A dissertation submitted for the degree of **Master of Science**

in the University of Glasgow

QUASISPECTRAL OPERATORS

by

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PREFACE

This dissertation is submitted in accordance with the regulations for the degree of **Master of Science** in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any other University.

The results contained in this dissertation are claimed as original except where indicated in the text.

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CONTENTS

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SUMMARY

The main subject in this thesis is the study of quasispectral operators, their basic properties, their roots and logarithms.

In Chapter One, which is not claimed as original, we include the main results on the spectral theory of linear operators and the basic properties of prespectral operators which will be required in other chapters.

In Chapter Two, we prove the existence of a complex Banach space Y and a homomorphism $E(\cdot)$ from Σ_p into a bounded Boolean algebra of projections on Y such that $E(\cdot)$ is not a spectral measure of any class.

In Chapter Three we prove that if S is a scalar-type operator and A is an operator that leaves invariant all the maximal spectral subspaces of S, then A commutes with S.

In Chapter Four, which is the main part of this thesis, we introduce the concept of quasispectral operator. We prove that a quasispectral operator of class Γ has a unique resolution of the identity of class Γ and a unique Jordan decomposition for resolutions of the identity of all classes. We show that every prespectral operator of class Γ is quasispectral of class Γ but that there exists a quasispectral operator which is not prespectral of any class. We show that a quasispectral operator is decomposable. We prove that if a quasispectral operator has a closed

range, then so does its scalar part. Finally in this chapter we consider further decompositions of quasispectral operators.

In Chapter Five we obtain analogues for quasispectral operators of results in Chapter 10 of [12] on logarithms and roots of prespectral operators. We give an affirmative answer to the following question. Does there exist a prespectral operator T of class Γ such that $f(T) = A$, where A is a given prespectral operator of class Γ?

Finally in Chapter Six, we prove a commutativity theorem for $\mathcal{U}$-scalar operators, where $\mathcal{U} = C(K)$ and K is a compact subset of the complex plane.

$\underset{\sim}{R}$	field of real numbers
$\underset{\sim}{C}$	field of complex numbers
$L(X)$	algebra of bounded linear operators on X
$C(K)$	algebra of all continuous complex-valued functions on a compact Hausdorff space K under the supremum norm
X^*	dual space of X
T^*	adjoint of an operator T
$T\|Y$	restriction of a mapping T to Y
$\|\|T\|\|$	norm of T
$R(T)$	range of T
$N(T)$	null-space of T
$\sigma(T)$	spectrum of T
$\rho(T)$	resolvent set of T
$\sigma_a(T)$	approximate point spectrum of T
$\sigma_c(T)$	continuous spectrum of T
$\sigma_r(T)$	residual spectrum of T
$\sigma_p(T)$	point spectrum of T
$\nu(T)$	spectral radius of T
$Y^\perp$	annihilator of a closed subspace Y of X
χ_τ	characteristic function of τ
Σ_p	σ-algebra of Borel subsets of $\underset{\sim}{C}$
$\overline{A}$ <u>or</u> clA	closure of the set A
A^o	interior of the set A
A'	complement of the set A
$\langle x,y \rangle$	value of the functional y in X^* at the point x of X
$\subseteq$	is contained in
$\subset$	is strictly contained in
$A \oplus B$	direct sum of A and B

CORRIGENDUM

A gap occurs in the proof of Theorem 4.1.5 on page 34. Define

$$K = \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}.$$

Let δ be a closed subset of $\mathbb{C} \setminus K$. Observe that $E(\delta)X$ is invariant under T and $\sigma(T|E(\delta)X) \subseteq \delta$. Also $\sigma(T|E(\delta)X) \subseteq K$, since $\|T|E(\delta)X\| \leq \|T\|$.

Hence $\sigma(T|E(\delta)X) = \emptyset$ and therefore $E(\delta) = 0$. There is a countable family $\{\tau_n\}$ of closed subsets of $\mathbb{C} \setminus K$. Since $E(\tau_n) = 0$ for each n it follows from the countable additivity of $E(\cdot)$ in the Γ-topology that $E(\mathbb{C} \setminus K) = 0$ and so $E(K) = I$. Now define

$$S_o = \int_K \lambda E(d\lambda), \quad N_o = T - S_o.$$

Then S_o is a scalar-type operator with resolution of the identity $E(\cdot)$ of class Γ and $S_oN_o = N_oS_o$, using Theorem 3.1.1. Since S_o and T commute, it is easy to see that, for δ closed, the spectral radius of $N_o|E(\delta)X$ does not exceed the diameter of δ. The argument given shows that N_o is quasinilpotent. It now follows that $\sigma(S_o) = \sigma(T)$ and hence that $S = S_o$, $N = N_o$. This completes the proof.

Let $A \subseteq X$. We denote by $\overline{A}, A^{\circ}$ and A' the norm closure, the interior and the complement of A in X respectively.

Let Y be a closed subspace of X. The quotient space of X by Y is denoted by X/Y.

Let $T \in L(X)$ and let $TY \subseteq Y$. The restriction of T to Y is denoted by $T|Y$. The operator induced by T on the quotient space X/Y will be denoted by T_Y.

We write N(T) and R(T) for the null-space and range respectively of the operator T. $\sigma(T)$ and $\rho(T)$ denote the spectrum and resolvent set of T respectively.

$\check{\mathcal{F}}(T)$ denotes the family of complex functions analytic on some open neighbourhood of $\sigma(T)$.

$\sigma_a(T)$ denotes the approximate point spectrum of T.

$\sigma_c(T)$ denotes the continuous spectrum of T.

$\sigma_p(T)$ denotes the point spectrum of T.

$\sigma_r(T)$ denotes the residual spectrum of T.

$\nu(T)$ denotes the spectral radius of T.

2. Spectral theory of linear operators

1. DEFINITION. Let $T \in L(X)$. The resolvent set $\rho(T)$ of T is the set of complex numbers λ for which $(\lambda I - T)$ is invertible in the Banach algebra L(X). The spectrum $\sigma(T)$ of T is the set $\mathbb{C} \setminus \rho(T)$. The function

$$\lambda \to (\lambda I - T)^{-1} \quad (\lambda \in \rho(T))$$

is called the resolvent of T.

2. DEFINITION. Let $T \in L(X)$. The spectral radius of T is defined by

$$\nu(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

3. <u>DEFINITION.</u> Let $T \in L(X)$. T is said to be <u>quasinilpotent</u> if and only if

$$\lim_{n \to \infty} ||T^n||^{1/n} = 0.$$

Let $T \in L(X)$. The resolvent set $\rho(T)$ is open. Also the spectrum $\sigma(T)$ is a compact set which is non-empty. The function

$$\lambda \to (\lambda I - T)^{-1} \quad (\lambda \in \rho(T))$$

is analytic. The spectral radius of T has the properties

$$\nu(T) = \lim_{n \to \infty} ||T^n||^{1/n} \leq ||T||.$$

It follows that T is quasinilpotent if and only if $\sigma(T) = \{0\}$.

The spectra of T and T* coincide. Moreover

$$((\lambda I - T)^{-1})^* = (\lambda I^* - T^*)^{-1} \quad (\lambda \in \rho(T)).$$

For proofs of these facts, see [12], pp. 3-7.

4. <u>DEFINITION.</u> Let $T \in L(X)$. Define

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not one-to-one}\};$$
$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is one-to-one}, \overline{(\lambda I - T)X} = X \text{ but } (\lambda I - T)X \neq X\};$$
$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is one-to-one but } \overline{(\lambda I - T)X} \neq X\}.$$

$\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are called respectively the <u>point spectrum</u>, the <u>continuous spectrum</u> and the <u>residual spectrum</u> of T. Clearly $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint and

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

If $\lambda \in \sigma_p(T)$ we say that λ is an <u>eigenvalue</u> of T. If this is the case there is a non-zero vector x in X such that $Tx = \lambda x$. Such a vector is called an <u>eigenvector</u> corresponding to the eigenvalue λ of T.

5. DEFINITION. Let $T \in L(X)$. Define

$\sigma_a(T) = \{\lambda \in \underset{\sim}{C}$: there is a sequence $\{x_n\}$ in X with

$$||x_n|| = 1 \text{ and } \lim_{n \to \infty} ||(\lambda I - T)x_n|| = 0.\}.$$

$\sigma_a(T)$ is called the approximate point spectrum of T.

6. DEFINITION. Let $T \in L(X)$. We denote by $\mathcal{F}(T)$ the family of all functions which are analytic on some open neighbourhood of $\sigma(T)$. Let $f \in \mathcal{F}(T)$ and let U be an open subset of $\underset{\sim}{C}$ whose boundary B consists of a finite number of rectifiable Jordan curves. We assume throughout that B is oriented so that

$$\int_B (\lambda-\mu)^{-1}\, d\lambda = 2\pi i \quad (\mu \in U),$$

$$\int_B (\lambda-\mu)^{-1}\, d\lambda = 0 \quad (\mu \notin U \cup B).$$

Suppose that $U \supseteq \sigma(T)$, and that $U \cup B$ is contained in the domain of analyticity of f. Then the operator $f(T)$ is defined by the equation

$$f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda I - T)^{-1} d\lambda.$$

The integral exists as a limit of Riemann sums in the norm of $L(X)$. It follows from Cauchy's theorem that $f(T)$ depends only on the function f and not on the open set U chosen to define this operator.

7. THEOREM. Let $T \in L(X)$ and let $f \in \mathcal{F}(T)$. Then $f(\sigma(T)) = \sigma(f(T))$.

For a proof of this result, the reader is referred to [12], p.13. This result is known as the spectral mapping theorem.

8. THEOREM. (i) Let $A \in L(X)$. Then $\exp A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$.

(ii) Let $A, B \in L(X)$ and $AB = BA$. Then

$$\exp(A+B) = \exp A . \exp B.$$

(iii) Let $A \in L(X)$. Then $\exp A . \exp(-A) = I$.

(iv) Let $A \in L(X)$. Then $\exp A$ is invertible in $L(X)$.

For a proof of this result see Theorem 1.23 of [12], p.14-15.

9. THEOREM. Let $S, N \in L(X)$, where $SN = NS$ and N is quasinilpotent. Then $\sigma(S+N) = \sigma(S)$. If f is analytic on a neighbourhood of $\sigma(S)$, then

$$f(S+N) = \sum_{n=0}^{\infty} \frac{f^{(n)}(S)N^n}{n!}.$$

For a proof of this result the reader is referred to Theorem 1.27 of [12], p.19.

Let Y be a closed subspace of X. Then Y is a Banach space under the norm of X. The annihilator $Y^{\perp}$ of Y is the closed subspace of X^* defined by

$$Y^{\perp} = \{f \in X^* : f(y) = 0 \text{ for all } y \text{ in } Y\}.$$

Now let $T \in L(X)$ and let Y be a closed subspace of X. Then Y is said to be invariant under T if and only if $TY \subseteq Y$. If this is the case we can define an operator $T|Y$ in $L(Y)$ by

$$(T|Y)y = Ty \quad (y \in Y).$$

$T|Y$ is called the restriction of T to Y. Next, we introduce an equivalence relation on X as follows

$$x_1 \sim x_2 \text{ if and only if } x_1 - x_2 \in Y.$$

The set of equivalence classes of elements of X corresponding to this equivalence relation is a complex vector space under the operations

$$[x_1]_Y + [x_2]_Y = [x_1+x_2]_Y,$$

$$\alpha[x]_Y = [\alpha x]_Y \qquad (\alpha \in \mathbb{C}).$$

This vector space is called the quotient space of X modulo Y and is denoted by X/Y. Define

$$||[x]_Y|| = \inf\{||x+y|| : y \in Y\}.$$

This is indeed a norm on X/Y and moreover X/Y is a complex Banach space under this norm. The mapping ϕ defined by

$$\phi(x) = [x]_Y$$

is called the <u>canonical mapping</u> of X onto X/Y. ϕ is continuous, linear and $||\phi|| \leq 1$. For a complete discussion of these facts the reader is referred to [4], pp.99-101 and 194.

Now let $T \in L(X)$ and let Y be a closed subspace of X invariant under T. The map

$$T_Y[x]_Y = [Tx]_Y$$

is well-defined. Moreover T_Y is a bounded linear operator on X/Y, ~~since it is the composition $\phi \circ T$ of two continuous linear mappings and~~ and $||T_Y|| \leq ||T||$. ~~$||T_Y|| \leq ||T||$.~~

10. <u>PROPOSITION.</u> Let Y be a closed subspace of X. Then there is a linear isometry J_1 of $(X/Y)^*$ onto $Y^\perp$ which is given by

$$\langle x, J_1 z\rangle = \langle [x]_Y, z\rangle$$

for all z in $(X/Y)^*$ and all x in X.

For a proof of this result, see Theorem 5.4.5 of [4], p.196.

11. <u>PROPOSITION.</u> Let Y be a closed subspace of X. Then there is a linear isometry J_2 of $X^*/Y^\perp$ onto Y^* which is given by

$$\langle y, J_2[z]_Y\rangle = \langle y, z\rangle$$

for all z in X^* and all y in Y.

For a proof of this result, see Theorem 5.4.4 of [4], p.195.

Let $E \in L(X)$. Then E is called a <u>projection</u> if and only if $E^2 = E$.

If E is a projection on X, there are closed subspaces X_1 and X_2 of X such that:

(i) X_1 is the range of E;

(ii) X_2 is the null-space of E;

(iii) $X = X_1 \oplus X_2$.

Conversely, let X_1 and X_2 be closed subspaces of X such that

$$X = X_1 \oplus X_2.$$

Then there is a projection E in L(X) whose range is X_1 and whose null space is X_2. Moreover E is uniquely determined by these conditions. For proofs of all these facts the reader is referred to [4], pp.336-340.

Now let $E,T \in L(X)$ and let E be a projection. Suppose that T leaves invariant the range of E. This is equivalent to

$$ETEx = TEx \quad (x \in X)$$

and hence ETE = TE.

Closed subspaces X_1 and X_2 are said to reduce T if $X = X_1 \oplus X_2$ and X_1, X_2 are both invariant under T. Let E be the unique projection whose range is X_1 and whose null space is X_2. Then the condition that X_1, X_2 reduce T is equivalent to ETE = TE and

$$(I-E)T(I-E) = T(I-E).$$

These conditions are equivalent to ET = TE.

12. PROPOSITION. Let $T \in L(X)$. Suppose that the closed subspaces X_1, X_2 of X reduce T. Then

$$\sigma(T) = \sigma(T|X_1) \cup \sigma(T|X_2).$$

For a proof of this result the reader is referred to Proposition 1.37 of [12].

Finally in this section we introduce the concept of spectral projection. Let $T \in L(X)$ and let τ be an open-and-closed subset of $\sigma(T)$. There is a function f in $\mathcal{F}(T)$ which is identically one on τ and which vanishes on the rest of $\sigma(T)$. We put $E(\tau,T) = f(T)$. If it is clear which operator T is being referred to we write $E(\tau)$ for convenience. It is clear from Cauchy's theorem that $E(\tau)$ depends only on τ and not on the particular f in $\mathcal{F}(T)$ chosen to define it. $E(\tau)$ is called the spectral projection corresponding to τ. If the open-and-closed set τ consists of the single point λ, the symbol $E(\lambda)$ will be used instead of $E(\{\lambda\})$. It will be convenient also to use the symbol $E(\tau)$ for any set τ of complex numbers for which $\tau \cap \sigma(T)$ is an open-and-closed subset of $\sigma(T)$. In this case we put

$$E(\tau) = E(\tau \cap \sigma(T)).$$

Thus $E(\tau) = 0$ if $\tau \cap \sigma(T)$ is empty.

13. THEOREM. Let $T \in L(X)$. Let Σ_o denote the Boolean algebra of open-and-closed subsets of $\sigma(T)$. If $\tau \in \Sigma_o$, then $E(\tau)$ is a projection, $TE(\tau) = E(\tau)T$ and $\sigma(T|E(\tau)X) = \tau$. The map $\tau \to E(\tau)$ is an isomorphism of Σ_o onto a Boolean algebra of projections in $L(X)$. This means that

(i) $E(\emptyset) = 0$,

(ii) $E(\sigma(T)\setminus\tau) = I - E(\tau) \qquad (\tau \in \Sigma_o)$,

(iii) $E(\tau_1 \cup \tau_2) = E(\tau_1)+E(\tau_2)-E(\tau_1)E(\tau_2)$

(iv) $E(\tau_1 \cap \tau_2) = E(\tau_1)E(\tau_2) \qquad (\tau_1,\tau_2 \in \Sigma_o)$.

For a proof of this result, see Theorem 1.39 of [12].

3. Prespectral operators

1. DEFINITION. A Boolean algebra $\underset{\sim}{B}$ of projections on X is a commutative subset of $L(X)$ such that

(i) $E^2 = E \quad (E \in \underset{\sim}{B})$;

(ii) $0 \in \underset{\sim}{B}$;

(iii) if $E \in \underset{\sim}{B}$, then $I-E \in \underset{\sim}{B}$;

(iv) if $E \in \underset{\sim}{B}$, $F \in \underset{\sim}{B}$, then

$$E \vee F = E+F-EF \in \underset{\sim}{B},$$

$$E \wedge F = EF \in \underset{\sim}{B}.$$

2. <u>DEFINITION.</u> A Boolean algebra $\underset{\sim}{B}$ of projections on X is said to be <u>bounded</u> if there is a real number M such that

$$||E|| \leq M \quad (E \in \underset{\sim}{B}).$$

3. <u>PROPOSITION.</u> Let $E_1, E_2, \ldots, E_n$ be a finite family of projections on X such that $E_r E_s = 0$ if $1 \leq r < s \leq n$. Suppose that $\alpha_1, \ldots, \alpha_n \in \underset{\sim}{C}$. Then

$$||\sum_{r=1}^{n} \alpha_r E_r|| \leq 4M \sup\{|\alpha_r| : 1 \leq r \leq n\},$$

where $M = \sup\{||E|| : E \in \underset{\sim}{B}\}$ and $\underset{\sim}{B}$ is the (necessarily finite) Boolean algebra of projections generated by $E_1, \ldots, E_n$.

For a proof of this result, see Proposition 5.3 of [12].

Next, we introduce the concept of spectral measure and then define a prespectral operator.

4. <u>DEFINITION.</u> A family $\Gamma \subseteq X^*$ is called <u>total</u> if and only if $x \in X$ and $\langle x,f \rangle = 0$ for all f in Γ together imply that $x = 0$.

5. <u>DEFINITION.</u> Let Σ be a σ-algebra of subsets of an arbitrary set Ω. Suppose that a mapping $E(\cdot)$ from Σ into a Boolean algebra of projections on X satisfies the following conditions:

(i) $E(\delta_1)+E(\delta_2)-E(\delta_1)E(\delta_2) = E(\delta_1 \cup \delta_2)$ $(\delta_1,\delta_2 \in \Sigma)$;

(ii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$ $(\delta_1,\delta_2 \in \Sigma)$;

(iii) $E(\Omega \setminus \delta) = I-E(\delta)$ $(\delta \in \Sigma)$;

(iv) $E(\Omega) = I$;

(v) there is $M > 0$ such that $||E(\delta)|| \leq M$ for all δ in Σ;

(vi) there is a total linear subspace Γ of X^* such that $\langle E(\cdot)x,f\rangle$ is countably additive on Σ, for each x in X and each f in Γ.

Then $E(\cdot)$ is called a spectral measure of class (Σ,Γ).

6. DEFINITION. An operator T in $L(X)$ is called a prespectral operator of class Γ if and only if the following two conditions are satisfied:

(i) there is a spectral measure $E(\cdot)$ of class (Σ_p,Γ) with values in $L(X)$ such that

$$TE(\delta) = E(\delta)T \qquad (\delta \in \Sigma_p),$$

where Σ_p denotes the σ-algebra of Borel subsets of the complex plane;

(ii) $$\sigma(T|E(\delta)X) \subseteq \overline{\delta} \qquad (\delta \in \Sigma_p).$$

The spectral measure $E(\cdot)$ is called a resolution of the identity of class Γ for T.

