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The object of this research is to examine two particular problems of transonic flow. The first problem of axisymmetric nature is solved in the physical plane. The second problem is of two dimensional character and the solution is obtained by a transformation to the hodograph plane. Thus part I of this thesis deals with the case of transonic flow past a slender pointed parabolic-arc body of revolution at zero angle of attack. Part II deals with the design of a straight walled wind tunnel with a finite porous section to gi reduced blockage interference in high subsonic compressible flow.

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PROBLEMS IN TRANSONIC FLOW

A thesis presented by

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in fulfilment of the requirements of the degree of Doctor of Philosophy of Glasgow University.

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results. Over the rear part of the body the values obtained by Spreiter and Alksne, and by Cole and Royce, and lower than those given by the experimental results while our values are in a good agreement with them.

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PART I.

CHAPTER 1.

Introduction.

The fact that it is not possible in general to obtain an exact analytic solution for the partial differential equation of transonic flow, as the equation is non-linear and of mixed character, has prevented the rapid increase of the analysis which has occurred in recent years with both subsonic and supersonic theory. A result of this is that the solutions for three dimensional transonic flow have to be found by using certain ? empirical rules. One of these is the transonic similarity rule which pertains to the pressures and forces on affinely related families of wings and bodies of revolution. Another is the emperically established rule by Whitcomb (1) which states that "near the speed of sound the zero-lift drag rise of a low-aspect-ratio wing and body combination is primarily dependent on the axial distribution of the cross-sectional area normal to the air stream." Heaslet and Spreiter (2) showed that Whitcomb's rule can be theoretically justified for bodies which are pointed at the front and taper to a point at the rear. This means that if the solution of the transonic equation can be found for a slender, pointed, non-lifting body of revolution the solution for any slender body having the same longitudinal distribution of cross-sectional area and tapering to a point at the rear is also determined. These rules and the frequent use of bodies of revolution in practical applications show that the solution of the transonic equation for slender pointed bodies of revolution in flight at zero angle of attack at Mach number near unity is of considerable importance to aircraft designers.

Since an exact analytic solution cannot be obtained for the transonic equation the designer has either to rely on experimental data obtained in wind tunnels or to use approximate methods for the solution. In the following analysis we shall consider a method which gives an approximate solution of the transonic flow equation.

The first approximate solutions for the transonic equation were used in the solution of problems in airfoil theory. The simplest methods, used by Munk, Prandtl and Glauert and others (3), are based on the complete linearisation of the equation. However this linearised theory has two significant limitations. The first is that the theory gives only a first approximation which is correct only for airfoils of small thickness ratio. The second is that the Mach numbers (of the velocity) cannot be close to unity anywhere in the field of flow.

The failure of the above methods to give a solution for high subsonic flows led to the use of the hodograph transformation by means of which the non-linear transonic equation is transformed into a linear differential equation, (for example Tricomi's Equation). This method has been applied with considerable success in the study of high subsonic flows around wedges and flat-plate airfoils and a number of specific results have been given in recent years by Guderly and Yoshihara(4), Helliwell and Mackie (5) among others. However it is very difficult to apply this method to calculate high subsonic flows around arbitrary airfoils with curved boundaries.

Other approximate methods of solving the non-linear equation for the case of transonic flow past a slender body are due to Oswatitsch and Keune (6). They suggested that the non-linear transonic small disturbance flow equation, which is the first order approximation of the exact transonic flow equation, can be linearised by replacing one of the partial differential coefficients in the non-linear term by a parameter. This method has been successfully applied to transonic flows past two-dimensional bodies by Maeder and Wood(7), Spreiter and Alksne (8). In the method used by Maeder and Wood the solution to the problem was found as a function of the parameter which was used to linearise the equation. The method used by Spreiter and Alksne is similar to the one used by Maeder and Wood. The difference is that after solving the linearised equation the parameter is given its correct value and the resulting differential equation is solved to give a solution to the problem while in Maeder and Wood's theory the parameter is replaced by a constant.

Since, in the two dimensional case, the method used by Spreiter and Alksne gave a better agreement with the experimental results than the one used by Maeder and Wood it seemed likely to assume that this method would give the better agreement with experimental results when applied to the solution of the flow past slender bodies of revolution at zero angle of attack in a free stream at Mach numbers near unity.

In the following problem an approximate solution to the transonic equation was found by using a method similar to the one used by Spreiter and Alksne in the two-dimensional case. However, while this work was being done Spreiter and Alksne (9) published their solution for the same problem. By this time we had found an approximate solution to the equation for accelerating flow which is of the same form as the one obtained by Spreiter and Alksne although a different analysis was used to obtain it. The difference in the complete solution to the problem for the flow at sonic speed past a slender pointed parabolic-arc body of revolution is that Spreiter and Alksne use three approximate solutions of the equation to give them a continuous solution while in this work two approximat solutions of the equation are used together with a shock relation to give a solution.

Spreiter and Alksne obtained a continuous solution for the fluid velocity o the surface of the body in the following manner. From the boundary conditions or the surface of the body they obtained a continuous expression for one of the velo city components. The expression for the other component was found from the solu tion of the transonic flow equation. In order to obtain an approximate solution of this equation Spreiter and Alksne linearised it using three different approaches in such a way that they obtained either a hyperbolic, a parabolic or an elliptic second order partial differential equation. The hyperbolic equation was to be used in regions where the flow was purely supersonic, the elliptic equation in regions where the flow was purely subsonic and the parabolic equatio in regions where the flow was of mixed character. These equations were reduced i a manner analogous to that of the present work to give first order ordinary diffe rential equations for the required velocity component on the surface of the body. In each case a family of solutions was obtained, but the members of a single family could not be fitted to yield a continuous solution over the complete surface of the body. The solution to the problem was now found by using a combination of the above families of solutions. Over the fore-body the parabolic equation may be used as the fluid velocity changes from subsonic to supersonic. Now the ordinary differential equation obtained in this case has a singular poin which occurs at a certain position on the fore-body. Spreiter and Alksne showed that there was only one solution from this parabolic family which was continuous through the singular point. This solution was then taken to yield the required velocity component over the fore-body. Over the centre of the body the hyperboli equation may be used as the fluid velocity is supersonic. This time the required solution was found by taking that which gave a smooth transition from the solution for the fore-body. Over the rear-part of the body the elliptic equation is able to be used as the fluid velocity is subsonic. Again the solution which gave a smooth transition from the solution for the centre of the body was taken, and thus a continuous solution was obtained for the fluid velocity on the surface of the body.

An alternative solution to the above problem has also been given by Cole and Royce (10). In this paper the transonic small disturbance flow equation is linearised by replacing the non-linear term by a linear term which is a good approximation for it near the surface of the body. This linearised equation is solved and it gives a continuous solution to the problem.

As it is not possible to give a rigorous mathematical justification for the various approximate methods which may be used, and in particular for the one we used in solving the transonic equation, the only way we can establish its validity is to compare the values we obtain for the coefficient of pressure on the surface of a slender body of revolution with experimental results. In this work such a comparison is made for the case of a slender parabolic-arc body of revolution in a free-stream in which the velocity far upstream is equal to that of sound. The agreement between theory and experiment is found to be satisfactory.

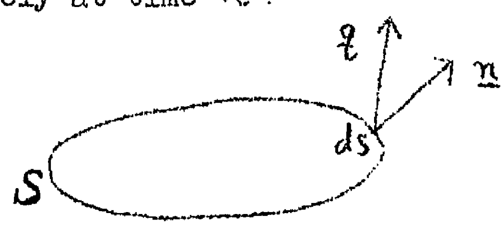
CHAPTER 2.

The Basic Equations of Transonic Flow.

Transonic flow is said to occur when the velocity of fluid particules in some regions of the field of flow is little different from the velocity of sound The basic equations which govern such a flow are derived in this section. The fluid is assumed to be a non-viscous, non-heat conducting perfect gas to which the adiabatic gas law applies. The motion is supposed to be steady, irrotational and dependent upon no external forces for its support.

The Equation of Continuity.

In a source and sink free region of the fluid consider a closed surface S enclosing a volume V. Let $\mathbf{n}$ be the outward-drawn normal vector to the element dS of the surface. Denote by ρ and $\mathbf{q}$ the density and velocity of the fluid respectively at time t.



The time rate of change of mass inside the finite surface is $\int_V \frac{\partial \rho}{\partial t}\, dV$ and the rate at which fluid flows across S into V is $-\int_S \rho\, \mathbf{q}\cdot\mathbf{n}\, dS$. Since no fluid is created or lost within S the mass can only be increased by flow across the boundary. Therefore the above two rates must be equal.

$$\therefore \int_V \frac{\partial \rho}{\partial t}\, dV + \int_S \rho\, \mathbf{q}\cdot\mathbf{n}\, dS = 0.$$

An application of Gauss' Theorem to the surface integral yields

$$\int_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{q}) \right] dV = 0 .$$

But V was an arbitrarily chosen volume. Therefore the integral vanishes identically. Hence.

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho q) = 0 . \qquad (1)$$

Thus for steady flow the equation of continuity is

$$\operatorname{div}(\rho q) = 0 . \qquad (2)$$

<u>The Equations of Motion.</u>

The fluid is again considered in an arbitrary volume V enclosed by a surface S .

The rate of change of momentum of any volume element is equal to the sum of forces acting on it. Application of this result to the volume V gives,

$$\frac{D}{Dt} \int_V \rho q \, dV = - \int_S p \, \underline{n} \, dS , \qquad (3)$$

where p is the pressure in the fluid and $\frac{D}{Dt} = q \cdot \nabla + \frac{\partial}{\partial t}$ denotes the time rate of change operator following a particle.

But $$\frac{D}{Dt} \int_V \rho q \, dV = \int_V \rho \frac{Dq}{Dt} dV + \int_V q \frac{D}{Dt}(\rho dV) .$$

Now if we consider a fluid particle of infinitesimal volume dV , then the mass of this fluid particle cannot change as it moves about.

$$\therefore \quad \frac{D}{Dt}(\rho \, dV) = 0 .$$

and $$\frac{D}{Dt} \int_V \rho q \, dV = \int_V \rho \frac{Dq}{Dt} dV .$$

On application of Gauss' Theorem to the surface integral and substituting $\int_V \rho \frac{Dq}{Dt} dV$ for $\frac{D}{Dt}\int_V \rho q \, dV$ equation (3) becomes

$$\int_V \left[\rho \frac{Dq}{Dt} + \nabla p\right] dV = 0.$$

But V was an arbitrarily chosen volume. Therefore the integrand vanished identically. Hence

$$\rho \frac{Dq}{Dt} + \nabla p = 0. \qquad (4)$$

For steady state flow the equation becomes

$$q \cdot \nabla q + \frac{1}{\rho} \nabla p = 0. \qquad (5)$$

Now p and ρ are related by the adiabatic gas law.

$$\therefore \; p = k \rho^{\gamma},$$

where k is a constant and γ is the ratio of the specific heat of the gas at constant pressure to its specific heat at constant volume.

$$\therefore \; \nabla p = \frac{p\gamma}{\rho} \nabla \rho,$$

$$= a^2 \nabla \rho.$$

where a is the local velocity of sound.

Substituting for ∇p in equation (5) we get

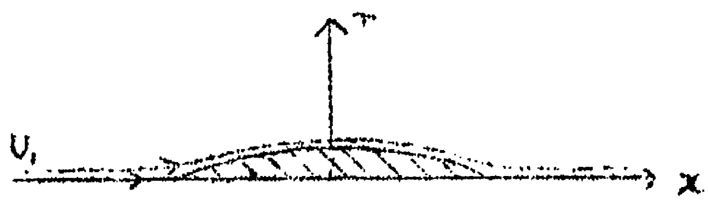
$$q \cdot \nabla q + \frac{a^2}{\rho} \nabla \rho = 0. \qquad (6)$$

This equation is now multiplied by $\underset{\sim}{q}$ scalarly and from equation (2) $\underset{\sim}{q} \cdot \nabla \rho$ can be replaced by $-\rho \nabla \cdot \underset{\sim}{q}$.

$$\therefore \quad \underset{\sim}{q} \cdot \nabla\left(\tfrac{1}{2} q^2\right) - a^2 \nabla \cdot \underset{\sim}{q} = 0. \tag{7}$$

Perturbation Velocities.

We will now assume that we have a uniform steady flow whose velocity is U_1, and direction of flow is parallel to the x-axis of the three dimensional co-ordinate system. If we now place in this flow, along the x-axis, a slender body of revolution with a smooth surface the disturbances caused to the uniform flow by the body will be small apart from the small region near the stagnation point at the nose of the body. Since the flow is the same in all meridian planes it will therefore be independent of the θ-co-ordinate in our cylindrical co-ordinates (x, r, θ).



Then we may write,

U = the component of velocity in x-direction $= U_1 + u$,

and V = the component of velocity in (r)-direction $= v$

$$\therefore \quad \underset{\sim}{q} = (U_1 + u)\,\underset{\sim}{i} + v\,\underset{\sim}{j},$$

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors in the direction of the x and r axes respectively.

The Equation of Motion in Terms of the Perturbation Velocities.

On substituting for $\underline{q}$ in equation (7) we obtain,

$$\left[(U_1+u)\underline{i} + v\underline{j}\right].\nabla\left[\tfrac{1}{2}\left\{(U_1+u)^2+v^2\right\}\right] - a^2\,\nabla.\left[(U_1+u)\underline{i} + v\underline{j}\right] = 0.$$

$$\therefore (U_1+u)\left[(U_1+u)\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right] + v\left[(U_1+u)\frac{\partial u}{\partial r} + v\frac{\partial v}{\partial r}\right] - \frac{a^2}{r}\frac{\partial(ur)}{\partial x} - \frac{a^2}{r}\frac{\partial(vr)}{\partial r} = 0.$$

But the flow is irrotational,

$$\text{i.e.}\quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial x} \qquad (8)$$

$$\therefore (U_1+u)^2\frac{\partial u}{\partial x} + 2v(U_1+u)\frac{\partial u}{\partial r} + v^2\frac{\partial v}{\partial r} - a^2\frac{\partial u}{\partial x} - a^2\frac{\partial v}{\partial r} - \frac{a^2 v}{r} = 0. \qquad (9)$$

In order to obtain a suitable expression for a^2 in terms of u and v we use equation (6) which gives,

$$\underline{q}.\nabla\underline{q} + k\rho^{\gamma-2}\gamma\nabla\rho = 0.$$

Therefore

$$\underline{q}.\nabla\underline{q} + \nabla\left[\frac{k\gamma}{\gamma-1}\rho^{\gamma-1}\right] = 0.$$

On multiplying both sides by $\underline{q}$ scalarly we get,

$$\underline{q}.\nabla\tfrac{1}{2}q^2 + \underline{q}.\nabla\left[\frac{k\gamma}{\gamma-1}\rho^{\gamma-1}\right] = 0,$$

and using the fact that $\frac{D}{Dt} = \underset{\sim}{q} \cdot \nabla$ we obtain

$$\frac{D}{Dt}\left(\frac{1}{2} q^2 + \frac{k\gamma}{\gamma-1} \rho^{\gamma-1}\right) = 0 .$$

$$\therefore \frac{1}{2} q^2 + \frac{k\gamma}{\gamma-1} \rho^{\gamma-1} = C ,$$

a constant following a particle ,

$$\text{or } \frac{1}{2} q^2 + \frac{a^2}{\gamma-1} = C , \qquad (10)$$

which is Bernoulli's Equation .

The flow conditions at infinity upstream , viz $\underset{\sim}{q} = U_1 \underset{\sim}{i}$ and $a = a_1$ are used to evaluated C and inserting the perturbation variables the equation becomes

$$\frac{1}{2} U_1^2 + \frac{a_1^2}{\gamma-1} = \frac{1}{2}\left[(U_1 + u)^2 + v^2\right] + \frac{a^2}{\gamma-1} . \qquad (11)$$

$$\therefore a^2 = a_1^2 - \frac{\gamma-1}{2}\left[2U_1 u + u^2 + v^2\right] .$$

Substitution of this expression in equation (9) gives

$$\frac{\partial u}{\partial x}\left[U_1^2 + 2U_1 u + u^2 - a_1^2 + \frac{\gamma-1}{2}(2U_1 u + u^2 + v^2)\right]$$

$$+ \frac{\partial v}{\partial r}\left[v^2 - a_1^2 + \frac{\gamma-1}{2}(2U_1 u + u^2 + v^2)\right] + 2v(U_1 + u)\frac{\partial u}{\partial r}$$

$$- \left[a_1^2 - \frac{\gamma-1}{2}(u^2 + v^2 + 2uU_1)\right]\frac{v}{r} = 0 . \qquad (12)$$

As u and v are small compared with U_1 and a_1, squares and higher powers of these quantities may be neglected by comparisons with U_1^2 and a_1^2, and on dividing by a_1^2 we get

$$\frac{\partial u}{\partial x}\left[M_1^2 - 1 + \frac{(\gamma+1)M_1^2 u}{U_1}\right] + \frac{\partial v}{\partial r}\left[\frac{(\gamma-1)M_1^2 u}{U_1} - 1\right]$$
$$+ 2\frac{M_1^2 v}{U_1}\frac{\partial u}{\partial r} - \left[1 - \frac{(\gamma-1)M_1^2 u}{U_1}\right]\frac{v}{r} = 0, \qquad (13)$$

where M is the local Mach number of the flow and suffix (1) here and hereafter refers to conditions infinitely far upstream.

<u>The Transonic Equation for Flows with Mach Number near Unity.</u>

Since the upstream velocity of the fluid is very close to the velocity of sound, $(1 - M_1^2)$ is small. Now let $\frac{u}{U_1}$ be of the order ε, $\frac{v}{U_1}$ be of the order ε^a, $\frac{\partial}{\partial x}$ be of the order ε^b, $\frac{\partial}{\partial r}$ be of the order ε^c, $\frac{1}{r}$ be of the order ε^d and $(1 - M_1^2)$ be of the order ε^e where ε is small and a and e are positive. Therefore on applying these orders of magnitude to the terms in equations (8) and (13) we see that

$$\varepsilon^{1+c} = \varepsilon^{a+b},$$

and
$$\varepsilon^{1+b+e} + \varepsilon^{2+b} + \varepsilon^{a+c+1} + \varepsilon^{a+c} + \varepsilon^{a+1+c} + \varepsilon^{a+d} + \varepsilon^{1+a+d} = 0.$$

In the second equation the third, fifth and seventh terms can be neglected as they are of a higher order than the fourth and sixth terms respectively. Thus

$$\varepsilon^{1+b+e} + \varepsilon^{2+b} + \varepsilon^{a+c} + \varepsilon^{a+d} = 0.$$

From the first equation we see that

$$1 + c = a + b$$

$$\therefore \qquad c = a + b - 1 . \qquad (14)$$

Hence the last equation becomes

$$\varepsilon^{1+b+e} + \varepsilon^{2+b} + \varepsilon^{2a+b-1} + \varepsilon^{a+d} = 0 .$$

In the most general case all four terms will be present.

Therefore a must be $\frac{3}{2}$, e must be 1, and e and d must be equal. These values for a and e show that u and $1 - M_1^2$ are of the same order of magnitude and that disturbances in the radial direction will be less than those in the longitudinal direction.

The transonic small disturbance equation for flows with Mach number near unity is thus

$$\frac{\partial u}{\partial x}\left[M_1^2 - 1 + \frac{(\gamma+1) M_1^2 u}{U_1} \right] - \frac{\partial v}{\partial r} - \frac{v}{r} = 0 . \qquad (15)$$

<u>The Velocity Potential.</u>

The condition for irrotational flow is

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial x} .$$

Since this is the condition that $u\,dx + v\,dr$ is a perfect differential, there exists a function ϕ such that

$$d\phi = u\,dx + v\,dr \tag{16}$$

$$\therefore \quad u = \frac{\partial \phi}{\partial x} \quad \text{and} \qquad v = \frac{\partial \phi}{\partial r} \tag{17}$$

ϕ is called the velocity potential.

Substituting in equation (15) for u and v in terms of the velocity potential we obtain

$$\frac{\partial^2 \phi}{\partial x^2}\left[M_1^2 - 1 + \frac{(\gamma+1)M_1^2}{U_1}\frac{\partial \phi}{\partial x}\right] - \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} = 0 \tag{18}$$

Boundary Conditions.

For a fluid there can be no flow through a solid surface, but for a non-viscous fluid slip past the solid surface may occur. If the equation of the surface is $f(x, r) = 0$, then the condition of zero velocity normal to the surface yields

$$q \cdot \nabla f(x,r) = 0.$$

$$\therefore (U_1 + u)\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial r} = 0.$$

$$\therefore \quad v = \left[U_1 \frac{\partial f}{\partial x} \middle/ \frac{\partial f}{\partial r} \right],$$ since u can be neglected as it is small compared with U_1.

$$\therefore \quad v = U_1 \frac{dR}{dx}, \qquad (19)$$

where $r = R(x)$ is the equation of the surface.

Since the body is slender, within the order of the approximations already made, the value of $v \left(r \frac{\partial \phi}{\partial r} \right)$ on the surface can be taken as the value of v on $r = 0$. To obtain an estimate of the velocity near the axis we see from equation (15) that

$$\frac{\partial}{\partial r}(r v) = r \left[M_1^2 - 1 + \frac{\gamma + 1}{U_1} M_1^2 u \right] \frac{\partial u}{\partial x}.$$

In general u and $\frac{\partial u}{\partial x}$ are not infinite, so that as $r \to 0$,

$$\frac{\partial}{\partial r}(r v) \to 0.$$

$$\therefore \quad r v = g(x).$$

This means that near the axis v behaves like $\frac{1}{r}$. Therefore the correct form for the approximate boundary conditions on the axis may be obtained from equation (19) as follows.

$$r v = U_1 \left(r \frac{dR}{dx} \right)_{r = R},$$

$$\text{or} \quad \lim_{r \to 0} (r v) = U_1 R \frac{dR}{dx}. \qquad (20)$$

Relations across a Stationery Shock Surface which is perpendicular to the free stream velocity in a flow with sonic conditions far upstream.

In deriving the basic equations of this section we have neglected the viscosity of the fluid. This assumption is not justified in certain problems and this is reflected in the nonexistence of a continuous solution. In the flow of a real fluid there may exist narrow regions in which a very rapid change of density and velocity take place due to viscous effects. In the theory of an ideal gas such a region is represented by a surface across which the velocity vector, the density and the pressure experience jumps. These jumps are governed by certain relations (shock conditions) which are derived from mechanical and thermodynamic considerations. In a steady state flow shocks only occur at supersonic speeds, though the flow becomes subsonic upon crossing the shock.

There are five shock conditions. The first, second and third state that the mass, momentum and energy are continuous across the shock. The fourth requires that the tangential component of the momentum is continuous across the shock surface. If there was a jump in the component it would have to be balanced by tangential forces acting on the surface. However all the forces on the fluid are pressure forces which are perpendicular to the surface against which they act, therefore the fourth condition must hold. Since the mass across the surface is continuous, the continuity of the tangential component of the momentum implies the continuity of the tangential component of the velocity across the shock surface. The fifth condition requires that the entropy of a particle increases upon crossing a shock surface. This condition means that the shock is compressive and the fluid velocity decreases upon crossing the shock surface so that the flow becomes subsonic behind.

We will now find the shock relations across a stationary shock surface perpendicular to the x-axis in a flow where the fluid velocity at infinity upstream is parallel to the x-axis and equal to the speed of sound. We will let the suffixes 2 and 3 refer to the conditions before and after the shock surface respectively. Then the shock relations are:

Conservation of Mass,

$$\rho_2(U_1 + u_2) = \rho_3(U_1 + u_3) , \tag{21}$$

Conservation of Momentum,

$$\rho_2(U_1 + u_2)^2 + p_2 = \rho_3(U_1 + u_3)^2 + p_3 , \tag{22}$$

Conservation of Energy,

$$\frac{1}{2} q_2^2 + \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} = \frac{1}{2} q_3^2 + \frac{\gamma}{\gamma-1} \frac{p_3}{\rho_3} ,$$

$$\text{or } \frac{1}{2}\left[(U_1 + u_2)^2 + v_2^2\right] + \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} = \frac{1}{2}\left[(U_1 + u_3)^2 + v_3^2\right] + \frac{\gamma}{\gamma-1} \frac{p_3}{\rho_3} , \tag{23}$$

Continuity of Tangential Component of Momentum,

$$v_2 = v_3 , \tag{24}$$

The Increase of Entropy gives ,

$$u_2 > u_3 \quad . \tag{25}$$

From equations (23) and (24) we see that

$$\frac{1}{2}(U_1+u_2)^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} = \frac{1}{2}(U_1+u_3)^2 + \frac{\gamma}{\gamma-1}\frac{p_3}{\rho_3} \quad . \tag{26}$$

From equations (21) and (22) we see that

$$p_3 = p_2 - \rho_3(U_1+u_3)^2 + \rho_3(U_1+u_2)(U_1+u_3) .$$

$$\therefore \frac{p_3}{\rho_3} = \frac{\rho_2}{\rho_3}\cdot\frac{p_2}{\rho_2} - (U_1+u_3)^2 + (U_1+u_2)(U_1+u_3) ,$$

$$= \frac{U_1+u_3}{U_1+u_2}\cdot\frac{p_2}{\rho_2} - (U_1+u_3)(u_3-u_2) .$$

On substituting in equation (26) this expression for $\frac{p_3}{\rho_3}$ we obtain

$$\frac{1}{2}(U_1+u_2)^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}$$
$$= \frac{1}{2}(U_1+u_3)^2 + \frac{\gamma}{\gamma-1}\left[\frac{U_1+u_3}{U_1+u_2}\frac{p_2}{\rho_2} - (U_1+u_3)(u_3-u_2)\right.$$

$$\therefore (u_2-u_3)\left[U_1 + \frac{1}{2}(u_2+u_3) + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}\frac{1}{U_1+u_2} - \frac{\gamma}{\gamma-1}(U_1+u_3)\right] = 0 .$$

But from equation (25)

$$u_2 - u_3 \neq 0 .$$

$$\therefore \; U_1 + \tfrac{1}{2}(u_2 + u_3) + \frac{\gamma}{\gamma-1}\,\frac{p_2}{\rho_2}\cdot\frac{1}{U_1 + u_2} - \frac{\gamma}{\gamma-1}(U_1 + u_2) = 0 \qquad (27)$$

Now from equation (10) we obtain

$$\frac{1}{2}U_1^2 + \frac{a_1^2}{\gamma-1} = \frac{1}{2}\left[(U_1 + u_2)^2 + v_2^2\right] + \frac{\gamma p_2}{(\gamma-1)\rho_2}$$

$$\therefore \; \frac{\gamma p_2}{(\gamma-1)\rho_2} = \frac{U_1^2}{\gamma-1} - U_1 u_2 - \frac{1}{2}\left(u_2^2 + v_2^2\right),$$

as $a_1 = U_1$ at infinity upstream.

On substituting this expression for $\dfrac{\gamma p_2}{(\gamma-1)\rho_2}$ in equation (27) we obtain

$$U_1 + \frac{1}{2}(u_2 + u_3) + \frac{1}{(U_1 + u_2)}\left[\frac{U_1^2}{\gamma-1} - U_1 u_2 - \frac{1}{2}(u_2^2 + v_2^2)\right] - \frac{\gamma}{\gamma-1}(U_1 + u_3) = 0 .$$

$$\therefore \; \frac{U_1}{2}(u_2 + u_3) + \frac{u_2}{2}(u_2 + u_3) - \frac{1}{2}(u_2^2 + v_2^2) - \frac{\gamma U_1}{\gamma-1}(u_2 + u_3) - \frac{\gamma u_2 u_3}{\gamma-1} = 0 .$$

As the second, third and fifth terms are of higher order than the first and fourth they can be neglected.

$$\therefore \; \frac{U_1}{2}(u_2 + u_3) - \frac{\gamma U_1}{\gamma-1}(u_2 + u_3) = 0 .$$

$$\therefore \quad U_1(u_2 + u_3)\frac{\gamma+1}{2(1-\gamma)} = 0.$$

$$\therefore \quad u_3 = -u_2 . \qquad (28)$$

<u>Change in Velocity Potential across a Shock Surface.</u>

From equation (16)

$$d\phi = u\,dx + v\,dr ,$$

$$\text{or} \quad \delta\phi = u\,\delta x + v\,\delta r .$$

Across a shock surface normal to the x-axis there is no change in r (i.e. $\delta r = 0$). Now the changes in u across the shock surface are finite. Therefore as $\delta x \to 0$, $\delta\phi \to 0$. This means that ϕ is continuous through a shock surface.

CHAPTER 3.

An Approximate Solution of the Problem of Transonic Flow past a Slender Pointed Body of Revolution.

The transonic small disturbance flow equation is

$$\frac{\partial^2\phi}{\partial x^2}\left[1 - M_1^2 - \frac{\gamma+1}{U_1} M_1^2 \frac{\partial\phi}{\partial x}\right] + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0 .$$

This equation is elliptic, parabolic or hyperbolic depending on whether $\left[1 - M_1^2 - \frac{\gamma+1}{U_1} M_1^2 \frac{\partial\phi}{\partial x}\right]$ is greater than, equal to or less than zero respectively. For values of M_1, near unity the type of solution depends on the sign of $\frac{\partial\phi}{\partial x}$. As the equation stands it is non-linear and very difficult to solve.

In the case of subsonic flow the flow equation is elliptic in character and in the case of supersonic flow it is hyperbolic in character. This leads one to use an approximate method for solving the transonic small disturbance flow equation which will make it parabolic in character, and thus intermediate between the elliptic and hyperbolic character of the subsonic and supersonic flows respectively.

The transonic small disturbance flow equation is now written in the following form so that the L.H.S. is parabolic in character,

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{\gamma+1}{U_1} M_1^2 \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} - (1-M_1^2)\frac{\partial^2\phi}{\partial x^2}.$$

However we must remember that although Oswatitch found that the form of the non-linear term in the transonic flow equation is unimportant the term itself is important. We must therefore have a term in the L.H.S. of our equation which represents the non-linear term and still keeps the character of our equation parabolic. This means that the term we introduce must not include $\frac{\partial^2 \phi}{\partial x^2}$. Therefore the simplest term which is closest in form to the non-linear term is $\frac{\partial \phi}{\partial x}$ times a parameter λ.

An Approximate Solution of the Transonic Equation for Accelerating Flow.

We therefore proceed in a manner analogous to that used by Spreiter and Alksne (8). We subtract $\lambda \frac{\partial \phi}{\partial x}$ from both sides of the equation, where λ is a positive parameter.

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \lambda \frac{\partial \phi}{\partial x} = \frac{\gamma+1}{U_1} M_1^2 \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} - (1-M_1^2)\frac{\partial^2 \phi}{\partial x^2} - \lambda \frac{\partial \phi}{\partial x},$$

or

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) - r\lambda \frac{\partial \phi}{\partial x} = r\left[\frac{\gamma+1}{U_1} M_1^2 \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} - (1-M_1^2)\frac{\partial^2 \phi}{\partial x^2} - \lambda \frac{\partial \phi}{\partial x}\right],$$

$$= r\, f(x, r), \tag{29}$$

where

$$f(x, r) = \frac{\gamma+1}{U_1} M_1^2 \frac{\partial \phi}{\partial x}\cdot\frac{\partial^2 \phi}{\partial x^2} - (1-M_1^2)\frac{\partial^2 \phi}{\partial x^2} - \lambda \frac{\partial \phi}{\partial x}. \tag{30}$$

We now use the Green's Function $G(x, x_0; r, r_0)$ which satisfies the equation

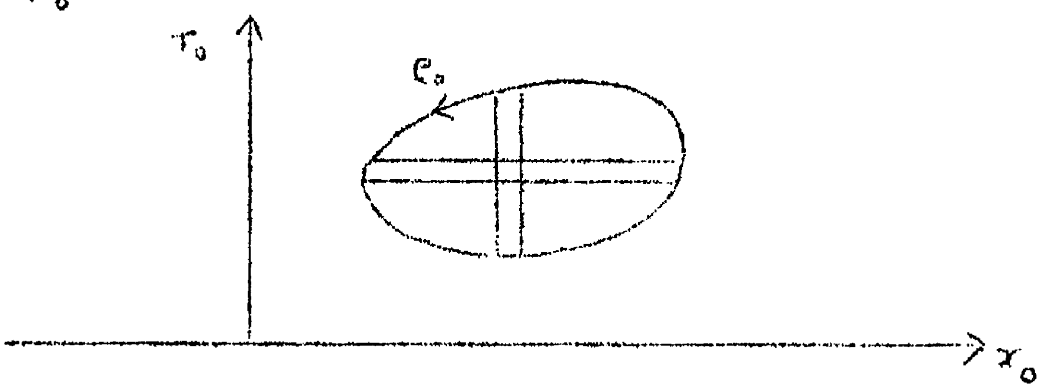
$$\frac{\partial}{\partial r_0}\left(r_0 \frac{\partial G}{\partial r_0}\right) + r_0 \lambda \frac{\partial G}{\partial x_0} = \delta(x - x_0)\,\delta(r - r_0), \qquad (31)$$

where $\delta(x - x_0)\,\delta(r - r_0)$ denotes a Dirac delta function, to help us in finding a solution to equation (29).

Equation (29) is multiplied by the Green's Function G and the current co-ordinates of equation (29) are changed to x_0 and r_0. Equation (31) is multiplied by ϕ and subtracted from the multiplied equation (29), and the reaulting equation is integrated over a region R_0 in which G has a finite value.

$$\therefore \iint_{R_0} \left[G \frac{\partial}{\partial r_0}\left(r_0 \frac{\partial \phi}{\partial r_0}\right) - \phi \frac{\partial}{\partial r_0}\left(r_0 \frac{\partial G}{\partial r_0}\right) - \lambda r_0 G \frac{\partial \phi}{\partial x_0} - \lambda r_0 \phi \frac{\partial G}{\partial x_0} \right] dR_0$$

$$= \iint_{R_0} \left[r_0 G f(x_0, r_0) - \phi\, \delta(x - x_0)\,\delta(r - r_0) \right] dR_0 \qquad (32)$$



The contour C_0 encloses the region R_0 in an anti-clockwise direction.

Now
$$\iint_{R_0}\left[\zeta\frac{\partial}{\partial r_0}\left(r_0\frac{\partial\phi}{\partial r_0}\right)-\phi\frac{\partial}{\partial r_0}\left(r_0\frac{\partial\zeta}{\partial r_0}\right)\right]dR_0$$
$$=\iint_{R_0}\left[\frac{\partial}{\partial r_0}\left(r_0\zeta\frac{\partial\phi}{\partial r_0}\right)-\frac{\partial}{\partial r_0}\left(r_0\phi\frac{\partial\zeta}{\partial r_0}\right)\right]dr_0dx_0,$$
$$=-\int_{C_0}\left[r_0\zeta\frac{\partial\phi}{\partial r_0}-r_0\phi\frac{\partial\zeta}{\partial r_0}\right]dx_0,$$

$$\iint_{R_0}\left[r_0\lambda\zeta\frac{\partial\phi}{\partial x_0}+r_0\lambda\phi\frac{\partial\zeta}{\partial x_0}\right]dR_0=\iint_{R_0}\left[r_0\lambda\frac{\partial}{\partial x_0}(\phi\zeta)\right]dr_0dx_0,$$
$$=\int_{C_0}r_0\lambda\phi\zeta\,dr_0,$$

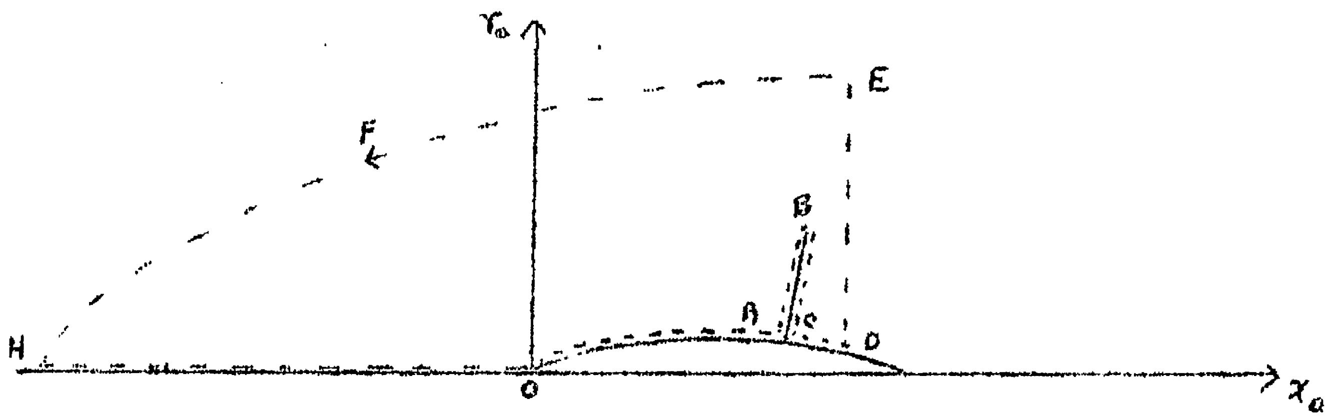
and
$$\iint_{R_0}\phi\,\delta(x-x_0)\,\delta(r-r_0)\,dR_0=\phi(x,r).$$

Therefore on substituting for the above terms in equation (32) we obtain,

$$-\int_{C_0}\left(r_0\zeta\frac{\partial\phi}{\partial r_0}-r_0\phi\frac{\partial\zeta}{\partial r_0}\right)dx_0-\int_{C_0}r_0\lambda\phi\zeta\,dr_0=\iint_{R_0}r_0 f(x_0,r_0)\zeta\,dR_0-\phi(x,r).$$

$$\therefore\ \phi(x,r)=\int_{C_0}\left(r_0\zeta\frac{\partial\phi}{\partial r_0}-r_0\phi\frac{\partial\zeta}{\partial r_0}\right)dx_0+\int_{C_0}r_0\lambda\phi\zeta\,dr_0+\iint_{R_0}r_0 f(x_0,r_0)\zeta\,dR_0 \quad (33)$$

Equation (33) is now applied to the region shown in the figure below.



The contour C_0 is the surface OABCDEFHO, and ABC is a shock surface. The contour surrounds the upstream flow field in the plans from $x_0 = -\infty$ to x excluding the body and the shock surface

Consider first ;

$$\int_{C_0} \left(r_0 \, G \frac{\partial \phi}{\partial r_0} - r_0 \, \phi \frac{\partial G}{\partial r_0} \right) dx_0 = \int_O^A + \int_A^B + \int_B^C + \int_C^D + \int_D^E + {}_F\!\int_E^H + \int_H^O .$$

Within the order of the approximations already made the boundary conditions on the surface of the body are referred to the line $r_0 = 0$. Thus on OA and CD r_0 is taken as zero, and from equation (20) we obtain the boundary condition $\left(r_0 \frac{\partial \phi}{\partial r_0} \right)_{r_0 \to 0} = S(x_0)$, say.

$$\therefore \int_O^A + \int_C^D = \int_O^{x_0} S(x_0) \, . \, G(x, x_0 ; r, r_0 = 0) \, dx_0 - \int_C^x \left(\phi \, r_0 \frac{\partial G}{\partial r_0} \right)_{r_0 = 0} dx_0 .$$

From D to E there is no change in x_0

$$\therefore \int_D^E = 0 .$$

On EFH, as the curve is an infinite distance from the body and G may be taken as zero.

$$\therefore {}_F\!\int_E^H = 0 .$$

Along HO ,

$$\left(r_0 \frac{\partial \phi}{\partial r_0} \right)_{r_0 = 0} = 0 .$$

$$\therefore \int_H^O = - \int_{-\infty}^{O} \left(r_0 \, \phi \frac{\partial G}{\partial r_0} \right)_{r_0 = 0} dx_0 .$$

$$\therefore \int_{C_0} \left(r_0 G \frac{\partial \phi}{\partial r_0} - r_0 \phi \frac{\partial G}{\partial r_0} \right) dx_0 = \int_0^x S(x_0)\, G(x, x_0;\ r, r_0 = 0)\, dx_0$$
$$- \int_0^x \left(\phi r_0 \frac{\partial G}{\partial r_0} \right)_{r_0 = 0} dx_0 - \int_{-\infty}^0 \left(\phi r_0 \frac{\partial G}{\partial r_0} \right)_{r_0 = 0} dx_0$$

+ integral round the shock surface .

$$\text{Now} \quad \int_{C_0} r_0 \lambda \phi G \, dr_0 = \int_0^A + \int_A^B + \int_B^C + \int_C^D + \int_D^E + {}_F\!\int_E^H + \int_H^0 .$$

From H to A and C to D there is no change in r_0

$$\therefore \int_H^0 = \int_0^A = \int_C^D = 0 .$$

Along the part EFH $\phi = 0$ and G may be taken as zero

$$\therefore {}_F\!\int_E^H = 0$$

Along DE we let the value of G be zero

$$\therefore \int_D^E = 0$$

$$\therefore \int_{C_0} r_0 \lambda \phi G \, dr_0$$ = value of theintegral round the shock surface .

$$\therefore \phi(x, r) = \int_0^x S(x_0)\, G(x, x_0;\ r, r_0 = 0)\, dx_0 - \int_0^x \left(\phi r_0 \frac{\partial G}{\partial r_0} \right)_{r_0 = 0} dx_0$$
$$- \int_{-\infty}^0 \left(\phi r_0 \frac{\partial G}{\partial r_0} \right)_{r_0 = 0} dx_0 + \text{values of integrals round the shock surface .} \qquad (34)$$

The Evaluation of the Function $G(x, x_0; r, r_0)$

The function G satisfies the equation

$$\frac{\partial}{\partial r_0}\left(r_0 \frac{\partial G}{\partial r_0}\right) + r_0 \lambda \frac{\partial G}{\partial x_0} = \delta(x - x_0)\, \delta(r - r_0), \qquad (31)$$

and has to satisfy the following conditions.

1) $G(x, x_0; r, r_0) = 0$ on $x_0 = x$,

2) $G(x, x_0; r, r_0)$ is finite on $r_0 = 0$

and (3) $G(x, x_0; r, r_0) \to 0$ as $r_0 \to \infty$

Let $y = x_0 - x$.

