The Electro-and Photoproduction of Pions

by

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CHAPTER 1.

INTRODUCTION.

The close relation between the interaction produced by a moving charged particle and that due to incident electromagnetic waves was first pointed out in 1924 by Fermi(1). Since pion production by electrons is induced by the action of this virtual electromagnetic field it is closely related to pion production by photons.

Weizsäcker(2) and Williams(3) considered in particular the field due to relativistic electrons with motion not appreciably affected by the process induced. They showed that in this case the field contained predominantly transverse components and were led to suggest the approximation that the longitudinal component be considered zero. Within the spirit of this approximation the total cross-section for the electroproduction of a pion from a nucleon was calculated by Feshbach and Lax(4) and by Strick and ter Haar(5), the latter investigating in particular its threshold behaviour.

Expressions for the meson spectrum , integrated over all angles, were derived by Sneddon and Touschek(6) using the fixed-source pion theory calculated in Born approximation. Nucleon recoil effects were neglected and the longitudinal terms in the matrix elements explicitly omitted.

The contributions to pion production by the longitudinal component of the field of the scattered electron were first investigated in detail by Dalitz and Yennie[7]. They discussed these contributions in terms of matrix elements specified in the pion-nucleon centre of mass system, both for various phenomenological contributions and for specific meson theories. The phenomenological analysis of charged and neutral pion photoproduction by Watson et al[8] indicated the behaviour of the transverse matrix elements on the energy shell where the electromagnetic field is real. Here the dominant terms leading to the resonant $I = 3/2$, $J = 3/2$ state are the magnetic dipole and electric quadrupole amplitudes.

Dalitz and Yennie also showed that the fixed-source pion theory indicated that near threshold the greater part of the longitudinal production of pions may be expected to arise from the monopole excitation. At higher energies, near the resonance, the quadrupole excitation may be expected to contribute to the longitudinal production and interfere with the dominant resonant excitations. However they showed that longitudinal contributions should form only a small part of the total inelastic scattering especially near threshold.

Another important difference between the electroproduction and photoproduction amplitudes, ignored up to this time, was the close dependence of the former on the electromagnetic structure of the nucleon and pion. Dalitz and Yennie showed that the importance of the dominant magnetic dipole excitations in the resonance region for large scattering angles of the electron depends critically upon the magnetic moment structure of the nucleon. They pointed out that information on inelastic electron-nucleon scattering at large momentum transfer from the scattered electron would provide a stringent test for any theory of the structure of the pion-nucleon system.

The interactions giving rise to pion production from nucleons by electrons and photons can be separated into two well known classes. Firstly there is the electromagnetic interaction associated with the electron or photon and secondly the strong interactions associated with the particles constituting the pion-nucleon system.

Up to the present time quantum electrodynamics has proved to be extremely successful in dealing with electromagnetic interactions. One expands the amplitude as a power series in $e^2 = 1/137$, the electron or photon coupling constant, and removes any divergences in the expansion by renormalisation. In principle one can calculate any matrix element to any required degree of accuracy and excellent agreement with experiment has always been obtained.

However attempts to treat strong interactions between elementary particles in a similar manner have not succeeded. The corresponding coupling constants turn out to be greater than one with the result that successive terms in the power series expansion rapidly increase.

In an attempt to overcome this difficulty Chew and Low[9] were led to propose a simplified model for pion-nucleon scattering. They considered the interaction as

$$f_r^0 \psi^* \left[\underset{\sim}{\sigma} \cdot \underset{\sim}{\nabla} (\underset{\sim}{\tau} \cdot \underset{\sim}{\phi}) \right] \psi \qquad \text{I.1}$$

with ψ and $\underset{\sim}{\phi}$ representing the nucleon and pion fields respectively, $\underset{\sim}{\sigma}$ the Pauli spin matrix, and $\underset{\sim}{\tau}$ an operator describing the charge state of the pion. Since anti-nucleon states are simply not contemplated in this theory it must be non-relativistic. Divergences also arise and it is necessary to introduce an arbitrary cut-off parameter to eliminate pions of too high a frequency.

Unsatisfactory as this theory may appear, Chew and Low found that the calculated low energy P-wave scattering phase shifts were in qualitative agreement with the corresponding experimental values. However this Yukawa theory[10] has no room for strongly interacting elementary particles other than pions and nucleons.

It was about this time that the importance of the analytic properties of S-matrix elements was first realised. Knowledge of the analytic properties of any function of one or more complex variables together with Cauchy's Integral Theorem leads to a representation of the function relating its real and imaginary parts. The simplest example of a dispersion relation is obtained by applying Cauchy's theorem to a function of a complex variable. Then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z' - z}\,dz'$$

where C is a simple closed contour within which f(Z) is analytic.

The remainder of this Chapter is devoted to a review of the history and rapid development of dispersion theory.

In Chapter 2 we investigate the electroproduction of a pion from a nucleon in the energy region where the I = 3/2, J = 3/2 resonance in the pion-nucleon system dominates. At the same time as much of this work was being done Dennery[11] investigated the electroproduction amplitude within the Cini-Fubini approximation. The formulation presented here has been derived quite independently, but although different in form from that of Dennery is equivalent throughout. The method of solution chosen, the N/D method for a two channel process, is different from that taken by Dennery. Unfortunately Dennery has presented no numerical results and therefore no direct comparison can be made between the two methods followed.

The problem is first formulated in a relativistic gauge invariant manner in terms of six Lorentz invariant amplitudes satisfying a Mandelstam representation. A decomposition of the complete amplitude into charge independent states of the orbital angular momentum l is performed and their analytic properties on the complex S-plane obtained (where S is the square of the barycentric energy of the final pion-nucleon state).

Three multipole amplitudes, magnetic dipole, electric quadrupole and longitudinal quadrupole, leading to a final P wave state are examined in detail. Their dynamical unphysical branch cuts are approximated by a carefully chosen set of poles with residues and positions dependent upon the invariant momentum transfer λ^2 from the scattered electron. Part of this dependence is exhibited explicitly through the electromagnetic form factors of the nucleon and pion. The nucleon form factors are taken from the electron scattering data of Hofstadter(12) and Wilson(13). The pion form factor is calculated from the values of the $I = 1$, $J = 1$ phase shift for pion-pion scattering obtained by Bransden and Moffat(14).

It is necessary at present to make use of other experimental information in making these pole approximations. We assume the existence and energy of the $I = 3/2$, $J = 3/2$ resonance in pion-nucleon scattering and the existence of strong pion-pion interactions, with their associated energies and widths, in the pion-pion system.

The method of solution used is the N/D method proposed by Chew and Mandelstam[15] and extended to multi-channel processes by Bjorken[16]. Within this method the close dependence, through unitarity, of the electroproduction amplitude upon pion-nucleon scattering is made clear. The effects of this process are investigated in two ways.

A set of poles, approximating the unphysical cuts in the P-wave pion-nucleon scattering amplitude, derived by Hendry[17] are first used to calculate the magnetic dipole and electric quadrupole amplitudes. The longitudinal amplitude is shown to be very small in the energy region considered. The differential cross-section for the electroproduction of pions from protons is calculated and compared with recent experimental data. Momentum transfers in the range $0 \leq \lambda^2 \leq 20 m_\pi^2$ are considered. Qualitative agreement is obtained. The effects of various pion-pion interactions on this cross-section are examined.

In order to improve the quantitative agreement with experiment the experimental values of the $I = 3/2$, $J = 3/2$ phase shift for pion-nucleon scattering were incorporated into the calculation and the above cross-section obtained. It was found necessary, in order to obtain reasonable agreement with experiment, to either introduce an extra unknown parameter into the calculation or assume some high energy behaviour for the phase shift. Both possibilities are examined.

In Chapter 3 we investigate the photoproduction of pions in a similar manner. The results obtained for the electroproduction amplitude in the limit $\lambda^2 \to 0$ are sufficient to calculate the contributions of the magnetic dipole and electric quadrupole amplitudes to the photoproduction cross-section. Similar methods of solution to the above are investigated.

1. Single Dispersion Relations.

Dispersion relations were first investigated in 1926-27 by Kronig[18] and Kramers[19] who were concerned with the classical dispersion of light and the relation between the real and imaginary parts of the index of refraction. Kramers showed that if the index of refraction n(w), viewed as a function of complex frequency w, was analytic in the upper half plane and approached unity at infinity, then it satisfied the dispersion relation

$$\mathrm{Re}\left(n(w)-1\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left(n(w')-1\right)}{w'-w}\,dw' \qquad \mathrm{I.2}$$

He showed further that any index of refraction with the above analytic properties then satisfied the fundamental requirement of causality.

There then followed a period of several years in which interest in the subject of dispersion theory declined. It was not until 1946 that Kronig[20] suggested the idea that the requirement of causality might restrict the form of the S-matrix for elementary particle scattering. Many attempts to impose this requirement together with the assumptions of local Hermitean interactions unitarity and completeness then followed. Gradually it was realized that a useful method of imposing causality on the theory was by requiring the transition amplitudes to be the boundary values of appropriate analytic functions of the energy variable.

Gell-Mann, Goldberger and Thirring(21) first derived dispersion relations for the forward scattering of light. They based their derivation upon a new form of the causality condition which is now called the principle of microscopic causality. This states that the commutator of two Heisenberg field operators at different space-time points vanishes if the separation between these points is spacelike.

Goldberger(22) applied this principle of microscopic causality to a representation of the amplitude for an arbitrary transition of particles with mass, proposed by Lehmann Symanzik and Zimmerman(23). This representation expresses the amplitude in terms of the Fourier transform of a matrix element of a commutator of two Heisenberg field operators. This enabled Goldberger to continue analytically the amplitude for the forward scattering of particles into the complex energy plane. A dispersion relation relating the real part of the scattering amplitude to its imaginary part followed immediately.

Many extensions of this work were made to allow for finite momentum transfer in the transition. Heuristic derivations of dispersion relations in the energy variable with the momentum transfer held constant were given by

Salam[24] and by Polkinghorne[25]. Rigorous proofs from the postulates of field theory were derived in order to establish the relations on a firm theoretical foundation. Dispersion relations for forward scattering were first proved rigorously by Symanzik[26]. Bogoliubov et al[27] extended Symanzik's proof to non-forward scattering when the mass of one of the particles was sufficiently negative and showed that the dispersion relation thus obtained could be analytically continued in this mass variable to the real case.

Other methods of proof have been presented. Lehmann[28] for example, based his proof on Dyson's integral representation[29] for the commutator of two Heisenberg field operators. He also succeeded in showing that the real and imaginary parts of the scattering amplitude, considered as functions of the momentum transfer with the energy variable held fixed, were analytic in certain elliptical areas on the complex momentum transfer plane.

One important consequence of these proofs was that the dispersion relations at fixed momentum transfer were found to be valid only if the momentum transfer was less than some maximum and if certain inequalities on the masses of the interacting particles were satisfied.

Single dispersion relations were written down for many problems. The first to be compared with experiment were the forward angle dispersion relations for pion-nucleon scattering derived by Goldberger et al.[30]. With them Haber-Schaim[31] determined the pion-nucleon coupling constant f^2 using experimentally observed cross-sections and Davidon and Goldberger[32] resolved the Fermi-Yang ambiguity by showing that only the Fermi phase shifts predicted a reasonable value for f^2.

Chew, Goldberger, Low and Nambu[33,34] on the basis of dispersion relations at finite momentum transfer, investigated the low energy behaviour of pion-nucleon scattering and pion photoproduction. Their treatment was perhaps the closest approach at that time to Heisenberg's concept of a dynamical S matrix theory. It incorporated all of the general principles which a correct theory must have namely: unitarity, analyticity, crossing symmetry and Lorentz covariance. In order to evaluate the equations they obtained, they assumed the experimentally observed dominance of the $I = 3/2$, $J = 3/2$ state and expanded their expressions in $1/M$ expansion, keeping only terms up to the first order. Their results were in good qualitative agreement with experiment up to the first resonance, but they were unable to determine the position of the (3/2, 3/2) resonance or the S wave scattering lengths.

The results obtained by Chew et al., are analogous to these obtained by Fubini, Nambu and Wataghin[35] for electroproduction apart from the inclusion of the nucleon electromagnetic form factors. As a first approximation the dispersion relations were solved in the static limit thus neglecting any appreciable recoil contributions. For the invariant momentum transfer λ^2 from the scattered electron greater than $(500 \text{ Mev/c})^2$ their solution cannot be trusted.

Another important application of the single dispersion relations was in the field of polology. The scattering amplitude can be considered as a function of the two variables W, the centre of mass energy, and $Z = \cos\theta$, θ being the scattering angle. Chew[36] has shown that many processes have a pole in the complex Z region close to the physical region $-1 \leq Z \leq 1$. The pole contributions can be readily calculated and separated out from the rest of the matrix element. Since the position of the pole is close to the physical region some suitably modified physical observable can be extrapolated in the variable Z to the pole position Z_o say, and the residues compared. For example, the differential cross-section $\frac{d\sigma}{d\Omega}$, modified by the factor $(Z - Z_o)^2$, extrapolated to $Z = Z_o$ can give some estimate as to the coupling between particles and as to their respective parities. This procedure has been followed by Cziffra and Moravcsik[37] for neutron-proton scattering and by Matthews and Salam[38] for K meson-nucleon scattering. Unfortunately in the latter case the uncertainty in the extrapolation at present is too large for the parity of the K-meson to be determined. Frazer[39] has also pointed out that extrapolation to the single pion pole in the electroproduction amplitude can determine the electromagnetic form factor of the pion. However present experimental data is insufficient to allow an accurate extrapolation outside the physical region.

Many other applications of dispersion relations at fixed momentum transfer were made with some success, particular examples being nucleon-nucleon scattering[40], the electromagnetic form factors of the nucleon[41], K- meson - nucleon scattering[42], and the decay of the pion[43]. It soon became clear that the restrictions on the momentum transfer did not allow the relations to provide a complete dynamical theory. However, these dispersion relations indicated the importance of considering the amplitudes as analytic functions of one or perhaps more complex variables with singularities whose presence and position could be determined from the masses and quantum numbers of the interacting particles. Knowledge of the positions and strengths of these singularities could perhaps completely determine the physical observables.

2. The Mandelstam Representation.

An important advance towards the realization of this conjecture was taken by Mandelstam[44] who considered the analytic continuation of the invariant amplitudes into the complex plane as functions simultaneously of energy and momentum transfer. It should be emphasised here that Mandelstam's ideas at present can only be applied to elements of the S matrix involving one or two particles.

For the general reaction involving four particles, two of which are incoming and two outgoing, as in Figure 1,

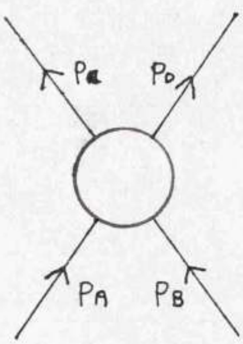


Figure 1

it is convenient to define the three invariants s,t,u by

$$\begin{aligned} s &= -(P_A + P_B)^2 \\ t &= -(P_A - P_C)^2 \\ u &= -(P_A - P_D)^2 \end{aligned} \qquad \text{I.3}$$

where P_A , P_B , P_C , P_D are the four-momenta of the particles.

The above diagram can be considered to describe three reactions

$$\begin{array}{ll} \text{I} & A + B \rightarrow C + D \\ \text{II} & A + C \rightarrow \bar{B} + D \\ \text{III} & A + D \rightarrow \bar{B} + C. \end{array} \qquad \text{I.4}$$

Each of the three invariants defined above then corresponds to the square of the total energy in the barycentric system for the corresponding reaction in Figure I. They are not independent, but are related by energy-momentum conservation by the equation

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 \qquad \text{I.5}$$

where the m_L $(L = A,B,C,D)$ are the masses of the interacting particles, given by $p_L^2 = -m_L^2$.

Mandelstam considered the set of invariant amplitudes $A^{(i)}$ (s,t,u) describing, say , reaction I. These amplitudes, he asserted, are then the set describing reactions II and III when the variables s, t, and u take their corresponding physical values for each reaction. This assertion, namely the general substitution rule extended to physical amplitudes, is of the utmost importance to dispersion theory in elementary particle physics and is a consequence of quantum field theory. From it, for example, if two or more of the interacting particles are identical, simple "crossing relations" can be obtained between the amplitudes $A^{(i)}(s,t,u)$ and the amplitude with two of the arguments interchanged.

Dispersion relations in the energy variable at fixed momentum transfer can now be derived for each of the invariant amplitudes when the variables s, t, and u take physical values for each of the three reactions. These physical amplitudes can then be considered as the boundary values of the analytic functions $A^{(i)}(s,t,u)$ when the arguments approach their physical values for each process. One can show that the physical regions for the three reactions do not overlap.

Under the assumption that the $A^{(i)}(s,t,u)$, considered now as functions of two complex variables, are analytic in the entire space of the two variables except for cuts along certain hyperplanes given by the single dispersion relations, Cauchy's Theorem then leads to the analytic representation

$$\begin{aligned} A^{(i)}(s,t,u) = {} & \frac{1}{\pi}\int ds'\,\frac{\rho_1(s')}{(s'-s)} + \frac{1}{\pi}\int dt'\,\frac{\rho_2(t')}{(t'-t)} \\ & + \frac{1}{\pi}\int du'\,\frac{\rho_3(u')}{(u'-u)} + \frac{1}{\pi^2}\int ds'\int du'\,\frac{\rho_{13}(s',u')}{(s'-s)(u'-u)} \\ & + \frac{1}{\pi^2}\int dt'\int du'\,\frac{\rho_{23}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\int ds'\int dt'\,\frac{\rho_{12}(s',t')}{(s'-s)(t'-t)} \end{aligned} \qquad \text{I.6}$$

The spectral functions ρ_k, ρ_{kj} are real and fail to vanish only when an argument is equal to the square of the mass of an actual physical system that has the quantum numbers of the corresponding reaction. Mandelstam has developed a general procedure for evaluating the boundaries of the regions in which these functions are non-zero.

The representation, eqn. I.6, will have this simple form only if the masses of the particles are such that no anomalous thresholds exist, that is the lower limits of the integrals are not determined simply by the masses of possible intermediate states. Also the amplitudes $A^{(i)}$ (s,t,u) must tend to zero as s, t, and u tend to infinity separately. In the latter case they may have to be modified by subtractions.

This representation however, has at present been derived from the single variable dispersion relations, valid provided the momentum transfer is sufficiently small and the masses of the interacting particles satisfy certain inequalities. The general principles of field theory contain much more information. In the absence of a rigorous treatment making use of all the information available from field theory Mandelstam[44] and others[45] examined the analytic structure of the invariant amplitudes in perturbation theory.

The contribution from an arbitrary diagram in perturbation theory may be written in the form[46]

$$f(s,t,u) = c' \int_0^1 d\alpha_1 \ldots d\alpha_n \frac{\delta(1-\sum_i \alpha_i)}{\left[F(\alpha_i; s,t,u)\right]^P} \qquad \text{I.7}$$

the α_i being the Feynman parameters used to write the contributions from the n lines of the graph in the form of a single denominator.

If f(s,t,u) is singular, Polkinghorne and Screaton[45] and also Tarski[45] have shown that each integration must either give rise to a singularity at an end point or a coincident singularity, that is two singularities "pinching" the contour, otherwise the singularities may be avoided by suitable distortion of the contours of integration.

Landau[47] first studied equation I.7 and showed that f(s,t,u) was singular when

$$F(\alpha_i ; s,t,u) = 0 \quad ; \sum_{i=1}^{n} \alpha_i = 1$$

and either

$$\alpha_i = 0 \qquad \text{I.8}$$

or

$$\frac{\partial F}{\partial \alpha_i} = 0 \qquad i = 1, \ldots n.$$

with the additional condition that the resultant singularities in the α_i integrations pinch the contour.

These equations allow determination of the curve Γ (s,t,u), the locus of possible singularities, in the scattering amplitude for any order in perturbation theory. It is necessary, in order to establish the representation I.6, to show that the complex points on Γ are not singular.

The properties of Γ for the general fourth order loop diagram were investigated in detail by Tarski[45] and extended by Landshoff, Polkinghorne and Taylor[48] and by Eden[49] to provide a proof of the representation for all orders in perturbation theory, subject to the absence of certain types of anomalous thresholds.

Of course these proofs are necessary but not yet sufficient. Ideally the theory of dispersion relations must deal only with diagrams with "free ends", that is with scattering amplitudes and their analytic continuation. At present however, the analytic properties of the amplitudes are derived by a Hamiltonian formalism which itself contains the concept of local fields.

The anomalous thresholds referred to above were first discovered by Karplus, Sommerfield and Wichman[50] and by Nambu[51]. They studied the fourth order loop diagram as a function of one of the invariants s, t, and u and determined the position of the first branch point on the real axis, the masses of the interacting particles being restrained to satisfy certain stability conditions. One might expect from the unitarity condition that the invariant amplitudes would have a branch point at the square of the mass of the lowest intermediate state of two or more particles allowed. However they found that some processes, such as

$$\gamma_v + N \longrightarrow N + \pi$$

where the virtual photon is produced by the annihilation of an electron-positron pair, exhibited the property that the first branch point was slightly different from the normal "threshold" branch point. Eden[49] has since shown that the absence of anomalous thresholds is ensured when they are absent from a limited number of lower order diagrams.

It turns out that constraints exist on the possible masses that can occur in intermediate configurations for one reaction, given the mass of the intermediate configuration for another. In other words, the region in which the double spectral functions ρ_{kj} fail to vanish is not rectangular but bounded by curves asymptotic to the square of the lowest mass of a multiparticle system with appropriate quantum numbers. When anomalous thresholds occur the curves cross their asymptotes and are bounded by horizontal or vertical tangents at the anomalous threshold value. Mandelstam[44] noted that the representation is valid until the masses are such that the points of tangency coincide whereupon complex points on Γ become singular.

It is useful to exhibit the connection between the Mandelstam representation and the single dispersion relations mentioned earlier. In the physical region for process I, say

$$\begin{aligned} \mathrm{Im}\, A^{(1)}(s,t,u) &= A_1(s,t,u) \\ &= \rho_1(s) + \frac{1}{\pi}\int dt' \frac{\rho_{12}(s,t')}{(t'-t)} + \frac{1}{\pi}\int du' \frac{\rho_{13}(s,u')}{(u'-u)} \end{aligned} \qquad \text{I.9}$$

In a similar manner A_2 and A_3 can be defined to be the Im $A^{(i)}$ (s,t,u) when the arguments take their corresponding physical values for reactions II and III. Then, for example, for fixed u

$$A(s,t,u) = \frac{1}{\pi}\int du' \frac{\rho_3(u')}{(u'-u)} + \frac{1}{\pi}\int ds' \frac{A_1(s',t,u)}{(s'-s)} + \frac{1}{\pi}\int dt' \frac{A_2(s,t',u)}{(t'-t)} \qquad \text{I.10}$$

Outside the physical region for reaction I $A_1(s,t,u)$, normally referred to as the absorptive part for reaction I, becomes complex. Equation I.9 can then be considered as defining the analytic continuation of $A_1(s,t,u)$.