The next proposition shows how to construct a prespectral operator from an arbitrary spectral measure

7. PROPOSITION. Let Ω be a set and Σ a σ-algebra of subsets of Ω. Let $E(\cdot)$ be a spectral measure of class (Σ,Γ) with values in $L(X)$ and let $f \in B(\Omega,\Sigma)$, where $B(\Omega,\Sigma)$ is the Banach algebra of bounded measurable functions on Ω under the supremum norm. Define

$$\psi(f) = \int_\Omega f(\lambda)E(d\lambda).$$

Then if $x \in X$, $y \in \Gamma$ and $\mu(\tau) = \langle E(\tau)x,y\rangle$ $(\tau \in \Sigma)$, we have

$$\langle \psi(f)x,y\rangle = \int_\Omega f(\lambda)\mu(d\lambda).$$

Define

$$F(\tau) = E(f^{-1}(\tau)) \qquad (\tau \in \Sigma_p).$$

Then $\psi(f)$ is a prespectral operator with a resolution of the identity $F(\cdot)$ of class Γ. Also

$$\psi(f) = \int_{\underset{\sim}{C}} \lambda F(d\lambda).$$

For a proof of this result, the reader is referred to Proposition 5.8 of [12].

8. DEFINITION. Let S be a prespectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ such that

$$S = \int_{\sigma(S)} \lambda E(d\lambda).$$

Then S is called a scalar-type operator of class Γ.

9. PROPOSITION. Let S be a scalar-type operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Define

$$\psi(f) = \int_{\sigma(S)} f(\lambda)E(d\lambda) \qquad (f \in C(\sigma(S))).$$

Then

(i) $\sigma(\psi(f)) = f(\sigma(S)) \qquad (f \in C(\sigma(S)))$

and (ii) ψ is a bicontinuous algebra isomorphism from $C(\sigma(S))$ into $L(X)$.

For a proof of this result, see Proposition 5.9 of [12].

10. THEOREM. (i) Let T be a prespectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda) \quad , \quad N = T-S.$$

Then S is a scalar-type operator with a resolution of the identity $E(\cdot)$ of class Γ, and N is a quasinilpotent operator commuting with $\{E(\tau): \tau \in \Sigma_p\}$. Moreover $\sigma(S) = \sigma(T)$.

(ii) Let S be a scalar-type operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Let N be a quasinilpotent operator on X commuting with $\{E(\tau) : \tau \in \Sigma_p\}$. Then S+N is a prespectral operator with a resolution of the identity $E(\cdot)$ of class Γ. Moreover $\sigma(S+N) = \sigma(S)$.

For a proof of this result, the reader is referred to Theorem 5.15 of [12].

11. DEFINITION. Let T be a prespectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda) \quad , \quad N = T-S.$$

Then S+N is called the Jordan decomposition of T corresponding to the resolution of the identity $E(\cdot)$. S is called the scalar part and N the radical part of the decomposition.

The following results summarize the fundamental properties of the class of prespectral operators.

12. THEOREM. Let T be a prespectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Let $A \in L(X)$ and let $AT = TA$. Then

$$A \int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)E(d\lambda)A \quad (f \in C(\sigma(T))).$$

For a proof of this result the reader is referred to Theorem 5.12 of [12].

13. THEOREM. Let T be a prespectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ.

(i) If $F(\cdot)$ is any resolution of the identity for T, then

$$\int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)F(d\lambda) \quad (f \in C(\sigma(T))).$$

(ii) T has a unique resolution of the identity of class Γ.

(iii) T has a unique Jordan decomposition for resolutions of the identity of all classes.

For a proof of this result see Theorem 5.13 of [12].

14. THEOREM. Let T be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $f \in \mathcal{J}(T)$. Then f(T) is a prespectral operator with resolution of the identity $F(\cdot)$ of class Γ given by

$$F(\tau) = E(f^{-1}(\tau)) \quad (\tau \in \Sigma_p).$$

For a proof of this result the reader is referred to Theorem 5.16 of [12].

15. THEOREM. Let K be a compact Hausdorff space, and let ψ be a continuous algebra homomorphism of C(K) into L(X) with $\psi(1) = I$. Let N, in L(X), be a quasinilpotent commuting with $\psi(f)$ for every f in C(K). Then there is a spectral measure $E(\cdot)$ of class (Σ_K, X), where Σ_K denotes the σ-algebra of Borel subsets of K, with values in $L(X^*)$ such that

$$\text{(i)} \quad \psi(f)^* = \int_K f(\lambda)E(d\lambda) \qquad (f \in C(K))$$

and (ii) $N^*E(\tau) = E(\tau)N^* \qquad (\tau \in \Sigma_K)$.

Moreover if $S \in \psi(C(K))$, then the adjoint of $T = S+N$ is prespectral of class X, and S^*+N^* is the Jordan decomposition of T^*.

For a proof of this result the reader is referred to Theorem 5.21 of [12].

16. THEOREM. Let $S,N \in L(X)$. Suppose that S is a scalar-type operator and N is a quasinilpotent commuting with S. Suppose also that

$A \in L(X)$ and A commutes with S+N. Then A commutes with each of S and N. Moreover, if $S+N = S_o+N_o$, where S_o is a scalar-type operator on X, N_o is a quasinilpotent and $S_oN_o = N_oS_o$ then $S = S_o$ and $N = N_o$.

For a proof of this result see Theorem 5.23 of [12].

17. <u>THEOREM.</u> Let S be a scalar-type operator on X. Let N, in L(X), be a quasinilpotent operator with SN = NS. Then if T = S+N is prespectral, every resolution of the identity for T is also a resolution of the identity for S. Also, T = S+N is the unique Jordan decomposition for T. Moreover N commutes with every resolution of the identity for T.

For a proof of this result the reader is referred to Theorem 5.24 of [12].

18. <u>THEOREM.</u> Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let N, in L(X), be quasinilpotent with SN = NS. Then S+N is prespectral of class Γ if and only if

$$NE(\tau) = E(\tau)N \qquad (\tau \in \Sigma_p).$$

For a proof of this result see Theorem 5.25 of [12].

19. <u>THEOREM.</u> Let T be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let τ be an open-and-closed subset of $\sigma(T)$. Then $E(\tau)$ is equal to the spectral projection corresponding to τ.

For a proof of this result the reader is referred to Theorem 5.27 of [12].

20. <u>THEOREM.</u> Let T be a prespectral operator on X. Then $\sigma_a(T) = \sigma(T)$.

For a proof of this result see Theorem 5.47 of [12].

CHAPTER TWO

Solution to a problem of T.A. Gillespie

The purpose of this brief chapter is to answer affirmatively the following question posed by Dr. T.A. Gillespie. Is it possible to construct a complex Banach space Y and a homomorphism $E(\cdot)$ from Σ_p into a bounded Boolean algebra of projections on Y, such that $E(\cdot)$ is not a spectral measure of any class?

1. The example

In order to construct an example of the type specified above, a preliminary result is required.

1. LEMMA. Let $T \in L(X)$. Let E,F be projections in L(X) such that EF = F and T,E,F commute. Then

$$\sigma(T|FX) \subseteq \sigma(T|EX) \subseteq \sigma(T).$$

PROOF. Let $\lambda \in \rho(T)$. Now E commutes with T and hence also with $(\lambda I-T)^{-1}$. Therefore $(\lambda I-T)^{-1}$ leaves EX invariant, and its restriction to that subspace is a bounded operator, clearly inverse to $(\lambda I-T)|EX$. Hence $\lambda \in \rho(T|EX)$ and $\sigma(T|EX) \subseteq \sigma(T)$. Similarly $T|EX$ commutes with $F|EX$, and $\sigma(T|FX) \subseteq \sigma(T|EX)$.

2. EXAMPLE. On the subspace of ℓ^∞ consisting of convergent sequences, the map which assigns to each such sequence its limit is a linear functional of norm one. Throughout this section L denotes a fixed linear functional on ℓ^∞ with $||L|| = 1$ such that for each convergent sequence $\{\xi_n\}$ we have

$$L(\{\xi_n\}) = \lim_{n \to \infty} \xi_n.$$

Define operators S and A on ℓ^∞ by

$$S\{\xi_n\} = \{\eta_n\},$$

where

$$\eta_n = \xi_n \qquad (n = 1,2),$$

$$= \frac{n-2}{n-1}\xi_n \qquad (n = 3,4,5, \ldots);$$

$$A\{\xi_n\} = \{L(\{\xi_n\}), L(\{\xi_n\}), 0,0,0, \ldots\}.$$

Clearly $||A|| = 1$ and $A^2 = 0$. Also

$$S\{\xi_n\} = \{\xi_n\} - \{\gamma_n\},$$

where

$$\gamma_n = 0 \qquad (n = 1,2),$$

$$= \frac{1}{n-1}\xi_n \qquad (n = 3,4,5, \ldots).$$

Since $L(\{\gamma_n\}) = 0$, then $AS\{\xi_n\} = A\{\xi_n\}$. It is easy to see that $SA\{\xi_n\} = A\{\xi_n\}$, and hence

$$AS = SA.$$

$\sigma(S)$ is the totally disconnected set consisting of 1 and the numbers $(n-2)/(n-1)$ for $n = 3,4,5, \ldots$. By regarding S as the adjoint of an operator on ℓ^1, it is easy to see that S is prespectral with a (unique) resolution of the identity $E(\cdot)$ of class ℓ^1 satisfying

$$E(\{1\})\{\xi_k\} = \{\xi_1,\xi_2,0,0,0, \ldots\},$$

$$E\left(\left\{\frac{n-2}{n-1}\right\}\right)\{\xi_k\} = \{\delta_{kn}\xi_k\} \qquad (n = 3,4, \ldots),$$

where δ_{kn} denotes the Kronecker delta. Define the sequence $\{\lambda_n\}$ by setting

$$\lambda_n = 1 \qquad (n = 1,2),$$

$$\lambda_n = \frac{n-2}{n-1} \qquad (n = 3,4,5, \ldots).$$

Then it is easy to see that for τ in Σ_p, $E(\tau)$ is the operator which multiplies the n^{th} term of a sequence by 1 if $\lambda_n \in \tau$ and by 0 if $\lambda_n \notin \tau$. The sequence $\{f_n\}$ of functions on $\sigma(S)$, given by

$$f_n(\lambda) = \lambda \quad \text{if} \quad \lambda < \frac{n-2}{n-1},$$

$$f_n(\lambda) = 1 \quad \text{if} \quad \lambda \geq \frac{n-2}{n-1},$$

for $n = 3,4,5, \ldots$ converges uniformly to the function identically equal to λ on $\sigma(S)$. One sees directly that

$$\int_{\sigma(S)} f_n(\lambda)E(d\lambda) \quad \text{converges to } S \text{ in the norm of } L(\ell^\infty),$$

and hence

$$S = \int_{\sigma(S)} \lambda E(d\lambda).$$

Observe that

(i) S is a scalar-type operator on ℓ^∞ of class ℓ^1;

(ii) S is the adjoint of a scalar-type spectral operator on ℓ^1;

(iii) $(\ell^\infty)^*$ is weakly complete by IV.8.16 and IV.9.9 of [14];

(iv) $\sigma(S)$ is totally disconnected.

We now construct another resolution of the identity for S. Define

$$F(\tau) = E(\tau)+AE(\tau)-E(\tau)A \quad (\tau \in \Sigma_p). \tag{1}$$

Using the relations $A^2 = 0$ and $AE(\tau)A = 0$ $(\tau \in \Sigma_p)$, it is easily verified that $F(\cdot)$ is a homomorphism from Σ_p into a Boolean algebra of projections on ℓ^∞ with $F(\sigma(S)) = I$. Clearly $||F(\tau)|| \leq 3$ for all τ in Σ_p.

For each positive integer n let e_n in ℓ^1 be given by

$$e_n = \{\delta_{nk}\}_{k=1}^{\infty}$$

and let e_n^* be the corresponding linear functional on ℓ^∞. Let Γ_1 be the total linear subspace in $(\ell^\infty)^*$ generated by e_1^*-L, e_2^*-L, and

$\{e_n^*: n = 3,4,5, \ldots\}$. Since for each τ in Σ_p and x in ℓ^∞

$$\left\langle F(\tau)x, e_n^* \right\rangle = \left\langle E(\tau)x, e_n^* \right\rangle \qquad (n = 3,4,5, \ldots),$$

$$\left\langle F(\tau)x, e_n^* - L \right\rangle = \chi_\tau(1)\left\langle x, e_n^* - L \right\rangle \qquad (n = 1,2),$$

it follows that $F(\cdot)$ is Γ_1-countably additive. Since $E(\cdot)$ and A commute with S, elementary algebra shows that

$$F(\tau)S = SF(\tau) \qquad (\tau \in \Sigma_p).$$

In order to prove that $F(\cdot)$ is a resolution of the identity for S it remains only to show that

$$\sigma(S|F(\tau)\ell^\infty) \subseteq \overline{\tau} \qquad (\tau \in \Sigma_p).$$

By virtue of Lemma 2.1.1 it suffices to prove this inclusion when τ is a closed subset of $\sigma(S)$. Again by Lemma 2.1.1, and the fact that $\sigma(S)$ is totally disconnected, it is sufficient to prove the inclusion for an open-and-closed subset τ of $\sigma(S)$. It is easy to see from the definition of $F(\cdot)$ that $E(\cdot)$ and $F(\cdot)$ agree on finite subsets of $\sigma(S)\setminus\{1\}$. Since every open-and-closed subset of $\sigma(S)$ is such a set or the complement in $\sigma(S)$ of such a set, $F(\cdot)$ and $E(\cdot)$ agree on open-and-closed subsets of $\sigma(S)$. Therefore

$$\sigma(S|F(\tau)\ell^\infty) = \sigma(S|E(\tau)\ell^\infty) \subseteq \tau$$

for τ open-and-closed in $\sigma(S)$. Observe that

$$AE(\{1\})\{1,1,1, \ldots\} = \{0,0,0, \ldots\}$$

and

$$E(\{1\})A\{1,1,1, \ldots\} = \{1,1,0, \ldots\}.$$

Hence, although A commutes with S, A does not commute with the resolution of the identity $E(\cdot)$. It now follows from (1) that $F(\{1\}) \neq E(\{1\})$. Therefore $F(\cdot)$ and $E(\cdot)$ are distinct resolutions of the identity for S.

Observe that $F^*(\cdot)$ and $E^*(\cdot)$ are Boolean algebra homomorphisms from Σ_p into $L(Y)$, where $Y = (\ell^\infty)^*$. Also $F^*(\cdot)$ and $E^*(\cdot)$ are bounded, and

$$F^*(\underset{\sim}{C}) = E^*(\underset{\sim}{C}) = I.$$

We observe that $F^*(\tau)S^* = S^*F^*(\tau)$ and then prove that

$$\sigma(S^*|F^*(\tau)Y) \subseteq \overline{\tau} \quad (\tau \in \Sigma_p).$$

By virtue of Lemma 2.1.1, it suffices to prove this inclusion when τ is a closed subset of $\sigma(S)$. Again by Lemma 2.1.1, and the fact that $\sigma(S)$ is totally disconnected, it is sufficient to prove the inclusion for an open-and-closed subset τ of $\sigma(S)$. We note that for such a τ

$$F(\tau) = \frac{1}{2\pi i} \int_B (\lambda I - S)^{-1} d\lambda,$$

where B is a suitable finite family of Jordan contours which enclose τ but exclude $\sigma(S)\setminus\tau$. We have

$$F^*(\tau) = \frac{1}{2\pi i} \int_B (\lambda I - S^*)^{-1} d\lambda$$

and so $F^*(\tau)$ is the spectral projection corresponding to the open-and-closed subset τ of $\sigma(S^*) = \sigma(S)$. Thus

$$\sigma(S^*|F^*(\tau)Y) = \tau \qquad (\tau \text{ open-and-closed})$$

and so

$$\sigma(S^*|F^*(\tau)Y) \subseteq \overline{\tau} \qquad (\tau \in \Sigma_p).$$

It was established earlier that $E(\cdot)$ and $F(\cdot)$ coincide on open-and-closed subsets of $\sigma(S)$. Hence $F^*(\cdot)$ and $E^*(\cdot)$ also coincide on open-and-closed subsets of $\sigma(S^*) = \sigma(S)$. Thus, the argument above establishes similarly that $S^*E^*(\tau) = E^*(\tau)S^*$ and

$$\sigma(S^*|E^*(\tau)Y) \subseteq \overline{\tau} \qquad (\tau \in \Sigma_p).$$

Now, S is prespectral operator and so, by Theorem 5.22 of [12], S^*

is a prespectral operator on Y. As noted earlier, Y is weakly complete and so, by Theorem 6.11 of [12], S^* is a scalar-type spectral operator. By Theorem 6.7 of [12], S^* has a unique resolution of the identity $G(\cdot)$ say. If $E^*(\cdot)$ and $F^*(\cdot)$ were both spectral measures, then they would form distinct resolutions of the identity for S^*, and so at least one, $H(\cdot)$ say, is distinct from $G(\cdot)$. Thus, there is a complex Banach space $Y = (\ell^{\infty})^*$ and a homomorphism $H(\cdot)$ from Σ_p into a bounded Boolean algebra of projections on Y such that $H(\cdot)$ is not a spectral measure of any class.

CHAPTER THREE

A commutativity theorem for a scalar-type operator

The purpose of this section is to show that if S is a scalar-type operator and A is an operator that leaves invariant all the maximal spectral subspaces of S, then A commutes with S. The method of proof is to first observe that the argument given in Theorem 1 of [6], p.526-530 is sufficient to deduce a special case of the result above. The general case then follows from a result of Colojoara and Foias.

1. The commutativity theorem

Prior to proving the first result in this section we require some additional notation. $\underset{\sim}{R}$ denotes the real line. Also, if $\tau \subseteq \underset{\sim}{C}$, and $z \in \underset{\sim}{C}$, then $\chi(\tau, z)$ denotes the characteristic function of the set τ evaluated at z.

1. THEOREM. Let S be a scalar-type operator of class Γ on X. Then S^* is a scalar-type operator on X^* with resolution of the identity $F(\cdot)$, say, of class X. Suppose that A, in L(X), has the property that

$$A^*F(\delta)X^* \subseteq F(\delta)X^*$$

for every closed subset δ of $\underset{\sim}{C}$. Then AS = SA.

PROOF. We first obtain some consequences of the hypothesis of the theorem. The first is merely a restatement of this hypothesis.

(1) If $\delta \subseteq \underset{\sim}{C}$ is closed, then $A^*F(\delta) = F(\delta)A^*F(\delta)$.

Next, we obtain an analogous result for open subsets of $\underset{\sim}{C}$.

(2) If $\tau \subseteq \underset{\sim}{C}$ is open, then $F(\tau)A^* = F(\tau)A^*F(\tau)$.

To see this, observe that $\underset{\sim}{C}\setminus\tau$ is closed and so by (1)

$$A^*(I^*-F(\tau)) = (I^*-F(\tau))A^*(I^*-F(\tau)).$$

On rearranging we obtain (2). We require also the following result.

(3) If $\delta \subseteq \underset{\sim}{C}$ is closed, $\tau \in \Sigma_p$ and $\overline{\tau} \cap \delta = \emptyset$, then $F(\delta)A^*F(\tau) = 0$.

To see this result observe that by (1) and hypothesis

$$F(\delta)A^*F(\overline{\tau}) = F(\delta)F(\overline{\tau})A^*F(\overline{\tau}) = 0.$$

Now, post-multiplying both sides of the equation $F(\delta)A^*F(\overline{\tau}) = 0$ by $F(\tau)$ gives the desired result.

