2) the surfaces $V(\underline{z})$ = constant form closed surfaces around the origin in Ω, and

(3) the total derivative of V w.r.t. time is sign definite (of opposite sign to V) in Ω

then all trajectories starting within Ω return to the origin, i.e. the system is asymptotically stable in Ω.

The quadratic function V of eqn. (4.1.13) can only produce stability regions of ellipsoidical shape surrounding the origin. If, for example, the true region of stability were the entire state space to one side of a plane $\sum_{i=1}^{n} k_i z_i$ = constant, the best result obtainable from a Liapunov function of the type (4.1.13) would be an ellipsoid touching this plane and centred on the origin. If however this closed surface were to include all physically realistic deviations of the state space, the result would be quite satisfactory.

It remains to investigate the Liapunov function (4.1.13) with these considerations in mind.

Differentiating V w.r.t. time gives

$$\dot{V} = -2cV + \left(\sum_{i=1}^{n} a_i z_i\right)^2 + 2\,\delta ky \sum_{i=1}^{n} a_i z_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j}$$

Since two transfer functions, R(p) and H(p), are involved, add to $\dot{V}$ the expression (see eqns. (4.1.11) and (4.1.12))

$$\delta ky \left(K\left(\sum_{i=1}^{n} \gamma_i z_i - \delta k\right) - \left(y + \sum_{i=1}^{n} \alpha_i z_i\right)\right) \equiv 0$$

where K is a real constant. The coefficients a_i may then be determined from the simultaneous quadratic eqns. (cf. section 1.1b, Chapter I)

$$2a_i \sum_{j=1}^{m} \frac{a_j}{w_i + w_j} = \alpha_i - K\gamma_i \quad (i = 1,2,\ldots n) \qquad (4.1.14)$$

and the derivative remains as

$$\dot{V} = -2cV + \left(\sum_{i=1}^{n} a_i z_i\right)^2 - \delta ky\,(K\,\delta k + y) \qquad (4.1.15)$$

where δk and y may be expressed in terms of the state variables from eqns. (4.1.11) and (4.1.12).

If eqns. (4.1.14) have appropriate roots, then for small perturbations V is negative definite and $\dot{V}$ is positive definite, thus proving stability of the equilibrium point $\underline{z} = \underline{\theta}$. Eqns. (4.1.14) should therefore ideally have appropriate roots for Re $(\lambda_i) < 0$ and no other restrictions. The actual restrictions on the system parameters are more severe (see Appendix II) and hence the introduction of the arbitrary constant K.

Consider now some general properties of the function $\dot{V}$. As previously mentioned, the system has two singularities, one at the origin and one at (see appendix VI)

$$z_i = \frac{G_2(0)}{\lambda_i} \quad (i = 1,2,\ldots n) \qquad (4.1.16)$$

It is useful to investigate the sign of $\dot{V}$ at this singularity. Substituting for z_i from eqn. (4.1.16) in eqn. (4.1.15) gives

$$\dot{V}(s) = -2cG_2^{\,2}(0) \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i\, a_j}{\lambda_i \lambda_j (w_i + w_j)}$$

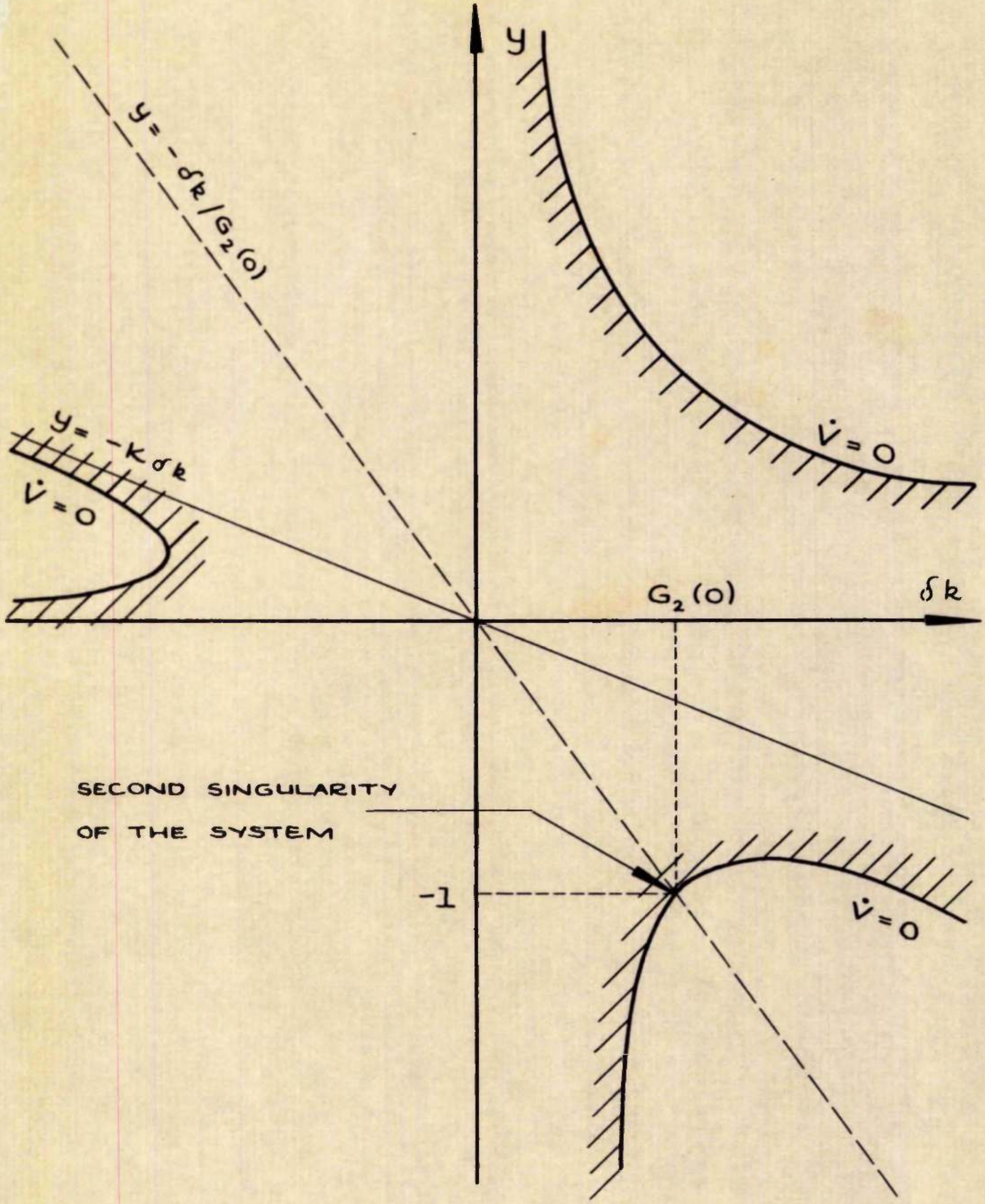

FIG. 9. GENERAL NATURE OF THE CURVE $\dot{V} = 0$ IN THE $\delta k, y$ PLANE. HATCHING IS DIRECTED TOWARDS REGIONS OF POSITIVE $\dot{V}$.

$$+ \left(\sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i}\right)\left(\sum_{i=1}^{n} \frac{\gamma_i}{\lambda_i}\right)\left(\sum_{i=1}^{n} \frac{K\gamma_i - \alpha_i}{\lambda_i}\right) G_2^3(0)$$

$$+ \left(\sum_{i=1}^{n} \frac{a_i}{\lambda_i}\right)^2 G_2^2(0).$$

Using the coefficient eqns. (4.1.14), this relationship becomes

$$\dot{V}(s) = G_2^2(0)\left[\sum_{i=1}^{n}\left(\frac{\alpha_i - K\gamma_i}{\lambda_i}\right) + \sum_{i=1}^{n}\left(\frac{K\gamma_i - \alpha_i}{\lambda_i}\right)\right] \equiv 0$$

for all K, $G_2(0)$.

The boundary of the region $\dot{V} > 0$ therefore passes, in general, through this singularity provided that the coefficients are of appropriate algebraic nature.

From eqn. (4.1.15), a necessary condition for this boundary to exist (i.e. $\dot{V} = 0$) is that

$$\delta ky\,(K\delta k + y) > 0 \qquad (4.1.17)$$

Inequality (4.1.17) then defines sectors of the δk, y plane in which $\dot{V}$ may be zero (see fig.9).

Furthermore, for very large $|\underline{z}|$, the curve $\dot{V} = 0$ has asymptotes parallel to the lines

$$\delta k = 0,\ y = 0,\ y + K\delta k = 0.$$

From the above information, the behaviour of the curve $\dot{V} = 0$ in the δk, y plane is of the nature shown in fig.9. In fig.9, the position of the second singularity of the system is indicated. The constant K must be chosen such that

$$K < 1/G_2(0) \tag{4.1.18}$$

otherwise (see fig.9) the curve $\dot{V} = 0$ cannot pass through the singularity, which can only be the case when equations (4.1.14) do not have appropriate roots.

Since y and δk represent planes in the $\underline{z}$ space, the general nature of the curve $\dot{V} = 0$ is three 'lobes' contained within the three sectors of space defined by inequality (4.1.17) (and fig.9), and convex in shape looking from the origin.

To determine the region of stability, the surface V = constant must be found which touches the innermost point of these lobes. A general digital computer programme to determine this surface is described in Appendix VI.

A specific example will now be considered to clarify the above application of the above analysis.

Example 4.1.

If delayed neutrons are neglected, then eqns. (4.1.1) and (4.1.2) become the one eqn.

$$\frac{dn}{dt} = \frac{1}{\ell} \qquad \delta kn.$$

Assume that the actuating device has a first order response. Then

$$\delta k = \frac{k}{1+Tp} \quad \left(\frac{n_o - n}{n_o}\right).$$

From eqns. (4.1.8) and (4.1.9),

$$R(p) = \frac{1/\ell}{p + \frac{k}{\ell(Tp+1)}} = \frac{Tp+1}{\ell T p^2 + \ell p + k},$$

$$H(p) = \frac{k}{\ell T p^2 \pm \ell p + k}.$$

A typical neutron lifetime is $\ell = 10^{-3}$ sec. Assume that the actuator rise time is 1/3 sec. and the gain 2/3. 10^{-3}.

Then

$$R(p) = \frac{10^3 (p+3)}{p^2 + 3p + 2} = \frac{2.10^3}{p+1} + \frac{-1.10^3}{p+2}$$

$$H(p) = \frac{2}{p^2 + 3p + 2} = \frac{2}{p+1} + \frac{-2}{p+2}$$

For convenience, define the new variable $r = 10^3\ \delta k$. This will eliminate the factor 10^3 in the numerator of R(p) (see eqns. (4.1.8) and (4.1.9)).

The canonic system parameters are then

$\lambda_1 = -1,\qquad \lambda_2 = -2,$

$\alpha_1 = -2,\qquad \alpha_2 = +1,$

$\gamma_1 = -2,\qquad \gamma_2 = +2,$

and the canonic eqns. are

$$\dot{z}_1 = -z_1 + ry, \quad y = 2z_1 - z_2$$

$$\dot{z}_2 = -2z_2 + ry, \quad r = -2z_1 + 2z_2.$$

From appendix VI, or directly from the above eqns. the relationship between the canonic variables z_1 and z_2 and the physical variables r and y are given by

$$z_1 = y + 1/2\ r$$

$$z_2 = y + r$$

The singularities of the system are then at

(1) $y = r = 0$, or $z_1 = z_2 = 0$

(2) $y = -1$, $r = 2/3$, or $z_1 = -2/3$, $z_2 = -1/3$.

From Appendix II, the necessary conditions for eqns. (4.1.14) to have appropriate real roots are

(a) $2K - c > 0$

(b) $3 - 2K - c > 0$

From inequality (4.1.18), K must be less than 3/2, and for real, negative definite V, c must be less than 1 (Re $(w_i) < 0$). Choosing $K = 1$, inequalities (a) and (b) reduce to

$c < 1$, and $c < 2$.

$K = 1$ therefore satisfies all the restrictions imposed. To determine the largest region of stability V = constant, the digital programme of Appendix VI is used for various values of c. Since c is already limited to the

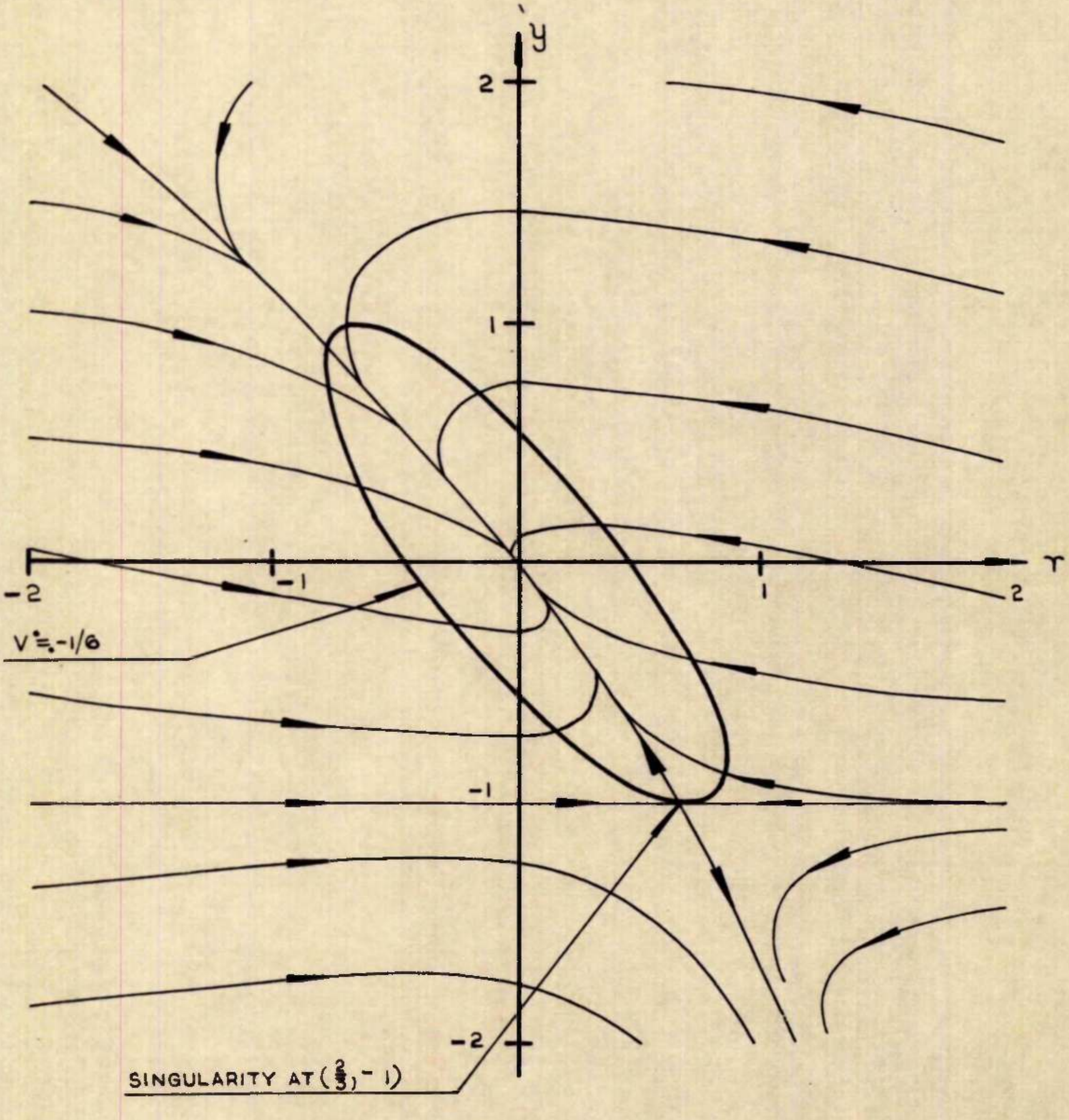

FIG. 11 CALCULATED REGION OF STABILITY $V \doteq -1/6$ SUPERIMPOSED ON THE PHASE PORTRAIT OF EXAMPLE 4·1

region $0 < c < 1$, the optimum value of c is readily found to be 0.6350. For this value, the following results were obtained:

Tangent point of V and W at

$y = -0.9973678$,

$r = 0.6664348$.

