Meridional circulation in the atmospheres of uniformly rotating stars of early spectral type

by

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Thesis

submitted to the

UNIVERSITY of GLASGOW

for the degree of

Ph.D.

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October 1967

CONTENTS

Page

Page

Page

SUMMARY

It has been recognised since 1925 (Eddington: "The Internal Constitution of the Stars" (C.U.P. 1930, p. 285) that von Zeipel's paradox for uniformly rotating stars could be resolved if a large-scale circulation were set up in meridian planes. Investigations since then have shown, amongst other things, that the velocity of the circulation currents is inversely proportional to the density. Thus, even though the currents are very slow deep inside a star, they become very fast near the surface. If simple, zero density boundary conditions are applied at the surface, there is a formal singularity there.

Although the surface layers of non-rotating stars are now understood in considerable detail, the same cannot be said for rotating stars. It appears that a detailed theory of the surface layers must take circulation into account. The main purpose of this thesis is to develop such a theory, with particular emphasis on the removal of the surface singularity.

This singularity must arise from the neglect of some important physical factor. It has normally been assumed that viscous and inertial forces are negligible, and this assumption must clearly be questioned when the theory predicts very large velocities. However, a preliminary investigation by the author (Smith: Z.für Astrophys. 63 166 1966) suggested that this assumption is valid arbitrarily near the surface if the rotation speed is slow enough.

An assumption which is certainly invalid whatever the rotation speed is that the photon mean free path is short near the surface. That assumption is implicit in the use of an equation for the radiative flux of a form normally used only in the theory of stellar interiors. Accordingly, a theory of the surface layers has been developed in this thesis which uses the non-local radiative transfer equation appropriate to the theory of stellar atmospheres.

It is found that, although the use of a non-local transfer equation does remove the formal singularity at the surface, the circulation speeds near the surface are still unrealistically large. When the assumption that viscous and inertial forces can be neglected is re-examined, it is found that, although inertial forces do become important near the surface, these forces are not sufficient to damp the speed of the flow. However, the circulation violates a stability criterion based on the Richardson number (see, for example, L. Prandtl, Essentials of Fluid Dynamics, Blackie 1952), and the flow becomes turbulent in a thin surface layer. Turbulence sets in when the flow speeds are of the order of the speed of sound, and turbulent viscosity then acts to prevent the speeds from further increasing. A qualitative model of the turbulent surface layer has been developed, on the basis of order-of-magnitude estimates. Although no detailed prediction is given for the emergent flux, it is concluded that the commonly used von Zeipel gravity-darkening cannot be correct when a turbulent layer is present.

PREFACE

In Chapter 1 of this thesis a general survey is given of previous work in the field of rotating stars, with particular reference to the problem of meridional circulation. In the first section of the chapter some relevant observations are discussed; in the second section a discussion is given of the theoretical results which led to the formulation of the present problem.

In order to make the problem tractable, it was necessary to make various assumptions and simplifications. These are discussed in Chapter 2.

The interior model adopted is that of Roxburgh, Griffith and Sweet (1965). These authors did not discuss the problem of meridional circulation, and the first stage of the present investigation was to derive the circulation in the outer layers of their model. This work, which has been published (Smith 1966), is described in Chapter 3.

The model described in Chapter 3 is not realistic, since the transfer of radiation is not properly treated. In Chapters 4 to 6 a model is developed which does give a proper treatment of radiative transfer, but which is based on the assumption that viscous and inertial forces are negligible.

It is found in Chapter 6 that the flow becomes turbulent near the surface. The effect of this turbulence is discussed in Chapter 7, where a qualitative model of a turbulent layer is developed, with particular reference to conditions at the surface. The final model is summarized in Chapter 8.

The work in Chapters 3 to 8 of the thesis was done by the author, with the exception of section 3 of Chapter 4, which is a discussion of various well-known methods of solution of the transfer equation, and Chapter 5, which is mostly a presentation of known theory in a form suitable for use in the present problem.

The work for this thesis was carried out while the author was a research student, and later a member of staff, in the Department of Astronomy in the University of Glasgow. One year of the research studentship was spent in the Department of Applied Mathematics and Theoretical Physics in the University of Cambridge, and the author is grateful to both Universities for provision of facilities. He also wishes to acknowledge the receipt of a grant from the Science Research Council for the period from October 1963 to October 1966.

It is a pleasure to thank Professor P.A. Sweet for his constant guidance and encouragement and for time spent in valuable discussion of the problem. The author is also very grateful to Professor L. Mestel, who supervised him while in Cambridge.

Finally, the author wishes to record his thanks to his wife for invaluable help in the preparation of the typescript.

CHAPTER 1

Introduction

"Wherefore, seeing we also are compassed about with so great a cloud of witnesses, let us .... run with patience the race that is set before us, ..."

The New Testament, Letter to the Hebrews, Ch.12, v.1.

I. Observations

1. Rotation

It has been known since the time of Galileo (1612), who recorded the motion of spots across the Sun's disk, that the Sun rotates about its axis. When it was realised that the stars were bodies similar to the Sun, it was reasonable to suppose that, in general, the stars also rotated. However, since a star does not present a visible disk, evidence of rotation is not so easily or directly obtained as it is for the Sun.

The method of measuring stellar rotational velocities appears to have been suggested first by Captain W. de W. Abney (1877), although it was thirty-two years before the first successful measurement was made, by Schlesinger at the Allegheny Observatory (1909). Abney's method used the fact that the radial velocity, measured by the Doppler effect, varies across the disk of a star if the star is rotating. Although it is not possible to measure the radial velocities at opposite limbs separately, as can be done for the

Sun, the variation across the disk will cause a broadening of lines in the star's spectrum. Abney illustrates this effect by supposing the stellar disk to be divided into three strips, one approaching the observer, one stationary and one receding. The light from the central strip gives rise to an unshifted line, while the lines produced by the two outer strips are Doppler-shifted in opposite directions. In a real star the velocity in the line of sight varies smoothly over the disk and one broadened line results. Measurement of the width of this line gives the line-of-sight component of the equatorial velocity of rotation. The measurements are complicated in practice by the need to estimate the contribution of other broadening mechanisms, such as turbulence and, especially in early type stars, the Stark effect.

Although Abney's method is the one in general use today, Schlesinger's observation was made in a rather different way, while observing the velocity-curve of δ Librae, an Algol-type eclipsing binary. The effect he observed (Schlesinger 1909, 1911; Forbes 1911) was due to the faint companion's successive obscuration of the limbs of the bright star at partial eclipse, causing the measured radial velocity to be first greater and then less than expected. He found the bright star to be rotating in the same direction as the orbital motion, with a line-of-sight component of velocity of 35 km/sec.

Since that first measurement, many observations have been made of stars of all types. The best recent measurements have been summarised by Allen (1963, p.204). An interesting, and still unexplained, feature of the results is the strong dependence of equatorial velocity on spectral type (Fig. 1), first recognised by Struve (1930). This dependence has been confirmed by detailed surveys by (among others) Su Shu Huang (1953) and Boiarchuk and Kopylov (1958). On average, stars of early spectral type rotate faster than those of later type, which rotate only slowly, if at all. The Sun, for example, of type G2V, has an equatorial velocity of about 2 km/sec.

In main sequence stars (luminosity class V) the division between fast and slow rotators occurs at about spectral type F, at a point where another division can be made. The structure of stars varies systematically with their central and surface temperatures. The central temperature determines the process of nuclear energy generation and the surface temperature determines the degree of ionisation of the surface gas. Late-type stars have outer convection zones (Fig. 2), caused by hydrogen ionisation, which become negligible at about spectral type F ($M/M_{\odot} > 1.7$; Strömgren 1965, Baker 1963). Early-type stars, whose fully-ionized outer layers are in radiative equilibrium, begin to develop convective cores at about the same spectral type (Strömgren 1965), due to the change-over from the proton-proton chain to the more temperature-sensitive CN cycle as a method of hydrogen-burning.

Thus it appears that, in general, stars with convective cores and radiative envelopes are fast rotators, while stars with radiative cores and outer convection zones (like the Sun) are slow

After

van den Heuvel 1965

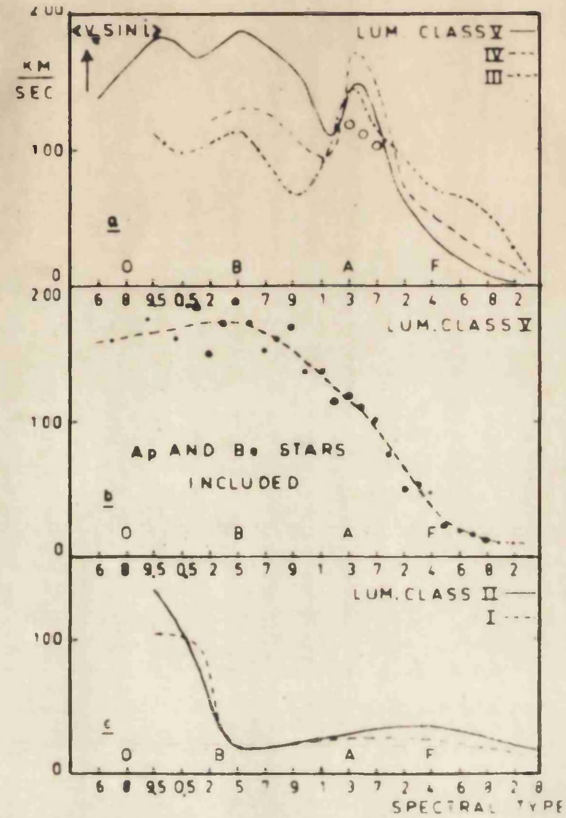

FIG. 1

The dependence of $< v_e \sin i >$ on spectral type is represented

(a) for stars of the luminosity classes III, IV and V (after Boiarchuk and Kopylov[1]). The open circles indicate values obtained after correction for the exclusion of Ap V or Am stars.

(b) for stars of the luminosity class V after correction for the exclusion of Ap or Am and Be stars.

(c) for stars of the luminosity classes I and II (after Boiarchuk and Kopylov[1]). [1] (1958)

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Schatzman 1962

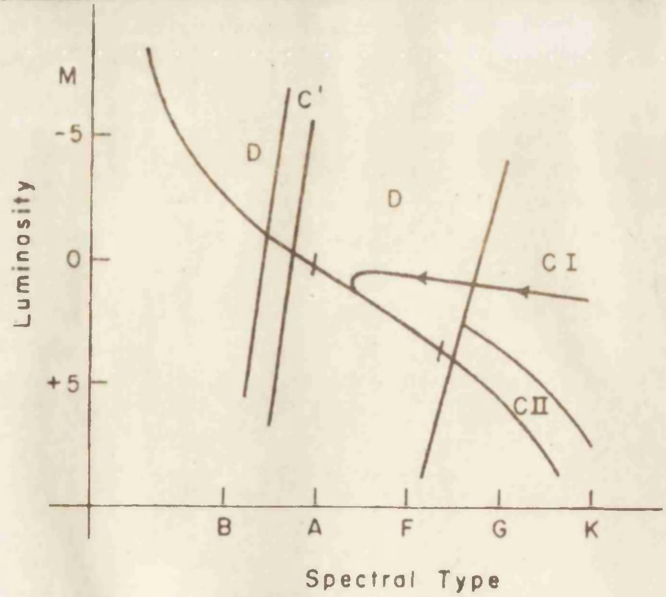

FIG. 2. — Regions in the Hertsprung-Russell Diagram. Regions C and C′ correspond to stars with a convective zone, D without a convective zone. In region CI stars present electromagnetic and equatorial mass loss ; in region CII, only electromagnetic mass loss.

rotators. E.P.J. van den Heuvel has given a recent discussion of this correlation (1965) and has suggested a solution in terms of the magnetic braking of late-type stars by a co-rotating corona, the braking occurring after the star has reached the main sequence. A recent paper on the solar wind by Brandt (1966) lends some support to this idea. On the other hand, Schatzman (1962) favours magnetic braking in the pre-main-sequence phase of evolution. His ideas have recently been supported by Wilson (1966) and by detailed calculations by Mestel (1967). Other less conventional solutions have been proposed (e.g. Gough 1966) but no explanation has yet gained general acceptance.

Whatever the explanation may be, the correlation enables a simplification to be made in the study of the effects of rotation. The effects will be greatest in early-type stars with radiative envelopes and small in stars with outer convection zones, which are notoriously more difficult to treat in detail. Accordingly, it is reasonable to restrict oneself, in the first instance, to considering the theory of meridional circulation, one of the effects of rotation, in radiative atmospheres. The theory is quite different in zones in convective equilibrium (Biermann 1951,1958; Kippenhahn 1959,1960,1963) and convective zones will not be considered in this thesis. A brief review of the theory in convective envelopes has been given by Mestel (1965). The problem is complicated by the need to introduce an anisotropic (tensor) viscosity, whose radial component is larger than the other components.

2. Turbulence

Measurements of rotation by Abney's method are complicated by the fact that lines may be broadened by turbulent motions as well as by rotation. Most observations of turbulence have been made of late-type stars (see, for example, Bell and Rodgers 1964,1967) and there is little evidence for turbulence in early-type stars. The phenomenon has been observed in early-type stars, however (Underhill 1967, personal communication), and, in view of the prediction of this thesis that there are turbulent motions in the surface layers of a rotating early-type star, it is worth considering very briefly the difficulties involved in observing turbulence.

First of all, the broadening due to turbulence must be distinguished from that due to rotation. In the paper in which the phenomenon of turbulence in stars was first conclusively demonstrated, Struve and Elvey (1934) showed that this could be done by using the curve of growth, since the gradient of the curve of growth is affected by turbulence but not by rotation.

Secondly, the size of the turbulent velocities can be estimated in several ways. These have been conveniently summarized by Su Shu Huang (1950), who distinguishes the following three methods:

(i) line-profile measurements; these refer to both large and small eddies, which broaden the line in different manners.

(ii) curve of growth measurements; this is the commonest method, and yields the most probable velocity of the small eddies. It gives no information about large eddies.

(iii) Doppler-shift measurements; these refer to individual large eddies.

In general, the results from the three methods differ. If measurements are available from both of the first two methods, the differences may be used to find the spectrum of the turbulence. Unfortunately, line-profile measurements are often not available and it is then impossible to say how the energy is distributed between various sizes of eddies. The interpretation of the observed turbulent velocity is uncertain in that case.

3. Magnetism and abundance anomalies

Like rotation, a magnetic field produces non-spherical distortions in a star and the effects have to be distinguished. The first stellar magnetic field was measured by Babcock (1947), who found a field of 1500 gauss in 78 Vir., a peculiar A-star (A2p), using the Zeeman effect. Babcock's later work suggests that stellar magnetic fields are probably ubiquitous(Babcock 1958a) and that strong coherent magnetic fields exist in all rapidly rotating stars with surface convection zones (Babcock 1958b). The measured fields are often of the order of kilogauss, and they all vary with time, some irregularly. However, Babcock's claim that rapid rotation and strong magnetic fields are correlated is based on circumstantial evidence, since he could not measure both in the same star - the Zeeman effect can only be measured in stars with narrow spectral lines, and not in those with lines broadened by rotation. Babcock's argument is based on two facts: that most of the magnetic stars which he observed were of spectral type A, and that it is in this spectral class that the mean angular velocity of stars reaches a maximum as a function of spectral type (Walker 1965b).

In order to test his claim, it is necessary to look more closely at the stars in which magnetic fields have been detected to see if there is any direct evidence for rotation. Two groups of sharp-lined A-stars are important for this purpose (Slettebak 1954). One is that of the metallic line stars (Am) defined by Roman, Morgan and Eggen (1948), of which a few possess weak magnetic fields (a few hundred gauss - Babcock 1958b). Abt (1961) has suggested that the Am stars are probably all binaries with inherently slow rotation. Strittmatter and Sargent (1966) give evidence to support the interpretation of the Am stars as intrinsically slow rotators, and use this interpretation to determine empirically the approximate position of the zero-rotation main sequence for clusters in which Am stars are found (see also next section).

The other group comprises the peculiar A-stars (Ap), discussed by, for example, Deutsch (1956), which all possess strong fields. There is disagreement as to whether these stars are intrinsically slow rotators or whether they are rapid rotators seen pole-on, as suggested by Babcock (1958b). Evidence for both possibilities has been discussed recently by Walker (1965a,b, 1966) who concludes finally (1966), from a study of abundances, that, although some Ap stars may be seen pole-on, Ap stars in general

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Walker 1965b

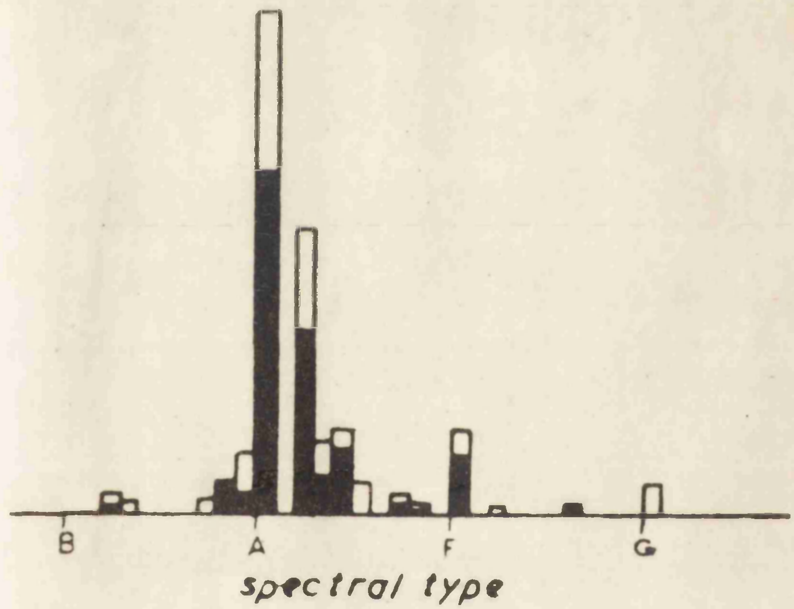

FIG. 3

Shows the incidence of stars found to possess magnetic fields by Babcock. Solid bars indicate magnetic stars, while open bars indicate probable magnetic stars.

cannot be rapidly rotating normal A-stars seen pole-on. A similar conclusion was reached by Sargent and Searle (1966), and by Sargent and Strittmatter (1966), who showed that the abnormally weak helium lines in certain B-stars cannot be explained as being produced by aspect effects in rapidly rotating stars, as had been suggested by Huang and Struve (1956) and by Guthrie (1964).

However, the main controversy (and the argument is by no means over) centres on the A-stars, in which spectral class Babcock found most stars with magnetic fields. Indeed, his results (Fig. 3) suggest that there are very few magnetic stars outside the range B5 to F2, but this conclusion is rather doubtful because of observational selection. For ease of measurement, sharp-lined stars, such as Ap or Am, are to be preferred, and Fig. 3 probably only means that Babcock observed far more stars of this kind than of any other, and therefore an unnaturally high proportion of A-stars. Indeed, he points this out in his catalogue (Babcock 1958a). There is no observational reason why BO-stars, for example, should not also possess large magnetic fields.

It is obviously important to obtain good theoretical models of rotating stars in order to be able to decide more definitely between the various interpretations of the observations of the Am and Ap stars. In particular, it is important to know what effect circulation currents could have near the surface, since some observed abundances might perhaps be explained if it were possible, for example, for material to be mixed throughout the star by the currents. This thesis confirms that that is unlikely. Circulation currents may also be expected to affect magnetic fields. However, it seems likely that in stars with strong magnetic fields there is no circulation (Mestel 1966, personal communication - see Chapter 7). Strongly magnetic stars will not, therefore, be considered in this thesis.

4. Rotational spread of the main sequence

Although magnetic fields undoubtedly exist and are of importance in many rotating stars, they are not themselves necessarily effects of rotation. The main intrinsic effect of rotation is to distort the figure of a rotating star from a sphere. The axis of rotation necessarily introduces a preferred direction in space and it is to be expected that various observed quantities will vary with the angle of inclination, i, of the rotation axis to the line of sight. Even the measured equatorial rotation velocity is a function of i, since, if v_e is the actual equatorial velocity, the measured quantity is $v_e \sin i$. Actual rotation velocities can be found only if i can be determined, as in eclipsing binary systems. Otherwise, statistical analysis is needed to find average values of v_e for a given spectral type, and it has not been possible until very recently to find v_e for a particular star. A method for doing so has now been described by Roxburgh, Sargent and Strittmatter (1966). The method makes use of the effect of rotation on the luminosity and spectral type (or colour) of a star.

The first quantitative estimate of this effect was made by

Sweet and Roy (1953), who showed that rotation could produce a spread in a Hertzsprung-Russell diagram of as much as half a magnitude for a given spectral type. This could account for at least part of the observed spread in the upper half of the main sequence of clusters (Strittmatter 1966), even though their results need some modification (Sweet 1965, personal communication). A more detailed discussion of the spread due to rotation is contained in Appendix I, where a criticism of a method of obtaining the zero-rotation main sequence for clusters is given. A discussion of theoretical results is given in section II 5 of this chapter.

II. Theory

1. The origin of meridian circulation

Circulation currents in the atmospheres of rotating stars cannot be directly observed, and their presence must be inferred from their effect on other quantities which can be observed. It is therefore essential to have a theoretical model which will give the effects of circulation on the surface conditions in stars. It is the aim of this thesis to provide such a model. In the rest of this chapter a summary will be given of the most important previous work on circulation in rotating stars.

The first important theoretical result is that due to von Zeipel (1924a,b), whose famous paradox concerns the rate of energy generation in a uniformly rotating star. Von Zeipel's result was that

$$\varepsilon \propto 1 - \frac{\Omega^2}{2\pi G\rho} \qquad (1.1)$$

where ε is the rate of energy generation at a point where the density is ρ, Ω is the constant angular speed of rotation and G is the gravitational constant. It is clear that, since the density decreases to zero in the outer layers of a star, the energy generation must become negative at some critical density near the surface, that is, energy will be absorbed rather than liberated. It appears that von Zeipel himself believed this result (Eddington 1925), but it is not now taken seriously, although there is still a place in the theory for a critical density, as will be seen later (Mestel 1966). The paradox was resolved almost simultaneously by Vogt (1925) and Eddington (1925). Eddington gives a very clear discussion of the problem in his book (1930, pp 282-288).

Von Zeipel's result depends on the strict maintenance of radiative equilibrium, that is, on the balancing of the divergence of the radiative flux only by (nuclear) energy generation. The paradox may be resolved by removing this strict condition. In that case, equilibrium must be maintained by some additional form of energy transport. An obvious transport process is convection. However, convection in this context has not the meaning usually understood in stellar structure. An atmosphere in radiative equilibrium is said to be unstable against (ordinary) convection if the temperature gradient is greater than the adiabatic temperature gradient (Schwarzschild's criterion (1906) - this criterion is altered in the presence of a magnetic field with a vertical component, which tends to stabilize the atmosphere (Gough and Tayler 1966)). Convection then starts, and the temperature gradient settles down to a value nearly equal to, and slightly greater than, the adiabatic value. In that case most of the energy is carried by the convection currents and radiative energy transport can be ignored.

However, if the atmosphere is radiatively stable, as is assumed in the present case, ordinary convection will not appear. Instead, large-scale laminar circulation currents are set up, caused by the break-down in radiative equilibrium, the temperature gradient remains subadiabatic and the radiative flux carries most of the energy. The large-scale "convection" currents carry only enough energy to maintain a steady state, and the structure is still essentially that of a region in radiative equilibrium. The circulation is confined to meridian planes and is generally referred to as meridian, or meridional, circulation. Circulation of this kind may also occur in zones of weak convection where much of the energy is carried by radiation. In that case, the flow is turbulent (Kippenhahn 1959).

2. Mixing

One effect of the circulation currents is to mix the material of the star. Calculations by Eddington (1929) suggested that the currents were fast enough to keep the star well-mixed and therefore of homogeneous composition. However, an inhomogeneous model was successful in explaining the existence of red giants (Hoyle and

Lyttleton 1942), and when detailed work on the evolution of well-mixed stars came to be done by the Bondis and others about 1950 it became apparent that there was some disagreement with observation (see Mestel's summary (1959)). This discrepancy was explained when Eddington's calculations were corrected by Sweet (1950) and Öpik (1951). They found† that the speed of the currents in the interior was a million times slower than Eddington had predicted, so that the time required for complete mixing is too great for stars to be even nearly homogeneous.

Further work on this problem was done by Mestel (1953), who investigated the effect of a non-constant molecular weight. It was found that inhomogeneities of that kind also cause circulation currents ("μ-currents"). These currents are in the opposite sense to the Eddington-Sweet circulation and further reduce the mixing effect of the latter. It is now generally agreed that the mixing by meridional circulation currents is negligible for the purposes of stellar evolution. This has been questioned by Porfir'ev (1963), who points out that the above results are based on the assumption of uniform rotation, which will be rapidly destroyed by the circulation itself. Although this is true in the absence of any constraint, the assumption may be justified if a suitable constraint is postulated, as will be seen later. Porfir'ev produces no quantitative theory to support his contention that the speed of the currents is fast enough in a non-uniformly rotating star to cause mixing, and his second section depends on the invalid assumption that the surface of a star is a

† Very similar work was done earlier by Gratton (1945).

streamline of the circulation. Also, his work on rotation without meridian circulation appears to be contradicted by that of Roxburgh (1964a,b). Schwarzschild's paper on the same subject (1947) is wrong because of the incorrect truncation of a series (Roxburgh 1964a) and it is likely that Porfir'ev's paper is wrong for a similar reason.

3. Steady state configurations

The circulation currents arising in a star in uniform rotation carry angular momentum. The resultant Coriolis forces will rapidly destroy the uniformity of the rotation unless there is some constraint. Such constraints will be considered shortly. In the absence of constraints, only two final steady states are possible. The star must settle down in a state of non-uniform rotation, either without meridian circulation or with meridian circulation and with the angular momentum per unit mass constant on stream lines of the circulation (Roxburgh 1964a). The latter case is extremely difficult to treat, as the rotation law is in general unknown if the stream lines are unknown while the form of the stream lines is itself determined by the rotation law. This kind of problem recurs in every case where steady circulation could arise, and no solution is yet known. A solution for the case of zero circulation has been given by Roxburgh (1964a,b). In that case the angular velocity is a function of the radius only and decreases outwards. His work replaces earlier (incorrect) work by Schwarzschild (1947) and two rather artificial models due to Rosseland (1936). Roxburgh has also shown(1966), by an extension of von Zeipel's theorem (1924a,b), that, in a rotating star with no constraints, there is no steady state configuration where the angular velocity depends only on the distance from the rotation axis.

Some iterative numerical work on the case of steady circulation has been attempted by M. Maheswaran (1966, personal communication) who finds that the circulation breaks up into several zones. This may be symptomatic of the result found by K. Fricke (1967, personal communication; to be published), who has been investigating the stability of steady-state configurations without constraints. He finds that Roxburgh's model (Roxburgh 1964a,b) is unstable to small perturbations. He has also investigated the problem of steady circulation in the radiative zone of a Cowling model. Using the Boussinesq approximation, he has been able to show that this configuration is also unstable. He concludes that no stable steady state configuration is possible for a non-uniformly rotating star without constraints.

The theoretician is therefore forced to consider the problem of rotation in the presence of constraints. Since constraints are necessary in any case, it seems best in the first instance to consider constraints which keep the rotation uniform. That case has the advantage of simplicity. Also, there is some observational evidence that uniform rotation may be the most realistic assumption (T.R. Stoeckley 1967, personal communication).

One plausible constraint is viscosity. In stellar conditions, however, molecular and radiative viscosities are both so small that viscous forces are generally negligible compared to centrifugal forces, even near the surface (Smith 1966). In the interior of a star the effect of viscous forces on the rotation law is certainly negligible (Jeans 1929). If there is turbulence, the situation is different. Turbulent viscosity is many times greater than radiative viscosity and may have an important effect. This situation has been considered by Kippenhahn (1959) and Osaki (1966) and will be discussed in detail in Chapter 7. The problem is greatly complicated by the still inadequate state of the theory of turbulence.

A process which may sometimes be effective is the braking of a star by the radiative transport of angular momentum (Jeans 1926, 1929). However, in most cases this effect is much smaller than that of the Coriolis forces and can be ignored (Mestel 1965).

The ineffectiveness of these processes and the prevalence of magnetic fields in stars, noted in section I 3, suggests that magnetic forces are the most likely ones to have an appreciable effect on rotation.* This has been supported by most of the recent work on magnetic stars. As in a non-magnetic star, strict uniformity of rotation is not possible because of the perturbing effect of the circulation. However, in a magnetic star two steady states with circulation are amenable to treatment (Roxburgh 1963). In one of

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*The rotation may also be affected by the "spin-down" mechanism discussed recently by Howard, Moore and Spiegel (1967). This mechanism should also tend to make the rotation uniform.

these the centrifugal force is taken as the dominant perturbation, with nearly uniform rotation maintained by a weak poloidal field. In the other case the dominant perturbation is a toroidal magnetic field of a particular form. While the first case may be relevant to early-type rotating stars, the second seems unlikely to have any application in view of theories of the origin of magnetic fields in stars (Roxburgh 1963, Mestel 1965), all of which suggest much stronger poloidal fields than the second case would allow. No solution has yet been obtained for any other case.

As before, a steady state without meridian circulation can also be considered. Various special models of non-uniformly rotating magnetic stars without circulation have been studied by Roxburgh and Strittmatter (Roxburgh 1966, Roxburgh and Strittmatter 1966a,b). These models differ from the non-magnetic models considered by Roxburgh (1964a,b) in that the angular velocity increases outwards rather than inwards. These models are not ruled out by Fricke's investigations on stability. However, the models used are rather artificial, since the magnetic field is purely toroidal, being built up by Biermann's "battery" effect (1950). It is known (Mestel and Roxburgh 1962) that even a very weak poloidal field is enough to prevent the "battery" from operating, and it therefore seems unlikely that the models of Roxburgh and Strittmatter are applicable to many real stars.

Thus, of the cases for which solutions exist, the only ones which seem likely to be relevant to real stars are the case of a non-magnetic star in steady non-uniform rotation without circulation and the case of a magnetic star with circulation and in nearly uniform rotation. The latter is therefore the only steady state model available as a basis for a closer study of meridian circulation.

4. Surface conditions

The construction of rotating model stars is so complex that until recently the simplest boundary conditions have been taken. In general this has meant taking simple zero pressure and temperature conditions at the surface. In this way a model can be constructed which gives a good description of the interior of the star but says very little about the outer layers. Boundary conditions for the velocity field are also required, and it is usually assumed that the velocities at the surface are finite and that there is no net outward flow of matter through any closed surface surrounding the star. (See, for example, Sweet 1950.)

Unfortunately, it seems that these boundary conditions for the velocity field are inconsistent with the simple zero pressure and temperature boundary conditions for the structure. That result is implicit in Öpik's paper (1951). Öpik used a perturbation theory to determine the circulation currents in a uniformly rotating star, employing an extension of von Zeipel's argument (Eddington 1930). Although his theory was accurate only to the first order in the ratio of centrifugal force to gravity, he retained a second order term, proportional to 1/density, which became dominant near the surface.

This term did not cause trouble in Öpik's model, which had a surface convection zone in which the theory was invalid. There was therefore no suggestion of a singularity at the surface.

However, certain difficulties present themselves even without considering Öpik's second order term. Mestel has noted (1953) that, although the radial component of the velocity field in Sweet's model (1950) is finite, the tangential component has singularities at the surface and at the boundary of a convection zone. A model of viscous dissipation was proposed to resolve this problem.

A more detailed study of the behaviour of the circulation near the surface has been published by Baker and Kippenhahn (1959). They showed that, near the surface of a non-uniformly rotating star, the radial component of the velocity field was proportional to 1/density, even using a first order perturbation theory. Their paper showed that the finite radial component in Sweet's work (1950) was due entirely to the use of uniform rotation, a special case of the more general class of rotation laws

$$\Omega = \sqrt{C_1 + \frac{C_2}{r^2 \sin^2 \theta}} \qquad (1.2)$$

(C_1, C_2 constants) for which a first order perturbation theory yields a finite radial component at the surface.

Although this result showed uniform rotation to be a special case when a first order perturbation theory is used, Öpik's result threw some doubt on the validity of a perturbation theory near the surface, since in his model the second order term dominates over the first order term as the surface is approached. To be sure why the singularity occurs, one should investigate the behaviour of the velocity field near the surface without resorting to a perturbation theory. This has recently been done by Mestel (1966). It is now clear that the 1/density dependence is a general feature of the velocity field for any rotation law, and that uniform rotation is a special case only in the sense that the 1/density terms are of second order in the ratio of centrifugal force to gravity due to the exact cancellation of the singular first order terms. A slightly less general form of the same result had already been found by the present author (Smith 1966) with the use of a Roche gravitational potential. This work is described in Chapter 3.

It is therefore clear that the use of a uniformly rotating model is valid but that a more realistic model of the surface layers is required. The present thesis describes such a model.

The main assumptions made in the above models are

(i) that viscous and inertial forces are negligible compared to the centrifugal forces due to rotation

(ii) that the local equation of energy transfer $\underset{\sim}{\mathcal{F}} \propto \text{grad } T$ ($\underset{\sim}{\mathcal{F}}$ = radiative flux, T = temperature) can be used.

While the author has shown that assumption (i) is consistent with the results obtained from a Roche model (Smith 1966), the second assumption is clearly dubious in the outer layers of a star, since it assumes that the photon mean free path is much less than the scale height. Mestel's work (1966) suggests that it is the form of the local equation which leads directly to the 1/density singularity, but even without that result one would be surprised if a local equation were to give realistic results in a region where the photon mean free path is comparatively long. All previous studies of stellar atmospheres have required to use the non-local transfer equation (which reduces to the local equation at great depths in the star - see, for example, Chandrasekhar 1939, pp 208-211) and one would hardly expect a rotating stellar atmosphere to be different in that respect.

The only reason that a non-local equation has not been used until recently in the study of rotating stars is that a local equation is easier to handle mathematically and gives perfectly adequate results for the overall structure of the star. Only when the structure of the atmosphere is considered is a non-local treatment necessary.

It is found that the use of a non-local transfer equation leads to the more realistic result that the velocity is finite at the surface. A formal proof of this result may be given very briefly, using known results; for example, it may be proved from p. 11 of Chandrasekhar's book on radiative transfer (1950 - hereafter referred to as R.T.) by using equation (9) of the author's paper (Smith 1966). However, to find the value of the velocity at the surface is more difficult, and requires a solution of the non-local transfer equation in a non-spherical atmosphere. An approximate solution is derived in this thesis (Chapters 4 to 6).

Recently, Osaki (1966) has also produced a theory of a rotating atmosphere with a non-local transfer equation. He assumes that the atmosphere is locally plane-parellel and uses the exact plane-parallel solution of the transfer equation to show that the condition of radiative equilibrium is grossly violated. Unfortunately, although his treatment adequately represents the variation of various quantities with latitude, he does not take any systematic account of the effect of curvature in the atmosphere and it is not clear how one would extend his model to include curvature effects. The equations in Chapters 4 and 5 of this thesis represent exactly the effects of curvature, although so far the equations have only been solved in the plane-parallel approximation.

The present author agrees with Osaki that the condition of radiative equilibrium is grossly violated, so that the non-local theory is singular in the sense that it predicts unrealistically large circulation speeds. This result was obtained independently of Osaki.

Osaki proposed two models of the surface layers which might be more realistic. In both models the rotation is non-uniform. In one model, the angular velocity is supposed redistributed in such a way that there is no circulation and the star is in radiative equilibrium. In the other model, the circulation speeds are supposed limited by turbulent dissipation. The present author rejects both these models. It is found that the flow is unstable, so that turbulence is certainly present. However, Osaki's turbulent model is internally inconsistent, for reasons which will explained in

Chapter 7, and the turbulent velocities turn out to be about one hundred times larger than those estimated by Osaki. It has not proved possible to obtain a quantitative model of the surface layers, but it is certain from the qualitative model given in Chapter 7 that turbulent velocities of the order of the speed of sound are to be expected near the surface of a rotating early-type star.

5. Observational consequences

The present thesis is the first detailed study of meridian circulation in an atmosphere with non-local radiative transfer. However, it is necessary to consider non-local effects to some extent if the variation of brightness over the surface of a rotating star is to be calculated, and several authors have used the theory of stellar atmospheres for this purpose, without considering circulation.

The surface variation of brightness was first studied by Sweet and Roy (1953) who used a rotating Cowling-model star with a local transfer equation and a limb-darkening coefficient of 0.6. More detailed work has been done recently by (for example) Collins (1963, 1965) and Roxburgh and Strittmatter (1965). These authors use a combination of non-local transfer theory in a plane-parallel atmosphere and the von Zeipel gravity-darkening, which is strictly true only for a local transfer theory. It is not obvious a priori that von Zeipel's result is a good approximation for a non-local theory and it is not assumed by either Osaki or the present author. Nonetheless, it is unlikely that their conclusions are much in error for stars without meridian circulation.

Both Collins and Roxburgh and Strittmatter use a Roche gravitational potential. The main difference between their approaches is that Roxburgh and Strittmatter use a rotating interior for their stellar model based on the rotating stellar model of Roxburgh, Griffith and Sweet (1965). The use of this interior model enables them to dispense with the assumptions made by Collins that the luminosity and polar radius are the same in a rotating star as in a non-rotating star of the same mass and chemical composition. Their results show these assumptions of Collins to be incorrect, but are otherwise in good agreement with Collins' results. The main qualitative difference is in the behaviour, for small rotation speeds, of stars inclined at a small angle to the line of sight. The initial decrease in luminosity as the rotation speed increases from zero, caused by a lower central temperature, is not found by Collins. Apart from this, Collins' assumptions give qualitatively correct results. In his later work, Collins (Collins and Harrington 1966) has used the Roxburgh, Griffith and Sweet rotating interior as a basis for the calculation of Hβ line profiles for families of rotating B stars.

The most important result of work on the variation of brightness over the surface of a rotating star is that the observed luminosity and so the absolute magnitude of a star of a given mass and chemical composition is a function of two things: the angle of inclination, i, of the rotation axis to the line of sight and the speed of rotation, Ω. The effective temperature, and so the colour, or

After Roxburgh and Strittmatter 1965

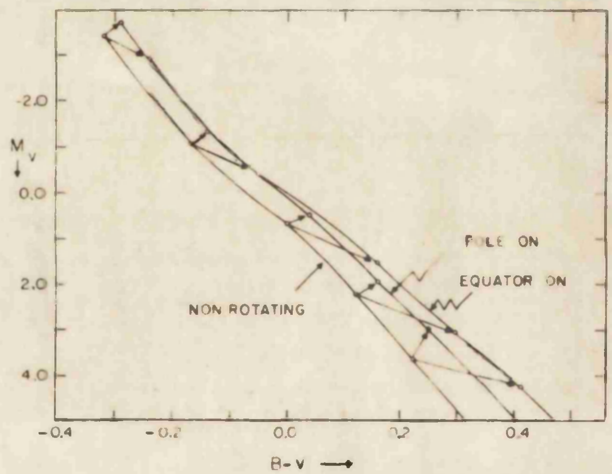

Fig. 4. Hertzsprung-Russell diagram for non-rotating and fully rotating stars

spectral type, is also a function of i and Ω. There is thus a spread in a colour-magnitude diagram, due to the rotation of the stars plotted on it. Rotating stars appear to the right of a notional main sequence for non-rotating stars. This was recognised by Sweet and Roy (1953) who found a limiting spread of about half a magnitude. Roxburgh and Strittmatter find a rather larger spread (Fig. 4).

Similar results have been found by Ireland (1965), who considers the two extreme cases of a Roche model and of a model of uniform density. He finds that these models give similar results despite their great physical difference. Although he does not specifically refer his results to a colour-magnitude diagram, he does show that rotation may change the spectral type of a star by as many as five sub-classes, a result more in agreement with Roxburgh and Strittmatter than with Sweet and Roy. It should be noted, however, that Ireland makes no allowance for limb-darkening.

All the above work is for stars in uniform rotation. Roxburgh and Strittmatter and Ireland have also considered non-uniformly rotating stars (Roxburgh and Strittmatter 1966b; Roxburgh 1963,1966; Ireland 1967). A similar spread is found in the HR diagram. Ireland (1967) finds that, for rapidly rotating stars, the spread in luminosity is more sensitive to the degree of non-uniformity of the rotation than to changes in the rotation speed itself. However, he uses a very special form of rotation law.

As noted by Roxburgh, Sargent and Strittmatter (Roxburgh, Sargent and Strittmatter 1966; Strittmatter 1966; Strittmatter and Sargent 1966), the theoretical predictions can, in principle, be combined with the observed spread in the main sequence to obtain a zero-rotaion main sequence for stellar clusters. A summary of the method they suggest is given in Appendix I, where it is shown that it is not at all clear that their method is reliable with the small number of currently available observations. Nonetheless, it is now possible, in principle, to obtain v as well as v sin i for the individual stars in a cluster.

If it is assumed that the results obtained by Strittmatter et al are substantially correct, an interesting fact emerges. The theoretical models which appear to agree most convincingly with the observations are the models of non-uniformly rotating stars without meridian circulation (Strittmatter and Sargent 1966). The uniformly rotating models predict a spread in luminosity for a given colour which is less than one fifth of that observed. This discrepancy appears to be too large to be explained by the uncertainties in the observations.

However, the uniformly rotating models must all contain circulation currents, the effect of which has been ignored in previous calculations (cf. Ireland 1965, p.66). It is possible that the inclusion of circulation currents in the treatment of the surface layers of uniformly rotating stars may lead to a better agreement with observation. At any rate, this thesis shows that the surface conditions are rather different from those assumed in previous uniformly rotating models, and there is little doubt that the von

Zeipel gravity-darkening assumed in these models is a poor approximation.

6. Binary stars

The theory of rotating stellar atmospheres may be applied, as indicated in section 5, to obtain the angle of inclination of the rotation axis to the line of sight. In binary stars, for which the angle of inclination can often be found by observing eclipses, the theory has another application. The problem described below was, in fact, the original motivation for the present thesis, and it is hoped to return to this problem at a later date.

In a close binary system, each star is illuminated by the other on the inward-facing hemisphere. The radiation from the other star will be absorbed, or scattered, and eventually re-emitted. This is the well-known reflection effect (see, for example, Kopal 1959). Recently (Ovenden 1963) some observations have been made of 57 Cygni which do not seem to be explicable in terms of the normal reflection effect (Napier 1966). It is hoped that the following considerations may throw some light on the problem.

The external illumination may be expected to set up further motions in the atmospheric gas. The present thesis shows that these motions will be turbulent, but the problem in a close binary system is complicated by the lack of axial symmetry and it is possible that new large-scale streaming may be set up within the turbulent region, which will redistribute the energy in the incident radiation over the surface of the star.

In the absence of external illumination, the boundary condition for the intensity of radiation is that the inward intensity is zero at the surface. For a close binary system, the external illumination on each star of the system would be represented by an inward intensity at the surface, varying in some prescribed way over the hemisphere facing the other star and zero over the other hemisphere. If the system is in synchronous rotation, so that the axes of rotation of the two stars are parallel to each other and to the axis of rotation of the whole system and the period of rotation of each star is the same as that of its revolution about the common centre of gravity, the two stars always present the same face to each other and the problem is independent of time. Nonetheless, the problem is more complicated than that of a single star, since the boundary condition is not axially symmetric and an extra independent variable is introduced. The non-axially-symmetric problem has not been attempted. A solution has been obtained only for the artificial axially-symmetric case of illumination parallel to the axis of rotation (Sweet 1965 - unpublished).

Only circular orbits are truly synchronous. Elliptical orbits, or general non-synchronous orbits, introduce the further complication of time variation, probably slightly simplified by the existence of a periodic solution. This time-dependent, non-axially-symmetric problem is well outwith the scope of the present thesis.

CHAPTER 2

Description of the basic model

"The basis or substratum - what you will -

Of the impending eighty thousand lines."

C.S. Calverley, The Cock and the Bull.

1. Assumptions

To make the problem of meridian circulation mathematically tractable, it is necessary that the basic stellar model be reasonably simple, although of course it must be sufficiently realistic that the results obtained are meaningful. In this chapter various simplifying assumptions are discussed and the basic equations and definitions are stated.

As mentioned in Chapter 1, only early-type stars will be considered, both because the effects of rotation are expected to be larger in such stars (on observational grounds - see, for example, van den Heuvel 1965) and because deep convection zones are not present in their atmospheres. Convection is not yet well enough understood for a simple description to be known to be adequate, and it was thought better to restrict the investigation to radiative atmospheres, such as are to be expected in stars of early spectral type.

Even in early-type stars there is a convection zone near the surface, associated with the ionization of helium in the same way as the deep convection zones of late-type stars are associated with the ionization of hydrogen. However, in stars earlier than about O8 the helium convection zone seems to have disappeared (Underhill 1950, 1951) and even when it is present (in stars later than O8) it is usually weak and only starts at several optical depths below the surface (Underhill 1950; Rudkjobing 1947). Helium ionization effects will therefore be ignored in this thesis*, for the sake of simplicity, and the atmosphere will be assumed to be stable against turbulent convection.

The role of helium in early-type atmospheres is further discussed in a later paper by Miss Underhill (1957), where she notes that the presence of a small amount of helium has little effect on the temperature and pressure distribution. That is, to find the structure of the atmosphere of an early-type star it is sufficient to assume that it consists of pure hydrogen. Since the distribution of molecular weight would in any case have been assumed to be homogeneous, the assumption of pure hydrogen does not cause to be missed any effect due to variation in molecular weight, such as the "μ-currents" investigated by Mestel (1953 - see Chapter 1, section II 2). Since the joint effect of the μ-currents and the meridian circulation can be found (to first order) simply by superposing the velocity fields, it would seem to add little to the investigation to consider the μ-currents as well. Besides, the velocities in the atmosphere of a star will be found to be such that the material near the surface may be expected to be extremely well-mixed. The μ-currents would then be negligible.

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*In a private conversation (1967, XIII IAU) Miss Underhill has confirmed that this is a good approximation.

The molecular weight will therefore be taken as a constant throughout, of value 1/2 since at the temperatures concerned the hydrogen will be virtually fully ionized.

A possible complication in early-type stars is the effect of radiation pressure on the effective gravity. This problem also has been discussed by Miss Underhill (1949). Comparison of Table 1 of her paper with the tables on pp 201, 207-8 of Allen's "Astrophysical Quantities" (1963) shows that, for main-sequence stars later than about 08, the radiation pressure gradient is less than 10% of the gas pressure gradient. While this could change the effective gravity by an amount comparable with the change due to rotation, the radiation pressure is still basically a small perturbation. To isolate the effect of a particular perturbation, it is useful to consider it as the only one acting. Therefore, since the radiation pressure clearly has no <u>dominant</u> effect, it will be omitted from the model discussed in this thesis. Further reasons for omitting it are that the effect of radiation pressure has already been considered (Underhill 1949) and that its inclusion unduly complicates the equations.

It should be noted that, for a detailed model atmosphere, either the radiation pressure (for O stars) or a helium ionization zone (for stars later than about 08) should be considered. In the light of the above discussion, however, the models with these effects included should not differ qualitatively from the models without the effects. The model used in this thesis may therefore be expected to represent roughly stars with spectral types in the range 05 to A0.

For definiteness, the data below for a BO star will be used when making numerical estimates.

Under the conditions of the Vogt-Russell theorem (see, for example, Chandrasekhar 1939), a star is uniquely determined by its mass and chemical composition. Since these parameters are not affected by rotation (unless the star is rotating so fast that there is mass loss at the equator or the central temperature changes sufficiently to alter significantly the rate of nuclear reactions), a rotating star may be uniquely specified by its rotation speed and the mass and chemical composition of its non-rotating counterpart, which will not in general be of the same spectral type (Ireland 1965). A typical non-rotating BO star (Allen 1963) has a mass of about $17M_{\odot}$ $(M = 3.4 \times 10^{34}\text{gm})$, a radius of about $7.6R_{\odot}$ $(R = 5.3 \times 10^{11}\text{cm})$ and a luminosity of about $1.3 \times 10^{4}L_{\odot}$ $(L_o = 4.9 \times 10^{37}\text{erg/sec})$. These values will be adopted for the non-rotating star of the same mass and chemical composition as the rotating model under consideration. A value of 2.2×10^{4}°K will be used for the effective temperature. This does not quite agree with Allen's value (2.1×10^{4}°K), but was chosen to satisfy approximately the relation $L_o = 4\pi R^2 \sigma T_e^4$ (see section 3 of this chapter).

