THE THEORY AND PERFORMANCE OF VALVE OSCILLATORS

with particular reference to the oscillation amplitude and the stability of oscillation.

by

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Preface.

The theory of oscillations has been the subject of many papers and books, and over the past 35 years a vast literature has grown up. At first sight it might appear that there could scarcely be any aspect of the subject which has not been thoroughly investigated, but a closer examination shows that this is far from the case. Existing theory is, in fact, severely limited both in its scope and in its methods.

Undoubtedly the most common form of oscillator is the regenerative type with grid-leak bias which is found in almost every radio receiver. It might be expected that this type would have attracted most attention, but on the contrary it has been largely neglected. Until 1949, when the author investigated the matter, there was no method of predicting the oscillation amplitude and no accurate quantitative theory of amplitude stability.

There is, of course, a large body of empirical knowledge concerning the operation of such oscillators, but such theory as exists has been directed mainly to the design of oscillators to produce a maximum power output. There appears to be two main reasons for this curious state of affairs.

Most of the theoretical development has been carried out by mathematicians who appear to have had no practical knowledge of the subject, and were content to study hypothetical systems which bore little resemblance to any practical oscillator. In the first development of any subject there is some justification for making drastic approximations, but the continued use of such methods is a tribute to the influence and prestige of the pioneers rather than to acumen of their followers.

The results have been unfortunate. A vast amount of energy has been devoted to the study of van der Pol's equation. Whatever the mathematical value of such studies may be, it must be admitted that, from the practical point of view, these efforts have been directed to solving the wrong problems.

Circuit designers have had little incentive to theoretical study. Small oscillators can be quickly and cheaply constructed, and the solution of oscillator problems is found more easily on the test-bench than in the study.

A serious gap thus exists in the present knowledge of the theory of oscillators which it is hoped this work may partly repair. Briefly the aim is to produce a theory which can be applied to predict quantitatively the performance of practical oscillators. Two main problems are considered - the calculation of the steady-state oscillation amplitude, and the conditions under which the oscillation is stable.

The work has been written in two parts which represent its chronological development. Though self-contained the parts are closely connected. Both treat the two main problems, but by very different methods and with different emphasis. Part 1 is concerned chiefly with the oscillation amplitude, but two kinds of amplitude instability are also considered. The treatment is discursive and the methods intuitive.

In Part 2 the main interest lies in the stability of oscillation, the steady-state performance being only one stage in the process. The problem is attacked from a more fundamental point of view, and the treatment is more general, concise and rigorous than in Part 1. More powerful methods are used which allow a very general theory of stability to be formulated. This includes as special cases all known forms of frequency and amplitude instability, and is applicable to any type of harmonic oscillator.

With the exception of the discussion on the effect of harmonics on the oscillation amplitude given in Sections 4 and 5, all the ground covered in Part 1 is included in Part 2. An Appendix to Part 2 shows that the results obtained in Part 1, though expressed in a different form, are in fact identical with the corresponding results in Part 2.

This method of presentation, though involving some repetition, is believed to be worth while. Part 1 offers an easier introduction to the subject than the more abstract treatment of Part 2. It presents a physical picture of the mechanism of instability, which is obscured in the more formal later treatment. But the powerful methods of Part 2 are needed to show how far these intuitive ideas can be trusted.

All the theoretical results have been fully confirmed by a comprehensive set of experiments.

Part 1 has been published under the title "Oscillation Amplitude in Simple Valve Oscillators" in the "Wireless Engineer" Vol. 26, 1949. Part 2 has been submitted for publication.

1. Introduction.

The oscillators to be considered are of the regenerative type shown in Fig. 1 in which the electrodes of a thermionic valve are coupled together by a frequency-selective network. For convenience the valve is shown as a triode, but the analysis is equally applicable to tetrode and pentode valves. The oscillation amplitude is limited by applying to the grid a negative biassing voltage produced by the flow of grid current through a high resistance R_g.

The main object of the investigation is to calculate the steady-state value of the oscillation amplitude; i.e. the alternating grid voltage. Existing methods[1] of calculating the performance of such oscillators begin by assuming certain operating conditions for the valve, including the amplitudes of the grid and anode alternating voltages and the grid bias voltage, and deduce therefrom the constants of the feedback network required to satisfy the assumed conditions. These conditions are usually such as to produce the maximum power output from the oscillator.

In many circumstances, however, the problem to be solved is the converse of this. The constants of the feedback network are fixed by considerations other than those of optimum operating conditions for the valve, and it is required to find the amplitude of oscillation which corresponds to the given network constants. An example of such ~~an oscillator~~ a problem is the oscillator in a superheterodyne receiver. Efficiency is not usually of first importance in these oscillators and may be small, as in the example quoted.

The problem could be solved, using existing methods, by calculating for several assumed values of oscillation amplitude the corresponding value of some parameter of the feedback network (e.g. the dynamic resistance of the oscillatory circuit, assuming this to be the variable), and constructing a graph from which the amplitude corresponding to any other

PART 1.

The Oscillation Amplitude in Simple Valve Oscillators.

Summary.

A method is developed of calculating the oscillation amplitude in simple valve oscillators of the regenerative type where grid-leak bias is used. The amplitude is found in terms of parameters which are functions of the valve and circuit constants, and the solution is presented in graphical form.

Two types of amplitude instability are studied and criteria for their existence deduced. The first type is periodic instability or squegging; the second type gives rise to the effect known as oscillation hysteresis.

The analysis is applicable to all the common types of oscillator circuit subject to the condition that the valve should always operate in the space-charge-limited region.

value of the dynamic resistance could be read off. If more than one parameter of the network were varied many sets of graphs would be required to cover all the possible combinations.

When an accurate solution is called for this is probably the only satisfactory method, but it is clearly very tedious, particularly when, as is usual, many parameters of the feedback network must be taken into account. In most problems of this kind, however, an approximate solution is all that is required. Precise calculations are seldom justified, as the primary data are not usually known with any great accuracy. An approximate general solution has been obtained in the form of a family of graphs which is applicable to all the simple types of circuit. The results are sufficiently accurate for all practical purposes.

The method of analysis is to examine separately the feedback network, the grid current and grid voltage, and the anode current. Finally the various quantities are grouped together to form a few parameters which are plotted as a family of curves. From these the oscillation amplitude can easily be found by the method explained in Section 6.

The calculated value of oscillation amplitude is attained only if this amplitude corresponds to a stable state of oscillation. Amplitude instability may take the form of a periodic modulation of the amplitude - an effect commonly known as squegging. As is well known, this can always be stopped by reducing the value of the grid capacitor C_g.

Another kind of instability is manifest in the effect known as oscillation hysteresis. As the coupling between anode and grid circuits is varied, points may be found where the oscillation amplitude jumps from one stable value to another, the critical points and the critical amplitudes being different according to the direction from which they are approached. The various forms of instability are studied in some detail and criteria for their existence are deduced.

Certain assumptions are made to simplify the analysis. Perhaps the most important of these is that the valve operates always in such a way that the space current and the electrode potentials are connected by the three-halves power law. The effect of this assumption is to restrict the validity of the analysis to oscillators in which the alternating anode voltage is not a very large fraction of the H.T. supply voltage, and thus to exclude oscillators of high efficiency. Various other assumptions are made concerning the valve and feedback network: these are introduced at the appropriate points in the text. Experiment shows that these assumptions do not lead to serious error.

2. Principal symbols.

b_a b_g = Constants of the valve. (See Section 6)

C_g = Grid bias capacitance.

c_{gk} = Grid-cathode capacitance of the valve.

i_a = Anode current.

i_{a1} = Anode current of oscillation frequency.

I_{a1} = Amplitude of anode current of oscillation frequency.

i_g = Grid current.

i_{g1} = Grid current of oscillation frequency.

I_{g1} = Amplitude of grid current of oscillation frequency.

I_g = Mean grid current.

K, N = Functions of the valve and circuit constants. (See equations (5.3) and (5.4))

ϕ = Conduction half-angle for the grid current.

R_g = Grid bias resistance.

r_g = Grid input resistance at the oscillation frequency.

μ = Amplification factor of the valve.

v_a = Anode voltage.

v_{a1} = Anode voltage of the oscillation frequency.

V_{a1} = Amplitude of anode voltage of oscillation frequency.

v_g = Grid voltage.

V_g = Mean grid voltage.

v_{g1} = Grid voltage of oscillation frequency.

V_{g1} = Amplitude of the grid voltage of oscillation frequency.

V_{ca} = Grid voltage at anode current cut-off point.

V_{cg} = Grid voltage at grid current cut-off point.

ω_o = Oscillation frequency. (Radians/sec.)

ω_d = Modulation frequency

X_g = $1/\omega_o(C_g + c_{gk})$

Z_T = $R_T + jX_T$ = Effective transfer impedance of the feedback network.

Z_I = $R_I + jX_I$ = Effective input impedance of the feedback network.

3. The feedback network.

The feedback network comprises the circuit elements lying between terminals 1,2 and 6,4 in Fig. 1, of which the most important is an oscillatory circuit. For convenience this is shown as a simple anti-resonant circuit with a coupling coil, but any of the other well-known arrangements such as the Hartley or Colpitts may be used. The object of this Section is to formulate relations between the currents and voltages in the network for use in later parts of the paper.

It is convenient to consider the interelectrode capacitances of the valve as belonging to the feedback network. Of these the most important is usually the grid-cathode capacitance c_{gk}. The anode-cathode and grid-anode capacitances are not shown in Fig.1. The former is equivalent to a capacitor connected between 1 and 2. In inductively coupled circuits the grid-anode capacitance has a negligible effect on the oscillation amplitude, but in circuits such as the tuned-grid tuned-anode type one or other of these capacitances plays a major part in producing regeneration and must therefore be included in the calculation of the network constants. The significance of c_{gk} for the present analysis is that it forms together with C_g a high-frequency potentiometer which may produce attenuation between terminals 5 and 6.

It is also convenient to consider the grid input resistance r_g of the valve, at the oscillation frequency, as if it were a physical resistor connected between 6 and 4 forming part of the feedback network. For the purpose of the analysis the most important parameters of the network are the effective transfer impedance $Z_T(j\omega)$ between 1,2 and 6,4, the effective input impedance $Z_I(j\omega)$ between 1 and 2, and the effective Q-factor of the anti-resonant circuit.

Elementary circuit theory shows that, at frequencies close to the resonant frequency ω_r, the transfer impedance (i.e the ratio of the voltage between 6,4 to the current flowing between

1 and 2), can be expressed for all kinds of simple networks in the form

$R_T/Z_T(j\omega) = 1 + 2jQ'(\omega/\omega_r - 1)$

where $Q' = R'/\omega_r L$, and R' is the effective parallel resistance of the anti-resonant circuit (including the effect of r_g).

It is assumed that the oscillation frequency ω_o is ~~in fact~~ equal to ω_r. In fact, for various reasons, this is never quite true, but the reactive component of Z_T is always very small at the oscillation frequency and has a negligible effect on the oscillation amplitude. It is therefore permissible to write

$$R_T/Z_T(j\omega) = 1 + 2jQ'(\omega/\omega_r - 1) \qquad (3.1)$$

R_T is the transfer resistance at the oscillation frequency, and in order that the grid voltage should be of the correct phase to maintain oscillation, R_T must be negative.

The current flowing into the network at 1 is equal and opposite to the anode current of the valve. (The currents and voltages shown in Fig. 1 are conventionally positive and the cathode potential is taken as zero) Hence if i_{a1} and v_{g1} are the anode current and grid voltage of fundamental frequency

$$v_{g1} = -i_{a1}R_T \quad \text{and} \quad V_{g1} = -I_{a1}R_T \qquad (3.2)$$

the capitals denoting values of the amplitudes.

At harmonic frequencies a difficulty arises because the harmonic grid currents depend mainly on the fundamental grid voltage and little on the harmonic grid voltages. The concept of grid input impedance is therefore nor a very useful one at harmonic frequencies. However it is not usually required to calculate the harmonic voltages with great accuracy, so for this purpose the harmonic grid currents can be neglected.

If the anode current of nth. harmonic frequency is i_{an} the corresponding grid voltage is

$$v_{gn} = -i_{an}Z_T(jn\omega_o) = -i_{an}(R_T(jn\omega_o) + jX_T(jn\omega_o))$$
$$= -i_{an}(R_{Tn} + jX_{Tn})$$

In practice the precise value of R_{Tn} may be difficult to determine, because of the variation of resistance with frequency, but in all the simple networks it is small compared with X_{Tn} and R_T and may be neglected. Then $v_{gn} = -ji_{an}X_{Tn}$

It is shown in Section 5 that the fundamental and harmonic anode currents reach their maxima almost simultaneously. The anode current may therfore be written approximately as

$$I_a + \sum_{1}^{\infty} I_{an}\cos n\omega_0 t \quad \ldots \quad \ldots \qquad (3.3)$$

and the alternating grid voltage is theerefore

$$v_{ga} = V_{g1}\cos\omega_0 t + \sum_{2}^{\infty} V_{gn}\sin n\omega_0 t \quad \ldots \qquad (3.4)$$

where $V_{gn} = I_{an}X_{Tn}$ and

$$V_{gn}/V_{g1} = -I_{an}X_{Tn}/I_{a1}R_T \quad \ldots \qquad (3.5)$$

The ratio X_{Tn}/R_T is inversely proportional to Q. It is assumed that Q is always large, so that the harmonic voltages are only a few percent of the fundamental.

So far as the input impedance $Z_I(j\omega)$ between 1 and 2 is concerned, feedback networks may be divided roughly into two classes. In networks of the first class Z_I is comparable with $-R_T$ at the oscillation frequency and is equal to a resistance R_I at that frequency. In networks of the second class, which are exemplified by an anti-resonant circuit connected between 3 and 4 with a coupling coil between 2 and 1, $Z_I(j\omega_0)$ is usually small compared with $-R_T$ and, depending on the coefficient of coupling between the coils, has a reactive component which may be negligible or comparable with the resistive component. For both classes of network it is easily shown that at frequencies near the oscillation frequency

$$Z_I = jX_a + R_I/(1 + 2jQ'(\omega/\omega_0 - 1)) \quad \ldots \qquad (3.6)$$

X_a is zero in networks of the first class.

The anode current of oscillation frequency is $i_{a1} = I_{a1}\cos\omega_0 t$. Then the anode voltage of oscillation frequency is $-(R_I + jX_a)i_{a1} = -V_{a1}\cos\omega_0 t + (V_{a1}X_a/R_I)\sin\omega_0 t$

where $\quad V_{a1} = R_I I_{a1} = -V_{g1}R_I/R_T \quad \ldots \qquad (3.7)$

At harmonic frequencies Z_I is almost wholly reactive. $Z_I(jn\omega_0) = jX_{In}$. Assuming the anode current to be given by (3.3) the alternating anode voltage is

$$v_{aa} = -V_{a1}\cos\omega_o t + (V_{a1}X_a/R_I)\sin\omega_o t + \sum_2^{\infty} V_{an}\sin n\omega_o t \qquad (3.8)$$

where $V_{an} = I_{an}X_{In}$

In connection with the study of amplitude instability it is necessary to know what effect changes in the grid input resistance have on the values of R_T and R_I.

Suppose that the power supplies to the valve are removed so that the grid input resistance becomes infinite. Let the impedance between terminals 6 and 4 at the oscillation frequency than be $R_o + jX_o$. Now let a current i of frequency ω_o flow into 1, and let the voltage at 6 be v. If a resistor of value r_g is then connected between 6 and 4 the voltage at 6 becomes, by Thévenin's theorem $vr_g/(r_g + R_o + jX_o)$. By definition the transfer impedance is $Z_T = R_T = (v/i)r_g/(r_g + R_o + jX_o)$

If r_g is changed to r_g' the transfer impedance becomes $Z_T' = (v/i)r_g'/(r_g' + R_o + jX_o)$. It is clear that Z_T' now has a reactive component so that the frequency at which Z_T' is wholly real is different from ω_o. It may be assumed that for all practical values of r_g this difference is very small , and also that X_o is small compared with $R_o + r_g$. Hence if R_T' is the transfer resistance corresponding to r_g'

$$R_T'/R_T = r_g'(r_g + R_o)/r_g(r_g' + R_o) \qquad \ldots \qquad \ldots \qquad (3.9)$$

Similarly if R_I' is the resistive component of $Z_I(j\omega_o)$ corresponding to r_g' it can be shown that

$$R_I'/R_I = r_g'(r_g + R_o)/r_g(r_g' + R_o) \qquad \ldots \qquad \ldots \qquad (3.10)$$

and so $R_I'/R_T' = R_I/R_T$

All the preceding formulae refer to steady sinusoidal currents, but in connection with squegging, which is treated in Section 7, amplitude -modulated currents and voltages are encountered. Suppose that the modulated grid ~~current~~ voltage at fundamental frequency is

$$v_{g1m} = V_{g1}(1 + a\cos\omega_d t)\cos\omega_o t \qquad (3.11)$$

Then from (3.1) and (3.6) the corresponding anode voltage is

$$v_{a1m} = v_{g1m}Z_I(j\omega)/Z_T(j\omega) = -V_{a1}(1 + a\cos\omega_d t)\cos\omega_o t$$
$$+ V_{a1}(X_a/R_I)(1 + a\cos\omega_d t - 2aQ'(\omega_d/\omega_o)\sin\omega_d t)\sin\omega_o t \qquad (3.12)$$

Of the two components of the anode voltage it is seen that the modulation on the first or in-phase component is unaltered in passing through the network. The modulation on the quadrature component is altered in both phase and amplitude.

The anode current of fundamental frequency associated with the voltage given by (3.12) is $i_{a1m} = -v_{g1m}/Z_T(j\omega)$. Substituting for $Z_T(j\omega)$ from (3.1) and for v_{g1m} from (3.11)

$$i_{a1m} = (V_{a1}/R_I)(1 + a\cos\omega_d t - 2aQ'(\omega_d/\omega_o)\sin\omega_d t)\cos\omega_o t \qquad (3.13)$$

This concludes the analysis of the feedback network.

Sometimes it is more convenient to connect the grid resistor R_g between 6 and 4 instaed of in the position shown. The only effect of this change is to put an additional path for high-frequency currents between 6 and 4 without affecting in any other way the operation of the oscillator.

4. Grid current.

The object of this Section is to calculate the grid current and thence the grid-bias voltage and the grid input resistance. If the alternating grid voltage is sinusoidal the work is simple, but there are two factors which give rise to harmonic grid voltages. Since the Q-factor of the network cannot be infinite, the harmonic anode and grid currents produce harmonic grid voltages, and, depending on the magnitude of these, the grid-bias voltage and the grid input impedance are more or less altered. The second factor is the time constant R_gC_g of the grid resistor and capacitor. Since this cannot be infinite the pulsating grid current produces harmonic grid voltages between 5 and 6.

To solve the problem the method which has been adopted is to carry through the calculation, first on the assumption that Q and R_gC_g are very large, and then to examine separately the effects of Q and R_gC_g having finite values. It is particularly important to find the minimum value of R_gC_g which will give satisfactory results, for the maximum value is limited by considerations of amplitude stability (squegging).

Let the mean grid current be I_g. This flows through R_g, and if the resistance to direct current between 3 and 4 is negligible compared with R_g the mean grid voltage is $-R_gI_g = V_g$ Assuming Q and R_gC_g to be infinite, harmonic voltages are zero and the grid voltage can be written

$$v_g = V_{g1}\cos\omega_0 t + V_g \quad \cdot\cdot \quad \cdot\cdot \quad (4.1)$$

It is found that in receiving valves and small transmitting valves having unipotential oxide-coated cathodes the relation between grid current and grid voltage is approximately linear for small positive grid voltages, and varies but little with anode or screen voltage provided these are considerably greater than the grid voltage. If anode and grid voltages become comparabke the grid current increases at a greater rate. This may happen if the anode load resistance is large and the valve is driven hard. It will be supposed that in the oscillators to which this analysis applies the excursion of the anode voltage is not a large fraction of the H.T. supply voltage.

When the grid voltage is negative the grid current characteristic is approximately exponential. This is important when very small values of oscillation amplitude have to be considered, and is discussed in Section 8 in connection with amplitude instability. Under normal steady-state conditions the oscillation amplitude is usually large enough to make the linear law an accurate approximation. The relation between grid current and voltage can then be written

$$i_g = b_g(v_g - V_{cg}) \quad \text{when } v_g > V_{cg}$$
$$= 0 \quad\quad v_g < V_{cg} \qquad (4.2)$$

b_g and V_{cg} are constants peculiar to the valve.

Substituting for v_g according to (4.1) gives

$$i_g = b_g(V_{g1}\cos\omega_0 t + V_g - V_{cg})$$

and current flows when $V_{g1}\cos\omega_0 t + V_g - V_{cg} > 0$, e.g. from $\omega_0 t = -\emptyset$ to $\omega_0 t = \emptyset$, where $\cos\emptyset = -(V_g - V_{cg})/V_{g1}$
In practical oscillators $|V_g|$ is much greater than V_{cg} and is comparable with V_{g1} so that current flows only for a fraction of each cycle. The mean grid current is

$$I_g = (b_g/\pi)\int_0^{\emptyset}(V_{g_1}\cos\theta + V_g - V_{cg})d\theta$$

From this and the preceding formula

$$-V_g = R_g I_g = V_{g1}b_g R_g(\sin\emptyset - \emptyset\cos\emptyset)/\pi = V_{g1}\cos\emptyset - V_{cg} \qquad (4.3)$$

$b_g R_g$ could be found from this equation in terms of V_g/V_{g1} and V_{cg}/V_{g1} and graphs of V_g/V_{g1} versus $b_g R_g$ could be drawn for various values of V_{cg}/V_{g1}. A simpler though approximate method is to proceed as follows:

Let $\emptyset_0$ be the value of $\emptyset$ corresponding to $V_{cg} = 0$. Let $\emptyset = \emptyset_0 + \emptyset_1$ and suppose that $\emptyset_1$ is so small that $\cos\emptyset_1 = 1$ and $\sin\emptyset_1 = \emptyset_1$. Then (4.3) becomes

$$V_{g1}b_g R_g(\sin\emptyset_0 - \emptyset_0\cos\emptyset_0 + \emptyset_0\emptyset_1\sin\emptyset_0)/\pi = V_{g1}\cos\emptyset_0 - V_{g1}\emptyset_1\sin\emptyset_0 - V_{cg}$$

Since $\phi = \phi_0$ when $V_{cg} = 0$

$$b_g R_g (\sin\phi_0 - \phi_0 \cos\phi_0)/\pi = \cos\phi_0 \qquad (4.4)$$

Substituting this into the above equation gives

$$V_{g1}\phi_1 \sin\phi_0 = -V_{cg}(1 - \phi_0 \cot\phi_0) \qquad (4.5)$$

Then $V_g = -V_{g1}\cos\phi_0 + V_{g1}\phi_1\sin\phi_0 + V_{cg}$

$$= -V_{g1}\cos\phi_0 + V_{cg}\phi_0\cot\phi_0 \qquad (4.6)$$

Fig. 2 shows $\cos\phi_0$ and $\phi_0\cot\phi_0$ plotted against $b_g R_g$. Exact values for V_g calculated from (4.3) indicate that for the range of $b_g R_g$ covered by Fig. 2 the error in formula (4.6) is less than about 1% for values of V_{cg}/V_{g1} between 0.3 and -0.6. In practice the values of V_{cg}/V_{g1} are much less than this.

Next consider the amplitude I_{g1} of the component of fundamental frequency in i_g. This is

$$I_{g1} = (2b_g/\pi)\int_0^{\phi}(V_{g1}\cos\theta + V_g - V_{cg})\cos\theta d\theta$$

$$= b_g V_{g1}(\phi - \sin\phi\cos\phi)/\pi$$

$$= V_{g1}(2b_g/3\pi)\phi^3(1 - \phi^2/5 + ..)$$

Similarly from (4.3)

$$I_g = V_{g1}(b_g/3\pi)\phi^3(1 - \phi^2/10 + ..)$$

Hence $I_{g1} = 2I_g(1 - \phi^2/10 + ..) \qquad (4.7)$

The grid input resistance at fundamental frequency is

$$r_g = V_{g1}/I_{g1} = \tfrac{1}{2}(V_{g1}/I_g)/(1 - \phi^2/10 + ..)$$

$$= -\tfrac{1}{2}R_g/(V_g/V_{g1})(1 - \phi^2/10 + ..)$$

Since ϕ is not very different from ϕ_0 and the term $\phi^2/10$ is in any case small, it is permissible to write

$$r_g = -\tfrac{1}{2}R_g/(V_g/V_{g1})(1 - \phi_0^2/10) \qquad (4.8)$$

The value of $1 - \phi_0^2/10$ is shown in Fig. 2. It is only at the smallest values of $b_g R_g$ that this term is appreciably different from 1. In most examples the difference can be neglected.

Now suppose that $R_g C_g$ is still infinite, but that, because of the finite value of Q, the grid voltage contains harmonic components, and that the mean grid voltage is thereby altered to $V_g + dV_g$. Then from (3.4)

$$v_g = V_{g1}\cos\omega_o t + V_g + \sum_2^\infty V_{gn}\sin n\omega_o t + dV_g$$

It will be assumed that V_{cg} is 0 so that $\phi = \phi_o$. The effect of V_{cg} on the value of dV_g can be shown to be negligibly small. Let conduction now begin when $\omega_o t = -(\phi_o + \phi_1)$ and end when $\omega_o t = \phi_o + \phi_2$. Then

$$V_{g1}\cos(\phi_o + \phi_1) - V_{g1}\cos\phi_o - \sum_2^\infty V_{gn}\sin n(\phi_o + \phi_1) + dV_g = 0$$

Assuming that $\sum_2^\infty V_{gn}\sin n\phi_o$ is small and that ϕ_1 is so small that $\sin\phi_1 = \phi_1$ and $\cos\phi_1 = 1 - \phi_1^2/2$,

$$\tfrac{1}{2}\phi_1^2(V_{g1}\cos\phi_o - \sum_2^\infty n^2 V_{gn}\sin n\phi_o) + \phi_1(V_{g1}\sin\phi_o + \sum_2^\infty nV_{gn}\cos n\phi_o) + \sum_2^\infty V_{gn}\sin n\phi_o - dV_g = 0$$

From which

$$\phi_1(V_{g1}\cos\phi_o - \sum_2^\infty n^2V_{gn}\sin n\phi_o) = -(V_{g1}\sin\phi_o + \sum_2^\infty nV_{gn}\cos n\phi_o)$$
$$+ \left((V_{g1}\sin\phi_o + \sum_2^\infty nV_{gn}\cos n\phi_o)^2 + 2(V_{g1}\cos\phi_o - \sum_2^\infty n^2V_{gn}\sin n\phi_o) \times (dV_g - \sum_2^\infty V_{gn}\sin n\phi_o)\right)^{\frac{1}{2}} \quad (4.9)$$

The expression for ϕ_2 is obtained from this by writing $-V_{gn}$ in place of V_{gn}.

The mean grid current is

$$I_g + dI_g = I_g - dV_g/R_g = (b_g/2\pi)\int_{-\phi_o - \phi_1}^{\phi_o + \phi_2}(V_{g1}\cos\theta + V_g + \sum_2^\infty V_{gn}\sin n\theta + dV_g)d\theta$$

$$= I_g - (b_g/4\pi)(V_{g1}(\phi_1^2 + \phi_2^2)\sin\phi_o + 2(\phi_1 - \phi_2)\sum_2^\infty V_{gn}\sin n\phi_o$$
$$+ (\phi_1^2 - \phi_2^2)\sum_2^\infty nV_{gn}\cos n\phi_o - 2dV_g(2\phi_o + \phi_1 + \phi_2))$$

Hence $dV_g((1 + b_gR_g\phi_o/\pi) + (\phi_1 + \phi_2)b_gR_g/2\pi)$

$$= (b_gR_g/4\pi)(V_{g1}(\phi_1^2 + \phi_2^2)\sin\phi_o + 2(\phi_1 - \phi_2)\sum_2^\infty V_{gn}\sin n\phi_o + (\phi_1^2 - \phi_2^2)\sum_2^\infty nV_{gn}\cos n\phi_o)$$

From (4.4) $1 + b_gR_g\phi_o/\pi = b_gR_g\tan\phi_o/\pi$

Hence $(dV_g/V_g)(\sin\phi_o + \frac{1}{2}(\phi_1 + \phi_2)\cos\phi_o) =$

$$\tfrac{1}{4}(2(\phi_2 - \phi_1)\sum_2^\infty (V_{gn}/V_{g1})\sin n\phi_o + (\phi_2^2 - \phi_1^2)\sum_2^\infty n(V_{gn}/V_{g1})\cos n\phi_o - (\phi_2^2 + \phi_1^2)\sin\phi_o)$$

Since dV_g is involved in the expressions for ϕ_1 and ϕ_2 this equation cannot be solved directly for dV_g. The method which has been adopted is to calculate ϕ_1 and ϕ_2 from (4.9) on the assumption that dV_g/V_g has certain small values (e.g. 1%, 2%) and to find the values of dV_g/V_g given by the above equation using these values of ϕ_1 and ϕ_2. The values of dV_g/V_g thus found are substituted back into (4.9) and the calculation repeated to obtain a better approximation.

The effect of harmonic voltages increases slightly as b_gR_g increases, but it is always small. As an example consider the extreme case in which $b_gR_g = 1000$, and the amplitudes of the second and third harmonic voltages are 5% and 3%, higher harmonics being negligible. The corresponding value of dV_g/V_g is found to be 1.6%. In practice the value of b_gR_g is usually much less than 1000 and the harmonic voltages are seldom as great as the values assumed.. It can be shown similarly that the effect of the harmonic voltages on the grid input impedance is also negligible.

The effect of a finite value of R_gC_g can now be considered. It will be supposed that the Q-factor of the oscillatory circuit is infinite so that the voltage between 3 and 4 is sinusoidal. The pulsating grid current, however, produces a non-sinusoidal voltage between 5 and 6 which changes the mean grid voltage and the grid input impedance. Another effect of a non-infinite value for R_gC_g is a small change in the amplitude and phase of the fundamental grid voltage. The aim of the following calculation is to find the value of V_g corresponding to any given value of R_gC_g, and hence to find the values of R_gC_g which produce certain specified small changes in the value of V_g.

Let the voltage between 3 and 4 be v_3. For the purpose of calculating the grid current the part of the feedback network to the right of terminals 3 and 4 may be replaced, according to Thévenin's theorem, by the equivalent circuit shown in Fig. 3 in which $C_g' = C_g + c_{gk}$ and

$v_{3e} = v_3(1 + j\omega_o C_g R_g)/(1 + j\omega_o C_g' R_g)$

In practical oscillators R_g is many times $1/\omega_o C_g$, and C_g is several times c_{gk}. Then to a high degree of accuracy $v_{3e} = v_3 C_g/C_g'$. For example, if C_g is 5 times c_{gk} and R_g is 15 times $1/\omega_o C_g$ the error in V_{3e} is only 0.07% and the phase error only 0.01 radian.

It is assumed initially that V_{cg} is zero. The effect of a finite value for V_{cg} is considered later and is shown to be small.

Let $v_{3e} = V_{3e}\cos\omega_o t$. Let v_b be the instantaneous value of the voltage across C_g', and let the grid current begin, in the cycle centered on $t = 0$, when $\omega_o t = -\phi_1$ and end when $\omega_o t = \phi_2$. In the following cycle current begins when $\omega_o t = 2\pi - \phi_1$. During the conduction period the current is $i_g = b_g(V_{3e}\cos\omega_o t + v_b)$. This must be equal to the sum of the currents flowing through R_g and C_g' which is $-v_b/R_g - C_g'(dv_b/dt)$

Hence $C_g' R_g(dv_b/dt) + (1 + b_g R_g)v_b = -b_g R_g V_{3e}\cos\omega_o t$

This differential equation is of a simple and well-known type. The solution is easily found to be

$$v_b = a\exp(-pb_g X_g \omega_o t) - V_{3e}\sin\theta\sin(\omega_o t + \theta)/p \qquad (4.10)$$

in which $p = 1 + 1/b_g R_g$

$$\tan\theta = pb_g X_g \qquad (4.11)$$

$$X_g = 1/\omega_o C_g'$$

and the constant a is determined by the initial conditions. The grid current, and therefore $V_{3e}\cos\omega_o t + v_b$ is zero when $\omega_o t = -\phi_1$ and when $\omega_o t = \phi_2$. Applying these conditions to (4.10) gives

$$\left.\begin{aligned} a\exp(pb_g X_g\phi_1) &= V_{3e}\sin\theta\sin(\theta - \phi_1)/p - V_{3e}\cos\phi_1 \\ a\exp(-pb_g X_g\phi_2) &= V_{3e}\sin\theta\sin(\theta + \phi_2)/p - V_{3e}\cos\phi_2 \end{aligned}\right\} \qquad (4.12)$$

Let $\phi_1 + \phi_2 = 2\phi_s$ and $\phi_1 - \phi_2 = 2\phi_d$

Two new equations may be formed by subtracting and adding the corresponding sides of equations (4.12) and the quotients of the corresponding sides of these new equations form a third equation, which, after substituting for ϕ_1 and ϕ_2 may be written

$$\tanh(pb_gX_g\phi_s) = \frac{p\sin\phi_d\sin\phi_s - \sin\theta\cos(\theta - \phi_d)\sin\phi_s}{\sin\theta\sin(\theta - \phi_d)\cos\phi_s - p\cos\phi_d\cos\phi_s}$$

$$= \tan\phi_s \frac{\sin\theta\cos\theta - (p - \sin^2\theta)\tan\phi_d}{\sin\theta\cos\theta\tan\phi_d + p - \sin^2\theta}$$

$$= \tan\phi_s(1 - q\tan\phi_d)/(q + \tan\phi_d)$$

where $q = (p - \sin^2\theta)/\sin\theta\cos\theta = 1/b_gX_g + pX_g/R_g$ (4.13)

The above equation may be rewritten as

$$\tan\phi_d = \frac{\tan\phi_s - q\tanh(pb_gX_g\phi_s)}{q\tan\phi_s + \tanh(pb_gX_g\phi_s)} \quad (4.14)$$

Between the end of one conducting period and the beginning of the next C_g' discharges exponentially through R_g. During this time $v_b = d\ \exp(-X_g\omega_o t/R_g)$ (4.15)

where d is a constant determined by the initial conditions. At the beginning and end of the discharge period $V_{3e}\cos\omega_o t + v_b$ is zero. Hence

$$\left.\begin{aligned} d\ \exp(-X_g\phi_2/R_g) &= -V_{3e}\cos\phi_2 \\ d\ \exp(X_g(\phi_1 - 2\pi)/R_g) &= -V_{3e}\cos\phi_1 \end{aligned}\right\} \quad (4.16)$$

By subtracting and adding the corresponding sides of these equations, and taking the quotients of the corresponding sides of the two new equations thus formed, one obtains

$$\tan\phi_d = \cot\phi_s\tanh(X_g(\pi - \phi_s)/R_g) \quad (4.17)$$

$\tan\phi_d$ may be eliminated between (4.14) and (4.17) to give an equation containing ϕ_s only

$$\tanh(X_g(\pi - \phi_s)/R_g)(q\tan\phi_s + \tanh(pb_gX_g\phi_s))$$
$$= \tan\phi_s(\tan\phi_s - q\tanh(pb_gX_g\phi_s))$$

This equation could be solved graphically, but an approximate analytical solution is less laborious and more accurate. A rough approximation for ϕ_s is ϕ_o, which can be found from Fig. 2. By the method of successive approximations the value of ϕ_s can then be calculated to any required degree of accuracy. ϕ_d follows at once from (4.17)

The mean grid voltage can now be calculated. This is done by integrating v_b over a cycle of oscillation. Expression (4.10) is integrated between $\omega_o t = -\phi_1$ and $\omega_o t = \phi_2$, and expression

(4.15) between $\omega_0 t = \phi_2$ and $\omega_0 t = 2\pi - \phi_1$. When the exponential terms have been eliminated by substitution from (4.12) and (4.16) the solution, denoted by V_g', can be written

$$V_g' = -(V_{3e}/\pi)\big((R_g/X_g)(1 - 1/pb_gR_g)\sin\phi_s\sin\phi_d$$
$$+ \sin\theta\sin\phi_s(\sin(\theta - \phi_d) + \cos(\theta - \phi_d)/pb_gX_g)/p\big)$$

On substituting for θ and p according to (4.11) this becomes

$$V_g' = -V_{3e}\sin\phi_s(\cos\phi_d + (R_g/X_g)\sin\phi_d)/p\pi$$

If R_g/X_g were infinite the mean grid voltage would be $V_g = -V_{3e}\cos\phi_o$. Hence

$$V_g'/V_g = (1 + (R_g/X_g)\tan\phi_d)\sin\phi_s\cos\phi_d/p\pi\cos\phi_o \qquad (4.18)$$

In Fig. 4 are shown the values of b_gR_g and R_g/X_g which correspond to values of $1 - V_g'/V_g$ of 0.5%, 1%, and 2%.

This problem has also been studied by Marique[2] whose method was to find ϕ_1 and ϕ_2 by solving graphically two simultaneous equations in ϕ_1 and ϕ_2. Marique also assumed that the mean value of v_b during the charging period is equal to the mean value during the discharge period. In spite of these limitations the values of V_g' obtained by Marique are in good agreement with those calculated from the exact formulae developed in this Section. Marique gave a table of values of b_gR_g and corresponding values of $R_g/2\pi X_g$ beyond which no appreciable increase in the bias voltage occurs as the value of X_g is reduced. The points corresponding to these values lie between the 0.5% and 1% graphs of Fig. 4

The effect of the grid current cut-off voltage V_{cg}, which so far has been neglected, can now be considered. It can be seen from (4.2) that, so far as the grid current is concerned, V_{cg} has the same effect as a generator of e.m.f. $-V_{cg}$, with zero impedance, connected in series with the grid. Suppose that for any particular values of R_g, C_g and V_g, the mean grid current is I_g. The mean grid current would be unchanged if the generator were removed and replaced by a resistor of value V_{cg}/I_g shunted by a very large capacitor. Since the alternating grid current flows almost entirely through the grid capacitor, the

the two resistors and two capacitors could be replaced, with little effect on the grid current, by a single resistor of value $R_g + V_{cg}/I_g = R_g(1 - V_{cg}/V_g)$ shunted by a capacitor of value C_g'. It is concluded that for the small changes of V_g which are of interest the graphs of Fig. 4 are valid if R_g is replaced by $R_g(1 - V_{cg}/V_g)$, but it is obvious from inspection of these graphs that this makes a negligible difference to the value of X_g.

