SUMMARY OF THESIS

EQUILIBRIUM DELAY DISTRIBUTIONS FOR QUEUES WITH RANDOM SERVICE

The problem which the thesis discusses is that of determining the probability of delay of a demand (customer) in a queueing system in whi service is random, i.e. on the completion of a service-time the server obtains the next customer for service by choosing at random from among those waiting. The system is assumed to be in statistical equilibrium, arrivals are assumed to follow the Poisson distribution, and two distin assumptions regarding service-time are made, (i) that it follows the negative exponential distribution, (ii) that it is constant.

For the case of negative exponential service-time, the work of a number of authors is reviewed:

(i) Molina (1927), who derived the equilibrium state probabilit of the system;

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be solved, the determination of the general form of the distribution by means of the first two moments, and the question of numerical computation;

(v) Pollaczek (1946), who used Laplace transforms and contour integration to find an exact expression for the delay distribution function, but in a form too complicated for actual computation;

(vi) Riordan (1953), who, in an attempt to check numerical valu obtained by means of a differential analyzer, found a met of evaluating exactly the moments of the distribution, a used them to approximate to the distribution function by sum of a few exponentials, thus obtaining numerical valu comparatively easily;

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The case of constant service-time has received comparatively litt attention, and the section dealing with this first reviews the work of Crommelin (1932), who derived equations satisfied by the equilibrium state probabilities and also obtained an expression for a generating function of these probabilities, and of Burke (1959), who gave a very clear analysis of the problem and obtained actual numerical values for the delay distribution, but only for the case of one server.

Burke's work appears to be capable of extension, and in the next section, which is new, it is shown that his methods can be used in the case of two servers.

There seems to be no record of a Monte Carlo investigation of the constant service-time case, and in the following section, which is als new, the method by which such an investigation was carried out, by mea of the Sirius computer, for one and for two servers is described. It shown that for one server good agreement with Burke's results was obta

Finally, it is pointed out that although Burke's methods can probably be extended to more than two servers, Monte Carlo methods offer easier way of dealing with this problem, and there seems to be no seri difficulty in using them to analyse not only larger systems but also cases in which more realistic assumptions are made regarding the arriv and service-time distributions.

EQUILIBRIUM DELAY DISTRIBUTIONS FOR QUEUES WITH RANDOM SERVICE

By

RAMANATHA THRIVIKRAMAN

A THESIS SUBMITTED TO THE UNIVERSITY OF GLASGOW FOR THE DEGREE OF M.Sc.

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A B S T R A C T

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MEANING OF SYMBOLS

c = number of lines

ν = number of calls in the system

P_ν = probability that there are ν calls in the system

h = average holding time

a = average number of calls arriving in time h

n = number of calls

α = a/c, the traffic intensity

$P_n(t)$ = probability that a delayed call waits longer than t, and that there are n others waiting with it at time t

$P(t)$ = probability that a delayed call waits longer than t

$F_n(t)$ = probability that a call, which arrives when there are n others waiting, is delayed more than time t

u = $\frac{ct}{h}$ new unit of delay time

$f(u)$ = probability that an arbitrary call is delayed more than t

k = number of places in the queue

$F(u)$ = conditional probability that a delayed call is delayed more than t

b_c = probability that there are not more than c calls in the system

$G(t/n)$ = conditional probability that a delayed call waits $\leq t$, given that the delayed call is with n others at the first departure epoch after its arrival

$G(t)$ = conditional probability that an arbitrary delayed call waits $\leq t$

EQUILIBRIUM DELAY DISTRIBUTION FOR QUEUES WITH RANDOM SERVICE

Introduction

Probabilistic methods have been fruitfully employed in recent years in problems involving queues. A queue arises when demands are made for service, not all the demands can receive service immediately on arrival, and the demands waiting for service are arranged in order of arrival, forming a waiting line. In practical cases there is a random element in the service-time or in the arrival times or in both. Many actual queueing systems are so complicated that precise mathematical treatment is impossible, but certain simplifying assumptions regarding service-time, arrival-time and other conditions of the system can be made which lead to a mathematically tractable problem which nevertheless gives a fairly good approximation to reality. Questions for which answers may be sought are

(i) the mean and distribution of the length of time for which a customer has to queue for service, i.e. the delay distribution;

(ii) the mean and distribution of the number of customers in the system at any instant;

(iii) the mean and distribution of the length of the server's busy periods, i.e. periods of uninterrupted activity.

In practical applications, we may require investigation of any one of these.

This thesis discusses the determination of the delay distribution with random selection from the queue, i.e. in the case where on the completion of a service-time, the server obtains the next customer for service by choosing at random from among those waiting and not necessarily by taking them in the order of their arrival. It is assumed that the system is in statistical equilibrium, i.e. that the probabilities of the states of which it is capable are independent of time. The arrivals are assumed to follow the Poisson distribution and the service-time is assumed either to have the negative exponential distribution or to be constant.

The general outline of the problem and basic assumptions are given in article 1. Article 2 is devoted to the case of Poisson arrivals and negative exponential service-time. In § 2.1 general comments on this case are made. In § 2.2 the equilibrium state probabilities are derived. Then the first attempt, that of Mellor in 1942, to get the delay distribution function with random selection from the queue is given in § 2.3, showing how it is defective in the formulation of the problem. This is followed by the correct formulation of the problem by Vaulot, in 1946, with the fundamental difference-differential equations, in § 2.4. In the next two sections, the methods by which Palm and Pollaczek solved Vaulot's fundamental equations theoretically are discussed. The results given by them are shown to be inconvenient for numerical computation.

In § 2.7 is given Riordan's rather more statistical approach, that of actual calculation of the moments of the delay distribution function and the use of these to fit the sum of a few exponential functions as an approximation. A similar method of dealing with the problem, using matrix notation, was given by Le Roy, and is described in the following section.

The problem involves the solution of an infinite number of differential equations with constant coefficients. Before the advent of computers, the solution of a finite number of equations to get an approximation for the infinite case could hardly be regarded as a practical method, because the maximum number of equations which could be dealt with was so small. Now the use of computers makes this a feasible method, and in § 2.9 are described the modifications to the equations for a queue of finite size k and the results of actually solving them by computer for $k = 20$, 40 and 60. In article 3, the problem with Poisson arrivals and constant service-time is discussed. Crommelin's method of evaluating the equilibrium state probabilities is given in § 3.2. Then Burke's investigation of the delay distribution function for a single line is given in detail in § 3.3. This suggested the extension of the problem to two or more lines. The mathematical derivation for two lines on the basis of Burke's argument is given in § 3.4.

Another way in which computers can make a contribution to the solution of statistical problems is in Monte Carlo methods, since they make it possible to study the experience of a large enough number of individuals to give an indication of the general form of the distribution. Such methods are obviously appropriate for this problem, and in § 3.5 is given a description of their use and of results which were obtained for one and for two lines.

1 Description of the Problem

In order to analyse problems on queues we must be in possession of three important data, viz. (i) the service mechanism, (ii) the arrival pattern of the customers and (iii) the rule by which customers are selected from the queue for service.

(i) In the case of the service mechanism of the system, the number of servers available to meet the demand and the service-time probability distribution are required. Information regarding the availability of the servers must also be given.

(ii) The arrival pattern of the customers is usually given by the probability distribution of the interval between successive demands for service.

(iii) The rule by which a server, when free, selects the next customer for service from among those waiting is usually called the queue discipline.

The natural case of queue discipline in many applications is "first come, first served". Contributions based on this discipline are numerous, as may be seen in the literature now available. Other types which have been considered are cases in which some demands have priority over others, "last come, first served", and random selection from the queue.

The last-mentioned case is interesting and is realised (at least approximately) in certain practical applications, but it appears that very few writers have devoted attention to it. As has been suggested by Palm, an eminent worker in this field, this may be due to its mathematical intractability.

The present thesis is a review of work done on queueing problems with selection at random from the queue, the object being to determine the probability distribution of delay when the system is in statistical equilibrium, together with a description of attempts made at various points to carry the discussion of the problem a little further.

The statement that the system is in statistical equilibrium means that the probabilities of the various states of which the system is capable are independent of time. The equilibrium state probabilities describe the non-transient behaviour of the system, i.e. the distribution to which it "settles down" a long time after it is started. If in such systems the initial distribution is the equilibrium one, the state probabilities remain constant, i.e. independent of time, so that this is often called the "stationary" distribution.

The number of demands present, either being served or waiting will be used as a description of the state of the system. Full availability of the servers will also be assumed.

The problem obviously has application to many different fields, but telephony is the one which most writers seem to have had in mind. For the greater part, therefore, the terminology of telephony will be used: the servers will be described as 'lines', the service-time as 'holding-time' and the demands or customers as 'calls'.

2 Poisson arrivals, negative exponential holding-time

2.1 Introduction: The simplest assumption which can be made about arrivals is that the probability that a demand for service is made in the infinitesimal time-interval $(t, t + dt)$ is proportional to dt, the constant of proportionality being independent of time, of other demands and of the number in the system. It is convenient to write this probability in the form $\frac{a}{h}.dt$, where h is the average holding-time and a is the average number of calls arriving (demands for service) in time h. This assumption implies that $e^{-\frac{a}{h}t}$ is the probability that the interval between successive demands exceeds t, and that the number of demands made in any finite interval of length T is a Poisson variable with mean $\frac{a}{h}T$.

Similarly, for holding-time the simplest assumption is that the probability that it exceeds t is $e^{-\frac{t}{h}}$: this implies that when a line is engaged the probability that it becomes free in the interval $(t, t + dt)$ is $\frac{dt}{h}$. In most of the theoretical treatment of queueing problems, it has been assumed, for simplicity, that either the arrival or the holding-time distribution is negative exponential.

2.2 The equilibrium state probabilities for this case were given by Molina in 1927 [7], his argument being as follows. Let c be the number of lines and P_ν the probability that there are ν calls in the system, $\nu = 0, 1, 2 \ldots$, (either being served or waiting). Only if $a < c$ can there be equilibrium. Let us consider the beginning and end of an infinitesimal time-interval of length dt. If at the end of the time-interval dt, there are ν calls in the system $(\nu = 1, 2, \ldots)$, then at the beginning of the time-interval there could have been $\nu - 1$, $\nu + 1$, or ν calls in the system (three mutually exclusive cases), followed by an arrival, a departure or neither respectively, during the time interval dt. The cases of more than one arrival or departure (release) in the interval dt are neglected, their probabilities being of higher order in dt.

The probability of an arrival in time dt is $\left(\frac{a}{h}\right)$ dt. When there are $(\nu + 1)$ calls in the system, the probability of a release is $\left(\frac{\nu+1}{h}\right)$ dt when $\nu < c$ and $\left(\frac{c}{h}\right)$ dt when $\nu \geq c$. When there are ν calls in the system, the probability that there will be neither an arrival nor a departure is $\left[1 - \left(\frac{a}{h}\right) dt - \left(\frac{\nu}{h}\right) dt\right]$ when $\nu < c$ and $\left[1 - \left(\frac{a}{h}\right) dt - \left(\frac{c}{h}\right) dt\right]$ when $\nu \geq c$.

Thus

$$P_\nu = P_{\nu-1}\left(\frac{a}{h}\right) dt + P_{\nu+1}\left(\frac{\nu+1}{h}\right) dt + P_\nu\left[1 - \left(\frac{a}{h}\right) dt - \left(\frac{\nu}{h}\right) dt\right]; \quad \nu = 1, 2, \quad (c-1).$$

$$P_\nu = P_{\nu-1}\left(\frac{a}{h}\right) dt + P_{\nu+1}\left(\frac{c}{h}\right) dt + P_\nu\left[1 - \left(\frac{a}{h}\right) dt - \left(\frac{c}{h}\right) dt\right]; \quad \nu = c, c+1, c+2, \cdots$$

By a similar argument, the case when $\nu = 0$ gives

$$P_0 = P_1\left(\frac{1}{h}\right) dt + P_0\left[1 - \left(\frac{a}{h}\right) dt\right],$$

which can be covered by the above equations, if we define P_ν to be zero when $\nu < 0$.

These equations give $\nu P_\nu = a P_{\nu-1}$; $\nu = 1, 2, \ldots c$ and $c P_{\nu+1} = a P_\nu$; $\nu = c, c+1, \ldots\ldots$, whose solution is

$$\left.\begin{aligned} P_\nu &= P_0 \frac{a^\nu}{\nu!}, \quad \nu = 1, 2, \ldots\ldots c, \\ P_\nu &= P_0 \frac{a^\nu}{c!\, c^{\nu - c}}, \quad \nu = c, c+1, \ldots \end{aligned}\right\} \qquad (1)$$

But $\sum_{\nu=0}^{\infty} P_\nu = 1$, being the sum of the probabilities of all the possible states, and it follows that

$$P_0^{-1} = \sum_{\nu=0}^{c-1} \frac{a^\nu}{\nu!} + \frac{a^c}{c!}\left(\frac{c}{c-a}\right)$$

Thus the probability that all the lines are engaged, i.e. that an arriving call will be delayed, is

$$\sum_{\nu=c}^{\infty} P_\nu = P_0 \frac{a^c}{c!}\left(\frac{c}{c-a}\right) = P_c\left(\frac{c}{c-a}\right),$$

a formula given by Erlang in 1917 [4] and denoted $C(c,a)$ by him.

