STUDIES IN THE DESIGN AND OPERATION OF CYCLIC ACCELERATORS

T H E S I S

for the degree

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by

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C O N T E N T S

PREFACE

The following thesis is written in the form of a monograph which attempts a unified, and somewhat generalised presentation of cyclic accelerator theory, with the associated design and operational problems. The larger fraction of the authors' original contributions to the subject are contained in the following reports and papers, copies of which are submitted with this monograph:

Papers published -

(A) 'A Note on Resonance Damping, at Injection, in Betatrons and Synchrotrons' Proc.Phys.Soc.,B,64,85(1951).

(B) 'On Resonance Damping at Injection in Betatrons and Synchrotrons'. Proc.Phys.Soc.B,64,579,(1951).

(C) 'Space Charge Forces in Strong-Focussing Synchrotrons' Phys. Rev.,93, 1378-1380, (1954).

(D) 'Regenerative Deflection as a Parametrically Excited Resonance Phenomenon'.Rev.Sci.Instr.,25,587-593,(1954).

(E) 'The use of an Electron Beam for the Accurate Measurement of Alternating Magnetic Field Strengths' Proc. I. E. E., 101, 441-449(Part II), (1954). (Co-author with K. Phillips, M.Sc., who was responsible for the construction of the instrument.)

Papers to be published -

(F) 'Construction and Characteristics of a 340 MeV Electron Synchrotron' Proc. I. E. E. (Co-author with F.R.Perry, J.B.Hansell, E.A.Finlay, and C.Ambasankaran. With E.A.Finlay, and the assistance of C.Ambasankaran, the author was concerned with measurements of the characteristics of the synchrotron magnet)

Papers to be published (continued)

(G) Preliminary Injection studies on the 340 MeV electron synchrotron at the University of Glasgow., Proc.I.E.E., co-author with Dr.W.McFarlane and D.L.Oldroyd. With Dr. McFarlane, and the assistance of D.L.Oldroyd, the author was concerned with initial experiments leading up to operation of the accelerator at the maximum possible excitation.

(H) Operational Characteristics of a 340 MeV electron synchrotron. - to be written by the author and Dr. W. McFarlane.

Unpublished Reports -

(J) 'An approach towards a formal, quantitative theory of Betatron-synchrotron Transition'.

(K) 'A beam extraction scheme for the 340 MeV electron synchrotron at Glasgow'.

(L) 'On First-order Azimuthal Inhomogeneities in the magnets of cyclic accelerators'.

(M) 'On the Injection of electrons into small-aperture, high energy particle accelerators' Parts I and II.

(N) Synchrotron Memorandom-12: 'On Injection into azimuthally Inhomogeneous guide fields'.

In this monograph, the essential results of these papers and reports have been inserted where relevant. In addition, the following sections of the monograph are original-

Chapter Two - The general theory of separable transverse motion in section 2.2.1; general formulae for calculation of momentum compaction coefficients in section 2.2.1; general focussing theory for 'smooth' parametric variations in section 2.2.3.2; theory of the second stability region in sections 2.2.3.1 and 2.2.3.2.

Chapter Three - Treatment of forced resonances and forced perturbations in 2.2 and 3.3 respectively. In the former it is shewn that the upper half of the harmonic range cannot give rise to forced resonances, and that the highest harmonic in this range gives rise to a change in orbit radius, in the way a change of particle momentum in a constant magnetic field does. Possible uses of this effect in assisting injection is discussed in Papers L and M.

Chapter Four - The whole of this chapter contains original comment from the author, and original experimental work performed in conjunction with Dr. W. McFarlane, as already stated.

Space has not been found in this monograph for the description of the experimental techniques and equipment for use with the synchrotron with which the author has been associated, and most of which will be described in Papers F, G, and H dealing, in detail, with all aspects of the synchrotron project.

I N T R O D U C T I O N

The Historical Development of Cyclic Accelerators -

In 1930 the first cyclic accelerator, known as the cyclotron, was successfully developed[1]. It was not until 1944, however, that a radial focussing mechanism accompanying its accelerating mechanism was recognised[2]. Use of this so-called 'phase-focussing' made possible the stable acceleration of particles whose angular velocity, $w = eBo/E = v/r$, in a constant magnetic field, B, varied with its momentum, $p = (1/c)(E^2-E_o^2)^{0.5}$, E and E_o representing the total energy and rest energy respectively of particles circulating on an orbit of radius, r, with linear velocity, v. Before the last war the rate of increase of particle energy was kept at an artificially high level to increase to a maximum the kinetic energy of particles of relativistically increasing mass using a fixed-frequency accelerating field. The maximum energy gained is proportional to the square root of the peak accelerating voltage, and this fact limited proton energies to about 20 MeV, deuteron energies to about 40 MeV, and so on. For the much lighter electrons much more energy could be obtained by straightforward application of electrostatic potential fields, than by acceleration in the cyclotron.

In 1940, however, Kerst successfully demonstrated the principle of Induction acceleration[3] in an electron accelerator called the Betatron[4]. It was unique in its time in two aspects: first, it accelerated particles on orbits of nearly constant radius in an increasing guide field that need therefore only be established over an annular region, and, second, the accelerating mechanism had a stability condition dependent only on particle momentum, p, not on v, and was therefore non-resonant in character. The first feature differed markedly from conditions obtaining in cyclotrons, where an increase in radius, r, is necessary for particles to gain energy, and the nearly constant guide field, B, has to be established over a circular region. This difference necessitated a serious investigation of guide field focussing mechanisms[5] to provide axial focussing without losing the radial stability inherent in induction acceleration. From the point of view of the development of very high energy cyclic accelerators it was this basic analysis of the stability condition for 'betatron' oscillations, rather than the successful application of the principle of induction acceleration that was important.

Subsequently, in 1944, as mentioned above, the 'resonant' accelerating mechanism of the cyclotron was extended by incorporating in it the radial focussing mechanism present if either the accelerating field frequency, $f_a = Cw/2\pi$, $C=1,2,3,\ldots$, and/or the guide field, B, is changed slowly with time to accomodate the change in w with energy, E. This generalisation of the 'resonant' accelerating mechanism provided an automatic tracking of the machine parameters, B and f_a, by the particle parameters v, and E (or p), and not the tracking of the particle parameters by the machine parameters as was originally envisaged. This radially stable 'resonant' accelerating mechanism, with betatron axial focussing was then used in a large number of high energy cyclic accelerators[6], those above about 1000 MeV using the annular, time-dependent 'synchrotron' guide field rather than the circular, time-independent 'cyclotron' guide field for economic reasons.

Despite this development, however, cyclic accelerators for energies above about 1000 MeV were becoming a formidable economic problem. In the summer of 1952 the solution to these economic problems was found with the generalisation of the principles of 'betatron' guide field focussing[7], providing much stronger radial and axial focussing of particle trajectories. The consequences of this development were a series of design studies of very small radial and axial aperture, large radius synchrotrons capable of accelerating protons and heavier particles to energies in the range $10^4 - 10^5$ MeV, the problems associated with which are still under consideration[8]. Electrons may not economically be accelerated to energies above about 5000 MeV in cyclic accelerators because of the large energy losses due to the near optical radiation emitted by them while being radially accelerated in order that their trajectories may be closed and approximately circular[9]. Several electron synchrotrons in the range 1 - 5 BeV are being designed[10], while electrons have been accelerated to between 500 - 1000 MeV in existing machines.[11]

Generalisation of design principles -

During the 25 years of cyclic accelerator development, progress has been sufficient for a generalisation of design principles to be attempted here. Throughout any period of cyclic acceleration the following general relation must be satisfied between particle parameters, total energy, E, and linear velocity, v, and the accelerator parameters, guide field, B, and orbit radius, r :

$$(E.v) \propto (B.r) \propto p = (E^2 - E_0^2)^{0.5}/c. \qquad (1)$$

Acceleration is obtained by increasing the momentum, p:

$$(dp/dt) \propto d(Br)/dt = r.\partial\langle B\rangle/\partial t + dr/dt(\langle B\rangle + r\partial\langle B\rangle/\partial r), \qquad (2)$$

of individual particles. Quantitative information for design calculations is generally obtained from the fundamental equations:

$$dp/dt = d(m.v)/dt = e.(E + B \wedge v/c), \qquad (3)$$

describing the motion of a <u>single</u> particle of charge e in an electromagnetic field - the domain of the vectors E and B in vacuo, satisfying the vector equations :

$$\nabla . B = \nabla . E = 0, \qquad E = E^{s} + E^{\beta} \qquad (4a)$$

$$\nabla \wedge E^{\beta} = -(1/c)(\partial B/\partial t), \qquad (4b)$$

$$\nabla \wedge B = (1/c)(\partial E/\partial t). \qquad (4c)$$

Motion of a charged particle in an electromagnetic field - the domain of the vector potential A, and scalar potential U, may also be described by either of the equations :

$$dp/dt = (d/dt)(\partial L_o/\partial v) + (d/dt)(\partial L_1/\partial v) = (\partial L_1/\partial q), \qquad (5a)$$

or

$$dp/dt = -\partial H/\partial q \quad \text{and} \quad dq/dt = \partial H/\partial p, \quad \text{where:} \qquad (5b)$$

$L_o = -m_o c^2(1 - (v/c)^2)^{1/2}$ is the free particle Lagrangian,

$L_1 = e.(A . v/c - U)$ is the interaction Lagrangian,

$H_o(p) = c(m_o^2 c^2 + p^2)^{1/2}$ is the Hamiltonian of a free particle of momentum $p = m_o v(1 - (v/c)^2)^{-1/2}$, and

$H = H_o(p - eA/c) + eU = p.v - L_o - L_1$ is the Hamiltonian of the particle in the electromagnetic field.

The Lagrangian and Hamiltonian formalisms (5) are equivalent to (3) through the following relations between the two descriptions of the electromagnetic field:

$$E^{\beta} = -\nabla U^{\beta} - \frac{1}{c}\partial A/\partial t, \qquad E^{s} = -\nabla U^{s}, \qquad (6)$$

$$B = \nabla \wedge A, \qquad U = U^{s} + U^{\beta}. \qquad (7)$$

The basic analytic problems presented are :

(a) Determination of the characteristics of the electromagnetic field necessary to give radial and axial stability to particle motion in approximately circular orbits, while providing the tangential forces required to increase particle momenta.

(b) Design of the electromagnetic field sources to provide the field characteristics necessary for stable particle acceleration, as determined in (a). With iron-cored electromagnets this problem simplifies to the determination of the pole profile necessary to provide these characteristics. It is this aspect of the problem that we shall be solely concerned about in this monograph.

The first class of problems are by far the most fundamental, their solutions establishing the essential contributions required from all necessary components of the accelerator installation, among them the magnet producing the guide field. We emphasize here the problems associated with the design of this unit merely because it is the most expensive single item in such installations.

These problems are most conveniently posed in the coordinates, r,z,θ, of the circular cylinder, and the following assumptions are made in order to clarify the exposition of generalised principles of cyclic acceleration:

(i) Absence of particle interactions - forces due to interaction between charged particles in a beam are taken to be negligible. This is implicit in the use of single particle equations of motion, and in the absence of charge and current density terms in Maxwell's equations (4). This assumption is not strictly valid, except for extreme relativistic particles for which $v = c$; such forces, called space charge forces because of the approximations usually adopted in estimating their effect on particle motion, are discussed later in greater detail.

(ii) Circular symmetry of the Electromagnetic field - $B_\theta = 0$, for all r, z, giving $A_r = A_z = 0$. If particles gain energy solely by tangential acceleration, then from (i)-

$$E_r = E_z = 0 \; ; \quad E_\theta = -\partial U^s/\partial \theta - (1/c)\partial A_\theta/\partial t. \tag{8}$$

Such symmetry obtains only in exactly circular magnets, in particular, in the absence of straight sections. Zero order conditions for exactly circular, and therefore closed particle trajectories are readily obtained by describing the motion in rotating coordinates, as shewn below. Substantial modifications prove to be necessary, however, in order to obtain the effect of assymetries, either deliberate (the introduction of straight sections) or unavoidable (alignment errors), on particle motion. This aspect of the problem is dealt with later in greater detail; such practical modifications are not essential to the exposition of fundamental principles attempted in this chapter.

(iii) Absence of radiation damping - This is a sufficiently valid assumption provided the energy loss during a single revolution is much smaller than the maximum available accelerating voltage, and the total radiation loss during acceleration is negligible compared with the final energy of the particle. Details of such effects will also be dealt with later.

With these assumptions, equations (3) may be written-

$$(d/dt)(m\dot{r}) - mr\dot{\theta}^2 = -er\dot{\theta}.B_z/c, \tag{9a}$$

$$(d/dt)(m\dot{z}) = er\dot{\theta}.B_r/c, \tag{9b}$$

$$(d/dt)(mr^2\dot{\theta} - e\phi/2\pi) = -e(\partial U^s/\partial \theta), \tag{9c}$$

where-

$$(1/2\pi)(d\phi/dt) = (d/dt).\int_0^r (r.B_z).dr = -d(A_\theta r)/dt, \tag{10}$$

Also,

$$r.E_\theta^s = -\frac{1}{c}\int_0^r r(\partial B_z/\partial t)dr = -r\partial A_\theta/c.\partial t, \tag{11}$$

$$r.B_r = -\int_0^r r(\partial B_z/\partial z)dr = -r\partial A_\theta/\partial z, \tag{12}$$

$$(\partial B_r/\partial z) - (\partial B_z/\partial r) = (-1/c)(dE_\theta/dt), \tag{13}$$

from (4b), (4a) and (4c) respectively.

The Lagrangian, L, and Hamiltonian, H, are now:

$$L = L_0 + L_1 = -m_0 c^2 \left(1 - (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)/c^2 \right)^{1/2} + erA_\theta \dot{\theta}/c - eU^s, \quad (14)$$

$$H = c\left(m_0^2 c^2 + p_r^2 + p_z^2 + \left(\frac{p_\theta}{r} - \frac{eA_\theta}{c}\right)^2 \right)^{1/2} + eU^s, \quad (15)$$

where –
$$p_r = \partial L/\partial \dot{r} = m\dot{r}, \quad (16a)$$
$$p_z = \partial L/\partial \dot{z} = m\dot{z}, \quad (16b)$$
$$p_\theta = \partial L/\partial \dot{\theta} = mr^2\dot{\theta} + erA_\theta/c, \quad (16c)$$

are the generalised momenta canonically conjugate to the position coordinates, r, z, θ respectively, in terms of which the equations of motion, (9), may be written:

$$\dot{p}_r = mr\dot{\theta}^2 - er\dot{\theta}.B_z/c = (e^2/2m)(\partial W/\partial r), \quad (17)a$$

$$\dot{p}_z = er\dot{\theta}.B_r/c = (e^2/2m)(\partial W/\partial z), \quad (17)b$$

$$\dot{p}_\theta = -e(\partial U^s/\partial \theta), \quad (17)c$$

from which, assuming acceleration in the tangential direction only, we have:

$$c^2(dm/dt) = er\dot{\theta}.E_\theta = (e^2/2m)(\partial W/\partial t), \quad (18)$$

where the potential function, W, is defined:

$$W = \left\{ (A_\theta/c) + \left(\int \frac{\partial U}{\partial \theta}.dt + C\right)/r \right\}^2, \quad (19)$$

where C is an integration constant.

We now have the basic equations of particle motion in a form readily adaptable for the application of the theory of small oscillations, in which framework the theory of that multiply periodic system – the cyclic accelerator – is best described. Consideration of the equilibrium conditions associated with these equations (zero-order approximation to the description of particle motion), suffices to enable one distinguish between the various types of accelerator practicable to this order of approximation. These zero-order equilibrium conditions are:

$$(\partial W/\partial r) = (\partial W/\partial z) = 0, \quad (20)$$

which provide a description of particle motion confined to the plane where $B_r = 0$, and to the circular orbit of radius, $r_e = (v/\mathring{\theta}) = (Ev/ecB_z)$, $E = mc^2$, (21)

in that plane. The potential U is provided by a r.f. field, of fundamental frequency $f = w_1/2\pi$ and peak voltage V, periodic in θ with period 2π, extending over a gap(or gaps of negligible angular extension, so that :

$$(\partial U/\partial \theta) = (V/2\pi) \sum_{k=-\infty}^{\infty} \sin(\int_0^t w_1 . dt - k\overline{\theta - \theta_e}), \qquad (22)$$

In the zero order approximation we consider the case where the angular velocity, $\mathring{\theta}$, is exactly equal to the highest submultiple of the angular radiofrequency, w_1. For k=1 therefore we have $(\partial U/\partial \theta) = (V/2\pi)\sin \theta_e$, and we have:

$$c^2(dm/dt) = ec(d/dt)(B_z/w_1) = ec(d/dt)(B_z r_e/v)$$

$$= (ew_1/2\pi)(V.\sin(\theta_e) + 2\pi r_e E^{\beta}_{\theta}). \qquad (23)$$

The accelerating voltage required per particle revolution may therefore be written:

$$(2\pi/w_1)(d/dt)(B_z/w_1) = (2\pi/w_1)(d/dt)(B_z r_e/v)$$

$$= (V.\sin(\theta_e) + 2\pi r_e E^{\beta}_{\theta})/c \qquad (24)$$

Two distinct classes of accelerating mechanism may be recognised - one in which θ is a cyclic coordinate, so that p_θ is a time-independent constant, and V = 0, and the other in which $V \neq 0$, so that p_θ is time-dependent, and θ no longer cyclic. The first method is known as

<u>Induction acceleration</u>:

Since $p_\theta = -(e/c)r_e^2(\partial A_\theta/\partial r)_{r=r_e}$, it is clear that zero order equilibrium conditions demand $p_\theta = 0$, giving r_e as the time-independent stable orbit provided:

$$(\partial \phi/\partial t) = 2\pi r_e^2(\partial B_z/\partial t) \qquad (25)$$

Since θ is cyclic, acceleration is independent of particle velocity, v and the method is essentially non-resonant in character.

<u>Synchronous Acceleration</u>:

With such mechanisms, E^{β}_{θ} is generally negligible compared to the radiofrequency voltage, V, and acceleration to the order of approximation considered here depends, as mentioned above, on exact equality between w_1 and $k\mathring{\theta}$, making it resonant in character. Even then the zero-order condition (24) is exact for particles with the correct initial conditions only if the rate of gain of energy is vanishingly small, i.e. Θ_e is almost zero. In order to examine the stability or otherwise of particle motion under a wider range of initial conditions and with more rapid acceleration we require a first order analysis, given later (Chapter I), with the result:

$$(d/dt)(B_z/w_1) = (d/dt)(B_z r_e/v) \ll w_1 V/2\pi c, \qquad (26a)$$

or,

$$\Theta_e \ll 1. \qquad (26b)$$

These relations state that particles may be accelerated, but slowly, only if B_z and/or w_1 are time-dependent. The distinction between implicit and explicit time-dependence is not yet made.

There is one unique condition which enables 'resonant' acceleration to take place under cyclic conditions without the synchronous stability conditions (24) and (26) being satisfied, i.e. when -

$$(d/dt)(B_z) = (d/dt)(w_1) = 0; \quad B_z \propto w_1 = \dot{\Theta}, \qquad (27)$$

a condition which arises during the acceleration of particles of infinite rest mass in the original 'conventional' or 'fixed-frequency cyclotron. The limiting condition - $B_z \propto \dot{\Theta}$ is most closely approached by protons and heavier nuclei with kinetic energies below 1 - 10 MeV, but energies about twice this may be obtained by sufficiently rapid acceleration[12] in existing conventional cyclotrons. As will be pointed out in Chapter I, this mechanism lacks the synchronous radial stability possessed by all other mechanisms satisfying (26). As a consequence of (27) heavy particles in conventional cyclotrons travelled round in orbits of increasing radii varying between very wide limits during the period t_o of acceleration. Cyclic accelerators allowing such variations will generally be referred to as

CYCLOTRONS when $dr_e/dt \gg (r_{inj}/t_o)$, (28)

where r_{inj} is the orbit radius when particles are injected into the cyclic accelerator. Accelerators which allow orbit variations between very narrow limits are

SYNCHROTRONS when $dr_e/dt \ll (r_{inj}/t_o)$. (29)

Synchronous accelerators may be of both types.

CYCLOTRONS - B_z is time-independent for economic reasons while w_1 (or $\dot{\theta}$) is made to decrease with time in order that the particles may be accelerated.

The Frequency-Modulated or Heavy particle Cyclotron[13] uses a slow continuous decrease of w_1 for the acceleration of heavy particles of finite mass whose total energy and velocity increase with time. This accelerator is thus the natural successor to the fixed frequency cyclotron for proton energies above 20 MeV.

The Electron Cyclotron or Microtron[14] allows $\dot{\theta}$ to decrease discontinuously by becoming increasing sub-multiples of w_1, which, with B_z, is kept constant.

SYNCHROTRONS - B_z must now be explicitly or implicitly time-dependent, and w_1 may be time-independent only if v is constant (as in the case of very light particles when v is almost c) when r_e is also almost constant, or if r_e is allowed to increase somewhat while still satisfying (29). In all synchrotrons constructed to date B_z is explicitly time-dependent, and r_e is kept almost constant during acceleration. They fall into the following groups:

Fixed-frequency or light particle synchrotrons[15] - Electrons are injected at velocities approaching that of light ($v/c = 0.98$ when total electron energy is about 2 MeV), using as injector:

(a) the induction accelerator - betatron starting,[16]

(b) the travelling wave linear accelerator or the electrostatic generator.[17]

(c) another synchronous accelerator, the microtron.[18]

Of these, the first tried was (a), which is the method used in the 340 MeV electron synchrotron at Glasgow. It is not, however, the most efficient method, although it is the most convenient. The more efficient method (b) is rapidly gaining preference, while (c) has not yet been tried.

Frequency-Modulated or Heavy particle synchrotrons-[19] In this case both the energy and velocity of the particles increase during acceleration so that w_1 has to be increased with time to keep r_e constant, and B_z has to increase still faster in order to provide stable synchronous conditions, cf (24) and (26). These conditions are generally applied to the acceleration of protons up to energies of 30,000 MeV at present, although the principle has been used in an electron synchrotron[20] not using one of the pre-accelerating mechanisms mentioned above to avoid problems associated with frequency modulation.

The construction of a synchrotron is much less expensive than the construction of a corresponding cyclotron for kinetic energies above about 500 MeV due to the finite flux density (about 20,000 gauss) at which most types of magnetic material saturate. This forces the maximum orbit radius for such particles to be much the same in both types of accelerator, when the annular character of the synchrotron guide field reduces quite considerably the energy stored in the magnetic field, and the mass of iron required to provide a low permeability path for it. Such considerations have, in fact, encouraged the latest developments in focussing theory, which, by providing much stronger focussing, have reduced by an order of magnitude at least the radial and axial dimensions of the annular field containing the particle orbits. The conditions for stable motion may, in one form, be written:

$$0 \leq |\langle \partial B_z/\partial r \rangle| << |\partial B_z/\partial r| \doteq m^2 B_z/r, \; m >> 1. \qquad (30)$$

The 'weak' or conventional focussing used in earlier accelerators discussed above was provided by:

$$0 < -\langle \partial B_z/\partial r \rangle = \partial B_z/\partial r < B_z/r. \qquad (31)$$

These relations, and their derivation, are examined in more detail in Chapter 2. However, for the purposes of the zero-order classification of cyclic accelerators presented here, the difference between (30) and (31) is very significant. Earlier, due to the axial focussing restriction, $\partial B_z/\partial r < 0$, B_z could not be made to increase significantly with time, by virtue of deliberate changes in r_e making B_z implicitly time dependent at the position of the accelerated particles. With the new focussing conditions (30) we may, however, have $\partial B_z/\partial t = 0$, and still keep orbits in a narrow annular region by altering w_1 and therefore forcing r_e to move into increasingly larger guide fields, the magnitude of this radial motion depending inversely on the magnitude of $\langle \partial B_z/\partial r \rangle$. Frequency modulation then controls the rate of increase of particle energy in much the same way as in the frequency-modulated cyclotron. This new synchronous accelerator is called the '<u>fixed-field synchrotron</u>',[21] because its guide field is annular, but not explicitly time-dependent, and owes its existence to the developments in guide field focussing theory discussed later.

Development of the Thesis -

Outlined above is the basic zero order analysis of particle motion in cyclic accelerators, and the types of cyclic accelerator it covers. It is the aim of this work to collect, and collate theoretical and experimental studies concerned with the design and operation of such machines, with the main purpose of serving as a guide to the design of very high energy cyclic accelerators. In presenting such a survey existing studies have been presented in a wider framework in order to bring out the underlying unity in the problems posed and the solutions offered. The limit of future developments for many years to come appear sufficiently clear for such a unified approach to be profitable. The emphasis here will therefore be more on general principles than on their application to detailed design studies, which fall outside the scope of this monograph.

In the design of large cyclic accelerators, preliminary theoretical and experimental studies of the more novel features of the equipment are of great importance. The more detailed such studies the greater the assurance that may be placed in the design; almost certainly it reduces considerably the time taken in the final stages of development concerned with, first the successful operation, and eventually the efficient operation of the accelerator. Where such careful preliminary investigation has been undertaken, the product is invariably a sound engineering proposition operating efficiently both on the basis of output beam intensity and on the basis of the ratio of time spent on operation and maintenance. The objection to such an approach is the development time factor, and several projects have adopted a 'cut and try' policy from the start. Certainly the design and construction of cyclic accelerators cannot be expected to await theoretical developments ensuring the certain success of the machine, and there are many accelerators operating successfully , and efficiently, under conditions that cannot be wholly explained by any theory or were not forseen at the time the machine was designed. However, with the huge financial investment necessary today in very large accelerators, a compromise is necessary, which usually takes the form of preliminary experimental investigations of acceleration conditions on fractional scale models.

CHAPTER ONE

ANALYTIC METHODS

C H A P T E R   O N E

ANALYTIC METHODS

1. INTRODUCTION:

The analytic basis of the two problems to be discussed here, namely-

(a) Determination of the characteristics of particle trajectories in various magnetic field configurations, and

(b) Design of the electromagnet to possess the field configuration necessary for stable and efficient acceleration,

have been outlined in the introductory discussion. In providing the zero-order analysis, the following basic assumptions were made:

Assumption (i) Absence of particle interactions. Further consequences of this assumption will be discussed quantitatively in 2.4 of this chapter, while accepting its validity for the present.

Assumption (ii) Absence of circular assymmetry. This we dispense with in what follows.

Assumption (iii) Absence of radiation damping. We shall discuss this matter further in 2.4 also, while accepting it for the present.

In this section we extend this analysis to the consideration of the following first order deviations from the equilibrium conditions described in the zero-order analysis:

$$r_e \quad \text{to} \quad r_e(1 + x_r), \tag{32a}$$

$$z_e \quad \text{to} \quad z_e + x_z r_e, x_z = \delta z / r_e, \tag{32b}$$

$$\dot{\theta} \quad \text{to} \quad w_e(1 + x_w), \quad w_e = w_1 \tag{32c}$$

p_e to $p_e(1 + x_p)$, (32d)
E_e to $E_e(1 + x_E)$, (32e)
v_e to $v_e(1 + x_v)$, (32f)
t_e to $t_e(1 + x_t)$, (32g)
m_e to $m_e(1 + x_m)$, (32h)

where p, E, v, t, m represent respectively the momentum, total energy, velocity, period of a single revolution, and mass of the accelerated particles. The following relations between the first order quantities, x_i, are valid for all types of cyclic accelerator -

$$x_E = (p_e^2c^2/E_e^2)x_p = ((E_e^2-E_o^2)/E_e^2)x_p = (v_e/c)^2x_p, \quad (33a)$$

$$x_v = (E_o^2/E_e^2)x_p = (E_o^2/p_e^2c^2)x_E = \frac{(1-(v/c)^2)}{(v/c)^2}.x_E \quad (33b)$$

$$x_w = k.x_t, \text{ and } x_m = x_E. \quad (33c)$$

If a circular, annular magnetic field of average radius, r_e, is separated into N_1 parts by straight sections, each of length, L, then, to the first order :

$$x_t = (1 + N_1L/2\pi r_e)^{-1}(\langle x_r\rangle - (x_v). \quad (34)$$

The very important relation between x_r and x_p emerges after the type of field in which the particle is to be accelerated is chosen. The spatial dependence of the radial and axial components of this field may, to the first order in x_r and x_z, be written :

$$B_z = \langle B_z\rangle_{r=r_e}(1 + f_o^z(\theta) - n_{zr}(\theta)x_r - n_{zz}(\theta)x_z), \quad (35a)$$

$$B_r = \langle B_z\rangle_{r=r_e}(f_o^r(\theta) - n_{rz}(\theta)x_z - n_{rr}(\theta)x_r), \quad (35b)$$

where -

$$(\langle B_z\rangle/r_e).n_{zr} = -\langle\partial B_z/\partial r\rangle_{r=r_e}(1 + f_1^z(\theta)), \quad (36a)$$

$$(\langle B_z\rangle/r_e).n_{zz} = -\langle\partial B_z/\partial z\rangle_{z=z_e}(1 + f_2^z(\theta)), \quad (36b)$$

$$(\langle B_z\rangle/r_e).n_{rz} = -\langle\partial B_r/\partial z\rangle_{z=z_e}(1 + f_1^r(\theta)),$$

$$(\langle B_z\rangle/r_e).n_{rr} = -\langle\partial B_r/\partial r\rangle_{r=r_e}(1 + f_2^r(\theta)), \quad (36c)$$

and

$$\langle F\rangle = (1/2\pi).\int_0^{2\pi} F(\theta).d\theta. \quad (36d)$$

The time-dependence of the first-order variables, x_i defined in (32), is obtained by substituting them in (9), which may then be written:

$$(d/dt)(\langle B_z\rangle \dot{x}_r/w_e) = \langle B_z\rangle w_e(-f_o^z + n_{zr}x_r + n_{zz}x_z + x_E + x_w), \quad (37a)$$

$$(d/dt)(\langle B_z\rangle \dot{x}_z/w_e) = \langle B_z\rangle w_e(f_o^r - n_{rz}x_z - n_{rr}x_r), \quad (37b)$$

$$(d/dt)\left\{ \begin{array}{l} (r_e^2/w_e)(d/dt)(\langle B_z\rangle \dot{x}_r/w_e) \quad + \\ \quad r_e^2\langle B_z\rangle(f_o + (1-n_{zr})x_r - n_{zz}x_z) \quad + \\ \qquad (r_e^2/c)(\langle B_z\rangle - (\emptyset/2\pi r_e^2)) \end{array} \right\} = (V/2\pi)\sin \theta', \quad (37c)$$

where $\dot{\theta}' = w_e . x_w$. Using (24), equation (37c) may be simplified to -

$$(d/dt)\left\{\frac{r_e^2}{w_e}.\frac{d}{dt}\left(\frac{\langle B_z\rangle \dot{x}_r}{w_e}\right) + r_e^2\langle B_z\rangle(f_o + \overline{1-n_{zr}}.x_r - n_{zz}x_z)\right\} = (V/2\pi)(\sin \theta' - \sin \theta_e), \quad (38)$$

where $\theta_e = 0$, and r_e time-independent when the betatron condition (25) is satisfied.

It is clear from (37) that the possibility of first-order coupling between radial and axial motion of particles during their acceleration exists, in general. We shall shortly consider, however, two special types of field configuration in which such first-order coupling is considerably reduced, if not completely removed. For the present, mention must be made of an important assumption implicit in the derivation of (37) and (38), namely:

<u>Assumption (iv)</u> Validity of linearisation of the equations, i.e, absence of any significant effects from second order deviations, x_i^2, x_ix_k, which make the equations non-linear. The consequences of this assumption will be investigated in 2.4 of this chapter.

One final assumption has now to be made to simplify the analysis of (37) and (38). It is one of the most soundly based, and is made in the form :

Assumption (v) Validity of the Adiabatic Theorem, under circumstances in which the fractional change in energy of a particle during a single revolution due to external accelerating torques, or forces, is extremely small. This leads, in the limiting case of no net acceleration, when the average orbit radius and the average guide field along it are time-independent, to Conservation of Linear and Angular Momentum, and the first-order relation:

$$\varkappa_m + \varkappa_w + \varkappa_r = \text{constant} = \varkappa_p. \tag{39}$$

From (37) it is clear that radial motion, in the absence of any net acceleration, or significant radial-axial coupling, is due to the guide field, $B_z(r,\theta)$, and the r.f. voltage V, i.e forces which, in practice, produce guide field focussing and synchronous acceleration respectively. To determine quantitatively the conditions under which these two causes of radial perturbation may be considered separately, we substitute (39) into (38), ignoring axial coupling terms. We have then:

$$\frac{d^4\theta'}{dt^4} + w_r^2 . \frac{d^2\theta'}{dt^2} + \frac{Vw_e^3 . \theta'}{2\pi\langle B_z\rangle r_e^2} = (Vw_e^2\theta_e / \langle B_z\rangle . 2\pi r_e^2) = 0, \tag{40}$$

assuming $|\theta' - \theta_e| << 1$, and $\langle B_z\rangle$ and w_e to be time-independent, and defining:

$$w_r = w_e(1 - n_{zr}(\theta))^{1/2}. \tag{41}$$

Substituting $\theta' \propto \exp(-iw_o t)$ into this linearised, constant coefficient differential equation, we have for w_o:

$$w_o^2 = w_r^2\left(\frac{1}{2} \mp \frac{1}{2}\left(1 - \frac{4V(1-n_{zr})^{-2}}{2\pi r_e^2\langle B_z\rangle w_e} \right)^{1/2} \right). \tag{42}$$

$$= w_r^2\left(\frac{1}{2} \mp \frac{1}{2}(1 - A)^{1/2}\right), \quad A = 2V(1-n_{zr})^{-2}/\pi r_e^2\langle B_z\rangle w_e.$$

Radial motion is therefore oscillatory if $A < 1$, and may be separable into two distinct types, which are uncoupled to a high degree of approximation if $A << 1$, which then satisfy the general equations:

$$(d/dt)(\langle B_z\rangle \dot{x}_r/w_e) + w_e\langle B_z\rangle(f_o^z + \overline{1-n}_{zr}\cdot x_r - n_{zz}x_z) = D, \quad (43a)$$

$$\langle B_z\rangle w_e\cdot(x_E + x_W + x_r) = D = \langle B_z\rangle w_e\cdot x_p, \quad (43b)$$

from (39), where D is time-independent, and

$$(d/dt)(r_e^2\langle B_z\rangle)(f_o^z + \overline{1-n}_{zr}\cdot x_r - n_{zz}x_z) = (V/2\pi)(\sin\theta' - \sin\theta_e), \quad (44a)$$

$$(n_{zr}x_r + n_{zz}x_z + x_E + x_W - f_o^z) = 0. \quad (44b)$$

The first pair of equations (43) describe the so-called 'Betatron oscillations', which accompany guide field focussing independently of synchronous acceleration, and even in its absence ($V=0$). The second pair of equations, (44), describe the much lower frequency radial 'synchrotron' oscillations accompanying the phase focussing mechanism in synchronous acceleration, and are obtained from (37a) and (38) by neglecting the radial acceleration term $-(d/dt)(\langle B_z\rangle \dot{x}_r/w_e)$- corresponding to the neglect of the term $-d^4\theta'/dt^4$ in (40). These two pairs of equations (43) and (44) may be rewritten:

The Betatron Radial Equation -

$$(\ddot{x}_r/w_e^2) + (1 - n_{zr})x_r - n_{zz}x_z = x_p - f_o^z . \quad (45)$$

The Synchrotron Equation -

$$(d/dt)(r_e^2\langle B_z\rangle)(x_r + x_E + x_W) = (V/2\pi)(\sin\theta' - \sin\theta_e) \quad (46)$$

We emphasize here the generality of these two equations describing the radial motion of particles in cyclic accelerators. Their main restriction lies in the neglect of second order deviations from equilibrium conditions. The final equation (37b) describing axial motion may be given here for completeness.

The Axial Equation -

$$(\ddot{x}_z/w_e^2) + n_{zr}x_z + n_{rr}x_r = f_o^r , \quad (47)$$

where we have: $n_{rz}(\theta) = n_{zr}(\theta)$, (48)

as a consequence of Assumption (v) and equation (13).

The analysis of equations (45),(46), and (47) for various combinations of $n_{ik}(\theta)$ and $f_o^{r,z}(\theta)$ constitute the basis of the mathematical theory of cyclic accelerators. Analytical methods adapted for this purpose are described in the following section 2, together with further consequences of assumptions (i),(iii),(iv) and (v) made in deriving the 3 basic equations above. In section 3 the analytical methods used in the solution of the lesser problems associated with the design of the guide field magnet are discussed.

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2. PARTICLE TRAJECTORIES AND THEIR STABILITY CHARACTERISTICS:

2.1 Statement of the problem - two dimensional approximations and asymptotic solutions:

Equations (45) - (47) may, by virtue of assumption (v) be divided into two groups -

Group I - The Betatron radial (45) and axial (47) equations fall naturally into one group, which define,-in

Case A - for intervals of the order of one revolution period, the radial and axial motion of particles of different initial momenta, which are coupled in general, independently of the mechanism of acceleration, or of the rate of acceleration, and in Case B - for intervals of the order of very many revolution periods, the asymptotic behaviour of such radial and axial motion, which depends now on the rate of acceleration, and also on the mechanism of acceleration. These 2 equations in group I are clearly dependent only, in case A on the spatial dependence, and in case B on the temporal dependence of the guide field as well. Radial and axial motion may thus, in case A, be treated independently of its slow azimuthal (tangential) acceleration, and the problem of motion in 3 dimensions resolves into one of motion in 2 dimensions.

The analysis of these two equations contain the whole first-order theory of the Betatron, subject to the assumptions (i),(iii),(iv) and (v).

Group II - This contains the remaining Synchrotron equation (46), which describes the independent (to a high degree of approximation) radial motion accompanying synchronous acceleration. Before this equation may be solved, however, the very important relation between $\langle x_r \rangle$ and x_p must be determined from the Group I equations. This relation, which is known as the 'momentum compaction equation', will depend on the spatial, and not the temporal dependence of the guide field. Solutions of this equation may also be obtained under the two special conditions mentioned above, namely, <u>case A</u> - for intervals of the order of one synchrotron oscillation period, when the radial motion is independent of the <u>average</u> rate of gain in energy, which is proportional to the time-dependence of the guide field, and in <u>case B</u> - for intervals of the order of very many synchrotron oscillation periods, the radial motion being again described by asymptotic solutions which depend on the average rate of gain in energy.

By virtue of its dependence on the momentum compaction relation, the behaviour of particles undergoing synchronous acceleration depends very largely on the spatial properties of the guide field, and the mechanism of guide field focussing. It is interesting also to note that while we have linearised the equations in Group I, we have not completely linearised the equation in Group II, which contains a term proportional to $\sin(\theta')$, where θ' will eventually be the dependent variable.

Assuming a momentum compaction relation of the form :

$$\varkappa_r = \alpha_m \cdot \varkappa_p , \tag{49}$$

and re-defining the dependent variables $\varkappa_r$, $\varkappa_z$ as follows -

$$\varkappa_r = \varkappa_1 , \ \mathring{\varkappa}_r = w_e \varkappa_2 , \ \varkappa_z = \varkappa_3 , \ \mathring{\varkappa}_z = w_e \varkappa_4 , \tag{50}$$

the equations of the two groups defined above may be written -

I. THE BETATRON EQUATION

$$\left[f(D) \right] \left[\varkappa \right] = \left[\eta \right] , \tag{51}$$

where,

$$\left[f(D) \right] = \begin{bmatrix} (d/dt)/w_e & -1 & 0 & 0 \\ (1-n_{zr}) & (d/dt)/w_e & -n_{zz} & 0 \\ 0 & 0 & (d/dt)/w_e & -1 \\ n_{rr} & 0 & n_{zr} & (d/dt)/w_e \end{bmatrix} , \tag{52}$$

$$\left[\varkappa \right] = \begin{bmatrix} \varkappa_1 \\ \varkappa_2 \\ \varkappa_3 \\ \varkappa_4 \end{bmatrix} , \quad \text{and} \quad \left[\eta \right] = \begin{bmatrix} 0 \\ \varkappa_p - f_o^z \\ 0 \\ f_o^r \end{bmatrix} ; \tag{53}$$

this matrix notation is particularly convenient for the analysis described in the following section.

