AN INTEGRAL EQUATION DESCRIBING A MAGNETIC CONFINEMENT SYSTEM

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PREFACE

In chapter one some of the work done on attempts to confine plasmas by means of magnetic fields is summarised. The non-adiabatic magnetic mirror machine which is the subject of this work is also described, along with how it came to be suggested.

In chapter two the general integral equation for a system of this type is derived. This work is due to Drs D.A. Dunnett and E.W. Laing. The solution of this general integral equation is a distribution function which is a function of the velocities of the particles in the system. The theory of how the mean containment time of the particles in this confinement system can be calculated is described.

In chapter three the equations necessary to calculate the kernel of the integral equation are derived from the general equations for a charged particle in a magnetic field. The method of calculating this kernel numerically from these equations is then described and discussed. Some of the results of kernels obtained in this way are given. This method, and the subsequent work are due to the author in collaboration with Drs D.A. Dunnett and E.W. Laing. Chapter four describes the method used to solve numerically the integral equation of chapter two. The method of solution is described algebraically and then the numerical calculations involved and some of the results of these calculations are given.

In chapter five more of the results produced by this method are presented. These are compared with the results obtained by a purely numerical method of dealing with the system, and with those obtained from a diffusion equation treatment of the integral equation of chapter two.

CHAPTER 1

MIRROR MACHINES

1.1 Introduction

Work has been in progress for a great many years on the study of the properties of plasmas, or hot ionised gases, but this study has been intensified in the last 20 years in the hope that a method will be found of confining plasmas for long enough to allow fusion of [redacted] nuclei to take place.

Many reactions involving light nuclei result in the liberation of energy, and these are the reactions which are utilised in attempts to produce energy by means of fusion. The reactions which are most practicable are those which do not need an extremely high temperature or a very high density of particles in order that the probability of the reaction occurring is large enough for the reaction to be at least self-sustaining, i.e. for it to produce enough energy for the reaction to occur again. Various measurements of probability have been made, and it has been concluded that the most practicable reactions are those involving the heavy isotopes of hydrogen, namely deuterium and tritium.

In order that the reactions can occur the particles must be contained in some type of system. Many possible methods of confinement by means of a magnetic field have been suggested, but these have been to some degree or another unsuccessful, in that various instabilities have built up before the correct criteria of density and temperature have been established.

1.2 Basic Principles of Mirror Machines

We shall be concerned in this work with plasmas which are confined by means of magnetic fields, and among the many devices which have been suggested for this purpose is the magnetic mirror machine. (A magnetic mirror is a region of locally high magnetic field.)

The so-called "mirror effect" is based on one particular quantity associated with a charged particle, namely its magnetic moment. The magnetic moment of a charged particle moving in a magnetic field is defined in a way analogous to that of a current flowing round a loop. If a current I is flowing round a loop of area dS, then the magnetic moment of the current loop is defined to be IdS.

In a uniform magnetic field a charged particle with mass m and charge e gyrates helically with radius r_L (Larmor radius) and frequency ω about a magnetic field line. The quantities ω and r_L are connected by the equation $v_\perp = r_L \omega$, where $v_\perp$ is the velocity of the particle perpendicular to the magnetic field line and ω is the gyro-frequency of the particle (Larmor frequency). The latter is related to the magnetic field strength H by $\omega = \frac{eH}{mc}$, where c is the speed of light.

If the gyrating particle is treated as a current loop, its magnetic moment μ may be defined as

$$\mu = \frac{e\omega}{2\pi c} \cdot \pi r_L^2 \qquad (1.1)$$

On substitution for r_L, equation (1.1) becomes

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{H}$$

$$\mu = \frac{W_\perp}{H} \tag{1.2}$$

In (1.2) W is the particle energy perpendicular to the magnetic field line.

The motion of a particle is said to be "adiabatic" if it is moving in a field which satisfies the following two criteria

$$\frac{1}{\omega H}\frac{\partial H}{\partial t} \ll 1 \tag{1.3}$$

if the field is time dependent,

or

$$\frac{r_L}{H}|\nabla H| \ll 1 \tag{1.4}$$

if the field is spatially variable.

These criteria are necessary because in a variable magnetic field particles tend to drift across magnetic field lines, but for fields satisfying (1.3) and (1.4) this drift is small and can be neglected.

It has been shown by Hellwig (1) that in any field satisfying these criteria, μ is, to a good degree of approximation, a constant. We shall consider only fields for which (1.3) and (1.4) hold, and therefore in which μ is an adiabatic invariant.

In such a field, and with no electric field present, $W = \frac{1}{2}mv^2$, the total energy of the particle, is a constant. If $W_{\parallel}$ is the energy corresponding to the particle's motion parallel to the magnetic field line then

$$W = W_\perp + W_\parallel$$

If H increases in such a way that conditions (1.3) and (1.4) are still obeyed then the constancy of μ implies that $W_{\perp}$ also increases such that

$$\frac{W_{\perp}}{H} = \frac{W_{o}}{H_{o}} \qquad (1.5)$$

where H_o is the initial magnetic field, and W_o is the initial energy perpendicular to the magnetic field. If the maximum value of H is H_{max} then

$$W_{\perp} \leq \frac{H_{max} W_o}{H_o}$$

In order that the particle should be reflected $W_{\parallel}$ must become zero, and this will only occur when

$$W < \frac{H_{max} W_o}{H_o} \qquad (1.6)$$

This is therefore the criterion for reflection. Substituting $\frac{1}{2}mv^2$ for W and $\frac{1}{2}mv_{\perp}^2$ for W_o, (1.6) can be written

$$\frac{v_{\perp}^2}{v^2} > \frac{H_o}{H_{max}}$$

If θ is the angle between the velocity vector $\underline{v}$ and the magnetic field line, and if the mirror ratio R is defined as $\frac{H_{max}}{H_o}$, the criterion for reflection becomes

$$\sin\theta \geq \sqrt{\frac{1}{R}}$$

The normalised magnetic moment of a particle is defined as $\xi = \frac{v_{\perp}^2}{v^2}$, and in terms of this the reflection criterion becomes $\xi \geq \sqrt{\frac{1}{R}}$.

A magnetic mirror machine consists basically of two regions of high magnetic field H_m, separated by a region of weaker magnetic field H_o, as is shown in figure 1.1.

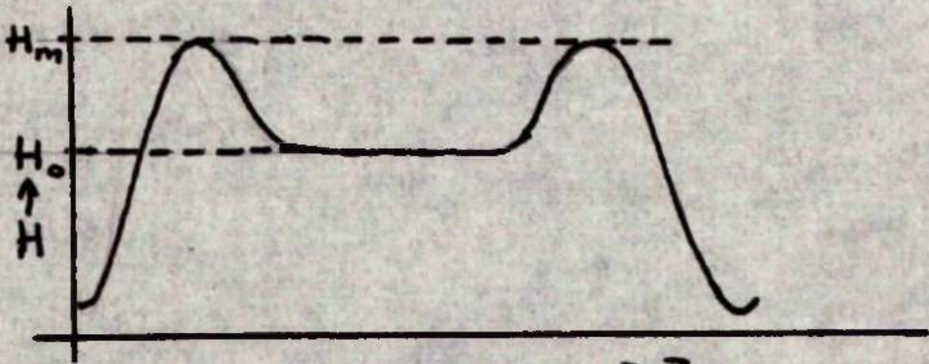


fig 1.1

The machine makes use of the adiabatic invariance of the magnetic moment of a charged particle in the following way. If in the central region of the machine the particle has total energy W and energy W_o perpendicular to the magnetic field then, using the invariance of the magnetic moment of the particle, it can be seen that the particle is reflected on reaching a mirror if $\mu = \frac{W_o}{H_o} > \frac{W}{H_m}$. In this way particles are trapped in the system.

1.3 Injection Mechanisms

One of the biggest problems of the simple mirror machine of fig.1.1 is the difficulty of injecting particles into it. If a particle has a sufficiently low magnetic moment to penetrate a mirror and enter the machine then it will be lost when it next encounters a mirror. Various injection devices have been suggested to overcome this problem, among them the following.

1. Time - Varying Field

There are various types of time-varying field (see 2), but the simplest of these is the field involving change of the mirror ratio.

The principle of this is that after a particle has entered the machine the mirror ratio of the machine is changed in accordance with (1.3), i.e. so that the magnetic moment of the particle is unchanged. There is then a non-zero probability that the particle will be reflected when it meets a mirror. The condition for trapping has been given by Post (3).

2. Collision Mechanisms

There will always be some collision between particles in a mirror machine, which will change the magnetic moments of some of the particles enough for them to be trapped, but this can be stimulated artificially by firing beams of particles simultaneously from both ends of the machine. This method traps a high proportion of the plasma (3).

3. Change of Charge - to - Mass Ratio

Particles can be trapped if once inside the machine they have their charge-to-mass ratio changed. This can be done by causing either neutral atoms or molecular ions to dissociate. Neutral atoms dissociating can produce ions with most of their energy in the transverse direction and therefore more liable to be reflected at a mirror.

These systems have been investigated both theoretically and experimentally and are described in various sources e.g. (2). The rest of this work will however be concerned with a different type of injection mechanism described in section 1.5.

1.4 The Sine - Wave Modulated Mirror Machine

A new system of injection was suggested and investigated experimentally by Sinelnikov et al (4). This consists of a mirror machine the field of which is modified in the central region by a small sine wave modulation.

In the central region the field is

$$H = H_o \left\{1 + h \sin \frac{2\pi z}{\lambda}\right\},$$

where $h \ll 1$ and λ is the wavelength of the modulation. This is shown diagrammatically in figure 1.2.

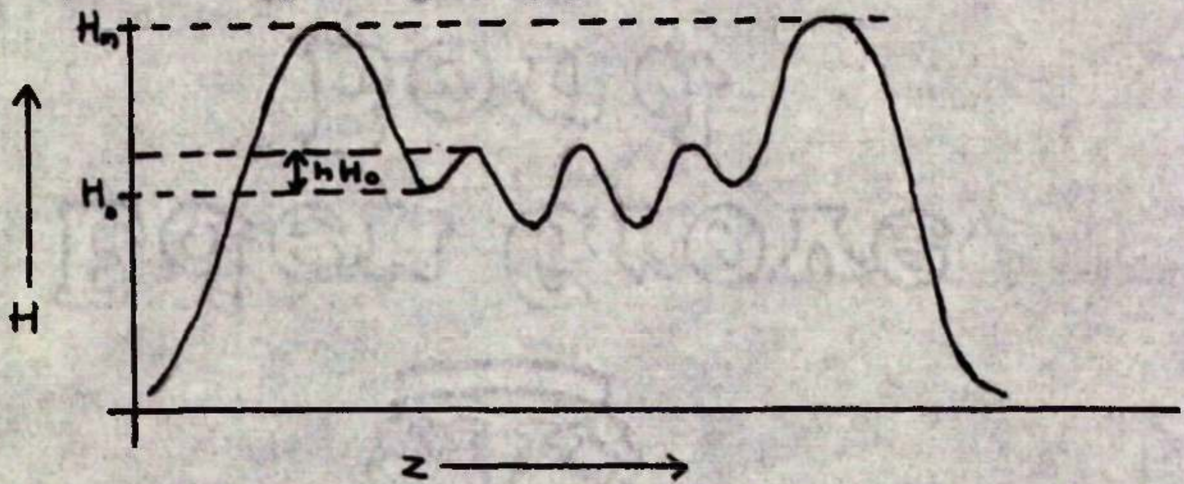


fig 1.2

This is a spatially varying field which deliberately does not satisfy the criterion (1.4), i.e. it is a non-adiabatic modification to the system.

Particles can be injected into this system along magnetic field lines, i.e. with zero magnetic moment.

Provided a particle's initial conditions satisfy

$$2\pi v = \omega\lambda,$$

where v is the injection velocity, ω is the Larmor frequency in the background field, i.e. $\omega = \frac{eH_o}{mc}$, and λ is the wavelength of the modulation, then there is a resonant interaction with the field, and parallel energy is transferred into transverse energy. This only occurs when the particle is injected off the axis of the system.

If the particle gains transverse energy then its magnetic moment is altered, and this means that there is now a non-zero probability that the particle will be reflected when it comes into contact with a magnetic mirror. The fact that the magnetic moment of the particle is changed every time it passes through the modulation means, however, that eventually the magnetic moment may become so small that the particle is lost when it next meets a mirror. This is how particles are lost from this device.

Laing and Robson (5) have made detailed numerical calculations of orbits in this type of trap, but found that because of rounding-off errors and the computing time necessary, it was not practicable to compute more than about 25 transits of the system. It was to overcome these difficulties caused by the fact that the modulations were sinusoidal that the model with which this thesis is concerned was suggested. This is described in the next section.

1.5 The Square - Wave Modulated Mirror Machine

In this model the sinusoidal modulation of the previous section is replaced by a step function which takes alternately the values $H_0(1 + h)$ and $H_0(1 - h)$ with $h \ll 1$. This is shown in figure 1.3.