The conditional probability that a delayed call will find n calls waiting is $\frac{1}{C(c,a)} P_{c+n}$ or

$$\frac{P_0\left(\frac{a^{c+n}}{c!\,c^n}\right)}{P_0\left(\frac{a^c}{c!}\right)\left(\frac{c}{c-a}\right)} \quad \text{or } (1-\alpha)\alpha^n,$$

$n = 0, 1, 2, \ldots$, where α is written for $\frac{a}{c}$; α is sometimes called the occupancy level or traffic intensity.

2.3 The first attempt to deal with the problem of the delay distribution when delayed calls are chosen at random from the queue appears to be that of Mellor (1942) [6]. His argument is as follows:-

Let $P_n(t)$ be the probability that a delayed call is delayed more than t and that there are n others waiting with it [(n + 1) waiting in all] at time t. The chance that a line becomes free in the infinitesimal interval (t,t+dt) and that the call in question is chosen for service is $\left(\frac{c}{h}\right) dt \ \frac{1}{n+1}$.

Then $P_n(t+dt)$ = chance that the call is delayed more than $t+dt$ and has n others waiting with it then $= P_n(t) \times$ [chance that the call is not served in interval $(t, t+dt)$]

$$= P_n(t) \left\{ 1 - \frac{c}{h(n+1)} dt \right\} \quad \text{.----} \quad (2)$$

This leads to the differential equation $\frac{dP_n}{dt} = -\frac{c}{h(n+1)} P_n$, which gives $P_n(t) = c_n e^{-\frac{ct}{h(n+1)}}$, c_n being the integration constant.

Now $P(t)$, the total probability that a delayed call is delayed more than t, is given by

$$P(t) = \sum_{n=0}^{\infty} P_{c+n+1} P_n(t) \quad ;$$

thus

$$P(t) = \sum_{n=0}^{\infty} P_{c+n+1} c_n e^{-\frac{ct}{h(n+1)}}$$

Putting $t = 0$, we get $P(0) = \sum_{n=0}^{\infty} P_{c+n+1} c_n$; but $P(0)$ is the probability that the call suffers some delay, which is $\sum_{n=0}^{\infty} P_{c+n}$. This can be written as $P_c \sum_{n=0}^{\infty} \left(\frac{a}{c}\right)^n$ or $\left(\frac{c}{a}\right) \sum_{n=1}^{\infty} P_{c+n}$, which shows that $c_n = \frac{c}{a}$ for all n.

Thus finally

$$P(t) = \frac{c}{a} \sum_{n=0}^{\infty} P_{c+n+1} e^{-\frac{ct}{h(n+1)}}$$

$$= \frac{c}{a} P_c \sum_{n=0}^{\infty} \left(\frac{a}{c}\right)^{n+1} e^{-\frac{ct}{h(n+1)}}$$

$$= P(0) \left(\frac{c-a}{a}\right) \sum_{n=0}^{\infty} \left(\frac{a}{c}\right)^{n+1} e^{-\frac{ct}{h(n+1)}}$$

Mellor's solution cannot be regarded as satisfactory, however, for the equation (2) assumes that if $(n + 1)$ calls are waiting at time $t + dt$, there must have been the same number at time t, whereas in fact there could have been n, $n + 2$ or $n + 1$ then, followed by an arrival, a departure or neither, respectively, in the interval $(t, t + dt)$. Nevertheless, Mellor's solution, as will be seen, is a useful approximation for large n.

2.4 The correct differential-difference equations for the problem were first given in 1946 by Vaulot [12] and Palm [8], almost simultaneously. They may be derived as follows. Let $F_n(t)$ be the probability that a call, which arrives to find n other calls waiting, is delayed more than time t. Consider the infinitesimal interval dt immediately succeeding the arrival of the call. In this interval there are only three possibilities: (i) another call arrives [probability $\left(\frac{a}{h}\right) dt$], (ii) a line becomes free [probability $\left(\frac{c}{h}\right) dt$], and (iii) neither of these two [probability $\left[1 - \frac{(a + c)}{h} dt\right]$]. In case (ii) the conditional probability that the call in question is not chosen for service, and is therefore still waiting at the end of the interval, is $\frac{n}{n+1}$. Thus

$$F_n(t) = \frac{a}{h} dt\, F_{n+1}(t - dt) + \frac{n}{n+1} \frac{c}{h} dt\, F_{n-1}(t - dt) + \left[1 - \frac{(a+c)}{h} dt\right] F_n(t - dt)$$

which leads, on letting $dt \to 0$, to

$$\frac{d\, F_n(t)}{dt} = \left(\frac{n}{n+1}\right) \frac{c}{h} F_{n-1}(t) - \frac{c+a}{h} F_n(t) + \frac{a}{h} F_{n+1}(t), \quad n = 0, 1, 2, \dots$$

Writing $\alpha = \frac{a}{c}$ and $u = \frac{ct}{h}$, we can express the above equation more simply in the new variable u as

$$\left.\begin{array}{r}\dfrac{dF_n(u)}{du} = \left(\dfrac{n}{n+1}\right)F_{n-1}(u) - (1+\alpha)F_n(u) + \alpha F_{n+1}(u) \\ n = 0, 1, 2 \cdots\cdots\end{array}\right\} \quad (3)$$

The boundary conditions are $F_n(0) = 1$, $F_n(\infty) = 0$ for all n; and $\lim_{n \to \infty} F_n(u) = 1$, for all u. The probability $f(u)$ that an arbitrary call will be delayed more than t is given by

$$f(u) = \sum_{n=0}^{\infty} P_{c+n} F_n(u) = P_c \sum_{n=0}^{\infty} \alpha^n F_n(u)$$

$$= C(c,a)\,(1-\alpha) \sum_{n=0}^{\infty} \alpha^n F_n(u),$$

and since $C(c,a)$ is the probability that a call will be delayed, the conditional probability $F(u)$ that a delayed call will be delayed more than t is given by

$$F(u) = (1-\alpha) \sum_{n=0}^{\infty} \alpha^n F_n(u) \qquad - (4)$$

Note that $F(0) = 1$.

Having derived equations (3), Vaulot used them to find the coefficients in the Maclaurin series for $F_n(u)$ and hence obtained the series for $f(u)$. The first two terms given by Vaulot for $f(u)$ are

$$f(u) = \frac{a^c}{(c-1)!\,(c-a)\,\Delta}\left[1 - u\,\frac{(1-\alpha)}{\alpha}\log\frac{1}{1-\alpha} + \cdots\cdots\right]$$

where $\alpha = \frac{a}{c}$ and $\Delta = \left[1 + \frac{a}{1} + \frac{a^2}{2!} + \cdot \quad + \frac{a^{c-1}}{(c-1)!} + \frac{a^c}{(c-1)!(c-a)}\right]$

The differential-difference equations (3) are the fundamental equations of the system and their solution is the principal problem. The theoretical solution of the equations has been discussed by both Palm and Pollaczek. Their methods are given below.

2.5 Palm, who was a pioneer in the problem appears to have drawn up as early as 1938 the theoretical solution of this problem but did not publish his work until 1946. He delayed the publication of his investigation in the hope of finding easier methods of numerical evaluation, a hope which was only partly realised. He pointed out that the mathematical treatment of the waiting time problem with a randomly served queue is appreciably more complicated than with an ordered queue.

He derived the fundamental equations (3) independently of Vaulot and also indicated how these equations are modified for the case of an ordered queue; the solution is then easily obtainable by a direct integration process.

The mathematical treatment of the problem by Palm starting from the fundamental equations (3) is as follows. Taking the fundamental difference-differential equations in the form

$$\frac{d\,F_n(t)}{dt} = \left(\frac{n}{n+1}\right)\frac{c}{h}F_{n-1}(t) - \frac{(c+a)}{h}F_n(t) + \frac{a}{h}F_{n+1}(t);$$
$$n = 0, 1, 2 \cdot \cdots\cdots$$

the following modifications are carried out for the case of an ordered queue.

(i) calls coming in later do not affect the waiting time, i.e. a = 0.

(ii) the call in question cannot get service unless it is at the head of the queue, i.e. the factor

$$\frac{n}{n+1} = 1 \ , \ n > 0 \quad \text{and} \quad \frac{n}{n+1} = 0 \ , \ n = 0$$

Thus with an ordered queue, we get the following relations for $n > 1$

$$\frac{d\,F_n(t)}{dt} = \frac{c}{h} F_{n-1}(t) - \frac{c}{h} F_n(t) \quad \cdots\cdots \quad (a)$$

and for $n = 1$

$$\frac{d\,F_1(t)}{dt} = -\frac{c}{h} F_1(t) \quad \cdots\cdots \quad (b)$$

Integrating equation (b) we get $F_1(t) = e^{-c/h^t}$ and then by recurrently integrating (a) we get

$$F_n(t) = \left[1 + \frac{c}{h}t + \frac{1}{2!}\left(\frac{c}{h}t\right)^2 + \cdots + \frac{1}{(n-1)!}\left(\frac{c}{h}t\right)^{n-1}\right] e^{-\frac{c}{h}t}$$

In the case of a randomly served queue, the solution is not so simple. Here Palm uses a method involving generating functions as shown below.

Let $$\phi(x, u) = \sum_{n=0}^{\infty} F_n(u)\, x^{n+1} \quad \cdots\cdots \quad (5)$$

We have

$$\phi(0, u) = 0 \ ; \quad \phi(x, 0) = \sum_{n=0}^{\infty} F_n(0)\, x^{n+1} = \frac{x}{1-x}$$

where $|x| < 1$

Expanding $\phi(x,u)$ in ascending powers of u, as

$$\phi(x, u) = \sum_{i=0}^{\infty} A_i(x)\, \frac{u^i}{i!} \ ; \quad \cdots\cdots \quad (6)$$

then $A_i(x) = \phi^{(i)}(x, 0) = \sum_{n=0}^{\infty} F_n^{(i)}(0)\, x^{n+1}$,

where the superscript (i) denotes the ith derivative with respect to u. From equations (3), $F_n^{(i)}(0)$ can be found by repeated differentiation with $F_n(0) = 1$. Hence for $i = 1$, we find $F_n^{(1)}(0)$ and so on. Thus $A_i(x)$ can be found, where $A_0(x) = \frac{x}{1 - x}$, and this gives a solution of the problem in principle. Finally

$$F(u) = \frac{(1-\alpha)}{\alpha} \phi(\alpha, u)$$

To calculate the values of $A_i(x)$, it is desirable to find the differential equation satisfied by $\phi(x, u)$. Multiplying both sides of (3) by $(n+1)\, x^{n+1}$ and summing for $n = 0, 1, 2, \ldots$, we find

$$\frac{\partial^2 \phi}{\partial x \partial u} + \frac{(1-x)(x-\alpha)}{x} \frac{\partial \phi}{\partial x} + \frac{\alpha}{x^2} \phi(x, u) = 0 \quad \ldots \text{(7)}$$

From (6) and (7) we get

$$\sum_{i=1}^{\infty} \frac{u^{i-1}}{(i-1)!} \frac{d A_i(x)}{dx} + \frac{(1-x)(x-\alpha)}{x} \sum_{i=0}^{\infty} \frac{u^i}{i!} \frac{d A_i(x)}{dx} + \frac{\alpha}{x^2} \sum_{i=0}^{\infty} \frac{u^i}{i!} A_i(x) = 0$$

Coefficients of each power of u must be zero if (6) is a solution of (7).

Therefore $\frac{d A_i(x)}{dx} + \frac{(1-x)(x-\alpha)}{x} \frac{d A_{i-1}(x)}{dx} + \frac{\alpha}{x^2} A_{i-1}(x) = 0$,

for $i = 1, 2, 3 \ldots$

Thus

$$A_i(x) = -\int_0^x \frac{(1-y)(y-\alpha)}{y}\, d A_{i-1}(y) - \alpha \int_0^x \frac{A_{i-1}(y)}{y^2}\, dy$$

Here $A_i(0)$ is zero from equation (6), since the boundary condition for $\phi(x,\mu)$ is $\phi(0,\mu) = 0$. Hence the lower limit of integration above is taken as zero. Then

$$A_i(x) = -\left[\frac{(1-y)(y-\alpha)}{y} A_{i-1}(y)\right]_0^x - \int_0^x A_{i-1}(y)\, dy$$

which finally gives

$$A_i(x) = \frac{-(1-x)(x-\alpha)}{x} A_{i-1}(x) - \alpha \lim_{x\to 0} \frac{A_{i-1}(x)}{x} - \int_0^x A_{i-1}(y)\, dy$$

with $A_0(x) = \frac{x}{1-x}$.