II. THE SYNCHROTRON EQUATION

$$(d/dt)\left(\frac{r_e^2 \langle B_z \rangle (d\theta'/dt)}{k w_e \left(\alpha_m' - \frac{E_o^2}{E_e^2} \right)} \right) = (V/2\pi)\left(\sin(\theta') - \sin(\theta_e) \right) , \tag{54}$$

where -

$$\alpha_m' = \alpha_m \left(1 + \frac{K_1 L}{2\pi r_e} \right)^{-1} , \tag{54'}$$

which may be derived from (46), using (39),(34) and (33).

It must be noted that conservation of linear momentum applies only to the betatron motion, and conservation of angular motion to synchrotron motion as a result of Assumption (v). Clearly conservation of angular momentum is an intrinsic property of induction acceleration *only in zero-order approximation* .

2.2 Linear Transformations in Phase Space:

In this section we shall consider solutions of the vector-matrix equation -

$$[dx/d\theta] = [A][x] + [\eta] , \tag{55}$$

which is identical to (51), with $D = (1/w_e)(d/dt)$, (56)

and

$$[ID - A][x] = [\eta] , \tag{57}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(1-n_{zr}) & 0 & n_{zz} & 0 \\ 0 & 0 & 0 & 1 \\ -n_{rr} & 0 & -n_{zr} & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{58}$$

If $A(\theta)$ is continuous in any interval, then there exists a unique solution of the homogeneous equation:

$$[dx/d\theta] = [A][x] , \tag{59}$$

which may be expressed in terms of the initial conditions-

$$[x] = [x(0)] , \quad \text{at} \quad \theta = 0, \tag{60}$$

in the form:

$$[x(\theta)] = [Y][x(0)]. \tag{61}$$

The matrix $[Y]$ may be regarded as an operator giving rise to a linear transformation in phase space converting initial values $[x(0)]$ into the final values determined by $[A]$. The transformation matrix, $[Y]$, is related to $[A]$ by the equations:

$$d[Y]/d\theta = [A(\theta)][Y] , \quad [Y(0)] = [I], \tag{62}$$

and its determinant is given by the equation:

$$|Y| = \exp\left(\int_0^\theta \mathrm{tr}(A).d\theta \right), = 1, \tag{63}$$

from which it is clear that $|Y|$ is independent of θ, and non-singular. The solution to the inhomogeneous equation may therefore be expressed in the form:

$$[x(\theta)] = [Y][x(0)] + [Y] \int_0^\theta [Y]^{-1}[\eta]\, d\theta. \tag{64}$$

The linear transformation operator, $[Y]$, has now to be constructed from a knowledge of the characteristics of $[A]$. Now an important inherent characteristic of $[A]$ in the theory of idealised cyclic accelerators is its periodicity of 2π, representing physically a complete revolution. In general, however, for an arbitrary $[A(\theta)]$ of any periodicity the equation (62) cannot be solved explicitly. Special cases, however, where $[A(\theta)]$ is constant for all θ, $0<\theta<2\pi$, or is piecewise constant over limited angular regions, $\delta\theta$, $\theta < \delta\theta < \theta + 2\pi/n$, fundamental period $4\pi/n$, may be given explicit solutions, as will be shewn later. Approximate solutions may be found for the more general cases, however, from an analysis of the properties of solutions when $[A]$ is nearly constant, but otherwise arbitrary, and when $[A]$ is an arbitrary periodic function of θ.

<u>$[A(\theta)]$ nearly constant, but otherwise arbitrary:</u>

It may be shewn that there exists a matrix T capable of changing $[A]$ by a collineatory transformation:

$$[A'] = [T]^{-1}[A][T] = \begin{bmatrix} \lambda_1 & b_{12} & b_{13} & b_{14} \\ 0 & \lambda_2 & b_{23} & b_{24} \\ 0 & 0 & \lambda_3 & b_{34} \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}, \tag{65}$$

to a triangle matrix, where the λ are the 4 roots of the determinantal equation -

$$|A - \lambda I| = 0, \tag{66}$$

called the characteristic equation of $[A]$, which are not necessarily distinct. Then the change of variable

$$[x] = [T][y] \tag{67}$$

converts (59) into the equation -

$$d[y]/d\theta = [A'][y] \quad , \tag{68}$$

and it is clear that in general each y_i is a linear combination of exponentials which have polynomials in θ as coefficients. The necessary and sufficient condition that all solutions of (59) tend to zero at $\theta = \infty$ is that all real parts of the characteristic values, λ_i, be negative. When the four λ_i are distinct, $[A']$ is a diagonal matrix, and the transformation matrix, $[Y]$ is given by the equation:

$$[Y] = [T]\begin{bmatrix} e^{\lambda_1\theta} & 0 & 0 & 0 \\ 0 & e^{\lambda_2\theta} & 0 & 0 \\ 0 & 0 & e^{\lambda_3\theta} & 0 \\ 0 & 0 & 0 & e^{\lambda_4\theta} \end{bmatrix}[T]^{-1} = \exp([A]\,\theta). \qquad (69)$$

$$= \sum_{i=1}^{4} (e^{\lambda_i\theta}) \frac{\prod_{j\neq i}(\lambda_j[I] - [A])}{\prod_{j\neq i}(\lambda_j - \lambda_i)}, \qquad (70)$$

using Sylvester's theorem.

These results apply to the cases in which $[A]$ is a constant matrix. However, if $[A]$ is nearly constant, and expressed in the form -

$$[A(\theta)] = [A'] + [B(\theta)] \qquad (71)$$

with $[B(\theta)] = 0$, $\theta = \infty$, and $\int^{\infty} \|dB/d\theta\|\, d\theta < \infty$, and the characteristic roots of $[A]$ are distinct, a matrix $[T(\theta)]$ converts $[dx/d\theta] = [A(\theta)][x]$ to

$$[dx/d\theta] = [T(\theta)]^{-1}[A(\theta)][T(\theta)]x - [T(\theta)]^{-1}\left[\frac{dT(\theta)}{d\theta}\right]x,$$

where $\int^{\infty}[T]^{-1}[dT/d\theta]\,d\theta < \infty$, $\int^{\infty} d\lambda_i(\theta)/d\theta\, d\theta < \infty$, and $\lambda_i(\theta)$ tends to λ_i as θ tends to infinity.

<u>$[A(\theta)]$ an arbitrary periodic function of period α:</u>

The solution of the matrix equation:

$$d\,[Y(\theta)]/d\theta = [A(\theta)][Y(\theta)], \quad [Y(0)] = [I], \qquad (62)$$

may be shewn to have the form:

$$[Y(\theta)] = [Q(\theta)][e^{B\theta}], \quad [Q(\theta+\alpha)] = [Q(\theta)] \qquad (72)$$

and $[B]$ is a constant, and in general complex matrix, if $[A(\theta+\alpha)] = [A(\theta)]$ is continuous for all θ. As a general consequence, we have- $[Y(\theta+\alpha)] = [Y(\theta)]\exp([B]\alpha)$, (73)

which is the general form of a theorem first formulated by Floquet, the solutions of the determinantal equation:

$$|Y(\alpha) - \beta I| = 0 \qquad (74)$$

providing the characteristic values of the transformation matrix $[Y(\alpha)]$. In the general case we can not find an explicit form for this matrix, but we may determine the form of general relations affecting the boundedness of solutions to (55) from the nature of its characteristic values β, which, from (74) may be expressed in terms of its elements. Since the $[Y(\alpha)]$ are unimodular (cf (63)) density in phase space is preserved (Liouvilles Theorem), and (74) has the general form :

$$\beta^4 - A\beta^3 + B\beta^2 - A\beta + 1 = 0, \tag{74}$$

where A and B are the real combinations of the elements of $[Y(\alpha)]$:

$$\begin{aligned} A &= \mathrm{Tr}\,[Y(\alpha)] = Y_{11}(\alpha)+Y_{22}(\alpha)+Y_{33}(\alpha)+Y_{44}(\alpha) \\ B &= 2 + (Y_{11}(\alpha)+Y_{22}(\alpha))(Y_{33}(\alpha)+Y_{44}(\alpha)). \end{aligned} \tag{75}$$

The 4 solutions of (74) may be put in the form -

$$\beta_1 = e^{i\nu'},\ \beta_2 = e^{-i\nu'},\ \beta_3 = e^{i\nu''},\ \beta_4 = e^{-i\nu''}, \tag{76}$$

where,

$$2\cos(\nu) = (A/2) \pm ((A^2/4) + 2 - B)^{1/2}, \tag{77}$$

and the solutions to (55) are bounded if:

$$-1 < \cos(\nu') < 1, \quad -1 < \cos(\nu'') < 1 . \tag{78}$$

The general problem of coupled radial-axial motion may thus be solved by considering the effect of the strength and character of the coupling on the inequalities:

$$\begin{gathered} -2 \leq (A/2) \pm ((A^2/4) + 2 - B)^{1/2} \leq 2, \\ (A/2)^2 + 2 - B \geq 0. \end{gathered} \tag{79}$$

The satisfaction of these inequalities is a prerequisite for bounded solutions and hence stable radial and axial motion during cyclic acceleration. The boundaries between stable and unstable zones are thus defined by lines on which ν' and ν'' are multiples of π.

The general rules to be satisfied for stable <u>coupled</u> motion have been given above , and suffice if such motion is described by the homogeneous equation (59). In general, however, the inhomogeneous equation must be used to describe particle motion, even in a 'perfect' magnet owing to the presence of the terms x_p in η. From (64) then, due to the finiteness of the particular integral, a stable closed orbit, i.e. $x(\theta) = x(\theta + 2\pi)$, is possible only if

$$|Y(2\pi) - I| \neq 0. \tag{80}$$

Approximation in practical applications:

As already mentioned the general theory outlined above may be used only if the transformation matrix, $[Y]$ can be derived explicitly from $[A]$. Two approximations are currently used for this purpose:

(I) In general the transformation matrix $[Y(\theta)]$ over the range $0 < \theta < 2\pi$ may be subdivided so as to be represented by the product of an arbitrary number of transformation matrices, $[Y_n(\delta\theta)]$, $\theta_n < \theta < \theta_n + \delta\theta$, over which range the matrix $[A]$ might be adequately represented by a constant coefficient matrix. This method is particularly suited to the design of strong focussing synchrotron in which $n_{ik}(\theta)$ is piecewise constant over limited angular ranges, when the only approximation involved in the use of this method of fabricating $[Y(2\pi)]$ is the assumption that $n_{ik}(\theta)$ changes discontinuously at the ends of these unit cells. This method will widely used in the discussion of focussing and accelerating mechanisms in Chapter Two.

(II) Reduction of the 4×4 transformation to a pair of 2×2 submatrices by partitioning, which is intimately connected with the problem of reducing to zero the coupling between radial and axial betatron oscillatory motion. From (58) it is clear that an obvious method of achieving this reduction is to have $n_{zz} = n_{rr} = 0$, i.e. making the guide field symmetrical about an axial plane, commonly known as the median plane. This is, however, not the only way of reducing the transformation matrix as will be shown in Chapter Two.

Associated with this reduction is the reduction of a quaternary quartic (or biquadratic) form to a pair of binary quadratic forms, all of which remain invariant throughout the cyclic motion of a given particle if the damping effects of acceleration are neglected. The existence of such an invariant multilinear form associated with the group of orthogonal, unimodular matrices to which $[Y(\theta)]$ belongs is a consequence of general Invariant theory, which will not concern us here. In the reduced form, however, the pair of invariant binary forms associated with the reduced 2×2 transformation matrices are of direct physical interest, defining respectively a tubular surface in x_1, x_2, θ radial phase space and x_3, x_4, θ axial phase space. The cross-section of these tubular surfaces at any azimuthal position is elliptical in shape, the equation defining its boundary being of the general form:

$$F x_1^2 + 2H x_1 x_2 + G x_2^2 = L^2 = \text{constant}, \quad (81)$$

where
$$FG - H^2 = \text{constant} \quad (82)$$

is an invariant associated with it. Explicit evaluation of these coefficients in terms of the elements of $[Y]$ is possible if we write this transformation matrix in the form:

$$[Y] = [S(\theta)]^{-1} [R(\nu)] [S(\theta)] \quad (83)$$

where $[R(\nu)]$ is an orthogonal, unimodular matrix representing rotation through an angle $\nu = \cos^{-1}(\mathrm{Tr}[Y]/2)$ which is independent of θ, and $[S(\theta)]$ is also an orthogonal matrix which describes the deformation of the ellipse in phase space as a function of θ. General differential equations describing this deformation as a function of θ for arbitrary periodic functions, $n_{ik}(\theta)$ can also be obtained in terms of the coefficients, F, G, and H, by differentiating (81) with respect to θ, when we have:

$$\partial F/\partial\theta = 2H(1 - n_{zr}), \text{ or } \partial F/\partial\theta = 2Hn_{zr}, \tag{84a}$$
$$\partial G/\partial\theta = -2H, \qquad \text{or } \partial G/\partial\theta = -2H, \tag{84b}$$
$$\partial H/\partial\theta = (1 - n_{zr})G - F, \text{or } \partial H/\partial\theta = n_{zr}G - F, \tag{84c}$$

in the cases of radial and axial motion respectively. It is easily shewn that both the coefficients, F, G, H, and their derivatives satisfy the invariant (82).

The differential equations (84) may also be obtained by using for $[S(\theta)]$ a matrix providing an infinitesimal shift in a unit cell providing a transformation in phase space represented by $[Y]$.

The application of the analytical techniques described in this section will be described in Chapter Two. The credit for the application of such methods to describe the characteristics of cyclic acceleration has been separately ascribed to Snyder[22], Caianiello[23] and Le Couteur[24].

2.3 Series expansions in Analytic functions:

The equations of radial and axial motion in the uncoupled approximation are of the form:

$$d^2\zeta/d\theta^2 + (a - 2q.R(\theta)).\zeta = \eta(\theta), \tag{85}$$

where a and q are real constants, and $R(\theta)$ is an arbitrary function, with a periodicity not exceeding 2π. These equations belong to a general class known as HILL'S equation, and are solved by expressing $R(\theta)$ as a fourier series, with the nature of the solution, bounded or unbounded, depending on the relationship existing between the two parameters, a and q. As a consequence of this approach, the analysis of the equation when $R(\theta)$ has a finite number of discontinuities is more conveniently, and accurately, performed using the matrix operator approach of the previous section, although by this method the phase trajectory is specified only at the discontinuities, its behaviour at other points having to be determined by the use of the characteristics of the transformation matrix associated with them. Under these circumstances, knowledge

of the behaviour of the invariant ellipse is particularly valuable, as it enables the azimuth at which the particle trajectories may deviate most from its average value to be easily determined, whatever the initial conditions.
In the absence of any discontinuities in $R(\theta)$, the method using a Fourier expansion is more convenient, although in principle, as has been shown by Minorski[25], an arbitrary, though finite, number of discontinuities may be used to approximate to a continuously varying function.

Two special cases of HILL'S equation (85) are MEISSNER'S equation[26] where R alternates between two constant values, which is more conveniently analysed using the techniques outlined in 22, and MATHIEU'S equation[27], when $R(\theta) = \cos(2\theta)$.

It has been shown that when $R(\theta)$ is given, the (a,q)-plane for the homogeneous part of (85), i.e. $\eta = 0$, may be divided into 'stable' and 'unstable' regions, the pair of solutions to the homogeneous equation being unstable (unbounded) if (a,q) is situated in an 'unstable' region, and stable (bounded) if (a,q) is situated in a 'stable' region. If (a,q) is situated on the boundary between these regions <u>one</u> solution has a period of π or 2π, provided $R(\theta)$ has a period π, the other independent solution being non-periodic and unbounded. The major problem in analysis lies in the determination of the equations governing these boundaries in a-q space for given $R(\theta)$. Although it is known, from the generalised statement of Floquet's theorem in 2.2 that every solution of the homogeneous part of (85) has the form:

$$\zeta = e^{\nu_1\theta} Q_1(\theta) + e^{\nu_2\theta} Q_2(\theta), \qquad (86)$$

where $Q_1(\theta)$ and $Q_2(\theta)$ are periodic of period π, there exists no simple method for obtaining the constants ν_1, ν_2 explicitly, given a, q, and $R(\theta)$. The following simple criterion for stability of the solutions, due to Liapounoff, is quite general, and therefore particularly valuable.

Some General Stability Rules -

THEOREM I : (Liapounoff) $R(\theta)$ continuous and periodic, period π.

All solutions of

$$d^2\zeta/d\theta^2 + (a - 2q.R(\theta)).\zeta = 0 \tag{87}$$

are bounded as $|\theta|$ tends to infinity provided

(a) $(a/q) \geq (2/\pi).\int_0^\pi R(\theta).d\theta$, (88a)

(b) $(|a|(\pi/2) - (2/\pi)) \leq \int_0^\pi |q.R(\theta)| d\theta$. (88b)

Another important general stability theorem may be stated in the form -

THEOREM II : $R(\theta)$ continuous, not necessarily periodic.

All solutions of (87) are bounded if, for all θ,

$$a - 2q.R(\theta) \geq m^2 > 0, \tag{89}$$

where m is any real constant.

An alternative, and less strict condition is

$$a - 2q.R(\theta) \geq 0, \quad \int^\infty (a - 2qR(\theta)).d\theta = \infty. \tag{90}$$

Another important related stability rule is

THEOREM III : $R(\theta)$ as in Theorem II.

If all solutions of

$$(d/d\theta)(k_1(\theta).\frac{du}{d\theta}) + \phi_1(\theta).u = 0 \tag{91}$$

are oscillatory as θ tends to infinity, and if

$$\phi_2 \geq \phi_1, \quad k_2 \geq k_1 > 0, \tag{92}$$

then all solutions of

$$(d/d\theta)(k_2(\theta).\frac{du}{d\theta}) + \phi_2(\theta).u = 0 \tag{93}$$

are also oscillatory.

An important general effect of the periodic $R(\theta)$ on the stability of solutions of (87) is clear from the first two theorems; a stable solution may exist if $(a - 2qR(\theta))$ is negative only if $R(\theta)$ is periodic. A particular consequence of this is that if $q=0$, solutions of (87) are unbounded if $a < 0$, while if $q \neq 0$, a periodic $R(\theta)$ may, in the same circumstances, make these solutions bounded, while a non-periodic $R(\theta)$ may not do so. This is very important in the theory of focussing and accelerating mechanisms described in Chapter Two.

The general stability theorem I given above provides very general, and hence unnecessarily restricted conditions for stability of the solutions of (87). However, explicit conditions for stability of these solutions may be obtained by defining the boundaries of the stable and unstable regions in a-q space. Using Floquet's theorem these boundaries in a-q space are given by the equations:

$$\cos(2\pi\nu) = \left(f_1(2\pi) + \frac{df_2}{d\theta}(2\pi) \right) = \pm 1, \tag{94}$$

where $f_1(\theta)$ and $f_2(\theta)$ are independent solutions of (87), satisfying the initial conditions -

$$f_1(0) = (df_2/d\theta)_{\theta=0} = 1, \tag{95a}$$

$$f_2(0) = (df_1/d\theta)_{\theta=0} = 0. \tag{95b}$$

The disadvantage is this method is that independent solutions of (87) must be known before such boundaries may be defined. The stability regions are consequently known in the greatest detail in the cases of Mathieu's and Meissner's equations respectively for which solutions are comparatively easily obtained.

Analytic solutions for HILL'S EQUATION -

A- The classical method, introduced by Hill, involves the solution of an infinite determinant, after the expansion of $R(\theta)$ in the Fourier cosine series:

$$q.R(\theta) = \sum_{r=1}^{\infty} \lambda_{2r}.\cos(2r\theta), \quad \sum_{r=1}^{\infty} \lambda_{2r} \text{ converging,} \tag{96}$$

which makes the assumption that $R(\theta)$ is symmetrical about $\theta=0$, i.e. even in θ. The solutions are obtained in the form-

$$z = e^{\nu\theta}.\sum_{r=-\infty}^{\infty} c_{2r}.e^{2ri\theta}, \tag{97}$$

ν being real, imaginary or complex, with the coefficients c_{2r}, λ_{2r} satisfying the infinite number of recurrence relations:

$$c_{2r} + (\lambda_2/\xi_{2r})(c_{2r-2} + c_{2r+2}) + (\lambda_4/\xi_{2r})(c_{2r-4} + c_{2r+4}) + \dots = 0, \quad (98)$$

for $r = \dots,-1,0,1,\dots,$ with $\xi_{2r} = a - (2r-i\nu)^2$.
For (97) to be a solution of (87), (98) must be satisfied simultaneously, requiring the infinite determinant

$$\Delta(i\nu) = \begin{vmatrix} \dots & \cdot & \cdot & \cdot & \dots \\ \dots & 1 & (\lambda_2/\xi_{-2}) & (\lambda_4/\xi_{-2}) & \dots \\ \dots(\lambda_2/\xi_0) & 1 & & (\lambda_2/\xi_0) & \dots \\ \dots(\lambda_4/\xi_2) & (\lambda_2/\xi_2) & & 1 & \dots \\ \dots & \cdot & \cdot & \cdot & \dots \end{vmatrix} = 0, \quad (99)$$

provided $\xi_{2r} \neq 0$.

It can be shewn that

$$\sin^2(i\nu\pi/2) = \Delta(0).\sin^2(\pi a^{1/2}/2), \quad (100)$$

so that ν is determined if $\Delta(0)$ can be evaluated. This determinantal method of finding ν from a and λ_{2r} is convenient only if all the $\lambda_{2r} << a$, when $\Delta(0)$ becomes:

$$\Delta(0) \doteq 1 + (\pi/4a^{1/2})\cot(\pi a^{1/2}/2).\left(\frac{\lambda_2^2}{1 - a} + \frac{\lambda_4^2}{4 - a} + \frac{\lambda_6^2}{9 - a} + \right). \quad (101)$$

Mathieu's equation provides the special case where $\lambda_4 = \lambda_6 = \lambda_8 = \dots = 0$, $\lambda_2 = q \neq 0$, but the method given above is just as difficult to use unless $q << a$.

B - The recurrence relations (98) may, however, be expressed in a transcendental equation for a in the form of an infinite continued fraction, following Ince[28], who first derived this for the special case of Mathieu's equation only. The transcendental equation in the general case for HILL'S[29] equation has not been derived because of its complexity, and is given here because of its usefulness in certain specific applications in later chapters.

Defining $v_{2r-2} = (c_{2r}/c_{2r-2})$, the recurrence relations, (98) may be expressed in the form:

$$v_{2r-2}\left\{ \xi_{2r} + \lambda_2 v_{2r}\left(1+ \frac{\lambda_4 v_{2r+2}}{\lambda_2} + \frac{\lambda_6 v_{2r+2} v_{2r+4}}{\lambda_2} + \dots\right)\right\} = X, \quad (102)$$

where $X = \lambda_2 (1 + \frac{\lambda_4/\lambda_2}{v_{2r-4}} + \frac{\lambda_6/\lambda_2}{v_{2r-4}} \cdot \frac{1}{v_{2r-6}} + \ldots).$ (103)

An alternative transcendental equation for v_{2r-2} may be derived, however, by replacing r by $(r-1)$ in the recurrence relations (98), and dividing throughout by $\lambda_2 c_{2r-2}$, when we obtain:

$$-v_{2r-2} = \frac{(\xi_{2r-2}/\lambda_2) + (1/v_{2r-4}) + \frac{\lambda_4/\lambda_2}{v_{2r-4}v_{2r-6}} + \frac{\lambda_6/\lambda_2}{v_{2r-4}v_{2r-6}v_{2r-8}} + \ldots}{1 + (\lambda_4 v_{2r}/\lambda_2) + (\lambda_6 v_{2r} v_{2r+2}/\lambda_2) + \ldots}, \quad (104)$$

with

$$-(1/v_{2r-4}) = \frac{1 + (\lambda_4 v_{2r-2}/\lambda_2) + (\lambda_6 v_{2r-2} v_{2r}/\lambda_2) + \ldots}{(\xi_{2r-4}/\lambda_2) + (1/v_{2r-6}) + (\lambda_4/\lambda_2 v_{2r-6} v_{2r-8}) + \ldots}, \quad (105)$$

which has to be substituted in (104). For $\lambda_{2r} << \lambda_2$, $r \geq 2$, an equation between a and λ_{2r} may be obtained by equating the expressions for v_{2r-2} derived from (102),(103) and (104),(105).

In the special case of Mathieu's equation the resulting transcendental equation, for $r=1$ takes the comparatively simple form:

$$\frac{-q/(2-i\nu)^2}{1 - a/(2-i\nu)^2 -} \; \frac{q^2/(2-i\nu)^2(4-i\nu)^2}{1 - a/(4-i\nu)^2 -} \; \frac{q^2/(4-i\nu)^2(6-i\nu)^2}{1 - a/(6-i\nu)^2 - \ldots} =$$

$$= \frac{a-\nu^2}{q} + \frac{q/(2+i\nu)^2}{1 - a/(2-i\nu)^2 -} \; \frac{q^2/(2-i\nu)^2(4-i\nu)^2}{1 - a/(4-i\nu)^2 - \ldots} \quad (106)$$

When ν is real the solutions (97) are unstable, and (106) is a relation between complex quantities, and equality between real and imaginary parts must be separately established, when ν has to be determined from a knowledge of a and q. When ν is imaginary, say $-i\nu=\beta$, β real, the solutions (97) are bounded and oscillatory, and may be rewritten:

$$\zeta = \sum_{r=-\infty}^{\infty} c_{2r} \cdot \cos(2r+\beta)\theta \equiv ce_{2n+\beta}(\theta, q), \quad (107)$$

which represents a Mathieu function of real fractional order, $2n+\beta$, $0 < \beta < 1$. The independent and coexistant solution of the same order is written:

$$se_{2n+\beta}(\theta,q) = \sum_{r=-\infty}^{\infty} a_{2r}.\sin(2r+\beta)\theta. \qquad (108)$$

Mathieu functions of integral order, $ce_{2n}(\theta,q)$ and $se_{2n+1}(\theta,q)$ are obtained in the limit as β tends to zero and unity respectively, the second solutions as given by (108) and (107) respectively vanishing in such limiting cases, being replaced by the unbounded functions $fe_{2n}(\theta,q)$ and $ge_{2n+1}(\theta,q)$. The pair of independent, coexistant solutions, ce_{2n}, fe_{2n} [29] satisfy Mathieu's equation when the parameters a, q satisfy (106) with $-i\nu = \beta = 2n$, $n=0,1,2,\ldots$, when the stable solution ce_{2n} has a period π. The pair of independent coexistant functions, se_{2n+1}, ge_{2n+1} satisfy Mathieu's equation simultaneously when the parameters a, q satisfy (106) with $-i\nu = \beta = 2n+1$, $n=0,1,2,\ldots$, when the stable half of the general solution, se_{2n+1} has the period 2π. These transcendental relations between a and q for ce_{2n}, se_{2n+1}, together with those for ce_{2n+1}, se_{2n} divide a-q space into regions in which the solutions of Mathieu's equation are bounded or unbounded, when β is respectively real or imaginary. For $q > 0$ β is real between the a-q curves corresponding to the functions ce_{2n}, se_{2n+1}, $n=0,1,2\ldots$; and se_{2n}, $n=1,2,\ldots$, ce_{2n+1}, $n=0,1,2\ldots$, i.e between the characteristic curves corresponding to solutions of period π and 2π respectively. For $q < 0$, β is real between the characteristic curves corresponding to ce_{2n}, ce_{2n+1}, $n \geq 0$; and se_{2n}, se_{2n+1}, $n \geq 1$. The regions in a-q space between the characteristic curves corresponding to functions both of period π or 2π give rise to unbounded solutions of Mathieu's equation.

Stable solutions, $ce_{m+\beta}, se_{m+\beta}$ of order $M+\beta$, $0<\beta<1$, have the period $2s\pi$ when $\beta = p/s$, a rational fraction less than unity. When β is irrational these functions are bounded, but not periodic. Once β or ν is determined, the recurrence relations (106) may be used to evaluate the coefficients, c_{2r}.

C - Solutions of the form:

$$\zeta = e^{\pm\nu\theta}.\phi(\theta,\pm\gamma), \tag{109}$$

ϕ being periodic in θ with a period of π or 2π, and γ a new parameter were first suggested for Mathieu's equation[30], and later generalised to apply to Hill's equation[31]. γ is defined in terms of a, and λ_{2r} by:

$$\begin{aligned} a = & \left(1 - \frac{\lambda_2^2}{4} - \frac{\lambda_4^2}{6} - \frac{11\lambda_2^2\lambda_4}{192} - \frac{13\lambda_2\lambda_4\lambda_6}{192} + \frac{\lambda_2^4}{48} - \frac{893\lambda_2^6}{22184} + \ldots\right) \\ & + \cos 2\gamma\left(\lambda_2 + \frac{\lambda_2\lambda_4}{4} + \frac{\lambda_4\lambda_6}{12} + \frac{5\lambda_2^2\lambda_6}{192} - \frac{\lambda_2^3}{64} - \frac{\lambda_2\lambda_4^2}{144} - \frac{175\lambda_2^3\lambda_4}{9612} + \ldots\right) \\ & + \cos 4\gamma\left(\frac{\lambda_2^2}{8} + \frac{5\lambda_2^2\lambda_4}{64} + \frac{13\lambda_2\lambda_4\lambda_6}{576} - \frac{11\lambda_2^4}{512} + \frac{9181\lambda_2^6}{1769472} + \ldots\right) \\ & + \ldots\ldots\ldots, \end{aligned} \tag{110}$$

from which ν may be obtained using the equation:

$$\begin{aligned} \nu = & \sin 2\gamma\left(\frac{\lambda_2}{2} + \frac{\lambda_2\lambda_4}{8} + \frac{\lambda_4\lambda_6}{24} - \frac{3\lambda_2^3}{128} + \frac{5\lambda_2^2\lambda_4}{384} + \frac{7\lambda_2\lambda_4^2}{288} - \ldots\right) \\ & + \sin 4\gamma\left(\frac{\lambda_2^2\lambda_4}{128} + \frac{\lambda_2\lambda_4\lambda_6}{1152} - \frac{3\lambda_2^4}{1024} + \frac{337\lambda_2^6}{442368} + \ldots\ldots\right) \\ & + \ldots\ldots\ldots, \end{aligned} \tag{111}$$

and

$$\begin{aligned} \phi(\theta,\gamma) = & \sin(\theta-\gamma) + \lambda_2 f_2(\theta,\gamma) + \lambda_4 f_4(\theta,\gamma) + \ldots \\ & + \lambda_2^2 g_2(\theta,\gamma) + \lambda_4^2 g_4(\theta,\gamma) + \ldots \\ & + \lambda_2\lambda_4 g_{24}(\theta,\gamma) + \ldots + \lambda_2^3 g_3(\theta,\gamma) + \ldots \end{aligned} \tag{112}$$

This form of solution would clearly be very difficult to use, even in the limiting case of Mathieu's equation, unless $a \gg \lambda_{2r} \ll 1$, and clearly applies only to a limited region of a-q space, i.e, where a has a value near unity. However, solutions of this form may be obtained to apply throughout a-q space by a suitable re-definition of the series for a, and ϕ, as has been done quite thoroughly in the case of Mathieu's equation. It is clear from (110) and (111) that the solutions are unbounded when any $\lambda_{2r} \neq 0$, and a sufficiently near unity for γ, and hence ν, to be real.

D - Solutions of HILL'S equation provided in A,B,C have used a Fourier expansion of $R(\theta)$. A different general approach consists of converting the differential equation (87) into an Integral equation:

$$\zeta(\theta) = f(\theta) + \lambda\cdot\int_0^\theta H(\theta,u).\zeta(u).du, \qquad (113)$$

where,

$$f(\theta) = \zeta(0).\cos(a^{1/2}\theta) + (d\zeta/d\theta)_{\theta=0}.a^{-1/2}\sin(a^{1/2}\theta),$$

$$H(\theta,u) = R(u).\sin a^{1/2}(\theta-u),$$

and $\lambda = 2qa^{-1/2}$,

or, alternatively,

$$f(\theta) = \zeta(0) + (d\zeta/d\theta)_{\theta=0}.\theta,$$

$$H(\theta,u) = (\theta-u)(\ (a/2q) - R(u)\ ),$$

$$\lambda = 2q.$$

Equation (113) is known as Volterra's Integral equation of the second kind, and is exactly equivalent to Fredholm's equation of the second kind if the nucleus, $K(\theta,u)$ in this latter equation is defined:

$$K(\theta,u) = H(\theta,u), \quad 0 \leq u \leq \theta,$$

$$= 0, \quad u \geq \theta.$$

The solution to the Fredholm equation may be expressed in the form:

$$\zeta(\theta) = f(\theta) + \lambda.\int_0^\theta K'(\theta,u).f(u).du, \qquad (114)$$

where $K'(\theta,u)$ may be expressed either as the Liouville-Neumann series:

$$K'(\theta,u) = \sum_{n=0}^{\infty} \lambda^n.K_{n+1}(\theta,u),$$

with $K_n(\theta,u) = \int_0^\theta K(\theta,y).K_{n-1}(y,u).dy, \quad K_1(\theta,u)=K(\theta,u),$

or, using a method due to Fredholm :

$$K'(\theta,u) = D(\theta,u)/D(\lambda),$$

where,

$$D(\theta,u) = K(\theta,u) + \sum_{n=1}^{\infty}(-1)^n . D_n(\theta,u) . \lambda^n / n! \quad ,$$

$$D(\lambda) = \sum_{n=0}^{\infty} (-1)^n . D_n . \lambda^n / n! ,$$

with the coefficients of these infinite series being found with the aid of the following recurrence relations:

$$D_m = \int_0^{\theta} D_{m-1}(\theta,\theta).d\theta ,$$

$$D_m(\theta,u) = K(\theta,u).D_m - m\int_0^{\theta} K(\theta,y).D_{m-1}(y,u).dy ,$$

and $\quad D_0(\theta,u) = K(\theta,u).$

In both expressions for the solution of (113), (114) is uniformly convergent unless $D(\lambda) = 0$. Explicit solutions of the inhomogeneous equation (113) in such cases may be obtained using the Schmidt-Hilbert method only if $K(\theta,u)$ is symmetrical, which does not apply in the cases of interest to us. If $f(\theta) = 0$, however, (113) becomes a homogeneous equation which possesses solutions not identically zero only if $D(\lambda) = 0$, the solutions to the former being known as the eigenfunctions, and solutions of the latter being known as eigenvalues of the homogeneous integral equation. It is interesting to note that $f(\theta)$ may vanish only if $\zeta(0) = (d\zeta/d\theta)_{\theta=0} = 0$, and in the complete absence of forcing terms in (87) which would make it inhomogeneous. It appears therefore that unbounded solutions of (113) may occur for λ_i satisfying the equation $D(\lambda) = 0$.

The analytic treatment of the HILL equation given above is far from comprehensive, and for a more general treatment reference must be made to the works of Strutt.[32] It has nevertheless been made clear that explicit solutions are necessary to define the boundaries between stable and unstable regions in a-q space, and vice versa. While it is possible to prove the existence of such continuous boundaries in a-q space in which solutions of (87) are stable or unstable, for arbitrary periodic $R(\theta)$, this is of little practical interest in cyclic accelerator theory.

Thus, while the special cases which comprise the Mathieu and Meissner equations can be completely solved, solutions of HILL'S equation are conveniently obtained only if all $\lambda_{2r} \ll a$. For this reason the following approximate solutions are of some practical significance.

Some Approximate Methods for Solving HILL'S Equation:

I - If any one particular harmonic component, say $\lambda_{2m}.\cos(2m\theta + \gamma_m)$, is much larger than all other $\lambda_{2r}.\cos(2r\theta + \gamma_r)$, then the HILL equation (87) may be put in the form:

$$d^2\zeta/d\theta^2 + (\ a - 2\lambda_{2m}.\cos(2m\theta+\gamma_m)\ ).\zeta =$$

$$= (\ \sum_{\substack{r=1 \\ r\neq m}}^{\infty} 2\lambda_{2r}.\cos(2r\theta+\gamma_r)\ ).\zeta,$$

which, with the change of independent variable-

$$(2m\theta+\gamma_m) = 2\theta',$$

may be written in the form:

$$d^2\zeta/d\theta'^2 + (\ a' - 2q.\cos(2\theta')\ ).\zeta = (\sum_{r\neq m} 2\lambda_{2r}\cos(\tfrac{2r\theta'}{m}+\gamma'_m))\zeta, \quad (115)$$

where, $$a' = a/m^2,\ \ q = \lambda_{2m}/m^2,\ \gamma'_r = \gamma_r-(r\gamma_m/m). \quad (116)$$

The method of solution is to treat solutions $\zeta=\zeta_0$ of the LHS, which is MATHIEU'S equation, as the first approximation to a solution of the complete equation, (115), and obtain a closer approximation by obtaining the particular integral, $\zeta = \zeta_1$ associated with (115) when ζ is replaced by ζ_0 on the RHS, and adding it to ζ_0 to give $\zeta = \zeta_0+\zeta_1$ as the second approximation to a solution of (115). The general solution of (115) may then be written:

$$\zeta = A\zeta_{01} + B\zeta_{02} + \zeta_{01}\int^{\theta'}\zeta_{02}(u)f(u)du - \zeta_{02}\int^{\theta'}\zeta_{01}(u)f(u)du, \quad (117)$$

where $$f(u) = (\sum_{r\neq m} 2\lambda_{2r}\cos(\tfrac{2ru}{m}+\gamma'_r)).\zeta_{01}, \quad (118)$$

and ζ_{01} and ζ_{02} are normalised to satisfy the equation-

$$\zeta_{02}(d\zeta_{01}/d\theta') - \zeta_{01}(d\zeta_{02}/d\theta') = 1. \quad (119)$$

Under the special terms of its validity given above, this method can be very useful, so that solutions inherently stable/unstable due to the predominant harmonic component MAY become unstable/stable in the presence of the additional, much smaller magnitude harmonic terms provided the point a'-q in (115) is very close to any boundary between stable and unstable regions in the appropriate a'-q space, i.e that associated with MATHIEU'S equation.

