Summary of a Ph.D. Thesis entitled

"The Numerical Solution of Certain Problems in Elastodynamics"

by J. M. Blair

The method of integral transforms can provide the solution of a differential equation satisfying prescribed boundary conditions if certain requirements involving the boundary conditions and the transforms are met. The three requirements are stated explicitly in the thesis for the differential equation $L \underset{\sim}{\varphi} = \underset{\sim}{f}$ on a finite domain R, where L is a matrix and $\underset{\sim}{\varphi}$ and $\underset{\sim}{f}$ are columns. In general not all the requirements can be satisfied for the equations of elasticity. If one particular requirement is relaxed, then the transform procedure may be applied in such a way as to reproduce in R a formal series solution of the above differential equation without reference to the boundary conditions, and the solution is a general solution in that sense. When the solution is applied to a particular set of boundary conditions there results an infinite system of simultaneous linear equations in an infinite number of unknowns, whose solution yields a solution of the differential equation. The infinite system is given formally in chapter I for the equations of elasticity.

Theoretical results pertaining to the solution of infinite systems of equations and approximate methods of solution are known, and chapter II is devoted to a statement and a discussion of those results which are relevant to the problems of the thesis. A deficiency in the existing theory is noted.

The application of the above approach to the solution of some specific /

specific vibration problems in elastodynamics is given in chapters III and IV. A study of the vibrating elastic parallelepiped with clamped edges provides an indication of the rate of convergence of the approximate numerical solution for different dimensions, and allows some conclusions to be drawn about the value of the method as a practical numerical procedure. The numerical results are compared with those obtained by another method due to V.V. Bolotin. The approximate solution of the infinite system of equations is justified in terms of the theory in chapter II.

Two problems of the axially symmetric vibrations of elastic rods are investigated in chapter IV. The first rod has all its bounding surfaces stress-free, while the second rod has one of its plane ends clamped and the remaining surfaces stress-free. Numerical results are presented for both problems. Those for the first case are compared with existing theoretical and experimental values, and they are shown to be the most accurate yet available. No other results for the second problem have been found in the literature. The infinite system of equations is studied in both cases, the conclusions being less satisfactory than for the parallelepiped, as certain questions remain unanswered.

In chapter V we consider an initial-value problem in which a stress pulse is suddenly applied to one end of an elastic rod. The solution is expressed as an infinite sum over the solutions for the free-free rod, using the method of eigenfunction expansions. The motion of the free end of the rod is computed using the finite set of eigenfunctions in /

in chapter IV, and the resulting solution is sufficiently accurate to show the successive reflections of the initial pulse as it traverses the rod. Some aspects of the solution are discussed by comparing it with the solution of the analogous problem for an infinite slab.

THE NUMERICAL SOLUTION OF CERTAIN PROBLEMS IN ELASTODYNAMICS

being a thesis presented by

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Preface

This thesis investigates the formal solution by integral transforms of the equations of elastodynamics, and uses it to compute new numerical solutions of some steady state vibration problems. The method is developed in chapters I and II, and is applied to vibrating elastic parallelepipeds and rods in chapters III and IV respectively. Certain conclusions about the effectiveness of the method are drawn from the results. An application of the solution is provided in chapter V, where a limited number of modes of vibration of a rod are used to construct the solution of an unsolved, initial-value problem.

I wish to express my thanks to my supervisor Professor D. C. Gilles for suggesting these problems to me and for his help throughout the course of the work. The possibility of using integral transforms to reduce the elastic rod problem to an infinite system of simultaneous linear equations was demonstrated to me by Professor B. Noble, who, in collaboration with T. Boag, had used a similar approach in investigating lap joints in beams. Professor Noble's help is gratefully acknowledged. The extension to the general boundary-value problem and to the elastic boundary-value problem in sections 3 - 7 resulted from a suggestion by Professor D.S. Butler. The computations were done on the Deuce computer at Glasgow University.

J. M. Blair

CONTENTS

Chapter I: A Solution of the Equations of Elasticity

1. Introduction

The mathematical equations governing the infinitesimal deformations of an elastic continuum have been studied by many workers since they were formulated by Cauchy in 1822, and there is a vast literature devoted to the subject. Because of the difficulty of constructing a general solution of the equations capable of satisfying an arbitrary set of boundary conditions, a variety of special methods and special types of solution have been developed. TEODORESCU [1964] gives a survey of the different methods which have been applied to plane problems, and indicates those methods which are capable of extension to more general three-dimensional problems.

In this thesis we concentrate our attention on certain dynamic and steady-state vibration problems, some of which have previously been solved by other methods and some which have not. Dynamic problems have been studied much less in general than static problems, although they are of considerable practical importance in the design of resonators and mechanical wave guides, and in other applications. The complete set of solutions of a steady-state vibration problem, as well as being of interest in its own right, might be expected to provide a solution of certain associated initial-value problems by the method of eigenfunction expansions, and the last chapter of the thesis uses this approach to predict the motion of a circular rod when a pressure pulse is applied to one end.

The method of separation of variables was one of the earliest methods to be applied to the equations of elasticity. A simple solution obtained in this way does not satisfy all the boundary conditions in general and an infinite series of simple solutions, each satisfying some of the boundary conditions, must be taken. If a sufficient number of arbitrary constants is incorporated in the infinite series, then we find that all the boundary conditions can be satisfied, although to do so we must solve an infinite system of simultaneous linear equations in an infinite number of unknowns.

The process of constructing the solution as an infinite series can be formalised by the use of integral transforms, and the first chapter of the thesis deals with the development of this technique and with the choice of the appropriate transforms for a finite, three-dimensional region. It is customary when using integral transforms to insert the transformed boundary conditions into the transformed equations of motion before solving for the transformed variables and inverting, but we show that by omitting to insert the boundary conditions and introducing instead certain arbitrary constants, a formal solution is obtained without reference to the boundary conditions. This solution is a general solution of the equations of motion in the sense that it may be used to solve a variety of sets of boundary conditions, although in each case it requires the solution of an infinite system of linear equations.

The second chapter is devoted to the theory of solution of infinite systems of equations. The known results for a particular class of infinite systems known as regular systems are stated without proofs, and are extended to deal with slightly more general systems called quasi-regular systems. An example of a system which does not belong to either category is given in section 16, and some properties of the solution are discussed. We shall find later that the system of equations occurring in the mixed boundary-value problem in section 40 has features in common with the example, and that the theoretical questions about the existence and uniqueness of the solution have not yet been settled.

In chapter III we apply the theory of the first chapter to the problem of the vibrating rectangular parallelepiped in plane strain. The transforms turn out to be finite Fourier transforms and the formal solution is a double Fourier series, which we reduce to a sum of eight single series by summing certain series analytically. The resulting solution is applied to a particular problem mentioned in SOMMERFELD [1950], namely the problem of a parallelepiped with clamped edges. An investigation of the resulting infinite system of equations shows that it is quasi fully regular, and the subsequent numerical solution is theoretically justified. Numerical values of some of the lower modes of vibration for three different rectangles are given, and a comparison of the observed rates of convergence in the three cases leads us to some conclusions about the effectiveness of the method.

This particular problem has been investigated by BOLOTIN [1961 b], using an asymptotic method developed in BOLOTIN [1961 a] for vibrating plate and shell problems. The assumption is that the solution may be represented by two terms, a basic solution applying over the whole region, and an edge effect whose influence is confined to the neighbourhood of the boundary. Bolotin takes a simple separation of variables solution as the basic solution, and uses it to generate the edge effect, so that both terms are solutions of the equations of motion. A different edge effect is taken for each distinct part of the boundary and is used to satisfy the boundary conditions there. The solution is approximate because each edge effect term, which is assumed to vanish on all the boundaries except one, in fact decays exponentially and is small but finite on all boundaries except one. The computed results in chapter III differ significantly from those given by the asymptotic method, and indicate that the assumptions of the latter are not valid for the equations of elasticity. A possible reason for this is given in section 27.

The axially symmetric vibrations of circular rods of finite length and radius are studied in chapter IV, The transforms for this region are shown to be finite Fourier transforms in the axial direction and finite Hankel transforms in the radial direction. By applying them in the way described in chapter I, the formal solution is constructed as a double series and reduced to a sum of six single series as before. Two particular problems are considered, in both of which the curved surface of the rod is stress-free. In the first the two ends are also stress-free, while in the second one end is clamped and the other free. The corresponding infinite system of equations is set up in both cases. The first system proves to be quasi-regular, and hence some properties of its solution are known. However the uniqueness of the solution, which is the other property necessary to justify a numerical solution, has not yet been proved. For the second problem the infinite system is not even quasi-regular, and we have not been able to give any theoretical justification of its solution. Numerical results to both problems are presented for one value of the length-to-radi ratio and one value of Poisson's ratio.

The lowest modes of vibration of a free-free elastic rod have been determined in MCNIVEN and PERRY [1962] by a different method based on the Pochhammer-Chree theory. POCHHAMMER [1876] and CHREE [1889] were the first to formulate the equations of vibration of an elastic rod, and they suggested a method of solution whereby a simple separation of variables solution is used to satisfy all the boundary conditions on the curved surface. The complete solutionfor a finite rod is then given by superposition. The curved surface boundary conditions lead to a transcendental equation, usually called the Pochhammer frequency equation, relating the wave number in the axial direction and the angular frequency of vibration, and it has been shown in ONOE, McNIVEN & MINDLIN [1962] that for each value of the frequency this equation has some real roots and an infinite number of complex roots. The roots vary continuously as the frequency varies, and a three-dimensional diagram can be constructed giving the position of all the roots for each frequency. The curves in this diagram are called the "branches" of the frequency equation. The real roots correspond to sinusoidal solutions and the complex roots to exponentially increasing and decreasing terms. Thus for a finite rod the solution for each frequency consists of an infinite sum over all the different branches. If this sum contains enough arbitrary constants, then the boundary conditions on the ends may be satisfied in principle. In practice, however, these conditions are not easily dealt with, since they involve series of Bessel functions of complex argument which lack the orthogonality property desirable from a theoretical and a computational point of view. To overcome these drawbacks Mindlin and his coworkers have developed an approximate solution consisting of the first few terms of an orthogonal function expansion. This solution is adjusted so as to reproduce the first three branches of the Pochhammer frequency equation at low frequencies, and so it may be regarded as equivalent to the first three terms of the Pochhammer solution. The details of the approximate method are given in McNIVEN & PERRY [1962]together with the numerical results, which are compared in section 39 with the results in section 38 and with experimental values.

The results in section 46 for the clamped-free rod appear to be new, as no references to published work on this problem have been found in the literature.

In chapter V we consider an initial-value problem in which one end of a short circular rod is acted upon by a pressure pulse applied symmetrically about the axis. Problems of this type are important in the practical design of mechanical wave-guides. We use the method of eigen-function expansions, where the eigenfunctions are the solutions for the free-free rod in section 38. This is a standard method, and is described in **COURANT and HILBERT** [1953]. It is often criticised as being unsuitable in practice because of the slow convergence of the series, but the results in section 54 indicate that the expected features of the solution are being reproduced reasonably well by the comparatively small number of eigenfunctions used. The computed solution predicts the arrival of the pulse travelling with the dilatational wave speed at the free end of the rod, and shows the subsequent reflected pulses.

MIKLOWITZ [1960] contains a historical survey of the literature dealing with transient pulses in rods. Evidently only a few of the theor-etical investigations use the exact equations of elasticity, and those which do base their approach on the PochhammerChree theory. Because of the difficulties of satisfying the end conditions with this theory and for other reasons which we shall mention shortly, the resulting solution applies only to semi-infinite rods, and deals only with the propagation of a pulse away from the source. It cannot deal with the multiple reflections which occur in a rod of finite length.

CURTIS [1960] typifies the general approach to these problems. The displacement vector is expanded as a Fourier series in θ, a Fourier integral in the frequency p, and an infinite series over the branches of the Pochhammer frequency spectrum. By relating the frequency and the wave numbers in this way we ensure that the solution automatically satisfies the curved surface boundary conditions. This form of the solution requires that the frequency spectrum be known for each different term of the Fourier series. So far the spectrum has been worked out in detail only for the first two terms. Curtis is able to satisfy end conditions of "mixed" type, in which either the normal component of displacement and the tangential component of stress or the normal component of stress and the tangential component of displacement are specified, but not of "pure" type, in which both components of either displacement or stress are given. The evaluation of the solution involves, for each term in the Fourier series, a curvilinear integration along each branch of the frequency spectrum. In practice the integrals are obtained by asymptotic methods which are accurate only at large distances from the end of the rod. Curtis uses only the first two or three branches in the frequency spectrum. Since the higher branches are most influential near the end of the rod, their omission should have little effect far from the end, although the solution will be inaccurate near the source of the disturbance. The dispersion of a pulse in a long rod is predicted by Curtis, and his results give good quantitative agreement with experiment for large distances of travel.

Of the more recent publications KAUL and McCOY [1964] uses the Mindlin approximation to the Pochhammer solution to include the case, not covered by Curtis' solution, of "pure" end conditions. ROSENFELD and MIKLOWITZ [1965] shows that the exact Pochhammer solution can be adapted to deal with "pure" end conditions, and also generalises the theory to rods of arbitrary cross-section.

Following a completely different line of approach LANGNER [1965] applies the method of finite differences to the motion of an elastic rod of finite length when one end undergoes a step wave displacement in time. A rectangular grid is used to cover the region, and the problem is treated as an initial-value problem in the usual way. The known solution at some instant t is used to compute the solution at some later time t + Δt by solving a system of simultaneous linear equations. Standard iterative methods are used to solve the equations on a computer. The method is applied to a variety of non-circular rods, and some numerical results are given. However since the magnitude of the various computational errors is unknown, and since there are no comparisons with other results or with experiment, the accuracy of the solution is uncertain.

The eigenfunction expansion of chapter V may be applied with very slight modifications to the analagous problem in viscoelasticity under certain conditions given in HUNTER [1965] , and it is hoped to extend the computations of chapter V to include this problem at some later date.

The appendix contains some standard series which we use in chapters III and IV. These series are taken from SNEDDON [1951].

1. The Equations of Elasticity

The dynamic equations of elasticity are derived in the textbooks e.g. SOKOLNIKOFF [1956], and are expressible in terms of the components of displacement and their derivatives. If we use vector notation and denote the displacement vector by $\underline{s}$, then it is shown in WEATHERBURN [1924] that the equations have the form

$$(\lambda + 2\mu)\ \text{grad div}\ \underline{s} - \mu \text{curl curl}\ \underline{s} = \rho \frac{\partial^2 \underline{s}}{\partial t^2} + \underline{f} \tag{1.1}$$

where λ and μ are the Lamé elastic constants, ρ is the density of the material, $-\underline{f}$ is the external force per unit mass, and t is the time variable. Let $\underline{s}$ have components (u, v, w) in the relevant coordinate system.

In rectangular cartesian coordinates (x, y, z) we have

$$\text{div}\ \underline{s} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad ,$$

and grad u and curl $\underline{s}$ are the vectors with components $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$ and $\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ respectively.

To express these vectors in a curvilinear orthogonal system of coordinates (q_1, q_2, q_3), write $x = x(q_1, q_2, q_3)$, $y = y(q_1, q_2, q_3)$ and $z = z(q_1, q_2, q_3)$, and define h_i as

$$h_i = \left[\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2\right]^{\frac{1}{2}}$$

for i = 1, 2 and 3. Then it is shown in WEATHERBURN [1924] that

$$\text{div}\ \underline{s} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 u) + \frac{\partial}{\partial q_2}(h_1 h_3 v) + \frac{\partial}{\partial q_3}(h_1 h_2 w)\right] ,$$

and that grad u and curl $\underline{s}$ have components $\left(\frac{1}{h_1}\frac{\partial u}{\partial q_1}, \frac{1}{h_2}\frac{\partial u}{\partial q_2}, \frac{1}{h_3}\frac{\partial u}{\partial q_3}\right)$ and

$\left(\frac{1}{h_2 h_3} \left\{ \frac{\partial}{\partial q_2} h_3 w - \frac{\partial}{\partial q_3} h_2 v \right\} , \frac{1}{h_3 h_1} \left\{ \frac{\partial}{\partial q_3} h_1 u - \frac{\partial}{\partial q_1} h_3 w \right\} , \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_1} h_2 v - \frac{\partial}{\partial q_2} h_1 u \right\} \right)$ respectively. These results enable us to express equation (1.1) in the coordinate system (q_1, q_2, q_3).

In particular for cylindrical polar coordinates (r, θ, z) we have $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$, so that

$$h_1 = \left[\cos^2 \theta + \sin^2 \theta + 0\right]^{\frac{1}{2}} = 1$$

$$h_2 = \left[r^2 \sin^2 \theta + r^2 \cos^2 \theta + 0\right]^{\frac{1}{2}} = r$$

$$h_3 = \left[0 + 0 + 1\right]^{\frac{1}{2}} = 1$$

Thus $\quad \operatorname{div} \underline{s} = \frac{1}{r} \frac{\partial}{\partial r} ru + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}$,

grad u has components $\left(\frac{\partial u}{\partial r}, \frac{1}{r} \frac{\partial u}{\partial \theta}, \frac{\partial u}{\partial z} \right)$, and curl $\underline{s}$ has components $\left(\frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \frac{1}{r} \frac{\partial}{\partial r} rv - \frac{1}{r} \frac{\partial u}{\partial \theta} \right)$.

It will be convenient for our subsequent treatment of equation (1.1) to rewrite it in matrix form. To do this we introduce matrices of differential operators $\underset{\sim}{d}$, $\underset{\sim}{g}$, and C to represent the divergence, gradient and curl respectively, and we denote by $\underset{\sim}{s}$ the column matrix $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$.

In the coordinate system (q_1, q_2, q_3) $\underset{\sim}{d}$, $\underset{\sim}{g}$, and C are defined by

$$\underset{\sim}{d} = \frac{1}{h_1 h_2 h_3} \begin{bmatrix} \frac{\partial}{\partial q_1} h_2 h_3 \\ \frac{\partial}{\partial q_2} h_1 h_3 \\ \frac{\partial}{\partial q_3} h_1 h_2 \end{bmatrix} , \quad \underset{\sim}{g} = \begin{bmatrix} \frac{1}{h_1} \frac{\partial}{\partial q_1} \\ \frac{1}{h_2} \frac{\partial}{\partial q_2} \\ \frac{1}{h_3} \frac{\partial}{\partial q_3} \end{bmatrix}$$

and
$$C = \begin{bmatrix} 0 & -\frac{1}{h_2h_3}\frac{\partial}{\partial q_3}h_2 & \frac{1}{h_2h_3}\frac{\partial}{\partial q_2}h_3 \\ \frac{1}{h_1h_3}\frac{\partial}{\partial q_3}h_1 & 0 & -\frac{1}{h_1h_3}\frac{\partial}{\partial q_1}h_3 \\ -\frac{1}{h_1h_2}\frac{\partial}{\partial q_2}h_1 & \frac{1}{h_1h_2}\frac{\partial}{\partial q_1}h_2 & 0 \end{bmatrix}$$

Equation (1.1) then has the form

$$(\lambda + 2\mu)\ \underset{\sim}{g}\ \underset{\sim}{d}^T\ \underset{\sim}{s} - \mu\, C\, C\, \underset{\sim}{s} = \rho\frac{\partial^2 s}{\partial t^2} + \underset{\sim}{f}\ , \qquad (1.2)$$

where $\underset{\sim}{d}^T$ is the transpose of $\underset{\sim}{d}$, and $\underset{\sim}{f}$ is the body force.

If the normal components of stress in this coordinate system are σ_1, σ_2 and σ_3 and the shear components are τ_{12}, τ_{13} and τ_{23}, we introduce the stress matrix T as

$$T = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix}$$

Then it can also be shown that

$$T = \lambda\Delta I + \mu\,(\underset{\sim}{g}\ \underset{\sim}{s}^T + \underset{\sim}{s}\ \underset{\sim}{g}^{T^*})\ ,$$

where Δ, the dilatation is given by $\Delta = \underset{\sim}{d}^T\underset{\sim}{s}$, and I is the unit matrix. The asterisk on $\underset{\sim}{g}^T$ indicates that it post-multiplies $\underset{\sim}{s}$.

3. The Method of Integral Transforms.

The method of integral transforms for solving a set of simultaneous linear differential equations is a method whereby the differential equations are reduced to simultaneous linear algebraic equations. The domain of the differential equations may be finite or infinite, and the details of the method differ slightly in the two cases. Since the problems considered in this thesis are all concerned with finite domains, we describe the method as it applies to this case.

We consider the finite region R bounded by the closed surface S, where the solution of the differential equation must satisfy prescribed boundary conditions at all points of S. We shall assume that the boundary conditions are homogeneous and that the differential equations are non-homogeneous. The problem can always be put in this form by an appropriate choice of the dependent variables.

Thus we wish to determine the solution $\underset{\sim}{\varphi}$ of the differential equation

$$L \underset{\sim}{\varphi} = \underset{\sim}{f} \qquad (1.3)$$

at all points of R, where $\underset{\sim}{\varphi}$ must satisfy the condition

$$M \underset{\sim}{\varphi} = \underset{\sim}{0} \qquad (1.4)$$

at all points of S. Here $\underset{\sim}{\varphi}$ and $\underset{\sim}{f}$ are column vectors of functions, $\underset{\sim}{0}$ is the zero column, and L and M are matrices of differential operators.

The first step in the method of integral transforms is to choose a suitable diagonal matrix of functions Ψ_n, to pre-multiply equation (1.3) by Ψ_n, and to integrate the resulting equation throughout R. This gives

$$\int_R \Psi_n L \underset{\sim}{\varphi}\, d\tau = \int_R \Psi_n \underset{\sim}{f}\, d\tau \quad . \qquad (1.5)$$

The choice of the matrix Φ_n will be discussed below. If L is a linear operator, integration by parts may be used to express the left side of (1.5) in the form

$$\int_R \Phi_n L \underset{\sim}{\varphi} d\tau = \int_S \underset{\sim}{g}_n dS + \int_R \Phi_n L^* \underset{\sim}{\varphi} \; d\tau ,$$

where L^*, the differential operator adjoint to L, operates on Φ_n. $\underset{\sim}{g}_n$ is a function of $\underset{\sim}{\varphi}$, Φ_n and their derivatives, of the form

$$\underset{\sim}{g}_n = H_n M \underset{\sim}{\varphi} + \underset{\sim}{k}_n$$

for some matrices H_n and $\underset{\sim}{k}_n$. Thus the transformed differential equation becomes

$$\int_S \underset{\sim}{g}_n \, dS + \int_R \Phi_n L^* \underset{\sim}{\varphi} \; d\tau = \int_R \Phi_n \underset{\sim}{f} \, d\tau$$

i.e. $$\int_R \Phi_n L^* \underset{\sim}{\varphi} \; d\tau = \int_R \Phi_n \underset{\sim}{f} \, d\tau - \int_S \underset{\sim}{g}_n \, dS \qquad (1.6.)$$

If the matrix Φ_n is correctly chosen it will satisfy the following three conditions

(i) $\{\Phi_n\}$ is a complete orthonormal set. It follows from this that $\underset{\sim}{\varphi}$ may be expressed as

$$\underset{\sim}{\varphi} = \sum_n \Phi_n \underset{\sim}{a}_n ,$$

where $\underset{\sim}{a}_n$ is a column vector of constants given by

$$\underset{\sim}{a}_n = \int_R \Phi_n \underset{\sim}{\varphi} \; d\tau$$

(ii) $$\Phi_n L^* = A_n \Phi_n ,$$

A_n being a square matrix of constant terms.

(iii) The column vector $\underset{\sim}{k}_n$ vanishes on S.

From condition (ii) we have

$$\int_R \Xi_n L^* \underset{\sim}{\varphi}\, d\tau = A_n \int_R \Xi_n \underset{\sim}{\varphi}\, d\tau = A_n \underset{\sim}{a}_n ,$$

the last part following from (i). Moreover from (1.4) and condition (iii) the surface integral is given by

$$\int_S \underset{\sim}{g}_n\, dS = \int_S (\Xi_n M \underset{\sim}{\varphi} + \underset{\sim}{k}_n)\, dS = \underset{\sim}{0}$$

Equation (1.6) thus becomes

$$A_n \underset{\sim}{a}_n = \int_R \Xi_n \underset{\sim}{f}\, d\tau .$$

Hence

$$\underset{\sim}{a}_n = A_n^{-1} \int_R \Xi_n \underset{\sim}{f}\, d\tau ,$$

if we assume that A_n is non-singular.

Since Ξ_n and $\underset{\sim}{f}$ are known functions and A_n is a known matrix, $\underset{\sim}{a}_n$ may be determined for each value of n. Hence the solution of equation (1.3) is given by

$$\underset{\sim}{\varphi} = \sum_n \Xi_n \underset{\sim}{a}_n = \sum_n \Xi_n A_n^{-1} \int_R \Xi_n \underset{\sim}{f}\, d\tau .$$

4. The Modified Method

In principle the method of the last section may be used to provide a solution of any boundary value problem of the type considered. In practice, however, the problem of finding a function Φ_n satisfying conditions (i), (ii) and (iii) may be comparable in difficulty with that of solving the original equations. In those circumstances it is possible to modify the method by choosing the set $\{\Phi_n\}$ to satisfy conditions (i) and (ii), but not condition (iii). The vector $\underset{\sim}{g}_n$ no longer vanishes on S, and equation (1.6) becomes

$$A_n \underset{\sim}{a}_n = \int_R \Phi_n \underset{\sim}{f}\, d\tau - \int_S \underset{\sim}{g}_n\, dS$$

$$= \int_R \Phi_n \underset{\sim}{f}\, d\tau - \underset{\sim}{b}_n ,$$

where $\underset{\sim}{b}_n = \int_S \underset{\sim}{g}_n\, dS$ is obtained by straightforward integration of $\underset{\sim}{g}_n$ without using the boundary conditions $M\underset{\sim}{\varphi} = \underset{\sim}{0}$. Hence $\underset{\sim}{b}_n$ is a column vector of unknown constants. Then

$$\underset{\sim}{a}_n = A_n^{-1} \int_R \Phi_n \underset{\sim}{f}\, d\tau - A_n^{-1} \underset{\sim}{b}_n ,$$

and the solution $\underset{\sim}{\varphi}$ may be expressed as

$$\underset{\sim}{\varphi} = \sum_n \Phi_n a_n = \sum_n \Phi_n \left(A_n^{-1} \int_R \Phi_n \underset{\sim}{f}\, d\tau - A_n^{-1} \underset{\sim}{b}_n \right) \tag{1.7}$$

Equation (1.7) is a formal expression for the solution of equation (1.3) for the region R, and is a general solution in the sense that it has been constructed without reference to the boundary conditions. It may thus be used to satisfy different sets of boundary conditions of the form (1.4).

To satisfy the particular conditions

$$M \underset{\sim}{\varphi} = \underset{\sim}{0}$$

we choose a complete orthonormal set $\{\Phi_n\}$, where Φ_n is a diagonal matrix, and use the fact that (1.4) will be satisfied if and only if

$$\int_S \Phi_m M \underset{\sim}{\varphi}\, dS = \underset{\sim}{0}$$

for all values of m. Thus (1.4) will be satisfied if

$$\int_S \Phi_m M \sum_n \Phi_n \left(A_n^{-1} \int_R \Phi_n \underset{\sim}{f}\, d\tau - A_n^{-1} \underset{\sim}{b}_n \right) dS = \underset{\sim}{0}$$

for all m. If we assume that the series may be differentiated term by term this condition becomes

$$\sum_n \int_S \Phi_m M \Phi_n\, dS \left(A_n^{-1} \int_R \Phi_n \underset{\sim}{f}\, d\tau - A_n^{-1} \underset{\sim}{b}_n \right) = \underset{\sim}{0}$$

for all m. That is

$$\sum_n D_{mn} \left(A_n^{-1} \int_R \Phi_n \underset{\sim}{f}\, d\tau - A_n^{-1} \underset{\sim}{b}_n \right) = \underset{\sim}{0}$$

where $D_{mn} = \int_S \Phi_m M \Phi_n\, dS$. If we assume further that the series are absolutely convergent, the condition becomes

$$\sum_n D_{mn} A_n^{-1} \underset{\sim}{b}_n = \sum_n D_{mn} A_n^{-1} \int_R \Phi_n \underset{\sim}{f}\, d\tau$$

for all m. This is an infinite set of simultaneous linear equations for the unknowns $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3, \ldots$, of the form

$$D_{11} A_1^{-1} \underset{\sim}{b}_1 + D_{12} A_2^{-1} \underset{\sim}{b}_2 + D_{13} A_3^{-1} \underset{\sim}{b}_3 + \cdots = \sum_n D_{1n} A_n^{-1} \int_R \Phi_n \underset{\sim}{f}\, d\tau$$

$$D_{21} A_1^{-1} \underset{\sim}{b}_1 + D_{22} A_2^{-1} \underset{\sim}{b}_2 + D_{23} A_3^{-1} \underset{\sim}{b}_3 + \ldots = \sum_n D_{2n} A_n^{-1} \int_R \underset{\sim}{I}_n \underset{\sim}{f} \, d\tau$$

$$D_{31} A_1^{-1} \underset{\sim}{b}_1 + D_{32} A_2^{-1} \underset{\sim}{b}_2 + D_{33} A_3^{-1} \underset{\sim}{b}_3 + \ldots = \sum_n D_{3n} A_n^{-1} \int_R \underset{\sim}{I}_n \underset{\sim}{f} \, d\tau$$

$$\vdots$$

If a solution $\{\underset{\sim}{b}_n\}$ of this system can be found such that the assumptions made earlier are valid and such that (1.7) converges at all points of R, then (1.7) is a solution of the problem.

5. Application of Integral Transforms to the Equations of Elasticity

We apply the method described in section 3 to the equations of elasticity, and show that it is necessary to use the modified method in general.

We showed in section 2 that the equations of elastodynamics have the form

$$L_D \underset{\sim}{\varphi} = \underset{\sim}{f} \quad ,$$

where $\underset{\sim}{\varphi} \equiv \underset{\sim}{s}$ represents the displacement, $-\underset{\sim}{f}$ is the external body force per unit mass, and L_D is given by

$$L_D \equiv (\lambda + 2\mu) \underset{\sim}{g} \underset{\sim}{d}^T - \mu CC - \rho I \frac{\partial^2}{\partial t^2} \; .$$

For steady state vibrations $\underset{\sim}{s}$ is of the form $\underset{\sim}{q} \; e^{ipt}$, p being the angular frequency of vibration and $\underset{\sim}{q}$ a function of the space variables only. The corresponding equations of motion are

$$L_V \underset{\sim}{\varphi} = \underset{\sim}{f} \, e^{-ipt} = \underset{\sim}{h} \quad , \qquad (1.8)$$

where $\underset{\sim}{\varphi} \equiv \underset{\sim}{q}$ and $L_V \equiv (\lambda + 2\mu) \underset{\sim}{g} \underset{\sim}{d}^T - \mu CC + \rho p^2 I$.

To apply integral transforms to the latter equations we must integrate by parts the integral

$$\int_R \Psi L_{\nu} \underset{\sim}{\varphi} \, d\tau ,$$

where Ψ is a diagonal matrix. This can be done with the help of some standard results in vector theory.