The singularities in the invariant amplitudes are completely determined by the vanishing of the denominators in equations I.9 and I.10. A general feature of the location of these singularities is that the higher the mass of the physical system associated with them the farther from the physical region they arise. Also the "strengths" of the singularities, that is the discontinuity across the branch cuts, are related to physical cross-sections and hence limited by the unitarity condition. Thus the dominant behaviour of the amplitudes tends to be controlled by the closest singularities. Chew[52] has compared this behaviour of the singularities to that of the potential due to point and line charges. Within this physical picture the distant singularities are associated with short range forces and states of low orbital angular momentum whereas the nearby singularities correspond to long range forces and states of high orbital angular momentum.

The discontinuities across the branch cuts are related directly to the absorptive parts $A_j(s,t,u)$, $j = 1,2,3$. The invariant amplitudes are normally defined to be proportional to and have the same phase as the matrix

$$T = \frac{1}{2i}(S - 1)$$

so that in the physical region for process I say,

$$A_1(s,t,u) = c_1 \sum_m \langle m | T | C, D \rangle^* \langle m | T | A, B \rangle \qquad \text{I.11}$$

the summation being over all intermediate states allowed.

It may well happen that, as in the reaction $\pi + \pi \rightarrow N + \bar{N}$ where the first branch point in the energy variable is at $4\,m_\pi^2$ and the physical region starts at $S = 4\,M_N^2$, it is necessary to extend the validity of equation I.11 to part of the unphysical region. Anomalous thresholds could also occur, but in either case Cutkosky(53) and Mandelstam(54) have shown that the extension can be made.

One can see from the unitarity equation that the calculation of the short range forces is immediately associated with the evaluation of multi-particle matrix elements. When treating the low orbital angular momentum contributions to the total amplitude it is therefore more convenient to examine each partial wave separately. Much simplicity is lost but it is in this region that arbitrary parameters may also enter the theory corresponding to an asymptotic behaviour in the amplitudes that requires subtractions in the representation.

Froissart[55] has derived asymptotic limits on the scattering of scalar particles. For the covariant amplitude A(s,t,u) he obtained

$$|A(s,t,u)| \leq s \log^2 s \qquad \text{I.12}$$

for forward and backward angles, and

$$|A(s,t,u)| \leq s^{3/4} \log^{3/2} s \qquad \text{I.13}$$

at any other fixed physical angle.

These bounds on the amplitude enabled Froissart to give a rigorous discussion on how many independent parameters are permitted when the double spectral functions are known. He found that, at the most , the S and P wave contributions in each of the related reactions and one overall subtraction constant can be independent.

Many of the problems investigated previously have been reinvestigated using the above ideas. Present calculations are limited by the lack of information about the spectral functions in the regions where multi-particle states contribute. It might well be that an extension of Mandelstam's ideas to processes involving more than two particles in each channel can be made. However present indications are that such a complete theory, dealing with functions of several complex variables, will require new and more powerful mathematical techniques.

At the present time the general approach to any problem is to calculate what contributions one can and with the help of any helpful experimental information (such as resonances and dominant states) try to obtain qualitative agreement with experiment.

What is perhaps the "simplest" of strong interaction problems investigated is that of pion-pion scattering. Chew and Mandelstam(15) have been intensively studying this problem looking for dominant S and P wave solutions, at low energies. Only two pion intermediate states were considered in the unitarity equation. The exchange of a pion pair should constitute the longest range force which in turn should dominate the low energy scattering. The singularities of the partial wave amplitudes on the q^2 plane (q being the barycentric momentum) were found to be the two cuts

$$0 \leq q^2 \leq \infty \qquad \text{and} \quad -\infty \leq q^2 \leq -4m^2_\pi$$

Chew and Mandelstam then expressed the partial wave amplitudes A_ℓ in the form N_ℓ/D_ℓ where N_ℓ contained the unphysical, left-hand, cut and D_ℓ the physical cut for the ℓ'th partial wave. Crossing symmetry relates the discontinuity across the left-hand cut to the physical amplitude. Their final solution therefore was obtained by solving a set of coupled integral equations relating the N_ℓ to the D_ℓ. The unknown subtraction constant λ involved is limited in its range of possible values since present experimental information seems to indicate that no pion-pion bound states occur.

A low energy resonance was obtained for dominant S wave solutions in the I = 0, J = 0 state. Unfortunately mathematical difficulties arose in handling the P-wave exchange force necessitating some cut-off procedure introducing further adjustable parameters.

The Chew-Mandelstam equations for pion-pion scattering have been obtained by Cini and Fubini[56] by a different, though equivalent, approximation. They expanded the double spectral integrals in the power series

$$\int ds' \int dt' \frac{\rho(s',t')}{(s'-s)(t'-t)} \simeq \int ds' \frac{\rho_1^0(s')}{(s'-s)} + \int dt' \frac{\rho_2^0(t')}{(t'-t)}$$

$$+ t\int ds' \frac{\rho_1'(s')}{(s'-s)} + s\int dt' \frac{\rho_2'(t')}{(t'-t)} \qquad \text{I.14}$$

$$+ \cdots$$

sufficient terms in the expansion being taken as are necessary to represent all the large phase shifts. The advantage of this procedure is that the approximations in the method are made in a clear manner in the beginning. The expansion is approximately valid at low energies but leads to an asymptotic behaviour of the amplitude inconsistent with unitarity.

Bransden and Moffat[14] found that consideration of the inverse amplitudes gave rise to a set of coupled integral equations more amenable to numerical solution. They also found that this procedure was such that the subtraction constant λ was sufficient for the determination of both the S and P wave amplitudes. Both S and P wave low energy resonances were obtained . Unfortunately some criticism has been raised about such a method involving inverse amplitudes. Other solutions for the low energy S and P wave amplitudes have been obtained by Ball and Wong[57] and by Desai[58].

Many other calculations have been made for other processes. Frazer and Fulco[59] applied a dispersion treatment for the process $\pi + \pi \longrightarrow N + \bar{N}$ to the electromagnetic structure of the nucleon. Reasonable agreement with experiment was obtained with the assumption of an $I = 1$, $J = 1$ resonance in the pion-pion system at an energy $\simeq 3.4\ m_\pi$. This resonance position is now considered too low, recent experiments in pion production predicting a resonance at an energy $\simeq$ 5.4 m_{π}.

However Frazer and Fulco showed that the introduction of a strong pion-pion interaction could enhance the spectral functions in such a manner that theory can become consistent with experimental data on the electromagnetic structure of nucleons. This led Clementel and Villi(60) to propose a simple model based on dispersion theory for the nucleon form factors. Fair agreement with experiment was obtained with the assumption of two resonances in the pion-pion system, the I = 1, J = 1 resonance at an energy of 4.5 m_{π} and an I = 0, J = 1 resonance at an energy of 3.2 m_{π}.

Pion-nucleon scattering has been investigated by Frazer and Fulco(61) and Frautschi and Walecka(62). The latter made numerical calculations for the dominant P-wave amplitude using a modified form of the N/D method of solution. They found that by approximating the left-hand cuts in the amplitude by a series of poles, the set of coupled integral equations reduced to a set of simultaneous equations for the N_{ℓ} and D_{ℓ}. Qualitative agreement with experiment was obtained with this form of solution for energies up to the first resonance. Pion-pion interaction in the I = 1, J = 1 state appeared to have little effect on the cross-sections.

Hendry(17) has since shown, by a similar procedure, that pion-pion interactions can provide a very probable mechanism for producing the higher resonances.

The effect of a low energy pion-pion interaction (a bipion in the I = 1, J = 1 state) on the photoproduction amplitude was investigated by Ball(63). His results differed only slightly from those obtained by Chew, Goldberger, Low and Nambu(34).

All of the above problems were also investigated(64) within the Cini-Fubini approximation. Of particular interest is the recent work by Dennery(11) on the electroproduction of pions. He showed that the amplitude consisted mainly of two parts which in the Cini-Fubini approximation are simply additive. One part describes the effects of a pion-pion resonance, while the other describes the I = 3/2, J = 3/2 resonant nucleon term. Most of his results have been derived independently on this thesis.

Considerable attention has also been paid to strange particle interactions(65) and non-relativistic scattering(66); double dispersion relations have been proved for a certain class of potentials.

Chew and Frautschi(67) recently introduced the strip approximation. This approximation consists of calculating only these portions of the double spectral functions which correspond to two particle intermediate states. Thus only the edges of the physical regions (corresponding to small momentum transfers at arbitrary energy) lying near such singularities can be treated reliably.

What may well turn out to be an important step towards the realisation of a complete theory of interactions between elementary particles has been taken by Regge[68]. He analytically continued solutions of the Schroedinger equation (under certain restrictions on the potential) into the complex angular momentum (ℓ) plane for Re $\ell > -\frac{1}{2}$. The only singularities are a finite number of poles in the upper half plane (Im $\ell > 0$) for physical kinetic energies moving as functions of the energy on to the real axis at negative energies. Regge showed that in a non-relativistic theory these poles are closely connected with bound states and resonances.

The existence of Regge poles in the relativistic S matrix has not yet been established, though Chew, Frautschi and Mandelstam[69] have argued that it is plausible that they should occur. If so, they may well shed greater light on theories of strong (and perhaps weak) interactions between elementary particles. Indeed, Chew and Frautschi[70] have stated they believe that a complete theory of strong interactions will follow from the ideas of Mandelstam and Regge.

CHAPTER 2.

The Electroproduction of a Pion From a Nucleon.

1. Kinematics.

The S matrix for the electroproduction of a pion from a nucleon may be written

$$S_{fi} = \delta_{fi} - \frac{i}{(2\pi)^{1/2}} \delta(s_1 + p_1 - s_2 - p_2 - q) \frac{m M}{\sqrt{\omega e_1 e_2 \epsilon_1 \epsilon_2}} T_{fi} \qquad \text{II.1}$$

where ϵ_1 and e_1 are the energies of the incident electron and nucleon of four-momenta s_1 and p_1, and ϵ_2, e_2 and ω are the energies of the outgoing electron nucleon and pion of four momenta s_2, p_2 and q respectively, in the laboratory frame.

By treating the interaction of the scattered electron to the lowest order in e, the electromagnetic coupling constant, Dalitz and Yennie(7) have shown that the T matrix element may be written

$$T_{fi} = \bar{u}(p_2) J_\mu u(p_1) \epsilon_\mu \qquad \text{II.2}$$

where

$$\epsilon_\mu = \frac{e\, \bar{u}(s_2) \gamma_\mu u(s_1)}{(s_2 - s_1)^2} \qquad \text{II.3}$$

and $\bar{u}(p_2) J_\mu u(p_1)$ is the matrix element for the production of a pion from a nucleon by a virtual photon(γ_v) of invariant four-momentum transfer

$$k_\mu^2 = \underset{\sim}{k}^2 - k_0^2 = (s_2 - s_1)^2 = \lambda^2 > 0 \qquad \text{II.4}$$

In the above we have chosen units such that $\hbar = c = m_\pi = 1$, $e^2 = 1/137$ and M = 6.72, where M is the nucleon mass.

The four vector ϵ_μ describing the electromagnetic interaction must satisfy the Lorentz gauge condition so that

$$k_\mu \epsilon_\mu = 0 \qquad \text{II.5}$$

The current J_μ is also limited in its form by charge conservation

$$k_\mu J_\mu = 0 \qquad \text{II.6}$$

Some interesting properties of the electroproduction amplitude can be deduced from the above equations. When the energy k_o of the virtual photon is zero, and this can occur in the physical region for $\lambda^2 \geqslant 2M+1 \simeq (500 \ \mathrm{Mev/c})^2$, the virtual photon is transverse in nature and longitudinal contributions to the total amplitude will vanish.

Equation II.6 also allows determination of the time component J_0 in terms of the space-like components of J_μ. However when $k_o = 0$ we must take great care in forming this relation in a meaningful manner. The method followed was first noted by Dalitz and Yennie(7). From equations II.5 and II.6 we have

$$J_\mu \epsilon_\mu = \underset{\sim}{J} \cdot \left(\underset{\sim}{\epsilon} - \frac{\underset{\sim}{k} \cdot \underset{\sim}{\epsilon}}{k_0^2} \underset{\sim}{k} \right) \qquad \text{II.7}$$

$$= \underset{\sim}{J} \cdot \underset{\sim}{a}$$

If instead of using $\underset{\sim}{\epsilon}$ as our polarization vector we use the vector $\underline{a}$, we then define the current $\underset{\sim}{J}$ to be well behaved at $k_o = 0$ and transfer the $^1/k_o$ dependence to the vector $\underline{a}$ where the limit can be taken correctly.

It is convenient to define the scalars

$$\begin{aligned} s &= -(p_1 + k)^2 \\ u &= -(k - p_2)^2 \\ t &= -(k - q)^2 \end{aligned} \qquad \text{II.8}$$

By energy-momentum conservation it follows that

$$s + t + u = 2M^2 + 1 - \lambda^2 \qquad \text{II.9}$$

We now limit our attention to the process

$$\gamma_v + N \longrightarrow \pi + N$$

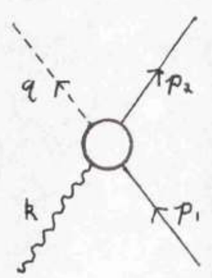


Figure 2

In the barycentric system of the final pion and nucleon

$$s = (K_o + E_1)^2 = (\Omega + E_2)^2 = W^2$$

$$u = M^2 - \lambda^2 - 2KQ\cos\theta \quad -2K_oE_2 \qquad \text{II.10}$$

$$t = 1 - \lambda^2 + 2KQ\cos\theta \quad - 2K_o\Omega$$

$$\cos\theta = \frac{\underset{\sim}{K} \cdot \underset{\sim}{Q}}{KQ}$$

We denote quantities in the barycentric system by capital letters and in the laboratory system by the corresponding small letters.

The energies and magnitudes of the momenta of the interacting particles are functions only of S, the square of the total centre of mass energy W (Appendix 1).

We now decompose the matrix $J_\mu \epsilon_\mu$ into a linear combination of relativistic, gauge invariant quantities. The procedure used is one proposed by Ball[63] for the photoproduction of pions. From general invariance principles.

$$J_\mu \epsilon_\mu = \sum_{i=1}^{8} B_i(s,t,u)\, N_i\,(p_1, p_2, k, \epsilon_\mu, \gamma_\mu) \qquad \text{II.11}$$

where the matrices N_i are all of the independent Lorentz-invariant forms which can be formed containing γ_u's and linear in ϵ_μ . They must also be linear in γ_5 since a pion occurs in the final state. A simple choice of N_i is

$$N_1 = i\,\gamma_5\,\gamma.\epsilon\,\gamma.k \qquad N_5 = \gamma_5\,\gamma.\epsilon$$
$$N_2 = 2i\,\gamma_5\,p.\epsilon \qquad N_6 = \gamma_5\,\gamma.k\;\;p.\epsilon$$
$$N_3 = 2i\,\gamma_5\,q.\epsilon \qquad N_7 = \gamma_5\,\gamma.k\;\;k.\epsilon \qquad \text{II.12}$$
$$N_4 = 2i\,\gamma_5\,k.\epsilon \qquad N_8 = \gamma_5\,\gamma.k\;\;q.\epsilon$$

$$p = \tfrac{1}{2}(p_1 + p_2)$$

The invariant amplitudes $B_i(s,t,u)$ thus defined are assumed to contain only the Mandelstam singularities (44) and no others. It can be easily checked that a Mandelstam representation can be written down for the total amplitude and that no anomalous thresholds occur. Comparison with lower order terms in perturbation theory indicates that the choice of amplitudes made above is such that no extra kinematical singularities are introduced. It is certainly true that the complete amplitude will contain complex singularities with contributions of order e^4. Since we have already made the approximation in treating the interaction between the scattered electron and nucleon to order e^2 they do not arise in the amplitudes we consider.

From equation II.6 replacement of ϵ_μ by k_μ in equation II.11 should make the right-hand side identically equal to zero. Since

$$A\,\gamma.k \;+\; B = 0$$

implies A = 0 and B = 0 we have the two equations

$$\lambda^2 B_1 + 2p.k\, B_2 + 2q.k\, B_3 + 2\lambda^2 B_4 = 0$$

$$B_5 + p.k\, B_6 + \lambda^2 B_7 + q.k\, B_8 = 0 \qquad \text{II.13}$$

Eliminating B_4 and B_7, and using the gauge condition, we make our final choice of six amplitudes which describe the process in a relativistic and gauge invariant manner and contain only the Mandelstam singularities. The matrix $J_\mu \epsilon_\mu$ can now be rewritten in the form

$$J_\mu \epsilon_\mu = \sum_{i=1}^{6} M_i\, A_i\, (s,t,u) \qquad \text{II.14}$$

with

$$M_1 = i\,\gamma_5\, \gamma.\epsilon\;\; \gamma.k \qquad M_4 = \gamma_5\;\; \gamma.\epsilon$$

$$M_2 = 2i\,\gamma_5\;\; p.\epsilon \qquad M_5 = \gamma_5\;\; \gamma.k\;\; p.\epsilon \qquad \text{II.15}$$

$$M_3 = 2i\,\gamma_5\; q.\epsilon \qquad M_6 = \gamma_5\;\; \gamma.k\;\; q.\epsilon$$

This decomposition is by no means unique. Dennery[11] made use of a different but equivalent set with the same properties. Unfortunately the amplitudes chosen by Fubini, Nambu and Wataghin[35] contain a kinematic singularity.

The isotopic spin dependence of the invariant amplitudes A_i (s,t,u) can be removed in the normal manner. Watson[71] has shown that the photon may be considered as an isotopic scalar together with the third component of an isotopic vector. For the vector part, we can treat the photon as a π^0 and therefore two charge independent forms are $\tau_\beta \tau_3$ and $\tau_3 \tau_\beta$, where β is the isotopic spin index of the final pion. For the scalar part of the photon we take the charge independent form τ_β . The three basic forms chosen are

$$g_\beta^+ = \delta_{\beta 3}$$

$$g_\beta^- = \frac{1}{2}\left[\tau_\beta \tau_3 - \tau_3 \tau_\beta\right] \qquad \text{II.16}$$

$$g_\beta^0 = \tau_\beta$$

We then define

$$A_i(s,t,u) = A_i^+ g_\beta^+ + A_i^- g_\beta^- + A_i^0 g_\beta^0 \qquad \text{II.17}$$

In future we shall denote the isotopic spin dependence of the amplitudes by the index T = +,-,0. Note that the amplitudes $A_i^{\,0}$ will contribute only to a final state with isotopic spin $\frac{1}{2}$.

Ball(63) has also shown that a form of crossing symmetry for the amplitudes $A_i^{\,T}(s,t,u)$ can be obtained by requiring that the amplitude for the production of a pion from a nucleon of momentum p_1 by a photon be equal to that for the production of the charge conjugate pion from an antinucleon of momentum p_1. This means that equation II.14 must be invariant under the transformation.

$$p_1 \longrightarrow -p_2 \; ; \qquad p_2 \longrightarrow -p_1 \; ; \qquad \gamma_\mu \longrightarrow -\gamma_\mu$$

and the order of the γ factors be interchanged.

Hence

$$A_i\ (s,t,u) = A_i(u,t,s) \qquad i = 1,2,5$$

and

$$A_i\ (s,t,u) = -A_i(u,t,s) \qquad i = 3,4,6. \qquad \text{II.18}$$

In isotopic spin space, since g_β^+ and g_β^o are hermitian and g_β^- antihermitian, the crossing relations become

$$\begin{aligned} A_i^{\overset{+}{o}}\ (s,t,u) &= A_i^{\overset{+}{o}}\ (u,t,s) && i = 1,2,5 \\ &= -\ A_i^{\overset{+}{o}}\ (u,t,s) && i = 3,4,6 \end{aligned} \qquad \text{II.19}$$

$$\begin{aligned} A_i^-\ (s,t,u) &= -\ A_i^-\ (u,t,s) && i = 1,2,5 \\ &= A_i^-\ (u,t,s) && i = 3,4,6. \end{aligned}$$

2. The Angular Momentum Decomposition.

The complete amplitude above can be reduced to Pauli spinor form by using the explicit form of the Dirac spinors

$$u_{r}(p) = \frac{M - i\gamma . p}{\sqrt{2M(E+M)}} \begin{pmatrix} \chi_r \\ 0 \end{pmatrix}$$

with $\chi_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ III.1

corresponding to the nucleon spin being up or down respectively in the rest system of the particles.

We define

$$\bar{u}_f J^T_\mu \epsilon_\mu u_i = \bar{\chi}_f \mathcal{F}^T \chi_i \qquad \text{III.2}$$

where

$$\mathcal{F}^T = i\underset{\sim}{\sigma}.\underset{\sim}{a}\,\mathcal{F}^T_1 + \frac{\underset{\sim}{\sigma}.\underset{\sim}{Q}\;\underset{\sim}{\sigma}.\underset{\sim}{K}\times\underset{\sim}{a}}{QK}\mathcal{F}^T_2 + \frac{i\underset{\sim}{\sigma}.\underset{\sim}{K}\;\underset{\sim}{Q}.\underset{\sim}{a}}{QK}\mathcal{F}^T_3$$
$$+ \frac{i\underset{\sim}{\sigma}.\underset{\sim}{Q}\;\underset{\sim}{Q}.\underset{\sim}{a}}{Q^2}\mathcal{F}^T_4 + \frac{i\underset{\sim}{\sigma}.\underset{\sim}{K}\;\underset{\sim}{K}.\underset{\sim}{a}}{K^2}\mathcal{F}^T_5 + \frac{i\underset{\sim}{\sigma}.\underset{\sim}{Q}\;\underset{\sim}{K}.\underset{\sim}{a}}{QK}\mathcal{F}_6 \qquad \text{III.3}$$

Note we are now in the centre of mass frame.

