# ON MICHELL'S THEOREM AND ITS PLACE IN THE STUDY OF FRAMEWORK OPTIMISATION

by

Alastair E. Caldwell

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of Glasgow

Department of Aeronautics
and Fluid Mechanics

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ProQuest LLC.
789 East Eisenhower Parkway
P.O. Box 1346
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## SUMMARY

The first theorem pertaining to the minimum weight layout of pin-jointed frameworks is due to J. C. Maxwell and shows that any framework which consists entirely of tension bars or entirely of compression bars is an optimum. However certain conditions must be satisfied by the external force system if such a framework is to be possible. Following Maxwell, A. G. M. Michell derived a theorem which gives sufficient conditions for a framework containing both tension and compression members to be an optimum: the bars must lie along lines of constant principal strain in a compatible virtual deformation of space, with the sign of the strain agreeing with the sign of the load in the bar. Displacement, strain and compatibility of strain are physical concepts which can be described by a mathematical model involving first and second order tensors. In Michell's theorem it is the mathematical rather than the physical model which is significant; this point is particularly important in connection with the optimum layout of frameworks on plane and curved surfaces.

Lines of principal strain can, in general, be represented by three orthogonal families of curves in three dimensional space. The general properties of orthogonal curvilinear systems, and the particular properties of those which define layouts for minimum weight frameworks, are readily derived by means of the tensor calculus. The equilibrium equations for a continuum can be adapted to apply to the optimum frameworks and, again, tensor calculus is the most convenient analytical technique. The layout and equilibrium equations so derived apply to optimum frameworks both in three dimensional space and on plane and curved surfaces.

The technique of linear programming provides another approach to the minimum weight design of pin-jointed frameworks. The method is more direct in that it finds an optimum framework for a given force system but is restricted by the fact that the framework nodes are selected from a predetermined set: however, given the restricted nodal pattern, the optimum framework still satisfies the Michell criterion. The formulation of the framework design problem as a linear program enables some simple buckling constraints to be included and also allows fixed points of reaction to be specified, so that the external force system is in part dependent on the form of the framework (this represents a departure from the conditions imposed in the theorems of Maxwell and Michell, in which the bars are all assumed to be at either the limiting compressive or tensile stress, these being taken as material constants, and the external force system is completely specified). Linear programming problems can be efficiently solved on a digital computer, although, in the case of the framework problem, a considerable amount of extra computation is involved if the data is to be read in, and the results printed out, in a convenient form.

Further developments in optimum layout of frameworks have been along the lines of mathematical programming, of which linear programming is a simple case, and away from Michell's theorem, which remains a unique and still fascinating contribution to structural design.

# CONTENTS

# CHAPTER 1

## INTRODUCTION

### THE DESIGN PROBLEM

Whereas in structural analysis both the externally applied force system and the structure supporting this system are completely defined and the object is to determine the stress distribution in the structure, in a design problem, the final form of the structure and its stress distribution have both to be determined. The extra variables introduced in this way require that more conditions than the usual ones of equilibrium and compatibility be supplied if the problem is to have a unique solution. The extra conditions are firstly that the stresses lie within prescribed safe limits and secondly that the structure be in some respect an optimum. The sense in which the structure is an optimum, and the degree to which it is initially undefined, are the main factors influencing the method of solution of the problem. In this particular study, optimum is synonymous with minimum weight, or minimum volume, and the only information given about the structure is that it is a pin-jointed framework of elastic, as opposed to plastic, design: the criterion of minimum weight is to be applied to determine the layout of the framework.

### HISTORICAL BACKGROUND

The first step towards the solution of this problem was taken by Maxwell (reference 1) in the latter half of the last century, but the most significant contribution is from Michell (reference 2) whose theorem, published in 1904, gives sufficient conditions for any pin-jointed framework to be an optimum, and embraces the theorem of Maxwell as a special case. The importance of this work was not realised until the 1950's when first Cox (reference 3) and then Hemp (reference 4) became interested and made further contributions to the subject. They succeeded in generating a flurry of interest in Michell's theorem both in this country and abroad (reference 5), until, in 1964, the publication of a paper by Dorn, Gomory and Greenberg (reference 13) set a trend in optimum layout studies which is still unbroken and which has largely put a stop to further research along the lines established earlier by Hemp. This paper described a completely original approach, based on a formulation in terms of linear programming and solution by a numerical analysis technique called the Simplex method, developed a few years earlier by Dantzig (reference 14) in an entirely different context. It was recognised by Hemp and Chan (reference 16) that the optimum frameworks derived by linear programming satisfied Michell's criterion and that the necessity of the criterion was thus established.

### SOME GENERAL POINTS

Maxwell's theorem shows that to obtain a minimum volume framework it is only necessary to minimise the volume of compression members or the volume of tension members, from which it follows that any framework which consists entirely of compression members or entirely of tension members is an optimum. The problem then is to determine under what conditions such a framework is possible, and some consideration is given to this in chapter two, where it is shown that the funicular polygon construction is of some relevance in the case of plane force systems. The latter part of chapter two is an account of Michell's theorem, a close look being taken at some of the mathematical relationships involved in its proof. According to the theorem, optimum frameworks have their members arranged along lines of principal strain in a compatible virtual deformation of space. The standard proof, which involves a comparison of the strain energy of various

frameworks subjected to the same virtual deformation, obscures to some extent the exact nature of the optimisation process, and thus a detailed examination of the proof is both interesting and instructive.

The virtual deformations of particular interest as regards Michell's theorem are those in which the principal strains are all of the same magnitude but not all of the same sign, and so the lines of principal strain, which are the lay-out curves for optimum frameworks, form orthogonal sets of curves. The general properties of orthogonal curvilinear systems are derived in chapter three, by means of the tensor calculus, which is the most appropriate mathematical tech-[5] nique in this case because the results appear in a form which makes them particularly suitable for application to the problem of compatibility of strain in chapter four. In the literature on tensor calculus, the main aspects of notation and terminology are standard, with some minor variations depending on individual authors. Where there is lack of uniformity, the conventions adopted herein are as follows:

a) A coordinate is denoted by the letter x with a single superscript e.g. $x^i$, $x^\alpha$, $x^2$.

b) Greek indices refer to surfaces and have the range 1-2; latin indices refer to the enveloping space and have the range 1-3, e.g. $x^i$ is a set of three space coordinates, $x^\alpha$ is a set of two surface coordinates: $[ij,k]$ is a space Christoffel symbol of the first kind, $[\alpha\beta,\gamma]$ is a surface Christoffel symbol of the first kind: $A^i B_i$ is the sum of three terms, $A^\alpha B_\alpha$ is the sum of two terms. When a number is used as an index it will be clear from the context which coordinate system is being employed. This notation is introduced because the theory of surfaces, from which certain results appear in chapter three, does not involve simply an investigation of the metric properties of two-dimensional Riemannian spaces. The idea of an enveloping space, with its own metric properties, is of paramount importance.

c) However certain relations between the fundamental quantities which describe a space can be stated without reference to either the dimensions of the space or the presence or absence of an enveloping space, and in such a case, capital letters are used as indices.

d) It is convenient to introduce subscripts and superscripts which do not possess tensorial significance. These are enclosed in brackets e.g. $g^{(1)}_{\alpha\beta}$ is a second order covariant tensor.

e) When an index representing a power is applied to a quantity with a tensor subscript or superscript, the quantity is enclosed in brackets to avoid confusion.

The brief account of some of the basic ideas of tensor calculus at the beginning of chapter three is intended to introduce and define the quantities to be used in the subsequent analysis and is not to be considered as a logically developed treatise on the subject. Details of standard theorems can be found in references [6, 7 and 8]; where details of a piece of analysis are given, the analysis is not part of the general theory, but has been undertaken specifically in connection with the present study.

As Borg has said in his book "Matrix-Tensor Methods in Continuum Mechanics", the significance of the equations of compatibility of strain is "occasionally a matter of some confusion", and their particular significance in the case of Michell's theorem has tended to be even more obscure than in other applications. Thus the emphasis in chapter four is on the mathematical processes which are

involved, and the point is underlined in Appendix A, where some of the layout equations are re-derived without reference to either strain or compatibility of strain. The layout and equilibrium equations of chapter four represent an extension of work done by Hemp, who first derived the equations appropriate to optimum plane frameworks (reference 4) using a different technique.

The design of optimum frameworks by linear programming, described in chapter five, is relevant to a study centred on Michell's theorem in that it demonstrates that any optimum framework satisfies Michell's criterion and the criterion is thus proved to be not only sufficient but necessary: it also sets the pattern* for further studies in minimum weight design leading to the solution of more realistic problems, and in this respect (and this respect only) is seen to be superior to Michell's approach. The presentation of the ideas in chapters five and seven is intended to show clearly how the formulation of the framework problem as a linear program, involving only equilibrium conditions (and not compatibility conditions) as constraints, is justified ultimately by the form of the solution, which is always a statically determinate framework: in general, interference with the basic problem breaks the logical circle, requiring a re-formulation which inevitably involves non-linear constraints. However there are two ways in which the problem can be elaborated without introducing non-linearities. The first is to allow the limiting compressive stress in framework members to depend on the length of the member, so that simple buckling constraints are effectively introduced: the second is not to specify the complete external force system, but give some forces and some points of reaction. Both devices are used in the computer program described in chapter six. The former, however, does have the obvious deficiency that two short compression members joined end to end are together lighter than one long compression member but are quite unacceptable as a design feature. The computer program detects this and other similar types of instability and tries to find an alternative solution.

The progress of structural optimisation from 1964 to the present day, in so far as it is based on the minimum weight layout of frameworks, is traced in chapter seven, and against the background of knowledge accumulated in this period, Michell's theorem is seen to be still a unique and extremely valuable contribution to the theory of design.

# CHAPTER 2

# THE THEOREMS OF MAXWELL AND MICHELL

## MAXWELL'S THEOREM

Maxwell's theorem states that in any fully-stressed pin jointed framework, the expression $V_{(t)}\sigma_{(t)} - V_{(c)}\sigma_{(c)}$ is a constant depending on the applied forces and their points of application, where $V_{(t)}$ is the total volume of tension members, $V_{(c)}$ the total volume of compression members, and $\sigma_{(t)}$ and $\sigma_{(c)}$ are the limiting stresses in tension and compression respectively, these latter two quantities being properties of the material from which the framework is constructed. Maxwell himself gave three proofs of this theorem (reference 1, pages 175-177) of which a variation of the third one, due to Cox (reference 3, page 83) is presented here.

Let the position vectors of the n nodes of a framework be $\underset{\sim}{r}_{(1)}, \underset{\sim}{r}_{(2)}, \ldots, \underset{\sim}{r}_{(n)}$, and the forces applied at these nodes be $\underset{\sim}{F}_{(1)}, \underset{\sim}{F}_{(2)}, \ldots, \underset{\sim}{F}_{(n)}$. The bar, of length $l_{(lm)}$ and cross sectional area $a_{(lm)}$, which joins nodes $l$ and m exerts on the node $l$ a force $\underset{\sim}{T}_{(lm)}$ given by:

$$\underset{\sim}{T}_{(lm)} = \frac{(\underset{\sim}{r}_{(m)} - \underset{\sim}{r}_{(l)})}{l_{(lm)}} a_{(lm)} \sigma_{(t)} \quad \text{- bar in tension}$$

$$\text{or: } \frac{-(\underset{\sim}{r}_{(m)} - \underset{\sim}{r}_{(l)})}{l_{(lm)}} a_{(lm)} \sigma_{(c)} \quad \text{- bar in compression}$$

The scalar product of $\underset{\sim}{T}_{(lm)}$ with the vector $(\underset{\sim}{r}_{(m)} - \underset{\sim}{r}_{(l)})$ is $a_{(lm)} l_{(lm)} \sigma_{(t)}$ or $-a_{(lm)} l_{(lm)} \sigma_{(c)}$, and the summation of such products over all the bars is $V_{(t)}\sigma_{(t)} - V_{(c)}\sigma_{(c)}$. The products may also be written as the sum of the two terms $-\underset{\sim}{T}_{(lm)} \cdot \underset{\sim}{r}_{(l)}$ and $-\underset{\sim}{T}_{(ml)} \cdot \underset{\sim}{r}_{(m)}$, and since for any node, $l$, say,

$$\sum_{p} \underset{\sim}{T}_{(lp)} + \underset{\sim}{F}_{(l)} = 0,$$

where the summation extends over all nodes connected to node $l$, the first summation is $\sum \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{r}_{(p)}$ and so:

$$V_{(t)}\sigma_{(t)} - V_{(c)}\sigma_{(c)} = \sum_{p=1}^{n} \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{r}_{(p)} \qquad \text{- (2.1)}$$

The quantity on the r.h.s. of equation (2.1) depends only on the external force system, and therefore the expression on the l.h.s. has the same value for all frameworks which support this force system. The total volume of a framework, V, is the sum of $V_{(t)}$ and $V_{(c)}$, and, using (2.1), it may be written in the alternative forms:

$$V = V_{(t)}\left(1 + \frac{\sigma_{(t)}}{\sigma_{(c)}}\right) - \frac{1}{\sigma_{(c)}} \sum_{p=1}^{n} \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{r}_{(p)} \qquad \text{- (2.2a)}$$

$$V = V_{(c)}\left(1 + \frac{\sigma_{(c)}}{\sigma_{(t)}}\right) + \frac{1}{\sigma_{(t)}} \sum_{p=1}^{n} \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{r}_{(p)} \qquad \text{- (2.2b)}$$

From these expressions it is evident that to minimise the volume of a framework, it is sufficient to minimise either the volume of tension members or the volume of compression members. Furthermore, if a framework can be found which consists entirely of compression members, or entirely of tension members, it is an optimum framework, and its volume is given by (2.2a) with $V_{(t)}$ equal to zero, or (2.2b) with $V_{(c)}$ equal to zero.

All-tension or all-compression frameworks are the simplest form of optimum, but their existence is dependent upon certain restrictions on the applied force system. Any force system for which $\sum_{\sim} F_{(p)} . \underset{\sim}{r}_{(p)}$ is zero, as, for example, in the case of a system of couples in which the line joining the points of application of a pair of equal and opposite forces is at right angles to the direction of the forces, precludes the possibility of designing such a structure. When $\sum_{\sim} F_{(p)} . \underset{\sim}{r}_{(p)}$ is positive, an all-tension structure may exist; when it is negative an all-compression structure may exist. The absolute value is of little significance: if the magnitude of each force in a given system was halved, and all distances were also reduced by a half, the directions and relative magnitudes of the forces and the distribution of points of application would still be the same and so therefore would be the form of any optimum framework that might exist, but $\sum_{\sim} F_{(p)} . \underset{\sim}{r}_{(p)}$ would be one quarter its original value.

The smallest convex polyhedron which can be formed by using certain of the points of application as vertices, and which encloses the remaining points of application, is of some importance. If the forces applied at any joint of a framework are drawn as a set of arrows all directed away from the joint, the arrows cannot all be to the one side of a plane passing through the joint if equilibrium is to be maintained. Thus a set of bars with the same type of loading (i.e. all-tension or all-compression), meeting at a free joint, must occupy more than a half space, and if the joint lies outside the polyhedron the bars cannot all be directed towards the polyhedron since it is convex. It follows that there can be no free joints and therefore no bars outside the polyhedron. Consider now the equilibrium of the joints at the vertices of the polyhedron. If the applied force at a vertex is to be reacted by a set of bars all having the same type of loading and all lying within the polyhedron, the line of action of the force must pass inside the polyhedron: if the force is directed towards the interior the bars are in compression, and if it is directed away from the interior the bars are in tension. These results follow directly from the statement made above regarding the equilibrium of a joint under a set of forces, and they give rise to the following necessary condition, to be satisfied by a force system which admits of an all-tension or all-compression framework:

Condition (a) The smallest convex polyhedron, formed by using some or all of the points of application of forces as vertices, and enclosing the remaining points of application, must be such that the lines of action of the forces at the vertices pass inside the polyhedron, the forces at the vertices all being directed inwards to, or all outwards from, the interior of the polyhedron.

For plane systems of forces, the polyhedron becomes a polygon, and if the polygon contains points of application only at its vertices, and not in the interior, the above condition is not only necessary, but sufficient. Consider figures 2.1a and 2.1b. The forces $\underset{\sim}{F}_{(1)}, \ldots, \underset{\sim}{F}_{(n)}$ are applied at the points $n_{(1)}, \ldots, n_{(n)}$ respectively, in figure 2.1a, and the corresponding force polygon, HIJKLM, is shown in figure 2.1b. A point O is chosen in the interior of the force polygon and joined to each vertex.

The line AB is then drawn parallel to MO, starting from an arbitrary point on the line of action of force $\underset{\sim}{F}_{(1)}$, BC is parallel, to HO, CD to IO, DE to JO, EG to KO, and GA to LO. That the points A and A' coincide to form a closed polygon, ABCDEG, may be proved as follows. The triangle LMO in figure 2.1b may be taken as a triangle of forces, the forces $\underset{\sim}{T}_{(16)}$ and $\underset{\sim}{T}_{(12)}$ being in equilibrium with $\underset{\sim}{F}_{(1)}$, and acting at the point A in figure 2.1a. Similarly, triangle MHO is the triangle of forces for equilibrium of $\underset{\sim}{T}_{(21)}$, $\underset{\sim}{T}_{(23)}$ and $\underset{\sim}{F}_{(2)}$ at B. But $\underset{\sim}{T}_{(21)}$ is represented in magnitude and direction by the line OM while $\underset{\sim}{T}_{(12)}$ is similarly represented by MO, and so the two forces are equal and opposite. If they are both applied along the line AB, their resultant is zero. The resultant of each of the pairs of equal and opposite forces along the lines BC, CD, DE and EG is likewise zero, and therefore the pair of forces $\underset{\sim}{T}_{(16)}$ and $\underset{\sim}{T}_{(61)}$ must be the equilibrant of the system of forces $\underset{\sim}{F}_{(1)}$, ..., $\underset{\sim}{F}_{(6)}$. It is evident that unless A' coincides with A, the pair of equal and opposite forces $\underset{\sim}{T}_{(16)}$ and $\underset{\sim}{T}_{(61)}$ constitute a couple, so that if the sum of the moments of the applied forces is zero, A' must be the same as A. The closed polygon ABCDEG is known as a funicular polygon. (For a plane system of forces, the construction of a force polygon and a funicular polygon determines whether or not the system is in equilibrium: if the former closes, there is no resultant force, and if the latter closes there is no resultant moment).

Provided that the funicular polygon ABCDEG in figure 2.1a lies entirely inside the polygon formed by joining the points of application of the forces, the layout of an all-tension framework is given by the funicular polygon itself, together with the lines joining each vertex to the point of application of the force acting through that vertex. The size of the funicular polygon depends on the point A from which the first line is drawn and since this point is arbitrary, the polygon can be reduced to any required size: however it may be necessary to redraw the force polygon at certain stages, because the order of the forces in the force polygon must be the same as in the funicular polygon, so that if, for example, the point B were to lie on the other side of the intersection of the lines of action of $\underset{\sim}{F}_{(2)}$ and $\underset{\sim}{F}_{(3)}$, the order of these two forces in the force polygon would need to be reversed. The funicular polygon is convex and therefore suitable as a layout for an all-tension framework provided that the conditions regarding the ordering of forces are satisfied, and provided also that the point O is chosen so that all the lines joining it to the vertices of the force polygon lie entirely inside the force polygon. An identical construction produces a compression framework for a system of inwardly directed forces.

In the manner described above, an optimum framework may be found for any system of forces which satisfies the necessary condition (a) and the further condition that all the points of application of forces are vertices of the smallest convex polygon. However this latter condition is unnecessarily restrictive, as figures 2.2a and 2.2b show. $\underset{\sim}{F}_{(5)}$ is applied at the interior point $n_{(5)}$, and a suitable funicular polygon, ABCDEG, can still be formed. As the point $n_{(5)}$ moves away from N, the point at which the force $\underset{\sim}{F}_{(5)}$ crosses the boundary of the outer polygon in the same direction (i.e. interior to exterior) as the forces at the vertices, it is impossible to keep $n_{(5)}$ outside the funicular polygon. For this particular construction, the point R, which is the intersection of the line of action of $\underset{\sim}{F}_{(5)}$ with the lines from S and T drawn parallel to LI and KI respectively, is the limiting position of $n_{(5)}$. However, by employing the system of figures 2.3a and 2.3b, in which the forces are taken in a different order when drawing the force

polygon, it is clear that $n_{(s)}$ can move as far as R', which is the intersection of the line of action of $\underset{\sim}{F}_{(s)}$ with the line from $n_{(z)}$ drawn parallel to IL, R'U being parallel to JL.

Thus what is important is the distance, d, from an interior point of application to the point at which the corresponding force crosses the boundary of the smallest convex polygon in the opposite direction to the forces at the vertices. The necessary condition (a) sets a limit of zero on d: a sufficient condition is that the value of d is not small relative to the dimensions of the smallest convex polygon. How small it may be cannot be stated precisely, but the limiting value for any given force system can easily be determined from the funicular polygon construction.

## MICHELL'S THEOREM

Michell's Theorem provides the conditions subject to which a framework containing both tension and compression members is an optimum. It states that there must exist a compatible virtual deformation of space in which the strain along compression bars is $-\varepsilon$, the strain along tension bars is $+\varepsilon$, and no linear element of space has a strain numerically greater than $\varepsilon$, $\varepsilon$ being a small positive number: if the compatible deformation can extend over the whole of space, the framework is a global optimum: if it extends only over a limited region of space the framework is an optimum relative to frameworks whose members are confined to the region of the compatible deformation. Michell himself only proved the sufficiency of these conditions (reference 3, pages 589-591) but they have since been shown to be necessary (see chapter 5).

The volume of a framework may be written in the form:

$$V = V_{(t)} + V_{(c)} = \frac{1}{2}\left(\frac{1}{\sigma_{(t)}} + \frac{1}{\sigma_{(c)}}\right)\left(V_{(t)}\sigma_{(t)} + V_{(c)}\sigma_{(c)}\right) + \frac{1}{2}\left(\frac{1}{\sigma_{(t)}} - \frac{1}{\sigma_{(c)}}\right)\left(V_{(t)}\sigma_{(t)} - V_{(c)}\sigma_{(c)}\right)$$

i.e., using equation (2.1)

$$V = \frac{1}{2}\left(\frac{1}{\sigma_{(t)}} + \frac{1}{\sigma_{(c)}}\right)\left(V_{(t)}\sigma_{(t)} + V_{(c)}\sigma_{(c)}\right) + \frac{1}{2}\left(\frac{1}{\sigma_{(t)}} - \frac{1}{\sigma_{(c)}}\right)\sum_{p} \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{r}_{(p)} \qquad - (2.3)$$

and therefore to minimise the volume, it is sufficient to minimise the expression $V_{(t)}\sigma_{(t)} + V_{(c)}\sigma_{(c)}$. Consider a framework which supports the system of forces $\underset{\sim}{F}_{(i)}$ applied at the points with position vectors $\underset{\sim}{r}_{(i)}$, and let the space occupied by the framework be subjected to a virtual deformation in which no linear strain is numerically greater than $\varepsilon$. The virtual work done by the applied forces is equal to the change in strain energy of the framework, i.e.

$$\sigma_{(t)} \sum_{(t)} a_{(lm)} l_{(lm)} \varepsilon_{(lm)} - \sigma_{(c)} \sum_{(c)} a_{(lm)} l_{(lm)} \varepsilon_{(lm)} = \sum_{p} \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{V}_{(p)} \qquad - (2.4)$$

where the $\varepsilon_{(lm)}$ are the strains imposed on the framework members, the $\underset{\sim}{v}_{(P)}$ are the vector displacements at the points of application of forces, and the first two summations are taken over tension members and compression members respectively. The $\varepsilon_{(lm)}$ all satisfy the inequality:

$$-\varepsilon \leqslant \varepsilon_{(lm)} \leqslant \varepsilon \qquad -(2.5)$$

and therefore

$$\sigma_{(t)} \sum_{(t)} a_{(lm)} l_{(lm)} \varepsilon_{(lm)} - \sigma_{(c)} \sum_{(c)} a_{(lm)} l_{(lm)} \varepsilon_{(lm)} \leqslant \sigma_{(t)} \sum_{(t)} a_{(lm)} l_{(lm)} \varepsilon + \sigma_{(c)} \sum_{(c)} a_{(lm)} l_{(lm)} \varepsilon$$
$$= \varepsilon \left( \sigma_{(t)} V_{(t)} + \sigma_{(c)} V_{(c)} \right).$$

Combining this inequality with equation (2.4) gives:

$$V_{(t)} \sigma_{(t)} + V_{(c)} \sigma_{(c)} \geqslant \frac{1}{\varepsilon} \sum \underset{\sim}{F}_{(P)} \cdot \underset{\sim}{v}_{(P)} \qquad -(2.6)$$

(2.6) sets a lower limit on the value of $V_{(t)}\sigma_{(t)} + V_{(c)}\sigma_{(c)}$ for any framework which supports the given force system and which has its members confined to the region of space in which the deformation is defined. This limit is attained by the framework (if it exists) for which the virtual strain is $+\varepsilon$ along tension members and $-\varepsilon$ along compression members, which means, by virtue of the inequality (2.5), that the members must lie along lines of constant principal strain. Such a framework is, by equation (2.3), an optimum.

The above proof is essentially the same as the one given by Michell (reference 3). Certain aspects of it are worth looking at more closely, because the equation of virtual work, which is represented by equation (2.4), and which is central to the proof, is inexact, in the sense that the expression on the l.h.s. of (2.4) is the change in strain energy of a framework only if powers of the $\varepsilon_{(lm)}$ higher than the first are ignored. However if the defining relation between the $\varepsilon_{(lm)}$ and the $\underset{\sim}{v}_{(P)}$ is taken to be that between small strains and small displacements, which is represented by the following tensor equation:

$$e_{IJ} = \frac{1}{2} \left( u_{I,J} + u_{J,I} \right) \qquad -(2.7)$$

$e_{IJ}$ being the strain tensor and $u_I$ the displacement vector, then equation (2.4) is mathematically exact. This becomes clear when the strain tensor is referred to Cartesian coordinates and written in the familiar form:

$$e_{IJ} = \begin{bmatrix} \frac{\partial u_1}{\partial x^1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x^3} + \frac{\partial u_3}{\partial x^1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial x^2}\right) & \frac{\partial u_2}{\partial x^2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x^3} + \frac{\partial u_3}{\partial x^2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x^1} + \frac{\partial u_1}{\partial x^3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x^2} + \frac{\partial u_2}{\partial x^3}\right) & \frac{\partial u_3}{\partial x^3} \end{bmatrix}$$

Then $\varepsilon_{(lm)}$ is the mean value of the expression $e_{ij}\, l^i_{(lm)}\, l^j_{(lm)}$ along the bar joining nodes $l$ and m, where $l^i_{(lm)}$ is the unit vector in the direction of the bar. But

$$e_{ij}\, l^i_{(lm)}\, l^j_{(lm)} = \left(l^1_{(lm)}\frac{\partial}{\partial x^1} + l^2_{(lm)}\frac{\partial}{\partial x^2} + l^3_{(lm)}\frac{\partial}{\partial x^3}\right)\left(l^1_{(lm)}u_1 + l^2_{(lm)}u_2 + l^3_{(lm)}u_3\right)$$

The expression on the r.h.s. of the above is the derivative in the direction of the bar of the component of virtual displacement along the bar, and $\varepsilon_{(lm)}$ is therefore the difference between the component of virtual displacement along the bar at one end and that at the other, divided by the length of the bar. (This is not necessarily the exact value of the mean strain in the bar as a result of the displacements $\underset{\sim}{y}_{(l)}$ and $\underset{\sim}{y}_{(m)}$ of its ends.) Each of the terms $\sigma_{(lm)}\, a_{(lm)}\, l_{(lm)}\, \varepsilon_{(lm)}$ or $-\sigma_{(lm)}\, a_{(lm)}\, l_{(lm)}\, \varepsilon_{(lm)}$ in equation (2.4) is then $-(\underset{\sim}{T}_{(lm)} \cdot \underset{\sim}{y}_{(l)} + \underset{\sim}{T}_{(ml)} \cdot \underset{\sim}{y}_{(m)})$ and the summation of all such terms is $\Sigma \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{y}_{(p)}$ as required. The exactness of equation (2.4) is thus dependent on the mathematical relationship between the nodal displacements $\underset{\sim}{y}_{(p)}$ and the linear strains $\varepsilon_{(lm)}$, and this relationship depends in turn on equation (2.7).

The property of the strain tensor which determines the existence of principal strains is a general property of all symmetric, covariant second order tensors. The expression $A_{IJ}\, l^I\, l^J$, where $A_{IJ}$ is an arbitrary symmetric tensor, varies in magnitude at a point as the unit vector $l^I$ changes direction. There are a set of extrema which are the roots $\lambda_{(K)}$, of the determinantal equation

$$\left|A_{IJ} - \lambda\, g_{IJ}\right| = 0 \qquad (2.8)$$

The $\lambda_{(K)}$, of which there are N in N-dimensional space, are all real if the metric of space is a definite quadratic form. Corresponding to each simple root there is a unique vector, $l^I_{(K)}$, and, in particular, if all the roots are simple, there are N vectors. These vectors form an orthogonal set and the directions of the vectors are called the principal directions determined by $A_{IJ}$. If the coordinate axes cooincide with the principal directions, the tensor which determined the principal directions is diagonal in form. (See reference 7, pages 47-50 and 12, 13). The principal directions determined by the strain tensor are, of course, the directions of principal strain and the corresponding $\lambda_{(K)}$ are the values of the principal strains. The problem of determining the layout of optimum frameworks is, by Michell's Theorem, equivalent to the problem of finding compatible strain fields in which the numerical value of the principal strains is everywhere the same, and this can be presented as the following mathematical problem:

To find the condition that the symmetric, covariant second order tensor $e_{IJ}$, for which the numerical value of the expression $e_{IJ}\, l^I\, l^J$ along the principal directions determined by $e_{IJ}$ is everywhere the same, is of the form $\frac{1}{2}(u_{I,J} + u_{J,I})$, where $u_I$ is an arbitrary vector.

Stating the problem in this form removes any need to discuss the order of magnitude of the quantities involved: what is important is the mathematical relationship between $e_{IJ}$ and $u_I$ as defined by equation (2.7). However if, $e_{IJ}$ is to be regarded as a strain tensor with constant principal strains, and $u_I$ the corresponding displacement vector, then the condition

that $e_{IJ}$ be of the form $\frac{1}{2}(u_{I,J} + u_{J,I})$ is the compatibility condition for small strains and small displacements. Any other compatibility condition, such as that for large deformations, is inappropriate. This point will be seen to be important when the layout of optimum frameworks on plane and curved surfaces is discussed.

For any tensor $e_{IJ}$, the principal directions are uniquely determined and are orthogonal except in the case when equation (2.8) has repeated roots. In this case, M vectors can be chosen, corresponding to the M identical $\lambda_{(K)}$, which are orthogonal to each other, and to the remaining N-M vectors. In the subspace spanned by the set of M vectors, the value of $e_{IJ}l^I l^J$ is the same in all directions, and in such a subspace, there is therefore no restriction on the layout of an optimum framework, although the members must all carry the same sign of loading. In particular, if $e_{IJ}$ is a constant multiple of the fundamental tensor, all the $\lambda_{(K)}$ are the same. In terms of a strain tensor this would imply a uniform dilatation of space, the strains being the same at all points and all directions, and would lead to optimum frameworks of the same type as established by Maxwell's Theorem.