If $\underline{\alpha}$ and $\underline{\beta}$ are any two vectors we have the following identity:

$$\underline{\beta}\cdot\text{grad div}\,\underline{\alpha} - \underline{\alpha}\cdot\text{grad div}\,\underline{\beta} \equiv \text{div}\,(\underline{\alpha}\,\text{div}\,\underline{\beta} - \underline{\beta}\,\text{div}\,\underline{\alpha}).$$

By integrating both sides throughout the region R and applying Gauss' divergence theorem to the right side, we obtain a result which can be interpreted in matrix form, and which leads to the relation

$$\int_R (\Psi \underset{\sim}{g}\, \underset{\sim}{d}^T \underset{\sim}{\varphi} - \Psi \underset{\sim}{d}^* \underset{\sim}{g}^{T*} \underset{\sim}{\varphi})\, d\tau = \int_S (\Psi \underset{\sim}{n}\, \underset{\sim}{d}^T \underset{\sim}{\varphi} - \Psi \underset{\sim}{d}^* \underset{\sim}{n}^T \underset{\sim}{\varphi})\, dS.$$

In this result an asterisk on $\underset{\sim}{g}$ and $\underset{\sim}{d}$ indicates that the operator post-multiplies the adjacent matrix Ψ; the elements of $\underset{\sim}{n}$ are the components of the unit normal vector.

The other identity which we use is

$$\underline{\beta}.\text{curl curl}\,\underline{\alpha} - \underline{\alpha}.\,\text{curl curl}\,\underline{\beta} \equiv \text{div}\left[(\text{curl}\,\underline{\alpha}) \times \underline{\beta} - (\text{curl}\,\underline{\beta}) \times \underline{\alpha}\right].$$

Integrating as before throughout R and using Gauss' divergence theorem on the right side leads to a second result which we convert to matrix notation to give finally

$$\int_R \left(\Psi C C \underset{\sim}{\varphi} - \Psi C^{T*} C^{T*} \underset{\sim}{\varphi}\right) d\tau = \int_S \left(\Psi_n C \underset{\sim}{\varphi} - \Psi C^{T*} \underset{\sim}{\varphi}_n\right) dS$$

Here $\underset{\sim}{\varphi}_n$ is the column matrix corresponding to the vector $\underline{\varphi}_n$ given by $\underline{\varphi}_n = \underline{\varphi} \times \underline{n}$, where $\underline{n}$ is the unit normal vector, and the rows $\underset{\sim}{\psi}^T_{ni}$ of Ψ_n are derived from the rows $\underset{\sim}{\psi}^T_i$ of Ψ in a similar way by the relation

$\underline{\Psi}_{ni} = \underline{\Psi}_i \times \underline{n}$

By combining these results together we find that

$$\int_R \Phi L_v \underset{\sim}{\varphi}\, d\tau = \int_R \Phi L^*_v \underset{\sim}{\varphi}\, d\tau$$
$$+ \int_S \left[(\lambda + 2\mu)(\Phi \underset{\sim}{n} \underset{\sim}{d}^T \underset{\sim}{\varphi} - \Phi \underset{\sim}{d}^* \underset{\sim}{n}^T \underset{\sim}{\varphi}) - \mu (\underline{\Phi_n} C \underset{\sim}{\varphi} - \Phi C^{T*} \underset{\sim}{\varphi}_n) \right] dS,$$

where $L^*_v \equiv (\lambda + 2\mu)\, \underset{\sim}{d}^* \underset{\sim}{g}^{T*} - \mu\, C^{T*} C^{T*} + \rho\, p^2 I$.

The transformed vibration equations are thus

$$\int_R \Phi L^*_v \underset{\sim}{\varphi}\, d\tau = \int_R \Phi \underset{\sim}{h}\, d\tau$$
$$+ \int_S \left[\mu(\Phi_n C \underset{\sim}{\varphi} - \Phi C^{T*} \underset{\sim}{\varphi}_n) - (\lambda + 2\mu)(\Phi \underset{\sim}{n}\, \underset{\sim}{d}^T \underset{\sim}{\varphi} - \Phi \underset{\sim}{d}^*\, \underset{\sim}{n}^T \underset{\sim}{\varphi}) \right] dS \quad (1.$$

If we can find a complete orthonormal set $\{\Phi_m\}$ such that each Φ_m is an eigenfunction of L^*_v , and such that the surface terms vanish when $\underset{\sim}{\varphi}$ satisfies the prescribed boundary conditions, then the solution can be found, by section 3.

6. Discussion of the Transformed Equations

By considering the particular case of the two-dimensional vibrations of a rectangular parallelepiped, we show that a set $\{\Psi_m\}$ satisfying conditions (i), (ii) and (iii) cannot be found in general.

For this problem $\underset{\sim}{\varphi}$ and Ψ are the matrices $\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ and $\begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}$ respectively. Take a rectangular coordinate system Oxy, such that the parallelepiped is bounded by x = 0, x = a, y = 0 and y = b. The column vectors $\underset{\sim}{d}$ and $\underset{\sim}{g}$ are then both given by $\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$ and L_v has the form

$$\mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) I + (\lambda+\mu)\begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix} + \rho p^2 I \quad .$$

The condition (ii) requiring that Ψ be an eigenfunction of L_v becomes in this case

$$\begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \begin{bmatrix} (\lambda+2\mu)\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2} + \rho p^2 & (\lambda+\mu)\frac{\partial^2}{\partial x \partial y} \\ (\lambda+\mu)\frac{\partial^2}{\partial x \partial y} & \mu\frac{\partial^2}{\partial x^2} + (\lambda+2\mu)\frac{\partial^2}{\partial y^2} + \rho p^2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}$$

That is, equating term by term,

$$(\lambda+2\mu)\frac{\partial^2\psi_1}{\partial x^2}+\mu\frac{\partial^2\psi_1}{\partial y^2}+\rho p^2\psi_1=a_{11}\psi_1$$

$$(\lambda+\mu)\frac{\partial^2\psi_1}{\partial x\,\partial y}=a_{12}\psi_2$$

$$(\lambda+\mu)\frac{\partial^2\psi_2}{\partial x\,\partial y}=a_{21}\psi_1$$

$$\mu\frac{\partial^2\psi_2}{\partial x^2}+(\lambda+2\mu)\frac{\partial^2\psi_2}{\partial y^2}+\rho p^2\psi_2=a_{22}\psi_2 \quad .$$

In practice the variables in these equations must be separable if ψ_1 and ψ_2 are to be readily obtainable. Hence ψ_1 must be of the form

$$\psi_1 = c_1 X(x)\,Y(y),$$

where $X = \cos(\xi x+\alpha)$, $Y = \cos(\eta y+\beta)$, and c_1, ξ, η, α, β, are constants. ψ_2 is then given by

$$\psi_2 = c_2\frac{dX}{dx}\,\frac{dY}{dy} \quad ,$$

where $c_2 = \dfrac{(\lambda+\mu)c_1}{a_{12}}$

By substituting these expressions for ψ_1 and ψ_2 in the surface integral in (1.9), we obtain the function which must vanish on the rectangle $x = 0$, $x = a$, $y = 0$, $y = b$. If we consider only the boundaries $x = 0$ and $x = a$ the relevant terms are

$$\begin{bmatrix} (\lambda+2\mu)\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\right)XY-(\lambda+2\mu)\varphi_1 X'Y+\mu\varphi_2 XY' \\ -(\lambda+2\mu)\varphi_1 X'Y''-\mu\left(\frac{\partial\varphi_1}{\partial y}-\frac{\partial\varphi_2}{\partial x}\right)X'Y'-\mu\varphi_2 X''Y' \end{bmatrix}$$

which must vanish for all values of y in the range $0 \leq y \leq b$. In the matrix the prime on X and Y denotes differentiation.

If one of the boundary conditions on both $x = 0$ and $x = a$ is

$$\varphi_1 = 0,$$

then the other boundary condition and the function X must be such that

$$\begin{bmatrix} (\lambda+2\mu)\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\right)XY+\mu\varphi_2 XY' \\ -\mu\left(\frac{\partial\varphi_1}{\partial y}-\frac{\partial\varphi_2}{\partial x}\right)X'-\mu\varphi_2 X'' \end{bmatrix}$$

vanishes. The first entry is zero for each value of y only if $X = 0$, since $\frac{\partial\varphi_1}{\partial x} \neq 0$ in general, in which case the second entry requires that $\frac{\partial\varphi_2}{\partial x} = 0$. If we shoose $X_m = \sin\frac{m\pi x}{a}$, where m is an integer, then $\{X_m\}$ is a complete orthogonal set of functions, and each satisfies the conditions $X_m(0) = X_m(a) = 0$. Thus if $\varphi_1(0) = \varphi_1(a) = 0$ is one boundary condition, then the other boundary condition must be $\frac{\partial\varphi_2}{\partial x} = 0$ for $x = 0$ and $x = a$, and X_m must be the function $\sin\frac{m\pi x}{a}$. A similar result is obtained for the boundaries $y = 0$ and $y = b$.

We conclude that the method of section 3 does not yield a solution of the second fundamental boundary-value problem of elasticity, in which the displacements vanish on S.

A similar argument shows that the first fundamental boundary-value problem, in which the stress vanishes on S, is also not soluble by this method. For the normal stress σ_x is of the form

$$\sigma_x = (\lambda + 2\mu)\frac{\partial\varphi_1}{\partial x} + \lambda\frac{\partial\varphi_2}{\partial y} ,$$

and if we again consider the boundaries x = 0 and x = a, then σ_x vanishes if

$$\frac{\partial\varphi_1}{\partial x} = \alpha\frac{\partial\varphi_2}{\partial y} ,$$

where $\alpha = -\dfrac{\lambda}{\lambda+2\mu}$. The surface terms which must vanish are thus

$$\left[\begin{array}{l} 2\mu\frac{\partial\varphi_2}{\partial y}XY - (\lambda+2\mu)\varphi_1 X'Y + \mu\varphi_2 XY' \\ -(\lambda+2\mu)\varphi_1 X'Y'' - \mu\left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right)X'Y' - \mu\varphi_2 X''Y' \end{array}\right]$$

and it is not difficult to show that the other requirements are $\varphi_2 = 0$ and $X' = 0$. Consequently if $\tau_{xy} = 0$ then $\frac{\partial\varphi_2}{\partial x} = -\frac{\partial\varphi_1}{\partial y}$, and we cannot find a function X which will ensure that both terms vanish.

This example of the vibrations of a rectangular parallelepiped serves to illustrate the general result that the method of integral transforms does not yield the solution of the equations of elasticity except for special sets of boundary conditions.

We now indicate the form of the solution by the modified method.

7. Formal Solution of the Equations of elasticity

If Φ_m is an eigenfunction of L_v^*, so that

$$\Phi_m L_v^* = A_m \Phi_m ,$$

then $\underset{\sim}{\varphi}$ is represented in the form

$$\underset{\sim}{\varphi} = \sum_m \Phi_m \underset{\sim}{a}_m ,$$

where $\underset{\sim}{a}_m = \int_R \Phi_m \underset{\sim}{\varphi}\, d\tau$. The transformed vibration equation becomes from (1.9),

$$A_m \underset{\sim}{a}_m = \int_R \Phi_m \underset{\sim}{h}\, d\tau$$

$$+ \int_S \left[\mu \left(\Phi_{m,n} C \underset{\sim}{\varphi} - \Phi_m C^{T^*} \underset{\sim}{\varphi}_n \right) - (\lambda + 2\mu) \left(\Phi_m \underset{\sim}{n} \underset{\sim}{d}^T \underset{\sim}{\varphi} - \Phi_m \underset{\sim}{d}^* \underset{\sim}{n}^T \underset{\sim}{\varphi} \right) \right] dS.$$

$$\therefore \underset{\sim}{a}_m = A_m^{-1} \int_R \Phi_m \underset{\sim}{h}\, d\tau$$

$$+ A_m^{-1} \int_S \left[\mu \left(\Phi_{m,n} C \underset{\sim}{\varphi} - \Phi_m C^{T^*} \underset{\sim}{\varphi}_n \right) - (\lambda + 2\mu) \left(\Phi_m \underset{\sim}{n} \underset{\sim}{d}^T \underset{\sim}{\varphi} - \Phi_m \underset{\sim}{d}^* \underset{\sim}{n}^T \underset{\sim}{\varphi} \right) \right] dS$$

Hence the solution of the equations is given by

$$\underset{\sim}{\varphi} = \sum_m \Phi_m \underset{\sim}{a}_m ,$$

with the above expression for $\underset{\sim}{a}_m$. We note that this solution involves two unknown functions, $\underset{\sim}{\varphi}$ and the normal derivative $\frac{\partial \underset{\sim}{\varphi}}{\partial n}$, and so is completely determined by specifying two boundary conditions on S.

Chapter II: Infinite Systems of Simultaneous Linear Equations

8. The Solution of Infinite Systems.

Infinite sets of equations were used as early as the 17th century for solving differential equations and other problems, but the theory of the solution was not studied until about 1900. Since then a great deal of literature has been published on the subject, and a review of the most important results may be found in KANTOROVICH & KRYLOV [1958]. It is remarked there that the subject is not yet in a completed form.

Most of the attention has been devoted to a particular class of infinite systems known as regular systems. This class includes many of the systems arising in practice. Conditions for the existence and uniqueness of the solution of a regular system are known, and methods of computing approximate solutions together with upper and lower bounds have been developed.

The theory of solution of regular systems is based essentially on the theory of absolutely convergent series, and consequently does not include certain systems involving conditionally convergent series. We shall give an example of an infinite system of equations which is not regular and whose solution exists and is obtainable by the method used for regular systems. We shall find later that the mixed boundary-value problem of section 40 gives rise to a non-regular system of equations which is not covered by the existing theory, involving as it does alternating series, and that the resulting computed solution cannot be justified theoretically.

9. Regular and Fully Regular Systems

This section and the next four together provide a summary of the known results concerning the solution of regular and fully regular systems, and are based on the material in KANTOROVICH and KRYLOV [1958]. The details of the proofs are given there.

An infinite system of equations is a system of the form

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots &= b_1 \\
a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \cdots &= b_2 \\
a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \cdots &= b_3
\end{aligned}
$$

The quantities x_1, x_2, $x_3 \ldots$ are the unknowns, and b_1, b_2, b_3, ... are the free terms. A sequence $\{x_i^*\}$ is said to be a solution of the infinite system if the left sides are convergent series when $x_i = x_i^*$, and if

$$\sum_{k=1}^{\infty} a_{ik} x_k^* = b_i$$

for $i = 1, 2, 3, \ldots$

By rearranging each equation, the system may be put into the form

$$x_i = \sum_{k=1}^{\infty} c_{ik} x_k + b_i \quad , \quad i = 1, 2, 3, \ldots \tag{2.1}$$

The latter system is defined to be regular if

$$\sum_{k=1}^{\infty} |c_{ik}| < 1$$

for $i = 1, 2, 3, \ldots$, and to be fully regular if there exists a number ϑ in the range $0 < \vartheta < 1$ such that

$$\sum_{k=1}^{\infty} |c_{ik}| \leq 1 - \vartheta$$

for $i = 1, 2, 3, \ldots$ In both cases we define ρ_i to be

$$\rho_i = 1 - \sum_{k=1}^{\infty} |c_{ik}| .$$

For a regular system it follows that $\rho_i > 0$, and for a fully regular system $\rho_i \geq \vartheta$.

The method of successive approximations for solving an infinite system is an iterative method in which a sequence of values $x_i^{(o)}$, $x_i^{(1)}$, $x_i^{(2)}$, ... is calculated by taking

$$x_i^{(o)} = 0$$

and
$$x_i^{(n+1)} = \sum_{k=1}^{\infty} c_{ik} x_k^{(n)} + b_i , \quad \begin{array}{l} i = 1, 2, 3, \ldots \\ n = 0, 1, 2, \ldots \end{array}$$

If the set $\left\{ x_i^{(n)} \right\}$ converges to a solution of the equations as $n \to \infty$, this solution is termed the principal solution of the system.

10. Existence of a Solution

The main result for regular and fully regular systems is the following one.

E.1. If there exists a constant K such that

$$|b_i| \leq K \rho_i$$

for all values of i, then the regular system (2.1) has a bounded solution x_i such that

$$|x_i| \leq K, \quad i = 1, 2, 3, \ldots,$$

and this solution may be found by the method of successive approximations.

11. Uniqueness of a Solution

In practice we often have some knowledge of the asymptotic behaviour of the unknowns x_i as $i \to \infty$, from a consideration of the original problem, and it is usually the case that the solution is either bounded or tends to zero. We can show by an example that a regular system can have more than one bounded solution, so that a stronger condition than regularity is required for uniqueness. The following results are concerned with the problem of uniqueness.

U 1. A homogeneous regular system cannot have a solution tending to zero and different from zero. i.e. if $\lim_{i\to\infty} x_i = 0$, then $x_i = 0$.

U 2. A regular system can have no more than one solution tending to zero. If the coefficients and free terms are positive, then the positive solution of it that tends to zero is its principal solution.

U 3. A fully regular system always has a unique bounded solution which is its principal solution.

U 4. If the principal solution X_i^* of the equations

$$X_i = \sum_k |c_{ik}| X_k + K \rho_i$$

is bounded below by a positive number, so that $X_i^* \geq \alpha > 0$, then the regular system (2.1) with bounded free terms has a unique bounded solution which is its principal solution.

U 5. If for any substitution of the form $x_i = M_i z_i (M_i \neq 0)$, where $M_i \to \infty$ as $i \to \infty$, in the regular system (2.1) the resulting system is regular, then the system (2.1) has a unique bounded solution.

12. The Method of Reduction

The method of reduction is an algorithm for computing an approximate solution of the infinite system. It is an iterative method at the Nth stage of which the solution $\overset{N}{x}_i$ of the equations

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = b_2 \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N = b_N \end{array}$$

is calculated. Successive approximations to x_i are $\overset{1}{x}_i, \overset{2}{x}_i, \overset{3}{x}_i, \ldots$, obtained by solving one, two, three, ... equations respectively. We have the following theoretical result.

R 1. The principal solution of the regular system (2.1), with free terms satisfying the condition $|b_i| \leq K \rho_i$, may be found by the method of reduction.

13. Upper and Lower Bounds for the Solution

It is possible to derive upper and lower bounds for the solution computed at any stage by the method of reduction, and the methods are described in the above reference. Since we have not used these techniques for the problems of the thesis, the details are omitted.

14. Quasi-Regular Systems

In this section we expand the treatment of quasi-regularity given in Kantorovich and Krylov, as the systems arising from vibration problems are of this type.

An infinite system of equations is qua si-regular if all equations of the system except a finite number satisfy the condition of regularity. Thus the system (2.1) is quasi-regular if

$$\sum_{k=1}^{\infty} |c_{ik}| < 1 \quad , \quad i = N+1, N+2, \ldots \tag{2.2}$$

and if

$$\sum_{k=1}^{\infty} |c_{ik}| < \infty \quad , \quad i = 1, 2, \ldots N$$

Consider now the system

$$x_i = \sum_{k=N+1}^{\infty} c_{ik} x_k + b_i + \sum_{k=1}^{N} c_{ik} x_k \quad , \quad i = N+1, N+2, \ldots \tag{2.3}$$

obtained by rewriting the equations (2.1) from the (N +1)th onwards. If we regard the unknowns as x_{N+1}, x_{N+2}, ..., and the free terms as $b_i + \sum_{k=1}^{N} c_{ik} x_k$, then the system (23) is regular. We write

$$\rho_i = 1 - \sum_{k=N+1}^{\infty} |c_{ik}| \quad , \quad i = N+1, N+2, \ldots$$

If /

If $|b_i| < K\rho_i$, then the system

$$x_i = \sum_{k=N+1}^{\infty} c_{ik}x_k + b_i \quad , \quad i = N+1, N+2, \ldots$$

has a solution x_i^*, by E1, which is unique if certain conditions are satisfied. Moreover the system

$$x_i = \sum_{k=N+1}^{\infty} c_{ik}x_k + c_{i\ell} \quad , \quad i = N+1, N+2, \ldots ; \ell = 1, 2, \ldots, N$$

is regular, and the free terms satisfy

$$|c_{i\ell}| \leqslant \rho_i ,$$

by (2.2), so the system has a solution. If it is unique, denote it by $x_{i\ell}^*$. Then the system

$$x_i = \sum_{k=N+1}^{\infty} c_{ik}x_k + c_{i\ell}x_\ell$$

has the unique solution $x_\ell x_{i\ell}^*$. Hence the system (2.3) has the unique solution

$$x_i = x_i^* + \sum_{\ell=1}^{N} x_\ell x_{i\ell}^* \quad , \quad i = N+1, N+2, \ldots$$

This solution is a solution of the original system (2.1) if it satisfies the first N equations of that system i.e. if $x_1, \ldots, x_N$ satisfy the equations

$$x_i = \sum_{k=1}^{N} c_{ik}x_k + b_i + \sum_{k=N+1}^{\infty} c_{ik}\left(x_k^* + \sum_{\ell=1}^{N} x_{k\ell}^* x_\ell\right) \quad , \quad i = 1, 2, \ldots, N .$$

If each of the solutions x_k^* and $x_{k\ell}^*$ is bounded, then the series $\sum_{k=N+1}^{\infty} c_{ik}x_k^*$ and $\sum_{k=N+1}^{\infty} c_{ik}x_{k\ell}^*$ converge, since $\sum_{k=1}^{\infty} |c_{ik}|$ converges. Thus the equations may be written in the form

$$x_i - \sum_{\ell=1}^{N}\left(c_{i\ell} + \sum_{k=N+1}^{\infty} c_{ik}x_{k\ell}^*\right)x_\ell = b_i + \sum_{k=N+1}^{\infty} c_{ik}x_k^* \quad , \quad i = 1, 2, \ldots, N .$$

Each distinct solution of this set of N equations gives rise to a solution of the infinite system (2.1). In particular if the system (2.1) is homogeneous, then its solution is determined by the solution of the finite homogeneous system

$$x_i - \sum_{l=1}^{N} \left(c_{il} + \sum_{k=N+1}^{\infty} c_{ik} x_{kl}^{*} \right) x_l = 0 \quad , \quad i = 1, 2, \ldots, N .$$

i.e. the system

$$x_i - \sum_{l=1}^{N} d_{il} x_l = 0 \quad , \quad i = 1, 2, \ldots, N$$

where

$$d_{il} = c_{il} + \sum_{k=N+1}^{\infty} c_{ik} x_{kl}^{*} \quad .$$

15. The Method of Reduction for a Quasi-Regular System.

We restrict our attention to the homogeneous system, and suppose that each coefficient in the infinite system is a function of a parameter p, so that $c_{ik} = c_{ik}(p)$. If we denote by $\overset{R}{x}_{il}$ the solution of the finite system

$$x_i = \sum_{k=N+1}^{N+R} c_{ik} x_k + c_{il} \quad , \quad i = N+1, \ldots, N+R ,$$

then the result R1 states that $\lim_{R \to \infty} \overset{R}{x}_{il} = x_{il}^{*}$.

Now consider the finite system

$$x_i - \sum_{l=1}^{N} \left(c_{il} + \sum_{k=N+1}^{N+R} c_{ik} \overset{R}{x}_{kl} \right) x_l = 0 \quad , \quad i = 1, 2, \ldots, N$$

i.e. the system

$$x_i - \sum_{\ell=1}^{N} \overset{R}{d}_{i\ell} x_\ell = 0 \quad , \quad i = 1, 2, \ldots, N ,$$

where $\overset{R}{d}_{i\ell} = c_{i\ell} + \sum_{k=N+1}^{N+R} c_{ik} \overset{R}{x}_{k\ell}$ Since $\overset{R}{x}_{i\ell} \to x^*_{i\ell}$ as $R \to \infty$, then $\overset{R}{d}_{i\ell} \to d_{i\ell}$ as $R \to \infty$.

The coefficients $d_{i\ell}$ and $\overset{R}{d}_{i\ell}$ are functions of p. Thus the infinite system has a solution provided the finite system

$$x_i - \sum_{\ell=1}^{N} d_{i\ell}(p) x_\ell = 0 , \qquad i = 1, 2, \ldots, N$$

has a solution. That is, provided that

$$\det\left[I - D(p) \right] = 0 ,$$

where D is the matrix $\left[d_{i\ell} \right]$ and $\underline{I}$ is the unit matrix. That is, provided that $\lambda_1(p) = 0$, where λ_1 is a latent root of the matrix I - D.

Let one such value of p be p^*, so that

$$\lambda_1(p^*) = 0.$$

By the same argument the system

$$x_i - \sum_{\ell=1}^{N} \overset{R}{d}_{i\ell} x_\ell = 0 , \qquad i = 1, 2, \ldots, N ,$$

has a solution provided $\overset{R}{\lambda}_1(p) = 0$, where $\overset{R}{\lambda}_1$ is a latent root of the matrix $I - \overset{R}{D}$, $\overset{R}{D}$ being the matrix $\left[\overset{R}{d}_{i\ell} \right]$. It follows from an earlier result that $\overset{R}{D} \to D$ as $R \to \infty$.

We now write

$$\overset{R}{D} = D + \overset{R}{E} ,$$

where $\overset{R}{E} = \left[\overset{R}{\varepsilon}_{ij} \right] = \overset{R}{D} - D$, and suppose that R is sufficiently large that $|\overset{R}{\varepsilon}_{ij}| < \varepsilon$ for all i, j. Then it can be shown (see, for example, WILKINSON [1965], chapter 2) that

$$\overset{R}{\lambda}_1 = \lambda_1 + \alpha\varepsilon + O(\varepsilon^2) ,$$

where α is a scalar whose value depends on the entries of D.

$$\therefore \; \overset{R}{\lambda}_1(p) = \lambda_1(p) + \alpha(p)\,\varepsilon + O(\varepsilon^2) .$$

Let the value of p for which the finite system has a solution be $\overset{R}{p} = p^* + \Delta \overset{R}{p}$. Then we have

$$\overset{R}{\lambda}_1(p^* + \Delta\overset{R}{p}) = 0$$

$$\therefore \; \lambda_1(p^* + \Delta\overset{R}{p}) + \alpha(p^* + \Delta\overset{R}{p})\,\varepsilon + O(\varepsilon^2) = 0 .$$

Now from Taylor's theorem, if λ_1 and α are twice differentiable,

$$\begin{aligned}\lambda_1(p^* + \Delta\overset{R}{p}) + \varepsilon\,\alpha(p^* + \Delta\overset{R}{p}) &= \lambda_1(p^*) + \varepsilon\,\alpha(p^*) + \Delta\overset{R}{p}\left[\lambda_1'(p^*) + \varepsilon\,\alpha'(p^*)\right] + O(\Delta\overset{R}{p}{}^2) \\ &= \varepsilon\,\alpha(p^*) + \Delta\overset{R}{p}\left[\lambda_1'(p^*) + \varepsilon\,\alpha'(p^*)\right] + O(\Delta\overset{R}{p}{}^2) ,\end{aligned}$$

since $\lambda_1(p^*) = 0$. Thus we have the result

$$\Delta\overset{R}{p} + O(\Delta\overset{R}{p}{}^2) = -\frac{\alpha(p^*)}{\lambda_1'(p^*) + \varepsilon\,\alpha'(p^*)}\,\varepsilon + O(\varepsilon^2)$$

from which it follows that $\Delta\overset{R}{p} \to 0$ as $R \to \infty$, since $\varepsilon \to 0$ as $R \to \infty$. Hence $\overset{R}{p} \to p^*$ as $R \to \infty$.

A similar result in matrix perturbation theory involving the latent vectors of a matrix may be used to show that, if $\overset{R}{x}_i$, i = 1, 2, ..., N, is the solution obtained by the method of reduction, $\overset{R}{x}_i \to x_i^*$ as $R \to \infty$, where x_i^* is the solution of the infinite system.

Thus the method of reduction gives a convergent sequence of approximations to the solution of a quasi-regular homogeneous system.

16. An Example of a Non-Regular System

We can show that the infinite system

$$\frac{\pi}{1!}x_1 - \frac{\pi^3}{3!}x_3 + \frac{\pi^5}{5!}x_5 - \frac{\pi^7}{7!}x_7 + \dots \quad = \quad 0$$

$$\frac{(2\pi)}{1!}x_1 - \frac{(2\pi)^3}{3!}x_3 + \frac{(2\pi)^5}{5!}x_5 - \frac{(2\pi)^7}{7!}x_7 + \dots \quad = \quad 0$$

$$\frac{(3\pi)}{1!}x_1 - \frac{(3\pi)^3}{3!}x_3 + \frac{(3\pi)^5}{5!}x_5 - \frac{(3\pi)^7}{7!}x_7 + \dots \quad = \quad 0$$

$$\cdot \quad \cdot \quad \cdot$$

has the solution

$x_1 = x_3 = x_5 = \quad \dots = 1$.

For the left side of each equation is the value of the function

$$f(x) = \frac{x}{1!}x_1 - \frac{x^3}{3!}x_3 + \frac{x^5}{5!}x_5 - \frac{x^7}{7!}x_7 + \dots$$

for $x = \pi$, 2π, 3π, ..., and the system of equations implies that $i(x)$ vanishes for $x = \pi$, 2π, 3π, ... Thus one possible function is

$$f(x) = \sin x,$$

giving the above solution.

This solution is not unique, since the values

$$x_1 = 2, \quad x_3 = 2^3, \quad x_5 = 2^5, \dots,$$

corresponding to $f(x) = \sin 2x$, also satisfy the equations.

However the bounded solution of the system is unique, apart from an arbitrary factor, and we can show that the method of reduction converges to this solution.

For, if we solve the first n equations for the unknowns $x_1, x_3, \ldots, x_{2n-1}$, setting all the other unknowns to zero, we define a function $f_n(x)$ of the form

$$f_n(x) = \frac{x}{1!}x_1 - \frac{x^3}{3!}x_3 + \frac{x^5}{5!}x_5 - \cdot \cdot \cdot + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}x_{2n-1}$$

which vanishes for $x^2 = \pi^2, (2\pi)^2, \ldots, (n\pi)^2$. Hence $f_n(x)$, apart from an arbitrary factor, must be the function

$$f_n(x) = x\prod_{k=1}^{n}\left(1 - \frac{x^2}{k^2\pi^2}\right).$$

Now $\lim_{n\to\infty} f_n(x) = \sin x$, and so the method of reduction gives the unique bounded solution of the infinite system.

We now show that the system is not regular or quasi-regular. For, if we consider the <u>nth</u> equation of the system, the ratio which must be less than unity to ensure regularity is

$$\left[\sum_{\substack{r=1\\ r\neq n}}^{\infty} \frac{(n\pi)^{2r-1}}{(2r-1)!} - \frac{(n\pi)^{2n-1}}{(2n-1)!}\right]\frac{(2n-1)!}{(n\pi)^{2n-1}} = \sinh n\pi \,.\, \frac{(2n-1)!}{(n\pi)^{2n-1}} - 1$$

$$\sim n^{1/2}c^n \text{ as } n\to\infty, \text{ where } c>1.$$

Thus the system is not quasi-regular.

The feature of this example which distinguishes it from a regular system is the conditional convergence of the infinite series involved. The theory of regular systems applies only when the series are absolutely convergent.

17. The Numerical Evaluation of Determinants

There are two algorithms for determinant evaluation which are suitable for use on a high speed computer and which are efficient in that they combine speed and accuracy. They both involve the reduction of a matrix to triangular form.

In the method of Gaussian elimination with interchanges, the matrix A is successively reduced by eliminating all the entries except one in the first column, then eliminating all except one of the resulting $n - 1$ entries in the second column, and so on until the matrix is finally in triangular form. The elimination at each stage is done by selecting as the pivotal row that particular row which has the largest element of those to be eliminated. Suitable multiples of the pivotal row are then subtracted from the other rows to effect the elimination. The resulting matrix is an upper triangular matrix, in which all the entries below the main diagonal are zero, with its rows interchanged. By the properties of determinants the determinant of A is equal to plus or minus the determinant of the final matrix, which is simply the product of the diagonal terms.