In a more detailed investigation, the radius, luminosity and effective temperature would emerge as results of an integration of the complete structure equations of the star. However, since stellar interiors are in general fairly well understood, and the present thesis is investigating essentially only the qualitative surface properties of the star, it did not seem worth repeating standard interior integrations. Of course, the rotation of the star will alter its internal structure to a certain extent. Fortunately, this effect has already been discussed (Roxburgh, Griffith and Sweet 1965) and it is possible to use an existing rotating interior model* as a basis for the new atmospheric investigation.

To obtain a simple, self-consistent, steady-state model, Roxburgh, Griffith and Sweet chose to consider a uniformly rotating star with a weak poloidal magnetic field. The role of this magnetic field (cf. Chapter 1, section II 3 and Roxburgh 1963) is solely to keep the rotation uniform by balancing the (toroidal) Coriolis forces due to the meridian circulation. In reality, the rotation is unlikely to be completely uniform. However, since uniform rotation does not seem to be a singular case, it is permissible to make the convenient idealisation that the rotation is strictly uniform. It is assumed that any meridian plane component of the magnetic force is negligible compared to the centrifugal forces due to the rotation. This assumption has the great merit that a magnetic field never appears explicitly in the structure equations.

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*The data quoted above do not quite agree with the mass/radius, mass/luminosity diagrams given by Roxburgh et al (1965). This is not important, since the only numerical results taken from that paper are on the variation of R and L with rotation speed (Roxburgh and Strittmatter 1965). This variation appears to be virtually independent of the exact model adopted.

As a first approximation, it will also be assumed that the speed of the meridian circulation is slow enough that viscous and inertial forces are negligible compared with the centrifugal forces (cf. Chapter 1, section II 3). This assumption is made in the first place as a mathematical convenience, but it will be seen to be at least partially justified. Only very near the surface (Chapters 6, 7) does viscosity have to be considered, and the inertial forces are always negligible.

The steady-state equation of motion for the stellar material (assumed to be a perfect gas) can then be written simply as

$$\frac{1}{\rho}\,\mathrm{grad}\,P = \mathrm{grad}\,\Phi + \Omega^2 \underset{\sim}{\varpi} \tag{2.1}$$

where P and ρ are the pressure and density, Φ is the gravitational potential, Ω is the rotation speed (a constant) and $\underset{\sim}{\varpi}$ is the vector distance from the axis of rotation. The gravitational potential of a gaseous, self-gravitating mass in rotation is rather complicated, particularly if the rotation is rapid. It would be convenient to find an approximation which simplified the mathematics while still retaining the essence of the physical situation.

Such an approximation has been developed by Roxburgh, Griffith and Sweet (1965 - this paper will hereafter be referred to as RGS). In their model the star is essentially divided into two regions. In the outer region the density is sufficiently low that the gravitational potential Φ can be taken as due entirely to the material of the inner region. It is therefore found by solving

Laplace's equation. Assuming that the inner region contains essentially all the mass of the star (an assumption which must, of course, be checked when the whole model has been assembled. However, Eddington's model stars (Eddington 1930) suggest that it is likely to be a good approximation), the solution is

$$\Phi = \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty} \frac{a_n}{r^n} P_n(\mu)\right] \qquad (2.2)$$

where G = gravitational constant, M = mass of inner region (= mass of star), r = distance from centre, $P_n(\mu)$ is the Legendre polynomial of order n, $\mu = \cos\theta$ where θ is the angle between the rotation axis and the radius vector, and the a_n are arbitrary constants. The a_n are determined by the degree of distortion from the spherical of the inner region. It was found by Monaghan and Roxburgh (1965) that the gravitational effect of the distortion of the inner region is small (in polytropes) compared with the centrifugal forces and that only the P_2 term needs to be considered. For present purposes, even this term can be ignored, that is, <u>all</u> the a_n may be taken as zero (as is in fact done in RGS) and Φ is represented simply by the Roche potential

$$\Phi = \frac{GM}{r} . \qquad (2.3)$$

In the inner region of the star, the mass is not negligible, but α, the ratio of centrifugal force to gravity, is small, even if this ratio is unity at the surface. This fact allows the use of a first order perturbation theory, similar to that used by Chandrasekhar (1933) and Sweet and Roy (1953). Details of this theory, and the criteria used to fit the two regions, are given fully in RGS, where it is shown that the approximations are likely to lead to errors of less than one per cent. The present theory of meridian circulation makes no explicit use of the structure of the inner region, and it will not be further considered here.

Various other assumptions are made. It is clearly reasonable to assume symmetry about the axis of rotation and about the equatorial plane. It is also reasonable to assume that there are no nuclear energy sources in the surface layers. Indeed, no energy production is likely by any means near the surface. It is also assumed that there is no energy dissipation in the surface layers. This assumption is re-examined in Chapter 7. The only plausible dissipative mechanism is viscosity, and the assumption of negligible viscous forces suggests that viscous dissipation is likely to be negligible as well.

As already mentioned, energy transfer is supposed to be solely by radiation, and by a large-scale laminar circulation in meridian planes. The transfer of radiation can be treated in two ways. In a "local" theory (normally used mainly in stellar interiors), the radiative flux $\underset{\sim}{\mathfrak{F}}$ can be written as

$$\underset{\sim}{\mathfrak{F}} = -\frac{16}{3}\frac{\sigma}{\kappa}\frac{T^3}{\rho}\,\mathrm{grad}\,T \qquad (2.4)$$

(see, for example, Schwarzschild 1958). When given by this formula, $\underset{\sim}{\mathfrak{F}}$ depends on local values of T, ρ and grad T (T is the temperature,
$\underset{\sim}{\mathfrak{F}}$ depends on local values of T,
σ is Stefan's constant and κ is the opacity). In the outer layers

of a star, where the photon mean free path is long, it would be surprising if such a local theory were to be adequate. Nonetheless, local thermodynamic equilibrium, which might also be expected to fail where the mean free path is long, will be assumed, and it seemed worth investigating the results of a local theory.*

In fact, it will be seen that the local theory does fail. The assumption of local thermodynamic equilibrium is not invalidated by this result, the accuracy of the LTE approximation holding remarkably close to the surface of a star (Kourganoff 1952 p.8), but it is necessary to use a non-local theory for the transfer of radiation. The relevant equations will be quoted in the summary in the next section.

Finally, the opacity must be considered. For mathematical simplicity, it is convenient to consider a gray atmosphere with either a Kramer's opacity ($\kappa \propto \rho^{\alpha} T^{\beta}$) or simply κ = constant, corresponding to an electron-scattering atmosphere. The latter is certainly the simpler and will be adopted throughout, although it must be recognised that this is not a particularly good approximation for the atmosphere of a B0 star. However, the mathematics becomes rather complex in the non-local theory and it seemed better in the

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*The local theory was investigated before Mestel (1966) had shown that the 1/density dependence was a general feature. However, even if that result had been known, it would still have been useful to have the results of the local theory for comparison with the non-local theory.

first place to aim for mathematical simplicity rather than for a highly accurate physical picture. The electron-scattering opacity should be good enough to give a qualitatively accurate result. It seems unlikely that a more accurate opacity law would alter the results of Chapters 6 and 7 enough to change their significance. For an atmosphere of pure hydrogen, κ has the value 0.38 (Allen 1963 p.94).

To summarise, then, the initial assumptions are as follows, in no particular order:

(i) Steady-state.

(ii) Uniform rotation, maintained against Coriolis forces by a weak magnetic field.

(iii) Star divided into two regions; Roche gravitational potential in outer region, the only region considered.

(iv) Axial and equatorial symmetry.

(v) Perfect gas - pure hydrogen atmosphere.

(vi) Radiation pressure negligible.

(vii) Non-rotating star of same mass and chemical composition would be BO.

(viii) Atmosphere stable against convection, other than large scale circulation.

(ix) Magnetic, viscous and inertial forces negligible compared with centrifugal forces.

(x) No energy production or dissipation in the atmosphere.

(xi) Local thermodynamic equilibrium.

(xii) Gray atmosphere with constant (electron-scattering) opacity.

2. Equations

With the above assumptions, the equations to be solved are relatively simple. Since the interior is not being considered, the only equations required are those which apply in the atmosphere, and these will now be summarised.

Consider first the equation of motion. In view of assumptions (i) and (ix), the hydrostatic approximation (equation (2.1)) may be used. Since uniform rotation and a Roche gravitational potential are being assumed (assumptions (ii) and (iii)), it is possible to define a joint potential Ψ by

$$\Psi = \frac{GM}{r} + \tfrac{1}{2}\Omega^2 r^2 \sin^2\theta \quad . \tag{2.5}$$

The meridian-plane component of the equation of motion may then be written in the convenient form

$$\text{grad } P = \rho \text{ grad } \Psi \tag{2.6}$$

which shows that surfaces of constant P and constant Ψ coincide, so that $P = P(\Psi)$. Hence also $\rho = \rho(\Psi)$ and equation (2.6) may be written in the alternative form

$$\frac{dP}{d\Psi} = \rho \; . \tag{2.6}$$

Because of assumptions (v) and (vi), the equation of state is simply

$$P = \frac{\mathcal{R}}{m}\rho T \tag{2.7}$$

($\mathcal{R}$ = gas constant, m = mean molecular weight = 1/2) which shows that T is also a function of Ψ only. These results show that it is convenient to use Ψ as a coordinate. A coordinate system based on Ψ is defined in the next section.

The coincidence of surfaces of constant pressure, density and temperature, which considerably simplifies the subsequent mathematics, depends crucially on assumptions (ii) and (ix). Some of the difficulties which arise when assumption (ix) is not valid are discussed in Chapter 7. A joint potential can still be defined if the rotation law is of the form $\Omega = \Omega(\varpi)$ (ϖ = distance from axis of rotation), but the only function of this form which leads to a self-consistent and tractable model is Ω = constant. Any more general function leads to a singularity on the axis if a magnetic field (with $\Omega\,\varpi^2$ constant on field lines) is invoked to maintain a steady state (cf. Mestel 1965, 1966).

Since the equation of motion is being used in the hydrostatic approximation, it gives no information about the circulation currents. The circulation velocity $\underline{v}$ has two, meridian-plane, components which are related by the continuity equation. For a compressible fluid in a steady state this may be written

$$\mathrm{div}(\rho\underline{v}) \quad = \quad 0 \qquad (2.8)(a)$$

or

$$\int \rho\underline{v}.\underline{dS} \quad = \quad 0 \qquad \text{over any closed surface.} \qquad (2.8)(b)$$

The component of $\underline{v}$ perpendicular to a Ψ-surface is determined by the equation of thermal equilibrium, which, by assumption (x), is

$$\frac{P}{\gamma-1}\,\underline{v}.\mathrm{grad}\,\log\,(P/\rho^{\gamma}) \quad = \quad -\,\mathrm{div}\,\underset{\sim}{\mathcal{F}} \qquad (2.9)$$

(cf. Sweet 1950). Here γ is the ratio of the principal specific heats of the gas and $\underset{\sim}{\mathfrak{F}}$ is the radiative flux. γ will be assumed throughout to be 5/3.

Whether the radiative flux is given by a local or a non-local equation, it must satisfy the energy balance equation

$$L = \int \underset{\sim}{\mathfrak{F}} . \underline{dS} \qquad \text{over a surface } \Psi = \text{constant.} \quad (2.10)$$

In this equation L is the total luminosity of the star and is determined by the interior solution. Its value depends on the rotation speed (Roxburgh and Strittmatter 1965; see also Appendix VI). Equation (2.10) takes its simple form because the surface of integration is a level surface. It is derived from the more general energy balance equation (discussed in Appendix III) by using equation (2.8')(b).

Equations (2.6) to (2.10) are five equations for three scalar and two vector functions. To complete the set, an expression for the vector function $\underset{\sim}{\mathfrak{F}}$ is necessary, and sufficient. The expression depends on whether a local or a non-local treatment is to be used. In the local theory, the expression for $\underset{\sim}{\mathfrak{F}}$ is simply (equation (2.4))

$$\underset{\sim}{\mathfrak{F}} = -\frac{16}{3}\frac{\sigma}{\varkappa}\frac{T^3}{\rho}\frac{dT}{d\Psi}\,\text{grad}\,\Psi \qquad (2.11)$$

where σ is Stefan's constant and $\varkappa$ is the opacity (constant, by assumption (xii)). In the non-local theory, one may write

$$\underset{\sim}{\mathfrak{F}} = (\mathfrak{F}_{\Psi} ; \mathfrak{F}_{\chi} ; \mathfrak{F}_{\phi}) \qquad (2.12)$$

where $\mathfrak{F}_{\Psi}$, $\mathfrak{F}_{\chi}$ and $\mathfrak{F}_{\phi}$ are the components of $\underset{\sim}{\mathfrak{F}}$ in the (Ψ, χ, ϕ) coordinate system defined in the next section. The first two components of $\mathfrak{F}$ are, respectively, parallel and perpendicular to grad Ψ and $\mathfrak{F}_{\phi}$ is the toroidal component. They are given by:

$$\mathfrak{F}_{\Psi} = -\int I \cos\lambda \, d\omega$$
$$\mathfrak{F}_{\chi} = \int I \sin\lambda \cos\eta \, d\omega \qquad (2.13)$$
$$\mathfrak{F}_{\phi} = \int I \sin\lambda \sin\eta \, d\omega$$

where: I is the intensity of radiation; λ and η are angles defining the direction of I, λ being the angle between the outward normal to the local Ψ surface and the direction of I and η being the angle between the meridian ϕ = constant and the projection of the direction of I on the surface Ψ = constant (see Figs. 9 and 10 in Chapter 4); $d\omega$ is an element of solid angle about the direction of I.

The intensity I is given by the differential equation

$$\frac{dI}{dl} = -\varkappa\rho\,(I - B) \qquad (2.14)$$

(the "transfer equation"; see, for example, Kourganoff 1952). Here the derivative is in the direction of I and B is the integrated Planck function (by assumption (xi)), given by

$$B = \frac{\sigma T^4}{\pi} \quad . \qquad (2.15)$$

3. Definitions

This chapter will be completed by defining two parameters and describing the coordinate systems used.

Equation (2.5) may be written:

$$\Psi = \frac{GM}{r}\left[1 + \frac{1}{2}\epsilon\frac{r^3}{R^3}\sin^2\theta\right] \qquad (2.16)$$

where

$$\epsilon = \frac{\Omega^2 R^3}{GM} \quad . \qquad (2.17)$$

The parameter ϵ is a measure of α, the ratio of centrifugal force to gravity at the surface. If $\epsilon \ll 1$, a perturbation theory may be used to solve the equations, as is shown later. Strictly, $\epsilon \neq \alpha$, but ϵ and α may be taken to be the same for most purposes. The relation between them is discussed in Appendix V.

A further parameter ϵ_1 is defined by

$$\epsilon_1 = \frac{H}{R} = \frac{\mathcal{R} T_e R}{mGM} \qquad (2.18)$$

where H = pressure scale height in an isothermal atmosphere of temperature T_e, and T_e is a mean effective temperature, defined by

$$L_o = 4\pi R^2 \sigma T_e^4 \quad . \qquad (2.19)$$

ϵ_1 is independent of the rotation speed, and is about 10^{-3} (8.5×10^{-4} for the assumed B0 star). Its significance will be discussed in Chapter 5.

In these formulae, R is the radius and L_o is the luminosity of the corresponding non-rotating star. The physical significance of R and T_e is blurred somewhat in a non-spherical star, but they are useful to define a scale for the system.

Finally, the two main coordinate systems used must be described. They are as follows.

(a) Spherical polar coordinates (r, θ, ϕ).

(b) The orthogonal set (Ψ, χ, ϕ). Here Ψ is defined by equation (2.5) or by equation (2.16) and the surfaces χ = constant are chosen to be orthogonal to the surfaces Ψ = constant. That is, they are chosen so that

$$\text{grad}\,\Psi \,.\, \text{grad}\,\chi = 0 \quad \text{everywhere} \qquad (2.20)$$

The general solution of this equation, with the condition χ = constant, is:

$$\frac{3}{2}(\cos\theta + \log\tan(\theta/2)) + \tfrac{1}{2}\epsilon\frac{r^3}{R^3}\cos^3\theta = f(\chi) \qquad (2.21)$$

where f is an arbitrary function of χ. An obvious particular solution is

$$f(\chi) = \chi \qquad (2.22)$$

and this is used for all the exact theory (Chapter 3). A solution which is approximate, but which gives more meaning to χ as a coordinate, is used in the perturbation theory. $f(\chi)$ is chosen in such a way that

$$\chi = \theta + \tfrac{1}{3}\epsilon\frac{r^3}{R^3}\sin\theta\cos\theta + \tfrac{1}{18}\epsilon^2\frac{r^6}{R^6}\sin\theta\cos\theta\,(1+\sin^2\theta) + O(\epsilon^3) \qquad (2.23)$$

Note that $\chi = \theta$ when $\theta = 0$, $\pi/2$ and π.

It should be noted that, since Ψ increases inwards, the coordinates (Ψ, χ, ϕ) form a left-handed coordinate system. Also, Ψ does not have the dimensions of a length, so that it is not a natural coordinate to work with. It is therefore convenient in the perturbation theory to define a new variable s , with the dimensions of a length, such that surfaces of constant s and constant Ψ coincide and such that s increases outwards. The coordinates (s, χ, ϕ) form a right-handed system, and therefore $\mathcal{J}_s = -\mathcal{J}_\Psi$. Reasons are given in Appendix IV, section 5, for choosing a particular definition for s which has the above properties. That definition is used throughout the main text.

CHAPTER 3

Meridian circulation

in an atmosphere with a local energy transport equation

"Singularity is almost invariably a clue."

Sir A. Conan Doyle, The Adventures of Sherlock Holmes.

1. The equations

In this chapter the meridian circulation velocity field in the atmosphere of the Roche model described in Chapter 2 is determined using the local energy transport equation for $\mathfrak{F}$. This work has been published as a short paper in Zeitschrift für Astrophysik (Smith 1966). It should be noted that in sections 1 and 2 there is no restriction on the value of Ω. The general expressions for the velocity would therefore be valid for rapidly rotating stars were it not for the conclusions of section 4.

The basic equations are equations (2.5) to (2.11) of Chapter 2, repeated here for convenience.

$$\Psi = \frac{GM}{r} + \frac{1}{2}\Omega^2 r^2 \sin^2\theta \tag{3.1}$$

$$\text{grad } P = \rho \text{ grad } \Psi \quad \text{or} \quad \frac{dP}{d\Psi} = \rho \tag{3.2}$$

$$P = \frac{\mathfrak{R}}{m}\rho T \tag{3.3}$$

$$\text{div } (\rho \underline{v}) = 0 \tag{3.4}$$

$$\frac{P}{\gamma-1}\,\underline{v}.\text{grad}\,\log\,(P/\rho^{\gamma}) \quad = \quad -\,\text{div}\,\tilde{\mathcal{J}} \qquad (3.5)$$

$$L \quad = \quad \int_{\Psi=\text{const.}} \tilde{\mathcal{J}}.\underline{dS} \qquad (3.6)$$

$$\tilde{\mathcal{J}} \;=\; -\,\frac{16}{3}\,\frac{\sigma^{*}}{\varkappa}\,\frac{T^{3}}{\rho}\,\frac{dT}{d\Psi}\,\text{grad}\,\Psi \qquad (3.7)$$

(σ^{*} is Stefan's constant; the asterisk is employed in this chapter to distinguish it from the dimensionless variable, σ, defined below.)

In the following formal theory, it is no more difficult to treat $\varkappa$ as variable than it is to take $\varkappa$ as a constant. In equation (3.7), therefore, $\varkappa$ is taken to be the Kramer's opacity

$$\varkappa \quad = \quad \varkappa_{o}\,\rho^{e-1}\,T^{-3-s} \qquad (3.8)$$

where $\varkappa_o$, e and s are constants. The constant $b = 6 + s + e$ is also used.

These equations are not in a convenient form for solution. The dimensionless variables of RGS will be used, with a slight change in notation. Where the notation differs from that of RGS, the following translation rules apply:

$$\sigma = x^{*}\,,\quad h = V^{*}\,,\quad \psi = \psi^{*}\,,\quad p = p^{*}\,,\quad t = t^{*}\,.$$

The dimensionless variables are defined by:

$$r = (2GM/\Omega^{2})^{1/3}\,\sigma$$

$$\Psi = (GM\Omega)^{2/3}\;2^{-1/3}\;\psi$$

$$P = \left[(64\pi\,\sigma^{*}/3L\varkappa_{o})(GM)^{\frac{2}{3}b+1}(m/\mathcal{R})^{b+1-e}(\Omega^{2}/2)^{\frac{1}{3}b}\right]^{1/e}\,p \;=\; A\,p \quad \text{say}$$

$$\rho = A\ (GM\Omega)^{-2/3}\ 2^{1/3}\ \rho^*$$

$$T = (m/\mathcal{R})\ (GM\Omega)^{2/3}\ 2^{-1/3}\ t$$

$$\underline{v} = (L/4\pi)\ (\Omega^2/2GM)^{2/3}\ A^{-1}\ \underline{u}$$

and $\underset{\sim}{\mathfrak{F}} = (L/4\pi)\ (\Omega^2/2GM)^{2/3}\ \underset{\sim}{\mathfrak{F}}^*$.

Using equation (3.8), and the second of equations (3.2), the equations in dimensionless form are:

$$\psi = \frac{1}{\sigma} + c^2 \sin^2\theta \tag{3.9}$$

$$\frac{dp}{d\psi} = \frac{p}{t} \tag{3.10}$$

$$p = \rho^* t \tag{3.11}$$

$$\mathrm{div}_\sigma(\rho^*\underline{u}) = 0 \tag{3.12}$$

$$\underset{\sim}{\mathfrak{F}}^* = -\frac{t^b}{p^e}\frac{dt}{d\psi}\,\mathrm{grad}_\sigma \psi \tag{3.13}$$

$$\frac{p}{\gamma-1}\,\underline{u}.\mathrm{grad}_\sigma \log(p/\rho^{*\gamma}) = -\,\mathrm{div}_\sigma\,\underset{\sim}{\mathfrak{F}}^* \tag{3.14}$$

$$\frac{dt}{d\psi} = \frac{p^e}{t^b h(\psi)} \tag{3.15}$$

Equation (3.15) is derived from equation (3.6) with the help of equation (3.13). h is defined by

$$h(\psi) = -\frac{1}{4\pi}\iint_{\psi = \text{const.}} \mathrm{grad}_\sigma \psi\,.\,\underline{dS}_\psi \tag{3.16}$$

The properties of the function h are considered in Appendix II. In terms of h, equation (3.13) can be re-written as:

$$\underset{\sim}{\mathcal{F}}^{*} = -\frac{1}{h(\psi)}\,\mathrm{grad}_{\sigma}\,\psi \quad . \qquad (3.17)$$

The subscript σ in the above equations denotes the use of the dimensionless spherical polar coordinates (σ, θ, ϕ), so that, for example,

$$\mathrm{grad}_{\sigma} \equiv \left(\frac{\partial}{\partial\sigma}\,,\ \frac{1}{\sigma}\frac{\partial}{\partial\theta}\,,\ 0\right)$$

since axial symmetry is being assumed. The subscript will be dropped in what follows.

The great mathematical advantage of this model arises from the decoupling of the structure equations from the equations for the meridian circulation. The run of temperature and pressure in the atmosphere can be found from equations (3.9), (3.10), (3.15) and (3.16), together with suitable boundary conditions, without reference to the circulation currents. That was done in RGS in the special case where the centrifugal force is exactly equal to surface gravity at the equator ($\alpha = 1$ in the notation of the previous chapter). The boundary conditions in this critical configuration, in which the star is on the verge of rotational break-up, are $p = t = 0$ at $\psi = \psi_{surface} = 3\times2^{-2/3}$. (See RGS and Appendix II.) It will be seen that in fact the theory is not valid for such large values of α.

2. Derivation and discussion of the circulation currents

The equations for h, p, t and ρ^{*} are such that these functions

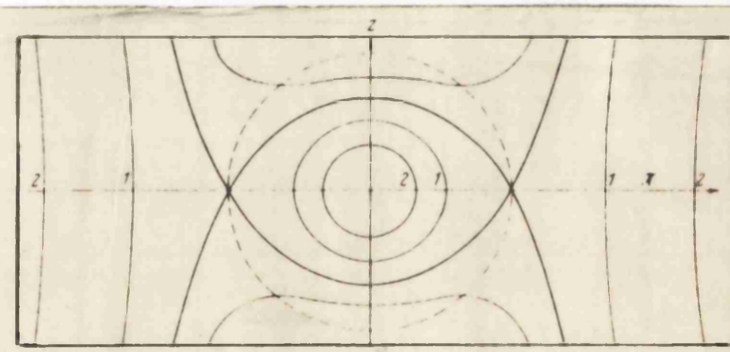

Fig. 65 Equipotentials for ROCHE's model. The dashed circle is the locus of the points where the centrifugal force balances exactly the horizontal component of the gravitational attraction. The heavy line represents the critical equipotential.

After P. Ledoux 1958 Handbuch der Physik Band LI p. 633

can in practice only be found by numerical integration. Since a procedure for the numerical solution of the equations has been laid down in RGS, and since the primary interest in this thesis is in the circulation currents, the functions h, p, t and ρ^* will be supposed to be known functions of ψ and the circulation velocity will be found in terms of these known functions, using equations (3.12), (3.14) and (3.17).

For this purpose, it is convenient to use the non-dimensional form of the orthogonal coordinate system (ψ, χ, ϕ) defined in the last chapter. In the dimensionless variables, ψ is defined by equation (3.9) and χ by

$$\chi = \frac{3}{2}(\cos\theta + \log\tan(\theta/2)) + \sigma^3\cos^3\theta \quad (3.18)$$

The surfaces ψ = constant are shown in Fig. 5. The χ-surfaces are orthogonal to them. The meridian circulation velocity $\underline{u}$ may then be expressed in terms of its components u_ψ, u_χ perpendicular and tangential respectively to the surfaces ψ = constant; i.e.

$$\underline{u} = u_\psi \frac{\text{grad}\,\psi}{|\text{grad}\,\psi|} + u_\chi \frac{\text{grad}\,\chi}{|\text{grad}\,\chi|} \quad (3.19)$$

The procedure for finding u_ψ, u_χ is as follows:

(1) Find grad log $(p/\rho^{*\gamma})$, which is proportional to grad ψ.

(2) Find u_ψ from equations (3.14) and (3.17).

(3) Find u_χ from equation (3.12).

First of all, it is easily shown that

$$\text{grad} \log (p/\rho^{*\gamma}) = \frac{\gamma - 1}{t} \left[\frac{\gamma}{\gamma - 1} \frac{dt}{d\psi} - 1 \right] \text{grad}\, \psi \tag{3.20}$$

using equations (3.10), (3.11). Hence from equations (3.11), (3.14), (3.17) and (3.19),

$$u_\psi = \frac{\frac{1}{h}\left[\frac{\nabla^2 \psi}{|\nabla \psi|} - \frac{1}{h}\frac{dh}{d\psi} |\nabla \psi| \right]}{\rho^* \left[\frac{\gamma}{\gamma - 1} \frac{dt}{d\psi} - 1 \right]} \tag{3.21}$$

Since ψ is known, a further step is possible. From equation (3.9),

$$\text{grad}\, \psi = \left(-\frac{1}{\sigma^2} + 2\sigma \sin^2\theta,\ 2\sigma \sin\theta \cos\theta,\ 0 \right)$$

so that

$$|\text{grad}\, \psi| = G(\sigma, \theta)/\sigma^{-2} \tag{3.22}$$

where

$$G(\sigma, \theta) = (1 - 4\sigma^3 \sin^2\theta + 4\sigma^6 \sin^2\theta)^{1/2} \tag{3.23}$$

Also it may be shown that

$$\nabla^2 \psi = 4 \tag{3.24}$$

Hence finally

$$u_\psi = \frac{\frac{1}{h}\left[\frac{4\sigma^{-2}}{G} - \frac{1}{h}\frac{dh}{d\psi}\frac{G}{\sigma^{-2}} \right]}{\rho^* \left[\frac{\gamma}{\gamma - 1} \frac{dt}{d\psi} - 1 \right]} \tag{3.25}$$

This should be compared with the more general formula derived recently by Mestel (1966).

To find u_χ, equation (3.12) must be solved. The boundary condition $u_\chi = 0$ when $\theta = 0$ is used. A formal solution is easily obtained in the coordinate system (ψ, χ, ϕ), in which equation

(3.12) can be written

$$\frac{\partial}{\partial\psi}\left[\rho^{*} u_{\psi} \frac{\sigma \sin\theta}{|\text{grad}\,\chi|}\right] = -\frac{\partial}{\partial\chi}\left[\rho^{*} u_{\chi} \frac{\sigma \sin\theta}{|\text{grad}\,\psi|}\right] \qquad (3.26)$$

u_{χ} is obtained from this by integrating with respect to χ over a surface ψ = constant. On such a surface

$$d\chi = \frac{\partial\chi}{\partial\sigma} d\sigma + \frac{\partial\chi}{\partial\theta} d\theta$$

$$0 = d\psi = \frac{\partial\psi}{\partial\sigma} d\sigma + \frac{\partial\psi}{\partial\theta} d\theta$$

and so, eliminating $d\sigma$ and using equations (3.9), (3.18), (3.23)

$$d\chi = \frac{3\cos^2\theta\, G(\sigma, \theta)}{2\sin\theta(1 - 2\sigma^3\sin^2\theta)}\, d\theta \text{ on } \psi = \text{constant}$$

Of course, on ψ = constant σ is a function of ψ and θ given by equation (3.9), which cannot be explicitly solved for σ. Hence

$$u_{\chi}(\psi, \sigma(\psi, \theta), \theta) =$$

$$\frac{-G(\sigma(\psi,\theta),\theta)}{\rho^{*}(\psi)\sigma^3(\psi,\theta)\sin\theta}\int_0^{\theta}\frac{3\cos^2 x\, G(\sigma(\psi,x),x)}{2\sin x(1-2\sigma^3(\psi,x)\sin^2 x)}\left\{\frac{\partial}{\partial\psi}\left[\frac{\rho^{*}u_{\psi}\,\sigma\sin\theta}{|\text{grad}\,\chi|}\right]\right\}_{(\psi,x)} dx \qquad (3.27)$$

This expression, albeit purely formal, shows the same 1/density dependence as is shown in the expression (3.25) for u_{ψ}. The boundary conditions p = t = 0 at the surface (discussed in Chapter 5) require $\rho^{*} = 0$ also at the surface (see the solution in RGS) and so the circulation velocity has a singularity at the surface. As explained in Chapter 1, this feature is not due to the use of a Roche potential (cf. Mestel 1966). However, neither can it be attributed to the use of a perturbation theory, as might have been thought because of the results of Öpik (1951) and Baker and Kippenhahn (1959). The singularity implies that one of the physical assumptions made is invalid.

The two least plausible assumptions made are

(1) the transfer of radiation can be described by a "local" equation

(2) inertial and viscous effects are negligible.

The first assumption is implicit in the use of equation (3.7) for $\mathfrak{F}$, which is a "local" equation in the sense that $\mathfrak{F}(\underline{r})$ depends only on the local values of ρ, T and grad T at the point $\underline{r}$. As noted in the last chapter, it would not really be surprising if this equation, the standard one in the theory of stellar interiors, were inadequate in the atmosphere of a star, where the radiative transfer is usually described by the "non-local" equations (2.12) to (2.15). It will be shown in Chapter 4 that the non-local theory does remove the surface singularity.

However, assumption (2) could also be wrong. Initially, it would seem likely that the velocities in a non-local theory would differ from the present velocities only in a fairly thin layer near the surface of the star, perhaps of the order of a scale height in thickness. Thus at, say, one optical depth below the surface the velocities might be expected to be almost as great* as on the local

---

*Note that this is a very pessimistic viewpoint. The fact, proved in Chapter 6, that the non-local theory predicts larger velocities at one optical depth than the local theory could hardly have been anticipated from the results so far.

theory.

In that case it is possible that viscous and inertial forces could be important near the surface. To test this, the expressions (3.25), (3.27) will be used to derive an order of magnitude estimate for the circulation velocities in terms of the rotation speed. It will be seen that assumption (2) may still be made if the rotation speed is sufficiently slow.

3. Order of magnitude of the velocities at the base of the atmosphere

Provisionally, slow rotation is assumed. This is defined by $\alpha = \alpha(R_{eq}) \ll 1$, where $\alpha(r) = \Omega^2 r^3/GM$ = ratio of centrifugal force to gravity in the equatorial plane, and R_{eq} is the equatorial radius of the rotating star (cf. Appendix V). The following values are taken for the constants involved:

$$\mathcal{R} = 8.3 \times 10^7 \quad \text{erg/deg/mole}$$

$$m = 1/2$$

$$G = 6.7 \times 10^{-8} \quad \text{dyne cm}^2/\text{gm}^2$$

$$\gamma = 5/3$$

The values taken for the physical parameters are those given in Chapter 2 (p. 32), assuming a BO star. No correction for rotation has been applied to R or L, since the calculation below is only very approximate. For simplicity an electron-scattering opacity is assumed (as in RCS), so that (cf. equation (3.8)) $e = 1$, $s = -3$, $b = 4$ and $\varkappa = \varkappa_0 \sim 0.38$ for pure hydrogen (Allen 1963).

The density and temperature at unit optical depth ($\tau = 1$)

are required. For slow rotation, it is approximately true that

$$\frac{dP}{dr} \sim - \rho g \qquad (3.28)$$

Also

$$\frac{d\tau}{dr} = - \rho \kappa \qquad (3.29)$$

by definition. Assuming $g \sim GM/R^2$ = constant, and that P = 0 at the surface (τ = 0)

$$P \sim \frac{GM}{\kappa R^2} \tau$$

Thus

$$\rho \sim \frac{m}{\mathcal{R}} \frac{GM}{\kappa R^2} \frac{\tau}{T}$$

The temperature at τ = 1 is near enough T_e. Hence

$$T_{\tau = 1} \backsim T_e \sim 2.2 \times 10^4 \ {}^{\circ}K \qquad (3.30)$$

$$\rho_{\tau = 1} \sim \frac{GMm}{\kappa R^2 \mathcal{R} T_e} = \frac{1}{\kappa \epsilon_1 R} \sim 5.8 \times 10^{-9} \ gm/cm^3 \qquad (3.31)$$

The awkward part of equation (3.25) is $h(\psi)$. However, in the case of slow rotation, an expansion in α can be used. Since

$$\Psi = \frac{GM}{r}\left[1 + \frac{1}{2}\alpha(r)\sin^2\theta\right]$$

the surface value of ψ is

$$\psi_s \sim \frac{2^{1/3}}{(GM\Omega)^{2/3}} \frac{GM}{R} = \left(\frac{2}{\alpha}\right)^{1/3} \gg 1$$

Thus, since $\psi > \psi_s$, $h(\psi)$ can be expanded in powers of $1/\psi$. It may be shown (cf. Appendix II) that

$$h(\psi) = 1 - \frac{4}{3}\frac{1}{\psi^3} - \frac{8}{3}\frac{1}{\psi^6} + O\left(\frac{1}{\psi^9}\right) \qquad (3.32)$$

so that

$$\frac{dh}{d\psi} = \frac{4}{\psi^4} + \frac{16}{\psi^7} + O\left(\frac{1}{\psi^{10}}\right) \qquad (3.33)$$

The expansion for $h(\psi)$ may be used in a series solution of equations (3.10) and (3.15) for small α. It may be shown that

$$\frac{dt}{d\psi} = \frac{1}{4} \quad \text{to zero order in } \alpha \qquad (3.34)$$

For large ψ, σ may be written (from (3.9)) as

$$\sigma = \frac{1}{\psi}\left[1 + \frac{1}{\psi^3}\sin^2\theta + \ldots\right]$$

so that

$$G(\sigma, \theta) = 1 - \frac{2}{\psi^3}\sin^2\theta + \ldots$$

These expansions may be used in equation (3.25) to find an expansion for u_ψ. The term in $1/\psi^2$ vanishes identically, leaving the leading term

$$u_\psi = \frac{512}{9}\frac{1}{\psi^5}\frac{1}{\rho^*}\left(1 - \frac{3}{2}\sin^2\theta\right) \qquad (3.35)$$

Note that the θ-dependence is just $P_2(\cos\theta)$ where P_2 is the Legendre polynomial of order 2.

u_χ may now be found by using the above expansions, and the fact that

$$|\text{grad}\,\chi| = \frac{3\cos^2\theta}{2\sigma\sin\theta}\,G(\sigma, \theta).$$

Equation (3.27) yields an expansion for u_χ whose leading term is

$$u_\chi = \frac{396}{9}\frac{1}{\psi^5}\frac{1}{\rho^*}\sin 2\theta \qquad (3.36)$$

To find numerical estimates, these expressions must be put in dimensional form. Thus, for example,

$$v_\Psi = \frac{L}{4\pi}\left(\frac{\Omega^2}{2GM}\right)^{2/3}\frac{1}{A}\,\frac{512}{9}\,\frac{1}{\psi^5}\,\frac{A\,2^{1/3}}{\rho(GM\Omega)^{2/3}}\left(1-\frac{3}{2}\sin^2\theta\right)$$

Using $\psi \sim \psi_s \sim (2/\alpha)^{1/3}$ and $\alpha \sim \Omega^2 R^3/GM$, this gives

$$\left.\begin{aligned} v_\Psi &\sim \frac{8}{9\pi}\,\frac{L}{GMR}\,\frac{1}{\rho}\,(1+3\cos 2\theta)\,\alpha^2 \\ \text{Similarly}\quad v_\chi &\sim \frac{8}{9\pi}\,\frac{L}{GMR}\,\frac{1}{\rho}\,7\sin 2\theta\,\alpha^2 \end{aligned}\right\} \qquad (3.37)$$

$$\left.\begin{aligned} \text{or}\quad v_\Psi &\sim 2.2\times 10^6\,(1+3\cos 2\theta)\,\alpha^2 \quad \text{cm/sec} \\ v_\chi &\sim 2.2\times 10^6\,7\sin 2\theta\,\alpha^2 \quad \text{cm/sec} \end{aligned}\right\} \qquad (3.38)$$

using the values given above. (See also Appendix VII, section 4.)

It is immediately apparent that v_Ψ and v_χ are comparable in size, which contradicts the usual approximation to the continuity equation

$$v_\chi \sim \frac{R}{H}\,v_\Psi \;,$$

found by assuming that ρv_Ψ changes appreciably in a scale height H while ρv_χ has a scale of variation of R. In fact, since $v_\Psi \propto 1/\rho$, ρv_Ψ is roughly constant in the atmosphere and the usual approximation is not valid. The approximation to the continuity equation will be found to be valid in the non-local theory, where the $1/\rho$ dependence does not appear.

The next point to note is that the velocity is proportional to α^2 to lowest order. Since a first order perturbation theory predicts non-zero velocity terms (see, for example, Mestel 1965, equation (21)), this result is at first rather surprising. The expansion has shown that first order terms appear, but cancel identically (the term in $1/\psi^2$ vanishes). This cancellation turns out to be a consequence of the use of the Roche potential. If the Roche potential is used in the derivation of Mestel's equation (21) instead of a general gravitational potential, the first order velocity is indeed zero. This result throws doubt on the use of the Roche potential, since there will, in general, be non-zero first order terms. However, for uniform rotation these will be finite, or at worst proportional to ρ'/ρ (Sweet 1950; Mestel 1965), and so the second order terms will dominate near the surface. The equations (3.37) may therefore be expected to give the right order of magnitude for the velocities in the surface layers.

It will be seen later that the vanishing of the first order terms is in part due to the use of a local theory. The non-local theory yields non-vanishing first order terms even with a Roche potential.

The third point to be noticed is the sign of the velocity components. Since u_ψ is defined by equation (3.19) to be in the direction of grad ψ, and since ψ increases inwards, the sign for v_ψ given by equation (3.37) means that the circulation rises at the equator and sinks at the poles, contrary to the usual first order result (Baker and Kippenhahn 1959; Sweet 1950). This circulation reversal in the outer layers was first mentioned* by Öpik (1951) and has recently been rediscussed by Mestel (1966). Since v_χ is

* In English. An earlier paper, in Italian, mentions the same phenomenon (Gratton 1945).

positive throughout, and χ increases with θ, the stream lines are as shown in Fig. 6, which agrees with Öpik's result. This is a rather odd result, which suggests that the polar regions continually lose mass. The non-local theory will show that this material in fact returns to the poles very near the surface.

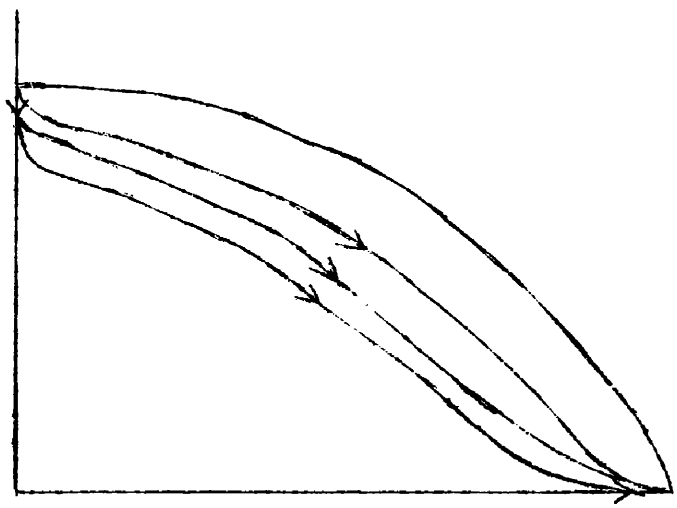

Fig. 6. Local theory stream lines.

4. Range of validity of the theory

The equations (3.38) can now be used to test the validity of assumption (2) of section 2. Consider first the full steady-state equation of motion (without the magnetic forces):

$$(\underline{V}.\underline{\nabla})\underline{V} + \frac{1}{\rho}\,\mathrm{grad}\,P = \mathrm{grad}\,\Phi + \frac{4}{3}\nu\,\mathrm{grad\,div}\,\underline{V} - \nu\,\mathrm{curlcurl}\,\underline{V} \quad (3.39)$$

where $\underline{V} = \underline{v} + \Omega\varpi\underline{t}$ is the general velocity of the stellar material

$\underline{v}$ = velocity in a meridian plane

$\underline{t}$ = unit toroidal vector

and ν = radiative (kinematic) viscosity

$$= \frac{16\sigma^{*}T^4}{15\kappa\rho^2c^2} \quad \text{(see, e.g., Cowling 1953)}$$

Since $\underline{v}$ changes rapidly near the surface, while ϖ does not, the meridian-plane component of equation (3.39) gives:

$$I = \text{inertial terms} \sim \frac{|\underline{v}|^2}{H}$$

$$V = \text{viscous terms} \sim \nu \frac{|\underline{v}|}{H^2}$$

$$C = \text{centrifugal terms} \sim \Omega^2 R$$

where H = scale height = $\epsilon_1 R = 4.5\times10^8$ cm.

An approximation for ν may be found as follows. Taking $T \sim T_e$,

$$\nu \sim \frac{16}{3}\frac{\sigma^*}{\kappa}\frac{T_e^3}{\rho}\frac{T_e}{H}\;\frac{1}{5}\frac{H}{\rho c^2}$$

But $\frac{16}{3}\frac{\sigma^*}{\kappa}\frac{T_e^3}{\rho}\left|\frac{dT}{dr}\right| \sim |\underline{\mathcal{F}}| \sim \frac{L}{4\pi R^2}$ for slow rotation, and (using

equation (3.34) and the definitions of t and ψ)

$$\left|\frac{dT}{dr}\right| \sim \frac{1}{4}\frac{T_e}{H}$$

Hence $$\nu \sim \frac{1}{5}\frac{H}{\rho c^2}\frac{L}{\pi R^2} \sim 9.6\times10^8 \text{ cm}^2/\text{sec} \qquad (3.40)$$

For $|\underline{v}|$, the root mean square value of $\underline{v}$ is taken, that is,

$$\left.\begin{aligned} |\underline{v}| &= \sqrt{\frac{1}{2}\int_0^\pi (v_\Psi{}^2 + v_\chi{}^2)\sin\theta\, d\theta} \\ &= \sqrt{\frac{88}{3}\frac{8}{9\pi}\frac{L}{GMR}\frac{1}{\rho}}\;\alpha^2 \quad \sim \quad 1.2\times10^7\,\alpha^2 \quad . \end{aligned}\right\} \qquad (3.41)$$

Then, using $\Omega^2 = \frac{GM\alpha}{R^3} \sim 1.5\times10^{-8}\,\alpha$,

$$\frac{I}{C} \sim 40\,\alpha^3 \qquad (3.42)$$

Inertial terms are therefore negligible compared with centrifugal terms if $40\,\alpha^3 \ll 1$, i.e. if

$$\alpha \ll 0.3 \quad . \qquad (3.43)$$

(In the author's paper (Smith 1966) slightly cruder estimates were used, leading to the slightly less stringent result $\alpha \ll 0.4$. The result is clearly qualitatively unchanged.)

In the same way it may be shown that

$$\frac{V}{C} \sim 7.2 \times 10^{-6} \, \alpha << 1 \quad \text{for} \quad \alpha << 1.4 \times 10^{5} \, . \tag{3.44}$$

Thus viscous terms can always be ignored, and inertial terms can be ignored if the rotation is slow enough.

It should be noted that these results mean that the stellar model used is not accurate for $\alpha = 1$, the value adopted in RGS, since in that case inertial terms are some 40 times greater than the centrifugal terms. However, the present order-of-magnitude calculation is itself inaccurate unless $\alpha << 1$, so that one is entitled to say only that the inertial and centrifugal terms are likely to be comparable near the surface for $\alpha = 1$ and that equation (3.2) can no longer be assumed to be valid for such rapidly rotating stars.

Fortunately, few stars are observed to rotate as fast as that. Allen (1963) quotes a mean value for α of about 10^{-1} for BO stars, so that to a reasonable approximation the above theory applies to the more slowly rotating B stars. This conclusion might appear to be in disagreement with the results of Walker (1965b), whose figures suggest that all BO stars are rotating on the verge of break-up. However, the present author's computations, on the same data, do not support Walker's result.

Whoever is right, there is still no doubt that some stars exist for which assumption (2) is valid. Attention from now on will be concentrated on such stars, for which only assumption (1) need be discarded. The details of the non-local theory which replaces that assumption will be given in Chapters 4 to 6.

However, it is appropriate to mention here a snag which will appear again later in more formidable guise. Mestel has pointed out (1965) that strong horizontal shearing can sometimes give rise to turbulence, the energy in the flow being sufficient to upset the otherwise stable density stratification. This is an example of the Kelvin-Helmholtz instability, and a sufficient criterion for <u>stability</u> is that the Richardson number*

$$J_R = \frac{g(-\rho'/\rho)}{(dv_\theta/dr)^2}$$

should be greater than about 1/4. (See, e.g., Chandrasekhar 1961.) If the flow did become turbulent, the effective viscosity of the gas would be much larger than the radiative value and viscous effects might not be negligible.

The results quoted here are derived in Appendices IV and VII, where the notation used is fully explained. To lowest order ($\epsilon \ll 1$)

$$J_R = \frac{g_o(-\rho_o'/\rho_o)}{(dv_\theta/dr)_o^2} \tag{3.45}$$

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*Strictly speaking, this expression for the Richardson number applies only in an incompressible fluid. However, the expression applicable in a compressible fluid differs from it only by a factor of order unity - see Chapter 6, section 5.

where $g_o = \frac{GM}{R^2}\frac{1}{y^2}$

$y = s/R$ ($s = r$ to lowest order, and $y = 1$ at the boundary)

$$\frac{\rho_o'}{\rho_o} = -\frac{3}{R}\frac{1}{y}\left(\frac{1}{1-y}\right)$$

and $$\left(\frac{dv_\theta}{dr}\right)_o = \frac{3584}{3}\epsilon_1^4 \frac{\kappa R}{GM}\sigma^* T_e^4 \sin 2\theta\, \epsilon^2 \left[\frac{8y^7}{(1-y)^3} + \frac{3y^8}{(1-y)^4}\right]$$

From these results

$$J_R = \frac{3^3\pi^2}{2^{14}7^2}\frac{(GM)^3}{\kappa^2\epsilon_1^8 RL^2}\frac{1}{\sin^2 2\theta\epsilon^4}\frac{(1-y)^7}{y^{17}(8-5y)^2} \qquad (3.46)$$

Assuming the values above for G, M, κ, ϵ_1, R and L, and taking $y = 1$ except in the term $(1-y)^7$,

$$J_R \doteq 8.6\times10^{15} \frac{(1-y)^7}{\sin^2 2\theta\epsilon^4}$$

The critical value of J_R is 1/4. $J_R = 1/4$ if $(1-y)^7 = \frac{\sin^2 2\theta\epsilon^4}{3.4\times10^{16}}$, i.e., taking $\sin^2 2\theta \sim 1/2$ and $\epsilon = 10^{-1}$, if

$$1-y \doteq 1.1\times10^{-3} \qquad (3.47)$$

Since $J_R > 1/4$ for 1-y greater than this value, instability is only possible in a very thin layer near the surface. $J_R < 1/4$ only in a boundary layer of thickness

$$\delta \doteq 1.1\times10^{-3} R = 1.3 H \qquad (3.48)$$

that is, about one scale height or about one optical depth. Thus instability is only possible (and the criterion does not require instability for $J_R < 1/4$) in the region where it has already been suggested that the local theory does not apply. It is therefore logically permissible to ignore for the present the possibility of a turbulent surface layer, in the hope that the non-local theory will show that the velocity gradient is small enough that turbulence can never appear. That is not in fact so, and the consequences will be discussed in Chapter 7.