Hence the A's can be found recursively, e.g.

$$A_1(x) = \frac{-(1-x)(x-\alpha)}{x} \cdot \frac{x}{1-x} - \alpha - \left[-x - \log_e(1-x)\right]$$

$$= \log_e(1-x)$$

This method of calculating $A_i(x)$ is excessively complicated; in the series for A_7, for example, there are more than 100 terms.

For numerical computation equations (8) and (9) given below can be used for recursive calculation, but even this is very laborious.

$$A_i(x) = \sum_{n=0}^{\infty} F_n^i(0)\, x^{n+1} \quad \text{- - - - - - -} \quad (8)$$

$$F_n^{i}(0) = \frac{n}{n+1} F_n^{i-1}(0) - (1+\alpha) F_n^{i-1}(0) + \alpha F_{n+1}^{i-1}(0) \quad \text{.....(9)}$$

As an illustration, for $A_1(x)$ we get

$$F_n^{1}(0) = \frac{n}{n+1} - (1+\alpha) + \alpha = -\frac{1}{n+1}$$

Hence

$$A_1(x) = \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1} = \log_e(1-x),$$

which is the same as is given by the other method.

Calculation of Moments and Form Factor

Palm evaluated the first and second moments of $F(u)$ and $F_n(u)$ and with the use of what he called the "Form Factor" investigated the nature of these distributions. He defined "form factor" ϵ of a distribution as the second moment of the distribution about the axis $t = 0$, divided by the square of the mean value of the distribution. Thus $\epsilon = 1 + \frac{\sigma^2}{s^2}$, where s is the mean and σ is the standard deviation. In the case of the negative exponential holding-time distribution $e^{-t/s}$, where s is the mean holding-time, the form factor ϵ is equal to 2, since $\sigma = s$.

Palm used the "form factor" to characterise the shape of one way limited distributions (i.e. from $t = 0$ to $t = \infty$, as in the case under investigation. He regarded the exponential distribution as the boundary between "steep" and "flat" distributions. For a steep distribution the form factor ϵ has values $1 \leq \epsilon < 2$ (ϵ can never be less than 1), and for a flat distribution it has values $2 < \epsilon$.

The μth moment for the generating function $\varphi(x,u)$ is defined by

$$M_\mu = -\int_0^\infty u^\mu \frac{\partial \phi(x,u)}{\partial u}\,du$$

$$= \mu \int_0^\infty u^{\mu-1} \phi(x,u)\,du \quad (\mu = 1, 2, \cdots)$$

From this, the first moment is

$$M_1 = \int_0^\infty \phi(x,u)\,du$$

Using the differential equation (7) and integrating the left-hand side of that equation with respect to u from 0 to ∞, we get

$$\frac{\partial}{\partial x}\int_0^\infty \frac{\partial \phi(x,u)}{\partial u}\,du + \frac{(1-x)(x-\alpha)}{x}\frac{\partial}{\partial x}\int_0^\infty \phi(x,u)\,du$$

$$+ \frac{\alpha}{x^2}\int_0^\infty \phi(x,u)\,du = 0$$

From equation (5) we have the boundary condition $\phi(x,0) = \frac{x}{1-x}$

$$\therefore \int_0^\infty \frac{\partial \phi(x,u)}{\partial u}\,du = \Big[\phi(x,u)\Big]_0^\infty = -\frac{x}{1-x}$$

With this value and the value of M_1, the above equation can be written

$$-\frac{d}{dx}\left(\frac{x}{1-x}\right) + \frac{(1-x)(x-\alpha)}{x}\frac{d}{dx}\left[M_1(x)\right] + \frac{\alpha}{x^2}M_1(x) = 0,$$

(i.e.) $$\frac{dM_1(x)}{dx} + \frac{\alpha M_1(x)}{x(1-x)(x-\alpha)} = \frac{x}{(1-x)^3(x-\alpha)}.$$

By integration we get

$$M_1(x) = \frac{1}{2-\alpha}\cdot\frac{x(2-x)}{(1-x)^2} + c\cdot x\cdot\frac{(1-x)^{\frac{\alpha}{1-\alpha}}}{(\alpha-x)^{\frac{1}{1-\alpha}}}$$

The integration constant c must have the value zero to make $M_1(x)$ finite for $\alpha < 1$.

Thus $$M_1(x) = \frac{1}{2-\alpha} \cdot \frac{x(2-x)}{(1-x)^2},$$

from which

$$M_1(\alpha) = \frac{\alpha}{(1-\alpha)^2}$$

Therefore the mean value of $F(u)$ [which is $\frac{1-\alpha}{\alpha} \phi(\alpha, u)$] is

$$\frac{\alpha}{(1-\alpha)^2} \cdot \frac{1-\alpha}{\alpha} = \frac{1}{1-\alpha}$$

By analogons methods the second moment of $F(u)$ can be worked out, and its value is $\frac{1}{(1-\alpha)^2} \frac{4}{2-\alpha}$. Then the form factor for $F(u)$ is

$$\epsilon = \frac{1}{(1-\alpha)^2} \cdot \frac{4}{2-\alpha} \cdot (1-\alpha)^2 = \frac{4}{2-\alpha}$$

For $0 < \alpha < 1$, it is clear that ϵ lies between 2 and 4. Hence the waiting-time distribution $F(u)$ is flat. For small values of α, the form factor approaches the value 2.

Palm obtained the form factor for $F_n(u)$ also and remarked that for $n > 1$, the distribution $F_n(u)$ is steep at least for small values of α.

Numerical computation

The numerical computation of the coefficients $A_i(x)$ obtained from the recurrence relation given above is not convenient because of alternating signs in the series expansions. To overcome this, Palm discussed other possibilities. He introduced coefficients $B_i(x)$ instead of $A_i(x)$ by defining

$$\varphi(x,u) = \frac{\alpha}{1-\alpha}\sum_{i=0}^{\infty} B_i(x)\,\frac{\left[\frac{1+\alpha}{\alpha}u\right]^i}{i!}\,e^{-\frac{1+\alpha}{\alpha}u}.$$

He got $B_o(x) = \frac{1-\alpha}{\alpha}\,\frac{x}{1-x}$ and a relation for $B_i(x)$:

$$\frac{1+\alpha}{\alpha}B_i(x) = \left(\frac{x}{\alpha}+\frac{1}{x}\right)B_{i-1}(x) - \underset{x\to 0}{\text{limit}}\,\frac{B_{i-1}(x)}{x} - \frac{1}{\alpha}\int_0^x B_{i-1}(\xi)\,d\xi.$$

It is then possible to determine all the $B_i(x)$. This however is also complicated for the higher i values. But a good feature here is that all the $B_i(\alpha)$ are positive and that they diminish numerically more rapidly than the corresponding $A_i(\alpha)$.

The treatment of the problem by Palm is ingenious. The recurrence relations for numerical computation are useful, but the expressions for the coefficients $A_i(x)$ are extremely complicated and involve large numbers of terms. Thus easier methods of numerical computation were sought by other writers, particularly Riordan in 1953.

2.6 Pollaczek, in his papers of 1946 and 1959, gave a complete mathematical solution of the fundamental equations (3), using generating functions, Laplace transforms and contour integration. His treatment of the problem is as given below.

Let us start from Vaulot's difference-differential equations for $F_n(t)$ in the form

$$F_n'(t) + \left(\frac{c+a}{h}\right)F_n(t) = \frac{n}{n+1}\cdot\frac{c}{h}F_{n-1}(t) + \frac{a}{h}F_{n+1}(t),$$

$$n = 0, 1, 2 \cdots\cdots$$

with the initial condition $F_n(0) = 1$; $n = 0, 1, 2 \ldots$. Recalling the definition of traffic intensity $\alpha = a/c$ and multiplying both sides of the above equation by $\frac{h}{\sqrt{ac}}\ \alpha^{n/2}\ e^{\frac{(c+a)}{h}t}$, we get

$$\frac{h}{\sqrt{ac}}\frac{d}{dt}\left[\alpha^{\frac{n}{2}} e^{\left(\frac{c+a}{h}\right)t} F_n(t)\right] = \frac{n}{n+1}\alpha^{\frac{n-1}{2}} e^{\left(\frac{c+a}{h}\right)t} F_{n-1}(t)$$
$$+\alpha^{\frac{n+1}{2}} e^{\left(\frac{c+a}{h}\right)t} F_{n+1}(t)$$

Then by substituting

$$t_0 = t\,\frac{\sqrt{ac}}{h} \quad \text{and} \quad f_n(t_0) = \alpha^{n/2} e^{\left(\frac{c+a}{h}\right)t} F_n(t)$$

we get

$$\frac{d f_n(t_0)}{dt_0} = \frac{n}{n+1} f_{n-1}(t_0) + f_{n+1}(t_0) \ldots\ldots (A),$$
$$n = 0, 1, 2 \ldots\ldots$$

with the initial condition

$$f_n(0) = \alpha^{n/2} \quad \ldots\ldots (n = 0, 1, 2 \ldots\ldots),$$

using a complex parameter Z for the Laplace transform of $f_n(t_0)$ in the form

$$\Phi_n(Z) = \int_0^\infty e^{-2Zt_0} f_n(t_0)\,dt_0,$$

which converges for $\mathcal{R}(z) > \frac{c+a}{2\sqrt{ac}} = \frac{1+\alpha}{2\sqrt{\alpha}}$ in view of the condition $0 \leq F_n(t) \leq 1$; we get by effecting this transform on equation (A)

$$2z\varphi_n(z) - \alpha^{n/2} = \frac{n}{n+1}\varphi_{n-1}(z) + \varphi_{n+1}(z),$$

$$(n = 0, 1, 2, \cdots\cdots)$$

Multiplying both sides of this equation by $(n+1)\,x^n$ and summing on n, we get

$$2z\sum_{n=0}^{\infty}(n+1)\,x^n\varphi_n(z) - \frac{1}{(1-x\sqrt{\alpha})^2} = \sum_{n=1}^{\infty} n\,x^n\varphi_{n-1}(z)$$

$$+\sum_{n=0}^{\infty}(n+1)\,x^n\varphi_{n+1}(z),$$

$$\left[\,|x| < \frac{1}{\sqrt{\alpha}}\,\right]$$

Now defining a generating function $\Phi(x,z)$ as

$$\Phi(x,z) = \sum_{n=0}^{\infty} x^n\varphi_n(z) = \sum_{n=0}^{\infty} x^n\int_0^{\infty} e^{-2zt_0} f_n(t_0)\,dt_0,$$

we deduce from the above equation that Φ satisfies the equation

$$2z\frac{d}{dx}(x\Phi) - \frac{1}{(1-x\sqrt{\alpha})^2} = x\frac{d}{dx}(x\Phi) + \frac{d\Phi}{dx},$$

which can be written

$$(x^2 - 2xz + 1)\frac{d\Phi}{dx} + (x - 2z)\Phi = -\frac{1}{(1 - x\sqrt{\alpha})^2}.$$

The problem now is to find a solution of this differential equation valid for $|x| < \frac{1}{\sqrt{\alpha}}$ and $R(z)$ sufficiently large.

Now putting the coefficient of $\frac{d\Phi}{dx}$ in the form

$$x^2 - 2xz + 1 = (x - x_1)(x - x_2), \text{ where}$$

$$x_1(z) = z + \sqrt{z^2 - 1} \text{ and } x_2(z) = z - \sqrt{z^2 - 1} = \frac{1}{2z} + \frac{1}{8z^3} \cdots\cdots$$

for $|z| > 1$,

and defining the sign of $\sqrt{z^2 - 1}$ here so that $\sqrt{z^2 - 1} \approx z$ at infinity on the z plane, cut from $z = -1$ to $z = 1$, we find the solution of the differential equation which vanishes for $x = x_2 \approx \frac{1}{2z}$ to be

$$\Phi(x, z) = (x_1 - x)^m (x_2 - x)^{-m-1} \int_x^{x_2} \frac{(x_2 - \xi)^m \, d\xi}{(x_1 - \xi)^{m+1}(1 - \xi\sqrt{\alpha})^2},$$

where $m = \frac{x_2}{x_1 - x_2} = \frac{z}{2\sqrt{z^2 - 1}} - \frac{1}{2}$

By permuting the operators $\sum_{n=0}^{\infty}$ and $\int_0^{\infty} dt_0$ in $\Phi(x, z)$, we get

$$\Phi(x, z) = \int_0^{\infty} e^{-2zt_0} \sum_{n=0}^{\infty} x^n f_n(t_0) \, dt_0 .$$

Using Fourier's theorem and re-substituting for $f_n(t_0)$ in terms of $F_n(t)$, we can write this

$$\sum_{n=0}^{\infty} x^n f_n(t_0) = e^{c/h(1+\alpha)t} \sum_{n=0}^{\infty} (x\sqrt{\alpha})^n F_n(t)$$

$$= \frac{1}{2\pi i}\int_{-i\infty+a}^{i\infty+a} e^{2zt_0}\,\Phi(x,z)\,2\,dz,$$

Thus we get

$$\sum_{n=0}^{\infty} (x\sqrt{\alpha})^n F_n(t) = e^{-c/h(1+\alpha)t}\cdot\frac{1}{\pi i}\int_{-i\infty+a}^{i\infty+a} e^{2\frac{c}{h}t\sqrt{\alpha}\,z}\,\Phi(x,z)\,dz \qquad \text{...(B)}$$

where $\left[\, a > \frac{1+\alpha}{2\sqrt{\alpha}} \;;\; t > 0 \right]$

The function $\Phi(x,z)$, considered as a function of z, is holomorphic in the complex plane (closed at infinity), cut along the segment $-1 \leq z \leq 1$, if we make $|x| < 1$ to ensure that $(x_1 - x) \neq 0$ in that plane [since $|x_1| \geq 1$]. The function admits, for z infinite, a Taylor series of the form $\frac{a_1}{z} + \frac{a_2}{z^2} + \ldots.$ and is bounded in that plane.