In the method of triangular decomposition the matrix A is expressed as the product

$$A = L\,U,$$

where U is upper triangular and L is lower triangular. That is all the entries of L above the main diagonal are zero, and by convention all the entries on the main diagonal are unity. Such a decomposition can be carried out in general, and is described in the standard texts on numerical analysis. The determinant of A is then given by the product of the diagonal entries of U, since the determinant of L is unity.

The computational efficiency of the two methods is examined in WILKINSON [1960], where it is shown that a process equivalent to pivoting in Gaussian elimination must be incorporated into the method of triangular decomposition to control the rounding-off errors. By carrying out a detailed error analysis of each of the processes it is also shown that triangular decomposition is an optimum method in the sense that the computational error in the determinant due to the reduction process is no greater than the maximum error caused by rounding off the exact entries of A to working accuracy. The corresponding error in Gaussian elimination is approximately $\sqrt{n}$ times the error in triangular decomposition, where n is the order of the determinant, so that the two methods are roughly comparable in accuracy for determinants of order ten. Both methods involve approximately the same amount of arithmetic for unsymmetric matrices.

The high accuracy of the decomposition method is achieved only if a certain part of the calculation is carried out to double-length accuracy, and some computers are designed to do this part to double-length automatically. If the facility is not available on a particular computer, then the two methods are exactly equivalent and have the same theoretical error bounds. It is always possible to obtain the facility by programming, but this may lengthen the computing time by a factor of four, and the advantage of the method is somewhat offset.

All the determinant evaluations in the thesis were done by the method of Gaussian elimination with interchanges, as it was the easier methed to programme for the Deuce computer. Since the largest determinant involved is of fourteenth order, there is very little to be gained by using the more accurate method, although for other problems requiring larger determinants it might be desirable to use triangular decompositian.

Chapter III. The Natural Modes of Vibration of a Rectangular Parallelepiped in Plane Strain

18. The Governing Equations of Motion.

We consider a parallelepiped of rectangular cross-section, and take a two-dimensional rectangular coordinate system with the origin O at one corner of the cross-section and the axes Ox and Oy along adjacent sides. The cross-section, of length a and breadth b, is then bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

Since we are considering two-dimensional free vibrations in the plane of the cross-section, the equations of motion are the equations (1.8) with $\underset{\sim}{\phi} = \begin{bmatrix} u \\ v \end{bmatrix}$ and $\underset{\sim}{h} = \underset{\sim}{0}$. It is convenient to divide each equation by the quantity $\lambda + 2\mu$, thus introducing the dimensionless constant $c_1 = \dfrac{\mu}{\lambda + 2\mu}$ and the frequency parameter $K^2 = \dfrac{\rho p^2}{\lambda + 2\mu}$. We also use $k^2 = \dfrac{K^2}{c_1}$. Since λ and μ are expressible in terms of Poisson's ratio ν, we can show that

$$c_1 = \frac{1 - 2\nu}{2 - 2\nu} .$$

Thus we take the equations of motion in the form

$$\frac{\partial^2 u}{\partial x^2} + c_1 \frac{\partial^2 u}{\partial y^2} + (1 - c_1) \frac{\partial^2 v}{\partial x \, \partial y} + K^2 u = 0$$

$$(1 - c_1) \frac{\partial^2 u}{\partial x \, \partial y} + c_1 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + K^2 v = 0$$

19. The Function Φ

By separating the variables in the equation

$$\Phi \overset{*}{L} = A \Phi$$

we find that a suitable choice for Φ is the matrix

$$\begin{bmatrix} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} & 0 \\ 0 & \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{bmatrix}$$

where m and n are integers. The expansion of u and v now involves a double summation over m and n. Thus we define

$$\bar{u}(m,n) = \int_0^a \sin\frac{m\pi x}{a}\int_0^b \cos\frac{n\pi y}{b}\, u(x,y)\,dy\,dx,$$

$$\bar{v}(m,n) = \int_0^a \cos\frac{m\pi x}{a}\int_0^b \sin\frac{n\pi y}{b}\, v(x,y)\,dy\,dx;$$

Then it is shown in SNEDDON [1951] that

$$u(x,y) = \frac{4}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon_n\, \bar{u}(m,n)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \tag{3.1}$$

$$v(x,y) = \frac{4}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon_m\, \bar{v}(m,n)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{3.2}$$

where

$$\varepsilon_n = \begin{cases} \frac{1}{2}, & n = 0 \\ 1, & n \neq 0 \end{cases}$$

20. The Formal Solution

With this choice of Φ and $\underset{\sim}{d}^T = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$, the transformed equations are given by equation (1.9). If we introduce the parameters

$$A_\xi(n) = \int_0^b u(\xi, y) \cos \frac{n\pi y}{b}\, dy ,$$

$$B_\eta(m) = \int_0^a \left(\frac{\partial u}{\partial y}\right)_{y=\eta} \sin \frac{m\pi x}{a}\, dx ,$$

$$C_\xi(n) = \int_0^b \left(\frac{\partial v}{\partial x}\right)_{x=\xi} \sin \frac{n\pi y}{b}\, dy ,$$

$$D_\eta(m) = \int_0^a v(x, \eta) \cos \frac{m\pi x}{a}\, dx ,$$

then the transformed equations have the form

$$\begin{bmatrix} \left(K^2 - \frac{m^2\pi^2}{a^2} - c_1\frac{n^2\pi^2}{b^2}\right) & -(1-c_1)\frac{m\pi}{a}\frac{n\pi}{b} \\ -(1-c_1)\frac{m\pi}{a}\frac{n\pi}{b} & \left(K^2 - c_1\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}\right) \end{bmatrix} \begin{bmatrix} \bar{u}(m,n) \\ \bar{v}(m,n) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{m\pi}{a}\cos m\pi . A_a(n) - \frac{m\pi}{a} . A_o(n) - c_1 \cos n\pi . B_b(m) + c_1 . B_o(m) \\ + (1-c_1)\frac{m\pi}{a}\cos n\pi . D_b(m) - (1-c_1)\frac{m\pi}{a} D_o(m) \\ (1-c_1)\frac{n\pi}{b}\cos m\pi . A_a(n) - (1-c_1)\frac{n\pi}{b} . A_o(n) - c_1 \cos m\pi . C_a(n) \\ + c_1 . C_o(n) + \frac{n\pi}{b}\cos n\pi . D_b(m) - \frac{n\pi}{b} D_o(m) \end{bmatrix} ,$$

where the right sides involve the eight unknown parameters

$A_a(n)$, $A_o(n)$, $B_b(m)$, $B_o(m)$, $C_a(n)$, $C_o(n)$, $C_b(m)$, $D_o(m)$,

To find the solution $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ of the original equations we must solve the transformed equations for $\begin{bmatrix} \bar{u}(m,n) \\ \bar{v}(m,n) \end{bmatrix}$, and substitute the resulting expressions into the double series (3.1) and (3.2). The summands in the double series which we obtain are linear combinations of the eight parameters above, and we find that, by taking the parameter into the outer series of the double sum in each case, we obtain an inner series which can be summed analytically. Hence the solution may be expressed as a sum of eight single series.

We illustrate the procedure by considering the term in $A_2(n)$ in $u(x,y)$, which is

$$\frac{4}{ab}\sum_m \sum_n \varepsilon_n \frac{\left[k^2 - \frac{m^2\pi^2}{a^2} - (2-c_1)\frac{n^2\pi^2}{b^2}\right]\frac{m\pi}{a}\cos m\pi}{\left(k^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}\right)\left(c_1k^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}\right)} \quad A_2(n)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$= \frac{2}{b}\sum_n \varepsilon_n \cos\frac{n\pi y}{b} A_2(n)\cdot\frac{2}{a}\sum_m \frac{\left[k^2 - \frac{m^2\pi^2}{a^2} - (2-c_1)\frac{n^2\pi^2}{b^2}\right]\frac{m\pi}{a}\cos m\pi}{\left(k^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}\right)\left(c_1k^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}\right)} \sin\frac{m\pi x}{a}$$

By splitting the summand into partial fractions we can write the inner series as

$$\frac{2}{a}\sum_m \left[-\frac{1}{k^2}\frac{n^2\pi^2}{b^2}\frac{\frac{m\pi}{a}\cos m\pi}{\gamma_n^2 + \frac{m^2\pi^2}{a^2}} + \frac{\gamma_n^2}{k^2}\frac{\frac{m\pi}{a}\cos m\pi}{\delta_n^2 + \frac{m^2\pi^2}{a^2}}\right]\sin\frac{m\pi x}{a}$$

$$= \frac{1}{k^2}\frac{n^2\pi^2}{b^2}\frac{\sinh\gamma_n x}{\sinh\gamma_n a} - \frac{\gamma_n^2}{k^2}\frac{\sinh\delta_n x}{\sinh\delta_n a},$$

where we have used series (A5) in the appendix. γ_n^2 and δ_n^2 are defined below. Thus the double series becomes

$$\frac{2}{b}\sum_n \varepsilon_n \frac{1}{k^2}\left(\frac{n^2\pi^2}{b^2}\frac{\sinh\gamma_n x}{\sinh\gamma_n a} - \gamma_n^2\frac{\sinh\delta_n x}{\sinh\delta_n a}\right)\cos\frac{n\pi y}{b}\cdot A_2(n).$$

By applying this procedure to each of the eight double series and using (A3), (A4), (A5) and (A6) in the appendix, we obtain the following solution .

$$
\begin{aligned}
k^2 u(x,y) = {} & \frac{2}{b}\sum_n \varepsilon_n\, \varphi_1(x) \cos\frac{n\pi y}{b} \cdot A_a(n) + \frac{2}{b}\sum_n \varepsilon_n\, \varphi_1(a-x)\cos\frac{n\pi y}{b} \cdot A_0(n) \\
& - \frac{2}{a}\sum_m \sin\frac{m\pi x}{a}\, \varphi_2(y) \,.\, B_b(m) + \frac{2}{a}\sum_m \sin\frac{m\pi x}{a}\, \varphi_2(b-y) \,.\, B_0(m) \\
& + \frac{2}{b}\sum_n \frac{n\pi}{b}\, \varphi_3(x) \cos\frac{n\pi y}{b} \cdot C_a(n) + \frac{2}{b}\sum_n \frac{n\pi}{b}\, \varphi_3(a-x) \cos\frac{n\pi y}{b} \cdot C_0(n) \\
& - \frac{2}{a}\sum_m \frac{m\pi}{a}\, \alpha_m^2 \sin\frac{m\pi x}{a}\, \varphi_4(y) \,.\, D_b(m) + \frac{2}{a}\sum_m \frac{m\pi}{a}\, \alpha_m^2 \sin\frac{m\pi x}{a}\, \varphi_4(b-y) \,.\, D_0(m)
\end{aligned}
$$

$$
\begin{aligned}
k^2 v(x,y) = {} & -\frac{2}{b}\sum_n \frac{n\pi}{b}\, \gamma_n^2\, \varphi_4(x) \sin\frac{n\pi y}{b} \cdot A_a(n) + \frac{2}{b}\sum_n \frac{n\pi}{b}\, \gamma_n^2\, \varphi_4(a-x) \sin\frac{n\pi y}{b} \cdot A_0(n) \\
& + \frac{2}{a}\sum_m \frac{m\pi}{a} \cos\frac{m\pi x}{a}\, \varphi_3(y) \,.\, B_b(m) + \frac{2}{a}\sum_m \frac{m\pi}{a} \cos\frac{m\pi x}{a}\, \varphi_3(b-y) \,.\, B_0(m) \\
& - \frac{2}{b}\sum_n \varphi_2(x) \sin\frac{n\pi y}{b} \cdot C_a(n) + \frac{2}{b}\sum_n \varphi_2(a-x) \sin\frac{n\pi y}{b} \cdot C_0(n) \\
& + \frac{2}{a}\sum_m \varepsilon_m \cos\frac{m\pi x}{a}\, \varphi_1(y) \,.\, D_b(m) + \frac{2}{a}\sum_m \varepsilon_m \cos\frac{m\pi x}{a}\, \varphi_1(b-y) \,.\, D_0(m) ,
\end{aligned}
$$

where

$$\alpha_m^2 = \frac{m^2\pi^2}{a^2} - k^2 ,$$

$$\beta_m^2 = \frac{m^2\pi^2}{a^2} - K^2 ,$$

$$\gamma_n^2 = \frac{n^2\pi^2}{b^2} - k^2 ,$$

$$\delta_n^2 = \frac{n^2\pi^2}{b^2} - K^2 ,$$

and

$$\varphi_1(x) = \frac{n^2\pi^2}{b^2}\, \frac{\sinh \gamma_n x}{\sinh \gamma_n a} - \gamma_n^2\, \frac{\sinh \delta_n x}{\sinh \delta_n a} ,$$

$$\varphi_2(y) = \alpha_m \frac{\cosh \alpha_m y}{\sinh \alpha_m b} - \frac{m^2\pi^2}{a^2} \frac{\cosh \beta_m y}{\beta_m \sinh \beta_m b} ,$$

$$\varphi_3(x) = \frac{\sinh \gamma_n x}{\sinh \gamma_n a} - \frac{\sinh \delta_n x}{\sinh \delta_n a} ,$$

$$\varphi_4(y) = \frac{\cosh \alpha_m y}{\alpha_m \sinh \alpha_m b} - \frac{\cosh \beta_m y}{\beta_m \sinh \beta_m b} .$$

In the functions φ_1, φ_2, φ_3, and φ_4, the quantities γ, δ, a and n are associated with the variable x, and α, β, b and m with y.

The above solution is a formal solution of the equations of motion for a rectangular region. A particular solution is obtained by choosing the eight parameters to satisfy the prescribed boundary conditions.

21. A Parallelepiped with Clamped Boundaries

We consider the special case in which the boundaries $x = 0$, $x = a$, $y = 0$ and $y = b$ are rigidly clamped. This problem is discussed in BOLOTIN [1961b], where a different method of solution is employed.

We shall find that the symmetry properties of the solution enable us to reduce the number of arbitrary constants, and hence to reduce the amount of computation in the approximate solution of the infinite set of equations. By observing the rate of convergence of successive approximations to the solution for rectangles having length-to-breadth ratios of 1, 2, and 4, we can draw certain conclusions about the effectiveness of the method.

Thus we wish to obtain a solution satisfying the boundary conditions

$$u(0,y) = 0 = u(a,y),$$
$$v(0,y) = 0 = v(a,y),$$
$$u(x,0) = 0 = u(x,b),$$
$$v(x,0) = 0 = v(x,b).$$

The first and last pairs are satisfied trivially by taking

$$A_o(n) = 0 = A_a(n),$$
$$D_o(m) = 0 = D_b(m),$$

giving the solution

$$k^2 u(x,y) = -\frac{2}{a}\sum_m \sin\frac{m\pi x}{a}\,\varphi_2(y).\,B_b(m) + \frac{2}{a}\sum_m \sin\frac{m\pi x}{a}\,\varphi_2(b-y).\,B_o(m)$$
$$+\frac{2}{b}\sum_n \frac{n\pi}{b}\,\varphi_3(x)\cos\frac{n\pi y}{b}.\,C_a(n) + \frac{2}{b}\sum_n \frac{n\pi}{b}\,\varphi_3(a-x)\cos\frac{n\pi y}{b}.\,C_o(n)$$

$$k^2 v(x,y) = \frac{2}{a}\sum_m \frac{m\pi}{a}\cos\frac{m\pi x}{a}\,\varphi_3(y).\,B_b(m) + \frac{2}{a}\sum_m \frac{m\pi}{a}\cos\frac{m\pi x}{a}\,\varphi_3(b-y)\,B_o(m)$$
$$-\frac{2}{b}\sum_n \varphi_2(x)\sin\frac{n\pi y}{b}.\,C_a(n) + \frac{2}{b}\sum_n \varphi_3(a-x)\sin\frac{n\pi y}{b}.\,C_o(n)$$

Now the symmetry of the body and of the boundary conditions implies that solutions must be either symmetrical or skew-symmetrical about $x = \frac{a}{2}$, and either symmetrical or skew-symmetrical about $y = \frac{b}{2}$. We consider only solutions having symmetry in both directions, and we refer to such solutions as symmetrical solutions.

22. Symmetrical Solutions.

The solution is symmetrical about $x = \frac{a}{2}$ if $v(x,y)$ is symmetrical and $u(x,y)$ skew-symmetrical. If $v(x,y)$ is symmetrical, then $\frac{\partial v}{\partial x}$ is skew-symmetrical. That is, for any value ξ in the range $[0,a]$,

$$\left(\frac{\partial v}{\partial x}\right)_{x=\xi} = -\left(\frac{\partial v}{\partial x}\right)_{x=a-\xi} .$$

Hence, from the definition of $C_\xi(n)$,

$$C_a(n) = -C_o(n)$$

for all integers n.

Moreover if $u(x,y)$ is skew-symmetrical about $x = \frac{a}{2}$, then so is $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial y} \sin \frac{m\pi x}{a}$ is also skew-symmetrical if m is odd. Thus, if m is odd, we have

$$\int_0^a \frac{\partial u}{\partial y} \sin \frac{m\pi x}{a}\, dx = 0$$

$$\therefore \; B_o(m) = 0 , \quad m = 1, 3, 5, 7 \ldots .$$

By a similar argument we deduce that, for symmetry about $y = \frac{b}{2}$,

$$B_b(m) = -B_o(m) , \quad m = 1, 2, 3, \ldots ,$$

$$\& \; C_o(n) = 0 , \qquad n = 1, 3, 5, 7, \ldots .$$

On making these substitutions and replacing φ_2 and φ_3 by the expressions on page 42, we obtain the symmetrical solutions

$$k^2 u(x,y) = \frac{2}{a}\sum_{m\ \mathrm{even}} \sin\frac{m\pi x}{a}\left[\frac{\alpha_m \cosh\alpha_m\left(\frac{b}{2}-y\right)}{\sinh\frac{\alpha_m b}{2}} - \frac{m^2\pi^2}{a^2}\frac{\cosh\beta_m\left(\frac{b}{2}-y\right)}{\beta_m \sinh\frac{\beta_m b}{2}}\right]\cdot B_0(m)$$

$$+\frac{2}{b}\sum_{n\ \mathrm{even}} \frac{n\pi}{b}\left[\frac{\sinh\gamma_n\left(\frac{a}{2}-x\right)}{\sinh\frac{\gamma_n a}{2}} - \frac{\sinh\delta_n\left(\frac{a}{2}-x\right)}{\sinh\frac{\delta_n a}{2}}\right]\cos\frac{n\pi y}{b}\cdot C_0(n) \quad ,$$

$$k^2 v(x,y) = \frac{2}{a}\sum_{m\ \mathrm{even}} \frac{m\pi}{a}\cos\frac{m\pi x}{a}\left[\frac{\sinh\alpha_m\left(\frac{b}{2}-y\right)}{\sinh\frac{\alpha_m b}{2}} - \frac{\sinh\beta_m\left(\frac{b}{2}-y\right)}{\sinh\frac{\beta_m b}{2}}\right]\cdot B_0(m)$$

$$+\frac{2}{b}\sum_{n\ \mathrm{even}} \left[\gamma_n\frac{\cosh\gamma_n\left(\frac{a}{2}-x\right)}{\sinh\frac{\gamma_n a}{2}} - \frac{n^2\pi^2}{b^2}\frac{\cosh\delta_n\left(\frac{a}{2}-x\right)}{\delta_n \sinh\frac{\delta_n a}{2}}\right]\sin\frac{n\pi y}{b}\cdot C_0(n)$$

23. The Infinite Set of Equations

It now remains to satisfy the boundary conditions

$v(0,y) = 0 = v(a,y)$

& $u(x,0) = 0 = u(x,b)$.

For a symmetrical solution it is sufficient to satisfy

$v(0,y) = 0$

& $u(x,0) = 0$,

and, by the completeness of the trigonometric functions, these conditions may be replaced by

$$\int_0^b \sin\frac{n\pi y}{b}\; v(0,y)\,dy = 0 \quad , \quad n = 1, 2, 3, \ldots$$

& $\int_0^a \sin\frac{m\pi x}{a}\, u(x,o)dx = 0 \quad , \qquad m = 1, 2, 3, \ldots$

respectively. The resulting infinite set of equations for $\left\{B_o(m)\right\}$ and $\left\{C_o(n)\right\}$ is

$$\frac{1}{2(1-c_1)k^2}\frac{a}{m\pi}\left[a_m\coth\frac{a_m b}{2}-\frac{m^2\pi^2}{a^2}\frac{1}{\beta_m}\coth\frac{\beta_m b}{2}\right]B_o(m)+\frac{2}{b}\sum_{n\text{ even}}\frac{\frac{n\pi}{b}}{\left(a_m^2+\frac{n^2\pi^2}{b^2}\right)\left(\beta_m^2+\frac{n^2\pi^2}{b^2}\right)}\cdot C_o(n)=0\,,$$

$$m=2,4,6,\ldots\quad,$$

$$\frac{2}{a}\sum_{m\text{ even}}\frac{\frac{m\pi}{a}}{\left(\gamma_n^2+\frac{m^2\pi^2}{a^2}\right)\left(\delta_n^2+\frac{m^2\pi^2}{a^2}\right)}\cdot B_o(m)+\frac{1}{2(1-c_1)k^2}\frac{b}{n\pi}\left[\gamma_n\coth\frac{\gamma_n a}{2}-\frac{n^2\pi^2}{b^2}\frac{1}{\delta_n}\coth\frac{\delta_n a}{2}\right]\cdot C_o(n)=0\,,$$

$$n=2,4,6,\ldots\quad.$$

If we take the unknowns in the order $B_o(2)$, $C_o(2)$, $B_o(4)$, $C_o(4)$, ... , then we have a homogeneous infinite system of the type considered in chapter II.

24. Regularity of the Infinite Set.

Since

$$B_o(m) = \int_0^a \left(\frac{\partial u}{\partial y}\right)_{y=0} \sin\frac{m\pi x}{a}\, dx$$

$$= \left[-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\left(\frac{\partial u}{\partial y}\right)_{y=0}\right]_0^a + \frac{a}{m\pi}\int_0^a\left(\frac{\partial^2 u}{\partial x\partial y}\right)_{y=0}\cos\frac{m\pi x}{a}\, dx\,,$$

it follows that

$$\frac{m\pi}{a}B_o(m) = \left(\frac{\partial u}{\partial y}\right)_{\substack{x=0\\y=0}} - \left(\frac{\partial u}{\partial y}\right)_{\substack{x=a\\y=0}} + O\left(\frac{1}{m}\right)\,, \quad m\to\infty\,.$$

Similarly

$$\frac{n\pi}{b} C_o(n) = \left(\frac{\partial v}{\partial x}\right)_{\substack{x=0\\y=0}} - \left(\frac{\partial v}{\partial x}\right)_{\substack{x=0\\y=b}} + O\left(\frac{1}{n}\right), \qquad n \to \infty .$$

Thus if we regard $\frac{m\pi}{a} B_o(m)$ and $\frac{n\pi}{b} C_o(n)$ as the unknowns in the infinite system, the solution we seek is bounded.

The infinite system now becomes

$$\frac{1}{2(1-c_1)k^2} \frac{a^2}{m^2\pi^2}\left[\alpha_m \coth\frac{\alpha_m b}{2} - \frac{m^2\pi^2}{a^2}\frac{1}{\beta_m}\coth\frac{\beta_m b}{2}\right]\cdot\frac{m\pi}{a}B_o(m) + \frac{2}{b}\sum_{n\ \mathrm{even}}\frac{1}{\left(\alpha_m^2+\frac{n^2\pi^2}{b^2}\right)\left(\beta_m^2+\frac{n^2\pi^2}{b^2}\right)}\cdot\frac{n\pi}{b}C_o(n) = 0,$$

$$m = 2, 4, 6, \ldots ,$$

$$\frac{2}{a}\sum_{m\ \mathrm{even}}\frac{1}{\left(\gamma_n^2+\frac{m^2\pi^2}{a^2}\right)\left(\delta_n^2+\frac{m^2\pi^2}{a^2}\right)}\cdot\frac{m\pi}{a}B_o(m) + \frac{1}{2(1-c_1)k^2}\frac{b^2}{n^2\pi^2}\left[\gamma_n\coth\frac{\gamma_n a}{2} - \frac{n^2\pi^2}{b^2}\frac{1}{\delta_n}\coth\frac{\delta_n a}{2}\right]\cdot\frac{n\pi}{b}C_o(n) = 0,$$

$$n = 2, 4, 6, \ldots .$$

If we confine our attention to the first of the above pair of equations, the diagonal term is the one involving $B_o(m)$. The condition for regularity is

$$\frac{2}{b}\sum_{n=2,4,\ldots}\left|\frac{1}{\left(\alpha_m^2+\frac{n^2\pi^2}{b^2}\right)\left(\beta_m^2+\frac{n^2\pi^2}{b^2}\right)}\right| < \frac{1}{2(1-c_1)k^2}\frac{a^2}{m^2\pi^2}\left|\alpha_m\coth\frac{\alpha_m b}{2} - \frac{m^2\pi^2}{a^2}\frac{1}{\beta_m}\coth\frac{\beta_m b}{2}\right|$$

for $m = 2, 4, 6, \ldots$.

Now for any value of K, no matter how large, α_m^2 and β_m^2 are positive if m is sufficiently large, in which case

$$\sum_{n=2,4,\ldots}\left|\frac{1}{\left(\alpha_m^2+\frac{n^2\pi^2}{b^2}\right)\left(\beta_m^2+\frac{n^2\pi^2}{b^2}\right)}\right| = \sum_{n=2,4,\ldots}\frac{1}{\left(\alpha_m^2+\frac{n^2\pi^2}{b^2}\right)\left(\beta_m^2+\frac{n^2\pi^2}{b^2}\right)}$$

$$= \frac{1}{k^2-K^2}\left[\sum_{n=2,4,\ldots}\frac{1}{\alpha_m^2+\frac{n^2\pi^2}{b^2}} - \sum_{n=2,4,\ldots}\frac{1}{\beta_m^2+\frac{n^2\pi^2}{b^2}}\right]$$

$$= \frac{b^2}{4(k^2-K^2)}\left[\frac{1}{\alpha_m b}\coth\frac{\alpha_m b}{2} - \frac{1}{\beta_m b}\coth\frac{\beta_m b}{2} - 2\left(\frac{1}{\alpha_m^2 b^2}-\frac{1}{\beta_m^2 b^2}\right)\right] ,$$

where we have used the series (A4) in the appendix. If $\underset{\smile}{n}$ is large we obtain the asymptotic form of this expression by putting

$$\alpha_m = \frac{m\pi}{a}\left(1-\frac{k^2a^2}{m^2\pi^2}\right)^{\frac{1}{2}} ,$$

$$\beta_m = \frac{m}{a}\pi\left(1-\frac{K^2a^2}{m^2\pi^2}\right)^{\frac{1}{2}} ,$$

and expanding each term binomially. This gives

$$\frac{2}{b}\sum_{n=2,4,\ldots}\left|\frac{1}{\left(\alpha_m^2+\frac{n^2\pi^2}{b^2}\right)\left(\beta_m^2+\frac{n^2\pi^2}{b^2}\right)}\right| = \frac{a^3}{4m^3\pi^3} + O\left(\frac{1}{m^4}\right) .$$

In the same way we obtain the result

$$\frac{1}{2(1-c_1)k^2}\frac{a^2}{m^2\pi^2}\left|\alpha_m \coth\frac{\alpha_m b}{2} - \frac{m^2\pi^2}{a^2}\frac{1}{\beta_m}\coth\frac{\beta_m b}{2}\right| = \frac{(1+c_1)}{(1-c_1)}\frac{a^3}{4m^3\pi^3} + O\left(\frac{1}{m^4}\right) .$$

The ratio of these asymptotic expressions is

$$\frac{(1 - c_1)}{(1 + c_1)} + O\left(\frac{1}{m}\right) .$$

From the symmetry of the two equations forming the infinite set we obtain exactly the same result for the second equation.

Now c_1 is positive for an elastic material. Thus

$$\frac{1 - c_1}{1 + c_1} < 1 .$$

Consequently the infinite system is quasi fully regular for each value of K, and so has a unique bounded solution for each value of K, whic h is given by the method of reduction. Thus all solutions of the infinite system obtained by the method of reduction correspond to solutions of the original problem.

25. Computational Details

The infinite system of equations may be written symbolically in the form

$$A(K)\, \underset{\sim}{x} = \underset{\sim}{0} ,$$

where $A(K)$ is the matrix of coefficients, $\underset{\sim}{x}$ is the column vector of unknowns, and $\underset{\sim}{0}$ is the zero vector. The solution of the system consists of the set of values of K for which

$$\det A(K) = 0,$$

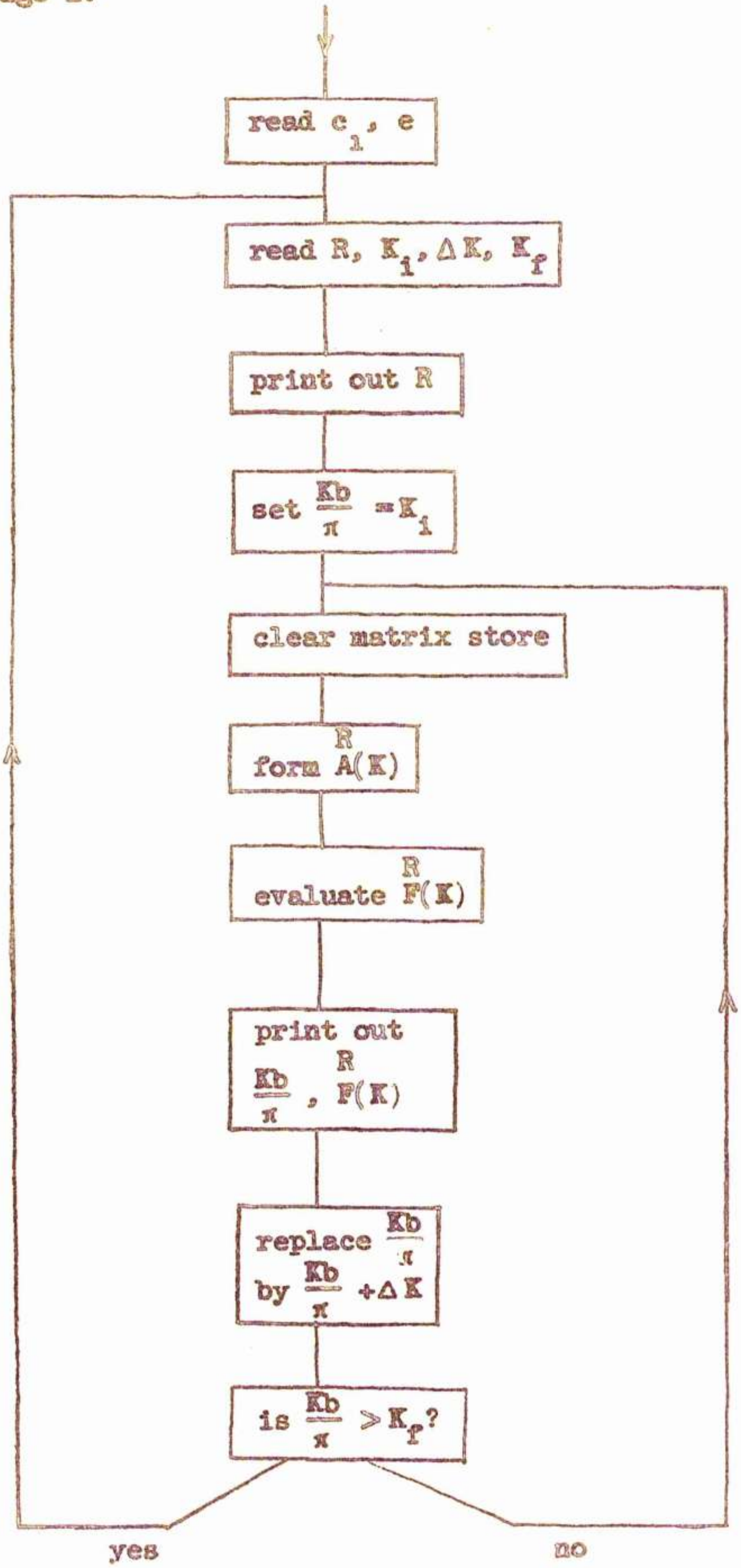
and the corresponding vectors $\underset{\sim}{x}$. We use the method of reduction, in which the equations

$$\overset{R}{A}(K)\,\underset{\sim}{\overset{R}{x}} = \underset{\sim}{0}$$

are solved for successively increasing values of R, where $\overset{R}{A}$ is the leading R x R submatrix of A and $\underset{\sim}{\overset{R}{x}}$ is the vector consisting of the first R rows of $\underset{\sim}{x}$.