Equation (31) now becomes

$$\frac{\partial}{\partial r_0}\left(r_0 \frac{\partial G}{\partial r_0}\right) + \lambda r_0 \frac{\partial G}{\partial y} = \delta(y)\, \delta(r - r_0).$$

Multiplying both sides of the above equation by e^{py} where p is a positive parameter, and integrating w.r.t y from $-\infty$ to 0 we obtain

$$\frac{d}{dr_0}\left(r_0 \frac{dg}{dr_0}\right) - p \lambda r_0 g = \delta(r - r_0), \qquad (35)$$

where $g = \int_{-\infty}^{0} G.\, e^{py} dy$

$$g = A.I_0(\sqrt{\lambda p}.\, r_0) + B.K_0(\sqrt{\lambda p}.\, r_0).$$

When $r_0 < r$, $g = A\, I_0(\sqrt{\lambda p}\,.\, r_0)$ because g is finite at $r_0 = 0$

when $r_0 > r$, $g = B\, K_0(\sqrt{\lambda p}\,.\, r_0)$ because $g \to 0$ as $r_0 \to \infty$,

and when $r_0 = r$ the solution is continuous .

$$\therefore \quad A\, I_0(\sqrt{\lambda p}\,.\, r) = B\, K_0(\sqrt{\lambda p}\,.\, r) . \tag{36}$$

Equation (35) is non integrated w.r.t. r_0 from $r - \varepsilon$ to $r + \varepsilon$

$$\therefore \int_{r-\varepsilon}^{r+\varepsilon} \frac{d}{dr_0}\left(r_0 \frac{dg}{dr_0}\right) dr_0 - \int_{r-\varepsilon}^{r+\varepsilon} \lambda r_0 p g \, dr_0 = \int_{r-\varepsilon}^{r+\varepsilon} \delta(r - r_0)\, dr_0 .$$

$$\therefore \left[r_0 \frac{dg}{dr_0} \right]_{r-\varepsilon}^{r+\varepsilon} - \int_{r-\varepsilon}^{r+\varepsilon} \lambda r_0 p g \, dr_0 = 1 .$$

ε is now allowed to tend to zero

$$\therefore \sqrt{\lambda p}\; r .\, B\, K_0'(\sqrt{\lambda p}\,.\, r) - \sqrt{\lambda p}\,.\, r \; A\, I_0'(\sqrt{\lambda p}\,.\, r) = 1 ,$$

$$\text{or} \quad -\sqrt{\lambda p}\,.\, r\, B.\, K_{-1}(\sqrt{\lambda p}\,.\, r) - \sqrt{\lambda p}\,.\, r .\; A\, I_{-1}(\sqrt{\lambda p}\,.\, r) = 1 . \tag{37}$$

Therefore on solving equations (36) and (37) for A and B and using the fact that

$$K_{-1}(\sqrt{\lambda p}\,.\, r)\, I_0(\sqrt{\lambda p}\,.\, r) + K_0(\sqrt{\lambda p}\,.\, r)\, I_{-1}(\sqrt{\lambda p}\,.\, r) = \frac{1}{\sqrt{\lambda p}\,.\, r} ,$$

we obtain

$$A = -K_0(\sqrt{\lambda p}\cdot r),$$

and

$$B = -I_0(\sqrt{\lambda p}\cdot r).$$

$$\therefore \quad g = -K_0(\sqrt{\lambda p}\cdot r)\, I_0(\sqrt{\lambda p}\cdot r_0) \quad \text{for} \quad r_0 < r, \qquad (38)$$

$$= -K_0(\sqrt{\lambda p}\cdot r_0)\, I_0(\sqrt{\lambda p}\cdot r) \quad \text{for} \quad r_0 > r.$$

Now $$G = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} g\, e^{-p\gamma}\, dp.$$

$$\therefore \quad G = \frac{1}{2(x_0 - x)}\, e^{\frac{\lambda(r^2 + r_0^2)}{4(x_0 - x)}}\, I_0\left[\frac{\lambda r r_0}{2(x - x_0)}\right] \quad \text{for} \quad x_0 < x,$$

$$= 0 \quad \text{for} \quad x_0 > x.$$

These expressions for G were obtained from reference (11)

The Evaluation of the Function ϕ

Insertion of the expressions for G given by equation (38) into equation (34) yields

$$\phi(x,r) = \int_0^x s(x_0)\, \frac{1}{2(x_0 - x)}\, e^{\frac{\lambda r^2}{4(x_0 - x)}}\, dx_0 + \iint_{R_0} G.\, f(x_0, r_0)\, dR_0$$

+ value of the integrals round the shock surface.

If we now assume that the shock surface is normal to the x_0 axis, the values of the integrals round the shock surface are zero as ϕ and $\frac{\partial \phi}{\partial r_0}$ are continuous across the shock surface.

$$\therefore \phi(x,r) = \int_0^x \frac{S(x_0)}{2(x_0-x)} e^{\frac{\lambda r^2}{4(x_0-x)}} dx_0 + \iint_{R_0} \varsigma \cdot r_0 \, f(x_0,r_0)\, dR_0 .$$

$$\therefore \frac{\partial \phi(x,r)}{\partial x} = \int_0^x \frac{\partial}{\partial x}\left[\frac{S(x_0)}{2(x_0-x)} e^{\frac{\lambda r^2}{4(x_0-x)}}\right] dx_0 + \left[\frac{S(x_0)}{2(x_0-x)} e^{-\frac{\lambda r^2}{4(x-x_0)}}\right]_{x_0=x}$$

$$+ \frac{\partial}{\partial x} \iint_{R_0} \varsigma \; r_0 \cdot f(x_0,r_0)\, dR_0 ,$$

$$= \int_0^x S(x_0)\left[\frac{1}{2(x_0-x)^2} e^{\frac{\lambda r^2}{4(x_0-x)}} + \frac{\lambda r^2}{8(x_0-x)^3} e^{\frac{\lambda r^2}{4(x_0-x)}}\right] dx_0$$

$$+ \frac{\partial}{\partial x} \iint_{R_0} r_0 \cdot \varsigma \cdot f(x_0,r_0)\, dR_0 ,$$

$$= \int_0^x \frac{S(x_0)}{2(x_0-x)^2} e^{\frac{\lambda r^2}{4(x_0-x)}} dx_0 + \left[-\frac{S(x_0)}{2(x_0-x)} e^{\frac{\lambda r^2}{4(x_0-x)}}\right]_{x_0=0}^{x_0=x}$$

$$+ \int_0^x e^{\frac{\lambda r^2}{4(x_0-x)}}\left[\frac{S'(x_0)}{2(x_0-x)} - \frac{S(x_0)}{2(x_0-x)^2}\right] dx_0 + \frac{\partial}{\partial x}\iint_{R_0} r_0 \cdot \varsigma \cdot f(x_0,r_0)\, dR_0 ,$$

$$= \int_0^x \frac{S'(x_0)}{2(x_0-x)} e^{\frac{\lambda r^2}{4(x_0-x)}} dx_0 + \frac{S(x_0=0)}{(-2x)} e^{-\frac{\lambda r^2}{4x}} + \frac{\partial}{\partial x}\iint_{R_0} r_0 \cdot \varsigma \cdot f(x_0,r_0)\, dR_0 .$$

But for pointed bodies $S(0) = 0$.

$$\therefore \frac{\partial \phi(x,r)}{\partial x} = \int_0^x \frac{S'(x_0)}{2(x_0-x)} e^{\frac{\lambda r^2}{4(x_0-x)}} dx_0 + \frac{\partial}{\partial x}\iint_{R_0} r_0 \cdot \varsigma \cdot f(x_0,r_0)\, dR_0 .$$

Now on the surface of the body $r = R(x)$

$$\frac{\partial \phi(x,R)}{\partial x} = \int_0^x \frac{S'(x_0)}{2(x_0-x)} e^{\frac{\lambda R^2}{4(x_0-x)}} dx_0 + \frac{\partial}{\partial x}\iint_{R_0} r_0 f(x_0,r_0)\,\zeta(x,x_0;R,r_0)\,dR_0,$$

$$= \int_0^x \frac{S'(x_0)-S'(x)}{2(x_0-x)} e^{\frac{\lambda R^2}{4(x_0-x)}} dx_0 + \int_0^x \frac{S'(x)}{2(x_0-x)} e^{\frac{\lambda R^2}{4(x_0-x)}} dx_0$$

$$+ \frac{\partial}{\partial x}\iint_{R_0} r_0 \cdot f(x_0,r_0).\,\zeta(x,x_0;R,r_0)\,dR_0,$$

$$\simeq \int_0^x \frac{S'(x_0)-S'(x)}{2(x_0-x)} dx_0 + \frac{S'(x)}{2}\int_0^x \frac{1}{(x_0-x)} e^{\frac{\lambda R^2}{4(x_0-x)}} dx_0$$

$$+ \frac{\partial}{\partial x}\iint_{R_0} r_0 f(x_0,r_0).\,\zeta(x,x_0;R,r_0)\,dR_0,$$

as $e^{\frac{\lambda R^2}{4(x_0-x)}} \simeq 1$ for $0 < x_0 < x$ and the integrand of the first integral is finite for all values of x_0 in the range $0 \le x_0 \le x$.

$$\therefore \frac{\partial \phi(x,R)}{\partial x} \simeq \int_0^x \frac{S'(x_0)-S'(x)}{2(x_0-x)} dx_0 - \frac{S'(x)}{2}\int_{\frac{\lambda R^2}{4x}}^{\infty} \frac{1}{y} e^{-y} dy$$

$$+ \frac{\partial}{\partial x}\iint_{R_0} r_0 f(x_0,r_0)\,\zeta(x,x_0;R,r_0)\,dR_0.$$

Now $\int_{\frac{\lambda R^2}{4x}}^{\infty} \frac{1}{y} e^{-y} dy \simeq -\log\left[e^C \cdot \frac{\lambda R^2}{4x}\right]$ as $\frac{\lambda R^2}{4x}$ is small and C is Euler's constant.

$$\therefore \frac{\partial \phi(x,R)}{\partial x} \simeq \int_0^x \frac{S'(x_0)-S'(x)}{2(x_0-x)} dx_0 + \frac{S'(x)}{2}\log\left[e^C \frac{\lambda R^2}{4x}\right]$$

$$+ \frac{\partial}{\partial x}\iint_{R_0} r_0 f(x_0,r_0)\,\zeta(x,x_0;R,r_0)\,dR_0.$$

The parameter λ is now replaced by $\frac{(\gamma+1) M_1^2}{U_1} \cdot \frac{\partial^2 \phi(x, r)}{\partial x^2}$ so that the above integral over the region R_0 can be evaluated and hence the expression for $\frac{\partial \phi}{\partial x}$ on the surface of the body can be found.

$$\therefore f(x, r) = \frac{(M_1^2 - 1) \lambda U_1}{(\gamma+1) M_1^2}$$

if the above expression is substituted for λ in equation (30).

It follows that

$$\frac{\partial}{\partial x} \iint_{R_0} r_0 . f(x_0, r_0) . G(x, x_0; R, r_0) \, d R_0$$

$$= \frac{\partial}{\partial x} \int_{x_0 = -\infty}^{x} \int_{r_0 = 0}^{\infty} \frac{(M_1^2-1)\lambda U_1}{(\gamma+1) M_1^2} \cdot \frac{r_0}{2(x_0 - x)} e^{\frac{\lambda(r_0^2 + R^2)}{4(x_0 - x)}} I_0\left[\frac{-\lambda R r_0}{2(x_0 - x)}\right] dr_0 \, dx_0,$$

$$= \frac{(M_1^2-1)U_1}{(\gamma+1) M_1^2} \frac{\partial}{\partial x} \int_{x_0 = -\infty}^{x} \frac{\lambda}{2(x_0 - x)} e^{\frac{\lambda R^2}{4(x_0 - x)}} \int_{r_0 = 0}^{\infty} r_0 \, e^{\frac{\lambda r_0^2}{4(x_0 - x)}} I_0\left[\frac{-\lambda R r_0}{2(x_0 - x)}\right] dr_0 \, dx_0.$$

The inner integral may be evaluated (see e.g. reference (12) page 394). Thus we find

$$\frac{\partial}{\partial x} \iint_{R_0} r_0 f(x_0, r_0) G(x, x_0; R, r_0) \, d R_0$$

$$= \frac{(M_1^2-1)U_1}{(\gamma+1) M_1^2} \frac{\partial}{\partial x} \int_{x_0 = -\infty}^{x} \frac{\lambda}{2(x_0 - x)} e^{\frac{\lambda R^2}{4(x_0 - x)}} \left[\frac{4(x - x_0)}{2\lambda} e^{\frac{\lambda^2 R^2}{4(x_0 - x)^2} \cdot \frac{4(x - x_0)}{4\lambda}}\right] dx_0,$$

$$= -\frac{(M_1^2-1)U_1}{(\gamma+1) M_1^2} \frac{\partial}{\partial x} \int_{x_0 = -\infty}^{x} dx_0 ,$$

$$= -\frac{(M_1^2-1)U_1}{(\gamma+1) M_1^2} .$$

$$\therefore \quad \frac{\partial \phi(x,R)}{\partial x} = \int_0^x \frac{S'(x_0) - S'(x)}{2(x_0 - x)} dx_0 + \frac{S'(x)}{2} \log\left[\frac{e^c R^2 (\gamma+1) M_1^2}{4x \; U_1} \frac{\partial^2 \phi(x,R)}{\partial x^2}\right] + \frac{(1-M_1^2) U_1}{(\gamma+1) M_1^2} \tag{39}$$

It will be noted that $\dfrac{\partial \phi(x,R)}{\partial x} = (u)_{body}$ and $\dfrac{\partial^2 \phi(x,R)}{\partial x^2} = \left(\dfrac{\partial u}{\partial x}\right)_{body}$

$$\text{Now} \quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial r} dr ,$$

$$= \frac{\partial u}{\partial x} dx + \frac{\partial v}{\partial x} dr , \quad \text{on using equation (8)}$$

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \frac{dr}{dx} .$$

On the surface of the body $r = R(x)$ and from equation (19) we obtain

$$\left(\frac{\partial u}{\partial x}\right)_{body} = \left(\frac{du}{dx}\right)_{body} - \left(\frac{dR}{dx}\right) \frac{S'(x)}{R} .$$

On substituting the expression in equation (39), and dropping the suffix body, we obtain

$$u = \int_0^x \frac{S'(x_0) - S'(x)}{2(x_0 - x)} dx_0 + \frac{(1-M_1^2) U_1}{(\gamma+1) M_1^2}$$
$$+ \frac{S'(x)}{2} \log\left[\frac{e^c R^2 (\gamma+1) M_1^2}{4x \; U_1} \left\{\frac{du}{dx} - \frac{S'(x)}{R} \frac{dR}{dx}\right\}\right] ,$$

$$\text{or} \quad \frac{du}{dx} = \frac{S'(x)}{R} \frac{dR}{dx}$$
$$+ \frac{4x U_1}{e^c R^2 (\gamma+1) M_1^2} \exp\left[\frac{2}{S'(x)} \left\{u - \frac{(1-M_1^2) U_1}{(\gamma+1) M_1^2} - \int_0^x \frac{S'(x_0) - S'(x)}{2(x_0 - x)} dx_0 . \right.\right. \tag{40}$$

Special Case of a Body with a Parabolic-Arc Profile in a Free-Stream with Sonic Velocity.

We shall consider the special form of the solution in the case where the longitudinal section of the body is symmetric with the upper boundary of the parabolic curve $r = R(x) = \delta x(1-x)$ where δ is a constant.

Then
$$S(x) = U_1 \delta^2 (1-2x)(1-x)(x),$$

$$S'(x) = U_1 \delta^2 (1-6x+6x^2),$$

and
$$\int_0^x \frac{S'(x_0) - S'(x)}{2(x-x_0)} dx_0 = \frac{U_1 \delta^2}{2} \int_0^x \frac{-6x_0 + 6x_0^2 + 6x - 6x^2}{(x_0 - x)} dx_0,$$

$$= \frac{U_1 \delta^2}{2} \int_0^x \left[-6 + 6(x_0 + x)\right] dx_0,$$

$$= \frac{U_1 \delta^2}{2} \left[-6x_0 + 3x_0^2 + 6x x_0\right]_0^x,$$

$$= \frac{U_1 \delta^2}{2} \left[9x^2 - 6x\right].$$

On application of these results to equation (40) remembering that $M_1 = 1$, it becomes,

$$\frac{du}{dx} = \frac{\delta^2 U_1 (1-6x+6x^2)(1-2x)}{x(1-x)}$$

$$+ \frac{4U_1}{(\gamma+1)\delta^2 x(1-x)^2 e^c} \exp\left[\frac{u - \frac{\delta^2 U_1}{2}(9x^2 - 6x)}{\frac{\delta^2 U_1}{2}(1-6x+6x^2)}\right]. \quad (41)$$

This equation has singular points at $x = 0, \frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6}$ and 1. We shall denote the values $\frac{1}{2} - \frac{\sqrt{3}}{6}$ and $\frac{1}{2} + \frac{\sqrt{3}}{6}$ by x_1 and x_2

respectively. However it is possible to find a continuous solution for u from $x = 0$ to $x = x_2$. This solution was found by giving u the value $\frac{U_1 \delta^2}{2}(9x_1^2 - 6x_1)$ at $x = x_1$. Then if m_1 is the value of $\frac{du}{dx}$ at $x = x_1$ we have

$$m_1 = \frac{4U_1}{(\gamma+1)\delta^2 x(1-x)^2 e} \exp\left[\frac{m_1 - \frac{U_1\delta^2}{2}(18x_1 - 6)}{\frac{U_1\delta^2}{2}(12x_1 - 6)}\right].$$

$$\therefore \log m_1 = \frac{2m_1}{U_1\delta^2(12x_1 - 6)} - \frac{3x_1 - 2}{2x_1 - 2} + \log\left[\frac{4U_1}{(\gamma+1)\delta^2(1-x_1)^2 x_1}\right],$$

$$= -Am_1 + B,$$ where A is a positive constant, and B is a constant.

Since this equation has only one real root it follows that there is only one solution of equation (41) that passes through the singular point at $x = x_1$. By means of a Taylor Series the values of u/U_1 slightly removed from the singular point can be calculated. Once these values are known the values of u/U_1 on the surface of the body in the range $0 < x < x_2$ can be found numerically. These values of u/U_1 for different values of δ are shown in tables I, II, III, IV, and V. (pages 53 to 57)

If however we use the values of u/U_1 obtained near the singular point $x = x_2$ we see that $\frac{du}{dx}$ is zero just upstream of it and infinite just downstream of it. This means that we cannot use this solution to obtain the values of u/U_1 over the rear-part of the body. The physical reason for this is that the fluid velocity decreases over the rear part of the body and this violates the condition that λ is a positive parameter.

However the values of u/U_1 over the entire body can be calculated by considering the solution in sections and joining the various results together. Thus we will now proceed to find a solution of equation (18) in which we impose the condition that $\frac{\partial^2 \phi}{\partial x^2}$ is always negative. This solution can be used in the region where the flow is decelerating and the above solution can be used in the region where the flow is accelerating.

An Approximate Solution of the Transonic Equation for Decelerating Flow.

The transonic small disturbance equation can be written in the following form,

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) = r\left[\frac{\gamma+1}{U_1} M_1^2 \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} - (1-M_1^2)\frac{\partial^2 \phi}{\partial x^2}\right].$$

We now proceed in a manner analogous to that of the preceding section, by adding $\lambda r \frac{\partial \phi}{\partial x}$ to both sides of the equation, where λ is a positive constant.

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + r\lambda \frac{\partial \phi}{\partial x} = r\left[\frac{\gamma+1}{U_1} M_1^2 \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} - (1-M_1^2)\frac{\partial^2 \phi}{\partial x^2} + \lambda \frac{\partial \phi}{\partial x}\right]$$

$$= r\, f(x, r), \qquad (42)$$

where $$f(x,r) = \frac{(\gamma+1)}{U_1} M_1^2 \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} - (1-M_1^2)\frac{\partial^2 \phi}{\partial x^2} + \lambda \frac{\partial \phi}{\partial x}. \qquad (43)$$

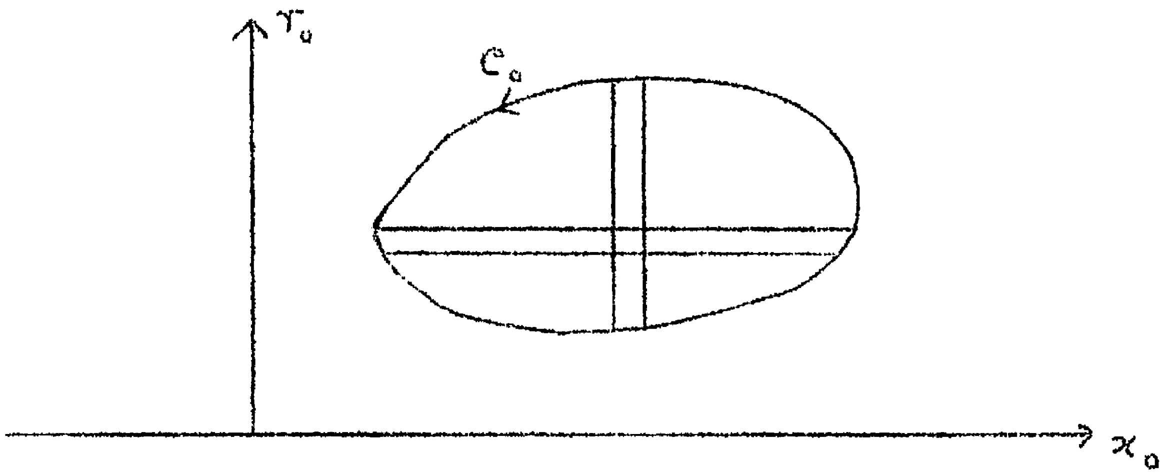
Equation (42) is now multiplied by the Green's Function $G(x, x_0; r, r_0)$ which satisfies the equation

$$\frac{\partial}{\partial r_0}\left(r_0 \frac{\partial \zeta}{\partial r_0}\right) - \lambda\, r_0 \frac{\partial \zeta}{\partial x_0} = \delta(x - x_0)\,\delta(r - r_0) \qquad (44)$$

where $\delta(x-x_0)\,\delta(r-r_0)$ denotes a Dirac delta function at $x = x_0$, $r = r_0$ and the running co-ordinates of equation (43) are changed to x_0 and r_0. Equation (44) is multiplied by ϕ and subtracted from the multiplied equation (43) and the resulting equation is integrated over a region R_0 in which ζ has a finite value

$$\iint_{R_0} \left[\zeta \frac{\partial}{\partial r_0}\left(r_0 \frac{\partial \phi}{\partial r_0}\right) - \phi \frac{\partial}{\partial r_0}\left(r_0 \frac{\partial \zeta}{\partial r_0}\right) + \zeta \cdot \lambda\, r_0 \frac{\partial \phi}{\partial x_0} + \phi \cdot \lambda \cdot r_0 \frac{\partial \zeta}{\partial x_0} \right] dR_0$$

$$= \iint_{R_0} \left[r_0 \cdot \zeta \cdot f(x_0, r_0) - \phi\, \delta(x - x_0)\,\delta(r - r_0) \right] dR_0 \,. \qquad (45)$$

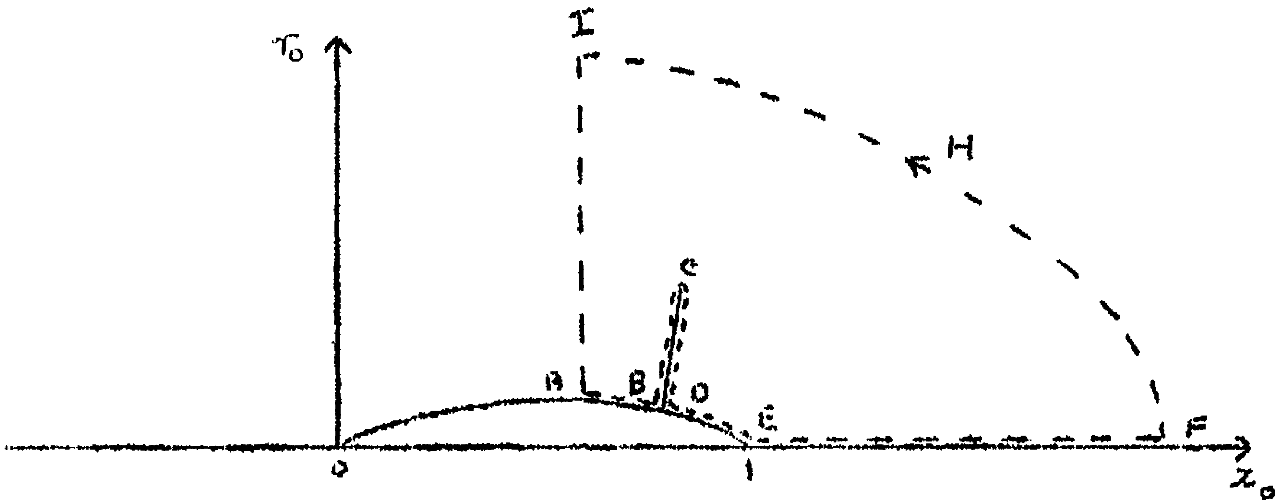


The curve C_0 encloses the region R_0 in an anti-clockwise direction.

Thus, exactly as in the previous analysis for accelerating flow, it follows

$$\phi(x,r) = \int_{C_0}\left[r_0\, G\, \frac{\partial \phi}{\partial r_0} - r_0\, \phi\, \frac{\partial G}{\partial r_0}\right] dx_0 - \int_{C_0} r_0 \lambda \phi G \, dr_0 + \iint_{R_0} r_0\, f(x_0, r_0)\, G \, dR_0$$

Equation (46) is now applied to the region shown in the figure below.



The contour C_0 is the curve ABCDEFHIA, and BCD is a shock surface. The contour surrounds the downstream flow field from $x = x_0$ to ∞ excluding the body, which we shall suppose to be of unit length, the wake and the shock surface.

As the analysis in the remaining part of this section and in the next two sections is formally identical apart from certain changes of sign and the regions of integration to that used in evaluating the velocity component u on the surface of the body for accelerating flow, only the important steps in the analysis are given.

On evaluating the integrals in equation (46) round the contour ABCDEFHIA we obtain

$$\phi(x,r) = \int_x^1 S(x_0)\, G(x,x_0;r,r_0=0)\,dx_0 - \int_x^1 \left(r_0\,\phi\,\frac{\partial G}{\partial r_0}\right)_{r_0=0} dx_0 - \int_1^\infty \left(r_0\,\phi\,\frac{\partial G}{\partial r_0}\right)_{r_0=0} dx_0$$

$$+\iint_{R_0} r_0\, f(x_0,r_0)\, G(x,x_0;r,r_0)\, dR_0 \quad \text{+ value of the integrals round the shock surface .} \qquad (47)$$

The Evaluation of the Function $G(x,x_0;r,r_0)$

The function $G(x,x_0;r,r_0)$ satisfies the equation

$$\frac{\partial}{\partial r_0}\left(r_0\,\frac{\partial G}{\partial r_0}\right) - \lambda\, r_0\,\frac{\partial G}{\partial x_0} = \delta(x-x_0)\,\delta(r-r_0)\,, \qquad (44)$$

and has to satisfy the following conditions

1) $G(x,x_0;\ r,r_0) = 0$ on $x_0 = x$,

2) $G(x,x_0;\ r,r_0)$ is finite on $r_0 = 0$,

and 3) $G(x,x_0;\ r,r_0) \rightarrow 0$ on $r_0 \rightarrow \infty$

Let $y = x - x_0$.

Equation (44) now becomes

$$\frac{\partial}{\partial r_0}\left(r_0\,\frac{\partial G}{\partial r_0}\right) + \lambda\, r_0\,\frac{\partial G}{\partial y} = \delta(y)\,\delta(r-r_0). \qquad (48)$$

The solution to the equation for the above boundary conditions has already been obtained in a previous section

$$\therefore \quad G = \frac{-1}{2(x_0-x)} e^{\frac{-\lambda(r^2+r_0^2)}{4(x_0-x)}} I_0\left[\frac{\lambda r r_0}{2(x_0-x)}\right] \quad \text{for} \quad x_0 > x, \tag{49}$$

$$= 0 \qquad\qquad \text{for} \quad x_0 < x.$$

The Evaluation of the Function ϕ

On assuming that the shock surface is normal to the x_0 axis and inserting the expressions for G given by equation (48) into equation (47) yields

$$\phi(x,r) = -\int_x^1 S(x_0)\cdot\frac{1}{2(x_0-x)} e^{\frac{-\lambda r^2}{4(x_0-x)}} dx_0 + \iint_{R_0} r_0 \cdot G \cdot f(x_0,r_0)\, dR_0 \tag{50}$$

Hence on the surface of the body $r = R(x)$ we obtain

$$\frac{\partial\phi(x,R)}{\partial x} \simeq -\int_x^1 \frac{S'(x_0)-S'(x)}{2(x_0-x)} dx_0 + \frac{S'(x)}{2}\log\left[\frac{\lambda R^2 e^c}{4(1-x)}\right] \tag{51}$$

$$+ \frac{\partial}{\partial x}\iint_{R_0} r_0 \cdot G(x,x_0;R,r_0)\, f(x_0,r_0)\, dR_0.$$

The parameter λ is now replaced by $\frac{(\gamma+1)M_1^2}{U_1}\frac{\partial^2\phi(x,r)}{\partial x^2}$ so that the above integral over the region R_0 can be evaluated and hence the following expression is obtained for $\frac{\partial\phi}{\partial x}$ on the surface of the body,

$$\frac{\partial \phi(x,R)}{\partial x} = -\int_x^1 \frac{S'(x_0) - S'(x)}{2(x_0 - x)}\, dx_0 + \frac{1 - M_1^2}{(\gamma+1) M_1^2}$$

$$+ \frac{S'(x)}{2} \log\left[- \frac{R^2 e^c (\gamma+1) M_1^2}{4(1-x) U_1} \frac{\partial^2 \phi(x,R)}{\partial x^2} \right] \quad (52)$$

It will be noted that $\dfrac{\partial \phi(x,R)}{\partial x} = (u)_{body}$ and

$$\frac{\partial^2 \phi(x,R)}{\partial x^2} = \left(\frac{\partial u}{\partial x}\right)_{body} = \left(\frac{du}{dx}\right)_{body} - \left(\frac{dR}{dx}\right) \cdot \frac{S'(x)}{R} .$$

On substituting the above expressions in equation (51) and dropping the suffix body we obtain

$$\frac{du}{dx} = \frac{S'(x)}{R} \cdot \frac{dR}{dx}$$

$$- \frac{4(1-x) U_1}{e^c R^2 (\gamma+1) M_1^2} \exp\left[\frac{2}{S'(x)} \left\{ u + \frac{(M_1^2 - 1) U_1}{(\gamma+1) M_1^2} + \int_x^1 \frac{S'(x_0) - S'(x)}{2(x_0 - x)} dx_0 \right\} \right] \quad (53)$$

<u>Special Case of a Body with a Parabolic-Arc Profile in a Free-Stream with Sonic Velocity</u>

We shall consider the special form of the solution in the case where the longitudinal section of the body is symmetric with the upper boundary of the parabolic curve $r = R(x) = \delta x(1 - x)$ where δ is a constant.

Then

$$S(x) = U_1 \delta^2 (1-2x)(1-x)(x),$$

$$S'(x) = U_1 \delta^2 (1-6x+6x^2),$$

and
$$\int_x^1 \frac{S'(x_0)-S'(x)}{2(x_0-x)}\,dx_0 = \frac{U_1\delta^2}{2}\Big[-6x_0+3x_0^2+6x x_0\Big]_x^1,$$
$$= -\frac{3U_1\delta^2}{2}(3x-1)(x-1).$$

On application of these results to equation (53) and remembering that $M_1 = 1$, it becomes,

$$\frac{du}{dx} = \frac{\delta^2 U_1(1-6x+6x^2)(1-2x)}{x(1-x)} - \frac{4U_1}{e^c\,\delta^2\,x^2(1-x)(\gamma+1)}\exp\left[\frac{u-\frac{3U_1\delta^2(3x-1)(x-1)}{2}}{\frac{U_1\delta^2}{2}(1-6x+6x^2)}\right] \qquad (54)$$

It is of interest to note that, if the transformation $y = 1-x$ is applied to the above equation, it becomes

$$\frac{du}{dx} = \frac{\delta^2 U_1(1-6y+6y^2)(1-2y)}{y(1-y)} + \frac{4U_1}{e^c\,\delta^2\,y(1-y)^2(\gamma+1)}\exp\frac{u-\frac{\delta^2U_1}{2}(9y^2-6y)}{\frac{U_1\delta^2}{2}(1-6y+6y^2)}, \qquad (55)$$

which is the same differential equation as the one obtained for u over the fore-body.

The Evaluation of u over the Complete Surface of the Parabolic-Arc Profile.

In the above two sections we have seen how expressions may be obtained for u over the fore-body and rear-body respectively. The problem still remains however of the manner in which these two solutions should be matched in order to give a solution over the complete body.

We consider first the solution for the after-body. From equation (54) it is easily seen that as $x \to x_2$ from the rear of the body if $u > \frac{3\delta^2 U_1}{2}(3x_2-1)(x_2-1)$ the value of $\frac{du}{dx}$ at $x = x_2$ is infinite. Furthermore for all values of $u < \frac{3\delta^2 U_1}{2}(3x_2-1)(x_2-1)$ the value of $\frac{du}{dx}$ is zero at $x = x_2$. But the solution for u over the fore-body which is valid in the range $0 < x < x_2$ gives the value of $\frac{du}{dx}$ at $x = x_2$ as zero. However it is not possible to have a continuous solution for u through the point $x = x_2$ by matching the fore-body solution to the after-body solution since the value of u from the fore-body solution is greater than $\frac{3\delta^2 U_1}{2}(3x_2-1)(x_2-1)$ at $x = x_2$.

As equation (55) is identical in form with equation (41) it follows that there is only one continuous solution for u in the range $x_1 < x < 1$. This solution is a mirror image about the point $x = 1/2$ of the continuous solution for u obtained from equation (41) wrich is valid in the range $0 < x < x_2$. From this solution we see that $\frac{du}{dx}$ is zero at $x = x_1$ and that the value of u is greater than $\frac{\delta^2 U_1}{2}(9x_1^2 - 6x_1)$ at $x = x_1$. This means that if we use this value for u in equation (41) the value of $\frac{du}{dx}$ is infinite at $x = x_1$. Hence we cannot have a continuous solution for u through the point $x = x_1$.

It is therefore impossible to find a continuous solution for u from equations (41) and (54) over the surface of a parabolic-arc body as $\frac{du}{dx}$ is positive for all regions where equation (41) is applicable and is negative for all regions where equation (54) is applicable.

Since we cannot obtain a continuous solution the only way to connect up the values of u over the fore-body and rear-body is to introduce a shock surface normal to the free-stream. From equations (28) and (25) the relationships between the perturbation velocities before u_2 and after u_3 the shock are

$$u_3 = -u_2 ,$$

and

$$u_2 > u_3$$

If now the curves of the two solutions, valid in the range $0 < x < x_2$ and $x_1 < x < 1$ are drawn (see figures 9 and 10) it is clear that there exists no value of x where the shock conditions can be satisfied. This implies that the shock surface must be at either $x = x_1$ or x_2 At $x = x_1$ the condition $u_2 > u_3$ cannot be satisfied. Therefore the shock surface must be placed at $x = x_2, \left(= \frac{1}{2} + \frac{\sqrt{3}}{6}\right)$.

The values of u over the rear-body may now be found by solving numerically equation (54) with the condition that at $x = x_2$ the value of u is equated to the negative of the corresponding value of u obtained from the solution of equation (41) valid in the range $0 < x < x_2$. The values of u over the complete surface of the parabolic arc body of revolution may thus be determined.. The valued of u/U_1 for different valies of δ are given in tables I. II, III, IV and V on pages 53 to 57

CHAPTER 4.

The Pressure Coefficient.

The pressure coefficient C_p is defined by

$$C_p = \frac{p - p_1}{\frac{1}{2}\rho_1 U_1^2}$$

From the adiabatic gas law we have

$$p - p_1 = p_1\left[\left(\frac{\rho}{\rho_1}\right)^{\gamma} - 1\right], \qquad (56)$$

and from equation (11) we have

$$\frac{1}{2}U_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}\left[(U_1+u)^2 + v^2\right] + \frac{\gamma}{\gamma-1}\frac{p}{\rho}.$$

$$\therefore \frac{1}{2}\left[U_1^2 - (U_1+u)^2 - v^2\right] = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left[\frac{p}{\rho}\cdot\frac{\rho_1}{p_1} - 1\right],$$

$$= \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left[\left(\frac{\rho}{\rho_1}\right)^{\gamma-1} - 1\right].$$

$$\therefore \left(\frac{\rho}{\rho_1}\right)^{\gamma-1} = 1 - \frac{\gamma-1}{\gamma}\cdot\frac{\rho_1}{2p_1}\left(2U_1 u + u^2 + v^2\right).$$

$$\therefore \left(\frac{\rho}{\rho_1}\right)^{\gamma} = \left[1 - \frac{\gamma-1}{\gamma}\cdot\frac{\rho_1}{2p_1}\left(2U_1 u + u^2 + v^2\right)\right]^{\frac{\gamma}{\gamma-1}}$$

$$\simeq 1 - \frac{\rho_1}{2p_1}\left(2U_1 u + u^2 + v^2\right) + \frac{1}{2\gamma}\frac{\rho_1^2}{p_1^2}U_1^2 u^2, \quad \text{to } O(u^2, v^2).$$

Therefore on substituting this expression for $\left(\frac{\rho}{\rho_1}\right)^{\gamma}$ in equation (56) we obtain

$$p - p_1 = -\frac{\rho_1}{2}\left(2U_1 u + u^2 + v^2\right) + \frac{1}{2\gamma}\frac{\rho_1^2}{p_1}U_1^2 u^2.$$

$$\therefore C_p = -\frac{1}{U_1^2}\left(2U_1 u + u^2 + v^2\right) + \frac{\rho_1}{\gamma p_1}u^2,$$

$$= -2\left(\frac{u}{U_1}\right) - \left(\frac{u}{U_1}\right)^2 - \left(\frac{v}{U_1}\right)^2 + \left(\frac{u}{a_1}\right)^2,$$

$$= -2\left(\frac{u}{U_1}\right) - (1 - M_1^2)\left(\frac{u}{U_1}\right)^2 - \left(\frac{v}{U_1}\right)^2.$$

Since the problem under investigation is one in which the fluid velocity at infinity upstream is sonic, we have

$$C_p = -2\left(\frac{u}{U_1}\right) - \left(\frac{v}{U_1}\right)^2, \quad \text{to } O(u^2, v^2). \qquad (57)$$

Now from our analysis of the magnitudes of the terms in the transonic equation (12) we found that if $\left(\frac{u}{U_1}\right)$ has a magnitude of ε, then the magnitude of

$\left(\frac{v}{U_1}\right)$ is $\epsilon^{3/2}$ where ϵ is a small parameter. However if we compare the values of $\left|\frac{u}{U_1}\right|$ and $\left|\frac{v}{U_1}\right|$ obtained on the surface of the body, see Table I, II III, IV, and V, from the solution of equation (15) and from the boundary conditions given by equation (19) respectively, we see that the above relation between their magnitudes does not hold at all points on the surface of the body. In fact over certain parts of the surface of the body the value of $\left(\frac{v}{U_1}\right)$ is greater than the value of $\left(\frac{u}{U_1}\right)$. This does not imply that the transonic small disturbance equation (12) used for the above analysis is not generally valid, but that in certain special small regions of the field of flow its use may be doubtful. We shall investigate this validity further in the remaining paragraph of this section. Before doing so we should remark that a consequence of this fact is that the term involving $\left(\frac{v}{U_1}\right)^2$ in equation (57) may be important in the calculation of pressure and should be retained.

We now look at the terms in equation (12) containing $\left(\frac{v}{U_1}\right)^2$ which we previously neglected to see if we can still justify their omission. These terms are $\frac{\gamma-1}{2} v^2 \frac{\partial u}{\partial x}$, $\frac{\gamma+1}{2} v^2 \frac{\partial v}{\partial r}$ and $\frac{\gamma-1}{2} \frac{v^3}{r}$. Referring back to equation (12) and using the notation employed in the order of magnitude discussion we note that the magnitude of these terms are ϵ^{2a+b+1}, ϵ^{4a+b-1} and ϵ^{4a+b-1} respectively. Now the magnitudes of the terms in equation (15) are ϵ^{2+b}, ϵ^{2+b}, ϵ^{2a+b-1} and ϵ^{2a+b-1} respectively. Therefore the first of the terms we neglected is ϵ^2 times the magnitude of the third and fourth terms in our transonic equation and the second and third terms we neglected are ϵ^{2a} or $\left(\frac{v}{U_1}\right)^2$ times the magnitude of the third and fourth terms in our transonic equation. This means that the terms we neglected are always of a much higher order than the third and fourth terms in our transonic equation and that we are still justified in neglecting them. It also means that our form for the transonic equation given by equation (15) still holds even if $\left(\frac{v}{U_1}\right)$ is not of the same order of magnitude as that given by our order of magnitude analysis, so long as it is not of zero order.

We can thus define the coefficient of pressure on the surface of our body with some degree of reliability, by

$$C_p = -2\left(\frac{u}{U_1}\right) - \left(\frac{v}{U_1}\right)^2 \qquad (57)$$

Graphs of C_p against x may now be drawn for different values of δ. These curves are shown in figures 1, 2,3,4 and 5 for $\delta = \frac{1}{3}, \frac{1}{3\sqrt{2}}, \frac{1}{5}, \frac{1}{6}$ and $\frac{1}{7}$ respectively. On the same figures are the curves obtained by Spreiter and Alksne (9) by Cole and Royce (10) and from experimental data (13,14).

CHAPTER 5.

CONCLUSIONS.

The curves of the coefficient of pressure shown in figures 1, 2, 3, 4 and 5, for different values of δ, calculated from our analysis are now compared with those obtained by Spreiter and Alksne (9), by Cole and Royce (10) and with the experimental results obtained from references (13) and (14). Over the fore-body our curves and the ones given by Spreiter and Alksne are identical because our solutions are identical for this region. In this region the curves given by Cole and Royce are almost identical with those we obtained and all three curves agree very well with the experimental results. Over the rear-body in the region $0.7 < x < 1$ the curves given by Cole and Royce, and by Spreiter and Alksne give values of the coefficient of pressure which are lower than those given by the experimental results while the curves we have obtained are in good agreement with the experimental results. In fact the use of a shock surface to connect up the solutions for u for the accelerating and decelerating regions of the flow gives a very good approximation for the rapid increase in the co-efficient of pressure which the experimental results indicate exists near the point $x = x_2 \left(= \frac{1}{2} + \frac{\sqrt{3}}{6}\right)$. However it should be noted that the steepness of this rise in the coefficient of pressure shown by the experimental results may be caused by the sting on the model which was tested.

However there is one slight anomaly in the present solution in that over a small region just after the shock surface the coefficient of pressure decreases a little before it increases; as one would expect a continuous increase from the shock surface to the tail of the body. Nevertheless the effect of this slight fall in the coefficient of pressure is insignificant because it is easily seen that in the calculation of the drag coefficient (see Appendix I) its effect is negligible.

We now investigate our analysis to see if we can find any reason for this fall in the coefficient of pressure immediately behind the shock surface. As we have already seen the order of magnitude of $\left(\frac{v}{U_1}\right)$ on the surface of the body is not the same as the order given by our order of magnitude analysis. Thus it is of interest to see how the magnitude of the term $\left(\frac{v}{U_1}\right)^2$ in equation (57) for the coefficient of pressure compares with the magnitude of the term $\frac{2u}{U_1}$. In figures 6,7 and 8 are shown the graphs of $v^2/2|u|U_1$ against x, for different values of δ, and on the same figures the graphs of $|u|/U_1$ against x are superimposed. The graphs of $|u|/U_1$ against x are useful, because in our order of magnitude analysis we neglected all the terms whose magnitudes were ϵ (or u/U_1) times the magnitude of any of the terms have retained in the transonic equation given by equation (15). From these figures we see that in the region near $x = x_2$, that is near the shock surface, $\frac{v^2}{2|u|U_1} \simeq \frac{|u|}{U_1}$. This means that in this region we have included a term in the evaluation of the coefficient of pressure which has the same order of magnitude as terms we have already neglected in our analysis. Therefore in order to obtain a more accurate value for the coefficient of pressure in this region we should retain all the second order terms in our transonic equation and in the relevant boundary conditions. However as we have already stated, this fall in the coefficient of pressure behind the shock surface is insignificant for the calculation of the drag coefficient and we do not believe that it justifies an attempt to obtain a more accurate solution than the one which we have presented.

APPENDIX I

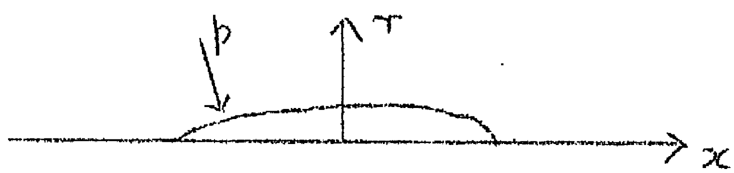
The Drag Coefficient

The drag coefficient (C_D) for a body of revolution is defined by

= Drag/free stream dynamic pressure x maximum cross-sectional area of the body,

$$= \frac{D}{\frac{1}{2}\rho_1 U_1^2 A} .$$

Consider a meridian plane of a slender body of revolution



It is customary to define the drag as the component in the free stream direction of the force exerted on the surface of the body by the excess pressure difference ($p - p_1$). Thus over an element δS of the surface there arises an element of drag δD given by

$$\delta D = (p - p_1).\delta S.\cos\left(\frac{\pi}{2} - \phi\right),$$

where ϕ is the angle between the tangent to the element δS of the surface and the x-axis.

$$\therefore \delta D = (p - p_1).\delta S \sin\phi,$$
$$= (p - p_1).\tan\phi.\delta x,$$
$$= (p - p_1)\frac{dR}{dx}.\delta x,$$

where $r = R(x)$ is the equation of the surface.