CHAPTER 4

Non-local radiative transfer in a non-spherical atmosphere.

"I do attend here on the general..."

W. Shakespeare, Othello, Act III Sc. iv.

1. The equation of transfer

The non-local theory differs from the local theory principally in its representation of the transfer of radiation through the atmosphere. The local equation for the radiative flux,

$$\underset{\sim}{\mathfrak{F}} = -\frac{16}{3}\frac{\sigma}{\kappa}\frac{T^3}{\rho}\,\mathrm{grad}\,T\ , \tag{4.1}$$

is replaced by the set of equations

$$\underset{\sim}{\mathfrak{F}} = (\mathfrak{F}_s, \mathfrak{F}_\chi, \mathfrak{F}_\phi) \tag{4.2}$$

where

$$\left.\begin{aligned} \mathfrak{F}_s &= \int I \cos\lambda\, d\omega \\ \mathfrak{F}_\chi &= \int I \sin\lambda \cos\eta\, d\omega \\ \mathfrak{F}_\phi &= \int I \sin\lambda \sin\eta\, d\omega \end{aligned}\right\} \tag{4.3}$$

$$\frac{dI}{dl} = -\kappa\rho(I-B) \tag{4.4}$$

and

$$B = \frac{\sigma T^4}{\pi}\ . \tag{4.5}$$

These equations have already been presented, in Chapter 2, where the notation was explained. The definition of s is given in Appendix IV.

Equation (4.2) merely defines $\mathfrak{J}_s$, $\mathfrak{J}_\chi$ and $\mathfrak{J}_\phi$ to be the components of $\underset{\sim}{\mathfrak{J}}$ in the directions of grad s, grad χ and grad ϕ respectively. Equation (4.3) relates $\underset{\sim}{\mathfrak{J}}$ to the intensity of radiation I, which satisfies the differential equation (4.4). Equation (4.5) identifies B as the integrated Planck function.

Since the temperature T appears in equation (4.5), this set of equations must, of course, be solved in conjunction with the usual structure equations. It is, however, useful to restrict attention for the moment to equations (4.2) to (4.5) and in particular to consider equation (4.4), the "equation of transfer", in more detail.

The intensity I is a function of direction as well as of position, a fact which complicates the general expression for dI/dl, the derivative of I in the direction of I. Consider first the simple case of a non-rotating, spherically symmetric star. It is customary in that case to take the atmosphere as stratified in plane parallel layers. Since the functions describing the structure of the atmosphere depend only on r, the distance from the centre of the star, and since the radius of curvature of the atmosphere is large compared with the mean free path of a photon, this is a very good approximation (see Chapter 5). In that case

$$dr = \cos\Lambda \, dl \qquad (4.6)$$

where Λ is the angle between the radius vector and the direction $\underline{dl}$ (Fig. 7). Since in this simple situation I must depend only on r and Λ, and Λ is constant along $\underline{dl}$, equation (4.4) then reduces

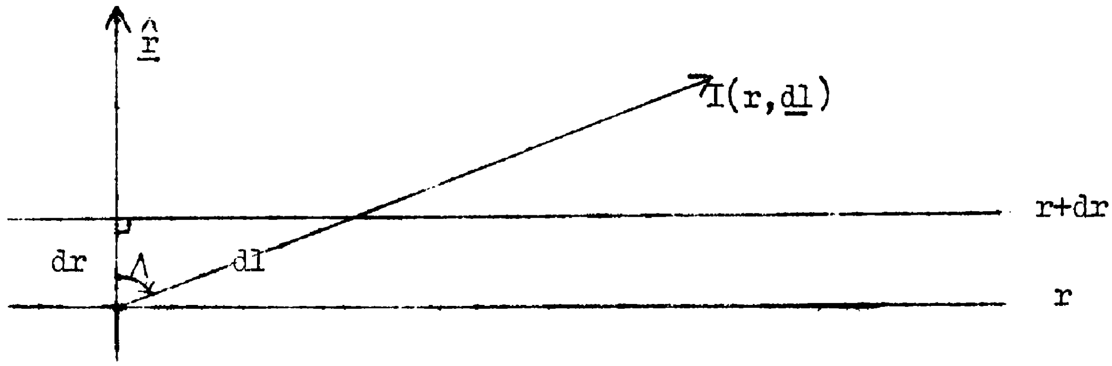

Fig. 7. See text.

immediately to

$$\left.\begin{aligned} \cos\Lambda \frac{dI}{d\tau} &= I - B \\ \text{where} \quad \tau &= -\int^{r} \kappa\rho \, dr' \end{aligned}\right\} \qquad (4.7)$$

Here τ is the "optical depth". The lower limit of integration is conventionally chosen to make $\tau = 0$ at the "surface" of the star (see Chapter 5).

If the mean free path of a photon is not small compared to the stellar radius, as will be the case for stars with extended envelopes (see, e.g., McCrea 1928, Chapman 1966), the plane parallel approximation no longer holds, and equation (4.6) must be supplemented

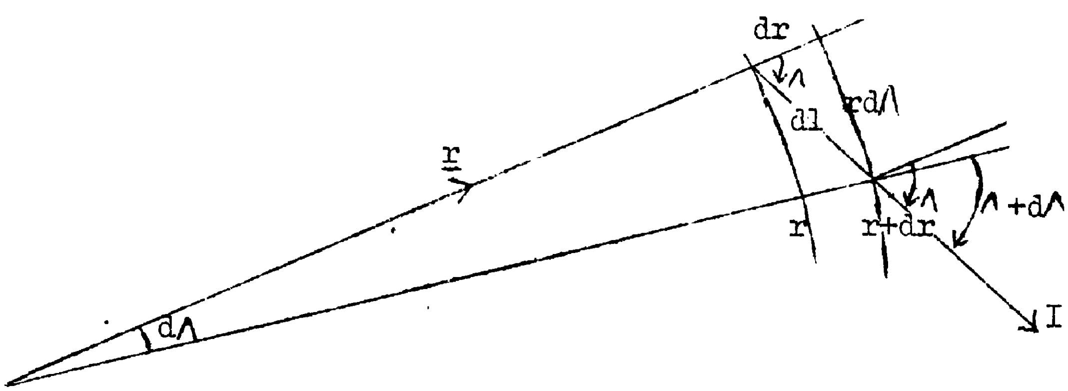

Fig. 8. See text.

by the equation

$$r\, d\Lambda = -\sin\Lambda\, dl \tag{4.8}$$

where the negative sign arises because Λ decreases along dl (Fig. 8). In that case, equation (4.4) becomes

$$\cos\Lambda \frac{\partial I}{\partial r} - \frac{\sin\Lambda}{r}\frac{\partial I}{\partial \Lambda} = -\kappa\rho\,(I - B) \tag{4.9}$$

(cf. R.T. p.23). This form of the transfer equation is valid at any level in the atmosphere of a non-rotating star.

In a rotating star, further generalization is necessary. Since the symmetry is axial, rather than spherical, I will depend on the co-latitude θ as well as on r, although it will still be independent of ϕ. Also, one angle is no longer sufficient to specify the direction dl. I is therefore in general a function of the four variables r, θ, Λ and H*, where H measures the direction of I with respect to a meridian plane, ϕ = constant (see Figs. 9 - 11, which will be explained shortly).

However, bearing in mind the results of Appendix IV, these variables are not the most suitable for a rotating star. It is better to choose as position coordinates the variables s, χ defined in Chapter 2 and Appendix IV and as angular variables the angles λ, η defined in Chapter 2 and by Figs. 9 - 11. The resulting equation will then be in the correct form for perturbation theory to be used in its

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*In this context, H is the Greek letter "capital eta". It should not be confused with H, the scale height.

solution.

If the correct form for equation (4.4) is to be obtained, it is necessary to find

$$\frac{dI}{dl}(s,\chi\,;\,\lambda,\eta) = \frac{\partial I}{\partial s}\frac{ds}{dl} + \frac{\partial I}{\partial \chi}\frac{d\chi}{dl} + \frac{\partial I}{\partial \lambda}\frac{d\lambda}{dl} + \frac{\partial I}{\partial \eta}\frac{d\eta}{dl} \quad . \tag{4.10}$$

To find the coefficients of the derivatives of I is a laborious procedure, which is most easily carried out in two stages. These are: (i) the calculation of dr/dl, $d\theta/dl$, $d\Lambda/dl$ and dH/dl in terms of r, θ, Λ, H. This will give the general form of the transfer equation in spherical polar coordinates.

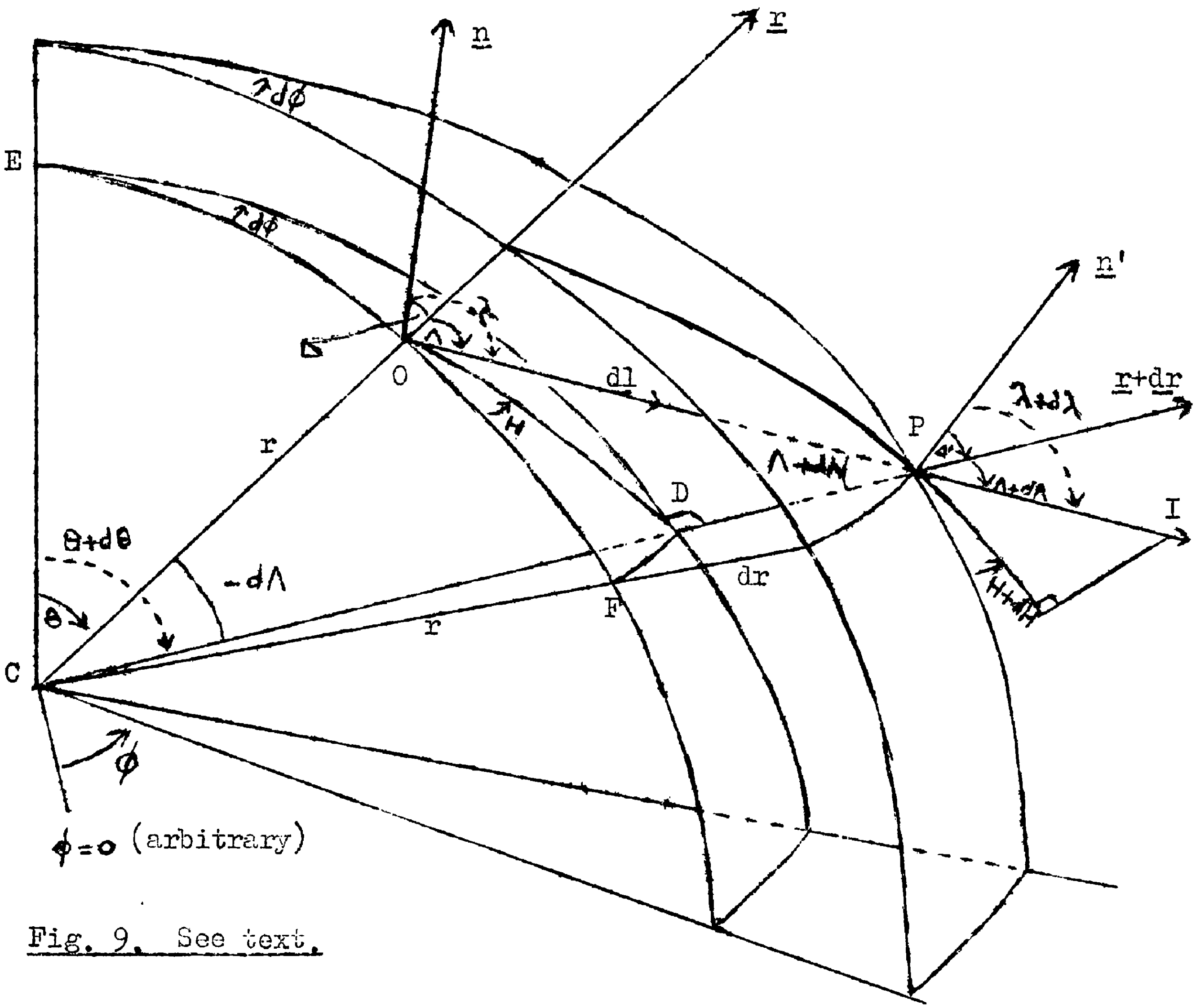


Fig. 9. See text.

(ii) using the relations between (s, χ, λ, η) and (r, θ, Λ, H), and using the results of stage (i), the calculation of ds/dl, $d\chi/dl$, $d\lambda/dl$ and $d\eta/dl$ in terms of s, χ, λ and η.

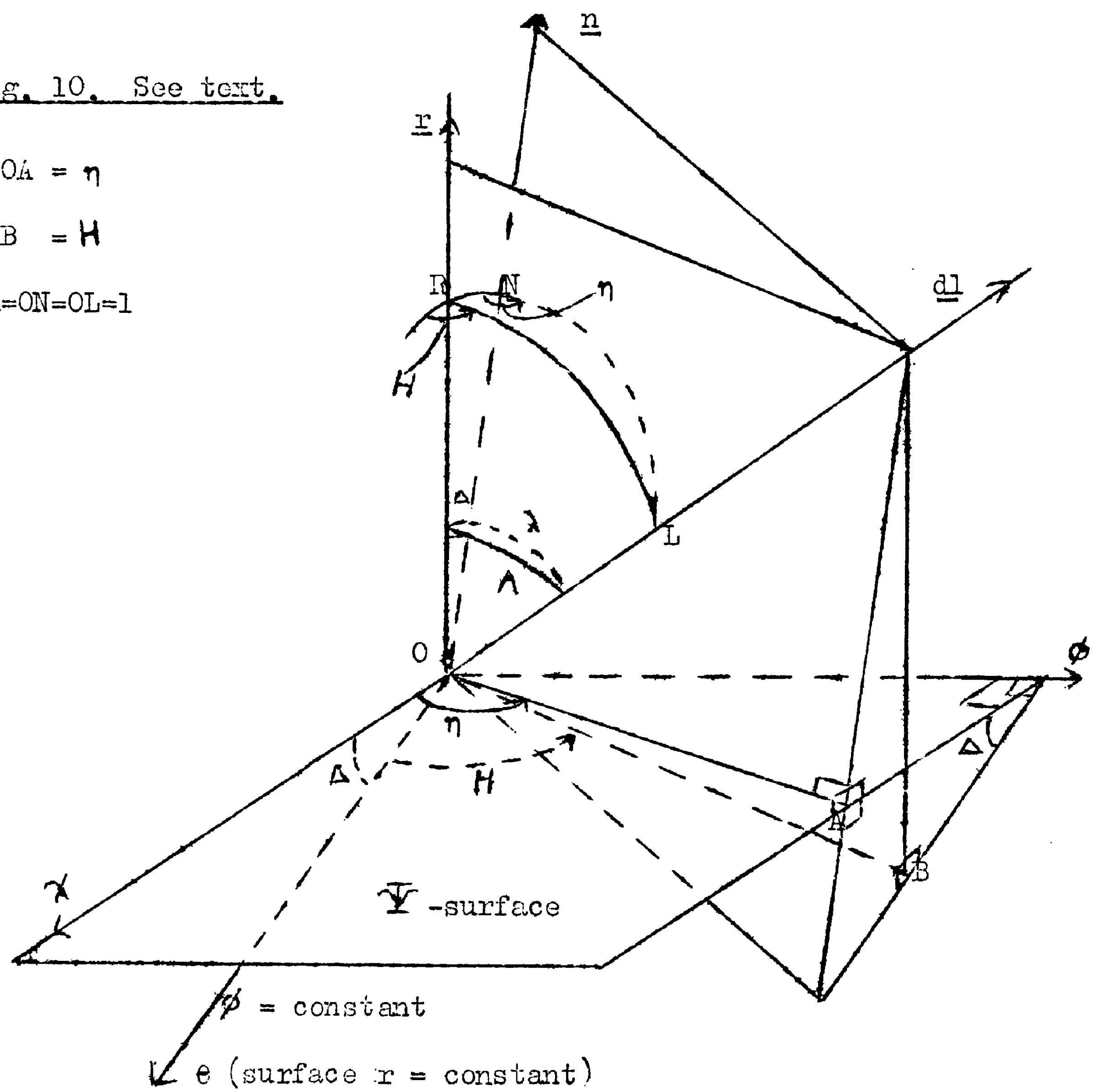
However, before proceeding to this calculation it is necessary to explain Figs. 9 - 11 in some detail. Fig. 9 shows portions of the spheres r = constant, r+dr = constant. The point P (r+dr, θ+dθ) is obtained from the point O (r, θ) by moving a distance dl in the direction of I, i.e. in the direction **dl** specified

Fig. 10. See text.

$\chi OA = \eta$

$\theta OB = H$

OR=ON=OL=1



by Λ and H or by λ and η. The angles Λ and λ are the angles between the direction $\underline{dl}$ and the normals at O to the surfaces of constant r and constant Ψ (or s) respectively. The angles H and η are, respectively, the angles between the meridian ϕ = constant through O and the projection of $\underline{dl}$ on to the surfaces of constant r and constant Ψ (or s) through O. The normal directions to these surfaces are shown by the arrows labelled $\underline{r}$, $\underline{n}$ respectively. $\underline{r} + \underline{dr}$, $\underline{n}'$ denote the normal directions to the corresponding surfaces through P.

The angle between the normals to the surfaces of constant Ψ and constant r through O is denoted by Δ. It is defined by

$$\cos\Delta = \hat{\underline{n}}.\hat{\underline{r}} \tag{4.11}$$

where $\hat{\underline{\ }}$ denotes a unit vector. By definition

$$\left.\begin{aligned} \hat{\underline{n}} &= -\frac{\text{grad}\,\Psi}{|\text{grad}\,\Psi|} \\ \hat{\underline{r}} &= (1,0,0) \quad \text{in spherical polars} \end{aligned}\right\} \tag{4.12}$$

Hence:

$$\cos\Delta = \frac{1 - \epsilon\frac{r^3}{R^3}\sin^2\theta}{(1 - 2\epsilon\frac{r^3}{R^3}\sin^2\theta + \epsilon^2\frac{r^6}{R^6}\sin^2\theta)} \tag{4.13}$$

It is not possible to express this exactly as an explicit function of s and χ. However, Δ may be found from this equation, to any desired order in ϵ, as a function of (r, θ) or of (s, χ). For present purposes, Δ is simply regarded as a known function and it is not explicitly evaluated.

Fig. 10 shows the region near O in more detail, to clarify the definition of η, and is largely self-explanatory. The points R, N and L are the points of intersection of a unit sphere centred on O with the directions $\underline{r}$, $\underline{n}$, $\underline{dl}$ respectively. The resulting spherical triangle, which is used to find relations between λ, η, Λ, H and Δ, is shown in more detail in Fig. 11.

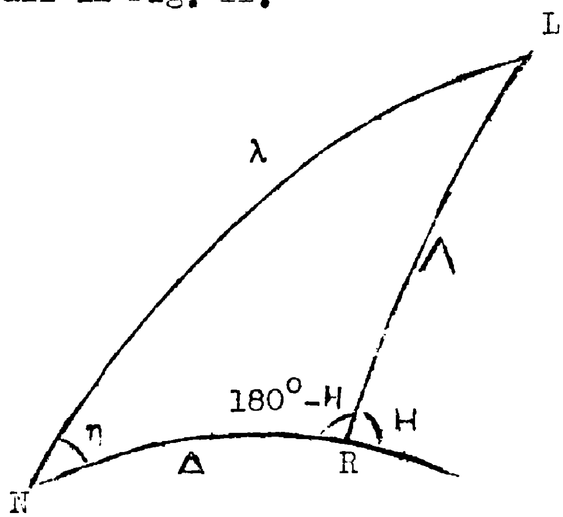


Fig. 11. See text.

It is now possible to calculate $\frac{dI}{dl}$.

Stage (i)

It is clear from Fig. 9 that

$$\underline{dr} = \underline{dl} \tag{4.14}$$

or, in components,

$$\left.\begin{aligned} dr &= dl \cos \Lambda \\ r\, d\theta &= dl \sin \Lambda \cos H \\ r \sin \theta\, d\phi &= dl \sin \Lambda \sin H \end{aligned}\right\} \tag{4.15}$$

Since $d\Lambda$ is negative, it follows from inspection of triangle COP and from the definition of Λ that

$$\hat{OCP} = -\, d\Lambda$$

Thus, since OD is a great circle

$$-\, r\, d\Lambda = OD = dl\, \sin\Lambda \quad . \tag{4.16}$$

Triangle ODE is a spherical triangle on the sphere r = const. in which, taking r as the unit of measurement, OE = θ, OD = $\theta + d\theta$, $\hat{EOD} = 180^\circ - H$ and $\hat{ODE} = H + dH$. Since dH, $d\theta$ are infinitesimal, the application of the sine formula of spherical trigonometry (Smart 1944) to these four elements gives:

$$dH \cos H \sin\theta = -\, d\theta \cos\theta \sin H \tag{4.17}$$

This result may also be put in the more easily visualizable form

$$dH = -\, d\phi \cos\theta \tag{4.18}$$

by making use of equation (4.15). Geometrically, this means that

$$\hat{DCF} = -\, dH .$$

This is a plausible, but not immediately obvious, result, although it is clear that dH must be negative. An alternative derivation of equations (4.16) and (4.18) is given in Appendix IX.

In summary, the results of stage (i) are:

$$\left.\begin{aligned} dr/dl &= \cos\Lambda \\ d\theta/dl &= \sin\Lambda \cos H / r \\ d\Lambda/dl &= -\sin\Lambda / r \\ dH/dl &= -\sin\Lambda \sin H \cot\theta / r \end{aligned}\right\} \tag{4.19}$$

Stage (ii)

This stage divides naturally into two parts - the transformation of the coordinates (Λ, H), and the transformation of the coordinates (r, θ). It will be seen that the second part cannot be completed in practice.

For the first part, the formulae of spherical trigonometry (see e.g. Smart 1944) can be applied to triangle NRL (Fig. 11) to give relations between λ, η, Λ, H and Δ. In principle only two formulae are needed to give (Λ, H) in terms of (λ, η, Δ). In practice, the cosine, sine and 4-parts formulae are all useful.

Some tedious algebra then leads to the expressions:

$$\left.\begin{aligned}
\frac{dr}{dl} &= \cos\lambda \cos\Delta + \sin\lambda \sin\Delta \cos\eta \\
\frac{d\theta}{dl} &= \frac{1}{r}\left[-\cos\lambda \sin\Delta + \sin\lambda \cos\Delta \cos\eta\right] \\
\frac{d\lambda}{dl} &= \frac{1}{r}\left[-\sin\lambda \cos\Delta + \cos\lambda \sin\Delta \cos\eta \right. \\
&\qquad \left. + \sin\lambda \sin\Delta \sin^2\eta \cot\theta + r \cos\eta \frac{d\Delta}{dl}\right] \\
\frac{d\eta}{dl} &= \frac{1}{r}\left[(-\sin\lambda \cos\Delta + \cos\lambda \sin\Delta \cos\eta) \sin\eta \cot\theta \right. \\
&\qquad \left. - \frac{\sin\eta \sin\Delta}{\sin\lambda} - r \cot\lambda \sin\eta \frac{d\Delta}{dl}\right]
\end{aligned}\right\} \quad (4.20)$$

These expressions are unsatisfactory for several reasons. Firstly, they still contain r and θ, and indeed the first two expressions are not yet the ones required. Further, the angle Δ appears explicitly. This would not be expected in a general expression in the coordinates (s, χ, λ, η) since, if the expression were obtainable directly, the angle Δ need never appear. It only appears here because equations (4.20) have been obtained via spherical polar coordinates.

The first two expressions may be replaced by ds/dl and $d\chi/dl$ by writing

$$\left.\begin{aligned} \frac{ds}{dl} &= \frac{\partial s}{\partial r}\frac{dr}{dl} + \frac{\partial s}{\partial \theta}\frac{d\theta}{dl} \\ \frac{d\chi}{dl} &= \frac{\partial \chi}{\partial r}\frac{dr}{dl} + \frac{\partial \chi}{\partial \theta}\frac{d\theta}{dl} \end{aligned}\right\} \qquad (4.21)$$

The relation between the (r, θ, ϕ) and (s, χ, ϕ) components of ∇s gives

$$\left.\begin{aligned} \frac{\partial s}{\partial r} &= \cos\Delta\,|\nabla s| \\ \frac{\partial s}{\partial \theta} &= -\,r\sin\Delta\,|\nabla s| \end{aligned}\right\} \qquad (4.22)$$

Similarly

$$\left.\begin{aligned} \frac{\partial \chi}{\partial r} &= \sin\Delta\,|\nabla \chi| \\ \frac{\partial \chi}{\partial \theta} &= r\cos\Delta\,|\nabla \chi| \end{aligned}\right\} \qquad (4.23)$$

These equations, together with equations (4.20), combine to give:

$$\left.\begin{aligned} \frac{ds}{dl} &= \cos\lambda\,|\nabla s| \\ \frac{d\chi}{dl} &= \sin\lambda\,\cos\eta\,|\nabla \chi| \end{aligned}\right\} \qquad (4.24)$$

These expressions are entirely satisfactory, in that neither Δ nor the coordinates (r, θ) appear explicitly. Also, the second expression is entirely independent of which solution of equation (2.21) is taken for χ.

Unfortunately, the methods used to obtain these expressions are not applicable to $\frac{d\lambda}{dl}$ or $\frac{d\eta}{dl}$. Since r, θ (and therefore Δ) are known only as <u>implicit</u> functions of s and χ, it seems to be impossible in practice to find $\frac{d\lambda}{dl}$ and $\frac{d\eta}{dl}$ as functions of λ, η, s, χ, $|\nabla s|$ and $|\nabla\chi|$ only. Of course it must be possible in principle, but the complexity of the functional relations between s, χ, r, θ and Δ is such that the author has had to admit defeat, albeit reluctantly. It is, therefore, not yet possible to give the general form of the transfer equation in the coordinates (s, χ, λ, η).

The difficulty is, of course, purely formal. If all functions are expanded in powers of ϵ, it is easy to find $\frac{d\lambda}{dl}$ and $\frac{d\eta}{dl}$ in terms of s, χ, λ and η to any order. As the transfer equation is only susceptible to solution by perturbation methods, the failure to find a general form for it is not a serious defect of the theory.

To summarize, the main results of this section are that the equation of transfer in a rotating star may be written in general as

$$\cos\Lambda \frac{\partial I}{\partial r} + \frac{\sin\Lambda \cos H}{r}\frac{\partial I}{\partial\theta} - \frac{\sin\Lambda}{r}\frac{\partial I}{\partial\Lambda} - \frac{\sin\Lambda \sin H \cot\theta}{r}\frac{\partial I}{\partial H} = -\kappa\rho(r, \theta)(I(r, \theta; \Lambda, H) - B(r, \theta)) \tag{4.25}$$

in spherical polar coordinates, or as

$$\cos\lambda\, |\nabla s| \frac{\partial I}{\partial s} + \sin\lambda \cos\eta |\nabla\chi| \frac{\partial I}{\partial\chi} + \frac{d\lambda}{dl}\frac{\partial I}{\partial\lambda} + \frac{d\eta}{dl}\frac{\partial I}{\partial\eta} = -\kappa\rho(s)(I(s, \chi; \lambda, \eta) - B(s)) \tag{4.26}$$

in the more appropriate coordinate system (s, χ, ϕ). $\frac{d\lambda}{dl}$ and $\frac{d\eta}{dl}$ are not obtainable in practice in terms of s, χ, $|\nabla s|$, $|\nabla\chi|$, λ and η only, but expressions for them in terms of r, θ, λ and η are given by equation (4.20).

2. A formal expression for v_s

Although it is not feasible to solve equation (4.25) or equation (4.26) exactly for I, it is possible to use these equations to obtain a formal expression for v_s, the component of $\underline{v}$ in the direction of ∇s.

It is known (R.T.p. 11) that, if the transfer equation is integrated over all directions $\underline{dl}$, an expression for div $\underline{\mathfrak{F}}$ is obtained. This result is only obvious in Cartesian coordinates, but it may be verified by integrating equation (4.25) over all solid angles about the normal to the sphere r = const. The use of this equation rather than equation (4.26) is justified by the fact that the result is independent of any particular coordinate system.

Since the element of solid angle is

$$d\omega = \sin\Lambda\, d\Lambda\, dH \tag{4.27}$$

the integral is

$$\int_0^{2\pi}\int_0^{\pi} (4.25) \sin\Lambda\, d\Lambda\, dH$$

which gives, after some manipulation,

$$\text{div}\, \underline{\mathfrak{F}} = -\, 4\pi\kappa\rho(J - B) \tag{4.28}$$

where

$$J = \frac{1}{4\pi} \int I \, d\omega . \qquad (4.29)$$

This result could also be obtained directly from energy considerations. If this expression for div $\underset{\sim}{\mathcal{F}}$ is substituted in equation (2.9), which may be written in the form

$$v_s \, |\nabla s| \frac{P}{\gamma - 1} \frac{d}{ds} \log \left(P/\rho^{\gamma}\right) = - \operatorname{div} \underset{\sim}{\mathcal{F}}, \qquad (4.30)$$

it is easy to show (cf. Chapter 3) that

$$v_s = \frac{4\pi\kappa(J - B)}{|\nabla s| \left[\frac{\gamma}{\gamma - 1} \frac{\mathfrak{R}}{m} \frac{dT}{ds} - \frac{d\Psi}{ds}\right]} . \qquad (4.31)$$

It is immediately clear that v_s $(= - v_{\Psi})$ does not have a 1/density dependence. Since the only difference between this formal theory and that of the previous chapter lies in the choice of expression for $\underset{\sim}{\mathcal{F}}$, this result substantiates the claim that the surface singularity in the local theory is due simply to the inadequacy of a local transfer equation near the surface.

Of course, from the form of the continuity equation v_{χ} has a term proportional to $(\rho'/\rho)v_s$. However, it will be seen in Chapter 6 that ρ goes to zero exponentially as the surface is approached, so that ρ'/ρ is finite in the non-local theory. The only other question is whether any factor still present in equation (4.31) could give rise to a singularity. However, κ, J and B must be finite everywhere on physical grounds and it may be shown that the denominator vanishes only for $(\epsilon r^3/R^3 = 1, \; \theta = \pi/2)$, the set of conditions which corresponds to the balance of centrifugal force and gravity at the equator of a star rotating on the verge of break-up. The bracket containing the temperature gradient is always positive by virtue of assumption (viii) of Chapter 2 that the atmosphere is stable against convection.

These considerations show conclusively that the velocity in a non-local theory is finite at the surface. However, it is not zero, and preliminary estimates of the size of v_s and v_χ at the surface are alarming. There is no reason, a priori, for assuming that the difference J-B is significantly smaller than either J or B, although J and B may be expected to be comparable in size. In any case, an upper limit for v_s may be obtained by assuming in the first instance that

$$J - B \approx B \approx \sigma T_e^4 \quad . \tag{4.32}$$

It follows at once from Appendix V that

$$|\underline{\nabla} s| \approx 1 \tag{4.33}$$

Since the expression involving the temperature gradient never vanishes, it is reasonable in the first instance to assume

$$\frac{\gamma}{\gamma-1}\frac{\mathcal{R}}{m}\frac{dT}{ds} - \frac{d\Psi}{ds} \approx \frac{d\Psi}{ds} \approx \frac{GM}{R^2} \tag{4.34}$$

It then follows, using the values quoted in Chapter 3, that

$$\left.\begin{aligned} v_s &\approx 10^9 \text{cm/sec} \\ v_\chi \approx \frac{R}{H} v_s &\approx 10^{12} \text{cm/sec} \end{aligned}\right\} \tag{4.35}$$

These results agree with Osaki (1966) in predicting speeds greater than those on the local theory by a factor of order $\frac{R}{H}$ (~ 1000) or more. If these estimates are borne out by more careful analysis, then the theory is clearly invalid and some further mechanism for damping the gas motions must be invoked. Such a mechanism will be discussed in Chapter 7.

However, it is not possible to say at this stage whether further analysis will confirm these figures or not. The difference J-B could be much less than B. It is at least a first order quantity (in ϵ), and could even be second order by analogy with the local theory (Chapter 3). The following analysis is therefore necessary, even though it does confirm the above estimates. Besides, there is no doubt that a non-local treatment is necessary, if not sufficient, for a proper description of the atmosphere and it is useful to consider the simplest non-local treatment first before becoming involved in the complications due to viscous and/or inertial effects.

3. Methods of solution of the transfer equation

In the expression (4.31) for v_s, J, B and T are unknown functions. In order to find these functions, and so to evaluate v_s more exactly, it is necessary to solve the transfer equation.

It is well known to be difficult to solve the transfer equation, even in the simple form of equation (4.7). In that case, the only one studied exhaustively, an exact solution is known for $B(\tau)$ and for $I(0,\mu)$ ($0 \leqslant \mu \leqslant 1$ - $\mu = \cos\Lambda$), the emergent intensity at $\tau = 0$, but no exact solution for $I(\tau, \mu)$ is known for general τ. This discourages any attempt at finding an exact solution for I in the present case, particularly as even the solution for $I(0,\mu)$ in the simple case involves sophisticated complex variable theory. What is required is a procedure which is known to give a satisfactory approximate solution for equation (4.7) and which can be extended to the solution of equation (4.25) or equation (4.26).

A great variety of methods has been developed to give approximate solutions for equation (4.7). Of the methods described by Kourganoff(1952), the most extensively used seem to be those based directly on the transfer equation. They are the moment method (a generalization of the Eddington approximations), the spherical harmonic method and the method of discrete ordinates. The merits and defects of these three methods, which Krook (1955) has shown to be formally equivalent, have been thoroughly discussed in the literature. This seemed a good reason for choosing one of them for use in the present problem.

In the spherical harmonic method, which is due to Eddington (1930, p.105), the intensity $I(\tau,\mu)$ is represented by a finite series of Legendre polynomials $P_j(\mu)$ whose coefficients $A_j(\tau)$ are determined by the transfer equation. In this simple form of the method, difficulties arise due to the impossibility of representing by a finite sum of continuous functions the function $I(0, \mu)$, which is discontinuous at $\mu = 0$. Kourganoff (1952 p. 101) mentions an elaboration of the method, due to Yvon, which partially removes this difficulty. The difficulty may be entirely removed by a modification of Yvon's method, due to Wilson and Sen (1963) who also give a useful discussion of the method.

The method of discrete ordinates was first suggested by Schuster (1905) and Schwarzschild (1906) and was generalized by Wick (1943) and by Chandrasekhar (R.T.), who has used the method extensivly. The radiation field is represented by 2n discrete streams of radiation, each associated with a particular value of μ, $\mu = \mu_i$ ($i = \pm 1, \ldots, \pm n$; $\mu_{-i} = - \mu_i$). Integrals can then be approximated by finite sums, using weights in the same way as in formulae for numerical quadrature. An excellent critical account of the method has been given by Kourganoff (1952), who discusses the relative merits of various quadrature formulae. Kourganoff prefers the Newton-Cotes formulae to the Gaussian formulae used by Chandrasekhar, but Sykes (1951) has shown that the Gauss method can be modified to give better results than either the standard Gauss or the Newton-Cotes formulae. Carlson (1955) has further refined the method of discrete ordinates in an application to the numerical solution of neutron diffusion problems and Grant (1963) has applied Carlson's "S_n-approximation" to the radiative transfer case.

However, the various modifications in these two methods of solution are all designed for more accurate solution of the problem. In the present case, accuracy was not, in the first instance, the primary concern. It seemed more important to find an approximate <u>analytical</u> expression for the velocity field which would show qualitatively how the velocity varied near the surface and which would also confirm or reject the numerical estimates made in the last section. Only if these numerical estimates turned out to be a gross overestimation would there be any point in improving the accuracy of the solution. The main concern, therefore, was to find the method which was most satisfactory in its lowest approximation.

Of the two methods discussed so far, the method of discrete ordinates has the advantage that Chandrasekhar has shown (R.T. p.364 et seq.) how it may be extended from plane-parallel geometry to spherical geometry; he has also estimated the error of the first approximation. However, in neither of the above methods is there any obvious way of extension to the non-spherical case, where I depends also on χ and η.

In a first attempt to solve the non-spherical case, the (χ, η) dependence of I was represented by a Fourier series for the η-dependence and a series of Legendre polynomials for the χ-dependence. Application of the symmetry conditions

$$\left.\begin{aligned} I(s,\chi;\lambda, \eta) &= I(s,\chi; \lambda, -\eta) \\ I(s,\chi;\lambda, \eta) &= I(s, \pi-\chi; \lambda, \pi-\eta) \end{aligned}\right\} \qquad (4.36)$$

enabled I to be written as

$$I(s,\chi; \lambda, \eta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} I_{nm}(s; \lambda)\, P_n(\cos\chi) \cos m\eta \qquad (4.37)$$

where m + n is even.

This approach has several difficulties. First of all, $\cos\chi$ and the second symmetry condition are meaningful only if $\chi = 0$ to lowest order (cf. equation (2.23)). This immediately necessitates the use of a perturbation theory, which is also required to enable the transfer equation to be written in the coordinates $(s, \chi; \lambda, \eta)$. This is unfortunate aesthetically, but the equation is so complicated that perturbation methods are inevitably necessary in any case. More seriously, it is not at all clear where to truncate the expression (4.37), and problems of consistency arise unless the equations for the I_{nm} are derived in general before truncation. This naturally involves much heavy algebra, and makes the method a cumbersome one. Nonetheless, the resulting equations for the $I_{nm}(s; \lambda)$ can be solved by the method of discrete ordinates, in principle to any degree of approximation, and this method would be the obvious choice if no more elegant one existed.

However, no account has yet been given of the moment method (Krook 1955) which is, in a certain sense, a generalization of the Eddington approximations. Consider the sequence of moment functions $(k = 0,1,2,\ldots)$

$$M_k(\tau) = \frac{1}{2}\int_{-1}^{+1} \mu^k I(\tau,\mu)\, d\mu \quad . \tag{4.38}$$

A corresponding sequence of moment equations can be obtained by integrating equation (4.7) over angle. The first 2n of these equations involve the 2n+1 moments of orders $0,1,\ldots,2n$. To obtain a closed set of equations, it is necessary to express the 2nth moment in terms of the lower moments. Krook (1955) shows that the appropriate relation is equivalent to the relation

$$\int_{-1}^{+1} P_{2n}(\mu)\, I(\tau,\mu)\, d\mu = 0 \qquad (4.39)$$

In the first approximation (n=1), this reduces to

$$K(\tau) = \frac{1}{3} J(\tau) \qquad (4.40)$$

in the more familiar notation in which $J = M_o$, $H = M_1$ and $K = M_2$. The Eddington approximations are just equation (4.40) and the boundary condition

$$J(0) = 2\, H(0) \;. \qquad (4.41)$$

Krook's more general method replaces the factor 2 in equation (4.41) by $\sqrt{3}$, but Eddington's boundary condition will be retained in what follows since it makes the detailed working simpler and that is a more important consideration in the first instance than accuracy, which is not in any case particularly good in the first approximation.

This method has the great advantage that there is an obvious generalization of the first approximation to the case of a non-spherical atmosphere. This generalization, which will be derived in the next section, is of a form which removes the difficulties associated with the expression of equation (4.26) in the appropriate coordinates. The general form of the resulting equations is elegant, and contains no reference to a particular coordinate system. Because of the integration over angle required to obtain the equations, there is no need to specify the η-dependence of I. Also, the general equations may be obtained without choosing a particular χ-dependence for I. Further, the general equations are obtainable without using a perturbation theory, and so are valid for rapidly rotating stars. That would be reason enough for choosing this method. The fact that the method is also simpler in its detailed working than the method of discrete ordinates makes it the obvious choice.

4. The generalized Eddington approximations

In the plane-parallel case considered by Krook (1955) and Eddington (1930), the moment equations in the first approximation are

$$\left.\begin{aligned} \frac{dH}{d\tau} &= J - B \\ \frac{dK}{d\tau} &= H \end{aligned}\right\} \qquad (4.42)$$

The first of these is obtained simply by integrating equation (4.7) over solid angle. The obvious generalization of this is to integrate equation (4.25) over solid angle. This was done in section 2 of this chapter, with the result

$$\operatorname{div} \underset{\sim}{\mathcal{F}} = -4\pi\kappa\rho(J - B) \qquad (4.43)$$

It is obvious that this reduces to the first of equations (4.42) in the simple plane-parallel case, where $\underset{\sim}{\mathcal{F}}$ has only a radial component, which is a function of r only.

The second moment equation is obtained by multiplying equation (4.7) by μ (= $\cos\Lambda$) and integrating over solid angle. If the same procedure is followed with equation (4.25), the integral of the right-hand side is $-\kappa\rho\mathfrak{F}_r$, since $\int\cos\Lambda\,d\omega = 0$ and the components of $\mathfrak{F}$ in spherical polars are given by

$$\left.\begin{aligned}\mathfrak{F}_r &= \int I\cos\Lambda\,d\omega\\ \mathfrak{F}_\theta &= \int I\sin\Lambda\cos H\,d\omega\\ \mathfrak{F}_\phi &= \int I\sin\Lambda\sin H\,d\omega\end{aligned}\right\}\qquad(4.44)$$

(cf. equations (4.3)).

Since the other components of $\mathfrak{F}$ are not in general zero, it is clear that the proper generalization of the second moment equation is to a vector equation whose right-hand side is $-\kappa\rho\mathfrak{F}$. This vector equation will have the three components

$$\left.\begin{aligned}&\int(4.25)\cos\Lambda\,d\omega\\ &\int(4.25)\sin\Lambda\cos H\,d\omega\\ \text{and}\quad&\int(4.25)\sin\Lambda\sin H\,d\omega\end{aligned}\right\}\qquad(4.45)$$

in spherical polar coordinates. As in section 2, the use of equation (4.25) rather than equation (4.26) is justified by the result's independence of any particular coordinate system.

If these integrations are performed, the left-hand side of the vector equation is not at first easy to interpret. The simplest way of proceeding is to guess at an interpretation, and then verify the guess.

In order to make an intelligent guess, it is necessary to consider the physical significance of the function K which appears in the second moment equation. This is less obvious than the significance of J and H, but it may be shown fairly easily (e.g. Chandrasekhar 1939 p. 192 or Kourganoff 1952 p. 14) that K represents $c/4\pi$ times the normal pressure of (integrated) radiation on each cm^2 of a given layer.

Now it is well known that in general the radiation pressure is a second order tensor with components given by

$$\underline{\underline{P_R}} = \frac{1}{c} \begin{bmatrix} \int I l^2 d\omega & \int I l m \, d\omega & \int I l n \, d\omega \\ \int I m l \, d\omega & \int I m^2 \, d\omega & \int I m n \, d\omega \\ \int I n l \, d\omega & \int I n m \, d\omega & \int I n^2 \, d\omega \end{bmatrix} \tag{4.46}$$

where (l,m,n) are the direction cosines of I in the orthogonal coordinate system considered (Chandrasekhar 1939 p. 195). In the spherical polar system used above

$$\left.\begin{aligned} l &= \cos\Lambda \\ m &= \sin\Lambda \cos H \\ n &= \sin\Lambda \sin H \end{aligned}\right\} \tag{4.47}$$

Comparison of equations (4.25) and equations (4.45) to (4.47) suggests that, if K is generalized to be the tensor

$$\underline{\underline{K}} = \frac{c}{4\pi} \underline{\underline{P_R}} \quad , \tag{4.48}$$

then some derivative of $4\pi\underline{\underline{K}}$ forms the left-hand side of (4.45). By analogy with (4.43), the most likely guess for equation (4.45) is therefore

$$4\pi \operatorname{div} \underline{\underline{K}} = -\varkappa\rho \underset{\sim}{\mathfrak{J}} . \tag{4.49}$$

This may be verified by using the general expression for the components of div $\underline{\underline{K}}$ (see e.g. Synge and Schild (1949)):

$$\operatorname{div}^i \underline{\underline{K}} = \frac{\partial K^{ij}}{\partial x^j} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} K^{jk} + \left\{ \begin{matrix} k \\ j \quad k \end{matrix} \right\} K^{ij} \tag{4.50}$$

(i = 1, 2, 3 and j, k summed from 1 to 3 (Einstein summation convention)),

remembering that the K^{ij} are contravariant components of $\underline{\underline{K}}$, related to the physical components (here written as K_{ij}, since covariant components will never appear) by the relation

$$K_{ij} = h_i h_j K^{ij} \quad \text{(no summation)} \tag{4.51}$$

where the line element of the metric is

$$dl^2 = g_{ij} dx^i dx^j \tag{4.52}$$

and

$$h_i = \sqrt{g_{ii}} \quad . \tag{4.53}$$

In spherical polar coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \quad . \tag{4.54}$$

Since equation (4.50) gives the contravariant components of div $\underline{\underline{K}}$, the right-hand side of equation (4.49) must also be expressed in contravariant components before agreement can be expected. For a vector with contravariant components X^i, the physical components X_i are given by

$$X_i = h_i X^i \quad . \tag{4.55}$$

Assuming that equations (4.44), (4.46) and (4.48) give the <u>physical</u> components of $\underline{\mathcal{F}}$ and $\underline{K}$ in spherical polar coordinates, some lengthy manipulation shows that the expressions (4.45) do indeed lead to equation (4.49), which is therefore the required generalization of the second moment equation, to which it reduces in the simple plane-parallel case.

To close the set of equations, it is now necessary to look for a generalization of the Eddington approximation, which may be written as

$$K = \frac{1}{4\pi} \int I \cos^2\Lambda \, d\omega = \frac{1}{3} J \tag{4.56}$$

(cf. equation (4.38), where the integral over η would be just 2π, since I is independent of H). This equation may be derived by assuming that $\cos^2\Lambda$ can be replaced by its average* value, i.e. that

$$\frac{1}{4\pi} \int I \cos^2\Lambda \, d\omega = \overline{\cos^2\Lambda} \, \frac{1}{4\pi} \int I \, d\omega = \overline{\cos^2\Lambda} \, J \; . \tag{4.57}$$

If $\underline{\underline{K}}$ is defined by equation (4.48), the components of $\underline{\underline{K}}$ in spherical polar coordinates are given by

$$\underline{\underline{K}} = \frac{1}{4\pi} \begin{bmatrix} \int I\cos^2\Lambda \, d\omega & \int I\cos\Lambda\sin\Lambda\cos H \, d\omega & \int I\cos\Lambda\sin\Lambda\sin H \, d\omega \\ \int I\cos\Lambda\sin\Lambda\cos H \, d\omega & \int I\sin^2\Lambda\cos^2 H \, d\omega & \int I\sin^2\Lambda\sin H\cos H \, d\omega \\ \int I\cos\Lambda\sin\Lambda\sin H \, d\omega & \int I\sin^2\Lambda\sin H\cos H \, d\omega & \int I\sin^2\Lambda\sin^2 H \, d\omega \end{bmatrix} \tag{4.58}$$

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* see equation (4.60) for a definition of the average.

Thus each component of $\underline{\underline{K}}$ is of the form

$$\frac{1}{4\pi}\int I\, f(\Lambda, H)\, d\omega .$$

The most obvious generalization of equation (4.57) is to write

$$\frac{1}{4\pi}\int I f(\Lambda, H) d\omega = \overline{f(\Lambda, H)}\,\frac{1}{4\pi}\int I\, d\omega = \overline{f(\Lambda, H)}.J \tag{4.59}$$

where

$$\overline{f(\Lambda, H)} = \frac{1}{4\pi}\int f(\Lambda, H)\, d\omega \tag{4.60}$$

When the various averages are evaluated, it is found, as might have been expected, that $\underline{\underline{K}}$ reduces to

$$\underline{\underline{K}} = \frac{1}{3} J\, \underline{\underline{1}} \tag{4.61}$$

where $\underline{\underline{1}}$ is the unit tensor, $\underline{\underline{1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is the proposed generalization of the Eddington approximation for $\underline{\underline{K}}$, and is valid in any coordinate system.