II - It is well known that a transformation of the dependent variable, ζ, of the type-

$$\zeta = e^{\int w.d\theta}, \tag{120}$$

transforms the second order differential equation,

$$d^2\zeta/d\theta^2 + \xi(\theta).\zeta(\theta) = 0, \tag{121}$$

into the generalised RICCATI equation:

$$dw/d\theta + w(\theta)^2 + \xi(\theta) = 0. \tag{122}$$

In the special case of HILL'S equation we then have-

$$\xi(\theta) = \xi_1^2(\theta) = a - 2q.R(\theta)$$

$$= a(1 - 2qR(\theta)/a) = 2q(\frac{a}{2q} - R(\theta)), \tag{123}$$

which is of the form-

$$\xi_1(\theta)^2 = \alpha(\beta - \gamma(\theta)). \tag{124}$$

The solution of (122) in the special cases where α is a 'large' parameter, i.e. $\alpha >> 0$, is particularly simple. Rewriting (120)

$$\zeta = e^{\sqrt{\alpha}\int w.d\theta}, \tag{120a}$$

(122) becomes

$$(1/\sqrt{\alpha})(dw/d\theta) + w^2 + (\beta - \gamma(\theta)) = 0, \tag{122a}$$

solutions of which may be expressed as formal (Laurent) series in $\alpha^{-1/2}$:

$$w(\theta) = \sum_{k=0}^{\infty} w_k(\theta).\alpha^{-k/2}. \tag{125}$$

Substituting this series into (122a), and equating the coefficients of $\alpha^{-k/2}$, k=0,1,2,...., we obtain a series of recurrence relations from which $w_k(\theta)$ can be determined. Neglecting the contribution of terms in (125) for $k \geq 3$, we have for ζ in (120a):

$$\xi \doteq C.\xi_1(\theta)^{-1/2}.e^{\pm i\alpha^{1/2}\int_0^\theta(\xi_{11})d\theta}, \qquad (126)$$

where C is a constant, and

$$\xi_{11}(\theta) = \xi_1(\theta) - (2\xi\xi_1''-3\xi_1'^2)/8\alpha\xi_1^3, \quad ' = (d/d\theta), '' = (d/d\theta)^2. \qquad (127)$$

In this approximation therefore, two independent solutions of (121) may be written:

$$\begin{matrix}\zeta_1\\ \zeta_2\end{matrix} \doteq C.\xi_1^{-1/2}.\begin{matrix}\cos\\ \sin\end{matrix}(\alpha^{1/2}\int_0^\theta \xi_{11}(\theta).d\theta). \qquad (128)$$

If ξ_{11} is real it is clear that the solutions (128) represent amplitude <u>and</u> frequency modulated, bounded oscillations of frequency, f, with

$$f = (2\pi)^{-1}.\alpha^{1/2}\xi_{11}(\theta), \qquad (129)$$

and amplitude proportional to $\xi_1^{-1/2}$. If ξ_{11} is complex, or imaginary the solutions are unbounded, and the motion it describes is unstable. The conditions for a stable (bounded) solution to exist may therefore be written:

$$\alpha > 0, \quad \beta > |\gamma(\theta)|_{max}, \gamma' - (\beta-\gamma(\theta))\gamma'' > 3(\gamma')^2/2,$$

and finally,

$$(2\alpha)^3(\beta - \gamma(\theta))^3 > (\gamma'/2) - 3(\gamma')^2/4 - (\beta-\gamma)\gamma''/2. \qquad (130)$$

The two special cases of interest are obvious from (123) and (124) as-

$$\alpha = a >> |q| > 0, \quad \text{and} \quad \alpha = 2q >> |a| > 0 .$$

We have outlined above many rigorously exact, and some deliberately approximate solutions to HILL'S equation, which is the general form of the equations of group I, eqn. (51), which are basically linear. The equation of group II describing synchronous acceleration is not included in the above discussion because it is basically non-linear. No general theory of non-linear differential equations can as yet be described, so that we shall content ourselves with the treatment in the following section of the effect on the stability of solutions of linear equations of both linear and non-linear perturbations.

2.4 The Perturbation of systems with Periodic solutions:

In sections 2.2 and 2.3 we have considered in some detail the behaviour of linear systems described by equations of the form -

$$[dx/d\theta] + [A][x] = [\eta(\theta)], \tag{55}$$

where $[A]$ is a constant or periodic matrix operator. We shall concern ourselves in this section with equations of the general form -

$$[dx/d\theta] + [A][x] = [\eta(\theta, x)], \tag{131}$$

where $[A]$ is constant or periodic as in (55), but where $[\eta(\theta,x)]$ is real, and, in general, a polynomial in θ and $[x]$, so that (131) is both non-linear and non-autonomous in general. We shall not attempt anything approaching a complete treatment of this equation, as, fortunately, particle motion in cyclic accelerators is almost completely described by the linear, non-autonomous system (betatron equations and focussing theory), and/or by the non-linear, autonomous system (synchrotron equation). Nevertheless, non-linear perturbations of the linear, non-autonomous system, and non-autonomous perturbations of the non-linear, autonomous system are of some importance in explaining important details of particle behaviour, or in avoiding catastrophic instability of their cyclic motion, both of which require some consideration of differential equations of the form (131). Before this is done, however, we shall briefly consider the class of non-linear, autonomous systems to which the synchrotron equation (54) belongs, under the conditions of Assumption (v).

Non-linear, Autonomous systems - These are of the form -

$$[dx/dt] + [A][x] = [\eta] \tag{132}$$

where the elements of $[A]$ are, in general, polynomials of the elements of $[x]$. In the special case of the synchrotron equation (54) we have:

$$[A] = \begin{bmatrix} 0 & -1 \\ (V/2\pi B)(1-\frac{x_1^2}{3!}+\frac{x_1^4}{5!}-\ldots) & 0 \end{bmatrix}, [\eta] = \begin{bmatrix} 0 \\ -V\sin(\theta_e)/2\pi B \end{bmatrix}, \quad (133)$$

when expressed in the form (132), or

$$[A] = \begin{bmatrix} 0 & -1 \\ -V/2\pi B & 0 \end{bmatrix}, [\eta(x)] = \begin{bmatrix} 0 \\ (-V/2\pi B)(\sin(\theta_e)-\frac{x_1^3}{3!}+\ldots) \end{bmatrix}, \quad (134)$$

when expressed in the form (131), where-

$$B = (B_z r_e^2/kw_e)(\alpha_m' - \frac{R_o^2}{E_e^2})^{-1} . \quad (135)$$

In the linear approximation clearly both forms are identical.

Alternatively, we may write (54) in the form:

$$x_2 . dx_2 = (V/2\pi B)(\sin(x_1) - \sin(\theta_e)).dx_1 ,$$

which may be integrated to read -

$$(x_2^2/2) + U(x_1) = C , \quad (136)$$

where

$$U(x_1) = -(V/2\pi B).\int(\sin x_1 - \sin \theta_e).dx_1$$

$$= (V/2\pi B)(\cos x_1 + x_1 .\sin \theta_e), \quad (137)$$

and C is the constant of integration representing the total energy of this conservative system, which is determined by the initial conditions.

Equation (136) represent the integral curves of (54) in $(x_1 - x_2)$, or phase space. Solutions in this form, in the conservative approximation, suffice for the purposes of synchrotron theory, where emphasis is placed on the amplitude of particle oscillatory motion, rather than on the frequency of the motion, which would require explicit solutions in (x_1, t) space such as is provided in 2.3, and will be provided in 2.5 in discussing asymptotic solutions. The positions of equilibrium of this system in phase space are those satisfying the equations -

$$(V/2\pi B)(\sin x_1 - \sin \theta_e) = x_2 = 0. \quad (138)$$

(x_1, x_2) satisfying (138) are also known as SINGULAR points.

The stability of these positions of equilibrium are decided by the following two fundamental theorems due to Lagrange and Liapounoff respectively.

STABILITY THEOREM (lagrange) - If the potential energy $U(x_1)$, is a minimum at the point of equilibrium, then the equilibrium is stable.

STABILITY THEOREM (Liapounoff) - If the potential energy is not a minimum at the point of equilibrium, then the equilibrium is unstable.

The positions of equilibrium of the synchrotron equation (54), or (136), for $-\pi \leq x_1 \leq \pi$, are clearly:

$$x_1 = \theta_e \ , \quad \text{and} \quad x_1 = (\pi - \theta_e), \quad 0 \leq \theta_e \leq \pi/2. \tag{139}$$

For $B > 0$, $x_1 = \theta_e$ is the position of the potial minimum, and this singular point is a focus, while $x_1 = \pi - \theta_e$ is the position of the potential maximum, and this singular point is known as a saddle-point. The limiting phase angles for stable acceleration of particles for which $x_2 = 0$, are then

$$x_1 = \pi - \theta_e, \quad \text{and} \quad x_1' \ , \text{ with } \ U(x_1') = U(\pi - \theta_e). \tag{140}$$

For $B < 0$, $x_1 = \theta_e$ is the saddle-point, and $x_1 = \pi - \theta_e$ is the focus, the singularity around which stable synchronous oscillations occur. The limiting phase angles, when $x_2 = 0$, are now

$$x_1 = \theta_e, \text{ and } x_1'', \text{ with } \ U(x'') = U(\theta_e). \tag{141}$$

When $B = \infty$, which occurs when

$$E_e = E_o(\alpha_m')^{-1/2}, \tag{142}$$

$U(x_1)$ vanishes identically, and phase stability does not exist. For energies below that given by (142), $B < 0$, and above this energy, $B > 0$, so that when a particle is accelerated through this critical energy the stable phase angle changes discontinuously from $(\pi - \theta_e)$ to θ_e. In other words the trough in the potential function, initially at $x_1 = \pi - \theta_e$, flattens out completely and reappears immediately at $\theta_e = x_1$, a shift in x_1 of $(\pi - 2\theta_e)$. Such a change is equivalent to a discontinuous shift in the phase of the RF accelerating voltage of the same magnitude.

The full implications of this discontinuity will be considered in greater detail in Chapter Two; it suffices for the present to note that, from an analytical point of view, this discontinuous change in the relative positions of the singularities, or what is equivalent, a discontinuous change in the magnitude and/or phase of the accelerating voltage, makes the system non-autonomous. In this particular problem the critical energy is approached so gradually thatthe use of asymptotic, or adiabatic solutions of the synchrotron equation (54), described in the following section, 2.5, is adequate for its consideration. We have now to discuss briefly various types of perturbation of the synchrotron and betatron equations.

Non-Autonomous Perturbations of the Synchrotron equation-

The conditions under which the discontinuous non-autonomous disturbance, treated above, appear are approached gradually, thus allowing the use of adiabatic approximations in evaluating its effects on synchronous stability and the amplitude of particle radial oscillations. Non-autonomous disturbances in general, and one important type in particular, do not, however, permit an analysis of their effects in this manner. The particular important problem concerns an evaluation of the effect on synchronous stability and radial amplitudes of changes in the peak value, V, of the RF accelerating voltage, which occur mainly very near the beginning and/or the end of the period of synchronous acceleration. As such it is one of the class of boundary-value problems associated with cyclic accelerators, and discussed in greater detail in Chapter Four of Part I. Past attempts at a solution of such problems are mentioned in PAPER J, accompanying this monograph, which introduces a fresh approach to the subject, using the integral curves of the autonomous theory discussed above and a limiting procedure based essentially on the linear theory of section 2.2.

Another type of non-autonomous perturbation of the Synchrotron equation arises due to the presence of coherent and non-coherent radiation damping, of which the former is much less important than the latter, which increases as the fourth power of the particle energy and is therefore significant only at the highest energies. An electron moving in a circular orbit radiates incoherently at the rate of

$$(6.03/r)(E/E_o)^4.10^{-9} \text{ eV/revolution.} \qquad (143)$$

where r is the orbit radius in metres, E the total energy and E_o the rest energy of the electron or proton in eV.[34] This energy loss is clearly much larger for electrons than for protons of the same kinetic energy circulating on an orbit of the same radius. In estimating the effects of such losses on the characteristics of the stable synchrotron orbits, however, it is sufficient to use asymptotic solutions of the synchrotron equation discussed in the following section.

Linear, and Non-Linear Perturbations of the Betatron equation-

In one of the early assumptions made in deriving the first-order betatron equation (51), namely, ASSUMPTION (i), forces due to interactions between charged particles in a beam of many particles were ignored. In practice, however, such forces are not negligible, and in the betatron, in particular, it is of some importance in obtaining suitable injection conditions (Cf Papers A and B). Such forces are taken into account without abandoning the basis of a single particle theory by assuming that this single particle moves in the 'smeared-out' space-charge field due to all the other particles in the beam. Such treatment of the problem is used in Paper C, and described in greater detail in Paper N. This treatment is very approximate; refinement of this theory is outside the scope of this monograph. In its approximate form this simple theory will be used extensively in subsequent discussions of space charge forces in both strong and weak focussing cyclic accelerators.

In investigating the effect of second order deviations from equilibrium conditions, which introduce non-linear perturbing terms into the betatron equations, we are clearly examining the validity of ASSUMPTION (iv). The general betatron equations including all second-order terms are equations (40) and (41) in PAPER P. The most important second order terms in these equations are those involving the functions describing the radial and axial guide field components and their spatial dependence. In general these additional terms are of not much importance, which, when combined with the difficulty of treating them in any general manner, will explain why such little space is accorded to them not only in this monograph, but more widely in literature dealing with cyclic accelerator theory. Such non-linear perturbations are of significance only in very special circumstances, cf PAPER D, and are rarely anything like so important as the linear perturbation due to space-charge forces discussed above.

Before concluding this brief discussion, however, we might introduce the following general stability theorems concerning the solutions of (55) in the presence of linear and/or non-linear perturbing terms which transform it to equations of the form (131), where $[A]$ is either constant or periodic in θ. Stability properties derived for the case where $[A]$ is constant carry over to the case where $[A]$ is periodic as a consequence of the canonical representation of solutions, equation (72), in the latter case.

STABILITY THEOREM FOR LINEAR PERTURBATIONS- All solutions of

$$[dx/d\theta] = ([A'] + [B(\theta)])[x],$$

are bounded if all solutions of

$$[dx/d\theta] = [A'][x]$$

are bounded, A' being either constant or periodic in θ, and $[B(\theta)] = 0$, $\theta = \infty$; $\int^{\infty} \| B(\theta) \| d\theta < \infty$.

In the case of non-linear perturbations the situation is much more complicated. Nevertheless, the stability of any solution of a non-linear equation can always be made to depend on the stability of the null(or trivial) solution $w = 0$, of a related equation, which may be linear. Thus, if x' is a real solution of $dx/d\theta = F(\theta, x)$ whose stability is to be investigated, let $y = x'+w$ be another solution, when we have

$$dw/d\theta = J(F, x')w + \ldots, \quad (144)$$

where $J(F, x')$ is the Jacobian matrix of F with respect to x'. The stability of the null solution of the linear approximation,

$$dw/d\theta = J(F, x') \quad (145)$$

determines in some cases the nature of the stability of the solution x' of the original non-linear equation. Then the stability theorem may be stated in the following form -

> IF THE CHARACTERISTIC EXPONENTS ASSOCIATED WITH THE EQUATION OF FIRST VARIATION WITH RESPECT TO THE SOLUTION $x = x'$,(145), ALL HAVE NEGATIVE REAL PARTS, THEN THE PERIODIC SOLUTION x' OF THE NON-LINEAR EQUATION IS ASYMPTOTICALLY STABLE AS θ TENDS TO INFINITY.

In the special case of the autonomous, non-linear equation, $dx/d\theta = F(x)$, the type of asymptotic stability above may not exist because one of the characteristic exponents(eigenvalues) associated with the equation of first variation may be taken as zero. It is possible, however, for a type of asymptotic stability to prevail which is of great importance. The solution $x = x'(\theta)$ may be regarded as a closed curve or orbit in x space to which any solution of the non-linear autonomous equation which comes near a point in this orbit tends as θ approaches infinity, provided all but one of the eigenvalues of the equation of first variation, which is zero, has negative real parts. The stability theorem governing this type of asymptotic orbital stability may be stated-

> LET ALL BUT ONE OF THE EIGENVALUES OF THE EQUATION OF FIRST VARIATION HAVE NEGATIVE REAL PARTS. THEN THERE EXISTS AN $\eta > 0$ SUCH THAT IF A SOLUTION ϕ OF THE CORRESPONDING AUTONOMOUS NON-LINEAR EQUATION

SATISFIES $\phi(\theta_1) - x'(\theta_0) < \eta$ FOR SOME θ_0 AND θ_1, THERE EXISTS A CONSTANT c SUCH THAT

$$\lim_{\theta=\infty} |\phi(\theta) - x'(\theta+c)| = 0.$$

Not only is there asymptotic orbit stability, but each solution near the orbit possesses an asymptotic phase, c. It is possible that in certain exceptional circumstances the asymptotic conditions prescribed by this last theorem may exist in cyclic accelerators[35].

The rather cursory treatment of the effects of perturbations afforded in this section is a direct result of the absence of any general treatment of non-linear equations. Of special solutions to particular problems much can be said which would take up too much space here. Further discussion of these points will be found later in Chapters 2,3 and 4.

2.5 Asymptotic Solutions:

As we have already mentioned, one of the consequences of the last Assumption (v) is that we may deal with the effects of particle acceleration on the amplitudes and frequencies of orbital motion with the aid of asymptotic solutions of the appropriate equations of motion, expressed with time, t, as the independent variable rather that θ, as in the description of the betatron oscillations (55). Further, in the case of the synchrotron equation (54), it is usual to consider the asymptotic behaviour of only the small amplitude phase oscillations, for the description of which the linearised version of (54) is a sufficiently accurate approximation. Both betatron equations and the synchrotron equation are then of the form -

$$d^2x/dt^2 + f(t).dx/dt + g(t).x = h(t), \qquad (146)$$

which is transformed by the substitution -

$$u = x.\exp(\int_0^t f(t)\, dt,$$

to an equation -

$$d^2u/dt^2 + F(t).u = 0, \qquad (147)$$

with $F(t) = g(t) - (df/dt)/2 - f(t)^2/4;$ (148)

we have neglected the forcing term and provided the equation governing solutions (asymptotic) of the homogeneous equation. An important feature of this transformation is that, provided $\int_0^t f(t)dt$ is finite for finite t, the zeros of x are the same as the zeros of u.

If now we define u by the equation,

$$u = \exp(\int^t \eta(t)dt),$$

we have:

$$\eta^2 + (d\eta/dt) + F(t) = 0, \qquad (148')$$

which may be solved by the iterative process in which it is assumed that $F(t)$ is a very slowly varying function of t. From the second iterative approximation we then have -

$$u \approx (F(t))^{-1/4}.\exp(\pm i\int^t (F(t))^{1/2} dt, \qquad (149)$$

from

$$\eta^2 = -F(t) + (i/2F^{1/2})(dF/dt). \qquad (150)$$

From (150) we see clearly the failure of this iterative expansion at the zeros of $F(t)$; the singularity of (149) at $F(t) = 0$ is a consequence of the failure of the series solution of (148'). Actually no such singularity exists, t_0, satisfying $F(t) = 0$, is clearly a regular point of the differential equation (147). Since the behaviour of physical systems are of interest in such circumstances, such as at the transition energy during synchronous acceleration when B, (54) and (135), tends to infinity, we may in the neighbourhood of $t=t_0$ approximate to $F(t)$ by the expression-

$$F(t) = C^2(t - t_0), \qquad (151)$$

when the resulting equation may be solved in terms of Bessel functions of 1/3 order, i.e.

$$u_{1,2} = (t-t_0)^{1/2} J_{\pm 1/3}(\frac{2C}{3}.\overline{t-t_0}^{3/2}), \qquad (152a)$$

$$= (w^{1/2}/F^{1/4}).J_{\pm 1/3}(w), \qquad (152b)$$

where,

$$w = \int_{t_0}^t F^{1/2}.dt, \quad t - t_0 \ll t_0 \qquad (152c)$$

In order to match the solutions (152) near the transition energy with these (149) valid at energies other than the transition energy we require the asymptotic form of (152) valid for $(t-t_o) \gg t_o$. This may be done by replacing the Bessel functions by the first term in their asymptotic expansions :

$$J_p(t) \simeq (2/\pi t)^{1/2}.\cos(\, t - \overline{p+1/2}.\pi \,) \qquad (153)$$

Use of these will be made in the discussion of synchronous acceleration in Chapter Two.

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3. DESIGN OF THE GUIDE FIELD MAGNET

3.1 Statement of the problems involved -

In the two sections above the major considerations involved in obtaining the spatial dependence of the guide field necessary for stable acceleration in an aperture of given dimensions have been defined by setting down the equations governing the particle motion and analysing the properties of its solutions. In the present section the problems of designing the magnet to meet the specifications resulting from such analysis applied to practical cases will be discussed, and methods of solving them will be introduced. We shall not be concerned here with the associated, and no less important, problems affecting the supply of electrical power to the magnet, and the associated electronic controls that are so necessary a part of an operating cyclic accelerator. Discussion of these will be found in Papers F,G,H.

Problems in guide field magnet design may broadly, and somewhat roughly, divided into the mechanical and the electromagnetic. Mecahnical design has, somewhat unfortunately, to satisfy the mutually exclusive requirements of flexibility and stability. The closer the tolerances on the desired spatial properties of the guide field the more stringent do both these requirements become, more especially in the strong focussing cyclic accelerators being designed than on any previous weak focussing accelerator. We shall not be concerned here with such mechanical features of magnet design; the necessary tolerances and adjustments to be built in to any design follow from a thorough examination of the types of orbit instability that may exist, their possible remedy, and the types of field disturbances that may give rise to them, which include the presence of remanent and eddy currents. All such features of the 'imperfect' magnet are discussed further in Chapters Three and Five.

We are concerned here with the brief description of analytic methods used to obtain a guide field with specified spatial properties using an iron-cored structure, which resolves itself into the problem of obtaining a pole-profile to do this at all flux densities. This problem is considerably easier, more especially in strong focussing accelerators, than the related problem of obtaining the same spatial properties with air-cored coils, despite the complication of flux transients and saturation which attend the use of iron cores. These problems, together with those concerned with the mechanical and electromagnetic 'imperfections' inherent in magnet design, which are mentioned above, may be considered as being the 'first-order' problems in magnet design. The problem of obtaining a pole profile to give the desired spatial dependence of the guide field in the absence of these effects may be considered as the 'zero order' problem in magnet design, having a similar relationship to first order problems as in the treatment of orbit characteristics. We shall discuss magnet design theory in the following two sections and in Chapter Five in this context.

3.2 The Zero-Order Problem in Magnet Design:

With the analytical techniques outlined in section 2 of this chapter we may specify the field properties along a given circular plane (which is annular in synchrotrons), necessary for stable cyclic acceleration, which may then be used to specify the field throughout the toroidal air space between the pole shoes in terms of either a vector potential, $\underline{A}$, or a scalar potential, ψ. Use of the scalar potential makes, implicitly, the assumption that the air space is devoid of electric current (ASSUMPTION (i)), which with quasi-static approximation (ASSUMPTION (v)) gives the vector relation -

$$\nabla \wedge \underline{B} = 0, \qquad (154)$$

making for the existence of a scalar potential, known as the magnetostatic potential, ψ, satisfying the equations:

$$\underline{B} = -\nabla\psi \quad \text{and} \quad \nabla.\underline{B} = \nabla.\nabla\psi = \nabla^2\psi = 0. \qquad (155)$$

In this approximation one assumes that the surfaces of highly permeable magnetic material are equipotentials for which ψ = constant, and with any of which the pole profiles may be exactly identified. The solution of LAPLACE'S equation (155) is generally obtained either in the two-dimensional approximation which neglects the curvature of the guide field, or in the three-dimensional case assuming axial symmetry. In both cases the solutions are separable into two independent functions each of one dimension.

Due to the finite annular width of the magnet poles, however, leakage effects at the edges are severe, and relaxation methods have generally to be used to determine the pole profile shape towards the edges necessary to counter the rapid weakening of the field due to such end leakage, and maintain the necessary spatial dependence over as large a radial region as possible compatible with the annular width of the poles. In this approximation it is possible to take into account the effects of the presence of the magnetising coil when it is sufficiently near the air gap to have sufficient effect on the field in that region.

3.3 <u>First Order Problems in Magnet Design:</u>

If the spatial distribution of the guide field is defined solely by the pole profile, perfectly aligned, under all conditions of excitation, the magnet is generally referred to as 'perfect'. It must, however, be clearly understood that this is a theoretical idealisation of the practical problem which always concerns 'imperfect' magnets. The first order problems introduced here have to deal with the nature, causes and effects of such imperfections under all realisable operating conditions, and in design one is concerned with reducing the magnitude of these 'imperfections' if not eliminating them entirely.

In describing quantitatively the 'imperfections' inherent in an electromagnet we have to distinguish between d.c magnets (as used in weak focussing conventional cyclotrons, synchrocyclotrons and microtrons, and the strong-focussing fixed-field alternating gradient synchrotron), and a.c magnets (as used in betatrons and both strong and weak focussing proton and electron synchrotrons). In the former all the additional field sources due to electromagnetic and mechanical disturbances give rise to 'imperfections' which are explicitly, but not necessarily implicitly, time-independent. The additional field sources giving rise to 'imperfections' in this case are due to remanent fields, and errors in pole profile manufacture and/or alignment. The variation in the properties of the iron path which give rise to remanent field variations are serious more from the field distortion arising at high flux densities due to saturation effects rather than because of the remanent field variations. Thus in d.c. magnets additional electromagnetic field sources are not of any great importance. In a.c. magnets, however, electromagnetic field sources due to remanence and induced(eddy) currents are of much greater importance due to the fact that particles are injected when the additional field sources, which we may refer to as 'error' field sources, are of the same order of magnitude as the field due to the magnetising current. These field sources in this case are explicitly, and <u>not</u> implicitly time-dependent. The time-dependence of one error field can differ very much from that of another. We are not, however, concerned here with the distinctive characteristics of these error fields other than in outlining a mathematical representation which will suffice for a description of their characteristics and their effect on particle orbits.

The spatial and time dependence of the q_i component of the magnetic field due to a source s in the immediate vicinity of the coordinates, q^o_{ij}, may be represented by the following series -

$$B^s_{q_i}(q_i, q_j, \theta, t) = B^s_{q_i}(q^o_i, q^o_j, \theta) + (\partial B^s_{q_i}/\partial q_i)_{q_i=q^o_i}(\delta q_i + \frac{\partial q_i}{\partial t}.\delta t) + $$

$$+ (\partial B^s_{q_i}/\partial q_j)_{q_j=q^o_j}(\delta q_j + \frac{\partial q_j}{\partial t}.\delta t) +$$

when the time dependence is implicit, and by the series -

$$B^s_{q_i}(q_i, q_j, \theta, t) = \left\{ \begin{array}{l} B^s_{q_i}(q^o_i, q^o_j, \theta) + (\partial B^s_{q_i}/\partial q_i)_{q_i=q^o_i}(\delta q_i) + .. \\ + (\partial B^s_{q_i}/\partial q_j)_{q_j=q^o_j}(\delta q_j) + .. \end{array} \right\} (F^s(t)),$$

where $q_{i,j} = q^o_{i,j} + \delta q_{i,j}$, and

$$B^s_{q_i}(q^o_i, q^o_j, \theta) = \langle B^s_{q_i}(q^o_i, q^o_j) \rangle + \sum_{m=1}^{\infty} (\delta B^s_{q_i})_m . \cos(m\theta + \gamma_m),$$

with $\langle .. \rangle = (1/2\pi).\int_0^{2\pi} .. d\theta$, when the time dependence is only explicit. The spatial and time dependence of the complete guide field is obtained by summing the contributions of the field sources due to the main magnetising current as well as all other sources of 'error' fields.

In general, we have -

$$\frac{\partial B^s_{q_i}}{\partial q^o_j}(q_i,q^o_j,\theta) = \left(\left\langle \frac{\partial B^s_{q_i}}{\partial q^o_j}(q_i,q^o_j)\right\rangle + \frac{\partial \langle B^s_{q_i}(q^o_i,q^o_j)\rangle}{\partial q^o_j} \right) +$$

$$+ \sum_{m=1}^{\infty} \left(\delta(\partial B^s_{q_i}/\partial q^o_j)_m + \partial(\delta B^s_{q_i})_m/\partial q_j \right)\cos \overline{m\theta+\gamma_m} \,)$$

As mentioned above the first order problems are more important in the case of a.c. magnets than in the case of d.c. magnets, so the analysis of magnet imperfections here, and in Chapters Three and Five, will be confined mainly to the a.c. magnet, when, for any one source, we may write -

$$B^s_{q_i}(q_i,q_j,\theta,t) = F^s(t)\langle B^s_{q_i}(q^o_i,q^o_j)\rangle \left\{ \begin{array}{l} 1 + \sum_{m=1}^{\infty} f^{q_i}_{om}(\theta) \\ -\langle n\rangle_{q_iq_i}\left(1 + \sum_{m=1}^{\infty} f^{q_i}_{2m}(\theta)\right) \frac{\rho q_i}{q^o_i} \\ -\langle n\rangle_{q_iq_j}\left(1 + \sum_{m=1}^{\infty} f^{q_i}_{1m}(\theta)\right) \frac{\rho q_j}{q^o_i} \end{array} \right\},$$

where

$$f^{q_i}_{om}(\theta) = (\delta B^s_{q_i})_m \cos \overline{m\theta+\gamma_m}/\langle B^s_{q_i}(q_i,q_j)\rangle,$$

$$f^{q_i}_{1m}(\theta) = (\delta n_{q_iq_j})_m . \cos \overline{m\theta+\gamma_m}/ \langle n\rangle_{q_iq_j}, \qquad (156)$$

$$f^{q_i}_{2m}(\theta) = (\delta n_{q_iq_i})_m . \cos \overline{m\theta+\gamma_m}/\langle n\rangle_{q_iq_i},$$

$$\langle n\rangle_{q_iq_j} = -(q^o_i/\langle B^s_{q_i}(q^o_i,q^o_j)\rangle)\left(\left\langle \frac{\partial B^s_{q_i}}{\partial q^o_j}(q_i,q_j)\right\rangle + \frac{\partial\langle B^s_{q_i}(q^o_i,q^o_j)\rangle}{\partial q^o_j} \right),$$

$$\delta n_{q_iq_j} = -(q^o_i/\langle B^o_{q_i}(q^o_i,q^o_j)\rangle)\left(\delta\left(\frac{\partial B^s_{q_i}}{\partial q^o_j}\right)_m + \frac{\partial(\delta B^s_{q_i})_m}{\partial q^o_j} \right),$$

and $(\partial/\partial q_j)_{q_j=q^o_j} = (\partial/\partial q^o_j)$.

This expression for the field source, s, terminates with the first term in the Taylor series. In applications we use cylindrical polar coordinates. If $q_i = r$, then $q_j = z$, and we have a description of the axial component, B_z, of the guide field. to the first order in the variables, x_r and x_z. If $q_i = z$, and $q_j = r$, we have a description, to the first order in the same variables, of the radial component, B_r, of the field.

The explicitly time-dependent functions, $F^s(t)$ may be a constant, or a circular or exponential function, depending on the conditions of excitation and magnet characteristics.

In the presence of many sources of magnetic field, we may write -

$$B_{q_i}(q_i, q_j, \theta, t) = \Sigma_s B^s_{q_i}(q_i, q_j, \theta, t), \qquad (157)$$

which may readily be put in the form of (156).

In any design of a cyclic accelerator magnet the magnitude of the additional 'error' field sources are reduced to a minimum, and methods of achieving such reductions are discussed later.

CHAPTER TWO

F O C U S S I N G   A N D   A C C E L E R A T I N G   M E C H A N I S M S

C H A P T E R   T W O

FOCUSSING AND ACCELERATING MECHANISMS

1. INTRODUCTION:

In the first chapter an adequate analytical basis for particle orbit studies and magnet design has been outlined. This chapter is concerned with putting it to use in design studies of focussing and accelerating mechanisms in 'perfect' cyclic accelerators.

It has been shewn that under adiabatic conditions we may treat analytically the focussing properties of both the guide field and the accelerating field, while under asymptotic conditions we may treat analytically the effect of particle acceleration on such focussing. We may treat the focussing properties of the guide field and accelerating fields independently in those cases where the periods of the bounded oscillations accompanying them are of different orders of magnitude(see pages 18 and 19), a condition satisfied in all the cases of interest to us. Given this independence, guide field focussing mechanisms are independent of the accelerating mechanism, be it induction or synchronous acceleration in all but very exceptional circumstances which we shall define below. Accelerating mechanisms, however, are very much dependent on the focussing mechanism (weak or strong) adopted. Accelerating mechanisms generally provide radial focussing only, while guide field focussing has to provide axial focussing without introducing radial defocussing, which would depend on the magnitude of radial focussing provided by the accelerating mechanism.

We shall discuss guide field focussing mechanisms in section 2, and accelerating mechanisms in section 3.

2. GUIDE FIELD FOCUSSING MECHANISMS :

2.1 Statement of the problems -

First emphasis in considering guide field focussing mechanisms is placed on the problem of ensuring as little first-order coupling between radial and axial motion. Until the development of a strong guide field focussing mechanism only one guide field configuration was known to provide such uncoupled motion. With the generalisation of guide field focussing theory to include strong focussing, further study has shewn that another field configuration may be used in special circumstances. The necessity for uncoupled transverse oscillations, and how it may be obtained, is discussed in 2.2.1, in the adiabatic approximation.

The next choice open to the designer is the strength of transverse focussing as related to the cost of the 'perfect' magnet and the magnitude of the beam of accelerated particles. These problems are discussed in 2.2.2, again in the adiabatic approximation, and will involve a quantitative treatment of momentum compaction.

These decisions are invariably made in the adiabatic approximation, in which the subsequent detailed examination of stability conditions may also be discussed (2.2.3). The asymptotic behaviour of particle motion in different focussing mechanisms in discussed in 2.3.

2.2 Adiabatic behaviour of guide field focussing mechanisms -

2.2.1 Uncoupled transverse motion: The general equation governing particle motion in the linear approximation is given by (55) and (58). We shall be concentrating here on the 'free' transverse oscillations described by (59). The conditions necessary for the transverse motion to be bounded is given by (79), where A,B are functions of the matrix elements of $[Y]$, which are derived from those of $[A]$ with the aid of (69) or (70).

The necessity for uncoupled transverse motion is directly related to the difficulties of analysis under 'coupled' conditions, and the many possibilities of such coupling providing resonance conditions and consequent instability. It is readily shewn that under uncoupled conditions, i.e. when the transverse motion is separable into two independent oscillatory modes, and $[A]$ expressed thus-

$$[A] = \left[\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array}\right], \qquad (158)$$

where 0 are 2x2 null matrices, and $[L_{1,2}]$ are 2x2 matrices of the form,

$$[L_{1,2}] = \begin{bmatrix} 0 & 1 \\ k^2_{1,2} & 0 \end{bmatrix}, \qquad (159)$$

then $[Y]$ is also expressed as a partitioned matrix operator,

$$[Y] = \left[\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right], \qquad (160)$$

with $[M_{1,2}]$ obtained from the equations -

$$[M_{1,2}] = \exp [L_{1,2}] .\theta \qquad (161)$$

$$= \begin{bmatrix} \cos(k_{1,2}.\theta) & (1/k_{1,2})(\sin k_{12}\theta) \\ -k_{12}\sin(k_{12}\theta) & \cos(k_{1,2}\theta) \end{bmatrix},$$

using Sylvester's theorem, (70). In these circumstances, (79) becomes-

$$-1 \leq \cos(k_{1,2}\theta) \leq 1, \qquad (79')$$

$$(\cos k_1\theta - \cos k_2\theta)^2 \geq 0,$$

when instability may only arise, in the presence of small perturbations, when $\cos k_1\theta \approx \cos k_2\theta$.

The advantages of uncoupled transverse modes, and methods through which it may be achieved are now obvious, i.e. $[A]$ in (58) has to be reduced to the form given by (158).

One obvious method is to have $n_{zz} = n_{rr} = 0$ in (58), making $k_1^2 = -(1-n_{zr})$, $k_2^2 = -n_{zr}$ in (159), and the resulting field configuration is that first introduced by Kerst for guide field focussing in betatrons, and subsequently used in all weak-focussing cyclic accelerators, and most of the strong focussing accelerators being planned today. We shall approach this subject in a more general manner by writing (59) in the transformed vector space -

$$[\bar{x}] = [S][x] , \tag{162}$$

when it becomes transformed into -

$$[d\bar{x}/d\theta] = [\bar{A}][\bar{x}] , \tag{59'}$$

where,
$$[\bar{A}] = [S][A][S]^{-1} . \tag{163}$$

The 4x4 transformation matrix operator, [S], is orthogonal, and consequently has the form:

$$[S] = \begin{bmatrix} \alpha E & \beta E \\ -\gamma E & \alpha E \end{bmatrix} , \text{ with } [S]^{-1} = \begin{bmatrix} \alpha E & -\beta E \\ \gamma E & \alpha E \end{bmatrix} , \tag{164}$$

where
$$\alpha^2 + \beta\gamma = 1, \ [E] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{165}$$

Using these expressions for [S], $[S]^{-1}$, we have from (163),

$$[\bar{A}] = \begin{bmatrix} a\alpha^2 + b\alpha\gamma + c\alpha\beta + d\gamma\beta & -a\alpha\beta + b\alpha^2 - c\beta^2 + d\alpha\beta \\ -a\alpha\gamma - b\gamma^2 + c\alpha^2 + d\alpha\gamma & a\beta\gamma - b\alpha\gamma - c\alpha\beta + d\alpha^2 \end{bmatrix} ,$$

where
$$a = \begin{bmatrix} 0 & 1 \\ k_1^2 & 0 \end{bmatrix}, \ d = \begin{bmatrix} 0 & 1 \\ k_2^2 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 0 & 0 \\ k_3^2 & 0 \end{bmatrix}, \ c = \begin{bmatrix} 0 & 0 \\ k_4^2 & 0 \end{bmatrix} .$$

The conditions under which $[\bar{A}]$ is of the form in (158) may therefore be written -

$$k_3^2\alpha^2 + (k_2^2 - k_1^2).\alpha\beta - k_4^2\beta^2 = 0 , \tag{166a}$$

$$k_4^2\alpha^2 + (k_2^2 - k_1^2).\alpha\gamma - k_3^2\gamma^2 = 0 . \tag{166b}$$

Equations (165) and (166) provide 3 equations to obtain 3 unknowns in terms of the constants k_1, k_2, k_3, k_4. It appears therefore that there are many possible field configurations which will give rise to uncoupled transverse motions in a transformed coordinate system defined by [S].

Certain relations between k_1, k_2, k_3, k_4 must, however, be satisfied if a Maxwellian field configuration is to be possible. These restrictions are -

$$k_2^2 - k_1^2 = 1 - 2n_{zr} \quad (167a)$$

and $$n_{rr} + n_{zz} = k_3^2 - k_4^2$$

$$= 0 \text{ in the 2-dimensional approx.}$$

$$=(B_r/B_z) \text{ in the axially symmetric approximation.} \quad (167b)$$

In the obvious case mentioned earlier, we have -

$$k_3 = k_4 = 0, \text{ giving } \alpha = 1, \quad \beta = \gamma = 0,$$

and [S] is the 4X4 unit matrix operator. One notices from (167b) that $B_r = 0$ in the plane of symmetry.

Another very interesting case is that obtained

when $$k_2^2 - k_1^2 = 0, \quad k_3^2 - k_4^2 = 0, \quad (168a)$$

giving $$\alpha^2 = \beta^2 = \gamma^2 = 1/2 . \quad (168b)$$

In this case the sub-matrices in (158) may be written-

$$[L_1] = \begin{bmatrix} 0 & 1 \\ (k_1^2+k_3^2 & 0 \end{bmatrix}, \quad [L_2] = \begin{bmatrix} 0 & 1 \\ (k_1^2-k_3^2) & 0 \end{bmatrix}. \quad (168c)$$

We may characterise the transformation operator, S , as a two-dimensional rotation operator, in which case it may be represented by a single parameter, ϵ, defined by

$$\alpha = \cos\epsilon, \quad \beta = \gamma = \sin\epsilon . \quad (169)$$

In the special case we are considering, S clearly rotates the transverse coordinate axes through 45°, in which two orthogonal planes the transverse motion is independent subject to (168). A coordinate system closely similar to this has been suggested by Sigurgeirsson of CERN[36], except that in his system $n_{zr} = 0$, so that $k_2 = 0$, $k_1^2 = -1$. Strictly, as seen from (168a), this system does not result in completely separable transverse motion, but in the limit of very strong focussing $k_3^2 = k_4^2 \gg k_1^2$, so that the coupling is small, being entirely due to the centrifugal force term in (45). The transverse motion is clearly separable, however, if the field configuration is altered so that $n_{zr} = 1/2$, instead of zero, and $n_{zz}=n_{rr}\neq 0$.

The method outlined here will enable one to discover many field confugurations which give rise to uncoupled transverse motion. It has been the purpose here not to discuss the merits and demerits of such different configurations, but to emphasize the importance of having uncoupled transverse motion, and to shew that, from the point of view of analysing orbital stability, all field configurations giving uncoupled motion are equivalent. This will allow us to discuss quite generally in this chapter the stability of uncoupled motion in 'perfect' accelerator magnets, and leave until Chapter Three the discussion of coupling instabilities in 'imperfect' magnets and to Chapter Five the relative merits of different field configurations giving such uncoupled motion.

2.2.2 Strength of Focussing and Momentum Compaction:

We turn now to a discussion of guide field focussing mechanisms as related to the annular aperture (defined as the ratio of the half width of the annular region containing particle orbits to its average radius) of a 'perfect' magnet, and the energy and intensity of the beam of injected particles. We are concerned here with relating the desirable characteristics of the focussing mechanisms so far discussed with the practical problem of injecting a charged particle beam of finite magnitude with a finite angular and momentum spread, into an 'imperfect' vacuum over a finite time interval. The conditions under which focussing may be obtained do not concern us here; we assume this now and discuss it further in 2.2.3. The strength of focussing and the extent to which it solves the problems mentioned above will be discussed here, to form a brief introduction to the general Initial Value problem analysed in greater detail in Chapter Four.