Finally the effect of a finite time constant on the grid input impedance can be calculated. The grid current of fundamental frequency can be expressed as $I_{g1c}\cos\omega_o t + I_{g1s}\sin\omega_o t$ or symbolically $(I_{g1c} - jI_{g1s})\exp j\omega_o t$ where

$$I_{g1c} = (b_g/\pi)\int_{-\phi_1}^{\phi_2}(V_{3e}\cos\theta + v_b)\cos\theta d\theta \qquad v_b \text{ is given by (4.10)}$$

The integral is equal to

$$I_{g1c} = (V_{3e}b_g/\pi(1 + p^2b_g^2X_g^2))\times$$
$$(\sin\phi_s\cos\phi_s((2X_g/R_g + b_gX_g)\sin2\phi_d - (1 - X_g^2/R_g^2 - b_gX_g^2/R_g)\times\cos2\phi_d)$$
$$+ \phi_s(1 + X_g^2/R_g^2 + b_gX_g^2/R_g))$$

Similarly

$$I_{g1s} = (V_{3e}b_g/\pi(1 + p^2b_g^2X_g^2))\times$$
$$(\sin\phi_s\cos\phi_s((1 - X_g^2/R_g^2 - b_gX_g^2/R_g)\sin2\phi_d$$
$$+ (2X_g/R_g + b_gX_g)\cos2\phi_d) - b_gX_g\phi_s)$$

The grid voltage of fundamental frequency is

$$(V_{3e} + jR_gX_g(I_{g1c} - jI_{g1s})/(R_g - jX_g))\exp j\omega_o t$$

Hence the grid input imped ance is

$$z_g = V_{3e}/(I_{g1c} - jI_{g1s}) + jR_gX_g/(R_g - jX_g) = r_g + jx_g$$

It is found that for the values of b_gR_g and R_g/X_g shown in Fig. 4 the values of r_g do not differ from those corresponding to an infinite value of R_gC_g by percentages greater than those shown in Fig. 4 for the change in V_g, and that x_g is always a small fraction of r_g. If it is assumed that errors in V_g and r_g less than say 1% can be neglected, then formulae (4.6) and (4.8) can be used, provided R_g/X_g is greater than the value shown in Fig. 4 corresponding to the assumed error.

5. Anode current.

The anode current is the difference between the cathode current and the sum of the currents flowing to all other electrodes. Thus for triodes $i_a = i_k - i_g$. In a triode the cathode current may be represented fairly accuractly, over a certain range, by the expression

$$i_k = b_a(v_g + v_a/\mu)^{3/2} \quad \text{when } v_g + v_a/\mu > 0$$
$$= 0 \quad\quad v_g + v_a/\mu < 0$$

in which v_a is the anode voltage and b_a and μ are constants peculiar to the valve. The errors in this expression occur mainly at large and small values of i_k. For large positive grid voltages especially if combined with low anode voltages the current is less than the theoretical value. This may happen if the anode-load impedance R_I is large and the valve is driven hard. It will be supposed that in the oscillators to which this analysis is applied the anode alternating voltage is not a large fraction of the H.T. supply voltage.

At small values of i_k errors may be produced by the fringing of the electrostatic field at the extremities of the electrodes, by the initial velocity of the electrons and other effects. The current is greater than the theoretical value, but these discrepancies are comparatively innocuous.

Under operating conditions v_a is the sum of a constant component V_a (the H.T. supply voltage) and an alternating component v_{aa}. Hence $i_a = b_a(v_g + v_{aa}/\mu - V_{ca})^{3/2} - i_g$ where V_{ca} is the value of v_g at which i_k becomes zero under static conditions. V_{ca} is always negative and is referred to as the anode current cut-off voltage. (The grid current is zero at this point)

In tetrode and pentode valves the anode current bears an almost constant ratio to the screen current provided the anode voltage is not too low. Let the anode current be k times, and the screen current 1 - k times, the sum of these currents. For many valves k is approximately 0.75 - o.8. If the screen voltage is fixed the expression for the anode current is similar to that given above for triodes. Thus

$$i_a = b_a(v_g + v_{aa}/\mu - V_{ca})^{3/2} - ki_g \quad . . \quad . . \qquad (5.1)$$

The constant b_a here refers, of course, to the anode current not to the cathode current. This expression may serve for all types of valves. For triodes $k = 1$. For pentodes and tetrodes μ is usually so large that v_{aa}/μ ban be neglected .

Taking account of harmonics, the grid voltage is $v_g = V_{g1}\cos\omega_0 t + V_g + \sum_2^\infty V_{gn}\sin n\omega_0 t$ and the alternating anode voltage is

$$v_{aa} = -V_{a1}\cos\omega_0 t + (V_{a1}(X_a/R_I)\sin\omega_0 t + \sum_2^\infty V_{an}\sin n\omega_0 t$$

Substituting these expressions into (5.1) and expanding by Taylor's theorem gives

$$i_a = b_a((V_{g1} - V_{a1}/\mu)\cos\omega_0 t + V_g - V_{ca})^{3/2} - ki_g$$
$$+ 1\tfrac{1}{2}b_a((V_{a1}(X_a/\mu R_I)\sin\omega_0 t + \sum_2^\infty (V_{gn} + V_{an}/\mu)\sin n\omega_0 t)$$
$$\times ((V_{g1} - V_{a1}/\mu)\cos\omega_0 t + V_g - V_{ca})^{\frac{1}{2}}$$

In the absence of the quadrature anode voltage and of the harmonic voltages, i_a could be represented by the series $I_a + \sum_1^\infty I_{an}\cos n\omega_0 t$. It is seen from the preceding expression for i_a that, since $V_{a1}(X_a/\mu R_I)$ and $V_{gn} + V_{an}/\mu$ are small compared with V_{g1}, the effect of adding the quadrature and harmonic voltages is to produce additional currents of small amplitudes in phase quadrature with the existing currents. These currents have therefore a negligible effect on the amplitudes and only a small effect on the relative phases of the total currents at the various harmonic frequencies. Their main result is a slight change in the frequency of oscillation.

The amplitude of the fundamental component in i_a is therefore

$$I_{a1} = (2b_a/\pi)\int_0^\pi ((V_{g1} - V_{a1}/\mu)\cos\theta + V_g - V_{ca})^{3/2}\cos\theta\, d\theta \; - kV_{g1}/r_g$$

where V_{g1}/r_g is the amplitude of the fundamental component of the grid current. The integral is evaluated in the Appendix. Two cases must be considered depending on whether $V_{g1} - V_{a1}/\mu$ is greater or less than $V_g - V_{ca}$.

If $V_{g1} - V_{a1}/\mu < V_g - V_{ca}$ anode current flows continuously, and the result of integration is

$$I_{a1} = 1\tfrac{1}{2}b_a(V_g - V_{ca})^{\frac{1}{2}}(V_{g1} - V_{a1}/\mu)(1 - (V_{g1} - V_{a1}/u)^2/32(V_g-V_{ca})^2 - ..)$$

$$- kV_{g1}/r_g \qquad (5.2)$$

From (3.2) $I_{a1} = - V_{g1}/R_T$, and from (3.7) $V_{a1} = -V_{g1}R_I/R_T$

Let $N = -1\tfrac{1}{2}b_a(-V_{ca})^{\frac{1}{2}}(R_T + R_I/\mu)/(1 - kR_T/r_g)$ (5.3)

$K = -(V_g/V_{g1})/(1 + R_I/\mu R_T)$ . . . . (5.4)

Then (5.2) can be written in the form

$$1 = N(1 - V_g/V_{ca})^{\frac{1}{2}}(1 - 1/32K^2(1 - V_{ca}/V_g)^2 - ..) \qquad (5.5)$$

If $V_{g1} - V_{a1}/\mu > V_g - V_{ca}$ anode current flows only when $-\beta < \omega_0 t < \beta$ where $\cos\beta = -(V_g-V_{ca})/V_{g1}(1+R_I/\mu R_T) = K(1-V_{ca}/V_g)$

The result of integration is then

$$I_{a1} = (3b_a/2^{\frac{1}{2}})V_{g1}^{3/2}(1 + R_I/\mu R_T)^{3/2}\sin^4\tfrac{1}{2}\beta$$

$$\times (1 - (1/4)\sin^2\tfrac{1}{2}\beta - (5/128)\sin^4\tfrac{1}{2}\beta - ..) - kV_{g1}/r_g$$

(5.6)

Substituting for N, K and I_{a1} as above this becomes

$$1 = \tfrac{1}{2}N(\tfrac{1}{2}V_g/KV_{ca})^{\frac{1}{2}}H^2(1 - H/8 - 5H^2/512) \quad . . \quad (5.7)$$

where $H = 2\sin^2\tfrac{1}{2}\beta = 1 - K + KV_{ca}/V_g$ . . (5.8)

From (5.5) and (5.7) N can be found in terms of V_g/V_{ca} and K. Fig. 5 shows V_g/V_{ca} plotted as a function of N for various values of K. The method of using Fig. 5 to calculate the oscillation amplitude is explained in the next Section.

If the graphs were continued upwards it would be found that for all values of K less than 1 there is a point on each graph at which its slope becomes infinite and thereafter negative. This behaviour is investigated in Section 8 where it is shown that all values of V_g/V_{ca} greater than these critical values correspond to unstable states.

For design purposes it is sometimes useful to know the mean anode current. This is

$$I_a = (b_a/\pi)\int_0^{\pi}(V_{g1}(1 + R_I/\mu R_T)\cos\theta + V_g - V_{ca})^{3/2}d\theta \quad + kV_g/R_g$$

From the integral eveluated in the Appendix

If $V_{g1}(1 + R_I/\mu R_T) < V_g - V_{ca}$

$I_a = b_a(V_g - V_{ca})^{3/2}(1 + 3/16K^2(1 - V_{ca}/V_g)^2 + ..) + kV_g/R_g$

If $V_{g1}(1 + R_I/\mu R_T) > V_g - V_{ca}$

$$I_a = 3b_a(\tfrac{1}{2}V_{g1})^{3/2}(1 + R_I/\mu R_T)^{3/2}\sin^4\tfrac{1}{2}\beta$$
$$\times (1 + (1/12)\sin^2\tfrac{1}{2}\beta + (3/128)\sin^4\tfrac{1}{2}\beta) + kV_g/R_g \qquad (5.9)$$

Now $r_g \doteqdot -\frac{1}{2}R_g(V_{g1}/V_g)$ and kV_g/R_g is small compared with I_a, which may therefore be written with little error as

$$I_a = -\tfrac{1}{2}(V_{g1}/R_T)(1 + H/6 + 7H^2/192 + ..) \quad . \quad . \qquad (5.10)$$

To calculate the grid harmonic voltages the amplitudes of the anode harmonic currents are required. The two of most importance are the second and the third. Neglecting grid current, the amplitude of the second harmonic current in i_a is

$$I_{a2} = (2b_a/\pi)\int_0^{\pi}(V_{g1}(1 + R_I/\mu R_T)\cos\theta + V_g - V_{ca})^{3/2}\cos 2\theta d\theta$$

It is usually only for large values of oscillation amplitude that the harmonic voltages have appreciable magnitudes, and so only the case in which $V_{g1}(1 + R_I/\mu R_T) > V_g - V_{ca}$ need be considered.

Then $I_{a2} = 6b_a(\tfrac{1}{2}V_{g1})^{3/2}(1 + R_I/\mu R_T)^{3/2}\sin^4\tfrac{1}{2}\beta$

$$\times (1 - (5/4)\sin^2\tfrac{1}{2}\beta + (35/128)\sin^4\tfrac{1}{2}\beta + ..)$$
$$= -(V_{g1}/R_T)(1 - \tfrac{1}{2}H + H^2/64) \quad . \quad . \quad . \quad . \qquad (5.11)$$

Similarly

$$I_{a3} = -(V_{g1}/R_T)(1 - 4H/3 + 85H^2/192 - H^3/96) \qquad (5.12)$$

6. Method of calculation and example.

The first step towards calculating the oscillation amplitude is to find the valve constants b_a, b_g, V_{ca}, V_g and μ if these are not already known. To find b_a and V_{ca} the two-thirds power of the anode current is plotted ~~against is plotted~~ for negative values of grid voltage, the screen and/or anode voltage being kept constant at the normal working value. A straight line is drawn through the points ignoring the deviation from linearity at small values of i_a. The intercept of this line on the voltage axis is V_{ca}, and $b_a = I_o(-V_{ca})^{-3/2}$ where I_o is the value of i_a when $v_g = 0$. Strictly a correction should be made for the grid current, but even when $v_g = 0$ this is a very small fraction of the anode current and may safely be neglected.

b_g and V_{cg} are found by plotting the grid current for small positive values of grid voltage (say up to about $-V_{ca}/3$) A straight line is drawn ignoring deviations from linearity near to $v_g = 0$. The slope of this line is b_g and the intercept on the voltage axis is V_{cg}. The values of grid current should be those obtained when the normal anode or screen voltages are applied. μ will usually be known , but if not it can be found by any of the standard methods.

The constants R_T and R_I of the feedback network should next be evaluated, taking account of the inter-electrode capacitances and the grid input resistance.

To find the grid input resistance from (4.8) it is necessary to find first from (4.6) the value of V_g/V_{g1}, which in turn depends on V_{cg}/V_{g1} which is not yet known. However, V_{cg}/V_{g1} is always small, so that a good approximation to the true value of V_g/V_{g1} can be obtained by assuming a rough value for V_{cg}/V_{g1} Approximate values for r_g, R_T and R_I can then be calculated. N and K are found from (5.3) and (5.4) If the valve is a triode $k = 1$, and if a pentode it will usually be sufficiently accurate to take $k = 0.75$ if the true value is not known. V_g/V_{ca} can then be read off from Fig. 5 and so V_{g1} can be calculated using the previously found value of V_g/V_{g1}.

An approximate value for V_{g1} having been obtained in this way, more accurate values for V_g/V_{g1} and N and K can be calculated, and, by repeating the procedure described above, a better approximation for V_{g_1} can be found. This will usually be sufficiently close to the first value to make further calculations unnecessary.

Should the harmonic voltages be of interest their approximate values can be found from (3.5), (5.11) and (5.12)

A check should be made to ensure that the assumptions on which the analysis is based are fulfilled. The ratio R_g/X_g should be compared with the values given by Fig. 4. If the anode alternating voltage is a large fraction of the H.T. supply voltage the maximum grid voltage $V_{g1} + V_g$ and the minimum anode voltage $V_a + V_{g1}R_I/R_T$ should be calculated; If the corresponding point lies below the "knee" on the characteristic curves for the valve, i.e. in the region where the anode current varies rapidly with anode volatage the analysis will not be valid because of the departure from the three-halves power law. Otherwise reasonable, accuracy in the values of V_{g1} can be expected.

Two general conclusions may be drawn from the graphs of Fig. 5 If a large amplitude is required K should be small. This implies that the grid-leak resistance R_g should be small and the ratio $-R_I/R_T$ should ~~be~~ also be small. On the other hand if good regulation (i.e. constancy of amplitude with respect to changes of load) is more important both these quantities should be large.

Example: The following data apply to the triode section of a triode-hexode frequency changer valve: $b_g = 0.00065$ mhos $V_{cg} = -0.13$ v. $b_a = 0.00056$ mho/ $V^{\frac{1}{2}}$, $V_{ca} = -9.1$ v, $\mu = 20$. The feedback network is as shown in Fig. 1 The mutual inductance between the coils is -0.5 times the inductance of the anode coil and the coefficient of coupling may be taken as 1.

$R_g = 10^5$ ohms, $C_g = 0.1\mu F$. Frequency of oscillation = 1 Kc/s

Find the oscillation amplitude when the dynamic resistance of the anti-resonant circuit by itself (i.e. with the power supplies to the valve cut off) has the values 10, 5, and 2.5 KΩ.

From the data, $b_g R_g = 65$. Taking as a first guess $V_{g1} = V_{ca}$ Fig. 2 and (4.6) give $V_g/V_{g1} = -0.89$, and from (4.8), $r_g = 57.6$ KΩ When the dynamic resistance of the circuit is 10 KΩ the input resistance R_I is equal to a resistance of 10 KΩ in parallel with a resistance of 57.6×0.5^2 KΩ = 230 KΩ. Hence $R_I = 9.58$ KΩ and $R_T = -9.58 \times 0.5 = -4.79$ KΩ. From (5.3) and (5.4) N = 10.1 and K = 0.99 Fig. 5 gives V_g/V_{ca} as 2.30 and so $V_{g1} = 23.5$ V. The second approximation gives $V_g/V_{g1} = -0.88$ but makes a negligible difference to the values of R_I, R_T and N. The value for K is now 0.98. Hence $V_g/V_{ca} = 2.35$ and $V_{g1} = 24.3$ V. Similarly for resistances of 5 and 2.5 KΩ the values of V_{g1} are 14.9 and 9.0 V. Since the ratio R_g/X_g is 63 the possible error in V_g from this cause is less than 0.5% (Fig. 4)

An experimental arrangement was set up having the constants specified above and the oscillation amplitude was measured in the three cases. The values found were 23.8, 14.9 and 8.2 V. It should be pointed out that this example is particularly unfavourable to the theory, as this type of valve, because of the unusually small length-to-diameter ratio of the electrodes shows a considerable tail in the anode-current grid-voltage characteristic i.e. the actual current is greater than the theoretical value for values of i_a less than about 25% of I_o

The anode current may be considered to be the sum of the theoretical current and an extra current. From the integral given in the preceding Section for the value of I_{a1} it can be seen that when V_{g1} is large the angle of flow β is small. The extra current then flows mainly when $\cos\theta$ is positive and the true value of I_{a1} and hence of V_{g1} is greater than the theoretical value, but the difference is not large because, for large values of V_{g1}, the extra current is a small fraction of the total current.

When V_{g1} is small most of the extra current flows when $\cos\theta$ is negative, and is relatively greater, both in magnitude and duration, than when V_{g1} is large. I_{a1} and thus V_{g1} is then less than the theoretical value, and the fractional error should be greater than for large values of V_{g1}. This is borne out in the example above by the comparatively large error of 10% at the smallest value of V_{g1}. At the largest value the error is of opposite sign to that expected, but since the error is small this discrepancy may reasonably be attributed to experimental errors and inaccuracies in the calculation.

7. Amplitude instability - Squegging.

In the previous Sections the oscillation amplitude was calculated on the assumption that the amplitude was stable, but this is not always the case. Two types of instability - periodic and aperiodic - will be discussed.

For instability of any kind to exist it is necessary that a small increase in the oscillation amplitude should cause the amplification A round the whole circuit at the oscillation frequency to increase from its steady-state value of 1, i.e. dA/dV_{g1} must be positive. The difference between the two forms of instability is that for the periodic type this fundamental requirement is satisfied only when certain relations exist between the rate of change of the oscillation amplitude, the time constant R_gC_g, and various other parameters of the network and amplifier. This type of instability is essentially dynamical.

In contrast, the aperiodic type of instability does not depend on any of the time factors mentioned above, though these may limit the speed with which the amplitude moves from an unstable value.

Periodic instability -or squegging as it is colloquially named - is of the greater practical importance and will be treated first. When this occurs the oscillation amplitude becomes periodically modulated. As is well known this kind of instability can be eliminated by reducing the time constant of the grid resistor and capacitor.

When the value of the grid capacitor is increased to the critical point where instability sets in, the modulation is often of small amplitude and sinusoidal waveform. This is used as the basis of the following analysis.

If the amplitude of modulation is small the modulation may be regarded as a pertubation of the steady state and the effects which it produces may be calculated by quasi-linear methods. The high-frequency oscillation may be regarded as a carrier wave whose function is to convey the modulation through the amplifier and feedback network, both of which alter the amplitude and phase of the modulation in a calculable manner. So far as the modulating wave is concerned the oscillator may be regarded as a low-frequency feedback amplifier with known amplitude-frequency and phase-frequency characteristics, and Nyquist's criterion[3] for the stability of such amplifiers may be applied. This method of determining the amplitude stability has also been described by Edson[4].

The physical mechanism of squegging is roughly as follows. If the grid voltage of fundamental frequency has a small sinusoidal modulation the grid-bias voltage is also modulated sinusoidally, but to a lesser extent and with a lagging phase which depend on the magnitude of the time constant R_gC_g. The modulation on the anode current of fundamental frequency is the resultant of the modulations produced by the grid voltage of oscillation frequency and by the grid-bias voltage. Since an increase in the magnitude of the grid-bias voltage reduces the anode current the resultant modulation on the anode current has a leading phase and may also, owing to the effect of the amplifier characteristic have an increased magnitude.

In its passage through the feedback network the modulation is retarded and reduced in amplitude to an extent depending on the Q-factor and the modulation frequency. At some frequency the phase shift produced by the network may be equal and opposite to that occurring in the amplifier so that the total phase shift is zero. If only one such frequency exists and if at this frequency the total amplification of the modulation round the circuit is greater than 1, the modulation tends to increase and the oscillation amplitude is therefore unstable. If the amplification is less than 1, or if no frequency exists at which the total phase shift is zero, the oscillation amplitude is stable.

The operation of the oscillator will now be analysed from this point of view. It will be supposed that R_g/X_g is larger than the values indicated in Fig. 4 , and that the modulation frequency is small compared with the oscillation frequency, so that the formulae developed in the first part of Section 4 can be applied to the instantaneous values of the oscillation amplitude and grid-bias voltage. The effect of harmonic voltages is neglected.

To investigate the stability of any particular amplitude suppose that the oscillator is actually operating with this amplitude. Then the problem is to find whether this condition represents a state of stable or unstable equilibrium. In low-frequency feedback amplifiers the method of investigating stability is to interrupt the circuit at some point and to find the amplification and phase shift round the circuit over the whole frequency range. The equivalent procedure for the oscillator is as follows.

Suppose that a signal generator of Zero impedance having a terminal voltage v_3 equal to that between 5 and 4 (Fig. 1) is connected between 5 and 4 and let an impedor of value equal to the impedance at fundamental frequency between 5 and 4 looking towards the grid be connected between 3 and 4. Then the current flowing between 3 and 5 is zero and the circuit may be broken at this point without altering the voltages in any part of the oscillator.

Now let the signal-generator voltage be modulated in amplitude to become $V_3(1 + a\cos\omega_d t)\cos\omega_o t$. In theory ω_d should be varied over the entire range from $-\infty$ to ∞ , but if the Q-factor is large the amplification of the modulation is practically zero for comparatively small values of ω_d. Thus in practice ω_d is much less than ω_o.

The grid voltage of fundamental frequency becomes

$$V_{g1m}\cos\omega_o t = V_{g1}(1 + a\cos\omega_d t)\cos\omega_o t \qquad \ldots \qquad (7.1)$$

The subscript m is used to denote the modulated value of a quantity.

Let the grid-bias voltage be

$$V_{gm} = V_g(1 + a_1\cos(\omega_d t - \theta_1)) \quad . . \quad . . \quad (7.2)$$

To simplify the calculations it will be supposed that $V_{cg} = 0$

From (4.3) the mean value of the grid current over a period long compared with $1/\omega_o$ but short compared with $1/\omega_d$ is

$I_{gm} = V_{g1m}(\sin\phi_m - \phi_m\cos\phi_m)b_g/\pi$ in which

$\cos\phi_m = -V_{gm}/V_{g1m} = \cos\phi_o(1 - a\cos\omega_d t + a_1\cos(\omega_d t - \theta_1))$

Let $\phi_m = \phi_o + \phi_1$, ϕ_1 being small, then $\cos\phi_m = \cos\phi_o - \phi_1\sin\phi_o$

Hence $\phi_1\sin\phi_o = \cos\phi_o(a\cos\omega_d t - a_1\cos(\omega_d t - \theta_1))$

The expression for I_{gm} becomes

$I_{gm} = V_{g1}(b_b/\pi)(1 + a\cos\omega_d t)(\sin\phi_o - \phi_o\cos\phi_o + \phi_1\phi_o\sin\phi_o)$

On substituting for $\phi_1\sin\phi_o$ and putting (from (4.3))

$V_{g1}(\sin\phi_o - \phi_o\cos\phi_o)b_g R_g/\pi = -V_g = V_{g1}\cos\phi_o$, I_{gm} can be written

$$I_{gm} = -(V_g/R_g)(1 + a(1 + \phi_o b_g R_g/\pi)\cos\omega_d t - a_1(\phi_o b_g R_g/\pi) \times \cos(\omega_d t - \theta_1)) \quad (7.3)$$

This current is equal to the sum of the steady and low-freqeency currents flowing through R_g and C_g' $(= C_g + c_{gk})$ which is

$$I_{gm} = -V_{gm}/R_g - C_g'(dV_{gm}/dt)$$

$$= -(V_g/R_g)(1 + a_1\cos(\omega_d t - \theta_1) - a_1\omega_d C_g' R_g\sin(\omega_d t - \theta_1)) \quad (7.4)$$

In (7.3) $\cos\omega_d t$ may be written as

$\cos\theta_1\cos(\omega_d t - \theta_1) - \sin\theta_1\sin(\omega_d t - \theta_1)$

Equating the coefficients of $\cos(\omega_d t - \theta_1)$ in (7.3) and (7.4) gives

$$a_1/a = \cos\theta_1 \quad . . \quad . . \quad (7.5)$$

and equating the coefficients of $\sin(\omega_d t - \theta_1)$ gives

$$a_1/a = \sin\theta_1(1 + \phi_o b_g R_g/\pi)/\omega_d C_g' R_g$$

The ratio of the two equations for a_1/a is

$$\tan\theta_1 = \omega_d C_g' R_g/(1 + \phi_o b_g R_g/\pi)$$

Let $S = 1 + b_g R_g \phi_o/\pi = (b_g R_g/\pi)\tan\phi_o \quad . . \quad (7.6)$

This quantity is shown in Fig. 6 as a function of $b_g R_g$

The expression for $\tan\theta_1$ can then be written as

$$\tan\theta_1 = \omega_d C_g' R_g/S \quad . . \quad . . \quad (7.7)$$

From (4.7) the amplitude of the grid current of fundamental frequency is

$$I_{g1m} = 2I_{gm}(1 - \phi^2_m/10 \;..) \doteq 2I_{gm}(1 - \phi_o^2/10 \;..)$$

From (7.4) and (4.7) this is

$$I_{g1m} = I_{g1}(1 + a_1\cos(\omega_d t - \theta_1) - a_1\omega_d C_g' R_g \sin(\omega_d t - \theta_1)) \qquad (7.8)$$

The grid input resistance is therefore

$$r_{gm} = V_{g1m}/I_{g1m}$$

$$= r_g(1 + a\cos\omega_d t - a_1\cos(\omega_d t - \theta_1) + a_1\omega_d C_g' R_g \sin(\omega_d t - \theta_1)) \qquad (7.9)$$

To calculate the anode current the anode voltage must first be found. It was shown in Section 3 that the anode voltage of fundamental frequency could be represented as the sum of two components, one in phase with, and the other in quadrature with the grid voltage. It was further shown in Section 5 that the quadrature component has a negligible effect on the amplitude of the anode current. It maty therefore be assumed that the effective part of the modulated anode voltage is

$-V_{a1}(1 + a_2\cos(\omega_d t - \theta_2))\cos\omega_o t$ where a_2 and θ_2 have yet to be determined.

It was shown in Section 3 that no change occurs in the amplitude or phase of the modulation as the voltage wave passes through the feedback network from anode to grid, and the same is true for intermediate points of the network. The modulated voltage between the (open) terminals 3 and 4 is therefore

$$V_3(1 + a_2\cos(\omega_d t - \theta_2))\cos\omega_o t$$

To determine the stability of oscillation this voltage is to be combpared with the voltage between 5 and 4 produced by the signal generator, which is $V_3(1 + a\cos\omega_d t)\cos\omega_o t$. One of the conditions for instability is that the two modulations should have the same phase i.e. $\theta_2 = 0$. Another condition is that the amplification of the modulation should be greater than 1, i.e. $a_2/a > 1$. It will appear later than there can be only one value of ω_d for which $\theta_2 = 0$. Consequently, by Nyquist's theory the two conditions above are sufficient to determine the stability or instability of the oscillation amplitude.

The anode current is

$$i_{am} = b_a(v_{gm} + v_{aam}/\mu - V_{ca})^{3/2} - ki_{gm}$$

Putting $v_{aam} = -V_{a1}(1 + a_2\cos\omega_d t)\cos\omega_o t$ (since $\theta_2 = 0$) substituting for v_{gm} from (7.1) and (7.2) and expanding by Taylor's theorem gives

$$i_{am} = b_a((V_{g1} - V_{a1}/\mu)\cos\omega_o t + V_g - V_{ca})^{3/2} - ki_{gm}$$
$$+ ((aV_{g1} - a_2V_{a1}/\mu)\cos\omega_d t\cos\omega_o t + a_1V_g\cos(\omega_d t - \theta_1))$$
$$\times 1\tfrac{1}{2}b_a((V_{g1} - V_{a1}/\mu)\cos\omega_o t + V_g - V_{ca})^{\frac{1}{2}}$$

Let $1\frac{1}{2}b_a((V_{g1} - V_{a1}/\mu)\cos\omega_o t + V_g - V_{ca})^{\frac{1}{2}}$

$$= G_o + 2G_1\cos\omega_o t + 2G_2\cos 2\omega_o t + ..$$

On substituting for I_{g1m} from (7.8) the modulated amplitude of the anode current of fundamental frequency can be expressed as

$$I_{a1m} = I_{a1} + (G_o + G_2)(aV_{g1} - a_2V_{a1}/\mu)\cos\omega_d t + 2a_1G_1V_g\cos(\omega_d t - \theta_1)$$
$$- ka_1(V_{g1}/r_g)(\cos(\omega_d t - \theta_1) - \omega_d C_g' R_g\sin(\omega_d t - \theta_1))$$

Putting $V_{a1} = -V_{g1}R_I/R_T$ and $V_{g1} = -I_{a1}R_T$

$$I_{a1m} = I_{a1}(1 - (G_o + G_2)(aR_T + a_2R_I/\mu)\cos\omega_d t$$
$$- a_1(2G_1R_TV_g/V_{g1} - kR_T/r_g)\cos(\omega_d t - \theta_1)$$
$$- ka_1(R_T/r_g)\omega_d C_g' R_g\sin(\omega_d t - \theta_1)) \qquad (7.10)$$

The coefficients G_o etc are given by the following expressions

$$G_o + G_2 = (3b_a/\pi)\int_0^{\pi}(V_{g1}(1 + R_I/\mu R_T)\cos\theta + V_g - V_{ca})^{\frac{1}{2}}\cos^2\theta d\theta$$

$$G_1 = (3b_a/2\pi)\int_0^{\pi}(V_{g1}(1 + R_I/\mu R_T) + V_g - V_{ca})^{\frac{1}{2}}\cos\theta d\theta$$

From the integral evaluated in the Appendix the following expressions are obtained.

When $V_{g1}(1 + R_I/\mu R_T) < V_g - V_{ca}$ i.e. $(V_{ca}/V_g - 1) > 1/K$

$$\left.\begin{aligned} G_o + G_2 &= 1\tfrac{1}{2}b_a(V_g - V_{ca})^{\frac{1}{2}}(1 - 3/32K^2(1 - V_{ca}/V_g)^2 - ..) \\ G_1 &= -(3b_a/8K)V_g(V_g - V_{ca})^{-\frac{1}{2}}(1 + 3/32K^2(1 - V_{ca}/V_g)^2 + ..) \end{aligned}\right\} \qquad (7.11)$$

When $(V_{ca}/V_g - 1) < 1/K$

$$G_o + G_2 = 1\tfrac{1}{2}b_a(-V_g/2K)^{\frac{1}{2}}H(1 - 7H/16 + 19H^2/256 + 65H^3/8192 ..)$$
$$G_1 = 1\tfrac{1}{2}b_a(-V_g/8K)H(1 - 3H/16 - 5H^2/256 - 35H^3/8192 ..) \qquad (7.12)$$

where H is given by (5.8)

Expression (7.10) for the anode current was derived from a consideration of the relations between the currents and voltages in the amplifier, but these currents and voltages must also satisfy certain relations determined by the properties ofthe feedback network. Some of these properties were examined in Section 3. Expression (3.13) shows that the amplitude of the anode current of fundamental frequency associated with the anode voltage $-V_{a1}(1 + a_2\cos\omega_d t)\cos\omega_o t$ is

$$(V_{a1}/R_I)(1 + a_2\cos\omega_d t - 2a_2Q'(\omega_d/\omega_o)\sin\omega_d t)$$

The derivation of this expression, however, was based on the assumption that R_I is constant, but since the grid input resistance is modulated the impedance which was connected between 3 and 4 must also be modulated, and hence R_I is modulated. From (3.10) and (7.9) the modulated value of R_I is

$$R_{Im} = R_I(1 + (R_o/(R_o + r_g))(a\cos\omega_d t - a_1\cos(\omega_d t - \theta_1) + a_1\omega_d C_g' R_g\sin(\omega_d t - \theta_1)))$$

It will be recalled that R_o is the resistive component of the network output impedance with the power supplies to the valve disconnected. Substituting R_{Im} for R_I in the above expression for the amplitude of the fundamental anode current gives

$$I_{a1m} = I_{a1}(1 + a_2\cos\omega_d t - 2a_2Q'(\omega_d/\omega_o)\sin\omega_d t - (R_o/(R_o + r_g))(a\cos\omega_d t - a_1\cos(\omega_d t - \theta_1) + a_1\omega_d C_g' R_g\sin(\omega_d t - \theta_1)))$$

The trigonometrical terms in this and in (7.10) can be expressed in terms of $\cos\omega_d t$ and $\sin\omega_d t$. When this is done the coefficients of $\cos\omega_d t$ and $\sin\omega_d t$ in the two expressions can be equated. Equating the coefficients of $\cos\omega_d t$ and substituting for a_1 according to (7.5) and for $\omega_d C_g' R_g$ according to (7.7) gives

$$(a_2/a)(1 + (G_o + G_2)R_I/\mu) = R_o/(R_o + r_g) - (G_o + G_2)R_T - (R_o/(R_o + r_g) - kR_T/r_g + 2G_1R_TV_g/V_{g1})\cos^2\theta_1 - S(R_o/(R_o + r_g) - kR_T/r_g)\sin^2\theta_1$$

Equating coefficients of $\sin\omega_d t$ and substituting for $\omega_d C_g' R_g$

$$a_2/a = (R_g/2Q'SX_g)(2G_1R_TV_g/V_{g1} - (S - 1)(R_o/(R_o + r_g) - kR_T/r_g)) \times \cos^2\theta_1$$

Solving these two equations for $\cos^2\theta_1$ and a_2/a

$$\cos^2\theta_1 (2G_1R_TV_g/V_{g1} - (S - 1)(R_o/(R_o + r_g) - kR_T/r_g))$$
$$\times (1 + (1 + (G_o + G_2)R_I/\mu)R_g/2Q'SX_g)$$
$$= R_o/(R_o + r_g) - (G_o + G_2)R_T - S(R_o/(R_o + r_g) - kR_T/r_g) \quad (7.13)$$

and

$$(a_2/a)(1 + (G_o + G_2)R_I/\mu + 2Q'SX_g/R_g)$$
$$= R_o/(R_o + r_g) - (G_o + G_2)R_T - S(R_o/(R_o + r_g) - kR_T/r_g) \quad (7.14)$$

From (7.7) $\omega_d/\omega_o = (SX_g/R_g)(\cos^{-2}\theta_1 - 1)^{\frac{1}{2}}$

This equation gives the modulation frequency at which the phase shift of the modulation is zero. If the value of $\cos^2\theta_1$ given by (7.13) is greater than 1 or less than 0 no real value of ω_d exists and the oscillation amplitude is stable. It is clear than ω_d can have only one real value since positive and negative frequencies are physically indistinguishable. Using expression (7.13) for $\cos^2\theta_1$ the criterion for a real value of ω_d can be written as

$$(1 + (G_o + G_2)R_I/\mu)R_g/2Q'SX_g >$$
$$\frac{-(G_o + G_2)R_T - (2G_1R_TV_g/V_{g1} - kR_T/r_g)}{2G_1R_TV_g/V_{g1} - (S - 1)(R_o/(R_o + r_g) - kR_T/r_g)} > -1 \quad (7.15)$$

The second requirement for instability is that the amplification a_2/a should be greater than 1. From (7.14) this requirement can be expressed in the form

$$2Q'SX_g/R_g < - r_g/(R_o + r_g) - (G_o + G_2)(R_T + R_I/\mu)$$
$$- S(R_o/(R_o + r_g) - kR_T/r_g) \quad (7.16)$$

Both (7.15) and (7.16) must be satisfied before instability can occur. However, it can be shown that if (7.16) is satisfied the left hand part of (7.15) is automatically satisfied, and if the right hand part of (7.15) is not satisfied then the right hand side of (7.16) is negative and so (7.16) cannot be satisfied It follows that stability and instability are determined entirely by (7.16).

It is clear that the most important factor is not the time constant R_gC_g' but the quantity $2Q'SX_g/R_g = 2Q'b_g\tan\phi_o/\pi\omega_oC_g'$ Instability may be stopped either by reducing C_g or by or by increasing Q' in the same ratio, all other parameters remaining constant. Reducing the grid resistor R_g has the effect of increasing $\tan\phi_0$ and thus improving stability, but as can be seen from Fig. 6, the rate of change of $\tan\phi_0$ with R_g is small so that a greater proportional change in R_g would be required.. The formulae also show that if the frequency of an oscillator is varied over a wide range instability is more likely to occur at the higher frequencies - a fact already well known from experiment. This tendency is offset to some extent by the fact that Q usually increases with frequency.