Value of V at tangent point = -0.1664389

The results were obtained to 11 significant figures and rounded off to 7. They are virtually indistinguishable from the ideal case when V touches the singularity at (-1, 2/3), and has a value of - 1/6.

Substuting for the above values in eqn. (4.1.13), the guaranteed region of asymptotic stability is given by

$$V = - 1/2\, y^2 - ry - 3/4\, r^2 = -0.1664389.$$

This region is shown superimposed upon the system's phase portrait in fig.11.

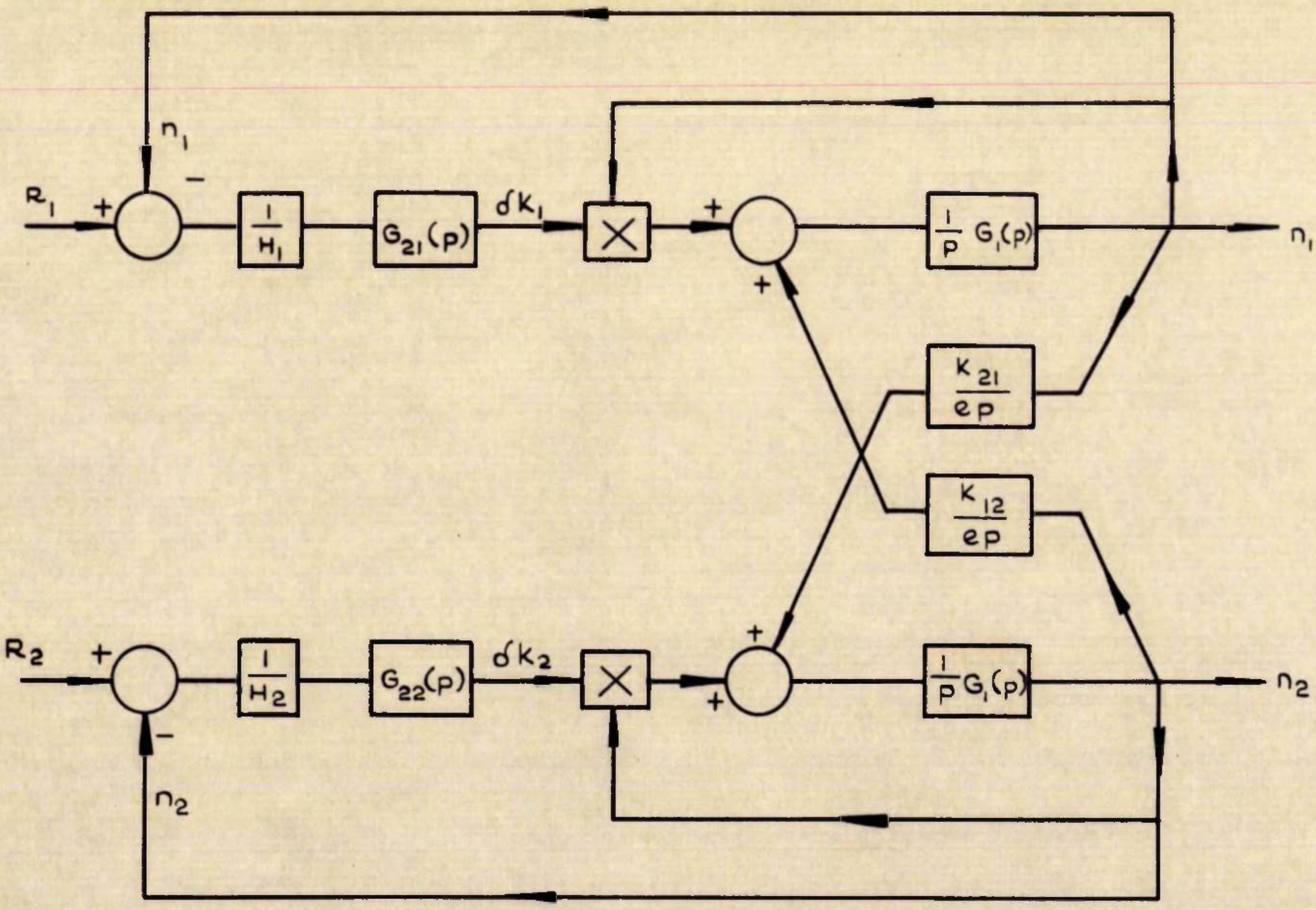

FIG. 12. A TWO - NODE REACTOR MODEL.

4.2. A multinode model

A more realistic spatial representation of an unmoderated nuclear reactor is one which assumes the reactor core to be a finite number of neutron sources, each affected by the neutron output of every other source. Such a model is constructed, for example, by finite difference representation of the partial differential equations describing spatial and temporal reactor kinetics (Harvey[10]). The equations describing an m-node model are then of the form

$$n_i = \frac{1}{p} G_1(p) \;\delta k_i \; n_i + \frac{1}{\ell p} \sum_{\substack{j=1 \\ (j \neq 1)}}^{m} k_{ij} \, n_j \qquad (4.2.1)$$

$(i = 1, 2, \ldots m)$

where $G_1(p)$ and ℓ are as defined in section 1, and k_{ij} $(i \neq j)$ are the interaction parameters between the fluxes n_i and n_j of the i th and j th nodes.

The equations of control for each node, based on the assumptions of section 1, are

$$\delta k_i = G_{2i}(p) \left(\frac{R_i - n_i}{H_i}\right) \qquad (i = 1,2,\ldots m) \qquad (4.2.2)$$

where R_i are the system inputs, and H_i are normalising coefficients as yet undecided. A two-node model is shown in block-diagram form in fig.12.

Unlike the single-node model, the above system of equations has many singularities. If in the steady state $n_i = N_i$ and $\delta k_i = \Delta K_i$, then the possible equilibrium states are given (from eqns. (4.2.1) and (4.2.2)) by the solutions of the equations

$$G_1(0)\,\Delta K_i N_i + \frac{1}{\ell} \sum_{\substack{j=1 \\ (j \neq i)}}^{m} k_{ij} N_j = 0 \qquad (4.2.3)$$

$$\Delta K_i = G_{2i}(0) \left(\frac{R_i - N_i}{H_i} \right) \qquad (4.2.4)$$

To determine these singularities in general proves impossible. Assume that for a particular system of interest these equations are solved, and the required equilibrium states $n_i = N_i^*$, $\delta k_i = \Delta K_i^*$ are selected from the solutions. The normalising coefficients are then chosen as the steady-state flux levels, i.e.

$$H_i = N_i^* . \qquad (4.2.5)$$

To transfer the origin to the point N_i^*, ΔK_i^*, define the new variables

$$y_i = \frac{n_i - N_i^*}{N_i^*} , \qquad r_i = \delta k_i - \Delta K_i^* . \qquad (4.2.6)$$

Substituting in eqns. (4.2.1) and (4.2.2) then gives

$$N_i^* y_i = \frac{1}{p} G_1(p)\,(\Delta K_i^* + r_i)(1 + y_i)\,N_i^*$$

$$+ \frac{1}{\ell p} \sum_{\substack{j=1 \\ (j \neq i)}}^{m} k_{ij} N_j^* (1 + y_j)$$

and

$$r_i + \Delta K_i^* = - G_{2i}(p)(1 + y - \frac{R_i}{N_i^*}).$$

Since N_i^* and ΔK_i^* satisfy the steady states of eqns. (4.2.3) and (4.2.4), subtract these equations from the above two equations. The dynamic behaviour about the new origin is then described by

$$y_i = \frac{1}{p} G_1(p)(r_i + \Delta K_i^* y_i + r_i y_i) + \frac{1}{\ell N_i^* p} \sum_{\substack{j=1 \\ (j \neq i)}}^{m} k_{ij} N_j^* y_j$$

$$r_i = -G_{2i}(p) y_i.$$

Rearranging these equations in a form similar to that of section 1 (see eqns. (4.1.8) and (4.1.9)),

$$y_i = g_{ii}(p) r_i y_i + \sum_{\substack{j=1 \\ (j \neq i)}}^{m} g_{ij}(p) y_j \qquad (4.2.7)$$

$$r_i = -h_{ii}(p) r_i y_i - \sum_{\substack{j=1 \\ (j \neq i)}}^{m} h_{ij}(p) y_j \qquad (4.2.8)$$

where

$$g_{ii}(p) = \frac{G_1(p)}{p + G_1(p)(G_{2i}(p) - \Delta K_i^*)} \qquad (4.2.9)$$

$$g_{ij}(p) = \frac{k_{ij} N_j^*}{\ell N_i^* (p + G_1(p)(G_{2i}(p) - \Delta K_i^*))} \qquad (4.1.10)$$

and $h_{ij}(p) = G_{2i}(p) g_{ij}(p)$ for all i,j.

A suitable canonic form of state equations representing the above system is then (cf. eqns. (4.1.10) -(4.1.12), section 1)

$$\underline{\dot{z}}^{(ii)} = \Lambda^{(ii)} \underline{z}^{(ii)} + \underline{ry}^{(ii)} + \sum_{\substack{j=1 \\ (j \neq i)}}^{m} \underline{b}^{(ij)} y_j \qquad (4.2.11)$$

$$y_i = - \underline{\alpha}^{(ii)T} \underline{z}^{(ii)} \qquad (4.2.12)$$

$$r_i = \underline{\gamma}^{(ii)T} \underline{z}^{(ii)} \qquad (i = 1,2,\ldots m) \qquad (4.2.13)$$

where the matrix notation of previous chapters is employed. $\Lambda^{(ii)}$ is therefore a matrix of the poles common to all (i,j)th transfer functions. $\underline{ry}^{(ii)}$ is a vector all of whose elements are the scalar $r_i y_i$. $\underline{\alpha}^{(ii)}$ is a vector of the residues (with sign reversed) of $g_{ii}(p)$, $\underline{\gamma}^{(ii)}$ is a vector of the residues (with sign reversed) of $h_{ii}(p)$, and the vectors $\underline{b}^{(ij)}$ $(i \neq j)$ are defined below.

The conditions for the above canonic form to be valid are given in Appendix VII. As always, one necessary condition is that there are no multiple poles in any transfer function.

As in section 1, it is easily shown that (see appendix VII)

$$\gamma_r^{(ii)} = G_{2i}(\lambda_r^{(ii)})\, \alpha_r^{(ii)} \qquad (4.2.14)$$

Furthermore, it is also proved in appendix VII that the ratio of the r th residue of $g_{ii}(p)$ to the r th residue of any $g_{ij}(p)$ $(i \neq j)$ is the same as the corresponding ratio between $h_{ii}(p)$ and $h_{ij}(p)$ $(i \neq j)$, and is given by

$$\frac{\alpha_r^{(ii)}}{\alpha_r^{(ij)}} = \frac{\gamma_r^{(ii)}}{\gamma_r^{(ij)}} = \frac{\ell N_i^*}{N_j^* k_{ij}} \quad G_1(\lambda_r) = \frac{1}{b_r^{(ij)}} \tag{4.2.15}$$

where $b_r^{(ij)}$ are the elements of the vectors $\underline{b}^{(ij)}$ in eqns. (4.2.9). It is therefore sufficient to evaluate the partial fraction expansions of only the transfer functions $g_{ii}(p)$, and the remaining vectors $\underline{\gamma}^{(ii)}$ and $\underline{b}^{(ij)}$ are determined from eqns. (4.2.14) and (4.2.15).

Take as a Liapunov function for the above system the form

$$V = \sum_{i=1}^{m} \underline{z}^{(ii)T} Q^{(ii)} \underline{z}^{(ii)} \tag{4.2.16}$$

where

$$q_{rs}^{(ii)} = \frac{a_r^{(ii)} a_s^{(ii)}}{w_r^{(ii)} + w_s^{(ii)}}, \quad w_r^{(ii)} = c + \lambda_r^{(ii)}.$$

The necessary condition (for real, negative definite V) that Re $(\lambda_r^{(ii)}) < 0$ for all r,i then requires that the closed-loop system when uncoupled ($k_{ij} = 0$ for $i \neq j$) is stable for small perturbations (see eqns. (4.2.7) and (4.2.8).)