Therefore the element of drag arising from an elemental ring of the body surface is $$2\pi R(p - p_1) \cdot \frac{dR}{dx} \cdot \delta x$$

Hence the total drag for a body of revolution of unit length is given by

$$D = \int_0^1 2\pi R(p - p_1) \frac{dR}{dx} \, dx .$$

$$\therefore \quad C_D = \frac{\int_0^1 2\pi R(p - p_1) \frac{dR}{dx} \, dx}{\frac{1}{2} \rho_1 U_1^2 A} ,$$

$$= \frac{2\pi}{A} \int_0^1 C_p \cdot R \cdot \frac{dR}{dx} \, dx . \qquad (58)$$

APPENDIX 2

TABLE I

$\delta = \frac{1}{3}$

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.0113	.3258	-.4027	.6992
.0313	.3125	-.2868	.4759
.0513	.2991	-.2318	.3742
.0713	.2858	-.1938	.3058
.0913	.2725	-.1638	.2534
.1113	.2591	-.1388	.2104
.1313	.2458	-.1169	.1735
.1513	.2325	-.0974	.1408
.1713	.2191	-.0797	.1114
.1913	.2058	-.0633	.0844
.2113	.1925	-.0481	.0592
.2313	.1791	-.0338	.0355
.2513	.1658	-.0203	.0131
.2713	.1525	-.0075	-.0083
.2913	.1391	+.0048	-.0289
.3113	.1258	.0165	-.0488
.3313	.1125	.0276	-.0679
.3513	.0991	.0383	-.0865
.3713	.0858	.0486	-.1046
.3913	.0725	.0584	-.1221
.4113	.0591	.0678	-.1391
.4313	.0458	.0767	-.1556
.4513	.0325	.0852	-.1716
.4713	.0191	.0933	-.1869
.4913	.0058	.1009	-.2018
.5113	-.0075	.1079	-.2159
.5313	-.0209	.1145	-.2294
.5513	-.0342	.1205	-.2422
.5713	-.0475	.1259	-.2541
.5913	-.0609	.1307	-.2652
.6113	-.0742	.1349	-.2753
.6313	-.0875	.1384	-.2846
.6513	-.1009	.1413	-.2918
.6713	-.1142	.1436	-.3002
.6913	-.1275	.1453	-.3069
.7113	-.1409	.1467	-.3133
.7313	-.1542	.1477	-.3192
.7513	-.1675	.1486	-.3253
.7713	-.1809	.1491	-.3309
.7887	-.1925	.1493	-.3357
.7887	-.1925	-.1493	+.2615
.8087	-.2058	-.1496	.2568
.8287	-.2191	-.1506	.2532
.8487	-.2325	-.1528	.2515
.8687	-.2458	-.1571	.2538
.8887	-.2591	-.1660	.2649
.9087	-.2725	-.1816	.2889
.9287	-.2858	-.2055	.3293
.9487	-.2991	-.2398	.3901
.9687	-.3125	-.2922	.4867
.9887	-.3258	-.4001	.6941

TABLE II

$\delta = \frac{1}{3\sqrt{2}}$

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.0113	.2304	-.2276	.4021
.0313	.2209	-.1671	.2854
.0513	.2115	-.1368	.2289
.0713	.2021	-.1149	.1889
.0913	.1927	-.0970	.1570
.1113	.1832	-.0817	.1299
.1313	.1738	-.0681	.1060
.1513	.1644	-.0558	.0846
.1713	.1549	-.0444	.0649
.1913	.1455	-.0339	.0466
.2113	.1361	-.0241	.0296
.2313	.1267	-.0148	.0135
.2513	.1172	-.0060	-.0016
.2713	.1078	+.0022	-.0161
.2913	.0984	.0101	-.0298
.3113	.0889	.0175	-.0429
.3313	.0795	.0246	-.0555
.3513	.0701	.0313	-.0675
.3713	.0609	.0376	-.0789
.3913	.0512	.0436	-.0898
.4113	.0418	.0492	-.1003
.4313	.0324	.0545	-.1101
.4513	.0229	.0595	-.1195
.4713	.0135	.0641	-.1284
.4913	.0041	.0683	-.1367
.5113	-.0051	.0722	-.1444

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.5313	-.0148	.0757	-.1515
.5513	-.0242	.0787	-.1581
.5713	-.0336	.0814	-.1639
.5013	-.0431	.0837	-.1693
.6113	-.0525	.0856	-.1740
.6313	-.0619	.0872	-.1782
.6513	-.0713	.0884	-.1819
.6713	-.0808	.0893	-.1851
.6913	-.0902	.0900	-.1881
.7113	-.0996	.0906	-.1911
.7313	-.1090	.0911	-.1941
.7513	-.1185	.0915	-.1970
.7713	-.1279	.0918	-.2000
.7887	-.1361	.0919	-.2023
.7887	-.1361	-.0919	+.1653
.8087	-.1455	-.0920	.1628
.8287	-.1549	-.0926	.1612
.8487	-.1644	-.0936	.1602
.8687	-.1738	-.0956	.1610
.8887	-.1832	-.0997	.1658
.9087	-.1927	-.1077	.1782
.9287	-.2021	-.1210	.2012
.9487	-.2115	-.1404	.2361
.9687	-.2209	-.1692	.2896
.9887	-.2304	-.2255	.3979

TABLE III

$\delta = \frac{1}{5}$

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p	x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.0113	.1955	-.1713	.3044	.5313	-.0125	.0608	-.1217
.0313	.1875	-.1287	.2223	.5513	-.0205	.0630	-.1265
.0513	.1795	-.1060	.1798	.5713	-.0285	.0650	-.1307
.0713	.1715	-.0891	.1487	.5913	-.0365	.0666	-.1345
.0913	.1635	-.0753	.1238	.6113	-.0445	.0679	-.1378
.1113	.1555	-.0632	.1023	.6313	-.0525	.0689	-.1406
.1313	.1475	-.0525	.0832	.6513	-.0605	.0697	-.1431
.1513	.1395	-.0427	.0659	.6713	-.0685	.0703	-.1454
.1713	.1315	-.0336	.0500	.6913	-.0765	.0708	-.1475
.1913	.1235	-.0252	.0352	.7113	-.0845	.0712	-.1496
.2113	.1155	-.0173	.0213	.7313	-.0925	.0716	-.1518
.2313	.1075	-.0099	.0083	.7513	-.1005	.0719	-.1539
.2513	.0995	-.0029	-.0041	.7713	-.1085	.0721	-.1560
.2713	.0915	+.0037	-.0157	.7887	-.1155	.0722	-.1577
.2913	.0835	.0099	-.0268	.7887	-.1155	-.0722	+.1311
.3113	.0755	.0159	-.0374	.8087	-.1235	-.0723	.1293
.3313	.0675	.0214	-.0474	.8287	-.1315	-.0726	.1279
.3513	.0595	.0267	-.0570	.8487	-.1395	-.0734	.1273
.3713	.0515	.0317	-.0661	.8687	-.1475	-.0748	.1278
.3913	.0435	.0364	-.0747	.8887	-.1555	-.0776	.1310
.4113	.0355	.0408	-.0829	.9087	-.1635	-.0835	.1403
.4313	.0275	.0449	-.0905	.9287	-.1715	-.0935	.1576
.4513	.0195	.0487	-.0977	.9487	-.1795	-.1084	.1846
.4713	.0115	.0522	-.1045	.9687	-.1875	-.1301	.2250
.4913	.0035	.0554	-.1107	.9887	-.1955	-.1715	.3078
.5113	-.0045	.0582	-.1165				

TABLE IV

$\delta = \frac{1}{6}$

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.0113	.1629	-.1278	.2291
.0313	.1562	-.0961	.1679
.0513	.1496	-.0795	.1366
.0713	.1429	-.0669	.1134
.0913	.1362	-.0565	.0944
.1113	.1296	-.0473	.0779
.1313	.1229	-.0391	.0631
.1513	.1162	-.0316	.0497
.1713	.1096	-.0246	.0372
.1913	.1029	-.0181	.0256
.2113	.0962	-.0120	.0148
.2313	.0896	-.0063	.0046
.2513	.0829	-.0009	-.0051
.2713	.0762	+.0042	-.0142
.2913	.0696	.0090	-.0228
.3113	.0629	.0135	-.0310
.3313	.0562	.0178	-.0388
.3513	.0496	.0218	-.0462
.3713	.0429	.0256	-.0531
.3913	.0362	.0292	-.0596
.4113	.0296	.0325	-.0658
.4313	.0229	.0355	-.0716
.4513	.0162	.0383	-.0769
.4713	.0096	.0409	-.0819
.4913	.0029	.0432	-.0864
.5113	-.0038	.0453	-.0906

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.5313	-.0104	.0471	-.0943
.5513	-.0171	.0487	-.0977
.5713	-.0238	.0500	-.1006
.5913	-.0304	.0511	-.1031
.6113	-.0371	.0520	-.1053
.6313	-.0438	.0526	-.1072
.6513	-.0504	.0531	-.1088
.6713	-.0571	.0535	-.1103
.6913	-.0638	.0539	-.1118
.7113	-.0704	.0541	-.1132
.7313	-.0771	.0544	-.1147
.7513	-.0838	.0546	-.1162
.7713	-.0904	.0547	-.1177
.7887	-.0962	.0548	-.1188
.7887	-.0962	-.0548	+.1003
.8087	-.1029	-.0548	.0991
.8287	-.1096	-.0551	.0982
.8487	-.1162	-.0556	.0978
.8687	-.1229	-.0566	.0981
.8887	-.1296	-.0585	.1001
.9087	-.1362	-.0626	.1065
.9287	-.1429	-.0700	.1195
.9487	-.1496	-.0810	.1396
.9687	-.1562	-.0969	.1693
.9887	-.1629	-.1264	.2262

TABLE V

$\delta = \frac{1}{7}$

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.0113	.1396	-.0986	.1778
.0313	.1339	-.0749	.1318
.0513	.1282	-.0621	.1078
.0713	.1225	-.0524	.0897
.0913	.1168	-.0442	.0747
.1113	.1111	-.0370	.0616
.1313	.1053	-.0304	.0498
.1513	.0996	-.0244	.0390
.1713	.0939	-.0189	.0290
.1913	.0882	-.0137	.0196
.2113	.0825	-.0088	.0109
.2313	.0768	-.0043	.0026
.2513	.0711	+.0000	-.0051
.2713	.0653	.0041	-.0125
.2913	.0596	.0079	-.0194
.3113	.0539	.0115	-.0260
.3313	.0482	.0149	-.0322
.3513	.0452	.0181	-.0380
.3713	.0368	.0211	-.0435
.3913	.0311	.0239	-.0487
.4113	.0253	.0264	-.0535
.4313	.0196	.0288	-.0580
.4513	.0139	.0310	-.0622
.4713	.0081	.0330	-.0660
.4913	.0025	.0347	-.0695
.5113	-.0032	.0363	-.0726

x	$\frac{v}{U_1}$	$\frac{u}{U_1}$	C_p
.5313	-.0089	.0376	-.0754
.5513	-.0147	.0388	-.0778
.5713	-.0204	.0398	-.0800
.5913	-.0261	.0406	-.0818
.6113	-.0318	.0412	-.0834
.6313	-.0375	.0416	-.0846
.6513	-.0432	.0420	-.0859
.6713	-.0489	.0423	-.0869
.6913	-.0547	.0425	-.0880
.7113	-.0604	.0427	-.0890
.7313	-.0661	.0429	-.0901
.7513	-.0718	.0430	-.0912
.7713	-.0775	.0431	-.0923
.7887	-.0825	.0432	-.0931
.7887	-.0825	-.0432	+.0795
.8087	-.0882	-.0432	.0786
.8287	-.0939	-.0434	.0780
.8487	-.0996	-.0438	.0777
.8687	-.1053	-.0445	.0778
.8887	-.1111	-.0458	.0793
.9087	-.1168	-.0488	.0840
.9287	-.1225	-.0545	.0941
.9487	-.1282	-.0631	.1098
.9687	-.1339	-.0753	.1327
.9887	-.1396	-.0975	.1754

References for Part I.

1. Whitcomb, R.T. — A Study of the Zero-Lift Drag-Rise Characteristics of Wing-Body Combinations near the Speed of Sound.
N.A.C.A., Rep. 1273, 1956

2. Heaslet, M.A. & Spreiter, J.R. — Three-Dimensional Transonic Flow Theory Applied to Slender Wings and Bodies.
N.A.C.A., Rep. 1318, 1957

3. Ward, G.N. — Linearised Theory of Steady High-Speed Flow.
Cambridge Univ. Press, 1955.

4. Guderley, G. & Yoshihara, H. — The Flow over a Wedge Profile at Mach Number 1
J. Aero. Sci., (1950), Vol. 17, pp.723-735.

5. Helliwell, J.B. & Mackie, A.G. — Two Dimensional Subsonic and Sonic Flow Past Thin Bodies.
J. Fl. Mech., Vol. 3, Part I, p. 93, Oct., 1957

6. Oswatitsch, K. & Keune, F. — The Flow around Bodies of Revolution at Mach Number One.
Proceedings of the Conference on High-Speed Aeronautics, Polytechnic Institute of Brooklyn, 1955.

7. Maeder, P.F. & Wood, A.D. — Linearised Transonic Flow Past Isolated Non-Lifting Airfoils.
Technical Rep. W.T. 24, Division of Engineering Brown University, 1957.

8. Spreiter, J.R. & Alksne, A.Y. — Thin Airfoil Theory Based on Approximate Solution of the Transonic Flow Equation.
N.A.C.A. Rep., 1359, 1958.

9. Spreiter J.R. & Alksne, A.Y. — Slender-Body Theory Based on Approximate Solution of the Transonic Flow Equation.
N.A.S.A., Tech. Rep., R-2, 1959

10. Cole, J.D. & Royce, W.W. — An Approximate Theory for the Pressure Distribution and Wave Drag of Bodies of Revolution at Mach Number One.
Proceedings of the Sixth Annual Conference on Fluid Mechanics, University of Texas, 1959.

11. Carslaw, H.S. & Jaegar, J.C. — Operational Methods in Applied Mathematics.
Oxford University Press, 1953.

12. Watson, G.N. — Theory of Bessel Functions.
Cambridge University Press, 1958.

13. Taylor, R.A. & McDevitt, J.B. — Pressure Distributions at Transonic Speeds for Parabolic-Arc Bodies of Revolution having Fineness Ratios of 10, 12, and 14.
N.A.C.A., T.N. 4234, 1958

14. Drougge, G. — Some Measurements on Bodies of Revolution at Transonic Speeds.
Ninth International Congress of Applied Mechanics, Univ. of Brussels, 1957

GRAPHS OF C_p AGAINST x ($\delta = 1/3$)

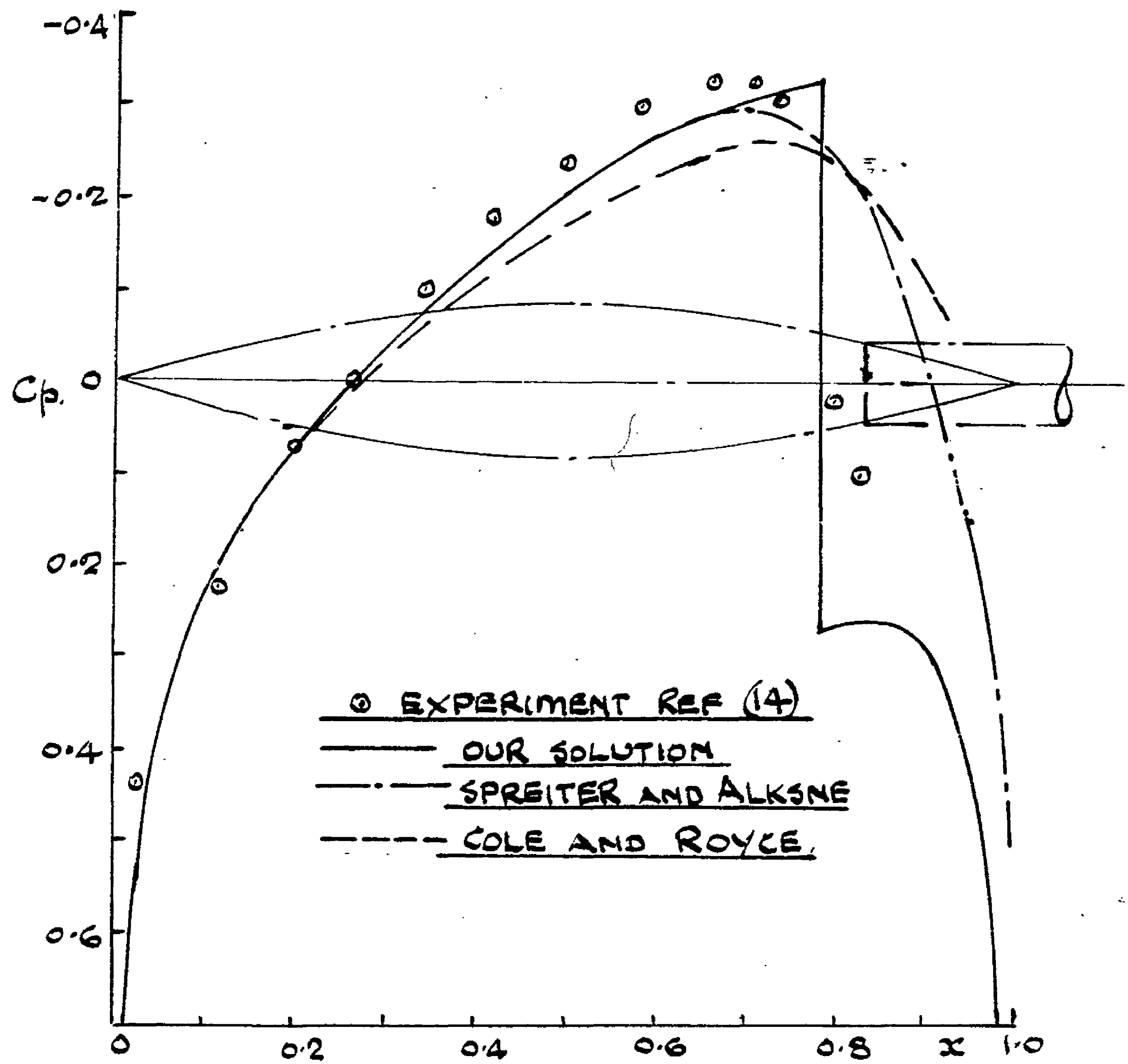


FIGURE 1.

GRAPHS OF C_p AGAINST x $\left(\delta = \frac{1}{3\sqrt{2}}\right)$

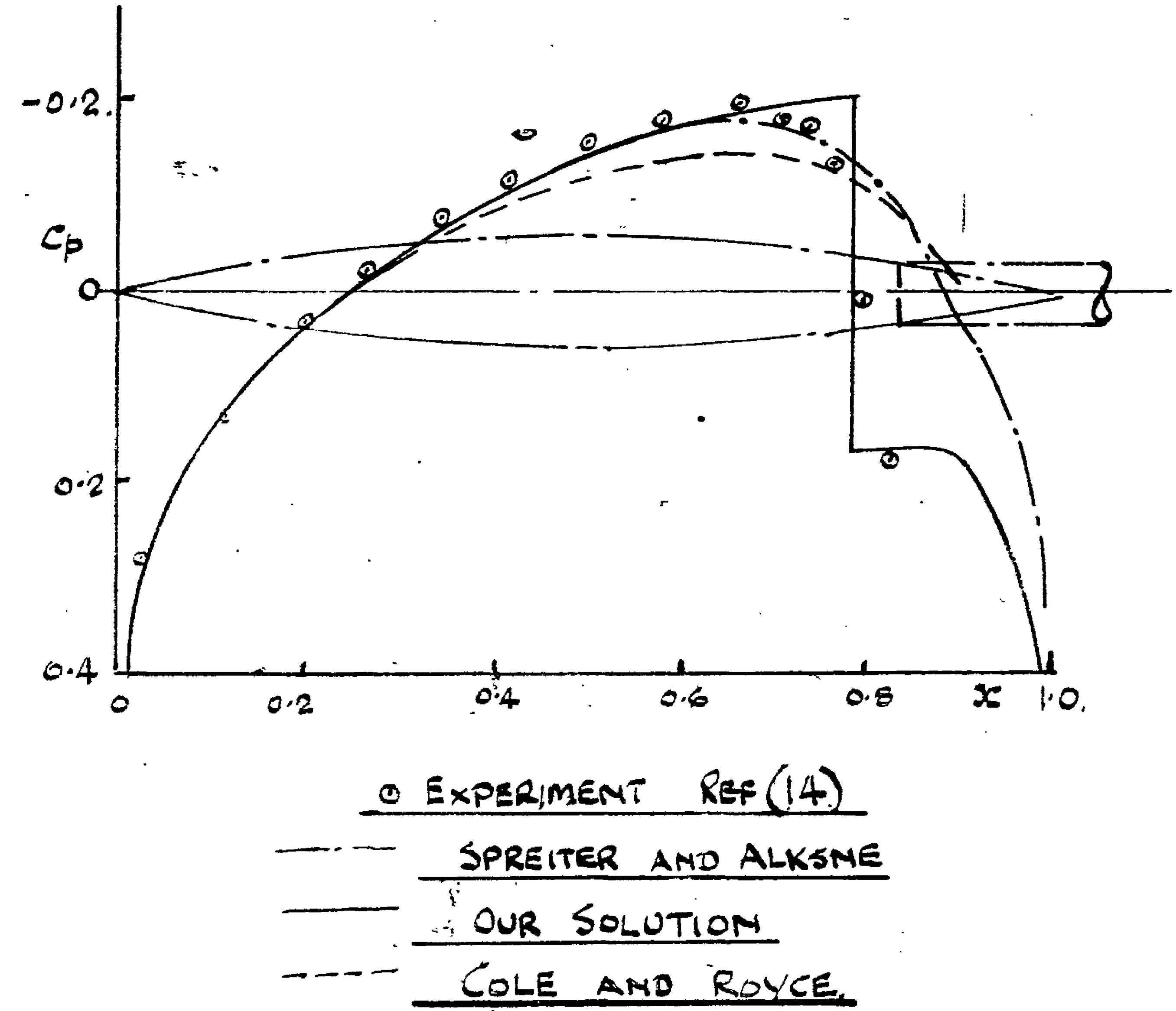


FIGURE 2

GRAPHS OF C_p AGAINST x. ($\delta = 1/5$)

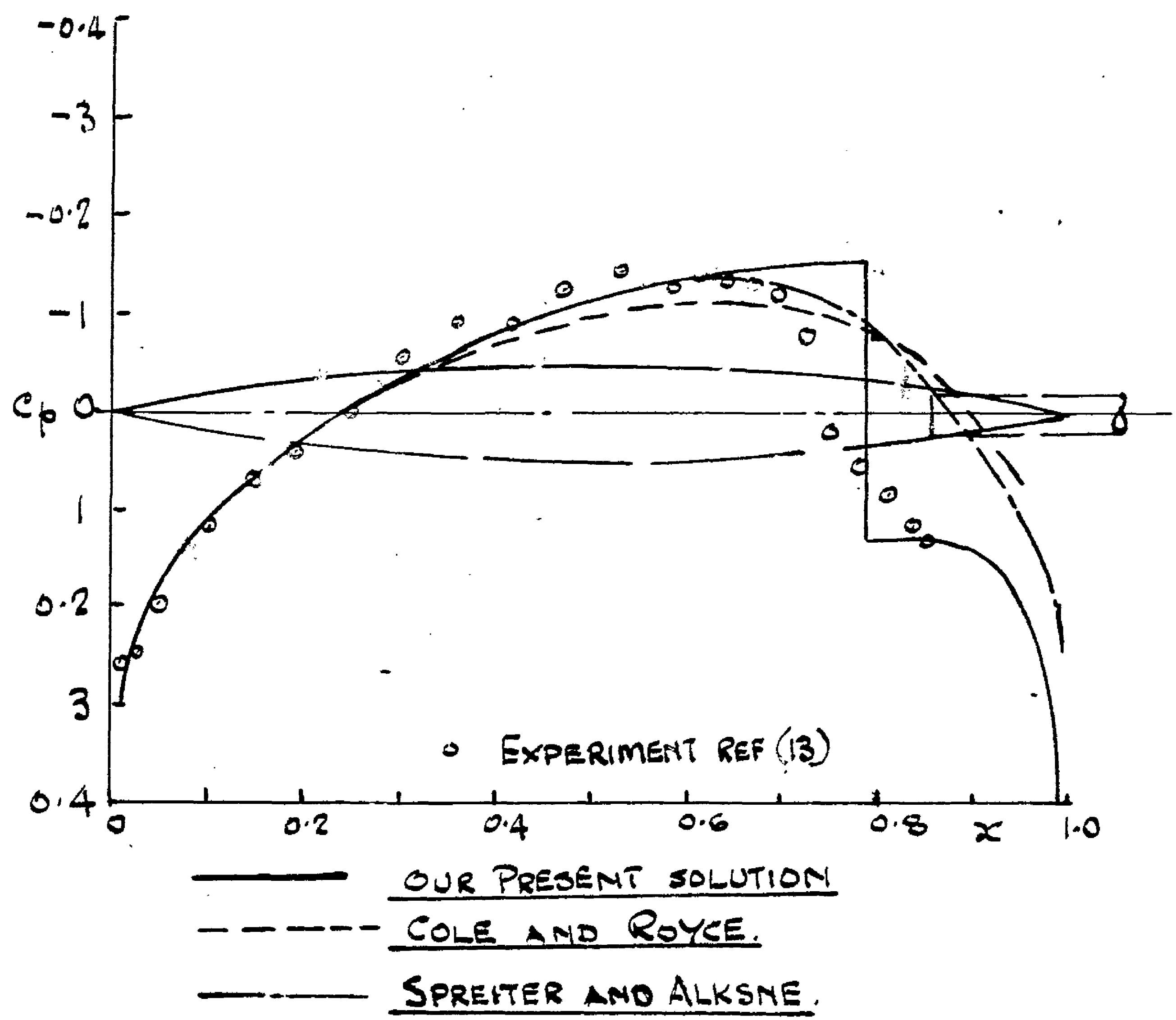
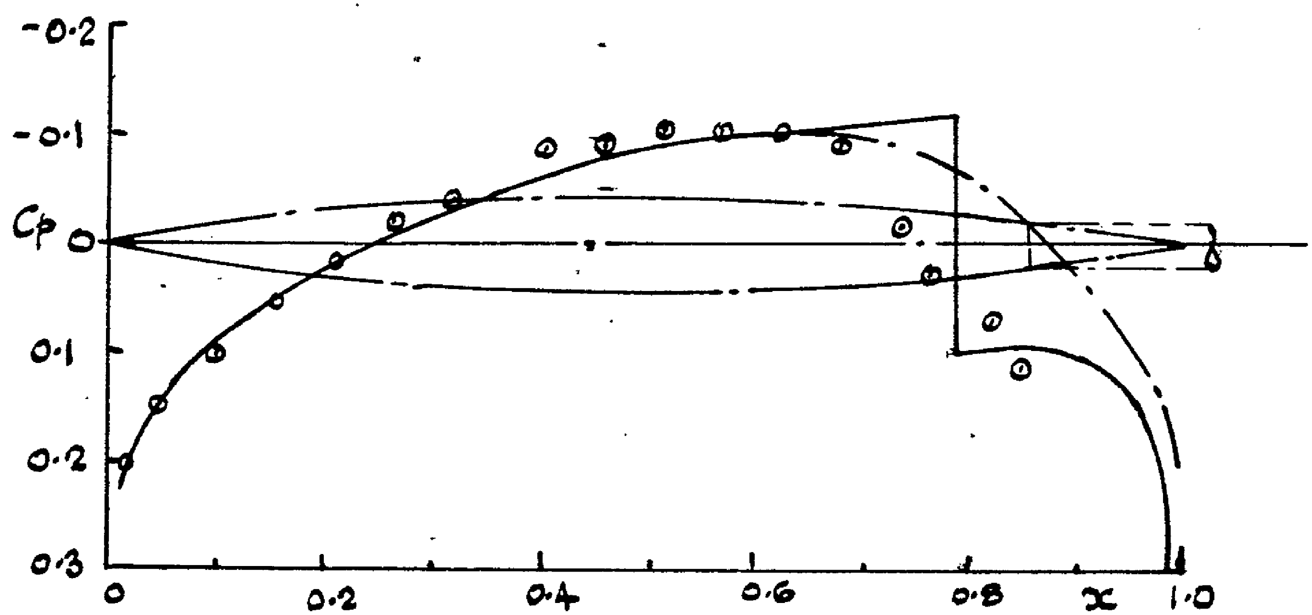


FIGURE 3.

GRAPHS OF C_p AGAINST x. $(\delta = \frac{1}{6})$



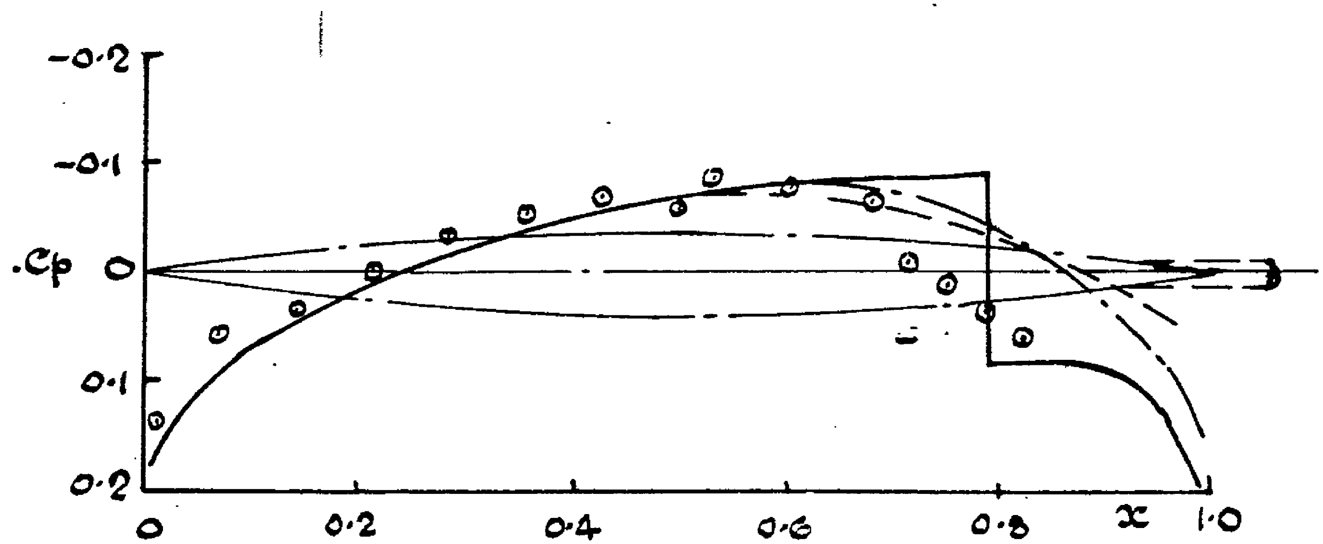
⊙ EXPERIMENT REF (13)

——— OUR SOLUTION

—·— SPREITER AND ALKSNE.

FIGURE 4

GRAPHS OF C_p AGAINST x $(\delta = \frac{1}{7})$



⊙ EXPERIMENT REF (13)

——— OUR SOLUTION.

- - - - COLE AND ROYCE.

—·— SPREITER AND ALKSNE

FIGURE 5.

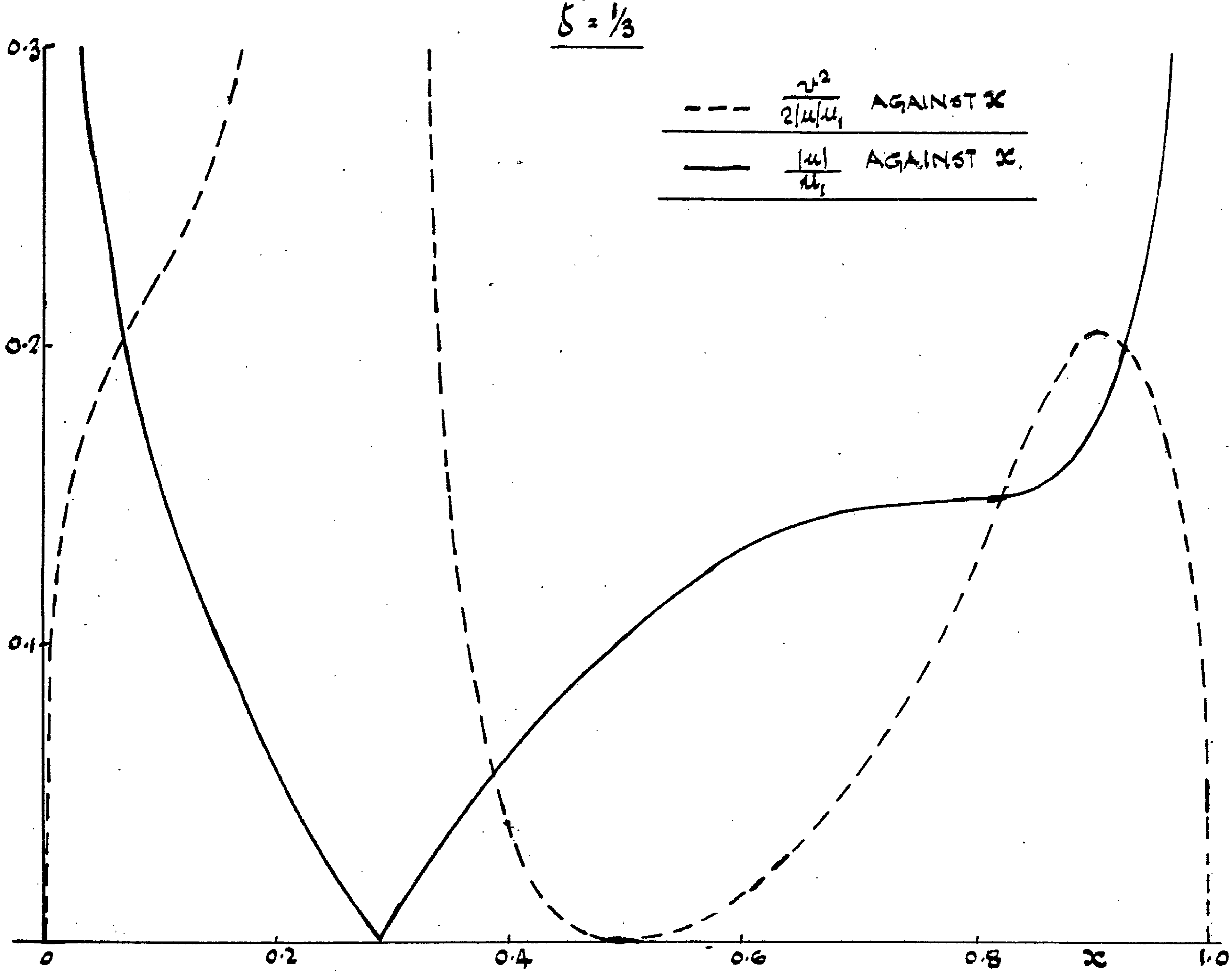

δ = 1/3
$\frac{v^2}{2|u|u_1}$ AGAINST x
$\frac{|u|}{u_1}$ AGAINST x.
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
x
1.0

FIGURE 6.

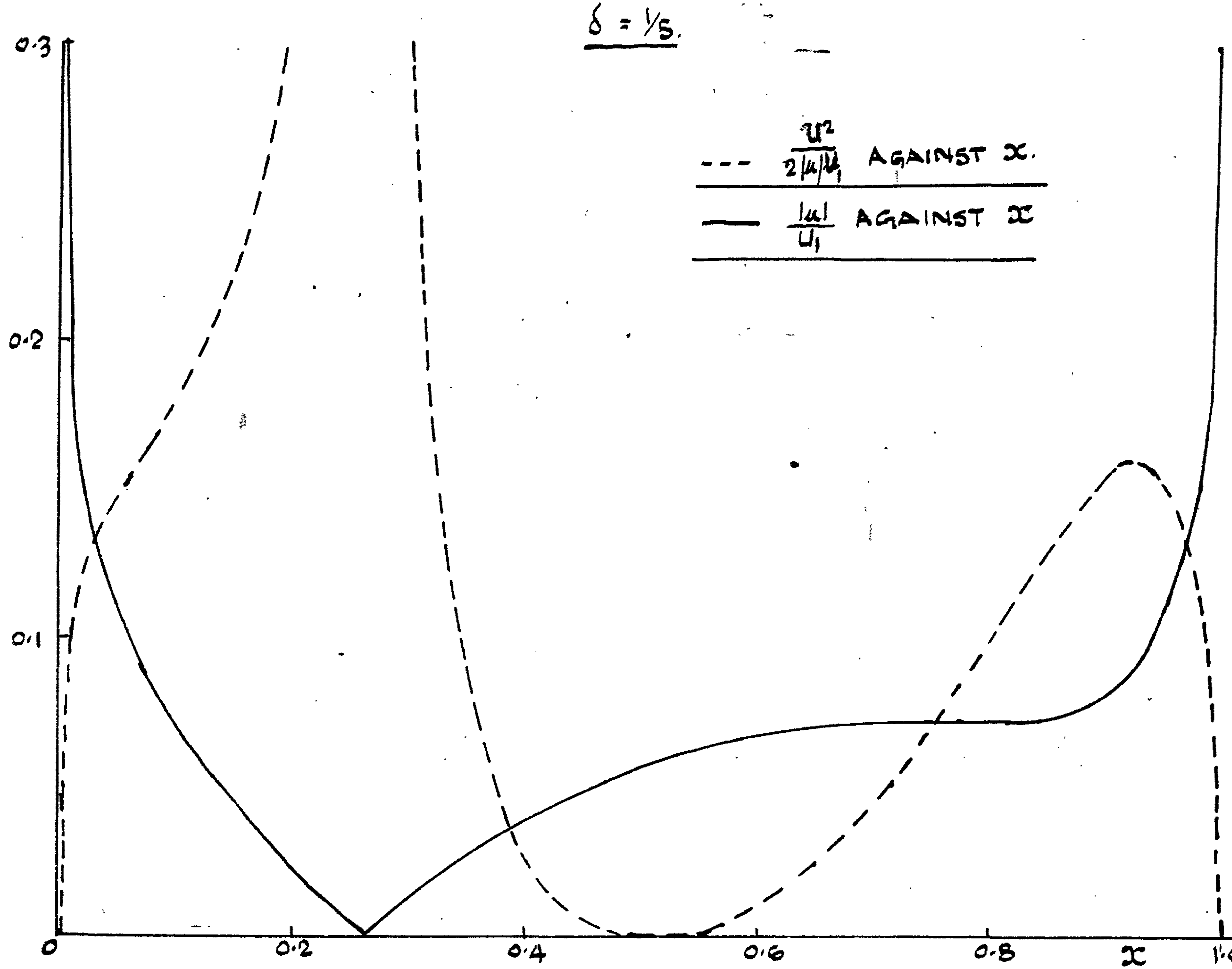


FIGURE 7.

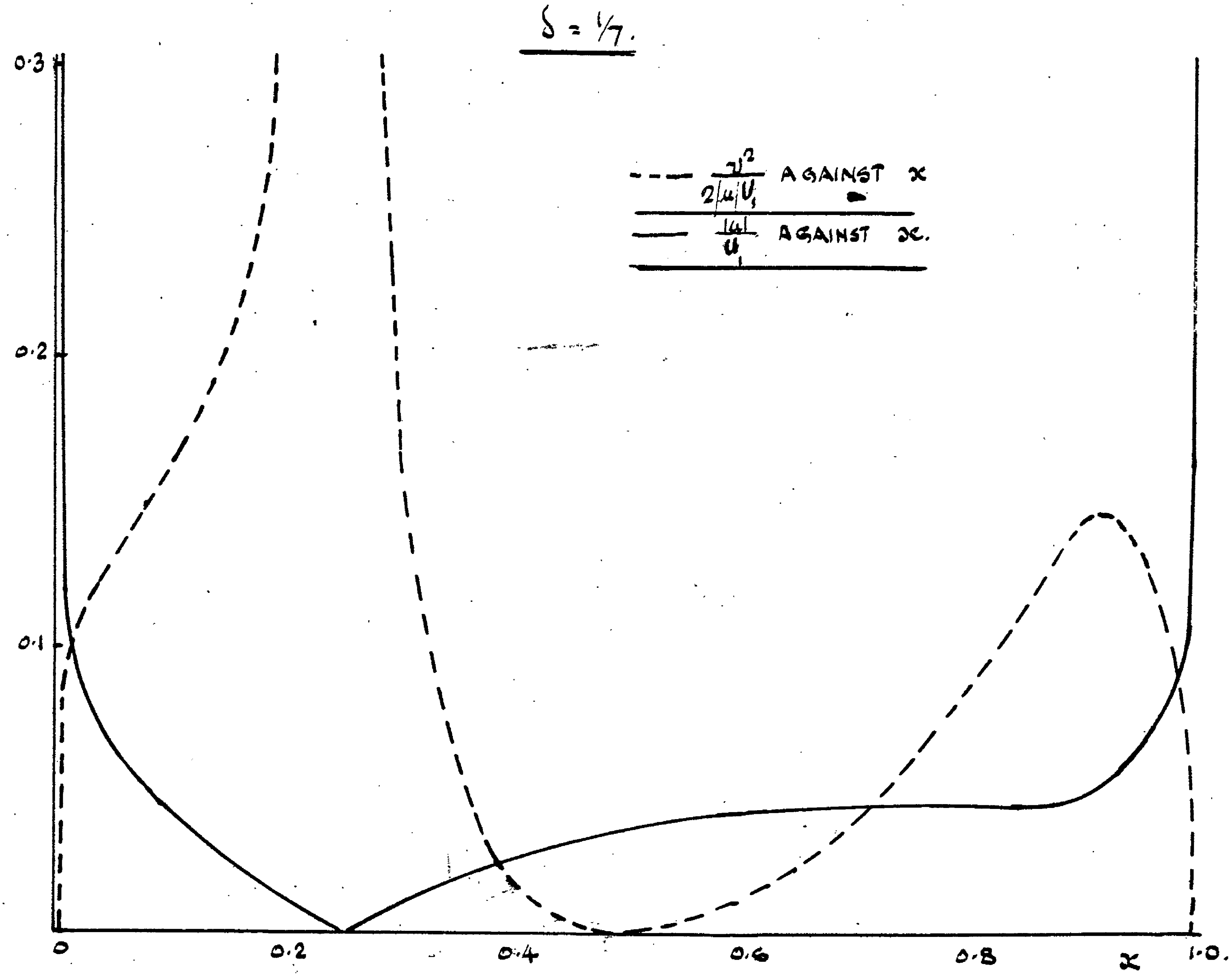

$\delta = 1/7$.
$\frac{v^2}{2|\mu|U_1}$ AGAINST x
$\frac{|\mu|}{U_1}$ AGAINST x.
0·3
0·2
0·1
0
0·2
0·4
0·6
0·8
1·0.
x

FIGURE 8.

