Commutative-by-finite Hopf algebras and their finite dual

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Declaration

I declare that, with the exception of results that are explicitly attributed to other mathematicians, the contents of this thesis are my original work and have not been submitted for any other degree at the University of Glasgow or any other institution.

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Abstract

During the last few decades, Hopf algebras have become a topic of great interest in mathematics, especially after the discovery in the 1980s of the examples now known as "quantum groups" by Drinfeld [30]. Since then Hopf algebras have been found to be quite relevant in many areas, such as mathematical physics, integrable systems, quantum computation as well as algebraic geometry, number theory and other areas of mathematics.

Throughout my PhD I have thoroughly studied Hopf algebras and the research my supervisor Professor Ken Brown and I have carried out focused mostly on a particular class of Hopf algebras we named *commutative-by-finite*. These are Hopf algebras H that are finitely generated modules over some Hopf subalgebra A which is both commutative and normal. Here normality is a generalization of the notion of a normal subgroup in group theory.

As the title suggests, the theme and purpose of this thesis is twofold.

First, we intend to study the properties and structure of commutative-by-finite Hopf algebras. More specifically, we investigate their homological properties, their primeness and semiprimeness, their representation theory, as well as many other structural features of these Hopf algebras. We also provide a variety of examples, among them being the well-known quantum groups whose parameter is a root of unit.

Second, we aim to understand the duals of these Hopf algebras. The dual of a Hopf algebra H is, as in the analytic sense, the set of k-linear functionals $H \to k$, where k is the base field. We research decompositions of the dual of commutative-by-finite Hopf algebras, breaking it down into "easier pieces" to compute. And the dual of commutative Hopf algebras, which nowadays is completely understood, motivates the study of two Hopf subalgebras of the dual of commutative-by-finite Hopf algebras.

The study of these Hopf algebras, and in particular their duals, allows one to tackle important open questions, in particular regarding their antipode and their Drinfeld double. This is left in the form of conjectures and questions at the end of the thesis for possibly future work.

Contents

Introduction

In algebra there has been over the past decades an increase in the research of Hopf algebras. These are algebras that possess a lot more extra structure than the basic ring and vector space structures, and their most widely known examples are the quantum groups.

This thesis concerns the study of a particular class of Hopf algebras we designated *commutative-by-finite*. These are Hopf algebras H that are finitely generated modules over some Hopf subalgebra A which is both commutative and normal; here normal should be thought of as a generalization of the notion of a normal subgroup into the Hopf setting.

We start in chapter 1 by providing an overview of basic definitions, notation, properties, results and examples one should be familiar with and bear in mind throughout the rest of the thesis. Most of these are Hopf algebra related and we give particular emphasis to the notion of a Hopf algebra and to the concept of Hopf dual, both of which are essential for understanding this dissertation.

Then, we move on to the study of commutative-by-finite Hopf algebras, the central focus of this thesis. These turn out to have many nice properties and in chapter 2 we answer, among others, the following questions: how big are their centres, that is are they finitely generated modules over their centres? How nice is the algebra extension $A \subseteq H$? Is it flat, projective or even free? What other homological properties do these Hopf algebras possess? For example, are they regular?

Moreover, this class of commutative-by-finite Hopf algebras may in fact be more general than one might think at first. We describe in detail a variety of examples, namely many quantum groups at roots of unity and two classifications of Hopf algebras of low Gelfand-Kirillov dimension.

In chapter 3 we delve deeper into understanding the structure of these Hopf algebras. We approach four distinct matters. First, we study the effect of the action of $\overline{H}$ on the spectrum of maximal ideals of the commutative Hopf algebra A; here $\overline{H} := H/A^+H$ denotes a quotient of H which in fact is a finite dimensional Hopf algebra. Second, we investigate the semiprimeness and primeness of commutative-by-finite Hopf algebras and the consequences for their structure. Specifically we tackle questions such as: what is the relation between the nilradicals of H and A? If A is semiprime (resp. prime), is H also semiprime (resp. prime)? Or vice-versa? Third, we look into the representation theory of these Hopf algebras and, in particular, find upper and lower bounds for the dimension of their irreducible modules. And fourth, we describe the structure of a subclass of commutative-by-finite Hopf algebras called commutative-by-(co)semisimple for which the Hopf algebra quotient $\overline{H}$ is both semisimple and cosemisimple (cosemisimple is the co-version of the notion of semisimple for coalgebras).

Next, we focus on the second main subject of this thesis, the dual of commutative-by-finite Hopf algebras. Our approach here is motivated by the commutative case. Let A be an affine commutative Hopf algebra over an algebraically closed field k of characteristic 0. Then, $A \cong \mathcal{O}(G)$ is the coordinate ring of some affine algebraic group G and it follows from an old result by Cartier-Gabriel-Kostant that its dual is the skew group ring

$$A^{\circ} = \mathcal{O}(G)^{\circ} \cong U(\mathfrak{g}) * kG, \tag{1}$$

where $\mathfrak{g} = \operatorname{Lie} G$ is the Lie algebra associated to G, which in turn acts on $U(\mathfrak{g})$ by conjugation. Moreover, G identifies with the group of characters of A (that is, algebra maps $A \to k$) and $U(\mathfrak{g})$ identifies with the set of functionals that vanish on some power of the augmentation ideal $A^{+} := \ker \epsilon_A$.

In chapter 4 our purpose is to describe the dual of a commutative-by-finite Hopf algebra H as best we can. Since H contains a commutative Hopf subalgebra A and it has a finite dimensional Hopf quotient $\overline{H}$, the idea is to decompose the dual of H into the duals of A and $\overline{H}$, both of which are theoretically easier to compute than H°, the former because A is commutative and the latter since $\overline{H}$ is finite dimensional. This is achieved in Theorem 4.1.5 under mild hypotheses, where we obtain the following smash product decomposition

$$H^{\circ} \cong \overline{H}^{*} \# A^{\circ}.$$

The hypotheses of this result are general enough that it decomposes the dual of all examples of Hopf algebras we list in chapter 2.

Furthermore, as per (1) the dual of A contains the two Hopf subalgebras $U(\mathfrak{g})$ and kG, and one would expect these would give rise to two Hopf subalgebras $\overline{H}^{*} \# U(\mathfrak{g})$ and $\overline{H}^{*} \# kG$ of H°. This is also achieved in chapter 4, where we describe two Hopf subalgebras of the dual of H, the tangential component $W(H)$ and the character component $\widehat{kG}$, which again under quite general conditions decompose as expected:

$$W(H) \cong \overline{H}^{*} \# U(\mathfrak{g}) \qquad \text{and} \qquad \widehat{kG} \cong \overline{H}^{*} \# kG.$$

This chapter is closed with several computations of all these objects in the dual for many of the examples we introduce in this thesis.

In the last chapter of this thesis we explore some connections between the duals of commutative-by-finite Hopf algebras and currently open questions regarding the antipode and the Drinfeld double. A few conjectures are proposed in chapter 5, all of which are supported by many examples and we even present a few partial results striving towards proving these conjectures.

Regarding originality, the introductory chapter 1 is not original, being a collection of widely known facts about Hopf algebras. The results in chapter 2 rely heavily on other mathematician's results which are referenced accordingly throughout the whole chapter. However, the notion of commutative-by-finite Hopf algebras introduced there seems to be the central focus of such intensive study for the first time. With exception to some parts of section 3.1, chapter 3 contains mostly original work due to the author and his supervisor Professor Ken Brown and has recently been written into a paper [11]. Chapter 4 is a clear generalization of research due to Jahn [44], who had already studied the dual of central-by-finite Hopf algebras, these being finitely generated modules over some central Hopf subalgebra. However, many results in section 4.3 and examples in section 4.4 are original work. The conjectures, partial results and examples computed in chapter 5 are completely original.

Chapter 1

Background

This introductory chapter gathers the most important notions and results that will feature throughout this thesis. We define most of the concepts we will be using but not all of them, as is the case for the notions of noetherian, artinian, (semi)prime and (semi)simple rings. Commutative ring theory can be found in [31], [47], [78]; and for noncommutative ring theory, see [37], [50], [64].

Basic concepts, examples and results on Hopf algebras are gathered in section 1.1. In subsection 1.1.5 we focus on a specific construction of Hopf algebras called crossed products, which provides countless examples of Hopf algebras.

In section 1.2 we introduce many homological notions and a few homological results for Hopf algebras.

In section 1.3 we briefly introduce duals of Hopf algebras and a few of their results. This introductory section is crucial for understanding the dual of commutative-by-finite Hopf algebras, which will be studied in chapter 4.

Lastly, in section 1.4 we introduce the Drinfeld double of finite dimensional Hopf algebras and some of its properties. Later on in chapter 5, we will study the Drinfeld double of commutative-by-finite Hopf algebras.

None of the concepts, examples or results in this chapter are original and they are all appropriately cited. Although Lemma 1.1.15 is quite well-known, we could not find a reference of its statement and proof so we present a proof here.

1.1 Hopf algebras

Throughout the whole thesis k will denote the base field of all vector spaces. Hopf algebras are algebras with a co-structure that is compatible with the algebra structure and they are endowed with a special map called the antipode. They are named after Heinz Hopf and, for more information on their history, check [5]. We proceed to define these concepts and illustrate them with a few examples.

An *algebra* A over a field k is a k-vector space with a multiplication, which is consistent with the vector addition and scalar multiplication. Throughout this thesis, all algebras are assumed to be associative and unital, with unit 1_A. In a seemingly more complicated fashion, an algebra is a k-vector space with two k-linear maps $m : A \otimes A \to A$ and $u : k \to A$ that satisfy

1. $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$ (associativity);

2. $m \circ (u \otimes \mathrm{id}) = s_l$ and $m \circ (\mathrm{id} \otimes u) = s_r$ (unit axiom),

where $s_l : k \otimes A \to A$ and $s_r : A \otimes k \to A$ denote respectively the scalar multiplication maps on the left and on the right.

In simple terms, a coalgebra is the dual version of this definition: a *coalgebra* C is a k-vector space with two k-linear maps $\Delta : C \to C \otimes C$ (the coproduct) and $\epsilon : C \to k$ (the counit) such that

1. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (coassociativity)

2. and $s_l \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = s_r \circ (\mathrm{id} \otimes \epsilon) \circ \Delta$ (counit axiom),

Throughout this thesis we will use Sweedler's notation, that is we denote the coproduct of an element $c \in C$ by

$$\Delta(c) = \sum c_1 \otimes c_2.$$

For example, under this notation the counit axiom can be rewritten as $\sum \epsilon(c_1)c_2 = c = \sum c_1\epsilon(c_2)$ for all $c \in C$.

A *bialgebra* B is both an algebra and a coalgebra in which both structures are compatible, that is the coalgebra structure maps Δ, ϵ are algebra maps. Note that this condition is equivalent to requiring the algebra structure maps m, u be coalgebra maps, [76, Lemma 5.1.1]. Since ϵ is an algebra map, its kernel is a maximal ideal of B (of codimension 1) which we shall denote by $B^+ := \ker \epsilon$. It is usually called the *augmentation ideal.*

A *Hopf algebra* H is a bialgebra with a k-linear map $S : H \to H$ called the antipode that satisfies

$$\sum S(h_1)h_2 = \epsilon(h)1_H = \sum h_1 S(h_2),$$

for all $h \in H$.

The antipode of an Hopf algebra is an antihomomorphism of algebras and coalgebras, that is, for any $h, k \in H$

$$S(hk) = S(k)S(h) \qquad \text{and} \qquad \sum S(h)_1 \otimes S(h)_2 = \sum S(h_2) \otimes S(h_1),$$

and $S(1) = 1$ and $\epsilon \circ S = \epsilon$. Another question pertaining to the antipode concerns its bijectivity. Although it is not bijective in general (see an example in [96]), it will be bijective for most of the Hopf algebras with which we will deal throughout this thesis.

Let H be a Hopf algebra. A subspace $I \subseteq H$ is a *coideal* if $\Delta(I) \subseteq I \otimes H + H \otimes I$ and $\epsilon(I) = 0$. A left coideal satisfies $\Delta(I) \subseteq H \otimes I$ and $\epsilon(I) = 0$. Right coideals are defined analogously. A subspace $I \subseteq H$ is a *Hopf ideal* if it is an ideal, a coideal and $S(I) \subseteq I$. When I is a Hopf ideal of H, the Hopf structure of H induces a Hopf structure in H/I, called the quotient Hopf algebra.

A k-linear map $f : H \to H'$ between Hopf algebras is a *Hopf map* if it is an algebra map, a coalgebra map and it preserves antipodes. That is, $m_{H'} \circ (f \otimes f) = f \circ m_H$ and $f \circ u_H = u_{H'}$ (preserves multiplication and units), $\Delta_{H'} \circ f = (f \otimes f) \circ \Delta_H$ and $\epsilon_{H'} \circ f = \epsilon_H$ (preserves coproduct and counit), and $S_{H'} \circ f = f \circ S_H$. The kernel of a Hopf map $f : H \to H'$ is a Hopf ideal of H and its image is a Hopf subalgebra of H'. A bijective Hopf map is an *isomorphism* of Hopf algebras.

For more details on basic concepts related to Hopf algebras, see for example [67], [76] or [94]. Here are a few basic examples of Hopf algebras.

Example 1.1.1 (Group algebras, [67, Examples 1.3.2, 1.5.3])**.** Let G be a group and consider the k-vector space with basis G

$$kG = \left\{ \sum_{\text{finite}} \lambda_g g : \lambda_g \in k \right\}.$$

The product in kG is determined by the group multiplication and the distributive property. The identity is 1_G. The coalgebra structure is as follows: for each element $g \in G$, $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ and this extends linearly to kG. It is easy to see these maps satisfy the coalgebra axioms and that they are algebra maps. The antipode is defined by $S(g) = g^{-1}$, for any $g \in G$.

On account of this example, an element h in a Hopf algebra H with $\Delta(h) = h \otimes h$ is called *grouplike*. It follows from the counit and antipode axioms that a grouplike element must be invertible and satisfy $\epsilon(h) = 1, S(h) = h^{-1}$. The grouplike elements of H form a group [76, Propositions 5.1.15(a), 7.6.3], which we shall denote by $G(H)$. This set is linearly independent over k [76, Lemma 2.1.12], hence the subalgebra of H generated by the grouplike elements is the group algebra $kG(H)$, which in fact is a Hopf subalgebra of H.

Example 1.1.2 (Enveloping algebras, [67, Examples 1.3.3, 1.5.4])**.** Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[,]$. Its tensor algebra $T(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ has a product determined by juxtaposition. Its coalgebra structure and antipode are given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \epsilon(x) = 0, \qquad S(x) = -x,$$

for every $x \in \mathfrak{g}$, and these extend canonically to n-tensors of $\mathfrak{g}^{\otimes n}$ for each n. The enveloping algebra of $\mathfrak{g}$ is

$$U(\mathfrak{g}) := T(\mathfrak{g})/I,$$

where $I := \langle xy - yx - [x, y] : x, y \in \mathfrak{g} \rangle$. It is easy to check that I is a Hopf ideal of $T(\mathfrak{g})$, hence the quotient $U(\mathfrak{g})$ is a Hopf algebra.

When $\mathfrak{g}$ is a finite dimensional Lie algebra with basis $\{x_1, \ldots, x_n\}$, then

$$U(\mathfrak{g}) = k\langle x_1, \ldots, x_n : x_i x_j - x_j x_i = [x_i, x_j], 1 \leq i \leq n\rangle.$$

This Hopf algebra is a noetherian domain [64, Corollaries 1.7.4, 1.7.5] with basis given by the following well-known result.

Theorem 1.1.3 (Poincaré-Birkhoff-Witt, [64, Theorem 1.7.5]). *If $\mathfrak{g}$ has a k-basis $\{x_1, \ldots, x_n\}$, then $U(\mathfrak{g})$ has a k-basis*

$$\{x_1^{t_1} \ldots x_n^{t_n} : t_i \geq 0\}.$$

In particular, if $\mathfrak{g}$ is an abelian Lie algebra (that is $[x, y] = 0$ for all $x, y \in \mathfrak{g}$), then $U(\mathfrak{g}) = k[x_1, \ldots, x_n]$ is the usual polynomial algebra in n variables.

An element h of a Hopf algebra H satisfying $\Delta(h) = h \otimes g + g' \otimes h$, with g, g' grouplike, is called skew-primitive or (g, g')-*primitive*; it follows from the counit and antipode axioms that skew primitive elements have $\epsilon(h) = 0$ and $S(h) = -g'^{-1}hg^{-1}$. In the particular case where $g = g' = 1$, h is called *primitive*. The subspace of primitive elements of a Hopf algebra H, which we denote by $P(H)$, is a Lie algebra whose brackets are given by commutators on H [76, Proposition 5.1.15(d)]. Therefore, the subalgebra of H generated by the primitive elements is $k\langle P(H)\rangle = U(P(H))$, a universal enveloping algebra, and it is a Hopf subalgebra of H.

The Hopf algebras kG and $U(\mathfrak{g})$ are commutative if and only if G is an abelian group and $\mathfrak{g}$ an abelian Lie algebra, respectively. However, both these examples are *cocommutative*, meaning that $\tau \circ \Delta = \Delta$, where $\tau : H \otimes H \to H \otimes H$ is the flip map. The first noncommutative and noncocommutative example is due to Sweedler (a Hopf quotient of the example in [94, pp. 89-90]) and later extended by Taft [95].

Example 1.1.4 (Taft algebras, [76, §7.3]). Let k be an algebraically closed field and let n, t be integers, $n \geq 2$, $1 \leq t < n$, and q a primitive nth root of unity. The Taft algebra with these parameters is

$$T_f(n, t, q) := k\langle g, x : g^n = 1, x^n = 0, xg = qgx\rangle,$$

where g is grouplike and x is $(1, g^t)$-skew primitive. The counit is $\epsilon(g) = 1$, $\epsilon(x) = 0$ and the antipode is $S(g) = g^{-1}$, $S(x) = -g^{-t}x$. This Hopf algebra has dimension n^2 and is clearly neither commutative nor cocommutative. In particular, Sweedler's example is the 4-dimensional Taft algebra: $T_f(2, 1, -1) = k\langle g, x : g^2 = 1, x^2 = 0, xg = -gx\rangle$.

There are also *infinite dimensional* Taft algebras which are defined analogously but we drop the relation $x^n = 0$,

$$T(n, t, q) = k\langle g, x : g^n = 1, xg = qgx\rangle$$

with the same coalgebra structure.

Example 1.1.5 (Coordinate Rings)**.** Let G be an algebraic variety in $\mathbb{A}_n(k)$ and $\mathcal{O}(G)$ its ring of coordinates (also known as ring of functions), that is

$$\mathcal{O}(G) := k[x_1, \ldots, x_n]/\{p \in k[x_1, \ldots, x_n] : p(G) = 0\}.$$

It is often also denoted by $k[G]$.

An *algebraic group* is an algebraic variety G which also possesses a group structure such that the maps

$$\begin{array}{rcl} m : G \times G & \to & G \\ (x, y) & \mapsto & xy \end{array} \quad \text{and} \quad \begin{array}{rcl} i : G & \to & G \\ x & \mapsto & x^{-1} \end{array}$$

are morphisms of algebraic varieties; see [1, section 4.2].

Lemma 1.1.6. *$\mathcal{O}(G)$ is a Hopf algebra if and only if G is an algebraic group.*

Proof. See [1, beginning of sections 3.4 and 4.2]. □

Provided G is an algebraic group, the Hopf structure of this algebra is as follows: for any $f \in \mathcal{O}(G)$, $\Delta(f) = \sum f_1 \otimes f_2$ is such that $\sum f_1(g)f_2(h) = f(gh)$ for all $g, h \in G$, $\epsilon(f) = f(1_G)$ and $S(f)$ is defined as $S(f)(g) = f(g^{-1})$ for all $g \in G$. Let us see some concrete examples of algebraic groups and their coordinate rings, [67, Example 1.5.7].

1. Consider the group $G = GL_n(k)$ of all $n \times n$ invertible matrices. We know from linear algebra that these are the matrices with nonzero determinant and, since the determinant *det* can be seen as a polynomial function on the entries of a matrix, the coordinate ring of GL_n is the ring of fractions

$$\mathcal{O}(GL_n) = k[X_{ij} : 1 \leq i, j \leq n][det^{-1}]$$

as an algebra, where each X_{ij} denotes the function sending a matrix to its (i, j)-entry. The coalgebra structure is $\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}$ and $\epsilon(X_{ij}) = X_{ij}(1_G) = \delta_{i,j}$.

2. The subgroup $G = SL_n(k)$ of $n \times n$ matrices of determinant 1 has coordinate ring

$$\mathcal{O}(SL_n) = k[X_{ij} : 1 \leq i, j \leq n]/\langle det - 1\rangle.$$

The coalgebra structure is similar to the one for $G = GL_n$.

Coordinate rings are important examples of Hopf algebras, in part because they are more general than one might think at first. They cover all affine commutative Hopf algebras in characteristic 0 by the following result. A k-algebra is said to be *affine* if it is finitely generated as a k-algebra.

Theorem 1.1.7 (Cartier)**.** *Let H be a commutative Hopf algebra.*

1. *If the field k has characteristic zero, then H is reduced (or, in other words, semiprime), that is it has no nonzero nilpotent elements.*

2. If H is affine reduced and k is algebraically closed, then

$$H \cong \mathcal{O}(G),$$

for the affine algebraic group $G \cong \operatorname{Maxspec}(H)$.

Proof. A proof of (1) can be found in [94, Theorem 13.1.2] or [99, Theorem 11.4]. And (2), if H is affine, commutative and reduced, by the Nullstellensatz it is the coordinate ring $\mathcal{O}(G)$ of the affine algebraic variety $G = \operatorname{Maxspec}(H)$ and, given H is Hopf, G must be an algebraic group by Lemma 1.1.6. $\square$

We exemplify the statement of this theorem with a few commutative Hopf algebras we have seen so far. Also, see Example 1.3.3 for the coordinate rings of cyclic groups.

Example 1.1.8. Consider the commutative group algebra $H = k\mathbb{Z}$. First, note that this Hopf algebra is isomorphic to the ring of Laurent polynomials $k[x^{\pm 1}]$ with x grouplike, with Hopf isomorphism determined by

$$\begin{array}{rcl} \varphi : k\mathbb{Z} & \to & k[x^{\pm 1}] \\ g & \mapsto & x \end{array},$$

where g is a generator of $\mathbb{Z}$.

Let k be an algebraically closed field. Since $k\mathbb{Z} \cong k[x^{\pm 1}]$ is an affine commutative domain, the previous result tells us $H = k\mathbb{Z}$ is the coordinate ring of the algebraic group $G = \operatorname{Maxspec}(k\mathbb{Z}) \cong \operatorname{Maxspec}(k[x^{\pm 1}]) = k^{\times}$, the multiplicative group of the base field; that is,

$$k\mathbb{Z} \cong k[x^{\pm 1}] = \mathcal{O}(k^{\times}).$$

Example 1.1.9. Let k be an algebraically closed field and consider the commutative Hopf algebra $H = k[x]$ from Example 1.1.2. Since H is a domain, it is the coordinate ring of the algebraic group $G = \operatorname{Maxspec}(k[x]) = (k, +)$, the additive group of the base field; that is

$$k[x] = \mathcal{O}(k, +).$$

Of course not all algebras are Hopf algebras. Any Hopf algebra contains a (maximal) ideal of codimension 1, the augmentation ideal $H^+ = \ker \epsilon$. Therefore, no simple k-algebra apart from k itself can have a Hopf (or even bialgebra) structure. Such is the case for the algebra of matrices $\mathcal{M}_n(k)$ (although it has a coalgebra structure) and the Weyl algebras in characteristic zero.

1.1.1 Invariants and coinvariants

Throughout this thesis we will see many instances in which Hopf algebras act or coact on algebras, hence the importance of the following definitions.

Let H be a Hopf algebra. A left H*-module algebra* is an algebra A which is also a left H-module and both structures are compatible, that is, for all $h \in H, a, b \in A$,

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \qquad \text{and} \qquad h \cdot 1_A = \epsilon(h) 1_A.$$

The right version is defined analogously. In this case, the set of H*-invariants* of A is the subalgebra

$$A^H = \{a \in A : h \cdot a = \epsilon(h)a, \forall h \in H\}.$$

There is also a co-version of the concept of modules. A right H*-comodule* is a k-vector space M with a k-linear map $\rho : M \to M \otimes H$ such that

(i) $(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho$;

(ii) $(\mathrm{id} \otimes \epsilon) \circ \rho = - \otimes 1_k$.

Left comodules are defined similarly.

A right H*-comodule algebra* is an algebra A that is also a right H-comodule such that the coaction $\rho : A \to A \otimes H$ is an algebra map. For example, any Hopf algebra H is canonically a comodule algebra over itself where the coaction $\rho = \Delta$ is the coproduct. Given a right H-comodule algebra A, the set of H*-coinvariants* of A is

$$A^{\mathrm{co}\, H} := \{a \in A : \rho(a) = a \otimes 1\}.$$

Left comodule algebras and left coinvariants ${}^{\mathrm{co}\, H}A$ are defined analogously.

For future purposes, we now present the following two examples.

Example 1.1.10. Let H, T be Hopf algebras with a Hopf surjection $\pi : H \to T$. Then, H is canonically a right T-comodule algebra with coaction $\rho_r = (\mathrm{id} \otimes \pi) \circ \Delta$. The set of coinvariants of this coaction

$$H^{\mathrm{co}\, T} = \left\{ h \in H : \sum h_1 \otimes \pi(h_2) = h \otimes 1 \right\}$$

is a left coideal subalgebra of H [29, comments at beginning of §4] (for a proof, check e.g. [36, Lemma 2.6.9]). Analogously, H has a canonical structure of left T-comodule with coaction $\rho_l = (\pi \otimes \mathrm{id}) \circ \Delta$. The set of coinvariants of this coaction ${}^{\mathrm{co}\, T}H$ is a right coideal subalgebra of H.

In particular, when $T = H/I$ for some Hopf ideal I of H and we denote $C = H^{\mathrm{co}\, H/I}$, we have

$$C^+ H \subseteq I. \tag{1.1}$$

For, an element of C^+ satisfies $\sum c_1 \otimes \pi(c_2) = c \otimes 1$ and $\epsilon(c) = 0$, and applying $s_l \circ (\epsilon \otimes \mathrm{id})$ to the first equation yields $\pi(c) = \pi(\sum \epsilon(c_1)c_2) = \epsilon(c)1 = 0$ in H/I, meaning that $c \in \ker \pi = I$. The equation now follows from I being an ideal of H.

Example 1.1.11. Let G be an algebraic group and $N \triangleleft G$ a normal subgroup. It is well-known that N acts on $\mathcal{O}(G)$ by regular representations, that is $(n \cdot f)(g) = f(n^{-1}g)$

for $n \in N, g \in G, f \in \mathcal{O}(G)$, and that the subring of invariants is

$$\mathcal{O}(G)^N = \mathcal{O}(G/N).$$

We can state this in terms of coactions as well. First note that $\mathcal{O}(G/N)$ embeds into $\mathcal{O}(G)$ via the identification $\mathcal{O}(G/N) = \{f \in \mathcal{O}(G) : f(gn) = f(g), \forall g \in G, \forall n \in N\}$ and we have a Hopf surjection $\pi : \mathcal{O}(G) \twoheadrightarrow \mathcal{O}(N)$ given by $f \mapsto f|_N$. Then, by the previous example $\mathcal{O}(G)$ is a right $\mathcal{O}(N)$-comodule algebra and the coinvariants are $f \in \mathcal{O}(G)$ such that $\sum f_1 \otimes \pi(f_2) = f \otimes 1$, that is, for all $g \in G, n \in N$

$$f(gn) = \sum f_1(g) f_2(n) = \sum f_1(g)\pi(f_2)(n) = f(g)1(n) = f(g).$$

Therefore,

$$\mathcal{O}(G)^{\mathrm{co}\,\mathcal{O}(N)} = \mathcal{O}(G/N).$$

1.1.2 Integrals

We introduce here the notions of integrals and unimodularity. These will be particularly important in section 1.4 and chapter 5.

Let H be a finite dimensional Hopf algebra. An element $t \in H$ is called a *left integral* of H if $ht = \epsilon(h)t$ for all $h \in H$. We denote the subspace of left integrals by $\int_H^l$. In fact, $\int_H^l$ is a one-dimensional two-sided ideal of H, over which H acts trivially on the left (that is, it acts by the counit ϵ) and possibly non-trivially on the right; see [67, Theorem 2.1.3(1)].

Similarly, a *right integral* is an element $t \in H$ such that $th = \epsilon(h)t$ for any $h \in H$, and the one-dimensional two-sided ideal of right integrals is denoted by $\int_H^r$. A finite dimensional Hopf algebra H is said to be *unimodular* if these ideals coincide - that is, $\int_H^l = \int_H^r$. For example, any finite dimensional commutative Hopf algebra is trivially unimodular.

Example 1.1.12. Let G be a finite group. Note that for any $h \in G$, $h\sum_{g\in G} g = \sum_{g\in G} g$, that is G acts by ϵ on the left on the element $\sum_{g\in G} g$. Thus, this element is a left integral of $H = kG$. Similarly, it is also a right integral, hence H is unimodular with integrals

$$\int_H = k\left(\sum_{g\in G} g\right).$$

Example 1.1.13. Let $H = T_f(n,t,q)$ be a finite dimensional Taft algebra from Example 1.1.4. It is easy to see that H acts trivially on the left on $(\sum_{i=0}^{n-1} g^i)x^{n-1}$ and trivially on the right on $x^{n-1}(\sum_{i=0}^{n-1} g^i)$. Thus, it is not unimodular with

$$\int_H^l = k\left(\sum_{i=0}^{n-1} g^i\right)x^{n-1} \qquad \text{and} \qquad \int_H^r = kx^{n-1}\left(\sum_{i=0}^{n-1} g^i\right).$$

We point out a very important result - the generalization of Maschke's theorem to finite dimensional Hopf algebras:

Theorem 1.1.14. *A finite dimensional Hopf algebra H is semisimple if and only if $\epsilon(\int_H^l) \neq 0$ if and only if $\epsilon(\int_H^r) \neq 0$.*

Proof. See [67, Theorem 2.2.1]. $\square$

1.1.3 Powers of a Hopf ideal

The following lemma is well-known and it will be used in this thesis, specifically in sections 3.2 and 3.4. Since I could not find it in the literature, I prove it here.

Lemma 1.1.15. *Let H be a Hopf algebra and I a Hopf ideal of H. Then, $\bigcap_{n\geq 1} I^n$ is a Hopf ideal of H.*

Proof. Clearly $J := \bigcap_{n\geq 1} I^n$ is an ideal of H contained in H^+. Moreover, since $S(I) \subseteq I$, we have $S(I^n) \subseteq S(I)^n \subseteq I^n$ for each $n \in \mathbb{N}$, hence $S(J) \subseteq J$. We need only prove it is a coideal. Since $\Delta(I) \subseteq I \otimes H + H \otimes I$, we have

$$\Delta(I^{2n}) \subseteq \Delta(I)^{2n} \subseteq [(I \otimes H) + (H \otimes I)]^{2n} \subseteq \sum_{i=0}^{2n} (I^i \otimes I^{2n-i}) \subseteq I^n \otimes H + H \otimes I^n,$$

because $I^i \otimes I^{2n-i} \subseteq I^n \otimes H$ for any $i \geq n$ and $I^i \otimes I^{2n-i} \subseteq H \otimes I^n$ for any $i < n$. Then,

$$\Delta(J) = \Delta\left(\bigcap_{n\geq 1} I^{2n}\right) \subseteq \bigcap_{n\geq 1} (I^n \otimes H + H \otimes I^n).$$

Therefore, it suffices to show that

$$\bigcap_{n\geq 1} (I^n \otimes H + H \otimes I^n) = J \otimes H + H \otimes J. \tag{1.2}$$

The inclusion $(\supseteq)$ is clear, so we prove the converse.

For each $i \geq 0$, choose $\{v_k^i\} \subseteq I^i$ such that $\{v_k^i + I^{i+1}\}$ is a k-basis of I^i/I^{i+1}. And let $\{v_k^\infty\}$ be a basis of J. Therefore, $\bigcup_{i=0}^{\infty}\{v_k^i\}$ is a k-basis of H and $\bigcup_{i,j=0}^{\infty}\{v_k^i \otimes v_l^j\}$ is a basis of $H \otimes H$. Let $x \in \bigcap_{n\geq 1}(I^n \otimes H + H \otimes I^n)$ and write

$$x = \sum_{finite} \lambda_{k,l}^{i,j} v_k^i \otimes v_l^j \tag{1.3}$$

in terms of the basis above. Suppose $x \notin J \otimes H + H \otimes J$, so there exists some nonzero $\lambda_{k,l}^{i,j}$ in (1.3) with $i, j \neq \infty$. Let

$$n = \min\{\max(i,j) : \text{for every } \lambda_{k,l}^{i,j} \neq 0 \text{ in (1.3) where } i, j \neq \infty\}.$$

Thus, for every $\lambda_{k,l}^{i,j} \neq 0$ in (1.3) with $i, j \neq \infty$ either $i \geq n$ or $j \geq n$ and so

$$x = \sum_{i \geq n \vee j \geq n} \lambda_{k,l}^{i,j} v_k^i \otimes v_l^j \in I^n \otimes H + H \otimes I^n.$$

But $x \in I^{n+1} \otimes H + H \otimes I^{n+1}$, hence

$$\sum_{i=n \vee j=n} \lambda_{k,l}^{i,j} v_k^i \otimes v_l^j = x - \sum_{i \geq n+1 \vee j \geq n+1} \lambda_{k,l}^{i,j} v_k^i \otimes v_l^j \in I^{n+1} \otimes H + H \otimes I^{n+1}.$$

But $\{v_k^i \otimes v_l^j : i \geq n+1 \vee j \geq n+1\}$ is a basis of $I^{n+1} \otimes H + H \otimes I^{n+1}$ and, even though the 2-tensors from the sum on the left-hand side are not in this basis, these all belong to $\{v_k^i \otimes v_l^j : i, j = 0, \dots, \infty\}$, which is a basis of $H \otimes H$. Therefore, we must have $\lambda_{k,l}^{i,j} = 0$ for all i, j, k, l where $i = n$ or $j = n$, which contradicts the definition of n. This proves equality in (1.2). $\square$

1.1.4 Coalgebra notions

We discuss here a few important coalgebra related notions, which will be mentioned throughout the thesis. For more on this subject, see [67, §5] or [76, §3.4, §3.7, §4].

A coalgebra is *simple* if it contains no nontrivial subcoalgebras, that is its only subcoalgebras are $\{0\}$ and itself. A coalgebra C is called *cosemisimple* if it is a direct sum of simple subcoalgebras. For example, group algebras kG are cosemisimple, since $kG = \bigoplus_{g \in G} kg$ and each subspace kg is a one-dimensional (hence simple) subcoalgebra. A Hopf algebra H is *involutory* if $S^2 = \mathrm{id}$. For example, commutative and cocommutative Hopf algebras are involutory [67, Corollary 1.5.12] and in characteristic zero a finite dimensional Hopf algebra is cosemisimple if and only if it is semisimple if and only if it is involutory, [52, Corollary 2.6, Theorem 3.3], [53, Theorem 3].

The *coradical* C_0 of a coalgebra C is the sum of its simple subcoalgebras. The *coradical filtration* of C is an ascending chain of subcoalgebras, in which the coradical C_0 is the smallest set and the other subcoalgebras are recursively defined as follows

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C),$$

for every $n \geq 1$.

A coalgebra C is *pointed* if every simple subcoalgebra is one-dimensional; equivalently, C is pointed if and only if its coradical $C_0 = kG(C)$ is the group algebra of the grouplike elements of C. A bialgebra generated as an algebra by grouplike and skew-primitive elements is pointed [76, Corollary 5.1.14].

A coalgebra C is called *irreducible* if any two nonzero subcoalgebras have nontrivial intersection. An *irreducible component* of a coalgebra C is a maximal irreducible subcoalgebra. The irreducible component of a Hopf algebra H containing the identity 1_H, which is denoted by H_1, is a Hopf subalgebra of H [67, Corollary 5.6.4(1)].

1.1.5 Smash and crossed products

Many examples of Hopf algebras can be constructed by smash products or crossed products of "smaller" Hopf algebras. We present these constructions here, as they will appear recurrently throughout the thesis.

Let A be an algebra and T a Hopf algebra. We say that T *acts weakly* on A if there is a k-linear map $\cdot : T \otimes A \to A$ such that $t \cdot 1_A = \epsilon(t)1_A$, $t \cdot (ab) = \sum (t_1 \cdot a)(t_2 \cdot b)$ and $1_T \cdot a = a$, for any $t \in T, a, b \in A$. In this case, a k-linear map $\sigma : T \otimes T \to A$ is called a *cocycle* if for all $s, t, u \in T$ we have $\sigma(t, 1_T) = \epsilon(t)1_A = \sigma(1_T, t)$ and

$$\sum (s_1 \cdot \sigma(t_1, u_1))\sigma(s_2, t_2u_2) = \sum \sigma(s_1, t_1)\sigma(s_2t_2, u)$$

and σ is convolution invertible. That is, there is a k-linear map $\sigma^{-1} : T \otimes T \to A$ such that $\sum \sigma(s_1, t_1)\sigma^{-1}(s_2, t_2) = \epsilon(s)\epsilon(t)1_A = \sum \sigma^{-1}(s_1, t_1)\sigma(s_2, t_2)$ for all $s, t \in T$. Moreover, A is said to be a *twisted T-module* with respect to σ if

$$s \cdot (t \cdot a) = \sum \sigma(s_1, t_1)(s_2t_2 \cdot a)\sigma^{-1}(s_3, t_3),$$

for all $s, t \in T, a \in A$.

Definition 1.1.16. Let T be a Hopf algebra that acts weakly on an algebra A and σ a cocycle such that A is a twisted T-module. The *crossed product* $A\#_\sigma T$ is the vector space $A \otimes T$ endowed with the product

$$(a\#s)(b\#t) = \sum a(s_1 \cdot b)\sigma(s_2, t_1)\#s_3t_2,$$

for any $a, b \in A, s, t \in T$. Note that in crossed products we write $a\#s$ for the tensor $a \otimes s$.

Crossed products are associative algebras with identity $1_A\#1_T$ [8, Lemmas 4.4, 4.5]. A *smash product* is the particular case of a crossed product in which the cocycle σ is trivial, that is $\sigma(s, t) = \epsilon(s)\epsilon(t)1_A$; in this case, A is an (untwisted) T-module algebra and we write their smash product as $A\#T$. Here multiplication is given by

$$(a\#s)(b\#t) = a(s_1 \cdot b)\#s_2t,$$

for all $a, b \in A, s, t \in T$. A smash product $A\#kG$ is often called a skew group algebra, and denoted by $A\#G$ or $A * G$.

A crossed product $A\#_\sigma T$ contains A as a subalgebra. In general T is not a subalgebra of $A\#_\sigma T$, but it is in the case of a smash product $A\#T$. Moreover, a crossed product $A\#_\sigma T$ is isomorphic to $A \otimes T$ as left A-modules and right T-comodules, that is the left A-action of $A\#_\sigma T$ and its T-coaction are respectively given by $a \cdot (b\#t) = ab\#t$ and $\rho(a\#t) = a\#t_1 \otimes t_2$, for all $a, b \in A, t \in T$. In particular, the crossed product $A\#_\sigma T$ is a free left A-module.

Most of the crossed products we will deal with arise from Hopf surjections. Suppose we have a Hopf surjection $\pi : H \to T$. Then, as mentioned in Example 1.1.10 H is a right T-comodule algebra with coaction $\rho = (\mathrm{id} \otimes \pi) \circ \Delta$ and coinvariants $H^{\mathrm{co}\,T} := \{h \in H : \rho(h) = h \otimes 1\}$. A right T-comodule map $\gamma : T \to H$ is called a *cleaving map* if it is convolution invertible, that is, there exists a linear map $\gamma^{-1} : T \to H$ such that for all $t \in T$

$$\sum \gamma(t_1)\gamma^{-1}(t_2) = \epsilon(t)1_H = \sum \gamma^{-1}(t_1)\gamma(t_2).$$

The following celebrated result by Doi and Takeuchi [28] encompasses a very nice criterion for when H decomposes into the crossed product $H^{\mathrm{co}\,T}\#_\sigma T$.

Theorem 1.1.17. *Let H, T be a Hopf algebras with a Hopf surjection $\pi : H \to T$. If there is a cleaving map, that is a convolution invertible right T-comodule map $\gamma : T \to H$, then writing $A := H^{\mathrm{co}\,T}$ we have*

$$H \cong A\#_\sigma T.$$

This is an isomorphism of algebras, left A-modules and right T-comodules. More specifically, the isomorphism is given by $\phi : A\#_\sigma T \to H, a\#t \mapsto a\gamma(t)$ and in $A\#_\sigma T$ the weak action of T on A is $t \cdot a = \sum \gamma(t_1)a\gamma^{-1}(t_2)$ and the cocycle is $\sigma(s,t) = \sum \gamma(s_1)\gamma(t_1)\gamma^{-1}(s_2t_2)$ for any $s, t \in T, a \in A$, where γ^{-1} denotes the convolution inverse of γ.

Conversely, any crossed product Hopf algebra $A\#_\sigma T$ has a cleaving map given by $\gamma : T \to A\#_\sigma T, t \mapsto 1\#t$. More specifically, its convolution inverse is $\gamma^{-1}(t) = \sum \sigma^{-1}(S(t_2), t_3)\#S(t_1)$.

Proof. See [67, Propositions 7.2.3, 7.2.7]. □

Notice that, in the decomposition of H into a crossed product from the previous result, $A = H^{\mathrm{co}\,T}$ is a left coideal subalgebra of H as in Example 1.1.10. However, in general A is not a Hopf subalgebra of H and we do not have an isomorphism $H \cong A \otimes T$ of coalgebras; see Example 1.1.21 below.

When such a cleaving map exists as in the previous result the extension $A \subseteq H$ is said to be *T-cleft*, see [67, Definition 7.2.1].

Remarks 1.1.18. There are two important situations in which we have cleaving maps.

1. If γ is an algebra map, then it is a cleaving map with convolution inverse $\gamma^{-1} = \gamma \circ S_T$. In this case, the cocycle is actually trivial and $H = A\#T$ is a smash product. See for example [8, Example 4.19] and [44, Lemma 2.30].

2. If γ is a coalgebra map that splits, meaning $\pi \circ \gamma = \mathrm{id}$, then it is a cleaving map with convolution inverse $\gamma^{-1} = S_H \circ \gamma$. For a proof, see for example [8, Theorem 4.14] or [44, Lemma 2.20]. In this situation, if A is a subcoalgebra of H, then H decomposes as a coalgebra into $A \otimes T$, the usual coalgebra structure of a tensor product of coalgebras.

Example 1.1.19 (Group algebras, [67, Example 7.1.6])**.** Let G be a group with a normal subgroup $N \lhd G$. We clearly have a Hopf surjection $\pi : kG \to k(G/N)$. It is easy to see the coinvariants are $kG^{\mathrm{co}\, k(G/N)} = kN$ and, defining a map $\gamma : k(G/N) \to kG, \bar{g} \mapsto g$ by some choice of coset representative, this is clearly a splitting coalgebra map. It follows that

$$kG \cong kN \#_\sigma k(G/N).$$

Moreover, as a coalgebra $kG \cong kN \otimes k(G/N)$.

Note that the crossed product decomposition above is in fact a smash product if and only if $\gamma : k(G/N) \to kG$ is an algebra map, that is G/N can be embedded as a subgroup of G. This holds precisely when G is the semidirect product $N \rtimes G/N$.

Example 1.1.20 (Enveloping algebras, [67, Corollary 7.2.8])**.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie ideal. The factor map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ into the quotient Lie algebra extends uniquely into a Hopf map $\pi : U(\mathfrak{g}) \to U(\mathfrak{g}/\mathfrak{h})$ and, similarly to the previous example, we have the decomposition

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \#_\sigma U(\mathfrak{g}/\mathfrak{h})$$

and the usual decomposition as coalgebras. As in the previous example, $U(\mathfrak{g})$ is in fact the smash product $U(\mathfrak{h}) \# U(\mathfrak{g}/\mathfrak{h})$ if and only if $\mathfrak{g}/\mathfrak{h}$ embeds into $\mathfrak{g}$ as a Lie subalgebra.