It is also clear that no matter how large $C_g'R_g$ may be the oscillation amplitude is stable if the right-hand side of (7.16) is negative. But if the time constant is infinitely large no change can take place in the grid-bias voltage, and so the grid resistor R_g could be replaced by a source of fixed potential of value V_g. It is sometimes claimed that stable operation of oscillators is not possible with a fixed grid-bias voltage, but it is clear from the above that this statement is not generally true.

The criteria for stability which have been developed here are based on the assumption that if a small modulation of the amplitude tends to die out, the amplitude is stable under all conditions. It is conceivable, however, that the amplitude may be stable with respect to small disturbances but may become unstable when large disturbances such as the switching of the H.T. supply voltage, takes place. Experimental evidence on this point is inconclusive but appears to indicate that any such effect does not make a great difference to the critical values of X_g.

As an example of a stability calculation the oscillator, the constants of which are given in Section 6, will be examined. It is required to find the maximum value of R_g/X_g consistent with stable operation when the dynamic resistance of the anti-resonant circuit by itself is 10 KΩ. The Q-factor of the circuit by itself is 10.

From the performance figures worked out in Section 6 $V_{g1} = 24.3$ V, $V_g = -21.4$ V, $K = 0.98$, $R_I = 9.58\ K\Omega$, $r_g = 58.2\ K\Omega$ $R_T = -4.79\ K\Omega$.

From Fig. 6, S = 11.5, from (7.12) and (5.8), $G_o + G_2 = 0.001$ mho, and $G_1 = 0.00055$ mho

. The grid input resistance reduces the Q-factor of the netwoerk to the value 10 x 9.58/10 = 9.58. Since the ratio of the mutual inductance between coils to the inductance of the anode coil is 0.5, the output resistance R_o is $10 \times 0.5^2 = 2.5\ K\Omega$ Substituting these values into (7.15) gives

$0.0067 R_g/X_g > 0.016 > -1$ which is satisfied if $R_g/X_g > 2.4$ Similarly (7.16) is satisfied if $R_g/X_g > 114$. The oscillation amplitude is therefore stable if $R_g/X_g < 114$. In this particular example there is not a large margin between the value of R_g/X_g which is small enough to satisfy the stability requirement and the value which is large enough to give a negligible error in the formulae for the grid-bias voltage and the grid-input reisistance. (Fig. 4)

8. Oscillation hysteresis.

There are two types of oscillation hysteresis which may occur in a self-biassed oscillator. The first and more familiar type is associated with small oscillation amplitudes and with values of coupling between anode and grid circuits just sufficient to start or maintain oscillation. The second type is peculiar to large amplitudes and is rarely encountered in practical oscillators. It will be discussed at the end of the Section.

If the coupling (e.g. mutual inductance) between anode and grid circuits is increased from a small value, a point is reached at which oscillation begins spontaneously. It might be expected that when the coupling is just greater than this critical value the oscillation amplitude would be very small, and that the oscillation would cease when the coupling is reduced below this value.

Sometimes, however, it is found that when the coupling is increased beyond the critical value the oscillation starts suddenly with an amplitude of perhaps several volts. If the coupling is then reduced the oscillation persists down to a second critical value of coupling less than that required to initiate oscillation, and at this point the oscillation, still perhaps with a comparatively large amplitude, ceases abruptly. Similar behaviour may result if the coupling is fixed and some other parameter, such as the H.T. voltage, is varied.

The effect obviously indicates a form of amplitude instability affecting all amplitudes below a certain value. The instability in this case is of quite a different character from that studied in the previous Section. The sole effect of the time constant and the Q-factor is now to limit the speed with which the oscillation amplitude passes from zero through the unstable values to its final stable condition. It will be shown that the effect is caused by the variation of the grid-input resistance with oscillation amplitude. At small amplitudes the resistance may be only a fraction of its value at large amplitudes. Consequently the effect is most pronounced in those oscillators where the grid-input resistance contributes largely to the total damping in the feedback network.

The hysteresis effect may be explained as follows. Oscillation begins spontaneously, with an initially small amplitude, when the coupling is raised to a value such that the total amplification round amplifier and feedback network is 1. The oscillation causes the grid-bias voltage to become slightly more negative and thus decreases the amplification of the valve, but at the same time the grid-input resistance is increased so that the attenuation in the feedback network is reduced. If the total amplification increases with increasing oscillation amplitude a state of instability exists, and the amplitude increases and continues to increase until a stable position is reached at which any further increase in amplitude would cause the total amplification to become less than 1. It is also obvious that the value of coupling required to maintain an oscillation, once started, at an amplitude corresponding to a high value of grid-input resistance may be less than that required to initiate oscillation.

To investigate the effect and to find the critical values of coupling and oscillation amplitude the operation of the valve with small values of grid voltage must be examined. In Section 4 it was assumed ~~thhat the relation~~ that the relation between grid current and grid voltage is a linear one. This approximation is justified when most of the grid current flows when the grid voltage is positive, but may be seriously in error for negative grid voltages.

When the grid voltage is more negative than about 0.3 V the relation between grid current and voltage is approximately exponential

$$i_g = I_0 \exp(v_g/V_0) \qquad \ldots \qquad \ldots \qquad (8.1)$$

I_0 and V_0 being constants of the valve. V_0 depends mainly on the cathode temperature, and for oxide-coated cathodes operated at normal rating is usually between 0.1 and 0.14 V. I_o depends also on the cathode temperature and the valve geometry.

It will be assumed as in Section 4 that the time constant C_gR_g is so large that the voltage across C_g and R_g due to the flow of grid current is substantially constant. Then $v_g = V_{g1}\cos\omega_ot + V_g$. Substituting this into (8.1) on the assumption that v_g is always negative, gives for the grid current

$$i_g = I_o\exp(V_g/V_o)(B_o(V_{g1}/V_o) + 2B_1(V_{g1}/V_o)\cos\omega_ot + 2B_2(V_{g1}/V_o)\cos2\omega_ot + ..)$$

in which B_o etc. are the modified Bessel coefficients of the first kind (usually denoted by I_o etc)

The mean grid current is

$$I_g = I_o\exp(V_g/V_o)B_o(V_{g1}/V_o) = -V_g/R_g \quad .. \quad (8.2)$$

The amplitude of the component of fundamental frequency is

$$I_{g1} = 2I_o\exp(V_g/V_o)B_1(V_{g1}/V_o)$$

Hence the grid-input resistance at fundamental frequency is

$$r_g = V_{g1}/I_{g1} = -\tfrac{1}{2}R_gV_{g1}B_o(V_{g1}/V_o)/V_g\,B_1(V_{g1}/V_o) \quad (8.3)$$

For any assumed value of V_{g1} (8.2) can be solved graphically to find V_g. This is most conveniently done by plotting the curve $y = x\exp x$. Then $V_g = -xV_o$ where $y = (R_gI_o/V_o)B_o(V_{g1}/V_o)$

r_g can then be calculated from (8.3). For small values of V_{g1}/V_o the following approximations are valid

$$B_o(V_{g1}/V_o) = 1 + (\tfrac{1}{2}V_{g1}/V_o)^2$$

$$B_1(V_{g1}/V_o) = \tfrac{1}{2}V_{g1}/V_o + \tfrac{1}{2}(\tfrac{1}{2}V_{g1}/V_o)^3$$

Let V_g' be the value of V_g when $V_{g1} = 0$, and let $V_g = V_g' + dV_g$. If V_{g1} is small it follows from (8.2) and the approximations for the Bessel coefficients that

$$dV_g = (\tfrac{1}{2}V_{g1}/V_o)^2V_g'/(1 - V_g'/V_o) \quad .. \quad .. \quad (8.4)$$

In the same way (8.3) becomes

$$r_g = r_g'(1 + hV_{g1}^2) \quad .. \quad (8.5)$$

in which $h = -(1 + V_g'/V_o)/8V_o^2(1 - V_g'/V_o)$ and $r_g' = -R_gV_o/V_g'$

Over the range of values of I_o, V_o and R_g usually found in practice the value of V_g'/V_o lies between -2.5 and -6.5. Thus the grid-input resistance for very small oscillation amplitudes may be much less than for large amplitudes at which it is approximately $-\frac{1}{2}R_g V_{g1}/V_g$.

The effective amplification round the oscillator circuit at the oscillation frequency may be defined in the following way. Suppose thtat the network is interrupted between 3 and 5 (Fig. 1), and that a generator of zero impedance having a terminal voltage $V_3 \cos\omega_o t$ is connected between 5 and 4. Between 3 and 4 is connected an impedor of value equal to the impedance measured between 5 and 4 looking towards the grid. Let the amplitude of the grid voltage of oscillation frequency be V_{g1} and let the anode current have a fundamental component of amplitude I_{a1}. This produces a voltage of amplitude V_{3a} between 3 and 4 such that $V_{3a} = -I_{a1}R_T V_3/V_{g1}$. The amplification is therefore $A = V_{3a}/V_3 = -I_{a1}R_T/V_{g1}$

In the steady state A = 1, and for any particular amplitude to be stable A must increase as V_{g1} decreases and decrease as V_{g1} increases. Since V_{g1} is assumed to be small expression (5.2) may be used to represent the anode current. This leads to

$$A = -1\tfrac{1}{2}b_a(R_T + R_I/\mu)(V_g - V_{ca})^{\frac{1}{2}}$$
$$\times (1 - V_{g1}^2(1 + R_I/\mu R_T)^2/32(V_g - V_{ca})^2) \quad + kR_T/r_g \qquad (8.6)$$

Let R_T' and R_I' be the values of R_T and R_I when $V_{g1} = 0$ and oscillation is just about to begin. As the oscillation amplitude increases R_T and R_I also change because r_g increases according to (8.5). The effect of changes in r_g on the values of R_T and R_I was examined in Section 3. Using formulae (3.9) and (3.10) and expression (8.5), the amplification can be written as

$$A = 1 + V_{g1}^2((1 - kR_T'/r_g')(hR_o/(R_o + r_g') + (V_g'/V_o)^2/8(1-V_g'/V_o)(V_g'-V_{ca}$$
$$- (1 + R_I'/\mu R_T')^2/32(V_g' - V_{ca})^2)$$
$$- hkR_T'/(R_o + r_g'))$$

Small amplitudes will be unstable if A increases with V_{g1} i.e if the coefficient of V_{g1}^2 is positive. On substituting for h and rearranging the terms, this criterion for the existence of hysteresis becomes

$$V_g'/(V_g' - V_{ca}) - (R_o + kr_g'/(k - r_g'/R_T'))(1 + V_g'/V_o)/(R_o + r_g')$$

$$> V_o^2(1 + R_I'/\mu R_T')^2(1 - V_g'/V_o)/4(V_g' - V_{ca})^2 \quad \ldots \quad \ldots \quad (8.7)$$

The critical values of R_T and R_I are also related by (8.6) with $A = 1$ and $V_{g1} = 0$. Thus

$$1 = -1\tfrac{1}{2}b_a(R_T' + R_I'/\mu)(V_g' - V_{ca})^{\frac{1}{2}} + kR_T'/r_g' \quad \ldots \quad (8.8)$$

When hysteresis exists the initial amplitude, if it is of the order of a few volts, may be calculated by the methods described in Section 6. The minimum stable amplitude is difficult to calculate but it could be found, if required, by graphical methods. In examples where the amplitude is neither large nor small there may be difficulty in deciding whether formulae (4.6) and (4.8) or formulae (8.2) and (8.3) should be used to calculate the grid-bias voltage and the grid-input resistance. If these quantities are calculated from both sets of formulae it will usually be found that there is a range of value of V_{g1} over which the two sets give substantially the same results, and it is then obvious which set should be used for any particular value of V_{g1}. This type of hysteresis effect has also been studied experimentally by Zepler[5]

<u>Example:</u> The feedback network of an oscillator consists of an anti-resonant circuit having a dynamic resistance of $10^5\,\Omega$ connected between 3 and 4 with a coupling coil connected between 1 and 2. $R_g = 10^5\,\Omega$. The oscillation amplitude is adjusted by the mutual inductance between anode and grid coils, The valve constants are $b_g = 0.00065$ mhos, $V_{cg} = -0.13$ V, $b_a = 0.00056$ mho/$V^{\frac{1}{2}}$, $V_{ca} = -9.1$ V, $\mu = 20$, $I_o = 10^{-4}$ A, $V_o = 0.143$ V. It is required to find if a hysteresis effect exists, and if so the initial stable oscillation amplitude. which is obtained when the mutual inductance is increased from zero.

Using the method described earlier, equation (8.2) can be solved to find $V_g' = -0.44$ V. Then $r_g' = 32.5$ KΩ. From (8.8)

$R_T' = -410\,\Omega$, and $R_I'/\mu R_T' = -0.00084$. On substituting these values into (8.7) it is found that the inequality is satisfied, and a hysteresis effect therefore exists.

From the above figures the ratio of the mutual inductance to the inductance of the grid coil is -0.0167 at the critical point where oscillation begins. Following the methods of Section 6 the stable oscillation amplitude corresponding to this value of coupling is 5.2 V.

The second and less familiar type of hysteresis effect is peculiar to large oscillation amplitudes, and takes the form of a jump from one stable amplitude to another. The critical amplitude at which instability occurs can be found as follows.

For large values of V_{g1} the amplitude of the anode current of fundamental frequency is given by (5.6) and the amplification can be expressed as

$$A = -\tfrac{3}{4}b_a R_T(\tfrac{1}{2}V_{g1})^{\frac{1}{2}}(1 + R_I/\mu R_T)^{3/2}\, H^2(1 - H/8 - 5H^2/512) + kR_T/r_g$$

If instability exists dA/dV_{g1} must be positive. It may be assumed that when V_{g1} is large the quantities R_T, R_I, r_g and K are substantially independent of V_{g1}. On carrying out the differentiation, the condition that dA/dV_{g1} should be positive becomes

$$4(1 - K) > 3H(1 - H/48 - 11H^2/1536 + ..)$$

Substituting for H according to (5.8) and reversing the series

$$V_{ca}/V_g < (1 - K)/3K + (1 - K)^2/27K + .. \qquad (8.9)$$

As all terms are small compared with the first this reduces to

$$V_g/V_{ca} > 3K/(1 - K) \quad .. \quad .. \quad (8.9)$$

It is obvious from this that instability can occur only if $K < 1$, since negative values of V_g/V_{ca} are inadmissible.

It can be shown by differentiating (5.7) that the condition for dA/dV_{g1} to be zero is the same as that for $d(V_g/V_{ca})/dN$ to be infinite. If the graphs of Fig. 5 were continued upwards it would be found that for all values of K less than 1 the slope would ultimately become infinite and thereafter negative. This tendency is most obvious in the graph for K = 0.7. All values of V_g/V_{ca} greater than the critical values given by (8.9) correspond to unstable states.

Once the oscillation amplitude has passed the critical point there is nothing in the theory to prevent it from increasing indefinitely. The amplitude is then limited by the fact that when the grid voltage is large and the anode voltage small the three-halves law is no longer followed and the division of current between the electrodes changes in such a way that the anode receives a rapidly decreasing share of the total current. In most oscillators indeed this limitation takes effect before the amplitude reaches the critical unstable value. Instability of this type has in fact been observed only under somewhat artificial conditions.

When K is greater than 1 it follows from (5.7) and (5.8) that as N increases V_g/V_{ca} is asymptotic to the value $K/(K - 1)$

References.

(1) Terman, F.E.: "Radio Engineering" 2nd. Ed., p.350. (McGraw Hill Book Co., New York, 1937)

(2) Marique, J.: "Notes on the theory of diode rectification," Wireless Engr. 12, p.17, 1935.

(3) Nyquist, H.: "Regeneration theory," Bell Syst. Tech. J., 11, p.126, 1932.

(4) Edson, W.A.: "Intermittent behaviour in oscillators," Bell Syst. Tech. J., 24, p.1, 1945.

(5) Zepler, E.E.: "Oscillation hysteresis in grid detectors," Wireless Engr., 23, p.222, 1946.

Appendix.

Consider the integral $I = \int_0^{\pi}(a\cos\theta + b)^n\cos^m\theta d\theta$ in which a and n are positive and m is a positive integer or 0. The integrand is to be taken as zero when $a\cos\theta + b$ is negative. Two cases arise depending on whether $a > b$ or $a < b$.

If $a < b$ the integrand exists for all values of θ and

$$I = b^n\int_0^{\pi}(1 + (a/b)\cos\theta)^n\cos^m\theta d\theta$$

$$= b^n\int_0^{\pi}(\cos^m\theta + (na/b)\cos^{m+1}\theta + (a^2/b^2)(n(n-1)/2!)\cos^{m+2}\theta + \ldots)d\theta$$

If m is even

$$I = 2b^n\int_0^{\frac{1}{2}\pi}(\cos^m\theta + (a^2/b^2)(n(n-1)/2!)\cos^{m+2}\theta + \ldots)d\theta$$

$$= \frac{b^n\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}(m+1))}{(\frac{1}{2}m)!}\left(1 + \frac{n(n-1)}{2!}\frac{(m+1)a^2}{(m+2)b^2} + \ldots\right)$$

If m is odd

$$I = 2b^n\int_0^{\frac{1}{2}\pi}((na/b)\cos^{m+1}\theta + (n(n-1)(n-2)a^3/b^3 3!)\cos^{m+3}\theta + \ldots)d\theta$$

$$= \frac{nab^{n-1}\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}(m+2))}{(\frac{1}{2}(m+1))!}\left(1 + \frac{(n-1)(n-2)}{3!}\frac{(m+2)a^2}{(m+3)b^2} + \ldots\right)$$

If $a > b$

$$I = \int_0^{\beta}(a\cos\theta + b)^n\cos^m\theta d\theta \quad \text{where } \cos\beta = -b/a$$

Then $I = 2^n a^n\int_0^{\beta}(\sin^2\tfrac{1}{2}\beta - \sin^2\tfrac{1}{2}\theta)^n\cos^m\theta d\theta$

Let $x = \sin^2\frac{1}{2}\theta/\sin^2\frac{1}{2}\beta$. Then

$$I = 2^n a^n\sin^{2n+1}\tfrac{1}{2}\beta\int_0^1(1-x)^n x^{-\frac{1}{2}}(1 - 2x\sin^2\tfrac{1}{2}\beta)^m(1 - x\sin^2\tfrac{1}{2}\beta)^{-\frac{1}{2}}dx$$

$$= 2^n a^n\sin^{2n+1}\tfrac{1}{2}\beta\int_0^1(1-x)^n(x^{-\frac{1}{2}} - (2m - \tfrac{1}{2})x^{\frac{1}{2}}\sin^2\tfrac{1}{2}\beta + x^{3/2}(2m^2 - 3m + 3/8)\sin^4\tfrac{1}{2}\beta - x^{5/2}(4m^3/3 - 5m^2 + 53m/12 - 5/16)\sin^6\tfrac{1}{2}\beta)dx$$

But the integral $\int_0^1 x^p(1-x)^n\,dx$ defines the Beta function

B(p+1, n+1) which can be expressed in terms of Gamma functions as $\Gamma(p+1)\Gamma(n+1)/\Gamma(p+n+2)$ Substituting this in the above integral gives

$$I = 2^n \pi^{\frac{1}{2}} a^n \sin^{2n+1}\tfrac{1}{2}\beta \frac{\Gamma(n+1)}{\Gamma(n+1\frac{1}{2})} \Big(1 - \frac{(4m-1)}{2(2n+3)}\sin^2\tfrac{1}{2}\beta$$

$$+ \frac{3(16m^2 - 24m + 3)}{8(2n+3)(2n+5)}\sin^4\tfrac{1}{2}\beta$$

$$- \frac{5(64m^3 - 240m^2 + 212m - 15)}{16(2n+3)(2n+5)(2n+7)}\sin^6\tfrac{1}{2}\beta \quad + \cdots \Big)$$

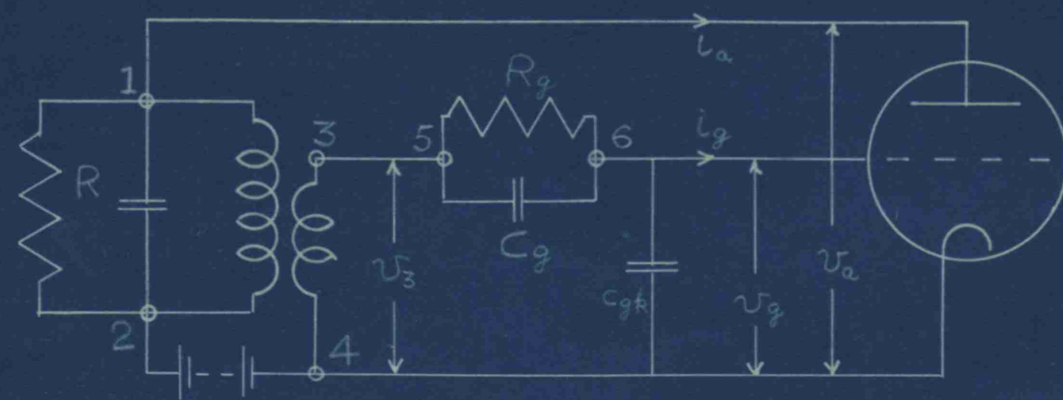

FIG 1 OSCILLATOR CIRCUIT

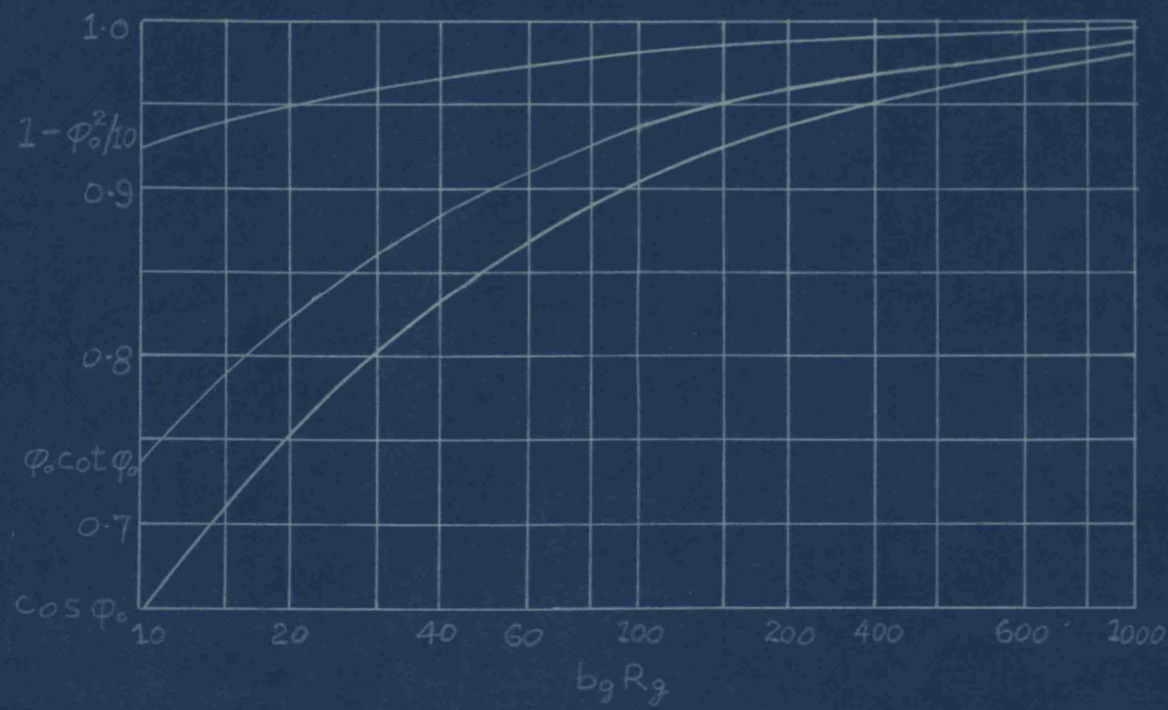

FIG 2 GRID BIAS AND GRID INPUT RESISTANCE FUNCTIONS

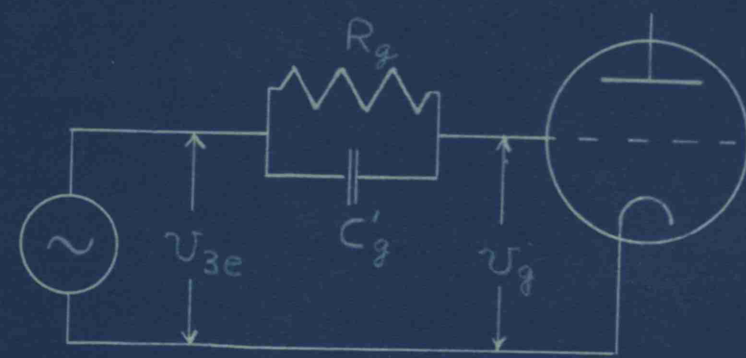

FIG 3 EQUIVALENT CIRCUIT FOR GRID CURRENT CALCULATION

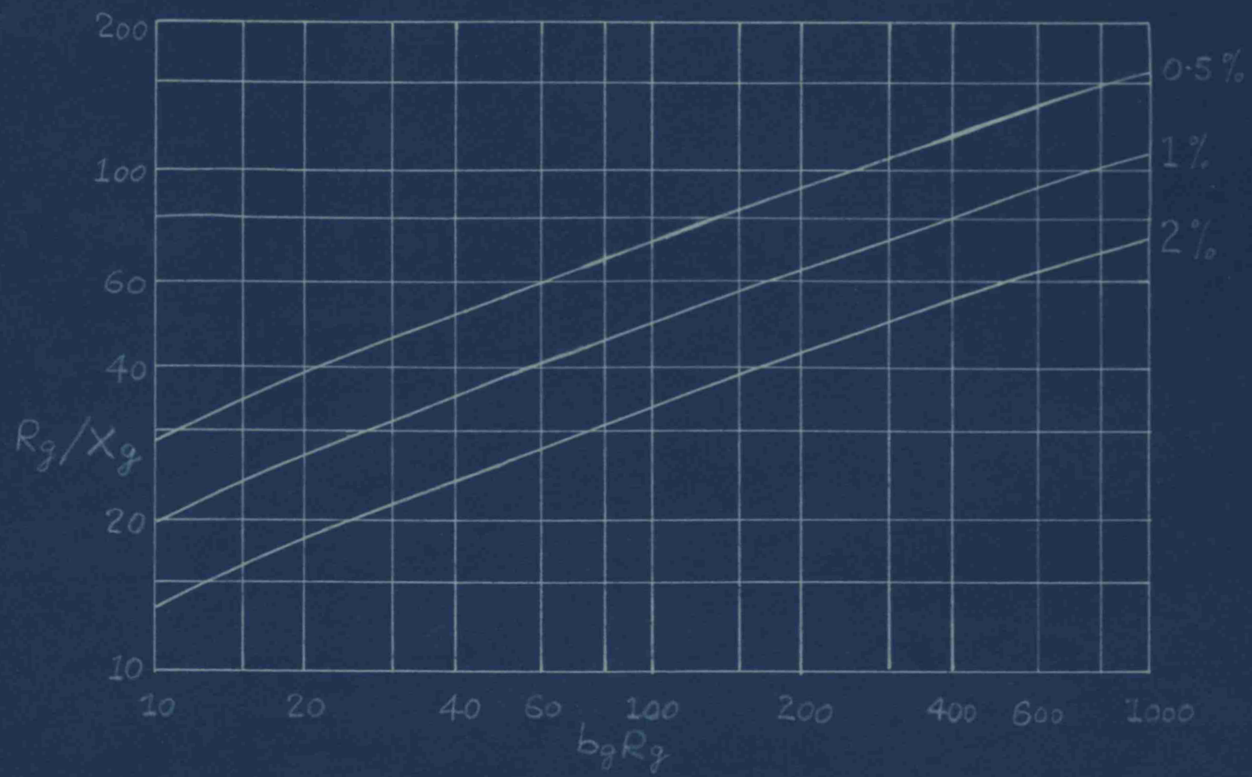

FIG 4 DECREASE IN GRID-BIAS VOLTAGE

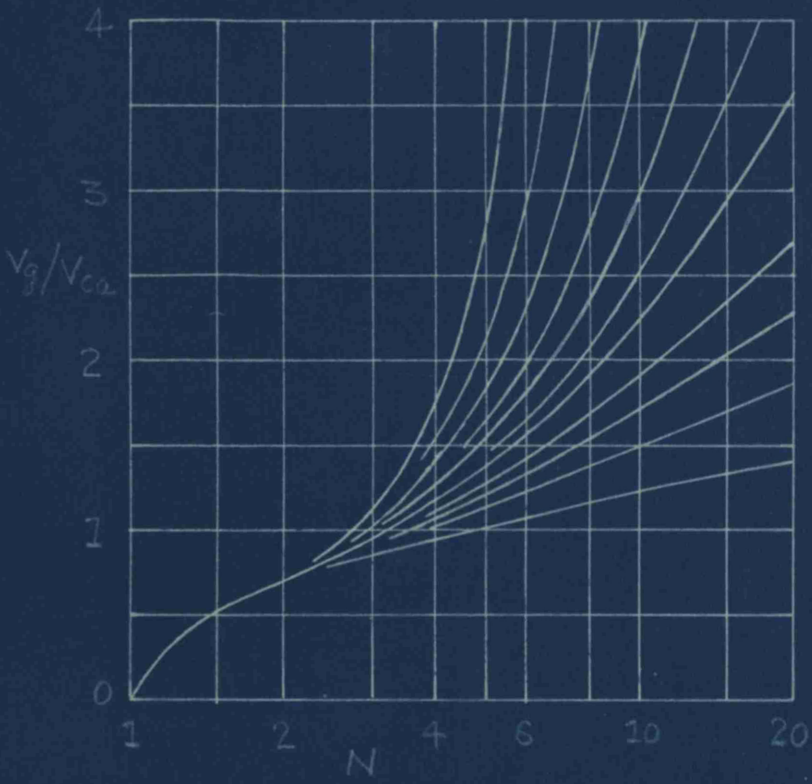

FIG 5 GRAPHS FOR GRID-BIAS VOLTAGE

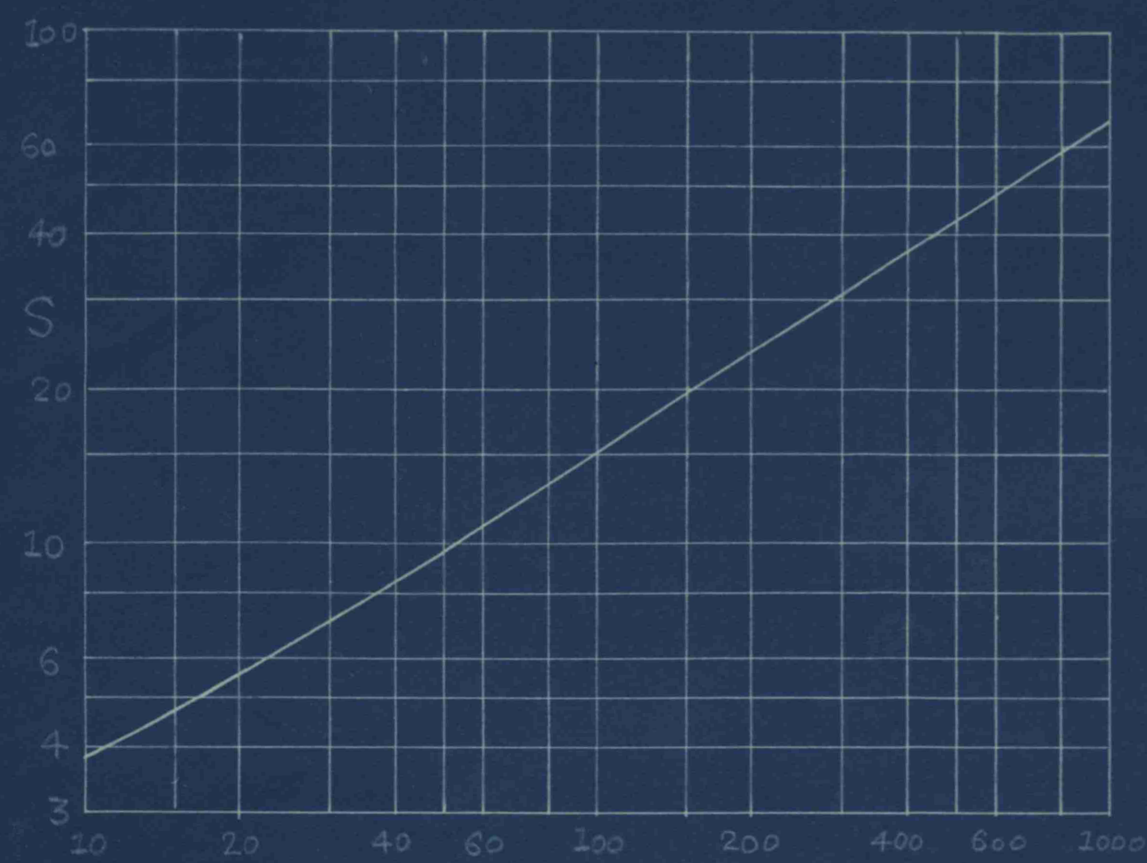

FIG 6 THE STABILITY CONSTANT S

PART 2.

Stability of Oscillation

in Valve Generators.

Summary.

A comprehensive theory of stability is developed which is applicable to a large class of harmonic oscillators. The paper is concerned mainly with the 4-terminal regenerative type of oscillator but the 2-terminal negative-resistance type is also considered. All known forms of instability appear as special cases of the general theory and some new forms are predicted. Stability is determined by the roots of a characteristic equation and stability criteria are obtained in the form of inequalities between the parameters of the amplifier and feedback network. When the feedback network is symmetrical with respect to the oscillation frequency the characteristic equation can be factorised to obtain independent criteria for frequency and amplitude stability. Hysteresis effects and periodic instability are analysed in detail. In addition to the general treatment, specific forms of the parameters appearing in the stability criteria are worked out for a three-halves-law amplifier with a semi-linear or exponential grid-current characteristic. Quantitative experimental confirmation is provided.

1. Introduction.

This investigation concerns the dynamical stability of free oscillations in generators producing a nearly-sinusoidal voltage waveform (harmonic oscillators) and is limited to those types known as "separable" oscillators which consist of a non-linear amplifier or resistance whose behaviour is independent of time, and a passive linear impedance network. Under certain conditions the frequency or amplitude of oscillation may become unstable and change to a new value or vary over a range of values. The object of this inquiry is to elucidate these conditions in their most general form.

Over the past 35 years the subject has attracted much attention. Early work on discontinuous amplitude changes was carried out by Appleton & van der Pol[1] and by Greaves[2]. Van der Pol also studied a form of discontinuous frequency instability[3]. Accounts of later work are given in recent publications[4-8].

All the early work, and most of the later, dealt with simple types of circuits, and it was usually assumed that the behaviour of the amplifier or non-linear resistance could be represented by a few terms of a power series. It was also supposed that the working point of the amplifier was independent of oscillation amplitude i.e. the non-oscillatory part of the voltage wave was of fixed amplitude.

In modern oscillators the amplifier characteristic can seldom be represented by a low-degree polynomial. Moreover grid-leak bias or some other form of amplitude control is invariably used. This causes the amplifier working point to vary with oscillation amplitude and may also give rise to a periodic instability sometimes known as "squegging".

Van Slooten[9,10] examined this type of instability by setting up a second order differential equation for the oscillation amplitude, stability being determined by the roots of the auxiliary equation. A simple form of circuit and valve characteristic was assumed.

A different approach was proposed by Edson[11] who suggested that the Nyquist criterion could be applied to determine the stability of oscillation amplitude by considering the changes in magnitude and phase experienced by a small sinusoidal modulation of the amplitude when transmitted through the amplifier and feedback network.

These two methods are limited to oscillators having feedback networks with a response characteristic symmetrical about the oscillation frequency e.g. a high-Q antiresonant circuit. In asymmetrical networks amplitude changes are accompanied by frequency changes which the methods are unable to take into account. Other suggestions, some of which are considered later, have been put forward. An account of these can be found in Edson's book.[12]

In their different ways all the methods so far proposed are restricted in scope, and the diversity of treatment obscures the fundamental unity of all forms of instability. The piecemeal treatment also suggests that there may exist types of instability hitherto overlooked. A new and more general theory seems desirable which would include all possible types of instability as special cases and be applicable to any kind of network or amplifier. It should also be possible to obtain numerical results for practical oscillators without having to make violent assumptions and drastic approximations. This would allow a theoretical and experimental check to be made on the validity of any proposed stability criterion.

2. List of symbols.

$A(p)$	= Amplitude stability function. See (8.2)
b_a b_g	= Constants of the valve. See (4.19) and (4.21)
$D(p)$	= Stability function for the oscillator. See (6.4)
$F(p)$	= Frequency stability function. See (8.2)
G_n	= Fourier coefficients of the grid-anode transconductance. See (5.10)
H	$= 1 - K + K/Y$
i_a i_g	= Anode and grid currents.
i_{a1} i_{g1}	= Anode and grid currents of fundamental frequency.
i_{ad} i_{gd}	= Perturbations of anode and grid currents.
k	= Fraction of total current reaching the anode.
K	$= -V_g/V_{g1}\left\|1 + Z_I/\mu Z_T\right\|$
N	= Parameter of the steady state. See (4.13) and (4.14)
p	$= a + j\omega_d$, complex frequency.
R_a R_g	= Anode and grid circuit D.C. resistances.
r_g	= Grid input resistance at fundamental frequency.
S_n	= Fourier coefficients of the grid input conductance. See (5.1)
u_n	= Complex amplitudes of transient grid voltage components. See (5.2)
v_a v_{a1}	= Anode voltage, anode voltage of frequency ω_o.
V_e	= Amplitude of equivalent grid voltage of frequency ω_o. See (4.5)
v_g V_g	= Grid voltage, mean grid voltage.
v_{g1} V_{g1}	= Grid voltage and amplitude of grid voltage at frequency ω_o.
v_{ad} v_{gd}	= Perturbations of anode and grid voltages.
V_{ca} V_{cg}	= Grid voltages required to cut off anode and grid currents.
Y	$= V_g/V_{ca}$
Z_i Z_o Z_t	$= R_i + jX_i$ etc. Open-circuit input, output and transfer impedances of the feedback network at frequency ω_o
Z_I Z_θ Z_T	= Modified values of Z_i etc. See (4.11)
Z_i^+ Z_i^o Z_i^-	$= Z_i(p + j\omega_o)$ $Z_i(p)$ $Z_i(p - j\omega_o)$ etc
Z_E	$= R_E + jX_E = (Z_T + Z_I/\mu)/(1 - kZ_T/r_g)$
θ	= Phase angle between anode current and grid voltage.
μ	= Amplification factor of the valve.
ω_o	= Oscillation frequency. (Fundamental)

3. Method of solution.

The method consists in examining the stability of the possible steady states of the oscillator. By a "possible steady state" is meant a condition in which the system is in equilibrium with a periodic wave of constant amplitude and frequency. Stability is determined by the nature of the disturbance produced when a possible steady state is momentarily perturbed by a small external force, equilibrium being stable if the amplitude of the disturbance decreases with time. If equilibrium is stable the possible steady state may be an actual steady state but if equilibrium is unstable the possible steady state cannot be realised physically.