Now let $E(\cdot)$ be the resolution of the identity of class Γ for S. Define

$$R = \int_{\sigma(S)} \mathrm{Re}\lambda E(d\lambda) \ , \ J = \int_{\sigma(S)} \mathrm{Im}\lambda E(d\lambda).$$

Observe that $S = R+iJ$. By Theorem 5.22 of [12], p.137, we have

$$\left(\int_{\sigma(S)} f(\lambda)E(d\lambda)\right)^* = \int_{\sigma(S)} f(\lambda)F(d\lambda) \quad (f \in C(\sigma(S))).$$

Using this in conjunction with 1.3.7, we see that R^* is a scalar-type operator on X^* with resolution of the identity $G(\cdot)$ of class X such that

$$R^* = \int_{\sigma(R)} \lambda G(d\lambda) \ , \ G(\underset{\sim}{C}\setminus\underset{\sim}{R}) = 0,$$

and for every real number ξ,

$$(4) \qquad G(\{\xi\}) = F(L_\xi),$$

where L_ξ is the line parallel to the imaginary axis through the point ξ. Let $x \in X$, $y \in X^*$. Define

$$g(\lambda) = \langle Ax, G((-\infty,\lambda])y\rangle \qquad (\lambda \in \underset{\sim}{R}),$$

$$h(\lambda) = \langle x, G((-\infty,\lambda])A^*y\rangle \qquad (\lambda \in \underset{\sim}{R}).$$

Now $\langle Ax, G(\cdot)y\rangle$ and $\langle x, G(\cdot)A^*y\rangle$ may be regarded as complex Borel measures on $\underset{\sim}{R}$. Hence g and h are right-continuous complex functions of bounded variation on $\underset{\sim}{R}$. Therefore the set D of points of $\underset{\sim}{R}$ at which either g or h is discontinuous is countable. If $\xi \in \underset{\sim}{R}\setminus D$ we have

$$\langle Ax, G(\{\xi\})y\rangle = \langle x, G(\{\xi\})A^*y\rangle = 0.$$

Hence, using (4) we obtain

$$(5) \qquad \langle x, A^*F(L_\xi)y\rangle = \langle x, F(L_\xi)A^*y\rangle = 0 \quad (\xi \in \underset{\sim}{R}\setminus D).$$

Now, $\sigma(S)$ is compact, and so there is a positive real number K such that

$$(6) \qquad \sigma(S) \subseteq \{z \in \underset{\sim}{C} : -K < \mathrm{Re}z < +K\}.$$

Let Ω denote the set on the right-hand side of (6). Observe that

$$(7) \qquad F(\underset{\sim}{C}\setminus\Omega) = F(\underset{\sim}{C}\setminus\overline{\Omega}) = 0.$$

Next, we construct a suitable sequence of functions converging uniformly to Rez on Ω. Let n be a positive integer. Since D is countable, $\underset{\sim}{R}\setminus D$ is dense in $\underset{\sim}{R}$ and so we may choose points $\{\xi_m : m = 0,1, \ldots, 2n+1\}$ in $\underset{\sim}{R}\setminus D$ such that the following two conditions hold:

$$(8) \qquad -K = \xi_o < \xi_1 < \ldots < \xi_{2n+1} = +K;$$

$$(9) \qquad |\xi_{m+1}-\xi_m-2K/(2n+1)| < 2K/(2n+1)^2 \quad (m = 0,1,2, \ldots, 2n).$$

We obtain immediately from (9)

$$(10) \qquad \xi_{m+1}-\xi_m < \frac{K}{n} \quad (m = 0,1, \ldots, 2n).$$

For m = 0,1, ..., 2n+1, let L_m be the line parallel to the imaginary axis through the point ξ_m. Define

$$(11) \qquad \tau_m = \{z \in \underset{\sim}{C} : \xi_{m-1} < \mathrm{Re}z < \xi_m\} \quad (m = 1, \ldots, 2n+1);$$

$$(12) \qquad \begin{aligned} \delta_m &= \{z \in \underset{\sim}{C} : (\xi_{m-1}+\xi_m)/2 < \mathrm{Re}z < \xi_m\} \quad (m = 1, \ldots, 2n+1); \\ f_n(z) &= \sum_{m=o}^{n} \xi_{2m+1}\chi(\tau_{2m+1},z) + \sum_{m=1}^{n} \xi_{2m}\chi(\overline{\tau}_{2m},z) \quad (z \in \overline{\Omega}). \end{aligned}$$

Observe that by (10), $f_n(z)$ converges to Rez uniformly on Ω and so as $n \to \infty$

$$(13) \qquad \int_\Omega f_n(\lambda)F(d\lambda) \to \int_\Omega \mathrm{Re}\lambda F(d\lambda) = \int_{\sigma(S)} \mathrm{Re}\lambda F(d\lambda) = R^*.$$

(The first equality follows from (6).) This leads us to consider the expression η defined by

$$(14) \qquad \eta = \left\langle x, \sum_{m=o}^{n} \xi_{2m+1}(A^*F(\tau_{2m+1})-F(\tau_{2m+1})A^*)y \right\rangle + \left\langle x, \sum_{m=1}^{n} \xi_{2m}(A^*F(\overline{\tau}_{2m})-F(\overline{\tau}_{2m})A^*)y \right\rangle.$$

Now, by (11),

$$\overline{\tau}_m = \tau_m \cup L_{m-1} \cup L_m \quad (m = 1, \ldots, 2n+1)$$

and the sets on the right-hand side of this equation are pairwise disjoint. Therefore

$$(15) \qquad F(\overline{\tau}_m) = F(\tau_m)+F(L_{m-1})+F(L_m) \quad (m = 1, \ldots, 2n+1).$$

However by (5)

$$\langle x, A^*F(L_m)y \rangle = \langle x, F(L_m)A^*y \rangle = 0 \quad (m = 1, \ldots, 2n+1)$$

and so (14) becomes

$$(16) \qquad \eta = \left\langle x, \sum_{m=1}^{2n+1} \xi_m(A^*F(\tau_m)-F(\tau_m)A^*)y \right\rangle.$$

Observe that by (1)

$$A^*F(\overline{\tau}_m) = F(\overline{\tau}_m)A^*F(\overline{\tau}_m) \quad (m = 1, \ldots, 2n+1).$$

Combining this with (15) gives for $m = 1, \ldots, 2n+1$,

$$A^*(F(\tau_m)+F(L_m)+F(L_{m-1})) = (F(\tau_m)+F(L_m)+F(L_{m-1}))A^*(F(\tau_m)+F(L_m)+F(L_{m-1})).$$

This may be rewritten as

$$(17) \qquad A^*F(\tau_m)-F(\tau_m)A^* = F(L_{m-1})A^*F(\tau_m)+F(L_m)A^*F(\tau_m)$$

by virtue of the equations

$$F(\tau_m)A^* = F(\tau_m)A^*F(\tau_m),\ A^*F(L_m) = F(L_m)A^*F(L_m),$$

$$A^*F(L_{m-1}) = F(L_{m-1})A^*F(L_{m-1}),$$

$$F(\tau_m)A^*F(L_m) = F(\tau_m)F(L_m)A^*F(L_m) = 0,$$

$$F(\tau_m)A^*F(L_{m-1}) = F(\tau_m)F(L_{m-1})A^*F(L_{m-1}) = 0,$$

$$F(L_m)A^*F(L_{m-1}) = F(L_m)F(L_{m-1})A^*F(L_{m-1}) = 0,$$

$$F(L_{m-1})A^*F(L_m) = F(L_{m-1})F(L_m)A^*F(L_m) = 0,$$

all of which follow from (1), (2) and (3). From (16) and (17) we obtain

$$(18) \quad \eta = \left\langle x, \sum_{m=1}^{2n+1} \xi_m (F(L_{m-1})A^*F(\tau_m)+F(L_m)A^*F(\tau_m))y \right\rangle.$$

We require two more formulae for η. To obtain the first of these, observe that by (6) and (8) we have $F(L_o) = F(L_{2n+1}) = 0$. By (1) and (5)

$$\left\langle x, F(L_m)A^*F(L_m)y \right\rangle = \left\langle x, A^*F(L_m)y \right\rangle = 0,$$

$$F(L_m)A^*F(C\setminus(\tau_m \cup \tau_{m+1} \cup L_m)) = 0.$$

It follows from the last two equations and (5) that

$$\left\langle x, F(L_m)A^*F(\tau_m)y \right\rangle + \left\langle x, F(L_m)A^*F(\tau_{m+1})y \right\rangle = \left\langle x, F(L_m)A^*y \right\rangle = 0.$$

From these facts we may rewrite equation (18) as follows

$$(19) \quad \eta = \left\langle x, \sum_{m=1}^{2n} (\xi_m - \xi_{m+1}) F(L_m)A^*F(\tau_m)y \right\rangle.$$

Now, it follows from (3) that $F(L_m)A^*F(\tau_m \setminus \delta_m) = 0$. Therefore (19) may be rewritten

$$(20)\qquad \eta = \left\langle x, \sum_{m=1}^{2n} (\xi_m - \xi_{m+1}) F(L_m) A^* F(\delta_m) y \right\rangle .$$

If $m \neq r$, then $\overline{\delta}_m \cap L_r = \emptyset$ and so it follows from (3) that

$$F(L_m) A^* F(\delta_r) = 0.$$

Also, if $m \neq r$, then $\delta_m \cap \delta_r = \emptyset$ and $L_m \cap L_r = \emptyset$. Hence

$$-\eta = \eta_1 + \eta_2 ,$$

where

$$\eta_1 = \left\langle x, (2K/(2n+1)) \sum_{m=1}^{2n} (F(L_m) A^* F(\delta_m)) y \right\rangle$$

$$= \left\langle x, (2K/(2n+1)) (F(\bigcup_{m=1}^{2n} L_m) A^* F(\bigcup_{m=1}^{2n} \delta_m)) y \right\rangle ,$$

$$\eta_2 = \left\langle x, \sum_{m=1}^{2n} (\xi_{m+1} - \xi_m - 2K/(2n+1)) F(L_m) A^* F(\delta_m) y \right\rangle .$$

Now let $M = \sup\{ ||F(\tau)|| : \tau \in \Sigma_p \}$. Then $M < \infty$ and

$$|\eta_1| \leq (2K/(2n+1)) ||A|| M^2 ||x|| \; ||y|| ,$$

$$|\eta_2| \leq (4nK/(2n+1)^2) ||A|| M^2 ||x|| \; ||y|| ,$$

using (9). Hence

$$|\eta| \leq (4K/(2n+1)) M^2 ||A|| \; ||x|| \; ||y||$$

$$(21)\qquad \leq (2KM^2/n) ||A|| \; ||x|| \; ||y|| .$$

From (10) we obtain

$$(22)\qquad \sup_{z \in \Omega} \left| \operatorname{Re} z - \sum_{m=o}^{n} \xi_{2m+1} \chi(\tau_{2m+1}, z) - \sum_{m=1}^{n} \xi_{2m} \chi(\overline{\tau}_{2m}, z) \right| \leq \frac{K}{n} .$$

Now, if f is any bounded Borel measurable function on $\sigma(S)$, $x_o \in X$ and $y_o \in X^*$, then we have

$$(23)\qquad \left| \left\langle x_o, \int_{\sigma(S)} f(\lambda) F(d\lambda) y_o \right\rangle \right| \leq 4M ||x_o|| \; ||y_o|| \sup_{\lambda \in \sigma(S)} |f(\lambda)| .$$

Take $x_o = Ax$, $y_o = y$ and

$$f(z) = \operatorname{Re} z - \sum_{m=o}^{n} \xi_{2m+1}\chi(\tau_{2m+1}, z) - \sum_{m=1}^{n} \xi_{2m}\chi(\overline{\tau}_{2m}, z) \quad (z \in \sigma(S)).$$

We get from (22) and (23)

$$\left\langle x, (A^*R^* - \sum_{m=o}^{n} \xi_{2m+1} A^*F(\tau_{2m+1}) - \sum_{m=1}^{n} \xi_{2m} A^*F(\overline{\tau}_{2m}))y \right\rangle$$

$$\leq (4MK/n)||A|| \; ||x|| \; ||y||.$$

Next, in (23) take $x_o = x$ and $y_o = A^*y$. Then we obtain

$$\left\langle x, (R^*A^* - \sum_{m=o}^{n} \xi_{2m+1} F(\tau_{2m+1})A^* - \sum_{m=1}^{n} \xi_{2m} F(\overline{\tau}_{2m})A^*)y \right\rangle$$

$$\leq (4MK/n)||A|| \; ||x|| \; ||y||.$$

From the last two inequalities and (14) we obtain

$$(24) \quad \left|\left\langle x, (A^*R^* - R^*A^*)y \right\rangle - \eta\right| \leq (8MK/n)||A|| \; ||x|| \; ||y||.$$

From (21) and (24) we get

$$\left|\left\langle x, (A^*R^* - R^*A^*)y \right\rangle\right| \leq (2MK||A|| \; ||x|| \; ||y||/n)\,(M+4).$$

Now n,x and y are arbitrary. Hence $A^*R^* = R^*A^*$. Similarly $A^*J^* = J^*A^*$. Since $S^* = R^* + iJ^*$, we deduce that $A^*S^* = S^*A^*$ and hence that $AS = SA$. This completes the proof of the theorem.

In order to prove the more general version of this theorem which we require, it is necessary to introduce a more general class of operators containing the prespectral operators.

2. DEFINITION. Let $T \in L(X)$. A closed subspace Y of X is called a maximal spectral subspace for T if

(i) Y is invariant under T,

and (ii) Z is another closed subspace of X, invariant under T, such that $\sigma(T|Z) \subseteq \sigma(T|Y)$,

then $Z \subseteq Y$.

3. <u>DEFINITION.</u> An operator T, in $L(X)$, is called <u>decomposable</u> if for every finite open covering $\{G_i : 1 \leq i \leq n\}$ of $\sigma(T)$ there exists a system $\{Y_i : 1 \leq i \leq n\}$ of maximal spectral subspaces for T such that

(i) $\sigma(T|Y_i) \subseteq G_i \qquad (1 \leq i \leq n)$,

(ii) every x in X can be expressed in the form $x = \sum_{i=1}^{n} y_i$, where $y_i \in Y_i$ for $i = 1, \ldots, n$.

4. <u>THEOREM.</u> A prespectral operator on X is decomposable.

<u>PROOF.</u> Let T be a prespectral operator with resolution of the identity $E(\cdot)$ of class Γ. Let δ be a closed subset of $\mathbb{C}$. It was shown in Lemma 14.3 of [12], p.266 that $E(\delta)X$ is the union of all closed subspaces Y of X with $TY \subseteq Y$ and $\sigma(T|Y) \subseteq \delta$. In other words, $E(\delta)X$ is a maximal spectral subspace for T.

Let $\{G_i : i = 1, \ldots, n\}$ be an open covering of $\sigma(T)$. Let $\lambda \in \sigma(T)$. Then $\lambda \in G_r$, for some $r = 1, \ldots, n$. There is an open disc Ω_λ with centre λ such that $\Omega_\lambda \subseteq G_r$. Let D_λ be the open disc centre λ and radius half that of Ω_λ. The discs $\{D_\lambda : \lambda \in \sigma(T)\}$ cover $\sigma(T)$ and so, by the compactness of $\sigma(T)$, there is a finite subcovering $D_1, \ldots, D_m$, say. We may assume without loss of generality that for $r = 1, \ldots, n$ each G_r contains at least one of the discs $D_1, \ldots, D_m$. Observe that for each $j = 1, \ldots, m$ we have

$$\overline{D}_j \subseteq G_i,$$

$$\overline{D}_j \cap (\mathbb{C} \setminus G_i) = \emptyset,$$

for some $i = 1, \ldots, n$. By Urysohn's lemma, there is a function f_j, in $C(\sigma(T))$, such that f_j takes the value 1 on $\overline{D}_j$ and the value 0 on $\mathbb{C} \setminus G_i$, and satisfies

$$0 \leq f_j(\lambda) \leq 1 \quad (\lambda \in \sigma(T)).$$

Define S_i to be the set of j such that supp $f_j \subseteq G_i$. (Here supp denotes the support of the function.) Then we define

$$F_i = \sum_{j \in S_i} f_j.$$

Now define $F = \sum_{i=1}^{n} F_i$ and observe that

$$F = \sum_{i=1}^{n} F_i \geq \sum_{j=1}^{m} f_j.$$

We note that for each x in $\sigma(T)$ we have

$$F(x) \geq \sum_{j=1}^{m} f_j(x) \geq 1$$

since, by construction, $f_j(x) = 1$ for at least one j. Now define

$$\phi_i = F_i / (\sum_{i=1}^{n} F_i) \quad (i = 1, \ldots, n).$$

Observe that

$$\phi_i \in C(\sigma(T)) \quad (i = 1, \ldots, n),$$

$$\sum_{i=1}^{n} \phi_i(\lambda) = 1 \quad (\lambda \in \sigma(T))$$

and

$$\delta_i = \text{supp } \phi_i \subseteq G_i \quad (i = 1, \ldots, n).$$

Then, for any x in X, we have

$$x = \sum_{i=1}^{n} x_i, \text{ where } x_i = \int_{\sigma(T)} \phi_i(\lambda) E(d\lambda)x \in E(\delta_i)X.$$

It suffices to take $Y_i = E(\delta_i)X$ $(i = 1, \ldots, n)$ in the definition 3.1.3 to establish that T is a decomposable operator.

We are now in a position to prove the main theorem of this chapter.

5. THEOREM. Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ and let $A \in L(X)$. Suppose that for each closed

set δ of $\underset{\sim}{C}$ we have $AE(\delta)X \subseteq E(\delta)X$. Then $AS = SA$.

PROOF. Define an operator $C(S,S)$ on $L(X)$ by

$$C(S,S)T = TS-ST \quad (T \in L(X)).$$

By the last theorem, S is a decomposable operator. It follows from the hypothesis of this theorem and Theorem 2.3.3 of [5], p.48 that

$$\lim_{n \to \infty} ||C(S,S)^n A||^{1/n} = 0.$$

Clearly

$$\lim_{n \to \infty} ||C(S^{**},S^{**})^n A^{**}||^{1/n} = 0,$$

where $C(S^{**},S^{**})$ is defined in an analogous way as an operator on $L(X^{**})$. Again, from Theorem 2.3.3 of [5], p.48, we conclude that

$$A^{**}G(\delta)X^{**} \subseteq G(\delta)X^{**}$$

for each closed set δ, where $G(\cdot)$ is the (unique) resolution of the identity of class X^* for the scalar-type operator S^{**}. (Note that, since S is a scalar-type operator, so are S^* and S^{**}.) We now apply Theorem 3.1.1 to the operators S^* and A^* to conclude that $A^*S^* = S^*A^*$. Hence $AS = SA$, completing the proof.

CHAPTER FOUR

Quasispectral operators

Ernst Albrecht introduced the class of quasispectral operators and established that a quasispectral operator of class Γ has a unique resolution of the identity of class Γ and a unique Jordan decomposition for resolutions of the identity of all classes. In this chapter we give a different proof of these results which is simpler in some respects than Albrecht's but less so in others. It is shown that every prespectral operator of class Γ is quasispectral of class Γ but that there exists a quasispectral operator which is not prespectral of any class. In the remainder of this chapter and in later chapters, we develop further properties of the class of quasispectral operators.

1. The basic properties of quasispectral operators

Albrecht [1] introduced the following class of operators.

1. DEFINITION. Let $T \in L(X)$. Then T is said to be a quasispectral operator of class Γ if there is a spectral measure $E(\cdot)$ of class (Σ_p, Γ) with values in $L(X)$ such that for all closed subsets δ of $\mathbb{C}$ we have

(i) $TE(\delta)X \subseteq E(\delta)X$,

and (ii) $\sigma(T|E(\delta)X) \subseteq \delta$.

The map $E(\cdot)$ is called a resolution of the identity of class Γ for T.

It is clear from this definition that if T is a prespectral operator with résolution of the identity $E(\cdot)$ of class Γ, then T is also a quasispectral operator with resolution of the identity $E(\cdot)$ of class Γ. As will be shown later in this chapter, the converse is false.

In order to establish properties of quasispectral operators it is necessary to introduce the concept of the single-valued extension property.

2. DEFINITIONS. Let $T \in L(X)$ and let $x \in X$. An X-valued function f_x, defined and analytic on an open subset $D(f_x)$ of $\mathbb{C}$ such that

$$(\zeta I-T)f_x(\zeta) = x \quad (\zeta \in D(f_x)),$$

is called a pre-imaging function for x and T. It is easy to see that

$$f_x(\zeta) = (\zeta I-T)^{-1} x \quad (\zeta \in \rho(T) \cap D(f_x)).$$

If for all x in X and all pairs $f_x^{(1)}$, $f_x^{(2)}$ of pre-imaging functions for x and T we have

$$f_x^{(1)}(\zeta) = f_x^{(2)}(\zeta) \quad (\zeta \in D(f_x^{(1)}) \cap D(f_x^{(2)})),$$

then T is said to have the single-valued extension property. In this case there is a unique pre-imaging function with maximal domain $\rho(x)$, an open set containing $\rho(T)$. The values of this function are denoted by $\{x(\zeta) : \zeta \in \rho(x)\}$. Let $\sigma(x) = \mathbb{C}\setminus\rho(x)$. Clearly $\sigma(x) \subseteq \sigma(T)$. $\rho(x)$ is called the resolvent set of x and $\sigma(x)$ is called the spectrum of x.

For a proof of the following result, the reader is referred to Theorem 5.31 of [12], p.143.

3. THEOREM. Let T be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then T has the single-valued extension

property.

The following elementary result proves to be quite important in the theory of quasispectral operators. Observe that it is converse to Theorem 3.1.5.

4. PROPOSITION. Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Suppose that $A \in L(X)$ and $AS = SA$. Then for every closed subset δ of $\mathbb{C}$ we have

$$AE(\delta)X \subseteq E(\delta)X.$$

PROOF. If δ is a closed subset of $\mathbb{C}$, then by Theorem 5.33 of [12], p.143 we have

$$E(\delta)X = \{x \in X : \sigma(x) \subseteq \delta\}.$$

Now if $x \in X$, we have

$$(\zeta I-S)Ax(\zeta) = A(\zeta I-S)x(\zeta) = Ax \quad (\zeta \in \rho(x)),$$

since $AS = SA$. Also the map $\zeta \to Ax(\zeta)$ is analytic in $\rho(x)$, and so we obtain successively

$$\rho(Ax) \supseteq \rho(x) \ ; \ \sigma(Ax) \subseteq \sigma(x).$$

Hence if $x \in E(\delta)X$, then also $Ax \in E(\delta)X$. The proof is complete.