To complete the generalization, it is necessary to consider what the general boundary condition will be which corresponds to equation (4.41). In the notation of this thesis, that equation is

$$\mathcal{J}_r = 2\pi J \quad \text{at the surface} \tag{4.62}$$

since $\mathcal{J}_r$ is the component of $\underset{\sim}{\mathcal{J}}$ in the direction of the outward normal and $\underset{\sim}{\mathcal{J}}$ is defined in such a way that $\mathcal{J}_r = 4\pi H$. (The Cartesian coordinate system in which H is defined is tangential to the sphere r = constant.) The obvious analogue of this equation in the (s, χ, ϕ) system is

$$\mathcal{F}_s = 2\pi J \quad \text{at the surface} \tag{4.63}$$

and boundary conditions are required also for $\mathcal{F}_\chi$ and $\mathcal{F}_\phi$.

At the surface

$$\left.\begin{aligned} \mathcal{F}_s &= \int_0^{2\pi}\int_0^{\pi/2} I \cos\lambda \sin\lambda \, d\lambda \, d\eta \\ J &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi/2} I \sin\lambda \, d\lambda \, d\eta \ , \end{aligned}\right\} \tag{4.64}$$

and

using the boundary condition $\quad I = 0 \quad$ at the surface $(\pi/2 \leq \lambda \leq \pi)$, (4.65)

and equation (4.63) follows if it is assumed that $\mathcal{F}_s$ may be written

$$\mathcal{F}_s = \overline{\overline{\cos\lambda}} \int_0^{2\pi}\int_0^{\pi/2} I \sin\lambda \, d\lambda \, d\eta = \overline{\overline{\cos\lambda}} . 4\pi J \tag{4.66}$$

where the average is now defined by

$$\overline{\overline{g(\lambda,\eta)}} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} g(\lambda,\eta) \sin\lambda \, d\lambda \, d\eta \tag{4.67}$$

Thus the required generalization of equation (4.63) would seem to be that, at the surface,

$$\left.\begin{aligned} \mathcal{F}_\chi &= \overline{\overline{\sin\lambda \cos\eta}} \ 4\pi J \\ \mathcal{F}_\phi &= \overline{\overline{\sin\lambda \sin\eta}} \ 4\pi J \end{aligned}\right\} \tag{4.68}$$

Evaluation of the averages shows that the required boundary conditions are

$$\mathcal{F}_\chi = \mathcal{F}_\phi = 0 \quad \text{at the surface .} \tag{4.69}$$

Thus the complete generalization of equations (4.40) to (4.42) is, to summarize:

$$\left.\begin{aligned} \operatorname{div} \underset{\sim}{\mathcal{F}} &= -4\pi\kappa\rho(J - B) \\ \operatorname{div}\left(\tfrac{1}{3}J\underline{\underline{1}}\right) &= -\frac{\kappa\rho}{4\pi}\underset{\sim}{\mathcal{F}} \end{aligned}\right\} \tag{4.70}$$

and

$$\left.\begin{aligned} \mathcal{F}_s &= 2\pi J \\ \mathcal{F}_\chi = \mathcal{F}_\phi &= 0 \end{aligned}\right\} \text{ at the surface} \tag{4.71}$$

Of course, equation (4.71) refers only to the coordinate system (s, χ, ϕ), but it is easy to see how it would be extended to another coordinate system.

5. The non-local equations in the (s, χ, ϕ) system

This chapter will be completed by expressing equations (4.70), (4.71) in the coordinates s, χ, ϕ and then writing down the full set of non-local equations in that system. Equation (4.61) shows that the physical components of $\underline{\underline{K}}$ are given by

$$K_{ij} = \frac{1}{3}\delta^i_j \, J \tag{4.72}$$

Before the expression (4.50) for div $\underline{\underline{K}}$ can be used, the contravariant components of $\underline{\underline{K}}$ are needed. In the (s, χ, ϕ) system, $x^1 = s$, $x^2 = \chi$ and $x^3 = \phi$. Since the coordinate system is orthogonal, the metric is therefore:

$$dl^2 = g_{11}ds^2 + g_{22}d\chi^2 + g_{33}d\phi^2 \tag{4.73}$$

where $g_{ii} = h_i^2$ and

$$h_1 = \frac{1}{|\nabla s|} \;; \qquad h_2 = \frac{1}{|\nabla \chi|} \;; \qquad h_3 = \frac{1}{|\nabla \phi|} = r \sin\theta \tag{4.74}$$

Hence, from equation (4.51),

$$\left.\begin{aligned} K^{ij} &= 0 \quad (i \neq j) \\ K^{ii} &= \frac{K_{ii}}{g_{ii}} = \frac{J}{3g_{ii}} \end{aligned}\right\} \quad \text{(no summation)} \tag{4.75}$$

and, using the usual expressions for the Christoffel symbols $\left\{ {i \atop j\;k} \right\}$ and for the contravariant components g^{ij} of the fundamental tensor, it may be shown that

$$\left\{ {i \atop j\;k} \right\} K^{jk} + \left\{ {k \atop j\;k} \right\} K^{ij} \equiv 0 \tag{4.76}$$

so that

$$\text{div}^i \underline{\underline{K}} = \frac{\partial K^{ii}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{1}{3}\frac{J}{g_{ii}}\right) . \tag{4.77}$$

Since these are the contravariant components of $\text{div}\,\underline{\underline{K}}$, they are equal to the contravariant components of $-(\varkappa\rho/4\pi)\underline{\mathfrak{J}}$. Using equations (4.55) and (4.74), equation (4.49) therefore reduces to

$$\frac{\partial}{\partial s}\left(|\underline{\nabla} s|^2 J\right) = -\frac{3\varkappa}{4\pi}\rho|\underline{\nabla} s|\mathfrak{J}_s \tag{4.78}$$

and

$$\frac{\partial}{\partial \chi}\left(|\underline{\nabla}\chi|^2 J\right) = -\frac{3\varkappa}{4\pi}\rho|\underline{\nabla}\chi|\mathfrak{J}_\chi \tag{4.79}$$

The third component simply gives $\mathfrak{J}_\phi \equiv 0$, since axial symmetry is assumed (Chapter 2).

The general expression for div $\underline{\mathfrak{J}}$ is well known, and in the (s,χ,ϕ) system equation (4.43) takes the form:

$$|\nabla s||\nabla\chi||\nabla\phi|\left[\frac{\partial}{\partial s}\left(\frac{\mathcal{F}_s}{|\nabla\chi||\nabla\phi|}\right) + \frac{\partial}{\partial\chi}\left(\frac{\mathcal{F}_\chi}{|\nabla s||\nabla\phi|}\right)\right] = -4\pi\kappa\rho\,(J-B) \qquad (4.80)$$

The three equations (4.78) to (4.80) for $\mathcal{F}_s$, $\mathcal{F}_\chi$ and J replace the transfer equation (4.4) and equation (4.3) for $\underline{\mathcal{F}}$. Equation (4.5) defining B $(=\sigma T^4/\pi)$ is still required. The structure equations (cf. Chapter 2) should now be written in the form

$$\frac{dP}{ds} = \rho\,\frac{d\Psi}{ds} \qquad (4.81)$$

and

$$P = \frac{\mathcal{R}}{m}\rho\,T \quad . \qquad (4.82)$$

v_s is given by equation (4.31), and v_χ by the continuity equation in the form

$$\frac{\partial}{\partial s}\left(\frac{\rho v_s}{|\nabla\chi||\nabla\phi|}\right) + \frac{\partial}{\partial\chi}\left(\frac{\rho v_\chi}{|\nabla s||\nabla\phi|}\right) = 0\ . \qquad (4.83)$$

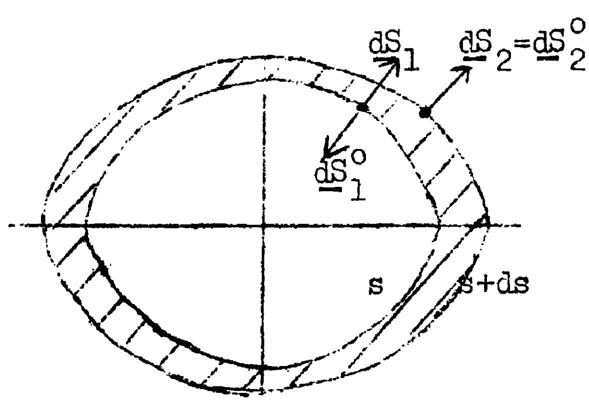
This gives 8 equations for 9 functions. The set of equations is closed formally by the condition

$$L = \int_{s=\text{const.}} \underline{\mathcal{F}}\cdot\underline{dS} \quad , \qquad (4.84)$$

but in the non-local theory this can be put in a more convenient form.

Consider two adjacent surfaces s = constant, s+ds = constant

Fig. 12. See text.



(Fig. 12), sufficiently far from the centre that, if equation (4.84) is applied to both surfaces, the value of L is the same for each. Then

$$\int_s \underline{\mathcal{J}}.\underline{dS}_1 = L = \int_{s+ds} \underline{\mathcal{J}}.\underline{dS}_2 \qquad (4.85)$$

or, taking normals directed outward from the volume between the surfaces,

$$\int_s \underline{\mathcal{J}}.\underline{dS}_1^o + \int_{s+ds} \underline{\mathcal{J}}.\underline{dS}_2^o = 0 \qquad (4.86)$$

Hence, by Gauss's theorem,

$$\int_V \operatorname{div} \underline{\mathcal{J}}\, dV = 0 \qquad (4.87)$$

where the integration is over the volume V (shaded in Fig. 12) between the two level surfaces. But

$$dV = \frac{ds\ d\chi\ d\phi}{|\nabla s||\nabla \chi||\nabla \phi|} \qquad (4.88)$$

and so, using equation (4.43),

$$\int_V \rho\ (J-B)\ \frac{ds\ d\chi\ d\phi}{|\nabla s||\nabla \chi||\nabla \phi|} = 0 \qquad (4.89)$$

Since the integrand is independent of ϕ, the integral over ϕ is just 2π. Also, the range of s is from s to s+ds, where ds is infinitesimal. Thus s is essentially constant throughout the integration, and the integral over s gives ds$\times$integrand. Hence, remembering that ρ is a function of s only, the integral reduces to

$$\int \frac{(J - B)}{|\nabla s||\nabla \chi||\nabla \phi|}\, d\chi = 0 \quad . \qquad (4.90)$$

The range of χ in this integral depends on the choice for $f(\chi)$ in equation (2.21). If $f(\chi) = \chi$, χ ranges from $-\infty$ (at the pole $\theta=0$)

through 0 (at the equator) to $+\infty$ (at the pole $\theta=\pi$) and the range of the integral is from $-\infty$ to $+\infty$. If $f(\chi)$ is chosen to make $\chi = \theta$ to lowest order in ϵ, the range of the integral is 0 to π, by the note after equation (2.23).

Note that, if equation (4.90) is used in place of equation (4.84), the equations of the non-local theory are independent of the value of L, which now appears only in the boundary conditions. This point will be discussed further in the next chapter.

CHAPTER 5

Boundary conditions and the formal solution of the non-local theory

"I could be bounded in a nut-shell, and count myself a king of infinite space."

W.Shakespeare, Hamlet, Act II Sc. ii.

1. General considerations

In the previous chapter virtually no mention was made of boundary conditions; in particular, there was (deliberately) no definition given of what was meant by the "surface" at which the boundary conditions of the generalized Eddington approximations were to be applied. This omission must be dealt with before any solution of the general equations of Chapter 4 is possible.

In the local theory, the simple conditions

$$P = T = 0 \quad \text{at} \quad s = R \tag{5.1}$$

were used for the structure variables. When the solution is completed, it is found that the density also vanishes at $s = R$, so that the star comes to an abrupt halt at a definite level surface. It is well-known, of course, that this is not a realistic picture, but it serves as an adequate approximation when only the overall structure of the star is required. In particular, the interior structure of a star is virtually independent of the exact boundary conditions applied at the surface, always assuming that proper account is taken of the major distinction between radiative and convective surface zones. The theory in this thesis refers only to stars with radiative atmospheres, for which it is known (see, e.g., Schwarzschild 1958) that the conditions (5.1) are entirely adequate for the determination of the luminosity and radius of a star of a given mass and chemical composition. It is for this reason that the results of RGS and Roxburgh and Strittmatter (1965) on the variation of luminosity and radius with rotation speed can be confidently used in the present model (cf. Appendix VI), although these results were obtained using the local theory.

At the same time, the conditions (5.1) are certainly not adequate for a description of the atmosphere of a star and they cannot be taken over to the non-local theory without modification. Before discussing the necessary modifications, however, it is useful to make a careful distinction between the <u>boundary</u> of a star, at which various mathematical conditions are imposed, and the <u>surface</u> of a star, which may be defined in several ways, each of which is an attempt to represent mathematically the visible surface of a star. Since this "surface" is really a transition zone of finite thickness between the photosphere and the chromosphere, no attempt to represent it as a mathematical surface of zero thickness can be completely satisfactory, and different representations are bound to give slightly different results. These differences are not usually important, but they can on occasion lead to controversy, as in recent exchanges over the determination of the exact shape of the Sun (Dicke and Goldenberg

1967a,b; Roxburgh 1967a), which will serve as a useful illustration of the different definitions.

Dicke and Goldenberg use a representation which depends for its success on the very rapid decrease of density with height in the outer atmosphere of a star. The surface is taken to be that layer of the star which is at optical depth unity in the line of sight. Because the line of sight is tangential to the atmosphere near the limb, a very small change in the radial optical depth corresponds to a large change in the tangential optical depth and it is possible to locate the limb very precisely. This definition of the surface is undoubtedly the most useful for the accurate determination of the shape of the Sun. Since the disk of the Sun in fact appears very sharp to the eye, it may be said that this definition is also the most realistic.

Nonetheless, Roxburgh's definition is also much used (see, e.g., Schwarzschild 1958) and it is perfectly adequate if the shape of a star is not the point at issue. In this representation, the surface is taken to be that on which the temperature is equal to the effective temperature. For convenience, this will be referred to as the "T-surface". In a spherical star, as will be shown shortly, this surface is one of constant radial optical depth, which is, therefore, parallel, at the limb, to the surface defined in terms of constant tangential optical depth, and the shape determined by either method will be the same, in principle. In a rotating star, as Roxburgh (1967a) rightly points out, this is no longer true.

The effective temperature T_{eff} (not to be confused with the mean effective temperature defined in Chapter 2) is defined by

$$\underset{\sim}{\mathcal{F}} \cdot \underline{dS} = \sigma T_{eff}^4 |\underline{dS}| \tag{5.2}$$

where $\underline{dS}$ is the normal to the T-surface (as yet undefined). For a spherical, non-rotating star, $\underset{\sim}{\mathcal{F}}$ has only a radial component, which is a function of r alone. T_{eff} is therefore a function of r alone and so, since T is also a function only of r, the surface $T = T_{eff}$ must be a sphere, which, in a spherical star, is a surface of constant optical depth.

However, in a rotating star, $\underset{\sim}{\mathcal{F}}$, and so T_{eff}, is a function of both s and χ, while (assuming there is no turbulence) T is a function of s only (cf. Chapter 2). In that case the equation

$$T(s) = T_{eff}(s,\chi) \tag{5.3}$$

defines the T-surface to be a surface $s = s(\chi)$, which is not in general a level surface. Since the surface, as defined by Dicke and Goldenberg, has a meridian-plane section at the limb which is essentially a contour of constant density, the visible profile of the Sun is that of a level surface (since $\rho = \rho(s)$). The two definitions will therefore be expected to give different results for the shape of the Sun if in each case the mathematical definition is supposed to represent the visible limb. There is little doubt that Dicke and Goldenberg have taken the correct definition for their purpose. The T-surface representation of the surface of a star is perhaps the more fundamental from a physical viewpoint, since the T-surface, or "photosphere", of the star is defined in such a way that it is the level in the atmosphere from which radiation just escapes from the star.

However, these definitions are made with the observer in mind rather than the builder of theoretical model atmospheres, and neither of the surfaces defined above is a suitable boundary for the star. The more realistic is not even axially symmetric, and both are defined in terms of functions which are themselves not uniquely determined until boundary conditions have been applied. If either of the above surfaces were to be taken as the boundary of the star, the boundary would itself be an unknown in the problem. That would clearly be unsatisfactory, and the theoretician must rather choose a boundary which is determined only by the rotation speed and, if necessary, by the overall structure of the star as found from the local theory.

Before defining such a boundary, it is worth considering in a little more detail how the boundary is defined in the local theory. Since $P = T = 0$ on the boundary, the star stops short there and the boundary is that surface which contains the total mass of the star. For a non-rotating star, this surface is a sphere and the condition that this sphere contains the total mass determines its radius, which is the radius of the star. For a rotating star, the boundary must be a level surface (since P and T are constant on a level surface), whose polar and equatorial radii are determined by the rotation speed and by the condition that the level surface contains the total mass. It is convenient in practice to relate the boundary of a rotating star to that of a non-rotating star of the same mass (cf. Appendix IV, section 6).

2. Non-rotating atmospheres - the plane-parallel approximation

In most stars, the atmosphere is a very thin outer skin whose thickness is a tiny fraction of the stellar radius. Since the mean free path of a photon in the atmosphere can hardly be greater than the height of the atmosphere, no photon will travel far enough between collisions to detect the curvature of the atmosphere, and it is a very good approximation to consider the atmosphere as stratified in plane parallel layers. The error in this approximation is of the order of the ratio of photon mean free path to stellar radius. The photon mean free path λ may be defined crudely by

$$\kappa\rho\lambda \sim 1 \ . \tag{5.3}$$

On the other hand, the pressure scale height H is defined by

$$H = \left|\frac{P}{dP/dr}\right| = \frac{\tau}{\kappa\rho} \tag{5.4}$$

where the second equality follows from the hydrostatic equation $\frac{dP}{dr} = \rho g$ (g = constant in atmosphere) and the definition of τ (equation (4.7)). Thus, at $\tau \sim 1$, $\lambda \sim H$ and it may equally well be said that the error in the plane parallel approximation is of order H/R. Another way of writing equation (5.4), again using the hydrostatic equation, is

$$H = \frac{\mathcal{R}T}{mg} \ . \tag{5.5}$$

Making the further approximations $T \sim T_e$ (valid at $\tau \sim \frac{2}{3}$ (see, e.g., Schwarzschild 1958)) and $g \sim \frac{GM}{R^2}$, the ratio $\frac{H}{R}$ becomes identical with the ratio ϵ_1 of Chapter 2 (equation (2.18)). This ratio appears naturally in the theory, and it is therefore a more convenient measure of the error in the plane-parallel approximation than is the more physically significant ratio $\frac{\lambda}{R}$. Since $\epsilon_1 \sim 10^{-3}$ for the stars considered, it is clear that the plane-parallel approximation is very good. It is much better than the Eddington approximations discussed in the previous chapter, which are known to be in error by as much as 15% near the surface (Kourganoff 1952).

Formally, the equations in the plane-parallel approximation may be obtained from the more general non-local equations by allowing ϵ_1 to tend to zero (corresponding to the infinite radius of curvature of a plane surface). To see how this formal procedure affects the boundary conditions, define a new coordinate ζ by

$$r = R(1 + \epsilon_1 \zeta) \tag{5.6}$$

and consider the range of ζ in the atmosphere, supposing for the moment that r=R represents some arbitrary level in the middle of the atmosphere and that r varies between R_{min} and R_{max} in the atmosphere. The corresponding range of ζ is

$$\zeta = \frac{R_{min} - R}{\epsilon_1 R} \quad \text{to} \quad \zeta = \frac{R_{max} - R}{\epsilon_1 R} .$$

Now let $\epsilon_1 \to 0$. The range of ζ then becomes the open interval $]-\infty, \infty[$.

Of course, ϵ_1 is really a constant, and the equations in the plane-parallel approximation are, more strictly, obtained by neglecting terms of order ϵ_1 in the more general equations. For finite ϵ_1, the range of ζ in the atmosphere is more like $]-5,5[$, so that to extend this range to the interval $]-\infty,\infty[$ seems a gross exaggeration. However, if one accepts the formal extension of the range, so that surface boundary conditions are to be applied as $\zeta \rightarrow +\infty$, it is found that the resulting pressure and temperature have virtually reached their surface values by the time ζ is as great as 5. Similarly, if the atmosphere is formally tied to the interior as $\zeta \rightarrow -\infty$, conditions at $\zeta = -5$ are found to be such that the local theory is valid (see Appendix VIII). This justifies the use of an infinite range of ζ to represent a physically finite atmosphere.

Notice that the formal procedure outlined above is not the same as a perturbation theory in ϵ_1. In a perturbation theory, ϵ_1 is essentially finite, though small, and an infinite range for ζ would not be allowable. Any attempt to use an infinite range for ζ in the first order equations of a perturbation theory in ϵ_1 leads to singularities. If an attempt is to be made to represent the effects of curvature, a different approach is required. Either the boundary must be taken at a finite radius or more stringent boundary conditions must be applied. This problem is discussed further in the next section.

In the plane-parallel approximation it is often convenient to use the optical depth τ as a coordinate. It arises naturally as a coordinate, and its physical significance provides some justification for the boundary conditions applied. If τ is defined by

$$\tau = \int_{\zeta}^{\infty} \varkappa \epsilon_1 R \rho \, d\zeta' \quad , \qquad (5.7)$$

then $\tau \to 0$ as $\zeta \to +\infty$ and $\tau \to \infty$ as $\zeta \to -\infty$. Consider the upper boundary first. Zero optical depth means no absorption. Thus there can be no absorbing matter beyond $\tau = 0$, and it is reasonable to take P=0 at $\tau = 0$. Also, if there is no absorbing matter beyond $\tau = 0$, there can be no emitting matter either and, assuming the star to be isolated, the second boundary condition must be that of no incident radiation. It will be seen that this condition requires the temperature to be finite at $\tau = 0$. The boundary condition of the Eddington approximation is based on no incident radiation at the "surface". This discussion shows that the "surface" must be taken as $\tau = 0$. Thus the surface boundary conditions are:

$$\left.\begin{aligned} P &= 0 \\ \mathcal{F}_r &= 2\pi J \end{aligned}\right\} \qquad \text{at } \tau = 0 \tag{5.8}$$

For a complete solution of a model star, it is necessary also to apply two boundary conditions at the centre. These conditions are represented in the local theory of the atmosphere by treating the luminosity L and radius R of the star as known - L appears in the energy balance equation and R appears in the boundary conditions. In the non-local theory, L and R no longer appear and the theory requires two more boundary conditions.

Now it is well known (see, for example, Chandrasekhar 1939, p.208) that the equations of the non-local theory reduce to those of the local theory at large optical depth (where the photon mean free path is small). For consistency, the solutions of the two sets of equations must also match as $\tau \to \infty$. This requirement provides the two extra boundary conditions needed for a unique solution of the atmosphere. They are most conveniently taken to be

$$\left. \begin{array}{l} |T_L - T_{NL}| \longrightarrow 0 \\ |\mathcal{F}_{rL} - \mathcal{F}_{rNL}| \longrightarrow 0 \end{array} \right\} \quad \text{as} \quad \tau \longrightarrow \infty \qquad (5.9)$$

where the subscripts L and NL denote local and non-local respectively. The reasons for choosing to match T and $\mathcal{F}_r$ rather than any other functions will become apparent in section 4, and in Chapter 6.

3. Effects of curvature

The case in which the mean free path of a photon is a non-vanishing fraction of the stellar radius has recently been considered by Chapman (1966), who briefly reviews earlier work. Since he considers a spherical atmosphere with no mass motions, the atmosphere is in radiative equilibrium and $J = B$. Normally, this allows the transfer equation to be solved, in terms of τ, without reference to the structure equations. Now, however, r appears explicitly in the transfer equation (equation (4.9)) and the solution depends on the form of the relation $\rho = \rho(r)$. As is done in this thesis, Chapman assumes a gray atmosphere and, to simplify the analysis, he considers the case $\kappa\rho = r^{-3}$, for which the Eddington approximations yield an analytic solution. The boundary is taken to be at infinity. In accordance with remarks in the previous section, it is necessary in that case to replace the condition of zero incident intensity by the more stringent condition

$$r^2 \, I(r,\mu) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty \tag{5.10}$$

Chapman produces the interesting result that, in an infinite spherical atmosphere, J can be written as the geometrical factor r^{-2} times what is essentially the result of the plane-parallel Eddington approximation.

However, it is not easy to see how his methods could be translated to fit the case of a rotating star. For one thing, it is no longer possible to solve the transfer equation on its own. Not only is $B \neq J$, so that the temperature must be considered, but also it is not permissible to prescribe the form of $\kappa\rho$ when one is looking for the velocity field, which depends on the atmospheric structure.

These are practical difficulties. A difficulty of principle arises in connection with the boundary conditions. The definition of Ψ is such that Ψ attains a minimum for a certain value of s, depending on ϵ. Only for values of s less than that are the level surfaces closed (cf. Fig. 5). For greater values of s the level surfaces are open, essentially because the limiting level surface is the one on which the centrifugal force balances gravity at the equator. If the star completely fills this level surface, in some sense, it starts to lose mass from the equator. There is therefore no closed level surface corresponding to the "sphere at infinity" on which Chapman applies his boundary conditions. It is necessary either to choose the boundary on some finite, closed level surface or to define a new family of surfaces which remain closed at infinity and have some physical significance. Neither alternative is attractive when examined in more detail.

These considerations strongly suggest that no attempt should be made in the first instance to include the effects of curvature on the atmosphere of a rotating star. Of course, variation with χ must be included, but that may be done within the framework of a plane-parallel approximation, as is shown in the next section.

4. Rotating atmospheres - formal solution in the plane-parallel case

The justification for using a plane-parallel approximation is the same for rotating stars as for non-rotating stars, and the error in the approximation will be of order ϵ_1 by the arguments of section 2. The difference between the two cases lies principally in the fact that in rotating stars some functions depend on the two variables s and χ and the χ-variation must be taken into account. It is therefore more correct to say that the atmosphere is treated as locally plane-parallel. The coordinate z which will be introduced to correspond to the ζ of the spherical star is constant on a level surface, so that surfaces of constant z are not parallel, being more closely spaced at the poles than at the equator. Nonetheless, the z-surfaces are all orthogonal to a particular surface of constant χ and may in that sense be regarded as locally parallel.

The coordinate z is defined in terms of s by

$$s = R(1 + \epsilon_1 z) \qquad (5.11)$$

and so corresponds to ζ in being zero on the boundary used in the local theory. This enables the local theory to be conveniently developed in the plane-parallel approximation for fitting purposes. Similar arguments to those in section 2 show that the appropriate range for z is $]-\infty, \infty[$. Again, "surface" boundary conditions are to be applied as $z \rightarrow +\infty$ and conditions are to be applied as $z \rightarrow -\infty$ which will ensure that the various functions fit smoothly to those of the local theory. By analogy with section 2, but omitting for the moment any reference to optical depth, the boundary conditions are:

$$\left.\begin{array}{r} P \rightarrow 0 \\ \mathfrak{F}_s - 2\pi J \rightarrow 0 \end{array}\right\} \quad \text{as } z \rightarrow +\infty \tag{5.12}$$

$$\left.\begin{array}{r} |T_L - T_{NL}| \rightarrow 0 \\ |\mathfrak{F}_{sL} - \mathfrak{F}_{sNL}| \rightarrow 0 \end{array}\right\} \quad \text{as } z \rightarrow -\infty \tag{5.13}$$

It will be seen shortly why no boundary condition for $\mathfrak{F}_\chi$ is included.

In the remainder of this chapter, the general equations of Chapter 4 will be simplified to the form valid in a locally plane-parallel atmosphere and the general method of solution will be outlined. This will indicate, among other things, the reason for choosing the functions T and $\mathfrak{F}_s$ in equation (5.13).

It is convenient to work with P, ρ and T in dimensionless form, since this immediately brings out orders of magnitude (assuming the dimensionless functions to be of order unity through most of the atmosphere, an assumption justified by the results). On the other hand, $\underline{\mathfrak{F}}$, J and B are all of the same order of magnitude and there is no great benefit in introducing dimensionless variables for them at this stage. The dimensionless variables p, ρ^* and t are defined by:

$$P = \frac{GM}{\kappa R^2} p \quad ; \qquad \rho = \frac{1}{\kappa \epsilon_1 R} \rho^* \quad ; \qquad T = T_e t \quad . \tag{5.14}$$

It can be shown that, to lowest order in ϵ_1,

$$\left.\begin{aligned} |\nabla s| &= A(\chi, \epsilon) \\ |\nabla \chi| &= \tfrac{1}{R} C(\chi, \epsilon) \\ |\nabla \phi| &= \tfrac{1}{R} D(\chi, \epsilon) \end{aligned}\right\} \tag{5.15}$$

where A, C and D are known functions (see Appendix V). Using these relations, and the definitions (5.14), it is easy to simplify the general equations by leaving out terms of order ϵ_1.

First consider equation (4.79). This immediately simplifies to

$$\mathcal{J}_\chi = -\frac{4\pi}{3} \epsilon_1 \frac{1}{\rho^* C} \frac{\partial}{\partial \chi} (C^2 J) \quad . \tag{5.16}$$

Thus, since $J \sim |\underline{\mathcal{F}}|$, $\mathcal{J}_\chi \sim \epsilon_1 \mathcal{J}_s$ and can be ignored. That is, in a locally plane-parallel atmosphere

$$\mathcal{J}_\chi \cong 0 \quad . \tag{5.17}$$

It is because of this result that no boundary condition for $\mathcal{J}_\chi$ is included in equations (5.12) and (5.13).

The one disquieting feature of equation (5.16) is the $1/\rho^*$ dependence, reminiscent of the trouble with v_s in the local theory. Examination of the general equations suggests that this feature is intrinsic and not an artifact of the plane-parallel approximation. It can only be hoped that a proper treatment of the effects of curvature would so alter the boundary conditions that $\mathcal{F}_\chi$ remained finite at the boundary despite the density being zero there. Alternatively, the $1/\rho^*$ dependence could conceivably be an artifact of the Eddington approximations, which are least accurate near the surface. If the exact theory required the χ-component of div $\underline{\underline{K}}$ to tend to zero as fast as ρ^*, $\mathcal{F}_\chi$ would remain finite at the surface. However, since ρ^* tends to zero exponentially as z tends to infinity (in the plane-parallel case), that explanation seems less convincing. Suffice it to say that the singularity can hardly be real, and that the presence of the factor ϵ_1 ensures that the trouble occurs only very near the surface. For the rest of this thesis equation (5.17) will be taken to be valid throughout the atmosphere.

The remaining equations give no trouble. Equation (4.78) reduces immediately to

$$|\underline{\nabla} s| \frac{\partial J}{\partial z} = -\frac{3}{4\pi}\rho^* \mathcal{F}_s \qquad (5.18)$$

and the use of equation (5.17) reduces equation (4.80) to

$$|\underline{\nabla} s| \frac{\partial \mathcal{F}_s}{\partial z} = -4\pi\rho^*(J-B) \quad . \qquad (5.19)$$

Since

$$\frac{d\Psi}{ds} = -\frac{GM}{R^2}E(\epsilon) + O(\epsilon_1) \qquad (5.20)$$

(E a known function of ϵ) the two structure equations reduce to

$$\frac{dp}{dz} = -\rho^{*}E \tag{5.21}$$

and

$$p = \rho^{*}t \tag{5.22}$$

The equations for $\underline{v}$ will not be considered here, since they are not needed for the solution for the structure of the atmosphere. The final equation, then, is equation (4.90), which reduces to

$$\int_0^{\pi} \frac{J - B}{A\ C\ D}\, d\chi = 0 \quad . \tag{5.23}$$

The limits are chosen to be 0 and π because these equations will be solved in detail in the next chapter using perturbation methods.

It is now clear that optical depth in this case should be defined by

$$\tau = \int_z^{\infty} \rho^{*} dz' \quad . \tag{5.24}$$

If the change of variables

$$\left.\begin{aligned} x &= \tau/|\nabla s| \\ \chi &= \chi \end{aligned}\right\} \tag{5.25}$$

is now made in equations (5.18), (5.19), J may be eliminated to give a second order linear differential equation for $\mathcal{J}_s$:

$$\frac{\partial^2 \mathcal{J}_s(x,\chi)}{\partial x^2} - 3\mathcal{J}_s(x,\chi) = -4\pi \frac{\partial B}{\partial x}(x,\chi) \tag{5.26}$$

whose general solution (in terms of τ) is

$$\mathcal{J}_s = A_+(\chi)e^{+\frac{\sqrt{3}\tau}{|\nabla s|}} + A_-(\chi)e^{-\frac{\sqrt{3}\tau}{|\nabla s|}} - \frac{2\pi}{|\nabla s|}\left[e^{+\frac{\sqrt{3}\tau}{|\nabla s|}}\int_0^{\tau} B(y)e^{-\frac{\sqrt{3}y}{|\nabla s|}}dy + e^{-\frac{\sqrt{3}\tau}{|\nabla s|}}\int_0^{\tau} B(y)e^{+\frac{\sqrt{3}y}{|\nabla s|}}dy\right] \tag{5.27}$$

Since $B = \sigma T^4/\pi$, two boundary conditions will determine $\mathcal{J}_s$ uniquely in terms of the temperature. One of these is the condition (5.13) that the flux should match the local flux as $\tau \rightarrow +\infty$. This partially explains the choice of $\mathcal{J}_s$ as one of the functions to be matched to the local theory. A combination of equations (5.12) and (5.19) gives a second condition, at $\tau = 0$:

$$\mathcal{J}_s(0,\chi) \quad = \quad 2\pi B(0) \;+\; \tfrac{1}{2}|\nabla s|\, \frac{\partial \mathcal{J}_s}{\partial \tau}(0,\chi) \; . \tag{5.28}$$

The condition is put in this form to eliminate any reference to J, which may then be determined from equation (5.19), in terms of B (and so as a function of temperature).

From equations (5.21) and (5.22), p and ρ^* may also be obtained as functions of temperature, using the boundary condition (5.12) for P. The temperature may therefore be regarded as a fundamental function, in terms of which all the other functions can be obtained. The solution for the temperature is obtained from equation (5.23) which gives B as a function of J, which is itself known in terms of B and $\mathcal{J}_s$. Equation (5.28) shows that this solution for the temperature involves the unknown constant B(0), which is determined by the condition (5.13) for T. One of the reasons why this was chosen as the other matching condition is now clear.

That completes the formal solution of the non-local equations for a rotating atmosphere. However, the method outlined above is difficult to apply in practice, since the final stage leads to the following integral equation for B:

$$B(\tau) = \frac{1}{\int_0^{\pi} \frac{d\chi}{|\nabla s|\, C\, D}} \int_0^{\pi} \frac{d\chi}{|\nabla s|\, C\, D} \Bigg[\frac{\sqrt{3}}{4\pi} \left(A_+(\chi)\, e^{+\frac{\sqrt{3}\tau}{|\nabla s|}} - A_-(\chi)\, e^{-\frac{\sqrt{3}\tau}{|\nabla s|}} \right)$$

$$- \frac{\sqrt{3}}{2|\nabla s|} \left(e^{+\frac{\sqrt{3}\tau}{|\nabla s|}} \int_0^{\tau} B(y)\, e^{-\frac{\sqrt{3}y}{|\nabla s|}} dy - e^{-\frac{\sqrt{3}\tau}{|\nabla s|}} \int_0^{\tau} B(y)\, e^{+\frac{\sqrt{3}y}{|\nabla s|}} dy \right) \Bigg] \qquad (5.29)$$

There is no hope of obtaining an analytical solution for this equation, particularly as $|\nabla s|$, C and D are known only as expansions in powers of ϵ. It is therefore necessary to impose the further restriction of slow rotation and to solve the non-local equations using perturbation methods. In the next chapter, the solution will be obtained to first order in ϵ.

CHAPTER 6

Solution of the non-local theory in the plane-parallel approximation and for slow rotation

"...the last state ... is worse than the first."

Gospel according to St. Matthew, Ch. 12, v.45.

1. Perturbation equations and boundary conditions

In this chapter, the non-local equations in the plane-parallel approximation will be solved to first order in ϵ by the use of perturbation methods. The equations have in fact been solved to second order in ϵ, but the second order theory is not essentially different from the first order theory, except that it is more cumbersome to handle and that the distinction between χ' and θ must be clearly made. It did not, therefore, seem to be useful to present the second order theory here, particularly as the velocity in the non-local theory has a non-vanishing first order term. Some of the second order results will, however, be quoted in section 4.

In this section the basic equations and boundary conditions needed to define the structure uniquely are gathered together. The equations for the velocity will be given after a solution for the structure has been obtained. The dimensionless variables used are defined by equations (5.11) and (5.14) and by

$$J = \frac{\sigma T_e^4}{\pi} J^* \ ; \quad B = \frac{\sigma T_e^4}{\pi} B^* \ ; \quad \hat{\underset{\sim}{\mathcal{J}}} = \sigma T_e^4 \ \hat{\underset{\sim}{\mathcal{J}}}^* . \tag{6.1}$$

In these variables, the complete set of non-local structure equations is:

$$\frac{dp}{dz} = -\rho^{*} E(\epsilon) \tag{6.2}$$

$$p = \rho^{*} t \tag{6.3}$$

$$B^{*} = t^{4} \tag{6.4}$$

$$|\nabla s| \frac{\partial \tilde{\mathcal{J}}_s^{*}}{\partial z} = -4\rho^{*}(J^{*} - B^{*}) \tag{6.5}$$

$$|\nabla s| \frac{\partial J^{*}}{\partial z} = -\frac{3}{4}\rho^{*} \tilde{\mathcal{J}}_s^{*} \tag{6.6}$$

and

$$\int_0^{\pi} (J^{*} - B^{*}) \frac{d\chi}{|\nabla s|\, CD} = 0 \tag{6.7}$$

where (see Appendix V and Chapter 5)

$$E(\epsilon) = 1 - \frac{2}{3}\epsilon\,(1 - P_2(0)) \tag{6.8}$$

$$|\nabla s| = 1 - \frac{4}{3}\epsilon\,(P_2(0) - P_2) + O(\epsilon^2) \tag{6.9}$$

$$C = 1 - \frac{1}{9}\epsilon\,(1 + 3P_2(0) - 7P_2) + O(\epsilon^2) \tag{6.10}$$

$$D = \frac{1}{\sin\chi}\left[1 + \frac{1}{9}\epsilon(1 - 3P_2(0) + 5P_2) + O(\epsilon^2)\right] \tag{6.11}$$

As in Appendices V and VII, $P_2(0)$ is used as a shorthand for $P_2(\cos\theta_o)$, θ_o being defined in Appendix IV; $P_2 \equiv P_2(\cos\chi)$.

These equations must be solved subject to the boundary conditions:

$$\left.\begin{array}{r} p \longrightarrow 0 \\ \tilde{\mathcal{J}}_s^{*} - 2J^{*} \longrightarrow 0 \end{array}\right\} \quad \text{as} \quad z \longrightarrow +\infty \tag{6.12}$$

$$\left.\begin{aligned} |t_L - t_{NL}| &\longrightarrow 0 \\ |\tilde{\mathcal{J}}^*_{sL} - \tilde{\mathcal{J}}^*_{sNL}| &\longrightarrow 0 \end{aligned}\right\} \quad \text{as} \quad z \longrightarrow -\infty \tag{6.13}$$

where t_L, $\tilde{\mathcal{J}}^*_{sL}$ are given by equations (A7.21) and (A7.32) of Appendix VII.

The first stage in the solution of these equations by perturbation methods is to develop the equations to the first order in ϵ by writing

$$\left.\begin{aligned} p &= p_o(z) + \epsilon\, p_1(z) + \ldots \\ \rho^* &= \rho^*_o(z) + \epsilon\, \rho^*_1(z) + \ldots \\ t &= t_o(z) + \epsilon\, t_1(z) + \ldots \\ B^* &= B^*_o(z) + \epsilon\, B^*_1(z) + \ldots \\ \tilde{\mathcal{J}}^*_s &= \tilde{\mathcal{J}}^*_{so}(z) + \epsilon\, \tilde{\mathcal{J}}^*_{s1}(z,\chi) + \ldots \\ J^* &= J^*_o(z) + \epsilon\, J^*_1(z,\chi) + \ldots \end{aligned}\right\} \tag{6.14}$$

It is immediately obvious that the zero order equations are:

$$\frac{dp_o}{dz} = -\rho^*_o \tag{6.15}$$

$$p_o = \rho^*_o t_o \tag{6.16}$$

$$B^*_o = t_o^4 \tag{6.17}$$

$$\frac{d\tilde{\mathcal{J}}^*_{so}}{dz} = -4\,\rho^*_o\,(J^*_o - B^*_o) \tag{6.18}$$

$$\frac{dJ^*_o}{dz} = -\frac{3}{4}\,\rho^*_o\,\tilde{\mathcal{J}}^*_{so} \tag{6.19}$$

and

$$J^*_o - B^*_o = 0 \tag{6.20}$$

with the boundary conditions (using Appendix VII)

$$p_o \longrightarrow 0 \quad \text{and} \quad \mathcal{J}^*_{so} - 2J^*_o \longrightarrow 0 \quad \text{as} \quad z \longrightarrow +\infty \tag{6.21}$$

$$t_o + \frac{1}{4} z \longrightarrow 0 \quad \text{and} \quad \mathcal{J}^*_{so} - 1 \longrightarrow 0 \quad \text{as} \quad z \longrightarrow -\infty \quad . \tag{6.22}$$

After some manipulation, the corresponding first order equations are found to be

$$\frac{dp_1}{dz} + \rho^*_1 = \frac{2}{3} \rho^*_o (1 - P_2(0)) \tag{6.23}$$

$$\frac{p_1}{p_o} = \frac{\rho^*_1}{\rho^*_o} + \frac{t_1}{t_o} \tag{6.24}$$

$$B^*_1 = 4\, t_o^3\, t_1 \tag{6.25}$$

$$\frac{\partial \mathcal{J}^*_{s1}}{\partial z} = -4\, \rho^*_o (J^*_1 - B^*_1) \tag{6.26}$$

$$\frac{\partial J^*_1}{\partial z} = -\frac{3}{4} \rho^*_o \mathcal{J}^*_{s1} - \frac{3}{4} \rho^*_1 \mathcal{J}^*_{so} - (P_2(0) - P_2)\rho^*_o \mathcal{J}^*_{so} \tag{6.27}$$

$$B^*_1 = \frac{1}{2} \int_0^{\pi} J^*_1 \sin\chi \, d\chi \tag{6.28}$$

The zero order equations have been used in obtaining these equations. The boundary conditions are (using Appendix VII)

$$p_1 \longrightarrow 0 \quad \text{and} \quad \mathcal{J}^*_{s1} - 2J^*_1 \longrightarrow 0 \quad \text{as} \quad z \longrightarrow +\infty \tag{6.29}$$

$$\left.\begin{aligned} & t_1 - \frac{1}{6} z\,(1 - P_2(0)) \longrightarrow 0 \\ & \mathcal{J}^*_{s1} - (\ell_1 - \frac{2}{3} P_2(0) + \frac{4}{3} P_2) \longrightarrow 0 \end{aligned}\right\} \quad \text{as} \quad z \longrightarrow -\infty \tag{6.30}$$

2. The zero order solution

It follows immediately from equations (6.18), (6.20) and (6.22) that

$$\mathcal{J}^{*}_{so} = 1 \qquad \text{for all } z. \tag{6.31}$$

Using this result, and equations (6.17) and (6.20), equation (6.19) reduces to

$$\frac{dt_o^4}{dz} = -\frac{3}{4}\rho_o^{*} \quad . \tag{6.32}$$

Comparison with Appendix VII now shows that the equations for p_o, ρ_o^{*} and t_o are exactly the same as in the local theory. However, the boundary conditions are different, and the solutions in the present case are not nearly so simple as the solutions of the local equations.

Using equations (6.17), (6.20) and (6.31), the second boundary condition may be written as

$$t_o^4 \longrightarrow \frac{1}{2} \quad \text{as} \quad z \longrightarrow +\infty \quad . \tag{6.33}$$

That is, the temperature is finite at the boundary, with boundary value $2^{-1/4}$. It will be seen that this makes a crucial difference to the solution.

Equations (6.15) and (6.32) may be used, as in the local theory, to obtain a relation between p_o and t_o. With the boundary conditions of the non-local theory the relation is

$$p_o = \frac{2}{3}(2t_o^4 - 1) \quad . \tag{6.34}$$

If this expression for p_o is substituted into equation (6.16) and then ρ_o^* is eliminated between the resulting equation and equation (6.32), a first order differential equation for t_o is obtained. The variables are separable and, with the change of variable

$$2^{1/4}\, t_o = u , \qquad (6.35)$$

the differential equation can be written as

$$\frac{u^4 du}{u^4 - 1} = -\frac{1}{4}\, 2^{1/4}\, dz . \qquad (6.36)$$

This can be integrated to give

$$u + \frac{1}{4}\left(\log \frac{u-1}{u+1} - 2 \tan^{-1} u\right) = -\frac{1}{4}\, 2^{1/4}\, z + A_1 \qquad (6.37)$$

where A_1 is an arbitrary constant to be determined by the condition (6.22) for t_o. A very similar result is given in Eddington's book (1930, p. 337).

Since the R.H.S. of equation (6.37) $\rightarrow +\infty$ as $z \rightarrow -\infty$, $u \rightarrow \infty$ as $z \rightarrow -\infty$; for large u

$$\frac{1}{4}\left(\log \frac{u-1}{u+1} - 2 \tan^{-1} u\right) \sim -\frac{\pi}{4} - \frac{1}{2}\frac{1}{u} . \qquad (6.38)$$

Application of the condition (6.22) for t_o (using equation (6.35)) therefore leads to the relation

$$-\frac{\pi}{4} = A_1 \qquad (6.39)$$

so that the solution for t_o is finally

$$t_o + \frac{1}{4}\, 2^{-1/4}\left(\log \frac{2^{1/4}t_o - 1}{2^{1/4}t_o + 1} - 2\tan^{-1}(2^{1/4}t_o)\right) = -\frac{1}{4}z - 2^{-1/4}\frac{\pi}{4} . \quad (6.40)$$

In principle, p_o can now be determined from equation (6.34) and then ρ_o^* from equation (6.16). In practice, it is clearly impossible to obtain p_o and ρ_o^* explicitly in terms of z for general z. Numerical tables, and graphs, of p_o and t_o as functions of z will be found in Appendix VIII.

However, asymptotic expressions can be found for p_o, ρ_o^* and t_o as $z \rightarrow \pm\infty$. The derivation of these is straightforward, and the method is outlined in Appendix VIII. The results are:

<u>(i) $z \longrightarrow +\infty$</u>

$$p_o = \frac{16}{3E}\, e^{-2^{1/4}z}\left(1 - \frac{2}{E}\, e^{-2^{1/4}z} + \ldots\right) \quad (6.41)$$

$$\rho_o^* = \frac{16.2^{1/4}}{3E}\, e^{-2^{1/4}z}\left(1 - \frac{4}{E}\, e^{-2^{1/4}z} + \ldots\right) \quad (6.42)$$

$$t_o = 2^{-1/4}\left[1 + \frac{2}{E}\, e^{-2^{1/4}z}\left(1 - \frac{5}{E}\, e^{-2^{1/4}z} + \ldots\right)\right] \quad (6.43)$$

where

$$E = e^{\pi/2 + 4} \doteq 263 \quad (6.44)$$

<u>(ii) $z \longrightarrow -\infty$</u>

$$p_o = \frac{4}{3}\left(\frac{z}{4}\right)^4\left[1 + \frac{1}{6}\left(\frac{4}{z}\right)^4 + O\left(\frac{4}{z}\right)^8\right] \quad (6.45)$$

$$\rho_o^* = -\frac{4}{3}\left(\frac{z}{4}\right)^3\left[1 + O\left(\frac{4}{z}\right)^8\right] \quad (6.46)$$

$$t_o = -\frac{z}{4}\left[1 + \frac{1}{6}\left(\frac{4}{z}\right)^4 + O\left(\frac{4}{z}\right)^8\right] . \quad (6.47)$$

Equations (6.45) and (6.46) show that, as expected, p_o and ρ_o^* also match on to the local results as $z \longrightarrow -\infty$.