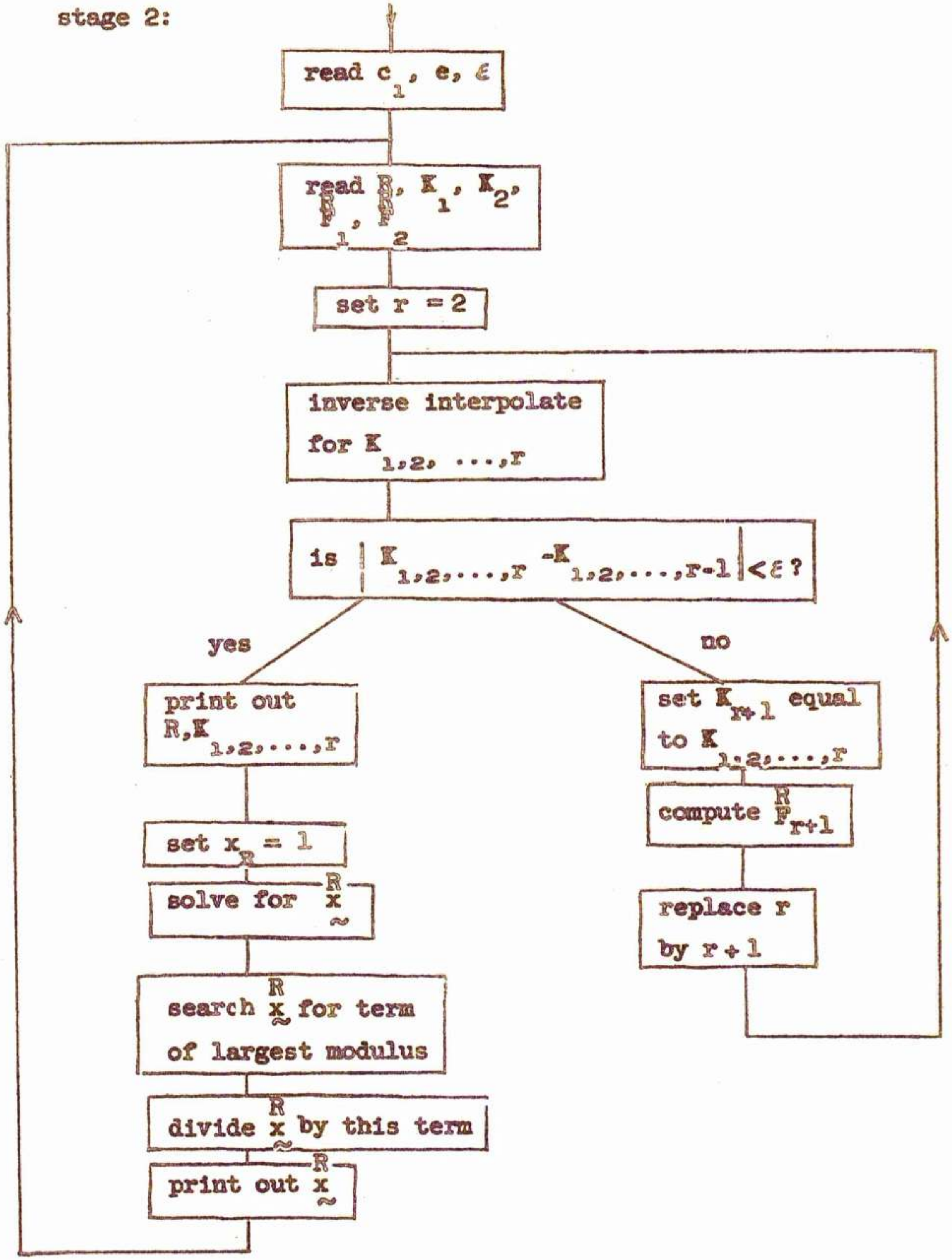
The computation is carried out in two stages, which we illustrate by flow diagrams. The ratio $\frac{b}{a}$ is denoted by e , and the dimensionless frequency parameter is taken as $\frac{Kb}{\pi}$.

stage 1:


read c_1, e
read R, K_i, ΔK, K_f
print out R
set $\frac{Kb}{\pi} = K_i$
clear matrix store
form $\overset{R}{A}(K)$
evaluate $\overset{R}{F}(K)$
print out
$\frac{Kb}{\pi}$, $\overset{R}{F}(K)$
replace $\frac{Kb}{\pi}$
by $\frac{Kb}{\pi} + \Delta K$
is $\frac{Kb}{\pi} > K_f$?
yes
no

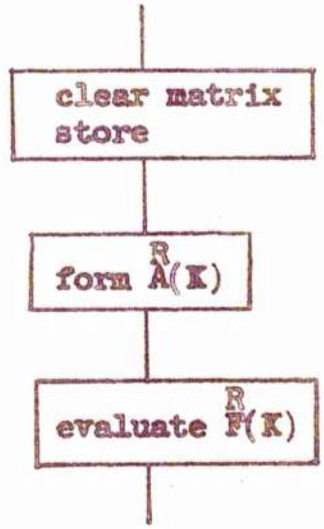
In the first stage the function $\overset{R}{F}(K) = \det \overset{R}{A}(K)$ is tabulated for a fixed value of R and a range of values of K, namely the range $\frac{Kb}{\pi} = K_i(\Delta K)K_f$. This tabulation is done for $R = 2, 4, 6, 8, \ldots$, and the approximate locations of the zeros of $\overset{R}{F}(K)$ are noted in each case. Apart from the determinant evaluation, which is discussed later, the bulk of the programming in this stage is concerned with the formation of $\overset{R}{A}(K)$. Only the non-zero entries are inserted into the matrix, so the matrix store must be cleared initially before forming $\overset{R}{A}(K)$, to ensure that all unused stores are empty. For a fixed value of $\frac{Kb}{\pi}$ the quantities $\alpha_m, \beta_m, \gamma_n$ and δ_n may be real or imaginary, and the programme must take account of this fact in the formation of $\coth \frac{\alpha_m b}{2}$ and other similar terms. A point worth noting is that as R increases the diagonal terms of $\overset{R}{A}(K)$ become less and less accurate due to cancellation of significant figures, which was indicated in section 24 by the asymptotic form. For small values of R this loss is not sufficiently important to merit special programming, but for larger matrices it should not be ignored.

stage 2:



In stage 2 inverse interpolation is used to calculate to a specified degree of accuracy the exact zero of $\overset{R}{F}(K)$ for each of the approximations noted in stage 1. K_1 and K_2 are two values of

$\frac{K_b}{\pi}$ straddling a zero of $\overset{R}{F}(K)$, and $\overset{R}{F}_1$ and $\overset{R}{F}_2$ are the corresponding values of $\overset{R}{F}(K)$. ε is the accuracy required in the computed zero. The computation of $\overset{R}{F}_{r+1}$ in stage 2 is equivalent to the section



in stage 1, and the same piece of programme is used in both cases. When the zero $K_{1,2,\ldots,r}$ has been obtained, the corresponding vector $\overset{R}{\underset{\sim}{x}}$ is found by solving the first $R - 1$ of the equations

$$\overset{R}{A}(K_{1,2,\ldots,r})\,\overset{R}{\underset{\sim}{x}} = \underset{\sim}{0}_R ,$$

with x_R, the Rth component of $\underset{\sim}{x}$, set equal to unity. This is done by a process of back-substitution in the reduced set of equations arising from the reduction of $\overset{R}{A}(K)$ to triangular form, and yields the entries of $\overset{R}{\underset{\sim}{x}}$ in the order $x_R, x_{R-1}, \ldots, x_2, x_1$. $\overset{R}{\underset{\sim}{x}}$ is then scaled to have its largest entry equal to unity.

The inverse interpolation is done by Aitken's iterative method of interpolation with the independent and dependent variables interchanged. The process is described by the flow diagram on the following page. Given the r pairs of numbers

F_1, K_1

F_2, $K_{1,2}$

F_3, $K_{1,2,3}$

.

.

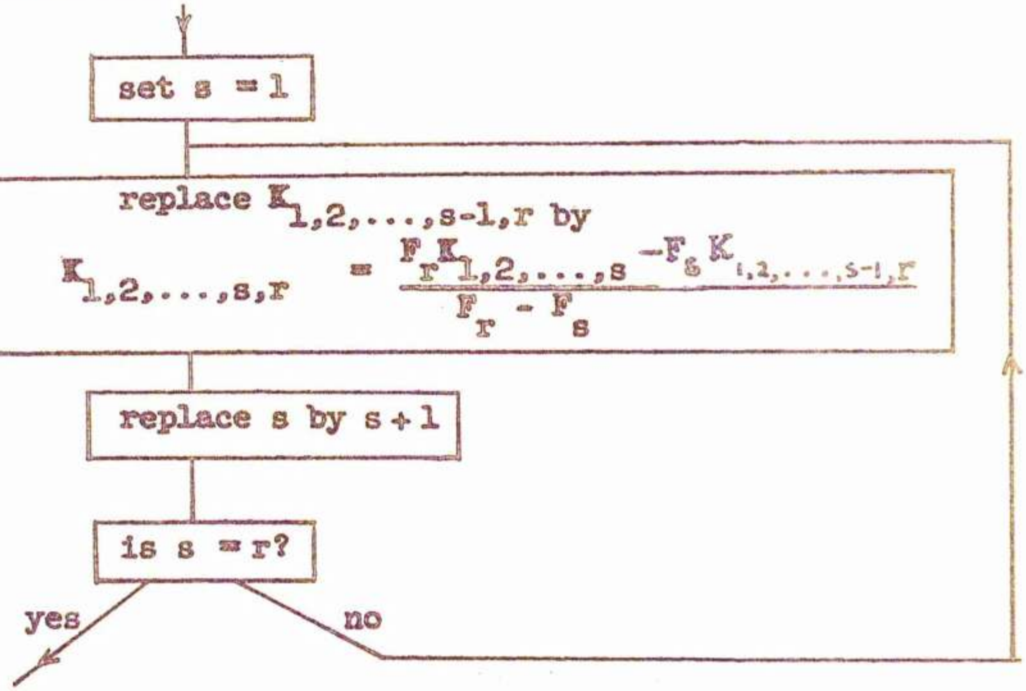
.

F_{r-1}, $K_{1,2,\ldots r-1}$

F_r, K_r ,

the routine calculates $K_{1,2,\ldots,r}$. Here $K_{1,2,\ldots,r}$ denotes the approximation to the zero of $F(K)$ obtained by interpolating the r points (F_1, K_1), (F_2, K_2),...., (F_r, K_r), with an (r-1)th degree polynomial, and $K_{1,2,\ldots,s-1,r} \equiv K_r$ when S = 1.

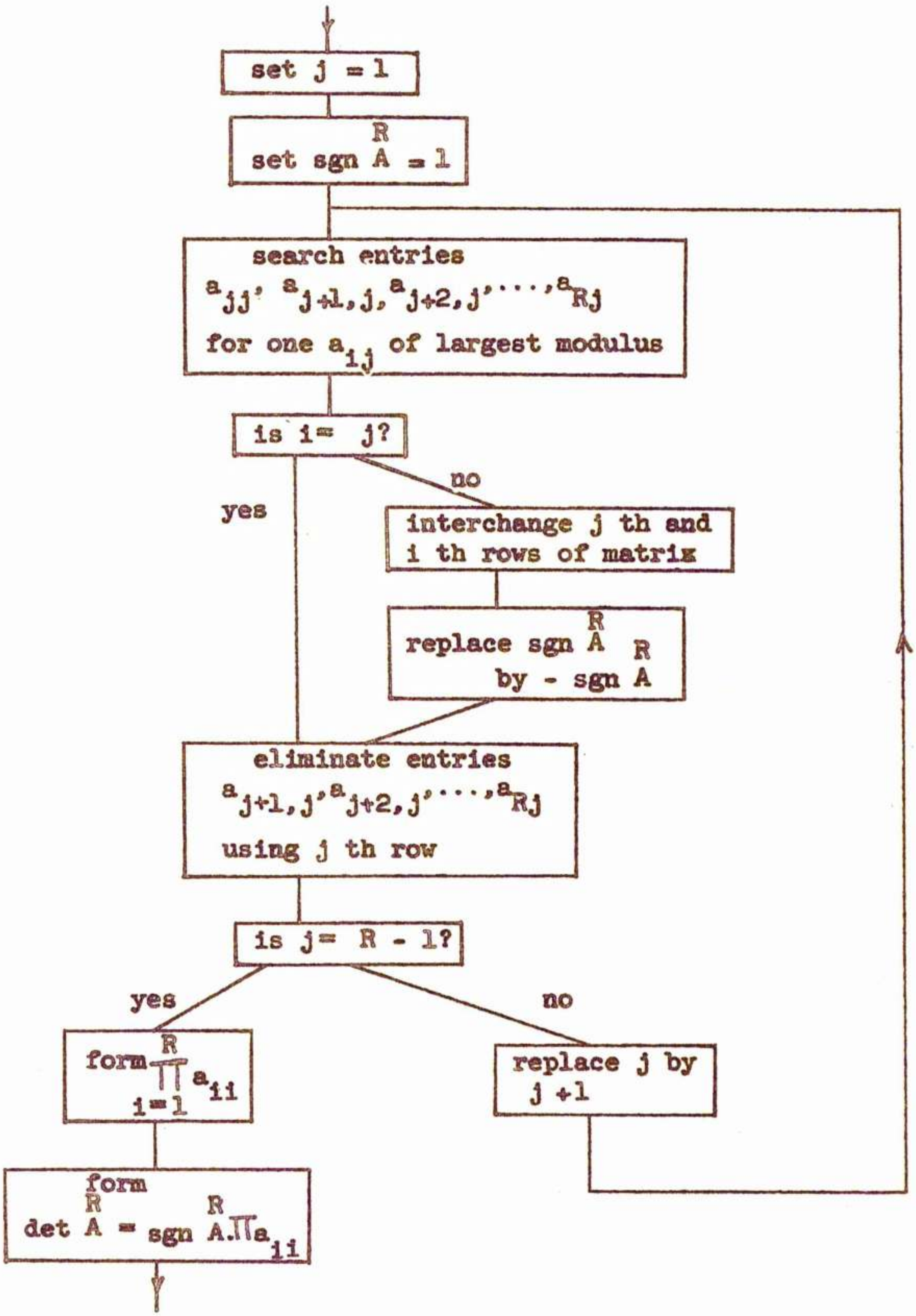
inverse interpolation:



A special determinant evaluation routine was written as a subroutine which could be incorporated into different computer programmes. This subroutine is used in both sections 1 and 2. A flow diagram for the routine, which uses the method of Gaussian elimination with interchanges, is given below.

In the flow diagram a_{ij} is used at each stage of the reduction to refer to the number currently occupying the store originally containing the i - jth entry of $\overset{R}{A}$.

determinant evaluation:



26. Results

The results in this section were obtained with $\nu = 0{\cdot}35$, which is one of the values taken by Bolotin.

The first set of results is for a square cross-section, when the additional condition of symmetry about the diagonals is imposed. For a square we have $e = 1$, and the extra condition implies that

$$B_o(m) = C_o(m) \,,$$
$$\gamma_m = \alpha_m \,,$$
$$\delta_m = \beta_m \,.$$

These relations simplify the form of the solution, and also reduce the infinite set of equations to the form

$$\frac{1}{2(1-c_1)k^2}\frac{b}{m\pi}\left[\alpha_m \coth\frac{\alpha_m b}{2} - \frac{m^2\pi^2}{b^2}\frac{1}{\beta_m}\coth\frac{\beta_m b}{2}\right]B_o(m) + \frac{2}{b}\sum_{n\ \mathrm{even}}\frac{\frac{n\pi}{b}}{\left(\alpha_m^2+\frac{n^2\pi^2}{b^2}\right)\left(\beta_m^2+\frac{n^2\pi^2}{b^2}\right)}\cdot B_o(n) = 0 \,,$$

$$m = 2, 4, 6, \ldots$$

For this system the method of reduction uses $R = 1, 2, 3, 4, \ldots$ successively, instead of $R = 2, 4, 6, 8, \ldots$, the sequence for the previous system.

Successive approximations to the lowest frequency of this type of vibration and the corresponding coefficients $B_o(2)$, $B_o(4)$, $B_o(6)$, ... are given in table 3.1.

Table 3.1

Order of Determinant	$\frac{Kb}{\pi}$	$B_0(2)$	$B_0(4)$	$B_0(6)$	$B_0(8)$	$B_0(10)$	$B_0(12)$	$B_0(14)$	$B_0(16)$	$B_0(18)$
1	2.04642	1.00000								
2	2.06385	.98066	1.00000							
3	2.06462	.96424	1.00000	.26847						
4	2.06485	.95918	1.00000	.27703	.16788					
5	2.06494	.95706	1.00000	.28122	.17326	.12063				
6	2.06498	.95602	1.00000	.28349	.17641	.12417	.09286			
7	2.06501	.95543	1.00000	.28480	.17834	.12647	.09533	.07468		
8	2.06502	.95507	1.00000	.28562	.17958	.12801	.09705	.07648	.06194	
9	2.06503	.95485	1.00000	.28615	.18041	.12907	.09827	.07780	.06330	.05256

As a check on the computations the residual displacement u(x,o) was calculated for each approximation, and the curves in figure 3.1 illustrate the results obtained

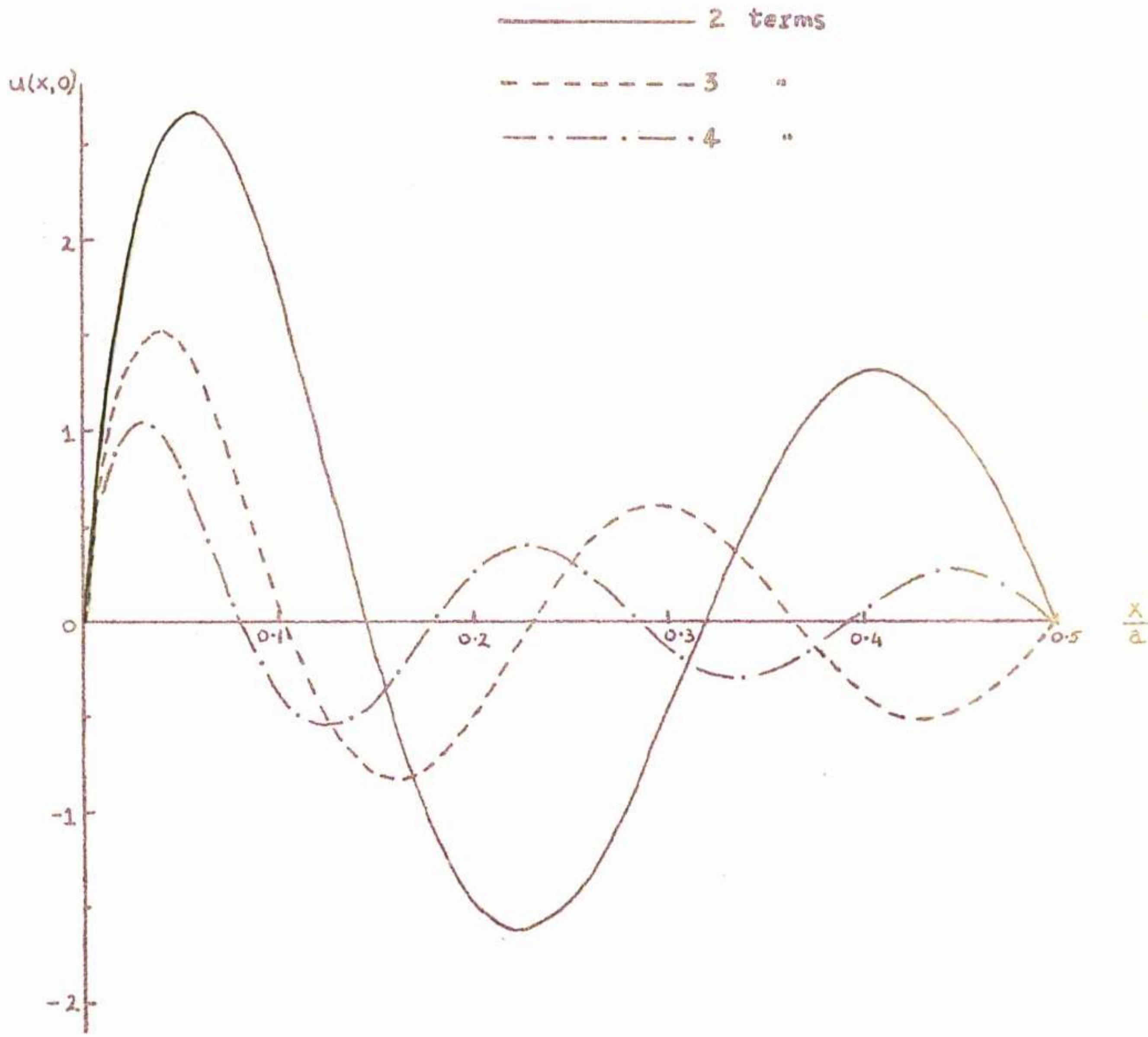


Figure 3.1

The maximum residual displacement is given as a percentage of the maximum displacement over the whole cross-section in table 3.2.

Table 3.2

order of determinant	1	2	3	4	5	6	7	8	9
$\frac{\max u(x,o)}{\max u(x,y)} \times 100$	21	5.2	3·1	2·1	1·6	1·3	1·0	0·88	0·76

Table 3.3 gives successive approximations to the first, fourth, seventh and tenth lowest natural frequencies, and in figure 3.2 the lower end of the computed spectrum of frequencies for motion symmetrical about the diagonals is indicated. Figure 3.3 has the corresponding spectrum predicted by Bolotin.

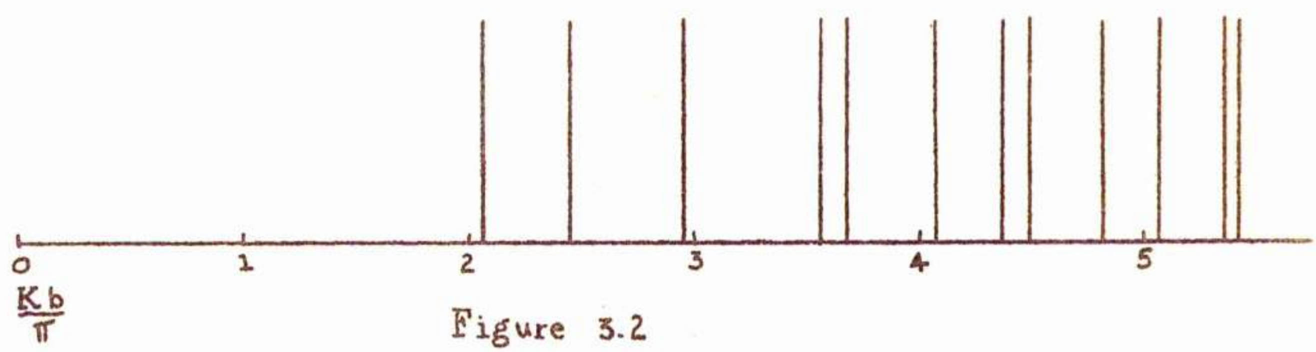


Figure 3.2

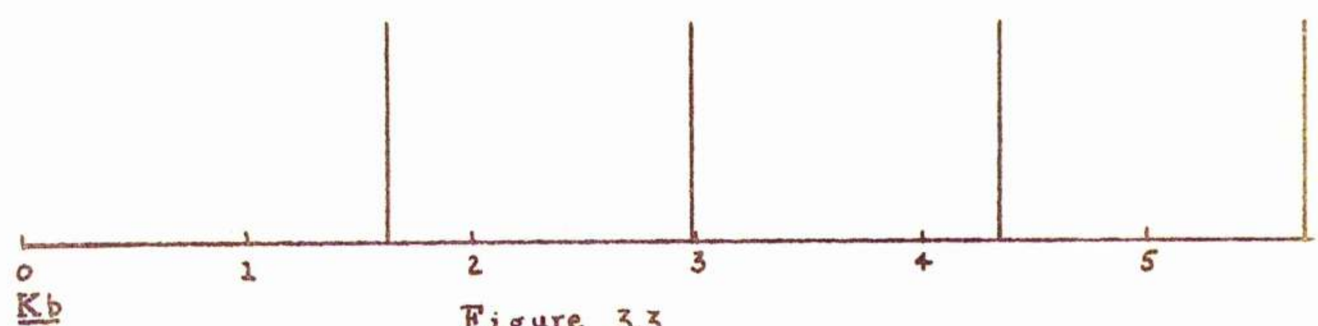


Figure 3.3

Table 3.3

order of determinant	$(\frac{Kb}{\pi})_1$	$(\frac{Kb}{\pi})_4$	$(\frac{Kb}{\pi})_7$	$(\frac{Kb}{\pi})_{10}$
1	2·04642	(3.54500)	(4.31932)	
2	2·06385	3·53628	4·31189	
3	2·06462	3·56645	4·31329	4·94751
4	2·06485	3·57122	4·36395	5·06370
5	2·06494	3·57146	4·36402	5·06464
6	2·06498	3·57152	4·36405	5·06465
7	2·06501	3·57155	4·36409	5·06465
8	2·06502	3·57156	4·36412	5·06465
9	2·06503	3·57157	4·36413	5·06466

It is not possible to say whether the bracketed frequencies in table 3.3 are the lowest approximations to the frequencies in the table or to neighbouring frequencies. The blank spaces in the column of $(\frac{Kb}{\pi})_{10}$ indicate that no approximation was found near this frequency.

In the second set of results symmetrical solutions were obtained for three different rectangles having e = 1, 2 and 4 respectively. The lowest frequency and the corresponding coefficients are given for the three cases in tables 3.4, 3.5 and 3.6, and the frequency spectra are given in figures 3.4, 3.5 and 3.6 respectively.

Table 3.4

Order of Determinant	$\frac{Kb}{\pi}$	$B_o(2)$	$C_o(2)$	$B_o(4)$	$C_o(4)$	$B_o(6)$	$C_o(6$	$B_o(8)$	$C_o(8)$	$B_o(10)$	$C_o(10)$
2	1·62762	-1·00000	1·00000								
4	1·63428	-·99999	1·00000	·24111	-·24111						
6	1·63457	1·00000	-.99996	.23755	.23756	-·06887	06888				
8	1·63462	1·00000	-1·00000	-·23686	·23686	-·06762	·06762	-·02923	·02923		
10	1.63462	1·00000	-1·00000	-·23668	·23668	-·06722	·06722	-·02871	·02871	-·01484	·01484

Table 3.5

Order of Determinant	$\frac{Kb}{\pi}$	$B_o(2)$	$C_o(2)$	$B_o(4)$	$C_o(4)$	$B_o(6)$	$C_o(6)$	$B_o(8)$	$C_o(8)$	$B_o(10)$	$C_o(10)$
2	1·03865	1·00000	·98426								
4	1·03962	·96131	1·00000	·38024	·08872						
6	1·03989	·94773	1·00000	·38245	·10716	·26198	·05041				
8	1·03998	·94343	1·00000	·38355	·11723	·26408	·05876	·18042	·03545		
10	1·04002	·94165	1·00000	·38415	·12268	·26526	·06427	·18200	·03996	·13133	.02712
12	1·04004	·94079	1·00000	·38450	·12576	.26597	·06789	·18297	·04325	·13246	·02987

Table 3.6

Order of determinant	$\frac{Kb}{\pi}$	$B_o(2)$	$C_o(2)$	$B_o(4)$	$C_o(4)$	$B_o(6)$	$C_o(6)$	$B_o(8)$	$C_o(8)$	$B_o(10)$	$C_o(10)$
2	.67110	1.00000	·07908								
4	·67114	1·00000	·08078	·01416	·01403						
6	·67118	1·00000	·08238	·01449	·01470	·02082	·00624				
8	·67120	1·00000	·08366	·01473	·01549	·02119	·00666	·02207	.00371		
10	·67122	1·00000	·08462	·01492	·01624	·02148	·00711	·02238	·00400	·02085	·00256
12	·67123	1·00000	·08524	·01504	·01688	·02167	·00754	·02260	·00428	·02107	·00275

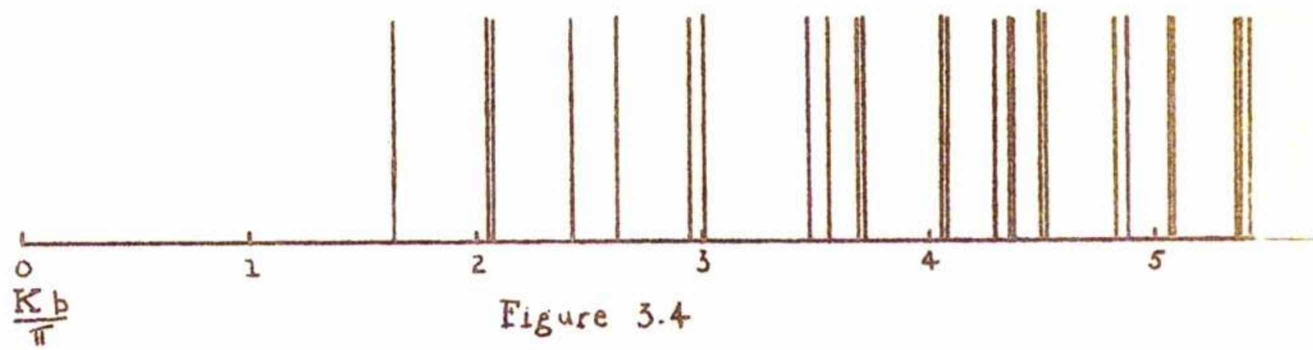


Figure 3.4

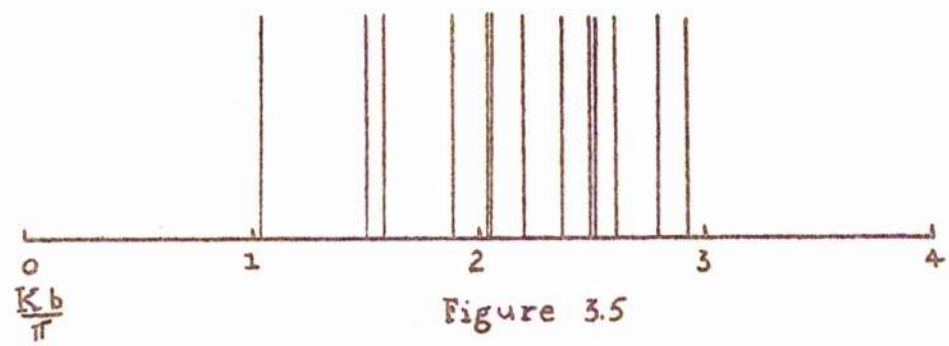


Figure 3.5

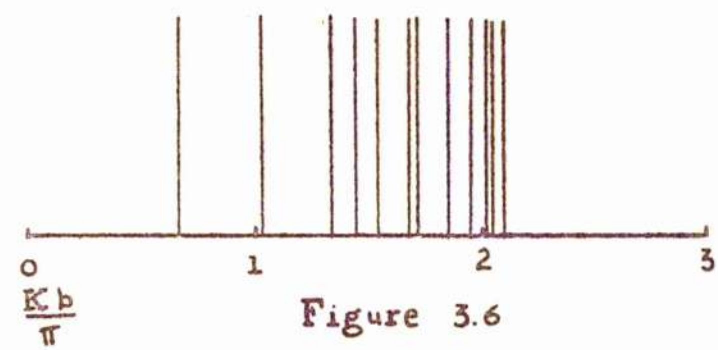


Figure 3.6

27. Discussion

Table 3.1 indicates that the sequence of approximations to the frequency is converging rapidly, and the extrapolated value obtained by Aitken's δ^2 process is 2·06505. Thus the ninth approximation is probably accurate to within one unit of the fourth decimal place.