Next, we require a stronger version of Theorem 5.10 of [12], pp.124-5. The proof of that theorem does not carry over directly to the present situation.

5. THEOREM. Let T be a quasispectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda), \quad N = T-S.$$

Then S is a scalar-type operator with resolution of the identity $E(\cdot)$ of class Γ and N is a quasinilpotent with $NS = SN$.

PROOF.* The first statement of the theorem is immediate. Now let $\varepsilon > 0$ be given. Then we can find a finite open covering of the compact set $\sigma(S)$ by sets $\{G_i : 1 \leq i \leq n(\varepsilon)\}$ each of diameter less than ε, where the diameter $d(G)$ of a set G is defined by

$$d(G) = \sup\{|\lambda-\mu| : \lambda \in G, \mu \in G\}.$$

Since S is a scalar-type operator, it is decomposable by Theorem 3.1.4. Hence there is a finite family $\{Y_i : 1 \leq i \leq n(\varepsilon)\}$ of maximal spectral subspaces for S such that $Y_i = E(\delta_i)X$ with δ_i closed,

$$\sigma(S|Y_i) \subseteq \delta_i \subseteq G_i \quad (1 \leq i \leq n(\varepsilon)), \tag{1}$$

and every x in X is of the form $\sum_{i=1}^{n(\varepsilon)} y_i$ with $y_i \in Y_i$.

Observe that by the hypothesis of the theorem we have

$$TY_i \subseteq Y_i \qquad (1 \leq i \leq n(\varepsilon)).$$

We now define

$$S_i = (S-\lambda_i I)|Y_i \quad (1 \leq i \leq n(\varepsilon)), \tag{2}$$

$$T_i = (T-\lambda_i I)|Y_i \quad (1 \leq i \leq n(\varepsilon)), \tag{3}$$

where $\lambda_i \in G_i$. It follows from the spectral mapping theorem and (1) that we have successively

$$\sigma(S_i) = \{\lambda-\lambda_i : \lambda \in \sigma(S|Y_i)\},$$

$$\sigma(S_i) \subseteq \{\lambda-\lambda_i : \lambda \in G_i\},$$

$$\sigma(S_i) \subseteq \{\mu: |\mu| < \varepsilon\} \quad (1 \leq i \leq n(\varepsilon)). \tag{4}$$

Similarly

* See Corrigendum.

$$\sigma(T_i) = \{\lambda-\lambda_i : \lambda \in \sigma(T|Y_i)\},$$

$$\sigma(T_i) \subseteq \{\lambda-\lambda_i : \lambda \in G_i\},$$

$$\sigma(T_i) \subseteq \{\mu : |\mu| < \varepsilon\} \quad (1 \leq i \leq n(\varepsilon)). \qquad (5)$$

As noted on p.3 of this thesis, the spectral radius formula

$$\lim_{n \to \infty} ||T^n||^{1/n} = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

is valid for all T in L(X). We therefore deduce from (4) and (5) that

$$\lim_{n \to \infty} ||S_i^n||^{1/n} = \sup\{|\lambda| : \lambda \in \sigma(S_i)\} \leq \varepsilon,$$

$$\lim_{n \to \infty} ||T_i^n||^{1/n} = \sup\{|\lambda| : \lambda \in \sigma(T_i)\} \leq \varepsilon,$$

valid for $1 \leq i \leq n(\varepsilon)$.

Since the terms of a convergent sequence are bounded, we deduce the existence of a real constant $M_\varepsilon > 0$ such that

$$||S_i^n|| < \varepsilon^n M_\varepsilon \text{ for every } n \geq 0 \text{ and } 1 \leq i \leq n(\varepsilon), \qquad (6)$$

$$||T_i^n|| < \varepsilon^n M_\varepsilon \text{ for every } n \geq 0 \text{ and } 1 \leq i \leq n(\varepsilon). \qquad (7)$$

Since T is a quasispectral operator with a resolution of the identity $E(\cdot)$ of class Γ, it follows from the definition 4.1.1 that

$$TE(\delta)X \subseteq E(\delta)X$$

for every closed subset δ of $\underset{\sim}{C}$. An application of Theorem 3.1.5 now yields the conclusion TS = ST. Hence SN = NS.

We deduce from (6) and (7) that for $1 \leq i \leq n(\varepsilon)$ and every positive integer n we have successively

$$||(T_i-S_i)^n|| = ||\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} T_i^k S_i^{(n-k)}||,$$

$$||(T_i-S_i)^n|| \leq \sum_{k=o}^{n} \binom{n}{k} ||T_i^k|| \, ||S_i^{n-k}||,$$

$$||(T_i-S_i)^n|| \leq M_\varepsilon^2 \sum_{k=o}^{n} \binom{n}{k} \varepsilon^k \varepsilon^{n-k} = M_\varepsilon^2 (2\varepsilon)^n, \qquad (8)$$

using the fact that $2^n = (1+1)^n = \sum_{k=o}^{n} \binom{n}{k}$.

Now define

$$Y = Y_1 \oplus \ldots \oplus Y_{n(\varepsilon)}.$$

Observe that the linear mapping

$$y_1 \oplus \ldots \oplus y_{n(\varepsilon)} \to y_1 + \ldots + y_{n(\varepsilon)}$$

from Y into X, being continuous and surjective, it follows from the closed graph theorem that there exists a constant $M \geq 0$ such that, for every x in X there is a $y_1 \oplus \ldots \oplus y_{n(\varepsilon)}$ with $x = y_1 + \ldots + y_{n(\varepsilon)}$ and

$$||y_1|| + \ldots + ||y_{n(\varepsilon)}|| \leq M||x||. \qquad (9)$$

From (2), (3), (8) and (9) it follows that for every x in X and all positive integers n we have successively

$$||(T-S)^n x|| = ||\sum_{i=1}^{n(\varepsilon)} (T-S)^n y_i||,$$

$$||(T-S)^n x|| \leq \sum_{i=1}^{n(\varepsilon)} ||(T-S)^n y_i||,$$

$$||(T-S)^n x|| \leq \sum_{i=1}^{n(\varepsilon)} ||(T_i-S_i)^n y_i||,$$

$$||(T-S)^n x|| \leq \sum_{i=1}^{n(\varepsilon)} ||(T_i-S_i)^n|| \, ||y_i||,$$

$$||(T-S)^n x|| \leq M_\varepsilon^2 (2\varepsilon)^n \sum_{i=1}^{n(\varepsilon)} ||y_i||$$

$$||(T-S)^n x|| \leq MM_\varepsilon^2 (2\varepsilon)^n ||x||,$$

whence

$$||(T-S)^n|| \leq MM_\varepsilon^2(2\varepsilon)^n.$$

It follows that

$$\overline{\lim_{n \to \infty}}||(T-S)^n||^{1/n} \leq \lim_{n \to \infty}(MM_\varepsilon^2)^{1/n} \cdot (2\varepsilon) = 2\varepsilon.$$

Now, if a sequence of non-negative real numbers fails to converge to 0, then its upper limit is positive. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{n \to \infty}||(T-S)^n||^{1/n} = 0,$$

and hence that N is quasinilpotent. The proof is complete.

Our next result is an improved version of the last result and contains the converse theorem.

6. THEOREM. (i) Let T be a quasispectral operator on X with a resolution of the identity E(•) of class Γ. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda), \quad N = T-S.$$

Then S is a scalar-type operator of class Γ and N is a quasinilpotent operator commuting with S. Moreover $\sigma(T) = \sigma(S)$.

(ii) Let S be a scalar-type operator on X and let N be a quasinilpotent operator commuting with S. Then S+N is a quasispectral operator. Moreover $\sigma(S+N) = \sigma(S)$.

PROOF. (i) The only statement not already proved in the last theorem is $\sigma(T) = \sigma(S)$. This follows from the quasinilpotence of N and Theorem 1.2.9.

(ii) Observe that if E(•) is a resolution of the identity for S, then for each closed set τ we have $SE(\tau)X \subseteq E(\tau)X$. Also, by

Proposition 4.1.4, we obtain $NE(\tau)X \subseteq E(\tau)X$. We deduce that

$$(S+N)E(\tau)X \subseteq E(\tau)X,$$

for each closed set τ. Finally, we deduce from Theorem 1.2.9 that

$$\sigma((S+N)|E(\tau)X) = \sigma(S|E(\tau)X) \subseteq \tau$$

for each closed set τ and the proof is complete.

This theorem leads us to the following definitions.

7. DEFINITIONS. Let T be a quasispectral operator on X. A sum $S+N$ such that $T = S+N$, $SN = NS$, S is a scalar-type operator and N is a quasinilpotent, is called a Jordan decomposition for T. S is called the scalar part and N the radical part of the decomposition.

Next, we prove that the adjoint of a quasispectral operator is a prespectral operator.

8. THEOREM. Let T be a quasispectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Then T^* is a prespectral operator on X^* of class X.

PROOF. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda), \quad N = T-S;$$

$$f(\lambda)E(d\lambda) \quad (f \in C(\sigma(T)))$$

$$\psi(f) = \int_{\sigma(T)} f(\lambda)E(d\lambda) \quad (f \in C(\sigma(T))).$$

It follows from Theorem 4.1.6 that N is a quasinilpotent operator and $NS = SN$. By Theorem 1.3.12, we have

$$\psi(f)N = N\psi(f) \qquad (f \in C(\sigma(T))).$$

It now follows from Theorem 1.3.15 that T^* is a prespectral operator on

X^* of class X and the proof is complete.

The following basic properties of quasispectral operators were established by Albrecht by a different method. See Theorem 4 of [1], p.302.

9. THEOREM. Let T be a quasispectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Let A, in $L(X)$, satisfy $AT = TA$. Then

(i) $$A \int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)E(d\lambda)A \quad (f \in C(\sigma(T))).$$

(ii) If $F(\cdot)$ is any resolution of the identity for T, then

$$\int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)F(d\lambda) \quad (f \in C(\sigma(T))).$$

(iii) T has a unique resolution of the identity of class Γ.

(iv) T has a unique Jordan decomposition for resolutions of the identity of all classes.

PROOF. By Theorem 4.1.8, T^* is prespectral of class X. There is a unique resolution of the identity $G(\cdot)$ of class X for T^* with the property that

$$\left(\int_{\sigma(T)} f(\lambda)E(d\lambda)\right)^* = \int_{\sigma(T)} f(\lambda)G(d\lambda) \quad (f \in C(\sigma(T))).$$

(See Theorem 5.22 of [12], p.137.) Since $A^*T^* = T^*A^*$, the first statement now follows from Theorem 1.3.12, the corresponding result for prespectral operators. Statement (ii) follows similarly from Theorem 1.3.13 and statement (iv) then follows immediately.

Now suppose that the resolution of the identity $F(\cdot)$ is also of class Γ. We have already established that

$$\int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)F(d\lambda) \quad (f \in C(\sigma(T))).$$

Let $x \in X$, $y \in \Gamma$. Define

$$\mu(\tau) = \langle E(\tau)x,y \rangle \qquad (\tau \in \Sigma_p),$$
$$\nu(\tau) = \langle F(\tau)x,y \rangle \qquad (\tau \in \Sigma_p).$$

By Proposition 1.3.7,

$$\int_{\sigma(T)} f(\lambda)\mu(d\lambda) = \left\langle \int_{\sigma(T)} f(\lambda)E(d\lambda)x,y \right\rangle,$$

$$\int_{\sigma(T)} f(\lambda)\nu(d\lambda) = \left\langle \int_{\sigma(T)} f(\lambda)F(d\lambda)x,y \right\rangle,$$

for all f in $C(\sigma(T))$. Hence

$$\int_{\sigma(T)} f(\lambda)\mu(d\lambda) = \int_{\sigma(T)} f(\lambda)\nu(d\lambda) \quad (f \in C(\sigma(T))).$$

$\mu(\cdot)$ and $\nu(\cdot)$ are finite countably additive measures with supports contained in $\sigma(T)$. Hence they are regular measures, and by the Riesz representation theorem $\mu = \nu$. It then follows that

$$\langle E(\tau)x,y \rangle = \langle F(\tau)x,y \rangle \quad (\tau \in \Sigma_p, x \in X, y \in \Gamma).$$

Since Γ is total, conclusion (iii) follows and the proof is complete.

As promised we now give an example to show that the class of quasispectral operators is strictly larger than the class of prespectral operators.

10. EXAMPLE. On the subspace of ℓ^∞ consisting of convergent sequences, the map which assigns to each such sequence its limit is a linear functional of norm 1. Throughout this section, L denotes a fixed linear functional on ℓ^∞ with $||L|| = 1$ such that for each convergent sequence $\{\xi_n\}$ we have

$$L(\{\xi_n\}) = \lim_{n \to \infty} \xi_n.$$

Define operators S and A on ℓ^∞ by

$$S\{\xi_n\} = \{\xi_1, \frac{1}{2}\xi_2, \frac{2}{3}\xi_3, \ldots\},$$

$$A\{\xi_n\} = \{L(\{\xi_n\}),0,0, \ldots\}.$$

Clearly $||A|| = 1$ and $A^2 = 0$. Also

$$S\{\xi_n\} = \{\xi_n\} - \{\eta_n\},$$

where $\eta_1 = 0$ and

$$\eta_r = r^{-1}\xi_r \quad (r = 2,3,4, \ldots).$$

Since $L(\{\eta_n\}) = 0$, we have $AS\{\xi_n\} = A\{\xi_n\}$. It is easy to see that $SA\{\xi_n\} = A\{\xi_n\}$, and hence

$$AS = SA.$$

$\sigma(S)$ is the totally disconnected set consisting of 1 and the numbers $1-n^{-1}$ for $n = 2,3,4, \ldots$. By regarding S as the adjoint of an operator on ℓ^1 we see that S is prespectral with a unique resolution of the identity $E(\cdot)$ of class ℓ^1 satisfying

$$E(\{1\})\{\xi_k\} = \{\xi_1,0,0,0, \ldots\},$$

$$E(\{\tfrac{n-1}{n}\})\{\xi_n\} = \{\delta_{kn}\xi_k\} \qquad (n = 2,3,4, \ldots).$$

Define the sequence $\{\lambda_n\}$ by setting $\lambda_1 = 1$ and

$$\lambda_n = \frac{n-1}{n} \qquad (n = 2,3,4, \ldots).$$

It is easy to see that for τ in Σ_p, $E(\tau)$ is the operator which multiplies the n^{th} term of a sequence by 1 if $\lambda_n \in \tau$ and by 0 if $\lambda_n \notin \tau$. The sequence $\{f_n\}$ of functions on $\sigma(S)$ given by

$$f_n(\lambda) = \lambda \qquad \text{if } \lambda < \frac{n-1}{n},$$

$$f_n(\lambda) = 1 \qquad \text{if } \lambda \geq \frac{n-1}{n},$$

for $n = 2,3,4, \ldots$ converges uniformly to the function identically equal to λ on $\sigma(S)$. One sees directly that

$$\int_{\sigma(S)} f_n(\lambda)E(d\lambda) \text{ converges to } S \text{ in the norm of } L(\ell^\infty)$$

and hence

$$S = \int_{\sigma(S)} \lambda E(d\lambda).$$

It follows that S is a scalar-type operator on ℓ^∞ of class ℓ^1. Since

$$AE(\{1\})\{1,1,1, \ldots\} = \{0,0,0, \ldots\}$$

and

$$E(\{1\})A\{1,1,1, \ldots\} = \{1,0,0, \ldots\},$$

it follows that A commutes with S but not with the resolution of the identity $E(\cdot)$. We now define $T = S+A$. Thus T is the sum of S and a nilpotent A that commutes with S. It follows from Theorem 4.1.6 (ii) that T is a quasispectral operator on ℓ^∞ of class ℓ^1. We show now that T is not prespectral of any class.

Suppose to the contrary that $G(\cdot)$ is a resolution of the identity of class Γ for the prespectral operator T. Then $G(\cdot)$ would also be a resolution of the identity for S, the scalar part of T, and A commutes with every value of $G(\cdot)$. Now, by Theorem 5.33 of [12], p.143, the projections $G(\{1\})$ and $E(\{1\})$ have the same range. Also

$$G(\{1\})\{1,1,1, \ldots\} \in E(\{1\})\ell^\infty$$

and

$$AG(\{1\})\{1,1,1, \ldots\} = \{0,0,0, \ldots\}.$$

However

$$A\{1,1,1, \ldots\} = \{1,0,0, \ldots\} \in E(\{1\})\ell^\infty = G(\{1\})\ell^\infty$$

and

$$G(\{1\})A\{1,1,1, \ldots\} = \{1,0,0, \ldots\}.$$

This gives a contradiction and so T is not prespectral of any class.

If A were to commute with any resolution of the identity for S, then T would be prespectral. A does not commute with any resolution of the identity for S. In fact every quasispectral operator which is not prespectral has the following property: its radical part does not commute with any resolution of the identity of its scalar part.

We have already tacitly used the following result in the course of proving Theorem 4.1.9. We record it for completeness.

11. NOTE. Let T, in L(X), be a quasispectral operator with a resolution of the identity E(•) of class Γ. Let ρ denote the union of all open subsets ν in $\underset{\sim}{C}$ such that E(ν) = 0. Then ρ can be expressed as the union of countably many such open sets: $\rho = \bigcup_n \nu_n$, say. Then for each n

$$\langle E(\nu_n)x,y\rangle = 0,$$

$$\langle E(\rho)x,y\rangle = 0 \qquad (x \in X,\ y \in \Gamma).$$

It follows that E(ρ) = 0. The complement of ρ is called the support of E(•). Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda).$$

Then, by Theorem 4.1.6 (i), we have σ(T) = σ(S). Also, by Note 5.7 of [12], p.121, the support of E(•) is equal to σ(S) = σ(T).

We now proceed to obtain some further basic properties of quasispectral operators. Let T be a quasispectral operator on X with resolution of the identity E(•) of class Γ. Recall that in Definition 1.2.6 we defined f(T), for each f in $\mathcal{F}$(T).

12. THEOREM. Let T be a quasispectral operator on X with resolution of the identity E(•) of class Γ. Let f ∈ $\mathcal{F}$ (T). Then f(T) is a quasi-

spectral operator with resolution of the identity $F(\cdot)$ of class Γ given by

$$F(\tau) = E(f^{-1}(\tau)) \quad (\tau \in \Sigma_p).$$

PROOF. The formula above clearly yields a spectral measure of class (Σ_p, Γ). Let τ be a closed subset of $\mathbb{C}$. Since f is continuous on some neighbourhood of $\sigma(T)$, $f^{-1}(\tau)$ is also a closed set. Also, since T is a quasispectral operator,

$$TE(f^{-1}(\tau))X \subseteq E(f^{-1}(\tau))X.$$

It follows from the equation $Tf(T) = f(T)T$ and Theorem 4.1.9 that $f(T)S = Sf(T)$, where S denotes the scalar part of T. Then from Proposition 4.1.4 we obtain

$$f(T)E(f^{-1}(\tau))X \subseteq E(f^{-1}(\tau))X.$$

Hence

$$f(T)F(\tau)X \subseteq F(\tau)X. \tag{1}$$

If $\lambda_o \in \mathbb{C}\setminus\tau$, the function h given by

$$h(\lambda) = (\lambda_o - f(\lambda))^{-1}$$

is analytic on a neighbourhood of $f^{-1}(\tau)$. Hence if C is a suitable finite family of rectifiable Jordan curves surrounding $f^{-1}(\tau)$ we have, by Definition 1.2.6 and Proposition 1.2.1

$$\left\{\frac{1}{2\pi i}\int_C h(\lambda)T_o(\lambda)d\lambda\right\}(\lambda_o I - f(T))E(f^{-1}(\tau)) = E(f^{-1}(\tau)),$$

where

$$T_o(\lambda) = \{(\lambda I - T)|E(f^{-1}(\tau))X\}^{-1}.$$

This shows that

$$\sigma(f(T)|F(\tau)X) \subseteq \tau. \tag{2}$$

From (1) and (2) we deduce that $F(\cdot)$ is a resolution of the identity for

T, and the proof is complete.