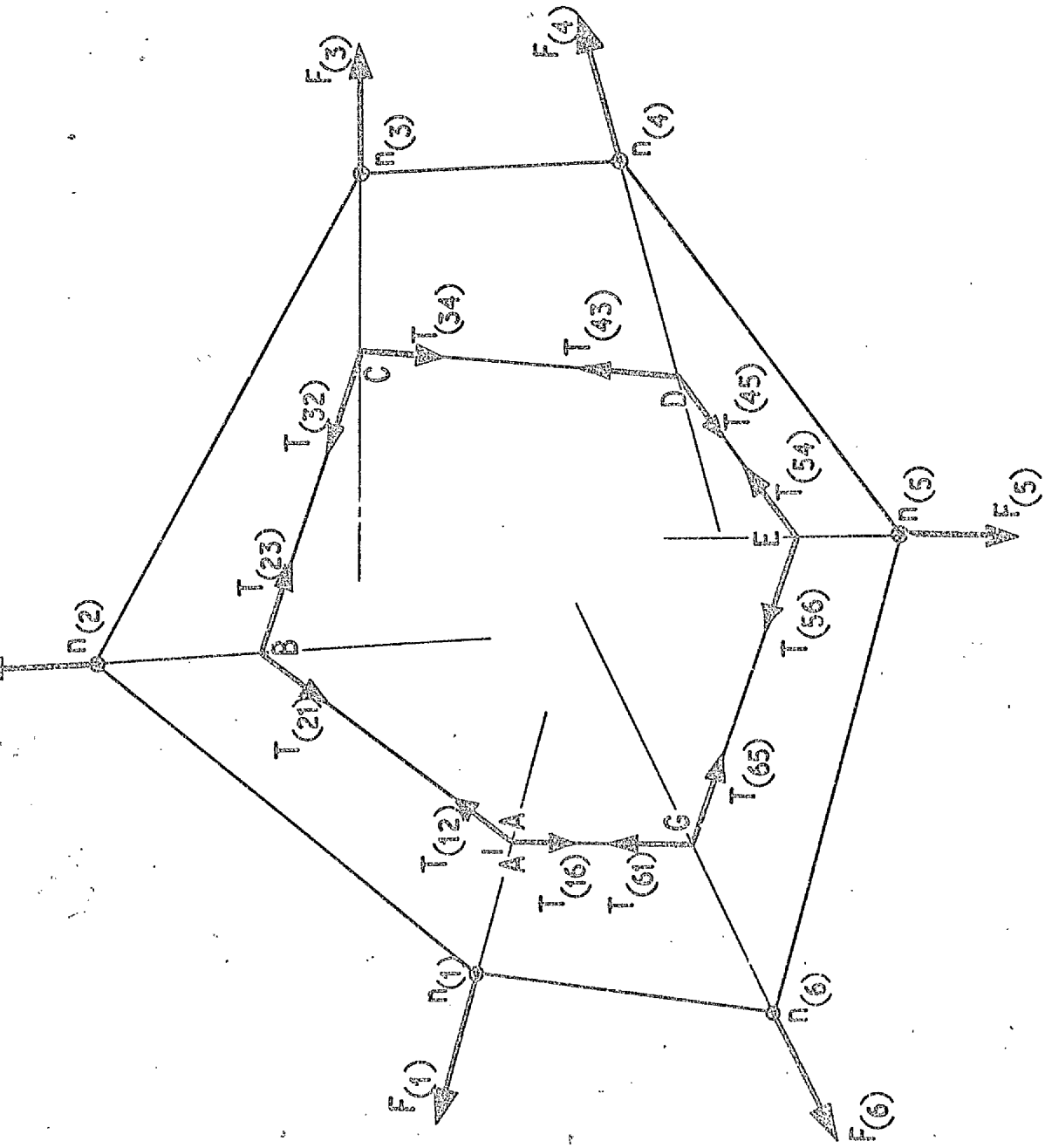
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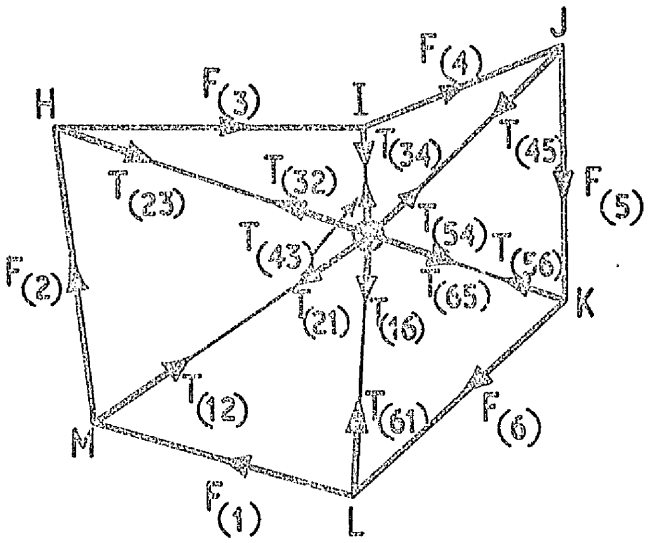
Figure 2.1a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T(61)
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M
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Figure 2·1b

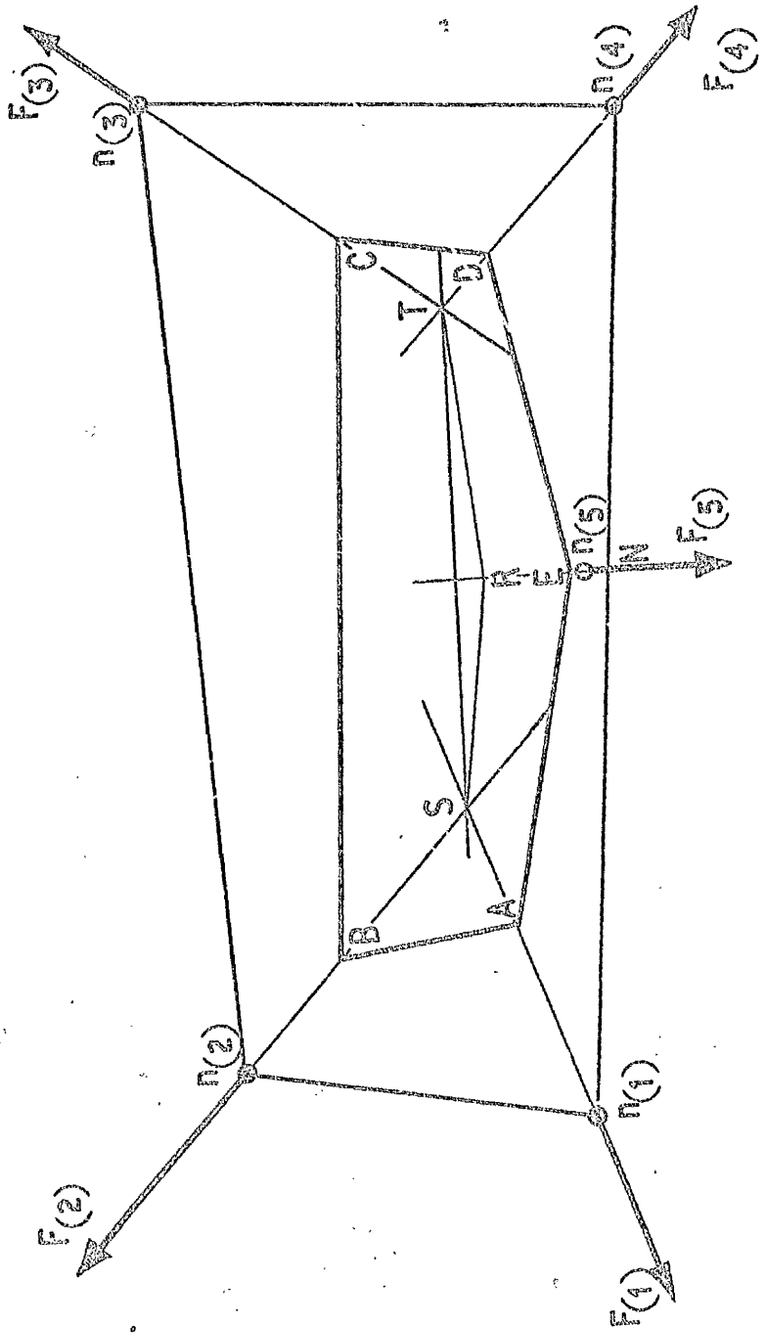



Figure 2.2a

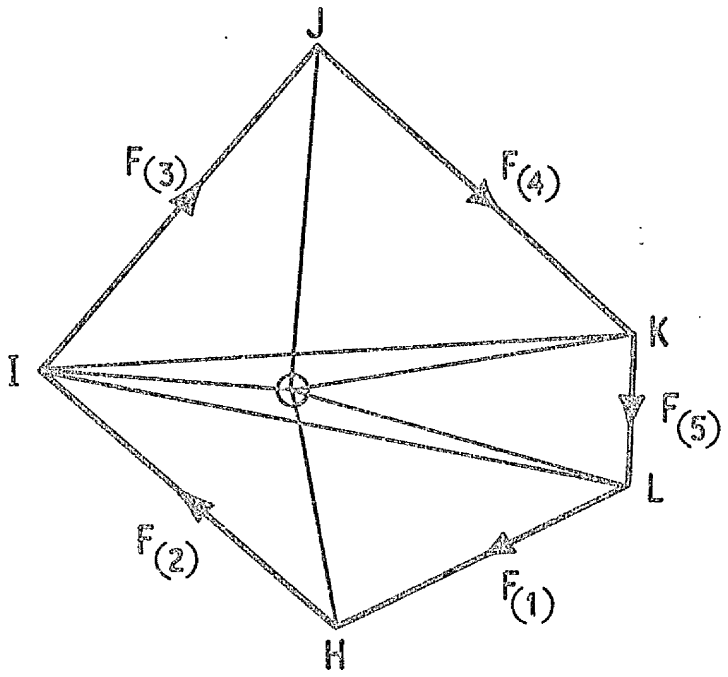



Figure 2·2b

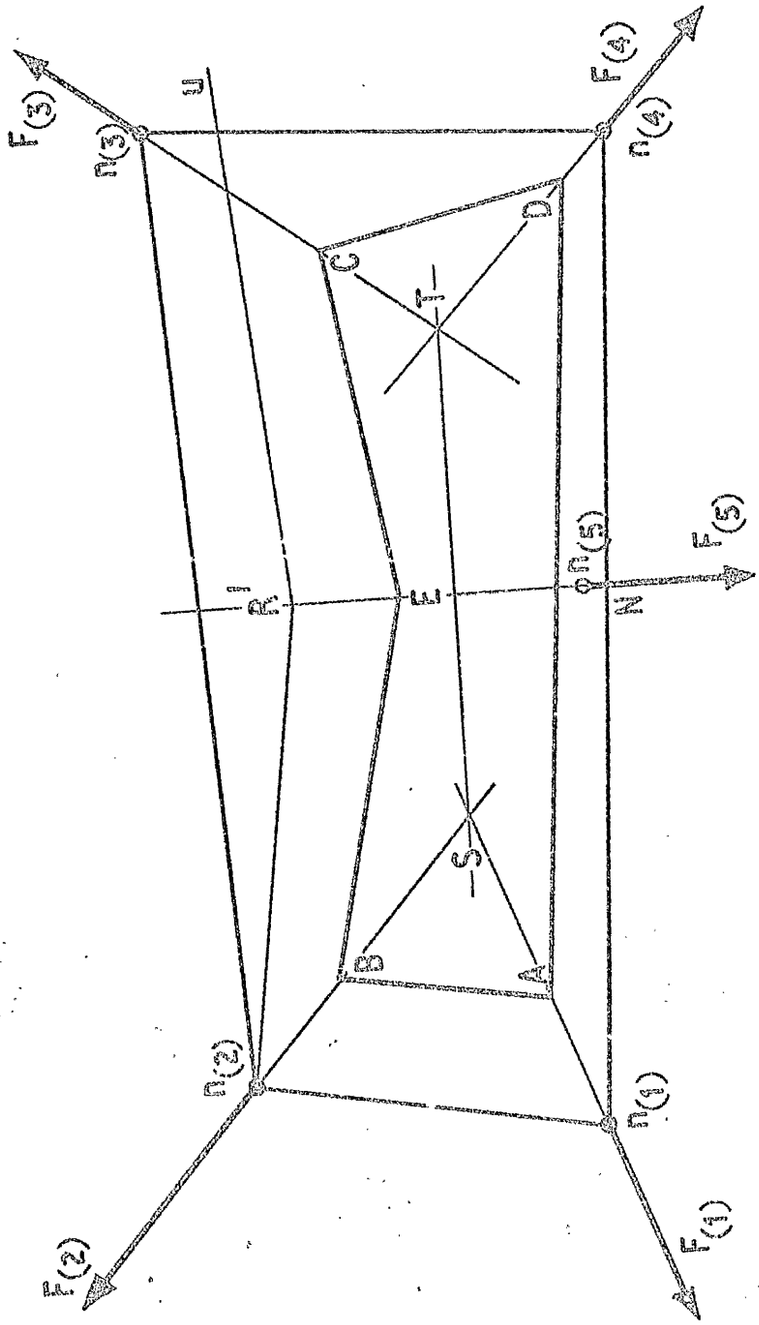



Figure 2.3a

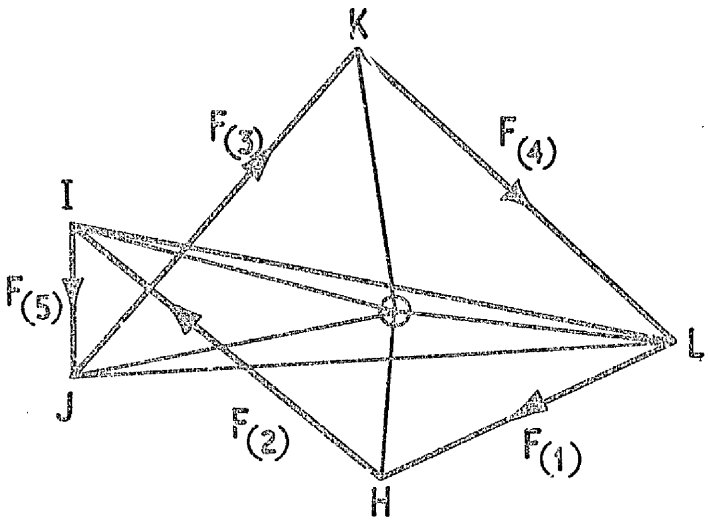



Figure 2·3b

# CHAPTER 3

## ORTHOGONAL CURVILINEAR CO-ORDINATES

As a preliminary to deriving the general equations of the layout curves for optimum frameworks, it is necessary to establish some general results concerning orthogonal systems of curves in three dimensional space. These results can be obtained most concisely by means of the Tensor calculus. It is not the purpose of this chapter to prove any of the basic theorems of Tensor calculus but merely to state them and then apply them to particular problems. Proofs of theorems can be found in references 6, 7 and 8. The question of notation has already been dealt with in chapter 1.

### The Fundamental Tensor

Riemannian space, of which Euclidean space is a special case, is characterised by the form of the expression for the line element. The square of the distance, ds, between the points with coordinates $x^I$ and $x^I + dx^I$ is invariant and is given by the following quadratic differential form:

$$ds^2 = g_{IJ}\, dx^I\, dx^J \qquad - (3.1)$$

where the symmetric covariant tensor $g_{IJ}$ is known as the fundamental tensor of the space, and the expression $g_{IJ}\, dx^I\, dx^J$ is called the metric. The fundamental tensor of Euclidean space referred to rectangular Cartesian coordinates is given by

$$g_{IJ} = 1, \quad I = J \; ; \quad g_{IJ} = 0, \quad I \neq J$$

The latter property, that of the off-diagonal components being zero, is a general property of the fundamental tensor of any Riemannian space referred to a system of orthogonal coordinates (reference 6, page 70). In particular, the fundamental tensor of three-dimensional Euclidean space, the metric of which is positive-definite, may be written in the form:

$$g_{ij} = \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & C^2 \end{bmatrix} \qquad - (3.2)$$

where A, B and C are real functions of the orthogonal coordinates $x^1$, $x^2$ and $x^3$. (The form of $g_{ij}$ given above is not in itself sufficient to ensure that the space it describes is Euclidean. There are certain equations to be satisfied by the functions A, B and C, these being equations (3.7) below.)

For a given value of one of the space coordinates the other two coordinates

can be thought of as a pair of surface coordinates. The corresponding fundamental tensor is also diagonal and for the surfaces determined by the space coordinates $x^2$ and $x^3$ , for example, is:

$$g^{(1)}_{\alpha\beta} = \begin{bmatrix} B^2 & 0 \\ 0 & C^2 \end{bmatrix} \qquad - (3.3a)$$

There are a further two sets of such coordinate surfaces, with fundamental tensors:

$$g^{(2)}_{\alpha\beta} = \begin{bmatrix} C^2 & 0 \\ 0 & A^2 \end{bmatrix} \qquad - (3.3b)$$

$$g^{(3)}_{\alpha\beta} = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix} \qquad - (3.3c)$$

The three sets of coordinate surfaces are mutually orthogonal.

The contravariant tensor $g^{IJ}$ is defined by the equation:

$$g_{IJ}\, g^{IK} = \delta^K_J$$

where the Kronecker delta, $\delta^K_J$ , which may be regarded as a mixed second order tensor (reference 6 page 9), is given by:

$$\delta^K_J = 1 \quad , \quad J = K$$
$$\delta^K_J = 0 \quad , \quad J \neq K$$

Therefore, for example:

$$g^{ij} = \begin{bmatrix} 1/A^2 & 0 & 0 \\ 0 & 1/B^2 & 0 \\ 0 & 0 & 1/C^2 \end{bmatrix} \qquad - (3.4)$$

Christoffel Symbols

The Christoffel symbols of the first and second kinds are defined respectively by the equations:

$$[I\ J,\ K] = \frac{1}{2}\left(\frac{\partial g_{IK}}{\partial x^J} + \frac{\partial g_{JK}}{\partial x^I} - \frac{\partial g_{IJ}}{\partial x^K}\right) \quad - (3.5a)$$

$$\left\{ \begin{matrix} I \\ JK \end{matrix} \right\} = g^{IL}\,[J\,K,\ L] \quad - (3.5b)$$

(Although the Christoffel symbols are not themselves tensors, the summation convention for repeated indices is taken to apply, the indices in the Christoffel symbols of the first kind all being treated as subscripts and those in the Christoffel symbols of the second kind being treated as one superscript and two subscripts).

For an orthogonal coordinate system in Euclidean space, with the fundamental tensor defined by equation (3.2), the form of the Christoffel symbols is particularly simple, and the three sets of coordinate surface Christoffel symbols are subsets of the set of space Christoffel symbols. They are contained in tables 1 and 2.

Curvature Tensor

The covariant curvature tensor is a fourth order tensor which is formed intirely from the components of the fundamental tensor and their derivatives up to the second order. It may be written in the form:

$$R_{IJKL} = \frac{1}{2}\left(\frac{\partial^2 g_{IL}}{\partial x^J \partial x^K} + \frac{\partial^2 g_{JK}}{\partial x^I \partial x^L} - \frac{\partial^2 g_{IK}}{\partial x^J \partial x^L} - \frac{\partial^2 g_{JL}}{\partial x^I \partial x^K}\right)$$

$$+ g^{TS}\left([J\,K,\ S][I\,L,\ T] - [J\,L,\ S][I\,K,\ T]\right) \quad - (3.6)$$

The condition that a space with the positive definite metric defined by equations (3.1) and (3.2) be Euclidean is that the curvature tensor be the zero tensor (reference 7, page 117). The number of independent components of the curvature tensor of an N-dimensional space which are not identically zero is $\frac{1}{12}N^2\,(N^2-1)$ (reference 6, page 53) and therefore the condition that $R_{ijkl}$ be the zero tensor gives six independent equations, viz:

$$R_{1212} = R_{2323} = R_{3131} = R_{1213} = R_{2123} = R_{3132} = 0$$

Using equations (3.2) and (3.6) and table 1, the first of these may be expanded as follows.

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{12}}{\partial x^2 \partial x^1} + \frac{\partial^2 g_{21}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\right)$$

$$+ \; g^{11}\left([21,1][12,1] - [22,1][11,1]\right) + \underset{\leq 0}{g^{12}}(\cdots) + \underset{\leq 0}{g^{13}}(\cdots)$$

$$+ \underset{\leq 0}{g^{21}}(\cdots) + g^{22}\left([21,2][12,2] - [22,2][11,2]\right) + \underset{\leq 0}{g^{23}}(\cdots)$$

$$+ \underset{\leq 0}{g^{13}}(\cdots) + \underset{\leq 0}{g^{23}}(\cdots) + g^{33}\left([21,3][12,3] - [22,3][11,3]\right)$$

$$= \frac{1}{2}\left(\frac{\partial^2(A^2)}{\partial x^2 \partial x^2} + \frac{\partial^2(B^2)}{\partial x^1 \partial x^1}\right) + \frac{1}{A^2}\left[\left(A\frac{\partial A}{\partial x^2}\right)^2 + AB\frac{\partial A}{\partial x^1}\frac{\partial B}{\partial x^1}\right]$$

$$+ \frac{1}{B^2}\left[\left(B\frac{\partial B}{\partial x^1}\right)^2 + AB\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2}\right] - \frac{1}{C^2}\left[AB\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right]$$

$$\therefore \quad R_{1212} = 0$$

$$\Rightarrow \quad -AB\frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial A}{\partial x^2}\right) - AB\frac{\partial}{\partial x^1}\left(\frac{1}{A}\frac{\partial B}{\partial x^1}\right) = \frac{A}{C}\frac{\partial A}{\partial x^3}\frac{B}{C}\frac{\partial B}{\partial x^3}$$

The other non-vanishing components of $R_{ijkl}$ may be similarly expanded to provide the set of equations:

$$-AB\frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial A}{\partial x^2}\right) - AB\frac{\partial}{\partial x^1}\left(\frac{1}{A}\frac{\partial B}{\partial x^1}\right) = \frac{A}{C}\frac{\partial A}{\partial x^3}\frac{B}{C}\frac{\partial B}{\partial x^3} \qquad - (3.7a)$$

$$-BC\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial B}{\partial x^3}\right) - BC\frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial C}{\partial x^2}\right) = \frac{B}{A}\frac{\partial B}{\partial x^1}\frac{C}{A}\frac{\partial C}{\partial x^1} \qquad - (3.7b)$$

$$-CA\frac{\partial}{\partial x^1}\left(\frac{1}{A}\frac{\partial C}{\partial x^1}\right) - CA\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial A}{\partial x^3}\right) = \frac{C}{B}\frac{\partial C}{\partial x^2}\frac{A}{B}\frac{\partial A}{\partial x^2} \qquad - (3.7c)$$

$$-\frac{\partial^2 B}{\partial x^1 \partial x^3} + \frac{1}{A}\frac{\partial B}{\partial x^1}\frac{\partial A}{\partial x^3} + \frac{1}{C}\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^1} = 0 \qquad - (3.7d)$$

$$-\frac{\partial^2 C}{\partial x^2 \partial x^1} + \frac{1}{B}\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^1} + \frac{1}{A}\frac{\partial C}{\partial x^1}\frac{\partial A}{\partial x^2} = 0 \qquad - (3.7e)$$

$$-\frac{\partial^2 A}{\partial x^3 \partial x^2} + \frac{1}{C}\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^2} + \frac{1}{B}\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^3} = 0 \qquad - (3.7f)$$

The curvature tensor of a two-dimensional space has only one independent component not identically zero, $R_{1212}$ . If $R_{1212}$ is calculated for the set of coordinate surfaces whose fundamental tensor is given by equation (3.3c), it is seen to differ from $R_{1212}$ of space only in that the term $g^{33}([2\ 1,\ 3][1\ 2,\ 3] - [2\ 2,\ 3][1\ 1,\ 3])$ is absent, there being no third dimension, and this is the term on the r.h.s. of equation (3.7a). Thus the expression on the l.h.s. of (3.7a) is $R_{1212}$ corresponding to $g^{(3)}_{\alpha\beta}$ , and similarly the expressions on the l.h.s.'s of (3.7b) and (3.7c) are the non-zero components of the two curvature tensors applying to the two sets of surfaces whose fundamental tensors are respectively $g^{(1)}_{\alpha\beta}$ and $g^{(2)}_{\alpha\beta}$ , as defined by equations (3.3a) and (3.3b).

It is clear that these expressions are not necessarily zero, surfaces being, in general, non-Euclidean spaces.

The Riemannian curvature, K, of a space, associated with the arbitrary, independent vectors $P^I$ and $Q^I$ , is defined to be the invariant

$$K = \frac{R_{IJKL}\, P^I P^K Q^J Q^L}{(g_{IK} g_{JL} - g_{IL} g_{JK}) P^I P^K Q^J Q^L}$$

The value of K at a point is unaltered if the two vectors are replaced by any linear combination (reference 6, page 56). Since there are only two independent vectors at any point in a two dimensional space, the Riemannian curvature of a surface is uniquely determined, and is given by:

$$K = \frac{R_{1212}}{g} \qquad - (3.8)$$

where g is the determinant formed from the components of the fundamental tensor of the surface.

From equation (3.8) it is evident that the Riemannian curvature of a surface is an intrinsic property of the surface, and does not rely on the concept of an enveloping space. Geometrically, the curvature of a surface is related to the change in direction of the tangent to the surface and this tangent is a space vector: hence the need for an enveloping space. This is not to say that the Riemannian curvature of a surface has no geometrical significance. It is the same as the Gaussian curvature (reference 8, page 96) and is the product of the two principal curvatures of the surface.

In order to develop expressions for the principal curvatures of a surface,

the quantities $x^i_\alpha$ are introduced, and are defined by:

$$x^i_\alpha = \frac{\partial x^i}{\partial x^\alpha}$$

The space coordinates, $x^i$, are considered to be functions of the surface coordinates, $x^\alpha$. The $x^i_\alpha$ may be regarded, for a given i, as the components of a covariant surface vector, and, for a given $\alpha$, as the components of a contravariant space vector. The tensor derivative of $x^i_\alpha$ is defined to be:

$$x^i_{\alpha;\beta} = \frac{\partial^2 x^i}{\partial x^\alpha \partial x^\beta} + \begin{Bmatrix} i \\ j\,k \end{Bmatrix} x^j_\alpha x^k_\beta - \begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix} x^i_\gamma \qquad - (3.9)$$

(reference 6, pages 67 and 77).

$x^i_{\alpha;\beta}$ is a contravariant space vector for any given $\alpha$ and $\beta$, and a covariant second order tensor for any given i. It may be shown (reference 6, page 77) that each of the vectors $x^i_{\alpha;\beta}$ is normal to the surface containing the $x^\alpha$, and therefore:

$$x^i_{\alpha;\beta} = b_{\alpha\beta}\xi^i \qquad - (3.10)$$

where $\xi^i$ is the unit normal to the surface. (The magnitude, P, of the arbitrary vector $P^I$ is defined by

$$P^2 = g_{IJ} P^I P^J \qquad - (3.11)$$

and therefore a unit vector $l^I$, satisfies the equation

$$g_{IJ} l^I l^J = 1 \qquad - (3.12)\ )$$

Equation (3.10) is the defining equation for the symmetric covariant surface tensor $b_{\alpha\beta}$.

The normal curvature of the surface in any direction is the curvature of the normal plane section of the surface in that direction, and may be shown to be $b_{\alpha\beta}\, t^\alpha\, t^\beta$ (reference 6, page 80) where $t^\alpha$ is the unit surface vector in the given direction. The quantity $b_{\alpha\beta}\, t^\alpha\, t^\beta$ possesses two extrema corresponding to the principal directions determined by the tensor $b_{\alpha\beta}$, and these are the principal curvatures of the surface. A line which is a direction of principal curvature at all its points is called a line of curvature. There is a relation between the curvature tensor and the tensor $b_{\alpha\beta}$, viz:

$$K_{\rho\alpha\beta\gamma} = b_{\alpha\gamma} b_{\rho\beta} - b_{\alpha\beta} b_{\rho\gamma}$$

which, in two dimensions, reduces to the single equation:

$$R_{1212} = b_{11} b_{22} - (b_{12})^2 = b \qquad - (3.13)$$

where b is the determinant of $b_{\alpha\beta}$ . Since $R_{1212}/g$ is invariant, so also is b/g, being the product of the two principal curvatures of the surface. Equation (3.13) is the equation of Gauss (reference 6, page 79).

(In three dimensional differential geometry, the components of the tensor $b_{\alpha\beta}$ , when the space coordinates are rectangular Cartesian, are called the second order magnitudes of the surface. The components of the fundamental tensor are called the first order magnitudes. The first order magnitudes involve only the metric properties of the surface itself, whereas the second order magnitudes depend also on the metric properties of the enveloping space (reference 9, pages 51-63). The significance of the Gauss equation is that the product of the two principal curvatures, which individually depend on the second order magnitudes, is given in terms of the first order magnitudes only.)

Consider now the $x^2$ , $x^3$ coordinate surfaces. If the space coordinates are chosen to be $x^1$ , $x^2$ , $x^3$ and the surface coordinates to be $x^2$ , $x^3$ , the quantities $x^i_\alpha$ are simply $\delta^i_\alpha$ . The $x^i_{\alpha;\beta}$ are, from equation (3.9) and table 2:

$$x^1_{2;3} = 0 \qquad x^1_{2;2} = -\frac{B}{A^2}\frac{\partial B}{\partial x^1} \qquad x^1_{3;3} = -\frac{C}{A^2}\frac{\partial C}{\partial x^1}$$

$$x^2_{2;3} = 0 \qquad x^2_{2;2} = 0 \qquad x^2_{3;3} = 0$$

$$x^3_{2;3} = 0 \qquad x^3_{2;2} = 0 \qquad x^3_{3;3} = 0$$

The unit normal to any surface is the unit vector along the $x^1$ coordinate direction, which, using equation (3.11), is given by:

$$\xi^i = \left(\frac{1}{A}, 0, 0\right)$$

and so, from equation (3.10)

$$b^{(1)}_{\alpha\beta} = \begin{bmatrix} -\frac{B}{A}\frac{\partial B}{\partial x^1} & 0 \\ 0 & -\frac{C}{A}\frac{\partial C}{\partial x^1} \end{bmatrix} \qquad - (3.14a)$$

The fact that $b^{(1)}_{\alpha\beta}$ is in diagonal form means that the coordinate directions are directions of principal curvature. The unit vectors in the two coordinate directions are (1/B, 0) and (0,1/C) and therefore the magnitudes of the

two principal curvatures are $-(1/AB)\partial B/\partial x^1$ and $-(1/AC)\partial C/\partial x^1$. The geometrical significance of equations (3.7a), (3.7b) and (3.7c) is now apparent: they are the Gauss equations for the three sets of coordinate surfaces.

$b^{(2)}_{\alpha\beta}$ and $b^{(3)}_{\alpha\beta}$, corresponding to the sets of surfaces defined by the pairs of space coordinates $x^3$, $x^1$ and $x^1$, $x^2$ respectively, are similarly diagonal in form:

$$b^{(2)}_{\alpha\beta} = \begin{bmatrix} -\frac{C}{B}\frac{\partial C}{\partial x^2} & 0 \\ 0 & -\frac{A}{B}\frac{\partial A}{\partial x^2} \end{bmatrix} \qquad - (3.14b)$$

$$b^{(3)}_{\alpha\beta} = \begin{bmatrix} -\frac{A}{C}\frac{\partial A}{\partial x^3} & 0 \\ 0 & -\frac{B}{C}\frac{\partial B}{\partial x^3} \end{bmatrix} \qquad - (3.14c)$$

The space coordinate curves form an arbitrary orthogonal network in space, and therefore the above analysis shows that in any such system, the coordinate curves are lines of curvature of the coordinate surfaces. The importance of this result in the context of what follows is that in concentrating attention on one particular coordinate surface it must be remembered that the two sets of space coordinate curves which lie on the surface do not form an arbitrary system of surface coordinates, but the particular system in which the coordinate curves are lines of curvature.

The latter three of the set of equations (3.7), just as the first three, represent specific properties of the sets of coordinate surfaces. The surface tensor $b_{\alpha\beta}$ may be shown (reference 6, page 79) to satisfy the following tensor equation:

$$b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0$$

which gives rise to only two independent equations, viz:

$$b_{11,2} - b_{12,1} = 0 \qquad - (3.15a)$$

$$b_{22,1} - b_{21,2} = 0 \qquad - (3.15b)$$

Equations (3.15) are the Codazzi equations. Consider the set of surfaces for which $b_{\alpha\beta}$ is given by equation (3.14a).

$$b^{(1)}_{11,2} = \frac{\partial b^{(1)}_{11}}{\partial x^2} - \begin{Bmatrix} 1 \\ 1\,2 \end{Bmatrix} b^{(1)}_{11} - \begin{Bmatrix} 2 \\ 1\,2 \end{Bmatrix} b^{(1)\prime}_{12} - \begin{Bmatrix} 1 \\ 1\,2 \end{Bmatrix} b^{(1)}_{11} - \begin{Bmatrix} 2 \\ 1\,2 \end{Bmatrix} b^{(1)\prime}_{12}$$

$\leq 0$ $\leq 0$

(The structure of the covariant derivative is given in reference 6, chapter 3.) The surface coordinates $x^1$ and $x^2$ are, in this case, the same as the space coordinates $x^2$ and $x^3$ respectively, and so the above equation becomes, using table 2:

$$b^{(i)}_{11,2} = \frac{\partial}{\partial x^3}\left(-\frac{B}{A}\frac{\partial B}{\partial x^1}\right) + \frac{2}{B}\frac{\partial B}{\partial x^3}\frac{B}{A}\frac{\partial B}{\partial x^1}$$

Similarly:

$$b^{(i)}_{12,1} = \frac{\partial b^{(i)}_{12}}{\partial x^1}^{\not=0} - \begin{Bmatrix}1\\11\end{Bmatrix} b^{(i)}_{12}{}_{\not=0} - \begin{Bmatrix}2\\11\end{Bmatrix} b^{(i)}_{22} - \begin{Bmatrix}1\\21\end{Bmatrix} b^{(i)}_{11} - \begin{Bmatrix}2\\21\end{Bmatrix} b^{(i)}_{12}{}_{\not=0}$$

$$= -\frac{B}{C^2}\frac{\partial B}{\partial x^3}\frac{C}{A}\frac{\partial C}{\partial x^1} + \frac{1}{B}\frac{\partial B}{\partial x^3}\frac{B}{A}\frac{\partial B}{\partial x^1}$$

Equation (3.15a) may be written:

$$\frac{\partial}{\partial x^3}\left(-\frac{B}{A}\frac{\partial B}{\partial x^1}\right) + \frac{1}{A}\frac{\partial B}{\partial x^3}\frac{\partial B}{\partial x^1} + \frac{B}{AC}\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^1} = 0$$

which is the same as (3.7d). In this fashion, all the equations of the type (3.15) corresponding to each of the tensors $b^{(i)}_{\alpha\beta}$, $b^{(2)}_{\alpha\beta}$ and $b^{(3)}_{\alpha\beta}$ can be shown to be identical to one of the three equations (3.7d), (3.7e) or (3.7f).

It has thus been proved that for any orthogonal network of coordinate curves in Euclidean three dimensional space, the condition that the curvature tensor be the zero tensor is equivalent to the condition that all the coordinate surfaces satisfy the Gauss-Codazzi equations.

## Strain in Orthogonal Curvilinear Coordinates

The strain tensor, $e_{ij}$, is given in terms of the displacement vector, $u_i$ by equation (2.7). Taking $e_{11}$ as an example, and expanding the covariant derivative gives the following:

$$e_{11} = u_{1,1} = \frac{\partial u_1}{\partial x^1} - \begin{Bmatrix}1\\11\end{Bmatrix} u_1 - \begin{Bmatrix}2\\11\end{Bmatrix} u_2 - \begin{Bmatrix}3\\11\end{Bmatrix} u_3$$

$$= \frac{\partial u_1}{\partial x^1} - \frac{1}{A}\frac{\partial A}{\partial x^1} u_1 + \frac{A}{B^2}\frac{\partial A}{\partial x^2} u_2 + \frac{A}{C^2}\frac{\partial A}{\partial x^3} u_3$$

Expressions for the other five independent components of the strain tensor may be derived in the above fashion. Thus:

$$e_{11} = \frac{\partial u_1}{\partial x^1} - \frac{1}{A}\frac{\partial A}{\partial x^1} u_1 + \frac{A}{B^2}\frac{\partial A}{\partial x^2} u_2 + \frac{A}{C^2}\frac{\partial A}{\partial x^3} u_3 \quad - (3.16a)$$

$$e_{22} = \frac{\partial u_2}{\partial x^2} - \frac{1}{B}\frac{\partial B}{\partial x^2} u_2 + \frac{B}{C^2}\frac{\partial B}{\partial x^3} u_3 + \frac{B}{A^2}\frac{\partial B}{\partial x^1} u_1 \quad - (3.16b)$$

$$e_{33} = \frac{\partial u_3}{\partial x^3} - \frac{1}{C}\frac{\partial C}{\partial x^3} u_3 + \frac{C}{A^2}\frac{\partial C}{\partial x^1} u_1 + \frac{C}{B^2}\frac{\partial C}{\partial x^2} u_2 \quad - (3.16c)$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x^2} - \frac{\partial u_2}{\partial x^1}\right) - \frac{1}{A}\frac{\partial A}{\partial x^2} u_1 - \frac{1}{B}\frac{\partial B}{\partial x^1} u_2 \quad - (3.16d)$$

$$e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x^3} - \frac{\partial u_3}{\partial x^2}\right) - \frac{1}{B}\frac{\partial B}{\partial x^3} u_2 - \frac{1}{C}\frac{\partial C}{\partial x^2} u_3 \quad - (3.16e)$$

$$e_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x^1} - \frac{\partial u_1}{\partial x^3}\right) - \frac{1}{C}\frac{\partial C}{\partial x^1} u_3 - \frac{1}{A}\frac{\partial A}{\partial x^3} u_1 \quad - (3.16f)$$

The components of a tensor, when transformed from a Cartesian system to another reference system may lose their physical significance and dimensionality (reference 6, page 102). To obtain the physical components of strain and displacement, it is necessary to transform the quantities $e_{ij}\, l^i m^j$ and $u_i l^i$, where $l^i$ and $m^j$ are the unit contravariant vectors in the directions of the $x^i$ and $x^j$ coordinate curves respectively. From equation (3.11):

$$l^i = \frac{\delta^i_k}{\sqrt{g_{kk}}} \quad ; \quad m^j = \frac{\delta^j_k}{\sqrt{g_{kk}}}$$

Therefore, denoting the physical strains by $\varepsilon_{ij}$ and the physical displacement vector by (u, v, w):

$$\varepsilon_{ij} = \frac{e_{ij}}{\sqrt{g_{ii}\, g_{jj}}} \quad \text{(no summation)} \quad - (3.17)$$

$$u = \frac{u_1}{A} \quad ; \quad v = \frac{u_2}{B} \quad ; \quad w = \frac{u_3}{C} \quad - (3.18)$$

Substituting in equations (3.16):

$$\varepsilon_{11} = \frac{1}{A}\frac{\partial u}{\partial x^1} + \frac{v}{AB}\frac{\partial A}{\partial x^2} + \frac{w}{AC}\frac{\partial A}{\partial x^3} \qquad - (3.19a)$$

$$\varepsilon_{22} = \frac{1}{B}\frac{\partial v}{\partial x^2} + \frac{w}{BC}\frac{\partial B}{\partial x^3} + \frac{u}{BA}\frac{\partial B}{\partial x^1} \qquad - (3.19b)$$

$$\varepsilon_{33} = \frac{1}{C}\frac{\partial w}{\partial x^3} + \frac{u}{CA}\frac{\partial C}{\partial x^1} + \frac{v}{CB}\frac{\partial C}{\partial x^2} \qquad - (3.19c)$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{B}{A}\frac{\partial}{\partial x^1}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial x^2}\left(\frac{u}{A}\right)\right] \qquad - (3.19d)$$

$$\varepsilon_{23} = \frac{1}{2}\left[\frac{C}{B}\frac{\partial}{\partial x^2}\left(\frac{w}{C}\right) + \frac{B}{C}\frac{\partial}{\partial x^3}\left(\frac{v}{B}\right)\right] \qquad - (3.19e)$$

$$\varepsilon_{31} = \frac{1}{2}\left[\frac{A}{C}\frac{\partial}{\partial x^3}\left(\frac{u}{A}\right) + \frac{C}{A}\frac{\partial}{\partial x^1}\left(\frac{w}{C}\right)\right] \qquad - (3.19f)$$

In obtaining the surface strains, it is not sufficient merely to consider the two dimensional form of equation (2.7), since this gives the strains as functions of a two dimensional surface displacement vector, whereas the actual displacement vector may be a three dimensional space vector. The disadvantage of equations (3.19) as they stand is that they are valid only for coordinate curves on a surface which are lines of curvature. However they may be generalised as follows

Take as an example the $x^2$ , $x^3$ coordinate surfaces, for which the relevant equations are (3.19b), (3.19c) and (3.19e). The displacement normal to the surfaces only appears in the equations for the direct strains, and reference to equation (3.14a) shows that they may be written:

$$\varepsilon_{11} = \frac{1}{B}\frac{\partial v}{\partial x^2} + \frac{w}{BC}\frac{\partial B}{\partial x^3} - \frac{u}{B^2}b^{(1)}_{11} \qquad - (3.20a)$$

$$\varepsilon_{22} = \frac{1}{C}\frac{\partial w}{\partial x^3} + \frac{v}{BC}\frac{\partial C}{\partial x^2} - \frac{u}{C^2}b^{(1)}_{22} \qquad - (3.20b)$$

and since $b^{(1)}_{12}$ is zero, (3.19e) can be re-written:

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{C}{B}\frac{\partial}{\partial x^2}\left(\frac{W}{C}\right) + \frac{B}{C}\frac{\partial}{\partial x^3}\left(\frac{V}{B}\right)\right] - \frac{u}{BC}\, b^{(1)}_{12} \quad (3.20c)$$

(The space strains $\varepsilon_{22}$ , $\varepsilon_{33}$ and $\varepsilon_{23}$ become, on the $x^2$ , $x^3$ surfaces, $e_{11}$ , $\varepsilon_{22}$ and $\varepsilon_{12}$ respectively).