This defining equation for the amplitudes $F^T_i(s,t,u)$ relates them in a unique manner to the invariant amplitudes as follows

$$
\begin{aligned}
\mathcal{F}_1^T &= \frac{O_1}{8\pi W}\left\{ (W-M)A_1^T - A_4^T \right\} \\
\mathcal{F}_2^T &= \frac{O_2}{8\pi W}\left\{ -(W+M)A_1^T - A_4^T \right\} \\
\mathcal{F}_3^T &= \frac{O_2(E_2+M)}{8\pi W}\left\{ -A_2^T + 2A_3^T + \tfrac{1}{2}(W+M)A_5^T - (W+M)A_6^T \right\} \\
\mathcal{F}_4^T &= \frac{O_1(E_2-M)}{8\pi W}\left\{ A_2^T - 2A_3^T + \tfrac{1}{2}(W-M)A_5^T - (W-M)A_6^T \right\} \\
\mathcal{F}_5^T &= G_5^T - \frac{K^2}{\lambda^2}\left\{ \mathcal{F}_1^T + x\,\mathcal{F}_3^T \right\} \\
\mathcal{F}_6^T &= G_6^T - \frac{K^2}{\lambda^2}\, x\, \mathcal{F}_4^T
\end{aligned}
\qquad \text{III.4}
$$

where

$$
\begin{aligned}
G_5^T = \frac{O_1(E_1-M)K_0}{8\pi W\lambda^2}\Big\{ &(W+M)A_1^T + (W+E_2)A_2^T + 2\Omega A_3^T \\
&+ A_4^T - \tfrac{1}{2}(W+M)(W+E_2)A_5^T - \Omega(W+M)A_6^T \Big\}
\end{aligned}
$$

$$
\begin{aligned}
G_6^T = \frac{O_2(E_1+M)K_0}{8\pi W\lambda^2}\Big\{ &-\frac{\lambda^2}{E_1+M}A_1^T - (W+E_2)A_2^T - 2\Omega A_3^T \\
&+ \frac{W+M}{E_1+M}A_4^T - \tfrac{1}{2}(W-M)(W+E_2)A_5^T - \Omega(W-M)A_6^T \Big\}
\end{aligned}
$$

and

$$
O_1 = \sqrt{(E_1+M)(E_2+M)} \quad ; \quad O_2 = \sqrt{(E_1-M)(E_2-M)} \quad ; \quad x = \cos\theta
$$

Later in the calculation we shall require the set of equations relating the invariant amplitudes to the $\mathcal{F}_i^T$ (s,t,u). It is

$$A_1^T = 4\pi\left\{\frac{1}{O_1}\mathcal{F}_1^T - \frac{1}{O_2}\mathcal{F}_2^T\right\}$$

$$A_2^T = \frac{2\pi}{W}\left\{-\frac{(W+M)}{O_1}\mathcal{F}_1^T + \frac{\lambda^2}{O_2(E_1+M)}\mathcal{F}_2^T - \frac{\Omega(W-M)}{O_2(E_2+M)}\mathcal{F}_3^T\right.$$
$$\left. + \frac{\Omega(W+M)}{O_1(E_2-M)}\mathcal{F}_4^T + \frac{\lambda^2(W-M)}{K_0 O_1(E_1-M)}G_5^T - \frac{\lambda^2(W+M)}{K_0 O_2(E_1+M)}G_6^T\right\}$$

III.5

$$A_3^T = \frac{\pi}{W}\left\{-\frac{(W+M)}{O_1}\mathcal{F}_1^T + \frac{\lambda^2}{O_2(E_1+M)}\mathcal{F}_2^T + \frac{(W-M)(W+E_2)}{O_2(E_2+M)}\mathcal{F}_3^T\right.$$
$$\left. - \frac{(W+E_2)(W+M)}{O_1(E_2-M)}\mathcal{F}_4^T + \frac{\lambda^2(W-M)}{K_0 O_1(E_1-M)}G_5^T - \frac{\lambda^2(W+M)}{K_0 O_2(E_1+M)}G_6^T\right\}$$

$$A_4^T = 4\pi\left\{-\frac{(W+M)}{O_1}\mathcal{F}_1^T - \frac{(W-M)}{O_2}\mathcal{F}_2^T\right\}$$

$$A_5^T = \frac{4\pi}{W}\left\{-\frac{1}{O_1}\mathcal{F}_1^T - \frac{(W+M)}{O_2(E_1+M)}\mathcal{F}_2^T + \frac{\Omega}{O_2(E_2+M)}\mathcal{F}_3^T\right.$$
$$\left. + \frac{\Omega}{O_1(E_2-M)}\mathcal{F}_4^T - \frac{\lambda^2}{K_0 O_1(E_1-M)}G_5^T - \frac{\lambda^2}{K_0 O_2(E_1+M)}G_6^T\right\}$$

$$A_6^T = \frac{2\pi}{W}\left\{-\frac{1}{O_1}\mathcal{F}_1^T - \frac{(W+M)}{O_2(E_1+M)}\mathcal{F}_2^T - \frac{(W+E_2)}{O_2(E_2+M)}\mathcal{F}_3^T\right.$$

$$-\frac{(W+E_2)}{O_1(E_2-M)}\tilde{\mathcal{F}}_4^T - \frac{\lambda^2}{K_0 O_1(E_1-M)} G_5^T - \frac{\lambda^2}{K_0 O_2(E_1+M)} G_6^T \Big\}$$

We can now proceed with the decomposition into states of definite angular momentum. The method followed was one proposed by Pearlstein and Klein(72) for the photoproduction amplitude.

The vector $\underset{\sim}{a}$ was written as

$$\underset{\sim}{a} = \underset{\sim}{a}_T + \underset{\sim}{a}_L \qquad \text{III.6}$$

$$\underset{\sim}{a}_L = \frac{\underset{\sim}{K} \cdot \underset{\sim}{a}}{K^2} \underset{\sim}{K}$$

The $\underset{\sim}{a}_T$ gives rise to the transverse transitions. The decomposition in this case will be just that for pion production by a real photon. The component $\underset{\sim}{a}_L$ however, is zero for the photoproduction amplitude. On the other hand, in the electroproduction amplitude it leads to transitions associated with the 'longitudinal polarization' of the virtual photon.

Since the nucleon in the final state can have its spin orientated in two directions the decomposition must be made in terms of six energy dependent amplitudes. Also since the virtual photon has variable mass they will be functions of the invariant momentum transfer λ^2. We denote them by $M_{\ell\pm}(s,\lambda^2)$, $E_{\ell\pm}(s,\lambda^2)$ and $Y_{\ell\pm}(s,\lambda^2)$, referring to transitions initiated by magnetic, electric and longitudinal type radiation respectively, leading to final states of orbital angular momentum ℓ and total angular momentum $J = \ell \pm \frac{1}{2}$. Note that under space inversion the longitudinal multipole amplitudes transform in a similar manner to the electric multipole amplitudes.

The decomposition is

$$
\begin{aligned}
\mathcal{F}_1 &= \sum_{\ell=0}^{\infty} \left[\ell M_{\ell+} + E_{\ell+}\right] P'_{\ell+1}(x) + \left[(\ell+1) M_{\ell-} + E_{\ell-}\right] P'_{\ell-1}(x) \\
\mathcal{F}_2 &= \sum_{\ell=1}^{\infty} \left[(\ell+1) M_{\ell+} + \ell M_{\ell-}\right] P'_{\ell}(x) \\
\mathcal{F}_3 &= \sum_{\ell=1}^{\infty} \left[E_{\ell+} - M_{\ell+}\right] P''_{\ell+1}(x) + \left[E_{\ell-} + M_{\ell-}\right] P''_{\ell-1}(x) \\
\mathcal{F}_4 &= \sum_{\ell=1}^{\infty} \left[M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-}\right] P''_{\ell}(x) \\
\mathcal{F}_5 &= \sum_{\ell=0}^{\infty} \left[(\ell+1) Y_{\ell+} P'_{\ell+1}(x) - \ell Y_{\ell-} P'_{\ell-1}(x)\right] - \mathcal{F}_1 - x\mathcal{F}_3 \\
\mathcal{F}_6 &= \sum_{\ell=1}^{\infty} \left[\ell Y_{\ell-} - (\ell+1) Y_{\ell+}\right] P'_{\ell}(x) - x\mathcal{F}_4
\end{aligned}
\qquad \text{III.7}
$$

The following orthogonality properties of the Legendre polynomials enable us to invert the above relations

$$\int_{-1}^{1} dx\, P_J(x)\left[P'_{L+1}(x) - P'_{L-1}(x)\right] = 2\,\delta_{LJ}$$

$$\int_{-1}^{1} dx\, P'_J(x)\left[\frac{P'_{L-2}(x) - P'_L(x)}{2L-1} - \frac{P'_L(x) - P'_{L+2}(x)}{2L+3}\right] = -2\,\delta_{LJ} \qquad \text{III.8}$$

This yields

$$M_{\ell+}(s,\lambda^2) = \frac{1}{2(\ell+1)}\int_{-1}^{1} dx \left[\mathcal{F}_1 P_\ell(x) - \mathcal{F}_2 P_{\ell+1}(x) - \mathcal{F}_3 \frac{P_{\ell-1}(x)-P_{\ell+1}(x)}{2\ell+1} \right]$$

$$E_{\ell+}(s,\lambda^2) = \frac{1}{2(\ell+1)}\int_{-1}^{1} dx \left[\mathcal{F}_1 P_\ell(x) - \mathcal{F}_2 P_{\ell+1}(x) \right.$$

$$\left. + \ell\, \mathcal{F}_3 \frac{P_{\ell-1}(x)-P_{\ell+1}(x)}{2\ell+1} + (\ell+1)\mathcal{F}_4 \frac{P_\ell(x)-P_{\ell+2}(x)}{2\ell+3} \right] \tag{III.9}$$

$$Y_{\ell+}(s,\lambda^2) = \frac{1}{2(\ell+1)}\int_{-1}^{1} dx \left[\mathcal{F}_1 P_\ell(x) + x\,\mathcal{F}_3 P_\ell(x) + x\,\mathcal{F}_4 P_{\ell+1}(x) \right.$$

$$\left. + \mathcal{F}_5 P_\ell(x) + \mathcal{F}_6 P_{\ell+1}(x) \right]$$

and for

$$M_{\ell-}(s,\lambda^2) = \frac{1}{2\ell}\int_{-1}^{1} dx \left[-\mathcal{F}_1 P_\ell(x) + \mathcal{F}_2 P_{\ell-1}(x) + \mathcal{F}_3 \frac{P_{\ell-1}(x)-P_{\ell+1}(x)}{2\ell+1} \right]$$

$$E_{\ell-}(s,\lambda^2) = \frac{1}{2\ell}\int_{-1}^{1} dx \left[\mathcal{F}_1 P_\ell(x) - \mathcal{F}_2 P_{\ell-1}(x) \right.$$

$$\left. -(\ell+1)\mathcal{F}_3 \frac{P_{\ell-1}(x)-P_{\ell+1}(x)}{2\ell+1} - \ell\,\mathcal{F}_4 \frac{P_{\ell-2}(x)-P_\ell(x)}{2\ell-1} \right]$$

$$Y_{\ell-}(s,\lambda^2) = \frac{1}{2\ell}\int_{-1}^{1} dx \left[\mathcal{F}_1 P_\ell(x) + x\,\mathcal{F}_3 P_\ell(x) + x\,\mathcal{F}_4 P_{\ell-1}(x) \right.$$

$$\left. + \mathcal{F}_5 P_\ell(x) + \mathcal{F}_6 P_{\ell-1}(x) \right]$$

3. The Analytic Properties of the Multipole Amplitudes.

The invariant amplitudes defined above satisfy the spectral representation proposed by Mandelstam(44)

$$
\begin{aligned}
A_i^T(s,t,u) = {} & \frac{{}^A\Gamma_i^T(\lambda^2)}{M^2-s} + \frac{{}^B\Gamma_i^T(\lambda^2)}{M^2-u} + \frac{Y_i^T(\lambda^2)}{1-t} \\
& + \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} ds' \int_{(M+1)^2}^{\infty} du' \, \frac{{}^T a_{12}^i(s',u')}{(s'-s)(u'-u)} \\
& + \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} ds' \int_{4}^{\infty} dt' \, \frac{{}^T a_{13}^i(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} du' \int_{4}^{\infty} dt' \, \frac{{}^T a_{23}^i(u',t')}{(u'-u)(t'-t)}
\end{aligned}
\qquad \text{IV.1}
$$

where

$$
\begin{aligned}
{}^A\Gamma_1^T(\lambda^2) &= \pm\,{}^B\Gamma_1^T(\lambda^2) = -4\pi g\left[F_1^T(\lambda^2) + 2M\,F_2^T(\lambda^2)\right] \\
{}^A\Gamma_2^T(\lambda^2) &= \pm\,{}^B\Gamma_2^T(\lambda^2) = 4\pi g\,F_1^T(\lambda^2) \\
{}^A\Gamma_3^T(\lambda^2) &= \mp\,{}^B\Gamma_3^T(\lambda^2) = 2\pi g\,F_1^T(\lambda^2) \\
{}^A\Gamma_4^T(\lambda^2) &= {}^B\Gamma_4^T(\lambda^2) = 0 \\
{}^A\Gamma_5^T(\lambda^2) &= \pm\,{}^B\Gamma_5^T(\lambda^2) = -8\pi g\,F_2^T(\lambda^2) \\
{}^A\Gamma_6^T(\lambda^2) &= \mp\,{}^B\Gamma_6^T(\lambda^2) = -4\pi g\,F_2^T(\lambda^2)
\end{aligned}
\qquad \text{IV.2}
$$

In the above $F_i^{\pm}(\lambda^2) = F_i^V(\lambda^2)$, ($i = 1$ or 2) are the isovector form factors of the proton and neutron and $F_i^{0}(\lambda^2) = F_i^S(\lambda^2)$ the isoscalar form factors defined by

$$F_i(\lambda^2) = F_i^s(\lambda^2) + \tau_3 F_i^v(\lambda^2)$$

They are normalized such that

$$F_1^{v,s}(0) = e/2 \quad ; \quad F_2^{v,s}(0) = \frac{e g_{v,s}}{2M}$$

where $g_{v,s}$ denotes the isovector or isoscalar part of the anomalous gyromagnetic ratio.

Also

$$Y_i^T(\lambda^2) = 0 \quad \text{except for } Y_3^-(\lambda^2) = -4\pi e g F_\pi(\lambda^2). \qquad \text{IV.3}$$

$F_\pi(\lambda^2)$ is the pion form factor normalized to unity at $\lambda^2 = 0$. We shall discuss these form factors in greater detail later.

The crossing relations now require

$$a_{12}^{i+0}(s,u) = \pm a_{12}^{i+0}(u,s)$$

$$a_{12}^{i-}(s,u) = \mp a_{12}^{i-}(u,s) \qquad \text{IV.3}$$

$$a_{13}^{i+0}(s,t) = \pm a_{23}^{i+0}(s,t)$$

and

$$a_{13}^{i-}(s,t) = \mp a_{23}^{i-}(s,t)$$

the upper sign to be taken for i = 1,2,5 and the lower for i = 3,4,6.

The regions in which the spectral functions $a_{j_k}^T$ are non-zero have been evaluated following the method outlined by Mandelstam[44]. The boundaries of these regions result from considering the lowest mass states possible in the three related processes

$$\gamma_v + N \longrightarrow \pi + N$$

$$\gamma_v + \bar{N} \longrightarrow \pi + \bar{N}$$

$$\gamma_v + \pi \longrightarrow N + \bar{N}$$

In the t spectrum we have to consider

$$\gamma_v + \pi \longrightarrow n\pi$$

Conservation of G parity (charge conjugation together with a rotation of π about the axis in isotopic spin space) requires that n be even for the (o) spectra and odd for the (+,-) spectra. Hence the regions in which a_{13}^{+-} and a_{23}^{+-} are non-zero will differ from those in which a_{13}^{o} and a_{23}^{o} are non-zero. The curves bounding these regions were found to be

for a_{13}^{+-} (s,t)

$$\left[s-(M+1)^2\right]\left[s-(M-1)^2\right](t-9)(t-1)+8(t-1)(1-3s-M^2) - 8\lambda^2 s(t-3)+8\lambda^2(t+1+2\lambda^2)(1-M^2)=0 \qquad \text{IV.5}$$

for a_{13}^{o} (s,t), the two curves

$$\left[s-(M+2)^2\right]\left[s-(M-2)^2\right]t(t-4)-2t(9s+31M^2-28) + t\lambda^2(8-14M^2)-4M^2+1-2\lambda^2 ts-8M^2\lambda^2+2\lambda^2 - 4M^2\lambda^4+\lambda^4=0 \qquad \text{IV.6}$$

and

$$\left[s-(M+1)^2\right]\left[s-(M-1)^2\right] t(t-16) - 8t(9s+M^2-1)$$

$$+ 8\lambda^2\left[2\lambda^2 - 2M^2\lambda^2 + 4 - 4M^2 + t - M^2 t - ts\right] - 16(M^2-1) = 0.$$

The boundaries for the spectral functions $a_{23}^{+-0}(u,t)$ are the above curves with the variable s replaced by u. For the spectral function $a_{12}^{+-0}(s,u)$ the boundary is

$$\left[s-(M+1)^2\right]\left[s-(M-1)^2\right]\left[u-(M+1)^2\right]\left[u-(M-1)^2\right]$$

$$-\left[4M^2-1\right]\left[2su - 2(M^2-1)(s+u) + 2M^4 - 1\right] \qquad \text{IV.7}$$

$$+ \lambda^2\left[2su - 4M^2 su - 2(1+M^2-2M^4)(s+u) + 8 - 12M^2\right.$$

$$\left. - 4M^6 - 2M^4 - 4M^2\lambda^2 + \lambda^2\right] = 0$$

If we define

$$A_b^i(s,u) = \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds' \frac{a_{12}^i(s',u)}{s'-s} - \frac{1}{\pi}\int_{-\infty}^{2M^2-\lambda^2-3-u} ds' \frac{a_{23}^i(s',u)}{s'-s} \qquad \text{IV.8}$$

and

$$A_c^i(s,t) = \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds' \frac{a_{13}^i(s',t)}{s'-s} - \frac{1}{\pi}\int_{-\infty}^{M^2-\lambda^2-2M-t} ds' \frac{a_{23}^i(s',t)}{s'-s}$$

then we can express the spectral representation, equation IV.1, in a single integral form for fixed s

$$A_i^T(s,t,u) = \frac{{}^A\Gamma_i^T}{M^2 - s} + \frac{{}^B\Gamma_i^T}{M^2 - u} + \frac{Y_i^T}{1 - t}$$

$$+ \frac{1}{\pi}\int du' \frac{{}^T A_b^i(s,u')}{u' - u} + \frac{1}{\pi}\int dt' \frac{{}^T A_c^i(s,t')}{t' - t} \qquad \text{IV.9}$$

The advantage of this representation is that the dependence of the invariant amplitudes upon $\cos\theta$ is made clear. It is now possible to deduce the analytic properties of the multipole amplitudes. To do this, we consider first the partial wave amplitude

$$_L A_i(s,\lambda^2) = \int_{-1}^{1} A_i(s,t,u)\, P_L(x)\, dx$$

$$= {}^A\Gamma_i(\lambda^2)\int_{-1}^{1}\frac{P_L(x)}{M^2 - s}dx + {}^B\Gamma_i(\lambda^2)\int_{-1}^{1}\frac{P_L(x)}{M^2 - u(s,x)}dx \qquad \text{IV.10}$$

$$+ Y_i(\lambda^2)\int_{-1}^{1}\frac{P_L(x)}{1 - t(s,x)}dx + \frac{1}{\pi}\int_{(M+1)^2}^{\infty} du'\, A_b^i(s,u')\int_{-1}^{1}\frac{P_L(x)}{u' - u(s,x)}dx$$

$$+ \frac{1}{\pi}\int_{4}^{\infty} dt'\, A_c^i(s,t')\int_{-1}^{1}\frac{P_L(x)}{t' - t(s,x)}dx$$

and consider its analytic properties on the complex S-plane.

<u>The Physical Cut</u>

$$(M + 1)^2 \leq S \leq \infty \qquad \text{IV.11}$$

This cut arises from the first terms in the defining equations for A_b^i and A_c^i. Since $-1 \leq x \leq 1$, it corresponds to the physical region for the production of a pion from a nucleon by a virtual photon. The second terms in A_b^i and A_c^i do not contribute any singularities. They are introduced artificially so that the representation may be written in a single integral form.

The Nucleon Pole.

$$s = M^2 \tag{IV.12}$$

This pole is contained in the term $\dfrac{1}{M^2 - s}$. However from the orthogonality properties of the Legendre polynomials this pole can occur only when L = 0.

The other terms in the partial wave amplitude have denominators linear in x by equations II.10. The singularities contained in each term must then be produced by logarithms of the form

$$\log \frac{a(s) + b(s)}{a(s) - b(s)}$$

and hence will give rise to cuts with branch points solutions of the equation

$$\lim_{y \to 1-} \left\{ a^2(s) - y\, b^2(s) \right\} = 0 \tag{IV.13}$$

In each case this equation has four roots, one always being infinite.

The Crossed Nucleon Cuts.

The term $\dfrac{1}{M^2 - u}$ gives rise to cuts with branch points solutions of the equation

$$\lim_{y \to 1-} \left\{ (\lambda^2 + 2K_0 E_2)^2 - 4y K^2 Q^2 \right\} = 0$$

which has roots at $s = -\infty$ and

$$s\left[M^2 s^2 + (\lambda^2 - 2M^4) s + 3M^2\lambda^2 - \lambda^2 + \lambda^4 + M^6 \right] = 0$$

This term then contains the two cuts

$$-\infty \leq s \leq 0$$

and

$$Re(s) = M^2 - \frac{\lambda^2}{2M^2} \qquad \text{IV.14}$$

$$\left| Im(s) \right| \leq \frac{\sqrt{\lambda^2(\lambda^2 + 4M^2)(4M^2 - 1)}}{2M^2}$$

The Pion Cuts.

In a similar manner the pole at $t = 1$ becomes two cuts on the complex s-plane with branch points solutions of

$$\lim_{y \to 1-} \left\{ (\lambda^2 + 2K_0 \Omega)^2 - 4y K^2 Q^2 \right\} = 0$$

with roots at $s = -\infty$ and

$$s\left[s^2 + s(\lambda^2 - 2M^2) + M^4 + 3M^2\lambda^2 - \lambda^2 + M^2\lambda^4 \right] = 0$$

These cuts are

$$-\infty \leq s \leq 0$$

and $$R_e(s) = M^2 - \lambda^2/2 \qquad \text{IV.15}$$

$$\left| I_m(s) \right| \leqslant \frac{\sqrt{\lambda^2(4M^2-1)(\lambda^2+4)}}{2}$$

<u>The Crossed Physical Cuts.</u>

These cuts arise from the vanishing of the u'-u denominator for $(M+1)^2 \leqslant u' \leqslant \infty$. As above the branch points will be solutions of the equations

$$\lim_{y \to 1-} \left\{ (u' - M^2 + \lambda^2 + 2K_0E_2)^2 - 4yK^2Q^2 \right\} = 0 \quad ; \; (M+1)^2 \leq u' \leq \infty$$

leading to the cuts

$$-\infty \leq s \leq 0 \qquad \text{IV.16}$$

$$\text{and} \quad -\infty \leq s \leq d(\lambda^2)$$

where $d(\lambda^2)$ is the maximum value of S which is a solution of the equation

$$u's^2 + s\left[u'(u' - 2M^2 - \lambda^2 - 1) - (M^2-1)(M^2+\lambda^2)\right]$$

$$- u'(M^2-1)(M^2+\lambda^2) + (2M^2 - 1 + \lambda^2)(M^4 + \lambda^2) = 0 \qquad \text{IV.17}$$

$$(M+1)^2 \leq u' \leq \infty$$

Note that $d(\lambda^2)$ is real.

The Pion-Pion Cuts.