Figure 3.1 shows that the residual displacement on $y = 0$ is oscillating more and more rapidly and with decreasing amplitude as R increases. The percentage error in table 3.2 is a measure of the accuracy of the computed eigenfunction for each value of R.

Since the eigenfunctions corresponding to the higher frequencies vary more rapidly over the region than those corresponding to the lower frequencies, we expect the computed values of the lower frequencies and eigenfunctions to be more accurate than those for the higher modes, for a fixed value of R . Table 3.3 illustrates that this is the case.

Bolotin's asymptotic method of solving vibrational problems is described in BOLOTIN [1961 a], where it is used successfully to compute the natural frequencies of plates and shells. It is applied to the problem of this chapter in BOLOTIN [1961 b] , from which the results in figure 3.3 were obtained. This method assumes that the solution of the problem consists of two distinct parts, the generating solution and the edge effect. The edge effect is small except near the boundaries of the region, while the generating solution is significant over the whole region, and constitutes the major part of the solution in the interior. We find in the plane strain problem that the generating solution corresponds to motion which is either purely dilatational or purely rotational, and that the edge effect exists for the latter case but not for the former. That is, when the generating solution is of dilatational type, a solution effective only at the boundaries cannot be found, and the edge effect is termed degenerate. Thus all the frequencies in figure 3.3 correspond to rotational motion. Now we can show, as KOLSKY [1953] does for a traction-free surface, that a plane dilatational or rotational wave incident on the clamped surface of a semi-infinite elastic slab generates on reflection both a dilatational and a rotational wave, except for special angles of incidence. This result suggests the unlikelihood of finding, in a medium bounded by clamped surfaces, standing waves of either dilatational or rotational type alone. Thus the initial assumption of the asymptotic method may not be valid for this particular problem, and this fact would explain why the method fails to predict fully the spectrum of frequencies. A disadvantage of the method is that there is no way of improving the approximate solution without altering the character of the method, and there is no error estimate.

Tables 3.4, 3.5 and 3.6 show that the successive approximations converge more and more slowly as e increases above the value 1. The same effect would occur if e were decreased below unity. The reason is that, when e is large, the solution given by integral transforms contains functions which vary slowly with x, typically $\sin \frac{m\pi x}{a}$, and other functions which vary rapidly, typically $\sinh \delta_n(\frac{a}{2} - x) \equiv \sinh \left(\frac{n^2\pi^2}{b^2} - K^2\right)^{\frac{1}{2}}(\frac{a}{2} - x)$, and the boundary conditions are satisfied by expanding the second function in a series of the form $\sum_m a_m \sin \frac{m\pi x}{a}$. The greater the value of e the more rapidly $\sinh \delta_n(\frac{a}{2} - x)$ varies relative to $\sin \frac{m\pi x}{a}$, and the more terms of the expansion are required to approximate $\sinh \delta_n(\frac{a}{2} - x)$ to a given degree of accuracy.

Thus the method gives optimum convergence when $e = 1$.

Chapter IV. The Natural Modes of Vibration of Circular Rods.

28. The Governing Equations for Axially-Symmetric Motion.

In this chapter we study the steady state vibrations of a homogeneous, isotropic, elastic rod of length ℓ and having a uniform, circular, cross-section of radius a, as shown in figure 4.1.

Figure 4.1

Take a system of cylindrical coordinates (r, θ, z) with the origin at the centre of one of the circular ends, and the z-axis along the axis of the rod. For simplicity we consider motion which is symmetrical about the axis of the rod, although the method may be extended to non-axially-symmetric motion. The components of displacement are then independent of θ, and there is no angular component of displacement. If the radial and axial components are denoted by $u(r, z)$ and $w(r, z)$ respectively, then the vector $\underset{\sim}{\varphi}$ is given by $\underset{\sim}{\varphi} = \begin{bmatrix} u \\ w \end{bmatrix}$.
$\underset{\sim}{d}$, $\underset{\sim}{g}$ and CC are the matrices

$$\begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial r} r \\ \\ \frac{\partial}{\partial z} \end{bmatrix} , \quad \begin{bmatrix} \frac{\partial}{\partial r} \\ \\ \frac{\partial}{\partial z} \end{bmatrix} , \quad \text{and} \begin{bmatrix} -\frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial r \partial z} \\ \\ \frac{1}{r}\frac{\partial^2}{\partial r \partial z} r & -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} \end{bmatrix}$$

respectively, and the equations of free vibration are

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r u + c_1 \frac{\partial^2 u}{\partial z^2} + (1-c_1)\frac{\partial^2 w}{\partial r \partial z} + K^2 u = 0$$

$$(1-c_1)\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial z} + c_1 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} + K^2 w = 0 ,$$

where $c_1 = \frac{\mu}{\lambda+2\mu} = \frac{1-2\nu}{2-2\nu}$, $K^2 = \frac{\rho p^2}{\lambda+2\mu}$, $k^2 = \frac{1}{c_1} K^2$

29. The Function Ψ

The matrix $\Psi = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}$ must be chosen so that ψ_1 and ψ_2 belong to complete sets of functions, and so that

$$\Psi L_v^* = A \Psi .$$

Now $C^{T^*} C^{T^*} = \begin{bmatrix} -\frac{\partial^2}{\partial z^2} & \frac{1}{r}\frac{\partial^2}{\partial r \partial z} r \\ \frac{\partial^2}{\partial r \partial z} & -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial z} \end{bmatrix}$, so that

$$L_v^* = \begin{bmatrix} (\lambda+2\mu)\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \mu\frac{\partial^2}{\partial z^2} + \rho p^2 & (\lambda+\mu)\frac{1}{r}\frac{\partial^2}{\partial r \partial z} r \\ (\lambda+\mu)\frac{\partial^2}{\partial r \partial z} & \mu\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + (\lambda+2\mu)\frac{\partial^2}{\partial z^2} + \rho p^2 \end{bmatrix} ,$$

and the condition

$$\Psi L_v^* = A \Psi$$

becomes, on dividing through by $\lambda + 2\mu$ and equating the matrices term by term,

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r\psi_1 + c_1 \frac{\partial^2 \psi_1}{\partial z^2} + K^2 \psi_1 = a_{11}\psi_1 \tag{4.1}$$

$$(1 - c_1)\frac{1}{r}\frac{\partial^2}{\partial r \partial z} r \psi_1 = a_{12}\psi_2 \tag{4.2}$$

$$(1 - c_1)\frac{\partial^2}{\partial r \partial z}\psi_2 = a_{21}\psi_1 \tag{4.3}$$

$$c_1 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial z^2} + K^2 \psi_2 = a_{22}\psi_2 \tag{4.4}$$

On separating the variables in (4.1) and (4.4) we find

$$\psi_1 = A\, J_1(\alpha_1 r) \cos(h_1 z + C_1)$$

$$\psi_2 = B J_0(\alpha_2 r) \cos(h_2 z + C_2)\,,$$

and for these to satisfy (4.2) and (4.3) we must have $\alpha_1 = \alpha_2$, $C_2 = \frac{\pi}{2} + C_1$ & $h_1 = h_2$.

For the expansions in complete sets of functions of the above forms we use double series expansions consisting of Fourier series in z and Fourier-Bessel and Dini series in r. The theory of the latter two expansions is given in WATSON [1922], where the basic theorems are derived. SNEDDON [1951] uses them to define finite Hankel transforms, and formulates the expansion theorems in these terms. For the sake of completeness we state the two relevant theorems.

Theorem 4.1

If $f(r)$ satisfies Dirichlet's conditions in the interval $[0, a]$, and if its finite Hankel transform is defined to be

$$\bar{f}(\xi_i) = \int_0^a r f(r) J_\mu(\xi_i r)\, dr$$

where ξ_i is a root of the transcendental equation

$$J_\mu(\xi_i a) = 0 \quad , \tag{4.5}$$

then at any point of $[0, a]$ at which the function $f(r)$ is continuous

$$f(r) = \frac{2}{a^2} \sum_i \bar{f}(\xi_i) \frac{J_\mu(\xi_i r)}{\left[J'_\mu(\xi_i a)\right]^2} \quad ,$$

where the sum i taken over all the positive roots of (4.5).

Theorem 4.2

If $f(r)$ satisfies Dirichlet's conditions in the closed interval $[a, a]$ and if its finite Hankel transform is defined to be

$$\bar{f}(\xi_i) = \int_0^a r f(r) J_\mu(\xi_i r)\, dr \quad ,$$

in which ξ_i is a root of the transcendental equation

$$\xi_i J'_\mu(\xi_i a) + h J_\mu(\xi_i a) = 0 \quad , \tag{4.6}$$

then, at each point of the interval at which $f(r)$ is continuous,

$$f(r) = \frac{2}{a^2} \sum_i \frac{\xi_i^2 \bar{f}(\xi_i)}{h^2 + \xi_i^2 - \frac{\mu^2}{a^2}} \quad \frac{J_\mu(\xi_i r)}{\left[J_\mu(\xi_i a)\right]^2} \quad ,$$

where the sum is taken over all the non-negative roots of (4.6).

We see from the theorems that the expansions in r of u and w must be of the forms $\sum_i a_i J_1(\xi_i r)$ and $\sum_i b_i J_0(\xi_i r)$ respectively, where the parameters ξ_i and H are chosen so that the two equati ons

$$J_j(\xi_i a) = 0$$

$$\xi_i a J_k'(\xi_i a) + HJ_k(\xi_i a) = 0$$

are satisfied simultaneously for either $j = 0$ and $k = 1$ or $j = 1$ and $k = 0$.

For the first case, since $xJ_1'(x) = xJ_0(x) - J_1(x)$, the equations are

$$J_0(\xi_i a) = 0$$

$$\xi_i a J_0(\xi_i a) + (H - 1)J_1(\xi_i a) = 0,$$

and hence the requirements are that $H = 1$ and $\xi_1, \xi_2, \xi_3, \ldots$ satisfy the equation

$$J_0(\xi_i a) = 0.$$

The equations in the second case are, since $J_0'(x) = - J_1(x)$,

$$J_1(\xi_i a) = 0$$

$$- \xi_i a\, J_1(\xi_i a) + HJ_0(\xi_i a) = 0,$$

which are both satisfied only if $H = 0$ and $\xi_1, \xi_2, \xi_3, \cdots$ are the roots of the equation

$$J_1(\xi_i a) = 0 \quad .$$

These two cases thus give rise to two different complete sets of functions which are eigenfunctions of L^*. In the subsequent analysis we shall use the second of the two sets, as it enables us to satisfy automatically one of the curved surface boundary conditions in the problems studied later in this chapter, namely the condition that the shear stress vanishes. The first set does not satisfy the condition that the normal component of stress vanishes, although it may be more convenient for some boundary conditions.

Thus we choose Ξ to be the matrix

$$\begin{bmatrix} J_1(\xi_i r)\cos\frac{m\pi z}{\ell} & 0 \\ 0 & J_0(\xi_i r)\sin\frac{m\pi z}{\ell} \end{bmatrix} ,$$

where $\xi_0 = 0$ and $\xi_1, \xi_2, \xi_3, \cdots$ are the positive roots of

$$J_1(\xi_i a) = 0$$

in increasing order of magnitude. For the expansion of u and w we define

$$\bar{u}(i,m) = \int_0^a \int_0^\ell r J_1(\xi_i r) \cos\frac{m\pi z}{\ell} u(r,z)\, dr\, dz \quad ,$$

$$\bar{w}(i,m) = \int_0^a \int_0^\ell r J_0(\xi_i r) \sin\frac{m\pi z}{\ell} w(r,z)\, dr\, dz \quad .$$

Then we have

$$u(r,z) = \frac{4}{a^2\ell} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon_m \bar{u}(i,m) \frac{J_1(\xi_i r)}{J_0^2(\xi_i a)} \cos \frac{m\pi z}{\ell} , \quad (4.7)$$

$$w(r,z) = \frac{4}{a^2\ell} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \bar{w}(i,m) \frac{J_0(\xi_i r)}{J_0^2(\xi_i a)} \sin \frac{m\pi z}{\ell} , \quad (4.8)$$

where, as before,

$$\varepsilon_m = \begin{cases} \frac{1}{2} , & m = 0 \\ 1 , & m \neq 0 \end{cases} .$$

30. The Formal Solution

We introduce the parameters

$$A_a(m) = \int_0^{\ell} u(a,z) \cos \frac{m\pi z}{\ell}\, dz,$$

$$B_a(m) = \int_0^{\ell} \left(\frac{\partial w}{\partial r}\right)_{r=a} \sin \frac{m\pi z}{\ell}\, dz ,$$

$$C_\zeta(i) = \int_0^{a} r J_1(\xi_i r) \left(\frac{\partial u}{\partial z}\right)_{z=\zeta} dr,$$

$$D_\zeta(i) = \int_0^{a} r J_0(\xi_i r)\, w(r,\zeta)\, dr.$$

The transformed equations, obtained by inserting the matrices $\underset{\sim}{d}$, $\underset{\sim}{g}$, $\mathbb{C}$ and $\mathbb{X}$ of the last two sections into equations (1.9), and integrating over the three surfaces $z = 0$, $r = a$ and $z = \ell$ are

$$\begin{bmatrix} \left(K^2 - \xi_i^2 - c_1 \frac{m^2\pi^2}{\ell^2}\right) & -(1-c_1)\xi_i \frac{m\pi}{\ell} \\ -(1-c_1)\xi_i \frac{m\pi}{\ell} & \left(K^2 - c_1\xi_i^2 - \frac{m^2\pi^2}{\ell^2}\right) \end{bmatrix} \begin{bmatrix} \bar{u}(i,m) \\ \bar{w}(i,m) \end{bmatrix}$$

$$= \begin{bmatrix} \xi_i a\, J_o(\xi_i a).A_a(m) - c_1 \cos m\pi . C_\ell(i) + c_1 C_o(i) + (1-c_1)\xi_i \cos m\pi . D_\ell(i) - (1-c_1)\xi_i . D_o(i) \\ (1-c_1)\frac{m\pi a}{\ell} J_o(\xi_i a).A_a(m) - c_1 a\, J_o(\xi_i a).B_a(m) + \frac{m\pi}{\ell}\cos m\pi . D_\ell(i) - \frac{m\pi}{\ell}.D_o(i) \end{bmatrix}$$

These equations involve the six parameters $A_a(m)$, $B_a(m)$, $C_\ell(i)$, $C_o(i)$, $D_\ell(i)$ and $D_o(i)$, corresponding to the six boundary conditions required to determine a solution.

We proceed as before to solve the equations for $\begin{bmatrix} \bar{u}(i,m) \\ \bar{w}(i,m) \end{bmatrix}$. On substituting the resulting expressions into the double series (4.7) and (4.8), splitting up each expansion into six separate series, and reducing each double series to a single series by rearranging if necessary the order of summation and summing the inner series analytically, we eventually obtain the solution

$$k^2 u(r,z) = \frac{2}{\ell}\sum_m \varepsilon_m \varphi_1(r;z;\frac{m\pi}{\ell};\alpha_m^2;m).A_a(m) + \frac{2}{\ell}\sum_m \varepsilon_m \varphi_1(r;z;1;-\frac{m\pi}{\ell};m).B_a(m)$$

$$+\frac{2}{a^2}\sum_i \varphi_2(r;z;-1;\xi_i;\cosh;\sinh \ell).C_\ell(i) - \frac{2}{a^2}\sum_i \varphi_2(r;\ell-z;-1;\xi_i;\cosh;\sinh \ell).C_o(i)$$

$$-\frac{2}{a^2}\sum_i \varphi_2(r;z;\xi_i;-\gamma_i^2;\cosh;\sinh \ell).D_\ell(i) + \frac{2}{a^2}\sum_i \varphi_2(r;\ell-z;\xi_i;-\gamma_i^2;\cosh;\sinh \ell).D_o(i)$$

(4.9)

$$k^2 w(r,z) = -\frac{2}{\ell}\sum_m \varphi_3(r;z;\tfrac{m\pi}{\ell};-\alpha_m^2;m).A_a(m) - \frac{2}{\ell}\sum_m \varphi_3(r;z;1;\tfrac{m\pi}{\ell};m).B_a(m)$$

$$+\frac{2}{a^2}\sum_i \varphi_4(r;z;1;-\xi_i;\sinh;\sinh\ell).C_\ell(i) + \frac{2}{a^2}\sum_i \varphi_4(r;\ell-z;1;-\xi_i;\sinh;\sinh\ell).C_o(i)$$

$$+\frac{2}{a^2}\sum_i \varphi_4(r;z;\xi_i;-\gamma_i^2;\sinh;\sinh\ell).D_\ell(i) + \frac{2}{a^2}\sum_i \varphi_4(r;\ell-z;\xi_i;-\gamma_i^2;\sinh;\sinh\ell).D_o(i)$$

(4.10)

where

$$\varphi_1(r;z;p_1;p_2;m) = \left[p_1\frac{m\pi}{\ell}\frac{J_1(\alpha_m r)}{J_1(\alpha_m a)} + p_2\frac{J_1(\beta_m r)}{J_1(\beta_m a)}\right]\cos\frac{m\pi z}{\ell}$$

$$\varphi_2(r;z;p_1;p_2;f;g(q)) = \frac{J_1(\xi_i r)}{J_0^2(\xi_i a)}\left[p_1\gamma_i\frac{f(\gamma_i z)}{g(\gamma_i q)} + p_2\frac{\xi_i}{\delta_i}\frac{f(\delta_i z)}{g(\delta_i q)}\right]$$

$$\varphi_3(r;z;p_1;p_2;m) = \left[p_1\alpha_m\frac{J_0(\alpha_m r)}{J_1(\alpha_m a)} + p_2\frac{m\pi}{\ell}\frac{J_0(\beta_m r)}{J_1(\beta_m a)}\right]\sin\frac{m\pi z}{\ell}$$

$$\varphi_4(r;z;p_1;p_2;f;g(q)) = \frac{J_0(\xi_i r)}{J_0^2(\xi_i a)}\left[p_1\xi_i\frac{f(\gamma_i z)}{g(\gamma_i q)} + p_2\frac{f(\delta_i z)}{g(\delta_i q)}\right]$$

and
$$\alpha_m^2 = k^2 - \frac{m^2\pi^2}{\ell^2},$$
$$\beta_m^2 = K^2 - \frac{m^2\pi^2}{\ell^2},$$

$$\gamma_i^2 = \xi_i^2 - k^2 ,$$

$$\delta_i^2 = \xi_i^2 - K^2 .$$

Series (A 1) - (A 6) in the appendix have been used in the reduction from double to single series.

The expressions (4.9) and (4.10) constitute an axially symmetric solution of the equations of elasticity for the cylindrical region $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq \ell$, and will be used as the basic solution for solving the problems in subsequent sections.

31. The Components of Stress

The stress components with which we are concerned are the normal stresses σ_r and σ_z, & the shear stress τ_{rz}. They are given in terms of the derivatives of the displacement and the elastic constants by the stress-strain relations on page 11 which are for axially-symmetric deformation,

$$\frac{1}{\lambda + 2\mu}\sigma_r = \frac{1}{r}\frac{\partial}{\partial r} ru - 2c_1\frac{u}{r} + (1-2c_1)\frac{\partial w}{\partial z} ,$$

$$\frac{1}{\lambda + 2\mu}\sigma_z = (1-2c_1)\frac{1}{r}\frac{\partial}{\partial r} ru + \frac{\partial w}{\partial z} ,$$

$$\frac{1}{\mu}\tau_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} .$$

To obtain the stresses corresponding to the formal solution given by (4.9) and (4.10) we differentiate the infinite series term by term. The resulting expressions are valid only at points where the infinite series converge. Since we shall have occasion later on to use these expressions to satisfy the boundary conditions and to compute the stress field, they are given in detail below.

$$\frac{k^2}{2c_1(\lambda+2\mu)}\,\sigma_r(r,z)$$

$$=\frac{2}{\ell}\sum_m \varepsilon_m\,\varphi_5\!\left(r;z;\frac{m^2\pi^2}{\ell^2};\alpha_m^2;m\right).A_a(m)+\frac{2}{\ell}\sum_m \varepsilon_m\,\varphi_5\!\left(r;z;\frac{m\pi}{\ell};-\frac{m\pi}{\ell};m\right).B_a(m)$$

$$+\frac{2}{a^2}\sum_i \varphi_6(r;z;1;\xi_i;\cosh;\sinh\ell).C_\ell(i)-\frac{2}{a^2}\sum_i \varphi_6(r;\ell-z;1;\xi_i;\cosh;\sinh\ell).C_o(i)$$

$$+\frac{2}{a^2}\sum_i \varphi_6(r;z;\xi_i;\gamma_i^2;\cosh;\sinh\ell).D_\ell(i)-\frac{2}{a^2}\sum_i \varphi_6(r;\ell-z;\xi_i;\gamma_i^2;\cosh;\sinh\ell).D_o(i)$$

$$\frac{k^2}{2c_1(\lambda+2\mu)}\,\sigma_z(r,z)$$

$$=\frac{2}{\ell}\sum_m \varepsilon_m\,\varphi_7\!\left(r;z;-\frac{m\pi}{\ell};\alpha_m^2;m\right).A_a(m)-\frac{2}{\ell}\sum_m \varepsilon_m\,\varphi_7\!\left(r;z;1;\frac{m\pi}{\ell};m\right).B_a(m)$$

$$+\frac{2}{a^2}\sum_i \varphi_8(r;z;1;\xi_i;\cosh;\sinh\ell).C_\ell(i)-\frac{2}{a^2}\sum_i \varphi_8(r;\ell-z;1;\xi_i;\cosh;\sinh\ell).C_o(i)$$

$$+\frac{2}{a^2}\sum_i \varphi_8(r;z;\xi_i;\gamma_i^2;\cosh;\sinh\ell).D_\ell(i)-\frac{2}{a^2}\sum_i \varphi_8(r;\ell-z;\xi_i;\gamma_i^2;\cosh;\sinh\ell).D_o(i)$$

$$\frac{k^2}{2\mu}\tau_{rz}(r;z)$$

$$=\frac{2}{\ell}\sum_{m}\varphi_9(r;z;\tfrac{m\pi}{\ell};-\alpha_m^2;m).A_2(m)+\frac{2}{\ell}\sum_{m}\varphi_9(r;z;1;\tfrac{m\pi}{\ell};m).B_2(m)$$

$$+\frac{2}{a^2}\sum_{i}\varphi_{10}(r;z;1;\xi_i;\sinh;\sinh\ell).C_\ell(i)+\frac{2}{a^2}\sum_{i}\varphi_{10}(r;\ell-z;1;\xi_i;\sinh;\sinh\ell).C_0(i)$$

$$+\frac{2}{a^2}\sum_{i}\varphi_{10}(r;z;\xi_i;\gamma_i^2;\sinh;\sinh\ell).D_\ell(i)+\frac{2}{a^2}\sum_{i}\varphi_{10}(r;\ell-z;\xi_i;\gamma_i^2;\sinh;\sinh\ell).D_0(i)\quad ,$$

where

$$\varphi_5(r;z;p_1;p_2;m)=\left[p_1\left\{\frac{\alpha_m J_0(\alpha_m r)}{J_1(\alpha_m a)}-\frac{1}{r}\frac{J_1(\alpha_m r)}{J_1(\alpha_m a)}\right\}+p_2\left\{\left(\frac{k^2}{2}-\frac{m^2\pi^2}{\ell^2}\right)\frac{J_0(\beta_m r)}{\beta_m J_1(\beta_m a)}-\frac{1}{r}\frac{J_1(\beta_m r)}{J_1(\beta_m a)}\right\}\right]\cos\frac{m\pi}{\ell}$$

$$\varphi_6(r;z;p_1;p_2;f;g(q))=-p_1\chi_1(r)\frac{\gamma_i f(\gamma_i z)}{g(\gamma_i q)}+p_2\chi_2(r)\frac{f(\delta_i z)}{\delta_i g(\delta_i q)}$$

$$\chi_1(r)=\xi_i\frac{J_0(\xi_i r)}{J_0^2(\xi_i a)}-\frac{1}{r}\frac{J_1(\xi_i r)}{J_0^2(\xi_i a)}$$

$$\chi_2(r)=\left[\xi_i^2+(1-2c_1)\frac{k^2}{2}\right]\frac{J_0(\xi_i r)}{J_0^2(\xi_i a)}-\frac{\xi_i}{r}\frac{J_1(\xi_i r)}{J_0^2(\xi_i a)}$$

$$\varphi_7(r;z;p_1;p_2;m)=\left[p_1\frac{m\pi}{\ell}\frac{\alpha_m J_0(\alpha_m r)}{J_1(\alpha_m a)}+p_2\left\{(1-2c_1)\frac{k^2}{2}+\frac{m^2\pi^2}{\ell^2}\right\}\frac{J_0(\beta_m r)}{\beta_m J_1(\beta_m a)}\right]\cos\frac{m\pi z}{\ell}$$

$$\varphi_8(r;z;p_1;p_2;f;g(q))=\frac{J_0(\xi_i r)}{J_0^2(\xi_i a)}\left[p_1\xi_i\frac{\gamma_i f(\gamma_i z)}{g(\gamma_i q)}+p_2\left(\frac{k^2}{2}-\xi_i^2\right)\frac{f(\delta_i z)}{\delta_i g(\delta_i q)}\right]$$

$$\varphi_9(r;z;p_1;p_2;m)=\left[p_1\left(\frac{k^2}{2}-\frac{m^2\pi^2}{\ell^2}\right)\frac{J_1(\alpha_m r)}{J_1(\alpha_m a)}+p_2\frac{m\pi}{\ell}\frac{J_1(\beta_m r)}{J_1(\beta_m a)}\right]\sin\frac{m\pi z}{\ell}$$

$$\varphi_{10}(r;z;p_1;p_2;f;g(q))=\frac{J_1(\xi_i r)}{J_0^2(\xi_i a)}\left[p_1\left(\frac{k^2}{2}-\xi_i^2\right)\frac{f(\gamma_i z)}{g(\gamma_i q)}+p_2\xi_i\frac{f(\delta_i z)}{g(\delta_i q)}\right]$$

32. A rod with Traction-free Surfaces

The solution in sections 30 and 31 is now used to determine the natural frequencies of axially-symmetric vibrati on of a rod whose ends $z = 0$ and $z = \ell$ and curved surface $r = a$ are stress-free. Such a situation is an ideal one, which is approximated in practice by a rod standing on end on a horizontal surface, since the gravitational forces on the rod are negligible compared with the stresses in the material.

This problem is discussed in LOVE [1944] page 289, where an approximate solution is derived. Detailed numerical results, obtained using the Mindlin approximation to the Pochhammer solution, are given in McNIVEN and PERRY [1962] for steel rods of various dimensions. McMAHON [1964] has an experimental study of this and related problems, and gives experimentally determined frequencies for steel and aluminium rods.

The boundary conditions of the problem are

$$\sigma_r = \tau_{rz} = 0 , \quad r = a, \quad 0 \leq z \leq \ell ,$$
$$\sigma_z = \tau_{rz} = 0 , \quad 0 \leq r \leq a, \quad z = 0 \text{ and } z = \ell .$$

33. The Transformed Boundary Conditions.

Some simplification of the formal solution (4.9) and (4.10) may be obtained by transforming certain of these conditions. Thus the condition $\tau_{rz} = 0$ gives

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0 \qquad r = a.$$

Multiplying by $\sin \frac{m\pi z}{\ell}$ and integrating with respect to z yields for $r = a$,

$$\int_0^{\ell} \frac{\partial w}{\partial r} \sin \frac{m\pi z}{\ell}\, dz = -\int_0^{\ell} \frac{\partial u}{\partial z} \sin \frac{m\pi z}{\ell}\, dz$$

$$= -\left[u \sin \frac{m\pi z}{\ell}\right]_0^{\ell} + \frac{m\pi}{\ell}\int_0^{\ell} u \cos \frac{m\pi z}{\ell}\, dz$$

$$= \frac{m\pi}{\ell}\int_0^{\ell} u \cos \frac{m\pi z}{\ell}\, dz .$$

i.e.

$$B_a(m) = \frac{m\pi}{\ell} A_a(m) . \tag{4.11}$$

By similar arguments the conditions $\tau_{rz}(r,0) = 0$ and $\tau_{rz}(r,\ell) = 0$ lead to the relations

$$C_o(i) = \xi_i D_o(i) \tag{4.12}$$

$$C_\ell(i) = \xi_i D_\ell(i) \tag{4.13}$$

If we transform the other conditions involving the normal components of stress we obtain relations between the parameters already defined and new parameters which we have to introduce, and the resulting solution still contains undetermined parameters. Consequently we postpone the normal stress conditions, and satisfy them at a later stage.

A solution of the equations satisfying three of the imposed boundary conditions and containing only three sets of parameters follows by replacing $B_a(m)$, $C_o(i)$ and $C_\ell(i)$ by the forms given above.

34. Symmetrical and Skew-Symmetrical Solutions

From the uniformity of the rod and the symmetry of the boundary conditions it follows that the normal modes must be either symmetrical or skew-symmetrical about the plane $z = \frac{\ell}{2}$. When these requirements are stated explicitly in terms of u and w they give rise to additional relations between the parameters in the solution.

For symmetrical solutions u, σ_r and σ_z are symmetrical about $z = \frac{\ell}{2}$, while w and τ_{rz} are skew-symmetrical. Hence we have $u(r, \ell - z) = u(r, z)$ and $w(r, \ell - z) = - w(r, \ell)$, and in particular

$$u(a, \ell - z) = u(a, z) \;,$$
$$w(r, \ell) = -w(r, 0).$$

It follows from the last two relations that

$$\left.\begin{aligned} A_a(m) &= 0 \;, \quad m \text{ odd} \;, \\ D_\ell(i) &= - D_o(i) \end{aligned}\right\} \qquad (4.14)$$

The corresponding conditions for skew-symmetrical solutions are that u, σ_r and σ_z are skew-symmetrical and w and τ_{rz} symmetri cal about $z = \frac{\ell}{2}$. These imply that

$$u(r, \ell - z) = - u(r, z)$$
$$w(r, \ell - z) = w(r, z)$$

leading to the conditions

$$\left.\begin{aligned} A_a(m) &= 0 \;, \qquad m \text{ even} \;, \\ D_\ell(i) &= D_o(i) \end{aligned}\right\} \qquad (4.15)$$

If equations (4.11), (4.12), (4.13) and either (4.14) or (4.15) are applied to the solution (4.9) and (4.10), the result is a solution of the equations involving only $A_a(m)$ and $D_o(i)$, which will be a solution of the problem if $A_a(m)$ and $D_o(i)$ are chosen so that the conditions $\sigma_r(a, z) = 0$ and $\sigma_z(r, 0) = 0$ are satisfied.