Thus, by Cauchy's Theorem, the path of integration of the integral $\frac{1}{\pi i}\int_{-i\infty+a}^{i\infty+a} e^{2\frac{c}{h}t\sqrt{\alpha}\,z}\,\Phi(x,z)\,dz$, where $t > 0$, can be taken to be a contour traversed in the positive sense, which follows the two edges of the cut $-1 \leq z \leq 1$.

If Φ_1 and Φ_2 be the values of Φ on the lower and upper edges of the cut, respectively, in passing from one edge of the cut to the other, x_1 and x_2 and also m and $(-m - 1)$ are permuted; thus the above integral is equal to

$$\lim_{\epsilon \to +0} \frac{1}{\pi i} \int_{-1+\epsilon}^{1-\epsilon} e^{2\frac{c}{h}t\sqrt{\alpha}z} \left[\Phi_1(x,z) - \Phi_2(x,z) \right] dz$$

$$= \lim_{\epsilon \to +0} \frac{1}{\pi i} \int_{-1+\epsilon}^{1-\epsilon} e^{2c/h\, t\sqrt{\alpha}z} (x_1-x)^m (x_2-x)^{-m-1} \cdot \left[\int_x^{x_2} \cdots \int_x^{x_1} \cdots d\xi \right] dz$$

$$= \lim_{\epsilon \to +0} \frac{1}{\pi i} \int_{-1+\epsilon}^{1-\epsilon} e^{2c/h\, t\sqrt{\alpha}z} (x_1-x)^m (x_2-x)^{-m-1} \cdot \int_{x_1}^{x_2} (x_1-\xi)^{-m-1} \cdot (x_2-\xi)^m \frac{d\xi}{(1-\xi\sqrt{\alpha})^2}\, dz .$$

The function $(x_1 - \xi)^{-m-1}(x_2 - \xi)^m$ is uniform in the ξ-plane cut between the points $\xi = x_1$ and $\xi = x_2$, and it is multiplied by $e^{-2\pi i(m+1)} = e^{-2\pi i m}$ when we pass from one edge of the cut to the other, revolving about the point $\xi = x_1$ in the positive sense. Taking a contour C_ξ surrounding this this cut and traversed in the positive sense, we have for the integral

$$\int_{x_1}^{x_2} (x_1-\xi)^{-m-1} (x_2-\xi)^m \frac{d\xi}{(1-\xi\sqrt{\alpha})^2} = -\frac{1}{1-e^{-2\pi i m}} \int_{C_\xi} (x_1-\xi)^{-m-1} \times (x_2-\xi)^m \frac{d\xi}{(1-\xi\sqrt{\alpha})^2} .$$

The integral on the right hand side behaves at infinity like $\frac{1}{\xi^3}$ and has only one pole, of the second order, $\xi = \frac{1}{\sqrt{\alpha}}$; evaluating the residue we get

$$\int_{x_1}^{x_2} (x_1-\ell)^{-m-1}(x_2-\ell)^{m}\frac{d\ell}{(1-\ell\sqrt{\alpha})^2} = 2\pi i\left(1-e^{-2\pi i m}\right)^{-1} \cdot \alpha^{-3/2}\left(x_1-\frac{1}{\sqrt{\alpha}}\right)^{-m-2}\left(x_2-\frac{1}{\sqrt{\alpha}}\right)^{m-1}.$$

Now if we use the substitution

$Z = \cos\tau$, $\sqrt{Z^2-1} = -i\sin\tau$, x_1 becomes $e^{-i\tau}$, x_2 becomes $e^{i\tau}$, and $m = \dfrac{x_2}{x_1-x_2}$ becomes $\dfrac{i}{2}\cot\tau - \dfrac{1}{2}$,

so that equation (B) may be written

$$\sum_{n=0}^{\infty}(x\sqrt{\alpha})^n F_n(t) = 2\int_0^{\pi} e^{(-1-\alpha+2\sqrt{\alpha}\cos\tau)\frac{c}{h}t} \cdot (e^{-i\tau}-x)^{m}(e^{i\tau}-x)^{-m-1} \cdot (1-\sqrt{\alpha}\,e^{i\tau})^{m-1}(1-\sqrt{\alpha}\,e^{-i\tau})^{-m-2}(e^{\pi\cot\tau}+1)^{-1}\sin\tau\, d\tau,$$

where $m = \dfrac{i}{2}\cot\tau - \dfrac{1}{2}$ and $|x| < 1$.

This is the formula for the generating function, and from it we can get the value of $F(u)$ by putting $x = \sqrt{\alpha}$ and multiplying both sides by $(1 - \alpha)$. Pollaczek used this formula to get an asymptotic expansion for large values of t intended for numerical computation. It is however highly complicated and appears to be inconvenient for practical computation.

In a later paper (1954) [13] Vaulot discussed transformations leading to a more convenient form for Pollaczek's integral and gave a numerical method of computing the delay distribution. Even in this case, however, the process is very complicated. He exemplified the method by evaluating F(u) for u = 140.

Pollaczek's paper can be regarded as giving a complete solution to the mathematical problem of determining the delay distribution. The method by which he inverts the Laplace transform is quite ingenious. The fact that the result is so complicated appears to be due to the intractability of the problem rather than to any deficiencies in his method.

2.7 Practical computation of the delay distribution. Riordan (1953).

The solutions of the problem by Palm and Pollaczek are mathematically complete but did not result in an easy method of calculation of the actual delay distribution. This probably led Riordan to take up the practical problem of computing the delay distribution curves.

In 1953 [11] he discussed this question. For small values of n, he solved equations (3) approximately using a differential analyser, and he pointed out that if we write $F_{n-1}(u) = F_n(u) = F_{n+1}(u)$ in (3) it reduces to $\frac{d F_n(u)}{d u} = - \frac{1}{(n+1)} F_n(u)$, which is Mellor's equation, so that Mellor's solution $F_n(u) = e^{-u/(n+1)}$ can be used as an approximation for large n. He also obtained, by repeated differentiation of (3) and evaluation at $u = 0$, the Maclaurin series for $F_n(u)$ in the form

$$F_n(u) \simeq 1 - \frac{u}{n+1} + \frac{\alpha}{2} \frac{u^2}{(n+1)^2} - \frac{\alpha(2\alpha-1)}{3!} \frac{u^3}{(n+1)^3} + \cdots ,$$

which happens to be the same as

$$F_n(u) = \left\{ 1 - \frac{(1-\alpha)u}{n+1} \right]^{\frac{1}{1-\alpha}}$$

As $\alpha \to 1$, this approaches Mellor's solution. This last form was used, as a better approximation than Mellor's, for large n.

To check the accuracy of his approximation, Riordan discussed the moments of the delay distribution. He calculated the first seven of these exactly, by deducing from equations (3) a corresponding relation between the moments, which he was able to solve. His method was as follows.
If we write

$$M_k = \int_0^\infty u^k \left[-F'(u)\right] du = k\int_0^\infty u^{k-1} F(u)\,du$$

for k = 1, 2, 3, 4, ...

and $$m_{n,k} = \int_0^\infty u^k \left[-F_n'(u)\right] du = k\int_0^\infty u^{k-1} F_n(u)\,du$$

for k = 1, 2, 3, ... ,

then $$M_k = (1-\alpha)\sum_{n=0}^{\infty} \alpha^n m_{n,k} .$$

Also $$m_{n,0} = \int_0^\infty -F_n'(u)\,du = F_n(0) = 1 \; ; \; M_0 = (1-\alpha)\sum_{n=0}^{\infty}\alpha^n = 1 .$$

Integrating both sides of (3) with respect to u from 0 to ∞ we get for k = 1, 2, 3, ...

$$-k(n+1)\,m_{n,k-1} = n\,m_{n-1,k} - (n+1)(1+\alpha)\,m_{n,k} + (n+1)\alpha\,m_{n+1,k} .$$

This relation was used to find the moments M_k for k = 1, 2, 3, ...
For M_1 the eq ation is multiplied by α^n and summed on n, giving the result

$$-L_{10} = \alpha L_{11} - (1+\alpha)L_{11} + L_{11} - L_{01} = -L_{01}$$

where

$$L_{01} = \sum \alpha^n m_{n,1} = (1-\alpha)^{-1} M_1,$$

$$L_{11} = \sum (n+1)\alpha^n m_{n,1},$$

$$L_{10} = \sum (n+1)\alpha^n = D\sum \alpha^{n+1} = (1-\alpha)^{-2},$$

$$D = \frac{d}{d\alpha}$$

Thus $-(1-\alpha)^{-2} = -(1-\alpha)^{-1} M_1$, which gives the value $M_1 = (1-\alpha)^{-1}$, which is the mean delay of delayed calls. This is the same as the mean delay in the case of service in order of arrival. Riordan then introduced functions L_{jk} defined by $L_{0,k} = \sum \alpha^n m_{n,k} = (1-\alpha)^{-1} M_k$ and $L_{jk} = \sum (n+1)(n+2)\cdots(n+j)\alpha^n m_{n,k}$ $(j = 0,1,2\cdots)$, and the quantity R_k, defined as the ratio of the moment M_k to the corresponding moment for service in order of arrival. Thus $R_k = L_{0k}(L_{k0})^{-1}$, which is equal to $(1-\alpha)^k \frac{M_k}{k!}$.

From the above recurrence relations between the moments, Riordan was able to deduce a corresponding relation between the L's which could be solved and which thus led to the determination of the R's. The first three values of R_k are found to be

$$R_1 = 1 \; ; \; R_2 = \frac{2}{2-\alpha} \text{ and } R_3 = \frac{2(2+\alpha)}{(2-\alpha)^2} .$$

In this way he was able to give the actual values of the first eight moments (including M_0).

Results suggested that $F(u)$ might be approximated by the sum of a few exponential terms, of the form

$$F(u) = A_1 e^{-\frac{(1-\alpha)u}{x_1}} + A_2 e^{-\frac{(1-\alpha)u}{x_2}} + \cdots ,$$

where $A_1, A_2 \ldots$ and $x_1, x_2 \ldots$ are parameters determined by fitting moments.

As an example, with 2 exponentials we require

$$A_1 + A_2 = 1 \ , \ A_1 x_1 + A_2 x_2 = 1 \ , \ A_1 x_1^2 + A_2 x_2^2 = R_2$$

and $$A_1 x_1^3 + A_2 x_2^3 = R_3$$

Using known values of R_2 and R_3 we find from these equations

$$x_1^{-1} = 2A_1 = 1 - \sqrt{\tfrac{\alpha}{2}} \ , \ x_2^{-1} = 2A_2 = 1 + \sqrt{\tfrac{\alpha}{2}} \ ,$$

(i.e.) the approximation for F(u), by taking two terms only is

$$\frac{1}{2}\left[\left(1-\sqrt{\tfrac{\alpha}{2}}\right) e^{-u(1-\alpha)(1-\sqrt{\alpha/2})} + \left(1+\sqrt{\alpha/2}\right) e^{-u(1-\alpha)(1+\sqrt{\alpha/2})}\right],$$

which is a good fit for $\alpha < 0.7$ roughly.

Riordan concluded that a small number of exponentials in the sum (≤ 3) is enough to give a good fit.

In an addendum to his paper Riordan stated that Pollaczek's integral for F(u), referred to in § 2.6, had been evaluated numerically by Rice for values of u up to 140, and that the agreement with his own values was close enough to suggest that approximation by the sum of a few exponentials is satisfactory.

Riordan's paper appears to be a major contribution to the work on this problem, firstly because he was able to evaluate the moments of the distribution exactly and secondly because he was able to get satisfactory numerical results by replacing the distribution function by such a simple expression as the sum of a few exponentials.

Wilkinson (1953) [14] carried out simulation tests to check agreement with Riordan's results. He examined the delays of 3000 calls with $c = 2$, $\alpha = 0.9$ and 1500 calls with $c = 10$, $\alpha = 0.8$, and found satisfactory agreement.

2.8 Le Roy, (1957) [5], gave a discussion of the problem in matrix notation. He first obtains a formal solution, in matrix form, which is effectively a Maclaurin series, of a system of first order linear differential equations with constant coefficients and with starting values for the functions at the origin. He then discusses the question of fitting by moments, i.e., he approximates to a required distribution function by means of a function of chosen form, having the same moments (up to a specified order) as the required function. The form he chose is $e^{-u}[d_o + d_1 u + d_2 \frac{u^2}{2!} + \dots + d_p \frac{u^p}{p!}]$, where d_p is a constant and p a positive integer, and to find the coefficients d_p, he uses Laguerre polynomials.