Now u, the physical displacement normal to the $x^2$ , $x^3$ surfaces is an invariant w.r.t. surface coordinate transformations. Denoting the normal displacement to any surface by $u^{(n)}$ , it is evident, from equations (3.16) and (3.20) that the surface strains are given by:

$$e_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) - u^{(n)} b_{\alpha\beta} \quad - (3.21)$$

The set of quantities $e_{\alpha\beta}$ must be a tensor because each of the products $e_{\alpha\beta}\lambda^\alpha\lambda^\beta$ , where $\lambda^\alpha$ is an arbitrary contravariant surface vector, is an invariant, being the same as $e_{ij}\lambda^i\lambda^j$ , where the space vector $\lambda^i$ has the same magnitude and direction as the surface vector $\lambda^\alpha$ . (This is an application of the quotient law in a form which is valid for a doubly-subscripted set of quantities, symmetrical in its subscripts, as $e_{\alpha\beta}$ is: see reference 6, page 13.) The r.h.s. of equation (3.21) is evidently in tensor form and therefore the equation is a tensor equation. It is valid for all surfaces and all coordinate systems. In particular for a general orthogonal coordinate system, the physical strains are given by:

$$\varepsilon_{11} = \frac{1}{A}\frac{\partial u}{\partial x^1} + \frac{v}{AB}\frac{\partial A}{\partial x^2} - s_{(1)}\, u^{(n)} \quad - (3.22a)$$

$$\varepsilon_{22} = \frac{1}{B}\frac{\partial v}{\partial x^2} + \frac{u}{AB}\frac{\partial B}{\partial x^1} - s_{(2)}\, u^{(n)} \quad - (3.22b)$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{B}{A}\frac{\partial}{\partial x^1}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial x^2}\left(\frac{u}{A}\right)\right] - s_{(3)}\, u^{(n)} \quad (3.22c)$$

where the physical (surface) displacement vector is (u, v), the fundamental tensor is given by equation (3.3c), $s_{(1)}$ and $s_{(2)}$ are the normal curvatures of the surface in the coordinate directions, and $s_{(3)}$ is $b_{12}/AB$. The equations for a plane surface are obtained by putting $s_{(1)}$ , $s_{(2)}$ and $s_{(3)}$ equal to zero. They are identical to the equations obtained by expanding the two dimensional form of equation (2.7).

| | SPACE CHRISTOFFEL SYMBOL | SURFACE CHRISTOFFEL SYMBOL | |
|---|---|---|---|
| SURFACE CONTAINING SPACE COORDINATES $x^1$ , $x^2$ | $\left[\begin{matrix}1\\1,1\end{matrix}\right]$ | $\left[\begin{matrix}1\\1,1\end{matrix}\right]$ | $A\partial A/\partial x^1$ |
| | $\left[\begin{matrix}1\\1,2\end{matrix}\right]$ | $\left[\begin{matrix}1\\1,2\end{matrix}\right]$ | $-A\partial A/\partial x^2$ |
| | $\left[\begin{matrix}1\\2,1\end{matrix}\right]$ | $\left[\begin{matrix}1\\2,1\end{matrix}\right]$ | $A\partial A/\partial x^2$ |
| | $\left[\begin{matrix}2\\1,1\end{matrix}\right]$ | $\left[\begin{matrix}2\\1,1\end{matrix}\right]$ | $A\partial A/\partial x^2$ |
| | $\left[\begin{matrix}1\\2,2\end{matrix}\right]$ | $\left[\begin{matrix}1\\2,2\end{matrix}\right]$ | $B\partial B/\partial x^1$ |
| | $\left[\begin{matrix}2\\2,1\end{matrix}\right]$ | $\left[\begin{matrix}2\\2,1\end{matrix}\right]$ | $-B\partial B/\partial x^1$ |
| | $\left[\begin{matrix}2\\1,2\end{matrix}\right]$ | $\left[\begin{matrix}2\\1,2\end{matrix}\right]$ | $B\partial B/\partial x^1$ |
| | $\left[\begin{matrix}2\\2,2\end{matrix}\right]$ | $\left[\begin{matrix}2\\2,2\end{matrix}\right]$ | $B\partial B/\partial x^2$ |
| SURFACE CONTAINING SPACE COORDINATES $x^1$ $x^3$ | $\left[\begin{matrix}1\\1,1\end{matrix}\right]$ | $\left[\begin{matrix}1\\1,1\end{matrix}\right]$ | $A\partial A/\partial x^1$ |
| | $\left[\begin{matrix}1\\1,3\end{matrix}\right]$ | $\left[\begin{matrix}1\\1,2\end{matrix}\right]$ | $-A\partial A/\partial x^3$ |
| | $\left[\begin{matrix}1\\3,1\end{matrix}\right]$ | $\left[\begin{matrix}1\\2,1\end{matrix}\right]$ | $A\partial A/\partial x^3$ |
| | $\left[\begin{matrix}3\\1,1\end{matrix}\right]$ | $\left[\begin{matrix}2\\1,1\end{matrix}\right]$ | $A\partial A/\partial x^3$ |
| | $\left[\begin{matrix}1\\3,3\end{matrix}\right]$ | $\left[\begin{matrix}1\\2,2\end{matrix}\right]$ | $C\partial C/\partial x^1$ |
| | $\left[\begin{matrix}3\\3,1\end{matrix}\right]$ | $\left[\begin{matrix}2\\2,1\end{matrix}\right]$ | $-C\partial C/\partial x^1$ |
| | $\left[\begin{matrix}3\\1,3\end{matrix}\right]$ | $\left[\begin{matrix}2\\1,2\end{matrix}\right]$ | $C\partial C/\partial x^1$ |
| | $\left[\begin{matrix}3\\3,3\end{matrix}\right]$ | $\left[\begin{matrix}2\\2,2\end{matrix}\right]$ | $C\partial C/\partial x^3$ |

TABLE 1

CHRISTOFFEL SYMBOLS OF THE FIRST KIND

| | SPACE CHRISTOFFEL SYMBOL | SURFACE CHRISTOFFEL SYMBOL | |
|---|---|---|---|
| SURFACE CONTAINING SPACE COORDINATES $x^2$ $x^3$ | $\left[\begin{matrix}2\\2,2\end{matrix}\right]$ | $\left[\begin{matrix}1\\1,1\end{matrix}\right]$ | $B\partial B/\partial x^2$ |
| | $\left[\begin{matrix}2\\2,3\end{matrix}\right]$ | $\left[\begin{matrix}1\\1,2\end{matrix}\right]$ | $-B\partial B/\partial x^3$ |
| | $\left[\begin{matrix}2\\3,2\end{matrix}\right]$ | $\left[\begin{matrix}1\\2,1\end{matrix}\right]$ | $B\partial B/\partial x^3$ |
| | $\left[\begin{matrix}3\\2,2\end{matrix}\right]$ | $\left[\begin{matrix}2\\1,1\end{matrix}\right]$ | $B\partial B/\partial x^3$ |
| | $\left[\begin{matrix}2\\3,3\end{matrix}\right]$ | $\left[\begin{matrix}1\\2,2\end{matrix}\right]$ | $C\partial C/\partial x^2$ |
| | $\left[\begin{matrix}2\\2,3\end{matrix}\right]$ | $\left[\begin{matrix}2\\2,1\end{matrix}\right]$ | $-C\partial C/\partial x^2$ |
| | $\left[\begin{matrix}3\\2,3\end{matrix}\right]$ | $\left[\begin{matrix}2\\1,2\end{matrix}\right]$ | $C\partial C/\partial x^2$ |
| | $\left[\begin{matrix}3\\3,3\end{matrix}\right]$ | $\left[\begin{matrix}2\\2,2\end{matrix}\right]$ | $C\partial C/\partial x^3$ |
| | $\left[\begin{matrix}1\\2,3\end{matrix}\right]$ | | 0 |
| | $\left[\begin{matrix}3\\1,2\end{matrix}\right]$ | | 0 |
| | $\left[\begin{matrix}2\\3,1\end{matrix}\right]$ | | 0 |
| | $\left[\begin{matrix}1\\3,2\end{matrix}\right]$ | | 0 |
| | $\left[\begin{matrix}2\\1,3\end{matrix}\right]$ | | 0 |
| | $\left[\begin{matrix}3\\2,1\end{matrix}\right]$ | | 0 |

TABLE 1 CONT'D

CHRISTOFFEL SYMBOLS OF THE FIRST KIND

| | SPACE CHRISTOFFEL SYMBOL | SURFACE CHRISTOFFEL SYMBOL | |
|---|---|---|---|
| SURFACE CONTAINING SPACE COORDINATES $x^1$ $x^2$ | $\left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\}$ | $\frac{1}{A}\frac{\partial A}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 1 \\ 1\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 1\ 2 \end{matrix} \right\}$ | $\frac{1}{A}\frac{\partial A}{\partial x^2}$ |
| | $\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\}$ | $\frac{1}{A}\frac{\partial A}{\partial x^2}$ |
| | $\left\{ \begin{matrix} 2 \\ 1\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 1\ 1 \end{matrix} \right\}$ | $-\frac{A}{B^2}\frac{\partial A}{\partial x^2}$ |
| | $\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\}$ | $-\frac{B}{A^2}\frac{\partial B}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\}$ | $\frac{1}{B}\frac{\partial B}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\}$ | $\frac{1}{B}\frac{\partial B}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 2 \\ 2\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 2\ 2 \end{matrix} \right\}$ | $\frac{1}{B}\frac{\partial B}{\partial x^2}$ |

TABLE 2

CHRISTOFFEL SYMBOLS OF THE SECOND KIND

| | SPACE CHRISTOFFEL SYMBOL | SURFACE CHRISTOFFEL SYMBOL | |
|---|---|---|---|
| SURFACE CONTAINING SPACE COORDINATES $x^1$ $x^3$ | $\left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\}$ | $\frac{1}{A}\frac{\partial A}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 1 \\ 1\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 1\ 2 \end{matrix} \right\}$ | $\frac{1}{A}\frac{\partial A}{\partial x^3}$ |
| | $\left\{ \begin{matrix} 1 \\ 3\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\}$ | $\frac{1}{A}\frac{\partial A}{\partial x^3}$ |
| | $\left\{ \begin{matrix} 3 \\ 1\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 1\ 1 \end{matrix} \right\}$ | $-\frac{A}{C^2}\frac{\partial A}{\partial x^3}$ |
| | $\left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\}$ | $-\frac{C}{A^2}\frac{\partial C}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 3 \\ 3\ 1 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\}$ | $\frac{1}{C}\frac{\partial C}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 3 \\ 1\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\}$ | $\frac{1}{C}\frac{\partial C}{\partial x^1}$ |
| | $\left\{ \begin{matrix} 3 \\ 3\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 2\ 2 \end{matrix} \right\}$ | $\frac{1}{C}\frac{\partial C}{\partial x^3}$ |

TABLE 2 CONT'D

CHRISTOFFEL SYMBOLS OF THE SECOND KIND

| | SPACE CHRISTOFFEL SYMBOL | SURFACE CHRISTOFFEL SYMBOL | |
|---|---|---|---|
| SURFACE CONTAINING SPACE COORDINATES $x^2$ $x^3$ | $\left\{ \begin{matrix} 2 \\ 2\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\}$ | $\frac{1}{B}\frac{\partial B}{\partial x^2}$ |
| | $\left\{ \begin{matrix} 2 \\ 2\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 1\ 2 \end{matrix} \right\}$ | $\frac{1}{B}\frac{\partial B}{\partial x^3}$ |
| | $\left\{ \begin{matrix} 2 \\ 3\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\}$ | $\frac{1}{B}\frac{\partial B}{\partial x^3}$ |
| | $\left\{ \begin{matrix} 3 \\ 2\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 1\ 1 \end{matrix} \right\}$ | $\frac{-B}{C^2}\frac{\partial B}{\partial x^3}$ |
| | $\left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\}$ | $\frac{-C}{B^2}\frac{\partial C}{\partial x^2}$ |
| | $\left\{ \begin{matrix} 3 \\ 3\ 2 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\}$ | $\frac{1}{C}\frac{\partial C}{\partial x^2}$ |
| | $\left\{ \begin{matrix} 3 \\ 2\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\}$ | $\frac{1}{C}\frac{\partial C}{\partial x^2}$ |
| | $\left\{ \begin{matrix} 3 \\ 3\ 3 \end{matrix} \right\}$ | $\left\{ \begin{matrix} 2 \\ 2\ 2 \end{matrix} \right\}$ | $\frac{1}{C}\frac{\partial C}{\partial x^3}$ |

TABLE 2 CONT'D

CHRISTOFFEL SYMBOLS OF THE SECOND KIND

| SPACE CHRISTOFFEL SYMBOL | SURFACE CHRISTOFFEL SYMBOL | |
|---|---|---|
| $\left\{ \begin{matrix} 1 \\ 2\ 3 \end{matrix} \right\}$ | | 0 |
| $\left\{ \begin{matrix} 3 \\ 1\ 2 \end{matrix} \right\}$ | | 0 |
| $\left\{ \begin{matrix} 2 \\ 3\ 1 \end{matrix} \right\}$ | | 0 |
| $\left\{ \begin{matrix} 1 \\ 3\ 2 \end{matrix} \right\}$ | | 0 |
| $\left\{ \begin{matrix} 2 \\ 1\ 3 \end{matrix} \right\}$ | | 0 |
| $\left\{ \begin{matrix} 3 \\ 2\ 1 \end{matrix} \right\}$ | | 0 |

TABLE 2 CONT'D

CHRISTOFFEL SYMBOLS OF THE SECOND KIND

# CHAPTER 4

## LAYOUT NETWORKS FOR OPTIMUM FRAMEWORKS

The layout lines for optimum frameworks are lines of constant principal strain $\pm\varepsilon$ in a compatible deformation of space. A deformation in which the principal strains are all of the same magnitude and also all of the same sign leads to frameworks whose members are either all in tension or all in compression, with no restriction on layout. Optimum frameworks containing both tension and compression members derive from deformations consisting of positive and negative principal strains and there are therefore certain conditions of orthogonality to be satisfied.

### Global Optima

The virtual deformation appropriate to a framework which is a global optimum must be compatible over the whole of 3-dimensional space. Lines of principal strain may be used as a system of orthogonal coordinate curves, referred to which the strain tensor may be written in the form:

$$e_{ij} = \begin{bmatrix} \varepsilon_{11}(x^1, x^2, x^3)A^2 & 0 & 0 \\ 0 & \varepsilon B^2 & 0 \\ 0 & 0 & -\varepsilon C^2 \end{bmatrix} \qquad - (4.1)$$

where $\varepsilon_{11}$ is a function of $x^1$, $x^2$ and $x^3$ and $\varepsilon$ is a constant (positive or negative). $\varepsilon_{11}$ and $\varepsilon$ are the values of the physical strains. For frameworks whose members are confined to a single surface, or family of non-intersecting surfaces, these surfaces may be used as $x^2$, $x^3$ coordinate surfaces and the condition on $\varepsilon_{11}$ is:

$$|\varepsilon_{11}| \leqslant |\varepsilon| \qquad - (4.2)$$

If the members are not so confined, the equality must be satisfied. In this case there is one set of coordinate surfaces on which the principal strains are the same in magnitude and sign, and on such surfaces framework members need not be constrained to lie along the (orthogonal) coordinate curves, but may take up any direction on a surface.

The tensor $e_{ij}$ must be of the form defined by equation (2.7). In this equation the components of the vector $u_i$ are single-valued functions of the coordinates and therefore $u_i^{(2)}$, the value of $u_i$ at the point $P_{(2)}$ whose coordinates are $x^i_{(2)}$, is related to $u_i^{(1)}$, the value of $u_i$ at the point $P_{(1)}$ with coordinates $x^i_{(1)}$, by the expression:

$$u_i^{(2)} = u_i^{(1)} + \int_{P_{(1)}}^{P_{(2)}} du_i$$

where the integral is taken along any continuous curve joing $P_{(0)}$ and $P_{(2)}$. Choosing rectangular Cartesian coordinates, so that covariant derivatives are simply partial derivatives, the above may be written:

$$u_i^{(2)} = u_i^{(0)} + \int_{P_{(0)}}^{P_{(2)}} u_{i,j}\, dx^j \qquad - (4.3)$$

(Equation (4.3) is not a tensor equation, but the tensor notation and summation convention are retained for convenience in writing). The expression under the integral sign may be reduced to one containing only the components of $e_{ij}$ and their derivatives by the following process. From equation (2.7):

$$u_{i,j} = e_{ij} + \tfrac{1}{2}(u_{i,j} - u_{j,i})$$

$$\therefore\ u_i^{(2)} = u_i^{(0)} + \int_{P_{(0)}}^{P_{(2)}} e_{ij}\, dx^j + \tfrac{1}{2}\int_{P_{(0)}}^{P_{(2)}} (u_{i,j} - u_{j,i})\, dx^j$$

$$= u_i^{(0)} + \int_{P_{(0)}}^{P_{(2)}} e_{ij}\, dx^j + \tfrac{1}{2}\int_{P_{(0)}}^{P_{(2)}} (u_{i,j} - u_{j,i})\, d(x^j - x^j_{(2)})$$

Integrating by parts:

$$u_i^{(2)} = u_i^{(0)} + \int_{P_{(0)}}^{P_{(2)}} e_{ij}\, dx^j + \left[\tfrac{1}{2}(u_{i,j} - u_{j,i})(x^j - x^j_{(2)})\right]_{P_{(0)}}^{P_{(2)}} - \int_{P_{(0)}}^{P_{(2)}} (x^j - x^j_{(2)})\, d\left[\tfrac{1}{2}(u_{i,j} - u_{j,i})\right]$$

$$= u_i^{(0)} + \tfrac{1}{2}(x^j_{(2)} - x^j_{(0)})(u^{(0)}_{i,j} - u^{(0)}_{j,i}) + \int_{P_{(0)}}^{P_{(2)}} \left[e_{il} + \tfrac{1}{2}(x^j_{(2)} - x^j)(u_{i,jl} - u_{j,il})\right] dx^l$$

Now

$$\tfrac{1}{2}(u_{i,jl} - u_{j,il}) = \tfrac{1}{2}(u_{i,jl} + u_{l,ij}) - \tfrac{1}{2}(u_{j,il} + u_{l,ij})$$

$$= \tfrac{1}{2}(u_{i,lj} + u_{l,ij}) - \tfrac{1}{2}(u_{j,li} + u_{l,ji})$$

$$= e_{il,j} - e_{jl,i} \qquad \text{(using (2.7))}$$

Therefore

$$u_i^{(2)} = u_i^{(0)} + \tfrac{1}{2}(x^j_{(2)} - x^j_{(0)})(u^{(0)}_{i,j} - u^{(0)}_{j,i}) + \int_{P_{(0)}}^{P_{(2)}} U_{il}\, dx^l$$

where $U_{il} = e_{il} + (x^j_{(2)} - x^j)(e_{il,j} - e_{jl,i})$

Since the value of $u_i^{(2)}$ must be independent of the path joining $P_{(1)}$ to $P_{(2)}$, $U_{il}\, dx^l$ must be a perfect differential i.e.

$$U_{il,k} = U_{ik,l}$$

and so:

$$\left[e_{il,k} - \delta^k_i (e_{il,j} - e_{jl,i}) - e_{ik,l} + \delta^j_i (e_{ik,j} - e_{jk,i})\right] + (x^j_{(2)} - x^j)(e_{il,jk} - e_{jl,ik} - e_{ik,jl} + e_{jk,il}) = 0$$

The expression in square brackets is identically zero, and since the equation must be true for any $(x^j_{(2)} - x^j)$:

$$e_{il,jk} + e_{jk,il} - e_{jl,ik} - e_{ik,jl} = 0 \qquad - (4.4)$$

Equation (4.4) was derived using a Cartesian coordinate system. However it is in tensor form and is therefore valid in any coordinate system, the partial derivatives being replaced by covariant derivatives. The only limitation of the derivation (due to Sokolnikoff - reference 10) is that it is valid only in spaces which admit of a Cartesian coordinate system i.e. Euclidean spaces. If $e_{ij}$ is a strain tensor, then equation (4.4) is the compatibility condition for small strains. However, while the physical ideas of strain and compatibility of strain are only represented approximately by the equations (2.7) and (4.4), it is emphasised that the mathematical derivation of (4.4) from (2.7) contains no approximations: (4.4) represents the sufficient condition that an arbitrary tensor field $e_{ij}$ can be expressed in terms of a vector field $u_i$ in the form given by (2.7). The necessity of the condition is easily shown by substituting from (2.7) into (4.4), the latter being identically satisfied.

Due to the symmetry of the pairs of indices il and jk, equation (4.4) provides only six independent equations:

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 0 \qquad - (4.5a)$$

$$e_{11,33} + e_{33,11} - 2e_{13,13} = 0 \qquad - (4.5b)$$

$$e_{22,33} + e_{33,22} - 2e_{23,23} = 0 \qquad - (4.5c)$$

$$e_{33,12} + e_{12,33} - e_{31,32} - e_{32,31} = 0 \qquad - (4.5d)$$

$$e_{11,23} + e_{23,11} - e_{12,13} - e_{13,12} = 0 \qquad - (4.5e)$$

$$e_{22,31} + e_{31,22} - e_{23,21} - e_{21,23} = 0 \qquad - (4.5f)$$

They may be expanded as follows. The general term in the equations is given by:

$$e_{ij,kl} = \frac{\partial e_{ij,k}}{\partial x^l} - \begin{Bmatrix} r \\ il \end{Bmatrix} e_{rj,k} - \begin{Bmatrix} r \\ jl \end{Bmatrix} e_{ir,k} - \begin{Bmatrix} r \\ kl \end{Bmatrix} e_{ij,r} \qquad - (4.6)$$

Thus it is necessary to form all the first order covariant derivatives, which may be done using the relation:

$$e_{ij,k} = \frac{\partial e_{ij}}{\partial x^k} - \begin{Bmatrix} l \\ ik \end{Bmatrix} e_{lj} - \begin{Bmatrix} l \\ jk \end{Bmatrix} e_{il}$$

and so

$$e_{11,1} = \frac{\partial e_{11}}{\partial x^1} - \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} e_{11} - \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} e'_{21} - \begin{Bmatrix} 3 \\ 11 \end{Bmatrix} e'_{31} - \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} e_{11} - \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} e'_{12} - \begin{Bmatrix} 3 \\ 11 \end{Bmatrix} e'_{13}$$

$$= \frac{\partial (\varepsilon_{11} A^2)}{\partial x^1} - \frac{2}{A} \frac{\partial A}{\partial x^1} \varepsilon_{11} A^2$$

(using equation (4.1) and Table 2)

$$= A^2 \frac{\partial \varepsilon_{11}}{\partial x^1}$$

The other first order covariant derivatives are similarly calculated and are listed in Table 3. From equation (4.6) and Tables 2 and 3:

$$e_{11,22} = \frac{\partial e_{11,2}}{\partial x^2} - \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} e_{11,2} - \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} e_{21,2} - \begin{Bmatrix} 3 \\ 12 \end{Bmatrix} e_{31,2}$$

$$- \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} e_{11,2} - \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} e_{12,2} - \begin{Bmatrix} 3 \\ 12 \end{Bmatrix} e_{13,2} - \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} e_{11,1} - \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} e_{11,2} - \begin{Bmatrix} 3 \\ 22 \end{Bmatrix} e_{11,3}$$

$$= \frac{\partial}{\partial x^2}\left(A^2 \frac{\partial \varepsilon_{11}}{\partial x^2}\right) - \frac{2}{A}\frac{\partial A}{\partial x^2} A^2 \frac{\partial \varepsilon_{11}}{\partial x^2} - \frac{2}{B}\frac{\partial B}{\partial x^1} B \frac{\partial B}{\partial x^1}(\varepsilon_{11} - \varepsilon)$$

$$+ \frac{B}{A^2}\frac{\partial B}{\partial x^1} A^2 \frac{\partial \varepsilon_{11}}{\partial x^1} - \frac{1}{B}\frac{\partial B}{\partial x^2} A^2 \frac{\partial \varepsilon_{11}}{\partial x^2} + \frac{B}{C^2}\frac{\partial B}{\partial x^3} A^2 \frac{\partial \varepsilon_{11}}{\partial x^3}$$

$$= A^2\frac{\partial^2\varepsilon_{11}}{\partial x^3\partial x^2} + B\frac{\partial B}{\partial x^1}\frac{\partial\varepsilon_{11}}{\partial x^1} - \frac{A^2}{B}\frac{\partial B}{\partial x^2}\frac{\partial\varepsilon_{11}}{\partial x^2} + \frac{BA^2}{C^2}\frac{\partial B}{\partial x^3}\frac{\partial\varepsilon_{11}}{\partial x^3} - 2\left(\frac{\partial B}{\partial x^1}\right)^2\varepsilon_{11} + 2\left(\frac{\partial B}{\partial x^1}\right)^2\varepsilon$$

Similarly:

$$\varepsilon_{22,11} = 2\left(\frac{\partial A}{\partial x^2}\right)^2\varepsilon_{11} - 2\left(\frac{\partial A}{\partial x^2}\right)^2\varepsilon$$

$$\varepsilon_{12,12} = B\frac{\partial B}{\partial x^1}\frac{\partial\varepsilon_{11}}{\partial x^1} - A\frac{\partial A}{\partial x^2}\frac{\partial\varepsilon_{11}}{\partial x^2} + \left[\left(\frac{\partial A}{\partial x^2}\right)^2 - \left(\frac{\partial B}{\partial x^1}\right)^2 - A\frac{\partial^2 A}{\partial x^2\partial x^2} + \frac{A}{B}\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2}\right]\varepsilon_{11}$$
$$-\left[\frac{AB}{C^2}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right]\varepsilon_{11} - \left[\left(\frac{\partial A}{\partial x^2}\right)^2 - \left(\frac{\partial B}{\partial x^1}\right)^2 - A\frac{\partial^2 A}{\partial x^2\partial x^2} + \frac{A}{B}\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2} + \frac{AB}{C^2}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right]\varepsilon$$

Substituting in (4.5a):

$$A^2\frac{\partial^2\varepsilon_{11}}{\partial x^2\partial x^2} + A^2\frac{\partial\varepsilon_{11}}{\partial x^2}\left(\frac{2}{A}\frac{\partial A}{\partial x^2} - \frac{1}{B}\frac{\partial B}{\partial x^2}\right) + \frac{\partial\varepsilon_{11}}{\partial x^3}\left(\frac{BA^2}{C^2}\frac{\partial B}{\partial x^3}\right)$$

$$- \frac{\partial\varepsilon_{11}}{\partial x^1}B\frac{\partial B}{\partial x^1} + 2AB\varepsilon_{11}\left[\frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial A}{\partial x^2}\right) + \frac{1}{C^2}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right]$$

$$- 2AB\varepsilon\left[\frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial A}{\partial x^2}\right) - \frac{1}{C^2}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right] = 0 \quad - (4.7a)$$

The remaining five of equations (4.5) may be expanded in the above fashion to give:

$$A^2\frac{\partial^2\varepsilon_{11}}{\partial x^3\partial x^3} + A^2\frac{\partial\varepsilon_{11}}{\partial x^3}\left(\frac{2}{A}\frac{\partial A}{\partial x^3} - \frac{1}{C}\frac{\partial C}{\partial x^3}\right) + \frac{\partial\varepsilon_{11}}{\partial x^2}\left(\frac{CA^2}{B^2}\frac{\partial C}{\partial x^2}\right)$$

$$- \frac{\partial\varepsilon_{11}}{\partial x^1}C\frac{\partial C}{\partial x^1} + 2AC\varepsilon_{11}\left[\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial A}{\partial x^3}\right) + \frac{1}{B^2}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2}\right]$$

$$+ 2AC\varepsilon\left[\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial A}{\partial x^3}\right) - \frac{1}{B^2}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2}\right] = 0 \quad - (4.7b)$$

$$\varepsilon_{11}\left[\frac{1}{A^2}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1}\right] - \varepsilon\left[2\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial B}{\partial x^3}\right) + \frac{1}{A^2}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1}\right] = 0 \quad - (4.7c)$$

$$\frac{\partial C}{\partial x^1}\frac{\partial\varepsilon_{11}}{\partial x^2} = 0 \quad - (4.7d)$$

$$A^2 \frac{\partial^2 \varepsilon_{11}}{\partial x^2 \partial x^3} + \left[ A \frac{\partial A}{\partial x^3} - \frac{A^2}{B} \frac{\partial B}{\partial x^3} \right] \frac{\partial \varepsilon_{11}}{\partial x^2}$$

$$+ \left[ A \frac{\partial A}{\partial x^2} - \frac{A^2}{C} \frac{\partial C}{\partial x^2} \right] \frac{\partial \varepsilon_{11}}{\partial x^3} = 0 \quad - (4.7e)$$

$$\frac{\partial B}{\partial x^1} \frac{\partial \varepsilon_{11}}{\partial x^3} = 0 \quad - (4.7f)$$

where equations (4.7d), (4.7e) and (4.7f) have been simplified using equations (3.7e), (3.7f) and (3.7d) respectively.

Restricted Optima

The conditions expressed by equations (4.2) and (4.1) are to be satisfied by the layout curves of frameworks which are global optima. The nature of the restrictions on the layout curves of frameworks which are the best only of those whose members are constrained to lie on a given surface is different, the reason being that surface strains, $e_{\alpha\beta}$, defined as in equation (3.21) in terms of a three dimensional displacement, are not in general required to satisfy any compatibility conditions: the components of the strain tensor may be specified as arbitrary functions of the surface coordinates and the result is always a compatible deformation, provided the tensor $b_{\alpha\beta}$ is not the zero tensor i.e. the surface is not plane. This may be proved as follows.

From equation (3.21):

$$e_{\alpha\beta} = \bar{e}_{\alpha\beta} - u^{(n)} b_{\alpha\beta} \quad - (4.8)$$

where $$\bar{e}_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad - (4.9)$$

Now $\bar{e}_{\alpha\beta}$ has the same form as $e_{ij}$, and the possibility presents itself of eliminating $\bar{e}_{\alpha\beta}$ from (4.8) using the two dimensional form of equation (4.4) i.e.:

$$e_{\alpha\beta\gamma\delta} + e_{\gamma\delta,\alpha\beta} - e_{\gamma\beta,\alpha\delta} - e_{\alpha\delta,\gamma\beta} = 0 \quad - (4.10)$$

Equation (4.10) is indentically satisfied, in Euclidean space, by expressions of the type defined by (4.9). However a curved surface is not a Euclidean space; the curvature tensor is non-zero, which means that covariant differentiation is not commutative (reference 6, page 51) e.g. $u_{\alpha,\beta\gamma}$ is not the same as $u_{\alpha,\gamma\beta}$, and therefore (4.10) is not identically satisfied by the $\bar{e}_{\alpha\beta}$. This difficulty may be overcome by concentrating attention on a particular point on the surface and making it the origin of a Riemannian coordinate system (reference 8, pages 59-62). At this point covariant derivatives become partial derivatives, all the Christoffel symbols being zero, and therefore equation (4.10) is satisfied

by the $\bar{e}_{\alpha\beta}$ . In two dimensions (4.10) gives only one independent equation:

$$\bar{e}_{11,22} + \bar{e}_{22,11} - 2\bar{e}_{12,12} = 0 \qquad - (4.11)$$

Thus, from (4.8), (4.9) and (4.11):

$$\frac{\partial^2}{\partial x^2 \partial x^2}\left(e_{11} + u^{(n)} b_{11}\right) + \frac{\partial^2}{\partial x^1 \partial x^1}\left(e_{22} + u^{(n)} b_{22}\right) - 2\frac{\partial^2}{\partial x^1 \partial x^2}\left(e_{21} + u^{(n)} b_{21}\right) = 0 \qquad - (4.12a)$$

Equation (4.12a) is a differential equation in $u^{(n)}$ , formed from the three independent equations represented by the tensor equation (4.8) by eliminating $u_1$ and $u_2$ . From (4.8) and (4.9):

$$\frac{\partial u_1}{\partial x^1} = e_{11} + u^{(n)} b_{11} \qquad - (4.12b)$$

$$\frac{\partial u_2}{\partial x^2} = e_{22} + u^{(n)} b_{22} \qquad - (4.12c)$$

The three equations (4.12) enable the three displacement components $u^{(n)}$ , $u_1$ and $u_2$ to be determined at the origin of a Riemannian coordinate system when the three components of the strain tensor $e_{\alpha\beta}$ are specified, and since any point may be made the origin of a Riemannian coordinate system, it is clear that, in general, there is no relation to be satisfied by the components of $e_{\alpha\beta}$ . However in the particular case of a plane surface, for which $b_{\alpha\beta}$ is the zero tensor, $e_{\alpha\beta}$ is the same as $\bar{e}_{\alpha\beta}$ and its components must satisfy equation (4.11), not only at the origin of a Riemannian coordinate system, but at any point in any coordinate system, since the curvature tensor is zero. The foregoing results are a direct consequence of the mathematical relationship between the tensor $e_{\alpha\beta}$ and the vector $u_i$ . They may be interpreted physically as demonstrating that a curved surface can deform to accommodate an arbitrarily specified strain pattern whereas for a plane surface and a solid the components of the strain tensor are not completely arbitrary. The reason for this is that a curved surface, which is a general Riemannian two-space (a Riemannian n-space may always be regarded as immersed in a Euclidean m-space provided that

$$m \geq \frac{1}{2} n(n+1) \qquad \text{(reference 7 page 51)}$$

which means that any Riemannian two-space is a surface in Euclidean 3-space) can deform to give a different space with different metric properties, whereas a solid must constitute the same Euclidean space before and after deformation, and likewise a plane surface, subject to the limitations imposed by small displacement theory. The compatibility equations for solids and plane surfaces are the conditions that the curvature tensor of the appropriate space be the zero tensor after a deformation (see, for example, reference 11 pages 38-42).

To obtain the compatibility condition for plane strain, consider the $x^2$ , $x^3$ coordinate surfaces, for which the fundamental tensor is given by equation (3.3a). The strain tensor is:

$$e_{\alpha\beta} = \begin{bmatrix} -\varepsilon B^2 & 0 \\ 0 & \varepsilon C^2 \end{bmatrix} \qquad - (4.13)$$

and the first order covariant derivatives $e_{\alpha\beta,\gamma}$ are the same as the corresponding space derivatives e.g. the surface component $e_{12,1}$ is the same as the space component $e_{22,2}$ as given in table 3. The second order covariant derivatives are given by a similar equation to (4.6). Thus:

$$e_{11,22} = \frac{\partial e'_{11,2}}{\partial x^3} - \left\{ {1 \atop 12} \right\} e'_{11,2} - \left\{ {2 \atop 12} \right\} e_{21,2} - \left\{ {1 \atop 12} \right\} e'_{11,2} - \left\{ {2 \atop 12} \right\} e_{12,2}$$
$$\neq 0 \qquad \neq 0 \qquad \neq 0$$
$$- \left\{ {1 \atop 22} \right\} e'_{11,1} - \left\{ {2 \atop 22} \right\} e'_{11,2}$$
$$\neq 0 \qquad \neq 0$$

$$= -\frac{2}{C}\frac{\partial C}{\partial x^2} \cdot 2\varepsilon C \frac{\partial C}{\partial x^2}$$

$$= -4\varepsilon\left(\frac{\partial C}{\partial x^2}\right)^2$$

Similarly:

$$e_{22,11} = 4\varepsilon\left(\frac{\partial B}{\partial x^3}\right)^2$$

$$e_{12,12} = 2\varepsilon\left(\frac{\partial B}{\partial x^3}\right)^2 - 2\varepsilon\left(\frac{\partial C}{\partial x^2}\right)^2 - 2\varepsilon B\frac{\partial^2 B}{\partial x^3 \partial x^3} + \frac{2B}{C}\varepsilon\left(\frac{\partial B}{\partial x^3}\right)\left(\frac{\partial C}{\partial x^3}\right)$$

Substituting in (4.11):

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial B}{\partial x^3}\right) = 0$$

The condition for flatness, $R_{1212} = 0$, is obtained by equating the l.h.s. of (3.7b) to zero, and therefore the layout equations for optimum plane frameworks are:

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial B}{\partial x^3}\right) = \frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial C}{\partial x^2}\right) = 0 \qquad - (4.14)$$

(An alternative derivation of these equations is given in Appendix A). Analytic solutions of (4.14) have been obtained by Hemp (reference 4) and Ghista and Resnikoff (reference 5). Cox (reference 3) describes a graphical method of obtaining solutions.