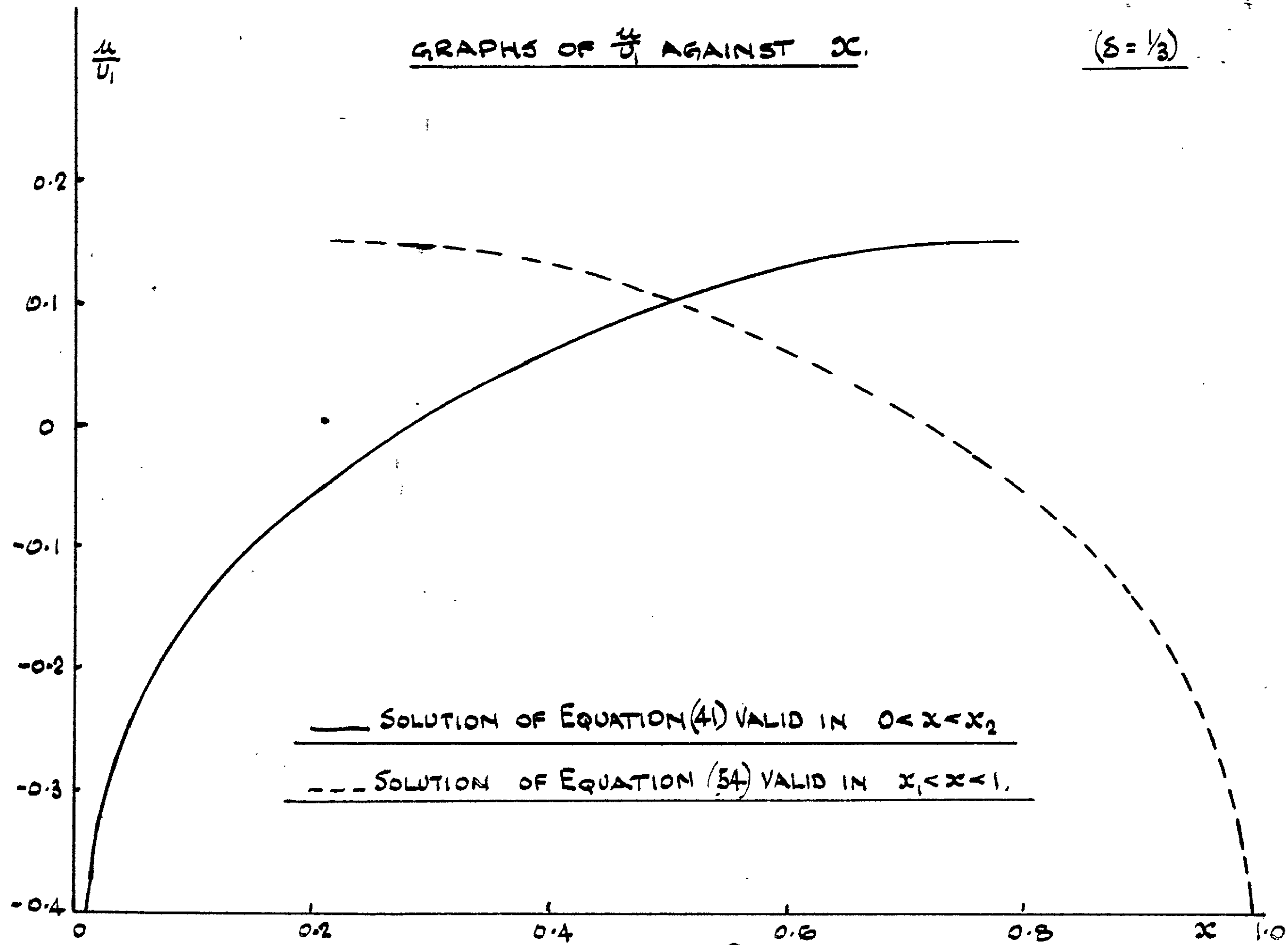

GRAPHS OF $\frac{u}{U_1}$ AGAINST x.
($\delta = 1/3$)
$\frac{u}{U_1}$
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.2
0.4
0.6
0.8
x
1.0
SOLUTION OF EQUATION (41) VALID IN $0 < x < x_2$
SOLUTION OF EQUATION (54) VALID IN $x_1 < x < 1$.

FIGURE 9

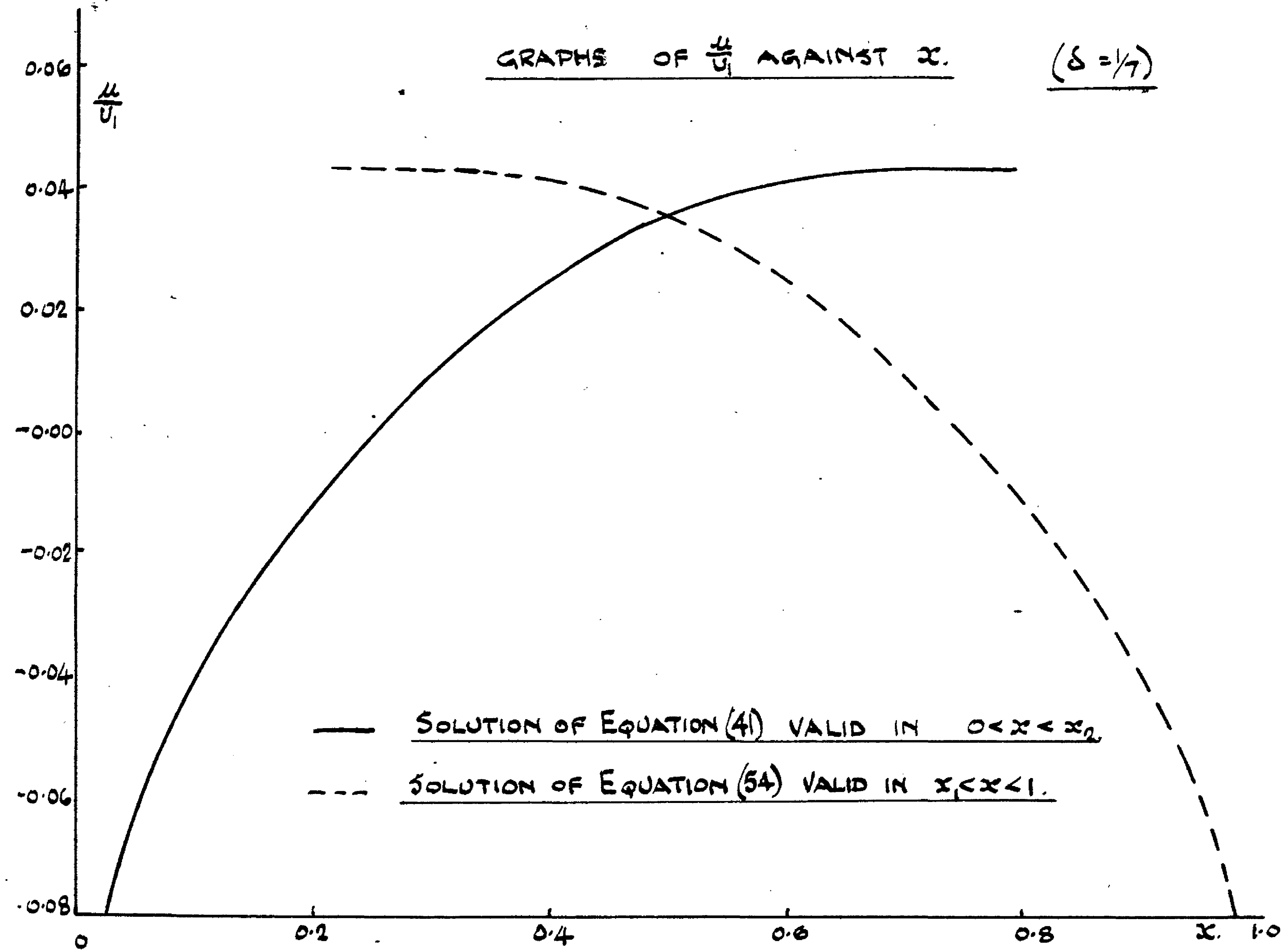

GRAPHS OF $\frac{u}{U_1}$ AGAINST x. ($\delta = 1/7$)
$\frac{u}{U_1}$
0.06
0.04
0.02
-0.00
-0.02
-0.04
-0.06
-0.08
0
0.2
0.4
0.6
0.8
x. 1.0
SOLUTION OF EQUATION (41) VALID IN $0 < x < x_2$
SOLUTION OF EQUATION (54) VALID IN $x_1 < x < 1$.

FIGURE 10

PART II.

CHAPTER I.

Introduction.

Since it is generally impossible by theoretical means to predict the performance of an aircraft the aircraft designer has to rely on experimental data obtained in wind tunnels. The designer can then establish the laws of similarity between the flow about the model in the wind tunnel and that about the actual aircraft by means of theoretical reasoning.

It is relatively easy to test the model in a closed tunnel or in a free jet of air when the speeds are well below sonic speed. In this case the significant similarity parameter is the Reynolds number. However in order to obtain complete similarity at high subsonic speeds a second similarity parameter, which is a function of the upstream Mach number and body thickness, must be considered. Tests must now be performed at a particular Mach and Reynolds number for a given body.

The principal difference between the flow about an aircraft flying in the atmosphere and the flow about its model in a wind tunnel, if the aerodynamic similarity parameters are the same, is caused by the finite lateral extension of the tunnel airstream. It can be shown that in a conventional wind tunnel this difference increases as the speed of sound is approached. Moreover, if a test section with solid walls is used, at a certain subsonic Mach number the model will cause the same effect as the throat of a Laval Nozzle. The speed of the upstream flow cannot be increased without change of upstream density, and the tunnel is then said to be choked. Therefore for an appreciable range of high subsonic speeds no testing is possible in a closed tunnel.

We will now consider the streamline patterns of flows at subsonic speeds about some body in free flight, in a solid straight walled tunnel and in a free jet. When the body is placed in a parallel flow of infinite extension it will cause the streamlines of the parallel flow to be deflected in such a way that the mass flow inside a stream tube remains constant, and the centrifugal forces caused by the streamline curvature are in equilibrium with the pressure forces. If the infinite parallel flow is replaced by a finite stream surrounded by solid straight walls, as will be the case in a wind tunnel, the streamlines forming the flow about the body are squeezed together more than they would be in free flight. This wall interference introduces changes in the pressure distribution over the body and leads in some cases to the phenomenon of choking. In the open jet type of wind tunnel the boundary is air at rest, which fact has the consequence that the curvature of the outside streamlines becomes greater than that of an infinite free stream, in order to balance the forces caused by the body since there is no longer any outside flow to resist the deformation. The flow pattern obtained is therefore one in which the streamlines are further apart than for free flight. This type of interference in which the distance between streamlines is effected by the boundary of the flow is called blockage interference.

In subsonic flow one can usually compensate for this effect by applying a correction factor. The body behaves as if it were tested at a higher or lower speed than that measured in the tunnel, depending on the type of boundary used. However in high subsonic flow the tunnel may be choked or a section of the desired speed range may be lost becuase of the change in character of the governing equations as the speed of sound is passed. It is therefore the removal of the blockage interference that is of the most importance in the design of a high subsonic wind tunnel.

One method used to eliminate blockage interference is the use of flexible walls which can be set to lie along the streamline that would occur in the infinite stream. The main disadvantage of this method lies in the time required to set the walls as a new setting is required for each Mach number.

Since solid straight walls and open jet boundaries influence the streamlines in opposite manners a possible solution of the problem for the elimination of the blockage interference should be obtained by using a boundary which is a combination of these two. This combination has led to the use of a straight walled tunnel with a finite slotted section along it and hence to a straight walled tunnel with a finite porous section along it.

A mathematical theory was developed by L.C. Woods (1) enabling wind tunnels with porous walls to be designed to give zero blockage interference in subsonic incompressible flow. The tunnel walls are taken to be porous over a finite range R and solid everywhere else, and a sealed jacket is placed over the porous section so that the pressure on the outside wall can be controlled. The porous wall is assumed to have the characteristic that the component of the velocity normal to the wall is proportional to the pressure drop across it, the constant of proportionality λ being termed the porosity of the wall. It has been shown by Preston and Rawcliffe (2) that it is possible to design a porous wall obeying this linear law. In the paper by L.C. Woods the relationship between the tunnel width H, the Mach number M, λ and R was found so that the blockage interference was zero. This relationship showed that for a given value of the porosity the length of the porous section must be reduced when the Mach number is increased to keep the zero blockage interference. Thus the tunnel needs to be fitted with adjustable sections of solid wall which can be moved across porous surfaces to reduce their effective length.

Since the flow patterns obtained for incompressible subsonic are similar to those for compressible subsonic flow the overall design characteristics of a high subsonic wind tunnel for zero blockage interference will be similar to the one obtained by L.C. Woods.

The problem considered in the following analysis is the design of a straight walled wind tunnel with a finite porous section to give zero blockage interference in high subsonic compressible flow.

Since we are dealing with the flow past a slender body, the transonic small disturbance flow equation, which is a first order approximation of the exact transonic flow equation is taken as the governing equation. The hodograph method is used in this problem to obtain a solution of this approximate equation. In this method the transonic small disturbance flow equation is transformed into a second order linear partial differential equation, viz. Tricomi's equation, by interchanging the dependent and independent variables. The solutions of the linear equation are first order approximations of the solutions for the flow of an ideal gas.

In the following analysis it is assumed that a solution to this problem can be determined by a perturbation from the solution obtained by Helliwell (4) for a channel with solid walls. This solution was obtained by using Tricomi's equation and is therefore a first order approximation of the solution for the flow of an ideal gas. Thus our solution may not be strictly justified for the flow of an ideal gas since the approximations made in obtaining the perturbations are of the same order as those made in deriving Tricomi's equation. However Tricomi's equation is the exact equation which governs the flow of a "Tricomi" gas. The properties of this gas in the neighbourhood of the speed of sound are very similar to those for an ideal gas. (see e.g. Bers(9)) For such a gas our solution will be strictly correct to first order.

CHAPTER 2

THE BASIC EQUATIONS OF TRANSONIC FLOW

Transonic flow is said to occur when the velocity of the fluid particles in some region of the flow is little different from the velocity of sound. The fluid we are considering is assumed to be a non-viscous, non-heat conducting perfect gas to which the adiabatic gas law applies. The motion is supposed to be steady, irrotational and dependent upon no external forces for its support.

The basic equations of continuity and momentum governing the flow have been obtained earlier in Part I of this thesis. In terms of the velocity ($\underset{\sim}{q}$), density (ρ), pressure (p) and speed of sound (a) in the fluid these equations take the form

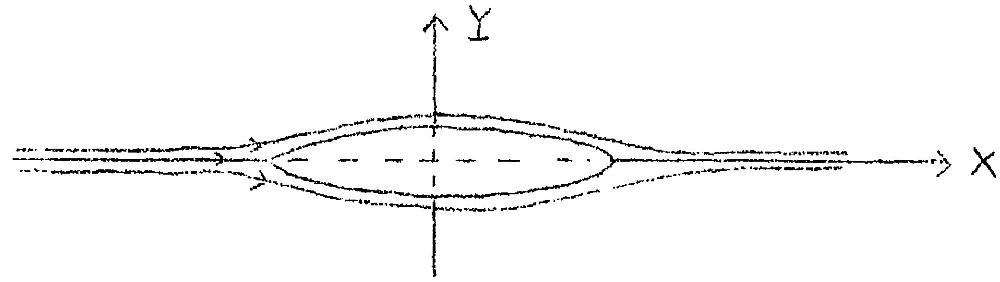
$$\operatorname{div}(\rho \underset{\sim}{q}) = 0 \quad , \tag{1}$$

$$\underset{\sim}{q} \cdot \nabla \underset{\sim}{q} + \frac{a^2}{\rho} \nabla \rho = 0 , \tag{2}$$

and
$$\underset{\sim}{q} \cdot \nabla \left(\tfrac{1}{2} \underset{\sim}{q}^2 \right) - a^2 \, \nabla \cdot \underset{\sim}{q} = 0 . \tag{3}$$

PERTURBATION VELOCITIES

We will now assume that we have a uniform steady flow whose velocity is U_1 and direction of flow is parallel to the X-axis of the two-dimensional coordinate system. If we now place in this flow, along the X-axis, a slender body with a smooth surface the disturbances caused to the uniform flow by the body will be small apart from the small region near the stagnation point at the nose of the body.



Then we may write,

U = the component of velocity in X-direction = $U_1(1 + u')$,

V = the component of velocity in Y-direction = $U_1 v'$.

Therefore $\quad q = U_1(1 + u')\,\underline{i} + U_1 v'\,\underline{j}$,

where $\underline{i}$ and $\underline{j}$ are the unit vectors in the direction of the X and Y axes respectively.

THE EQUATION OF MOTION IN TERMS OF THE PERTURBATION VELOCITIES

On using the condition that the flow is irrotational we obtain the relationship

$$\frac{\partial v'}{\partial X} = \frac{\partial u'}{\partial Y} . \tag{4}$$

From equation (2) and using the flow conditions at infinity upstream, viz $q = U_1 \underline{i}$ and $a = a_1$, we obtain Bernoulli's Equation

$$\frac{1}{2} q^2 + \frac{a^2}{\gamma - 1} = \frac{1}{2} U_1^2 + \frac{a_1^2}{\gamma - 1} . \tag{5}$$

On substituting for q in equation (3) and using equations (4), (5) and the fact that as $u'U$ and $v'U$ are small compared with U_1 and a_1, squares and higher powers of them may be neglected by comparison with U_1^2 and a_1^2. By a similar development to that of Part I we obtain,

$$\frac{\partial u'}{\partial X}\left[(M_1^2 - 1) + (\gamma+1) M_1^2 u'\right] + \frac{\partial v'}{\partial Y}\left[(\gamma-1) u' M_1^2 - 1\right] = -2 v' M_1^2 \frac{\partial u'}{\partial Y} \tag{6}$$

where M is the local Mach number of the flow, and suffix (1) here and hereafter refers to conditions infinitely far upstream.

THE TRANSONIC EQUATION FOR FLOWS WITH HIGH SUBSONIC VELOCITIES

We will now consider the particular case where the upstream velocity of the fluid is a little less than the velocity of sound, so that $M_1 < 1$ and $1 - M_1^2$ is small. Now let u' be of the order ε, v' be of the order of ε^a, $\frac{\partial}{\partial X}$ be of the order ε^b, $\frac{\partial}{\partial Y}$ be of the order ε^d, and $(1 - M_1^2)$ be of the order ε^e where ε is small and a, b, d and e are positive. Therefore on applying these orders of magnitude to the terms in equations (4) and (6) we see that

$$\varepsilon^{a+b} = \varepsilon^{1+d},$$

$$\varepsilon^{1+e+b} + \varepsilon^{2+b} + \varepsilon^{1+a+d} + \varepsilon^{a+d} = \varepsilon^{1+a+d}.$$

In the second equation as the two terms whose orders of magnitude are ε^{1+a+d} are of a higher order than the term whose order of magnitude is ε^{a+d} they can be neglected. Thus

$$\varepsilon^{1+e+b} + \varepsilon^{2+b} + \varepsilon^{a+d} = 0.$$

From the first equation we see that

$$a + b = 1 + d.$$

$$\therefore \qquad d = a + b - 1.$$

Hence the last equation becomes

$$\varepsilon^{1+e+b} + \varepsilon^{2+b} + \varepsilon^{2a+b-1} = 0.$$

The index b is common, thus

$$\varepsilon^{1+e} + \varepsilon^{2} + \varepsilon^{2a-1} = 0.$$

In the most general case all 3 terms will be present. Therefore <u>a</u> must be $\frac{3}{2}$ and <u>e</u> must equal 1. These values for <u>a</u> and <u>e</u> show that u' and $1 - M_1^2$ are of the same order of magnitude and that the disturbances in the transverse direction will be less than those in the horizontal direction.

The transonic small disturbance equation for two dimensional flow is

$$\frac{\partial u'}{\partial X}\left[1 - M_1^2 - (\gamma+1) M_1^2 u'\right] + \frac{\partial v'}{\partial Y} = 0 \qquad (7)$$

Now let $$u = 1 - M_1^2 - (\gamma+1) M_1^2 u' , \qquad (8)$$

and $$v = (\gamma+1) M_1^2 v' . \qquad (9)$$

Substitution of these values in (4) and (7) gives

$$\frac{\partial u}{\partial Y} = - \frac{\partial v}{\partial X} , \qquad (10)$$

$$u \frac{\partial u}{\partial X} - \frac{\partial v}{\partial Y} = 0 . \qquad (11)$$

It is not possible to obtain an exact analytic solution of the above equations. However it is possible to transform the equations into linear ones. This is accomplished by the hodograph transformation in which the dependent and independent variables are interchanged.

Thus we take

$$X = X(u,v) , \qquad (12)$$

and $$Y = Y(u,v) . \qquad (13)$$

Solving the simultaneous equations in $\frac{\partial u}{\partial X}$ and $\frac{\partial v}{\partial X}$ obtained by differentiating equations (12) and (13) w.r.t. X we find

$$\frac{\partial u}{\partial X} = \frac{\partial Y}{\partial v} / \Delta ,$$

and $$\frac{\partial v}{\partial X} = - \frac{\partial Y}{\partial u} / \Delta ,$$

where $$\Delta = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix} .$$

In a similar manner we find $\dfrac{\partial u}{\partial Y} = -\dfrac{\partial X}{\partial v} \Big/ \Delta$,

and $\dfrac{\partial v}{\partial Y} = \dfrac{\partial X}{\partial u} \Big/ \Delta$.

Therefore substituting these values in (10) and (11) we get

$$\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial u} = 0 , \tag{14}$$

and $$u \frac{\partial Y}{\partial v} - \frac{\partial X}{\partial u} = 0 , \tag{15}$$

so long as $\Delta \neq 0$. The special case $\Delta = 0$ is referred to later.

To eliminate X from (14) and (15) we differentiate (14) w.r.t. u and (15) w.r.t. v and add. Thus

$$\frac{\partial^2 Y}{\partial u^2} + u \frac{\partial^2 Y}{\partial v^2} = 0 . \tag{16}$$

This is the equation of Tricomi and is elliptic, parabolic or hyperbolic according as u is greater than, equal to or less than zero. The relationship between u and M is now found.

RELATIONSHIP BETWEEN u and M

We have previously defined $U = U_1(1 + u')$ and $V = U_1 v'$.

$$\therefore M^2 = \frac{U_1^2(1+u')^2 + U_1^2 v'^2}{a^2} ,$$

$$\simeq \frac{U_1^2}{a^2}(1 + 2u') ,$$

$$= \frac{M_1^2 a_1^2}{a^2}(1 + 2u') . \tag{17}$$

To evaluate $\dfrac{a^2}{a_1^2}$ we return to equation (6) which gives

$$a^2 + \tfrac{1}{2}(\gamma - 1) q^2 = a_1^2 + \tfrac{1}{2}(\gamma - 1) U_1^2 .$$

$$\therefore\ a^2 + \tfrac{1}{2}(\gamma-1)a^2M^2 = a_1^2 + \tfrac{1}{2}(\gamma-1)U_1^2 .$$

$$\therefore\ \frac{a^2}{a_1^2} + \tfrac{1}{2}(\gamma-1)\frac{a^2}{a_1^2}M^2 = 1 + \tfrac{1}{2}(\gamma-1)M_1^2 . \tag{18}$$

Substituting for $\frac{a^2}{a_1^2}$ from equation (17) we obtain

$$(1+2u')\frac{M_1^2}{M^2}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right] = 1+\tfrac{1}{2}(\gamma-1)M_1^2 .$$

Now

$$u' = -\frac{u-(1-M_1^2)}{M_1^2(\gamma+1)} .$$

$$\therefore\ \left[1+2\left\{\frac{1-M_1^2-u}{(\gamma+1)M_1^2}\right\}\right]\frac{M_1^2}{M^2}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right] = 1+\tfrac{1}{2}(\gamma-1)M_1^2 .$$

$$\therefore\ u = (1-M^2)\cdot\frac{1+\frac{1}{2}(\gamma-1)M_1^2}{1+\frac{1}{2}(\gamma-1)M^2} . \tag{19}$$

From equation (17) to zero order of approximation we see that $\frac{a^2}{a_1^2} = \frac{M_1^2}{M^2}$.
Therefore equation (18) to zero order is

$$\frac{M_1^2}{M^2} + \tfrac{1}{2}(\gamma-1)M_1^2 = 1 + \tfrac{1}{2}(\gamma-1)M_1^2 .$$

Therefore to zero order approximation $M_1 = M$, and hence

$$u = 1 - M^2 . \tag{20}$$

As in subsequent development we shall be interested in flows which have velocities no higher than sonic velocity u will always have a non-negative value.

Now $\Delta = 0$ corresponds to a limiting line in the mapping from the hodograph plane back to the physical plane. It is known that the limiting line is the envelope of one set of characteristics of the governing equation for the flow. Thus in flows which are entirely subsonic limit lines cannot occur, $\Delta \neq 0$ except possibly at exceptional points, and the hodograph transformation is valid since the relevant differential equation is everywhere elliptic.

ELEMENTARY SOLUTIONS OF TRICOMI'S EQUATION

If a new variable r defined by

$$\tau = \frac{2}{3} \mu^{3/2} = \frac{2}{3}(1-M^2)^{3/2}, \qquad (21)$$

is introduced simple solutions of Tricomi's Equation can be obtained by separation of variables. Equation (16) now becomes

$$\frac{\partial^2 Y}{\partial \tau^2} + \frac{1}{3\tau}\frac{\partial Y}{\partial \tau} + \frac{\partial^2 Y}{\partial \nu^2} = 0 . \qquad (22)$$

By setting $Y = \tau^{1/3} f_n(\tau, \nu)$ solutions of equation (22) can be found by separation of the variables and lead to solutions of the type

$$Y = \tau^{1/3} e^{\pm\lambda\nu} \mathcal{B}_{\pm 1/3}(\lambda\tau) , \qquad (23)$$

where $\mathcal{B}_{\pm 1/3}(\lambda\tau)$ is any linear combination of Bessel Functions of order $\pm 1/3$ and λ is a constant, either real or imaginary.

BOUNDARY CONDITIONS

I SOLID SURFACES

For a fluid there can be no flow through a solid surface, but for a non-viscous fluid slip past the solid surface may occur. If the equation of the surface is f(X,Y) = 0, then the condition of zero velocity normal to the surface yields

$$q \cdot \nabla f(X, Y) = 0 .$$

$$\therefore U_1(1+u')\frac{\partial f}{\partial X} + U_1 v' \frac{\partial f}{\partial Y} = 0$$

$$\therefore v' = -\frac{\partial f}{\partial X} \Big/ \frac{\partial f}{\partial Y} ,$$ since u' can be neglected as it is small compared with unity

$$= \frac{dY}{dX}$$

$$=$$ slope of surface. (24)

but $$v = (\gamma+1) M_1^2 v'$$

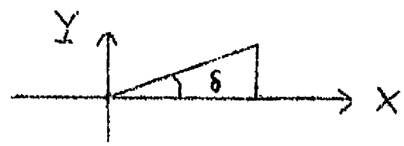
Therefore on all solid surfaces the boundary condition is given by

$$v = (\gamma+1) M_1^2 \times \text{ the slope of the surface.}$$

(a) If the surfaces are parallel to the X axis the boundary condition is given by

$$v = 0 \tag{25}$$

(b) For a slender wedge whose semi-angle is δ and whose nose is at the origin of the coordinate system the equation of the face is given by $Y \simeq \delta X$.

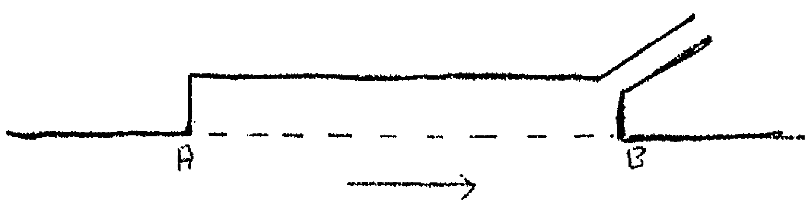


Therefore the boundary condition on the face of the wedge is given by

$$v = (\gamma+1) M_1^2 \delta ,$$

$$= v_0 \quad , \text{ (say)} \tag{26}$$

II PORCUS WALLS



Consider a wall bounding the fluid and lying parallel to the flow far upstream, in which there occurs a porous section from A to B. Let there be a chamber behind the porous wall in which the fluid can be maintained at a constant pressure p_e by means of pumps. It has been shown [6] that the boundary condition on the porous wall is homogeneous because of the viscous effects of the fluid there and that the pressure drop across the wall is proportional to the normal component of the fluid velocity at the porous wall. Thus along the section AB we shall assume the following relationship between the velocity and pressure.

$$v = \eta (p - p_c) \ , \tag{27}$$

where η is a constant of proportionality,

$$= \eta \left[(p - p_1) - (p_c - p_1) \right] . \tag{28}$$

From the adiabatic gas law we have

$$p - p_1 = p_1 \left[\left(\frac{\rho}{\rho_1} \right)^{\gamma} - 1 \right] , \tag{29}$$

and from equation (5) we have

$$\frac{1}{2} q^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{1}{2} U_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} .$$

$$\therefore \frac{1}{2} \left[U_1^2 - q^2 \right] = \frac{\gamma}{\gamma - 1} \cdot \frac{p_1}{\rho_1} \left[\frac{p}{\rho} \cdot \frac{\rho_1}{p_1} - 1 \right] ,$$

$$= \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} \left[\left(\frac{\rho}{\rho_1} \right)^{\gamma - 1} - 1 \right] .$$

$$\therefore \left(\frac{\rho}{\rho_1} \right)^{\gamma - 1} = 1 + \frac{\gamma - 1}{2\gamma} \cdot \frac{\rho_1}{p_1} \left[U_1^2 - q^2 \right] . \tag{30}$$

$$\therefore \left(\frac{\rho}{\rho_1} \right)^{\gamma} = \left[1 + \frac{\gamma - 1}{2\gamma} \frac{\rho_1}{p_1} (U_1^2 - q^2) \right]^{\frac{\gamma}{\gamma - 1}} ,$$

$$\simeq 1 + \frac{\rho_1}{2 p_1} (U_1^2 - q^2) .$$

Therefore on substituting the expression for $\left(\frac{\rho}{\rho_1} \right)^{\gamma}$ in equation (29) we obtain

$$p - p_1 \simeq \frac{\rho_1}{2} (U_1^2 - q^2) ,$$

$$= \frac{\rho_1}{2} \left[U_1^2 - U_1^2 (1 + u')^2 - U_1^2 v'^2 \right] ,$$

$$= - \rho_1 U_1 u' , \quad \text{as } u'^2 \text{ and } v'^2 \text{ can be neglected}$$

$$= \frac{\rho_1 U_1^2}{M_1^2 (\gamma + 1)} \left[u - u_1 \right] , \tag{31}$$

where $\mu_1 = 1 - M_1^2$,

$$= \left(\frac{3}{2}\right)^{2/3} \frac{\rho_1 U_1^2}{M_1^2(\gamma+1)} \left[\tau^{2/3} - \tau_1^{2/3} \right] . \tag{32}$$

Therefore on substituting for this expression in equation (28) we find

$$v = \eta \left[\left(\frac{3}{2}\right)^{2/3} \frac{\rho_1 U_1^2}{M_1^2(\gamma+1)} \left(\tau^{2/3} - \tau_1^{2/3} \right) - (p_c - p_1) \right],$$

$$= \eta \left[k_1 \tau^{2/3} - k_2 \right], \tag{33}$$

$$= \eta . F(\tau), \tag{34}$$

where $k_1 = \left(\frac{3}{2}\right)^{2/3} \frac{\rho_1 U_1^2}{(\gamma+1) M_1^2}$,

$k_2 = k_1 \tau_1^{2/3} + k_3$,

and $k_3 = p_c - p_1$.

THE STREAM FUNCTION

From the equation of continuity (1) in the case of two dimensional flow we have

$$\frac{\partial(\rho U)}{\partial X} = - \frac{\partial(\rho V)}{\partial Y} .$$

Since this is the condition that $\rho U dY - \rho V dX$ is a perfect differential there exists a function ψ such that

$$d\psi = \rho U dY - \rho V dX . \tag{35}$$

$$\therefore \quad \rho U = \frac{\partial \psi}{\partial Y} \quad \text{and} \quad - \rho V = \frac{\partial \psi}{\partial X} .$$

In steady motion the particle paths coincide with the streamlines, and on a streamline

$$\frac{dX}{\rho U} = \frac{dY}{\rho V} ,$$

$$\therefore \quad \rho V \, dX = \rho U \, dY .$$

Therefore on a streamline $\quad d\psi = 0$,

i.e. ψ = constant.

Therefore ψ = constant gives the streamlines, and ψ is called the stream-function.

Now equation (35) gives

$$d\psi = \rho U_1 (1 + u') \, dY - \rho U_1 v' \, dX ,$$

$$= \rho U_1 \, dY + \rho U_1 u' \, dY - \rho U_1 v' \, dX .$$

In the above equation the second and third terms can be neglected as they are small when compared with the first term.

$$\therefore \quad d\psi = \rho U_1 \, dY .$$

$$\therefore \quad \psi = \int_A^Y \rho U_1 \, dY ,$$ where A is any constant.

Since at any point in the flow the velocity differs by terms of first order from that at infinity upstream the density ρ will also differ by first order terms from ρ_1. Thus to terms of zero order ρ may be replaced by ρ_1 in in the above integral.

$$\therefore \quad \psi \simeq \rho_1 U_1 \int_A^Y dY .$$

Now if the line $Y = 0$ is taken as $\psi = 0$,

$$\psi = \rho_1 U_1 Y .$$

$$\therefore \quad \psi \propto Y . \qquad (36)$$

FREE STREAMLINES

Bernoulli's equation along a streamline is $\frac{1}{2} q^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = C$.
Now if a streamline divides the flow into two regions $\mathfrak{D}$ and D, we have

$$\frac{1}{2}q^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = C_{\mathcal{D}} \quad \text{and} \quad \frac{1}{2}q^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = C_D$$

as the conditions of flow on the streamlines in the regions $\mathcal{D}$ and D respectively. If P is a point on the dividing streamline, and we approach P from the region $\mathcal{D}$ the pressure takes a value p_1, the velocity a value q_1 and the density a value ρ_1. Similarly by approaching P from the region D the corresponding values are p_2, q_2 and ρ_2. Therefore at P the equations are

$$\frac{1}{2}q_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = C_{\mathcal{D}},$$

$$\text{and} \quad \frac{1}{2}q_2^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} = C_D.$$

But the pressure must be continuous at P,

$$\text{i.e.} \quad p_1 = p_2.$$

Now if $p_1 = p_2$ then ρ_1 must equal ρ_2 in view of the adiabatic gas law. Thus

$$q_1^2 - q_2^2 = \text{a constant.}$$

This equation shows that the velocities in the two regions are not continuous unless the constant in the above equation is zero. In particular if one of the regions is at rest (i.e. $q_2 = 0$) the streamline which separates the fluid in motion from the fluid at rest is called a free streamline. The properties of a free streamline are that the pressure, velocity, density and stream-function are constant along it.

THE PRESSURE COEFFICIENT

The pressure coefficient C_p is defined by

$$C_p = \frac{p - p_1}{\frac{1}{2}\rho_1 U_1^2}.$$

On substitution for $p - p_1$ from equation (31) we obtain

$$C_p = \frac{2}{(\gamma+1)M_1^2}\left[u - u_1\right]. \qquad (37)$$

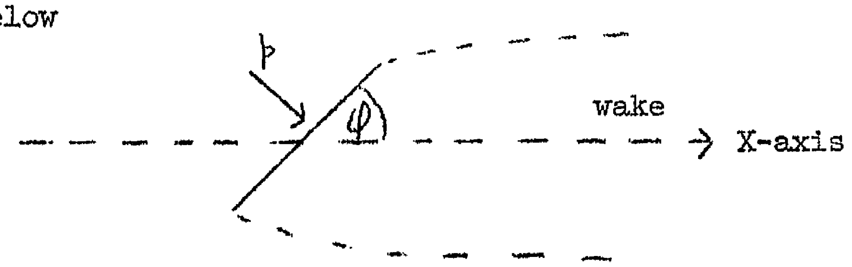
THE DRAG COEFFICIENT

The drag coefficient over a surface C_D is defined by

$$C_D = \frac{\text{Drag over the surface}}{\text{Free stream dynamic pressure x a suitable length}},$$

$$= \frac{D}{\frac{1}{2}\rho_1 u_1^2 . \ell}$$

Consider a plane surface inclined at an angle ϕ to the -axis as shown in the figure below



It is customary to define the drag over the surface as the force exerted upon it by the excess pressure difference $(p - p_1)$. Thus over an element δS of the surface there arises an element of drag δD given by

$$\delta D = (p - p_1) . \delta S . \sin\phi ,$$
$$= (p - p_1) \tan\phi . \delta X ,$$

which for a slender body, since $\tan \phi = \frac{dY}{dx}$ where $Y = g(x)$ in the equation of the surface, becomes

$$\delta D = (p - p_1) . \frac{dg}{dX} \delta X .$$

For a slender wedge whose semi-angle is δ and whose nose is at the origin of the coordinate system the equation of the face is given by $Y \simeq \delta X$.

Therefore the drag over a single face of the wedge of length ℓ is given by

$$D = \int_0^\ell (p - p_1) . \delta . dX .$$

$$\therefore C_D = \frac{\int_0^\ell (p - p_1) . \delta \; dX}{\frac{1}{2}\rho_1 u_1^2 \ell} ,$$

$$= \frac{1}{\ell} \int_0^\ell C_p . \delta . dX .$$

Thus for a wedge of unit length

$$C_D = \int_0^1 c_p . \delta . dx \ ,$$

$$= \frac{2\delta}{(\gamma+1)M_1^2} \int_0^1 (u - u_1)\, dx \ , \qquad \text{from equation (37).}$$

But $\qquad x = x(r, \nu)$.

$$\therefore dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \nu} d\nu .$$

But on the face of a wedge

$$\nu = \nu_0$$

$$\therefore \quad dx = \left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} dr .$$

If the values of r at the nose and shoulder of the wedge are r_n and r_S respectively, then

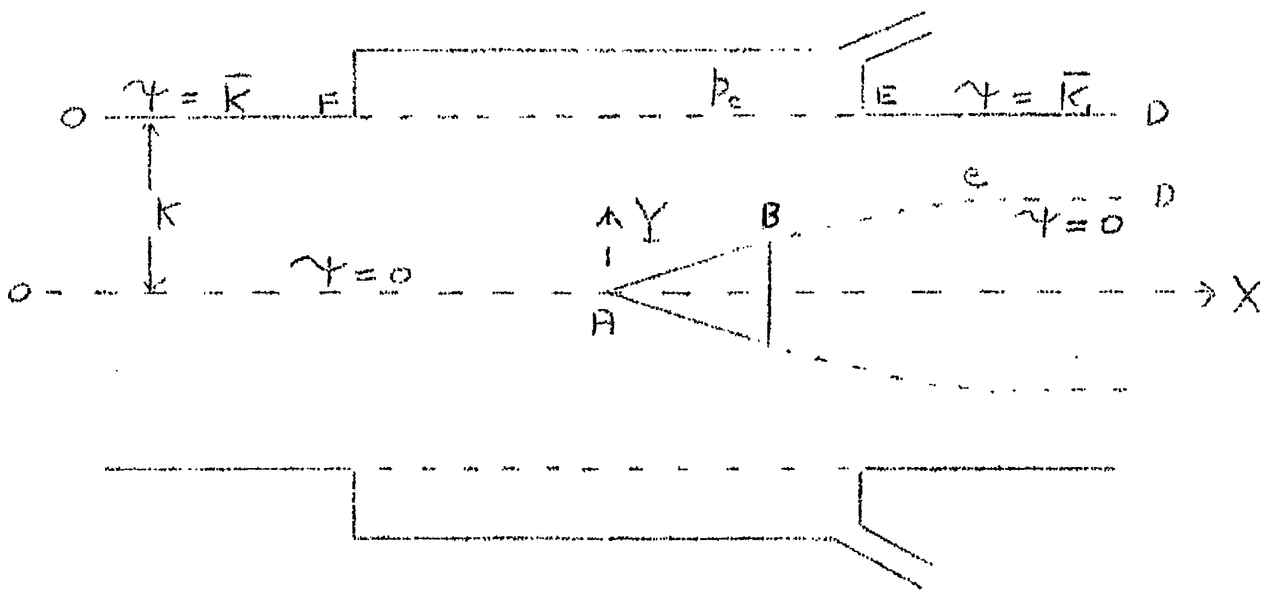
$$C_D = -\frac{2(\frac{3}{2})^{2/3}\delta}{(\gamma+1)\, M_1^2}\left[\int_{r_s}^{r_n} r^{2/3}\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} dr + r_1^{2/3}\right] .$$

Now it is an experimental fact that for transonic flow past a wedge sonic velocity occurs at the shoulder. Therefore at the shoulder u = 0, and hence $r_s = 0$.

At the nose of the wedge there is a stagnation point. This means that M = 0 and the value of u is of the order ε^0. Therefore as the magnitude of u at the stagnation point is of lower order than that assumed for u in the theory, the value of u at the nose of the wedge is taken as ∞. Hence $r_n = \infty$ and

$$C_D = -\frac{2(\frac{3}{2})^{2/3}\delta}{(\gamma+1)\, M_1^2}\left[\int_0^{\infty} r^{2/3}\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} dr + r_1^{2/3}\right] . \qquad (38)$$

An Investigation of High Subsonic Flow past a Wedge in a Channel of which a Section of the Walls is Porous.



Physical Plane.

The channel has a semi-width K and the porous section of the channel walls is from F to E. The chamber behind the porous wall has been designed so that the pressure of the fluid can be kept at a constant value p_e by means of pumps. The slender wedge of unit length and semi-angle δ is symmetrically placed at zero angle of attack in the channel with its nose at the origin (A) of the co-ordinate system (X,Y)

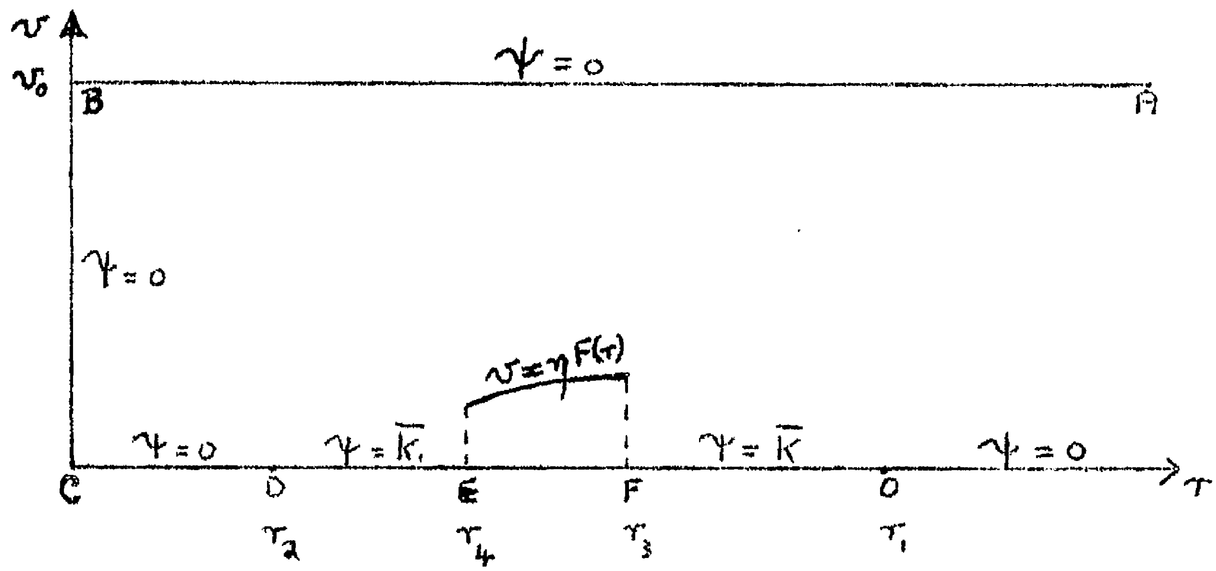
The dividing streamline $\psi = 0$ comes from infinity upstream (O) where the fluid has Mach number M_1 and associated $\tau = \tau_1$ and is parallel to the channel wall until it reaches the nose of the wedge at the stagnation point A. The streamline then divides and as the flow is symmetrical about the X-axis only the upper part of the streamline is now considered. After A the streamline goes along the face of the wedge to the shoulder at B where the fluid velocity becomes sonic. The streamline then breaks away freely from the wedge and the fluid velocity remains sonic until the streamline again becomes parallel to the channel wall at C. It then continues parallel to the channel wall to infinity downstream and the fluid velocity decreases so that its Mach number and associated τ value at infinity downstream (D) become M_2 and τ_2 respectively. This form for the streamline leaving the shoulder has been taken because it has been shown by Roshko (3) in studies of flow of incompressible fluids that it gave a better aggrement with experimental results than the standard form used by Kirchkoff and Helmholtz in which the flow breaks away from the shoulder with the shoulder velocity and keeps this velocity to infinity downstream. Since subsonic compressible flows and incompressible flows have similar behaviour it is assumed that the Roshko form will again give the better agreement.