The branch points on the complex S-plane arising from the vanishing of the denominator t'-t are solutions of

$$\lim_{y \to 1-} \left\{ (t' - 1 + \lambda^2 + 2 K_0 \Omega)^2 - 4 y K^2 Q^2 \right\} = 0$$

and the cuts are

$$-\infty \leq s \leq 0$$

and a cut in the complex plane given by the locus of points satisfying the equation

$$s^2 + s(t' - 1 + \lambda^2 - 2M^2) + (M^2 - 1)(M^2 + \lambda^2) + \frac{M^2(\lambda^2 + 1)^2}{t'} = 0 \qquad \text{IV.18}$$

$4 \leq t' \leq \infty$ for isoscalar amplitudes.

$9 \leq t' \leq \infty$ for isovector amplitudes.

As one would expect the singularities on the complex S-plane move about as functions of the invariant momentum transfer λ^2.

We now consider in detail the analytic properties of the three multipole amplitudes magnetic dipole, electric quadrupole and longitudinal quadrupole contributing to the resonant final state. In order to retain the correct threshold behaviour in these amplitudes as $K^2 \longrightarrow 0$ and $Q^2 \longrightarrow 0$ we consider the three amplitudes

$$m_{1+} = \frac{s}{QK} \quad M_{1+}$$

$$e_{1+} = \frac{s}{QK} \quad E_{1+} \qquad \text{IV.19}$$

$$\text{and} \quad y_{1+} = \frac{1}{QK\left[(W+M)^2 - 1\right]} \quad Y_{1+}$$

The additional factors are introduced to remove the possibility of a pole in the amplitudes at $s = 0$ and to improve the convergence of integrals introduced later. This redefinition of the amplitudes also simplifies their analytic properties.

Each of the three amplitudes defined above is related to the invariant amplitudes by equations III.4 and III.9. They can therefore be expressed as a linear combination of the partial wave amplitudes and hence must contain the above singularities apart from any modifications introduced by kinematical factors.

Consider first the nucleon pole. We noted that it only arose when L = 0. It is produced by the diagram

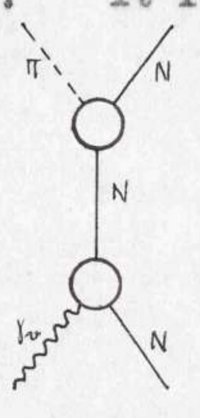


Figure 3.

Selection rules do not allow this intermediate state to contribute to the amplitudes under consideration and therefore they do not contain the pole at $s = M^2$.

All other cuts in the partial wave amplitudes are contained in the multipole amplitudes. The only cut contained by the kinematical factors is the

<u>Irrationality Cut</u>

$$-\infty \leq s \leq 0 \qquad \text{IV.20}$$

since they contain $\sqrt{S}$.

One might expect branch cuts arising from $O_1(s)$ and $O_2(s)$ since

$$2WO_1 = \sqrt{[(W+M)^2-1][(W+M)^2+\lambda^2]}$$

$$2WO_2 = \sqrt{[(W-M)^2-1][(W-M)^2+\lambda^2]}$$

If however we take that sheet such that $\sqrt{S} > 0$ then $O_1^{\,2}(s)$ is always greater than zero. Also in the complete expression for each of the three amplitudes only even powers of $O_2(s)$ occur. The poles at $O_2^2(s) = 0$ cannot arise since by inspection the amplitudes are finite at that point.

All of the cuts contained in the amplitudes are indicated in Figure 4 for $\lambda^2 \simeq 10\, m_\pi^2$.

We now take a Cauchy integral round all of these cuts,

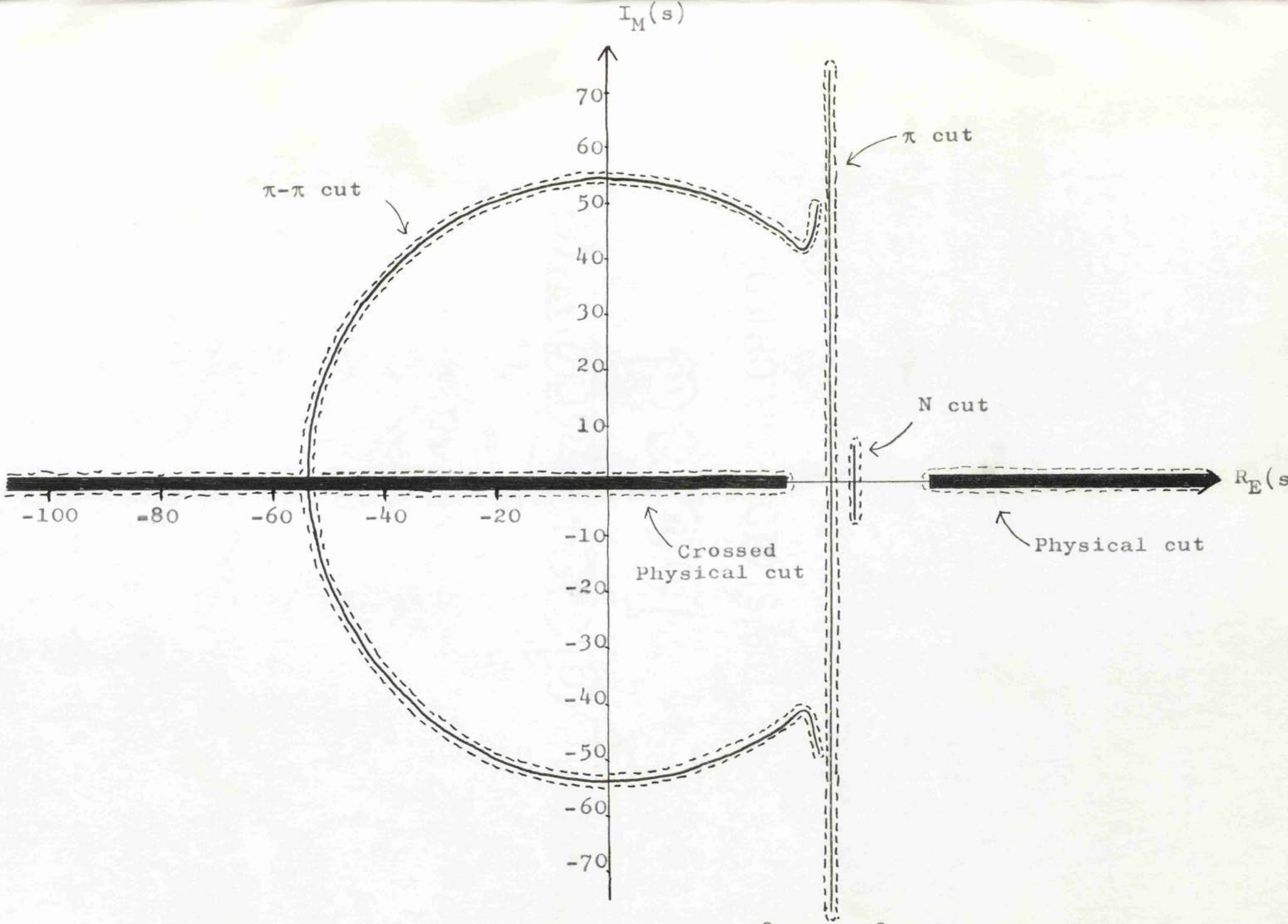


Figure 4 $\lambda^2 = 10\ m_\pi^2$

the contour of integration being as shown.

The contribution to an arbitrary amplitude $n_{\ell\pm}(s,\lambda^2)$ say, from any unphysical cut on the left will be of the form

$$\frac{1}{2\pi i}\int_{C_i} \frac{n_{\ell\pm}(s',\lambda^2)}{s'-s}\,ds' = \frac{1}{\pi}\int \frac{\text{Abs}\, n_{\ell\pm}(s',\lambda^2)}{s'-s}\,ds' \qquad \text{IV.21a}$$

where C_i is a clockwise contour round the cut and the absorptive part of $n_{\ell\pm}(s,\lambda^2)$ is

$$\text{Abs}\, n_{\ell\pm}(s,\lambda^2) = \frac{1}{2i}\left[n_{\ell\pm}(s_+,\lambda^2) - n_{\ell\pm}(s_-,\lambda^2)\right] \qquad \text{IV.21b}$$

Here $n_{\ell\pm}(s_+,\lambda^2)$ is its value on one side of the cut and $n_{\ell\pm}(s_-,\lambda^2)$ its value on the other.

In order to simplify the N/D method of solution we shall adopt, it is convenient to try and approximate the left-hand cuts in the multipole amplitudes by a series of poles of the form

$$\sum_i \frac{r_i(\lambda^2)}{s - s_i(\lambda^2)} = \frac{1}{\pi}\int \frac{\text{Abs}\, n_{\ell\pm}(s',\lambda^2)}{s'-s}\,ds' \qquad \text{IV.22}$$

for values of S in the physical energy region, $s \geq (M+1)^2$. To do this we must examine the discontinuity across each cut in turn.

The range of λ^2 considered is

$$0 \leq \lambda^2 \leq 60\, m_\pi^2 \qquad \text{IV.23}$$

At present, experiments on the electroproduction of pions from hydrogen have not reached such high momentum transfers. However it is of interest at the present time to investigate which effects should be important at high momentum transfer and to examine the dependence of the cross-section in this region upon pion-pion interactions and the nucleon and pion form factors.

4. <u>The Crossed Nucleon and Pion Cuts.</u>

The pion pole term contributes to the three amplitudes under consideration. Its only contribution is through the invariant amplitude $A_3^{-}(s,t,u)$ with

$$A_3^{-\pi}(s,t,u) = -4\pi \frac{eg F_\pi(\lambda^2)}{1 - t(s,x)} \qquad \text{V.1}$$

Writing $1 - t = -c - b$

where $c = -\lambda^2 - 2K_o\Omega$ and $b = 2\,QK = 2\,O_1O_2$

and substituting in equations III.4 , III.8 and IV.19, we obtain its contribution to the magnetic dipole amplitude,

$$m_{1+}^{\overset{+}{\underset{0}{-}}\pi} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} eg F_\pi(\lambda^2) \frac{W(E_2+M)}{8O_1b} \int_{-1}^{1} \frac{x^2-1}{x + c/b} dx \qquad \text{V.2}$$

The integration with respect to x can be easily carried out giving rise to terms containing $\log\left(\frac{c+b}{c-b}\right)$ and it is precisely this logarithm which gives rise to the two pion cuts.

Now the discontinuity across the finite complex cut is contained only in the logarithm and the discontinuity across this cut due to the logarithm will be just $\pm 2\pi i$. The sign of the discontinuity across this cut however is slightly more difficult to obtain. For convenience, and we are at liberty to do so, we chose the cut to be the straight line joining the two branch points. On the other hand the theory indicates that the cut should be taken to be the locus of points satisfying

$$c^2(s) = x^2 b^2(s) \qquad ; \qquad 0 \leq x \leq 1 \qquad \text{V.3}$$

a quartic in s, two of the solutions being real and corresponding to the other cut

$$-\infty \leq s \leq 0$$

The sign of the discontinuity across this more appropriate cut is found in the normal manner. One considers a value of s lying just above the cut and tests whether the denominator $x + c/b$ acquires a small positive or negative imaginary part. The identity

$$\frac{1}{x + a \pm i\epsilon} = P\frac{1}{x + a} \mp i\pi\,\delta(x + a) \qquad \text{V.4}$$

is then sufficient to determine the sign of the discontinuity. This procedure was carried out numerically.

The absorptive part across the finite pion cut for the magnetic dipole amplitude was calculated to be

$$\frac{1}{\pi}\,\mathrm{Abs}\, m_{1+}^{-\pi}(s, \lambda^2) = \pm\, e g F_\pi(\lambda^2)\, \frac{W(E_2 + M)(b^2 - c^2)}{64\, O_1^4\, O_2^3} \qquad \text{V.5}$$

the upper sign being taken for Im (s) > 0 and the lower for Im (s) < 0. The integral in equation III.22 was computed for values of S and λ^2 in the regions

$$(M + 1)^2 \leq s \leq 95$$
$$0 \leq \lambda^2 \leq 60 \qquad \text{V.6}$$

Note we can now take the contour of integration to be the straight line joining the two branch points.

For each value of λ^2 considered the integral III.22 was approximated by a pole on the real axis of the form

$$e g F_\pi(\lambda^2) \frac{r}{s - s_0} \qquad \text{V.7}$$

in the range of S considered. The dependence of the residues and pole positions on λ^2 were then described by some suitable polynomial in λ^2.

The absorptive parts of $e^{-\pi}_{1+}$ and $y^{-\pi}_{1+}$ were evaluated in a similar manner and the finite cut in each case approximated by a λ^2 dependent pole on the real axis of the form

$$n^{-\pi}_{1+} = e g F_\pi(\lambda^2) \frac{r_1(\lambda^2)}{s - s_1(\lambda^2)} \; ; \; n^{\overset{+}{0}\pi}_{1+} = 0 \qquad \text{V.8}$$

The residues and pole positions for each amplitude are given in Table 1.

The cut along the negative real s axis is complicated by irrationality. Performing the integration in equation V.2

$$m^{\overset{\pm}{0}\pi}_{1+} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e g F_\pi(\lambda^2) \frac{W(E_2+M)}{8 O_1 b^2} \left\{ -2c - \frac{(b^2-c^2)}{b} \log \frac{c+b}{c-b} \right\} \qquad \text{V.9}$$

which is of the form

$$h(W)\left[d(W) + e(W)\log Z(W)\right] \qquad \text{V.10}$$

For $s = -w^2$ on the upper side of the cut $W = iw$ and on the lower $W = -iw$. Also above the cut

$$\log Z(W) = \log \left|Z(W)\right| \pm i\pi$$

the sign being determined in a manner similar to that outlined above. Thus the discontinuity across the cut is just

$$2i\,\mathcal{I}_m\left\{h(i\omega)\left[d(i\omega) + e(i\omega)\left(\log\left|Z(i\omega)\right| \pm i\pi\right)\right]\right\} \qquad \text{V.11}$$

For the absorptive part of the magnetic dipole amplitude across this cut one then obtains

$$\mathrm{Abs}\, m_{1+}^{-}(s,\lambda^2) = e g F_\pi(\lambda^2)\, \mathcal{I}_m \frac{W(E_2+M)}{8b^2 O_1}\left\{-2c + \frac{c^2-b^2}{b}\left[\log\left|\frac{c+b}{c-b}\right| + i\pi\right]\right\} \qquad \text{V.12}$$

The absorptive parts of the electric quadrupole and longitudinal quadrupole amplitudes are calculated in a similar manner.

We have approximated this cut by three poles in each amplitude so that together with the pole approximating the short cut, each amplitude is equal to

$$n_{1+}^{-\pi}(s,\lambda^2) = e g F_\pi(\lambda^2)\sum_{i=1}^{4} \frac{r_i(\lambda^2)}{s - s_i(\lambda^2)} \qquad \text{V.13}$$

$$n_{1+}^{\overset{+}{0}\pi}(s,\lambda^2) = 0$$

The crossed nucleon cuts were treated in a similar way. In this case we split the contributions into two parts each associated with the charge or magnetic form factors of the nucleon. Again the short cut was approximated by two poles on the real axis and the infinite cut by ten poles in each amplitude so that

$$n^{\tau N}_{1+}(s,\lambda^2) = R^\tau(\lambda^2)\sum_{i=5}^{10}\frac{r_i(\lambda^2)}{s - s_i(\lambda^2)} + S^\tau(\lambda^2)\sum_{i=11}^{16}\frac{r_i(\lambda^2)}{s - s_i(\lambda^2)} \qquad \text{V.14}$$

where

$$R^+ = -R^- = g\,F_1^V(\lambda^2) \quad ; \quad R^0 = g\,F_1^S(\lambda^2)$$

$$S^+ = -S^- = g\,F_2^V(\lambda^2) \quad ; \quad S^0 = g\,F_2^S(\lambda^2)$$

Such a large number of poles were taken to approximate the infinite cut in this case since in the energy region considered it turns out that they contain the dominant contributions to the total amplitudes. Also it was hoped that the approximation of an infinite cut by poles would improve as the number of poles taken was increased.

The above representations are in good agreement with those obtained by Dennery[11] for the magnetic dipole and electric quadrupole amplitudes. They are accurate to within 3% in the regions $60 \leq s \leq 90$ and $0 \leq \lambda^2 \leq 60$.

As one would expect, the dominant contribution to the total electroproduction amplitude in these regions appears to come from the magnetic dipole amplitude. There is a small but appreciable contribution from electric quadrupole excitations, but longitudinal contributions are small and may be considered negligible.

5. The Crossed Physical Cut.

It can be shown that when

$$0 \leq s \leq d(\lambda^2) \qquad \text{VI.1}$$

the variables are in the physical region for the process $\gamma_v + \bar{N} \longrightarrow \pi + \bar{N}$, namely

$$u \geqslant (M+1)^2 \quad \text{and} \quad -1 \leq x^x \leq 1 \qquad \text{VI.2}$$

where x^x is the cosine of the scattering angle.
Therefore this part of the crossed physical cuts is related by crossing to physical virtual photon production of a pion.

To determine its contribution to the amplitudes we use the procedure adopted by Frautschi and Walecka[62] when investigating pion-nucleon scattering. We introduce the functions

$$A_i^{xT}(s, x) = \pm \frac{1}{\pi} \int_{(M+1)^2}^{\infty} \frac{\mathrm{Abs}\, A_i^T(u', s)}{u' - u(s, x)}\, du' \qquad \text{VI.3}$$

analytic in the complex s-plane except for a branch cut along the crossed physical cuts. The sign in equation VI.3 is taken such that $A_i^x(s, x)$ has the same discontinuity across the cut as the associated invariant amplitude.
Equations III.5 and III.7 enable the invariant amplitudes to be expanded in terms of the multipole amplitudes. It can be proved by examining the boundaries of the regions in which the spectral functions are non-zero that this expansion converges

for o $\leq$ s $\leq$ d(λ^2) . If we assume the dominance of the magnetic dipole amplitude leading to the resonant I = 3/2, J = 3/2 state, this expansion takes the simple form

$$A_i^0(s,t,u) = 0$$

$$A_1^{\pm}(s,u) = 4\pi\left[\frac{3x}{O_1} \quad \frac{2}{O_2}\right]M_{1+}^{\pm}(s,\lambda^2)$$

$$A_2^{\pm}(s,u) = \frac{2\pi}{W}\left[-\frac{3(W+M)x}{O_1} + \frac{2\lambda^2}{O_2(E_2+M)} + \frac{3\Omega(W-M)}{O_2(E_2+M)}\right]M_{1+}^{\pm}(s,\lambda^2)$$

$$A_3^{\pm}(s,u) = \frac{\pi}{W}\left[-\frac{3(W+M)x}{O_1} + \frac{2\lambda^2}{O_2(E_1+M)} - \frac{3(W-M)(W+E_2)}{O_2(E_2+M)}\right]M_{1+}^{\pm}(s,\lambda^2) \qquad \text{VI.4}$$

$$A_4^{\pm}(s,u) = 4\pi\left[\frac{-3(W+M)x}{O_1} - \frac{2(W-M)}{O_2}\right]M_{1+}^{\pm}(s,\lambda^2)$$

$$A_5^{\pm}(s,u) = \frac{4\pi}{W}\left[-\frac{3x}{O_1} - \frac{2(W+M)}{O_2(E_1+M)} - \frac{3\Omega}{O_2(E_2+M)}\right]M_{1+}^{\pm}(s,\lambda^2)$$

$$A_6^{\pm}(s,u) = \frac{2\pi}{W}\left[-\frac{3x}{O_1} - \frac{2(W+M)}{O_2(E_1+M)} + \frac{3(W+E_2)}{O_2(E_2+M)}\right]M_{1+}^{\pm}(s,\lambda^2)$$

The reasons for assuming the dominance of this state are twofold. Firstly analysis of photoproduction data from experiment indicates the importance of this state for energies around the resonance and secondly Fubini, Nambu and Wataghin[35] obtained reasonable agreement with experiment within the magnetic dipole approximation. They also obtained a formula relating the resonant magnetic dipole amplitude to the resonant amplitude $f_{1+}^{3/2}(s)$ in pion-nucleon scattering

$$M_{1+}^{3/2}(s,\lambda^2) = \frac{M\mathcal{U}^N(\lambda^2)}{g}\ \frac{K}{Q}\ f_{1+}^{3/2}(s) \qquad \text{VI.5}$$

with

$$\mathcal{U}^N(\lambda^2) = \frac{1}{M}F_1^V(\lambda^2) + F_2^V(\lambda^2)$$

Further calculations are simplified to a large extend if we also assume the resonance as being very sharp. Chew, Goldberger, Low and Nambu[33] have shown that this corresponds to the approximation.

$$\mathcal{I}m\, f_{1+}^{3/2}(s) = \frac{2\pi W_R g^2}{3M^2}\ Q^2\,\delta(s-W_R^2) \qquad \text{VI.6}$$

where W_R is the barycentric energy of the pion-nucleon $I = 3/2$, $J = 3/2$ resonance. Hence

$$\mathcal{I}m\, M_{1+}^{\pm}(s,\lambda^2) = \frac{2\pi}{g}\begin{bmatrix}2\\-1\end{bmatrix}\frac{g\,\mathcal{U}^N(\lambda^2)}{M}\,KQW_R\,\delta(s-W_R^2) \qquad \text{VI.7}$$

The absorptive parts of $A_i^T(u,s)$ across this cut can now be deduced by inserting equation VI.7 in equation VI.4 and making use of the crossing relations. One obtains

$$A_i^{x\,0}(s,x) = 0$$

$$A_1^{x\,\pm}(s,x) = \begin{bmatrix}2\\1\end{bmatrix}K^R(\lambda^2)\left\{\frac{2}{O_2^R} - \frac{3x_R^x}{O_1^R}\right\}\frac{1}{u-W_R^2} \qquad \text{VI.8}$$

$$A_2^{x\,\pm}(s,x) = \begin{bmatrix}2\\1\end{bmatrix}K^R(\lambda^2)\left\{\frac{3x_R^x(W_R+M)}{2W_R O_1^R} - \frac{\lambda^2}{W_R O_2^R(E_1^R+M)} - \frac{3\Omega^R(W_R-M)}{2W_R O_2^R(E_2^R+M)}\right\}\frac{1}{u-W_R^2}$$

$$A_3^{x\pm}(s,x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} K^R(\lambda^2)\left\{ -\frac{3x_R^x(W_R+M)}{4W_R\,O_1^R} + \frac{\lambda^2}{2W_RO_2^R(E_1^R+M)} - \frac{3(W_R-M)(W_R+E_2^R)}{4W_R(E_2^R+M)O_2^R}\right\}\frac{1}{u-W_R^2}$$

$$A_4^{x\pm}(s,x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} K^R(\lambda^2)\left\{ -\frac{3x_R^x(W_R+M)}{O_1^R} - \frac{2(W_R-M)}{O_2^R}\right\}\frac{1}{u-W_R^2}$$

$$A_5^{x\pm}(s,x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} K^R(\lambda^2)\left\{ \frac{3x_R^x}{W_R\,O_1^R} + \frac{2(W_R+M)}{W_R(E_1^R+M)O_2^R} + \frac{3\Omega^R}{W_R(E_2^R+M)O_2^R}\right\}\frac{1}{u-W_R^2}$$

$$A_6^{x\pm}(s,x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} K^R(\lambda^2)\left\{ -\frac{3x_R^x}{2W_R\,O_1^R} - \frac{(W_R+M)}{W_R(E_1^R+M)O_2^R} + \frac{3(W_R+E_2^R)}{2W_R(E_2^R+M)O_2^R}\right\}\frac{1}{u-W_R^2}$$

where x_R^x is given by

$$x_R^x = \frac{M^2 - \lambda^2 - 2K_0^R E_2^R - s}{2K^R Q^R}$$

and $K^R(\lambda^2)$ by

$$K^R(\lambda^2) = \frac{8\pi g\,\mathcal{M}^N(\lambda^2)\,W_R K^R Q^R}{9M}$$

The suffix R denotes an energy or momenta evaluated at $s = W_R^2$.