Differentiating V w.r.t. time,

$$\dot{V} = 2\sum_{i=1}^{m} \underline{z}^{(ii)T} Q^{(ii)} (\Lambda^{(ii)} \underline{z}^{(ii)} + \underline{ry} + \sum_{\substack{j=1 \\ (j\neq i)}}^{m} \underline{b}^{(ij)} y_j)$$

$$= -2cV + (\sum_{i=1}^{m} \underline{a}^{(ii)T} \underline{z}^{(ii)})^2 + 2\sum_{i=1}^{m} r_i y_i \underline{z}^{(ii)T} Q^{(ii)} \underline{e}^{(ii)}$$

$$+ 2 \sum_{\substack{i=1 \\ (i\neq j)}}^{m} \sum_{j=1}^{m} \underline{z}^{(ii)T} Q^{(ii)} \underline{b}^{(ij)} y_j$$

where $\underline{a}^{(ii)}$ is a vector of the coefficients $a_r^{(ii)}$ associated with the matrix $Q^{(ii)}$, and $\underline{e}^{(ii)}$ is the unit vector (cf. eqn. (3.2.2) Chapter III.)

Add to $\dot{V}$ the expression (see section 1, and eqns. (4.2.12) and (4.2.13))

$$-\sum_{i=1}^{m} r_i y_i (y_i + \underline{\alpha}^{(ii)T} \underline{z}^{(ii)} + K_i r_i - K_i \underline{\gamma}^{(ii)T} \underline{z}^{(ii)}) \equiv 0.$$

The coefficient equations are then

$$2Q^{(ii)} \underline{e}^{(ii)} = \underline{\alpha}^{(ii)} - K_i \underline{\gamma}^{(ii)} \qquad (i = 1,2,\ldots m) \qquad (4.2.17)$$

and the derivative remains as

$$\dot{V} = -2cV + (\sum_{i=1}^{m} \underline{a}^{(ii)T} \underline{z}^{(ii)})^2 - \sum_{i=1}^{m} r_i y_i (K_i r_i + y_i)$$

$$- 2 \sum_{\substack{i=1 \\ (i\neq j)}}^{m} \sum_{j=1}^{m} \underline{z}^{(ii)T} Q^{(ii)} \underline{b}^{(ij)} \underline{\alpha}^{(jj)T} \underline{z}^{(jj)} \qquad (4.2.18)$$

The effects of the system interactions now appear in the last term of eqn. (4.2.18).

The region of stability can be determined, as in section 1, by finding the surface V = constant which lies within the region $\dot{V} \geq 0$. The complexity of the model does not allow any general investigation of this region, and a specific example will now be considered.

Example 4.3. A two-node model.

If delayed neutrons are again neglected, then $G_1(p) = 1/\ell$. Assume that the system is symmetric, and that

$R_1 = R_2 = R, \quad H_1 = H_2 = H, \quad k_{12} = k_{21} = k,$

$G_{21} = G_{22} = K.$

Eqns. (4.2.3) and (4.2.4) defining possible steady states are then

$$\Delta K_1 N_1 + kN_2 = 0 \qquad \Delta K_2 N_2 + kN_1 = 0$$

$$\Delta K_1 = \frac{K}{H}(R-N_1) \qquad \Delta K_2 = \frac{K}{H}(R-N_2)$$

or eliminating ΔK_1, and ΔK_2,

$$\frac{KR}{H} N_1 - \frac{K}{H} N_1^2 + kN_2 = 0 \tag{4.2.19}$$

$$\frac{KR}{H} N_2 - \frac{K}{H} N_2^2 + kN_1 = 0 \tag{4.2.20}$$

The solutions of eqns. (4.2.19) and (4.2.20) are easily found to be

(1) $N_1 = N_2 = 0$ (4.2.21)

(2) $N_1 = N_2 = R + \frac{kH}{K}$ (4.2.22)

(3) $$\left.\begin{aligned} 2KN_1 &= (KR - kH) \pm \sqrt{(KR-kH)(KR+3kH)} \\ 2KN_2 &= (KR-kH) \mp \sqrt{(KR-kH)(KR+3kH)} \end{aligned}\right) \quad (4.2.23)$$

The desired operating state is selected to be

$N_1^* = N_2^* = R + \frac{kH}{K} = H$ (see eqn. (2.4.5))

Therefore

$$H = \frac{RK}{K-k}$$

and

$$\Delta K_1^* = \Delta K_2^* = \frac{K}{H}\left(R-R-\frac{kH}{K}\right) = -k.$$

The singularities of the system are then, from eqns. (4.2.21) to (4.2.23)

(1) $n_1 = n_2 = 0$, or $y_1 = y_2 = -1$, $r_1 = r_2 = K$

(2) $n_1 = n_2 = \frac{RK}{K-k}$, or $y_1 = y_2 = r_1 = r_2 = 0$

(3) $n_{1,2} = \frac{R(K-2k)}{2(K-k)} \mp \frac{R}{2(K-k)} \sqrt{(K-2k)(K+2k)}$

or $y_{1,2} = -1 + \frac{(K-2k)}{2K} \mp \frac{1}{2K} \sqrt{(K-2k)(K+2k)}$

$r_{1,2} = \frac{(K+2k)}{2} \pm \frac{1}{2K} \sqrt{(K-2k)(K+2k)}$

Assume that $K = 10^{-3}$, and $k = 0.6 \times 10^{-3}$. Then for a typical value of $\ell = 10^{-3}$, the above singularities are

(1) $y_1 = y_2 = -1, \quad r_1 = r_2 = 10^{-3}$ (4.2.24)

(2) $y_1 = y_2 = r_1 = r_2 = 0$ (4.2.25)

The solutions of (3) above are imaginary.

The relevant system parameters are then

$$\lambda_1^{(11)} = \lambda_1^{(22)} = -\frac{1}{\ell} (K - \Delta K^*) = -10^3 (10^{-3} + 0.6 \times 10^{-3}) = -1.6$$

$$\alpha_1^{(11)} = \alpha_1^{(22)} = -1/\ell = -10^3$$

$$\gamma_1^{(11)} = \gamma_1^{(22)} = -K/\ell = -1$$

$$b_1^{(12)} = b_1^{(21)} = k = 0.6 \times 10^{-3}$$

The canonic eqns. (4.2.11) - (4.2.13) are therefore

$$\dot{z}_1^{(11)} = -1.6z_1^{(11)} + r_1 y_1 + 0.6 \times 10^{-3} y_2$$

$$\dot{z}_1^{(22)} = -1.6z_1^{(22)} + r_2 y_2 + 0.6 \times 10^{-3} y_1$$

$$y_1 = 10^3 z_1^{(11)} \qquad y_2 = 10^3 z_1^{(22)}$$

$$r_1 = -z_1^{(11)} \qquad r_2 = -z_1^{(22)}$$

or, defining the new variables, $x_1 = 10^3 z_1^{(11)}$, $x_2 = 10^3 z_1^{(22)}$,

$$\dot{x}_1 = -1.6x_1 - x_1^2 + 0.6x_2$$

$$\dot{x}_2 = -1.6x_2 - x_2^2 + 0.6x_1$$

$$y_1 = x_1 \qquad y_2 = x_2$$

$$r_1 = -10^{-3}x_1 \qquad r_2 = -10^{-3}x_2.$$

The singularities in the (x_1, x_2) phase plane are then at (0,0) and (-1,-1).

From the coefficient equations (4.2.17), putting $K_1 = K_2 = C$,

$$\frac{a_1^{(11)2}}{w_1^{(11)}} = \frac{a_1^{(22)2}}{w_1^{(22)}} = (C - 10^3) \qquad (4.2.26)$$

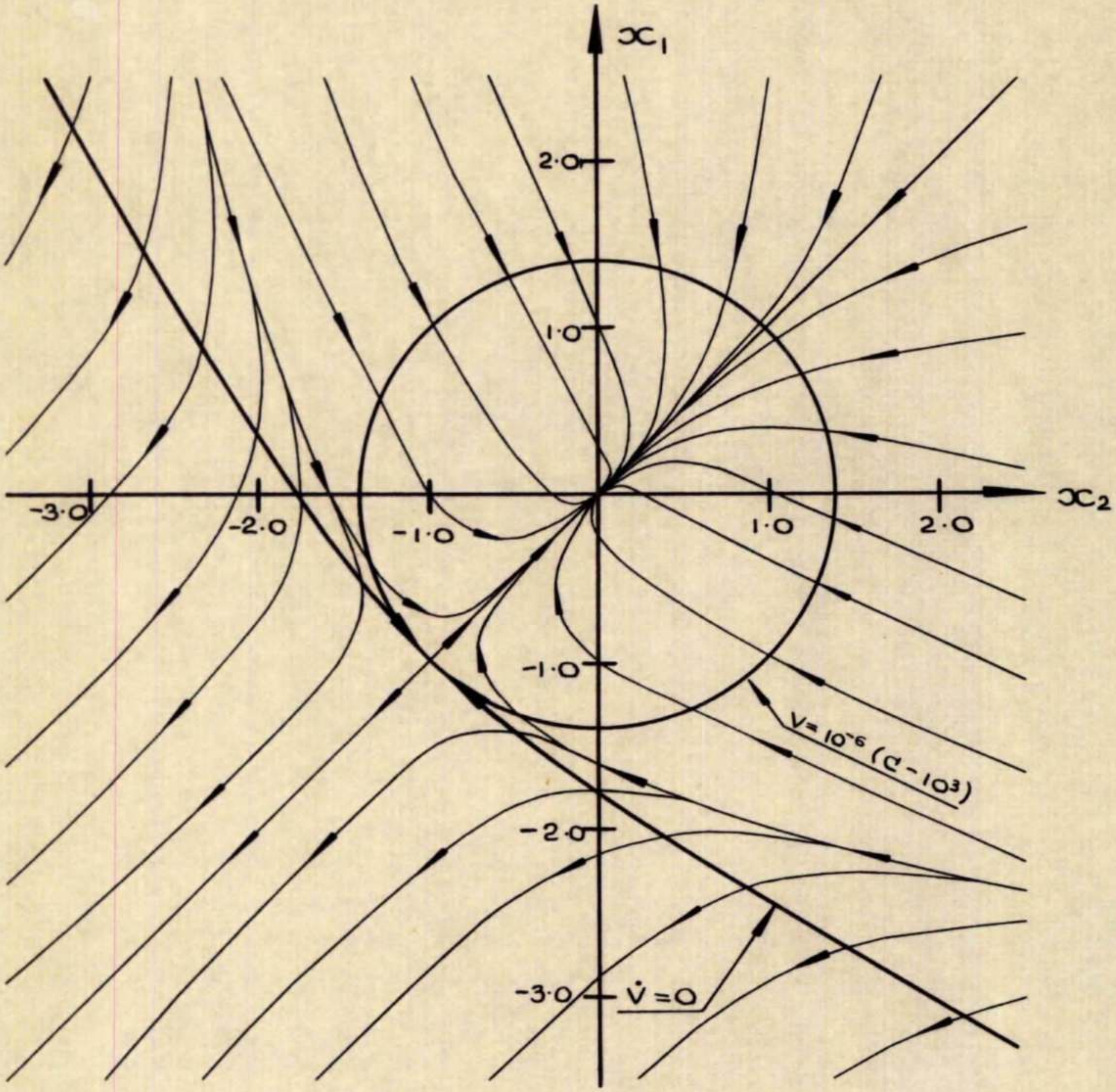

FIG. 1[illegible]. PHASE PORTRAIT OF TWO-NODE EXAMPLE AND THE CALCULATED REGION OF STABILITY. $V = 10^{-6}(C - 10^3)$

From eqns. (4.2.16) and (4.2.26)

$$V = \frac{a_1^{(11)2}}{2w_1^{(11)}} z_1^{(11)2} + \frac{a_1^{(22)2}}{2w_1^{(22)}} z_1^{(22)2}$$

$$= \frac{10^{-6}}{2} (C - 10^3)(x_1^{\ 2} + x_2^{\ 2}). \qquad (4.2.27)$$

From eqn. (4.2.18),

$$\dot{V} = \frac{\lambda_1^{(11)} \cdot a_1^{(11)2}}{w_1^{(11)}} z_1^{(11)2} + \frac{\lambda_1^{(22)} \cdot a_1^{(22)2}}{w_1^{(22)}} z_1^{(22)2}$$

$$+ \ 10^3 k \left(\frac{a_1^{(11)2}}{w_1^{(11)}} + \frac{a_1^{(11)2}}{w_1^{(22)}} \right) z_1^{(11)} z_1^{(22)}$$

$$- \ 10^{-6} (C - 10^3)(x_1^{\ 3} + x_2^{\ 3})$$

$$= 10^{-6} (10^3 - C)(1.6x_1^{\ 2} - 1.2x_1x_2 + 1.6x_2^{\ 2} + x_1^{\ 3} + x_2^{\ 3})$$

Provided that $C < 10^3$, then for $\dot{V} \geq 0$,

$$1.6x_1^{\ 2} - 1.2x_1x_2 + 1.6x_2^{\ 2} + x_1^{\ 3} + x_2^{\ 3} \geq 0. \qquad (4.2.28).$$

In figure 13 the relevant portion of the curve $\dot{V} = 0$ (from inequality (4.2.28)) is superimposed on the system's phase portrait. The curve passes through the singularity (-1,-1), and since $\dot{V}$ is positive 'to the right of and above' this curve, the calculated region of stability is given by the surface V = constant which passes through the point (-1,-1). From eqn. (4.2.27)

$$V = \frac{10^{-6}}{2} (C - 10^3)(x_1^2 + x_2^2) = 10^{-6} (c - 10^3) \text{ at } (-1,-1).$$

The stability region is therefore simply the circle

$$x_1^2 + x_2^2 = 2$$

shown in figure 13. The choice of the constants C and c do not affect this result, provided that $C < 10^3$ and $c < 1.6$. Since the established region of stability includes zero power level (-1,-1) and twice full power level (1,1) it includes all physically sensible deviations from the operating state (0,0).

Comment on Chapter IV.

Reduction of the nuclear reactor equations to a special canonic form has again proved useful for stability analysis. In the one node model, this form utilises the poles of the closed-loop process when linearised. In the multinode model, the form utilises the poles of the uncoupled closed loops when linearised. Liapunov functions are then chosen based on the prerequisite that these loops are small-perturbation stable.