On the solid walls of the channel from O to F and E to D the value of the stream function ψ is $\overline{k}$ and $\overline{k}_1$ respectively. Along the porous section FE the stream function will be a variable function of position depending essentially upon the rate at which fluid passes from the channel into the chamber behind FE.

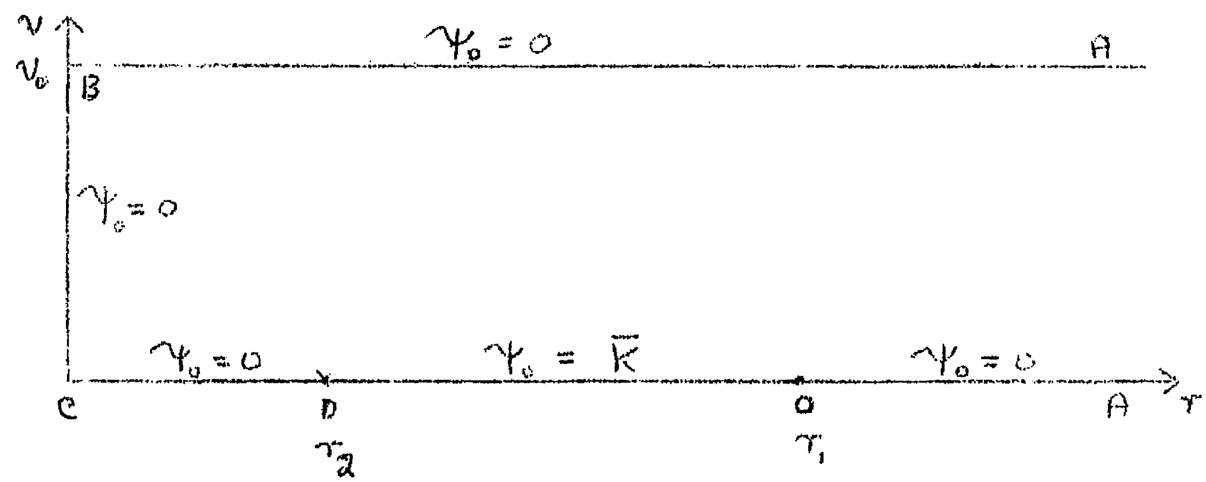
From equation (27) we see that the boundary condition on the solid wall of the channel from O to F and E to D, and on the dividing streamline $\psi = 0$ from O to A and C to D is given by $v = 0$. The boundary condition on the face of the wedge (A to B) is given by $\nu = \nu_0$ as shown by equation (28). Along the porous section of the channel wall from F to E the boundary condition is given by $\nu = \eta \cdot F(\tau)$ as shown by equation (34).

It is known that for the flow past a wedge in a channel with solid walls the fluid accelerates along the channel wall from infinity upstream to infinity downstream. As we will be determining the solution of the porous wall problem as a perturbation from the solution in the case of a solid wall we impose the condition that the fluid in the channel with porous walls still accelerates along the channel wall from infinity upstream to infinity downstream. Therefore, if the values of τ at F and E are τ_3 and τ_4 respectively, $\tau_1 > \tau_3 > \tau_4 > \tau_2$

The boundary value problem is now set up in the hodograph plane which is shown in the fugure below



For a wedge of the same dimensions placed in a similar position in a channel with solid walls, and using the same notation, the hodograph plane and boundary values are shown in the diagram as described in an earlier paper (4) by J.B. Helliwell. The suffix (0) here and hereafter will be associated with the solution for the wedge in the channel with solid walls.



The boundary conditions are as follows,

$$\psi_0 = 0 \qquad \tau \geq 0 \qquad \nu = \nu_0 \quad ,$$

$$\psi_0 = 0 \qquad \tau = 0 \qquad 0 \leq \nu \leq \nu_0 \quad ,$$

$$\psi_0 = 0 \qquad 0 \leq \tau < \tau_2 \,,\ \tau > \tau_1 \qquad \nu = 0 \quad ,$$

$$\psi_0 = \bar{K} \qquad \tau_2 < \tau < \tau_1 \qquad \nu = 0 \quad .$$

On comparison of these two figures we see that the boundary value problem for the wedge in a channel with porous walls is similar to the boundary-value problem for the wedge in a channel with solid walls.

We now look more closely at the boundary conditions on the wall of the channel with the porous section. Now as some fluid has either been pushed into or sucked out of the chamber behind the porous wall the streamline

$\Psi = \bar{K}$ will either have gone into this chamber or have been pushed out into the main stream. It has been shown by L.C. Woods (1) that in the case of an incompressible flow the value of Ψ could be taken as $\bar{K}$ on the wall of the channel from infinity downstream to infinity upstream without incurring more than a second order error term provided the amount of fluid pumped into or sucked out of the chamber is of first order magnitude when compared with the amount of fluid passing through the channel. Since subsonic compressible flows and incompressible flows have similar behaviour the value of Ψ on the channel wall is taken as $\bar{K}$ (See Appendix I.)

It has already been shown by equation (36) that $\Psi \propto Y$. Therefore the boundary conditions in terms of Y for the wedge in a channel with porous walls are

$$Y = 0 \qquad \tau \geq 0 \qquad \nu = \nu_0 ,$$

$$Y = 0 \qquad \tau = 0 \qquad 0 < \nu < \nu_0 ,$$

$$Y = 0 \qquad 0 \leq \tau < \tau_2 ,\ \tau > \tau_1 \qquad \nu = 0 ,$$

$$Y = K \qquad \tau_2 < \tau < \tau_4 ,\ \tau < \tau < \tau_1 \qquad \nu = 0 ,$$

$$Y = K \qquad \tau_4 < \tau < \tau_3 \qquad \nu = \eta . F(\tau) .$$

In the case of the wedge in a channel with solid walls the boundary conditions in terms of Y_0 are

$$Y_0 = 0 \qquad \tau \geqslant 0 \qquad \nu = \nu_0 ,$$

$$Y_0 = 0 \qquad \tau = 0 \qquad 0 < \nu < \nu_0 ,$$

$$\Upsilon_0 = 0 \qquad 0 \le \tau < \tau_2, \;\; \tau > \tau_1 \qquad \nu = 0 ,$$

$$\Upsilon_0 = K \qquad \tau_2 < \tau < \tau_1 \qquad \nu = 0 .$$

Since η is small it means that the boundary conditions are very similar. Thus it will be assumed that the solution of the present problem may be determined by a perturbation from the solution for a channel with solid walls, and that we may write

$$\Upsilon = \Upsilon_0 + \eta \gamma ,$$

where η is the small parameter already defined.

By means of a Taylor Series in ν the value of Υ on $\nu = 0$ between E and F can now be found from $\Upsilon(\tau,\nu)_{\nu=\eta F(\tau)} = K$.

$$\therefore \;\; \Upsilon(\tau,0) + \left(\frac{\partial \Upsilon}{\partial \nu}\right)_{\nu=0} \eta F(\tau) + \cdots = K.$$

$$\therefore \;\; \Upsilon(\tau,0) + \left[\frac{\partial(\Upsilon_0 + \eta\gamma)}{\partial \nu}\right]_{\nu=0} \cdot \eta \cdot F(\tau) + \cdots = K.$$

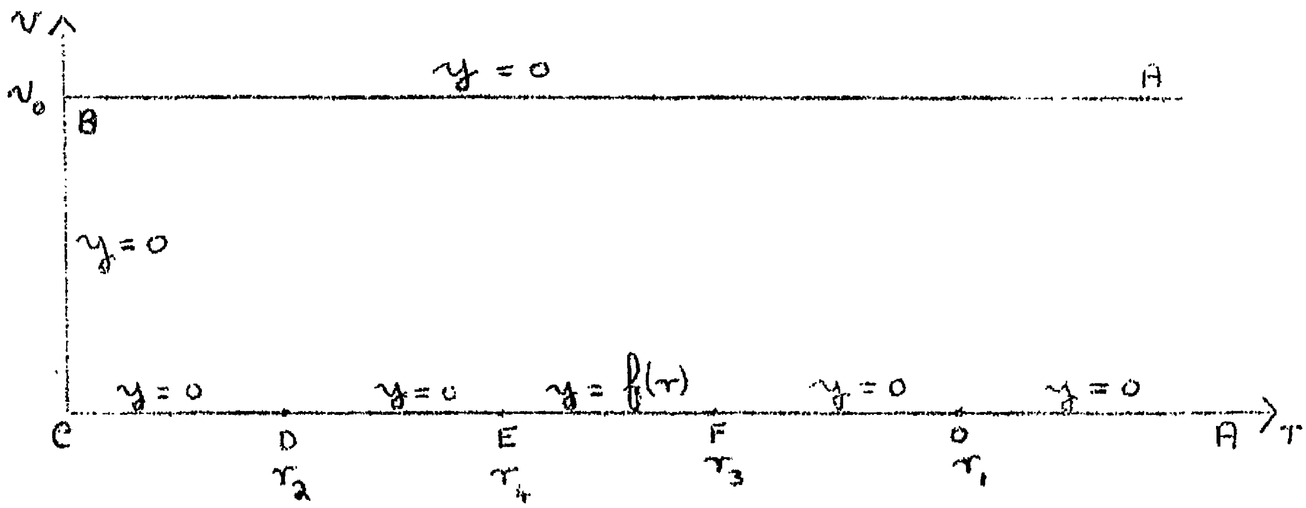
$$\therefore \;\; \Upsilon(\tau,0) = K + \left[-\left(\frac{\partial \Upsilon_0}{\partial \nu}\right)_{\nu=0} \cdot \eta \cdot F(\tau)\right] , \text{ to first order of } \eta$$

$$= K + \eta \beta(\tau) ,$$

where

$$\beta(\tau) = -\left(\frac{\partial \Upsilon_0}{\partial \nu}\right)_{\nu=0} \cdot F(\tau) .$$

The boundary value problem in the hodograph plane may now be given for y. It is shown in the following figure



The boundary conditions are as follows,

$y = 0$	$\tau \geq 0$	$\nu = \nu_0$,
$y = 0$	$\tau = 0$	$0 \leq \nu \leq \nu_0$,
$y = 0$	$0 \leq \tau < \tau_4$	$\nu = 0$,
$y = 0$	$\tau > \tau_3$	$\nu = 0$,
$y = f(\tau)$	$\tau_4 < \tau < \tau_3$	$\nu = 0$.

Since Y and Y_0 satisfy Triconi's equation so also must y. Therefore from equation (26) and using the boundary conditions on $\nu = \nu_0$ and $\tau = 0$ the solution for y may be written in the form

$$y = \int_0^\infty g(\lambda)\, \tau^{1/3}\, J_{\frac{1}{3}}(\lambda\tau)\, \frac{\sinh\lambda(\nu_0 - \nu)}{\sinh\lambda\nu_0}\, d\lambda. \qquad (39)$$

The function $g(\lambda)$ is found from the boundary condition on $\nu = 0$.

Thus

$$\int_0^\infty g(\lambda) \, r^{1/3} J_{\frac{1}{3}}(\lambda r) \, d\lambda \;\; = 0, \qquad 0 \le r < r_4$$
$$= f(r), \qquad r_4 < r < r_3$$
$$= 0, \qquad r > r_3 .$$

By means of the Hankel Inversion Formula we find

$$g(\lambda) = \int_{r_4}^{r_3} r^{2/3} \lambda \, f(r) \, J_{\frac{1}{3}}(\lambda r) \, dr . \qquad (39')$$

The series expression for Y_0 valid in the range $r_2 < r < r_1$ can be obtained from the paper (4) by J.B. Helliwell so that the expressions for $f(r)$ and hence $g(\lambda)$ and y can be found. The series form for Y_0 is taken rather than the integral form to avoid difficulties in the subsequent manipulations. Thus

$$Y_0 = -\frac{2Kr^{1/3}}{\nu_0} \sum_{n=1}^{\infty} \left\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi r}{\nu_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{\nu_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi r}{\nu_0}\right) \right\} \sin\left(\frac{n\pi \nu}{\nu_0}\right)$$
$$+ K\left(1 - \frac{\nu}{\nu_0}\right),$$

and

$$f(r) = \frac{k_1 K}{\nu_0}\left[r^{2/3} + 2r \sum_{n=1}^{\infty} \left(\frac{n\pi}{\nu_0}\right) \left\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi r}{\nu_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{\nu_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi r}{\nu_0}\right) \right\} \right]$$
$$- \frac{k_2 K}{\nu_0}\left[1 + 2r^{1/3} \sum_{n=1}^{\infty} \left(\frac{n\pi}{\nu_0}\right) \left\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi r}{\nu_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{\nu_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi r}{\nu_0}\right) \right\} \right].$$

(40)

Hence

$$g(\lambda) = k_1 \frac{K\lambda}{v_0} \int_{r_4}^{r_3} \left[r^{4/3} J_{\frac{1}{3}}(\lambda r) + 2\sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \left\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) \right\} r^{5/3} J_{\frac{1}{3}}(\lambda r) \right] dr$$

$$- \frac{k_2 K\lambda}{v_0} \int_{r_4}^{r_3} \left[r^{2/3} J_{\frac{1}{3}}(\lambda r) + 2\sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \left\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) \right\} r J_{\frac{1}{3}}(\lambda r) \right] dr$$

and

$$y = \frac{k_1 K r^{1/3}}{v_0} \int_0^{\infty} \lambda J_{\frac{1}{3}}(\lambda r) \frac{\sinh \lambda (v_0 - v)}{\sinh \lambda v_0} \int_{r_4}^{r_3} \left\{ z^{4/3} J_{\frac{1}{3}}(\lambda z) + 2\sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \left[r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \right.\right.$$

$$\left.\left. + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \right] z^{5/3} J_{\frac{1}{3}}(\lambda z) \right\} dz \, d\lambda.$$

$$- \frac{k_2 K r^{1/3}}{v_0} \int_0^{\infty} \lambda J_{\frac{1}{3}}(\lambda r) \frac{\sinh \lambda (v_0 - v)}{\sinh \lambda v_0} \int_{r_4}^{r_3} \left\{ z^{2/3} J_{\frac{1}{3}}(\lambda z) + 2\sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \left[r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \right.\right.$$

$$\left.\left. + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \right] z J_{\frac{1}{3}}(\lambda z) \right\} dz \, d\lambda.$$

As the above expression for y could not be evaluated directly the orders of integration and summation of the series were interchanged

$$\therefore y = \frac{k_1 K r^{1/3}}{v_0} \left[\int_{r_4}^{r_3} z^{4/3} L(r,v,z) dz + 2\sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \left\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4}^{r_3} z^{5/3} K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) L(r,v,z) \, dz \right.\right.$$

$$\left.\left. + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4}^{r_3} z^{5/3} I_{1/3}\left(\frac{n\pi z}{v_0}\right) L(r,v,z) \, dz \right\} \right]$$

$$- \frac{k_2 K r^{1/3}}{v_0} \left[\int_{r_4}^{r_3} z^{2/3} L(r,v,z) dz + 2\sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \left\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4}^{r_3} z K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) L(r,v,z) \, dz \right.\right.$$

$$\left.\left. + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4}^{r_3} z I_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) L(r,v,z) \, dz \right\} \right],$$

where $$L(r, v, z) = \int_0^\infty \lambda J_{\frac{1}{3}}(\lambda r) J_{\frac{1}{3}}(\lambda z) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} d\lambda .$$

Series forms of the integral $L(r, v, z)$

Let $\lambda v_0 = \pi t$.

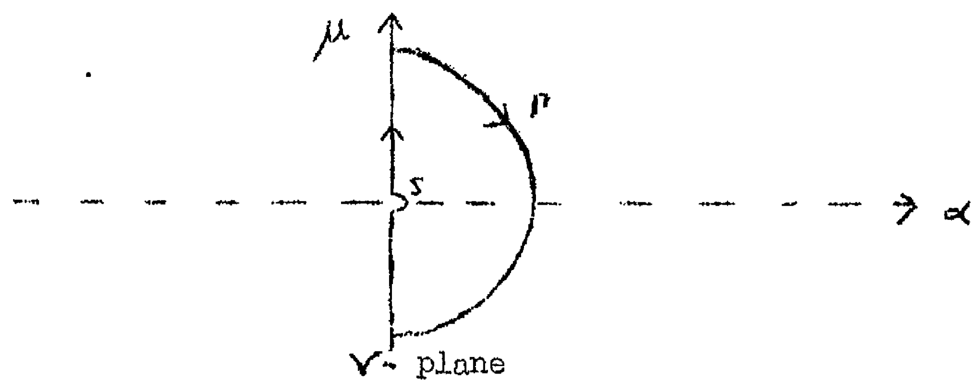
$$\therefore L = \int_0^\infty \left(\frac{\pi t}{v_0}\right) J_{\frac{1}{3}}\left(\frac{\pi r t}{v_0}\right) J_{\frac{1}{3}}\left(\frac{\pi z t}{v_0}\right) \frac{\sinh \pi t(1 - \frac{v}{v_0})}{\sinh \pi t} \cdot \frac{\pi}{v_0} dt ,$$

$$= \frac{\pi^2}{v_0^2} \int_0^\infty t \; J_{\frac{1}{3}}\left(\frac{\pi r t}{v_0}\right) J_{\frac{1}{3}}\left(\frac{\pi z t}{v_0}\right) \frac{\sinh \pi t(1 - \frac{v}{v_0})}{\sinh \pi t} dt .$$

For $r < z$ let us consider the contour integral

$$W = D \int_C \nu I_{\frac{1}{3}}\left(\frac{\pi r \nu}{v_0}\right) K_{\frac{1}{3}}\left(\frac{\pi z \nu}{v_0}\right) \frac{e^{i \nu \pi (1 - \frac{v}{v_0})}}{\sin \nu \pi} d\nu ,$$

where D is a constant and the contour C in the complex ν -plane goes from $-i\infty$ to $+i\infty$ along the imaginary axis, is indented at the origin by the small semi-circle (s) and is closed by the right hand infinite semi-circle(Γ)



$$W = D\int_{-\infty}^{0}(\ \ )\,i\,d\mu + D\int_{0}^{\infty}(\ \ )\,i\,d\mu + D\int_{S}(\ \ )\,d\nu + D\int_{\Gamma}(\ \ )\,d\nu.$$

On the infinitesimal semi-circle set $\nu = \rho e^{i\Theta}$ then

$$\int_S (\ )\,d\nu = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho e^{i\Theta} I_{\frac{1}{3}}\left(\frac{\pi r \rho e^{i\Theta}}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{\pi z \rho e^{i\Theta}}{\nu_0}\right) \frac{e^{i\rho e^{i\Theta}\pi(1-\frac{z}{\nu_0})}}{\sin(\rho e^{i\Theta}\pi)}\, i\rho e^{i\Theta} d\Theta,$$

For small values of ρ the integral becomes

$$\frac{i\rho^2 e^{2i\Theta}\left(\frac{\pi r\rho e^{i\Theta}}{2\nu_0}\right)^{1/3}}{\Gamma\left(\frac{4}{3}\right)} \quad \frac{\pi}{2\sin\left(\frac{\pi}{3}\right)}\left(\frac{\pi z\rho e^{i\Theta}}{2\nu_0}\right)^{-1/3} \cdot \frac{1}{\Gamma\left(\frac{2}{3}\right)}\ \frac{1}{\sin(\rho e^{i\Theta}\pi)},$$

$$= i\,\frac{\rho e^{i\Theta}\pi}{\sin(\rho e^{i\Theta}\pi)}\cdot \rho e^{i\Theta}\left(\frac{r}{z}\right)^{1/3}\ \frac{1}{2\sin\left(\frac{\pi}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}.$$

As $\rho \to 0$ the value of the integral $\to 0$

Therefore the contribution to W from the indentation at the origin is zero

On the infinite semi-circle (Γ) set $\nu = Re^{i\Theta}$ then

$$\int_\Gamma (\ \ )\,d\nu = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} Re^{i\Theta} I_{\frac{1}{3}}\left(\frac{\pi r Re^{i\Theta}}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{\pi z Re^{i\Theta}}{\nu_0}\right) \frac{e^{iRe^{i\Theta}\pi(1-\frac{\nu}{\nu_0})}}{\sin(Re^{i\Theta}\pi)}\, iRe^{i\Theta} d\Theta$$

For large values of R the integral becomes

$$iR^2 e^{2i\Theta}\ \frac{e^{\frac{\pi r Re^{i\Theta}}{\nu_0}}}{\left(2\pi\frac{\pi r Re^{i\Theta}}{\nu_0}\right)^{1/2}}\cdot\frac{\pi^{1/2} e^{-\frac{\pi z Re^{i\Theta}}{\nu_0}}}{\left(\frac{2\pi z Re^{i\Theta}}{\nu_0}\right)^{1/2}}\cdot\frac{e^{iR\pi(1-\frac{\nu}{\nu_0})e^{i\Theta}}}{\frac{e^{iRe^{i\Theta}\pi}-e^{-iRe^{i\Theta}\pi}}{2}},$$

$$= \frac{-R\nu_0 e^{i\theta} e^{-\frac{\pi R}{\nu_0}(z-r)\cos\theta} e^{-\frac{i\pi R}{\nu_0}\sin\theta(z-r)} e^{iR\pi(1-\frac{\nu}{\nu_0})\cos\theta} e^{-R\pi(1-\frac{\nu}{\nu_0})\sin\theta}}{\pi (rz)^{1/2}\left[e^{iR\pi\cos\theta} e^{-R\pi\sin\theta} - e^{iR\pi\cos\theta} e^{R\pi\sin\theta}\right]}$$

Now at $\theta = 0$ for large values of R the integrand is

$$\frac{R\nu_0 e^{-\frac{\pi R}{\nu_0}(z-r)} e^{iR\pi(1-\frac{\nu}{\nu_0})}}{\pi (r.z)^{1/2}\left[e^{iR\pi} - e^{-iR\pi}\right]},$$

$$= \frac{R\nu_0 e^{-\frac{\pi R}{\nu_0}(z-r)} e^{-iR\pi\frac{\nu}{\nu_0}}}{\pi (r.z)^{1/2}\left[1 - e^{-2iR\pi}\right]}.$$

Therefore the integrand $\to$ 0 as R $\to \infty$ through non-integer values at $\theta = 0$.

In the range $0 < \theta \leq \pi/2$, $\sin\theta = a$ and $\cos\theta = b$ where a and b are both +ve.

$$\therefore \ |\text{integrand}| \quad \leq \frac{R\nu_0 e^{-\frac{\pi R}{\nu_0}(z-r)b} e^{-R\pi(1-\frac{\nu}{\nu_0})a}}{\pi (r.z)^{1/2}\left[e^{-R\pi a} - e^{R\pi a}\right]},$$

$$= \frac{R\nu_0 e^{-\frac{\pi R}{\nu_0}(z-r)b} e^{-R\pi(2-\frac{\nu}{\nu_0})a}}{\pi (r.z)^{1/2}\left[e^{-2\pi a R} - 1\right]}$$

$\therefore$ the |integrand| $\to 0$ as $R \to \infty$ for $0 < \theta \leq \frac{\pi}{2}$

if $\nu < 2\nu_0$.

In the range $-\frac{\pi}{2} \leq \theta < 0$, $\sin\theta = -a$ and $\cos\theta = b$

where a and b are +ve .

$$|\text{integrand}| \leq \frac{R\nu_0 \, e^{-\frac{\pi R}{\nu_0}(z-r)b + R\pi(1-\frac{\nu}{\nu_0})a}}{\pi (rz)^{1/2}\left[e^{R\pi a} - e^{-R\pi a}\right]} ,$$

$$= \frac{R\nu_0 \, e^{-\frac{\pi R}{\nu_0}(z-r)b - R\pi a\frac{\nu}{\nu_0}}}{\pi (rz)^{1/2}\left[1 - e^{-2\pi R a}\right]} .$$

$\to 0$ as $R \to \infty$ for $-\frac{\pi}{2} \leq \theta < 0$

if $\nu > 0$.

Therefore the contribution to w from theinfinite semi-circle Γ is zero as long as $0 < \nu < 2\nu_0$.

$$\int_{-\infty}^{0}(\quad)\, i\, d\mu + \int_{0}^{\infty}(\quad)\, i\, d\mu$$

$$= \int_{-\infty}^{0} i\mu\, I_{\frac{1}{3}}\left(\frac{\pi r i\mu}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{\pi z i\mu}{\nu_0}\right) \frac{e^{-\mu\pi(1-\frac{\nu}{\nu_0})}}{\sin(i\mu\pi)}\, i\, d\mu$$

$$+ \int_{0}^{\infty} i\mu\, I_{\frac{1}{3}}\left(\frac{\pi r i\mu}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{\pi z i\mu}{\nu_0}\right) \frac{e^{-\mu\pi(1-\frac{\nu}{\nu_0})}}{\sin(i\mu\pi)}\, i\, d\mu$$

$$= -\int_{0}^{\infty} i\mu\, I_{\frac{1}{3}}\left(-\frac{\pi r i\mu}{\nu_0}\right) K_{\frac{1}{3}}\left(-\frac{\pi z i\mu}{\nu_0}\right) \frac{e^{\mu\pi(1-\frac{\nu}{\nu_0})}}{\sin(-i\mu\pi)}\, i\, d\mu$$

$$+ \int_{0}^{\infty} i\mu\, I_{\frac{1}{3}}\left(\frac{\pi r i\mu}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{\pi z i\mu}{\nu_0}\right) \frac{e^{-\mu\pi(1-\frac{\nu}{\nu_0})}}{\sin(i\mu\pi)}\, i\, d\mu$$

$$= \int_0^\infty \frac{i\mu}{\sinh(\mu\pi)} \Big[I_{\frac{1}{3}}\Big(-\frac{\pi r i\mu}{\nu_0}\Big) K_{\frac{1}{3}}\Big(-\frac{\pi z i\mu}{\nu_0}\Big) e^{\mu\pi(1-\frac{\nu}{\nu_0})} + I_{\frac{1}{3}}\Big(\frac{\pi r i\mu}{\nu_0}\Big) K_{\frac{1}{3}}\Big(\frac{\pi z i\mu}{\nu_0}\Big) e^{-\mu\pi(1-\frac{\nu}{\nu_0})} \Big] d\mu,$$

$$= \int_0^\infty \frac{i\mu}{\sinh(\mu\pi)} \Big[I_{\frac{1}{3}}\Big(-\frac{\pi r i\mu}{\nu_0}\Big) \frac{\pi}{2\sin\frac{\pi}{3}} \Big\{ I_{-\frac{1}{3}}\Big(-\frac{\pi z i\mu}{\nu_0}\Big) - I_{\frac{1}{3}}\Big(-\frac{\pi z i\mu}{\nu_0}\Big)\Big\} e^{\mu\pi(1-\frac{\nu}{\nu_0})} + I_{\frac{1}{3}}\Big(\frac{\pi r i\mu}{\nu_0}\Big) \frac{\pi}{2\sin\frac{\pi}{3}} \Big\{ I_{-\frac{1}{3}}\Big(\frac{\pi z i\mu}{\nu_0}\Big) - I_{\frac{1}{3}}\Big(\frac{\pi z i\mu}{\nu_0}\Big)\Big\} e^{-\mu\pi(1-\frac{\nu}{\nu_0})} \Big] d\mu,$$

as $K_\nu(z) = \dfrac{\pi}{2\sin\nu\pi}\Big[I_{-\nu}(z) - I_\nu(z)\Big]$,

$$= \int_0^\infty \frac{i\mu}{\sinh(\mu\pi)} \Big[e^{-\frac{i\pi}{6}} J_{\frac{1}{3}}\Big(\frac{\pi r\mu}{\nu_0}\Big) \Big\{ e^{\frac{i\pi}{6}} J_{-\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big) - e^{-\frac{i\pi}{6}} J_{\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big)\Big\} e^{\mu\pi(1-\frac{\nu}{\nu_0})} + e^{\frac{i\pi}{6}} J_{\frac{1}{3}}\Big(\frac{\pi r\mu}{\nu_0}\Big) \Big\{ e^{-\frac{i\pi}{6}} J_{-1/3}\Big(\frac{\pi z\mu}{\nu_0}\Big) - e^{\frac{i\pi}{6}} J_{\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big)\Big\} e^{-\mu\pi(1-\frac{\nu}{\nu_0})} \Big] \frac{\pi\, d\mu}{2\sin\frac{\pi}{3}}$$

as $I_\nu(z e^{\pm\frac{i\pi}{2}}) = e^{\pm\frac{i\nu\pi}{2}} J_\nu(z)$,

$$= \int_0^\infty \frac{i\mu\pi\, J_{\frac{1}{3}}\big(\frac{\pi r\mu}{\nu_0}\big)}{2\sinh(\mu\pi)\sin\frac{\pi}{3}} \Big[\Big\{ J_{-\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big) - \cos\Big(\frac{\pi}{3}\Big) J_{\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big)\Big\} e^{\mu\pi(1-\frac{\nu}{\nu_0})} + \Big\{ J_{-\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big) - \cos\Big(\frac{\pi}{3}\Big) J_{\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big)\Big\} e^{-\mu\pi(1-\frac{\nu}{\nu_0})} + i\sin\Big(\frac{\pi}{3}\Big) J_{\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big) e^{\mu\pi(1-\frac{\nu}{\nu_0})} - i\sin\Big(\frac{\pi}{3}\Big) J_{\frac{1}{3}}\Big(\frac{\pi z\mu}{\nu_0}\Big) e^{-\mu\pi(1-\frac{\nu}{\nu_0})} \Big] d\mu,$$

$$= -\int_0^\infty \frac{\mu\pi\, J_{\frac{1}{3}}\big(\frac{\mu\pi r}{\nu_0}\big) J_{\frac{1}{3}}\big(\frac{\mu\pi z}{\nu_0}\big)}{2\sinh(\mu\pi)} \Big[e^{\mu\pi(1-\frac{\nu}{\nu_0})} - e^{-\mu\pi(1-\frac{\nu}{\nu_0})} \Big] d\mu$$

$+$ imaginary part,

$$= -\int_0^\infty \mu\pi \, J_{\frac{1}{3}}\left(\frac{\mu\pi r}{v_0}\right) J_{\frac{1}{3}}\left(\frac{\mu\pi z}{v_0}\right) \frac{\sinh \mu\pi(1-\frac{v}{v_0})}{\sinh \mu\pi} \, d\mu + \text{imaginary part} .$$

Now by Cauchy's Theorem W is equal to $-2\pi i$ times the sum of the residues of the integrand at its poles. The pole of the integrand for W occur at $\nu = 1, 2, 3 \ldots \ldots\ldots$

$$\therefore W = -2\pi i \, D \sum_{n=1}^{\infty} \frac{n \, I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) e^{in\pi(1-\frac{v}{v_0})}}{\pi \cos(n\pi)} ,$$

$$= -2\pi i \, D \sum_{n=1}^{\infty} \left(\frac{n}{\pi}\right) (-1)^n I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \left\{ \cos\left[n\pi(1-\frac{v}{v_0})\right] + i \sin\left[n\pi(1-\frac{v}{v_0})\right] \right\} .$$

Now $L(r, v, z) = -\dfrac{\pi}{D v_0^2}$ x Real part of W

$$= -\frac{\pi}{D v_0^2} \left[2\pi D \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{\pi}\right) I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \sin\left\{n\pi(1-\frac{v}{v_0})\right\} \right] ,$$

$$= -\frac{2\pi^2}{v_0^2} \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{\pi}\right) I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \left[\sin(n\pi) \cos\left(\frac{n\pi v}{v_0}\right) - \cos(n\pi) \sin\left(\frac{n\pi v}{v_0}\right) \right] ,$$

$$= \frac{2\pi}{v_0^2} \sum_{n=1}^{\infty} n \, I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \sin\left(\frac{n\pi v}{v_0}\right) .$$

$$\therefore \int_0^\infty \lambda \, J_{\frac{1}{3}}(\lambda r) J_{\frac{1}{3}}(\lambda z) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} \, d\lambda = \frac{2\pi}{v_0^2} \sum_{n=1}^{\infty} n \, I_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) \sin\left(\frac{n\pi v}{v_0}\right) .$$

if $r < z$ and $0 < v < 2v_0$.

But $L(r, v, z)$ is symmetrical in r and z.

$$\therefore \int_0^\infty \lambda J_{\frac{1}{3}}(\lambda r) J_{\frac{1}{3}}(\lambda z) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} d\lambda = \frac{2\pi}{v_0^2} \sum_{n=1}^{\infty} n I_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi r}{v_0}\right) \sin\left(\frac{n\pi v}{v_0}\right)$$

if $r > z$ and $0 < v < 2v_0$,

The x, y Co-ordinates

On returning to the co-ordinates (x,y) of the field of flow, insertion of the series form for $L(r, v, z)$ into the previous expressions for y yields

$$y = \frac{2\pi r^{1/3}}{v_0} \sum_{p=1}^{\infty} p \cdot I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \sin\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) \cdot K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz \qquad (40)$$

for $0 \leq r < r_4$,

$$= \frac{2\pi r^{1/3}}{v_0} \sum_{p=1}^{\infty} p \sin\left(\frac{p\pi v}{v_0}\right) \left[K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \int_{r_4}^{r} z f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz + I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \int_{r}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz \right], \qquad (42)$$

for $r_4 < r < r_3$,

$$= \frac{2\pi r^{1/3}}{v_0} \sum_{p=1}^{\infty} p \cdot K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \sin\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz, \qquad (43)$$

for $r > r_3$.

where the function $f(z)$ is given by equation (40)

Now x and y are related by the equations (14) and (15) which on the substitution of $\left(\frac{3}{2}\right)^{2/3} r^{2/3}$ for μ become:

$$\frac{\partial x}{\partial v} = -\left(\frac{3}{2}\right)^{1/3} r^{1/3} \frac{\partial y}{\partial r}, \tag{44}$$

and

$$\frac{\partial x}{\partial r} = \left(\frac{3}{2}\right)^{1/3} r^{1/3} \frac{\partial y}{\partial v} \quad \text{respectively}. \tag{45}$$

Therefore considering the range $0 \leq r < r_4$ we obtain from equations (41) and (45)

$$\frac{\partial x}{\partial r} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \left(\frac{p\pi}{v_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz. \tag{46}$$

$$x = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \, I_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz + V(v),$$

and

$$\frac{\partial x}{\partial v} = -\left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \left(\frac{p\pi}{v_0}\right) I_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \sin\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz + V'(v)$$

where $V(v)$ is any continuous function of v.

But from equations (41) and (46)

$$\frac{\partial x}{\partial v} = -\left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \left(\frac{p\pi}{v_0}\right) I_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \sin\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz.$$

$$\therefore \quad V'(v) = 0.$$

$$\therefore \quad V(v) = \text{a constant} = A.$$

$$\therefore \ x = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot I_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz + A. \quad (47)$$

In the range $r > r_3$ we obtain from equation (43) and (44)

$$\frac{\partial x}{\partial v} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \left(\frac{p\pi}{v_0}\right) K_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \sin\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz .$$

$$\therefore \ x = -\left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot K_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz + R(r),$$

and $$\frac{\partial x}{\partial r} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot \left(\frac{p\pi}{v_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz + R'(r),$$

where $R(r)$ is any continuous function of r.

But from equations (43) and (45)

$$\frac{\partial x}{\partial r} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot \left(\frac{p\pi}{v_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz .$$

$$\therefore \ R'(r) = 0 .$$

$$\therefore \ R(r) = \text{a constant} = B .$$

$$\therefore \ x = -\left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot K_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz + B. \quad (49)$$

For the range $\tau_4 < \tau < \tau_3$ we obtain from equation (42)

$$\frac{\partial y}{\partial \tau} = -\tau^{1/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\left(\frac{p\pi}{v_0}\right)K_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\int_{\tau_4}^{\tau} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz$$

$$+\tau^{1/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\, K_{\frac{1}{3}}\left(\frac{p\pi\tau}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\tau^{2/3} f(\tau)\, I_{\frac{1}{3}}\left(\frac{p\pi\tau}{v_0}\right)$$

$$+\tau^{1/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\, I_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\left(\frac{p\pi}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\int_{\tau}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz$$

$$-\tau^{1/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\, I_{\frac{1}{3}}\left(\frac{p\pi\tau}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\tau^{2/3} f(\tau)\, K_{\frac{1}{3}}\left(\frac{p\pi\tau}{v_0}\right),$$

$$= -\tau^{1/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\left(\frac{p\pi}{v_0}\right)K_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\int_{\tau_4}^{\tau} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz$$

$$+\tau^{1/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\left(\frac{p\pi}{v_0}\right)I_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\int_{\tau}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz,$$

as the two apparently divergent sums cancel each other. Therefore using equation (44) we obtain

$$\frac{\partial x}{\partial v} = \left(\frac{3}{2}\right)^{1/3}\tau^{2/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\left(\frac{p\pi}{v_0}\right)K_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\int_{\tau_4}^{\tau} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz$$

$$-\left(\frac{3}{2}\right)^{1/3}\tau^{2/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\left(\frac{p\pi}{v_0}\right)I_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\sin\left(\frac{p\pi v}{v_0}\right)\int_{\tau}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz.$$

$$\therefore\ x = -\left(\frac{3}{2}\right)^{1/3}\tau^{2/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\, K_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\cos\left(\frac{p\pi v}{v_0}\right)\int_{\tau_4}^{\tau} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz$$

$$+\left(\frac{3}{2}\right)^{1/3}\tau^{2/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\, I_{-\frac{2}{3}}\left(\frac{p\pi\tau}{v_0}\right)\cos\left(\frac{p\pi v}{v_0}\right)\int_{\tau}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz + R(\tau),$$

and $$\frac{\partial x}{\partial r} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot \left(\frac{p\pi}{v_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$$

$$- \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot K_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) r^{2/3} f(r)\, I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right)$$

$$+ \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot \left(\frac{p\pi}{v_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r}^{r_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$$

$$- \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \; I_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \cdot r^{2/3} f(r)\, K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) + R'(r),$$

where $R(r)$ is any continuous function of r.

$$\therefore \frac{\partial x}{\partial r} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot \left(\frac{p\pi}{v_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$$

$$+ \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \cdot \left(\frac{p\pi}{v_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r}^{r_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$$

$$- \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} r^{2/3} \cdot p \cdot f(r) \cdot \cos\left(\frac{p\pi v}{v_0}\right) \left[I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) K_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) + I_{-\frac{2}{3}}\left(\frac{p\pi r}{v_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \right] + R'(r);$$

$$= \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \left(\frac{p\pi}{v_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r_4}^{r} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$$

$$+ \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{v_0^2}\right) \sum_{p=1}^{\infty} p \left(\frac{p\pi}{v_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi r}{v_0}\right) \cos\left(\frac{p\pi v}{v_0}\right) \int_{r}^{r_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$$

$$-\left(\frac{3}{2}\right)^{1/3} \tau^{4/3}\left(\frac{2\pi}{\nu_0^2}\right) f(\tau) \sum_{p=1}^{\infty} \cos\left(\frac{p\pi\nu}{\nu_0}\right) \cdot \left(\frac{\nu_0}{\pi\tau}\right) \quad + R'(\tau)$$

as

$$I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z} .$$

$$\therefore \left(\frac{\partial x}{\partial \tau}\right) = \left(\frac{3}{2}\right)^{1/3} \tau^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p \left(\frac{p\pi}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi\tau}{\nu_0}\right)\cos\left(\frac{p\pi\nu}{\nu_0}\right)\int_{\tau_4}^{\tau} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+\left(\frac{3}{2}\right)^{1/3} \tau^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p \left(\frac{p\pi}{\nu_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi\tau}{\nu_0}\right)\cos\left(\frac{p\pi\nu}{\nu_0}\right)\int_{\tau}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+\left(\frac{3}{2}\right)^{1/3} \frac{\tau^{1/3}}{\nu_0} f(\tau) \quad + R'(\tau) ,$$

as

$$\sum_{p=1}^{\infty} \cos\left(\frac{p\pi\nu}{\nu_0}\right) = -\frac{1}{2} .$$

But from equations (42) and (45)

$$\frac{\partial x}{\partial \tau} = \left(\frac{3}{2}\right)^{1/3} \tau^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p \left(\frac{p\pi}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi\tau}{\nu_0}\right)\cos\left(\frac{p\pi\nu}{\nu_0}\right)\int_{\tau_4}^{\tau} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+\left(\frac{3}{2}\right)^{1/3} \tau^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p \left(\frac{p\pi}{\nu_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi\tau}{\nu_0}\right)\cos\left(\frac{p\pi\nu}{\nu_0}\right)\int_{\tau}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz .$$

$$\therefore R'(\tau) = -\left(\frac{3}{2}\right)^{1/3} \frac{\tau^{1/3}}{\nu_0} f(\tau) .$$

$$\therefore R(\tau) = -\left(\frac{3}{2}\right)^{1/3} \frac{1}{\nu_0} \int_{D}^{\tau} q^{1/3} f(q)\, dq ,$$

where D is a constant.

$$\therefore\ x = -\left(\frac{3}{2}\right)^{1/3} \tau^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p.\cos\left(\frac{p\pi\nu}{\nu_0}\right) K_{-\frac{2}{3}}\left(\frac{p\pi\tau}{\nu_0}\right)\int_{\tau_4}^{\tau} z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+\left(\frac{3}{2}\right)^{1/3} \tau^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p.\cos\left(\frac{p\pi\nu}{\nu_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi\tau}{\nu_0}\right)\int_{\tau}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$-\left(\frac{3}{2}\right)^{1/3}\cdot\frac{1}{\nu_0}\int_{0}^{\tau} q^{1/3}\cdot f(q)\, dq. \qquad (50)$$

Now $X = X_0 + \eta x$ and at the nose of the wedge $X = 0$, $X_0 = 0$, $\tau = \infty$ and hence $x = 0$. Therefore from equation (49)

$$0 = 0 + B$$

$$\therefore B = 0 .$$

Now X_0 and X are continuous for all values of τ. Therefore x is continuous for all values of τ. From equation (47) the value x tends to as $\tau \to \tau_4$ is

$$\left(\frac{3}{2}\right)^{1/3}\tau_4^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p\cos\left(\frac{p\pi\nu}{\nu_0}\right) I_{-\frac{2}{3}}\left(\frac{p\pi\tau_4}{\nu_0}\right)\int_{\tau_4}^{\tau_3} z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+ A,$$

and from equation (50) the value x tends to as $r \to r_4$ is

$$\left(\frac{3}{2}\right)^{1/3} r_4^{2/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p.\cos\left(\frac{p\pi v}{v_0}\right) I_{-\frac{2}{3}}\left(\frac{p\pi r_4}{v_0}\right)\int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz - \left(\frac{3}{2}\right)^{1/3}\frac{1}{v_0}\int_D^{r_4} q^{1/3} f(q)\,dq$$

But these two values of x must be equal

$$\therefore\ A = -\left(\frac{3}{2}\right)^{1/3}\frac{1}{v_0}\int_D^{r_4} q^{1/3} f(q)\,dq .$$

As $r \to r_3$ the value x tends to from equation (50) is

$$-\left(\frac{3}{2}\right)^{1/3} r_3^{2/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p.\cos\left(\frac{p\pi v}{v_0}\right) K_{-\frac{2}{3}}\left(\frac{p\pi r_3}{v_0}\right)\int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz - \left(\frac{3}{2}\right)^{1/3}\frac{1}{v_0}\int_D^{r_3} q^{1/3} f(q)\,dq .$$

and the value x tends to from equation (49) is

$$-\left(\frac{3}{2}\right)^{1/3} r_3^{2/3}\left(\frac{2\pi}{v_0^2}\right)\sum_{p=1}^{\infty} p\cos\left(\frac{p\pi v}{v_0}\right) K_{-\frac{2}{3}}\left(\frac{p\pi r_3}{v_0}\right)\int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right)dz .$$

But these two values of x must be equal

$$\therefore\ D = r_3 ,$$

and hence

$$A = -\left(\frac{3}{2}\right)^{1/3}\frac{1}{v_0}\int_{r_3}^{r_4} q^{1/3}.f(q).dq .$$

Collecting the various expressions, we have the following forms for χ which are all valid in the range $0 < \nu < 2\nu_0$.