The details of this solution are given below, since they are required for t he computer programmes on which the computations are based.

$$\frac{k^2}{2} u(r,z) = \frac{2}{\ell} \sum_s \epsilon_s \varphi_1\left(r; z; \frac{s\pi}{\ell}; \frac{k^2}{2} - \frac{s^2\pi^2}{\ell^2}; s\right) . A_a(s)$$

$$+ \frac{2}{a^2} \sum_i \varphi_2\left(r; \frac{\ell}{2} - z; \xi_i; \frac{k^2}{2} - \xi_i^2; g; f\left(\frac{\ell}{2}\right)\right) . D_o(i)$$

$$\frac{k^2}{2} w(r,z) = -\frac{2}{\ell} \sum_s \epsilon_s \varphi_3\left[r; z; \frac{s\pi}{\ell}; -\left(\frac{k^2}{2} - \frac{s^2\pi^2}{\ell^2}\right); s\right] . A_a(s)$$

$$+ \frac{2}{a^2} \sum_i \varphi_4\left[r; \frac{\ell}{2} - z; \xi_i; \frac{k^2}{2} - \xi_i^2; f; f\left(\frac{\ell}{2}\right)\right] . D_o(i)$$

$$\frac{k^2}{4\mu} \sigma_r(r,z) = \frac{2}{\ell} \sum_s \epsilon_s \varphi_5\left(r; z; \frac{s^2\pi^2}{\ell^2}; \frac{k^2}{2} - \frac{s^2\pi^2}{\ell^2}; s\right) . A_a(s)$$

$$+ \frac{2}{a^2} \sum_i \varphi_6\left[r; \frac{\ell}{2} - z; -\xi_i; \frac{k^2}{2} - \xi_i^2; g; f\left(\frac{\ell}{2}\right)\right] . D_o(i)$$

$$\frac{k}{4\mu}\sigma_z(r,z) = \frac{2}{\ell}\sum_s \varepsilon_s \varphi_7\left(r;z;-\frac{s\pi}{\ell};\frac{k}{2}-\frac{s\pi}{\ell^2};s\right).A_2(s)$$

$$-\frac{2}{a^2}\sum_i \varphi_8\left[r;\frac{\ell}{2}-z;\xi_i;-\left(\frac{k^2}{2}-\xi_i^2\right);g;f\left(\frac{\ell}{2}\right)\right].D_o(i)$$

$$\frac{k^2}{4\mu}\tau_{rz}(r,z) = \frac{2}{\ell}\sum_s \varepsilon_s \varphi_9\left[r;z;\frac{s\pi}{\ell};-\left(\frac{k^2}{2}-\frac{s^2\pi^2}{\ell^2}\right);s\right].A_2(s)$$

$$+\frac{2}{a^2}\sum_i \varphi_{10}\left[r;\frac{\ell}{2}-z;\xi_i;-\left(\frac{k^2}{2}-\xi_i^2\right);f;f\left(\frac{\ell}{2}\right)\right].D_o(i)$$

where $\varphi_1, \ldots, \varphi_{10}$ are the functions defined earlier. For symmetrical solutions $f(x) \equiv \sinh x$, $g(x) \equiv \cosh x$, and s takes the values 0, 2, 4, 6, ..., while for skew-symmetrical solutions $f(x) \equiv \cosh x$, $g(x) = \sinh x$, and s takes the values 1, 3, 5, 7, ...

35. The remaining Boundary Conditions.

The solution of the last section is a solution of the problem if it satisfies the boundary conditions $\sigma_r(a, z) = 0$, $\sigma_z(r,o)=0$ and $\sigma_z(r, \ell) = 0$. From the symmetry and skew-symmetry of σ_z, the last condition is redundant.

Now the sets of functions $\{\cos\frac{n\pi z}{\ell}\}$ and $\{J_0(\xi_i r)\}$ are complete sets, so that the relevant boundary conditions are equivalent to the conditions

$$\int_0^{\ell} \sigma_r(a,z)\cos\frac{n\pi z}{\ell}\,dz = 0 \qquad , \; n = 0, 1, 2, 3, \ldots ,$$

$$\int_0^{a} r J_0(\xi_i r)\,\sigma_z(r,0)\,dr = 0 \qquad , \; i = 0, 1, 2, 3, \ldots \; .$$

Substituting the series for σ_r and σ_z and integrating term by term produces the infinite set of equations given below.

$$\varphi_5\left(a;0;\frac{s^2\pi^2}{\ell^2};\frac{k^2}{2}-\frac{s^2\pi^2}{\ell^2};s\right).A_2(s) + \frac{4}{a^2}k^2\sum_i \varphi_{11}(s).D_0(i) = 0 \qquad (4.16)$$

$$\frac{2}{\ell}ak^2\sum_s \varepsilon_s\varphi_{12}(s).A_2(s) - \varphi_8\left[a;\frac{\ell}{2};\xi_i;-\left(\frac{k^2}{2}-\xi_i^2\right);g;f\left(\frac{\ell}{2}\right)\right].D_0(i) = 0 \qquad (4.17)$$

The first equation applies for all relevant values of s and the second for $i = 0, 1, 2, 3, \ldots$. The functions $\varphi_{11}(s)$ and $\varphi_{12}(s)$ are defined by

$$\varphi_{11}(s) = \frac{\left[(1-2c_1)\frac{k^2}{4}-(1-c_1)\frac{s^2\pi^2}{\ell^2}\right]\xi_i^2-(1-2c_1)\frac{k^2}{4}\alpha_s^2}{(\xi_i^2-\alpha_s^2)(\xi_i^2-\beta_s^2)}\cdot\frac{1}{J_0(\xi_i a)}$$

$$\varphi_{12}(s) = \frac{\left[(1-c_1)\xi_i^2-(1-2c_1)\frac{k^2}{4}\right]\frac{s^2\pi^2}{\ell^2}-(1-2c_1)\frac{k^2}{4}\gamma_i^2}{\left(\gamma_i^2+\frac{s^2\pi^2}{\ell^2}\right)\left(\delta_i^2+\frac{s^2\pi^2}{\ell^2}\right)}$$

Regularity of the Infinite System

Before investigating the infinite system in detail we note that $B_a(m) = 0\left(\frac{\ell}{m\pi}\right)$ as $m \to \infty$, and $C_o(i) = 0\left[\frac{J_o(\xi_i a)}{\xi_i}\right]$ as $i \to \infty$. Hence $A_a(m) = 0\left(\frac{\ell^2}{m^2\pi^2}\right)$ and $D_o(i) = 0\left[\frac{J_o(\xi_i a)}{\xi_i^2}\right]$.

The main result which must be established is that the method of reduction for the system (4.16) and (4.17) converges to the solution given above. This result follows if the system is quasi-regular with bounded free terms, and if the solution above is the principal solution. The last part is true if the system has a unique bounded solution.

To prove the regularity take the equations in the form

$$\frac{\ell^2}{s^2\pi^2}\varphi_5\left(a;0;\frac{s^2\pi^2}{\ell^2};\frac{k^2}{2}-\frac{s^2\pi^2}{\ell^2};s\right)x_s + \frac{4}{a^2}k^2\sum_i \frac{J_o(\xi_i a)}{\xi_i^2}\varphi_{11}(s)y_i = 0$$

$$\frac{2}{\ell}ak^2\sum_s \varepsilon_s\frac{\ell^2}{s^2\pi^2}\varphi_{12}(s)x_s - \frac{J_o(\xi_i a)}{\xi_i^2}\varphi_8\left[a;\frac{\ell}{2};\xi_i;-\left(\frac{k^2}{2}-\xi_i^2\right);g;f\left(\frac{\ell}{2}\right)\right]y_i = 0 ,$$

where $x_s = \frac{s^2\pi^2}{\ell^2}A_o(s)$ and $y_i = \frac{\xi_i^2}{J_o(\xi_i a)}D_o(i)$.

By the asymptotic expressions given above the solution of the latter system corresponding to the solution of the physical problem is bounded.

Consider the second equation of the latter system, since it is slightly easier to deal with than the first. We must show that the ratio

$$\frac{\frac{2}{\ell} ak^2 \sum_s \varepsilon_s \frac{\ell^2}{s^2\pi^2} \left|\varphi_{12}(s)\right|}{\frac{1}{\xi_i^2} \left|J_0(\xi_i a)\, \varphi_8\right|}$$

is less than unity for i sufficiently large.

It is clear that, for a fixed value of k^2, $\varphi_{12}(s)$ is ultimately positive as $s \to \infty$, provided ξ_i is sufficiently large. Thus

$$|\varphi_{12}(s)| = \varphi_{12}(s)$$

for $s > N(\xi_i)$, say. This fact is used to write

$$\sum_s \varepsilon_s \frac{\ell^2}{s^2\pi^2} \left|\varphi_{12}(s)\right| = \sum_s \varepsilon_s \frac{\ell^2}{s^2\pi^2} \varphi_{12}(s) - 2\sum_s^{N} \varepsilon_s \frac{\ell^2}{s^2\pi^2} \varphi_{12}(s)$$

where the second series contains only positive terms. The first series may be summed analytically by expressing the summand in partial fractions and summing each part separately using standard series.

An asymptotic form of the result can now be derived for large i by elementary methods. If we assume that the first N terms are omitted from the above series, since we are concerned only with quasi-regularity, the result we obtain, after some algebra, is

$$\frac{2}{\ell} a k^2 \sum_s \xi_s \frac{\ell^2}{s^2 \pi^2} \left| \varphi_{12}(s) \right| = (1-c_1) \frac{a k^2}{4} \frac{1}{\xi_i} - O\left(\frac{1}{\xi_i^2}\right) .$$

The asymptotic form of the denominator can also be obtained without difficulty, and is

$$(1 - c_1) \frac{k^2}{2} \frac{1}{\xi_1} - O\left(\frac{1}{\xi_1^3} \right).$$

To compare the numerator and the denominator we must multiply the latter by $\frac{a}{2}$. When this is done the regularity ratio is

$$\frac{1 - O\left(\frac{1}{\xi_i}\right)}{1 - O\left(\frac{1}{\xi_i^2}\right)} = 1 - O\left(\frac{1}{\xi_i}\right) < 1$$

$$\therefore \rho_i = O\left(\frac{1}{\xi_i}\right)$$

When both sides of the second equation are divided by the coefficient of y_i and the resulting equation put into non-homogeneous form, the free terms are $O\left(\frac{1}{\xi_1} \right)$. Thus the free terms are bounded by ρ_1.

The first equation is dealt with in a similar way. In this case we make use of the asymptotic expressions for the modified Bessel functions $I_0(x)$ and $I_1(x)$ to produce the series

$$\frac{I_0(x)}{I_1(x)} \sim 1 + \frac{1}{2x} + \frac{3}{8x^2} + \frac{3}{8x^3} + O\left(\frac{1}{x^4}\right)$$

The algebra is a little more complicated, but it can be shown eventually, with the same assumptions as before, that the asymptotic forms of the numerator and denominator are

$$(1-c_1)\frac{k^2}{a}\frac{\ell}{s\pi} - \frac{3(1-c_1)k^2}{a^2}\frac{\ell^2}{s^2\pi^2} + O\left(\frac{\ell^3}{s^3\pi^3}\right)$$

and $$(1-c_1)\frac{k^2}{2}\frac{\ell}{s\pi} - \frac{c_1 k^2}{2a}\frac{\ell^2}{s^2\pi^2} + O\left(\frac{\ell^3}{s^3\pi^3}\right)$$

respectively. Since $0 \leq c_1 \leq \frac{1}{2}$, then $3(1 - c_1) > c_1$. Hence, after multiplying the numerator by $\frac{a}{2}$, the ratio is

$$1 - O\left(\frac{1}{s}\right),$$

which is less than unity. It follows that

$$\rho_s = O\left(\frac{1}{s}\right),$$

and the free terms are bounded, by the argument used previously.

Thus the infinite system of equations is quasi-regular and has bounded free terms. It follows by E 1 that a solution exists, and the method of reduction converges to the principal solution, by R 1.

The question of the uniqueness of the solution of the system has not been settled. If we transform the unknowns by the transformation $x_s = \frac{s\pi}{\ell} x_s'$, $y_i = \xi_i y_i'$, then the second of the resulting equations satisfies the condition of regularity, but the first does not. The transformation

$x_s = \dfrac{s^2\pi^2}{\ell^2} x''_s$, $y_i = \xi_i^2 y''_i$ produces two equations, neither of which satisfies the condition. Other transformations which attempt to establish the result by U 5 give rise to systems which are more difficult to test for regularity.

The other approach to this question suggested in the literature is to use the result in U 4. This involves a study of the solutions generated by the method of successive approximations, and requires an estimate of the way in which the solutions tend to their limiting value. An attempt has been made along these lines, but so far without success.

In the absence of a complete mathematical justification we rely on the fact that the corresponding physical problem has a unique solution in general, for each frequency, to justify the computed solution.

37. Computational details.

The computational problem is essentially the one described in section 25, and involves the solution of singular systems of simultaneous equations. The only difference is in the details of the coefficients in the simultaneous equations. To compute the Bessel functions J_0, J_1, I_0, and I_1 occurring in the di agonal terms from the first equation, a special computer routine was written using Chebyshev series expansions over part of the range of the argument and asymptotic series over the remainder. The procedure is described in CLENSHAW [1962], from which the coefficients in the expansions were obtained.

The computation of the frequencies and coefficients is done in two stages, as before, and consists of a tabulation followed by an iteration. The same determinant evaluation routine is used. The dimensionless frequency is taken as $\frac{Ka}{\pi}$ and the data for the tabulation are the same as before, with $e = \frac{a}{\ell}$.

Different computer programmes are used to deal with symmetric and skew-symmetric solutions, although only minor differences are involved. The unknowns are taken in the order $A_a(s)$, $D_o(0)$, $A_a(s+2)$, $D_o(1)$, ..., with $s = 0$ & 1 for symmetrical and skew-symmetrical solutions respectively, and are scaled so that the largest term is unity.

38. Results

Numerical results have been obtained for a steel rod, taking $\nu = 0{\cdot}29$, whose length and radius are equal. From the previous experience with the method we expect these relative dimensions to give the best rate of convergence.

The first eight symmetrical and the first eight skew-symmetrical modes have been computed, and the combined frequency spectrum of values of $\frac{Ka}{\pi}$ is presented in figure 4.2. Figures 4.3, 4.4 and 4.5 give the mode shapes and the lines of zero displacement for the first fourteen modes. The full lines and dashed lines show the undeformed and deformed cross-sections

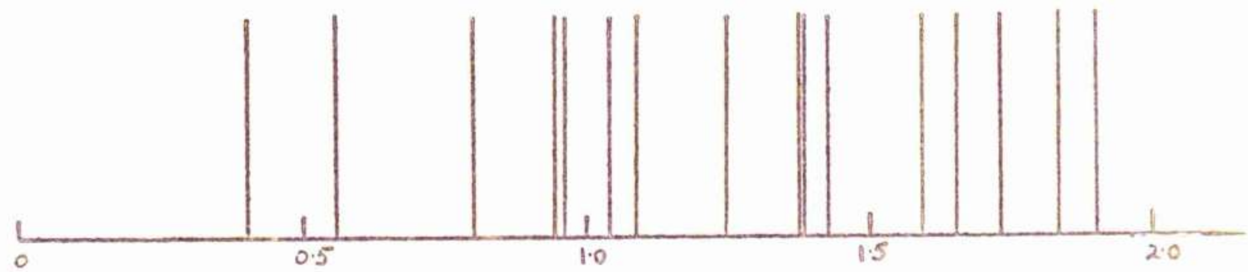


Figure 4.2

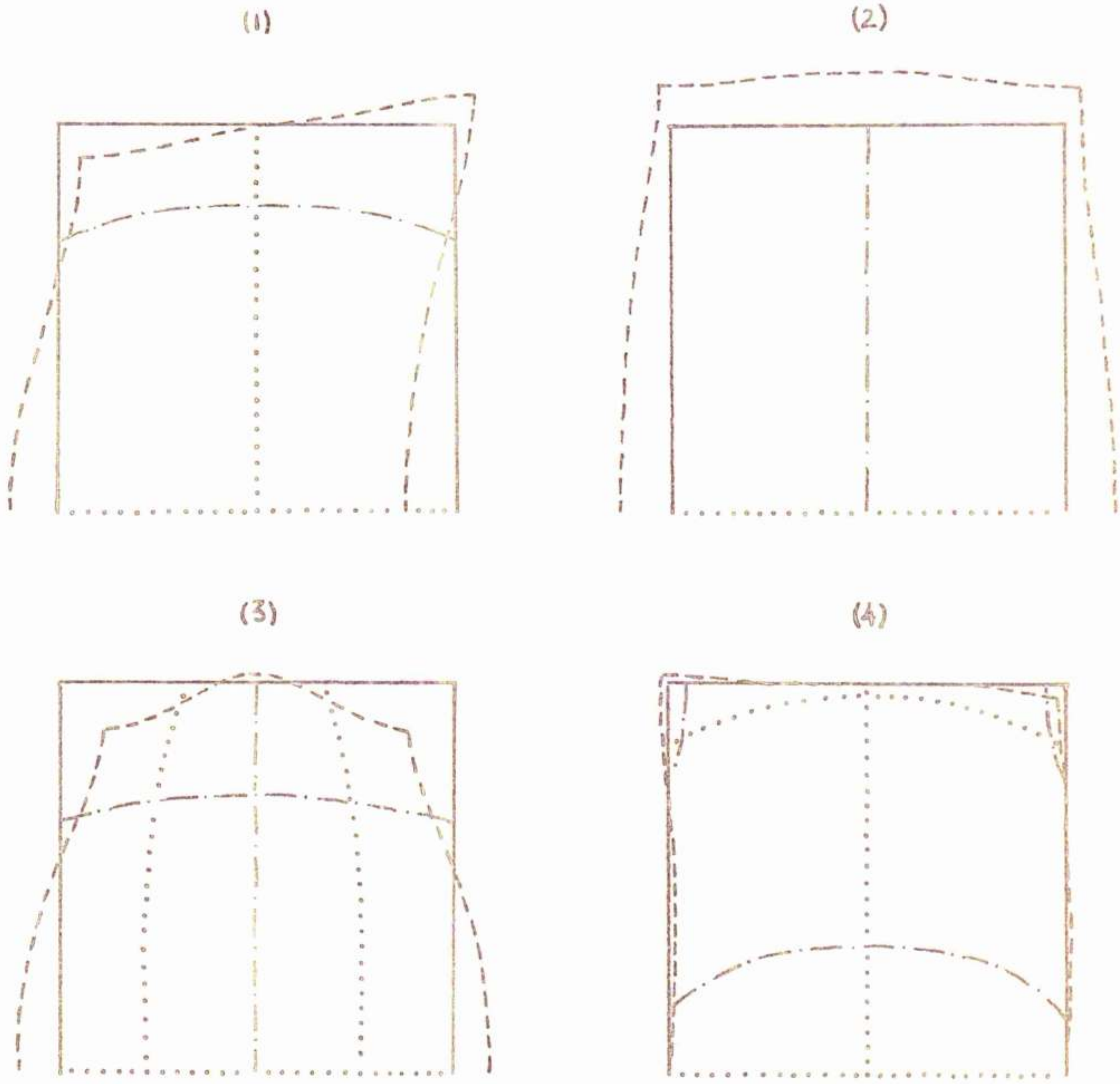


Figure 4.3

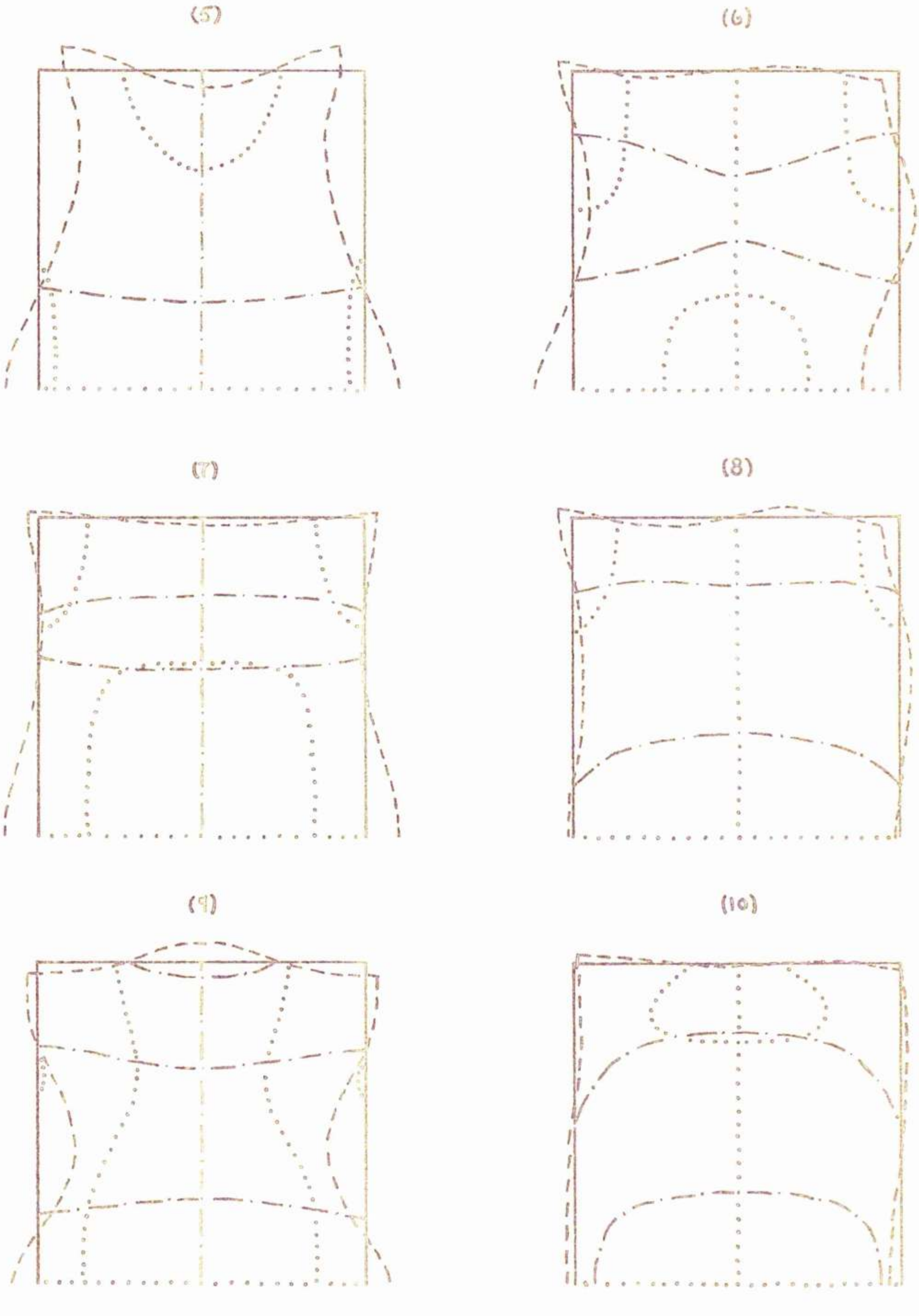


Figure 4.4

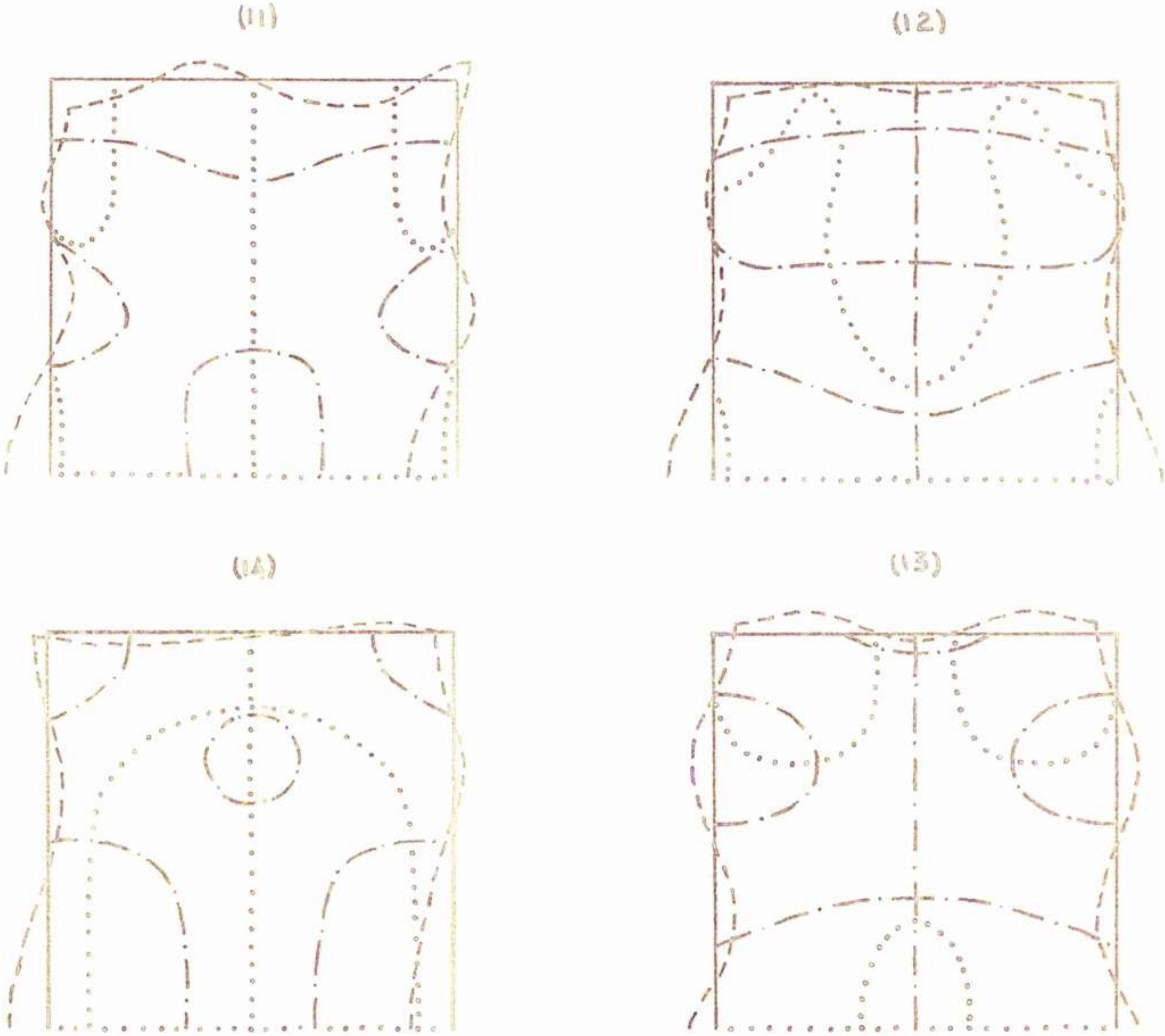


Figure 4.5

respectively, the dotted lines are the lines of zero u, and the dashed and dotted lines are the lines of zero w. Table 4.1 consists of the successive approximations to the frequency and coefficients for the first symmetrical mode.

As a check on the computations the residual stresses have been computed on the surface of the rod. Since the infinite series for the stresses are divergent when $r = a$ and $z = 0$, they exhibit the Gibbs phenomenon at this point, and we cannot use the maximum value of the residual stress over the whole surface as an estimate of the error. Instead we use, for symmetrical solutions, the residual normal stresses at the centre of the two surfaces $r = a$ and $z = 0$, namely the stresses $\sigma_r(a, \frac{\ell}{2})$ and $\sigma_z(0, 0)$, since the points $(a, \frac{\ell}{2})$ and $(0,0)$ are the antinodes for the successive approximations. Table 4.2 gives values of the two residual normal stresses for the successive approximations to the first symmetrical mode.

Table 4.1

order of det	$\frac{Ka}{\pi}$	$A_a(0)$	$D_o(0)$	$A_a(2)$	$D_o(1)$	$A_a(4)$	$D_o(2)$	$A_a(6)$	$D_o(3)$	$A_a(8)$	$D_o(4)$
2	·57157	1	·28537								
4	·56015	1	·28361	−·06089	·04298						
6	·55888	1	·28384	−·07406	·04473	−·01097	−·00772				
8	·55843	1	·28396	−·07963	·04548	−·01331	−·00820	−·00454	·00300		
10	·55822	1	·28403	−·08228	·04585	−·01478	−·00845	−·00533	·00315	−·00248	−·00152
12	·55810	1	·28407	−·08370	·04606	−·01571	−·00859	−·00591	·00323	−·00284	−·00158
14	·55804	1	·28409	−·08453	·04619	−·01631	−·00867	−·00634	·00329	−·00312	−·00162

Table 4.2

order of determinant	2	4	6	8	10
$\dfrac{\lvert\sigma_r(a,\frac{\ell}{2})\rvert}{\max\lvert\sigma_r(r,z)\rvert} \times 100$	4.1	2.9	1.9	1.4	1.2
$\dfrac{\lvert\sigma_z(0,0)\rvert}{\max\lvert\sigma_z(r,z)\rvert} \times 100$	51	24	20	17	15

Table 4.3

computed frequency	experimental value	Pochhammer value
1.45	1.48	1.62
2.01	2.02	1.98
2.88	2.88 (2.95)	2.27

In table 4.3 a comparison is given for the lowest three frequencies of vibration obtained by different methods. The first column has the values taken from figure 4.2, those in the second column are taken from McMAHON[1964], and those in the third column from McNIVEN & PERRY [1962]. The quantity tabulated in table 4.3 is not $\frac{Ka}{\pi}$ but $\frac{pa}{c}$, where p is the angular frequency and c is the speed of sound in the rod. For a material of density ρ we have $c = \sqrt{\frac{E}{\rho}}$, where E is Young's modulus. Hence

$$\left(\frac{pa}{c}\right)^2 = \frac{\lambda + 2\mu}{E}(Ka)^2 .$$

If we use the fact that $E = 2\mu(1+\nu)$ where μ is the shear modulus and ν is Poission's ratio, then

$$\frac{E}{\lambda + 2\mu} = \frac{(1+\nu)(1-2\nu)}{1-\nu} .$$

Taking $\nu = 0.29$, the value for steel, gives

$$\frac{pa}{c} = 3.60 \frac{Ka}{\pi} .$$

The third experimental value in table 4.3 is an estimate obtained by extrapolating an experimental curve, and is of uncertain accuracy. The adjacent figure in brackets is the corresponding value for an aluminium rod, and we observe that frequencies for steel are generally lower than those for aluminium.

39. Discussion

Table 4.1 indicates that the sequence of approximate frequencies is converging rapidly, and we would accept the last value quoted as being correct to within one or two units of the fourth decimal place. The accuracy of the coefficients is not so certain, but the discrepancies are not likely to be more than one or two units in the third decimal place for the A's and the fourth for the D's.

The values of the residual stresses in table 4.2 show that these stresses are decreasing monotonically, and so confirm the numerical solution as being an approximate solution of the problem. The two sequenc es in table 4.2 appear to be converging at approximately the same rate. Although the rate of convergence is rather slow, it is reasonable to assume that the residual stress would be arbitrarily small by taking a sufficient number of terms in the solution.

The experimental frequencies in table 4.3 are in agreement with the computed values. There is no indication given of the magnitude of the experimental error, but it is unlikely to be less than 2 per cent. If we accept this figure, then the two lowest pairs of frequencies agree to within the experimental error. We cannot properly compare the third pair, because of the uncertainty in the experimental value.