It would appear at this stage as though any orthogonal network on a curved surface could be used as a layout network for an optimum framework. However considerations of equilibrium lead to certain restrictions, as will be demonstrated later. These restrictions do not apply to plane frameworks and, in this case, equations (4.14) are the only ones to be satisfied.

Particular Cases of Equations (4.7)

Consider equations (4.7) with $\varepsilon_{11}$ equal to $\varepsilon$. They reduce to:

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial A}{\partial x^3}\right) = 0 \qquad \text{- (4.15a)}$$

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial B}{\partial x^3}\right) = 0 \qquad \text{- (4.15b)}$$

$$\frac{\partial B}{\partial x^3}\frac{\partial A}{\partial x^3} = 0 \qquad \text{- (4.15c)}$$

(An alternative derivation of these equations is given in Appendix A). Now $(1/AC)\partial A/\partial x^3$ and $(1/BC)\partial B/\partial x^3$ are, by equation (3.14c), the principal curvatures of the $x^1$, $x^2$ surfaces, which, in this case, are the surfaces with principal strains equal in sign and magnitude. Thus optimum frameworks in three dimensions are to be laid out with all the members having one particular loading sign on coordinate surfaces for which one principal curvature is zero, and the members with the opposite sign of loading along the normals to these surfaces.

Suppose that the $x^2$, $x^3$ surfaces are parallel planes and the $x^1$ curves the straight lines normal to these planes. The $x^2$, $x^3$ surfaces have both principal curvatures zero and the other two sets of surfaces have one principal curvature zero, namely that along the $x^1$ direction. Therefore, from equations (3.14):

$$\frac{\partial B}{\partial x^1} = \frac{\partial C}{\partial x^1} = \frac{\partial A}{\partial x^2} = \frac{\partial A}{\partial x^3} = 0 \qquad \text{- (4.16)}$$

Substituting equations (4.16) into equations (3.7) and (4.15) provides the conditions that the $x^1$, $x^2$, $x^3$ curves can be strained $+\varepsilon, +\varepsilon, -\varepsilon$ respectively, and the result is the same as equations (4.14). Optimum plane frameworks are thus global optima.

Equations (4.15) need only be satisfied for frameworks with members not confined to one particular set of coordinate surfaces. If the members are so confined, $\varepsilon_{11}$ need only satisfy the inequality (4.2) and, in particular, it may be zero. Equations (4.7) can then be written, using equations (3.7):

$$\frac{\partial}{\partial x^1}\left(\frac{1}{A}\frac{\partial C}{\partial x^1}\right) = 0 \qquad \text{- (4.17a)}$$

$$\frac{\partial}{\partial x^1}\left(\frac{1}{A}\frac{\partial B}{\partial x^1}\right) = 0 \qquad \text{- (4.17b)}$$

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial B}{\partial x^3}\right) + \frac{1}{A^2}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1} = 0 \qquad \text{- (4.17c)}$$

Consider a plane surface whose coordinate curves, denoted $x^2$, $x^3$, are such that equations (4.14) are satisfied, and suppose the pattern of curves possesses a line of symmetry. The plane can be rotated about this line to produce a three dimensional system whose third set of coordinate curves, $x^1$, is a set of circles. The derivatives of the components of the fundamental tensor with respect to $x^1$ are all zero, and therefore equations (4.17) are satisfied. Optimum frameworks based on such axially symmetric systems must, in general, have members confined to radial surfaces, because the strains have only been shown to be compatible when the strain along the radii is zero.

Equilibrium

As a limiting case of a pin-jointed framework, as the number of joints tends to infinity, a continuum is considered, in which the framework members lie along lines of principal stress. The equilibrium equations obtained in this way will apply to a framework whose members lie along orthogonal sets of curves. The forces in the framework members can be associated with principal stresses in the continuum due to the absence of shear stress in the directions of principal stress. The stress equilibrium equations in general tensor form are:

$$E^{ij}{}_{,i} = 0 \qquad \text{- (4.18)}$$

where $E^{ij}$ is the stress tensor (reference 6, page 101), which is taken in the form:

$$E^{ij} = \begin{bmatrix} \sigma_{11}/A^2 & 0 & 0 \\ 0 & \sigma_{22}/B^2 & 0 \\ 0 & 0 & \sigma_{33}/C^2 \end{bmatrix} \qquad \text{- (4.19)}$$

the $\sigma_{ij}$ being the physical components of stress and satisfying the relations:

$$\sigma_{ij} = E^{ij} l_i l_j$$

$$g^{ij} l_i l_j = 1$$

The general first order covariant derivative of the contravariant tensor $E^{ij}$ is:

$$E^{ij}{}_{,k} = \frac{\partial E^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ lk \end{Bmatrix} E^{lj} + \begin{Bmatrix} j \\ lk \end{Bmatrix} E^{il}$$

Thus:

$$E^{11}{}_{,1} = \frac{\partial E^{11}}{\partial x^1} + \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} E^{11} + \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \overset{1}{E}{}^{21}_{\neq 0} + \begin{Bmatrix} 1 \\ 31 \end{Bmatrix} \overset{1}{E}{}^{31}_{\neq 0}$$

$$+ \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} E^{11} + \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \overset{1}{E}{}^{12}_{\neq 0} + \begin{Bmatrix} 1 \\ 31 \end{Bmatrix} \overset{1}{E}{}^{13}_{\neq 0}$$

$$= \frac{\partial}{\partial x^1}\left(\frac{\sigma_{11}}{A^2}\right) + \frac{2}{A}\frac{\partial A}{\partial x^1}\frac{\sigma_{11}}{A^2}$$

$$= \frac{1}{A^2}\frac{\partial \sigma_{11}}{\partial x^1}$$

Similarly:

$$E^{21}{}_{,2} = \frac{1}{A^2 B}\frac{\partial B}{\partial x^1}(\sigma_{11} - \sigma_{22})$$

$$E^{31}{}_{,3} = \frac{1}{A^2 C}\frac{\partial C}{\partial x^1}(\sigma_{11} - \sigma_{33})$$

From equation (4.18), with j equal to 1:

$$E^{11}{}_{,1} + E^{21}{}_{,2} + E^{31}{}_{,3} = 0$$

Substituting from the above expressions:

$$\frac{\partial}{\partial x^1}(BC\sigma_{11}) - \frac{1}{A}\frac{\partial B}{\partial x^1}(CA\sigma_{22}) - \frac{1}{A}\frac{\partial C}{\partial x^1}(AB\sigma_{33}) = 0 \quad - (4.20a)$$

The other two equilibrium equations are obtained in the same manner as above by giving j the values 2 and 3. They are:

$$\frac{\partial}{\partial x^2}(CA\sigma_{22}) - \frac{1}{B}\frac{\partial C}{\partial x^2}(AB\sigma_{33}) - \frac{1}{B}\frac{\partial A}{\partial x^2}(BC\sigma_{11}) = 0 \quad - (4.20b)$$

$$\frac{\partial}{\partial x^3}(AB\sigma_{33}) - \frac{1}{C}\frac{\partial A}{\partial x^3}(BC\sigma_{11}) - \frac{1}{C}\frac{\partial B}{\partial x^3}(CA\sigma_{22}) = 0 \quad - (4.20c)$$

The element of area on the $x^2$ , $x^3$ surfaces is $BCdx^2 dx^3$, and therefore $\sigma_{11} BCdx^2 dx^3$ is the net force normal to the element of area. In a framework this is carried by a bar passing through the element $dx^2$ $dx^3$ , but the stress in the bar is not the same as $\sigma_{11}$ : it is the maximum allowable stress in the $x'$ direction, $\sigma_{(11)}$ , which is constant. The cross sectional area of the bar, $a_{(i)}$, is not the same as the element of area, but is given by:

$$a_{(i)} = \left| \frac{\sigma_{11}}{\sigma_{(11)}} \right| BC\, dx^2 dx^3 \qquad - (4.21)$$

The total volume of the bars along the $x'$ coordinate curves is therefore given by:

$$V_{(i)} = \int a_{(i)} . A\, dx' = \iiint \left| \frac{\sigma_{11}}{\sigma_{(11)}} \right| ABC\, dx' dx^2 dx^3$$

from which the volume of the whole framework is:

$$V = \iiint \left( \left| \frac{\sigma_{11}}{\sigma_{(11)}} \right| + \left| \frac{\sigma_{22}}{\sigma_{(12)}} \right| + \left| \frac{\sigma_{33}}{\sigma_{(13)}} \right| \right) ABC\, dx' dx^2 dx^3 \qquad - (4.22)$$

where $\sigma_{(11)}$ , $\sigma_{(12)}$ and $\sigma_{(13)}$ are all either $\sigma_{(t)}$ or $\sigma_{(c)}$ , the limiting stresses in tension and compression respectively.

Two of the equilibrium equations for a surface are obtained from the two dimensional form of equation (4.18). The equation of equilibrium in the direction normal to a surface may be obtained as follows. Equation (4.20a) is the equation of equilibrium normal to the $x^2$ , $x^3$ coordinate surfaces. Dividing all through by ABC gives:

$$\frac{1}{BC\, dx^2 dx^3} \frac{\partial}{A \partial x'} (\sigma_{11} BC\, dx^2 dx^3) + E^{11} b^{(i)}_{11} + E^{22} b^{(i)}_{22} = 0 \qquad - (4.23)$$

where $E^{11}$ and $E^{22}$ are the components of the surface stress tensor

$$E^{\alpha\beta} = \begin{bmatrix} \sigma_{22}/B^2 & 0 \\ 0 & \sigma_{33}/C^2 \end{bmatrix}$$

and $b^{(i)}_{\alpha\beta}$ is defined by equation (3.14a). The first term of equation (4.23) may be interpreted as P/t where P is force per unit area normal to a surface, and t is the surface thickness. The equilibrium of a surface is thus represented by the tensor equations:

$$E^{\alpha\beta}{}_{,\gamma} = 0 \qquad - (4.24a)$$

$$E^{\alpha\beta} b_{\alpha\beta} = -P/t \qquad - (4.24b)$$

For the arbitrary orthogonal coordinate system of equations (3.22), equations (4.24) give:

$$\frac{\partial}{\partial x^1}(B\sigma_{11}) - \frac{\partial B}{\partial x^1}\sigma_{22} = 0 \qquad - (4.25a)$$

$$\frac{\partial}{\partial x^2}(A\sigma_{22}) - \frac{\partial A}{\partial x^2}\sigma_{11} = 0 \qquad - (4.25b)$$

$$s_{(1)}\sigma_{11} + s_{(2)}\sigma_{22} = -P/t \qquad - (4.25c)$$

Corresponding to equation (4.21), the cross-sectional area of the bar which transmits the force $\sigma_{11}\, tBdx^2$ normal to the line element $Bdx^2$ is given by:

$$a_{(1)} = \left|\frac{\sigma_{11}}{\sigma_{(11)}}\right| t\, B\, dx^2 \qquad - (4.26)$$

and the total volume of the framework is therefore:

$$V = t\iint\left(\left|\frac{\sigma_{11}}{\sigma_{(11)}}\right| + \left|\frac{\sigma_{22}}{\sigma_{(12)}}\right|\right) AB\, dx^1 dx^2 \qquad (4.27)$$

Except in the case of a plane surface, for which P must be zero and equation (4.25c) is identically satisfied, the three equations (4.25) imply a restriction on the orthogonal coordinate curves if equilibrium is to be possible. The actual form of the equation to be satisfied by A and B is rather complicated. It may be derived as follows.

Assume that neither $s_{(1)}$ nor $s_{(2)}$ is zero and eliminate $\sigma_{22}$ from equations (4.25a) and (4.25b) using (4.25c) to give two first order linear differential equations in $\sigma_{11}$ which may be written:

$$\frac{\partial \sigma_{11}}{\partial x^1} + f_{(11)}(x^1, x^2)\sigma_{11} - f_{(12)}(x^1, x^2) = 0 \qquad (4.28a)$$

$$\frac{\partial \sigma_{11}}{\partial x^2} + f_{(21)}(x^1, x^2)\sigma_{11} - f_{(22)}(x^1, x^2) = 0 \qquad (4.28b)$$

where

$$f_{(11)} = \frac{1}{B}\frac{\partial B}{\partial x^1}\left(1 + \frac{s_{(1)}}{s_{(2)}}\right) \qquad (4.29a)$$

$$f_{(12)} = -\frac{1}{tB}\frac{\partial B}{\partial x^1}\frac{P}{S_{(2)}} \qquad - (4.29b)$$

$$f_{(21)} = \frac{1}{A}\frac{\partial A}{\partial x^2}\left(1 + \frac{S_{(2)}}{S_{(1)}}\right) + \frac{S_{(2)}}{S_{(1)}}\frac{\partial}{\partial x^2}\left(\frac{S_{(1)}}{S_{(2)}}\right) \qquad - (4.29c)$$

$$f_{(22)} = -\frac{S_{(2)}}{tS_{(1)}}\frac{\partial}{\partial x^2}\left(\frac{P}{S_{(2)}}\right) - \frac{1}{tA}\frac{\partial A}{\partial x^2}\frac{P}{S_{(1)}} \qquad - (4.29d)$$

$$( S_{(1)} \neq 0 \;,\; S_{(2)} \neq 0 )$$

Differentiating the first of (4.28) w.r.t. $x^2$ and substituting from the second for $\partial\sigma_{11}/\partial x^2$, and differentiating the second w.r.t. $x^1$ and substituting from the first for $\partial\sigma_{11}/\partial x^1$ gives:

$$\frac{\partial^2\sigma_{11}}{\partial x^1 \partial x^2} + \sigma_{11}\left(\frac{\partial f_{(21)}}{\partial x^2} - f_{(11)}f_{(21)}\right) + \left(f_{(11)}f_{(22)} - \frac{\partial f_{(12)}}{\partial x^2}\right) = 0 \qquad (4.30a)$$

$$\frac{\partial^2\sigma_{11}}{\partial x^1 \partial x^2} + \sigma_{11}\left(\frac{\partial f_{(21)}}{\partial x^1} - f_{(21)}f_{(11)}\right) + \left(f_{(21)}f_{(12)} - \frac{\partial f_{(22)}}{\partial x^1}\right) = 0 \qquad (4.30b)$$

These equations can be treated as simultaneous equations in $\sigma_{11}$ and $\partial^2\sigma_{11}/\partial x^2 \partial x^1$. The differential relationship between these two quantities finally gives the equation to be satisfied by the metric.

If one of the curvatures is zero, say $s_{(1)}$, then (4.25c) gives:

$$\sigma_{22} = \frac{-P}{tS_{(2)}}$$

which can be substituted into (4.25b) to give

$$\sigma_{11} = -\frac{\partial}{\partial x^2}\left(\frac{AP}{S_{(2)}}\right)\Big/ t\frac{\partial A}{\partial x^2}$$

Replacing $\sigma_{11}$ and $\sigma_{22}$ in equation (4.25a) by the above expressions provides the following equation:

$$\frac{\partial}{\partial x^1}\left[\frac{B}{t}\frac{\partial}{\partial x^2}\left(\frac{AP}{S_{(2)}}\right)\Big/\frac{\partial A}{\partial x^2}\right] - \frac{P}{tS_{(2)}}\frac{\partial B}{\partial x^1} = 0 \qquad - (4.31)$$

$$( S_{(1)} = 0 \;,\; S_{(2)} \neq 0 )$$

An analysis of two examples quoted by Michell in reference 2, using the equations derived in chapters 3 and 4, is contained in Appendix B.

| SYMBOL | | VALUE |
|---|---|---|
| $e_{11,1}$ | | $A^2\partial\varepsilon_{11}/\partial x^1$ |
| $e_{11,2}$ | | $A^2\partial\varepsilon_{11}/\partial x^2$ |
| $e_{11,3}$ | | $A^2\partial\varepsilon_{11}/\partial x^3$ |
| $e_{22,1}$ | | 0 |
| $e_{22,2}$ | | 0 |
| $e_{22,3}$ | | 0 |
| $e_{33,1}$ | | 0 |
| $e_{33,2}$ | | 0 |
| $e_{33,3}$ | | 0 |
| $e_{12,1}$ | $(e_{21,1})$ | $A\partial A/\partial x^2(\varepsilon-\varepsilon_{11})$ |
| $e_{12,2}$ | $(e_{21,2})$ | $B\partial B/\partial x^1(\varepsilon_{11}-\varepsilon)$ |
| $e_{12,3}$ | $(e_{21,3})$ | 0 |
| $e_{23,1}$ | $(e_{32,1})$ | 0 |
| $e_{23,2}$ | $(e_{32,2})$ | $-2\varepsilon B\partial B/\partial x^3$ |
| $e_{23,3}$ | $(e_{32,3})$ | $2\varepsilon C\partial C/\partial x^2$ |
| $e_{31,1}$ | $(e_{13,1})$ | $-A\partial A/\partial x^3(\varepsilon+\varepsilon_{11})$ |
| $e_{31,2}$ | $(e_{13,2})$ | 0 |
| $e_{31,3}$ | $(e_{13,3})$ | $C\partial C/\partial x^1(\varepsilon+\varepsilon_{11})$ |

TABLE 3

FIRST ORDER COVARIANT DERIVATIVES OF THE SPACE STRAIN COMPONENTS

# CHAPTER FIVE

## NUMERICAL APPROACH

The analytical techniques developed in the previous chapters serve only as an inverse method of obtaining optimum frameworks; that is, a compatible strain field is first derived, which leads to layout curves for tension and compression members of possible optimum frameworks, and finally, by drawing boundaries and using the condition of static equilibrium at the boundaries, an appropriate applied force system is found for each framework. The problem of finding an optimum framework for a given force system is one which cannot readily be solved analytically. In its simplest form it may be stated in the following terms: to find quantities A, B, C, $t_{(1)}$ , $t_{(2)}$ , $t_{(3)}$ ($t_{(1)}$ , $t_{(2)}$ , $t_{(3)} \geqslant 0$) which satisfy the layout equations (4.7) (with (4.4)) and equations (3.7) together with the appropriate differential equations of equilibrium e.g. equations (4.20) or (4.25), and also the boundary equilibrium conditions. Since the associated virtual displacement field must be continuous, so also must be A, B, C, $t_{(1)}$ , $t_{(2)}$ and $t_{(3)}$ but they need only be piecewise continuously differentiable i.e. they may be described by different analytic functions in different regions, as long as there is a unique value of each associated with every point. The problem is further complicated by the fact that there may be regions in which the limiting stresses $\sigma_{(11)}$, $\sigma_{(12)}$ and $\sigma_{(13)}$ are either all $\sigma_{(t)}$ or all $\sigma_{(c)}$ , and in this case there is no restriction on layout.

The numerical method described in this chapter approaches the problem of designing an optimum framework for a given force system from an entirely different direction to the analytical method suggested above. It does, however, lead to frameworks which are, in a limited sense, Michell frameworks. In this approach, originally described in a paper by Dorn, Gomory and Greenberg (reference 13) the design problem is formulated as a problem in linear programming which can then be solved by means of the Simplex method (due to Dantzig - reference 14). The duality theory of linear programming establishes the connection between the optimum framework so obtained and Michell frameworks.

The formulation is as follows. A grid of nodes is defined which represent possible nodes of a framework. A framework bar can join any node to any other node so that if there are n nodes, there are $n(n-1)/2$ possible bars. The framework which contains every possible bar is referred to as the ground structure. The next step is to set up the equations of joint equilibrium for the ground structure, there being between 3n and (3n-6) such equations, which may be represented in the following way:

$$\begin{aligned}
\sum d_{(1j)} T_{(j)} \quad &+ \quad F_{(1)} \quad = 0 \\
\sum d_{(2j)} T_{(j)} \quad &+ \quad F_{(2)} \quad = 0 \\
&\vdots \\
\sum d_{(mj)} T_{(j)} \quad &+ \quad F_{(m)} \quad = 0
\end{aligned} \qquad - (5.1)$$

The first equation is the equation of equilibrium of node 1 in the x-direction (x, y, z being rectangular Cartesian coordinates), the second represents equilibrium of node 1 in the y-direction, and so on, to the last equation which represents equilibrium of node n in the z-direction. The $T_{(j)}$ are the tensions in the bars of the framework and each of the coefficients $d_{(ij)}$ is either one of the direction cosines of the $j^{th}$ bar if the $j^{th}$ bar is connected to the node to which the $i^{th}$ equation applies, or zero otherwise. The $F_{(j)}$ are the components of the applied forces at the various nodes. At nodes where there are constraints in any of the coordinate directions the corresponding equilibrium equations are missed out, since the reactive forces at such points may have any value. The number of redundant equations in the set (5.1) is from zero to six. In any correctly formulated problem there are at least as many variables as indepent equations, and in most cases these are considerably more variables than independent equations. The former case produces a unique solution whereas the latter gives rise to a multiplicity of solutions from which must be chosen the one which minimises the structural volume. If the structural volume can be expressed as a linear function of the variables $T_{(j)}$, then the problem defined above is one in linear programming.

It is necessary now to consider some aspects of the general linear programming problem, which may be presented as follows:

Find $x_{(1)} \geqslant 0$, $x_{(2)} \geqslant 0$, ...., $x_{(n)} \geqslant 0$ that maximise z where

$$
\begin{aligned}
c_{(1)}\,x_{(1)} &+ c_{(2)}\,x_{(2)} + \cdots + c_{(n)}\,x_{(n)} = z \\
a_{(11)}\,x_{(1)} &+ a_{(12)}\,x_{(2)} + \cdots + a_{(1n)}\,x_{(n)} = a_{(10)} \\
a_{(21)}\,x_{(1)} &+ a_{(22)}\,x_{(2)} + \cdots + a_{(2n)}\,x_{(n)} = a_{(20)} \qquad - (5.2) \\
&\vdots \\
a_{(m1)}\,x_{(1)} &+ a_{(m2)}\,x_{(2)} + \cdots + a_{(mn)}\,x_{(n)} = a_{(m0)}
\end{aligned}
$$

The coefficients $c_{(j)}$ are called the prices of the variables $x_{(j)}$ and the linear form z is referred to as the objective function. The remaining m equations are the constraint equations. The general problem may contain inequality constraints, but these are converted to equality constraints by the addition of "slack" and "surplus" variables (reference 15, page 72). The variables $x_{(j)}$ are defined to be non-negative and therefore any variable which in practice is unrestricted in sign must be represented in the linear programming problem as the difference between two variables, both of which are non-negative (reference 15, page 168). Thus, in the framework problem, in which all the variables $T_{(j)}$ are unrestricted in sign, the number of variables in the corresponding linear program is twice the number of bars in the ground structure. The reason for the non-negativity restrictions appears to be that the Simplex algorithm for solving linear programs was originally developed to deal with problems arising in the field of economics (as is well illustrated by the terminology associated with linear programming) and in such problems the variables were, by their nature, required to be non-negative. The computational efficiency of the Simplex algorithm, and its variants, is such that it is better to retain the non-negativity restrictions at the expense of introducing extra variables than to attempt to make use of an alternative procedure in which the non-negativity restrictions are removed. Any solution of the constraint

equations which satisfies the non-negativity restrictions is called a feasible solution.

The general problem has been presented in terms of maximisation but minimisation problems can easily be dealt with by taking $-\sum c_{(j)}\, x_{(j)}$ as the objective function. A feasible solution which maximises the objective function is called an optimal feasible solution.

The fundamental theorem of linear programming asserts that if an optimal feasible solution exists, then there exists an optimal feasible solution in which the number of non-zero variables is not greater than the number of independent constraints. This result is proved in reference 15. The non-zero variables together with enough of the zero variables to make a total of m, m being the total number of constraints, are referred to as basic variables. The corresponding solution is called a basic optimal solution. The Simplex algorithm operates by first establishing a basic feasible solution i.e. a feasible solution in which at least (n-m) variables are zero where n is the number of variables, and then allowing a non-basic variable to become non-zero while simultaneously driving to zero one of the basic variables, a process which effectively exchanges a basic and a non-basic variable to give a new set of basic variables. The new basic variable is chosen in such a way as to give an improved value (or at least a value not less than the previous one) to the objective function. Since there are a finite number of basic solutions, the optimal solution, if one exists, is obtained in a finite number of iterations.

The iterations are carried out with equations (5.2) in the following canonical form:

$$
\begin{aligned}
-Z \quad & + \bar{c}_{(m+1)}\, x_{(m+1)} + \cdots + \bar{c}_{(n)}\, x_{(n)} = -Z_{(0)} \\
x_{(1)} \quad & + \bar{a}_{(1\,m+1)}\, x_{(m+1)} + \cdots + \bar{a}_{(1n)}\, x_{(n)} = \bar{a}_{(10)} \\
x_{(2)} \quad & + \bar{a}_{(2\,m+1)}\, x_{(m+1)} + \cdots + \bar{a}_{(2n)}\, x_{(n)} = \bar{a}_{(20)}
\end{aligned}
\qquad - (5.3)
$$

$$
x_{(m)} + \bar{a}_{(m\,m+1)}\, x_{(m+1)} + \cdots + \bar{a}_{(m\,n)}\, x_{(n)} = \bar{a}_{(m0)}
$$

with $\bar{a}_{(10)}, \ldots, \bar{a}_{(m0)} \geq 0$

The basic variables have been denoted $x_{(1)}, \ldots, x_{(m)}$ (although, of course, they are not necessarily the first m variables of the original linear program) and the basic feasible solution is $z = z_{(0)}$, $x_{(1)} = \bar{a}_{(10)}$, $x_{(2)} = \bar{a}_{(20)}$, $\ldots$, $x_{(m)} = \bar{a}_{(m0)}$. The transformation of the equations (5.2) to the canonical form (5.3) constitutes Phase 1 of the solution of the linear program. Equations (5.3) are clearly linearly independent because each contains a variable that does not occur in any other equation. If there were p redundancies in the original constraints then p of the equations are vacuous i.e. all the coefficients $\bar{a}_{(j0)}, \bar{a}_{(j\,m+1)}, \ldots, \bar{a}_{(jn)}$ are zero in these equations. The basic variables associated with vacuous equations are artificial variables not belonging to the set of variables originally defined for the linear program. Phase 1 will be discussed in more detail after Phase 2 has been presented.

Phase 2 begins with the feasible canonical form (5.3) and its associated

basic feasible solution. This solution is optimal if all the $\bar{c}_{(j)}$ are less than or equal to zero because in that case any increase in the value of the non-basic variables either produces a decrease in the value $\bar{z}_{(o)}$ of the objective function or, for those $x_{(j)}$ corresponding to the zero $\bar{c}_{(j)}$, gives an alternative optimal solution. Suppose that $\bar{c}_{(m+k)}$ is positive: then by increasing $x_{(m+k)}$ from zero to some positive value a new solution, with an increased value of z, is given by:

$$\begin{aligned} z &= z_{(o)} + \bar{c}_{(m+k)} x_{(m+k)} \\ x_{(1)} &= \bar{a}_{(1o)} - \bar{a}_{(1\,m+k)} x_{(m+k)} \\ &\vdots \\ x_{(m)} &= \bar{a}_{(mo)} - \bar{a}_{(m\,m+k)} x_{(m+k)} \end{aligned} \qquad - (5.4)$$

The maximum value to which $x_{(m+k)}$ can be increased is the minimum of the terms $\bar{a}_{(io)} / \bar{a}_{(i\,m+k)}$ for which $\bar{a}_{(i\,m+k)}$ is positive. Thus one basic variable, $x_{(l)}$, say, becomes zero while all the rest remain positive. A new feasible canonical form with $x_{(m+k)}$ a basic variable and $x_{(l)}$ a non-basic variable is obtained by dividing every term in the $l^{th}$ equation by $\bar{a}_{(l\,m+k)}$ - the pivot element - and then subtracting multiples of this equation from all the others, including the z -equation, so that the column of coefficients of $x_{(m+k)}$ - the pivot column - has all its elements zero except the one corresponding to the $l^{th}$ constraint equation, which is one. The row of coefficients of the $l^{th}$ constraint equation is called the pivot row. The choice of the variable to enter the basis (i.e. become non-basic) is made by determining the largest positive $\bar{c}_{(m+k)}$. This does not necessarily produce the largest possible increase in the value of the objective function since the maximum permissible value of $x_{(m+k)}$ may be very small, even zero. However it has been determined experimentally that maximising the increase in the objective function at each stage (method of steepest ascents) does not lead to a significant decrease in the number of iterations required to reach an optimal solution in a given linear program, and the extra computation involved is therefore not worth while (reference 15, page 111).

If the minimum $(\bar{a}_{(io)} / \bar{a}_{(i\,m+k)})$ is not unique, then basic variables other than the one to be exchanged become zero and the new basis is said to be degenerate. From equations (5.4) it is clear that at the next iteration, if the value of $\bar{a}_{(i\,m+k)}$, corresponding to a basic variable, $x_{(i)}$, with the value zero (i.e. $\bar{a}_{(io)}$ is zero), is positive, the maximum permissible value of $x_{(m+k)}$ is zero, so that the value of the objective function is unchanged and the new basic solution is still degenerate. In this situation it is conceivable that several iterations might produce a series of degenerate solutions, leading back to a set of basic variables identical to a previous set. If a set of basic variables is repeated, the canonical form is exactly repeated because there can be only one set of linearly independent equations containing a given set of basic variables. Thus the sequence of iterations is repeated ad infinitum. This process, known as cycling, can be avoided by a particular choice of the minimum $(\bar{a}_{(io)} / \bar{a}_{(i\,m+k)})$ in the event of a tie, which ensures that no basis is ever repeated. The method of making the choice is straight-forward; the minimum of the quantities $(\bar{a}_{(i\,m)} / \bar{a}_{(i\,m+k)})$ for the values of i which produced the same $(\bar{a}_{(io)} / \bar{a}_{(i\,m+k)})$ determines the pivot row. If there is still a tie, then the minimum of $(\bar{a}_{(i\,m+1)} / \bar{a}_{(i\,m+k)})$ is determined, and so on until a decision is reached. The theory underlying this method, which is the perturbation

method of Charnes, is described in Chapter 6 of reference 15.

In the absence of degeneracy, the value of the objective function improves at every iteration and therefore there is no possibility of a basis being repeated. When degeneracy occurs the method just described ensures that no basis is repeated and so the optimal solution, if it exists, is reached in a finite number of iterations, whether degeneracy occurs or not. In some problems there may not be a basic optimal solution: from equations (5.4) it is evident that if all the $\bar{a}_{(i\,m+k)}$ are negative, $x_{(m+k)}$ can be made arbitrarily large, producing an unbounded solution in which the number of non-zero variables can be one greater than the number of basic variables.

Phase 1 begins with the problem in the form of equations (5.2). In order to obtain a feasible canonical form, an augmented program is formed by the addition of artificial variables, $x_{(n+i)}$, to give the equations:

$$\begin{aligned} -Z \qquad & + c_{(1)}x_{(1)} + \cdots + c_{(n)}x_{(n)} = 0 \\ x_{(n+1)} \qquad & + a_{(11)}x_{(1)} + \cdots + a_{(1n)}x_{(n)} = 0 \\ x_{(n+2)} \qquad & + a_{(21)}x_{(1)} + \cdots + a_{(2n)}x_{(n)} = 0 \end{aligned} \qquad - (5.5)$$

$$x_{(n+m)} + a_{(m1)}x_{(1)} + \cdots + a_{(mn)}x_{(n)} = 0$$

The artificial variables are initially taken as basic variables. An auxiliary objective function, u, is formed, which is minus the sum of the artificial variables. Using equations (5.5) this can be written:

$$-u + \sum_{i=1}^{m} a_{(i1)}x_{(1)} + \sum_{i=1}^{m} a_{(i2)}x_{(2)} + \cdots + \sum_{i=1}^{m} a_{(in)}x_{(n)} = \sum_{i=1}^{m} a_{(i0)} \qquad - (5.6)$$

Equations (5.5) and (5.6) constitute a feasible canonical form of the augmented problem. By maximising u, using the Simplex algorithm as in Phase 2, the sum of the artificial variables is made as small as possible. At each iteration an artificial variable is removed from the basis and a legitimate variable inserted, until the optimality criterion is satisfied i.e. all the coefficients in the auxiliary objective function are negative or zero. If the value of u is zero at this stage, then all the artificial variables are zero and a basic feasible solution to the original problem has been obtained. There may still be artificial variables present in the basis, and unless they belong to vacuous equations, they must be eliminated to prevent them becoming non-zero during Phase 2. This can be done simply by exchanging with non-basic legitimate variables. The solution is unaffected because all the variables concerned are zero. The presence of vacuous equations indicates redundancy in the original constraints. The corresponding artificial variables cannot be eliminated from the basis, but equally they cannot become non-zero, since the equations containing them will remain vacuous during Phase 2. Other artificial variables can be dropped from the list of non-basic variables and Phase 2 initiated. In Phase 2 the auxiliary objective function is ignored. If the auxiliary objective function cannot be reduced to zero in Phase 1, there is no basic feasible solution to the original problem, and therefore no optimal solution.

Returning now to the framework problem, an optimal solution to equations (5.1) can be obtained using the above method provided that the objective function is a linear function of the variables $T_{(j)}$ . There is then a basic optimal solution with the number of load-carrying members not greater than the number of independent equilibrium equations. Also, from the form of any basic solution, it is clear that none of the basic variables can be expressed as a linear combination only of the remaining basic variables. In the first place, this means that the optimum framework is statically determinate, and in the second, it ensures that only one of the two bounded variables corresponding to each unbounded variable $T_{(j)}$ can appear in any basis. Since the optimum framework is statically determinate, the omission of any elastic compatibility conditions in the constraint equations (5.1) is justified. The cross-sectional areas of the framework bars, $a_{(j)}$ , are thus not included as variables in the constraint equations and can be chosen to satisfy the criterion of minimum volume only. The objective function for minimum volume is then:

$$\min V = \sum_j a_{(j)} l_{(j)} = \frac{1}{\sigma_{(t)}} \sum_j T^+_{(j)} l_{(j)} + \frac{1}{\sigma_{(c)}} \sum_j T^-_{(j)} l_{(j)} \quad -(5.7)$$

where $T_{(j)} = T^+_{(j)} - T^-_{(j)} \; ; \; T^+_{(j)}, T^-_{(j)} \geqslant 0$

A computer program for solving the framework problem (with some elaborations) using the Simplex method is described in the next chapter.

Related to every linear programming problem there is a dual problem (reference 15, chapter 8), which, in the particular case of the framework problem, provides the link with Michell frameworks. If the original (primal) problem is presented in terms of maximisation, then the dual problem is one of minimisation, and vice versa. The importance of the dual problem is that if there is an optimal solution to the primal problem, there is also an optimal solution to the dual problem, and both solutions are the same. In the framework problem, the dual variables may be interpreted as a set of virtual nodal displacements, the dual objective function as virtual work of the external forces, and the dual constraint equations as a limit on the linear strain between any two nodes (reference 16). At the optimal solution the virtual work is maximised while the virtual strain between any two nodes joined by a load-carrying member is either $+ \varepsilon_{(t)}$ or $- \varepsilon_{(c)}$, depending on whether the load is tensile or compressive, and the value of the virtual strain between all other pairs of nodes lies between $- \varepsilon_{(c)}$ and $+ \varepsilon_{(t)}$ . There is a slight difference from the criterion satisfied by Michell frameworks in that the limiting virtual strains $\varepsilon_{(t)}$ and $- \varepsilon_{(c)}$ do not necessarily possess the same numerical value (they only turn out to be the same if $\sigma_{(t)}$ and $\sigma_{(c)}$ are the same). However this strain system is removed from that of Michell only by a pure dilatation of space, and therefore any framework which admits of the one, admits also of the other. (Dorn, Gomory and Greenberg, who first formulated the problem of optimum framework design as one in linear programming, were apparently unaware of Michell's theorem and interpreted the dual problem in terms of the actual displacements of the framework nodes under the given applied force system, and the related strains in the framework bars.)

The solution of the framework problem by linear programming is quite general, in that given a system of applied forces and points of reaction such that equilibrium of all the external forces is possible, an optimum framework can be determined, and any such optimum framework satisfies Michell's criterion. Of course since there are only a finite number of possible nodes defined for the primal linear program, the virtual displacements arising from the dual, which are determined at the nodes only, are not given as a continuous vector field, as in Michell's theorem. Likewise, the virtual strains are only defined

between possible nodes, and not over the whole of the space occupied by the framework. These differences are merely a result of the way in which the linear programming problem is formulated; by increasing the number of nodes in the ground structure, the virtual strains and displacements could theoretically be made as close to continuous tensor fields as desired. It may therefore be said that the necessity of the Michell criterion for optimum frameworks has been established.

# CHAPTER SIX

## COMPUTER PROGRAM

The program, which is listed at the end of this chapter, finds the lightest framework of those whose nodes are confined to specified points, which can equilibrate a given force system and which can satisfy certain conditions of yielding and buckling. The programming language used is Fortran.