When the disturbance is small the oscillator behaves towards it like a linear network with time-varying parameters. Now the transients in any linear system can be represented as the sum of a number of independent normal modes and to find the form of these modes in an oscillator it is sufficient to consider the simplest example.

Suppose that the oscillator consists of a 2-terminal linear passive network having an admittance $Y(D)$, $(D=d/dt)$, connected across a suitable non-linear resistance element. Let ω_0 be the frequency of steady oscillation. If a small disturbance v_d is added to the steady state voltage then, so far as v_d is concerned, the non-linear resistance behaves like a linear resistance with a conductance g varying periodically at the oscillation frequency. Thus

$$g = \sum_{0}^{\infty} g_r \cos(r\omega_0 t + \phi_r) \quad . \quad . \quad \quad . \quad . \qquad (3.1)$$

The currents produced by v_d in the linear network and non-linear resistance must be equal and opposite. i.e.

$$Y(D)v_d = -gv_d \quad . \quad . \quad \quad . \quad . \qquad (3.2)$$

In a linear network with constant parameters the general transient is the sum of a number of elementary waves of the form $\exp.\alpha t\cos(\omega_d t + \theta)$. It is clear that the elementary transient in an oscillator must be different from this, for although the current in the linear network would be of the same form, the current in the non-linear resistance would be the sum of an infinite number of waves of different frequencies. Inspection of equation (3.2) suggests that a possible solution would be

$$v_d = \exp.at \sum_{-\infty}^{\infty} V_n \cos((n\omega_0+\omega_d)t + \theta_n) \quad . \quad . \qquad (3.3)$$

The currents in both linear and non-linear branches would then have components of all the frequencies $n\omega_0+\omega_d$ and by choosing suitable values for the parameters these currents could be made equal and opposite.

From the analytical point of view, Y(D) is a rational function of D and so (3.2) is a linear differential equation with periodic coefficients. Floquet's theory of such equations[16] shows that the solutions are of the form (3.3). Solutions in which $n\omega_0$ is replaced by $nk\omega_0/m$, k and m being integers, are also theoretically possible but it can be shown that these require exact relations between the coefficients g_r and must therefore be excluded on physical grounds.

(3.3) represents the terminal voltage for one transient normal mode but equation (3.2) can usually be satisfied for more than one value of a and ω_d. The general disturbance can therefore be represented as the sum of a set of normal-mode voltages each similar to (3.3). In the more convenient complex notation (3.3) becomes

$$v_d = V_d \sum_{-\infty}^{\infty} u_n \exp(p + jn\omega_0)t \qquad (3.4)$$

where $p = a + j\omega_d$ and the coefficients u_n are complex numbers. The actual voltage is the real part of this expression With each normal mode is associated a characteristic complex frequency p, and the criterion for stability is therefore that all possible values of p should have negative real parts since this ensures that v_d ultimately vanishes.

By writing n + m for n in (3.4) it is seen that if p is a solution then so is $p + jm\omega_0$, m being any integer. The imaginary part of p may therefore be restricted to the range $-\frac{1}{2}\omega_0 < \omega_d < \frac{1}{2}\omega_0$ Also, by writing -n for n it follows that if p is a solution then so is p* (complex conjugate)

To calculate p, the currents of like frequencies in the network and non-linear resistance are equated. At any particular (complex) frequency $p + jq\omega_o$ the current in the linear network is of (complex) amplitude $V_d u_q Y(p + jq\omega_o)$, but the current in the non-linear resistance depends on all the values of $V_d u_n$. Thus an infinite set of linear simultaneous equations for the unknowns u_n is obtained, the coefficients of u_n being functions of p and the quantities g_r and $\emptyset_r$ occuring in (3.1) When these equations are written in canonical form the condition for compatibility[18] is that the infinite determinant formed by the coefficients should vanish. This gives a determinantal equation from which the values of p could in principle be calculated. This equation is a more general form of Hill's determinantal equation[17] However, to make the calculation practicable the equation must be simplified.

In many oscillators the steady-state voltage is nearly sinusoidal because the network impedance at the harmonic frequencies is very small. In calculating the transient behaviour it should therefore be permissible to neglect all terms associated with harmonic frequencies. This means that the expression for v_d can be restricted to three terms

$$v_d = V_d \sum_{-1}^{1} u_n \exp(p + jn\omega_o)t \quad . . \quad (3.5)$$

Only the first three terms in the Fourier series for g need then be taken into account and the infinite determinant is reduced to its three central rows and columns. (A similar approximation was used by Hill)

When (3.5) is adopted the previously-noted restriction on the range of ω_d becomes obligatory in order to confine the transient frequencies to the appropriate range.

For regenerative oscillators of the type shown on Fig.1 (which are the main subject of the paper) the procedure is similar but the algebra is naturally more complicated. The amplifier has input, output, and transfer conductances each of the form (3.1) and three functions are needed to specify the performance of the feedback network namely, the open-circuit input impedance Z_i between terminals 1 and 3, the open-circuit output impedance Z_o between 2 and 3, and the open-circuit transfer impedance Z_t between 1,3 and 2,3.

Both anode and grid transient voltages are of the form (3.5), and since the input and output currents and voltages of the amplifier are related through the network impedances, all currents and voltages can be expressed in terms of a single quantity - the amplifier input (grid) voltage.

For simplicity the power supplies and the grid-bias arrangement are not shown, and a triode is depicted although the analysis applies equally to tetrodes and pentodes. The values of Z_i and Z_o for D.C. are taken as R_a and R_g, and these will usually be equal to the H.T. decoupling resistance and the grid-leak resistance. Z_t is zero for D.C. i.e. there is no path for D.C. between the anode and grid except via the cathode terminal.

Before stability can be investigated the possible steady states of the oscillator must be calculated.

4. Amplitude and frequency in the steady state.

The method used here is an extension of a previous analysis[19] and the general procedure - sometimes known as the method of equivalent linearisation - is to consider only the constant and fundamental frequency components of the currents and voltages. When, as is usual, the amplifier input voltage is nearly sinusoidal the method yields a good approximation even when the currents are markedly non-sinusoidal.

The grid-cathode voltage is first considered. Let this be

$$v_g = V_{g1}\cos\omega_o t + V_g \quad . \ . \quad . \ . \qquad (4.1)$$

V_g is the grid-bias voltage which may be derived either from a fixed source of e.m.f. or from the flow of grid current through the grid-leak resistance R_g. The second arrangement is examined first. If the anode voltage is not too small or does not vary too greatly, the grid current is well approximated by a single-valued function of v_g only.

$$i_g = e(v_g)$$

When v_g is periodic i_g is also periodic. Thus

$$i_g = I_g + I_{g1}\cos\omega_o t + \text{etc.}$$

The constant component I_g flows through R_g to produce the bias voltage. If all the bias is produced in this way Fourier analysis gives

$$V_g = -I_g R_g = -(R_g/\pi)\int_o^{\pi} e(V_{g1}\cos x + V_g)dx \qquad (4.2)$$

For any given value of V_{g1} this is an equation for V_g. Since i_g has harmonic components, the assumption that v_g is sinusoidal implies that R_g is shunted by a capacitor of suitable size.

Due to the flow of grid current the amplifier has an A.C. input resistance r_g given by the ratio of the fundamental components of grid voltage and current. Thus

$$V_{g1} = r_g I_{g1} = (2r_g/\pi)\int_o^{\pi} e(V_{g1}\cos x + V_g)\cos x dx \qquad (4.3)$$

from which r_g may be determined. If the function $e(v_g)$ is such that V_g is proportional to V_{g1} then r_g is constant and conversely. When the bias voltage is obtained from a fixed source and is sufficient to prevent the flow of grid current, $r_g = \infty$. When the fixed bias voltage is insufficient to stop grid current r_g is still given by (4.3), but (4.2) is not then valid.

To calculate the anode current i_a it is assumed that the cathode current i_k is a function of $v_g + v_a/\mu$, v_a being the anode-cathode voltage and μ a constant. For a triode $i_a = i_k - i_g$ and for multigrid valves i_a is usually a fairly constant fraction of this. Thus in all cases

$$i_a = f(v_g + v_a/\mu) - ki_g$$

where $f(v)$ is a single-valued function of v. For triodes $k = 1$ and for many pentodes $k \doteqdot 0.8$. It is more convenient to write

$$i_a = f(v_g + v_{aa}/\mu - V_{ca}) - ki_g \qquad (4.4)$$

where v_{aa} is the alternating component of v_a and V_{ca} depends on the mean anode voltage. If the anode circuit contains a decoupling resistance, V_{ca} will vary with the oscillation amplitude. This is discussed in Appendix 3. Meanwhile V_{ca} is assumed to be constant.

The alternating anode voltage is assumed to be sinusoidal.

Let $$v_g + v_{aa}/\mu = V_g + V_e\cos(\omega_0 t + \theta) \quad . . \qquad (4.5)$$

Then $$i_a = f(V_e\cos(\omega_0 t + \theta) + V_g - V_{ca}) - ki_g \qquad (4.6)$$

Let $K = -V_g/V_e$ and $Y = V_g/V_{ca}$. By Fourier analysis the component of i_a of fundamental frequency is

$$i_{a1} = gh(K,Y)V_e\cos(\omega_0 t + \theta) - kV_{g1}\cos\omega_0 t/r_g \qquad (4.7)$$

where $$V_e gh(K,Y) = (2/\pi)\int_0^{\pi} f(V_e\cos x + V_g - V_{ca})\cos x\,dx \qquad (4.8)$$

g is an arbitrary positive constant having the measure of conductance which can be chosen to give a suitable scale for the function $h(K,Y)$.

The actual currents and voltages are now replaced by their complex amplitudes, the reference quantity being the grid alternating voltage. Using the same symbols as for the real voltages (4.7) becomes

$$i_{a1} = (v_{g1} + v_{a1}/\mu)gh(K,Y) - kv_{g1}/r_g \qquad (4.9)$$

Now in the impedance network of Fig.1. the currents and voltages are connected by the following relations:

$$
\left.\begin{aligned}
i_{g1} &= v_{g1}/r_g \\
i_{a1} &= -v_{g1}/Z_t - i_{g1}Z_o/Z_t = -v_{g1}(1 + Z_o/r_g)/Z_t \\
v_{a1} &= v_{g1}Z_i/Z_t + Z_n i_{g1} = v_{g1}(Z_i/Z_t + Z_n/r_g) \\
\text{where } Z_n &= Z_i Z_o/Z_t - Z_t
\end{aligned}\right\} \qquad (4.10)
$$

Suppose that the network is modified by connecting a resistance r_g between 2 and 3. The new network could be replaced by a three-element network similar to the original but with modified values for the elements. Denoting the new elements by Z_I Z_Θ Z_T

$$
\left.\begin{aligned}
Z_I &= Z_i - Z_t^2/(r_g + Z_o) = (Z_i + Z_n Z_t/r_g)/(1 + Z_o/r_g) \\
Z_\Theta &= Z_o/(1 + Z_o/r_g) \\
Z_T &= Z_t/(1 + Z_o/r_g) \\
Z_N &= Z_I Z_\Theta/Z_T - Z_T = Z_n
\end{aligned}\right\} \qquad (4.11)
$$

It is noted that $Z_\Theta/Z_T = Z_o/Z_t$, and if $Z_n = 0$ then also $Z_I/Z_T = Z_i/Z_t$.

From (4.10) and (4.11) $i_{a1} = -v_{g1}/Z_T$ and $v_{a1} = v_{g1}Z_I/Z_T$

Substituting these into (4.9) gives

$$1 = -gh(K,Y)(Z_T + Z_I/\mu) + kZ_T/r_g$$

and finally putting

$$Z_E = (Z_T + Z_I/\mu)/(1 - kZ_T/r_g) \qquad (4.12)$$

gives $\quad 1 = -gh(K,Y)Z_E$

This is the equation defining the steady state. Writing $Z_E = R_E + jX_E$, and similarly for Z_I and Z_T, and equating real and imaginary parts of the equation gives

$$1 = Nh(K,Y) \qquad (4.13)$$

where $\quad N = -gR_E = -g(R_T + R_I/\mu)/(1 - kR_T/r_g) \qquad (4.14)$

and $\quad X_E = 0$

or $\quad X_T + X_I/\mu + k(R_I X_T - R_T X_I)/\mu r_g = 0 \qquad (4.15)$

If $f(v)$ is an increasing function of v then $gh(K,Y)$ is always positive. Hence R_E and so also R_T must be negative.

Equation (4.14) gives the amplitude, and (4.15) the frequency of oscillation but the equations are not completely independent. r_g will usually vary more or less with the oscillation amplitude and μ and k may also vary slightly. The frequency of oscillation will generally be affected by all these changes. For the frequency to be independent of amplitude and of changes in the amplifier it is necessary that Z_i, Z_o and Z_t should be entirely resistive at the frequency ω_o. If this condition is satisfied (4.11) shows that $X_T = X_I = 0$ whatever the value of r_g and the frequency equation (4.15) is then independent of μ k and r_g.

In simple oscillators only one frequency is possible but with more complicated networks two or more values of ω_o may be found to satisfy (4.15). These values will be frequencies of possible steady states only if corresponding values of Y can be found to satisfy the amplitude equation (4.14).

The parameters K and θ are found as follows. When expressed in complex form the voltage $V_e \cos(\omega_o t + \theta)$ has an amplitude $V_e \exp j\theta$. On comparing (4.7) and (4.9) it is seen that

$$V_e \exp j\theta = v_{g1} + v_{a1}/\mu = v_{g1}(1 + Z_I/\mu Z_T)$$

Hence $\quad K = -V_g/V_e = -V_g/V_{g1}\left|1 + Z_I/\mu Z_T\right| \qquad (4.16)$

and
$$\left.\begin{aligned} \exp j\theta &= (1 + Z_I/\mu Z_T)/\left|1 + Z_I/\mu Z_T\right| \\ \exp -j\theta &= (1 + Z_I^*/\mu Z_T^*)/\left|1 + Z_I/\mu Z_T\right| \end{aligned}\right\} \qquad (4.17)$$

When Z_T is resistive Z_I must also be resistive and then

$$K = -V_g/V_{g1}(1 + R_I/\mu R_T) \qquad \theta = 0 \qquad (4.18)$$

To obtain working formulae, particular forms of the functions $e(v_g)$ and $f(v)$ must be studied. The process of solution will therefore be illustrated with reference to those forms which occur most often in practice. Details of the calculations are given in Appendix 1.

It is first assumed that the grid-bias voltage is obtained by a grid-leak and condenser arrangement. When the oscillation amplitude is fairly large grid current flows mainly when v_g is positive, and if the minimum anode voltage is well above the maximum grid voltage a fair approximation to i_g is

$$\left.\begin{aligned} i_g = e(v_g) &= b_g(v_g - V_{cg}) \quad \text{when } v_g > V_{cg} \\ &= 0 \quad \text{when } v_g \leqslant V_{cg} \end{aligned}\right\} \qquad (4.19)$$

b_g and V_{cg} are measurable constants. Then

$$\left.\begin{aligned} V_g &= V_g'(1 - V_{cg}/V_{g1}) \\ r_g &= r_g'(1 + V_{cg}/V_{g1}) \end{aligned}\right\} \qquad (4.20)$$

These formulae are valid for small values of V_{cg}/V_{g1}. Fig.2 shows $-V_g'/V_{g1}$ and r_g'/R_g as functions of $b_g R_g$. These are the familiar curves for the diode rectifier.

When the amplifier operates in the space-charge-limited condition the cathode current follows a three-halves power law

$$\left.\begin{aligned} f(v) &= b_a v^{3/2} \quad \text{when } v \geq 0 \\ &= 0 \quad \text{when } v \leq 0 \end{aligned}\right\} \quad (4.21)$$

with $\quad v = V_e \cos(\omega_0 t + \theta) + V_g - V_{ca}$

b_a is a constant and V_{ca} is the grid voltage required to reduce the anode current to zero when the anode voltage is constant. The quantity g in (4.7) is taken as

$$g = 1\tfrac{1}{2} b_a (-V_{ca})^{\frac{1}{2}}$$

Then $\quad N = -1\tfrac{1}{2} b_a (-V_{ca})^{\frac{1}{2}} R_E \quad \ldots \quad \ldots \quad (4.22)$

Fig.3 shows N as a function of K and Y calculated from (4.8).

To find the amplitude and frequency of oscillation approximate values of V_g/V_{g1} and r_g are found from Fig.2 by neglecting V_{cg}. Z_I and Z_T are then calculated and ω_0 is found by solving (4.15). K and N can next be evaluated and finally Y is read off from Fig.3. A value for V_{g1} is thus obtained from which V_{cg}/V_{g1} can be found, and by repeating the procedure taking V_{cg} into consideration, better approximations for V_{g1} and ω_0 can be obtained.

The results will be correct only if the expressions for i_a and i_g are valid. Slight departures from the three-halves law at low currents produce little error, but large deviations may occur if the minimum anode voltage approaches the "knee" in a pentode or is comparable with the maximum grid voltage in a triode.

Large amplitudes require a small value of K, but good regulation is obtained when K is large i.e. the change of amplitude with load is small. Large values of R_g and small values of μ give large values of K. Since $f(v) = 0$ when $v < 0$ it follows that $V_e + V_g - V_{ca} > 0$ for otherwise the anode current would always be zero. Hence if $K > 1$ the upper limit of Y is $K/(K - 1)$ when grid current is neglibly small. However the three-halves law would almost certainly not be followed for these extreme values. Other features of the graphs are discussed in later Sections.

When the grid-bias voltage is obtained from a fixed source and is large enough to stop the flow of grid current Fig.3 can again be used to find V_{g1}. Y is now a constant, N can be evaluated as before and the corresponding value of K read off from the graphs. V_{g1} follows from K.

If the grid-bias voltage is fixed but insufficient to stop grid current a direct solution (by successive approximations) is laborious, but indirect solutions are easily obtained by observing that in an oscillator with grid-leak bias the steady state is unchanged if the grid leak is replaced by a source of e.m.f. equal to $-I_gR_g$.

In studying behaviour at the threshold of oscillation the values of V_g and r_g for vanishingly small amplitudes are required. The grid current characteristic has the exponential form

$$i_g = I_0 \exp(v_g/V_0) \qquad (4.23)$$

where I_0 and V_0 are positive constants. Then

$$r_g = -R_gV_0/V_g \qquad (4.24)$$

Fig.4 shows $-V_g/V_0$ as a function of R_gI_0/V_0

It is seen that the value of r_g may be many times less than for the semi-linear form of characteristic.

5. Transient grid and anode currents.

Let a small disturbance v_{gd} be added to the steady-state grid voltage v_g. The grid current is

$$e(v_g + v_{gd}) = e(v_g) + v_{gd}e'(v_g)$$

The change in grid current produced by v_{gd} is $i_{gd} = v_{gd}e'(v_g)$. Since $v_g = V_{g1}\cos\omega_0 t + V_g$, $e'(v_g)$ can be expressed as a cosine Fourier series in $\omega_0 t$

$$e'(v_g) = \sum_{-\infty}^{\infty} S_n \exp jn\omega_0 t$$

where $$S_n = (1/\pi)\int_0^{\pi} e'(V_{g1}\cos x + V_g)\cos nx\,dx = S_{-n} \qquad (5.1)$$

Following Section 3 the transient voltage is written as

$$v_{gd} = V_d \sum_{-1}^{1} u_n \exp(p + jn\omega_0)t \qquad (5.2)$$

The transient current has an infinite number of terms but only those having the same frequencies as v_{gd} need be considered. Thus

$$i_{gd} = V_d \sum_{-1}^{1} w_n \exp(p + jn\omega_0)t \qquad (5.3)$$

w_n being complex numbers. Equating (5.3) to the terms of like frequencies in the product $v_{gd}e'(v_g)$ gives three equations connecting the values of w_n and u_n. These are conveniently expressed in matrix notation

$$\begin{bmatrix} w_1 \\ w_0 \\ w_{-1} \end{bmatrix} = \begin{bmatrix} S_0 & S_1 & S_2 \\ S_1 & S_0 & S_1 \\ S_2 & S_1 & S_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_0 \\ u_{-1} \end{bmatrix} \qquad (5.4)$$

From (4.2) (4.3) and (5.1) the following relations are obtained.

$$\left.\begin{aligned} (S_0 - S_2)r_g &= 1 \\ S_1 R_g &= -(1 + S_0 R_g)\,dV_g/dV_{g1} \\ (1+S_0R_g)(S_0+S_2) - 2S_1^2 R_g &= (1+S_0R_g)(1-(V_{g1}/r_g)\,dr_g/dV_{g1})/r_g \end{aligned}\right\} \qquad (5.5)$$

The first expression is found by integrating (4.3) by parts, and the second and third by differentiating (4.2) and (4.3) with respect to V_{g1}. These equations are independent of the form of $e(v_g)$, but it may be assumed that $e'(v_g)$ is always positive, for a negative slope would indicate an unstable form of characteristic. With this restriction it can be shown that

$$S_0 > 0 \quad S_0 + S_2 > 0 \quad S_0 - S_2 > 0 \quad S_0(S_0 + S_2) - 2S_1^2 > 0 \qquad (5.6)$$

The first three inequalities follow at once from the definition of S_n and the fourth can be proved by applying Schwarz's inequality[21].

When the grid-bias voltage is supplied from a fixed source and is large enough to stop grid current, $S_0 = S_1 = S_2 = 0$. If the grid-current characteristic has the semi-linear form of (4.19) it is shown in Appendix 2 that

$$\left.\begin{aligned} S_2/(S_0 - S_2) &= S_2 r_g = (1 + 0.72V_{cg}/V_{g1})S_2' r_g' \\ S_2/S_1 &= -(1 - V_{cg}/V_{g1})V_g'/V_{g1} + V_{cg}/V_{g1} \\ S_0 R_g &= S_0' R_g(1 - 0.36V_{cg}/V_{g1}) \end{aligned}\right\} \quad (5.7)$$

Fig.5 shows $S_0' R_g$ and $S_2' r_g'$ plotted against $b_g R_g$. S_2/S_1 is found by using Fig.2.

At the threshold of oscillation when V_{g1} is vanishingly small, the differential coefficients of V_g and r_g appearing in (5.5) are of major importance. It is shown in Appendix 1 that when the grid current has the exponential form (4.23)

$$\left.\begin{aligned} &dV_g/dV_{g1} = nV_{g1}/V_g \qquad dr_g/dV_{g1} = mr_g V_{g1}/V_g^2 \\ &n = (V_g/V_0)^2/2(1 - V_g/V_0) \\ &m = -(V_g/V_0)^2(1 + V_g/V_0)/4(1 - V_g/V_0) \end{aligned}\right\} \quad (5.8)$$

Next the transient anode current is considered. Let small disturbances v_{gd} and v_{ad} be added to the steady-state grid and anode voltages, and let the change in anode current so produced be i_{ad}. Then from (4.4) and (4.5)

$$i_{ad} = (v_{gd} + v_{ad}/\mu)f'(V_e\cos(\omega_0 t + \theta) + V_g - V_{ca}) - ki_{gd}$$

f' can obviously be expressed as a cosine Fourier series in $\omega_0 t + \theta$. Hence

$$i_{ad} + ki_{gd} = (v_{gd} + v_{ad}/\mu)\sum_{-\infty}^{\infty} G_n \exp jn(\omega_0 t + \theta) \qquad (5.9)$$

where
$$G_n = (1/\pi)\int_0^{\pi} f'(V_e\cos x + V_g - V_{ca})\cos nx\,dx \qquad (5.10)$$

As before only terms of the three principal frequencies in i_{ad} and v_{ad} need be considered. Thus

$$\left.\begin{aligned} i_{ad} &= V_d\sum_{-1}^{1} x_n \exp(p + jn\omega_0)t \\ v_{ad} &= V_d\sum_{-1}^{1} y_n \exp(p + jn\omega_0)t \end{aligned}\right\} \quad (5.11)$$

Using (5.2), (5.3), and (5.11), $i_{ad} + ki_{gd}$ is equated to the terms of like frequency on the R.H.S. of (5.9) to give

$$\begin{bmatrix} x_1 + kw_1 \\ x_0 + kw_0 \\ x_{-1} + kw_{-1} \end{bmatrix} = \begin{bmatrix} G_0 & G_1 q & G_2 q^2 \\ G_1 q^{-1} & G_0 & G_1 q \\ G_2 q^{-2} & G_1 q^{-1} & G_0 \end{bmatrix} \begin{bmatrix} u_1 + y_1/\mu \\ u_0 + y_0/\mu \\ u_{-1} + y_{-1}/\mu \end{bmatrix} \quad (5.12)$$

in which $q = \exp j\theta$ .. .. .. (5.13)

Expressions for q and q^{-1} are given by (4.17)

Many of the stability criteria can be simply expressed in terms of the coefficients G_n. The following relations are required in later Sections.

$$G_0 - G_2 = g/N = -1/R_E = -(1 - kR_T/r_g)/(R_T + R_I/\mu) \quad (5.14)$$

This is obtained by integrating (4.8) by parts and using (4.14) and (5.10). It is independent of the form of $f(v)$. If the restriction that $f'(v)$ should not be negative is imposed, the following inequalities can be proved in the same way as for S_n.

$$G_0 > 0 \quad G_0 + G_2 > 0 \quad G_0 - G_2 > 0 \quad G_0(G_0 + G_2) - 2G_1^2 > 0 \quad (5.15)$$

In Appendix 2 the values of G_0, G_1 and G_2 for a three-halves-law amplifier are calculated. Fig.6 shows $G_2/(G_0 - G_2) = -G_2R_E$ and $-G_1R_E$ plotted as functions of the parameter $H = 1 - K + K/Y$. From these graphs the values of G_2/G_1 and G_0R_E are easily obtained. When V_{g1} is very small

$$\left.\begin{aligned} G_0 &= g(1 - Y)^{\frac{1}{2}} = -1/R_E \qquad G_1 = -Y/4KR_E(1 - Y) \\ G_2 &= Y^2/32K^2R_E(1 - Y)^2 \end{aligned}\right\} \quad (5.16)$$

When V_{g1} is large

$$G_2/G_1 = (1-15H/16+35H^2/256+..)/(1-3H/16-5H^2/256+..) \quad (5.17)$$

It is also shown in the Appendix that

$$G_1 > 0 \quad \text{and} \quad G_1^2 - G_0G_2 > 0 \quad .. \quad .. \quad (5.18)$$

In the foregoing analysis it has been tacitly assumed that when the amplifier is a tetrode or pentode the screen-grid voltage remains constant. If this condition is not satisfied the theory must be modified in the way discussed in Appendix 3.

6. The determinental equation for p.

In the preceding Section the transient grid and anode currents were expressed in terms of the corresponding voltages and the amplifier parameters, but these currents and voltages are also related through the impedances of the feedback network. Written in operational form these relations are

$$\left.\begin{aligned} i_{ad} &= -(1/Z_t(D))v_{gd} - (Z_o(D)/Z_t(D))i_{gd} \\ v_{ad} &= (Z_i(D)/Z_t(D))v_{gd} + Z_n(D)i_{gd} \end{aligned}\right\} \quad (6.1)$$

Now $Z(D)\ \exp(p + jn\omega_o)t = \exp(p + jn\omega_o)t\ Z(p + jn\omega_o)$

Substituting in these equations for v_{gd}, i_{gd}, v_{ad} and i_{ad} according to (5.2), (5.3), and (5.11), gives

$$x_n = -u_n/Z_t(p + jn\omega_o) - w_n Z_o(p + jn\omega_o)/Z_t(p + jn\omega_o)$$

$$y_n = u_n Z_i(p + jn\omega_o)/Z_t(p + jn\omega_o) + w_n Z_n(p + jn\omega_o)$$

In order to conserve space the following notation is adopted

$Z_i(p + j\omega_o) = Z_i^+ \quad Z_i(p) = Z_i^o \quad Z_i(p - j\omega_o) = Z_i^-$

with similar expressions for Z_o, Z_t etc.

It is convenient to refer to Z_i^+, Z_i^- etc. as the high-frequency impedances and to Z_i^o etc. as the low-frequency impedances. Impedances without superscripts are meant, as in previous Sections, to refer to the oscillation frequency, e.g. $Z_T = Z_T(j\omega_o)$ When $p = 0$, $Z_i^o = R_a$ and $Z_o^o = R_g$. When $p = \pm\frac{1}{2}j\omega_o$ the "low" frequencies merge with the "high" frequencies, but in most practical oscillators all the impedances are then negligible.

The above expressions for x_n and y_n are now substituted in (5.12) and both sides of the equation premultiplied by the diagonal matrix

$$\begin{bmatrix} Z_t^+ & 0 & 0 \\ 0 & Z_t^o & 0 \\ 0 & 0 & Z_t^- \end{bmatrix}$$

The result is

$$\begin{bmatrix} u_1 + w_1(Z_o^+ - kZ_t^+) \\ u_o + w_o(Z_o^o - kZ_t^o) \\ u_{-1} + w_{-1}(Z_o^- - kZ_t^-) \end{bmatrix} +$$

$$\begin{bmatrix} G_o Z_t^+ & G_1 Z_t^+ q & G_2 Z_t^+ q^2 \\ G_1 Z_t^o q^{-1} & G_o Z_t^o & G_1 Z_t^o q \\ G_2 Z_t^- q^{-2} & G_1 Z_t^- q^{-1} & G_o Z_t^- \end{bmatrix} \begin{bmatrix} u_1(1 + Z_i^+/\mu Z_t^+) + w_1 Z_n^+/\mu \\ u_o(1 + Z_i^o/\mu Z_t^o) + w_o Z_n^o/\mu \\ u_{-1}(1 + Z_i^-/\mu Z_t^-) + w_{-1} Z_n^-/\mu \end{bmatrix} = 0$$

Finally the w's are substituted according to (5.4) to give a set of equations for u_n.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_o & b_o & c_o \\ a_{-1} & b_{-1} & c_{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_o \\ u_{-1} \end{bmatrix} = 0 \tag{6.2}$$

The coefficients are as follows:

$$\begin{aligned}
a_1 &= 1 + G_o(Z_t^+ + Z_i^+/\mu) + S_o(Z_o^+ - kZ_t^+) + G_oS_oZ_n^+Z_t^+/\mu \\
&\quad + G_1S_1qZ_n^oZ_t^+/\mu + G_2S_2q^2Z_n^-Z_t^+/\mu \\
b_1 &= G_1qZ_t^+(1 + Z_i^o/\mu Z_t^o) + S_1(Z_o^+ - kZ_t^+) + G_oS_1Z_n^+Z_t^+/\mu \\
&\quad + G_1S_oqZ_n^oZ_t^+/\mu + G_2S_1q^2Z_n^-Z_t^+/\mu \\
c_1 &= G_2q^2Z_t^+(1 + Z_i^-/\mu Z_t^-) + S_2(Z_o^+ - kZ_t^+) + G_oS_2Z_n^+Z_t^+/\mu \\
&\quad + G_1S_1qZ_n^oZ_t^+/\mu + G_2S_oq^2Z_n^-Z_t^+/\mu \\
a_o &= G_1q^{-1}Z_t^o(1 + Z_i^+/\mu Z_t^+) + S_1(Z_o^o - kZ_t^o) + G_oS_1Z_n^oZ_t^o/\mu \\
&\quad + G_1S_oq^{-1}Z_n^+Z_t^o/\mu + G_1S_2qZ_n^-Z_t^o/\mu \\
b_o &= 1 + G_o(Z_t^o + Z_i^o/\mu) + S_o(Z_o^o - kZ_t^o) + G_oS_oZ_n^oZ_t^o/\mu \\
&\quad + G_1S_1Z_t^o(q^{-1}Z_n^+ + qZ_n^-)/\mu \\
c_o &= G_1qZ_t^o(1 + Z_i^-/\mu Z_t^-) + S_1(Z_o^o - kZ_t^o) + G_oS_1Z_n^oZ_t^o/\mu \\
&\quad + G_1S_oqZ_n^-Z_t^o/\mu + G_1S_2q^{-1}Z_n^+Z_t^o/\mu \\
a_{-1} &= G_2q^{-2}Z_t^-(1 + Z_i^+/\mu Z_t^+) + S_2(Z_o^- - kZ_t^-) + G_oS_2Z_n^-Z_t^-/\mu \\
&\quad G_1S_1q^{-1}Z_n^oZ_t^-/\mu + G_2S_oq^{-2}Z_n^+Z_t^-/\mu \\
b_{-1} &= G_1q^{-1}Z_t^-(1 + Z_i^o/\mu Z_t^o) + S_1(Z_o^- - kZ_t^-) + G_oS_1Z_n^-Z_t^-/\mu \\
&\quad + G_1S_oq^{-1}Z_n^oZ_t^-/\mu + G_2S_1q^{-2}Z_n^+Z_t^-/\mu \\
c_{-1} &= 1 + G_o(Z_t^- + Z_i^-/\mu) + S_o(Z_o^- - kZ_t^-) + G_oS_oZ_n^-Z_t^-/\mu \\
&\quad + G_1S_1q^{-1}Z_n^oZ_t^-/\mu + G_2S_2q^{-2}Z_n^+Z_t^-/\mu
\end{aligned} \tag{6.3}$$

Neglecting the trivial case of $u_1 = u_o = u_{-1} = 0$, the condition for the consistency of equations (6.2) is that the determinant of the coefficient matrix should vanish[18]. The auxiliary condition that at least one of the minors should not vanish is always satisfied. Since all the coefficients are functions of p the determinant is also a function of p.

$$D(p) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_o & b_o & c_o \\ a_{-1} & b_{-1} & c_{-1} \end{vmatrix} \tag{6.4}$$

The condition for consistency is

$$D(p) = 0 \quad . . \quad . . \quad . . \quad (6.5)$$

This is the characteristic equation of the oscillator the roots of which are the characteristic frequencies of the transient normal modes. When the determinant is multiplied out the general expression for D(p) is of extreme length even allowing for some cancellation of terms. In most practical examples however the expression can be greatly simplified, in one way or another, by making suitable approximations, but these depend on the particular form of oscillator and the type of instability being studied.

7. Criteria for stability.

The oscillation is stable if all the roots of the equation (6.5) have negative real parts, for this ensures that the amplitude of any transient decreases with time. In general a direct solution of the equation is not feasable: only in certain special cases, some of which are considered later, can the expression for D(p) be sufficiently simplified to permit the direct calculation of p. However, a complete solution is unnecessary since all that need be known is whether or not the real parts of all the roots are negative, and this information can be obtained without actually solving the equation. Two methods will be discussed. The first is an application of the Routh-Hurwitz stability rules.

It is known from electric circuit theory[22] that any impedance function Z(p) associated with a linear network is a real rational function of p i.e. the quotient of two polynomials with real coefficients. Functions like $Z(p + jn\omega_o)$ will also be rational functions of p but with complex coefficients. Since D(p) consists of sums products and quotients of such functions it is also a rational function. It remains to find the nature of the coefficients.

Now for any real impedance[24] $Z(p^* + jn\omega_o) = Z^*(p - jn\omega_o)$ Inspections of the expressions for a_1 etc. shows that if p^* is substituted for p the following transformations take place.

$$\left.\begin{array}{lll} a_1 \rightarrow c_{-1}^* & b_1 \rightarrow b_{-1}^* & c_1 \rightarrow a_{-1}^* \\ a_o \rightarrow c_o^* & b_o \rightarrow b_o^* & c_o \rightarrow a_o^* \\ a_{-1} \rightarrow c_1^* & b_{-1} \rightarrow b_1^* & c_{-1} \rightarrow a_1^* \end{array}\right\} \quad (7.1)$$

Hence $D(p^*) = D^*(p)$ and from this it follows that the coefficients of the polynomials are real. It can also be shown by using (4.11), (4.12), (4.15), and (4.17) that when $p = 0$

$$a_1 = c_1 \qquad a_o = c_o \qquad a_{-1} = c_{-1} \qquad (7.2)$$

Hence $D(0) = 0$, and D(p) can therefore be written as $D(p) = pP_1(p)/P_2(p)$, in which $P_1(p)$ and $P_2(p)$ are real polynomials. The root $p = 0$ corresponds to an oscillation of constant amplitude and frequency ω_0, i.e. the steady state. All other roots of $D(p) = 0$ are the same as the roots of $P_1(p) = 0$. Routh[26], and later Hurwitz[28], investigated the conditions for all the roots of such equations to have negative real parts. The Routh-Hurwitz stability criteria take the form of a number of inequalities between the coefficients of the polynomial. The details are given in later Sections where the method is applied to particular problems.

The second method originates also in the work of Routh[27], and is based on a theorem of Cauchy[29] (Routh also made use of the theorem in deriving the first method) One way of stating the theorem is as follows: "If $D(p) = u + jv$ is analytic except for a finite number of poles inside and on a closed contour, then the number of times which the locus of $D(p)$ encircles the origin when p moves once round the contour is $N - P$, where N is the number of zeros and P the number of poles of $D(p)$ inside the contour."

