1961 - 62

1997

Some problems involving partial differential equations with mixed boundary conditions.

C. C. Bartlett

Summary.

A function ϕ satisfies the equation

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0$$

in a region R bounded by a closed curve C on which mixed boundary conditions are specified, for example $\phi = 0$ on a part A of the boundary and $\frac{\partial\phi}{\partial n} = 0$ on a part B, where C = A + B. It is required to find the values of k for which the equation possesses solutions satifying the mixed boundary conditions.

Two variational principles are given for these eigenvalues, and conditions are obtained under which these two principles would give upper and lower bounds for the lowest eigenvalue. Transcendental equations, obtained for the determination of the lowest eigenvalue, are shown to be functions of an unknown function which is, for example, the value of ϕ on the part B of the boundary, or of $\partial\phi/\partial n$ on part A. If a first order approximation to this function is made, it appears that the resulting approximation to the eigenvalue is of second order.

The general theory, obtained for a simple closed curve, is extended to investigate a curve enclosing a certain type of composite region, and it is shown that

the conditions under which the two variational principles would give upper and lower bounds are similar to those of the simpler problem.

Several problems are worked out in detail and some numerical results are obtained for comparison with results obtained by other authors by other methods. It is shown that in all of the chosen problems for which the two variational principles can be given, the conditions that they give upper and lower bounds are satisfied.

An alternative method of solution of these problems is given, using conformal transformations in a slightly modified form of Schwinger's "equivalent static problem" method.

The paper concludes with a brief note on the application of variational principles to mixed boundary problems in potential theory.

SOME PROBLEMS INVOLVING ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH MIXED BOUNDARY CONDITIONS

being a THESIS presented by

CLIFFORD CHARLES BARTLETT

to the University of Glasgow in

application for the degree of

DOCTOR OF PHILOSOPHY

D E C L A R A T I O N

I hereby declare that the following thesis is a record of original work, that it has been composed by me, and that it has not been accepted for any other degree.

C C Barton

D E C L A R A T I O N

I certify that Mr. C.C.Bartlett has fulfilled the conditions of the Ordnance and Regulations for the presentation of the following thesis.

D C Pack

PERSONAL FOREWORD

The work for this thesis was started in September 1957, and completed in September 1961. The first three years were carried out in the Mathematics Department, Royal College of Science and Technology, Glasgow, under the supervision of D. C. Pack and B. Noble. The fourth year was completed in the Mathematics Department, the University of New Mexico, Albequerque, N.M.

Paragraph 18 has been published in the Proceedings of the Edinburgh Mathematical Society, June 1961, under the title "A Derivation of Certain Variational Principles for Mixed Boundary Value Problems in Potential Theory" by C. C. Bartlett and B. Noble.

The substance of the first six paragraphs has been submitted for publication to Applied Scientific Research under the title "A Variational Method for the Solution of Eigenvalue Problems Involving Mixed Boundary Conditions" by C. C. Bartlett and B. Noble.

The substance of paragraph 14 will shortly be submitted for publication by C. C. Bartlett.

I wish to thank Professor D. C. Pack for providing me with the facilities for carrying out this work and Mr. Ben Noble for introducing me to this subject and for his constant encouragement and guidance.

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Two variational principles are given for these eigenvalues, and conditions are obtained under which these two principles would give upper and lower bounds for the lowest eigenvalue. A number of problems are worked out in detail and it is shown that in most of them the conditions for obtaining upper and lower bounds are satisfied.

An alternative method of solution is given, using conformal transformations in a slightly modified form of Schwinger's 'equivalent static problem' method.

The paper concludes with a brief note on the application of variational principles to mixed boundary problems in potential theory.

§ 1. Introduction.

The type of eigenvalue problem specified in the summary has an extensive literature.

In the Rayleigh-Ritz variational approach the unknown function ϕ is replaced by the subclass of functions ψ having the form $\psi = c_1 \bar{\Phi}_1 + c_2 \bar{\Phi}_2 + \ldots + c_s \bar{\Phi}_s$ where the trial functions $\bar{\Phi}$ satisfy the specified boundary and continuity conditions, but not the differential equation. The method gives only an upper bound for the eigenvalues with no information about the accuracy of the bound. Closer approximations are obtainable, but the Ritz minimizing process becomes very complicated, especially for higher modes. One is usually reduced to the somewhat haphazard approach of applying the method to problems whose results are already known accurately, and comparing the approximate results with the accurate. This is not satisfactory because the examples chosen may happen to be particularly favorable. Modifications which simplify the minimizing process have been suggested, notably by Galerkin (1) but they do not overcome the lack of a lower bound.

The Courant-Trefftz method (2)(3) is effectively the counterpart of the Rayleigh-Ritz method in that trial functions are chosen to satisfy the differential equation but less restrictive boundary and continuity conditions. A lower bound is obtained for the eigenvalues. This bound can be improved by imposing more stringent boundary conditions. Indeed, the method can theoretically be made to yield an upper bound as well, by imposing boundary conditions more stringent than in the original problem. The algebraic process is, however, even more complicated than in the Ritz method.

A method has been given by Weinstein (4) which will give both upper and lower bounds. Unfortunately the expression to be minimized is very complicated. Only very simple trial functions can be used, limiting the closeness of approach of the two bounds. Hahn (5), Goddard (6) and others have solved related problems by reducing them to the solution of an infinite set of simultaneous linear equations, which must then be solved numerically. Hansen (7) has obtained an approximate solution by determining k so that certain assumed field distributions satisfy the specified boundary conditions on certain points of the boundary. Further applications of a similar method can be found, for instance, in (8) pp. 1435, 1859. Related problems occur in connection with the resonance frequencies in microwave cavities and a convenient method for computing these frequencies in practice is given in the chapter by Marcuwitz in (9).

Hansen and Chu (10) and Chu (11) have used an extension of Schwinger's integral-equation variational method. An exposition of Schwinger's method applied to propagation in wave guides is given, for instance, in (12). The method used in this paper is also essentially Schwinger's method modified to apply to eigenvalue problems, but we have approached the problem from a point of view which is somewhat different from either Schwinger's original approach in propagation problems, or the approach in (11) and (10). The argument in (11) is qualitative. The proofs in (10) refer to impedance boundary condition $\partial \phi/\partial n + \sigma \phi = 0$ and it seems to be difficult to apply the argument to cases where $\phi = 0$ on part of the boundary and $\partial \phi/\partial n = 0$ on the remainder. The advantage of the variational method over the methods of Hahn and Hansen mentioned in the last paragraph is that it involves much less labor for a specified degree of accuracy. A considerable number of problems of practical interest can be formulated in terms of mixed boundary-value problems and some of them are considered below in detail. We shall not, however, be concerned with the formulation of the partial differential equation of the problems investigated, being content to state the problem and to quote the appropriate equation and boundary conditions. After a development of the general theory a simple problem is examined in some detail. This problem is of little practical interest, but serves to introduce and demonstrate several aspects which recur in the succeeding problems, all of which are direct or indirect developments of this first example.

§ 2. General Theory.

We use the notation

$$(f,g) = \iint_R f(x,y)\,g(x,y)\,dx\,dy,$$
$$(\nabla f, \nabla g) = \iint_R \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} \right) dx\,dy . \qquad (2)$$

Green's theorem states that

$$(\nabla f, \nabla g) = -(f, \nabla^2 g) + \int_C f \frac{\partial g}{\partial n}\,ds$$

where $\partial/\partial n$ denotes differentiation normal to the boundary in an outward direction. Suppose that ϕ satisfies $\nabla^2 \phi + k^2 \phi = 0$ in a closed region R bounded by a curve $C = A + B$ with $\phi = 0$ on the part A of the boundary and $\partial\phi/\partial n = 0$ on the part B. Let k_1 be the lowest eigenvalue of this problem, with corresponding eigenfunction $\phi_1(x,y)$ normalized so that

$$(\phi_1, \phi_1) = 1 \qquad (3)$$

From Green's theorem

$$(\nabla\phi_1, \nabla\phi_1) = -(\phi_1, \nabla^2\phi_1) + \int_C \phi_1 \frac{\partial\phi_1}{\partial n}\,ds$$
$$= k_1^2(\phi_1, \phi_1) = k_1^2 \qquad (4)$$

We proceed to obtain an upper bound k_+ for the lowest eigenvalue k_1. We take a function $\bar{\phi}(x,y)$ which approximates $\phi_1(x,y)$ and show that if $\bar{\phi}$ is a first order approximation to ϕ_1, then the resulting k_+ is a second order approximation to k_1 and $k_+^2 \geq k_1^2$.

Assume that $\Phi(x,y)$ is normalized, so that

$$(\Phi, \Phi) = 1 \tag{5}$$

and that Φ is continuous in R, possesses continuous first-order derivatives in R and piecewise continuous second-order derivatives. We define an error function $\delta(x,y)$ by the equation $\Phi = \phi_1 + \delta$. From (5),

$$(\phi_1 + \delta, \phi_1 + \delta) = 1,$$ and on expanding and using (5),

$$2(\phi_1, \delta) = -(\delta, \delta). \tag{6}$$

In order to obtain an upper bound for the lowest eigenvalue k_1 suppose that Φ satisfies the following conditions.

(i) $\nabla^2 \Phi + k_+^2 \Phi = 0$ in R for some constant k_+,

(ii) $\Phi = 0$ on part A of the boundary,

(iii) $\int_B \Phi \frac{\partial \Phi}{\partial n} ds = 0$.

It will be seen later that in the application we consider, condition (iii) provides a transcendental equation from which k_+ can be determined.

As in the derivation of (4), using Green's theorem, equation (5), and conditions (i) - (iii), we have

$$(\nabla\Phi, \nabla\Phi) = k_+^2. \tag{7}$$

Also $$(\nabla\Phi, \nabla\Phi) = (\nabla\phi_1, \nabla\phi_1) + 2(\nabla\phi_1, \nabla\delta) + (\nabla\delta, \nabla\delta), \tag{8}$$

$$(\nabla\delta, \nabla\phi_1) = -(\delta, \nabla^2\phi_1) + \int_c \delta \frac{\partial \phi_1}{\partial n} ds$$

$$= k_1^2 (\delta, \phi_1). \tag{9}$$

Using (4), (6), (7), (9) in (8) we find

$$k_+^2 = k_1^2 + \left\{ (\nabla\delta, \nabla\delta) - k_1^2(\delta,\delta) \right\}. \tag{10}$$

This shows that if Φ is a first order approximation to ϕ_1, then k_+ is a second order approximation to k_1. From Rayleigh's principle it is well known that if δ is any function continuous in R, with piecewise continuous first-order derivatives in R, and such that $\delta = 0$ when $\phi = 0$ on C, then

$$\frac{(\nabla\delta, \nabla\delta)}{(\delta,\delta)} \geqslant k_1^2 \tag{11}$$

A brief account of Rayleigh's principle can be found in (22) page 368. A more detailed account appears in (33) Ch III. From the definitions of Φ and ϕ, the conditions of Rayleigh's principle are satisfied by δ, so that on using (11) in (10),

$$k_+^2 \geqslant k_1^2. \tag{12}$$

Another variational principle for the lowest eigenvalue can be obtained by considering a function Ψ which satisfies conditions

(i)′ $\nabla^2 \Psi + k_-^2 \Psi = 0$ in R for some constant k_-,

(ii)′ $\partial\Psi/\partial n = 0$ on part B of the boundary,

(iii)′ $\int_A \Psi \frac{\partial \Psi}{\partial n} ds = 0.$

On repeating the argument leading to (7) but using (i)′ - (iii)′ we find

$$(\nabla\Psi, \nabla\Psi) = k_-^2. \tag{13}$$

As before, we define an error function $\varepsilon(x,y)$ such that $\Psi = \phi_1 + \varepsilon$. The result (9) with ε in place of δ is no longer true since Ψ and therefore ε is no longer, in general, zero on part A of the boundary. Instead of (9) we write

$$(\nabla\varepsilon, \nabla\phi_1) = -(\phi_1, \nabla^2\varepsilon) + \int_C \phi_1 \frac{\partial\varepsilon}{\partial n} ds$$

$$= -(\phi_1, \nabla^2 \Psi - \nabla^2 \phi_1)$$

$$= k_-^2 (\phi_1, \Psi) - k_1^2$$

$$= k_-^2 - k_1^2 + k_-^2 (\phi_1, \varepsilon) \ . \tag{14}$$

On substituting (4), (6), (13), (14) in (18) we have

$$k_-^2 = k_1^2 - \left\{ (\nabla\varepsilon, \nabla\varepsilon) - k_-^2 (\varepsilon, \varepsilon) \right\} \tag{15}$$

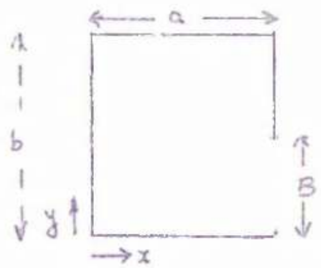
This result should be compared with (10). It shows that if Ψ is a first-order approximation to ϕ_1 then k_- is a second-order approximation to k_1. Unfortunately Rayleigh's principle no longer applies since Ψ and therefore ε is not in general zero on those parts of the boundary where ϕ_1 is zero. Hence the result (11) with ε in place of δ is no longer available and we cannot deduce from (15) that $k_- < k_1$. (Attempts to prove results analogous to (11) for ε, by the method used to prove (11) in (13) p. 164, ran into the non-uniform convergence difficulties mentioned in (13) pp 102-104.) However, the result that $k_- < k_1$ will be proved for some applications in this paper.

§3 A Problem involving a rectangular region.

To provide a simple demonstration of the application of the theory thus far, we consider a rectangular region $0 \leq x \leq a$, $0 \leq y \leq b$ with $\partial\phi/\partial n = 0$ on all sides except $x = a$, $0 \leq y \leq b$, on which

$$\partial\phi/\partial x = 0,\ B < y \leq b;\quad \phi = 0,\ 0 \leq y \leq B \tag{16}$$

Figure I



This will be referred to as the Simple Problem.

By separation of variables a solution of $\phi_{xx} + \phi_{yy} + k^2\phi = 0$ which satisfies the boundary conditions on $x = 0$, $y = 0$, $y = b$ is

$$\phi(x,y) = \sum_{n=0}^{\infty} \epsilon_n' A_n \cosh\gamma_n x \cos n\pi y/b, \tag{17a}$$

where

$$\left.\begin{aligned} \epsilon_n' &= \tfrac{1}{2} \quad \text{when } n = 0 \\ &= 1 \quad \text{when } n > 0 \end{aligned}\right\},$$

$$\gamma_n = \left[(n\pi/b)^2 - k^2\right]^{\frac{1}{2}} = -i\left[k^2 - (n\pi/b)^2\right]^{\frac{1}{2}}. \tag{17b}$$

We assume a time factor $\exp(-i\omega t)$ and this is the reason for the minus sign in the last equation. We can obtain approximations to the eigenvalues in either of two ways which will be distinguished throughout as cases I and II.

Case I. Consider a function Φ defined by

$$\Phi(x,y) = \sum_{n=0}^{\infty} \epsilon_n' B_n \cosh\Gamma_n x \cos n\pi y/b \tag{18}$$

where

$$\Gamma_n = \left[(n\pi/b)^2 - k_+^2\right]^{\frac{1}{2}} = -i\left[k_+^2 - (n\pi/b)^2\right]^{\frac{1}{2}}, \tag{19}$$

so that Φ satisfies $\nabla^2\Phi + k_+^2\Phi = 0$ and hence condition (i) in § 2. We shall assume that

$$\begin{aligned} \Phi &= 0 \quad, \quad x = a \quad, \quad 0 \le y \le B, \\ &= F(y) \quad, \quad x = a \quad, \quad B \le y \le b, \end{aligned} \tag{20}$$

so that Φ satisfies condition (ii) in § 2. The function $F(y)$ is unknown. If we use (18) to give a form for $\Phi(a,y)$, multiply it by $\cos \frac{m\pi y}{b}$ and integrate from 0 to b with respect to y, it follows, using (20), that

$$B_n \cosh \Gamma_n a = \frac{2}{b} \int_B^b F(\eta) \cos \frac{n\pi\eta}{b} \, d\eta. \tag{21}$$

On substituting this result in (18), taking the derivative with respect to x, and setting $x = a$, we have

$$\left(\frac{\partial \Phi}{\partial x}\right)_{x=a} = \frac{2}{b} \sum_{n=0}^{\infty} \varepsilon_n' \Gamma_n \tanh \Gamma_n a \int_a^b F(\eta) \cos \frac{n\pi\eta}{b} \, d\eta \cos \frac{n\pi y}{b} \tag{22}$$

Condition (iii) in § 2 gives

$$\int_B^b \left\{ \Phi \left(\frac{\partial \Phi}{\partial n} \right) \right\}_{x=a} dy = 0.$$

On substituting (20), (22) in this expression we find

$$\sum_{n=0}^{\infty} \varepsilon_n' \Gamma_n \tanh \Gamma_n a \left\{ \int_B^b F(y) \cos \frac{n\pi y}{b} \, dy \right\}^2 = 0. \tag{23}$$

This is a transcendental equation for the determination of k_+. The analysis in § 2 indicates that if we choose as $F(y)$ a first order approximation to the value of ϕ on $x = a$, then the smallest root of this equation gives a second-order approximation to the lowest eigenvalue k_1. Also k_+ is greater than k_1.

Case II. Consider a function Ψ defined by

$$\Psi(x,y) = \sum_{n=0}^{\infty} \varepsilon_n' C_n \cosh \Theta_n x \cos n\pi y/b \tag{24}$$

where $\Theta_n = \left[(n\pi/b)^2 - k_-^2 \right]^{\frac{1}{2}} = -i \left[k_-^2 - (n\pi/b)^2 \right]^{\frac{1}{2}}$, (25) so that $\nabla^2 \Psi + k_-^2 \Psi = 0$ and hence Ψ satisfies condition (i)' of § 2. We shall assume that

$$\begin{aligned} \partial\Psi/\partial x &= G(y) \quad , \quad x = a \ , \ 0 \leq y < \beta , \\ &= 0 \quad , \quad x = a \ , \ \beta < y \leq b , \end{aligned} \tag{26}$$

so that Ψ also satisfies condition (ii)' in § 2. The function $G(y)$ is unknown. The constants C_n in (24) can be expressed in terms of $G(y)$:

$$C_n \Theta_n \sinh \Theta_n a = \frac{2}{b} \int_0^\beta G(\eta) \cos(n\pi\eta/b)\, d\eta . \tag{27}$$

On substituting this result in (24) we find, for $x = a$,

$$\Psi(a, y) = \frac{2}{b} \sum_{n=0}^{\infty} \varepsilon_n' \left\{ \Theta_n \tanh \Theta_n a \right\}^{-1} \int_0^\beta G(\eta) \cos \frac{n\pi}{b}\eta \, d\eta \cos \frac{n\pi y}{b}. \tag{28}$$

Condition (iii)' in § 2 gives

$$\int_0^\beta \left\{ \Psi \frac{\partial \Psi}{\partial x} \right\}_{x=a} dy = 0.$$

Substitution of (26) and (28) in this expression gives

$$\sum_{n=0}^{\infty} \varepsilon_n' \left\{ \Theta_n \tanh \Theta_n a \right\}^{-1} \left\{ \int_0^\beta G(y) \cos \frac{n\pi y}{b} dy \right\}^2 = 0. \tag{29}$$

This is a transcendental equation for the determination of k_-. The analysis in § 2 indicates that if $G(y)$ is a first-order approximation to the value of $\partial\phi/\partial x$ on $x = a$, then the smallest root of (29) gives a second order approximation to the lowest eigenvalue k_1. It is shown in § 5 that k_- is less than k_1.

§ 4. A derivation by means of integral equations.

In this section we derive the transcendental equations (23) and (29) by an integral equation approach. If we apply the boundary conditions (16) to the solution (17) we obtain the following 'dual series':

$$\sum_{n=0}^{\infty}{}' \varepsilon_n' \gamma_n A_n \sinh \gamma_n a \cos n\pi y/b = 0, \qquad B < y \leq b, \tag{30a}$$

$$\sum_{n=0}^{\infty}{}' \varepsilon_n' A_n \cosh \gamma_n a \cos n\pi y/b = 0, \qquad 0 \leq y < B. \tag{30b}$$

Constants A_n satisfying these relations exist only for special values of k, and the problem is to find these values of k and the corresponding values of A_n. Equations (30) can be reduced to integral equations in either of two ways, and a variational expression can be derived from each of the integral equations, corresponding to cases I and II in § 3.

Case I'. Suppose that the left-hand side of (30b) equals the (unknown) function $f(y)$, $B < y \leq b$. The constants A_n can be expressed in terms of $f(y)$ [c.f. (21)] and if the result is substituted in (30a) we have

$$\sum_{n=0}^{\infty}{}' \varepsilon_n' \gamma_n \tanh \gamma_n a \int_B^b f(\eta) \cos \frac{n\pi\eta}{b}\, d\eta \cos \frac{n\pi y}{b} = 0, \qquad B < y \leq b \tag{31}$$

This is an integral equation for $f(y)$ which possesses solutions only for certain k. The difficulty in solving this equation exactly is that we have to find simultaneously k and $f(y)$ so that the equation is satisfied in $B < y \leq b$. If we knew $f(y)$, we could multiply (31) by $f(y)$, integrate over y and obtain

$$\sum_{n=0}^{\infty}{}' \varepsilon_n' \gamma_n \tanh \gamma_n a \left\{ \int_B^b f(y) \cos \frac{n\pi y}{b}\, dy \right\}^2 = 0 \tag{32}$$

This has removed the y-dependence and gives a straight forward transcendental equation for k. Of course we do not know $f(y)$ exactly. But if we guess a reasonable approximation to $f(y)$, say $F(y)$, and use this in place of $f(y)$ in (32), we should expect that the resulting equation would give a reasonable approximation to k. The meaning of "reasonable" will become clear later.

<u>Case II'.</u> Similarly suppose that the left hand side of (30a), i.e. $\partial\phi/\partial n$ at $x = a$, equals the (unknown) function $g(y)$, $0 \leq y \leq B$. The constants A_n can be expressed in terms of $g(y)$ (c.f.(27) and on substituting the result in (30b) we have the integral equation

$$\sum_{n=0}^{\infty} \varepsilon_n' \left\{ \gamma_n \tanh \gamma_n a \right\}^{-1} \int_0^B g(\eta) \cos\frac{n\pi\eta}{b}\, d\eta \cos\frac{n\pi y}{b} = 0, \qquad 0 \leq y < B. \tag{33}$$

This possesses solutions only for certain values of k. As before, we can remove the y-dependence by multiplying by $g(y)$ and integrating over y. If in the resulting equation we replace $g(y)$ by an approximation to $g(y)$, say $G(y)$, we obtain the transcendental equation (29).

§5 Further general theory.

In this section we shall generalize (23) and (29) and consider the stationary nature of the resulting expressions directly. Consider the solution of any second order partial differential equation in curvilinear coordinates (x,y) in the region $0 \le x \le a$, $0 \le y \le b$. Assume that a solution satisfying the boundary conditions on all boundary surfaces except $x = a$ can be found in the form

$$\phi = \sum_{n=0}^{\infty}{}' A_n X_n(x) Y_n(y) \tag{34}$$

where $\int_0^b Y_m(y) \cdot Y_n(y)\, dy = 0, \quad m \neq n \; ; \; = 1, \quad m = n.$

Suppose that mixed boundary conditions hold on $x = a$:

$$\phi = 0, \quad 0 \le y < B, \qquad \partial\phi/\partial x = 0, \quad B < y \le b.$$

Then proceeding exactly as in §3 or §4 it is easily shown that we obtain the following results in which

$$p_n(k) = X_n'(a)/X_n(a) \tag{35}$$

Case I. The integral equation for $f(y)$, the exact value of ϕ on $x = a$, $B < y \le b$, is

$$\sum_{n=0}^{\infty}{}' p_n(k) \int_B^b f(\eta) Y_n(\eta)\, d\eta \; Y_n(y) = 0, \quad B < y \le b. \tag{36}$$

The variational expression generalizing (23) is

$$\sum_{n=0}^{\infty}{}' p_n(k_+) \left\{ \int_B^b F(y)\, Y_n(y)\, dy \right\}^2 = 0 \tag{37}$$

where $F(y)$ is an approximation to the value of ϕ on $x = a$, $B < y \le b$.

Case II. The integral equation for $g(y)$, the exact value of $\partial\phi/\partial n$ on $x = a$, $0 \le y < b$, is

$$\sum_{n=0}^{\infty}{}' \frac{1}{p_n(k)} \int_0^B g(\eta) Y_n(\eta)\, d\eta \; Y_n(y) = 0, \quad 0 \le y < b. \tag{38}$$

The variational expression generalizing (29) is

$$\sum_{n=0}^{\infty} \frac{1}{p_n(k_-)} \left\{ \int_0^B G(y)\, Y_n(y)\, dy \right\}^2 = 0 \qquad (39)$$

where $G(y)$ is an approximation to the value of $\partial\phi/\partial n$ on $x = a$, $0 \leq y < B$.

In order to investigate the stationary property of (37) suppose that

$$F(y) = \alpha f(y) + \delta \xi(y) \qquad (40)$$

where α is an arbitrary constant which is at our disposal since the magnitudes of F and f are not fixed - f satisfies the homogeneous integral equation (36) so that f and F can be multiplied by any constant. (Similarly in § 2 the magnitudes of ϕ_1 and Φ are fixed arbitrarily). Once α has been chosen it is assumed in the usual way that $\xi(y)$ is a function which gives the shape of the error function and δ is a small constant. The variational nature of any expression containing $F(y)$ is investigated by varying δ. It will appear later that it is convenient to fix α by assuming that

$$\int_B^b \xi(y)\, Y_0(y)\, dy = 0 . \qquad (41)$$

From (40) this means that

$$\alpha = -\int_B^b F(y)\, Y_0(y)\, dy \Big/ \int_B^b f(y)\, Y_0(y)\, dy .$$

The constant α does not depend on δ and therefore we can ~~take α = 1 without loss of generality, so that~~ simply write

$$F(y) = f(y) + \delta \xi(y) \qquad (42)$$

where the $f(y)$ in this equation is α times the $f(y)$ in (40)

~~and equation (41) is still true.~~

On multiplying (36) by any function $\chi(y)$ and integrating over y we have

$$\sum_{n=0}^{\infty} p_n(k) \int_B^b f(y)\, Y_n(y)\, dy \int_B^b \chi(y)\, Y_n(y)\, dy = 0 . \qquad (43)$$

Expansion of $p_n(k_+)$ in (37) in a Taylor series gives

$$p_n(k_+) = p_n(k) + (k_+ - k)\, p_n'(k) + O(k_+ - k)^2 . \qquad (44)$$

If we now insert (42), (44) in (37) and use (41) and (43) with $\mathcal{X} = f$ and $\mathcal{X} = \xi$ we can show that

$$k_+ - k = -\frac{\sum_{n=1}^{\infty} p_n(k)\left\{\int_B^b \xi(y)\, Y_n(y)\,dy\right\}^2}{\sum_{n=0}^{\infty} p_n'(k)\left\{\int_B^b f(y)\, Y_n(y)\,dy\right\}^2}\,\delta^2 + O(\delta^3). \tag{45}$$

Hence the difference between k_+ and k is second order. If further it is assumed that

$$\text{a)} \quad p_n(k) > 0 \quad , \quad n \geqslant 1 \,, \tag{46a}$$

$$\text{b)} \quad p_n'(k) < 0 \quad , \quad n \geqslant 0 \,, \tag{46b}$$

then (44) gives the result

$$k_+ \geqslant k.$$

It should be noted that although conditions (46) are sufficient, they are not necessary.

In the example in § 3, on comparing (17) and (34),

$$p_n(k) = \gamma_n \tanh \gamma_n a.$$

Suppose we are investigating the lowest eigenvalue. Then, on physical grounds, it is evident that $0 < k < \pi/2a$. This means that γ_n is real for $n \geq 1$ so that assumption (a) is satisfied. Note that, from (17b), $p_0(k) = -k \tan ka < 0$. This is the reason for arranging that the relation (41) is true so that the sum in the numerator of (45) starts from $n = 1$. We have also

$$p_n'(k) = -ka\left[(\gamma_n a)^{-1} \tanh \gamma_n a + \operatorname{sech}^2 \gamma_n a\right],$$

so that condition (b) is satisfied. Hence the transcendental equation (23) gives an upper limit for the smallest eigenvalue.

In order to investigate the stationary property of (39) we proceed in a similar way. As in the analysis leading to (41), (42) we can arrange that

$$G(y) = g(y) + \varepsilon\,\eta(y)$$

where $\displaystyle\int_0^B \eta(y)\, Y_0(y)\,dy = 0$

Expansion of the term $1/p_n(k_-)$ and use of these results in (39) gives

$$k_- - k = \frac{\sum_{n=1}^{\infty} \frac{1}{p_n(k)} \left\{ \int_0^B g(y) Y_n(y)\,dy \right\}^2}{\sum_{n=0}^{\infty} \frac{p_n'(k)}{[p_n(k)]^2} \left\{ \int_0^B g(y) Y_n(y)\,dy \right\}^2} \varepsilon^2 + O(\varepsilon^3) \qquad (\cancel{x})$$

Hence the difference between k_- and k is second order. If in addition conditions (a) and (b) are satisfied, then

$$k_- \leq k.$$

Since conditions (a) and (b) are satisfied for the problem of § 3 the transcendental equation (29) will give a lower limit for the smallest eigenvalue.

We have thus established that we can obtain upper and lower bounds for the lowest eigenvalue in this particular problem. We are able to do so, also, in several other problems. Further, in the problems computed we show that Cases I and II give $k_- < k_+$ with the differences between k_- and k_+ very small. The computations in every case, however, involve the summation of infinite series by approximation methods and the results may yet be upset by the introduction of these approximations.

§ 6. Numerical results for the Simple Problem.

Choice of $F(y)$, $G(y)$. To obtain numerical approximations for the lowest eigenvalue and the corresponding lowest eigenfunction of the problem of § 3 we must first choose suitable trial functions $F(y)$, $G(y)$. The substitution of any suitable approximation function $F(y)$ in (23) results in a value k_+ which is no less than the lowest eigenvalue k_1. We may further substitute into (23) an eligible function $F(y)$ which depends upon one or more parameters $k_1\ k_2 \ldots k_N$, and minimize with respect to the parameters. The minimum so achieved will give a lowest upper bound k_+ with respect to the N parameters. The larger the number N, the wider is the class of functions so defined and so, in general, the lower is the computed upper bound for λ_1.