Although simple quadratic Liapunov functions are used, in the two examples considered an adequate region of stability is obtained from physical considerations. For the one node model, the general nature of the Liapunov function derivative indicates equally satisfactory results for systems of arbitrary order. Although the multinode model is considerably more complicated, it is to be hoped that the similar approach taken will also provide adequate results in general.

It is possible to apply the 'λ-transformation' methods of Chapters II and III to the nuclear reactor analysis, to obtain estimates of transient response to an initial deviation. Since the stability region obtained is only a small fraction of the true region (although adequate from physical considerations), such estimates are bound to be poor. For example, for an initial deviation near to the point (1,1) in fig.13, any estimate must dictate a very great time to return to the origin, since the deviation is near to the theoretical stability limit. In practice, the point (1,1) is virtually an infinite distance from the stability boundary in the upper half of the phase plane. Criteria for estimating transient response are therefore not included here.

Use of a digital computer is unavoidable for any but the simplest of systems. The obtained region of stability, however, is always of a quadratic form. Linear transformation from canonic variables to a system's physical variables (see appendices VI and VII) then provides a simple algebraic surface within which stability is guaranteed.

The criteria represent only sufficient conditions, and furthermore since the systems considered are not totally stable, no information is given about stability under time-varying forced inputs to the systems (cf. Class 1 and Class 2 systems).

CHAPTER V

SELF-OSCILLATORY SYSTEMS

Introduction

In chapters I, II, and III stability criteria were derived for systems containing functional nonlinearities. These criteria established either total stability or instability of such systems. The criteria must fail, however, for systems which are of a self-oscillatory nature, since such systems are globally stable in the sense of Liapunov. Their response is finitely bounded but is not asymptotic to the origin.

To derive Liapunov functions for such systems is extremely difficult. For a system which possesses a stable limit cycle, for example, the derivative of any negative definite function chosen would have to be negative definite within the region of state space enclosed by the limit cycle (thus proving that all trajectories within the cycle move away from the origin) and positive definite outwith the limit cycle (thus proving that all trajectories outside the cycle move inwards towards it).

Assuming that such functions could be constructed, the task of finding the appropriate regions of inward and outward turning trajectories would be virtually impossible in the multidimensional case.

The problem is an important one, however, since unless systems physically destroy themselves, unstable modes are often of an oscillatory nature. If a given system should by accident assume an oscillatory mode, it is desirable to be able to predict the amplitude and frequency of such a mode, and therefore the degree to which the system will be damaged.

The common technique of estimating such modes from system equations is very elementary, and in essence consists of determining whether or not a sine wave will satisfy the equations. Since most real systems have linear transfer functions which have attenuative properties, it is possible to neglect any harmonic distortion of the sine wave as it passes through any nonlinear device present.

For single loop systems containing one nonlinearity, the graphical describing function technique (see e.g. Gibson[4]) is commonly used, and a vast amount of literature has been published on this method.

For multivariable systems containing many nonlinearities, this graphical method is not easily applicable, since it would involve plotting frequency characteristics for many different values of describing function amplitudes. An analytic approach must be taken (see e.g. Popov[14]). In this chapter a particular system of class 2 configurations (see Chap.III) is examined for oscillatory modes using such a technique.

5.1. Oscillatory modes of a two-variable system.

Consider the following class 2 system:

$$y_1 = g_{11}(p)\ (X_1 - 0.01y_1{}^2) + Ky_2 \qquad (5.1.1)$$

$$y_2 = Ky_1 + g_{22}(p)\ (X_2 - 0.01y_2{}^2) \qquad . \qquad (5.1.2)$$

where

$g_{11}(p) = g_{22}(p) = \dfrac{1}{p^3 + p^2 + p}$, and X_1 and X_2 are constant reference signals. Assume that the system has an equilibrium state (Y_1,Y_2).

Then defining the new variables

$$r_1 = y_1 - Y_1 , \; r_2 = y_2 - Y_2 ,$$

eqns. (5.1.1) and (5.1.2) may be written as

$$X_1 - 0.01Y_1^2 + KY_2 = 0$$

$$X_2 - 0.01Y_2^2 + KY_1 = 0$$

$$r_1 = g_1(p)[-0.01r_1^2 + Kr_2]$$

$$r_2 = g_2(p)\,[-0.01r_2^2 + Kr_1]$$

where

$$g_1(p) = \frac{1}{p^3 + p^2 + p + 0.02Y_1}$$

$$g_2(p) = \frac{1}{p^3 + p^2 + p + 0.02Y_2} \; .$$

For Y_1 and $Y_2 < 50$, $g_1(p)$ and $g_2(p)$ have roots with negative real parts,and the criteria of Chapter III could possibly be applied to establish system stability under these equilibrium conditions.

For Y_1 and $Y_2 > 50$, $g_1(p)$ and $g_2(p)$ each have one positive real root and a complex conjugate pair of roots with negative real parts. Since the functions r_1^2 and r_2^2 cannot have their input axes rotated without intersecting the characteristics, the pole-shifting technique cannot be applied to this system. Consequently the criteria of Chapter III cannot be applied for Y_1 and $Y_2 > 50$, since the condition Re $(\lambda_i) < 0$ for all i can never be satisfied.

Assume that solutions exist of the form

$$y_1 = Y_1 + a_1 \sin wt = Y_1 + y_{1P}$$

$$y_2 = Y_2 + a_2 \sin (wt + \emptyset) = Y_2 + y_{2P}$$

If all harmonics higher than the fundamental are assumed to be heavily attenuated, then the outputs of the nonlinear devices are

$$y_1^2 \doteq Y_1^2 + 2Y_1 a_1 \sin wt + \frac{a_1^2}{2}$$

$$= Y_1^2 + \frac{a_1^2}{2} + 2Y_1 y_{1P}$$

$$y_2^2 \doteq Y_2^2 + 2Y_2 a_2 \sin (wt + \emptyset) + \frac{a_2^2}{2}$$

$$= Y_2^2 + \frac{a_2^2}{2} + 2Y_2 y_{2P}.$$

Substituting for y_1, y_1^2 and y_2^2 in eqns. (5.1.1) and (5.1.2) gives

$$(p^3 + p^2 + p + 0.02Y_1)\, y_{1P} = X_1 - 0.01Y_1^2 - 0.01\,\frac{a_2^2}{2} + KY_2 + Ky_{2P} \quad (5.1.3)$$

$$(p^3 + p^2 + p + 0.02Y_2)\, y_{2P} = X_2 - 0.01\, Y_2^2 - 0.01\,\frac{a_2^2}{2} + KY_1 + Ky_{1P} \quad (5.1.4)$$

Equating the constant terms in eqns. (5.1.3) and (5.1.4) gives

$$0 = X_1 - 0.01\, Y_1^2 - 0.01\,\frac{a_1^2}{2} + KY_2 \quad (5.1.5)$$

$$0 = X_2 - 0.01\, Y_2^2 - 0.01\,\frac{a_1^2}{2} + KY_1 \quad (5.1.6)$$

and equating sinusoidal terms gives

$$(p^3 + p^2 + p + 0.02\, Y_1)\, y_{1P} = Ky_{2P}, \quad (5.1.7)$$

$$(p^3 + p^2 + p + 0.02Y_2)\, y_{2P} = Ky_{1P} \quad (5.1.8)$$

Eliminating y_{2P} from eqns. (5.1.7) and (5.1.8),

$$(p^6 + 2p^5 + 3p^4 + (2 + \alpha)\, p^3 + (1 + \alpha)\, p^2$$

$$+ \alpha p + \beta - K^2)\, y_{1P} = 0 \quad (5.1.9)$$

where

$$\alpha = 0.02(Y_1 + Y_2), \qquad \beta = 4.10^{-4}\, Y_1 Y_2.$$

The condition that eqn. (5.1.9) be satisfied for all t may effectively be found by replacing p by jw (w real), since y_{1P} is a sinusoidal function (see e.g. Popov[14], chaps. XVI and XVII). Equating real and imaginary parts in eqn. (5.1.9) then gives

Re: $w^2 = 1$ or $w^2 = 1/2\ \alpha$

Im: $\alpha = \beta + 1 - K^2$ for $w^2 = 1$

or $-1/8\ \alpha^3 + 1/4\ \alpha^2 - 1/2\ \alpha + \beta - K^2 = 0$ for $w^2 = 1/2\ \alpha$.

The latter equation has real roots for negative α only, and since $w^2 = 1/2\ \alpha$ in this case no real solutions exist for $w^2 = 1/2\ \alpha$. The necessary and sufficient conditions for oscillations to be present in the system are therefore, from the above equations,

$w^2 = 1$, $\alpha = \beta + 1 - K^2$.

The theory therefore predicts that all oscillatory modes should have a frequency of 1 radian per second, and that the relationships between the constant parts of the outputs, Y_1 and Y_2, should be defined by the equation

$$(0.02Y_1 - 1)(0.02Y_2 - 1) = K^2 \qquad (5.1.10)$$

K = 0.1

X_1	X_2	Y_1	Y_2	a_1	a_2
0	25	23.9	51.4	4.4	24.2
0	30	23.4	51.4	7.8	40.7
0	35	23.4	52.4	10.2	52.5
0	40	22.9	53.3	12.2	61.0
5	25	32.2	50.8	8.3	28.6
5	30	31.6	50.8	14.2	43.1
5	35	31.2	51.8	15.6	53.9
5	40	30.5	53.3	17.3	62.0
10	20	39.0	48.5	4.9	9.7
10	25	38.5	50.4	15.1	34.0
10	30	37.0	51.4	19.5	45.5
10	40	36.1	53.3	24.4	63.0
15	20	43.4	49.5	29.3	49.5
15	25	41.9	50.0	24.4	36.4
15	30	40.0	49.5	29.3	49.5
15	35	39.5	50.5	31.7	57.3
15	40	39.0	52.3	33.1	64.0
20	10	48.5	39.0	9.7	4.9
20	15	49.5	43.4	22.3	18.0
20	20	49.9	46.8	32.2	28.9
20	25	43.4	48.5	35.6	40.8
20	30	42.5	49.0	38.5	50.0
20	35	41.0	50.0	40.8	58.6
20	40	41.5	52.3	42.5	66.0
25	0	51.4	23.9	24.2	4.4
25	5	50.8	32.2	28.6	8.3
25	10	50.4	38.5	34.0	15.1
25	15	50.0	41.9	36.4	24.4

Table 1. Results from simulated system.

K = 0.1 (continued)

X_1	X_2	Y_1	Y_2	a_1	a_2
25	20	48.5	43.4	40.8	35.6
25	25	45.4	47.5	44.3	42.7
25	30	43.9	49.0	47.8	52.0
25	35	43.4	49.0	49.2	59.6
25	40	42.9	51.9	50.8	66.5
30	0	51.4	23.4	40.7	7.8
30	5	50.8	31.6	43.1	14.2
30	10	51.4	37.0	45.5	19.5
30	15	49.5	40.0	49.5	29.3
30	20	49.0	42.9	50.0	38.5
30	25	49.0	43.9	52.0	47.8
30	30	45.8	47.0	55.0	53.8
30	35	45.3	48.5	57.0	61.1
30	40	44.8	51.4	58.5	67.0
35	0	52.4	23.4	52.5	10.2
35	5	51.8	31.2	53.9	15.6
35	15	50.5	39.5	57.3	31.7
35	20	50.0	41.0	58.6	40.8
35	25	49.0	43.4	59.6	49.2
35	30	48.5	45.3	61.1	57.0
35	35	46.3	48.5	64.8	62.2
40	0	53.3	22.9	61.0	12.2
40	5	53.3	30.5	62.0	17.3
40	10	53.3	36.1	63.0	24.4
40	15	52.3	39.0	64.0	33.1
40	20	52.3	41.5	66.0	42.5
40	25	51.9	42.9	66.5	50.8
40	30	51.4	44.8	67.0	58.5
40	40	49.2	50.9	70.7	68.3

Table 1 (continued). Results from simulated system.

K = 0.2

X_1	X_2	Y_1	Y_2	a_1	a_2
0	15	30.7	46.1	4.4	8.2
0	20	29.2	47.5	15.6	33.9
0	25	26.8	48.0	23.9	54.8
0	30	25.4	49.0	25.8	62.5
5	10	37.8	41.2	1.2	2.4
5	15	35.6	44.6	19.0	27.2
5	20	33.6	45.1	24.8	39.2
5	25	32.4	45.6	29.0	49.6
5	30	31.7	46.5	31.6	58.3
10	5	41.2	37.8	2.4	1.2
10	15	38.1	42.7	29.7	34.0
10	20	38.1	44.1	33.6	44.0
10	25	35.2	44.6	37.5	52.4
10	30	34.0	45.8	39.8	60.3
15	0	46.1	30.7	8.2	4.4
15	5	44.6	35.6	27.2	19.0
15	10	42.7	38.1	34.0	29.7
15	15	40.5	41.7	40.0	38.3
15	20	38.2	43.1	43.1	47.0
15	30	37.5	46.0	47.3	62.7
20	0	47.5	29.2	33.9	15.6
20	5	45.1	33.6	39.2	24.8
20	10	44.1	38.1	44.0	33.6
20	15	43.1	38.2	47.0	43.1
20	20	40.9	42.0	50.7	49.7
20	30	38.5	44.6	55.0	64.0
25	0	48.0	26.8	54.8	23.9
25	5	45.6	32.4	49.6	29.0
25	10	44.6	35.2	52.4	37.5
25	25	40.1	42.7	60.2	59.2
25	30	38.5	43.1	60.8	65.5

Table 1(continued). Results from simulated system.

K = 0.2 (continued)

X_1	X_2	Y_1	Y_2	a_1	a_2
30	0	49.0	25.4	62.5	25.8
30	5	46.5	31.7	58.3	31.6
30	10	45.8	34.0	60.3	39.8
30	15	46.0	37.5	62.7	47.3
30	20	44.6	38.5	64.0	55.0
30	25	43.1	38.5	65.5	60.8

K = 0.3

X_1	X_2	Y_1	Y_2	a_1	a_2
0	5	32.7	36.3	14.6	17.0
0	10	30.8	39.0	23.9	31.3
0	15	28.5	40.0	29.3	41.5
0	20	25.6	41.7	32.4	49.5
5	0	36.3	32.7	17.0	14.6
5	5	33.1	33.0	27.3	25.2
5	10	33.4	37.8	33.4	37.8
5	15	31.9	38.8	37.3	46.6
5	20	30.2	40.5	40.0	55.7
10	0	39.0	30.8	31.3	23.9
10	5	37.8	33.4	37.8	33.4
10	10	33.1	33.0	41.0	40.7
10	15	33.6	36.8	44.3	48.5
15	0	40.0	30.8	31.3	23.9
15	5	38.8	31.9	46.6	37.3
15	10	36.8	33.6	48.5	44.5

Table 1(continued). Results from simulated system.