3. The first order solution

Since J_1^* and $\mathcal{J}_{s1}^*$ are functions of two variables, the first order equations inevitably lead, in general, to an integral equation for B_1^*, as indicated in Chapter 5. This difficulty can only be surmounted by assuming a form for the χ-dependence of J_1^* and $\mathcal{J}_{s1}^*$, a procedure which would seem to be entirely arbitrary.

However, it is well-known that, under reasonably general conditions, any function of χ can be expressed as an expansion in an infinite series of Legendre polynomials. That is, it is true in general that

$$J_1^*(z,\chi) = \sum_{n=0}^{\infty} J_{1n}^*(z)\, P_n(\cos\chi) \qquad (6.48)$$

and

$$\mathcal{J}_{s1}^*(z,\chi) = \sum_{n=0}^{\infty} \mathcal{J}_{s1n}^*(z)\, P_n(\cos\chi) \quad . \qquad (6.49)$$

If these expressions are substituted into equations (6.26) and (6.27), an infinite set of ordinary differential equations for the J_{1n}^* and $\mathcal{J}_{s1n}^*$ is obtained by equating coefficients of the P_n. At first sight, this might not appear to be much of a simplification over solving an integral equation. However, since, apart from J_1^* and $\mathcal{J}_{s1}^*$, the only χ-dependence in the equations is either P_o ($\equiv 1$) or P_2, there are only three basic types of equation to be solved. These are:

$$\frac{d\mathcal{J}^*_{s1o}}{dz} = -4\,\rho^*_o(J^*_{1o} - B^*_1)$$
$$\frac{dJ^*_{1o}}{dz} = -\frac{3}{4}\rho^*_o\mathcal{J}^*_{s1o} - \frac{3}{4}\rho^*_1 - P_2(0)\,\rho^*_o \qquad (6.50)$$

$$\frac{d\mathcal{J}^*_{s12}}{dz} = -4\,\rho^*_o J^*_{12}$$
$$\frac{dJ^*_{12}}{dz} = -\frac{3}{4}\rho^*_o\mathcal{J}^*_{s12} + \rho^*_o \qquad (6.51)$$

and

$$\frac{d\mathcal{J}^*_{s1n}}{dz} = -4\,\rho^*_o J^*_{1n}$$
$$\frac{dJ^*_{1n}}{dz} = -\frac{3}{4}\rho^*_o\mathcal{J}^*_{s1n} \qquad \text{for } n=1 \text{ and } n\geqslant 3 \qquad (6.52)$$

It is useful at this point to introduce the optical depth as a variable. In a perturbation theory, it is convenient to use a zero order optical depth, defined by

$$\tau = \int_z^\infty \rho^*_o\,dz' \qquad (6.53)$$

(cf. equation (5.24)). In terms of this variable, equations (6.52) are easily solved. Application of the boundary conditions (6.29) and (6.30) as $\tau \longrightarrow 0$ and as $\tau \longrightarrow \infty$ respectively then shows that

$$\mathcal{J}^*_{s1n} = J^*_{1n} = 0 \quad \text{for all } z, \text{ for } n=1 \text{ and } n\geqslant 3 \;. \qquad (6.54)$$

This proves that the χ-dependence of $\mathcal{J}^*_s$ and J^* consists of P_o and P_2 terms only, a result that would be expected intuitively from the form of the equations. It is now obvious that the use of the expansions

(6.48) and (6.49) considerably simplifies the first order theory, without any loss of generality. It can be shown that in the pth order theory Legendre polynomials of even order up to 2p are required to represent J^* and $\mathcal{J}_s^*$. An expansion in Legendre polynomials is therefore not a simplification in the general theory, which is essentially an infinite order perturbation theory, and in which, therefore, all the even Legendre polynomials would be required.

$\mathcal{J}_{s12}^*$ and J_{12}^* are the only first order functions in equations (6.51), which can therefore be solved independently of the other first order equations. In terms of the variable τ, the equations are

$$\frac{d\mathcal{J}_{s12}^*}{d\tau} = 4\,J_{12}^* \qquad (6.55)$$

$$\frac{dJ_{12}^*}{d\tau} = \frac{3}{4}\,\mathcal{J}_{s12}^* - 1 \qquad (6.56)$$

It is easy to obtain from these equations the second order differential equation for $\mathcal{J}_{s12}^*$

$$\frac{d^2\mathcal{J}_{s12}^*}{d\tau^2} - 3\,\mathcal{J}_{s12}^* = -4 \qquad (6.57)$$

whose general solution is

$$\mathcal{J}_{s12}^* = \frac{4}{3} + A_2\,e^{+\sqrt{3}\tau} + A_3\,e^{-\sqrt{3}\tau} \,. \qquad (6.58)$$

The boundary conditions are (equations (6.29) and (6.30))

$$\mathcal{J}_{s12}^* = 2\,J_{12}^* \quad \text{at} \quad \tau = 0\,; \qquad \mathcal{J}_{s12}^* \rightarrow \frac{4}{3} \text{ as } \tau \longrightarrow \infty \qquad (6.59)$$

The condition as $\tau \longrightarrow \infty$ requires $A_2 = 0$. Equation (6.55) then gives

$$J_{12}^{*} = -\frac{\sqrt{3}\,A_3}{4}\, e^{-\sqrt{3}\,\tau} \qquad (6.60)$$

and it is easily shown that the condition at $\tau = 0$ requires $A_3 = -\frac{8}{3}(2-\sqrt{3})$, so that

$$\mathcal{J}_{s12}^{*} = \frac{4}{3}\left(1 - 2(2-\sqrt{3})\, e^{-\sqrt{3}\,\tau}\right) \qquad (6.61)$$

$$J_{12}^{*} = \frac{2}{\sqrt{3}}(2-\sqrt{3})\, e^{-\sqrt{3}\,\tau} \qquad (6.62)$$

The equations for $\mathcal{J}_{slo}^{*}$ and J_{1o}^{*} involve ρ_1^{*} and B_1^{*} and so must be solved in conjunction with the other four first order equations. It follows at once from equations (6.28) and (6.48) that

$$B_1^{*} = J_{1o}^{*} \quad . \qquad (6.63)$$

This simplifies the first of equations (6.50) which, together with the boundary condition (6.30), now gives

$$\mathcal{J}_{slo}^{*} = \ell_1 - \frac{2}{3} P_2(0) \quad . \qquad (6.64)$$

This result may be combined with the boundary condition (6.29) for J_1^{*} and equations (6.63) and (6.25) to give

$$\frac{t_1}{t_o} \rightarrow \frac{1}{4}\ell_1 - \frac{1}{6} P_2(0) \qquad \text{as } z \rightarrow +\infty . \qquad (6.65)$$

The functions J_{1o}^{*}, B_1^{*} may be eliminated from equations (6.63), (6.50) and (6.25) to give (using also equation (6.64))

$$\frac{d}{dz}(4t_o^3 t_1) = -\frac{3}{4}\rho_1^{*} - \left(\frac{3}{4}\ell_1 + \frac{1}{2} P_2(0)\right)\rho_o^{*} \qquad (6.66)$$

This equation must now be solved in conjunction with equations (6.23) and (6.24).

Elimination of ρ_1^* between equations (6.23) and (6.66), and the use of equation (6.15) and the boundary conditions for p_1, p_o, t_1 and t_o as $z \longrightarrow +\infty$ leads to the relation

$$16\, t_o^4 \frac{t_1}{t_o} = 3\, p_1 + (2+3\ell_1)\, p_o + 2\ell_1 - \frac{4}{3} P_2(0). \quad (6.67)$$

If ρ_1^* is now eliminated between equations (6.23) and (6.24), the relation (6.67) may be used in the resulting equation to obtain the following differential equation involving only p_1 and zero order functions:

$$\frac{dp_1}{dz} + p_1 \rho_o^* \left(\frac{1}{p_o} - \frac{3}{16 t_o^4} \right) = \rho_o^* \left[\frac{(2+3\ell_1)p_o + 2\ell_1 - \frac{4}{3} P_2(0)}{16 t_o^4} + \frac{2}{3}(1 - P_2(0)) \right] \quad (6.68)$$

The use of equations (6.15) and (6.32) shows that $\frac{1}{\rho_o^*}$ is an integrating factor for this equation. When the equation has been divided through by ρ_o^*, the R.H.S. can be expressed as a function of t_o only by using equation (6.34) for p_o. The R.H.S. may then be integrated by using t_o as variable of integration, the change from z to t_o being effected through equation (6.32) (using also equations (6.16) and (6.34)). The integral which occurred in section 2 appears again, and equation (6.40) may be used to simplify the final result for p_1 to

$$p_1 = \rho_o^* \left[A_4 + (\ell_1 - \tfrac{2}{3}P_2(0))2^{-1/4}\frac{\pi}{4} - \tfrac{2}{3}(1+P_2(0))t_o + \tfrac{2}{3}(1 + \tfrac{3}{8}\ell_1 - \tfrac{5}{4}P_2(0))\, z \right] \qquad (6.69)$$

where A_4 is an arbitrary constant. This constant must be determined by a condition as $z \longrightarrow -\infty$, since the conditions as $z \longrightarrow +\infty$ have already been invoked and are satisfied for any value of A_4. The only condition available is condition (6.30) for t_1. If equations (6.69) and (6.67) are combined to give an expression for t_1, the application of condition (6.30) shows that $A_4 = -(\ell_1 - \frac{2}{3}P_2(0))\, 2^{-1/4}\frac{\pi}{4}$. It may then be shown that

$$p_1 = \tfrac{2}{3}\left(1 + \tfrac{3}{8}\ell_1 - \tfrac{5}{4}P_2(0)\right) z \frac{p_o}{t_o} - \tfrac{2}{3}(1 + P_2(0))\, p_o \qquad (6.70)$$

$$\rho_1^* = \tfrac{1}{2}\left(1 + \tfrac{3}{8}\ell_1 - \tfrac{5}{4}P_2(0)\right) z \frac{\rho_o^*}{t_o}\left(1 + \tfrac{1}{6}\frac{1}{t_o^4}\right) - \left(\tfrac{2}{3} + \tfrac{1}{4}\ell_1 + \tfrac{1}{2}P_2(0)\right)\rho_o^* \qquad (6.71)$$

$$t_1 = \tfrac{1}{8}\left(1 + \tfrac{3}{8}\ell_1 - \tfrac{5}{4}P_2(0)\right) z \frac{p_o}{t_o^4} + \left(\tfrac{1}{4}\ell_1 - \tfrac{1}{6}P_2(0)\right) t_o \qquad (6.72)$$

It then follows at once that

$$J_{1o}^* = B_1^* = \tfrac{1}{2}(1 + \tfrac{3}{8}\ell_1 - \tfrac{5}{4}P_2(0)) z \frac{p_o}{t_o} + (\ell_1 - \tfrac{2}{3}P_2(0))\, t_o^4 \quad . \qquad (6.73)$$

That completes the solution of the first order equations. However, before using these results to find the first order velocity, one feature of the solution should be noted. Although all the first order functions are finite as $z \longrightarrow +\infty$, the same is not true for the ratios p_1/p_o and ρ_1^*/ρ_o^*, which tend to infinity with z. This result follows at once from equations (6.70) and (6.71), since $t_o \longrightarrow 2^{-1/4}$ as $z \longrightarrow \infty$. The ratio t_1/t_o causes no trouble, because p_o tends exponentially to zero as $z \longrightarrow \infty$ (see equation (6.41)).

The singularities in these ratios mean that it is not valid to use perturbation methods throughout the atmosphere, and it is important to have an estimate of where the analysis breaks down. It can be shown that $\dfrac{p_1/p_o}{\rho_1^*/\rho_o^*} \longrightarrow 1$ as $z \longrightarrow +\infty$, so that consideration of either ratio will give the same result. For simplicity, consider $\dfrac{p_1}{p_o}$. If the values for $P_2(0)$ and ℓ_1 given in Appendix VI are put into equation (6.70), it gives

$$\frac{p_1}{p_o} = 0.35\,\frac{z}{t_o} - 0.8 \quad . \tag{6.74}$$

Since t_o is a monotonically decreasing function of z (see Appendix VIII), it is clear that p_1/p_o increases monotonically with z. Also, it has a lower bound (= -2.2), to which it tends asymptotically as $z \longrightarrow -\infty$, since $t_o \sim -\frac{1}{4} z$ as $z \longrightarrow -\infty$. p_1/p_o therefore has the form shown in Fig. 13. If perturbation theory is to be assumed valid for all

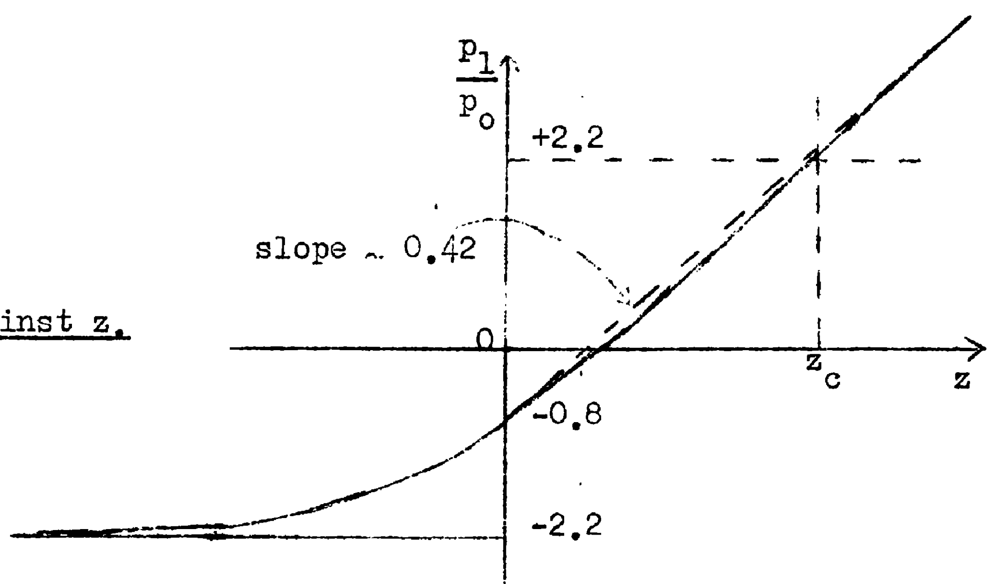


Fig. 13.

Graph of p_1/p_o against z.

negative* z, then it must, for consistency, be assumed valid also at least up to $z = z_c$, where $p_1/p_o = +2.2$. That value of z will give a lower bound to the values of z for which the perturbation theory breaks down. Since $t_o \doteq 2^{-1/4} = 0.84$ for all $z \gtrsim 1$ (see Appendix VIII), it follows that

$$\frac{p_1}{p_o} = +2.2 \quad \text{at} \quad z = z_c \doteq 7.2 \tag{6.75}$$

and so the perturbation theory is only invalid outside the range of z in which significant changes occur (cf. Chapter 5). It can be shown that z = 7.2 corresponds to an optical depth of 3.6×10^{-6}, so that the perturbation method breaks down only in a physically insignificant fraction of the atmosphere. The above theory may therefore be used to find the circulation currents in the bulk of the atmosphere.

4. The circulation currents

The exact equations for the circulation currents are equations (4.31) and (4.83). In the plane-parallel approximation these equations reduce to

$$v_s = 4\,\frac{\mu R^2}{GM}\sigma T_e^4 \frac{(J^* - B^*)}{|\nabla s|\left[\frac{\gamma}{\gamma-1}\frac{dt}{dz} + 1 - \frac{2}{3}\epsilon(1-P_2(0))\right]} \tag{6.76}$$

and

$$v_\chi = -\frac{D|\nabla s|}{\epsilon_1 \rho^*}\int_0^\chi \frac{1}{C\;D}\,\frac{\partial}{\partial z}(\rho^* v_s)\,d\chi' \tag{6.77}$$

---

* For $\epsilon \sim 0.1$, $|\epsilon p_1/p_o| \lesssim 0.22$ for all negative z, a result consistent with the assumptions of perturbation theory.

where $|\nabla s|$, C and D are given by equations (6.9) to (6.11). Since $J_o^* = B_o^*$ (equation (6.20)), v_s and v_χ can be obtained correct to first order in ϵ without using the first order structure. That is,

$$v_{s1} = 4\,\frac{\varkappa R^2}{GM}\sigma T_e^4\,\frac{(J_1^* - B_1^*)}{\left[\frac{\gamma}{\gamma-1}\frac{dt_o}{dz} + 1\right]} \tag{6.78}$$

and

$$v_{\chi 1} = -\frac{1}{\sin\chi\,\rho_o^*\,\epsilon_1}\int_0^\chi \sin\chi'\frac{\partial}{\partial z}(\rho_o^* v_{s1})\,d\chi' \tag{6.79}$$

Equations (6.63) and (6.62) can be used to obtain $J_1^*-B_1^*$ $(= J_{12}^* P_2)$ as a function of τ; also, although it is not possible to write dt_o/dz explicitly in terms of z, it can be shown that $\frac{dt_o}{dz} = -\frac{3}{16}\,p_o/t_o^4$. It is therefore possible to write v_s as a function of τ and χ, since it follows at once from equations (6.15) and (6.53) that

$$p_o = \tau \tag{6.80}$$

and then from equation (6.34) that

$$t_o^4 = \tfrac{1}{2}(1 + \tfrac{3}{2}\tau) . \tag{6.81}$$

If γ is taken to be 5/3, it follows that

$$v_{s1} = \frac{8}{\sqrt{3}}(2-\sqrt{3})\,\frac{\varkappa R^2}{GM}\sigma T_e^4\,\frac{(1+3\tau/2)}{(1+9\tau/16)}\,e^{-\sqrt{3}\tau}P_2(\cos\chi) . \tag{6.82}$$

It may then be shown that

$$v_{\chi 1} = \frac{4(2-\sqrt{3})}{\sqrt{3}\,2^{-1/4}}\,\frac{\varkappa R^2}{GM}\frac{\sigma T_e^4}{\epsilon_1}\sin\chi\cos\chi\;\frac{e^{-\sqrt{3}\tau}\left[1+\frac{3}{2}\tau+\frac{9}{8}\tau(1+\frac{9}{16}\tau) - \sqrt{3}\tau(1+\frac{9}{16}\tau)(1+\frac{3}{2}\tau)\right]}{(1+\frac{9}{16}\tau)^2(1+\frac{3}{2}\tau)^{1/4}} \tag{6.83}$$

These expressions are, as expected, non-singular at $\tau = 0$. They are not zero there, but before calculating the surface values of $\underline{v}$ it is worth considering some properties of the flow pattern.

The signs of v_{s1} and $v_{\chi 1}$ are such that the circulation currents rise at the poles and sink at the equator. Since $v_{\chi 1}$ changes sign (at $\tau = 0.637$, $z \doteq -3$), the mass flow to the equator near the surface returns at greater depths. Although there is no net mass flow across the surface $\tau = 0$, this surface is not a stream line of the flow, since v_{s1} is finite there. However, since $v_{s1} \sim \epsilon_1 v_{\chi 1}$, the ratio of the normal mass flow to the horizontal mass flow is small at any point on the surface $\tau = 0$. The flow pattern therefore approximates to a single closed cell in each hemisphere.

This result must be compared with that of the local theory, since it would be expected that the two results would agree at great depths. There is no doubt that the formal, exact non-local expression for v_s, given by equation (4.31), reduces to the corresponding local expression, given by equation (3.25), as $z \to -\infty$. This has been verified by applying the method used by Chandrasekhar (1939 p. 208) to prove the well-known fact that the non-local transfer equation reduces to the local one at great depths. It is further guaranteed by the boundary conditions for the non-local theory, which ensure that the non-local structure variables fit asymptotically to their local counterparts for large negative z.

Nonetheless, the flow pattern predicted by the local theory is qualitatively different from that described above. It is easily seen from equations (A7.41), (A7.42) that, on the local theory, the circulation currents sink at the poles and rise at the equator (Fig. 6). Also, since v_{χ} does not change sign, the local theory contains no mechanism for returning material to the poles. (This defect of the local theory has not previously been mentioned, as it is of lesser importance than the singularity at $z = 0$. It is, however, another reason for seeking a better description of the circulation.)

At first sight, the explanation of the disagreement would seem to be that the non-local expressions are first order in ϵ, while the local expressions are second order in ϵ, the first order terms being identically zero. Since v_{s1} and $v_{\chi 1}$ on the non-local theory die out exponentially for large τ, there is agreement between the theories to first order in ϵ. One would imagine, then, that, if the non-local velocity were obtained to second order in ϵ, the second order terms would tend asymptotically to the lowest order local expressions. This would require that the second order non-local expressions had terms which behaved like $1/|z|^3$ for large negative z. With this in mind, the second order theory was calculated. However, it turns out that, when the plane-parallel approximation is used, the second order expressions for the non-local velocity also die out exponentially for large τ (N.B. $\tau = O(z^4)$ as $z \rightarrow -\infty$). This immediately rules out the obvious explanation of the disagreement.

The real reason for the disagreement is that the non-local expression for the velocity is singular in the sense used in Osaki's paper (1966). That is, order of magnitude estimates of the non-local and local velocities show that the non-local velocities are greater than the local ones by a factor of the order of $1/\epsilon_1$. Thus, if expressions for the velocities are derived which neglect terms of order ϵ_1 times the terms retained, the lowest order non-local expressions cannot be expected to fit the lowest order local expressions for large τ. The terms neglected in the non-local expressions are precisely those which will tend asymptotically to the local expressions at great depths, where the lowest order terms will be negligible because of their exponential decrease.

The plane-parallel approximation is therefore inadequate for a full description of the non-local velocity field, since it fails to take account of terms which, though negligible near the surface, become important at large optical depths. Unfortunately, it is not a simple matter to include such terms, which represent the effect of curvature, in the non-local theory, as was pointed out in Chapter 5. If terms of relative order ϵ_1 are to be found by using perturbation methods, it is no longer possible to apply boundary conditions as $z \rightarrow \pm\infty$. No obvious alternative presents itself immediately; besides, any attempt to extend the theory to higher order in ϵ_1 rapidly produces equations which are soluble only by numerical methods. Such an attempt was therefore abandoned, since this thesis is concerned only with obtaining a qualitative picture of the surface layers, and a qualitative picture can be obtained without including higher order terms in the non-local theory. Notice, incidentally, that to obtain a complete picture the non-local theory would have to be solved to second order in ϵ_1, since $v_\chi \sim \frac{1}{\epsilon_1} v_s$ and so v_χ is of order $\left(\frac{1}{\epsilon_1}\right)^2$ times the local v_χ. Terms of order ϵ_1^2 times the lowest order expression would be needed before v_χ would be seen to fit to the local v_χ at large depths.

However, a judicious combination of the local and plane-parallel non-local theories suffices to show qualitatively what is the flow pattern of the circulation near the surface. The plane-parallel non-local theory predicts a single closed cell. However, since the sign of v_s on the local theory is opposite to that on the non-local theory, it is clear that there must in fact be a circulation reversal near the surface. In a more accurate treatment of the non-local theory, the terms of order ϵ_1 must be of the opposite sign to the lowest order terms. Since they decrease more slowly with depth than the lowest order terms (as they must to fit the local theory), there must be a depth at which the first order and lowest order terms balance and the circulation reverses. This depth cannot be found exactly without solving the non-local theory to higher order in ϵ_1, but, since the first order terms tend asymptotically to the lowest order local expression, the depth of the reversal may be roughly estimated by equating the lowest order expressions for v_s on the local and non-local theories (equations (6.82) and (A7.41)). The first order expression (in ϵ) can be used for the non-local theory, since it will always be greater, to lowest order in ϵ_1, than the lowest order ϵ^2 term.

It is found that the two expressions for v_s are roughly equal at $\tau = 5.20$ ($z = -5.55$). This means that the circulation reversal occurs much nearer the surface than that predicted by the work of Öpik (1951) and Mestel (1966). That reversal,* which occurs at a depth given by $\rho = \Omega^2/2\pi G$, is not predicted by the present theory, since the Roche approximation is not valid at such a depth.

The present reversal is therefore a qualitatively new result, stated here for the first time, although the contradiction between the local and non-local theories has been noticed by Osaki (1966, and private communication). Taken in conjunction with the Öpik-Mestel reversal, the new result suggests that the final flow pattern should be as shown in Fig. 14, which is not drawn to scale.

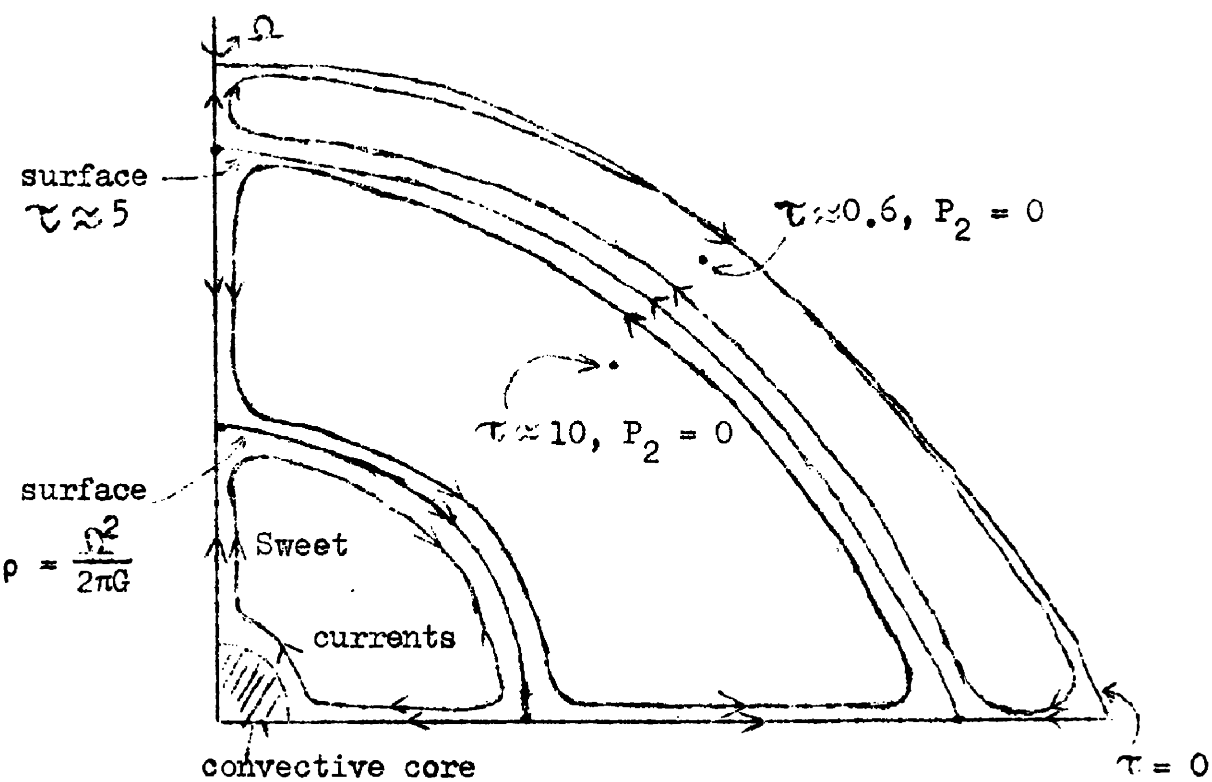


Fig. 14. Circulation pattern in the absence of turbulence.

* See also Gratton (1945).

Now that the pattern of the flow has been determined, it is necessary to find the speed of the flow. Since this is obviously largest at the surface, it is convenient first to evaluate v_s and v_χ at $\tau = 0$. From equations (6.82), (6.83) and (2.19)

$$v_s(0) = \frac{2}{\sqrt{3}}(2-\sqrt{3}) \frac{\varkappa L_o}{\pi GM} \epsilon \, P_2(\cos\chi) \qquad (6.84)$$

and

$$v_\chi(0) = \frac{2^{1/4}}{\sqrt{3}}(2-\sqrt{3}) \frac{\varkappa L_o}{\pi GM} \frac{\epsilon}{\epsilon_1} \sin\chi\cos\chi \qquad (6.85)$$

to lowest order in ϵ and ϵ_1. Using the values quoted in Chapter 3 for the other quantities, one finds that

$$v_s(0) = 8.1\times10^8 \epsilon \, P_2(\cos\chi) \text{cm/sec} \qquad (6.86)$$

and

$$v_\chi(0) = 2.8\times10^{11} \epsilon \sin 2\chi \text{ cm/sec} \quad . \qquad (6.87)$$

While the result for $v_s(0)$ is almost plausible for ϵ as large as 0.1, the result for $v_\chi(0)$ is clearly unacceptable unless ϵ is 10^{-4} or less. Such values are uninterestingly small, and it is now necessary to ask what additional mechanism can be invoked near the surface to damp the circulation.

Since the predicted speeds are considerably larger than the speeds estimated in Chapter 3, using the local theory, it is no longer necessarily true that viscous and inertial forces are negligible, and that assumption should be re-examined. Further, the discussion at the end of Chapter 3 showed that the flow might be turbulent even when its speed was low enough that viscous and inertial forces could be ignored. In that case, turbulent viscosity could be an important damping agent. It seems even more likely now that turbulence will appear, and that possibility will be tested first by considering the stability of the flow.

5. The Richardson number

The exponential decrease of v_χ with depth shows that large gradients are present in the horizontal flow. The strong shear forces implied by such a flow might be expected to give rise to instability, in particular to the Kelvin-Helmholtz instability, whose source lies in the energy stored in the kinetic energy of relative motion of different layers. Inertia will prevent this energy from making the flow unstable so long as the Richardson number J_R is greater than 1/4 (see, e.g., Chandrasekhar 1961), the Richardson number being defined (cf. Chapter 3) by

$$J_R = \frac{g_o\left(-\frac{d\rho_o/ds}{\rho_o}\right)}{(dv_\chi/ds)_o^2} \quad . \tag{6.88}$$

If, however, J_R is less than 1/4 the inertia may not be able to prevent different layers of the flow from intermingling. In that case, the flow will very rapidly become turbulent.

It is possible to evaluate J_R, using the results of this chapter. Equations (6.16), (6.80), (6.81) and (6.83) may be combined with the result that $g_o = GM/R^2$ to give

$$J_R = \frac{GM\,\epsilon_1}{Rv_\chi^2(0)2^{1/4}} \frac{(1+\frac{9}{8}\tau)(1+\frac{9}{16}\tau)^6(1+\frac{3}{2}\tau)^{7/4}\; e^{2\sqrt{3}\tau}}{\tau^2\;(\,A(\tau) + B(\tau)\,)^2} \qquad (6.89)$$

where

$$A(\tau) = (1+\tfrac{9}{16}\tau)(1+\tfrac{3}{2}\tau)(\tfrac{21}{8} - 2\sqrt{3} + (\tfrac{273}{64} - \tfrac{27\sqrt{3}}{4})\tau + (\tfrac{99}{16} - \tfrac{405\sqrt{3}}{128})\tau^2 + \tfrac{81}{32}\tau^3\,)$$

and

$$B(\tau) = -\tfrac{3}{2}(1+\tfrac{81}{64}\tau)(1+(\tfrac{21}{8} - \sqrt{3})\tau + \tfrac{3}{16}(\tfrac{27}{8} - 11\sqrt{3})\tau^2 - \tfrac{27\sqrt{3}}{32}\tau^3\,).$$

It is not immediately obvious what value this quantity has. However, near $\tau = 0$ the expression reduces to

$$J_R \sim \frac{1.6\times10^{-9}}{\tau^2} \qquad (6.90)$$

where the usual numerical values have been used (including $\epsilon = 0.1$, and taking $\sin^2\chi \sim \cos^2\chi \sim \frac{1}{3}$). J_R is therefore amply greater than 1/4 for small enough τ, but

$$J_R \lesssim \frac{1}{4} \quad \text{if} \quad \tau \gtrsim 8\times10^{-5}\,. \qquad (6.91)$$

Numerical investigation shows that J_R first becomes very much less than 1/4 for larger τ, but increases again as τ increases beyond about 1 and that $J_R \longrightarrow \infty$ as $\tau \longrightarrow \infty$. It is found that

$$J_R \simeq \frac{1}{4} \quad \text{when} \quad \tau \doteq 7.9 \;(z = -6.2). \qquad (6.92)$$

Thus between $\tau \doteq 8\times10^{-5}$ and $\tau \doteq 8$, J_R is less than 1/4, becoming at least as small as 4×10^{-7} (at $z = -4$, $\tau = 4/3$).

As pointed out by Chandrasekhar (1961, Chapter XI), instability need not occur when J_R is less than 1/4. However, J_R is so much smaller than 1/4 throughout a large part of the outer layers of the star that the restraining influence of inertia on the flow must be almost negligible. It therefore seems very likely that the flow becomes turbulent and that turbulent viscosity serves to damp the otherwise excessive speeds.

It is clear, at any rate, that viscous and/or inertial terms can no longer be omitted from the equation of motion. In the next chapter, the relative sizes of the possible additional terms are discussed, and a more detailed development of the most likely model is given. It is found that inertial terms can be important, but that the main damping agent is turbulent viscosity.

As a postscript to this discussion, it should be noted that, strictly speaking, the definition of the Richardson number given in equation (6.88) is valid only in an incompressible medium. However, a similar argument to that of Chandrasekhar (1961, p.491) may be applied to the case of a compressible medium. It is then found (Prandtl 1952, p.382) that the Richardson number should be written

$$J_R = \frac{g\left[\frac{\partial\rho}{\partial s} - \left(\frac{\partial\rho}{\partial s}\right)_{ad}\right]}{\rho\,(\partial v_\chi/\partial s)^2} \qquad (6.93)$$

where $(\partial\rho/\partial s)_{ad}$ is the density gradient for an adiabatic fluid. For a polytropically stratified fluid, the change in the definition is equivalent to replacing $\frac{1}{\rho}\frac{\partial\rho}{\partial s}$ by $\frac{1}{H}\left(\frac{1}{n} - \frac{1}{\gamma}\right)$ where H is the pressure scale height and n is the polytropic index ($p \propto \rho^n$). Near the surface of a star, $n \doteq 3$, so that $\frac{1}{H}\left(\frac{1}{n} - \frac{1}{\gamma}\right) \doteq -0.27 \frac{1}{H}$. This should be compared with $\frac{1}{\rho_o}\frac{d\rho_o}{ds} \doteq -\frac{1}{H}$. Thus the Richardson number for a compressible medium is smaller than that for an incompressible medium, but it is of the same order of magnitude and the correction for compressibility makes little difference to the results found above. No correction for compressibility will be made in the following qualitative model.

CHAPTER 7

Discussion of the turbulent surface layer

Second Witch: "... Like a hell-broth boil and bubble.

All: Double, double toil and trouble,

Fire, burn; and, caldron, bubble."

W. Shakespeare, Macbeth, Act IV Sc. i.

1. General consequences of the failure of the hydrostatic approximation

In the previous chapter it was seen that, if viscous and inertial terms are omitted from the equation of motion, the flow speeds predicted by the non-local theory are unrealistically large, except for rotation speeds which are too slow to be of interest. This strongly suggests that inertial and/or viscous forces cannot be neglected, since these forces might be expected to damp the flow. Since the flow is probably turbulent, according to the Richardson criterion, the dominant damping agent is presumably turbulent viscosity. However, it is worth also considering the influence of radiative viscosity and of inertial effects.

If the assumption is retained that the magnetic field exerts negligible forces in meridian planes, the meridian-plane component of the full steady-state equation of motion is (see, for example, Landau and Lifshitz 1959)

$$(\underline{v}.\underline{\nabla})\underline{v} + \frac{1}{\rho}\underline{\nabla}P = \underline{\nabla}\underline{\Phi} + \Omega^2\underline{\varpi} + \frac{\eta}{\rho}\nabla^2\underline{v} + \frac{1}{\rho}(\zeta + \frac{\eta}{3})\underline{\nabla}(\mathrm{div}\,\underline{v}) \quad (7.1)$$

where ζ and η are coefficients of viscosity. It is usually assumed that $\zeta \ll \eta$, and that assumption will be adopted throughout this chapter. The radiative viscosity is then represented by the single coefficient η, defined (as in Chapter 3) by

$$\eta = \rho\nu = \frac{16\sigma T^4}{15\kappa\rho c^2} \quad . \qquad (7.2)$$

It will be assumed that, in the turbulent case, the effect of the turbulence on the mean flow can be represented also by equation (7.1), where $\underline{v}$ is the <u>mean</u> flow and η is now a coefficient representing turbulent viscosity (ζ is again assumed negligible). This assumption will be further discussed later.

The relative sizes of the various terms in equation (7.1) may be estimated, as in Chapter 3. However, in the present case v_χ and v_Ψ have quite different sizes, since

$$v_\Psi \sim \epsilon_1 v_\chi \quad . \qquad (7.3)$$

It is therefore necessary to be more careful in estimating the sizes of the inertial and viscous terms. Careful investigation shows that the relative sizes of these terms depend on whether the Ψ-component or the χ-component of equation (7.1) is considered. In both cases, it is found that

$$I = \text{inertial terms} \sim \frac{v_\chi^2}{R} \quad . \qquad (7.4)$$

However, the other terms differ, it being found (see Note (a) in Appendix X) that

$$\left.\begin{aligned}
C_{\Psi} &= \text{centrifugal}^{*} \text{ terms in } \Psi\text{-component} \sim \epsilon \frac{GM}{R^2} \\
C_{\chi} &= \quad " \quad " \quad " \ \chi\text{-component} \sim \epsilon \epsilon_1 \frac{GM}{R^2} \\
V_{\Psi} &= \text{viscous} \quad " \quad " \ \Psi\text{-component} \sim \frac{\eta}{\rho} \frac{v_{\Psi}}{H^2} \\
V_{\chi} &= \quad " \quad " \quad " \ \chi\text{-component} \sim \frac{\eta}{\rho} \frac{v_{\chi}}{H^2}
\end{aligned}\right\} \qquad (7.5)$$

It is therefore necessary to treat the two components separately. This is one reason for the difference, which will be found later, between the present model and that of Kippenhahn (1959), who considered only the r-component of the equation of motion.

First of all, a value for η is required. In the case of radiative viscosity, η may be estimated from equation (7.2). Taking $T \sim T_e$ and $\rho \sim \frac{1}{\kappa \epsilon_1 R}$,

$$\left.\begin{aligned}
\eta &\sim \frac{4L\epsilon_1}{15\pi R c^2} \sim 7.4 \text{ gm/cm/sec} \\
\text{and} \qquad \frac{\eta}{\rho} &\sim 1.3 \times 10^9 \ \text{cm}^2/\text{sec}
\end{aligned}\right\} \qquad (7.6)$$

In the case of turbulent viscosity, the order of magnitude estimate

$$\eta \sim \rho \, v_t \, h \qquad (7.7)$$

will be used (see, e.g., Landau and Lifshitz 1959), where h is a length whose size is of the order of the dimensions of the largest eddies in the flow and v_t is of the order of the fluctuation in the mean velocity over the distance h. The choice of values for v_t and h will be discussed later. The usual values will be taken for the

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* See Note (b) in Appendix X.

other parameters of the problem; in particular, the value 0.1 will be taken for ϵ and η will throughout be taken as $1/\kappa\epsilon_1 R$.

If the sizes of the various terms are estimated using

$$\left.\begin{aligned} v_\chi &\approx v_\chi(0)_{max} \sim 2.8\times10^{11}\epsilon \\ v_\Psi &\approx v_s(0)_{max} \sim 8.1\times10^{8}\epsilon \end{aligned}\right\} \qquad (7.8)$$

it is found that, for example,

$$\frac{I}{C_\Psi} \sim 1.8\times10^{6} \quad . \qquad (7.9)$$

This merely emphasises the extent of the contradiction which has been produced by assuming that inertial and viscous forces can be neglected near the surface. A more useful procedure is to estimate at what speeds the inertial and viscous terms become comparable with the centrifugal terms.

It is reasonable to suppose that, whatever the details of the model, the flow speeds increase as the surface is approached until either the inertial or the viscous terms become comparable with the centrifugal terms, and that from this point to the surface there is no further increase in the order of magnitude of the flow speeds. With this in mind, note that the inertial or viscous terms always become important first in the χ-component of the equation of motion, since

$$\left.\begin{aligned} \frac{I}{C_\chi} &\sim \frac{1}{\epsilon_1}\frac{I}{C_\Psi} \gg \frac{I}{C_\Psi} \\ \text{and} \qquad \frac{V_\chi}{C_\chi} &\sim \frac{1}{\epsilon_1^2}\frac{V_\Psi}{C_\Psi} \ggg \frac{V_\Psi}{C_\Psi} , \end{aligned}\right\} \qquad (7.10)$$

the latter result holding both for radiative and for turbulent viscosity since it is independent of η. It is therefore sufficient to restrict one's attention to the χ-component, since, if the flow speeds do not increase significantly after I/C_χ or V_χ/C_χ have become of order unity, inertial and viscous terms will never be important in the Ψ-component.

Calculation shows that, for radiative viscosity,

$$\frac{I}{V_\chi} \sim \frac{\epsilon_1 H v_\chi}{\eta/\rho} \gtrsim 1 \quad \text{if} \quad v_\chi \gtrsim 3.5 \times 10^5 \text{cm/sec} \tag{7.11(a)}$$

Also

$$\frac{I}{C_\chi} \sim \frac{R}{\epsilon \epsilon_1 GM} v_\chi^2 \sim 1 \quad \text{if} \quad v_\chi \sim 6.1 \times 10^5 \text{ cm/sec} \tag{7.11(b)}$$

Thus, if only the case of radiative viscosity is considered, the inertial terms in the equation of motion appear to determine the flow, damping it to a quite reasonable speed.

However, although the viscous terms are negligible in the equation of motion, it must presumably be viscosity which dissipates the energy of the flow and it is necessary to consider the effect of viscous dissipation on the thermal balance in the surface layers. The thermal balance equation must now be written:

$$\frac{P}{\gamma-1} \underline{v}.\text{grad} \log (P/\rho^\gamma) = - \text{div} \underset{\sim}{\mathcal{F}}_{rad} + \rho\varepsilon_{dissip} \tag{7.12}$$

where

$$\rho\varepsilon_{dissip} = \frac{1}{2} \eta \left[\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3} \delta_{jk} \frac{\partial v_m}{\partial x_m} \right]^2 \tag{7.13}$$

is the viscous dissipation in Cartesian coordinates. It is not easy to transform this expression into the (s, χ, ϕ) coordinate system, but there is little doubt that the largest term will be $\frac{1}{2}\eta\,(\partial v_\chi/\partial s)^2$.

It is easy to show that

$$\text{L.H.S. of (7.12)} \sim \frac{P v_\chi}{R} \sim H\rho\,\frac{GM}{R^2}\,\frac{v_\chi}{R} \quad , \tag{7.14}$$

using the approximation grad $P = \rho$ grad Φ to obtain the second expression. If v_χ is given by equation (7.11)(b), and it is assumed that $\rho \sim 1/\kappa\epsilon_1 R$, it is found that

$$\text{L.H.S. of (7.12)} \sim 2.5 \times 10^{-2} \tag{7.15}$$

whereas

$$\text{div}\,\underset{\sim}{\mathfrak{F}} \sim \epsilon\frac{\sigma T_e^{4}}{H}{}^{*} \sim 3.1 \times 10^{3} \;\; \text{for } \epsilon = 0.1 \; . \tag{7.16}$$

This means that equation (7.12) can now be written approximately as

$$\text{div}\,\underset{\sim}{\mathfrak{F}} = \rho\varepsilon_{dissip} \quad . \tag{7.17}$$

However, if the viscosity is assumed to be radiative, so that $\eta = 7.4$, it is found that with $v_\chi = 6.1 \times 10^5$ cm/sec

$$\rho\varepsilon_{dissip} \sim \frac{1}{2}\eta\left(\frac{v_\chi}{H}\right)^2 \sim 6.9 \times 10^{-6} \quad . \tag{7.18}$$

This result shows that the equation of thermal balance cannot be satisfied on this model.

Presumably, then, the speeds continue to increase (but see Note (c) in Appendix X), even though the inertial and centrifugal

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* This estimate holds only in the non-local theory. In the local theory, $\text{div}\,\underset{\sim}{\mathfrak{F}} \sim \epsilon\sigma T_e^4/R$. However, even in that case the L.H.S. of equation (7.12) is less than $\text{div}\,\underset{\sim}{\mathfrak{F}}$.

terms are comparable, until the Richardson number becomes less than 1/4. The flow will then become turbulent, and turbulent viscosity will damp the flow. It will be found that turbulent viscosity is a much more efficient dissipative agent and that equation (7.17) can be satisfied at quite reasonable speeds.

The Richardson number may be taken to be (see Chapter 6)

$$J = \frac{g(-\rho'/\rho)}{\left(\frac{\partial v_\chi}{\partial s}\right)^2}$$

or, assuming $g \sim GM/R^2$, $\rho'/\rho \sim 1/H$ and $\partial v_\chi/\partial s \sim v_\chi/H$,

$$J \sim \frac{\epsilon_1 GM/R}{v_\chi^2} \qquad (7.19)$$

It is well known that in a region in thermal convection the temperature gradient stays very close to the value at which the region becomes unstable. It seems reasonable to assume, by analogy, that the flow in the present case is damped by turbulence to such an extent that the Richardson number never becomes much less than 1/4. In that case, v_χ may be estimated roughly by putting $J = 1/4$, so that

$$v_\chi^2 \sim 4\epsilon_1 \frac{GM}{R} \qquad (7.20)$$

and

$$v_\chi \sim 3.8 \times 10^6 \text{ cm/sec} \quad . \qquad (7.21)$$

The flow is therefore supersonic, since the speed of sound, $\bar{c}$, is given by

$$\bar{c}^2 \sim \gamma \frac{P}{\rho} \sim \frac{5}{3} H \frac{GM}{R^2} \sim \frac{5}{3} \epsilon_1 \frac{GM}{R}, \qquad (7.22)$$

and so

$$\bar{c} \sim 2.5 \times 10^6 \text{ cm/sec} \quad \text{and} \quad v_\chi \sim 1.5\ \bar{c} \ . \tag{7.23}$$

However, the factor 1.5 is probably not significantly different from 1, since no correction has been made for compressibility and also the critical Richardson number is slightly uncertain.

This result differs appreciably from that of Osaki (1966), who found $v_\chi \sim 10^{4.5}$ cm/sec. However, it is clear from the present discussion that such a flow would not violate the Richardson criterion, and so could not be unstable. Osaki's model is therefore internally inconsistent, in that he finds that turbulent viscosity damps the flow to such an extent that it is no longer turbulent. This inconsistency would have shown itself if Osaki had considered in more detail the equation of thermal balance. The differences between his model and the present one will be further discussed in the next section.

A point on which the two models are basically in agreement is the treatment of the turbulence. This is a vexed question, to which there is, as yet, no satisfactory answer, even in the usual problem of a convectively unstable region. In the present problem, that of turbulence in a region with a radiatively stable temperature gradient, it is even less clear how to proceed. In the absence of any recognised procedure, it seems best to represent the turbulence by the simplest possible model. At the very least the solution may give some information about the validity of the model. It is to be hoped that it also gives a qualitative description of the turbulent region.

When the time-dependent equation of motion which describes the instantaneous state of the turbulent flow is averaged over time, one obtains a time-independent equation for the averaged quantities which is of the form of equation (7.1) except that the viscous terms are replaced by a number of terms involving the fluctuations due to the turbulence. The simplest way of dealing with these terms is to define a turbulent viscosity $\eta_{effective}$ by the condition that the extra terms can be written in exactly the same way as the viscous terms involving η in equation (7.1) except that the coefficient of radiative viscosity is replaced by $\eta_{effective}$. In principle, this definition determines $\eta_{effective}$, although the exact expression would be complicated and not very useful. In practice, $\eta_{effective}$ is simply assumed to be a constant. Dimensional arguments (see, e.g., Landau and Lifshitz 1959, p.119) show that $\eta_{effective}$ can be represented by an expression of the form of equation (7.7); v_t and h can then be determined by order of magnitude considerations. This will be done in the next section.