### PRESENTATION OF DATA

The pattern of possible nodes and the method of numbering them is shown in figure 6.1. NX, NY, NZ are the first three numbers presented as data and they specify the nodal pattern completely. A bar may join any node to any other node. The bars are numbered such that bar 1 joins nodes 1 and 2, bar 2 joins nodes 1 and 3, and bar NJ-1 joins nodes 1 and NJ, where NJ is the total number of nodes; bar NJ joins nodes 2 and 3, bar $[(NJ-1) + (NJ-2)]$ joins nodes 2 and NJ, and so on. With this convention, the number of the bar joining nodes I and J, where $I < J$, is given by:

$$K1 = J-I*(I+1)/2 + NJ*(I-1)$$

Two other parameters, CB and SR, are read in along with NX, NY and NZ. SR is the ratio of the tensile yield stress to the compressive yield stress: CB is such that if the square of the length of a bar is greater than CB * SR, it will fail by buckling before it reaches the compressive yield stress, or, if CB is zero, failure by buckling is not considered. The introduction of CB, a constant for all bars, embodies the assumption that the second moment of area of a bar is directly proportional to its cross-sectional area (CB is, in fact, the constant of proportionality multiplied by $\pi^2 E/\sigma(t)$ where E is Young's Modulus and $\sigma(t)$ is the yield stress in tension). This in turn implies that the buckling load is directly proportional to the cross-sectional area, and the volume of a fully-stressed member is thus always a constant times the member load. Another assumption which maintains the linearity between volume and maximum load, is that the slenderness ratio of all bars is the same. Then all bars fail at the same compressive stress, whether by buckling or yielding, and in this case CB can be set to zero and an appropriate value given to SR. The parameters CB and SR only affect the values of the coefficients in the objective (merit) function.

The first data card, therefore, contains five quantities, of which the first three are integers and the last two real numbers e.g.

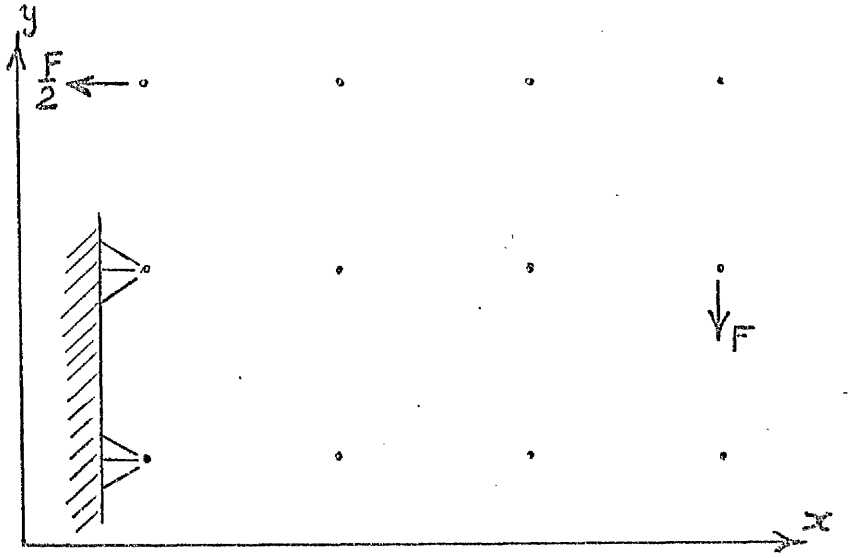
1st Data card: 4 3 1 8.0 1.0

The above means that there are four nodes in the x-direction, three in the y-direction and one in the z-direction i.e. plane frameworks only are considered; for any bar whose length is greater than $2\sqrt{2}$, the Euler buckling stress is less than the compressive yield stress; the yield stress in tension is the same as the yield stress in compression.

The next data supplies information about points of reaction and applied forces. Each of the nodes has three integers and up to three real numbers associated with it. The integers are either one or zero, one representing a constraint e.g. 0 1 0 implies a constraint in the y-direction. The real numbers are the components of applied force at the node, the number of such components being, in general, three minus the number of nodal constraints. However in the case where the third integer of the first card is one, as above, a storage saving

device in the program comes into operation and this means that the third equilibrium equation of each node is missed out, all the coefficients being zero. The number of components of force at a node is thus two minus the number of nodal constraints. This only applies when the third integer of the first card is one. If one of the others is one, then the problem is treated as 3-dimensional, resulting in a wastage of storage space.

The sets of three integers, taken in order of nodes, are presented to the computer first, then the real numbers. The following example illustrates these points.




2nd data card: 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0
3rd data card: 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
4th data card: 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
5th data card: 0.0 -1.0 -0.5 0.0 0.0 0.0 0.0 0.0 0.0 0.0

These integers and real numbers must be such that an external force system in equilibrium is possible. If this is not the case, the program detects it and prints out "NO BASIC FEASIBLE SOLUTION".

A final data card, containing either a one or a zero, controls the form of the print out. A one causes intermediate stages of the calculation to be printed out, whereas a zero gives only the final results.

## OUTLINE OF MODE OF OPERATION

The data is processed by the main program, in conjunction with subroutines MP6 and MP9, to give a set of equations (equilibrium equations) and an objective function (minimum volume). This linear program is presented to subroutine Simplx which, with the aid of subroutines MP1, MP2, MP3 and MP4 solves it and passes the results on to MP5. MP5 interprets the results as a particular framework layout, this information being given first to MP8 to order and print out, then to MP7 to analyse and detect nodal instabilities. A node is deemed unstable if all the members meeting at it are in compression and lie in a plane along the normal to which the node is unconstrained: in the case of compression members in a straight line, the node must be constrained in two directions. A number is assigned to each node indicating the type of instability, if any, which is present, and the infomation returned to the main program, where nodal perturbations are calculated such that the unstable nodes are displaced in the direction of instability. A check is made to determine whether or not an alternative optimum may exist: if so, the program returns to the

beginning and the process begins again.

Termination occurs when a framework is found in which there are no nodal instabilities (in the case of plane nodal patterns, when there are no in-plane nodal instabilities) or when the test for alternative optima proves negative.

A flowchart showing the essential operations of the program is given in figure 6.6.

## OUTPUT OF RESULTS

During each cycle of the complete program, the output consists of the cycle number, the nodal perturbations for the cycle, a list of bar tensions with the numbers of the bars to which they apply and a zero or a one depending on whether the limiting compressive stress in the bar is the compressive yield stress or the Euler buckling stress, the optimum value of the objective function, the values of the reactive forces at the constraints, the number of iterations in each of the two phases of the optimisation process, and details of unstable nodes. If no further improvement can be made, the message "NO RE-CYCLE. OPTIMUM SOLUTION OBTAINED" is printed.

## SOME DETAILED COMPUTATIONAL ASPECTS

### 1. STORAGE

The array A contains the coefficients of the objective function, the constraint equations and the auxiliary objective function. The doubly subscripted quantities $a(ij)$ are stored in a one-dimensional array using the formula:

$$a(ij) = A(I*IROW + J*JCOL+1)$$

where IROW has the value 1 and JCOL has a value equal to 1 plus the maximum value of the subscript i - in other words the length of a column of A. The storage is organised in this way to suit the needs of subroutine Simplx, which was developed from a program by Kunzi, Tzschach and Zehnder (reference 17). The authors of the original program claimed increased efficiency in handling two-dimensional arrays by such a device. However the effect is not likely to be very pronounced since the result is to achieve what the computer compiler does anyway. The efficiency of the compiler will determine how much computing time, if any, is saved. The addition of 1 to the A subscripts is to allow the values of I and J to be zero, zero subscripts not being permitted in Fortran. The zeroth row of A contains the objective function; in the notation of chapter five:

$$z_{(0)} = A(1)$$
$$c_{(k)} = A(K*JCOL+1)$$

The auxiliary objective function is stored in the $(1+m)^{th}$ row, and in the following rows are the coefficients of inadmissible equations i.e. equations applying to directions in which the corresponding nodes are constrained.

The ordering of the coefficients within A for the general linear programming problem of equations (5.2), deriving from a framework problem in which the number of nodal constraints is $l$, is as follows:

| ROW | COLUMN 0 | 1 | 2 | . . . . . | n |
|---|---|---|---|---|---|
| 0 | 0 | $c_{(1)}$ | $c_{(2)}$ | . . . . . | $c_{(n)}$ |
| 1 | $-a_{(10)}$ | $a_{(11)}$ | $a_{(12)}$ | . . . . . | $a_{(1n)}$ |
| 2 | $-a_{(20)}$ | $a_{(21)}$ | $a_{(22)}$ | . . . . . | $a_{(2n)}$ |
| . | . | | | | |
| . | . | | | | |
| . | . | | | | |
| m | $-a_{(m0)}$ | $a_{(m1)}$ | $a_{(m2)}$ | . . . . . | $a_{(mn)}$ |
| m+1 | $\sum_{i=1}^{m} a_{(i0)}$ | $-\sum_{i=1}^{m} a_{(i1)}$ | $-\sum_{i=1}^{m} a_{(i2)}$ | . . . . . | $-\sum_{i=1}^{m} a_{(in)}$ |
| m+2 | | $a_{(m+1\ 1)}$ | $a_{(m+1\ 2)}$ | . . . . . | $a_{(m+1\ n)}$ |
| . | | | | | |
| . | | | | | |
| m+l+1 | | $a_{(m+l\ 1)}$ | $a_{(m+l\ 2)}$ | . . . . . | $a_{(m+l\ n)}$ |

The inadmissible equations are not necessarily, of course, the last $l$ equations of the set. The value of $(m+l)$ is either 2*NJ or 3*NJ depending on whether the problem is two- or three-dimensional. The value of n is twice the number of admissible bars in the ground structure (as a result of the non-negativity restrictions - see chapter five); not all the bars in the ground structure are admissible, those constrained along their own directions at both ends being discounted.

Lists LNB and LB contain respectively the indices of the non-basic and basic variables corresponding to any given column or row of the array A. To start with, only the artificial variables are basic, and these are given the indices (N+1) to (N+M), the $(N+K)^{th}$ artificial belonging to the $K^{th}$ equation. Thus, initially:

$$LB(K) = N+K$$

The original variables are all non-basic: initially, the coefficient of the $K^{th}$ variable is in the $K^{th}$ column of the array, and so:

$$LNB(K) = K$$

The lists LNB and LB are amended after each iteration of the Simplex algorithm.

L1 contains the indices of columns for consideration as the pivot column. Initially this comprises all the columns of the array A, but, during phase 1, as artificial variables become non-basic, the indices of columns which correspond to artificial variables are dropped. L2 contains the indices of rows for consideration as the pivot row i.e. all the rows 1...m of the array: it remains unaltered.

Lists LE1, LE2, LE3, and LE4 are involved only in data processing, for the correlation of the variables and equations of the original framework problem with those of the final linear programming problem. XL, DX, DY, DZ contain the lengths and direction cosines of the bars in the ground structure. LBC(I) takes the value one or zero depending on whether the limiting compressive stress in the $I^{th}$ bar is the Euler buckling stress or the compressive yield stress. LNC contains the nodal constraints. AB and LI are used in the

interpretation of the zeroth column of A as a particular framework, the tensions in the members being stored in AB and the corresponding subscripts in LI. MA is employed in the investigation of the stability of framework nodes If the bar joining nodes I and J, J>I, is a load-carrying member in an optimum framework, then the bar subscript is stored in the MA(I,J-1) and MA(J,I), being positive if the bar is in tension and negative if the bar is in compression. LN1 and LN2 contain the numbers of the two nodes to which any bar is joined. Thus if the $K^{th}$ bar joins nodes I and J, J>I, then:

LN1(K)=I, LN2(K)=J

LCN (I) contains a number which indicates the direction of instability, if any, at node I. PERT (I,1), PERT (I,2) and PERT (I,3) are the nodal perturbations in the x, y and z coordinate directions at node I. Finally, B is a list of the values of the applied external forces.

The maximum dimensions of all the arrays used in the program are given in table 4. The actual dimensions quoted in the program are calculated for a plane system containing a maximum of 20 nodes, with the number of admissible bars in the ground structure limited by the constraints to a maximum of 125.

## 2. CALCULATION OF THE LENGTHS AND DIRECTION COSINES OF THE BARS FORMING THE GROUND STRUCTURE

This section begins at statement 40. The coordinates of the nodes I and J which the bar joins are determined by subroutine MP9, the basis of which is the library function XMODF. XMODF (K,L), where K and L are integer arguments, gives the remainder when K is divided by L. Use is also made of the fact that an integer divided by another integer is rounded down to the nearest integer. The coordinates assigned to the nodes are in accordance with the numbering system shown in figure 6.1. From the coordinates of the nodes and the corresponding perturbations, if any, the lengths and direction cosines of the bars are determined by subroutine MP6. Any bar which joins two nodes both of which have a constraint along the bar direction is ruled out as a suitable structural member, and its length is set to zero.

## 3. THE LISTS LE1, LE2, LE3 and LE4

LE1(K1) determines the number of bars rejected up to and including bar number K1, where K1 depends only on the numbers of the nodes which the bar joins. LE2(K2) performs a complementary function; K2 steps by one each time a bar is accepted and LE2(K2) contains the corresponding K1 value. LE3 and KE serve the same purpose in the ordering of equations as LE1 and K1 in the ordering of bars. LE4 and NC are not quite the same as LE2 and K2, since NC steps by one each time an equation is rejected rather than accepted. Thus LE4 contains the KE values of the inadmissible equations.

## 4. FORMATION OF THE EQUILIBRIUM EQUATIONS

All the non-zero coefficients in the equilibrium equations are direction cosines of bars, which have already been calculated, so that this section of the program involves only the transference of the numbers in the lists DX, DY and DZ to the correct locations in A. Each iteration of the DO-loop ending at statement 210 deals essentially with the coefficients of the equilibrium equations applying to the node whose number is the same as the number of the iteration. However, since each direction cosine is used again, with the opposite sign, in the equation applying to the node at the other end of the corresponding bar, it is worthwhile inserting this value in the appropriate location in A at the same time. This means that at the $K^{th}$ iteration of the DO-loop, only the direction cosines of the bars which join node K to nodes K+1, K+2, ...., NJ are involved, because the direction cosines of the bar

joining nodes J and K, where J< K, were entered at the $J^{th}$ iteration.

## 5. CALCULATION OF THE COEFFICIENTS IN THE OBJECTIVE FUNCTION

The objective function is not exactly the same as given in equation (5.7), but is multiplied by $\sigma_{(t)}$ to give:

$$\min \sigma_{(t)} V = l_{(1)} T^{+}_{(1)} + l_{(2)} T^{+}_{(2)} + \cdots + l_{(k)} T^{+}_{(k)} + \frac{\sigma_{(t)}}{\sigma_{(c)}} l_{(1)} T^{-}_{(1)}$$

$$+ \frac{\sigma_{(t)}}{\sigma_{(c)}} l_{(2)} T^{-}_{(2)} + \cdots + \frac{\sigma_{(t)}}{\sigma_{(c)}} l_{(k)} T^{-}_{(k)}$$

If $l^{2}_{(j)} >$ SR * CB, then:

$$l^{2}_{(j)} > \frac{\sigma_{(t)}}{\sigma_{(c)}} \frac{\pi^{2} E I}{\sigma_{(t)} a_{(j)}}$$ , by the definition of CB

i.e. $$\sigma_{(c)} > \frac{\pi^{2} E I}{a_{(j)} l^{2}_{(j)}} = \sigma_{(\varepsilon)}$$

where $\sigma_{(\varepsilon)}$ is the Euler buckling stress. In this case the coefficient of $T^{-}_{(j)}$ in the objective function is not $(\sigma_{(t)} / \sigma_{(c)}) l_{(j)}$ but $(\sigma_{(t)} / \sigma_{(\varepsilon)}) l_{(j)}$ .

$$\frac{\sigma_{(t)}}{\sigma_{(c)}} l_{(j)} = \text{SR*XL(J)}$$

$$\frac{\sigma_{(t)}}{\sigma_{(\varepsilon)}} l_{(j)} = \text{XL(J)**3/CB}$$

All the coefficients in the objective function are stored as the negative of their actual values since subroutine Simplx is a maximisation process, whereas the problem to be solved is one of minimisation.

## 6. SUBROUTINE SIMPLX

The solution of the linear program is carried out by subroutine Simplx, with subroutines MP1, MP2, MP3 and MP4, the method being in strict agreement with the theory presented in chapter five. The $T^{+}_{(j)}$ become the variables 1 to KR1 of the linear program, the $T^{-}_{(j)}$ the variables KR1 to N, and the artificial variables the variables N+1 to N+M3, where KR1 is the number of admissible bars, N is 2*KR1 and M3 is the number of admissible equations.

During a re-cycle of the program, the print control parameter, PC, is given the value 1 when the magnitude of the objective function attains the smallest value not less than 10% above the optimum magnitude of the objective function in the first cycle, so that details of the layout for the current iteration and all subsequent iterations are printed out. The reason for this is that the method of nodal perturbations is designed to force a search for an alternative optimum layout with the same weight as the initial optimum but

without the nodal instabilities. However, the effect of perturbing the nodes may be to produce a slight increase or decrease in the weight of any alternative optimum that may exist, if such an alternative optimum makes use of a perturbed node. For this reason it is necessary to have details of as many layouts as possible with weights in the region of the first cycle optimum weight.

The standard technique for determining alternative optimal solutions to linear programs is to look for zero values in the coefficients of the objective function when the optimal solution is obtained. From equations (5.4) if $\bar{c}_{(m+k)}$ is zero, then $x_{(m+k)}$ can be made any value and the value of the objective function is unchanged. In the framework problem it would be necessary to increase $x_{(m+k)}$ to a value which drove one of the basic variables to zero in order to ensure that the resulting framework would be statically determinate: in other words, once the pivot column has been chosen by determining the zero $\bar{c}_{(m+k)}$, the pivot row is determined by the same criterion as is used in the Simplex algorithm. If the initial optimal basis is degenerate, the new basis may also be degenerate, so that the solution is effectively the same. In the framework problem, because of the fact that not every available node is used, often more than half of the basic variables in an optimal solution are zero, and therefore in searching for alternative optima, it is possible to perform a large number of iterations before any change in the pattern of load-carrying members is produced: in fact it often happens that the optimum layout is unique in terms of load-carrying members, although at the optimal solution to the linear program there are zero coefficients in the objective function. In these circumstances the method of nodal perturbations seems a more satisfactory method of searching for alternative optima. It has the disadvantage that not all possible layouts with weights in the region of the original optimum weight are necessarily produced and therefore an alternative optimum could still be by-passed. Nevertheless, in terms of the amount of computation involved, it provides a more realistic approach than the other method.

The re-cycle is only initiated if the maximum coefficient of the objective function is zero at the end of phase 2, because otherwise the possibility of alternative optima can be ruled out.

## 7. SUBROUTINES MP5 AND MP8

From the indices of the basic variables, contained in list LB, the framework bars to which these variables refer, and the sign of the load in each bar, are determined in subroutine MP5. The bar numbers are stored in list LI and the bar loads (positive tension, negative compression) and value of the objective function are stored in list AB. The reason for not simply storing this information in LB and the zeroth column of A is that the existing values in these locations must be preserved in the event of the parameter PC being 1 and subroutine MP5 being called after every iteration in phase 2 of subroutine Simplx. However the values of the reactive forces at the constraints can be put in the zeroth column of A, in the last NC locations, NC being the number of nodal constraints, since the last NC rows contain the coefficients of inadmissible equations and are thus not involved in the optimisation process. The indices of the constraint forces are in list LE4, the indices of forces being the same as the numbers of the equations to which they apply.

The details of the framework layout are printed out by subroutine MP8. Although the value of the objective function is labelled "WEIGHT", it is not the actual weight of the framework, but a non-dimensional parameter from which the weight may be obtained by multiplying by a factor $\rho Fd/\sigma_{(t)}$ where $\rho$ is the material density, d is the distance between two adjacent nodes and F the magnitude, in units consistent with $\sigma_{(t)}$, of the unit of force used in the data.

## 8. SUBROUTINE MP7

Each row of the two-dimensional array MA, in subroutine MP7, contains the numbers of the load-carrying members attached to a particular node of the optimum framework. The bar number is entered as positive if the bar is in tension and negative if the bar is in compression. The rows of MA are then scanned until one is found in which all the non-zero elements are negative. The arrangement of bars and constraints at the appropriate node is examined to determine if the node is unstable, and a number indicating the type of instability, if any, stored in list LCN. Thus:

| | | |
|---|---|---|
| | LCN(I) = 0: | Node I is stable |
| Three or more compression bars or constraints in the same plane | LCN(I) = 1: | Direction of instability at Node I has a component in the x coordinate direction |
| | LCN(I) = 2: | Direction of instability at Node I has a component in the y coordinate direction |
| | LCN(I) = 3: | Direction of instability at Node I has a component in the z coordinate direction |
| Two compression bars, or one bar and one constraint, in line | LCN(I) = 4: | Directions of instability at Node I have components in the y and z coordinate directions |
| | LCN(I) = 5: | Directions of instability at Node I have components in the z and x coordinate directions |
| | LCN(I) = 6: | Directions of instability at Node I have components in the x and y coordinate directions |

## 9. CALCULATION OF NODAL PERTURBATIONS

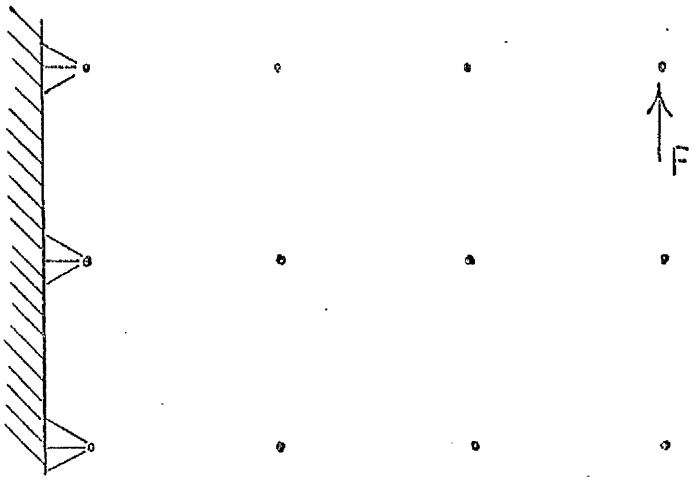
Unstable nodes are displaced in the direction of instability so that the arrangement of bars which caused the instability is no longer in equilibrium and cannot be repeated in the next cycle of the program. The direction of the displacement at a node (i.e. whether it is along the x, y or z coordinate direction) depends on the corresponding value in LCN: the sign of the displacement (i.e. whether in the positive or negative direction of the coordinate axis) depends on the number of the cycle, being alternately positive and negative in successive cycles. This minimises the possibility of an unstable arrangement being repeated due to a set of nodes all being displaced in the same direction. In the case of an instability due to two in-line compression bars, a check is made to see if they can be replaced by a single bar without altering the structural weight. Such a replacement is only possible if the limiting compressive stress in all the bars involved is the compressive yield stress and not the Euler buckling stress.

## ILLUSTRATIVE EXAMPLES

Due to the limited amount of storage available on Glasgow University's KDF9 computer, three-dimensional examples with a realistic number of nodes could not be run. In the early stages of development of the program, a single cell of two nodes in each coordinate direction was used to check that the various sections of the program were functioning correctly. The results themselves are of little significance. However some interesting results can be obtained using plane systems and consideration will now be given to several examples of this type.

### EXAMPLE 1

Cantilever framework to transmit force to rigid boundary:



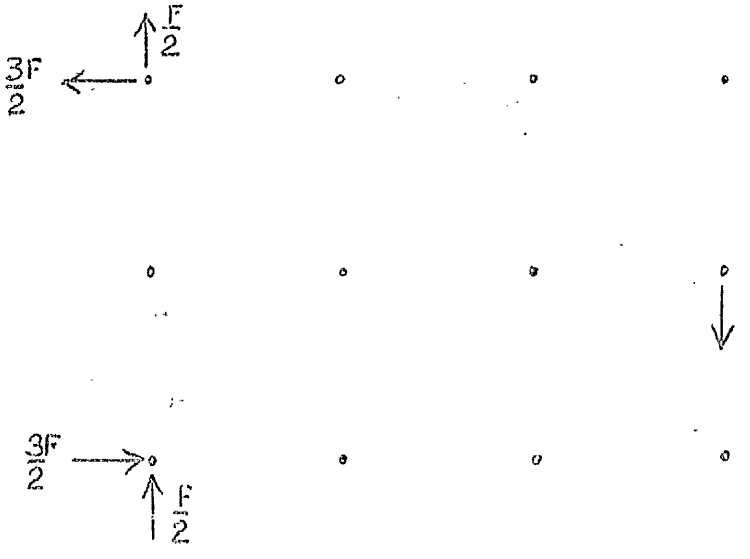

The data cards are as follows:

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1st | data card: | 4 | 3 | 1 | 8.0 | 1.0 | | | | | | | | | | | | |
| 2nd | data card: | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 3rd | data card: | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4th | data card: | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | | | | | | | | | |
| 5th | data card: | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | | | | | | | | | |
| 6th | data card: | 0 | | | | | | | | | | | | | | | | | |

For this example, the computer output for the first cycle and the beginning of the second cycle is given at the end of the chapter. The frameworks produced at the end of each of the three cycles of the program are shown in figures 6.2a, 6.2b and 6.2c. The C or T against the bar numbers indicates compression or tension respectively. All the frameworks produced in cycles 2 and 3 with weights within 5% of the minimum weight of 10.0 relied on a compressive load path between nodes 9 and 12, the unstable nodes 10 and 11 being suitably braced. The best alternative arrangement to be found by the computer is shown in figure 6.2d. Over the three cycles of the program there were 118 iterations of the Simplex algorithm, including both phases 1 and 2, and the total amount of computing time was 6 minutes 55 seconds.

EXAMPLE 2

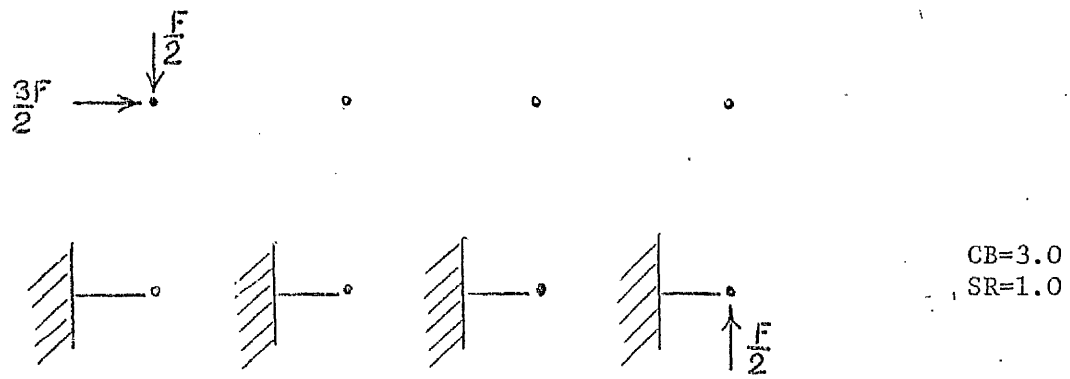
In this example the given forces form a system in equilibrium and there are no nodal constraints.



1st data card: 4 3 1 3.0 1.0
2nd data card: 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3rd data card: 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
4th data card: 1.5 0.5 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
5th data card: 0.0 0.0 0.0 -1.0 -1.5 0.5 0.0 0.0 0.0 0.0 0.0 0.0
6th data card: 0

The framework produced at the end of the first cycle is shown in figure 6.3a. Nodes 2 and 6 are unstable and therefore a second cycle is required, which merely introduces the bracing shown in figure 6.3b. Apart from the bracing members for nodes 2 and 6, the framework is very nearly symmetric about a horizontal centre line. The symmetry is only spoiled by the long tensile bar 27, which could be replaced by bars 48 and 24, with the additional advantage of stabilising node 6. Pairs of corresponding bars, one above and one below the line of symmetry, each carry the same magnitude of loading, one bar being in tension and the other in compression. Thus the problem can be dealt with by the method of the next example.

EXAMPLE 3




1st data card: 4 2 1 3.0 1.0
2nd data card: 1 0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0
3rd data card: 0.0 0.0 0.0 0.5 1.5 -0.5 0.0 0.0 0.0 0.0 0.0 0.0
4th data card: 0

The optimum solution is shown in figure 6.4: apart from the bracing bars 16 and 17 for the unstable node 6, the layout is the same as one half of the symmetric framework of example 2. By duplicating the layout below the line of constrained nodes and assuming the same magnitude but opposite sign of loading in corresponding bars, the reactive forces at the constraints become zero, and the force system and optimum layout of example 2 are produced. The advantage of using this method, for force systems with the necessary symmetry, is that a finer grid of nodes can be used for any given amount of available computer storage than is possible by solving the full problem. The disadvantage is that the assumption of symmetry in the framework layout will inevitably lead to a weight penalty in some examples although it did not in this particular example. If, however, the directions of all the forces are reversed, one optimum layout is simply the tensile bar 19 joining nodes 4 and 5. Because of the value 3.0 for CB, a bar of the same length as bar 19, carrying an equal and opposite load, is considerably heavier, and the weight of the two bars together is 21.6667. Thus in this case the assumption of symmetry in the

framework leads to a weight penalty of 117% over the absolute minimum. As a general rule it is best to base the whole framework on the half framework solution with the shortest compression members, which means solving that half of the problem in which the main load carrying members are likely to be in compression.

No nodal instabilities can arise along the line of symmetry because the constraints in the half framework problem permit only a horizontal reaction, which means that bars meeting at a node on the line of symmetry cannot all carry the same sign of loading.

EXAMPLE 4

The force system is similar to that of example 2 except that the points of reaction for the tip load are twice as far apart. The solution, obtained by the method of example 3, is shown in figure 6.5. To have solved the full problem with the same nodal pattern would have resulted in a ground structure of 276 bars, and a linear program with 600 variables and 48 equations.

NJ=Number of nodes

NC=Number of constraints

| ARRAY | MAXIMUM DIMENSION IN 3-D PROBLEM | MAXIMUM DIMENSION IN 2-D PROBLEM |
|---|---|---|
| A | ((3*NJ+2)*(NJ*(NJ-1)+1)) | ((2*NJ+2)*(NJ*(NJ-1)+1)) |
| LNB | (NJ*(NJ-1)) | (NJ*(NJ-1)) |
| LB | (3*NJ) | (2*NJ) |
| L1 | (NJ*(NJ-1)) | (NJ*(NJ-1)) |
| L2 | (3*NJ) | (2*NJ) |
| LE1 | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| LE2 | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| LE3 | (3*NJ) | (2*NJ) |
| LE4 | (NC) | (NC) |
| XL | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| DX | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| DY | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| DZ | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| LBC | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| LNC | (3*NJ) | (3*NJ) |
| AB | (3*NJ-NC+1) | (2*NJ-NC+1) |
| LI | (3*NJ-NC) | (2*NJ-NC) |
| MA | (NJ,NJ-1) | (NJ,NJ-1) |
| LN1 | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| LN2 | (NJ*(NJ-1)/2) | (NJ*(NJ-1)/2) |
| LCN | (NJ) | (NJ) |
| PERT | (NJ,3) | (NJ,3) |
| B | (3*NJ-NC) | (2*NJ-NC) |

TABLE 4

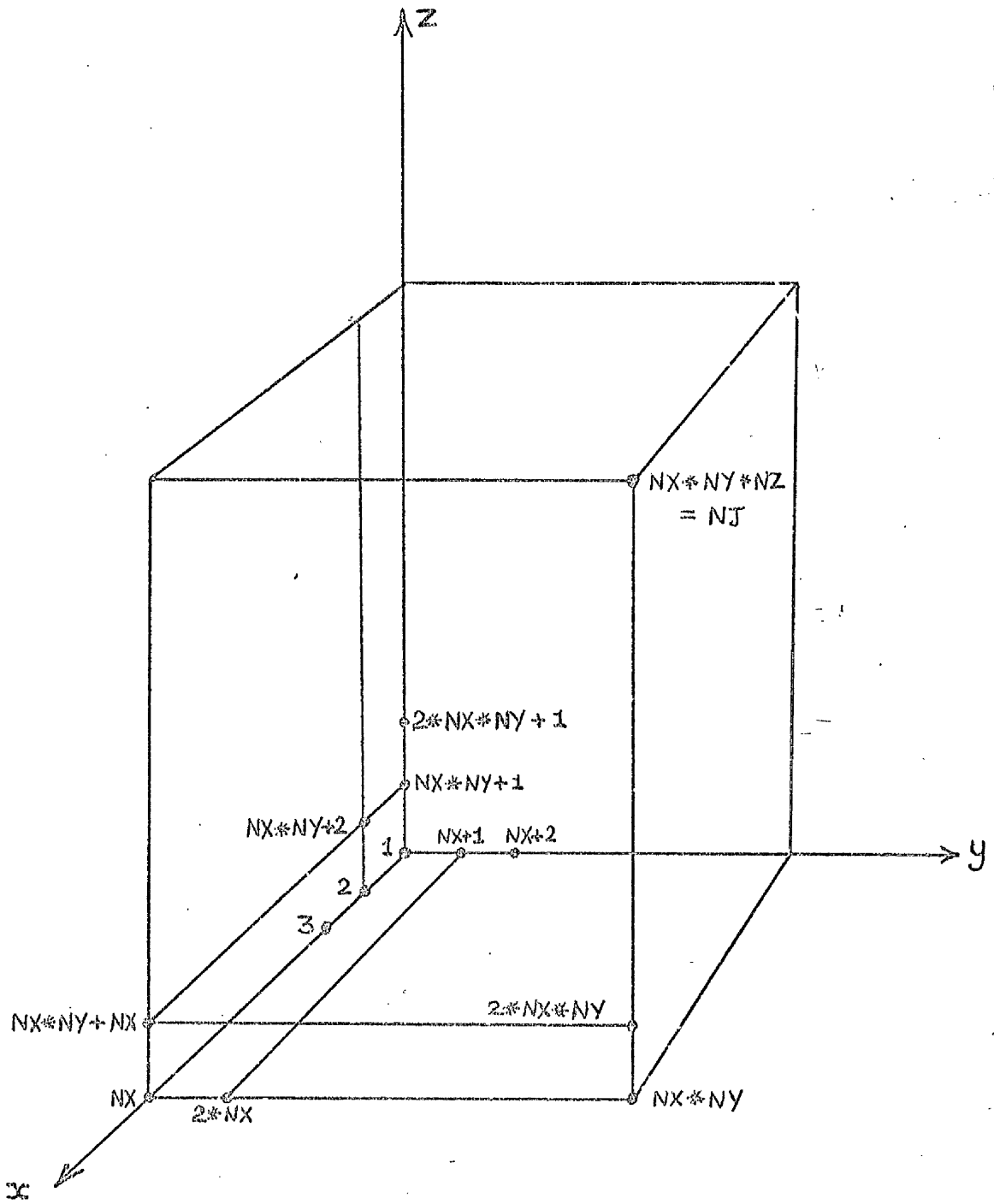
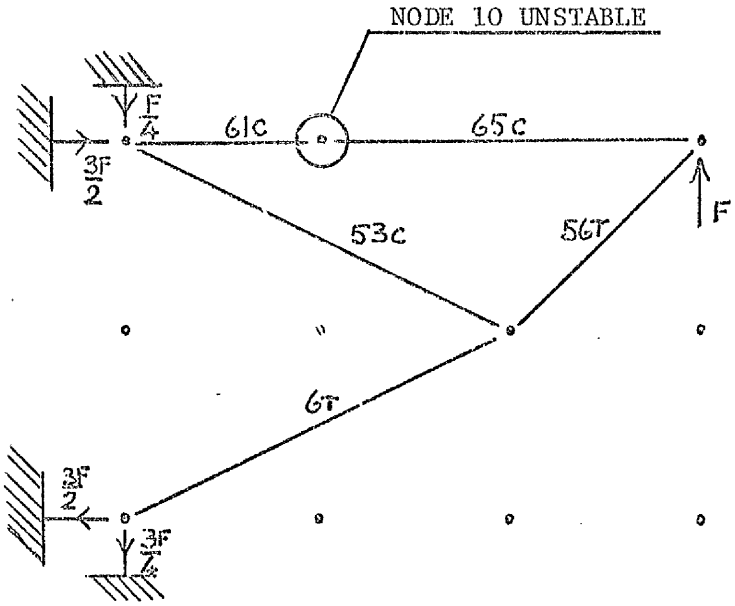
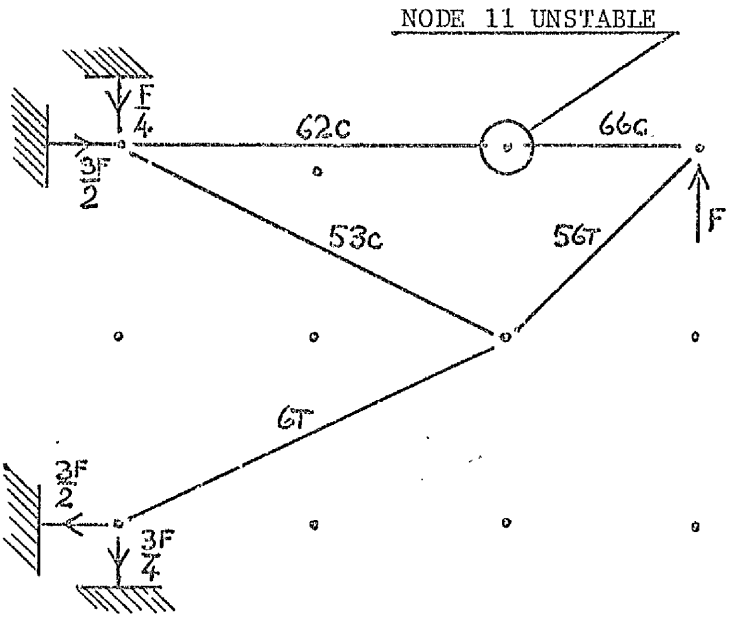