Example 1.1.21 (Taft algebras)**.** Recall the definition of a Taft algebra $H = T(n,t,q)$ from Example 1.1.4. It decomposes into a crossed product in more than one way.

On one hand, since x and g skew-commute, the right ideal xH is a 2-sided ideal of H and, x being $(1, g^t)$-primitive, xH is a Hopf ideal. Therefore, we have a quotient Hopf map $\pi : H \to H/xH \cong kC_n$, where $C_n = \langle \bar{g} \rangle$. The coinvariants are $H^{\mathrm{co}\, kC_n} = k[x]$ and the natural map $\gamma : kC_n \to H, \bar{g} \mapsto g$ is both an algebra and a splitting coalgebra map, hence H decomposes into the smash product

$$T(n,t,q) \cong k[x] \# kC_n$$

with action $\bar{g} \cdot x = q^{-1}x$. Moreover, the coalgebra decomposition is the usual, that is $T(n,t,q) \cong k[x] \otimes kC_n$.

We point out that finite dimensional Taft algebras $T_f(n,t,q)$ can similarly be decomposed into crossed products,

$$T_f \cong k\langle x : x^n = 0 \rangle \# kC_n.$$

Note however that the subalgebra $k\langle x : x^n = 0\rangle$ is commutative but not reduced, hence in characteristic 0 it is not a Hopf subalgebra of $T_f(n,t,q)$ by Theorem 1.1.7.

On the other hand, it is not hard to see $x^{n'}$ is a primitive element, where $n' := n/(n,t)$, hence we have the Hopf quotient $\pi : H \to H/x^{n'}H$, whose corresponding coinvariants are $k[x^{n'}]$. The Hopf algebra $H/x^{n'}H = k\langle \bar{g}, \bar{x} : \bar{g}^n = 1, \bar{x}^{n'} = 0, \bar{x}\bar{g} = q\bar{g}\bar{x} \rangle$, which we denote by $U(n,t,q)$, will be discussed further in the next paragraph. We have a natural coalgebra map $\gamma : U(n,t,q) \to H, \bar{x}^i\bar{g}^j \mapsto x^i g^j$ with $0 \le i < n', 0 \le j < n$, whence

$$T(n,t,q) \cong k[x^{n'}]\#_\sigma U(n,t,q),$$

where the x acts trivially and g acts by $g \cdot x^{n'} = q^{-n'}x^{n'}$; and in general the cocycle σ is nontrivial. The coalgebra structure is the usual one.

For future purposes, we record here a decomposition of the Hopf algebra $U := U(n,t,q)$ itself. Note that, when $(n,t) = 1$, U is just the finite dimensional Taft algebra $T_f(n,t,q)$, and when $(n,t) \ne 1$ it decomposes into a crossed product of well-known Hopf algebras. Let $d := (n,t), t' := t/d$. One easily sees that $I := (\bar{x}, \bar{g}^d - 1)U$ is a Hopf ideal of U, so we get a Hopf surjection $\pi' : U \to U/I \cong kC_d$. Moreover, the coinvariants of the canonical right kC_d-comodule structure of U are $k\langle \bar{x}, \bar{g}^d\rangle = T_f(n', t', q^d)$. The natural map $\gamma : kC_d \to U$ is a splitting coalgebra map, hence cleaving. Therefore, U decomposes as the crossed product

$$U(n,t,q) \cong T_f(n', t', q^d)\#_\tau kC_d.$$

In general neither the action of C_d on $T_f(n', t', q^d)$ nor the cocycle τ are trivial. The coalgebra structure is the usual one.

An extension of the notion of crossed products, or cleft extensions, is the concept of *Galois extensions*, which will feature throughout the thesis. Let T be a Hopf algebra and H be a right T-comodule algebra with coaction $\rho : H \to H \otimes T$. The extension $H^{\mathrm{co}\,T} \subseteq H$ is *right T-Galois* if the map $\beta : H \otimes_{H^{\mathrm{co}\,T}} H \to H \otimes T, a \otimes b \mapsto (a \otimes 1)\rho(b)$ is bijective.

This notion can be regarded as an extension of the notion of cleftness as per the following result.

Theorem 1.1.22. *Let T be a Hopf algebra and H a right T-comodule algebra. Let $A = H^{\mathrm{co}\,T}$. Then, the following are equivalent:*

1. *$A \subseteq H$ is T-cleft.*
2. *$A \subseteq H$ is T-Galois and has the normal basis property, that is, $H \cong A \otimes T$ as left A-modules and right T-comodules.*

Proof. See [67, Theorem 8.2.4]. □

1.1.6 Cocommutative Hopf algebras

So far the only examples we have seen of cocommutative Hopf algebras are group algebras kG and enveloping algebras $U(\mathfrak{g})$. It turns out that over an algebraically closed field of characteristic 0 any cocommutative Hopf algebra is built up as a smash product of these Hopf algebras.

Theorem 1.1.23 (Cartier-Gabriel-Kostant). *Let H be a cocommutative Hopf algebra over an algebraically closed field k.*

1. *Then, as Hopf algebras*
$$H \cong H_1 * G,$$
where $G := G(H)$ acts on the irreducible component of the identity H_1 by conjugation and this skew group ring has the tensor coalgebra structure, that is $H \cong H_1 \otimes kG$ as coalgebras.

2. *If additionally k has characteristic 0, then $H_1 \cong U(\mathfrak{g})$ where $\mathfrak{g} := P(H)$ and as Hopf algebras*
$$H \cong U(\mathfrak{g}) * G.$$

Proof. See [67, Corollary 5.6.4(3), Theorem 5.6.5]. □

Remark 1.1.24. Part (2) does not hold in characteristic $p > 0$; see [67, Example 5.6.8] for a counterexample. In this case, the structure of H_1 is more complicated; see [67, Theorem 5.6.9] for a partial result on its coalgebra structure.

1.2 Homological notions

Throughout this thesis we will often mention and apply homological concepts. We sum up a few definitions and results in this section. We also mention how some of these notions simplify for noetherian or affine algebras, which will be the case for most of the Hopf algebras studied in this thesis. For more on homological algebra see for example [49], [77], [100]. Throughout this section assume k is algebraically closed.

A ring R is a *polynomial identity ring* (or PI ring) if R satisfies a monic polynomial $p \in \mathbb{Z}\langle x_1, \ldots, x_n\rangle$ for some $n \in \mathbb{N}$, that is, for every $r_1, \ldots, r_n \in R$ we have $p(r_1, \ldots, r_n) = 0$. The *minimal degree* of R, denoted by $\mathrm{min.deg}(R)$, is the smallest degree of a monic polynomial identity of R. For more on PI rings, see [64, §13].

Let R be a commutative k-algebra. The *classical Krull dimension* of R, denoted $\mathrm{Kdim}\, R$, is the largest positive integer n such that $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ is a strict chain of prime ideals of R. If R contains chains of prime ideals of arbitrary length, $\mathrm{Kdim}\, R = \infty$. See [78, Chapter 6]. The definition of Krull dimension for noncommutative noetherian rings can be found in [64, Chapter 6] and it coincides with the classical notion for noetherian PI rings [64, 6.4.7, Theorem 6.4.8]. In particular, these two notions coincide for the class of affine commutative-by-finite Hopf algebras, which will be the focus of this thesis; see Theorem 2.1.3.

Another way to measure the growth of k-algebras is the *Gelfand-Kirillov dimension* (or in short GK-dimension), denoted $\mathrm{GKdim}\, R$. Its definition can be found in [49, §2]. For a noetherian PI algebra R, $\mathrm{GKdim}\, R$ is either a nonnegative integer or infinity and, provided R is also affine, $\mathrm{GKdim}\, R = \mathrm{Kdim}\, R$ [49, Corollary 10.16].

For any ring R, a left (resp. right) R-module M is *injective* if, for any left (resp. right) R-modules $A \subseteq B$, any homomorphism $A \to M$ extends to a homomorphism $B \to M$. A *projective* left (resp. right) module is the dual notion of an injective module, that is, for any left (resp. right) R-module surjection $B \twoheadrightarrow A$, any homomorphism $M \to A$ can be lifted to a homomorphism $M \to B$. For example, any free module is projective [77, Theorem 3.14].

A left or right artinian ring R that is injective as a module over itself is called *quasi-Frobenius*. A finite dimensional algebra R is called *Frobenius* if there is a non-degenerate bilinear form $\sigma : R \times R \to k$ such that $\sigma(ab, c) = \sigma(a, bc)$ for all $a, b, c \in R$; see [51, Theorem 3.15] for equivalent definitions. As one would expect, a Frobenius algebra is quasi-Frobenius; see [51, Proposition 3.14] or [77, Theorem 4.39]. Moreover, any finite dimensional Hopf algebra is Frobenius [67, Theorem 2.1.3]; and any commutative quasi-Frobenius k-algebra is Frobenius [101, Remark 1.3].

A left (resp. right) R-module M is *flat* if the functor $- \otimes_R M$ (resp. $M \otimes_R -$) is exact, meaning that, for any short exact sequence $0 \to A \to B \to C \to 0$ of right (resp. left) R-modules, $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is an exact sequence of groups. A flat R-module is *faithfully flat* provided the functor $- \otimes_R M$ is exact and faithful, that is the sequence $0 \to A \to B \to C \to 0$ is exact if and only if $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is exact. Projective modules are flat [77, Corollary 3.46] and free modules are faithfully flat. See [77] for more on these modules and [99, §13] for more on faithful flatness.

Let M be a left R-module. The left *injective dimension* of M, denoted $\operatorname{injdim} M$, is shortest length among injective resolutions of M, that is the smallest n such that $0 \to M \to E_0 \to E_1 \to \ldots \to E_n \to 0$ is an exact sequence of injective R-modules E_i. If no such finite resolution exists, we say $\operatorname{injdim} M = \infty$. If R has finite left and right injective dimension as a module over itself, we say R is *Gorenstein*. For example, affine commutative Hopf algebras are Gorenstein [10, 2.3, Step 1] and an important result by Wu and Zhang says that noetherian PI Hopf algebras are Gorenstein [102, Theorem 0.1].

Dually, the *projective dimension* of M, denoted $\operatorname{prdim}(M)$, is the smallest n such that $0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ is an exact sequence with projective modules P_i. If no such finite sequence exists, we say $\operatorname{prdim}(M) = \infty$.

The *left global dimension* of R is

$$\operatorname{gldim} R := \sup\{\operatorname{prdim}(M) : \text{ left } R\text{-module } M\}.$$

The left global dimension also coincides with the supremum of the injective dimensions of all left R-modules, [77, Theorem 9.10]. The *right* global dimension is defined similarly. If R is a (left and right) noetherian ring, the left and right global dimensions coincide [77, Corollary 9.23]; and if R is a noetherian ring with finite global dimension, then the global dimension is the supremum of the projective dimensions of irreducible

R-modules [64, Corollary 7.1.14]. For a Hopf algebra H, we have

$$\operatorname{gldim} H = \operatorname{prdim}(k),$$

that is, its global dimension is controlled by the trivial H-module $H/H^{+} \cong k$ [56, § 2.4]. A noetherian ring is named *regular* if it has finite global dimension. For example, affine commutative Hopf algebras in characteristic 0 are regular [99, §11.4, §11.6, §11.7].

We now present some strengthened versions of the Gorenstein and regularity notions for noncommutative rings. The definition and basic properties of Ext-groups can be found in [77, § 6, § 7]. Let R be a (left and right) noetherian ring that is finitely generated over its centre Z. We say R is right *injectively homogeneous* if the upper grade $\sup\{i : \operatorname{Ext}^{i}(R/I_R, R_R) \neq 0\}$ is constant among maximal ideals I of R with the same intersection with Z. And R is said to be *homologically homogeneous (hom. hom.)* over its centre if it has finite global dimension and $\operatorname{prdim}(R/I)$ is constant among all maximal ideals I of R with the same intersection with Z. Moreover, R is *Z-Macaulay* if $G(\mathbf{m}, R) = \operatorname{Kdim}_{Z_{\mathbf{m}}}(R_{\mathbf{m}}) = \operatorname{Kdim}(Z_{\mathbf{m}})$ for every maximal ideal $\mathbf{m}$ of Z, where $R_{\mathbf{m}}$ and $Z_{\mathbf{m}}$ are respectively the localizations of R and Z at $\mathbf{m}$ and the grade $G(\mathbf{m}, R)$ is the largest integer n for which there exist elements $x_1, \ldots, x_n \in \mathbf{m}$ such that $x_i + \sum_{j=1}^{i-1} x_j R$ is a nonzero divisor of $R/\sum_{j=1}^{i-1} x_j R$ for each $1 \leq i \leq n$. For more, see [16], [17].

We now present two more concepts that generalize the notion of Gorenstein, which have also been extensively studied in the context of Hopf algebras. Let H be a noetherian Hopf k-algebra. We say H is *Artin-Schelter-Gorenstein* (or in short AS-Gorenstein) if

(i) H has finite injective dimension d (on the left and on the right);
(ii) $\operatorname{Ext}^{i}_{H}({}_{H}k, {}_{H}H) = \operatorname{Ext}^{i}_{H}(k_H, H_H) = 0$ for all $i \neq d$;
(iii) and $\operatorname{Ext}^{d}_{H}({}_{H}k, {}_{H}H) \cong \operatorname{Ext}^{d}_{H}(k_H, H_H) \cong k$.

Moreover, if H also has finite global dimension, we say H is AS-regular. We say H is *Auslander-Gorenstein* if

(i) H has finite injective dimension d (on the left and on the right);
(ii) for all $0 \leq j \leq d$, every noetherian left H-module M and right H-submodule $N \subseteq \operatorname{Ext}^{j}({}_{H}M, {}_{H}H)$, one has $\operatorname{Ext}^{i}(N_H, H_H) = 0$ for all $i < j$;
(iii) and (ii) holds for right H-modules.

If, additionally, H has finite global dimension, we say H is Auslander-regular.

Let H be a noetherian Hopf algebra of finite GK-dimension. The *grade* of a left and right noetherian H-module M is $j(M) = \min\{i : \operatorname{Ext}^{i}_{H}(M, H) \neq 0\}$. We say H is *GK-Cohen-Macaulay* if for any left (and right) noetherian H-module M we have

$$\operatorname{GKdim} M + j(M) = \operatorname{GKdim} H.$$

A noetherian ring H with finite Krull-dimension is called *Krull-Macaulay* if $\operatorname{Kdim} M +$

$j(M) = \operatorname{Kdim} H$ for every finitely generated left or right H-module M.

Note that a noetherian Hopf algebra H which is both AS-Gorenstein and GK-Cohen-Macaulay has $\operatorname{injdim} H = \operatorname{GKdim} H$, since H being AS-Gorenstein implies the grade of the trivial H-module k equals $\operatorname{injdim} H$, hence the equality follows by the GK-Cohen-Macaulay property.

It is not hard to show that any finite dimensional Hopf algebra, being Frobenius, must be AS-Gorenstein, Auslander-Gorenstein and GK-Cohen Macaulay of injective dimension 0. More recently Wu and Zhang proved the following crucial result.

Theorem 1.2.1 (Wu-Zhang, [102, Theorems 0.1, 0.2])**.** *An affine noetherian PI Hopf algebra is AS-Gorenstein, Auslander-Gorenstein and GK-Cohen-Macaulay.*

An Auslander-Gorenstein GK-Cohen-Macaulay noetherian ring has an artinian quasi-Frobenius classical ring of fractions [2, Theorem 6.1]. The *classical ring of fractions* is constructed by inverting the set of all regular elements of the ring. This extends the construction of the quotient field of a commutative domain but for noncommutative rings it requires some conditions; see [37, Chapter 6] with particular emphasis on the Ore condition and Goldie's theorem.

1.3 Hopf dual

We now introduce the notion of duality in Hopf algebras. This is a key concept in this thesis, as we will thoroughly study the duals of commutative-by-finite Hopf algebras in chapter 4.

Let $(H, m, u, \Delta, \epsilon, S)$ be a finite dimensional Hopf algebra. Its dual

$$H^* := \operatorname{Hom}_k(H, k)$$

is the k-vector space of functionals of H, that is k-linear maps $H \to k$. It is endowed with an algebra (resp. coalgebra, antipode) structure, which is obtained by transposing the coalgebra (resp. algebra, antipode) structure of H; that is, $(H^*, \Delta_H^*, \epsilon_H^*, m_H^*, u_H^*, S_H^*)$ is a Hopf algebra as follows.

More specifically, the product in H^* is given by

$$(fg)(h) = \sum f(h_1)g(h_2),$$

for any $f, g \in H^*, h \in H$. It is usually named *convolution product*. The identity of H^* is the functional ϵ_H, the counit of H. Direct computations show that $\epsilon f = f = f\epsilon$ for any $f \in H^*$.

The coproduct of H^* is defined as follows. First, $H^* \otimes H^* \cong (H \otimes H)^*$ are canonically isomorphic as in Lemma 1.3.2(3) below. So, $\Delta_{H^*}(f) = \sum f_1 \otimes f_2 \in H^* \otimes H^*$ is completely determined by

$$f(ab) = \sum f_1(a) f_2(b),$$

for any $f \in H^*, a, b \in H$. The counit of H^* is $\epsilon_{H^*} : H^* \to k$ with

$$\epsilon_{H^*}(f) = f(1_H)$$

for all $f \in H^*$. And the antipode of H^* is $S_{H^*} : H^* \to H^*$ such that

$$S_{H^*}(f)(h) = f(S(h)),$$

for any $f \in H^*, h \in H$.

Theorem 1.3.1. *Let H be a finite dimensional Hopf algebra. Then, H^* is a Hopf algebra with the above structure.*

Proof. This is a particular case of Theorem 1.3.5(2). □

Here are a few properties of the dual Hopf algebra.

Lemma 1.3.2. *Let H, T be finite dimensional Hopf algebras.*

1. *H^* is commutative (resp. cocommutative) if and only if H is cocommutative (resp. commutative).*
2. *H and H^* have the same dimension.*
3. *$(H \otimes T)^*$ and $H^* \otimes T^*$ are canonically isomorphic as Hopf algebras.*
4. *$(H^*)^*$ and H are canonically isomorphic as Hopf algebras.*

Proof. (1) This follows from the fact that the algebra and coalgebra structures of H give rise to the coalgebra and algebra structures of H^* respectively.

(2) This is a well-known fact about duals of vector spaces.

(3) The Hopf isomorphism is

$$\begin{aligned} \phi : H^* \otimes T^* &\to (H \otimes T)^* \\ f \otimes g &\mapsto \phi(f \otimes g) : [a \otimes b \mapsto f(a)g(b)] \end{aligned}.$$

(4) The Hopf isomorphism is

$$\begin{aligned} \varphi : H &\to (H^*)^* \\ h &\mapsto \varphi_h : [f \mapsto f(h)] \end{aligned}.$$

□

Example 1.3.3 (Group algebras)**.** Let G be a finite group. For each $g \in G$ let $g^* : kG \to k, g' \mapsto \delta_{g,g'}$, that is $\{g^* : g \in G\}$ is the dual basis of G for kG, hence it is a basis of $(kG)^*$. It is easy to see that $g^*h^* = \delta_{g,h}g^*$, that is these elements are idempotents and pairwise orthogonal, and the identity is $\epsilon = \sum_{g\in G} g^*$. Thus, as algebras

$$(kG)^* \cong k^{|G|}.$$

Its coalgebra structure is as follows:

$$\Delta(g^*) = \sum_{uv=g} u^* \otimes v^*$$

and $\epsilon(g^*) = \delta_{g,1_G}$. The antipode is given by $S(g^*) = (g^{-1})^*$.

In the particular case of finite cyclic groups C_n, we can say a bit more. Let k be an algebraically closed field whose characteristic does not divide n. Then, the group algebra kC_n is self-dual, meaning

$$(kC_n)^* \cong kC_n$$

as Hopf algebras. Let q be a primitive nth root of unity and g a generator of C_n. The isomorphism is

$$\begin{array}{rcl} \varphi : kC_n & \to & (kC_n)^* \\ g^i & \mapsto & f_i : [g^j \mapsto q^{ij}] \end{array}.$$

A proof can be found in [44, Example 1.20].

In more geometric terms, C_n can be identified with the cyclic group of nth roots of unity of $\mathbb{A}^1(k)$. Comparing the formulas of the coproduct, counit and antipode of coordinate rings and duals, it is clear we have the identification

$$\mathcal{O}(C_n) = (kC_n)^*.$$

Note that not all group algebras are self-dual: for any nonabelian group G, kG is not commutative but it is cocommutative, so $(kG)^*$ is commutative.

Example 1.3.4 (Taft algebras)**.** Let $T_f(n,t,q)$ be a finite dimensional Taft algebra as in Example 1.1.4. These Hopf algebras are self-dual when $(n,t) = 1$. This can be shown directly, the isomorphism being

$$\begin{array}{rcl} \varphi : T_f(n,t,q) & \to & T_f(n,t,q)^* \\ x & \mapsto & X : [x^i g^j \mapsto \delta_{i,1}] \\ g & \mapsto & G : [x^i g^j \mapsto \delta_{i,0} q^{-t^{-1} j}] \end{array},$$

where t^{-1} is the inverse of t modulo n. For suggestions on the proof, see [76, Exercise 7.4.3].

For future purposes we also record here the dual of

$$U(n,t,q) = k\langle g, x : g^n = 1, x^{n'} = 0, xg = qgx\rangle,$$

where g is grouplike, x is $(1, g^t)$-primitive and $n' = n/(n,t)$. Recall it from Example

1.1.21, where we saw that $U(n,t,q) \cong T_f(n',t',q^d)\#_\sigma kC_d$, where $d=(n,t), t'=t/d$. However, as a coalgebra it is $T_f(n',t',q^d) \otimes kC_d$, hence as an algebra

$$U(n,t,q)^* \cong T_f(n',t',q^d)^* \otimes (kC_d)^*$$

and, since $(n',t')=1$, both these Hopf algebras are self-dual as stated above. Let G and X respectively denote the invertible and nilpotent generators of T_f^* and α denote the generator of $(kC_d)^*$. From Example 1.3.3 and the first part of this example, it is easy to see that these functionals are defined as follows: $G(x^i g^j) = \delta_{i,0} q^{-t'^{-1}dk}$, $X(x^i g^j) = \delta_{i,1}$ and $\alpha(x^i g^j) = \delta_{i,0} q^{n'r} = \delta_{i,0} q^{n'j}$, where $j = dk + r$ with $0 \le r < d$ and t'^{-1} is the inverse of t' modulo n'.

1.3.1 Finite dual

We now discuss the dual of infinite dimensional Hopf algebras.

In infinite dimension the dual is not as easy to define. The problem lies in dualizing the product of H: the dual of $m : H \otimes H \to H$ is a map $m^* : H^* \to (H \otimes H)^*$ and, while in finite dimensions $(H \otimes H)^* \cong H^* \otimes H^*$ making m^* eligible for a well-defined coproduct of H^*, the same does not hold in infinite dimension. In fact, the inclusion $H^* \otimes H^* \subseteq (H \otimes H)^*$ is actually strict for all infinite dimensional Hopf algebras. The way we go around this problem is to restrict the set of functionals $H \to k$ we look at, considering only those which are not affected by this problem, that is

$$H^\circ := \{f \in H^* : m^*(f) \in H^* \otimes H^*\}.$$

There are many equivalent ways of defining H°. In the following result we give a more insightful characterization of this subspace of H^*.

Theorem 1.3.5. *Let H be a Hopf algebra.*

1. *Let $f \in H^*$. The following are equivalent:*
   - *$m^*(f) \in H^* \otimes H^*$;*
   - *$f(I) = 0$ for some left ideal I of H with finite codimension;*
   - *$f(I) = 0$ for some right ideal I of H with finite codimension;*
   - *$f(I) = 0$ for some ideal I of H with finite codimension.*
2. *H° is a Hopf algebra, by restricting all structure maps of H^* to it.*

Proof. See [67, Lemma 9.1.1, Theorem 9.1.3]. □

Definition 1.3.6. Let H be a Hopf algebra. The Hopf algebra H° described in the previous result is called the *finite dual* of H.

Note that in finite dimension H° is clearly just H^* but in infinite dimension the inclusion $H^\circ \subset H^*$ is strict; for example, check [44, Example 1.27].

Before we compute some examples, let us look at the following important result on the finite dual of commutative reduced Hopf algebras.

Theorem 1.3.7. *Let H be an affine commutative Hopf algebra over an algebraically closed field k. Suppose H is reduced.*

1. *Then, $H \cong \mathcal{O}(G)$ for some algebraic group G and its dual is*

$$\mathcal{O}(G)^\circ \cong H' * G$$

as a Hopf algebra. Here $H' := \{f \in H^\circ : f((H^+)^n) = 0, \text{ for some } n \geq 1\}$ is the subspace of functionals that vanish on some power of the augmentation ideal H^+; the action of G on H' is given by conjugation; and $\mathcal{O}(G)^\circ$ has the tensor coalgebra structure.

2. *The functionals in G are the grouplike elements of H°, that is the algebra homomorphisms $H \to k$, or in other words the characters of H. Hence, the functionals contained in G are precisely the functionals that vanish on maximal ideals of H.*

Assume further that k has characteristic 0. Then,

3. *$H' \cong U(\mathfrak{g})$, where $\mathfrak{g} = \operatorname{Lie} G$ is the Lie algebra of G, and so*

$$\mathcal{O}(G)^\circ \cong U(\mathfrak{g}) * G.$$

The Lie algebra $\mathfrak{g}$ is also the subspace of primitive elements of H° and has the following description:

$$\mathfrak{g} \cong (H^+/(H^+)^2)^*$$

is the set of functionals on H^+ that vanish on $(H^+)^2$. The Lie brackets of $\mathfrak{g}$ are given by the commutator in H°.

Proof. First, since H is a reduced affine commutative algebra, $H \cong \mathcal{O}(G)$ is the coordinate ring of the affine algebraic group $G = \operatorname{Maxspec}(H)$ by Theorem 1.1.7(2).

(1),(2) Since $H = \mathcal{O}(G)$ is commutative, its dual is cocommutative and, by Theorem 1.1.23(1), $H^\circ \cong (H^\circ)_\epsilon * G(H^\circ)$ as Hopf algebras, where $(H^\circ)_\epsilon$ is the irreducible component of H° containing $1_{H^\circ} = \epsilon$. By [67, Proposition 9.2.5], $(H^\circ)_\epsilon$ is just H', as defined in the statement.

Moreover,

$$G(H^\circ) = \operatorname{Alg}(H, k) \cong \operatorname{Maxspec}(H) = G,$$

that is the grouplikes of H° are the algebra maps $H \to k$, which are clearly in bijective correspondence with $\operatorname{Maxspec}(H) = G$.

(3) By Theorem 1.1.23(2), $H' = (H^\circ)_\epsilon = U(\mathfrak{g})$ with $\mathfrak{g} := P(H^\circ)$. It follows from [1, Corollary 4.3.2, § 4.3.3], or [43, § 5.1], that

$$\mathfrak{g} := P(H^\circ) = \operatorname{Lie} G \cong (H^+/(H^+)^2)^*.$$

And the Lie brackets of $\mathfrak{g}$ are given by commutators on H° by [43, § 9.3]. □

This allows us to easily decompose the dual of many examples.

Example 1.3.8 (Polynomial ring)**.** Let k be algebraically closed of characteristic zero. Consider the commutative Hopf algebra $H = k[x]$. As we mentioned in Example 1.1.9 its corresponding algebraic group is $G = \operatorname{Maxspec}(H) \cong (k, +)$, the additive group of the base field. Its Lie algebra is

$$\mathfrak{g} \cong (H^+/(H^+)^2)^* = (xH/x^2H)^* = (k\overline{x})^* = kf,$$

where f is the dual basis of $\overline{x}$. It is easy to see that f extends to the functional of H given by $f(x^i) = \delta_{i,1}$ for all $i \geq 0$.

Since H is also cocommutative, its dual is commutative and the Lie brackets of $\mathfrak{g}$ are trivial, hence its enveloping algebra is $U(\mathfrak{g}) = k[f]$, and also the action of G on $U(\mathfrak{g})$ is trivial. Therefore,

$$k[x]^\circ \cong k[f] \otimes k(k, +),$$

where for each $\lambda \in (k, +)$ the corresponding character of H is given by $\chi_\lambda(x^i) = \lambda^i$ for all $i \geq 0$.

This is easily extended to polynomial rings on many variables:

$$k[x_1, \dots, x_m]^\circ \cong k[f_1, \dots, f_m] \otimes k(k^m, +).$$

Remark 1.3.9. In positive characteristic, the dual of $H = k[x]$ is slightly different. It still decomposes into $H^\circ \cong H' * (k, +)$ by Theorem 1.3.7(1), but H' is no longer a polynomial algebra. In this case, it is a divided power Hopf algebra

$$H' = k[f^{(n)} : n \geq 0],$$

where each functional $f^{(n)}$ is given by $f^{(n)}(x^i) = \delta_{i,n}$; see [67, Example 9.1.7] for the proof of this fact and [67, Example 5.6.8] for the definition of a divided power Hopf algebra. This can also be easily generalized for many variables, and

$$k[x_1, \dots, x_m]^\circ \cong k[f_1^{(n)}, \dots, f_m^{(n)} : n \geq 0] \otimes k(k^m, +).$$

Example 1.3.10 (Laurent polynomial ring)**.** Let k be an algebraically closed field of characteristic zero. Recall from Example 1.1.8 the commutative Hopf algebra $H = k[x^{\pm 1}]$ of Laurent polynomials. Its affine algebraic group is $G = \operatorname{Maxspec}(H) = k^\times$, the multiplicative group of the base field. The Lie algebra

$$\mathfrak{g} = (H^+/(H^+)^2)^* = ((x-1)H/(x-1)^2H)^* = (k\,\overline{x-1})^* = kf,$$

where f is the dual basis of $\overline{x-1}$. By induction f extends to the functional of H given by $f(x^i) = i$ for all $i \in \mathbb{Z}$.

Since H is cocommutative, H° is commutative and the Lie brackets of $\mathfrak{g}$ are trivial, so $U(\mathfrak{g}) = k[f]$, and the action of G on $U(\mathfrak{g})$ is also trivial. Thus,

$$k[x^{\pm 1}]^\circ = k[f] \otimes k(k^\times, \cdot),$$

where for each $\lambda \in k^\times$ the corresponding character is defined by $\chi_\lambda(x^i) = \lambda^i$ for $i \in \mathbb{Z}$.

In a similar fashion, the dual of the Laurent polynomial algebra with multiple indeterminates is

$$k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^\circ \cong k[f_1, \ldots, f_n] \otimes k(k^\times, \cdot)^n.$$

Example 1.3.11. Let k be an algebraically closed field and $G = (k, +) \rtimes k^\times$, where $k^\times$ acts on $(k, +)$ by multiplication. Its coordinate ring is $H = \mathcal{O}(G) = k[x, y^{\pm 1}]$, where x and y are the polynomial functions on G defined by $x(\alpha, \beta) = \alpha$ and $y(\alpha, \beta) = \beta$ for any $(\alpha, \beta) \in G$. As for the coalgebra structure of $\mathcal{O}(G)$, x is $(1, y)$-primitive and y is grouplike. For, recall from Example 1.1.5 that by definition $\Delta(x)$ is the function $G \times G \to k$ such that for any $(\alpha, \beta), (\gamma, \delta) \in G$

$$\Delta(x)((\alpha, \beta), (\gamma, \delta)) = x((\alpha, \beta)(\gamma, \delta)) = x(\alpha + \beta\gamma, \beta\delta) = \alpha + \beta\gamma.$$

Thus, $\Delta(x) = x \otimes 1 + y \otimes x$, and similarly $\Delta(y)$ is such that $\Delta(y)((\alpha, \beta), (\gamma, \delta)) = \beta\delta$, hence $\Delta(y) = y \otimes y$.

When k has characteristic 0, Theorem 1.3.7 yields

$$\mathcal{O}((k, +) \rtimes k^\times)^\circ \cong U(\mathfrak{g}) * G.$$

Here $\mathfrak{g}$ is a 2-dimensional Lie algebra, say with basis $\{f, f'\}$, and it is easy to deduce that these functionals are defined by $f(x^i y^j) = \delta_{i,1}$ and $f'(x^i y^j) = \delta_{i,0} j$; and for each $(\alpha, \beta) \in G$ the corresponding character is defined by $\chi_{(\alpha,\beta)} = \alpha^i \beta^j$. The product is given by

$$f'f - ff' = f, \qquad \chi_{(\alpha,\beta)} f = \beta f \chi_{(\alpha,\beta)}, \qquad \chi_{(\alpha,\beta)} f' = f' \chi_{(\alpha,\beta)};$$

see Appendix, §A.4. In particular, $\mathfrak{g}$ is the 2-dimensional nonabelian solvable Lie algebra.

Example 1.3.12. Let k be an algebraically closed field of characteristic zero and $H = \mathcal{O}(SL_n)$ the coordinate ring of the special linear group, the group of matrices with determinant 1. Its corresponding Lie algebra is $\mathfrak{sl}_n$, the space of matrices with trace 0 [43, §9.4], hence the dual of this Hopf algebra is

$$\mathcal{O}(SL_n)^\circ \cong U(\mathfrak{sl}_n) * SL_n.$$

We should mention that there are many results in the literature on duals of other Hopf algebras, such as the enveloping algebras of solvable and semisimple Lie algebras which were studied by Hochschild; see [39], [40] and [41].

Examples 1.3.8 and 1.3.10 show that in general H° does not preserve many notions of dimension in infinite dimensional Hopf algebras, such as being noetherian or affine or having finite Gelfand-Kirillov dimension. This is clear even for the basic example of $H = \mathbb{C}[x]$, which is noetherian and affine and has GK-dimension one. But, due to the second tensorand of $H^\circ = \mathbb{C}[f] \otimes \mathbb{C}(\mathbb{C}, +)$, the dual of $\mathbb{C}[x]$ is neither noetherian nor affine and has infinite GK-dimension. There are also some examples in which the opposite situation occurs; see for example [44, Example 1.28] for an infinite dimensional Hopf algebra whose dual is the trivial Hopf algebra. These examples also show that in general $(H^\circ)^\circ$ is no longer isomorphic to H.

In the following result we gather a few properties H° possesses.

Lemma 1.3.13. *Let H, T be Hopf algebras.*

1. *If H is commutative (resp. cocommutative), then H° is cocommutative (resp. commutative).*

2. *$(H \otimes T)^\circ \cong H^\circ \otimes T^\circ$ as Hopf algebras.*

3. *If I is an ideal (resp. coideal) of H, then the subspaces of H°*

$$\{f \in H^\circ : f(I) = 0\} \quad \textit{and} \quad \{f \in H^\circ : f(I^n) = 0, \textit{ for some } n > 0\}$$

are subcoalgebras (resp. subalgebras) of H°. In particular, they are Hopf subalgebras of H° when I is a Hopf ideal of H.

Proof. (1) This follows from the fact that the algebra and coalgebra maps of H respectively induce the coalgebra and algebra structures of H°.

(2) Similarly to Lemma 1.3.2, the isomorphism is given by

$$\begin{array}{rcl} \phi : H^\circ \otimes T^\circ & \to & (H \otimes T)^\circ \\ f \otimes g & \mapsto & \phi(f \otimes g) : [a \otimes b \mapsto f(a)g(b)] \end{array}.$$

We will only check well-definiteness, since it is easy to prove ϕ is a Hopf isomorphism. Let $f \in H^\circ, g \in T^\circ$, say $f(I) = 0 = g(J)$ for some ideals I of H and J of T with finite codimension. Then, $\phi(f \otimes g)$ vanishes at the ideal $I \otimes T + H \otimes J$ of $H \otimes T$, which has finite codimension because

$$(H \otimes T)/(I \otimes T + H \otimes J) \cong (H/I) \otimes (T/J). \tag{1.4}$$

For, there exists an obvious epimorphism $\psi : H \otimes T \twoheadrightarrow (H/I) \otimes (T/J)$ and $H \otimes I + J \otimes T \subseteq \ker \psi$. A dimension argument and the isomorphism theorem now yield that (1.4) holds. Therefore, $\phi(f \otimes g) \in (H \otimes T)^\circ$.

(3) The proof for the second subspace can be found in [67, Lemma 9.2.1] and it can easily be adapted to prove the statements for the first subspace. □

Dualizing maps

As I mentioned before, we will be studying duals of Hopf algebras quite extensively in this thesis. In doing so, we will often dualize or transpose maps between Hopf algebras. It is then important to point out a few features and issues that arise in doing this.

Given a k-linear map $\phi : H \to T$ between Hopf algebras, its dual (also known as transpose or pullback) is

$$\begin{array}{rcl} \phi^\circ : T^\circ & \to & H^\circ \\ f & \mapsto & f \circ \phi \end{array}.$$

This is not always a well-defined map, since we may have $f \circ \phi \in H^* \setminus H^\circ$ for some $f \in T^\circ$; see [44, Example 3.3] for an example of a map whose dual map is not well-defined and see [44, Lemma 3.5] for a nice criterion on well-definiteness of the dual map.

Provided ϕ° is well-defined, we may deduce a few properties as per the following result.

Proposition 1.3.14. *Let $\phi : H \to T$ be a Hopf algebra map and suppose $\phi^\circ : T^\circ \to H^\circ$ is a well-defined map. Then,*

1. *ϕ° is a Hopf algebra map.*
2. *If ϕ is surjective, ϕ° is injective.*

Proof. See [44, Theorem 3.12]. □

We note that it is not always true that ϕ° is surjective when ϕ is injective. See [44, Example 3.8] for an example where this statement does not hold and [44, Lemma 3.10] for a simple criterion.

1.4 The Drinfeld double in finite dimension

In this section we introduce the notion of the Drinfeld double for finite dimensional Hopf algebras, and discuss a few of its properties and a few examples. The Drinfeld double was first introduced by Drinfeld in 1986 in [30]. For more information on it, see for example [67, § 10], [76, § 13].

Throughout this section let $(H, m, u, \Delta, \epsilon, S)$ be a finite dimensional Hopf algebra. Then, its dual H^* is also a finite dimensional Hopf algebra and, as in section 1.3, we denote its antipode by S^*. Recall that the antipode of a finite dimensional Hopf algebra is bijective, [67, Theorem 2.1.3].

In order to define the multiplication in the Drinfeld double of H, we must first define the left and right actions, $\rightharpoonup$ and $\leftharpoonup$, of H on H^*. It is easy to see that H^* is a left and right H-module as follows: for any $f \in H^*, h \in H$

$$h \rightharpoonup f : k \mapsto f(kh) \qquad \text{and} \qquad f \leftharpoonup h : k \mapsto f(hk).$$

Let us now define the *Drinfeld double* $D(H)$ of H. As a vector space it is

$$D(H) = H^* \otimes H,$$

that is, if H is n-dimensional with basis $\{h_i : 1 \leq i \leq n\}$ and $\{h_i^* : 1 \leq i \leq n\}$ denotes the dual basis of H^*, then $D(H)$ is n^2-dimensional with basis $\{h_i^* \otimes h_j : 1 \leq i, j \leq n\}$. The product on $D(H)$ is defined as follows: for any $f, \varphi \in H^*, h, k \in H$

$$(f \otimes h)(\varphi \otimes k) = \sum f(h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}h_3) \otimes h_2 k. \tag{1.5}$$

The identity element is the obvious one $1_{H^*} \otimes 1_H = \epsilon_H \otimes 1_H$.

As a coalgebra

$$D(H) = (H^*)^{cop} \otimes H,$$

meaning that its coproduct is the usual one in H and it is twisted in H^* as follows:

$$\Delta_{D(H)}(f \otimes h) = \sum (f_2 \otimes h_1) \otimes (f_1 \otimes h_2)$$

for any $f \in H^*, h \in H$. Moreover, the counit is the usual one

$$\epsilon_{D(H)}(f \otimes h) = f(1)\epsilon_H(h),$$

for any $f \in H^*, h \in H$.

Lastly, since the antipode of $(H^*)^{cop}$ is $(S^{-1})^*$ [67, Lemma 1.5.11], the antipode of $D(H)$ (being an algebra anti-homomorphism) is given by

$$S_{D(H)}(f \otimes h) = S_{D(H)}[(f \otimes 1)(1 \otimes h)] = (1 \otimes Sh)((S^{-1})^* f \otimes 1).$$

Lemma 1.4.1. *Let H be a finite dimensional Hopf algebra. Then,*

1. *$D(H)$ is a Hopf algebra with the above structure.*

2. *H and $(H^*)^{cop}$ are Hopf subalgebras of $D(H)$ by the obvious inclusions.*

Proof. See [76, Theorem 13.1.2(a),(b)]. □

The multiplication in the double is quite different than the usual product on the tensor of algebras, so the double is usually denoted with a bowtie $\bowtie$ instead of a tensor symbol $\otimes$,

$$D(H) = (H^*)^{cop} \bowtie H,$$

as are its elements. However, considering H and $(H^*)^{cop}$ are Hopf subalgebras of $D(H)$, in this thesis we will drop either symbol and write fh instead of $f \bowtie h$ for a generic element of $D(H)$.

The first part of the following lemma gives another formula for the product in $D(H)$, this time in terms of the left and right actions of H^* on H. It is easy to see that H is a left and right H^*-module as follows: for any $f \in H^*, h \in H$

$$f \rightharpoonup h = \sum h_1 f(h_2) \qquad \text{and} \qquad h \leftharpoonup f = \sum f(h_1) h_2. \tag{1.6}$$

Lemma 1.4.2. *Let H be a finite dimensional Hopf algebra. For any $h \in H, f \in H^*$,*

1. $hf = \sum f_2((S^{-1})^*(f_1) \rightharpoonup h \leftharpoonup f_3)$.

2. $S(fh) = \sum (Sh_3 \rightharpoonup (S^{-1})^*(f) \leftharpoonup h_1) Sh_2$.

Proof. For a proof of (1), see [67, Lemma 10.3.11]. And (2) is a consequence of (1.5) as follows:

$$\begin{aligned} S_{D(H)}(fh) &:= S(h)(S^{-1})^*(f) = \sum ((Sh)_1 \rightharpoonup (S^{-1})^*(f) \leftharpoonup S^{-1}(Sh)_3)(Sh)_2 \\ &= \sum (Sh_3 \rightharpoonup (S^{-1})^*(f) \leftharpoonup h_1) Sh_2. \end{aligned}$$

□

We now compute the Drinfeld double of a few examples.

Example 1.4.3. Consider the group algebra of a finite group $H = kG$. Note that $S(g) = g^{-1}$ for each $g \in G$, so S is its own inverse. Recall its dual $(kG)^*$ from Example 1.3.3 and let $\{g^* : g \in G\}$ denote the dual basis of $(kG)^*$.

The product in $D(kG)$ is determined by

$$gk^* = (g \rightharpoonup k^* \leftharpoonup S^{-1}g)g,$$

for any $g, k \in G$. But for any $l \in G$ we have

$$(g \rightharpoonup k^* \leftharpoonup g^{-1})(l) = k^*(g^{-1}lg) = \delta_{g^{-1}lg,k} = \delta_{l,gkg^{-1}} = (gkg^{-1})^*(l),$$

hence

$$gk^* = (gkg^{-1})^* g.$$

Therefore, as an algebra $D(kG)$ has the structure of a skew group ring

$$D(kG) = (kG)^* * G.$$

The coproduct is $\Delta(h^*g) = \sum_{uv=h} v^*g \otimes u^*g$ and by Lemma 1.4.2(2) the antipode is

$$S(h^*g) = (Sg \rightharpoonup (S^{-1})^*(h^*) \leftharpoonup g)Sg = (g^{-1} \rightharpoonup (h^{-1})^* \leftharpoonup g)g^{-1} = (g^{-1}h^{-1}g)^*g^{-1}.$$

We thus may note the following special cases.