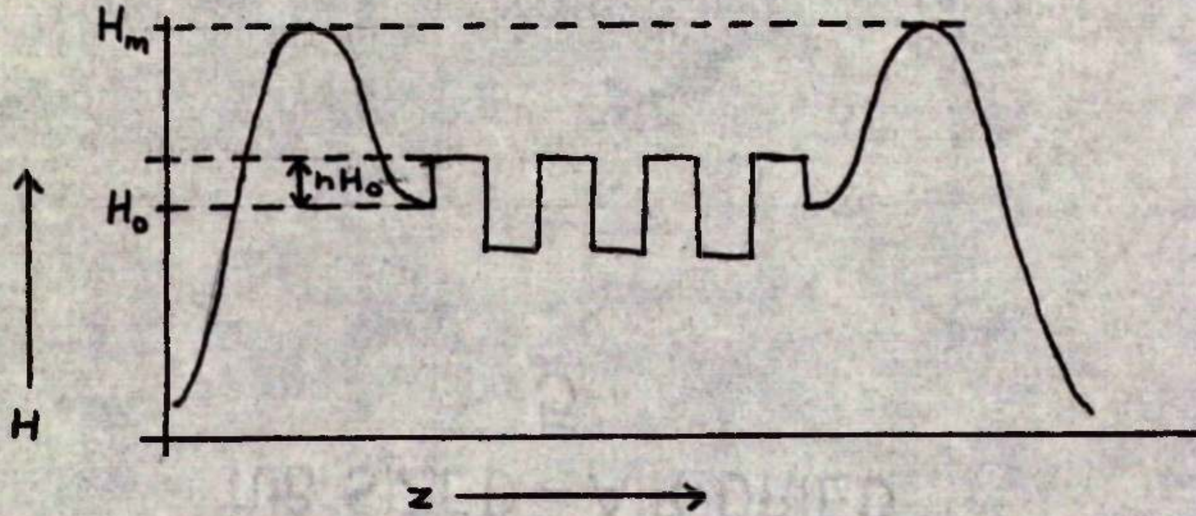


fig 1.3

This system is unphysical, owing to the discontinuities in the field, but it has nevertheless been shown by Dunnett et al. (6) to have closely similar properties to the sinusoidal system. This is because when the characteristic length over which the field varies is already of the order of the Larmor orbit diameter, further reduction in the characteristic length does not significantly alter the effect of the field change on the magnetic moment of the particles (7).

This square wave system has been investigated numerically (6) and the theory of this type of trap has been studied by Dunnett and Laing (8). In this thesis some of this theoretical treatment is given and then the work done in solving numerically the resulting integral equation is described.

CHAPTER 2

THE GENERAL INTEGRAL EQUATION

2.1 Derivation of General Integral Equation

The magnetic mirror machine described in section 1.5 can be represented symbolically by the system shown in figure 2.1.

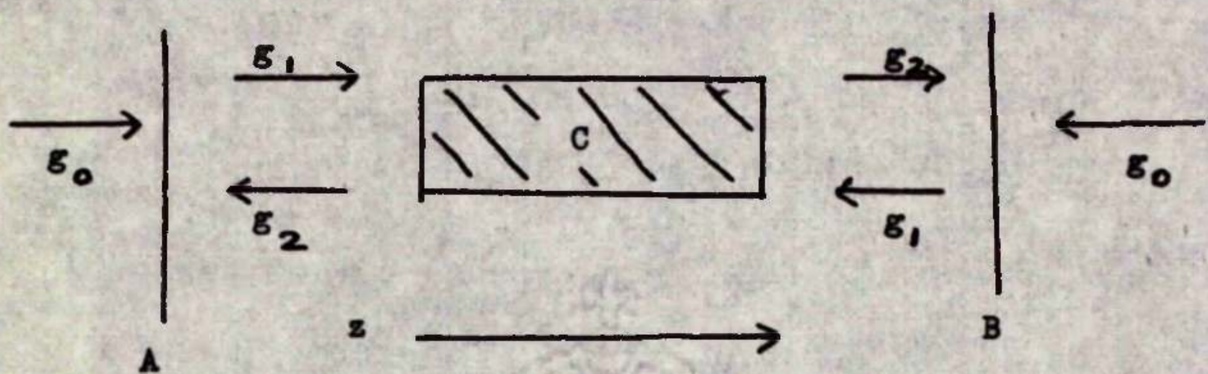


fig 2.1

In this system A and B are magnetic mirrors both with mirror ratio M, and C is some type of symmetric modulation in which the magnetic moment of a particle is changed. This is the system considered in (8)

On coming in contact with a mirror a particle will be transmitted if its magnetic moment is less than 1/M, and will be reflected with its magnetic moment unchanged if it is greater than 1/M.

The quantity $g_0(\xi)d\xi$ is the number of particles per second reaching a mirror with magnetic moment in the range $(\xi, \xi + d\xi)$. This flux is incident from outside the machine and it is assumed that it has such a magnetic moment that it will all be transmitted.

The quantity $g_1(\xi)d\xi$ is the number of particles per second entering the modulation across a plane at right angles to the z-axis in the range $(\xi, \xi + d\xi)$ and $g_2(\xi)d\xi$ is the number of particles per second leaving the modulation in the range $(\xi, \xi + d\xi)$.

The behaviour of a particle in the modulation is dependent not only on its magnetic moment but also on the phase with which it enters the modulation. The phase of a particle is a measure of the position which the particle has reached in its circular orbit. It is therefore necessary to consider the effect which the phase of the particles has on the functions $g_0(\xi)$, $g_1(\xi)$ and $g_2(\xi)$. The phase of the particles forming $g_0(\xi)$ is a matter of choice, as this is an input function, but to calculate the phases of the particles forming $g_1(\xi)$ and $g_2(\xi)$ it is necessary to consider the change of phase of a particle passing through a mirror or being reflected by one.

Laing and Robson have given a method of calculating the phase change on reflection by an adiabatic mirror in an appendix to (5). For a parabolic mirror, which is taken as representative, this method shows that the phase change is proportional to L, the length in which the field doubles in strength. If L is sufficiently large the phase change is a large multiple of 2π, and this is the usual condition for a mirror to be adiabatic.

If this criterion is satisfied then a group of particles with magnetic moment between ξ and $\xi + d\xi$ will have a spread in phase of greater than 2π after reflection at a mirror if the following condition, which is derived in (6), holds

$$L \frac{d\xi}{v} > 4\xi^{\frac{\xi}{2}}(3+\xi)^{-1}$$

In any physical experiment L usually satisfies this condition, as in order that the mirror should be adiabatic L must be reasonably large.

Therefore a small spread in magnetic moment produces phase randomisation on reflection from a mirror. We can therefore assume that the phases of the particles in any flux which has been reflected by a mirror are completely random. This assumption has been made by Wingerson et al. (9) and by Dunnett et al.(6).

Then $g_1(\xi)$ and $g_2(\xi)$ are independent of the phases of the parti particles forming them. We therefore choose that the phases of the particles forming the input flux $g_0(\xi)$ are completely random.

$Q(\xi, \xi')\,d\xi$ is defined to be the probability that a particle entering C with magnetic moment in the range $(\xi, \xi + d\xi)$ will be transmitted through C and emerge with magnetic moment equal to ξ'.

$P(\xi, \xi')\,d\xi$ is defined to be the probability that a particle entering C with magnetic moment in the range $(\xi, \xi + d\xi)$ will be reflected in C and emerge with magnetic moment equal to ξ'.

Then $P(\xi, \xi')\,d\xi + Q(\xi, \xi')\,d\xi$ is the probability that a particle entering C with magnetic moment in the range $(\xi, \xi + d\xi)$ will emerge, having been either transmitted or reflected, with magnetic moment equal to ξ'.

Therefore

$$\int_0^1 \left[P(\xi, \xi') + Q(\xi, \xi') \right] d\xi' = 1$$

Since the system is entirely symmetric, then $P(\xi, \xi') = P(\xi', \xi)$, that is, the probability that a particle entering C with magnetic moment ξ is transmitted with magnetic moment ξ' is the same as the probability that a particle entering C with magnetic moment ξ' is transmitted with magnetic moment ξ.

Also $Q(\xi, \xi') = Q(\xi', \xi)$.

Using these definitions we can write

$$g_1(\xi) = g_0(\xi) + \theta(\xi - 1/M)\, g_2(\xi) \tag{2.1}$$

and
$$g_2(\xi') = \int_0^1 Q(\xi,\xi')\, g_1(\xi)\, d\xi + \int_0^1 P(\xi,\xi')\, g_1(\xi)\, d\xi \qquad (2.2)$$
where
$$\theta(\xi - 1/M) = \begin{cases} 0 & \xi < 1/M \\ 1 & \xi > 1/M \end{cases}$$

Then, substituting for $g_1(\xi)$, (2.2) becomes

$$g_2(\xi') = \int_0^1 \left[P(\xi,\xi') + Q(\xi,\xi')\right] g_0(\xi)\, d\xi + \int_0^1 \left[P(\xi,\xi') + Q(\xi,\xi')\right] \theta(\xi + 1/M)\, g_2(\xi)\, d\xi$$

This can be written

$$g_2(\xi') = G_0(\xi') + \int_0^1 K(\xi,\xi')\, \theta(\xi - 1/M)\, g_2(\xi)\, d\xi \qquad (2.3)$$

where $K(\xi,\xi') = P(\xi,\xi') + Q(\xi,\xi')$

and $G_0(\xi') = \int_0^1 K(\xi,\xi')\, g_0(\xi)\, d\xi$

Incorporating $\theta(\xi - 1/M)$ in the limits of integration (2.3) becomes

$$g_2(\xi') = G_0(\xi') + \int_{\frac{1}{M}}^1 K(\xi,\xi')\, g_2(\xi)\, d\xi \qquad (2.4)$$

This is the general integral equation which has to be solved for the system.

2.2 Calculation of Containment Time

The most useful quantity which can be derived from the solution of (2.4) is the mean containment time of a particle in the machine, and this can be done as follows.

The number of particles in the region between a mirror and the modulation C is

$$\int_0^1 \frac{\ell\, g_2(\xi)}{u(\xi)}\, d\xi \qquad (2.5)$$

where $u(\xi)$ is the axial velocity of a particle with magnetic moment ξ, and ℓ is the length of the region between a mirror and the modulation.

Then (2.5) becomes

$$\int_0^1 \frac{\ell\, g_2(\xi)}{v\,(1-\xi)^{1/2}}\, d\xi$$

The number of particles in a length of the injected beam equal to the length between a mirror and the modulation is ℓ/v if the particles have zero magnetic moment.

Therefore the density increase in the machine is

$$\int_0^1 \frac{g_2(\xi)}{(1-\xi)^{1/2}}\, d\xi$$

This is equivalent to the containment time normalised to τ_0, which is the time a particle would take to traverse the system along the axis if its velocity was all parallel to the field.

Therefore the mean containment time $= \tau_0 \int_0^1 \frac{g_2(\xi)}{(1-\xi)^{1/2}}\, d\xi$

CHAPTER 3

THE KERNEL

3.1 Introduction

As defined in section 2.1, the kernel of the general integral equation (2.4) is

$$K(\xi, \xi') = P(\xi, \xi') + Q(\xi, \xi')$$

Then $K(\xi, \xi')\, d\xi$ is the probability that a particle entering the modulation with magnetic moment in the range $(\xi, \xi + d\xi)$ will leave the modulation with magnetic moment equal to ξ' having been either reflected or transmitted.

To calculate this probability it is necessary to consider in detail the actual form of the modulation. In this chapter the equations necessary to calculate this probability are derived, and the numerical calculation of the probability is described.

3.2 Equations of Motion

These have been derived by Laing and Robson (5) for the sine wave modulation of section 1.4, and by Dunnett, Laing, Robson and Roberts (6) for the square wave modulation of section 1.5.

Using cylindrical polar coordinates (r, θ, z), then the non-adiabatic magnetic field can be represented by the vector potential $\underline{A} = (0, \frac{1}{2} r H_0 f(z), 0)$. For the sine wave model $f(z) = 1 + h \sin z$, and for the square wave model $f(z) = 1 + h$ or $f(z) = 1 - h$ depending on the region of the modulation.

Then the magnetic field is $\underline{H} = (-\frac{1}{2} r H_0 \frac{df}{dz}, 0, H_0 f(z))$.

Then as in (5), the equations in such a field for a charged particle of charge e mass m are

$$\frac{d^2r}{dt^2} = \frac{p_\theta^2}{m^2r^3} - \frac{e^2H_o^2}{4m^2c^2}\, rf^2 \qquad (3.1)$$

$$\frac{d^2z}{dt^2} = -\left\{\frac{p_\theta}{mr} - \frac{erH_of}{2mc}\right\} \frac{erH_o}{2mc}\frac{df}{dz} \qquad (3.2)$$

p_θ, a constant of the motion, is given by

$$p_\theta = mr^2\frac{d\theta}{dt} + \frac{e}{2c}\, r^2H_of \,. \qquad (3.3)$$

By substituting $\tau = \frac{1}{2}\omega t$, where $\omega = \frac{eH_o}{mc}$, (3.3) becomes

$$C = \frac{2p_\theta}{m\omega} = r^2\left(\frac{d\theta}{d\tau} + f\right)$$

therefore C is also a constant of the motion.

Equations (3.1) and (3.2) have as a first integral

$$\left(\frac{dr}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 + \left(\frac{C}{r} - rf\right)^2 = v^2 \qquad (3.4)$$

where $v = \frac{2|\underline{V}|}{\omega}$, if $\underline{V}$ = total velocity of the particle.

These are the general equations for any f_i; in the next section these equations are applied to the square wave model.

3.3 Equations for Square Wave Model

If these equations are applied to the system of figure 1.3, the central modulations are given by

$$f_i = \begin{cases} 1 + h & i \text{ even} \\ 1 - h & i \text{ odd} \end{cases}$$

as in figure (3.1).

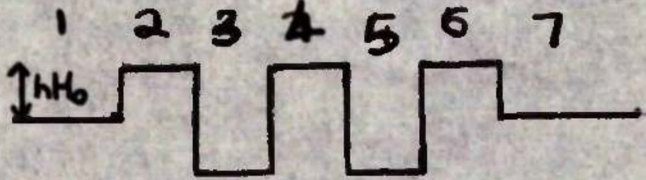


fig 3.1

Then for the i th region the equation for radial motion is

$$\frac{d^2 r}{d\tau^2} = \frac{C^2}{r^3} - r f_i^2$$

and this has the general solution

$$r^2 = \alpha_i + \beta_i \cos(2f_i \tau + \psi_i) \qquad (3.5)$$

where $\alpha_i^2 - \beta_i^2 = C^2/f_i^2$

At the boundary between the i th and the i+1 th regions both r and $\frac{dr}{d\tau}$ are continuous. $\frac{dr}{d\tau}$ is continuous because the discontinuity is a singular radial field, which does not affect motion parallel to it. Then r^2 and $r\frac{dr}{d\tau}$ are continuous.