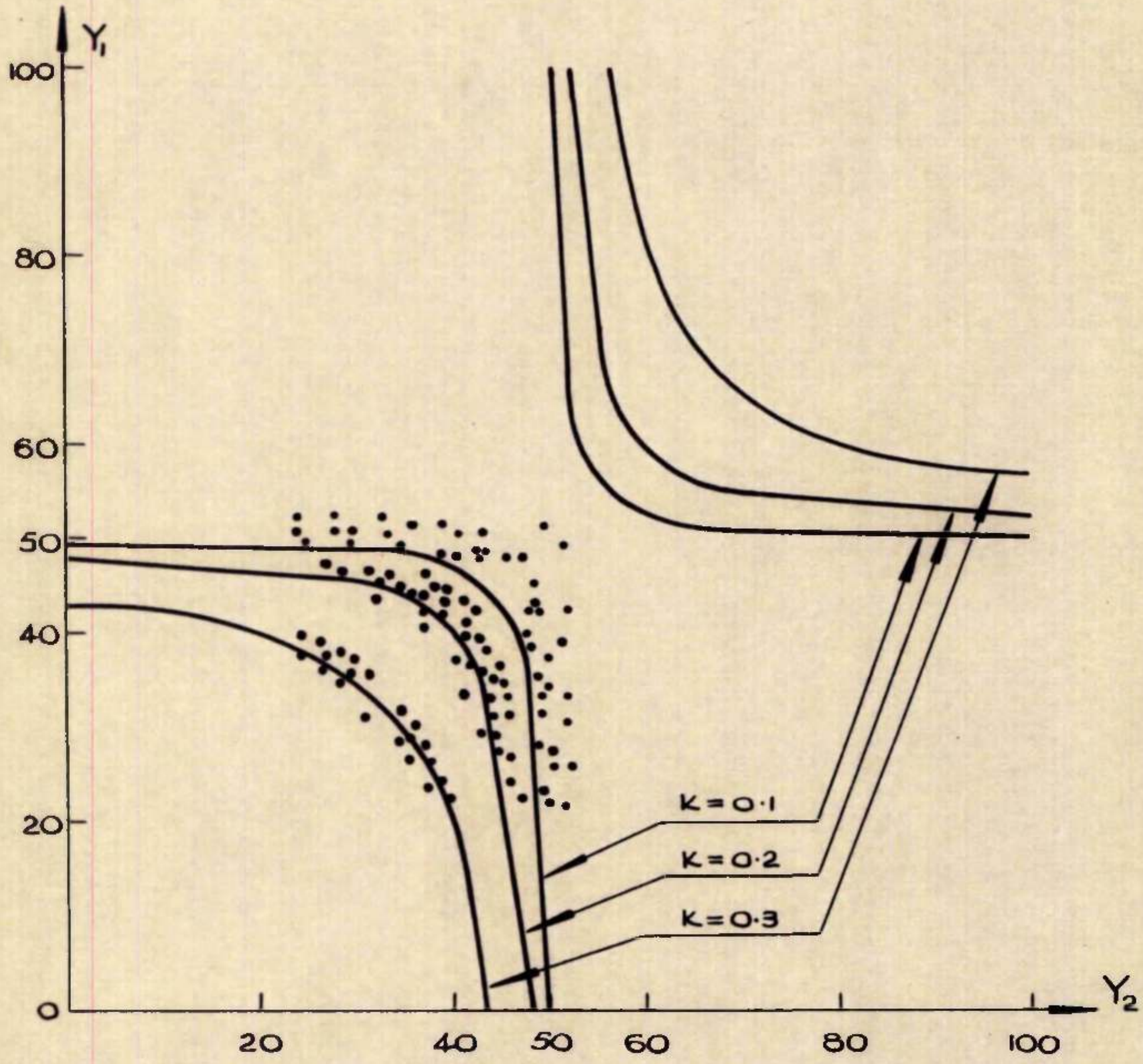

FIG. 15. THEORETICAL AND EXPERIMENTAL VALUES OF Y_1 AND Y_2 WHEN THE SYSTEM IS IN OSCILLATION.

—— THEORETICAL VALUES.

• • EXPERIMENTAL VALUES FOR VARIOUS INPUTS

From eqn. (5.1.10), curves may be drawn in the (Y_1, Y_2) plane which represent possible values of these parameters when oscillations occur. These are shown in fig. 15, together with experimental values obtained by simulation of the system on an analogue computer. The experimental values were obtained by setting y_1, y_2 and their derivatives equal to zero at $t = 0$, and selecting different constant inputs X_1 and X_2. The full set of results obtained is given in table 1.

From eqns. (5.1.7) and (5.1.8)

$$\frac{y_{1P}}{y_{2P}}(j1) = \frac{a_1}{a_2} = \frac{K}{0.02Y_1-1} = \frac{0.02Y_2-1}{K} \qquad (5.1.11)$$

and

$$\frac{y_{1P}}{y_{2P}}(j1) = \emptyset = \text{sign}\left[\frac{K}{0.02Y_1-1}\right] = \text{sign}\left[\frac{0.02Y_2-1}{K}\right] \qquad (5.1.12)$$

From eqn. (5.1.12), it is easily shown that Ø is 180° on the lower branches of the curves of fig. 15 and 0° on the upper branches.

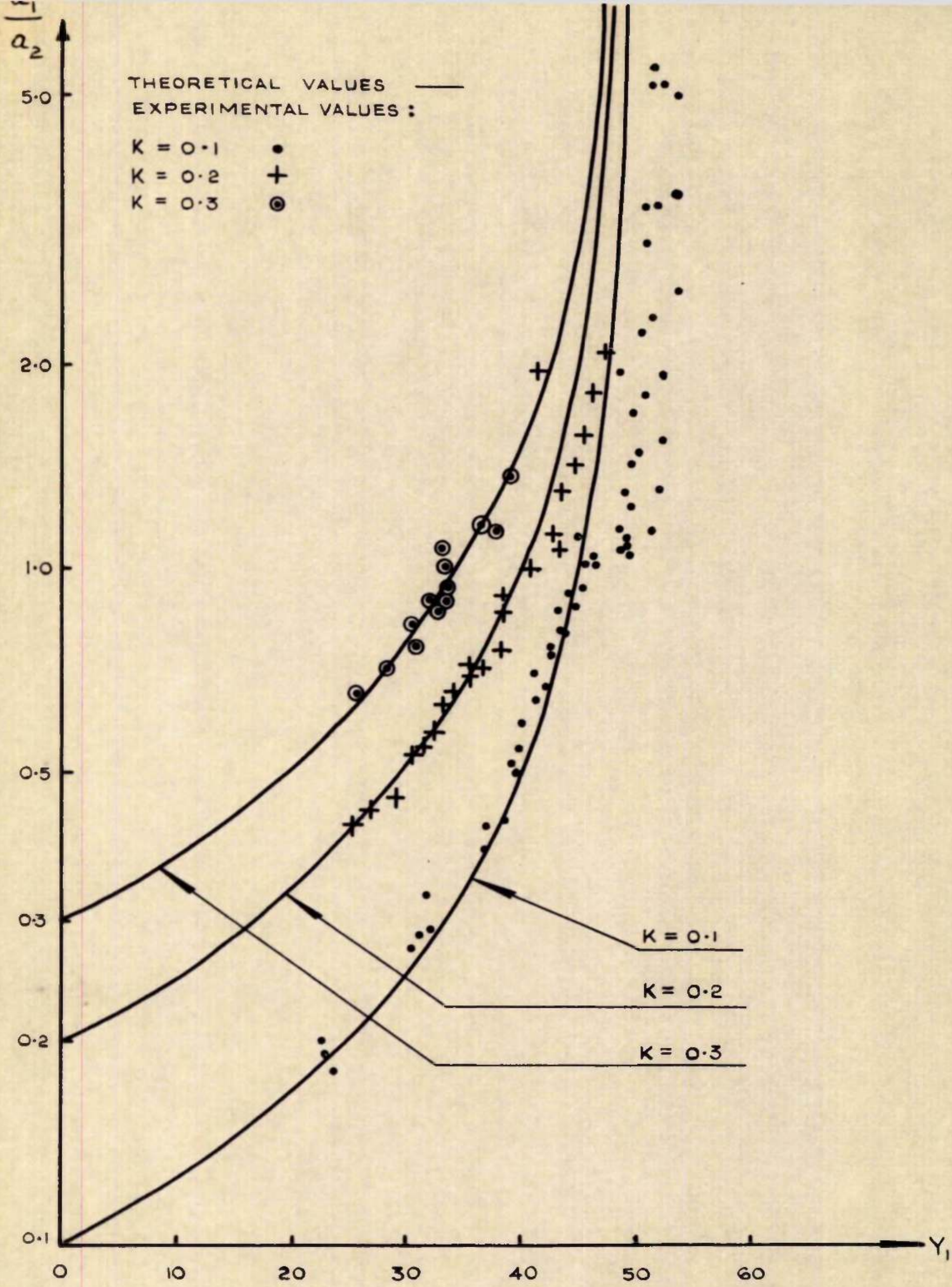

FIG. 18. RELATIONSHIPS BETWEEN RATIO OF AMPLITUDES a_1/a_2 AND OUTPUT VALUE Y_1

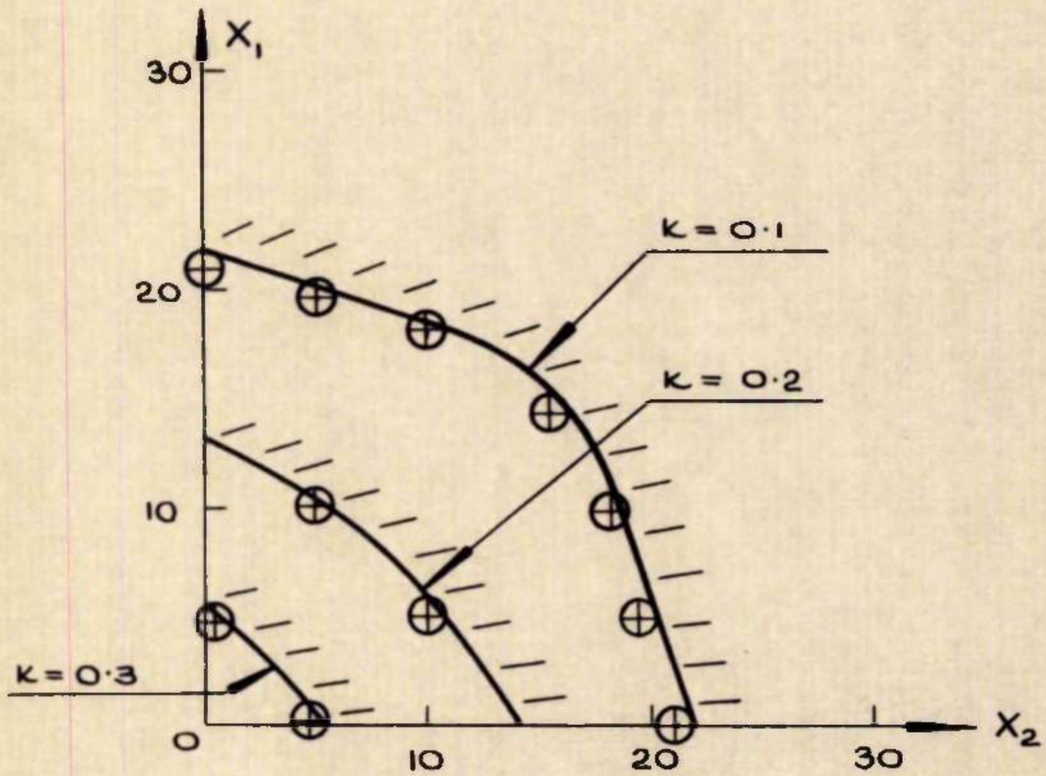

FIG. 17. THEORETICAL AND EXPERIMENTAL BOUNDARIES BETWEEN OSCILLATORY AND NON-OSCILLATORY RESPONSE. HATCHING IS DIRECTED TOWARDS REGIONS OF PREDICTED OSCILLATION.