1. Let G be a (finite) abelian group and k be an algebraically closed field whose characteristic does not divide $|G|$. Then, as an algebra

$$D(kG) \cong k(G \times G).$$

   Since G is a finite abelian group, it suffices to prove this for $G = C_n$. But since group algebras of cyclic groups are self-dual by Example 1.3.3, we have

$$D(kC_n) = (kC_n)^* \otimes C_n \cong kC_n \otimes C_n \cong k(C_n \times C_n).$$

2. If $G = D_n = \langle a, b : a^2 = 1, b^n = 1, aba = b^{-1} \rangle$ a finite dihedral group, then the product in $D(kD_n)$ is determined by

$$aa^* = a^*a, \qquad bb^* = b^*b, \qquad ba^* = (ab^{-2})^*b, \qquad ab^* = (b^{-1})^*a.$$

Example 1.4.4. Consider the finite dimensional Taft algebras $T_f(n, t, q)$ from Example 1.1.4 and assume that $(n, t) = 1$. These Taft algebras are self-dual as in Example 1.3.4; more specifically the invertible and nilpotent generators of T_f^* are given by $G(x^ig^j) = \delta_{i,0}q^{-t^{-1}j}, X(x^ig^j) = \delta_{i,1}$, where t^{-1} is the inverse of t modulo n.

The product in $D(T_f)$ is determined by the relations in T_f and T_f^* and

$$gG = Gg, \qquad gX = qXg, \qquad xG = q^{-1}Gx, \qquad xX = Xx + G^t - g^t.$$

We record the calculations here for future purposes in chapter 5. Since g is grouplike, (1.5) yields

$$g\varphi = (g \rightharpoonup \varphi \leftharpoonup g^{-1})g$$

for any $\varphi \in T_f^*$. The first two relations follow from

- $(g \rightharpoonup G \leftharpoonup g^{-1})(x^ig^j) = G(g^{-1}x^ig^jg) = q^iG(x^ig^j) = \delta_{i,0}q^{-t^{-1}j} = G(x^ig^j)$;
- and $(g \rightharpoonup X \leftharpoonup g^{-1})(x^ig^j) = q^iX(x^ig^j) = q\delta_{i,1} = qX(x^ig^j)$.

The third relation follows from Lemma 1.4.2(1) and from G being grouplike:

$$xG = G(G^{-1} \rightharpoonup x \leftharpoonup G) = G(G(x) + G(g^t)x + G(g^t)g^tG^{-1}(x)) = q^{-1}Gx.$$

And by (1.5) the fourth relation is given by

$$xX = (x \rightharpoonup X) + (g^t \rightharpoonup X)x + (g^t \rightharpoonup X \leftharpoonup -xg^{-t})g^t$$

and

- $(x \rightharpoonup X)(x^i g^j) = X(x^i g^j x) = q^{-j} X(x^{i+1} g^j) = q^{-j}\delta_{i+1,1} = q^{-j}\delta_{i,0} = G^t(x^i g^j)$;
- $(g^t \rightharpoonup X)(x^i g^j) = X(x^i g^{j+t}) = \delta_{i,1} = X(x^i g^j)$;
- $(g^t \rightharpoonup X \leftharpoonup xg^{-t})(x^i g^j) = X(xg^{-t} x^i g^j g^t) = q^{it} X(x^{i+1} g^j) = \delta_{i,0} = 1(x^i g^j)$.

Remark 1.4.5. The Drinfeld double can be seen as a particular case of bicrossproducts, a construction due to Majid [59] where two Hopf algebras act or coact on each other.

1.4.1 Properties of the Drinfeld double

We now look at some properties of the Drinfeld double, the most important of which are the fact that $D(H)$ is unimodular and quasitriangular.

The Drinfeld double is quite a symmetric object, being "built up" by a Hopf algebra and its dual, and often a property and the corresponding co-property are equivalent, as is the case for commutativity.

Proposition 1.4.6. *Let H be a finite dimensional Hopf algebra. The following are equivalent:*

1. *$D(H)$ is commutative;*
2. *H and H^* are commutative;*
3. *H and H^* are cocommutative;*
4. *$D(H)$ is cocommutative.*

And, in this case $D(H) = H^* \otimes H$ as Hopf algebras.

Proof. (2) $\Leftrightarrow$ (3) This is Lemma 1.3.2(1).

(1) $\Leftrightarrow$ (2) Clearly if $D(H)$ is commutative, so are its subalgebras H and H^*. Conversely, if H and H^* are commutative, H is commutative and cocommutative by (3), hence $S^{-1} = S$ [67, Corollary 1.5.12] and (1.5) gives

$$hf = \sum(h_1 \rightharpoonup f \leftharpoonup Sh_3)h_2 = \sum(h_1 Sh_3 \rightharpoonup f)h_2 = \sum(h_1 Sh_2 \rightharpoonup f)h_3 = fh$$

for all $h \in H$ and $f \in H^*$.

(3) $\Leftrightarrow$ (4) This follows easily from the fact that $D(H) = (H^*)^{cop} \otimes H$ as a coalgebra and that $(H^*)^{cop}$ is cocommutative if and only if H^* is. $\square$

Recall the notions of integrals and unimodularity from subsection 1.1.2.

Theorem 1.4.7. *Let H be a finite dimensional Hopf algebra. If $0 \neq t \in \int_H^r$ and $0 \neq T \in \int_{H^*}^l$, then Tt is a left and right integral of $D(H)$. In particular, $D(H)$ is unimodular.*

Proof. See [67, Theorem 10.3.12] or [76, Proposition 13.2.2]. $\square$

Example 1.4.8. Let $H = T_f(n,t,q)$ be a finite dimensional Taft algebra with $(n,t) = 1$ and recall its Drinfeld double from Example 1.4.4. By Examples 1.1.13 and 1.3.4, $D(H)$ is unimodular with integral

$$\int_{D(H)} = k \left(\sum_{i=0}^{n-1} G^i \right) X^{n-1} x^{n-1} \left(\sum_{i=0}^{n-1} g^i \right).$$

The following result on the semisimplicity of the double $D(H)$ is a consequence of the previous result coupled with Theorem 1.1.14. Recall the notion of a cosemisimple Hopf algebra from subsection 1.1.4.

Corollary 1.4.9. *Let H be a finite dimensional Hopf algebra. The following are equivalent:*

1. *$D(H)$ is semisimple;*
2. *H and H^* are semisimple;*
3. *H and H^* are cosemisimple;*
4. *$D(H)$ is cosemisimple.*

Proof. See [67, Corollary 10.3.13] or [76, Corollary 13.2.3]. $\square$

The following example includes work due to H. Chen, [22] and [23].

Example 1.4.10. Consider again the Taft algebras $T_f(n,t,q)$ from Example 1.1.4. We know they are not semisimple, as they possess the nilpotent ideal xH. Thus, by Corollary 1.4.9 their double $D(T_f)$ is also not semisimple. We then investigate its nontrivial radical $J = \mathrm{rad}(D(T_f))$ and the corresponding semisimple quotient $D(T_f)/J$. For this we make use of the following result by Chen and assume for the rest of this example that $t = 1$.

Theorem 1.4.11 (Chen, [22, Theorems 2.5, 2.6])**.** *Let $T_f(n,1,q)$ denote a finite dimensional Taft algebra. For each $l = 1,\dots,n$, there are exactly n non-isomorphic irreducible $D(T_f)$-modules $V(l,r)$ of dimension l, for all $r = 1,\dots,n$.*

For each $1 \leq l, r \leq n$, we have $D(T_f)/\mathrm{Ann}V(l,r) \cong \mathcal{M}_l(k)$ by Jacobson's density theorem [50, Corollary 11.17], hence by the Chinese Remainder Theorem the semisimple quotient of $D(T_f)$ is

$$D(T_f)/J \cong D(T_f)/\bigcap_{l,r} \mathrm{Ann}V(l,r) \cong \bigoplus_{l,r} D(T_f)/\mathrm{Ann}V(l,r) \cong \bigoplus_{l=1}^{n} \mathcal{M}_l(k)^{\oplus n}.$$

In particular, for Sweedler's 4-dimensional example $H_4 = T_f(2,1,-1) = k\langle g, x : g^2 = 1, x^2 = 0, xg = -gx\rangle$, we have

$$D(H_4)/J \cong k \oplus k \oplus \mathcal{M}_2(k) \oplus \mathcal{M}_2(k).$$

Therefore, the semisimple quotient $D(H_4)/J$ has dimension 10 and $J = \mathrm{rad}(D(H_4))$ is 6-dimensional.

Another important property of the double $D(H)$ of a finite dimensional Hopf algebra H is the fact that it is a quasitriangular Hopf algebra. A Hopf algebra H is said to be quasitriangular if its antipode is bijective and there is an invertible element $R = \sum_i a_i \otimes b_i$ in $H \otimes H$ such that for any $h \in H$

$$R\Delta(h)R^{-1} = \tau\Delta(h),$$

where $\tau : H \otimes H \to H \otimes H$ is the flip map, and

$$\sum_i \Delta a_i \otimes b_i = \sum_{i,j} a_i \otimes a_j \otimes b_i b_j \quad \text{and} \quad \sum_i a_i \otimes \Delta b_i = \sum_{i,j} a_i a_j \otimes b_j \otimes b_i.$$

Theorem 1.4.12. *Let H be a finite dimensional Hopf algebra. Then, $D(H)$ is quasitriangular with $R = \sum_i h_i \otimes h_i^*$, where $\{h_i\}$ is a basis of H and $\{h_i^*\}$ is its dual basis of H^*.*

Proof. See [67, Theorem 10.3.6] or [76, Theorem 13.2.1]. □

On one hand, quasi-triangular Hopf algebras have a certain symmetry in their representation theory, see [67, Lemma 10.1.2]. On the other hand, the second and fourth powers of their antipodes are inner automorphisms; see [67, Proposition 10.1.4, Theorem 10.1.13] for respective simplifying formulas.

Chapter 2

Commutative-by-finite Hopf algebras

Throughout my thesis I will focus on the study of a class of Hopf algebras named *commutative-by-finite* Hopf algebras. These Hopf algebras are finitely generated as modules over some Hopf subalgebra which is both commutative and normal. Throughout this chapter k denotes an algebraically closed field.

In section 2.1 we study many basic properties of these Hopf algebras. In particular, we investigate their centre and several homological properties. We sum up some of these features in the following result.

Theorem 2.0.1. *Let H be an affine commutative-by-finite Hopf algebra, finite over the commutative normal Hopf subalgebra A. Then,*

1. *H is a PI ring;*
2. *the antipode of H is bijective;*
3. $\mathrm{GKdim}\, H = \mathrm{Kdim}\, H = \mathrm{GKdim}\, A = \mathrm{Kdim}\, A := d$ *is finite;*
4. *H is a finitely generated module over its affine centre;*
5. *H is Auslander-Gorenstein, AS-Gorenstein and GK-Cohen-Macaulay with injective dimension d;*
6. *H has an artinian quasi-Frobenius classical ring of fractions $Q(H)$.*

Furthermore, we discuss in subsection 2.1.2 the extension $A \subseteq H$ which, under various hypotheses (for example, when k has characteristic 0), possesses very nice properties, namely H is a faithfully flat projective left and right A-module. In subsection 2.1.3 we discuss the regularity of these Hopf algebras, providing conditions that guarantee their regularity and stating several properties these regular Hopf algebras possess.

Moreover, in section 2.2 we illustrate these properties with many well-known examples of commutative-by-finite Hopf algebras, namely quantum groups at roots of unity and Hopf algebras with low GK-dimension.

As a last note, we point out that most of the results in this chapter rely heavily on other mathematicians' work which is referenced accordingly.

2.1 Definition and basic properties

We proceed to define commutative-by-finite Hopf algebras and study many of their properties. As mentioned in the introduction to the chapter, k denotes an algebraically closed field in this chapter.

A subalgebra A of a Hopf algebra H is *normal* if it is invariant under the left and right adjoint actions of H; that is, for all $a \in A$ and $h \in H$,

$$(ad_l h)(a) := \sum h_1 a S(h_2) \in A \qquad \text{and} \qquad (ad_r h)(a) := \sum S(h_1) a h_2 \in A.$$

Note that a central subalgebra A is clearly normal. For example, in a group algebra the subalgebra generated by a normal subgroup is normal in this sense.

The following result states a crucial consequence of normality, which leads to the important quotient Hopf algebra in Theorem 2.1.3(2) that will be carried on throughout the thesis.

Proposition 2.1.1. *Let A be a Hopf subalgebra of a Hopf algebra H. If A is normal, then $A^+H = HA^+$ is a Hopf ideal of H.*

Proof. See [67, Lemma 3.4.2]. □

The following class of Hopf algebras is the key object of study of this thesis.

Definition 2.1.2. A Hopf k-algebra H is *commutative-by-finite* if it is a finitely generated (left or right) module over a commutative normal Hopf subalgebra A.

Commutative Hopf algebras are clearly commutative-by-finite, by taking $A = H$; and all finite dimensional Hopf algebras are also commutative-by-finite, just by taking the trivial Hopf subalgebra $A = k1$. Moreover, any Hopf algebra that is finitely generated over a central Hopf subalgebra, as is often the case for quantum groups at roots of unity, is clearly commutative-by-finite. We explore more examples of commutative-by-finite Hopf algebras in section 2.2.

The following result lists a few properties of these Hopf algebras. A crucial property is that a commutative-by-finite Hopf algebra can be thought of as an extension of a commutative Hopf algebra by a finite dimensional Hopf algebra, which is made precise in part (2) of the following result.

Theorem 2.1.3. *Let H be a commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A with augmentation ideal A^+.*

1. *The following are equivalent:*

   (a) *H is noetherian.*

(b) H is affine.

(c) A is affine.

(d) A is noetherian.

2. *The quotient $\overline{H} := H/A^+H$ is a finite dimensional Hopf algebra.*

3. *The left (resp. right) adjoint action of H on A factors through $\overline{H}$, so that A is a left (resp. right) $\overline{H}$-module algebra.*

4. *H is a PI ring.*

Proof. (1) (c) $\Leftrightarrow$ (d): Clearly, any affine commutative ring is noetherian by Hilbert's Basis Theorem; see for example [37, Corollary 1.10]. Conversely, a commutative noetherian Hopf algebra is affine by Molnar's theorem [66].

(d) $\Leftrightarrow$ (a): If A is noetherian, then H is a noetherian A-module [37, Corollary 1.4], and hence *a fortiori* satisfies ACC on its left and right ideals. The converse follows from [34], according to which a commutative subring A of a noetherian ring H, with H a finitely-generated A-module, must also be noetherian.

(b) $\Leftrightarrow$ (c): That (c) $\Rightarrow$ (b) is trivial: denoting the k-generators of A by $a_1, \ldots, a_r$ and the A-generators of H by $h_1, \ldots, h_s$, then H is generated as a k-algebra by $a_1, \ldots, a_r, h_1, \ldots, h_s$. The converse follows from a generalized version of the Artin-Tate lemma attributed to Small, which states that, if $A \subseteq H$ is any extension of k-algebras where H is affine and is a finitely generated left module over a commutative subalgebra A, then A is affine. A proof can be found at [79, Lemma 1.3].

(2) Since A is normal, A^+H is a Hopf ideal by Proposition 2.1.1 and $\overline{H}$ is finite dimensional since H is a finitely generated A-module.

(3) It is easy to see that A is a left (resp. right) H-module algebra with the left (resp. right) adjoint action of H. Moreover, for all $h \in H, a \in A^+, b \in A$ we have

$$(ha) \cdot b = \sum h_1 a_1 b S(a_2) S(h_2) = \epsilon(a) h \cdot b = 0,$$

since A is commutative. Hence, $A^+H = HA^+$ acts trivially on A and the left adjoint action factors through an $\overline{H}$-action on A. Thus, A is a left $\overline{H}$-module algebra. The right action of H on A factors similarly.

(4) Every k-algebra which is a finitely generated module over a commutative subalgebra satisfies a PI by [64, Corollary 13.1.13(iii)]. $\square$

Throughout the thesis I will use the notation introduced in the previous result for the finite dimensional Hopf quotient

$$\overline{H} := H/A^+H.$$

I will also denote the Hopf algebra surjection by $\pi : H \to \overline{H}$. This induces a canonical structure of $\overline{H}$-comodule algebra on H, as discussed in Example 1.1.10.

The following result lists a few more important properties of commutative-by-finite Hopf algebras under the equivalent hypotheses of Theorem 2.1.3(1).

Theorem 2.1.4. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A.*

1. *The antipode S of H is bijective.*

2. $\operatorname{GKdim} H = \operatorname{GKdim} A = \operatorname{Kdim} H = \operatorname{Kdim} A < \infty$.

Proof. (1) Every affine noetherian PI Hopf algebra has bijective antipode by [85, Corollary 2].

(2) Since H is a finitely generated A-module, we have $\operatorname{GKdim} H = \operatorname{GKdim} A$ by [49, Proposition 5.5]. By Theorem 2.1.3(1) A and H are both affine and noetherian. So $\operatorname{GKdim} A = \operatorname{Kdim} A$ by [49, Theorem 4.5] and it follows from Noether's normalization theorem that any affine commutative algebra has finite Krull dimension [78, Theorem 6.10, Corollary 6.33]. At last, $\operatorname{GKdim} H = \operatorname{Kdim} H$ by [49, Corollary 10.16]. □

Remarks 2.1.5. Keep the notation of Theorems 2.1.3 and 2.1.4.

1. Parts (1) and (4) of Theorem 2.1.3 and parts (1) and (2) of Theorem 2.1.4 are valid (with the same proofs) without the hypothesis that A is normal in H.

2. It is easy to show that for affine commutative-by-finite Hopf algebras it is enough to require that the commutative Hopf subalgebra A be invariant under left (or right) adjoint action only, as left and right normality are equivalent here.

   Proof. First note that by Theorem 2.1.4(1) the antipode S of H is bijective and by [67, Corollary 1.5.12] the antipode S_A of A is also bijective (an involution in fact). Let $a \in A, h \in H$. Then,

$$\begin{aligned} ad_r(h)(Sa) &= \sum S(h_1)S(a)h_2 = S\left(\sum S^{-1}(h_2)ah_1\right) \\ &= S\left(\sum S^{-1}(h)_1 a S(S^{-1}(h)_2)\right) = S(ad_l(S^{-1}h)(a)). \end{aligned}$$

   The equivalence between invariance under left or right adjoint action now follows from bijectivity of S_A and S_H. □

3. Notice that any affine commutative-by-finite Hopf algebra always contains a *reduced* commutative normal Hopf subalgebra A'. This is clear if A is reduced itself, as is the case in characteristic 0 by Theorem 1.1.7. If A is not reduced, we may assume without loss of generality that $\operatorname{char} k = p > 0$. Since A is affine commutative, its nilradical ideal $N(A)$ is finitely generated, hence it is nilpotent, say $N(A)^m = 0$ for some $m > 0$.

Taking r such that $p^r > m$, let

$$A' = \{a^{p^r} : a \in A\}.$$

Since $(a+b)^{p^r} = a^{p^r} + b^{p^r}$, A' is closed under addition. Since k is algebraically closed, it is closed under scalar multiplication, hence it is a subalgebra of A. Moreover, we know that

$$\Delta(a^{p^r}) = \left(\sum a_1 \otimes a_2\right)^{p^r} = \sum (a_1 \otimes a_2)^{p^r} = \sum a_1^{p^r} \otimes a_2^{p^r} \in A' \otimes A'$$

and it follows easily that A' is a Hopf subalgebra of H. Furthermore, A' is clearly commutative and reduced, and H is a finitely generated A'-module: if $a_1, \dots, a_n$ denote the algebra generators of A, then A is a finite A'-module spanned by $\{a_1^{i_1} \dots a_n^{i_n} : 0 \leq i_k < p^r\}$, hence H is also a finite A'-module.

2.1.1 Finiteness over the centre

A very important property of affine commutative-by-finite Hopf algebras is their finiteness over their centres.

Recall from Theorem 2.1.3(3) that the left adjoint action of A on H factors through an $\overline{H}$-action, and A is an $\overline{H}$-module algebra. As per subsection 1.1.1, we denote the invariants of this action by $A^{\overline{H}}$. Their relation with the centre $Z(H)$ of H is the following:

$$Z(H) \cap A = A^{\overline{H}}. \tag{2.1}$$

Proof. First, $A^{\overline{H}} = A^H$ by Theorem 2.1.3(3). If $a \in A$ is central in H, then $h \cdot a = \sum h_1 a S(h_2) = \epsilon(h) a$ for all $h \in H$, proving $A \cap Z(H) \subseteq A^H$. Conversely, any invariant is central: if $a \in A^H$,

$$ha = \sum h_1 a S(h_2) h_3 = \sum (h_1 \cdot a) h_2 = a \sum \epsilon(h_1) h_2 = ah,$$

for any $h \in H$. □

We now prove that commutative-by-finite Hopf algebras are finitely generated modules over their centres, which follows easily from an important result by Skryabin on integrality over invariants.

Theorem 2.1.6 (Skryabin, [84, Proposition 2.7]). *Let T be a finite dimensional Hopf algebra and A an affine commutative left T-module algebra. Then, A is a finitely generated module over A^T.*

Proof. On one hand, if $\operatorname{char} k > 0$, A is clearly $\mathbb{Z}$-torsion, so [84, Proposition 2.7(b)] applies to show that A is integral over A^T. If, on the other hand, $\operatorname{char} k = 0$, then A is semiprime by Theorem 1.1.7, so that, in the terminology of [84], A is T-reduced, and again [84, Theorem 2.5, Proposition 2.7(a)] give A integral over A^T. Since A is affine and an integral A^T-module, it is a finitely generated A^T-module; for, if $x_1, \dots, x_n$ denote the algebra generators of A, then $A = A^T\langle x_1, \dots, x_n\rangle$ and, since each x_i is integral over A^T, A is a finitely generated A^T-module by [31, Corollary 4.5]. □

Corollary 2.1.7. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A and let $Z(H)$ be the center of H. Then,*

1. *A is a finitely generated $A^{\overline{H}}$-module;*

2. *H is a finitely generated $Z(H)$-module;*

3. *$Z(H)$ is affine.*

Proof. (1) This is Theorem 2.1.6.

(2) Since H is a finitely generated A-module, it is a finitely generated $A^{\overline{H}}$-module by (1). Since $A^{\overline{H}} \subseteq Z(H)$ by (2.1), the result follows.

(3) $Z(H)$ is affine by the Artin-Tate lemma, [64, Lemma 13.9.10]. □

2.1.2 Homological properties and consequences

In this subsection we study homological properties of affine commutative-by-finite Hopf algebras. Recall these homological concepts from section 1.2.

As we have mentioned before, affine commutative Hopf algebras and finite dimensional Hopf algebras are examples of commutative-by-finite Hopf algebras. Both these classes exhibit important homological properties. On one hand, affine commutative Hopf algebras are Gorenstein, meaning they have finite injective dimension [10, §2.3, Step 1], and in characteristic 0 they are regular, that is they have finite global dimension [99, §11.4, §11.6, §11.7]. On the other hand, finite dimensional Hopf algebras are Frobenius [67, Theorem 2.1.3], and so in particular self-injective. We review in the following result how these features partially extend to the commutative-by-finite case.

Theorem 2.1.8. *Let H be an affine commutative-by-finite Hopf algebra, finite over the commutative normal Hopf subalgebra A. Let* $\operatorname{GKdim} H = d$.

1. *H is AS-Gorenstein and Auslander-Gorenstein, of injective dimension d.*

2. *H is GK-Cohen-Macaulay.*

3. *H is left and right GK-pure; that is, every non-zero left or right ideal of H has GK-dimension d.*

4. *H is injectively homogeneous. Thus, H is $Z(H)$-Macaulay.*

5. *H, $Z(H)$ and A^H each have artinian classical rings of fractions, $Q(H)$, $Q(Z(H))$ and $Q(A^H)$.*

6. The regular elements $\mathcal{Z}$ of $Z(H)$ and $\mathcal{A}$ of A^H are also regular in H, and

$$Q(H) = H[\mathcal{Z}]^{-1} = H[\mathcal{A}]^{-1}.$$

7. $Q(H)$ is quasi-Frobenius.

Proof. (1),(2) By results of Wu and Zhang [102, Theorems 0.1, 0.2], every affine noetherian PI Hopf algebra is AS-Gorenstein, Auslander-Gorenstein and GK-Cohen-Macaulay. It is immediate from the definitions that a Hopf algebra which is both AS-Gorenstein and GK-Cohen-Macaulay must have its injective dimension equal to its GK-dimension.

(3) is an immediate consequence of (2). For any non-zero left or right ideal I of H,

$$\mathrm{Ext}^0_H(I, H) = \mathrm{Hom}_H(I, H) \neq 0,$$

hence the grade $j(I) = 0$. Since H is GK-Cohen-Macaulay, $\mathrm{GKdim}\,(I) = \mathrm{GKdim}\,(H)$.

(4) First note that, since H is noetherian and finitely generated over its centre $Z(H)$ by Corollary 2.1.7, it is a fully bounded noetherian ring [37, Proposition 9.1]. By Theorem 2.1.4(2) $\mathrm{GKdim}\, H = \mathrm{Kdim}\, H$ and by (2) H is GK-Cohen-Macaulay, hence H is Krull-Macaulay by [91, Lemma 6.1]. Now results of Brown and Macleod show that H is $Z(H)$-Macaulay [17, Theorem 4.8] and injectively homogeneous [17, Theorem 5.3].

(5) It is known that GK-pure noetherian PI algebras have artinian rings of fractions [64, Corollary 6.8.16], so H has a classical artinian ring of fractions by (3). The arguments for $Z(H)$ and for $A^{\overline{H}}$ are identical; we deal here with $Z(H)$. As with H, it suffices to prove that $Z(H)$ is GK-pure.

Since H is a finite $Z(H)$-module by Corollary 2.1.7, [49, Proposition 5.5] yields $\mathrm{GKdim}\, Z(H) = \mathrm{GKdim}\, H = d$. Let I be a nonzero ideal of $Z(H)$. Then, by (3) $\mathrm{GKdim}\,(IH)_H = d$. Since IH is a $(Z(H), H)$-bimodule and finitely generated over both H and $Z(H)$ by noetherianity of $Z(H)$ as in Corollary 2.1.7(3), it follows from [49, Corollary 5.4] that

$$\mathrm{GKdim}_{\,Z(H)}(IH) = \mathrm{GKdim}\,(IH)_H = d.$$

Again since H is a finite $Z(H)$-module, IH is the homomorphic image of a finite direct sum of copies of I as a $Z(H)$-module. Hence by [49, Proposition 5.1(a),(b),(d)] we have

$$d = \mathrm{GKdim}_{\,Z(H)}(IH) \leq \mathrm{GKdim}_{\,Z(H)}(I) \leq \mathrm{GKdim}\, Z(H) = d.$$

Therefore, $\mathrm{GKdim}_{\,Z(H)}(I) = d$. This proves $Z(H)$ is GK-pure.

(6) Once again we deal with $\mathcal{Z}$ but the argument is similar for $\mathcal{A}$. Let $z \in \mathcal{Z}$, a regular element of $Z(H)$. If $zh = 0$ for some $h \in H$, then

$$\mathrm{GKdim}_{\,Z(H)} Z(H)h = \mathrm{GKdim}_{\,Z(H)/Z(H)z} Z(H)h \leq \mathrm{GKdim}\, Z(H)/Z(H)z < d$$

by [49, Proposition 5.1(c),(d), Proposition 3.15] respectively. As in the proof of (5), Hh is a homomorphic image of a finite direct sum of copies of $Z(H)h$, so $\mathrm{GKdim}_H(Hh) = \mathrm{GKdim}_{Z(H)}(Hh) \leq \mathrm{GKdim}_{Z(H)} Z(H)h < d$. Since H is GK-pure by (3), we must have $h = 0$, that is z is regular in H.

Therefore, we can form the ring of fractions $H[\mathcal{Z}]^{-1}$, a subring of $Q(H)$. By Corollary 2.1.7(2), $H[\mathcal{Z}]^{-1}$ is a finitely generated $Z(H)[\mathcal{Z}]^{-1}$-module. Since $Z(H)[\mathcal{Z}]^{-1} = Q(Z(H))$ is artinian by part (5), $H[\mathcal{Z}]^{-1}$ is an artinian $Z(H)[\mathcal{Z}]^{-1}$-module, hence *a fortiori* $H[\mathcal{Z}]^{-1}$ is an artinian ring [37, Corollary 4.7]. But a regular element in an artinian ring is a unit, thus the set of regular elements of H must be $\mathcal{Z}$, that is, $H[\mathcal{Z}]^{-1} = Q(H)$.

(7) This follows from (1),(2) and [2, Corollary 6.2]. □

Further homological properties

In the following theorem we gather results from the literature on the homological properties of the extension $A \subseteq H$, which in particular is often faithfully flat and projective.

Theorem 2.1.9. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Suppose one of the following hypotheses holds:*

(i) A is reduced;

(ii) $\mathrm{char}\, k = 0$*;*

(iii) A is central;

(iv) H is pointed.

Then:

1. *$A \subseteq H$ is a left and right faithfully flat $\overline{H}$-Galois extension.*

2. *A equals the right and the left $\overline{H}$-coinvariants of the $\overline{H}$-comodule H; that is,*

$$H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H = A.$$

3. *H is a finitely generated projective left and right A-module.*

4. *A is a left (and right) A-module direct summand of H.*

Proof. Notice that hypothesis (ii) is a particular case of (i) by Theorem 1.1.7.

(1),(2) Assume (i) holds. Then, A has finite global dimension by [99, §11.6, §11.7]. By Theorem 2.1.8(1) and Theorem 2.1.4(2),

$$\mathrm{injdim}\,(H) = \mathrm{GKdim}\,(H) = \mathrm{GKdim}\,(A) = \mathrm{injdim}\,(A).$$

Hence, [102, Theorem 0.3] yields that H is a projective left and right A-module. A flat extension of Hopf algebras with bijective antipodes is faithfully flat, [63, Corollary 2.9]. Together with Theorem 2.1.4(1) and [67, Corollary 1.5.12], this proves right and left faithful flatness.

If (iii) holds, then H is also a faithfully flat A-module by a result of Schneider [83, Theorem 3.3]. Under (iv), H is even a free A-module by [73].

The fact that $A = H^{\mathrm{co}\,\overline{H}} = {}^{\mathrm{co}\,\overline{H}}H$ and $A \subseteq H$ is $\overline{H}$-Galois follows from faithful flatness and $A^+H = HA^+$, as is shown in the proof of [67, Proposition 3.4.3].

(3) This follows from (1) by [63, Corollary 2.9].

(4) The left A-module H/A is in the category ${}_A M^H$, in the notation of [63]. Hence H/A is left A-projective by (1) and [63, Corollary 2.9], so the exact sequence

$$0 \longrightarrow A \longrightarrow H \longrightarrow H/A \longrightarrow 0$$

of left A-modules splits, as required. The argument on the right is identical. $\square$

Remark 2.1.10. Note that the inclusion $A \subseteq H^{\mathrm{co}\,\overline{H}} \cap {}^{\mathrm{co}\,\overline{H}}H$ always holds: clearly the restriction of $\pi : H \to H/A^+H$ to A is just ϵ_A, hence for any $a \in A$ we have $\sum a_1 \otimes \pi(a_2) = \sum a_1 \otimes \epsilon(a_2)\overline{1} = a \otimes \overline{1}$, so $A \subseteq H^{\mathrm{co}\,\overline{H}}$. The argument is similar for left coinvariants.

We know of no examples of commutative-by-finite Hopf algebras that do not satisfy the conclusions of the previous theorem. In fact, A is reduced in all examples in section 2.2. Note that if H is a flat A-module, then by [63, Corollary 2.9] and Theorem 2.1.4(1) H is a faithfully flat A-module, and the other statements of the theorem will follow with the same proof as above. So we propose the following question:

Question 2.1.11. For an affine commutative-by-finite Hopf algebra H, can the normal commutative Hopf subalgebra A be chosen so that H is not flat over A?

Freeness of H over A

From Theorem 2.1.9 we know that under certain hypotheses H is both a projective and faithfully flat A-module. The obvious next question is under which conditions H is in fact a free A-module. Some positive cases are provided in the following result. Recall the notions of coradical and pointed from section 1.1.4.

Proposition 2.1.12. *Let H be an affine commutative-by-finite Hopf algebra with commutative normal Hopf subalgebra A. Then, H is a free A-module when*

1. *H is pointed;*
2. *or A contains the coradical of H.*

Moreover, H decomposes into the crossed product $A\#_\sigma \overline{H}$ when

1. *H is pointed;*

or, more generally,

2. *the coradical of H is contained in $AG(H)$.*

Proof. The two first statements follow from [73] and [72, Corollary 2.3]. The two last statements are proved in [82, Corollary 4.3]. □

However, in general H is *not* a free A-module, as shown by the following example due to Radford [75].

Example 2.1.13. Consider the commutative Hopf algebra $H = \mathcal{O}(SL_2(k))$.

Let A be the subalgebra generated by the monomials of even degree. Then, it is easy to see that A is a Hopf subalgebra with $A \cong \mathcal{O}(PSL_2(k))$ and the Hopf quotient $\overline{H} = (kC_2)^*$. In fact, denoting the algebra generators of H by E_{ij}, where $i, j = 1, 2$, then in $\overline{H} = H/A^+H$ we have $\overline{E_{11}}^2 = \epsilon(E_{11})^2 = 1$ and $\overline{E_{11}E_{12}} = \epsilon(E_{11})\epsilon(E_{12}) = 0$, which implies $\overline{E_{12}} = 0$, and similarly $\overline{E_{21}} = 0$ and $\overline{E_{22}} = \overline{E_{11}}$. Hence,

$$\overline{H} = k\langle \overline{E_{11}} : \overline{E_{11}}^2 = 1\rangle = (kC_2)^*,$$

where C_2 is the subgroup of SL_2 generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. And by Example 1.1.11 and Theorem 2.1.9(2)

$$A = H^{\operatorname{co}\overline{H}} = \mathcal{O}(SL_2)^{\operatorname{co}\mathcal{O}(C_2)} = \mathcal{O}(SL_2/C_2) = \mathcal{O}(PSL_2).$$

Clearly H is a finite A-module. However, Radford proved that H is not a free A-module [75], by showing that H decomposes into $A \oplus M$ as A-modules and, if H is a free A-module, then M must be a free A-module of rank 1; lastly he proved M is not a cyclic A-module.

2.1.3 Regularity

In this subsection we discuss when commutative-by-finite Hopf algebras have finite global dimension (or, in other words, when they are regular) and the properties they possess when they exhibit regularity.

On one hand, it is commonly known that a finite dimensional Hopf algebra has finite global dimension if and only if it is semisimple, and semisimple finite dimensional Hopf algebras are completely characterized by Theorem 1.1.14. For, a finite dimensional Hopf algebra H is Frobenius [67, Theorem 2.1.3], thus self-injective, and if its global dimension is finite, then $\operatorname{gldim} H = \operatorname{injdim} H = 0$ by [6, Proposition 4.2]; and having global dimension 0 is equivalent to being semisimple by [64, 7.1.8].

On the other hand, an affine commutative Hopf algebra has finite global dimension if and only if it has no non-zero nilpotent elements [99, §11.6, §11.7]; in particular, by

Theorem 1.1.7 any affine commutative Hopf algebra in characteristic 0 has finite global dimension.

It is thus natural to look for an easily checked necessary and sufficient criterion for an affine commutative-by-finite Hopf algebra to have finite global dimension. Examples suggest there may be no such simple condition, but sufficient conditions for smoothness are not hard to obtain, as follows.

Proposition 2.1.14. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A.*

1. *If A is semiprime and $\overline{H}$ is semisimple, then H has finite global dimension.*

2. *If* $\operatorname{char}(k) = 0$ *and $\overline{H}$ is semisimple, then H has finite global dimension.*

3. *If H has finite global dimension and H is A-flat, then A has finite global dimension and A is semiprime.*

Proof. (1) As mentioned above, since A is affine and semiprime, it has finite global dimension by [99, §11.6, §11.7]. In particular, the trivial A-module k has a finite projective resolution $0 \to P_n \to \ldots \to P_0 \to {}_A k \to 0$. By Theorem 2.1.9(1), H_A is faithfully flat, hence we have an exact sequence

$$0 \to H \otimes_A P_n \to \ldots \to H \otimes_A P_0 \to H \otimes_A k \to 0 \tag{2.2}$$

and $H \otimes_A k \cong H \otimes_A (A/A^+) \cong H/HA^+ = \overline{H}$.

We claim (2.2) is a projective resolution of $\overline{H}$ by H-modules. In fact, since each P_i is a finitely generated projective left A-module, then $P_i \oplus M_i = A^{\oplus n_i}$ for some left A-module M_i and positive integer n_i. Thus,

$$(H \otimes_A P_i) \oplus (H \otimes_A M_i) \cong H \otimes_A (P_i \oplus M_i) \cong H \otimes_A A^{\oplus n_i} \cong H^{\oplus n_i},$$

whence $H \otimes_A P_i$ is a projective H-module, as claimed.

Therefore, $\operatorname{prdim}_H \overline{H} < \infty$. Since $\overline{H}$ is semisimple, it is a direct sum of $\overline{H}$-simple modules, one of which is the trivial $\overline{H}$-module k. In particular, this is a decomposition as H-modules and, since $\operatorname{prdim}_H \overline{H} < \infty$, then so must be the projective dimension of its H-direct summand ${}_H k$. But by [56, Section 2.4] the global dimension of a Hopf algebra is determined by the projective dimension of its trivial module, so $\operatorname{gldim} H = \operatorname{prdim}_H(k) < \infty$.

(2) This follows from (1) and Theorem 1.1.7.

(3) Since H is A-flat, it is also A-projective by the proof of Theorem 2.1.9(1),(3). If H has finite global dimension, then the finite projective resolution of the trivial H-module is also a (finite) projective resolution of A-modules by transitivity of projectivity, and again by [56, Section 2.4] A has finite global dimension. Lastly, A is semiprime by [99, §11.6, §11.7]. □

Remarks 2.1.15.

1. Not all commutative-by-finite Hopf algebras are regular, even when they contain no nonzero nilpotent elements and k has characteristic zero. For example, the Hopf algebras $B = B(n, p_0, \dots, p_s, q)$ constructed by Goodearl and Zhang in [38] and discussed in §2.2.6 below are affine commutative-by-finite Hopf domains with $\operatorname{GKdim} B = 2$, but have infinite global dimension.

2. The converses of Propositions 2.1.14(1) and (2) are false even when A is central or H is cocommutative.

   For the case when A is central one can take the quantized enveloping algebra $U_\epsilon(\mathfrak{g})$ of any simple Lie algebra $\mathfrak{g}$ at a root of unity ϵ. They are regular with $\operatorname{gldim} U_\epsilon(\mathfrak{g}) = \dim_k \mathfrak{g}$ [20, Theorem XIII.8.2], but $\overline{H} = u_\epsilon(\mathfrak{g})$ is a (finite dimensional) restricted quantized enveloping algebra, which in general is not semisimple. For example, consider the special linear Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(k)$. See §2.2.1 for more details.

   As for the cocommutative case, consider the torsion-free polycyclic group

   $$G := \langle x, y : x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2} \rangle$$

   discussed in [70, Lemma 13.3.3]. As explained by Passman, G has a normal subgroup $N = \langle x^2, y^2, (xy)^2 \rangle$ which is free abelian of rank 3, with G/N a Klein 4-group. Moreover, by construction $(*)$ G does *not* have any normal abelian subgroup W of finite index with $|G : W|$ prime to 2. Take a field k of characteristic 2 and let $H = kG$. Then, H is an affine commutative-by-finite domain by [70, Theorem 13.4.1], and $\operatorname{gldim} H = 3$ by Serre's theorem on finite extensions, [70, Theorem 10.3.12]. Thus, the commutative normal Hopf subalgebras of H over which H is a finitely generated module have the form $A = kW$ for some normal abelian subgroup W of finite index in G and Maschke's theorem says that the group algebra $\overline{H} = H/A^+H = k(G/W)$ is semisimple if and only if $\operatorname{char}(k) = 2$ does not divide the order $|G/W|$, which in view of property $(*)$ cannot happen here. Therefore, H does not have a normal commutative Hopf subalgebra A with $\overline{H}$ semisimple.

We now gather results from the literature to show that smooth affine commutative-by-finite Hopf algebras share many of the attractive properties of commutative noetherian rings of finite global dimension.

Theorem 2.1.16. *Let H be an affine commutative-by-finite Hopf k-algebra of finite global dimension d.*

1. *H is Auslander-regular, AS-regular and GK-Cohen-Macaulay with GK-dimension d.*

2. H is homologically homogeneous over its centre $Z(H)$.

3. H is a finite direct sum of prime rings,

$$H = \bigoplus_{i=1}^{t} H_i$$

where each H_i is a prime hom. hom. algebra of Gelfand-Kirillov and global dimension d.

4. $Z(H) = \bigoplus_{i=1}^{t} Z(H_i)$, where $Z(H_i)$ is an affine integrally closed domain of GK-dimension d for all i.

Proof. (1) It follows from Theorem 2.1.8(1),(2) that H is Auslander-regular, AS-regular and GK-Cohen-Macaulay with $\operatorname{GKdim} H = \operatorname{injdim} H$. And by [6, Proposition 4.2] $\operatorname{injdim} H = \operatorname{gldim} H$.

(2) First, H is Krull-Macaulay as in the proof of Theorem 2.1.8(4). The statement now follows from (1) and [17, Theorem 4.8 and Corollary 5.4].

(3) Since H is a noetherian PI ring by Theorem 2.1.3, it follows from (1) and [91, Theorem 5.4] that H decomposes into a direct sum of prime rings. Then, there exist idempotent pairwise orthogonal elements $e_1, \ldots, e_t$ in H (that is, $e_i e_j = \delta_{i,j} e_i$) such that each $H_i = e_i H$ is idempotently generated and $1_H = e_1 + \ldots + e_t$. By GK-pureness of H as in Theorem 2.1.8(3), each ideal H_i has $\operatorname{GKdim}(H_i) = \operatorname{GKdim}(H) = d$.

Now fix $1 \leq i \leq t$ and let V be an irreducible H_i-module. We claim that

$$\operatorname{prdim}_{H_i}(V) = \operatorname{prdim}_H(V). \tag{2.3}$$

Let $0 \to P_n \to \ldots \to P_0 \to V \to 0$ be a minimal projective resolution of V by H_i-modules. Since P_i is H_i-projective and H_i is H-projective (for it is a direct summand of H), then P_i is H-projective and the above is a projective resolution of V by H-modules. Therefore, $\operatorname{prdim}_H(V) \leq \operatorname{prdim}_{H_i}(V)$. Conversely, consider a minimal projective resolution $0 \to P_k \to \ldots \to P_0 \to V \to 0$ of V by H-modules. Since $H_i = e_i H$ is H-projective (hence H-flat), applying $e_i H \otimes_H -$ yields the exact sequence $0 \to e_i P_k \to \ldots \to e_i P_0 \to e_i V \to 0$ of H_i-modules. Since V is an irreducible H_i-module, $V = e_i V$. And for each $1 \leq j \leq k$, we can decompose $P_j = 1_H \cdot P_j = \bigoplus_i e_i P_j$ and, since P_j is H-projective, there exists some H-module Q_j such that $P_j \oplus Q_j = H^{\oplus r_j}$ for some positive integer r_j, hence $e_i P_j \oplus e_i Q_j = (e_i H)^{\oplus r_j} = H_i^{\oplus r_j}$, that is $e_i P_j$ is a projective H_i-module. Therefore, the above is a projective resolution of V by H_i-modules, so $\operatorname{prdim}_{H_i}(V) \leq \operatorname{prdim}_H(V)$ and (2.3) follows.

Any irreducible H_i-module V can be regarded as an irreducible H-module (by letting the other direct summands act as zero). Since H is hom. hom. by (2), we have $\operatorname{prdim}_H(V) = \operatorname{gldim}(H) = d$. Moreover, by (2.3) $\operatorname{prdim}_{H_i}(V) = \operatorname{prdim}_H(V) = d$.

Therefore, by [64, Corollary 7.1.14],

$$\operatorname{gldim}(H_i) = \sup\{\operatorname{prdim}_{H_i}(V) : V_{H_i} \text{ irreducible}\} = d.$$

In particular, $\operatorname{prdim}_{H_i}(V) = \operatorname{gldim}(H_i)$ for every irreducible H_i-module V, that is, H_i is a hom. hom. ring.