In the range $0 \le r < r_4$

$$\chi = \left(\frac{3}{2}\right)^{1/3} r^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p\cdot\cos\left(\frac{p\pi\nu}{\nu_0}\right) I_{-\frac{2}{3}}\left(\frac{p\pi r}{\nu_0}\right)\int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$-\left(\frac{3}{2}\right)^{1/3}\cdot\frac{1}{\nu_0}\int_{r_3}^{r_4} q^{1/3} f(q)\, dq\,, \qquad (51)$$

$r_4 < r < r_3$,

$$\chi = -\left(\frac{3}{2}\right)^{1/3} r^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p\cdot K_{-\frac{2}{3}}\left(\frac{p\pi r}{\nu_0}\right)\cos\left(\frac{p\pi\nu}{\nu_0}\right)\int_{r_4}^{r} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+\left(\frac{3}{2}\right)^{1/3} r^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p\cdot I_{-\frac{2}{3}}\left(\frac{p\pi r}{\nu_0}\right)\cos\left(\frac{p\pi\nu}{\nu_0}\right)\int_{r}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$-\left(\frac{3}{2}\right)^{1/3}\cdot\frac{1}{\nu_0}\int_{r_3}^{r} q^{1/3} f(q)\, dq\,, \qquad (52)$$

and $r > r_3$,

$$\chi = -\left(\frac{3}{2}\right)^{1/3} r^{2/3}\left(\frac{2\pi}{\nu_0^2}\right)\sum_{p=1}^{\infty} p\, K_{-\frac{2}{3}}\left(\frac{p\pi r}{\nu_0}\right)\cos\left(\frac{p\pi\nu}{\nu_0}\right)\int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz.$$

$$(53)$$

The integrals $\int z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$, $\int z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{v_0}\right) dz$ and $\int q^{1/3} f(q)\, dq$ can be evaluated analytically but as the expressions are very long and complex they have not been shown here.

The x - co-ordinate at the wedge shoulder.

At the shoulder of the wedge where $\tau = 0$ and $v = v_0$ the value of x there, say x_s , could be found from equation (51). However this expansion for x_s is very complex. In order to find a less complex form for x_s we go back to the integral form for x .

Now equation (39) gives

$$y = \int_0^\infty g(\lambda)\, \tau^{1/3}\, J_{\frac{1}{3}}(\lambda\tau)\, \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0}\, d\lambda ,$$

and x and y are related by equations (44) and (45).

From equations (39) and (45) we obtain

$$\frac{\partial x}{\partial \tau} = -\left(\frac{3}{2}\right)^{1/3} \tau^{2/3} \int_0^\infty \lambda\, g(\lambda)\, J_{\frac{1}{3}}(\lambda\tau)\, \frac{\cosh \lambda(v_0 - v)}{\sinh \lambda v_0}\, d\lambda$$

$$\therefore\ x = \left(\frac{3}{2}\right)^{1/3} \tau^{2/3} \int_0^\infty g(\lambda)\, J_{-\frac{2}{3}}(\lambda\tau)\, \frac{\cosh \lambda(v_0 - v)}{\sinh \lambda v_0}\, d\lambda + V(v) ,$$

and $$\frac{\partial x}{\partial v} = -\left(\frac{3}{2}\right)^{1/3} \tau^{2/3} \int_0^\infty \lambda \cdot g(\lambda)\, J_{-\frac{2}{3}}(\lambda\tau)\, \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0}\, d\lambda + V'(v),$$

where $V(v)$ is any continuous function of v .

But from equations (39) and (44)

$$\frac{\partial x}{\partial v} = -\left(\frac{3}{2}\right)^{1/3} r^{2/3} \int_0^\infty \lambda . g(\lambda) . J_{-\frac{2}{3}}(\lambda r) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} d\lambda .$$

$$\therefore V'(v) = 0 .$$

and hence $V(v)$ = a constant = A .

$$\therefore x = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \int_0^\infty g(\lambda) . J_{-\frac{2}{3}}(\lambda r) \frac{\cosh \lambda(v_0 - v)}{\sinh \lambda v_0} d\lambda + A$$

Now on the face of the wedge $v = v_0$ and onsubstituting for $g(\lambda)$ the expression given by equation (39′) we obtain

$$x = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \int_0^\infty J_{-\frac{2}{3}}(\lambda r) \operatorname{cosech}(\lambda v_0) \int_{r_4}^{r_3} z^{2/3} \lambda f(z) J_{\frac{1}{3}}(\lambda z) dz d\lambda . + A .$$

In order to simplify the integration in the above expression the order of integration has been interchanged. Thus

$$x = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \int_{r_4}^{r_3} z^{2/3} f(z) \int_0^\infty \lambda J_{\frac{1}{3}}(\lambda z) J_{-\frac{2}{3}}(\lambda r) \operatorname{cosech}(\lambda v_0) \, d\lambda . dz + A . \qquad (54)$$

The constant A is determined from the fact that at the nose of the wedge $x = 0$ and $r = \infty$. The value of x at the shoulder where $r = 0$ can then be found by substituting this value of r in equation (54)

Evaluation of the Integral $T(r, z) = r^{2/3} \int_0^\infty \lambda J_{\frac{1}{3}}(\lambda z) J_{-\frac{2}{3}}(\lambda r) \operatorname{cosech}(\lambda v_0)\, d\lambda$.

We will first consider the value of $T(r, z)$ as $r \to \infty$

Let $t = \lambda r$

$$\therefore T(r, z) = r^{-4/3} \int_{\lambda=0}^{\lambda=\infty} t\, J_{-\frac{2}{3}}(t)\, J_{\frac{1}{3}}\left(\frac{tz}{r}\right) \operatorname{cosech}\left(\frac{t v_0}{r}\right) dt.$$

At $\lambda = \infty$, $t = \infty$,

and at $\lambda = 0$, $t \geqslant 0$

If $t = \infty$ at $\lambda = 0$,

$$T(\infty, z) = 0.$$

For all values of t in the range $0 < t < \infty$ the value of the integrand is finite. In the neighbourhood of $t = 0$, $T(r, z)$ behaves like $\int t^{-1/3} dt$. Therefore the contribution to $T(r, z)$ from the lower limit is zero. Near $t = \infty$, $T(r, z)$ behaves like $r^{-\frac{4}{3}} \int t\, e^{-\frac{t v_0}{r}}\, dt$. Therefore as $r \to \infty$ the contribution to $T(r,z)$ from the upper limit is zero. Hence $T(r,z)$ tends to zero as r tends to infinity .

$$\therefore\ T(\infty, z) = 0 .$$

We will now find the value of $T(r,z)$ at $r = 0$.

At $r = 0$, $\quad r^{2/3} J_{-\frac{2}{3}}(\lambda r) = \dfrac{2^{2/3} \lambda^{-2/3}}{\Gamma(\frac{1}{3})}$.

$$\therefore\ T(0,z) = \frac{2^{2/3}}{\Gamma(\frac{1}{3})} \int_0^\infty \lambda^{1/3} J_{\frac{1}{3}}(\lambda z)\, \mathrm{cosech}(\lambda v_0)\, d\lambda ,$$

$$= \frac{2^{2/3}}{\Gamma(\frac{1}{3})} \int_0^\infty \sum_{p=0}^\infty \frac{\lambda^{1/3} (-1)^p \left(\frac{\lambda z}{2}\right)^{2p+\frac{1}{3}}}{p!\ \Gamma(p+\frac{4}{3})}\, \mathrm{cosech}(\lambda v_0)\, d\lambda ,$$

as

$$J_{\frac{1}{3}}(\lambda z) = \sum_{p=0}^\infty \frac{(-1)^p \left(\frac{\lambda z}{2}\right)^{2p+\frac{1}{3}}}{p!\ \Gamma(p+\frac{4}{3})} .$$

In order to find an analytic expression for $T(0,z)$ the integration is done before the series is summed

$$\therefore\ T(0,z) = \sum_{p=0}^\infty \frac{(-1)^p\, 2^{\frac{1}{3}-2p}\, z^{\frac{1}{3}+2p}}{\Gamma(\frac{1}{3}) .\ p!\ \Gamma(p+\frac{4}{3})} \int_0^\infty \lambda^{\frac{2}{3}+2p}\, \mathrm{cosech}(\lambda v_0)\, d\lambda ,$$

$$= \sum_{p=0}^\infty \frac{(-1)^p\, 2^{\frac{1}{3}-2p}\, z^{\frac{1}{3}+2p}}{\Gamma(\frac{1}{3})\ p!\ \Gamma(p+\frac{4}{3})\, v_0^{\frac{5}{3}+2p}} \int_0^\infty \mathrm{cosech}(t) .\ t^{\frac{2}{3}+2p}\, dt ,$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p \, 2^{1/3-2p} \, z^{1/3+2p}}{\Gamma(\frac{1}{3}) \, p! \, \Gamma(p+\frac{4}{3}) \, v_0^{5/3+2p}} \; 2\,\Gamma(2p+\tfrac{5}{3})\,(1-2^{-2p-\frac{5}{3}})\,\zeta(2p+\tfrac{5}{3}),$$

where ζ is the Riemann Zeta Function, and the expression for the evaluation of the $\int_0^\infty (\quad)\,dt$ was obtained from page 32 of reference (7)

$$T(0,z) = \sum_{p=0}^{\infty} H(p) \frac{z^{1/3+2p}}{v_0^{5/3+2p}} \;,$$

where

$$H(p) = \frac{(-1)^p \, 2^{4/3-2p} \, \Gamma(\frac{5}{3}+2p) \, (1-2^{-2p-\frac{5}{3}}) \, \zeta(2p+\frac{5}{3})}{\Gamma(\frac{1}{3}) \, p! \, \Gamma(p+\frac{4}{3})} \,.$$

Since $T(\infty, z) = 0$ and $x = 0$ at $\tau = \infty$ the constant A must be zero. At the shoulder of the wedge where $\tau = 0$ and $x = x_s$ equation (54) gives

$$x_s = \left(\frac{3}{2}\right)^{1/3} \sum_{p=0}^{\infty} H(p) \, v_0^{-2p-\frac{5}{3}} \int_{\tau_4}^{\tau_3} z^{1+2p} f(z)\, dz \qquad (55)$$

The Shoulder Condition.

Now the wedge is of unit length. Therefore $X = 1$ at the shoulder and

$$1 = X_{0s} + \eta \, x_s \;,$$

where X_{o_s} is the value of X_o at the shoulder of the wedge in the channel with solid walls and x_s is the value of x at the shoulder of the wedge in the channel with a section of the walls porous. The expression for X_{o_s} was obtained from the paper (4) by J.B. Helliwell and is

$$X_{o_s} = \frac{2(3)^{1/3} K v_0^{1/3}}{\pi^{1/2}\,\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \Gamma(n-\tfrac{1}{6})\,(2^{2n-\frac{1}{3}}-1)\,\zeta(2n-\tfrac{1}{3}) \left\{ \left(\frac{r_1}{2v_0}\right)^{2n} - \left(\frac{r_2}{2v_0}\right)^{2n} \right\},$$

and the expression for x_s is given by equation (55)

$$\therefore\ 1 = \frac{2(3)^{1/3} K v_0^{1/3}}{\pi^{1/2}\,\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \Gamma(n-\tfrac{1}{6})(2^{2n-\frac{1}{3}}-1)\,\zeta(2n-\tfrac{1}{3}) \left\{ \left(\frac{r_1}{2v_0}\right)^{2n} - \left(\frac{r_2}{2v_0}\right)^{2n} \right\}$$

$$+ \left(\frac{3}{2}\right)^{1/3} \eta \sum_{p=0}^{\infty} H(p)\, v_0^{-2p-\frac{5}{3}} \int_{r_4}^{r_3} z^{1+2p} f(z)\,dz.$$

On substituting the expression for $f(z)$ given by equation (40) we obtain

$$1 = \frac{2(3)^{1/3} K v_0^{1/3}}{\pi^{1/2}\,\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \Gamma(n-\tfrac{1}{6})\,(2^{2n-\frac{1}{3}}-1)\,\zeta(2n-\tfrac{1}{3}) \left\{ \left(\frac{r_1}{2v_0}\right)^{2n} - \left(\frac{r_2}{2v_0}\right)^{2n} \right\}$$

$$+ \left(\frac{3}{2}\right)^{1/3} \frac{\eta k_1 K}{v_0} \sum_{p=0}^{\infty} \frac{H(p)}{v_0^{2p+5/3}} \int_{r_4}^{r_3} \Big[z^{2p+\frac{5}{3}} + 2 z^{2+2p} \sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \Big\{ r_2^{2/3} I_{-\frac{2}{3}}\!\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}\!\left(\frac{n\pi z}{v_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\!\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}\!\left(\frac{n\pi z}{v_0}\right) \Big\} \Big] dz$$

$$- \left(\frac{3}{2}\right)^{1/3} \frac{\eta k_2 K}{v_0} \sum_{p=0}^{\infty} \frac{H(p)}{v_0^{2p+\frac{5}{3}}} \int_{r_4}^{r_3} \Big[z^{1+2p} + 2 z^{4/3+2p} \sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \Big\{ r_2^{2/3} I_{-\frac{2}{3}}\!\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}\!\left(\frac{n\pi z}{v_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\!\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}\!\left(\frac{n\pi z}{v_0}\right) \Big\} \Big] dz$$

$$= \frac{2(3)^{1/3} K v_0^{1/3}}{\pi^{1/2}\,\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \Gamma(n-\tfrac{1}{6}).(2^{2n-\frac{1}{3}}-1)\,\zeta(2n-\tfrac{1}{3})\left\{\left(\frac{r_1}{2v_0}\right)^{2n}-\left(\frac{r_2}{2v_0}\right)^{2n}\right\}$$

$$+\left(\frac{3}{2}\right)^{1/3}\eta k_1 K \sum_{p=0}^{\infty} H(p)\int_{r_4/v_0}^{r_3/v_0}\Big[z^{5/3+2p}+2z^{2+2p}\sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)K_{\frac{1}{3}}(n\pi z)$$

$$+\left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}(n\pi z)\Big\}\Big]dz$$

$$-\left(\frac{3}{2}\right)^{1/3}\frac{\eta k_2 K}{v_0^{2/3}} \sum_{p=0}^{\infty} H(p)\int_{r_4/v_0}^{r_3/v_0}\Big[z^{1+2p}+2z^{4/3+2p}\sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)K_{\frac{1}{3}}(n\pi z)$$

$$+\left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z)\Big\}\Big]dz.$$

Now $k_2 = k_1 r_1^{2/3} + k_3$ and if we let $\alpha = \dfrac{\eta k_1}{v_0}$

and $\beta = \dfrac{\eta k_3}{v_0}$ we obtain the following expression for

$\dfrac{1}{K v_0^{1/3}}$ from the above equation.

$$\frac{1}{K v_0^{1/3}} = \frac{2(3)^{1/3}}{\pi^{1/2}\,\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \Gamma(n-\tfrac{1}{6})\,(2^{2n-\frac{1}{3}}-1)\,\zeta(2n-\tfrac{1}{3})\left\{\left(\frac{r_1}{2v_0}\right)^{2n}-\left(\frac{r_2}{2v_0}\right)^{2n}\right\}$$

$$+\left(\frac{3}{2}\right)^{1/3}\alpha \sum_{p=0}^{\infty} H(p)\Bigg[\left[\frac{z^{2p+\frac{8}{3}}}{2p+\frac{8}{3}}-\left(\frac{r_1}{v_0}\right)^{2/3}\frac{z^{2+2p}}{2p+2}\right]_{r_4/v_0}^{r_3/v_0}$$

$$+2\int_{r_4/v_0}^{r_3/v_0}\Big[z^{2+2p}-\left(\frac{r_1}{v_0}\right)^{2/3} z^{4/3+2p}\Big]\sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)K_{\frac{1}{3}}(n\pi z)$$

$$+\left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z)\Big\}dz\Bigg]$$

$$-\left(\frac{3}{2}\right)^{1/3}\beta \sum_{p=0}^{\infty} H(p)\int_{r_4/v_0}^{r_3/v_0}\Big[z^{1+2p}+2z^{4/3+2p}\sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)K_{\frac{1}{3}}(n\pi z)$$

$$+\left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z)\Big\}\Big]dz \qquad (56)$$

The Drag Coefficient.

From equation (38) the drag coefficient is given by

$$C_D = -\frac{2\left(\frac{3}{2}\right)^{1/3}\delta}{(\gamma+1)M_1^2}\left[\int_0^{\infty} r^{2/3}\left(\frac{\partial X}{\partial r}\right)_{v=v_0} dr + r_1^{2/3}\right].$$

Now $$X = X_0 + \eta x .$$

$$\therefore C_D = -\frac{2\left(\frac{3}{2}\right)^{1/3}\delta}{(\gamma+1)M_1^2}\left[\int_0^\infty \tau^{2/3}\left(\frac{\partial X_0}{\partial \tau} + \eta\frac{\partial x}{\partial \tau}\right)_{\nu=\nu_0} d\tau + \tau_1^{2/3}\right],$$

$$= C_{D_0} + \eta C_{D_1},$$

where
$$C_{D_0} = -\frac{2\left(\frac{3}{2}\right)^{1/3}\delta}{(\gamma+1)M_1^2}\left[\int_0^\infty \tau^{2/3}\left(\frac{\partial X_0}{\partial \tau}\right)_{\nu=\nu_0} d\tau + \tau_1^{2/3}\right],$$

and
$$C_{D_1} = \frac{-2\left(\frac{3}{2}\right)^{1/3}\delta}{(\gamma+1)M_1^2}\int_0^\infty \tau^{2/3}\left(\frac{\partial x}{\partial \tau}\right)_{\nu=\nu_0} d\tau .$$

The form of C_{D_0} is that of the drag coefficient relevant to the channel whose walls are solid. Therefore from the paper (4) by J.B. Helliwell we obtain

$$C_{D_0} = \left(\frac{3}{2}\right) K\delta^2\left[\left(\frac{\tau_1}{\nu_0}\right)^2 - \left(\frac{\tau_2}{\nu_0}\right)^2\right] - \frac{2\left(\frac{3}{2}\right)^{2/3}\delta^{5/3}\left(\frac{\tau_1}{\nu_0}\right)^{2/3}}{(\gamma+1)^{1/3}M_1^{2/3}} .$$

However it is to be noted that, for given values of K, δ and r_1, r_2 is implicitly dependent on η.

Evaluation of e_{D_1}

From equation (46) we see that for $z > r$,

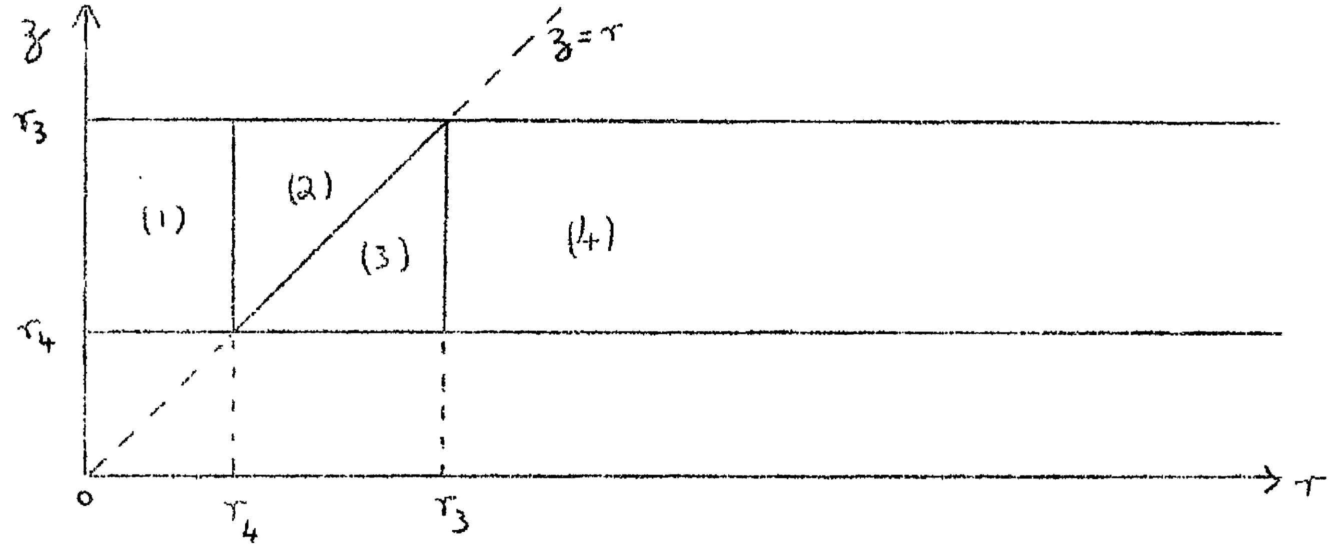
$$\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{\nu_0^2}\right) \sum_{p=1}^{\infty} (-1)^p \, p \left(\frac{p\pi}{\nu_0}\right) I_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz,$$

and from equation (48) we see that for $z < r$,

$$\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} = \left(\frac{3}{2}\right)^{1/3} r^{2/3} \left(\frac{2\pi}{\nu_0^2}\right) \sum_{p=1}^{\infty} (-1)^p \, p \left(\frac{p\pi}{\nu_0}\right) K_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) \int_{r_4}^{r_3} z^{2/3} f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz.$$

Since the expressions for $\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0}$ have been found in the form of integrals over the range $z = r_4$ to r_3 the value of e_{D_1} is now found from the double integral over the rectangle from $r = 0$ to ∞ and $z = r_4$ to r_3. As the expressions for $\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0}$ have been found for $r > z$ and $r < z$ the area of the double integration has been split into four parts so that the analytic expression for e_{D_1} can be found.

The four parts are shown in the following diagram



$$\therefore \int_0^\infty r^{2/3}\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} dr = \left(\frac{3}{2}\right)^{1/3}\left(\frac{2\pi}{\nu_0^2}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\, p \cdot \left(\frac{p\pi}{\nu_0}\right) z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right)\int_{r=0}^{r_4} r^{4/3}\, I_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr\, dz$$

$$+\left(\frac{3}{2}\right)^{1/3}\left(\frac{2\pi}{\nu_0^2}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\, p \cdot \left(\frac{p\pi}{\nu_0}\right) z^{2/3} f(z)\, K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right)\int_{r=r_4}^{z} r^{4/3}\, I_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr\, dz$$

$$+\left(\frac{3}{2}\right)^{1/3}\left(\frac{2\pi}{\nu_0^2}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\, p \cdot \left(\frac{p\pi}{\nu_0}\right) z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right)\int_{r=z}^{r_3} r^{4/3}\, K_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr\, dz$$

$$+\left(\frac{3}{2}\right)^{1/3}\left(\frac{2\pi}{\nu_0^2}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\, p \cdot \left(\frac{p\pi}{\nu_0}\right) z^{2/3} f(z)\, I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right)\int_{r=r_3}^{\infty} r^{4/3}\, K_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr\, dz \qquad (57)$$

The inner integrals were now evaluated, and are

$$\int_{r=0}^{r_4} r^{4/3} I_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr = \left(\frac{\nu_0}{p\pi}\right) r_4^{4/3} I_{4/3}\left(\frac{p\pi r_4}{\nu_0}\right) ,$$

$$\int_{r=r_4}^{z} r^{4/3} I_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr = \left(\frac{\nu_0}{p\pi}\right) z^{4/3} I_{4/3}\left(\frac{p\pi z}{\nu_0}\right) - \left(\frac{\nu_0}{p\pi}\right) r_4^{4/3} I_{4/3}\left(\frac{p\pi r_4}{\nu_0}\right) ,$$

$$\int_{r=z}^{r_3} r^{4/3} K_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr = \left(\frac{\nu_0}{p\pi}\right) z^{4/3} K_{4/3}\left(\frac{p\pi z}{\nu_0}\right) - \left(\frac{\nu_0}{p\pi}\right) r_3^{4/3} K_{\frac{4}{3}}\left(\frac{p\pi r_3}{\nu_0}\right) ,$$

and $$\int_{r=r_3}^{\infty} r^{4/3} K_{\frac{1}{3}}\left(\frac{p\pi r}{\nu_0}\right) dr = \left(\frac{\nu_0}{p\pi}\right) r_3^{4/3} K_{4/3}\left(\frac{p\pi r_3}{\nu_0}\right) .$$

On substitution of these values into equation (57) we obtain

$$\int_0^{\infty} \left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} dr = \left(\frac{3}{2}\right)^{1/3} \left(\frac{2\pi}{\nu_0^2}\right) \int_{z=r_4}^{r_3} \sum_{p=1}^{\infty} (-1)^p \cdot p \cdot r_4^{4/3} I_{4/3}\left(\frac{p\pi r_4}{\nu_0}\right) \cdot z^{2/3} f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+ \left(\frac{3}{2}\right)^{1/3} \left(\frac{2\pi}{\nu_0^2}\right) \int_{z=r_4}^{r_3} \sum_{p=1}^{\infty} (-1)^p \cdot p \cdot z^2 I_{4/3}\left(\frac{p\pi z}{\nu_0}\right) f(z) K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$- \left(\frac{3}{2}\right)^{1/3} \left(\frac{2\pi}{\nu_0^2}\right) \int_{z=r_4}^{r_3} \sum_{p=1}^{\infty} (-1)^p p \; r_4^{4/3} I_{4/3}\left(\frac{p\pi r_4}{\nu_0}\right) z^{2/3} f(z) \cdot K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+ \left(\frac{3}{2}\right)^{1/3} \left(\frac{2\pi}{\nu_0^2}\right) \int_{z=r_4}^{r_3} \sum_{p=1}^{\infty} (-1)^p p \cdot z^2 K_{4/3}\left(\frac{p\pi z}{\nu_0}\right) f(z) I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$-\left(\frac{3}{2}\right)^{1/3}\left(\frac{2\pi}{\nu_0^2}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\; p\; r_3^{4/3}\, K_{4/3}\left(\frac{p\pi r_4}{\nu_0}\right) z^{2/3} f(z)\, I_{1/3}\left(\frac{p\pi z}{\nu_0}\right) dz$$

$$+\left(\frac{3}{2}\right)^{1/3}\left(\frac{2\pi}{\nu_0^2}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\; p\; r_3^{4/3}\, K_{4/3}\left(\frac{p\pi r_4}{\nu_0}\right) z^{2/3} f(z)\, I_{1/3}\left(\frac{p\pi z}{\nu_0}\right) dz\,,$$

$$=\left(\frac{3}{2}\right)^{1/3}\left(\frac{2\pi}{\nu_0^2}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\; p\; z^2 f(z)\left[I_{4/3}\left(\frac{p\pi z}{\nu_0}\right)K_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right) + I_{\frac{1}{3}}\left(\frac{p\pi z}{\nu_0}\right)K_{4/3}\left(\frac{p\pi z}{\nu_0}\right)\right] dz$$

$$=\left(\frac{3}{2}\right)^{1/3}\left(\frac{2}{\nu_0}\right)\int_{z=r_4}^{r_3}\sum_{p=1}^{\infty}(-1)^p\; z\; f(z)\, dz$$

In the preceding pages orders of integration and summation have been frequently interchanged to permit the evaluation to proceed. The occurrence of the non-convergent sum $\sum_{p=1}^{\infty}(-1)^p$ must therefore be interpreted at this stage in the Cesarò sense, its value is then finite and definite, namely $-\frac{1}{2}$.

$$\therefore \int_0^{\infty}\left(\frac{\partial x}{\partial r}\right)_{\nu=\nu_0} dr = -\left(\frac{3}{2}\right)^{1/3}\cdot\frac{1}{\nu_0}\int_{z=r_4}^{r_3} z\, f(z)\, dz\,,$$

and

$$C_{D_1} = \frac{3\,\delta}{(\gamma+1)M_1^2\nu_0}\int_{z=r_4}^{r_3} z\, f(z)\, dz\,.$$

Hence substituting for $f(z)$ the expression given by equation (40) we find:

$$C_{D_1} = \frac{3\,\delta\, k_1 K}{(\gamma+1)M_1^2\nu_0^2}\int_{z=r_4}^{r_3}\left[z^{5/3} + 2z^2\sum_{n=1}^{\infty}\left(\frac{n\pi}{\nu_0}\right)\left\{r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{\nu_0}\right)K_{\frac{1}{3}}\left(\frac{n\pi z}{\nu_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{\nu_0}\right) I_{1/3}\left(\frac{n\pi z}{\nu_0}\right)\right\}\right] dz$$

$$- \frac{3\delta k_2 K}{(\gamma+1) M_1^2 v_0^2} \int_{z=r_4}^{r_3} \Big[z + 2 z^{4/3} \sum_{n=1}^{\infty} \left(\frac{n\pi}{v_0}\right) \Big\{ r_2^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right) + r_1^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}\left(\frac{n\pi z}{v_0}\right)\Big\}\Big] dz ,$$

$$= \frac{3\delta k_1 K v_0^{2/3}}{(\gamma+1) M_1^2} \Big\{ \Big[\frac{3 z^{8/3}}{8} - \left(\frac{r_1}{v_0}\right)^{2/3}\left(\frac{z^2}{2}\right)\Big]_{r_4/v_0}^{r_3/v_0}$$
$$+ 2 \int_{z=r_4/v_0}^{r_3/v_0} \Big[z^2 - \left(\frac{r_1}{v_0}\right)^{2/3} z^{4/3}\Big]\Big[\sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z) + \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z)\Big\}\Big] dz \Big\}$$

$$- \frac{3\delta k_3 K}{(\gamma+1) M_1^2} \Big\{ \Big[\frac{z^2}{2}\Big]_{r_4/v_0}^{r_3/v_0} + 2\int_{r_4/v_0}^{r_3/v_0} z^{4/3} \sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z) + \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z)\Big\} dz \Big\} .$$

$\therefore$ Substituting the expression for C_{D_1} and C_{D_0} into the equation for C_D and letting $\alpha = \frac{\eta k_1}{v_0}$ and $\beta = \frac{\eta k_3}{v_0}$ we obtain

$$C_D = \left(\frac{3}{2}\right) K \delta^2 \Big[\left(\frac{r_1}{v_0}\right)^2 - \left(\frac{r_2}{v_0}\right)^2\Big] - \frac{2\left(\frac{3}{2}\right)^{2/3} \delta^{5/3} \left(\frac{r_1}{v_0}\right)^{2/3}}{(\gamma+1)^{1/3} M_1^{2/3}}$$

$$+ \frac{3\delta K v_0 \alpha}{(\gamma+1) M_1^2} \Big\{ \Big[\frac{3 z^{8/3}}{8} - \left(\frac{r_1}{v_0}\right)^{2/3} \frac{z^2}{2}\Big]_{r_4/v_0}^{r_3/v_0}$$
$$+ 2\int_{z=r_4/v_0}^{r_3/v_0} \Big[z^2 - \left(\frac{r_1}{v_0}\right)^{2/3} z^{4/3}\Big]\Big[\sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z) + \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}(n\pi z)\Big\}\Big] dz \Big\}$$

$$- \frac{3\delta K v_0 \beta}{(\gamma+1) M_1^2} \Big\{ \Big[\frac{z^2}{2}\Big]_{r_4/v_0}^{r_3/v_0} + 2\int_{r_4/v_0}^{r_3/v_0} z^{4/3} \sum_{n=1}^{\infty}(n\pi)\Big\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z) + \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z)\Big\} dz \Big\} ,$$

$$= \left(\frac{3}{2}\right) K \delta^2 \left\{ \left(\frac{\tau_1}{\nu_0}\right)^2 - \left(\frac{\tau_2}{\nu_0}\right)^2 \right\} - \frac{2\left(\frac{3}{2}\right)^{2/3} \delta^{5/3} \left(\frac{\tau_1}{\nu_0}\right)^{2/3}}{(\gamma+1)^{1/3} \, M_1^{2/3}}$$

$$+ 3 K \delta^2 \alpha \left\{ \left[\frac{3 z^{8/3}}{8} - \left(\frac{\tau_1}{\nu_0}\right)^{2/3} \frac{z^2}{2} \right]_{\tau_4/\nu_0}^{\tau_3/\nu_0} \right\}$$

$$+ 6 K \delta^2 \alpha \int_{z=\tau_4/\nu_0}^{\tau_3/\nu_0} \left[z^2 - \left(\frac{\tau_1}{\nu_0}\right)^{2/3} z^{4/3} \right] \left[\sum_{n=1}^{\infty} (n\pi) \left\{ \left(\frac{\tau_2}{\nu_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi\tau_2}{\nu_0}\right) K_{\frac{1}{3}}(n\pi z) \right.\right.$$

$$\left.\left. + \left(\frac{\tau_1}{\nu_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi\tau_1}{\nu_0}\right) I_{\frac{1}{3}}(n\pi z) \right\} \right] dz$$

$$- 3 K \delta^2 \beta \left[\left[\frac{z^2}{2}\right]_{\tau_4/\nu_0}^{\tau_3/\nu_0} + 2 \int_{\tau_4/\nu_0}^{\tau_3/\nu_0} z^{4/3} \sum_{n=1}^{\infty} (n\pi) \left\{ \left(\frac{\tau_2}{\nu_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi\tau_2}{\nu_0}\right) K_{\frac{1}{3}}(n\pi z) \right.\right.$$

$$\left.\left. + \left(\frac{\tau_1}{\nu_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi\tau_1}{\nu_0}\right) I_{1/3}(n\pi z) \right\} dz \right] . \qquad (58)$$

<u>The X-Co-ordinates at the Ends of the Porous Section.</u>

At the ends of the porous section on the channel wall X has the values X_E and X_F. Now $X = X_0 + \eta x$. Therefore on approximating to zero order the values of X_E and X_F are these valued found from the expression for X_0 when $\nu = 0$ and $\tau = \tau_4$ and τ_3 respectively. The series form for X_0 valid in the range $\tau_2 < \tau < \tau_1$ with $\nu = 0$ was obtained from the paper

(4) by J.P. Helliwell and is

$$X_0 = 2\left(\frac{3}{2}\right)^{1/3} K\nu_0^{1/3}\left(\frac{r}{\nu_0}\right)^{2/3}\sum_{n=1}^{\infty}\left\{\left(\frac{r_2}{\nu_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{\nu_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r}{\nu_0}\right) - \left(\frac{r_1}{\nu_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{\nu_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r}{\nu_0}\right)\right\}$$

$$+\left(\frac{3}{2}\right)^{4/3} K\nu_0^{1/3}\left[\left(\frac{r_1}{\nu_0}\right)^{4/3} - \left(\frac{r}{\nu_0}\right)^{4/3}\right] \tag{59}$$

$$\therefore X_E = 2\left(\frac{3}{2}\right)^{1/3} K\nu_0^{1/3}\left(\frac{r_4}{\nu_0}\right)^{2/3}\sum_{n=1}^{\infty}\left\{\left(\frac{r_2}{\nu_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{\nu_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r_4}{\nu_0}\right)\right.$$

$$\left. - \left(\frac{r_1}{\nu_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{\nu_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r_4}{\nu_0}\right)\right\}$$

$$+\left(\frac{3}{2}\right)^{4/3} K\nu_0^{1/3}\left\{\left(\frac{r_1}{\nu_0}\right)^{4/3} - \left(\frac{r_4}{\nu_0}\right)^{4/3}\right\}, \tag{60}$$

and $$X_F = 2\left(\frac{3}{2}\right)^{1/3} K\nu_0^{1/3}\left(\frac{r_3}{\nu_0}\right)^{2/3}\sum_{n=1}^{\infty}\left\{\left(\frac{r_2}{\nu_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{\nu_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r_3}{\nu_0}\right)\right.$$

$$\left. - \left(\frac{r_1}{\nu_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{\nu_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r_3}{\nu_0}\right)\right\}$$

$$+\left(\frac{3}{2}\right)^{4/3} K\nu_0^{1/3}\left[\left(\frac{r_1}{\nu_0}\right)^{4/3} - \left(\frac{r_3}{\nu_0}\right)^{4/3}\right]. \tag{61}$$

The length of the porous section = $X_E - X_F$. (62)

CHAPTER 4.

NUMERICAL ANALYSIS

We now turn to consider the evaluation of the following expressions which are obtained from the general results given in the previous section.

The Channel Width (K)

The expression (56) which defined the channel width may be written in the form.

$$\frac{1}{K v_0^{1/3}} = \mathfrak{f}_1 + \alpha\,\mathfrak{f}_2 + \beta\,\mathfrak{f}_3 , \tag{63}$$

where
$$\mathfrak{f}_1 = \frac{2(3)^{1/3}}{\pi^{1/2}\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\, \Gamma(n-\tfrac{1}{6})\left(2^{2n-\frac{1}{3}}-1\right).\,\zeta(2n-\tfrac{1}{3}).\left[\left(\frac{r_1}{2v_0}\right)^{2n} - \left(\frac{r_2}{2v_0}\right)^{2n}\right],$$

$$\mathfrak{f}_2 = \left(\frac{3}{2}\right)^{1/3} \sum_{p=0}^{\infty} H(p) \Bigg\{ \left[\frac{z^{2p+\frac{8}{3}}}{(2p+\frac{8}{3})} - \left(\frac{r_1}{v_0}\right)^{2/3} \frac{z^{2p+2}}{(2p+2)} \right]_{r_4/v_0}^{r_3/v_0}$$
$$+ 2\int_{r_4/v_0}^{r_3/v_0} \left[z^{2+2p} - \left(\frac{r_1}{v_0}\right)^{2/3} z^{4/3+2p} \right] \sum_{n=1}^{\infty} (n\pi) \Big\{ \left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z)$$
$$+ \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z) \Big\} dz \Bigg\},$$

$$\mathfrak{f}_3 = -\left(\frac{3}{2}\right)^{1/3} \sum_{p=0}^{\infty} H(p). \int_{r_4/v_0}^{r_3/v_0} \Big[z^{1+2p} + 2 z^{4/3+2p} \sum_{n=1}^{\infty} (n\pi) \Big\{ \left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z)$$
$$+ \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}(n\pi z) \Big\} \Big] dz,$$

and
$$H(p) = (-1)^p\, 2^{4/3-2p}\, \Gamma(\tfrac{5}{3}+2p).\left(1-2^{-2p-\frac{5}{3}}\right).\ \zeta(2p+\tfrac{5}{3}) \Big/ \Gamma(\tfrac{1}{3}).\Gamma(p+\tfrac{4}{3}).\ p!\ .$$

A typical evaluation of the functions $\int_1$, $\int_2$ and $\int_3$ is shown in the next section. The relationships between $\int_1$, $\int_2$ and $\int_3$ and the transonic similarity parameter $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$, $\left(=\left[\frac{3}{2}\right]^{2/3}\left[\frac{r_1}{v_0}\right]^{2/3}\right)$, are shown on figures 1, 2 and 3 respectively in which the quantities $\frac{r_2}{v_0}$, $\frac{r_3}{v_0}$ and $\frac{r_4}{v_0}$ are taken as additional parameters.

The Drag Coefficient C_D

The expression (58) which defines the drag coefficient may be written in the following form

$$C_D = K\delta^2 \int_4 + \frac{\delta^{5/3}}{(\gamma+1)^{1/3} M_1^{2/3}} \int_5 + K\delta^2 \alpha \int_6 + K\delta^2 \beta \int_7 , \qquad (64)$$

where

$$\int_4 = \left(\frac{3}{2}\right)\left[\left(\frac{r_1}{v_0}\right)^2 - \left(\frac{r_2}{v_0}\right)^2\right],$$

$$\int_5 = -2\left(\frac{3}{2}\right)^{2/3}\left(\frac{r_1}{v_0}\right)^{2/3},$$

$$\int_6 = 3\left[\frac{3z^{8/3}}{8} - \left(\frac{r_1}{v_0}\right)^{2/3}\frac{z^2}{2}\right]_{r_4/v_0}^{r_3/v_0} + 6\int_{r_4/v_0}^{r_3/v_0}\left[z^2 - \left(\frac{r_1}{v_0}\right)^{2/3} z^{4/3}\right]\sum_{n=1}^{\infty}(n\pi)\left\{\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z) + \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}(n\pi z)\right\} dz ,$$

and

$$\int_7 = -3\left\{\left[\frac{z^2}{2}\right]_{r_4/v_0}^{r_3/v_0} + 2\int_{r_4/v_0}^{r_3/v_0} z^{4/3}\sum_{n=1}^{\infty}(n\pi)\left[\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{1}{3}}(n\pi z) + \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{\frac{1}{3}}(n\pi z)\right] dz\right\}.$$

On substituting for K from equation (63) and using the fact that $\nu_0 = (\gamma+1) M_1^2 \delta$ we obtain

$$C_D = \frac{\delta^{5/3}}{(\gamma+1)^{1/3} M_1^{2/3}} \left[\frac{I_4 + \alpha I_6 + \beta I_7}{I_1 + \alpha I_2 + \beta I_3} + I_5 \right].$$

$$\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_D = \frac{I_4 + \alpha I_6 + \beta I_7}{I_1 + \alpha I_2 + \beta I_3} + I_5 \tag{65}$$

$$= \left(I_4 + \alpha I_6 + \beta I_7\right) \frac{1}{I_1}\left(1 - \frac{\alpha I_2 + \beta I_3}{I_1}\right) + I_5, \quad \text{to } O(\eta)$$

$$= \left(\frac{I_4}{I_1} + I_5\right) + \alpha\left(\frac{I_6}{I_1} - \frac{I_4 I_2}{I_1^2}\right) + \beta\left(\frac{I_7}{I_1} - \frac{I_4 I_3}{I_1^2}\right),$$

$$= \frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_2} + \frac{(\gamma+1)^{1/3} \alpha M_1^{2/3}}{\delta^{5/3}} C_{D_3} + \frac{(\gamma+1) M_1^{2/3}}{\delta^{5/3}} \beta C_{D_4} \tag{66}$$

A typical evaluation of the functions $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_2}$, $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_3}$ and $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_4}$ is shown in the following section. The relationships between $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_2}$, $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_3}$ and $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_4}$ and the transonic similarity parameter

$\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$ are shown in figures 4, 5 and 6 respectively in which $\left(\frac{r_2}{v_0}\right)$, $\left(\frac{r_3}{v_0}\right)$ and $\left(\frac{r_4}{v_0}\right)$ are taken as additional parameters.

The Position of the Porous Section.

The X -co-ordinates of the ends of the porous section are X_E and X_F where $X_E = X\left(\frac{r}{v_0} = \frac{r_4}{v_0}, v = 0\right)$ and $X_F = X\left(\frac{r}{v_0} = \frac{r_3}{v_0}, v = 0\right)$.

Now equations (60) and (61) give the zero order expression for X_E and X_F. Therefore on substituting the zero order value of $K v_0^{1/3}$ from equation (63) we obtain

$$X\left(\frac{r}{v_0}, v=0\right) = 2\left(\frac{3}{2}\right)^{1/3} \frac{1}{f_1} \left(\frac{r}{v_0}\right)^{2/3} \sum_{n=1}^{\infty} \left[\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r}{v_0}\right) - \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r}{v_0}\right)\right.$$
$$\left. + \left(\frac{3}{2}\right)^{4/3} \frac{1}{f_1}\left[\left(\frac{r_1}{v_0}\right)^{4/3} - \left(\frac{r}{v_0}\right)^{4/3}\right]\right]$$

A typical evaluation of X_E and X_F is shown in the next section. The relationship between X_E (or X_F) and the transonic similarity parameter is shown in figure 8, in which $\frac{r_2}{v_0}$ is taken as an additional parameter.