We can now set about approximating this part of the crossed physical cut in a manner similar to that used in the approximation of the pion and crossed nucleon cuts. We substitute equation VI.8 in equation III.4 and then calculate the three amplitudes m_{1+}, y_{1+} and ℓ_{1+} from equation III.9. After performing the integration over x we are again left with terms involving logarithms of the form

$$\log\frac{\lambda^2 - M^2 + 2K_0E_2 + W_R^2 + 2KQ}{\lambda^2 - M^2 + 2K_0E_2 + W_R^2 - 2KQ}$$

The cut has been reduced in size as a result of the sharp resonance approximation. The discontinuity of the logarithm across the cut was evaluated as before and the Cauchy integrals round the cut performed for the range of physical values of s and λ^2, equation V.6.

This part of the crossed physical cuts was approximated by one pole in each amplitude

$$n_{1+}^{T}(s, \lambda^2) = \frac{U^T(\lambda^2)\, \tau_{17}(\lambda^2)}{s - s_{17}(\lambda^2)} \qquad \text{VI.9}$$

with $$U^+(\lambda^2) = 2U^-(\lambda^2) = \frac{g}{2M}\left[F_2^V(\lambda^2) + \frac{1}{M}F_1^V(\lambda^2)\right]$$

$$U^0(\lambda^2) = 0$$

As expected the dominant contribution from this cut arises from the magnetic dipole amplitude. Contributions from the electric and longitudinal quadrupole amplitudes are negligible.

It is impossible at present to estimate the contribution from the negative real axis part of the crossed physical cuts. Firstly the Legendre polynominal expansion fails on this part of the cut and secondly the absorptive parts of the amplitudes across the cut depend upon the double spectral functions about which very little is known at the present time. We neglect its contribution in further calculations.

6. <u>The Pion-Pion Cuts.</u>

The absorptive parts of the amplitudes across the pion-pion cuts are related to the amplitude for the process

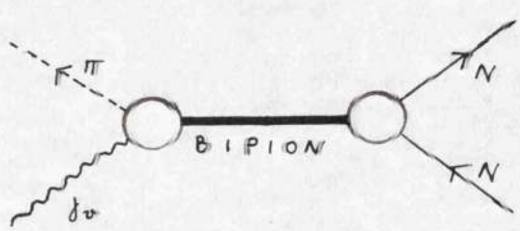
$$\gamma_v + \pi \longrightarrow N + \bar{N}$$

Conservation of G parity allows only intermediate states with an even number of pions to contribute to the isoscalar amplitudes and an odd number of pions to the isovector amplitudes. Consider firstly the contributions to the isoscalar amplitudes.

The lowest mass intermediate state contributing to these states is the I = 1, J = 1 bipion. This state leads to a branch cut in the electroproduction amplitude

$$4 \leq t \leq \infty$$

If the interaction is strong enough this cut, to a good approximation, can be replaced by a pole at the resonance energy t_R^v. In such a case we have just to add contributions from the diagram



to the electroproduction amplitude. This method has been followed by Gourdin, Lurie and Martin[73] and by de Tollis, Ferrari and Munczck[74] for the photoproduction amplitude and we make use of their results.

The contribution to the electroproduction amplitude can be written

$$\mathcal{M}^{\circ} = \frac{e\Lambda^{v}(\lambda^{2})\, t_{R}^{v}\, \phi(t_{R}^{v})}{8\sqrt{2}} \Bigg\{ A_{0}^{'} M_{1} - \left(\frac{g^{v} b^{v} t}{M} + 2Ma^{v} \right) \frac{M_{1}}{t_{R}^{v} - t}$$

$$+ \frac{g^{v} b^{v}}{M} Q.K \frac{M_{2}}{t_{R}^{v} - t} - \frac{g^{v} b^{v}}{M} P.K \frac{M_{3}}{t_{R}^{v} - t} \qquad \text{VII.1}$$

$$+ 2a^{v} P.K \frac{M_{4}}{t_{R}^{v} - t} - 2a^{v} \frac{M_{5}}{t_{R}^{v} - t} \Bigg\}$$

In the above $\Lambda^{v}(\lambda^{2})$ and $\phi\,(t_{R}^{v})$ are functions related to the $\gamma_{v} + \pi \longrightarrow \pi + \pi$ vertex. a^{v} and b^{v} are associated with the isovector form factors of the nucleon and are discussed in detail below. The constant $A_{0}^{'}$ is introduced by a subtraction made in the amplitude $A_{1}(s,t,u)$ to allow for its dominant high energy behaviour. In the centre of mass frame of the $N\bar{N}$ system this amplitude has an extra energy factor associated with it.

The contribution of the above diagram to each of the invariant amplitudes is now easily obtained

$$A_i^{\pm\pi\pi} = 0$$

$$A_1^{0\pi\pi} = \frac{e\Lambda^v(\lambda^2)\,t_R^v\,\phi(t_R^v)}{8\sqrt{2}}\left\{A_0' - \left(\frac{g^v b^v t}{M} + 2Ma^v\right)\right\}\frac{1}{t_R^v - t}$$

$$A_2^{0\pi\pi} = \frac{e\Lambda^v(\lambda^2)\,t_R^v\,\phi(t_R^v)}{8\sqrt{2}}\;\frac{g^v b^v}{2M}\;\frac{(t+\lambda^2-1)}{t_R^v - t}$$

$$A_3^{0\pi\pi} = \frac{e\Lambda^v(\lambda^2)\,t_R^v\,\phi(t_R^v)}{8\sqrt{2}}\;\frac{g^v b^v}{4M}\;\frac{(\lambda^2-2M^2-1+2s+t)}{t_R^v - t} \qquad \text{VII.2}$$

$$A_4^{0\pi\pi} = \frac{e\Lambda^v(\lambda^2)\,t_R^v\,\phi(t_R^v)}{8\sqrt{2}}\;\frac{a^v}{2}\;\frac{(2M^2+1-\lambda^2-2s-t)}{t_R^v - t}$$

$$A_5^{0\pi\pi} = -\frac{e\Lambda^v(\lambda^2)\,t_R^v\,\phi(t_R^v)}{8\sqrt{2}}\;\frac{2a^v}{t_R^v - t}$$

$$A_6^{0\pi\pi} = 0$$

The lowest mass state contributing to the continuum of intermediate states in the isovector amplitudes is the three pion state, the tripion, with $I = 0$, $J = 1$. Since they can be considered as particles with the same spin the contributions of this state to the isovector amplitudes take a similar form to the above. In this case however the tripion contributes only to the (+) amplitudes. Similar equations to equations VII.2 are obtained for the (+) amplitudes with $\Lambda^v(\lambda^2)$, a^v, b^v, g^v and t_R^v replaced by $\Lambda^s(\lambda^2)$, a^s, b^s, g^s and t_R^s respectively. The resonance in the three pion system is at t_R^s and $\Lambda^s(\lambda^2)$ is associated with the vertex $\Upsilon_v + \pi \longrightarrow \pi + \pi + \pi$.

The constants $a^{v,s}$, $b^{v,s}$ arise because in deriving the above equations we have made use of a similar approximation to describe the electromagnetic structure of the nucleon. In this model, first proposed by Hofstadter and Herman(12), the nucleon is considered as consisting of two Yukawa clouds surrounding a delta function core. The two clouds are approximated by the bipion and tripion. This leads to the representations for the nucleon form factors

$$
\begin{aligned}
F_1^v(\lambda^2) &= \frac{e}{2}\left(1 - a^v + \frac{a^v t_R^v}{t_R^v + \lambda^2}\right) \\
F_1^s(\lambda^2) &= \frac{e}{2}\left(1 - a^s + \frac{a^s t_R^s}{t_R^s + \lambda^2}\right) \\
F_2^v(\lambda^2) &= \frac{e g_v}{2M}\left(1 - b^v + \frac{b^v t_R^v}{t_R^v + \lambda^2}\right) \\
F_2^s(\lambda^2) &= \frac{e g_s}{2M}\left(1 - b^s + \frac{b^s t_R^s}{t_R^s + \lambda^2}\right)
\end{aligned}
\qquad \text{VII.3}
$$

$$
g^v = \mu_p' - \mu_N = 3\cdot 69 \quad ; \quad g^s = \mu_p' + \mu_N = -\cdot 13
$$

Where μ_p' and μ_N are the anomalous static proton and neutron moments respecrively. The parameters in the representation VII.3 are obtained from electron scattering experiments and are discussed in the next section.

We neglect other contributions to the electroproduction amplitude from higher mass intermediate states since we have no certain way of calculating them.

The pole approximations for the pion-pion cuts can now be obtained as before. This time, however, we require to know the region of convergence of the multipole expansion on the complex cut.

The expansion will converge inside an ellipse on the complex y-plane with foci at $y = \pm 1$, where y is the cosine of the angle of scattering in the process $\gamma_v + \pi \longrightarrow N + \bar{N}$. The size of the ellipse will depend upon the position of the first singularity in $A_c^T(s,t)$ which in turn depends upon the boundaries of the regions in which the spectral functions $a_{13}^T(s,t)$ and $a_{23}^T(s,t)$ are non-zero. The detailed calculation is outlined in Appendix 2.

It was found that the expansion was valid for values of t^1 in equation IV.18 in the regions

$$4 < t_R^v \leq t' \leq 61{\cdot}25 + 4{\cdot}35\lambda^2 \qquad \text{isoscalar amplitudes}$$

$$9 < t_R^s \leq t' \leq 37{\cdot}8 + 5{\cdot}66\lambda^2 \qquad \text{isovector amplitudes} \qquad \text{VII.4}$$

We therefore considered only the contributions from these parts of the cut to the multipole amplitudes and approximated them by eight poles.

$$n_{1+}^{0\pi\pi}(s,\lambda^2) = H^v(\lambda^2)\frac{t_{20}(\lambda^2)+i\,t_{21}(\lambda^2)}{s-s_{20}(\lambda^2)-i\,s_{21}(\lambda^2)} + J^v(\lambda^2)\frac{t_{18}(\lambda^2)+i\,t_{19}(\lambda^2)}{s-s_{18}(\lambda^2)-i\,s_{19}(\lambda^2)} \qquad \text{VII.6a}$$

+ complex conjugate

$$n_{1+}^{+\pi\pi}(s,\lambda^2) = H^s(\lambda^2)\frac{t_{24}(\lambda^2)+i\,t_{25}(\lambda^2)}{s-s_{24}(\lambda^2)-i\,s_{25}(\lambda^2)} + J^s(\lambda^2)\frac{t_{22}(\lambda^2)+i\,t_{23}(\lambda^2)}{s-s_{22}(\lambda^2)-i\,s_{23}(\lambda^2)} \qquad \text{VII.6b}$$

with

$$H^{v,s}(\lambda^2) = \frac{e\Lambda^{v,s}(\lambda^2)\,t_R^{v,s}\,\phi(t_R^{v,s})\,g^{v,s}\,b^{v,s}}{256\sqrt{2}\,\pi\,M}$$

$$J^{v,s}(\lambda^2) = \frac{e\Lambda^{v,s}(\lambda^2)\,t_R^{v,s}\,\phi(t_R^{v,s})\,a^{v,s}}{256\sqrt{2}\,\pi} \qquad \text{VII.7}$$

No calculation was made for the longitudinal quadrupole amplitude since in further calculations its contribution is being neglected.

The residues and pole positions approximating the cuts in each amplitude are given in Table 1.

Table 1

The notation followed in this table is

$${}^{\lambda_1^2}r_i^{\lambda_2^2} \text{ valid for } \lambda_1^2 < \lambda^2 \le \lambda_2^2$$

$$r_j \text{ valid for } 0 < \lambda^2 \le 60\, m_\pi^2$$

MAGNETIC DIPOLE

${}^{0}r_1^{13}$	$.215\lambda^4 - 1.23\lambda^2 + 16.1$	s_1	$.169\lambda^4 - 3.54\lambda^2 + 45.16$
${}^{13}r_1^{60}$	15.33	s_1	27.01
r_2	$.0148\lambda^2 + 3.336$	s_2	$-10\sqrt{6.88\lambda^2 + 537.1}$
r_3	$.00829\lambda^2 + 13.68$	s_3	$-1475 + .55\lambda^2$
r_4	77.62	s_4	-27110
r_5	$219/(\lambda^2 + 96.75)$	s_5	$-.124\lambda^2 + 45.16$
r_6	$\sqrt{23.06 - .269\lambda^2}$	s_6	-170
r_7	$\sqrt{113 - .454\lambda^2}$	s_7	-605
r_8	$10\sqrt{11.09 - .0167\lambda^2}$	s_8	-1885
r_9	$10\sqrt{36.2 + .0095\lambda^2}$	s_9	-10760
r_{10}	89.2	s_{10}	-48450
r_{11}	$2927/(\lambda^2 + 96)$	s_{11}	$-.126\lambda^2 + 45.16$
r_{12}	$21.47 - .611\lambda^2$	s_{12}	$-100\sqrt{3.12 - .049\lambda^2}$
r_{13}	$10\sqrt{79.12 + .069\lambda^2}$	s_{13}	$-100\sqrt{37.35 + .012\lambda^2}$
r_{14}	$10\sqrt{933 + .917\lambda^2}$	s_{14}	-1910
r_{15}	580	s_{15}	-10800

r_{16}	876	s_{16}	-48510
r_{17}	$10\sqrt{50.64 - .305\lambda^2}$	s_{17}	$10\sqrt{4.898 + .012\lambda^2}$
r_{18}	$-.209\lambda^4 + 2.03\lambda^2 + 53.03$	s_{18}	$.021\lambda^4 - 1.37\lambda^2 + 21.2$
r_{19}	$-.148\lambda^4 + 7.52\lambda^2 - 6.14$	s_{19}	$-.024\lambda^4 + .75\lambda^2 + 34.2$
r_{20}	$-.518\lambda^4 + 21.81\lambda^2 - 67.36$	s_{20}	$.016\lambda^4 - .56\lambda^2 + 26$
r_{21}	$.067\lambda^4 + 1.9\lambda^2 - 394.5$	s_{21}	$.807\lambda^2 + 33.52$
r_{22}	$-.079\lambda^4 - 1.066\lambda^2 + 157.6$	s_{22}	$.0045\lambda^4 - .63\lambda^2 + 31.2$
r_{23}	$-.237\lambda^4 + 11.83\lambda^2 - 40.6$	s_{23}	$-.024\lambda^4 + 1.57\lambda^2 + 23.9$
r_{24}	$-.044\lambda^4 + 12.78\lambda^2 - 189$	s_{24}	$-.225\lambda^2 + 38.95$
r_{25}	$.183\lambda^4 - 12.94\lambda^2 - 307.1$	s_{25}	$1.489\lambda^2 + 21.22$

Electric Quadrupole

r_1	$-.00013\lambda^4 - .09\lambda^2 + 4.53$	s_1	$.093\lambda^4 - 3.96\lambda^2 + 56.3$
r_2	$.00364\lambda^2 + 1.033$	s_2	$.0029\lambda^4 - 1.16\lambda^2 - 234$
r_3	$-.0026\lambda^2 - 4.49$	s_3	-1472
r_4	25.85	s_4	-27150
r_5	$.00013\lambda^4 - .003\lambda^2 + .038$	s_5	$-.124\lambda^4 + .29\lambda^2 + 55.1$
r_5	$.00013\lambda^4 - .003\lambda^2 + .038$	s_5	$-9.54\lambda^2 + 125.9$
$r_6^{0\ 20}$	$.593 - .0022\lambda^2$	s_6	$-1.432\lambda^2 - 195.8$
$r_6^{20\ 60}$	$.593 - .0022\lambda^2$	s_6	$-5.55\lambda^2 - 113.5$
r_7	$.0044\lambda^2 + 1.125$	s_7	-586.2
r_8	$.338 + .00305\lambda^2$	s_8	$-1.853\lambda^2 - 943$
r_9	-8.57	s_9	-12150
r_{10}	-20	s_{10}	-49780
r_{11}	0	s_{11}	$-$
$r_{12}^{0\ 20}$	$\sqrt{279.3 - .687\lambda^2}$	s_{12}	$-.751\lambda^2 - 188.3$
$r_{12}^{20\ 60}$	$\sqrt{279.3 - .687\lambda^2}$	s_{12}	$-1.16\lambda^2 - 180.2$
r_{13}	$.108\lambda^2 + 33.05$	s_{13}	-600.7
r_{14}	$\sqrt{20.53\lambda^2 + 7563}$	s_{14}	-1822
r_{15}	119.3	s_{15}	-10420
r_{16}	141.5	s_{16}	-47720

r_{17}	$-\lambda^2 + 2.676$	s_{17}	$.0546\lambda^4 - 2.07\lambda^2 + 59.2$
r_{18}	$-.0113\lambda^4 + .064\lambda^2 - 3.85$	s_{18}	$.905\lambda^2 + 29.99$
r_{19}	$-.0245\lambda^4 + .77\lambda^2 - 12.7$	s_{19}	$.068\lambda^4 + 1.01\lambda^2 + 33.29$
r_{20}	$-6.24\lambda^2 - 40.13$	s_{20}	$.083\lambda^4 + 1.3\lambda^2 + 29.6$
r_{21}	$-.506\lambda^4 + 10.6\lambda^2 - 169$	s_{21}	$2.4\lambda^2 + 30.94$
r_{22}	$-.015\lambda^4 + .714\lambda^2 - 7.93$	s_{22}	$-.228\lambda^4 + 6.2\lambda^2 + 40.1$
r_{23}	$-.013\lambda^4 - .015\lambda^2 + 2.88$	s_{23}	$.0434\lambda^6 - 1.69\lambda^4 + 17.3\lambda^2 + 4.63$
r_{24}	$.0622\lambda^6 - 2.04\lambda^4 + 14.87\lambda^2 - 130.2$	s_{24}	$-.381\lambda^4 + 3.17\lambda^2 + 44$
r_{25}	$-.454\lambda^4 + 4.93\lambda^2 + 31.7$	s_{25}	$-.0455\lambda^6 + 1.272\lambda^4 - 5.31\lambda^2 + 38.79$

Longitudinal Quadrupole

$r_1^{0\ 22}$	$-.0024\lambda^4+.049\lambda^2-.234$	s_1	$-.174\lambda^4+7.32\lambda^2-56.9$
$r_1^{22\ 36}$	$-.0047\lambda^4+.296\lambda^2-4.58$	s_1	$-.48\lambda^4+27.6\lambda^2-389$
$r_1^{36\ 60}$	$.0218$	s_1	$-.48\lambda^4+27.6\lambda^2-389$
r_2	0	s_2	-
r_3	0	s_3	-
r_4	0	s_4	-
r_5	$10^{-5}(.435\lambda^4+5.3\lambda^2)$	s_5	49.7
r_6	$10^{-2}(-.0018\lambda^4+.11\lambda^2-3.3)$	s_6	$.00067\lambda^4-.056\lambda^2+2.86$
r_7	0	s_7	-
r_8	0	s_8	-
r_9	0	s_9	-
r_{10}	0	s_{10}	-
r_{11}	0	s_{11}	-
r_{12}	$-8.46/(\lambda^2+7.57)$	s_{12}	$.00326\lambda^4+.23\lambda^2-7.76$
r_{13}	0	s_{13}	-
r_{14}	$69.1/(\lambda^2+.55)$	s_{14}	-12360
r_{15}	$611/(\lambda^2+.62)$	s_{15}	-118520
r_{16}	0	s_{16}	-
r_{17}	0	s_{17}	-

7. The N/D Solution and Results.

The method of solution used is the N/D method first proposed by Chew and Mandelstam[15]. Bjorken[16] has generalised this method to deal with multi-channel processes.

In the energy region we consider the present problem has only two possible channels. We label them

$\gamma_v + N$	Channel 1
$\pi + N$	Channel 2

and introduce the two by two matrix $g(s)$ with elements $g_{ij}(s)$ the transition amplitudes of specified angular momentum and isotopic spin from channel i to channel j. We assume that $g(s)$ may be written

$$g(s) = D^{-1}(s)\,N(s) \qquad \text{VIII.1}$$

where the elements of $D(s)$, $D_{ij}(s)$, contain only the physical cut for the process channel i $\longrightarrow$ channel j and the $N_{ij}(s)$ all other cuts. The unphysical cuts in the transition amplitudes $g_{ij}(s)$ are taken to be approximated by a set of poles of the form $\dfrac{a_{ij}^n}{s - s_{ij}^n}\ (n = 1, \cdots)$
Each complex pole must be accompanied by its complex conjugate so that the Riemann-Schwarz reflection principle

$$g^*(s) = g(s^*) \qquad \text{VIII.2}$$

is satisfied. It follows that since $N(s)$ is real on the right-hand cuts

$$D^*(s) = D(s^*) \qquad \text{VIII.3}$$

From the above properties of $N_{ij}(s)$ and $D_{ij}(s)$ we have

$$N_{ij}(s) = \sum_{k,n} D_{ik}(s^n_{kj}) \frac{a^n_{kj}}{s - s^n_{kj}} \qquad \text{VIII.4}$$

and
$$D_{ij}(s) = \delta_{ij} + \frac{(s-s_0)}{\pi} \int_{\sigma_{ij}}^{\infty} ds' \frac{\mathcal{I}m\, D_{ij}(s')}{(s'-s)(s'-s_0)} \qquad \text{VIII.5}$$

We have chosen the normalisation of D_{ij} such that

$$D_{ij}(s_0) = \delta_{ij}$$

The range of integration in equation VIII.5 is over the physical cut $\sigma_{ij} \leq s \leq \infty$ contained in $D_{ij}(s)$.

The unitarity condition for the inverse of $g(s)$ is

$$\mathcal{I}m\, (g^{-1})_{ij} = - k_i\, \theta_i\, \delta_{ij} \qquad \text{VIII.6}$$

where k_i is the barycentric momentum of the ith channel and θ_i is the step function 0 or 1 according as W is less than or greater than the threshold energy for channel i. The unitarity condition will have to be modified to allow for any redefinition of the transition amplitudes such as in equation IV.19 .