(4) From (3) we have $Z(H) = \bigoplus_{i=1}^{t} Z(H_i)$. Each $Z(H_i)$ is affine, being a factor of the affine algebra $Z(H)$, and each $Z(H_i)$ is a domain, since H_i is prime. Since H is a finite $Z(H)$-module by Corollary 2.1.7, each H_i is a finitely generated $Z(H_i)$-module. Thus, $\operatorname{GKdim} Z(H_i) = \operatorname{GKdim} H_i = d$ by [49, Proposition 5.5] and (3). And since each H_i is hom. hom. by (3), each $Z(H_i)$ is integrally closed by [15, Theorem 6.1]. □

2.2 Examples

The obvious examples of commutative-by-finite Hopf algebras are the commutative Hopf algebras (where $A = H$) and the finite dimensional Hopf algebras (where $A = k1$ is the trivial Hopf subalgebra). In this section we list other important families of examples, such as quantum groups at roots of unity and Hopf algebras of low GK-dimension. We also give a non-example in §2.2.7.

2.2.1 Quantized enveloping algebras at a root of unity

Our first examples of commutative-by-finite Hopf algebras in this section are quantized enveloping algebras at a root of unity. We do not define them here but they can be found in the literature; see for example [13, I.6.3], [21, Section 9.1A], [26, Section 9.1]. And we assume k to be an algebraically closed field of characteristic zero in this subsection.

Quantized enveloping algebras $U_q(\mathfrak{g})$ of semisimple finite dimensional Lie algebras $\mathfrak{g}$ are noetherian Hopf algebras, [21, Section 9.1A], [26, Section 9.1]. When $q = \epsilon$ is a primitive lth root of unity, $U_\epsilon(\mathfrak{g})$ is a free module of rank $l^{\dim \mathfrak{g}}$ over a central Hopf subalgebra Z_0, [26, Corollary and Theorem 19.1], [21, Propositions 9.2.7 and 9.2.11], [13, Theorem III.6.2]. Its finite dimensional Hopf quotient

$$\overline{H} = U_\epsilon(\mathfrak{g})/Z_0^+ U_\epsilon(\mathfrak{g}) =: u_\epsilon(\mathfrak{g})$$

is the restricted quantized enveloping algebra of $\mathfrak{g}$. Hence, these Hopf algebras are commutative-by-finite and smooth of global dimension $\dim \mathfrak{g}$, [20, Theorem XIII.8.2].

For future purposes, we give a few more details on two examples.

Example 2.2.1 (Quantized enveloping algebras of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$)**.** Let ϵ be a primitive lth root of unity, with l odd.

The quantum group $U_\epsilon(\mathfrak{sl}_2)$ is generated by $E, F, K^{\pm 1}$ and subject to the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = \epsilon^2 EK, \quad KF = \epsilon^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{\epsilon - \epsilon^{-1}}.$$

Moreover, K is grouplike, E is $(1, K)$-skew primitive and F is $(K^{-1}, 1)$-skew primitive [13, I.6.2]. This quantum group is a free module over the central Hopf subalgebra

$$A = k[E^l, F^l, K^{\pm l}]$$

with basis $\{F^i K^j E^k : 0 \leq i, j, k < l\}$ [13, III.6.2]. By Theorem 1.1.7, A is the coordinate ring of the affine algebraic group

$$G = \mathrm{Maxspec}(A) = ((k, +) \times (k, +)) \rtimes k^\times,$$

a semidirect product in which $k^\times$ acts on $(k, +)^2$ by multiplication: more specifically, since E^l is $(1, K^l)$-primitive, F^l is $(K^{-l}, 1)$-primitive and K is grouplike, the multiplication in G is given by

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \gamma\alpha', \beta\gamma'^{-1} + \beta', \gamma\gamma')$$

for any $\alpha, \beta, \alpha', \beta' \in k, \gamma, \gamma' \in k^\times$. Moreover, the associated Hopf quotient is the restricted quantized enveloping algebra of $\mathfrak{sl}_2$,

$$\overline{H} = U_\epsilon(\mathfrak{sl}_2)/(E^l, F^l, K^l - 1) =: u_\epsilon(\mathfrak{sl}_2),$$

which clearly is not semisimple.

The quantum group $U_\epsilon(\mathfrak{sl}_3)$ is generated by E_i, F_i, K_i (for $i = 1, 2$) with relations

$$K_i E_j = \begin{cases} \epsilon^2 E_j K_i, & \text{if } i = j \\ \epsilon^{-1} E_j K_i, & \text{if } i \neq j \end{cases}, \qquad K_i F_j = \begin{cases} \epsilon^{-2} F_j K_i, & \text{if } i = j \\ \epsilon F_j K_i, & \text{if } i \neq j \end{cases},$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{\epsilon - \epsilon^{-1}}, \qquad E_i^2 E_j - (\epsilon + \epsilon^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad (i \neq j)$$

$$K_i K_j = K_j K_i, \qquad F_i^2 F_j - (\epsilon + \epsilon^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad (i \neq j).$$

Moreover, K_i is grouplike, E_i is $(1, K_i)$-skew primitive and F_i is $(K_i^{-1}, 1)$-skew primitive [13, I.6.2]. Introducing the nonsimple roots $E_3 = E_1 E_2 - \epsilon^{-1} E_2 E_1$ and $F_3 = F_1 F_2 - \epsilon^{-1} F_2 F_1$, it is known that $U_\epsilon(\mathfrak{sl}_3)$ is a free module over the central Hopf subalgebra

$$A = k[K_1^{\pm l}, K_2^{\pm l}, E_1^l, E_2^l, E_3^l, F_1^l, F_2^l, F_3^l]$$

with A-basis $\{F_1^{r_1} F_2^{r_2} F_3^{r_3} K_1^{s_1} K_2^{s_2} E_1^{t_1} E_2^{t_2} E_3^{t_3} : 0 \leq r_i, s_i, t_i < l\}$ [13, III.6.2]. This central

Hopf subalgebra is the coordinate ring of the affine algebraic group

$$G = \operatorname{Maxspec}(A) = (k, +)^6 \rtimes (k^\times)^2$$

by Theorem 1.1.7. And it is easy to see that for each $i = 1, 2$ E_i^l is $(1, K_i^l)$-primitive and F_i^l is $(K_i^{-l}, 1)$-primitive, E_3^l is $(1, K_1^l K_2^l)$-primitive, F_3^l is $(K_1^{-l} K_2^{-l}, 1)$-primitive and K_1, K_2 are grouplike, hence the multiplication in G is given by

$$\begin{aligned}&(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2)(\alpha_1', \alpha_2', \alpha_3', \beta_1', \beta_2', \beta_3', \gamma_1', \gamma_2')\\ &= (\alpha_1 + \gamma_1\alpha_1', \alpha_2 + \gamma_2\alpha_2', \alpha_3 + \gamma_1\gamma_2\alpha_3', \beta_1\gamma_1'^{-1} + \beta_1', \beta_2\gamma_2'^{-1} + \beta_2', \beta_3\gamma_1'^{-1}\gamma_2'^{-1} + \beta_3', \gamma_1\gamma_1', \gamma_2\gamma_2').\end{aligned}$$

Furthermore, the associated Hopf quotient

$$\overline{H} = U_\epsilon(\mathfrak{sl}_3)/(E_1^l, E_2^l, E_3^l, F_1^l, F_2^l, F_3^l, K_1^l - 1, K_2^l - 1) =: u_\epsilon(\mathfrak{sl}_3)$$

is the restricted quantized enveloping algebra of $\mathfrak{sl}_3$.

2.2.2 Quantized coordinate rings at a root of unity

Our next examples of commutative-by-finite Hopf algebras are quantized coordinate rings (also known as quantum rings of functions) at a root of unity. We also do not define them here but they can be found in [13, §I.7], [25, Section 4.1]. We continue to assume that k is an algebraically closed field of characteristic zero.

Quantized coordinate rings $\mathcal{O}_q(G)$ of connected, simply connected, semisimple Lie groups G are noetherian Hopf algebras, [25, Sections 4.1 and 6.1]. If $q = \epsilon$ is a primitive lth root of unity, $\mathcal{O}_\epsilon(G)$ contains a central Hopf subalgebra isomorphic to $\mathcal{O}(G)$, [25, Proposition 6.4], [13, Theorem III.7.2], over which $\mathcal{O}_\epsilon(G)$ is a free module of rank $l^{\dim(G)}$ [14]. In fact, $\mathcal{O}(G) \subseteq \mathcal{O}_\epsilon(G)$ is a cleft extension, with a coalgebra splitting cleaving map; see [4, Remark 2.18(b)]. In the notation of [4] such a cleaving map can be obtained by dualizing the algebra projection $\phi : \Gamma_\epsilon(\mathfrak{g}) \twoheadrightarrow u_\epsilon(\mathfrak{g})$ where $\mathfrak{g} = \operatorname{Lie} G$.

The finite dimensional Hopf quotient

$$\overline{H} = \mathcal{O}_\epsilon(G)/\mathcal{O}(G)^+\mathcal{O}_\epsilon(G) =: o_\epsilon(G)$$

is usually known as the restricted quantized coordinate ring. Note that $o_\epsilon(G) \cong u_\epsilon(\mathfrak{g})^*$ is the dual of the restricted quantized enveloping algebra of $\mathfrak{g} = \operatorname{Lie} G$, [13, III.7.10].

The algebras in this family are thus commutative-by-finite, and smooth of global dimension $\dim(G)$, [12, Theorem 2.8].

2.2.3 Enveloping algebras of Lie algebras in positive characteristic

For our next example let k be an algebraically closed field of characteristic $p > 0$ and recall the enveloping algebras introduced in Example 1.1.2.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Its universal enveloping algebra $U(\mathfrak{g})$ is a noetherian Hopf algebra [64, Corollary 1.7.4] and in positive characteristic it is finitely generated over its centre [46]. More specifically, let $\mathfrak{g} = \bigoplus_{i=1}^{m} kx_i$ be a finite dimensional Lie algebra. For each $1 \leq i \leq m$, there exists in $U(\mathfrak{g})$ a central p-polynomial y_i on x_i, that is, a central polynomial in $U(\mathfrak{g})$ of the form

$$y_i = a_0 x_i + a_1 x_i^p + \ldots + a_n x_i^{p^n},$$

for some $n \geq 0$ and $a_0, \ldots, a_n \in k$ [46, Proposition 1]. Moreover, since each x_i is primitive in $U(\mathfrak{g})$ and k has characteristic p, each p-polynomial y_i is also primitive. Hence, $A = k\langle y_1, \ldots, y_m \rangle$ is a central Hopf subalgebra of $U(\mathfrak{g})$ and, in fact, it is a polynomial algebra on these primitive generators. Lastly, $U(\mathfrak{g})$ is a free A-module of finite rank with basis given by the monomials

$$\{x_1^{i_1} \ldots x_m^{i_m} : 0 \leq i_j < d_j\},$$

where $d_j = \deg(y_j)$, [46, Proposition 2]. Therefore, the enveloping algebra $U(\mathfrak{g})$ is commutative-by-finite and by [20, Theorem XIII.8.2] it is a smooth domain with

$$\operatorname{gldim}(U(\mathfrak{g})) = \operatorname{GKdim}(U(\mathfrak{g})) = \dim \mathfrak{g}.$$

For restricted Lie algebras, we can be more precise about the p-polynomials y_i. A Lie algebra $\mathfrak{g}$ is *restricted* if it possesses a map $x \mapsto x^{[p]}$ that satisfies for all $x, y \in \mathfrak{g}$ and $\lambda \in k$,

1. $ad(x^{[p]}) = ad(x)^p$, meaning $[x^{[p]}, y] = [x, [\ldots [x, y] \ldots]]$.
2. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$.
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i}$, where $s_i(x,y)$ is the coefficient of t^{i-1} in $ad(tx+y)^{p-1}(x)$.

Example 2.2.2. The Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(k)$, consisting of $n \times n$ matrices, is restricted over a field of characteristic p with the map $A = (a_{ij}) \mapsto A^{[p]} = (a_{ij}^p)$. Moreover, any Lie subalgebra of $\mathfrak{gl}_n(k)$ which is closed under this map, such as $\mathfrak{sl}_n(k)$, is also a restricted Lie algebra.

If $\mathfrak{g} = \bigoplus_{i=1}^{m} kx_i$ is a restricted Lie algebra, the p-polynomials are $y_i = x_i^p - x_i^{[p]}$ for each $1 \leq i \leq m$, thus $U(\mathfrak{g})$ is a free module of finite rank p^m over the central Hopf subalgebra $A = k\langle x_i^p - x_i^{[p]} : 1 \leq i \leq m \rangle$, which is a polynomial algebra on these primitive generators, [45, Section 2.3], [13, Theorem I.13.2(8)]. Its Hopf quotient $\overline{H} = U(\mathfrak{g})/A^+U(\mathfrak{g})$ is the restricted enveloping algebra of $\mathfrak{g}$ and is usually denoted by $u^{[p]}(\mathfrak{g})$.

2.2.4 Group algebras of finitely generated abelian-by-finite groups

We now present a large class of commutative-by-finite Hopf algebras coming from groups. Here k is an algebraically closed field of arbitrary characteristic.

Let G be a finitely generated group with an abelian normal subgroup N of finite index. Then, N is also finitely generated, so G is polycyclic-by-finite, and hence kG is an affine cocommutative Hopf algebra, which is noetherian by [70, Corollary 10.2.8]. Since kG is a finite kN-module, it is commutative-by-finite. As we saw in Example 1.1.19 these extensions are cleft, and these algebras decompose into the crossed products $kG \cong kN\#_\sigma k(G/N)$, with a splitting coalgebra cleaving map.

The set N_0 of elements of N of finite order is a normal subgroup of N, usually known as the torsion subgroup of N, and the quotient group N/N_0 is finitely generated abelian and free [70, Lemma 4.1.5]. Let d denote the rank of N/N_0, commonly referred to as the torsion-free rank of N; it coincides with the Hirsch length of G, see [70, remarks on p. 380, Lemma 10.2.10]. Moreover, by [18, Theorem 6.7] we have

$$\operatorname{injdim}(kG) = d.$$

By [70, Theorem 10.3.13], kG has finite global dimension if and only if $\operatorname{char} k = 0$, or $\operatorname{char} k = p > 0$ and G has no elements of order p. In this case, we have $\operatorname{gldim}(kG) = \operatorname{injdim}(kG) = d$ by [6, Proposition 4.2].

2.2.5 Prime regular affine Hopf algebras of GK-dimension 1

Our next examples are Hopf algebras arriving from a classification in GK-dimension 1. The prime regular affine Hopf algebras of GK-dimension 1 were completely classified when by Ding, Liu and Wu in [27], building on [19] and [58]. Here k is an algebraically closed field of characteristic 0.

By a fundamental result of Small, Stafford and Warfield [90, Theorem] a semiprime affine algebra of GK-dimension one is a finitely generated module over its centre. But in fact more is true for these Hopf algebras - they are all commutative-by-finite, which we checked on a case-by-case basis.

The classification consists of 2 finite families and 3 infinite families, as follows.

(I) The commutative algebras $k[x]$ and $k[x^{\pm 1}]$.

(II) A single cocommutative noncommutative example, the group algebra of the infinite dihedral group

$$D = \langle a, b : a^2 = 1, aba = b^{-1} \rangle.$$

(III) The family of Taft algebras $T(n, t, q)$;

(IV) The family of generalized Liu algebras $B(n, w, q)$;

(V) and Ding-Liu-Wu's examples $D(m, d, q)$.

We explore these examples further in the following pages.

(II) Group algebra of the dihedral group

The infinite dihedral group D is abelian-by-finite, with abelian normal subgroup $N = \langle b \rangle$. Thus, $H = kD$ is commutative-by-finite over the normal commutative Hopf subalgebra $A = k\langle b \rangle$, with the Hopf quotient $\overline{H} = H/(b-1)H \cong kC_2$. Moreover, H decomposes into the smash product

$$kD \cong A * kC_2,$$

where C_2 acts on A by conjugation.

(III) Taft algebras

Recall the infinite dimensional Taft algebras $H = T(n, t, q)$ from Example 1.1.4:

$$T(n, t, q) = k\langle g, x : g^n = 1, xg = qgx \rangle,$$

where g is grouplike and x is $(1, g^t)$-primitive.

We claim that the commutative subalgebra $A = k[x^{n'}]$ is a normal Hopf subalgebra, where $d = (n, t), n' = n/d$. As mentioned in Example 1.1.21 $x^{n'}$ is a primitive element, so A is a Hopf subalgebra. Moreover, the left adjoint action of H on A is given by $g \cdot x^{n'} = gx^{n'}g^{-1} = q^{-n'}x^{n'}$ and

$$x \cdot x^{n'} = x^{n'+1} + g^t x^{n'}(-g^{-t}x) = x^{n'+1} - q^{-n't}x^{n'+1} = 0,$$

since $q^{-n't} = q^{-nt'} = 1$ where $t' = t/d$. Normality of A now follows from Remark 2.1.5(2).

As we saw in Example 1.1.21, these algebras decompose as crossed products into

$$T(n, t, q) \cong A\#_\sigma \overline{H} \cong k[x^{n'}]\#_\sigma(T_f(n', t', q^d)\#_\tau kC_d).$$

(IV) Generalized Liu algebras

Let n and w be positive integers and let q be a primitive nth root of 1. We define the generalized Liu algebras $B(n, w, q)$ as follows,

$$B(n, w, q) := k\langle x^{\pm 1}, g^{\pm 1}, y : x \text{ central}, yg = qgy, g^n = x^w = 1 - y^n \rangle,$$

with x and g grouplike and y $(1, g)$-primitive. Clearly, $A := k[x^{\pm 1}]$ is a central Hopf subalgebra over which $H = B(n, w, q)$ is a free module of rank n^2 [19, (3.4.4)]. The corresponding Hopf quotient is

$$\overline{H} = k\langle \overline{g}, \overline{y} : \overline{g}^n = 1, \overline{y}^n = 0, \overline{yg} = q\overline{gy} \rangle = T_f(n, 1, q).$$

Moreover, $B(n, w, q)$ decomposes as a crossed product. Consider the natural map $\gamma : \overline{H} \to H, \bar{g}^i \bar{y}^j \mapsto g^i y^j$ for any $0 \leq i, j < n$. Although it is neither an algebra nor a coalgebra map, it is easy to check that it is convolution invertible with inverse $\gamma^{-1}(\bar{g}^i \bar{y}^j) = (-1)^j q^{-\binom{j}{2}} g^{-j} y^j g^{-i}$. Therefore, since k has characteristic 0, $H^{\mathrm{co}\,\overline{H}} = A$ by Theorem 2.1.9(2) and by Theorem 1.1.17 we have

$$B(n, w, q) \cong A \#_\sigma \overline{H} = k[x^{\pm 1}] \#_\sigma T_f(n, 1, q)$$

as algebras, for some nontrivial cocycle σ.

(V) Ding-Liu-Wu's examples

Let m and d be positive integers with $(1+m)d$ even, and let q be a primitive $2m$th root of 1. The Hopf algebra $D(m, d, q)$ is defined as follows: as an algebra it is generated by $x^{\pm 1}, g^{\pm 1}, y, u_0, u_1, \ldots, u_{m-1}$ with relations

$$xg = gx, \quad xy = yx, \quad yg = \gamma gy, \quad x^w = g^m = 1 - y^m,$$

$$xu_i = u_i x^{-1}, \quad yu_i = \phi_i u_{i+1} = qx^d u_i y, \quad u_i g = \gamma^i x^{-2d} g u_i,$$

$$u_i u_j = (-1)^j q^{j^2} \frac{1}{m} x^{-\frac{m+1}{2} d} [i, m-2-j] \, y^{i+j \mod m} g,$$

where $\gamma = q^2$, $w = md$, $\phi_i = 1 - \gamma^{-(i+1)} x^d$ and the brackets $[,]$ are defined as

$$[s, t] = \begin{cases} \phi_{\bar{s}} \phi_{\bar{s}+1} \ldots \phi_{\bar{t}} & \text{if } \bar{s} \leq \bar{t} \\ 1 & \text{if } \bar{s} = \bar{t} + 1 \\ \phi_{\bar{s}} \ldots \phi_{m-1} \phi_0 \ldots \phi_{\bar{t}} & \text{if } \bar{s} \geq \bar{t} + 2 \end{cases}.$$

The coalgebra structure is as follows: x and g are grouplike, y is $(1, g)$-primitive and

$$\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} x^{-jd} g^j u_{i-j} \otimes u_j,$$

using the convention $u_s := u_{s \bmod m}$; $\epsilon(x) = \epsilon(g) = \epsilon(u_0) = 1$ and $\epsilon(y) = \epsilon(u_i) = 0$ for $1 \leq i < m$. As for the antipode, $S(x) = x^{-1}, S(g) = g^{-1}, S(y) = -g^{-1}y$ and

$$S(u_i) = (-1)^i q^{i(i+2)} u_i x^{id+\frac{3}{2}(1-m)d} g^{m-i-1}.$$

Note that the subalgebra generated by $x^{\pm 1}, g^{\pm 1}, y$ is the generalized Liu algebra $B(m, w, \gamma)$, in the notation of 2.2.5(IV). Furthermore, the product of any two elements u_i, u_j belongs to this subalgebra.

Let $A = k[x^{\pm 1}]$. This commutative Hopf subalgebra is central (hence normal) in $k\langle x, g, y\rangle \cong B(m, w, \gamma)$ and the left adjoint action of u_i on x is

$$\begin{aligned}
& \sum_{j=0}^{m-1} \gamma^{j(i-j)} x^{-jd} g^j u_{i-j} x S(u_j) \\
= & \sum_{j=0}^{m-1} (-1)^j q^{2j(i-j)+j(j+2)} x^{-jd-1} g^j u_{i-j} u_j x^{jd+\frac{3}{2}(1-m)d} g^{m-j-1} \\
= & \sum_{j=0}^{m-1} q^{j(2i-j+2)+j^2} \frac{1}{m} x^{-jd-1} g^j x^{-\frac{m+1}{2}d} \left[i-j, m-2-j\right] y^i g x^{jd+\frac{3}{2}(1-m)d} g^{m-j-1} \\
= & \frac{1}{m} x^{-1+d(1-2m)} \sum_{j=0}^{m-1} q^{j(2i+2)} \left[i-j, m-2-j\right] g^j y^i g^{m-j} \\
= & \frac{1}{m} x^{-1+d(1-2m)} \sum_{j=0}^{m-1} q^{j(2i+2)+2i(m-j)} \left[i-j, m-2-j\right] g^m y^i \\
= & \frac{1}{m} x^{-1+d(1-m)} \left(\sum_{j=0}^{m-1} q^{2j} \left[i-j, m-2-j\right] \right) y^i \\
= & \frac{1}{m} q^{2i} x^{-1+d(1-m)} \left(\sum_{j=0}^{m-1} q^{-2j} \left[j, m-2-(i-j)\right] \right) y^i.
\end{aligned}$$

By [27, Lemmas 3.2 and 3.5] the parenthesised sum simplifies to $\delta_{i,0} m x^{(m-1)d}$, hence

$$ad_l(u_i)(x) = \frac{1}{m} x^{-1+d(1-m)} \delta_{i,0} m x^{(m-1)d} = x^{-1}\delta_{i,0}.$$

And by Remark 2.1.5(2) A is normal.

Moreover, by [27, Equation 4.8] $H = D(m, d, q)$ is a free A-module of finite rank,

$$H = \left(\bigoplus_{i,j=0}^{m-1} A g^i y^j \right) \oplus \left(\bigoplus_{i,k=0}^{m-1} A g^i u_k \right). \tag{2.4}$$

We now investigate the corresponding Hopf quotient.

Lemma 2.2.3. *Let $H = D(m, d, q)$ and $A = k[x^{\pm 1}]$ as above, and consider the Hopf quotient $\overline{H} = H/A^+H$.*

1. *As a vector space,*

$$\overline{H} = \left(\bigoplus_{j,k=0}^{m-1} k\overline{g}^j\overline{y}^k\right) \oplus \left(\bigoplus_{j,k=0}^{m-1} k\overline{g}^j\overline{y}^k\overline{u_0}\right).$$

2. *As an algebra $\overline{H}$ is generated by $\overline{g}, \overline{y}, \overline{u_0}$ with relations $\overline{y}^m = 0, \overline{g}^m = 1, \overline{yg} = \gamma\overline{gy}, \overline{u_0 g} = \overline{g u_0}, \overline{u_0 y} = q^{-1}\overline{y u_0}, \overline{u_0}^2 = \overline{g}$. Moreover, $\overline{g}$ is grouplike, $\overline{y}$ is $(1, \overline{g})$-primitive and*

$$\Delta(\overline{u_0}) = \overline{u_0} \otimes \overline{u_0} + \sum_{j=1}^{m-1} \alpha_j \overline{g}^j \overline{y}^{m-j}\overline{u_0} \otimes \overline{y}^j\overline{u_0},$$

where $\alpha_j = \gamma^{-j^2}(1-\gamma^{-j})^{-1}\dots(1-\gamma^{-1})^{-1}(1-\gamma^{-(m-j)})^{-1}\dots(1-\gamma^{-1})^{-1}$.

3. *As an algebra $\overline{H}$ decomposes as the crossed product*

$$\overline{H} \cong T_f(m, 1, \gamma)\#_\sigma kC_2.$$

Proof. (1) We know from (2.4) that $\overline{H} = \left(\bigoplus_{j,k=0}^{m-1} k\overline{g}^j\overline{y}^k\right) \oplus \left(\bigoplus_{j,l=0}^{m-1} k\overline{g}^j\overline{u_l}\right)$ and each $\overline{u_l}$ can be "reduced to $\overline{u_0}$" by iterative use of the relation $\overline{u_{l+1}} = (1-\gamma^{-(l+1)})^{-1}\overline{y u_l}$. Therefore, $\{\overline{g}^j\overline{y}^k\overline{u_0}^l : 0 \le j,k < m, 0 \le l \le 1\}$ generates $\overline{H}$ and, by a dimensional argument, it must be a basis of $\overline{H}$.

(2) The relation $\overline{u_0}^2 = \overline{g}$ is the only one that needs justification. We know $\overline{u_0}^2 = \frac{1}{m}(1-\gamma^{-1})\dots(1-\gamma^{-(m-1)})\overline{g}$. And since γ is a primitive mth root of unity, so is γ^{-1}, hence $\gamma^{-1}, \dots, \gamma^{-(m-1)}$ are all the distinct mth roots of unity except 1; but, since $\prod_{i=0}^{m-1}(z-\gamma^{-i}) = z^m - 1$ and $\prod_{i=1}^{m-1}(z-\gamma^{-i}) = \frac{z^m-1}{z-1} = z^{m-1} + \dots + z + 1$, then $\prod_{i=1}^{m-1}(1-\gamma^{-i}) = 1^{m-1} + \dots + 1 + 1 = m$ and the relation follows.

(3) Consider the ideal $I = (\overline{g}-1, \overline{y})\overline{H}$. It is a coideal, since $\epsilon(I) = 0$ and $\Delta(\overline{g}-1) = (\overline{g}-1)\otimes\overline{g} + 1\otimes(\overline{g}-1)$ and $\Delta(\overline{y}) = \overline{y}\otimes\overline{g} + 1\otimes\overline{y}$ both belong to $I\otimes\overline{H} + \overline{H}\otimes I$. As $S(\overline{g}-1) = \overline{g}^{-1} - 1 = -\overline{g}^{-1}(\overline{g}-1)$ and $S(\overline{y}) = -\overline{g}^{-1}\overline{y}$ are both in I, it is a Hopf ideal. Thus, we have a Hopf surjection $\pi : \overline{H} \to \overline{H}/I \cong kC_2$.

This Hopf surjection π induces a right kC_2-comodule algebra structure on $\overline{H}$ with coaction $\rho = (\mathrm{id}\otimes\pi)\Delta_{\overline{H}}$. Moreover, the map $\gamma : kC_2 \to \overline{H}$ given by $\gamma(1) = 1, \gamma(a) = \overline{u_0}$ (where a denotes a generator of C_2) is a convolution invertible kC_2-comodule map, with inverse given by $\gamma^{-1}(1) = 1, \gamma^{-1}(a) = \overline{u_0}\,\overline{g}^{-1} = \overline{u_0}\,\overline{g}^{m-1}$. Hence, γ is a cleaving map and by Theorem 1.1.17

$$\overline{H} \cong \overline{H}^{\mathrm{co}\,kC_2}\#_\sigma kC_2.$$

Lastly, we claim the coinvariants of this coaction are the elements of $T_f(m, 1, \gamma)$, the subalgebra generated by $\overline{g}$ and $\overline{y}$. In fact, $\rho(\overline{g}) = \overline{g}\otimes\pi(\overline{g}) = \overline{g}\otimes 1$, $\rho(\overline{y}) = \overline{y}\otimes 1 + \overline{g}\otimes\pi(\overline{y}) = \overline{y}\otimes 1$ and, since $\overline{H}^{\mathrm{co}\,kC_2}$ is a subalgebra of $\overline{H}$, $T_f(m,1,\gamma) \subseteq \overline{H}^{\mathrm{co}\,kC_2}$. And, since both $T_f(m,1,\gamma)$ and $\overline{H}^{\mathrm{co}\,kC_2}$ have the same dimension, namely m^2, they must equal. $\square$

Note that in this decomposition the action of kC_2 on T_f is trivial everywhere except $a \cdot \overline{y} = \xi^{-1}\overline{y}$ and the cocycle is trivial everywhere except in $\sigma(a,a) = \overline{g}$. Moreover, as a coalgebra $\overline{H}$ does not decompose into $T_f(m,1,\gamma) \otimes kC_2$, because $\overline{u_0}$ is not a grouplike.

Remark 2.2.4. Note that in [19] and [27] the skew-primitive generator of the Taft algebras is $(g^t,1)$-primitive, unlike here where we defined it to be $(1,g^t)$-primitive. This difference, however, is not as important as it might appear at first, since this only depends on the generators of $T(n,t,q)$ we consider. For, if $T(n,t,q)$ denotes a Taft algebra where the skew-primitive generator x is $(g^t,1)$-primitive, then $x' = g^{-t}x$ is $(1,(g')^t)$-primitive where $g' = g^{-1}$ and g', x' generate $T(n,t,q^{-1})$. An analogous remark holds for finite dimensional Taft algebras.

In a similar fashion, the generalized Liu algebras were defined in [19] as above but with y being $(g,1)$-primitive. Once again, this depends only on the generators one considers, for if $B(n,w,q)$ is a generalized Liu algebra with y being $(1,g)$-primitive, then $g' = g^{-1}$ and $y' = (-1)^n g^{-1} y$ generate the generalized Liu algebra $B(n,w,q^{-1})$ in which y' is $(g',1)$-primitive.

In [27] the new Hopf algebras $D(m,d,q)$ were also defined slightly differently: the algebra structure is the same as above but the coproduct requires the generator y to be $(g,1)$ and the coproduct of u_i is also different from above. Therefore, we present here $D(m,d,q)^{cop}$, the Hopf algebra with the same algebra structure as $D(m,d,q)$ and coproduct $\Delta^{cop} = \tau \circ \Delta$, where τ is the flip map. Given the classification in [27], we must have an isomorphism

$$D(m,d,q)^{cop} \cong D(m,d,q'),$$

for some primitive root q', which therefore allows us to consider $D(m,d,q)^{cop}$ here. On a more technical note, the counit of $D(m,d,q)^{cop}$ is easily computable from the counit axiom and we used [67, Lemma 1.5.11] and the antipode in [27] to compute S^{cop}.

Summary

These Hopf algebras are all free over their respective normal commutative Hopf subalgebras. All families except $D(m,d,q)$ are pointed, thus decompose as crossed products $H \cong A\#_\sigma \overline{H}$ by Proposition 2.1.12. The key feature of $D(m,d,q)$ is the fact they are not pointed, [27, Proposition 4.9]; still they are free over their commutative normal Hopf subalgebras but I do not know whether they decompose as a crossed product $A\#_\sigma \overline{H}$.

More recently, there has been some work [55] towards the classification of certain prime affine Hopf algebras of GK-dimension one that satisfy weaker hypotheses than regularity; see the classification in [55, Theorem 7.1]. The new Hopf algebras constructed in this paper are fraction versions of Taft algebras, generalized Liu algebras and the Hopf algebras $D(m,d,q)$. Each of these is commutative-by-finite and actually Liu conjectures that every prime Hopf algebra of GK-dimension one over an algebraically closed field of characteristic zero is commutative-by-finite [55, Conjecture 7.19, Remark 7.20].

2.2.6 Noetherian PI Hopf domains of GK-dimension two

Our last examples come from another classification of Hopf algebras, this time of GK-dimension 2. In this subsection we again assume that k is algebraically closed of characteristic 0.

Let H be a noetherian Hopf algebra domain with $\mathrm{GKdim}\,(H) = 2$. Such Hopf algebras were classified in [38, Theorem 0.1] under the additional assumption that

$$\mathrm{Ext}^1_H({}_Hk, {}_Hk) \neq 0. \qquad (\sharp)$$

By [38, Lemma 3.1], the hypothesis ($\sharp$) is equivalent to the assumption that $(H^+)^2 \neq H^+$; and by [38, Proposition 3.8(c)] it is also equivalent to the fact that H has an infinite dimensional commutative Hopf factor. There are 5 classes of such Hopf algebras and the commutative-by-finite ones are precisely the ones that satisfy a polynomial identity; see [38, proof of Proposition 0.2(b)]. These are as follows.

(I) The group algebras of the groups $\mathbb{Z} \times \mathbb{Z}$ and

$$\mathbb{Z} \rtimes \mathbb{Z} = \langle a, b : aba^{-1} = b^{-1} \rangle.$$

The group $\mathbb{Z} \rtimes \mathbb{Z}$ is abelian-by-finite over the normal abelian subgroup $\mathbb{Z} \times 2\mathbb{Z} = \langle a^2, b \rangle$, hence $k(\mathbb{Z} \rtimes \mathbb{Z})$ is commutative-by-finite;

(II) The enveloping algebra of the 2-dimensional abelian Lie algebra, that is $k[x, y]$;

(III) The family of Hopf algebras $A(l, n, q)$;

(IV) and the family of Hopf algebras $B(n, p_0, p_1, \dots, p_s, q)$.

(III) The localized quantum plane $A(l, n, q)$ at a root of unity

Let $l \in \mathbb{N}, n \in \mathbb{Z}$ and q a primitive lth root of 1. Consider the Hopf algebra

$$A(l, n, q) = k\langle x^{\pm 1}, y : xy = qyx \rangle,$$

with x grouplike and y $(1, x^n)$-primitive.

Let $A = k\langle (x^l)^{\pm 1}, y^{l'} \rangle$, where $d = (n, l), l' = l/d, n' = n/d$. A is a Hopf subalgebra of $H = A(l, n, q)$, because x^l is grouplike and, since q^n is a l'th primitive root of unity, $y^{l'}$ is $(1, x^{nl'})$-primitive with $x^{nl'} = x^{ln'} \in A$. Furthermore, A is commutative because x^l is central. To see that A is normal, note first that x and y act trivially on x^l, again because it is central. Moreover, $(ad_l x)(y^{l'}) = xy^{l'}x^{-1} = q^{l'}y^{l'} \in A$ and

$$(ad_l y)(y^{l'}) = yy^{l'} - x^n y^{l'} x^{-n} y = y^{l'+1} - q^{nl'} y^{l'} y = y^{l'+1} - q^{ln'} y^{l'+1} = 0.$$

And H is clearly a finitely generated A-module, hence H is commutative-by-finite.

Let us also compute the Hopf quotient $\overline{H}$. Following the notation from Example 1.1.21, we have

$$\overline{H} = k\langle \overline{x}^{\pm 1}, \overline{y} : \overline{xy} = q\overline{yx}, \overline{x}^l = 1, \overline{y}^{l'} = 0\rangle = U(l, n, q^{-1}) \cong T_f(l', n', q^{-d})\#_\sigma kC_d$$

as algebras; and as coalgebras $\overline{H} = T_f \otimes kC_d$.

These Hopf algebras also decompose into crossed products. Consider the natural map $\gamma : \overline{H} \to H$ given by $\gamma(\overline{x}^i\overline{y}^j) = x^iy^j$ for $0 \leq i < l, 0 \leq j < l'$; even though it is neither an algebra nor a coalgebra map, it is convolution invertible with inverse $\gamma^{-1}(\overline{x}^i\overline{y}^j) = (-1)^j q^{-\binom{j}{2}} x^{-nj}y^jx^{-i}$. Therefore, since k has characteristic 0, $H^{\mathrm{co}\,\overline{H}} = A$ by Theorem 2.1.9(2), and Theorem 1.1.17 implies that as algebras

$$H \cong A\#_\sigma\overline{H} = k[(x^l)^{\pm 1}, y^{l'}]\#_\tau[T_f(l', n', q^{-d})\#_\sigma kC_d].$$

(IV) The Hopf algebras $H = B(n, p_0, \ldots, p_s, q)$

Let $s \geq 2$, let $n, p_0, \ldots, p_s$ be positive integers with $p_0 \mid n$ and $\{p_i : i \geq 1\}$ strictly increasing and pairwise coprime and coprime with p_0, and let q be a primitive lth root of 1 where $l = (n/p_0)p_1 \ldots p_s$. Consider the subalgebra $B(n, p_0, \ldots, p_s, q)$ of the localized quantum plane $A(l, n, q) = k\langle x^{\pm 1}, y\rangle$ from 2.2.6(III) generated by $x^{\pm 1}$ and $\{y_i := y^{m_i} : 1 \leq i \leq s\}$, where $m_i := \Pi_{j\neq i}p_j$, that is

$$B(n, p_0, \ldots, p_s, q) = k\langle x^{\pm 1}, y_1, \ldots, y_s : xy_i = q^{m_i}y_ix, y_1^{p_1} = \ldots = y_s^{p_s}, y_iy_j = y_jy_i\rangle.$$

This supports a Hopf structure with x being grouplike and y_i being $(1, x^{m_in})$-primitive and it has a k-basis given by

$$\{y_1^{i_1} \ldots y_s^{i_s}x^j : i_1 \geq 0, j \in \mathbb{Z}, 0 \leq i_k < p_k \text{ for } k = 2, \ldots, s\}.$$

Let $A = k\langle y_1^{p_1}, \ldots, y_s^{p_s}, (x^l)^{\pm 1}\rangle = k[y^{p_1\cdots p_s}, x^{\pm l}]$. It is commutative, since x^l is central in $A(l, n, q)$ and therefore in $H = B(n, p_0, \ldots, p_s, q)$. A is a Hopf subalgebra of H, because x^l is grouplike and, since q^{m_in} is a primitive p_ith root of unity,

$$\Delta(y_i^{p_i}) = \sum_{k=0}^{p_i} \binom{p_i}{k}_{q^{m_in}} y_i^{j-k}x^{m_ink} \otimes y_i^k = y_i^{p_i} \otimes 1 + (x^l)^{p_0} \otimes y_i^{p_i} \in A \otimes A.$$

Moreover H is clearly a finite A-module and A is normal: x and y_j act trivially on x^l, because it is central, and $(ad_lx)(y_i^{p_i}) = xy_i^{p_i}x^{-1} = q^{m_ip_i}y_i^{p_i} \in A$ and, since m_ip_in is a multiple of l,

$$(ad_ly_j)(y_i^{p_i}) = y_jy_i^{p_i} + x^{m_jn}y_i^{p_i}(-x^{-m_jn}y_j) = y_jy_i^{p_i} - q^{m_ip_im_jn}y_i^{p_i}y_j = 0.$$

We now focus on the Hopf quotient $\overline{H}$.

Lemma 2.2.5. *Let $H = B(n, p_0, \ldots, p_s, q)$ be defined as above. Consider the Hopf subalgebra $A = k\langle y_1^{p_1}, \ldots, y_s^{p_s}, (x^l)^{\pm 1}\rangle$ and the Hopf quotient $\overline{H} = H/A^+H$. As an algebra, $\overline{H}$ decomposes into*

$$\big[[\ldots[T_f(p_1, p_0, \xi_1)\#_{\sigma_1}T_f(p_2, p_0p_1, \xi_2)]\,\#_{\sigma_2}\ldots]\,\#_{\sigma_{s-1}}T_f(p_s, p_0p_1\ldots p_{s-1}, \xi_s)\big]\,\#_\tau kC_{n/p_0},$$

where $\xi_i = q^{-(n/p_0)m_ip_{i+1}\ldots p_s}$ is a primitive p_ith root of unity. Moreover, as coalgebras

$$\overline{H} \cong T_f(p_1, p_0, \xi_1) \otimes T_f(p_2, p_0p_1, \xi_2) \otimes \ldots \otimes T_f(p_s, p_0p_1\ldots p_{s-1}, \xi_s) \otimes kC_{n/p_0}.$$

Proof. We know that

$$\overline{H} = k\langle \overline{x}^{\pm 1}, \overline{y_1}, \ldots, \overline{y_s} : \overline{x}^l = 1, \overline{y_i}^{p_i} = 0, \overline{xy_i} = q^{m_i}\overline{y_ix}, \overline{y_iy_j} = \overline{y_jy_i}\rangle,$$

where $\overline{x}$ is grouplike and $\overline{y_i}$ is $(1, \overline{x}^{m_in})$-primitive. Recall that $l = (n/p_0)p_1\ldots p_s$ and $m_i := \Pi_{j\neq i}p_j$.

We have a Hopf quotient $\overline{H} \twoheadrightarrow \overline{H}/(\overline{x}^{n/p_0} - 1, \overline{y_1}, \ldots, \overline{y_s}) \cong kC_{n/p_0}$, which induces a kC_{n/p_0}-comodule structure on $\overline{H}$. The natural map $\gamma : kC_{n/p_0} \to \overline{H}$ that sends a fixed generator of the cyclic group to $\overline{x}$ is a splitting coalgebra map, hence it is a cleaving map and by Theorem 1.1.17

$$\overline{H} \cong \overline{H}^{\mathrm{co}\, kC_{n/p_0}}\#_\tau kC_{n/p_0}.$$

Moreover, it is easy to see that $\overline{x}^{n/p_0}, \overline{y_1}, \ldots, \overline{y_s}$ are coinvariants of this coaction, and upon comparing dimensions we conclude that $\overline{H}^{\mathrm{co}\, kC_{n/p_0}}$ is the subalgebra generated by $\overline{x}^{n/p_0}, \overline{y_1}, \ldots, \overline{y_s}$, that is,

$$\overline{H}^{\mathrm{co}\, kC_{n/p_0}} = k\langle g, \overline{y_1}, \ldots, \overline{y_s} : \overline{y_i}^{p_i} = 0, g^{p_1\ldots p_s} = 1, \overline{y_i}g = \xi_ig\overline{y_i}, \overline{y_iy_j} = \overline{y_jy_i}\rangle,$$

where $g = \overline{x}^{n/p_0}$ and $\xi_i = q^{-(n/p_0)m_i}$ is a primitive p_ith root of unity. Note that g is grouplike and $\overline{y_i}$ is $(1, \overline{x}^{m_in}) = (1, g^{p_0m_i})$-primitive.

We now prove by induction on s that the Hopf algebra

$$T := k\langle g, y_1, \ldots, y_s : y_i^{p_i} = 0, g^{p_1\ldots p_s} = 1, y_ig = q_igy_i, y_iy_j = y_jy_i\rangle,$$

where g is grouplike, y_i is $(1, g^{p_0m_i})$-primitive and q_i is a primitive p_ith root of unity, decomposes as the crossed product of s Taft algebras as

$$T \cong [\ldots[T_f(p_1, p_0, \xi_1)\#_{\sigma_1}T_f(p_2, p_0p_1, \xi_2)]\,\#_{\sigma_2}\ldots]\,\#_{\sigma_{s-1}}T_f(p_s, p_0p_1\ldots p_{s-1}, \xi_s),$$

where $\xi_i = q_i^{p_{i+1}\ldots p_s}$.