A similar treatment of turbulence was given by Kippenhahn (1959), who considered meridional circulation in zones of weak convection. However, his results agree neither with Osaki's nor with those discussed here, although his estimate of v_t is nearer to the result of this chapter than to Osaki's result. The difference between his result and the present one arises for three main reasons. Firstly, Kippenhahn considered only the r-component of the equation of motion, whereas it is clear from the present work that the χ-component is more important. He also assumed that v_t was of the order of v_r, rather than v_χ. Since he used a local theory of radiative transfer, he could assume $v_\chi \sim v_r$, but this in itself is a difference between the models used by Kippenhahn and the writer. Finally, Kippenhahn, like Osaki, did not consider what was the origin of the turbulence which he postulated. Thus, although the turbulence was introduced as a braking mechanism near the boundaries of the convective zone, the instability which could cause the turbulence was not considered. Assuming that $v_\chi \sim v_r$, Kippenhahn's flow does not violate the Richardson criterion and so his model, like Osaki's, is internally inconsistent. Nonetheless, it is not clear what effect the presence of thermal convection would have on the flow, and the writer does not wish to claim that the present model necessarily applies to Kippenhahn's case without modification.

One other point must be considered here. In the presence of turbulence, it is unlikely that a magnetic field can remain undistorted. No theory of turbulence is sufficiently detailed to enable one to say exactly what happens to the field, but the numerical experiments conducted by Weiss (1966) strongly suggest that magnetic field lines are expelled from turbulent regions. If that is true, as will be assumed here, there is no need to consider magnetic forces in the equation of motion. This is a simplification in one respect, since with such large flow speeds the magnetic field required to balance the Coriolis force in the toroidal component of the equation of motion would be so large as to have significant effects in the poloidal component as well (cf. Mestel 1965). On the other hand, if a magnetic field is no longer present the Coriolis force is free to destroy the uniformity of the rotation, and it is necessary to complicate the model by allowing for non-uniform rotation. This is equivalent to allowing the flow to have a non-constant ϕ-component, v_ϕ . Presumably the gradient of v_ϕ will be limited by the Richardson criterion in the same way as the gradient of v_χ . This will prevent the non-uniformity of the rotation from becoming too large.

It might also be argued that, since, in the absence of any external forces, the rotation law would be determined by the conservation of angular momentum, the rotation law is likely to adjust itself to be as close to angular momentum conservation as possible. However, the turbulence is driven by a powerful force, capable of producing motions in a non-viscous medium of the order of the speed of light, and it is therefore likely that the effect of turbulence will be to prevent the rotation law from adjusting to angular momentum conservation. Besides, it is easy to see that, if streamlines of the flow enter and leave the underlying uniformly rotating region, the angular momentum $\Omega \varpi^2$ cannot be a constant on a streamline since, on the boundary of the uniformly rotating region, Ω is constant but ϖ is not.

2. A self-consistent order-of-magnitude model

The order-of-magnitude estimates for the terms in the equation of motion (equations (7.4),(7.5)) were based on the assumption that the rotation was uniform. Suppose now that

$$\Omega = \Omega_0 + \Omega_1 \tag{7.24}$$

where Ω_0 is a constant and Ω_1 is a function of s and χ. Then there are various extra terms, containing Ω_1, in the equation of motion. So long as $\Omega_1 < \Omega_0$, it can be shown that these terms can be ignored in the Ψ-component. In the χ-component, an extra term appears, whose relative size cannot be determined without further information. It is therefore necessary to write

$$c_\chi \sim \epsilon \frac{GM}{R^2}\left(\epsilon_1 + \frac{\Omega_1}{\Omega_0}\right) \qquad (7.25)$$

In the ϕ-component, which has not previously been considered, the non-uniform rotation contributes a viscous force which, in the absence of magnetic forces, just balances the Coriolis force due to the circulation. The ϕ-component can be written exactly as

$$\left.\begin{aligned} \underline{v}.\mathrm{grad}(\Omega\, r^2\sin^2\theta) &= \frac{\eta}{\rho}\nabla_1^2(\Omega_1 r^2\sin^2\theta) \\ \text{where} \quad \nabla_1^2 &\equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\,\frac{\partial}{\partial\theta}\right) \end{aligned}\right\} \qquad (7.26)$$

Ω_1 may be written instead of Ω on the R.H.S. of this equation because $\nabla_1^2(\Omega_0 r^2\sin^2\theta) \equiv 0$. This equation may therefore be used to estimate Ω_1 by writing it as

$$v_\chi \Omega_0 R \sim \frac{\eta}{\rho} R^2 \frac{\Omega_1}{H^2} \quad . \qquad (7.27)$$

To obtain this approximate form of equation (7.26) it has been assumed that the scale height H is the scale for the vertical variation of Ω_1, that $v_\Psi \sim \epsilon_1 v_\chi$ and that $\Omega_1 < \Omega_0$.

It is now necessary to decide what value to take for η, which has been defined by

$$\eta \sim \rho \, v_t \, h \quad . \tag{7.7}$$

The turbulence breaks the flow up into eddies, and v_t and h are typical values for the variation in speed across an eddy and the size of an eddy respectively. It is not necessarily reasonable to suppose, as Osaki does, that $h \sim H$; with a driving force capable of producing speeds up to the speed of light, the flow must be expected to break up into many eddies, with a typical size considerably less than the extent of the turbulent region, which is a few scale heights deep. It is more reasonable to suppose that the typical eddy size is determined by the need to maintain thermal balance. The typical eddy size is presumably the suitable scale for the estimation of the viscous dissipation, so that one may write

$$\rho\varepsilon_{\text{dissip}} \sim \frac{1}{2}\,\eta\left(\frac{\partial v_\chi}{\partial s}\right)^2 \sim \frac{1}{2}\,\eta\left(\frac{v_t}{h}\right)^2 \tag{7.28}$$

This is a different expression from the one used in the case of radiative viscosity. When turbulence is present, the bulk of the dissipation is due to the random motion in the eddies, so that $\rho\varepsilon_{\text{dissip}}$ must depend on the velocity gradient across an eddy, typically v_t/h, not on the mean velocity gradient, typically v_χ/H. Dimensional arguments (see, for example, Landau and Lifshitz 1959, pp 119-120) lead to the same result, except for the factor 1/2 .

Since $\text{div}\,\tilde{\mathcal{J}} \sim \epsilon\sigma\, T_e^4/H$ (assuming the non-local theory - see the footnote on p.147), h will then be approximately determined by

$$\epsilon \frac{\sigma T_e^4}{H} \sim \frac{1}{2} \rho \, v_t \, h \left(\frac{v_t}{h}\right)^2 \quad . \tag{7.29}$$

Equations (7.27) and (7.29) are two equations to determine the quantities Ω_1, v_t and h. Assuming that inertial terms can be ignored, an assumption justified by later checking for consistency, a third equation is obtained by writing the χ-component of the equation of motion as

$$\epsilon \frac{GM}{R^2} \left(\epsilon_1 + \frac{\Omega_1}{\Omega_0}\right) \sim v_t \, h \, \frac{v_\chi}{H^2} \quad . \tag{7.30}$$

Equation (7.27) is more conveniently written

$$\frac{\Omega_1}{\Omega_0} \sim \epsilon_1^2 \, \frac{v_\chi R}{v_t \, h} \quad . \tag{7.31}$$

It is not immediately obvious which term on the L.H.S. of equation (7.30) is the larger. However, if the term $\epsilon \epsilon_1 GM/R^2$ is assumed to be the larger it is easy to show from equations (7.30), (7.31) and (7.20) that $\Omega_1/\Omega_0 \sim 4/\epsilon > 1$. This contradicts the assumption that the first term is larger, which requires $\Omega_1/\Omega_0 < \epsilon_1$.

This result should be compared with Osaki's model (1966), in which the first term is tacitly assumed to be the larger. That is, only the contribution from the horizontal pressure gradient is considered by Osaki. This is incorrect if v_χ is determined by the condition $J \sim 1/4$, whatever assumptions are made about v_t and h. Osaki also assumes that $v_t \sim v_\chi$ and $h \sim H$. The first of these assumptions turns out to be fairly good, but the second is completely inconsistent with the present model. It is this which accounts for the great difference between Osaki's results and the present ones. The neglect of the Ω_1 term in the χ-component of the equation of motion is consistent with Osaki's value for Ω_1, which gives $\Omega_1/\Omega_0 \sim \epsilon_1$. The inconsistency in Osaki's model arises only in the thermal balance equation, which he does not consider in detail, but which cannot be satisfied with his assumptions.

It has been established that the Ω_1 term is the important one on the L.H.S. of equation (7.30). Equations (7.30) and (7.31) can therefore be combined to give

$$(v_t h)^2 \sim H^2 \epsilon_1^2 \epsilon \frac{GM}{R} \tag{7.32}$$

Using equation (7.20), this may be written

$$v_t h \sim \frac{1}{2}\sqrt{\epsilon\epsilon_1}\; v_\chi H \tag{7.33}$$

and so, from equation (7.31)

$$\frac{\Omega_1}{\Omega_0} \sim 2\sqrt{\frac{\epsilon_1}{\epsilon}} \sim 1.8\times10^{-1} \quad . \tag{7.34}$$

This is large, but consistent with the assumption $\Omega_1 < \Omega_0$ on which the order of magnitude estimates are based. Also, since $\Omega_0 = (\epsilon GM/R^3)^{1/2} \sim 3.9\times10^{-5}$, the variation in rotation speed over the stellar surface is approximately

$$\Omega_1 R \sim 0.18\,\Omega_0 R \sim 3.7\times10^6 \text{ cm/sec} ; \tag{7.35}$$

that is, of the same order of magnitude as v_χ. This suggests that the Richardson number for v_ϕ will about 1/4, as suggested at the end of the last section. This is easily verified, since the Richardson number for v_ϕ is

$$J_\phi = \frac{g\left(\frac{-\rho'}{\rho}\right)}{\left(\frac{\partial v_\phi}{\partial s}\right)^2} \tag{7.36}$$

where, as before, $g \sim GM/R^2$ and $-\rho'/\rho \sim 1/H$. The v_ϕ gradient is

$$\frac{\partial v_\phi}{\partial s} = \frac{\partial}{\partial s}(\Omega \varpi) \sim \frac{\Omega_1}{H} R \quad . \tag{7.37}$$

The result $J_\phi \sim 1/4$ then follows from equation (7.34) and the definition of ϵ.

To find v_t and h separately, it is necessary to use equation (7.29), which gives

$$\frac{h}{H} \sim \frac{\rho v_t^3}{2\epsilon\sigma T_e^4} \tag{7.38}$$

If this expression is substituted into equation (7.32), and it is assumed that $\rho \sim 1/\kappa\epsilon_1 R$, it is found that

$$v_t^8 \sim 4\epsilon_1^4 \epsilon^3 \kappa^2 (\sigma T_e^4)^2 \; GMR \tag{7.39}$$

so that $$v_t \sim 1.7 \times 10^6 \text{ cm/sec} \sim 0.45 v_\chi \tag{7.40}$$

This shows that the mean flow, v_χ, is subject to variations of the order of 45%. The typical distance over which these variations occur is found from equations (7.38) and (7.40) to be given by

$$\frac{h}{H} \sim 1.0 \times 10^{-2} \tag{7.41}$$

(cf. Osaki's assumption: $h \sim H$).

This result is perhaps rather surprising. While one would expect $h/H < 1$, a factor of 100 between the scale height and the eddy size seems rather large. Since there is no doubt that such a factor is required to make the present model self-consistent, one is led to ask if the model is entirely realistic. There is little doubt that turbulence occurs, in which case v_χ is unambiguously determined by the criterion $J \sim 1/4$ and Osaki's model is certainly wrong somewhere. However, it is possible that in fact $h \sim H$ and that thermal balance is achieved by some other, more efficient, process. A possible candidate is radiative dissipation, that is, the direct conversion of mechanical energy into radiation. This alternative model has not been investigated in any detail, but rough estimates suggest that radiative dissipation is less efficient than viscous dissipation. It therefore appears that the turbulent eddies are indeed very small in comparison with the scale height. This presumably means simply that the turbulence is very strong, which is consistent with the known strength of the driving force.

The final stage in this order of magnitude model is to check the validity of the assumptions (i) that inertial terms are negligible in the χ-component of the equation of motion and (ii) that both inertial and viscous terms are negligible in the Ψ-component. It is found that

$$\frac{I}{V_\chi} \sim 2\sqrt{\frac{\epsilon_1}{\epsilon}} \sim 1.8 \times 10^{-1} \qquad (7.42)$$

$$\frac{I}{C_{\Psi}} \sim \frac{\epsilon_1}{\epsilon} \sim 8.5 \times 10^{-3} \qquad (7.43)$$

and

$$\frac{V_{\Psi}}{I} \sim \frac{1}{2}\sqrt{\epsilon\epsilon_1} \sim 4.6 \times 10^{-3} \qquad (7.44)$$

Assumption (i) is therefore a reasonably good approximation and assumption (ii) is an extremely good approximation.

3. The model in more detail

So far the turbulent surface layer has been considered in isolation. In this section the relation of the turbulent layer to the overall stellar model will be considered and the detailed equations of the model will be written down, taking into consideration the order-of-magnitude discussion of section 2.

In the previous chapters it has been shown that a non-local theory of energy transport must be used in a stellar atmosphere if a proper description of the fluid motions in the outermost layers is required. The full non-local theory developed in Chapter 4 is valid to any depth, although it becomes indistinguishable from the local theory at large optical depth. However, the approximate solution found in Chapter 6 only holds down to about $\tau = 5$, where the neglected terms in v_s become comparable with those retained.

Since turbulence has already set in below this point, at about $\tau = 8$ (see Chapter 6), it may not appear necessary to use the non-local theory at all outside the turbulent region. It is certainly wrong to use only the approximate non-local theory to calculate v_s outside the turbulent region. However, a mixture of the local and non-local theories, or the full non-local theory when this has been worked out, must be used to calculate v_χ outside the turbulent layer, since at $\tau \sim 8$ the non-local v_χ is still more than ten times the local value ($v_{\chi NL} \approx 4.8 \times 10^5$ cm/sec; $v_{\chi L} \approx 2.8 \times 10^4$ cm/sec). This makes it difficult to decide what boundary conditions should be applied at the lower edge of the turbulent layer, since the boundary occurs in the transition region between the domains of validity of the local and approximate non-local theories, a region which cannot be adequately described by the methods of this thesis.

The problem is further complicated by the fact that the order-of-magnitude estimates suggest that inertial terms become important before turbulence sets in. If that is so, it may also be necessary to take account of magnetic forces, which must somehow be increased in order to balance the growing Coriolis force as the turbulent region is approached. One may imagine, perhaps, that the required increase in field strength is provided by concentration of the field lines expelled from the turbulent layer. If this process does not sufficiently increase the magnetic forces, it may be necessary to consider the possibility of slight non-uniform rotation outside the turbulent layer.

It is also possible that the magnetic field will be expelled from the region where inertial terms are important, since Weiss's (1966) model applies to any cellular convection, without specific reference to turbulence. The question of time-scales then arises, and it is possible that turbulence is a more efficient expulsion mechanism than laminar circulation, so that although the field has been expelled from the turbulent region it may still be present immediately below. Nonetheless, it seems likely that the circulation speeds in this region are greater than the diffusion speed of the field through the medium. It is therefore unlikely that a steady state exists in which the rotation is kept uniform by the magnetic field.

In any case, the mathematical problems involved in the treatment of the transition zone between the outer turbulent region and the part of the inner non-turbulent region where local theory can safely be applied are formidable. No attempt at solving these problems will be made in this thesis and for simplicity the transition zone will be assumed to rotate uniformly. Since uniform rotation does not seem to be a singular case, the results obtained with that assumption should be qualitatively correct.

In the turbulent region also, various problems arise which must be passed over in this thesis. According to Chapter 6, there would appear to be an optically thin layer very near the surface which is not turbulent and it is not obvious how this should be included in the model, particularly as the non-local theory of Chapter 6 breaks down within this layer. It will be assumed for simplicity that some form of convective overshoot ensures that the star is turbulent right to the surface. The details of the overshooting will not be considered.

Near the lower boundary, the approximate non-local theory of radiative transport, described in Chapter 6, probably begins to fail.

However, this problem is intimately bound up with the treatment of the transition region and for the rest of this chapter the plane-parallel approximation will be used for any detailed calculation. This is certainly valid near the surface ($\tau = 0$) which is the only region treated in any detail.

With all these approximations and assumptions the model considered below is inevitably a crude one, but it is an improvement on the order-of-magnitude model. In particular, it is possible to say something about the variation of angular velocity over the surface of the star.

The equations which will now be written down do not appear to assume the plane-parallel approximation, since they are written in terms of s, not of z. However, when the order-of-magnitude estimates were made, terms of order ϵ_1 were dropped. The present equations contain only those terms which were retained and therefore the plane-parallel approximation has already been applied. As usual, therefore, the equations will be solved in terms of z.

The s-component of the equation of motion is virtually unchanged, being

$$\frac{\partial P}{\partial s} = \rho \frac{\partial \Psi}{\partial s} \tag{7.45}$$

The only difference is that P, ρ, and so T are no longer functions of s only. Also Ψ is now defined in terms of Ω_0, that is,

$$\Psi = \frac{GM}{r} + \frac{1}{2}\Omega_0^2 \, r^2 \sin^2\theta \tag{7.46}$$

Since Ω is not a constant, Ψ can no longer be interpreted as a joint potential. Nonetheless, Ψ, or s, is still useful as a coordinate. For simplicity, ϵ will be redefined by

$$\epsilon = \frac{\Omega_o^2 R^3}{GM} \tag{7.47}$$

so that $\Psi(r,\theta,\epsilon)$ is the same as before. However, the equations are no longer obtained by a simple expansion of the full equations to first order in ϵ.

The χ- and ϕ-components of the equation of motion are respectively:

$$-2\Omega_o\Omega_1 s \sin\chi\cos\chi = \frac{\eta}{\rho}\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial v_\chi}{\partial s}\right) \tag{7.48}$$

and
$$2\Omega_o v_\chi s \cos\chi = \frac{\eta}{\rho}\sin\chi\frac{\partial^2}{\partial s^2}(\Omega_1 s^2) \quad . \tag{7.49}$$

The other two unfamiliar equations are the thermal balance equation

$$\operatorname{div}\underline{\mathcal{F}} = \rho\ \varepsilon_{dissip} \tag{7.50}$$

and the flux integral

$$L = \int \underline{\mathcal{F}}_{tot}\cdot \underline{dS} \quad . \tag{7.51}$$

The latter must now be written in terms of the total flux, since P, ρ and T are no longer constant on level surfaces (cf. Appendix III).

The remaining equations are the same as before. The equation of state is

$$P = \frac{\mathcal{R}}{m}\rho\ T \tag{7.52}$$

The equation of continuity is still

$$\text{div}\,(\rho \underline{v}) = 0 \tag{7.53}$$

although now it is used to find v_s, not v_χ, since v_χ is found from equations (7.48) and (7.49) and v_s can no longer be found from the equation of thermal balance.

Finally, assuming the Eddington approximations are still adequate, the radiative transfer is described by the equations

$$\text{div}\,\underline{\mathcal{F}} = -\,4\pi\kappa\rho\,(J - B) \tag{7.54}$$

and

$$\text{div}(J\underline{\underline{1}}) = -\frac{3\kappa}{4\pi}\,\rho\,\underline{\mathcal{F}} \tag{7.55}$$

where

$$B = \frac{\sigma T^4}{\pi} \tag{7.56}$$

and $\underline{\underline{1}}$ is the unit tensor.

The equations for $\underline{\mathcal{F}}$ are much more complicated than before, because of the need to use $\underline{\mathcal{F}}_{tot}$ in the flux integral, and no attempt will be made to solve them. Some suggestions on the effect of the turbulent region on $\underline{\mathcal{F}}$ will be found at the end of this chapter.

The structure equations can only be solved in conjunction with the radiative equations, because of equation (7.56); it is therefore not possible to find P, ρ and T without finding $\underline{\mathcal{F}}$. However, equations (7.48), (7.49) and (7.53) can be decoupled from the remaining equations if ρ is approximated by ρ_o, since ρ_o is unaffected by the turbulence and may be taken from the results of Chapter 6. It is therefore possible to solve for Ω_1, v_χ and v_s simply by assuming a value for η.

The equations will first be put in dimensionless form, using the results of the order-of-magnitude model as a guide. Write

$$s = R\,(1 + \epsilon_1 z) \tag{7.57}$$

as usual, and also

$$\left.\begin{aligned} \rho &= \frac{\rho^*}{\varkappa \epsilon_1 R} \\ \eta &= \frac{\epsilon_1}{\varkappa}\sqrt{\epsilon \frac{GM}{R}} \\ v_\chi &= \sqrt{\epsilon_1 \frac{GM}{R}}\, U_\chi \\ v_s &= \epsilon_1 \sqrt{\epsilon_1 \frac{GM}{R}}\, U_s \\ \Omega_1 &= \sqrt{\frac{\epsilon_1}{\epsilon}}\, \Omega_0 \omega_1 \end{aligned}\right\} \tag{7.58}$$

It may then be shown that, to lowest significant order, the equations (7.48) and (7.49) become:

$$2\omega_1 \sin\chi \cos\chi = -\frac{1}{\rho_0^*}\frac{\partial^2 U_\chi}{\partial z^2} \tag{7.59}$$

$$2\, U_\chi \cos\chi = \frac{1}{\rho_0^*}\ \sin\chi \frac{\partial^2 \omega_1}{\partial z^2} \tag{7.60}$$

The continuity equation will be considered later.

If these equations are to be solved, some assumptions must be made about the χ-dependence of U_χ and ω_1. Suppose that one starts with the general assumption that

$$\left.\begin{aligned} \omega_1 &= \sum_{n=0}^{\infty} \omega_{1n}(z)\ P_n(\cos\chi) \\ U_\chi &= \sum_{n=0}^{\infty} u_{\chi n}(z)\ P_n(\cos\chi)\sin\chi\cos\chi \end{aligned}\right\} \tag{7.61}$$

Then it is easy to show that equation (7.59) yields the equations

$$2\omega_{1n} = -\frac{1}{\rho_o^*}\frac{d^2u_{\chi n}}{dz^2} \quad n = 0, 1, 2, \ldots \qquad (7.62)$$

whereas equation (7.60) becomes

$$\frac{1}{\rho_o}\frac{\partial^2}{\partial z^2}(\omega_{10} + \omega_{11}P_1 + \omega_{12}P_2 + \ldots) = \frac{2}{3}(1 + P_2)(u_{\chi o} + u_{\chi 1}P_1 + u_{\chi 2}P_2 + \ldots) . \qquad (7.63)$$

If the R.H.S. of this equation is written as a series of the form $\sum_{m=0}^{\infty} a_m P_m$, it may be shown that each a_m involves <u>all</u> the $u_{\chi n}$ with $n \geqslant m - 2$, since the product P_2P_n is of the form

$$P_2P_n = \sum_{r=0}^{n+2} b_r P_r \quad . \qquad (7.64)$$

It is therefore not possible in general to decouple the equations obtained by equating coefficients of P_m in equation (7.63). The only case in which the equations decouple is when

$$u_{\chi n} \equiv 0 \text{ for } n \geqslant 1 \quad . \qquad (7.65)$$

There is no reason to believe that this is a particularly good approximation for U_χ. However, the series for U_χ has to be truncated at some point if a solution is to be obtained in practice, and it seems reasonable as a first approximation to truncate the series in the simplest way. The model is not sufficiently reliable to justify the use of a more elaborate representation for U_χ.

The series for ω_1 may be similarly truncated, except that ω_{12} must be non-zero if a non-trivial solution is to be obtained. Then, dropping the subscript from $u_{\chi o}$,

$$\frac{d^2\omega_{10}}{dz^2} = \frac{2}{3}\rho_o^* u_\chi \tag{7.66}$$

$$\frac{d^2\omega_{12}}{dz^2} = \frac{4}{3}\rho_o^* u_\chi \tag{7.67}$$

and

$$\omega_{1n} \equiv 0 \quad \text{for } n = 1, \ n \geqslant 3 \tag{7.68}$$

Equation (7.62) is then an identity for $n = 1$, $n \geqslant 3$. For $n = 2$, this equation and equation (7.65) give $\omega_{12} \equiv 0$, which contradicts equation (7.67) (rejecting the uninteresting solution $u_\chi \equiv 0$). This contradiction simply arises from the truncation of the series for ω_1 and U_χ. One of the equations (7.62) ($n = 2$) and (7.67) is redundant, and the uninteresting one, $\omega_{12} = 0$, will be dropped. Equation (7.62) therefore gives only one useful equation, viz:

$$2\omega_{1o} = -\frac{1}{\rho_o^*}\frac{d^2u_\chi}{dz^2} \tag{7.69}$$

Because of the presence of ρ_o^*, which is known only numerically as a function of z, it is not possible to solve equations (7.66), (7.67) and (7.69) without the aid of a computer. In view of the many approximations in the model, and the uncertainties in the boundary conditions (which will be discussed shortly), it did not seem justifiable to spend computer time finding a complete solution before a more accurate model had been developed. The present thesis does not attempt to improve the model outlined above, and the rest of this chapter is therefore confined to such discussion of the equations as is possible without the help of a computer. In particular, the form of the solution near the surface is obtained.

First of all, equations (7.66) and (7.67) may be combined and integrated to give

$$\omega_{12} = 2\,\omega_{1o} + C\,z + D \tag{7.70}$$

C and D are arbritrary constants, to be determined by the boundary conditions, which must now be discussed. Six conditions are required, two for each of ω_{1o}, ω_{12} and u_χ.

In reality, the boundary between the turbulent and non-turbulent regions is not well-defined, and it is not at all clear what conditions are valid there. For simplicity, it will be supposed that the turbulent region starts abruptly at a definite value of z, $z = z_o$. The results of Chapter 6 suggest that $z_o \sim -6$, but this need not be assumed for present purposes. Below $z = z_o$, it will be assumed that the rotation is strictly uniform, so that two of the boundary conditions must be

$$\omega_{1o} = \omega_{12} = 0 \quad \text{at } z = z_o \tag{7.71}$$

The value of v_χ at $z = z_o$ may be calculated from the theory of Chapter 6, once the value of z_o has been decided. The solution in this chapter will not go far enough to require this value, so this boundary condition will simply be written

$$u_\chi = u_{\chi o} \quad \text{at} \quad z = z_o \tag{7.72}$$

where $u_{\chi o}$ is known in principle.

The values of ω_{12}, ω_{1o} and u_χ at the surface cannot be imposed in advance, as will be seen later. For the moment, it is sufficient to demand that

$$\omega_{1o},\ \omega_{12} \text{ and } u_\chi \text{ are finite as } z \longrightarrow +\infty \qquad (7.73)$$

If possible, one would like to require u_χ to vanish at the surface. This point will be discussed later.

It is now possible to find the constants C and D. The conditions at the surface clearly require C = 0 and then, if the condition (7.72) is applied, D is also zero. Thus

$$\omega_{12} = 2\omega_{1o} \qquad (7.74)$$

This already gives some information about surface conditions. Using this result, the rotation law can be written as

$$\frac{\Omega - \Omega_o}{\Omega_o} = 3\sqrt{\frac{\epsilon_1}{\epsilon}}\,\omega_{1o}\cos^2\chi \qquad (7.75)$$

where Ω_o can now be identified with the equatorial angular velocity. Unfortunately, the sign of ω_{1o} at the surface is not yet known. If it turns out to be negative, then equation (7.75) would predict an equatorial acceleration and the angular velocity would decrease towards the poles according to a law not unlike the observed solar rotation law (see, for example, Brandt and Hodge 1964). If ω_{1o} were to be - 0.74 at the surface, equation (7.75) would give a curve which fell between the two solar curves for $(\Omega - \Omega_o)/\Omega_o$ found by observing sunspots and by observing the Doppler effect at the limb. However, this cannot be verified without a complete solution of the equations; it is more likely in any case that the agreement is coincidental since the present model is of an early type star and the Sun is a late type star.

With the help of the asymptotic expression for ρ_o^* as $z \rightarrow +\infty$ (equation (6.42)), it is possible to solve equations (7.66) and (7.69) for u_χ near the surface. In general, these equations combine to give

$$\frac{3}{4}\frac{1}{\rho_o^*}\frac{d^2}{dz^2}\left(\frac{1}{\rho_o^*}\frac{d^2u_\chi}{dz^2}\right) + u_\chi = 0 \tag{7.76}$$

However, near the surface, ρ_o^* is given by an expression of the form

$$\rho_o^* = \sum_{n=1}^{\infty} a_n \frac{e^{-nh}}{E^n} , \tag{7.77}$$

where $h = 2^{1/4}z$, $a_1 = \frac{16}{3}2^{1/4}$ and $a_2 = -4a_1$. Equation (7.76) may then be written in the form

$$\sum_{m=0}^{\infty}\left[\frac{d^4u_\chi}{dh^4}A_{4m}\frac{e^{-mh}}{E^m} + \frac{d^3u_\chi}{dh^3}A_{3m}\frac{e^{-mh}}{E^m} + \frac{d^2u_\chi}{dh^2}A_{2m}\frac{e^{-mh}}{E^m} + \frac{e^{-2h}}{E^2}u_\chi A_{om}\frac{e^{-mh}}{E^m}\right] = 0 \tag{7.78}$$

which can be solved by writing

$$u_\chi = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{nm}h^{r-m}\frac{e^{-nh}}{E^n} . \tag{7.79}$$

The index r can be determined from an indicial equation obtained by first equating coefficients of e^{-Nh} for each N and then, in these coefficients, equating coefficients of powers of h.

The general solution of equation (7.78) is complicated. However, for large h the series for ρ_o^* and u_χ converge quickly and only the first few terms need be considered. If only the first two terms are taken in equation (7.77) (cf. equation (6.42)), equation (7.78) reduces to

$$(1 + \frac{4}{E} e^{-h})\, u_\chi^{iv} + 2\, u_\chi^{iii} + u_\chi^{ii} = 0 \tag{7.80}$$

which has the general solution

$$u_\chi = b_{oo} h + b_{o1} + (b_{1o} h + b_{11}) \frac{e^{-h}}{E} + O\left(\frac{e^{-2h}}{E^2}\right) \tag{7.81}$$

where b_{oo}, b_{o1}, b_{1o} and b_{11} are arbitrary constants to be determined by the boundary conditions. The condition (7.73) requires that $b_{oo} = 0$. Equation (7.69) may be used to find ω_{1o} to lowest order in e^{-h}/E. It is found that

$$\omega_{1o} = -\frac{3}{32}\, 2^{1/4}(b_{1o} h + b_{11} - 2b_{1o}) + \ldots \tag{7.82}$$

Condition (7.73) then requires that $b_{1o} = 0$, so that

$$\omega_{1o} = -\frac{3}{32}\, 2^{1/4} b_{11} + O\left(\frac{e^{-h}}{E}\right) \text{ as } h \rightarrow \infty \tag{7.83}$$

There is therefore equatorial acceleration if b_{11} is positive and the rotation law will agree with that of the Sun if $b_{11} = 6.6$.

Since $b_{1o} = 0$, u_χ is given by

$$u_\chi = b_{o1} + \frac{b_{11}}{E} e^{-2^{1/4} z} + \ldots \text{ as } z \longrightarrow +\infty \tag{7.84}$$

The constants b_{o1} and b_{11} can only be determined by the boundary conditions at the lower boundary, $z = z_o$, and so they cannot be found without the numerical integration of equation (7.76), using equation (7.84) as a starting series. In practice, the integration would be performed for various trial values of b_{o1} and b_{11} until values were found which enabled the solution to satisfy the boundary conditions at $z = z_o$. Since there are two conditions to be satisfied there, it is not possible to demand $u_\chi = 0$ at the surface, since that would require $b_{o1} = 0$ and would leave only one constant to be determined by two conditions. In general, therefore, one would expect that u_χ does not vanish at the surface.

This affects the behaviour of U_s near the surface, which can now be found from the continuity equation. To lowest order, the continuity equation can be written

$$\frac{\partial}{\partial z}(\rho_o^* U_s) = -2\rho_o^* u_\chi(z) P_2(\cos\chi) \qquad (7.85)$$

which has the general solution

$$\rho_o^* U_s = -2P_2 \int^z \rho_o^* u_\chi \, dz' + A(\chi) \qquad (7.86)$$

where A is an arbitrary function of χ. Assuming the boundary condition

$$U_s \text{ is finite as } z \rightarrow +\infty, \qquad (7.87)$$

and writing $U_s = u_s(z)P_2(\cos\chi)$, it can be shown that

$$u_s = \frac{1}{\rho_o^*} \int_z^\infty 2\rho_o^* u_\chi \, dz' \qquad (7.88)$$

if the integral converges and tends to zero faster than ρ_o^* as $z \rightarrow +\infty$. That these conditions are satisfied can be seen by substituting the asymptotic expressions for ρ_o^* and u_χ into equation (7.88) and integrating. It is found that

$$u_s = 2^{3/4}\left(b_{o1} + \frac{b_{11}}{2E} e^{-2^{1/4}z} + \ldots\right) \text{ as } z \rightarrow +\infty, \qquad (7.89)$$

a result which satisfies the condition (7.87). Like u_χ, u_s does not, in general, vanish at the surface, although the case $b_{o1} = 0$ is not excluded. It is a feature of all theories of meridian circulation, up to and including the present one, that one is not able to demand that $\underline{v}$ should be zero at the surface. Nonetheless, there is no mass outflow at the surface, since the density tends to zero there, so this feature is not too serious a defect.

4. Qualitative discussion of the emergent flux

One of the objects of this thesis was to discover whether or not von Zeipel's result, that the radiative flux at the surface is proportional to the surface gravity, is a good approximation. Although this question could have been answered in the negative if the theory of Chapter 6 had been valid at the surface, the presence of a turbulent layer throws doubt on this conclusion, and the theory presented above is not adequate to remove the doubt.

This is a serious shortcoming of the theory, since the only way at present of obtaining the actual rotation speeds of individual single stars, even in principle, is by the effect of rotation on the star's position in the H-R diagram (see Appendix I). This is crucially affected by the distribution of brightness over the surface of the star, which is usually assumed to be given by the von Zeipel gravity-darkening. If that assumption is seriously wrong, many current models of rotating stellar atmospheres will have to be re-calculated.

Von Zeipel's result is trivially true for a uniformly rotating star if the local theory of transfer is assumed to hold, since in that case $\underset{\sim}{\mathcal{F}} \propto \underline{\nabla}\Psi$ and the flux is automatically proportional to the surface gravity. It is therefore useful to use the local theory as a comparison, to see how far the flux in other theories departs from proportionality to gravity.

In the local theory it was found that the flux only had an s-component, given (to lowest order in ϵ_1) by

$$\mathcal{F}_s = \sigma T_e^4 \left[1 + \epsilon\left(\ell_1 - \tfrac{2}{3}P_2(0) + \tfrac{4}{3}P_2(\cos\chi)\right) + \ldots\right] \quad (7.90)$$

(see equations (A7.6) and (A7.22)), where ℓ_1 and $P_2(0)$ have the values given in Appendix VI. In the non-local theory, on the other hand, the flux still only had an s-component, to lowest order in ϵ_1, but its value at the surface ($\tau = 0$) would be

$$\mathcal{F}_s = \sigma T_e^4 \left[1 + \epsilon\left(\ell_1 - \tfrac{2}{3}P_2(0) + \tfrac{4}{3}(1 - 2(2 - \sqrt{3}))P_2(\cos\chi)\right) + \ldots\right] \quad (7.91)$$

if the theory were valid there (see equations (6.1), (6.31), (6.61) and (6.64)). The effect of the non-local theory is therefore to make the flux more spherically symmetric than predicted by the local theory. The differences are summarised in the following table, in which the entries are $(\mathcal{F}_s - \sigma T_e^4)/\sigma T_e^4 \epsilon$.

	Pole	Equator
Local	+0.60	-1.40
Non-local	-0.12	-1.04

It is immediately obvious that the difference between the pole and the equator is appreciably less on the non-local theory. (The fact that $\mathcal{F}_s < \sigma T_e^4$ at both pole and equator is simply a result of the decrease of luminosity produced by rotation - T_e is defined in terms of L_o, the luminosity of the corresponding non-rotating star.)

The question to which one would like an answer now is : How much effect does turbulence have on this result? No definitive answer is yet possible, but the following considerations suggest that the effect of turbulence is to reverse the effect produced by the non-local theory, that is, the flux from the surface may be expected to be less spherically symmetric than would be predicted by the local theory.

It is a curious feature of flow in a region with a radiatively stable temperature gradient that the energy carried by the flow is propagated in the opposite direction to the velocity vector. On the local theory, the velocity at the poles is mainly directed away from the surface. The circulation therefore carries energy towards the surface at the poles. In the absence of dissipation, this energy is not available to be radiated away and it contributes nothing to the observed radiation. However, if the energy in the circulation were available, the total flux radiated away would presumably be even less spherically symmetric than the local radiative flux alone.

In the turbulent layer, much of the circulation energy will be dissipated. Of course, the simple local theory is not valid in the turbulent layer. However, the turbulent layer is fairly thin and it might be a reasonable approximation to suppose that the effect of the turbulence is to convert the circulation energy into radiation in situ, without redistributing it over χ. This would mean that the total flux, radiative plus convective, would be essentially constant, for a given χ, throughout the turbulent region. If that were so, the value of the constant would be given by the total flux entering the turbulent region from below. At the base of the turbulent layer, all quantities, except possibly v_χ, can be adequately represented by the local theory, and so the total flux would be less spherically symmetric than required by gravity-darkening, for the reasons given in the last paragraph.

It is easy to see how large the effect would be. The expression for the convective flux is given in Appendix III (equation (A3.4)) and it may be evaluated using the results of Appendix VII. Since $\underline{v}$ has a χ-component, the total flux also has a χ-component, so that now the flux is not even normal to the surface, except at poles and equator where $v_\chi = 0$. Assuming the local theory throughout, and taking $\epsilon = 0.1$, it can be shown that

$$\frac{\mathcal{F}_s - \sigma T_e^4}{\epsilon \sigma T_e^4} = -0.73 + 2.75\, P_2(\cos\chi) = \begin{cases} 2.02 & \text{(poles)} \\ -2.11 & \text{(equator)} \end{cases} \tag{7.92}$$

and

$$\frac{\mathcal{F}_\chi}{\epsilon \sigma T_e^4} = -2.49 \sin 2\chi \tag{7.93}$$

This is not even approximately the same as the von Zeipel gravity-darkening.

However, the present result is based on the assumption that the flux which enters the turbulent region at a particular value of χ leaves it at approximately the same value of χ. It is not certain whether that assumption can be reconciled with the fact that the flux entering the region has an appreciable χ-component - about 25% of the s-component at $\chi = \pi/4$. If the non-local theory were to be used for v_χ, as might be more appropriate, the χ-component of the flux could even be larger than the s-component. It is difficult to believe that this will be dissipated in such a way that the total flux emerges unaltered at the surface. Nonetheless, it is just as unlikely that the emergent flux will adjust itself to be proportional to the surface gravity. The above model may not be valid, but the discussion shows that the von Zeipel gravity-darkening is by no means the only possibility and that it should not be used without more justification than is usually given.

No further results can be obtained for the turbulent region without prolonged computation which, as mentioned earlier, does not seem justified for such a crude model. It is clear that much work is still necessary before a detailed model of the extreme outer layers of an early type star can be constructed; in particular, before the emergent flux can be predicted with any confidence. The present chapter is intended to indicate tentatively the kind of approach which will be needed to produce such a model.

CHAPTER 8

Summary and conclusions

"Begin at the beginning", the King said, very gravely, "and go on till you come to the end: then stop."

Lewis Carroll, Alice in Wonderland, Ch. 12.

Other authors (see Chapter 1) had found that the meridian-plane velocity field in rotating stars possessed a 1/density singularity at the surface. This result follows from the use of a local theory for the transfer of radiation, as is shown in Chapter 3. It has now been found that it is necessary to take into account the non-local nature of radiative transfer. When that is done, the singularity is no longer present, as is shown in Chapter 4. Nonetheless, that model still does not represent reality since, although there is no longer a formal singularity in the velocity field, the non-local theory is also singular in the sense that it predicts unrealistically large circulation speeds at the surface (see Chapter 6). This result was obtained independently by Osaki (1966).

It has been found (see Chapters 6 and 7) that the flow becomes unstable, and that turbulence develops, when the horizontal speed is of the order of the speed of sound. The flow breaks up into eddies, in which its energy is dissipated, and the mean speed of the flow increases no further. This model appears to be physically realistic when examined qualitatively and it is hoped that it will be possible later to develop the model quantitatively.

In the meantime, two conclusions of a qualitative nature can be drawn. The main conclusion is that one should expect to observe turbulence in rotating early-type stars, with turbulent velocities of the order of the speed of sound. It was therefore encouraging to learn that Miss Underhill (1967, personal communication) has observed turbulence in B stars. The turbulent velocities she has observed are, at about 10 km/sec, a little smaller than those predicted by the present theory, but the discrepancy is not significant in view both of the uncertainties involved in this kind of observation and of the qualitative nature of the theory.

The second conclusion is that the distribution of radiative flux over the surface of a rotating early-type star departs grossly from the von Zeipel gravity-darkening. It was originally hoped that it would be possible to find a quantitative expression for the flux distribution which could be compared with the von Zeipel distribution. It would then have been possible to re-assess the work of Roxburgh and Strittmatter (1965, 1966 a and b) and others (see Chapter 1) on the effect of rotation on a star's position in the HR diagram. Unfortunately, the presence of a turbulent region near the surface puts a quantitative comparison beyond the scope of this thesis. It is not even possible to say definitely whether the flux distribution is more or less spherically symmetric than the von Zeipel gravity-darkening, although some tentative suggestions are made in Chapter 7. Nonetheless, there is no doubt that it is not a good approximation to assume that the emergent flux is proportional to gravity. It is interesting to note that Roxburgh (1967b) comes to a similar conclusion for highly distorted stars whose outer layers are in convective equilibrium. He finds that von Zeipel's result must be replaced by the approximate relation

$$\text{flux} \propto (\text{gravity})^{0.6} \tag{8.1}$$

However, he claims that von Zeipel's result is reasonably accurate for stars with radiative outer layers. This is presumably because he does not consider circulation and so his models contain no turbulent layer at the surface.

It is appropriate to conclude by considering briefly what can be said qualitatively about the overall structure of the outer layers of a rotating early-type star. It is convenient to distinguish five zones, although in practice the dividing lines are probably not well-defined. In the lowest zone, which fits directly on to the interior solution, the photon mean free path is short compared with the scale height and the local theory may safely be used. The zone is assumed to be maintained in uniform rotation by a weak magnetic field, which is not significantly disturbed by the comparatively slow laminar circulation in meridian planes. The zone is above the circulation reversal discussed by Öpik (1951) and Mestel (1966) and so the mass flow is downward at the poles.

Consider what happens to the flow as it rises again at the equator. As the density decreases, so the speed of the flow increases, until it is greater than the speed at which the magnetic field lines can diffuse through the gas. This marks the beginning of the second zone. It is not clear what happens to the magnetic field in this zone, but it is certain that it can no longer simply be taken for granted as an agent to keep the rotation uniform. It may well have been expelled altogether from the zone if the time-scale for expulsion is short enough. If not, it will need to be considered explicitly. In either case, non-uniform rotation must appear.

Further up in the atmosphere, the speed of the flow, though still subsonic, becomes large enough that the inertia of the flow begins to exert forces which are an appreciable fraction of the centrifugal forces due to the rotation of the star. There is therefore a third zone, in which inertial effects are important but the flow is still laminar. It seems likely that there will be no magnetic field in this zone, which is again in non-uniform rotation.

As the flow is followed still higher in the atmosphere, it enters a zone in which the photon mean free path has become so long that non-local effects must be considered. This fourth zone is otherwise similar to the third zone.

The flow which passed from pole to equator in the first three zones returns to the pole near the top of the fourth zone at speeds which are now approaching the speed of sound. The horizontal shear becomes turbulent because of the Kelvin-Helmholtz instability. This marks the boundary of the fifth zone, which extends from here to the surface of the star. Since the instability occurs at about the speed of sound, complicated shock phenomena may occur near the boundary. The possibility of shock waves arising in the outer layers of rotating stars with circulation currents has previously been suggested by Sweet (1965, personal communication) and Kippenhahn (1959).

Within the turbulent zone, the flow is rapidly broken up into eddies which are small compared to the scale-height. No coherent magnetic field can exist in such conditions. The rotation is non-uniform, and the differential flow round the rotation axis is also unstable. It therefore contributes further to the turbulence which can be regarded as being isotropic in the χ- and ϕ-directions. The large divergence in the radiative flux is now balanced, not by the divergence of the convective flux, as in the other zones, but by the dissipation of the energy of the flow in turbulent eddies. This energy eventually reaches the surface and contributes to the emergent flux of radiation, whose distribution over the surface is certainly not given by the von Zeipel gravity-darkening. The actual distribution must, however, remain a subject for future investigation.

APPENDIX I

Zero-rotation main sequence

1. Theory

The position of a star in a colour-magnitude diagram is not determined uniquely by its mass and chemical composition, but depends also on its rotation speed. The locus of the positions in the diagram of non-rotating stars of various masses and compositions may be termed the zero-rotation main sequence. For a star of a given mass and chemical composition, the displacement, due to rotation, from the zero-rotation main sequence depends on v, the equatorial rotation speed, and i, the angle of inclination of the rotation axis to the line of sight. This is illustrated in Fig.15, which shows, among other things, the displacement of a star of given mass and composition in the two extreme cases when the star is rotating pole-on and equator-on to the observer.

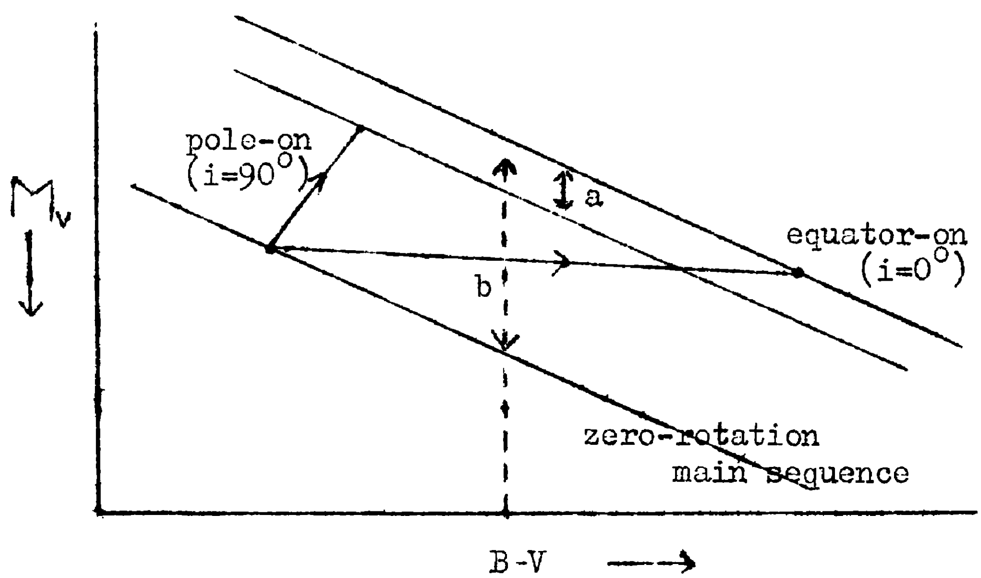

Fig. 15. Effect of rotation on a star's position in the HR diagram.

The locus of stars with a given v and i, but varying masses, is a line roughly parallel to the zero-rotation main sequence, but somewhat above it at a given B-V. Although the value of i makes a large difference to the position of a given star on the HR diagram, the difference in magnitude, at a given colour, between the loci defined by $(v, i = 0)$ and $(v, i = 90^\circ)$ is in general small compared with the mean displacement of the loci above the zero-rotation main sequence at the same colour. That is, in Fig.15, $a \ll b$. It is therefore possible, for each value of v, to define a "rotating main sequence", whose displacement from the zero rotation main sequence, at a given colour, depends essentially only on v. The equation of this rotating main sequence is

$$M_v(B-V) - M_v^{(0)}(B-V) = \kappa v^2 \qquad (A1.1)$$

where $M_v(B-V)$ = absolute magnitude of a star of given B-V and v

$M_v^{(0)}(B-V)$ = absolute magnitude of a star of same B-V and zero rotation

v = equatorial speed of rotation,

and κ is a quantity which is essentially a constant. It varies slightly with i and is a slowly varying function of mass and radius along the main sequence. These variations are small enough to be ignored in the first instance (Strittmatter 1966).

The value of κ can be calculated theoretically, and this has been done (Sweet and Roy 1953, Roxburgh and Strittmatter 1966b, Strittmatter 1966). It is of some interest to see if the value of $\varkappa$ can also be obtained from observational data with sufficient accuracy to be used as a check on the theoretical values so far obtained. Strittmatter (1966) has recently examined the data available for Praesepe and he claims to be able to estimate a value for $\varkappa$ from this data. In section 4 of this appendix his method is examined in some detail and it is shown that his claim rests on a very shaky foundation. Sections 2 and 3 contain necessary preliminary discussion.