Let us write

$$F(\eta) = \sum_{r=0}^{m} c_r f_r(\eta) \tag{47}$$

where c_r is a parameter and $f_r(\eta)$ is an eligible approximation function for $f(\eta)$. Substituting $F(\eta)$ for $f(\eta)$ in (31) and writing $\Gamma_n \tanh \Gamma_n a = \lambda_n$ where Γ_n is the resulting approximation to γ_n, we obtain the equation

$$\sum_{n=0}^{\infty} \varepsilon_n' \lambda_n \sum_{r=0}^{m} c_r I_{rn} \cos n\pi y/b = 0$$

where $\qquad I_{rn} = \int^{b} f_r \cos n\pi\eta/b \, d\eta$.

Multiplying by $\int^{2} f_r \cos n\pi\eta/b \, d\eta$ over y to remove the y-dependence, we obtain the set of $(m + 1)$ simultaneous equations

$$\sum_{n=0}^{\infty} \sum_{r=0}^{m} \varepsilon_n' \lambda_n c_r I_{rn} I_{sn} = 0 \quad , \ s = 0, 1, \ldots m. \tag{48}$$

For the lowest eigenvalue, λ_0 will be the predominant term. We therefore rewrite (48), separating out the λ_0 terms:

$$\tfrac{1}{2}\lambda_0 \sum_{r=0}^{m} c_r I_{r0} I_{s0} = -\sum_{n=1}^{\infty} \sum_{r=0}^{m} \lambda_n c_r I_{rn} I_{sn}$$

$$= -\sum_{r=0}^{m} c_r S_{rs} \quad \text{where } S_{ab} = \sum_{n=1}^{\infty}{}' \lambda_n I_{an} I_{bn} .$$

We have therefore the system of $(m + 1)$ equations with $(m + 1)$ coefficients c_r,

$$\sum_{r=0}^{m} c_r \left[\frac{\lambda_0}{2} I_{r0} I_{s0} + S_{rs} \right] = 0 \qquad s = 1, 2, \ldots m.$$

These will be consistent if the determinant $\left| \frac{\lambda}{2} I_{ro} I_{so} + S_{rs} \right|$ vanishes.

With some manipulation, this determinant may be written

$$\begin{vmatrix} \frac{1}{2}\lambda_0 I_{00}^2 + S_{00} & \frac{1}{2}\lambda_0 I_{10} I_{00} + S_{10} & \frac{1}{2}\lambda_0 I_{20} I_{00} + S_{20} & \cdots \\ -S_{00} I_{10} + S_{01} I_{00} & -S_{10} I_{10} + S_{11} I_{00} & \cdots & \cdot\cdot \\ -S_{00} I_{20} + S_{02} I_{00} & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdots & \cdots & & \cdot \\ \vdots & & & \end{vmatrix} = 0. \tag{49}$$

which is linear in λ_0 and can, at least theoretically, be solved for k_+ for any m in (48). The computational labor for $m > 0$ becomes great very rapidly as m increases. We shall indicate numerically that if f_0 is suitably chosen the numerical results of taking $f_r = 0$, $r \geq 1$ are sufficiently accurate for most practical purposes and that taking f_0, f_1, $f_2 \neq 0$, $f_r = 0$, $r \geq 3$, while increasing the labor enormously, does not improve the results appreciably.

If $f_r = 0 \quad r \geq 1$, we require to solve

$$\tfrac{1}{2}\lambda_0 I_{00}^2 + S_{00} = 0 .$$

that is, $\quad \frac{1}{2}\lambda_0 I_{00}^2 + \sum_{n=1}^{\infty} \lambda_n I_{0n}^2 = 0.$

which is idential with (23).

Similarly if we write $G(\eta) = \sum_{r=0}^{n} d_r g_r(\eta)$ (50)

where d_r is a parameter and $g_r(\eta)$ an eligible approximation function for $g(\eta)$, we arrive at a determinantal equation similar to (49) which reduces to (29) when $n = 0$.

Choice of functions $f_0(y)$ and $g_0(y)$ can be made by considering the edge behavior of ϕ and its derivatives near the point $x = a$, $y = B$. A summary of the literature on edge conditions will be found in (14) pp. 75-76.

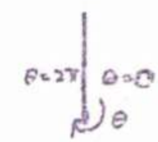
In the immediate neighborhood of a sharp edge or corner the equation $\nabla^2\phi + k^2\phi = 0$ is well approximated by Laplace's equation $\nabla^2\phi = 0$. In terms of polar coordinates with the pole at the sharp edge or corner, this has solution

$$\phi = (Ar^p + Br^{-p})(C\cos p\theta + D\sin p\theta) \quad ,$$

where p is any number and A, B, C, D, constant multipliers. For ϕ to be finite at $r = 0$ we must have $B = 0$. We consider two cases:

(i) At a sharp edge the boundary conditions are

$$\partial\phi/\partial\theta = 0 \ , \quad \theta = 0 \quad \text{and} \quad \theta = 2\pi \ . \quad ,$$

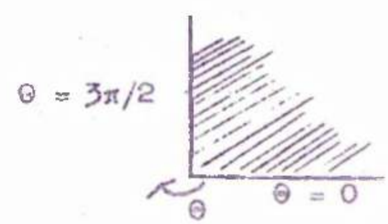


This is satisfied when $\theta = 0$ if $D = 0$. When $\theta = 2\pi$ we require $-pAr^p \sin 2\pi p = 0$ of which the first non-trivial value of p is 1/2.

Hence $$\phi = Ar^{\frac{1}{2}}\cos\tfrac{1}{2}\theta \quad \text{and} \quad \partial\phi/\partial r = \tfrac{1}{2}Ar^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta \qquad (51a)$$

That is, the field is proportional to $r^{-1/2}$ and the potential is proportional to $r^{1/2}$, when r is small, r being the linear distance from the edge. Thus, if ~~whether~~ we are considering an aperture, whether in a straight line, a circle or any other curve, the field and potential near the sharp edge of the aperture are approximately inversely and directly proportional respectively to the square root of the linear distance from the edge.

(ii) At a right angled corner the boundary conditions are



$$\frac{\partial\phi}{\partial\theta} = 0 \ , \quad \theta = 0 \text{ and } \theta = 3\pi/2.$$

With those conditions, D is again zero and $\sin\frac{3\pi p}{2} = 0$.

The lowest non-trivial value of p is therefore 2/3, and for this value,

$$\phi = Ar^{2/3}\cos\tfrac{2}{3}\theta \quad \text{and} \quad \partial\phi/\partial r = \tfrac{2}{3}Ar^{-\frac{1}{3}}\cos\tfrac{2}{3}\theta \qquad (51b)$$

A problem involving such a corner is considered in § 10.

It follows that in the simple problem the mixed boundary conditions (16) imply that on $x = a$,

$$\phi \sim c_1 (y-B)^{\frac{1}{2}} \quad \text{as } y \to B_{+0} \quad , \tag{52a}$$

$$\frac{\partial \phi}{\partial x} \sim c_2 (B-y)^{-\frac{1}{2}} \quad \text{as } y \to B_{-0} \tag{52b}$$

Also on $x = a$, $\partial\phi/\partial x$ is finite at $y = 0$ and ϕ is finite at $y = b$. Hence suitable trial functions will be

$$F(y) = \left[(b-B)^2 - (b-y)^2 \right]^{\frac{1}{2}}, \tag{53a}$$

$$G(y) = (B^2 - y^2)^{-\frac{1}{2}}. \tag{53b}$$

These have been chosen firstly because they satisfy the edge conditions (52) and secondly because the resulting integrals can be evaluated simply - an important consideration.

<u>Limiting cases</u>: We shall compute eigenvalues for the range of values of the ratio $B/b = 0(\frac{1}{8})1$. The limiting cases $B/b = 0$, $B/b = 1$ will have the respective boundary conditions

a) $\partial\phi/\partial x = 0 \quad , \quad x = 0, a$
$\quad\; \partial\phi/\partial y = 0 \quad , \quad y = 0, b$

b) $\dfrac{\partial\phi}{\partial x} = 0 \quad , \quad x = 0$

$\quad\; \dfrac{\partial\phi}{\partial y} = 0 \quad , \quad y = 0, b$

$\quad\; \phi = 0 \quad , \quad x = a .$

By separation of variables, solutions of $\phi_{xx} + \phi_{yy} + k^2\phi = 0$ satisfying these conditions give the eigenvalues

a) $k_{rs} = \left[\left(\frac{r\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2 \right]^{\frac{1}{2}},$

b) $k_{rs} = \left[(r+\frac{1}{2})^2\left(\frac{\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2 \right]^{\frac{1}{2}}$

respectively, where r, s are integers. The lowest eigenvalues for the two limiting cases are therefore $k_{00} = 0$ and $k_{00} = \pi/2a$.

Before using trial functions (53) we can make a very rough approximation by taking the simple trial function

$$G(y) = 1. \tag{54}$$

Then
$$\int_0^B G(y)\cos n\pi y/b \; dy = (b/n\pi)\sin(n\pi B/b) \quad , \; n \geq 1$$
$$= B \quad , \; n = 0$$

If we make the further approximations

$$\coth \Theta_n a \sim 1 , \quad \Theta_n b \sim n\pi , \quad n \geq 1 ,$$

equation (29) reduces to

$$\frac{\cot(k_- a)}{k_- b} = \frac{2}{B^2}\left(\frac{b}{\pi}\right)^3 \frac{1}{b} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin^2(n\pi B/b). \tag{55}$$

To sum the infinite series, let

$$F(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin^2(n\alpha).$$

Differentiating twice with respect to α,

$F''(\alpha) = 2\sum_1^{\infty} \frac{1}{n}\cos 2n\alpha \sim -2\log 2\alpha$, quoting a well known result.

Therefore integrating twice successively,

$$F(\alpha) \sim -\alpha^2\left[\log 2\alpha - 3/4\right]$$
$$= -\alpha^2 \log\left[\frac{2\alpha}{e^{3/4}}\right] \tag{56}$$

Using (56) in (55) we obtain a first approximation formula

$$\frac{\cot k_- a}{k_- b} = -\frac{2}{\pi}\log\frac{2.23 b}{\pi B} . \tag{57}$$

Solutions for $k_- a$ over a range of values of the ratio B/b are shown in Table I(a). As must be expected for such a crude approximation, the method fails when B/b is large.

Returning to trial functions (53), the resulting integrals can be evaluated explicitly by Sonine's first integral [(23) p 46 equation (5)] :

$$\int_B^b \left[(b-B)^2 - (b-y)^2\right]^{\frac{1}{2}} \cos\frac{n\pi y}{b}\, dy = \frac{b(b-B)}{2n}(-1)^n J_1\left(\frac{n\pi[b-B]}{b}\right) , \quad n \geq 1 , \tag{58a}$$
$$= \tfrac{1}{4}\pi (b-B)^2 , \qquad n = 0 ,$$

$$\int_0^B (B^2-y^2)^{-\frac{1}{2}} \cos(n\pi y/b)\, dy = \tfrac{1}{2}\pi J_0\left(\tfrac{n\pi B}{b}\right), \quad n \geq 0. \tag{58b}$$

The approximations k_+, k_- to the smallest eigenvalue k are the smallest roots of the following transcendental equations, obtained by using (58a,b) in (23),(29):

$$\tfrac{1}{2}k_+ b \tan k_+ a = \left[\frac{2b}{\pi(b-B)}\right]^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \Gamma_n b \tanh(\Gamma_n a)\, J_1^2\left(\frac{n\pi(b-B)}{b}\right), \tag{59a}$$

$$\tfrac{1}{2}(k_- b \tan k_- b)^{-1} = \sum_{n=1}^{\infty} (\Theta_n b \tanh \Theta_n a)^{-1} J_0^2(n\pi B/b). \tag{59b}$$

To solve equation (59b) for k_- we rewrite it in the form

$$\tfrac{1}{2}(k_- b \tan k_- b)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n\pi} J_0^2(n\pi B/b) + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Theta_n b \tanh \Theta_n a} - \frac{1}{n\pi} \right\} J_0^2(n\pi B/b). \tag{60}$$

The first series on the right of (60) can be summed by writing

$$\sum_{n=1}^{\infty} \frac{1}{n} J_0^2(nz) = \sum_{n=1}^{N-1} \frac{1}{n} J_\nu^2(nz) + \sum_{n=N}^{\infty} \frac{1}{n} J_\nu^2(nz) = S_1 + S_2 \tag{61}$$

where N is a suitable integer. S_1 is evaluated numerically and N is chosen large enough so that the Bessel functions in the second summation can be replaced by their asymptotic expansion:

$$S_2 \sim \frac{2}{\pi z} \sum_{n=N}^{\infty} \frac{1}{n^2} \cos^2\left(nz - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right).$$

This is evaluated by using the approximation

$$\sum_{n=N}^{\infty} f(n) \sim \int_{N-\frac{1}{2}}^{\infty} f(x)\, dx. \tag{62}$$

More accurate formulae can be used for evaluating this sum. See for example, (15) p. 156 and appendix I at the end of this paper. However, taking N = 16, (62) is sufficiently accurate for the present analysis. The accuracy of the final sum (61) can be checked by repeating the calculation with, say, N = 12.

We can now solve (60) in the following way. First we make the approximations

$$\left.\begin{aligned} \Theta_n b &\sim n\pi \\ \tanh \Theta_n b &\sim 1 \end{aligned}\right\} \quad n \gg 1. \tag{63}$$

Then the second sum of (60) is zero; and the resulting transcendental equation is easy to solve. We obtain a first approximation to k_-, shown in table I(b). We can then use this first approximation for k_- to form a second estimate of $\Theta_n b$ and $\tanh \Theta_n a$ to replace (63) for the first N terms. Hence we can estimate the sum of the first N terms of the second series of (60), a series which is quickly convergent, and solve to find a second estimate of k_-. This procedure is repeated until no change is found in the value of k_-. Equation (59a) can be solved for k_+ in exactly the same way. Final results are shown in Table I(c) and Fig. IV, which shows that the upper and lower limits are extremely close. The accuracy depends on how closely F(y) and G(y) approximate the true values of f(y) and g(y). Physically we should therefore expect k_+a to be more accurate when B is nearly equal to b, and k_-a when B is small. Even ~~the~~ using approximations (63) ~~(b) is~~ the results shown in Table I(b) are remarkably accurate though, of course, this no longer gives upper and lower limits.

Finally we can choose the parameter laden trial functions

$$F(y) = \sum_{r=0}^{m} c_r \left[(b-B)^2 - (b-y)^2 \right]^{r+\frac{1}{2}},$$

$$G(y) = \sum_{r=0}^{\infty} d_r \left[B^2 - y^2 \right]^{r-\frac{1}{2}}$$

corresponding to (47), (50). The resulting integrals, taking forms similar to (58), are easily evaluated. Taking m = 2, we require to solve a third order determinant (49). The labor involved is enormously increased and in view of the closeness of the upper and lower bounds in Table I(c), the results cannot be greatly improved. In the particular case when B/b = 1/8, k_-a increases from 0.50699 to 0.50701 .

An alternative solution of this problem, using a conformal mapping technique is outlined in §16. Values of k_- computed by this method are shown in Table I(d).

Values of the potential function $\phi(x,y)$:

It is readily shown from preceding formulae that approximate values for the potential $\phi(x,y)$ are given by:

Case I:
$$\frac{1}{4}\pi\left(\frac{b-\beta}{2b}\right)^2\frac{\cos k_+x}{\cos k_+a}+\cancel{(b-\beta)}\,2\sum_{n=1}^{\infty}\frac{(-)^n}{n}\frac{\cosh\Gamma_n x}{\cosh\Gamma_n a}J_1\left(\frac{n\pi[b-\beta]}{b}\right)\cos\frac{n\pi y}{b}, \qquad (64)$$

Case II:
$$-\frac{\pi}{2k_-b}\frac{\cos k_-x}{\sin k_-a}+\pi\sum_{n=1}^{\infty}\frac{1}{\Theta_n b}\frac{\cosh\Theta_n x}{\sinh\Theta_n a}J_0(n\pi\beta/b)\cos\frac{n\pi y}{b}. \qquad (65)$$

We compute a pattern of values of ϕ for both cases I and II at the points

$$x = 0(\tfrac{a}{4})a, \qquad y = 0(\tfrac{b}{2})b,$$

for the particular case $a/b = 7/8$. At the computed points where $x < a$ the ratios $\cosh\Gamma_n x/\cosh\Gamma_n a$ and $\cosh\Theta_n x/\sinh\Theta_n a$ both tend rapidly to zero as n increases, so that (64) and (65) are easily summed to a high degree of accuracy. When $x = a$ these two ratios tend rapidly to unity and we therefore require to evaluate the infinite sums

$$\sum_{N+1}^{\infty}\frac{(-)^n}{n}J_1\left(\frac{n\pi[b-\beta]}{b}\right)\cos\frac{n\pi y}{b}, \qquad (66a)$$

$$\sum_{N+1}^{\infty}\frac{\pi}{\Theta_n b}J_0\left(\frac{n\pi\beta}{b}\right)\cos\frac{n\pi y}{b}. \qquad (66b)$$

Taking N large enough, we replace $J_0(x)$, $J_1(x)$ by their asymptotic values

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}\right),$$

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{3\pi}{4}\right),$$

and use approximation (63), viz $\theta_n b \sim n\pi$. The Series (66) are then approximately

$$\frac{1}{\pi}\left(\frac{2b}{b-B}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \cos\left(\frac{n\pi B}{b}-\frac{\pi}{4}\right)\cos\frac{n\pi y}{b}, \tag{67a}$$

$$-\frac{1}{\pi}\left(\frac{2b}{b-B}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \cos\left(\frac{n\pi B}{b}-\frac{\pi}{4}\right)\cos\frac{n\pi y}{b}. \tag{67b}$$

They therefore differ only by a constant multiplier and are obviously convergent, although rather slowly so, when y is small compared with b. They may be summed using equation (62).

Figure II shows that the potentials obtained from these formulae agree satisfactorily for $B = \frac{1}{2}b$. For $B > \frac{1}{2}b$ we should expect (64) to give more accurate results and for $B < \frac{1}{2}b$, (65) should be more accurate. The agreement for $B = \frac{1}{2}b$ indicates that satisfactory results can be obtained over the whole range of B using the appropriate equation (64) or (65). Figure III shows that the two sets are in satisfactory agreement for $B = \frac{1}{8}b$. The value of the potential should be zero at $x = a$, over the range $0 \leq y \leq B$. With the approximations $G(y)$ and $F(y)$ this will not be so, but it should average zero approximately. Computing a set of values for ϕ in case I over this range shows that it is so.

In order to illustrate the stationary nature of the estimates k_+ and k_- using approximations (53) we have investigated the effect of varying ν in the approximations

$$F(y) = \left[(b-B)^2-(b-y)^2\right]^{\nu}, \qquad \nu > 0, \tag{68a}$$

$$G(y) = (B^2-y^2)^{-\nu}, \qquad -\nu > 0. \tag{68b}$$

The integrals (58) are replaced by

$$\int_B^b \left[(b-B)^2-(b-y)^2\right]^{\nu} \cos n\pi y/b \, dy = \frac{\sqrt{\pi}}{2}\,\Gamma(\nu+1)\left(\frac{2b}{n\pi[b-B]}\right)^{\frac{2\nu+1}{2}}(b-B)^{2\nu+1} J_{\frac{2\nu+1}{2}}\left(\frac{n\pi[b-B]}{b}\right), \quad n \geq 1,$$

$$= \frac{\sqrt{\pi}}{2}\,\Gamma(\nu+1)\,(b-B)^{2\nu+1}\Big/\Gamma\left(\nu+\tfrac{3}{2}\right), \qquad n = 0.$$

$$\int_0^B \left(B^2-y^2\right)^{-\nu} \cos n\pi y/b \, dy = B^{1-2\nu}\frac{\sqrt{\pi}}{2}\,\Gamma(1-\nu)\left(\frac{2b}{n\pi B}\right)^{\frac{1-2\nu}{2}} J_{\frac{1-2\nu}{2}}\left(\frac{n\pi B}{b}\right), \quad n \geq 1$$

$$= B^{1-2\nu}\frac{\sqrt{\pi}}{2}\,\Gamma(1-\nu)/\Gamma(3/2-\nu), \quad n = 0.$$

The infinite series corresponding to those in equations (59a), (59b) are of the forms $A\sum_{n=1}^{\infty}\frac{1}{n^{2\nu+1}}\cos^2\alpha$ and $B\sum_{n=1}^{\infty}\frac{1}{n^{3-2\nu}}\cos^2\beta$, the first of which converges if $\nu > 0$, and the second if $\nu < 1$. Near these limits the convergence is slow. The results are given in Table II, Figure V. It is seen that on varying ν the resulting values of k_+ pass through a minimum and the values of k_- pass through a maximum. The results confirm that the value $\nu = \frac{1}{2}$ which has been used in (58) is very close to the optimum value of ν.

Table I. Solutions of the Simple problem.

Values of $\frac{2}{\pi}k_+a$ and $\frac{2}{\pi}k_-a$, with $a/b = 7/8$, $B/b = 0(\frac{1}{8})1$.

a) Solutions of equation (57),

b) Solutions of (59) using approximations (63),

c) "Exact" solutions of (59),

d) First approximations from the Equivalent Static Problem, §16.

	B/b	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
(a)			.503	.602	.700	.811	.914	fails	-	
(b)	$\frac{2}{\pi}k_+a$		.524	.631	.724	.811	.889	.950	.988	1.000
	$\frac{2}{\pi}k_-a$	0	.512	.610	.714	.802	.880	.927	.954	
(c)	$\frac{2}{\pi}k_+a$		$.507_6$	$.608_7$	$.700_1$	$.789_2$	$.873_4$	$.942_3$	$.985_9$	1.000
	$\frac{2}{\pi}k_-a$	0	$.507_0$	$.608_6$	$.700_1$	$.788_8$	$.870_8$	$.931_1$	.947	
(d)	$\frac{2}{\pi}k_-a$		.512	.610	.714	.804	.888	.949	.985	

Table II. Variations of k_+a and k_-a with ν, using the approximations (68) and (63) in (59), $a/b = 7/8$, $B/b = 1/8$.

ν	0	1/4	1/2	3/4	1
$\frac{\pi}{2}k_+a$	-	.545	.524	.536	1.796
$\frac{\pi}{2}k_-a$	0.500	.506	.512	.442	-

Figure II. Values of the potential $\Phi(x,y)$, Simple problem.

$a/b = 7/8$, $B/b = 1/2$

Upper figures are for k_+ (Case I)

Lower figures are for k_- (Case II)

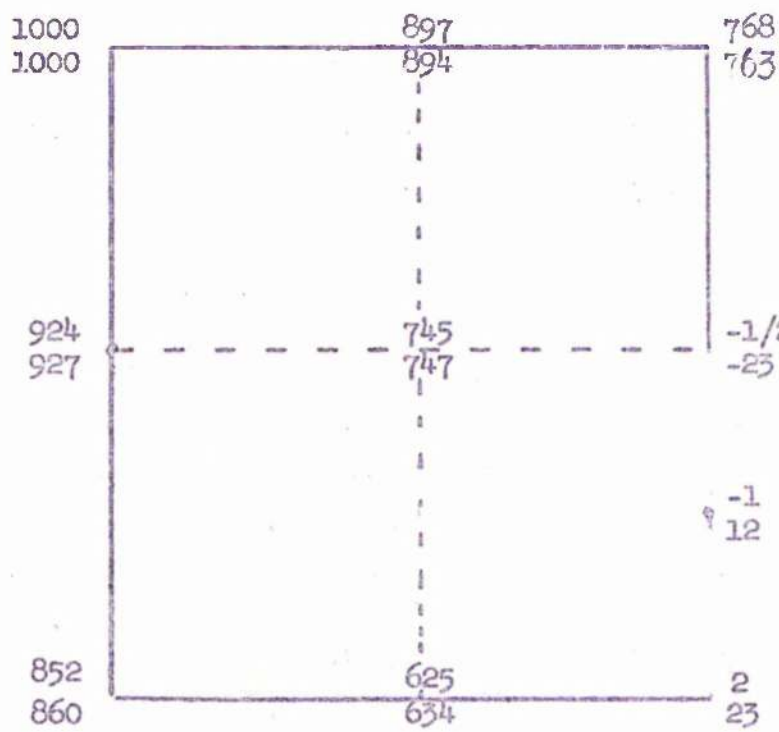


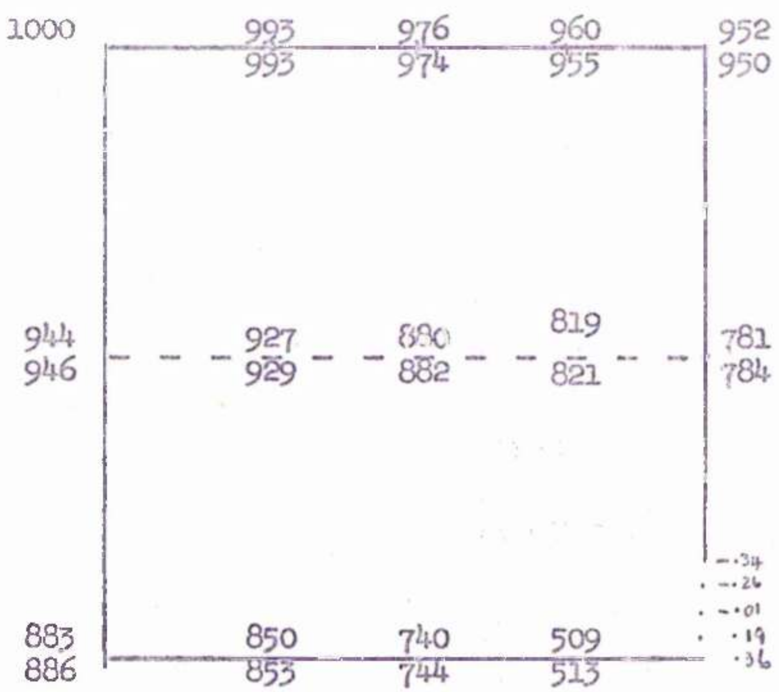
Figure III. Values of the potential $\bar{\Phi}(x,y)$, Simple problem.

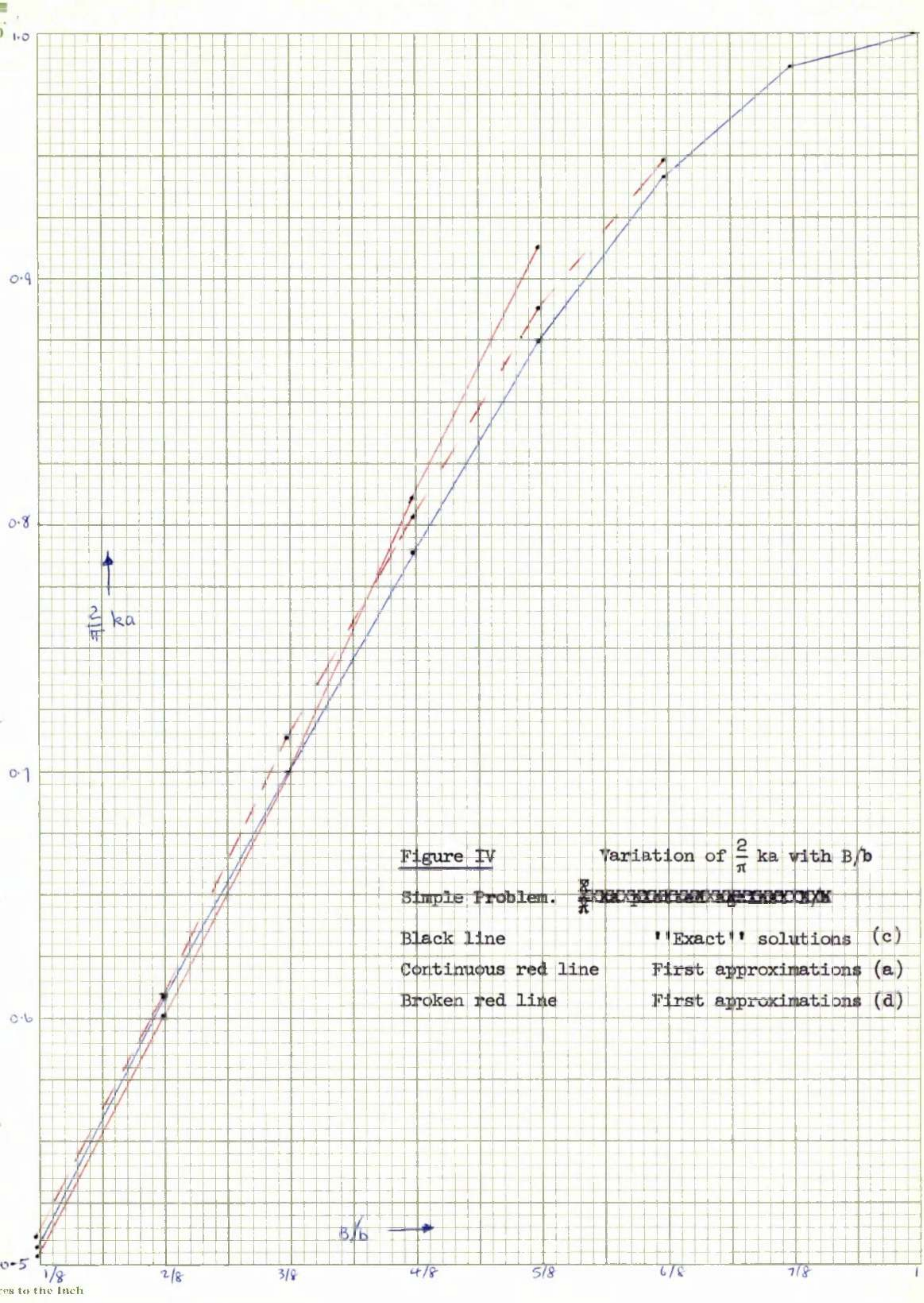
$a/b = 7/8$, $B/b = 1/2$

Upper figures are for k_+ (Case I)

Lower figures are for k_- (Case II)

The figures in the range $x = a$, $y = 0(b/32)\, b/8$ are for Case II only.