For $s = 1$, $T = T_f(p_1, p_0, q_1)$ as required. Suppose the decomposition above holds for $s-1$ and consider $T = k\langle g, y_1, \ldots, y_s : y_i^{p_i} = 0, g^{p_1\ldots p_s} = 1, y_ig = q_igy_i, y_iy_j = y_jy_i\rangle$, where q_i is a primitive p_ith root of 1, g is grouplike and y_i is $(1, g^{p_0m_i})$-primitive. We have the Hopf quotient

$$T \twoheadrightarrow T/(g^{p_s}-1, y_1, \dots, y_{s-1}) \cong T_f(p_s, p_0 m_s, q_s) = T_f(p_s, p_0 p_1 \dots p_{s-1}, q_s).$$

The natural map $\gamma : T_f(p_s, p_0 p_1 \dots p_{s-1}, q_s) \to T$ given by $\gamma(\overline{y_s}^i \overline{g}^j) = y_s^i g^j$, for $0 \leq i, j < p_s$, is a splitting coalgebra map, so it is a cleaving map and by Theorem 1.1.17

$$T \cong T^{\operatorname{co} T_f} \#_{\sigma_1} T_f(p_s, p_0 p_1 \dots p_{s-1}, q_s). \tag{2.5}$$

It is easy to see that $y_1, \dots, y_{s-1}, g^{p_s}$ are coinvariants of this coaction and by comparing dimensions we conclude that $T^{\operatorname{co} T_f}$ is the subalgebra generated by these elements, that is,

$$T^{\operatorname{co} T_f} = k\langle h, y_1, \dots, y_{s-1} : h^{p_1 \dots p_{s-1}} = 1, y_i^{p_i} = 0, y_i h = q_i^{p_s} h y_i, y_i y_j = y_j y_i \rangle,$$

where $h = g^{p_s}$ is grouplike and each y_i is $(1, h^{p_0 m_i / p_s})$-primitive. The induction hypothesis yields

$$T^{\operatorname{co} T_f} \cong T_f(p_1, p_0, \xi_1) \#_{\sigma_1} \dots \#_{\sigma_{s-2}} T_f(p_{s-1}, p_0 p_1 \dots p_{s-2}, \xi_{s-1}),$$

where $\xi_i = (q_i^{p_s})^{p_{i+1} \dots p_{s-1}} = q_i^{p_{i+1} \dots p_s}$. Together with (2.5) this completes the induction step.

Regarding the coalgebra structure of $\overline{H}$, every crossed product is obtained from a cleaving map which is a splitting coalgebra map, hence the coalgebra structure is the usual one by Remark 1.1.18(2). $\square$

It is easy to see that H itself decomposes into a crossed product: the natural map $\gamma : \overline{H} \to H$ given by $\gamma(\overline{x}^i \overline{y_1}^{j_1} \dots \overline{y_s}^{j_s}) = x^i y_1^{j_1} \dots y_s^{j_s}$, where $0 \leq i < l, 0 \leq j_l < p_l$, is neither an algebra nor a coalgebra map but it is convolution invertible, with inverse

$$\gamma^{-1}(\overline{x}^i \overline{y_1}^{j_1} \dots \overline{y_s}^{j_s}) = (-1)^{j_1 + \dots + j_s} q^{-\binom{j_1}{2} - \dots - \binom{j_s}{2}} x^{-n m_s j_s} y_s^{j_s} \dots x^{-n m_1 j_1} y_1^{j_1} x^{-i}.$$

Therefore, since k has characteristic 0 Theorem 2.1.9(2) gives $H^{\operatorname{co} \overline{H}} = A$, and Theorem 1.1.17 guarantees that as algebras

$$B(n, p_0, \dots, p_s, q) \cong A \#_\sigma \overline{H} \cong k[(x^l)^{\pm 1}, y^{p_1 \dots p_s}] \#_\sigma [T_f \#_{\sigma_{s-1}} \dots \#_{\sigma_1} T_f \#_\tau k C_{n/p_0}].$$

Summary

As was shown in [38, Proposition 0.2(a) and proof of Proposition 1.6], all the above algebras have global dimension 2, except for the family (IV), whose members have infinite global dimension, being free over the coordinate ring $k\langle y^{m_i} : 1 \leq i \leq s \rangle$ of a singular curve.

More recently, in [98] it was shown that not all noetherian Hopf k-algebra domains of Gelfand-Kirillov dimension 2 satisfy ($\sharp$). More precisely, a key discovery of [98] was an infinite family of noetherian Hopf algebra domains of GK-dimension 2, with $\mathrm{Ext}^1_H({}_Hk, {}_Hk) = 0$ for all members of the family. All these algebras satisfy polynomial identities - in fact, by [98, Theorem 2.7], each of them is a free module of finite rank over a central Hopf subalgebra which is the coordinate ring of a 2-dimensional solvable group. In particular, they are all commutative-by-finite. Almost all of them have infinite global dimension, some of them actually being free modules of finite rank over an algebra in (IV).

2.2.7 A counterexample

For an algebraically closed field k of characteristic 0, S. Gelaki and E. Letzter gave an example of a prime noetherian Hopf k-algebra U of Gelfand-Kirillov dimension 2 which is *not* commutative-by-finite [35].

Their example is as follows: let $U = U(\mathfrak{g}) * C_2$ be the Hopf algebra generated by x, y, u, v, t with relations

$$x \text{ central}, \quad yu - uy = u, \quad yv - vy = -v, \quad uv + vu = x, \quad u^2 = v^2 = 0$$

$$tx = xt, \quad ty = yt, \quad tu = -ut, \quad tv = -vt, \quad t^2 = 1,$$

where t is a generator of C_2. U is a Hopf algebra with coalgebra structure given by x, y being primitive, u, v being $(1, t)$-primitive and t being grouplike.

Note that, unlike throughout the rest of this thesis, here $\mathfrak{g}$ does not denote a Lie algebra but the Lie superalgebra $\mathrm{pl}(1,1)$, namely $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ consists of the 2×2 matrices with basis

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where x, y form a basis of $\mathfrak{g}_0$ and u, v a basis of $\mathfrak{g}_1$. See [81] for more on Lie superalgebras. Often U is referred to as the "bosonization" of the enveloping algebra of $\mathrm{pl}(1,1)$.

This Hopf algebra U is noetherian, affine, PI and prime but it is not finitely generated as a module over any of its normal commutative Hopf subalgebras [35, Theorem 3.2], hence U is not commutative-by-finite.

Note that U contains a nonzero nilpotent element, namely u, forming part of a PBW basis of U. Thus, although U has GK-dimension 2, it is not a domain, so it does not feature in the list in §2.2.6. Moreover, U is a free $k\langle u\rangle$-module, so that

$$\mathrm{prdim}_U(k) \geq \mathrm{prdim}_{k\langle u\rangle}(k) = \infty,$$

hence $\mathrm{gldim}\,(U) = \infty$ by [56, § 2.4].

Chapter 3

Their structure

In this chapter, we delve deeper into structural properties of affine commutative-by-finite Hopf algebras. Let H be an affine commutative-by-finite Hopf algebra with commutative normal Hopf subalgebra A. We continue to assume in this chapter that the base field k is algebraically closed.

In section 3.1 we study the effect of the left and right $\overline{H}$-actions (introduced in Theorem 2.1.3(3)) on the ideals of A; and we will mostly focus on its maximal ideals. Moreover, we introduce the important notion of *orbital semisimplicity*, which will play a crucial role when studying the duals of commutative-by-finite Hopf algebras, more specifically in section 4.3. Note that it has some implications in section 3.2 as well.

In section 3.2 we study the nilradicals of A and H and obtain a surprising result on the relation between the (semi)primeness of H and A. More specifically, we proved:

Theorem 3.0.1. *Let H be an affine commutative-by-finite Hopf algebra, with commutative normal Hopf subalgebra A. Assume that $N(A)$ is $\overline{H}$-stable.*

1. *If H is semiprime, then so is A.*

2. *If H is prime, then A is a domain.*

In section 3.3 we study the representation theory of affine commutative-by-finite Hopf algebras. In particular, we find upper bounds for the dimension of the irreducible H-modules as follows:

Theorem 3.0.2. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Suppose that A is a domain. Then, for any simple H-module V,*

$$\dim_k(V) \leq \dim_k(\overline{H}).$$

In the last section 3.4, we study the structure of commutative-by-finite Hopf algebras whose Hopf quotient $\overline{H}$ is semisimple and cosemisimple. This is achieved in Theorem 3.4.3. Due to the extensive and intricate structure presented in this result, we highlight here its implications on prime commutative-by-(co)semisimple Hopf algebras. According to this result, these Hopf algebras are in simple terms extensions of affine commutative domains by group algebras.

Corollary 3.0.3. *Let H be a prime commutative-by-finite Hopf algebra, finite over the affine normal commutative Hopf subalgebra A. Suppose that $\overline{H}$ is semisimple and cosemisimple. Then, there exists a left coideal subalgebra D of H and a finite group Γ such that*

1. *A and D are affine commutative domains with $A \subseteq D$ and Γ acts faithfully on D via the adjoint action;*
2. *$H/D^+H \cong k\Gamma$ and the order of Γ is a unit in k;*
3. *Suppose in addition that H is pointed. Then, $H \cong D\#_\sigma k\Gamma$ for some cocycle σ.*

In subsections 3.1.1 and 3.1.2 we mostly use results discovered by other mathematicians such as Montgomery, Schneider [68] and Skryabin [87]. However, throughout the rest of the chapter most of the results are original, due to the author and his supervisor, and have recently been written into a paper [11].

3.1 Stability under Hopf actions

In this section, we study the effect of the left and right $\overline{H}$-actions on the ideals of A, especially on its maximal ideals. In subsections 3.1.1 and 3.1.2 we work in the broader setting of a finite dimensional Hopf algebra T acting on a (often commutative) T-module algebra A. We return to the study of the class of commutative-by-finite Hopf algebras in subsection 3.1.3, where we apply the results of the previous subsections to the case where T is the finite dimensional Hopf algebra $\overline{H}$ acting on the commutative normal Hopf subalgebra A of H.

3.1.1 Hopf orbits of maximal ideals

The action of finite dimensional Hopf algebras on algebras has been previously studied by many mathematicians, including for example Montgomery, Schneider [68] and Skryabin [87]. In fact, these mathematicians have studied the stability of the ideals of an algebra under a Hopf action, this being precisely what we will do in this section.

The definition of stability under a Hopf action varies throughout the literature, for example it differs in [68, Definition 2.3(2)] from the one in [87]. Therefore, we begin by defining the notion of stability in the way most suitable for our purposes.

Definition 3.1.1. Let T be a Hopf algebra and A a left T-module algebra.

1. A subspace V of A is T-*stable* if $t \cdot v \in V$ for all $t \in T$ and $v \in V$.
2. The T-*core* of an ideal I of A is the subspace

$$I^{(T)} \;=\; \{x \in I : t \cdot x \in I,\, \forall t \in T\}.$$

Note that, recalling the notation of invariants from §1.1.1, we have the inclusion

$$A^T \cap I \subseteq I^{(T)},$$

since for any $x \in A^T \cap I$ we have $t \cdot x = \epsilon(t)x \in I$ for any $t \in T$. However, in general, $I^{(T)}$ will be strictly larger than the subspace $A^T \cap I$ of T-invariants of I, hence the use of brackets in our notation.

Lemma 3.1.2. *Let T be a Hopf algebra, A a left T-module algebra and I an ideal of A. Then,*

1. *$I^{(T)}$ is the largest T-stable subspace of A contained in I.*
2. *$I^{(T)}$ is an ideal of A.*

Proof. (1) We first prove that $I^{(T)}$ is T-stable. Let $x \in I^{(T)}$ and $t \in T$. Then, for any $s \in T$

$$s \cdot (t \cdot x) = (st) \cdot x \in I,$$

that is, $t \cdot x \in I^{(T)}$. Now take any T-stable subspace $J \subseteq I$ of A. Then, for any $t \in T$ and $x \in J$ we have $t \cdot x \in J \subseteq I$, hence $x \in I^{(T)}$ by definition.

(2) Clearly $I^{(T)}$ is closed under addition. Let $a \in A, x \in I^{(T)}, t \in T$. Then,

$$t \cdot (ax) = \sum (t_1 \cdot a)(t_2 \cdot x) \in I,$$

since $t_2 \cdot x \in I$ and I is an ideal of A. Thus, $ax \in I^{(T)}$ and similarly so does xa. This proves $I^{(T)}$ is an ideal of A. □

Let A be a commutative left T-module algebra. The notion of the T-core of an ideal of A leads to an equivalence relation on the prime spectrum of A and to the notion of orbits under this action, as discussed in a more general setting by Skryabin in [87] and in a slightly different setting by Montgomery and Schneider in [68]. However, in view of the applications of this action in this thesis, we limit our attention to the spectrum of the *maximal* ideals of a commutative T-module algebra A.

Definition 3.1.3. Let T be a Hopf algebra and A a commutative left T-module algebra.

1. We define an equivalence relation $\sim^{(T)}$ on $\mathrm{Maxspec}(A)$ as follows: for $\mathbf{m}, \mathbf{m}' \in \mathrm{Maxspec}(A)$, $\mathbf{m} \sim^{(T)} \mathbf{m}'$ if and only if $\mathbf{m}^{(T)} = \mathbf{m}'^{(T)}$.
2. For each $\mathbf{m} \in \mathrm{Maxspec}(A)$, its T-*orbit* is the $\sim^{(T)}$-equivalence class of $\mathbf{m}$, that is the set of maximal ideals of A with the same T-core as $\mathbf{m}$,

$$\mathcal{O}_{\mathbf{m}} = \{\mathbf{m}' \in \mathrm{Maxspec}(A) : \mathbf{m}'^{(T)} = \mathbf{m}^{(T)}\}.$$

The following key result is essentially due to Skryabin [87, Theorems 1.1, 1.3]. It states a few properties of this equivalence relation and its orbits; in particular, it gives a very important way of describing these orbits.

Proposition 3.1.4 (Skryabin, [87]). *Let T be a finite dimensional Hopf algebra, A an affine commutative left T-module algebra and $\mathbf{m} \in \mathrm{Maxspec}(A)$.*

1. *$A/\mathbf{m}^{(T)}$ is a T-simple algebra, that is, its only T-stable ideals are $\{0\}$ and $A/\mathbf{m}^{(T)}$.*
2. *$A/\mathbf{m}^{(T)}$ is a (finite dimensional) Frobenius algebra.*
3. *$\mathcal{O}_{\mathbf{m}} = \{\mathbf{m}' \in \mathrm{Maxspec}(A) : \mathbf{m}^{(T)} \subseteq \mathbf{m}'\}$.*
4. *$\mathcal{O}_{\mathbf{m}}$ is finite.*

Proof. (1),(2) Since k is algebraically closed and A is commutative affine, we have $A/\mathbf{m} \cong k$. Then, $\mathbf{m} \cap A^T$ is a maximal ideal of A^T and $A^T/(\mathbf{m} \cap A^T) \cong k$. Thus, by Theorem 2.1.6 $A/(\mathbf{m} \cap A^T)A$ is a finite dimensional algebra, and therefore so is its factor algebra $A/\mathbf{m}^{(T)}$.

Now (1) follows from [87, Proposition 3.5]. In particular, in the language of [87], $A/\mathbf{m}^{(T)}$ is T-semiprime, meaning it has no nontrivial nilpotent T-stable ideals. Thus, by [87, Theorem 1.3] the classical ring of quotients $Q(A/\mathbf{m}^{(T)})$ is quasi-Frobenius. But by the previous paragraph $A/\mathbf{m}^{(T)}$ is finite dimensional, hence artinian, so it equals its classical ring of quotients, $A/\mathbf{m}^{(T)} = Q(A/\mathbf{m}^{(T)})$. And finally a commutative quasi-Frobenius k-algebra is actually Frobenius [101, Remark 1.3].

(3) If $\mathbf{m}' \in \mathcal{O}_{\mathbf{m}}$, $\mathbf{m}^{(T)} = \mathbf{m}'^{(T)} \subseteq \mathbf{m}'$ and so $\mathcal{O}_{\mathbf{m}} \subseteq \{\mathbf{m}' \in \mathrm{Maxspec}(A) : \mathbf{m}^{(T)} \subseteq \mathbf{m}'\}$. For the reverse inclusion, suppose some $\mathbf{m}' \in \mathrm{Maxspec}(A)$ contains $\mathbf{m}^{(T)}$, so that $\mathbf{m}^{(T)} \subseteq \mathbf{m}'^{(T)} \subseteq \mathbf{m}'$. Hence, by (1) $A/\mathbf{m}^{(T)}$ is T-simple and so $\mathbf{m}'^{(T)}/\mathbf{m}^{(T)}$ must be $\{0\}$, thus $\mathbf{m}'^{(T)} = \mathbf{m}^{(T)}$; that is, $\mathbf{m}' \in \mathcal{O}_{\mathbf{m}}$.

(4) By (3) there exists a bijection $\mathcal{O}_{\mathbf{m}} \cong \mathrm{Maxspec}(A/\mathbf{m}^{(T)})$. The statement now follows from (2) and the fact that an artinian ring has finitely many maximal ideals. □

3.1.2 Orbital semisimplicity

When the Hopf algebra T acting on a commutative T-module algebra A is a group algebra kG of a finite group G, the setting is the familiar one of classical invariant theory. In particular, G acts by k-algebra automorphisms on A, hence the orbit of a maximal ideal $\mathbf{m} \in \mathrm{Maxspec}(A)$ is the set of ideals of the form $\mathbf{m}^g := \{g \cdot a : a \in \mathbf{m}\}$ for some $g \in G$, and the kG-core of $\mathbf{m}$ is

$$\mathbf{m}^{(kG)} = \bigcap_{g \in G} \mathbf{m}^g,$$

so that $A/\mathbf{m}^{(kG)}$ is a finite direct sum of copies of k.

We extend this property to any Hopf algebra T. This is a very important notion which will play a part in sections 3.2 and 4.3.

Definition 3.1.5. Let T be a finite dimensional Hopf algebra and A an affine commutative left T-module algebra. Then, A is T-*orbitally semisimple* if $A/\mathbf{m}^{(T)}$ is semisimple for every $\mathbf{m} \in \mathrm{Maxspec}(A)$.

The prefix T will be omitted when this is clear from the context.

In view of Proposition 3.1.4(3), A is T-orbitally semisimple if and only if for every $\mathbf{m} \in \operatorname{Maxspec}(A)$

$$\mathbf{m}^{(T)} = \bigcap_{\mathbf{m}' \in \mathcal{O}_{\mathbf{m}}} \mathbf{m}'. \tag{3.1}$$

The following result is due to Montgomery and Schneider, [68, Theorem 3.7, Corollary 3.9, Lemma 2.5] and generalizes an earlier result of Chin [24, Lemma 2.2]. It states that when T is pointed the T-orbits of maximal ideals of A are determined by the grouplike elements of T.

Proposition 3.1.6 (Montgomery-Schneider, [68])**.** *Let T be a finite dimensional Hopf algebra and A an affine commutative T-module algebra. Let $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_m = T$ be the coradical filtration of T, and let $\langle T_0 \rangle$ be the Hopf subalgebra of T generated by T_0. Then,*

1. *For an ideal I of A, $\left(I^{(\langle T_0 \rangle)}\right)^{m+1} \subseteq I^{(T)}$.*
2. *For all $\mathbf{m}, \mathbf{m}' \in \operatorname{Maxspec}(A)$, $\mathbf{m}^{(T)} = \mathbf{m}'^{(T)}$ if and only if $\mathbf{m}^{(\langle T_0 \rangle)} = \mathbf{m}'^{(\langle T_0 \rangle)}$.*
3. *Suppose T is pointed. For any $\mathbf{m}, \mathbf{m}' \in \operatorname{Maxspec}(A)$, $\mathbf{m}^{(T)} = \mathbf{m}'^{(T)}$ if and only if $\mathbf{m}' = g \cdot \mathbf{m}$ for some grouplike element g of T.*

Even though this might at first glance suggest that orbital semisimplicity holds when T is pointed, the following example shows that orbital semisimplicity does not always hold (even when T is pointed).

Example 3.1.7. Consider the n^2-dimensional Taft algebra defined in Example 1.1.4

$$T = T_f(n, 1, q) := k\langle g, x : g^n = 1, x^n = 0, xg = qgx \rangle,$$

with g grouplike and x $(1, g)$-primitive. Let A be the polynomial algebra $k[u, v]$. As shown by Allman, [3, §3], A is a left T-module algebra with the action defined by

$$g \cdot u = u, \qquad g \cdot v = qv, \qquad x \cdot u = 0, \qquad x \cdot v = u.$$

This action is not orbitally semisimple. Indeed, consider the maximal ideals of the form $\mathbf{m}_a := \langle u - a, v \rangle$, for some $a \in k^\times$. First note that, being generated by grouplike and skew-primitive elements, $T = T_f(n, 1, q)$ is pointed by [76, Corollary 5.1.14], with coradical $T_0 = kG(T) = k\langle g \rangle$. By Proposition 3.1.6(3), the orbit of $\mathbf{m}_a$ is determined by the action of the group $G(T) = \langle g \rangle$. Since $g \cdot (u - a) = u - a$ and $g \cdot v = qv$, we have $g \cdot \mathbf{m}_a = \mathbf{m}_a$, hence each maximal ideal $\mathbf{m}_a$ constitutes its own orbit. However, none of these ideals is T-invariant, since $x \cdot v = u \notin \mathbf{m}_a$. Therefore, $\mathbf{m}_a^{(T)}$ must be strictly smaller than $\mathbf{m}_a$ and by (3.1) orbital semisimplicity does not hold.

In fact, we have

$$\mathbf{m}_a^{(T)} = \langle u - a, v^n \rangle. \tag{3.2}$$

Clearly, $g \cdot v^n = v^n$, $g \cdot (u-a) = u-a$ and $x \cdot (u-a) = 0$. By induction, we have $x \cdot v^k = (1+q+\ldots+q^{k-1})uv^{k-1}$ for every $k \geq 1$. In particular, since q is a primitive nth root of 1, $x \cdot v^n = 0$. Therefore, $\langle u-a, v^n \rangle \subseteq \mathbf{m}_a^{(T)}$. For the converse, it suffices to prove that $A/\langle u-a, v^n \rangle$ is T-simple.

First note that $R := A/\langle u-a, v^n \rangle \cong k\langle \bar{v} : \bar{v}^n = 0 \rangle$. Let I be a nonzero T-stable ideal of R and pick a nonzero element $w \in I$, say $w = \lambda_0 + \lambda_1 \bar{v} + \ldots + \lambda_m \bar{v}^m$ with $\lambda_m \neq 0$. By induction, it follows that $x^k \cdot \bar{v}^k = a^k \prod_{i=1}^{k-1}(1+q+\ldots+q^i)$ and $x^j \cdot \bar{v}^k = 0$ for every $j > k$. Thus, $x^m \cdot w = a^m \prod_{i=1}^{m-1}(1+q+\ldots+q^i) \in I$, by T-stability of I, and it is a nonzero scalar. Therefore, $I = R$ and (3.2) follows.

However, we do know orbital semisimplicity holds in a variety of situations. This result depends crucially on work done by other mathematicians, namely Linchenko [54], Skryabin and van Oystaeyen [89]. Recall the notions of cosemisimple and involutory from §1.1.4.

Theorem 3.1.8. *Let T be a finite dimensional Hopf algebra and A an affine commutative T-module algebra. Then, A is T-orbitally semisimple in each of the following cases:*

1. *the action is trivial;*
2. *the action factors through a group;*
3. *T is cosemisimple;*
4. *T is involutory and $\operatorname{char} k = 0$ or $\operatorname{char} k = p > \dim_k(A/\mathbf{m}^{(T)})$ for all $\mathbf{m} \in \operatorname{Maxspec}(A)$.*

Proof. (1) This is trivial: if T acts by ϵ on A, then $\mathbf{m}^{(T)} = \mathbf{m}$ for all maximal ideals $\mathbf{m}$ of A, hence $A/\mathbf{m}^{(T)} \cong k$ is trivially semisimple.

(2) Suppose that the T-action on A factors through the group Γ, that is there exists a Hopf ideal I of T that acts trivially on A and such that $T/I \cong k\Gamma$. Then, for each $\mathbf{m} \in \operatorname{Maxspec}(A)$,

$$\mathbf{m}^{(T)} = \mathbf{m}^{(k\Gamma)} = \bigcap_{\gamma \in \Gamma} \mathbf{m}^{\gamma}$$

and $A/\mathbf{m}^{(T)}$ is clearly semisimple.

(3) Let $\mathbf{m} \in \operatorname{Maxspec}(A)$. By Proposition 3.1.4(1) $A/\mathbf{m}^{(T)}$ is T-simple. Thus, $A/\mathbf{m}^{(T)}$ is semiprime by [89, Theorem 0.5(ii)], hence semisimple [50, Theorem 10.24].

(4) If k has characteristic 0 this is simply a restatement of (3), since in this case T is involutory if and only if it is cosemisimple if and only if it is semisimple, [52, Corollary 2.6, Theorem 3.3] and [53, Theorems 3 and 4]. Suppose that $\operatorname{char} k = p > 0$ and let $\mathbf{m} \in \operatorname{Maxspec}(A)$. If $\dim_k(A/\mathbf{m}^{(T)}) < p$ then its Jacobson radical is T-stable, by [54, Theorem]. By the T-simplicity of $A/\mathbf{m}^{(T)}$, ensured by Proposition 3.1.4(1), this forces $A/\mathbf{m}^{(T)}$ to be semisimple, as required. □

Remark 3.1.9. There is a considerable amount of overlap between cases (2), (3) and (4) in the above result. In characteristic 0, as already noted in the proof, being semisimple, cosemisimple and involutory are equivalent conditions on T. Moreover, when A is a commutative T-module domain and T is cosemisimple, Skryabin showed in [88, Theorem 2] that the action factors through a group. This extends earlier work of Etingof and Walton [33].

3.1.3 $\overline{H}$-stability

The equivalence relation of Definition 3.1.3 was studied by Montgomery and Schneider [68], before Skryabin [87], in the special setting of a faithfully flat T-Galois extension $R \subseteq S$. In fact, the equivalence relation as defined on $\operatorname{Spec}(R)$ in [68, Definition 2.3(2)] is different from the one given above, in that they define an ideal I of R to be T-stable if $IS = SI$, and then use this to define their equivalence relation.

In the following lemma we discuss the relation between these two notions of stability.

Lemma 3.1.10. *Let H be a Hopf algebra with bijective antipode and A a normal Hopf subalgebra of H. Write $\overline{H} := H/A^+H$. Let I be an ideal of A.*

1. *If $(ad_l h)(x) \in I$ for all $h \in H, x \in I$, then $HI \subseteq IH$, and IH is an ideal of H.*
2. *If $(ad_r h)(x) \in I$ for all $h \in H, x \in I$, then $IH \subseteq HI$, and HI is an ideal of H.*

If the extension $A \subseteq H$ is faithfully flat $\overline{H}$-Galois, the following are equivalent:

(i) $(ad_l h)(x) \in I$, *for all* $h \in H, x \in I$.

(ii) $(ad_r h)(x) \in I$, *for all* $h \in H, x \in I$.

(iii) $HI = IH$.

Proof. (1) If I is invariant under left adjoint action, then for all $x \in I, h \in H$ we have

$$hx = \sum h_1 x S(h_2) h_3 = \sum (ad_l h_1)(x) h_2 \in IH,$$

proving $HI \subseteq IH$. It follows that IH is a (2-sided) ideal of H. (2) is proved similarly.

In addition, assume that $A \subseteq H$ is faithfully flat $\overline{H}$-Galois.

$(i) \Leftrightarrow (iii)$ If I is invariant under left adjoint action, then $HI \subseteq IH$ by (1). We now prove this actually implies the equality $HI = IH$, as in [68, Remark 1.2(ii)]. Since the extension $A \subseteq H$ is faithfully flat $\overline{H}$-Galois, we have a bijective correspondence between the categories of left A-modules ${}_A\mathcal{M}$ and $(H, \overline{H})$-Hopf modules ${}_H\mathcal{M}^{\overline{H}}$ [68, §1, p.191] as follows:

$$\begin{array}{rcl} {}_A\mathcal{M} & \longleftrightarrow & {}_H\mathcal{M}^{\overline{H}} \\ M & \longmapsto & H \otimes_A M \cong HM \\ N^{\mathrm{co}\,\overline{H}} & \longleftarrow & N \end{array}.$$

Since $HI \subseteq IH$, IH is a left H-module. Therefore,

$$\begin{array}{rcll} IH & = & H(IH)^{\mathrm{co}\,\overline{H}} & \text{by the previous bijective correspondence,} \\ & \subseteq & H(IH \cap A) & \text{since } H^{\mathrm{co}\,\overline{H}} = A, \\ & = & HI & \text{since } A \subseteq H \text{ is faithfully flat.} \end{array}$$

Conversely, suppose that $IH = HI$, and let $a \in I$ and $h \in H$. Then,

$$(ad_l h)(a) = \sum h_1 a S(h_2) \in IH \cap A,$$

since A is normal. But the extension $A \subseteq H$ being faithfully flat gives $IH \cap A = I$, that is, I is invariant under left adjoint action.

The equivalence $(ii) \Leftrightarrow (iii)$ is proved analogously. $\square$

Returning to our primary focus, H once again denotes an affine commutative-by-finite Hopf k-algebra and A a commutative normal Hopf subalgebra over which H is a finitely generated module. Let $\overline{H}$ denote the finite dimensional Hopf factor H/A^+H. Recall from Theorem 2.1.3(3) that the left and right adjoint actions of H on A factor over the ideal A^+H, so that A is a left and right $\overline{H}$-module algebra.

Using the terminology introduced at Definition 3.1.1(1), we'll refer to an ideal I of A as *left $\overline{H}$-stable* if it is invariant under left adjoint action, that is $(ad_l h)(x) \in I$ for all $x \in I, h \in \overline{H}$. The notion of an *right* $\overline{H}$-stable ideal is defined analogously.

Notice that, when A is semiprime (as is the case for all examples in section 2.2), $A \subseteq H$ is a faithfully flat $\overline{H}$-Galois extension by Theorem 2.1.9(1), hence left and right stable are equivalent notions by Lemma 3.1.10 and we refer to I simply as $\overline{H}$-stable. The non-semiprime case will be dealt with in section 3.2.

As in Definition 3.1.1(2), the *left* $\overline{H}$-core of an ideal I of A is

$$^{(\overline{H})}I = \{x \in I : (ad_l h)(x) \in I, \forall h \in H\},$$

which is the largest left $\overline{H}$-stable ideal of A contained in I. Right $\overline{H}$-cores are denoted by $I^{(\overline{H})}$ and defined similarly. The equivalence relation $\sim^{(\overline{H})}$ on $\mathrm{Maxspec}(A)$ and its associated orbits are defined just as in subsection 3.1.1.

Proposition 3.1.4 presents a nice description of these orbits and states that, for each maximal ideal $\mathbf{m}$ of A, the commutative algebras $A/\mathbf{m}^{(\overline{H})}$ are finite dimensional and Frobenius. In Theorem 3.3.1 we shall obtain an upper bound for the dimensions of these algebras, and hence for the cardinalities of the orbits $\mathcal{O}_{\mathbf{m}}$.

Example 3.1.11. Consider the group algebra $H = kD$ of the infinite dihedral group from §2.2.5. Its commutative normal Hopf subalgebra is $A = k\langle b \rangle$ and the Hopf quotient is $\overline{H} = kC_2$. Let $\mathbf{m}_\lambda := (b - \lambda) \in \mathrm{Maxspec}(A)$ for some $\lambda \in k^\times$. Since $(ad_l a)(b - \lambda) = a(b - \lambda)a^{-1} = b^{-1} - \lambda = -\lambda b^{-1}(b - \lambda^{-1})$, we have $a \cdot \mathbf{m}_\lambda = \mathbf{m}_{\lambda^{-1}}$ and the equivalence relation $\sim^{(\overline{H})}$ is determined by $\lambda \sim \lambda^{-1}$. In particular, $\mathcal{O}_{\mathbf{m}_\lambda} = \{\mathbf{m}_\lambda, \mathbf{m}_{\lambda^{-1}}\}$

for $\lambda \neq \pm 1$, and $\mathbf{m}_1 = A^+$ and $\mathbf{m}_{-1}$ each constitute their own orbit. Since $\overline{H} = kC_2$ is a group algebra, the $\overline{H}$-cores are $\mathbf{m}_\lambda^{(\overline{H})} = \mathbf{m}_\lambda \cap \mathbf{m}_{\lambda^{-1}}$ for $\lambda \neq \pm 1$, and $\mathbf{m}_1^{(\overline{H})} = \mathbf{m}_1$ and $\mathbf{m}_{-1}^{(\overline{H})} = \mathbf{m}_{-1}$.

In view of Definition 3.1.5, we say A is *left* orbitally semisimple if $A/{}^{(\overline{H})}\mathbf{m}$ is semisimple for every $\mathbf{m} \in \text{Maxspec}(A)$. One similarly obtains a notion of *right* orbital semisimplicity. This left and right distinction when dealing with orbital semisimplicity becomes obsolete by the following lemma.

Lemma 3.1.12. *Let H be an affine commutative-by-finite Hopf algebra, with commutative normal Hopf subalgebra A.*

1. *Let V be a subspace of A. Then,*
$$S\left({}^{(\overline{H})}V\right) = S(V)^{(\overline{H})}.$$

2. *A is left orbitally semisimple if and only if it is right orbitally semisimple.*

3. *Let I be an ideal of A such that $S(I) = I$. Then, I is left $\overline{H}$-stable if and only if it is right $\overline{H}$-stable.*

Proof. (1) This is a generalization of the argument in Remark 2.1.5(2). Let $v \in V$ and $h \in H$. Then,

$$\begin{aligned} ad_r(h)(Sv) &= \sum S(h_1)S(v)h_2 = S\left(\sum S^{-1}(h_2)vh_1\right) \\ &= S\left(\sum S^{-1}(h)_1 v S(S^{-1}(h)_2)\right) = S(ad_l(S^{-1}h)(v)). \end{aligned}$$

Thus, if $v \in {}^{(\overline{H})}V$, $ad_r(h)(Sv) = S(ad_l(S^{-1}h)(v)) \in S(V)$ for all $h \in H$, that is, $Sv \in S(V)^{(\overline{H})}$; and, conversely, if $S(v) \in S(V)^{(\overline{H})}$ for some $v \in V$, then bijectivity of S yields $ad_l(S^{-1}h)(v) \in S^{-1}(S(V)) = V$ for all $h \in H$, hence $v \in {}^{(\overline{H})}V$.

(2) By (1), S induces an isomorphism between the commutative algebras $A/{}^{(\overline{H})}\mathbf{m}$ and $A/S(\mathbf{m})^{(\overline{H})}$. Considering S acts as a permutation on Maxspec(A), the algebras $A/{}^{(\overline{H})}\mathbf{m}$ are semisimple if and only if the algebras $A/\mathbf{m}^{(\overline{H})}$ are too.

(3) Suppose $S(I) = I$. If I is left $\overline{H}$-stable, then $I^{(\overline{H})} = S(I)^{(\overline{H})} = S({}^{(\overline{H})}I) = S(I) = I$; and conversely, if I is right $\overline{H}$-stable, ${}^{(\overline{H})}I = S^{-1}(S(I)^{(\overline{H})}) = S^{-1}(I^{(\overline{H})}) = S^{-1}(I) = I$. $\square$

Given part (2) of the previous lemma, the adjectives left and right will from now on be omitted from orbital semisimplicity. This notion will play an important role in sections 3.2 and 4.3.

In view of Example 3.1.7 it seems probable that not all affine commutative-by-finite Hopf algebras are orbitally semisimple. However, we know of no such example at present, as the following result shows.

Proposition 3.1.13. *All the examples of affine commutative-by-finite Hopf algebras described in §2.2 satisfy orbital semisimplicity.*

Proof. Commutative Hopf algebras, such as (I) in 2.2.5 and (II) in 2.2.6, are trivially orbitally semisimple: for these the Hopf subalgebra A equals H, hence $\mathbf{m}^{(\overline{H})} = \mathbf{m}$ for any maximal ideal $\mathbf{m}$ of H, and $H/\mathbf{m} \cong k$ is trivially semisimple.

The Hopf algebras in §2.2.1, §2.2.2, §2.2.3 and the family (IV) in §2.2.5 are orbitally semisimple by Theorem 3.1.8(1), since their corresponding normal commutative Hopf subalgebra A is central.

The group algebras in §2.2.4 (including examples (II) in §2.2.5 and (I) in §2.2.6) are special cases of Theorem 3.1.8(2), since their corresponding Hopf quotients $\overline{H}$ are actually group algebras.

Consider now the family (III) of Taft algebras $T(n,t,q)$ in §2.2.5. There we showed that the left adjoint action of $H = T(n,t,q)$ on $A = k[x^{n'}]$ was determined by $g \cdot x^{n'} = q^{-n'}x^{n'}$ and $x \cdot x^{n'} = 0$. We know that this action reduces to an $\overline{H}$-action on A. But since x acts trivially on A, $\bar{x} \in \overline{H}$ also acts trivially on A, hence this action reduces to an action by

$$\overline{H}/\bar{x}\overline{H} \cong k\langle \bar{g} : \bar{g}^n = 1\rangle = kC_n,$$

a group action. By Theorem 3.1.8(2) this example exhibits orbital semisimplicity.

Recall the family (V) $D(m,d,q)$ in §2.2.5. Its corresponding commutative normal Hopf subalgebra $A = k[x^{\pm 1}]$ is central in $k\langle x,g,y\rangle \subseteq D(m,d,q)$, so x,g,y act trivially on A, and $u_i \cdot x = x^{-1}\delta_{i,0}$. Therefore, the H-action on A reduces to an action by

$$H/((x-1)H + (g-1)H + yH) \cong \overline{H}/((\bar{g}-1)\overline{H} + \bar{y}\overline{H}) \cong k\langle \bar{u_0}\rangle = kC_2$$

by Lemma 2.2.3(3). Thus, $D(m,d,q)$ is orbitally semisimple by Theorem 3.1.8(2).

Consider the family (III) $A(l,n,q)$ in §2.2.6. As we calculated before, the action of $H = A(l,n,q)$ on its commutative Hopf subalgebra $A = k[(x^l)^{\pm 1}, y^{l'}]$ is given by $x \cdot x^l = x^l$, $x \cdot y^{l'} = q^{l'}y^{l'}$ and $y \cdot A = 0$. So the $\overline{H}$-action on A reduces to an action by

$$\overline{H}/\bar{y}\overline{H} \cong k\langle \bar{x} : \bar{x}^l = 1\rangle = kC_l.$$

Theorem 3.1.8(2) now gives orbital semisimplicity.

And lastly, for the family (IV) $B(n,p_0,\dots,p_s,q)$ in §2.2.6, each y_j acts trivially on A, hence the action of $\overline{H}$ on A reduces to an action by

$$\overline{H}/(\bar{y_1},\dots,\bar{y_s}) \cong k\langle \bar{x} : \bar{x}^l = 1\rangle = kC_l.$$

Thus, orbital semisimplicity follows from Theorem 3.1.8(2). □

3.2 The nilradical and primeness

In this section we study the primeness and semiprimeness of commutative-by-finite Hopf algebras, starting with the classical commutative case in §3.2.1 and moving on to the commutative-by-finite case in §3.2.2.

3.2.1 The commutative case

The following result lists a few properties of commutative Hopf algebras regarding their minimal prime ideals and nilradical. Recall Theorem 1.1.7 which states that commutative Hopf algebras are reduced in characteristic 0, hence any statements about the nilradical of commutative Hopf algebras are actually only relevant in positive characteristic.

Lemma 3.2.1. *Let A be an affine commutative Hopf algebra with* $\mathrm{Kdim}\,(A) = d$*. Then,*

1. *The nilradical $N(A)$ of A is a Hopf ideal.*
2. *The left coideal subalgebra $C := A^{\mathrm{co}\,A/N(A)}$ of A has the following properties:*
   (i) C is a (finite dimensional) Frobenius subalgebra of A with $C^+A \subseteq N(A)$.
   (ii) C is local, that is it has a unique maximal ideal C^+.
   (iii) A is a free C-module.
3. *$A/N(A)$ has finite global dimension d, which also equals its Krull and GK-dimension. Thus, it is a finite direct sum of commutative affine domains,*
$$A/N(A) = \bigoplus_{i=1}^{m} A_i,$$
each of Krull, Gelfand-Kirillov and global dimensions d.
4. *There is a unique minimal prime ideal P of A contained in A^+.*
5. *$P = \bigcap_n (A^+)^n + N(A)$ is a Hopf ideal.*
6. *There exists an idempotent $f \in A$ such that $\bigcap_n (A^+)^n = fA$.*

Proof. Let A be an affine commutative Hopf algebra. First, an affine commutative algebra has finite Krull dimension and it equals its GK-dimension; see [78, Theorem 6.10, Corollary 6.33] and [49, Theorem 4.5(a)].
(1) Since every prime ideal of A is an intersection of maximal ideals [78, Proposition 6.37], the nilradical and Jacobson radical coincide, that is,

$$N(A) = \bigcap_{\mathbf{m}\in \mathrm{Maxspec}(A)} \mathbf{m} = \bigcap_{\text{irreducible } A\text{-module } V} \mathrm{Ann}(V). \tag{3.3}$$

Since k is algebraically closed, any irreducible A-module is one-dimensional. Thus, if V and W are irreducible A-modules, then $V \otimes W$, being one-dimensional, is also irreducible. But A acts on $V \otimes W$ through Δ, hence by (3.3)

$$\Delta(N(A))(V \otimes W) = N(A) \cdot (V \otimes W) = 0.$$

Since $V \cong A/\mathbf{m}$ and $W \cong A/\mathbf{n}$ for some maximal ideals $\mathbf{m}$ and $\mathbf{n}$, we have $\Delta(N(A)) \subseteq \mathrm{Ann}_{A\otimes A}(V \otimes W) = \mathbf{m} \otimes A + A \otimes \mathbf{n}$ by [92, Lemma 10.2]. This holds for any two irreducible A-modules, hence

$$\begin{aligned} \Delta(N(A)) &\subseteq \bigcap_{\mathbf{m},\mathbf{n}\in \mathrm{Maxspec}(A)} (\mathbf{m} \otimes A + A \otimes \mathbf{n}) \\ &= \left(\bigcap_{\mathbf{m}\in \mathrm{Maxspec}(A)} \mathbf{m}\right) \otimes A + A \otimes \left(\bigcap_{\mathbf{n}\in \mathrm{Maxspec}(A)} \mathbf{n}\right) \\ &= N(A) \otimes A + A \otimes N(A). \end{aligned}$$

Since $N(A)$ is nilpotent, clearly $\epsilon(N(A)) = 0$. And S is an automorphism of A, so it permutes maximal ideals of A and $S(N(A)) = N(A)$.

(2) Let $C = A^{\mathrm{co}\, A/N(A)}$. By Example 1.1.10 and in particular equation (1.1), C is a left coideal subalgebra of A with $C^+A \subseteq N(A)$. In particular, C^+ is a nilpotent ideal of C and, since maximal ideals are prime, C^+ must be contained in any maximal ideal of C, and so C^+ is the unique maximal ideal of C, that is C is local.

A commutative Hopf algebra is flat over its coideal subalgebras by [63, Theorem 3.4], hence A is a flat C-module. By [99, §13.2, Theorem 2], A_C is faithfully flat if and only if $IA \neq A$ for every maximal ideal I of C; and clearly $C^+A \neq A$, thus A is a faithfully flat C-module.

A commutative Hopf algebra is faithfully flat over a left (or right) coideal subalgebra if and only if it is projective [63, Corollary 3.5], hence A is a projective C-module. And any projective module over a local ring is free [77, Theorem 4.44 and subsequent comments], thus A is a free C-module.

Moreover, since A is faithfully flat over C, any strictly ascending chain of ideals of C induces a strictly ascending chain of ideals of A, thus A being noetherian implies C is too. In particular, C^+ is a finitely generated nilpotent maximal ideal. Taking n such that $(C^+)^n = 0$, C decomposes as a vector space into

$$C/C^+ \oplus C^+/(C^+)^2 \oplus \ldots \oplus (C^+)^{n-1}$$

and, since C^+ is finitely generated, each vector space $(C^+)^i/(C^+)^{i+1}$ is finite dimensional, hence so is C.