Applying these last two conditions and using (3.5) gives

$$\left.\begin{aligned} \alpha_i + \beta_i \cos\phi_i &= \alpha_{i+1} + \beta_{i+1} \cos\psi_{i+1} \\ f_i \beta_i \sin\phi_i &= f_{i+1} \beta_{i+1} \sin\psi_{i+1} \end{aligned}\right\} \qquad (3.6)$$

letting $\tau = 0$ on the boundary.

In (3.6) ϕ_i = phase at the end of the i th region

ψ_i = phase at beginning of the i th region

Then $\phi_i = \psi_i + \theta_i$, where θ_i is the phase change in the i th region.

θ_i is given by $\theta_i = \frac{2l_i f_i}{u_i}$ where l_i is the length of the i th region, and u_i is the axial velocity in the i th region. In this treatment the unit of length is $\lambda/2\pi$, where λ is the wavelength of the corresponding sinusoidal modulation. Therefore in order to match the two modulations $l_i = \pi$.

Also $\quad u_i^2 = \left(\frac{dz}{d\tau}\right)^2$

But by (3.4)

$$u_i^2 = v^2 - \left(\frac{C}{r} - rf_i\right)^2 - \left(\frac{dz}{d\tau}\right)^2$$

Differentiating (3.5) with respect to τ this reduces on substitution to

$$u_i^2 = v^2 + 2Cf_i - 2a_i f_i^2$$

Then

$$\theta_i = \frac{2\pi f_i}{\sqrt{v^2 + 2Cf_i - 2a_i f_i^2}} \tag{3.7}$$

Substituting $\zeta_i = \beta_i \cos\psi_i$ and $\eta_i = \beta_i \sin\psi_i$, then equations (3.6) become

$$\left.\begin{aligned} a_{i+1} + \zeta_{i+1} &= a_i + \zeta_i \cos\theta_i - \eta_i \sin\theta_i \\ \eta_{i+1} &= k_i(\zeta_i \sin\theta_i + \eta_i \cos\theta_i) \end{aligned}\right\} \tag{3.8}$$

where $k_i = f_i/f_{i+1}$

If σ_i is defined by

$$\sigma_i = a_i + \zeta_i \cos\theta_i - \eta_i \sin\theta_i$$

then (3.8) gives

$$\left.\begin{aligned} \zeta_{i+1} &= \tfrac{1}{2}\left[\sigma_i - \frac{1}{\sigma_i}\left(\eta_{i+1}^2 + \frac{C^2}{f_{i+1}^2}\right)\right] \\ a_{i+1} &= \tfrac{1}{2}\left[\sigma_i + \frac{1}{\sigma_i}\left(\eta_{i+1}^2 + \frac{C^2}{f_{i+1}^2}\right)\right] \end{aligned}\right\} \tag{3.9}$$

These equations also hold for a particle going through the modulation from right to left, because of the symmetry of the system, but with i + 1 replaced by i - 1.

The normalised magnetic moment ξ of the particle is given by

$$\xi = \frac{v_i^2}{v^2} = 1 - \frac{\left(\frac{dz}{d\tau}\right)^2}{v^2}$$

Then in the i th region

$$\xi_i = 1 - \frac{u_i^2}{v^2}$$

Substituting for u_i this becomes

$$\xi_i = \frac{-2Cf_i + 2a_i f_i^2}{v^2} = \frac{2f_i}{v^2}\left(f_i a_i - C\right)$$

Therefore

$$a_i = \frac{v^2}{2f_i^2}\xi_i + \frac{C}{f_i} \qquad (3.10)$$

Since the change of phase in passing through each region can be calculated from the magnetic moment in that region, it is therefore possible to calculate the magnetic moment of any particle leaving the modulation knowing its magnetic moment and phase on entering the modulation.

3.4 Internal Reflection at a Boundary

Since ξ_i is a normalised magnetic moment, then by definition $\xi_i \leq 1$.

Therefore

$$a_i \leq \frac{v^2}{2f_i^2} + \frac{C}{f_i} \qquad (3.11)$$

Then, if on calculation of some value of a_i from the a of the previous region the value of a_i is greater than $\frac{v^2}{2f_i^2} + \frac{C}{f_i}$; where f_i is the value of f in the i th region, then physically this means that the particle is reflected at one of the internal discontinuities.

Particles are reflected from an internal boundary with an unchanged magnetic moment but with a change in phase. This change of phase is the change produced by the particle travelling through the region before the boundary from which it is reflected. The particle is trapped between two successive discontinuities until its phase is such that it can penetrate through into one of the adjacent regions. It has been postulated that it is this process which traps most of the long-lived particles in the machine.

3.5 Calculation of Kernel

The theory developed in the previous sections makes it possible to calculate the magnetic moment of a particle in any region from the magnetic moment and phase of the particle in the previous region.

For one particular input magnetic moment i.e. the magnetic moment with which the particles enters the modulation, it is possible to calculate the output magnetic moment i.e. the magnetic moment with which the particle leaves the modulation, for each input phase, and the variation of this is shown in figs. F1 and F2. The region in fig. F2 which has output magnetic moment constant and equal to the input magnetic moment corresponds to the phases which are such that, by (3.11), the particle is reflected at the first barrier, and therefore never enters the modulation. This category of particles requires special treatment which will be described in a later section.

For a particle entering the modulation with magnetic moment ξ, what is required is $K(\xi, \xi')$, where $K(\xi, \xi')\, d\xi$ is the probability that the particle entering the modulation with magnetic moment in the range $(\xi, \xi + d\xi)$ will leave the modulation with magnetic moment ξ', and this can be calculated from the graph of ξ' against input phase ϕ. (This is referred to in future as a (ξ', ϕ) diagram.

Suppose that a particle enters the modulation with magnetic moment ξ, and random phase, i.e. all phases have an equal probability of occuring.

Suppose for simplicity that the (ξ', ϕ) diagram of this input ξ is as shown in fig. 3.2.

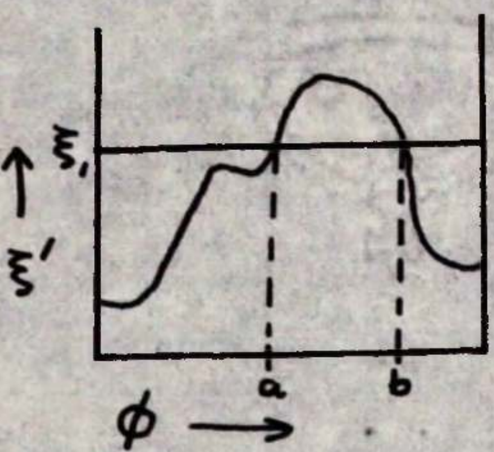


fig. 3.2

$K(\xi, \xi_1)\, d\xi_1$ is the probability that a particle entering with magnetic moment ξ will leave with magnetic moment in the region $(\xi_1, \xi_1 + d\xi_1)$. Defining $PR(\xi, \xi_1)$ as the probability that a particle entering with magnetic moment ξ will leave with magnetic moment greater than ξ_1, then

$$K(\xi, \xi_1)\, d\xi_1 = PR(\xi, \xi_1) - PR(\xi, \xi_1 + d\xi_1)$$

Then

$$K(\xi, \xi_1) = \lim_{d\xi_1 \to 0} - \left[\frac{PR(\xi, \xi_1 + d\xi_1) - PR(\xi, \xi_1)}{d\xi_1} \right]$$

i.e.
$$K(\xi,\xi_1) = -\left[\frac{d\,PR(\xi,\xi')}{d\xi'}\right]_{\xi'=\xi_1} \qquad (3.12)$$

Since all phases have the same probability of occuring, then

$$PR(\xi,\xi_1) = \frac{b-a}{2\pi} \qquad (3.13)$$

We can generalise (3.13) to the case where ξ becomes greater than ξ_1 at $a_1, a_2, \ldots\ldots\ldots a_n$ and becomes less than ξ_1 at $b_1, b_2, \ldots\ldots\ldots b_n$, with $a_1 < b_1 < a_2 < b_2 \ldots\ldots\ldots < a_n < b_n$.

Then

$$PR(\xi,\xi_1) = \frac{b_1-a_1}{2\pi} + \frac{b_2-a_2}{2\pi} + \ldots\ldots\ldots + \frac{b_n-a_n}{2\pi}$$
$$= \sum_{i=1}^{n} \frac{b_i - a_i}{2\pi}$$

$PR(\xi,\xi_1)$ is a function of ξ_1 which varies from 1 when $\xi_1 = 0$ to 0 when $\xi_1 = 1$.

We thus have a procedure for calculating $K(\xi,\xi_1)$ which can be used numerically, and how this is done is described in the following section.

3.6 Method of Numerical Calculation

Using equations (3.7) to (3.10) the output magnetic moment ξ' of a particle for any input magnetic moment ξ and phase $\emptyset$ can be calculated. This is done by first calculating α_1 from the input magnetic moment ξ by 3.10, then calculating $\zeta_1 = \beta_1 \cos\emptyset$ and $\eta_1 = \beta_1 \sin\emptyset$, where $\beta_1^2 = \alpha_1^2 - C^2$, and C is the constant defined in 3.4 and is equal to 4. Θ_1 is also calculated from 3.7. Using these, η_2 and then α_2 are calculated by 3.8 and 3.9. α_2 is compared with $\frac{Y^2}{2Z_2^2} + \frac{C}{Z_2}$, and if it is greater than this the particle is considered to have been reflected at the first boundary. If $a_2 \leq \frac{v^2}{2f_2^2} + \frac{C}{f_2}$, the particle will penetrate into the second region, and ς_2 and θ_2 are calculated. The whole process is repeated until the last boundary is crossed, then $\xi' = \frac{2}{v^2}(a_n - C)$, where a_n is the a in the region after the last boundary. If the particle is reflected at any internal boundary the process is repeated in reverse until the initial region is reached, or another reflection occurs.

Initially a regular interval $d\xi$ of magnetic moment, and a regular interval $d\emptyset$ of phase is chosen, and the output magnetic moment is calculated for ξ going from 0 to 1 by steps of $d\xi$, and for $\emptyset$ going from 0 to 2π by steps of $d\emptyset$. The size of $d\xi$ and $d\emptyset$ can be varied, and in practice they are made progressively smaller until the results obtained converge to a limit. Then for any particular input magnetic moment we have a series of output magnetic moments corresponding to different input phases, and this is what is shown graphically in fig.F1 and fig. F2. For each ξ, $FR(\xi, \xi')$ is calculated for ξ' going from 0 to 1 in steps of $d\xi$, i.e. the values of ξ' are the same as the values of ξ already used. For a fixed ξ the values of output magnetic moment are in ascending order of the phase giving rise to them, i.e. they go from $\emptyset = 0$ to $\emptyset = 2\pi$. Then to calculate one entry of the square matrix $FR(\xi, \xi')$ we consider the magnetic moments for a fixed ξ. The first of these is compared with ξ', and if it is less than ξ' the second is compared with ξ', and so on until an entry is found which is greater than ξ'. Then to find a_1, which is the phase at which the magnetic moment is equal to ξ', linear interpolation is used between the point just reached and the one before it. The comparison is then continued until an entry is found which is less than ξ', and b_1 is found by linear interpolation. This process is continued until the last entry is reached and then $\sum_{i=1}^{n} \frac{b_i - a_i}{2}$ is calculated. If the first entry is greater than ξ', then $a_1 = 0$ is inserted, and if the last entry is greater than ξ' then $b_n = 2\pi$ is inserted. This gives $PR(\xi, \xi')$.

One problem which arises in calculating this probability is what to do if, as in fig. F2, many successive entries give an output equal to the input magnetic moment and to ξ', then since a computer cannot recognise the absolute equality of numbers which are not integers the results obtained do not make sense, as the output cannot be classed strictly as less than or greater than ξ'. To cope with this we calculate $PR(\xi, \xi_1)$ and $PR(\xi, \xi_2)$, where $\xi_1 = \xi' + 0.0001$ and $\xi_2 = \xi' - 0.0001$. We take $PR(\xi, \xi')$ to be $PR(\xi, \xi_1)$ and store the difference

$$PA(\xi) = PR(\xi, \xi_1) - PR(\xi, \xi_2)$$

to be dealt with separately, as this corresponds to the probability that a particle will be reflected without entering the modulation, and therefore with its magnetic moment unchanged.

The result of having calculated $PR(\xi, \xi')$ is a square array of numbers, each row of which (corresponding to a constant input magnetic moment) has first entry equal to 1 and last entry equal to zero. For a fixed ξ, the graph of PR against ξ' is of the form shown in fig. 3.3, and the graphs of various PR's are shown in figs. F3 and F4. What is really required is the derivative of the function at each entry in the row, and this has to be done numerically.

This is done by different methods in different regions of the range of ξ', because of the shape of the curve.

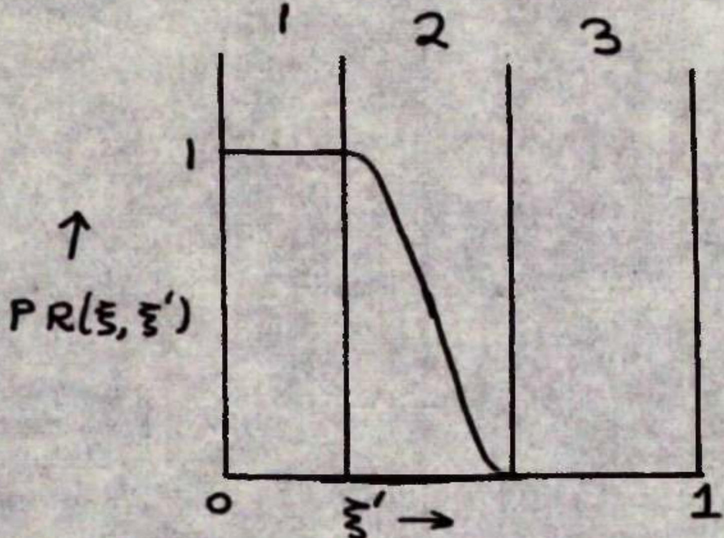


fig. 3.3

In region 1 $PR(\xi,\xi') = 1$ for all ξ', therefore

$$K(\xi,\xi') = -\frac{d\,PR(\xi,\xi')}{d\xi'} = 0$$

Similarly, in region 3, $PR(\xi,\xi') = 0$ for all ξ', therefore $K(\xi,\xi') = 0$. In region 2 $K(\xi,\xi')$ is calculated using the first three terms of Stirling's formula, as given on page 208 of (11). Various formulae were considered, but for this particular shape of curve this formula gave the most rapid convergence for the smallest number of terms. In this way the array $K(\xi,\xi') = -\frac{d\,PR(\xi,\xi')}{d\xi'}$ is formed. This is a square matrix of numbers, with the number of terms in it controlled only by the size of the interval $d\xi$.