If we assume that the computed frequencies are correct, then the errors in the Pochhammer values are 12, 1.5 and 21 per cent respectively. Such errors are not altogether unexpected, in view of the fact that the Pochhammer solution has been obtained by truncating the eigenfunction series expansion after the first three terms and approximating the resulting solution by a simpler expression. Very little is known about the rate of convergence of the eigenfunction expansion, as no systematic computations using it have been carried out. However the results in JOHNSON & LITTLE [1965], where a similar eigenfunction expansion is used to solve a static problem for the semi-infinite elastic strip, show that the boundary conditions are satisfied to within 63 per cent with ten branches of the frequency spectrum and to within 21 percent with twenty branches. These percentage errors are obtained by dividing the maximum difference between the true boundary condition and the computed boundary condition by the maximum value of the true boundary condition, and multiplying the ratio by 100 percent .

The sketches in figures 4.3, 4.4, and 4.5 give some information about the various modes of vibration, but it is necessary to refer to the computed displacement field in order to identify the type of motion involved in each case.

It is clear that mode (2) is almost purely dilatational, characterised by the fact that along a normal to any surface the normal component of displacement varies trigonometrically while the tangential component is constant. We find that the computed displacements satisfy these criteria to a good degree of approximation.

It is known that Rayleigh surface waves can travel along the surface of a semi-infinite elastic region, and we expect them to exist also in a finite region. For such waves the amplitude of the motion decays with depth. In particular the normal component of displacement increases to a maximum and then decreases exponentially, while the tangential component decreases and changes sign before passing through a turning point and tending to zero. A graph of the displacements for steel is given on page 22 of KOLSKY [1953] . We see there that the tangential component changes sign at a depth $d \doteqdot 0.2\Lambda$, where Λ is the wavelength of the plane surface wave.

A closer inspection of the displacements in mode (2) indicates that the main dilatation wave is coupled with a surface wave, and the resultant motion is a superposition of the two.

Mode (3) has the typical features of a surface wave , and we observe that parallel to each surface is a surface at which the tangential displacement changes sign. The wave length in the radial direction appears to be 2a which, for a single wave, gives the depth d $\fallingdotseq$ 0.4a. Since however there are surface waves on each end, there is interaction between the two, and we can show that the depth d is modified to approximately .25a, agreeing with the diagram.

The other symmetrical modes, namely (5), (7), (9), (12) and (13) are predominantly dilatational in character, although in each case the dilatation wave is coupled with a surface wave and with a distortion wave. It is difficult to separate out the various effects for the higher modes, but the dilatation wave in modes (2) and (5) is estimated to have wavelengths 4a and 2a respectively in the radial direction and 2a in both cases in the axial direction.

An inspection of the displacements for the skew-symmetrical modes leads to the conclusion that they are predominantly distortion modes. This is most clearly seen for mode (1) where the shear surface occurs at a depth of .25a approximately.

We find also in modes (4), (8) and (10) that the displacement of the surfaces is mainly tangential, and that the distorted cross-section is almost identical with the undistorted section, the areas being very nearly equal. Again the distortion waves are coupled with surface waves and with dilatation waves, and an esti mate of their wave length is not easy. Modes (1) and (4) seem to have wavelengths 4a and $\frac{4}{3}$a respectively radially, and 2a and a respectively axially.

An interesting question arises in connection with the node patterns in figures 4.3, 4.4 and 4.5. We find that, for simpler eigenvalue problems which can be solved explicitly, the node patterns consist of intersecting families of nodes, with each family associated with a certain direction in space. For example the rectangular membrane has one family parallel to each side of the rectangle. In such cases, where the problem is solvable explicitly, the variables are separable, and the question arises as to whether the node pattern will be of the same type when the variables are not separable. In our present problem we can detect a family of lines of zero w parallel to the curved surface, but there is no indication of such a family parallel to the ends, apart from the single line in the symmetrical modes. The same is true for lines of zero u. Indeed for the higher modes the lines appear to join together to form a single family of contours rather than distinct intersecting families. Thus the evidence seems to point to the conclusion that sy stems of intersecting families of nodes occur only in separable problems. However in one of the higher modes which we have computed, but which is not presented here because of doubts about its accuracy, there are indications of the presence of lines of zero w parallel to the ends of the cross-section . Thus the answer to our question must await a more extensive evaluation of the eigensystem.

40. A Mixed Boundary-value Problem

As a second example of the use of the basic solution for a cylindrical region, we consider a rod with a free curved surface, one of whose ends is free and the other rigidly fixed ("encastré"). This problem differs from those studied previously in that the displacements are specified over a part of the boundary and the stresses over the other part. As far as the writer is aware no previous estimates of the natural frequencies of vibration of this system have been obtained, either experimentally or theoretically, although VALOV [1962] has considered the static deflection of a rod under similar boundary conditions.

We take the boundary conditions to be

$$\sigma_r = \tau_{rz} = 0, \quad r = a, \quad 0 \leq z \leq \ell \quad ,$$
$$u = w = 0, \quad 0 \leq r \leq a, \quad z = 0,$$
$$\sigma_z = \tau_{rz} = 0, \quad 0 \leq r \leq a, \quad z = \ell \; .$$

41. Transformed Boundary Conditions

By the argument of section 33 the conditions $\tau_{rz}(a,z) = 0$, $w(r,0) = 0$ and $\tau_{rz}(r,\ell) = 0$ are transformed into the relations

$$B_a(m) = \frac{m\pi}{\ell} A_a(m), \tag{4.18}$$

$$D_o(i) = 0 \quad , \tag{4.19}$$

$$C_\ell(i) = \xi_1 D_\ell(i) \tag{4.20}$$

respectively. The other three boundary conditions are ineffective when transformed, and so they are ignored at this stage.

42. The Solution

If we eliminate $B_a(m)$, $D_o(i)$ and $C_\ell(i)$ from the basic solution in sections 30 and 31 using (4.18), (4.19) and (4.20), we obtain a solution of the equations of motion satisfying the three boundary conditions stated above. This solution is given by

$$\frac{k^2}{2} u(r,z) = \frac{2}{\ell} \sum_m \varepsilon_m \varphi_1\left[r; z; \frac{m\pi}{\ell}; \frac{k^2}{2} - \frac{m^2\pi^2}{\ell^2}; m\right] . A_a(m)$$

$$- \frac{2}{a^2} \sum_{i=0}^{\infty} \varphi_2\left[r; z; \xi_i; \frac{k^2}{2} - \xi_i^2; \cosh; \sinh \ell\right] . D_\ell(i)$$

$$+ \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{1}{2} \varphi_2\left[r; \ell - z; 1; -\xi_i; \cosh; \sinh \ell\right] . C_o(i)$$

$$\frac{k^2}{2}\, w(r,z) = -\frac{2}{\ell}\sum_m \varepsilon_m\, \varphi_3\left[r;z;\frac{m\pi}{\ell};-\left(\frac{k^2}{2}-\frac{m^2\pi^2}{\ell^2}\right);m\right].\,A_a(m)$$

$$+\frac{2}{a^2}\sum_{i=0}^{\infty} \varphi_4\left[r;z;\xi_i;\frac{k^2}{2}-\xi_i^2;\sinh;\sinh\ell\right].\,D_\ell(i)$$

$$+\frac{2}{a^2}\sum_{i=0}^{\infty} \frac{1}{2}\,\varphi_4\left[r;\ell-z;1;-\xi_i;\sinh;\sinh\ell\right].\,C_o(i)$$

$$\frac{k^2}{4\mu}\,\sigma_r(r,z) = \frac{2}{\ell}\sum_m \varepsilon_m\, \varphi_5\left[r;z;\frac{m^2\pi^2}{\ell^2};\frac{k^2}{2}-\frac{m^2\pi^2}{\ell^2};m\right].\,A_a(m)$$

$$-\frac{2}{a^2}\sum_{i=0}^{\infty} \varphi_6\left[r;z;-\xi_i;\frac{k^2}{2}-\xi_i^2;\cosh;\sinh\ell\right].\,D_\ell(i)$$

$$+\frac{2}{a^2}\sum_{i=1}^{\infty} \frac{1}{2}\,\varphi_6\left[r;\ell-z;-1;-\xi_i;\cosh;\sinh\ell\right].\,C_o(i)$$

$$\frac{k^2}{4\mu}\,\sigma_z(r,z) = \frac{2}{\ell}\sum_m \varepsilon_m\, \varphi_7\left[r;z;-\frac{m\pi}{\ell};\frac{k^2}{2}-\frac{m^2\pi^2}{\ell^2};m\right].\,A_a(m)$$

$$+\frac{2}{a^2}\sum_{i=0}^{\infty} \varphi_8\left[r;z;\xi_i;-\left(\frac{k^2}{2}-\xi_i^2\right);\cosh;\sinh\ell\right].\,D_\ell(i)$$

$$-\frac{2}{a^2}\sum_{i=1}^{\infty} \frac{1}{2}\,\varphi_8\left[r;\ell-z;1;\xi_i;\cosh;\sinh\ell\right].\,C_o(i)$$

$$\frac{k^2}{4\mu}\,\tau_{rz}(r,z) = \frac{2}{\ell}\sum_m \varepsilon_m\, \varphi_9\left[r;z;\frac{m\pi}{\ell};-\left(\frac{k^2}{2}-\frac{m^2\pi^2}{\ell^2}\right);m\right].\,A_a(m)$$

$$+\frac{2}{a^2}\sum_{i=0}^{\infty} \varphi_{10}\left[r;z;\xi_i;-\left(\frac{k^2}{2}-\xi_i^2\right);\sinh;\sinh\ell\right].\,D_\ell(i)$$

$$+\frac{2}{a^2}\sum_{i=1}^{\infty} \frac{1}{2}\,\varphi_{10}\left[r;\ell-z;1;\xi_i;\sinh;\sinh\ell\right].\,C_o(i)$$

43. The Remaining Boundary Conditions

The remaining bou ndary conditions are satisfied by imposing on the above solutions the conditions

$$\int_0^{\ell} \sigma_r(a,z) \cos \frac{m\pi z}{\ell}\, dz = 0 , \qquad m = 0, 1, 2, 3, \ldots ,$$

$$\int_0^{a} r\, J_1(\xi_i r) u(r,0)\, dr = 0 , \qquad i, = 1, 2, 3, \ldots ,$$

$$\int_0^{a} r\, J_0(\xi_i r) \sigma_z (r,\ell)\, dr = 0 , \qquad i = 0, 1, 2, 3, \ldots ,$$

which lead to the infinite set of equations

$$\varphi_5\left[a;0;\frac{m^2\pi^2}{\ell^2};\frac{k^2}{2}-\frac{m^2\pi^2}{\ell^2};m\right].A_a(m)$$

$$-\frac{2}{a^2}\cos m\pi \sum_{i=0}^{\infty} \varphi_{13}\left[(1-2c_1)\frac{k^2}{4}-(1-c_1)\frac{m^2\pi^2}{\ell^2};(1-2c_1)\frac{k^2}{4}\left(\frac{m^2\pi^2}{\ell^2}-k^2\right)\right].D_\ell(i)$$

$$-\frac{2}{a^2}\frac{1}{2}\sum_{i=1}^{\infty} \frac{\xi_i}{2}\varphi_{13}\left[1;(3-2c_1)\frac{m^2\pi^2}{\ell^2}-k^2\right].C_0(i) = 0 , \qquad m=0,1,2,3,\ldots \qquad (4.21)$$

$$\frac{2}{\ell}\sum_m \varepsilon_m \varphi_{14}\left[(3-2c_1);\gamma_1^2\right].A_a(m) + \frac{1}{\xi_i}\varphi_8\left[a;0;1;\xi_i;\cosh;\sinh \ell\right].D_\ell(i)$$

$$-\frac{1}{2\xi_i}\varphi_8\left[a;\ell;\frac{1}{\xi_i};-\frac{\xi_i^2}{\left(\frac{k^2}{2}-\xi_i^2\right)};\cosh;\sinh \ell\right].C_0(i) = 0 , \qquad i=1,2,3,\ldots \qquad (4.22)$$

$$\frac{2}{\ell}\sum_{m} \varepsilon_m \varphi_{14}\left[(1-c_1)\xi_i^2-(1-2c_1)\frac{k^2}{4};-(1-2c_1)\frac{k^2}{4}\left(\xi_i^2-k^2\right)\right]\cos m\pi . A_2(m)$$

$$+\varphi_8\left[a;\ell;\xi_i;-\left(\frac{k^2}{2}-\xi_i^2\right);\cosh;\sinh \ell\right].D_\ell(i)$$

$$-\frac{1}{2}\varphi_8\left[a;0;1;\xi_i;\cosh;\sinh \ell\right].C_0(i)=0 \quad , \quad i=0,1,2,3,\ldots \tag{4.23}$$

where $$\varphi_{13}\left[p_1;p_2\right]=k^2\,\frac{(p_1\xi_i^2+p_2)}{(\xi_i^2-\alpha_m^2)(\xi_i^2-\beta_m^2)}\,\frac{1}{J_0(\xi_i a)}$$

$$\varphi_{14}\left[p_1;p_2\right]=a k^2\,\frac{\left(p_1\frac{m^2\pi^2}{\ell^2}+p_2\right)}{\left(\gamma_i^2+\frac{m^2\pi^2}{\ell^2}\right)\left(\delta_i^2+\frac{m^2\pi^2}{\ell^2}\right)} \quad .$$

44. The Infinite System

If we choose as the unknowns the quantities

$$\frac{m^2\pi^2}{\ell^2} \cos m\pi . A_a(m) \quad ,$$

$$\frac{\xi_i^2}{J_0(\xi_i a)} \quad D_\ell(i) \quad ,$$

$$\frac{\xi_i}{J_0(\xi_i a)} \quad C_0(i) \; ,$$

then equations (4.21) and (4.23) are of the same form as the equations in section 36, and satisfy the same conditions. Equation (4.22), however, is of an altogether different form, and we find that the ratio which must be less than unity for regularity has the asymptotic form

$$\frac{\frac{2}{3}\, a\ell k^2 \frac{1}{\xi_i^2} + O\left(\frac{1}{\xi_i^3}\right)}{(1 + c_1)\frac{k^2}{4}\frac{1}{\xi_i^3} + O\left(\frac{1}{\xi_i^5}\right)} \quad .$$

This ratio tends to infinity as $i \to \infty$

The feature which distinguishes equation (4.22) from all the other equations studied so far is that it involves an alternating series, whereas the other series are ultimately positive or negative. The example in section 16 illustrates that such a system of equations need not satisfy the condition of regularity in order to be soluble by the method of reduction.

Thus the infinite set of equations is not included in the existing theory of solution, and we cannot give a mathematical justification of the numerical solution in section 46. However the existence and uniqueness of the solution of the physical problem, and the completeness properties of the original expansions of the solution provide some justification that the computed solution is a valid solution of the problem.

45 Computational Details.

The computational problem is the same as before, except that the determinant of coefficients is different. The unknown constants in the equations are introduced in the order $A_a(0)$, $D_\ell(0)$, $C_o(1)$, $A_a(1)$, $D_\ell(1)$, $C_o(2)$, $A_a(2)$, $D_\ell(2)$, .., and successive determinants of orders 5, 8, 11, 14, .. are used in applying the method of reduction.

46. Results and Discussion.

Results have been obtained for $\nu = 0{\cdot}29$ and $e = 1$. Table 4.4 has the successive approximations to the first mode of vibration, and figure 4.6 has the first seven frequencies in the spectrum.

Fewer modes were computed for this problem because of the slower rate of convergence of the infinite series. In the previous problem symmetry properties enabled us to reduce the number of infinite sets of unknowns to two, and successive approximations to the solution, using each time an additional unknown from each set, involved determinants of orders 2, 4, 6, 8, 10, ... For the clamped-free rod there are three sets of unknowns, and successive determinants have orders 2, 5,,8, 11, ..., so that, for a fixed number of terms, the solution for the free-free rod is more accurate than that for the clamped-free rod. The successive frequencies in table 4.4 converge more slowly than the corresponding values in table 4.1, and the coefficients also decay

Table 4.4

order	$\frac{Ka}{\pi}$	$A_a(0)$	$D_\ell(0)$	$C_o(1)$	$A_a(1)$	$D_\ell(1)$	$C_o(2)$	$A_a(2)$	$D_\ell(2)$	$C_o(3)$	$A_a(3)$	$D_\ell(3)$
5	·43497	−·69599	1	−·42893	·00594	−·00614						
8	·43725	−·67188	1	−·35000	·01893	·00677	·22052	·08768	−·00480			
11	·43865	−·66460	1	−·31560	·03104	·00524	·18504	·09522	−·00390	−·14250	·02187	·00150
14	·43901	−·66032	1	−·30418	·03173	·00748	·16593	·10177	−·00494	−·12161	·02337	·00215

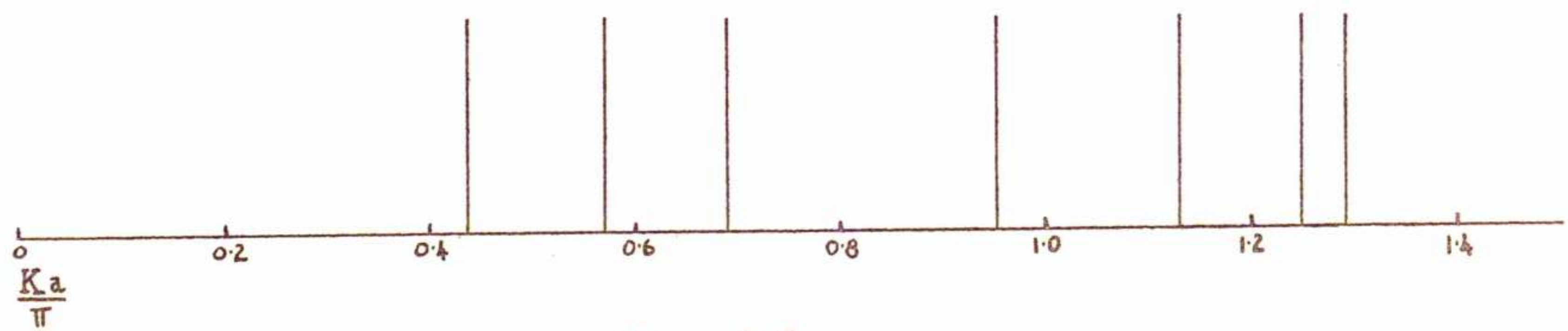


Figure 4.6

less rapidly.

An aspect of this problem which has not been investigated here is the stress singularity at the edge of the clamped end. Sternberg refers to the existence of such a singularity in his review of VALOV [1962], and points out that a different form of the solution must be developed in the neighbourhood of the singularity. A similar situation is investigated in KARP & KARAL [1962], [1964], where the nature of the static solution at corners in elastic media is considered. The form of the solution taken there may be adaptable to the present problem.

More work remains to be done on this problem, in particular to extend the spectrum of eigenvalues and to assess the accuracy of those given in figure 4.3, to study the theoretical problem of the convergence of the method of reduction for alternating series, and to investigate the solution in the neighbourhood of the singularity.

Chapter V: An Initial-value Problem

47. The Problem

The problem which is the subject of this chapter is illustrated in figure 5.1

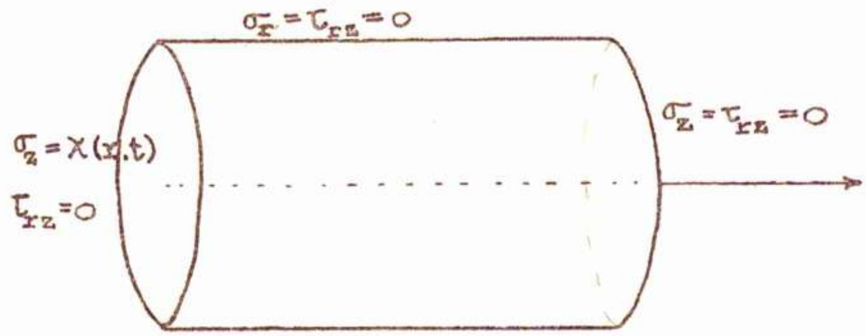


Figure 5.1

A circular rod of radius a and length ℓ, at rest initially in an unstressed state, is set in motion by a normal axially-symmetric, pressure pulse

$$\sigma_z = X(r,t)$$

applied suddenly to the end $z = 0$. The other surfaces are assumed to remain stress-free, and there is no body force. We derive a formal solution of the problem as an eigenfunction expansion when X is an arbitrary function of r and t, and compute some values for a particular case.

This investigation is motivated by an experiment carried out by KOLSKY [1954], in which a small piece of lead azide is detonated at the centre of one face of a steel rod, the motion of the other end being measured by a condenser microphone. The rod is 10·4 cm. long and has a radius of 7·6 cm; the microphone is 1 cm in diameter. The detonation time is estimated at 2 - 3 μsec, and $\chi(r,t)$ has the approximate form

$$\chi(r,t) = -P\,\delta(r)\left[H(t) - H(t-T)\right],$$

where P is a constant, $\delta(r)$ is the Dirac delta function, $H(t)$ is the Heaviside function, and T is the detonation time. Kolsky gives a qualitative interpretation of the experimental results on the basis of the optical ray theory, but no accurate quantitative solution has been obtained.

It can be shown that the above form of the function $\chi(r,t)$ corresponds to a very slowly convergent series of eigenfunctions, in which a large number of accurately known eigenfunctions are necessary for an accurate calculation. Consequently we take a slightly simpler form to obtain better convergence, and we hope to be able to compute the solution of the more difficult problem when a fuller set of eigenfunctions is available.

48. The Governing Equations

The equations of motion have the form

$$L_1 \underset{\sim}{s} = \rho \frac{\partial^2 \underset{\sim}{s}}{\partial t^2} \tag{5.1}$$

in matrix notation, where $L_1 = L_D + \rho I \frac{\partial^2}{\partial t^2}$ and L_D is defined in section 5, $\underset{\sim}{S}$ is the displacement column vector, and ρ is the density. For axially symmetric motion we have $\underset{\sim}{S} = \begin{bmatrix} u \\ w \end{bmatrix}$

and

$$L_1 = \begin{bmatrix} (\lambda+2\mu)\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \mu\frac{\partial^2}{\partial z^2} & (\lambda+\mu)\frac{\partial^2}{\partial r\partial z} \\ (\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial z} & \mu\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + (\lambda+2\mu)\frac{\partial^2}{\partial z^2} \end{bmatrix}$$

Since the initial disturbance of the rod is axially-symmetric, then the subsequent motion is axially-symmetric.

The boundary conditions are

$$\sigma_r = \tau_{rz} = 0, \quad r = a, \quad 0 \leq z \leq \ell,$$

$$\sigma_z = X(r,t) \text{ and } \tau_{rz} = 0, \quad 0 \leq r \leq a, \quad z = 0,$$

$$\sigma_z = \tau_{rz} = 0, \quad 0 \leq r \leq a, \quad z = \ell,$$

for all time $t > 0$, and the initial conditions at $t = 0$ are

$$u = w = 0, \quad \frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} = 0,$$

for $0 \leq r \leq a$ and $0 \leq z \leq \ell$.

49. The System of Eigenfunctions

We take as the eigenfunctions the solutions of the problem of the free-free vibrating rod in chapter IV, i.e. the set $\{\underset{\sim}{s}_i\}$ satisfying the equation

$$L_1 \underset{\sim}{s}_i = -\rho p_i^2 \underset{\sim}{s}_i \tag{5.2}$$

and the boundary conditions

$$\sigma_r = \tau_{rz} = 0, \quad r = a, \quad 0 \leqq z \leqq \ell,$$
$$\sigma_z = \tau_{rz} = 0, \quad 0 \leqq r \leqq a, \quad z = 0,$$
$$\sigma_z = \tau_{rz} = 0, \quad 0 \leqq r \leqq a, \quad z = \ell.$$

In this set $\underset{\sim}{s}_0$ denotes the eigenfunction with p = 0. It corresponds to a rigid displacement of the rod, for which $\underset{\sim}{s}_0 = \begin{bmatrix} 0 \\ C \end{bmatrix}$ where C is a constant.

It can be shown that the eigenfunctions are orthogonal in the sense that

$$\int_R \underset{\sim}{s}_i^T \underset{\sim}{s}_j \, d\tau = 0$$

if $i \neq j$. For we have

$$\underset{\sim}{s}_i^T L_1^* = -\rho p_i^2 \underset{\sim}{s}_i^T$$

and

$$L_1 \underset{\sim}{s}_j = -\rho p_j^2 \underset{\sim}{s}_j$$

from which it follows, by combining these results together and integrating, that

$$-\rho(p_i^2 - p_j^2)\int_R \underset{\sim}{s}_i^T \underset{\sim}{s}_j \, d\tau = \int_R \left(\underset{\sim}{s}_i^T L_1^* \underset{\sim}{s}_j - \underset{\sim}{s}_i^T L_1 \underset{\sim}{s}_j\right) d\tau.$$

An argument similar to the one used in section 5 gives the result

$$\int_R \left(\underset{\sim}{s}_i^T L_1^* \underset{\sim}{s}_j - \underset{\sim}{s}_i^T L_1 \underset{\sim}{s}_j \right) d\tau$$

$$= \int_S \left[\mu \left(\underset{\sim}{s}_{in}^T C \underset{\sim}{s}_j - \underset{\sim}{s}_i^T C^T \overset{*}{\underset{\sim}{s}}_{jn} \right) - (\lambda + 2\mu) \left(\underset{\sim}{s}_i^T \underset{\sim}{n} \underset{\sim}{d}^T \underset{\sim}{s}_j - \underset{\sim}{s}_i^T \underset{\sim}{d}^* \underset{\sim}{n}^T \underset{\sim}{s}_j \right) \right] dS ,$$

and if we substitute the boundary conditions on $\underset{\sim}{s}_i$ and $\underset{\sim}{s}_j$ we find that the surface integral vanishes. Hence

$$\int_R \underset{\sim}{s}_i^T \underset{\sim}{s}_j \, d\tau = 0$$

if $i \neq j$, and the eigenfunctions are orthogonal.

To obtain an orthonormal system each eigenfunction $\underset{\sim}{s}_i$ must be scaled by a constant c_i so that

$$\frac{1}{c_i^2} \int_R \underset{\sim}{s}_i^T \underset{\sim}{s}_i \, d\tau = 1 \quad .$$

Hence c_i is given by

$$c_i^2 = \int_R \underset{\sim}{s}_i^T \underset{\sim}{s}_i \, d\tau .$$

The eigenfunctions $\underset{\sim}{s}_i$ in chapter IV are in the form of infinite series, and the formal expression for c_i^2 in terms of these series is given in detail below, since it is used in constructing a computer programme to calculate c_i^2 . However in computing c_i^2 only a finite number of terms of the series is used, since only a finite number of coefficients is known for each approximation.

$$\frac{k^4}{4}\int_0^a\int_0^\ell r u^2\,dr\,dz$$

$$= \frac{2}{\ell}\sum_s \varepsilon_s \left[\frac{s^4\pi^4}{\ell^4}\left\{ \frac{a^2}{2}\frac{J_0^2(\alpha_s a)}{J_1^2(\alpha_s a)} - \frac{a}{\alpha_s}\frac{J_0(\alpha_s a)}{J_1(\alpha_s a)} + \frac{a^2}{2} \right\} + \left(\frac{k^2}{2} - \frac{s^2\pi^2}{\ell^2}\right)^2 \left\{ \frac{a^2}{2}\frac{J_0^2(\beta_s a)}{J_1^2(\beta_s a)} - \frac{a}{\beta_s}\frac{J_0(\beta_s a)}{J_1(\beta_s a)} + \frac{a^2}{2} \right\} \right.$$

$$\left. + \frac{2}{(1-c_1)k^2}\frac{s^2\pi^2}{\ell^2}\left(\frac{k^2}{2} - \frac{s^2\pi^2}{\ell^2}\right)\left\{ \beta_s a\frac{J_0(\beta_s a)}{J_1(\beta_s a)} - \alpha_s a\frac{J_0(\alpha_s a)}{J_1(\alpha_s a)} \right\} \right] \cdot A_a^2(s)$$

$$+ \frac{2}{a^2}\sum_i \frac{\xi_i^2}{J_0^2(\xi_i a)} \left[\frac{\gamma_i^2}{2f^2\left(\frac{\gamma_i \ell}{2}\right)}\left(\frac{\sinh \gamma_i \ell}{\gamma_i} \pm \ell\right) + \frac{\left(k^2/2 - \xi_i^2\right)^2}{2\delta_i^2 f^2\left(\frac{\delta_i \ell}{2}\right)}\left(\frac{\sinh \delta_i \ell}{\delta_i} \pm \ell\right) \right.$$

$$\left. - \frac{4\gamma_i\left(\frac{k^2}{2} - \gamma_i^2\right)}{(1-c_1)k^2\delta_i}\left\{ \gamma_i g\left(\frac{\delta_i \ell}{2}\right) - \delta_i g\left(\frac{\gamma_i \ell}{2}\right) \right\} \right] \cdot D_0^2(i)$$

$$+ \frac{4k^4}{a\ell}\sum_i \frac{\xi_i^2}{J_0(\xi_i a)} D_0(i) \sum_s \varepsilon_s \frac{\left[\frac{s^2\pi^2}{\ell^2} - (1-2c_1)\gamma_i^2\right]\left[(3-2c_1)\frac{s^2\pi^2}{\ell^2} + \gamma_i^2\right]}{\left(\gamma_i^2 + \frac{s^2\pi^2}{\ell^2}\right)^2\left(\delta_i^2 + \frac{s^2\pi^2}{\ell^2}\right)^2} \cdot A_a(s)$$

$$\frac{k^4}{4}\int_0^a\int_0^\ell r w^2\,dr\,dz$$

$$= \frac{2}{\ell}\sum_s \varepsilon_s \frac{s^2\pi^2}{\ell^2}\left[\alpha_s^2\frac{a^2}{2}\left\{ \frac{J_0^2(\alpha_s a)}{J_1^2(\alpha_s a)} + 1 \right\} + \left(\frac{k^2}{2} - \frac{s^2\pi^2}{\ell^2}\right)^2 \frac{a^2}{2\beta_s^2}\left\{ \frac{J_0^2(\beta_s a)}{J_1^2(\beta_s a)} + 1 \right\} \right.$$

$$\left. - \frac{2\alpha_s\left(\frac{k^2}{2} - \frac{s^2\pi^2}{\ell^2}\right)}{(1-c_1)k^2\beta_s}\left\{ \alpha_s a\frac{J_0(\beta_s a)}{J_1(\beta_s a)} - \beta_s a\frac{J_0(\alpha_s a)}{J_1(\alpha_s a)} \right\} \right] \cdot A_a^2(s)$$

$$+ \frac{2}{a^2}\sum_i \frac{1}{J_0^2(\xi_i a)} \left[\frac{\xi_i^4}{2f^2\left(\frac{\gamma_i \ell}{2}\right)}\left(\frac{\sinh \gamma_i \ell}{\gamma_i} \mp \ell\right) + \frac{\left(\frac{k^2}{2} - \xi_i^2\right)^2}{2f^2\left(\frac{\delta_i \ell}{2}\right)}\left(\frac{\sinh \delta_i \ell}{\delta_i} \mp \ell\right) \right.$$

$$\left. - \frac{4\xi_i^2\left(\frac{k^2}{2} - \xi_i^2\right)}{(1-c_1)k^2}\left\{ \gamma_i g\left(\frac{\gamma_i \ell}{2}\right) - \delta_i g\left(\frac{\delta_i \ell}{2}\right) \right\} \right] \cdot D_0^2(i)$$

$$- \frac{4k^4}{a\ell}\sum_i \frac{1}{J_0(\xi_i a)} D_0(i) \sum_s \varepsilon_s \frac{s^2\pi^2}{\ell^2} \frac{\left[\frac{s^2\pi^2}{\ell^2} + (3-2c_1)\xi_i^2 - k^2\right]\left[(1-2c_1)\frac{s^2\pi^2}{\ell^2} - \xi_i^2 - (1-2c_1)k^2\right]}{\left(\gamma_i^2 + \frac{s^2\pi^2}{\ell^2}\right)^2\left(\delta_i^2 + \frac{s^2\pi^2}{\ell^2}\right)^2} \cdot A_a(s)$$

For symmetric modes $s = 0, 2, 4, 6, \ldots, f(x) = \sinh x$, $g(x) = \coth x$, and the upper sign is taken with ℓ, and for skew-symmetric modes $s = 1, 3, 5, \ldots, f(x) = \cosh x$, $g(x) = \tanh x$, and the lower sign applies.

c_i^2 is obtained for each eigenfunction by evaluating

$$\int_0^a \int_0^\ell r(u^2 + w^2)\,dr\,dz .$$

It is not difficult to show that the normalised eigenfunction $\underset{\sim}{s}_0$ is given by

$$\underset{\sim}{s}_0 = \begin{bmatrix} 0 \\ \cdot \\ a\sqrt{2e} \end{bmatrix}, \qquad \text{where } e = \frac{a}{\ell} .$$

50. The Eigenfunction Expansion

Let $\underset{\sim}{s}$ be the solution of the initial value problem and $\{\underset{\sim}{s}_i\}$ the orthonormal set of eigenfunctions. That is, the $\underset{\sim}{s}_i$ are assumed to be scaled so that

$$\int_R \underset{\sim}{s}_i^T \underset{\sim}{s}_j \, d\tau = \delta_{ij} \tag{5.3}$$

We expand $\underset{\sim}{s}$ in terms of the $\underset{\sim}{s}_i$ and write

$$\underset{\sim}{s} = \sum_i a_i(t)\, \underset{\sim}{s}_i \tag{5.4}$$

where $\underset{\sim}{s}$ is a function of the space variables and time, while $\underset{\sim}{s}_i$ is a function of the space variables only. Strictly speaking such an expansion should be justified by proving that the system $\{\underset{\sim}{s}_i\}$ is complete, but we shall assume that this is so. From (5.3) and (5.4) we have

$$a_i(t) = \int_R \underset{\sim}{s}_i^T \underset{\sim}{s}\, d\tau \ .$$

By operating with $\int_R \underset{\sim}{s}_i^T \ . \ . \ . \ d\tau$

on equation (5.1) we obtain

$$\int_R \underset{\sim}{s}_i^T L_1 \underset{\sim}{s}\, d\tau = \rho \int_R \underset{\sim}{s}_i^T \frac{\partial^2 \underset{\sim}{s}}{\partial t^2}\, d\tau$$

$$= \rho \frac{d^2}{dt^2} \int_R \underset{\sim}{s}_i^T \underset{\sim}{s}\, d\tau$$

$$= \rho \frac{d^2 a_i}{dt^2} \quad . \qquad (5.5)$$

Now

$$\int_R \underset{\sim}{s}_i^T L_1 \underset{\sim}{s}\, d\tau = \int_R \underset{\sim}{s}^T L_1 \underset{\sim}{s}_i\, d\tau$$

$$+ \int_S \Big[(\lambda + 2\mu)(\underset{\sim}{s}_i^T \underset{\sim}{n}\, \underset{\sim}{d}^T \underset{\sim}{s} - \underset{\sim}{s}_i^T \underset{\sim}{d}^{*} \underset{\sim}{n}^T \underset{\sim}{s}) - \mu(\underset{\sim}{s}_{in}^T C \underset{\sim}{s} - \underset{\sim}{s}_i^T C^T \overset{*}{\underset{\sim}{s}}_n) \Big] dS \qquad (5.6)$$

and from equation (5.2) we have

$$\int_R \underset{\sim}{s}^T L_1 \underset{\sim}{s}_i\, d\tau = -\rho p_i^2 \int_R \underset{\sim}{s}^T \underset{\sim}{s}_i\, d\tau$$

$$= -\rho p_i^2 a_i \quad .$$

The boundary conditions on $\underset{\sim}{s}$ and $\underset{\sim}{s}_1$ are now used to eliminate certain terms in the surface integral in equation (5.6). When this is done the integral becomes

$$-\frac{1}{a^3}\int_0^a r w_i(r,o)\chi(r,t)dr \quad ,$$

where $w_i(r,z)$ is the axial component of $\underset{\sim}{s}_i$.