At last, we prove that C is Frobenius. Note that A, being an affine commutative Hopf algebra, is Gorenstein [10, Proposition 2.3, Step 1]. Let $0 \to A \to E_0 \to \ldots \to E_n \to 0$ be an injective resolution of A. We claim that each E_i is an injective C-module.

Let I be an ideal of C and $f : I \to E_i$ a C-homomorphism. Since A is C-free, f extends uniquely to an A-linear map $f' : AI \to E_i$. Since E_i is an injective A-module, f' in turn extends to an A-linear map $\widehat{f} : A \to E_i$ by Baer's criterion [77, Theorem 3.20]. Thus, $\widehat{f}\Big|_C : C \to E_i$ extends f and again Baer's criterion yields that E_i is an injective C-module.

Therefore, the C-module A has finite injective dimension. Since A is C-free, we have $A \cong \bigoplus_i C_i$ where $C_i \cong C$ as C-modules. Since $C_i \cong C$ is a direct summand of A, it also has finite injective dimension [64, 7.1.7], hence C is Gorenstein. But (local) commutative Gorenstein finite dimensional rings are self-injective by [31, Proposition 21.5], thus quasi-Frobenius. Finally, commutative quasi-Frobenius algebras are Frobenius [101, Remark 1.3].

(3),(4) In view of (1), $A' := A/N(A)$ is a semiprime Hopf algebra. Hence, A' has finite global dimension by [99, §11.6, §11.7]. By (the commutative case of) Theorem 2.1.16(1) and [49, Theorem 4.5], $\operatorname{gldim}(A') = \operatorname{GKdim}(A') = \operatorname{Kdim}(A')$. And, given the bijective correspondence between the prime ideals of A and A', $\operatorname{Kdim}(A') = \operatorname{Kdim}(A) = \operatorname{GKdim}(A) = d$. Let us prove that A' is a finite direct sum of domains.

First A contains finitely many minimal prime ideals by [37, Theorem 3.4], say $P_1, \dots, P_m$, and their intersection equals $N(A)$ [37, Proposition 3.10]; in particular, the intersection of their images in A' is zero. Since A' is regular, so is the localization $A'_{\mathbf{m}}$ at every maximal ideal $\mathbf{m}$ of A' [77, Theorem 9.52], hence $A'_{\mathbf{m}}$ is an integral domain by [31, Corollary 10.14]. In particular, $A'_{\mathbf{m}}$ contains a unique minimal prime, namely $\{0\}$, thus every maximal ideal of A contains a unique minimal prime ideal.

Therefore, $P_1, \dots, P_m$ are pairwise comaximal and there exists a unique minimal prime ideal, say $P := P_1$, contained in the augmentation ideal A^+. Moreover, by comaximality the Chinese Remainder theorem yields

$$A/N(A) \cong \bigoplus_{i=1}^{m} A/P_i \tag{3.4}$$

as algebras, where each $A_i := A/P_i$ is an integral domain; see [47, Theorem 168]. By (the commutative case of) Theorem 2.1.16(3) and [49, Theorem 4.5], each A_i has Krull, Gelfand-Kirillov and global dimension equal to $\operatorname{GKdim}(A') = d$.

(5) Since A/P is a commutative affine domain, Krull's Intersection Theorem [31, Corollary 5.4] yields

$$\bigcap_n (A^+)^n \subseteq P.$$

Conversely, given decomposition (3.4), there exist orthogonal idempotents $\overline{e_1}, \dots, \overline{e_m} \in A'$ such that $1_{A'} = \overline{e_1} + \dots + \overline{e_m}$ and each direct summand $A_i = \overline{e_i}A'$ is idempotently generated. In particular, $P/N(A) = \bigoplus_{i=2}^m A_i = \overline{e}A'$ is idempotently generated by $\overline{e} = \overline{e_2} + \dots + \overline{e_m}$. Thus, $P/N(A) = \overline{e}A' = \overline{e}^n A' = P^n/N(A) \subseteq [(A^+)^n + N(A)]/N(A)$

for all $n \geq 1$, hence

$$P \subseteq \bigcap_n (A^+)^n + N(A).$$

Combining this with the previous inclusion yields the desired equality. But $\bigcap_n (A^+)^n$ and $N(A)$ are Hopf ideals by Lemma 1.1.15 and (1), then so is P.

(6) As above, the minimal prime ideal $P/N(A)$ of $A/N(A)$ is generated by an idempotent element $\overline{e} = e + N(A)$, that is $P/N(A) = eA/N(A)$. By [70, Lemma 2.3.7], there exists an idempotent f of A such that $f + N(A) = e + N(A)$. This is usually known as lifting idempotents. Thus, in A we have $P = fA + N(A)$.

Since $N(A)$ is nilpotent and f is idempotent, $P^k = fA$ for some $k \in \mathbb{N}$, hence $\bigcap_n P^n = fA$. In particular, fA is a Hopf ideal of A by (5) and Lemma 1.1.15, and it is contained in $\bigcap_n (A^+)^n$. It suffices to prove the reverse inclusion.

Since the minimal prime ideals of A are comaximal, as in the proof of (3),(4), and $fA = \bigcap_n P^n$, the only minimal prime of A containing fA is P. Consider the Hopf algebra $\overline{A} := A/fA$ and its unique minimal prime $\overline{P} := P/fA$. By Krull's intersection theorem [31, Corollary 5.4], there exists some $\overline{x} \in \overline{A}^+$ such that

$$(1 - \overline{x}) \left(\bigcap_n (\overline{A}^+)^n \right) = 0. \tag{3.5}$$

Moreover, the commutative Hopf algebra $\overline{A}$ has an artinian quotient ring (by the commutative case of Theorem 2.1.8(5)). Hence, by Small's theorem [64, Corollary 4.1.4] the regular elements of $\overline{A}$ are the elements that are regular modulo $N(\overline{A})$. But $N(\overline{A}) = \overline{P}$, because $\overline{P}$ is the unique minimal prime of $\overline{A}$, and, since $\overline{A}/\overline{P}$ is a domain, the set of regular elements of $\overline{A}$ is $\overline{A} \setminus \overline{P}$. In particular, $1 - \overline{x} \notin \overline{A}^+$, so it must be regular and (3.5) proves the claim. $\square$

Following the notation of the previous result, in general C is not a Hopf subalgebra of A, as illustrated by the following example [9, Example 2.5.3].

Example 3.2.2. Let k be a field of characteristic $p > 0$ and fix a positive integer n.

Consider the coordinate ring of $G = (k, +) \rtimes k^\times$ as in Example 1.3.11 and recall that $\mathcal{O}(G) = k[x, y^{\pm 1}]$, where x is $(1, y)$-primitive and y is grouplike. Since k has characteristic p, x^{p^n} is $(1, y^{p^n})$-primitive, hence it generates a Hopf ideal of $\mathcal{O}(G)$. Let

$$A = \mathcal{O}(G)/(x^{p^n}) = k[x, y^{\pm 1}]/(x^{p^n}).$$

This affine commutative Hopf algebra has nilradical $N(A) = \overline{x}A$: clearly, $\overline{x}$ is nilpotent so $\overline{x}A \subseteq N(A)$ and equality follows from the fact that $A/\overline{x}A \cong k[\overline{y}^{\pm 1}]$ is semiprime. And the set of coinvariants associated to $A \twoheadrightarrow A/N(A)$,

$$C = A^{\operatorname{co} A/N(A)} = k\langle \overline{x} \rangle,$$

is the subalgebra generated by $\overline{x}$; clearly, any element in this subalgebra is a coinvariant and an easy calculation yields that these are the only coinvariants. And the left coideal subalgebra C is clearly not a Hopf subalgebra of A.

Note that in this example $N(A)$ is a prime ideal of A, because $A/N(A) \cong k[\overline{y}^{\pm 1}]$ is a domain. Hence, $N(A)$ is the unique minimal prime ideal of A; in particular, $P = N(A)$ and it is easy to see that $\bigcap_n (A^+)^n = 0$. $\square$

Remark 3.2.3. An unsatisfactory gap in our knowledge lies on the inclusion $C^+A \subseteq N(A)$. We know of no examples where this inclusion is strict; note that equality even holds for Example 3.2.2. And the equality is known to hold when the coradical of A is cocommutative (in particular, if A is pointed) by [60, Theorem]. But we do not know whether it is an equality in general.

Thus, the following question remains unanswered:

Question 3.2.4. Is there some affine commutative Hopf algebra A such that, in the notation of Lemma 3.2.1, $C^+A \subsetneq N(A)$?

We now introduce the affine algebraic group G associated to A. This definition is quite important and will be used throughout the thesis, especially in section 4.3.

Definition/Lemma 3.2.5. Let A be an affine commutative Hopf algebra. Since $N(A)$ is a Hopf ideal by Lemma 3.2.1(1), $A/N(A)$ is an affine commutative semiprime Hopf algebra, so it is the coordinate ring of an affine algebraic group G by Theorem 1.1.7(2), that is

$$A/N(A) \cong \mathcal{O}(G).$$

Thus G may be identified with the set $\mathrm{Maxspec}(A)$ of maximal ideals of A.

Remarks 3.2.6. Recall the subalgebra C and minimal prime P of A from Lemma 3.2.1 and the notation of the previous definition/lemma.

1. The minimal prime ideal P is just the defining ideal for the connected component G° of the identity for the group G, that is $A/P \cong \mathcal{O}(G^\circ)$; see [1, end of §4.2.3], [43, §7.3], [99, §6.7] for the definition and properties of G°.

2. The affine algebraic group $G = \mathrm{Maxspec}(A)$ also identifies with the set $\mathrm{Alg}(A, k)$ of algebra maps $A \to k$, also known as characters of A. These functionals act on A on the left by $\rightharpoonup$; see (1.6) in section 1.4. The subalgebra C from Lemma 3.2.1(2) is the set of left invariants A^G of A under the left G-action $\rightharpoonup$.

   Proof. For each $g \in G$, let $\mathbf{m}_g$ and χ_g respectively denote the corresponding maximal ideal and character of A. Note that, since $\ker \chi_g = \mathbf{m}_g$ contains $N := N(A)$ for every $g \in G$, each χ_g induces a character $\overline{\chi_g}$ of A/N given by $\overline{\chi_g}(a+N) = \chi_g(a)$ for any $a \in A$.

If $c \in C := A^{\operatorname{co} A/N}$, then for every $g \in G$

$$\chi_g \rightharpoonup c = \sum c_1 \chi_g(c_2) = \sum c_1 \overline{\chi_g}(c_2 + N) = c\overline{\chi_g}(1 + N) = c.$$

Thus, $C \subseteq A^G$.

For the converse, first note that, since A is affine and commutative, the nilradical N is the intersection of maximal ideals of A [78, Proposition 6.37]. We have the following map

$$\phi : A \otimes A/N \to \prod_{g \in G} A$$

given by $a \otimes (b + N) \mapsto (a\overline{\chi_g}(b + N))_{g\in G} = (a\chi_g(b))_{g\in G}$. In fact, this is an embedding: if $\{a_i : i \in I\}$ is a k-basis of A and $\sum_i a_i \otimes (b_i + N) \in \ker \phi$ for some $b_i \in A$, we have $\sum_i a_i \chi_g(b_i) = 0$ for all $g \in G$, then $\chi_g(b_i) = 0$ for every $i \in I$ and $g \in G$, in which case every $b_i \in \bigcap_g \ker \chi_g = \bigcap_g \mathbf{m}_g = N$.

Suppose $c \in A^G$. Then,

$$\phi\left(\sum c_1 \otimes (c_2 + N)\right) = \left(\sum c_1 \chi_g(c_2)\right)_{g\in G} = (\chi_g \rightharpoonup c)_{g\in G} = (c)_{g\in G}.$$

But this is also the image of $c \otimes (1 + N)$, hence injectivity of ϕ implies $\sum c_1 \otimes (c_2 + N) = c \otimes (1 + N)$, meaning that $c \in C$. □

Similarly, the left coinvariants ${}^{\operatorname{co} A/N(A)}A$ correspond to the right invariants of the right G-action $\leftharpoonup$ on A.

We now introduce another important subalgebra B of A.

Lemma 3.2.7. *Let A be an affine commutative Hopf algebra. Recall the ideal P and the left coideal subalgebra C from Lemma 3.2.1 and the notation from Definition 3.2.5. Define $B := A^{\operatorname{co} A/P}$, a left coideal subalgebra of A with $C \subseteq B$, over which A is flat.*

1. *$(B + N(A))/N(A) = \mathcal{O}(G/G^\circ)$, the coordinate ring of the* discrete part of *G;*

2. *$(B+N(A))/N(A)$ is a finite dimensional semisimple Hopf subalgebra of $A/N(A)$ with $P = B^+A + N(A)$;*

3. *$A/N(A)$ is a free $(B + N(A))/N(A)$-module.*

4. *Suppose that either A is reduced or its coradical is cocommutative (for example, A is pointed). Then,*

   (i) B is a (finite dimensional) Frobenius subalgebra of A with $P = B^+A$.

   (ii) B is semilocal, that is it has finitely many maximal ideals.

   (iii) A is a free B-module and B is a free C-module.

Proof. By Example 1.1.10 B is a left coideal subalgebra of A, which contains C because $A \twoheadrightarrow A/P$ factors through $A \twoheadrightarrow A/N(A)$. Flatness of A over B follows from [63, Theorem 3.4].

(1) This follows from Example 1.1.11 as follows:

$$(B + N(A))/N(A) = (A/N(A))^{\mathrm{co}\, A/P} = \mathcal{O}(G)^{\mathrm{co}\, \mathcal{O}(G^\circ)} = \mathcal{O}(G/G^\circ).$$

(2) Since G° has finite index in G, [1, end of §4.2.3] or [43, Proposition 7.3], $(B + N(A))/N(A)$ is a finite dimensional Hopf subalgebra of $A/N(A)$. And a finite dimensional semiprime algebra is semisimple [50, Theorem 10.24]. Since G° is a normal subgroup of G [43, Proposition 7.3], then in the language of [97] $P/N(A)$ is a (co)normal Hopf ideal of $A/N(A)$ [97, §5] with $B^+A + N(A) = P$ by [97, Theorem 4.3].

(3) A commutative Hopf algebra is free over its finite dimensional Hopf subalgebras [74, Section 2, Theorem 1].

(4) If A is reduced, by (2),(3) A is a free module over the finite dimensional subalgebra B and $P = B^+A$. Suppose now the coradical of A is cocommutative. Then, $C^+A = N(A)$ by [60, Theorem] and so $P = B^+A$ by (2). Moreover,

$$N(B) = N(A) \cap B = C^+A \cap B = C^+B,$$

thus, B/C^+B is finite dimensional by (2). Furthermore, C^+B is nilpotent, so let n be such that $(C^+B)^n = (C^+)^n B = 0$. Hence, as a vector space

$$B = B/C^+B \oplus C^+B/(C^+)^2B \oplus \ldots \oplus (C^+)^{n-1}B,$$

and since C^+B is a finitely generated ideal of B, each quotient ring $(C^+)^iB/(C^+)^{i+1}B$ is finite dimensional and, therefore, so is B. A commutative Hopf algebra is free over any finite dimensional left (or right) coideal subalgebra [62, Theorem 3.5(iii)], which proves A is B-free.

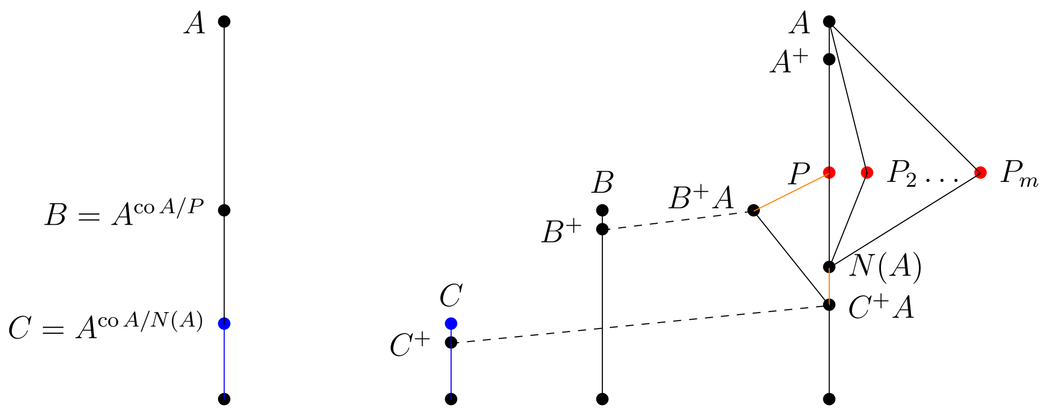
Moreover, by (1),(2) $B/C^+B \cong \mathcal{O}(G/G^\circ)$ is a commutative semisimple algebra, hence $B/C^+B \cong k^{\oplus t}$ where $t = |G : G^\circ|$. By lifting idempotents [70, Lemma 2.3.7], B decomposes as an algebra into a direct sum $B = \bigoplus_{i=1}^t B_i$ where each $B_i \cong C$ as C-modules. This proves B is C-free and semilocal [50, Example 20.5], and a direct sum of Frobenius algebras is Frobenius. ☐

Remarks 3.2.8.

1. In general, the left coideal subalgebra B from the previous lemma is not a Hopf subalgebra of A. For instance, consider Example 3.2.2 in which the nilradical $N(A)$ is the unique minimal prime ideal of A, hence $P = N(A)$ and $B = C$. However, when A is reduced (for example, when the characteristic of k is zero by Theorem 1.1.7) B is a Hopf subalgebra of A by part (2) of the previous result.

2. The previous lemma is more complicated than we would like due to the gap in our knowledge mentioned in Remark 3.2.3. In particular, we know no examples where the statements from part (4) do not hold.

We draw the following diagram, in order to visualize and better understand the prime and semiprime structure of an affine commutative Hopf algebra A, namely the subalgebras B and C and the corresponding ideals of A. The blue lines and bullet points denote finite dimensional algebras, the red bullet points correspond to minimal prime ideals and the orange lines are conjecturally equalities.



3.2.2 The commutative-by-finite case

We now move on to the study of the nilradical and primeness of affine commutative-by-finite Hopf algebras. We start with a few considerations on the $\overline{H}$-stability of the nilradical $N(A)$ of the commutative normal subalgebra A. Then, assuming $N(A)$ is $\overline{H}$-stable, we prove many properties on the (semi)primeness of H.

Recall the notion of orbital semisimplicity from section 3.1.3.

Proposition 3.2.9. *Let H be an affine commutative-by-finite Hopf algebra, with commutative normal Hopf subalgebra A.*

1. *$N(A)$ is left $\overline{H}$-stable if and only if it is right $\overline{H}$-stable.*
2. *If A is orbitally semisimple, then $N(A)$ is $\overline{H}$-stable.*

Proof. (1) Note that $S(N(A)) = N(A)$ as in the proof of Lemma 3.2.1(1). The statement now follows from Lemma 3.1.12(3).

(2) Recall from Lemma 3.1.12(2) that left and right orbital semisimplicity are equivalent. Since A is orbitally semisimple,

$$N(A) = \bigcap_{\mathbf{m}\in\mathrm{Maxspec}(A)} \mathbf{m} = \bigcap_{\mathbf{m}\in\mathrm{Maxspec}(A)} \mathbf{m}^{(\overline{H})} = \bigcap_{\mathbf{m}\in\mathrm{Maxspec}(A)} {}^{(\overline{H})}\mathbf{m},$$

and, since each ideal $\mathbf{m}^{(\overline{H})}$ is right $\overline{H}$-stable and each ${}^{(\overline{H})}\mathbf{m}$ is left $\overline{H}$-stable, $N(A)$ is $\overline{H}$-stable. □

Given part (2) of the previous Lemma and Lemma 3.1.13, we know of no examples where $N(A)$ is not $\overline{H}$-stable, so the following question remains:

Question 3.2.10. Let H be an affine commutative-by-finite Hopf algebra with commutative normal Hopf subalgebra A. Is the nilradical $N(A)$ of A always $\overline{H}$-stable?

Remark 3.2.11. It seems unlikely that this will always hold. Notice that even if one considers a finite dimensional commutative T-module algebra R, with T a finite dimensional Hopf algebra, then $N(R)$ may not be T-stable. For instance, consider Example 3.1.7 and in its notation take $T = T_f(n, 1, q)$ to be the n^2-dimensional Taft algebra and R to be $A/\mathbf{m}_a^{(T)} \cong k\langle \overline{v} : \overline{v}^n = 0\rangle$ for some $a \in k^\times$; its nilradical is $N(R) = \overline{v}R$ and it is not T-stable because $x \cdot \overline{v} = a \notin N(R)$.

We carry the observations and notation of the previous subsection into the next result, which provides parallel but not definitive results for commutative-by-finite Hopf algebras.

Proposition 3.2.12. *Let H be an affine commutative-by-finite Hopf algebra, with commutative normal Hopf subalgebra A. Let P and C, G and B be as in Lemma 3.2.1, Definition 3.2.5 and Lemma 3.2.7 respectively. Assume that $N(A)$ is $\overline{H}$-stable.*

1. *The minimal prime ideal P is an $\overline{H}$-stable Hopf ideal of A, so $N(A)H$ and PH are Hopf ideals of H. Moreover,*
$$N(A)H \cap A = N(A) = N(H) \cap A, \tag{3.6}$$
where $N(H)$ denotes the nilradical of H, and
$$PH \cap A = P. \tag{3.7}$$
2. *If H is semiprime, then so is A.*
3. *C is invariant under the left adjoint action of H.*
4. *Let $Q_1, \ldots, Q_r$ be the minimal prime ideals of H which are contained in H^+. For all $i = 1, \ldots, r$,*
$$Q_i \cap A = P, \tag{3.8}$$
and
$$N(A)H \subseteq PH \subseteq \bigcap_{i=1}^{r} Q_i. \tag{3.9}$$
If $H/N(A)H$ has finite global dimension, then $r = 1$.
5. *If H is prime, then A is a domain.*
6. *Assume that either A is reduced or H is pointed.*

(i) The set of right coinvariants of H induced by the Hopf quotient $H \twoheadrightarrow H/N(A)H$ is

$$H^{\operatorname{co} H/N(A)H} = A^{\operatorname{co} A/N(A)} = C,$$

a local (finite dimensional) Frobenius left coideal subalgebra of A over which H is a free module and $C^+A = N(A)$.

(ii) The set of right coinvariants of H induced by the Hopf quotient $H \twoheadrightarrow H/PH$ is

$$H^{\operatorname{co} H/PH} = A^{\operatorname{co} A/P} = B,$$

a semilocal (finite dimensional) Frobenius left coideal subalgebra of A over which H is a faithfully flat projective module and $B^+H = PH$. Moreover, $B/C^+B = \mathcal{O}(G/G^\circ)$ is a (finite dimensional) semisimple normal Hopf subalgebra of $H/N(A)H$, over which $H/N(A)H$ a free module.

Proof. (1) By Lemma 3.2.1(5) $P = \bigcap_n (A^+)^n + N(A)$ and, since A^+ is $\overline{H}$-stable, so is $\bigcap_n (A^+)^n$. Thus, P is left and right $\overline{H}$-stable. By Lemma 3.1.10(1),(2) PH and $N(A)H$ are ideals and, since both P and $N(A)$ are Hopf ideals of A by Lemma 3.2.1(1),(5), they generate Hopf ideals of H.

On one hand, since $N(A)$ is left and right $\overline{H}$-stable, $N(A)H = HN(A)$ by Lemma 3.1.10, hence $N(A)H$ is nilpotent and we have $N(A)H \cap A \subseteq N(A)$. The reverse inclusion is clear. On another hand, $N(A)H$ being nilpotent implies that $N(A)H \subseteq N(H)$, and so $N(A) \subseteq N(H) \cap A$. The reverse inclusion is obvious.

Moreover, since P is a minimal prime ideal of A, it follows from [47, Theorem 86] that P is the annihilator of some nonzero ideal I of A. Thus, we have $I(PH \cap A) = 0$, and so $PH \cap A \subseteq \operatorname{Ann}_A(I) = P$. The reverse inclusion is clear.

(2) It follows from (3.6).

(3) Consider the Hopf surjection $\pi' : A \to A/N(A)$ and denote the corresponding right $A/N(A)$-coaction of A by ρ. Take $h \in H$ and $c \in C$. Thus, denoting the left adjoint action by $\cdot$ we have

$$\rho(h \cdot c) = \sum h_1 c_1 S(h_4) \otimes \pi'(h_2 c_2 S(h_3)) = \sum h_1 c_1 S(h_3) \otimes \pi'(h_2 \cdot c_2).$$

By Lemma 3.2.1(2) C is a left coideal with $C^+ \subseteq N(A)$ and we are assuming $N(A)$ is left $\overline{H}$-stable, then in the sum above each c_2 can be decomposed into $c_2' + \epsilon(c_2)1$ with $c_2' \in C^+$ and $h_2 \cdot c_2' \in N(A)$, so

$$\rho(h \cdot c) = \sum h_1 c_1 S(h_3) \otimes \pi'(h_2 \cdot \epsilon(c_2)1) = \sum h_1 c_1 S(h_3) \otimes \epsilon(h_2)\epsilon(c_2) = (h \cdot c) \otimes 1.$$

Therefore, C is invariant under left adjoint action of H.

(4) Since H is noetherian, it has finitely many minimal prime ideals and there is a finite product of minimal prime ideals that equals zero [37, Theorem 3.4], so the maximal

(hence prime) ideal H^+ must contain some minimal prime. Let $Q_1, \ldots, Q_r$ be as stated.

If $H/N(A)H$ has finite global dimension, then it is a finite direct sum of prime rings by Theorem 2.1.16(3). In particular, its minimal prime ideals are comaximal, hence H^+ contains only one minimal prime.

Everything to be proved concerns objects which contain $N(A)H$, and by (3.6) $H/N(A)H$ is a commutative-by-finite Hopf algebra, finite over the normal commutative semiprime Hopf subalgebra $A/N(A)$. Thus we can factor by the Hopf ideal $N(A)H$ of H, and hence assume in proving (4) that A is semiprime.

Let $P = P_1, \ldots, P_m$ be the minimal prime ideals of A. Since A is semiprime, it is a finite direct sum of domains by Lemma 3.2.1(3). Thus, the ideal $I := \bigcap_{i=2}^m P_i$ is not contained in A^+. But $IP \subseteq I \cap P = \{0\}$; and, since P is $\overline{H}$-stable by part (1), Lemma 3.1.10 gives $PH = HP$, thus, for each $i = 1, \ldots, r$,

$$IHP = \{0\} \subseteq Q_i. \tag{3.10}$$

But I is not contained in Q_i since $Q_i \cap A \subseteq A^+$, so the primeness of Q_i and (3.10) yield

$$P \subseteq Q_i \cap A, \tag{3.11}$$

thus $PH \subseteq Q_i$ for every $1 \leq i \leq r$ and (3.9) follows.

The reverse inclusion to (3.11) requires more justification. Comparing dimensions, we have

$$\mathrm{GKdim}\,(A) \geq \mathrm{GKdim}\,(A/P) \geq \mathrm{GKdim}\,(A/Q_i \cap A) = \mathrm{GKdim}\,(H/Q_i). \tag{3.12}$$

The inequalities follow from [49, Lemma 3.1] and (3.11) and the equality follows from H being a finitely generated A-module and [49, Proposition 5.5]. We now aim to prove that

$$\mathrm{GKdim}\,(H/Q_i) = \mathrm{GKdim}\,(H). \tag{3.13}$$

First, observe that H has an artinian classical ring of fractions $K = Q(H)$ by Theorem 2.1.8(5). We claim that KQ_i is a proper ideal of K for each $i = 1, \ldots, r$. Let $Q_1, \ldots, Q_t$ be *all* minimal prime ideals of H, $I_i = \bigcap_{j \neq i} Q_j$ and $N = N(H)$. We know that $Q_i I_i \subseteq \bigcap_{j=1}^t Q_j = N$ by [37, Proposition 3.10]. By Small's theorem [64, Theorem 4.1.4], the set $\mathcal{C}(0)$ of regular elements of H coincides with the set $\mathcal{C}(N)$ of regular elements of H/N. Since K is a flat H-module [37, Corollary 10.13],

$$K/KN \cong K \otimes_H H/N = H[\mathcal{C}(N)]^{-1} \otimes_H H/N = Q(H/N).$$

Since $(Q_i/N)(I_i/N) = 0$ in H/N, then $KQ_i \neq K$, otherwise $0 = KQ_iI_i/N = KI_i/N$ and so I_i would be N, a contradiction.

Now observe that the quotient ring $K = Q(H)$ is quasi-Frobenius by Theorem 2.1.8(7). Therefore, the (left, say) annihilator J of Q_i in H is non-zero, since this is true for all proper ideals in a quasi-Frobenius ring, [93, Chapter XIV, §3, Definition and Proposition 3.1]. Thus, J can be regarded as a right H/Q_i-module, hence $\mathrm{GKdim}\,(J) \leq \mathrm{GKdim}\,(H/Q_i) \leq \mathrm{GKdim}\,(H)$ by [49, Proposition 5.1(d), Lemma 3.1], and so (3.13) follows from H being GK-pure as in Theorem 2.1.8(3).

Since $\mathrm{GKdim}\,(A) = \mathrm{GKdim}\,(H)$ by Theorem 2.1.4(2), (3.12) and (3.13) yield

$$\mathrm{GKdim}\,(A/P) = \mathrm{GKdim}\,(A/Q_i \cap A).$$

Since any proper quotient of the affine domain A/P must have a strictly lower GK-dimension [49, Proposition 3.15], we must have $Q_i \cap A = P$.

(5) This is a special case of (4): if H is prime, its unique minimal prime is $Q_1 = \{0\}$, which forces $P = \{0\}$ by (3.8) and A is a domain.

(6) Assume A is reduced or H is pointed. Since the Hopf surjection $H \twoheadrightarrow \overline{H} = H/A^+H$ factors through $H \twoheadrightarrow H/PH$ which in turn factors through $H \twoheadrightarrow H/N(A)H$, then

$$H^{\mathrm{co}\, H/N(A)H} \subseteq H^{\mathrm{co}\, H/PH} \subseteq H^{\mathrm{co}\, \overline{H}} = A,$$

the last equality following from Theorem 2.1.9(2).

(i) By (3.6) $A/N(A)$ embeds into $H/N(A)H$ and so $H^{\mathrm{co}\, H/N(A)H} = A^{\mathrm{co}\, A/N(A)} = C$. It is a local Frobenius subalgebra of A by Lemma 3.2.1(2). When H is pointed, $C^+A = N(A)$ by [60, Theorem] (and it is trivial when A is reduced). By Theorem 2.1.9(3) and Lemma 3.2.1(2) H is a projective C-module, and, given C is local, H is a free C-module [77, Theorem 4.44 and subsequent comments].

(ii) By (3.7) A/P embeds into H/PH, thus $H^{\mathrm{co}\, H/PH} = A^{\mathrm{co}\, A/P} = B$. By Lemma 3.2.7(4) B is a semilocal Frobenius subalgebra of A with $B^+A = P$. By Theorem 2.1.9(1),(3) and Lemma 3.2.7(4), H is a faithfully flat projective B-module.

By Lemma 3.2.7(2) $B/C^+B = \mathcal{O}(G/G^\circ)$ is a finite dimensional semisimple Hopf subalgebra of $A/N(A)$. By (1) and Lemma 3.1.10, $B^+H = HB^+$. By Theorem 2.1.9(1) and Lemma 3.2.7(3) $H/N(A)H$ is faithfully flat over its Hopf subalgebra B/C^+B, hence B/C^+B is normal by [67, Proposition 3.4.3]. A Hopf algebra is free over any finite dimensional normal Hopf subalgebra [83, Theorem 2.1(2)], thus $H/N(A)H$ is a free B/C^+B-module. □

Remarks 3.2.13.

1. The previous result does not completely answer the question of semiprimeness of H, and indeed the structure of the nilradical $N(H)$ is a delicate question. We know from Proposition 2.1.14(1) and Theorem 2.1.16(3) that if A is semiprime and $\overline{H}$ is semisimple then H is semiprime, and in fact it is a direct sum of prime algebras; but the converse is easily seen to be false - see counterexamples in Remark 2.1.15(2). And even the question as to when a smash product $A\#\overline{H}$ of a commutative $\overline{H}$-module algebra A by a finite dimensional Hopf algebra $\overline{H}$ is

semiprime has been the subject of much research and remains currently unresolved - see for example [89].

2. Lemma 3.2.7 yields a short exact sequence of Hopf algebras in the sense of [83, Definition 1.5] for the coordinate ring A of an affine algebraic group G, namely

$$0 \longrightarrow B = A^{\operatorname{co} A/P} = \mathcal{O}(G/G^{\circ}) \longrightarrow A = \mathcal{O}(G) \longrightarrow A/P = \mathcal{O}(G^{\circ}) \longrightarrow 0.$$

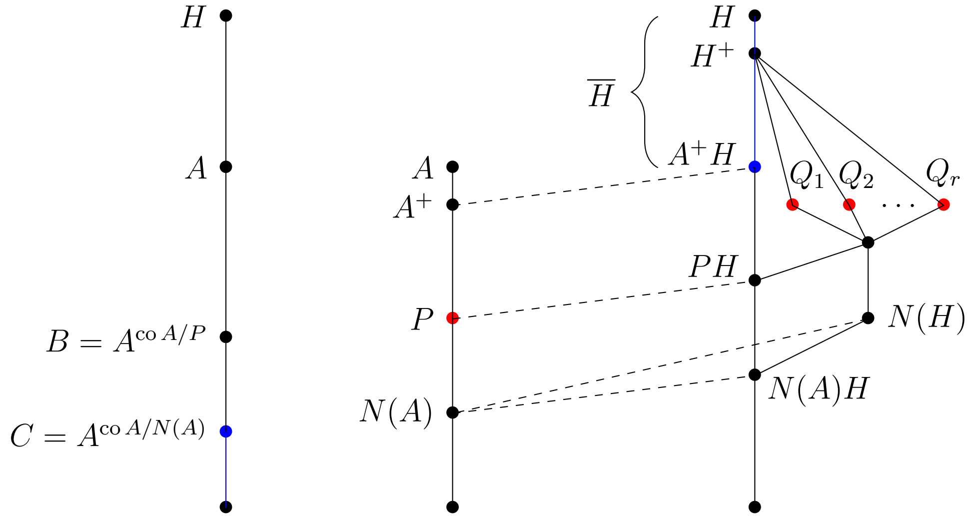
Lu, Wu and Zhang proposed that a similar exact sequence should be valid more widely [58, §6, Theorem 6.5, Remark 6.6]. They in fact prove a partial version of this suggestion for noetherian affine regular Hopf k-algebras of GK-dimension 1, with k of arbitrary characteristic and not necessarily algebraically closed, [58, Theorem 6.5].

However, for an affine commutative-by-finite Hopf algebra H with a normal commutative reduced Hopf subalgebra A, the exact sequence of Hopf algebras given by Proposition 3.2.12,

$$0 \longrightarrow B \longrightarrow H \longrightarrow H/PH \longrightarrow 0,$$

fails to realize this picture, because in general PH is *not* a prime ideal of H. For instance, this occurs for any finite dimensional Hopf algebra H: in the notation of the proposition, $r = 1$ and $Q_1 = H^+$, and considering $A = k$, we have $\overline{H} = H$, $P = \{0\}$.

Once again, we summarize the prime and semiprime structure of a commutative-by-finite Hopf algebra H in the following diagram. Finite dimensional subalgebras and finite dimensional quotient rings are coloured in blue bullet points and minimal prime ideals are coloured in red.



3.3 Their representation theory

In this section we study the representation theory of affine commutative-by-finite Hopf algebras. We start with a few general considerations on the PI and minimal degrees of prime factors of these Hopf algebras in subsection 3.3.1 and find bounds for the dimension of their simple modules in subsection 3.3.2.

3.3.1 Background facts

Let H be an affine commutative-by-finite Hopf algebra, and recall the assumption that the field k is algebraically closed. Recall from section 1.2 the notions of PI ring and minimal degree.

Since H is a finitely generated module over its affine centre $Z := Z(H)$ by Corollary 2.1.7, its simple modules are finite dimensional vector spaces over k, by Kaplansky's theorem [13, I.13.3]. More specifically, let V be a simple, say left, H-module and $M = \mathrm{Ann}(V)$ its annihilator, which is a (two-sided) primitive ideal of H. Since H is finitely generated over its centre, it is right and left fully bounded [37, Proposition 9.1], thus every primitive ideal of H (and, in particular, M) is maximal. Moreover, H is a finitely generated Z-module, hence H/M is a finitely generated module over $(Z+M)/M \cong Z/(Z \cap M)$. Since $Z \cap M$ is maximal in Z and k is algebraically closed, H/M is a finite dimensional k-vector space and, in particular, so is V.

A noetherian ring has finitely many minimal prime ideals, [37, Theorem 3.4], so let $Q_1, \ldots, Q_t$ be the minimal prime ideals of H. Every maximal ideal contains some minimal prime, so every simple H-module is annihilated by at least one Q_i. Therefore, let us first focus on the simple H/Q_i-modules.

Posner's theorem [13, I.13.3] states that for each i the prime algebra H/Q_i has a central simple quotient ring of fractions, and the *PI-degree* of H/Q_i is defined to be the square root of the dimension of $Q(H/Q_i)$ over its centre. The same theorem yields

$$n_i := \text{PI-deg}(H/Q_i) = \frac{1}{2}\text{min.deg}(H/Q_i). \tag{3.14}$$

Moreover, for each i the dimension of the simple H/Q_i-modules is bounded by n_i [13, Theorem I.13.5], and this bound is actually attained,

$$n_i = \max\{\dim_k(V) : V \textit{ a simple } H/Q_i\textit{-module}\},$$

by [13, Lemma III.1.2(2)]. In fact, "most" simple H/Q_i-modules have dimension n_i, in the sense that the intersection of the annihilators of these maximal simple H/Q_i-modules is Q_i, whereas the intersection of the annihilators of the smaller simple modules strictly contains Q_i, [13, Lemma III.1.2].

Given the above we make the following definition for affine commutative-by-finite

Hopf algebras. Let the *representation theoretic PI-degree of* H be

$$\text{rep.PI.deg}(H) := \max\{n_i : 1 \leq i \leq t\}.$$

Since each simple H-module is annihilated by some minimal prime, we have

$$\text{rep.PI.deg}(H) = \max\{\dim_k(V) : V \textit{ a simple } H\textit{-module}\}.$$

Clearly, $\text{min.deg}(H/Q_i) \leq \text{min.deg}(H)$ for every i, hence it follows from (3.14) that

$$\text{rep.PI.deg}(H) \leq \frac{1}{2}\text{min.deg}(H). \tag{3.15}$$

Note that in general this inequality is strict, but it is an equality if H is semiprime. For, if H is semiprime, we have an embedding $H \hookrightarrow \bigoplus_{i=1}^{t} H/Q_i$, thus $\text{min.deg}(H) \leq \max\{\text{min.deg}(H/Q_i) : 1 \leq i \leq t\}$. But $\text{min.deg}(H) \geq \text{min.deg}(H/Q_i)$ for every $1 \leq i \leq t$, so we have the equality $\text{min.deg}(H) = \max\{\text{min.deg}(H/Q_i) : 1 \leq i \leq t\} = 2\text{rep.PI.deg}(H)$.

3.3.2 Bounds on dimensions of simple modules

We now use the concepts mentioned in the previous subsection to find upper bounds for the simple modules of commutative-by-finite Hopf algebras. The approach to this problem was in part inspired by Clifford's results on the representation theory of finite group algebras [70, Theorem 7.2.16].

Note that we also find upper bounds for the factor algebras $A/\mathbf{m}^{(\overline{H})}$ discussed in section 3.1.3, and consequently upper bounds for the size of the orbits of maximal ideals of A which were discussed in sections 3.1.1 and 3.1.3.

Theorem 3.3.1. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A, and V a simple left H-module. Keep the notation of §3.3.1, let $d_A(H)$ denote the minimal number of generators of H as an A-module and let*

$$n(V) := \min\{n_i : Q_i \cdot V = 0,\ 1 \leq i \leq t\}.$$

Then,

1. *We have*
$$\dim_k(V) \leq n(V) \leq \text{rep.PI.deg}(H) \leq \frac{1}{2}\text{min.deg}(H).$$

2. *There exists* $\mathbf{m} \in \text{Maxspec}(A)$ *such that* $\text{Ann}_A(V) = \mathbf{m}^{(\overline{H})}$.

3. *There is an embedding $A/\mathbf{m}^{(\overline{H})} \hookrightarrow V$ of A-modules. Hence,*
$$|\mathcal{O}_{\mathbf{m}}| \leq \dim_k(A/\mathbf{m}^{(\overline{H})}) \leq \dim_k(V).$$

4. If ${}_AH$ is projective (as is the case under any of the hypotheses of Theorem 2.1.9), we also have

$$\frac{1}{2}\text{min.deg}(H) \leq d_A(H).$$

Proof. (1) As discussed in §3.3.1, $\dim_k(V) \leq n_i$ for every Q_i such that V is a simple H/Q_i-module, that is every Q_i such that $Q_i \cdot V = 0$. Thus,

$$\dim_k(V) \leq n(V) \leq \text{rep.PI.deg}(H) \leq \frac{1}{2}\text{min.deg}(H).$$

(2) It follows from (1) that V is finite dimensional, so it contains a simple A-submodule V_0 with annihilator $\mathbf{m} \in \text{Maxspec}(A)$.

First, $\text{Ann}_A(V) \subseteq \text{Ann}_A(V_0) = \mathbf{m}$ and it is left $\overline{H}$-stable, since A is normal and for every $a \in \text{Ann}_A(V), h \in H$ and $v \in V$ we have

$$(ad_l h)(a) \cdot v = \sum h_1 \cdot (a \cdot (Sh_2 \cdot v)) = 0.$$

Similarly, $\text{Ann}_A(V)$ is right $\overline{H}$-stable. Thus, $\text{Ann}_A(V) \subseteq \mathbf{m}^{(\overline{H})}$ by Lemma 3.1.2.

Conversely, since $H\mathbf{m}^{(\overline{H})}$ is a 2-sided ideal of H, $\{v \in V : (H\mathbf{m}^{(\overline{H})})v = 0\}$ is a H-submodule of V, which is non-zero since it contains V_0. By simplicity, $(H\mathbf{m}^{(\overline{H})})V = 0$. In particular, $\mathbf{m}^{(\overline{H})} \subseteq \text{Ann}_A(V)$.

(3) By definition, $\mathbf{m}^{(\overline{H})} \subseteq \bigcap_{\mathbf{m}' \in \mathcal{O}_\mathbf{m}} \mathbf{m}'$ and so $A/\bigcap_{\mathbf{m}' \in \mathcal{O}_\mathbf{m}} \mathbf{m}' \cong \bigoplus_{\mathbf{m}' \in \mathcal{O}_\mathbf{m}} A/\mathbf{m}' \cong k^{\oplus|\mathcal{O}_\mathbf{m}|}$ is a factor algebra of $A/\mathbf{m}^{(\overline{H})}$. Thus, the first inequality follows.

Now let $\{v_1, \ldots, v_r\}$ be a k-basis of V and consider the map

$$\iota : A \to V^{\oplus r} \ : \ a \mapsto (av_1, \ldots, av_r).$$

Since its kernel is $\text{Ann}_A(V) = \mathbf{m}^{(\overline{H})}$ by (2), $A/\mathbf{m}^{(\overline{H})}$ embeds in $V^{\oplus r}$ via ι. Since $A/\mathbf{m}^{(\overline{H})}$ is a Frobenius algebra by Proposition 3.1.4(2), it is self-injective. Therefore, $A/\mathbf{m}^{(\overline{H})}$ is (isomorphic to) a direct summand of the $A/\mathbf{m}^{(\overline{H})}$-module $V^{\oplus r}$ by [37, Corollary 5.5].

The commutative algebra $A/\mathbf{m}^{(\overline{H})}$ is finite dimensional, hence artinian, so it is a finite direct sum of non-isomorphic indecomposable submodules [50, Corollary 19.22], and each of these must be a direct summand of ${}_AV$. Therefore, $A/\mathbf{m}^{(\overline{H})}$ embeds in ${}_AV$. The second inequality now follows from this embedding.