In Routh's application the contour of p was the imaginary axis from $-j\infty$ to $j\infty$ and a semi-circle of infinite radius, centered on the origin, lying in the right-hand half-plane. This contour encloses all values of p having positive real parts. As the functions considered by Routh had no poles (except at ∞) the stability criterion was that the locus of $D(p)$ should not enclose the origin. What Routh in fact considered was the number of times which the ratio u/v passed through 0 and changed in sign from positive to negative and vice versa, but this is simply another way of specifying the number of encirclements. Bode[23] reached the same conclusion also by way of Cauchy's theorem, and Nyquist[30] had previously obtained a similar result by another method.

The Routh-Nyquist criterion, as it may properly be named, is more general than the Routh-Hurwitz in so far as it applies to any analytical function and not merely to polynomials with real coefficients, but the criterion fails if the function has any poles within the contour. This difficulty can be overcome by using a slightly more sophisticated definition[31] of "encirclement", but for the present purpose this refinement is unnecessary. The impedances Z_i, Z_o, Z_t, are those of a passive network and so can have no poles in the right-hand half-plane, but Z_t, being a transfer function, may have zeros. Since Z_t occurs in the denominators of some terms in the expressions for a_1 etc. these terms will have poles where Z_t has zeros. It can be shown, however, that all such terms cancel in the final result. Hence D(p) has no poles in the right-hand half-plane.

Two difficulties prevent the Routh-Nyquist criterion from being applied directly to oscillators. First $D(0) = 0$ i.e. the locus of D(p) passes through the origin. This can be avoided by indenting the p-plane contour with a small semi-circle about the origin in the right-hand half-plane. The second point, which was discussed in Section 3, is that the domain of p must be restricted so that its imaginary part lies between $-\frac{1}{2}\omega_0$ and $\frac{1}{2}\omega_0$. Instead of the Routh-Nyquist contour the contour shown in Fig.7 must therefore be used. For convenience the contour is described in the clockwise direction. The stability criterion can then be stated as:

"The oscillation is stable if the locus of D(p) does not enclose the origin when p describes the contour of Fig.7" Any encirclement - which indicates instability - will be in the clockwise direction.

The Routh-Nyquist criterion is more usually expressed in terms of encirclement of the point 1,0. With this convention the oscillation is stable if the locus of $1 - D(p)$ does not encircle the point 1,0. In the language of feedback amplifier theory D(p) corresponds to the "return difference" and $1 - D(p)$ to the "loop transmission", although there is no physical loop in the oscillator corresponding to this function. In the present application it is more convenient to adhere to the expression in terms of D(p).

Although stability can be discussed completely in terms of either the Routh-Hurwitz or the Routh-Nyquist criteria it is advantageous to use both. The Routh-Nyquist locus diagram is valuable in illustrating points which are not immediately obvious from the Routh-Hurwitz criteria, and in deriving numerical relations for the simpler types of instability. With the more complicated forms of instability the Routh-Hurwitz rules are the only practicable method of obtaining numerical results.

Some general features of the locus are now considered. Earlier it was shown that $D(p^*) = D^*(p)$. This means that the locus has mirror symmetry with respect to the real axis. Also, for very large values of p all the network impedances vanish because of the shunting effect of stray capacitances. Then $a_1 = b_0 = c_{-1} = 1$, and all the other coefficients are 0. Hence $D(p) \rightarrow 1$ as $p \rightarrow \infty \pm \frac{1}{2}j\omega_0$. In most oscillators $D(p) = 1$ when $p = \pm\frac{1}{2}j\omega_0$, so only imaginary values of p, i.e. real frequencies need be considered.

Fig.8 shows loci corresponding to (a) stability (b) instability - one zero, and (c) instability- two zeros. When p is small $D(p) = D(0) + pD'(0) = pD'(0)$ and because of symmetry $D'(0)$ is a real number. As p traverses the small semicircle near the origin $D(p)$ also describes a semicircle, and this lies in the right-hand half-plane if $D'(0)$ is positive, and in the left-hand half-plane if $D'(0)$ is negative. These remarks together with obvious topological considerations lead to the following conclusions.

(1) If $D'(0)$ is positive the locus of $D(p)$ encircles the origin an even number of times or not at all.

(2) If $D'(0)$ is negative $D(p)$ makes an odd number of encirclements.

It follows that

(3) The oscillation can be stable only if $D'(0) > 0$

The possibility that $D'(0) = 0$ is excluded, for this would imply that $D(p)$ had a double zero at the origin and this would corresponding to a transient of frequency ω_0 with linearly increasing amplitude. Condition (3) therefofe covers all possibilities. Double or multiple zeros can exist only in theory. They represent a critical adjustment of the network and amplifier parameters which cannot be achieved in practice. This remark applies also to any zero, simple or multiple, occurring at any other point on the contour.

The stability criteria derived here indicate only whether a given possible steady state is stable or unstable with respect to small disturbances. They cannot be used to predict whether the given state, if stable, will in fact be realised, nor can they be made to reveal what course the oscillation will take when it departs from an unstable steady state.

Corresponding to a given set of parameters, an oscillator may have two or more stable steady states, and which of these is realised depends on the history of the oscillator. A disturbance of sufficient magnitude may change the oscillation from one stable mode to another. Each possible steady state has associated with it a different set of transient normal modes, and although the constants of the network and amplifier may remain unchanged, yet each steady state represents in effect a different system, and requires its own characteristic equation to describe stability.

8. Symmetrical networks.

Symmetrical networks are special cases of general (asymmetrical) networks and the reason for considering the particular before the general is one of convenience. Because of the great simplification resulting from symmetry, the analysis of symmetrical networks can be carried out more easily and in greater detail than is feasable for the general type. Also the feedback networks of many practical oscillators are symmetrical.

An impedance is symmetrical with respect to a frequency ω_o if the real part has even symmetry, and the imaginary part odd symmetry, about the line $p = j\omega_o$ in the p plane. Thus

$$Z(p^* + j\omega_o) = Z^*(p + j\omega_o)$$

No real impedance can be symmetrical about a frequency other than 0, but some networks such as high-Q resonant circuits have approximate symmetry over a limited range of **p**. For practical purposes a symmetrical impedance is therefore defined as one for which $\left|Z(p^* + j\omega_o) - Z^*(p + j\omega_o)\right|$ is negligible compared with the maximum value of $\left|Z(p + j\omega_o)\right|$ over the appropriate range of p. Since for all real impedances, $Z(p^* + j\omega_o) = Z^*(p - j\omega_o)$, it follows that for symmetrical impedances $\left|Z(p + j\omega_o) - Z(p - j\omega_o)\right|$ is also negligible.

If the impedances of an oscillator feedback network are symmetrical with respect to ω_o then

$Z_i^- = Z_i(p - j\omega_o) = Z_i(p + j\omega_o) = Z_i^+$, with similar expressions for Z_o^+ etc. It follows that the modified impedances Z_I^+ etc. are also symmetrical. When $p = 0$ Z_I etc are resistances, and so $\theta = 0$ and $q = \exp j\theta = 1$. Inspection of expressions (6.3) shows that $a_1 = c_{-1}$ $b_1 = b_{-1}$ $c_1 = a_{-1}$ $a_o = c_o$. The determinant for D(p) becomes, from (6.4)

$$D(p) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_o & b_o & a_o \\ c_1 & b_1 & a_1 \end{vmatrix}$$

This can be factorised as follows:

$$D(p) = F(p)A(p) \quad \ldots \quad (8.1)$$

where
$$\left.\begin{aligned} F(p) &= a_1 - c_1 \\ A(p) &= b_o(a_1 + c_1) - 2a_ob_1 \end{aligned}\right\} \quad (8.2)$$

$D(p)$ is zero if either $F(p)$ or $A(p)$ is zero and the number of encirclements of the origin made by the locus of $D(p)$ is the sum of the encirclements by $F(p)$ and $A(p)$. Since neither of these factors has a pole within the contour of p any encirclement must be clockwise. Hence the Routh-Nyquist stability criterion is that the locus of neither $F(p)$ nor $A(p)$ should enclose the origin.

To see the physical significance of these two conditions, the transient grid voltage, which is the real part of (5.3), is added to the steady state voltage (4.1). The high-frequency part of the total voltage is

$$V_{g1}(1 + m_1 \exp.at\cos(\omega_d t + \phi_1)) \times \cos(\omega_o t + m_2 \exp.at \cos(\omega_d t + \phi_2)) \quad \ldots \quad (8.3)$$

where
$$\left.\begin{aligned} m_1 \exp j\phi_1 &= (u_1 + u_{-1})V_d/V_{g1} \\ m_2 \exp j\phi_2 &= -j(u_1 - u_{-1})V_d/V_{g1} \end{aligned}\right\} \quad (8.4)$$

(8.3) is the expression for a wave modulated in both amplitude and phase (or frequency). The two modulations have the same complex frequency $p = a + j\omega_d$, and the coefficients of modulation may be taken as the complex amplitudes (8.4)

Using expression (8.2), the equations (6.2) for u_n can be transformed to the equivalent set

$$\left.\begin{aligned} (u_1 - u_{-1})F(p) &= 0 \\ (u_1 + u_{-1})A(p) &= 0 \\ (u_1 + u_{-1})a_o &= -b_ou_o \end{aligned}\right\} \quad (8.5)$$

Suppose that for some particular value, p_1, $F(p_1) = 0$ but $A(p_1) \neq 0$. Then $u_1 + u_{-1} = 0$ and from (8.4) $m_1 = 0$. Also $u_o = 0$, so there are no voltages or currents of frequency p. (8.3) shows that the transient disturbance takes the form of a frequency modulation of the steady-state oscillation. Hence if $F(p)$ encircles the origin p_1 has a positive real part and the oscillation frequency is unstable.

Similarly if $A(p_2) = 0$ but $F(p_2) \neq 0$ then $u_1 - u_{-1} = 0$ and $m_2 = 0$. This corresponds to amplitude modulation of the steady-state oscillation, and if $A(p)$ encircles the origin the amplitude is unstable. The third equation in (8.5) shows that there is now a low-frequency component in the total voltage, i.e. the grid-bias voltage is also modulated.

Thus in a symmetrical-network oscillator the transient takes the general form of independent modulations of frequency and amplitude, the characteristic modulation frequencies being the roots of $F(p) = 0$ and $A(p) = 0$ respectively. This independence holds only for small disturbances. If the frequency is unstable the disturbance will eventually become large enough to produce a sensible change of amplitude. On the other hand large changes of amplitude can take place without affecting the frequency.

In the previous Section it was pointed out that $a_1 = c_1$ when $p = 0$. Hence and from (8.2), $F(0) = 0$. It can also be shown that $F(p) \rightarrow 1$ and $A(p) \rightarrow 1$ when $p \rightarrow \infty \pm \frac{1}{2}j\omega_0$. The loci of $F(p)$ have the same general form as those of Fig.8 and using the same argument as for $D(p)$, the frequency can be stable only if $F'(0) > 0$. Typical loci for $A(p)$ are shown in Fig.9. $A(0)$ is either positive or negative, for the condition $A(0) = 0$ would require a critical adjustment of parameters. Inspection of Fig.9 shows that the amplitude can be stable only if $A(0) > 0$. The locus then encircles the origin an even number of times or not at all. Similarly if $A(0) < 0$ $A(p)$ makes an odd number of encirclements.

In terms of $D(p)$, $D'(0) = F'(0)A(0)$. $D'(0)$ is positive if $F'(0)$ and $A(0)$ have the same sign and it is clear that this must be positive, for otherwise both amplitude and frequency would be unstable and $D(p)$ would make at least two encirclements.

The various forms of instability may conveniently be classified according to the signs of $F'(0)$ and $A(0)$. Starting with $A(0)$, the simplest type of instability occurs when $A(0) < 0$ and the locus makes a single encirclement. This means that the equation $A(p) = 0$ has one real positive root and the disturbance therefore takes the form of a unidirectional movement of the amplitude away from the steady-state value. Instability of this type will be described as "aperiodic".

The next simplest kind of instability occurs when $A(0) > 0$ and the locus makes two encirclements. The equation $A(p) = 0$ has now two real positive roots or two complex conjugate roots with positive real parts. In the latter case the amplitude is modulated by an exponentially expanding sine wave, and in the former the amplitude changes have the highly non-sinusoidal form typical of relaxation oscillations. In both cases the instability is "periodic". A triple encirclement would indicate a combination of periodic and aperiodic instabilities, and so on for more complicated loci. ?

Similarly the frequency has aperiodic instability if $F'(0) < 0$, and periodic instability if $F(p)$ makes two encirclements.

Aperiodic instability results in the various hysteresis effects some of which were studied by Van der Pol and others. The various forms of frequency and amplitude instability which may occur in symmetrical-network oscillators are now studied in some detail.

9. Frequency stability.

The frequency stability function $F(p) = a_1 - c_1$ may be written in terms of the network and amplifier parameters by substituting in expressions (6.3) for $S_o - S_2$ according to (5.5), for Z_t^+ etc after the manner of (4.11), and for $G_o - G_2$ according to (5.14). This gives

$$F(p) = (1 + (Z_o^+ - kZ_t^+)/r_g)(1 - Z_E^+/R_E) \qquad (9.1)$$

where $Z_E^+ = (Z_T + Z_I/\mu)/(1 - kZ_T^+/r_g)$

The low-frequency impedances Z_i^o etc. do not appear. This is to be expected since if only the frequency is modulated there are no voltages or currents of frequency p. Also the only parameters of the amplifier which appear are k, μ and the constants which help to determine r_g. It would seem that frequency stability is independent of the non-linear amplifier characteristic. However, in deriving these results it was assumed that the amplifier currents were single-valued functions of the voltages, and the conclusions will therefore be valid only for such types of amplifier.

From (9.1), $F'(0) = -(1 + (R_o - kR_t)/r_g)Z_E'/R_E$,

and since R_t and R_E are negative the criterion for aperiodic stability is $Z_E' > 0$. This can be expressed in terms of real frequencies as follows:

$Z_E^+ = R_E^+ + jX_E^+$, but the components are not themselves functions of p. However, as Z_E^+ is analytic it follows from the Cauchy-Riemann definition that at any point on the real frequency axis,

$$(d/dp)Z_E(p + j\omega_o) = (d/dj\omega_d)R_E(j\omega_o + j\omega_d) + (d/dj\omega_d)jX_E(j\omega_o + j\omega_d)$$

Since Z_E^+ is symmetrical, R_E is an even function of ω_d and its derivative at $\omega_d = 0$ is therefore 0. Hence $Z_E' = X_E'$ and

$$F'(0) = -(1 + (R_o - kR_t)/r_g)X_E'/R_E \qquad (9.2)$$

where X_E' denotes the derivative of X_E with respect to ω_d at $\omega_d = 0$. The criterion for aperiodic stability is then

$$-X_E'/R_E > 0 \quad \text{or} \quad X_E' > 0 \qquad (9.3)$$

Another useful form is $F'(0) = \lim_{p \to o} F(p)/p$ (9.4)

Inspection of Fig. 8b shows that the simplest type of network showing aperiodic instability has a transmission characteristic with two peaks of maximum response and a minimum at the oscillation frequency. It can be shown that there are three possible steady states, the frequencies of the other two corresponding to the point f, and that these are stable. (See Section 14) When the oscillation frequency departs from the unstable value ω_0 it finally settles down at one of these points.

The double encirclement of Fig. 8c represents a new type of instability. Assuming the corresponding roots of $F(p) = 0$ to be complex conjugate, the oscillation frequency is modulated by an exponentially expanding sine wave. The transmission characteristic has three maxima the smallest being at ω_0. There are five possible steady states but only the stability of the central frequency can be discussed here since it is only with respect to this frequency that the network is symmetrical. Of the other frequencies it can be shown that the two corresponding to the point h are unstable, and the two corresponding to g are stable.

Under suitable conditions a sustained periodic frequency modulation of small magnitude can be produced. When the parameters are adjusted to well beyond the critical values the inevitable slight asymmetry favours one or other of the component frequencies $\omega_0 \pm \omega_d$, and the oscillation eventually settles down at one of the stable points g.

The network of Fig. 10, though not representing any practical oscillator, is the simplest in which the two forms of instability can be demonstrated. The central circuit resonates at ω_0 and has a shunt resistance nR; the other two resonate at $\omega_0 \pm \omega_m$ and have a resistance R. For simplicity it is assumed that the coils have the same Q-factor, that the coefficient of coupling between the two coils in each of the three pairs is 1, and that the voltage transformation ratio r is the same for each pair. No grid bias arrangement is shown as the result is independent of the particular way in which the bias voltage is obtained. Then $Z_t^+ = Z_i^+/r$, $Z_o^+ = Z_i^+/r^2$, $Z_n^+ = 0$.

Let $T = 2Q/\omega_o$ and $c = T\omega_m$ . . (9.5)

Then

$$Z_i^+/R = ((2+n)(1+pT)^2 + nc^2)/(1+pT)((1+pT)^2 + c^2) \qquad (9.6)$$

This is an approximation valid for large values of Q. Substituting for Z_i^+ etc. according to (9.5) and (9.6)

$$F(p) = 1 - Z_i^+/R_i$$

$$= \frac{pT(p^2T^2 + 2pT(1 - bc^2) + 1 + (1 - 4b)c^2)}{(1+pT)((1+pT)^2 + c^2)} \qquad (9.7)$$

where $1/b = 2 + n + nc^2$

The simplest oscillator in which aperiodic instability can exist has a two-circuit feedback network obtained by eliminating the central coil in Fig.10. This is equivalent to putting $n = 0$. Then from (9.4) $F'(0) = T(1 - c^2)/(1 + c^2)$. For aperiodic stability $c^2 < 1$ i.e. $2Q\omega_m/\omega_o < 1$ (9.8)

Similarly for the triple-circuit network the criterion for aperiodic stability is $1 + (1 - 4b)c^2 > 0$. This is more conveniently expressed as a criterion for instability by substituting for b, thus

$$2(c^2 - 1)/(c^2 + 1)^2 > n > -2/(c^2 + 1) \qquad (9.9)$$

A negative value of n could be realised by reversing the leads to the central coil, but when this is done the equations remain valid only if μ and r_g are very large.

Since the numerator of (9.7) is a quadratic it is not difficult to find the conditions for the roots of $F(p) = 0$ to have negative real parts. More directly, the Routh-Hurwitz stability criteria are: $1 + (1 - 4b)c^2 > 0$ $1 - bc^2 > 0$

The first is simply the condition for aperiodic stability, the second is the condition for periodic stability. Substituting for b and using (9.9) this can be more easily written as a criterion for periodic instability:

$$(c^2 - 2)/(c^2 + 1) > n > 2(c^2 - 1)/(c^2 + 1)^2 \qquad (9.10)$$

Instability can exist only if $c > \sqrt{3}$.

Further examination of (9.7) reveals the nature of these unstable states. If $n > (c^2 + 1)^{-\frac{1}{2}} - (c^2 + 1)^{-1}$ the roots of $F(p) = 0$ are complex conjugates, and for smaller values of n both roots are real.

When $n < -2/(c^2 + 1)$ it can be shown that R_t is positive and no oscillation is then possible. From (9.9) and (9.10) the stability criterion is therefore

$$n > (c^2 - 2)/(c^2 + 1) \qquad (9.11)$$

All other possible values of n correspond to either periodic or aperiodic instability.

10. Amplitude stability - Hysteresis.

In Section 8 it was shown that the condition for the amplitude to have aperiodic stability is $A(0) > 0$. The behaviour of the oscillator is investigated by writing $A(0)$ in terms of the network and amplifier parameters. It is first supposed that the grid-bias voltage derives entirely from the flow of grid current, so that S_1 etc. may be substituted according to (5.5). Using also (4.11), (5.14), (6.3) and (8.2), $A(0)$ becomes

$$\begin{aligned} A(0) = {} & 2(1 + S_oR_g)(1 + R_o/r_g)\Big[(R_T + R_I/\mu)(G_2 + (G_oG_2 - G_1^2)R_a/\mu) \\ & \qquad + G_1R_T(1 - kR_a/\mu R_g)(dV_g/dV_{g1})\Big] \\ & - (1 + S_oR_g)(1 + R_o/r_g)(V_{g1}/r_g)(dr_g/dV_{g1}) \\ & \times \Big[(1 + G_oR_a/\mu)(R_\theta - kR_T)/r_g \\ & \qquad ((G_o + G_2)(1 + G_oR_a/\mu) - 2G_1^2R_a/u)R_NR_T/\mu r_g\Big] \end{aligned} \tag{10.1}$$

The network elements appearing in this expression are all resistances. Time constants have no influence on aperiodic stability: their sole effect is to limit the speed with which the amplitude moves away from an unstable value.

$A(0)$ has been written in this form in order to show that an important factor in stability is the manner in which the grid-bias voltage and grid input resistance vary with oscillation amplitude. The derivatives dV_g/dV_{g1} and dr_g/dV_{g1} are strictly defined only when V_{g1} changes infinitely slowly, but this does not mean that the criterion is valid only for slow changes. The derivatives appear because they are related to S_o, S_1, and S_2 through the equations defining the steady-state values of V_g and r_g, and it is these relations which have been used in (10.1).

In many oscillators the grid-current characteristic is such that when V_{g1} is moderately large V_g is almost proportional to V_{g1}, and r_g is almost constant. Then $dV_g/dV_{g1} = V_g/V_{g1}$ and $dr_g/dV_{g1} = 0$

Substituting for K according to (4.18), and for R_E and then R_T from (4.14) and (4.11) gives

$$A(0) = -2(1 + S_oR_g)(1 + (R_o - kR_t)/r_g)R_E$$

$$\times (KG_1 - G_2 + (G_1^2 - G_oG_2 - kKG_1/R_g)R_a/\mu) \qquad (10.2)$$

Since R_E is negative the criterion for stability is

$$KG_1 - G_2 + (G_1^2 - G_oG_2 - kKG_1/R_g)R_a/\mu > 0 \qquad (10.3)$$

If R_a is small or μ large this reduces to

$$KG_1 - G_2 > 0 \quad \cdot\cdot \quad \cdot\cdot \qquad (10.4)$$

This criterion can be interpreted in terms of the slope of the graph of Y plotted against N of which the curves of Fig.3 are particular examples. By differentiating equations (4.8) and (4.13) with respect to Y, keeping K and V_{ca} constant, and using also (5.10) and (5.14), it can be shown that

$$\partial N/\partial Y = 2N^2(KG_1 - G_2)/gY$$

Since g is positive, $\partial N/\partial Y$ has the same sign as $KG_1 - G_2$, and so for stability, $\partial N/\partial Y > 0$. In deriving this result it was assumed that V_{ca} is independent of V_{g1}, but this is true only if $R_a = 0$ or $\mu = \infty$, for otherwise a change in V_{g1} would change the mean anode current which would change the mean anode voltage which in turn would change V_{ca}. A straightforward, but tedious calculation shows that when V_{ca} varies, $\partial N/\partial Y$ has the same sign as the L.H.S. of the more general criterion (10.3)

These results are independent of the form of the amplifier characteristic provided only V_g/V_{g1} is constant. For a three-halves-law amplifier G_1 and G_2 can be expressed in terms of a parameter $H = 1 - K + K/Y$. As G_1 is now positive (10.4) can be written $G_2/G_1 < K$, and the corresponding value of H obtained from Fig.6. A more direct method is to substitute ~~the~~ for G_2/G_1 according to (5.17) and to reverse the series thus obtained. (10.4) then becomes

$$1/Y > (1 + (1 - K)/9 + ..)(1 - K)/3K \quad \cdot\cdot \qquad (10.5)$$

Since Y cannot be negative the amplitude is stable for all values of Y when $K > 1$. When $K < 1$ the stable values of Y are less than the critical value given by turning the inequality (10.5) into an equation.

In terms of the graphs of Y against N, the greatest stable amplitude occurs when $\partial N/\partial Y = 0$, i.e. where N is also a maximum. It can be shown that this value is

$$N_{max} = (27/32)^{\frac{1}{2}}(1 - K)^{-3/2}(1 + (1 - K)/6 + ..) \qquad (10.6)$$

These tendencies are suggested in the graphs of Fig.3 and shown clearly in Fig.11a. The full line is the theoretical value of Y for K = 0.6 Once Y has passed the critical point b there is nothing in the theory to prevent it increasing indefinitely. Actually the amplitude is then limited by the diversion of anode current to the grid or screen. The broken line shows a possible form of Y due to this effect. In most oscillators this limitation occurs before the critical value of Y is reached.

A hysteresis effect exists in the region abcd. As N is varied Y moves round the boundary in the direction shown, the portions bc and da being irreversible. The second critical point d does not lie on the theoretical curve and the calculation of the second critical amplitude is beyond the scope of the present analysis. Although points on both branches ab and cd represent stable states it is obvious that an impressed force of sufficient magnitude could shift the operating point from one branch to another.

A second kind of hysteresis effect can exist when r_g and V_g/V_{g1} vary considerably with change of V_{g1}. This variation is most marked for small values of V_{g1} i.e. near to the threshold of oscillation. Fig.11b shows the effect of varying a parameter of the feedback network, e.g. the mutual inductance M between anode and grid circuits. Oscillation begins when M is increased to the critical point b, but small amplitudes are unstable and V_{g1} immediately moves to the point c. When M is reduced a second critical point d is reached at which V_{g1} suddenly falls to zero. The effect is due mainly ot the variation of r_g. If this increases with V_{g1} the loop gain of the amplifier and feedback network, measured at the oscillation frequency, may also increase and become sufficient to maintain an oscillation of large amplitude for a value of M less than that needed to iniatiate oscillation.

The condition for avoiding this kind of hysteresis effect is that vanishingly small amplitudes should be stable. It is sufficient to consider the particular case of a three-halves-law amplifier (4.21) with an exponential grid-current characteristic (4.23). The values of G_n for small values of V_{g1} are given by (5.16), and the derivatives of V_g and r_g by (5.8). Substituting these into (10.1), using also (4.18) and (5.14), and neglecting powers of V_{g1} higher than the square gives

$$A(0) = (b + c + d)f \quad . . \quad . . \quad (10.7)$$

where
$$b = (1 - 3R_a/\mu R_E)(1 + R_I/\mu R_T)^2/32(1/Y - 1)^2$$
$$c = n(1 - kR_a/\mu R_g)/4(1/Y - 1)$$
$$d = \tfrac{1}{2}m(1 - R_a/\mu R_E)(R_E(k - R_\theta/R_T) + R_N/\mu)/(1 + R_I/\mu R_T)r_g$$
$$f = 2(V_{g1}/V_g)^2(1 + S_oR_g)(1 + (R_o - kR_t)/r_g)$$

Since $f > 0$ the stability criterion is $b + c + d > 0$. Substituting for m and n from (5.8) this becomes

$$1-kR_a/\mu R_g + (1-3R_a/\mu R_E)(1+R_I/\mu R_T)^2(1-V_g/V_o)(V_o/V_g)^2/4(1/Y-1)$$
$$- (1+V_g/V_o)(1/Y-1)(1-R_a/\mu R_E)(R_E(k-R_\theta/R_T)+R_N/\mu)/(1+R_I/\mu R_T)r_g > 0 \quad (10.8)$$

The values of the parameters are those at the threshold of oscillation. In many oscillators R_N is negligible. If also μ is large, then on substituting for R_E according to (5.16) the criterion simplifies to

$$1 + (1 - V_g/V_o)(V_o/V_g)^2/4(1/Y -1)$$
$$+ (k - R_\theta/R_T)(1 + V_g/V_o)(1 - Y)^{\frac{1}{2}}/gYr_g > 0 \quad (10.9)$$

In this expression R_T is negative, Y is small and positive, and g is also positive. It follows that instability can exist only if $1 + V_g/V_o < 0$. Referring to (5.8) it is seen that this is also the condition that r_g should increase with V_{g1}.

In (10.9), k, V_o and g are constants of the valve, and V_g, Y and r_g, which are given by (4.23) and Fig. 4, depend only on R_g and the valve constants. Hence the only two independently adjustable parameters are R_g and the ratio R_θ/R_T. In most practical situations $1 + V_g/V_o < 0$. Stability is then assisted by making R_c/R_T small. The effect of varying R_g is less easy to follow, but it is clear that stability is obtained with all sufficiently large values of R_g, for V_g changes much less rapidly than r_g and so the L.H.S. of (10.9) can be made to approach 1. Stability is also assured for values of R_g small enough to make $1 + V_g/V_o > 0$ (Fig.4). This form of instability has been studied experimentally by Zepler[32].

In the next type of oscillator to be considered the grid-bias voltage is fixed and large enough to stop grid current. The stability criterion follows at once from (10.1) by putting $dV_g/dV_{g1} = dr_g/dV_{g1} = 0$, and is (since R_g is negative)

$$-G_2 + (G_1^2 - G_oG_2)R_a/\mu > 0 \qquad (10.10)$$

If R_a is small or μ large this reduces to

$$G_2 < 0 \qquad (10.11)$$

Now for a three-halves-law amplifier $G_1^2 - G_oG_2 > 0$ (see 5.18) Hence whatever the value of R_a stability is assured if (10.11) is satisfied and whatever the value of G_2 (10.10) can be satisfied by choosing a sufficiently large value for R_a.

Criterion (10.11) can be interpreted in terms of the graphs of N plotted as a function of Y and K. Since $R_a/\mu = 0$ V_{ca} is constant, and as V_g has been assumed constant Y is also constant. By differentiating (4.8) and using (5.10) and (5.14) it can be shown that $\partial N/\partial K = 2N^2G_2/gK$. Since g and K are positive $\partial N/\partial K$ has the same sign as G_2. If $R_a/\mu \neq 0$, V_{ca} varies with V_{g1} i.e. with K, and a more lengthly calculation shows that $-\partial N/\partial K$ has then the same sign as the L.H.S. of (10.10)

From Fig.3 it would seem that for the three-halves-law amplifier $\partial N/\partial K$ is always positive, and all amplitudes therefore unstable. However, for small values of Y and N the graphs cross one another and $\partial N/\partial K$ becomes negative. This cannot be shown in Fig.3 because the ordinates of the graphs would be too close to be distinguished.

Returning to (10.11), Fig.6 shows that $G_2 < 0$ when

$H = 1 - K + K/Y > 1.42$. Since K is positive this inequality can be satisfied only if $Y < 1$, i.e. the grid-bias voltage must not exceed the cut-off value V_{ca}. Hence for stability $K > 0.42Y/(1 - Y)$, and the corresponding stable amplitudes are

$$V_{g1} < (V_g - V_{ca})/0.42(1 + R_I/\mu R_T) \qquad (10.12)$$

All oscillations of stable amplitude will also be self-starting, for if this inequality is satisfied for the steady-state value it is satisfied for all smaller values of V_{g1}. A second restriction is imposed by the requirement that no grid current should flow. With the semi-linear form of grid characteristic (4.19) this means that

$$V_{g1} < V_{cg} - V_g \qquad (10.13)$$

Finally the amplifier may operate with a fixed grid-bias voltage insufficient to prevent the flow of grid current. This mode of operation, like the previous one, is little used in practice but it is of theoretical importance in connection with periodic instability. If the bias voltage is to be independent of V_{g1} then $R_g = 0$. A(0) can be obtained from (10.1) by substituting for dV_g/dV_{g1} and dr_g/dV_{g1} according to (5.5) and then letting $R_g \rightarrow 0$. For the present purpose it is sufficient to consider the simplified case where $R_a = R_N = 0$. Then

$$A(0) = 2(G_2(R_t + R_i/u) + S_2(R_o - kR_t)) \qquad (10.14)$$

Using (5.14) the stability criterion can be expressed as

$$G_2R_E + (S_2r_g + G_2R_E)(R_o - kR_t)/r_g > 0 \qquad (10.15)$$

11. Amplitude stability - Squegging.

A form of instability consisting in a periodic modulation of the oscillation amplitude is well-known by the colloquial name of "squegging". In the simplest example periodic instability is represented by the double encirclement shown in Fig.9c This implies that the equation $A(p) = 0$ is at least of the second degree, and since the measure of p is the reciprocal of time there must exist at least two time constants in the feedback network. These may be associated with the low-frequency impedances Z_i^o etc, with the high-frequency impedances Z_i^+ etc. or with both. In what follows it is assumed that the oscillation has aperiodic stability, for otherwise there would be no point in discussing periodic instability.

Instability due entirely to low-frequency time constants would mean that the oscillator had been constructed to generate both low- and high-frequency waves simultaneously. Instability due entirely to high-frequency time constants may be examined by putting $Z_i^o = Z_o^o = Z_t^o = 0$. This ensures that no low-frequency voltages exist in the network. The grid-bias voltage is then fixed, and it is sufficient to consider the case of no grid current, so that $S_o = S_1 = S_2 = 0$ and $r_g = \infty$ Then from (6.3) and (8.2)

$$A(p) = (1 - bZ_E^+/R_E) \tag{11.1}$$

where $b = (G_o + G_2)/(G_o - G_2)$

Since aperiodic stability is assumed, $b < 1$.

Now when $r_g = \infty$ the frequency stability function $F(p)$ given by (9.1) differs from (11.1) only in that $b = 1$. Consideration of Figs. 8b and 9c then shows that if $A(p)$ encircles the origin $F(p)$ must also do so. Periodic amplitude instability is therefore accompanied by aperiodic frequency instability, and it is clear that when the network parameters are adjusted towards critical values frequency instability always precedes amplitude instability. Since this form of amplitude instability can have no separate existence it is unnecessary to consider it further.

There remains the third possibility, which is the only one of practical or theoretical importance, that instability is due to the existence of suitable time constants associated with both low- and high-frequency impedances. The coupling between the two is provided by the frequency-changing property of the amplifier. In the ~~simplest~~ example shown in Fig.12 the feedback network is an anti-resonant circuit with a coupling coil, and the grid-bias is obtained by a grid-leak and condenser. A decoupling resistance and condenser are also included in the anode circuit. This network has one high-frequency and two low-frequency time constants.

In order that the network should be symmetrical the Q-factor must be large, the coefficient of coupling between the two coils must be 1, and the impedances of C_a and C_g must be small at the oscillation frequency. It can be shown by the more general methods of Section 15 that departures from these conditions produce only second-order errors.

Since the coefficient of coupling is 1, Z_i^+ and Z_o^+ are proportional to Z_t^+, and $Z_n^+ = 0$. If the Q-factor is very large $Z_t^o = 0$ and Z_i^o and Z_o^o are the impedances of the combinations R_aC_a and R_gC_g. Then from (6.3) and (8.2)

$$A(p) = (1 + G_oZ_i^o/\mu)((1 + S_oZ_o)(1 + BZ_t^+/R_t) - DZ_o^oZ_t^+/R_gR_t) - (E + FZ_o^o/R_g)Z_i^oZ_t^+/R_aR_t \qquad (11.2)$$

$$\left.\begin{aligned} \text{where}\quad B &= (G_o + G_2)(R_t + R_i/\mu) + (S_o + S_2)(R_o - kR_t) \\ D &= 2S_1R_g(G_1R_t + S_1(R_o - kR_t)) \\ E &= 2(G_1(R_t + R_i/\mu) - kS_1R_t)G_1R_a/\mu \\ F &= 2(G_1S_o(R_t + R_i/\mu) - G_oS_1R_t)G_1R_aR_g/\mu \end{aligned}\right\} \qquad (11.3)$$

$$\left.\begin{aligned} \text{Also}\quad R_a/Z_i^o &= 1 + pT_a \\ R_g/Z_o^o &= 1 + pT_g \\ R_t/Z_t^+ &= 1 + pT_t \end{aligned}\right\} \qquad (11.4)$$

where $T_a = R_aC_a$ $T_g = R_gC_g$ $T_t = 2Q/\omega_o = 2R/\omega_o^2L$

The expression for R_t/Z_t^+ is an approximation valid for large values of Q.

Substituting (11.4) into (11.2), the equation $A(p) = 0$ becomes

$$a_o p^3 + a_1 p^2 + a_2 p + a_3 = 0 \qquad (11.5)$$

with

$$\left.\begin{aligned} a_o &= T_a T_g T_t \\ a_1 &= (1 + G_o R_a/\mu) T_g T_t + M T_a T_g \\ a_2 &= L T_a + M(1 + G_o R_a/\mu) T_g - E T_g \\ a_3 &= L(1 + G_o R_a/\mu) - E - F \\ \text{and} \quad L &= (1 + S_o R_g)(1 + B) - D \\ M &= (1 + S_o R_g) T_t/T_g + 1 + B \end{aligned}\right\} \qquad (11.6)$$

Since a_o is positive the Routh-Hurwitz stability criteria are

$$a_1 > 0 \quad a_2 > 0 \quad a_3 > 0 \quad a_1 a_2 - a_o a_3 > 0 \qquad (11.7)$$

Now the L.H.S. of (11.5) is in fact the numerator of $A(p)$, the denominator being $(1 + pT_a)(1 + pT_g)(1 + pT_t)$. Hence $A(0) = a_3$ The third inequality in (11.7) is therefore the condition for aperiodic stability which has been assumed to be satisfied. Inspection of (11.7) shows that only three of the inequalities are independent, for if the last three are satisfied the first is automatically satisfied, and similarly the second is satisfied if the other three are satisfied. It is therefore sufficient to consider the last and one or other of the first two.