2. Application of theory.

For a given set of observations, $M_V^{(0)}$ is not known a priori. In equation (A1.1), only M_V is directly observable, and so this equation cannot be used immediately to find $\varkappa$. For a particular star, the observable quantities are M_V, B-V and v sin i.

Suppose that the stars are randomly oriented, so that the distribution of i among the observed stars is random and

$$p(i)\,di = \text{probability of } i \text{ being in range } i, i+di = \sin i \, di \qquad (A1.2)$$

Then (Chandrasekhar and Münch 1950)

$$\langle (v \sin i)^2 \rangle = \tfrac{2}{3}\langle v^2 \rangle \qquad (A1.3)$$

where $$\langle v^2 \rangle = \frac{\int_0^\infty f(v)\, v^2\, dv}{\int_0^\infty f(v)\, dv}, \qquad (A1.4)(a)$$

$$\langle (v \sin i)^2 \rangle = \frac{\int_0^\infty \phi(v \sin i)(v \sin i)^2 d(v \sin i)}{\int_0^\infty \phi(y)\, dy} , \qquad \text{(A1.4)(b)}$$

$f(v)\,dv$ is the number of stars with v in the range $v, v + dv$ and $\phi(v \sin i)d(v \sin i)$ is the number of stars with $v \sin i$ in the range $v \sin i$, $v \sin i + d(v \sin i)$. The distribution ϕ can be observed. The distribution f cannot, but equation (A1.3) enables $\langle v^2 \rangle$ to be found from the observations. In general, ϕ and f will depend on B-V, so that the averages are only meaningful if taken in a sufficiently restricted range of colour. Denote the average at a particular B-V by $\langle \ \rangle_c$. Then for a given B-V the average of equation (A1.1) over v is

$$\langle M_v \rangle_c - M_v^{(0)} = \kappa \langle v^2 \rangle_c , \qquad \text{(A1.5)}$$

since $M_v^{(0)}$ is independent of v. In this equation,

$$\langle M_v \rangle_c = \frac{\int_0^\infty f_c(v)\, M_v(v^2)\, dv}{\int_0^\infty f_c(v)\, dv} \qquad \text{(A1.6)}$$

Eliminating $M_v^{(0)}$ between equations (A1.1) and (A1.5), and using equation (A1.3), gives

$$M_v - \langle M_v \rangle_c = \tfrac{3}{2}\kappa \left(\tfrac{2}{3}v^2 - \langle (v \sin i)^2 \rangle_c\right) \qquad \text{(A1.7)}$$

Since M_v is observable and the averages are averages of observable quantities, the value of v for a star with an observed M_v can be found from this equation if $\varkappa$ is assumed known from theory. The only problem is to evaluate the averages in practice.

It is obviously legitimate to approximate $\langle (v \sin i)^2 \rangle_c$ by an arithmetic mean:

$$\langle (v \sin i)^2 \rangle_c = \frac{1}{N_c} \sum_{j=1}^{N_c} (v \sin i)_j^2 \qquad \text{(A1.8)}$$

where N_c is the total number of observations at a given colour. $\langle M_v \rangle_c$ can be similarly approximated for the following reason. Suppose

$f_c(v)\, dv$ = no. of stars of given B-V in range $v, v + dv$

and $g_c(M_v) dM_v$ = no. of stars of same B-V in corresponding range $M_v,\ M_v + dM_v$

where the corresponding range is defined by equation (A1.1), which shows that M_v is a function of v only, for a given colour. Then

$$f_c(v)\, dv = g_c(M_v)\, dM_v \qquad \text{(A1.9)}$$

and

$$\langle M_v \rangle_c = \frac{\int_0^\infty g_c(M_v)\, M_v\, dM_v}{\int_0^\infty g_c(M_v)\, dM_v} \qquad \text{(A1.10)}$$

which may, now obviously, be approximated by

$$\langle M_v \rangle_c = \frac{1}{N_c} \sum_{j=1}^{N_c} M_{v_j} \ . \qquad \text{(A1.11)}$$

Once v has been found for a particular star, $M_V^{(0)}$ can be found from equation (A1.1). In principle, each star of the same colour will give the same value of $M_V^{(0)}$. In practice, there will be a slight scatter, due mainly to observational error. The true value of $M_V^{(0)}$ for each colour may be estimated by taking an arithmetic mean. The zero-rotation main sequence is then defined by finding $M_V^{(0)}$ for several colour intervals.

3. A method of finding κ

The discussion of section 2 is only useful if the theoretical results are known to be reliable. That is not the case and it is desirable to have a method of determining κ from the observations. The following method has a sounder theoretical basis than that used by Strittmatter (1966), but it will be seen to require more data than is at present available.

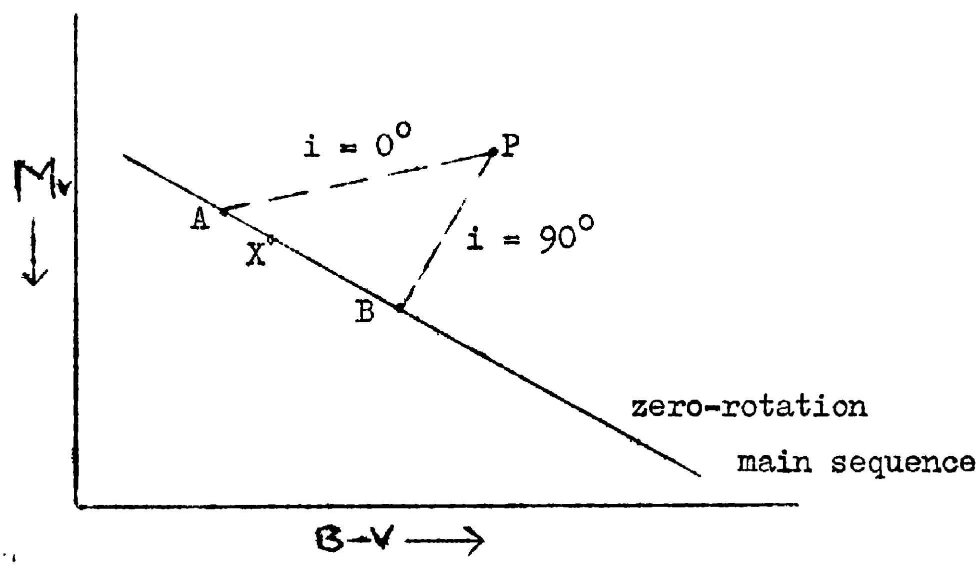

Fig.16 . Possible values of i for a star with a given M_V, B-V.

Consider the point P in fig. 16. A star in this position has a definite absolute magnitude and colour and so, from equation (A1.1), it has a definite value of v. However, it may have any value of i, depending on its mass. Any star in the mass range corresponding to the portion AB of the zero-rotation main sequence could be displaced to P if it had the correct value of i.

Suppose a number of stars with the same (M_v, B-V) but differing v sin i are observed. Then an average may be taken, using the formula for <u>fixed</u> v

$$\left\langle (v \sin i)^2 \right\rangle_c = \frac{2}{3} v^2 \qquad \text{(A1.12)}$$

which corresponds to (A1.3); this will immediately give the value of v^2 for the observed M_v. According to equation (A1.1), a plot of M_v against v^2 should give a straight line with slope $\varkappa$ and intercept $M_v^{(0)}$ on the M_v axis. Such a line can be obtained in principle for each colour, so that $M_v^{(0)}$ can be found as a function of colour and the zero-rotation main sequence can be determined. Also, since $\varkappa$ should be the same for each colour, the lines will be roughly parallel and $\varkappa$ can be accurately found by superimposing the lines and taking the mean slope. Alternatively, if the data were good enough, the slight variation of $\varkappa$ with B-V could be detected.

There are two difficulties in this method, one of principle and one of practice. Equation (A1.12) assumes that the axes of the stars at P are randomly oriented. That is reasonable only if the corresponding stars on the main sequence are supposed uniformly distributed with spectral type over the range AB (Fig.16). If, for example, the main sequence contained no stars earlier than X, for evolutionary or other reasons, stars appearing at P could only have i in the range $i_X \leq i \leq 90^o$. In that case, $p(i) \neq \sin i$ as assumed, since $p(i) = 0$ for $0 \leq i \leq i_X$. However, so long as an actual cut-off like that is not present, the variation of number of stars is probably slow enough, at least for early-type stars, that equation (A1.12) is a good approximation.

The second difficulty is that, for the method to be used, several stars in each interval of B-V must have the same M_V. In practice, not enough stars are available for this method to be useful. Certainly the data available for Praesepe, the cluster studied by Strittmatter, is insufficient for this method to be applied.

4. Strittmatter's method for finding $\varkappa$

One way of making up for the lack of data available is to plot $M_V - M_V^{(0)}$ against v^2, since this difference is independent of colour and the results from all the colour intervals can be superimposed. Of course, $M_V^{(0)}$ is not known as a function of colour. However, it is usually possible to estimate at least the slope of the zero-rotation main sequence by drawing a line through those stars in the HR diagram which have the smallest brightness for a given colour. This should give a reference line, r(B-V), which differs from $M_V^{(0)}$(B-V) only by a constant magnitude difference, so that a plot of

$M_V - r$ against v^2 will have a slope $\varkappa$ and the same intercept ($= M_V^{(o)} - r$) for all colours. It may also be assumed that a line parallel to the line $r(B-V)$ will be made up of stars of the same v, though varying v sin i. It is then possible to find this value of v by averaging over the stars on this line. Since stars of all colours can now be considered, more stars are available for the average and a meaningful value for v^2 can be obtained, although even with this method barely enough stars are available in practice for the larger values of v.

Strittmatter uses essentially this method, but combines the data in a slightly different manner, which obviates any averaging problem for large v but which introduces fresh difficulties. Instead of plotting $M_V - r$ against v^2, Strittmatter plots $M_V - \langle M_V \rangle_c$ against $Q = (v \sin i)^2 - \langle (v \sin i)^2 \rangle_c$. He expects a mean relation between these quantities, and therefore fits a best straight line to the plotted points, using a least squares analysis, whose slope is claimed to be proportional to $\varkappa$. It is not clear that this procedure is justified. From equation (A1.1),

$$M_V - \langle M_V \rangle_c = \tfrac{3}{2}\varkappa Q + \tfrac{3}{2}\varkappa v^2 \left(\tfrac{2}{3} - \sin^2 i\right) \qquad \text{(A1.13)}$$

so that the relation between $M_V - \langle M_V \rangle_c$ and Q is certainly not linear; the scatter in Strittmatter's diagram is intrinsic rather than due to random error and a least squares analysis is not strictly applicable. It is, therefore, not surprising that Strittmatter finds a root mean square error of about 18% in the determination of the slope.

However, when the theoretically correct method of analysing the plot of $M_V - \langle M_V \rangle_c$ against Q is considered, it is found not to be practicable. From equation (A1.7), stars with a given $\Delta M_V = M_V - \langle M_V \rangle_c$ have a fixed $X = \frac{2}{3} v^2 - \langle (v \sin i)^2 \rangle_c$. From the plot of ΔM_V against Q, it can be found what values of Q correspond to a particular value of X.

The points which give these values of Q will lie in a horizontal strip of the plot, defined by a small range in ΔM_V. The value of X corresponding to this range can be estimated by taking the arithmetic mean of the Q-values of the points in the horizontal strip, although it is not clear that this is the correct average to take. It is then possible to plot ΔM_V against X, which should give a straight line of slope $\frac{3}{2}\kappa$. However, for a given small range of ΔM_V there are not really sufficient points to give a meaningful average value for X. It is therefore probably just as accurate, in practice, to take an average over all the points in the plot by doing a least squares analysis, as Strittmatter does. The main snag of such a procedure is that it is not at all clear what is the theoretical relation between the line given by a plot of ΔM_V against X and the line obtained by a least squares analysis. There is therefore no way of knowing how good an approximation for κ is given by the least squares analysis.

It seems, then, that the only practicable method of finding κ from observations rests on a rather insecure theoretical foundation while the methods which are theoretically sound, and simple in principle, are unworkable in practice through lack of data. The only definite conclusion one can reach from this study is that, while the zero-rotation main sequence can be found if the theoretical values of κ are accepted, not enough information is at present available to enable κ to be found sufficiently accurately from observations to provide a meaningful check on the theory.

APPENDIX II

Properties of $h(\psi)$

In the local theory there appears a function h defined by

$$h(\psi) = -\frac{1}{4\pi}\int_{\psi = \text{const.}} \text{grad}\,\psi\,.\,\underline{dS}_{\psi} \qquad \text{(A2.1)}$$

where the integral is taken over a surface of constant ψ. In terms of the dimensionless radius, σ, and the colatitude θ, ψ is defined by (equation (3.9))

$$\psi = \frac{1}{\sigma} + \sigma^2 \sin^2\theta \qquad \text{(A2.2)}$$

Because of the lack of sphericity of the ψ surfaces (see Fig. 5) the function $h(\psi)$ is extremely complicated, particularly near the surface of the star and for stars rotating at the limit of stability. The theory of Chapter 3 has been applied, by Roxburgh, Griffith and Sweet (1965), to find the structure of a star rotating at break-up speed. In that treatment an expansion for $h(\psi)$ near the surface was required to start the numerical integration and, for convenience, this expansion was simply obtained from the numerical determination of h by fitting a curve to the numerical values (Roxburgh, 1966, private communication). The expression found was (RGS, equation (4.23))

$$h(\psi) = 0.639275 + 0.849(\psi - \psi_s)^{4/5} + \ldots \qquad \text{(A2.3)}$$

where $\psi_s = 3 \times 2^{-2/3}$ is the surface value of ψ at break-up. This result is quoted for later comparison with a series for $h(\psi)$ obtained by expansion of the integral.

Before attempting the expansion, it is necessary to put the integral in scalar form. The first requirement for this is to know the expression for an element of area on a ψ-surface. Let Δ be the angle between a normal $\underline{n}$ to the ψ-surface and the radius vector through the same point (Fig. 17). Then

$$\tan\Delta = \frac{dr}{r d\theta} = \frac{d\sigma}{\sigma d\theta} \qquad (A2.4)$$

and the dimensionless element of area is

$$\left|\underline{dS}_{\psi}\right| = \frac{\sigma d\theta}{\cos\Delta} \cdot \sigma \sin\theta \, d\phi = \sigma^2 \sin\theta \, d\theta \, d\phi \left[1 + \frac{1}{\sigma^2}\left(\frac{d\sigma}{d\theta}\right)^2_{\psi=\text{const}}\right]^{1/2} \qquad (A2.5)$$

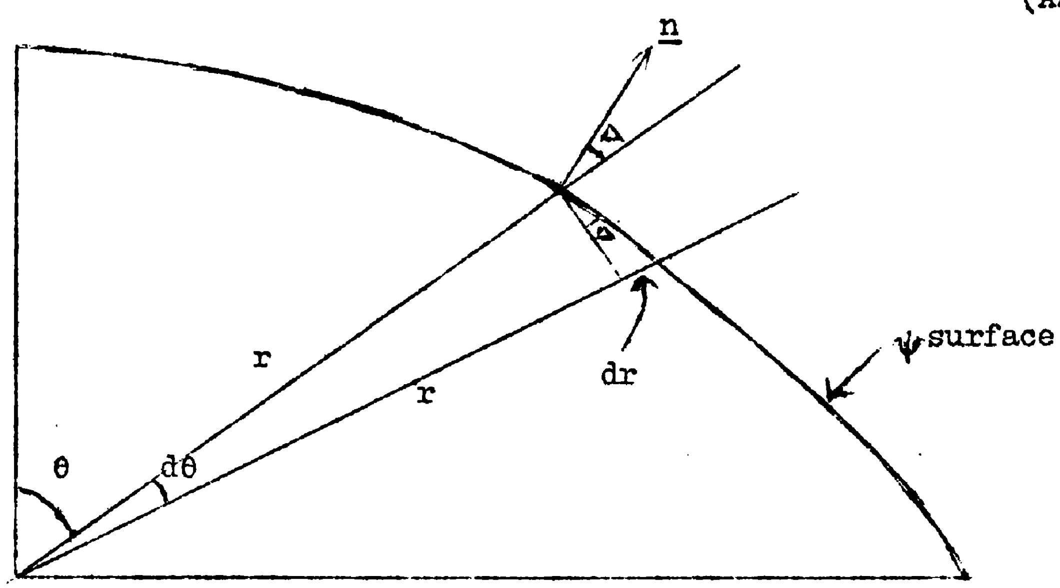

Fig. 17. See text.

From the definition of ψ it is now possible to express the integrand as a function of σ and $\mu = \cos\theta$. On a surface of constant ψ, $\mu = \mu(\sigma)$ and the integrand must be expressed as a function of σ only by writing throughout

$$\mu = \left(\frac{\sigma^3 - \psi\sigma + 1}{\sigma^3}\right)^{1/2} . \tag{A2.6}$$

From the form of equation (A2.2), this is more convenient than eliminating σ from the integrand. Also for convenience, the variable of integration should now be changed to x, where

$$x = \frac{1}{\sigma} \tag{A2.7}$$

so that $\mu = (x^3 - \psi x^2 + 1)^{1/2}$. The limits of integration for μ are 0 and 1 (this gives the integral over half the surface, which must be multiplied by 2 to give $h(\psi)$); the corresponding limits for x are:

$$\left.\begin{array}{ll} \mu = 1 & x = \psi \\ \mu = 0 & x = x_1 \\ \multicolumn{2}{l}{\text{where } x_1 \text{ is the largest root of } x^3 - \psi x^2 + 1 = 0} \end{array}\right\} \tag{A2.8}$$

It should be noted that $x_1 < \psi$ for all ψ. Some manipulation now gives

$$h(\psi) = +\frac{1}{2}\int_{x_1}^{\psi} \frac{(5x^4 - 4\psi x^3 - 4x + 4\psi)\,dx}{(x^3 - \psi x^2 + 1)^{1/2}\quad x^2} \tag{A2.9}$$

Alternatively, $h(\psi)$ may be written as

$$h(\psi) = -\frac{1}{3} + \frac{2}{3}\psi I_1 - 2I_{-1} \tag{A2.10}$$

where $$I_m = \int_{x_1}^{\psi} \frac{x^m\,dx}{(x^3 - \psi x^2 + 1)^{1/2}}$$

$h(\psi)$ is an elliptic integral, and it may be expressed in terms of elementary functions and the standard Jacobi elliptic integrals (see, for example, Whittaker and Watson, 1927, p.512 et seq.). However, the resulting formula, which is given below, is not particularly useful, because of its complexity. It is

$$h(\psi) = -\frac{1}{3} + \frac{2A}{3q}(\psi pq - 3)\left[K(k^2) + F(k^2,\phi)\right]$$

$$- \frac{2A\psi(p-q)}{3(1-k^2)}\left[E(k^2) + E(k^2,\phi)\right] + \frac{2A(p-q)}{pq}\left[\Pi\left(-\frac{q^2}{p^2}, k^2\right) + \Pi\left(-\frac{q^2}{p^2},k^2,\phi\right)\right]$$

$$+ A(p-q)\left[\frac{2\psi}{3(1-k^2)}\left(\frac{1-k^2Y}{X}\right)^{1/2} - \frac{\log\left(\frac{1-R}{1+R}\right)}{\sqrt{(p^2-q^2)(k^2p^2-q^2)}}\right]$$

where: $A = \dfrac{2x_1}{\sqrt{2x_1^3 - 1 + 2X}}$; $X = \sqrt{x_1^3(x_1^3 - 2)}$; $Y = \left(\dfrac{1 - X}{1 + X}\right)^2$;

$$p = x_1 + \sqrt{\frac{x_1^3 - 2}{x_1}} \;; \quad q = x_1 - \sqrt{\frac{x_1^3 - 2}{x_1}} \;;$$

$$k = \frac{2x_1^3 - 1 - 2X}{\sqrt{4x_1^3 + 1}} \;; \quad R = \sqrt{\frac{(p^2-q^2)(1-k^2Y)}{(k^2p^2-q^2)(1-Y)}} \;; \qquad \text{(A2.11)}$$

$$F(k^2,\phi) = \int_0^{\phi}\frac{d\alpha}{(1-k^2\sin^2\alpha)} \;; \qquad K(k^2) = F(k^2,\pi/2);$$

$$E(k^2) = E(k^2,\pi/2);$$

$$E(k^2,\phi) = \int_0^{\phi}\sqrt{(1-k^2\sin^2\alpha)}\,d\alpha \;; \quad \Pi(-\frac{q^2}{p^2},k^2) = \Pi(-\frac{q^2}{p^2},k^2,\pi/2);$$

$$\Pi\left(-\frac{q^2}{p^2},k^2,\phi\right) = \int_0^{\phi}\frac{d\alpha}{(1-\frac{q^2}{p^2}\sin^2\alpha)\sqrt{(1-k^2\sin^2\alpha)}} \;; \quad \text{and} \quad \sin\phi = \frac{1 - X}{1 + X}.$$

The functions F, E and Π are the Jacobi elliptic integrals of the first, second and third kind respectively. Note that these integrals are not defined when $k^2 = 1$, which is the case at the surface of a star rotating at break-up speed, where $x_1^3 = 2$ (this result follows from $\nabla\psi = \underline{0}$ at $\theta = \pi/2$). However, the value of h when $x_1^3 = 2$ can be calculated directly from equation (A2.10), since in that case the I_m can be integrated directly in terms of elementary functions. The result is

$$h_s = \frac{11}{3} - 2\sqrt{3} + 2 \ln \frac{2+\sqrt{3}}{3} = 0.639256 \qquad \text{(A2.12)}$$

This result was first derived by Griffith(1962). Comparison with equation (A2.3) shows that the value given in RGS is correct to 4 significant figures.

In order to test the accuracy of the numerical expansion used in RGS it is desirable to expand $h(\psi)$ about $\psi = \psi_s$. This is a very complicated procedure, since $\frac{dh}{d\psi}$ has a logarithmic singularity at $\psi = \psi_s$. However, by writing

$$\psi = \psi_s + \xi \qquad \text{(A2.13)}$$

and using the expression (A2.10) for $h(\psi)$, it can be shown, by reducing the problem to one of solving a second order differential equation, that

$$\begin{aligned} h(\psi) = h_s &+ 2^{-1/3} 3^{-1/2} \xi \ln(1/\xi) \\ &+ 2^{-1/3}\left[\frac{8}{3} + 3^{-1/2}\left(-5 - \frac{5}{12}\ln 2 + \frac{9}{8}\ln 3 - \frac{1}{4}\ln(2+\sqrt{3})\right)\right]\xi \\ &+ O(\xi^{3/2}\ln(1/\xi)) \qquad \text{(A2.14)} \\ = \; & 0.639256 + 0.45825\,\xi\ln(1/\xi) + 0.1084\,\xi + \ldots \end{aligned}$$

Comparison of this result with that of RGS (equation (A2.3)) shows that the results agree to within one part in six hundred for $\xi \leq 0.02$.

Even at $\xi = 0.1$, the results differ by less than 3%.

Except in Chapter 3, it has been possible in this thesis only to study slowly rotating stars. In that case ψ is large compared with 1, even near the surface of the star, and it is more useful to expand $h(\psi)$ in powers of $1/\psi$. It is convenient to do this by writing

$$\left.\begin{aligned} \psi &= x_1(1 + \varepsilon) \\ \text{where} \quad \varepsilon &= \frac{1}{x_1^3} \longrightarrow 0 \quad \text{as} \quad \psi \longrightarrow \infty \end{aligned}\right\} \tag{A2.15}$$

and expanding $h(\psi)$ (given by equation (A2.10)) in powers of ε first. It is then easy to show that

$$h(\psi) = 1 - \frac{4}{3}\frac{1}{\psi^3} - \frac{8}{3}\frac{1}{\psi^6} - \frac{188}{15}\frac{1}{\psi^9} + O\left(\frac{1}{\psi^{12}}\right) . \tag{A2.16}$$

APPENDIX III

The general energy balance equation

In Chapter 2, the energy balance equation was quoted in the form(equation (2.10))

$$L = \int_{\Psi=\text{const}} \underset{\sim}{\mathcal{F}} \cdot \underline{dS} \qquad \text{(A3.1)}$$

where the integration is over a surface of constant Ψ and $\underset{\sim}{\mathcal{F}}$ is the radiative flux, given by

$$\underset{\sim}{\mathcal{F}}_{rad} = -\frac{16}{3}\frac{\sigma}{\kappa}\frac{T^3}{\rho}\underline{\nabla}T \quad . \qquad \text{(A3.2)}$$

More generally, if $\underset{\sim}{\mathcal{F}}_{tot}$ is the total flux of energy through the surface layers, the equation

$$L = \int_{\Sigma} \underset{\sim}{\mathcal{F}}_{tot} \cdot \underline{dS} \qquad \text{(A3.3)}$$

is valid for <u>any</u> closed surface Σ containing the energy-generating core of the star. (It is assumed that there are no energy sources in the atmosphere.) In a radiative atmosphere with circulation currents, the total flux is the sum of the radiative flux, $\underset{\sim}{\mathcal{F}}_{rad}$, and the convective flux, given by

$$\underset{\sim}{\mathcal{F}}_{conv} = \left(\frac{\gamma}{\gamma - 1} P - \rho\Psi \right) \underline{v} \qquad \text{(A3.4)}$$

(cf. for example, RGS, equation (2.8)).

If the surface Σ is a surface of constant Ψ, then, since P, ρ and T are functions of Ψ only, it may be shown (using the equation of state) that

$$\int_{\Sigma} \underset{\sim}{\mathcal{F}}_{conv} \cdot \underline{dS} = \left(\frac{\gamma}{\gamma - 1} \frac{\mathcal{R}}{m} T - \Psi \right) \int_{\Psi = const} \rho \underline{v} \cdot \underline{dS} = 0 \qquad (A3.5)$$

The second equality follows from the assumption that there is no net flow of mass over any closed surface containing the centre of the star (cf. equation (2.8)(b)). In that case, equation (A3.3) reduces to

$$L = \int_{\Psi = const} \underset{\sim}{\mathcal{F}}_{rad} \cdot \underline{dS} \qquad (A3.6)$$

which is just equation (A3.1).

For a general surface, which is not a level surface, this result is not true and equations (A3.2) to (A3.4) must be used. If, on the other hand, the surface Σ is such that its distance from a given level surface is everywhere of order ϵ, an approximate result may be obtained. In that case, equation (A3.3) may be written as

$$L = \int_{\Sigma} \underset{\sim}{\mathcal{F}}_{rad} \cdot \underline{dS} + \left(\int_{\Sigma} - \int_{\Psi = const} \right) \underset{\sim}{\mathcal{F}}_{conv} \cdot \underline{dS} \qquad (A3.7)$$

by making use of equation (A3.5). The second integral is the integral of div $\underset{\sim}{\mathcal{F}}_{conv}$ over the volume between the two surfaces. Since the distance between the surfaces is, by hypothesis, of order ϵ, the volume element must also be of order ϵ. At the same time, the integrand involves $\underline{v}$, which is known to be at least of first order in ϵ (second order in the local theory). The integral is therefore a product of two quantities of order ϵ and must be itself of order ϵ^2(ϵ^3 in the local theory). Hence equation (A3.3) may be written approximately as

$$L = \int_{\Sigma_\epsilon} \underset{\sim}{\mathcal{F}}_{rad} \cdot \underline{dS} \qquad (A3.8)$$

where the error is of order ϵ^2. The subscript ϵ is intended to be a reminder that this result holds only for surfaces Σ_ϵ which are nearly level surfaces.

APPENDIX IV

Singular Perturbation Theory.

1. The problem stated.

Once it has been proved that the 1/density dependence of the velocity field in the local theory is a general result (cf Chapter 3 and Mestel 1966), there is no reason why perturbation methods should not be used even in the surface layers of a star, provided the rotation is slow enough. Indeed, that was implicitly assumed in expanding u_ψ and u_χ in powers of $1/\psi$ in Chapter 3.

However, care is required in using a perturbation theory, because of the distortion of the rotating star from a spherical shape. A naive use of spherical polar coordinates leads to first order terms which are comparable with, or greater than, the zero order terms near the surface, and it is necessary to introduce a new coordinate system to surmount this difficulty.

The problem is most easily illustrated by using the local theory, for which an exact solution is known to exist, but it also occurs in the non-local theory, and the new coordinates must be used in that case also.

2. General results.

In any problem in which perturbation methods are applicable, there exists a small parameter. In this case it is ϵ, defined in Chapter 2. Then a general function Q may be written

$$Q = Q_0 + \epsilon Q_1 + \dots \qquad (A4.1)$$

If one supposes that spherical polar coordinates are appropriate, then $Q = Q(r, \theta, \epsilon)$. Also, since $\epsilon = 0$ corresponds to a spherical star, $Q_0 = Q_0(r)$. In general, $Q_1 = Q_1(r, \theta)$.

In a rotating star, in which equation (2.6) is valid, P, ρ and T are functions of Ψ (and ϵ) only. Now it is possible to write, for a general function Q of Ψ and ϵ,

$$\begin{aligned} Q(\Psi, \epsilon) &= Q(\Psi(r, \theta, \epsilon); \epsilon) \\ &= Q_0(\Psi(r, \theta, 0); 0) + \epsilon\left[\frac{\partial Q}{\partial \Psi}\frac{\partial \Psi}{\partial \epsilon} + \frac{\partial Q}{\partial \epsilon}\right]_{\epsilon = 0} + \dots \end{aligned} \qquad (A4.2)$$

But $\Psi(r, \theta, 0) = \Psi_0(r)$, $\qquad \left(\frac{\partial Q}{\partial \Psi}\right)_{\epsilon = 0} = \frac{dQ_0}{d\Psi_0}$,

$$\left(\frac{\partial \Psi}{\partial \epsilon}\right)_{\epsilon = 0} = \Psi_1 \text{ and } \left(\frac{\partial Q}{\partial \epsilon}\right)_{\epsilon = 0} = Q'(\Psi(r, \theta, 0); 0) = Q'(r).$$

It then follows from the definition of Ψ (equation (2.16)) that

$$Q(\Psi, \epsilon) = Q_0(r) + \epsilon\left[Q_{100}(r)\sin^2\theta + Q_{11}(r)\right] \qquad (A4.3)$$

where
$$Q_{100}(r) = -\frac{1}{2}\frac{r^4}{R^3}\frac{dQ_0}{dr} \qquad (A4.4)$$

Thus, once the zero order theory is complete, the θ-dependence of $Q_1(r, \theta)$ is known immediately. This is a considerable simplification, since it means that the differential equations in the first order theory are ordinary rather than partial.

It is more convenient to express $Q_1(r, \theta)$ in terms of Legendre polynomials, $P_n(\cos\theta)$ ($n = 0, 1, 2\dots$). Equations(A4.3), (A4.4) show that only P_o and P_2 appear and that the coefficient of P_2 is

$$Q_{12} = - \frac{2}{3} Q_{100} = \frac{1}{3} \frac{r^4}{R^3} \frac{dQ_o}{dr} \qquad (A4.5)$$

This expression may be verified by writing

$$Q_1(r, \theta) = \sum_{i=1}^{\infty} Q_{1i}(r) P_i(\cos\theta) \qquad (A4.6)$$

for the functions P, ρ and T and determining the $Q_{1i}(r)$ from the hydrostatic equation and the equation of state (equations (2.6),(2.7)). Only the Q_{10} and the Q_{12} are non-zero, and the Q_{12} are indeed given by equation (A4.5). The Q_{10} must be determined by using the equations (2.10),(2.11). However, to demonstrate the problem raised by using spherical polar coordinates, it is not necessary to find the Q_{10}, since the Q_{12}, which can be found immediately from the zero order theory, sufficiently show the nature of the problem.

3. The problem illustrated.

The complete equations for the local theory are equations (3.1) to (3.7). Equations (3.4),(3.5) are not required here, and it is easily seen that, when $\epsilon = 0$, the other equations reduce to

$$\Psi_o = \frac{GM}{r} \qquad (A4.7)$$

$$\frac{dP_o}{dr} = - \rho_o \frac{GM}{r^2} \qquad (A4.8)$$

$$P_o = \frac{\mathcal{R}}{m} \rho_o T_o \qquad (A4.9)$$

$$\mathcal{F}_{or} = -\frac{16}{3}\frac{\sigma}{\varkappa}\frac{T_o^3}{\rho_o}\frac{dT_o}{dr} \tag{A4.10}$$

$$L_o = \int_{r=const.} \mathcal{F}_o \cdot d\underline{S} \tag{A4.11}$$

Equations (A4.10) and (A4.11) may be combined to give a differential equation for T_o:

$$\frac{dT_o^4}{dr} = -\frac{3}{4}\varkappa R^2 T_e^4 \frac{\rho_o}{r^2} \tag{A4.12}$$

where the definition (2.19) has been used to give L_o in terms of R and T_e. Equations (A4.9) and (A4.12), together with the boundary conditions

$$P_o = T_o = 0 \quad \text{on } r = R \tag{A4.13}$$

then uniquely determine P_o, ρ_o and T_o as functions of r. It is easy to show that the solutions of these equations are

$$P_o = \frac{GM}{\varkappa R^2}\frac{4}{3(4\epsilon_1)^4}\left(\frac{R}{r} - 1\right)^4 \tag{A4.14}$$

$$\rho_o = \frac{1}{\varkappa\epsilon_1 R}\frac{4}{3(4\epsilon_1)^3}\left(\frac{R}{r} - 1\right)^3 \tag{A4.15}$$

$$T_o = T_e \frac{1}{4\epsilon_1}\left(\frac{R}{r} - 1\right) \tag{A4.16}$$

if the definition (2.18) is used for ϵ_1. Similar results have previously been obtained for a more general opacity law by Eddington (1930a) and Chandrasekhar (1939,p300). It is now possible to find P_{12}, ρ_{12} and T_{12} immediately from equation (A4.6). The results are:

$$3\frac{T_{12}}{T_o} = \frac{3}{4}\frac{P_{12}}{P_o} = \frac{\rho_{12}}{\rho_o} = -\frac{r^2}{R^2}\frac{1}{\left(\frac{R}{r}-1\right)} \longrightarrow -\infty \text{ as } r \to R \qquad \text{(A4.17)}$$

Since P_{12}, ρ_{12} and T_{12} are all finite or zero at $r = R$, the singularity may not appear disastrous from a physical point of view. However, it is quite clearly inadmissible mathematically since it violates the basic principle of a perturbation analysis, that the zero order terms should be dominant everywhere.

As was mentioned in section 1, essentially the same problem, though in a less severe form, occurs in the non-local theory,* so that the difficulty is not caused by using the local theory. To see the nature of the trouble, consider the following rough physical argument, based on fig. 18.

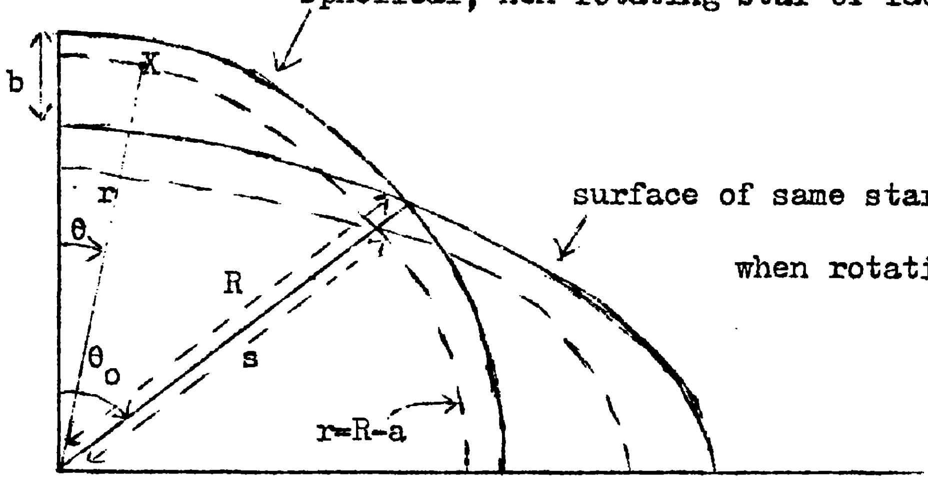

Fig. 18. See text. (s and θ_o are defined in later sections.)

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*In that case, using the plane parallel approximation (see Chapter 5),

$$\frac{T_{12}}{T_o} \to 0 \text{ as } z \to \infty \text{, but } \left.\begin{matrix}\frac{P_{12}}{P_o}\\ \frac{\rho_{12}}{\rho_o}\end{matrix}\right\} \to -\frac{2^{1/4}}{3\epsilon_1} \sim -10^3 \text{ as } z \longrightarrow \infty$$

This follows from equation (A4.5) and the zero order results of Chapter 6.

Take a spherical, non-rotating star of radius R, and let the pressure be given by $P_o(r)$, where $P_o = 0$ at $r = R$ (and, implicitly, $P_o \equiv 0$ outside the star). Now suppose the star is set rotating. It will be distorted in such a way that the equatorial radius becomes greater than the polar radius. Suppose that the polar radius is now less than the radius of the non-rotating star by an amount b (Fig. 18). For the models of RGS, b is positive. (More generally, this may not be true and the following argument would then need to be modified.)

Consider the sphere $r = R - a$, $a < b$. At the point X on this sphere, chosen so as to be outside the rotating star, the value of the pressure P has changed from $P_o(R - a)$ to zero and equation (A4.1) gives (to first order in $\in$)

$$0 = P_o(R - a) + \in P_1 (R - a, \theta) \qquad \text{(A4.18)}$$

which immediately implies that $\in P_1 \sim P_o$. Of course, if $a > b$ this argument no longer holds and so trouble is to be expected only near the surface.

The above argument shows that the trouble arises essentially because the perturbation is about a fixed point in space. That is, the change, due to rotation, in the pressure, density or temperature at a fixed point in space is supposed small. The above argument shows why this is a poor assumption near the surface. It would be more correct to suppose that P, ρ and T remain approximately constant for any <u>fluid element</u> of the star, which may of course be displaced by rotation. Thus the coordinates of a fluid element, as well as its pressure, density and temperature, may be expected to be affected by rotation. This crude physical discussion is made precise in the next section.

4. Lighthill's method

Mathematically, the above problem requires the methods of singular perturbation theory for its solution. The author is indebted to Professor Milton Van Dyke for drawing his attention to Lighthill's method of strained coordinates, which is admirably suited to the present situation (see, for example, Van Dyke 1964, Chapter 6).

The principle of Lighthill's method is to introduce new coordinates differing only slightly from the old ones and obtained by expanding the old coordinates in powers of the perturbation parameter. In the present case, the new coordinates, (s,χ^*) say, will be defined in terms of (r,θ) by

$$\left.\begin{aligned} r &= s + \epsilon r_1(s,\chi^*) + \ldots \\ \theta &= \chi^* \end{aligned}\right\} \tag{A4.19}$$

Only the coordinate r need be changed, since there is no evidence that the θ-dependence is causing any trouble. Later it will be seen to be more appropriate to replace θ by the χ defined in Chapter 2. For the moment, χ^* will be written as θ throughout.

If all other functions, Q say, are written

$$Q = Q_0(s) + \epsilon Q_1(s,\theta) + \ldots \tag{A4.20}$$

the function $r_1(s,\theta)$ may be determined by the condition that

$$\frac{Q_1}{Q_0} \lesssim 1 \quad \text{everywhere,} \tag{A4.21}$$

in particular near the surface. This procedure has been carried out, using the local theory, and a function $r_1(s,\theta)$ has been found which makes $\frac{P_1}{P_0}$, $\frac{\rho_1}{\rho_0}$ and $\frac{T_1}{T_0}$ finite at the surface of the star. It was found that there is a simple physical interpretation for s which makes s constant on a level surface. In view of the exact theory of Chapter 3, the use of such a coordinate appears entirely natural, and it is not surprising that this choice for s removes the trouble.

For this reason, and because the details of the process are tedious, the method of obtaining r_1 will only be outlined very briefly. The results of a local perturbation theory are required to provide boundary conditions for the non-local theory. However, the results are only required in the plane-parallel approximation and are more appropriately derived separately (Appendix VII).

5. A solution for r_1

To apply Lighthill's method, the coordinates (r,θ) are written as in equation (A4.19) and all other functions are expanded as in equation (A4.20). Working to first order in ϵ, one can write the gradient operator as

$$\left.\begin{aligned} \text{grad} &\equiv \left(\frac{\partial}{\partial r}\,;\ \frac{1}{r}\frac{\partial}{\partial\theta}\,;\ 0\right) \\ \text{where}\quad \frac{\partial}{\partial r} &\equiv \left(1-\epsilon\frac{\partial r_1}{\partial s}\right)\frac{\partial}{\partial s} \\ \text{and}\quad \frac{\partial}{\partial\theta} &\equiv \frac{\partial}{\partial\chi^*}-\epsilon\frac{\partial r_1}{\partial\chi^*}\frac{\partial}{\partial s} \end{aligned}\right\} \tag{A4.22}$$

and then the equations (3.1) to (3.3), (3.6) and (3.7) may be written in the new coordinates, to first order in ϵ.

It is intuitively obvious that, since r = s to lowest order, the zero order equations will be the same as in section 3, except that r will be replaced by s throughout. The zero order pressure, density and temperature are therefore given by equations (A4.14) to (A4.16) with r replaced by s.

To write down the first order equations in general is a lengthier procedure. However, if it is assumed, as seems reasonable, that r_1 can be expressed as

$$r_1(s,\theta) = r_{10}(s) + r_{12}(s)P_2(\cos\ \theta) \tag{A4.23}$$

the methods of section 2 show that, as before, P_1, ρ_1 and T_1 only have P_o and P_2 terms and that in this case the coefficients of P_2 are given by expressions of the form

$$Q_{12}(s) = \left(r_{12}(s) + \frac{1}{3}\frac{s^4}{R^3} \right)\frac{dQ_o}{ds} \tag{A4.24}$$

The difference between this equation and equation (A4.5) arises from the different expression for $\nabla\psi_1$ in the new coordinates.

It then follows at once that

$$\frac{3T_{12}}{T_o} = \frac{3}{4}\frac{P_{12}}{P_o} = \frac{\rho_{12}}{\rho_o} = -\frac{s^2}{R^2}\frac{(1 + 3r_{12}\frac{R^3}{s^4})}{(\frac{R}{s} - 1)}, \tag{A4.25}$$

a result which should be compared with equation (A4.17). In this case it is possible to prevent a singularity at s = R by choosing r_{12} suitably. An obvious choice is

$$r_{12} = -\frac{1}{3}\frac{s^4}{R^3} \qquad (A4.26)$$

which makes $P_{12} \equiv \rho_{12} \equiv T_{12} \equiv 0$. This appears to be a very drastic solution to the problem, but it is justified by the very natural interpretation of s to which it leads.

Of course, r_1 is not fully determined until a suitable function has been chosen for r_{10}. Such a function was originally found by solving the first order equations for P_{10}, ρ_{10} and T_{10} and applying the condition (A4.21). It is unnecessary to go into the details of this process, and it is simply noted that one of the possible solutions for r_{10} is consistent with the interpretation of s given below, which is introduced on a more intuitive basis.

6. The interpretation of s

The result of the choice (A4.26) for r_{12} is to make P_1, ρ_1 and T_1 functions of s only. It is already known that P, ρ and T are functions of Ψ only. This immediately suggests that s should be chosen in such a way as to be constant on a surface of constant Ψ (for a given ϵ). One way of doing this is as follows. Suppose that the polar radius of a rotating star is less than R, the radius of the non-rotating star of the same mass. This is certainly true of the models considered here, which are based on RGS. Then it is clear that, whatever the rotation speed, there will always be a small circle on the surface (centred on the axis of rotation) whose distance from the centre of the star is R (Fig. 18). Let θ_o be the angular radius of this circle. The circle then determines a cone of semi-angle θ_o whose axis is the rotation axis. This cone will cut any surface of constant Ψ in a small circle of angular radius θ_o. s will be defined to be the distance of this small circle from the centre of the star. It is then immediately clear from Fig. 18 that the surface of the star is given by s = R, and also that

$$\Psi = \frac{GM}{s}\left(1 + \frac{1}{2}\epsilon\frac{s^3}{R^3}\sin^2\theta_o \right) . \qquad (A4.27)$$

From equation (2.16) for Ψ in terms of r, and using the fact that $\sin^2\theta = \frac{2}{3}(1 - P_2(\cos\theta))$, it may then be shown that

$$r_1(s,\theta) = \frac{1}{3}\frac{s^4}{R^3}\left[P_2(\cos\theta_o) - P_2(\cos\theta)\right] \qquad (A4.28)$$

in agreement with equation (A4.26).

With this choice for s, the perturbation is about a point essentially fixed on a level surface and the effect of distortion, introduced by the rotation, is included in the zero order functions, which can no longer strictly be thought of as the functions appropriate to a non-rotating star, although they reduce to these as $\epsilon \to 0$. Since the boundary conditions are now to be applied on a surface of constant s, s = R, it is clear that no point with a constant value of s can ever cross the boundary and the argument of section 3 no longer applies. For this reason, it is virtually certain, not only that there will be no trouble in the non-local theory when it is expressed in terms of s, but also that no trouble will arise in higher order terms.

Since s is constant on a level surface, the coordinates

(s, θ, ϕ) do not form an orthogonal set. It is therefore more appropriate to express the results in terms of the orthogonal coordinates (s, χ, ϕ) where χ is defined as in Chapter 2. For use in a perturbation theory, the choice of χ which makes $\chi = \theta$ to lowest order (equation (2.23)) is most convenient. Then, since θ only appears in the first order theory, the difference between θ and χ only affects the second order theory.

Finally, the value of the angle θ_o must be considered. It is not immediately clear how this angle varies with rotation speed. In principle, θ_o is determined by the condition that the mass within the surface $s = R$ is the same as the mass of the non-rotating star of radius R. This condition can only be applied if the complete structure of the star is known. In practice, it is convenient to make use of previously constructed models and to determine θ_o from the polar radius of the rotating star, which has been found in terms of R by Roxburgh, Griffith and Sweet (Roxburgh and Strittmatter 1965). The variation of θ_o with rotation speed is discussed in Appendix VI.

Of course, if the polar radius, R_p, of the rotating star is greater than R, θ_o cannot be defined at all. In that case the best definition for s is probably to take s to be the polar radius of the Ψ surface on which s is to be constant. It is easily shown that with that definition

$$r = s + \frac{1}{3} \frac{s^4}{R^3} (1 - P_2) + \ldots \qquad (A4.29)$$

which is still in agreement with equation (A4.26). However, in that case the boundary of the star (on the local theory) would be $s = R_p$ and so would depend on the rotation speed. Since s appears in the zero order theory, that means introducing an unknown variation with ϵ into the zero order theory. That seems less satisfactory than the presence of θ_0 in the above definition of s, which only introduces an unknown variation with ϵ into the first order theory. It therefore seems preferable to use the definition of s in terms of R and θ_0 where this is possible.

APPENDIX V

Some relations involving Ψ and χ.

1. Algebraic relations

In this section, only Ψ is considered. The definition of Ψ in spherical polar coordinates is

$$\Psi = \frac{GM}{r} + \frac{1}{2}\Omega^2 r^2 \sin^2\theta \quad . \tag{A5.1}$$

By definition of s and θ_o (see Appendix IV), this may also be written

$$\Psi = \frac{GM}{s} + \frac{1}{2}\Omega^2 s^2 \sin^2\theta_o \quad . \tag{A5.2}$$

Since the boundary of a star, on the local theory, is given by s = R, where R is the radius of the non-rotating star of the same mass, the boundary value of Ψ is

$$\Psi_b = \frac{GM}{R} + \frac{1}{2}\Omega^2 R^2 \sin^2\theta_o \tag{A5.3}$$

Suppose that the equatorial and polar radii of this level surface are R_{eq} and R_p respectively. Then, from equation (A5.1), other expressions for Ψ_b are:

$$\frac{GM}{R_p} = \Psi_b = \frac{GM}{R_{eq}} + \frac{1}{2}\Omega^2 R_{eq}^{\ 2} \tag{A5.4}$$

For a star of a given mass, Ω can only take values up to a certain maximum value at which the centrifugal force balances gravity at the equator. When Ω has this maximum value, the equatorial radius is also a maximum and the star is on the verge of rotational break-up. If the maximum equatorial radius is denoted by $R_{eq_{max}}$, the maximum value of Ω is given by the condition

$$\frac{\Omega_{max}^2 R_{eq_{max}}^3}{GM} = 1 \qquad (A5.5)$$

If the parameter α is defined in general by

$$\alpha = \frac{\Omega^2 R_{eq}^3}{GM} \qquad (A5.6)$$

it is clear that α measures the ratio of centrifugal force to gravity at the equator of the star when it is rotating with angular speed Ω and that $\alpha \leq 1$, taking the value 1 when the star is on the verge of break-up.