1.0
0.9
0.8
0.7
0.6
0.5
$\frac{2}{\pi}$ ka
B/b
1/8
2/8
3/8
4/8
5/8
6/8
7/8
1
Figure IV Variation of $\frac{2}{\pi}$ ka with B/b
Simple Problem.
Black line "Exact" solutions (c)
Continuous red line First approximations (a)
Broken red line First approximations (d)

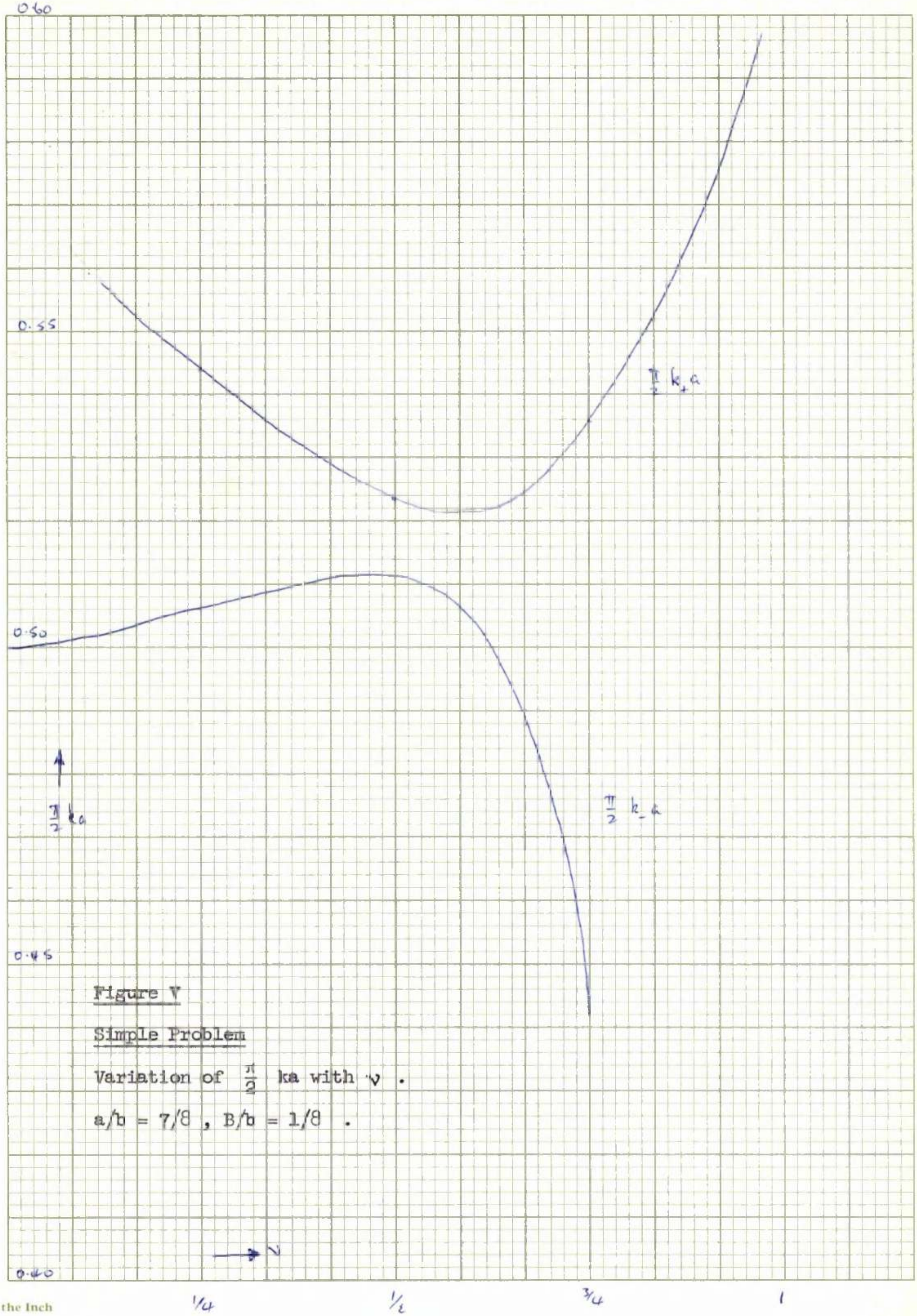


Figure V

Simple Problem

Variation of $\frac{\pi}{2}$ ka with ν .

$a/b = 7/8$, $B/b = 1/8$.

§ 7 Extension of The General Theory to a composite region.

Consider next the following problem involving a composite region. The function ϕ satisfies $\nabla^2\phi + k^2\phi = 0$ in a closed region R bounded by a curve C on which $\phi = 0$ on some part of C and $\frac{\partial\phi}{\partial n} = 0$ on the remainder. Suppose that R is subdivided into two regions S and T, $R = S + T$, with a common boundary D on part of which $\partial\phi/\partial n = 0$ and ϕ is continuous across the remainder. The significance of D is simply that in the applications considered later S and T are chosen so that the boundary value problems can be solved by separation-of-variables in S and T separately, but not in R as a whole. We extend the notation (2) in an obvious way by writing, for example,

$$(f, g)_S = \iint_S f(x,y)\, g(x,y)\, dx\, dy.$$

As before, let k_1 be the lowest eigenvalue with corresponding normalized eigenfunction ϕ_1. We introduce again a normalized function Φ which approximates ϕ, and an error function $\delta(xy)$ so that $\Phi = \phi_1 + \delta$ where Φ has the same properties as before except that this function and its derivatives may be discontinuous across D.

We consider a region $R = S + T$ as in Fig. VI bounded by C, the simple closed curve LUQQ'VL' with LPQ the common boundary between S and T. On this common boundary we suppose that $\frac{\partial\phi}{\partial n} = 0$ on LP and ϕ is continuous across PQ.

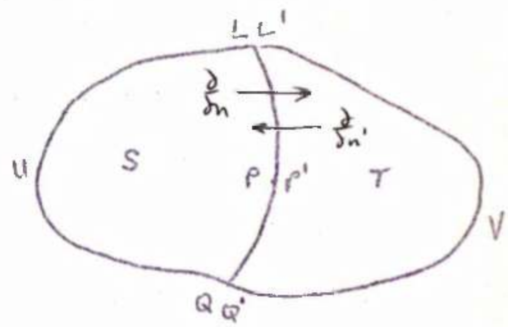


Fig. VI.

From Green's Theorem

$$(\nabla\Phi, \nabla\Phi)_S = -(\Phi, \nabla^2\Phi)_S + \int_{QPL} \Phi \frac{\partial\Phi}{\partial n}\, ds,$$

$$(\nabla\Phi, \nabla\Phi)_T = -(\Phi, \nabla^2\Phi)_T + \int_{L'P'Q'} \Phi \frac{\partial\Phi}{\partial n'}\, ds.$$

where the direction of the normal derivatives $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial n'}$ are shown in Fig VI, so that $\frac{\partial}{\partial n'} = -\frac{\partial}{\partial n}$.

Therefore adding,

$$(\nabla\Phi, \nabla\Phi)_R = -(\Phi, \nabla^2\Phi)_R + \left[\int_{QP} \Phi \frac{\partial\Phi}{\partial n} ds - \int_{Q'P'} \Phi \frac{\partial\Phi}{\partial n} ds\right] \tag{69}$$
$$+ \left[\int_{PL} \Phi \frac{\partial\Phi}{\partial n} ds - \int_{P'L'} \Phi \frac{\partial\Phi}{\partial n} ds\right].$$

In order to obtain an upper bound k_+ for the lowest eigenvalue k_1, suppose that Φ satisfies the following conditions:

(i) $\nabla^2\Phi + k_+{}^2\Phi = 0$ in S and T separately for some constant k_+,

(ii) Φ satisfies the same conditions as ϕ on C, (~~70~~)

(iii) Φ is continuous across PQ,

(iv) $\int \Phi \frac{\partial\Phi}{\partial n} ds = 0$ on PL, P'L'.

Then (69) reduces to

$$(\nabla\Phi, \nabla\Phi)_R = k_+^2 \quad \text{provided} \int_{QP} \Phi\left[\left(\frac{\partial\Phi}{\partial n}\right)_{QP} - \left(\frac{\partial\Phi}{\partial n}\right)_{Q'P'}\right] ds = 0 \qquad (70a)(\cancel{71})$$

Also $(\nabla\delta, \nabla\phi_1)_R = k_1^2(\delta, \phi_1)$ with the same proviso. (70b)($\cancel{72}$)

Using exactly the same reasoning that lead to equations (10),(11),(12) it follows that, if (70a) ($\cancel{71}$) is satisfied, $k_+{}^2 \geq k_1{}^2$, where if Φ is a first order approximation to ϕ_1, then k_+ is a second order approximation to k_1.

A second variational principle for the lowest eigenvalue can be obtained by considering a function Ψ which satisfies the conditions

(i)' $\nabla^2\Psi + k_-{}^2\Psi = 0$ in S and T separately for some constant k_-,

(ii)' Ψ satisfies the same conditions as ϕ on C,

(iii)' $\partial\Psi/\partial n = 0$ on PL, P'L',

(iv)' $\int \Psi \frac{\partial\Psi}{\partial n} ds = 0$ on QP, Q'P' .

We also define an error function $\Sigma(x,y)$ such that $\Psi = \phi_1 + \Sigma$.

Repeating the argument leading to (70a),(70b) ($\cancel{71}$),($\cancel{72}$) but using (i)' - (iv)', we find

$$(\nabla\Psi, \nabla\Psi)_R = k_-^2 \qquad (\cancel{73})$$

provided $\int_{QP} \Psi[(\partial\Psi/\partial n)_{QP} - (\partial\Psi/\partial n)_{Q'P'}] ds = 0$, ($\cancel{74}$)

and $(\nabla\Sigma, \nabla\phi_1)_R = k_-^2 - k_1^2 + k_-^2(\phi_1, \Sigma)$ as in (14).

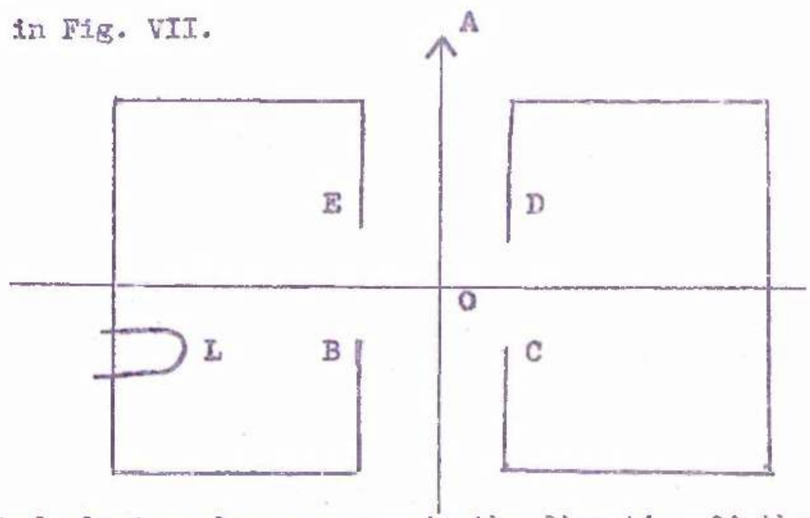
It follows again that if Ψ is a first order approximation to ϕ_1, then k_- is a second order approximation to k_1, but not necessarily that $k_-^2 \leq k_1^2$. The conditions that k_+ and k_- should give upper and lower bounds for the lowest eigenvalue k_1 will be exactly the same as those in §5. It is shown in the problem of §9 that these conditions are satisfied and that therefore $k_- \leq k_1$.

§ 8 Klystron Resonators.

A direct extension of the Simple problem leads to a problem of practical interest, that of determining the natural modes and frequencies of Klystron resonators. A Klystron resonator is a type of electron vacuum tube employed for the amplification and generation of microwave frequencies. Physical descriptions, applications and operational requirements are given, for example, in (16), (17), (18). Extensive bibliographies are given in (19) and (20) and detailed numerical results are to be found in (21).

The basis of operation of a Klystron amplifier is the interaction between an electron beam and a resonant cavity. A simple form of such a resonator is a re-entrant cylinder of circular cross section, and cross section through its axis OA as shown in Fig. VII.

Figure VII



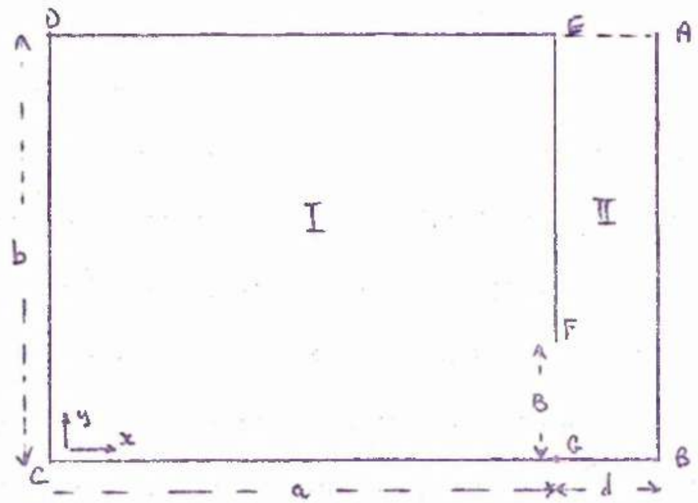
A velocity modulated electron beam passes in the direction OA through a conducting grid structure in the region BCDE, and excites electro-magnetic oscillations in the cavity. It is required in practice to know relative dimensions of lengths BC and CD for maximum efficiency. Coupling to the resonator is made with a small loop L. The positioning of this loop is determined by the position of current nodes in the internal surface. A study of such re-entrant cavities is given in (21) chapter VIII.

9 Southwell's Problem.

We shall consider first a simple two-dimensional problem, regarding Fig. VII as the plane cross section of an infinite tube. This has been dealt with in some detail by Southwell in (22) by a Relaxation method. There are two major objections to using Relaxation methods for this type of problem, firstly that the numerical work can become excessive, and secondly that relaxation becomes ~~unstable~~ inaccurate in the neighborhood of an edge or corner. Comparison of the results obtained in this paper, with those given in (22) shows a discrepancy in the eigenvalue computations of up to 10%. This is most probably due to the ~~instability~~ inaccuracy of the relaxation procedure near the re-entrant edge.

Utilizing the symmetry properties of Fig. VII it is sufficient to consider a rectangular region $0 \leq x \leq a + d$, $0 \leq y \leq b$, as in Fig. VIII.

Fig. VIII.



We specify the boundary conditions governing a potential function ϕ:

(i) $\phi = 0$ on AB,

(ii) $\frac{\partial\phi}{\partial n} = 0$ on BC, CD, DE, EF,

(iii) $\frac{\partial\phi}{\partial n} = 0$ on EA ~~(X)~~ (71)

(iv) ϕ continuous across FG

(v) $\frac{\partial\phi}{\partial n}$ continuous across FG.

(We shall be interested in the configuration in which d is small compared with a. When this is so the results are very little effected whether we use condition (iii) or replace it by $\phi = 0$ on EA.)

We divide the figure into two regions.

$$\text{I:} \quad 0 \leq x \leq a \;, \quad 0 \leq y \leq b \,,$$

$$\text{II:} \quad a \leq x \leq a + d, \quad 0 \leq y \leq b,$$

with corresponding potential functions ϕ_1 and ϕ_2. By separation of variables, solutions of $\phi_{xx} + \phi_{yy} + k^2\phi = 0$ in the two regions, satisfying the boundary conditions are

$$\phi_1(x,y) = \frac{2}{b} \sum_{n=0}^{\infty} \varepsilon_n' A_n \cosh \gamma_n x \cos(n\pi y/b),$$

$$\phi_2(x,y) = \frac{2}{b} \sum_{n=0}^{\infty} \varepsilon_n' B_n \sinh[\gamma_n(x-a-d)] \cos(n\pi y/b),$$

where $\gamma_n = \left[(n\pi/b)^2 - k^2\right]^{\frac{1}{2}} = -i\left[k^2 - (n\pi/b)^2\right]^{\frac{1}{2}}$, and ε_n' is defined as in (17b). We can obtain approximations to the eigenvalues, as in the Simple problem, in either of two ways:

<u>Case I.</u> We consider functions Φ_1 and Φ_2 defined in regions I and II respectively by

$$\Phi_1(x,y) = \frac{2}{b} \sum_{n=0}^{\infty} \varepsilon_n' A_n \cosh \Gamma_n x \cos(n\pi y/b), \tag{72a}$$

$$\Phi_2(x,y) = \frac{2}{b} \sum_{n=0}^{\infty} \varepsilon_n' B_n \sinh[\Gamma_n(x-a-d)] \cos(n\pi y/b), \tag{72b}$$

where $\Gamma_n = \left[(n\pi/b)^2 - k_+^2\right]^{\frac{1}{2}} = -i\left[k_+^2 - (n\pi/b)^2\right]^{\frac{1}{2}}$,

so that $\nabla^2\Phi_1 + k_+^2\Phi_1 = \nabla^2\Phi_2 + k_+^2\Phi_2 = 0$.

We introduce unknown functions $F_1(y)$, $F_2(y)$, $F(y)$ by writing

$$\Phi_1(a,y) = F_1(y), \quad B \leq y \leq b$$

$$\Phi_2(a,y) = F_2(y), \quad B \leq y \leq b$$

$$\left.\begin{aligned} \Phi_1(a,y) - \Phi_2(a,y) &= F_1(y) - F_2(y) = F(y), && B \leq y \leq b \,, \\ &= 0 && 0 \leq y \leq b \,. \end{aligned}\right\} \tag{73}$$

so that $\Phi(x,y)$ satisfies condition (iv) of (71), where $\Phi = \Phi_1, \Phi_2$, in I, II respectively.

Condition (v) of (71) requires that $\frac{\partial \Phi_1}{\partial n} - \frac{\partial \Phi_2}{\partial n} = 0, \; x = a, \; 0 \le y \le B$

That is, $\sum_{n=0}^{\infty} \varepsilon_n' (A_n \sinh \Gamma_n a - B_n \cosh \Gamma_n d) \cos(n\pi y/b) = 0, \; 0 \le y \le b$, using (ii) of (71),

so that $\quad B_n \cosh \Gamma_n d = A_n \sinh \Gamma_n a.$

Substituting for B_n in (72b) and using (73) we obtain the expression for A_n,

$$A_n = \frac{\cosh \Gamma_n d}{\cosh \Gamma_n (a+d)} \int_B^b F(\xi) \cos \frac{n\pi \xi}{b} \, d\xi \; . \tag{74}$$

But condition (ii) of (71) requires $\quad \partial \Phi_1 / \partial n = 0, \; x = a, \; B \le y \le b.$

If we were to use (72a), (74) in this condition we would obtain the following integral equation to determine the exact value of k:

$$\sum_{n=0}^{\infty} \varepsilon_n' \Gamma_n \frac{\sinh \Gamma_n a \cosh \Gamma_n d}{\cosh \Gamma_n (a+d)} \int_B^b F(\xi) \cos \frac{n\pi\xi}{b} \, d\xi \cos \frac{n\pi y}{b} = 0 , \quad B \le y \le b.$$

Multiplying by F(y) and integrating with respect to y to remove the y dependence, as for the Simple Problem,

$$\sum_{n=0}^{\infty} \varepsilon_n' \Gamma_n \frac{1}{\coth \Gamma_n a + \tanh \Gamma_n d} \left[\int_B^b F(\xi) \cos \frac{n\pi \xi}{b} \, d\xi \right]^2 = 0 \tag{75}$$

which corresponds to (23) of the Simple Problem.

<u>Case II</u> Consider functions Ψ_1 and Ψ_2 defined by

$$\Psi_1(x,y) = (2/b) \sum_{n=0}^{\infty} \varepsilon_n' C_n \cosh \Theta_n x \cos(n\pi y/b), \tag{76a}$$

$$\Psi_2(x,y) = (2/b) \sum_{n=0}^{\infty} \varepsilon_n' D_n \sinh[\Theta_n (x-a-d)] \cos(n\pi y/b), \tag{76b}$$

where Ψ and $\partial\Psi/\partial n$ are continuous across $x = a, \; 0 \le y \le B$, and where

$$\Theta_n = [(n\pi/b)^2 - k_-^2]^{\frac{1}{2}} = -i [k_-^2 - (n\pi/b)^2]^{\frac{1}{2}}$$

so that

$$\nabla^2 \Psi_1 + k_-^2 \Psi_1 = \nabla^2 \Psi_2 + k_-^2 \Psi_2 = 0 .$$

We introduce an unknown function G(y) by writing

$$\frac{\partial \Psi_1}{\partial n} = \frac{\partial \Psi_2}{\partial x} = G(y) \quad , \; x = a, \; 0 \le y \le B . \tag{77}$$

$$= 0 \; , \quad x = a, \; B \le y \le b .$$

From (76) and (77) we can quickly derive the forms

$$C_n = \frac{-1}{\Theta_n \sinh \Theta_n a} \int_0^B G(\xi) \cos \frac{n\pi\xi}{b} \, d\xi , \tag{78a}$$

$$D_n = \frac{1}{\Theta_n \cosh \Theta_n d} \int_0^B G(\xi) \cos \frac{n\pi\xi}{b} \, d\xi , \tag{78b}$$

and hence, substituting (78) in (76) and using the continuity condition,

$$\sum_{n=0}^{\infty} \varepsilon_n' \frac{1}{\Theta_n} \left(\coth \Theta_n a + \tanh \Theta_n d \right) \left[\int_0^B G(\xi) \cos \frac{n\pi\xi}{b} \, d\xi \right]^2 = 0 \tag{79}$$

corresponding to (29) of the Simple problem.

Numerical results for Southwell's problem.

To be able to compare results with those of (22) pages 392, 394, we choose the dimensions of Fig. VIII to be

$b/d = 8$, $a/d = 7$, $B/d = 0(1)8$.

The eigenvalues in the limiting cases $B/d = 0,8$ are exactly as in the Simple problem,

$$k_{rs} = \left[\left(\frac{r\pi}{a} \right)^2 + \left(\frac{s\pi}{b} \right)^2 \right]^{\frac{1}{2}} \quad , \quad B/d = 0 , \tag{80a}$$

$$k_{rs} = \left[(r+\tfrac{1}{2})^2 \left(\frac{\pi}{a} \right)^2 + \left(\frac{s\pi}{b} \right)^2 \right]^{\frac{1}{2}} \quad , \quad B/d = 8 . \tag{80b}$$

and the lowest eigenvalue k_{00} for $B/d = 1(1)7$ will therefore be between zero and $\pi/2$.

The process of finding numerical approximations to the lowest eigenvalues of this problem is almost identical with that which dealt with the Simple problem. The conditions for choosing suitable trial functions $F(y)$ and $G(y)$ are identical in the two problems, and we therefore take

$$F(\eta) = \left[(b-B)^2 - (b-\eta)^2 \right]^{\frac{1}{2}} , \tag{81a}$$

$$G(\eta) = \left[B^2 - \eta^2 \right]^{-\frac{1}{2}} \tag{81b}$$

A slight simplification is obtained by writing $b - B = C$ in (81a), and $b - \eta = \xi$, so that our trial functions will be

$$F(\xi) = (c^2 - \xi^2)^{\frac{1}{2}}, \tag{82a}$$

$$G(\eta) = (B^2 - \eta^2)^{-\frac{1}{2}}. \tag{82b}$$

Substituting (82a,b) in (75) and (79) respectively, we obtain the approximate transcendental equations for the determination of k_+ and k_-,

$$\sum_{n=0}^{\infty} \varepsilon_n' P_n b \frac{1}{\coth P_n a + \tanh P_n d} \left[\int_0^c F(\xi) \cos \frac{n\pi\xi}{b} d\xi \right]^2 = 0, \tag{83a}$$

$$\sum_{n=0}^{\infty} \varepsilon_n' \frac{1}{\Theta_n b} \left(\coth \Theta_n a + \tanh \Theta_n d \right) \left[\int_0^B G(\xi) \cos \frac{n\pi\xi}{b} d\xi \right]^2 = 0. \tag{83b}$$

A very crude first approximation, corresponding to (57), can again be found by using the trial function $G(\eta) = 1$ in (83b), together with the approximations

$$\begin{aligned} \coth \Theta_n a + \tanh \Theta_n d &\sim 2, \quad n \geq 1, \\ \Theta_n &\sim \frac{n\pi}{b}, \quad n \geq 1. \end{aligned} \tag{84}$$

Substitution of (84) in (83b) reduces it to

$$\frac{1}{bk_-} \left[\cot k_- a - \tan k_- d \right] = \frac{2}{B^2} \left(\frac{b}{\pi} \right)^3 \frac{1}{b} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin^2 \left(\frac{n\pi B}{b} \right),$$

for the right hand side of which, being identical with the right hand side of (55), we know the approximate sum. The first approximation to (83b) is therefore

$$\frac{1}{bk_-} \left[\cot k_- a - \tan k_- d \right] = \frac{2}{\pi} \log \frac{2.23 b}{\pi B}. \tag{85}$$

Values of $k_- a$ obtained from this equation, over the range of values of B/b, are shown in Table III(a).

The next approximations for k_+ and k_- to the smallest eigenvalue are obtained by using the trial functions (82) in equations (83). The resulting transcendental equations are almost identical with equations (59) and the successive steps in their approximate solution are precisely the same. We therefore omit the details. Resulting values of $k_- b$ and $k_+ b$ are shown in Table III(b).

Finally using the trial function

$$G(\xi) = \sum_{r=0}^{m} D_r \left[B^2 - \xi^2 \right]^{r-\frac{1}{2}}$$

we obtain the following results for the case when $b/d = 8$, $c/d = 7$, $B/d = 1$

m	0	1	2
$\frac{2}{\pi} k_- b$	$.4735_4$	$.4735_9$	$.4736_4$

indicating slightly improved results for enormously increased labor.

<u>Table III.</u> Solution of Southwell's problem by the variational method.

Values of $\frac{2}{\pi} k_+ b$ and $\frac{2}{\pi} k_- b$, with $b/d = 8$, $a/d = 7$, $B/d = 0(1)8$.

(a) Solutions of equations (85)

(b) Solutions of equations (82) using approximating functions (81)

	B/d	0	1	2	3	4	5	6	7	8
(a)	$\frac{2}{\pi} k_- b$	-	.550	.652	.738	.835	.927	fails	-	-
(b)	$\frac{2}{\pi} k_+ b$	-	$.479_8$	$.585_7$	$.678_1$	$.766_5$	$.851_4$	$.924_4$	$.978_4$	1.000
	$\frac{2}{\pi} k_- b$	0	$.473_5$	$.582_2$	$.675_1$	$.762_4$	$.839_5$	$.901_1$	$.929_4$	-

<u>Values of Ψ_1 and Ψ_2</u> can be obtained from formulae (76a) and (76b).

The forms for Case II are

$$\Psi_1 = \frac{1}{2} \frac{\cos k_- x}{k_- d \sin k_- a} - \sum_{n=1}^{\infty} \frac{1}{\Theta_n d} \frac{\cosh \Theta_n x}{\sinh \Theta_n a} J_0\left(\frac{n\pi B}{b}\right) \cos \frac{n\pi y}{b}, \tag{86a}$$

$$\Psi_2 = -\frac{1}{2} \frac{\sin k_- (x-a-d)}{k_- d \cos k_- d} - \sum_{n=1}^{\infty} \frac{1}{\Theta_n d} \frac{\sinh[\Theta_n (x-a-d)]}{\cosh \Theta_n d} J_0\left(\frac{n\pi B}{b}\right) \cos \frac{n\pi y}{b}. \tag{86b}$$

The infinite series in (86) again converge very rapidly, and can therefore be summed easily, except on the line x = a. When x = a the infinite series may be approximated by

$$\sum_{n=1}^{N} \frac{1}{\Theta_n d} \coth \Theta_n a \, J_0\left(\frac{n\pi\beta}{b}\right) \cos\frac{n\pi y}{b} + \frac{b}{\pi a} \sum_{n=N+1}^{\infty} \frac{1}{n} J_0\left(\frac{n\pi\beta}{b}\right) \cos\frac{n\pi y}{b}\,, \tag{87a}$$

and

$$\sum_{n=1}^{N} \frac{1}{\Theta_n d} \tanh \Theta_n d \, J_0\left(\frac{n\pi\beta}{b}\right) \cos\frac{n\pi y}{b} + \frac{b}{\pi a} \sum_{n=N+1}^{\infty} \frac{1}{n} J_0\left(\frac{n\pi\beta}{b}\right) \cos\frac{n\pi y}{b}\,. \tag{87b}$$

The infinite tails of these two expressions are identical and are the same as (66b). Their values are therefore known.

Figure IX shows values of ϕ computed for Case II. These are compared with Southwell's figures (22) page 382, in Figure X. Southwell has used the value $\frac{2}{\pi}(kb) = .497_1$ compared with our $\frac{2}{\pi}k\,b = .473_3$. It is of interest to compare Figures IX and X with Figure XI which gives values of ϕ computed from our expressions (87) but using Southwell's value, $\frac{2}{\pi}(kb) = .497_1$.

Figures IX, X, XI. Values of the potential $\phi(x,y)$.

$b/d = 8$, $a/d = 7$, $B/d = 1$.

Figure IX. Computed from (87), with $\frac{2}{\pi}k\,b = .473_3$

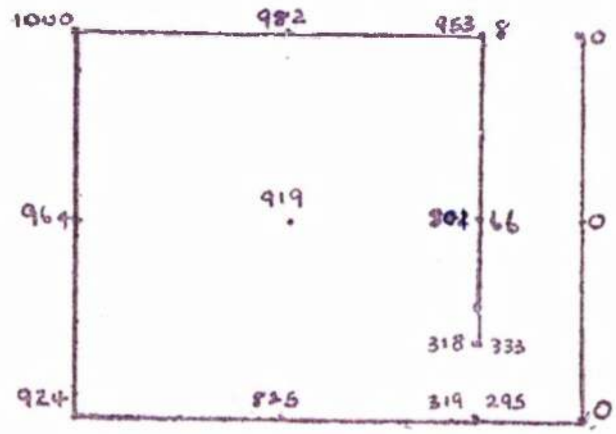


Figure X. Southwell's results, given in (22), page 392, with $\frac{2}{\pi}k\frac{b}{a} = .497_1$

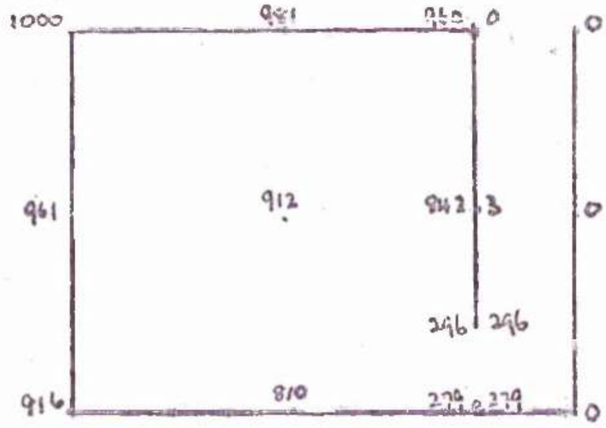
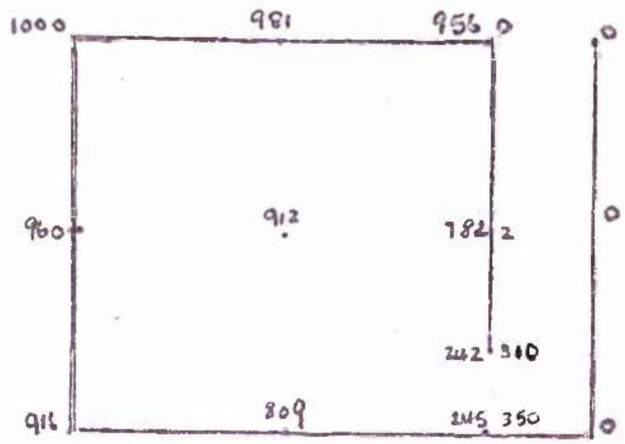


Figure XI. Computed from (87), with $\frac{2}{\pi}k\,b = .497_1$



Values of ϕ_1 and ϕ_2 should, of course, be equal on the line $x = a$, $0 \leq y \leq B$. With the approximations $F(y)$ and $G(y)$ this will not be so, but the average difference should approximate zero. The results shown in Table IV agree with this.