13. THEOREM. Let T be a quasispectral operator on X with resolution of the identity E(•) of class Γ. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda) \text{ , } N = T-S.$$

Let $f \in \mathcal{F}(T)$. Then

$$f(T) = \sum_{n=o}^{\infty} \frac{N^n}{n!} \int_{\sigma(T)} f^{(n)}(\lambda)E(d\lambda),$$

the series converging in the norm of L(X).

PROOF. It follows from Theorem 4.1.6 that σ(T) = σ(S) and so $\mathcal{F}(T) = \mathcal{F}(S)$. The present theorem will follow immediately from Theorem 1.2.9 as soon as we show that

$$f(S) = \int_{\sigma(S)} f(\lambda)E(d\lambda) \qquad (f \in \mathcal{F}(S)).$$

Let $x \in X$ and $y \in \Gamma$. Then if $\mu(\tau) = \langle E(\tau)x,y\rangle \ (\tau \in \Sigma_p)$ we have

$$\left\langle \int_{\sigma(S)} g(\lambda)E(d\lambda)x,y \right\rangle = \int_{\sigma(S)} g(\lambda)\mu(d\lambda) \tag{3}$$

for every g in C(σ(S)). Observe that, if C is a suitable finite family of rectifiable Jordan curves surrounding σ(S) and if $f \in \mathcal{F}(S)$, then by Proposition 1.3.7

$$\langle f(S)x,y\rangle = \frac{1}{2\pi i} \int_C f(\lambda) \langle (\lambda I-S)^{-1}x,y\rangle \, d\lambda$$

$$= \frac{1}{2\pi i} \int_C f(\lambda) \left\{\int_{\sigma(S)} (\lambda-\xi)^{-1}\mu(d\xi)\right\} d\lambda.$$

A standard argument involving Fubini's theorem shows that we may interchange the order of integration in the double integral to get

$$\left\langle f(S)x,y\right\rangle = \int_{\sigma(S)} f(\lambda)\mu(d\lambda). \tag{4}$$

Since Γ is total, it follows from (3) and (4) that

$$f(S) = \int_{\sigma(S)} f(\lambda)E(d\lambda) \quad (f \in \mathcal{J}(S)).$$

This completes the proof.

14. DEFINITION. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Define

$$S = \int_{(T)} \lambda E(d\lambda)\ ,\ N = T-S.$$

T is said to be of type m if and only if

$$f(T) = \sum_{n=o}^{m} \frac{N^n}{n!} \int_{\sigma(T)} f^{(n)}(\lambda)E(d\lambda) \quad (f \in \mathcal{J}(T)).$$

15. PROPOSITION. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda)\ ,\ N = T-S.$$

(i) T is of type m if and only if $N^{m+1} = 0$.

(ii) T is a scalar-type operator if and only if it is of type 0.

PROOF. If $N^{m+1} = 0$, then clearly the formula of Theorem 4.1.13 reduces to the formula of Definition 4.1.14. Conversely, if T is of type m, we see by putting

$$f(\lambda) = \lambda^{m+1}/(m+1)!$$

in these two formulae that

$$0 = N^{m+1} \int_{\sigma(T)} E(d\lambda) = N^{m+1}.$$

This completes the proof of (i). Statement (ii) follows immediately.

16. PROPOSITION. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Define

$$S = \int_{\sigma(T)} \lambda E(d\lambda) \text{ , } N = T-S.$$

Let $f \in \mathcal{F}(T)$. The scalar part of the Jordan decomposition of f(T) is equal to f(S). If T is of type m, then f(T) is also of type m.

PROOF. By Theorem 4.1.13 and its proof, we have

$$f(T) = \sum_{n=o}^{\infty} f^{(n)}(S) \frac{N^n}{n!} ,$$

$$f(S) = \int_{\sigma(T)} f(\lambda)E(d\lambda).$$

Define

$$F(\tau) = E(f^{-1}(\tau)) \quad (\tau \in \Sigma_p).$$

By Theorem 4.1.12 and Proposition 1.3.7, $F(\cdot)$ is a resolution of the identity of class Γ for both f(T) and f(S). Thus if we can show that the operator

$$N_1 = \sum_{n=1}^{\infty} f^{(n)}(S) \frac{N^n}{n!}$$

is quasinilpotent, the first statement will be proved. Let $\mathcal{Q}$ be the closed commutative subalgebra of L(X) generated by N and

$$\{\int_{\sigma(T)} f(\lambda)E(d\lambda) : f \in C(\sigma(T))\}.$$

Observe that the radical of $\mathcal{Q}$ is a closed ideal of $\mathcal{Q}$. Hence N_1 is quasinilpotent. To prove the second statement, note that $N^{m+1} = 0$ and so

$$N_1 = \sum_{n=1}^{m} f^{(n)}(S) \frac{N^n}{n!} .$$

Hence $N_1^{m+1} = 0$ and the proof is complete.

We have already seen that a quasispectral operator may have distinct resolutions of the identity corresponding to distinct total linear subspaces of the dual space. However, as we now show, the projection corresponding to an open-and-closed subset of the spectrum always coincides with the spectral projection. Also, the ranges of the projections corresponding to a closed set are equal.

17. PROPOSITION. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let τ be an open-and-closed subset of $\sigma(T)$. Then $E(\tau)$ is equal to the spectral projection corresponding to τ.

PROOF. Observe that

$$\sigma(T|E(\tau)X) \subseteq \tau \text{ and } \sigma(T|(I-E(\tau))X) \subseteq \mathbb{C}\setminus\tau.$$

The desired conclusion now follows from Proposition 1.53 of [12], p.37.

In order to prove that the ranges of the projections corresponding to a closed set are equal, some preliminary results are required.

18. LEMMA. Let $T \in L(X)$ and let Y be a closed subspace of X invariant under T. If T has the single-valued extension property, then so does $T|Y$.

PROOF. Let $y \in Y$, and let f and g be analytic Y-valued functions, defined on open sets $D(f)$ and $D(g)$ respectively, such that

$$(\xi I-T)f(\xi) = y \quad (\xi \in D(f)),$$

$$(\xi I-T)g(\xi) = y \quad (\xi \in D(g)).$$

Since T has the single-valued extension property,

$$f(\xi) = g(\xi) \quad (\xi \in D(f)\cap D(g)).$$

Therefore $T|Y$ has the single-valued extension property.

19. <u>THEOREM.</u> Let T be a quasispectral operator on X. Then T has the single-valued extension property.

<u>PROOF.</u> It follows from Theorem 4.1.8 that T^* is a prespectral operator. By Theorem 5.22 of [12], p.137, T^{**} is also a prespectral operator and so by Theorem 5.31 of [12], p.143, T^{**} has the single-valued extension property. Now T is the restriction of T^{**} to its closed invariant subspace X. It now follows from the preceding lemma that T also has the single-valued extension property and so the proof is complete.

20. <u>LEMMA.</u> Let T, in L(X), have the single-valued extension property, and let $x \in X$. The spectrum $\sigma(x)$ of x is empty if and only if $x = 0$.

<u>PROOF.</u> The function $\xi \to x(\xi)$ is entire. If $|\xi| > ||T||$, then

$$\langle x(\xi), y\rangle = \langle(\xi I-T)^{-1}x, y\rangle \qquad (y \in X^*)$$

and the right-hand side tends to 0 as $\xi \to \infty$. By Liouville's theorem $x(\xi) = 0$ $(\xi \in \mathbb{C})$ and so $x = (\xi I-T)x(\xi) = 0$.

21. <u>THEOREM.</u> Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let δ be a closed subset of $\mathbb{C}$. Then

$$E(\delta)X = \{x \in X : \sigma(x) \subseteq \delta\}.$$

<u>PROOF.</u> Let $x \in E(\delta)X$ so that $E(\delta)x = x$. Since $\sigma(T|E(\delta)X) \subseteq \delta$, we see from the relation

$$((\xi I-T)|E(\delta)X)^{-1}E(\delta)x = ((\xi I-T)|E(\delta)X)^{-1}x$$

that for ξ in $\mathbb{C}\setminus\delta$ the left-hand side is a pre-imaging function for x and T. Hence $\rho(x) \supseteq \mathbb{C}\setminus\delta$ and so $\sigma(x) \subseteq \delta$.

Conversely, assume that $\sigma(x) \subseteq \delta$. Observe that there is an increasing sequence $\{\tau_n\}$ of closed subsets of the set $\sigma(T) \cap (\mathbb{C}\setminus\delta)$ with

$$\bigcup_{n=1}^{\infty} \tau_n = \sigma(T) \cap (\mathbb{C}\setminus\delta).$$

By Urysohn's lemma, there is a function h_n in $C(\sigma(T))$ such that

$$h_n(\lambda) = 1 \quad (\lambda \in \tau_n), \quad h_n(\lambda) = 0 \quad (\lambda \in \delta \cap \sigma(T)),$$

$0 \leq h_n(\lambda) \leq 1$ $(\lambda \in \sigma(T))$ and the support of h_n is disjoint from δ. As in Proposition 4.1.4, the spectrum of the element $y = \int_{\sigma(T)} h_n(\lambda)E(d\lambda)x$ is contained in $\sigma(x)$ and in the support on h_n. By Lemma 4.1.20, $y = 0$. Let $y' \in \Gamma$ and $\mu(\tau) = \langle E(\tau)x, y'\rangle$ $(\tau \in \Sigma_p)$. By Lebesgue's theorem of dominated convergence and Note 4.1.11

$$\langle (I-E(\delta))x, y'\rangle = \lim_{n \to \infty} \int_{\sigma(T)} h_n(\lambda)\mu(d\lambda) = 0$$

and so, since Γ is total, $E(\delta)x = x$.

Finally, we show that the spectrum and the approximate point spectrum of a quasispectral operator coincide. For the definition and properties of the approximate point spectrum the reader is referred to Definition 1.15 and Theorem 1.16 of [12], pp.8-9.

22. THEOREM. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then $\sigma_a(T) = \sigma(T)$.

PROOF. Let $G = \sigma(T) \setminus \sigma_a(T)$. Now G is open, since it is equal to the intersection of $\mathbb{C} \setminus \sigma_a(T)$ and the interior of $\sigma(T)$. If δ is any compact subset of G, then $\sigma_a(T|E(\delta)X) = \emptyset$ since

$$\sigma_a(T|E(\delta)X) \subseteq \sigma(T|E(\delta)X) \cap \sigma_a(T) \subseteq \delta \cap \sigma_a(T) = \emptyset.$$

Hence, by Theorem 1.16 of [12], p.9, the boundary of the compact set $\sigma(T|E(\delta)X)$ is empty and so $\sigma(T|E(\delta)X) = \emptyset$. Therefore $E(\delta)X = \{0\}$, and so for all x in X and y in Γ we have $\langle E(\delta)x, y\rangle = 0$. Each measure $\langle E(\cdot)x, y\rangle$ is countably additive on Σ_p and so $\langle E(G)x, y\rangle = 0$. Since Γ is total $E(G) = 0$. Hence, by Note 4.1.11, $E(\rho(T) \cup G) = 0$. Again by Note 4.1.11, $\rho(T)$ is the largest open set on which the spectral measure

$E(\cdot)$ vanishes. Therefore $G = \emptyset$ and $\sigma_a(T) = \sigma(T)$.

2. Restrictions of quasispectral operators.

The main purpose of this brief section is to show that a quasispectral operator is decomposable.

1. PROPOSITION. Let T be a quasispectral operator on X and let Y be a closed subspace of X invariant under T. If $T|Y$ is a quasispectral operator, then $\sigma(T|Y) \subseteq \sigma(T)$.

PROOF. By Theorem 4.1.22, $\sigma_a(T) = \sigma(T)$ and $\sigma_a(T|Y) = \sigma(T|Y)$. Also, by Theorem 1.16 (i) of [12], p.9 we have $\sigma_a(T|Y) \subseteq \sigma_a(T)$ and so the desired conclusion follows.

Next, we show that if T is a quasispectral operator on X with a resolution of the identity $E(\cdot)$, then $E(\delta)X$ is a maximal spectral subspace for T for each closed subset δ of $\mathbb{C}$.

2. PROPOSITION. Let T, in L(X), be a quasispectral operator with resolution of the identity $E(\cdot)$ of class Γ and let δ be a closed subset of $\mathbb{C}$. Then $E(\delta)X$ is the union of all closed subspaces Y of X with $TY \subseteq Y$ and $\sigma(T|Y) \subseteq \delta$.

PROOF. Let Y be a closed subspace of X with $TY \subseteq Y$ and $\sigma(T|Y) \subseteq \delta$. Lemma 4.1.18 shows that $T|Y$ has the single-valued extension property. Let y_X and y_Y be respectively the maximal X-valued and Y-valued analytic functions which satisfy

$$(\xi I-T)y_X(\xi) = y, \quad \text{for all } \xi \text{ in } \rho_X(y),$$
$$(\xi I-T)y_Y(\xi) = y, \quad \text{for all } \xi \text{ in } \rho_Y(y),$$

where $\rho_X(y)$ and $\rho_Y(y)$ are the domains of definition of these functions. Let $\sigma_X(y)$ and $\sigma_Y(y)$ be the complements of these sets. y_Y may be regarded as X-valued and so by the maximality of $\rho_X(y)$ we have $\rho_Y(y) \subseteq \rho_X(y)$. Therefore

$$\sigma_X(y) \subseteq \sigma_Y(y) \subseteq \sigma(T|Y) \subseteq \delta.$$

By Theorem 4.1.21 we have

$$E(\delta)X = \{x \in X : \sigma_X(x) \subseteq \delta\}$$

and so $Y \subseteq E(\delta)X$. Finally $\sigma(T|E(\delta)X) \subseteq \delta$ and so the proof is complete.

In view of the preceding proposition, the argument of the proof of Theorem 3.1.4 suffices to prove the following more general result.

3. THEOREM. A quasispectral operator is decomposable.

3. Relationships between a quasispectral operator and its scalar part.

The purpose of this section is to present analogues for quasispectral operators of results of Foguel [19] on the relationships between a spectral operator and its scalar part. The corresponding problem for prespectral operators was investigated by Dowson [9] and Nagy [20].

Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Throughout this section

$$S = \int_{\sigma(T)} \lambda E(d\lambda), \quad N = T-S.$$

1. LEMMA. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then S is in the closed subspace of $L(X)$ generated by $\{E(\tau) : \tau \in \Sigma_p, 0 \notin \overline{\tau}\}$.

PROOF. Let $\varepsilon > 0$ be given. By the definition of the integral there is a partition $\tau_o, \tau_1, \ldots, \tau_n$ of $\sigma(S)$ into Borel sets with the point 0 in at most one of the closures $\overline{\tau}_i$ and with

$$\left\| S - \sum_{i=0}^{n} \lambda_i E(\tau_i) \right\| < \varepsilon$$

for any choice of the complex numbers λ_i in τ_i. If $0 \notin \sigma(S)$ this proves the lemma. If $0 \in \sigma(S)$ we may without loss of generality take $0 \in \tau_o$ and $\lambda_o = 0$ in the inequality above, which proves the lemma in this case too.

2. THEOREM. Let T be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let T belong to the right (left) ideal J in L(X). Then every projection $E(\tau)$ with $0 \notin \overline{\tau}$ belongs to J. If J is closed, then S and N also belong to J.

PROOF. Suppose that $\tau \in \Sigma_p$ and $0 \notin \overline{\tau}$. Let $T_\tau = T|E(\tau)X$. Since $\sigma(T|E(\tau)X) \subseteq \overline{\tau}$, it follows that $0 \in \rho(T_\tau)$, and hence T_τ^{-1} exists as a bounded linear operator on the space $E(\tau)X$. Let V_τ, in L(X), be defined by the equation

$$V_\tau x = T_\tau^{-1} E(\tau) x \quad (x \in X).$$

Then $TV_\tau = E(\tau) = V_\tau T$, which proves that $E(\tau) \in J$. It follows from Lemma 4.3.1 that S and hence N also belong to J if J is closed.

3. THEOREM. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let T belong to the closed right (left) ideal J of L(X). Then every projection $E(\tau)$ with $0 \notin \overline{\tau}$ belongs to J. Also S and N belong to J.

PROOF. By Theorem 4.1.8, T^* is a prespectral operator on X^* with resolution of the identity $F(\cdot)$ of class X such that

$$\left(\int_{\sigma(T)} f(\lambda) E(d\lambda) \right)^* = \int_{\sigma(T)} f(\lambda) F(d\lambda) \quad (f \in C(\sigma(T))).$$

It follows that the Jordan decomposition of T^* is $S^* + N^*$. Observe that the family $J^* = \{A^* \in L(X^*) : A \in J\}$ is a closed left (right) ideal of $L(X^*)$.

By the previous theorem $S^* \in J^*$ and so clearly we have $S \in J$. Now observe that S is a scalar-type operator and so prespectral. Another application of the previous theorem shows that if $0 \notin \overline{\tau}$, then $E(\tau) \in J$. Finally, since J is an ideal, $T \in J$ and $S \in J$, we have also $N = T-S \in J$. The proof is complete.

4. COROLLARY. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. If T is compact, then so are S,N and every projection $E(\tau)$ with $0 \notin \overline{\tau}$.

PROOF. By Corollary 2.9 of [12], p.48, the compact operators on X form a closed two-sided ideal of L(X).

5. COROLLARY. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. If T is weakly compact, then so are S,N and every projection $E(\tau)$ with $0 \notin \overline{\tau}$.

PROOF. By Corollary V1.4.6 of [14], p.484, the weakly compact operators on X form a closed two-sided ideal in L(X).

Observe that if Y is a closed subspace of X, the set $\{A \in L(X) : AX \subseteq Y\}$ is a closed right ideal of L(X). Hence we can deduce the following result from Theorem 4.3.3.

6. COROLLARY. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then the ranges of S,N and $E(\tau)$ with $0 \notin \overline{\tau}$ are contained in the closure of the range of T.

Let A_o, in L(X), be fixed. Then the sets

$$\{A \in L(X) : A_o A = 0\},$$
$$\{A \in L(X) : AA_o = 0\},$$

are respectively closed right and left ideals of L(X). Hence our next result also follows from Theorem 4.3.3.

7. <u>COROLLARY.</u> Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. If $A_o T = 0$ (respectively $TA_o = 0$), then $A_o S = A_o N = A_o E(\tau) = 0$ if $0 \notin \bar{\tau}$ (respectively $SA_o = NA_o = E(\tau)A_o$ if $0 \notin \bar{\tau}$).

For the definition of a quasispectral operator of finite type m the reader is referred to Definition 4.1.

8. <u>COROLLARY.</u> Let T be a quasispectral operator on X with radical part N. Then T is of finite type if and only if $N^p T = 0$ for some positive integer p.

<u>PROOF.</u> If T is of finite type, then $N^n = 0$ for some positive integer n and so $N^n T = TN^n = 0$. Conversely if some power of N annihilates T, say

$$N^p T = TN^p = 0,$$

then it follows from Corollary 4.3.7 that $N^{p+1} = 0$ and so T is of finite type.

9. <u>COROLLARY.</u> Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $x \in X$. If $Tx = 0$ and $0 \notin \bar{\tau}$, then

$$Sx = Nx = E(\tau)x = 0.$$

<u>PROOF.</u> Observe that for a given x in X, the set

$$\{A \in L(X) : Ax = 0\}$$

is a closed left ideal of L(X). The result now follows at once from Theorem 4.3.3.

10. <u>COROLLARY.</u> Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $\{x_n\}$ be a bounded sequence in X for which $\{Tx_n\}$ is convergent. Then

(i) if $0 \notin \bar{\tau}$, $\{E(\tau)x_n\}$ is convergent;

(ii) $\{Sx_n\}$ is convergent;

(iii) $\{Nx_n\}$ is convergent.

PROOF. The set of all A, in L(X), for which $\{Ax_n\}$ is convergent, is a left ideal of L(X) which is closed because $\{x_n\}$ is bounded. The result now follows at once from Theorem 4.3.3.

11. COROLLARY. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $\{x_n\}$ be a bounded sequence in X for which $\{Tx_n\}$ is convergent to 0. Then

(i) if $0 \notin \bar{\tau}$, $\{E(\tau)x_n\}$ is convergent to 0;

(ii) $\{Sx_n\}$ is convergent to 0;

(iii) $\{Nx_n\}$ is convergent to 0.

PROOF. The set of all A, in L(X), for which $\{Ax_n\}$ is convergent to 0, is a left ideal of L(X) which is closed because $\{x_n\}$ is bounded. The result now follows at once from Theorem 4.3.3.

12. THEOREM. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $A \in L(X)$ and let $N_o = NE(\{0\})$. Then $TA = 0$ if and only if $A - E(\{0\})A = N_oA = 0$.