He prefers this form to the sum of exponentials chosen by Riordan, because he considers the calculations simpler. In fact there seems to be little difference between the two methods in mathematical complexity.

He discusses also the following question: given the function

$$F(u) = e^{-u}\left[d_0 + d_1 u + d_2 \frac{u^2}{2!} + \dots + d_p \frac{u^p}{p!}\right] \qquad (1)$$

(i.e. d_o, d_1, ... d_p are known) and its first q moments M_1, M_2, ... M_q, it is required to find d_j ($j = p + 1, p + 2, \dots p + q$) in terms of d_o, d_1, ... d_p, M_1, M_2, ... M_q, such that $e^{-u} \sum_{j=0}^{(p+q)} d_j \frac{u^j}{j!}$ has the same moments as $F(u)$ defined in (1).

He now expresses Vaulot's fundamental equations (3) in matrix form and gives their formal solution, effectively a Maclaurin series. For numerical calculation, however, he prefers the form

$F(u) = e^{-u} [d_o + d_1 u + \ldots + d_p \frac{u^p}{p!}]$ and he calculates d_o (=1), d_1 and d_2 from the differential equations. He also finds a recurrence relation between the d_n in a form similar to that of Palm's. He shows too that there is a recurrence relation between the moments, which can be solved, and calculates M_1, M_2, M_3 and M_4.

Using matrix notation, he discusses Riordan's method of solving the recurrence relation for the moments, and shows that it is effectively the same as his own.

As a numerical example, he takes the values of d_o, d_1, d_2, M_1, M_2, M_3, M_4 already found, and $\alpha = \frac{1}{2}$, and finds d_3, d_4, d_5 and d_6 in the expression (1) for F(u). Then he evaluates F(u) for u = 1.

The use of matrix algebra adopted by Le Roy to solve the problem is interesting. For practical calculations, however, his method seems to be complicated, as he was able to get only 7 of the d-coefficients in his numerical example. Since Riordan seems to get a good fit with a sum of only 2 or 3 exponentials, the disadvantages of his method are not important in practice.

2.9 Finite queue

A situation which clearly leads to similar equations and which is perhaps more realistic, but which does not seem to have been discussed before, is that in which calls are chosen at random from a queue which is restricted to a finite size k. It is true that such a case is not likely to be suitable for theoretical treatment, but the fact that equations (3) now become finite in number makes it possible to solve them by computer for values of k large enough to be practical and to give an approximation to the result for the infinite queue.

Both the equilibrium state probabilities and the differential equations for the delay distribution must be modified for this case. In the transition equations of § 2.2 above, ν now takes values from 0 to c + k; equations (1) become

$$P_\nu = P_0 \frac{a^\nu}{\nu!}, \quad \nu = 1, 2, \cdots c; \quad P_\nu = P_0 \frac{a^\nu}{c!\, c^{\nu - c}},$$
$$\nu = c, (c+1), \cdots (c+k),$$

where $\sum_{\nu=0}^{c+k} P_\nu = 1$, so that

$$P_0^{-1} = \sum_{j=0}^{c-1} \frac{a^j}{j!} + \frac{a^c}{c!}\left[\frac{1 - \left(\frac{a}{c}\right)^{k+1}}{1 - a/c}\right].$$

We are interested in what happens to a call which is delayed and <u>is admitted to the queue</u>; the probability that an arriving call is delayed but is admitted is $\sum_{\nu=c}^{c+k-1} P_\nu = P_c\left[\frac{1-(a/c)^k}{1 - a/c}\right]$,

and the conditional probability that such a call, on its arrival, finds n others waiting is

$$\frac{1-\alpha}{P_c(1-\alpha^k)} P_{c+n} \quad \text{or} \quad \frac{1-\alpha}{1-\alpha^k}\alpha^n, \quad n = 0, 1, \cdots (k-1),$$
$$\left(\alpha = \frac{a}{c}\right)$$

Equations (3) are now k in number (n = 0, 1, ... k - 1), because we are considering only calls which are delayed but are admitted to the system. Since n is the number of calls waiting when the call in question arrives, the maximum value of n is k - 1. The first (k - 1) equations are unchanged, but the last one becomes

$$\frac{dF_{k-1}(u)}{du} = \frac{k-1}{k} F_{k-2}(u) - F_{k-1}(u).$$

Therefore the k equations now are

$$\left.\begin{aligned} \frac{dF_n(u)}{du} &= \frac{n}{n+1}F_{n-1}(u) - (1+\alpha)F_n(u) + \alpha F_{n+1}(u) \\ &\qquad\qquad \text{for } n = 0, 1, 2 \cdots\cdots k-2 \\ \text{and} \quad \frac{dF_{k-1}(u)}{du} &= \frac{k-1}{k}F_{k-2}(u) - F_{k-1}(u). \end{aligned}\right\} \quad (3')$$

Thus finally, the probability that a call which is delayed but is admitted will be delayed more than t is given by

$$F(u) = \frac{1-\alpha}{1-\alpha^k}\sum_{n=0}^{k-1}\alpha^n F_n(u) \quad - - - \cdots\cdots \quad (4')$$

with $F(0) = 1$.

By means of the Sirius computer, the equations (3') were solved for $\alpha = 0.9$ and $k = 20$, 40 and 60 and for values of u up to 50. The results are given below.

Solution of finite set of differential equations

The above process involved the numerical solution of 20, 40 and 60 linear differential equations. The right hand side terms are relatively simple in that they never have more than three non-zero terms. A programme was written which constructed the right hand sides and then the equations were solved by a Runge-Kutta process. With 20 equations, each cycle took less than 1 minute, while 40 and 60 equations took for each cycle less than 2 and 3 minutes respectively. The results obtained are shown in Table No. 1 attached. Riordan's results for $\alpha = 0.9$ for the infinite queue are also given for comparison.

A system with a finite number of places in the queue is in a sense not directly comparable with one with an infinite number, for in the former our attention is restricted to calls which gain admission to the system, while in the latter the question of admission does not arise. Nevertheless a comparison between Riordan's results and ours may be of interest as indicating how good an approximation the infinite-number case gives for practical systems, which necessarily have a finite number of places.

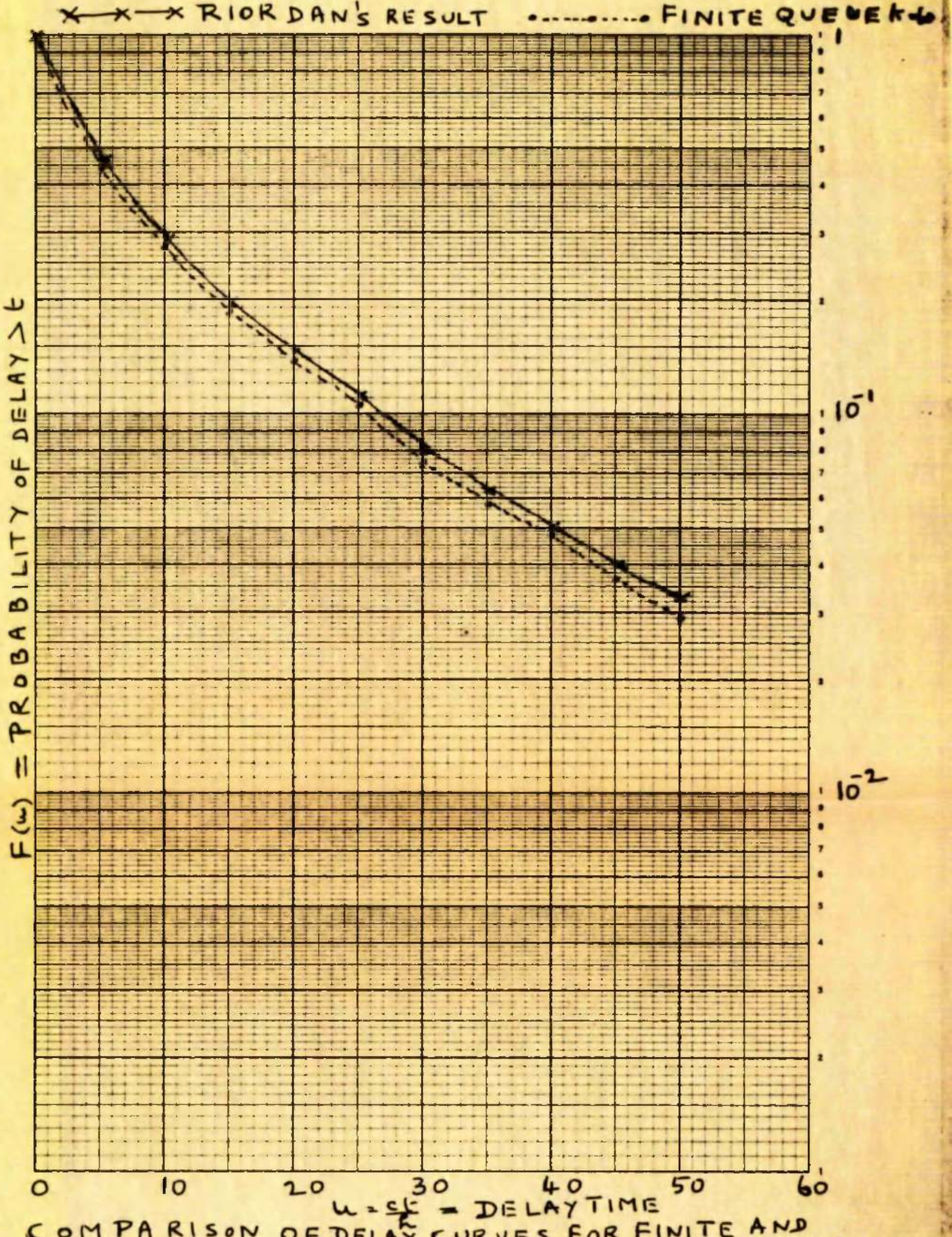
Thus from Table No. 1 we see that the values of $F(u)$ for $k = \infty$ and $k = 60$ differ by less than 10^{-3} when $u = 15$, whereas at that value of u the values for $k = \infty$ and $k = 40$ differ by nearly 8×10^{-3}; on the other hand, when u is as low as 5 the values of $F(u)$ for $k = \infty$ and $k = 20$ differ by nearly 6×10^{-2}. We can interpret these results for practical cases by remembering that $u = \frac{ct}{h}$: the value $u = 15$ corresponds to 15 average holding-times when $c = 1$, to 1.5 when $c = 10$ and to 0.3 when $c = 50$. In fact from the table it appears that the infinite-queue results give a good approximation, even for values of u as high as 50, for practical cases in which the queue is limited to size 40 or more, whereas the approximation is rather inaccurate for u as low as 5 when the queue is limited to size 20.

Graph No. 1 gives the delay distribution curve for the finite queue when $k = 60$ in comparison with that for the infinite queue given by Riordan.

TABLE No. 1

u / F(u)	0	5	10	15	20	25	30	35	40	45	50
Riordan's result for infinite queue	1.0000	0.4700	0.2860	0.1940	0.1450	0.1100	0.0800	0.0650	0.0500	0.0400	0.0330
Finite queue k = 20	1.0000	0.41179	0.22913	0.13934	0.08867	0.05796	0.03854	0.02591	0.01755	0.01195	0.00816
F.Q k = 40	1.0000	0.45693	0.27888	0.18636	0.13091	0.09502	0.07061	0.05341	0.04098	0.03179	0.02489
F.Q k = 60	1.0000	0.46268	0.28576	0.19339	0.13771	0.10144	0.07659	0.05893	0.04604	0.03642	0.02912

Finite queue results compared with Riordan's result, for infinite queue.



COMPARISON OF DELAY CURVES FOR FINITE AND INFINITE QUEUE

FIG. No. 1

3. Poisson arrivals, constant holding-time

3.1 Introduction

An assumption which is often made about holding-time is that it is constant. This is at the opposite extreme to exponential holding-time. It has a practical application in telephony, being approximately realised with long-distance calls.

3.2 The equilibrium state probabilities for this case were given by Crommelin in 1932 [2], his argument being as follows. Take the constant holding-time as unity. The probability that the system is in a given state at the beginning of a unit time interval is equal to the probability of the same state at the end of the interval. The equations satisfied by the P_v may be obtained by the following argument.

Let us consider the states of the system at the beginning and at the end of a unit interval of time. If the number of calls in the system at the beginning is c or less and there are no arrivals during the unit time interval, the number of calls at the end of the interval will be zero (the holding-time being unity in this case). Thus $P_0 = b_c\, e^{-a}$, where $b_c = \sum_{v=0}^{v=c} P_v$ = probability of not more than c calls in the system and a is the parameter of the Poisson arrival input.