Table IV. Values of ϕ_1, ϕ_2 in Southwells problem, on the line

$x = a$, $B/a = 1$, $b/a = 8$ $\quad y = 0(b/32)B_y \quad \frac{2}{\pi}k_- b = \cdot 473_3$

y	0	b/32	2b/32	3b/32	b/8 (=B)
ϕ_1	319_3	319_1	318_8	318_5	318_2
ϕ_2	294_7	297_8	304_9	315_8	332_7

Higher eigenvalues k_{rs} are easily found. For example, from (80) we see that the second lowest symmetrical eigenvalue k_{11} will lie somewhere between the values $\left[(\pi/a)^2 + (\pi/b)^2\right]^{1/2}$ and $\left[(3\pi/2a)^2 + (\pi/b)^2\right]^{1/2}$,

that is $\quad \pi^2 + (\pi b/a)^2 < (k_{11}b)^2 \leq \pi^2 + (3\pi b/2a)^2 \qquad$ (88)

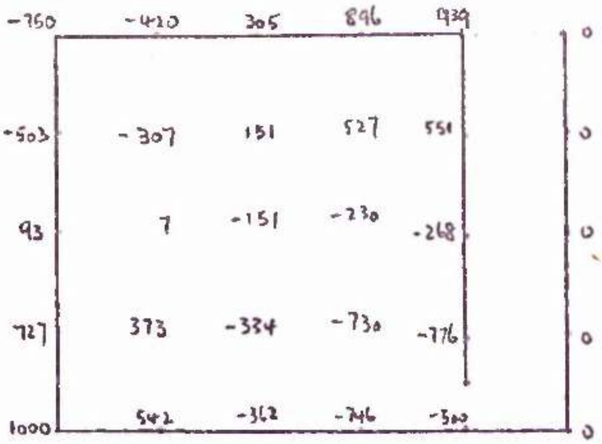
when $0 < B/b < 1$. It follows that $\Theta_n b = \left[(n\pi)^2 - (k_{11})^2 b\right]^{1/2}$ is imaginary for $n = 0, 1$ and also that the function Θ_n which is most dependent upon k_{11} will be Θ_1. Separating out the first two terms of (83b), using approximations (82b) and result (58b), and putting $\Theta_0 = i\Theta_0'$, $\Theta_1 = i\Theta_1'$, so that Θ_0' and Θ_1' are real,

$$\frac{1}{2}\frac{1}{\Theta_0' b}\left(\cot\Theta_0' a - \tan\Theta_0' d\right) + \frac{1}{\Theta_1' b}\left(\cot\Theta_1' a - \tan\Theta_1' d\right) J_0^2\left(\frac{n\pi B}{b}\right)$$
$$= \sum_{n=2}^{\infty}\left(\frac{\coth\Theta_n a + \tanh\Theta_n d}{\Theta_n b}\right) J_0^2\left(\frac{n\pi B}{b}\right) \qquad (89)$$

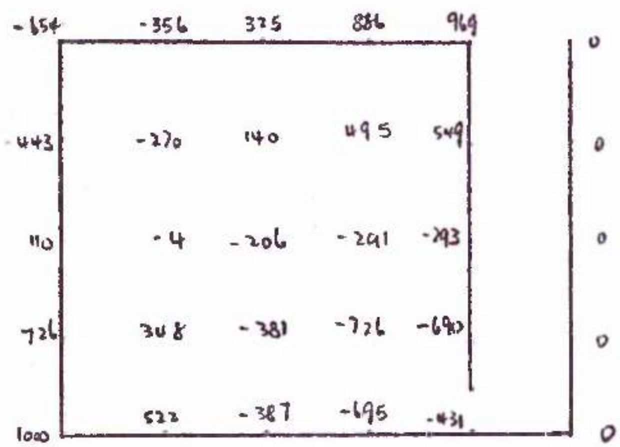
A first estimate of k_{11} for given B/b is best found by assuming a linear relationship between $k_{11}b$ and B/b, using (88). Using the results in the first term of (89), and approximations (84) in the right side of (89), a second estimate is obtained. An iterative process now easily improves the approximation to $k_{11}b$ to any desired accuracy.

We have computed $k_{11}b$ and corresponding values of ϕ_1 in a particular case. The results, shown in Figure XII can be compared directly with Southwell's results, Figure XIII, for the same configuration. Our value of $k_{11}b = 5.039$ is to be compared with Southwells, $k_{11}b = 5.15$.

<u>Figure XII.</u> Southwell's problem, values of ϕ for second lowest symmetrical eigenvalue $k_{11} = 5.039$, $a/d = 7$, $b/d = 8$, $B/d = 1$



<u>Figure XIII.</u> Southwell's results (<u>22</u>) page 394, with the same configuration as Figure XII. Eigenvalue $k_{4} \sim 5.15$



<u>Upper and Lower bounds</u> for Southwell's problem.

In determining whether the inequalities $k_- \leq k_1 \leq k_+$ are true for the lowest eigenvalue k_1 of Southwell's problem, the form of the function $p_n(k)$ of expressions (46) is

$$p_n(k) = \gamma_n \left[\coth \gamma_n a + \tanh \gamma_n d\right]^{-1} \tag{90}$$

where $$\gamma_n = \left[(n\pi/b)^2 - k^2\right]^{\frac{1}{2}} = -i\left[k^2 - (n\pi/b)^2\right]^{\frac{1}{2}}.$$

It is obviously true that $p_n(k) > 0$ for $n \geq 1$. We also require $\frac{d}{dk}(p_n(k)) \equiv p_n'(k)$ to be negative for $n \geq 0$.

When $n \geq 1$: $$p_n'(k) = \frac{-k}{\coth \gamma_n a + \tanh \gamma_n d}\left[\frac{1}{\gamma_n d} - E_n\right] \tag{91}$$

where $$E_n = \frac{(1 - \tanh^2 \gamma_n d) - (a/d)(\coth^2 \gamma_n a - 1)}{\coth \gamma_n a + \tanh \gamma_n d}$$

$$< \frac{1 - \tanh^2 \gamma_n d}{\coth \gamma_n a + \tanh \gamma_n d} < 1 - \tanh \gamma_n d \quad \text{since } \coth \gamma_n a > 1. \tag{92}$$

Therefore $p_n'(k)$ is negative provided $\frac{1}{\gamma_n d} > 1 - \tanh \gamma_n d$, which is so.

When $n = 0$: $\gamma_o = -ik$ and $p_o(k) = -k(\cot ka - \tan kd)^{-1}$.

Therefore $p_o'(k) = -(\cot ka - \tan kd)^{-1} - k(a \operatorname{cosec}^2 ka + d \sec^2 kd)$.

But we already know that for the lowest eigenvalue, $0 \leq ka \leq \pi/2$. Therefore with the dimensions $ka = 7kd$, it follows that $\cot ka > \tan kd$ and hence that $p_o'(k)$ is negative. Cases I and II do therefore give true upper and lower bounds respectively for the lowest eigenvalue of Southwell's problem.

§ 10. Cylindrical Klystron.

The problem of the last section is easily extended to that of a three dimensional resonator with circular axial symmetry, as shown in Figure XIV

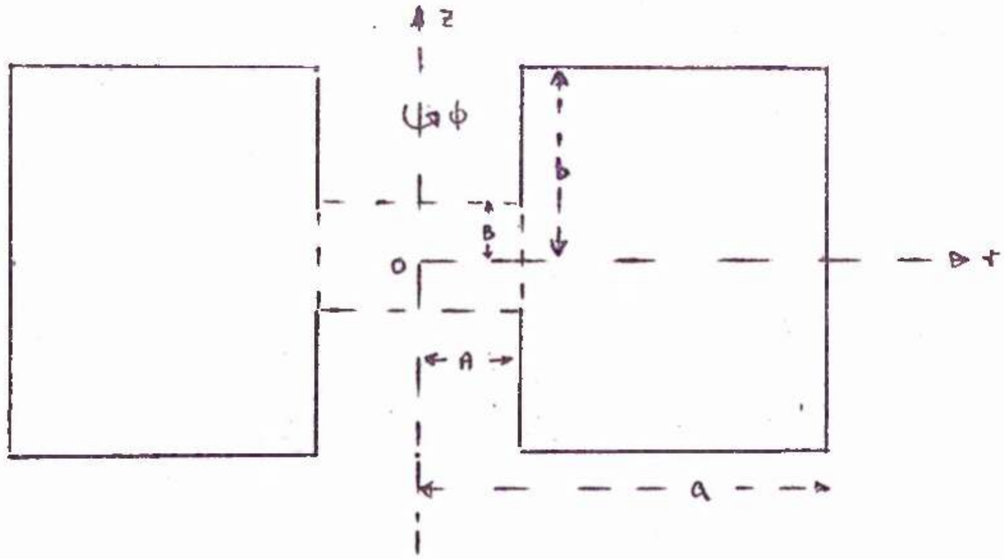


Figure XIV

A method for the solution of this problem is given by L. S. Goddard in (6), where the problem is reduced to the solution of an infinite set of simultaneous linear equations which must be solved numerically. Electric and magnetic field vectors $\underline{E} \equiv (E_r, E_z, E_\phi)$ and $\underline{H} \equiv (H_r, H_z, H_\phi)$ satisfy the conditions of symmetry

$$E_\phi \equiv 0, \qquad H_r \equiv H_z \equiv 0 \quad , \tag{93}$$

and Maxwell's equations

$$iKE_r = -\partial H_\phi / \partial z \, ,$$
$$iKE_z = \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) \tag{94}$$

where $\dfrac{\partial^2 H_\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial H_\phi}{\partial r} + \dfrac{\partial^2 H_\phi}{\partial z^2} + \left(K^2 - \dfrac{1}{r^2}\right) H_\phi = 0.$

We choose the boundary conditions

$$E_z = 0 \;,\; r = a \;,\; 0 \leq |z| \leq b \,,$$
$$= 0 \,,\; r = A \;,\; B \leq |z| \leq b \,,$$
$$E_r = 0 ,\; |z| = B \text{ and } |z| = b \;,\; 0 \leq r \leq A \,,\; A \leq r \leq a \,, \tag{95}$$
$$E_z \text{ and } H_\phi \text{ continuous at } r = A \,,\; |z| < B.$$

If we write $H_\phi = \psi$ so that $ikE_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$, then we require ψ to satisfy the equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + \left(k^2 - \frac{1}{r^2}\right)\psi = 0. \tag{96}$$

The symmetry conditions make it sufficient to consider the region indicated in Figure XV, which we divide into two regions,

I: $0 \leq r \leq A$, $0 \leq z \leq B$

II: $A \leq r \leq a$, $0 \leq z \leq b$

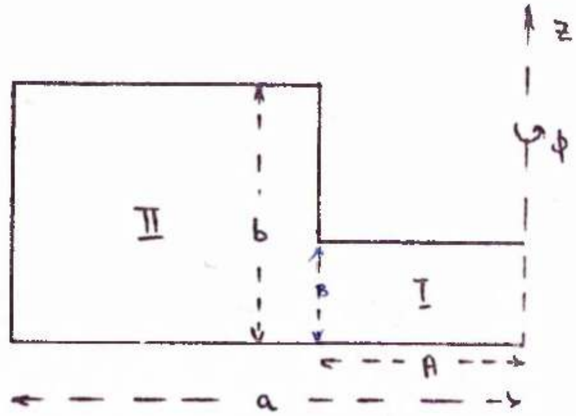


Figure XV

Effectively we are dividing the resonator into a cylindrical space of radius A and a toroidal space radii A and a, of rectangular cross section. Solutions ψ_1 and ψ_2 of (96) in those two spaces, subject to the boundary conditions, are obtained by separation of variables to give

$$\text{I}: \quad \psi_1 = \left(\frac{2}{B}\right)\sum_{n=0}^{\infty} \varepsilon_n' A_n \frac{k}{\alpha_n} L(\alpha_n r)\cos(n\pi z/B) \tag{97a}$$

$$\text{II} \quad \psi_2 = \left(\frac{2}{b}\right)\sum_{n=0}^{\infty} \varepsilon_n' B_n \frac{k}{\beta_n} M(\beta_n r)\cos(n\pi z/b)$$

where
$$\alpha_n = \left[k^2 - (n\pi/B)^2\right]^{\frac{1}{2}} = -i\left[(n\pi/B)^2 - k^2\right]^{\frac{1}{2}}, \tag{97b}$$

$$\beta_n = \left[k^2 - (n\pi/b)^2\right]^{\frac{1}{2}} = -i\left[(n\pi/b)^2 - k^2\right]^{\frac{1}{2}},$$

$$L(\alpha_n r) = J_1(\alpha_n r),$$

$$M(\beta_n r) = J_1(\beta_n r)Y_0(\beta_n a) - J_0(\beta_n a)Y_1(\beta_n r).$$

Then
$$\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_1) = \frac{2}{B}\sum_{n=0}^{\infty}\varepsilon_n' A_n k L'(r\alpha_n)\cos(n\pi z/B), \tag{98a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_2) = \frac{2}{b}\sum_{n=0}^{\infty}\varepsilon_n' B_n k M'(r\beta_n)\cos(n\pi z/b), \tag{98b}$$

where
$$L'(r\alpha_n) = J_1'(r\alpha_n) + \frac{1}{r\alpha_n}J_1(r\alpha_n) = J_0(r\alpha_n)$$

$$M'(r\beta_n) = J_0(r\beta_n)Y_0(a\beta_n) - J_0(a\beta_n)Y_0(r\beta_n).$$

We can find approximations to the eigenvalues if we use the method of Case II of the earlier problems, and write, on $r = A$,

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_1) = \frac{1}{r}\frac{\partial}{\partial r}(r\Psi_2) &= f(z) \qquad 0 \le z \le B\\ &= 0 \qquad B \le z \le b\end{aligned}\right\}. \tag{99}$$

$f(z)$ is the unknown function for which we shall require to choose a trial function. From (98) and (99) we obtain expressions for the constants A_n and B_n,

$$kA_n L'(A\alpha_n) = \int_0^B f(z)\cos(n\pi z/B)\,dz = \mathfrak{F}_n(B), \text{ say},$$

$$kB_n M'(A\beta_n) = \int_0^B f(z)\cos(n\pi z/b)\,dz = \mathfrak{F}_n(b).$$

and therefore expressions for the potential function in the two regions are

$$\Psi_1 = \frac{2}{B}\sum_{n=0}^{\infty}\varepsilon_n'\frac{1}{\alpha_n}\frac{L(\alpha_n r)}{L'(\alpha_n A)}\mathfrak{F}_n(B)\cos(n\pi z/B), \tag{100a}$$

$$\Psi_2 = \frac{2}{b}\sum_{n=0}^{\infty}\varepsilon_n'\frac{1}{\beta_n}\frac{M(\beta_n r)}{M'(\beta_n A)}\mathfrak{F}_n(b)\cos(n\pi z/b). \tag{100b}$$

The transcendental equation for the determination of k_- found by using the continuity of Ψ at $r = A$, $0 \le z \le B$, is therefore

$$\sum_{n=0}^{\infty}\varepsilon_n'\left[\frac{1}{B\alpha_n}\frac{L(\alpha_n A)}{L'(\alpha_n A)}\mathfrak{F}_n^2(B) - \frac{1}{b\beta_n}\frac{M(\beta_n A)}{M'(\beta_n A)}\mathfrak{F}_n^2(b)\right] = 0. \tag{101}$$

<u>Numerical results for the cylindrical klystron.</u>

In choosing a form for the unknown $f(z)$, we have at the point $r = A$, $z = B$, an angle of $3\pi/2$ instead of a sharp edge. Result (51b) suggests a trial function of the form

$$f(z) = C(B^2 - z^2)^{-\frac{1}{3}}, \tag{102}$$

so that
$$\left.\begin{aligned}\mathfrak{f}_n(b) &= C\int_0^b (B^2-z^2)^{-\frac{1}{3}}\cos(n\pi z/b)\,dz = (2bB/n\pi)^{1/6}\,\frac{\sqrt{\pi}}{2}\,\Gamma(\tfrac{1}{3})\,J_{1/6}(n\pi B/b),\\ \mathfrak{f}_n(B) &= C\int_0^B (B^2-z^2)^{-\frac{1}{3}}\cos(n\pi z/B)\,dz = (2B^2/n\pi)^{1/6}\,\frac{\sqrt{\pi}}{2}\,\Gamma(\tfrac{2}{3})\,J_{1/6}(n\pi),\\ \mathfrak{f}_0(b) &= \mathfrak{f}_0(B) = B^{\frac{1}{3}}\,\frac{\sqrt{\pi}}{2}\,\Gamma(\tfrac{2}{3})\big/\Gamma(7/6).\end{aligned}\right\} \tag{103}$$

Solution of (101) for the lowest eigenvalue k_- follows the same lines as in the previous problems. To obtain first approximation we write

$$\alpha_n B \sim \beta_n b \sim n\pi\,, \quad n \geqslant N, \tag{104a}$$

$$\frac{L(\alpha_n A)}{L'(\alpha_n A)} \sim \frac{I_1(n\pi A/B)}{I_0(n\pi A/B)} \sim 1\,, \quad n \geqslant N\,, \ \text{since}\ n\pi A/B \gg 1\,, \tag{104b}$$

$$\frac{M(\beta_n A)}{M'(\beta_n A)} \sim \frac{I_1\left(\frac{n\pi A}{b}\right)K_0\left(\frac{n\pi a}{b}\right) - I_0\left(\frac{n\pi a}{b}\right)K_1\left(\frac{n\pi A}{b}\right)}{I_0\left(\frac{n\pi A}{b}\right)K_0\left(\frac{n\pi a}{b}\right) + I_0\left(\frac{n\pi a}{b}\right)K_0\left(\frac{n\pi A}{b}\right)} \sim -\frac{K_1\left(\frac{n\pi A}{b}\right)}{K_0\left(\frac{n\pi A}{b}\right)} \sim -1\,, \ n \geqslant N. \tag{104c}$$

Separating the first term of (101), using approximations (104) with $N = 1$, and integrals (103), we require to solve for k_- the equation

$$\frac{1}{2k_-A}\left[\frac{A}{b}\,\frac{J_1(Ak_-)Y_0(ak_-) - J_0(ak_-)Y_1(Ak_-)}{J_0(Ak_-)Y_0(ak_-) - J_0(ak_-)Y_0(Ak_-)} - \frac{A}{B}\,\frac{J_1(Ak_-)}{J_0(Ak_-)}\right].$$

$$\sim \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{\frac{1}{3}}\Gamma(2/3)^2\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}\left[B^{\frac{1}{3}}J_{1/6}^{2}(n\pi) + b^{\frac{1}{3}}J_{1/6}^{2}\left(\frac{n\pi B}{b}\right)\right]. \tag{105}$$

The infinite series can be summed by the methods in (61) and (62) and then the solution of (105) is easily found. As before, if we use this first approximation for k_- to form a second estimate of the functions in (104) for the first N terms, we can solve a modified form of (105) to obtain a better estimate of k_-.

To be able to compare results with those given by Goddard we have taken the dimensions of Figure XV as $b = 2A$, $B = \frac{1}{4}A$, $a = 2A(2A)8A$. Results, together with those of Goddard and Warnecke (21) are given in Table V. This problem is one which lends itself readily to solution by the Equivalent Static method. Such a solution is given in §(17), the results being shown in Table V.

Table V. Cylindrical Klystron.

Values of $\lambda = \frac{2\pi A}{\underline{k}A}$ with $b = 2A$, $B = \frac{1}{4}A$, $a = 2A(2A)8A$

		$a = 2A$	$4A$	$6A$	$8A$
Equivalent Static Method		13.43	20.07	24.45	28.34
Variational Method	1st Approximation	13.57	20.27	24.54	28.53
	2nd Approximation	14.09	20.82	25.18	29.15
Goddard		13.99	20.72	25.07	29.01
Warnecke (21) page 226		14.20	21.00		

Values of the potential ψ. Formulae for the exact values of the function in the two regions are given by expressions (100). Making use of (102) and (104) with a suitable choice of N, the infinite series in (100) are found to converge very rapidly except when $r = A$. This is because the expression

$$\frac{M_1(\alpha_n r)}{M_1'(\alpha_n a)} = \frac{I_1(\beta_n r)K_0(\beta_n a) + I_0(\beta_n a)K_1(\beta_n r)}{I_0(\beta_n A)K_0(\beta_n a) - I_0(\beta_n a)K_0(\beta_n A)} \sim -\frac{K_1(\beta_n r)}{K_0(\beta_n A)}$$

$$\sim 0 \quad \text{when } n > 4 \text{ if } r > A,$$

$$\sim 1 \quad \text{when } n > 4 \text{ if } r = A. \tag{106}$$

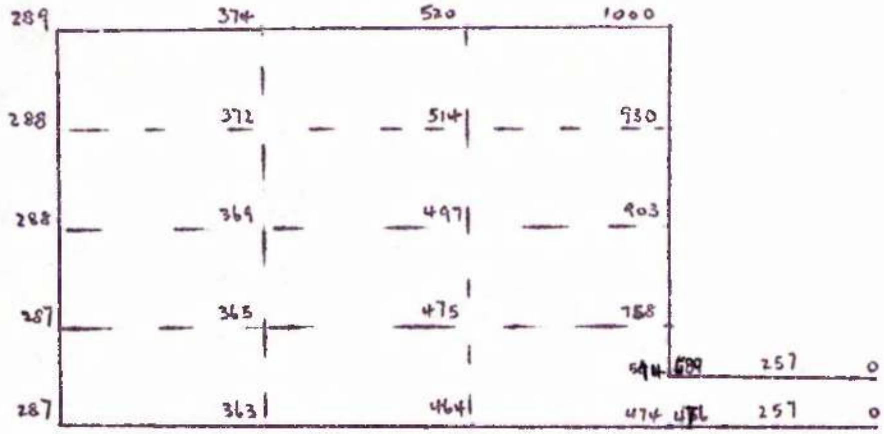
In the case when $r = A$, if we approximate this ratio by unity when $r > 4$, we are required to sum the infinite series

$$\sum_{n=5}^{\infty} \frac{1}{n^{4/3}} J_{1/4}(n\pi/8)\cos(n\pi z/b). \tag{107}$$

This is effectively the same as (66b) and can be evaluated by methods used for the earlier series. Results for the configuration $b = 2A$, $B = \frac{1}{4}A$, $a = 4A$ are shown in Figure XVI

Figure XVI Cylindrical Klystron. Values of the potential ψ .

$b = 2A, \quad B = \frac{1}{4}A, \quad a = 4A$



§ 11. An infinite cylinder of circular cross-section, with a slit.

A problem of considerable importance is that of the Helmholtz resonator, the simplest form of which is a hollow sphere with a circular aperture. Before considering this we will look at a slightly simpler problem, that of finding the eigenvalues of an infinite cylinder of circular cross-section with a slit. This is effectively the Helmholtz resonator reduced to two dimensions, and is somewhat similar to the Simple problem of § 3.

We require a potential function ϕ to satisfy the conditions

$$\left.\begin{array}{ll} \text{(i)} & \nabla^2\phi + k^2\phi = 0, \\ \text{(ii)} & \phi = 0,\ r = a,\ -\beta \leq \theta \leq \beta, \\ \text{(iii)} & \dfrac{\partial\phi}{\partial r} = 0,\ r = a,\ \beta \leq \theta \leq 2\pi - \beta, \\ \text{(iv)} & \phi \text{ finite at } r = 0. \end{array}\right. \qquad (108)$$

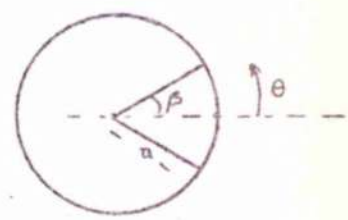


Figure XVII

Writing (i) in polar form,

$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + k^2\phi = 0,$$

a solution, obtained by separation of variables, satisfying (i) and (iv), is

$$\phi = \sum_{n=0}^{\infty}{}' \epsilon_n' \left(A_n \cos n\theta + B_n \sin n\theta\right) J_n(kr)/J_n(ka) \qquad (109)$$

We can again obtain approximations to the eigenvalues in either of two ways:

<u>Case I</u> Consider a function Φ defined by

$$\Phi(x,y) = \sum_{n=0}^{\infty}{}' \epsilon_n' \left(A_n \cos n\theta + b_n \sin n\theta\right) J_n(k_+ r)/J_n(k_+ a) \qquad (110)$$

so that Φ satisfies $\nabla^2\Phi + k_+^{\,2}\,\Phi = 0$, and therefore condition (i). If we assume that

$$\left.\begin{aligned} \Phi &= 0, \quad r = a, \quad -\beta \leq \theta \leq \beta, \\ &= F(\theta), \quad r = a, \quad \beta \leq \theta \leq 2\pi - \beta \end{aligned}\right\} \qquad (111)$$

then Φ satisfies condition (ii). From (110), (111),

$$A_n = \frac{1}{\pi}\int_{\beta}^{2\pi-\beta} F(\theta)\cos n\theta \, d\theta = \frac{1}{\pi} I_1(n), \text{ say},$$

$$B_n = \frac{1}{\pi}\int_{\beta}^{2\pi-\beta} F(\theta)\sin n\theta \, d\theta = \frac{1}{\pi} I_2(n).$$

Substituting these results in (110), taking the derivative with respect to r and setting r = a, we find

$$\left(\frac{\partial\Phi}{\partial r}\right)_{r=a} = \frac{k}{\pi}\sum_{n=0}^{\infty} \epsilon_n'\left(I_1(n)\cos n\theta + I_2(n)\sin n\theta\right) J_n'(k_+a)/J_n(k_+a)$$

which, by condition (111), is zero. Multiplying by $F(\theta)$ and integrating, we obtain the transcendental equation for the evaluation of k_+,

$$\sum_{n=0}^{\infty} \epsilon_n' \frac{J_n'(k_+a)}{J_n(k_+a)}\left[I_1(n)^2 + I_2(n)^2\right] = 0 \tag{112}$$

Case II. Consider a function Ψ defined by

$$\Psi(x,y) = \sum_{n=0}^{\infty} \epsilon_n'\left(C_n\cos n\theta + D_n \sin n\theta\right) J_n(kr)/J_n(ka). \tag{113}$$

If we assume that $(\partial\Psi/\partial r) = 0$, $r = a$, $\beta \leq \theta \leq 2\pi - \beta$,

$$= G(\theta),\ r = a,\ -\beta \leq \theta \leq \beta \tag{114}$$

we arrive by a similar argument at the transcendental equation for the evaluation of k_-,

$$\sum_{n=0}^{\infty} \epsilon_n' \frac{J_n(k_-a)}{J_n'(k_-a)}\left[I_3^2(n) + I_4^2(n)\right] = 0 \tag{115}$$

where $\displaystyle I_3(n) = \int_{-\beta}^{\beta} G(\theta)\cos n\theta \, d\theta$,

$$I_4(n) = \int_{-\beta}^{\beta} G(\theta)\sin n\theta \, d\theta.$$

In choosing trial functions $F(\theta)$ and $G(\theta)$ we are concerned with the immediate neighborhood of the points $r = a$, $\theta = \pm\beta$, where there is effectively the same sharp edge configuration as in the Simple problem of § 3. Therefore, if we first rearrange $I_1(n)$ and $I_2(n)$ into the forms

$$I_1(n) = (-)^n \int_{-\gamma}^{\gamma} F(\theta) \cos n\theta \, d\theta ,$$

$$I_2(n) = (-)^{n+1} \int_{-\gamma}^{\gamma} F(\theta) \sin n\theta \, d\theta ,$$

where $\gamma = \pi - \beta$, we can choose trial functions

$$F(\theta) = C(\gamma^2 - \theta^2)^{\frac{1}{2}},$$

$$G(\theta) = D(\beta^2 - \theta^2)^{-\frac{1}{2}},$$

exactly as before. It follows at once that $I_2(n) = I_4(n) = 0$, and

$$I_1(n) = (-)^n \frac{C\pi\gamma}{n} J_1(n\gamma), \qquad I_1(0) = C\gamma^2\pi/2,$$

$$I_3(n) = D\pi J_0(n\beta), \qquad I_3(0) = D\pi .$$

The exact transcendental equations (112) (115) are therefore replaced by the approximate equations

$$\frac{1}{2}\frac{\gamma^2}{4}\frac{J_0'(ka)}{J_0(ka)} + \sum_{n=1}^{\infty} \frac{J_n'(ka)}{J_n(ka)} \frac{1}{n^2} J_1^2(n\gamma) = 0 , \tag{116}$$

$$\frac{1}{2}\frac{J_0(ka)}{J_0'(ka)} + \sum_{n=1}^{\infty} \frac{J_n(ka)}{J_n'(ka)} J_0^2(n\beta) = 0 . \tag{117}$$

<u>The limiting cases are:</u>

a) $\frac{\partial\phi}{\partial r} = 0$, $r = a$ for all θ.

That is, from (109), $\sum_{n=0}^{\infty} \varepsilon_n'(A_n \cos n\theta + B_n \sin n\theta) J_n'(ka) = 0$.

For the lowest eigenvalue k_1, this gives $J_0'(k_1a) = 0$ i.e. $J_1(k_1a) = 0$.

Therefore $k_1a = 0$.

b) $\phi = 0$, $r = a$ for all θ

That is, $\sum_{n=0}^{\infty} \varepsilon_n'(A_n \cos n\theta + B_n \sin n\theta) J_n(ka) = 0$.

For the lowest eigenvalue k_1, $J_0(k_1a) = 0$, $k_1a = 2.405$.

Thus for intermediate cases, $0 < k_1a < 2.405$

The summation of the infinite series of (116)(117), using the approximation

$$J_n'(ka)/J_n(ka) \sim \frac{n}{ka} - \frac{ka/2}{n+1} , \quad n > N , \tag{118}$$

is very similar to earlier summations. Resulting values of ka are given for Cases I and II in Table VI.

Table VI Infinite cylinder with a slit. Values of ka with $\beta = 0(\pi/8)\pi$.

β	0	$\pi/8$	$2\pi/8$	$3\pi/8$	$4\pi/8$	$5\pi/8$	$6\pi/8$	$7\pi/8$	π
Case I	-	-	.880	1.048	1.245	1.490	1.875	2.290	2.405
Case II	0	.715	.875	1.046	1.257	1.471	1.835	-	-

The analysis of Case II breaks down for $\beta > \frac{6\pi}{8}$. This is because $J_1{}'(ka)/J_1(ka) \longrightarrow 0$ as $ka \longrightarrow 1.840$, so that the sum of the series in (117) becomes infinite. This fact becomes significant when we consider the problem of upper and lower bounds.