The transverse oscillations resulting from guide field focussing may be written as a Fourier series -

$$x(\theta) = \Sigma_m A_m \cos(m\theta+\gamma_m).$$

In general, however, the amplitude of one component, say $m = m'$, is likely to be predominant, and $x(\theta) = A_{m'}\cos(m'\theta+\gamma_{m'})$. Electrons (or protons) may be injected, or deflected by elastic collisions with atoms of the gas in the vacuum chamber, at an angle, $\pm\eta$, at $\theta = 0$, where

$$\pm\eta = \mp Am'.\sin(\gamma_{m'}) = \mp Am'(1 - (\Delta r/A)^2)^{1/2}, \quad (170)$$

and Δr is the amplitude of the oscillation at the moment of collision or injection. The maximum amplitude of the oscillation occurs at the azimuth-

$$\theta = n\pi - (\sin^{-1}(\eta/Am'))/m' \simeq n\pi - \eta/Am'^2, \ \eta << 1, \quad (171)$$

and $n=0,1,\ldots$, with its magnitude defined by the equation-(170).

If we confine particle motion within the limits, r' and r'', $r'' > r_e > r'$, the amplitude Δr, and angular deviation, η, must lie within the area in phase space defined by the equations -

$$\begin{aligned}(\Delta r/r_e)^2 + (\eta/m')^2 &= (r'' - r_e)^2, \ \Delta r > 0, \\ &= (r_e - r')^2, \ \Delta r < 0.\end{aligned} \quad (172)$$

From such approximate analysis (which is exact only in the weak focussing limit) it is clear that the maximum permissable η for a given aperture is proportional to m', the fundamental frequency of the oscillation, or from (170),

$$|A| \propto |\eta/m'| \quad (170a)$$

Strong focussing is merely a method whereby the fundamental frequency of stable betatron oscillations is increased. From the analysis presented earlier(Chapter One,section 2), it is clear that this is achieved by allowing the parameters, k, in $[L_{1,2}]$ to have a large amplitude, high frequency periodicity in θ. For example, if we assume that 2N magnet sectors provide the k in $[L_{1,2}]$ with a oscillatory component, sinusoidal and of period $2\pi/N$, then a change in the independent variable converts the betatron eqns. to the form (85), and the eventual solution written (cf(107))

$$x = \sum_{r=-\infty}^{+\infty} c_{2r}.\cos N(2r+\beta)\frac{\theta}{2} ; 0 < \beta < 1,$$

and β is obtained in terms of the k by using the transcendental equation (106). Thus, if $\beta = 1/2$, and we take c_0 as the predominant coefficient in the series, then $x = c_0.\cos(N\theta/4)$, and the period of the oscillation is $8\pi/N$, i.e., the oscillation repeats itself after 8 sectors, and the amplitude is reduced by a factor inversely proportional to N. The magnitude of this parametric variation is related to its period by stability conditions to be described in 2.2.3.

Such motion is characteristic of particle behaviour where the parameters, k, are arbitrary periodic functions of θ . A more general treatment of the behaviour of particles under stable conditions has been described briefly in 2.2, using the operational rather than the analytical approach. It is clear that if $[Y]$ is in the form -

$$[Y] = \begin{bmatrix} \cos(k\theta) & (1/k)\sin(k\theta) \\ -k\sin(k\theta) & \cos(k\theta) \end{bmatrix} \qquad (161a)$$

then, associated with the vector matrix equation,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [Y]\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix},$$

is the invariant bilinear form,

$$U = x_1^2 + (x_2/k)^2 = x_1^{0^2} + (x_2^0/k)^2 = \text{constant}, \qquad (173)$$

whose principal axes form the frame of reference. The maximum displacement, $(x_1)_{max} = kU$, and occurs when $x_2 = 0$. The expression for $[Y]$ given in(161a) is however, associated with a matrix $[A]$ independent of θ . For an arbitrarily varying, but still basically periodic $[A]$ we may obtain an approximate $[Y]$ composed of the products $[Y_1][Y_2]$ .., each matrix being associated with an azimuthal region, infinitesimal if necessary, over which $[A]$ is substantially independent of θ, or over which an average value for the θ dependent parameters in A over that range may be chosen. This is the approximation discussed on page 27(I). This composite $[Y]$ satisfying (61) will give rise to a general bilinear form of the type (81). However, a co-linear transformation of the matrix, $[Y]$, by $[S]$,

$$[\bar{Y}] = [S]\,[Y]\,[S]^{-1} , \tag{174}$$

leaves (61) invariant with $[\bar{x}] = [S]\,[x]$, (174a)
and may reduce Y to the form (161a), so that the associated bilinear form is referred to its principal axes. If we use S matrices of the form (164), the following parametrisation of Y is obtained-

$$[Y] = \begin{bmatrix} \alpha & -\beta \\ \gamma & \alpha \end{bmatrix} \begin{bmatrix} \cos(k\theta) & (1/k).\sin(k\theta) \\ -k.\sin(k\theta) & \cos(k\theta) \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\gamma & \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta) + \alpha(\beta k - \frac{\gamma}{k})\sin(k\theta) ; & (\frac{\alpha^2}{k} + \beta^2 k)\sin(k\theta) \\ -(\frac{\gamma^2}{k} + \alpha^2 k)\sin(k\theta) ; & \cos(k\theta) - \alpha(\beta k - \frac{\gamma}{k})\sin(k\theta) \end{bmatrix} \tag{175}$$

This parametrisation may be simplified to the following -

$$[Y] = \begin{bmatrix} \cos(k\theta) + \alpha\sin(k\theta) & \beta'\sin(k\theta) \\ (1+\alpha^2)/\beta'^2 \sin(k\theta) & \cos(k\theta) - \alpha\sin(k\theta) \end{bmatrix} \tag{175a}$$

by having $(\beta k - \frac{\gamma}{k}) = 1$, $(\frac{\alpha^2}{k} + \beta^2 k) = \beta'$,

or to

$$[Y] = \begin{bmatrix} \cos(k\theta) - \sin(k\theta)\sinh\psi\sin2\chi ; & \sin(k\theta)(\sinh\psi\cos2\chi + \cosh\psi) \\ \sin(k\theta)(\sinh\psi\cos2\chi - \cosh\psi) ; & \cos k\theta + \sin k\theta\sinh\psi\sin2\chi \end{bmatrix} \tag{175b}$$

where $\alpha(\beta k - \frac{\gamma}{k}) = -\sin2\chi\sinh\psi$,

$$(\alpha^2/k) + \beta^2 k = \sinh\psi\cos2\chi + \cosh\psi ,$$

$$-\alpha^2 k - (\gamma^2/k) = \sinh\psi\cos2\chi - \cosh\psi .$$

If the composite transformation operator is put in the form of any of the 3 alternatives provided by (175), the S matrix is immediately determined, and the bilinear form readily expressable relative to its principle axes. An alternative method to determine the set of parameters as functions of θ, is to use the differential equations (84).

In terms of the parameters α and β', these equations are

$$\partial\beta'/\partial\theta = -2\alpha \,,\ \partial\alpha/\partial\theta = k^2\beta' - (1+\alpha^2)/\beta' \,, \quad (176)$$

with

$$F = (1+\alpha^2)/\beta' \,,\ G = \beta \,,\ H = \alpha, \quad (176a)$$

which are consistent with (84) if $(1-n_{zr})$ or n_{zr} is replaced by k^2. Maximum displacement is given by,

$$(x_1)_{max} = \beta' U. \quad (176b)$$

In terms of the parameters ψ, χ these equations are -

$$2\sin 2\chi\left(\frac{\partial\chi}{\partial\theta} + 1\right) = (\partial\psi/\partial\theta)(\coth\psi\cos 2\chi + 1),$$
$$(\partial\psi/\partial\theta) = k^2\sin 2\chi \,, \quad (177)$$

$$F = e^{-\psi}\cos^2\chi + e^{\psi}\sin^2\chi \,,\ G = e^{-\psi}\sin^2\chi + e^{\psi}\cos^2\chi \,,$$
$$H = -\sin\chi\cos\chi\,(2\sinh\psi). \quad (177a)$$

From the operational point of view it is more convenient to use the parameters, α,β', while the parameters, ψ, χ are superior from the geometrical point of view as we readily visualise χ as the angle through which phase space is rotated, and ψ as a measure of the stretching or contraction of the axes. The maximum transverse displacement then occurs when $\chi = 0$, and ψ is a maximum, for a given value of U. Again we notice from (176b) that, given U, the maximum displacement is proportional to β', where, from (175a),

$$\beta' = \beta + (1/k) \,,\ k \propto n^{1/2}. \quad (178)$$

Thus, the presence of a finite β tends to reduce the strength of focussing for a given k. Reasons for the existance of such terms are given in 2.2.3.

We have discussed, at first on a specialised basis, and then on a generalised basis, the strength of focussing as related to the spatial and angular spread of the beam. The causes of such spreading are gas scattering, injection conditions, space charge forces and the presence of radial oscillations accompanying the phase focussing mechanism in synchronous acceleration. This last factor may, however, be regarded as not only reducing the available aperture, but, by introducing a momentum spread in the accelerated beam, as also modifying the focussing strength of the guide field. Momentum spread also exists in the injected beam, and we turn now to a discussion of the effects of momentum spread on the spatial distribution of the beam, and its relation with the focussing streingth of the guide field, that is, the subject - momentum compaction.

Momentum Compaction - By this term we refer to equation (49), where the coefficient α_m relates the momentum spread, x_p, to the average (with respect to θ) radial spread, $\langle x_r \rangle$. We note the presence of α_m in the synchrotron equation (54), which is a measure of the interference of the characteristics of the guide field focussing mechanism with the synchronous acceleration mechanism. It can be seen from (55) and (64) that α_m is closely related to the particular integral of the betatron equation, which may be used to evaluate it in those cases where (62) is soluble and $[Y]$ obtained explicitly in terms of the elements of $[A]$. Before we use this method, however, we must specify more explicitly the influence of a momentum spread x_p on the parameters in the betatron radial equation (45), assuming the conventional configuration giving separable transverse motion, i.e. with $n_{zz} = 0$.

Assume that the instantaneous orbit, defined as the orbit whose radius of curvature, r_e, satisfies the equation, $eB(r_e)r_e = p_e$, of a particle of momentum p_e is completely circular. In a 'perfect' magnet which we are considering the field along this radius, $B(r_e)$ is constant, i.e. f_o in (53) is zero. At any azimuthal angle, $\theta = \theta_o$ say, the radius of curvature, r_i of the instantaneous orbits of particles of different momenta $p = p_e(1 + x_p)$ must satisfy

$$\overset{o}{x}_r = (r_i - r_e)/r_e = \overset{o}{\alpha}_m \cdot x_p \; ; \; \overset{o}{\alpha}_m = (1 - n(\theta_o))^{-1} \quad (179)$$

In fact, if n were independent of θ, the particular integral of (55) would be exactly

$$x_r(\text{all } \theta) = \alpha_m x_p = x_p/(1 - n). \quad (180)$$

The motion of particles of momentum $p = p_e(1 + \varkappa_p)$, about $r = r_e$, the instantaneous orbit radius of particles of momentum p_e, is described by the inhomogeneous equation-

$$d^2\varkappa_r/d\theta^2 + (1 - n_{zr}(p))\cdot\varkappa_r = \varkappa_p . \quad (45')$$

By having $f_o^z = -\varkappa_p$, in (45) we introduce a field inhomogeneity at $r = r_e$ which has the same effect on a particle of momentum p_e as if its momentum were $p_e(1 + \varkappa_p)$, and its motion described about $r=r_e$. It is worth noticing that if we have $f_o^z = \varkappa_p \neq 0$, we describe the motion of particles of momentum $p_e(1+\varkappa_p)$ about its <u>own</u> instantaneous orbit $r_i = r_e(1 + \varkappa_p/(1-n))$, by altering the field at $r=r_e$ to $B_z(r_e)(1 - \varkappa_p)$. In the first instance, however, we introduce a field inhomogeneity of opposite polarity,

$$B_z(r_e)(1 + \varkappa p).$$

The effective field index, $n(p)$ in this modified field is

$$n_{zr}(p) = (r_e/B_z(r_e))(B_z/r)(1 + \varkappa p) = (1+\varkappa_p)\cdot n_{zr}(p_e). \quad (181)$$

Thus, the field index, $n_{zr}(p)$ associated with particles of increased momentum is proportionately increased, i.e

$$\delta n/n = \varkappa_p, \text{ and } n_{zr}(p) = (1+\varkappa p)n_{zr}(p_e). \quad (181')$$

For n_{zr} = constant therefore, the momentum compaction relation (49) is a transcendental of the form -

$$\langle\varkappa r\rangle = \alpha_m\varkappa_p , \quad \alpha_m = (1 - n(p_e)(1 + \varkappa p))^{-1}, \quad (49')$$
$$= (1 + \varkappa_r n_{zr}(p_e)) / (1 - n_{zr}(p_e)).$$

When $n(p_e)$ is azimuthally dependent, and if the instantaneous orbit of a particle of momentum $p = p_e$ is exactly circular, then the instantaneous orbits of particles with different momenta are not circular, i.e, $r_i = r_i(\theta)$. The momentum compaction relation is still, however, obtained from the particular integral of (45'). The analytical result for $n_{zr}(p_e)$ = constant is (49'), which is an obvious particular integral of (45'). The operational approach using the equation (64) is less obvious, and is described briefly in the following paragraph, in a form from which extensions to more complicated cases may readily be made.

Presuming stable conditions, the solution Y of (62) may be put in the form

$$[Y(\theta)] = \begin{bmatrix} \cos k\theta & (1/k)\sin k\theta \\ -k.\sin k\theta & \cos k\theta \end{bmatrix}, \quad k^2 = (1 - n_{zr}(p)),$$

so that $[Y(\theta)][Y(\theta')]^{-1} = [Y(\theta - \theta')]$,

and the latter part of the right hand side of (64) representing the particular integral of (55) put in the form-

$$P\,I = x_p \begin{bmatrix} (1/k)(& (\cos k\theta)/k & + & \sin k\theta \\ & \cos k\theta & - & (1/k)\sin k\theta \end{bmatrix}.$$

The maximum and minimum of $x_r = x_1$ occurs when $x2 = 0$, i.e. $\tan k\theta = k$, or $\cos k\theta = \pm(1 + k^2)^{-1}$, giving

$$\langle\alpha 1\rangle = x_p/k^2 \quad . \qquad (49'')$$

The analytical approach, when $n = n(\theta)$, for obtaining the particular integral in terms of the eigenfunctions of the homogeneous equation is a standard one, and is outlined briefly in section 3 of PAPER N.

Considerable space may be devoted to evaluating the momentum compaction coefficients in a variety of field configurations. The standard cases treated by others are those in which the field characteristics change discontinuously from one sector to the next, for treating which the operational approach is ideally suitable. The number of possible arrangements of such discontinuous changes is large, and must be governed by practical considerations and, ultimately, by the net focussing provided for radial and axial oscillations under 'perfect' conditions. The coefficients associated with 'smooth' changes in the field gradient have not received any consideration. In the next section we shall discuss the stability conditions associated with both 'smooth' and 'discontinuous' field gradient changes, and obtain simultaneously the momentum compaction coefficients appropriate to the conditions.

It suffices here to point out that to a first approximation a weak azimuthally dependent perturbation of the field gradient index, n, does <u>not</u> modify the 'constant n' momentum compaction relation (49'). However, as is shewn in PAPERS L and N, accompanying this monograph, the presence of any field inhomogeneity contributing to f_0^z in (35a) <u>does</u> alter the momentum compaction relationship. Thus, if we assume that angular velocity is conserved and,

$$f_0^z(\theta) = \sum_{j=1}^{\infty} \alpha_j \cos(j\theta + \gamma j), \tag{182}$$

then (49') is modified to -

$$\langle \alpha_r \rangle = \alpha_m x_p + \sum_{j=1}^{\infty} \alpha_j^2\, n(1+x_p)/2(\ 1 - n(1+x_p)\ )(j^2 - 1 + n(1+x_p))$$

If we assume linear velocity to be conserved then this relation has to be modified to read - (183)

$$\langle \alpha r \rangle = \alpha_m x_p + \sum_{j=1}^{\infty} \alpha_j^2 (2 - n(1+x_p))/2(1-n(1+x_p))(j^2 - 1 + n(1+x_p)).$$

The conditions which make valid such particular integrals are -

$$x_r << 1,\ (\alpha_j^2 / j^2)_{j\neq 0} << 1,\ \ j^2 \not\approx 4(1 - \langle n \rangle). \tag{184}$$

This last condition in (184) implies the absence of resonance conditions, under which circumstances particle motion is quite different; these conditions are discussed in PAPERS A,B,D and K, and later in Chapter Three. For $x_p << 1$, the additional terms in (183) are very slightly dependent on x_p, and may be considered as constant additives to $\langle \alpha_r \rangle$ for particlesof all momenta, i.e. the linear relationship beween $\langle \alpha_r \rangle$ and x_p is further distorted.

The effect of momentum compaction on injection and acceleration conditions is quite considerable. In those cases where $|n| >> 1$ it means that particle beams with a considerable momentum spread may be confined in a comparatively small radial aperture. This possibility may give rise to stable focussing, as will be shewn in the following section, only in the presence of a strong focussing, alternating gradient field, when $\langle n \rangle$ may be considerably larger than unity, and of either sign. This possibility has been cleverly exploited in a design of strong focussing accelerators known as FIXED-FIELD ALTERNATING GRADIENT(FFAG)

SYNCHROTRON. The average field $\langle B_z(r)\rangle$ may be written -

$$\langle B_z(r)\rangle = \langle B_z(r_0)\rangle(1 - \langle n\rangle x_r + \langle n\rangle(\langle n\rangle+1)x_r^2 - \langle n\rangle(\langle n\rangle+1)(\langle n\rangle+2) + \ldots\ldots$$

$$\doteq \langle B_z(r_0)\rangle(1 + \langle n\rangle x_r)^{-1}, \text{ if } \langle n\rangle x_r \gg 1, \tag{185}$$

where

$$\langle B_z(r)\rangle = \langle B_z(r_0)\rangle(r/r_0)^{-\langle n\rangle},$$

$$\text{or} \quad \langle B_z / r\rangle_{r=r_0} = \langle B_z(r_0)\rangle\langle n\rangle/r_0 . \tag{186}$$

(185) may be rewritten, $x_p - x_r = -\langle n\rangle x_r$,

giving

$$x_p = (1 - \langle n\rangle).x_r . \tag{187}$$

The FFAG Synchrotron differs in one important aspect from the more conventional cyclic accelerators. Particles are accelerated by a synchronous mechanism in which the changing frequency of the RF accelerating field forces their orbits to move radially into an increasing average guide field, i.e.the guide field <u>at the orbit</u> is implicitly time-dependent (cf. equation (2)). Two forms of such accelerators have been proposed[8] -

(a) The Mark I accelerator using a reversed d.c. field to provide the alternating field gradients, and

(b) The Mark V accelerator using spirally ridged poles to provide the alternating field gradients with d.c fields of the same polarity throughout, thereby reducing the circumference of the magnet.

This synchrotron is only a development of the general principles of accelerator design already outlined in which the momentum compaction aspect of strong focussing has been emphasized and exploited in order to make possible the use of a d.c. magnet as a strong focussing synchrotron. It has already been pointed out that the design of a d.c. magnet is much simpler than that of a corresponding a.c. magnet due to the many error field sources associated with the latter. In the a.c. synchrotron the effect of strong focussing on momentum compaction is incidental, and not necessary, and the coefficient α_m is generally larger than that associated with FFAG synchrotrons. This means that the transition energy, E_{tr}, at which phase instability occurs(cf (54)),

$$E_{tr}^2 = E_o^2(1 + \frac{N_1 L}{2\pi r_e})/\alpha_m, \qquad (188)$$

is reduced, making it more likely to lie below the maximum energy to which <u>protons</u> are accelerated. In electron synchrotrons the much smaller rest energy ($E_o \approx 0.5$ MeV) makes it convenient to have as small a transition energy as possible, in order to make easier injection at energies above it. From this point of view a strong focussing electron synchrotron is preferably an a.c. synchrotron, and a strong focussing proton synchrotron preferably a d.c. synchrotron.

The effect of momentum compaction on injection conditions is considerable, and will be dealt with in Chapter Four. A very small momentum compaction coefficient, while increasing the magnitude of the current injected into a given aperture, makes considerably more difficult efficient injection in a.c. synchrotrons and betatrons where the equilibrium orbit is substantially constant throughout acceleration.

2.2.3. Stability Conditions for Transverse Focussing:

Using the operator approabh, and assuming uncoupled transverse oscillatory modes, the stability conditions are expressed, using (79') and (161), thus-

$$-1 < \cos \nu_1 = (1/2)\mathrm{Tr}\, M_1 < 1,$$
$$-1 < \cos \nu_2 = (1/2)\mathrm{Tr}\, M_2 < 1, \qquad (79'')$$

where M_1 and M_2 are products of 'constant parameter' matrix operators. Where M_1 and M_2 describe the effects of field configurations on particle orbits exactly, that is, when parametricc excitation is piece-wise constant with a finite number of discontinuities, the stability conditions is readily obtained in the form (79"). The stable region in k_1,k_2 space is then defined by the curves obeying the equations -

$$|\cos \nu_1| = |\cos \nu_2| = 1. \qquad (189)$$

These conditions are quite general, and apply to all cases of uncoupled transverse modes referred to their appropriate principal axes.

When the number of discontinuities is infinite we have the limiting case of smooth changes in the parameter, k, for which cases it is difficult to state the stability conditions exactly, either analytically or using the operator techniques, except in very special cases, such as when the azimuthal dependence of $k_{1,2}$ is expressable as a circular function or as a 'sawtooth' function. In general, however, the operator methods are far more flexible, and some attention has been given by others to determine how to approximate to 'smooth' parametric variations with piece-wise constant parametric variations including a small number of discontinuities, the fewer the better. In the following discussion of stability conditions on a quantitative basis we shall be concerned with this problem

2.2.3.1 Field Configurations with a finite number of discontinuities -

(a) No discontinuities - weak focussing.

In this case we have,

$$\left[M_{1,2}(2\pi)\right] = \begin{bmatrix} \cos(2k_{1,2}\pi) & (1/k_{1,2})(\sin 2k_{1,2}\pi) \\ -k_{1,2}\sin(2k_{1,2}\pi) & \cos(2k_{1,2}\pi) \end{bmatrix},$$

giving,

$$\left|\cos(2k_{1,2}\pi)\right| < 1 \qquad (190)$$

as the stability condition. With $k_1 = (1-n_{zr})^{1/2}$, and $k_2 = (n_{zr})^{1/2}$, (190) can be rewritten in the form,

$$0 < n_{zr} < 1, \qquad (190')$$

in which case it is the standard Kerst stability condition for transverse oscillations in a betatron.

(b) Weak focussing with straight sections.

The transformation matrix for a straight section is- $\begin{bmatrix} 1 & \theta_s \\ 0 & 1 \end{bmatrix}$, θ_s being the angle in radians subtended by the straight section at the centre of the magnet. If the magnet is composed of straight sections of dif-

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-ferent angular lengths, θ_{s1} and θ_{s2} separating weak focussing curved sections of angular length θ_3 and θ_4, with $\theta_{s1}+\theta_{s2}+\theta_3+\theta_4 = 2\pi/N$, $N = 1,2,3,\ldots$, then the overall transformation matrix operator, $[Y(2\pi/N)]$ is written -

$$[Y(2\pi/N)] = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}, \quad \text{where -}$$

$$Y_{11} = \cos k_{1,2}\theta_3 \cos k_{1,2}\theta_4 - \sin k_{1,2}\theta_3 \sin k_{1,2}\theta_4 - k_{1,2}\theta_{s1}\sin k_{1,2}\theta_4 \cos k_{1,2}\theta_3,$$

$$Y_{12} = (\sin k_{1,2}\theta_3 \sin k_{1,2}\theta_4)(-\theta_{s2}) + (\theta_{s2})(\cos k_{1,2}\theta_3 \cos k_{1,2}\theta_4) + (k_{1,2}^{2}-\theta_{s2})(\sin k_{1,2}\theta_3 \cos k_{1,2}\theta_4) + (k_{1,2}-k_{1,2}\theta_{s2})(\sin k_{1,2}\theta_4 \cos k_{1,2}\theta_3)$$

$$Y_{21} = -k_{1,2}\sin k_{1,2}\theta_3 \cos k_{1,2}\theta_4 - k_{1,2}\sin k_{1,2}\theta_4 \,(\cos k_{1,2}\theta_3 - k_{1,2}\theta_{s1}\sin k_{1,2}\theta_3),$$

$$Y_{22} = \cos k_{1,2}\theta_3 \cos k_{1,2}\theta_4 - \sin k_{1,2}\theta_3 \sin k_{1,2}\theta_4 (1 - k^2_{1,2}\theta_{s1}\theta_{s2}) - k_{1,2}\theta_{s2}\sin k_{1,2}\theta_4 \cos k_{1,2}\theta_3 - k_{1,2}\sin k_{1,2}\theta_3 \cos k_{1,2}\theta_4(\theta_{s1}+\theta_{s2}).$$

For $N=1$, $\theta_{s1}=\theta_{s2}=\theta_4 = 0$, $\theta_3 = 2\pi$, this reduces to

$$|\cos(2\pi k_{1,2})| < 1,$$

as in case (a) above. It is interesting to notice here that this inequality is violated if $k_{1,2} = 0,1,2,3,\ldots$, the two smallest integers defining the weak focussing boundary values, $n = 0,1$. In addition, however, the inequality is violated also when $k_{1,2} = 1/2, 3/2, 5/2, \ldots$. All values but the first give conditions violating (190'). The first value, $k_1 = 1/2$, gives <u>inherent</u> radial instability at $n_{zr} = 3/4$, and <u>inherent</u> axial instability at $n_{zr} = 1/4$. These are the well-known <u>parametric resonances</u> in weak-focussing guide fields which we discuss in greater detail in the following Chapter Four, or the π, or half integral resonances of strong focussing theory. By 'inherent' we are stressing that such instability does not lead to anti-damping of the guide field induced oscillations in the absence of a definitive periodicity of 2π, as will be shewn now.

For the case where $\theta_{s1} = \theta_{s2} = (2\pi/N).a$, and $\theta_3 = \theta_4 = (2\pi/N)(1-a)$, $N = 1,2,3,\ldots$, we have -

$$[Y(2\pi/N)] = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

These *operators* may be put in the form (175a) if we define new parameters, $k'_{1,2}$, α', β' satisfying the equations:

$$\cos(2\pi k'_{1,2}/N) = \cos(2\pi k_{1,2}(1-a)/N) - (\pi k_{1,2}a/N)\sin(2\pi k_{1,2}(1-a)/N),$$

$$\alpha'\sin(2\pi k'_{1,2}/N) = (\pi k_{1,2}a/N)\sin(2\pi k_{1,2}(1-a)/N),$$

$$\beta'\sin(2\pi k'_{1m2}/N) = (2\pi a/N)\cos(2\pi k_{1,2}(1-a)/N) + (1/k_{1,2})\sin(2\pi k_{1,2}(1-a)/N).$$

For $a \ll 1$, an approximate solution is,

$$k'_{1,2} = k_{1,2}(1-a), \quad \alpha' = (\pi k_{1,2}a/N), \quad \beta' = (1/k_{1,2});$$

and the stability condition becomes -

$$|\cos(2\pi k_{1,2}(1-a)/N)| < 1 \qquad (190'')$$

leading to (190').

The momentum compaction relationship is derived from

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \int_0^\theta \left[Y(\theta-\theta')\right]\begin{bmatrix} 0 \\ x_p \end{bmatrix} d\theta'$$

$$= x_p \begin{bmatrix} (1/k)(\sin k'\theta - (\cos k'\theta)/k') \\ \cos k'\theta(1-\alpha/k') + \sin k'\theta(\alpha + 1/k') \end{bmatrix}$$

Extrema in x_1 occur when $x_2 = 0$, i.e., $\tan(k'\theta) = (\alpha - k')/(1+\alpha k')$, giving, in this approximation,

$$\langle x_1 \rangle = (1/k_1 \; k'_1)x_p = \left[(1+a)/(1-n_{zr}(1+x_p))\right]x_p \qquad (191)$$

These results shew that while the magnitude of the stability region is not changed to the order of approximation considered here, the frequency of both radial and axial oscillations is reduced, so reducing the focussing provided by the guide field, and the radial spread for a given momentum spread is also increased proportionally to the increase in circumferense due to the presence of straight sections. It is also clear, from (190") that the parametric resonance mentioned above occurs <u>only</u> if $N = 1$, i.e. when there is a definite 2π periodicity provided by the presence of only <u>one</u> straight section, *and* when -

$$(1-n_{zr})(1-2a) = 1/4 \quad \text{(radial case)}$$
$$(n_{zr})(1-2a) = 1/4 \quad \text{(axial case)}. \qquad (192)$$

In these circumstances we may regard the straight section as providing a single 'negative' bump in n_{zr}, so that a similar resonance instability must be expected for a <u>single</u> 'positive' bump in n_{zr}. If $N \geq 2$ in (190'), i.e. two or more straight sections of <u>equal</u> length are present, then parametric resonance may not occur for values of $k_{1,2}$ in the range $0 < k_{1,2} < 1$, which is equivalent to (190'). However, if the length of any one section is <u>not</u> equal to the others, then 2π periodicity exists and parametric instability recurs when (192) is satisfied. In its analytical form this problem has been discussed in great detail in Papers A, B, D, K, and will be analysed further in Chapter Three.

(c) <u>Large number of discontinuities - Strong Focussing Limit</u> :

As has already been shewn, strong focussing is obtained by creating guide field conditions which substantially increase the frequency of guide field induced 'free' oscillations, without destroying their boundedness. We shall now analyse stability conditions when the field gradient, n_{zr} alternates between values n_1 and n_2, $n_1 > 0$, $n_2 < 0$, over angular regions $\pi(1+a)/N$ and $\pi(1-a)/N$ respectively, giving an overall periodicity of $2\pi/N$.

The transformation operator in these circumstances is written,

$$[Y(2\pi/N)]_{\text{radial}} = \begin{bmatrix} \cosh A\cos B-(\frac{1}{N^2}-V^2)^{\frac{1}{2}}(U^2-\frac{1}{N^2})^{-\frac{1}{2}}\sinh A\sin B\,; & \ldots \\ \ldots & ;\cosh A\cos B+(U^2-\frac{1}{N^2})^{1/2}(\frac{1}{N^2}-V^2)^{-\frac{1}{2}}\sinh A\sin B \end{bmatrix}$$

$$[Y(2\pi/N)]_{\text{axial}} = \begin{bmatrix} \cosh C\cos D+\{\,iV/U\,\}\sinh C\sin D\,; & \ldots\ldots \\ \ldots\ldots & ;\ \cosh C\cos D-(\,U/iV\,)\sinh C\sin D\,. \end{bmatrix}$$

where $U^2 = n_1/N^2$, $V^2 = n_2/N^2$, $C = -iV\pi(1-a)$, $D = U\pi(1+a)$,

$$A = \pi(1+a)(U^2-(1/N)^2)^{1/2},\quad B = \pi(1-a)(-V^2+(1/N)^2)^{1/2}.$$

The boundaries of stable regions in U - V space are defined by the equations -

$$\cosh \pi(1-a)(-V^2)^{1/2} . \cos\pi(1+a)U \; - \; (\tfrac{1}{2}(U^2+V^2)/U(-V^2)^{1/2}) \sinh \pi(1-a)(-V^2)^{1/2} . \sin \pi(1+a)U = \pm 1, \qquad (193)$$

which defines the region of axial stability, and

$$\cosh \pi(1+a)(U^2-\tfrac{1}{N^2})^{1/2} . \cos\pi(1-a)(\tfrac{1}{N^2} - V^2)^{1/2} \; + \; \tfrac{1}{2}((N^2(U^2+V^2) - 1)/(U^2N^2-1)^{1/2}(1-N^2V^2)^{1/2} . \sinh \pi(1+a)(U^2-\tfrac{1}{N^2})^{1/2} \sin\pi(1-a)(\tfrac{1}{N^2}-V^2)^{1/2} = \pm 1 \qquad (193')$$

which defines the region of radial stability.

The standard case, originally treated by Courant, Livingstone and Snyder,[37] is that in which $a = 0$. In general strong focussing is only obtained if $N >> 1$, so that with these two simplifications, (163) and (163') reduce to -

$$\cosh(\pi U)\cos(\pi i V) + \left(\frac{U^2+V^2}{2iUV}\right).\sin(\pi i V).\sinh(\pi U), \text{ - radial} \qquad (193a)$$

$$\cosh(\pi i V)\cos(\pi U) - \left(\frac{U^2+V^2}{2iUV}\right)\sinh(\pi i V)\sin(\pi U), \text{ - axial.} \qquad (193b)$$

It will be noticed that these last two equations are equivalent if U^2 and $-V^2$ in either is replaced by $-V^2$ and U^2 respectively.

When the idea of strong focussing was first proposed, these boundaries were thought to enclose a region throughout which radial and axial stability existed in a 'perfect' magnet. It is now known,[38] however, that this region is subdivided into smaller stable regions by three types of resonance phenomena :

(a) the forced, or 2π resonances,
(b) the parametric, or π resonances, and
(c) the resonances due to coupling between the transverse oscillatory modes.

As a result, the statement that regions enclosed by(193) provide stable conditions can only apply to perfectly designed <u>and</u> constructed magnets, for while all three resonance phenomena may arise through constructional errors, as will be shewn in Chapter Three, design errors may also introduce such instabilities. Accepting the existence of finite constructional errors as inevitable, the region in U,V space enclosed by (193) has not the significance it possessed originally. The curved boundaries satisfy equations of the form-

$$f(x,y) = \cosh(x)\cos(y) + \left(\frac{x^2-y^2}{2xy}\right)\sinh(x)\sin(y) = \pm 1, \qquad (194)$$

and may be obtained most conveniently by points in x,y space at which they intersect with the straight lines -

$$x = \gamma . y$$

Thus, for $\gamma = i$, both (193a,b) give the following transcendental for x :

$$\cosh(x).\cos(x) = \pm 1 , \qquad (195)$$

and the smallest of the infinite number of values satisfying (195) are-

$$x = 0,\ 1.87,\ 4.67,\ 4.72,\ \ldots\ldots\ldots, \qquad (195')$$

or, where $n_1 = -n_2$, with N representing the number of identical units comprising the magnet (2N is then the least number of magnet sectors necessary), (195') may be written-

$$(n_1/N^2) = 0,\ 0.354;\ 2.21,\ 2.25\ ;\ \ldots\ldots \qquad (195'')$$

The first two values define the corners of the CLS (Courant-Livingstone - Snyder) stable region lying on the diagonal in $n_1,-n_2$ space (points O and A in Fig(Pg.86) The second pair of values define the corners of a second stable region lying on the same diagonal, but at very much larger values of n_1 and $-n_2$. It is shewn as an insert in Fig(Pg 86) from which its much smaller size when compared with the CLS stable region is obvious. While the existence of this, and even smaller stable regions is probably recognized, it has never been explicitly emphasised, despite the much stronger focussing that it affords; the frequency of radial and axial oscillations induced by the guide field is at least 2.5 times that in the CLS region, for the same periodicity in guide field characteristics, i.e. the same value of N. The answer to this lies in the treatment of resonance phenomena and magnet imperfections discussed in Chapter Three. Stable regions for $a \neq 0$ are not depicted.

The momentum compaction coefficient, α_m, is given by -

$$2\alpha_m = \frac{n_1+n_2-2}{(n_1-1)(1-n_2)} + \frac{n_1-n_2}{(1-n_2)(n_1-1)} \cdot F(n_1,n_2),$$

where,

$$F(n_1,n_2) = \frac{(1-n_2)^{\frac{1}{2}}\sin(2\pi/N)(1-n_2)^{\frac{1}{2}} + (n_1-1)^{\frac{1}{2}}\sinh(2\pi/N)(n_1-1)^{\frac{1}{2}}}{\left\{\begin{array}{l}(1-n_2)^{1/2}\cosh(2\pi/N)(n_1-1)^{1/2}.\sin(2\pi/N)(1-n_2)^{1/2} - \\ - (n_1-1)^{1/2}\sinh(2\pi/N)(n_1-1)^{1/2}.\cos(2\pi/N)(1-n_2)^{1/2}\end{array}\right\}}$$

when $a = 0$. When $a \neq 0$, the magnitude of this coefficient is again obtained using (64).

(d) <u>Strong focussing - with straight sections</u>:

The effect of straight sections on the stability region and on the compaction coefficient is again a standard problem whose solution commences with the construction of the overall transformation operator, [Y] , which is straightforward, if tedious, when all parameters are piecewise constant, and all changes are discontinuous. The necessity for straight sections in pactical accelerators with cyclic orbits is obvious, as are the possible methods of introducing them. In all cases where the length of the straight sections are comparable with the circumferential lenth of the curved sections the stability region is substantially reduced in size, and the possibility of π (parametric) and 2π (forced) resonances seriously increased. The optimum situation amd magnitude of straight sections is consequently a very important part of detailed magnet design studies which do not fall within the framework of this monograph. The effect on the stability region of certain particular combinations of straight and curved sections is illustrated in the paper by Wernholm et al.[39] If the stability condition, however complicated, is reduced to the form-

$$-1 < |\cos(N\nu)| < 1, \qquad (196)$$

then the number of betatron oscillations per revolution is

$$Q = (N/2\pi).\nu \qquad (197)$$

2.2.3.2 Smooth parametric variations - the limiting case of an infinite number of infinitesimal discontinuities.

In the limited space available we shall discuss a weak focussing limit when the smooth paramtric variations are in the form of perturbations, the strong focussing limit when the smooth parametric variations are dominant, and the effect in this case of the introduction of straight sections, and finally a brief analysis of the operational approach using various conventions to relate the smooth variations to 'equivalent' discontinuous variations. The particular case of smooth variations has been the subject of analysis in several papers by the author (A,B,C,D, K, L, M, N), and has recently assumed increased importance in cyclic accelerator theory due to the introduction of the FFAG synchrotron with spirally ridged poles.

(a) Weak focussing limit - 'smooth' parametric perturbations.

Parametric perturbations in this limiting case may occur as the result of field and/or field gradient azimuthal variations, due to error fields (Papers D,K,L,M,N), or to space charge forces (PapersA,B,C). From the analysis contained there it is clear that the stability region in the presence of parametric variations of the form -

$$(a - 2qR(\varkappa)) , \quad qR(\varkappa) = \lambda_2 . \cos(2\varkappa), \quad \text{of (96)},$$

or,

$$n(\varkappa) = \langle n \rangle + \alpha_j \cos(j\theta + \gamma_j) = \langle n \rangle + \alpha_j . \cos(2\varkappa),$$

where $a = 4(1-\langle n \rangle)/j^2$, $q = \lambda_2 = (2\alpha_j/j^2)$, radial case,

and $a = 4\langle n \rangle/j^2$, $q = \lambda_2 = -2\alpha_j/j^2$, axial case.

$j = 1,2,3,\ldots\ldots$ $\lambda_2 \ll 1$, is defined by the requirement that ν in (109) is real, or $0 < \beta < 1$ in (107).

It is a consequence of (106) that guide field oscillations under such circumstances are unstable when $a = 0, 1, 4, 9, \ldots$ unless parametric variation coefficient, λ_2 vanishes. In the following equations we define the boundaries of stable and unstable regions in a, λ_2 space, for $\lambda_2 \ll 1$.

$a \simeq 0$: $a = -(\lambda_2^{\,2}/2) + (7\lambda_2^{\,4}/128) - \ldots,$

giving $\langle n\rangle = -(\alpha_j^2/2j^2)$ in the axial case,

and $\langle n\rangle = 1 + (\alpha_j^2/2j^2)$ in the radial case.