Since the expressions for f_2, f_3, $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_3}$ and $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_4}$ can be expressed in the form $Z\left(\frac{r_3}{v_0}\right) - Z\left(\frac{r_4}{v_0}\right)$ for given values of $\left(\frac{\tau_1}{\nu_0}\right)$ and $\left(\frac{\tau_2}{\nu_0}\right)$ where Z stands for the appropriate functions, additional curves in figures 2, 3, 5 and 6 can easily be obtained for other values of the parameters $\left(\frac{\tau_3}{\nu_0}\right)$ and $\left(\frac{\tau_4}{\nu_0}\right)$ in the range $0{\cdot}4 \geqslant \frac{\tau_3}{\nu_0}$, $\frac{\tau_4}{\nu_0} \geqslant 0{\cdot}1$ by means of addition and subtraction and hence interpolation, as shown in the following example

$$Z(0{\cdot}3) - Z(0{\cdot}1) = \left[Z(0{\cdot}3) - Z(0{\cdot}2)\right] + \left[Z(0{\cdot}2) - Z(0{\cdot}1)\right],$$

$$Z(0{\cdot}4) - Z(0{\cdot}1) = \left[Z(0{\cdot}4) - Z(0{\cdot}3)\right] + \left[Z(0{\cdot}3) - Z(0{\cdot}1)\right].$$

Hence the value of $Z(0{\cdot}35) - Z(0{\cdot}15)$ could be found by interpolation.

Now from equations (56), (60) and (61) we see that for given conditions at infinity upstream

$$\frac{1}{K\nu_0^{1/3}} = \frac{1}{K\nu_0^{1/3}}\left(\frac{\tau_2}{\nu_0}, \frac{\tau_3}{\nu_0}, \frac{\tau_4}{\nu_0}, \eta, p_c\right),$$

$$X_E = X_E\left(\frac{\tau_2}{\nu_0}, \frac{\tau_4}{\nu_0}\right),$$

and $X_F = X_F\left(\frac{\tau_2}{\nu_0}, \frac{\tau_3}{\nu_0}\right)$ respectively.

Therefore for given values of X_E, X_F, η, p_c and $\frac{1}{k\nu_0^{1/3}}$ we can find the values of $\left(\frac{\tau_2}{\nu_0}\right)$, $\left(\frac{\tau_3}{\nu_0}\right)$ and $\left(\frac{\tau_4}{\nu_0}\right)$ from the curves shown in figures 1, 2, 3 and 8. Hence the value of C_D can be found from the curves shown in figures 4, 5 and 6.

We now consider the difference between the drag coefficient of a given wedge in a channel of given width with a finite porous section and the drag coefficient of the same wedge in a chcked channel of the same width with solid walls when the conditions at infinity upstream in both channels are the same.

If β_1^*, β_4^*, β_5^*, M_1^*, C_D^* and $\left(\frac{1}{k\nu_0^{1/3}}\right)^*$ are the values of β_1, β_4, β_5, M_1, C_D and $\left(\frac{1}{k\nu_0^{1/3}}\right)$ respectively when $\left(\frac{\tau_1}{\nu_0}\right) = \left(\frac{\tau_1}{\nu_0}\right)^*$, $\left(\frac{\tau_2}{\nu_0}\right) = 0$ and there is no porous section, we obtain from equation (63) and (65) that

$$\left(\frac{1}{k\nu_0^{1/3}}\right)^* = \beta_1^* ,$$

and

$$\frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}} C_D^* = \frac{\beta_4^*}{\beta_1^*} + \beta_5^* .$$

We now find the value of $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_D$ for the same wedge in a channel of the same width with a finite porous section when $\left(\frac{r_1}{\nu_0}\right) = \left(\frac{r_1}{\nu_0}\right)^*$ and $\left(\frac{1}{K\nu_0^{1/3}}\right) = \left(\frac{1}{K\nu_0^{1/3}}\right)^*$.

From equation (63) we see that

$$\frac{1}{K\nu_0^{1/3}} = \int_1^* = \int_1 + \alpha \int_2 + \beta \int_3 . \qquad (67)$$

$$\therefore \alpha = \frac{\int_1^* - \int_1}{\int_2} - \frac{\int_3}{\int_2} \beta . \qquad (68)$$

On substitution for (α) and $\left(\int_1 + \alpha \int_2 + \beta \int_3\right)$ from equations (68) and (67) in equation (65) we obtain

$$\frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}} C_D = \frac{\int_4}{\int_1^*} + \frac{\int_6}{\int_1^*}\left[\frac{\int_1^* - \int_1}{\int_2} - \frac{\int_3}{\int_2}\beta\right] + \frac{\int_7}{\int_1^*}\beta + \int_5^* .$$

$$\frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}}\left[C_D - C_D^*\right] = \frac{\int_4 - \int_4^*}{\int_1^*} + \frac{\int_6}{\int_1^*}\left(\frac{\int_1^* - \int_1}{\int_2}\right) + \left(\frac{\int_2 \int_7 - \int_3 \int_6}{\int_1^* \int_2}\right)\beta . \qquad (69)$$

Therefore on approximating to zero order we obtain

$$\frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}}\left[C_D - C_D^*\right] = \frac{\beta_4 - \beta_4^*}{\beta_1^*} + \frac{\beta_6(\beta_1^* - \beta_1)}{\beta_2} . \qquad (70)$$

A typical evaluation of $\frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}}\left[C_D - C_D^*\right]$ is shown in the next section. The relationship between $\frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}}\left[C_D - C_D^*\right]$ and $\frac{1-M_2^2}{\left[(\gamma+1) M_1^2 \delta\right]^{2/3}} = \left(\frac{3}{2}\right)^{2/3}\left(\frac{\tau_2}{\nu_0}\right)^{2/3}$ is shown in figure 7 in which $\left(\frac{1}{k\nu_0^{1/3}}\right)^*$, $\left(\frac{\tau_1}{\nu_0}\right)^*$, $\left(\frac{\tau_3}{\nu_0}\right)$ and $\left(\frac{\tau_4}{\nu_0}\right)$ are taken as additional parameters.

CHAPTER 5.

NUMERICAL CALCULATION

In this section we present a typical evaluation of the various functions which have been defined in the previous section. The values of $\left(\frac{\tau_1}{v_0}\right)$, $\left(\frac{\tau_2}{v_0}\right)$ $\left(\frac{\tau_3}{v_0}\right)$ and $\left(\frac{\tau_4}{v_0}\right)$ are taken as $\frac{2}{\pi}$, 0.05, 0.3 and 0.2 respectively.

The values of the Bessel Functions $I_{\pm\nu}(z)$ can be found from the tables given in reference (8). The values of the Bessel Function $K_\nu(z)$ were found from the values of $I_{\pm\nu}(z)$ by using the relationship

$$K_\nu(z) = \frac{2}{\sin \nu\pi}\left[I_{-\nu}(z) - I_\nu(z)\right].$$

For large values of z the values of $I_{\pm\nu}(z)$ and $K_\nu(z)$ were found from their asymptotic expansions. These expansions are

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1 + \frac{(4\nu^2-1)}{8z} + \cdots\right]$$

and

$$I_{\pm\nu}(z) \sim \frac{1}{(2\pi z)^{1/2}} e^{+z}\left[1 - \frac{(4\nu^2-1)}{8z} + \cdots\right].$$

Since the integrals $\int_{\tau_4/v_0}^{\tau_3/v_0} z^{2+2p} I_{\frac{1}{3}}(n\pi z)\,dz$, $\int_{\tau_4/v_0}^{\tau_3/v_0} z^{4/3+2p} I_{\frac{1}{3}}(n\pi z)\,dz$ $\int_{\tau_4/v_0}^{\tau_3/v_0} z^{2+2p} K_{\frac{1}{3}}(n\pi z)\,dz$ and $\int_{\tau_4/v_0}^{\tau_3/v_0} z^{4/3+2p} K_{\frac{1}{3}}(n\pi z)\,dz$ could not be evaluated to give simple analytic expressions their values were obtained satisfactorily by means of Simpson's Rule. The values of the above integrals for different values of p and n are shown in tables 1, 2, 3 and 4 respectively.

The evaluation of the function $F(p)$ defined by

$$F(p) = \frac{(-1)^p \, 2^{3-2p} \, \Gamma(2p+\frac{2}{3}) . (1 - 2^{-2p-\frac{5}{3}}) . \zeta(2p+\frac{5}{3})}{p! \quad \Gamma(p+\frac{4}{3})} ,$$

for different values of p is shown in table 5.

We are now able to calculate the values of the various functions which have already been defined in the previous section.

Evaluation of the function f_1

The function f_1 can be defined by

$$f_1 = \frac{2(3)^{1/3}}{\pi^{1/2} \, \Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \left[R(n) . \left(\frac{r_1}{2v_0}\right)^{2n} - R(n) . \left(\frac{r_2}{2v_0}\right)^{2n} \right] ,$$

where

$$R(n) = \frac{(-1)^{n+1}}{n!} . \Gamma(n-\tfrac{1}{6}) . (2^{2n-\frac{1}{3}} - 1) \; \zeta(2n-\tfrac{1}{3}) .$$

The evaluation of the series $\sum_{n=1}^{\infty} R(n) \left(\frac{r}{2v_0}\right)^{2n}$ for $\left(\frac{r}{v_0}\right) = \left(\frac{r_1}{v_0}\right)$ and $\left(\frac{r}{v_0}\right) = \left(\frac{r_2}{v_0}\right)$ is given in table 6

$$\therefore \; f_1 = \frac{2(3)^{1/3}}{\pi^{1/2}\,\Gamma(\frac{1}{3})}\left[0\,4794 - 0\cdot 0033\right],$$

$$= 0\cdot 2892$$

Evaluation of the function f_2

The function f_2 can be defined by

$$f_2 = \left(\frac{3}{2}\right)^{1/3}\frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})}\sum_{p=0}^{\infty}F(p)\left[\frac{z^{2p+\frac{8}{3}}}{(2p+\frac{8}{3})}\right]_{r_4/v_0}^{r_3/v_0}$$

$$-\left(\frac{3}{2}\right)^{1/3}\frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})}\left(\frac{r_1}{v_0}\right)^{2/3}\sum_{p=0}^{\infty}F(p)\left[\frac{z^{2p+2}}{(2p+2)}\right]_{r_4/v_0}^{r_3/v_0}$$

$$+2\pi\left(\frac{r_2}{v_0}\right)^{2/3}\left(\frac{3}{2}\right)^{1/3}\frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})}\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}n\,.\,F(p)\;I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0}z^{2+2p}K_{\frac{1}{3}}(n\pi z)\,dz$$

$$+2\pi\left(\frac{r_1}{v_0}\right)^{2/3}\left(\frac{3}{2}\right)^{1/3}\frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})}\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}n\,.\,F(p)\;K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0}z^{2+2p}I_{\frac{1}{3}}(n\pi z)\,dz$$

$$-2\pi\left(\frac{r_1}{v_0}\right)^{2/3}\left(\frac{r_2}{v_0}\right)^{2/3}\left(\frac{3}{2}\right)^{1/3}\frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})}\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}n\,.\,F(p)\,.\,I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0}z^{4/3+2p}K_{\frac{1}{3}}(n\pi z)\,dz$$

$$-2\pi\left(\frac{r_1}{v_0}\right)^{4/3}\left(\frac{3}{2}\right)^{1/3}\frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})}\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}n\,.\,F(p)\,.\,K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0}z^{4/3+2p}I_{\frac{1}{3}}(n\pi z)\,dz$$

The evaluation of $\sum_{p=0}^{\infty}F(p)\,.\left[\frac{z^{2p+\frac{8}{3}}}{2p+\frac{8}{3}}\right]_{r_4/v_0}^{r_3/v_0}$, $\sum_{p=0}^{\infty}F(p)\left[\frac{z^{2p+2}}{2p+2}\right]_{r_4/v_0}^{r_3/v_0}$,

$$\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}n\,F(p).\,I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0}z^{2+2p}K_{\frac{1}{3}}(n\pi z)\,dz,$$

$$\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}n\,.\,F(p)\;K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0}z^{2+2p}I_{\frac{1}{3}}(n\pi z)\,dz,$$

$$\sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \, . \, F(p) \, I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3+2p} \, K_{\frac{1}{3}}(n\pi z) \, dz \, ,$$

$$\text{and} \quad \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \, . \, F(p) \, K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3+2p} \, I_{\frac{1}{3}}(n\pi z) \, dz \, .$$

is given in tables 7, 8, 9, 10, 11 and 12 respectively.

$$\therefore \; f_2 = \left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.05689\right] - \left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left(\frac{2}{\pi}\right)^{2/3} \left[0.1423\right]$$

$$+ \, 2\pi \, . \, (0.05)^{2/3} \left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.08277\right] + 2\pi \, . \left(\frac{2}{\pi}\right)^{2/3}\left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.006189\right]$$

$$- \, 2\pi \, . \left(\frac{2}{\pi}\right)^{2/3} (0.05)^{2/3} \left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.1925\right] - 2\pi \left(\frac{2}{\pi}\right)^{4/3}\left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.01407\right]$$

$$= \; - \, 0.03202 \, .$$

<u>Evaluation of the function f_3</u>

The function f_3 can be defined by

$$f_3 = -\left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \, . \sum_{p=0}^{\infty} F(p) \, . \left[\frac{z^{2+2p}}{2+2p} \right]_{r_4/v_0}^{r_3/v_0}$$

$$-(2\pi) \, . \left(\frac{3}{2}\right)^{1/3} \left(\frac{r_2}{v_0}\right)^{2/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \, . \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \, . \, F(p) \, . \, I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3+2p} \, K_{\frac{1}{3}}(n\pi z) \, dz \, ;$$

$$-(2\pi) \, . \left(\frac{3}{2}\right)^{1/3} \left(\frac{r_1}{v_0}\right)^{2/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \, . \, F(p) \, . \, K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3+2p} \, I_{\frac{1}{3}}(n\pi z) \, d.$$

$$\therefore \, \mathfrak{f}_3 = -\left(\frac{3}{2}\right)^{1/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.1423\right] - 2\pi . \left(\frac{3}{2}\right)^{1/3} (0.05)^{2/3} \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.1925\right]$$

$$- 2\pi . \left(\frac{3}{2}\right)^{1/3} . \left(\frac{2}{\pi}\right)^{2/3} . \frac{1}{2^{2/3}\,\Gamma(\frac{1}{3})} \left[0.01407\right],$$

$$= -0.1001 .$$

Evaluation of the function $\mathfrak{f}_4$

The function $\mathfrak{f}_4$ is defined by

$$\mathfrak{f}_4 = \left(\frac{3}{2}\right) \left[\left(\frac{r_1}{v_0}\right)^2 - \left(\frac{r_2}{v_0}\right)^2 \right] .$$

$$\therefore \, \mathfrak{f}_4 = \left(\frac{3}{2}\right) \left[\left(\frac{2}{\pi}\right)^2 - (0.05)^2 \right] ,$$

$$= 0.6041 .$$

Evaluation of the function $\mathfrak{f}_5$

The function $\mathfrak{f}_5$ is defined by

$$\mathfrak{f}_5 = -2\left(\frac{3}{2}\right)^{2/3} \left(\frac{r_1}{v_0}\right)^{2/3} .$$

$$\therefore \, \mathfrak{f}_5 = -2\left(\frac{3}{2}\right)^{2/3} . \left(\frac{2}{\pi}\right)^{2/3} ,$$

$$= -1.939 .$$

Evaluation of the function $\mathfrak{f}_6$

The function $\mathfrak{f}_6$ can be defined by

$$\mathfrak{f}_6 = 3 \left[\frac{3}{8} z^{8/3} - \left(\frac{r_1}{v_0}\right)^{2/3} . \frac{3}{2} z^2 \right]_{r_4/v_0}^{r_3/v_0}$$

$$+6\pi\left(\frac{r_2}{v_0}\right)^{2/3}\sum_{n=1}^{\infty} n\ I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^2 K_{\frac{1}{3}}(n\pi z)\,dz$$

$$+6\pi\left(\frac{r_1}{v_0}\right)^{2/3}\sum_{n=1}^{\infty} n\ K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^2 I_{\frac{1}{3}}(n\pi z)\,dz$$

$$-6\pi\left(\frac{r_1}{v_0}\right)^{2/3}\left(\frac{r_2}{v_0}\right)^{2/3}\sum_{n=1}^{\infty} n\ I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3} K_{\frac{1}{3}}(n\pi z)\,dz$$

$$-6\pi\left(\frac{r_1}{v_0}\right)^{4/3}\sum_{n=1}^{\infty} n\ K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3} I_{\frac{1}{3}}(n\pi z)\,dz.$$

The evaluation of $\sum_{n=1}^{\infty} n\ I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^2 K_{\frac{1}{3}}(n\pi z)\,dz$,

$\sum_{n=1}^{\infty} n\,K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^2 I_{\frac{1}{3}}(n\pi z)\,dz$, $\sum_{n=1}^{\infty} n\,I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3} K_{\frac{1}{3}}(n\pi z)\,dz$

and $\sum_{n=1}^{\infty} n\ K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3} I_{\frac{1}{3}}(n\pi z)\,dz$ is shown in tables 13, 14, 15 and 16 respectively.

$$\therefore f_6 = 3\left[\frac{3z^{8/3}}{8} - \left(\frac{2}{\pi}\right)^{2/3}\frac{z^2}{2}\right]_{0.2}^{0.3} + 6\pi(0.05)^{2/3}\left[0.01452\right]$$

$$+6\pi\left(\frac{2}{\pi}\right)^{2/3}\left[0.001089\right] - 6\pi\left(\frac{2}{\pi}\right)^{2/3}(0.05)^{2/3}\left[0.03377\right]$$

$$-6\pi\left(\frac{2}{\pi}\right)^{4/3}\left[0.002445\right]$$

$$= -0.06263.$$

Evaluation of the function $\int_4$

The function $\int_4$ can be defined by

$$\int_4 = -3\left[\frac{z^2}{2}\right]_{r_4/v_0}^{r_3/v_0} - 6\pi\left(\frac{r_2}{v_0}\right)^{2/3}\sum_{n=1}^{\infty} n\, I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3} K_{\frac{1}{3}}(n\pi z)\,dz$$

$$-6\pi\left(\frac{r_1}{v_0}\right)^{2/3}\sum_{n=1}^{\infty} n\, K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3} I_{\frac{1}{3}}(n\pi z)\,dz$$

$$\therefore \int_4 = -3\left[\frac{z^2}{2}\right]_{0.2}^{0.3} - 6\pi(0.05)^{2/3}\left[0.03337\right] - 6\pi\left(\frac{2}{\pi}\right)^{2/3}\left[0.002445\right],$$

$$= -0.1959.$$

Evaluation of $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_2}$

$$\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_2} = \frac{\int_4}{\int_1} + \int_5 .$$

$$\therefore \frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_2} = \frac{0.6041}{0.2892} - 1.939,$$

$$= 0.150.$$

Evaluation of $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_3}$

$$\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_3} = \frac{\beta_6}{\beta_1} - \frac{\beta_4 \beta_2}{\beta_1^2} .$$

$$\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_3} = -\frac{0.06263}{0.2892} + \frac{(0.6041)(0.03202)}{(0.2892)^2},$$

$$= 0.0144 .$$

Evaluation of X_E and X_F

$$X_E = X\left(\frac{r}{r_0} = \frac{r_4}{r_0}\right) \quad \text{and} \quad X_F = X\left(\frac{r}{r_0} = \frac{r_3}{r_0}\right)$$

Now
$$X\left(\frac{r}{r_0}\right) = 2\left(\frac{3}{2}\right)^{1/3} \frac{1}{\beta_1} \left(\frac{r}{r_0}\right)^{2/3} \sum_{n=1}^{\infty} \left[\left(\frac{r_2}{r_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{r_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r}{r_0}\right) - \left(\frac{r_1}{r_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{r_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r}{r_0}\right)\right]$$
$$+ \left(\frac{3}{2}\right)^{4/3} \frac{1}{\beta_1} \left[\left(\frac{r_1}{r_0}\right)^{4/3} - \left(\frac{r}{r_0}\right)^{4/3}\right].$$

The evaluation of $\sum_{n=1}^{\infty} \left[\left(\frac{r_2}{r_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{r_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r}{r_0}\right) - \left(\frac{r_1}{r_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{r_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r}{r_0}\right)\right]$

for $\frac{r}{r_0} = \frac{r_4}{r_0}$ and $\frac{r}{r_0} = \frac{r_3}{r_0}$ is shown in tables 17 and 18 respectively

$$\therefore X_E = 2\left(\frac{3}{2}\right)^{1/3} \frac{1}{(0\cdot 2892)}(0\cdot 2)^{2/3}(0\cdot 2554)$$

$$+\left(\frac{3}{2}\right)^{4/3}\frac{1}{(0\cdot 2892)}\left[\left(\frac{2}{\pi}\right)^{4/3} - (0\cdot 2)^{4/3}\right],$$

$$= 1\cdot 971,$$

and $$X_F = 2\left(\frac{3}{2}\right)^{1/3} \frac{1}{(0\cdot 2892)} \cdot (0\cdot 3)^{2/3}(0\cdot 0405)$$

$$+\left(\frac{3}{2}\right)^{4/3}\frac{1}{0\cdot 2892}\left[\left(\frac{2}{\pi}\right)^{4/3} - (0\cdot 3)^{4/3}\right],$$

$$= 1\cdot 279.$$

Evaluation of $\dfrac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}}\left[C_D - C_D^*\right]$

$$\frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}}\left[C_D - C_D^*\right] = \frac{\beta_4 - \beta_4^*}{\beta_1^*} + \frac{\beta_6}{\beta_1} \cdot \frac{(\beta_1^* - \beta_1)}{\beta_2}$$

Now $$\beta_1^* = \frac{2(3)^{1/3}}{\pi^{1/2}\Gamma(\frac{1}{3})}\sum_{n=1}^{\infty} R(n) \cdot \left(\frac{r_1}{2v_0}\right)^{2n},$$

$$= \frac{2(3)^{1/3}}{\pi^{1/2}\Gamma(\frac{1}{3})} \cdot \left[0\cdot 4494\right],$$

$$= 0\cdot 2912,$$

$$\beta_1^* - \beta_1 = \frac{2(3)^{1/3}}{\pi^{1/2}\Gamma(\frac{1}{3})}\sum_{n=1}^{\infty} R(n)\left(\frac{r_2}{2v_0}\right)^{2n},$$

$$= \frac{2(3)^{1/3}}{\pi^{1/2}\Gamma(\frac{1}{3})}\left[0\cdot 003268\right],$$

$$= 0\cdot 001984,$$

and $$f_4 - f_4^* = -\left(\frac{3}{2}\right)\left(\frac{r_2}{v_0}\right)^2,$$

$$= -\left(\frac{3}{2}\right)(0 \cdot 05)^2,$$

$$= -0 \cdot 00375.$$

$$\therefore \frac{(\gamma+1)^{1/3} M_1^{*\,2/3}}{\delta^{5/3}}\left[C_D - C_D^*\right] = -\frac{0 \cdot 003750}{0 \cdot 2912} + \frac{0 \cdot 06263}{0 \cdot 2912} \cdot \frac{0 \cdot 001984}{0 \cdot 03202},$$

$$= -\frac{0 \cdot 003750}{0 \cdot 2912} + \frac{0 \cdot 003880}{0 \cdot 2912},$$

$$= 0 \cdot 0004 \ldots$$

CONCLUSIONS

The problem considered in this work was the design of a straight walled and tunnel with a finite porous section to give a reduced blockage interference in high subsonic compressible flow. The blockage interference will be reduced if we can choose values for the various parameters so as to make the value of the drag coefficient evaluated from equation (58) more nearly equal to the drag coefficient for the same wedge in a free stream when the upstream conditions are the same.

The relationship between the drag coefficient for a wedge in a free stream and the transonic similarity parameter was found by J. B. Helliwell and A. G. Mackie [5] and is shown in figure 9.

Now from equations (58), (56), (60) and (61) we see that for given conditions at infinity upstream

$$\frac{(\gamma+1)^{1/3} M_1^{2/3} C_D}{\delta^{5/3}} = f_n\left(\frac{\tau_2}{v_0}, \frac{\tau_3}{v_0}, \frac{\tau_4}{v_0}, \eta, p_e\right),$$

$$\frac{1}{K v_0^{1/3}} = f_n\left(\frac{\tau_2}{v_0}, \frac{\tau_3}{v_0}, \frac{\tau_4}{v_0}, \eta, p_e\right),$$

$$X_E = f_n\left(\frac{\tau_2}{v_0}, \frac{\tau_4}{v_0}\right)$$

$$\text{and } X_F = f_n\left(\frac{\tau_2}{v_0}, \frac{\tau_3}{v_0}\right) \qquad \text{respectively.}$$

On substituting for C_D the value of $C_{D\ free}$ obtained from figure 9 with the given value of $\left(\frac{\tau_1}{v_0}\right)$ we see that theoretically, for given values of K, δ, p_e, X_E and X_F, on elimination of $\left(\frac{\tau_2}{v_0}\right)$, $\left(\frac{\tau_3}{v_0}\right)$ and $\left(\frac{\tau_4}{v_0}\right)$ from the above equations a value of η can be found which will give zero blockage interference. However this value is of considerably larger magnitude than the values of η for which the present perturbation theory is valid. This means that it is not possible to design a straight walled tunnel with a finite porous section to give zero blockage interference in high subsonic compressible flow on the basis of a solution to the problem derived by a perturbation from the solution in the case of the channel with solid walls. However figure 7 shows the relationship between the difference of the drag coefficient of a given wedge in a channel of given width with a finite porous section and the drag coefficient of the same wedge in a choked channel of the same width with solid walls, when the conditions at infinity upstream in both channels are the same, and the downstream parameter is $\left(\frac{3}{2}\right)^{2/3}\left(\frac{\tau_2}{\nu_0}\right)^{2/3}$ in the porous case. These curves show that the difference in the drag coefficients increases as the value of $\left(\frac{3}{2}\right)^{2/3}\left(\frac{\tau_2}{\nu_0}\right)^{2/3}$ increases and the increase is greater when the porous section is upstream of the wedge nose, which means that some of the blockage interference has been eliminated. The curves also suggest that it should be possible to eliminate the blockage interference entirely by the use of larger values of η than those for which the present theory is valid. This follows because we could then suck out more fluid from the channel and thereby increase the value of $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$, hence increasing the value of $\frac{(\gamma+1)^{1/3}M_1^{2/3}}{\delta^{5/3}}C_D$.

APPENDIX I

The Change in the Value of the Stream Function (ψ) across the Porous Section

From equation (35) we obtain

$$d\psi = \rho U_1 \, dY + \rho U_1 u' \, dY - \rho U_1 v' \, dX ,$$

and from equation (30) we obtain

$$\left(\frac{\rho}{\rho_1}\right) = \left[1 + \frac{\gamma - 1}{2\gamma} \frac{\rho_1}{p_1}\left(U_1^2 - q^2\right)\right]^{\frac{1}{\gamma-1}} ,$$

$$\simeq 1 + \frac{1}{2}\left(\frac{\rho_1}{\gamma p_1}\right)\left(U_1^2 - q^2\right) ,$$

$$= 1 + \frac{1}{2}\left(\frac{1}{a^2}\right)\left(U_1^2 - U_1^2 - 2u'U_1^2 - u'^2U_1^2 - v'^2U_1^2\right) ,$$

$$\simeq 1 - u' M_1^2 ,$$ as u'^2 and v'^2 can be neglected when compared with u'.

$$\therefore d\psi \simeq \rho_1 U_1 \left[1 - u' M_1^2 + u' - u'^2 M_1^2 \right] dY - \rho_1 U_1 \left[v' - u'v' M_1^2 \right] dX ,$$

$$= \rho_1 U_1 \left[1 + u'(1 - M_1^2) - u'^2 M_1^2 \right] dY - \rho_1 U_1 v' \left[1 - u' M_1^2 \right] dX ,$$

In the coefficient of dY the second and third terms can be neglected as they are of magnitude ε^3 compared with the first term. In the coefficient of dX the second term can be neglected as it is of higher order than the first.

$$\therefore \quad d\psi \simeq \rho_1 U_1 \, dY - \rho_1 U_1 v' \, dX .$$

Now along the porous section from E to F dY = 0 and from equation (33) we obtain

$$v' = \frac{\eta}{(\gamma+1)M_1^2}\left[k_1 \tau^{2/3} - k_2\right].$$

$$\therefore \quad d\psi = -\frac{\rho U_1 \eta}{(\gamma+1)M_1^2}\left[k_1 \tau^{2/3} - k_2\right] dX ,$$

$$= -\frac{\rho_1 U_1 \eta}{(\gamma+1)M_1^2}\left[k_1 \tau^{2/3} - k_1 \tau_1^{2/3} - k_3\right] dX ,$$

$$= -\frac{\rho_1 U_1}{(\gamma+1)M_1^2}\left[\alpha v_0^{1/3} \tau^{2/3} - \alpha v_0^{1/3} \tau_1^{2/3} - \beta v_0\right] dX ,$$

$$= -\frac{\rho_1 U_1}{(\gamma+1)M_1^2}\left[\alpha . v_0 \left(\frac{\tau}{v_0}\right)^{2/3} - \alpha v_0 \left(\frac{\tau_1}{v_0}\right)^{2/3} - \beta v_0\right] dX ,$$

$$= -\rho_1 U_1 \delta \left[\alpha . \left(\frac{\tau}{v_0}\right)^{2/3} - \alpha \left(\frac{\tau_1}{v_0}\right)^{2/3} - \beta\right] dX .$$

Thus since α, β and δ are of first order it follows that the change in ψ across the finite porous section is of second order, and may be neglected within the order of approximation in the main body of the thesis.

APPENDIX II

Table 1. Values of $\int_{\tau_4/\nu_0}^{\tau_3/\nu_0} z^{2+2p}\, I_{1/3}(n\pi z)\, dz$.

n \ p	0	1	2	3
1	5.9071×10^{-3}	3.991×10^{-4}	2.826×10^{-5}	2.088×10^{-6}
2	1.0395×10^{-2}	7.124×10^{-4}	5.113×10^{-5}	3.827×10^{-6}
3	1.9328×10^{-2}	1.349×10^{-3}	9.845×10^{-5}	7.484×10^{-6}
4	3.8084×10^{-2}	2.708×10^{-3}	2.012×10^{-4}	1.554×10^{-5}
5	7.8239×10^{-2}	5.668×10^{-3}	4.283×10^{-4}	
6	1.6563×10^{-1}	1.222×10^{-2}	9.388×10^{-4}	
7	3.6072×10^{-1}	2.706×10^{-2}	2.110×10^{-3}	
8	7.9199×10^{-1}	6.048×10^{-2}		
9	1.7755	1.378×10^{-1}		
10	4.0346	3.178×10^{-1}		
11	9.2765	7.411×10^{-1}		
12	2.1566×10			
13	5.0590×10			
14	1.1947×10^{2}			
15	2.8575×10^{2}			
16	6.8553×10^{2}			

Table 2. Values of $\int_{\tau_4/\nu_0}^{\tau_3/\nu_0} z^{4/3+2p} I_{\frac{1}{3}}(n\pi z)\, dz$

n \ p	0	1	2	3
1.	1.3442×10^{-2}	8.986×10^{-4}	6.320×10^{-5}	4.656×10^{-6}
2	2.3584×10^{-2}	1.601×10^{-3}	1.142×10^{-4}	
3	4.369×10^{-2}	3.024×10^{-3}	2.197×10^{-4}	
4	8.5793×10^{-2}	6.060×10^{-3}	4.486×10^{-4}	
5	1.7575×10^{-1}	1.266×10^{-2}	9.545×10^{-4}	
6	3.7120×10^{-1}	2.727×10^{-2}		
7	8.0686×10^{-1}	6.080×10^{-2}		
8	1.7692	1.348×10^{-1}		
9	3.9619			
10	8.9964			
11	2.0674×10			
12	4.8047×10			
13	1.1269×10^{2}			
14	2.6609×10^{2}			
15	6.3646×10^{2}			
16	1.5270×10^{3}			

Table 3. Values of $\int_{\tau_4/\nu_0}^{\tau_3/\nu_0} z^{2+2p} K_{\frac{1}{3}}(n\pi z)\,dz$.

n \ p	0	1	2	3
1	3.7684×10^{-3}	2.434×10^{-4}	1.652×10^{-5}	1.173×10^{-6}
2	1.2422×10^{-3}	7.857×10^{-5}	5.228×10^{-6}	3.643×10^{-7}
3	4.6873×10^{-4}	2.905×10^{-5}	1.895×10^{-6}	
4	1.8819×10^{-4}	1.143×10^{-5}	7.310×10^{-7}	
5	7.8433×10^{-5}	4.664×10^{-6}	2.926×10^{-7}	
6	3.3768×10^{-5}	1.967×10^{-6}	1.210×10^{-7}	
7	1.4819×10^{-5}	8.443×10^{-7}	5.088×10^{-8}	
8	6.6311×10^{-6}	3.702×10^{-7}		
9	3.0079×10^{-6}	1.642×10^{-7}		
10	1.3882×10^{-6}	7.423×10^{-8}		
11	6.4491×10^{-7}	3.373×10^{-8}		
12	3.0421×10^{-7}			
13	1.4424×10^{-7}			
14	6.9367×10^{-8}			
15	3.4979×10^{-8}			
16	1.6127×10^{-8}			
17	8.1428×10^{-9}			
18	4.0771×10^{-9}			

Table 4. Values of $\int_{\tau_4/\nu_0}^{\tau_3/\nu_0} z^{4/3+2p} K_{\frac{1}{3}}(n\pi z)\,dz$

n \ p	0	1	2	3
1	8.6957×10^{-3}	5.529×10^{-4}	3.713×10^{-5}	2.621×10^{-6}
2	2.8865×10^{-3}	1.794×10^{-4}	1.179×10^{-5}	8.157×10^{-7}
3	1.0985×10^{-3}	6.672×10^{-5}	4.290×10^{-6}	
4	4.4516×10^{-4}	2.642×10^{-5}	1.662×10^{-6}	
5	1.8730×10^{-4}	1.086×10^{-5}	6.689×10^{-7}	
6	8.1458×10^{-5}	4.619×10^{-6}		
7	3.6125×10^{-5}	2.001×10^{-6}		
8	1.6347×10^{-5}	8.855×10^{-7}		
9	7.5067×10^{-6}	3.971×10^{-7}		
10	3.5044×10^{-6}	1.814×10^{-7}		
11	1.6476×10^{-6}			
12	7.8599×10^{-7}			
13	3.7686×10^{-7}			
14	1.8323×10^{-7}			
15	9.2814×10^{-8}			
16	4.3358×10^{-8}			
17	2.2142×10^{-8}			
18	1.1182×10^{-8}			

TABLE 5 VALUES OF F(p)

p	0	1	2	3	4	5
$1-2^{-5/3-2p}$	6.8502×10^{-1}	9.2125×10^{-1}	9.8031×10^{-1}	9.9508×10^{-1}	9.9877×10^{-1}	9.9969×10^{-1}
2^{3-2p}	8	2	5×10^{-1}	1.25×10^{-1}	3.125×10^{-2}	7.8125×10^{-3}
$\Gamma\left(2p+\frac{2}{3}\right)$	1.3541	1.5046	1.4711×10	3.8903×10^{2}	1.9884×10^{4}	1.6658×10^{6}
$\zeta\left(2p+\frac{5}{3}\right)$	2.1320	1.1090	1.0223	1.0052	1.0013	1.0003
p!	1	1	2	6	2.4×10	1.2×10^{2}
$\Gamma(p+\frac{1}{3})$	2.6789	8.9297×10^{-1}	1.1906	2.7781	9.2604	4.0128×10
F(p)	5.9057	-3.4429	3.0957	-2.9181	2.7960	-2.7025

p	6	7	8	9	10	11
$1-2^{-5/3-2p}$	9.9992×10^{-1}	9.9998×10^{-1}	1.0000	1.0000	1.0000	1.0000
2^{3-2p}	1.9531×10^{-3}	4.8828×10^{-4}	1.2207×10^{-4}	3.0518×10^{-5}	7.6294×10^{-6}	1.9074×10^{-6}
$\Gamma(2p+\frac{2}{3})$	2.0730×10^{8}	3.5887×10^{10}	8.2460×10^{12}	2.4280×10^{15}	8.9134×10^{17}	3.9912×10^{20}
$\zeta(2p+\frac{5}{3})$	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
p!	7.2×10^{2}	5.04×10^{3}	4.032×10^{4}	3.6288×10^{5}	3.6288×10^{6}	3.9917×10^{7}
$\Gamma(p+\frac{1}{3})$	2.1402×10^{2}	1.3554×10^{3}	9.9399×10^{3}	8.2833×10^{4}	7.7310×10^{5}	7.9889×10^{6}
F(p)	2.6274	-2.5652	2.5122	-2.4653	2.4239	-2.3868

TABLE 6

$$\sum_{n=1}^{\infty} R(n) \left(\frac{r}{2v_o}\right)^{2n} \quad \text{for } r = r_1 \text{ and } r_2$$

n	1	2	3	4	5	6	7	8
$2n - \frac{1}{3}$	1.6667	3.6667	5.6667	7.6667	9.6667	11.6667	13.6667	15.6667
$2^{2n-\frac{1}{3}} - 1$	2.1748	1.1699×10	4.9797×10	2.0219×10^{2}	8.1175×10^{2}	3.2500×10^{3}	1.3003×10^{4}	5.2015×10^{4}
$\Gamma\left(n - \frac{1}{6}\right)$	1.1287	9.4060×10^{-1}	1.7240	4.8860	1.8730×10	9.0520×10	5.2810×10^{2}	3.6080×10^{3}
$\zeta(2n - \frac{1}{3})$	2.1320	1.1090	1.0223	1.0052	1.0013	1.0003	1.0001	1.0000
$n!$	1	2	6	2.4×10	1.2×10^{2}	7.2×10^{2}	5.04×10^{3}	$4.032 \times 10^{[illegible]}$
$R(n)$	5.2334	-6.1018	1.4628×10	-4.1377×10	1.2687×10^{2}	-4.0872×10^{2}	1.362×10^{3}	-4.6545×10^{3}
$\left(\frac{r_1}{2v_o}\right)^{2n}$	1.0132×10^{-1}	1.0266×10^{-2}	1.0402×10^{-3}	1.0539×10^{-4}	1.0678×10^{-5}	1.0819×10^{-6}	1.0962×10^{-7}	1.1107×10^{-8}
$\left(\frac{r_2}{2v_o}\right)^{2n}$	6.25×10^{-4}	3.906×10^{-7}						
$R(n).\left(\frac{r_1}{2v_o}\right)^{2n}$	0.5302	-0.0626	0.0152	-0.0044	0.0014	-0.0004	0.0001	-0.0001
$R(n).\left(\frac{r_2}{2v_o}\right)^{2n}$	0.003271	-0.000003	0.000000					

$$\therefore \quad \sum_{n=1}^{\infty} R(n).\left(\frac{r_1}{2v_o}\right)^{2n} = \underline{\underline{0.4794}},$$

$$\text{and} \quad \sum_{n=1}^{\infty} R(n).\left(\frac{r_2}{2v_o}\right)^{2n} = \underline{\underline{0.003268}}.$$

TABLE 7 $\sum_{p=0}^{\infty} F(p) . \frac{1}{\left(2p + \frac{8}{3}\right)} \left[z^{2p+\frac{8}{3}} \right]_{r_4/v_o}^{r_3/v_o}$

p	0	1	2	3	4
$2p + \frac{8}{3}$	2.6667	4.6667	6.6667	8.6667	10.6667
$\left(\frac{r_4}{v_o}\right)^{2p+\frac{8}{3}}$	1.3680×10^{-2}	5.4719×10^{-4}	2.1887×10^{-5}	8.7550×10^{-7}	3.5020×10^{-8}
$\left(\frac{r_3}{v_o}\right)^{2p+\frac{8}{3}}$	4.0333×10^{-2}	3.6230×10^{-3}	3.2670×10^{-4}	2.9402×10^{-5}	2.6462×10^{-6}
$\left[z^{2p+\frac{8}{3}} \right]_{r_4/v_o}^{r_3/v_o}$	2.6653×10^{-2}	3.0758×10^{-3}	3.0481×10^{-4}	2.8526×10^{-5}	2.6112×10^{-6}
$\left(\frac{F(p)}{2p+\frac{8}{3}}\right) \left[z^{2p+\frac{8}{3}} \right]_{r_4/v_o}^{r_3/v_o}$	.05903	-.00227	.00014	-.00001	.00000

$$\therefore \sum_{p=0}^{\infty} F(p) . \frac{1}{\left(2p + \frac{8}{3}\right)} \left[z^{2p+8/3} \right]_{r_4/v_o}^{r_3/v_o} = \underline{\underline{.05689}}$$

TABLE 8 $\sum_{p=0}^{\infty} F(p).\ \frac{1}{(2p+2)} \left[z^{2p+2} \right]_{r_4/v_o}^{r_3/v_o}$

p	0	1	2	3
$2p + 2$	2	4	6	8
$\left(\frac{r_4}{v_o}\right)^{2p+2}$	4×10^{-2}	1.6×10^{-3}	6.4×10^{-5}	2.56×10^{-6}
$\left(\frac{r_3}{v_o}\right)^{2p+2}$	9×10^{-2}	8.1×10^{-3}	7.29×10^{-4}	$6{,}561 \times 10^{-5}$
$\left[z^{2p+2} \right]_{r_4/v_o}^{r_3/v_o}$	5×10^{-2}	6.5×10^{-3}	6.65×10^{-4}	6.305×10^{-5}
$F(p)\ \frac{1}{2p+2} \left[z^{2p+2} \right]_{r_4/v_o}^{r_3/v_o}$	0.1476	-0.0056	0.0003	-0.0000

$$\therefore \quad \sum_{p=0}^{\infty} F(p)\ \frac{1}{(2p+2)} \left[z^{2p+2} \right]_{r_4/v_o}^{r_3/v_o} = \underline{\underline{0.1423}}$$

TABLE 9. $\sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \cdot F(p) \cdot I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{2+2p} K_{\frac{1}{3}}(n\pi z)\,dz$

Values of $n \cdot F(p) \cdot I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{2+2p} K_{\frac{1}{3}}(n\pi z)\,dz$

n \ p	0	1	2	3	4
1	.04613	-.00174	.00011	-.00001	.00000
2	.02021	-.00075	.00004	-.00000	
3	.00951	-.00034	.00002		
4	.00469	-.00017	.00001		
5	.00240	-.00008	.00000		
6	.00127	-.00004			
7	.00069	-.00002			
8	.00038	-.00001			
9	.00021	-.00001			
10	.00012	-.00000			
11	.00007				
12	.00004				
13	.00002				
14	.00001				
15	.00001				
16	.00000				
Col. Totals	.08576	-.00316	.00018	-.00001	.00000

$$\therefore \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \cdot F(p) \cdot I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{2+2p} K_{\frac{1}{3}}(n\pi z)\,dz = .08277$$

TABLE 10 $\sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \cdot F(p) \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{2+2p} I_{1/3}(n\pi z)\, dz.$

Values of $n \cdot F(p) \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{2+2p} I_{1/3}(n\pi z)\, dz.$

n \ p	0	1	2	3	4
1	.004354	-.000171	.000011	-.000001	.000000
2	.001440	-.000058	.000004	-.000000	
3	.000441	-.000018	.000001		
4	.000135	-.000006	.000000		
5	.000042	-.000002			
6	.000013	-.000001			
7	.000004	-.000000			
8	.000001				
9	.000000				
Col. Totals	.006430	-.000256	.000016	-.000001	.000000

$$\therefore \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \cdot F(p) \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{2+2p} I_{1/3}(n\pi z)\, dz = \underline{.006189}$$

TABLE 11 $\sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \cdot F(p) \, I_{-2/3}\left(\frac{n\pi r_2}{\nu_0}\right) \int_{r_4/\nu_0}^{r_3/\nu_0} z^{4/3+2p} K_{\frac{1}{3}}(n\pi z)\, dz$

Values of $n \cdot F(p) \, I_{-2/3}\left(\frac{n\pi r_2}{\nu_0}\right) \int_{r_4/\nu_0}^{r_3/\nu_0} z^{4/3+2p} K_{\frac{1}{3}}(n\pi z)\, dz$

n \ p	0	1	2	3
1	.1065	-.0039	.0002	.0000
2	.0470	-.0017	.0001	
3	.0223	-.0008	.0000	
4	.0111	-.0004		
5	.0057	-.0002		
6	.0030	-.0001		
7	.0017	-.0001		
8	.0009	-.0000		
9	.0005			
10	.0003			
11	.0002			
12	.0001			
13	.0001			
14	.0000			
Col. Totals	.1994	-.0072	.0003	-.0000

$$\therefore \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n \cdot F(p) \, I_{-2/3}\left(\frac{n\pi r_2}{\nu_0}\right) \int_{r_4/\nu_0}^{r_3/\nu_0} z^{4/3+2p} K_{\frac{1}{3}}(n\pi z)\, dz = \underline{0.1925}$$

TABLE 12

$$\sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n.\ F(p).\ K_{2/3}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3+2p}\, I_{1/3}(n\pi z)\, dz$$

Values of $n.\ F(p).\ K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3+2p}\, I_{\frac{1}{3}}(n\pi z)\, dz$

n \ p	0	1	2	3
1	.00991	-.00039	.00002	-.00000
2	.00327	-.00013	.00001	
3	.00100	-.00004	.00000	
4	.00030	-.00001		
5	.00009	-.00000		
6	.00003			
7	.00001			
8	.00000			
Col. Totals	.01461	-.00057	.00003	-.00000

$$\therefore \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} n\ F(p).\ K_{2/3}\left(\frac{n\pi r_1}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^{4/3+2p}\, I_{1/3}(n\pi z)\, dz = \underline{.01407}$$

TABLE 13 $$\sum_{n=1}^{\infty} n \cdot I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^2 K_{\frac{1}{3}}(n\pi z)\, dz$$

n	$n\, I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^2 K_{\frac{1}{3}}(n\pi z)\, dz$
1	.007811
2	.003423
3	.001610
4	.000795
5	.000407
6	.000215
7	.000116
8	.000064
9	.000036
10	.000020
11	.000011
12	.000007
13	.000004
14	.000002
15	.000001
16	.000001
17	.000000

$$\therefore \sum_{n=1}^{\infty} n \cdot I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right)\int_{r_4/v_0}^{r_3/v_0} z^2 K_{\frac{1}{3}}(n\pi z)\, dz = \underline{.01452}$$

TABLE 14

$$\sum_{n=1}^{\infty} n \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^2 \, I_{\frac{1}{3}}(n\pi z)\, dz$$

n	$n \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^2 \, I_{\frac{1}{3}}(n\pi z)\, dz$
1	.0007372
2	.0002439
3	.0000747
4	.0000229
5	.0000071
6	.0000022
7	.0000007
8	.0000002
9	.0000001
10	.0000000

$$\therefore \sum_{n=1}^{\infty} n \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^2 \, I_{1/3}(n\pi z)\, dz = \underline{.0010890}$$

TABLE 15 $$\sum_{n=1}^{\infty} n \cdot I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3} K_{\frac{1}{3}}(n\pi z)\, dz$$

n	$n\ I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3} K_{\frac{1}{3}}(n\pi z)\, dz$
1	.01803
2	.00795
3	.00377
4	.00188
5	.00097
6	.00052
7	.00028
8	.00016
9	.00009
10	.00005
11	.00003
12	.00002
13	.00001
14	.00001
15	.00000

$$\therefore \sum_{n=1}^{\infty} n \cdot I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3} K_{\frac{1}{3}}(n\pi z)\, dz = \underline{.03377}$$

TABLE 16

$$\sum_{n=1}^{\infty} n \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3} I_{1/3}(n\pi z)\, dz$$

n	$n\, K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3} I_{1/3}(n\pi z)\, dz$
1	.001677
2	.000553
3	.000169
4	.000052
5	.000016
6	.000005
7	.000002
8	.000001
9	.000000

$$\therefore \sum_{n=1}^{\infty} n \cdot K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \int_{r_4/v_0}^{r_3/v_0} z^{4/3} I_{1/3}(n\pi z)\, dz = .002475$$

TABLE 17 $\sum_{n=1}^{\infty}\left[\left(\frac{r_2}{r_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{r_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r_4}{r_0}\right) - \left(\frac{r_1}{r_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{r_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r_4}{r_0}\right)\right]$

n	(1) $\left(\frac{r_2}{r_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{r_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r_4}{r_0}\right)$	(2) $\left(\frac{r_1}{r_0}\right)^{2/3} K_{2/3}\left(\frac{n\pi r_1}{r_0}\right) I_{-2/3}\left(\frac{n\pi r_4}{r_0}\right)$	(1) - (2)
1	0.2618	0.0975	0.1643
2	0.0613	0.0105	0.0525
3	0.0225	0.0017	0.0208
4	0.0095	0.0003	0.0092
5	0.0044	0.0001	0.0043
6	0.0022	0.0000	0.0022
7	0.0011		0.0011
8	0.0006		0.0006
9	0.0003		0.0003
10	0.0002		0.0002
11	0.0001		0.0001
12	0.0001		0.0001
13	0.0000		0.0000

$$\therefore \sum_{n=1}^{\infty}\left[\left(\frac{r_2}{r_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{r_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r_4}{r_0}\right) - \left(\frac{r_1}{r_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{r_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r_4}{r_0}\right)\right] = \underline{0.2557}$$

TABLE 18 $\sum_{n=1}^{\infty}\left[\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r_3}{v_0}\right) - \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r_3}{v_0}\right)\right]$

	(1)	(2)	
n	$\left(\frac{r_2}{v_0}\right)^{2/3} I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right) K_{2/3}\left(\frac{n\pi r_3}{v_0}\right)$	$\left(\frac{r_1}{v_0}\right)^{2/3} K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) I_{-2/3}\left(\frac{n\pi r_3}{v_0}\right)$	(1) - (2)
1	0.1522	0.0981	0.0541
2	0.0270	0.0158	0.0112
3	0.0070	0.0037	0.0033
4	0.0022	0.0010	0.0012
5	0.0007	0.0003	0.0004
6	0.0003	0.0001	0.0002
7	0.0001	0.0000	0.0001
8	0.0000		0.0000

$$\therefore \sum_{n=1}^{\infty}\left[\left(\frac{r_2}{v_0}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{n\pi r_2}{v_0}\right) K_{\frac{2}{3}}\left(\frac{n\pi r_3}{v_0}\right) - \left(\frac{r_1}{v_0}\right)^{2/3} K_{\frac{2}{3}}\left(\frac{n\pi r_1}{v_0}\right) I_{-\frac{2}{3}}\left(\frac{n\pi r_3}{v_0}\right)\right] = \underline{0.0705}$$

References for Part II.