Upper and Lower Bounds for the lowest eigenvalue k_1

The question as to whether $k_- \leq k_1 \leq k_+$ depends on the signs of the function $p_n(k)$ and $p_n{}'(k)$, where

$$p_n(k) = J_n{}'(ka)/J_n(ka) = \frac{n}{ka} - \frac{J_{n+1}(ka)}{J_n(ka)} \qquad (119)$$

where

$$\frac{J_{n+1}(x)}{J_n(x)} = \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{n+2k+1}}{k!\,\Gamma(n+k+2)} \Bigg/ \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

$$= \frac{x}{2} \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{2k}}{k!\,(n+k+1)\,\Gamma(n+k+1)} \Bigg/ \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{2k}}{k!\,\Gamma(n+k+1)} .$$

$$< \frac{x}{2} \cdot \frac{1}{n+1}$$

Therefore $p_n(k) > \frac{n}{ka} - \frac{ka}{2} \cdot \frac{1}{n+1}$

$$> 0 \text{ if } (ka)^2 < 2n(n+1).$$

This is so if $n > 1$ since $ka < 2{\cdot}5$

When $n = 1$, we require $(ka)^2 < 4$. Although this is not so for all values of the lowest eigenvalue in the range $0 \leq \beta \leq \pi$, it is so for the range of values of β over which we are able to compare k_- and k_+, as shown in Table VI.

Also, $$p_n'(k) = \frac{d}{dk}\left(p_n(k)\right) = -\frac{na}{(ka)^2} - \frac{J_{n+1}' J_n - J_{n+1} J_n'}{J_n^2} \cdot a$$

$$= -a\left[\left(\frac{J_{n+1}}{J_n}\right)^2 + \frac{n}{(ka)^2} + 1 - \frac{2n+1}{ka}\,\frac{J_{n+1}}{J_n}\right]. \qquad (120)$$

But $$1 - \frac{2n+1}{ka}\,\frac{J_{n+1}(ka)}{J_n(ka)} > 1 - \frac{2n+1}{ka}\cdot\frac{ka}{2n+2} > 0$$

Therefore $\quad p_n'(k) < 0 \;, \; n \geqslant 0.$

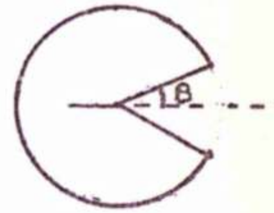
Hence cases I and II give upper and lower bounds for k_1 over the range of values of β where they can both be obtained.

§ 12. The Helmholtz Resonator

An early investigation of the absorbtion and scattering of a plane wave incident upon a spherical shell with a circular aperture is given by Rayleigh in (24). Solving the equation $\nabla^2 \psi + k^2 \psi = 0$ on such a shell with boundary condition $\partial \psi / \partial r = 0$, $r = a$, except on the aperture, an expression is obtained for ψ in terms of the normal sound velocity $u = \partial \psi / \partial r$ in the aperture. An approximate form for u is assumed and the continuity of pressure at the aperture is examined.

An approximate solution of the eigenvalue problem for the simplest Helmholtz resonator is $f_o = \frac{c}{2\pi}\sqrt{K/V_o}$, where f_o is the lowest normal frequency, V_o is the volume of the resonator and K is the diameter of the aperture. This result which is approximately true only when K is small compared with V_o, is given, for example, in (34) and (35).

We consider a hollow sphere of radius a on whose surface



(i) $\partial\phi/\partial r = 0, \quad r = a, \quad \beta \leq \theta \leq \pi$

(ii) $\phi = 0, \quad r = a, \quad 0 \leq \theta \leq \beta$.

A solution of $\nabla^2\phi + k^2\phi = 0$ which satisfies these boundary conditions is

$$\phi = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) j_n(kr) \tag{121}$$

in terms of Legendre polynomials and spherical Bessel functions.

<u>Case I.</u> Consider a function $\Phi(x,y) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) j_n(k_+ r)$.

If we write

$$\begin{aligned} \Phi &= 0, \quad r = a, \quad \gamma \leq \theta \leq \pi , \\ &= F(\theta), \quad r = a, \quad 0 \leq \theta \leq \gamma , \end{aligned} \tag{122}$$

where $\gamma = \pi - \beta$.

Then Φ satisfies $\nabla^2 \Phi + k_+^2 \Phi = 0$.

From (121) and (122), multiplying by $P_n(\cos\theta)\sin\theta$ and integrating with respect to θ from $-\pi$ to $+\pi$, and using the result

$$\int_{-\pi}^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta = \delta_m^n \frac{2}{2n+1} ,$$

$$A_n = \frac{1}{j_n(ka)} \frac{2n+1}{2} \int_0^{\gamma} F(t)\, P_n(\cos t) \sin t \, dt .$$

Then to satisfy condition (i),

$$\left(\frac{\partial \Phi}{\partial r}\right)_{r=a} = k \sum_{n=0}^{\infty} \frac{j_n'(ka)}{j_n(ka)} \frac{2n+1}{2} P_n(\cos\theta) \int_0^{\gamma} F(t)\, P_n(\cos t) \sin t \, dt$$

$$= 0 \quad , \; 0 \leq \theta \leq \gamma .$$

Multiplying by $F(\theta)\sin\theta$ and integrating from 0 to γ with respect to θ, we obtain a transcendental equation for the determination of k_+,

$$\sum_{n=0}^{\infty} \frac{j_n'(ka)}{j_n(ka)} \frac{2n+1}{2} \left[I_1(n) \right]^2 = 0 \quad , \quad 0 \leq \theta \leq \gamma \tag{123}$$

where $I_1(n) = \int_0^{\gamma} F(t)\, P_n(\cos t) \sin t \, dt .$

Case II. The corresponding transcendental equation for a function

$$\Psi(x,y) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)\, j_n(kr) \tag{124}$$

satisfying

$$\left(\frac{\partial \Psi}{\partial r}\right)_{r=a} = 0 , \qquad \beta \leq \theta \leq \pi ,$$
$$= G(\theta) , \qquad 0 \leq \theta \leq \beta , \tag{125}$$

and therefore $\nabla^2 \Psi + k_-^2 \Psi = 0$,

is

$$\sum_{n=0}^{\infty} \frac{j_n(ka)}{j_n'(ka)} \frac{2n+1}{2} \left[I_2(n) \right]^2 = 0 , \quad 0 \leq \theta \leq \beta , \tag{126}$$

where $I_2(n) = \int_0^{\beta} G(t)\, P_n(\cos t) \sin t \, dt .$

For the approximation functions $F(\theta)$ and $G(\theta)$ we choose

$$F(\theta) = A(\cos\theta - \cos\gamma)^{1/2} \quad , \quad \gamma = \pi - \beta ,$$

$$G(\theta) = B(\cos\theta - \cos\beta)^{-1/2} .$$

The choice is determined equally by the fact that these functions fit the edge conditions and that the resulting integrals can be evaluated simply. To evaluate the integrals we use the results given in (8) p. 1326.

$$\sqrt{2}\sum_{0}^{\infty}\sin\left[(n+\tfrac{1}{2})\beta\right]P_n(\cos\theta) = \begin{cases} 0, & 0\le\beta\le\theta\le\pi, \\ 1/(\cos\theta-\cos\beta)^{\frac{1}{2}} & 0\le\theta\le\beta\le\pi, \end{cases} \tag{127}$$

$$\int_0^{\pi} P_m(\cos t)\,P_n(\cos t)\sin t\,dt = \frac{2}{2n+1}\,\delta_m^n . \tag{128}$$

Then with $F(\theta) = (\cos\theta - \cos\gamma)^{1/2}$,

$$I_1(n) = \int_0^{\gamma}(\cos t-\cos\gamma)^{\frac{1}{2}}P_n(\cos t)\sin t\,dt$$

$$= \int_0^{\gamma}\frac{P_n(\cos t)\sin t\cos t\,dt}{(\cos t-\cos\gamma)^{\frac{1}{2}}} - \cos\gamma\int_0^{\gamma}\frac{P_n(\cos t)\sin t\,dt}{(\cos t-\cos\gamma)^{1/2}}, \tag{129}$$

where, using (127), (128),

$$\int_0^{\gamma}\frac{P_n(\cos t)\sin t\,dt}{(\cos t-\cos\gamma)^{1/2}} = \sqrt{2}\sum_{m=0}^{\infty}\sin\left[(n+\tfrac{1}{2})\beta\right]\int_0^{\gamma}P_n(\cos t)P_m(\cos t)\sin t\,dt$$

$$= \sqrt{2}\sin\left[(n+\tfrac{1}{2})\beta\right]\cdot\frac{2}{2n+1}. \tag{130}$$

To evaluate the 1st integral on the right of (129), we use (127) and make the substitution $\cos t = z$ to obtain

$$\sqrt{2}\sum_{m=0}^{\infty}\sin\left[(m+\tfrac{1}{2})\gamma\right]\int_{-1}^{1}P_n(z)\,z\,P_m(z)\,dz \tag{131}$$

where $\quad zP_m(z) = \frac{1}{2m+1}\left[(m+1)P_{m+1}(z) + mP_{m-1}(z)\right].$

Hence using (128), (131) reduces to

$$\frac{2\sqrt{2}}{2n+1}\left[\frac{n}{2n-1}\sin\left[(n-\tfrac{1}{2})\gamma\right] + \left(\frac{n+1}{2n+3}\right)\sin\left[(n+\tfrac{3}{2})\gamma\right]\right].$$

The form for $I_1(n)$ is therefore

$$I_1(n) = \frac{2\sqrt{2}}{2n+1}\Bigg[\frac{n}{2n-1}\sin\left[(n-\tfrac{1}{2})\gamma\right] + \frac{n+1}{2n+3}\sin\left[(n+\tfrac{3}{2})\gamma\right] - \cos\gamma\sin\left[(n+\tfrac{1}{2})\gamma\right]\Bigg]$$

$$= \frac{\sqrt{2}}{2n+1}\left[\frac{1}{2n-1}\sin\left[(n-\tfrac{1}{2})\gamma\right] - \frac{1}{2n+3}\sin\left[(n+\tfrac{3}{2})\gamma\right]\right].$$

With $G(\theta) = B(\cos\theta - \cos\beta)^{-1/2}$,

$$I_1(n) = \int_0^\beta (\cos\theta - \cos\beta)^{-\frac{1}{2}} P_n(\cos t)\sin t\,dt$$

$$= \sqrt{2}\,\sin\left[(n+\tfrac{1}{2})\beta\right] \cdot \frac{2}{2n+1}\,.$$

The approximations to the transcendental equations (123), (126) are therefore

$$\sum_{n=0}^{\infty}{}' \frac{1}{2n+1}\frac{j_n'(ka)}{j_n(ka)}\left[\frac{1}{2n-1}\sin[(n-\tfrac{1}{2})\gamma] - \frac{1}{2n+3}\sin\left[(n+\tfrac{3}{2})\gamma\right]\right]^2 = 0\,, \tag{132}$$

$$\sum_{n=0}^{\infty}{}' \frac{1}{2n+1}\frac{j_n(ka)}{j_n'(ka)}\sin^2\left[(n+\tfrac{1}{2})\beta\right] = 0\,. \tag{133}$$

Solution of these equations for k_+a and k_-a follows the same lines as in previous problems. It is tedious but straightforward. For large n we replace $j_n'(ka)/j_n(ka)$ by $J_{n-\frac{1}{2}}(ka)/J_{n+\frac{1}{2}}(ka) - (n+1)/ka$ which, when n is large enough, is approximately equal to $\frac{2}{ka}(n+\frac{1}{2}) - \frac{n+1}{ka} = \frac{n}{ka}$. (134)

The limiting cases for the lowest eigenvalue are:

a) $\frac{\partial\phi}{\partial r} = 0$ for all θ, $r = a$.

That is $\frac{d}{dr}\left[\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}(kr)\right]_{r=a} = 0\,.$

For the lowest eigenvalue, n = 0, so that we require

$$\frac{d}{dr}\left[\frac{1}{\sqrt{r}}J_{\frac{1}{2}}(kr)\right]_{r=a} = 0$$

which reduces to tan ka = ka, the first solution being $k_1a = 0$.

b) $\phi = 0$ for all θ, $r = a$.

That is, from (121), $\sum_{n=0}^{\infty}{}' A_n P_n(\cos\theta)\, j_n(ka) = 0\,.$

Therefore for the lowest eigenvalue, $j_0(ka) = 0$,

that is, $\sqrt{\frac{2}{\pi ka}}\sin ka = 0$ of which the first solution is $k_1a = \pi$.

Therefore for values of β intermediate between 0 and π

$$0 < k_1a < \pi.$$

Values of $\frac{ka}{\pi}$ obtained by solving (123), (126), are given in Table VII.

Table VII Helmholtz resonator, radius a, $\beta = 0(\pi/8)\pi$.

Values of ka/π .

β	0	$\pi/8$	$2\pi/8$	$3\pi/8$	$4\pi/8$	$5\pi/8$	$6\pi/8$	$7\pi/8$	π
Case I	-	.224	.290	.386	.500	.658	.875	.990	1.000
Case II	0	.200	.272	.385	.497	.636	$\sim$.589	-	-

Upper and Lower limits for the lowest eigenvalue.

Whether or not the two cases give upper and lower limits for k depends on the function $p_n(k) = j_n'(ka)/j_n(ka)$ where

$$j_n'(ka) = \left[\frac{d}{dr} j_n(kr) \right]_{r=a} .$$

a) $p_n(k) = j_n'(ka)/j_n(ka)$. The Spherical Bessel function $j_n(ka)$ can be expressed in terms of ordinary Bessel functions of the first kind,

$$j_n(ka) = \left(\frac{2}{\pi ka} \right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(ka) \tag{135}$$

The first zero of $J_{1/2}(ka)$, and therefore of $j_0(ka)$ is at $ka = \pi$ and the first zeros of $J_{n+1/2}(ka)$ and therefore of $j_n(ka)$, where n is an integer greater than zero, is at $ka > \pi$. See, for example, (30) p. 479. Therefore in the range of values of the lowest eigenvalue ka in which we are interested, that is $0 < k < \pi$, it follows that $j_n(ka)$ is greater than zero. Thus for $p_n(k)$ to be greater than zero we require $j_n'(ka) > 0$. We shall require the recurrence relationships

$$j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x), \tag{136a}$$

$$\frac{n}{x} j_n(x) - j_{n+1}(x) = j_n'(x). \tag{136b}$$

Using (135) and the series expansion $J_n(x) = \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{2k+n}}{k!\,\Gamma(k+n+1)}$, it follows from (136) that

$$j_n'(x) = \sqrt{\frac{\pi}{2x}} \cdot \left(\frac{x}{2}\right)^{n-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{2k}}{k!\,\Gamma(k+n+3/2)} \left(\frac{n}{2} - \frac{x^2}{4(k+n+3/2)}\right)$$

which is greater than zero if $\frac{n}{2} > \frac{x^2}{4(k+n+3/2)}$.

That is, $j_n'(x) > 0$ if $2n(n+3/2) > x^2$, $0 < x < \pi$. This will be so if $n \geq 2$. When $n = 1$ it is easy to show that

$$j_1'(x) > 0 \;, \quad 0 \leq x < L \;, \qquad (137a)$$

$$< 0 \;, \quad L < x \leq \pi \;, \qquad (137b)$$

where $L \sim 2.08$. Thus $p_n(k)$ is greater than zero, $n \geq 1$, provided $0 \leq x < L$ and is less than zero, $L < x \leq \pi$.

(b) $$\frac{1}{a}\frac{d}{dk}\left[p_n(k)\right] = \frac{j_n(ka)\,j_n''(ka) - j_n'^2(ka)}{j_n^2(ka)}$$

which we require to be negative, $n \geq 0$.

From (136) $$j_n''(x) = j_n(x)\left[\frac{n^2-n}{x^2} - 1\right] + \frac{2}{x}\,j_{n+1}(x).$$

Therefore, putting $ka = x$,

$$\frac{1}{a}\frac{d}{dx}\left[p_n(x)\right] = j_n(x)\,j_{n+1}(x)\left[\frac{2(n+1)}{x} - \left(\frac{n}{x^2}+1\right)\frac{j_n(x)}{j_{n+1}(x)} - \frac{j_{n+1}(x)}{j_n(x)}\right]. \qquad (138)$$

Since $j_r(x) > 0$, $r \geq 0$, $0 \leq x < \pi$, we require to show that

$$\frac{2n+1}{x} < \left(\frac{n}{x^2}+1\right)\frac{j_n(x)}{j_{n+1}(x)} + \frac{j_{n+1}(x)}{j_n(x)}.$$

But $$\frac{j_n(x)}{j_{n+1}(x)} = \frac{J_{n+\frac{1}{2}}(x)}{J_{n+\frac{3}{2}}(x)} = \frac{2}{x} \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{2k}}{k!\,\Gamma(k+n+3/2)} \Bigg/ \sum_{k=0}^{\infty} \frac{(-)^k (x/2)^{2k}}{k!\,\Gamma(k+n+5/2)}$$

$$> \frac{2}{x}\left[\frac{1}{\Gamma(n+3/2)} - \frac{(x/2)^2}{\Gamma(n+5/2)}\right]\Gamma(n+5/2), \quad n \geqslant 0,\ x > 0$$

$$= \frac{2}{x}\left[n + 3/2 - (x/2)^2\right]$$

$$\geqslant \frac{2}{x}\left[n + \tfrac{1}{2}\right], \quad 0 \leq x \leq 2.$$

Therefore $\frac{d}{dk}\left[p_n(k)\right] < 0$ if $\frac{2(n+1)}{x} < (n+1/2)(\frac{n}{x^2}+1)$, $0 < x \leq 2$, since $\frac{j_{n+1}(x)}{j_n(x)} > 0$. This is true when $n \geq 2$.

It is easy to show numerically that $\frac{d}{dk}\left[p_n(k)\right] < 0$ if $n = 0, 1$, $0 \leq x \leq 2$.

We have thus shown that $p_n(k) > 0$, $n \geq 1$ $0 \leq x \leq 2$,

$$p_n{}^0(k) < 0, \quad n \geq 0 \quad 0 \leq x \leq 2,$$

and we have upper and lower bounds for k for this range of values of x. Expression (45) shows that a possible alternative set of conditions to (46) could be

$$p_n(k) < 0, \quad n \geq 1 \tag{139a}$$

$$p_n{}^0(k) > 0, \quad n \geq 0 \tag{139b}$$

However we have already seen that

$$j_n{}^0(x) > 0 \quad n \geq 2 \quad 0 \leq x \leq \pi$$

and $\quad j_1{}^0(x) < 0 \qquad L < x \leq \pi, \quad L \sim 2.08.$

Thus neither conditions (46) nor (139) give an answer as to whether we have upper and lower bounds for the lowest eigenvalue in the range $L < x \leq \pi$. This does not mean that we do not indeed have such bounds, merely that we have not established that we do.

§ 13 Diffraction by a plane angular sector.

An important problem which proves amenable to the present variational approach is that of a plane angular sector. Its importance in diffraction theory and in electrostatics is due to the fact that such a sector is the simplest flat strip with a sharp corner.

An approximate solution of this problem is given by Noble in (31) using a method developed in (36). Eigenvalue equations are obtained in the form of dual series equations. These are approximated by ignoring part of the infinite series after which an unknown function $f(\phi)$ is introduced. The dual series are reduced to a single integral equation which can be solved exactly for $f(\phi)$. Approximations to the eigenvalue can now be found and a method for improving these approximations is given. Other references to this problem are given in (31).

We consider such a sector, containing an angle 2α, lying in $0 < r < \infty$, $\theta = \pi/2$, $-\alpha < \phi < \alpha$, where r, θ, ϕ are spherical polar coordinates. We shall consider only those eigenfunctions which are symmetrical about $\phi = 0, \pi$, and about $\theta = \pi/2$, and we shall require a potential function Θ to satisfy the boundary conditions

(i) $\Theta = 0, \quad \theta = \pi/2, \quad 0 \leq \phi < \alpha, \quad \text{all } r,$

(ii) $\dfrac{\partial \Theta}{\partial \theta} = 0, \quad \theta = \pi/2, \quad \alpha < \phi \leq \pi, \quad \text{all } r,$

Figure XVIII

and the equation $\nabla^2 \phi + k^2 \phi = 0$. (140)

We express (140) in spherical coordinates and solve it by separation of variables. Making the substitution $\Theta = R(r)\, S(\theta,\phi)$ in the equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} + k^2 r^2 \psi = 0, \tag{141}$$

we obtain the equations

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[r^2k^2 - \mu(\mu+1)\right] R = 0, \tag{142}$$

and $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} + \mu(\mu+1)S = 0, \tag{143}$$

where (142) has solutions

$$R = r^{-\frac{1}{2}}\left[A\, J_{\mu+\frac{1}{2}}(kr) + B\, Y_{\mu+\frac{1}{2}}(kr)\right].$$

Again, putting $S = U(\theta)V(\phi)$ in (143) we find

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{dU}{d\theta}\right) + \left[\mu(\mu+1)\sin^2\theta - m^2\right] U = 0, \tag{144}$$

and $$\frac{d^2V}{d\phi^2} + m^2 V = 0. \tag{145}$$

For eigenvalues symmetrical about $\phi = 0, \pi$ and about $\theta = \pi/2$, (145) yields

$$V = C \cos m\phi \tag{146}$$

Substitution of $\cos\theta = z$ reduces (144) to Legendre's equation with solutions

$$U = P_\mu^m(\cos\theta). \tag{147}$$

Thus the general solution of (143) satisfying the required symmetry conditions is

$$S(\theta, \phi) = \sum_{m=0}^{\infty}{}' \alpha_m P_\mu^m(\cos\theta)\cos m\phi, \tag{148}$$

which we shall require to solve for eigenvalues μ.

Using (148), the boundary conditions (i) and (ii) can be expressed

$$\text{(i)}' \quad \sum_{m=0}^{\infty} a_m \cos m\phi = 0, \quad 0 \le \phi < \alpha,$$

$$\text{(ii)}' \quad \sum_{m=0}^{\infty} a_m' \cos m\phi = 0, \quad \alpha < \phi \le \pi,$$

where $$a_m = \alpha_m P_\mu^m(0), \tag{149a}$$

and $$a_m' = \alpha_m \frac{d}{d\theta}\left[P_\mu^m(\cos\theta)\right]_{\theta=\pi/2}. \tag{149b}$$

Case I Consider a function Φ defined by

$$\Phi(x,y) = \sum_{m=0}^{\infty} \alpha_m P_{\mu_+}^m(\cos\theta)\cos m\phi \tag{150}$$

so that Φ satisfies $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \mu_+(\mu_++1)\Phi = 0,$

where μ_+ is an approximation to μ.

If we assume that

$$\bar{\Phi} = \sum_{m=0}^{\infty} A_m \cos m\phi = 0 \quad , \beta \le \phi < \pi, \quad \theta = \pi/2 \qquad (151)$$
$$= F(\phi) \; , \; 0 \le \phi \le \beta \; , \quad \theta = \pi/2$$

where $A_m = \alpha_m P^m_{\mu+}(0)$, $F(\phi)$ is an unknown function of ϕ, and $\beta = \pi - \alpha$,

it follows at once that

$$A_m = \left(\frac{2}{\pi}\right) \int_0^{\beta} F(\phi) \cos m\phi \, d\phi, \qquad (152a)$$

and $$A_0 = \left(\frac{1}{\pi}\right) \int_0^{\beta} F(\phi) \, d\phi . \qquad (152b)$$

If then $\bar{\Phi}$ also satisfies $\partial\bar{\Phi}/\partial\theta = 0$, $\theta = \pi/2$, $\alpha \le \phi \le \pi$, that is if

$$\frac{\partial\bar{\Phi}}{\partial\theta} = \sum_{m=0}^{\infty} A_m' \cos m\phi = 0 \; , \; 0 \le \phi \le \beta \, ,$$

where $A_m' = \frac{d}{d\theta}\left[\alpha_m P^m_{\mu+}(\cos\theta)\right]_{\theta = \pi/2}$ we obtain the integral formulation

$$\sum_{m=0}^{\infty} \varepsilon_m' \frac{A_m'}{A_m} I_1(m) \cos m\phi = 0, \quad 0 \le \phi \le \beta, \qquad (153)$$

and hence the variational form

$$\sum_{m=0}^{\infty} \varepsilon_m' \frac{A_m'}{A_m} I_1^2(m) = 0 \; , \qquad (154)$$

where $I_1(m) = \int_0^{\beta} F(\phi) \cos m\phi \, d\phi$, $m = 0, 1, 2, \ldots$

Case II In a similar way we define a function $\Psi(\theta,\phi)$ in terms of μ_-,

another approximation to μ, and obtain a formulation corresponding to (153)

$$\sum_{m=0}^{\infty} \varepsilon_m' \frac{B_m'}{B_m} I_2(m) \cos m\phi = 0 \qquad 0 \le \phi \le \alpha \, , \; \alpha = \pi - \beta .$$

where $B_m = \alpha_m P^m_{\mu-}(0)$, and a variational form

$$\sum_{m=0}^{\infty} \varepsilon_m' \frac{B_m'}{B_m} I_2^2(m) = 0 \qquad (155)$$

where $I_2(m) = \int_0^{\alpha} G(\phi) \cos m\phi \, d\phi$ $m = 0, 1, 2, \ldots$ is in terms of an unknown function $G(\phi)$.

The limiting cases for the lowest eigenvalue are

a) When $\alpha = \pi$, $\Theta(r,\theta,\phi)$ is zero everywhere in the plane $\theta = \pi/2$ so that from (150) $\sum_{m=0}^{\infty} \alpha_m P^m_{\mu+}(0) \cos m\phi = 0$ for all ϕ.

The lowest eigenvalue is therefore given by $P_\mu(0) = 0$. The solution which is finite at $\theta = 0, \pi$, and odd about $\theta = \pi/2$, is $\mu = 1$.

b) When $\alpha = 0$, $\partial\Theta/\partial\theta$ is zero everywhere in the plane $\theta = \pi/2$, giving $\mu = 0$.

Numerical Results.

The case when $\alpha = \pi/2$ corresponds to the case of a half plane for which $\mu = 1/2$ exactly. It has been conjectured (31) that $\mu = \alpha/\pi$ holds for $0 \leq \alpha \leq \pi$, and this is certainly consistent with those three known values $\mu = 0, 1/2, 1$ when $\alpha = 0, \pi/2, \pi$ respectively. It proves not to be correct but we can conveniently use the linear relationship to give a set of first approximations to μ for specified values of α in this interval. A set of values of μ for $\alpha = 0\ (\pi/10)\pi$ is obtained in (31) and, for purposes of comparison we choose the same set of values of α.

Suitable trial functions will be

$$F(\phi) = C(\beta^2 - \phi^2)^{1/2},$$

$$G(\phi) = D(\alpha^2 - \phi^2)^{-\frac{1}{2}},$$

giving

$$\left.\begin{aligned} I_1(m) &= \frac{1}{2m}\pi\beta\, J_1(m\beta), \qquad & m \geq 1 \\ &= \pi\beta^2/4, & m = 0 \end{aligned}\right\} \tag{156a}$$

$$I_2(m) = (\pi/2)J_0(m\alpha) \qquad m \geq 0. \tag{156b}$$

We therefore replace the exact equations (154), (155) using (156) by

$$\frac{A_0}{A_0'} = -\frac{\beta^2}{8}\left[\sum_{m=1}^{\infty}{}' \frac{A_m'}{A_m} \frac{J_1^2(m\beta)}{m^2}\right]^{-1}, \tag{157}$$

and

$$\frac{B_0}{B_0'} = -2\sum_{m=1}^{\infty}{}' \frac{B_m}{B_m'} J_0^2(m\beta), \tag{158}$$

where $\dfrac{A_m}{A_m'} = -2\left[\dfrac{P_{\mu_+}^{m}(\cos\theta)}{\dfrac{d}{d\theta}\left\{P_{\mu_+}^{m}(\cos\theta)\right\}}\right]_{\theta=\pi/2}$ and $\dfrac{B_m}{B_m'}$ is the same but with μ_+ replaced by μ_-.

These expressions can be put into a more convenient form, using equations (20) and (23) of (23) Vol. I p. 145.

After some manipulation we obtain

$$\frac{A_m}{A_m'} = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}\mu_+\right)\Gamma\left(\frac{1}{2}m-\frac{1}{2}\mu_+\right)}{\Gamma\left(1+\frac{1}{2}m+\frac{1}{2}\mu_+\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}\mu_+\right)} . \tag{159}$$

For large x, $\Gamma(x)/\Gamma(x+\frac{1}{2}) \sim 1/x^{1/2}$ and therefore

$$\begin{aligned} \frac{A_m}{A_m'} &\sim \frac{1}{\left(1+\frac{1}{2}m+\frac{1}{2}\mu_+\right)^{\frac{1}{2}}} \cdot \frac{1}{\left(\frac{1}{2}m-\frac{1}{2}\mu\right)^{\frac{1}{2}}} \\ &= (2/m)\left(1+\frac{1+\mu_+}{m}\right)^{-\frac{1}{2}}\left(1-\frac{\mu_+}{m}\right)^{-\frac{1}{2}} \\ &\sim \frac{2}{m}\left(1-\frac{1}{2m}\right) \end{aligned} \tag{160}$$

We therefore approximate (157) and (158), using (160), by

$$\frac{A_0(\mu)}{A_0'(\mu)} = -\frac{\beta^2}{8}\left[\sum_{n=1}^{N}\frac{A_m'(\mu)}{A_m(\mu)}\cdot\frac{J_1^2(n\beta)}{n^2} + \sum_{N+1}^{\infty}\frac{1}{2m-1}J_1^2(m\beta)\right]^{-1} \tag{161}$$

and

$$\frac{A_0(\mu)}{A_0'(\mu)} = -2\sum_{n=1}^{N}\frac{B_m(\mu)}{B_m'(\mu)}J_0^2(m\alpha) - 4\sum_{N+1}^{\infty}\frac{1}{m}\left(1-\frac{1}{2m}\right)J_0^2(m\alpha) \tag{162}$$

where, in this particular problem we choose $N = 6$.