The general problem with three time constants is dealt with in the following Section: two simpler cases are considered here. It is first supposed that the decoupling resistance $R_a = 0$ This is a situation often approximated in practice. Then $E = F = a_o = 0$. The characteristic equation (11.5) reduces to a quadratic, and the criterion for periodic stability is simply $a_2 > 0$, which, from (11.6) is

$$(1 + S_o R_g) T_t/T_g > -(1 + B)$$

$$\text{or} \qquad (1 + S_o R_g) Q X_g/R_g > -\tfrac{1}{2}(1 + B) \qquad (11.8)$$

where $X_g = 1/\omega_o C_g$, and from (11.3), (4.11), (5.5), and (5.14)

$$\tfrac{1}{2}(1 + B) = G_2 R_E + (S_2 r_g + G_2 R_E)(R_o - kR_t)/r_g \qquad (11.9)$$

Stability is aided by large values of Q, $S_o R_g$, and X_g/R_g. Regarding R_g and X_g as the disposable parameters this requires R_g and X_g/R_g to be large. However a large value of X_g/R_g leads to a decrease in grid-bias voltage, to the production of harmonic voltages and to the appearance of a reactive component in the grid-input impedance[19]. To avoid these undesirable effects X_g/R_g must not exceed a certain maximum value which can be shown to be a function of b_gR_g. Taking this maximum as the value[19] corresponding to a reduction of grid-bias voltage of 1%, it is found that $(1 + S_oR_g)X_g/R_g$ varies from 0.2 to 0.62 when b_gR_g varies from 10 to 1000. It would appear that stability is aided by making R_g as large as possible, the value of C_g being choosen to suit. A lower limit to C_g is set by the shunting effect of the valve and wiring capacitance, and an upper limit to R_g because of grid emission and ionisation. Any further improvement can be obtained only by increasing the grid-cathode conductance b_g.

A second way of improving stability is to make $-\frac{1}{2}(1 + B)$ small. From (11.9) this means that $-G_2R_E$ should be small and $S_2(R_o - kR_t)$ large. Fig.6 shows that $-G_2R_E$ is small when H is large and this requires a large value of K and a small value of Y, i.e. a small oscillation amplitude. From the formulae of Appendix 2 it can be shown that S_2 is large when R_g is small. This conflicts with the previous requirement that R_g should be small. Hence depending on the relative values of the two terms in $-\frac{1}{2}(1 + B)$ stability may be improved by increasing or decreasing R_g (with corresponding adjustment of C_g). If C_g is fixed Stability is always improved by decreasing R_g.

When $\frac{1}{2}(1 + B) > 0$ the stability criterion (11.8) is satisfied whatever the values of Q, R_g, and C_g. On comparing (11.9) and (10.15) it is seen that this is identical with the condition for periodic stability with fixed grid-bias voltage and with grid current flowing. The physical explanation of this correspondence is obvious. If periodic stability is independent of T_g then T_g can be made infinite. The grid-bias voltage could then change only infinitely slowly, and so the grid leak and condenser could be replaced by a generator of e.m.f. V_g.

Thus there exists a critical amplitude below which the oscillation is unconditionally stable. If $r_g \gg R_o - kR_t$ the critical amplitude is that which makes $G_2 = 0$. i.e. for which $H = 1.42$ (Fig.6). Hence, and using (4.18)

$$V_{g1} = -V_{oa}/(K + 0.42)(1 + R_I/\mu R_T) \qquad (11.10)$$

For a given oscillation amplitude, stability is therefore improved by using a valve ~~with~~ for which V_{oa} is large and μ small. Inspection of (11.9) shows that when r_g is not very small its effect is to strengthen the inequality and thus to increase the maximum amplitude of absolute stability.

In this oscillator the grid-bias voltage varies with V_{g1} but the "anode bias" or mean anode voltage is fixed. This suggests that periodic instability might also exist in an oscillator with a fixed grid-bias voltage if the mean anode voltage varies with V_{g1} i.e. if R_a is finite. It will be assumed that the grid-bias voltage is sufficient to stop grid current flow. Then $S_n = 0$ and $r_g = \infty$. Again the characteristic equation (11.5) reduces to a quadratic and the stability criterion is $a_2 > 0$. From (11.3), (11.6), and (11.9) this is

$$\left.\begin{aligned} &(1 + G_oR_a/\mu)T_t/T_a > -(1 + B) \\ \text{or} \quad &(1 + G_oR_a/\mu)QX_a/R_a > -G_2R_E \end{aligned}\right\} \qquad (11.11)$$

where $X_a = 1/\omega_o C_a$

If $G_2 < 0$ the amplitude is stable for all values of T_t and T_a. This also guarantees aperiodic stability (10.11). If $G_2 > 0$ periodic instability will exist for suitable values of T_t and T_a, but a restriction (10.10) must then be imposed on R_a to ensure aperiodic stability.

When grid current flows and V_g is constant, S_o and S_2 are finite but $Z_o^o = 0$. The stability criterion is the same as the first form of (11.11) but the full expression (11.9) for $1 + B$ must now be used.

12. Amplitude stability - 3 time constants.

When all three time constants in the circuit of Fig. 12 are finite the increase in the complexity of the analysis makes a general discussion much more difficult. One method of approach is to compare the oscillator with another which differs from it only in that $R_a = 0$. For such an oscillator the stability criterion issimply $a_2 > 0$, or from (11.6), $M > 0$. The problem is therefore to find whether this inequality is strengthened or weakened by making R_a finite.

Of the stability criteria (11.7) the last and one of the first two must be satisfied. Using (11.6) the first can be written

$$M > -(1 + G_oR_a/\mu)T_t/T_a \qquad (12.1)$$

Since G_o is always positive this inequality is strengthened for all values of R_a. The last inequality in (11.7) is more complicated, but a simplification can be made by assuming that R_a is small and neglecting terms in R_a^2 and T_a^2. Using (11.3) and (11.6) this leads to

$$M^2T_a/T_t + M(1 + 2G_oR_a/\mu) - E > 0$$

If $E < 0$ stability is enhanced by increasing R_a from 0. Inspection of (11.3) shows that $E < 0$ if

$$G_1(1 + R_i/\mu R_t) > kS_1$$

For a three-halves-law amplifier G_1 is positive and is usually much greater than S_1. Only for very small amplitudes of oscillation is $E > 0$ and such amplitudes are in any case absolutely stable. The effect of R_a is therefore to improve stability.

When R_a is not very small no simplification is possible. The criterion $a_1a_2 - a_oa_3 > 0$ is a quadratic in M with coefficients depending on T_a, T_t and T_g. By substituting for M according to (11.6) this can be made a quadratic in T_g from which the critical value of T_g can be calculated. The expressions for the coefficients in this inequality are rather lengthy and in numerical calculations it is more convenient to deal directly with the expressions for a_o etc given by (11.6).

The labour of computation is slightly reduced by expressing L, E and F in the following forms. It is assumed that V_g is proportional to V_{g1}.

$$L = -2(1 + S_oR_g)(1 + R_o/r_g)(1 - kR_T/r_g)(KG_1 - G_2)R_E$$

$$E = 2(1 + R_o/r_g)(G_1R_E - (S_1r_g + G_1R_E)kR_T/r_g)G_1R_a/\mu$$

$$E + F = -2(1 + S_oR_g)(1 + R_o/r_g)(R_E(KG_o - G_1)(1 - kR_T/r_g) + (S_2/S_1)kR_T/r_g)G_1R_a/\mu \qquad (12.2)$$

The second type of oscillator to be discussed is the Hartley circuit shown in Fig. 13. This differs essentially from Fig. 12 in that the low-frequency transfer impedance Z_t^o is no longer 0. As before the characteristic equation is a cubic. The work is straightforward but the algebra is excessively tedious. Some simplification is achieved by assuming that μ is very large. Then with the same assumptions as before

$$A(p) = (1 + S_oZ_o^o + (G_o - kS_o)Z_t^o)(1 + BZ_t^+/R_t) - (Z_o^o/R_g + (G_1/S_1 - k)Z_t^o/R_g)DZ_t^+/R_t \qquad (12.3)$$

B and D are given by (11.3), and R_t/Z_t^+ by (11.4). Since the impedance of the antiresonant circuit is neglegible at low frequencies

$$R_g/Z_o^o = (1 + p(T_a + T_g + R_aC_g) + p^2T_aT_g)/(1 + p(T_a + R_aC_g))$$

$$R_a/Z_t^o = (1 + p(T_a + T_g + R_aC_g) + p^2T_aT_g)/pT_g$$

Substituting these expressions into (12.3) the coefficients in the cubic equation for p are obtained as

$$a_o = T_aT_gT_t$$

$$a_1 = MT_aT_g + T_gT_t(1 + R_a(G_o + 1/R_g) + S_oR_a(1 - k))$$

$$a_2 = MT_g + L(T_a + R_aC_g) + (1 + B)(G_o - kS_o)R_aT_g - D(G_1/S_1 - k)R_aC_g$$

$$a_3 = L \qquad (12.4)$$

M and L are given by (11.6) and (12.2), and 1 + B and D by (11.9) and (11.3).

The stability criteria are again taken as the first and last inequalities in (11.7). The first is

$$M > -(1 + R_a(G_o + 1/R_g) + S_oR_a(1 - k))T_t/T_a \qquad (12.5)$$

Since G_o, S_o and 1 - k are positive the effect of R_a is to strengthen the inequality. The other inequality can be simplified by assuming that R_a is small, that k = 1, and that r_g is large compared with R_o and R_t. This gives

$$M^2 T_a/T_t + M(1 + G_o R_a) - 2(G_1^2 - G_o G_2) R_a R_E > 0$$

For a three-halves-law amplifier $G_1^2 - G_o G_2 > 0$, and the effect of R_a is to strengthem this inequality also. The stabilising influence of R_a is due to the strong degenerative feedback which exists at low frequencies.

When R_a is not very small the full expressions for a_o etc must be used. As for the previous oscillator M can be substituted according to (11.6) to obtain the criterion in the form of a quadratic in T_g.

When $T_g = \infty$ the stability criteria become $a_1 > 0$, $a_2 > 0$ With suitable(fixed) values for the other parameters these inequalities give two values of R_a between which the amplitude is stable.

13. Asymmetrical networks.

An asymmetrical impedance is one which does not satisfy the symmetry equation $Z(p^* + j\omega_0) = Z^*(p + j\omega_0)$. Since no impedance function is truly symmetrical a more practical definition of asymmetry is that $|Z(p^* + j\omega_0) - Z^*(p + j\omega_0)|$ should not be negligible compared with the maximum value of $|Z(p + j\omega_0)|$ over the relevent range of p.

When the impedances in a feedback network are asymmetrical the expressions (6.3) for a_1 etc cannot in general be simplified, and the full expression for D(p) would run to hundreds of terms. One effect of asymmetry is therefore to increase greatly the complexity of the analysis. Only in a few special cases is it possible to make simple statements about stability comparable with those for symmetrical networks. Otherwise simplifying assumptions must usually be introduced to obtain a managable solution.

Physically the complication is due to the fact that when a carrier wave modulated in either amplitude or frequency is applied to a network which is asymmetrical with respect to the carrier frequency, the transmitted wave is modulated in both amplitude and frequency. Hence if the impedances of an oscillator feedback network are asymmetrical with respect to the oscillation frequency any change in oscillation amplitude must be accompanied by a change in frequency, and vice versa.

This is the reason why the determinant for D(p) cannot now be factorised to obtain independent criteria for frequency and amplitude stability. For the same reason asymmetrical-circuit oscillators, such as the resistance-capacitance type, cannot be keyed without "chirping" i.e. without a change of frequency during the periods of build-up and decay of the oscillation.

Most asymmetrical networks fall into one or other of two classes - those in which asymmetry is a slight and unavoidable imperfection, and those which are inherently highly asymmetrical. Also a network may have local symmetry near to ω_0 but be quite asymmetrical when considered over the total range of p. In near-symmetrical networks the coupling between amplitude and frequency changes is small: one type of modulation is dominant and the other simply concomitant. The stability criteria are only slightly different from those of a symmetrical network. In highly asymmetrical networks amplitude and frequency modulations may be of comparable magnitude and the distinction between amplitude and frequency stability becomes blurred, though in many cases it will still be possible to say that instability is mainly of one kind or the other.

Using (7.1) the determinant for D(p) can be written as

$$D(p) = \tfrac{1}{2}(F(p)A(p) + F^*(p^*)A^*(p^*)) \qquad (13.1)$$

where $F(p) = a_1 - c_1 - (a_o - c_o)b_1/b_o$

$$A(p) = b_o(a_{-1} + c_{-1}) - b_{-1}(a_o + c_o) \qquad (13.2)$$

These expressions are more general forms of the frequency and amplitude stability functions used in previous Sections, but they no longer have the same physical significance. D(p) can be written in terms of the symmetrical and anti-symmetrical parts of these functions.

Let $F_s(p) = \tfrac{1}{2}(F(p) + F^*(p^*))$

$$F_a(p) = \tfrac{1}{2}(F(p) - F^*(p^*))$$

be the symmetrical and anti-symmetrical parts of F(p), with similar expressions for A(p). Then

$$D(p) = F_s(p)A_s(p) + F_a(p)A_a(p)$$

Let p_1 be a root of $F(p) = 0$ and let $p_1 + d$ be a corresponding root of $D(p) = 0$. Then if d is small

$$d = -F_aA_a/(F_s'A_s + F_a'A_a + F_aA_a')$$

the values of the functions being taken at p_1. If the asymmetry of both F(p) and A(p) is small, then d is of the second order of smallness, except possibly at certain critical points.

Aperiodic stability is first considered. This requires $D'(0) > 0$. From (7.2), $F(0) = 0$, hence

$$D'(0) = \tfrac{1}{2}(F'(0)A(0) + F'^*(0)A^*(0)) \qquad (13.3)$$

Although D'(0) is thus known when F'(0) and A(0) are known the individual values of these quantities do not by themselves indicate stability or instability, for they are no longer related to the number of possible encirclements which the loci of $F(p)$ and $A(p)$ can make, and the sum of these encirclements is not necessarily equal to the number of encirclements made by $D(p)$ However, in two special cases it is possible to express $D'(0)$ as the product of two independent factors which can be interpreted as frequency and amplitude stability functions.

It is observed that $D'(0)$ depends entirely on the behaviour of the oscillator for very small values of p, so that although amplitude and frequency changes may be very rapid, the stability criterion is the same as if these changes were infinitely slow and may therefore be expressed in terms of the steady-state behaviour discussed in Section 4. The amplitude and frequency are defined by $R_E gh(K,Y) = -1$ and $X_E = 0$. Slow changes of amplitude and frequency can take place independently if either

(1) The steady-state frequency is independent of small amplitude changes.

or (2) The steady-state amplitude is independent of small frequency changes.

For if (1) is satisfied (2) is automatically satisfied and vice versa. If μ and k are assumed constant the only amplitude-dependent quantity in the expression (4.15) for X_E is r_g. The frequency-dependent quantities in the amplitude equation are R_E and K. Condition (1) or (2) is therefore satisfied if

$$\left.\begin{array}{ll} & \text{(a) } r_g \text{ is constant.} \\ \text{or if} & \text{(b) } X_E \text{ is independent of } r_g \\ \text{or if} & \text{(c) } R_E'/R_E + 2G_2R_EK'/K = 0 \end{array}\right\} \quad (13.4)$$

The last is the total derivative of the amplitude equation with respect to frequency, and is obtained via (4.8), (5.10) and (5.14).

r_g is constant if the grid-bias voltage is fixed and sufficient to stop grid current ($r_g = \infty$), or if the grid-current characteristic has the semi-linear form of (4.19) with $V_{cg} = 0$

Using (4.11), (4.12), and (4.15) conditions (b) and (c) can be expressed in terms of the primary quantities R_t, X_t etc., but the general formulae are rather awkward. If $\mu = \infty$ (b) simplifies to $X_\theta = 0$ and (c) to $R_E^1 = 0$

Although amplitude and frequency changes are coupled the stability criteria become uncoupled if (13.4) is satisfied. This point is illustrated and developed in the next Section.

14. Aperiodic stability.

Two modes of operation will be considered - first with r_g constant, and second with a very small oscillation amplitude and with $\mu = \infty$. If r_g is constant V_g is proportional to V_{g1}, and from (13.2), (13.3), (6.3), etc.

$$D'(0) = P_o P_f P_a$$

where
$$P_o = 2(1 + S_o R_g)\left|1 + (Z_o - kZ_t)/r_g\right|^2$$
$$P_f = X_E'$$
$$P_a = KG_1 - G_2 + (G_1^2 - G_o G_2 - kKG_1/R_g)R_a/\mu$$

Comparing this with (9.2) and (10.2) it is seen that $D'(0)$ is the same as the product of $F'(0)$ and $A(0)$ for a symmetrical-network oscillator but with $|1 + (Z_o - kZ_t)/r_g|$ in place of $1 + (R_o - kR_t)/r_g$. Since $P_o > 0$, $D'(0)$ is positive if P_f and P_a have the same sign. The conditions $P_f > 0$, $P_a > 0$ are identical with the criteria (9.3) and (10.3) for frequency and amplitude stability in symmetrical networks. The second possibility that both P_f and P_a are negative must be rejected, since in the limit of vanishing asymmetry the criteria must become those for a symmetrical network.

Hence it seems reasonable to say that if $P_f > 0$ and $P_a < 0$ the instability is essentially of amplitude, the frequency change being merely concomitant, and vice versa if $P_f < 0$ and $P_a > 0$. Independent criteria are obtained because condition (13.4a) is fulfilled. If the grid-bias voltage is fixed and sufficient to stop grid current the same formulae are obtained but with $r_g = \infty$ and $K = 0$.

The frequency stability criterion $X_E' > 0$ will now be used to examine a hysteresis effect which occurs in the oscillator shown in Fig.14. When the resonant frequency of the secondary circuit is varied by adjusting the condenser C_s, the oscillation frequency varies as shown in Fig.15. For simplicity the grid-bias voltage is fixed and sufficient to stop grid current, but an identical result is obtained with grid-leak bias (constant r_g) provided the resistance of the primary circuit is adjusted to allow for the extra damping due to grid current.

$$\left.\begin{array}{l} \text{Let} \quad C_s = (1+s)C, \quad \omega_r^2 LC = 1, \quad \omega_0 = \omega_r + \omega, \quad Q = R/\omega_r L \\ \quad M = bL, \quad u = b^2Q^2 - 1, \quad x = 2Q\omega/\omega_r + sQ \end{array}\right\} \quad (14.1)$$

If Q is large and b and s are small

$$R/Z_i = (2 + u - x(x - sQ) + j(2x - sQ))/(1 + jx)$$

Since a phase reversal in the feedback network is necessary to maintain oscillation, Z_t and so also Z_E, is proportional to $-Z_i$. The possible steady-state frequencies are those which make Z_i real. Equating the imaginary part of Z_i to 0 gives

$$x^3 - sQx^2 - ux - sQ = 0 \qquad (14.2)$$

The behaviour of the roots of this equation can be examined by the standard methods[33]. If $u < 0$, i.e. if $bQ < 1$ the equation has only one real root, and so there is only one possible steady-state frequency whatever the value of C_s. If $u > 0$ there are three real roots for a certain range of values of sQ, and one real root outside this range.

The criterion for aperiodic frequency stability, $X_E' > 0$, is equivalent to $-X_i' > 0$. Now $-X_i$ can be written as a fraction of which the numerator is the L.H.S. of (14.2) and the denominator is real and positive. For large values of ω, X_i has the same sign as ω, and since X_i is also a continuous single-valued function of ω it follows that when equation (14.2) has three real roots, then $-X_i' > 0$ for the smallest and largest values of ω, and $-X_i' < 0$ for the intermediate value. Hence of the three possible steady-state frequencies two are stable and the third unstable. The dotted line in Fig.15 shows the unstable value.

At the points a and c where irreversible jumps take place equation (14.2) has two equal roots. The critical values of sQ are given by

$$\left.\begin{array}{l} 8(sQ)^2 = y + (y^2 + 64u^3)^{\frac{1}{2}} \\ \text{where} \quad y = u^2 - 18u - 27 \end{array}\right. \qquad (14.3)$$

The corresponding critical frequencies are

$$\omega/\omega_r = -s((1 + 3u/s^2Q^2)^{\frac{1}{2}} + 2)/6 \qquad (14.4)$$

At the points b and d the (stable) frequencies are

$$\omega/\omega_r = s((1 + 3u/s^2Q^2)^{\frac{1}{2}} - 1)/3 \qquad (14.5)$$

When $s = 0$ the two circuits are tuned to the same frequency ω_r which is also the oscillation frequency if $u < 0$. If $u > 0$ the two stable frequencies are

$$\omega/\omega_r = \pm\tfrac{1}{2}u^{\frac{1}{2}}/Q \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14.6)$$

The maximum and minimum frequencies at f and g are obtained when $sQ = \pm\frac{1}{2}(1 - u)$, and are

$$\omega/\omega_r = (1 + u)/4Q = b^2Q/4 \qquad (14.7)$$

This type of hysteresis effect has been studied from a somewhat different point of view by van der Pol[3].

Instability at the threshold of oscillation, i.e. for very small values of V_{g1} is next considered. As for the symmetrical network treated in Section 10, the analysis will be restricted to a three-halves-law amplifier with an exponential grid-current characteristic. It is also assumed that $\mu = \infty$. This makes $q = 1$ and $Z_T = R_T$ since equation (4.15) for the steady-state frequency is now $X_T = 0$. Using (13.2), (13.3) etc. the result is

$$D'(0) = P_o(P_fP_a + P_c)$$

where

$$\left.\begin{aligned} P_o &= 2(V_{g1}/V_g)^2(1 + S_oR_g)\left|1 + (Z_o - kZ_t)/r_g\right|^2 \\ P_f &= -X_E'/R_E \\ P_a &= Y^2/32(1 - Y)^2 + nY/4(1 - Y) + \tfrac{1}{2}m(k - R_\theta/R_T)R_E/r_g \\ P_c &= -\tfrac{1}{2}mX_\theta R_E'/R_Tr_g \end{aligned}\right\} \qquad (14.8)$$

The values of the parameters are those at the inception of oscillation and m and n are given by (5.8). From (9.2) and (10.7) it is seen that $P_oP_fP_a$ is the same as $F'(0)A(0)$ for a symmetrical network but with $|1 + (Z_o - kZ_t)/r_g|$ in place of $1 + (R_o - kR_t)/r_g$. Since $P_o > 0$ the first requirement for stability is that

$$P_fP_a + P_c > 0 \quad . \quad . \quad . \quad . \qquad (14.9)$$

In a symmetrical network the criteria for aperiodic stability are $P_f > 0$, $P_a > 0$. The term P_c therefore represents a coupling which is 0 if m or X_θ or R_E' is 0. If $m = 0$ then $dr_g/dV_{g1} = 0$ and condition (13.4a) is satisfied. $X_\theta = 0$ and $R_E' = 0$ are the forms taken by (13.4b) and (13.4c) when $\mu = \infty$.

In an asymmetrical network aperiodic instability may exist even when P_f and P_a are positive. Conversely the oscillation may be stable when either is negative, but the possibility of stability with both negative is excluded for the same reason as before. A sharp distinction between frequency and amplitude instability is now impossible.

Oscillators of this type are rare, for in most feedback networks the zeros of X_E lie near to maximum or minimum values of R_E, and R_E' is therefore small. The effects are most easily demonstrated by choosing a rather artificial type of circuit such as that shown in Fig.16. X_a is a reactance, positive or negative, which is large compared with the impedance of the tuned circuit coupled through the mutual inductance -M. It is assumed also that the Q-factor of the circuit is large and that the reactance of C_g is small.

The oscillation frequency differs from the resonant frequency of L and C by an amount depending on the ratio X_a/R_a.

Let $\quad R_1 = Rr_g/(R + r_g), \quad M = bL \quad . \quad . \qquad (14.10)$

Straightforward algebra gives the following results

$$\left.\begin{aligned} &R_T = -bR_1/(1 + X_a^2/R_a^2), \quad R_T/R_\theta = -b, \quad X_\theta/R_\theta = X_a/R_a \\ &X_E'/R_E = 2R_E/\omega_0^2 M, \quad R_E' = -X_E' X_a/R_a, \quad R_T = R_E/(1 + kR_E/r_g) \end{aligned}\right\}(14.11)$$

Since $P_f > 0$ the stability criterion (14.9) can be written $P_a > -P_c/P_f$, and using (14.11) and (14.8) this becomes

$$P_a > -\tfrac{1}{2}mR_E(X_a/R_a)^2/br_g$$

$$\text{or} \quad P_a > \tfrac{1}{2}m(R_E + bR_1(1 + kR_E/r_g))/br_g \qquad (14.12)$$

The second form is more convenient for computation and the first shows how the stability is affected by X_a. As for the symmetrical network, instability is possible only if $m > 0$. The effect of the coupling term P_c is to reduce the stability margin.

15. Periodic stability.

In view of the complexity of any general treatment only the simplest example will be considered. The oscillator of Fig.16 is again chosen. As before it is assumed that μ, Q and X_a are large, and in addition that r_g is large compared with Z_o and Z_t and is independent of V_{g1}. Then $R_t = R_T = R_E$. From (13.2)

$$F(p) = bpT_t/(1 + bpT_t)$$

$$A(p) = \frac{(1 + S_oR_g + pT_g)(2G_2R_t + b^*pT_t) - 2G_1R_tS_1R_g}{(1 + pT_g)(1 + b^*pT_t)}$$

where $b = (1 + jX_a/R_a)$, $T_g = R_gC_g$, $T_t = 2Q/\omega_o$

Using (13.1) D(p) can be written as a rational function of p, the numerator of which is

$$pT_t(a_1p^2 + a_2p + a_3)$$

where $a_1 = T_gT_t$

$$a_2 = (1 + S_oR_g)T_t + 2G_2R_tT_g + T_tX_a^2/R_a^2$$

$$a_3 = 2(1 + S_oR_g)G_2R_t - 2G_1R_tS_1R_g$$

The Routh-Hurwitz stability rules are that a_1, a_2 and a_3 should have the same sign. The condition $a_3 > 0$ is simply the criterion for aperiodic stability which is assumed to be satisfied. For periodic stability $a_2 > 0$, which can be written as

$$(1 + S_oR_g + X_a^2/R_a^2)T_t/T_g > -2G_2R_E \qquad (15.1)$$

This may be compared with (11.8) to which it reduces when $X_a = 0$ When X_a/R_a is small the asymmetry results in only a second order error when the symmetrical-network criterion is used. This is in agreement with the general result obtained in Section 13. It is clear from the form of the criterion that any instability is primarily of amplitude.

16. Two-terminal oscillators.

Much attention has been given in the literature to the two-terminal oscillator, for it is the simplest to treat mathematically, and the equation to a four-terminal oscillator (with no grid current and no anode decoupling impedance) can be reduced to the same form. The oscillator consists of a passive linear network of impedance Z(p) connected in parallel with a non-linear resistance element which has a negative slope over part of its range.

It is assumed that Z(p) has a significant value only near to the oscillation frequency, so that any direct or low-frequency voltage across Z(p) is negligible. The steady-state voltage is then $v = V\cos\omega_0 t$. If the current-voltage relation for the non-linear resistance is $i = f(v)$, the current of fundamental frequency is $I\cos\omega_0 t$ where

$$I = (2/\pi)\int_0^{\pi} f(V\cos x)\cos x\,dx \;=\; Vh(V) \qquad \ldots \qquad (16.1)$$

This must be equal and opposite to the current of fundamental frequency in the linear network. Hence if $Z = R + jX$, the steady-state equations are

$$X = 0 \quad \text{and} \quad 1 = -Rh(V) \qquad \ldots \qquad (16.2)$$

A small disturbance v_d is added to v and the sum of the currents due to v_d in Z(p) and in the non-linear resistance is equated to 0. The analysis follows that of Section 5 and gives

$$\begin{bmatrix} a_1 & c_1 \\ a_{-1} & c_{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_{-1} \end{bmatrix} = 0$$

where $a_1 = 1 + G_0Z^+$, $c_1 = G_2Z^+$, $a_{-1} = G_2Z^-$, $c_{-1} = 1 + G_0Z^-$

and $G_n = (1/\pi)\int_0^{\pi} f'(V\cos x)\cos nx\,dx \qquad G_0 - G_2 = -1/R \qquad (16.3)$

The stability function is $D(p) = \begin{vmatrix} a_1 & c_1 \\ a_{-1} & c_{-1} \end{vmatrix}$ Proceeding on the lines of Section 13, $D(p) = \frac{1}{2}(F(p)A(p) + F^*(p^*)A^*(p^*))$

where $F(p) = 1 - Z^+/R$, $A(p) = 1 + (G_0 + G_2)Z^-$

The first criterion for aperiodic stability is

$D'(0) = -2G_2X' > 0$. But stability is obtained only if the frequency and amplitude are separately stable, and this requires

$$-X' > 0 \quad \text{and} \quad G_2 > 0 \qquad (16.4)$$

This may be expressed in another way. If an effective conductance for the non-linear resistance at fundamental frequency is defined as $Y_e = I/V$, then from (16.1), (16.3) and (16.2), $dY_e/dV = 2G_2/V$. Let the admittance of the linear network be $Y = G + jB$. Then $-X' = R^2B'$. Hence the stability criteria can be expressed as $dB/d\omega > 0$ and $dY_e/dV > 0$. It is not sufficient for the product of these quantities to be positive[13]

In the conventional treatments of the problem $f(v)$ is usually represented as a power series $f(v) = \sum_0^\infty g_n v^n$. From (16.1) and (16.2) the steady-state amplitudes are given by

$$1/R + \sum_0^\infty g_{2n+1}(\tfrac{1}{2}V)^{2n}(2n+1)!/n!(n+1)! = 0 \qquad (16.5)$$

and from (16.3) and (16.4) the criterion for amplitude stability is

$$\sum_1^\infty g_{2n+1}(\tfrac{1}{2}V)^{2n}(2n+1)!/(n+1)!(n-1)! > 0 \qquad (16.6)$$

Adding (16.5) and (16.6) gives

$$1/R + \sum_0^\infty g_{2n+1}(\tfrac{1}{2}V)^{2n}(2n+1)!/(n!)^2 > 0$$

which agrees with the results obtained by Appleton and Van der Pol[1]

It is implicit in the analysis that $f(v)$ is a single-valued function of v, i.e. the non-linear resistance is the "voltage-controlled" type. For the "current-controlled" type of resistance the voltage is a single-valued function of current. The analysis can be carried out in exactly the same way as before by replacing all the circuit elements by their duals.

Periodic instability can be treated in the same way as in previous Sections. Since the low-frequency impedance of the network is assumed to be zero ~~the~~ only frequency instability is of interest. The criteria (16.4) are therefore not sufficient to guarantee stability. If the low-frequency impedance is made finite periodic amplitude instability can also exist.

17. Other stability criteria.

In this Section some of the methods proposed by previous investigators are critically examined. Most writers, following Appleton and van der Pol[1,20], have postulated variations of amplitude and phase which are assumed to be so slow that terms in the variational equation containing time derivatives higher than the second could be neglected.

Now, in fact, the changes of amplitude and phase may be very rapid so that the fundamental assumption of the analysis is not always justified. Nevertheless the criteria derived are valid for all cases of aperiodic instability however rapid the changes may be. The reason for this curious result lies in the fact, discussed in Section 13, that aperiodic stability is determined entirely by the behaviour of the characteristic function near to $p = 0$, and small values of p correspond to slow variations in the amplitude and frequency of oscillation.

Periodic instability is beyond the scope of these methods and various other proposals have been put forward. Edson[11] has suggested that Nyquist's criterion may be used to determine amplitude stability by interrupting the feedback network at some suitable point. A signal generator with a terminal voltage equal to that of the possible steady-state oscillation is connected across the grid side of the break, and across the other side is connected an impedance equal to that on the grid side. The steady voltage appearing across this impedance is equal to the signal-generator voltage.

If the applied voltage is now amplitude modulated the voltage returned to the other side of the break is also modulated but with a different magnitude and phase due to the action of the amplifier and feedback network. The complex ratio of the two modulations is analogous to the loop transmission function in feedback-amplifier theory and the oscillation should be stable if the locus of the ratio does not encircle the point 1,0 when the modulation frequency is varied from $-\infty$ to ∞.

It was shown in Section 7 that the use of this contour leads to difficulties, the appropriate one being that of Fig.7. Another point is that if low-frequency currents flow from terminals 1 to 2 of Fig.1 it may be impossible to interrupt the network without destroying the relations between the low-frequency currents and voltages. This difficulty can be avoided in the following way. A generator of zero impedance is inserted in any branch of the network, its e.m.f. being such that when added to the steady oscillation the result is an amplitude-modulated wave. The current of oscillation frequency flowing through the generator is similarly modulated, and the complex ratio of the modulations of voltage and current defines the impedance into which the generator operates. Stability is obtained if the locus of this impedance does not enclose the origin. This is equivalent to the closed-circuit form of Nyquist's criterion.[25]

It is convenient to consider the more general case in which the generator e.m.f represents modulations of amplitude, of frequency, and of the mean grid voltage. Let a generator of e.m.f. $e = V_d \sum_{-1}^{1} m_n \exp(p + jn\omega_o)t$ be inserted in the lead between the grid and terminal 2 in Fig.1. The calculation proceeds on the lines of Section 6 with $v_{gd} - e$ written in place of v_{gd} in the expressions for i_{ad} and v_{ad}. Since only symmetrical networks can be treated by this method, $q = 1$ and $Z_t^- = Z_t^+$, $c_{-1} = a_1$ etc. The equations (6.2) for u_1, u_o and u_{-1} become

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_o & b_o & a_o \\ c_1 & b_1 & a_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_o \\ u_{-1} \end{bmatrix} = \begin{bmatrix} h_1 \\ h_o \\ h_{-1} \end{bmatrix} \qquad (17.1)$$

where
$$\left.\begin{aligned} h_1 &= m_1(1 + G_o Z_1^+/\mu) + m_o G_1 Z_t^+ Z_1^o/\mu Z_t^o + m_{-1} G_2 Z_1^+/\mu \\ h_o &= m_o(1 + G_o Z_1^o/\mu) + (m_1 + m_{-1}) G_1 Z_t^o Z_1^+/\mu Z_t^+ \\ h_{-1} &= m_{-1}(1 + G_o Z_1^+/\mu) + m_o G_1 Z_t^+ Z_1^o/\mu Z_t^o + m_1 G_2 Z_1^+/\mu \end{aligned}\right\} \qquad (17.2)$$

If $m_1 = m_{-1} = 1$ and $m_o = 0$ the sum of the real part of e and the steady oscillation is an amplitude-modulated wave. In calculating the impedance into which the generator operates it is sufficient, by virtue of symmetry, the consider the voltage and current at only one of the high frequencies, say $p + j\omega_o$

Equation (17.1) can be solved in the usual way for u_1, u_0 and u_{-1} and the grid current of frequency $p + j\omega_0$ can then be calculated from (5.3) and (5.4). The impedance is given by the ratio of $V_d \exp(p + j\omega_0)t$ to this current, and is

$$Z = A(p)/M(p) \quad . \quad . \quad . \quad . \quad . \quad (17.3)$$

where $A(p)$ is the amplitude stability function of Section 8 and

$$M(p) = (S_0 + S_2)(h_1 b_0 - h_0 b_1) + S_1(h_0(a_1 + c_1) - 2h_1 a_0) \quad (17.4)$$

Now the amplitude is unstable or stable as the locus of $A(p)$ does or does not enclose the origin. If the locus of Z is to be a true criterion of stability it must enclose the origin the same number of times and in the same sense as that of $A(p)$. It follows that the locus of $M(p)$ must not enclose the origin. The simplified case of $\mu = \infty$ is first considered. Substituting for h_1, a_1 etc according to (17.2) and (6.3),

$$M(p) = S_0 + S_2 + (S_0(S_0 + S_2) - 2S_1^2)Z_0^0$$
$$+ (G_0(S_0 + S_2) - 2G_1 S_1 - k(S_0(S_0 + S_2) - 2S_1^2))Z_t^0$$

The impedances Z_0^0 and Z_t^0 are passive driving-point and transfer functions and thus can have no poles in the right-hand half-plane. Only zeros of $M(p)$ need therefore be considered. (5.6) shows that $S_0 + S_2$ and $S_0(S_0 + S_2) - 2S_1^2$ are always positive. Also, since Z_0^0 is a passive driving point impedance its real part is always positive. However, there is nothing to prevent the coefficient of Z_t^0 from being negative, and the real part of Z_t^0 may itself be negative. Hence it is possible for $M(p)$ to have a zero in the right-hand half-plane. A necessary and sufficient condition for $M(p)$ not to enclose the origin is therefore $Z_t^0 \equiv 0$. This ensures that the real part of $M(p)$ is always positive.

For the general case where μ is finite the application of this condition gives

$$M(p) = (1 + G_0 Z_1^0/u)(1 + (G_0 + G_2)Z_1^+/u)$$
$$\times \ (S_0 + S_2 + (S_0(S_0 + S_2) - 2S_1^2)Z_0^0)$$

(5.15) shows that G_0 and $G_0 + G_2$ are always positive. Z_1^0 and Z_1^+ are passive driving-point functions and so have positive real parts. Hence the loci of none of the three factors in this expression can enclose the origin and so neither can that of M(p). The condition $Z_t^o \equiv 0$ means that low-frequency currents cannot be transmitted through the network. When this condition is satisfied it is possible to select a point where the network may be interrupted to apply the open-circuit form of the Nyquist test in the manner proposed by Edson. The results are the same as for the closed-circuit test.

In the foregoing analysis the stability rule was expressed in terms of the high-frequency components of voltage and current, but Edson has suggested[14] that the Nyquist open- and closed-circuit tests may be used also at low frequencies. This method, if valid, would be most valuable, for it could be applied also to asymmetrical networks. It will be sufficient to examine the simplest case of the closed-circuit form of test with a symmetrical network and with $\mu = \infty$

The generator e.m.f. is now $e = V_d \exp . pt$, i.e. $m_1 = m_{-1} = 0$ and $m_0 = 1$. The impedance into which the generator operates is the ratio of e to the grid current of frequency p. It is calculated in the same way as before and is $Z = A(p)/L(p)$, where A(p) is the amplitude stability function and

$$L(p) = S_o + (S_o(S_o + S_2) - 2S_1^2)Z_o^+ \\ + (S_o(G_o + G_2) - 2S_1G_1 - k(S_o(S_o + S_2) - 2S_1^2))Z_t^+$$

For the impedance locus to be a valid criterion of amplitude stability L(p) must not enclose the origin. S_o and the real part of the first term are always positive, but the coefficient of Z_t^+ may be positive or negative. The necessary and sufficient condition for L(p) not to enclose the origin is that $Z_t^+ \equiv 0$, but this implies that there is no transmission of high-frequency current through the network and therefore no oscillation. It follows that stability cannot be determined by applying the Nyquist test at low frequencies.