$a \simeq 1$: $a = 1 + \lambda_2 - \lambda_2^2/8 + \ldots\ldots$

$a = 1 - \lambda_2 - \lambda_2^2/8 + \ldots\ldots$

giving, $\langle n\rangle = (j/2)^2 \pm (\alpha_j/2)$, in the axial case,

and $\langle n\rangle = \left(1 - \dfrac{j^2}{4}\right) \pm \dfrac{\alpha_j}{2}$, in the radial case.

$a \simeq 4$: $a = 4 - (\lambda_2^{\,2}/12) + \ldots\ldots$

$a = 4 + (5\lambda_2^2/12) - \ldots\ldots$

giving,
$$\begin{aligned}\langle n\rangle &= j^2(\,1 - (\lambda_2^2/48)\,)\\ &= j^2(\,1 + (5\lambda_2^2/48)\,)\end{aligned} \quad \text{in the axial case,}$$

and
$$\begin{aligned}\langle n\rangle &= 1 - j^2(\,1 - (\lambda_2^2/48)\,)\\ &\ 1 - j^2(\,1 + (5\lambda_2^2/48)\,)\end{aligned} \quad \text{in the radial case.}$$

These boundaries are plotted in $\langle n\rangle, \alpha_j$ space, $j=1,2,\ldots$ in the accompanying figure, from which it is clear that this region is subdivided into at least 15 major regions, with each of which may be associated different modes of radial and/axial stability or instability. Thus, we have-

Region A - Axially stable, but radially unstable in all modes.

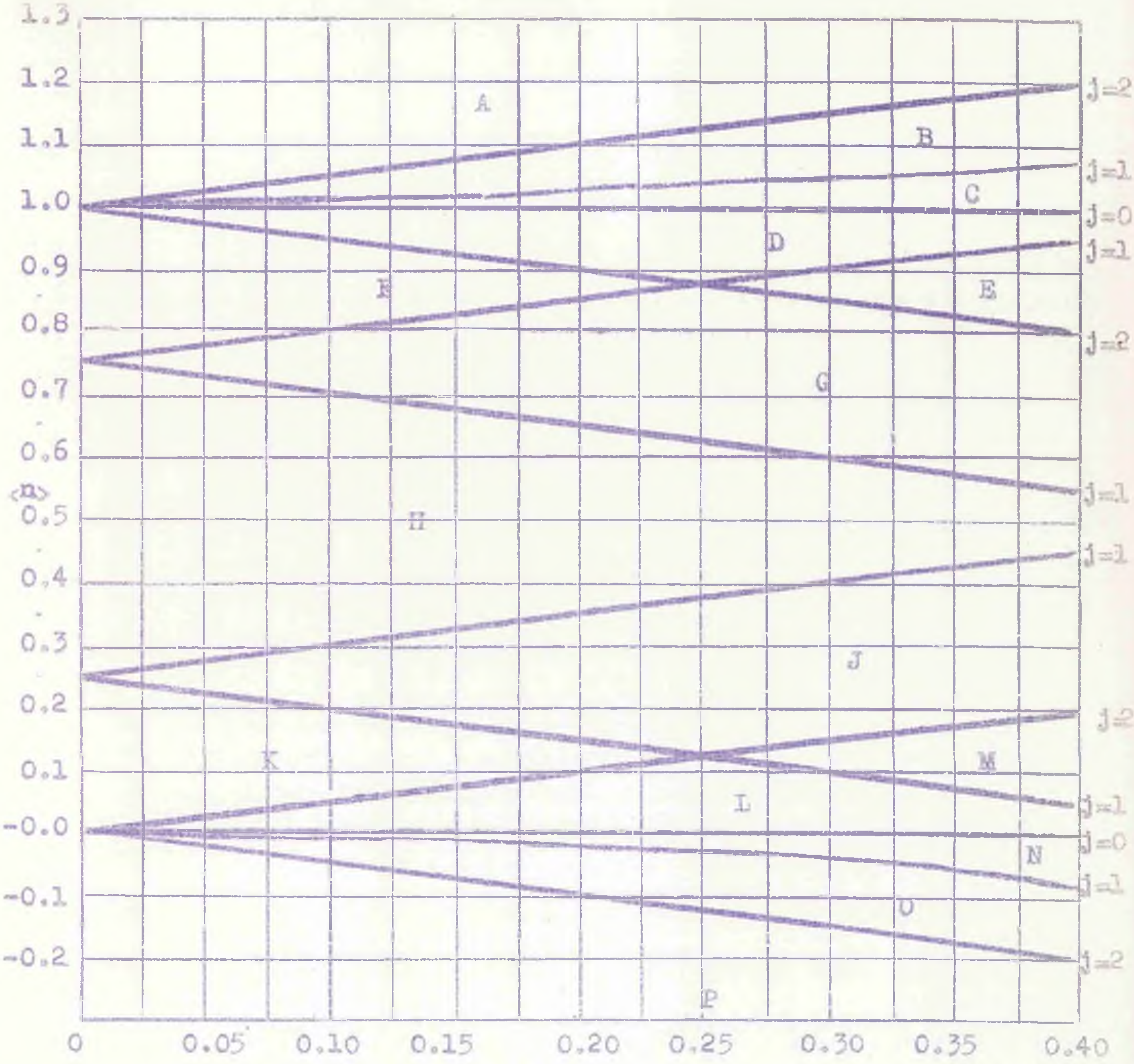
Region P - Radially stable, but axially unstable in all modes.

Region B - Axially stable in $j=0,1$ modes, but not in the $j=2$ mode, and radially unstable in all modes as in A.

Region O - Radially stable in the $j=0,1$ modes, but not in the $j=2$ mode, and axially unstable in all modes as in P.

Region C - Axially stable in $j=0,1$ modes, but not in the $j=2$ mode, as in B, and radially unstable in the $j=0$ mode, but radially stable in the $j=1$ mode. The presence of additional j terms slightly increases the size of this region.

Region N - Radially stable in the $j=0,1$ modes, but not in the $j=2$ mode as in O, and axially unstable in



α_j

RADIAL AND AXIAL STABLE REGIONS IN WEAK FOCUSSING GUIDE FIELDS WITH PARAMETRIC PERTURBATIONS.

It must be noted that the instabilities discussed opposite, and depicted above are parametric resonances. Additional coupling resonances further subdivide the regions above into stable and unstable regions. These phenomena are treated in the following chapter.

the $j=0$ mode, but axially stable in the $j=1$ mode. Again, as in C, the presence of additional j terms slightly increases the size of this region.

Region D - Radially stable in the $j=0,1,2$ modes, and axially stable in the $j=0,1$ modes, but axially unstable in the $j=2$ mode. In the immediate vicinity of $\langle n \rangle = 1$ there is $j=1$ axial instability which we have neglected because of its relatively small magnitude, compared to C and D.

Region L - Axially stable in the in the $j=0,1,2$ modes, and radially stable in the $j=0,1$ modes, but radially unstable in the $j=2$ mode. As in the case of regions C and D, there is $j=1$ radial instability in the immediate vicinity of $\langle n \rangle=0$, which has been neglected because of the relatively small size of this region compared with regions L and N.

Region F)
Region H) Radially and axially unstable in all modes of
Region K) parametric excitation.

Region E - Radially stable when $j=0,2$ and unstable when $j=1$, and axially stable when $j=0,1$, and unstable when $j=2$.

Region M - Axially stable when $j=0,2$ and unstable when $j=1$, and radially stable when $j=0,1$ and unstable when $j=2$.

Region G - Axially stable when $j=0,1,2$ and radially stable when $j=0,2$, but radially unstable when $j=1$.

Region J - Radially stable when $j=0,1,2$ and axially stable when $j=0,2$ but axially unstable when $j=1$.

The modification to this diagram in the presence of coupling resonances will be dealt with in the following chapter, where they will be shewn further to subdivide regions F, H, and K where parametric stability of both radial and axial oscillations exists.

Regions D,C and L,N are of especial interest because they provide axial instability in the presence of radial stability, and radial instability in the presence of axial instability for $j=1,2\ldots.$, i.e. in the presence of parametric excitation. The significance of this is discussed in Chapter Four in connection with extraction mechanisms for synchrocyclotrons.

For $\nu \neq 1,2,3 \ldots.,$

and $q^2 \ll 2\nu^2(\nu^2 - 1),$

we have $a^{\frac{1}{2}} \approx \nu = 2(1 - \langle n \rangle)^{1/2}/j$,

and the momentum compaction relationship is given by -

$$\varkappa_r = \varkappa_p\left((1/\nu) - (q^2/16)(2\nu.\nu^2+5)/(\nu^2-4)(\nu^2-1)^2 \right). \quad (198)$$

(b) Strong Focussing Limit- 'smooth' parametric variations-

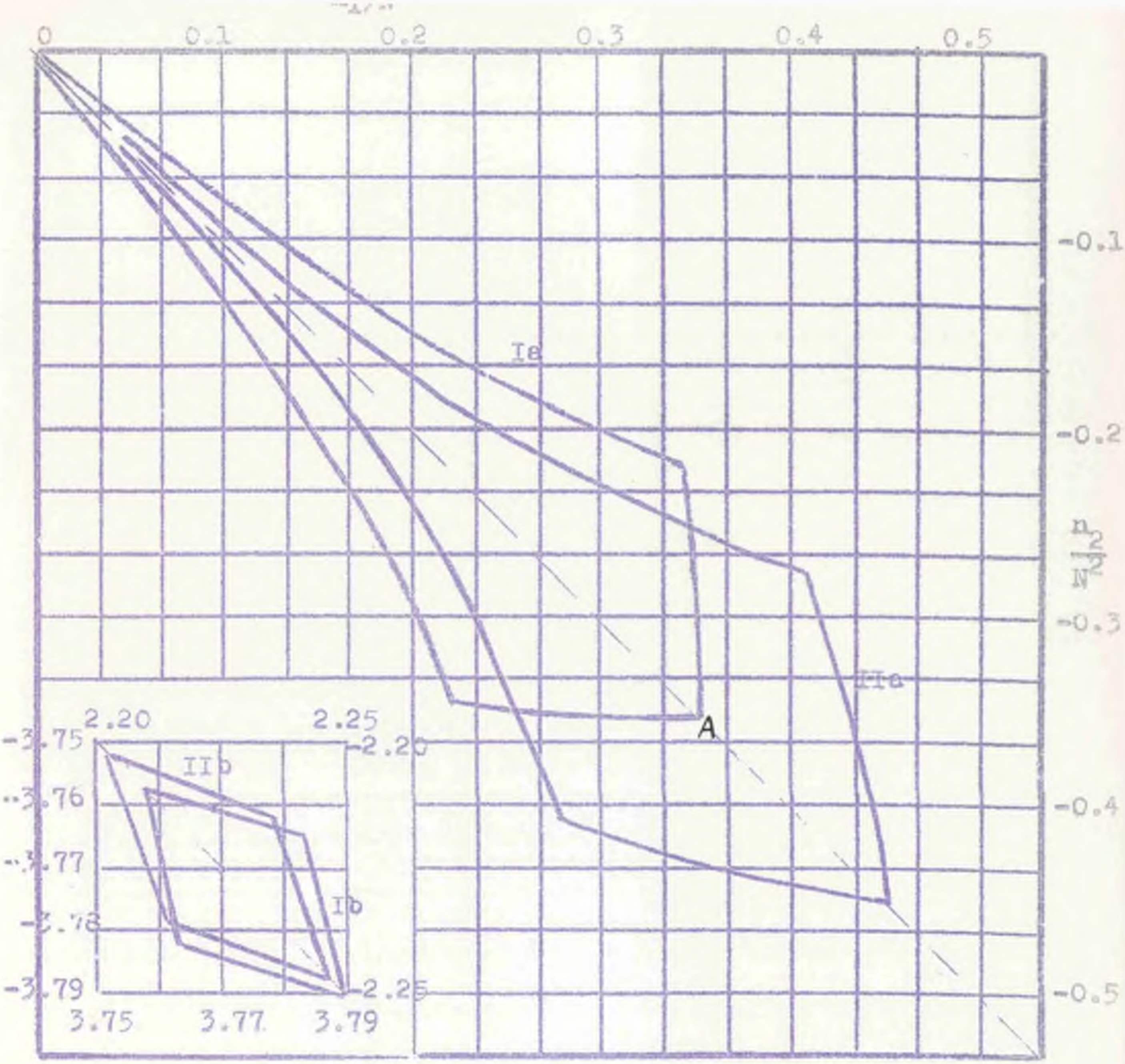
In this limit the following inequality is satisfied -

$$1 \ll |a| \ll |2\lambda_2| \ll 5 , \quad (199)$$

when solutions are of the form given by (107) and (108), and β supplied by the transcendental equation (106), which simplifies under the conditions given by the second half of the inequality in (199) to the equation -

$$(\beta_r^2/4) - (\lambda_2^2/2)(\beta_r^2+4)/(\beta_r^2-4)^2 = (-\beta_z^2/4) + (\lambda_2^2/2)(\beta_z^2+4)(\beta_z^2-4)^{-2}$$
$$= - \langle n \rangle /N^2. \quad (200)$$

This is identical, except for a small change in notation, with equation (7) in Paper C . Boundaries of the region in which both radial and axial motion is stable is given by (200) with $\beta_r = \beta_z = 0,1$, and is given on the Figure on page 86. The stable region thus defined is the equivalent of the CLS stable region, which is also given on the figure for purposes of comparison. As in the strong focussing magnet with discontinuous parametric changes, 2.2.3.1(c), there is a family of such stable regions, obtained at successively larger values of $\langle n \rangle$, and becoming increasingly small.



RADIAL AND AXIAL STABLE REGIONS IN STRONG FOCUSSING GUIDE FIELDS USING PARAMETRIC VARIATIONS

Ia,b Parametric variations piecewise constant, and changing discontinuously.

IIa,b Parametric variations sinusoidal.

In the insert, stable region IIa is referred to the scale on the top and right, while stable region IIb is referred to the scale at the bottom and left.

In the largest of these additional stable regions, radial and axial motion may be described by series of the form -

$$x = \sum_{r=-\infty}^{\infty} c_{2r} \cdot \cos(2r+1+\beta)(N\theta/2) \text{ , and} \qquad (107')$$

$$x = \sum_{r=-\infty}^{\infty} c_{2r} \cdot \sin(2r+1+\beta)(N\theta/2) , \qquad (108')$$

the coefficients satisfying recurrence relations similar to the Mathieu form of (102) or (103). The parameter β is given by the equation -

$$\lambda_2^2\left(\frac{1}{(\beta-1)^2-(4\langle n\rangle/N^2)-\lambda_2^2\cdot\delta'_{-1}} + \frac{1}{(\beta+3)^2-(4\langle n\rangle/N^2)-\lambda_2^2\cdot\delta_1} \right) = (\beta+1)^2 - (4\langle n\rangle/N^2) , \qquad (201a)$$

in the case of axial motion, with ,

$$(2r+5+\beta)^2\delta_{2r+1} = \left(1 - \frac{4\langle n\rangle/N^2}{(2r+5+\beta)^2} - \lambda_2^2\cdot\delta_{2r+3}\right)^{-1} , \qquad (201b)$$

and, $$(2r-3+\beta)^2\cdot\delta'_{2r-1} = \left(1 - \frac{4\langle n\rangle/N^2}{(2r-3+\beta)^2} - \lambda_2^2\cdot\delta'_{2r-3}\right)^{-1} \qquad (201c)$$

In the radial case, $\langle n\rangle$ above is to be replaced with $(1 - \langle n\rangle)$, and these two sets of equations then define the boundaries of the stable region when β is given the special values - 0, 1. It is plotted on the Figure on page 86 on a slightly larger scale in the insert, where it may be compared with its counterpart when parametric changes are discontinuous.

Equations (107), (108) which are suitable for describing transverse particle motion in the larger stable region are rapidly converging infinite series in which the $r=0$ term is predominant. Thus, as λ_2 tends to zero only c_0 remains finite, and both the equation and its solutions degenerate to those corresponding to the harmonic oscillator. To the first order therefore these eigenfunctions are those of a harmonic oscillator of frequency - $\beta Nw/2$ cycles/sec., where w is the rotational frequency of the particles. This oscillation frequency is of the order of βN times that obtaining in a weak focussing synchrotron, N representing the least number of identical units comprising the magnet.

As already mentioned therefore, for a given angular spread from the injector the amplitude of these free oscillations is decreased by the factor $-(\beta N)^{-1}$. In the case of the eigenfunctions (107') and (108'), however, we have a more slowly converging series which degenerates into a harmonic oscillator eigenfunction, i.e. only c_1 remains finite, as λ_2 tends to zero. To a less valid approximation therefore the eigenfunctions are those of a harmonic oscillator of frequency $(\beta+1)Nw/2$ cycles/sec., which is $(\beta+1)N$ times that obtaining in ordinary synchrotrons with weak focussing, and $(\beta+1)/\beta$ times that obtained in the larger stable region. For $\beta = 0.5$ the focussing strength in the smaller of the two stable regions is 3 times greater. A big disadvantage, however, is the much smaller size of the stable region, and the relative merits of these two regions will be discussed in Chapter Three taking into account the presence of magnet imperfections and resonance mechanisms.

The appropriate momentum compaction coefficient may be written -

$$\frac{\langle \alpha_r \rangle}{\alpha_p} = 2c_0^2 \left\{ \frac{(c_{-2}/c_0)^2}{N(\beta-2)} + \frac{(c_2/c_0)^2}{N(\beta+2)} + \frac{1}{N\beta} \right\}, \qquad (202)$$

where c_0 has to satisfy the normalising equation -

$$(1/c_0)^2 - (\beta N/2) = (c_2/c_0)^2(\beta+2)(N/2) + (c_{-2}/c_0)^2 N(\beta-2)/2 + N(c_2/c_0)(1+\beta)\cos(2\theta) + N(c_{-2}/c_0)(\beta-1)\cos 2\theta + \beta N(c_2 c_{-2}/c_0^2)\cos(4\theta) . \qquad (202a)$$

The ratios c_2/c_0, c_{-2}/c_0 are obtained from recurrence relations of the form (102).

(c) Strong focussing using 'smooth' parametric variations - introduction of equivalent parametric discontinuities -

The difficulty and complexity of even simple calculations using the analytical approach when parametric variations are continuous throughout will be obvious in the strong focussing example given above. It is of some importance therefore to determine the equivalent discontinuities that may be introduced to produce the same effect as the sinusoidal parametric variations. As a first apprixination zero field regions may be introduced between regions with field gradients which are piecewise constant, but of opposite polarity. If the zero field regions are of angular dimensions, $\theta = a\pi/N$, and the curved regions each of angular dimensions, $\theta = (1-a)\pi/N$, the stability condition may be written -

$$\left| \begin{array}{l} \cos A\cosh B - \dfrac{(n_1-n_2)}{2(n_1n_2)^{1/2}}.\sin A\sinh B - \dfrac{(n_1n_2)^{1/2}}{2}C\sin A\sinh B \\ + C(n_2^{1/2}\cos A\sinh B - n_1^{1/2}\sin A\cosh B \end{array} \right| < 1, \qquad (200')$$

where $A = \pi(1-a)n_1^{1/2}/N$, $B = \pi(1-a)n_2^{1/2}/N$, $C = \pi a/N$, $N>>1$. The boundaries are defined by the LHS equated to ± 1, the extreme corners of this region lying along the $A = B$ diagonal in n_1, n_2 space being defined by the equation -

$$\left(\begin{array}{l} \cos A\cosh A + A(a/\sqrt{1-a})(\cos A\sinh A - \sin A\cosh A) - \\ \qquad - (A^2/2)(a/\sqrt{1-a})\sin A\sinh A \end{array} \right) = \pm 1, \qquad (203)$$

which reduces to (195) when $a = 0$. Solutions of (203) for different values of a are given below:

$a = 0$: $A = 0, 1.87, \ldots,$ $(n_1/N^2) = 0, 0.354, \ldots$

$a = 0.04$: $A = 0, 1.8, \ldots,$ $(n_1/N^2) = 0, 0.355, \ldots$

$a = 0.10$: $A = 0, 1.7, \ldots,$ $(n_1/N^2) = 0, 0.360, \ldots$

$a = 0.18$: $A = 0, 1.6, \ldots,$ $(n_1/N^2) = 0, 0.382, \ldots$

$a = 0.23$: $A = 0, 1.5, \ldots,$ $(n_1/N^2) = 0, 0.39, \ldots$

$a = 0.40$: $A = 0, 1.25, \ldots,$ $(n_1/N^2) = 0, 0.44, \ldots$

The trend here is obvious; zero-field sections of angular length about 0.4 times the basic periodicity, i.e 2/3 the angular length of a curved section, are required to simulate the effect of 'smooth' sinusoidal parametric variations. This is a very rough estimate based only on normalising the diagonal lengths of the two stable regions defined by (200) and (200') above. The introduction of field-free regions as part of curved sections is therefore a useful approximation to a 'smooth' sinusoidal variation of field gradient. This is not identical to the problem of introducing physical field-free sections between curved sections as discussed in 2.2.3.1(d), when the length of the field-free sections included is independent of the angular length of the curved sections, and the behaviour of the stable region with increasing length of straight sections is that of the parameter A in (203), i.e., the diagonal length of the stable region, in units of n/N^2, decreases as the square of A as 'a' is increased.

2.3 Asymptotic behaviour of guide field focussing mechanisms -

We have dwelt at some length in section 2.2 on the adiabatic behaviour of transverse oscillatory motion in guide fields of different types. The asymptotic behavior of all transverse oscillations in all such fields is, however, as similar as their adiabatic behaviour is different, and much less space will be devoted to their treatment here.

The asymptotic behaviour of solutions to the betatron equations under all circumstances of interest are given in Chap.I.2.5, equations (146),(148),(149). From (37a) and (37b) it is then clear that the amplitude of these transverse oscillations varies as $(\langle B_z\rangle/W_e)^{-1/2}$ or $m^{-1/2}$, ie. positive damping occurs as particles are accelerated, although the damping is much greater for relativistic than for non-relativistic particles.

The asymptotic behaviour of the 'forcing' terms, i.e. due to the vector, $[\eta]$, in equations (51), (53), caused by the momentum spread of the injected beam and/or the injection of a monoenergetic beam of particles into a time dependent guide field, and also to any azimuthal variations of the guide field, is inversely proportional to the relativistic momentum, $p = e\langle B_z\rangle r_e$, i.e. proportional to $\langle B_z\rangle^{-1}$ as compared with the asymptotic behaviour of the 'free' oscillation amplitudes which are proportional to $\langle B_z \rangle^{-1/2}$ as shewn above. This asymptotic behaviour due to κ_p is essentially a motion of the 'instantaneous' orbits about which both the 'free' and 'forced' transverse oscillations occur. In the absence of any accelerating mechanisms, the radius of these orbits contract steadily as shewn with increasing guide field. This behaviour is of some use in solving some initial value problems, i.e injection, which is discussed in Chapter Four.[40]

3. ACCELERATING MECHANISMS AND THEIR FOCUSSING PROPERTIES:

3.1 Statement of problems -

Guide field focussing in an elemental form was originally introduced in fixed-frequency cyclotrons by pole face shimming in order to introduce some degree of axial focussing.[41] It has retained this essential purpose in all designs of cyclic accelerator, as the accelerating mechanism generally provides radial focussing, which is the subject of discussion in this section. Guide field focussing was first introduced, and, essentially, remains to provide axial focussing without introducing too much radial defocussing, and, certainly, without introducing the dangers of radial instability during the period of acceleration.

We are concerned here with the two known forms of accelerating mechanism - INDUCTION ACCELERATION, discussed in 3.2, and the far more flexible SYNCHRONOUS ACCELERATION, discussed in 3.3.

3.2 Mechanism of Induction Acceleration -

3.2.1 Introduction: The theory of this mechanism was first treated as a consistent whole by Kerst and Serber,[5] and first put to practical use in Betatrons by Kerst[3,4]. In the theory, treatment was given of guide field focussing mechanisms for the special case when the field gradient index, n, is independent of azimuthal angle, which proved the most important development from the point of view of cyclic accelerators in general. The principle of induction acceleration itself had previously been stated by others[42], and concerned only the forcing term, F, given by (37c) and (39), as -

$$F = \varkappa_p = (\delta\emptyset/\emptyset_e) + \text{constant},$$

where $\delta\emptyset = \emptyset_e - \emptyset$, $\emptyset_e = 2\pi r_e^2 \langle B_z \rangle$, $\emptyset$ being the magnetic flux threading the orbit of radius $r = r_e$. In the weak focussing approximation we then have -

$$\langle \varkappa_r \rangle = \alpha_m \left(\frac{\delta\emptyset}{\emptyset_e} + \text{constant} \right), \quad \alpha_m = 1/(1-n). \qquad (204)$$

In the adiabatic limit we may then have $\langle \varkappa_r \rangle = 0$ only if

$$\delta\emptyset/\emptyset_e + \text{constant} = 0, \qquad (204')$$

i.e. in this limit the radius r_e defined by -

$$\emptyset = 2\pi r_e^2 \langle B_z(r_e) \rangle = \delta\emptyset = \text{constant} \qquad (204'')$$

is time-independent. These are the 'Instantaneous orbits' of Kerst and Serber's theory[5]. The presence of the central flux, $\emptyset$, in (204) allows the forcing term, F, to be made zero over a range of orbit radii determined by the magnitude of the constant, <u>in the adiabatic approximation</u>, so keeping these radii constant in this approximation. This is the essence of the prinniple of induction acceleration. In a slowly varying field, however, only one of these orbits are <u>asymptotically</u> time-independent, i.e. that defined by -

$$\emptyset - 2\pi r_e^2 \langle B_z(r_e) \rangle = \delta\emptyset = 0. \qquad (25')$$

3.2.2. <u>The focussing mechanism</u> : - Focusing during induction acceleration is described therefore by the asymptotic behaviour of instantaneous orbits of radii $r_i \neq r_e$, and, in general, is expressed in terms of the momentum compaction coefficient α_m. The quantitative relation for the asymptotic radial focussing that occurs during induction acceleration is obtained from (204) in the form -

$$-\frac{(d\langle \varkappa_r \rangle/dt)}{\langle \varkappa_r \rangle} = \frac{d\langle B_z \rangle/dt}{\langle B_z \rangle} + \frac{dr_e/dt}{r_e/2},$$

or,

$$-\frac{d(\delta r)/dt}{\delta r} = \frac{d\langle B_z \rangle/dt}{\langle B_z \rangle} + \frac{dr_e/dt}{r_e}. \qquad (205)$$

Stated in this form it is obvious that the physical quantity of interest is the instantaneous orbit radius, r_i, about which the guide field oscillations occur. While we have assumed r_e to be time-independent above, it is rarely so in practice. The non-linear dependence of magnetic flux upon current makes ϕ, and hence p_θ time-dependent, which for very slow and small changes may be looked upon as a time-dependence of r_e, i.e. $dr_e/dt \neq 0$. As will be mentioned in Paper F, measurements of this parameter are taken together with those of $d\langle B_z\rangle/dt$, so that it is always possible to determine dr_i/dt. This is particularly important in the case of betatron-started electron synchrotrons where the betatron flux bars are deliberately designed to saturate during the period of electron acceleration, just after transition to synchronous acceleration. We might regard this inherent imperfection of magnetic material in not having a linear flux-current characteristic as giving rise to defocussing of the particle beam, in the sense that it increases the probability of losing particles to the walls of the containing vacuum chamber. In this connection it might, somewhat belatedly, be appropriate to define as a defocussing mechanism any mechanism that increases this probability, and as a focussing mechanism any mechanism that decreases this probability. While this is not strictly the general description of focussing and defocussing, it most certainly is consistent with all the uses to which such terms are put in cyclic accelerator theory.

3.3 Synchronous acceleration Mechanisms -

3.3.1 Introduction: While the mechanism of induction acceleration maintains an equilibrium orbit of nearly constant radius throughout the period of acceleration-the elegant simplicity of the method would have to be sacrificed for the radius to be made deliberately time-dependent - the method of synchronous acceleration is sufficiently flexible for the radius to have any desired time-dependence by merely programming the frequency of the accelerating voltage. In addition the mechanism of induction acceleration is suitable only for accelerating electrons to energies below the level at which radiation losses become excessive, while synchronous acceleration *is* suitable, in different forms, for accelerating all charged particles over a wider range of energies.

The effect of synchronous acceleration on particle motion is, in general, to provide radial focussing independently of guide field focussing. Indeed, using the definition of focussing and defocussing given above guide field axial focussing produces a measure of radial defocussing. The behaviour of radial motion during synchronous acceleration is described by equation (46) or (54). The behaviour of solutions of this equation under adiabatic conditions determine particle motion when the net acceleration is zero, or negligible during a single period of rotation. This aspect of the mechanism is discussed in 3.3.2.

The asymptotic behaviour of particle motion and its stability characteristics are, however, much more complicated than the comparable problem associated with guide field focussing and induction acceleration discussed above, and is discussed in 3.3.3.

3.3.2 Adiabatic characteristics of particle motion during synchronous acceleration -

The essential characteristics of this non-linear equation in the absence of non-autonomous perturbations has been treated adequately by phase plane approach in Chapter One, 2.4. The solution in this form clearly demonstrates the existence of phase stability during synchronous acceleration. Of much greater interest, however, is the behaviour of these solutions in the presence of non-autonomous perturbations such as exist during the start of synchronous acceleration. For this reason we pass on to a discussion of asymptotic behaviour and return to adiabatic behaviour in Chapter Four in discussing some Initial value problems.

Before this, however, we might relate the general synchrotron equation (54), and its associated stable region with that existing in weak focussing guide fields. The synchrotron equation may be written -

$$(d/dt)\left(I.\frac{d\theta'}{dt}\right) - (V/2\pi)(\sin\theta' - \sin\theta_e) = 0, \qquad (54)$$

where,
$$I = (r_e^2\langle B_z\rangle/kw_e)\left(\alpha_m' - \frac{E_o^2}{E_e^2}\right)_v^{-1} = B \text{ (cf (135))}. \qquad (54a)$$

In a weak focussing synchrotron without straight sections $\alpha_m' = \alpha_m = (1-n)^{-1}$, and I reduces to the conventional form-

$$I = (E_e/kw_e^2)(1/K);\ K = 1 + (n/\overline{1-n})(c^2/v^2).$$

'I' may be looked upon as the moment of inertia of the 'phase' pendulum, and is positive as long as $\alpha_m' > (E_o/E_e)^2$, i.e as long as $\alpha_m' > 1$. Now in weak focussing synchrotrons without straight sections $\alpha_m = (1-n)^{-1} \approx 4$. However, if the length of straight sections introduced is 1/4 the circumferential length of the curved sections, then, again for $n=3/4$, $\alpha_m' \doteq (4/5)(4) = 3.2$, and the moment of inertia, I, is increased. The area in phase space in which stable synchronous acceleration may take place is enclosed by the curve satisfying the equation -

$$I=B>0,\quad (d\theta'/dt) = (V/\pi I)^{1/2}(\cos\theta' + \cos\theta_e + \sin\theta_e(\theta'+\theta_e-\pi))^{1/2},$$
$$I=B<0,\quad (d\theta'/dt) = (V/\pi I)^{1/2}(\cos\theta' - \cos\theta_e + \sin\theta_e(\theta'-\theta_e))^{1/2},$$

which follow from the treatment in Chapter One, 2.4.
The maximum phase velocity, that occurs in the stable system when $B>0,\ \theta' = \theta_e$; $B<0,\ \theta' = \pi-\theta_e$, has the magnitudes-

$$B>0:\quad (d\theta'/dt)_{max} = (2V/\pi I)^{1/2}(\cos\theta_e - \sin\theta_e(\pi/2 - \theta_e))^{1/2},$$
$$B<0:\quad (d\theta'/dt)_{max} = (2V/\pi I)^{1/2}(\cos\theta_e + \sin\theta_e(\tfrac{\pi}{2} - \theta_e))^{1/2},$$

for the two cases of interest. In both cases it is clear that an increase in the moment of inertia, I, reduces the magnitude of the stable region. Where this occurs as a result of strong guide field focussing it is the result of strong momentum compaction as would be expected. Now the relation between the amplitude of phase oscillation $\theta'-\theta_e$, and $\langle x_r \rangle$ may, in general, be written -

$$x_W = (d\theta/dt)/w_e = (k\langle x_r\rangle/\alpha_m)(\alpha_m' - (E_o/E_e)^2).$$

Again, in the weak focussing guide field without straight sections this reduces to

$$x_W = k\langle x_r\rangle(n + (1-n)(v/c)^2).$$

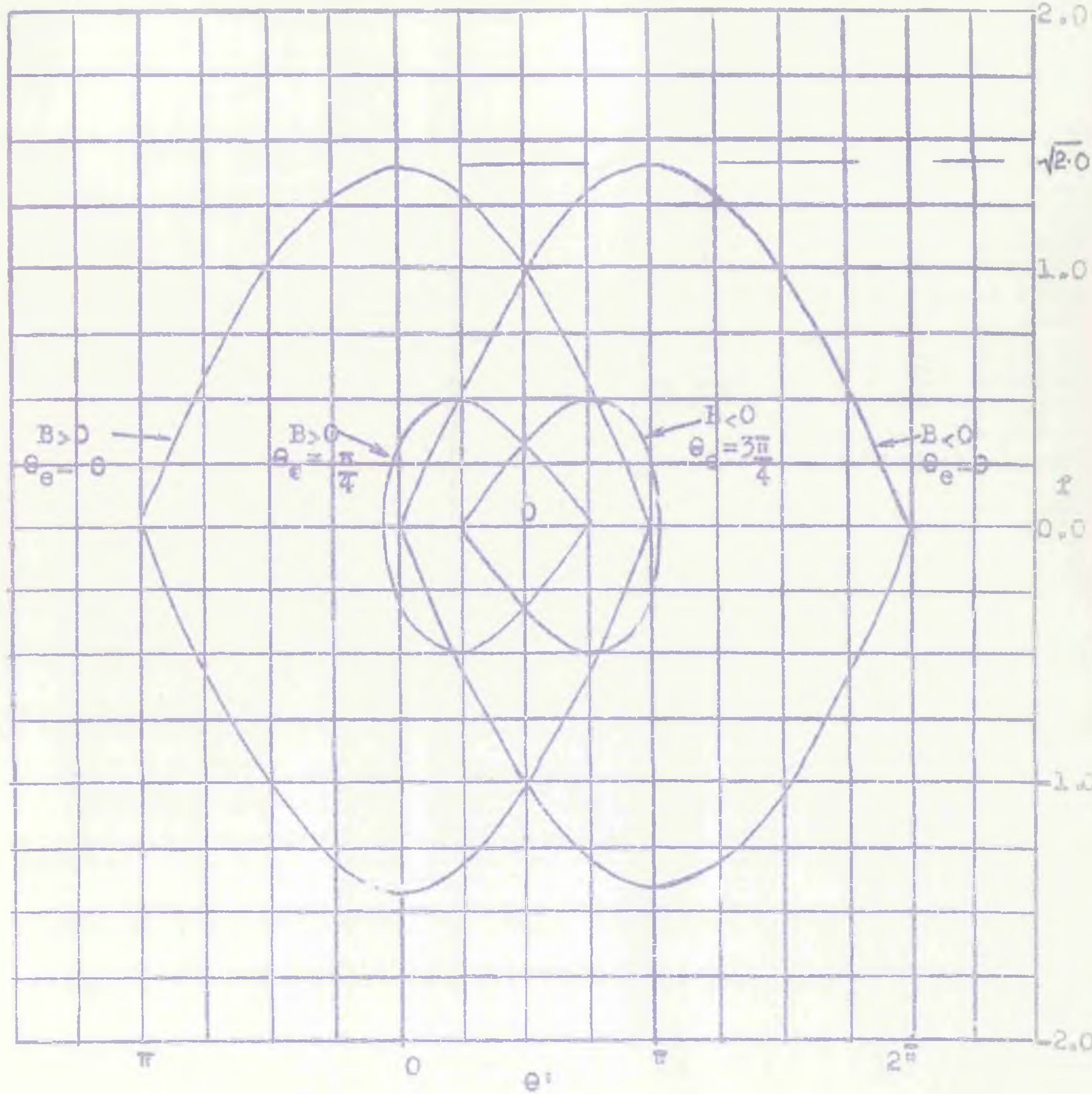
Radial motion as a result of phase oscillations is therefore

$$\langle x_r\rangle = (\alpha_m/v)(w_e V/\pi k\langle B_z\rangle)^{\frac{1}{2}}(\alpha_m' - (\frac{E_o}{E_e})^2)^{-1/2} f(\theta',\theta_e), \qquad (206)$$

where $f(\theta',\theta_e)$ has the two different forms given above for $B \lessgtr 0$. The use of the azimuthally averaged x_r, i.e $\langle x_r\rangle$, in the description of synchronous oscillation amplitude is a direct consequence of the assumption that the period of such oscillations is many times the period of a single revolution, a sufficiently accurate approximation for our purposes in first-order theory. It is interesting to note that the shape of the synchronous stable region is dependent only on θ_e, which is a measure of the minimum accelerating voltage required per revolution to keep the cyclic particle motion in synchronism with the frequency of the accelerating voltage, and on the polarity of $I = B$, which depends on the characteristics of the guide field focussing mechanism. The figure on page 97 illustrates the stable regions for several θ_e, and for both polarities of B. It is also clear from (206) that the magnitude of radial synchronous oscillations is rapidly increased as the energy of particles approaches the transition energy defined by (142), despite the fact that the phase oscillation amplitude is reduced as rapidly, a measure of the rapid disappearance of synchronous focussing. We pass now to a discussion of asymptotic behaviour, in which more discussion of particle behaviour as this transition energy is approached is provided.

3.3.3 Asymptotic behaviour of particle motion during Synchronous acceleration -

While adiabatic behaviour(in the absence of any significant net acceleration of the particle) is very much



TYPICAL STABLE REGIONS ASSOCIATED WITH TWO DIFFERENT MODES OF SYNCHRONOUS ACCELERATING MECHANISMS.

The $B < 0$ mode is the linear accelerator mode, while the $B > 0$ is the weak focusing cyclic accelerator mode.

Only in strong focussing cyclic accelerators do both types exist, the $B>0$ mode following the $B<0$ mode.

dependent on initial conditions, asymptotic behaviour is not so dependent. Nevertheless, there is considerable difference between asymptotic behaviour of synchronous oscillations in weak and strong focussing guide fields due to the possibility of transition between the two modes of behaviour discussed in 3.3.2. At the instant of transition between these two modes, the moment of inertia $I = B = \infty$, and the function, $F(t)$ in the normal asymptotic solution, given by (149) as -

$$x_r = F^{-1/4} . e^{i\int^t F^{1/2} . dt} , \qquad (149)$$

vanishes, making this form of asymptotic solution invalid. As shewn in 2.5, Chapter One, however, an alternative form of solution (152) may be found to overlap (149) sufficiently away from the transition energy for a solution to be obtained at all energies.

There is no space to deal more fully with this phenomenon. It is sufficient to note that if a change in RF phase of $(\pi - 2\theta_e)$ occurs at the instant of transition, the width of the stable region is not altered, and the amplitude of phase oscillations is not dangerously increased subsequently.

We shall discuss this two-mode behaviour again in Chapter Four.

4. CONCLUSIONS:

We have discussed in this chapter from a generalised theoretical basis outlined in Chapter One, the various types of Guide field focussing and Accelerating mechanisms. From the analysis presented it is possible to see the particular advantages of different combinations of these two distinct aspects of accelerator theory. It is because of the generalised approach adopted that this distinction can be drawn so clearly. Also we have neglected inherent imperfections and the many resonance mechanisms that follow as a natural consequence. Classification of the many imperfections and resonances that may occur is the purpose of the following chapter.

CHAPTER THREE

MAGNET IMPERFECTIONS AND RESONANCE MECHANISMS

C H A P T E R   T H R E E

MAGNET IMPERFECTIONS AND RESONANCE MECHANISMS

1. INTRODUCTION:

Resonance mechanisms in general give rise to radial and/or axial defocussing. In the linear approximation, to which we confined ourselves in Chapter Two and continue to do so in this chapter, this leads in almost all cases to a build-up of oscillation amplitudes to infinity in favourable circumstances. In very special circumstances resonance mechanisms are used to give a controlled focussing or defocussing effect; this aspect of the subject lies within the province of Chapter Four. We shall confine discussion here to giving, in section 2 a general classification of resonance mechanisms in terms of their causes and general characteristics.