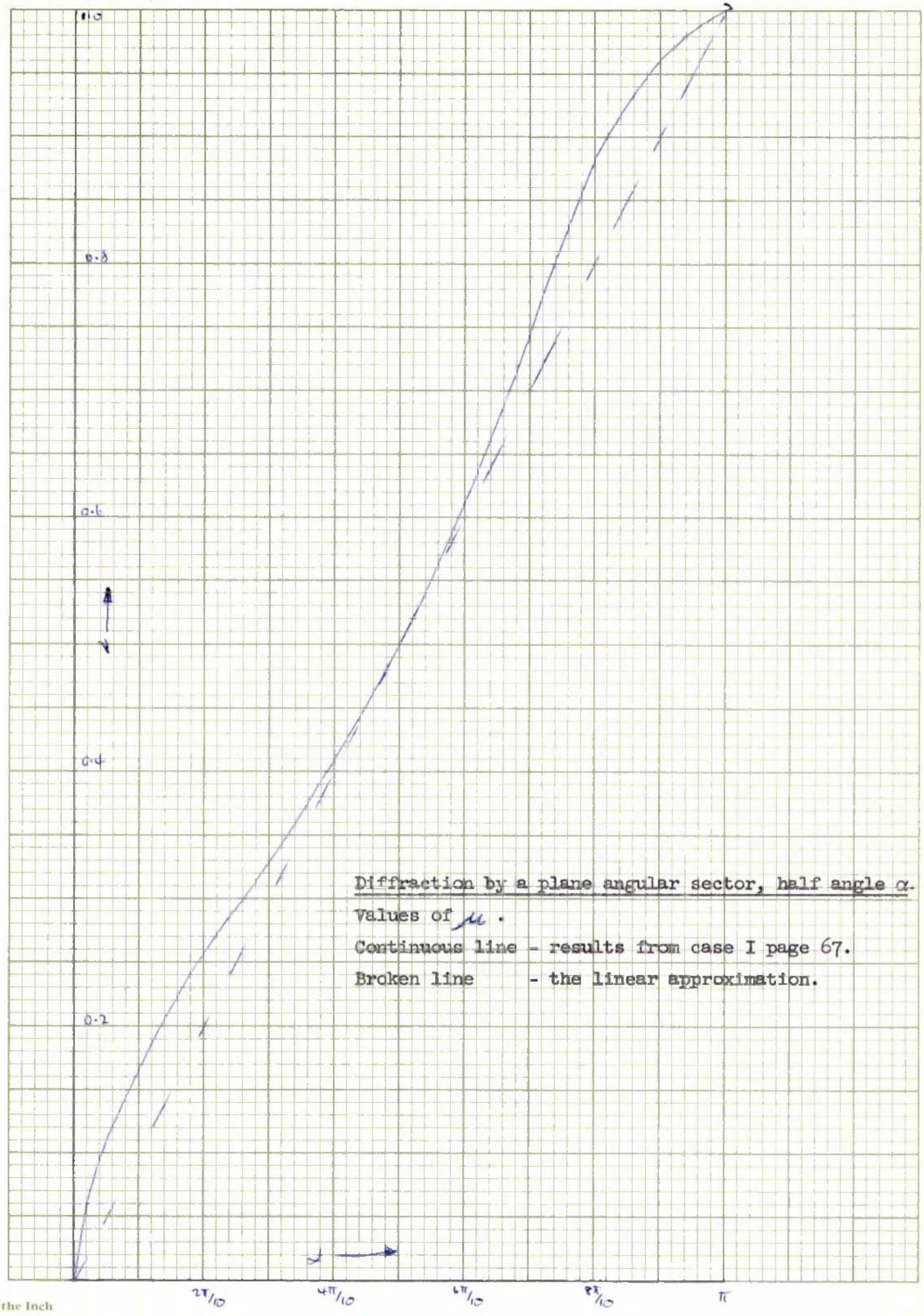
We compare results with the 'linear variation' conjecture and with those of Noble given in (31).

Table VIII Diffraction by a plane angular sector, half angle $\alpha = 0(\pi/10)\pi$.

Values of μ.

α	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
Linear Variation	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
Noble	0	.195	.264	.332	.410	.505	.620	.753	.892	.975	1
Case I	-	196_0	263_6	331_8	409_8	505_4	612_7	745_0	881_0	969_4	1
Case II	0	193_8	263_2	331_5	409_7	505_2	611_3	743_0	837_2	-	-

Agreement with Noble's results is good for all values of α. If, however, we compare the results of Case I for large values of α, that is for values of α where Case I should be most accurate, with the corresponding figures of Noble the differences are slightly greater than elsewhere.


1.0
0.8
0.6
0.4
0.2
μ
α
$2\pi/10$
$4\pi/10$
$6\pi/10$
$8\pi/10$
π
Diffraction by a plane angular sector, half angle α.
Values of μ .
Continuous line - results from case I page 67.
Broken line - the linear approximation.

This may be due to Noble's having used the approximation $A_m/A_m{}' = 2/m$ where we have used the slightly more accurate one $A_m/A_m{}' = \frac{2}{m}(1 - \frac{1}{2m})$.

Upper and Lower bounds for the lowest eigenvalue.

$$\text{If } p_m(\mu) = \frac{a_m'}{a_m} = \frac{\Gamma(1+\frac{1}{2}m+\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}m-\frac{1}{2}\mu)}, \tag{163}$$

then cases I and II give upper and lower bounds for μ if

$$p_m(\mu) > 0, \quad m \geq 1, \quad 0 < \mu < 1, \tag{164a}$$

$$\frac{d}{d\mu}[p_m(\mu)] < 0, \quad m \geq 0, \quad 0 < \mu < 1. \tag{164b}$$

If we write $a = \frac{1}{2}(m + \mu)$, $b = \frac{1}{2}(m - \mu)$ we note that a and b are both greater than zero when $m \geq 1$, $0 < \mu < 1$.

Therefore $p_m(\mu) = \dfrac{\Gamma(1+a)\,\Gamma(\frac{1}{2}+b)}{\Gamma(\frac{1}{2}+a)\,\Gamma(b)} > 0, \quad m \geq 1, \quad 0 < \mu < 1,$

Since $\Gamma(x) > 0$ when $x > 0$. Condition (164a) is therefore satisfied.

$$\text{Also } 2\frac{d}{d\mu}\left(p_m(\mu)\right) = \left\{ \begin{aligned} &\Gamma(\tfrac{1}{2}+a)\Gamma(b)\left[\Gamma'(1+a)\Gamma(\tfrac{1}{2}+b) - \Gamma(1+a)\Gamma'(\tfrac{1}{2}+b)\right] \\ &\quad - \Gamma(1+a)\Gamma(\tfrac{1}{2}+b)\left[\Gamma'(\tfrac{1}{2}+a)\Gamma(b) - \Gamma(\tfrac{1}{2}+a)\Gamma'(b)\right] \end{aligned} \right\} \left[\Gamma(\tfrac{1}{2}+a)\Gamma(b)\right]^{-2}.$$

If we write $\Gamma'(z) = \psi(z)\Gamma(z)$, this reduces to

$$2\frac{d}{d\mu}\left(p_m(\mu)\right) = \frac{\Gamma(1+a)\Gamma(\frac{1}{2}+b)}{\Gamma(\frac{1}{2}+a)\,\Gamma(b)}\left[\psi(1+a) - \psi(\tfrac{1}{2}+a) - \psi(\tfrac{1}{2}+b) + \psi(b)\right]. \tag{165}$$

When $m \geq 1$ all the gamma functions in (165) are greater than zero and therefore

$$\frac{d}{d\mu}\left(p_m(\mu)\right) < 0 \text{ if } E = \left[\psi(1+a) - \psi(\tfrac{1}{2}+a)\right] - \left[\psi(\tfrac{1}{2}+b) - \psi(b)\right] < 0.$$

But by (23) Vol. I, equation (1) p. 15 and (1), (2) p.20,

$$\psi(\tfrac{1}{2}+\tfrac{1}{2}x) - \psi(\tfrac{1}{2}x) = 2\int_0^1 \frac{t^{x-1}}{1+t}\,dt, \quad x > 0.$$

Therefore $E = 2\displaystyle\int_0^1 (1+t)^{-1}(t^{2a} - t^{2b-1})dt$ which is less than zero if $t^{2a} - t^{2b-1} < 0$, $0 < t < 1$. This will be so if $2a > 2b - 1$, that is, if $2\mu + 1 > 0$ which is true if $0 < \mu < 1$.

When m = 0 , a = $\mu/2$, b = $-\mu/2$ and therefore,

$$2\frac{d}{d\mu}\left[P_0(\mu)\right] = \frac{\Gamma(1+\mu/2)\Gamma(\frac{1}{2}-\mu/2)}{\Gamma(\frac{1}{2}+\mu/2)\Gamma(-\mu/2)}\left[\Psi(1+\tfrac{\mu}{2}) - \Psi(\tfrac{1}{2}-\mu/2) - \Psi(\tfrac{1}{2}+\mu/2) + \Psi(-\mu/2)\right]$$

$$= R(\mu)\left\{\left[\Psi(1+\mu/2) - \Psi(\tfrac{1}{2}+\mu/2)\right] - \left[\Psi(\tfrac{1}{2}-\mu/2) - \Psi(-\mu/2)\right]\right\}$$

where $R(\mu)$ is a negative constant, $0 < \mu < 1$.

Using (23) Vol. I, equation (8) P. 16 and (1), (2), P(20),

$$\Psi(\tfrac{1}{2}-\mu/2) - \Psi(-\mu/2) = \left[\Psi(\tfrac{3}{2}-\tfrac{\mu}{2}) - \frac{1}{\frac{1}{2}-\mu/2}\right] - \left[\Psi(1-\mu/2) + \frac{1}{\mu/2}\right]$$

$$= 2\int_0^1 \frac{t^{1-\mu}}{1+t}\,dt - \frac{2}{2-\mu}.$$

Therefore $\quad 2\frac{d}{d\mu}\left[P_0(\mu)\right] = 2R(\mu)\left\{\int_0^1 \frac{t^{\mu}-t^{1-\mu}}{1+t}\,dt + \frac{1}{2-\mu}\right\}.$

But $\quad \int_0^1 \frac{t}{1+t}\,dt < \int_0^1 \frac{t^{a}}{1+t}\,dt < \int_0^1 \frac{1}{1+t}\,dt, \quad 0 < a < 1,$

That is $\quad 1-\log 2 < \int_0^1 \frac{t^{a}}{1+t}\,dt < \log 2, \quad 0 < a < 1.$

Also $\quad \frac{1}{2-\mu} > \frac{1}{2}, \quad 0 < \mu < 1.$

It follows that

$$\left\{\int_0^1 \frac{t^{\mu}-t^{1-\mu}}{1+t}\,dt + \frac{1}{2-\mu}\right\} > \frac{1}{2} + (1-2\log 2) > 0,$$

and therefore $\quad \frac{d}{d\mu}\left[P_0(\mu)\right] < 0, \quad 0 < \mu < 1$

and generally, $\quad \frac{d}{d\mu}\left[P_m(\mu)\right] < 0, \; m \geq 0, \; 0 < \mu < 1.$

Both conditions (164) being satisfied, cases I and II do indeed give upper and lower bounds for μ.

§ 14 A Partially Clamped circular plate.

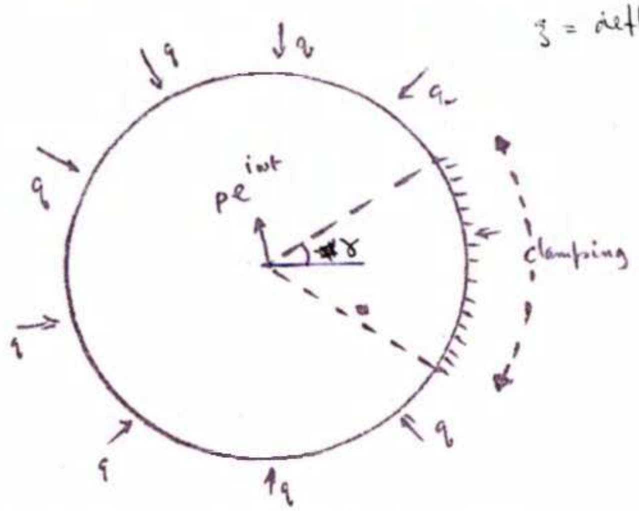
As a final problem we examine the vibrations of a circular plate of radius a, clamped along part of its edge and simply supported on the remainder. If the plate is subject to a ~~compressive load q~~ uniformly distributed constant compressive load q per unit length of the circumference acting in the middle surface of the plate and to a periodic load $p(\rho,\theta)e^{i\omega t}$ perpendicular to the plate, the general differential equation of the problem is

$$N\nabla^4 \zeta(\rho,\theta,t) + \mu h \frac{\partial^2 \zeta}{\partial t^2} + q\nabla^2 \zeta = p(\rho,\theta)e^{i\omega t} \qquad (175)$$

where N, μ, h are physical constants of the system, and $\rho = r/a$.

ζ = deflection of plate

Figure XIX



An analysis of the bending and buckling of plates is given in (25) where solutions of problems of this type are approached by finite difference methods. Weinberger (26) has developed finite difference methods to give both upper and lower bounds, the upper bound being obtained by using a grid which is smaller than the region R of the problem, and the lower by using a grid which is slightly larger. These processes are further developed in, for instance, (27) and (28).

This specific problem is discussed in detail in (29) by a method involving the approximate solution of an infinite number of simultaneous linear homogeneous algebraic equations. The eigenvalues obtained in (29) do not agree with those obtained here.

If we make the substitutions $\bar{W} = we^{i\omega t} = w(\rho, \theta)e^{i\omega t}$, and $\frac{\mu h \omega^2}{N} = k^2$, and consider free vibrations, putting $p = 0$, we can rewrite (175) as

$$\left[\nabla^4 + (q/N)\nabla^2 - k^2 \right] w = 0 \tag{176}$$

where $\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}$.

We specify the boundary conditions

(i) $w = 0, \quad 0 \leq \theta \leq 2\pi, \quad \rho = 1,$

(ii) $\frac{\partial w}{\partial \rho} = 0, \quad -\gamma < \theta < \gamma, \quad \rho = 1$ (due to the clamping),

(iii) $\left[\frac{\partial^2 w}{\partial \rho^2} + \nu\left(\frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \theta^2} \right) \right] = 0, \quad \gamma \leq \theta \leq 2\pi - \gamma,$

where ν is Poisson's ratio.

If we apply a Fourier transform

$$W_s(n\rho) = \int_0^{2\pi} w(\theta, \rho) \sin n\theta \, d\theta .$$

$$W_c(n\rho) = \int_0^{2\pi} w(\theta, \rho) \cos n\theta \, d\theta ,$$

$$w(\theta, \rho) = \frac{1}{\pi} \sum_{n=0}^{\infty} \varepsilon_n' \left[W_c(n, \rho) \cos n\theta + W_s(n\rho) \sin n\theta \right] \tag{177}$$

to (176) we obtain

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2} \right]^2 W_t + \frac{q}{N}\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2} \right] W_t - k^2 W_t = 0 \tag{178}$$

where t stands for either s or c.

As a solution, we try $W_t = A_n J_n(\rho K)$ where K is some constant, and $J_n(\rho K)$ satisfies $\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + (K^2 - \frac{n^2}{\rho^2}) \right] J_n(K\rho) = 0$

Then $\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2} \right] J_n(K\rho) = -K^2 J_n(K\rho)$

and hence $\nabla^4 J_n(K\rho) = K^4 J_n(K\rho)$.

(178) therefore reduces to

$$\left[K^4 - (q/N)K^2 - k^2 \right] J_n(k\rho) = 0 \tag{178a}$$

which determines values of K:

$$K^2 = \alpha^2 = \tfrac{1}{2}\left[q/N + \left(q^2/N^2 + 4k^2 \right)^{\frac{1}{2}} \right] \tag{179a}$$

and
$$K^2 = -\beta^2 = \tfrac{-1}{2}\left[\left(q^2/N^2 + 4k^2 \right)^{\frac{1}{2}} - q/N \right] \tag{179b}$$

Solutions of (178) can thus be written in the form

$$W_t = A_t(n) J_n(\rho\alpha) + B_t(n) J_n(\rho i \beta).$$

Condition (i) requires that $W_t = 0$ on $\rho = 1$, and therefore

$$A_t(n) J_n(\alpha) = -B_t(n) J_n(i\beta) = C_t(n) \quad (\text{say}). \tag{180}$$

Substitution of (180) in (177) gives the equation

$$w = \frac{1}{\pi} \sum_{n=0}^{\infty} \varepsilon_n' \left[\frac{J_n(\rho\alpha)}{J_n(\alpha)} - \frac{J_n(\rho i\beta)}{J_n(i\beta)} \right] \left[C_c(n) \cos n\theta + C_s(n) \sin n\theta \right]. \tag{181}$$

Applying conditions (ii) and (iii) successively to (181) we finally obtain the two equations

$$\sum_{n=0}^{\infty} \varepsilon_n' L_n \left[C_c(n) \cos n\theta + C_s(n) \sin n\theta \right] = 0 \qquad -\gamma \le \theta \le \gamma, \ \rho = 1, \tag{182a}$$

$$\sum_{n=0}^{\infty} \varepsilon_n' M_n \left[C_c(n) \cos n\theta + C_s(n) \sin n\theta \right] = 0 \qquad \gamma \le \theta \le 2\pi - \gamma, \ \rho = 1, \tag{182b}$$

where
$$L_n = \alpha \frac{J_n'(\alpha)}{J_n(\alpha)} - i\beta \frac{J_n'(i\beta)}{J_n(i\beta)} = \alpha \frac{J_n'(\alpha)}{J_n(\alpha)} - \beta \frac{I_n'(\beta)}{I_n(\beta)}, \tag{183a}$$

and
$$M_n = \alpha^2 \frac{J_n''(\alpha)}{J(\alpha)} - \beta^2 \frac{I_n''(\beta)}{I_n(\beta)} + \nu L_n. \tag{183b}$$

<u>Case I</u> We define a function $\Phi(\rho, \theta)$ satisfying the conditions

(i)° $\Phi = 0 \qquad 0 \le \theta \le 2\pi, \ \rho = 1$

(ii)° $\partial\Phi/\partial\rho = 0 \qquad \alpha \le \theta \le 2\pi - \alpha, \ \rho = 1$ where $\alpha = \pi - \gamma$,

(iii)° $\partial^2\Phi/\partial\rho^2 + \nu\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2} \right) = 0, \qquad -\alpha \le \theta \le \alpha,$

(iv)° $\left(\nabla^4 + (\ell/n)\nabla^2 - k_1^2 \right) \Phi = 0.$

(Note: We are using α in this problem with two different meanings, firstly in (179a) and secondly in (ii)° and (iii)° above. This does not lead to any confusion).

If we further let $\frac{\partial \Phi}{\partial \rho} = F(\theta)$, $-\alpha \leq \theta \leq \alpha$, $\rho = 1$ where $F(\theta)$ is an unknown function of θ, then (182a) gives

$$\sum_{n=0}^{\infty} \varepsilon_n' L_n \left[C_c(n) \cos n\theta + C_s(n) \sin n\theta \right] = 0 \qquad \alpha \leq \theta \leq 2\pi - \alpha, \ \rho = 1, \tag{184}$$
$$= F(\theta) \qquad -\alpha \leq \theta \leq \alpha, \ \rho = 1.$$

Solving for $C_c(n)$ and $C_s(n)$ and substituting in (182b) we obtain the familiar form of the transcendental equation to be solved for k_+,

$$\sum_{n=0}^{\infty} \varepsilon_n' \frac{M_n}{L_n} \left[I_1^2(n) + I_2^2(n) \right] = 0 \tag{185}$$

where $I_1(n) = \int_{-\alpha}^{\alpha} F(\xi) \cos n\xi \, d\xi$,

$$I_2(n) = \int_{-\alpha}^{\alpha} F(\xi) \sin n\xi \, d\xi .$$

<u>Case II</u> The equation for k_- corresponding to (185) is

$$\sum_{n=0}^{\infty} \varepsilon_n' \frac{L_n}{M_n} \left[I_3^2(n) + I_4^2(n) \right] = 0 \tag{186}$$

where $I_3(n) = \int_{-\gamma}^{\gamma} G(\xi) \cos n\xi \, d\xi$,

$$I_4(n) = \int_{-\gamma}^{\gamma} G(\xi) \sin n\xi \, d\xi ,$$

in terms of the unknown function $G(\theta)$, $-\gamma \leq \theta \leq \gamma$.

<u>Approximations to L_n and M_n:</u>

In order to be able to sum the infinite series in (185), (186) we require approximations for the ratio L_n/M_n for large values of n. From expressions (183) we find that
$$\frac{M_n}{L_n} = \frac{\alpha^2 J_n''(\alpha) I_n(\beta) - \beta^2 I_n''(\beta) J_n(\alpha)}{\alpha J_n'(\alpha) I_n(\beta) - \beta I_n'(\beta) J_n(\alpha)} + \nu$$
which, after some manipulation, reduces to

$$\frac{M_n}{L_n} = (\nu - 1) + (\alpha^2 + \beta^2)\left[\alpha \frac{J_{n+1}(\alpha)}{J_n(\alpha)} + \beta \frac{I_{n+1}(\beta)}{J_n(\beta)} \right]^{-1}. \tag{187}$$

We shall consider the following [redacted] problems arising from (175):

1) $p = 0$, $q = 0$, and therefore from (179), $\alpha = \beta = \sqrt{k} = \ell$, say.
This is the problem of free oscillation with no compressive load.

2) $p = 0$, $q \neq 0$, $\omega = 0$ and therefore $k = 0$. Then $\alpha^2 = q/N$, $\beta = 0$.
This is the buckling problem for the plate.

3) $p \neq 0$, $\omega = 0$, $q = 0$. Then $\alpha = \beta = 0$. This is the problem of the static bending of the plate.

For n large compared with x we use the asymptotic forms, given in (30) or (31),

$$J_n(x) \sim \frac{(\frac{1}{2}x)^n}{n!}\left[1 - \frac{(\frac{1}{2}x)^2}{n+1}\right],$$

$$I_n(x) \sim \frac{(\frac{1}{2}x)^n}{n!}\left[1 + \frac{(\frac{1}{2}x)^2}{n+1}\right].$$

Then

$$\frac{J_{n+1}(x)}{J_n(x)} \sim \frac{\frac{1}{2}x}{n+1}\left[1 + \frac{x^2}{4(n+1)(n+2)}\right], \tag{188a}$$

$$\frac{I_{n+1}(x)}{I_n(x)} \sim \frac{\frac{1}{2}x}{n+1}\left[1 - \frac{x^2}{4(n+1)(n+2)}\right]. \tag{188b}$$

Using (188) in the three different problems we find:

Problem 1)

$$\frac{M_n}{L_n} = \nu - 1 + 2\ell\left[\frac{J_{n+1}(\ell)}{J_n(\ell)} + \frac{I_{n+1}(\ell)}{I_n(\ell)}\right]^{-1}$$

$$\sim \nu - 1 + 2\ell\left[\frac{\ell}{n+1}\right]^{-1} = 2n + \nu + 1.$$

Problem 2)

$$\frac{M_n}{L_n} = \nu - 1 + \alpha\left[\frac{J_{n+1}(\alpha)}{J_n(\alpha)}\right]^{-1}$$

$$\sim \nu - 1 + \alpha\left[\frac{\frac{1}{2}\alpha}{n+1}\right]^{-1} = 2n + \nu + 1.$$

Problem 3)

$$\frac{M_n}{L_n} = \nu - 1 + \lim_{\alpha,\beta \to 0}\left\{(\alpha^2 + \beta^2)\left[\alpha\frac{J_{n+1}(\alpha)}{J_n(\alpha)} + \beta\frac{I_{n+1}(\beta)}{I_n(\beta)}\right]^{-1}\right\}$$

$$= \nu - 1 + \lim_{\alpha \to 0}\left\{2\alpha\left[\frac{J_{n+1}(\alpha)}{J_n \alpha} + \frac{I_{n+1}(\alpha)}{I_n(\alpha)}\right]^{-1}\right\}$$

$$= \nu - 1 + \lim_{\alpha \to 0}\left\{2\alpha\left[\frac{\alpha}{n+1}\right]^{-1}\right\} = 2n + \nu + 1.$$

We shall therefore be able, very conveniently, to approximate the 'tails' of the infinite series in (185) and (186) respectively, for the two [redacted] problems simultaneously. The approximate forms will be

$$\sum_{n=N+1}^{\infty} \frac{M_n}{L_n}\left[I_1(n)^2 + I_2(n)^2 \right] \sim \sum_{n=N+1}^{\infty} (2n+\nu+1)\left[I_1(n)^2 + I_2(n)^2 \right], \tag{189a}$$

and
$$\sum_{n=N+1}^{\infty} \frac{L_n}{M_n}\left[I_3(n)^2 + I_4(n)^2 \right] \sim \sum_{n=N+1}^{\infty} \frac{1}{2n+\nu+1}\left[I_3(n)^2 + I_4(n)^2 \right]. \tag{189b}$$

Numerical results for the partially clamped circular plate.

In choosing suitable trial functions $F(\theta)$ and $G(\theta)$, the configuration of this problem is effectively the same as that in § 11. We can therefore choose

$$F(\theta) = A(\alpha^2-\theta^2)^{\frac{1}{2}}, \quad -\alpha \le \theta \le \alpha, \tag{190a}$$

$$G(\theta) = B(\gamma^2-\theta^2)^{-\frac{1}{2}}, \quad -\gamma \le \theta \le \gamma, \quad \gamma = \pi - \alpha. \tag{190b}$$

The functions being symmetrical about $\theta = 0$, integrals $I_2(n)$ and $I_4(n)$ are all zero and

$$I_1(n) = A\int_{-\alpha}^{\alpha}(\alpha^2-\xi^2)^{\frac{1}{2}}\cos n\xi\, d\xi = \frac{A\pi\alpha}{2n}J_1(n\alpha), \quad n \geq 1, \tag{191a}$$
$$= A\pi\alpha^2/4, \quad n = 0,$$

$$I_3(n) = B\int_{-\gamma}^{\gamma}(\gamma^2-\xi^2)^{-\frac{1}{2}}\cos n\xi\, d\xi = B\frac{\pi}{2}J_0(n\gamma), \quad n \geq 0. \tag{191b}$$

Finally substituting from (189), (190), (191) in (185), (186) and using asymptotic approximations for $J_0(n\alpha)$ and $J_1(n\gamma)$ for large n, the approximation equations to be solved numerically are

$$\frac{\alpha^2}{8}\frac{M_0}{L_0} + \sum_{n=1}^{N}\frac{M_n}{L_n}\frac{1}{n^2}J_1^2(n\alpha) + \frac{1}{\pi\alpha}\sum_{n=N+1}^{\infty}\frac{2n+\nu+1}{n^3}\left[1-\sin 2n\alpha\right] = 0 \tag{192a}$$

$$\frac{1}{2}\frac{L_0}{M_0} + \sum_{n=1}^{N}\frac{L_n}{M_n}J_0^2(n\gamma) + \frac{1}{\pi\gamma}\sum_{n=N+1}^{\infty}\frac{1}{2n+\nu+1}\cdot\frac{1}{n}\left[1+\sin 2n\gamma\right] = 0 \tag{192b}$$

Limiting cases.

a) If the plate is clamped along the whole of its edge, $\alpha = 0$,

$$\frac{\partial w}{\partial \rho} = 0 \quad \text{for all } \theta, \ \rho = 1.$$

Then (184) reduces to

$$\sum_{n=0}^{\infty} \varepsilon_n' L_n \left[C_c(n) \cos n\theta + C_s(n) \sin n\theta \right] = 0 \quad \text{for all } \theta. \qquad (193a)$$

The lowest eigenvalue will be given by $L_0 = 0$, that is

$$\alpha \frac{J_1(\alpha)}{J_0(\alpha)} + \beta \frac{I_1(\beta)}{I_0(\beta)} = 0 \qquad (194a)$$

b) If the plate is simply supported all around with no clamping, $\gamma = 0$,

$$\frac{\partial^2 w}{\partial \rho^2} + \nu \left(\frac{1}{\rho} \frac{\partial w}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 w}{\partial \theta^2} \right) = 0 \quad \text{for all } \theta, \rho = 1,$$

which gives

$$\sum_{n=0}^{\infty} \varepsilon_n' M_n \left[C_c(n) \cos n\theta + C_s(n) \sin n\theta \right] = 0 \quad \text{for all } \theta. \qquad (193b)$$

The lowest eigenvalue will be given by $M_0 = 0$ where, from (187), (194b)

$$\frac{M_0}{L_0} = (\nu - 1) + (\alpha^2 + \beta^2) \left[\alpha \frac{J_1(\alpha)}{J_0(\alpha)} + \beta \frac{I_1(\beta)}{I_0(\beta)} \right]^{-1}.$$

Problem 1) Partially clamped circular plate. Free oscillations with no compressive load. ($p = 0$, $q = 0$).

Solutions of the limiting case equations (194a) and (194b) when $\alpha = \beta = \sqrt{k} = \ell$ are $\ell = 3.196$ and $\ell = 2.205$ respectively. (194c)

Solutions of (192), putting Poisson's Ratio $\nu = 1/4$, are given in Table VIII.

Table VIII. Free oscillations of a partially clamped circular plate. Lowest eigenvalue.

Values of $\ell = \sqrt{k}$ where $k^2 = \mu h \omega^2/N$, $\nu = 1/4$.

γ	0	$\pi/8$	$2\pi/8$	$3\pi/8$	$4\pi/8$	$5\pi/8$	$6\pi/8$	$7\pi/8$	π
Case I	-	2.423	2.520	2.623	2.740	2.869	3.020	3.144	3.196
Case II	2.205	2.417	2.517	2.620	2.735	2.857	2.980	3.021	-

The result in (29) for $\gamma = 2\pi/8$ is $\ell = 3.98$. This lies outside the range for the lowest eigenvalue given by the limiting cases (194c). An investigation of the next lowest symmetrical eigenvalue gives the range of values of ℓ as $3.720 \leq \ell \leq 4.611$ for $0 \leq \gamma \leq \pi$. It seemed possible that the value $\ell = 3.98$ for $\gamma = 2\pi/8$ might, in fact, be this next eigenvalue. We therefore have investigated these second eigenvalues more closely.

The limiting cases are obtained from (193) with $L_1 = 0$, $M_1 = 0$. The resulting values are $\ell = 4.611$, $\ell = 3.720$ respectively. In choosing trial functions $F(\theta)$ and $G(\theta)$ we require them to be zero at $\theta = 0$ and antisymmetric about $\theta = 0$. We therefore choose

$$F(\theta) = A\,\frac{\theta}{\alpha}\left(\alpha^2 - \theta^2\right)^{\frac{1}{2}}, \quad -\alpha \leq \theta \leq \alpha,$$

$$G(\theta) = B\,\frac{\theta}{\gamma}\left(\gamma^2 - \theta^2\right)^{-\frac{1}{2}}, \quad -\gamma \leq \theta \leq \gamma.$$

These being odd functions, the integrals $I_1(n)$ and $I_3(n)$ are now zero and equations (185), (186) reduce to

$$\sum_{n=1}^{\infty} \frac{M_n}{L_n}\,\frac{1}{n^2}\,J_2^{\,2}(n\alpha) = 0\,,$$

$$\sum_{n=1}^{\infty} \frac{L_n}{M_n}\,J_1^{\,2}(n\gamma) = 0\,.$$

Using the same approximations as in the earlier solution we obtain the values for ℓ, given in Table IX.