z
NX*NY*NZ
= NJ
2*NX*NY + 1
NX*NY+1
NX*NY+2
1
NX+1
NX+2
y
2
3
NX*NY+NX
2*NX*NY
NX
2*NX
NX*NY
x

FIGURE 6.1



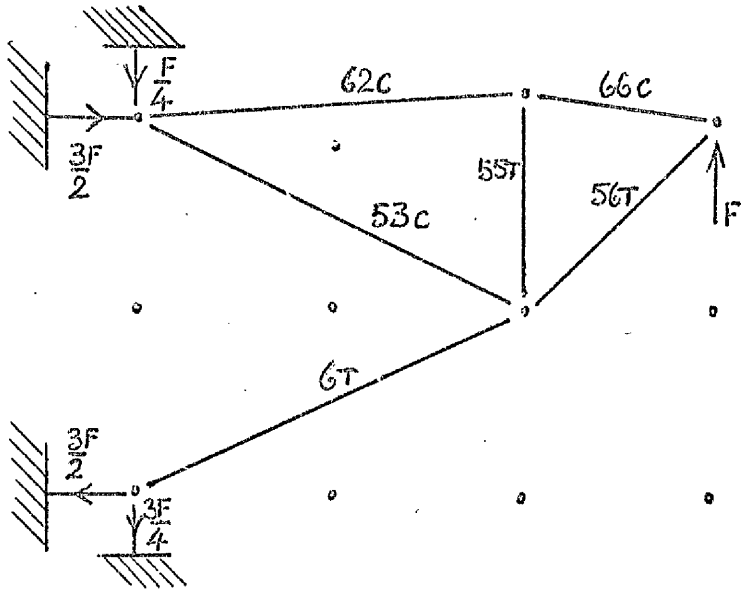

WEIGHT: 10.0000

FIGURE 6.2a



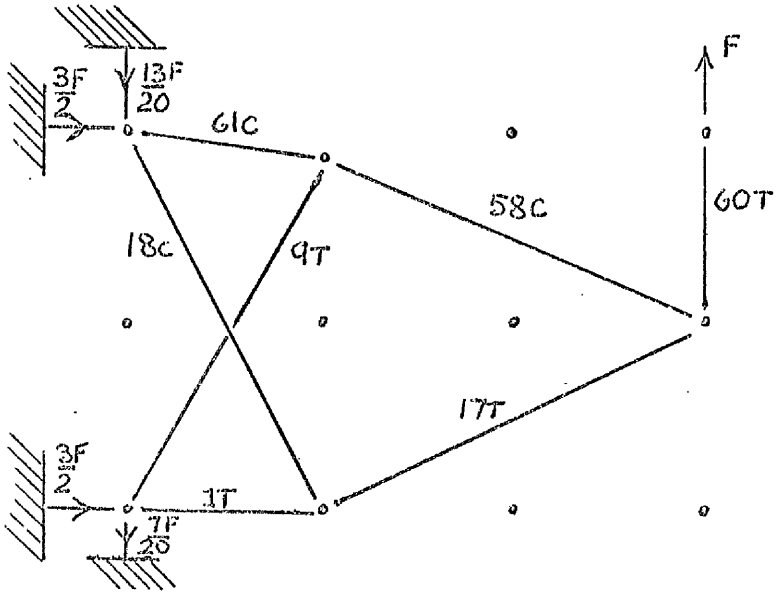

WEIGHT: 10.0000

FIGURE 6.2b



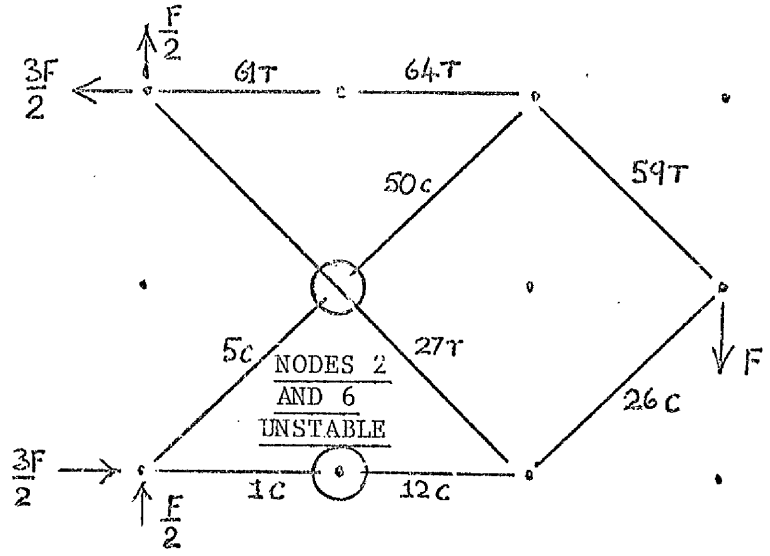

WEIGHT: 9.9364

FIGURE 6.2c



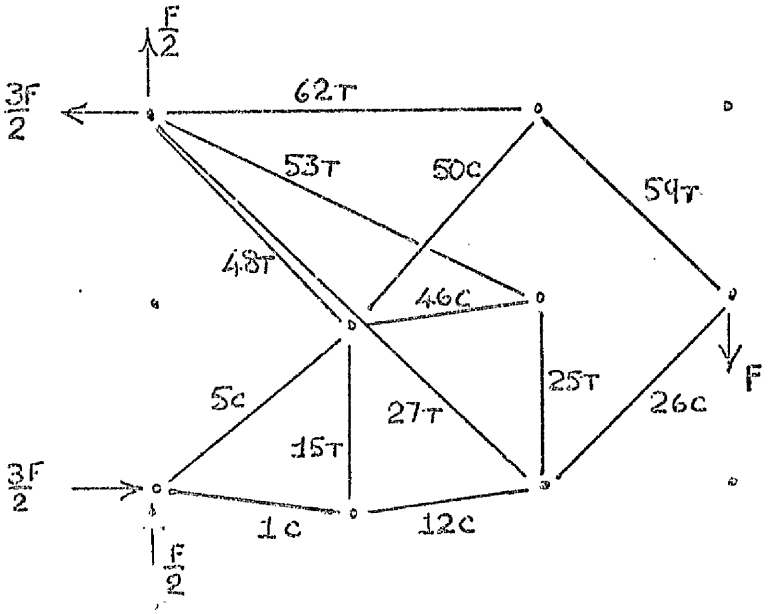

WEIGHT: 10.8932
(11.0000 WITH NODE
10 UNPERTURBED)

FIGURE 6.2d



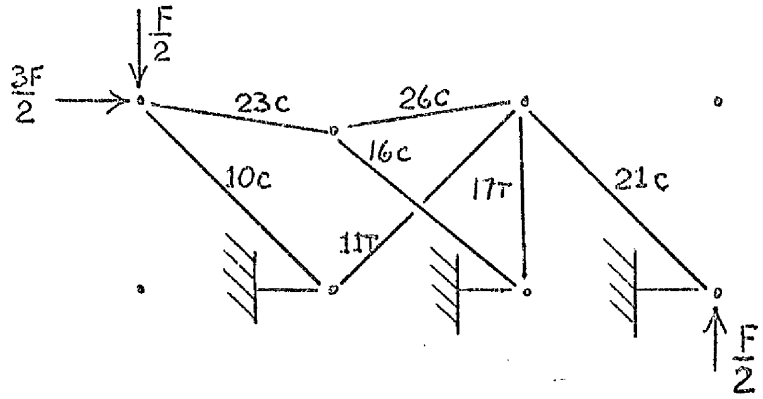

WEIGHT: 10.0000

FIGURE 6.3a



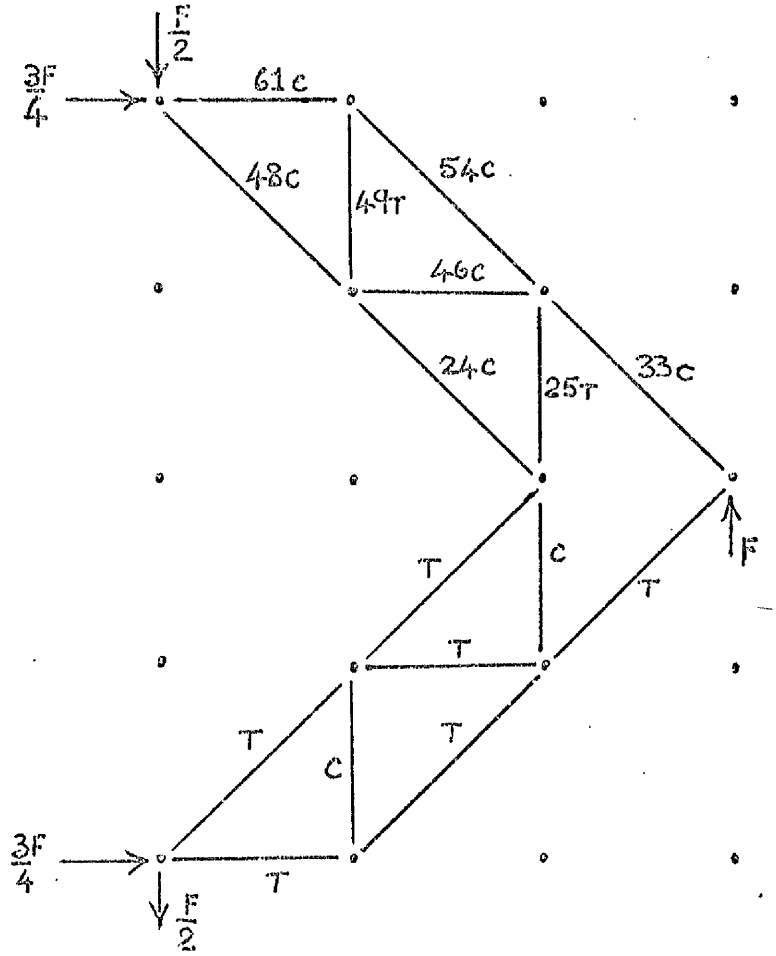

WEIGHT: 10.0149

FIGURE 6.3b




WEIGHT: 5.1778

FIGURE 6.4




WEIGHT: 8.0000

FIGURE 6.5

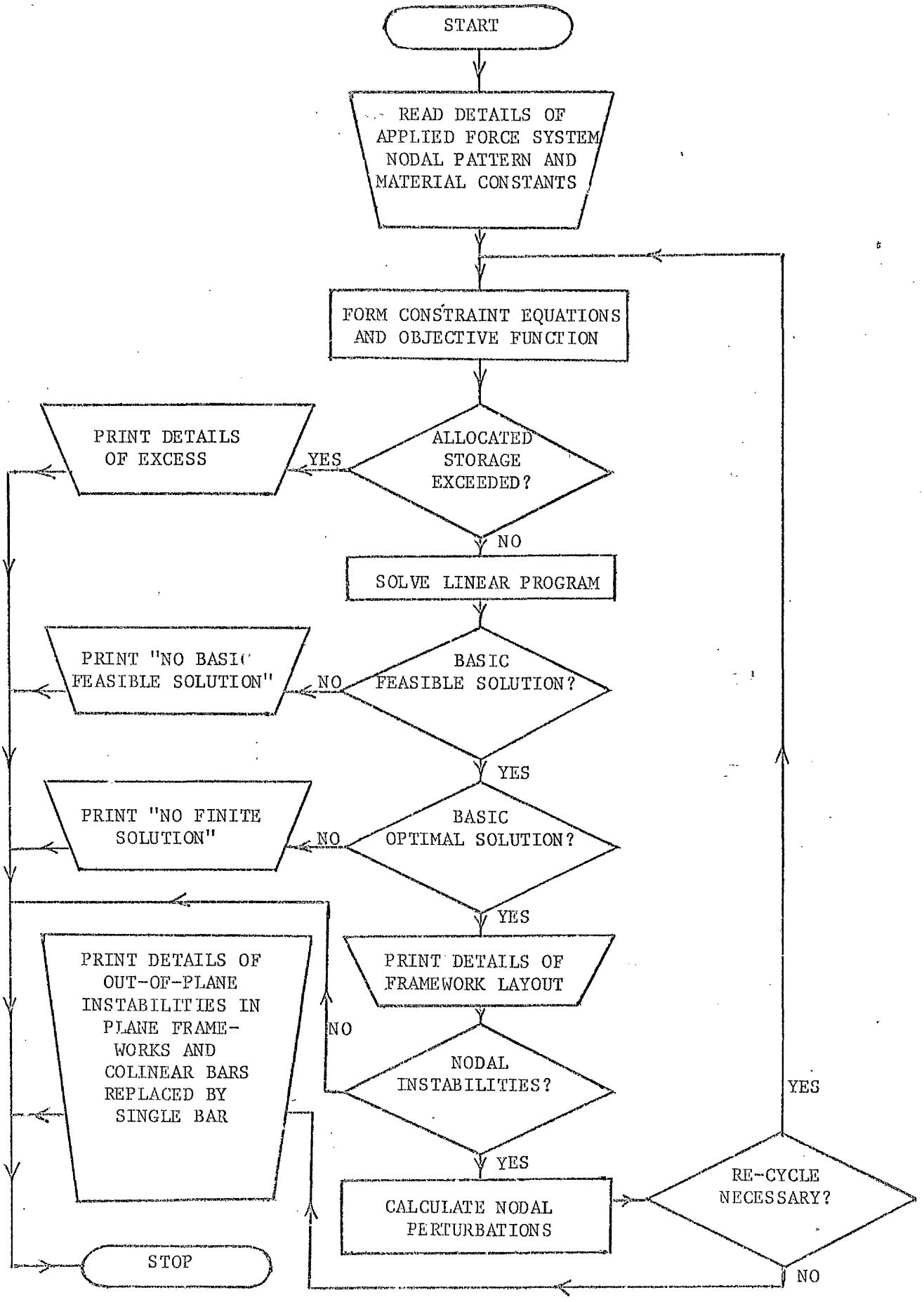

START
READ DETAILS OF APPLIED FORCE SYSTEM NODAL PATTERN AND MATERIAL CONSTANTS
FORM CONSTRAINT EQUATIONS AND OBJECTIVE FUNCTION
ALLOCATED STORAGE EXCEEDED?
YES
PRINT DETAILS OF EXCESS
NO
SOLVE LINEAR PROGRAM
BASIC FEASIBLE SOLUTION?
NO
PRINT "NO BASIC FEASIBLE SOLUTION"
YES
BASIC OPTIMAL SOLUTION?
NO
PRINT "NO FINITE SOLUTION"
YES
PRINT DETAILS OF FRAMEWORK LAYOUT
NODAL INSTABILITIES?
NO
PRINT DETAILS OF OUT-OF-PLANE INSTABILITIES IN PLANE FRAME-WORKS AND COLINEAR BARS REPLACED BY SINGLE BAR
YES
CALCULATE NODAL PERTURBATIONS
RE-CYCLE NECESSARY?
YES
NO
STOP

FIGURE 6.6

CYCLE 1

NODAL PERTURBATIONS

| | | | | |
|---|---|---|---|---|
| NODE | 1 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 2 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 3 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 4 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 5 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 6 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 7 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 8 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 9 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 10 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 11 | 0.0000 | 0.0000 | 0.0000 |
| NODE | 12 | 0.0000 | 0.0000 | 0.0000 |

LAYOUT AT END OF PHASE 2

| | | |
|---|---|---|
| T 1 | 0.0000 | 0 |
| T 15 | -0.0000 | 0 |
| T 34 | 0.0000 | 0 |
| T 24 | -0.0000 | 0 |
| T 7 | 0.0000 | 1 |
| T 33 | 0.0000 | 0 |
| T 60 | 0.0000 | 0 |
| T 5 | 0.0000 | 0 |
| T 48 | -0.0000 | 0 |
| T 53 | -0.5590 | 0 |
| T 26 | 0.0000 | 0 |
| T 59 | -0.0000 | 0 |
| T 64 | -0.0000 | 0 |
| T 49 | 0.0000 | 0 |
| T 6 | 1.6771 | 0 |
| T 61 | -1.0000 | 0 |
| T 56 | 1.4142 | 0 |
| T 65 | -1.0000 | 0 |

COMPUTER OUTPUT
FOR EXAMPLE 1

WEIGHT 10.0000

FORCES AT CONSTRAINTS

| | |
|---|---|
| F 1 | -1.5000 |
| F 2 | -0.7500 |
| F 9 | 0.0000 |
| F 10 | 0.0000 |
| F 17 | 1.5000 |
| F 18 | -0.2500 |

NUMBER OF ITERATIONS IN PHASE 1
19
NUMBER OF ITERATIONS IN PHASE 2
12

NODE 10 IS UNSTABLE

CYCLE 2

NODAL PERTURBATIONS

| | | | |
|---|---|---|---|
| NODE 1 | 0.0000 | 0.0000 | 0.0000 |
| NODE 2 | 0.0000 | 0.0000 | 0.0000 |
| NODE 3 | 0.0000 | 0.0000 | 0.0000 |
| NODE 4 | 0.0000 | 0.0000 | 0.0000 |
| NODE 5 | 0.0000 | 0.0000 | 0.0000 |
| NODE 6 | 0.0000 | 0.0000 | 0.0000 |
| NODE 7 | 0.0000 | 0.0000 | 0.0000 |
| NODE 8 | 0.0000 | 0.0000 | 0.0000 |
| NODE 9 | 0.0000 | 0.0000 | 0.0000 |
| NODE 10 | 0.0000 | -0.1000 | 0.0000 |
| NODE 11 | 0.0000 | 0.0000 | 0.0000 |
| NODE 12 | 0.0000 | 0.0000 | 0.0000 |

PHASE 2, ITERATION 10

| | | |
|---|---|---|
| T 1 | 0.6667 | 0 |
| T 15 | -0.3333 | 0 |
| T 34 | 0.0000 | 0 |
| T 25 | 0.0000 | 0 |
| T 2 | 0.0000 | 0 |
| T 49 | 0.0000 | 0 |
| T 60 | 1.0000 | 0 |
| T 5 | 0.4714 | 0 |
| T 46 | 0.3333 | 0 |
| T 53 | -0.9317 | 0 |
| T 17 | 0.7454 | 0 |
| T 59 | -0.9428 | 0 |
| T 61 | 0.0000 | 0 |
| T 66 | 0.0000 | 0 |
| T 6 | 0.5590 | 0 |
| T 62 | -0.6667 | 0 |
| T 36 | 0.0000 | 0 |
| T 55 | 0.6667 | 0 |

COMPUTER OUTPUT
FOR EXAMPLE 1
CONTINUED

WEIGHT 11.3333

FORCES AT CONSTRAINTS

| | |
|---|---|
| F 1 | -1.5000 |
| F 2 | -0.5833 |
| F 9 | 0.0000 |
| F 10 | 0.0000 |
| F 17 | 1.5000 |
| F 18 | -0.4167 |

PHASE 2, ITERATION 11

| | | |
|---|---|---|
| T 1 | 0.6667 | 0 |
| T 15 | -0.3333 | 0 |
| T 34 | 0.0000 | 0 |
| T 25 | 0.0000 | 0 |
| T 2 | 0.0000 | 0 |
| T 49 | 0.0000 | 0 |
| T 60 | 1.0000 | 0 |
| T 5 | 0.2357 | 0 |

```
T 48    -0.2357     0
T 53    -0.7454     0
T 17     0.7454     0
T 59    -0.9428     0
T 61     0.0000     0
T 66     0.0000     0
T  6     0.7454     0
T 62    -0.6667     0
T 36     0.0000     0
T 55     0.6667     0

WEIGHT    11.0000

FORCES AT CONSTRAINTS

F  1   -1.5000
F  2   -0.5000
F  9    0.0000
F 10    0.0000
F 17    1.5000
F 18   -0.5000

PHASE 2, ITERATION 12

T  1     0.0000     0
T 15    -0.0000     0
T 34     0.0000     0
T 25     0.0000     0
T  2     0.0000     0
T 49     0.0000     0
T 60     0.0000     0
T  5     0.0000     0
T 48    -0.0000     0
T 53    -0.5590     0
T 17     0.0000     0
T 59    -0.0000     0
T 61     0.0000     0
T 66    -1.0000     0
T  6     1.6771     0
T 62    -1.0000     0
T 36     0.0000     0
T 56     1.4142     0

WEIGHT    10.0000

FORCES AT CONSTRAINTS

F  1   -1.5000
F  2   -0.7500
F  9    0.0000
F 10    0.0000
F 17    1.5000
F 18   -0.2500
```