(4) Assume H is a projective A-module. For every maximal ideal $\mathbf{m}$ of A, $H_\mathbf{m} := A_\mathbf{m} \otimes_A H$ is a projective module over the local ring $A_\mathbf{m}$, hence it is free by [77, Theorem 4.44 and following comments], clearly of rank at most $d_A(H)$. The right action of H on $H_\mathbf{m}$ yields the homomorphism

$$\begin{array}{rcl} \psi_\mathbf{m} : H & \to & \text{End}(H_\mathbf{m}) \\ h & \mapsto & [x \in H_\mathbf{m} \mapsto xh] \end{array}.$$

We claim that the intersection of the kernels of these maps as $\mathbf{m}$ ranges through

Maxspec(A) is $\{0\}$. This intersection consists of elements $h \in H$ such that $1_{H_{\mathbf{m}}}h = 0$ in $H_{\mathbf{m}}$ for all $\mathbf{m} \in$ Maxspec(A). Let $\phi_{\mathbf{m}} : H \to H_{\mathbf{m}} := A_{\mathbf{m}} \otimes_A H$ be the natural map and $J = \bigcap_{\mathbf{m}\in \mathrm{Maxspec}(A)} \ker \phi_{\mathbf{m}}$. We need only show $J = 0$. First, $\ker \phi_{\mathbf{m}} = \{a \in A : ac = 0, \text{ for some } c \in A \setminus \mathbf{m}\}$ [77, Theorem 3.71]. If $J \neq 0$, its annihilator $\mathrm{Ann}_A(J)$ is a proper ideal of A, hence it is contained on some maximal ideal $\mathbf{m}'$ of A. But $J \subseteq \ker \phi_{\mathbf{m}'}$, thus $\mathrm{Ann}_A(J)$ contains some element $c \in A \setminus \mathbf{m}'$, which is a contradiction. This proves the claim.

Since $H_{\mathbf{m}}$ is a free $A_{\mathbf{m}}$-module of rank at most $d_A(H)$, we have the embedding

$$H \hookrightarrow \prod_{\mathbf{m}\in \mathrm{Maxspec}(A)} \mathrm{End}_{A_{\mathbf{m}}}(H_{\mathbf{m}}) \hookrightarrow \prod_{\mathbf{m}\in \mathrm{Maxspec}(A)} \mathcal{M}_{d_A(H)\times d_A(H)}(A_{\mathbf{m}}).$$

By the Amitsur-Levitski's theorem [64, Theorem 13.3.3(ii)], $\mathrm{min.deg}(H) \leq 2d_A(H)$, as required. $\square$

In the following corollary we improve the upper bound $d_A(H)$ in case A is a domain. Recall from Proposition 3.2.12(5) that when $N(A)$ is $\overline{H}$-stable (for example, if k has characteristic zero) A is a domain, provided H is prime.

Corollary 3.3.2. *Let H be an affine commutative-by-finite Hopf algebra with commutative normal Hopf subalgebra A. Suppose that A is a domain. Let V be a simple H-module and $\mathbf{m}$ the annihilator of a simple A-submodule of V. Retain the notation of Theorem 3.3.1. Then,*

$$|\mathcal{O}_{\mathbf{m}}| \leq \dim_k(A/\mathbf{m}^{(\overline{H})}) \leq \dim_k(V) \leq \mathrm{rep.PI.deg}(H) \leq \frac{1}{2}\mathrm{min.deg}(H) \leq \dim_k(\overline{H}).$$

Proof. First note that H is A-projective by Theorem 2.1.9(3), so all the inequalities of Theorem 3.3.1 are valid and we need only justify the last inequality, $\frac{1}{2}\mathrm{min.deg}(H) \leq \dim_k(\overline{H})$. By [77, Theorem 4.44 and following comments] H is a locally free A-module, that is, for every maximal ideal $\mathbf{m}$ of A, $H_{\mathbf{m}} := H \otimes_A A_{\mathbf{m}}$ is a free $A_{\mathbf{m}}$-module, say $H \otimes_A A_{\mathbf{m}} \cong A_{\mathbf{m}}^{r_{\mathbf{m}}}$ for some positive integer $r_{\mathbf{m}}$.

We claim that the rank $r_{\mathbf{m}}$ of $H_{\mathbf{m}}$ is the same for every maximal ideal $\mathbf{m}$ of A. Let Q be the quotient field of A. First note that, since H is A-projective, it is a direct summand of a free A-module. In particular, the nonzero elements of A are regular in H, meaning that $A_{\mathbf{m}}$ is contained in Q. Then,

$$H \otimes_A Q \cong (H \otimes_A A_{\mathbf{m}}) \otimes_{A_{\mathbf{m}}} Q \cong A_{\mathbf{m}}^{r_{\mathbf{m}}} \otimes_{A_{\mathbf{m}}} Q \cong Q^{r_{\mathbf{m}}}.$$

Therefore, $r_{\mathbf{m}} = \dim_Q(H \otimes_A Q)$, which is independent of $\mathbf{m}$. We call this constant rank r.

Moreover, such rank must be in particular given by

$$r = \mathrm{rank}_{A_{A^+}}(H \otimes_A A_{A^+}) = \dim_k(\overline{H}).$$

The second equality follows from Nakayama's lemma [78, Remark 8.25], according to which a minimal set of generators of the finitely generated module $M = H \otimes_A A_{A^+}$ over the local ring $R = A_{A^+}$ can be obtained from a basis of $M/A^+M \cong (H \otimes_A A_{A^+})/(A^+H \otimes_A A_{A^+}) \cong H/A^+H = \overline{H}$ over $R/A^+ = A_{A^+}/A^+A_{A^+} \cong A/A^+ \cong k$.

Therefore, via left multiplication of elements of H on $H \otimes_A Q$, H now embeds into

$$\mathrm{End}_Q(H \otimes_A Q) \cong \mathcal{M}_r(Q),$$

and again by Amitsur-Levitski's theorem min.deg$(H) \leq 2r$. $\square$

3.4 Commutative-by-(co)semisimple Hopf algebras

This section concerns the study of commutative-by-finite Hopf algebras H whose corresponding Hopf quotient $\overline{H}$ is semisimple and cosemisimple. We continue to assume that k is an algebraically closed field.

We start with a few examples and an auxiliary result before we prove the main result of this section, Theorem 3.4.3, that describes in detail the structure of these Hopf algebras. We also give a corollary explaining the structure of prime commutative-by-(co)semisimple Hopf algebras and finish this section with a few questions for future work.

3.4.1 Preliminaries

We start with some obvious examples of such Hopf algebras. Recall that any finite dimensional Hopf algebra in characteristic 0 is cosemisimple if and only if it is semisimple if and only if it is involutory [52, Corollary 2.6, Theorem 3.3], [53, Theorems 3 and 4].

Examples 3.4.1.

1. Take the coordinate ring $A = \mathcal{O}(\Lambda)$ of an algebraic group Λ over a field k, and a finite group Γ whose order is a unit in k with a group homomorphism $\alpha : \Gamma \to \mathrm{Aut}(\Lambda)$. So Γ acts on A by $(\gamma \cdot f)(\lambda) = f(\alpha(\gamma)^{-1}(\lambda))$, for $\gamma \in \Gamma, f \in \mathcal{O}(\Lambda), \lambda \in \Lambda$, and it extends to an action by $T = k\Gamma$. We can now form the smash product

$$H = A\#T = A * \Gamma.$$

This is a Hopf algebra, with the usual coproduct of tensor of coalgebras; and it is clearly commutative-by-(co)semisimple.

For example, $k[x_1, \ldots, x_n] * S_n$ is commutative-by-(co)semisimple, where the permutation group S_n acts on the polynomial ring $k[x_1, \ldots, x_n]$ by permuting indeterminates, meaning $\sigma \cdot x_i = x_{\sigma(i)}$ for all $\sigma \in S_n, 1 \leq i \leq n$.

2. Another large collection of examples is provided in §2.2.4, by those group algebras $H = kG$ where G has a finitely generated abelian normal subgroup N of finite index, such that G/N contains no elements of order $\operatorname{char} k$.

The main result of this section, Theorem 3.4.3, suggests that these examples may go some way towards exhausting all the possibilities for commutative-by-(co)semisimple Hopf algebras. Before we delve into that result we give an auxiliary definition and an additional lemma, both of which are needed for its proof.

We first introduce the notion of a polycentral ideal: an ideal I of a noetherian ring R is *polycentral* if there are elements $x_1, \ldots, x_t \in I$ with $I = \sum_{i=1}^{t} x_i R$, such that x_1 is in the centre $Z(R)$ of R and, for $j = 2, \ldots, t$, $x_j + \sum_{i=1}^{j-1} x_i R \in Z(R/\sum_{i=1}^{j-1} x_i R)$. Polycentral ideals share many of the properties of ideals of commutative noetherian rings, [64, Chapter 4, §2], [70, Chapter 11, §2].

This auxiliary lemma is a result of Skryabin [86], that extends another by Masuoka [61], and we briefly explain how this follows from the results in these papers.

Proposition 3.4.2 (Skryabin, [86])**.** *Every coideal subalgebra of a finite dimensional semisimple Hopf algebra is semisimple.*

Proof. Every left or right coideal subalgebra of a finite dimensional Hopf algebra is Frobenius [86, Theorem 6.1]. A coideal subalgebra of a semisimple Hopf algebra is Frobenius if and only if it is separable by [61, Theorem 2.1]. And a separable finite dimensional algebra is semisimple [70, Theorem 7.3.9]. □

3.4.2 Their structure

We now present our results on the structure of commutative-by-semisimple Hopf algebras.

Theorem 3.4.3. *Let H be a commutative-by-finite Hopf algebra, finite over the affine normal commutative Hopf subalgebra A, with* $\operatorname{GKdim}(H) = d$. *Suppose that $\overline{H} = H/A^+H$ is semisimple and cosemisimple. Let $N(A)$ denote the nilradical of A and P be the unique minimal prime ideal of A with $P \subseteq A^+$, as in Proposition 3.2.12. Let $G^\circ \subseteq G$ be the algebraic groups such that $\mathcal{O}(G) = A/N(A)$ and $\mathcal{O}(G^\circ) = A/P$, as in Definition 3.2.5. Then,*

1. *$N(A)$ is an $\overline{H}$-stable Hopf ideal of A, so the Hopf ideal $N(A)H$ is the nilradical of H,*
$$N(A)H = N(H).$$
Moreover, P is also an $\overline{H}$-stable Hopf ideal of A, hence PH is a Hopf ideal of H.

2. *$H/N(A)H$ and H/PH are semiprime commutative-by-semisimple Hopf algebras, of GK-dimension and global dimension d. They are faithfully flat $\overline{H}$-Galois extensions of $A/N(A)$ and A/P respectively.*

3. *Let $Q_1, \dots, Q_t$ be the minimal prime ideals of H. Precisely one minimal prime ideal, say Q_1, is contained in H^+, and this minimal prime contains PH. Reorder the remaining Q_i and fix s, $1 \leq s \leq t$, so that $P \subseteq Q_i$ if and only if $i \leq s$. Then,*

$$N(A)H = \bigcap_{i=1}^{t} Q_i, \qquad PH = \bigcap_{i=1}^{s} Q_i,$$

and for $j = 1, \dots, s$

$$Q_j \cap A = P.$$

Thus,

$$H/N(A)H \cong \bigoplus_{i=1}^{t} H/Q_i \quad \text{and} \quad H/PH \cong \bigoplus_{i=1}^{s} H/Q_i$$

are direct sums of prime algebras of GK-dimension and global dimension d.

4. *There are subalgebras*

$$C \subseteq B \subseteq A \subseteq D \subseteq H, \tag{3.16}$$

such that:

   (i) $C := A^{\mathrm{co}\, A/N(A)}$ is a local (finite dimensional) Frobenius left coideal subalgebra of A with $C^+A \subseteq N(A)$, and A is a free C-module. Moreover, C is invariant under the left adjoint action of H.

   (ii) $B := A^{\mathrm{co}\, A/P}$ is a left coideal subalgebra of A, over which A is flat and such that

$$P = B^+A + N(A).$$

   Moreover, $(B+N(A))/N(A) = \mathcal{O}(G/G^\circ)$ is a finite dimensional semisimple Hopf subalgebra of $A/N(A)$, normal in $H/N(A)H$, over which $A/N(A)$ is a free module.

   (iii) There is a factor group algebra $k\Gamma$ of $\overline{H}$ with Hopf epimorphism $\alpha : H \to k\Gamma$, such that the left and right adjoint actions of $\overline{H}$ on A/P both factor through an inner faithful $k\Gamma$-action. Thus, $D := H^{\mathrm{co}\, k\Gamma}$ is a left coideal subalgebra of H, it is a finitely generated A-module and H is left and right faithfully flat over D. Moreover, D is invariant under the left adjoint action of H, $D^+H = HD^+ = \ker \alpha$ is a Hopf ideal of H and

$$H/D^+H \cong k\Gamma. \tag{3.17}$$

5. *D/A^+D is semisimple.*

6. *For all $a \in A$ and $d \in D$,*

$$ad - da \in DP.$$

7. *$D/N(A)D$ has GK-dimension and global dimension d, so it is homologically ho-*

mogeneous and is a direct sum of prime algebras, each of GK-dimension and global dimension d.

8. *There is a unique minimal prime ideal L of D with $L \subseteq D^+$. Moreover*

$$L \cap A = P \tag{3.18}$$

and

$$L = \bigcap_{i \geq 1} (D^+)^i + N(A)D. \tag{3.19}$$

9. *D/L is an affine commutative domain of GK-dimension and global dimension d.*

10. *L is a left H-stable ideal of D, so LH is a Hopf ideal of H. Thus,*

    (i) *D/L is a left coideal subalgebra of H/LH and a finitely generated module over the Hopf subalgebra A/P of H/LH.*

    (ii) *the left adjoint action of H on D induces a left adjoint action of H/LH on D/L, and this action factors through an inner faithful group action for some group Λ which maps surjectively to Γ.*

11. *We have the inclusions*

$$N(A)H \subseteq PH \subseteq LH \subseteq Q_1,$$

and

$$Q_1 \cap D = L.$$

12. *$E := H^{\mathrm{co}\, H/LH}$ is a left coideal subalgebra of H such that*

$$B \subseteq E \subseteq D. \tag{3.20}$$

Moreover, $E = D^{\mathrm{co}\, D/L}$ and it is invariant under the left adjoint action of H.

13. *Assume further that H is pointed. Then, H is a faithfully flat left and right E-module and*

$$Q_1 = LH.$$

Also, H and H/LH are crossed products as such

$$H \cong A\#_\sigma \overline{H} \qquad \text{and} \qquad H/LH \cong (D/L)\#_\tau \Gamma,$$

for some cocycles σ and τ.

Proof. (1),(2),(3) Since $\overline{H}$ is cosemisimple, A is $\overline{H}$-orbitally semisimple by Theorem 3.1.8(3), hence $N(A)$ is $\overline{H}$-stable by Proposition 3.2.9 and $N(A)H$ is a nilpotent Hopf ideal of H as in Proposition 3.2.12(1).

By Proposition 3.2.12(1) we have $N(A)H \cap A = N(A)$, so $H/N(A)H$ is commutative-by-finite with commutative normal Hopf subalgebra $A/N(A)$; the corresponding Hopf quotient is $(H/N(A)H)/((A/N(A))^+) \cong \overline{H}$. Since $A/N(A)$ is reduced, the extension is faithfully flat $\overline{H}$-Galois by Theorem 2.1.9(1). And since $A/N(A)$ is reduced and the Hopf quotient $\overline{H}$ is semisimple, Proposition 2.1.14(1) and Theorem 2.1.16(3) yield that

$$H/N(A)H \cong \bigoplus_{i=1}^{t} H/Q_i$$

is a finite direct sum of prime rings, where $Q_1, \ldots, Q_t$ are *all* minimal prime ideals of H. In particular, these ideals are comaximal, so H^+ contains exactly one of them. And $H/N(A)H$ is semiprime, thus $N(A)H$ is the nilradical of H.

As for the GK-dimension, Theorem 2.1.4(2) and Proposition 3.2.1(3) yield

$$\mathrm{GKdim}\,(H/N(A)H) = \mathrm{GKdim}\,(A/N(A)) = \mathrm{GKdim}\,(A) = \mathrm{GKdim}\,(H) =: d. \quad (3.21)$$

The fact that the global dimension of $H/N(A)H$ is d follows from Theorem 2.1.16(1) and each H/Q_i also has GK-dimension and global dimension d by Theorem 2.1.16(3).

Since $N(A)$ is $\overline{H}$-stable, P is also $\overline{H}$-stable and PH is a Hopf ideal of H by Proposition 3.2.12(1). Since $PH \cap A = P$ by Proposition 3.2.12, the properties of H/PH follow for the same reasons as for $H/N(A)H$.

Note that $\mathrm{GKdim}\,(H/Q_i) = \mathrm{GKdim}\,(A/P) = d$ for all i, and by [49, Lemma 4.3] $\mathrm{GKdim}\,(A/(Q_i \cap A)) = \mathrm{GKdim}\,(H/Q_i) = d$. Since a proper factor of an affine commutative domain must have strictly smaller GK-dimension [49, Proposition 3.15], the inclusion $P \subseteq Q_i \cap A$ must be an equality for every $i = 1, \ldots, s$.

(4) (i) This was proved in Lemma 3.2.1(2) and Proposition 3.2.12(3).

(ii) This was proved in Lemma 3.2.7 and Proposition 3.2.12(6).

(iii) Since P is left and right $\overline{H}$-stable by (1), the $\overline{H}$-actions on A restrict to $\overline{H}$-actions on A/P. Consider first the *right* adjoint action of $\overline{H}$ on A/P. By [88, Theorem 2] the right adjoint action of the cosemisimple Hopf algebra $\overline{H}$ on the commutative domain A/P factors through a group algebra $k\Gamma$, that is there exists a Hopf ideal I of $\overline{H}$ that annihilates A/P under right adjoint action and such that $H/I \cong k\Gamma$, with $k\Gamma$ acting inner faithfully on A/P. Consider the Hopf epimorphism $\alpha : H \twoheadrightarrow \overline{H} \twoheadrightarrow k\Gamma$ and the right coinvariants

$$D := H^{\mathrm{co}\, k\Gamma}.$$

D is a left coideal subalgebra of H by Example 1.1.10, and is invariant under the left adjoint action of H by [67, Lemma 3.4.2(2)]. Since α factors through $\pi : H \to \overline{H}$, it follows from Remark 2.1.10 that

$$A \subseteq H^{\mathrm{co}\, \pi} \subseteq H^{\mathrm{co}\, k\Gamma} = D.$$

Since D is an A-submodule of H and H is a noetherian A-module, then D is a finitely generated A-module.

By [69, Corollary 1.5], H is left and right faithfully flat over D, and $\ker \alpha = D^+H$, whence D^+H is a Hopf ideal and (3.17) follows. By Koppinen's lemma [69, Lemma 1.4] and the fact that D^+H is an ideal of H, $S(D^+H) = HD^+ \subseteq HD^+H = D^+H$. Since $S(A^+H) = HA^+$, S induces a bijection on the finite dimensional space $\overline{H}$, so that $HD^+ = D^+H$.

Repeating the above argument for the *left* adjoint action of $\overline{H}$ on A/P yields a Hopf epimorphism $\beta : H \twoheadrightarrow \overline{H} \twoheadrightarrow k\Lambda$ for some finite group algebra $k\Lambda$ acting inner faithfully on A/P. It suffices to prove that $\ker \alpha = \ker \beta$, for in this case $k\Gamma \cong H/\ker \alpha = H/\ker \beta \cong k\Lambda$ and the two groups are isomorphic as required.

Let us prove the claim. Recall from the proof of Lemma 3.1.12(1) that for any $h \in H$ and $v \in A/P$ we have

$$ad_r(h)(Sv) = S(ad_l(S^{-1}h)(v)).$$

Thus, if $h \in \ker \alpha$, then $S(ad_l(S^{-1}h)(v)) = ad_r(h)(Sv) = 0$ and, since the antipode is bijective, $S^{-1}h \in \ker \beta$. Hence, $S^{-1}(\ker \alpha) \subseteq \ker \beta$. A similar argument proves the reverse inclusion. Since $\ker \alpha = D^+H$ satisfies $S(D^+H) = D^+H$ as above, we have

$$\ker \beta = S^{-1}(\ker \alpha) = S^{-1}(D^+H) = D^+H = \ker \alpha.$$

(5) Since H is faithfully flat over D by (4)(iii), we have $D/A^+D = D/(A^+H \cap D) \cong (D + A^+H)/A^+H \subseteq \overline{H}$. Since $\overline{H}$ is semisimple, (5) follows from Proposition 3.4.2.

(6) Since $D^+ \subseteq \ker \alpha$ annihilates A/P, for all $a \in A, d \in D$ we have $d = \epsilon(d)1 + d'$ for some $d' \in D^+$ and

$$(ad_r d)(a) = \epsilon(d)a + (ad_r d')(a) \equiv \epsilon(d)a \mod P;$$

that is, the right adjoint action of D on A/P is given by ϵ. Thus, for any $a \in A, d \in D$,

$$ad = \sum d_1 S(d_2) a d_3 = \sum d_1 \,(ad_r d_2)(a) \equiv \sum d_1 \epsilon(d_2) a = da \mod HP,$$

since D is a left coideal of H. Since H is faithfully flat over D by (4)(iii), $HP \cap D = DP$.

(7) Since H is right faithfully D-flat, $N(A)H \cap D = N(A)D$. And by Proposition 3.2.12 $N(A)H \cap A = N(A)$. Thus, $H' = H/N(A)H$ is commutative-by-finite with normal commutative Hopf subalgebra $A' = A/N(A)$; and it contains the left coideal subalgebra $D' = D/N(A)D$. By [49, Lemma 3.1] $\mathrm{GKdim}\,(A') \le \mathrm{GKdim}\,(D') \le \mathrm{GKdim}\,(H')$, hence $\mathrm{GKdim}\,(D') = d$ by (3.21).

First we prove that $\mathrm{gldim}\,(D') \le d$. To see this, note that $\mathrm{gldim}\,(H') = d$ by (2) and, since $H'/(D')^+H' \cong H/D^+H \cong k\Gamma$ by (3.17) which is cosemisimple, it follows that $\mathrm{gldim}\,(D') \le \mathrm{gldim}\,(H') = d$ by [48, Lemma 9]; the key point in the proof of this result is that the cosemisimplicity of $H'/(D')^+H'$ ensures that the left D'-module direct sum decomposition $H' = D' \oplus U$ of [63, Corollary 2.9] can be achieved as D'-bimodules.

We now prove that any irreducible left D'-module has projective dimension d. This shows that D' is homologically homogeneous and $\operatorname{gldim}(D') = d$ by [64, Corollary 7.1.14]. Let V be an irreducible left D'-module. Since $\operatorname{gldim}(D') \leq d$, V must have finite projective dimension over D', say $t := \operatorname{prdim}_{D'}(V) \leq d$. It suffices to prove that $t \geq d$.

Let $0 \to P_t \to \ldots \to P_0 \to V \to 0$ be a D'-projective resolution of V. As explained above, D' is a left D'-direct summand of H' and, since H' is a projective A'-module by Theorem 2.1.9(3), D' is also A'-projective. In particular, each P_i is A'-projective, hence $\operatorname{prdim}_{A'}(V) \leq t$. And, V can be regarded as an irreducible H'-module (by letting the direct complement of D' act trivially), hence V is finite dimensional by Kaplansky's theorem - see beginning of §3.3.1.

Moreover, by Lemma 3.2.1(3) $A' = \bigoplus_{i=1}^{m} A_i$ is a finite direct sum of commutative affine domains, each of global dimension d. We now claim that all the irreducible A'-modules have projective dimension d. First, an irreducible A'-module W is an irreducible A_i-module for some $1 \leq i \leq m$, since each commutative domain $A_i = A'e_i$ is generated by an idempotent e_i of A', so W decomposes into submodules as $W = \bigoplus_{i=1}^{m} e_i W$ and by irreducibility $W = e_i W$ for some i. Second, $W \cong A_i/\mathbf{m}$ for some maximal ideal $\mathbf{m}$ of the commutative domain A_i. Thus, upon localizing at $\mathbf{m}$, [47, Theorem 176] yields that the projective dimension of $(A_i)_{\mathbf{m}}/\mathbf{m}(A_i)_{\mathbf{m}} \cong A_i/\mathbf{m} \cong W$ equals height of $\mathbf{m} = \operatorname{Kdim}(A_i)_{\mathbf{m}} = \operatorname{Kdim}(A_i) = d$ by [78, (8.2)]. This proves the claim.

Therefore, all the finite dimensional A'-modules (and, in particular, V) have projective dimension d, so $d \leq t$, as required.

The direct sum decomposition of D' now follows from [15, Theorem 5.3], and by (3) each direct summand has GK-dimension and global dimension d.

(8) $D/N(A)D$ is semiprime by (7), hence $N(D) \subseteq N(A)D$ and, since by (1) $N(A)$ is $\overline{H}$-stable, $N(A)D$ is nilpotent, thus $N(A)D = N(D)$. Moreover, by the decomposition of $D/N(A)D$ into a direct sum of prime rings in (7), the minimal prime ideals of D are comaximal, hence there is a unique minimal prime ideal L of D contained in D^+.

If $P = N(A)$ then clearly $P \subseteq N(A)D \subseteq L$. Suppose on the other hand that $N(A) \subsetneq P$. Let $P = P_1, \ldots, P_m$ be the minimal prime ideals of A, as in the proof of Lemma 3.2.1. Since P is the unique minimal prime contained in A^+ and A^+ is maximal (hence prime), $\bigcap_{i=2}^{m} P_i$ is not contained in A^+. But $P(\bigcap_{i=2}^{m} P_i) \subseteq \bigcap_{i=1}^{m} P_i = N(A)$, thus in particular there exists $y \in A \setminus A^+$ with $yP \subseteq N(A)$. By stability of P as in (1),

$$yDP \subseteq yPD \subseteq N(A)D \subseteq L.$$

But $y \notin L$ because $L \cap A \subseteq A^+$, hence primeness of L yields

$$P \subseteq L \cap A. \tag{3.22}$$

For the reverse inclusion, note that D/L is a finitely generated (left, say) $A/(L \cap A)$-module, so that

$$\mathrm{GKdim}\,(A/(L \cap A)) = \mathrm{GKdim}\,(D/L) = d$$

by [49, Proposition 5.5] and (7). A proper factor of an affine commutative domain has strictly lower GK-dimension [49, Proposition 3.15] and, since $\mathrm{GKdim}\,(A/P) = d$ by the proof of (2), we must have equality in (3.22).

We now prove (3.19). By (7) the minimal prime $L/N(A)D$ of $D/N(A)D$ is generated by an idempotent (similarly to P in the proof of Lemma 3.2.1), and since $L \subseteq D^+$ we have

$$L \subseteq \bigcap_{i \geq 1} (D^+)^i + N(A)D.$$

We prove the reverse inclusion. The image of A in D/L is $(A + L)/L \cong A/P$ by (3.18) and it is central in D/L by (6) and the fact that $DP \subseteq L$. Now the maximal ideal D^+/L of D/L contains the central ideal generated by A^+/P, and the quotient $(D^+/L)/((A/P)^+) \cong D^+/A^+D$ is a maximal ideal of D/A^+D, a semisimple algebra by (5), hence D^+/A^+D is generated by a central idempotent. Thus, D^+/L is a polycentral ideal of D/L. By the version of Krull's Intersection Theorem for polycentral ideals, [70, Theorems 11.2.8 and 11.2.13], since D/L is a prime ring, we have

$$\bigcap_{i \geq 1} (D^+)^i \subseteq L.$$

This proves the required equality.

(9) We first prove that D/L is commutative. By (3.19), it suffices to prove that $D/(D^+)^i$ is commutative for each $i \geq 1$.

Choose elements $a_1, \ldots, a_m$ of A^+ whose images form a k-basis of $A^+/(A^+)^2$. Let $e \in D^+$ be such that $e + A^+D$ is the central idempotent generator of D^+/A^+D guaranteed by (5). Then,

$$D^+/(D^+)^2 = \left(De + \sum_j Da_j + (D^+)^2\right)/(D^+)^2 = \left(ke + \sum_j ka_j + (D^+)^2\right)/(D^+)^2.$$

Since $A^+ = \sum ka_j + (A^+)^2$, we have

$$A^+D \subseteq \sum a_jD + (A^+)^2D \subseteq \sum ka_j + (D^+)^2.$$

Thus, the quotient $D^+/(\sum_j ka_j + (D^+)^2)$ is idempotently generated, because it is a factor of D^+/A^+D which is idempotently generated by $e + A^+D$. But $D^+/(\sum_j ka_j + (D^+)^2)$ is also a factor of $D^+/(D^+)^2$, hence it must be zero, so

$$D^+ = \sum_j ka_j + (D^+)^2. \tag{3.23}$$

Therefore, for each $i \geq 2$,

$$D^+/(D^+)^i \text{ is spanned by monomials of length at most } i-1 \text{ in } a_1, \dots, a_m.$$

Hence $D/(D^+)^i$ is commutative, as required.

Therefore, D/L is a commutative affine domain. Since L is a minimal prime of D, by (7) D/L has GK-dimension and global dimension d.

(10) First note that $N(A)$ and D^+ are left $\overline{H}$-stable by (1) and (4)(iii) respectively, hence L is left H-stable by (3.19). Moreover, $N(A)H$ and D^+H are Hopf ideals of H again by parts (1) and (4)(iii). Since H is faithfully D-flat and $HD^+ = D^+H$, we have

$$\bigcap_i (D^+H)^i = \bigcap_i (D^+)^i H = \left(\bigcap_i (D^+)^i\right) H$$

and this is a Hopf ideal of H by Lemma 1.1.15. Therefore, by (3.19),

$$LH = N(A)H + \bigcap_i (D^+H)^i$$

is a Hopf ideal of H.

(i) By faithful flatness of H over D, we have $LH \cap D = L$, so D/L is a left coideal subalgebra of H/LH. By (3.18), A/P embeds as a Hopf subalgebra into H/LH. More, it is contained in D/L which is a finitely generated A/P-module, since D is a finite A-module.

Throughout the rest of the proof of (10), let $A' = A/P$, $D' = D/L$ and $H' = H/LH$, so that $A' \subseteq D' \subseteq H'$ as per the previous paragraph with A' a Hopf subalgebra and D' a left coideal subalgebra of H', and A' and D' are affine commutative domains by (9).

(ii) Since D is invariant under the left adjoint action of H by (4)(iii) and L is H-stable as above, then D' is also invariant under left adjoint action of H'. Since D' is commutative by (9) and A' is a Hopf subalgebra contained in D', then for all $a' \in A'^+$ and $d' \in D'$

$$ad_l(a')(d') = \sum a'_1 d' S(a'_2) = \sum a'_1 S(a'_2) d' = \epsilon(a')d' = 0.$$

Thus, the left adjoint action of H' on D' factors through $H'A'^+ = A'^+H'$.

Since $H'/A'^+H' \cong \overline{H}$ is semisimple and cosemisimple and D' is a commutative domain, this action in turn factors through an inner faithful group action [88, Theorem 2], say $H'/I \cong k\Lambda$ for some finite group Λ and some Hopf ideal I of H' that annihilates D' under the left adjoint action. However, by (4)(iii) the left adjoint action of H on A' factors through D^+H with $H/D^+H \cong k\Gamma$ acting inner faithfully. And, since $A' \subseteq D'$, it follows that $I \subseteq D'^+H'$ and we have a Hopf epimorphism $k\Lambda \twoheadrightarrow k\Gamma$.

(11) We need only prove $LH \subseteq Q_1$ and $Q_1 \cap D = L$. We use the same argument as applied to P in (8).

Since L is the unique minimal prime ideal of D contained in D^+, there exists some element $z \in D \setminus D^+$ with $zL \subseteq N(D)$, for the same reason as for P in (8). But $N(D) = N(A)D \subseteq N(A)H = N(H)$ is contained in any minimal prime ideal of H, so left $\overline{H}$-stability of L and Lemma 3.1.10 yield

$$zHL = zLH = N(D)H \subseteq Q_1.$$

Since $z \notin Q_1$, primeness of Q_1 gives $L \subseteq Q_1 \cap D$. In particular, $LH \subseteq Q_1$.

Since H is a finitely generated D-module, H/Q_1 is a finitely generated $D/(Q_1 \cap D)$-module, hence [49, Proposition 5.5], (3) and (9) respectively yield

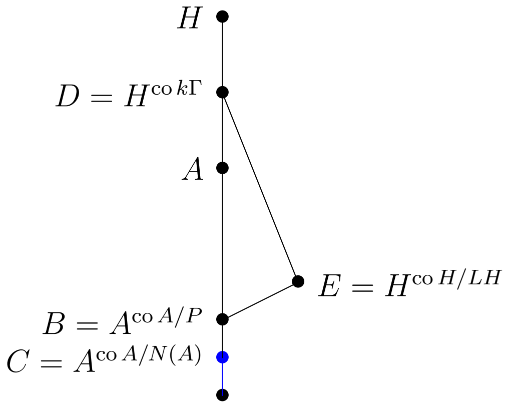
$$\operatorname{GKdim}(D/(Q_1 \cap D)) = \operatorname{GKdim}(H/Q_1) = d = \operatorname{GKdim}(D/L).$$

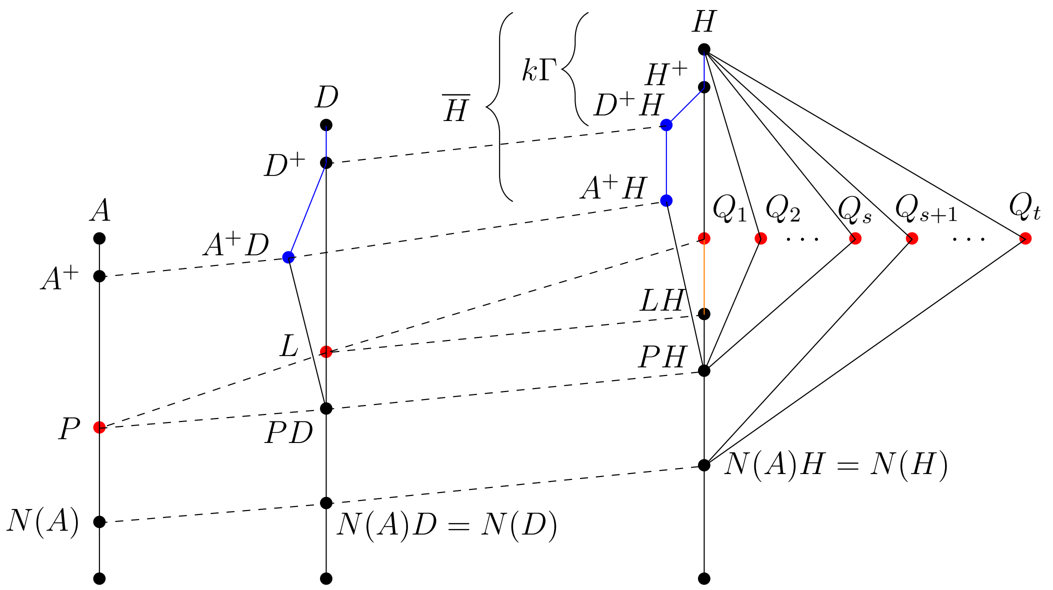
But, since a proper factor of an affine commutative domain has strictly lower GK-dimension [49, Proposition 3.15], we must have $Q_1 \cap D = L$.

(12) Since LH is a Hopf ideal by (10), we may consider the coinvariants E of the Hopf surjection $H \twoheadrightarrow H/LH$. On one hand, we have $LH \subseteq D^+H$, hence $H \twoheadrightarrow H/D^+H \cong k\Gamma$ factors through $H \twoheadrightarrow H/LH$ and $E = H^{\operatorname{co} H/LH} \subseteq H^{\operatorname{co} k\Gamma} = D$ by (3.17). On the other hand, since $D \subseteq H$ is faithfully flat, it follows from (3.18) that $LH \cap A = P$, hence $B = A^{\operatorname{co} A/P} \subseteq H^{H/LH} = E$. Furthermore, E is invariant under left adjoint action by [67, Lemma 3.4.2(2)].

(13) Suppose that H is pointed. By [60, Theorem] H is a faithfully flat left and right module over its left coideal subalgebra E. Then, as in Proposition 2.1.12, by [82, Corollary 4.3] $H \cong A\#_\sigma\overline{H}$, for some cocycle σ. Similarly, H/LH is pointed, so the inclusion $D/L \subseteq H/LH$ yields the decomposition $H/LH \cong D/L\#_\sigma k\Gamma$ for some cocycle τ. Moreover, this crossed product of a finite group acting faithfully on the commutative domain D/L is prime by [71, Corollary 12.6]. And minimality of Q_1 yields $Q_1 = LH$. □

In the following diagrams we summarize the prime and semiprime structure of commutative-by-(co)semisimple Hopf algebras explained in the previous theorem. Finite dimensional subalgebras of quotient algebras are coloured in blue, minimal prime ideals in red and conjectural equalities in orange.





As it turns out, this theorem leads to the rather striking conclusion that prime commutative-by-(co)semisimple Hopf algebras are in simple terms extensions of affine commutative domains by group algebras.

Corollary 3.4.4. *Let H be a prime commutative-by-finite Hopf algebra, finite over the affine normal commutative Hopf subalgebra A. Suppose that $\overline{H} = H/A^+H$ is semisimple and cosemisimple. Then,*

1. *$H/D^+H \cong k\Gamma$ for a finite group Γ whose order is a unit in k;*
2. *A and D are affine commutative domains;*
3. *There exists a group Λ which acts faithfully on D via the left adjoint action, and the group algebra $k\Lambda$ maps surjectively onto $k\Gamma$;*
4. *Suppose in addition that H is pointed. Then, H is a crossed product of D by $k\Gamma$, that is, $H \cong D\#_\sigma k\Gamma$ for some cocycle σ.*

Proof. The hypothesis that H is prime is equivalent to saying that, in the notation of Theorem 3.4.3, $Q_1 = \{0\}$. Thus, P and L are both $\{0\}$, and A and D are domains, so parts (1) and (2) follow from the definitions of D and Γ. And parts (3) and (4) follow from (10) and (13) of that theorem. □

The next example illustrates the intricate structure of commutative-by-semisimple Hopf algebras. It also shows that the inclusions of (3.16) and (3.20) can all be strict.

Example 3.4.5. Let k be algebraically closed of characteristic 0 and

$$G = (\langle x \rangle \times S_3) \rtimes C_2,$$

where S_3 is the group of permutations on 3 symbols and $\langle x \rangle$ is the infinite cyclic group. Let σ and β be respectively a 3-cycle and a 2-cycle in S_3 and let a be a generator of C_2. The action is as follows: C_2 acts trivially on S_3 and acts on $\langle x \rangle$ as $a \cdot x = x^{-1}$.

Let $H = kG$ and define $A = k(\langle x\rangle \times \langle\sigma\rangle)$. Clearly A is a commutative Hopf subalgebra of H and by Theorem 1.1.7 A is reduced. And A is normal, because $N = \langle x\rangle \times \langle\sigma\rangle$ is a normal subgroup of G:

$$\beta x\beta^{-1} = x, \qquad \beta\sigma\beta^{-1} = \sigma^{-1}, \qquad axa^{-1} = x^{-1}, \qquad a\sigma a^{-1} = \sigma.$$

Thus, H is commutative-by-finite with Hopf quotient $\overline{H} = k(\langle\beta\rangle \times C_2)$, which is semisimple (by Theorem 1.1.14) and cosemisimple. Note that H is not prime, since G contains nontrivial finite normal subgroups [70, Theorem 4.2.10], for instance S_3 and C_2.

Focusing on the structure of A, we now compute its unique minimal prime ideal P contained in A^+ and the respective coinvariants B. By Lemma 3.2.1(5) $P = \bigcap_{n\geq 1}(A^+)^n$ and $A^+ = A(x-1) + A(\sigma-1)$. Note that $A(\sigma-1)^3 = A(\sigma-1)^2 = A(\sigma-1)$ and $A(x-1)(\sigma-1) + A(\sigma-1) = A(\sigma-1)$, thus $(A^+)^n = A(x-1)^n + A(\sigma-1)$ and

$$P = \bigcap_{n\geq 1} A(x-1)^n + A(\sigma-1) = A(\sigma-1).$$

And it is easy to see the corresponding coinvariants are

$$B = A^{\operatorname{co} A/P} = A^{\operatorname{co} k\langle x\rangle} = k\langle\sigma\rangle.$$

Since the only element of G that acts nontrivially on x is $a \in C_2$, the H-action on $A/P \cong k\langle x\rangle$ factors through the inner faithfully action of $\Gamma = C_2$. Thus, it is easy to see that

$$D = H^{\operatorname{co} kC_2} = k(\langle x\rangle \times S_3).$$

And, similarly to the computation of P, one easily concludes that the unique minimal prime ideal of D contained in D^+ is

$$L = PD + (\beta-1)D = (\sigma-1)D + (\beta-1)D.$$

Note the isomorphism $A/P \cong D/L \cong k\langle x\rangle$ and, since H is pointed, it follows from Theorem 3.4.3(13) that

$$Q_1 = LH = PH + (\beta-1)H.$$

Moreover, it is very easy to prove that

$$E = H^{\operatorname{co} H/LH} = H^{k(\langle x\rangle \rtimes C_2)} = kS_3.$$

Thus in this case

$$C \subsetneq B \subsetneq A \subsetneq D \subsetneq H \quad \text{and} \quad B \subsetneq E \subsetneq D$$

with

$$PH \subsetneq LH = Q_1, \quad \text{and} \quad H \cong A\#\overline{H} \cong D\#\Gamma.$$

However, for most examples in section 2.2 the Hopf quotient $\overline{H}$ is not semisimple or cosemisimple but its action on A/P often reduces to a semisimple action, that is there exists a Hopf ideal I of $\overline{H}$ that annihilates A/P and such that $\overline{H}/I$ is semisimple; see proof of Proposition 3.1.13. Therefore, we leave the following rather broad question, possibly for future work.

Question 3.4.6. Let H be an affine commutative-by-finite Hopf algebra with commutative normal Hopf subalgebra A. Suppose the action of $\overline{H}$ on A/P reduces to a semisimple action. What is the structure of H?

We also leave a question regarding the representation theory of these Hopf algebras. Since the Hopf quotient $\overline{H}$ factors through a group algebra $k\Gamma$, it seems sensible to look into the representation theory of finite group algebras. An important result on this subject, Clifford's theorem [70, Theorem 7.2.16], provides a method that often allows one to find the simple modules of a finite group algebra using simple modules of the group algebras of its normal subgroups.

Question 3.4.7. Can we extend that idea to commutative-by-(co)semisimple Hopf algebras?

Chapter 4

Their finite dual

The object of study of this chapter is the finite dual of affine commutative-by-finite Hopf algebras. Recall the notion of finite dual from section 1.3. Throughout this chapter k denotes an algebraically closed field.

A commutative-by-finite Hopf algebra H contains the commutative normal Hopf subalgebra A and its Hopf quotient $\overline{H} := H/A^+H$ is finite dimensional. Thus, in section 4.1 we aim to describe the dual H° in terms of the duals A° and $\overline{H}^*$, both of which are easier to compute, the former since A is commutative and the latter because $\overline{H}$ is finite dimensional. This is accomplished in Theorem 4.1.5 which in part states the following:

Theorem 4.0.1. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Suppose that we have a decomposition $H = A \oplus X$ as right A-modules. If X can be chosen to be a coideal of H, then H° decomposes as the smash product*

$$H^\circ \cong \overline{H}^* \# A^\circ,$$

as left $\overline{H}^*$-modules, right A°-comodules and algebras.

This theorem requires the existence of a decomposition of $H = A \oplus X$ as right A-modules; in practice we know this holds under any of the hypotheses of Theorem 2.1.9(i)-(iv), that is when A is central in H, or A is reduced, or H is pointed. However, these hypotheses are not as restrictive as one might think at first. In fact, we know no examples where A cannot be chosen so that it satisfies the hypotheses above. In particular, most of the commutative-by-finite examples in section 2.2 satisfy these hypotheses and their dual decomposes as in this theorem; see section 4.4.