PROOF. Let $N_oA = A - E(\{0\})A = 0$. Then we have

$$E(\tau)A = E(\tau)E(\underset{\sim}{C}\setminus\{0\})A = 0$$

if $0 \notin \bar{\tau}$. It follows from Lemma 4.3.1 that $SA = 0$. Moreover

$$\begin{aligned} NA &= N[E(\{0\}) + E(\underset{\sim}{C}\setminus\{0\})]A \\ &= NE(\{0\})A + NE(\underset{\sim}{C}\setminus\{0\})A \\ &= N_oA + N[A - E(\{0\})A] \\ &= 0. \end{aligned}$$

It follows that TA = SA+NA = 0.

Conversely, let TA = 0. Observe that the set

$$\{B \in L(X) : BA = 0\}$$

is a closed left ideal of L(X). Hence, by Corollary 4.3.7, we have $E(\tau)A = 0$ if $0 \notin \overline{\tau}$. Now, for each x in X and y in Γ we have

$$\langle E(\underset{\sim}{C} \setminus \{0\})Ax, y \rangle = \lim_{n \to \infty} \langle E(\tau_n)Ax, y \rangle,$$

where

$$\tau_n = \{z : |z| \geq n^{-1}\}$$

for each positive integer n. Since Γ is total, A-E({0})A = 0 and the proof is complete.

13. <u>COROLLARY.</u> Let T be a quasispectral operator on X with resolution of the identity E(·) of class Γ. If E({0}) = 0, then TA = 0 if and only if A = 0.

<u>PROOF.</u> By Theorem 4.3.12, TA = 0 if and only A = 0 in this case.

The following result is valid for quasispectral operators but for its proof we require the special case of the result for prespectral operators. We include the proof of this for completeness. See Theorem 11.12 of [12], pp.218-219.

14. <u>THEOREM.</u> Let T be a prespectral operator on X with resolution of the identity E(·) of class Γ. Let $x \in X$ and let n be a positive integer. Then $(\lambda I-T)^n x = 0$ if and only if $E(\{\lambda\})x = x$ and $N^n x = 0$.

<u>PROOF.</u> Let $(\lambda I-T)^n x = 0$ and let τ be a closed subset of $\underset{\sim}{C}$ such that $\lambda \in \underset{\sim}{C} \setminus \tau$. Let $T_\tau = T|E(\tau)X$ and $I_\tau = I|E(\tau)X$. Then $\lambda \in \rho(T_\tau)$ and so

$$E(\tau)x = (\lambda I_\tau - T_\tau)^{-n}(\lambda I-T)^n E(\tau)x,$$
$$E(\tau)x = (\lambda I_\tau - T_\tau)^{-n} E(\tau)(\lambda I-T)^n x = 0.$$

Let $\tau_n = \{z : |z-\lambda| \geq n^{-1}\}$ and let $y \in \Gamma$. Then

$$\langle E(\underset{\sim}{C}\setminus\{\lambda\})x,y\rangle = \lim_{n \to \infty}\langle E(\tau_n)x,y\rangle = 0.$$

Since Γ is total we obtain

$$E(\underset{\sim}{C}\setminus\{\lambda\})x = 0 \ , \ E(\{\lambda\})x = x.$$

Hence

$$Sx = SE(\{\lambda\})x = \int_{\{\lambda\}} \mu E(d\mu)x = \lambda E(\{\lambda\})x = \lambda x,$$

which shows that

$$(\lambda I-T)x = -Nx$$

and hence that

$$0 = (\lambda I-T)^n x = (-1)^n N^n x.$$

This proves the necessity of the conditions. Now, conversely, suppose that $E(\{\lambda\})x = x$ and $N^n x = 0$. It follows as above that $(\lambda I-S)x = 0$ and hence that

$$(\lambda I-T)^n x = (-1)^n N^n x.$$

Therefore $(\lambda I-T)^n x = 0$.

In order to prove the more general version of the last result applicable to quasispectral operators, some preliminary results are required. Prior to proving these, we introduce some notation.

Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $f \in C(\sigma(S))$. Define

$$\operatorname{supp} f = \operatorname{cl}\{\lambda \in \sigma(S) : f(\lambda) \neq 0\}$$

and

$$\psi(f) = \int_{\sigma(S)} f(\lambda)E(d\lambda).$$

15. <u>LEMMA.</u> Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $x \in X$ and $f \in C(\sigma(S))$. Then

$$\sigma(\psi(f)x) \subseteq \text{supp } f.$$

PROOF. Suppose that $\xi \in \mathbb{C}\setminus\text{supp } f$. Then the function g_ξ defined by

$$g_\xi(\lambda) = \frac{f(\lambda)}{\xi-\lambda} \quad \text{for } \lambda \neq \xi,$$

$$g_\xi(\xi) = 0,$$

is in $C(\sigma(S))$. Moreover the function

$$\xi \to \psi(g_\xi) \qquad (\xi \in \mathbb{C}\setminus\text{supp } f)$$

is an analytic operator-valued function. Furthermore

$$(\xi I-S)\psi(g_\xi)x = \psi(f)x$$

and so $\xi \in \rho(\psi(f)x)$. The result follows.

16. PROPOSITION. Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $x \in X$ and $f \in C(\sigma(S))$. Suppose that

$$\sigma(x) \cap \text{supp } f = \emptyset.$$

Then $\psi(f)x = 0$.

PROOF. Let $\xi \to x(\xi)$ be the (unique) X-valued analytic function defined on $\rho(x)$ and satisfying

$$(\xi I-S)x(\xi) = x \quad (\xi \in \rho(x)).$$

It follows that

$$(\xi I-S)\psi(f)x(\xi) = \psi(f)(\xi I-S)x(\xi) = \psi(f)x \quad (\xi \in \rho(x))$$

and so $\rho(x) \subseteq \rho(\psi(f)x)$. Taking complements we obtain

$$\sigma(\psi(f)x) \subseteq \sigma(x).$$

However, by the previous lemma,

$$\sigma(\psi(f)x) \subseteq \text{supp } f.$$

Hence

$$\sigma(\psi(f)x) = \emptyset.$$

We deduce from Lemma 4.1.20 that $\psi(f)x = 0$.

17. THEOREM. Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let δ be a closed subset of $\mathbb{C}$. Then $x \in E(\delta)X$ if and only if $\psi(f)x = 0$ for all f in $C(\sigma(S))$ with support disjoint from δ.

PROOF. Let $x \in E(\delta)X$. By Theorem 4.1.2., we have $\sigma(x) \subseteq \delta$. If $f \in C(\sigma(S))$ and $\delta \cap \text{supp } f = \emptyset$, then

$$\sigma(x) \cap \text{supp } f = \emptyset$$

and, by the preceding proposition, we have $\psi(f)x = 0$.

Conversely, assume that $\psi(f)x = 0$ for all f in $C(\sigma(S))$ with support disjoint from δ. Observe that there is an increasing sequence $\{\tau_n\}$ of closed subsets of the set $\sigma(S) \cap (\mathbb{C} \setminus \delta)$ such that

$$\bigcup_{n=1}^{\infty} \tau_n = \sigma(S) \cap (\mathbb{C} \setminus \delta).$$

Let n be fixed. By Urysohn's lemma, there is a function h_n in $C(\sigma(S))$ such that

$$h_n(\lambda) = 1 \qquad (\lambda \in \tau_n),$$

$$h_n(\lambda) = 0 \qquad (\lambda \in \delta \cap \sigma(S)),$$

and $0 \leq h_n(\lambda) \leq 1$, for all λ in $\sigma(S)$. Observe that

$$\lim_{n \to \infty} h_n(\lambda) = 1 \qquad (\lambda \in \sigma(S) \cap (\mathbb{C} \setminus \delta)),$$

$$\lim_{n \to \infty} h_n(\lambda) = 0 \qquad (\lambda \in \sigma(S) \cap \delta).$$

By Note 4.1.11, the support of the spectral measure $E(\cdot)$ is $\sigma(S)$.

Let $y \in \Gamma$. It follows that

$$\langle (I-E(\delta))x,y \rangle = \langle E((\underset{\sim}{C} \setminus \delta) \cap \sigma(S))x,y \rangle.$$

Let $\mu(\tau) = \langle E(\tau)x,y \rangle$ $(\tau \in \Sigma_p)$. Observe that the measure μ is finite and so, by Lebesgue's theorem of dominated convergence,

$$\langle E((\underset{\sim}{C} \setminus \delta) \cap \sigma(S))x,y \rangle = \lim_{n \to \infty} \int_{\sigma(S)} h_n(\lambda)\mu(d\lambda)$$

$$= \lim_{n \to \infty} \langle \psi(h_n)x,y \rangle = 0,$$

by hypothesis. Hence

$$\langle (I-E(\delta))x,y \rangle = 0 \qquad (y \in \Gamma)$$

and, since Γ is total, $E(\delta)\mathfrak{X} = x$. This completes the proof.

18. <u>THEOREM.</u> Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $x \in X$ and let n be a positive integer. Then $(\lambda I-T)^n x = 0$ if and only if $E(\{\lambda\})x = x$ and $N^n x = 0$.

<u>PROOF.</u> First, we note that it follows from Theorem 4.1.8 that T^* is a prespectral operator on X^* with resolution of the identity $F(\cdot)$ of class X such that

$$\left(\int_{\sigma(T)} f(\lambda)E(d\lambda) \right)^* = \int_{\sigma(T)} f(\lambda)F(d\lambda) \quad (f \in C(\sigma(T)).$$

Also, by Theorem 5.22 of [12], p.137, T^{**} is a prespectral operator on X^{**} with resolution of the identity $G(\cdot)$ of class X^* such that

$$\left(\int_{\sigma(T)} f(\lambda)F(d\lambda) \right)^* = \int_{\sigma(T)} f(\lambda)G(d\lambda) \quad (f \in C(\sigma(T))).$$

It follows that

$$\left(\int_{\sigma(T)} f(\lambda)E(d\lambda) \right)^{**} = \int_{\sigma(T)} f(\lambda)G(d\lambda) \quad (f \in C(\sigma(T))).$$

In particular, the Jordan decomposition of T^* is S^*+N^* and the Jordan decomposition of T^{**} is $S^{**}+N^{**}$. Let

$$x \to \tilde{x}$$

denote the canonical embedding of X into X^{**}.

Suppose that $(\lambda I-T)^n x = 0$. Then from above

$$(\lambda I^{**}-T^{**})^n \tilde{x} = 0$$

and so, by applying Theorem 4.3.14 to the prespectral operator T^{**}, we deduce that

$$(N^{**})^n \tilde{x} = 0,$$

$$G(\{\lambda\})\tilde{x} = \tilde{x}.$$

Hence $N^n x = 0$ and $\tilde{x} \in G(\{\lambda\})X^{**}$. Suppose that f, in $C(\sigma(T))$, has the property that $\lambda \in \tilde{C}\setminus \text{supp } f$. By the previous theorem and the discussion above we have successively

$$\int_{\sigma(T)} f(\lambda)G(d\lambda)\tilde{x} = 0,$$

$$\left(\int_{\sigma(T)} f(\lambda)E(d\lambda)\right)^{**}\tilde{x} = 0,$$

$$\int_{\sigma(T)} f(\lambda)E(d\lambda)x = 0.$$

Hence $x \in E(\{\lambda\})X$, again using Theorem 4.3.17.

Now conversely, suppose that $E(\{\lambda\})x = x$ and $N^n x = 0$. We have

$$Sx = SE(\{\lambda\})x = \int_{\{\lambda\}} \mu E(d\mu)x = \lambda E(\{\lambda\})x = \lambda x,$$

which shows that

$$(\lambda I-T)x = -Nx$$

and hence that

$$(\lambda I-T)^n x = (-1)^n N^n x = 0.$$

Thus, the conditions are necessary and sufficient. This completes the proof.

Observe that in the course of proving the last theorem, we have established the following result.

19. COROLLARY. Let T be a quasispectral operator on X with scalar part S. Then $\sigma_p(T) \subseteq \sigma_p(S)$.

For completeness we include the special versions of Theorems 4.3.14 and 4.3.18 applicable to scalar-type operators.

20. THEOREM. Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then $\lambda \in \sigma_p(S)$ if and only if $E(\{\lambda\}) \neq 0$. Moreover, if for some x in X and some positive integer n we have $(\lambda I - S)^n x = 0$, then $Sx = \lambda x$. Thus if $\lambda \in \sigma_p(S)$ the ascent of the operator $\lambda I - S$ is one.

21. THEOREM. Let X be separable and let T, in L(X), be a quasispectral operator with resolution of the identity $E(\cdot)$ of class Γ. Then $\sigma_p(T)$ is countable.

PROOF. By Theorem 4.3.18, we have $\sigma_p(T) \subseteq \{\lambda : E(\{\lambda\}) \neq 0\}$. There is $M > 0$ such that

$$||E(\tau)|| \leq M < \infty \qquad (\tau \in \Sigma_p).$$

Let $x_\lambda \in X$, $x_\mu \in X$ with the properties $||x_\lambda|| = ||x_\mu|| = 1$, $E(\{\lambda\})x_\lambda = x_\lambda$ and $E(\{\mu\})x_\mu = x_\mu$. Then if μ and λ are distinct points of $\sigma_p(T)$ we have

$$||x_\lambda - x_\mu|| \geq M^{-1}||E(\{\lambda\})(x_\lambda - x_\mu)|| = M^{-1}||x_\lambda|| = M^{-1}.$$

Since X is separable, it follows that $\sigma_p(T)$ is countable.

For a proof of the next result, the reader is referred to Theorem 11.19 of [12], p.221.

22. <u>THEOREM.</u> Let T, in L(X), be a prespectral operator of class Γ. If T has a closed range, then so does S, the scalar part of T.

We deduce from this a more general version applicable to the class of quasispectral operators.

23. <u>THEOREM.</u> Let T, in L(X), be a quasispectral operator of class Γ. If T has a closed range, then so does S, the scalar part of T.

<u>PROOF.</u> It follows from Theorem 4.1.8 that T^* is a prespectral operator on X^* with Jordan decomposition S^*+N^*. By Theorem V1.6.2. of [14], p.487, since T has a closed range, so also does T^*. It follows from Theorem 4.3.22 that S^* has a closed range. By Theorem V1.6.4 of [14], pp.488-489, S has a closed range and so the proof is complete.

For a proof of the next result, the reader is referred to Theorem 11.20 of [12], pp.221-223.

24. <u>THEOREM.</u> Let T be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then T has a closed range if and only if the following two conditions hold.

(i) Either $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$.

(ii) The operator $TE(\{0\})$ has a closed range.

We deduce from this a more general version applicable to the class of quasispectral operators.

25. <u>THEOREM.</u> Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then T has a closed range if and only if the following two conditions hold.

(i) Either $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$.

(ii) The operator $TE(\{0\})$ has a closed range.

PROOF: Suppose that T has a closed range. It follows from Theorem 4.1.8 that T^* is a prespectral operator on X^*. By Theorem VI.6.2 of [14], p.487, since T has a closed range so also does T^*. It follows by applying the preceding theorem to the prespectral operator T^* that either $0 \in \rho(T^*)$ or 0 is an isolated point of $\sigma(T^*)$. Since $\sigma(T) = \sigma(T^*)$, we have proved that either $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$. In the first case $E(\{0\}) = 0$ and the second condition (ii) is proved. Suppose now that 0 is an isolated point of $\sigma(T)$. Let C be the positively oriented circle centre 0 and radius $\varepsilon > 0$ so small that $\sigma(T)\setminus\{0\}$ lies outside C. Then

$$E(\{0\}) = \frac{1}{2\pi i} \int_C (\lambda I-T)^{-1} d\lambda$$

and so $TE(\{0\}) = E(\{0\})T$. Let y be in the closure of the range of $TE(\{0\})$ and let $\{x_n\}$ be a sequence in X such that $TE(\{0\})x_n \to y$. Since T has a closed range, there is an x in X with $Tx = y$ and so

$$TE(\{0\})x = E(\{0\})Tx = E(\{0\})y = y,$$

which proves (ii).

Conversely we assume (i) and (ii). If $0 \in \rho(T)$, then $TX = X$, and so we may assume that $0 \in \sigma(T)$. Let $y \in \overline{TX}$ and let $\{x_n\}$ be a sequence in X such that $Tx_n \to y$. Since 0 is an isolated point of $\sigma(T)$, the argument above shows that $TE(\{0\}) = E(\{0\})T$ and so

$$TE(\{0\})x_n = E(\{0\})Tx_n \to E(\{0\})y$$

and, since the range of $TE(\{0\})$ is closed, there is a vector w with

$$TE(\{0\})w = E(\{0\})y.$$

Again, using the fact that 0 is an isolated point of $\sigma(T)$, we deduce that $0 \in \rho(T|E(\tilde{C}\setminus\{0\})X)$ and so for some z in $E(\tilde{C}\setminus\{0\})X$ we have $Tz = E(\tilde{C}\setminus\{0\})y$. Hence

$$T(x+E(\{0\})w) = E(\mathbb{C}\setminus\{0\})y+E(\{0\})y = y,$$

which proves that the range of T is closed.

4. Decompositions of quasispectral operators

The purpose of this section is to prove analogues, valid for quasispectral operators, of two results of Dunford pertaining to algebras of spectral operators; namely Theorem 13 of [13], pp.343-344 and Theorem 14 of [13], pp.344-345. We then proceed to consider various decompositions of quasispectral operators.

If $T,U, \ldots, V \in L(X)$, the symbol $\mathfrak{A}(T,U, \ldots, V)$ denotes the smallest subalgebra of $L(X)$, which is closed in the norm topology of $L(X)$, which contains $T,U, \ldots, V$ and I, and which contains the inverse W^{-1} of any of its elements, provided that the inverse exists as an element of $L(X)$. If K is a compact subset of $\mathbb{C}$, $R(K)$ denotes the uniform closure of complex rational functions with poles in $\mathbb{C}\setminus K$. Clearly $R(K)$ is a closed subalgebra of $C(K)$. Two Banach algebras are said to be equivalent in case they are topologically and algebraically isomorphic.

1. THEOREM. Let T, in $L(X)$, be a quasispectral operator of class Γ. Let $T = S+N$ be the Jordan decomposition of T. Then $\mathfrak{A}(T,S)$ is a commutative Banach algebra and

$$\mathfrak{A}(T,S) = \mathfrak{A}(S) \oplus J,$$

where J is the radical of $\mathfrak{A}(T,S)$. Furthermore, $\mathfrak{A}(S)$ is equivalent to $R(\sigma(T))$, and every operator in $\mathfrak{A}(T,S)$ is quasispectral of class Γ.

PROOF. Since $\mathfrak{A}(T,S)$ is the norm closure in $L(X)$ of elements of the form $p(T,S)[q(T,S)]^{-1}$, where p,q are polynomials, it follows that $\mathfrak{A}(T,S)$ is commutative. Let $E(\cdot)$ be the resolution of the identity of class Γ for T. There is M such that

$$||E(\tau)|| \leq M < \infty \qquad (\tau \in \Sigma_p).$$

If f is rational and analytic on $\sigma(T) = \sigma(S)$, then by the spectral mapping theorem we have $f(\sigma(S)) = \sigma(f(S))$ and hence

$$\sup_{\lambda \in \sigma(S)} |f(\lambda)| \leq ||f(S)|| \leq 4M \sup_{\lambda \in \sigma(S)} |f(\lambda)|.$$

Thus $\mathcal{A}(S)$ is equivalent to $R(\sigma(S))$. Since $\mathcal{A}(S)$ has no non-zero quasinilpotent elements, we see that $\mathcal{A}(S) + J$ is a direct vector sum contained in $\mathcal{A}(T,S)$. To get an inclusion in the opposite direction, we first observe that, since S+N is a Jordan decomposition for T we have SN = NS and

$$S = \int_{\sigma(T)} \lambda E(d\lambda).$$

Define

$$\psi(f) = \int_{\sigma(T)} f(\lambda)E(d\lambda) \qquad (f \in C(\sigma(T))).$$

Observe that it follows from Theorem 1.3.12 that

$$N\psi(f) = \psi(f)N \qquad (f \in C(\sigma(T))).$$

By Theorems 4.1.12 and 4.1.16, f(T) is a quasispectral operator of class Γ with Jordan decomposition

$$f(T) = f(S)+N_1,$$

where N_1 is a quasinilpotent commuting with S. Hence, in particular, if T^{-1} exists then

$$T^{-1} = S^{-1}+N_2. \qquad (1)$$

Also

$$T^n = S^n+N_3,$$

$$T^nS^m = S^{n+m}+N_4$$

and so

$$p(T,S) = p(S,S)+N_5, \qquad (2)$$

where p is a polynomial and N_5 is a quasinilpotent operator which commutes with S because it is a polynomial in S and N. If q is also a polynomial in two variables, the operator

$$r(T,S) = p(T,S)[q(T,S)]^{-1}$$

will be defined as an element of $\mathfrak{A}(T,S)$ if and only if $q(\lambda,\lambda) \neq 0$ for all λ in $\sigma(T)$. In this case we see from (1) and (2) that

$$r(T,S) = r(S,S)+N_6,$$

where N_6 is a quasinilpotent operator commuting with S. An arbitrary U in $\mathfrak{A}(T,S)$ is a limit of rational functions r_n in T and S. Let $\mathfrak{M}$ be the maximal ideal space of $\mathfrak{A}(T,S)$. Since

$$\sigma(T) = \sigma(S) = S(\mathfrak{M}) = T(\mathfrak{M})$$

and

$$T(m) = S(m) \qquad (m \in \mathfrak{M}),$$

we obtain

$$\sup_{\lambda \in \sigma(S)} |r_n(\lambda,\lambda)-r_p(\lambda,\lambda)| = \sup_{m \in M} |\{r_n(T,S)-r_p(T,S)\}(m)|$$

$$\leq ||r_n(T,S)-r_p(T,S)||.$$

Hence there is f in $R(\sigma(S))$ such that $r_n(\lambda,\lambda) \to f(\lambda)$ uniformly on $\sigma(S)$. Thus $r_n(S,S) \to f(S)$ in $\mathfrak{A}(S)$. Also $\{r_n(T,S)-r_n(S,S)\}$ is a Cauchy sequence of elements of J, each of which commutes with S. Since J is a closed two-sided ideal of $\mathfrak{A}(T,S)$, it follows that $U \in \mathfrak{A}(S) \oplus J$. Finally each operator in $\mathfrak{A}(T,S)$ is the sum of an operator of the form $\psi(g)$ for some g in $C(\sigma(S))$ and a quasinilpotent which commutes with S. By Theorem 1.3.12, 1.3.7, it follows that each operator in $\mathfrak{A}(T,S)$ is the sum of a scalar-type operator and a commuting quasinilpotent operator. By Theorem 4.1.6 (ii) such an operator is quasispectral of class Γ and the proof is complete.