3.7 Particles Reflected without entering the Modulation

Section 3.6 describes how, for certain input phases, particles are reflected at the first boundary of the modulation and therefore leave the modulation with magnetic moment unchanged. This phenomenon only occurs at fairly high magnetic moments, unless the step is very high. This range of θ for which output ξ is equal to ξ'

creates a problem, as it produces in effect a discontinuity of height $PA(\xi)$ in the graph of $PR(\xi, \xi')$ for this value of ξ, as shown in fig. F4, and complicates the numerical differentiation. This is solved in the following way.

$PA(\xi)$ is the probability that a particle with magnetic moment ξ will be reflected without entering the modulation and with magnetic moment unchanged.

We define $PR'(\xi, \xi')$ so that $PR'(\xi, \xi')\, d\xi$ is the probability that a particle which enters the modulation with magnetic moment in the region $(\xi, \xi + d\xi)$ leaves it with magnetic moment greater than ξ'.

Then for $\xi' < \xi$

$$PR'(\xi, \xi') = PR(\xi, \xi') - PA(\xi)$$

This removes the discontinuity in $PR(\xi, \xi')$, and the curve $PR'(\xi, \xi')$ has the same derivative as $PR(\xi, \xi')$ with respect to ξ', except at the point $\xi = \xi'$.

Also, a new matrix $R(\xi, \xi')$ is defined by

$R(\xi, \xi')\, d\xi'$ is the probability that a particle which enters the modulation with magnetic moment ξ will be reflected or transmitted with magnetic moment in the range $(\xi', \xi' + d\xi')$.

Then the entries of R are given by

$$R(\xi, \xi') = - \frac{d\, PR'(\xi, \xi')}{d\xi'}$$

This matrix is the same as $K(\xi, \xi')$, with the exception of the diagonal terms, which are connected by the equation

$$R(\xi, \xi) + PA(\xi) = K(\xi, \xi).$$

In this way $K(\xi, \xi')$ can be found for all values of ξ and ξ', and we can proceed to solve the general integral equation 2.4 for any particular input parameters. H ow this is done is described in the next chapter.

3.8 Normalisation of the Kernel

As a particle which enters the modulation with magnetic moment ξ must emerge with some value of magnetic moment, and conversely a particle emerging from the modulation with magnetic moment ξ must have entered with some value of magnetic moment, then the following normalisation conditions must hold for K.

$$\int_0^1 K(\xi, \xi')\, d\xi' = 1 \qquad (3.14)$$

$$\int_0^1 K(\xi, \xi')\, d\xi = 1 \qquad (3.15)$$

These can be checked for the calculated kernels using various techniques for numerical integration, and they have been found to be approximately correct, but it is important for the solution of the integral equation that this normalisation should be exact.

It is impossible to achieve complete normalisation of both rows and columns of the matrix, but it will be shown in the next chapter that it is more important that (3.14) should be satisfied than that (3.15) should be satisfied.

Satisfying equation (3.14) is equivalent to normalising the rows of the matrix $K(\xi, \xi')$.

The method of normalisation is to integrate the row using the trapezoidal rule, as this is the integration procedure used in the subsequent work. The reason for choosing this integration procedure is explained in the next chapter. All the entries in the row are then divided by the integral of the row. Equation (3.14) is then satisfied.

3.9 Results

The kernel has been calculated for many values of the initial parameters, but to show the general characteristics we choose a typical case. This is the system shown in fig. 3.4 with n = 11.

fig. 3.4

We choose the other initial parameters h = 0.042, v = 2.21. Calculations have shown that the kernel has converged when $d\xi \leq 0.025$ and $d\phi \leq 0.1$.

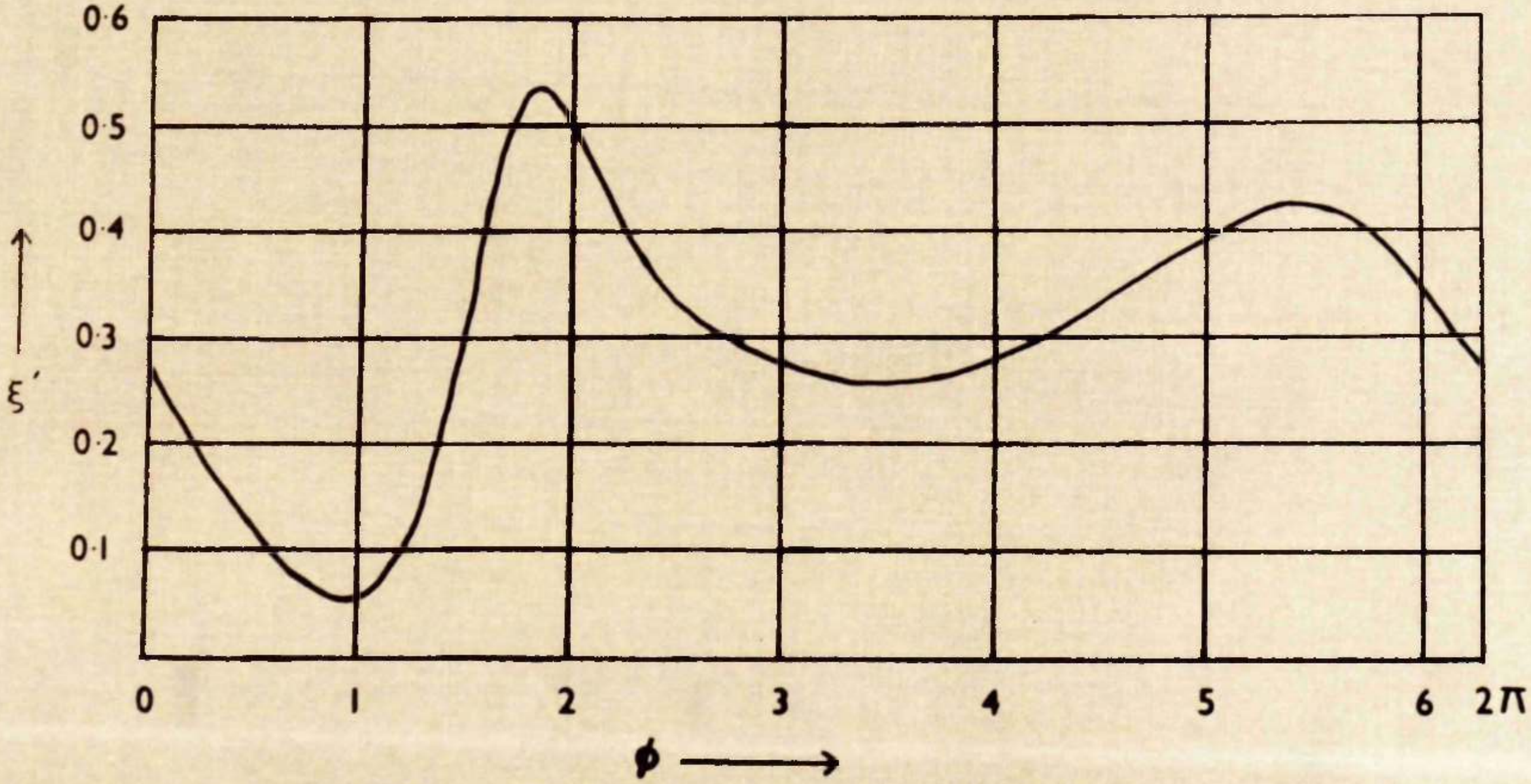
Figs. F1 and F2 show two typical sets of output ξ''s for a fixed input ξ and varying input phase ϕ.

Figs. F3 and F4 show FR (ξ, ξ') for the same input ξ's as in F1 and F2.

Figs. F5, F6 and F7 show three rows of the final kernel R (ξ, ξ') for fixed input ξ's.

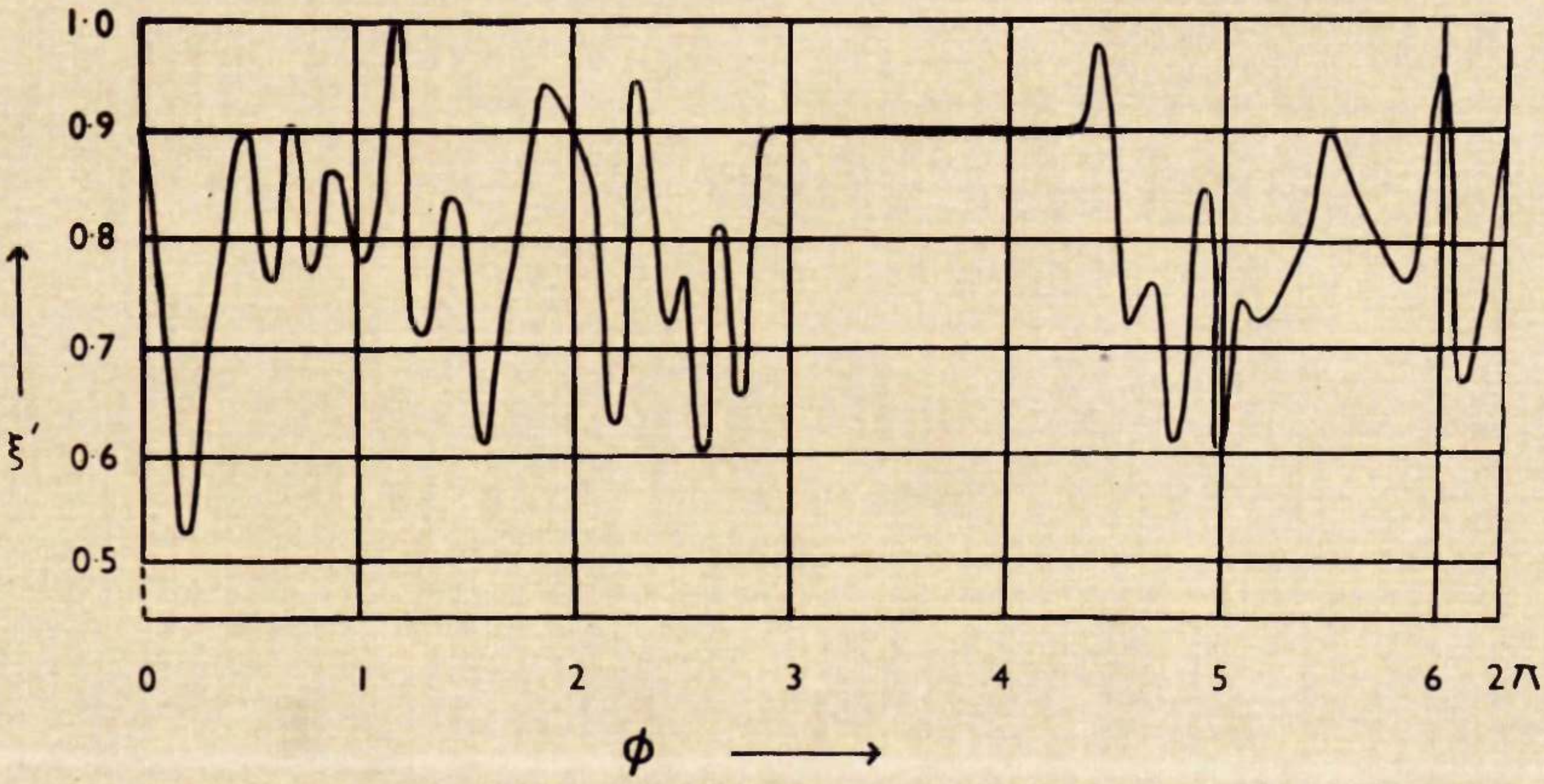
Now that we have developed a technique for calculating the kernel for any set of input parameters we can proceed to use this in the solution of the integral equation derived in chapter 2. This is done in the next chapter.