—— THEORETICAL VALUES

⊕ EXPERIMENTAL VALUES

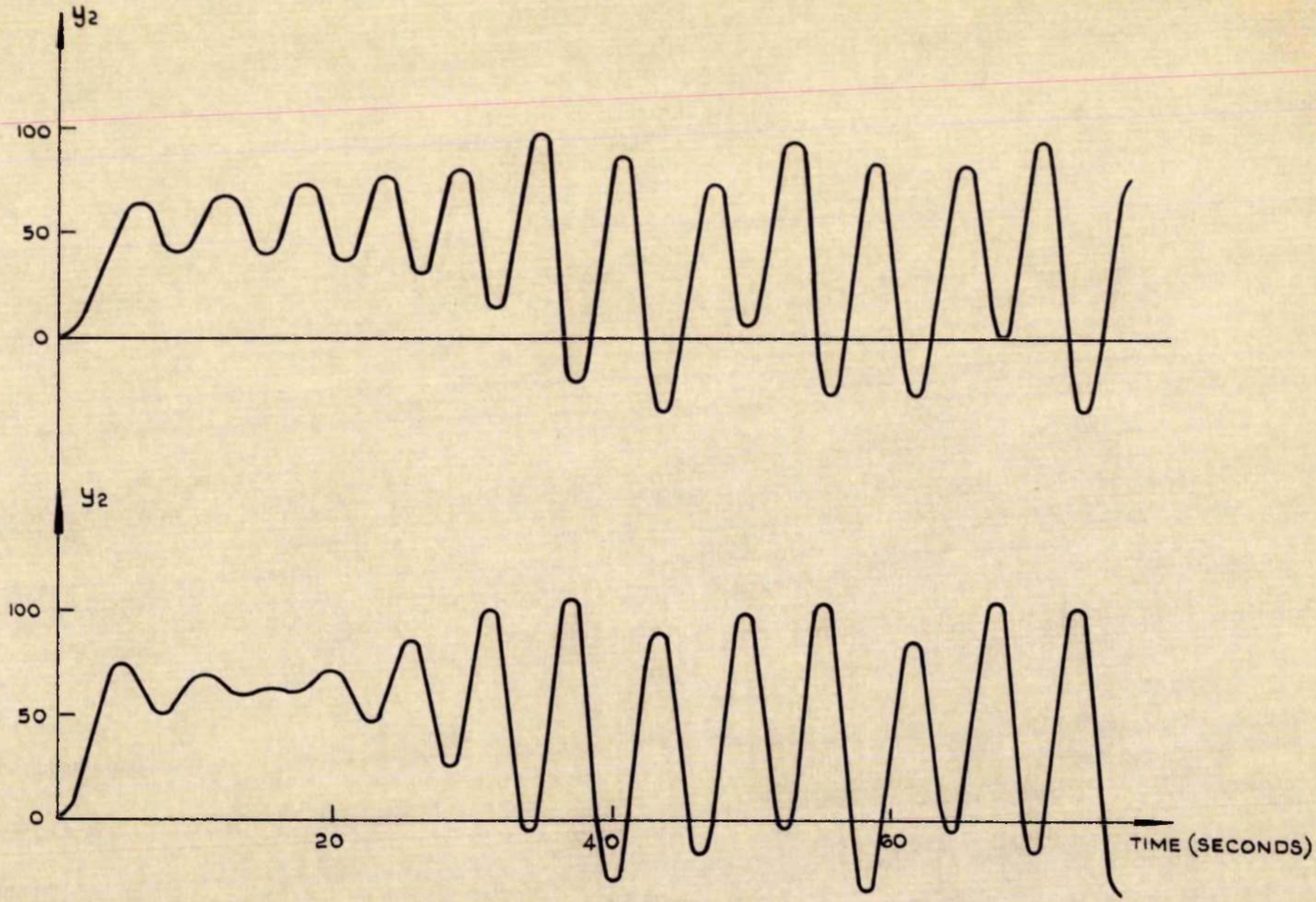

FIG. 16 EXPERIMENTAL RESPONSE FOR $X_1 = 15$, $X_2 = 20$, $K = 0{\cdot}3$

In all experimental results, the phase angle was never more than 10° from the value of 180°. The frequency of oscillation was indistinguishable from 1 radian/second, but in some cases, particularly when the amplitudes of y_1 and y_2 were of similar magnitude, subharmonic effects became noticeable, as shown in figure 16.

It is theoretically possible, from eqns. (5.1.5), (5.1.6), (5.1.10) and (5.1.11), to determine for given inputs X_1 and X_2 the output variables Y_1, Y_2, a_1 and a_2. In fact a fifth order polynomial is involved, and no analytic solutions can be found. At best, one may determine the limiting values of X_1 and X_2 necessary to maintain oscillations.

For real a_1 and a_2, from eqns. (5.1.5) and (5.1.6),

$$\begin{aligned} X_1 &> 0.01Y_1^2 - KY_2 \\ X_2 &> 0.01Y_2^2 - KY_1 \end{aligned} \qquad (5.1.13)$$

Combining eqns. (5.1.13) with eqn. (5.1.10) the boundary of oscillatory response in the (X_1, X_2) plane may be constructed for various values of K as shown in fig. 17, together with the experimental results obtained from the simulated system.

Finally, in fig. 18 are shown the theoretical and experimental dependence of the ratio of system amplitudes, a_1/a_2, to the output level Y_1. Since Y_1 and Y_2 are uniquely related by eqn. (5.1.10), it is irrelevant whether Y_1 or Y_2 is used as a coordinate.

5.2. Comment on Chapter V

Since the two transfer functions of the system are highly attenuative even at 1 rad/sec., it is to be expected that the method will be reasonably successful for the example chosen, as in fact is seen from figs. 15, 16 17 and 18. The worst discrepancies between theory and reality apparently occur when the loops are weakly coupled (K small) and also when the two outputs y_1 and y_2 have similar amplitudes. These would seem to be significant features in the approximations of the theory.

Certain values of output levels have been predicted (the upper branches of the curves in fig.15) as existing in oscillatory modes. These modes could not be found in the simulated system and are assumed to be physically nonexistent. Unfortunately, the method does not guarantee stability or otherwise of any predicted oscillatory modes. It is possible that the upper branches of the curves of fig. 15 represent an unstable limit cycle; no evidence of this could be found from the simulated system.

There are several existing techniques (West,[15], Cosgriff[16], Bonnen[17], Brensted[18]) for investigating stability of oscillatory systems by means of perturbation techniques. An attempt was made to apply Cosgriff's method (ref. 16, Chap.9) but no meaningful results were obtained for the system of section 1.

In a recent paper by Gurel and Lapidus[19] Liapunov techniques are applied to second-order oscillatory systems. Although use of a digital computer is required even for such low order systems, the method may be capable of generalisation.

Since the Lurie technique has proved fairly successful for the systems considered in previous chapters, it is to be hoped that some further modifications will enable oscillatory systems to be dealt with. No such modifications have yet been found by the author.

CHAPTER VI

CONCLUSIONS AND COMMENT

6.1. Class 1 and class 2 systems.

The principal aid to using Liapunov's method for nonlinear systems is apparently the choice of state variables used to describe the system. If the system state variables are not chosen to be the special canonic forms, the choice of general Liapunov functions can be extremely difficult.

By use of the parameter c, the criteria obtained are applicable, in general, to all systems whose linear equivalents are stable, or unstable, as the case may be (see comments on Chaps. II and III). Class 1 criteria are very little more difficult to apply than the single loop criteria of Chapter I. Class 2 criteria are more difficult to apply, but the calculations involved are in general less laborious than those incurred by transferring to the equivalent class 1 systems and using the relevant common row pole criteria.

It is worthwhile to compare some of the results obtained here with frequency domain criteria based on the work of Popov[20]. A recent paper by Jury and Lee[21] gives a frequency domain criterion for systems which include class 1 (and hence by transformation, class 2). For an n-variable system, the criterion reduces to satisfaction of the inequality (see eqn. 5.13, ref. 21).

$$H.[K^{-1} + (I + jwQ)\, G\, (jw] > 0 \qquad (6.1.1)$$

for all $-\infty < w < +\infty$, where

H is the hermitian operator

G(p) is the system transfer matrix, which must be stable (as in the criteria developed here)

Q is an $n \times n$ diagonal matrix of arbitrary constants q_{ii}

K is a matrix of the constants K_i which confine the nonlinear characteristics to the sectors $[0, K_i]$.

In general, inequality (6.1.1) must be at least as difficult to apply as any of the criteria proposed here, since it must be examined for all values of w, and various values of the n constants q_{ii}, until the inequality is satisfied. If some system parameters are unspecified, optimum values of the q_{ii} must be found to give the least restraint on these parameters for stability.

The frequency domain criterion has the distinct advantage, however, of being applicable to systems involving time delays, and systems whose frequency responses are only experimentally known.

As a comparison of the two approaches, consider first example 2.1, Chapter II. Inequality (6.1.1) then reduces to

$$4K^2(1 + q_{11} w^2)(2 + q_{22} w^2) - k^2 [(1 + q_{11}^2 w^2)(4 + w^2)$$

$$+ 2(2 - w^2)(1 - q_{11} q_{22} w^2) + 3 (q_{11} + q_{22}) w^2 + (1 + w^2)(1 + q_{22}^2 w^2)] > 0$$

for all w.

By choosing $q_{11} = 1$, $q_{22} = 1/2$, this reduces to

$$(1 + w^2)(4 + w^2)[2K^2 - \frac{9}{4} k^2] \quad > \quad 0.$$

It is therefore sufficient that

$$\frac{k^2}{K^2} \quad < \frac{8}{9} \doteq. \ 0.889$$

as compared to the result $\frac{k^2}{K^2} < 0.788$ obtained in example 2.1.

Secondly, consider example 3.2. This system must first be transformed to the equivalent class 1 system before inequality (6.1.1) may be applied. The resulting inequality contains an eighth order polynomial in w, but for comparison it is sufficient to consider the case w = 0. To further simplify calculations, put $K_{11} = K_{22} = K$ and $K_{12} = K_{21} = k$. Inequality (6.1.1) is then satisfied, provided that

$$1 - \frac{k^2 K^2}{320\,(k_1 K-6)} > 0 \tag{6.1.2}$$

$$\text{and } (k_1 K-6) - \frac{81}{(160)^2} k^2 K^2 > 0 \tag{6.1.3}$$

For $K = 10^3$, $k = 10$ and $k_1 = 3$, the left hand sides of inequalities (6.1.2) and (6.1.3) are -104.4 and -313306 respectively, and the frequency domain criterion fails.

From example 3.2, the relevant leading minors of W are

$\Delta_1 = 2.394.10^6$, $\Delta_2 = 2.3702.10^9$, $\Delta_3 = 3.702.10^{11}$, $\Delta_4 = 5.591178.10^{17}$

and the system is therefore established as totally stable.

From the above comparisons, it is clear that neither approach has provided more than sufficient conditions for stability. For a single loop system, the frequency domain criterion may be applied graphically, but for multivariable systems a computer is almost always required (see Jury and Lee[21], p.7).

For high order systems, a computer is also required for application of the criteria developed here. In terms of computation, there appears to be little to choose between the two methods. It remains to be seen whether in general one or the other can provide more satisfactory results.

The next step in improving the criteria of Chaps. II and III might be to investigate the use of a set of n parameters c_i, one for each row of the transfer matrix G (cf. the n parameters q_{ii} in the frequency domain criterion). Although this would increase the number of arbitrary constants by(n-1), the ability to select these constants as desired might provide better results.

6.2. The nuclear reactor

The use of special canonic forms of state variables also plays an important part in applying Liapunov's second method to systems containing multiplicative nonlinearities. A simple approach has been adopted which produces quadratic regions of stability regardless of the shape of the true stability region. From a physical point of view, this approach is adequate provided that the quadratic surfaces include most deviations likely to occur in a given system. For the particular examples considered this was the case.

Even for simple reactor models, use of a digital computer is required, but only one arbitrary parameter is involved, and rapid convergence to optimum results is assured.

Kerr[13] has considered the stability of a second order one node reactor model using Liapunov techniques, and has obtained a stability region somewhat greater than the quadratic surfaces defined in Chapter IV. Although the analysis was confined to a second order system, the proposed 'method of undetermined coefficients' (p.120, ref. 13) for constructing Liapunov functions may prove useful in improving the general quadratic functions used in Chapter IV.

One further example of systems involving multiplications is to be found in ref. 22 (Buruvoy and Slepov) which examines the stability of some second order thermochemical processes by phase portrait methods. In particular, the phase portrait of fig. 18 ref. 22, is remarkably similar to that of example 4.2. Such processes should therefore be capable of analysis by the type of criteria developed in Chapter IV.

6.3. Self oscillatory systems

While the possibly oscillatory modes of a given system may be determined by the elementary methods of Chapter V, the question of existence or stability of such modes is still largely an unsolved problem.

The only promising approach using Liapunov techniques found in the literature appears to be that of Zubov[23], in which a Liapunov function may be obtained from an iterative, convergent series for an arbitrary system. Unfortunately, the series thus constructed is not, in general monotonically convergent, and a great number of terms may have to be included before a satisfactory stability region is obtained.

To determine the stability or otherwise of a predicted oscillatory mode, it is possible to consider deviations about this mode, when a set of differential equations with periodic coefficients are obtained. Techniques such as the Dual input Describing function (West[15]) and the incremental Nyquist plot (Gibson[4]) may then be applied, but for multivariable systems these procedures become very laborious.

Frequency domain criteria, similar to that mentioned in section 1, have been established for systems with time-varying coefficients. These criteria are only valid when the coefficients are positive for all time. The differential equations obtained from deviations about a self-oscillatory mode inevitably contain sinusoidal coefficients, and these criteria therefore fail (see e.g. Naumov[24])

6.4. Some general comments

The simultaneous quadratic equations which appear in all criteria of this thesis have, in general, n^2 sets of roots, where n is the number of system outputs. More than one of these sets may be of the appropriate algebraic nature, and the question arises as to whether any one set will give better results than any other.

In the examples considered throughout this thesis, no evidence has been found to support this proposition, but many more examples must be dealt with before a general conclusion can be reached. In criteria 1.1 and 1.5, the sign of the relevant functions W is independent of the quadratic coefficients. For these two criteria, therefore, the same results are obtained for any appropriate set of roots.

In section 6.1, some comparison was made between the stability criteria of Chaps. II and III and frequency domain criteria. The frequency criteria prove rather unwieldy for estimation of transient response. While the concepts of gain and phase margins are applicable, in the general case these margins must be found in an n-dimensional space of the arbitrary parameters q_{ii}.

Finally, note that the parameter c is necessarily confined in all stability criteria to lie within the range $0 < c < |\lambda_{min}|$ where $|\lambda_{min}|$ is the smallest modulus of system poles. A search for the optimum value of c is therefore also confined to this region, which considerably reduces computation as compared to the frequency criterion.

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APPENDIX I

Liapunov's total stability theorem for autonomous systems.

Given the general system of differential equations

$$\underline{\dot{x}} = \underline{f}\ (\underline{x}), \qquad \text{(A.1.1)}$$

then if there exists a function $V(\underline{x})$, continuous together with its first partial derivatives in the entire state space $\underline{x}$, such that

(1) $V(\underline{x})$ is sign definite in the entire space, i.e.

$V(x)$ is of one sign for all $|\ \underline{x}\ | \neq 0$, $V(\underline{0}) = 0$;

(2) $\lim\ V(\underline{x}) \to \infty$ as $|\ \underline{x}\ | \to \infty$;

(3) $\frac{dV}{dt}$ is sign definite of the opposite sign to V in the entire space,

then every trajectory of the system (A.1.1) leads to the origin, and the system is said to be globally asymptotically stable or <u>totally stable</u>.