For S in the physical range $\sigma_{ij} \leq s \leq \infty$ we have

$$\mathcal{I}m\, D_{ij}(s) = - k_j\, N_{ij}(s)$$

Hence

$$D_{ij}(s) = \delta_{ij} + \frac{s - s_0}{\pi} \sum_{k,n} D_{ik}(s^n_{kj})\, a^n_{kj} \int_{\sigma_{ij}}^{\infty} \frac{k_j(s')\, ds'}{(s'-s)(s'-M^2)(s'-s^n_{kj})} \qquad \text{VIII.7}$$

We next investigate the order of magnitude of the D_{ij} and N_{ij} to powers in the electromagnetic coupling constant and ignore these of order e^2. It can be deduced that

$$g_{12}(s) = \sum_k \frac{D_{22}(s^k_{12})\, a^k_{12}}{D_{22}(s)\,(s - s^k_{12})} \qquad \text{VIII.8}$$

It depends only upon knowledge of the denominator function for pion-nucleon scattering and the pole approximations for the unphysical cuts in the electroproduction amplitude.

Knowledge of the residues in the three amplitudes considered is not complete until we have determined the form factors of the pion and nucleon.

The pion form factor, $F_\pi(\lambda^2)$ is at present not known. In the vertex

$$\gamma_v \longrightarrow \pi + \pi$$

only intermediate states of an even number of pions can contribute. For low momentum transfer from the electron the only intermediate state which contributes is therefore the two pion state. In this region it can be shown that $F_\pi(\lambda^2)$ must have the same phase as pion-pion scattering in the resonant J = 1, I = 1 state, which we shall designate as δ_1. Also the pion form factor is an analytic function in the entire complex λ^2 plane except for a branch cut along the real axis for $\lambda^2 \leq -4$. Cauchy's integral theorem leads to the representation

$$F_\pi(\lambda^2) = 1 - \frac{\lambda^2}{\pi}\int_4^\infty dt' \frac{\mathrm{Im}\, F_\pi(t')}{t'(t'+\lambda^2)} \qquad \text{VIII.9}$$

for which there exists the approximate Omnes solution

$$F_\pi(\lambda^2) \simeq e^{-u(\lambda^2)} \qquad \text{VIII.10}$$

where

$$u(\lambda^2) = \frac{\lambda^2}{\pi}\int_4^\infty dt' \frac{\delta_1(t')}{t'(t'+\lambda^2)}$$

The results of Bransden and Moffat(14) predict an effective range formula for the phase shift $\delta_1(t)$

$$\tan\delta_1(t) = \frac{4\Gamma}{t - t_R^v}\sqrt{\frac{t_R^v\,(t-4)^3}{t\,(t_R^v-4)^3}} \qquad \text{VIII.11}$$

(Γ being the width of the resonance which we take to be approximately equal to .7 m_π)

The pion form factor can now be calculated from equations VIII.10 and VIII.11. The results are shown in Figure 5.

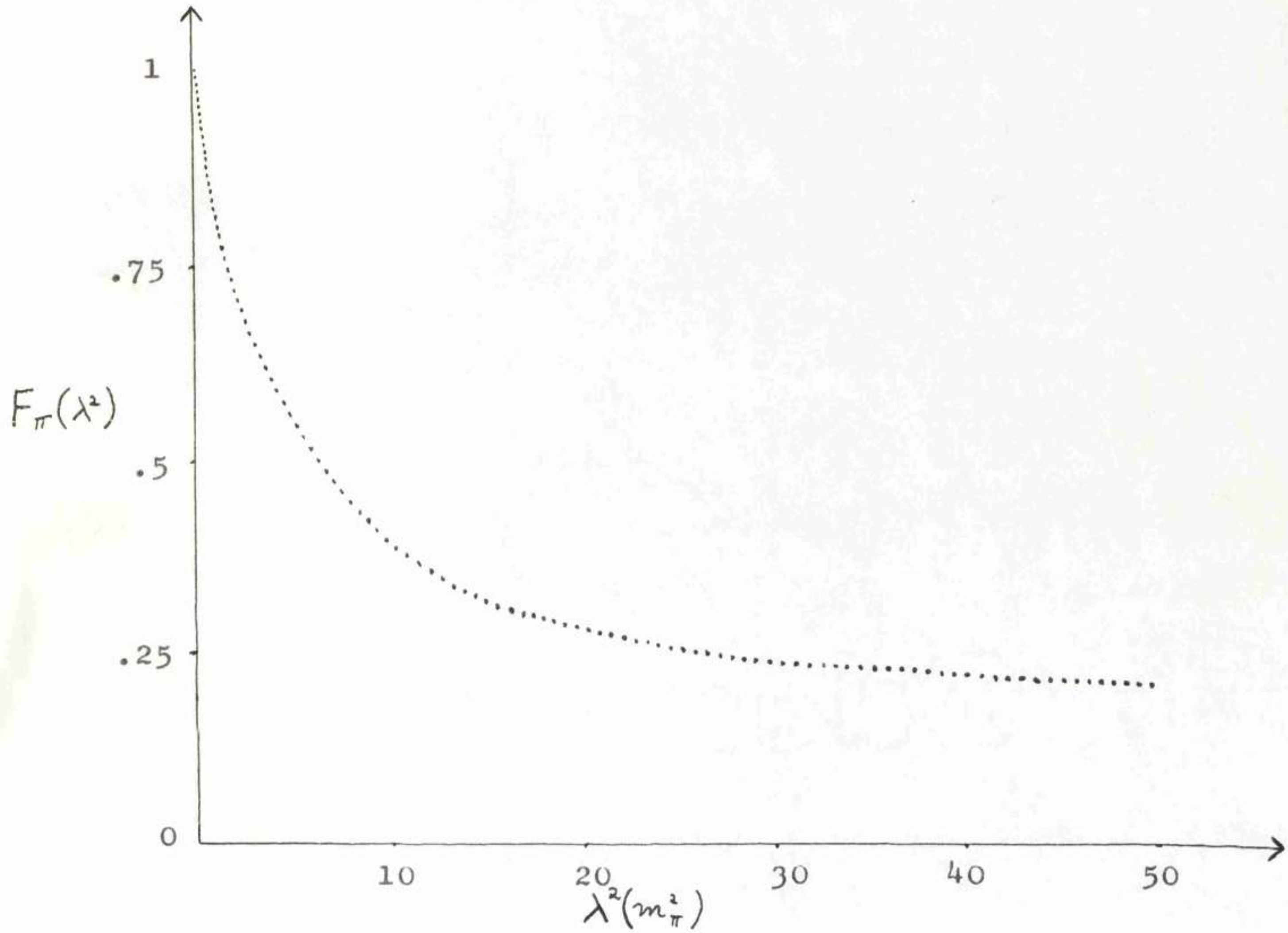


Figure 5.

The Pion Form Factor.

Calculation of the charge and magnetic form factors of the proton and neutron is simplified because we can refer to the experimental results of Hofstader[12] and Wilson[13] for comparison. The isovector and isoscalar form factors of the nucleon have the representations

$$F^{v}_{1,2}(\lambda^2) = F^{v}_{1,2}(0) - \frac{\lambda^2}{\pi}\int_4^\infty \frac{g^{v}_{1,2}(t')}{t'(t'+\lambda^2)}\,dt' \qquad \text{VIII.12}$$

and $$F^{s}_{1,2}(\lambda^2) = F^{s}_{1,2}(0) - \frac{\lambda^2}{\pi}\int_9^\infty \frac{g^{s}_{1,2}(t')}{t'(t'+\lambda^2)}\,dt'$$

The spectral functions, $g(t')$ are connected through unitarity to the processes $\gamma_v \longrightarrow n\pi$ and $n\pi \longrightarrow N + \bar{N}$, n being even in the isovector case and odd for the isoscalar amplitudes. Present experimental evidence indicates that there is a strong concentration of the spectral functions in the regions where the bipion and tripion resonances are dominant and indeed the theoretical treatment of nucleon electromagnetic structure by Frazer and Fulco[59] showed that such an enhancement could give good agreement with experiment. We are thus led to the representations VII.3 for the nucleon form factors, proposed by Clemental and Villi[60].

The experimental fits of these representations made both at Stanford and at Cornell give for the masses of the bipion and tripion the values

$$t^{v}_{R} \simeq 20\, m_\pi^2 \quad ; \quad t^{s}_{R} \simeq 10\, m_\pi^2 \qquad \text{VIII.13}$$

with the associated parameters taking the values

$$a^v = b^v = 1{\cdot}2 \quad ; \quad a^s = {\cdot}56 \quad , \quad b^s = -3$$

The mass of the bipion predicted in this manner is not too different from that obtained by the meson experiments of Erwin, March, Walker and West(76), which give $t_R^v \simeq 30m_\pi^2$. Hofstadter has noted that a choice of $t_R^v \simeq 28m_\pi^2$ and $t_R^s \simeq 20m_\pi^2$ fits present experimental data on electron scattering equally as well.

In the calculations below the set of values VIII.13 were used.

The cross-section calculated was that for the process

$$e^- + p \longrightarrow e^- + \begin{cases} p + \pi^0 \\ n + \pi^+ \end{cases} \qquad \text{VIII.14}$$

It can be written

$$\frac{d^2\sigma}{d\Omega\, d\epsilon_2} = \frac{e^2}{4\pi^3} \frac{Q^3 K^2 s_2 M}{s_1 W^5 \lambda^4} \Big\{ |m_{1+}^a|^2 + |m_{1+}^b|^2 + 3|e_{1+}^a|^2 + 3|e_{1+}^b|^2 \Big\} \Big\{ 2 s_1^2 \sin^2\sigma + \lambda^2 \Big\} \qquad \text{VIII.15}$$

where

$$n_{1+}^a = \sqrt{\frac{2}{3}} \left(n_{1+}^+ - n_{1+}^- \right) \qquad \text{VIII.16a}$$

and

$$n_{1+}^b = \frac{1}{\sqrt{3}} \left(n_{1+}^+ + 2n_{1+}^- + 3n_{1+}^0 \right) \qquad \text{VIII.16b}$$

are the amplitudes leading to final states of total isotopic spin 3/2 and 1/2 respectively. Contributions to this cross-section from the longitudinal quadrupole amplitudes have been neglected. The reason for calculating this cross-section is that, up to the present time, only experiments[77] in which the final electron is observed have been performed.

We state here for completeness, the relations between the centre of mass energy W, the invariant momentum transfer λ^2 and the energies of the incoming and outgoing electrons

$$\lambda^2 = -2m_e^2 - 2s_1 s_2 \cos\sigma + 2\epsilon_1\epsilon_2$$

$$s = M^2 - \lambda^2 + 2M(\epsilon_1 - \epsilon_2) \qquad \text{VIII.17}$$

Three methods of using the N/D method of solution for the multipole amplitudes have been attempted. The first makes use of a theoretical description of the pion-nucleon channel and the other two the known experimental pion-nucleon scattering phase shifts in the I = 3/2, J = 3/2 state. We discuss each in turn.

Hendry[17] has derived a set of poles on the complex S-plane approximately the left-hand cuts contained in the P-wave pion-nucleon scattering amplitude $g_{1+}(s) = \dfrac{s\, e^{i\delta_{1+}} \sin\delta_{1+}}{Q^3}$ He found that the contributions from the nucleon pole term could be represented by three poles.

$$g_{1+}^{I=3/2}(s) = \frac{68.7}{s - 45.2} + \frac{350}{s + 140} + \frac{320}{s + 520}$$

$$g_{1+}^{I=1/2}(s) = \frac{-34.4}{s - 45.2} + \frac{-175}{s + 140} + \frac{-160}{s + 520} \qquad \text{VIII.18a}$$

Other contributions, from the crossed physical cut and pion-pion cut, were calculated where possible with the corresponding pole approximations

$$g_{1+}^{I=3/2}(s) = \frac{2.1}{s - 25} + \frac{7.8 - 30.9i}{s - 38.6 - 8.8i} + \frac{7.8 + 30.9i}{s - 38.6 + 8.8i}$$

$$g_{1+}^{I=1/2}(s) = \frac{8.4}{s - 25} + \frac{-15.6 + 61.8i}{s - 38.6 - 8.8i} + \frac{-15.6 - 61.8i}{s - 38.6 + 8.8i} \qquad \text{VIII.18b}$$

Only contributions from a possible bipion intermediate state with energy $\simeq 3.4\ m\pi$ are contained in the above.

We note firstly that this representation of the pion-nucleon P wave scattering amplitude leads to a cross-section with only qualitative agreement with experiment. The position of the I = 3/2, J = 3/2 resonance is rather lower than presently accepted, occuring at s $\simeq 64m_\pi^2$. Also the cross-section is a factor of two too large. This lack of quantitative agreement with experiment must be due both to the approximation of the cuts by poles and to the lack of knowledge of contributions from multiparticle intermediate states. Since we have treated the electroproduction process in a similar fashion we must expect correspondingly poor agreement with experiment. We should however be able to reproduce the qualitative features of the process.

Given the incident electron energy (we took ϵ_1 = 550 Mev) and the electron scattering angle σ, for each value of the outgoing electron energy ϵ_2 the variables s and λ^2 can be calculated from equation VIII.17. Hendry found it convenient to redefine his amplitudes such that they became

$$g_{1+}^{I} = \frac{s\, e^{i\delta_{1+}^{I}} \sin \delta_{1+}^{I}}{Q^3}$$

Here Q is the barycentric momentum in the pion-nucleon system

$$Q^2 = \frac{[s-(M+1)^2][s-(M-1)^2]}{4s}$$

The unitarity condition then becomes

$$\mathcal{I}m\left(g^{-1}\right)_{22} = -\frac{Q^3}{s}\,\theta_2 \qquad \text{VIII.19}$$

leading, as above, to

$$D_{22}(s) = 1 + \frac{M^2-s}{\pi}\sum_n D_{22}(s_{22}^n)\, a_{22}^n \int_{(M+1)^2}^{\infty} \frac{Q^3(s')\,ds'}{s'(s'-s_{22}^n)(s'-M^2)(s'-s)} \qquad \text{VIII.20}$$

Equation VIII.20 gives a set of n simultaneous equations for the $D_{22}(s_{22}^n)$ and hence we can solve for $D_{22}(s)$ and $D_{22}(s_{12}^k)$. The cross-section is then calculated from equations VIII.8 and VIII.15.

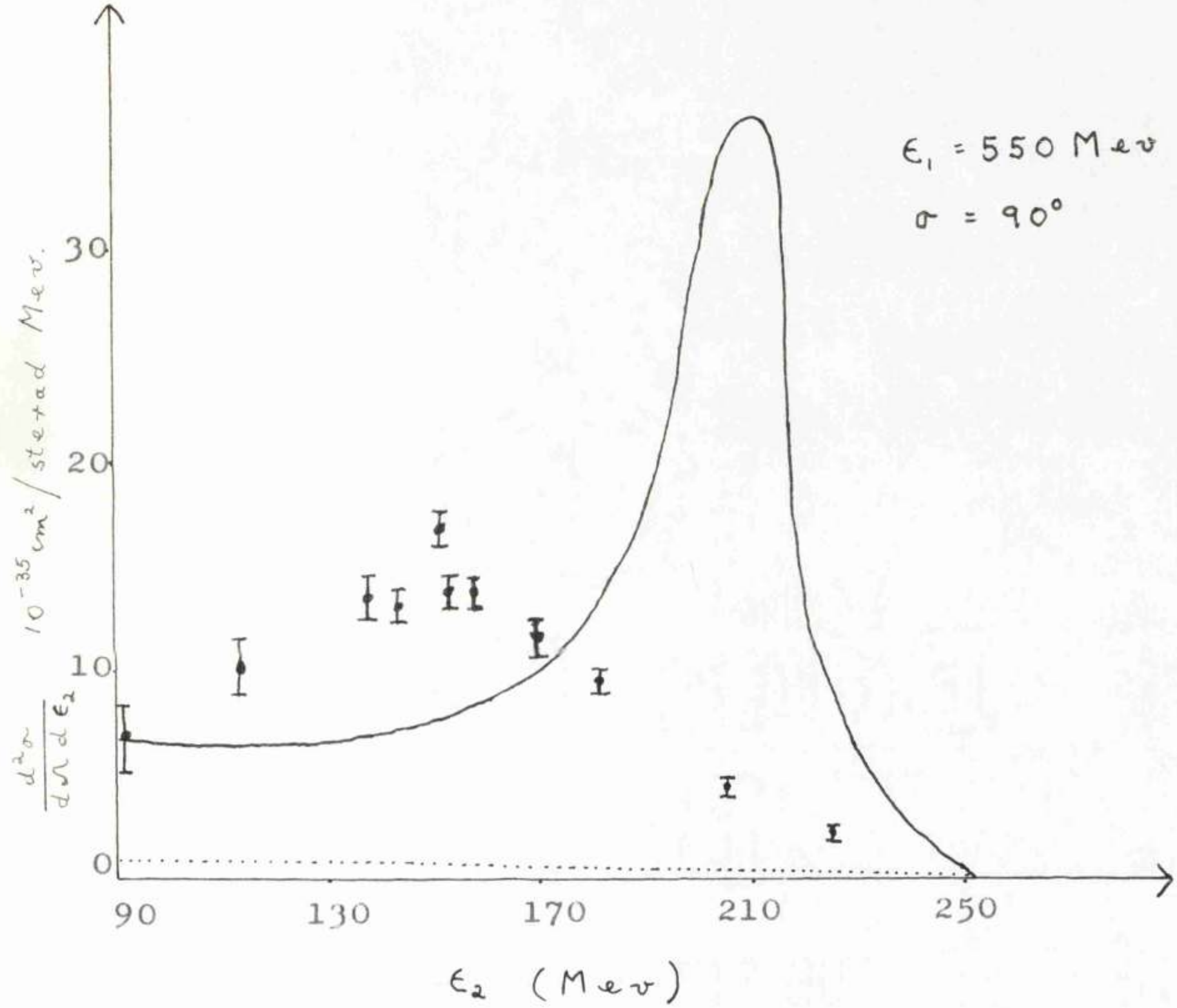


Figure 6.

Magnetic Dipole contribution to cross-section

____ I = 3/2 -------I = 1/2

I Experimental Points from Ohlsen(77)

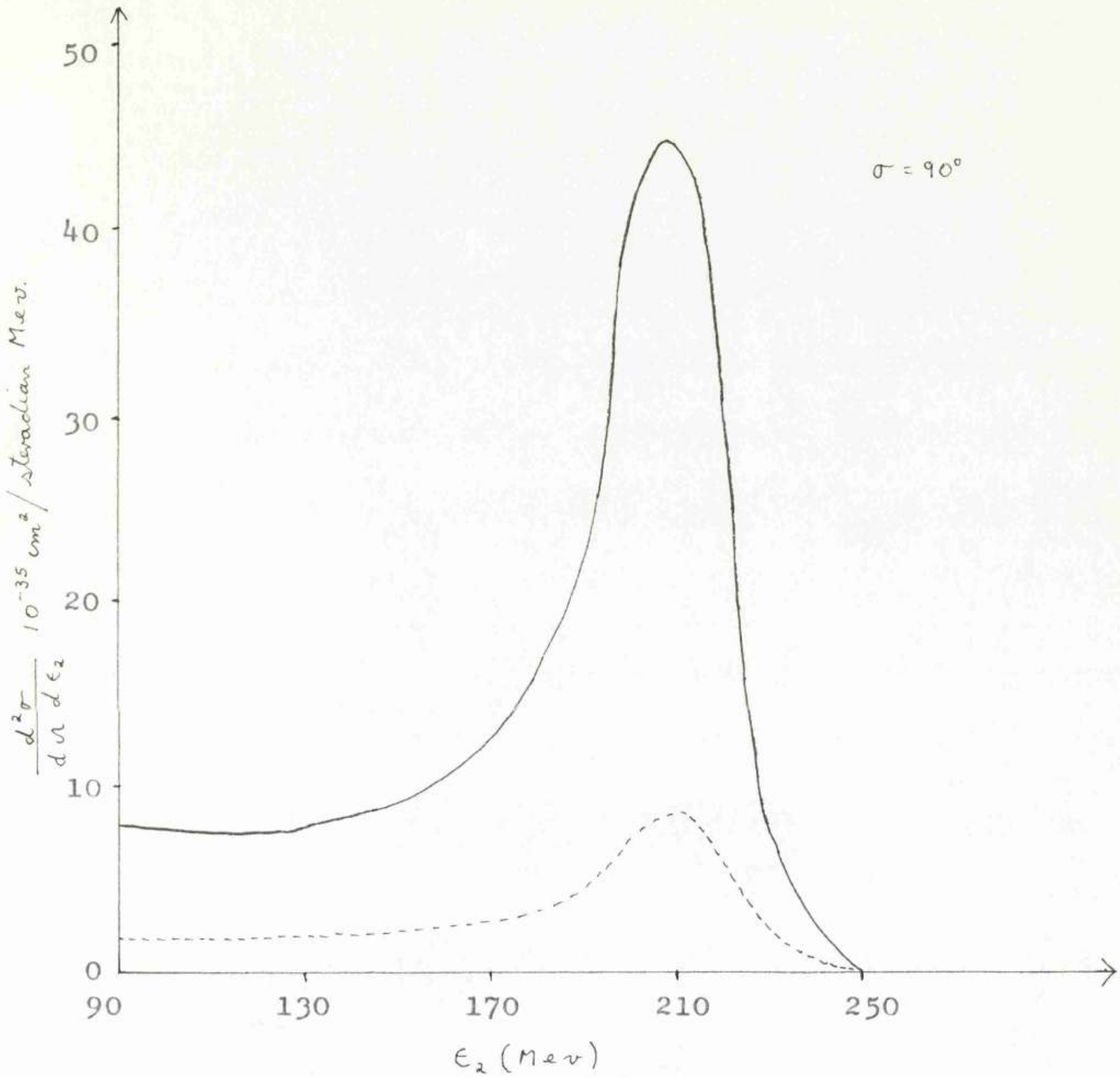


Figure 7

——— Differential electroproduction cross-section with magnetic dipole and electric quadrupole contributions

- - - - Electric quadrupole contribution

The contribution of the magnetic dipole amplitude to this cross-section is shown in Figure 6. As one would expect the $I = 3/2$ amplitude is dominant, contributions from the $I = 1/2$ state being negligible. The qualitative features of the cross-section have been reproduced. However the position of the resonance is rather nearer to threshold than the experimental one. Also the cross-section is a factor of two too large.

The inclusion of a pion-pion interaction (a tripion) in the $I = 3/2$ amplitude has very little effect on the cross-section. Contributions from the $I = 1$, $J = 1$ bipion modify only the $I = 1/2$ amplitude and have therefore been ignored. The importance of tripion contributions must depend upon the constant $\phi\ (t^s_R)$ and the coupling constant $\Lambda^s(\lambda^2)$, which at present are not known. In this calculation we considered $\phi\ (t^s_R) \simeq .75$ and $\Lambda\ (\lambda^2)$ to be constant and equal to 1,0, or -1. We note that a change in the mass of the tripion will make little difference to this cross-section in the energy region considered.

The other enhanced amplitude in this energy region is the electric quadrupole amplitude. We see from Figure 7 that at resonance it contributes about one quarter of the total cross-section. Again only the qualitative features of this contribution have been obtained. A possible tripion contribution again makes little difference to this amplitude, being the order of 2 or 3%.