Although Edson was concerned only with amplitude stability his method may be adapted to determine also frequency stability in symmetrical networks. Considering the closed-circuit form of test, let $-m_{-1} = m_1 = 1$ and $m_0 = 0$. The sum of the real part of e and the steady oscillation is then a frequency-modulated wave. The impedance into which the test generator operates is taken as the ratio of voltage to current at the frequency $p + j\omega_0$ and is $Z = F(p)/N(p)$, where $F(p)$ is the frequency stability function of (8.2) and $N(p) = (1 - Z_i^{\dagger}/\mu R_E)/r_g$. Since R_E is negative and $Z_i^{\dagger}$ has a positive real part the locus of $N(p)$ cannot enclose the origin, and the locus of Z therefore makes the same number of encirclements as that of $F(p)$. Both the open- and closed-circuit forms of the Nyquist test may be used without restriction to determine frequency stability.

Other criteria for periodic amplitude stability proposed by Edson and by van Slooten are examined in the following Section.

18. Experimental results and conclusions.

A series of tests was carried out to check the main points of the theory. Two amplifiers - a triode and a pentode - were used of which the parameters occurring in expressions (4.4), (4.19), (4.22) and (4.23) were as follows.

Amplifier	g mhos	V_{ca} volts	μ	k
1	$6.57\ 10^{-3}$	-8.10	20.6	1
2	$6.40\ 10^{-3}$	-2.70	2200	0.78
	b_g mhos	V_{cg} volts	I_o amps	V_o volts
1	$1.05\ 10^{-3}$	0.05	$1.27\ 10^{-4}$	0.103
2	$1.98\ 10^{-3}$	-0.16	$9.9\ 10^{-4}$	0.105

These figures were obtained from plots of the anode and grid currents. At small values of i_a the three-halves law is not followed, and for small values of v_g the grid-current curve is not linear. V_{ca} and V_{cg} were therfore found by extrapolating the linear parts of the i_g and $i_a{}^{2/3}$ graphs. I_o and V_o were obtained in a similar way. The effects of departures from the theoretical forms are discussed later.

Dust-cored coils were used to obtain a high coupling factor between the two windings. This makes the voltage ratio between anode and grid equal to $-R_i/R_t = -R_I/R_T$, and also $R_n = 0$. The inductance of the main coil was 1.015 mH. Unless otherwise stated the oscillation frequency was 50.76 Kc/s.

<u>Test 1</u> Oscillation amplitude. (Section 4)

For tests (b) and (c) the circuit of Fig. 12 was used with $R_a = 0$; for test (a) the same but with the anti-resonant circuit and the coupling coil interchanged. The oscillation amplitude was varied by adjusting the resistance R and the number of turns of the coupling coil. Amplifier 1 was used throughout. The method of calculation is described in Section 4.

Test	R KΩ	N	V_{g1} volts calc.	meas.
(a) K = 0.75	2.09	2.32	8.90	8.80
R_g = 17.5 KΩ	4.57	4.23	18.0	18.0
$-R_I/R_T$ = 0.205	6.90	5.50	27.3	25.5
	12.5	7.52	——	39.6
(b) K = 1.00	1.06	2.49	7.50	7.40
R_g = 66.6 KΩ	2.09	4.86	12.4	12.7
$-R_I/R_T$ = 2.44	4.57	10.3	20.9	21.1
	6.90	15.2	27.2	26.7
(c) K = 1.83	6.89	2.44	6.48	6.22
R_g = 614 KΩ	12.5	4.43	8.40	7.78
$-R_I/R_T$ = 9.75	27.7	9.76	10.4	9.76

With the exception of the last result of test (a) the calculated and measured amplitudes agree within the limits of experimental error. According to Fig. 3 there is no value of Y corresponding to N = 7.52 and K = 0.75. Departures from the three-halves law at small values of i_a would produce only slight errors unless the valve had an exceptionally long "tail". However, for large positive grid voltages the anode current may be much less than the theoretical value. This would account for the last result of test (a) for the maximum grid voltage there is 10.2 volts. The slightly higher errors in test (c) may be attributed to the fact that the quantity $1 + R_I/\mu R_T$, which is the denominator of K, is approximately 0.5, so that in calculating K any errors in R_I/R_T and μ are doubled.

Departures of the grid-current law from the semi-linear form at small voltages produce usually little error. When R_g is small the grid is driven well into the linear region unless the oscillation amplitude is very small. If V_{g1} is excessively large the grid may be driven beyond the linear region. The effective value of b_g and thus of K is then increased. When R_g is large the ratio V_g/V_{g1} is almost -1 and is practically independent of changes in the value of b_g.

Test 2. Frequency stability. (Section 9)

The circuit used was a modification of that shown in Fig. 10. Three identical coils were used with equal shunt resistances, the effect of varying the shunt resistance nR of the central coil being simulated by a potentiometer connected across the coupling coil. With $\mu = \infty$ and a large grid-leak resistance, the equation for the modified circuit is the same as for the original, n being now the potentiometer ratio. The circuits were tuned to frequencies of 45, 50 and 55 Kc/s. i.e. $\omega_m/\omega_o = 0.1$. Starting with the potentiometer set at maximum (n = 1) and the oscillation frequency at 50 Kc/s., the setting was reduced to the critical value at which instability began.

Q	c	n calc	n meas
10	2	0.40	0.38
15	3	0.70	0.66
20	4	0.82	0.79

The theoretical values of n were calculated from (9.11) and (9.5). This theory is strictly valid only when Q is very large and ω_m/ω_o very small. It may be shown that the true theoretical value of n is less than that of the approximate theory, and that with the values of parameters used in the experiment the difference is of the same order as the observed error.

At the critical value of n instability began as a small sustained frequency modulation. As ~~the~~ n was further decreased the oscillation frequency changed suddenly to a new stable value.

Test 3. Amplitude stability - hysteresis. (Section 10)

The circuit was that of Fig. 12 with $R_g = 0$. Instability of the first kind - a change from one large amplitude to another - could not be detected with Amplifier 1. Using Amplifier 2 it was produced by lowering the screen voltage (normally 150) to 10 V, and making $R_g = 5.43$ KΩ. The critical amplitude corresponded to $V_g = -0.45$ V. At such low voltages the parameters

b_a etc have little or no meaning, and the confirmation is thus only qualitative. In all practical amplifiers this form of instability is inhibited by the departure from the three-halves law at large grid voltages.

For the second type of hysteresis effect - the sudden starting and stopping of oscillation - the anti-resonant circuit and coupling coil in Fig. 12 were interchanged. The resistance R was adjusted till oscillation just began. This was done for various values of R_g to find the critical value at which hysteresis appeared. From the measured value of R_g the quantities Y, r_g/R_E and V_g/V_0 can be found by using (4.24), (5.16) and Fig. 4. The stability criterion (10.8) can then be used to calculate the ratio $R_\theta/R_T = R_T/R_I$. Amplifier 1 was used with $R_a = 0$.

R_g KΩ	V_g/V_0	Y	$-R_\theta/R_T$ calc	$-R_\theta/R_T$ meas
500	-4.84	0.061	9.8	9.8
300	-4.42	0.056	6.5	4.9

It is a matter of some difficulty to decide whether or not a hysteresis effect exists, for at the critical value of R_g the initial stable amplitude is vanishingly small, and it is not easy to distinguish between a small hysteresis effect and a smooth start of oscillation. The first result was obtained by taking the average of the two values of R_g at which it was certain that hysteresis did and did not exist. The second result is a single estimation of R_g.

<u>Test 4.</u> Periodic amplitude instability. (Section 11)

The circuit of Fig. 12 was used with $R_a = 0$. For various values of R and R_g the grid capacitance C_g was increased until instability set in. To restore stability C_g had often to be reduced to a value less than the original critical value.

Using the measured values of R, R_g and V_g, the values of S_0R_g, S_2r_g and G_2R_E are obtained from Figs. 2, 5 and 6. The stability criterion (11.8) can then be used to calculate the critical value of C_g. Amplifier 1 was used and $-R_I/R_T = 2.44$

Test	R	$-V_g$	Q	C_g $\mu F \times 10^{-3}$			
	KΩ	volts		calc	meas	van S.	Edson
(a)	1.15	6.60	3.55	∞	∞	∞	0.75
R_g = 23.1 KΩ	2.09	12.2	6.47	11.3	11.8	30.0	1.4
	4.57	22.0	14.2	9.40	10.5	23.6	3.0
	6.90	28.8	21.3	10.5	13.2	27.0	4.5
(b)	1.06	6.46	3.27	∞	∞	∞	0.27
R_g = 66.6 KΩ	2.09	11.2	6.47	8.00	8.31	22.7	0.54
	4.57	18.5	14.2	6.50	6.50	16.4	1.2
	6.90	23.6	21.3	6.95	6.85	17.5	1.8
(c)	0.99	6.60	3.06	61.0	∞	∞	0.09
R_g = 194 KΩ	2.09	10.7	6.47	5.55	5.49	17.2	0.20
	6.90	20.3	21.3	5.20	5.36	12.9	0.65
	12.5	27.5	38.6	4.82	4.97	15.5	1.2

With the exception of the last result in Test (a) the calculated and measured values of C_g agree well. The error is probably connected with the large value (8 V) of the maximum grid voltage. Departures of the grid-current law from the semi-linear form produce errors in S_0 and S_2 which are likely to be serious for large values of R_g and small values of V_{g1} for the grid is then driven only slightly positive. Errors must also be expected with small values of R_g and large values of V_{g1}.

The last two columns in the table are calculated from formulae given by van Slooten[9] and Edson[15]. Van Slooten's criterion is

$$1.4Q(b_g R_g)^{2/3} X_g/R_g > (Y - 1)/(1 - Y(1 + V_{g1}/V_g))$$

Edson's criterion is $2QX_g/R_g > -V_{g1}/V_g$

In each test the first result gives the maximum amplitude for which stability is independent of C_g. In van Slooten's theory all values of $-V_g$ less than $-V_{ca}$ (8.1 V in this case) represent stable states.

Test 5. Periodic amplitude stability with fixed grid bias. (Section 11)

The circuit used was that of Fig. 12 with the grid leak replaced by a source of e.m.f. sufficient to stop grid current. The H.T. supply voltage was adjusted to keep the mean anode voltage at its normal value of 150 v. With all other parameters fixed the decoupling capacitance C_a was adjusted to the point of instability.

Using the measured values of R, R_a, V_g and V_{g1} a value for C_a can be calculated from (11.11) by way of Fig. 6, (5.14) etc. Amplifier 1 was used and $-R_I/R_T = 2.44$.

R_a KΩ	R KΩ	$-V_g$ volts	V_{g1} volts	Q	C_a μF x 10^{-3} calc	meas
19.8	1.22	7.00	5.55	3.78	22.7	14.1
19.8	1.71	8.00	5.87	5.30	10.5	8.6
11.4	1.50	7.50	5.23	4.65	18.9	14.2

This experiment was difficult to carry out because the oscillation amplitude changes very rapidly with variations of R and V_g. Another source of error lies in the fact that the value of C_a is about the same as that of the tuning capacitance C and that Q is small. This means that the impedance of C_a at the oscillation frequency is not negligible compared with R as assumed in the theory.

Test 6. Periodic amplitude stability with 3 time constants. (Section 12)

The circuit of Fig. 12 was used with amplifier 1, the H.T. supply voltage being adjusted to keep the mean anode voltage constant. For various values of R_a and R_g the value of C_g was adjusted to the unstable point. The other parameters, which were constant throughtout, were R = 2.09 KΩ, C_a = 0.00905 μF Q = 6.47, $-R_I/R_T = 2.44$.

Using the measured values of V_g, values for C_g can be calculated from (11.6), (11.7) and (12.2) as described in Section 12.

R_g	R_a	$-V_g$	C_g $\mu F \times 10^{-3}$	
KΩ	KΩ	volts	calc	meas
	0		11.3	11.8
23.1	2.14	12.2	18.1	18.1
	4.83		25.9	24.8
	0		8.0	8.3
66.6	2.14	11.2	12.7	13.5
	4.83		17.5	17.9
	0		5.60	5.50
194	2.14	10.7	8.70	8.50

The value of 0.00905 μF for C_a is much less than would normally be used, being in fact about equal to the tuning capacitance C. This value was chosen in order to demonstrate clearly the effect of the decoupling network. With values of C_a which would be used in practice the effect on the critical value of C_g was too small to be measured accurately.

Test 7. Hartley oscillator. (Section 12)

Fig. 13 shows the circuit used. For the first two tests R_a was fixed and C_g adjusted to the critical value, and in the last two C_g was fixed (at ∞) and R_a was varied. The other parameters, which were fixed throughout, were R_g = 53.1 KΩ, R = 4.57 KΩ, Q = 14.2, $-V_g$ = 8.4 v, C_a = 0.0585 μF, $-R_I/R_T$ =2.44 Amplifier 2 was used.

The values of C_g were calculated from $a_1a_2 = a_0a_3$, and those of R_a from $a_2 = 0$, $a_1 = 0$, using the expressions for a_0 etc given by (12.4)

R_a KΩ		C_g $\mu F \times 10^{-3}$	
meas	calc	calc	meas
0	—	8.82	9.74
1.00	—	26.4	29.5
1.59	1.68	—	∞
5.66	4.25	—	∞

In accordance with the ideas put forward in Section 12 there is a range of values of R_a over which the amplitude is stable when $C_g = \infty$. When the values of R_a, C_a and C_g are compared with those of the previous test the powerful stabilising effect of the degenerative feedback at low frequencies becomes evident. A similar effect could be obtained with the circuit of Fig. 12 if the lower terminal of C_g were connected to the upper terminal of C_a instead of to the cathode.

The errors are somewhat larger than in previous experiments but the reason for this was not discovered.

<u>Test 8.</u> Frequency hysteresis (Section 14)

~~Test 8.~~ The circuit of Fig. 14 was used with amplifier 2. Grid bias was obtained by grid leak and condenser, the damping resistance R being adjusted to suit. The mutual inductance was realised by using a third coil. After M and R had been set to suitable values the secondary tuning capacitance C_s was varied and the frequencies of the various critical points in Fig. 15 were measured. Owing to the rapid variation of frequency with C_s it was not possible to find the frequencies at a and c. The values of C_s at b and d were also measured, but no values could be found corresponding to f and g because of the extreme flatness of the maximum and minimum.

$R = 6.7\ K\Omega$, $b = M/L = 0.1$, $C = 9800\ \mu\mu F$

With $M = 0$, oscillation frequency = 48.5 Kc/s. $Q = 20$

From (14.1), $u = 3$.

From (14.3), values of sQ at points b and d are ∓ 1.18

Hence " C_s " " $9800 \mp 580\ \mu\mu F$

Measured values $= 9800 \mp 600\ \mu\mu F$

From (14.5) frequencies at points b and d are 48.50 ∓ 1.65 Kc/s

Measured values $= 48.50 \mp 1.66$ Kc/s

From (14.7) frequencies at points f and g are 48.50 ± 2.43 Kc/s

Measured values $= 48.50 \pm 2.49$ Kc/s

<u>Test 9.</u> Aperiodic stability with asymmetrical network. (Section 12)

Fig. 16 shows the circuit used. With R_a fixed R was adjusted till oscillation just began and it was observed whether or not a hysteresis effect was present. This was repeated for different values of R_a to find the value of R at which hysteresis was just detectable. Amplifier 2 was used and $b = -M/L = 0.205$ With R_g = 349 KΩ the critical value of R was 3.92 KΩ.

Using (4.24), (5.8), (5.16), (14.10) and Fig. 4, the quantity P_a given by (14.8) is found to be 0.16. P_a is also the L.H.S. of the stability criterion (14.12). Using the above figures the R.H.S. of this inequality is found to be 0.18. The discrepancy may be attributed to the experimental difficulty of deciding if a hysteresis effect exists.

<u>Test 10.</u> Effect of R_a on oscillation amplitude. (Appendix 3)

The circuit of Fig. 12 was used, the H.T. supply voltage being kept constant while R_a was varied. The other parameters were R = 2.09 KΩ, R_g = 66.6 KΩ, $-R_I/R_T = 2.44$. Values of the oscillation amplitude were calculated from (A3.2) and Fig. 17. Amplifier 1 was used.

R_a	V_{g1} volts	
KΩ	calc	meas
0	12.4	12.7
4.83	9.50	9.20
11.4	6.85	7.07
19.8	5.43	5.37

These experiments have shown that the theory, so far as it goes, is substantially correct. Errors arise mainly when the actual behaviour of the amplifier and network depart from the assumed forms. It seems safe to say that the general theory is always valid provided the true values of the various parameters are used.

From the practical point of view a serious drawback is the stability criteria are valid only for small disturbances, whereas to be of any practical use an oscillator must have stability under all conditions of operation which may include large switching surges. A theory to take account of large disturbances would need to be a non-linear theory, in contrast to the linear theory of this paper. Such a theory would be very complicated and each particular oscillator would have to be treated as a separate problem.

A second disadvantage is that the stability criteria are not very simple: it would often be quicker to find the solution by experiment. However, it must be remembered that the analysis has been carried out in detail with the object of testing as accurately as possible the basic theory. Such refinements would not be justified in practical design calculations for the primary data would seldom be known with any great accuracy.

References.

(1) Appleton, E.V. and van der Pol, B.: "On a type of oscillation hysteresis in a simple triode oscillator," Phil. Mag. 43, p.177, 1922.

(2) Greaves, W.H.M.: "On the stability of the periodic states of the triode oscillator," Proc. Camb. Phil. Soc. 22, p.16, 1923.

(3) van der Pol, B.: "On oscillation hysteresis in a triode generator with two degrees of freedom," Phil. Mag. 43, p.700, 1922.

(4) Minorsky, N.: "Introduction to non-linear mechanics," (J.W.Edwards, Ann Arbor, 1947)

(5) Kryloff, N. and Bogoliuboff, N.: "Introduction to non-linear mechanics" (Princeton University Press, 1947)

(6) Andronow, A.A. and Chaikin, C.E.: "Theory of oscillations" (Princeton University Press, 1949)

(7) Stoker, J.J.: "Non-linear vibrations in mechanical and electrical systems" (Interscience Publishers Inc., New York, 1950)

(8) Lefschetz, S. (Editor): "Contributions to the theory of non-linear oscillations" (Princeton University Press, Vol.1, 1950, Vol.2, 1952)

(9) van Slooten, J.: "The stability of a triode oscillator with grid condenser and leak," Wireless Engr. 16, p.16, 1939.

(10) van Slooten, J.: "Stability and instability in triode oscillators," Philips Tech. Rev. 7, pp.40 and 171, 1942

(11) Edson, W.A.: "Intermittent behaviour in oscillators," Bell Syst. Tech. J. 24, p.1, 1945.

(12) Edson, W.A.: "Vacuum-tube oscillators" (John Wiley & Sons Inc., New York, 1953)

(13) Ibid. p.440.

(14) Ibid. Ch.10.

(15) Ibid. p.230.

(16) Whittaker, E.T. and Watson, G.N.: "Modern analysis" 4th. Ed. p.412. (Cambridge University Press, 1940)

(17) Ibid. p.415.

(18) Aitken, A.C.: "Determinants and matrices," 2nd. Ed. Ch.3, (Oliver & Boyd, Edinburgh, 1942)

(19) Gladwin, A.S.: "Oscillation amplitude in simple valve oscillators," Wireless Engr. 26, pp.159 and 201, 1949.

(20) van der Pol, B.: "Forced oscillations in a circuit with non-linear resistance," Phil. Mag. 3, p.65, 1927.

(21) Hardy, G.H.: "A course of pure mathematics," 9th. Ed. p.340. (Cambridge University Press, 1948)

(22) Bode, W.H.: "Network analysis and feedback amplifier design," p.24. (D. van Nostrand Co. Inc., New York, 1945)

(23) Ibid. p.148

(24) Ibid. p.106

(25) Ibid. p.165

(26) Routh, E.J.: "A treatise on the stability of a given state of motion particularly steady motion," p.28. (Macmillan & Co., London, 1877)

(27) Ibid. p.23.

(28) Hurwitz, A.: "Ueber die Bedingungen unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitz," Math. Ann. 46, p.273, 1895.

(29) Copson, E.T.: "Theory of functions of a complex variable," p.119. (Oxford University Press, 1935)

(30) Nyquist, H.: "Regeneration theory," Bell Syst. Tech. J. 11, p.126, 1932.

(31) Dzung, L.S.: "The stability criterion," in "Automatic and manual control," p.13. (Butterworth's Scientific Publications, London, 1952)

(32) Zepler, E.E.: "Oscillation hysteresis in grid detectors," Wireless Engr. 23, p.222, 1946.

(33) Turnbull, H.W.: "Theory of equations," Ch.9. (Oliver & Boyd, Edinburgh, 1939)

Appendix 1. The steady-state oscillation.

Let the grid-current characteristic have the semi-linear form

$$i_g = e(v_g) = b_g(v_g - V_{cg}) \quad \text{when} \quad v_g > V_{cg}$$
$$= 0 \qquad\qquad v_g \leqslant V_{cg} \qquad (A1.L)$$

b_g and V_{cg} are parameters of the valve.

Let $$v_g = V_{g1}\cos\omega_o t + V_g \quad . . \quad . . \quad (A1.2)$$

It is assumed that current flows during only part of each cycle. The conduction angle 2ϕ is defined by

$$\cos\phi = (V_{cg} - V_g)/V_{g1} \qquad (A1.3)$$

If V_{g1} is constant V_g and ϕ vary with V_{cg}. Let ϕ_o be the value of ϕ when $V_{cg} = 0$, and let $\phi = \phi_o + \phi_1$. If ϕ_1 is so small that $\cos\phi_1 = 1$, $\sin\phi_1 = \phi_1$ it follows from (4.2) and the above equations that

$$\left.\begin{aligned} 1 &= (\tan\phi_o - \phi_o)b_gR_g/\pi \\ \phi &= \phi_o(1 - \pi V_{cg}\cot\phi_o/b_gR_gV_{g1}\phi_o\sin\phi_o) \\ 1 - V_{cg}\sec\phi/V_{g1} &= (\tan\phi - \phi)b_gR_g/\pi \\ V_g &= -\cos\phi_o(V_{g1} - V_{cg}\phi_o/\sin\phi_o) \end{aligned}\right\} (A1.4)$$

The first equation gives ϕ_o in terms of b_gR_g. ϕ and V_g follow from the others. As b_gR_g varies from 10 to 1000 the coefficient of V_{cg}/V_{g1} in the second equation varies from 0.40 to 0.34, and $\phi_o/\sin\phi_o$ in the fourth equation varies from 1.13 to 1. Since V_{cg}/V_{g1} is always small in practice it is sufficiently accurate to take average values at say $b_gR_g = 100$ namely, 0.36 and 1. This approximation gives

$$\left.\begin{aligned} \phi &= \phi_o(1 - 0.36V_{cg}/V_{g1}) \\ V_g &= V_g'(1 - V_{cg}/V_{g1}) \end{aligned}\right\} (A1.5)$$

where $V_g' = -V_{g1}\cos\phi_o$

Using the first equation of (A1.4) $-V_g'/V_{g1}$ can be found as a function of b_gR_g (Fig.2)

The grid input resistance is found from (4.3), (A1.1) and (A1.3) as $r_g = -R_g(V_{g1}/V_g)(\sin\phi - \phi\cos\phi)/(\phi - \sin\phi\cos\phi)$

Using the value of ϕ given by (A1.3) it can be shown that the fractional error produced by writing ϕ_o for ϕ in this equation is of the order $0.07\phi^2V_{cg}/V_{g1}$ which is negligible. Substituting

for V_{g1}/V_g according to (A1.5) gives $r_g = r_g'(1 + V_{cg}/V_{g1})$
where $r_g' = R_g(\sin\phi_o - \phi_o\cos\phi_o)/\cos\phi_o(\phi_o - \sin\phi_o\cos\phi_o)$
Using (A1.4) r_g'/R_g can be calculated as a function of b_gR_g (Fig.2)

When V_g/V_{g1} is known the oscillation amplitude could be found by solving equation (4.13) to obtain Y in terms of K and N. An easier method is to make use of the relation between N and $G_o - G_2$ (5.14) namely $G_o - G_2 = g/N$. When the valve operates in the space-charge-limited condition the anode current (in the absence of grid current) follows the three-halves law

$$f(v) = b_a v^{3/2} \quad v > 0$$
$$= 0 \qquad v \leqslant 0 \qquad \text{(A1.6)}$$

and $v = V_e\cos(\omega_o t + \theta) + V_g - V_{ca}$
In the following Appendix the values of G_o and G_2 corresponding to this state are found in terms of the parameter $H = 1 - K + K/Y$
Using these values N can be expressed as follows.
If $H > 2$

$$1/N = (8/15\pi)(HY/K)^{\frac{1}{2}}((3 + (H - 1)^2)E - (H - 1)(H - 2)F) \qquad \text{(A1.7)}$$

where F and E are the complete elliptic integrals of the first and second kinds. In the usual notation

$$F = F_1(k,\tfrac{1}{2}\pi) \quad E = E_1(k,\tfrac{1}{2}\pi) \quad \text{where } k = (2/H)^{\frac{1}{2}} \qquad \text{(A1.8)}$$

When $H < 2$

$$1/N = (4/15\pi)(2Y/K)^{\frac{1}{2}}((6 + 2(H - 1)^2)E - (H - 2)(H - 4)F) \qquad \text{(A1.9)}$$

$$F = F_1(k',\tfrac{1}{2}\pi) \quad E = E_1(k',\tfrac{1}{2}\pi) \quad \text{where } k' = (H/2)^{\frac{1}{2}} \qquad \text{(A1.10)}$$

N can thus be calculated as a function of K and Y (Fig.3)
When $Y \to 0$, $H \to K/Y$ and it can be shown from (A1.7) that $N \to 1$.
When $Y \to \infty$, $H \to 1 - K$, and since a negative value for H is inadmissible, Y must be finite for all values of N when $K > 1$.
When $K < 1$ it is evident from (A1.9) that as $Y \to \infty$ then $N \to 0$.

For small values of grid voltage and when the effects of grid emission and ionisation can be neglected, the grid current is approximately

$$i_g = e(v_g) = I_o\exp(v_g/V_o) \qquad \text{(A1.11)}$$

where I_o and V_o are constants of the valve. Equation (4.2) then gives $V_g = -I_oR_g\exp(V_g/V_o)B_o(V_{g1}/V_o)$ and (4.3) gives

$r_g = -\frac{1}{2}R_g(V_{g1}/V_g)B_o(V_{g1}/V_o)/B_1(V_{g1}/V_o)$. In these expressions B_o and B_1 are the modified Bessel functions of the first kind (usually denoted by I_o and I_1). If V_{g1} is small these functions can be replaced by the first two terms of their series expansions $B_n(x) = (\frac{1}{2}x)^n(1 + x^2/4(n + 1))/n!$. Then as $V_{g1} \rightarrow 0$

$$V_g = -I_oR_g\exp(V_g/V_o) \qquad r_g = -R_gV_o/V_g \quad . \quad . \quad . \quad . \qquad (A1.12)$$

Since I_o and V_o are known V_g/V_o can be found as a function of I_oR_g/V_o. (Fig.4)

The rates of change of V_g and r_g with V_{g1} are also required. Using the above approximations for the Bessel functions it is easily shown that as $V_{g1} \rightarrow 0$

$$dV_g/dV_{g1} = nV_{g1}/V_g \qquad dr_g/dV_{g1} = mr_gV_{g1}/V_g^2 \qquad (A1.13)$$

where
$$\left.\begin{aligned} n &= (V_g/V_o)^2/2(1 - V_g/V_o) \\ m &= -(V_g/V_o)^2(1 + V_g/V_o)/4(1 - V_g/V_o) \end{aligned}\right\} \qquad (A1.14)$$

Appendix 2. Evaluation of S_n and G_n.

Let the grid current have the semi-linear form of (A1.1)

Then $$e'(v_g) = b_g \quad \text{when } v_g > V_{cg}$$
$$= 0 \qquad v_g \leq V_{cg}$$

From the definition (5.1) of S_n, $S_o = b_g\phi/\pi$, $S_1 = b_g\sin\phi/\pi$, $S_2 = b_g\sin2\phi/2\pi$. Using expressions (A1.5)

$S_oR_g = S_o'R_g(1 - 0.36V_{cg}/V_{g1})$ where $S_o'R_g = b_gR_g\phi_o/\pi$

$S_2/(S_o - S_2) = S_2r_g = S_2'r_g(1 + 0.72V_{cg}/V_{g1})$

where $S_2'r_g = \sin2\phi_o/(2\phi_o - \sin2\phi_o)$

Also $S_2/S_1 = \cos\phi = -(1 - V_{cg}/V_{g1})V_g'/V_{g1} + V_{cg}/V_{g1}$

Using the first equation of (A1.4), ϕ_o and thence $S_2'r_g$ and $S_o'R_g$ can be calculated as functions of b_gR_g (Fig.5)

When f(v) follows the three-halves law of (A1.6)

$$f'(v) = 1\tfrac{1}{2}b_av^{\frac{1}{2}} \quad v > 0$$
$$= 0 \qquad v \leq 0$$

Putting $g = 1\frac{1}{2}b_a(-V_{ca})^{\frac{1}{2}}$ the definition of G_n (5.10) becomes

$G_n = (g/\pi)\int_o^\theta (1 - Y + (Y/K)\cos x)^{\frac{1}{2}}\cos nx\,dx$ where $\theta = \pi$ if $Y < K/(1 + K)$, and $\cos\theta = K(1 - 1/Y)$ if $Y > K/(1 + K)$.

Writing $H = 1 - K + K/Y$, the integral becomes

$$G_n = (2g/\pi)(HY/K)^{\frac{1}{2}}\int_o^{\frac{1}{2}\theta} (1 - (2/H)\sin^2x)^{\frac{1}{2}}\cos 2nx\,dx \qquad \text{(A2.1)}$$

If $H > 2$, $\theta = \pi$, and if $H < 2$ the transformation $\sin y = (H/2)^{\frac{1}{2}}\sin x$ makes the upper limit of integration again $\frac{1}{2}\pi$. In both cases the result can be expressed in terms of the complete elliptic integrals.

$$\left.\begin{aligned} \text{If } H > 2, \quad G_o &= (2g/\pi)(HY/K)^{\frac{1}{2}}E \\ G_1 &= (2g/3\pi)(HY/K)^{\frac{1}{2}}((H - 1)E - (H - 2)F) \\ G_2 &= (2g/15\pi)(HY/K)^{\frac{1}{2}}((3 - 4(H - 1)^2)E + 4(H - 1)(H - 2)F) \end{aligned}\right\} \text{(A2.2)}$$

where F and E are given by (A1.8)

$$\left.\begin{aligned} \text{If } H < 2 \quad G_o &= (g/\pi)(2Y/K)^{\frac{1}{2}}(2E + (H - 2)F) \\ G_1 &= (g/3\pi)(2Y/K)^{\frac{1}{2}}(2(H - 1)E - (H - 2)F) \\ G_2 &= (g/15\pi)(2Y/K)^{\frac{1}{2}}((6 - 8(H - 1)^2)E + (H - 2)(4H - 1)F) \end{aligned}\right\} \text{(A2.3)}$$

where F and E are given by (A1.10)

The following approximations are obtained from power series expansions of F and E. When V_{g1} is very small and V_g does not tend to zero with V_{g1} (e.g. exponential grid-current law), K and H are large, and then

$$\left.\begin{aligned} G_0 = g(1-Y)^{\frac{1}{2}} = -1/R_E \qquad G_1 &= -Y/4K(1-Y)R_E \\ G_2 &= Y^2/32K^2(1-Y)^2R_E \end{aligned}\right\} \quad (A2.4)$$

$$\text{When } H < 2 \qquad \left.\begin{aligned} G_0 &= \tfrac{1}{2}g(Y/2K)^{\frac{1}{2}}H(1 + H/16 + 3H^2/256 \,..) \\ G_1 &= \tfrac{1}{2}g(Y/2K)^{\frac{1}{2}}H(1 - 3H/16 - 5H^2/256 \,..) \\ G_2 &= \tfrac{1}{2}g(Y/2K)^{\frac{1}{2}}H(1 - 15H/16 + 35H^2/256 \,..) \end{aligned}\right\} \quad (A2.5)$$

For a three-halves-law amplifier $f(v) = (2/3)vf'(v)$. From this it can easily be shown that $G_0 - 5G_2 = 4(H - 1)G_1$. Since $G_0 - G_2 = -1/R_E$, $-G_2R_E = G_2/(G_0 - G_2)$, and from the above $-G_1R_E = (1 + 4G_2R_E)/4(H - 1)$. Hence $-G_1R_E$ and $-G_2R_E$ can be calculated as functions of H (Fig.6)

Since $(V_e\cos x + V_g - V_{ca})^{\frac{1}{2}}$ is a positive decreasing function of x in the range $0 < x < \pi$, it follows that for the three-halves-law amplifier $G_1 > 0$. The inequality $G_1^2 - G_0G_2 > 0$ is also of some importance. Since $G_0 > 0$ the inequality is satisfied when $G_2 < 0$. Only values of $H < 1.42$ need therefore be considered since it is only in this range that $G_2 > 0$ (Fig.6) Using (A2.5) it is easily shown that the inequality is satisfied for all sufficiently small values of H, and the graphs of Fig.6 can be used to show that it is also satisfied when $H > 0.2$

Appendix 3. The effect of anode-circuit D.C. resistance on the oscillation amplitude.

If I_a is the mean anode current and R_a the D.C. resistance, the mean anode voltage is less than the H.T. supply voltage by an amount I_aR_a. Let V'_{ca} be the grid voltage for anode current cut-off when the mean anode voltage equals the supply voltage. Then

$$V_{ca} - V'_{ca} = I_aR_a/\mu = (1 - n^2)V_{ca} \qquad (A3.1)$$

where $n^2 = V'_{ca}/V_{ca}$

$$\text{Now } I_a = (1/\pi)\int_0^{\pi} f(V_e \cos x + V_g - V_{ca})dx \quad + kV_g/R_g$$

When f(v) follows the three-halves law the integral can be expressed in terms of G_o and G_1 by writing $f(v) = (2/3)vf'(v)$.

Then $I_a = (2/3)(V_eG_1 + (V_g - V_{ca})G_o) + kV_g/R_g$

Substituting this into (A3.1) gives

$$n^2 - 1 = -(\phi(H)/R_E + kK/R_g)YR_a/\mu K \qquad (A3.2)$$

where $\phi(H) = -(2/3)(G_o(H - 1) + G_1)R_E$

This function is shown graphically in Fig.17.

The oscillation amplitude can be found as follows. V'_{ca} is a known constant. Let $N' = -1\tfrac{1}{2}b_a(-V'_{ca})^{\frac{1}{2}}R_E$. Then $N = N'/n$. Assuming some plausible value for n, a value for Y can be read off from Fig.3. This is used to find H and hence $\phi(H)$. A value for n can then be calculated from (A3.2). This will differ more or less from the assumed value. Using the new value of n, the calculation is repeated to find a third value, and the process is continued until no further improvement is obtained.

Appendix 4. The effect of an impedance in the screen-grid circuit.

In tetrodes and pentodes the grid-anode amplification factor is usually so large that, provided the valve operates above the "knee" of the characteristic, the anode voltage has a negligible influence on anode current. If the screen-grid voltage varies the anode current can be expressed as $i_a = f(v_g + v_{as}/\mu_s - V_{ca}) - ki_g$, where v_{as} is the alternating screen voltage and μ_s is the grid-screen amplification factor. Usually the impedance in the screen circuit is a decoupling network having a negligible impedance at the oscillation frequency, so that in the steady state v_{as} is practically zero. Then $\theta = 0$ and $q = 1$.

The transient current is

$$i_{ad} = (v_{gd} + v_{sd}/\mu_s)f'(V_{g1}\cos\omega_0 t + V_g - V_{ca}) - ki_{gd}$$

and $$i_{ad} + ki_{gd} = (v_{gd} + v_{sd}/\mu_s)\sum_{-\infty}^{\infty} G_n \exp jn\omega_0 t \qquad \text{(A4.1)}$$

where v_{sd} is the transient screen voltage. Since the screen and anode currents are in the ratio $(1 - k)/k$ it follows that $v_{sd} = -Z_s(D)i_{sd} = -Z_s(D)i_{ad}(1 - k)/k$, where $Z_s(D)$ is the screen circuit impedance. Substituting for i_{ad} from (6.1)

$$v_{gd} + v_{sd}/\mu_s = (1 + Z_s(D)/\mu_1 Z_t(D))v_{gd} + (Z_s(D)Z_o(D)/\mu_1 Z_t(D))i_{gd} \qquad \text{(A4.2)}$$

in which $\mu_1 = \mu_s k/(1 - k)$

Now the corresponding quantity which is used in the equations of Sections 5 and 6 is

$$v_{gd} + v_{ad}/\mu = (1 + Z_i(D)/\mu Z_t(D))v_{gd} + Z_n(D)i_{gd}/\mu$$

Comparing this with (A4.2) and comparing (A4.1) with (5.9) it is seen that the equations of Sections 5 and 6 will be valid under the new conditions if μ is replaced by μ_1, $Z_i(D)$ by $Z_s(D)$, and $Z_n(D)$ by $Z_s(D)Z_o(D)/Z_t(D)$. Also Z_s^+ and Z_s^- will be negligible. Only Z_s^o will have a significant value.

Appendix 5.

Relations between stability criteria obtained in Parts 1 and 2

In Part 1 criteria for periodic and aperiodic amplitude instability (with $R_a = 0$) were derived by methods very different from those of Part 2, but the results, though expressed in a different forms are in fact identical.