FIGURE F1



NORMALISED MAGNETIC MOMENT ξ'
VS INPUT PHASE ϕ
$h = 0{\cdot}042$, $n = 11$, $\xi = 0{\cdot}1$

FIGURE F2

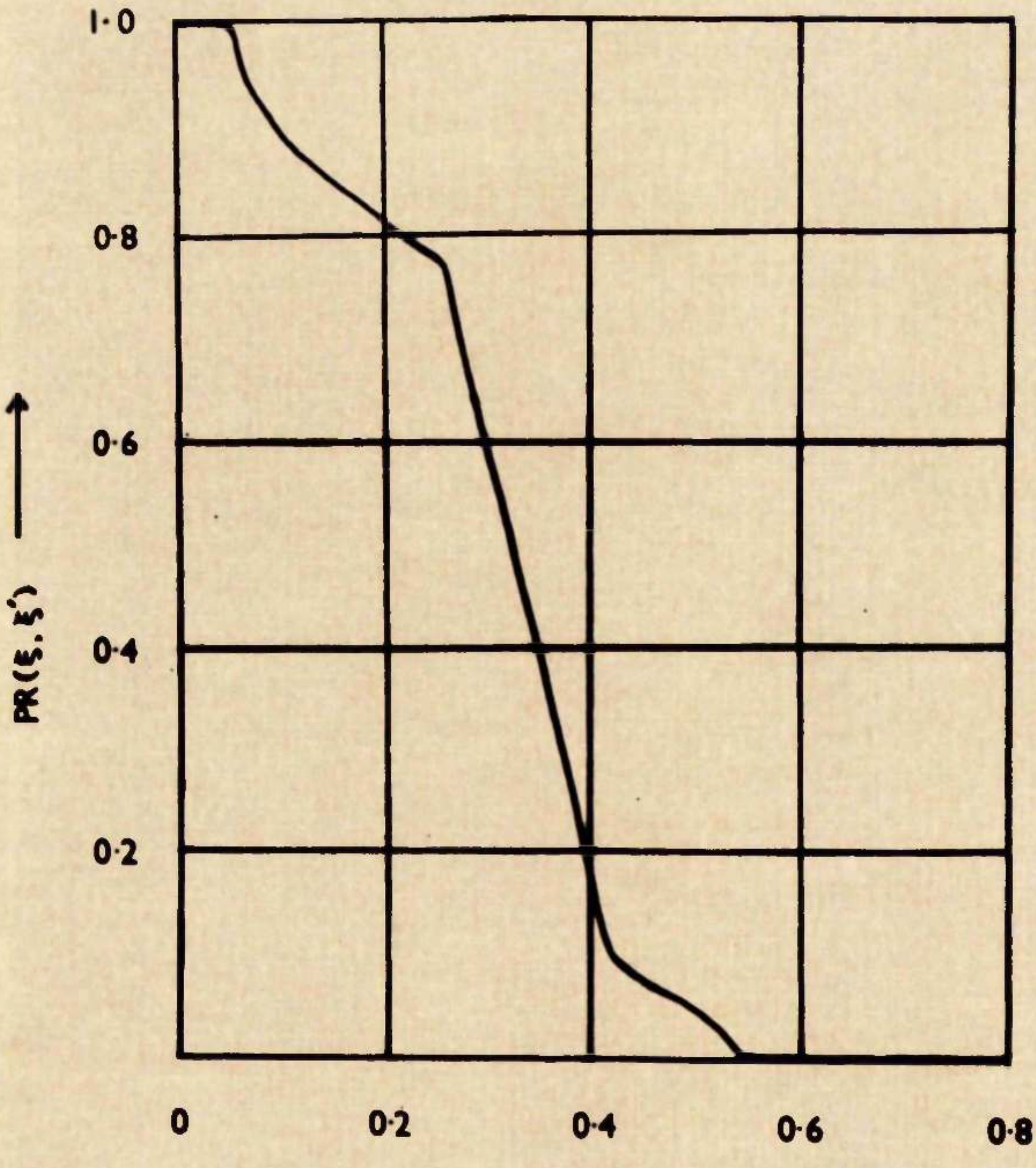


NORMALISED MAGNETIC MOMENT ξ'

VS INPUT PHASE ϕ

$h = 0{\cdot}042$, $n = 11$, $\xi = 0{\cdot}9$

FIGURE F3

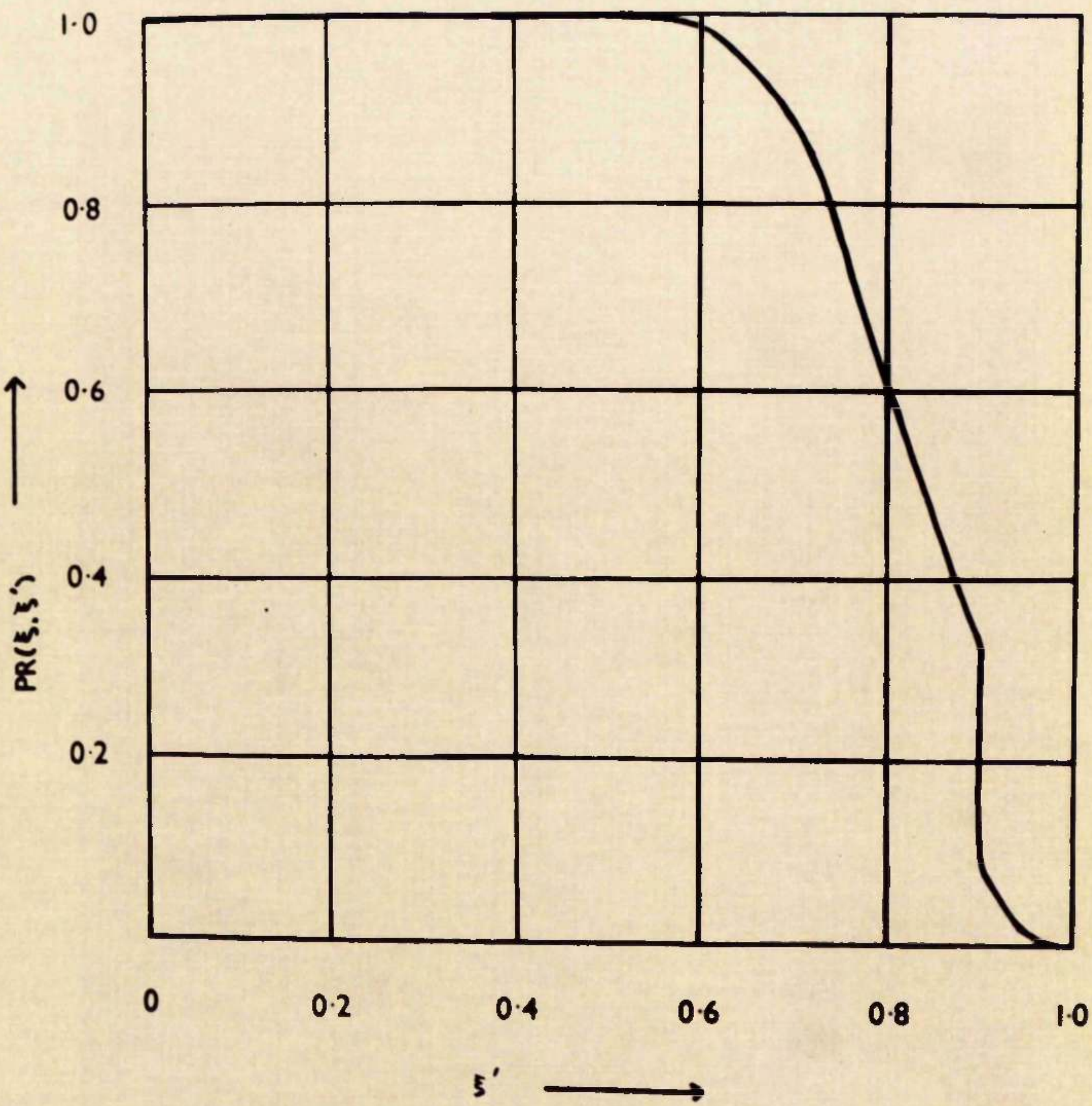


PROBABILITY PR(ξ, ξ′)

VS OUTPUT MAGNETIC MOMENT ξ′

$h = 0{\cdot}042$, $n = 11$, $\xi = 0{\cdot}1$

FIGURE F4

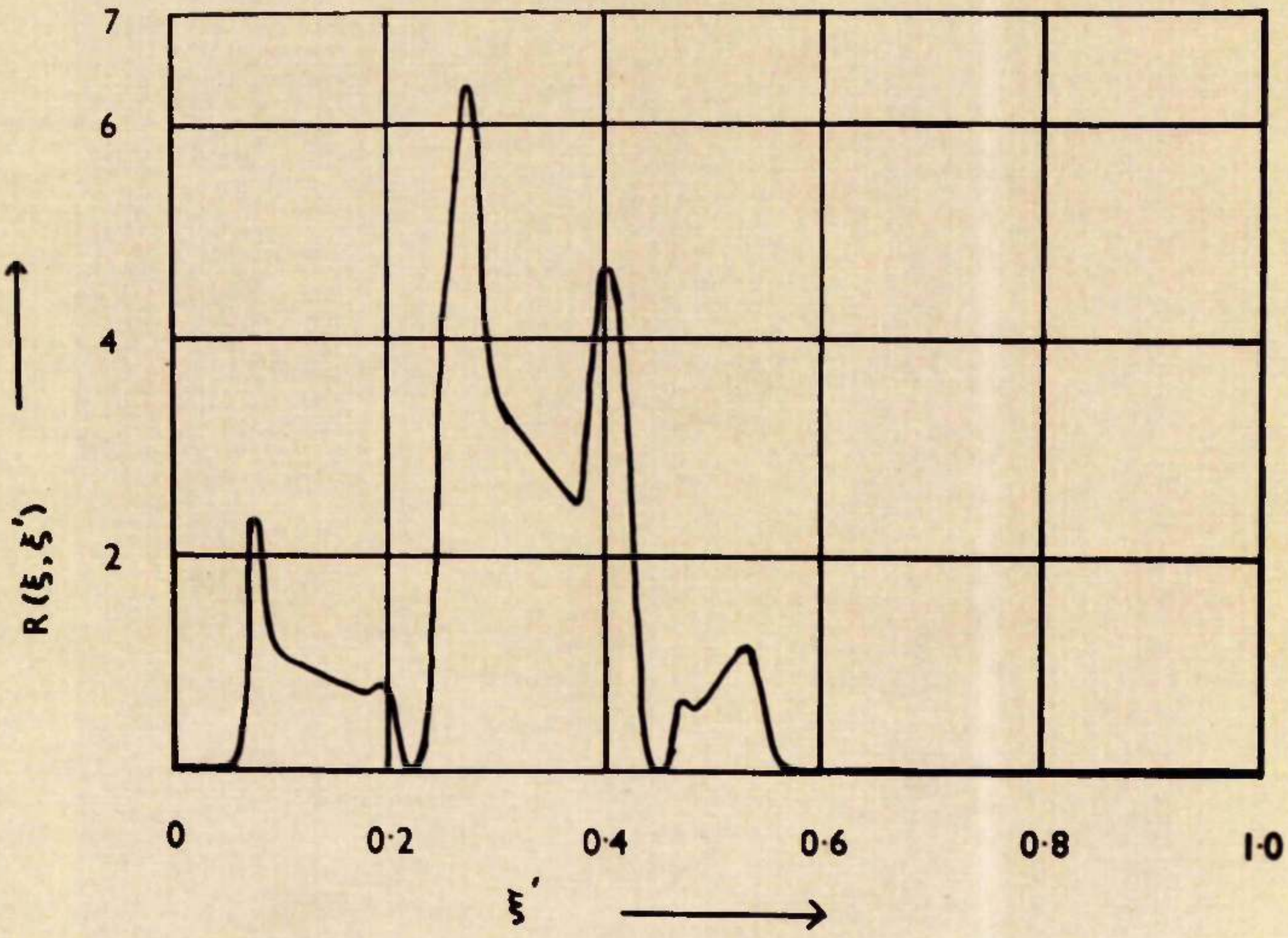


PROBABILITY PR (ξ . ξ')

VS OUTPUT MAGNETIC MOMENT ξ'

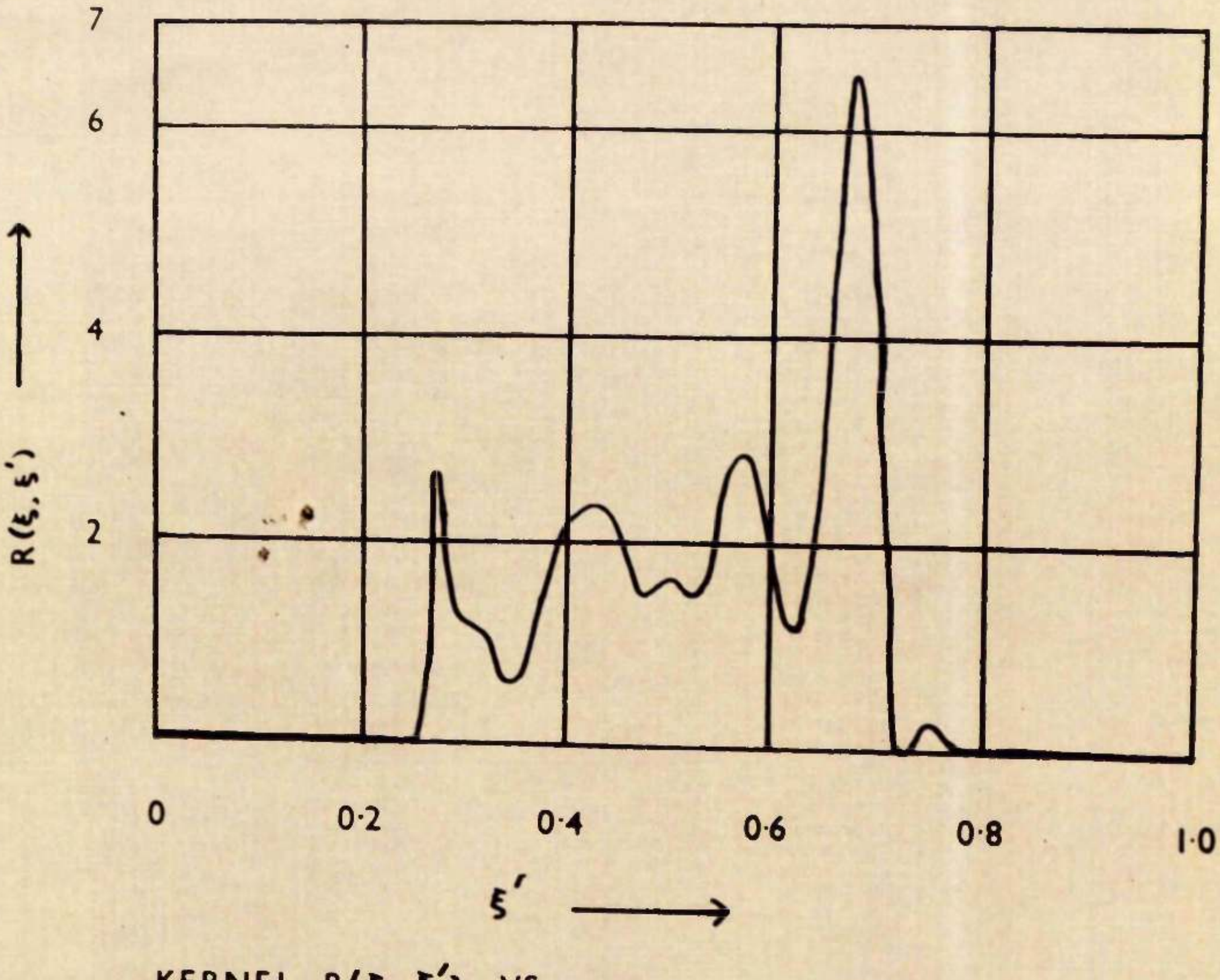
$h = 0{\cdot}042$, $n = 11$, $\xi = 0{\cdot}9$