Note that condition (2) is always satisfied by sign definite quadratic functions.

APPENDIX II

II.1. Some necessary conditions for appropriate solutions of coefficient equations.

The general equation

$$2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = 2\sqrt{R}\, a_i + \gamma_i \quad (i = 1,2,\ldots n) \qquad \text{(A.2.1)}$$

includes all other types of equation encountered in the stability criteria. The conditions given below may be found derived in detail in Letov[1] and Rekasius[2].

Multiplying each of eqns. (A.2.1) by w_i and adding gives

$$\left(\sum_{i=1}^{n} a_i\right)^2 = 2\sqrt{R} \sum_{i=1}^{n} a_i w_i + \sum_{i=1}^{n} w_i \gamma_i \qquad \text{(A.2.2)}$$

Dividing each of eqns. (A.2.1) by w_i and adding gives

$$\left(\sum_{i=1}^{n} \frac{a_i}{w_i}\right)^2 = 2\sqrt{R} \sum_{i=1}^{n} \frac{a_i}{w_i} + \sum_{i=1}^{n} \frac{\gamma_i}{w_i}$$

or

$$\sum_{i=1}^{n} \frac{a_i}{w_i} = \sqrt{R} \pm \sqrt{R + \sum_{i=1}^{n} \frac{\gamma_i}{w_i}} \qquad \text{(A.2.3)}$$

Straightforward addition of eqns. (A.2.1) gives

$$2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{w_i + w_j} = 2\sqrt{R} \sum_{i=1}^{n} a_i + \sum \gamma_i \qquad \text{(A.2.4.)}$$

For the case n = 2, eqns. (A.2.2) and (A.2.3) yield the further relation

$$[a_1 + a_2 - \sqrt{R}\,(w_1 + w_2)]^2 = \gamma_1 w_1 + \gamma_2 w_2 + R(w_1 + w_2)^2 - 2\, w_1 w_2 \sqrt{R^2 + R \sum_{i=1}^{2} \frac{\gamma_i}{w_i}} \qquad \text{(A.2.5.)}$$

From eqns. (A.2.2) to (A.2.5) the following necessary conditions may be set down for each type of coefficient equations.

(1) The set $2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = \alpha_i$.

require that

(a) $r = -\sum_{i=1}^{n} \alpha_i > 0$

(b) $\sum_{i=1}^{n} \alpha_i w_i \geq 0$

(c) $\sum_{i=1}^{n} \frac{\alpha_i}{w_i} \geq 0$

(2) The set $2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = 2\sqrt{r}a_i + \beta_i$

require that

(a) $r = -\sum_{i=1}^{n} \alpha_i > 0$

(b) $-\sum_{i=1}^{n} \frac{\alpha_i}{w_i} \geq 0$

(c) <u>for n = 2</u>

$$(\alpha_1 w_1 + \alpha_2 w_2)^2 + 2r\,(\alpha_1 w_1 - \alpha_2 w_2)(w_1^2 - w_2^2) + r^2\,(w_1^2 - w_2^2)^2 \geq 0$$

(3) The set $2a_i \sum_{j=1}^{n} \frac{a_j}{w_i + w_j} = 2\sqrt{R}\, a_i + \alpha_i$

require that

(a) $R + \sum_{i=1}^{n} \frac{\alpha_i}{w_i} \geq 0$

(b) for n = 2

$$(\alpha_1 w_1 + \alpha_2 w_2)^2 + 2R\,(\alpha_1 w_1 - \alpha_2 w_2)(w_1^2 - w_2^2) + R^2(w_1^2 - w_2^2) \geq 0$$

For all three cases, the conditions given are necessary and sufficient for n = 1 and n = 2.

II.2. A digital method of solution

If the general equations (A.2.1) are written in the form

$f_i(a_1, a_2, \ldots a_n) = 0 \; (i = 1, 2, \ldots n)$

the solution of the equations may be expressed as the minimum of the positive function

$$F = \sum_{i=1}^{n} f_i^2. \qquad (A.2.6)$$

Any optimisation or hill-climbing digital technique may be used to find the minimum of the function F in eqn. (A.2.6). If complex coefficients are involved, the computer programme must include complex algebra subroutines. If, and only if, solutions exist of the appropriate nature, the minimum of F will be identically zero.

APPENDIX III

A conditionally positive semidefinite function.

The quadratic function (see page 31)

$$\dot{V} = (\sum_{i=1}^{n} a_i x_i)^2 + r_{11} f_1^2 (y_1) + r_{22} f_2^2 (y_2) + (r_{12} + r_{21}) f_1(y_1) f_2(y_2)$$

$$+ f_2(y_2) \sum_{i=1}^{n} a_i \left(2a_i \sum_{j=1}^{n} \frac{a_j u_{2j}}{\lambda_i + \lambda_j} - \beta_{2i}\right)$$

may only be at best positive semidefinite, and only when certain relationships exist between the system parameters.

Define

$$2b_i = 2a_i \sum_{j=1}^{n} \frac{a_j u_{2j}}{\lambda_i + \lambda_j} - \beta_{2i} \qquad (i = 1,2,\ldots n).$$

$\dot{V}$ may then be written in the quadratic form $\underline{X}^T A \underline{X}$, where

$$\underline{X}^T = [x_1, x_2, x_3 \ldots x_n, f_1(y_1), f_2(y_2)]$$

and

$$A = \begin{bmatrix} a_1^2 & a_1 a_2 & \cdots & a_1 a_n & 0 & b_1 \\ a_1 a_2 & a_2^2 & \cdots & a_2 a_n & 0 & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_1 a_n & a_2 a_n & \cdots & a_n^2 & 0 & b_n \\ 0 & 0 & \cdots & 0 & r_{11} & \frac{1}{2}(r_{12}+r_{21}) \\ b_1 & b_2 & \cdots & b_n & \frac{1}{2}(r_{12}+r_{21}) & r_{22} \end{bmatrix}$$

Since the first n leading minors of A are zero, A is at best only a positive semidefinite matrix, and only when (see, for example Fraser, Duncan and Collar) *

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}, \;\ldots\ldots\; \frac{a_r}{a_{r+1}} = \frac{b_r}{b_{r+1}}, \;\ldots\ldots\; \frac{a_{n-1}}{a_n} = \frac{b_{n-1}}{b_n} .$$

or in general, when

$$\frac{a_r}{a_{r+1}} = \frac{2a_r \sum_{j=1}^{n} \frac{a_j u_{2j}}{\lambda_r + \lambda_j} - \beta_{2r}}{2a_{r+1} \sum_{j=1}^{n} \frac{a_j u_{2j}}{\lambda_{r+1} + \lambda_j} - \beta_{2,r+1}} \qquad \text{(A.3.1)}$$

Only for n = 1 is the matrix A possibly sign definite without the restrictions (A.3.1).

* FRASER, DUNCAN and COLLAR. Elementary Matrices. Cambridge Univ. Press, 1946.

APPENDIX IV.

Conditions for validity of class 1 canonic forms.

IV.1. No common row poles

The canonic variables $\underline{z}^{(ij)}$ $(i,j = 1,2,\ldots n)$ of eqn. (2.2a.1) may be expressed in terms of the physical variables of the system as follows.

The k th differential of eqn. (2.2a.2) is

$$p^k y_i = \sum_{j=1}^{n} \underline{\alpha}^{(ij)T} [\Lambda^{(ij)}]^k \underline{z}^{(ij)} + \sum_{j=1}^{n} \underline{\alpha}^{(ij)T} \Gamma_k f_j(y_j)$$

where

$$\Gamma_k = \sum_{m=1}^{k} p^{k-m} [\Lambda^{(ij)}]^{m-1} \underline{e}^{(ij)} - R_{ij} p^k \qquad \text{(A.4.1)}$$

Since eqn. (2.2a.2) involves only the state variables pertaining to the ith row of G, it is sufficient to differentiate each eqn. (2.2a.2) (r-1) times, where there are altogether r poles, and therefore state variables, in the ith row of G.

Defining

$$F_k = p^k y_i - \sum_{j=1}^{n} \underline{\alpha}^{(ij)T} \Gamma_k f_j (y_j)$$

eqn. (A.4.1) may be written as

$$\sum_{j=1}^{n} \underline{\alpha}^{(ij)T} [\Lambda^{(ij)}]^k \underline{z}^{(ij)} = F_k \qquad \text{(A.4.2)}$$

k = 0,1,2,...r-1).

or in matrix notation,

$$B_i \underline{Z} = \underline{F}$$

where $\underline{Z}$ is a column vector of all the state variables of the ith row of G. For the canonic form to be valid, it is sufficient that $| B_i | \neq 0$ for all i.

$$| B_i | = \prod_{\text{all } j,k} \alpha_k^{(ij)} \begin{vmatrix} 1 & \cdots & 1 \\ \Lambda^{(i1)} & & \Lambda^{(in)} \\ [\Lambda^{(i1)}]^2 & & [\Lambda^{(in)}]^2 \\ \vdots & & \vdots \\ [\Lambda^{(i1)}]^{r-1} & \cdots & [\Lambda^{(in)}]^{r-1} \end{vmatrix}$$

The above determinant is the Vandermonde determinant (see e.g. Stoll)* of all the poles in the ith row, i.e.

$$| B_i | = \prod_{\text{all } j,k} \alpha_k^{(ij)} \cdot \prod_{\text{all } m,j,r} (\lambda_m^{(ij)} - \lambda_r^{(ij)})$$

For $| B_i | = 0$, it is therefore sufficient that

(1) No transfer function in G contains multiple poles, and

(2) No transfer function may have the same pole as any other in the same row of G.

* STOLL, R.R. Linear algebra and matrix theory. McGraw-Hill 1952.

IV. 2. Common row poles

The kth differential of eqn. (2.2b.3) is

$$p^k y_i = \sum_{j=1}^{n} \underline{\alpha}^{(ij)T} [\Lambda^{(ij)}]^k \underline{z}^{(ij)} + \underline{e}^{(i)T} [\Lambda^{(i)}]^k \underline{z}^{(i)}$$

$$+ \sum_{j=1}^{n} \Gamma_k f_j (y_j) \qquad \text{(A.4.3)}$$

where

$$\Gamma_k = \sum_{m=1}^{k} \underline{\alpha}^{(ij)T} [\Lambda^{(ij)}]^{m-1} p^{k-m} - R_{ij} p^k$$

$$+ \sum_{m=1}^{k} \underline{\hat{\alpha}}^{(ij)T} [\Lambda^{(i)}]^{m-1} p^{k-m} .$$

Defining

$$F_k = p^k y_i - \sum_{j=1}^{n} \Gamma_k \ f_j (y_j),$$

eqn. (A.4.3) may be written as

$$\sum_{j=1}^{n} \underline{\alpha}^{(ij)T} [\Lambda^{(ij)}]^k \underline{z}^{(ij)} + \underline{e}^{(i)T} [\Lambda^{(i)}]^k \underline{z}^{(i)} = F_k \qquad \text{(A.4.4)}$$

k = 0,1,2,...r-1)

where r is the total number of common and non-common poles in the ith row of G. Writing eqn. (A.4.4) in matrix form,

$$B_i \underline{Z} = \underline{F}$$

where Z is a column vector of all the state variables of the ith row of G. For the canonic form to be valid, it is sufficient that $|B_i| \neq 0$ for all i.

$$|B_i| = \prod_{\text{all } j,k} \alpha_k^{(ij)} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Lambda^{(i1)} & & \Lambda^{(in)} & \Lambda^{(i)} \\ [\Lambda^{(i1)}]^2 & & [\Lambda^{(in)}]^2 & [\Lambda^{(i)}]^2 \\ \vdots & & \vdots & \vdots \\ [\Lambda^{(i1)}]^{r-1} & \cdots & [\Lambda^{(in)}]^{r-1} & [\Lambda^{(i)}]^{r-1} \end{vmatrix}$$

which is simply the Vandermonde determinant among all the common and non-common row poles of the ith row of G. For $|B_i| \neq 0$, it is sufficient that no transfer function in G contains multiple poles.

APPENDIX V

The kth differential of eqn. (3.1.4), Chapter III is

$$p^k y_i = \alpha^{(ii)T} \Lambda^{(ii)k} \underline{z}^{(ii)} + \sum_{\substack{j=1 \\ (j \neq i)}}^{n} \left[\underline{b}_k^{(ij)T} \underline{z}^{(ij)} + {}_j f_k \right] \qquad \text{(A.5.1)}$$

where

$$\underline{b}_k^{(ij)T} = \underline{\alpha}^{(ii)T} \sum_{\ell=1}^{k} \underline{\lambda}^{(ii)k-\ell} \cdot \underline{\alpha}^{(ij)T} \Lambda^{(ij)\ell-1} - R_{ii} \underline{\alpha}^{(ij)T} \Lambda^{(ij)k},$$

$${}_j f_k = \underline{\alpha}^{(ii)T} \sum_{\ell=1}^{k} \Lambda^{(ii)k-\ell} p^{\ell-1} \underline{f}_i^{(ii)} - R_{ii}\left[p^k f_i(y_i) + p^{k-1} \underline{\alpha}^{(ij)T} \underline{y}_j^{(ij)} - R_{ij} p^k y_j\right]$$

$$+ \underline{\alpha}^{(ii)T} \sum_{\ell-1}^{k} \underline{\lambda}^{(ii)k-\ell} [- \underline{\alpha}^{(ij)T} \sum_{q=1}^{\ell-1} \Lambda^{(ij)\ell-1-q} p^{q-1} +$$

$$+ R_{ij} p^{\ell-1}] y_j$$

If there are a total of m poles in the ith row the transfer matrix $G = G_1 + G_2$ (see section 3.1, Chapter III) the relationships between the m state variables $\underline{z}^{(ij)}$ (j = 1,2,...n, i constant) may be found by using equation (A.5.1) for k = 0,1,2,...m-1. In matrix form,

$$B_i \underline{Z} = \underline{F}$$

where $\underline{F}$ is a column vector whose general element is

$$p^r y_i - f_r$$

and B_i is the matrix

$$
\begin{bmatrix}
\underline{\alpha}^{(11)T} , & \underline{b}_o^{(12)T} , & \cdots\cdots & \underline{b}_o^{(1n)T} \\
\underline{\alpha}^{(11)T} \Lambda^{(11)} , & \underline{b}_1^{(12)T} , & \cdots\cdots & \underline{b}_1^{(1n)T} \\
\vdots & \vdots & & \vdots \\
\underline{\alpha}^{(11)T} \Lambda^{(11)m-1} , & \underline{b}_{m-1}^{(12)T} , & \cdots\cdots & \underline{b}_{m-1}^{(1n)T}
\end{bmatrix}
$$

For the canonic form to be valid, it is sufficient that $| B_i | \neq 0$ for all i. Attempts to find $| B_i |$ in general have failed. It is evident from the vectors $\underline{b}_k^{(ij)}$, however, that one necessary condition is that no transfer function can have multiple poles. For given systems, the determinants $| B_i |$ must be evaluated numerically.