COMPUTER OUTPUT
FOR EXAMPLE 1
CONTINUED

FORTRAN PROGRAM TO DETERMINE THE LIGHTEST PIN-JOINTED FRAMEWORK WHICH CAN SAFELY EQUILIBRATE A GIVEN FORCE SYSTEM

```
      DIMENSION A(10550),LNB(250),LB(40),L1(250),L2(40),LE1(190),
     *LE2(190),LE3(40),LE4(40),XL(190),DX(190),DY(190),DZ(190),
     *LBC(190),LNC(60),AB(40),LI(40),MA(20,19),LN1(190),LN2(190),
     *LCN(20),PERT(20,3),B(40)
      INTEGER PC
      READ 10,NX,NY,NZ,CB,SR
   10 FORMAT (8OZ)
      NJ=NX*NY*NZ
      KK=3*NJ
      READ 10,(LNC(K),K+1,KK)
C
C     LNC(K) EQUAL TO 1 INDICATES A NODAL CONSTRAINT OR POINT OF
C     REACTION. THE CORRESPONDING EQUILIBRIUM EQUATION IS SUBSEQUENTLY
C     MISSED OUT
C
      NJ1=NJ-1
      MON5=0
C
C     MON5 COUNTS THE NUMBER OF CYCLES OF THE MAIN PROGRAM
C
      K2=0
      DO 30 I=1,NJ
      DO 20 J=1,3
   20 PERT(I,J)=0.0
   30 CONTINUE
C
C     CALCULATION OF THE DIRECTION COSINES OF THE BARS FORMING THE
C     GROUND STRUCTURE.
C
   40 DO 160 I=1,NJ1
      CALL MP9 (I,NX,NY,NZ,IX,IY,IZ)
      I1=I+1
      DO 150 J=I1,NJ
      IF (LCN(I).EQ.0.AND.LCN(J).EQ.0.AND.MON5.NE.0) GO TO 150
C
C     AFTER THE FIRST CYCLE THE DIRECTION COSINES NEED ONLY BE
C     RE-COMPUTED FOR PERTURBED NODES
C
      CALL MP9 (J,NX,NY,NZ,JX,JY,JZ)
      K1=J-I*(I+1)/2+NJ*(I-1)
C
C     K1 IS THE NUMBER OF THE BAR JOINING NODES I AND J
C
      IF (MON5.NE.0) GO TO 120
      LN1(K1)=I
      LN2(K1)=J
      IF (K1.LE.190) GO TO 120
      PRINT 110
  110 FORMAT (25HBAR SUBSCRIPT EXCEEDS 190)
      GO TO 450
  120 DELTAX=JX+PERT(J,1)-IX-PERT(I,1)-
      DELTAY=JY+PERT(J,2)--IY--PERT(I,2)
      DELTAZ=JZ+PERT(J,3)-IZ-PERT(I,3)
      CALL MP6 (DELTAX,DELTAY,DELTAZ,SXL,SDX,SDY,SDZ,LNC,I,J)
C
C     MP6 CALCULATES THE LENGTH AND DIRECTION COSINES OF THE BAR
C     IF THE BAR IS INADMISSIBLE THESE QUANTITIES ARE SET TO ZERO
```

```
      IF (MON5.NE.0) GO TO 140
      IF (SXL.LE.10.0E-10) GO TO 130
C
C     THERE ARE TWO NUMBERS ASSOCIATED WITH EACH BAR. ONE(K1)DEPENDS
C     ON THE NUMBERS OF THE NODES WHICH THE BAR JOINS,AND THE OTHER ON
C     THE NUMBER OF ADMISSIBLE BARS ALREADY DETERMINED. LISTS LE1 AND
C     LE2 CORRELATE THESE TWO NUMBERS
C
      K2=K2+1
      LE2(K2)=K1
      IF (K1.EQ.1) LE1(K1)=0
      IF (K1.NE.1) LE1(K1)=LE1(K1-1)
      GO TO 140
  130 IF (K1.EQ.1) LE1(K1)=1
      IF (K1.NE.1) LE1(K1)=LE1-1)+1
  140 XL(K1)=SXL
      DX(K1)=SDX
      DY(K1)=SDY
  150 DZ(K1)=SDZ
  160 CONTINUE
      IF (MON5.NE.0) GO TO 185
      IROW=1
C
C     AS WITH THE BARS,THERE ARE TWO NUMBERS ASSOCIATED WITH EACH
C     EQUILIBRIUM EQUATION. ONE DEPENDS ON THE NUMBER OF THE NODE AND
C     THE COORDINATE DIRECTION TO WHICH THE EQUATION APPLIES,AND THE
C     OTHER ON THE NUMBER OF ADMISSIBLE EQUATIONS ALREADY DETERMINED
C     LISTS LE3 AND IE4 CORRELATE THESE TWO NUMBERS
C
      KE=0
      DO 165 K=1,KK
      IF (NZ.EQ.1.AND.XMODF(K,3).EQ.0) GO TO 165
      KE=KE+1
      IF (LNC(K).EQ.0.AND.K.NE.1) LE3(KE)=LE3(KE-1)
      IF (LNC(K).EQ.0.AND.K.EQ.1) LE3(KE)=0
      IF (LNC(K).EQ.1.AND.K.NE.1) LE3(KE)=LE3(KE-1)+1
      IF (LNC(K).EQ.1.AND.K.EQ.1) LE3(KE)=1
      NC=LE3(KE)
      IF (LNC(K).EQ.1) LE4(NC)=KE
  165 CONTINUE
      IF (NZ.EQ.1) KK1=2*NJ
      IF (NZ.NE.1) KK1=3*NJ
      M3=KK1-NC
      JCOL=KK1+2
C
C     FOR PLANE SYSTEMS THE X-Y PLANE SHOULD BE USED TO SAVE
C     STORAGE SPACE
C
      KR=NJ*(NJ-1)/2
      KR1=KR-LE1(KR)
      IF (KR1.LE.125) GO TO 180
      PRINT 170
  170 FORMAT (46HNUMBER OF BARS IN GROUND STRUCTURE EXCEEDS 125)
      GO TO 450
  180 KA=JCOL*(2*KR1+1)
      READ 10,(B(K),K=1,M3)
```

```
C     THE APPLIED FORCE SYSTEM IS STORED IN B AND TRANSFERRED INTO A
C     DURING EACH CYCLE OF THE MAIN PROGRAM
C
 185  DO 190 K=1,KA
 190  A(K)=0.0
C
C     FORMATION OF EQUILIBRIUM EQUATIONS
C
C     IF AN EQUATION IS INADMISSIBLE THE COEFFICIENTS ARE STORED IN THE
C     BOTTOM ROWS OF THE MATRIX A IN ORDER TO CALCULATE SUBSEQUENTLY
C     THE REACTIONS AT THE CONSTRAINTS
C
      DO 210 I=1,NJ1
      J1=NJ*I-I*(I+1)/2
      J2=J1-LE1(J1)
      J3=NJ*(I-1)-(I-1)*I/2
      IF (J3.EQ.0) J4=1
      IF (J3.NE.0) J4=J3-LE1(J3)+1
      DO 200 J=J4,J2
      I1=3*I-2
      I2=3*I-1
      I3=3*I
      IF (NZ.EQ.1) GO TO 191
      IA=I1
      IB=I2
      GO TO 192
 191  IA=2*I-1
      IB=2*I
 192  IF (LNC(I1).EQ.0) K1=(IA-LE3(IA))*IROW+J*JCOL+1
      IF (LNC(I1).EQ.1) K1=(M3+1+LE3(IA))*IROW+J*JCOL+1
      IF (LNC(I2).EQ.0) K2=(IB-LE3(IB))*IROW+J*JCOL+1
      IF (LNC(I2).EQ.1) K2=(M3+1+LE3(IB))*IROW+J*JCOL+1
      IF (NZ.EQ.1) GO TO 193
      IF (LNC(I3).EQ.0) K3=(I3-LE3(I3))*IROW+J*JCOL+1
      IF (LNC(I3).EQ.1) K3=(M3+1+LE3(I3))*IROW+J*JCOL+1
 193  K4=LE2(J)
      II=K4+I*(I+1)/2+NJ*(1-I)
C
C     II IS THE NUMBER OF THE NODE AT THE OTHER END OF BAR K4
C     FROM NODE I
C
      II1=3*II-2
      II2=3*II-1
      II3=3*II
      IF (NZ.EQ.1) GO TO 194
      IIA=II1
      IIB=II2
      GO TO 195
 194  IIA=2*II-1
      IIB=2*II
 195  IF (LNC(II1).EQ.0) K5=(IIA-LE3(IIA))*IROW+J*JCOL+1
      IF (LNC(II1).EQ.1) K5=(M3+1+LE3(IIA))*IROW+J*JCOL+1
      IF (LNC(II2).EQ.0) K6=(IIB-LE3(IIB))*IROW+J*JCOL+1
      IF (LNC(II2).EQ.1) K6=(M3+1+LE3(IIB))*IROW+J*JCOL+1
      IF (NZ.EQ.1) GO TO 196
      IF (INC(II3).EQ.0) K7=(II3-LE3(II3))*IROW+J*JCOL+1
      IF (LNC(II3).EQ.1) K7=(M3+1+LE3(II3))*IROW+J*JCOL+1
```

```
 196  A(K1)=DX(K4)
      A(K2)=DY(K4)
      IF (NZ.EQ.1) GO TO 197
      A(K3)=DZ(K4)
 197  A(K5)=-A(K1)
      A(K6)=-A(K2)
      IF (NZ.EQ.1) GO TO 200
      A(K7)=-A(K3)
 200  CONTINUE
 210  CONTINUE
      DO 230 I=1,M3
      DO 220 J=1,KR1
      K1=I*IROW+J*JCOL+1
      K2=I*IROW+(J+KR1)*JCOL+1
 220  A(K2)=-A(K1)
 230  CONTINUE
C
C     THE VARIABLES ARE UNRESTRICTED IN SIGN
C
      DO 235 K=1,M3
 235  A(K+1)=B(K)
      DO 250 I=1,M3
      K=I+1
      IF (A(K).GE.0.0) GO TO 250
      J1=2*KR1+1
      DO 240 J=1,J1
      K=I*IROW+(J-1)*JCOL+1
 240  A(K)=-A(K)
C
C     THE CONSTANT TERMS IN THE EQUATIONS MUST BE POSITIVE FOR
C     SUBROUTINE SIMPLX
C
 250  CONTINUE
C
C     CALCULATION OF THE COEFFICIENTS IN THE MERIT FUNCTION
C
      NV=KR+M3
      DO 253 K=1,NV
 253  LBC(K)=0
      DO 260 J=1,KR1
      J1=LE2(J)
      K1=J*JCOL+1
      K2=(J+KR1)*JCOL+1
      SIGMA=CB*SR/XL(J1)**2
      IF (CB.NE.0.0.AND.SIGMA.LT.1.0) GO TO 255
C
C     IF BUCKLING CONSTRAINTS ARE INCLUDED AND THE BUCKLING STRESS
C     IS LOWER THAN THE LIMITING COMPRESSIVE STRESS,THE BUCKLING
C     CONSTRAINT IS ACTIVE
C
      A(K2)=-XL(J1)*SR
      GO TO 260
 255  A(K2)=-XL(J1)**3/CB
      LBC(J1)=1
 260  A(K1)=-XL(J1)
      N=2*KR1
      IF (MON5.NE.0) GO TO 265
```

```
      READ 10,PC
C
C     PC CONTROLS THE FORM OF THE PRINT OUT.IF PC EQUALS ONE THE
C     STRUCTURAL LAYOUT AND CONSTRAINT FORCES ARE CALCULATED AND
C     PRINTED OUT,BY SUBROUTINES MP5 AND MP8,AT EACH ITERATION IN PHASE
C     2 OF SUBROUTINE SIMPLX.IF PC EQUALS ZERO,ONLY THE STRUCTURES
C     OBTAINED AT THE COMPLETION OF PHASE 2 IN EACH CYCLE AND THE
C     CORRESPONDING CONSTRAINT FORCES ARE PRINTED OUT
C
 265  MON5=MON5+1
      PRINT 267,MON5,(I,(PERT(I,J),J=1,3),I=1,NJ)
 267  FORMAT (6H1CYCLE,I3//19HNODAL PERTURBATIONS//(4HNODE,I4,3F10.4))
      CALL SIMPLX (A,IROW,JCOL,N,M3,KR,KR1,NJ,NC,PC,MON2,MON3,MON4,MON5,
     *LNB,LB,L1,L2,LE2,LE4,LBC,LI,AB,LN1,LN2,MA,DX,DY,DZ,LNC,LCN)
C
C     SIMPLX PERFORMS THE OPTIMISATION PROCESS
C
      PRINT 270,MON3,MON4
 270  FORMAT (//31HNUMBER OF ITERATIONS IN PHASE 1/I3/
     *31HNUMBER OF ITERATIONS IN PHASE 2/I3)
      IF (MON2.EQ.1) GO TO 450
      NP=0
C
C     IN THE FOLLOWING SECTION PERTURBATIONS ARE CALCULATED SO THAT
C     CRITICAL NODES ARE DISPLACED IN THE DIRECTION OF INSTABILITY
C
      DO 430 I=1,NJ
      IF (LCN(I).EQ.0) GO TO 430
      IF (LCN(I).LE.3) GO TO 330
      DO 280 J=1,NJ1
 280  IF (MA(I,J).NE.0) GO TO 290
 290  IF (J.EQ.NJ1) GO TO 330
      K1=-MA(I,J)
      J1=J+1
      DO 300 J=J1,NJ1
 300  IF (MA(I,J).NE.0) GO TO 310
 310  IF (J.EQ.NJ1.AND.MA(I,J).EQ.0) GO TO 330
      K2=-MA(I,J)
      STL=(XL(K1)+XL(K2))**2
      IF (STL.GT.(CB*SR).AND.CB.NE.0.0) GO TO 330
      N1=LN1(K1)
      N2=LN2(K2)
      K3=N2-N1*(N1+1)/2+NJ*(N1-1)
      PRINT 320,I,K1,K2,K3
 320  FORMAT (//19HINSTABILITY AT NODE,I3,1X,14HMAY BE REMOVED/
     *17HBY REPLACING BARS,I3,1X,3HAND,I3,1X,6HBY BAR,I3)
      GO TO 420
C
C     IT MAY BE POSSIBLE TO REPLACE TWO COLINEAR BARS WITH A SINGLE BAR
C     WITHOUT INCURRING A WEIGHT PENALTY
C
 330  LCNI=LCN(I)
      GO TO (340,350,360,370,380,390),LCNI
 340  IF (NX.EQ.1) GO TO 400
      PERT(I,1)=0.1*((-1.0)**MON5)
      NP=NP+1
      GO TO 400
 350  IF (NY.EQ.1) GO TO 400
```

```
      PERT(I,2)=0.1*((-1.0)**MON5)
      NP=NP+1
      GO TO 400
  360 IF (NZ.EQ.1) GO TO 400
      PERT(I,3)=0.1*((-1.0)**MON5)
      NP=NP+1
      GO TO 400
  370 IF (NZ.EQ.1) PERT(I,2)=0.1*((-1.0)**MON5)
      IF (NZ.NE.1) PERT(I,3)=0.1*((-1.0)**MON5)
      NP=NP+1
      GO TO 400
  380 IF (NX.EQ.1) PERT(I,3)=0.1*((-1.0)**MON5)
      IF (NX.NE.1) PERT(I,1)=0.1*((-1.0)**MON5)
      NP=NP+1
      GO TO 400
  390 IF (NY.EQ.1) PERT(I,1)=0.1*((-1.0)**MON5)
      IF (NY.NE.1) PERT(I,2)=0.1*((-1.0)**MON5)
      NP=NP+1
  400 PRINT 410,I
  410 FORMAT (//4HNODE,I3,1X,11HIS UNSTABLE)
  420 IF (ABS(PERT(I,1)+PERT(I,2)+PERT(I,3)).LE.10.0E-10) LCN(I)=0
  430 CONTINUE
      IF (NP.NE.0.AND.MON2.NE.2) GO TO 40
      PRINT 440
  440 FORMAT (//38HNO RE-CYCLE. OPTIMUM SOLUTION OBTAINED)
  450 CALL EXIT
      END



      SUBROUTINE SIMPLX (A,IROW,JCOL,N,M3,KR,KR1,NJ,NC,PC,MON2,MON3,
     *MON4,MON5,LNB,LB,L1,L2,LE2,LE4,LBC,LI,AB,LN1,LN2,MA,DX,DY,DZ,LNC,
     *LCN)
C
C     ROUTINE FOR THE OPTIMISATION OF A LINEAR PROGRAM USING THE
C     SIMPLEX METHOD
C     DEGENERACIES ARE TAKEN INTO ACCOUNT
C
      DIMENSION A(1),LNB(1),LB(1),L1(1),L2(1),LE2(1),LE4(1),LBC(1),
     *LI(1),AB(1),LN1(1),LN2(1),MA(NJ,1),DX(1),DY(1),DZ(1),LNC(1),
     *LCN(1)
      INTEGER PC
      MON1=1
      MON2=1
      MON3=0
      MON4=0
C
C     MON1 AND MON2 TAKE THE VALUE ZERO WHEN PHASES 1 and 2 ARE
C     TERMINATED. MON2 EQUAL TO ZERO MEANS THERE ARE NO ALTERNATIVE
C     OPTIMA. MON3 AND MON4 COUNT THE ITERATIONS IN PHASES 1 AND 2
C
C     INITIALISATION OF THE INDEX LISTS
C
      DO 10 K=1,N
      L1(K)=K
   10 LNB(K)=K
      NL1=N
      DO 20 I=1,M3
      L2(I)=I
   20 LB(I)=N+I
      NL2=M3
```

```
      IPC=PC
C
C     COMPUTATION OF THE AUXILIARY OBJECTIVE FUNCTION IN PREPARATION
C     FOR PHASE 1
C
   45 N1=N+1
      DO 60 JJ=1,N1
      J=JJ-1
      Q1=0.0
      DO 50 I=1,M3
      K=I*IROW+J*JCOL+1
   50 Q1=Q1+A(K)
      K=(M3+1)*IROW+J*JCOL+1
   60 A(K)=-Q1
C
C     COMPUTATION OF A BASIC FEASIBLE SOLUTION BY MEANS OF THE SIMPLEX
C     METHOD. MP3 DETERMINES THE MAXIMUM COEFFICIENT OF THE AUXILIARY
C     OBJECTIVE FUNCTION
C
   70 CALL MP3 (A,M3+1,IROW,JCOL,JP,L1,NL1,RMAX)
      K=(M3+1)*IROW+1
      IF (RMAX.GT.10.0E-10.OR.A(K).GT.-10.0E-10) GO TO 80
      PRINT 75
   75 FORMAT (26HNO BASIC FEASIBLE SOLUTION)
      RETURN
C
C     THE AUXILIARY OBJECTIVE FUNCTION CANNOT BE REDUCED TO ZERO. NO
C     BASIC FEASIBLE SOLUTION EXISTS
C
   80 IF (RMAX.GT.10.0E-10.OR.ABS(A(K)).GE.10.0E-10) GO TO 120
C
C     IF THE AUXILIARY OBJECTIVE FUNCTION HAS THE VALUE ZERO A BASIC
C     FEASIBLE SOLUTION HAS BEEN FOUND. ARTIFICIAL VARIABLES
C     REMAINING IN THE BASIS AT A ZERO LEVEL,AND NOT BELONGING TO
C     VACUOUS EQUATIONS MUST BE ELIMINATED TO PREVENT THEM BECOMING
C     NON-ZERO DURING PHASE 2
C
      DO 90 IP=1,M3
      IF (LB(IP).NE.IP+N) GO TO 90
      CALL MP4 (A,IP,IROW,JCOL,L1,NL1,JP,RMAX)
      IF (ABS(RMAX).GT.10.0E-10) GO TO 130
   90 CONTINUE
      MON1=0
      GO TO 210
  120 CALL MP1 (A,L2,NL2,IP,IROW,JCOL,JP,Q1,N)
C
C     MP1 CHOOSES THE PIVOT ROW
C
  130 CALL MP2 (A,0,M3+1,0,N,IP,JP,IROW,JCOL)
C
C     MP2 TRANSFORMS THE MATRIX OF COEFFICIENTS
C
      MON3=MON3+1
      IF (LB(IP).LT.N+1) GO TO 200
      DO 140 K=  ,NL1
  140 IF (L1(K).EQ.JP) GO TO 150
  150 NL1=NL1-1
      DO 160 NM=K,NL1
  160 L1(NM)=L1(NM+1)
```

```
C     ONCE AN ARTIFICIAL VARIABLE IS REMOVED FROM THE BASIS ITS INDEX
C     IS ELIMINATED FROM L1,I.E.IT IS NOT CONSIDERED FOR RE-ENTRY TO
C     THE BASIS
C
C     IN THE FOLLOWING SECTION UP TO 210 THE LISTS LB AND LNB ARE
C     AMENDED
C
 200  NM=LNB(JP)
      LNB(JP)=LB(IP)
      LB(IP)=NM
      IF (MON1.NE.0) GO TO 70
C
C     PHASE 2.OPTIMISATION SECTION
C
      IF (PC.NE.1) GO TO 210
 205  CALL MP5 (A,LB,LE2,AB,LI,M3,KR1,KR,IROW,JCOL,NC)
      CALL MP8 (A,LE4,LBC,LI,AB,NC,M3,MON4,PC)
 210  CALL MP3 (A,0,IROW,JCOL,IP,L1,NL1,RMAX)
      IF (RMAX.GT.10.0E-10) GO TO 220
      MON2=0
C
C     IF THE MAXIMUM COEFFICIENT OF THE OBJECTIVE FUNCTION IS LESS
C     THAN OR EQUAL TO ZERO,THE OPTIMUM SOLUTION HAS BEEN OBTAINED
C
      IF (RMAX.LT.-10.0E-10) MON2=2
      IF (PC.EQ.1) GO TO 215
      CALL MP5 (A,LB,LE2,AB,LI,M3,KR1,KR,IROW,JCOL,NC)
      CALL MP8 (A,LE4,LBC,LI,AB,NC,M3,MON4,PC)
      GO TO 250
 215  PRINT 217
 217  FORMAT (//17HPHASE 2 COMPLETED)
      PC=IPC
      GO TO 250
 220  CALL MP1 (A,L2,NL2,IP,IROW,JCOL,JP,Q1,N)
 230  IF (IP.NE.0) GO TO 240
      PRINT 235
 235  FORMAT (18 HNO FINITE SOLUTION)
      RETURN
 240  CALL MP2 (A,0,M3,0,N,IP,JP,IROW,JCOL)
      MON4=MON4+1
      IF (MON5.EQ.1) W1=-1.1*A(1)
      IF (MON5.EQ.1.OR.PC.EQ.1) GO TO 200
      W2=-A(1)
      IF (W2.GE.W1) GO TO 200
      CALL MP2 (A,0,M3,0,N,IP,JP,IROW,JCOL)
      MON4=MON4-1
      PC=1
      GO TO 205
 250  CALL MP7 (A,MA,LI,LN1,LN2,AB,DX,DY,DZ,LNC,LCN,M3,KR,NJ)
      RETURN
      END


      SUBROUTINE MP1 (A,L2,NL2,IP,IROW,JCOL,JP,Q1,N)
C
C     MP1 DETERMINES THE MINIMUM OF ALL THE QUOTIENTS
C     -A(I*IROW)/A(I*IROW+JP*JCOL) FOR WHICH THE A(I*IROW+JP*JCOL)
C     ARE LESS THAN ZERO.DEGENERACY IS TAKEN INTO CONSIDERATION
C
      DIMENSION A(1),L2(1)
```

```
      IP=0
      IF (NL2.LT.1) RETURN
      DO 1 I=1,NL2
      K=L2(I)*IROW+1
      K1=K+JP*JCOL
    1 IF (A(K1).LT.-10.0E-10) GO TO 2
      RETURN
    2 Q1=-A(K)/A(K1)
      IP=L2(I)
      I1=I+1
      IF (I1.GT.NL2) RETURN
      DO 6 I=I1,NL2
      K=L2(I)*IROW+1
      K1=K+JP*JCOL
      IF (A(K1).GE.-10.0E-10) GO TO 6
      Q=-A(K)/A(K1)
      IF (Q.GE.Q1) GO TO 3
      IP=L2(I)
      Q1=Q
      GO TO 6
    3 IF (Q.NE.Q1) GO TO 6
C
C     HERE IT IS DETERMINED WHICH OF TWO ROWS WITH EQUAL QUOTIENT
C     SHOULD BE USED AS PIVOT ROW TO ENSURE THAT CYCLING DOES NOT OCCUR
C
      IO=L2(I)
      DO 4 J=1,N
      KO=IP*IROW+J*JCOL+1
      K2=IP*IROW+JP*JCOL+1
      K=IO*IROW+J*JCOL+1
      QP=-A(KO)/A(K2)
      QO=-A(K)/A(K1)
      IF (QP.LT.QO) GO TO 6
    4 IF (QO.LT.QP) GO TO 5
    5 IP =IO
    6 CONTINUE
      RETURN
      END



      SUBROUTINE MP2 (A,IO,I1,JO,J1,IP,JP,IROW,JCOL)
C
C     MP2 EXCHANGES A BASIC AND A NONBASIC VARIABLE AND TRANSFORMS THE
C     MATRIX OF COEFFICIENTS
C
      DIMENSION A(1)
      K=IP*IROW+JP*JCOL+1
      PIV=1.0/A(K)
      II0=IO+1
      II1=I1+1
      JJO=JO+1
      JJ1=J1+1
      DO 2 II=II0,II1
      I=II-1
      IF (I.EQ.IP) GO TO 2
```

```
      KO=I*IROW+JP*JCOL+1
      A(KO)=A(KO)*PIV
      DO 1 JJ=JJO,JJ1
      J=JJ-1
      IF (J.EQ.JP) GO TO 1
      K1=I*IROW+J*JCOL+1
      K2=IP*IROW+J*JCOL+1
      A(K1)=A(K1)-A(K2)*A(KO)
    1 CONTINUE
    2 CONTINUE
      DO 3 JJ=JJO,JJ1
      J=JJ-1
      K2=IP*IROW+J*JCOL+1
    3 IF (J.NE.JP) A(K2)=-A(K2)*PIV
      A(K)=PIV
      RETURN
      END



      SUBROUTINE MP3 (A,I1,IROW,JCOL,JP,L1,NL1,RMAX)
C
C     MP3 DETERMINES THE MAXIMUM OF THOSE COEFFICIENTS FOR WHICH THE
C     INDEX IS CONTAINED IN THE LIST L1
C
      DIMENSION A(1),L1(1)
      K=I1*IROW+L1(1)*JCOL+1
      RMAX=A(K)
      JP=L1(1)
      IF (NL1.LT.2) RETURN
      DO 1 J=2,NL1
      K=I1*IROW+L1(J)*JCOL+1
      IF (A(K).LE.RMAX) GO TO 1
      RMAX=A(K)
      JP=L1(J)
    1 CONTINUE
      RETURN
      END



      SUBROUTINE MP4 (A,I1,IROW,JCOL,LIST,NLIST,JP,RMAX)
C
C     MP4 DETERMINES THE NUMBER WITH THE LARGEST ABSOLUTE VALUE WHICH
C     BELONGS TO ROW I1 AND TO ONE OF THE COLUMNS WHOSE INDEX IS
C     CONTAINED IN LIST
C
      DIMENSION A(1),LIST(1)
      JP=LIST(1)
      K=I1*IROW+1
      KO=LIST(1)*JCOL+K
      RMAX=A(KO)
      IF (NLIST.LT.2) RETURN
      DO 1 J=2,NLIST
      KO=K+LIST(J)*JCOL
```

```
      IF (ABS(RMAX).GE.ABS(A(KO))) GO TO 1
      JP=LIST(J)
      RMAX=A(KO)
    1 CONTINUE
      RETURN
      END



      SUBROUTINE MP5 (A,LB,LE2,AB,LI,M3,KR1,KR,IROW,JCOL,NC)
C
C     MP5 CALCULATES THE STRUCTURAL LAYOUT AND CONSTRAINT FORCES
C     CORRESPONDING TO THE CURRENT VALUES OF THE ELEMENTS IN THE
C     MATRIX A
C
      DIMENSION A(1),LB(1),LE2(1),AB(1),LI(1)
      DO 10 K=1,NC
      K1=M3+2+K
   10 A(K1)=0.0
      DO 60 I=1,M3
      IF (LB(I).LE.KR1) GO TO 30
      I1=LB(I)-KR1
      IF (I1.GT.KR1) I2=I1-KR1+KR
C
C     IF THE VARIABLE HAS AN INDEX GREATER THAN TWICE THE NUMBER OF
C     ADMISSIBLE BARS,IT IS AN ARTIFICIAL VARIABLE
C
      IF (I1.LE.KR1) I2=LE2(I1)
      AB(I+1)=-A(I+1)
C
C     IF THE INDEX IS GREATER THAN THE NUMBER OF ADMISSIBLE BARS,THE
C     CORRESPONDING BAR IS IN COMPRESSION
C
      GO TO 40
   30 AB(I+1)=A(I+1)
      I1=LB(I)
      I2=LE2(I1)
   40 LI(I)=I2
      IF (NC.EQ.0) GO TO 60
C
C     CALCULATION OF THE REACTIVE FORCES
C
      DO 50 K=1,NC
      IF (ABS(AB(I+1)).LE.10.0E-10) GO TO 50
      K1=(M3+1+K)*IROW+1
      K2=K1+I1*JCOL
      A(K1)=A(K1)-A(K2)*AB(I+1)
   50 CONTINUE
   60 CONTINUE
      AB(1)=-A(1)
      RETURN
      END
```

```
      SUBROUTINE MP6 (DELTAX,DELTAY,DELTAZ,SXL,SDX,SDY,SDZ,LNC,I,J)
C
C     MP6 CALCULATES THE LENGTH AND DIRECTION COSINES OF EACH BAR
C
      DIMENSION LNC(1)
      SXL=(DELTAX**2+DELTAY**2+DELTAZ**2)**0.5
      SDX=DELTAX/SXL
      SDY=DELTAY/SXL
      SDZ=DELTAZ/SXL
      LCX=0
      LCY=0
      LCZ=0
      II1=3*I-2
      II2=3*I-1
      II3=3*I
      JJ1=3*J-2
      JJ2=3*J-1
      JJ3=3*J
      IF (LNC(II1).EQ.1.AND.LNC(JJ1).EQ.1) LCX=1
      IF (LNC(II2).EQ.1.AND.LNC(JJ2).EQ.1) LCY=1
      IF (LNC(II3).EQ.1.AND.LNC(JJ3).EQ.1) LCZ=1
      IF (LCX.EQ.1.AND.ABS(SDX-1.0).LE.10.0E-10) GO TO 5
      IF (LCY.EQ.1.AND.ABS(SDY-1.0).LE.10.0E-10) GO TO 5
      IF (LCZ.EQ.1.AND.ABS(SDZ-1.0).LE.10.0E-10) GO TO 5
      IF ((LCX+LCY).EQ.2.AND.ABS(SDZ).LE.10.0E-10) GO TO 5
      IF ((LCY+LCZ).EQ.2.AND.ABS(SDX).LE.10.0E-10) GO TO 5
      IF ((LCZ+LCX).EQ.2.AND.ABS(SDY).LE.10.0E-10) GO TO 5
      IF ((LCX+LCY+LCZ).EQ.3) GO TO 5
C
C     IF THE BAR IS CONSTRAINED ALONG ITS OWN DIRECTION AT BOTH ENDS
C     IT IS INADMISSIBLE
C
      GO TO 6
    5 SXL=0.0
      SDX=0.0
      SDY=0.0
      SDZ=0.0
    6 RETURN
      END


      SUBROUTINE MP7 (A,MA,LI,LN1,LN2,AB,DX,DY,DZ,LNC,LCN,M3,KR,NJ)
C
C     MP7 TESTS FOR UNSUPPORTED COMPRESSION NODES
C
      DIMENSION A(1),MA(20,1),LI(1),LN1(1),LN2(1),AB(1),DX(1),DY(1),
     *DZ(1),LNC(1),LCN(1)
      NJ1=NJ-1
      DO 20 I=1,NJ
      LCN(I)=0
      DO 10 J=1,NJ1
   10 MA(I,J)=0
   20 CONTINUE
      DO 30 I=1,M3
      I2=LI(I)
      IF (I2.GT.KR) GO TO 30
```

```
      II1=LN1(I2)
      II2=LN2(I2)
      J1=II2-1
      J2=II1
      IF (AB(I+1).GT.10.0E-10) MA(II1,J1)=I2
      IF (AB(I+1).LT.-10.0E-10) MA(II1,J1)=-I2
      MA(II2,J2)=MA(II1,J1)
   30 CONTINUE
C
C     THE NUMBERS OF THE BARS AT EACH NODE ARE STORED IN MATRIX MA.
C     THE BAR SUBSCRIPT GOES IN AS POSITIVE FOR A TENSION MEMBER,
C     NEGATIVE FOR A COMPRESSION MEMBER
C
      DO 180 I=1,NJ
      I1=3*I-2
      I2=3*I-1
      I3=3*I
      DO 40 J=1,NJ1
   40 IF (MA(I,J).GT.0) GO TO 180
C
C     A NODE IS STABLE IF IT HAS AT LEAST ONE TENSION MEMBER
C
      JJ=0
      DO 50 J=1,NJ1
   50 IF (MA(I,J).LT.0) JJ=JJ+1
      IF (JJ.EQ.0) GO TO 180
      DO 60 J=1,NJ1
   60 IF (MA(I,J).NE.0) GO TO 70
   70 K1=-MA(I,J)
      IF (JJ.EQ.1) GO TO 170
      J1=J+1
      DO 80 J=J1,NJ1
   80 IF (MA(I,J).LT.0) GO TO 90
   90 K2=-MA(I,J)
      PERPX=DY(K1)*DZ(K2)-DZ(K1)*DY(K2)
      PERPY=DZ(K1)*DX(K2)-DX(K1)*DZ(K2)
      PERPZ=DX(K1)*DY(K2)-DY(K1)*DX(K2)
C
C     THE VECTOR PRODUCT DETERMINES THE NORMAL TO THE PLANE CONTAINING
C     THE TWO BARS,UNLESS THEY ARE COLINEAR
C
      IF (JJ.EQ.2) GO TO 120
      J1=J+1
      DO 100 J=J1,NJ1
  100 IF (MA(I,J).NE.0) GO TO 110
  110 K3=-MA(I,J)
  120 IF (ABS(PERPX).LE.10.0E-10.AND.ABS(PERPY).LE.10.0E-10.AND.
     *ABS(PERPZ).LE.10.0E-10) GO TO 130
      IF (JJ.EQ.2) GO TO 160
      STP=DX(K3)*PERPX+DY(K3)*PERPY+DZ(K3)*PERPZ
      GO TO 140
  130 IF (JJ.EQ.2) GO TO 170
      PERPX=DY(K1)*DZ(K3)-DZ(K1)*DY(K3)
      PERPY=DZ(K1)*DX(K3)-DX(K1)*DZ(K3)
      PERPZ=DX(K1)*DY(K3)-DY(K1)*DX(K3)
C
C     IF THERE ARE MORE THAN TWO BARS,AND THE FIRST TWO ARE COLINEAR,
```

```
C     THE FIRST AND THIRD CANNOT BE COLINEAR
C
      STP=DX(K2)*PERPX+DY(K2)*PERPY+DZ(K2)*PERPZ
 140  IF (ABS(STP).GT.10.0E-10) GO TO 180
C
C     IF THE SCALAR TRIPLE PRODUCT IS NON-ZERO,THE BARS ARE NOT CO-
C     PLANAR AND THE NODE IS STABLE
C
      J1=J+1
      DO 150 J=J1,NJ1
      IF (MA(I,J).EQ.0) GO TO 150
      K4=-MA(I,J)
      STP=DX(K4)*PERPX+DY(K4)*PERPY+DZ(K4)*PERPZ
      IF (ABS(STP).GT.10.0E-10) GO TO 180
 150  CONTINUE
 160  Z1=LNC(I1)*PERPX
      Z2=LNC(I2)*PERPY
      Z3=LNC(I3)*PERPZ
      IF (ABS(Z1).GT.10.0E-10.OR.ABS(Z2).GT.10.0E-10.OR.
     *ABS (Z3).GT.10.0E-10) GO TO 180
C
C     IF THE BARS ARE CO-PLANAR AND IF THERE IS NOT A NODAL CONSTRAINT
C     WITH A COMPONENT IN THE DIRECTION OF THE NORMAL TO THE PLANE,
C     THE NODE IS UNSTABLE
C
 165  IF (ABS(PERPX).GT.10.0E-10) LCN(I)=1
      IF (ABS(PERPY).GT.10.0E-10.AND.LCN(I).EQ.0) LCN(I)=2
      IF (ABS(PERPZ).GT.10.0E-10.AND.LCN(I).EQ.0) LCN(I)=3
      GO TO 180
 170  NS=LNC(I1)+LNC(I2)+LNC(I3)+1
      GO TO (171,172,173,180),NS
 171  IF (ABS(DX(K1)).GT.10.0E-10) LCN(I)=4
      IF (ABS(DY(K1)).GT.10.0E-10.AND.LCN(I).EQ.0) LCN(I)=5
      IF (ABS(DZ(K1)).GT.10.0E-10.AND.LCN(I).EQ.0) LCN(I)=6
      GO TO 180
 172  PERPX=DY(K1)*LNC(I3)-DZ(K1)*LNC(I2)
      PERPY=DZ(K1)*LNC(I1)-DX(K1)*LNC(I3)
      PERPZ=DX(K1)*LNC(I2)-DY(K1)*LNC(I1)
      IF (ABS(PERPX).LE.10.0E-10.AND.ABS(PERPY).LE.10.0E-10.AND.
     *ABS(PERPZ).LE.10.0E-10) GO TO 171
      GO TO 165
 173  PERPX=FLOAT(LNC(I1)-1)
      PERPY=FLOAT(LNC(I2)-1)
      PERPZ=FLOAT(LNC(I3)-1)
      Z1=DX(K1)*PERPX
      Z2=DY(K1)*PERPY
      Z3=DZ(K1)*PERPZ
      IF (ABS(Z1).LE.10.0E-10.AND.ABS(Z2).LE.10.0E-10.AND.
     *ABS(Z3).LE.10.0E-10) GO TO 165
C
C     A NODE AT WHICH THERE IS ONLY ONE BAR OR TWO COLINEAR BARS,MUST
C     BE CONSTRAINED IN TWO DIRECTIONS PERPENDICULAR TO A BAR OR THE
C     NODE IS UNSTABLE
C
 180  CONTINUE
      RETURN
      END
```

```
      SUBROUTINE MP8 (A,LE4,LBC,LI,AB,NC,M3,MON4,PC)
C
C     MP8 PRINTS OUT THE LAYOUT DETAILS CALCULATED IN MP5
C
      DIMENSION A(1),LE4(1),LBC(1),LI(1),AB(1)
      INTEGER PC
      IF (PC.NE.1) GO TO 40
      PRINT 10,MON4
   10 FORMAT (//17HPHASE 2,ITERATION,I3/)
      GO TO 60
   40 PRINT 50
   50 FORMAT (//24HLAYOUT AT END OF PHASE 2/)
   60 DO 80 I=1,M3
      I2=LI(I)
      PRINT 70,I2,AB(I+1),LBC(I2)
   70 FORMAT (1HT,I3,F10.4,5X,I1)
   80 CONTINUE
      PRINT 90,AB(1)
   90 FORMAT(//6HWEIGHT,F10.4)
      IF (NC.EQ.0) GO TO 130
      PRINT 100
  100 FORMAT (//21HFORCES AT CONSTRAINTS/)
      DO 120 K=1,NC
      K1=M3+2+K
      PRINT 110,LE4(K),A(K1)
  110 FORMAT (1HF,I3,F10.4)
  120 CONTINUE
  130 RETURN
      END



      SUBROUTINE MP9 (K,NX,NY,NZ,KX,KY,KZ)
C
C     MP9 DETERMINES THE COORDINATES (KX,KY,KZ) OF NODE K
C
      NXY=NX*NY
      IF (XMODF(K,NXY).EQ.0) GO TO 20
      KZ=K/NXY
      KXY=XMODF(K,NXY)
      IF (XMODF(KXY,NX).EQ.0) GO TO 10
      KX=XMODF(KXY,NX)-1
      KY=KXY/NX
      GO TO 30
   10 KX=NX-1
      KY=KXY/NX-1
      GO TO 30
   20 KX=NX-1
      KY=NY-1
      KZ=K/NXY-1
   30 RETURN
      END
```

# CHAPTER SEVEN

## FURTHER DEVELOPMENTS

The minimum weight design of a pin-jointed framework for a single loading case has been shown to be a problem in linear programming. However any elaboration of the basic constraints gives rise to problems in non-linear programming, the solution of which is considerably more complicated. The non-linearity arises in the first place, simply as the result of the addition of non-linear constraints such as deflection constraints or buckling constraints in which the second moment of area is not directly proportional to the cross-sectional area: it arises, in the second place, because any increase in the number of independent constraints, whether by the addition of linear or non-linear types, over the minimum number, which is the number of independent equilibrium equations, destroys the argument that the optimum framework must be statically determinate, and therefore, from the point of view of elastic design at any rate, compatibility conditions are necessary. The inclusion of compatibility conditions means that there is no direct proportionality between load and cross-sectional area in structural members, so that the two sets of variables $T_{(j)}$ and $a_{(j)}$ are not interchangeable and must both be included as variables in the constraints. In the compatibility constraints they occur in non-linear groupings. In plastic design, where a structure is said to be acceptable if any statically admissible stress distribution can be found such that the stresses in the members are less than or equal to the corresponding yield stresses, the question of compatibility does not arise. In this case, provided that additional constraints are themselves linear, as, for example, in the multi-load problem, where there are several sets of equilibrium equations, one for each loading case, the problem can be kept within the realms of linear programming.

The most direct method of forming compatibility conditions is to apply Castigliano's theorem of compatibility to the ground structure. Thus if the ground structure possesses k degrees of redundancy, k self-equilibrating systems are chosen and the following equations formed:

$$\frac{\partial U^{(i)}}{\partial R^{(i)}} = 0 , \quad i = 1, \ldots, k. \qquad - (7.1)$$

where $U^{(i)}$ and $R^{(i)}$ are respectively the strain energy and force in the redundant member of the $i^{th}$ self-equilibrating system. Equations (7.1) can be written:

$$\sum_j \frac{T_{(j)}^{(i)} l_{(j)}^{(i)}}{E\, a_{(j)}^{(i)}} \frac{\partial T_{(j)}^{(i)}}{\partial R^{(i)}} = 0 , \quad i = 1, \ldots, k \qquad - (7.2)$$

where the summation extends over all the members of the $i^{th}$ system, including the redundant member. The disadvantage of equations (7.2) is that they are only meaningful as compatibility equations if none of the $T_{(j)}^{(i)}$ becomes zero. If, for example, $T_{(j)}^{(m)}$ is zero, then the $m^{th}$ equation of the set (7.2) can easily be shown to mean that there is zero strain between the nodes which the $j^{th}$ bar joins, a condition which is superfluous: if more than one of the terms in any equation vanish, the constraint is quite meaningless. Accepting that there are going to be some non-linear constraints in any case, a better approach is to formulate the equilibrium equations in terms of the elastic stiffness matrix, thus making all the equilibrium constraints non-linear and introducing the nodal

displacements as variables in place of the bar tensions.

In recent years, a lot of attention has been given to the solution of non-linear programming problems, and their application to optimum design of frameworks. The earliest work was done by Schmit (reference 18), but applied only to simple statically indeterminate frameworks of fixed layout. Dobbs and Felton (reference 19) considered optimum layout of pin-jointed frameworks subject to multiple loading cases. They also gave some attention to the problem of buckling, by specifying that framework members be tubular in design and individually optimised for simultaneous occurrence of Euler and local buckling modes, thus giving rise to a stress constraint for each member of the ground structure in which appeared the square root of the cross-sectional area of that member. They did not consider buckling modes involving more than one member and evidently did not encounter the problem of an optimum design with two compressive members at, or close to, their respective buckling stresses, pin-jointed end to end and unsupported at the joint. Even in the context of non-linear programming, it is difficult to see how constraints could be included to guard against this sort of instability. The stability analysis of a given structure, by means of the geometric stiffness matrix and the associated linear stability determinant (the meaning of linear in this case is far removed from its meaning in mathematical programming), is a fairly complex procedure and not one which lends itself readily to the formulation of constraints for a design problem. The best approach would seem to be to adopt a basic structural form which is less inherently unstable than a pin-jointed framework, and in this connection the optimisation of rigid-jointed frameworks is worthy of attention. Majid and Elliot (reference 20) have considered the optimum layout of pin-jointed frameworks subject to deflection constraints, but have ignored buckling.

Since the early analytical work by Hemp (reference 4), little else has been done on the analytical side of optimum layout studies. Subsequent to the publication in 1964 of the paper by Dorn, Gomory and Greenberg, the emphasis has been almost entirely on the numerical side of the work, and the advancement of techniques in mathematical programming has continually widened the scope of the problems which can be solved. The claim is usually made, not without some justification, that progress is being made towards greater realism and that results are being produced which are of value to the practising engineer, although so long as layout studies are based on pin-jointed frameworks with zero joint weight and a tendency to buckle, there is some ground for doubting the validity of this claim. What can be said about the results is that because of the nature of the numerical approach, in which each problem is treated individually, no universal trend has been established and no fundamental truth even hinted at, and because of the restrictions imposed by computer storage on the extent of ground structures, the optimum frameworks produced have been unremarkable from an engineering point of view, and prosaic from an aesthetic point of view.

Both as regards basic philosophy and aesthetic appeal Michell's theorem is still the most fascinating aspect of the work done to date on optimum layout of structures. The associated mathematical problem of determining the classes of curves satisfying the Michell criterion, which has been considered in some detail herein, is one to which further attention could usefully be given. Now that the link has been established between Michell's theorem and the linear programming approach to optimum layout, there is nothing to be gained by attempting numerical solutions of the analytical problem posed at the beginning of chapter five. The Simplex algorithm is simple and efficient: the bulk of the computer program presented in chapter six is concerned with the processing of data and results. Moreover mathematical programming has shown itself to be flexible and potentially capable of providing realistic solutions to structural design problems, which means that any approximate solution of the layout equations of chapter four is redundant, since it does not provide a structure of any practical value. However,

exact solutions, of which relatively few exist, are, if nothing else, of philosophical interest and for that reason, further work on the analytical side is to be recommended. The results could turn out to have some influence on practical design: the inverse approach to the solution of analytical problems in engineering has proved useful in the past.

# APPENDIX A

## ALTERNATIVE DERIVATION OF THE LAYOUT EQUATIONS

It was shown in Chapter 2 that the important properties of the strain tensor, as far as Michell's Theorem is concerned, are firstly that there are principal directions in which the quantities $e_{IJ}l^I l^J$ are extrema, and secondly that it derives from a vector field in such a way as to establish the relation

$$e_{IJ}\,l^I l^J = \left(l^K \frac{\partial}{\partial x^K}\right)\left(l^M u_M\right)$$

The above equation is still true if $e_{IJ}$ is related to $u_I$ by the equation:

$$e_{IJ} = u_{I,J} \qquad \text{- (A.1)}$$

The tensor $e_{IJ}$ is not now symmetric and it cannot therefore be diagonalised in a similar manner to the strain tensor. However in N-dimensional space there still exist N orthogonal directions in which the quantities $e_{IJ}l^I l^J$ are extrema, because

$$\lambda = e_{IJ}l^I l^J = \tfrac{1}{2}(e_{IJ}+e_{JI})l^I l^J + \tfrac{1}{2}(e_{IJ}-e_{JI})l^I l^J$$
$$\leq 0$$
$$= \bar{e}_{IJ}\,l^I l^J \qquad \text{- (A.2)}$$

where $\bar{e}_{IJ}$ is the symmetric part of $e_{IJ}$. Thus the maxima and minima of $\lambda$ lie along the principal directions determined by the symmetric tensor $\bar{e}_{IJ}$.

The condition that the tensor $e_{IJ}$ form the symmetric part of $u_{I,J}$ (the small strain compatibility condition) is more complicated than the condition that $e_{IJ}$ be of the form $u_{I,J}$. From the derivation of equation (4.4) at the beginning of chapter 4, it is clear that the latter condition implies that the expressions $e_{IJ}\,dx^J$ are perfect differentials, and so the $e_{IJ}$ must satisfy the tensor equation:

$$e_{IJ,K} = e_{IK,J} \qquad \text{- (A.3)}$$

The two dimensional form of equation (A.3) will now be used to derive equations (4.14), which must be satisfied by the layout lines of optimum plane frameworks. The layout lines are used as an orthogonal coordinate system, the coordinates being denoted $x^2$ and $x^3$. The fundamental tensor is $g^{(1)}_{\alpha\beta}$ as given in equation (3.3a). The diagonal components of the tensor $e_{\alpha\beta}$ are

$$e_{11} = \varepsilon B^2, \qquad e_{22} = -\varepsilon C^2$$

as before. The off-diagonal components of $\bar{e}_{\alpha\beta}$ must be zero, and therefore the off-diagonal components of $e_{\alpha\beta}$ must be equal and opposite. Thus

$$e_{\alpha\beta} = \begin{bmatrix} \varepsilon B^2 & BCf \\ -BCf & -\varepsilon C^2 \end{bmatrix} \qquad - (A.4)$$

where f is an arbitrary function of the coordinates. Equation (A.3) provides two independent equations in two dimensions, viz:

$$e_{12,1} = e_{11,2}$$

$$e_{21,2} = e_{22,1}$$

which, on expanding the covariant derivatives, become respectively:

$$\frac{2\varepsilon}{C}\frac{\partial B}{\partial x^3} + \frac{\partial f}{\partial x^2} = 0$$

$$\frac{2\varepsilon}{B}\frac{\partial C}{\partial x^2} + \frac{\partial f}{\partial x^3} = 0$$

Differentiating the first w.r.t. $x^3$ and the second w.r.t. $x^2$ and subtracting gives:

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial B}{\partial x^3}\right) - \frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial C}{\partial x^2}\right) = 0$$

This equation, combined with the equation $R_{1212}=0$ gives equations (4.14).

In three dimensions the tensor $e_{ij}$ contains three arbitrary functions which have to be eliminated from the nine equations produced by (A.3). The process is fairly lengthy, but it serves as a useful check on some of the three dimensional layout conditions derived in chapter 4. The following is an outline of an alternative derivation of equations (4.15).