FIGURE F5



KERNEL $R(\xi . \xi')$ VS
OUTPUT MAGNETIC MOMENT ξ'
$h = 0{\cdot}042$, $n = 11$, $\xi = 0{\cdot}1$

FIGURE F6

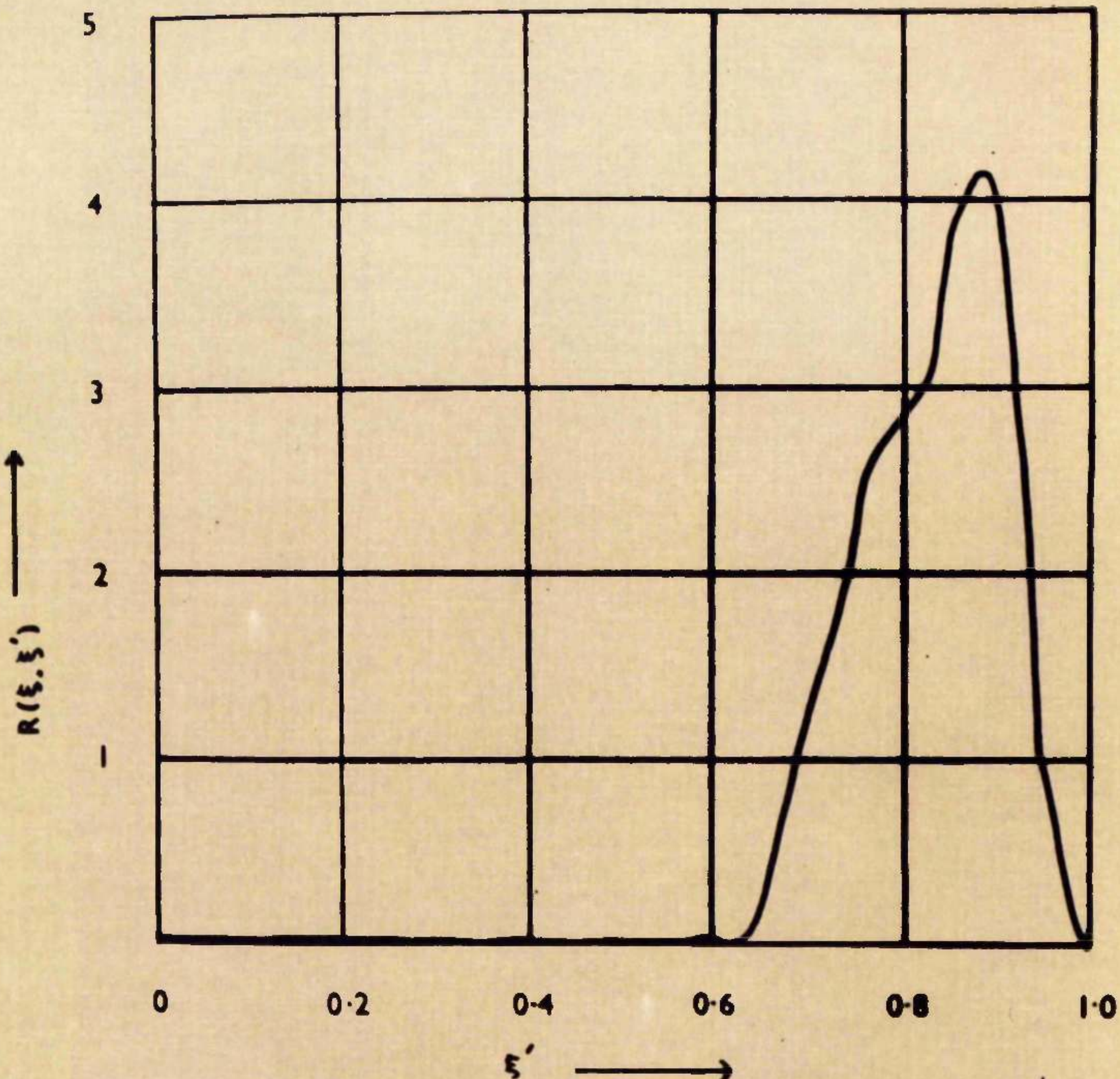


KERNEL R(ξ, ξ') VS

OUTPUT MAGNETIC MOMENT ξ'

$h = 0{\cdot}042$, $n = 11$, $\xi = 0{\cdot}55$

FIGURE F7



KERNEL R(ξ, ξ′) VS

OUTPUT MAGNETIC MOMENT ξ′

h = 0 042 , n = 11 , ξ = 0·9

CHAPTER 4

SOLUTION OF THE GENERAL INTEGRAL EQUATION

In chapter 2 the general integral equation for this non-adiabatic mirror machine was derived, and in this chapter the method which was used to solve it is described.

4.1 Solution of the Homogeneous Equation

The general integral equation for the system is

$$g_2(\xi') = G_o(\xi') + \int_{\frac{1}{M}}^{1} K(\xi,\xi')\, g_2(\xi)\, d\xi \qquad (4.1)$$

where $g_2(\xi')\, d\xi'$ is the number of particles leaving the modulation in the range $(\xi', \xi' + d\xi')$.

We solve first the equation

$$g_2(\xi') = \int_{\frac{1}{M}}^{1} K(\xi,\xi')\, g_2(\xi)\, d\xi \qquad (4.2)$$

Since $K(\xi,\xi')$ is a symmetric matrix the solution of the general equation (4.1) can be calculated quite simply from the solution of this equation. The connection between these two solutions is derived in the next section.

The method used in the solution of this equation is that described on page 442 of (10). This is the method of finding characteristic values or eigenvalues, and characteristic functions or eigenfunctions, for the kernel. This method discovers the characteristic value with the smallest absolute value and then, by subtracting out the corresponding eigenfunction gives successively larger eigenvalues.

If the eigenvalues are $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_r, \ldots$ arranged in order, with $\lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots\ldots < \lambda_r < \ldots\ldots$ and the corresponding eigenfunctions are $g^{(1)}(\xi), g^{(2)}(\xi), \ldots\ldots\ldots, g^{(r)}(\xi), \ldots\ldots\ldots$ then any trial function $y_1(\xi)$ can be expressed as a linear combination of these eigenfunctions plus a function $p(\xi)$ which satisfies

$$\int_{\frac{1}{m}}^{1} K(\xi, \xi')\, p(\xi)\, d\xi = 0$$

Then
$$y_1(\xi) = \sum_{r=1}^{\infty} a_r g^{(r)}(\xi) + p(\xi)$$

where each $g^{(r)}(\xi)$ satisfies the integral equation

$$g^{(r)}(\xi') = \lambda_r \int_{\frac{1}{m}}^{1} K(\xi, \xi')\, g^{(r)}(\xi)\, d\xi \qquad (4.3)$$

Substituting this trial function into the right-hand side of equation (4.2), this gives a new trial function

$$y_2(\xi') = \sum_{r=1}^{\infty} \int_{\frac{1}{m}}^{1} K(\xi, \xi')\, a_r g^{(r)}(\xi)\, d\xi$$

using equations (4.3) this gives

$$y_2(\xi) = \sum_{r=1}^{\infty} \frac{a_r g^{(r)}(\xi)}{\lambda_r}$$

Continuing this process of substitution gives, on the n th iteration

$$y_n(\xi) = \sum_{r=1}^{\infty} \frac{a_r g^{(r)}(\xi)}{(\lambda_r)^n}$$
$$= \frac{1}{(\lambda_1)^n} \left(a_1 g^{(1)}(\xi) + \left(\frac{\lambda_1}{\lambda_2}\right)^n a_2 g^{(2)}(\xi) + \left(\frac{\lambda_1}{\lambda_3}\right)^n a_3 g^{(3)}(\xi) + \ldots \right)$$

As n increases then $\left(\frac{\lambda_1}{\lambda_r}\right)^n$ tends to zero for all $r > 1$.

Then the ratio $\frac{y_n(\xi)}{y_{n+1}(\xi)}$ tends to λ_1.

A more efficient ratio to calculate λ_1 is

$$\lambda_1 = \frac{\int_{\frac{1}{M}}^{1} y_n(\xi)\, y_{n+1}(\xi)\, d\xi}{\int_{\frac{1}{M}}^{1} \left[y_{n+1}(\xi) \right]^2 d\xi} \quad .$$

which can be derived using a variational principle, and this is the ratio which was used in this problem.

When λ_1 has converged to the accuracy which is needed then the eigenfunction corresponding to it is normalised. The normalised eigenfunction is known as $y^{(1)}(\xi)$. This eigenfunction is subtracted out from the trial function and the whole process is repeated, this time giving λ_2. It is necessary to take the additional precaution of subtracting out any remnants of $y^{(1)}(\xi)$ from each successive iteration produced.

The process is repeated to give successive eigenvalues and corresponding eigenfunctions until enough eigenvalues have been obtained. This depends entirely on the relative magnitudes of the eigenvalues, as will be shown in the next section.

This method becomes successively more clumsy as the number of eigenfunctions to be subtracted out increases. At this stage the method is not accurate, and the number of iterations necessary for convergence increases greatly at each step. It therefore takes too much computer time to calculate more than about 10 eigenvalues, but this is usually considerably more than is necessary.

Before proceeding, it is necessary to show why it is more important that (3.14) should be satisfied than (3.15).

$y^{(n)}(\xi)$ satisfies the integral equation

$$y^{(n)}(\xi') = \lambda_n \int_{\frac{1}{M}}^{1} K(\xi, \xi')\, y^{(n)}(\xi)\, d\xi$$

Integrating this from 0 to 1 with respect to ξ'

$$\int_0^1 y^{(n)}(\xi')\,d\xi' = \lambda_n \int_0^1 d\xi' \int_{\frac{1}{M}}^1 K(\xi,\xi')\, y^{(n)}(\xi)\, d\xi$$

$$= \lambda_n \int_{\frac{1}{M}}^1 y^{(n)}(\xi)\, d\xi \int_0^1 K(\xi,\xi')\, d\xi'$$

Then λ_n is inversely proportional to $\int_0^1 K(\xi,\xi')\,d\xi'$.

Therefore the eigenvalues of (4.2) are directly dependent on $\int_0^1 K(\xi,\xi')\,d\xi'$, and so we ensure that this integral is equal to 1.

It can also be shown in the following way that the smallest eigenvalue λ_1 is greater than 1.

The eigenfunction $g^{(1)}(\xi)$ satisfies the equation

$$g^{(1)}(\xi') = \lambda_1 \int_{\frac{1}{M}}^1 K(\xi,\xi')\, g^{(1)}(\xi)\, d\xi$$

Then integrating this over ξ' from 0 to 1

$$\int_0^1 g^{(1)}(\xi')\,d\xi' = \lambda_1 \int_0^1 d\xi' \int_{\frac{1}{M}}^1 K(\xi,\xi')\, g^{(1)}(\xi)\, d\xi.$$

Changing the order of integration

$$\int_0^1 g^{(1)}(\xi')\,d\xi' = \lambda_1 \int_{\frac{1}{M}}^1 g^{(1)}(\xi)\,d\xi \int_0^1 K(\xi,\xi')\,d\xi'.$$

But $\int_0^1 K(\xi,\xi')\,d\xi' = 1$,

so $\int_0^1 g^{(1)}(\xi')\,d\xi' = \lambda_1 \int_{\frac{1}{M}}^1 g^{(1)}(\xi)\,d\xi.$

Now $g^{(1)}(\xi)$ must be positive, as the trial function used to obtain it is always positive, and also the kernel is positive. Therefore, if $g^{(1)}(\xi)$ is positive for every value of ξ, then

$$\int_0^1 g^{(1)}(\xi)\,d\xi \geqslant \int_{\frac{1}{M}}^1 g^{(1)}(\xi)\,d\xi.$$

Therefore

$$\lambda_1 = \frac{\int_0^1 g^{(1)}(\xi)\, d\xi}{\int_{\frac{1}{m}}^1 g^{(1)}(\xi)\, d\xi} \geqslant 1$$

Then $\lambda_1 > 1$ unless $\int_0^{\frac{1}{m}} g^{(1)}(\xi)\, d\xi = 0$

4.2 Solution of General Integral Equation

This method is described in (10). If the eigenvalues of (4.2) are $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n, \ldots$ with corresponding normalised eigenfunctions $y^{(1)}(\xi)$, $y^{(2)}(\xi)$, ..., $y^{(n)}(\xi)$, ..., then the general solution of (4.2) is

$$g_2(\xi) = \sum_{n=1}^{\infty} a_n y^{(n)}(\xi).$$

$a_n = \int_{\frac{1}{m}}^1 g_2(\xi)\, y^{(n)}(\xi)\, d\xi$, since the eigenfunctions are orthogonal and normalised, i.e. they satisfy

$$\int_{\frac{1}{m}}^1 y^{(i)}(\xi)\, y^{(j)}(\xi)\, d\xi = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Equation (4.1) is

$$g_2(\xi') = G_0(\xi') + \int_{\frac{1}{m}}^1 K(\xi, \xi')\, g_2(\xi)\, d\xi.$$

Then

$$g_2(\xi') - G_0(\xi') = \int_{\frac{1}{m}}^1 K(\xi, \xi')\, g_2(\xi)\, d\xi.$$

The solution of this is

$$g_2(\xi') - G_0(\xi') = \sum_{n=1}^{\infty} a_n y^{(n)}(\xi') \qquad (4.4)$$

where

$$a_n = \int_{\frac{1}{m}}^1 \left[g_2(\xi') - G_0(\xi')\right] y^{(n)}(\xi')\, d\xi'.$$

Let $\quad e_n = \int_{\frac{1}{m}}^1 g_2(\xi')\, y^{(n)}(\xi')\, d\xi'$

and $\quad f_n = \int_{\frac{1}{m}}^1 G_0(\xi')\, y^{(n)}(\xi')\, d\xi'$

Then $\qquad a_n = c_n - f_n \qquad\qquad (4.5)$

Multiplying (4.1) by $y^{(n)}(\xi')$ and integrating from 1/M to 1 with respect to ξ', it becomes

$$\int_{\frac{1}{M}}^{1} g_2(\xi')\, y^{(n)}(\xi')\, d\xi' = \int_{\frac{1}{M}}^{1} G_o(\xi')\, y^{(n)}(\xi')\, d\xi' + \int_{\frac{1}{M}}^{1} y^{(n)}(\xi') \left[\int_{\frac{1}{M}}^{1} K(\xi,\xi')\, g_2(\xi)\, d\xi \right] d\xi' \qquad (4.6)$$

Changing the order of integration in the last integral, and using the relationship

$$y^{(n)}(\xi) = \lambda_n \int_{\frac{1}{M}}^{1} K(\xi,\xi')\, y^{(n)}(\xi')\, d\xi'$$

the equation (4.6) becomes

$$c_n = f_n + \frac{c_n}{\lambda_n} \qquad (4.7)$$

Eliminating c_n between (4.5) and (4.7) gives

$$a_n = \frac{1}{\lambda_n - 1} f_n$$

On substituting this value for a_n in (4.4) the solution becomes

$$g_2(\xi) = G_o(\xi) + \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - 1} y^{(n)}(\xi) \qquad (4.8)$$

where $\quad f_n = \int_{\frac{1}{M}}^{1} G_o(\xi)\, y^{(n)}(\xi)\, d\xi.$

This gives the complete solution of the general integral equation (4.1) in terms of the eigenfunctions of the homogeneous integral equation (4.2).