Thus equation (5.5) becomes

$$\rho\frac{d^2a_i}{dt^2} = -\rho p_i^2 a_i - \frac{1}{a^3}\int_0^a r w_i(r,o)\chi(r,t)dr$$

i.e. $$\frac{d^2a_i}{dt^2} + p_i^2 a_i = F(t) \quad ,$$

where $$F(t) = -\frac{1}{\rho a^3}\int_0^a r w_i(r,o)\chi(r,t)dr \quad .$$

When the function $\chi(r,t)$ is given, then $F(t)$ is a known function, and equation (5.1) has been reduced to a non-homogeneous linear differential equation for a_i.

The initial conditions on $\underset{\sim}{s}$ become

$$\sum_i a_i \underset{\sim}{s}_i = \underset{\sim}{0} \quad ,$$

$$\sum_i \frac{da_i}{dt}\underset{\sim}{s}_i = \underset{\sim}{0} \, ,$$

from which it follows, by using condition (5.3), that

$$a_i = \frac{da_i}{dt} = 0$$

at t = 0.

The solution of the differential equation satisfying these conditions may be found by the method described in CODDINGTON & LEVINSON [1955] page 74, and is

$$a_i(t) = \frac{1}{p_i}\int_0^t F(t')\sin p_i(t-t')dt'$$

$$= -\frac{1}{\rho p_i a^5}\int_0^t\int_0^a r w_i(r,o)\chi(r,t')\sin p_i(t-t')dr\,dt'$$

for $i \neq 0$. The zero frequency term is found in the same way and can be shown to be

$$a_o(t) = -\frac{1}{\rho a^5}\int_0^t\int_0^a r w_o(r,o)\chi(r,t')\,(t-t')dr\,dt' \quad .$$

If $\chi(r,t)$ is a known function then a_i is known, at least in principle, and we have thus obtained a solution of the initial-value problem.

51. A Particular Case - Step Function Loading

If the applied pressure is in the form of a step-wave in time applied over a circle of radius R, then $\chi(r,t)$ has the form

$$\chi(r,t) = -P\left[H(r) - H(r-R)\right]H(t) \quad .$$

This gives, for $j \neq 0$,

$$a_j(t) = \frac{2P}{\rho p_j^2 a^3} \sin^2 \frac{p_j t}{2} \int_0^R r\, w_j(r,o)\, dr$$

$$= \left(\frac{2P}{\lambda + 2\mu}\right) \frac{1}{(K_j a)^2} \sin^2 \frac{p_j t}{2}\ \frac{1}{a^3} \int_0^R r\, w_j(r,o)\, dr\ ,$$

where K_j is the frequency parameter corresponding to p_j.

From the infinite series for $w(r,z)$ in section 34 we obtain

$$\int_0^R r\, w_j(r,o)\, dr = \left(\frac{R}{a}\right)^2 D_{oj}(0) + 2\,\frac{R}{a^2} \sum_{i=1}^{\infty} \frac{J_1(\xi_i r)}{\xi_i J_0^2(\xi_i a)} D_{oj}(i)\ , \qquad j \neq 0,$$

where $D_{oj}(i)$ are the constants in the series for $\underset{\sim}{s}_j$. We see from this expression that the smaller the ratio $\frac{R}{a}$ becomes the more accurate the evaluation of the infinite series must be, in order to maintain accuracy in the result. Thus any finite computation using a fixed number of terms of the series gives the best accuracy when $R = a$.

When $j = 0$,

$$a_0(t) = \frac{P R^2 t^2 \sqrt{e}}{\rho a^5 2\sqrt{2}} = \left(\frac{2P}{\lambda + 2\mu}\right)\left(\frac{R}{a}\right)^2 \left(\frac{c_d t}{a}\right)^2 \frac{\sqrt{e}}{4\sqrt{2}}\ ,$$

where $\qquad c_d = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} \qquad$ is the dilatational wave speed.

For the special case of $R = a$ and $a = \ell$ the expressions are

$$\int_0^R r\, w_j(r,0)\,dr = D_{0j}(0) \quad ,$$

$$a_0(t) = \left(\frac{2P}{\lambda+2\mu}\right)\left(\frac{c_d t}{a}\right)^2 \frac{1}{4\sqrt{2}} \quad , \tag{5.7}$$

$$a_j(t) = \left(\frac{2P}{\lambda+2\mu}\right)\frac{1}{(K_j a)^2}\sin^2\frac{p_j t}{2}\;\frac{D_{0j}(0)}{a^3} \quad , \; j \neq 0 \quad . \tag{5.8}$$

52. The Solution for an Infinite Slab.

When the initial pressure front arrives from the source we expect the centre region of the free end of the rod to behave like the surface of an infinite slab, at least for a short time, and we have consequently derived the solution for the infinite slab, as a check on the computed solution.

For a slab bounded by the planes $z = 0$ and $z = \ell$ subjected to a step-wave pressure pulse applied uniformly at time $t = 0$ to the face $z = 0$, the subsequent motion is given by

$$w(z,t) = \frac{P}{\lambda+2\mu}\left[\frac{(c_d t)^2}{2\ell} + \frac{2}{\ell}\sum_{n=1}^{\infty}\frac{\ell^2}{n^2\pi^2}\left(1 - \cos\frac{n\pi c_d t}{\ell}\right)\cos\frac{n\pi z}{\ell}\right] ,$$

where $w(z,t)$ is the displacement in the z -direction. In particular at $z = \ell$ there results

$$\left(\frac{\lambda+2\mu}{2P}\right)\frac{w(\ell,t)}{\ell} = \begin{cases} 0 & , \quad 0 < c_d t < \ell \ , \\ \dfrac{c_d t}{\ell} - 1 & , \quad \ell < c_d t < 3\ell \ , \\ 2\left(\dfrac{c_d t}{\ell} - 2\right) & , \quad 3\ell < c_d t < 5\ell \ , \\ 3\left(\dfrac{c_d t}{\ell} - 3\right) & , \quad 5\ell < c_d t < 7\ell \ , \\ \vdots & \end{cases}$$

The graph of this displacement is given in figure 5.2

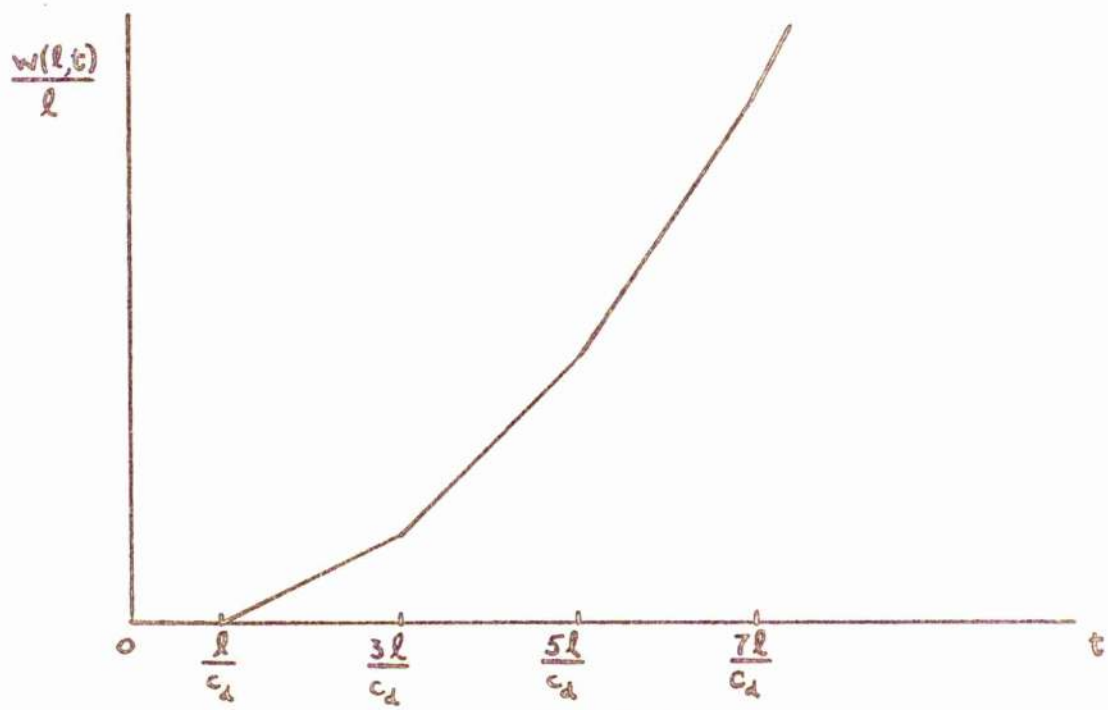


Figure 5.2

The interpretation of this solution is that the wave front propagates into the slab with the dilatational wave speed c_d. The surface $z = \ell$ is motionless until the front arrives, whereupon it moves with uniform speed. The front is reflected and travels back into the material, experiencing another reflection at $z = 0$. After the second reflection the front reinforces the external pressure, doubling its magnitude, and the cycle is repeated. At the beginning of each cycle the current magnitude of the pressure front is incremented by the external pressure at $z = 0$, and results in an increase of the speed of the surface $z = \ell$. This incrementation of the speed is illustrated in figure 5.2 by an increase in the slope of the curve.

53. Computational Details

The first part of the computation is to normalise each of the eigenfunctions, and a computer programme is written to calculate c_1^2, given $\frac{Ka}{\pi}$ and the approximate coeffieients, using the expressions in section 49. The scaled coefficients are printed out, and are used as data for the second programme, which computes the displacement at any point (r, z) in the rod for that mode.

Detailed computations are done for a steel rod of length and radius 10·4 cm. and for the pressure pulse of section 51 with R = a. KOLSLY[1953] gives the following material constants for steel:

$$\lambda = 11.2 \times 10^{10} \text{ newtons}/m^2$$

$$\mu = 8.1 \times 10^{10} \quad " \quad "$$

$$\rho = 7.8 \times 10^3 \text{ kgm}/m^3 .$$

The circular frequency p has the dimensions of T^{-1}, and for the above constants it is given in terms of K by the relation

$$p = .056989 \text{ Ka } (\mu s)^{-1} ,$$

where μs denotes "micro-second".

A further programme uses as data the displacements for the different modes, and calculates the displacement of a specified point in the rod for a range of values of t using the expansion (5.4) and the coefficients (5.7) and (5.8). The quantity $\left(\frac{2P}{\lambda+2\mu} \right)$ is a multiplicative constant in the solution, and is set equal to unity.

54. Results and Discussion

Computations have been done forthe case of R = a. The first sixteen terms of the eigenfunction expansion are used, since only th ese ones are known to any accuracy. With this solution the axial displacement of the free end has been calculated over an interval of time covering several reflections of the initial front, and the results are given below.

Figure 5.3 gives the computed displacement of three different points on the end $z = \ell$ as the initial front arrives. The infinite slab displacement is given for comparison. Figure

5.4 gives the displacement of the point (o, ℓ) together with the infinite slab displacement, for a longer interval of time covering four reflections. Finally figure 5.5 gives the shape of the end section $z = \ell$ at a particular instant in time, $t = 27\mu s$, after the pressure front has arrived.

In figure 5.3 we can take as a measure of the error in each of the computed displacements the maximum deviation from zero of the appropriate curve between $t = 0$ and $t = \dfrac{\ell}{c_d}$.

The error defined in this way is more or less uniform in magnitude over the cross-section from $r = 0$ to $r = a$, and varies from 0.08 units at $r = 0$ to $- 0.07$ units at $r = a$. After $t = \dfrac{\ell}{c_d}$

the curve for $r = 0$ tends towards the straight line, and eventually becomes approximately parallel to it, displaced vertically upwards by about 0.02 units. This deviation is well within the error bound noted above. The other curves also become approximately parallel to the straight line, but their deviations, both of approximately 0.15 units vertically downwards, are outwith the computational error bounds, and represent genuine differences, which are illustrated in a different way in figure 5.5.

On the smaller scale of figure 5.4 we see that the displacement $w(0, \ell)$ follows closely the displacement of the infinite slab over a comparatively long period of time, and that the rod displacement oscillates about the infinite slab displacement. This oscillation is not entirely due to the computational

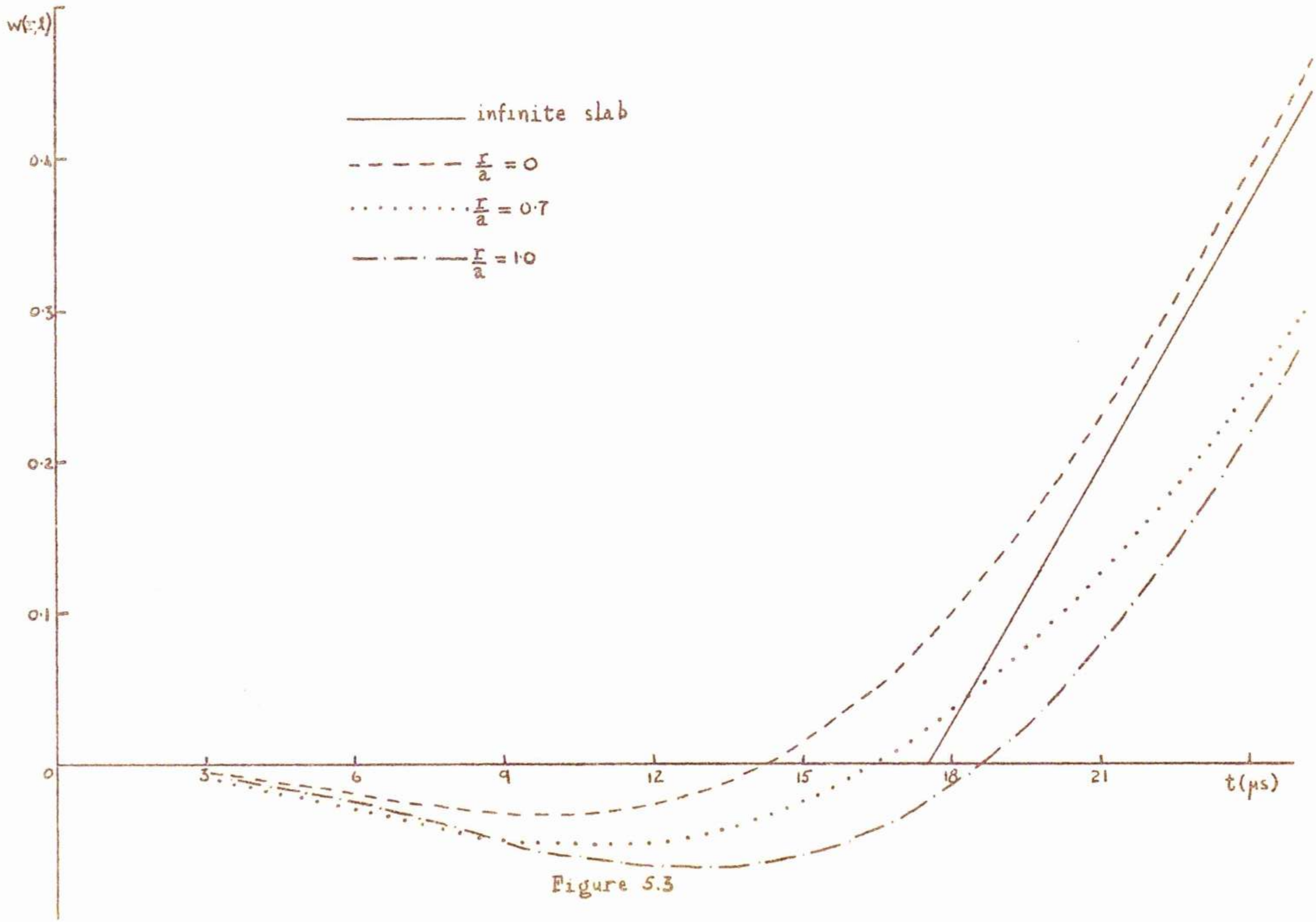


Figure 5.3

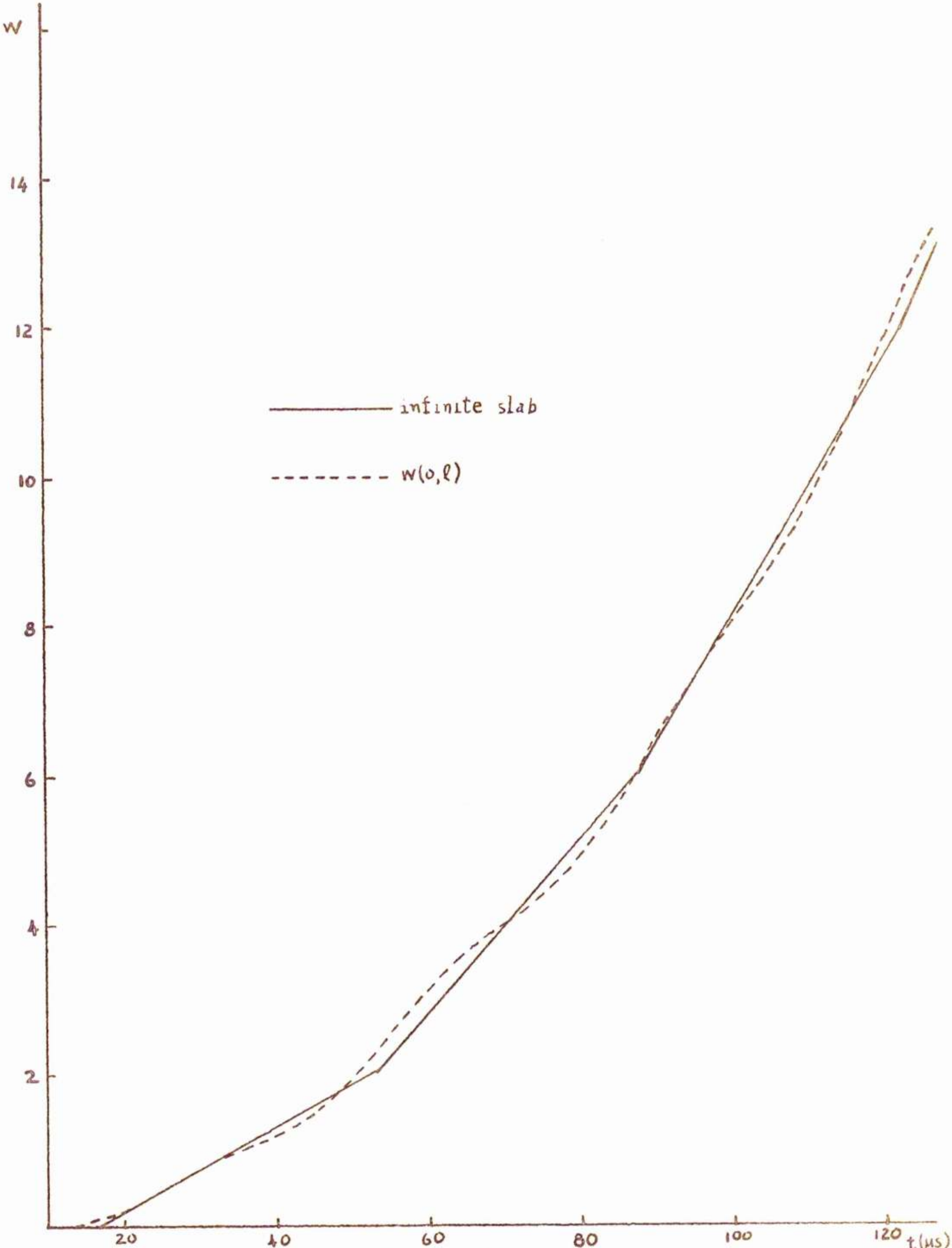


Figure 5.4

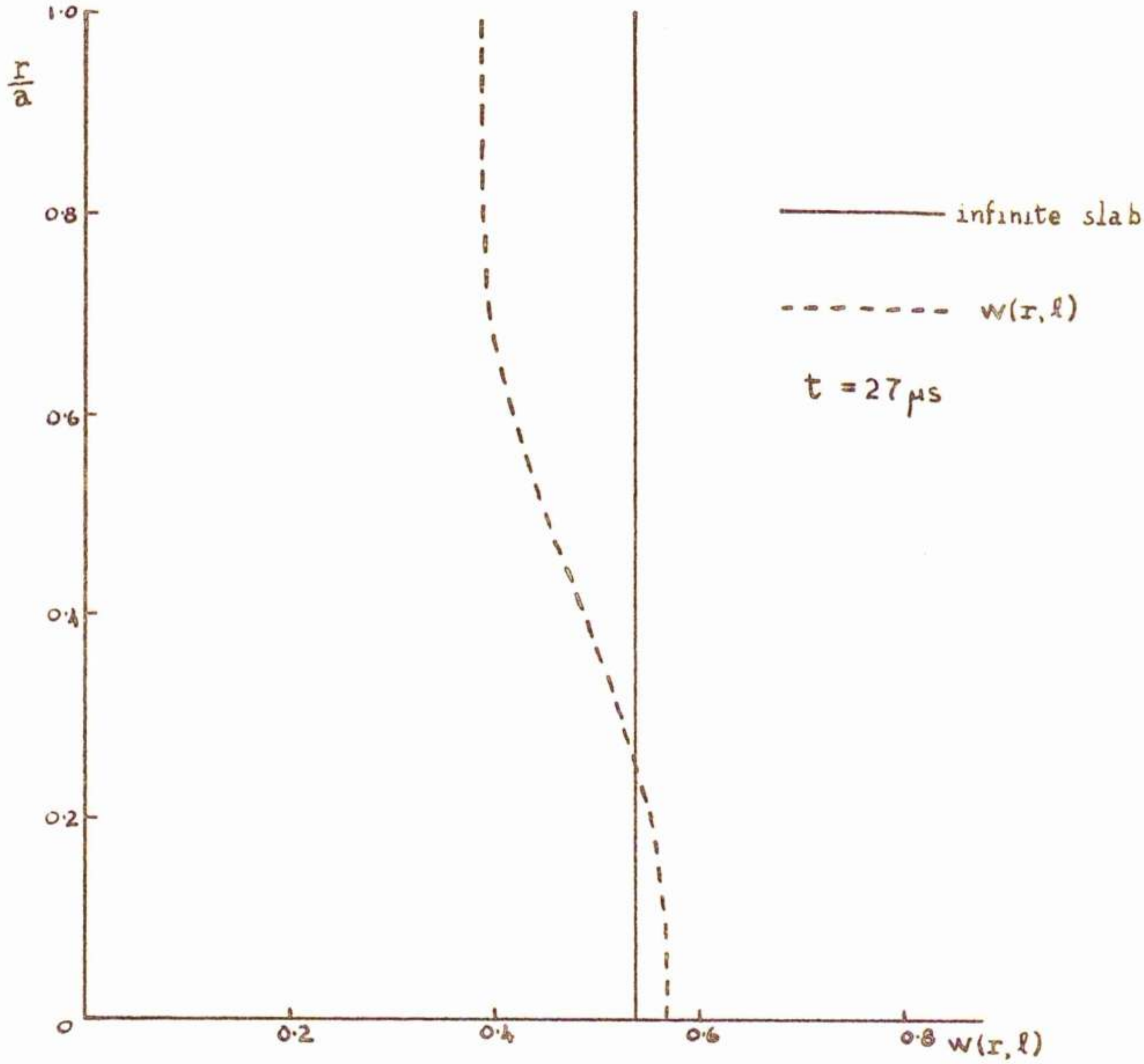


Figure 5.5

error, although a proper interpretation of it cannot be made until more accurate numerical results are available.

As the pressure front travels along the rod it produces a distortion of the material, and the points in the rod are free to move in both the axial and radial directions, in contrast to the situation in the slab where the motion is confined to the direction of propagation. The curved surface undergoes the greatest radial displacement, while the displacement of points near the axis is predominantly axial. Figure 5.5 illustrates this, and shows that the axial motion of the free end diminishes with distance from the axis. The end is no longer plane after the pressure front arrives, but develops a "bulge" at the centre. Another difference between the motion of the rod and the infinite slab is due to the surface wave which propagates along the curved surface after the initial impact. This wave is reflected inwards by the corners and travels along the free end, converging on the centre, where it reinforces the displacement caused by the dilatational front. This reinforcement can be seen in figure 5.4, occurring at time $t \doteqdot 60\,\mu s$. Since the surface wave speed for steel is approximately $\frac{c_d}{2}$, we expect the reinforcement to occur at $t \doteqdot \frac{4\ell}{c_d} \doteqdot 70\,\mu s$.

In theory the results obtained for this problem can be used to give the solution when the applied pressure pulse has the form

$$X(r,t) = -P\left[H(r)-H(r-a)\right]\left[H(t)-H(t-T)\right]$$

i.e. when the pulse is applied for a finite time T. However for a pulse of very short duration the numerical inaccuracies render the resulti ng solution meaningless. This is seen byconsidering the coefficients $a_i(t)$ in the eigenfunction expansion. They contain the factor

$$\int_0^T \sin p_i(t-t')dt',$$

which starts to decrease with i only when $p_iT > \pi$, and which is small to some order of magnitude if $p_iT \gg \pi$. For T = $3\mu s$ the condition is

$$\frac{Ka}{\pi} = \frac{1}{0.057} \frac{p}{\pi}\mu s$$

$$\gg \frac{1}{0.057} \times \frac{1}{3}$$

$$\doteqdot 6$$

It is clear from this condition that a system of eigenfunctions for which the greatest value of $\frac{Ka}{\pi}$ is 1.9 cannot provide an accurate solution, and that a much more extensive system is necessary for short-duration pulses.

Appendix

$$\frac{J_1(\alpha r)}{J_1(\alpha a)} = \frac{2}{a^2}\sum_{i=0}^{\infty} \frac{\xi_i a\, J_0(\xi_i a)}{\alpha^2 - \xi_i^2}\, \frac{J_1(\xi_i r)}{J_0^2(\xi_i a)}, \qquad J_1(\xi_i a) = 0 \tag{A1}$$

$$\frac{J_0(\alpha r)}{\alpha J_1(\alpha a)} = \frac{2}{a^2}\sum_{i=0}^{\infty} \frac{a\, J_0(\xi_i a)}{\alpha^2 - \xi_i^2}\, \frac{J_0(\xi_i r)}{J_0^2(\xi_i a)}, \qquad J_1(\xi_i a) = 0 \tag{A2}$$

$$\frac{\cosh bz}{b \sinh b\ell} = \frac{2}{\ell}\sum_{m=0}^{\infty} \frac{\varepsilon_m \cos m\pi}{b^2 + \frac{m^2\pi^2}{\ell^2}} \cos\frac{m\pi z}{\ell} \tag{A3}$$

$$\frac{\cos b(\ell - z)}{b \sinh b\ell} = \frac{2}{\ell}\sum_{m=0}^{\infty} \frac{\varepsilon_m}{b^2 + \frac{m^2\pi^2}{\ell^2}} \cos\frac{m\pi z}{\ell} \tag{A4}$$

$$-\frac{\sinh bz}{\sinh b\ell} = \frac{2}{\ell}\sum_{m=0}^{\infty} \frac{m\pi/\ell \,\cos m\pi}{b^2 + \frac{m^2\pi^2}{\ell^2}} \sin\frac{m\pi z}{\ell} \tag{A5}$$

$$\frac{\sinh b(\ell - z)}{\sinh b\ell} = \frac{2}{\ell}\sum_{m=0}^{\infty} \frac{m\pi/\ell}{b^2 + \frac{m^2\pi^2}{\ell^2}} \sin\frac{m\pi z}{\ell} \tag{A6}$$

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