Magnet imperfections do, in general, also give rise to radial and/or axial defocussing either by producing forced oscillations or by inducing resonance mechanisms. In section 3 a general classification of magnet imperfections is described.

2. RESONANCE MECHANISMS - A GENERAL CLASSIFICATION:

2.1 Some general definitions -

The stability regions of guide field focussing mechanisms discussed in Chapter Two are surrounded in Parameter space (the parameters are the radial and axial field gradients) by regions of instability where the oscillation amplitude may, in the linear approximation and in a guide field of infinite transverse dimensions, tend to infinity.

Mechanisms giving rise to build-up of infinite amplitude oscillations under such circumstances are resonance mechanisms in general, and parametric resonances in particular. These particular resonances also belong to the general class of 'continuum' resonances, because the values of parameters at which they can occur occupy an area in parameter space. There is another class of resonances which occur for values of a parameter lying along a line in parameter space, which may appropriately be referred to as 'line' resonances. Parametric resonances, which may either be 'line' or 'continuum' resonances, are characterised by the fact that they are induced by spatial variations in the parameters of guide field focussing mechanisms, i.e radial and axial field gradients, as already mentioned. Which particular field component it is whose axial or radial gradient is involved depends on the oscillatory modes into which the transverse guide field induced oscillations are separated.

In addition to parametric resonances there are 'forced' resonances and 'coupling' resonances. The former are predominantly line resonances, and are due to instability of forced oscillations, i.e those described by the particular integral of the betatron equation, at special values of parameters lying along lines in parameter space. The latter, i.e, coupling resonances, are closely related to parametric resonances, and are due to coupling between the nearly independent oscillatory modes produced by guide field focussing.

We deal first, in 2.2, with forced resonances, followed in 2.3 by a treatment of parametric resonances, and followed finally in 2.4 by discussion of coupling resonances.

2.2 Forced Resonances -

In the operational approach introduced in 2.2 Chapter One, it was shewn that the particular integral could be written in terms of the transformation operatory, [Y].

It was also pointed out there that for the presence of a stable closed orbit in the presence of forcing terms in the betatron equations (55), i.e, for forced resonances to be absent, $[Y(2\pi)]$ must satisfy the inequality -

$$|Y(2\pi) - I| \neq 0, \text{ or } \left(1 - \frac{Tr\{Y(2\pi)\}}{2}\right) \neq 0. \qquad (207)$$

Now the shift in phase in betatron oscillations after one complete revolution, $k\tilde{\nu}$, is given by: $\cos(k\tilde{\nu}) = \frac{Tr\,[(Y)]}{2}$, so that the condition necessary for forced resonances may be written-

$$k\tilde{\nu} = 2j\pi, \quad j = 0,1,2,3,\ldots.. \qquad (207')$$

In other words, for a forced resonance to occur it is necessary for the phase shift, $k\tilde{\nu}$, once round to be an integral multiple of 2π, or there must be an integral number of betatron radial and axial oscillatory periods in 2π. We have stressed the fact that the condition (207') is necessary, it does not follow immediately that it is sufficient. For any particular value of j in (207') it is necessary for a forcing term of the appropriate frequency to be also present. This clearly emerges from the analysis of the equivalent problem of forced resonances when the parametric variation is sinusoidal. It can then be shewn that in the presence of forcing terms of the form

$$f_o = \alpha_p.\cos(p\theta+\lambda_p), \qquad (208)$$

with the parametric variation in the form -

$$n(\theta) = \langle n\rangle + \alpha_k.\cos(k\theta+\gamma_k), \qquad (208')$$

forced resonances may occur for values of the mode parameter, β, satisfying the equation -

$$\beta k = \pm\ 2p\ ,\ 0 < \beta < 1\ . \qquad (209)$$

β in this case is defined by (200), and is identified with points in parameter space IIa in the Figure, page 86. It is related to the mode parameter, $\tilde{\nu}$, in (207') by the relation-

$$(2 - \beta) \doteq 2\tilde{\nu}\ . \qquad (210)$$

It is clear at once that forcing terms with periods less than that corresponding to $p = k/2$, cannot give rise to forced resonances, and, more important, even if β does satisfy the condition (209), say for $p = p'$, such a forced resonance is absent if $\alpha_{p'} = 0$. Another important feature which emerges from the analysis of sinuisoidal parametric variations is that in stable region IIa(Fig.on page 86) only forced oscillations of frequences giving p values in the range $0 \leq p \leq (k/2)$ may give rise to forced resonances, i.e. constant terms and the lower half of the available range of harmonics of the frequency of revolution. Since the mode parameter, β, will be about 0.5, forced oscillations of greatest potential danger will be those with $p = k/4$.

The condition for forced resonances in stable region IIb (Fig. on page 86) may similarly be written -

$$(\beta - 1) = -2p/k, \text{ or } \beta = 1 - 2(p/k);\ 0 < \beta < 1. \quad (211)$$

Again, resonances may not occur for $(k/2) < p < k$, i.e for the upper half of the available harmonic range. The difference with conditions in region IIa is, however, that for very low harmonics, $p \ll k$, resonances, $\beta \approx 0$ in IIa, while $\beta \approx 1$ in IIb. For $\beta \approx 0.5$, however, the resonance frequences in both regions are roughly equal. Now, the larger we make β the stronger is guide field focussing; this means that in stable region IIa the stronger the focussing the higher the forced oscillation frequencies giving rise to forced resonances, while in region IIb the stronger the focussing the lower the frequencies giving rise to forced resonances. The significance of this will be discussed later in this chapter.

Forced radial and axial resonances of all available frequences up to $p=k/2$ nevertheless subdivide <u>both</u> stable regions I and II into $(k/2)^2$ smaller regions, so that the larger we make k, the smaller are the elemental regions in which both radial and axial motion remains stable. For equal k we have shewn already (Page 87-88) that focussing strength in IIb is about 3 times that in IIa, so that for equal focussing strength $k_{IIb} \approx k_{IIa}/3$, i.e the number, N, of magnet sectors is reduced by a factor of about 3, making the number of regions into which IIb is subdivided by forced resonances about 1/9 of the number into which IIa is divided.

Thus the elemental stable regions in IIa and IIb are of roughly the same size in (n/k^2) space. The n value in both cases is roughly the same, however, so that in n-space the elemental stable regions in IIa remain an order of magnitude larger than those in IIb. Perturbations changing the n value are thus far more serious in IIb than they are in IIa.

It is clear from the analysis above that forced resonances are 'line' type resonances. They are also known as '2π - resonances'.

2.3 Parametric Resonances -

In operational notation the general stability condition is written (cf equation (79")) :

$$\left|Tr.Y(\tfrac{2\pi}{k})\right| < 1, \qquad (79''')$$

where k, as usual, represents the basic periodicity of the parametric variation. If this ideal periodicity is destroyed, and replaced by the periodicity, $k' = k/j$, $j=1,2,3..k$. Then (79"') must be replaced by the stability condition -

$$\left|Tr.Y(2\pi j/k)\right| < 1. \qquad (212)$$

Now it is easily shewn that

$$Tr.Y^{k/j}(2\pi j/k) = Tr.Y(2\pi),$$

so that stability is ensured for all values of j if

$$\left|Tr.Y(2\pi)\right| < 1, \text{ or } \left|\cos \bar{\nu}k\right| < 1, \qquad (213)$$

giving the condition for parametric resonances as -

$$\bar{\nu}k = j\pi .$$

(Note that the forced resonance stability condition is -

$$(\cos \bar{\nu}k) < 1.$$

A case of parametric resonance has been introduced already in Chapter Two, section 2.2.3(b), pg 75, which treats from the operational standpoint the characteristics of weak focussing guide fields mechanisms. It is shewn there that parametric resonances occur in the radial case when $n_{zr} = 3/4$, and in the axial case when $n_{zr} = 1/4$., in the limit of vanishing parametric variation. The analytical approach to the same problem is discussed in Paper B, and from both approaches it is clear (a) that parametric resonances in the weak focussing limit are continuum resonances. As parametric excitation increases these regions of parametric instability become the unstable borders around stable regions I and II in the Figure (pg.86). Parametric resonances in general are continuum resonances, as opposed to forced resonances, which are inherently 'line' resonances as long as the magnitude of forcing terms is finite. Parametric resonances are line resonances only in the limiting case of vanishing parametric excitation. In the particular case of the strong focussing limit, this occurs when $Y(2\pi)$ tends to $Y^k(2\pi/k)$.

(b) parametric resonances in this limiting case occur when the frequency of the guide field oscillations is half the frequency of the term giving rise to the parametric excitation. This is only obvious from the treatment of the weak focussing case. A similar proof for the strong focussing case may be obtained using the approximate solutions to HILL'S equation outlined in pages 39-41.

2.4 Coupling Resonances -

The basis of all analysis of coupling resonances is that the coupling is in the nature of a perturbation, and that the transverse oscillatory modes produced by guide field focussing are almost separable. An analytic method for determining the possible separable modes in maxwellian field configurations has been outlined in Chapter Two(2.2.1). It was pointed out there that even in a perfectly separated system coupling instability may occur when the frequency of radial and axial oscillations is almost equal. Thus, in the weak focussing case, such coupling resonance occurs when $(1-n) = n$, or $n=0.50$. In the presence of coupling perturbations, the inequalities (79) must be satisfied for resonance instability to be absent. Such conditions are unfortunately not very suitable for practical application to coupled systems in general, and we state the results obtained by Bell below.[43]

It is clear from (79') that even when the transverse oscillatory modes are completely separated there is the possibility of resonant coupling when-

$$\cos(k\nu') = \cos(k\nu''), \text{ or} \\ 2j\pi \pm k\nu' = k\nu'', \; j = 0,1,2,\ldots, \tag{214}$$

where ν', ν'', $k\nu'$, $k\nu''$ are defined in terms of the elements of $[Y(2\pi)]$, and $[Y(2\pi/k)]$, which may be written -

$$[Y(2\pi/k)] = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{bmatrix}, \tag{215a}$$

$$[Y(2\pi)] = \begin{bmatrix} A_2 & B_2 & 0 & 0 \\ C_2 & D_2 & 0 & 0 \\ 0 & 0 & A_2 & B_2 \\ 0 & 0 & C_2 & D_2 \end{bmatrix} = Y^k(2\pi/k). \tag{215b}$$

by the relations,

$$\cos(\nu') = (a_1+d_1)/2, \; \cos(\nu'') = (a_2+d_2)/2, \\ \cos(k\nu') = (A_1+D_1)/2, \cos(k\nu'') = (A_2+D_2)/2. \tag{215c}$$

The general problem is to determine which of the possibilities in (214) give coupling instability when $a_1+d_1 = a_2+d_2$. Bell[43] has shewn that under these circumstances, and in a slightly perturbed system, the combined system is stable unless either of the possibilities in (214) is satisfied at the same time that C_1 and C_2 are of opposite polarity, where,

$$C_1 = ((\sin k\nu')/\sin \nu').c_1, \text{ and} \\ C_2 = ((\sin k\nu'')/\sin \nu'').c_2.$$

Thus, if c_1 and c_2 have the same sign coupling instability may arise only if $\sin k\nu' = -\sin k\nu''$, or $k\nu' = 2j\pi - k\nu''$, i.e. $k\nu'+k\nu''$=integral multiple of 2π. The other case, when $k\nu'-k\nu''=2j\pi$, is stable under these circumstances. It only becomes unstable in the alternative case when c_1 and c_2 have the opposite polarity.

In most of the proposed strong focussing designs c_1 and c_2 have the same polarity, so that the condition for coupling instability is

$$k\nu'+k\nu''=2j\pi. \tag{214}$$

The general theory of nearly coupled transverse whose results were briefly described above was a first-order theory, i.e. the radial and axial displacements entered only to the first power in the betatron equation, and other coupling terms were neglected as being of higher order. Such a coupling term exists in the axial equation (Paper B, equation (4)), and the consequences of this second order coupling in weak focussing accelerator theory is presented in this paper(section 3). The results presented there shew that second order resonances occur for $\langle n \rangle \doteq 0.2$ and 0.5, of which the former is much the stronger because of the larger area of the continuum in parameter space over which it may occur. The other resonances at $\langle n \rangle = 0.7, 0.8, ..$ are of higher order still, and much less likely to cause serious instability.

A great many more resonances would be brought into view if we considered in greater detail the effect of the many higher order terms in the betatron equations, which make them non-linear. It is not intended to consider such problems here, where accelerator theory is based essentially on the first-order or linear approximation.

3. MAGNET IMPERFECTIONS - A GENERAL CLASSIFICATION:

3.1 Introduction -

By magnet imperfections we include all causes of perturbation which make the guide field depart from exact characteristics demanded by the focussing and resonance coupling theory presented so far. Such imperfections that we shall discuss will be analysed on the basis of the first order perturbations that they produce.

Such imperfections are caused always through mechanical and/or electromagnetic disturbances which give rise to forcing terms(such as azimuthal field variations described in (156) by functions f_0), to perturbation of parametric variations(disturbance of field gradient variations due, among other causes, to space charge forces, and described by f_1), and finally to coupling perturbations (introduction of additional unwanted field gradients described by functions f_2 in (156). These three types of disturbances have been shewn above to give rise under suitable conditions to transverse oscillation instability due to resonance conditions being established. In addition, forcing terms produce a forced oscillation whose magnitude is given by the particular integral of the betatron equations, and, far from resonant conditions, independent of the nature and magnitude of parametric variations. Indeed, as was shewn in section 2.2 of this chapter, there are a whole group of forced oscillation frequencies which may not in any circumstances give rise to resonant instability, but do give rise to stable forced oscillations. In the following sections we list the causes of all such perturbations, and discuss the relation between their magnitude and the orbit perturbation that results.

3.2 <u>The Perturbation of parametric variations</u> -

As has been shewn above, perturbed parametric variations give rise to parametric resonances or coupling resonances depending on whether the parameter(i.e. field gradient) considered is producing guide field focussing or coupling between the transverse oscillatory modes. Thus, in the conventional betatron, synchrocyclotrons and synchrotrons the parameter producing guide field focussing is $n_{zr} \propto \partial B_z/\partial r$, which is, in the absence of perturbation, independent of azimuthal angle, θ. The parameter producing first order coupling is $n_{rr} \propto \partial B_r/\partial r$. In an alternative separable system suggested in Chapter Two, the parameter producing guide field focussing is $n_{rr} \propto \partial B_r/\partial r$, while the parameter producing coupling is $n_{zr} \propto \partial B_z/\partial r$. We are concerned here with the possible causes of parametric perturbations, whether it produces focussing or coupling.

3.3.1 <u>Electromagnetic Error Fields</u> - These fields are due to spatially dependent remanence in the iron and/or to spatially dependent induced currents in a.c. magnets. The spatial dependence of remanent fields are caused by varying iron properties, different magnetic path lengths, non-uniform flux distributions; the spatial dependence of induced(eddy) currents depends on the variation of interlamination resistances in the iron paths and the presence of metal objects in the vicinity of the air-gap. The time dependence of these error fields is also of importance, depending on the time dependence of the MMF across the magnet gap. Transient conditions are fairly common in modern accelerators, and considerably modify the flux distributions in a.c. magnets due to the finite time constant associated with the build-up of magnetic flux. The whole problem is one of great complexity, and is nowhere more so than in the large betatron started electron synchrotrons, where a combination of all these problems arise. Paper F gives some thought, and much careful measurement to determining the parameter perturbations produced in a typical weak focussing accelerator of this type.

We might also consider as an electromagnetic disturbance of guide field parameters the space charge forces in charged particle beams. These are not imperfections in the true sense of the word, but they nevertheless modify considerably the spatial and time dependence of the field gradients. Papers B and C are treatments of this problem in weak and strong guide fields respectively. An important point to emphasize here concerns the first harmonic variation with azimuthal angle produced by the injected currents. The importance of this was emphasized in Paper B in connection with injection into weak focussing synchrotrons and betatrons. A similar azimuthal variation will exist in any cyclic accelerator at injection owing to the fact that space charge density builds up in front of the injector to a value larger than that behind the injector. This provides an inherent source of excitation of the parametric or π resonances even in a perfectly assembled magnet with no electromagnetic field disturbances other than those due to space charge forces.

Apart from qualitative calculations of the effect of space charge on guide field parameters (Paper C) it is not convenient to determine, even qualitatively, the effect of remanent fields or eddy currents inside the magnet core on the guide field. As a consequence a great deal of emphasis is placed on measuring the guide field characteristics under a variety of excitation conditions; such a comprehensive programme is described in Paper F, where some rough qualitative discussion of the effect of transient conditions on guide field characteristics is given.

3.2.2 Mechanical Misalignments - The following constructional errors give rise to perturbing parametric variations:

(a) Variations in shape of pole profile;

(b) Differences in the length of straight sections;
These errors are common to strong and weak focussing synchrotrons.

(c) Differences in the length of curved sections; such errors are important only in strong focussing synchrotrons where alternate sections have very different spatial properties.

(d) Twist of magnet sections about the median plane. This misalignment error introduces a coupling term if the oscillatory modes in the perfectly aligned magnet are uncoupled. Thus, if $|\partial B_z/\partial r| >> 0$, $\partial B_z/\partial z = 0$ in a perfectly aligned system, then a twist of $\delta\theta$, gives

$$\delta(|\partial B_z/\partial z|) = 2|n_{zr}|.|\delta\theta|, \qquad (216)$$

while in a magnet designed so that $|\partial B_z/\partial z| >> 0$, $\partial B_z/\partial r = 0$, the same twist, $\delta\theta$, gives

$$\delta(|\partial B_z/\partial r|) = 2|n_{rr}|.|\delta\theta|. \qquad (216')$$

(e) Finally, all mechanical disturbances giving rise to forcing terms give rise also to parametric variations if one takes into account variation of angular velocity, with conservation of linear velocity, in the betatron equation. (Paper B, equation (3) - note the presence of the $h_o(\theta)$ term on the LHS). The relation between mechanical misalignments and the forcing terms it produces are discussed in the following section 3.3.

3.3 Introduction of 'forced' perturbations -

3.3.1 The particular integral of the betatron equation-

As already mentioned, the effect of 'forced' perturbation on particle motion is obtained quantitatively from the effect of $f_o^{r,z}(\theta)$ on the r.h.s of the betatron equation, (51), on its particular integral. In the general case when the betatron equations are of the class of HILL'S equation, the matrix operational approach, using piecewise constant parametric variation to approximate to any given variation, is probably the most readily applied to obtain quick numerical results. In this section, however, we shall obtain the particular integrals of the betatron equations when they are of the form of MATHIEU'S equation.

For Mathieu eigenfunction written in the form (107) and (108), with the forcing term,

$$f_o = \alpha_p . \cos(p\theta + \lambda_p),\ 0 < p < k,$$

and the parametric variation of the form,

$$n(\theta) = \langle n \rangle + \alpha_k . \cos(k\theta + \gamma_k),$$

the particular integral is written in the form -

$$x_{P.I} = \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} (kp\alpha_p c_{2r_1} c_{2r_2}) . (F' + F''), \qquad (217)$$

where $F' = (4p^2 - k^2(2r_1+\beta))^{-1} (\sin X . \sin Y + \frac{k}{p}(r_1 + \frac{\beta}{2}) \cos X \cos Y),$

$$- F'' = (4p^2 - k^2(2r_2+\beta))^{-1} (\sin X . \sin Y - \frac{k}{p}(r_2 + \frac{\beta}{2}) \cos X \cos Y),$$

$$X = (p\theta + \lambda_p), \quad Y = (r_1 - r_2)(k\theta + \gamma_k).$$

A reasonable first approximation is obtained by consideration of the single term for $r_1 = r_2 = 0$. The general normalising equation-

$$\sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} c_{2r_1} c_{2r_2} (2r_1 + \beta) = (2/k)^2 \qquad (217a)$$

reduces to $c_o^2 = (1/\beta)$, and $x_{P.I}$ may be written in the simplified form-

$$x_{P.I} = \frac{\alpha_p (2/k)^2}{((2p/k)^2 - \bar{\beta}^2)} . \cos(p\theta + \lambda_p) \qquad (217b)$$

It is clear from (217) that the forced resonance condition is $\beta = \pm (2p/k)$. (209)

Since $0 < \beta < 1$ in the stable region it is important to notice, as has been mentioned already, that forced resonances may not occur for $(k/2) < p < k$. Of the frequencies of forcing terms in this range, one of particular interest is that for which $p = k$, i.e. the frequency of the forcing term is equal to the frequency of parametric variation. In weak focussing synchrotrons, this is a natural consequence of the effect described in 3.2.2(e) above. In strong focussing synchrotrons this effect is dominated by the large parametric variation required for the focussing mechanism when the frequency of this variation is equal to that of the forcing term. For the weak focussing case, this effect is described and analysed in Papers L, M(Part 1), and N. M(Part 2) has general treatment of the problem, with results for the three special cases (i) harmonic oscillator approximation,

(ii) Weak focussing approximation, and

(iii) Strong focussing approximation.

This problem was first investigated by the author in the second special case for reasons associated with the existence of large, high harmonic variations of the guide field described in Paper N. Previous to this investigation the criterion for validity of solutions providing the effect of forcing terms, i.e. the particular integral, was $\alpha_p << 1$, for all p. During preliminary investigations of guide field variations in the 340 MeV Glasgow synchrotron it was found that non-uniform flux distributions due to the presence of the betatron flux bars gave large amplitude, $\alpha_p \sim 1$, high harmonic, $p >> 1$, remanent field variations. The analysis that was performed showed that the proper criterion for smallness of perturbation should be (α_p/p^2), and not α_p. On this basis it was found that apart from a unidirectional orbit shift, the origional particular integral remained unaltered provided $\alpha_p/p^2 << 1$. The subsequent papers M generalised the problem, and sought to make use of the orbit shift to assist injection efficiency. This aspect is discussed further in Chapter Four.

3.3.2 Electromagnetic Error fields - All the electromagnetic disturbances mentioned in 3.2.1, with the exception of space charge effects, give rise to forced oscillations, and, for appropriate values of the guide field parameter, to the 'forced' resonances already discussed. In large weak focussing electron synchrotrons it has been general practice to inject particles at relatively low energies (~100 keV) and to modulate the frequency of the accelerating voltage, or use betatron acceleration to provide stable acceleration along an orbit of nearly constant radius during the period when the electron velocity is substantially below that of light. For 100 keV electrons, $\langle B_z \rangle_{inj} . r_{inj} = 1155$ gauss.cm.; in the Glasgow synchrotron $r_{inj} = 125$ cm., so that $\langle B_z \rangle_{inj} = 9.2$ gauss. The particular integral, (217), in the weak focussing approximation $(\beta = 4(1-\langle n \rangle)/k^2)$, is written-

$$x_{p.I} = \frac{\alpha_p}{(p^2 - (1-\langle n \rangle))} . \cos(p\theta + \lambda_p); \qquad (217')$$

and for $p=1$, $x_{p.I} \not> 0.01$, $\langle n \rangle = 0.7$, α_p must not exceed about 0.008, i.e. the guide field must be uniform to within about 0.07 gauss. Paper F gives a full account of the efforts made in measurement and construction to maintain adequate field uniformity in this particular accelerator.

For weak focussing machines in general, $0 < \langle n \rangle < 1$, so that there is not any possibility of forced resonances, and the $p = 1$ harmonic is the most serious harmonic to contend with. It is made even more important to eliminate because, as pointed out in 3.2.2(e) such azimuthal variation may also give rise to parametric resonances when $\langle n \rangle = 3/4$ (radial instability) or $\langle n \rangle = 1/4$ (axial instability). The significance of the $\langle n \rangle = 3/4$ parametric response is not diminished by the fact that most weak focussing synchrotrons have been designed with $\langle n \rangle$ in the range $0.5 < \langle n \rangle < 0.75$. The effects of this resonance on particle behaviour near injection is discussed in Paper B and in the following Chapter Four.

3.3.3 Mechanical Misalignments - The most important of such misalignments which give rise to azimuthal field variations (not field gradients as discussed in 3.2.2) are:

(a) Relative radial displacement of magnet sections- The radial displacement of a magnet sector, δr_c, produces a field change, δB_z, at a particular orbit of radius, r, given by,

$$-\delta B_z/\langle B_z\rangle = n(\theta).(\delta r_c/r). \qquad (218)$$

It is interesting to notice that in weak focussing synchrotrons a small first harmonic radial variation of magnet sections corresponds just to a sideways shift of the magnet which carries particle orbits with it. Thus, $(\delta r_c/r)=\alpha_1/\langle n\rangle = \varkappa^r_{P.I}$. In the strong focussing synchrotron this phenomenon is extended to all forced oscillations with $p \ll k$, provided conditions for forced resonances do not exist.

(b) Relative axial displacement of magnet sections - The axial displacement of a magnet sector, δz_c, produces a field change, δB_r, at a particular orbit of radius, r, given by,

$$-\delta B_r/\langle B_z\rangle = n(\theta).(\delta z_c/r). \qquad (218')$$

In this case an upward shift of all sectors by the same amount, gives $\varkappa^z_{P.I} = \delta z_c/r$; as one expects the displaced orbit moves with the magnet. A first harmonic axial displacement, however, produces an axial oscillation several times the maximum sector displacement in amplitude, in the case of weak focussing accelerators. In strong focussing accelerators it is easily shewn that, as for radial displacements, the very low harmonic axial displacements will carry the disturbed orbits with them.

(c) Twist of magnet sectors about the radial centre of the median plane. This produces a radial field, δB_r related to the twist, $\delta\theta$, $\delta\theta \ll 1$, by the equation,

$$\delta B_r/\langle B_z\rangle = \delta\theta. \qquad (219)$$

It is pointed out in Paper F that for weak focussing synchrotrons this gives the most severe alignment problem, even though the problem of coupling instability is not a serious one.

In all the analysis given above only the conventional guide field, which is symmetric about the median plane, i.e. $\partial B_z/\partial z = 0, \partial B_r/\partial r = 0$, has been considered. In an alternative guide field configuration to give separable oscillatory modes (Chapter Two, 2.2.1, page 62), $\partial B_z / \partial r = \partial B_r / \partial z = 0$, a radial displacement error gives rise to axial field errors and therefore radial forced oscillations, and vice versa, so that slowly varying radial and axial misalignment errors of magnet sections are very much more serious, when compared with the behaviour of forced oscillations under similar conditions in the more conventional type of guide field.

CHAPTER FOUR

INITIAL - VALUE PROBLEMS

C H A P T E R   F O U R

INITIAL - VALUE PROBLEMS

1. INTRODUCTION :

In the earlier chapters we have emphasized the general mathematical theory of steady-state accelerating conditions in cyclic accelerators. In this chapter we deal with problems in which boundary(or initial) conditions in one form or another are involved, i.e. with 'transient' ra ther than with steady-state phenomena. We cannot hope to deal with all aspects of the three types of initial value problem - (i) Injection Mechanisms, section 2.

(ii) Ejection Mechanisms, section 3.

(iii) Transition Mechanisms, section 4.

so we turn from the general to the particular and deal with the detailed characteristics of such mechanisms in two weak focussing cyclic accelerators with which the author has been associated during the past five years. One is the 20 MeV betatron at Metropolitan-Vickers Electrical Co.,Ltd., Manchester,[44] and the other is the 340 MeV betatron-started electron synchrotron in the Department of Natural Philosophy, University of Glasgow.[45] The experimental results obtained from the operational characteristics of the betatron were obtained in collaboration with D. Major, and the results obtained from the synchrotron were obtained in collaboration with Dr. W. McFarlane, and the valuable assistance of J.Mellor, W. simpson and T.Elder.

Sections 2, 3, and 4 deal respectively with injection, ejection and transition mechanisms. The observations and interpretations of significant characteristics is original work, which was stimulated by discussion with the above mentioned individuals, E. A . Finlay and C.Ambasankaran.

2 INJECTION MECHANISMS:

2.1 Betatron injection -

2.1.1 The control of important parameters-

In any weak or strong guide field focussing mechanism, the parameters of importance are $f_o^{r,z}(\theta)$, $f_1(\theta)$, and $f_2(\theta)$, which in the conventional guide field configuration are terms which produce forced, parametric and coupling excitation or resonances respectively. In the induction acceleration mechanism there is also the asymptotic movement of 'instantaneous' orbits towards the 'equilibrium orbit, radius r_e, defined by (25). Control of these parameters is important; in the case of the functions, f, above, this is especially important during the period during, and immediately subsequent to particle injection, and happily, such control is particularly convenient to provide during this period when the guide field is weakest. Control of r_e throughout the period of induction acceleration is also not as important as control during the period of injection; usually the most suitable orbit during the injection interval is satisfactory for the rest of the period of acceleration, except in very exceptional circumstances, which arise during betatron flux core saturation or during a period dominated by transient flux disturbances. In the rest of this section we consider, briefly, the control of these five parameters, $f_o^z(\theta)$, $f_o^r(\theta)$, $f_1(\theta)$, $f_2(\theta)$, and r_e.

$f_o^z(\theta)$: Although there is no danger of forced resonances in weak focussing synchrotrons and betatrons, small amplitude, low harmonic azimuthal variations of B_z, described by $f_o^z(\theta)$, can produce large radial oscillations, and can, in suitable circumstances ($\langle n \rangle = 3/4, 1/4$) induce a parametric resonance. f_o^z is controlled near guide field zero (when injection occurs) by quadrature currents induced in coils enclosing all the guide field flux over a region of azimuthal extension, $2\pi/p$. These p coils control all harmonic variations up to the $(p/2)$th. In the betatron, with its rectangular yoke, p=8 coils were used on the pole face for this purpose. In the 340 MeV synchrotron, p=20 coils were used, one around each of the 20 sections of its distributed yoke. The large 20th harmonic caused by the non-uniform flux distribution in the presence of betatron flux bars (Paper F gives a fuller description of this point) was not corrected as a result of conclusions based on the analysis given in Paper L,N.

$f_0^r(\theta)$ and $f_1(\theta)$ Control of both these parameters may be conveniently obtained using currents through coils concentric with particle orbits attached either to the pole faces or to the toroidal vacuum chamber(doughnut). From standard electromagnetic theory we have for the radial proâuced in air at the radius $r = r_c + \delta r$, $\delta r << 1$, on the plane equidistant between those of 2 single turn coils, of equal radius, r_c, with centres on the same axis spaced 2z cm. apart, the expression -

$$B_r = \frac{2(I_1 \mp I_2)z}{2r_c^2(r_c+\delta r)^{3/2}\left(1 + \frac{(z^2/4r_c)}{(r_c+\delta r)}\right)^{1/2}}\left(-K + \left(1 + \frac{2r_c^2}{z^2}\right)E\right), \qquad (220)$$

where I_1 and I_2 are the magnitudes of the currents in the two coils, and the negative sign in (220) applies if the direction of current flow is the same in both coils, and vice versa, and K and E are complete elliptic integrals of the first and second kind. In most cases of interest, especially in the case of the larger radius accelerators, $z >> r_c$, and (220) may be simplified to the form -

$$B_r \approx 2(I_1 \mp I_2)/z\left(1 + \frac{3\delta r_c}{2r_c}\right), \qquad (220')$$

since $2(r_c/z)^2 >> K > 1$, and $E \approx 1$.

If these coils are attached to the pole faces of infinitely permeable iron, and we assume that the pole faces are very nearly parallel and have an annular width comparable with its mean radius, then (220') has to be rewritten-

$$B_r \doteq \frac{2(I_1 \mp I_2)}{\left(1 + \frac{3\delta r_c}{2r_c}\right)} \cdot \left(\frac{2}{z} - \frac{1}{3z} + \frac{1}{5z} + \cdots\right), \qquad (220'')$$

or, alternatively, and expressing I in amperes -

$$B_r = (I_1 \pm I_2)(0.36/z). \qquad (220'')$$

Similarly, we have,

$$\partial B_r/\partial z = -(I_1 \pm I_2)(1 + \pi^2/8)/5z^2, \qquad (221)$$

where I is again expressed in amperes, and the positive sign applies to the case when the direction of currents in both coils are the same. From (221) we then have -

$$\delta n = (r_c/\langle B_z\rangle)(0.46/z^2)(I_1 \pm I_2). \qquad (222)$$

It is clear from these expressions that if the polarities and magnitudes of these currents are separately controlled then the same set of pole face coils may be made simultaneously to control $f_o^r(\theta)$ and $f_1(\theta)$. From (220') it is readily shewn that using these coils the effect on $f_2(\theta)$ is negligible; more exactly,

$$\partial B_r/\partial r \sim -B_r/r_c. \qquad (223)$$

In the 20 MeV betatron a very coarse control only of $\langle n\rangle$ is available with the aid of a single coil in the centre of each pole face, i.e. a single control of the unidirectional current in both coils is provided which may alter both their magnitude and polarity simultaneously.

In the 340 MeV synchrotron a much more extensive control of the guide field is provided. Five concentric coils on each pole face, with independent control for magnitude and polarity of current in each coil, control $\langle n\rangle$ and $\langle f_o^r\rangle$. In addition facilities are provided for the control of the azimuthal variation of both n and f_o^r with another set of pole face coils. With a sufficient number of controls of magnitude and polarity of current in each coil, harmonics up to the 10th may be altered, although as used only harmonics up to the 2nd. may be affected. Results of experiments with such an arrangement of coils is described in the following sections.

$f_2(\theta)$: This parameter is clearly little effected by current in the pole face coil, and would require currents in coils in the median plane. No such control has been used.

2.2.2 The characteristics of a 20 MeV Betatron-

The following are significant experiments betraying characteristics of the injection mechanisms in small betatrons.

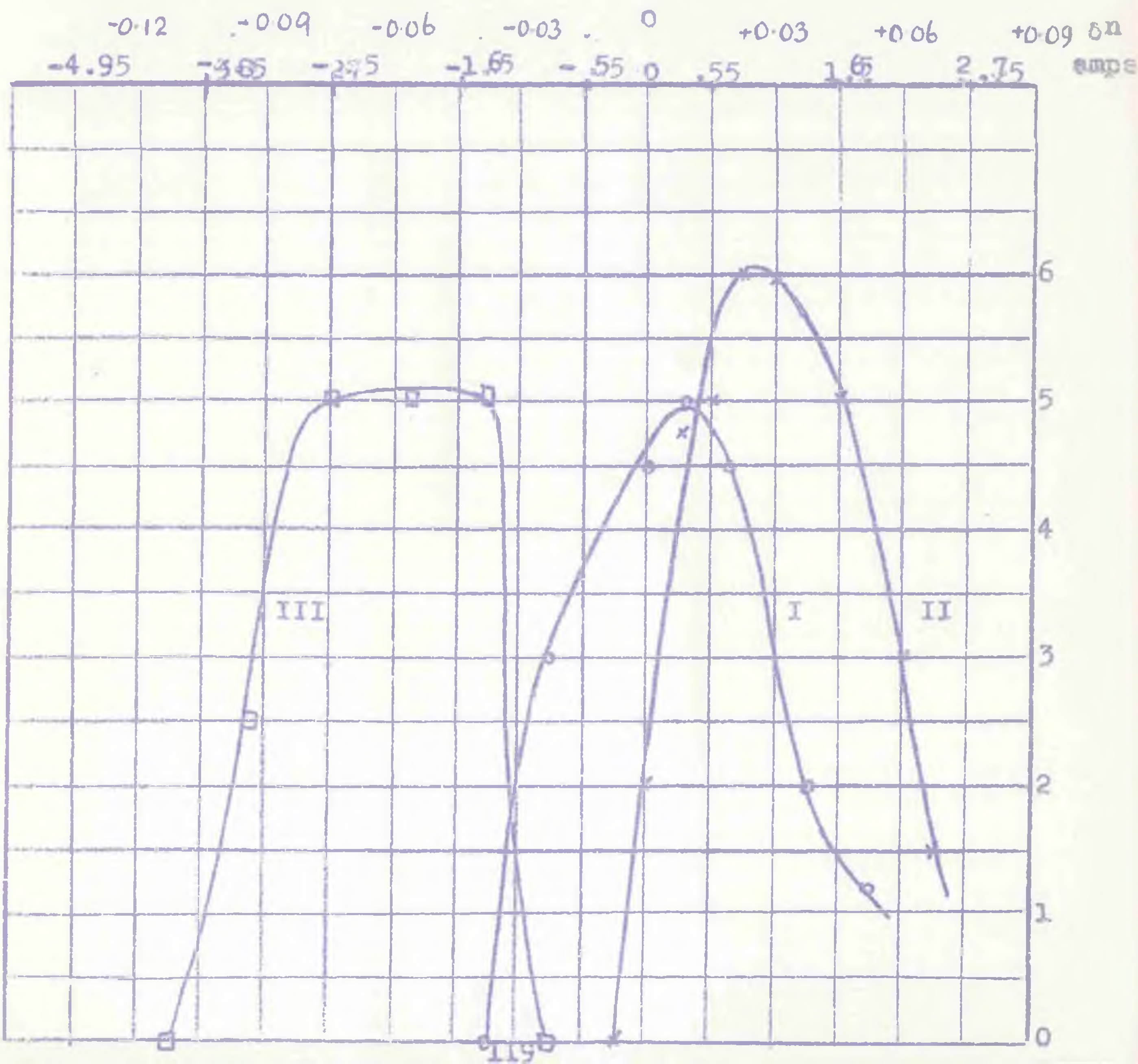
EXPERIMENT - Effect on betatron output of the value of $\langle n \rangle$ during and just after the injection interval.

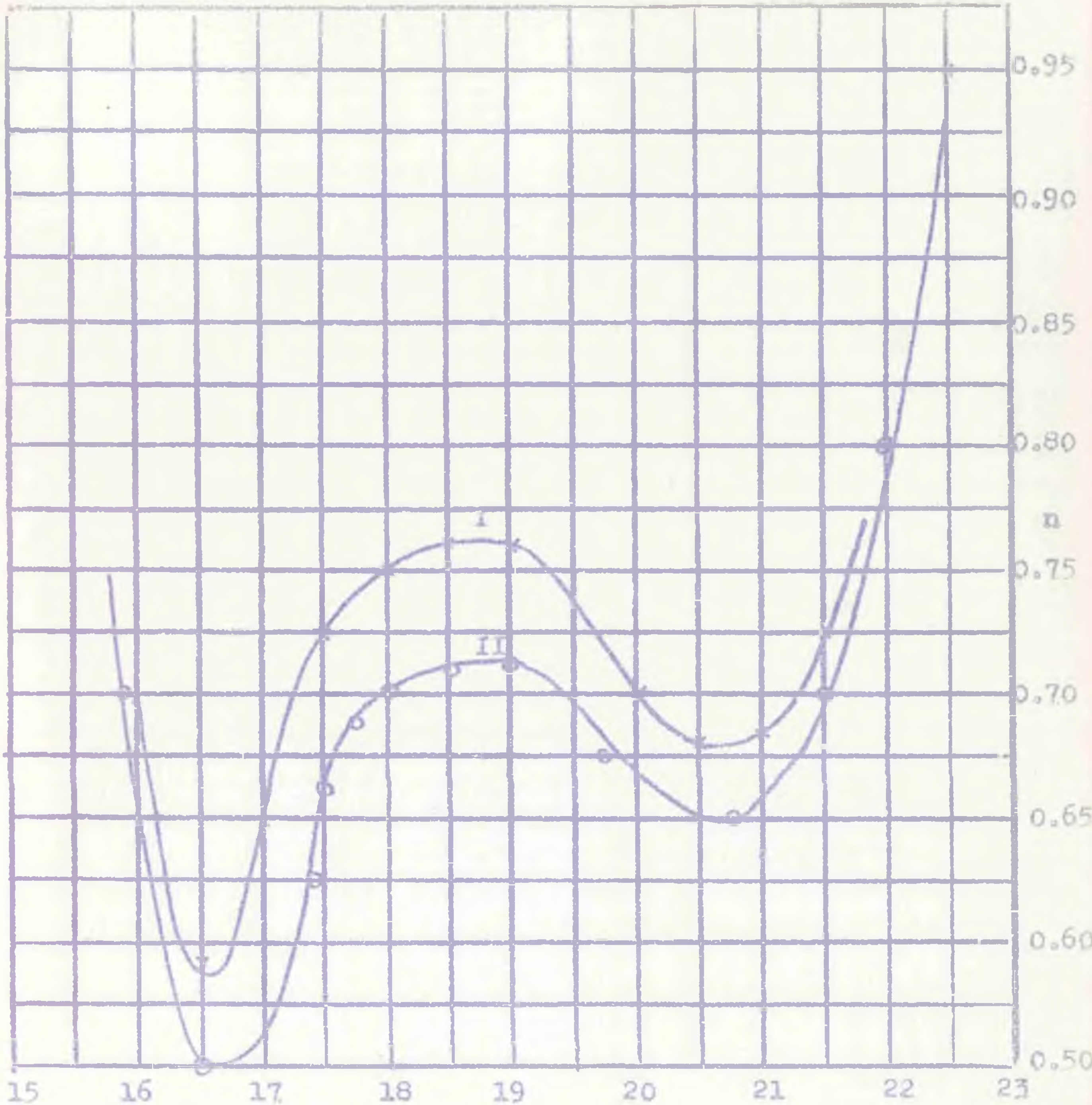
Curve I : $\langle n \rangle$ characteristic before introduction of shimming wires on pole faces.