4.3 Method of Solution of (4.1)

The integral on the right hand side of equation (4.1) is an integral over the range 1/M to 1 and the function $g_2(\xi')$ is defined over the total range 0 to 1. Consideration of the integral equation shows that the values of $g_2(\xi')$ for ξ' less than 1/M are entirely dependent on the values of $g_2(\xi')$ for ξ' greater than 1/M. Because of this the equation (4.1) can be divided into two parts, one for ξ' greater than 1/M and the other for ξ' less than 1/M.

We can define

$$g_2(\xi') = g_3(\xi) + g_4(\xi')$$

where

$$g_3(\xi') = \begin{cases} g_2(\xi') & 0 \le \xi' < 1/M \\ 0 & 1 \ge \xi' \ge 1/M \end{cases}$$

and

$$g_4(\xi') = \begin{cases} g_2(\xi') & 1/M \le \xi' \le 1 \\ 0 & 0 \le \xi' < 1/M \end{cases}$$

$G_0(\xi')$ can be similarly subdivided into

$$G_0(\xi') = G_3(\xi') + G_4(\xi')$$

where

$$G_3(\xi') = \begin{cases} G_0(\xi') & 0 \le \xi' < 1/M \\ 0 & 1/M \le \xi' \le 1 \end{cases}$$

and

$$G_4(\xi') = \begin{cases} G_0(\xi') & 1/M \le \xi' \le 1 \\ 0 & 0 = \xi' < 1/M \end{cases}$$

Then equation (4.1) becomes

$$g_3(\xi') + g_4(\xi') = G_3(\xi') + G_4(\xi') + \int_{\frac{1}{M}}^{1} K(\xi, \xi') \left[g_3(\xi) + g_4(\xi) \right] d\xi. \tag{4.9}$$

But $\int_{\frac{1}{M}}^{1} K(\xi, \xi')\, g_3(\xi)\, d\xi = 0$, since $g_3(\xi) = 0$ over the range of integration, so (4.9) becomes

$$g_3(\xi') + g_4(\xi') = G_3(\xi') + G_4(\xi') + \int_{\frac{1}{M}}^{1} K(\xi,\xi')\, g_4(\xi)\, d\xi \qquad (4.10)$$

If $1/M \leq \xi' \leq 1$, (4.10) becomes

$$g_4(\xi') = G_4(\xi') + \int_{\frac{1}{M}}^{1} K(\xi,\xi')\, g_4(\xi)\, d\xi \qquad (4.11)$$

and if $0 \leq \xi' < 1/M$, (4.10) becomes

$$g_3(\xi') = G_3(\xi') + \int_{\frac{1}{M}}^{1} K(\xi,\xi')\, g_4(\xi)\, d\xi \qquad (4.12)$$

From (4.12) it can be seen that the solution for ξ' in the range (0, 1/M) is entirely dependent on the solution for ξ' in the range (1/M, 1).

In practice the eigenvalues and eigenfunctions of the homogeneous equation corresponding to (4.11)

i.e. $$g_4(\xi') = \int_{\frac{1}{M}}^{1} K(\xi,\xi')\, g_4(\xi)\, d\xi$$

are found, and then the solution of (4.11) calculated using the formula (4.8). From this solution $g_3(\xi')$ is calculated using (4.12). This gives the total solution of $g_2(\xi')$ for all ξ' in the range (0,1).

From this solution the mean containment time of the particles in the modulation can be calculated by means of the formula derived in section 2.2.

4.4 The Source Function $g_o(\xi)$

There remains the problem of the initial input distribution function $g_o(\xi)$, and the function $G_o(\xi)$ which is derived from it.

In section 2.1 we made the restriction that $g_o(\xi)$ consists of particles which will all enter the modulation. Then $g_o(\xi) = 0$ for $\xi \geq 1/M$. Also, it has been shown in (5) that the maximum gain in magnetic moment at each step for a particle going through the modulation is of the order of h. Therefore if a particle has too low a magnetic moment on entering the modulation then it may not gain sufficient magnetic moment to be trapped, and may therefore pass straight through the machine.

The choice of the distribution function is arbitrary, but usually we choose a unit flux function with a fixed magnetic moment and completely random phase. This fixed magnetic moment ξ_1 is less than 1/M, but is high enough for some of the particles to reach, in one transit of the system, a magnetic moment such that they are trapped.

Then $$g_o(\xi) = \delta(\xi - \xi_1)$$

where $\delta(\xi - \xi_1)$ is the Dirac Delta Function.

$$G_o(\xi') = \int_0^1 K(\xi,\xi')\, g_o(\xi)\, d\xi$$

then $$G_o(\xi') = K(\xi_1,\xi')$$

From this, each of the f_n can be calculated.

4.6 Numerical Solution of the General Integral Equation

A trial function is chosen, and the iterative procedure described in section 4.1 is used to find successive approximations to the first eigenvalue λ_1. The trial function used is $y_1(\xi) = \xi$ This is considered to have converged sufficiently when the difference between the λ's produced by two successive iterations is less than 10^{-10}. The eigenfunction corresponding to this eigenvalue is then normalised, and subtracted out from the trial function, and the process repeated to give the second eigenvalue.

This process can be repeated to give as many eigenvalues as are desired, but in practice no more than about 5 are necessary. For example, for a typical case with M = 10, h = 0.042, n = 11, and v = 2.21, then the first five eigenvalues are

λ_1 = 1.02489

λ_2 = 1.19638

λ_3 = 1.49471

λ_4 = 1.65988

λ_5 = 2.14303

The term depending on λ_n is proportional to $\frac{1}{\lambda_n - 1}$, so that it is the relative magnitudes of $\frac{1}{\lambda_n - 1}$ for increasing n which is important. In this case these ratios are

40.17; 5.093; 2.026; 1.516; 0.8748.

Therefore the relative importance of the terms drops off sharply with each successive eigenvalue.

Another problem which arises in this numerical calculation is the choice of an integration procedure. Some of the quantities to be integrated have sharp peaks at certain points, and therefore any integration procedure which weights points unequally is liable to give answers which depend on the choice of weighting factors. Simpson's formula (see page 234 of (11)) which uses unequal weighting was rejected for this reason after it had proved to be numerically impracticable to obtain a fine enough mesh to overcome the problem. It was finally found by experiment that the procedure which gave the most consistent results was the simple trapezoidal rule (see page 232 of (11)). Since this method of integration was used in the solution of the integral equation it was also chosen as the integration procedure for normalising the kernel, in order to obtain consistency between the two parts of the solution. Since the first eigenvalue is usually very close to 1, anything which changes the value of λ_1 slightly makes correspondingly a much larger change in $\frac{1}{\lambda_1 - 1}$. Since the total solution depends on this ratio it is extremely important that the normalisation, which affects λ_1, should be correct.

There remains the problem of integrating $K(\xi, \xi') g(\xi)$ with respect to ξ, since

$$K(\xi, \xi') = R(\xi, \xi') + Ph(\xi) \delta(\xi - \xi')$$

where $\delta(\xi - \xi')$ is the Dirac Delta function.

Then

$$\int_{\frac{1}{M}}^{1} K(\xi, \xi') g(\xi) d\xi = \int_{\frac{1}{M}}^{1} \left[R(\xi, \xi') + Ph(\xi) \delta(\xi - \xi') \right] g(\xi) d\xi$$

$$= \int_{\frac{1}{M}}^{1} R(\xi, \xi') g(\xi) d\xi + Ph(\xi') g(\xi') \qquad (4.13)$$

Then (4.13) gives the method of integration of $K(\xi, \xi') g(\xi)$.

From the solution of the general integral equation the mean containment time of particles in the machine is calculated, and it is this mean containment time which can be compared with the results obtained by other methods of dealing with the system. The results of this treatment and those with which they can be compared are given in the next chapter.

TABLE 1

Comparison of results from this treatment with those from the statistical treatment

$\xi_1 = 0 \cdot 5$

n_1	n_2	h	v	τ_1	τ_2
11	10	0.042	2.21	30.5 τ_0	27 τ_0
7	6	0.064	2.16	23.1 τ_0	22 τ_0
15	14	0.033	2.24	37.5 τ_0	42 τ_0

n_1 = no. of steps in this treatment

n_2 = no. of steps in statistical treatment

τ_1 = mean containment time in this treatment

τ_2 = mean containment time in statistical treatment

TABLE 2

Comparison of results from this treatment with those from the diffusion equation treatment

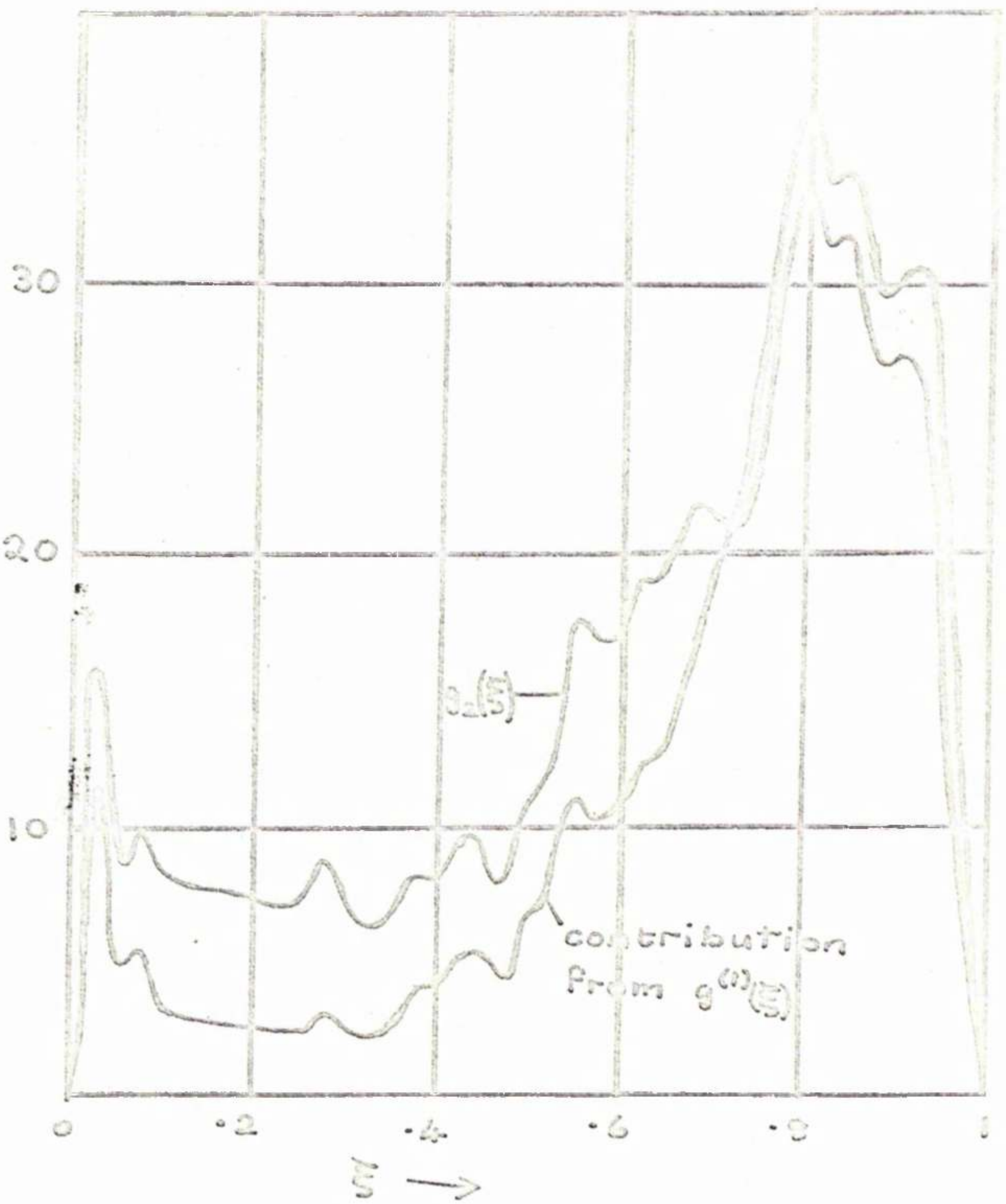
$v = 2.15$ $h = 0.039$

n	M	τ_1	Range of τ_2
11	10	$6.1\ \tau_0$	$18\tau_0 - 48\tau_0$
7	10	$5.2\ \tau_0$	$32\tau_0 - 35\tau_0$
7	8	$3.8\ \tau_0$	$14\tau_0 - 16\tau_0$