By similar arguments we get

$$\left.\begin{aligned} P_1 &= b_c\, a\, e^{-a} + P_{c+1} e^{-a} \\ P_2 &= b_c \frac{a^2}{2!} e^{-a} + P_{c+1}\, a\, e^{-a} + P_{c+2} e^{-a} \\ &- - - - - - - - - - - - \\ &- - - - - - - - - - - - \\ P_v &= b_c \frac{a^v}{v!} e^{-a} + P_{c+1} \frac{a^{v-1}}{(v-1)!} e^{-a} + \dots + P_{c+v} e^{-a}. \end{aligned}\right\} \quad (8)$$

Using the generating function technique, the values of P_ν are obtained as follows

$$\text{Let } f(z) = \sum_{\nu=0}^{\infty} z^\nu P_\nu, \text{ where } f(1) = \sum_{\nu=0}^{\infty} P_\nu = 1$$

Since $0 \le P_\nu \le 1$, $f(z)$ is regular and bounded for $|z| < 1$. Multiplying equations (8) by z^0, z^1 ..... respectively and adding, we get

$$f(z) = b_c e^{-a} e^{az} + z P_{c+1} e^{-a} e^{az} + z^2 P_{c+2} e^{-a} e^{az} + \dots$$

$$= e^{a(z-1)} \left\{ b_c + \sum_{\nu=1}^{\infty} z^\nu P_{c+\nu} \right\}$$

$$\therefore z^c f(z) = e^{a(z-1)} \left\{ z^c b_c + \sum_{\nu=c+1}^{\infty} z^\nu P_\nu \right\}$$

$\therefore$ if $Q_c(z) = \sum_{\nu=0}^{c} z^\nu P_\nu$ (so that $Q_c(1) = b_c$),

then $z^c f(z) = e^{a(z-1)} \left(z^c b_c + f(z) - Q_c(z) \right)$

$$\therefore f(z) = \frac{Q_c(z) - z^c b_c}{1 - z^c e^{a(1-z)}}$$

If this is to give a determinate solution for P_0, P_1 ... P_c then $\phi(z) \equiv 1 - z^c e^{a(1-z)}$ must have the same number of roots in and on the unit circle as $Q_c(z) - z^c b_c$.

Crommelin gives an analytical proof also of the fact that the equation $\phi(z) = 0$ has c roots in and on the unit circle and that the only root of unit modulus is unity. Note that $z = 1$ is a root of $Q_c(z) - z^c b_c = 0$, since $Q_c(1) = b_c$. So we can write now

$$Q_c(z) - z^c b_c = k(z-1)(z-\lambda_1)\cdots\cdots(z-\lambda_{c-1}),$$

where λ_n ($n = 1, 2, \ldots c-1$) is a root of $\phi(z) = 0$ such that $|\lambda_n| < 1$

Thus $$f(z) = \frac{k(z-1)(z-\lambda_1)\cdots\cdots(z-\lambda_{c-1})}{\phi(z)}, \quad ----(9)$$

and k must be such that $f(1) = 1$.

Hence
$$k = -\frac{(c-a)}{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_{c-1})}$$

The required P_v are the coefficients of the powers of z in the expansion of (9) when $|z| < 1$. For practical determination of the values of the P's, however it appears to be more convenient to solve the equations (8) numerically.

Crommelin worked out the delay distribution for order-of-arrival service using generating functions and gave a numerical example for 10 lines and for traffic intensity .4. Curves showing the probabilities of delays exceeding various specified times were also given and compared with the corresponding results for exponential holding-times.

In a later paper, 1934 [3] he extended this work.

3.3 Burke, in 1959 [1], took up the problem of the equilibrium delay distribution for one channel with constant holding-time, Poisson arrivals and random service. The problem for order-of-arrival service had already been widely investigated, in particular by Pollaczek in 1930 and 1959 and by Crommelin in 1932 and 1934; but Burke's appears to be the first attempt to consider the case of random service with constant holding-time. He discussed the problem for one line.

He assumes statistical equilibrium and points out that the delay consists of two parts, a fractional part at the beginning, followed by an integral part. The first part is the time from the arrival instant of the call in question to the first departure epoch after the arrival of this call. The second part is the time from the first departure epoch to the time the call in question gains service. Since at the first departure epoch random selection from the waiting calls is adopted, the second part of the delay is statistically independent of the first part. The probability distribution for the first part is a uniform distribution over the interval 0 to 1 because of the assumption of random arrivals.

The state of the system at the first departure epoch after the arrival of the call in question, which is delayed, is the basis on which the problem is built. The service-time is taken as unity. Burke's arguments are as given below.

Let p_n = equilibrium probability that a delayed call has n other calls waiting with it at first departure epoch after its arrival, $n = 0, 1, \ldots$ etc. There can be n others waiting with it then only if

(i) at the last departure <u>before</u> its arrival there were j calls in the system and during the holding-time $(n + 1 - j)$ other calls arrived $(j = 1, 2, \ldots\ n+1)$ (at a departure epoch, the call just finishing service is not included in the counting);

(ii) at the last departure before its arrival there were zero calls in system and during the holding-time n arrived; thus

$$p_n = \sum_{j=1}^{n+1} P_j e^{-a} \frac{a^{n-j+1}}{(n-j+1)!} + P_0 e^{-a} \frac{a^n}{n!}$$

(i.e.) $$p_n = (P_0+P_1) e^{-a} \frac{a^n}{n!} + P_2 e^{-a} \frac{a^{n-1}}{(n-1)!} + \cdots\cdots + P_{n+1} e^{-a},$$

which is the same as P_n obtained from Crommelin's equations (8) for the case $c = 1$.

If, following Burke, we write $G(t/n)$ as the conditional probability of a delay $\leq t$, given that the delayed call is with n others at the first departure epoch after its arrival, and $G(t)$ for the delay distribution for an arbitrary call, then $G(t) = \sum_{n=0}^{\infty} p_n G(t/n)$ - - - - - - - - (10)
Let the actual delay be T (a random variable), consisting of a fractional part T" followed by an integral part T'.

(i) T' is independent of T", because of the random choice at the first departure epoch after the arrival of the call in question

(ii) T" has a uniform distribution over (0 to 1) because of the random arrival of the calls.

Now let us assume for a given time t, as in the case of T, that the fractional part is t" and the integral part is t'.

Then $$G(t/n) = \text{Probability } (T \leq t|n)$$
$$= Pr(T' < t'|n) + Pr[T' = t' \text{ and } T'' \leq t''|n]$$
$$= Pr(T' < t'|n) + t'' Pr(T' = t'|n)$$
$$= \sum_{i=0}^{t'-1} Pr(T' = i|n) + t'' Pr(T' = t'|n) \text{.....(11)}$$

If we write Probability $(T' = i/n) \equiv Q_i(n)$, then $Q_0(n) = \frac{1}{n+1}$ and

$$Q_i(n) = \left\{1 - Q_0(n)\right\} \sum_{j=0}^{\infty} e^{-a}\left(\frac{a^j}{j!}\right) Q_{i-1}(n+j-1) \qquad \ldots (12)$$
$$i = 1, 2, \ldots$$

It is possible to write an expression for $Pr.(T' = i/n)$ by direct probability arguments, but for computation the recurrence relation (12) is more convenient. From (9) we get $P_0 = 1 - a$ and then from P_0 the values of the P_n can be obtained by recurrence from (8). Then $G(t)$ is obtained by substituting in (10) the values of $G(t/n)$ and P_n obtained by the recurrence relations stated above.

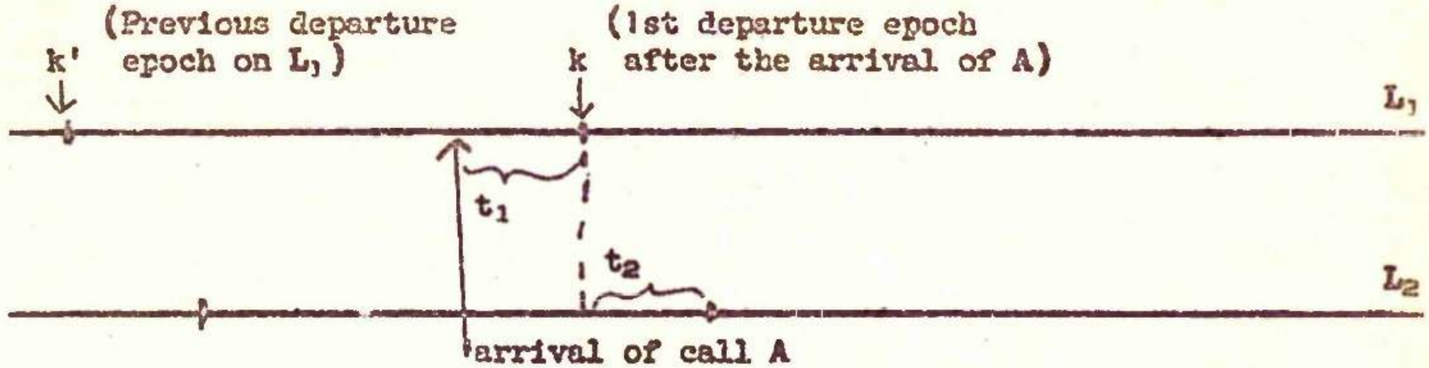
Burke gives graphs for the distribution of delay up-to 130 holding-times and compares the results with those for random-selection queues for exponential holding-times given by Wilkinson, for large occupancy levels.

Burke's paper is notable for its clear statement of the assumptions underlying the solution, for its emphasis on the fact that the delay distribution depends essentially on the state of the system at the next departure epoch following the arrival of the call in question, and for its derivation of the equilibrium state probabilities at that instant. As will be seen immediately, Burke's method can up-to a point be extended to systems with more than one line.

3.4 <u>Extension of Burke's formulae for two lines</u>

(i) <u>Equilibrium state probabilities for $c = 2$ - constant holding time</u>.

Let us consider the case of two lines L_1 and L_2 as shown in the figure below, to get the equilibrium state probability p_n.



A call A arrives and finds both lines engaged. The first departure epoch after A's arrival will be denoted by k, and the line on which it occurs will be called L_1. The probability that at k there will be $(n + 1)$ calls waiting, including A [i.e. $(n + 2)$ altogether in the system - the departing call is not included in the counting] will be denoted by p_n.

To find a relation involving the p_n in the equilibrium case we fix attention on k', the last departure epoch on L_1 before the arrival of A. Then if at k' the number of calls in the system exceeds one, k' is exactly one unit of time earlier than k, but if at k' the number in the system is one or none, k' is more than one unit earlier than k. In that case, since both lines are engaged when A arrives, there is certain to have been an arrival (of a call B, say) exactly one unit earlier than k: in arriving at the relation involving the p_n we must take into account arrivals during the (unit) service-time of B. Similarly, if the number in the system at k' is two [(i.e.) a call just starts service on L_1 at k' and a call is in process of being served on L_2], there is certain to have been an arrival between k' and the arrival of A, otherwise L_2 would not be engaged when A arrives.

It follows then that if there are n calls waiting with A at k, there could have been j in the system at k' [$j = 3, 4, \ldots (n + 3)$]

with $(n + 3 - j)$ arrivals (in addition to that of A) in the service-time ending at k, or 2, 1, 0 at k', with n arrivals in the service-time ending at k. Thus, since in equilibrium the state-probabilities are independent of the time of k',

$$p_n = (P_0 + P_1 + P_2 + P_3)\frac{a^n}{n!}e^{-a} + P_4\frac{a^{n-1}}{(n-1)!}e^{-a} + \cdots\cdots + P_{n+3}e^{-a}.$$

This is Crommelin's formula for P_n for $c = 3$. Therefore the conditional probability p_n for $c = 2$ is identical with the state-probability P_n for $c = 3$, and can be obtained by solving Crommelin's equations (8).

(ii) Delay distribution

To get the conditional probability of a delay $\leq t$, given that the delayed call is waiting with n others at the first departure epoch after its arrival, we proceed as follows.

A call A arrives and finds both the lines L_1 and L_2 are engaged. Let the first release be at time t_1 after its arrival, on line L_1, and let there be n other calls waiting with it then. The second release is on L_2, at time t_2 after the first release on L_1.

Then $0 < t_1 < 1$, $(0 < t_2 < 1)$; $0 < t_1 + t_2 < 1$ (i.e.), $0 < t_2 < 1 - t_1$. If we write $G(t/n, t_1, t_2)$ for the conditional probability of a delay $\leq t$ in these circumstances, and $G(t/t_1, t_2)$ as the delay distribution for an arbitrary delayed call, which arrives a time t_1 before the first release on L_1, which is itself time t_2 before the first release on L_2, then

$$G(t \mid t_1, t_2) = \sum_{n=0}^{\infty} p_n G(t \mid n, t_1, t_2) \quad \text{----} \quad (C)$$

The delay T consists of a fractional part T" at the beginning (up to the first release) and a part T' after the first release, which may be integral, (if it gains service on line L_1) or partly integral, partly fractional (if it gains service on line L_2). These can be seen clearly from the figure shown below.