Curve II : $\langle n \rangle$ characteristic with steel shimming wires on pole faces.

Curve III: $\langle n \rangle$ characteristic after removal of steel wires on pole faces.

Peak emission in all cases was about 0.5 amps.





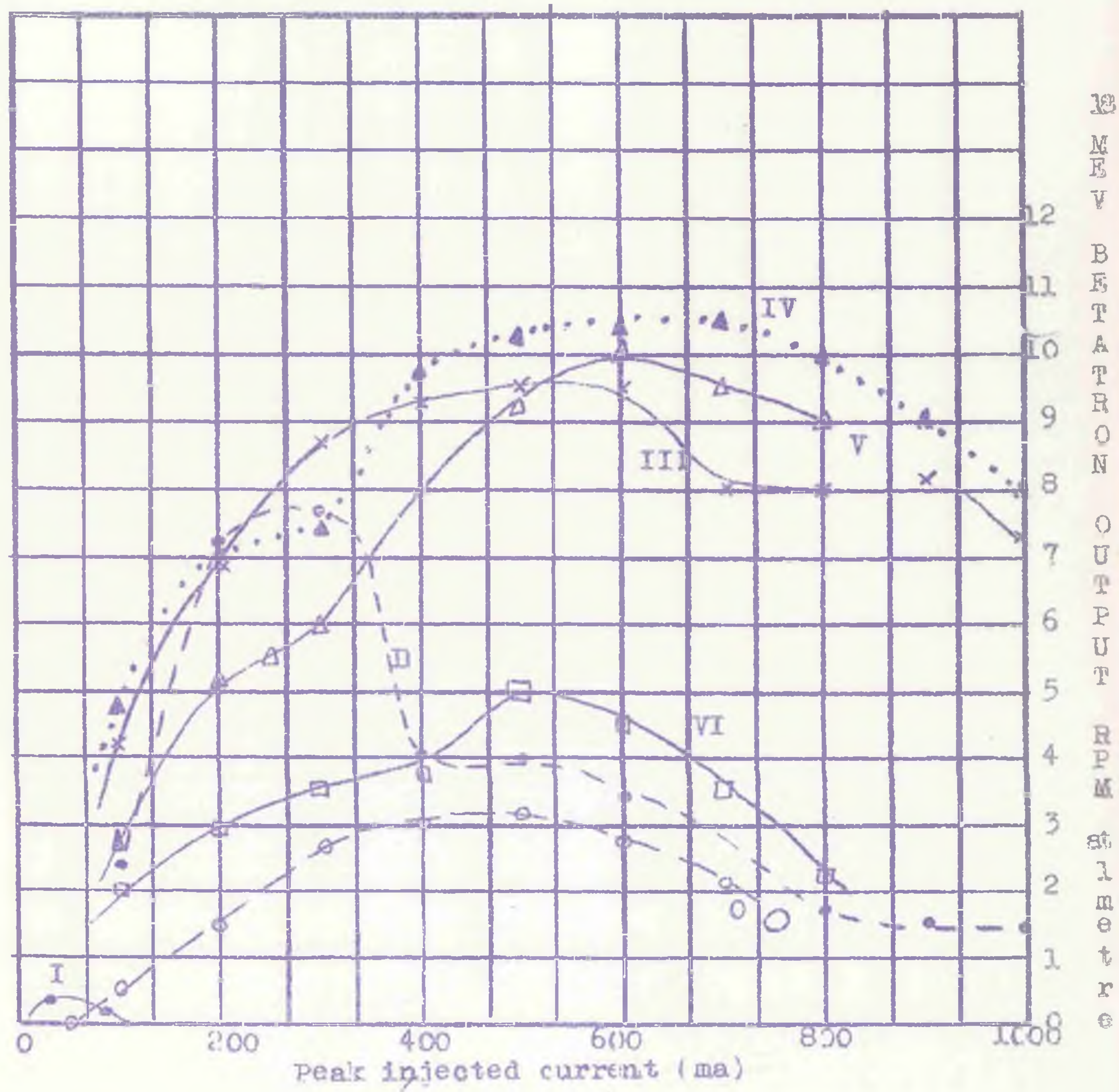
Experiment - Radial dependence of the focussing parameter, n, at the injector azimuth.

Curve I Normal radial variation of n (without steel wire shims)

Curve II Variation of n with radius with a current of -2.8 amp.pk. through a single coil on the upper pole face. From this measurement, we have in the central region immediately below the coil(radius 19 cm.):

$$\delta n \approx -0.02 \text{ / amp.pk.}$$

The theoretical formula gives $\delta n = -0.03$ /amp.pk. (equation (222)).



EXPERIMENT - Variation of output with intensity of injected electron current of kinetic energy 30 keV.

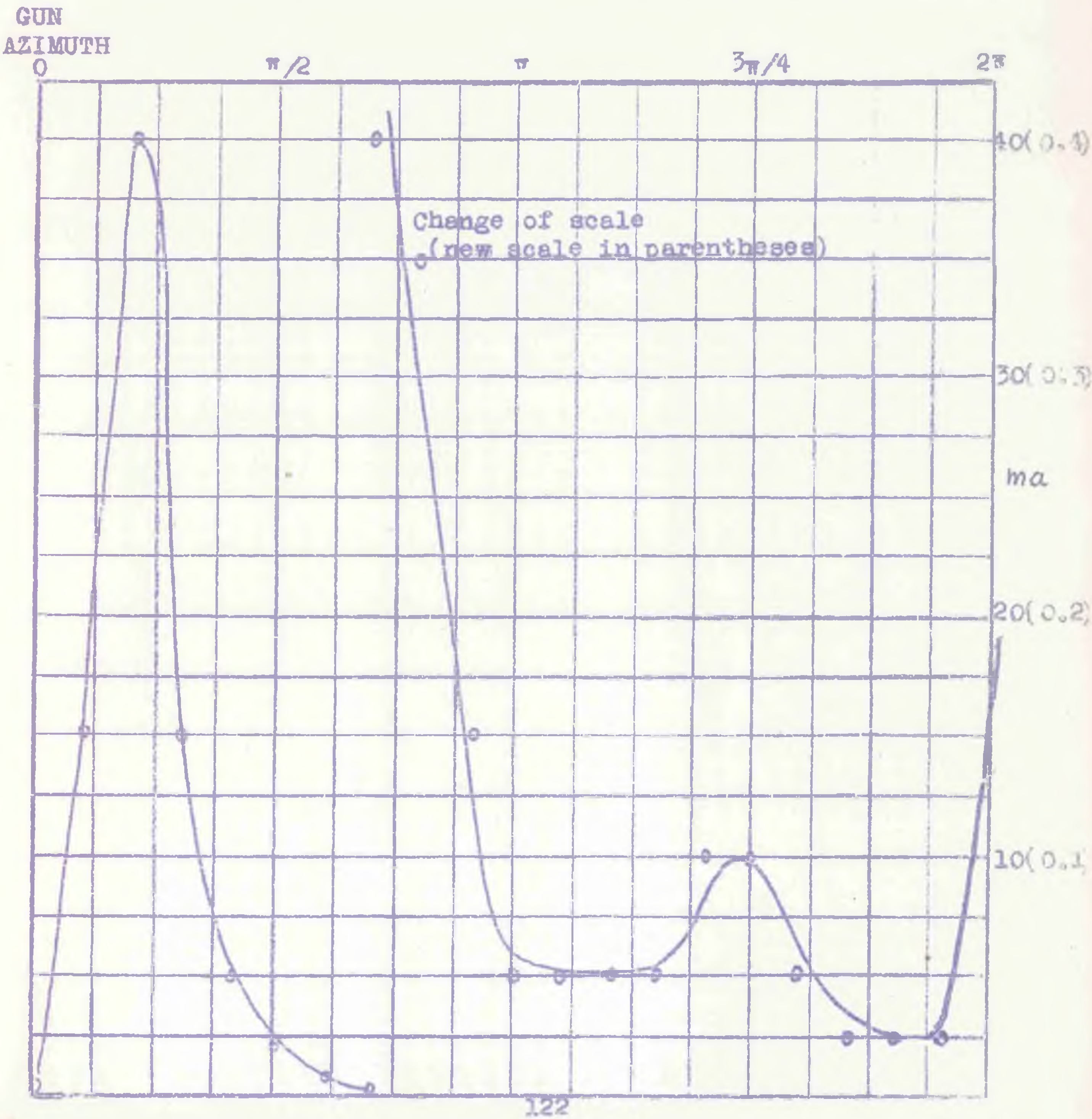
Other parameters - Betatron equilibrium radius 20 cm.
Injection gun radius 22 cm.

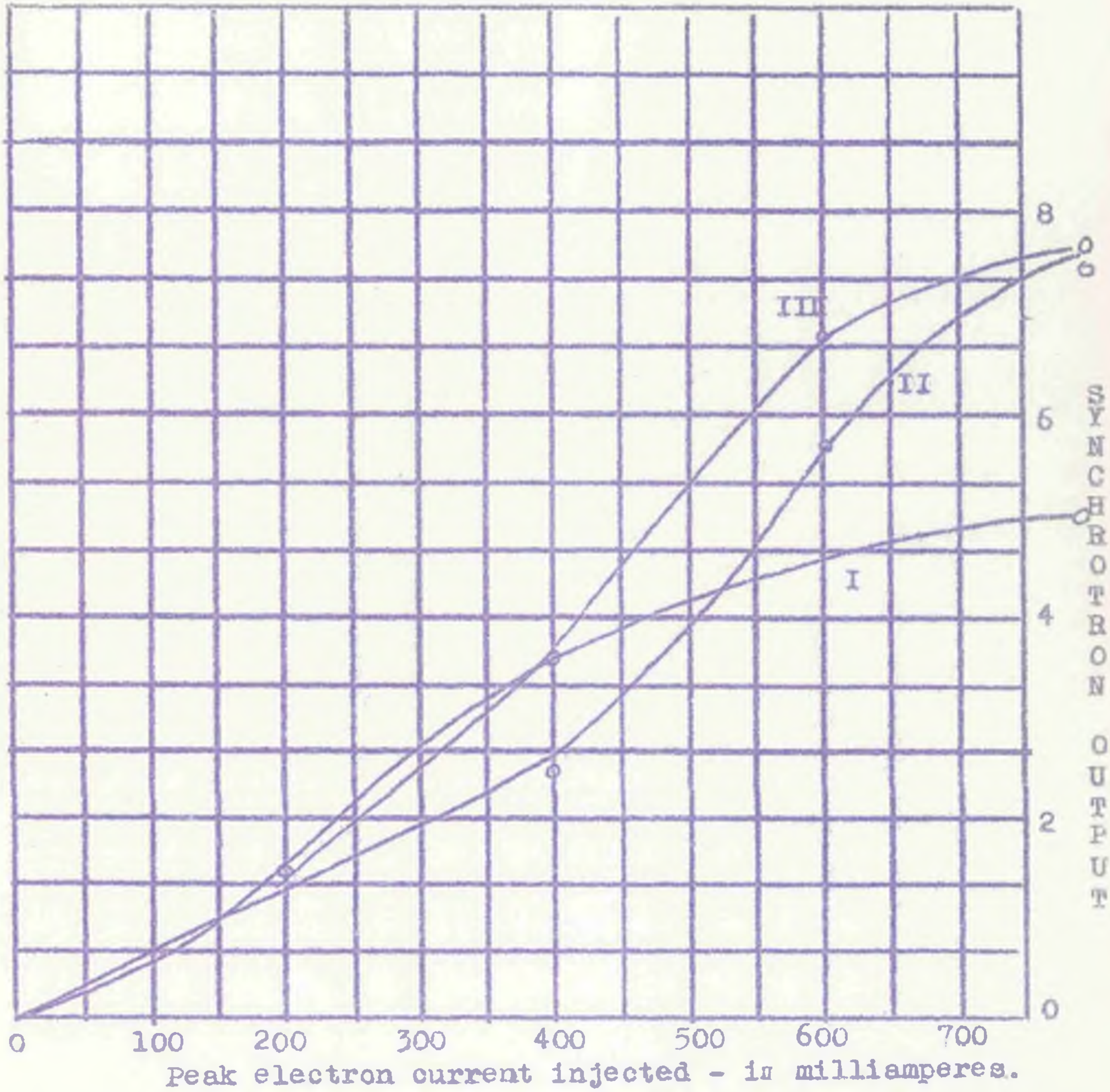
Curve 0 - Emmission characteristic with steel wire shims on poles.
Curve I - Emission characteristic without steel wire shims.
Curve II-Characteristic with no shims, current through single coil at 19 cm. = -0.7 amp., $\delta n \approx -0.015$.
Curve II -Characteristic, no shims, $\delta n \approx -0.03$.
Curve IV-Characteristic, no shims, $\delta n \approx -0.045$.
Curve V -Characteristic, no shims, $\delta n \approx -0.06$.
Curve VI-Characteristic , no shims, $\delta n \approx -0.075$.

2.1.3 <u>Some characteristics of a 340 MeV electron synchrotron</u> -

The following experiments give evidence of significant characteristics of the injection mechanism in a comparatively large betatron-started electron synchrotron.

EXPERIMENT - A typical azimuthal distribution of electron current at a given instant during the injection interval, under conditions which give the maximum intensity of circulating current(comprising electrons which complete at least one revolution.



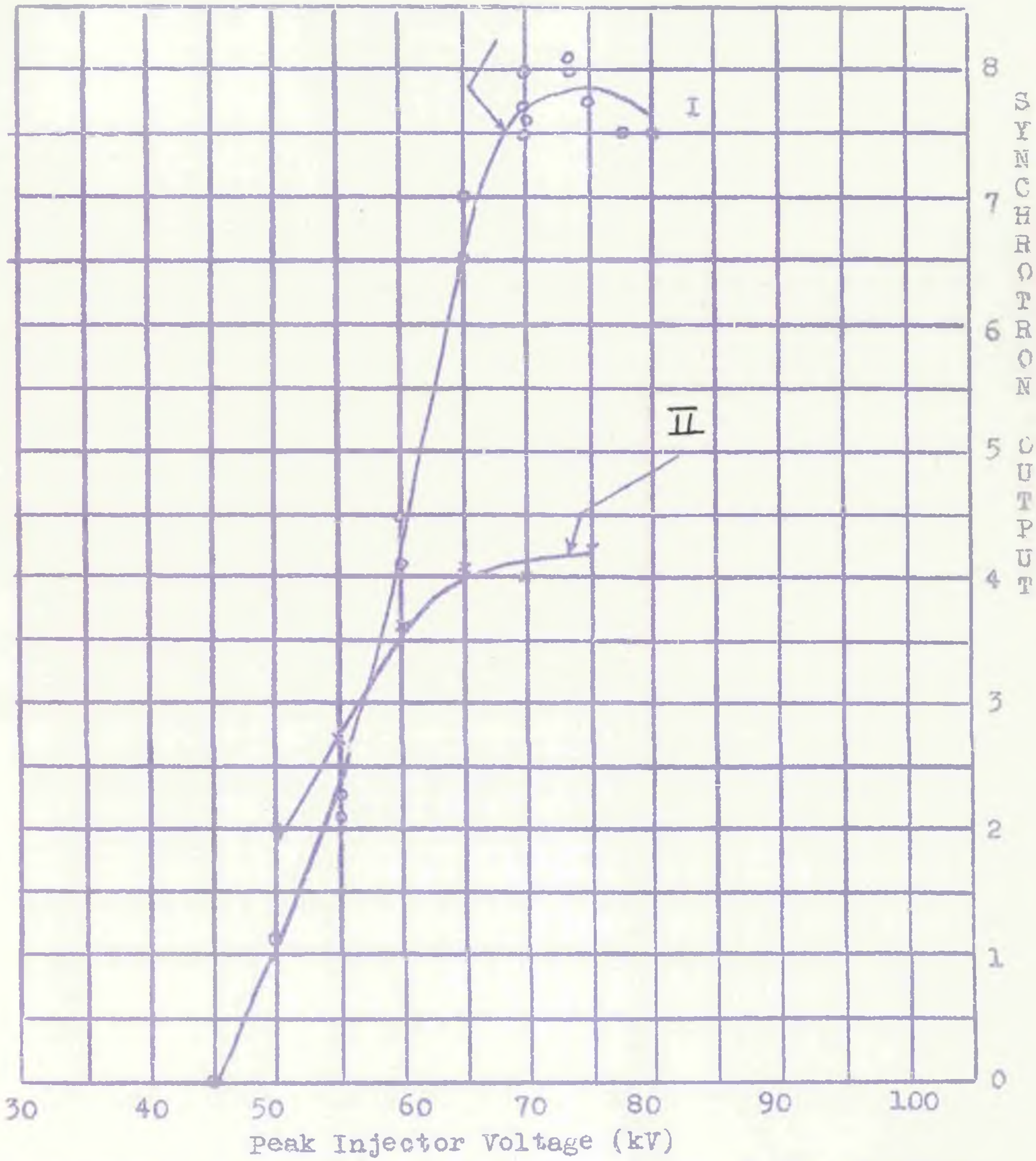


EXPERIMENT - Effect on output of the intensity of the injected beam of kinetic energy 75keV.

Curve I - Low harmonic field variations adjusted to give maximum output with peak emission 400 ma.

Curve II- Same optimisation as in curve I, this time made when the peak emission was 800 ma.

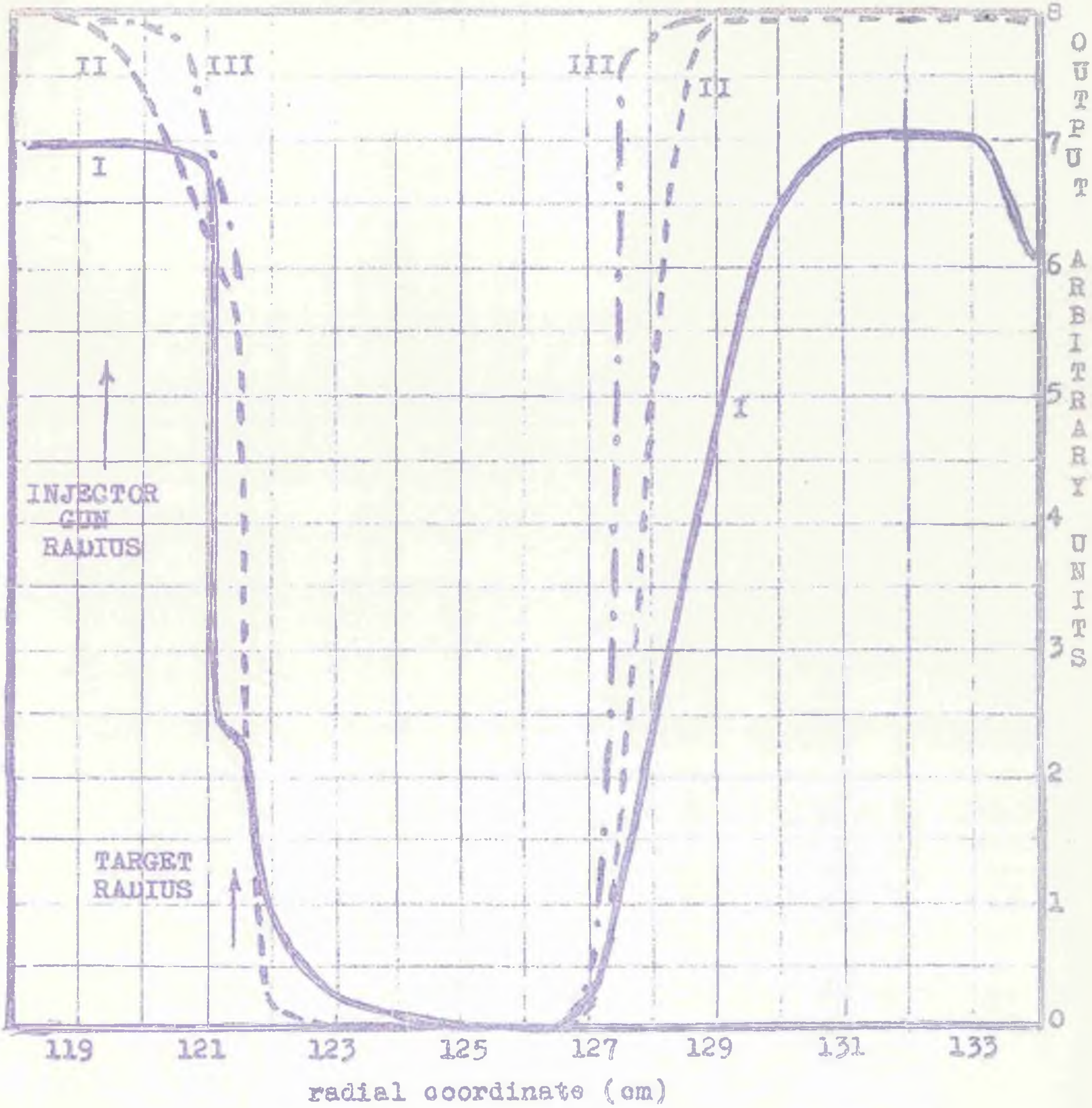
Curve III - Same adjustments made when the peak emission was 600 ma.



EXPERIMENT - The effect on output intensity of the electron kinetic energy at injection.

Curve I - Peak electrun current injected - 800 ma. All other parameters were unaltered as the injector voltage was increased.

Curve II - Electron current injected (max) 400 ma. Adjus-low harmonic field variations for each measured point to obtain maximum output, which is, as a consequence much less critically dependent on injector voltage. All other parameters had been adjusted to give maximum output with a current of 800 ma. 75 keV electrons.



EXPERIMENT VARIATION OF USEFUL RADIAL APERTURE WITH AZIMUTH.

I Radial limiters at $\theta = \pi$.

Injector voltage 70 kV, peak emission 600 ma.

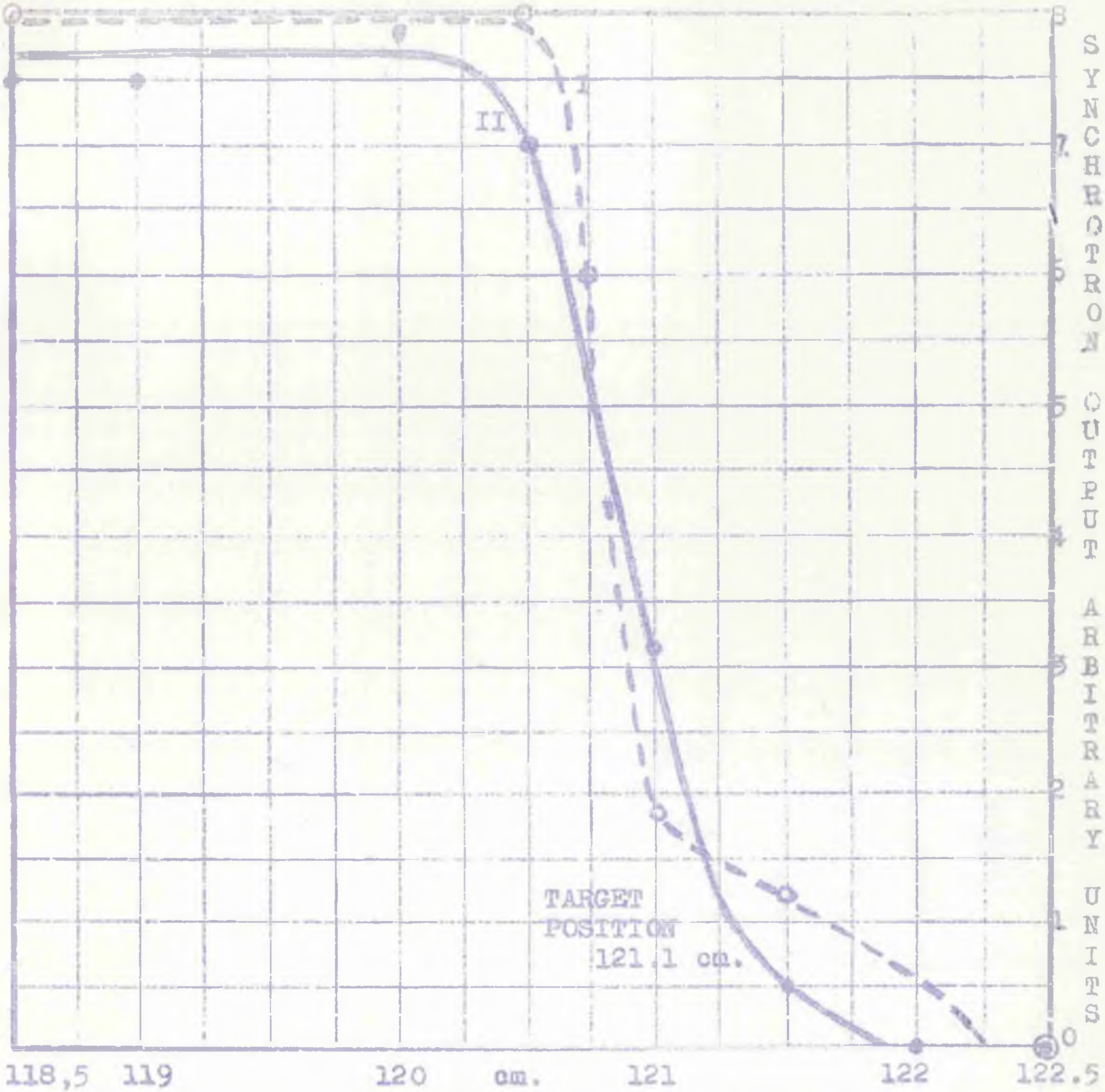
II Radial limiters at $\theta = 1.70\ \pi$.

Injector voltage 70 kV, peak emission 600 ma.

III Radial limiters at $\theta = 1.7\ \pi$.

Injector voltage 80 kV, peak emission 800 ma.

These are typical curves obtained when the accelerator is producing its best output.



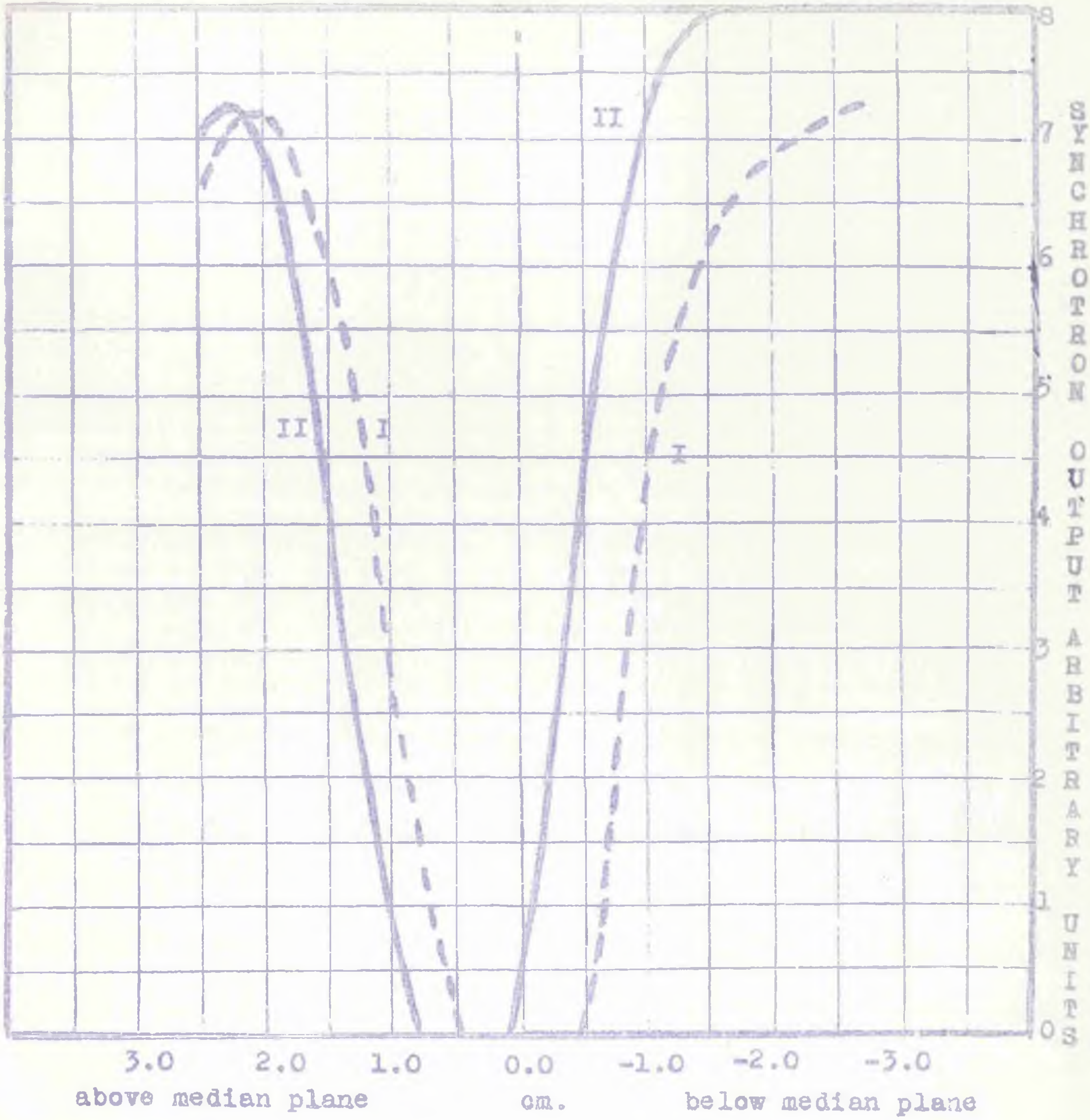
Radial position of the tip of the internal probe.

EXPERIMENT BEAM INTENSITY AS A FUNCTION OF THE RADIAL POSITION OF THE INTERNAL PROBE

I Probe inserted at $\Theta = \pi$.

II Probe inserted at $\Theta = 1.7\ \pi$.

The rapid decrease in intensity at about 120.7 cm. is due to the interception by the probe of high energy electrons before they lose any appreciable fraction of their energy in the target.



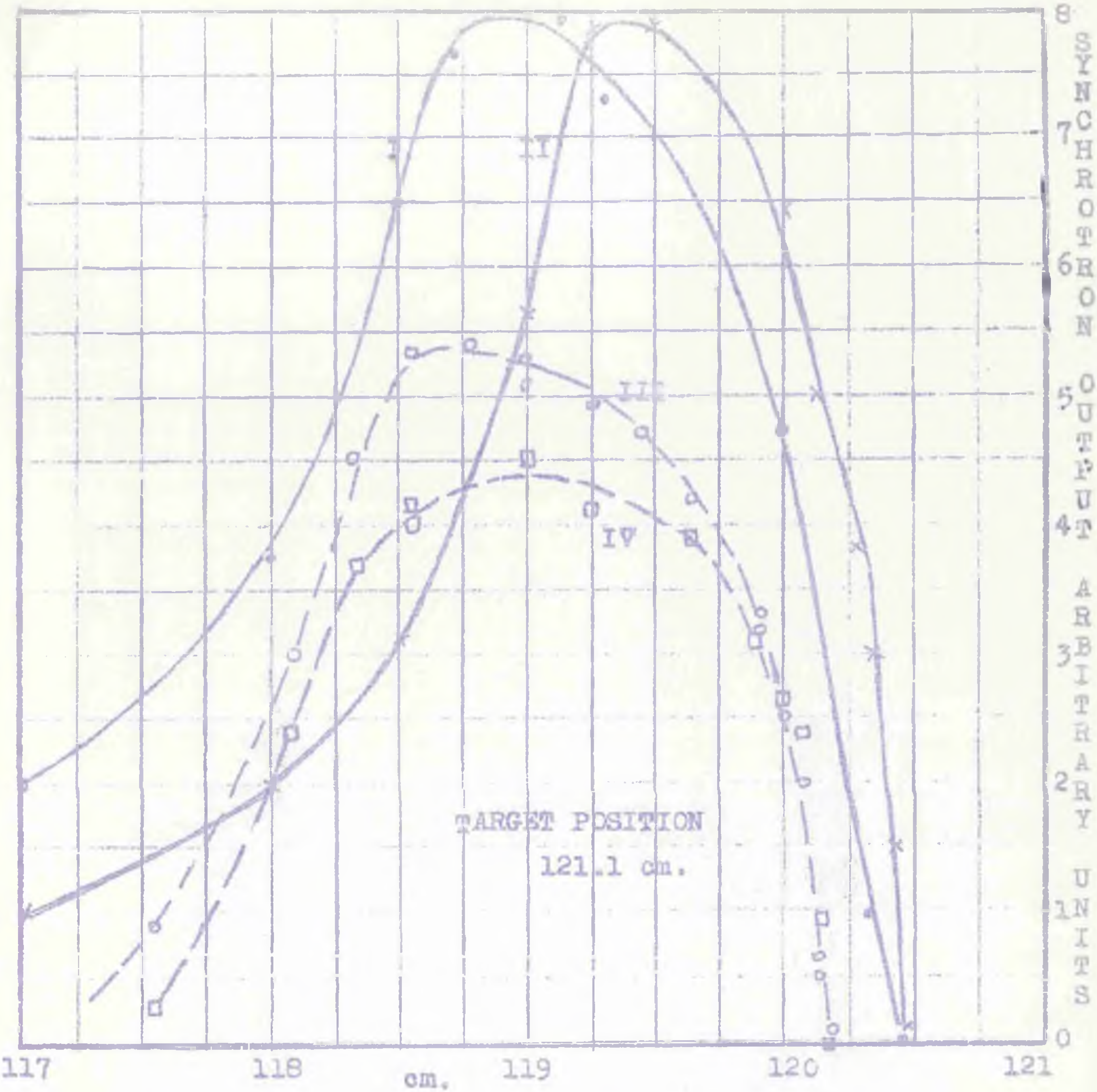
EXPERIMENT   VARIATION OF USEFUL AXIAL APERTURE WITH AZIMUTH.

Injector voltage 70 kV, peak emission 600 ma.

I   Aperture limiters at $\theta = \pi$.

II   Aperture limiters at $\theta = 1.7\pi$.

These are typical results obtained when the accelerator is producing its best output.

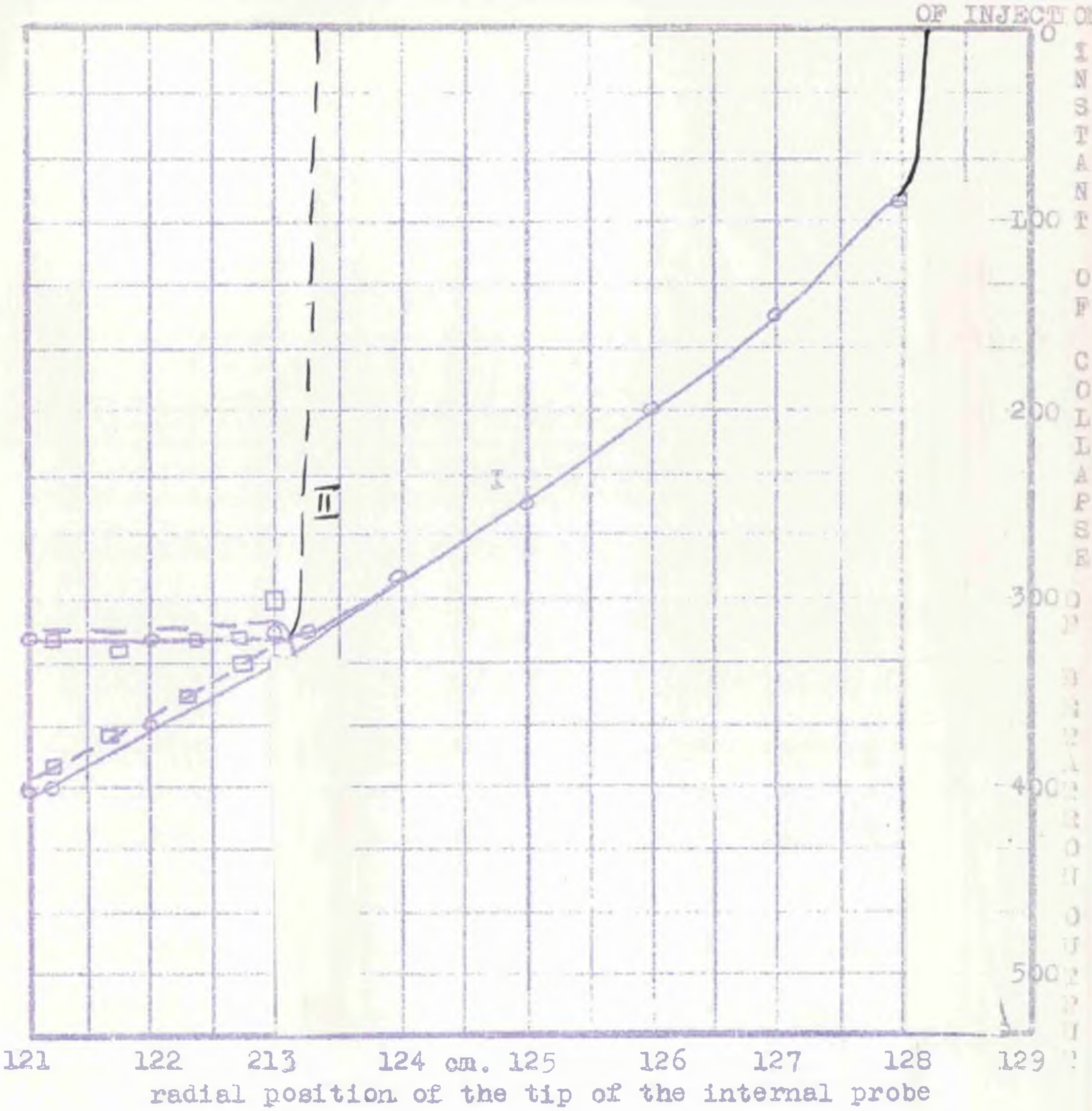


Radial position of injector filament.

EXPERIMENT BEAM INTENSITY AS A FUNCTION OF THE RADIAL POSITION OF THE INJECTOR.

I
II Injector voltage 70 kV, peak emission 70 ma.
Curve II was obtained one week after curve I, with everything except the magnitude of corrector coil currents unchanged.

III Injector voltage 73 kV, peak emission 800 ma.
IV Curve III was measured with the early timing peak, and curve IV with the late timing peak.