τ_1 = mean containment time for diffusion equation treatment

τ_2 = mean containment time for this treatment

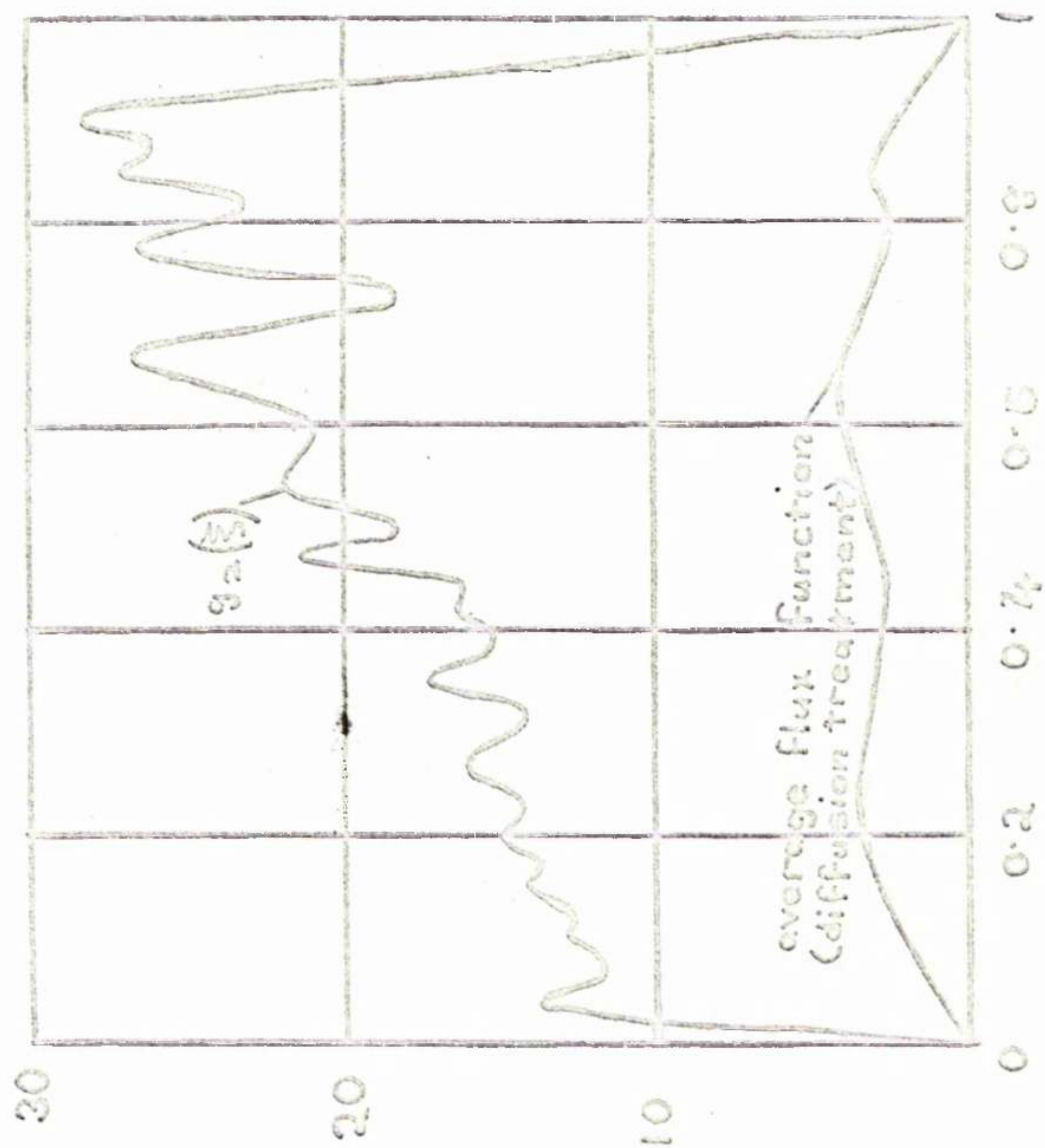
FIGURE F8



$g_2(\xi)$ and contribution to $g_2(\xi)$ from $g^{(1)}(\xi)$ vs ξ

$n = 11$, $h = 0{\cdot}042$, $v = 2{\cdot}21$, $M = 10$, $\xi_1 = 0{\cdot}5$

FIGURE F9



n = 7, h = 0.039, σ = 2.15, M = 10

CHAPTER 5

RESULTS

In order to discuss the significance of the results obtained using the method described in the previous chapters, it is necessary to describe briefly the other methods which have been used to treat this type of system, and the results which have been obtained using those methods.

These other methods give results with which to compare the results obtained in this thesis. The purely numerical treatment will be described first, and then the diffusion equation treatment.

5.1 A Statistical Treatment

This is described in (6). The method consists basically of taking a particle with a fixed input magnetic moment and a completely random phase, and tracing the history of the particle using the equations derived in chapter 3. The assumption is made that the phase of the particle is completely randomised every time it is reflected at a mirror: this is shown in (6) to be a justifiable assumption. By tracing the path of the particle while it is in the machine the containment time of the particle can be calculated.

In order to find the general behaviour of particles in this system, a statistically significant number of particles is taken, i.e. 1000 particles or more, each with the same input magnetic moment and with a phase chosen by some method producing random numbers. From the containment times of each of these particles, the mean containment time of the particles in the system is calculated, for this input magnetic moment.

It is not possible to compare exactly the results obtained by this method and those obtained by the method described in this thesis. This is because those in (6) have been obtained for a non-symmetric modulation, i.e. for a complete number of periods of modulation, whereas the method described in this thesis depends entirely on the consideration of a symmetric system which produces a symmetric kernel for the integral equation of chapter 2.

We can, however, compare the results obtained in (6) with the results from the method in this thesis for a system with one fewer step, i.e. we compare n = 11 for this thesis with n = 5 periods (10 steps) for (6).

With this condition, the comparable results are shown in table 1.

5.2 A Diffusion Eq uation Treatment

Another method of dealing with this type of system has been devised by Dunnett and Laing, and is described in (8).

This involves initially deriving a set of coupled integral equations, each one of which is similar to equation (4.1), but refers only to transmission from one region of the modulation to the next region, and not to transmission through and reflection from the whole modulation.

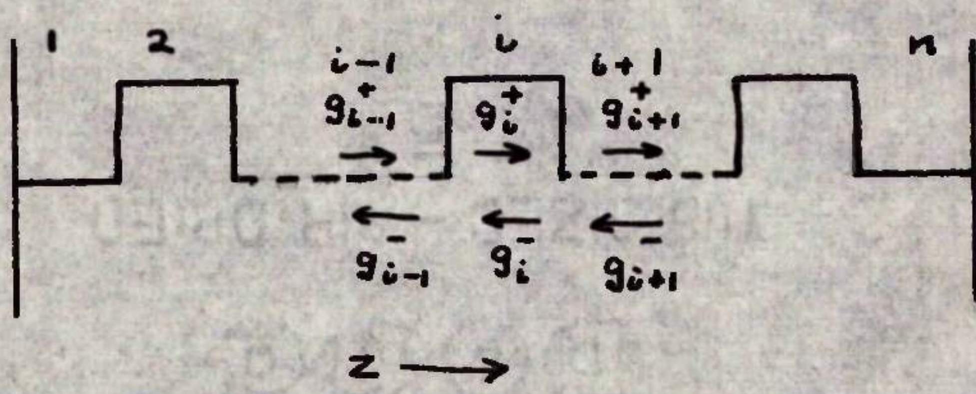


fig. 5.1

If the distribution functions g in the various regions are those indicated in fig. 5.1, then a typical member of the set of integral equations is as follows.

$$g^-_i(\xi) = \int_0^\infty Q_{i\,i+1}(\xi,\xi')\, g^-_{i+1}(\xi')\, d\xi' + P_{i+1\,i}(\xi)\, g^+_i(\xi) \tag{5.1}$$

Since the modulation is totally symmetric, and since a total symmetry is imposed on the system by injection through both mirrors of the system, then

$$g^+_i(\xi) = g^-_{n+1-i}(\xi)$$

so equation 5.1 becomes

$$g^-_i(\xi) = \int_0^\infty Q_{i\,i+1}(\xi,\xi')\, g^-_{i+1}(\xi')\, d\xi' + P_{i+1\,i}(\xi)\, g^-_{n+1-i}(\xi) \tag{5.2}$$

Where $P_{i+1\,i}(\xi)$ is the probability that a particle with magnetic moment ξ, in the i th region is reflected at the boundary between the i th and the i + 1 th region.

$Q_{i\,i+1}(\xi,\xi')\,d\xi'$ is the probability that a particle with magnetic moment in the range $(\xi', \xi'+d\xi')$ will be transmitted into the $i+1$ th region with magnetic moment ξ.

It can be shown that $\xi' - \xi$ is small compared with ξ' for all ξ' which contribute to the integral in 5.2. Then by letting $y = \xi' - \xi$, (5.2) becomes

$$g_i^-(\xi) = \int_0^\infty Q_{ii+1}(\xi,\xi')\, g_{i+1}^-(\xi + y)\, d\xi' + P_{i+1i}(\xi)\, g_{n+1-i}^-(\xi) \quad (5.3)$$

The assumption is made that $g_{i+1}^-(\xi')$ is a slowly varying function of ξ', and as a result $g_{i+1}^-(\xi + y)$ can be expanded in a Taylor series about ξ, i.e.

$$g_{i+1}^-(\xi + y) = g_{i+1}^-(\xi) + y\frac{dg_{i+1}^-(\xi)}{d\xi} + \tfrac{1}{2}y^2\frac{d^2 g_{i+1}^-(\xi)}{d\xi^2} + \ldots\ldots$$

Then equation (5.3) becomes

$$g_i^-(\xi) = \alpha_{ii+1} g_{i+1}^-(\xi) + \beta_{ii+1}\frac{dg_{i+1}^-(\xi)}{d\xi} + \tfrac{1}{2}\gamma_{ii+1}\frac{d^2 g_{i+1}^-(\xi)}{d\xi^2} + P_{i+1i}(\xi)\, g_{n+1-i}^-(\xi)$$

where

$$\begin{bmatrix}\alpha_{ii+1}\\ \beta_{ii+1}\\ \gamma_{ii+1}\end{bmatrix} = \int_0^\infty Q_{ii+1}(\xi,\xi')\begin{bmatrix}1\\ (\xi'-\xi)\\ (\xi'-\xi)^2\end{bmatrix} d\xi'$$

The set of equations thus produced is capable of being solved numerically, whereas the solution of the original set of coupled integral equations becomes very complicated for more than 3 regions.

Since the diffusion equations depend on a symmetric modulation, the results can be compared with those obtained for the same number of steps by the method of this thesis.

The results obtained using the diffusion equation treatment were calculated with a more complicated source function than the simple source function used in this treatment, but it is nevertheless possible to compare the orders of magnitude of the results obtained by the two methods. This is done in table 2.

5.3 Discussion of Results

It is interesting to look at the eigenvalues produced by this method, and at the form of the corresponding eigenfunctions and of the resulting distribution function.

As we have seen in chapter 4, the eigenvalues are greater than 1, and become progressively larger. We have also shown that the relative importance of the term involving λ_n is proportional to $\frac{1}{\lambda_n - 1}$, and not directly proportional to λ_n. For small values of h, the eigenvalues are reasonably close to one another, but this is compensated for by the fact that the smallest eigenvalue λ_1 is very close to 1, and it therefore makes a very large contribution to the resulting distribution function. For large h the eigenvalues become comparatively large quite rapidly, but the smallest eigenvalue is usually not so close to 1. Thus the smaller h is, the greater the contribution the first eigenfunction makes to the total distribution function.

It is interesting to look at the shape of the distribution function $g_2(\xi)$. A typical one of these is shown in fig.(F8) and also shown in the figure is the contribution made to the distribution function by the first eigenfunction $g^{(1)}(\xi)$. It can be seen that the contribution of $g^{(1)}(\xi)$ is less than the total distribution function in the lower ranges of ξ, i.e. $\xi < \cdot 7$, but is greater in the higher ranges of ξ. The peak at high values of ξ is almost certainly due to particles of high magnetic moment being internally reflected in the modulation and thus spending longer in the modulation than those with lower magnetic moments. It is this high peak which contributes greatly to the high containment times obtained by this method.

It seems likely from the shapes of the eigenfunctions that the first eigenfunction represents almost all the particles which spend a long time in the machine, as there is no sign of this peak at high values of ξ appearing in any of the subsequent eigenfunctions. This is a possible area for further investigation.

Another interesting feature which appears in the distribution function $g_2(\xi)$ is a peak at a very low value of ξ, in the case of this example at $\xi = 0.25$. This is probably due to the fact that there is a reflecting barrier in ξ at $\xi = 0$. This will always be a small peak, as these particles have such small magnetic moments that they will be lost when they meet a mirror.

5.4 Comparison of Results

From table 2 it can be seen that the results from this method are very different from those obtained by the diffusion equation treatment. The difference is one of an order of magnitude, and not just of a few times τ_0. Since the results of the diffusion equation treatment have been calculated, Dunnett and others have postulated that the low values obtained by this method may be due to the fact that the distribution functions $g_i(\xi)$ vary too rapidly for high values of ξ for them to be accurately expressed by means of a Taylor series expansion. This is borne out by comparison of an average flux function derived from the diffusion equation treatment with a total flux function for the method outlined in the previous chapters with the same initial parameters (fig.F9).

From the results in table 1 it can be seen that, allowing for the additional step in the results from (6), the results from the statistical treatment and from the method described in this work are very similar. Therefore, assuming that the statistical results are correct, the method outlined in this thesis gives an accurate description of the system, and also a means of finding mean containment times for particles in a modulated mirror machine, with the particles treated as an ensemble, rather than simply as individual particles.

5.5 Further Work

There are various further investigations which might be made along the lines of the method outlined in this thesis. It has been suggested that it might be instructive to make some investigation into the structure of the eigenvalues produced. Another possibility would be to look at the results obtained by varying the parameters h and v.

There are various other problems to which a diffusion equation approach has yielded results which are incompatible with those obtained by experimental methods. It is possible that an approach similar to that described in this thesis may give results compatible with the experimental data for such problems.

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