Writing $Y = V_g/V_{ca}$ in (10.5) gives the criterion for aperiodic instability with large oscillation amplitudes as

$$V_{ca}/V_g < (1 - K)/3K + (1 - K)^2/27K + \ldots$$

which is the same as expression (8.9) of Part 1.

Putting $R_a = R_N = 0$, $R_E = (R_T + R_I/\mu)/(1 - kR_T/r_g)$, and $R_\theta = R_o r_g/(R_o + r_g)$ in (10.8) gives the criterion for the existence of a hysteresis effect at vanishingly small amplitudes as

$$V_g/(V_g - V_{ca}) - (R_o + kr_g/(k - r_g/R_T))(1 + V_g/V_o)/(R_o + r_g)$$

$$> V_o^2(1 + R_I/\mu R_T)^2(1 - V_g/V_o)/4(V_g - V_{ca})^2$$

whichmis the same as expression (8.7) of Part 1.

The criterion (11.8) for periodic stability can be rewritten as follows.

Let $Q' = Qr_g/(R_o + r_g)$, i.e. Q' is the Q-factor of the circuit when the extra damping due to the grid-input resistance is taken into account.

Let $1 + S_oR_g = S$. From Fig. 5 it is seen that over a large range of values of b_gR_g, $S_2r_g \doteqdot \frac{1}{2}S_oR_g$. Hence $S_2r_g = \frac{1}{2}(S - 1)$

Substituting these transformations into (11.8) and using also (4.11) and (5.14) the criterion for periodic instability is

$$2SQ'X_g/R_g < -r_g/(R_o + r_g) - (G_o + G_2)(R_T + R_I/\mu)$$
$$- S(R_o/(R_o + r_g) - kR_T/r_g)$$

This is the same as expression (7.16) of Part 1.

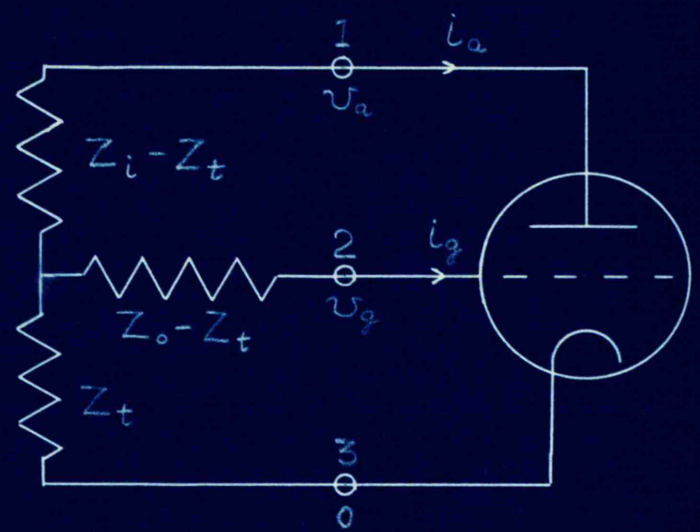

FIG 1 OSCILLATOR CIRCUIT

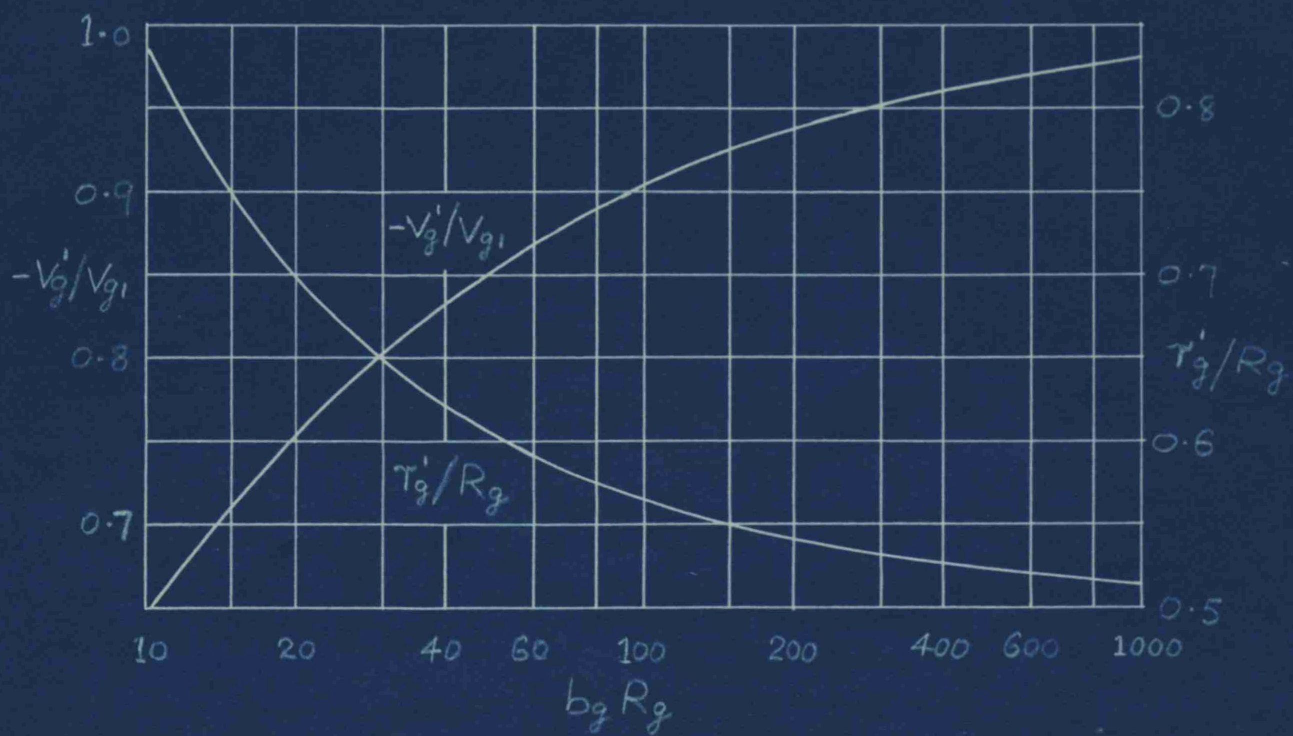

FIG 2 GRID BIAS AND GRID RESISTANCE

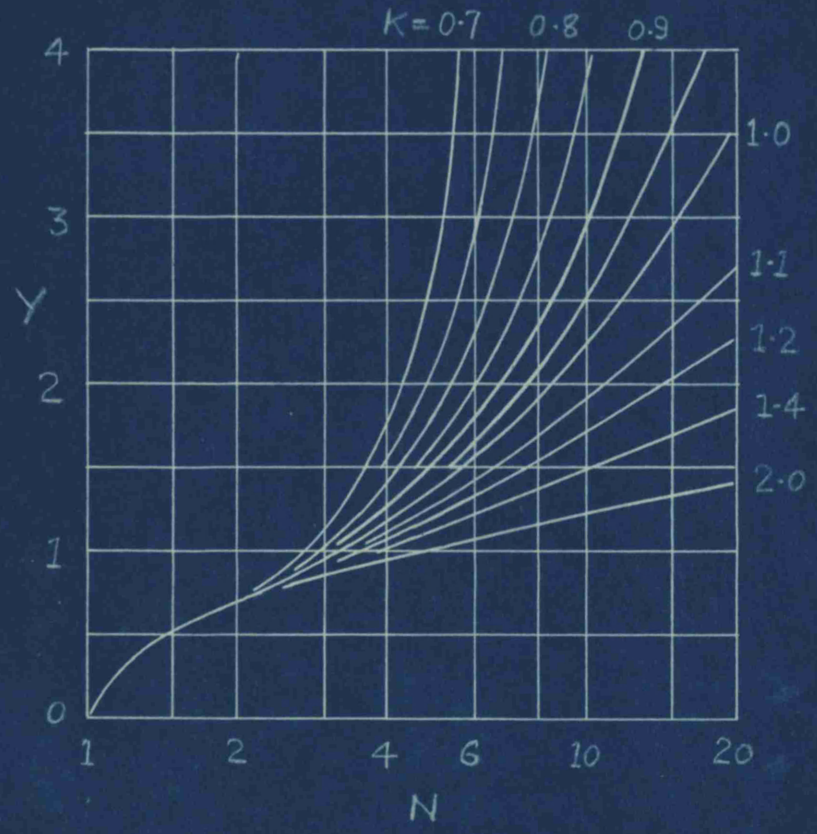

FIG 3 GRAPHS FOR OSCILLATION AMPLITUDE

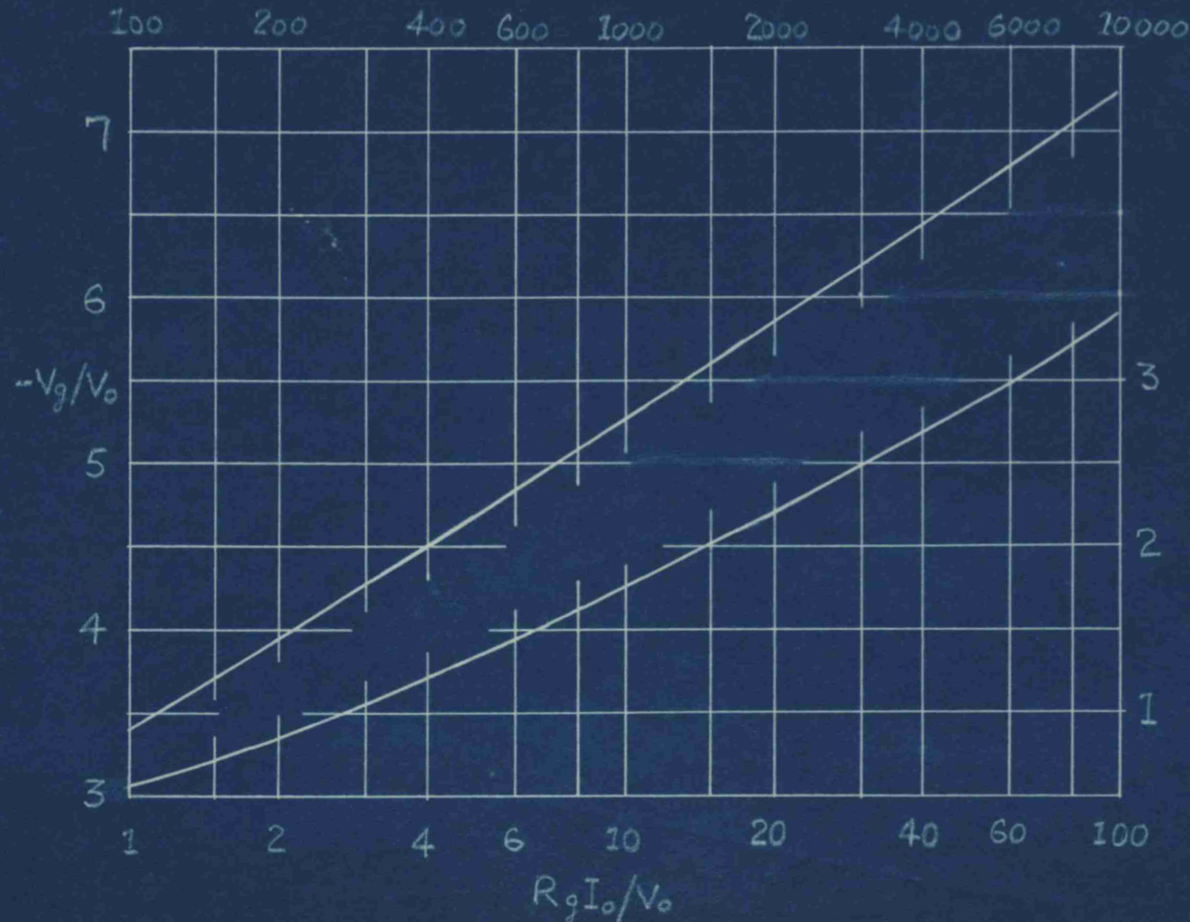

FIG 4 GRID BIAS (EXPONENTIAL CHARACTERISTIC)

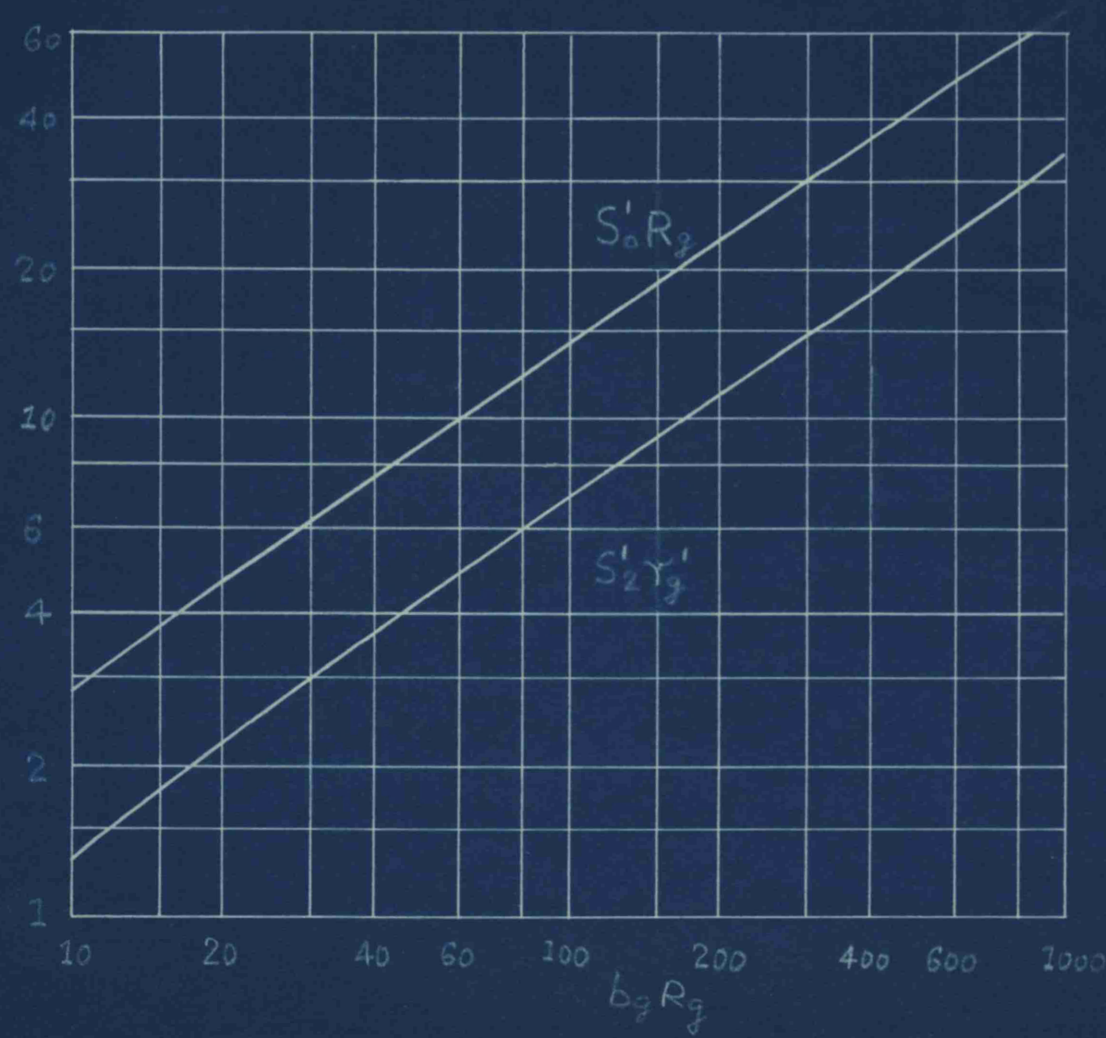

FIG 5 GRID

CONDUCTANCE FUNCTIONS

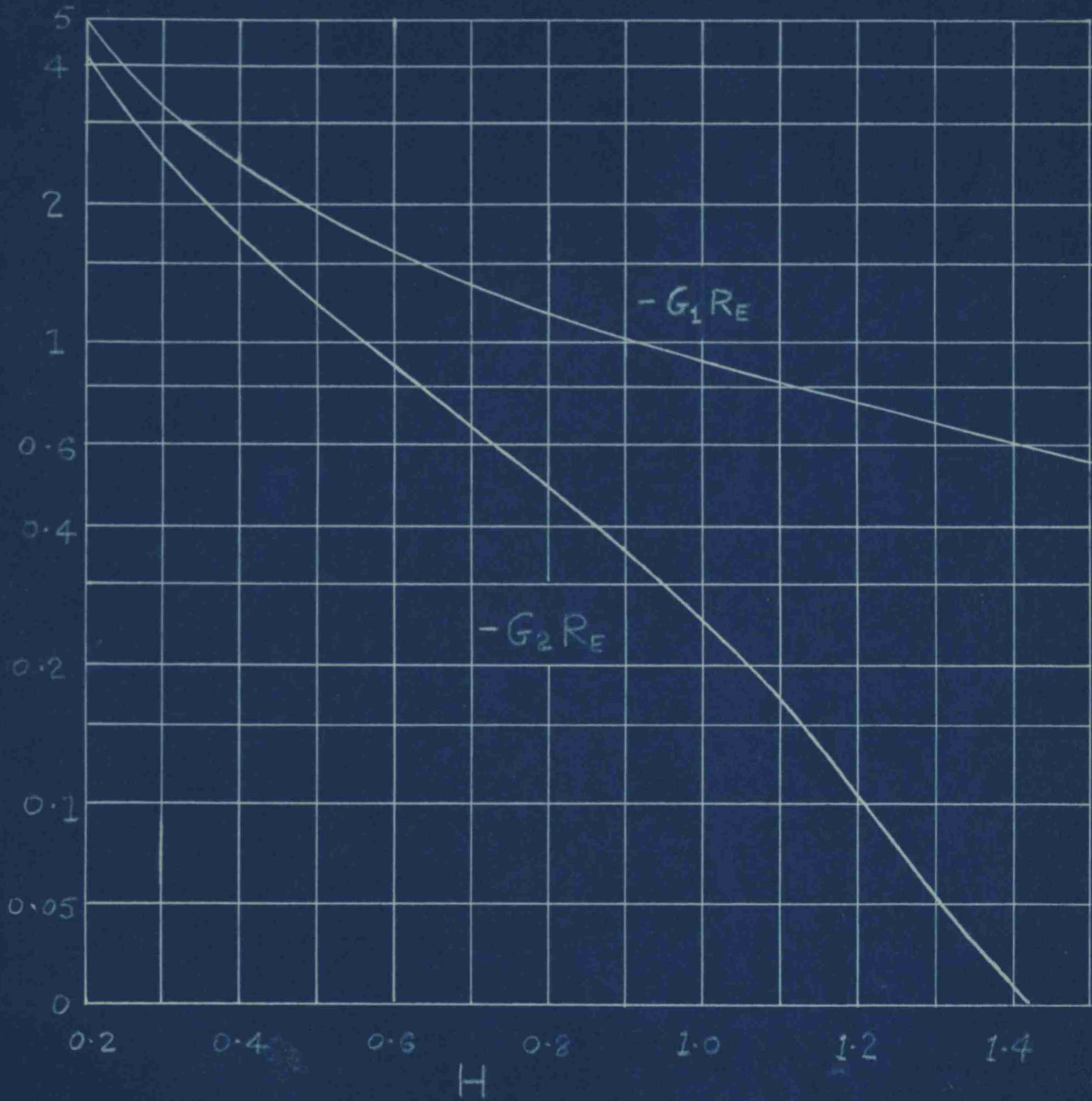

FIG. 6 ANODE CONDUCTANCE FUNCTIONS

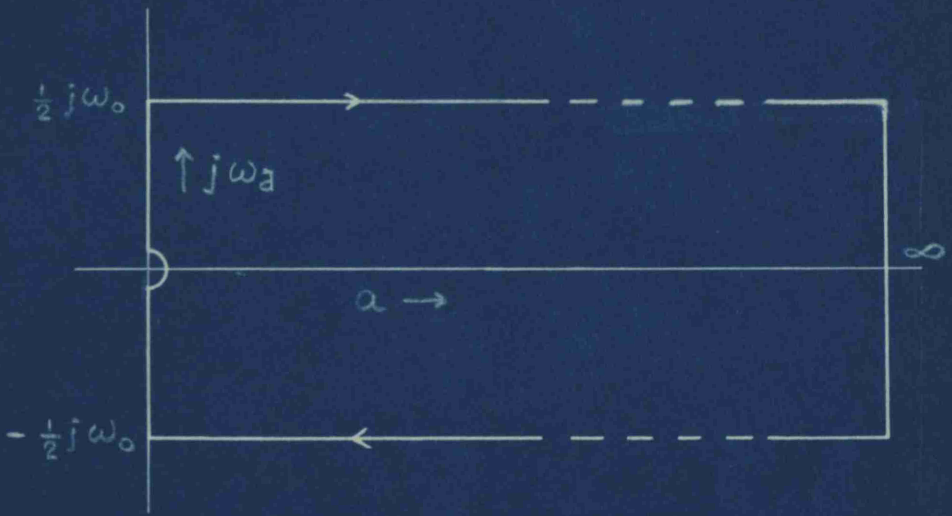

FIG 7 p - PLANE CONTOUR

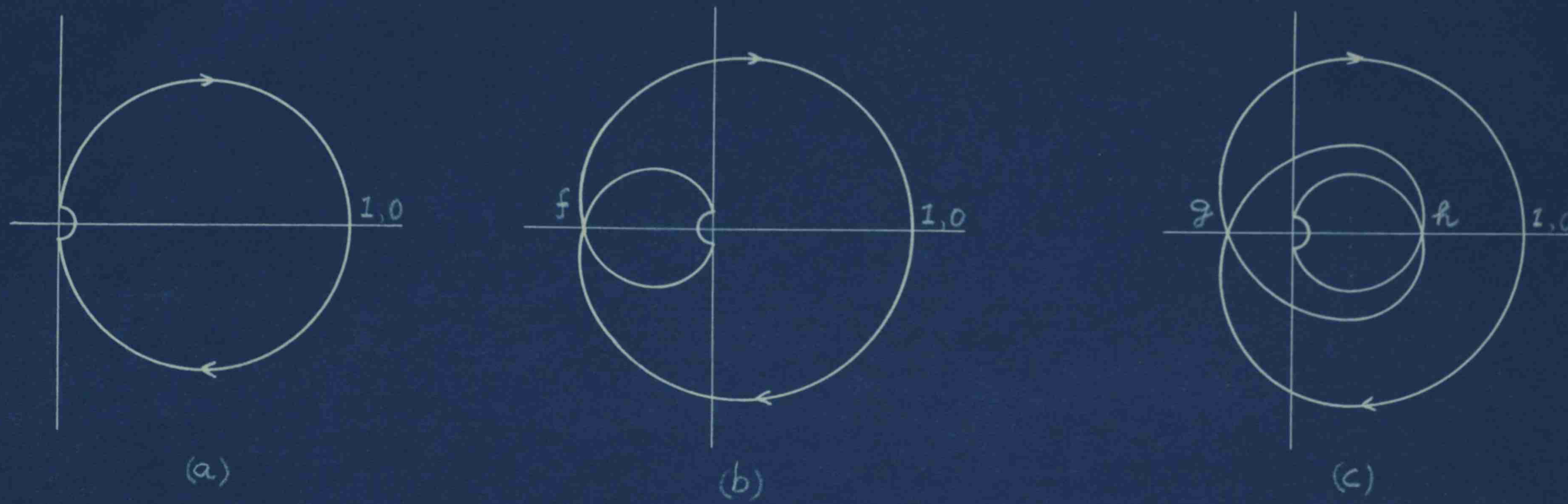

FIG 8 LOCI OF D(p) OR F(p)

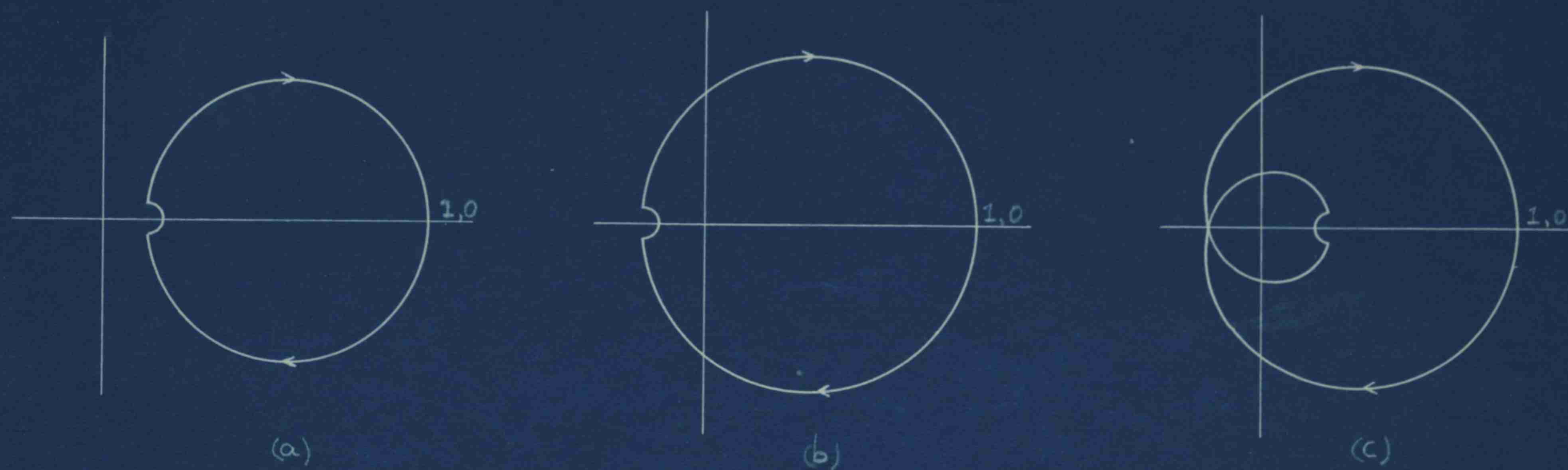

FIG 9 LOCI OF A(p)

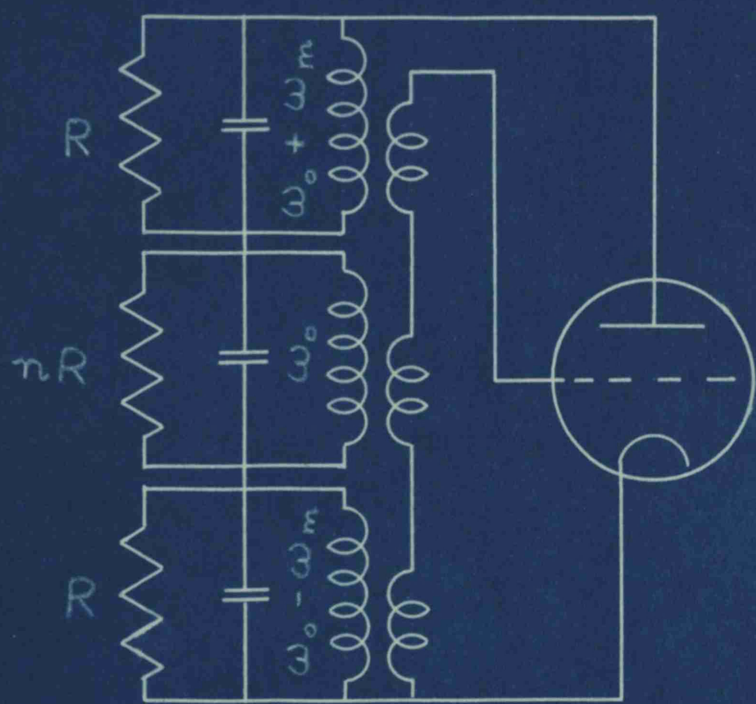

FIG 10 CIRCUIT FOR FREQUENCY INSTABILITY

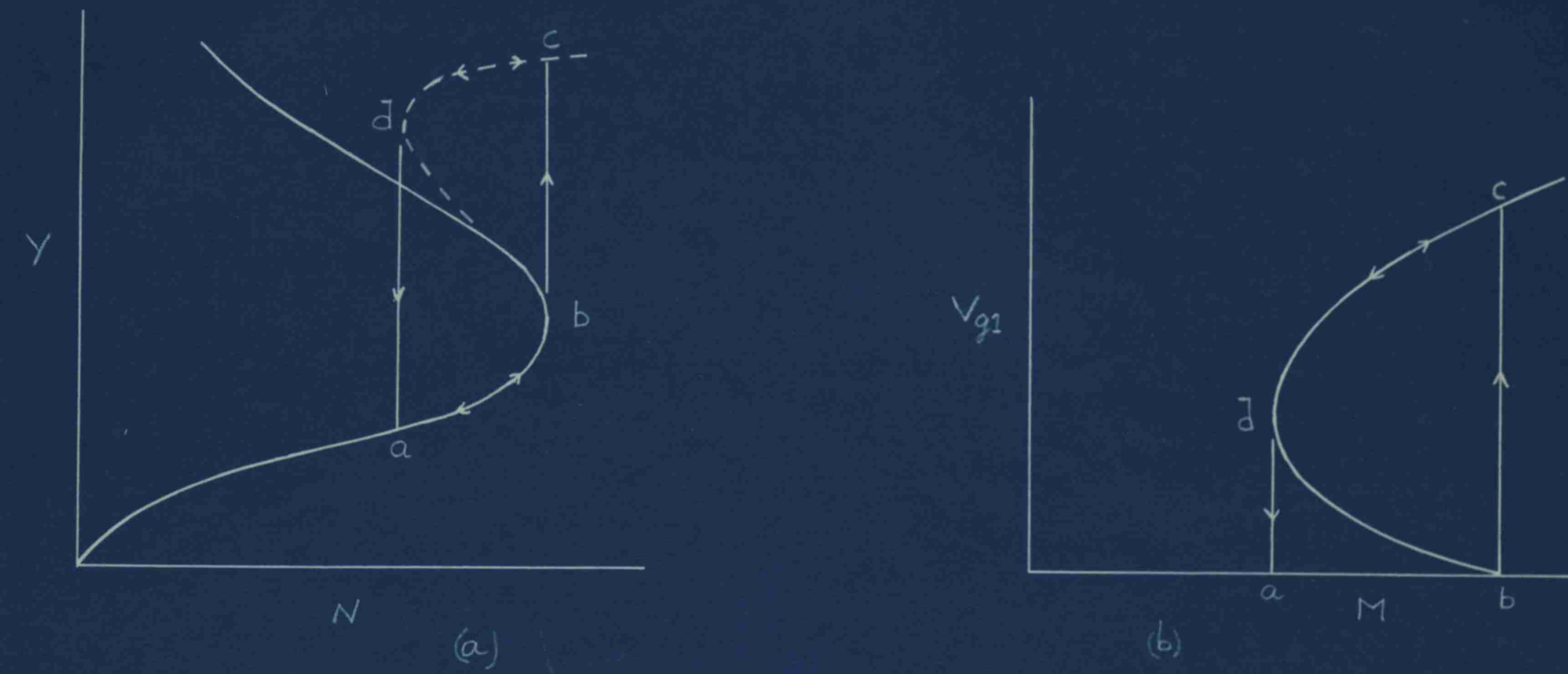

FIG 11 AMPLITUDE HYSTERESIS

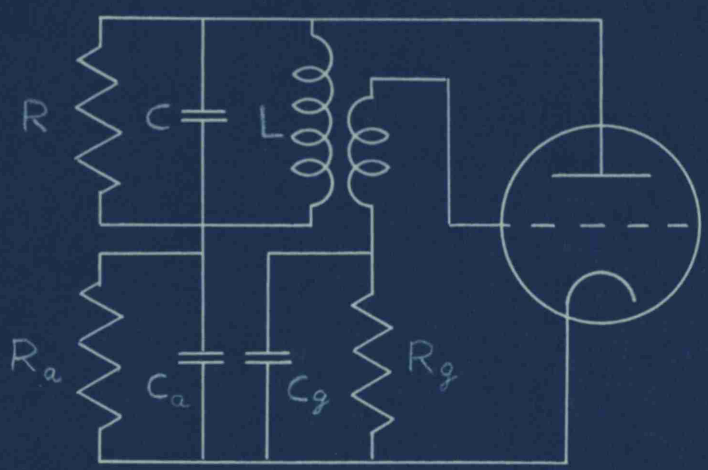

FIG 12 OSCILLATOR WITH
3 TIME CONSTANTS

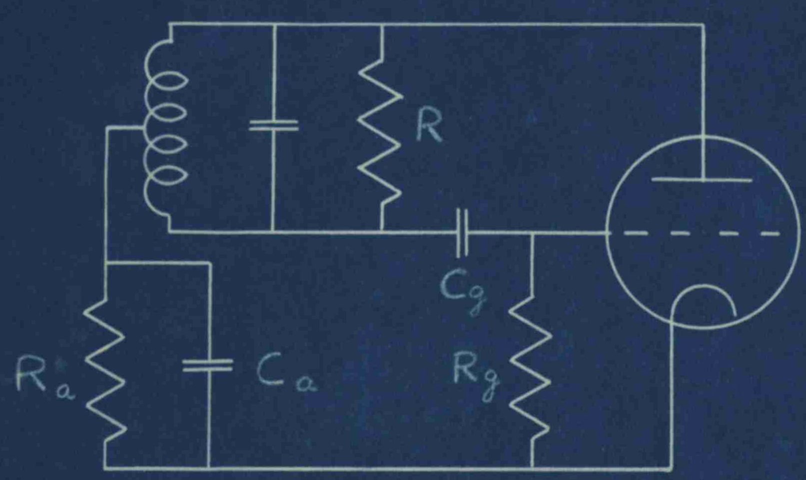

FIG 13 HARTLEY OSCILLATOR

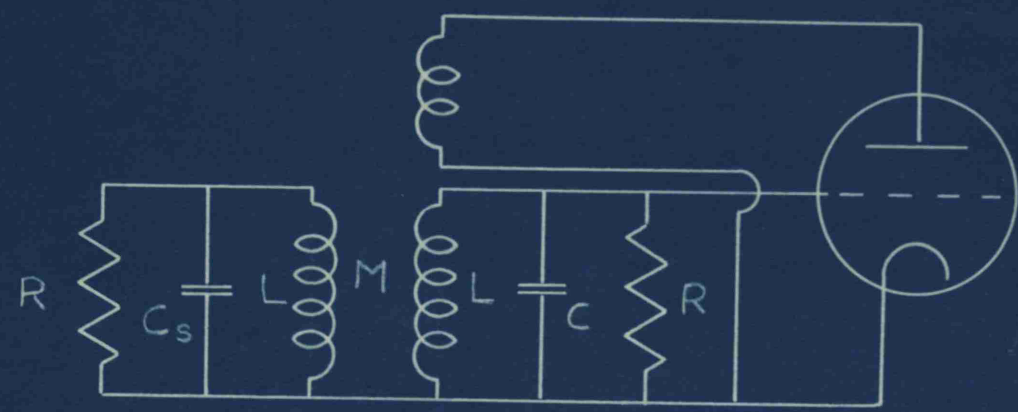

FIG 14 CIRCUIT FOR FREQUENCY HYSTERESIS

FIG 15 FREQUENCY HYSTERESIS

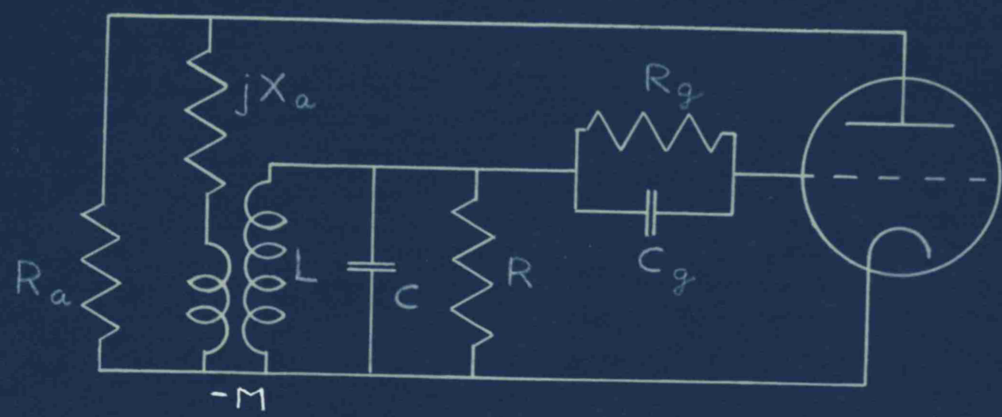

FIG 16 ASYMMETRICAL NETWORK OSCILLATOR

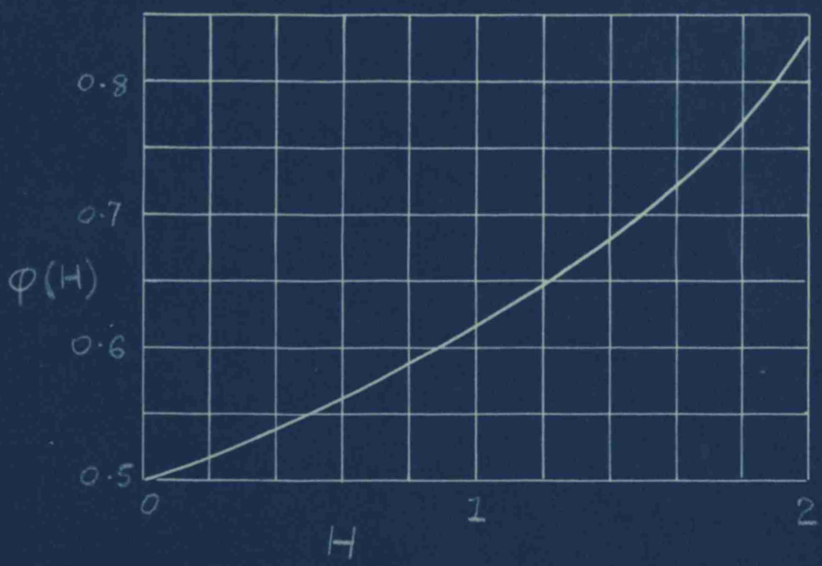

FIG 17 THE FUNCTION φ(H)