1. Woods, L.C. — On the Theory of Two-Dimensional Wind Tunnels with Porous Walls. Proceedings Royal Society of London, Series A, Vol. 233, 1956.

2. Preston, H.H. and Rawcliffe, A.G. — Aero. Res. Comm., No. 9, 1950

3. Roshko, A. — 1954 Nat. Adv. Comm. Aero., Wash., Tech. Noto No. 3168

4. Helliwell, J.B. — Two Dimensional Flow Patterns at High Subsonic Speeds past Wedges in Channels with Parallel Walls.
J. Fluid. Mech., Vol. 3, part 4, Jan., 1958.

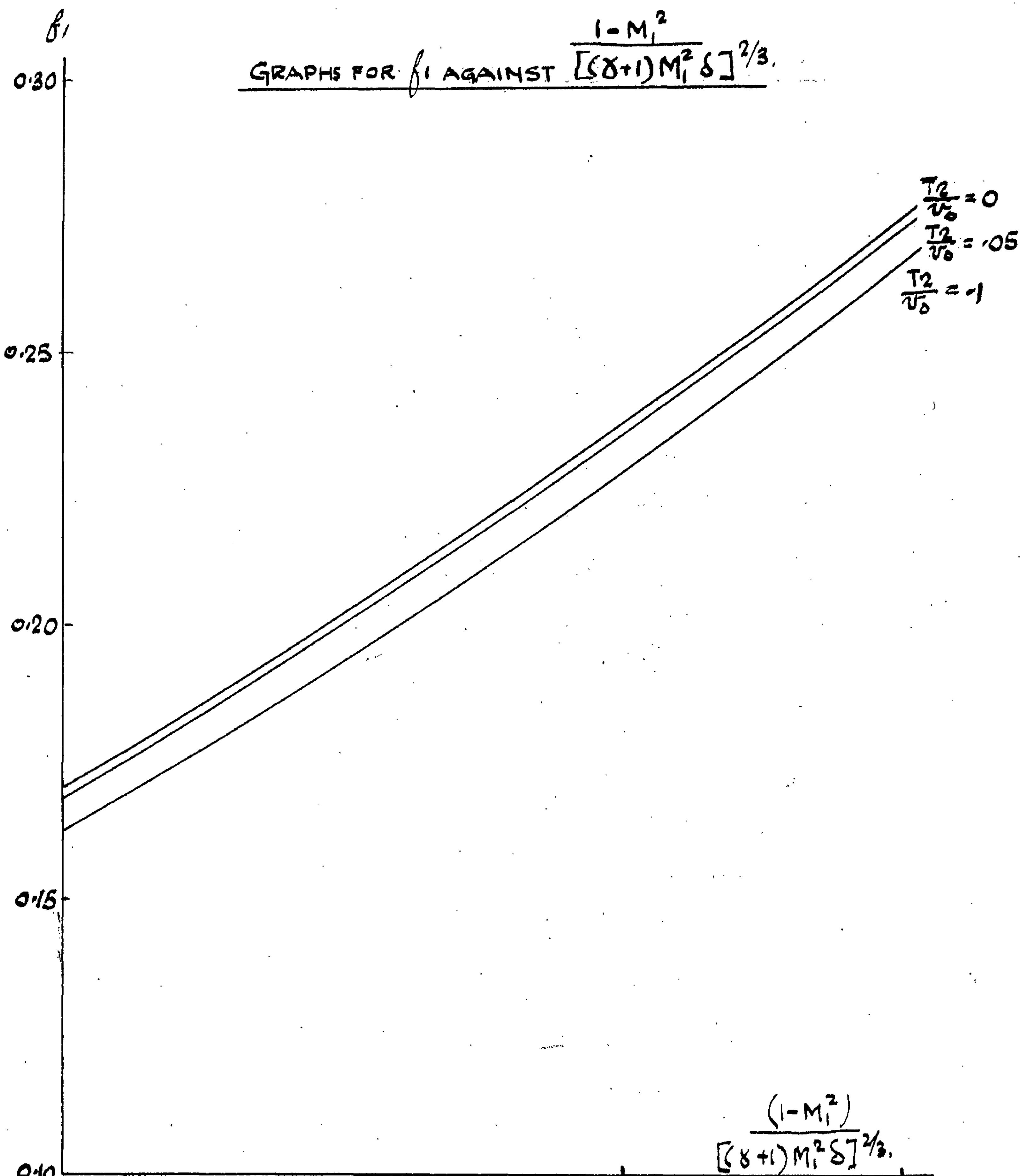
5. Helliwell, J.B. and Mackie, A.G. — Two Dimensional Subsonic and Sonic Flow Past Thin Bodies.
J. Fluid Mech., Vol. 3, part 1, Oct., 1957

6. Maeder, P.F. and Wood, A.D. — Transonic Wind Tunnel Test Sections. Zeitschrift Für Angewandte Mathematik Und Physik, Vol. VII, 1956

7. Erdelyi, A. (Ed) — Higher Transcendental Functions. Vol. II McGraw-Hill Book Company Inc. 1953.

8. ~~Erdelyi, A. (Ed.)~~ — Tables of Bessel Functions of Fractional Order. Vol. II. New York, Columbia University Press, 1949

9. Bers, L. — Mathematical Aspects of Subsonic and Transonic Gas Dynamics.
John Wiley and Sons, Inc. New York, 1958



f_1
GRAPHS FOR f_1 AGAINST $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$.
0·30
0·25
0·20
0·15
0·10
$\frac{T_2}{v_0}=0$
$\frac{T_2}{v_0}=\cdot05$
$\frac{T_2}{v_0}=\cdot1$
$\frac{(1-M_1^2)}{[(\gamma+1)M_1^2\delta]^{2/3}}$.
0·8
0·9
0·95

FIGURE 1

GRAPHS FOR f_2 AGAINST $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$.

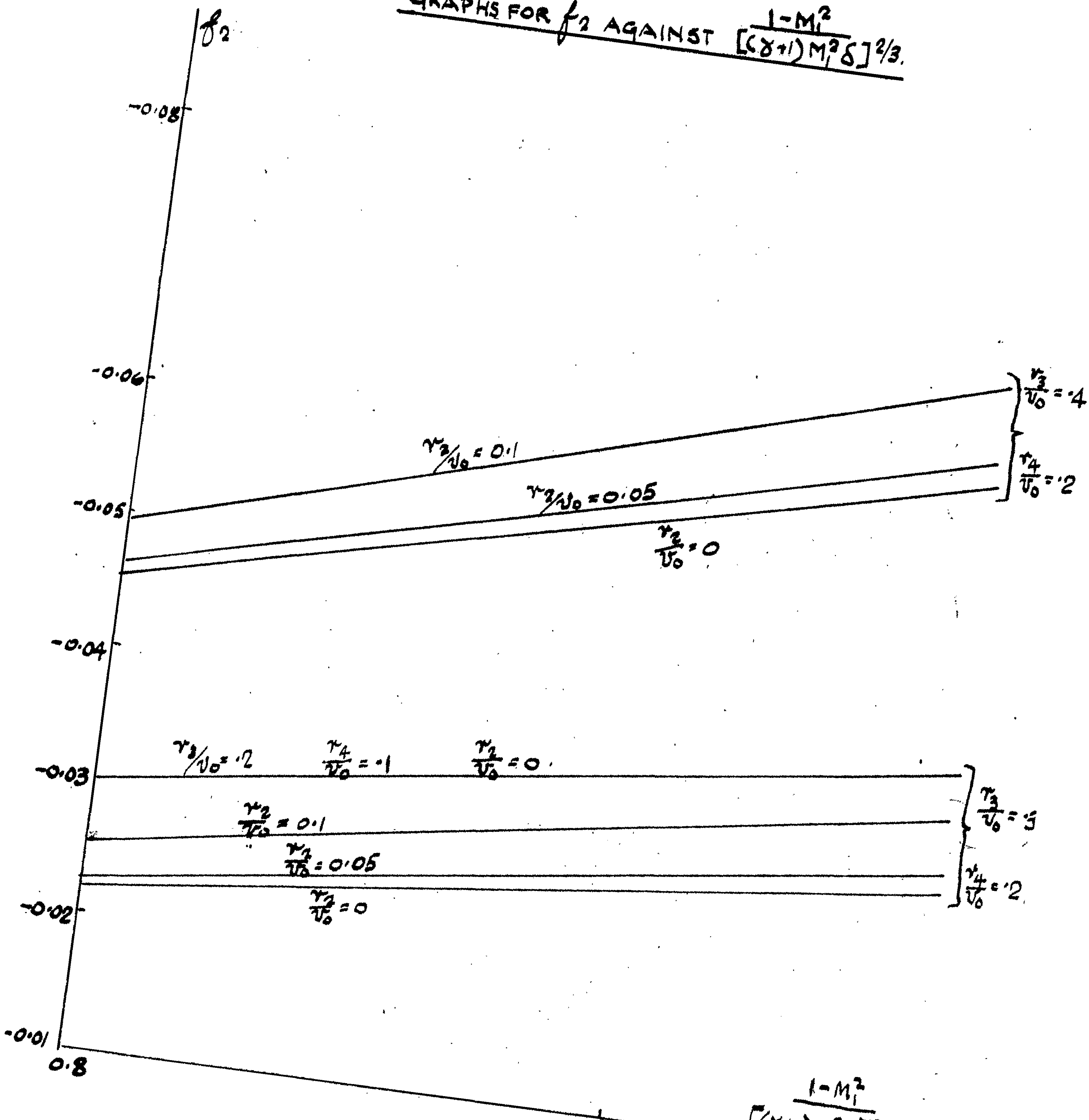
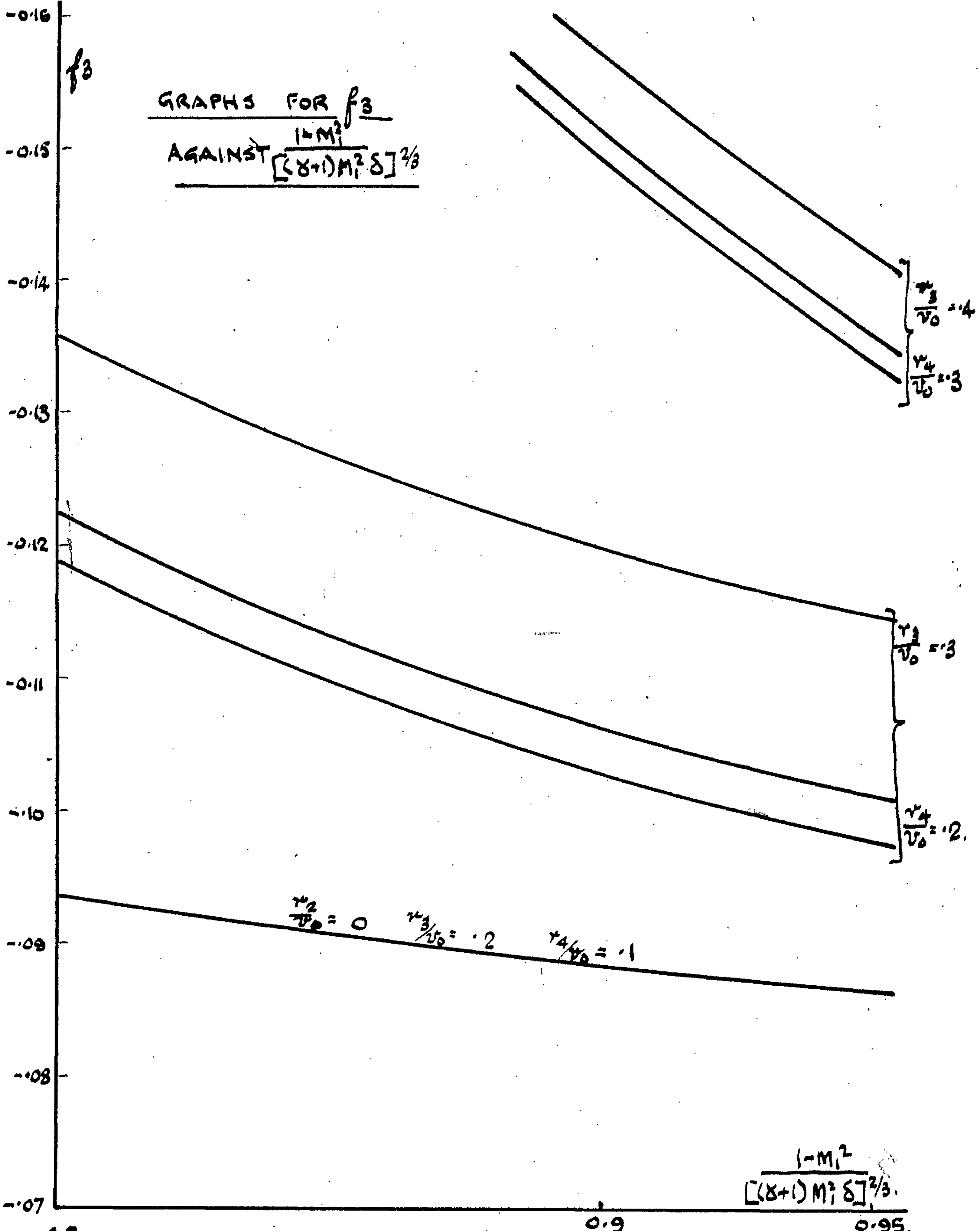


FIGURE 2.



-0.16
-0.15
-0.14
-0.13
-0.12
-0.11
-.10
-.09
-.08
-.07
0.8
0.9
0.95.
f3
GRAPHS FOR f3
AGAINST $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$
$r_3/v_0 = .4$
$r_4/v_0 = .3$
$r_3/v_0 = .3$
$r_4/v_0 = .2$
$r_2/v_0 = 0$ $r_3/v_0 = .2$ $r_4/v_0 = .1$
$\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$

FIGURE 3.

GRAPHS OF $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_2}$ AGAINST $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$.

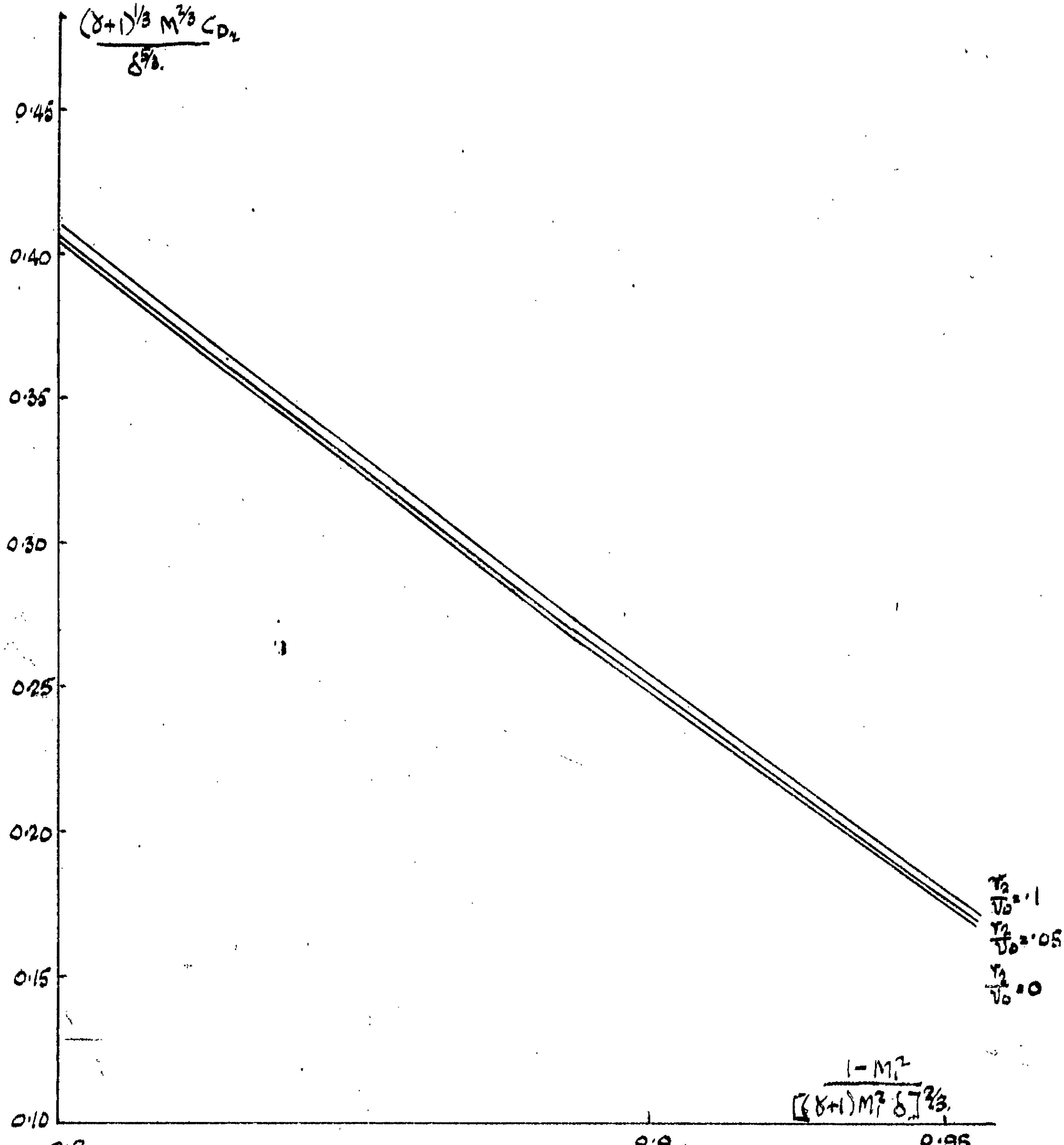


FIGURE 4

GRAPHS OF $\frac{(\gamma+1)^{1/3} M_1^{2/3} C_{D_3}}{\delta^{5/3}}$ AGAINST $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$

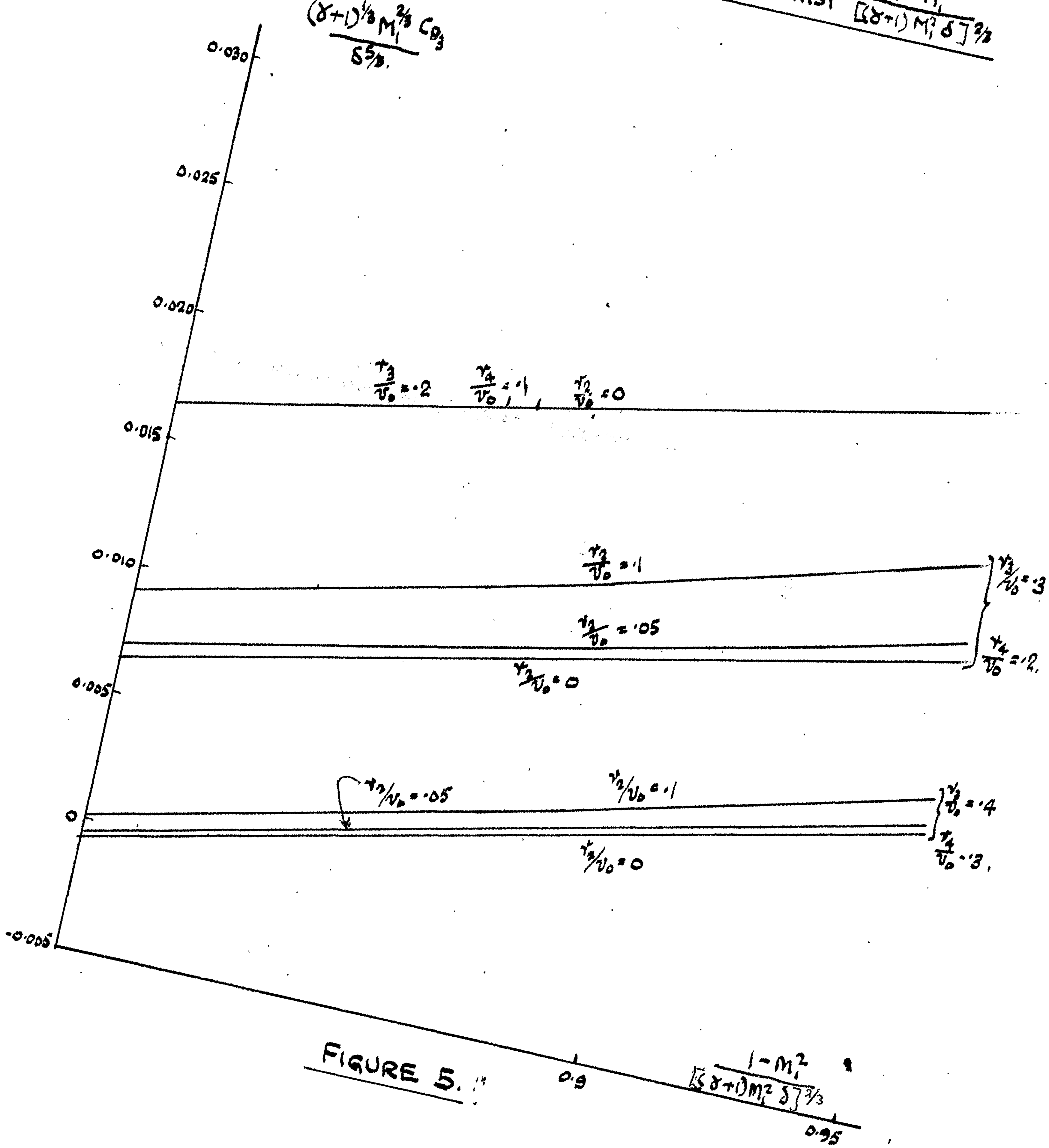


FIGURE 5.

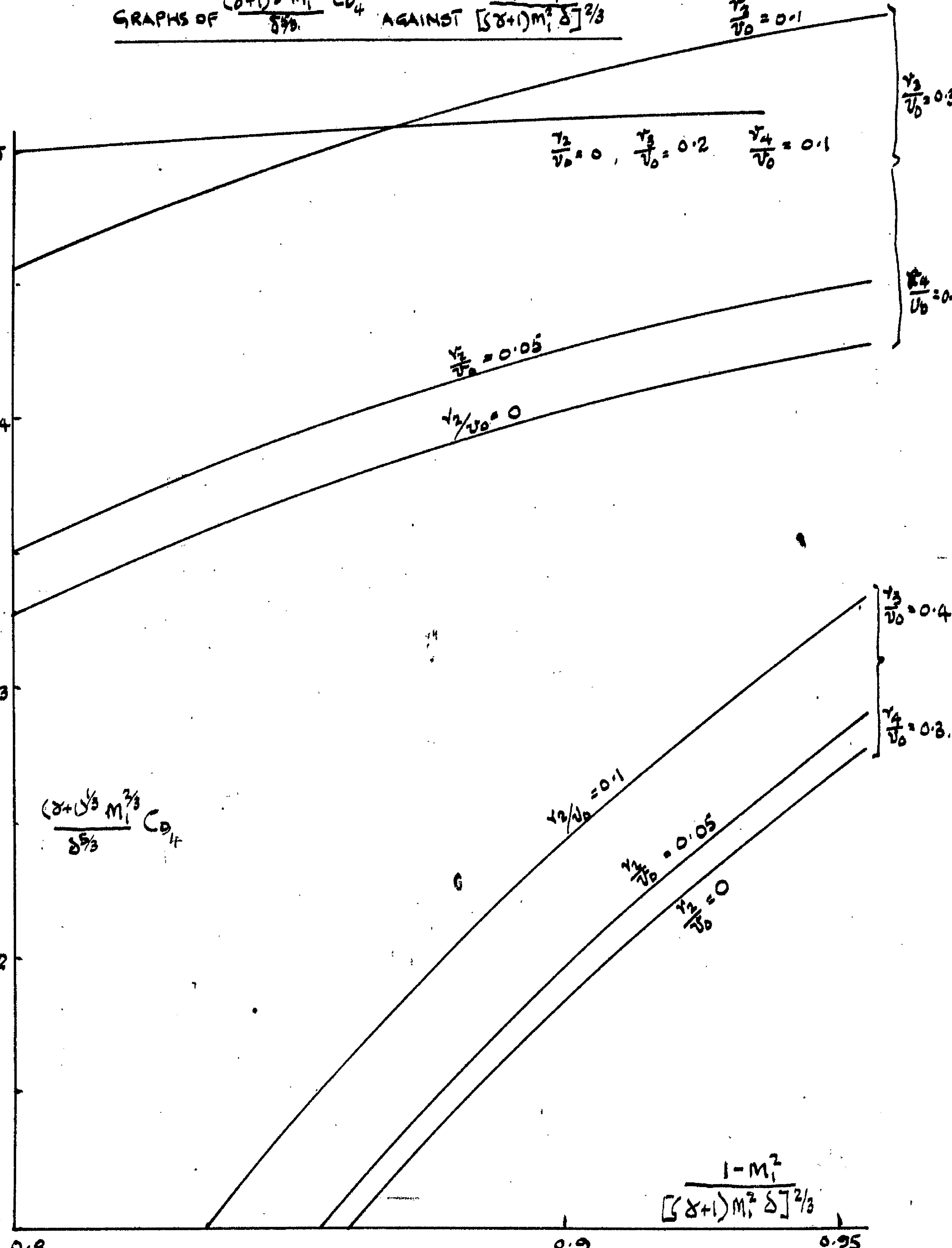

GRAPHS OF $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_4}$ AGAINST $[(\gamma+1)M_1^2\delta]^{2/3}$
$\frac{v_3}{v_0} = 0.1$
$\frac{v_2}{v_0} = 0$, $\frac{v_3}{v_0} = 0.2$ $\frac{v_4}{v_0} = 0.1$
$\frac{v_2}{v_0} = 0.05$
$v_2/v_0 = 0$
$\frac{v_3}{v_0} = 0.4$
$\frac{v_4}{v_0} = 0.3$
$v_2/v_0 = 0.1$
$v_2/v_0 = 0.05$
$\frac{v_2}{v_0} = 0$
$\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D_4}$
$\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$
·05
·04
·03
·02
·01
0·8
0·9
0·95

FIGURE 6.

GRAPHS OF $\frac{(\gamma+1)^{1/3} M_1^{x\,2/3}}{\delta^{5/3}}\left[C_D - C_D^x\right]$ AGAINST $\left(\frac{3}{2}\right)^{2/3}\left(\frac{\nu_2}{\upsilon_0}\right)^{2/3}$.

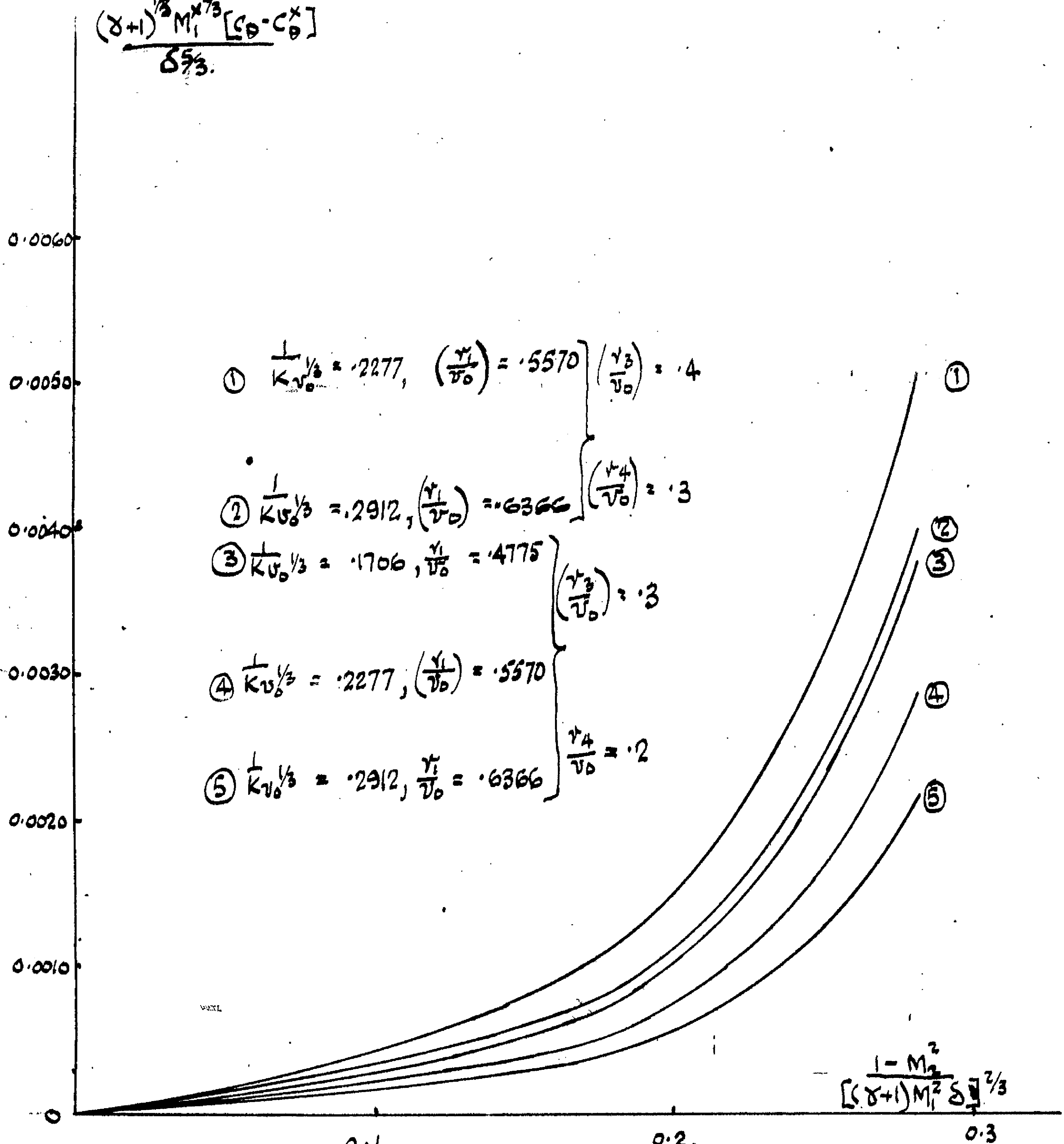


FIGURE 7.

GRAPHS SHOWING POSITIONS OF ENDS OF POROUS SECTION FOR DIFFERENT VALUES OF $\frac{r_1}{v_0}$, $\frac{r_2}{v_0}$, $\frac{r_3}{v_0}$ AND $\frac{r_4}{v_0}$

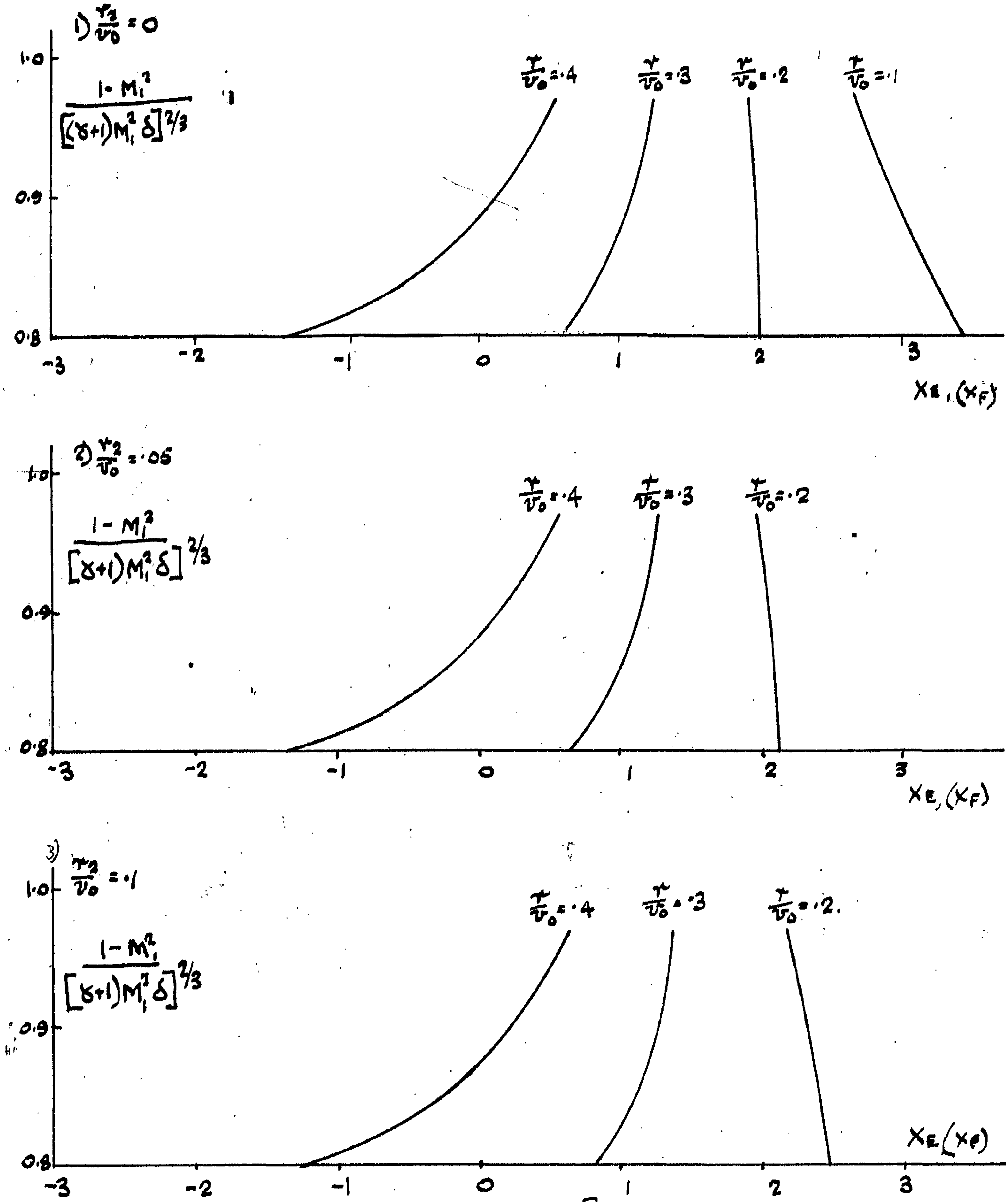


FIGURE 8.

GRAPH OF $\frac{(\gamma+1)^{1/3} M_1^{2/3}}{\delta^{5/3}} C_{D\,FREE}$ AGAINST $\frac{1-M_1^2}{[(\gamma+1)M_1^2\delta]^{2/3}}$

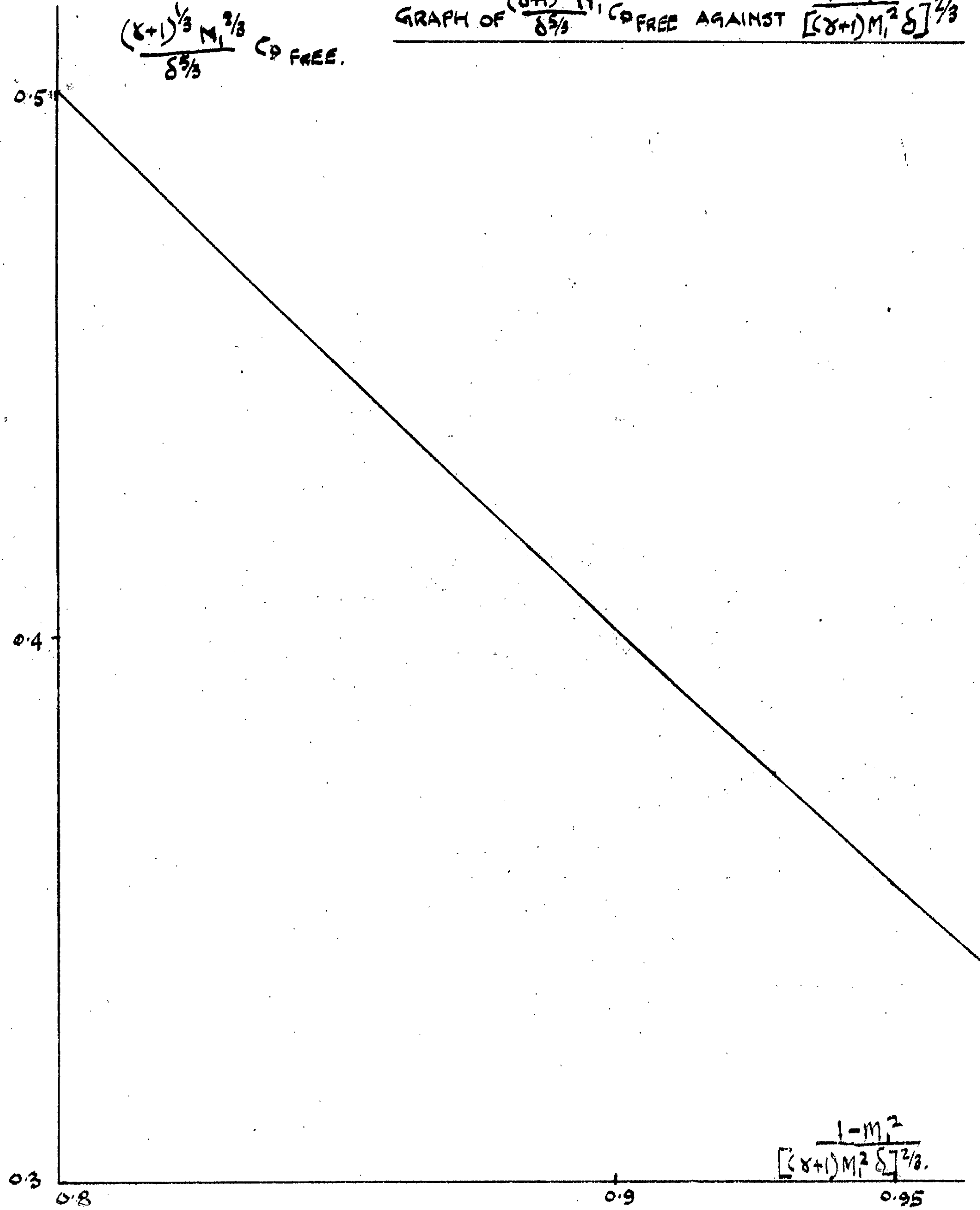


FIGURE 9