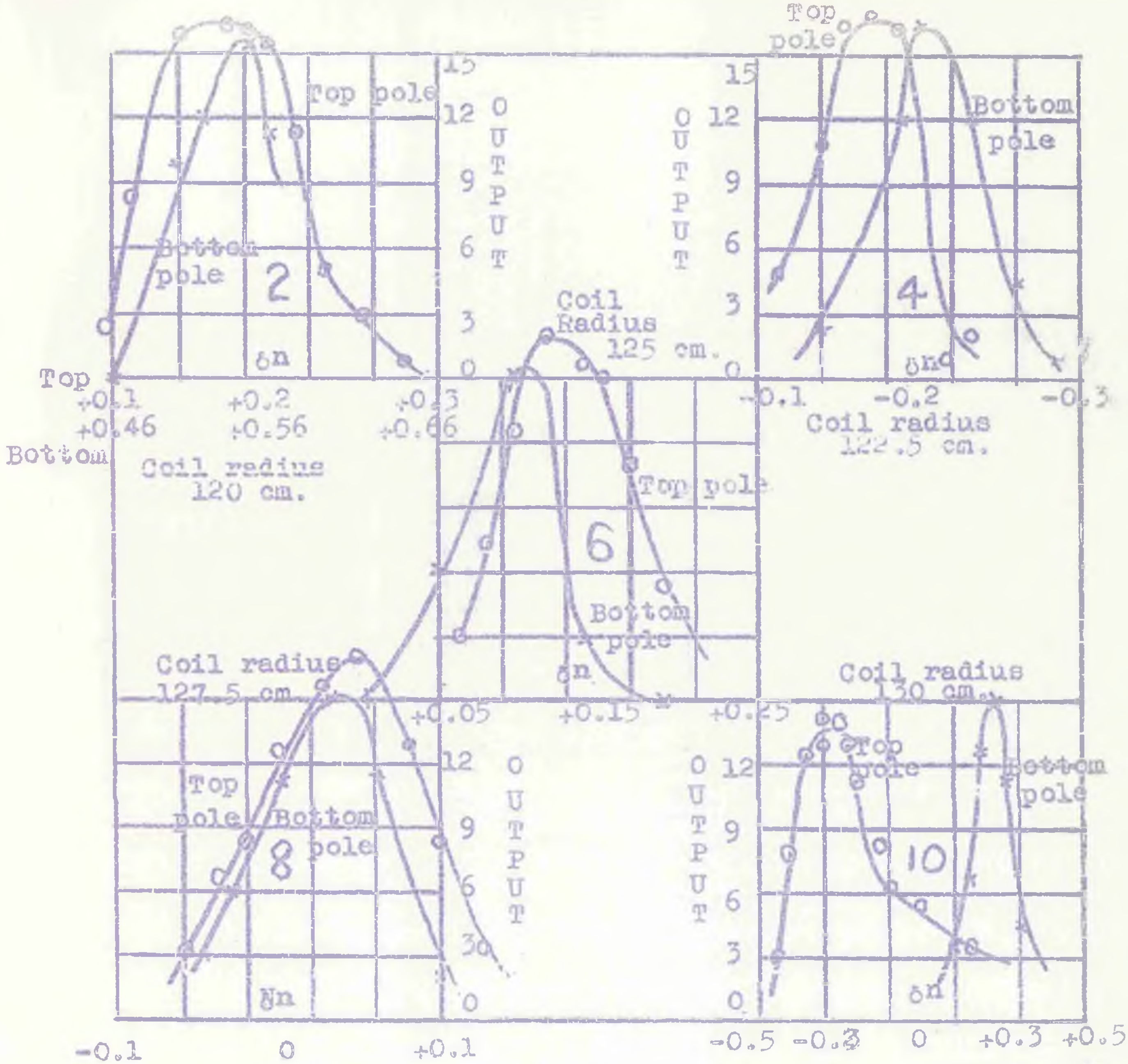


<u>EXPERIMENT</u> INSTANT OF COLLAPSE OF THE BETATRON OUTPUT AS A FUNCTION OF THE RADIAL POSITION OF THE INTERNAL PROBE.

I. Internal probe at $\theta = \pi$

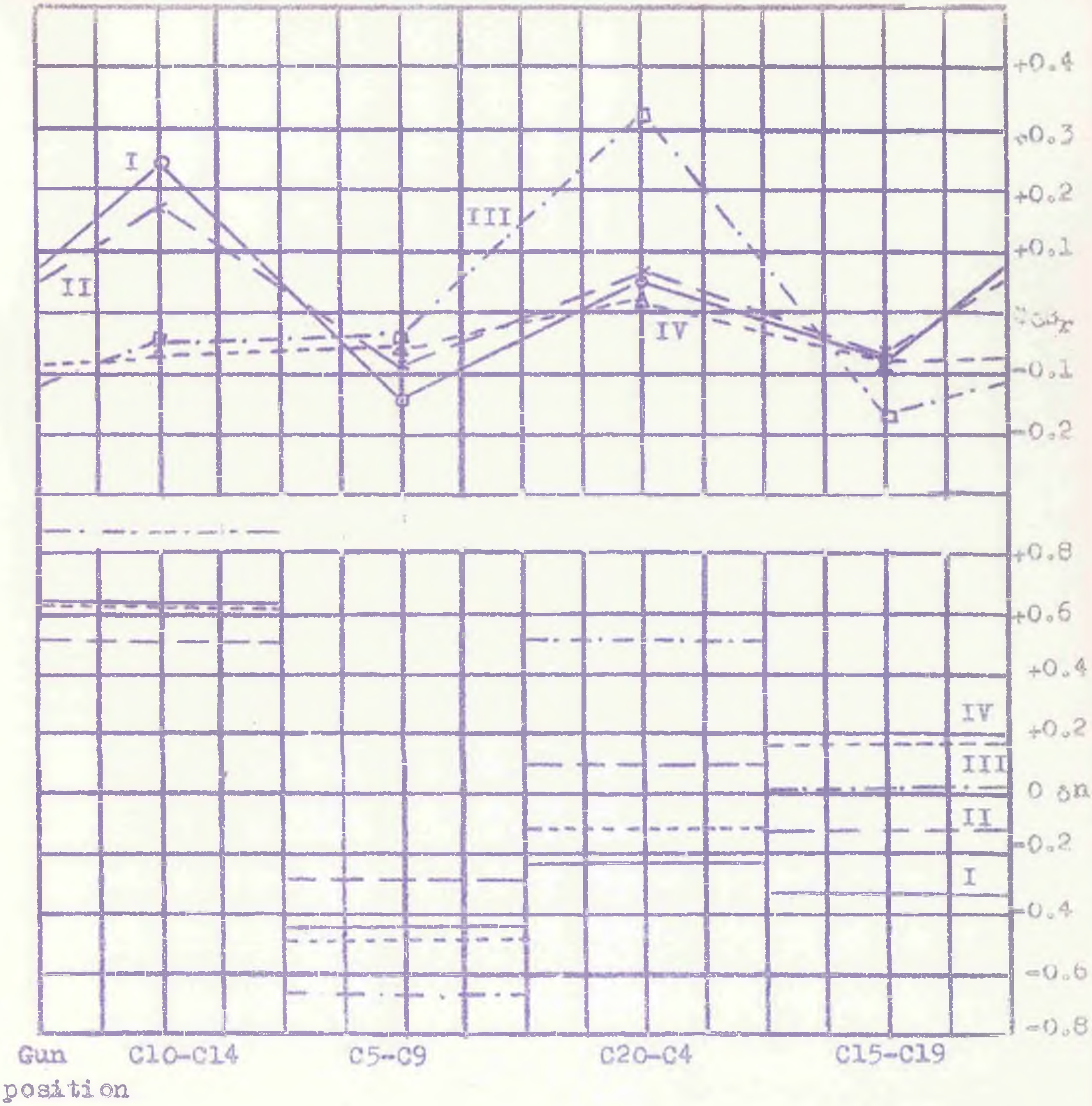
II. Internal probe at $\theta = 1.7\,\pi$.

In the case of curve I, the intensity of the betatron pulses increase when the internal probe is moved in beyond 121 cm(the target radius), but starts to decrease when the tip of the probe moves in beyond 122 cm.
An early 'loss' peak (50 microseconds after injection), is accompanied by a second about 10 microseconds later when the probe moves beyond about 127 cm. at $\theta = \pi$.



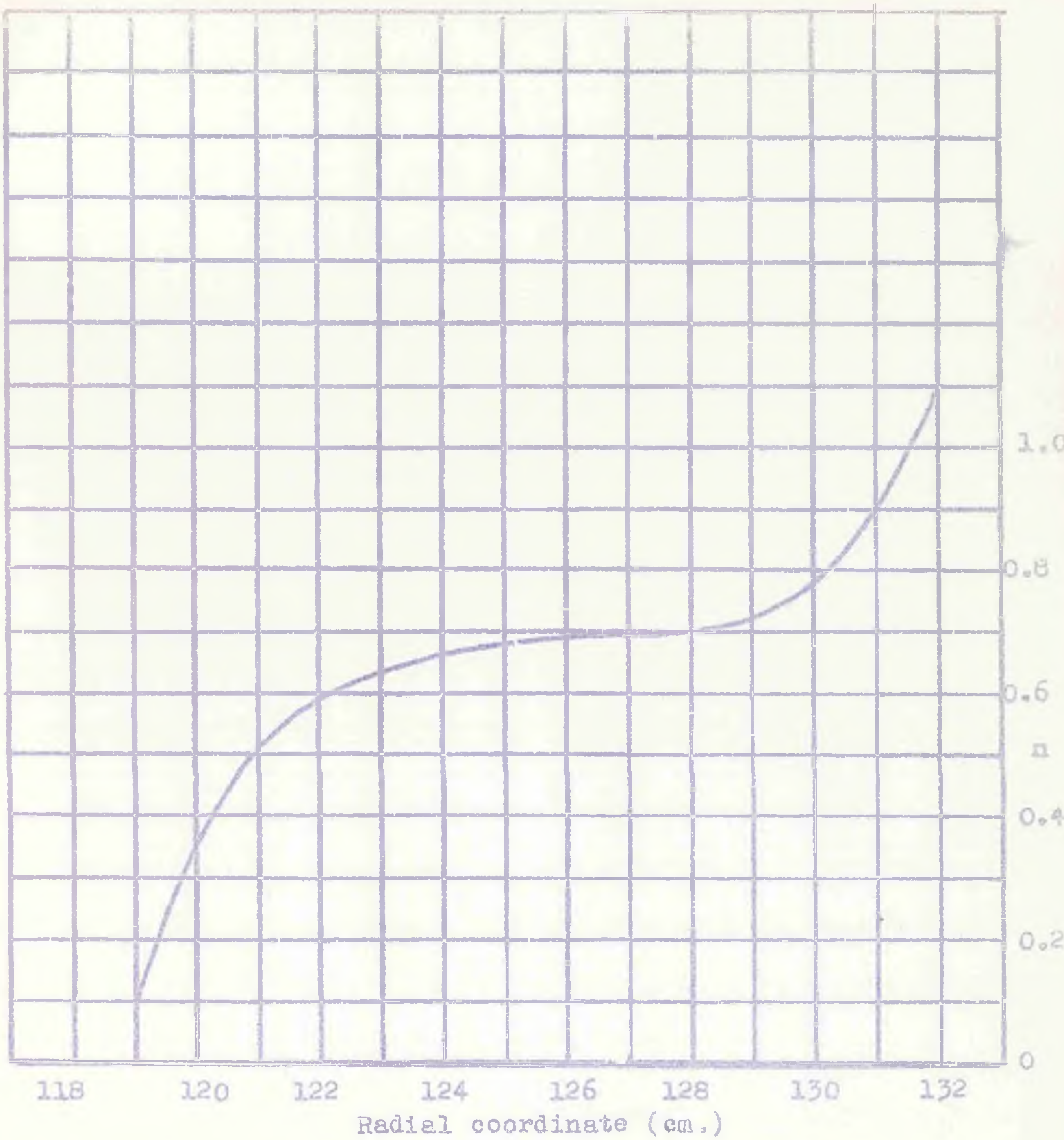
EXPERIMENT - Effect on synchrotron output intensity of current through individual pole face coils at different radial positions attached to upper and lower pole faces.

In all cases the current in 9 coils was kept constant, as were all other parameters, while the current in the remaining coil was varied, so that optimum output is obtained with currents corresponding to the maxima of all 10 curves. The magnitude of the current is given in terms of the change in field gradient index, δn, in the median plane at a radial position immediately below the coil, so that the total change δn_T at this position for a given output is the sum of the changes due to both upper and lower coils, plus smaller effects due to the other 8 coils. General formulae for calculating such effects are given in section 2.1.3 of this chapter (Four).



Measurement of the quadrant pole face coil currents at radii 123.74(Curves I and II) and 128.8 cm (Curves III and IV), expressed in terms of the changes δn, and δB_r they produce.

These two sets of azimuthal distributions were measured at different periods when the accelerator output was near its maximum. The different results obtained have to be correlated with the different currents in the $\langle n \rangle$ coils. Curves I and III were measured when the $\langle n \rangle$ coil currents were those corresponding to the maxima of the curves in Fig(page 130).



Radial variation of the guide field focussing parameter, n, in the median plane, 340 MeV electron synchrotron.) Paper F.

If the limits of the stable region are defined as the radii at which $n=0.1$, then these radial limits are at $r = 118.5$ and $r = 131.5$, an annular region of about 13 cm.

2.1.4 Some observations on, and interpretations of the characteristics of injection mechanisms in betatrons -

The problem is essentially that of low energy injection into weak focussing guide fields. The difference between injection into an induction accelerator and injection directly into a weak focussing synchrotron is that in the latter the accelerating mechanism is started independently of the guide field focussing mechanism. Thus particles may be injected on 'instantaneous' orbits whose radii are contracting comparatively rapidly as the guide field increases, and the accelerating voltage may commence at a time most suitable to trap in stable orbits the largest number of particles in such contracting orbits. The problem of getting particles to miss the injector structure is therefore rather straightforward on paper (contraction rates are sufficiently large to ensure some confidence on this point), but, in practice, as shewn by Cosmotron and Birmingham experience, efficient injection is not so readily obtained. We shall not discuss any further here the problem of direct injection into a synchrotron except to note that it is a special case of the transition problem discussed in section 4.

While betatron-synchrotron transition is very efficient, and easily put into operation, as our experience on the Glasgow synchrotron has shewn, the facts of successful injection in betatrons have no established theoretical basis, making the task of getting any one accelerator to work for the first time an extremely difficult one. It was the conviction of the author that while a comprehensive system of controlling the spatial distribution of the guide field is essential during the 'optimising' period, when both the magnet excitation and efficiency of injection would have to be considerably increased, such controls should be reduced to a minimum, and the magnet operated continuously at the lowest possible excitation during the period occupied by initial injection experiments, when effort would be concentrated on improving techniques for detecting low energy electrons and determining the number of revolutions completed by a fraction of the electrons within the acceptance interval. This programme, its consequences and the initial injection experiments are fully described in Paper G.

We turn now to an outline of the major characteristics of betatron injection mechanisms common to most efficiently, and reliably operating betatrons and electron synchrotrons.

CHARACTERISTIC I - Space charge forces: The probability of any single electron, injected at any angle and at any instant during the acceptance interval, avoiding the injector structure in completing successive revolutions during this interval is usually negligible, unless the spatial and/or time dependence of the guide field is specially adjusted in order to make this probability significant. This probability is considerably increased if the single electron is accompanied by other electrons, the increase being proportional to I^k, $k > 1$, where I is the magnitude of the injected current during the acceptance interval. As I is further increased a maximum (k=0) is reached, after which output decreases with increasing emission from the injector. The locus of the maxima of such curves for different injector voltages does not appear to reach a maximum for voltages up to about 100 kV.

CHARACTERISTIC II - In all accelerators the focussing index, n, lies in the range $0.5 \leq n \leq 0.75$. The greater fraction have $0.65 < n < 0.75$.

CHARACTERISTIC III - Successful injection is obtained with the injector at the radius, r_{inj} $> r_e$ 'external' injection,
$< r_e$ 'internal' injection,
where r_e is the betatron equilibrium radius.

CHARACTERISTIC IV - Injection efficiency is not sensitive to mechanical restriction of the radial aperture immediately behind the injector, such as may be obtained by extending the injector anode structure, or using separate aperture limiters.

Thus, Fig(pg.121) demonstrates characteristic I, Figs (pg.120 and 132) demonstrate characteristic II, the fact that the 20 MeV betatron uses external injection and the 340 MeV synchrotron uses internal injection demonstrates characteristic III, and finally Fig(pg.125) demonstrates characteristtic IV.

Indeed, this last characteristic is the most remarkable; output decreases by only about 10 percent when a vertical wire, part of a specially designed aperture limiter, was moved a whole centimetre beyond the radial tip of the injector anode.

In all cases the original Kerst-Serber theory[5] is quite indequate for explaining these basic common features of the injection mechanism.

Before going on to emphasize the distinctive features of the 2 accelerators with which the author has been associated, a brief survey of efforts made to explain the common characteristics mentioned above will be given.

All theories put forward have to include an explanation of how the defocussing effects of space charge forces assist injection efficiency (characteristic I), and produce characteristics III and IV. There are two classes of theories based on this space charge defocussing effect to provide an injection mechanism, one using a plausible time dependence of the space charge density during the acceptance interval, while the other uses a plausible spatial dependence of these forces. The two theories in the first class are due to Kerst[46] and Wideroe[47], the theory in the second class is due to the author.

Kerst[46] suggested that during injection the circulating current increases with time, storing electromagnetic and electrostatic energy in the beam which results in a loss of kinetic energy of electrons first injected, and a consequent shrinkage of their orbits. There is, in addition, an increase in radial oscillation amplitude during this period , and it is the net effect which is significant, a fact not emphasized by Kerst. Furthermore, there is the problem of explaining how the circulating space charge builds up for a sufficiently long period to give the required contraction , the fact that the effect is reversible when charge density eventually decreases, and the damaging fact that the orbit shrinkage(expansion) is of the same order as the oscillation amplitude build-up(damping).

When faced with the formidable features III and IV, the conclusion reached by several investigators[48] is that the operative injection mechanism is much more powerful than that suggested by Kerst.

The other theory in this class, due to Wideroe,[47] suggests that the doughnut fills up with primary and secondary electrons, and that it is the collapse of this accumulated charge at the end of the acceptance interval which produces the rapid oscillation damping. The only difference between Wideroe's theory and that of Kerst is that the latter based his estimate of oscillation damping on linear, azymptotic theory, while the former bases his estimate on non-linear theory using the potential concept introduced into betatron theory by Rajchman and Cherry.[49] Both theories are therefore subject to the same criticisms, from which one concludes that a more powerful mechanism than that produced by time dependent space charge effects is necessary.

A more powerful injection mechanism is based on the spatial dependence of the space charge density, in particular, a first harmonic component in the variation of charge density with azimuthal angle. This theory, due to the author, is presented in Papers A and B, where it is shewn to exhibit all the common characteristics mentioned earlier. Depending on a parametric resonance phenomenon near $\langle n \rangle = 0.75$, and bearing in mind that space charge forces increase n from the point of view of the radial oscillations, it appears plausible that conditions for its occurrence are extremely favourable provided the first harmonic azimuthal variation of the correct phase is present, and $\langle n \rangle$ is sufficiently close to 0.75, but never greater than this value. The reason for this was not emphasised elsewhere, but is nevertheless reasonably obvious. If, during the acceptance interval, space charge forces increase $\langle n \rangle$ above 0.75 where parametric de-excitation of radial oscillations may occur, then at the end of this interval successfully trapped electrons would have to be accelerated through a period during which $\langle n \rangle$ became exactly 0.75 as dissipation of charge density commenced, which would lead to their complete loss through radial instability. It would be impossible to guarantee that coupling inhomogeneities, other than that caused by spatial distribution of charge density, did not exist in magnitudes sufficient to give rise to serious radial instability in such circumstances. Removal of space charge defocussing, when, in the presence of space charge, $\langle n \rangle \leq 0.75$, would rapidly result in radial stability if the change in charge density was sufficient, even though the presence of some amplitude modulation of the radial oscillations in such circumstances would result in the amplitude of some previously damped oscillations subsequently increasing in amplitude. The magnitude of this amplitude modulation may be calculated from equation (10) of Paper B. Thus, an important feature of this mechanism is that it should be very sensitive to $\langle n \rangle$ near 0.75, with $\langle n \rangle \lesssim 0.75$ in the absence of charge, and as uniform as possible over the guide field annulus, especially over that half of it containing 'instantaneous' orbits. In the betatron, with external injection, this would be the outer half, while in the case of the synchrotron, using internal injection, this would be the inner half.

The effective δn produced by a current I amperes, of charged particles, forming a ring of circular cross-section, with a radius ζ times the mean radius of the ring, and uniform density is written -

$$\delta n_{zr} = -\delta n_{rz} = 6(e/m)v^{-3}\zeta^{-2}I(1-\frac{v^2}{c^2}).10^{-9} ,$$

where v is the particle velocity. Thus, for electrons of kinetic energy 25 keV($v=0.3c$), $\zeta=0.15$, $I=0.5$, $\delta n = 0.07$, while for electrons of kinetic energy 100 keV($v=0.55c$), $\zeta=0.15$, $I=0.5$, $\delta n = 0.012$, or for $\zeta=0.05$, $I=0.5$, $\delta n \simeq 0.1$.

Having briefly outlined the major theories that suggest possible injection mechanisms, and having partially concluded that, of these, only the resonance damping theory offers a mechanism sufficiently powerful to explain all the characteristics of the injection mechanism common to most betatrons, we turn now to some interpretations of the special characteristics of the injection mechanism in the two accelerators described in sections 2.1.2 and 2.1.3 of this chapter.

Figs (pages 119 and 130) provide some evidence of how critical injection efficiency is to changes in $\langle n \rangle$. Thus, in the betatron a change $\delta n \approx 0.03$ reduces output by half, and a change $\delta n \not< 0.05$ reduces output to zero. In the synchrotron a change of $\delta n \approx 0.04$ reduces output by half, and the change $\delta n \approx 0.1$ reduces output to zero. These results are unfortunately complicated by the fact that the change δn is accompanied by an axial orbit shift related to δn by the equation-

$$\delta \langle z \rangle \approx z . \delta n,$$

which may be derived from (220") and (222), where $2z$ is the mean axial height of the gap between pole faces. For the betatron, $z = 3$ cm., so that the axial shift for $\delta n = 0.05$ is 0.15 cm., while for the synchrotron, $z = 5$ cm. so that the axial shift is 0.5 cm. for $\delta n = 0.1$. These axial shifts are too large to be ignored, and we may only assume that some small fraction of the effect is due to axial shift of the median plane.

If we assume that the doughnut is uniformly filled with space charge at injection then the azimuthal distribution in the Fig(page 122), which was measured by noting the magnitude of the electron currents collected by individual sectors of the doughnut, gives a reasonable measure of the spatial distribution which provides the coupling inhomogeneity for parametric resonance at $\langle n \rangle_{eff} = 0.75$.

We turn finally to the two most interesting and significant sets of results described in the previous two sections. In the emmission characteristics displayed in Fig(Page 121) it is clear that there are more than one maximum, or, at the least, a maximum accompanied by a point of inflexion, in each characteristic. When these experimental results are observed to occur in a guide field whose focussing index has the radial dependence depicted in the Fig(pg.120), some very plausible conclusions favouring the resonance damping theory of injection may be deduced. It is suggested that the first maxima (or points of inflaxion) in the emission characteristics occur for instantaneous orbits with radii in the range 18 to 19 cm.

where $\langle n \rangle$ is closest to 0.75, while the second maxima (or points of inflexion) occur for instantaneous orbits with radii in the range 20 to 21.5 cm. where $\langle n \rangle$ is much lower - about 0.67 - and would require a larger charger charge density to bring $\langle n \rangle$ to about 0.75. Thus, in the case of curve II, the first maxima occurs for a peak emission of 250 ma, and the second for a peak emission of about 500 ma. The difference, 250 ma., corresponds to a change, $\delta n \approx 0.04$, while the difference between the maxima and minimum of the curve I (pg 120) at radial positions 19 cm. and 21 cm. is about 0.08. This difference is not serious because the δn due to the space charge is very sensitive to the value of ζ assumed. The important point to stress is that the qualitative nature of the emission characteristic and the radial dependence of the focussing parameter, n, make such a suggestion extremely plausible and support any theory which states that injection efficiency is at a maximum for a given value of the focussing parameter.

The other very interesting result is shewn in the Fig(page 129). Here we have the very remarkable fact that a betatron beam is detected some 100 microseconds after injection even when the tip of the probe is inserted so that its extremity reaches 128 cm., i.e. only the radial region between 128cm. and about 130 cm. is left for electrons to circulate throughout the acceptance interval and for the subsequent 100 microseconds, at the azimuthal angle $\theta = \pi$ which is diametrically opposite the gun. Indeed, the single vertical wire which comprises the radial limiter when used in the azimuthal position would not completely remove the betatron beam whatever its position within the stable region, and, as shewn in the Fig.(pg125) completely removed the synchrotron beam only over the radial region between 125 cm. and 126.5 cm. Similar experiments at $\theta = 1.7\,\pi$ (just behind the injector) produced quite different results; the betatron output was reduced drastically to zero when the tip of the probe, or the vertical radial limiter wire was moved beyond about 123 cm. These two striking experimental facts suggest the following observations:

(a) Curve I represents the variation of the radius, r_i, of some instantaneous orbits which follow closely the variation of the betatron equilibrium orbit radius, r_e, with flux density. Due to the slow contraction of the latter, and the relation (205) between r_i and r_e, we have $r_i \geq r_e$. The time scale has to be converted to a flux scale before it may be compared with the betatron orbit characteristics described in Paper F. Thus, 100 microseconds here represents a guide field flux density increase of about 80 gauss, or an increase in electron energy of about 3 MeV, and it appears that a betatron beam $\gtrsim$ 12 MeV may be obtained. A rather curious feature is the splitting of the betatron beam at about 123.5 cm. and an energy of about 9 MeV, so that one part spirals in very rapidly, while the other part spirals in rather more slowly. This cannot at present be explained.

(b) The drastic decrease of injection efficiency to zero when the internal probe is inserted beyond about 123 cm. behind the injector, and the fact that a similar probe may be inserted well beyond the centre of the stable region (125 cm.) diametrically opposite the injector without reducing injection efficiency to zero, suggests that during the injection interval the amplitude of the radial oscillations may be large only in the vicinity of the injector and very small diametrically opposite the injector. This can occur only if rapid oscillation damping is accompanied by very slow precession rates. Small damping rates with fast or slow precession, or rapid damping with rapid precession rates will not exhibit such characteristics as readily as the first combination of damping and precession rates, which are easily obtained with the resonance damping mechanism. Furthermore, it is interesting to note that that particles oscillating about orbits outside the conventionally defined acceptance interval (about 119 cm to 125 cm.), i.e. those orbits with radii between 125 cm. and 128 cm., are accepted for induction acceleration. The assumption implicit in these observations is that the effects observed experimentally at $\theta = \pi$ appear also in the vicinity of this azimuthal position, in other words it is not due to any local distortion of the instantaneous orbits in this position. However, it is unlikely that any such changes in the shape of instantaneous orbits will explain these remarkable results.

All such observation on the injection mechanism must take into account the possible existence of errors in field gradients due to field disturbances which become significant at the low guide fields existing during the injection process. Thus, measurements described in Paper F indicate errors in $n_{zr} \sim 0.1$.

3. EJECTION MECHANISMS :

3.1 Statement of problems -

There is a large variety of ejection mechanisms, all of which depend on providing a perturbation either of the guide field focussing or the accelerating field focussing mechanisms. The difficulty of extracting a well defined beam from a cyclic accelerator, however, increases considerably if the beam is to consist of charged particles(electrons or protons), and it is much simpler to make experimental use of the high energy charged particles within the accelerating chamber, or to convert them into an electrically neutral particle beam (neutrons or photons) which is then readily extracted in a well defined beam .

In addition, there is also the problem of extracting such charged or neutral particle beams over extremely narrow or extremely wide time intervals depending on the type of experiment one is performing, although with some experiments the width of the extracted beam is of little interest. Very narrow output pulses are required either for experiments in which the lifetime of short lived particles (e.g. muons) or excited nuclear levels is to be measured accurately in the absence of background radiation from the accelerator, or, if a charged particle beam as monoenergetic as possible is required.

1 Very wide output pulses are generally needed when neutral particle beams are used, and the need for a monoenergetic charged particle beam is not as important as the improvement in counting statistics that results from the ratio -

beam on / beam off

being as large as conveniently possible.

3.2 Ejection of charged particle beams -

Much effort, and many different mechanisms have been applied to this problem. However, if it is essential to extract efficiently a well-defined charged particle beam from cyclic accelerators, some form of magnetic shunt is essential. Such shunts have been successfully used only in cyclotrons so far where a thick-walled magnetic channel is used with an ejection mechanism powerful enogh to separate the orbits of particles in successive revolutions sufficiently to enable a significant number to enter the channel without striking its walls. Perturbations of the accelerating field mechanism (such as by shifting the equilibrium orbits by rapidly changing freqnency) are not sufficiently quick acting. In such cases a particularly convenient mechanism is the excitation of a radial parametric resonance. For the cyclotrons, where $\langle n \rangle \approx 0$ a convenient radial resonance is obtained in regions L, N (Fig. on page 83), which gives radial instability with a second harmonic variation $n(\theta) = \alpha_2 . \cos(2\theta + \gamma_2)$, without axial instability provided certain coupling resonances are avoided. Paper D gives a full analysis of this mechanism, and all source references.

The application of parametric resonances of this class to synchrotrons, where $\langle n \rangle \approx 0.7$, is discussed in Paper K.

It may be concluded that for efficient charged particle ejection a magnetic shunt and a suitable parametric resonance mechanism are essential, a conclusion fortified by the successful application of such principles to the University of Liverpool syhchrocyclotron.[50] It might be added that in these circumstances it would be natural to obtain a narrow output pulse, and the modification of the mechanism to obtain wide output pulses would be difficult, though not impossible.

3.3 Ejection of neutral particle beams -

This is a much simpler problem, and a great many solutions are known, which is not worth cataloguing here. The major problems here are not efficiency and well defined beams; over these factors we have little control other than altering the physical size and material of the converter used. Instead, the problem is that of obtaining when required very narrow or very wide pulses. For the former a parametric resonance is again the obvious solution, and the mechanism described in Paper K has been used successfully in America to produce output pulses of photons from a large synchrotron over time intervals ≤ 0.2 microseconds.[51]

The problem in obtaining wide output pulses is that of maintaing uniform intensity during the duration of the pulse. Thus, while it is easy to separate photons coming at the start of the pulse from photons coming at the end of the pulse by intervals ~ 1 to 2 milliseconds in the Glasgow synchrotron, it is much more difficult to ensure that they emerge at the same intensity throughout this period. The mechanism used to produce such wide pulses in Glasgow consists of perturbing synchronous stability conditions by slowly reducing the peak accelerating voltage, and non-uniform distributions which vary from pulse to pulse are common, and control is difficult.

The problem here is that at the time of ejection the electron bunch has been reduced considerably in size through asymptotic damping, and has to be moved through ~ 3 cm. before electrons begin to strike the converter (target) and subsequently through ~ 1 cm. before all electrons are converted to photons, or lost through striking the walls of the doughnut. Ideally the electron beam should be moved radially comparatively quickly at first and subsequently much more slowly as the first electrons begin to strike the converter. Flux/core saturation, which is used for beam ejection in large and small betatrons, produces the opposite effect, i.e. radial motion becomes increasingly rapid with time. The time behaviour of the orbit suggested as ideal above, if wide output pulses are to be obtained in a controlled manner, can be obtained with the aid of an inductive transient started late through the accelerating period and used to provide induction acceleration or decelleration (expansion or contraction of particle orbits). A small ignitron may be used to provide the switch starting the transient current build-up in the ejector coils, which may also derive their power from the main tuned circuit exciting the synchrotron magnet.

4. <u>TRANSITION MECHANISMS</u> :

Discussion here will be confined to the betatron-synchrotron transition mechanism which occurs in betatron-started synchrotrons. Treatment is, however, readily generalised to deal with all types of transition mechanism, a collective term describing particle behaviour during the interim period between the start of one form of accelerating mechanism and the end of another, when the two types of accelerating mechanism co-exist. Thus, Betatron-synchrotron transition mechanism deals with particle motion in the interval between the start of synchronous acceleration and the end of induction acceleration , where the former is taking over from the latter. The efficiency of this transition mechanism is a measure of the number of particles which are accelerated in stable synchronous orbits after the end of betatron acceleration relative to the number in stable betatron orbits before the synchronous mechanism was started.

The efficiency of transition is equal to the capture efficiency into phase stable orbits provided the radial excursions occuring as a result of the largest phase oscillation, when superimposed on the radial oscillations about the betatron equilibrium orbit as a result of guide field focussing and azimuthal field variations, allow particles to remain within the stable region of the guide field, defined pessimistically as the annular region of the guide field between $n=0$ and $n=1$ (see figure on page 132). Thus, displacement of the betatron orbit r_e, from the centre of the stable region inevitably increases the probability of losing particles undergoing the larger amplitude phase oscillations, although from the analysis in Chapter Two, section 3.2.2(Page 92) it is shewn that the 'instantaneous' orbits about which radial oscillations occur never coincides exactly with a time-dependent betatron orbit except for the instant when they may intersect one another. Also, it is shewn in equation (206) that the amplitude of the radial excursions accompanying phase oscillations is proportional to $V^{1/2}$, where V is the peak value of the r.f. accelerating voltage, which is assumed time-independent. The main problem is that during the transition period we are discussing, V is alwaps time-dependent, rising more or less rapidly to its maximum value, which is usually arranged to remain constant during the period of synchronous acceleration, depending on the power in the r.f. oscillator and the Q of the resonant circuit (the quarter-wave resonator which forms part of the dougnut). Particle behaviour during this transition period when V is increasing, and the stable phase angle θ_e slowly increasing as the magnitude of induction acceleration slowly decreases (gradual saturation of the betatron flux bars causing a gradual contraction of the radius of the betatron equilibrium orbit radius), is described therefore by a non-linear and non-autonomous synchrotron equation, with whose analysis one is concerned in order to estimate quantitatively the efficiency of capture during this period. Many earlier attempts to solve it, under a variety of simplifying conditions, are referred to in Paper J,[52,53] which also contains a new approach to the problem, and quantitative estimates of capture efficiency.

In the analysis presented in Paper J it is assumed that V rises to its maximum value in a series of discontinuous steps, so that during any interval to be made as small as desirable, both V and $\theta_e = \sin^{-1}(v'/V)$. A whole family of integral curves (phase trajectories) of the general form (136) is thus obtained, each characterised by different, but constant, values of V and v'. Initial conditions fix the values of θ'_o, and $(d\theta'/dt)_o = \psi_o$, and the values of V and v' for the first trajectory, and subsequent values are obtained under the stipulation that physical continuity of elementary phase trajectories is maintained, although, using such methods, the trajectory as a whole may not be analytic at these points. The result of such analysis shews that a particle is captured in phase stable orbits if it lies within the region in phase space defined by-

$$\psi_o^{\,2} = (\pi I)^{-1}\left\{ F(\theta'_o) \mp f(v') + V_M\cos(\theta'_e) + v'_m(\theta'_o + \theta'_e - (1+2\beta')\pi \right), \qquad (224)$$

where it is assumed that the peak r.f. voltage, V increases from an initial value, V_1, to V_M, while v' increases from v'_1 to v'_M during the same period, and subsequently remaining constant, with $\theta'_e = \sin^{-1}(v'_M/V_M)$. Also,

$$F(\theta'_o) = V_1\cos(\theta'_o) + \sum_{\nu=1}^{M-1}(V_{\nu+1} - V_\nu).\cos\left(\theta'_o - \frac{2\nu\pi}{M}\right), \qquad (224a)$$

$$f(v') = (2\pi/m)\left\{ \sum_{\nu=1}^{M-1} v'_\nu - (M-1)v'_M \right\}, \text{ and} \qquad (224B)$$

$$-2\beta'\pi + \pi - \theta'_e \lesssim -2(M-1)\pi/m + \theta'_o\,,\ \beta' = \text{positive integer.} \qquad (224c)$$

The limiting cases of this general treatment occur for $M=0$, and $M \gg m$, corresponding to the instantaneous rise of V to its maximum value, the assumption made in Goward's[52] treatment, and the infinitely slow rise of V to its maximum value, the assumption which makes Kaiser's[53] treatment exact when $v'=0$ throughout. In these limiting cases we have-

$M = 0$: $f(v') = \beta' = 0$, $F(\theta'_0) = V_M \cos(\theta)_0$.

Capture efficiency, assuming that particles are uniformly distributed in a rectangular region in phase space stretching from $\theta'_0 = -\pi$ to $\theta'_0 = \pi$, approaches 100 per cent only if $V_m \gg v'_M$, conditions under which stable phase oscillations are sufficiently large to make particles spiral out into the unstable regions of the guide field. Experimental results do not agree with predictions from this limiting case of general theory.

$M \gg m$: $\beta' = M = \infty$, $F(\theta'_0) = 0$.

The stability condition now becomes -

$$\psi_0^2 = (V_M/\pi I). \qquad (225)$$

The remarkable feature of (225), first noticed by Kaiser using a more approximate analysis, was that it was independent of θ'_0, and that the ψ_0 dimension of the <u>rectangular</u> stable region defined by (225) is is $(1/2)^{1/2}$ times the maximum height of the stable region associated with voltage, V_M in the limiting case $M=0$. Both these facts mean that 100 per cent capture efficiency is obtained if the r.f. voltage rises infinitely slowly to a value $(2)^{1/2}$ times the voltage whose stable region has a maximum radial width corresponding to the radial spread of particles at the start of the transition period.

A general treatment of the problem may be made using the results in equations (224) for any function $V(\theta')$, and quantative capture efficiencies obtained. In paper J this analysis has been applied to the case where V increases linearly with phase angle θ'. The results given there show that capture efficiencies rapidly converge to 100 per cent, for $1 < \beta' \leq 10$.

In the case of the Glasgow synchrotron this range of β' corresponds to rise times of the r.f. peak voltage, V from zero to a maximum value of the order of 3000 volts while the electron energy is of the order of 3 MeV, of from 3 - 30 microseconds. It has been found that deliberate attempts to increase the rise time above about 10 - 20 microseconds have little effect on the synchrotron beam intensity, and hence on capture efficiency. Thus there is good agreement between the theoretical predictions and experimental results although the former was based on a linear increase of V with <u>phase angle</u>, θ', while the experimental results were obtained with r.f. shaping which gave a linear voltage increase with <u>time</u>. This is some proof that the exact manner in which the voltage rises is not of importance to the first order as is the period over which it rises to its maximum value. Theoretical proof of this could be obtained from the theory presented here, although it would hardly be worth the effort in view of the order of the effect one is expecting.

C H A P T E R   F I V E

CONCLUSIONS -

THEORY OF MEGNET DESIGN AND PRACTICE OF ACCELERATOR OPERATION

In this monograph the majority of space has been devoted to general theoretical principles, and the mathematical theory required for their solution (Chapters One to Three), while in Chapter Four we discuss the facts of injection into betatrons and electron synchrotrons, among many other initial value problems, and find that in 15 years since the development of the betatron all theories past and present do not wholly succeed in explaining them. These two aspects of accelerator physics, i.e. the theoretical idealised aspect and the operational aspect, are not inconsistent, but they do indicate a principle relating to economy of effort in this field of endeavour.

A line has to be drawn between over-emphasizing the importance of experimental measurement of guide field characteristics in the absence of electron beam effects (or proton beam effects in proton accelerators), and relying entirely on injection experiments in commissioning an accelerator. We must stress the reference to experimental measurements and the exclusion of theoretical efforts, because the latter do not hold up injection experiments, while the former do. This is especially important in the case of the strong focussing magnets at present in the design stage. All measurement of field characteristics, at all excitations up to the maximum, and in the absence of space charge effects, should be done as individual sections of the magnet are completed, the form taken by the excitation being identical to that envisaged for the complete structure. As soon as the magnet is assembled as one complete unit, and the mechanical stability, structural alignment and electrical characteristics(voltage transients etc.) are within the required limits, then injection experiments should commence, supplemented by field measurements as they appear necessary. In view of the experience with the two accelerators discussed in Chapter Four, this procedure is strongly recommended.

Another point to stress is that provision for control of every important parameter must be included in the design, and brought gradually into use during the initial injection experiments. Strong focussing accelerators are more complex dynamical structures than weak focussing accelerators, and while such controls near the injection interval only suffice for the latter, programmed controls that may be applied throughout the period of acceleration will be necessary for the former.

With these few comments on accelerator programme planning, we conclude this thesis.

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