The last result in this section is an analogue of Theorem 14 of [13], pp.344-345. It is proved by a method similar to that of the last theorem and so we merely state it.

2. <u>THEOREM.</u> Let T, in L(X), be a quasispectral operator with resolution of the identity $E(\cdot)$ of class Γ. Define

$$\psi(f) = \int_{\sigma(T)} f(\lambda)E(d\lambda) \qquad (f \in C(\sigma(T)).$$

Then $\mathcal{A} = \mathcal{A}\{T,\psi(f):(f \in C(\sigma(T))\}$ is a commutative Banach algebra and

$$\mathcal{A} = \{\psi(f) : f \in C(\sigma(T))\} \oplus J,$$

where J is the radical of $\mathcal{A}$. Furthermore, every operator in $\mathcal{A}$ is quasi-spectral of class Γ.

Earlier in this chapter we obtained the Jordan decomposition of a quasispectral operator. We now consider some other decompositions of quasispectral operators.

3. <u>LEMMA.</u> Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then S can be expressed in the form

$$S = S_1 + iS_2,$$

where S_1 and S_2 are scalar-type operators of class Γ such that

(i) $S_1S_2 = S_2S_1$,

(ii) $\sigma(S_1)$ and $\sigma(S_2)$ are sets of real numbers,

(iii) the Boolean algebra of projections generated by the resolutions of the identity of S_1 and S_2 is bounded.

<u>PROOF.</u>

$$S = \int_{\sigma(S)} \lambda E(d\lambda) = \int_{\sigma(S)} \mathrm{Re}\lambda E(d\lambda) + i\int_{\sigma(S)} \mathrm{Im}\lambda E(d\lambda)$$

$$= \int_{(\underset{\sim}{R})} \lambda E_1(d\lambda) + i\int_{(\underset{\sim}{R})} \lambda E_2(d\lambda),$$

where S_1 is the operator on the left and iS_2 the operator on the right, and

$$E_1(\tau) = E'(\{z : z = x+iy \text{ and } x \in \tau\}),$$

$$E_2(\tau) = E'(\{z : z = x+iy \text{ and } y \in \tau\}.$$

Conditions (i), (ii) and (iii) are easily verified.

4. THEOREM. Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then there exist operators R and J on X such that

(i) $T = R+iJ$ and $RJ = JR$;

(ii) $\sigma(R)$ and $\sigma(J)$ are sets of real numbers;

(iii) R is a scalar-type operator of class Γ;

(iv) J is a quasispectral operator of class Γ;

(v) the Boolean algebra of projections generated by the resolutions of the identity of R and J is bounded.

If R_1 and J_1 satisfy conditions (i) and (ii), then they are quasispectral operators of class Γ and there exists a quasinilpotent operator Q such that

$$R_1 = R+Q, \quad J_1 = J+iQ.$$

If R_1 and J_1 satisfy conditions (i), (ii) and (iii), then $R = R_1$ and $J = J_1$.

PROOF. Let S+N be the Jordan decomposition of T. Using the notation of Lemma 4.4.3, define

$$R = S_1, \quad J = S_2-iN.$$

Conditions (i)-(v) then follow from Lemma 4.4.3 and Theorem 4.1.6 (ii).

Now suppose that R_1 and J_1 satisfy conditions (i) and (ii). Then $R_1T = TR_1$ and $J_1T = TJ_1$. It follows from Theorem 1.3.12 that R_1, R, J_1 and

J all commute. Let $\mathcal{A} = \mathcal{A}(R,R_1,J,J_1)$ and let $\mathcal{M}$ be the maximal ideal space of this algebra. Observe that

$$0 = (T-T)(m) = (R-R_1)(m)+i(J-J_1)(m) \qquad (m\in\mathcal{M}).$$

However $(R-R_1)(m)$ and $(J-J_1)(m)$ are real numbers by condition (ii) and hence

$$(R-R_1)(m) = (J-J_1)(m) = 0 \qquad (m\in\mathcal{M}).$$

Thus if $Q = R_1-R$, then Q is a quasinilpotent and $J_1 = J+iQ$. Another application of Theorem 4.1.6 shows that R_1 and J_1 are quasispectral operators of class Γ.

Finally, suppose that in addition R_1 is a scalar-type operator of class Γ. Then, since $\sigma(R_1)$ and $\sigma(R)$ are sets of real numbers, we deduce from Theorem 5.40 of [12], p.154, that after appropriate equivalent renormings of X, both R_1 and R become hermitian operators. By Theorem 4.18 of [12], p.109, there is an equivalent renorming of X under which R and R_1 become simultaneously hermitian. Assume that this renorming has been carried out. Then $R-R_1$ is hermitian and quasinilpotent. By Sinclair's theorem, $Q = 0$, $R = R_1$ and $J = J_1$. (See Theorem 4.10 of [12], p.105.) The proof is complete.

<u>NOTE.</u> The argument given in the proof above shows that if we merely assume that R_1 and J_1 satisfy conditions (i), (ii) and the condition

(iii) R_1 is a scalar-type operator,

then the conclusion $R = R_1$ and $J = J_1$ remains true.

5. <u>LEMMA.</u> Let S be a scalar-type operator on X with resolution of the identity E(•) of class Γ. Then there are scalar-type operators S_1 and S_2 both of class Γ satisying

(i) $S_1S_2 = S_2S_1 = S$,

(ii) $\sigma(S_1)$ is a set of non-negative real numbers,

(iii) $\sigma(S_2)$ is a subset of the unit circle,

(iv) the Boolean algebra of projections generated by the resolutions of the identity of S_1 and S_2 is bounded.

PROOF. It follows from the operational calculus for a scalar-type operator that

$$S = \int_{\sigma(S)} \lambda E(d\lambda) = \int_{\sigma(S)} |\lambda| E(d\lambda) \int_{\sigma(S)} \mathrm{sgn}\lambda E(d\lambda),$$

where

$$\mathrm{sgn}\ \lambda = \frac{\lambda}{|\lambda|} \quad \text{if } \lambda \neq 0$$

and

$$\mathrm{sgn}\ 0 = 0.$$

Thus $S = S_1S_2$, where

$$S_1 = \int_{\sigma(S)} |\lambda| E(d\lambda) = \int_{\underset{\sim}{R}} \mu E_1(d\mu),$$

where $E_1(\cdot)$ is defined by

$$E_1(\tau) = E(\{\lambda : |\lambda| \in \tau\})$$

and

$$S_2 = \int_{\sigma(S)} \mathrm{sgn}\lambda E(d\lambda) = \int_C \mu E_2(d\mu),$$

where C is the unit circle in $\underset{\sim}{C}$ and

$$E_2(\tau) = E(\{\lambda : \mathrm{sgn}\ \lambda \in \tau\}).$$

Conditions (i)-(iv) are easily verified.

6. THEOREM. Let T be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then there exist two operators P and U such

that ·

(i) $T = PU = UP$,

(ii) $\sigma(P)$ is a set of non-negative real numbers,

(iii) $\sigma(U)$ is a subset of the unit circle in $\underset{\sim}{C}$,

(iv) U is a scalar-type operator of class Γ,

(v) P is a prespectral operator of class Γ,

(vi) the Boolean algebra of projections generated by the resolutions of the identity of P and U is bounded.

<u>PROOF.</u> Let $T = S+N$ be the Jordan decomposition of T. Using the notation of the preceding lemma, put $U = S_2$ and $P = S_1+S_2^{-1}N$. Observe that it follows from Theorem 1.3.10 that $S_2N = NS_2$. Conditions (i)-(vi) are easily verified from Lemma 4.4.5 and Theorem 1.3.10.

7. <u>THEOREM.</u> Let T be a quasispectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Suppose that T is invertible and power-bounded in the sense that

$$||T^n|| \leq K < \infty \qquad (n \in \underset{\sim}{Z}).$$

Then T is a scalar-type operator of class Γ.

<u>PROOF.</u> Let $T = S+N$ be the Jordan decomposition of T. By Theorem 4.1.8, T^* is a prespectral operator on X^* of class X and its Jordan decomposition is $T^* = S^*+N^*$. Moreover, T^* is invertible and power-bounded. It follows from Theorem 10.17 of [12], pp.212-213 that T^* is a scalar-type operator; that is $N^* = 0$. Hence $N = 0$ and the proof is complete.

8. <u>THEOREM.</u> Let S_1 and S_2 be commuting scalar-type operators. If S_1+S_2 is a quasispectral operator, then it is of scalar-type.

<u>PROOF.</u> Let $E_1(\cdot)$ and $E_2(\cdot)$ be resolutions of the identity of S_1 and S_2 respectively. Define

$$R_1 = \int_{\sigma(S_1)} \mathrm{Re}\lambda E_1(d\lambda) \ , \ J_1 = \int_{\sigma(S_1)} \mathrm{Im}\lambda E_1(d\lambda),$$

$$R_2 = \int_{\sigma(S_2)} \mathrm{Re}\lambda E_2(d\lambda) \ , \ J_2 = \int_{\sigma(S_2)} \mathrm{Im}\lambda E_2(d\lambda).$$

It follows from Theorem 1.3.12 that R_1, J_1, R_2, J_2 all commute. From Theorem 5.40 of [12], p.154, we deduce that, after appropriate equivalent renormings of X, both R_1, J_1 and R_2, J_2 become hermitian operators.

Let $E(\cdot)$ be a resolution of the identity of the quasispectral operator S_1+S_2. Define

$$S = \int_{\sigma(S_1+S_2)} \lambda E(d\lambda) \ , \ Q = S_1+S_2-S,$$

$$R = \int_{\sigma(S_1+S_2)} \mathrm{Re}\lambda E(d\lambda) \ , \ J = \int_{\sigma(S_1+S_2)} \mathrm{Im}\lambda E(d\lambda).$$

Again by Theorem 5.40 of [12], p.154, it follows that there is an equivalent renorming of X under which R and J become hermitian operators. By Theorem 4.1.9, the seven operators $R, J, R_1, J_1, R_2, J_2, Q$ all commute. By Theorem 4.18 of [12], p.109, there is an equivalent renorming of X under which R, J, R_1, J_1, R_2, J_2 become simultaneously hermitian. Assume that this renorming has been carried out. Define

$$\mathcal{A} = \mathcal{A}(R, J, R_1, J_1, R_2, J_2, Q),$$

and let $\mathcal{M}$ be the maximal ideal space of $\mathcal{A}$. We have

$$R_1(m)+R_2(m)+iJ_1(m)+iJ_2(m) = R(m)+iJ(m)+0 \quad (m \in \mathcal{M}).$$

Therefore

$$(R_1+R_2-R)(m)+i(J_1+J_2-J)(m) = 0 \qquad (m \in \mathcal{M}).$$

It follows that

$$R_1+R_2-R = 0 \ , \ J_1+J_2-J = 0,$$

and so $Q = 0$. Therefore S_1+S_2 is of scalar-type and the proof of the theorem is complete.

CHAPTER FIVE

Roots and logarithms of quasispectral operators

The purpose of this chapter is twofold. In the first section we obtain analogues of results in Chapter 10 of [12] on logarithms and roots of prespectral operators. In the other section we solve affirmatively the following problem. Let A be a prespectral operator of class Γ. Does there exist a prespectral operator T of class Γ such that $f(T) = A$?

1. Roots and logarithms of quasispectral operators

1. DEFINITIONS. If T is an operator on X, an operator A on X such that $\exp A = T$ is called a logarithm of T. Also, if m is a positive integer and B is an operator on X which satisfies $B^m = T$, then B is called an m^{th} root of T.

2. LEMMA. Let $T \in L(X)$. Suppose that

$$X = X_1 \oplus \dots \oplus X_n,$$

where each X_r is a closed subspace of X invariant under T and $T|X_r$ is quasi-spectral of class Γ_r ($r = 1, \dots, n$), where each Γ_r is a subspace of X_r^* ($r = 1, \dots, n$). Then T is quasispectral of class Γ, where

$$\Gamma = \Gamma_1 \oplus \dots \oplus \Gamma_n.$$

PROOF. Let $E_r(\cdot)$ be the resolution of the identity of class Γ_r for $T|X_r$. If $x = \sum_{r=1}^{n} x_r$ with $x_r \in X_r$, we define for every τ in Σ_p

$$E(\tau)x = \sum_{r=1}^{n} E_r(\tau)x_r.$$

Clearly $E(\cdot)$ is a spectral measure of class (Σ_p,Γ). Let δ be a closed subset of $\underset{\sim}{C}$. By hypothesis

$$TE_r(\delta)x_r = E_r(\delta)TE_r(\delta)x_r \qquad (r = 1, \ldots, n),$$

and so we obtain

$$\sum_{r=1}^{n} TE_r(\delta)x_r = \sum_{r=1}^{n} E_r(\delta)TE_r(\delta)x_r,$$

from which we deduce that for each closed subset δ of $\underset{\sim}{C}$ and all x in X we have

$$TE(\delta)x = E(\delta)TE(\delta)x$$

and so $TE(\delta)X \subseteq E(\delta)X$. Now let $\lambda \in \underset{\sim}{C}\setminus\delta$. By hypothesis we have

$$\sigma(T|E_r(\delta)X_r) \subseteq \delta \qquad (r = 1, \ldots, n)$$

and so it follows that

$$\lambda \in \rho(T|E_r(\delta)X_r) \qquad (r = 1, \ldots, n).$$

Hence $\lambda I-T$ maps each of the subspaces $E_r(\delta)X_r$ in a one-to-one manner onto all of itself. Therefore $\lambda I-T$ maps $E(\delta)X$ in a one-to-one manner onto all of itself. This shows that $\lambda \in \rho(T|E(\delta)X)$ and hence that

$$\sigma(T|E(\delta)X) \subseteq \delta \qquad (\delta \text{ closed}),$$

which completes the proof that T is a quasispectral operator of class Γ.

3. LEMMA. Let S be a scalar-type operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let $P \in L(X)$. Suppose that $P^2 = P$ and

$$PE(\tau) = E(\tau)P \qquad (\tau \in \Sigma_p).$$

Then $S|PX$ is a scalar-type operator of class Γ.

PROOF. Define $G(\tau) = E(\tau)|PX$ $(\tau \in \Sigma_p)$. Observe that Γ is a total set for the Banach space PX. Hence $G(\cdot)$ is a spectral measure of class (Σ_p, Γ) with values in $L(PX)$. It follows from Lemma 2.1.1 that

$$\sigma(S|G(\tau)PX) \subseteq \sigma(S|E(\tau)X) \subseteq \overline{\tau} \qquad (\tau \in \Sigma_p).$$

Clearly

$$S|PX = \int_{\underset{\sim}{C}} \lambda G(d\lambda)$$

and so $S|PX$ is a scalar-type operator of class Γ.

4. THEOREM. Let S_o be a scalar-type spectral operator on X with finite spectrum. Assume that $S_o \neq 0$.

(i) Let T be a quasispectral operator on X with resolution of the identity $G(\cdot)$ of class Γ. Suppose that $TS_o = S_oT$ and

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then $T+S_o$ is quasispectral of class Γ.

(ii) Let S be a scalar-type operator on X with resolution of the identity $G(\cdot)$ of class Γ. Suppose that

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then $S+S_o$ is a scalar-type operator of class Γ.

PROOF. Let $E(\cdot)$ be the resolution of the identity of S_o and let $\{\lambda_r : r = 1, \ldots, n\}$ be the non-zero points of $\sigma(S_o)$. Then

$$S_o = \sum_{r=1}^{n} \lambda_r E(\lambda_r),$$

where

$$I = E(0) + \sum_{r=1}^{n} E(\lambda_r)\cdot$$

and

$$X = E(0)X \oplus E(\lambda_1)X \oplus \dots \oplus E(\lambda_n)X. \qquad (1)$$

Let Y be one of the (n+1) subspaces on the right-hand side of (1) and let P be the projection of X on to Y. Then, by the commutativity theorem for spectral operators.

$$PG(\tau) = G(\tau)P \qquad (\tau \in \Sigma_p).$$

Let T = S+N be the Jordan decomposition of T. By the previous lemma, $S|Y$ is a scalar-type operator of class Γ. Therefore, by Proposition 1.3.7, the operator $(S+S_o)|Y$, which is $S|Y$ plus a scalar multiple of the identity on Y, is also a scalar-type operator of class Γ. Moreover, since $NS_o = S_oN$, we have NP = PN and $N|Y$ is a quasinilpotent operator that commutes with $(S+S_o)|Y$. It now follows from Theorem 4.1.6 that $(S+S_o+N)|Y$ is a quasispectral operator of class Γ. An application of Lemma 5.1.2 suffices to complete the proof of the theorem.

A similar argument establishes the following result.

5. THEOREM. Let S_o be a scalar-type spectral operator on X with finite spectrum. Assume that $S_o \neq 0$.

(i) Let T be a quasispectral operator on X with resolution of the identity G(•) of class Γ. Suppose that $TS_o = S_oT$ and

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then S_oT is quasispectral of class Γ.

(ii) Let S be a scalar-type operator on X with resolution of the identity G(•) of class Γ. Suppose that

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then S_oS is a scalar-type operator of class Γ.

6. THEOREM. Let A, in L(X), be a quasispectral operator of class Γ. Suppose that the point 0 lies in the unbounded component of $\rho(A)$. Then there is T, in L(X), with the following properties.

(i) T is quasispectral of class Γ and exp T = A.

(ii) If B, in L(X), commutes with A, then B commutes with T.

PROOF. For each λ in $\sigma(A)$, there is an open disc $\Omega(\lambda)$ with $\lambda \in \Omega(\lambda)$ but $0 \notin \Omega(\lambda)$. A finite family $\Omega(\lambda_1), \ldots, \Omega(\lambda_n)$ of these discs cover $\sigma(A)$. Let Ω be the open set formed by taking the union of $\bigcup_{r=1}^{n} \Omega(\lambda_r)$ and the bounded components of $\rho(A)$. Then Ω is simply connected and so by Theorem 13.18(g) of [21], pp.262-263 there is f analytic in Ω such that

$$\exp f(\lambda) = \lambda \qquad (\lambda \in \Omega).$$

Since $\sigma(A) \subseteq \Omega$, $f \in C(\sigma(A))$. Let $E(\cdot)$ be the resolution of the identity of class Γ for A. Define

$$S = \int_{\sigma(A)} \lambda E(d\lambda),$$

$$N = A-S,$$

$$S_o = \int_{\sigma(A)} f(\lambda)E(d\lambda).$$

Note that $\exp(S_o) = S$. Define

$$Q = N \exp(-S_o).$$

Since A = S+N is the Jordan decomposition of A we have NS = SN. It then follows from Theorem 1.3.12 that $NS_o = S_oN$. Hence Q is quasinilpotent.