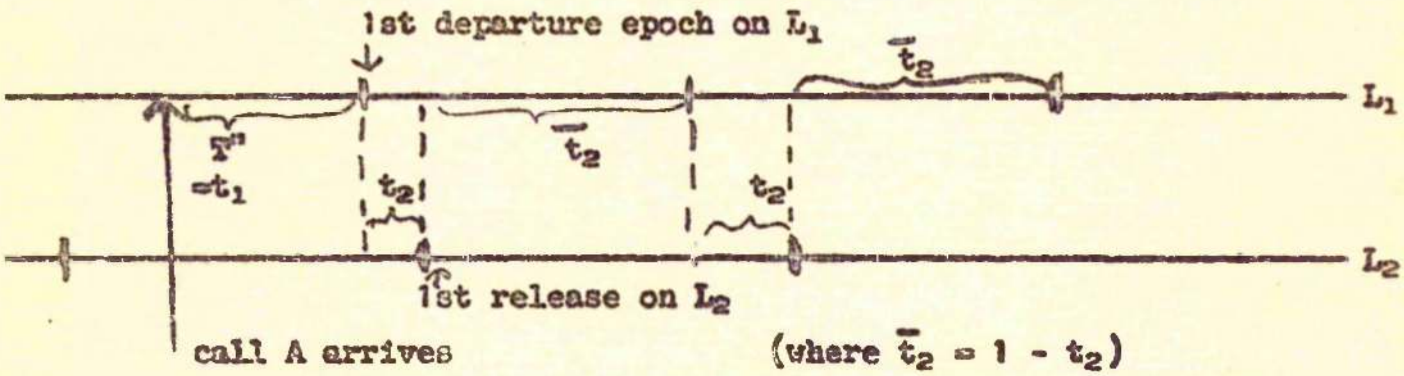


FIGURE 1

Let [T'] denote the integer part of T' and let us assume for a given time t that the fractional part is t" and the integral part t'.

Then $G(t/n, t_1, t_2)$ = Probability $\{[T'] < t'/n\}$

+ Probability $\{[T'] = t'$ and fractional part of $T \leq t''/n\}$

Let us define $P_n(i/t_2)$ as the probability that the call in question is present with n others just before a departure epoch, the next release being at t_2 (on the other line) and is delayed i intervals of time. By an interval we mean the time between successive departures on lines L_1 and L_2 starting from the first departure epoch on L_1. That is the time intervals are t_2 and $\bar{t}_2$ alternately, until the call gains service. Note that $(t_2 + \bar{t}_2) = 1$ as shown in Figure 1 above.

$$\therefore G(t|n, t_1, t_2) = \Pr\left[i \leq (2t'-1) \middle| n\right] + \Pr\left[i = 2t' \text{ and } t_1 \leq t'' \middle| n\right] + \Pr\left[i = (2t'+1) \text{ and } (t_1+t_2) \leq t'' \middle| n\right]$$

(where i = number of intervals of time t_2 and $\bar{t}_2$ taken alternately from first release to actual departure)

$$= \sum_{i=0}^{2t'-1} P_n(i|t_2) + \Pr(t_1 \leq t'')\, P_n(2t'|t_2) + \Pr\left[(t_1+t_2) \leq t''\right] P_n\left[(2t'+1)\middle| t_2\right]$$

where t_1 and t_2 have uniform distribution over 0 to 1 since the arrival of the call is random.

The first release can be on either line and hence multiplying the result by 2, we get

$$G(t|n) = 2\left\{ \int_0^1 dt_1 \int_0^{1-t_1} \sum_{i=0}^{2t'-1} P_n(i|t_2)\,dt_2 + \int_0^{t''} dt_1 \int_0^{1-t_1} P_n(2t'|t_2)\,dt_2 + \int_0^{t''} dt_1 \int_0^{t''-t_1} P_n\left[(2t'+1)\middle| t_2\right] dt_2 \right\}$$

The value of $P_n(i/t_2)$ in the integrand can be evaluated for suitable values of n, i, t_2 by means of the recurrence formula given below and then a numerical integration process can be used to get the value of $G(t/n)$.

$$P_n(0|t_2) = \frac{1}{n+1} \; ; \; n = 0, 1, 2, \cdots\cdots$$

$$P_n(i|t_2) = \sum_{j=0}^{\infty} e^{-\lambda t_2} \frac{(\lambda t_2)^j}{j!} P_{n+j-1}(i-1|\bar{t}_2)\left(\frac{n}{n+1}\right).$$

where $\bar{t}_2 = 1 - t_2$, $i = 1, 2, \cdots$, $n = 0, 1, 2, \cdots$, λ is Poisson arrival input parameter.

It appears, therefore, that Burke's argument can be extended to the case of 2 lines, giving a practical numerical procedure. It was felt, however, that an attempt to solve the problem by Monte Carlo methods would be not only simpler but easier to apply to cases with more than two lines.

3.5 Monte Carlo Method

(i) It is clear that the problem lends itself to investigation by Monte Carlo methods. As a check, it was decided first to examine the case c = 1 in this way, to see if good agreement with Burke's results could be obtained.

A programme for the Sirius computer was written to find the state probabilities in the case c = 1. Using Crommelin's equations (8) when c = 1, the values of P_1, P_2 .... are obtained by a step-by-step process, starting with the value of $P_0 = 1 - \alpha$,

Let $R_n = P_0 + P_1 + \cdots + P_n = \sum_{i=0}^{n} P_i$; then

$$P_n = R_n - R_{n-1} \; (n > 0) \text{ and } P_0 = R_0$$

The values of R_0, R_1, R_2 ... were obtained up to the stage at which the value was .9999 correct to 4 places of decimals.

A Monte Carlo programme was written in which four digit random numbers (interpreted as probabilities to 4 decimal places) were used to determine

(i) how many calls are present with the given call at the first departure epoch after its arrival, using the values of

R_0, R_1, R_2 ... already obtained;

(ii) whether the given call is chosen for service at a release;

(iii) if it is not chosen, how many other calls arrive before the next release.

Thus the history of each call could be followed. The results obtained for the delay times of calls were arranged in a frequency distribution and the delay distribution was worked out for comparison with Burke's results.

For $\alpha = 0.9$, three sets of independent data, containing respectively 500, 500 and 428 calls were obtained, to examine the consistency in the results. They are given below in Table II along with Burke's for comparison. It is evident that the results are fairly consistent and that they agree satisfactorily with Burke's results, as read from his graph. The attached graph No. 2 shows the curves of Burke's result and the average value of the three sets obtained by the Monte Carlo method. From the curves it is seen that there is good agreement between the two for values of u up to 30. For higher values of $\cancel{\mu}$ u the Monte Carlo method gave very few observations and this probably accounts for the difference between the results.

The results are given in terms of $\alpha[1 - G(t)]$, as is usual in delay problems.

TABLE II

DELAY DISTRIBUTION FOR c = 1 BY MONTE CARLO METHOD

Delay greater than t / 0.9 [1-G(t)]	1	2	3	4	5	6	7	8	9	10
1st set of 500 items	0.6372	0.4896	0.4014	0.3204	0.2754	0.2286	0.1854	0.1476	0.1278	0.1044
2nd set of 500 items	0.5994	0.4464	0.3762	0.3060	0.2412	0.2032	0.1818	0.1602	0.1422	0.1224
3rd set of 428 items	0.6030	0.4410	0.3570	0.2830	0.2370	0.1970	0.1630	0.1480	0.1340	0.1170
Average of the three	0.6138	0.4607	0.3794	0.3044	0.2520	0.2111	0.1777	0.1525	0.1348	0.1147
Burke's result	0.61	0.48	0.38	0.31	0.27	0.22	0.19	0.17	0.15	0.12

TABLE II (CONTINUATION)

Delay greater than t / 0.9 [1-G(t)]	11	12	13	14	15	16	17	18	19	20
I Set	0.0918	0.0810	0.0738	0.0666	0.0612	0.0558	0.0540	0.0522	0.0486	0.0486
II Set	0.1102	0.0954	0.0882	0.0792	0.0666	0.0612	0.0540	0.0486	0.0450	0.0396
III Set	0.1110	0.0940	0.0880	0.0790	0.0640	0.0520	0.0500	0.0500	0.0470	0.0450
Average of the three	0.1040	0.0901	0.0832	0.0762	0.0643	0.0567	0.0529	0.0504	0.0473	0.0447
Burke's result	0.110	0.090	0.080	0.076	0.070					0.045

TABLE II (CONTINUED)

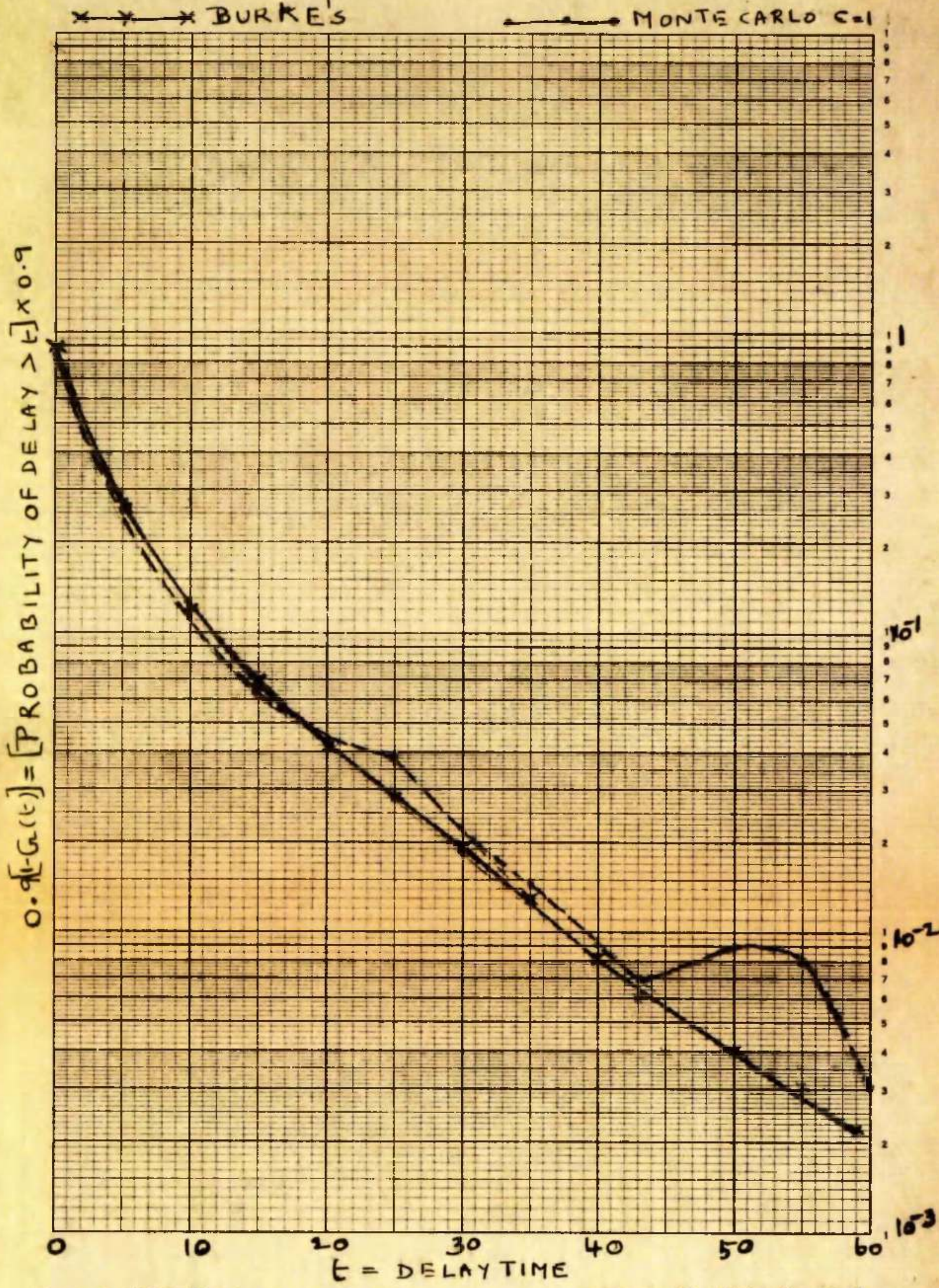
Delay greater than t / 0.9 [1-G(t)]	21	22	23	24	25	27	28	29	30	32	33	35
I Set	0.0450	0.0432	0.0396	0.0396	0.0396	0.0378	0.0342	0.0306	0.0252	0.0234	0.0198	0.0162
II Set	0.0360	0.0324	0.0270	0.0234	0.0234	0.0234	0.0216	0.0180	0.0162	0.0162	0.0144	0.0126
III Set	0.0440	0.0410	0.0410	0.0300	0.0250	0.0250	0.0250	0.0220	0.0200	0.0180	0.0180	0.0180
Average of the three	0.0416	0.0391	0.0359	0.0315	0.0296	0.0290	0.0271	0.0239	0.0208	0.0195	0.0176	0.0158
Burke's result					0.0300				0.0190			0.0140

TABLE II (CONTINUED)