At higher momentum transfer from the scattered electron (that is at higher incident electron energy or greater electron scattering angle) the cross-section will decrease in magnitude but exhibit the same features. However from Table 1 we see that contributions from the crossed nucleon pole term are rapidly decreasing functions of λ^2 whereas the single pion pole term contribution becomes constant apart from a slight decrease in the pion form factor. Accurate measurement of this cross-section in this region may therefore provide further information on pion structure.

In view of the poor quantitative agreement with experiment obtained with this method of solution, due partly to an inadequate description of pion-nucleon scattering, a method of incorporating the experimental data about this process into the calculation was investigated.

One possible way of doing this is to consider the set of pole approximations to the unphysical cuts in the pion-nucleon scattering amplitude as parameters and to vary them until reasonable agreement with experiment is obtained. However there may well be more than one such set of pole approximations describing pion-nucleon scattering and there is therefore little to be gained by this method. It is better to try and incorporate any experimental data directly into the calculation.

In the energy region considered, below the threshold for inelastic processes, we can consider pion-nucleon scattering to be a one channel process and write the explicit form of the denominator function $D_{22}(S)$

$$D_{22}^{I,\ell\pm}(s) = e^{-v(s)}$$

$$v(s) = \frac{M^2 - s}{\pi} \int_{(}^{\infty} \frac{\phi^{I,\ell\pm}(s')\, ds'}{(s' - M^2)(s' - s)} \qquad \text{VIII.21}$$

This solution for the denominator function was first given by Omnes[78]. Here $\phi^{I,\ell\pm}(s)$ is the phase of the amplitude $g^{I}_{\ell\pm}$ for pion-nucleon scattering. In the low energy region $\phi^{I,\ell\pm}$ will be identically equal to $\delta^{I}_{\ell\pm}$. At energies above the inelastic threshold $\delta^{I}_{\ell\pm}$ can become complex and the relation becomes

$$\tan\phi = \frac{e^{2b} - \cos 2a}{\sin 2a} \qquad \text{VIII.22}$$

where

$$a = \mathrm{Re}\, \delta^{I}_{\ell\pm}$$

$$b = \mathrm{Im}\, \delta^{I}_{\ell\pm}$$

The previous calculation indicates that the dominant contributions to the cross-section, in the energy region considered, come from the I = 3/2, J = 3/2 P wave amplitudes. We therefore limit our attention to only these states.

Woolcock[79] has tabulated $\delta_{1+}^{3/2}$ in the energy region $(M+1)^2 \leq s \leq 100$ where it is real (or has very small complex part). At higher energies it becomes complex and is not known. The convergence of the integral in equation VIII.21 is not good enough to permit a cut-off at $s' = 100$ since at this energy ϕ or δ is quite large, the order of 150^0. Two possibilities are therefore open: either to make a subtraction and hence introduce a further unknown parameter into the calculation or to extrapolate the phase shift ϕ to some high energy behaviour.

If we make a subtraction we have

$$D_{22}(s) = e^{u(s)}$$

$$u(s) = (s - M^2)\,u'(M^2) - \frac{(s-M^2)^2}{\pi}\int_{(M+1)^2}^{\infty} \frac{\phi(s')\,ds'}{(s'-M^2)^2(s'-s)} \qquad \text{VIII.23}$$

where$^{(1)}$ denotes differentiation with respect to s.

With a cut-off in equation VIII.23 at $s' = 100$ the integral on the right is determined to within 10% for $(M + 1)^2 \leq S \leq 75$.

The parameter $U^1(M^2)$ can be determined by requiring that the modulus of the denominator function $D_{22}(s)$ has a minimum turning value when the corresponding amplitude goes through a resonance. If this were the case, since the P wave, I = 3/2 amplitude has a resonance at $S = S_R \simeq 76.5\ m_\pi^2$, the subtraction constant would then be given by

$$U'(M^2) = \frac{(S_R - M^2)^2}{\pi} \int_{(M+1)^2}^{\infty} \frac{s'[f(s') - f(S_R)] - S_R(s' - S_R) f'(S_R)}{s'(s' - S_R)^2} ds'$$

$$+ \frac{2(S_R - M^2)}{\pi} \int_{(M+1)^2}^{\infty} \frac{s' f(s') - S_R f(S_R)}{s'(s' - S_R)} ds' + \frac{\delta'(S_R)}{\pi} \log \frac{(M+1)^2}{S_R - (M+1)^2} \qquad \text{VIII.24}$$

$$+ \frac{.5}{(M+1)^2 - S_R}$$

with

$$f(s) = \frac{\phi^{3/2,3/2}(s)}{(s - M^2)^2}$$

As one might expect the solution of this equation depends sensitively upon the high-energy behaviour of the phase shift. Calculation of $U^1(M^2)$ using Woolcock's phase shifts with a cut-off in the integrals at $S^1 = 100$ leads to a cross-section for the electroproduction of pions from hydrogen which is far too large.

It is therefore more convenient to determine $U^1(M^2)$ from one value of the electroproduction cross-section taken from experiment. The differential cross-section was calculated varying $U^1(M^2)$ at the resonance energy of the outgoing electron for $\epsilon_1 = 550$ Mev and $\sigma = 90^o$. A value of $-.002$ for $U^1(M^2)$ was found to give a good fit at this energy. The differential cross-section was then calculated in the normal manner from equations VIII.8, VIII.15,

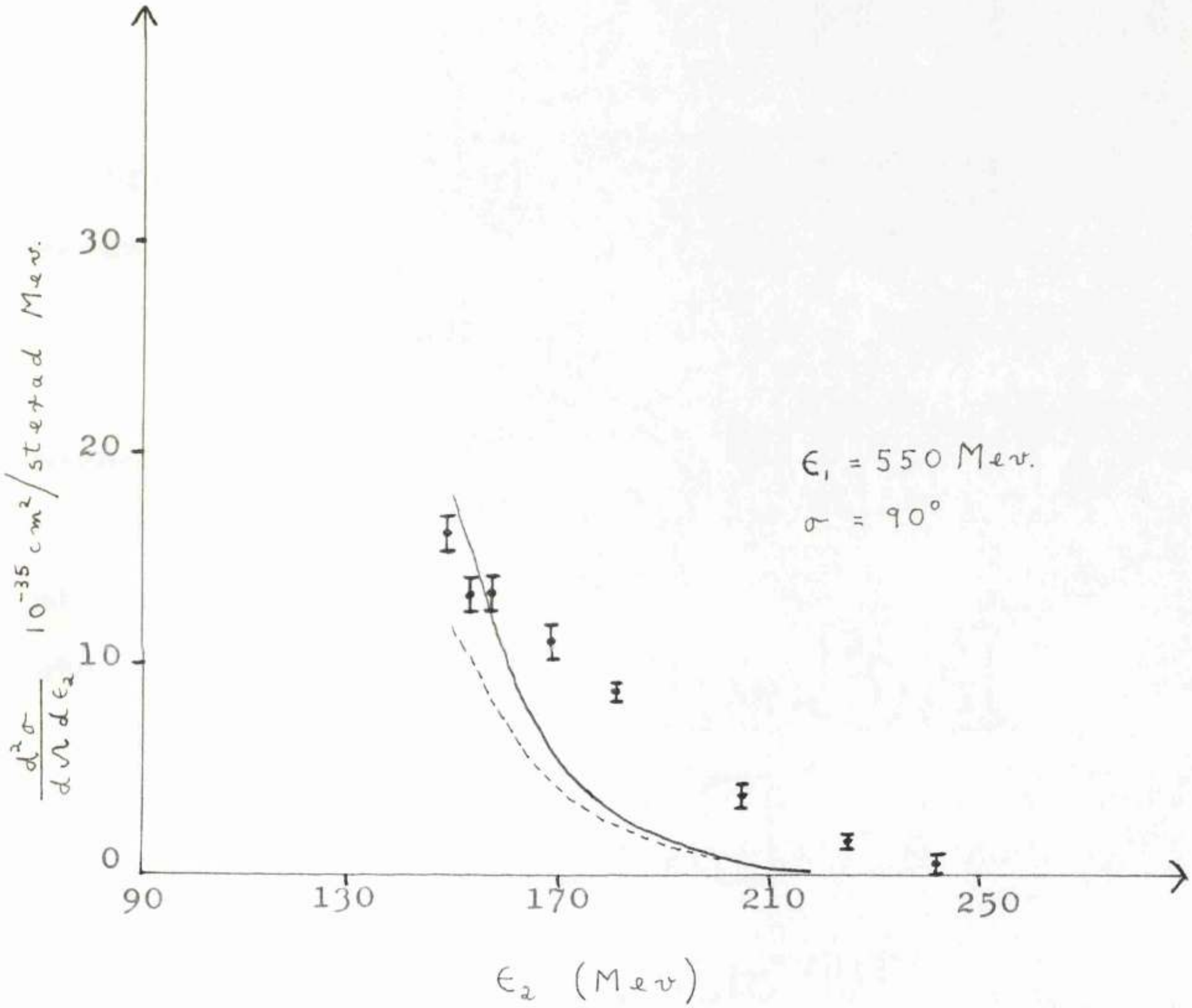


Figure 8

---- Magnetic Dipole contribution

—— with Electric Quadrupole contributions
$U^1(M^2) = -.002$

and VIII.23. The results are shown in Figure 8.

Good agreement with experiment can only be expected from threshold up to the resonance energy. At higher energies the high energy behaviour of the phase shift becomes important again and further subtractions are necessary.

One can conclude from this calculation that the magnetic dipole amplitude is dominant at resonance with the electric quadrupole amplitude contributions a factor of four smaller. This is in good agreement with the phenomenological calculation of Dalitz and Yennie(7) and the calculation of Gartenhaus and Lindner(80) using the theory of Fubini, Nambu and Wataghin(35).

We also note that near threshold there appears to be appreciable contributions to the cross-section from some other multipole amplitudes, most likely the electric dipole and longitudinal monopole amplitudes. The results of Dalitz and Yennie(7) also indicate this possibility.

The other method of solution suggested earlier is to assume some high energy behaviour for the phase ϕ. The previous method was only applicable in a limited energy region for the final pion-nucleon system whereas if we assume some high energy behaviour of ϕ we can now examine the theoretical differential cross-section for the electroproduction of pions at higher energies.

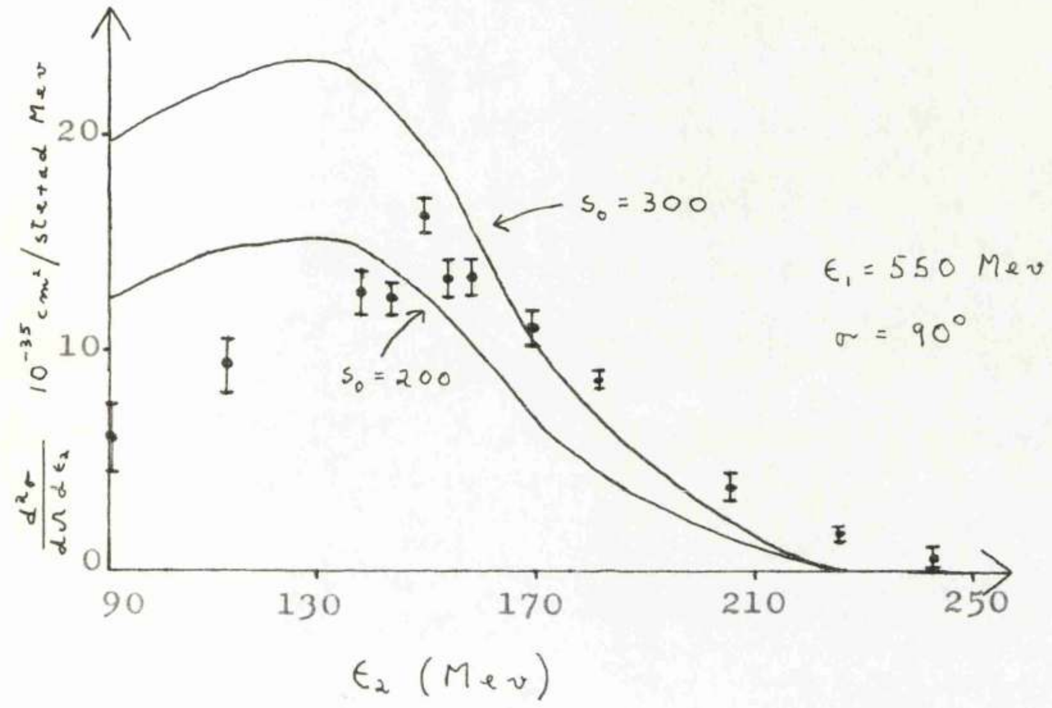


Figure 9

Contribution of Magnetic Dipole, I = 3/2.
Extrapolation to $\phi = 0^{\circ}$ at $s = s_o$

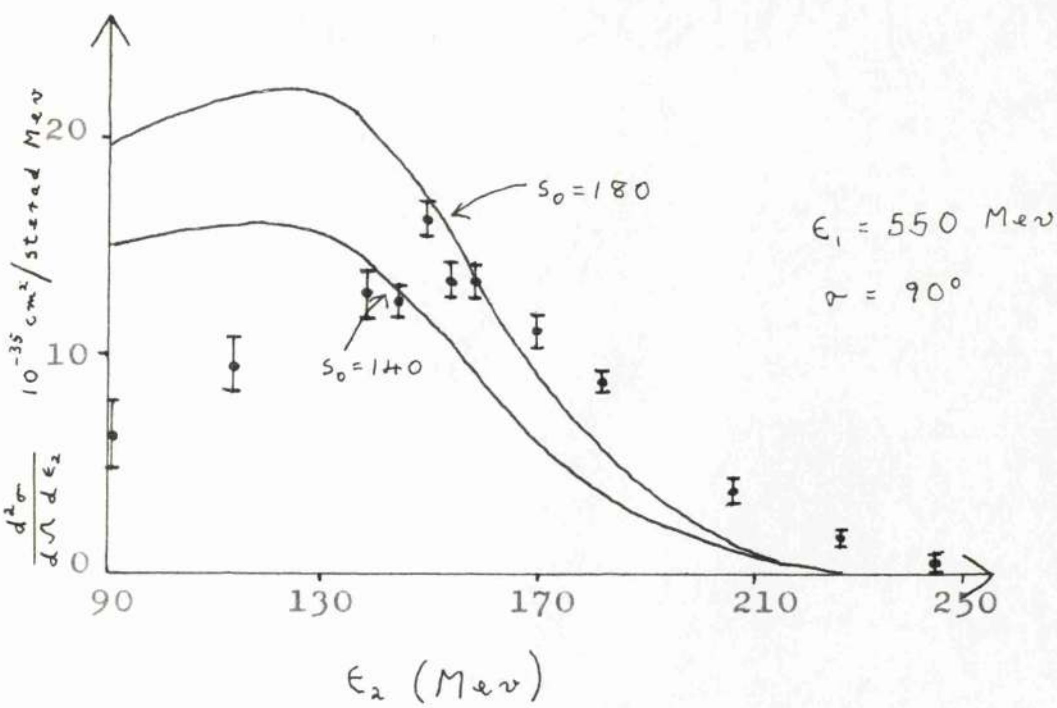
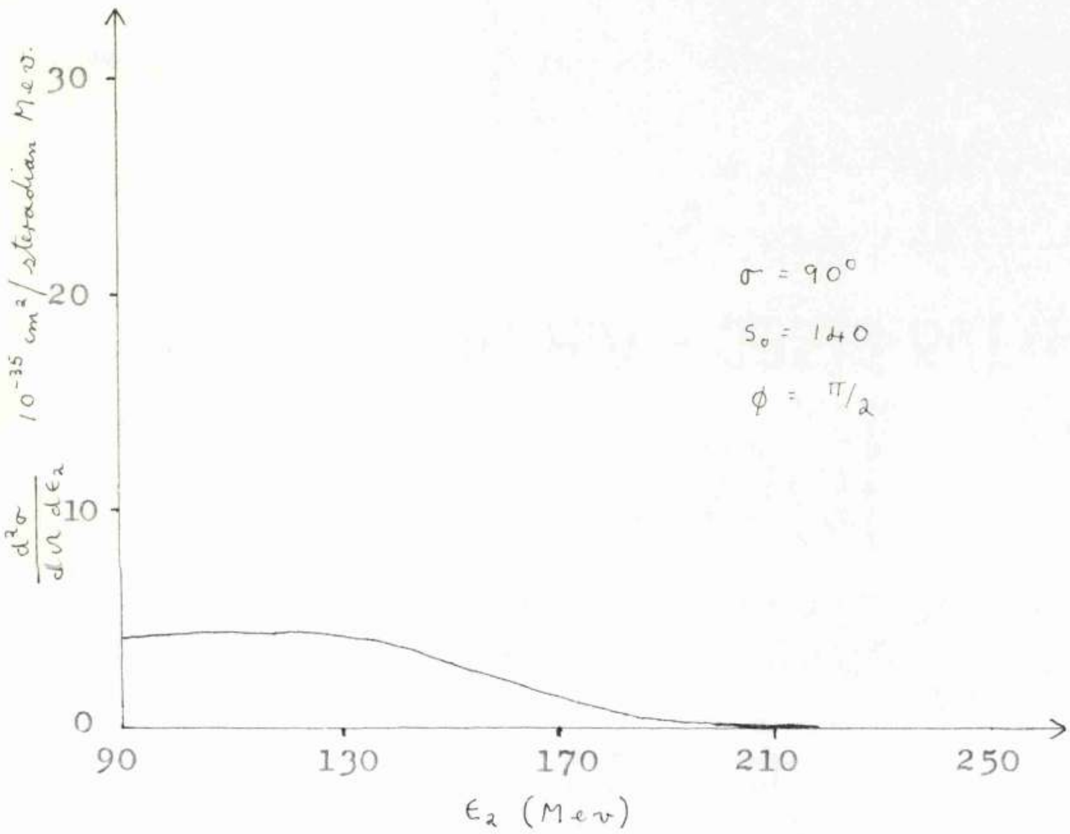


Figure 10

Contribution of Magnetic Dipole, I = 3/2
Extrapolation to $\phi = \pi/2$ at $s = s_o$



<u>Figure 11</u>

Electric Quadrupole contribution to differential cross-section, I = $3/2$.

The high energy behaviour of the phase ϕ was taken to be a linear continuation of Woolcock's phase shifts at $s \simeq 100\ m_\pi^2$ to either 0 or $\pi/2$ at some energy s_o say. The previous calculation gave no indication of a resonance in the differential cross-section but we see from Figures 9, 10, and 11, where the results of this calculation are presented, that the simple extrapolations chosen for ϕ indicate a resonance at the required energy. The agreement with the observed cross-section above resonance is not good but the indications are that with greater knowledge of the phase ϕ for pion-nucleon scattering this method of solution could lead to an adequate description of the electroproduction of pions in this region.

The same qualitative features of the cross-section have been predicted as before. However the results obtained do not appear to be good enough to allow a detailed investigation into the electromagnetic structure of the proton, neutron and pion. Fair agreement with experiment can be obtained with the nucleon form-factors obtained by Hofstadter(12) and Wilson(13).

Chapter 3.

The Photoproduction of a Pion from a Nucleon.

1. Introduction.

Many of the results obtained in the previous Chapter are relevant to the study of the photoproduction amplitude when we take the limit $\lambda^2 \rightarrow 0$. Since the photon in this case is real (i.e. on the mass-shell) and hence transversely polarized with the condition

$$\underset{\sim}{k} \cdot \underset{\sim}{\epsilon} = 0 \qquad \text{IX.1}$$

the longitudinal amplitudes $Y_{\ell\pm}(s,o)$ will not contribute to the total amplitude. Also the pion and nucleon electromagnetic form-factors, normalised to the case of a real photon with $\lambda^2 = 0$, are constant throughout.

We are therefore interested in the transverse multipole amplitudes $M_{\ell\pm}(s,o)$ and $E_{\ell\pm}(s,o)$ contributing to the photoproduction amplitude. From Chapter 2 we observe that they have the analytic properties on the complex S-plane, namely

The Physical Cut.

$$(M + 1)^2 \leq s \leq \infty$$

The Crossed Nucleon Cut.

$$-\infty \leq s \leq 0$$

and a pole at $s = M^2$

The Pion Cut.

$$-\infty \leq s \leq 0$$

and a pole at $s = M^2$

The Crossed Physical Cuts.

$$-\infty \leq s \leq 0$$

and

$$-\infty \leq s \leq \frac{M}{M+1}\left(M^2 - M - 1\right)$$

The Pion-Pion Cut.

$$-\infty \leq s \leq 0$$

and that formed by the locus of points

$$s = \frac{1}{2}\left(2M^2 + 1 - t'\right) \pm \frac{1}{2}\left(t' - 1\right)\sqrt{\left(1 - \frac{4M^2}{t'}\right)}$$

with

$4 \leq t' \leq \infty$ for the isoscalar amplitudes

$9 \leq t' \leq \infty$ for the isovector amplitudes.

The Irrationality Cut.

$$-\infty \leq s \leq 0$$

However when λ^2 was not equal to zero we found before that the crossed nucleon and pion cuts arose from the logarithms

$$\log \frac{\lambda^2 + 2K_0E_2 + 2KQ}{\lambda^2 + 2K_0E_2 - 2KQ} \quad \text{and} \quad \log \frac{\lambda^2 + 2K_0\Omega + 2KQ}{\lambda^2 + 2K_0\Omega - 2KQ}$$

respectively. In the limit of $\lambda^2 \longrightarrow 0$ these terms become

$$\log \frac{E_2 + Q}{E_2 - Q} \quad \text{and} \quad \log \frac{\Omega + Q}{\Omega - Q}$$

respectively, each containing only the cut

$$-\infty \leq s \leq 0$$

The pole at $s = M^2$ contained in the photoproduction amplitude arises from the vanishing of

$$2 W O_2 = (W-M)\sqrt{(W-M)^2 - 1}$$

contained in the crossed nucleon and pion pole term contributions to each multipole amplitude (such as that contribution to the magnetic dipole amplitude exhibited in equation V.9).

It follows therefore that the set of poles approximating the unphysical cuts in the magnetic dipole and electric quadrupole amplitudes for $\lambda^2 > 0$ does not approximate to the left-hand cuts contained in the corresponding photoproduction amplitudes. However there exists a simple relation between the differential cross-section for the electroproduction of a pion and the total cross-section for the photoproduction of a pion from a nucleon;

$$\sigma_{ph}(s) = \frac{4\pi^2 \epsilon_1}{\alpha} (1 - \cos\sigma) \lim_{\lambda^2 \to 0} \frac{d^2\sigma}{d\Omega\, d\epsilon_2} \qquad \text{IX.2}$$

where σ_{ph} is the total photoproduction cross-section at a centre of mass energy squared S.

This means that we shall be unable to calculate any angular distributions for the photoproduction of pions.