<u>Table IX</u>. Free oscillations of a partially clamped plate. Second symmetrical eigenvalues. Values of $\ell = \sqrt{k}$ where $k^2 = \mu h \omega^2/N$, $\nu = 1/4$

γ	0	$\pi/8$	$2\pi/8$	$3\pi/8$	$4\pi/8$	$5\pi/8$	$6\pi/8$	$7\pi/8$	π
Case I	—	—	3.855	3.991	4.191	4.411	4.532	4.606	4.611
Case II	3.72	3.78	3.857	4.000	4.140	4.363	-	-	-

The value for $\gamma = \pi/4$, $\ell \sim 3.86$, still does not agree well with the result in (29), $\ell \sim 3.98$.

Upper and Lower bounds for problem 1).

We require $p_n(k) > 0$, $n \geq 1$, (195a)

$$\frac{d}{dk}\left[p_n(k)\right] < 0, \quad n \geq 0 \tag{195b}$$

where
$$p_n(k) = \nu - 1 + 2\ell\left[\frac{J_{n+1}(\ell)}{J_n(\ell)} + \frac{I_{n+1}(\ell)}{I_n(\ell)}\right]^{-1},$$
$$\ell = \sqrt{k}, \quad 2.2 < \ell < 3.2.$$

Using the series expansions

$$J_n(\ell) = (\ell/2)^n \sum_{k=0}^{\infty} \frac{(-)^k (\ell/2)^{2k}}{k!\,(k+n)!} \quad ; \quad I_n(\ell) = (\ell/2)^n \sum_{k=0}^{\infty} \frac{(\ell/2)^{2k}}{k!\,(k+n)!} \tag{196}$$

it follows quickly that

$$\frac{J_{n+1}(\ell)}{J_n(\ell)} < \frac{\ell/2}{n+1-(\ell/2)^2} \quad ; \quad \frac{I_{n+1}(\ell)}{I_n(\ell)} < \frac{\ell/2}{n+1},$$

and therefore that

$$Q(\ell) \equiv \frac{J_{n+1}(\ell)}{J_n(\ell)} + \frac{I_{n+1}(\ell)}{I_n(\ell)} < \frac{\ell}{n+1-(\ell/2)^2}$$

Hence
$$p_n(k) = \nu - 1 + 2\ell/Q(\ell) \tag{197}$$
$$> \nu - 1 + 2\left[n+1-(\ell/2)^2\right]$$
$$> \nu - 1 + 2\left[n+1-2.56\right] \quad \text{since } 2.2 < \ell < 3.2$$
$$> 0 \quad \text{when } n \geq 2,\ \nu = 1/4.$$

It is easy to show numerically that $p_1(k) > 0$, $2.2 < \ell < 3.2$.

Therefore condition (195a) is satisfied.

To investigate (195b) we use expansions (196) to obtain the forms

$$\frac{2(n+1)}{\ell}\cdot\frac{J_{n+1}(\ell)}{J_n(\ell)} = 1 + \frac{(\ell/2)^2}{(n+1)(n+2)} + \frac{2(\ell/2)^4}{(n+1)^2(n+2)(n+3)} + \cancel{\frac{(\ell/2)^6}{O(n^6)}} + O\left(\frac{1}{n^6}\right), \tag{198a}$$

$$\frac{2(n+1)}{\ell}\cdot\frac{I_{n+1}(\ell)}{I_n(\ell)} = 1 - \frac{(\ell/2)^2}{(n+1)(n+2)} + \frac{2(\ell/2)^4}{(n+1)^2(n+2)(n+3)} + \cancel{\frac{(\ell/2)^6}{O(n^6)}} + O\left(\frac{1}{n^6}\right). \tag{198b}$$

Differentiating (197) with respect to k, we find

$$2\sqrt{k}\ \frac{d}{dk}\left[p_n(k)\right] = \frac{2}{Q^2}\left(Q - \ell Q'\right)$$

where $Q' = \dfrac{dQ}{d\ell} = 2 + \left[\dfrac{J_{n+1}(\ell)}{J_n(\ell)}\right]^2 - \left[\dfrac{I_{n+1}(\ell)}{I_n(\ell)}\right]^2 - \dfrac{2n+1}{\ell}Q$

Therefore $\dfrac{d}{dk}\left[p_n(k)\right] < 0$ if $\ell Q'/Q - 1 > 0$.

But $$\ell Q'/Q - 1 = \frac{2\ell}{Q} + \ell\left[\frac{J_{n+1}(\ell)}{J_n(\ell)} - \frac{I_{n+1}(\ell)}{I_n(\ell)}\right] - 2(n+1) \qquad (199)$$

where, using (198).

$$\frac{2\ell}{Q} = \frac{2(n+1)}{1 + \dfrac{2(\ell/2)^4}{(n+1)^2(n+2)(n+3)} + \dfrac{(\ell/2)^6}{O(n^5)}} > 2(n+1)\left[1 - \frac{2(\ell/2)^4}{(n+1)^2(n+2)(n+3)}\right], \qquad (200a)$$

$$\ell\left[\frac{J_{n+1}(\ell)}{J_n(\ell)} - \frac{I_{n+1}(\ell)}{I_n(\ell)}\right] > \frac{4(\ell/2)^4}{(n+1)^2(n+2)}, \qquad (200b)$$

and therefore using (200) in (199),

$$\ell Q'/Q - 1 > \frac{8(\ell/2)^4}{(n+1)^2(n+2)(n+3)} > 0 \quad , n \geq 0.$$

Thus condition (195b) is also satisfied, and the Cases I and II do therefore give upper and lower bounds for the lowest eigenvalue.

Problem 2) Partially clamped circular plate. The buckling problem.

$(p = 0,\ q \neq 0,\ \omega = 0,\ \alpha^2 = q/N\ ,\ \epsilon = 0)$.

From (194a, b) the limiting cases for the lowest eigenvalue are given by

$\alpha \dfrac{J_1(\alpha)}{J_o(\alpha)} = 0$ of which the first non-zero solution is $\alpha = 3.8316$,

$$\frac{\alpha}{J_o(\alpha)}\left[J_1(\alpha) - \alpha J_o(\alpha) - \nu J_1(\alpha)\right] = 0 \quad \text{with solution } \alpha = 2.017.$$

Solving equations (192) with the appropriate forms for L_n and M_n, we obtain, over a set of values of the angle γ, values of α and therefore of the ratio q/N at which buckling will take place. The values are given in Table X.

Table X Buckling of a partially clamped circular plate.

Values of $\alpha = (q/N)^{\frac{1}{2}}$ for $\gamma = 0(\pi/8)\pi$.

γ	0	$\pi/8$	$2\pi/8$	$3\pi/8$	$4\pi/8$	$5\pi/8$	$6\pi/8$	$7\pi/8$	π
Case I	--	2.370_9	2.525_0	2.690_2	2.888	3.144	3.471	3.750	3.832
Case II	2.017	2.370_8	2.525_0	2.689_9	2.887	3.142	3.463	--	--

The result for $\gamma = 2\pi/8$ given in (29) page 257, is $\alpha \sim 3.13$.

Upper and lower bounds for problem 2)

To determine whether cases I and II give true upper and lower bounds for α, and therefore for the buckling force q, we require

$$p_n(q) > 0 \quad , \quad n \geq 1, \tag{201}$$

$$\frac{d}{dq}\left[p_n(q)\right] < 0 \quad , \quad n \geq 0, \tag{202}$$

where $\quad p_n(q) = \nu - 1 + \alpha J_n(\alpha)/J_{n+1}(\alpha)$

and $\quad \alpha = (q/N)^{\frac{1}{2}}$.

From p. 78 we have that

$$\alpha J_n(\alpha)/J_{n+1}(\alpha) > 2(n+1) - \alpha^2/2.$$

Condition (201a) is therefore satisfied if

$$2n + \nu + 1 - \alpha^2/2 > 0 \,, \qquad (\nu = \tfrac{1}{4}), \tag{203}$$

where we require $2.017 < \alpha < 3.832$.

It is clear that (203) is satisfied for all values of α in this range if $n > 3$. It is easy to show numerically that (201) is also satisfied for $n = ,2, 3$ for all the given α, and also for $n = 1$ provided $\alpha < 3.608$.

To investigate (202b) we observe that

$$\frac{d}{dq}\left[\rho_n(q)\right] = \frac{\alpha}{2q}\frac{d\rho_n}{d\alpha}$$

$$= \frac{\alpha}{2q}\left[2(n+1)\frac{J_n(\alpha)}{J_{n+1}(\alpha)} - \alpha\left(\frac{J_n^2(\alpha)}{J_{n+1}^2(\alpha)}+1\right)\right]$$

which is less than zero if

$$\frac{\alpha/2}{n+1}\left[\frac{J_n(\alpha)}{J_{n+1}(\alpha)} + \frac{J_{n+1}(\alpha)}{J_n(\alpha)}\right] > 1$$

provided $J_n(\alpha)/J_{n+1}(\alpha) > 0$ in the given range of α, which is so.

From (198a)

$$\frac{\alpha/2}{n+1}\frac{J_{n+1}(\alpha)}{J_n(\alpha)} = \frac{(\alpha/2)^2}{(n+1)^2}\left[1 + \frac{(\alpha/2)^2}{(n+1)(n+2)} + O\left(\frac{1}{n^4}\right)\right] > \frac{(\alpha/2)^2}{(n+1)^2} \tag{204}$$

and $$\frac{\alpha/2}{n+1}\frac{J_n(\alpha)}{J_{n+1}(\alpha)} = \left[1 + \frac{(\alpha/2)^2}{(n+1)(n+2)} + O\left(\frac{1}{n^4}\right)\right]^{-1} > 1 - \frac{(\alpha/2)^2}{(n+1)(n+2)} \tag{205}$$

Therefore, adding (204), (205),

$$\frac{\alpha/2}{n+1}\left[\frac{J_{n+1}}{J_n} + \frac{J_n}{J_{n+1}}\right] > 1 + \frac{(\alpha/2)^2}{n+1}\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= 1 + \frac{(\alpha/2)^2}{(n+1)^2(n+2)}$$

$$> 1, \quad n \geqslant 0$$

Conditions (201), (202) are therefore satisfied over the range of values of for which we can make comparison, and cases I and II do give upper and lower bounds for the buckling force q.

§ 15 Equivalent Static Method

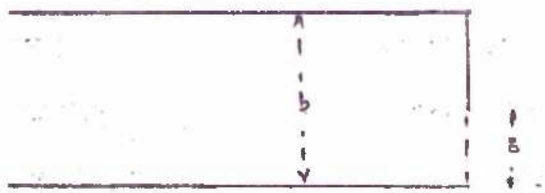
We investigate briefly an alternative approach to the solution of the type of problem already considered, by a method utilizing conformal mapping. The use of conformal mapping implies, of course, a limitation of this approach to problems which are essentially two dimensional, that is, truly two dimensional, or possessing properties of symmetry which enable the dynamic equations to be reduced to two dimensional equations. This approach, proposed by Schwinger (39) is referred to as the equivalent static method. Applying this method to problems of wave propagation and scattering, Schwinger has shown that by comparing the solution of a static problem with that of a dynamic problem of identical cross section, it is possible to obtain in principle a vigorous solution of the dynamic problem. We shall show that by considering a static problem with a cross section similar to that of a dynamic problem we can find approximations to the eigenvalues of the dynamic problem. This approach leads easily to first approximations which could then be used as a starting point for obtaining better approximations by the variational methods already described. We shall apply the method to two of the problems investigated earlier in the paper and compare results.

§16 The Simple Problem by the equivalent static method.

Defining the dynamic problem exactly as before, we obtain the integral equation formulation's (31) and (33), from Cases I and II respectively. To simplify the presentation we shall carry through the analysis using Case II only. The integral equation (33) can be written

$$\frac{1}{k b \tan k a}\int_0^B g(\eta)\,d\eta = 2\sum_{n=1}^{\infty}\frac{1}{\gamma_n b \tanh \gamma_n a}\int_0^B g(\eta)\cos\frac{n\pi\eta}{b}\,d\eta\,\cos\frac{n\pi y}{b}. \tag{206}$$

We choose for an approximately equivalent static problem a semi-infinite channel $x \leq 0$ of width b, with an aperture $0 \leq y \leq B$ in the otherwise closed end at $x = 0$.



In order to obtain equivalence between the static and dynamic problems we must permit the existence of sources at infinity which excite higher static "incident waves" of arbitrary amplitudes. If, then, the boundary conditions are

(i) $\dfrac{\partial\phi}{\partial y} = 0$, $y = 0$, $y = b$

(ii) $\dfrac{\partial\phi}{\partial x} = 0$, $x = 0$, $B < y \leq b$,

(iii) $\phi = 0$, $x = 0$, $0 \leq y < B$,

the stream function ϕ can be expressed in the form

$$\phi = V_0 x + \sum_{n=0}^{\infty} b_n \cos\frac{n\pi y}{b}\, e^{n\pi x/b} \tag{207}$$

where we are permitting a source at $x = -\infty$.

Then
$$\left(\frac{\partial\phi}{\partial x}\right)_{x=0} = V_0 + \frac{\pi}{b}\sum_{n=1}^{\infty} n b_n \cos\frac{n\pi y}{b}.$$

If we now write
$$\left.\begin{aligned}(\partial\phi/\partial x)_{x=0} &= 0, \quad B \leq y \leq b \\ &= f(y), \quad 0 \leq y \leq B\end{aligned}\right\}$$

where $f(y)$ is an unknown function of y, it follows that

$$V_0 = 1/b \int_0^B f(\eta)\,d\eta, \tag{208a}$$

$$b_n = \frac{2}{n\pi}\int_0^B f(\eta)\cos\frac{n\pi\eta}{b}\,d\eta, \tag{208b}$$

and hence, substituting for V_0 and b_n in (207) and putting $x = 0$,

$$\phi_{x=0} = b_0 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^B f(\eta)\cos\frac{n\pi\eta}{b}\,d\eta\cos\frac{n\pi y}{b}$$

$$= 0, \quad 0 \leq y \leq B.$$

that is

$$-\pi b_0 = 2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^B f(\eta)\cos\frac{n\pi\eta}{b}\,d\eta\cos\frac{n\pi y}{b}, \quad 0 \leq y \leq B. \tag{209}$$

The right hand members of (206) and (209) are very similar. If indeed we use our earlier approximations $\tanh\gamma_n a \sim 1$, $\gamma_n b \sim n\pi$, (206) is approximated by

$$\frac{\pi}{k_- b\tan k_- a}\int_0^B g(\eta)\,d\eta = 2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^B g(\eta)\cos\frac{n\pi\eta}{b}\,d\eta\cos\frac{n\pi y}{b} \tag{210}$$

and the right hand members of (209) and (210) are identical in form.

Defining $F(y) = -f(y)/b_0$ and $G(y) = cg(y)$ where c is a constant such that

$$\frac{c}{k_- b\tan k_- a}\int_0^B g(\eta)\,d\eta = 1, \tag{211a}$$

we find $F(y) = G(y)$ and hence, using (208a),

$$\int_0^B G(\eta)\,d\eta = -bV_0/b_0\,. \tag{211b}$$

Thus we obtain from (211) a simple approximation equation for k_-,

$$k_- b\tan k_- a = -bV_0/b_0\,. \tag{212}$$

We have therefore reduced the problem to one of finding values of b_0 and V_0.

We proceed to do so by mapping the region of the original problem conformally into a geometrically simpler region over which a solution can be found by means of complex function theory. The procedure yields valid results since a solution of Laplace's equation in a transformed channel is also a solution in the original channel, if the transformation is conformal.

We find a suitable transformation in two steps, first a transformation which maps the channel periphery in the z-plane into the real axis of the w-plane, second a transformation which maps an infinite channel with parallel sides in the t-plane into the real axis of the w-plane. This gives us a transformation from the z- to the t-plane which maps the original channel into an infinite channel with parallel sides.

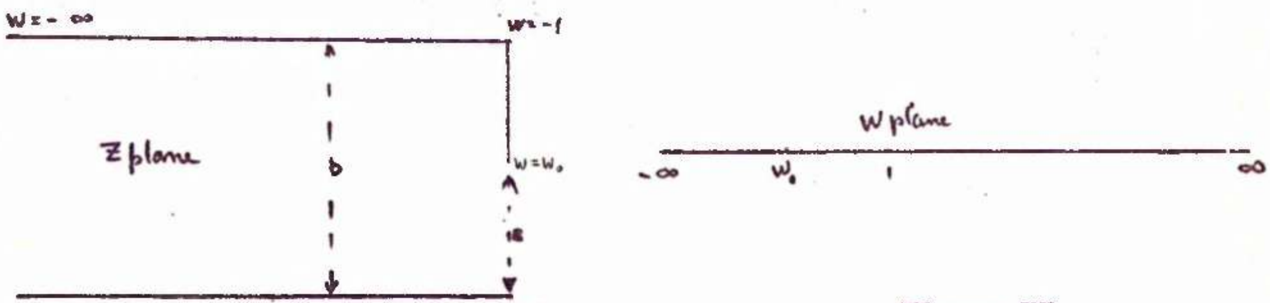


Figure XX

The Schwartz-Christoffel transformation from the z-plane to the w-plane will be given by $\frac{dz}{dw} = \frac{A}{(w^2-1)^{1/2}}$

so that $z = A\cosh^{-1} w + \alpha$

If we choose values of w such that when
$$\left.\begin{array}{ll} w = 1, & z = 0 \\ w = -1, & z = ib \\ w = w_0, & z = iB \end{array}\right\}$$

then $z = \frac{b}{\pi}\cosh^{-1} w$ and $w_0 = \cos(\pi B/b)$. (213)

The second transformation from the t-plane to the w-plane

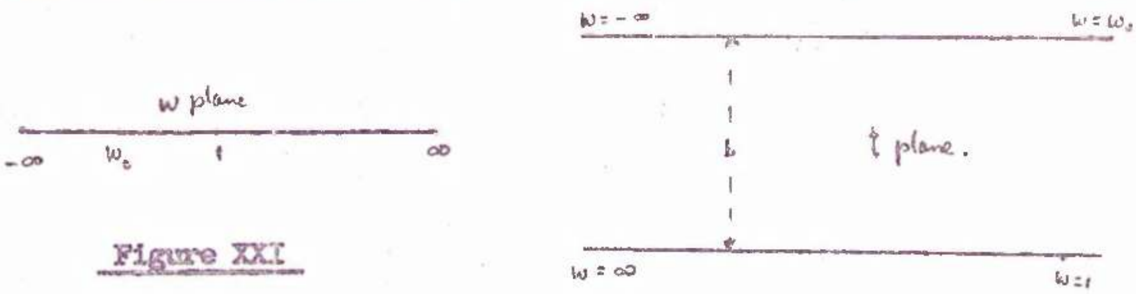


Figure XXI

is given by $\dfrac{dt}{dw} = \dfrac{A}{\sqrt{(w-1)(w-w_0)}}$,

so that $\quad t = A\cosh^{-1}\left[\dfrac{2}{(w_0-1)}\left(w - \dfrac{w_0+1}{2}\right)\right] + B$

If when $\left.\begin{array}{ll} w = w_0, & t = ib \\ w = 1, & t = 0 \end{array}\right\}$

then $\quad t = \dfrac{-b}{\pi}\cosh^{-1}\left[\dfrac{2}{(w_0-1)}\left(w - \dfrac{w_0+1}{2}\right)\right] + ib$

and therefore $\quad w = \dfrac{w_0+1}{2} - \dfrac{w_0-1}{2}\cosh\dfrac{\pi t}{b}$

$$= D + C\cosh\frac{\pi t}{b} \quad \text{where } C = \sin^2(\pi B/2b), \quad D = \cos^2(\pi B/2b) \tag{214}$$

Finally the transformation from the z- to the t-plane is obtained from

(213), (214) $\quad D + C\cosh(\pi t/b) = \cosh(\pi z/b)$,

that is,
$$C e^{-\pi t/b} = e^{-\pi z/b} + e^{\pi z/b} - 2D - C e^{\pi t/b}. \tag{215}$$

This gives t as an implicit function of z which, because of its complexity, is difficult to express explicitly. It can in fact be obtained explicitly using a theorem due to Lagrange (<u>40</u>) pages 132-133, but considering the crudity of the approximation we have used above we can proceed in a simpler way.

As we are considering only negative values of the real parts of z and t, a first approximation $t^{(1)}$ to t can be obtained at once from (215):

$$C e^{-\pi t^{(1)}/b} = e^{-\pi z/b} + e^{\pi z/b} - 2D$$
$$= e^{-\pi z/b}\left[1 - 2D e^{\pi z/b} + e^{2\pi z/b}\right].$$

Hence $\quad \log C - \pi t^{(1)}/b = -\pi z/b + \log\left(1 - 2D e^{\pi z/b} + e^{2\pi z/b}\right)$

giving $t^{(1)}$ explicitly in terms of z,

$$\pi t^{(1)}/b = \log C + \pi z/b + \sum_{n=1}^{\infty} \alpha_n e^{n\pi z/b} \tag{216a}$$

where the α_n's are real constant coefficients.

A second approximation $t^{(2)}$ follows from (215) by writing

$$C e^{-\pi t^{(2)}/b} = C e^{-\pi t^{(1)}/b} - C e^{\pi t^{(1)}/b}, \tag{216b}$$

and further approximations can be found in an obvious way.

The set of solutions of Laplace's equation in the t-plane which satisfies the proper boundary conditions is $W = U + iV \sim \begin{cases} t, \\ e^{2u\pi t/b} \end{cases}$ where u is any integer. We examine the particular solution pW = t where p is a constant of proportionality. The corresponding solution in the z-plane is, from (216),

$$\pi p W/b = \pi z/b + \log C + \sum_{n=1}^{\infty} \beta_n e^{n\pi z/b}$$

where the β_n's are also real constant coefficients.

Equating real and imaginary parts,

$$\mathrm{Re}\left(\pi p W/b\right) = \log C + \pi x/b + \sum_{n=1}^{\infty} \beta_n e^{n\pi x/b} \cos(n\pi y/b).$$

That is,

$$\mathrm{Re}\,(pW) = \frac{2b}{\pi}\log\left(\sin\frac{\pi\beta}{2b}\right) + x + \frac{b}{\pi}\sum_{n=1}^{\infty} \beta_n e^{n\pi x/b}\cos(n\pi y/b) \tag{217}$$

If we now identify (217) with (207), we obtain the results:

$$V_0 = 1,$$

$$b_0 = \frac{2b}{\pi}\log\left(\sin\frac{\pi\beta}{2b}\right),$$

As we are considering only negative values of the real parts of z and t, a first approximation $t^{(1)}$ to t can be obtained at once from (215):

$$C e^{-\pi t^{(1)}/b} = e^{-\pi z/b} + e^{\pi z/b} - 2D$$
$$= e^{-\pi z/b}\left[1 - 2D e^{\pi z/b} + e^{2\pi z/b}\right].$$

Hence $\log C - \pi t^{(1)}/b = -\pi z/b + \log\left(1 - 2D e^{\pi z/b} + e^{2\pi z/b}\right)$

giving $t^{(1)}$ explicitly in terms of z,

$$\pi t^{(1)}/b = \log C + \pi z/b + \sum_{n=1}^{\infty} \alpha_n e^{n\pi z/b} \qquad (216a)$$

where the α_n's are real constant coefficients.

A second approximation $t^{(2)}$ follows from (215) by writing

$$C e^{-\pi t^{(2)}/b} = C e^{-\pi t^{(1)}/b} - C e^{\pi t^{(1)}/b}, \qquad (216b)$$

and further approximations can be found in an obvious way.

The set of solutions of Laplace's equation in the t-plane which satisfies the proper boundary conditions is $W = U + iV \sim \begin{cases} t \\ e^{2n\pi t/b} \end{cases}$ where U is any integer. We examine the particular solution pW = t where p is a constant of proportionality. The corresponding solution in the z-plane is, from (216),

$$\pi p W/b = \pi z/b + \log C + \sum_{n=1}^{\infty} \beta_n e^{n\pi z/b}$$

where the β_n's are also real constant coefficients.

Equating real and imaginary parts,

$$\mathrm{Re}\left(\pi p W/b\right) = \log C + \pi x/b + \sum_{n=1}^{\infty} \beta_n e^{n\pi x/b} \cos(n\pi y/b).$$

That is,

$$\mathrm{Re}(pW) = \frac{2b}{\pi}\log\left(\sin\frac{\pi\beta}{2b}\right) + x + \frac{b}{\pi}\sum_{n=1}^{\infty} \beta_n e^{n\pi x/b} \cos(n\pi y/b) \qquad (217)$$

If we now identify (217) with (207), we obtain the results:

$$V_0 = 1,$$

$$b_0 = \frac{2b}{\pi}\log\left(\sin\frac{\pi\beta}{2b}\right),$$

and hence from (212) the approximation to the eigenvalue equation is

$$\frac{1}{k b \tan k a} = \frac{2}{\pi} \log\left(\sin \frac{\pi B}{2b}\right). \tag{218}$$

This result, being obtained from Case II, is most accurate for small values of B. If B is small compared with b and we make the approximation $\sin \frac{\pi B}{2b} \sim \frac{\pi B}{2b}$, the right hand member of (218) is approximately equal to $(-\frac{2}{\pi})\log(\frac{2b}{\pi B})$. This differs very little from the right hand member of (57), $(-\frac{2}{\pi})\log(\frac{2 \cdot 23b}{\pi B})$.

Numerical solutions of (218) are shown in Table I(d) and can be compared with solution of (57) in Table I(a). See page 27.

The theoretical extension of this analysis to obtain closer approximations is quite straightforward. The practical manipulation becomes immediately very involved and the problem of estimating the order of approximation very difficult. The useful application of the method is therefore that it yields quickly first approximations which can be used as a starting point for the variational method of solution.

§ 17 Equivalent Static method applied to the Cylindrical Klystron

As a second demonstration of the application of the equivalent static method we return to the cylindrical Klystron problem of § 10.

Equations (100) are

$$\psi_1 = \frac{2}{B}\sum_{n=0}^{\infty} \varepsilon_n' \frac{1}{\alpha_n} \frac{L(\alpha_n r)}{L'(\alpha_n A)} \int_0^B f(\eta)\cos\frac{n\pi\eta}{B}\,d\eta\,(\cos n\pi z/B),$$

$$\psi_2 = \frac{2}{b}\sum_{n=0}^{\infty} \varepsilon_n' \frac{1}{\beta_n} \frac{M(\beta_n r)}{M'(\beta_n A)} \int_0^B f(\eta)\cos\frac{n\pi\eta}{b}\,d\eta\,(\cos n\pi z/b),$$

where $\alpha_0 = \beta_0 = k$.

The requirement that ψ be continuous at $r = A$, $0 \leq z \leq B$, then gives

$$-\left[\frac{1}{kB}\cdot\frac{L(kA)}{L'(kA)} - \frac{1}{kb}\frac{M(kA)}{M'(kA)}\right]\int_0^B f(\eta)\,d\eta \tag{219}$$

$$= 2\sum_{n=1}^{\infty}\int_0^B f(\eta)\left[\frac{1}{B\alpha_n}\frac{L(\alpha_n A)}{L'(\alpha_n A)}\cos(n\pi\eta/B)\cos(n\pi z/B) - \frac{1}{b\beta_n}\frac{M(\beta_n A)}{M'(\beta_n A)}\cos(n\pi\eta/b)\cos\left(\frac{n\pi z}{b}\right)\right]d\eta .$$

Using the first approximations (104a,b,c) and putting

$H(k) \equiv -\left[\frac{1}{kB}\frac{L(kA)}{L'(kA)} - \frac{1}{kb}\frac{M(kA)}{M'(kA)}\right]$ in (219), we find

$$H(k)\int_0^B f(\eta)\,d\eta \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^B f(\eta)\left[\cos\frac{n\pi\eta}{B}\cos\frac{n\pi z}{B} + \cos\frac{n\pi\eta}{b}\cos\frac{n\pi z}{b}\right]d\eta, \tag{220}$$

where the right hand member is a dimensionless constant. If we choose a constant c such that $cf(\eta) = F(\eta)$ is a solution of the equation

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^B F(\eta)\left[\cos\frac{n\pi\eta}{B}\cos\frac{n\pi z}{B} + \cos\frac{n\pi\eta}{b}\cos\frac{n\pi z}{b}\right]d\eta = 1, \tag{221}$$

then we can write the approximation equation(220) in the form

$$H(k)\int_0^B F(\eta)\,d\eta = 1, \tag{222}$$

where we shall employ the equivalent static method to find a value for

$\int_0^B F(\eta)\,d\eta$.

We look for a static problem with a cross section similar to that of the cross section of Figure XV, which leads to an integral equation like (221). We consider an infinite channel with parallel sides with a step change in height at $x = 0$ as in Figure XXII.

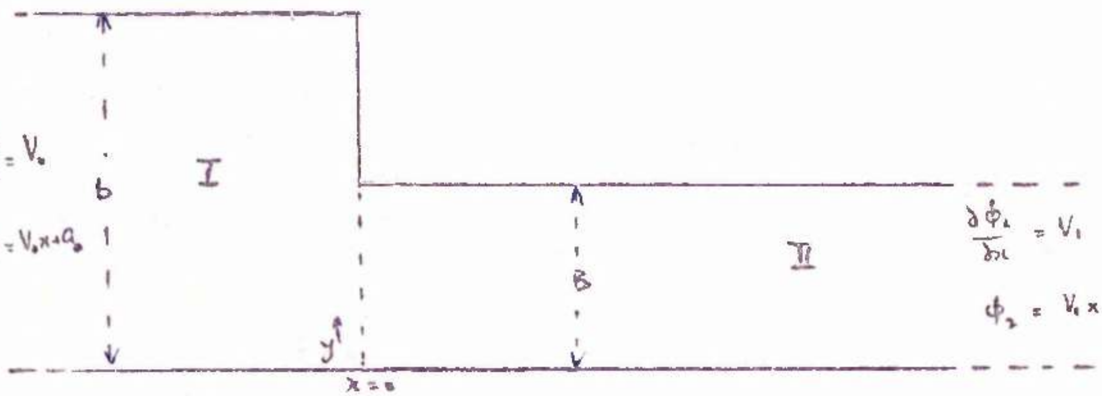


Figure XXII

Proceeding as before we define a stream function $\phi = \phi_1, \phi_2$ in regions I, II to the left and right of $x = 0$ respectively, where ϕ satisfies $\nabla^2\phi + k^2\phi = 0$. In order to obtain equivalence between the static and dynamic problems we again permit the existence of sources at infinity. Let us suppose that the conditions are

$$\phi_1 = V_0 x + a_0 \qquad \text{at} \quad x = -\infty$$

$$\phi_2 = V_1 x + b_0 \qquad \text{at} \quad x = +\infty \ ,$$

and that the normal derivatives of ϕ are zero on the solid boundaries. The condition of continuity will require that $V_0 b = V_1 B$. The stream function can then be expressed in the form of infinite series:

$$\phi_1 = V_0 x + \sum_{n=0}^{\infty} a_n \cos(n\pi y/b)\, e^{n\pi x/b} , \tag{223a}$$

$$\phi_2 = V_0 \frac{b}{B} x + \sum_{n=0}^{\infty} b_n \cos(n\pi y/B)\, e^{-n\pi x/B} . \tag{225b}$$

In the aperture at $x = 0$, $0 < y \leq B$.