APPENDIX VI

Analysis of a one node reactor model

VI.1. Conditions for validity of the canonic form

The kth differentials of eqns. (4.1.11) and (4.1.12) are respectively

$$p^k y = -\sum_{i=1}^{n} \alpha_i \lambda_i^k z_i + \sum_{m=1}^{k} c_m \delta ky \qquad \text{(A.6.1)}$$

and

$$p^k \delta k = \sum_{i=1}^{n} \gamma_i \lambda_i^k z_i + \sum_{m=1}^{k} d_m \delta ky \qquad \text{(A.6.2)}$$

where

$$c_m = p^{k-m} \sum_{i=1}^{n} \alpha_i \lambda_i^{m-1},$$

$$d_m = p^{k-m} \sum_{i=1}^{n} \gamma_i \lambda_i^{m-1}.$$

Assume that the order of $G_1(p)$ is r, and the order of $G_2(p)$ is q.

To determine the relationships between the state variables z_i and the system physical variables δk and y, it is sufficient to differentiate eqn. (4.1.11) r times and eqn. (4.1.12) (q-1) times.

Define

$$-F_{1k} = p^k y + \sum_{m=1}^{k} c_m \delta ky,$$

$$F_{2k} = p^k \delta k - \sum_{m=1}^{k} d_m \delta ky$$

Then from eqns. (A.6.1) and (A.6.2),

$$\sum_{i=1}^{n} \alpha_i \lambda_i^k z_i = F_{1k} \quad (k = 0,1,2,\ldots r) \tag{A.6.3}$$

$$\sum_{i=1}^{n} \gamma_i \lambda_i^k z_i = F_{2k} \quad (k = 0,1,2,\ldots q-1) \tag{A.6.4}$$

Expressing eqns. (A.6.3) and (A.6.4) in matrix form,

$$B \underline{Z} = \underline{F}$$

where $\underline{Z}$ is a column vector of all the state variables z_i $(i = 1,2,\ldots n)$. For the canonic form to be valid, it is sufficient that $| B | \neq 0$.

Suppose that in eqns. (4.1.3) and (4.1.5)

$$G_1(p) = \frac{a(p)}{b(p)}, \qquad G_2(p) = \frac{c(p)}{d(p)} \tag{A.6.5}$$

where a, b, c and d are polynomials of p. The transfer functions R(p) and H(p) (see eqns. (4.1.8) and (4.1.9) may then be written as

$$R = \frac{ad}{bdp + ac}, \qquad H = \frac{ac}{bdp + ac}$$

Assume that

$$a = \prod_{j=1}^{r} (p - w_{1j}) \qquad c = \prod_{j=1}^{s} (p - w_{2j}),$$

$$d = \prod_{j=1}^{t} (p - \lambda_{2j}), \quad bdp + ac = \prod_{j=1}^{n} (p - \lambda_j).$$

Then applying the residue theorem,

$$\alpha_i = - \frac{\prod_{j=1}^{r} (\lambda_i - w_{1j}) \prod_{j=1}^{t} (\lambda_i - \lambda_{2j})}{\prod_{\substack{j=1 \\ (j \neq i)}}^{n} (\lambda_i - \lambda_{ij})} \qquad \text{(A.6.6)}$$

and

$$\gamma_i = - \frac{\prod_{j=1}^{r} (\lambda_i - w_{1j}) \prod_{j=1}^{S} (\lambda_i - w_{wj})}{\prod_{\substack{j=1 \\ (j \neq i)}}^{n} (\lambda_i - \lambda_j)} \qquad \text{(A.6.7.)}$$

Dividing eqn. (A.6.7) by eqn. (A.6.6),

$$\frac{\gamma_i}{\alpha_i} = \frac{\prod_{j=1}^{S} (\lambda_i - w_{wj})}{\prod_{j=1}^{t} (\lambda_i - \lambda_{2j})} = G_2(\lambda_i) \text{ (from eqns. (A.6.5))}$$

Therefore $\gamma_i = G_2(\lambda_i).\ \alpha_i\ (i = 1,2,\ldots n)$ (A.6.8).

Substituting for γ_i from eqn. (A.6.8), the determinant of B is given by

$$|B| = \prod_{\text{all } i} \alpha_i \begin{vmatrix} G_2(\lambda_1), & G_2(\lambda_2), & \cdots & G_2(\lambda_n) \\ \lambda_1 G_2(\lambda_1), & \lambda_2 G_2(\lambda_2), & \cdots & \lambda_n G_2(\lambda_n) \\ \vdots & \vdots & & \vdots \\ \lambda_1^{q-1} G_2(\lambda_1), & \lambda_2^{q-1} G_2(\lambda_2), & \cdots & \lambda_n^{q-1} G_2(\lambda_n) \\ -1, & -1, & \cdots & -1 \\ -\lambda_1, & -\lambda_2, & \cdots & -\lambda_n \\ \vdots & \vdots & & \vdots \\ -\lambda_1^r, & -\lambda_2^r, & \cdots & \lambda_n^r \end{vmatrix}$$

Evidently if any $\lambda_i = \lambda_j$ $(i \neq j)$ the above determinant vanishes. Attempts to find the general determinant have failed, and $|B|$ should therefore be evaluated for each system under consideration.

VI.2. Singularities of the model.

In terms of the state variables, the singularities of the system are at (see eqn. 4.1.10)

$$Z_i = -\frac{1}{\lambda_i} \delta ky \quad (i = 1,2,\ldots n) \qquad \text{(A.6.6)}$$

Multiplying eqn. (A.6.6) by α_i and summing over i gives

$$\sum_{i=1}^{n} \alpha_i z_i = -y = \sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i} \delta k y \qquad \text{(A.6.7)}$$

Similarly

$$\sum_{i=1}^{n} \gamma_i z_i = \delta k = \sum_{i=1}^{n} \frac{\gamma_i}{\lambda_i} \delta k y \qquad \text{(A.6.8)}$$

From eqns. (A.6.7) and (A.6.8), the system singularities are at

(1) $\delta k = y = 0$: $z_i = 0$ for all i

(2) $\delta k = -1/ \sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i}$, $y = 1/ \sum_{i=1}^{n} \frac{\gamma_i}{\lambda_i}$:

$$z_i = \frac{1}{\lambda_i \sum_{i=1}^{n} \frac{\gamma_i}{\lambda_i} \sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i}}$$

But $\sum_{i=1}^{n} \frac{\gamma_i}{\lambda_i} \sum_{i=1}^{n} \frac{\gamma_i}{\lambda_i} = R(0).\ H(0) = \frac{1}{G_2(0)}$

So the second singularity is at

$$z_i = \frac{G_2(0)}{\lambda_i} \qquad (i = 1,2,\ldots n)$$

VI.3. <u>A digital method of determining the region of stability</u>

Mathematically, the problem may be stated as

V = minimum subject to the restriction $\dot{V} \geq 0$. Using the method of undetermined multipliers, the required tangent point may be found from the solution of the equations

$$\dot{V} = 0, \qquad \frac{\partial V}{\partial Z_i} + \lambda \frac{\partial \dot{V}}{\partial Z_i} = 0 \quad (i = 1,2,\ldots n)$$

where λ is the undetermined multiplier.

From eqn. (4.1.13),

$$\frac{\partial V}{\partial z_i} = 2a_i \sum_{j=1}^{n} \frac{a_j z_j}{w_i + w_j} \tag{A.6.9}$$

From eqn. (4.1.15),

$$\frac{\partial \dot{V}}{\partial z_i} = 2a_i \sum_{j=1}^{n} \frac{a_j (\lambda_i + \lambda_j) z_j}{w_i + w_j} + \alpha_i \sum_{i=1}^{n} \gamma_i z_i \sum_{i=1}^{n} (2\alpha_i - K\gamma_i) z_i$$

$$+ \gamma_i \sum_{i=1}^{n} \alpha_i z_i \sum_{i=1}^{n} (\alpha_i - 2k\gamma_i) z_i \tag{A.6.10}$$

From eqns. (A.6.9) and (A.6.10),

$$\sum_{i=1}^{n} \left(\frac{\partial V}{\partial z_i} + \lambda \frac{\partial \dot{V}}{\partial z_i} \right) = 0$$

$$= 2V + \lambda \, (2\dot{V} + \delta ky \, (y + K \delta k))$$

$$= 2V + \lambda \, \delta ky \, (y + K \delta k)$$

$$\therefore \quad \lambda = - \frac{2V}{\delta ky \, (y + K \delta k)}$$

and the tangent point is given by the solutions of the equations

$$F_i(\underline{z}) = \frac{\partial V}{\partial z_i} - \frac{2V}{\delta ky \, (y + K \delta k)} \cdot \frac{\partial \dot{V}}{\partial z_i} = 0.$$

A digital optimisation procedure may therefore be applied, to find the solutions as the minimum of the function

$$W = \sum_{i=1}^{n} F_i^2 (\underline{z}).$$

APPENDIX VII

Properties of the multinode reactor equations

VII.1. Some relationships between the residues.

In eqns. (4.2.7) and (4.2.8), the transfer functions of the system may be written as

$$g_{ii}(p) = \frac{\prod_j (p-w_{1j}) \prod_j (p-\lambda_{2j})}{\prod_j (p-\lambda_j)}$$

$$g_{ij}(p) = \frac{k_{ij} N_j^*}{\ell N_i^*} \frac{\prod_j (p-w_{1j}) \prod_j (p-w_{2j})}{\prod_j (p-\lambda_j)}$$

$$h_{ii}(p) = \frac{\prod_j (p-w_{1j}) \prod_j (p-w_{2j})}{\prod_j (p-\lambda_j)}$$

$$h_{ij}(p) = \frac{k_{ij} N_j^*}{\ell N_i^*} \cdot \frac{\prod_j (p-w_{2j}) \prod_j (p-\lambda_{1j})}{\prod_j (p-\lambda_j)}$$

where λ_j are the poles of these transfer functions

λ_{1j} are the poles of $G_1(p)$

λ_{2j} are the poles of $G_{2i}(p)$

w_{1j} are the zeroes of $G_1(p)$

w_{2j} are the zeroes of $G_{2i}(p)$

Applying the residue theorem, it follows that

$$\frac{\gamma_r^{(ii)}}{\alpha_r^{(ii)}} = \frac{\prod_j (\lambda_r - w_{2j})}{\prod_j (\lambda_r - \lambda_{2j})} = G_{2i}(\lambda_r)$$

and

$$\frac{\alpha_r^{(ii)}}{\alpha_r^{(ij)}} = \frac{\gamma_r^{(ii)}}{\gamma_r^{(ij)}} \quad (i \neq j) = \frac{\ell N_i^*}{k_{ij} N_j^*} \cdot \frac{\prod_j (\lambda_r - w_{1j})}{\prod_j (\lambda_r - \lambda_{1j})}$$

$$= \frac{\ell N_i^*}{k_{ij} N_j^*} \; G_1(\lambda_r)$$

From eqns. (4.2.7), (4.2.8) and (4.2.11), the above ratio, in order that the canonic form be correct, is also the inverse of the elements $b_r^{(ij)}$ $(i \neq j)$.

V11.2. Conditions for validity of the canonic form

The kth differential of eqn. (4.2.12) is

$$p^k y_i = - \underline{\alpha}^{(ii)T} \Lambda^{(ii)k} \underline{z}^{(ii)} - f_{1k} \qquad \text{(A.7.1)}$$

where $f_{1k} = \underline{\alpha}^{(ii)T} \sum_{\ell=1}^{k=1} p^{\ell-1} (\underline{ry} + \sum_{\substack{j=1 \\ (j \neq i)}}^{m} \underline{b}^{(ij)} y_j)$.

The kth differential of eqn. (4.2.13) is

$$p^k r_i = \underline{\gamma}^{(ii)T} \Lambda^{(ii)k} \underline{z}^{(ii)} - f_{2k} \qquad \text{(A.7.2)}$$

where

$$f_{2k} = - \underline{\gamma}^{(ii)T} \sum_{\ell=1}^{k-1} p^{\ell-1} (\underline{ry} + \sum_{\substack{j=1 \\ (j \neq i)}}^{m} \underline{b}^{(ij)} y_j).$$

Assume that there are m poles in $G_1(p)$ and q poles in $G_2(p)$. Then by using eqn. (A.7.1) for k = 0,1,2...m and eqn. (A.7.2) for k = 0,1,2,...q-1 the canonic variables $\underline{z}^{(ii)}$ are found from the eqn.

$$B_i \underline{Z}^{(ii)} = \underline{F}$$

where $\underline{F} =$

$$\begin{bmatrix} y_i \\ py_i + f_{11} \\ \vdots \\ p^m y_i + f_{1m} \\ r_i \\ pr_i + f_{21} \\ \vdots \\ p^{q-1} r_i + f_{2,q-1} \end{bmatrix}$$

and B_i is the matrix

$$
\begin{bmatrix}
-\underline{\alpha}^{(ii)T} \\
\underline{\alpha}^{(ii)T}\Lambda^{(ii)} \\
\vdots \\
\underline{\alpha}^{(ii)T}\Lambda^{(ii)m} \\
\gamma^{(ii)T} \\
\gamma^{(ii)T}\Lambda^{(ii)} \\
\vdots \\
\gamma^{(ii)T}\Lambda^{(ii)q-1}
\end{bmatrix}
$$

Using the relationships of VII.1, a necessary condition for nonsingular B_i is that there are no multiple poles in any transfer function. Attempts to find the general determinant of the matrices B_i have failed, and each particular system must be dealt with numerically to check validity of the canonic transformation.