It is of interest to examine the pole approximations to the unphysical cuts in the region of small λ^2 to see which contributions are dominant in the photoproduction cross-section. From the previous Chapter we have the approximations for

The Pion Cuts

$$m_{1+}^{\overset{+}{\underset{0}{-}}}(s,0) = \begin{bmatrix}0\\1\\0\end{bmatrix}\left\{\frac{5\cdot 33}{s-45\cdot 16}+\frac{1\cdot 11}{s+232}+\frac{4\cdot 53}{s+1475}+\frac{25\cdot 69}{s+27110}\right\}$$

$$\ell_{1+}^{\overset{+}{\underset{0}{-}}}(s,0) = \begin{bmatrix}0\\1\\0\end{bmatrix}\left\{\frac{150}{s-56\cdot 28}+\frac{\cdot 34}{s+234}+\frac{-1\cdot 48}{s+1472}+\frac{8\cdot 56}{s+27150}\right\}$$

The Crossed Nucleon Cut

$$m_{1+}^{\overset{+}{\underset{0}{-}}}(s,0) = \begin{bmatrix}1\\-1\\1\end{bmatrix}\left\{\frac{\cdot 374}{s-45\cdot 16}+\frac{\cdot 79}{s+170}+\frac{1\cdot 75}{s+605}+\frac{5\cdot 53}{s+1885}\right.$$

$$\left.+\frac{9\cdot 96}{s+10760}+\frac{14\cdot 76}{s+48450}\right\}$$

$$\begin{bmatrix}g_v\\-g_v\\g_s\end{bmatrix}\left\{\frac{\cdot 751}{s-45\cdot 16}+\frac{\cdot 53}{s+177}+\frac{2\cdot 19}{s+611}+\frac{7\cdot 53}{s+1910}\right.$$

$$\left.+\frac{18\cdot 82}{s+10800}+\frac{21\cdot 57}{s+48510}\right\}$$

$$\ell_{1+}^{\overset{+}{\underset{0}{-}}}(s,0) = \begin{bmatrix}1\\-1\\1\end{bmatrix}\left\{\frac{\cdot 0065}{s-55\cdot 17}+\frac{\cdot 098}{s+196}+\frac{\cdot 186}{s+586}+\frac{\cdot 056}{s+943}\right.$$

$$+ \frac{-1\cdot 42}{s+12150} + \frac{-3\cdot 32}{s+49780} \Bigg\}$$

$$\begin{bmatrix} g_v \\ -g_v \\ g_s \end{bmatrix} \Bigg\{ \frac{\cdot 412}{s+188} + \frac{\cdot 814}{s+600} + \frac{2\cdot 14}{s+1822} + \frac{2\cdot 94}{s+10420}$$

$$+ \frac{3\cdot 48}{s+47720} \Bigg\}$$

The Crossed Physical Cut.

$$m^{\pm}_{1+}{}^{0}(s, 0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \frac{\cdot 068}{s - 22\cdot 13}$$

$$e^{\pm}_{1+}{}^{0}(s, 0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \frac{\cdot 0026}{s - 59\cdot 2}$$

The Pion-Pion Cuts.

We considered in this contribution to the photoproduction amplitude only the effects of possible bipion and tripion states. The pole approximations are

$$m^{+}_{1+}(s,0) = e\Lambda^{s}(0)\,\phi(t^{s}_{R}) \Bigg\{ \frac{\cdot 776 - \cdot 2i}{s - 31\cdot 22 - 23\cdot 87i} + \frac{-\cdot 097 - \cdot 157i}{s - 38\cdot 95 - 21\cdot 22i}$$

$$+ \text{complex conjugate} \Bigg\}$$

$$m^{0}_{1+}(s,0) = e\Lambda^{v}(0)\,\phi(t^{v}_{R}) \Bigg\{ \frac{\cdot 26 - \cdot 03i}{s - 21\cdot 22 - 34\cdot 21i} + \frac{\cdot 034 - \cdot 2i}{s - 26\cdot 02 - 33\cdot 52i}$$

$$+ \text{complex conjugate} \Bigg\}$$

$$e^{+}_{1+}(s,0) = e\Lambda^{s}(0)\,\phi(t^{s}_{R}) \Bigg\{ \frac{-\cdot 039 + \cdot 014i}{s - 40\cdot 05 - 4\cdot 63i} + \frac{-\cdot 066 + \cdot 016i}{s - 44\cdot 02 - 38\cdot 8i}$$

$$+ \text{ complex conjugate} \Big\}$$

$$e^{0}_{1+}(s,0) = e\Lambda^{v}(0)\,\phi(t^{v}_{R})\left\{\frac{-.019-.063i}{s-30-33.29} + \frac{-.02-.086i}{s-29.6-30.94i}\right.$$

$$\left. + \text{ complex conjugate}\right\}$$

Contributions from the irrationality cut are contained in each of the above contributions.

In the study of the electroproduction amplitude we found that the I = 3/2 amplitudes gave the dominant contributions to the differential cross-section. For photoproduction we therefore neglect the I = 1/2 amplitudes. The amplitudes for the electroproduction of a pion from a proton can be found from

$$\begin{aligned} \langle \gamma_v p \mid \pi^0 p \rangle &= \sqrt{\tfrac{2}{3}}\,\langle \gamma_v p \mid (\pi N)^{I=3/2} \rangle \\ &= \tfrac{2}{3}\left[n^{+}_{\ell\pm} - n^{-}_{\ell\pm} \right] \end{aligned} \qquad \text{IX.3a}$$

and

$$\begin{aligned} \langle \gamma_v p \mid \pi^{+} n \rangle &= -\tfrac{1}{\sqrt{3}}\,\langle \gamma_v p \mid (\pi N)^{I=3/2} \rangle \\ &= -\tfrac{\sqrt{2}}{3}\left[n^{+}_{\ell\pm} - n^{-}_{\ell\pm} \right] \end{aligned} \qquad \text{IX.3b}$$

Hence within this approximation the total cross-section for the photoproduction of a π^0 from a proton is twice that for a π^+ from a proton.

From the above we see that the contributions from the pion pole at $s = M^2$ and that part of the crossed nucleon pole at $s = M^2$ due to the magnetic moment structure of the nucleon are by far the most important contributions to the total photoproduction cross-section in the energy region around the $I = 3/2$, $J = 3/2$ resonance. Contributions from that part of the crossed physical cut considered and the pion-pion cut are very small, the magnitude of the latter again depending upon the unknown parameters $\Lambda^{s,v}(o)$ and $\phi\ (t_R^{v,s})$.

2. Results

The numerical calculation of the differential cross-section for the electroproduction of a pion from a proton was made as outlined in the previous Chapter. In this case we considered the momentum transfer from the scattered electron as constant and equal to $.01\,m_\pi^2$. With a fixed incident electron energy, the barycentric total energy squared was varied over the energy region

$$(M + 1)^2 \leq s \leq 80$$

and the outgoing electron energy and the electron scattering angle calculated from equations VIII.17.

The total cross-section for the photoproduction of a pion from a nucleon was then found from equation IX.2. The results are shown in Figures 12 and 13 for the photoproduction of a π^0 from protons. The π^+ cross-section will be a factor of two smaller.

The results shown in Figure 12 were obtained assuming a linear extrapolation of Woolcock's[79] phase shifts, for pion nucleon scattering in the $I = 3/2$, $J = 3/2$ state, to 90° at $s = 140\,m_\pi^2$. As before the behaviour of the cross-section above the resonance energy is not in agreement with experiment. This must be due to a poor extrapolation of the phase shift and contributions from the distant parts of the unphysical cuts in the electroproduction amplitude

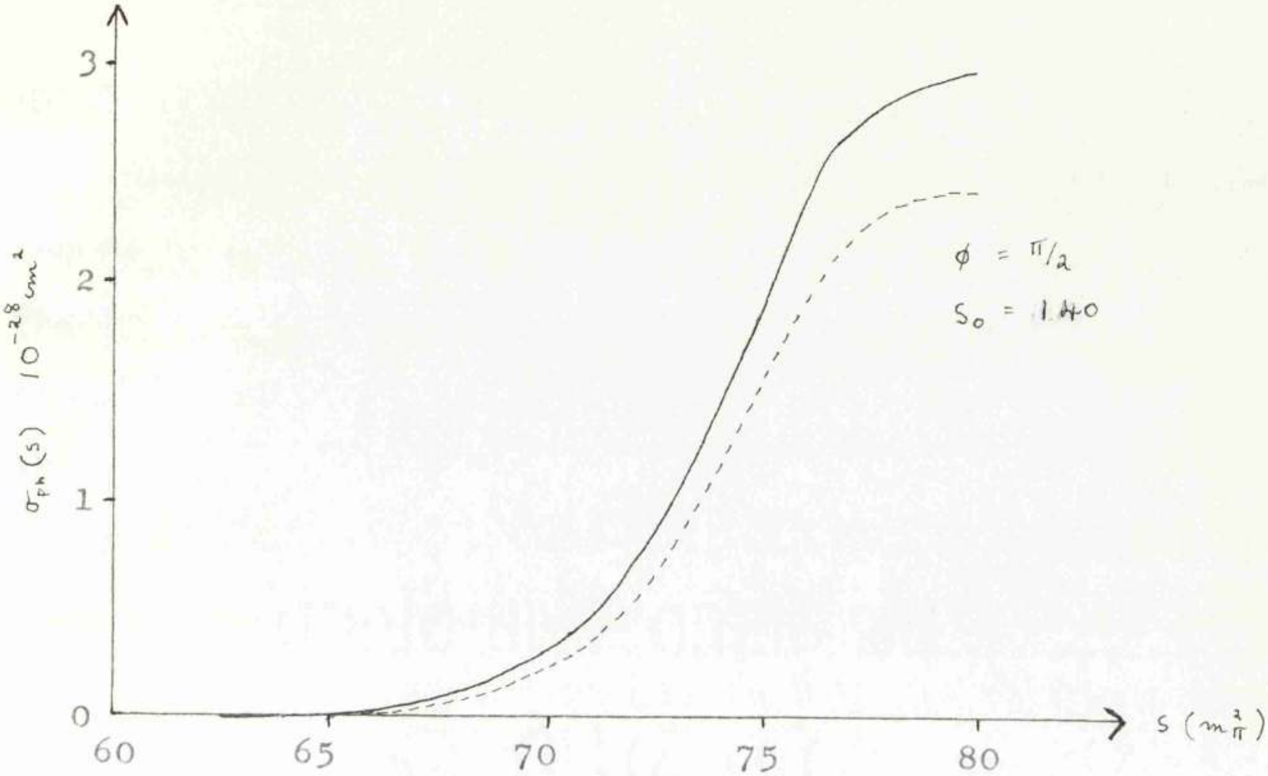


Figure 12

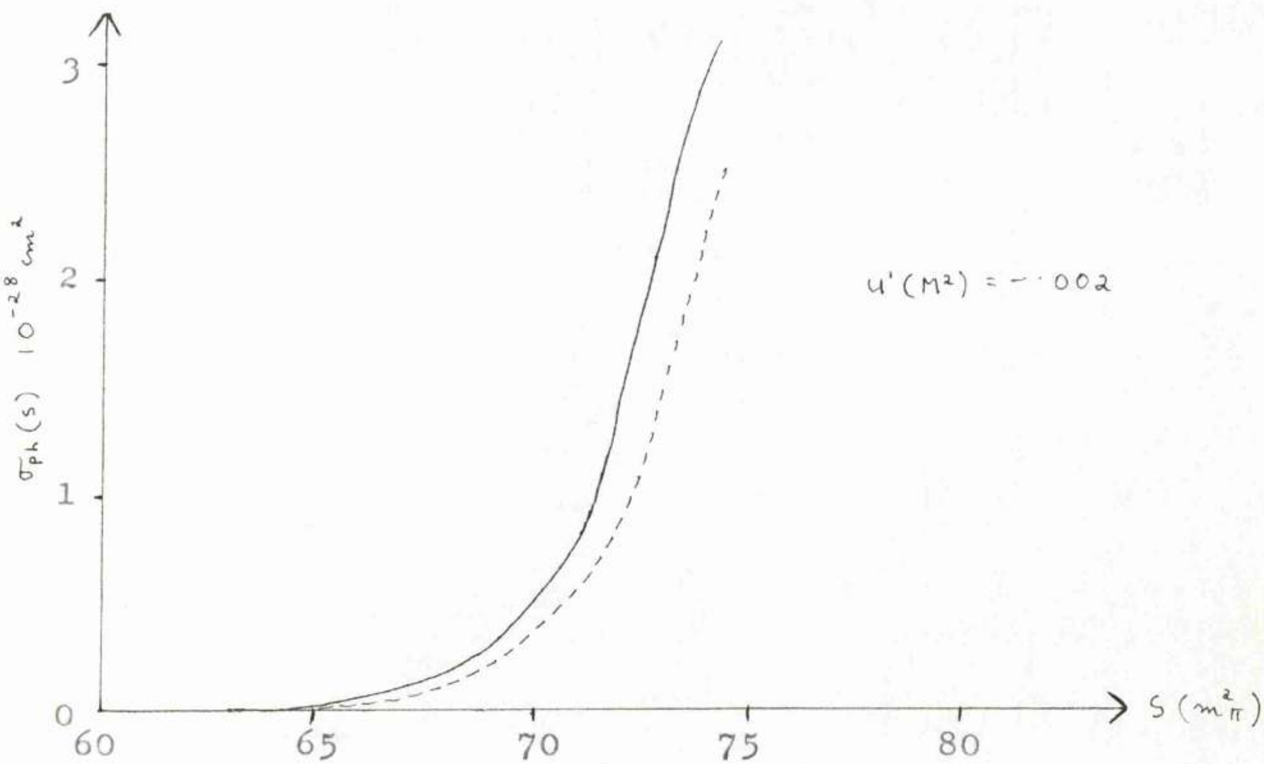


Figure 13

---- Magnetic dipole contributions to π^0 total cross-section from protons

—— Magnetic dipole and electric quadrupole contributions to π^0 cross-section which could not be approximated by poles.

A subtraction in the Omnes solution for the denominator function $D_{22}(s)$ leads to the cross-section shown in Figure 13. Here we can only expect agreement up to the resonance energy. The subtraction constant $u'(M^2)$ with the value of -.002 predicts the correct magnitude of the cross-section at resonance.

Since the photoproduction cross-section is independent of the pion and nucleon electromagnetic form factors an accurate value for $u'(M^2)$ could be obtained by detailed comparison with experiment and used to calculated the electroproduction cross-section at high momentum transfer from the scattered electron.

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Appendix 1.

We list here some useful kinematical relations relating the energies and momenta of the interacting particles to the variable S, the square of the barycentric energy in the pion nucleon system

$$E_1 = \frac{s + M^2 + \lambda^2}{2\sqrt{s}} \qquad K_0 = \frac{s - M^2 - \lambda^2}{2\sqrt{s}}$$

$$E_2 = \frac{s + M^2 - 1}{2\sqrt{s}} \qquad \Omega = \frac{s - M^2 + 1}{2\sqrt{s}}$$

$$Q^2 = \frac{[s-(M+1)^2][s-(M-1)^2]}{4s}$$

$$K^2 = \frac{[s-(M+i\lambda)^2][s-(M-i\lambda)^2]}{4s}$$

Appendix 2.

We summarize here the method followed in investigating the region of convergence of the Legendre polynomial expansion in the unphysical region. In particular we describe in detail the procedure followed to find the region of convergence on the pion-pion cut. We use the theorem[81]:

If $F(Z)$ is analytic on the complex Z plane inside an ellipse with foci at $Z = \pm 1$, then it can be expanded in a Legendre polynomial series

$$F(z) = \sum_{\ell} a_\ell P_\ell(z)$$

inside the ellipse.

In the crossed process

$$\gamma_v + \pi \longrightarrow N + \bar{N}$$

the cosine of the scattering angle in the centre of mass system is given by

$$y(t,s) = \cos\phi = \frac{(2M^2 - 2s - t + 1 - \lambda^2)\sqrt{t}}{\sqrt{(t^2 + 1 + \lambda^4 - 2t + 2t\lambda^2 + 2\lambda^2)(t - 4M^2)}}$$

Here t is the centre of mass total energy squared and S can be considered as the momentum transfer. We are interested in the case of complex momentum transfer in this process.

The pion-pion cut in the electroproduction amplitude arises from the term

$$\frac{1}{\pi}\int_4^\infty dt' \, \frac{{}^{T}A_c^{i}(s,t')}{t' - t}$$

Consider the singularities in the function

$$^{\pm}A_c^i(s,t) = \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds' \, \frac{^{\pm}a_{13}^i(s',t)}{s'-s} - \frac{1}{\pi}\int_{-\infty}^{M^2-\lambda^2-2M-t} ds' \, \frac{^{\pm}a_{23}^i(s',t)}{s'-s} \qquad \text{X.1}$$

As noted by Frautschi and Walecka(62) we need only investigate the first term in equation X.1 since the second term is obtained from it through the crossing transformation $s \longleftrightarrow u$ at fixed t. Under this transformation

$$y(t,s)^{x} = -y(t,s)$$

Thus the singularities from the first term for positive y in equation X.1 will give rise to symmetric singularities at - y in the second term.

The denominator of the first term vanishes for real s in the region

$$(M+1)^2 \leq s \leq \infty$$

We assume that the lowest value of s for which it can vanish (s_0) is given by equation IV.5, that is by the boundary equation of the region in which $^{\pm}a_{13}^i(s,t)$ is non-zero

$$s_0^2(t-9)(t-1) + s_0\left[-2(M^2+1)(t-9)(t-1) - 24(t-1) - 8\lambda^2(t-3)\right]$$

$$+ (M^2-1)^2(t-9)(t-1) - 8(t-1)(M^2-1) - 8\lambda^2(t+1+2\lambda^2)(M^2-1) = 0$$

It follows that, on the complex y plane, the Legendre polynomial expansion will converge

(i) for $4 \leq t \leq 4M^2$

inside the ellipse with minor axis of length

$$x_0 = \frac{(2s_0 - 2M^2 + t - 1 + \lambda^2)\sqrt{t}}{\sqrt{(t^2 + 1 + \lambda^4 - 2t + 2t\lambda^2 + 2\lambda^2)(4M^2 - t)}}$$

and foci at $y = \pm 1$

(ii) for $4M^2 \leq t \leq \infty$

inside the ellipse with major axis of length

$$y_0 = \frac{(2M^2 - 2s_0 - t + 1 - \lambda^2)\sqrt{t}}{\sqrt{(t^2 + 1 + \lambda^4 - 2t + 2t\lambda^2 + 2\lambda^2)(t - 4M^2)}}$$

and foci at $y = \pm 1$

For a given point on the pion-pion cut (which might well be complex) corresponding to a momentum transfer, s^1 say, in the crossed process

$$Y_v + \pi \longrightarrow N + \bar{N}$$

and a value of the energy t in the physical region for this process $4 \leq t \leq \infty$ we calculated the angle of scattering $y(t, s^1)$. The Legendre polynomial expansion will then converge at this point, if, for all energies t, the point $y\ (t, s^1)$ lies inside the ellipse

(i) for $4 \leq t \leq 4M^2$

$$\frac{\xi^2}{x_0^2 + 1} + \frac{\eta^2}{x_0^2} = 1$$

(ii) for $4M^2 \leq t \leq \infty$

$$\frac{\xi^2}{y_0^2} + \frac{\eta^2}{y_0^2 - 1} = 1$$

where $\xi = \text{Re } y(t,s^1)$ and $\eta = \text{Im } y(t,s^1)$

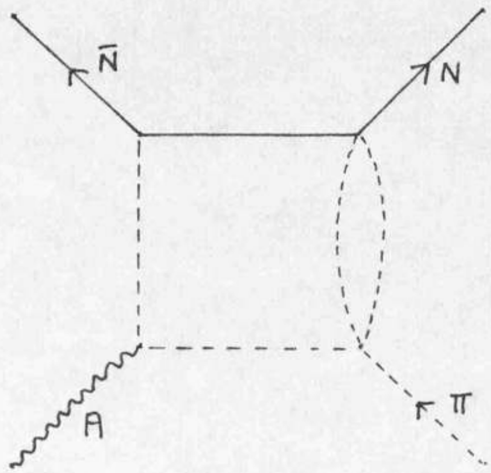
The results of the calculation indicate that the Legendre polynomial expansion for the isovector amplitudes converges on the pion-pion cut for values of t^1 in equation IV.18 in the range

$$9 \leq t^1 \leq 37 \cdot 8 + 5 \cdot 66 \lambda^2$$

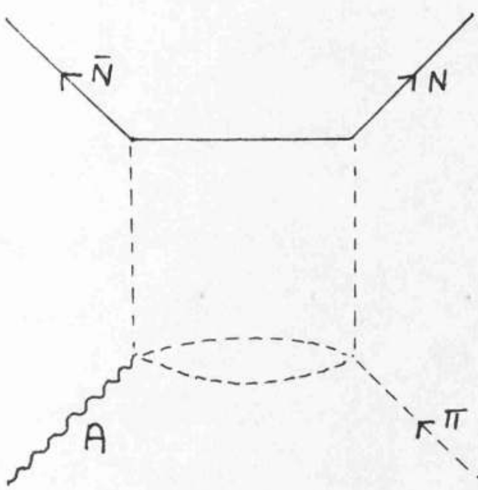
A similar calculation was performed for the isoscalar expansion. It was found that it converged on the pion-pion cut in the range

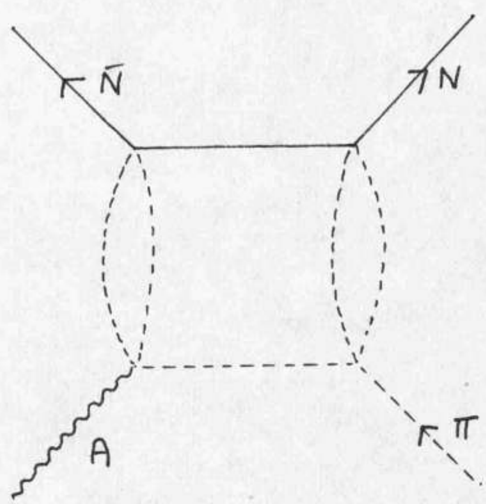
$$4 \leq t^1 \leq 61.25 + 4.35\lambda^2.$$

It is of interest to note that these results are also applicable to processes in which the lowest mass intermediate states contributing to the continuum in a crossed process are of the types



and





that is, if the quantum numbers of the particle A allow the diagrams. For example we could consider the particle A as a pion. The last two diagrams are then allowed and correspond to the diagrams contributing to the circular pion-pion cut in the pion-nucleon scattering amplitude. It follows that a Legendre polynomial expansion of the pion-nucleon scattering will converge on the circular cut for

$$4 \leq t' \leq 56.9$$

$$s^2 + s(t' - 2 - 2M^2) + (M^2 - 1)^2 = 0$$

i.e. on the circle $s = (M^2 - 1)\, e^{i\theta}$ for

$$|\theta| \leq 66^\circ$$

This is in excellent agreement with an earlier calculation of Frazer and Fulco[59].

A similar calculation could be performed to investigate the region of convergence of the Legendre polynomial expansion on the crossed physical cut

$$-\infty \leq s \leq d(\lambda^2).$$

In the actual calculation however we investigated the size of this region within the 'strip' approximation of Frautschi and Walecka[62] which indicates that the expansion does converge on the reduced cut actually considered.

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