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{x=0} = \left(\frac{\partial \phi_2}{\partial x}\right)_{x=0}$$

If we set them both equal to an unknown function $g(y)$ over the interval $0 \leq y \leq B$, then

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{x=0} \equiv V_0 + \sum_{n=0}^{\infty} a_n \cos(n\pi y/b) = 0 \quad B < y \leq b ,$$
$$= g(y) \quad 0 \leq y \leq B ,$$

$$\left(\frac{\partial \phi_2}{\partial x}\right)_{x=0} \equiv V_0 \frac{b}{B} - \sum_{n=0}^{\infty}{}' b_n \cos(n\pi y/B) = g(y) \quad 0 \leq y \leq B ,$$

and therefore

$$V_0 b = \int_0^B g(\eta) d\eta , \tag{224a}$$

$$a_n = \frac{2}{n\pi} \int_0^B g(\eta) \cos(n\pi\eta/b) d\eta, \quad n \geq 1 , \tag{224b}$$

$$b_n = -\frac{2}{n\pi} \int_0^B g(\eta) \cos(n\pi\eta/B) d\eta , \quad n \geq 1 . \tag{224c}$$

The integral equation obtained from the requirement that $\phi(x,y)$ be continuous across the line $x = 0$ is

$$\phi_1(0,y) - \phi_2(0,y) = 0 \qquad 0 \leq y \leq B.$$

That is $\quad b_0 - a_0 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^B g(\eta) \left[\cos\frac{n\pi\eta}{b} \cos\frac{n\pi y}{b} + \cos\frac{n\pi\eta}{B} \cos\frac{n\pi y}{B}\right] d\eta .$

Then $G(\eta) = g(\eta)/(b_0 - a_0)$ is a solution of the equation

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^B G(\eta) \left[\cos\frac{n\pi\eta}{b} \cos\frac{n\pi y}{b} + \cos\frac{n\pi\eta}{B} \cos\frac{n\pi y}{B}\right] d\eta = 1 . \tag{225}$$

(221) and (225) are identical if $F(\eta) \equiv G(\eta)$ (226)

Therefore, using (224a) and (226) in (222), we obtain the approximation equation for the evaluation of the lowest eigenvalue k:

$$H(k) \sim \frac{b_0 - a_0}{V_0 b} . \tag{227}$$

[Note: H(k) is defined in the line preceding (220).]

Evaluation of the Static Parameters:

In order to solve (227) we evaluate a_o, b_o, V_o by conformal mapping. We again use two transformations, mapping the original polygon in the z-plane onto the real axis of the w-plane and thence onto an infinite channel with parallel sides in the t-plane.

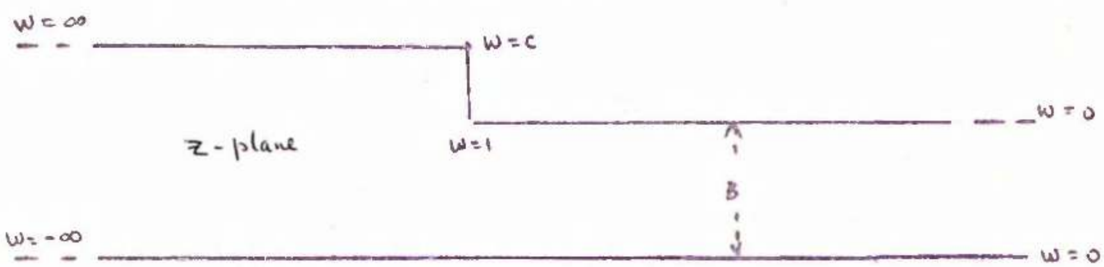


FIGURE XXIII

Giving to the vertices in Figure XXIII the w coordinates shown, the Schwartz-Christoffel transformation from the z- to the w-plane is given by

$$\frac{dz}{dw} = \frac{A(w-1)^{\frac{1}{2}}}{w(w-c)^{\frac{1}{2}}}$$

where A and c must be determined. The result of integrating,

$$\frac{z}{A} = ch^{-1}\left(\frac{2w-c-1}{c-1}\right) - \frac{1}{\sqrt{c}}\, ch^{-1}\left(\frac{(c+1)w-2c}{(c-1)w}\right) + D + iE \qquad (228)$$

must satisfy the following:

$w = c$ when $z = ib$

$w = 1$ when $z = iB$

z is real when $w \rightarrow -\infty$

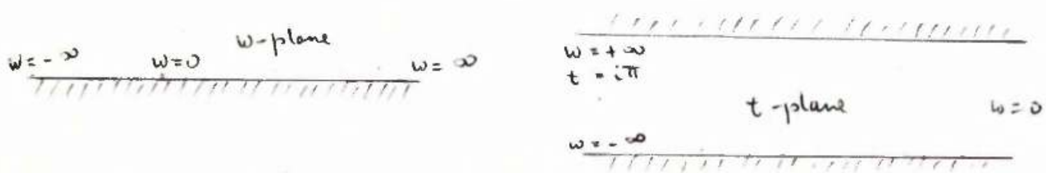
$z = x + ib$ when $w \rightarrow +\infty$.

For these to be satisfied we find

$$c = (b/B)^2, \quad A = -b/\pi, \quad D = 0, \quad E = b/A = -\pi,$$

and therefore the transformation (228) reduces to

$$-\frac{\pi z}{b} = \cosh^{-1}\left(\frac{2w-c-1}{c-1}\right) - \frac{1}{\sqrt{c}}\, ch^{-1}\left(\frac{[c+1]w-2c}{[c-1]w}\right) - i\pi.$$



The transformation mapping the real axis of the w-plane into the infinite channel in the t-plane is $w = -e^{-t}$ and therefore, combining the two, the final transformation is

$$-\frac{\pi z}{b} = \cosh^{-1}\left(\frac{2e^{-t}}{1-c} + \frac{c+1}{1-c}\right) - \frac{1}{\sqrt{c}}\cosh^{-1}\left(\frac{c+1}{c-1} + \frac{2c}{c-1}e^{t}\right) - i\pi \qquad (229)$$

Again, the set of solutions of Laplace's equation in the t plane which satisfies the proper boundary conditions is $W = U + iV = \begin{cases} t \\ e^{\pm n\pi t/b} \end{cases}$ $n = 0, 1, 2, \ldots$

If we examine the particular solution $pW = t$ where p is a constant, substitution for t in (229) gives W implicitly as a function of z. To express W explicitly as a function of z we proceed as before:

a) when t is large and negative;

Let $\quad \zeta_1 = \frac{2e^{-t}}{1-c} + \frac{c+1}{1-c} = \frac{2e^{-t}}{1-c}\left[1 + \frac{c+1}{2}e^{t}\right].$

Then $\quad \zeta_1 + [\zeta_1^2 - 1]^{\frac{1}{2}} = \frac{4e^{-t}}{1-c}\left[1 + \left(\frac{c+1}{2}\right)e^{t} + O(e^{2t})\right].$

Therefore $\quad \cosh^{-1}\zeta_1 = \log\left(\zeta_1 + [\zeta_1^2 - 1]^{\frac{1}{2}}\right) \sim \log\frac{4}{1-c} - t + \frac{c+1}{2}e^{t}. \qquad (230a)$

Similarly let $\zeta_2 = \frac{c+1}{c-1} + \frac{2c}{c-1}e^{t}.$

Then $\quad \cosh^{-1}\zeta_2 = \log\left(\zeta_2 + [\zeta_2^2 - 1]^{\frac{1}{2}}\right) \sim \cosh^{-1}\left(\frac{c+1}{c-1}\right) + \sqrt{c}\,e^{t}. \qquad (230b)$

Therefore when t is large and negative, substituting (230) in (229),

$$-\frac{\pi z}{b} \sim \left(\log\frac{4}{1-c} - t + \frac{c+1}{2}e^{t}\right) - \frac{1}{\sqrt{c}}\left[\cosh^{-1}\left(\frac{c+1}{c-1}\right) + \sqrt{c}\,e^{t}\right] - i\pi$$

$$= \log\frac{4}{1-c} - \frac{1}{\sqrt{c}}\cosh^{-1}\left(\frac{c+1}{c-1}\right) - t + \frac{c-1}{2}e^{t} - i\pi. \qquad (231a)$$

b) when t is large and positive we find similarly that

$$-\frac{\pi z}{b} \sim \sqrt{c}\cosh^{-1}\left(\frac{c+1}{c-1}\right) - \log\frac{4c}{c-1} - t + \frac{c-1}{2c}e^{-t} - i\pi. \qquad (231b)$$

Hence, substituting the particular solution $pW = t$ in (231):

a) When t is large and negative a first approximation to the explicit expression for W_1 in region I is

$$pW_1^{(1)} = \log\left(\frac{4}{1-c}\right) - R - i\pi + \pi z/b \quad \text{where } R = \frac{1}{\sqrt{c}}\cosh^{-1}\left(\frac{c+1}{c-1}\right).$$

A second approximation is then

$$pW_1^{(2)} = pW_1^{(1)} + \frac{c-1}{2}e^{pW_1^{(1)}}$$

$$= \log\left(\frac{4}{1-c}\right) - R - i\pi + \pi z/b - \frac{c-1}{2}\cdot\frac{4}{1-c}e^{-R}e^{\pi z/b}$$

and therefore $R_e\, pW_1^{(2)} = \log\left(\frac{4}{c-1}\right) - R + \frac{\pi x}{b} + 2e^{-R}e^{\pi x/b}\cos(\pi y/b)$. (232a)

b) when t is large and positive we find similarly that in region II,

$$R_e\, pW_2^{(2)} = cR - \log\left(\frac{4c}{c-1}\right) + \frac{\pi x}{B} - 2e^{-cR}e^{-\pi x/b}\cos(\pi y/B). \quad (232b)$$

Identifying W with ϕ, equations (223) and (231) give

$$a_0 = \frac{1}{p}\left[\log\frac{4}{c-1} - R\right] \quad ; \quad b_0 = \frac{1}{p}\left[cR - \log\frac{4c}{c-1}\right];$$

$$V_0 = \pi/pb \quad ; \quad a_1 = 2e^{-R}/p \quad ; \quad b_1 = -2e^{-cR}/p.$$

and hence equation (227) reduces to the simple transcendental equation, giving a first approximation to the lowest eigenvalue k,

$$H(k) \sim \frac{1}{\pi}\left[\frac{c+1}{\sqrt{c}}\cosh^{-1}\left(\frac{c+1}{c-1}\right) - 2\log\frac{4\sqrt{c}}{c-1}\right] \text{ where } \sqrt{c} = b/B. \quad (233)$$

Numerical results.

Choosing the same dimensions as in §10, we have $\sqrt{c} = \frac{b}{B} = 8$. The right hand side of (233) can therefore be computed at once, and the equation can be written

$$\frac{1}{ka}\left[\frac{a}{c}\,\frac{M(ka)}{M'(ka)} - \frac{a}{d}\,\frac{L(ka)}{L'(ka)}\right] = 1.0804.$$

Solutions for a = 2A(2A)8A are shown in Table V page 49 for comparison with the earlier ones.

§ 18 A note on the application of variational principles for mixed boundary-value problems in potential theory.

The methods used in § 2 can be used in considering the following problem: A potential function ϕ satisfies the equation $\phi_{xx} + \phi_{yy} = 0$ in a region R bounded by a closed curve C on which mixed boundary conditions are specified,

i) $\phi = f(s)$ on a part A of the boundary.

ii) $\frac{\partial \phi}{\partial n} = g(s)$ on a part B.

where $C = A + B$ and distance along C is denoted by s. Electrostatic problems of this type have been solved approximately in (37) and (38) by formulating them in terms of integral equations and then applying variational principles to the integral equations.

We use the notation (2) of § 2, and Green's Theorem in the form

$$(\nabla f, \nabla g) = -(f, \nabla^2 g) + \int_C f \, \partial g/\partial n \, ds . \tag{241}$$

where $\partial/\partial n$ denotes differentiation normal to the boundary in an outward direction. If a potential function ϕ satisfies the conditions specified above, then using (241),

$$(\nabla \phi, \nabla \phi) = -(\phi, \nabla^2 \phi) + \int_C \phi \frac{\partial \phi}{\partial n} ds$$

$$= \int_A f(s) \, \partial \phi/\partial n \, ds + \int_B g(s) \phi(s) \, ds . \tag{242}$$

Upper and Lower Limits.

We proceed to find upper and lower limits by a method very similar to that of § 5, using two functions Φ and Ψ which approximate to ϕ.

a). Suppose Φ is a function which approximates to ϕ, so that we can set $\Phi = \phi + \delta$ where δ is an error function, small compared with ϕ. We let Φ satisfy the conditions

(i)' $\partial \Phi/\partial n = g(s)$ on B, so that $\partial \delta/\partial n = 0$ on B,

(ii)' $\nabla^2 \Phi = 0$ in R, so that $\nabla^2 \delta = 0$ in R.

Then $(\nabla\Phi, \nabla\Phi) = (\nabla\phi, \nabla\phi) + 2(\nabla\delta, \nabla\phi) + (\nabla\delta, \nabla\delta)$, (243)

and $(\nabla\delta, \nabla\Phi) = -(\phi, \nabla^2\delta) + \int_c \phi \frac{\partial\delta}{\partial n} ds$

$$= \int_A f(s) \frac{\partial\Phi}{\partial n} ds - \int_A f(s) \frac{\partial\phi}{\partial n} ds. \qquad (244)$$

Substituting (244) in (243),

$$(\nabla\Phi, \nabla\Phi) - 2\int_A f(s) \frac{\partial\Phi}{\partial n} ds = (\nabla\phi, \nabla\phi) - 2\int_A f(s) \frac{\partial\phi}{\partial n} ds + (\nabla\delta, \nabla\delta). \qquad (245)$$

If we define $I(\chi) = \int_B g(s)\chi(s)ds - \int_A f(s) \frac{\partial\chi}{\partial n} ds$, (246)

then (245) can be written

$$(\nabla\Phi, \nabla\Phi) - 2\int_A f(s) \frac{\partial\Phi}{\partial n} ds = I(\phi) + (\nabla\delta, \nabla\delta)$$

i.e. $(\nabla\Phi, \nabla\Phi) - 2\int_A f(s) \frac{\partial\Phi}{\partial n} ds \geqslant I(\phi)$ since $(\nabla\delta, \nabla\delta) \geqslant 0$. (247)

If δ is a first order quantity, the difference between the two sides of (247) is second-order. We have

$$(\nabla\Phi, \nabla\Phi) = -(\Phi, \nabla^2\Phi) + \int_c \Phi \frac{\partial\Phi}{\partial n} ds$$

$$= \int_A \Phi(s) \frac{\partial\Phi}{\partial n} ds + \int_B g(s)\Phi(s) ds.$$

Hence (247) gives

$$\int_B g(s)\Phi(s) ds + \int_A \Phi(s) \frac{\partial\Phi}{\partial n} ds - 2\int_A f(s) \frac{\partial\Phi}{\partial n} ds \geqslant I(\phi). \qquad (248)$$

b) Suppose Ψ is a function which approximates of ϕ, so that we can write $\Psi = \phi + \varepsilon$ where ε is an error function, small compared with ϕ. We let Ψ satisfy

(i)'' $\Psi = f(s)$ on A so that $\varepsilon = 0$ on A

(ii)'' $\nabla^2\Psi = 0$ in R so that $\nabla^2\varepsilon = 0$ in R.

Then $(\nabla\Psi, \nabla\Psi) = (\nabla\phi, \nabla\phi) + 2(\nabla\varepsilon, \nabla\phi) + (\nabla\varepsilon, \nabla\varepsilon)$, (249)

and $(\nabla\varepsilon, \nabla\phi) = -(\varepsilon, \nabla^2\phi) + \int_c \varepsilon \frac{\partial\phi}{\partial n} ds$

$$= \int_B g(s)\Psi(s) ds - \int_B g(s)\phi(s) ds. \qquad (250)$$

Combining (249) and (250) and using (242) and (246), we find that

$$-(\nabla\Psi,\nabla\Psi)+2\int_B g(s)\Psi(s)\,ds = I(\phi)-(\nabla\varepsilon,\nabla\varepsilon),$$

i.e. $$-(\nabla\Psi,\nabla\Psi)+2\int_B g(s)\Psi(s)\,ds \le I(\phi) \text{ since } (\nabla\varepsilon,\nabla\varepsilon)>0. \tag{251}$$

But, using condition (ii)°°,

$$(\nabla\Psi,\nabla\Psi)=\int_A f(s)\frac{\partial\Psi}{\partial n}ds+\int_B \Psi\frac{\partial\Psi}{\partial n}ds.$$

Hence (251) gives

$$-\left[\int_A f(s)\frac{\partial\Psi}{\partial n}ds+\int_B \Psi\frac{\partial\Psi}{\partial n}ds-2\int_B g(s)\Psi(s)ds\right]\le I(\phi). \tag{252}$$

Equations (248) and (252) give upper and lower bounds for $I(\phi)$ in terms of line integrals along the boundary of the region. However, the expressions on the left of (248) and (252) are not very useful as they stand. In (248) for example, we have used condition (i)° and assumed that $\partial\phi/\partial n = g(s)$ on B. This means that Φ cannot be chosen arbitrarily on A and B. We proceed to derive the variational principles.

a) In addition to conditions (i)° and (ii)°, suppose that Φ satisfies the condition (iii)° $\partial\Phi/\partial n = G(s)$ on A, where $G(s)$ is a chosen approximation to the unknown value of $\partial\Phi/\partial n$ on A.

Then $\partial\Phi/\partial n$ is known on the whole of the boundary C, and by Green's Theorem, using suitable Green's functions, we can determine expressions for the unknown function $\Phi(s)$ on B in terms of $g(s)$ and $G(s)$:

$$\Phi(s) = \int_A K(s,t)G(t)dt + \int_B K(s,t)g(t)dt \tag{253}$$

where $K(s,t)$ is assumed to be a known function so that all functions on the right of (253) are known. On substituting in (248) we find

$$\int_B\int_B K(s,t)g(s)g(t)\,ds\,dt = J(G) \ge I(\phi) \tag{254}$$

where $J(G) = 2\left\{\int_A f(s)G(s)ds + \int_A\int_B K(s,t)G(s)g(t)dsdt\right.$

$$\left. - \int_A\int_A K(s,t)G(s)G(t)dsdt\right\}, \tag{255}$$

and in the derivation we have used the fact that $K(s,t) = K(t,s)$ since K is derived from a Green's function.

b) In addition to condition (i)'', (ii)'', suppose that Ψ satisfies the condition (iii)'' $\Psi = F(s)$ on B, where $F(s)$ is a chosen approximation to the unknown value of ϕ on B.

Then Ψ is known on the whole of C and by Green's theorem, using suitable Green's functions, we can deduce an expression for the unknown function $\partial\Psi/\partial n$ on A in terms of $f(s)$ and $F(s)$:

$$\frac{\partial\Psi}{\partial n} = \int_A L(s,t)f(t)dt + \int_B L(s,t)F(t)dt,$$

where $L(s,t)$ is assumed known. On substituting in (252) we find

$$H(F) - \int_A\int_A L(s,t)f(s)f(t)ds\,dt \leq I(\phi) \tag{256}$$

where

$$H(F) = 2\left\{\int_B g(s)F(s)ds - \int_A\int_B L(s,t)f(s)F(t)ds\,dt\right.$$

$$\left. - \int_B\int_B L(s,t)F(s)F(t)ds\,dt\right\}. \tag{257}$$

Expressions (254), (256) yield the required variational principles. By choosing a form for $G(s)$ which contains arbitrary parameters and by minimizing the left side of (254) with respect to these parameters we can find an upper bound for $I(\phi)$. Similarly by choosing $F(s)$ so as to maximize the left side of (256) we can find a lower bound for $I(\phi)$.

We can briefly derive integral equations from the variational expressions. Suppose that in (254), (255) we have

$$G(s) = \Gamma(s) + \eta\gamma(s)$$

where $\Gamma(s)$ is the exact value of $\partial\phi/\partial n$ on A, and $\eta\gamma(s)$ is the error where η is a small parameter. Then (14) gives

$$J(G) = J(\Gamma) + 2\eta P(\Gamma,\gamma) - \eta^2 \int_A \int_A K(s,t)\gamma(s)\gamma(t)ds\, dt$$

where, on using the symmetry of $K(s,t)$,

$$P(\Gamma,\gamma) = \int_A \gamma(s)\left\{ f(s) - \int_B K(s,t)g(t)dt - \int_A K(s,t)\Gamma(t)dt \right\} ds.$$

If the inequality (254) is to be true for any choice of $G(s)$ this implies that $P(\Gamma,\gamma) = 0$ for any $\gamma(s)$ and hence

$$\int_A K(s,t)\Gamma(t)dt = f(s) - \int_B K(s,t)g(t)dt \quad , \quad (s \text{ on } A). \tag{258}$$

This is an integral equation for $\Gamma(t)$.

Similarly from (256), (257), if $F(s) = \Theta(s) + \eta\theta(s)$ where $\Theta(s)$ is the exact value of ϕ on B, we find the integral equation for Θ:

$$\int_B L(s,t)\Theta(t)dt = g(s) - \int_A L(s,t)f(t)dt, \qquad (s \text{ on } B). \tag{259}$$

Special cases of the integral equations (18), (19) form the starting point for the analysis in (37) and (38).

§ 19 Appendix I. Summation of Series.

Two forms of infinite series occur in the problems of paragraphs 6, 8, 11, 13, and, slightly modified, in paragraphs 10, 12, 14. They are

$$\sum_{n=N+1}^{\infty} \frac{1}{n} J_{\mu}(n\alpha) J_{\nu}(n\alpha), \tag{261}$$

where μ and ν are positive integers or zero, and α is some constant,

$$\sum_{n=N+1}^{\infty} \frac{1}{n} J_{\nu}(n\alpha) \cos(n\beta y) \tag{262}$$

where ν is a positive integer or zero, β and y are constants. Series (261) appears in the approximate solution of the eigenvalue equations, series (262) in the equations for the evaluation of the potential function ϕ. Their summation can be carried to any degree of accuracy.

a) $$\sum_{n=k}^{\infty} \frac{1}{n} J_{\mu}(n\alpha) J_{\nu}(n\alpha) = \alpha \sum_{n=k}^{\infty} \frac{J_{\mu}(n\alpha) J_{\nu}(n\alpha)}{n\alpha}$$

$$= \alpha \sum_{m=0}^{\infty} \frac{J_{\mu}([m+k]\alpha) J_{\nu}([m+k]\alpha)}{[m+k]\alpha}$$

$$= \alpha \sum_{m=0}^{\infty} f(x_0 + mh)$$

where $x_0 = k\alpha$, $h = \alpha$, and $f(x) = \dfrac{J_{\mu}(x) J_{\nu}(x)}{x}$.

We use a result, given in (15) page 156,

$$\sum_{m=0}^{\infty} f(x_0 + mh) = \frac{1}{h}\int_{x_0}^{\infty} f(x)\,dx + \frac{1}{2} f(x_0) - \frac{h}{6} f'(x_0) + \frac{1}{360} h^3 f'''(x_0) \ldots \tag{263}$$

Then $$\sum_{n=k}^{\infty} \frac{1}{n} J_{\mu}(n\alpha) J_{\nu}(n\alpha) = \int_{k\alpha}^{\infty} \frac{1}{x} J_{\mu}(x) J_{\nu}(x)\,dx + \frac{J_{\mu}(k\alpha) J_{\nu}(k\alpha)}{2k}$$

$$- \frac{\alpha}{6}\left[\frac{d}{dx}\left(\frac{J_{\mu}(x) J_{\nu}(x)}{x}\right)\right]_{x=k\alpha} + \cdots \tag{264}$$

The coefficients of successive terms on the right of (264) decrease rapidly and the sum of the series can be computed simply, (if laboriously), to any desired

accuracy if the value of the integral $I_{\mu\nu} = \int_{k\alpha}^{\infty} \frac{1}{x} J_\mu(x) J_\nu(x)dx$ is known.
By (23) page 92 equation (52), the following ~~this~~ is a special case of the Weber-Schafheitlin integral where, if $(\mu+\nu) > 0$,

$$\int_0^\infty \frac{J_\mu(x) J_\nu(x)}{x} dx = \frac{2}{\pi} \frac{\sin[(\nu-\mu)\pi/2]}{(\nu^2-\mu^2)} .$$

Thus, since the value of the integral $\int_0^{k\alpha} \frac{1}{x} J_\mu(x) J_\nu(x)dx$ can be obtained by any one of a number of numerical approximation methods, the infinite series can be summed if $\mu + \nu > 0$.

If $\mu = \nu = 0$ we require to know the value of $\int_{k\alpha}^{\infty} \frac{J_0^{\,2}(x)dx}{x}$

We use (23) page 47,

$$\frac{1}{2} z^{-\nu} \pi^{\frac{1}{2}} \Gamma(\nu+\tfrac{1}{2}) J_\nu^{\,2}(z) = \int_0^{\pi/2} J_\nu(2z\sin\theta)(\sin\theta)^\nu(\cos\theta)^{2\nu} d\theta ,$$

Where, if $\nu = 0$,

$$\frac{\pi}{2} J_0^{\,2}(\eta\alpha) = \int_0^{\pi/2} J_0(2\eta\alpha\sin\theta)d\theta .$$

Dividing by η and integrating with respect to η,

$$\int_a^\infty \frac{1}{\eta} J_0^{\,2}(\eta\alpha)\, d\eta = \frac{2}{\pi}\int_a^\infty\int_0^{\pi/2} \frac{1}{\eta} J_0(2\eta\alpha\sin\theta)d\theta\, d\eta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\int_a^\infty J_0(2\eta\alpha\sin\theta)\frac{d\eta}{\eta}\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\int_{2a\alpha\sin\theta}^\infty \frac{J_0(v)}{v} dv\, d\theta . \qquad (265)$$

But by (41) pages 33, 34

$$\int_x^\infty \frac{J_0(v)dv}{v} = -\gamma + \log 2 - \log x + \sum_{n=1}^\infty (-)^{n-1}\left(\frac{x}{2}\right)^{2n} \Big/ 2n(n!)^2 .$$

Therefore replacing x by $2a\alpha \sin\theta$, we find from (265)

$$\int_a^{\infty} \frac{1}{\eta} J_0^2(\eta\alpha)d\eta = -\gamma + \log 2 - \log(2a\alpha) + \log 2 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-)^{n-1}(a\alpha)^{2n}}{2n(n!)^2}\int_0^{\pi/2}\cos^{2n}\theta\, d\theta\right]. \tag{266}$$

The infinite series in (266) converges rapidly giving any desired approximation to $\int_a^{\infty} \frac{1}{\eta} J_0^2(\eta\alpha)\,d\eta$ and hence to $\sum_{n=k}^{\infty}{}' \frac{1}{n} J_0^2(n\alpha)$.

b) Proceeding as before, using (15) page 156, we can express (262) in the form

$$\sum_{n=k}^{\infty}{}' \frac{1}{n} J_\nu(n\alpha)\cos(n\beta y) = \int_{k\alpha}^{\infty} \frac{1}{x} J_\nu(x)\cos\left(\frac{\beta y}{\alpha}x\right)dx + \frac{\alpha}{2}\left[\frac{J_\nu(x)\cos(k\beta y)}{k\alpha}\right] + \cdots$$

If $\nu > 0$, we have by (30) page 405,

$$\begin{aligned}\int_0^{\infty} \frac{J_\nu(at)}{t}\cos bt\, dt &= \frac{1}{\mu}\cos\mu \arcsin(b/a), \quad a>b, \\ &= \frac{a^{\mu}\cos(\mu\pi/2)}{\mu[b+(b^2-a^2)^{\frac{1}{2}}]^{\mu}}, \quad b>a,\end{aligned}\Bigg\} \quad a, b \text{ real}, \geq 0.$$

We can therefore sum the series (262).

If $\nu = 0$ (which is the important case in the problems considered),

$$\sum_{n=k}^{\infty}{}' \frac{1}{n} J_\nu(n\alpha)\cos(n\beta y) = \int_{k\alpha}^{\infty} \frac{1}{x} J_0(x)\cos\left(\frac{\beta y}{\alpha}x\right)dx + \frac{\alpha}{2}\left[\frac{J_0(k\alpha)\cos(k\beta y)}{k\alpha}\right] + \cdots$$

If we write

$$\begin{aligned}\int_k^{\infty} \frac{J_0(at)\cos bt}{t}\, dt &= \int_K^{\infty}\frac{\cos bt}{t}dt - \int_0^K \frac{J_0(at)-1}{t}\cos bt\, dt + \int_0^{\infty}\frac{J_0(at)-1}{t}\cos bt\, dt \\ &= I_1 + I_2 + I_3,\end{aligned}$$

then $I_1 = \int_k^{\infty} \frac{\cos bt}{t}\,dt$ which is fully tabulated,

and $I_2 = \int_0^K \frac{J_0(at)-1}{t}\cos bt\, dt$ which can be evaluated numerically for any given K.

To evaluate $I_3 = \int_0^{\infty} \frac{J_0(at)-1}{t}\cos bt\, dt$ we observe that

$$\frac{d}{da}\left[J_0(at)\right] = -t\, J_1(at).$$

Therefore $\int_0^a J_1(xt)dx = -\left[\frac{1}{t}J_0(xt)\right]_0^a = -\frac{1}{t}\left[J_0(at) - 1\right]$

and hence $\int_0^\infty \frac{J_0(at) - 1}{t} \cos bt \, dt = -\int_0^\infty \int_0^a J_1(xt) \cos bt \, dx \, dt$

$$= -\int_0^a \int_0^\infty J_1(xt) \cos bt \, dt \, dx.$$

But by (30) page 405,

$$\int_0^\infty J_\mu(at) \cos bt \, dt = \frac{-a^\mu \sin(\frac{1}{2}\mu\pi)}{(b^2-a^2)^{\frac{1}{2}}[b+(b^2-a^2)^{\frac{1}{2}}]^\mu}, \quad b > a.$$

Therefore $I_3 = \displaystyle\int_0^a \frac{x \, dx}{(b^2-x^2)^{1/2}[b+(b^2-x^2)^{\frac{1}{2}}]}$

$$= -\left\{ \log[b + (b^2-x^2)^{\frac{1}{2}}] \right\}_0^a$$

$$= \log\left[\frac{2b}{b+(b^2-a^2)^{\frac{1}{2}}}\right].$$

The summation of (262) when $\nu = 0$ can therefore also be completed to any required accuracy.

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