In terms of α, equation (A5.4) reads

$$\frac{R_{eq}}{R_p} = 1 + \frac{1}{2}\alpha \qquad (A5.7)$$

so that $R_{eq}/R_p = 3/2$ for a star rotating on the verge of break-up (cf. RGS, section 3).

However, the parameter α is not really suitable for use in a perturbation theory, since R_{eq} depends on Ω^2. There are two ways of overcoming this difficulty. The way adopted in this thesis, which is suitable if $R_p < R < R_{eq}$ (so that θ_o and s may be defined), is to use the parameter ϵ, defined by

$$\epsilon = \frac{\Omega^2 R^3}{GM} \qquad (A5.8)$$

This has the disadvantage of not being equal to one when $\Omega = \Omega_{max}$, although of course $\epsilon < 1$ for all Ω. The advantage of this parameter is that it is expressed entirely in terms of quantities known a priori. The alternative parameter ω^2, used by Roxburgh and Strittmatter (1965) and defined by

$$\omega^2 = \frac{\Omega^2 R_{eq_{max}}^3}{GM} \quad , \qquad (A5.9)$$

does not have this advantage, since $R_{eq_{max}}$ is not known until the solution for the star is complete. On the other hand, when $\Omega = \Omega_{max}$, $\omega^2 = 1$. However, ϵ is a more suitable parameter for use in a perturbation theory, where Ω never gets as large as Ω_{max} and $R_{eq_{max}}$ can not be determined. Also, since R_{eq} tends to R as the rotation speed tends to zero, ϵ is more nearly a measure of the ratio of centrifugal force to gravity than is ω^2 for slow rotation.

In terms of ϵ, Ψ may be written as

$$\frac{GM}{r}\left(1 + \frac{1}{2}\epsilon\frac{r^3}{R^3}\sin^2\theta\right) = \Psi = \frac{GM}{s}\left(1 + \frac{1}{2}\epsilon\frac{s^3}{R^3}\sin^2\theta_o\right) \qquad (A5.10)$$

and also

$$\frac{GM}{R_{eq}}\left(1 + \frac{1}{2}\epsilon\frac{R_{eq}^3}{R^3}\right) = \Psi_b = \frac{GM}{R}\left(1 + \frac{1}{2}\epsilon\sin^2\theta_o\right) \qquad (A5.11)$$

From equation (A5.11) it follows that

$$\frac{R_{eq}}{R} = 1 + \frac{1}{2}\epsilon\cos^2\theta_o + O(\epsilon^2) \qquad (A5.12)$$

but this relation is slightly unsatisfactory as θ_o also depends on ϵ. It is better to use equation (A5.4) to obtain the exact relation

$$\frac{R}{R_p} = 1 + \frac{1}{2}\epsilon\sin^2\theta_o \qquad (A5.13)$$

This relation can be used to find θ_o as a function of ϵ if R/R_p is known from the interior solution. Values for θ_o are given in Appendix VI, where it will be found necessary to have a relation between ϵ and ω^2. Let the value of R_p when $\Omega = \Omega_{max}$ be $R_{p_{min}}$. Then, using equation (A5.7),

$$\epsilon = \frac{8}{27}\left(\frac{R}{R_{p_{min}}}\right)^3 \omega^2 \quad . \qquad \text{(A4.14)}$$

This is the required relation.

2. Differential relations

In this section, the formulae for the various scalar and vector derivatives of Ψ, s and χ are summarized. Since Ψ is defined by equation (A5.1), its gradient in spherical polar coordinates is

$$\nabla\Psi = -\frac{GM}{r^2}\left[1 - \epsilon\frac{r^3}{R^3}\sin^2\theta;\ -\epsilon\frac{r^3}{R^3}\sin\theta\cos\theta;\ 0\right] \qquad \text{(A5.15)}$$

It follows easily that

$$|\nabla\Psi| = \frac{GM}{r^2}\left[1 - 2\epsilon\frac{r^3}{R^3}\sin^2\theta + \epsilon^2\frac{r^6}{R^6}\sin^2\theta\right]^{\frac{1}{2}} \qquad \text{(A5.16)}$$

From the definition for χ

$$\chi = \frac{3}{2}\left(\cos\theta + \log\tan\frac{\theta}{2}\right) + \frac{1}{2}\epsilon\frac{r^3}{R^3}\cos^3\theta \qquad \text{(A5.17)}$$

it follows that

$$\nabla\chi = \frac{3}{2}\frac{\cos^2\theta}{\sin\theta}\frac{1}{r}\left[\epsilon\frac{r^3}{R^3}\sin\theta\cos\theta;\ 1 - \epsilon\frac{r^3}{R^3}\sin^2\theta;\ 0\right] \qquad \text{(A5.18)}$$

and so

$$|\nabla\chi| = \frac{3}{2}\frac{\cos^2\theta}{\sin\theta}\frac{1}{r}\left[1 - 2\epsilon\frac{r^3}{R^3}\sin^2\theta + \epsilon^2\frac{r^6}{R^6}\sin^2\theta\right]^{\frac{1}{2}} \qquad \text{(A5.19)}$$

Notice that each of the expressions for $|\nabla\Psi|$, $|\nabla\chi|$ has the same square root appearing in it. Also, the expressions for the gradients in the coordinate system (Ψ, χ, ϕ) are

$$\nabla\Psi = (|\nabla\Psi|;\ 0\ ;\ 0) \quad ; \quad \nabla\chi = (0\ ;|\nabla\chi|;\ 0) \tag{A5.20}$$

The above expressions are exact. In a perturbation theory, it is more useful to work in terms of (s,χ,ϕ) (see Appendix IV) where s is defined in terms of Ψ by equation (A5.2). Then

$$|\nabla\Psi| = \frac{GM}{s^2}\left(1 - \frac{2}{3}\epsilon\frac{s^3}{R^3}\left(1 + P_2(0) - 2P_2(\cos\chi)\right) + O(\epsilon^2)\right) \tag{A5.21}$$

where $P_2(0)$ is a shorthand for $P_2(\cos\theta_o)$. Since $\nabla s = \frac{ds}{d\Psi}\nabla\Psi$

and
$$\frac{d\Psi}{ds} = -\frac{GM}{s^2}\left(1 - \frac{2}{3}\epsilon\frac{s^3}{R^3}(1 - P_2(0))\right) \tag{A5.22}$$

it follows that

$$|\nabla s| = 1 - \frac{4}{3}\epsilon\frac{s^3}{R^3}\left(P_2(0) - P_2(\cos\chi)\right) + O(\epsilon^2) \tag{A5.23}$$

In the equations (A5.21), (A5.23) the χ which appears is that defined by (cf. equation (2.23))

$$\chi = \theta + \frac{1}{3}\epsilon\frac{r^3}{R^3}\sin\theta\cos\theta + O(\epsilon^2) \tag{A5.24}$$

For this definition of χ, it follows that

$$|\nabla\chi| = \frac{1}{s}\left(1 - \frac{1}{9}\epsilon\frac{s^3}{R^3}\left(1 + 3P_2(0) - 7P_2(\cos\chi)\right) + O(\epsilon^2)\right) \tag{A5.25}$$

If second order terms are required, it must be remembered that $\chi \neq \theta$ and so the expressions

$$\left.\begin{aligned} \cos\theta &= \cos\chi\left(1 + \frac{2}{9}\epsilon\frac{s^3}{R^3}\left(1 - P_2(\cos\chi)\right)\right) \\ P_2(\cos\theta) &= P_2(\cos\chi) + \frac{2}{9}\epsilon\frac{s^3}{R^3}\left(1 + P_2(\cos\chi) - 2(P_2(\cos\chi))^2\right) \\ (P_2(\cos\chi))^2 &= \frac{18}{35}P_4(\cos\chi) + \frac{2}{7}P_2(\cos\chi) + \frac{1}{5} \end{aligned}\right\} \quad (A5.26)$$

must be used. These expressions are also needed to work out $r\sin\theta$, which appears because $|\nabla\phi|$ does (Chapters 5 and 6) and

$$|\nabla\phi| = \frac{1}{r\sin\theta} \quad (A5.27)$$

It is found that

$$r\sin\theta = s\sin\chi\left[1 - \frac{1}{9}\epsilon\frac{s^3}{R^3}\left(1 - 3P_2(0) + 5P_2(\cos\chi)\right) + O(\epsilon^2)\right] \quad (A5.28)$$

APPENDIX VI

Numerical values for θ_o and ℓ_1

1. The value of θ_o

It was shown in Appendix V that θ_o could be found if R/R_p were known as a function of rotation speed. An expression for R_p/R can be found from the results of RGS, and is quoted by Roxburgh and Strittmatter (1965) as

$$R_p = R(1 - 0.1087\omega^2) \qquad (A6.1)$$

where ω^2 is defined in Appendix V.

The constant in this formula was calculated for $\omega^2 = 1$, so that equation (A6.1) strictly only holds for $\omega^2 = 1$ (Strittmatter 1967, personal communication). However, work done since this formula was published has indicated that the constant does not vary much with ω^2. Since no details of the variation were available, the author has assumed in what follows that equation (A6.1) holds for all values of ω^2. The following results are therefore subject to variation when more detailed information becomes available.

It is worth noting here that there is a mistake in section 2 of the paper from which equation (A6.1) is taken (Roxburgh and Strittmatter 1965). Equation (2.7) in that section should read

$$\frac{1}{x} + \frac{4}{27}\omega^2 x^2(1 - \mu^2) = \frac{R_{p_{min}}}{R_p} = \frac{0.8913}{1 - 0.1087\omega^2} \qquad (A6.2)$$

where $x = r/R_{p_{min}}$. In the derivation of their equation (2.7), Roxburgh and Strittmatter took $x = r/R_p$ and confused R_p and $R_{p_{min}}$, so that the R.H.S. of the equation was just 1. This was probably due to the fact that in RGS, whose results they were using, only the case $\omega^2 = 1$ was considered, so that R_p and $R_{p_{min}}$ were the same. The author is grateful to Dr. Strittmatter for pointing out this mistake.

Equation (A6.1) will now be used to derive numerical values for θ_o. First of all, the relation (A5.14) between ϵ and ω^2 must be introduced into equation (A5.13). This gives:

$$\frac{R}{R_p} = 1 + \frac{1}{2}\frac{8}{27}\left(\frac{R}{R_{p_{min}}}\right)^3 \omega^2 \sin^2\theta_o \tag{A6.3}$$

From equation (A6.1), $R_{p_{min}} = 0.8913\ R$. This is an exact result.

Hence

$$\epsilon = 0.4184\omega^2 \tag{A6.4}$$

exactly and also

$$\sin^2\theta_o = \frac{\dfrac{1}{1 - 0.1087\omega^2} - 1}{0.2092\,\omega^2} \quad . \tag{A6.5}$$

It then follows at once that

$$\left.\begin{aligned} \sin^2\theta_o &= 0.5831 \\ \theta_o &= 49^\circ\ 47' \\ P_2(\cos\theta_o) &= 0.1254 \end{aligned}\right\} \quad \text{for } \omega^2 = 1 \tag{A6.6}$$

These results are exact. If it is assumed that equation (A6.1) holds also for small ω^2, then it can be shown that

$$\sin^2\theta_o = 0.5196 + 0.0565\,\omega^2 + O(\omega^4) \tag{A6.7}$$

so that

$$\left.\begin{array}{r}\theta_o \rightarrow 46^\circ\ 7' \\ P_2(\cos\theta_o) \rightarrow 0.2206\end{array}\right\} \quad \text{as } \omega^2 \rightarrow 0 \qquad \text{(A6.8)}$$

These results are likely to be at least qualitatively correct, and of the right order of magnitude. In particular, it is clear that θ_o does not vary much with rotation speed and that the limiting value of θ_o as $\omega^2 \rightarrow 0$ is fairly large ($\sim 45^\circ$). Unfortunately, since P_2 has a zero at about 55° ($\cos\theta = 1/\sqrt{3}$), $P_2(\cos\theta_o)$ does vary appreciably with rotation speed. For purposes of calculation, the values

$$\left.\begin{array}{rcl}\epsilon & = & 0.1 \\ P_2(\cos\theta_o) & = & 0.2\end{array}\right\} \qquad \text{(A6.9)}$$

will be taken. These are approximately self-consistent. ($P_2(\cos\theta_o) = 0.2 \Rightarrow \epsilon = 0.101$ if equation (A6.7) is used.) For this value of ϵ, $\alpha \doteq 0.107$, so that ϵ is a good measure of the ratio of centrifugal force to gravity.

2. The value of ℓ_1

Roxburgh and Strittmatter (1965) also give a formula for the variation of luminosity with rotation speed. In their notation it is

$$L = L_u(1 - 0.247\omega^2) \qquad \text{(A6.10)}$$

Like equation (A6.1), this formula is strictly correct only for $\omega^2 = 1$ (Strittmatter 1967, personal communication). However, it will be assumed to hold approximately for small ω^2, as equation (A6.1) does.

L_u is the luminosity of the non-rotating star of the same mass as the rotating star, and corresponds to L_o in the notation of Chapter 2. Using equation (A6.4), the formula may therefore be written as

$$L = L_o(1 - 0.605\epsilon) \qquad \text{(A6.11)}$$

Hence, since ℓ_1 is defined by $L = L_o(1 + \epsilon\,\ell_1)$ (see Appendix VII),

$$\ell_1 = -0.60 \qquad \text{(A6.12)}$$

This result is given only to two significant figures because of uncertainty as to the accuracy of equation (A6.10) for slow rotation.

APPENDIX VII

Solution of the local equations by perturbation methods

1. The local equations in the plane-parallel approximation

The only justification for solving the local equations by the approximate methods of a perturbation theory is that the results are required to provide boundary conditions for the non-local equations. Since the boundary conditions will be applied in the plane-parallel approximation (see Chapter 5), the local equations will only be solved in that approximation. A more exact solution has been derived, and, for completeness, the results found will be quoted at the end of this appendix.

The hydrostatic equation and the equation of state are the same in the local theory as in the non-local theory. It therefore follows immediately from Chapter 5 that

$$\frac{dp}{dz} = -\overset{*}{\rho}E(\epsilon) \tag{A7.1}$$

and

$$p = \overset{*}{\rho}t \tag{A7.2}$$

where z is defined by

$$s = R(1 + \epsilon_1 z) \tag{A7.3}$$

and, using equations (5.20) and (A5.22),

$$E(\epsilon) = 1 - \tfrac{2}{3}\epsilon(1 - P_2(0)). \tag{A7.4}$$

As in Appendix V, $P_2(0)$ is used as a shorthand for $P_2(\cos\theta_o)$, where θ_o is defined as in Appendix IV.

Since boundary conditions for $\underline{v}$ are not required from the local theory, the only other equations of interest are equations (3.6) and (3.7). In terms of s, equation (3.7) is

$$\underline{\mathcal{F}} = -\frac{16}{3}\frac{\sigma}{\kappa}\frac{T^3}{\rho}\frac{dT}{ds}\underline{\nabla}s \tag{A7.5}$$

Before expressing this in terms of z, it is convenient to write it in terms of the dimensionless variables p, ρ^*, t and $\mathcal{F}^*$ defined by equations (5.14) and (6.1). Since $\underline{\nabla}s = (|\underline{\nabla}s|;\ 0\ ;\ 0)$, it follows that

$$\mathcal{F}^*_s = -\frac{16}{3}\frac{t^3}{\rho^*}\frac{dt}{dz}|\underline{\nabla}s| \tag{A7.6}$$

and that $\mathcal{F}_\chi \equiv \mathcal{F}_\phi \equiv 0$.

In terms of s, equation (3.6) is

$$L = \int_0^{2\pi}\int_0^{\pi}\frac{\mathcal{F}_s\, d\chi\, d\phi}{|\nabla\chi||\nabla\phi|} \tag{A7.7}$$

Defining ℓ by

$$L = 4\pi R^2\sigma T_e^4\,\ell, \tag{A7.8}$$

using the definitions (5.15) for C and D, and remembering that all functions are ϕ-independent, this equation reduces to

$$\ell = \frac{1}{2}\int_0^{\pi}\frac{\mathcal{F}^*_s\, d\chi}{CD} \tag{A7.9}$$

in the plane-parallel approximation. From Appendix V it follows that

$$C = 1 - \frac{1}{9}\epsilon(1 + 3P_2(0) - 7P_2) + O(\epsilon^2)$$

and

$$D = \frac{1}{\sin\chi}(1 + \frac{1}{9}\epsilon(1 - 3P_2(0) + 5P_2) + O(\epsilon^2)) \tag{A7.10}$$

That Appendix also gives $|\underline{\nabla}s|$, to lowest order in ϵ_1:

$$\left|\nabla s\right| = 1 - \tfrac{4}{3}\epsilon(P_2(0) - P_2) + O(\epsilon^2) \tag{A7.11}$$

Equations (A7.6) and (A7.9) combine to give

$$\ell = -\frac{2}{3}\frac{1}{\rho^*}\frac{dt^4}{dz}\int_0^\pi \frac{|\nabla s|}{CD}\, d\chi \tag{A7.12}$$

The equations must now be developed to the first order in ϵ by writing

$$\ell = 1 + \epsilon\ell_1 + \ldots \tag{A7.13}$$

and p, ρ^*, t and $\mathcal{F}_s^*$ similarly, as in equation (6.14). The zero order value for ℓ follows immediately from definition (2.19) for T_e. The value of the first order quantity ℓ_1 is approximately -0.6 (see Appendix VI).

It is easy to show that equations (A7.1), (A7.2) and (A7.12) yield the zero order equations

$$\frac{dp_o}{dz} = -\rho_o^* \tag{A7.14}$$

$$p_o = \rho_o^* t_o \tag{A7.15}$$

$$\frac{dt_o^4}{dz} = -\frac{3}{4}\rho_o^* \tag{A7.16}$$

After some manipulation, the corresponding first order equations are found to be

$$\frac{dp_1}{dz} + \rho_1^* = \frac{2}{3}\rho_o^*(1 - P_2(0)) \tag{A7.17}$$

$$\frac{p_1}{p_o} = \frac{\rho_1^*}{\rho_o^*} + \frac{t_1}{t_o} \tag{A7.18}$$

$$\frac{dt_1/dz}{dt_o/dz} + \frac{3t_1}{t_o} - \frac{\rho_1^*}{\rho_o^*} = \ell_1 + \frac{2}{3} P_2(0) \tag{A7.19}$$

These equations may now be solved, using the boundary conditions

$$p_o = p_1 = t_o = t_1 = 0 \quad \text{at } z = 0, \tag{A7.20}$$

to give p, ρ^* and t to first order in ϵ. (These boundary conditions are just the usual conditions P = T = 0 at s = R expressed in the present approximations.) The radiative flux, however, follows immediately from equation (A7.6) by using equations (A7.16) and (A7.19). It is

$$\mathcal{F}_s^* = 1 + \epsilon\left(\ell_1 - \frac{2}{3} P_2(0) + \frac{4}{3} P_2(\cos\chi)\right). \tag{A7.21}$$

2.The zero order solutions

It follows easily from equations (A7.14), (A7.16) and the boundary conditions that

$$p_o = \frac{4}{3} t_o^4 \tag{A7.22}$$

With this result, equations (A7.15),(A7.16) give

$$t_o = -\frac{1}{4} z \tag{A7.23}$$

It then follows at once from equations (A7.22), (A7.15) that

$$p_o = \frac{4}{3}\left(\frac{z}{4}\right)^4 \tag{A7.24}$$

$$\rho_o^* = -\frac{4}{3}\left(\frac{z}{4}\right)^3 \tag{A7.25}$$

3. The first order solutions

Using equation (A7.23) for t_o, equations (A7.18), (A7.19) give an expression for p_1:

$$p_1 = p_o \left(\frac{4t_1}{t_o} - 4 \frac{dt_1}{dz} - \mathcal{l}_1 - \frac{2}{3} P_2(0) \right) \tag{A7.26}$$

If this result is substituted into both of equations (A7.17), (A7.18) and if ρ_1^* is eliminated between the two resulting equations, a second order differential equation for t_1 is obtained (using equations (A7.23) to (A7.25)):

$$\frac{d^2t_1}{dz^2} + \frac{4}{z}\frac{dt_1}{dz} = \frac{2}{3}\frac{1}{z}(1 - P_2(0)) \tag{A7.27}$$

This can be integrated once using the integrating factor z^4 and the resulting first order differential equation can be integrated immediately to give

$$t_1 = \frac{1}{6} z (1 - P_2(0)) - \frac{B_1}{3z^3} + B_2 \tag{A7.28}$$

where B_1 and B_2 are arbitrary constants. The boundary conditions $p_1 = t_1 = 0$ at $z = 0$ require $B_1 = B_2 = 0$ so that finally

$$t_1 = \frac{1}{6} z (1 - P_2(0)) \tag{A7.29}$$

It then follows from equation (A7.26) that

$$p_1 = -\frac{4}{3}\left(\frac{z}{4}\right)^4 \left(\mathcal{l}_1 + \frac{10}{3} - \frac{8}{3} P_2(0)\right) \tag{A7.30}$$

and from equation (A7.18) that

$$\rho_1^* = \frac{4}{3}\left(\frac{z}{4}\right)^3 \left(\mathcal{l}_1 + \frac{8}{3} - 2 P_2(0)\right) \tag{A7.31}$$

That completes the local perturbation theory in the form required for the non-local boundary conditions. For that purpose, the only important equations are (A7.21) for $\mathfrak{F}^*_s$ and (A7.23) and (A7.29) for t. The last two may be more concisely written as

$$t = -\frac{1}{4} z \left(1 - \frac{2}{3}\epsilon(1 - P_2(0))\right) \qquad (A7.32)$$

4.The "exact" solution

It is shown in Appendix IV that an analytic solution can be obtained for the zero order local equations even when the effects of curvature are considered. The simplicity of the plane-parallel results, even in the first order theory, strongly suggests that it should also be possible to solve the first order equations with curvature effects included. Such a solution would be "exact" in the sense that the plane-parallel approximation had not been used.

A solution of this kind has been obtained and will be recorded here for interest and completeness, although no use is made of it. It would find a use for itself if a method were developed for the solution of the non-local equations which did not invoke the plane-parallel approximation.

The results for the structure variables P, ρ and T are rather cumbersome. They are:

$$P = \frac{4}{3}\frac{1}{(4\epsilon_1)^4}\frac{GM}{\varkappa R^2}\left(\frac{R}{s} - 1\right)^4\left[1 + \epsilon\left(4A\left(\frac{s}{R}\right) - B\left(\frac{s}{R}\right)\right)\right] \qquad (A7.33)$$

$$\rho = \frac{4}{3}\frac{1}{(4\epsilon_1)^3}\frac{1}{\varkappa\epsilon_1 R}\left(\frac{R}{s} - 1\right)^3\left[1 + \epsilon\left(3A\left(\frac{s}{R}\right) - B\left(\frac{s}{R}\right)\right)\right] \qquad (A7.34)$$

$$T = \frac{1}{4\epsilon_1} T_e \left(\frac{R}{s} - 1\right) \left[1 + \epsilon A\left(\frac{s}{R}\right)\right] \tag{A7.35}$$

where

$$A(y) = \frac{1}{3}\frac{y}{1-y}\left[P_2(0)(1-y^2) + \frac{32}{3} + \frac{8}{(1-y)^3}\left(y^3 \log y + \frac{1}{2}(3-y)(1-y) - 3(1-y)^2 + \frac{1}{2}y(1-y)^3\right)\right] \tag{A7.36}$$

and

$$B(y) = \frac{8}{3}\frac{y^3}{(1-y)^4}\left[1 + \frac{3}{2}y - 3y^2 + \frac{1}{2}y^3 + 3y\log y\right] + \ell_1 \tag{A7.37}$$

Although it is not immediately obvious, these results reduce to the plane-parallel results when s/R is written as $1 + \epsilon_1 z$ and the functions are expanded to lowest order in ϵ_1.

The result for the radiative flux is much simpler. It is

$$\mathcal{F}_s = \sigma T_e^4 \frac{R^2}{s^2}\left[1 + \epsilon\left(\ell_1 - \frac{2}{3}\frac{s^3}{R^3}P_2(0) + \frac{4}{3}\frac{s^3}{R^3}P_2(\cos\chi)\right)\right] \tag{A7.38}$$

and it is immediately obvious that this reduces to the plane-parallel result when $s = R$.

Finally, the velocity on the local theory has been derived to lowest order in ϵ in terms of s and χ, with the help of Mestel's formula (Mestel 1966), and used as a check on the results of section 3 of Chapter 3. It was found that

$$v_{\underline{\Psi}} \; (= -v_s) = \frac{2048}{3}\epsilon_1^4 \frac{\varkappa R^2}{GM}\sigma T_e^4 \frac{(s/R)^8}{(1-s/R)^3} P_2(\cos\chi)\epsilon^2 \tag{A7.39}$$

and

$$v_\chi = \frac{3584}{3}\epsilon_1^4 \frac{\varkappa R^2}{GM}\sigma T_e^4 \frac{(s/R)^8}{(1-s/R)^3} \sin 2\chi \, \epsilon^2 \tag{A7.40}$$

With a suitable change of variables, and using equation (A7.34), these results can be shown to agree with equations (3.37), derived from the exact solution.

Writing $s = R(1 + \epsilon_1 z)$, the results become

$$v_s = -\frac{2048}{3}\epsilon_1 \frac{\kappa R^2}{GM}\sigma T_e^4 \frac{P_2(\cos\chi)}{|z|^3}\epsilon^2 \tag{A7.41}$$

and

$$v_\chi = \frac{3584}{3}\epsilon_1 \frac{\kappa R^2}{GM}\sigma T_e^4 \frac{\sin 2\chi}{|z|^3}\epsilon^2 \tag{A7.42}$$

to lowest order in ϵ_1. From the results in this form, it can be seen that the order of magnitude estimates for $\underline{v}$ in Chapter 3 hold at about $z = -3.5$, that is, $\tau \doteq 1$ (see Appendix VIII). This is a useful check on the correctness of the estimates in Chapter 3.

It is worth noting, in conclusion, that the local velocity is not only second order in ϵ^2 but is also first order in ϵ_1. This point is discussed in Chapter 6.

APPENDIX VIII

Behaviour of the zero order functions

Since the zero order structure of the star is prescribed, and is unaffected by the details of the perturbations induced by rotation, the solutions of the zero order equations can be tabulated once and for all. In the non-local theory, solutions have only been obtained in the plane-parallel approximation, and it is these results which will be tabulated here.

The dimensionless zero order functions are p_o, ρ_o^*, t_o, B_o^*, J_o^* and $\mathcal{F}_{so}^*$ (see Chapter 6). However, $\mathcal{F}_{so}^* = 1$, $J_o^* = B_o^* = t_o^4$ and $p_o = \rho_o^* t_o$ so that only p_o and t_o need be tabulated. p_o is chosen rather than ρ_o^* because the optical depth is given by $\tau = p_o$. The functions p_o and t_o are given, in terms of the height in the atmosphere, z, by the equations (cf. Chapter 6)

$$p_o = \frac{2}{3}(2t_o^4 - 1) \tag{A8.1}$$

and

$$t_o + \frac{1}{4}\,2^{-1/4}\left(\log\frac{2^{1/4}t_o - 1}{2^{1/4}t_o + 1} - 2\tan^{-1}(2^{1/4}t_o)\right) = -\frac{1}{4}z - 2^{-1/4}\,\frac{\pi}{4} \tag{A8.2}$$

for the non-local theory, and by the equations (cf. Appendix VII)

$$p_{oL} = \frac{4}{3}\left(\frac{z}{4}\right)^4 \qquad (z \leq 0) \tag{A8.3}$$

$$t_{oL} = -\frac{1}{4}z \qquad (z \leq 0) \tag{A8.4}$$

for the local theory. The local solutions are not defined for $z > 0$.

The graph of t_{oL} against z is a straight line, and so t_{oL} need not be tabulated. The other three solutions are tabulated in tables I and II. t_o^4 is also tabulated for convenience.

Table I Local p_o against z

z	p_{oL}
0.0000	0.0000
- 1.0000	0.0052
- 2.0000	0.0832
- 3.0000	0.4212
- 4.0000	1.3312
- 5.0000	3.2500
- 6.0000	6.7392

The four solutions are graphed in Graphs I and II, together with the asymptotic expressions for p_o and t_o as $z \rightarrow +\infty$, which are derived below and tabulated in Table III. It is also useful to have asymptotic expressions for p_o and t_o as $z \rightarrow -\infty$. The boundary conditions ensure that, for example, $t_o - t_{oL} \rightarrow 0$ as $z \rightarrow -\infty$, but it may sometimes be necessary to know how fast $t_o - t_{oL}$ tends to zero. Expressions for p_o and t_o which give this information are derived below, and the quantities $(p_o - p_{oL})/p_{oL}$, $(t_o - t_{oL})/t_{oL}$ are tabulated in Table III, but not graphed.

Consider first the expressions as $z \rightarrow +\infty$. It is clear from Table II that t_o is very close to $2^{-1/4}$ for all z greater than zero. Thus, as $z \rightarrow \infty$, t_o may be written

Table II Non-local p_o and t_o against z

z	p_o (= τ)	t_o	t_o^4
+ ∞	0	0.8408964	1/2
3.6420	0.000328	0.8410	0.500246
1.5496	0.0035	0.8420	0.5026
0.9640	0.0067	0.8430	0.5050
0.6200	0.0099	0.8440	0.5074
0.3752	0.0131	0.8450	0.5098
0.1884	0.0164	0.8460	0.5123
+ 0.0284	0.0196	0.8470	0.5147
- 0.1044	0.0228	0.8480	0.5171
- 0.2196	0.0261	0.8490	0.5196
- 0.3216	0.0293	0.8500	0.5220
- 0.9652	0.0627	0.8600	0.5470
- 1.3464	0.0972	0.8700	0.5729
- 1.6200	0.1329	0.8800	0.5997
- 1.8388	0.1699	0.8900	0.6274
- 2.0208	0.2081	0.9000	0.6561
- 3.1168	0.6667	1.0000	1.0000
- 3.5344	1.00000	1.05737	1.2500
- 3.8048	1.2855	1.1000	1.4641
- 4.3668	2.0981	1.2000	2.0736
- 4.8712	3.1415	1.3000	2.8561
- 5.3424	4.4555	1.4000	3.8416
- 5.7940	6.7500	1.5000	5.0625
- 7.9156	20.667	2.0000	16.0000
- 11.975	107.33	3.0000	81.0000
- 15.990	340.67	4.0000	256.0000
- ∞	+ ∞	+ ∞	+ ∞

$$t_o = 2^{-1/4}(1 + \delta) \qquad \text{where } \delta \ll 1 \tag{A8.5}$$

It may be shown that

$$\tan^{-1}(1 + \delta) = \frac{\pi}{4} + \frac{1}{2}\delta + O(\delta^2) \tag{A8.6}$$

It then follows from equation (A8.2), in which z has been replaced for convenience by $h = 2^{1/4}z$, that

$$\log\frac{\delta}{2} = -h - \frac{\pi}{2} - 4 - \frac{5}{2}\delta \tag{A8.7}$$

Hence $$\frac{\delta}{2} = e^{-h}e^{-(\frac{\pi}{2}+4)}e^{-\frac{5}{2}\delta}$$

or, expanding $e^{-\frac{5}{2}\delta}$ in a power series in δ,

$$\delta = \frac{2e^{-h}}{E}\left(1 - \frac{5e^{-h}}{E} + O\left(\frac{e^{-2h}}{E^2}\right)\right) \tag{A8.8}$$

where $$E = e^{\frac{\pi}{2}+4} = 262.65 \doteq 263 \tag{A8.9}$$

Thus
$$\left.\begin{aligned} t_o &= 2^{-1/4}\left[1 + \frac{2e^{-h}}{E}\left(1 - \frac{5e^{-h}}{E} + O\left(\frac{e^{-2h}}{E^2}\right)\right)\right] \\ &= 0.840896\left[1 + 0.00762\, e^{-1.189z}(1 - 0.019e^{-1.189z} + \ldots\ldots)\right] \end{aligned}\right\} \tag{A8.10}$$

and then, from equation (A8.1),

$$\left.\begin{aligned} p_o &= \frac{16}{3}\frac{e^{-h}}{E}\left(1 - \frac{2e^{-h}}{E} + O\left(\frac{e^{-2h}}{E^2}\right)\right) \\ &= 0.0203e^{-1.189z}(1 - 0.0076e^{-1.189z} + \ldots) \end{aligned}\right\} \tag{A8.11}$$

It is clear from Table II and the graphs that these expressions give very good values for p_o and t_o for all positive z.

For large negative z, the expressions are simpler. Table II shows that $t_o \to \infty$ and so the expression for t_o, given by equation (A8.2), can be expanded as a power series in $1/t_o$. It can be shown that

$$\left.\begin{aligned} \tan^{-1} A &\sim \frac{\pi}{2} - \frac{1}{A} + \frac{1}{3}\frac{1}{A^3} \\ \log \frac{1 - 1/A}{1 + 1/A} &\sim -\frac{2}{A} - \frac{2}{3}\frac{1}{A^3} \end{aligned}\right\} \text{ as } A \to \infty \qquad (A8.12)$$

It then follows that

$$t_o + \frac{1}{4} z \sim -\frac{1}{6}\left(\frac{4}{z}\right)^3 \qquad \text{as } z \to -\infty$$

so that

$$\left.\begin{aligned} t_o &= -\frac{1}{4} z \left[1 + \frac{1}{6}\left(\frac{4}{z}\right)^4 + O\left(\frac{4}{z}\right)^8 \right] \\ p_o &= \frac{4}{3}\left(\frac{z}{4}\right)^4 \left[1 + \frac{1}{6}\left(\frac{4}{z}\right)^4 + O\left(\frac{4}{z}\right)^8 \right] \end{aligned}\right\} \text{as } z \to -\infty \quad (A8.13)$$

The tabulated quantity is

$$\frac{p_o - p_{oL}}{p_{oL}} = \frac{t_o - t_{oL}}{t_{oL}} = \frac{1}{6}\left(\frac{4}{z}\right)^4 + \ldots \qquad (A8.14)$$

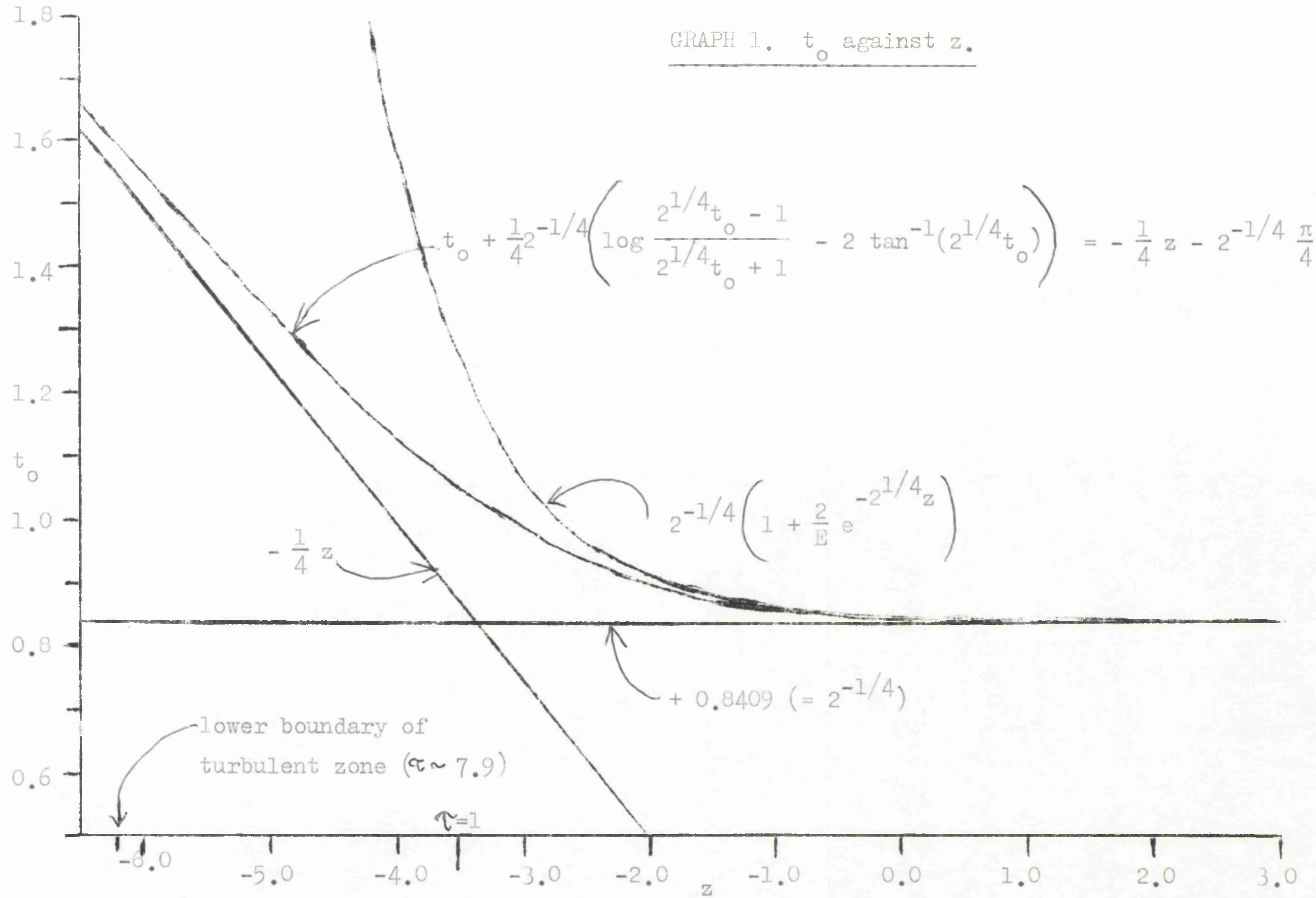
and is tabulated only for $|z| > 3$. It can be seen from Table III that the non-local solution agrees with the local solution to within about 1% for $z < -8$.

The tabular values for z in Tables II and III were chosen for convenience in the calculations, which were carried out on a desk calculator.

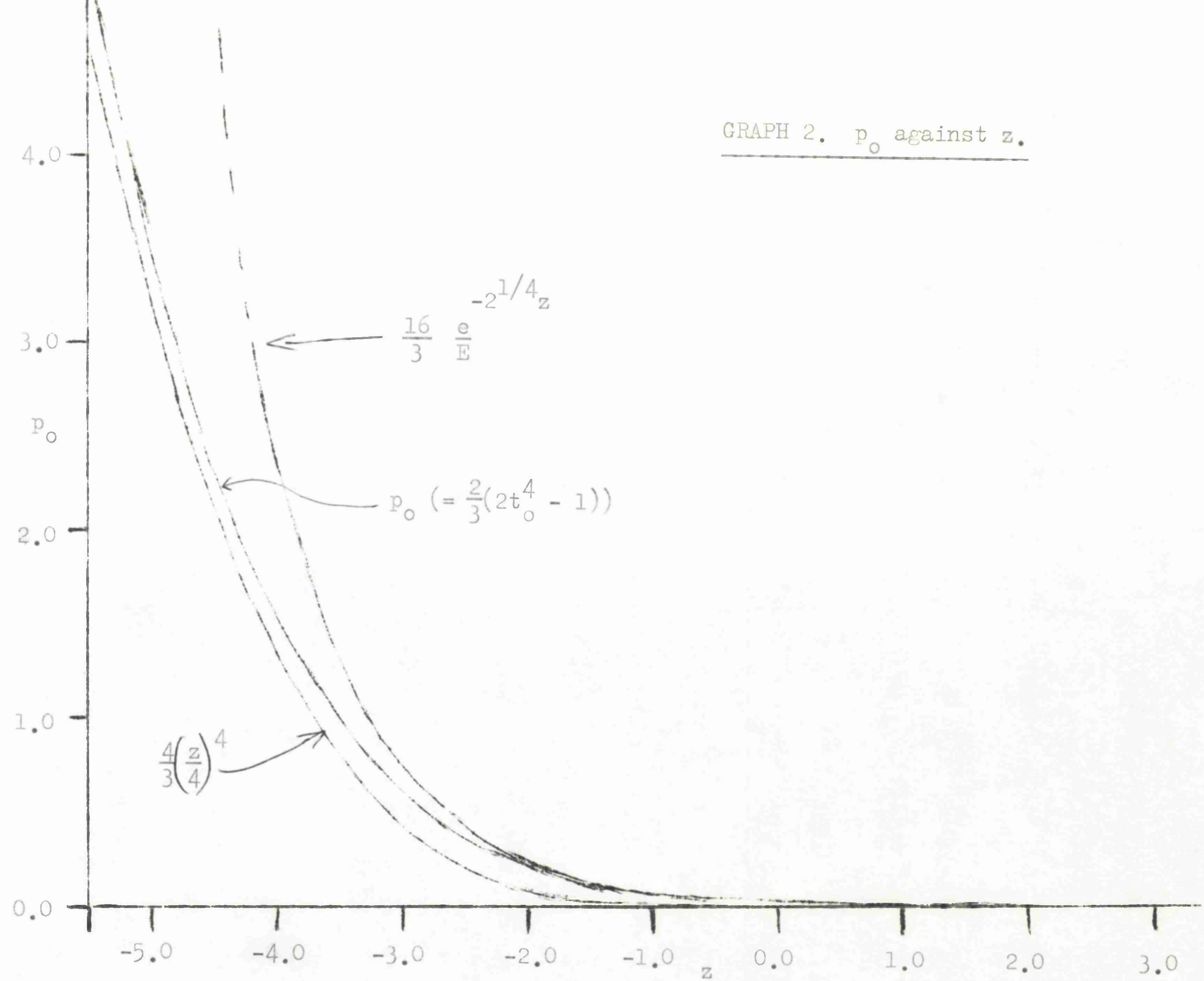
Table III Asymptotic expressions

$h=2^{1/4}z$	z	$\frac{16}{3E}e^{-2^{1/4}z}$	$2^{-1/4}(1+\frac{2}{E}e^{-2^{1/4}z})$	$\frac{1}{6}\left(\frac{4}{z}\right)^4$
6	5.0454	0.000051	0.8409	
4	3.3636	0.000372	0.8410	
2	1.6818	0.00275	0.8418	
0	0.0000	0.0203	0.8473	
- 1	- 0.8409	0.0552	0.8583	
- 2	- 1.6818	0.1501	0.8882	
- 3	- 2.5227	0.4078	0.9695	
- 4	- 3.3636	1.1098	1.1908	0.3333
- 5	- 4.2045	3.0313	1.7968	0.1365
- 6	- 5.0454	8.1239	3.4027	0.0658
	- 6.0000			0.0329
	- 8.0000			0.0104
	- 10.0000			0.0043
	- 12.0000			0.0021
	- 14.0000			0.0011
	- 16.0000			0.0007
	- 20.0000			0.0003
	- 24.0000			0.0001

GRAPH 1. t_o against z.



GRAPH 2. p_o against z.



APPENDIX IX

Alternative derivation of equations (4.16) and (4.18)

In Chapter 4 (pp 72-3) a derivation was given for equations (4.16) and (4.18). These equations may also be derived as follows. Choose local right-handed Cartesian axes (Oxyz), (Px'y'z') such that Oz and Pz' are parallel to the rotation axis of the star and Ox and Px' are in the meridian planes through O and P respectively. Then, as regards orientation, the axes (Px'y'z') are equivalent to the axes (Oxyz) rotated in the positive direction about the z-axis through an angle $d\phi$. Suppose that (l,m,n) and (l',m',n') are the direction cosines of $\underline{dl}$ in the two coordinate frames. The condition that $\underline{dl}$ is constant in direction in a *fixed* coordinate frame then leads to:

$$\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} \cos(d\phi) & -\sin(d\phi) & 0 \\ \sin(d\phi) & \cos(d\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l' \\ m' \\ n' \end{bmatrix} \qquad (A9.1)$$

where, of course, (l,m,n) and (l',m',n') are known in terms of (Λ, H, θ) and $(\Lambda+d\Lambda, H+dH, \theta+d\theta)$ respectively. Only two of the three equations arising from equation (A9.1) are linearly independent. These two equations reduce to equations (4.16) and (4.18) after some manipulation.

APPENDIX X

Notes to Chapter 7

Note (a) (see p. 143)

In making the order of magnitude estimates which give rise to equations (7.5), it has been assumed that the vertical scale for the variation of $\underline{v}$ is the scale height H and that the horizontal scale of variation is the stellar radius R. This is certainly true in the absence of turbulence. When turbulence is present, the typical size of an eddy is another parameter affecting the flow. However, it seems reasonable to suppose that the eddy size is the scale of variation of the fluctuations in the flow and that the mean flow still varies on a scale of H vertically and R horizontally.

Note (b) (see p. 144)

It should be noted that, in equation (7.5), the phrase "centrifugal terms" is used, rather loosely, to cover the three terms $\frac{1}{\rho}\underline{\nabla} P$, $\underline{\nabla}\underline{\Phi}$ and $\Omega^2\underline{\Theta}$. For the moment, Ω is assumed constant, so that the last two terms are simply $\underline{\nabla}\underline{\Psi}$. The χ-component therefore arises solely from $\frac{1}{\rho}\underline{\nabla} P$. This has a non-zero χ-component because P, ρ and T are no longer functions only of $\underline{\Psi}$ when inertial and/or viscous terms are included in the equation of motion.

Note (c) (see p. 147)

It has been suggested by Mestel (1966, personal communication) that, in stars with strong magnetic fields, a mechanism other than turbulent viscosity may operate to damp the circulation. It is possible that, at a depth below that at which inertial terms become important, the magnetic forces have become comparable with the centrifugal forces, which decrease with the density. Then the magnetic field would become the dominant perturbation and it is possible that this would cause the meridional circulation (which is rotationally driven) to die out, the structure of the field being determined in such a way that div $\mathfrak{F} = 0$. Once set up, this structure would probably be stable to changes in the field, which does not transport angular momentum. For this model to be valid, one would require fields given by

$$\frac{B^2}{8\pi} \sim \frac{1}{2} \rho \Omega^2 R^2 ; \qquad (A10.1)$$

that is, more than 1000 gauss. Such fields appear to exist in some strongly magnetic stars, in which, therefore, there is possibly no atmospheric circulation.

In stars with weaker fields the turbulent model described in Chapter 7 is more likely to be valid, with the field mostly expelled from the turbulent zone. Any field lines expelled to the stellar surface would be unattached to the internal field and would probably be blown away by the stellar wind. The field in such stars would then be mostly trapped beneath the surface.

One would therefore expect turbulence and strong magnetic fields to be anti-correlated in the observations. It is interesting, therefore, that Babcock (1958a, and Fig. 3) finds little evidence for magnetic fields in early B stars. It might also be expected that no magnetic fields would be detected of a size less than some limiting value at which turbulence becomes dominant - but this value might be below the limits of observation.

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Meridional circulation

in the atmospheres of uniformly rotating stars

of early spectral type

by

Robert Connon Smith

Errata

p. 94 : Equation (4.76) - (4.79) should read

$$\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} K^{jk} + \left\{ \begin{matrix} k \\ j \quad k \end{matrix} \right\} K^{ij} \equiv -\frac{1}{3} J \frac{\partial}{\partial x^i} \left(\frac{1}{g_{ii}} \right) \tag{4.7}$$

$$\mathrm{div}^i \underline{\underline{K}} = \frac{\partial K^{ii}}{\partial x^i} - \frac{1}{3} J \frac{\partial}{\partial x^i} \left(\frac{1}{g_{ii}} \right) = \frac{1}{g_{ii}} \frac{\partial}{\partial x^i} \left(\frac{1}{3} J \right) \tag{4.7}$$

$$|\underline{\nabla} s| \frac{\partial J}{\partial s} = -\frac{3\kappa}{4\pi} \rho \mathcal{F}_s \tag{4.7}$$

$$|\underline{\nabla} \chi| \frac{\partial J}{\partial \chi} = -\frac{3\kappa}{4\pi} \rho \mathcal{F}_\chi \tag{4.7}$$

p. 111 : Equation (5.16) should read

$$\mathcal{F}_\chi = -\frac{4\pi}{3} \epsilon_1 \frac{c}{\rho_*} \frac{\partial J}{\partial \chi} \tag{5.1}$$