Delay greater than t / 0.9 [1-G(t)]	39	40	41	42	43	47	50	53	55	59	66	73
I Set	0.0144	0.0144	0.0144	0.0144	0.0108	0.0108	0.0090	0.0090	0.0090	0.0090	0.0090	0.0090
II Set	0.0090	0.0090	0.0072	0.0072	0.0072	0.0054	0.0054	0.0036	0.0018	0.0018	0.0018	0.0018
III Set	0.0180	0.0160	0.0160	0.0120	0.0090	0.0090	0.0090	0.0080	0.0080	0.0060	0.0030	0.0018
Average of the three	0.0139	0.0132	0.0126	0.0113	0.0095	0.0088	0.0082	0.0069	0.0063	0.0057	0.0050	0.0045
Burke's result		0.008					0.004		0.003			0.001

TABLE II (CONTINUED)

Delay greater than t / 0.9 [1-G(t)]	77	80	81	83	94
I Set	0.0072	0.0054	0.0054	0.0018	0.0018
II Set	0.0018	0.0018	0.0018	0.0018	
III Set	0.0018	0.0018			
Average of the three	0.0038	0.0032			
Burke's result		0.0007			



COMPARISON OF DELAY CURVES C=1 ; α = 0.9

FIG. No. 2

52-A

(ii) In Crommelin's equations, when $c > 1$, the equations can no longer be solved by a step-by-step process. $P_v \to 0$ as $v \to \infty$, however, so that all the P's from a certain stage onward will be negligible to a given degree of accuracy; if the higher P's are equated to zero, the equations can be solved for the lower P's as a finite set of linear algebraic equations. It was convenient on the computer to put P_{36} and higher P's as zero and solve the resulting 35 equations. In fact it was found that P_{15} and onwards were negligible. The values of R_0, R_1 ... up to the one which, correct to 4 places of decimals, was equal to .9999 were obtained from the P's. The Monte Carlo programme for this case was as follows: Four digit random numbers interpreted as decimal fractions were obtained from the computer and used to represent the probabilities of the various random occurrences in the problem. Of the first two random numbers, the smaller one was taken as t_1 and the difference between the two as t_2, so that $t_1 + t_2 < 1$. Random numbers in succession were then used to represent the probabilities of the problem at each stage to see whether the arbitrary call A obtained service on line L_1 or L_2 and to get the delay time. Thus the programme finds, by means of random numbers,

(i) how many calls are waiting with the given call A at the first departure epoch on L_1 after its arrival, using the values of R_0, R_1, R_2 ... already obtained;

(ii) whether call A gains service on L_1 at the first departure epoch;

(iii) if the call A is not chosen for service at the first departure epoch on L_1, how many other calls arrive in time t_2 before the next release on line L_2;

(iv) whether call A gains service on L_2 at the first departure epoch on L_2;

(v) again if the call A is not chosen for service on line L_2 at its first departure epoch, how many other calls arrive in time $\bar{t}_2$ before the next release on line L_1.

The above processes (ii) to (v) are repeated to trace the history of each call and to find its delay time. The arrivals in (iii) and (v) follow Poisson distributions with mean a t_2 and a $\bar{t}_2$ alternately, and the programme had to be constructed to take account of this alternation.

For two independent sets of 1000 calls each, the delay times were obtained and the delay distributions were worked out. The two delay distributions are shown in Table III. It is clear from the figures that there is consistency in the results. Graph III shows the two curves. In Graph IV the results of the two sets combined are given. There appear to be no results available, even for service in order of arrival, against which these can be directly compared. In his 1934 paper, Crommelin gives results for delays up to 6 holding-times, for constant holding-time and service in order of arrival, for 2 lines but for $\alpha = 0.8$. His results are shown in graph IV. The agreement with ours appears to be satisfactory and, as is to be expected, his graph lies above ours for small values of t and below for large values of t.

It seems reasonable to conclude that systems involving random service are best examined by Monte Carlo methods, for there appears to be no difficulty in applying the methods not only to larger systems of the types we have discussed but also to others in which perhaps more general, or more realistic, assumptions are made regarding the arrival and service-time distributions.

Further work on these lines would appear to be worthwhile. In addition, the extension, given in § 3.4, of Burke's work can probably be applied to more than two lines, and it seems possible that this might lead to a practical method of numerical evaluation of the distribution.

TABLE III

I set of 1000 items   Delay time Distribution for c = 2 by Monte Carlo method

Delay greater than t	1	2	3	4	5	6	7	8	9	10	11	20
[1 - G(t)] Probability of delay > t	0.354	0.159	0.089	0.061	0.040	0.050	0.019	0.014	0.011	0.009	0.006	0.001

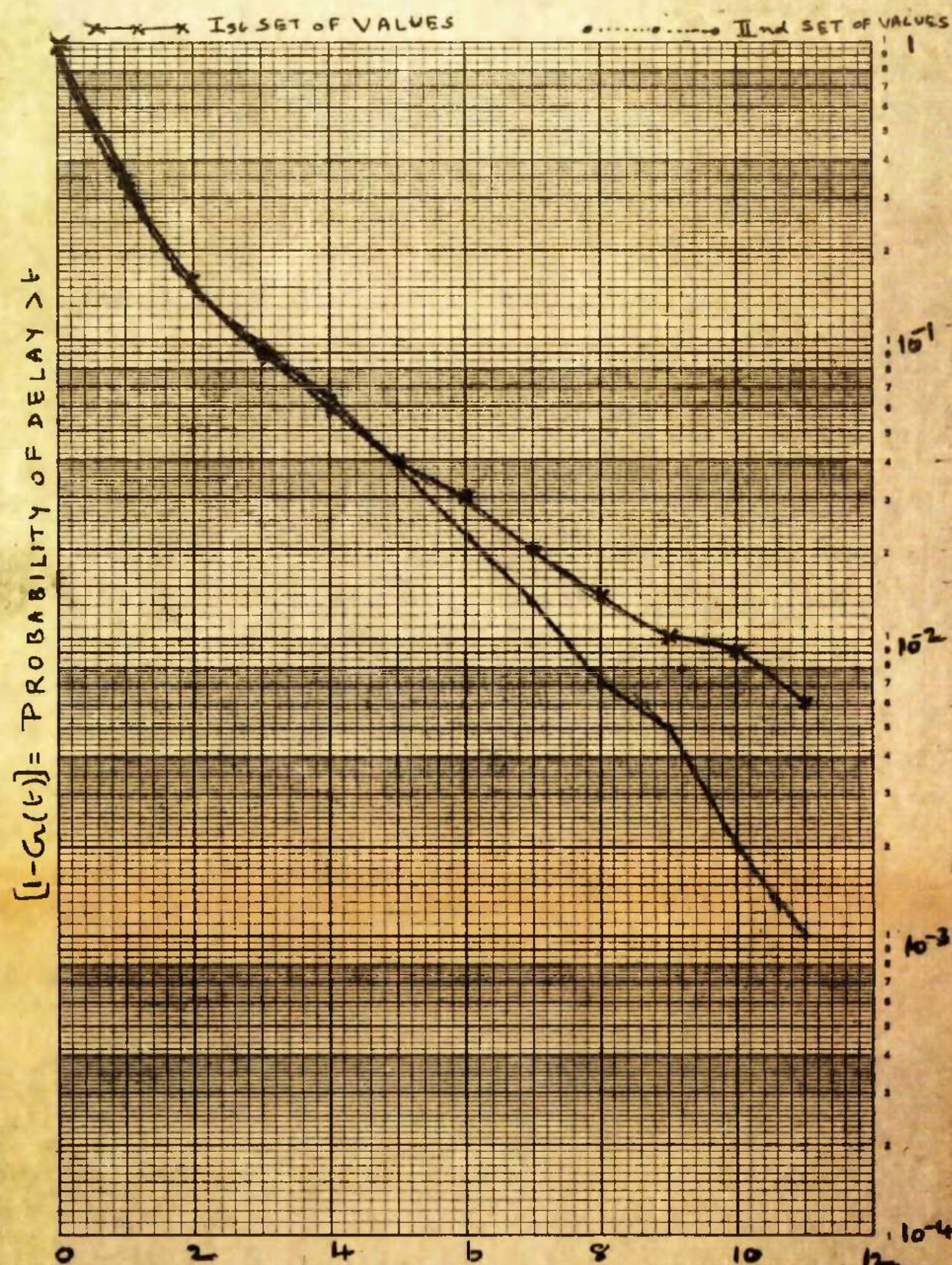
II set of 1000 items

Delay greater than t	1	2	3	4	5	6	7	8	11	12	13
[1 - G(t)] Probability of delay > t	0.358	0.154	0.091	0.065	0.038	0.022	0.013	0.007	0.005	0.002	0.001

TABLE IV

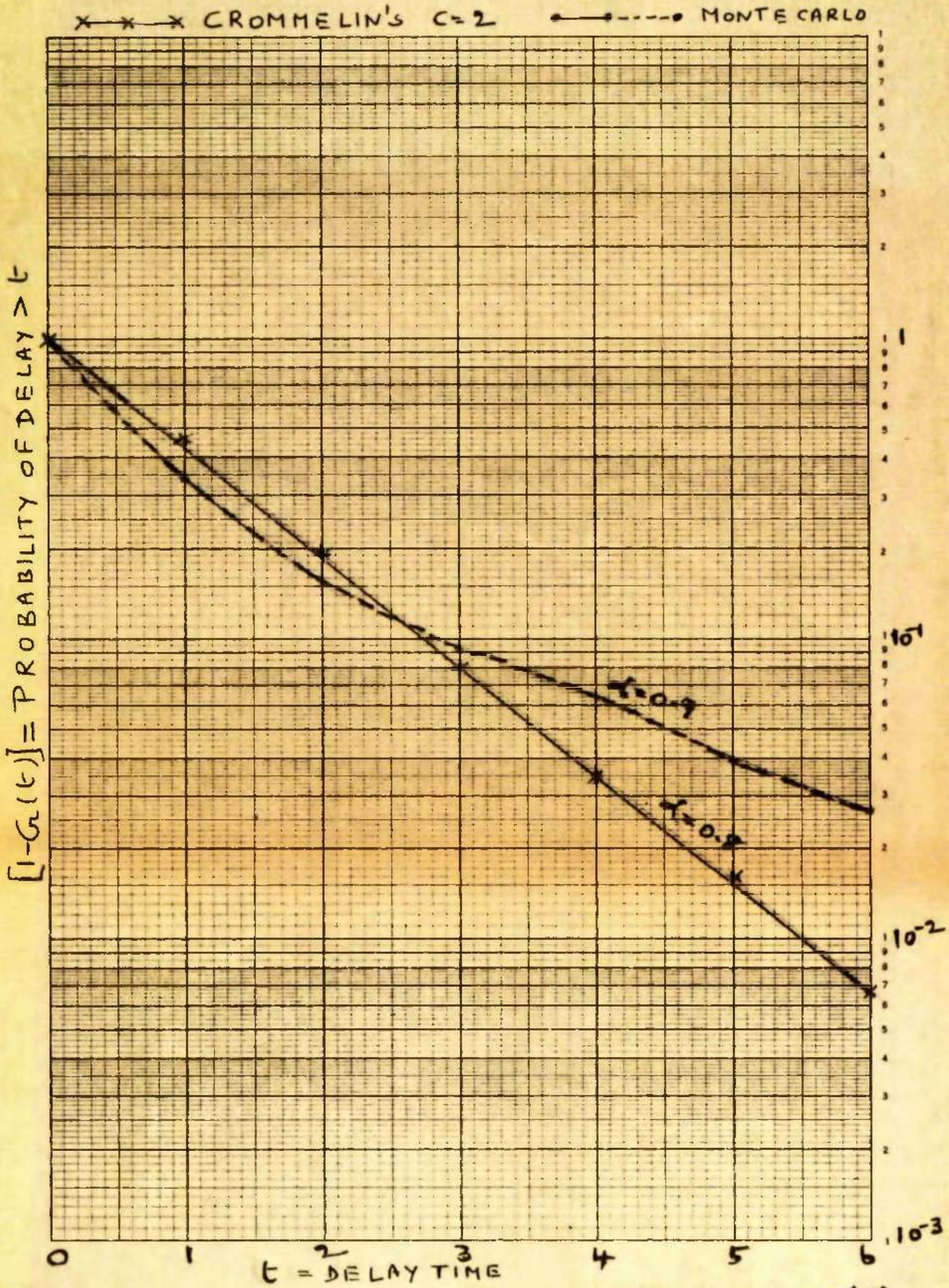
Comparison of the Monte Carlo result with that of Crommelin's (for $\alpha = 0.8$ and for service in order of arrival)

Delay greater than t	1	2	3	4	5	6
[1-G(t)] of 2000 items	0.346	0.156	0.090	0.063	0.039	0.026
[1-G(t)] Crommelin's for $\alpha = 0.8$; c=2	0.470	0.194	0.081	0.035	0.017	0.006



COMPARISON BETWEEN TWO SETS OF VALUES; C = 2

FIG. NO. 3



COMPARISON OF DELAY CURVES MONTE CARLO AND CROMMELIN'S

FIG. NO. 4

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