Let C denote the circle, centre the origin, radius $\frac{1}{2}$, described once counterclockwise. The operator

$$N_o = \frac{1}{2\pi i} \int_C (\lambda I - Q)^{-1} \log(1+\lambda)\, d\lambda$$

is well-defined. Also $\sigma(N_o) = \{0\}$, and so N_o is quasinilpotent. Further

$$\exp N_o = I+Q.$$

Since $N_o S_o = S_o N_o$ and $NS_o = S_o N$, we obtain

$$\begin{aligned}\exp(S_o+N_o) &= \exp S_o \exp N_o = (\exp S_o)(I+Q) \\ &= (\exp S_o) + (\exp S_o)N \exp(-S_o) \\ &= S+N = A.\end{aligned}$$

Define $T = S_o+N_o$. Observe that T is the sum of a scalar-type operator of class Γ and a commuting quasinilpotent. Hence by Theorem 4.1.6 (ii) T is quasispectral of class Γ.

Suppose now that BA = AB. It follows from Theorem 4.1.9 that BS = SB, BN = NB, and $BS_o = S_oB$. Hence BQ = QB,

$$B(\lambda I-Q)^{-1} = (\lambda I-Q)^{-1}B \qquad (\lambda \in \rho(Q)),$$

and so $BN_o = N_oB$. Therefore BT = TB and the proof is complete.

NOTE. Clearly the hypothesis of the last theorem could be weakened. Only the existence of a continuous logarithm on $\sigma(A)$ is required.

7. THEOREM. Let $m \geq 2$ be a positive integer. Let A, in L(X), be a quasispectral operator of class Γ. Suppose that the point 0 lies in the unbounded component of $\rho(A)$. Then there is T, in L(X), with the following properties.

(i) T is quasispectral of class Γ and $T^m = A$.

(ii) If B, in L(X), commutes with A, then B commutes with T.

PROOF. By Theorem 5.1.6, there is T_o, in L(X), with the following properties.

(a) T_o is quasispectral of class Γ and $\exp T_o = A$.

(b) If B, in L(X), commutes with A, then B commutes with T_o.

Define $T = \exp(m^{-1}T_o)$. Then T has the required properties.

NOTE. Clearly the hypothesis of the last theorem could be weakened. Only the existence of a continuous m^{th} root on $\sigma(A)$ is required.

2. Roots of prespectral operators

We note first that two of the theorems in [12], pp.196-199 have essential hypotheses omitted. The correct statements of these results are as follows.

1. THEOREM. Let S_o be a scalar-type spectral operator on X with finite spectrum. Assume that $S_o \neq 0$.

(i) Let T be a prespectral operator on X with resolution of the identity $G(\cdot)$ of class Γ. Suppose that $TS_o = S_oT$ and

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then $T+S_o$ is prespectral of class Γ.

(ii) Let S be a scalar-type operator on X with resolution of the identity $G(\cdot)$ of class Γ. Suppose that

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then $S+S_o$ is a scalar-type operator of class Γ.

2. THEOREM. Let S_o be a scalar-type spectral operator on X with finite spectrum. Assume that $S_o \neq 0$.

(i) Let T be a prespectral operator on X with resolution of the identity $G(\cdot)$ of class Γ. Suppose that $S_oT = TS_o$ and

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then S_oT is prespectral of class Γ.

(ii) Let S be a scalar-type operator on X with resolution of the identity $G(\cdot)$ of class Γ. Suppose that

$$G(\tau)S_o = S_oG(\tau) \qquad (\tau \in \Sigma_p).$$

Then S_oS is a scalar-type operator of class Γ.

3. THEOREM. Let A be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let f be a function analytic on a region Ω, such that $\sigma(A) \subseteq f(\Omega)$ and $f'(\lambda) \neq 0$ for all λ in Ω. Then there is an operator T on X such that T is prespectral of class Γ, $\sigma(T) \subseteq \Omega$ and $f(T) = A$.

PROOF. Let $\lambda \in \sigma(A)$. Then there exists a point ζ in Ω such that $f(\zeta) = \lambda$. By Theorem 10.34 of [21], p.217, there exist open neighbourhoods V_ζ and W_λ such that f is one-to-one mapping of V_ζ onto W_λ. The set W_λ is open and $\lambda \in W_\lambda$. Hence we can find an open disc D_λ which is properly contained in W_λ and has λ as its centre. Let $\delta(\lambda)$ be the open disc with centre λ and radius half that of D_λ. As λ runs through $\sigma(A)$, the corresponding discs

$$\{\delta(\lambda) : \lambda \in \sigma(A)\}$$

cover $\sigma(A)$. Since $\sigma(A)$ is compact, there is a finite subcovering; that is

$$\sigma(A) \subseteq \bigcup_{r=1}^{n} \delta(\lambda_r).$$

For brevity, let δ_r denote $\delta(\lambda_r)$ and let W_r denote the open neighbourhood corresponding to λ_r. Let g_r be the inverse of f on W_r. Define

$$\tau_1 = \delta_1 \cap \sigma(A),$$

$$\tau_2 = (\delta_2 \setminus \delta_1) \cap \sigma(A),$$

$$\text{------------------}$$

$$\tau_n = (\delta_n \setminus [\bigcup_{r=1}^{n-1} \delta_r]) \cap \sigma(A).$$

Observe that $\{\tau_r : r = 1, \ldots, n\}$ is a family of pairwise disjoint sets such that $\tau_r \in \Sigma_p$,

$$\tau_r \subseteq \overline{\delta}_r \subseteq W_r \qquad (r = 1, \ldots, n).$$

$$\sigma(A) \subseteq \bigcup_{r=1}^{n} \tau_r.$$

Define

$$T = \bigoplus_{r=1}^{n} g_r(A|E(\tau_r)X).$$

We know that

$$\sigma(A|E(\tau_r)X) \subseteq \overline{\tau}_r \qquad (r = 1, \ldots, n).$$

Also by Theorem 14.2 of [12], pp.265-266, $A|E(\tau_r)X$ is a prespectral operator with resolution of the identity $E(\cdot)|E(\tau_r)X$ of class Γ.

Now, by Theorem 10.34 of [21], p.217, g_r is analytic on W_r, and so it follows from Theorem 5.16 of [12], pp.130-131 that $g_r(A|E(\tau_r)X)$ is also prespectral of class Γ. It follows from Theorem 5.16 that the resolution of the identity of class Γ for $g_r(A|E(\tau_r)X)$ is $F_r(\cdot)$, where

$$F_r(\delta) = E(g_r^{-1}(\delta))|E(\tau_r)X$$

$$= E(g_r^{-1}(\delta) \cap \tau_r)|E(\tau_r)X \quad (\delta \in \Sigma_p;\ r = 1, \ldots, n).$$

Define

$$F(\delta) = \sum_{r=1}^{n} F_r(\delta) \qquad (\delta \in \Sigma_p).$$

We wish to prove that T is a prespectral operator with resolution of the identity $F(\cdot)$ of class Γ and that $f(T) = A$.

Let $\delta_1, \delta_2 \in \Sigma_p$. Observe that

$$\begin{aligned} F(\delta_1 \cap \delta_2) &= \sum_{r=1}^{n} F_r(\delta_1 \cap \delta_2) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\delta_1 \cap \delta_2) \cap \tau_r) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\delta_1) \cap \tau_r \cap g_r^{-1}(\delta_2) \cap \tau_r) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\delta_1) \cap \tau_r) E(g_r^{-1}(\delta_2) \cap \tau_r) \end{aligned}$$

It follows that

$$F(\delta_1 \cap \delta_2) = \sum_{r=1}^{n} F_r(\delta_1) F_r(\delta_2) = F(\delta_1) F(\delta_2) \qquad (\delta_1, \delta_2 \in \Sigma_p),$$

using the fact that $\tau_1, \ldots, \tau_n$ are pairwise disjoint. Also

$$\begin{aligned} F(\underset{\sim}{C}) &= \sum_{r=1}^{n} F_r(\underset{\sim}{C}) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\underset{\sim}{C}) \cap \tau_r) \\ &= \sum_{r=1}^{n} E(\tau_r) \\ &= E(\sigma(A)) = I. \end{aligned}$$

Now let $\delta \in \Sigma_p$. Observe that

$$F(\underset{\sim}{C}\setminus\delta) = \sum_{r=1}^{n} F_r(\underset{\sim}{C}\setminus\delta)$$

$$= \sum_{r=1}^{n} E(g_r^{-1}(\underset{\sim}{C}\setminus\delta) \cap \tau_r)$$

$$= \sum_{r=1}^{n} \left[E(g_r^{-1}(\underset{\sim}{C}) \cap \tau_r) - E(g_r^{-1}(\delta) \cap \tau_r)\right]$$

$$= \sum_{r=1}^{n} E(\tau_r)\left[I - E(g_r^{-1}(\delta) \cap \tau_r)\right]$$

$$= I - \sum_{r=1}^{n} (g_r^{-1}(\delta) \cap \tau_r)$$

$$= I - F(\delta).$$

Let $\delta_1, \delta_2 \in \Sigma_p$. Then

$$F(\delta_1 \cup \delta_2) = F(\underset{\sim}{C}\setminus(\underset{\sim}{C}\setminus\delta_1) \cap (\underset{\sim}{C}\setminus\delta_2))$$

$$= I - F((\underset{\sim}{C}\setminus\delta_1) \cap (\underset{\sim}{C}\setminus\delta_2))$$

$$= I - F(\underset{\sim}{C}\setminus\delta_1)F(\underset{\sim}{C}\setminus\delta_2)$$

$$= I - (I - F(\delta_1))(I - F(\delta_2))$$

$$= F(\delta_1) + F(\delta_2) - F(\delta_1)F(\delta_2).$$

Also if $||E(\tau)|| \leq M < \infty$, then clearly $||F(\tau)|| \leq nM < \infty \quad (\tau \in \Sigma_p)$.

We deduce that if $\{\delta_m\}$ is a pairwise disjoint sequence of sets in Σ_p, then

$$F(\bigcup_{m=1}^{k} \delta_m) = \sum_{m=1}^{k} F(\delta_m).$$

Hence if $x \in X$ and $y \in \Gamma$, then we obtain

$$\left\langle F\left(\bigcup_{m=1}^{k} \delta_m\right)x,y\right\rangle = \sum_{m=1}^{k} \left\langle F(\delta_m)x,y\right\rangle$$

$$= \sum_{m=1}^{k} \sum_{r=1}^{n} \left\langle F_r(\delta_m)x,y\right\rangle$$

$$= \sum_{m=1}^{k} \sum_{r=1}^{n} \left\langle E(g_r^{-1}(\delta_m) \cap \tau_r)x,y\right\rangle$$

$$= \sum_{r=1}^{n} \left\langle E\left(g_r^{-1}\left(\bigcup_{m=1}^{k} \delta_m\right) \cap \tau_r\right)x,y\right\rangle .$$

Let $k \to \infty$. Using the properties of the inverse image and the countable additivity of $\langle E(\cdot)x,y\rangle$ on Σ_p for x in X and y in Γ, we deduce that

(a) $\lim_{k \to \infty} \left\langle F\left(\bigcup_{m=1}^{k} \delta_m\right)x,y\right\rangle$ exists $(x \in X,\ y \in \Gamma)$

(b) $\lim_{k \to \infty} \left\langle F\left(\bigcup_{m=1}^{k} \delta_m\right)x,y\right\rangle = \left\langle F\left(\bigcup_{m=1}^{\infty} \delta_m\right)x,y\right\rangle \qquad (x \in X,\ y \in \Gamma).$

This completes the proof that $F(\cdot)$ is a spectral measure of class (Σ_p,Γ).

Since g_r is analytic on a neighbourhood of $\overline{\tau}_r$,

$$g_r(A|E(\tau_r)X) = \frac{1}{2\pi i} \int_B g_r(\lambda)((\lambda I-A)|E(\tau_r)X)^{-1} d\lambda,$$

where B is a suitable finite family of contours in $\rho(A|E(\tau_r)X)$. Since A is prespectral, it follows that

$$AE(\tau) = E(\tau)A \qquad (\tau \in \Sigma_p)$$

and so $A|E(\tau_r)X$ commutes with $E(\cdot)|E(\tau_r)X$. We deduce readily from this that $F_r(\cdot)$ commutes with $g_r(A|E(\tau_r)X)$ and consequently

$$TF(\tau) = F(\tau)T \qquad (\tau \in \Sigma_p).$$

Also

$$F(\delta)X = \bigoplus_{r=1}^{n} E(g_r^{-1}(\delta) \cap \tau_r)X \qquad (\delta \in \Sigma_p).$$

Each of the subspaces on the right-hand side reduces T and so, by Proposition 1.37 of [12], p.25-26

$$\sigma(T|F(\delta)X) = \bigcup_{r=1}^{n} \sigma(T|E(g_r^{-1}(\delta) \cap \tau_r)X)$$

$$= \bigcup_{r=1}^{n} \sigma((g_r(A|E(\tau_r)X)|E(g_r^{-1}(\delta) \cap \tau_r)X)$$

$$\subseteq \bar{\delta} \qquad (\delta \in \Sigma_p)$$

by Theorem 4.1.12. It follows that F(•) is a resolution of the identity for T. Finally

$$f(T) = f(\bigoplus_{r=1}^{n} g_r(A|E(\tau_r)X))$$

$$= \bigoplus_{r=1}^{n} (f \circ g_r)(A|E(\tau_r)X)$$

$$= \bigoplus_{r=1}^{n} A|E(\tau_r)X$$

$$= A,$$

and so the proof is complete.

4. <u>COROLLARY.</u> Let A, in L(X), be a scalar-type operator of class Γ. Let Ω be a region and let f be a function analytic on Ω such that $\sigma(A) \subseteq \Omega$ and $f'(\lambda) \neq 0$ for all λ in Ω. Then there is a scalar-type operator S of class Γ such that ~~f(S)=A~~ $\sigma(S) \subseteq \Omega$ and $f(S) = A$.

<u>PROOF.</u> By Lemma 5.1.3, $A|E(\tau_r)A$ is a scalar-type operator of class Γ for each r = 1, ..., n, in the notation of the proof of the last theorem. It follows from Proposition 1.3.7 that $g_r(A|E(\tau_r)X)$ is also a scalar-type operator of class Γ and hence that T is a scalar-type operator of class Γ. This completes the proof.

Finally, to round off the considerations in this section we state the special case of Theorem 5.3.3 for spectral operators together with a result of Apostol [2].

5. THEOREM. Let A, in L(X), be a spectral operator. Let Ω be a region and let f be a function analytic on Ω such that $\sigma(A) \subseteq f(\Omega)$ and $f'(\lambda) \neq 0$ for all λ in Ω. Then there is a spectral operator T_o on X such that $\sigma(T_o) \subseteq \Omega$ and $f(T_o) = A$. Moreover, if $T \in L(X)$ and $f(T) = A$, then T is a spectral operator.

CHAPTER SIX

A commutativity theorem for certain $\mathfrak{A}$-scalar operators

The purpose of the final chapter of this thesis is to prove a commutativity theorem for $\mathfrak{A}$-scalar operators, where $\mathfrak{A} = C(K)$ and K is a compact subset of the complex plane.

1. The commutativity theorem

Let K be a compact subset of $\underset{\sim}{C}$. Let S be an $\mathfrak{A}$-scalar operator on X, in the sense of Foias and Colojoara, where $\mathfrak{A} = C(K)$. It follows that there is a continuous algebra homomorphism ψ from C(K) into L(X) such that

$$\psi(f_o) = I \ , \ \psi(f_1) = S,$$

where

$$f_o(\lambda) = 1 \quad (\lambda \in K) \ , \ f_1(\lambda) = \lambda \qquad (\lambda \in K).$$

Hence there is a real constant M such that

$$||\psi(f)|| \leq M||f|| \qquad (f \in C(K)),$$

where $||f||$ denotes the supremum norm on f. We note the following properties of such an operator S.

(i) S has the single-valued extension property.

(ii) Let δ be a closed subset of $\underset{\sim}{C}$. Then

$$X_\delta = \{x \in X : \sigma(x) \subseteq \delta\}$$

is a closed subspace of X and is a maximal spectral subspace for S.

(iii) S is a decomposable operator.

For the definition of a decomposable operator the reader is referred to Definition 3.1.5. For the definition and properties of $\mathfrak{A}$-spectral operators and $\mathfrak{A}$-scalar operators, the reader is referred to [5] pp.59-67.

1. THEOREM. Let K be a compact subset of $\mathbb{C}$ and let $\mathfrak{A} = C(K)$. Let S be an $\mathfrak{A}$-scalar operator on X. Let A, in L(X), have the property that

$$AX_\delta \subseteq X_\delta$$

for every closed subset δ of $\mathbb{C}$. Then AS = SA.

PROOF. By Theorem 1.3.15, S^* is a scalar-type operator on X^* of class X. It then follows from Theorem 5.22 of [12], p.137 that S^{**} is a scalar-type operator on X^{**} of class X^*.

Define an operator C(S,S) on L(X) by

$$C(S,S)T = TS-ST \qquad (T \in L(X)).$$

As stated previously, S is a decomposable operator. It follows from the hypothesis of this theorem and Theorem 2.3.3 of [5], p.48 that

$$\lim_{n \to \infty} ||C(S,S)^n A||^{1/n} = 0.$$

Clearly

$$\lim_{n \to \infty} ||C(S^{**},S^{**})^n A^{**}||^{1/n} = 0,$$

where $C(S^{**},S^{**})$ is defined as an analogous way as an operator on $L(X^{**})$. Again, from Theorem 2.3.3 of [5], p.48, we conclude that

$$A^{**}G(\delta)X^{**} \subseteq G(\delta)X^{**},$$

for each closed set δ, where $G(\cdot)$ is the (unique) resolution of the identity of class X^* for the scalar-type operator S^{**}. We now apply Theorem 3.1.1 to the operators S^* and A^* to conclude that $A^*S^* = S^*A^*$. Hence AS = SA, completing the proof.

REMARK. It is an interesting but unsolved problem as yet to characterize those algebras $\mathfrak{A}$ for which the conclusion of Theorem 6.1.1 holds.

2. <u>COROLLARY</u>. Let K be a compact subset of $\mathbb{C}$ and let $\mathfrak{A} = C(K)$. Let X_1 and X_2 be two non-zero complex Banach spaces. Let S_1 and S_2 be $\mathfrak{A}$-scalar operators on X_1 and X_2 respectively. Let A be a bounded linear mapping from X_1 into X_2 such that

$$AX_{S_1}(\delta) \subseteq X_{S_2}(\delta),$$

for every closed subset δ of $\mathbb{C}$, where

$$X_{S_1}(\delta) = \{x \in X_1 : \sigma(x) \subseteq \delta\},$$

$$X_{S_2}(\delta) = \{x \in X_2 : \sigma(x) \subseteq \delta\}.$$

Then $AS_1 = S_2A$.

<u>PROOF</u>. There are continuous algebra homomorphisms ψ_1 and ψ_2 such that

$$\psi_1(f_o) = I \ , \ \psi_1(f_1) = S_1, \ \psi_2(f_o) = I \ , \ \psi_2(f_1) = S_2$$

in the notation of the theorem. Let $X = X_1 \oplus X_2$ and

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix},$$

$$\psi(f) = \begin{bmatrix} \psi_1(f) & 0 \\ 0 & \psi_2(f) \end{bmatrix} \qquad (f \in C(K)).$$

Then ψ is a continuous algebra homomorphism from $C(K)$ into $L(X)$ with $\psi(f_o) = I$ and $\psi(f_1) = S$. By Proposition 1.1.3 of [5], p.3

$$\sigma_S((x_1,x_2)) = \sigma_{S_1}(x_1) \cup \sigma_{S_2}(x_2) \qquad ((x_1,x_2) \in X).$$

If δ is a closed subset of $\mathbb{C}$, then by Proposition 1.4 of [5], p.4, we have

$$X_{S_1}(\delta) \oplus X_{S_2}(\delta) = (X_1 \oplus X_2)_S(\delta),$$

where the term on the right-hand side is the maximal spectral subspace corresponding to δ. Now $\tilde{A}$ can be represented on X by

$$\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}.$$

Hence by the hypothesis of the theorem

$$\check{A}X_S(\delta) \subseteq X_S(\delta)$$

for all closed subsets δ of $\underset{\sim}{C}$. By the previous theorem $\tilde{A}S = S\tilde{A}$. This is equivalent to

$$\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$$

which implies that

$$\begin{bmatrix} 0 & 0 \\ AS_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ S_2A & 0 \end{bmatrix}$$

and hence that $AS_1 = S_2A$.

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