The appropriate form of $e_{ij}$ is:

$$e_{ij} = \begin{bmatrix} \varepsilon A^2 & ABf & ACg \\ -ABf & \varepsilon B^2 & BCh \\ -ACg & -BCh & -\varepsilon C^2 \end{bmatrix} \qquad - (A.5)$$

where f, g and h are arbitrary functions of the coordinates. The nine equations from (A.3) are:

$$e_{11,2} = e_{12,1}$$

$$e_{22,3} = e_{23,2}$$

$$e_{33,1} = e_{31,3}$$

$$e_{22,1} = e_{21,2}$$

$$e_{33,2} = e_{32,3}$$

$$e_{11,3} = e_{13,1}$$

$$e_{12,3} = e_{13,2}$$

$$e_{21,3} = e_{23,1}$$

$$e_{31,2} = e_{32,1}$$

The above set of equations can be reduced, after some manipulation to the following:

$$\frac{\partial f}{\partial x^1} = \frac{1}{C}\frac{\partial A}{\partial x^3} h$$

$$\frac{\partial f}{\partial x^2} = \frac{-1}{C}\frac{\partial B}{\partial x^3} g$$

$$\frac{\partial f}{\partial x^3} = \frac{1}{B}\frac{\partial C}{\partial x^2} g - \frac{1}{A}\frac{\partial C}{\partial x^1} h$$

$$\frac{\partial g}{\partial x^1} = \frac{-1}{B}\frac{\partial A}{\partial x^2} h + \frac{2\varepsilon}{C}\frac{\partial A}{\partial x^3}$$

$$\frac{\partial g}{\partial x^2} = \frac{1}{C}\frac{\partial B}{\partial x^3} f + \frac{1}{A}\frac{\partial B}{\partial x^1} h$$

$$\frac{\partial g}{\partial x^3} = \frac{-1}{B}\frac{\partial C}{\partial x^2} f + \frac{2\varepsilon}{A}\frac{\partial C}{\partial x^1}$$

$$\frac{\partial h}{\partial x^1} = -\frac{1}{C}\frac{\partial A}{\partial x^3} f + \frac{1}{B}\frac{\partial A}{\partial x^2} g$$

$$\frac{\partial h}{\partial x^2} = -\frac{1}{A}\frac{\partial B}{\partial x^1} g + \frac{2\varepsilon}{C}\frac{\partial B}{\partial x^3}$$

$$\frac{\partial h}{\partial x^3} = \frac{1}{A}\frac{\partial C}{\partial x^1} f + \frac{2\varepsilon}{B}\frac{\partial C}{\partial x^2}$$

The partial derivatives can be eliminated using the nine equations of the type:

$$\frac{\partial^2 (f, g, h)}{\partial x^i \partial x^j} = \frac{\partial^2 (f, g, h)}{\partial x^j \partial x^i} \qquad - (A.6)$$

It is found that the coefficients of the functions f, g and h in equations (A.6) reduce to zero, and $\varepsilon$ is therefore a factor of every term. Four of the equations are identically satisfied, three reduce to equation (4.15c) and the other two give equations (4.15a) and (4.15b). For example:

$$\frac{\partial}{\partial x^3}\left(\frac{\partial g}{\partial x^1}\right) = -h\frac{\partial}{\partial x^3}\left(\frac{1}{B}\frac{\partial A}{\partial x^2}\right) - \frac{1}{B}\frac{\partial A}{\partial x^2}\left(\frac{1}{A}\frac{\partial C}{\partial x^1} f + \frac{2\varepsilon}{B}\frac{\partial C}{\partial x^2}\right)$$

$$+ 2\varepsilon\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial A}{\partial x^3}\right)$$

$$\frac{\partial}{\partial x^1}\left(\frac{\partial g}{\partial x^3}\right) = -f\frac{\partial}{\partial x^1}\left(\frac{1}{B}\frac{\partial C}{\partial x^2}\right) - \frac{1}{B}\frac{\partial C}{\partial x^2}\left(\frac{1}{C}\frac{\partial A}{\partial x^3} h\right)$$

$$+ 2\varepsilon\frac{\partial}{\partial x^1}\left(\frac{1}{A}\frac{\partial C}{\partial x^1}\right)$$

$$\therefore\ h\left[-\frac{\partial}{\partial x^3}\left(\frac{1}{B}\frac{\partial A}{\partial x^2}\right) + \frac{1}{BC}\frac{\partial C}{\partial x^2}\frac{\partial A}{\partial x^3}\right]$$

$$+ f\left[\frac{\partial}{\partial x^1}\left(\frac{1}{B}\frac{\partial C}{\partial x^2}\right) - \frac{1}{AB}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^1}\right]$$

$$= 2\varepsilon\left[\frac{1}{B^2}\frac{\partial C}{\partial x^2}\frac{\partial A}{\partial x^2} - \frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial A}{\partial x^3}\right) + \frac{\partial}{\partial x^1}\left(\frac{1}{A}\frac{\partial C}{\partial x^1}\right)\right]$$

The coefficient of h is zero by equation (3.7f); that of f is zero by equation (3.7d); that of $\mathcal{E}$ is $-4\partial/\partial x^3((1/C)\partial A/\partial x^3)$ by equation (3.7c), and so:

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial A}{\partial x^3}\right) = 0$$

which is equation (4.15a)

# APPENDIX B

## TWO EXAMPLES

This Appendix contains an analysis of two optimum frameworks discussed by Michell (reference 3), making use of equations derived in chapters 3 and 4. This provides a check on the equations, since results derived from them can be compared with those quoted by Michell, and also illustrates the application of the equations to particular examples.

### 1. Layout Based on Equiangular Spirals

Figure B.1. shows the form of the layout network. It produces a plane framework which can be used as a cantilever beam to transmit the force F, applied at the intersection of the two outer spirals, to a circle, or part of a circle, centred on the origin of the spirals.

Equiangular spirals are curves which make a constant angle with the radius vector, the angle in this case being $\pm 45^\circ$. Thus, in plane polars:

$$\frac{dr}{r\,d\theta} = \tan \pm 45^\circ = \pm 1$$

The equations of the two sets of spirals can therefore be written in the form:

$$r = x^1 e^{\theta} \qquad \text{- (B.1a)}$$

$$r = x^2 e^{-\theta} \qquad \text{- (B.1b)}$$

The constants of integration, $x^1$ and $x^2$, are now used as coordinates, the two sets of spirals being the coordinate curves.

$$r = (x^1 x^2)^{\frac{1}{2}} \qquad \text{- (B.2a)}$$

$$\theta = \tfrac{1}{2} \log (x^1/x^2) \qquad \text{- (B.2b)}$$

$$ds^2 = dr^2 + r\,d\theta^2$$

$$= \left(\frac{\partial r}{\partial x^1} dx^1 + \frac{\partial r}{\partial x^2} dx^2\right)^2 + r^2 \left(\frac{\partial \theta}{\partial x^1} dx^1 + \frac{\partial \theta}{\partial x^2} dx^2\right)^2$$

$$= \frac{1}{2} \frac{x^2}{x^1} (dx^1)^2 + \frac{1}{2} \frac{x^1}{x^2} (dx^2)^2$$

Therefore the fundamental tensor referred to the equiangular spiral coordinate system is:

$$g_{\alpha\beta} = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x^2/x^1 & 0 \\ 0 & \frac{1}{2}x^1/x^2 \end{bmatrix} \qquad - (B.3)$$

The functions A and B defined by (B.3) satisfy equations (4.14) in which B, C, $x^2$ and $x^3$ are replaced by A, B, $x^1$ and $x^2$ respectively. The equiangular spirals therefore provide a suitable layout for an optimum framework. The equilibrium equations (4.25) become, with $b_{\alpha\beta}$ and P zero:

$$\frac{\partial \sigma_{11}}{\partial x^1} + \frac{1}{2x^1}(\sigma_{11} - \sigma_{22}) = 0 \qquad - (B.4a)$$

$$\frac{\partial \sigma_{22}}{\partial x^2} + \frac{1}{2x^2}(\sigma_{22} - \sigma_{11}) = 0 \qquad - (B.4b)$$

Boundary Conditions

The concentrated force F is converted to a distributed force along the edges of the beam by means of two concentrated edge members. Consider the upper one, $x^1 = a$ (figure B.1); it can be conveniently imagined as part of an $x^2, x^3$ surface, where the $x^3$ are parallel straight lines perpendicular to the $x^1, x^2$ plane. The function C, where, $Cdx^3$ is the element of distance along the $x^3$ lines can be taken as simply unity. The $x^2$ curve is a line of curvature of the $x^2, x^3$ surface and the normal curvature of the surface in that direction is $-(1/AB)\partial B/\partial x^1$, by equation (3.14a). The other principal curvature is zero. Equations (4.25) can be applied with B, C, $x^2, x^3$, $\sigma_{22}$, 0, $-\sigma_{11}$ replacing A, B, $x^1$, $x^2$, $\sigma_{11}$, $\sigma_{22}$ and P respectively. The first gives:

$$\frac{\partial \sigma_{22}}{\partial x^2} = 0$$

and therefore, from considerations of equilibrium at the tip of the beam:

$$\sigma_{22} = \frac{F}{\sqrt{2}\, t_{(3)} t_{(1)}}$$

where $t_{(3)}$ is the thickness of the $x^1, x^2$ plane and $t_{(1)}$ is the thickness of the $x^2, x^3$ surface. The second equation is identically satisfied and the third gives:

$$\sigma_{11} = \frac{-F}{\sqrt{2}\, t_{(3)} AB} \frac{\partial B}{\partial x^1} = \frac{-F}{2t_{(3)}(x^1 x^2)^{1/2}} = \frac{-F}{2t_{(3)} r} \qquad (B.5a)$$

Similarly on the lower boundary, $x^2 = a$;

$$\sigma_{22} = \frac{F}{2t_{(3)}(x'x^2)^{1/2}} = \frac{F}{2t_{(3)}r} \qquad \text{- (B.5b)}$$

Solution of Equilibrium Equations

By making the substitutions

$$\sigma_{11}' = r\sigma_{11} = (x'x^2)^{1/2}\sigma_{11}$$

$$\sigma_{22}' = r\sigma_{22} = (x'x^2)^{1/2}\sigma_{22}$$

equations (B.4) become:

$$\frac{\partial \sigma_{11}'}{\partial x'} = \frac{1}{2x'}\sigma_{22}'$$

$$\frac{\partial \sigma_{22}'}{\partial x^2} = \frac{1}{2x^2}\sigma_{11}'$$

and the boundary conditions (B.5) become:

$$\sigma_{11}' = -\frac{F}{2t_{(3)}} \quad \text{on} \quad x' = a$$

$$\sigma_{22}' = \frac{F}{2t_{(3)}} \quad \text{on} \quad x^2 = a$$

By changing the independent variables to x and y, where

$$x = \frac{1}{2}\log\frac{a}{x'}$$

$$y = \frac{1}{2}\log\frac{a}{x^2}$$

the problem can be stated as:

$$\left.\begin{array}{l} \dfrac{\partial \sigma_{11}'}{\partial x} = -\sigma_{22}' \;:\; \dfrac{\partial \sigma_{22}'}{\partial y} = -\sigma_{11}' \\ \sigma_{11}'(0, y) = -\dfrac{F}{2t_{(3)}} \;:\; \sigma_{22}'(x, 0) = \dfrac{F}{2t_{(3)}} \end{array}\right\} \text{- (B.6)}$$

$$\text{i.e. } \frac{\partial^2 \sigma_{11}'}{\partial x \partial y} = \sigma_{11}' \quad : \quad \sigma_{11}'(0, y) = \frac{-F}{2t_{(3)}}$$

$$\left(\frac{\partial \sigma_{11}'}{\partial x}\right)_{(x,0)} = \frac{-F}{2t_{(3)}} \qquad \text{- (B.7)}$$

The above can be solved by the use of the Laplace Transform, defined by:

$$\mathcal{L}\,[u(x,y) : y \to p] = \int_0^\infty u(x,y) e^{-py} dy = \bar{u}(x,y) \qquad \text{- (B.8a)}$$

Thus

$$\left(\frac{\partial u}{\partial x}\right)^{-} = \frac{\partial}{\partial x}(\bar{u}) \qquad \text{- (B.8b)}$$

$$\left(\frac{\partial u}{\partial y}\right)^{-} = p\bar{u} - u(x,0) \qquad \text{- (B.8c)}$$

Using equations (B.8):

$$\bar{\sigma}_{11}' = \left(\frac{\partial^2 \sigma_{11}'}{\partial x \partial y}\right)^{-} = \frac{\partial}{\partial x}\left(\frac{\partial \sigma_{11}'}{\partial y}\right)^{-} = \frac{\partial}{\partial x}\left[p\bar{\sigma}_{11}' - \sigma_{11}(x,0)\right]$$

$$\therefore \quad p\frac{\partial \bar{\sigma}_{11}'}{\partial x} - \bar{\sigma}_{11}' = -\frac{F}{2t_{(3)}} \quad \text{by (B.7)}$$

$$\therefore \quad \bar{\sigma}_{11}' = \frac{F}{2t_{(3)}} + f(p)\, e^{x/p} \qquad \text{- (B.9)}$$

where f is an arbitrary function. But

$$\bar{\sigma}_{11}'(0, p) = \int_0^\infty e^{-py} \sigma_{11}'(0,y)\, dy = -\frac{F}{2t_{(3)}p} \qquad \text{- (B.10)}$$

by (B.7). From (B.9) and (B.10):

$$f(p) = -\frac{F}{2t_{(3)}}\,\frac{(p+1)}{p}$$

Therefore $$\sigma_{11}'(x,y) = \mathcal{L}^{-1}\left[\frac{-F}{2t_{(3)}}\left(e^{x/p}-1\right) - \frac{F}{2t_{(3)}}\frac{1}{p}e^{x/p}\right]$$

From reference 12, page 244 $$\mathcal{L}^{-1}\left[e^{x/p}-1\right] = x^{\frac{1}{2}}y^{-\frac{1}{2}}I_{(1)}(2x^{\frac{1}{2}}y^{\frac{1}{2}})$$

page 245 $$\mathcal{L}^{-1}\left[p^{-\nu-1}e^{x/p}\right] = x^{-\frac{1}{2}\nu}y^{\frac{1}{2}\nu}I_{(\nu)}(2x^{\frac{1}{2}}y^{\frac{1}{2}})$$

and so

$$\sigma_{11}' = \frac{-F}{2t_{(3)}}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}}I_{(1)}(2x^{\frac{1}{2}}y^{\frac{1}{2}}) + I_{(0)}(2x^{\frac{1}{2}}y^{\frac{1}{2}})\right] \quad \text{- (B.11)}$$

where $I_{(\nu)}$ is the modified Bessel function of order $\nu$, defined by:

$$I_{(\nu)}(z) = \sum_{r=0}^{\infty}\frac{(z/2)^{\nu+2r}}{\Gamma(r+1)\Gamma(\nu+r+1)}$$

The following recurrence relations are easily established:

$$z\frac{d(I_{(\nu)}(z))}{dz} = \nu I_{(\nu)}(z) + I_{(\nu+1)}(z) \quad \text{- (B.12a)}$$

$$z\frac{d(I_{(0)}(z))}{dz} = -\nu I_{(\nu)}(z) + z I_{(\nu+1)}(z) \quad \text{- (B.12b)}$$

Using (B.6) and (B.12):

$$\sigma_{22}' = \frac{F}{2t_{(3)}}\left[\left(\frac{y}{x}\right)^{\frac{1}{2}}I_{(1)}(2x^{\frac{1}{2}}y^{\frac{1}{2}}) + I_{(0)}(2x^{\frac{1}{2}}y^{\frac{1}{2}})\right]$$

Thus $\sigma_{11}$ and $\sigma_{22}$ are:

$$\sigma_{11} = \frac{-F}{2t_{(3)}}(x'x^2)^{-\frac{1}{2}}\left[\left(\frac{\log a/x'}{\log a/x^2}\right)^{\frac{1}{2}}.I_{(1)}\left\{(\log a/x'.\log a/x^2)^{\frac{1}{2}}\right\} + I_{(0)}\left\{(\log a/x'.\log a/x^2)^{\frac{1}{2}}\right\}\right] \quad \text{- (B.13a)}$$

$$\sigma_{22} = \frac{F}{2t_{(3)}}(x'x^2)^{-\frac{1}{2}}\left[\left(\frac{\log a/x^2}{\log a/x'}\right)^{\frac{1}{2}}.I_{(1)}\left\{(\log a/x'.\log a/x^2)^{\frac{1}{2}}\right\} + I_{(0)}\left\{(\log a/x'.\log a/x^2)^{\frac{1}{2}}\right\}\right] \quad \text{- (B.13b)}$$

As $x^2 \to a$, $\log a/x^2 \to 0$, and $I_{(1)}\left\{(\log a/x^2.\log a/x^2)^{\frac{1}{2}}\right\} \to 0$

But $$\lim_{x^2 \to a}\left[\left(\frac{\log a/x'}{\log a/x^2}\right)^{\frac{1}{2}} I_{(1)}\left\{(\log a/x' \log a/x^2)^{\frac{1}{2}}\right\}\right]$$

$$= \lim_{x \to 0}\left(\frac{y}{x}\right)^{\frac{1}{2}} I_{(1)}(2x^{\frac{1}{2}} y^{\frac{1}{2}}) = \lim_{x \to 0}\left(\frac{y}{x}\right)^{\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(x^{\frac{1}{2}} y^{\frac{1}{2}})^{1+2r}}{\Gamma(r+1)\,\Gamma(r+2)}$$

$$= y = \frac{1}{2}\log\frac{a}{x'}$$

$$\therefore \quad \sigma_{11}(x', a) = \frac{-F}{2t_{(3)}}(x'a)^{-\frac{1}{2}}\left[\log\frac{a}{x'} + 1\right]$$

$$\sigma_{22}(a, x^2) = \frac{F}{2t_{(3)}}(ax^2)^{-\frac{1}{2}}\left[\log\frac{a}{x^2} + 1\right]$$

The results of the above analysis are shown in figure B.2. The dimensionless parameter $-qT_\alpha/F$ where $T_\alpha$ is $t_{(3)}\sigma_{11}$, is plotted against angular position, $\Theta$. The coordinates $x^1$ and $x^2$ have been denoted $\alpha$ and $\beta$ for clarity in printing. The origin coincides with the origin of the spiral system, and the line $\Theta = 0$ is the axis of symmetry of the beam as shown in figure B.1. The steep lines are lines of constant $x^1$ or $x^2$, and the curves joining the intersections of the first set are curves of constant radius. Of these latter curves, the uppermost (labelled "root circle") denotes the circle for which $\Theta_{(\omega)}$, in figure B.1., has the value $\pi$ i.e. the edge members intersect on this circle. The curves also give the value of $aT_\beta/F$ when $\Theta$ is replaced by $-\Theta$.

Volume of Beam

Equation (4.27) is not very useful in this case, due to the difficulty of evaluating the double integral analytically. However there is another way of calculating the volume of an optimum framework. From equations (2.3) and (2.6):

$$V = \frac{1}{2\varepsilon}\left(\frac{1}{\sigma_{(t)}} + \frac{1}{\sigma_{(c)}}\right)\sum \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{V}_{(p)} - \frac{1}{2}\left(\frac{1}{\sigma_{(t)}} - \frac{1}{\sigma_{(c)}}\right)\sum \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{T}_{(p)} \qquad \text{(B.14)}$$

In order to make use of (B.14) it is necessary to integrate the strain-displacement equations (3.22). Using equations (B.2) and (B.3), the first two of (3.22) become:

$$\frac{1}{r}\frac{\partial u}{\partial \Theta} - \frac{\partial u}{\partial r} - \frac{v}{r} = \sqrt{2}\,\varepsilon \qquad \text{- (B.15a)}$$

$$\frac{1}{r}\frac{\partial v}{\partial \Theta} + \frac{\partial v}{\partial r} + \frac{u}{r} = \sqrt{2}\,\varepsilon \qquad \text{- (B.15b)}$$

The physical displacement vector referred to plane polars, $(u', v')$ is:

$$u' = \frac{1}{\sqrt{2}}(u - v) \quad : \quad v' = \frac{1}{\sqrt{2}}(u + v)$$

Substituting in (B.15) provides the following two equations:

$$\frac{1}{r}\frac{\partial u'}{\partial \theta} + \frac{\partial v'}{\partial r} - \frac{v'}{r} = 2\varepsilon$$

$$\frac{1}{r}\frac{\partial v'}{\partial \theta} + \frac{\partial u'}{\partial r} + \frac{u'}{r} = 0$$

The symmetry of the spiral layout demands a solution independent of $\theta$,

$$\therefore \quad u' = \frac{k_{(2)}}{r} \quad : \quad v' = r\log(k_{(1)} r)^{2\varepsilon}$$

where, for $u' = v' = 0$ at the origin, $k_{(2)} = 0$ (since $\lim_{r \to 0}(r \log r) = 0$). The part of $v'$ proportional to r (i.e. $2\varepsilon r \log k_{(1)}$) describes a pure rotation and may be ignored.

$$\therefore \quad \underset{\sim}{v} = (u', v') = (0, 2\varepsilon r \log r)$$

The tangential forces at the root circle, integrated round the circumference (on which $\underset{\sim}{v}$ is constant) must equal $Fa/r_{(o)}$. The radial forces contribute nothing to $\sum_i \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{v}_{(p)}$

$$\therefore \quad \frac{1}{\varepsilon}\sum_i \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{v}_{(p)} = 2Fa\log a - \frac{2Fa}{r_{(o)}} r_{(o)} \log r_{(o)}$$

$$= 2Fa \log \frac{a}{r_{(o)}} \qquad \text{- (B.16)}$$

The contribution to $\sum_i \underset{\sim}{F}_{(p)} \cdot \underset{\sim}{r}_{(p)}$ from the tip load and the loads at the root applied to the edge members is obviously zero. The contribution from the framework members terminating on the root circle is:

$$-2\varepsilon r_{(o)} \log r_{(o)} \int_{-\theta_{(o)}}^{\theta_{(o)}} t_{(3)}\, \sigma_{rr}\, r_{(o)}\, d\theta = -\varepsilon r_{(o)}^2 t_{(3)} \log r_{(o)} \int_{-\theta_{(o)}}^{\theta_{(o)}} (\sigma_{11} + \sigma_{22})\, d\theta$$

$$= 0$$

since $\sigma_{22}(\theta) = -\sigma_{11}(-\theta)$

(The physical components of stress transform as Cartesian tensors. Thus the radial stress $\sigma_{rr}$ is given in terms of $\sigma_{11}$ and $\sigma_{22}$ by the expression:

$$\sigma_{rr} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad )$$

From the above, $\sum_{\sim} \underset{\sim}{F}_{(\phi)} \cdot \underset{\sim}{r}_{(\phi)} = 0$ and the volume of the beam is, by equations (B.14) and (B.16):

$$V = \left( \frac{1}{\sigma_{(t)}} + \frac{1}{\sigma_{(c)}} \right) \left( F a \log \frac{a}{r_{(\phi)}} \right)$$

## 2. Layout Based on Rhumb Lines on the Surface of a Sphere

This is an example of a global optimum in 3 dimensions, with members confined to one coordinate surface. The layout lines are the two sets of 45° rhumb lines on the surface of a sphere. A compatible deformation exists in which one set of rhumb lines, together with corresponding lines on concentric spheres, is strained by $+\varepsilon$, and the other sets of rhumb lines are strained by $-\varepsilon$. The radii from the centre of the spheres are unstrained. Taking the normal loading to be zero, it is found that the resultant of the surface stresses around a line of latitude is a pure torque about a line joining the poles of the sphere.

Rhumb lines are lines bearing a constant angle to the merideans of a sphere. In this case the angle is $\pm 45°$, and so, in spherical polar coordinates, $r, \theta, \phi$:

$$\frac{r\, d\theta}{r \sin\theta\, d\phi} = \tan \pm 45° = \pm 1$$

$$\therefore \quad \frac{d\phi}{d\theta} = \pm \frac{1}{\sin\theta}$$

The two sets of rhumb lines are therefore given by

$$\phi = \log \left( x^2 \left| \tan \theta/2 \right| \right)$$

$$\phi = -\log \left( x^3 \left| \tan \theta/2 \right| \right)$$

The constants of integration, $x^2$ and $x^3$, are coordinates in the system in which the rhumb lines are coordinate curves. Taking $x^1$ to be the same as r, the equations defining the transformation from the $r, \theta, \phi$ coordinate system to the $x^1, x^2, x^3$ coordinate system are:

$$r = x^1 \qquad \text{- (B.17a)}$$

$$\phi = \frac{1}{2}\log\left(\frac{x^2}{x^3}\right) \qquad \text{- (B.17b)}$$

$$\Theta = 2\tan^{-1}\left[\frac{1}{(x^2 x^3)^{1/2}}\right] \qquad \text{- (B.17c)}$$

Using a similar method to the last example, the fundamental tensor referred to the $x^1, x^2, x^3$ coordinates is found to be:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{2(x^1)^2 x^3}{x^2(1+x^2x^3)^2} & 0 \\ 0 & 0 & \dfrac{2(x^1)^2 x^2}{x^3(1+x^2x^3)^2} \end{bmatrix}$$

i.e.

$$A = 1 \qquad \text{- (B.18a)}$$

$$B = \left(\frac{x^3}{x^2}\right)^{1/2} \frac{\sqrt{2}\,x^1}{1 + x^2x^3} \qquad \text{- (B.18b)}$$

$$C = \left(\frac{x^2}{x^3}\right)^{1/2} \frac{\sqrt{2}\,x^1}{1 + x^2x^3} \qquad \text{- (B.18c)}$$

The above expressions satisfy equations (3.7), the conditions for an orthogonal coordinate system in three dimensional Euclidean space, and equations (4.17), the compatibility conditions for a $0, +\varepsilon, -\varepsilon$ strain system.

The appropriate equilibrium equations are equations (4.25) with P equal to zero and $\sigma_{22}$, $\sigma_{33}$, B, C, $x^2$, $x^3$ replacing $\sigma_{11}$, $\sigma_{22}$, A, B, $x^1$ and $x^2$ respectively. Since $f_{(12)}$ and $f_{(22)}$ are both zero, the condition that a solution to the equilibrium equations exists is, from equations (4.30):

$$\frac{\partial f_{(11)}}{\partial x^3} = \frac{\partial f_{(21)}}{\partial x^2}$$

$s_{(1)}/s_{(2)}$ is unity and therefore the above condition may be written:

$$\frac{\partial}{\partial x^3}\left(\frac{1}{C}\frac{\partial C}{\partial x^2}\right) = \frac{\partial}{\partial x^2}\left(\frac{1}{B}\frac{\partial B}{\partial x^3}\right)$$

which is satisfied by the given expressions for B and C. Equation (4.25c) becomes

$$\sigma_{33} = -\sigma_{22}$$

and substituting for $\sigma_{33}$ in equations (4.25a) and (4.25b) gives:

$$\frac{\partial \sigma_{22}}{\partial x^2} + \frac{2}{C}\frac{\partial C}{\partial x^2}\sigma_{22} = 0$$

$$\frac{\partial \sigma_{22}}{\partial x^3} + \frac{2}{B}\frac{\partial B}{\partial x^3}\sigma_{22} = 0$$

These integrate in the form:

$$\sigma_{22} = \frac{h_{(1)}(x^3)}{C^2} = \frac{h_{(2)}(x^2)}{B^2}$$

where $h_{(1)}$ and $h_{(2)}$ are arbitrary functions of $x^3$ and $x^2$ respectively, and therefore:

$$\sigma_{22} = -\sigma_{33} = \frac{k(x^1)^2 x^2 x^3}{(1+x^2x^3)^2} \qquad \text{- (B.19)}$$

where k is a constant. In spherical polar coordinates (B.19) is:

$$\sigma_{22} = -\sigma_{33} = \frac{k}{r^2 \sin^2\theta} = \frac{k}{R^2} \qquad \text{- (B.20)}$$

where R is the radius of the line of latitude through the current point. The stress in the $\theta$ coordinate direction, $\sigma_{\theta\theta}$, and the stress in the $\phi$ coordinate direction, $\sigma_{\phi\phi}$, are:

$$\sigma_{\theta\theta} = 0$$

$$\sigma_{\phi\phi} = \frac{k}{R^2}$$

The resultant of $\sigma_{\phi\phi}$ around a line of latitude is a pure torque of magnitude T, say, where

$$T = 2\pi k t$$

t being the surface thickness. Thus:

$$\sigma_{22} = -\sigma_{33} = \frac{T}{2\pi t}\,\frac{1}{r^3\sin^2\theta}$$

and, from equation (4.27), the total volume of the framework, V, is given by:

$$V = \left(\frac{1}{\sigma_{(c)}} + \frac{1}{\sigma_{(t)}}\right)\iint \frac{T}{2\pi r^2}\,\frac{BC\,dx^2\,dx^3}{\sin^2\theta} \qquad - (B.21)$$

If the torque is applied around the lines of latitude $\theta = \pi/2 \pm \lambda$, (B.21) becomes:

$$V = \left(\frac{1}{\sigma_{(c)}} + \frac{1}{\sigma_{(t)}}\right)\int_0^{2\pi} d\phi \int_{\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}+\lambda} \frac{T}{2\pi r^2 \sin^2\theta}\, r^2 \sin\theta\, d\theta$$

i.e.

$$V = 2T\left(\frac{1}{\sigma_{(c)}} + \frac{1}{\sigma_{(t)}}\right)\log\tan\left(\frac{\pi}{4} + \frac{\lambda}{2}\right)$$

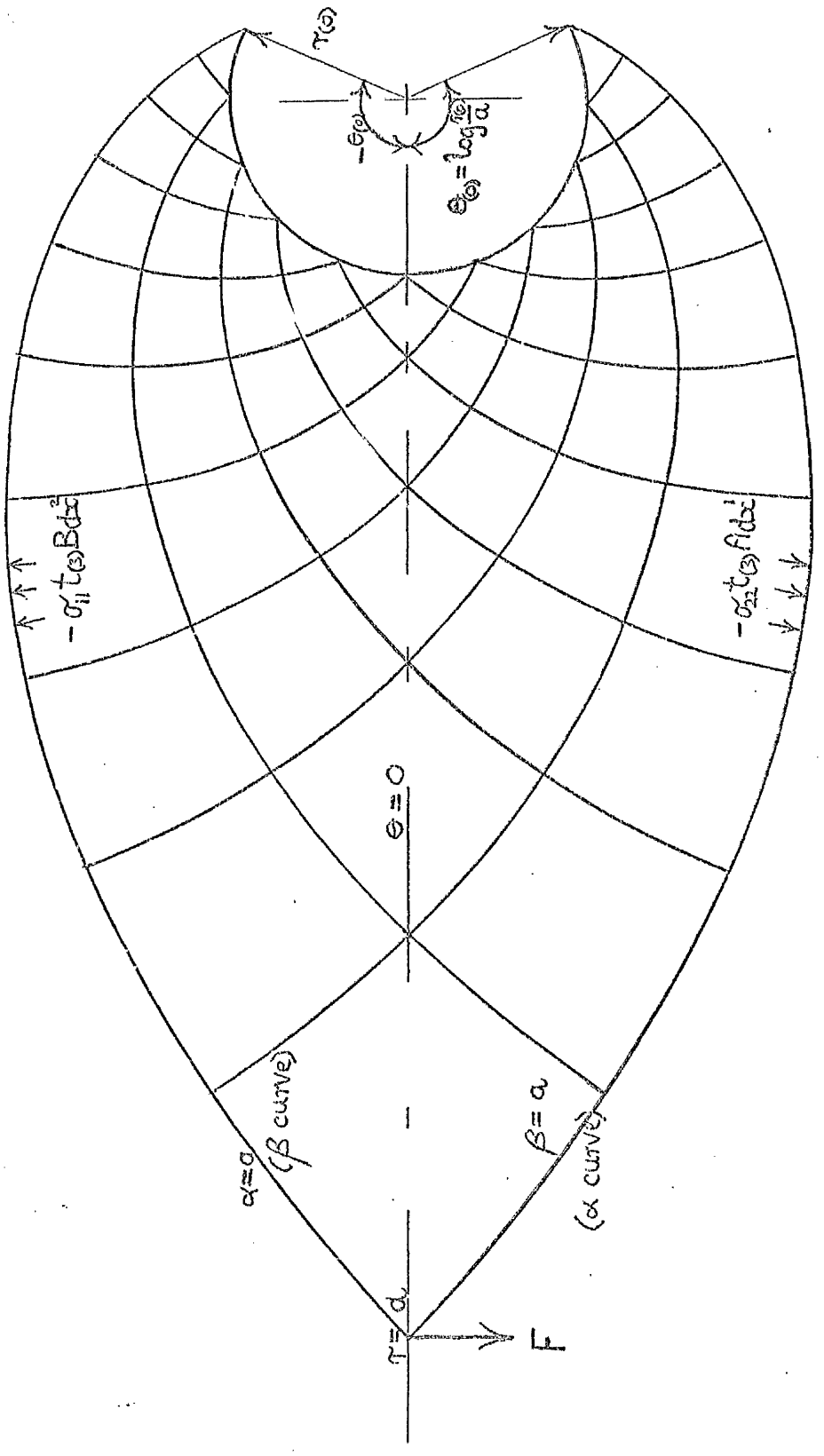



FIGURE B.1

EQUIANGULAR SPIRAL LAYOUT

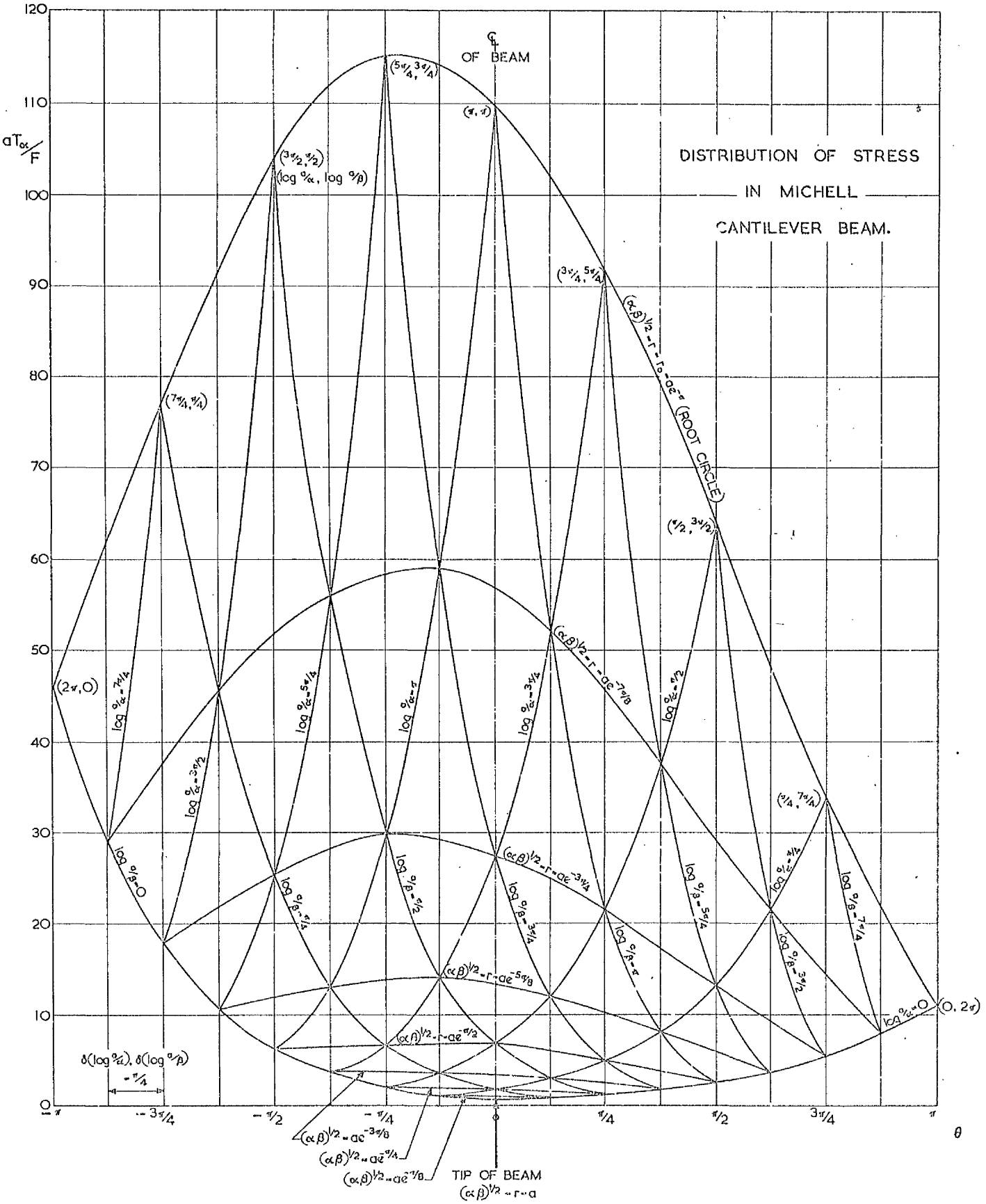

DISTRIBUTION OF STRESS
IN MICHELL
CANTILEVER BEAM.
℄
OF BEAM
$\alpha T_\alpha / F$
θ
TIP OF BEAM
$(\alpha\beta)^{1/2} = r = a$
$(\alpha\beta)^{1/2} = ae^{-3\pi/8}$
$(\alpha\beta)^{1/2} = ae^{-\pi/4}$
$(\alpha\beta)^{1/2} = ae^{-\pi/8}$
$(\alpha\beta)^{1/2} = r = ae^{-\pi/2}$
$(\alpha\beta)^{1/2} = r = ae^{-5\pi/8}$
$(\alpha\beta)^{1/2} = r = ae^{-3\pi/4}$
$(\alpha\beta)^{1/2} = r = ae^{-7\pi/8}$
$(\alpha\beta)^{1/2} = r = r_0 = ae^{-\pi}$ (ROOT CIRCLE)
$\delta(\log \alpha), \delta(\log \beta) = \pi/4$
(2π, 0)
(7π/4, π/4)
(3π/2, π/2)
(log α, log β)
(5π/4, 3π/4)
(π, π)
(3π/4, 5π/4)
(π/2, 3π/2)
(π/4, 7π/4)
(0, 2π)

FIGURE B.2

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