When A is reduced, its dual is well-understood and given by $A^\circ = A' \# kG$ according to Theorem 1.3.7, where G is the algebraic group such that $A \cong \mathcal{O}(G)$. And, although $\overline{H}^*$ is a Hopf subalgebra of H°, in general A° is not; under the conditions of the previous theorem, A° is a subalgebra of H° but it generally will not be a subcoalgebra. We study its coalgebra structure in H° as follows. We construct in sections 4.2 and 4.3 two subcoalgebras of H°, the tangential component $W(H)$ and the character component

$\widehat{kG}$, which respectively extend to H° the roles of the Hopf subalgebras A' and kG of A°. And we prove that (under certain hypotheses) these decompose as expected. Recall the notion of orbital semisimplicity from Definition 3.1.5.

Theorem 4.0.2. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. In addition, assume that A is reduced and let G denote its corresponding affine algebraic group. Then,*

1. *$W(H)$ is a Hopf subalgebra of H° that decomposes into the crossed product*

$$W(H) \cong \overline{H}^* \#_\sigma A',$$

for some cocycle σ and action of A' on $\overline{H}^*$.

2. *Moreover, if $A \subseteq H$ is orbitally semisimple and $H^\circ \cong \overline{H}^* \#_\tau A^\circ$ decomposes as a crossed product, then $\widehat{kG}$ is a Hopf subalgebra of H° that decomposes into*

$$\widehat{kG} \cong \overline{H}^* \#_{\tau|_{kG \otimes kG}} kG.$$

To sum up, one should have in mind the following picture of the dual of affine commutative-by-finite Hopf algebras, with a normal commutative reduced Hopf subalgebra.

$$\begin{tikzcd}
 & H^\circ \cong \overline{H}^* \#_\sigma (A' \# kG) \arrow[dl, dash] \arrow[dr, dash] & \\
\overline{H}^* \#_\sigma A' \cong W(H) \arrow[dr, dash] & & \widehat{kG} \cong \overline{H}^* \#_\sigma kG \arrow[dl, dash] \\
 & \overline{H}^* \arrow[d, dash] & \\
 & \bullet &
\end{tikzcd}$$

Lastly, in section 4.4 we use the previous results to decompose the dual of most of the examples of commutative-by-finite Hopf algebras from section 2.2, as well as their corresponding tangential and character components.

Many results in this chapter are generalizations of theorems discovered by Astrid Jahn [44, § 5], where she studied the dual of Hopf algebras that are finitely generated over affine central Hopf subalgebras (or, in other words, central-by-finite Hopf algebras); see Remark 4.4.10 at the end of the chapter. Having said that, most of section 4.3 is original, as are many computations in section 4.4.

4.1 Decompositions of the dual

In this section we aim to decompose the dual of an affine commutative-by-finite Hopf algebra H in terms of the duals of its normal commutative Hopf subalgebra A and the dual of its finite dimensional Hopf quotient $\overline{H}$. We start by revisiting the dual of commutative reduced Hopf algebras in §4.1.1. We then study the relationship between $\overline{H}^*$, A° and H°, and obtain a decomposition of H° in Theorem 4.1.5. We leave the computation of examples to section 4.4. Moreover, throughout most of the chapter, we fix the following notation:

Notation. $\iota : A \hookrightarrow H$ denotes the Hopf embedding and $\pi : H \twoheadrightarrow \overline{H}$ denotes the Hopf surjection. When in presence of any of the hypotheses of Theorem 2.1.9(i)-(iv), $\Pi : H = A \oplus X \twoheadrightarrow A$ denotes the projection of right A-modules along X. Moreover, as introduced in Definition 3.2.5, whenever A is reduced, G denotes the algebraic group such that $A \cong \mathcal{O}(G)$.

4.1.1 The dual of A - revisited

As we have done so far, let A denote the normal commutative Hopf subalgebra of a commutative-by-finite Hopf algebra H.

Assume A is reduced. Following the notation introduced in Definition 3.2.5, $A \cong \mathcal{O}(G)$ for the affine algebraic group $G = \mathrm{Maxspec}(A)$. Theorem 1.3.7 describes the well understood structure of the Hopf dual of the commutative Hopf subalgebra A,

$$A^\circ \cong A' \# kG. \tag{4.1}$$

Here

$$A' := \{f \in A^\circ : f((A^+)^n) = 0, \text{ for some } n > 0\}$$

and

$$G = \mathrm{Maxspec}(A) \cong \mathrm{Alg}(A, k) = G(A^\circ)$$

is the affine space of maximal ideals of A, which bijectively corresponds to the group of characters of A (that is, algebra maps $A \to k$). Hence, we can think of its group algebra as

$$kG = \{f \in A^\circ : f(\mathbf{m}_1 \cap \ldots \cap \mathbf{m}_r) = 0, r \geq 1, \mathbf{m}_i \in \mathrm{Maxspec}(A)\},$$

the space of functionals vanishing on some finite intersection of maximal ideals of A.

Also recall from Theorem 1.3.7(3) that, when k has characteristic 0, $A' \cong U(\mathfrak{g})$ is the enveloping algebra of the Lie algebra

$$\mathfrak{g} := \mathrm{Lie}\, G = P(A^\circ) \cong (A^+/(A^+)^2)^*,$$

that is, $\mathfrak{g}$ is the subspace of functionals on A^+ that vanish on $(A^+)^2$.

4.1.2 Subspaces and quotient spaces of the dual

We now discuss the relationship between the duals of H and $\overline{H}$.

Lemma 4.1.1. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Then,*

1. *There is an embedding of Hopf algebras*

$$\pi^\circ : \overline{H}^* \hookrightarrow H^\circ : f \mapsto f \circ \pi.$$

2. *$\pi^\circ(\overline{H}^*)$, which we identify with $\overline{H}^*$, is a normal Hopf subalgebra of H°.*

3. *H° is a free (right and left) $\overline{H}^*$-module.*

Proof. (1) We first show that π° is well-defined: given $f \in \overline{H}^*$, $\pi^\circ(f) = f \circ \pi$ obviously vanishes at A^+H which is an ideal of H of finite codimension by Theorem 2.1.3(2), thus $\pi^\circ(f) \in H^\circ$. Moreover, π° is an injective Hopf map by Proposition 1.3.14.

(2) We first note that

$$\pi^\circ(\overline{H}^*) = \{f \in H^\circ : f(A^+H) = 0\}. \tag{4.2}$$

Since A^+H is a Hopf ideal of H, $\pi^\circ(\overline{H}^*)$ is a Hopf subalgebra of H° by Lemma 1.3.13(3) and, since π° is injective, $\pi^\circ(\overline{H}^*)$ is isomorphic as a Hopf algebra to $\overline{H}^*$.

We claim that

$$\pi^\circ(\overline{H}^*) = \{f \in H^\circ : f(ah) = \epsilon(a)f(h) = f(ha), \forall a \in A, \forall h \in H\}.$$

Clearly any functional of the right-hand subspace is in $\pi^\circ(\overline{H}^*)$. Conversely, let $f \in \pi^\circ(\overline{H}^*)$ and $a \in A, h \in H$. Then, $a = \epsilon(a)1 + a'$ with $a' \in A^+$ and

$$f(ah) = \epsilon(a)f(h) + f(a'h) = \epsilon(a)f(h),$$

proving the first equality. The second equality follows from the fact that, since A is normal, $HA^+ = A^+H$ by Proposition 2.1.1.

To deduce normality of $\pi^\circ(\overline{H}^*)$, let $f \in \pi^\circ(\overline{H}^*)$ and $\varphi \in H^\circ$. For any $a \in A^+, h \in H$,

$$\begin{aligned}
(ad_r\varphi)(f)(ah) &= \left[\sum S^\circ(\varphi_1)f\varphi_2\right](ah) \\
&= \sum S^\circ(\varphi_1)(a_1h_1)f(a_2h_2)\varphi_2(a_3h_3) \\
&= \sum \varphi_1(S(h_1)S(a_1))\epsilon(a_2)f(h_2)\varphi_2(a_3h_3) \\
&= \sum \varphi(S(h_1)S(a_1)\epsilon(a_2)a_3h_3)f(h_2) \\
&= \epsilon(a)\sum \varphi(S(h_1)h_3)f(h_2) = 0.
\end{aligned}$$

By equation (4.2), $\pi^\circ(\overline{H}^*)$ is closed under right adjoint action of H°. Similarly, since $HA^+ = A^+H$, $\pi^\circ(\overline{H}^*)$ is closed under the left adjoint action.

(3) Since $\overline{H}^*$ is a finite dimensional and normal Hopf subalgebra of H°, H° is a free $\overline{H}^*$-module by [83, Theorem 2.1(2)]. □

Considering the previous result, from now on we identify $\overline{H}^*$ with

$$\pi^\circ(\overline{H}^*) = \{f \in H^\circ : f(A^+H) = 0\}.$$

Unfortunately, the relationship between the duals of H and A is not as "nice" as the one with the dual of $\overline{H}$. We look into it in the following result.

Lemma 4.1.2. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Then,*

1. *There is a surjective Hopf algebra map*
$$\iota^\circ : H^\circ \twoheadrightarrow A^\circ : f \mapsto f \circ \iota = f|_A .$$
2. *Via ι°, H° is canonically a right and left A°-comodule algebra, with coinvariants*
$$(H^\circ)^{\mathrm{co}\, A^\circ} = {}^{\mathrm{co}\, A^\circ}(H^\circ) = \overline{H}^*.$$
3. *The Hopf ideal $(\overline{H}^*)^+H^\circ$ of H° is contained in* $\ker \iota^\circ$.

Proof. (1) We first show that ι° is well-defined. Let $f \in H^\circ$, that is $f(J) = 0$ for some ideal J of H of finite codimension. Then, $J \cap A$ is an ideal of A of finite codimension, because $A/(J\cap A) \cong (A+J)/J \subseteq H/J$. Hence, $f|_A (J\cap A) = 0$, and $\iota^\circ(f) = f|_A \in A^\circ$.

Since ι is a Hopf algebra map, then so is the restriction map $\iota^\circ : f \mapsto f|_A$ by Proposition 1.3.14(1).

We now show it is surjective. Let $f \in A^\circ$, that is $f(I) = 0$ for some ideal I of A of finite codimension. We aim to find some functional $\phi \in H^\circ$ such that $\phi|_A = f$. Since A is commutative and noetherian, all its ideals satisfy the Artin-Rees property [64, Proposition 4.2.6]. In particular, there is a positive integer n such that

$$A \cap HI^n \subseteq I.$$

We claim the left ideal $J = HI^n$ has finite codimension in H. Since H is a finitely generated A-module, H/J is a finitely generated $A/(J \cap A)$-module and we need only show $J\cap A$ has finite codimension in A. And since $I^n \subseteq HI^n \cap A$, it suffices to show that A/I^n is finite dimensional. We prove this for A/I^2 and by induction it follows for any positive integer n. Since A is noetherian, both ideals I and I^2 are finitely generated, hence I/I^2 is a finitely generated A-module. But since I/I^2 is annihilated by I, it is in fact a finitely generated A/I-module. Now, since I has finite codimension, I/I^2 must be finite dimensional and, since $(A/I^2)/(I/I^2) \cong A/I$, A/I^2 is finite dimensional.

Now as a vector space we can decompose

$$H = (A + J) \oplus X,$$

for some subspace X. Let $\phi : H \to k$ be such that $\phi(J) = \phi(X) = 0$ and $\phi(a) = f(a), \forall a \in A$. Then, ϕ is well-defined, because $A \cap J \subseteq I$ and $f(I) = 0$. Thus, $\phi \in H^\circ$ by the previous claim and Theorem 1.3.5(1), and clearly $\phi|_A = f$ as required. This finishes the proof of surjectivity of ι°.

(2) H° is canonically a right A°-comodule algebra, with right coaction $\rho := (\mathrm{id} \otimes \iota^\circ) \circ \Delta$; see Example 1.1.10. The right coinvariants are the functionals f such that

$$\rho(f) = \sum f_1 \otimes (f_2 \circ \iota) = f \otimes \epsilon_A,$$

that is, the functionals such that $f(ha) = f(h)\epsilon(a)$ for all $h \in H, a \in A$. These are precisely the functionals in $\overline{H}^*$, as shown in the proof of Lemma 4.1.1(2). The left case is analogous.

(3) First, $\overline{H}^*$ is a normal Hopf subalgebra of H° by Lemma 4.1.1(2), so by Proposition 2.1.1 $(\overline{H}^*)^+ H^\circ$ is a Hopf ideal of H°. Let $f \in (\overline{H}^*)^+$. Then, $f(A^+H) = 0$ by definition of $\overline{H}^*$, and $\epsilon_{H^\circ}(f) = f(1) = 0$. Thus $f(A) = 0$, so that $f \in \ker \iota^\circ$. Since $\ker \iota^\circ$ is an ideal of H°, this proves that $(\overline{H}^*)^+ H^\circ \subseteq \ker \iota^\circ$. □

Remarks 4.1.3.

1. The proof of surjectivity of ι° is much simpler when the extension $A \subseteq H$ is faithfully flat, as is the case under any of the hypotheses in Theorem 2.1.9. For, let $f \in A^\circ$ with $f(I) = 0$ for some ideal I of H with finite codimension. Under faithful flatness we know $HI \cap A = I$, so H/HI is a finitely generated A/I-module, hence HI has finite codimension. Now proceeding as in the proof of (1), one constructs a functional $\phi \in H^\circ$ with $\phi|_A = f$.

2. An unsatisfactory gap in our knowledge lies on the fact that we do not know whether the inclusion $(\overline{H}^*)^+ H^\circ \subseteq \ker \iota^\circ$ is ever strict. It is an equality in an abundance of situations as we will see in Theorem 4.1.5(4). In particular, the equality holds for most of the examples in §2.2; see section 4.4.

The following result concerns the dual of the right A-module projection $\Pi : H \twoheadrightarrow A$, provided that H decomposes into $A \oplus X$ as right A-modules. In practice, we know such hypothesis holds under any of the conditions of Theorem 2.1.9(i)-(iv); in particular, it holds when k has characteristic 0.

Lemma 4.1.4. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Suppose that we have a right A-module decomposition $H = A \oplus X$, and consider the corresponding right A-module projection $\Pi : H \twoheadrightarrow A$. Recall from Lemma 4.1.2 that H° is a canonical right A°-comodule via ι°. Then,*

1. *There is a map of right A°-comodules*

$$\Pi^\circ : A^\circ \longrightarrow H^\circ : f \mapsto f \circ \Pi,$$

that is, $\Pi^\circ(f)|_A = f$ *and* $\Pi^\circ(f)(X) = 0$.

2. *We have* $\iota^\circ \circ \Pi^\circ = \mathrm{id}_{A^\circ}$. *Thus,* Π° *is injective.*

3. *We have*
$$(\overline{H}^*)^+ H^\circ \oplus \Pi^\circ(A^\circ) \subseteq \ker \iota^\circ \oplus \Pi^\circ(A^\circ) = H^\circ. \tag{4.3}$$

Proof. (1) We first prove that Π° is well-defined. Let $f \in A^\circ$, with $f(I) = 0$ for an ideal I of A of finite codimension. Since $H = A \oplus X$ decomposes as right A-modules, then $HI = I \oplus XI$ and, in particular, $HI \cap A = I$. Thus, H/HI is a finitely generated module over A/I, hence it is finite dimensional. Therefore, $\Pi^\circ(f)(HI) = f(HI \cap A) = f(I) = 0$ and the finite codimension of HI yields $\Pi^\circ(f) \in H^\circ$, proving well-definiteness of Π°.

The right A°-comodule structures of A° and H° are respectively given by Δ_{A° and $\rho := (\mathrm{id} \otimes \iota^\circ) \circ \Delta$. Let $f \in A^\circ$. Then for all $a \in A, h \in H$ we have

$$\begin{aligned}(\rho \circ \Pi^\circ)(f)(h \otimes a) &= \rho(f \circ \Pi)(h \otimes a) = \left[\sum (f \circ \Pi)_1 \otimes (f \circ \Pi)_2 \circ \iota\right](h \otimes a)\\ &= \sum (f \circ \Pi)_1(h)(f \circ \Pi)_2(a) = (f \circ \Pi)(ha) = f(\Pi(h)a),\end{aligned}$$

since Π is a right A-module map, and

$$(\Pi^\circ \otimes \mathrm{id})\Delta(f)(h \otimes a) = \left[\sum (\Pi^\circ \circ f_1) \otimes f_2\right](h \otimes a) = \sum f_1(\Pi(h))f_2(a) = f(\Pi(h)a).$$

Thus Π° is a right A°-comodule map.

(2) By construction, $(\iota^\circ \circ \Pi^\circ)(f) = \Pi^\circ(f)|_A = f$.

(3) Let $f \in \ker \iota^\circ \cap \Pi^\circ(A^\circ)$. Then, $f = \Pi^\circ(\varphi)$ for some $\varphi \in A^\circ$, hence

$$0 = \iota^\circ(f) = \iota^\circ \circ \Pi^\circ(\varphi) = \varphi$$

by (2), so $f = 0$. This proves the sum $\ker \iota^\circ + \Pi^\circ(A^\circ)$ is direct. The first inclusion in (4.3) is Lemma 4.1.2(3).

Let $f \in H^\circ$. Then, $(\Pi^\circ \circ \iota^\circ)(f) \in \Pi^\circ(A^\circ)$ and by (2)

$$\iota^\circ(f - (\Pi^\circ \circ \iota^\circ)(f)) = 0,$$

so that $H^\circ = \ker \iota^\circ \oplus \Pi^\circ(A^\circ)$, as required. □

4.1.3 The main result

Now that we have studied the basic relations between the duals of H, A and $\overline{H}$, we turn to study when H° can be decomposed into a smash or crossed product of A° and $\overline{H}^*$. Our results on this are gathered in the following theorem. It requires the existence of a right A-module decomposition $H = A \oplus X$, which in practice we only know holds under any of the hypotheses of Theorem 2.1.9(i)-(iv).

Theorem 4.1.5. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Suppose that we have a decomposition $H = A \oplus X$ as right A-modules, and consider the corresponding right A-module projection $\Pi : H \twoheadrightarrow A$.*

1. *Suppose that X can be chosen to be a coideal of H. Then, $\Pi^\circ : A^\circ \to H^\circ$ is an algebra map and H° decomposes as the smash product*

$$H^\circ \cong \overline{H}^* \# A^\circ, \tag{4.4}$$

as left $\overline{H}^*$-modules, right A°-comodules and algebras.

2. *Suppose that as an algebra $H = A\#_\sigma \overline{H}$ is a crossed product whose cleaving map $\gamma : \overline{H} \to H$ is a coalgebra map, and*

   (i) either A is central in H;

   (ii) or γ commutes with the antipodes, that is $S_H \circ \gamma = \gamma \circ S_{\overline{H}}$.

   For example, such is the case if $H = A\#\overline{H}$ decomposes as a smash product. Then, (1) applies and the action of A° on $\overline{H}^*$ is trivial, that is

$$H^\circ \cong \overline{H}^* \otimes A^\circ$$

as left $\overline{H}^*$-modules, right A°-comodules and algebras.

3. *Suppose X can be chosen to be an A-bimodule right (or left) ideal of H. Then, X is a two-sided ideal of H and H° decomposes as a crossed product,*

$$H^\circ \cong \overline{H}^* \#_\sigma A^\circ, \tag{4.5}$$

as left $\overline{H}^*$-modules, right A°-comodules and algebras. Moreover, the above isomorphism induces a coalgebra isomorphism $H^\circ \cong \overline{H}^* \otimes A^\circ$.

4. *In both cases (1) and (3), $\overline{H}^*$ is embedded into H° as the coefficient subalgebra of the smash or crossed product via π°, and the kernel of $\iota^\circ : H^\circ \to A^\circ$ is $(\overline{H}^*)^+ H^\circ$.*

Proof. (1) Since $X = \ker \Pi$ is a coideal, Π is a coalgebra map, and so the right A°-comodule map Π° of Lemma 4.1.4(1) is an algebra map. In particular, by Remark 1.1.18(1) it is a cleaving map with convolution inverse $\Pi^\circ \circ S_{A^\circ}$. By Theorem 1.1.17

$$H^\circ \cong \overline{H}^* \#_\sigma A^\circ$$

for some action of A° on $\overline{H}^*$ and cocycle σ. Since the cleaving map Π° is an algebra map, the cocycle σ is trivial.

(2) First, $H = A\gamma(\overline{H})$ as left A-modules. If (i) A is central, clearly $H = \gamma(\overline{H})A$ as right A-modules. And the same holds for (ii): since the antipodes S_H and S_A are bijective

and γ is such that $S_H \circ \gamma = \gamma \circ S_{\overline{H}}$, then

$$H = S(A\gamma(\overline{H})) = S\gamma(\overline{H})S(A) = \gamma S(\overline{H})A \subseteq \gamma(\overline{H})A.$$

Thus, in both cases $H = \gamma(\overline{H})A = A \oplus \gamma(\overline{H}^+)A$ as right A-modules. Since γ is a coalgebra map, $\epsilon(\gamma(\overline{H}^+)A) = \gamma(\epsilon(\overline{H}^+))A = 0$ and

$$\Delta(\gamma(\overline{H}^+)A) = \Delta\gamma(\overline{H}^+)\Delta(A) \subseteq \gamma(\overline{H}^+)A \otimes H + H \otimes \gamma(\overline{H}^+)A,$$

so $\gamma(\overline{H}^+)A$ is a coideal of H and (1) applies. Note that the corresponding projection $\Pi : H = \gamma(\overline{H})A \twoheadrightarrow A$ is given by $\Pi(\gamma(\bar{h})a) = \epsilon(\bar{h})a$ for all $\bar{h} \in \overline{H}, a \in A$.

It remains to prove the action of A° on $\overline{H}^*$ is trivial, that is in H°

$$\Pi^\circ(f')\pi^\circ(f) = \pi^\circ(f)\Pi^\circ(f'),$$

for all $f \in \overline{H}^*, f' \in A^\circ$.

We first claim that $\pi \circ \gamma = \mathrm{id}_{\overline{H}}$. Let $\bar{h} \in \overline{H}$. Since $\gamma : \overline{H} \to H$ is a right $\overline{H}$-comodule coalgebra map (where the coaction of $\overline{H}$ on H is $\rho = (\mathrm{id} \otimes \pi)\Delta$), we have

$$\sum \gamma(\bar{h_1}) \otimes \pi\gamma(\bar{h_2}) = \sum \gamma(\bar{h})_1 \otimes \pi(\gamma(\bar{h})_2) = \rho\gamma(\bar{h}) = (\gamma \otimes \mathrm{id})\Delta(\bar{h}) = \sum \gamma(\bar{h_1}) \otimes \bar{h_2}$$

and now applying $\epsilon \otimes \mathrm{id}$ to both sides yields $\pi\gamma(\bar{h}) = \bar{h}$, as claimed.

Let $f \in \overline{H}^*, f' \in A^\circ, a \in A, \bar{h} \in \overline{H}$. We have

$$\begin{aligned} \pi^\circ(f)\Pi^\circ(f')(\gamma(\bar{h})a) &= \sum \pi^\circ(f)(\gamma(\bar{h_1})a_1)\Pi^\circ(f')(\gamma(\bar{h_2})a_2) \\ &= \sum f\pi(\gamma(\bar{h_1})a_1)f'\Pi(\gamma(\bar{h_2})a_2) = \sum f\pi\gamma(\bar{h_1})\epsilon(a_1)\epsilon(\bar{h_2})f'(a_2) \\ &= \sum f(\bar{h_1})\epsilon(a_1)\epsilon(\bar{h_2})f'(a_2) = f(\bar{h})f'(a). \end{aligned}$$

One analogously proves that $\Pi^\circ(f')\pi^\circ(f)(\gamma(\bar{h})a) = f(\bar{h})f'(a)$. This completes the proof of (2).

(3),(4) Since $X = \ker \Pi$ is an A-bimodule and a right ideal of H, it is a two-sided ideal of H, as for any $x \in X$ and $h = a' + x' \in H = A \oplus X$ we have $hx = a'x + x'x \in X$. Then, Π is an algebra map and Π° is a coalgebra embedding that splits, for $\iota^\circ \circ \Pi^\circ = \mathrm{id}_{A^\circ}$ by Lemma 4.1.4(2). Then, by Remark 1.1.18(2) it is a cleaving map with convolution inverse $S^\circ \circ \Pi^\circ$ and by Theorem 1.1.17

$$H^\circ \cong \overline{H}^* \#_\sigma A^\circ,$$

for some action of A° on $\overline{H}^*$ and some cocycle σ.

In both cases (1) and (3), the isomorphism from $\overline{H}^* \# A^\circ$ or $\overline{H}^* \#_\sigma A^\circ$ to H° is given by $\zeta = m_{H^\circ} \circ (\pi^\circ \otimes \Pi^\circ)$. When restricting to $\overline{H}^*$, this map reduces to the injective Hopf map π°, so under this embedding $\overline{H}^*$ is a Hopf subalgebra of H°. In the second case, Π° is a coalgebra map and hence so is ζ, proving the final part of (3).

For the final claim, $(\overline{H}^*)^+ H^\circ \subseteq \ker \iota^\circ$ by Lemma 4.1.4(3). Fix a k-basis $\{f_i : i \in \mathcal{I}\}$ of A°. Then observe that, from the right side of either isomorphism (4.4) or (4.5), H° is a free left $\overline{H}^*$-module with basis $\{\Pi^\circ(f_i) : i \in \mathcal{I}\}$. Let $f \in \ker \iota^\circ$ and let us write $f = \sum \varphi_i \Pi^\circ(f_i)$, with $\varphi_i \in \overline{H}^*$. Then, for any $a \in A$

$$0 = f(a) = \sum \varphi_i(a_1)\Pi^\circ(f_i)(a_2) = \sum \epsilon(a_1)\varphi_i(1) f_i(a_2) = \left[\sum \varphi_i(1) f_i\right](a),$$

so $\sum \varphi_i(1) f_i = 0$ in A° and by linear independence we have $\varphi_i(1) = 0$ for all i, that is $f \in (\overline{H}^*)^+ H^\circ$. This proves the equality in the statement. □

Remarks 4.1.6.

1. As was mentioned before the theorem, the hypothesis that H decomposes into $A \oplus X$ as right A-modules is satisfied under any of the hypothesis of Theorem 2.1.9(i)-(iv). In particular, it holds for all examples from §2.2.

2. In general the decomposition (4.4) is not an isomorphism of Hopf algebras, not even of coalgebras. The coalgebra structure of H° will be clarified later on in sections 4.2 and 4.3. For instance, consider the group algebra of the dihedral group $H = kD$ defined in §2.2.5. It decomposes into the smash product $A \# \overline{H} = k\langle b \rangle * kC_2$, hence by Theorem 4.1.5(2), Example 1.3.3 and Example 1.3.10

$$H^\circ \cong \overline{H}^* \otimes A^\circ = kC_2 \otimes (k[f] \otimes k(k^\times, \cdot)).$$

   In A° the functional f is primitive and, for each $\lambda \in k^\times$, the associated functional χ_λ is grouplike. However, in H° neither the functional $\Pi^\circ(f)$ is primitive nor the functionals $z_\lambda := \Pi^\circ(\chi_\lambda)$ are grouplike in H°. In fact, $\Pi^\circ(f)$ is $(\alpha, 1)$-primitive and

$$\Delta(z_\lambda) = \frac{1}{2}\left(1 \otimes (1 + \alpha) + z_{\lambda^{-2}} \otimes (1 - \alpha)\right)(z_\lambda \otimes z_\lambda),$$

   where α is the generator of $\overline{H}^* = kC_2$; see Corollary 4.4.6(II). Still $kC_2 \otimes k[f]$ and $kC_2 \otimes k(k^\times, \cdot)$ are Hopf subalgebras of H° and in fact these will constitute the tangential and character components for this example.

3. Assume the hypotheses of Theorem 4.1.5(1). Since $\Pi^\circ : A^\circ \to H^\circ$ is a cleaving map with convolution inverse $\Pi^\circ \circ S_{A^\circ}$, by the formulas in Theorem 1.1.17 the action of A° on $\overline{H}^*$ in the smash product (4.4) is given by

$$f \cdot \varphi = \sum \Pi^\circ(f_1)\varphi\Pi^\circ(S_{A^\circ} f_2),$$

   for any $f \in A^\circ, \varphi \in \overline{H}^*$.

   When A is reduced, $A \cong \mathcal{O}(G)$ and its dual is $A^\circ \cong A' * G$ as in (4.1). The functionals indexed by G are algebra homomorphisms $\chi_g : A \to k$ and they are grouplike in A°, hence χ_g acts by conjugation over $\overline{H}^*$, that is for any $\varphi \in \overline{H}^*$

$$\chi_g \cdot \varphi = \Pi^\circ(\chi_g)\varphi\Pi^\circ(\chi_{g^{-1}}),$$

despite in general $\Pi^\circ(\chi_g)$ not being a grouplike in H°.

Moreover, if the base field k has characteristic 0, then $A' \cong U(\mathfrak{g})$ by Theorem 1.3.7(3). In this case, the elements of $\mathfrak{g}$ are primitive in A°, hence any $f \in \mathfrak{g}$ acts as a derivation on $\overline{H}^*$, that is, for any $\varphi \in \overline{H}^*$

$$f \cdot \varphi = \Pi^\circ(f)\varphi - \varphi\Pi^\circ(f),$$

even though in general $\Pi^\circ(f)$ will not be a primitive element of H°.

4. Note that equation (4.4) says that the Hopf dual of a commutative-by-finite Hopf algebra (that satisfies the hypothesis that X can be chosen to be a coideal) is a finite-by-cocommutative Hopf algebra, since it contains the finite dimensional normal Hopf subalgebra $\overline{H}^*$ and the quotient $H^\circ/(\overline{H}^*)^+H^\circ \cong A^\circ$ is cocommutative.

5. We discuss further the hypothesis of Theorem 4.1.5(1) that X can be chosen to be a coideal in the A-module decomposition $H = A \oplus X$. We first note that the existence of a right A-module coideal X is equivalent to the existence of a right A-module coalgebra projection $\Pi : H \twoheadrightarrow A$.

   In [80] Schauenburg introduced the notion of Π-crossed products, a generalization of the notion of crossed products discussed in subsection 1.1.5. It turns out that, if such a right A-module coalgebra projection Π exists and one of the conditions of Theorem 2.1.9(i)-(iv) is satisfied, then H decomposes as a Π-crossed product $A\#_\sigma\overline{H}$.

Proof. Consider the map $\gamma : \overline{H} \to H$ given by

$$\gamma(\overline{h}) = \sum h_1 S\Pi(h_2).$$

First, γ is well-defined, for if $h \in A^+H = HA^+$, say $h = \sum h_i a_i$ with $a_i \in A^+$, then

$$\sum h_{i1}a_{i1}S\Pi(h_{i2}a_{i2}) = \sum h_{i1}a_{i1}S(a_{i2})S\Pi(h_{i2}) = \sum \epsilon(a_i)\gamma(h_i) = 0.$$

Second, γ is a left H-comodule map, where $\overline{H}$ is a left H-comodule with coaction given by $\rho(\overline{h}) = \sum h_1 S\Pi(h_3) \otimes \overline{h_2}$; for we have

$$(\mathrm{id} \otimes \gamma)\rho(\overline{h}) = \sum h_1 S\Pi(h_3) \otimes \gamma(\overline{h_2}) = \sum h_1 S\Pi(h_4) \otimes h_2 S\Pi(h_3) = \Delta_H\gamma(\overline{h}).$$

Third, it is easy to check that γ is convolution invertible with inverse given by $\gamma^{-1}(\overline{h}) = \sum \Pi(h_1)S(h_2)$. Therefore, by [80, Theorem 5.14] H decomposes into the Π-crossed product $H^{\operatorname{co}\overline{H}}\#_\sigma\overline{H}$ and by Theorem 2.1.9(2) $H^{\operatorname{co}\overline{H}} = A$. $\square$

In particular, provided X can be chosen to be a coideal, H is a free A-module. We know this does not hold in general; see Example 2.1.13. Even so, this example is a commutative Hopf algebra, thus its dual is already well understood (see Theorem 1.3.7).

6. We know of no examples where (4.4) does not hold. This is illustrated in section 4.4, where we compute the Hopf dual of many of the examples of commutative-by-finite Hopf algebras introduced in §2.2. Therefore, the following question stands:

   Question 4.1.7. When H is affine commutative-by-finite, can A always be chosen so that H° is a smash product of $\overline{H}^*$ by A°?

We now turn our attention to the study of important subcoalgebras of the dual of commutative-by-finite Hopf algebras. More specifically we will construct two subcoalgebras of H°, $W(H)$ and $\widehat{kG}$, and study their properties. This will be done respectively in sections 4.2 and 4.3. These subcoalgebras shed some light on the coalgebra structure of the dual of commutative-by-finite Hopf algebras, of which we have not said much so far.

4.2 The tangential component $W(H)$

Recall that when A is reduced its dual decomposes into $A^\circ \cong A' * G$ as in (4.1), and its Hopf subalgebra A' is described in terms of functionals as

$$A' := \{f \in A^\circ : f((A^+)^n) = 0, \text{ for some } n > 0\}. \tag{4.6}$$

In this section we extend this construction to a Hopf subalgebra $W(H)$ of H°, which we call *tangential component*. We study some of its properties and we prove a decomposition in the final result of this section. In section 4.4 we compute the tangential component for many of the examples of commutative-by-finite Hopf algebras from §2.2. Throughout this section k continues to denote an algebraically closed field and A will often be reduced.

4.2.1 Definition and basic properties

In this subsection we define an analogue of the Hopf subalgebra A' for the finite dual of a commutative-by-finite Hopf algebra.

Recalling (4.6), the functionals in the Hopf subalgebra A' of A° are precisely the ones that vanish on some power of the augmentation ideal A^+. In order to extend this idea to the dual of H, we must consider functionals that vanish on some ideal of H. The most naive approach seems to be to consider the functionals that vanish on some power of the ideal A^+H of H. This turns out to be precisely the right approach.

Definition 4.2.1. Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. The *tangential component* of H° is

$$W(H) := \{f \in H^\circ : f((A^+H)^n) = 0, \text{ for some } n > 0\}.$$

This subspace $W(H)$ turns out to be a Hopf subalgebra of H° and it satisfies normality, as we state in the following result.

Lemma 4.2.2. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. Then,*

1. *$W(H)$ is a normal Hopf subalgebra of H°.*

2. *$\overline{H}^*$ is a normal Hopf subalgebra of $W(H)$.*

Proof. (1) First, since A is normal, by Proposition 2.1.1 A^+H is a Hopf ideal of H and by Lemma 1.3.13(3) $W(H)$ is a Hopf subalgebra of H°. We now prove its normality, in a similar fashion to the proof of normality of $\overline{H}^*$ in Lemma 4.1.1(2).

Let $\varphi \in H^\circ$ and $f \in W(H)$, so f vanishes at $(A^+H)^n$ for some positive integer n. Notice it also follows from Proposition 2.1.1 that $(A^+H)^n = (A^+)^nH$. Take $a_1, \ldots, a_n \in A^+$ and $h \in H$. Then,

$$\begin{aligned}
&\left[\sum S^\circ(\varphi_1)f\varphi_2\right](a_1 \ldots a_nh) \\
= &\sum S^\circ(\varphi_1)(a_{1,1} \ldots a_{n,1}h_1)f(a_{1,2} \ldots a_{n,2}h_2)\varphi_2(a_{1,3} \ldots a_{n,3}h_3) \\
= &\sum \varphi_1(S(h_1)S(a_{n,1}) \ldots S(a_{1,1}))f(a_{1,2} \ldots a_{n,2}h_2)\varphi_2(a_{1,3} \ldots a_{n,3}h_3) \\
= &\sum \varphi(S(h_1)S(a_{n,1}) \ldots S(a_{1,1})a_{1,3} \ldots a_{n,3}h_3)f(a_{1,2} \ldots a_{n,2}h_2) \\
= &\sum \varphi(S(h_1)S(a_{n,1})a_{n,3} \ldots S(a_{1,1})a_{1,3}h_3)f(a_{1,2} \ldots a_{n,2}h_2),
\end{aligned}$$

since A is commutative. For each i we write $a_{i,2} = \epsilon(a_{i,2})1 + a'_{i,2}$, where $a'_{i,2} \in A^+$. Writing $[1,n]$ for $\{1,2,\ldots,n\}$, by the binomial theorem we have

$$f(a_{1,2} \ldots a_{n,2}h_2) = \sum_{\mathcal{P}\subseteq[1,n]} \prod_{i\in\mathcal{P}} \epsilon(a_{i,2}) f\left(\prod_{i\in[1,n]\setminus\mathcal{P}} a'_{i,2}h_2\right).$$

Therefore,

$$
\begin{aligned}
& \left[\sum S^\circ(\varphi_1) f \varphi_2\right](a_1 \dots a_n h) \\
= & \sum \sum_{\mathcal{P}\subseteq[1,n]} \varphi(S(h_1)S(a_{n,1})a_{n,3}\dots S(a_{1,1})a_{1,3}h_3) \prod_{i\in\mathcal{P}} \epsilon(a_{i,2}) f \left(\prod_{i\in[1,n]\setminus\mathcal{P}} a'_{i,2}\, h_2\right) \\
= & \sum \sum_{\mathcal{P}\subseteq[1,n]} \varphi\left(S(h_1)\prod_{i\in\mathcal{P}} S(a_{i,1})\epsilon(a_{i,2})a_{i,3} \prod_{i\in[1,n]\setminus\mathcal{P}} S(a_{i,1})a_{i,3}\, h_3\right) f \left(\prod_{i\in[1,n]\setminus\mathcal{P}} a'_{i,2} h_2\right) \\
= & \sum \sum_{\mathcal{P}\subseteq[1,n]} \prod_{i\in\mathcal{P}} \epsilon(a_i)\, \varphi\left(S(h_1) \prod_{i\in[1,n]\setminus\mathcal{P}} S(a_{i,1})a_{i,3}\, h_3\right) f \left(\prod_{i\in[1,n]\setminus\mathcal{P}} a'_{i,2} h_2\right).
\end{aligned}
$$

The summands where $\mathcal{P} \neq \emptyset$ vanish due to $\epsilon(a_i) = 0$ for all $i \in [1, n]$ and the summands where $\mathcal{P} = \emptyset$ vanish because f is zero on $(A^+H)^n = (A^+)^n H$. The proof for the left adjoint action is similar, so normality of $W(H)$ in H° is proved.

(2) Clearly, $W(H)$ contains $\overline{H}^*$ by the identification of the latter subalgebra following Lemma 4.1.1. Since $\overline{H}^*$ is normal in H° by Lemma 4.1.1(2), it is obviously normal in the Hopf subalgebra $W(H)$. □

4.2.2 Decomposition of $W(H)$

In the following result we decompose the tangential component $W(H)$ into the crossed product $\overline{H}^* \#_\sigma A'$ when A is a reduced Hopf subalgebra of H. Recall the notation introduced at the beginning of §4.1, and from Lemma 4.1.2 the Hopf algebra surjection $\iota^\circ : H^\circ \to A^\circ, f \mapsto f|_A$ and (under certain hypotheses) from Lemma 4.1.4 the right A°-comodule map $\Pi^\circ : A^\circ \to H^\circ$ given by $\Pi^\circ(f)|_A = f, \Pi^\circ(f)(X) = 0$, where the decomposition $H = A \oplus X$ as right A-modules is afforded by Theorem 2.1.9(4).

Theorem 4.2.3. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative Hopf subalgebra A. In addition, assume that A is reduced, so that A° decomposes as $A' \# kG$ as in (4.1). Then,*

1. *$\iota^\circ(W(H)) = A'$.*
2. *$W(H)$ decomposes into the crossed product*
$$W(H) \cong \overline{H}^* \#_\sigma A', \tag{4.7}$$
for some cocycle σ and action of A' on $\overline{H}^*$.
3. *$W(H)^{\mathrm{co}\, A'} = \overline{H}^*$.*
4. *Suppose that X can be chosen to be a coideal in the A-module decomposition $H = A \oplus X$ of Theorem 2.1.9(4). Then, σ is trivial, that is (4.7) is a decomposition*

of $W(H)$ as the smash product

$$W(H) \cong \overline{H}^* \# A'. \tag{4.8}$$

When k has characteristic zero, $A' \cong U(\mathfrak{g})$ and the action of $U(\mathfrak{g})$ on $\overline{H}^*$ is given by

$$f \cdot \varphi = \Pi^\circ(f)\varphi - \varphi\Pi^\circ(f),$$

for $f \in \mathfrak{g}, \varphi \in \overline{H}^*$.

Proof. (1) First recall the description (4.6) of A'. Since ι° is just restriction of functionals to the subdomain A, it is clear that $\iota^\circ(W(H)) \subseteq A'$. To prove equality, first observe that Theorem 2.1.9(4) gives a right A-module decomposition $H = A \oplus X$. Given any $f \in A'$, with say $f((A^+)^n) = 0$, normality of A implies

$$(A^+H)^n = (HA^+)^n = H(A^+)^n = (A^+)^n \oplus X(A^+)^n$$

and by Lemma 4.1.4(1) $\Pi^\circ(f)$ vanishes at $(A^+H)^n$, that is $\Pi^\circ(f) \in H^\circ$ is actually in $W(H)$. Since $\iota^\circ \circ \Pi^\circ = \mathrm{id}_{A^\circ}$ by Lemma 4.1.4(2), $\iota^\circ(\Pi^\circ(f)) = f$ and the result follows.

(2),(3) By Lemma 4.1.2(2), H° is a right A°-comodule algebra with $\rho = (\mathrm{id} \otimes \iota^\circ)\Delta$. In particular, since $W(H)$ is a Hopf subalgebra of H° by Lemma 4.2.2(1), and using part (1) of the present theorem,

$$\rho(W(H)) \subseteq W(H) \otimes \iota^\circ(W(H)) = W(H) \otimes A'. \tag{4.9}$$

That is, the structure of right A°-comodule of H° restricts to a structure of right A'-comodule on $W(H)$.

We have noted in the proof of (1) that the injective right A°-comodule map $\Pi^\circ : A^\circ \to H^\circ$ maps A' into $W(H)$. Thus, in light of (4.9), $\Pi^\circ|_{A'} : A' \to W(H)$ is an injection of right A'-comodules.

A map is convolution invertible if and only if its restriction to the coradical is convolution invertible, see [96, Lemma 14] or [67, Lemma 5.2.10]. As pointed out in the proof of Theorem 1.3.7, A' is the irreducible component $(A^\circ)_\epsilon$ of A° containing $1_{A^\circ} = \epsilon$ and, since A° is cocommutative, it is pointed [67, Remarks at beginning of §5.6]. Hence, A' is irreducible and pointed, and therefore connected [67, Remarks after Definition 5.6.1], that is, its coradical is k and $\Pi^\circ|_k$ is trivially convolution invertible. Therefore, $\Pi^\circ|_{A'}$ is a cleaving map and by Theorem 1.1.17,

$$W(H) \cong W(H)^{\mathrm{co}\,A'} \#_\sigma A'.$$

Since the A'-comodule structure of $W(H)$ is the restriction of the A°-comodule structure of H°, it follows from Lemma 4.1.2(2) and Lemma 4.2.2(2) that

$$W(H)^{\mathrm{co}\,A'} = W(H) \cap (H^\circ)^{\mathrm{co}\,A^\circ} = W(H) \cap \overline{H}^* = \overline{H}^*.$$

(4) This follows from (2) and Theorem 4.1.5(1), since the cocycle σ in (2) is in this case trivial by the latter result. The last part is a special case of Remark 4.1.6(3). $\square$

Remarks 4.2.4.

1. Recall from Theorem 1.3.7(3) that when k has characteristic zero, $A' \cong U(\mathfrak{g})$ is the enveloping algebra of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. In this case, the previous theorem decomposes $W(H)$ as the crossed product $\overline{H}^* \#_\sigma U(\mathfrak{g})$. These are usually named "differential crossed products" and more information about them can be found in [67, Theorem 7.1.10].

2. As we mentioned in Remark 4.1.6(2), the isomorphism (4.8) is not of coalgebras. Even in characteristic zero where $A' \cong U(\mathfrak{g})$ is an enveloping algebra, any functional $f \in \mathfrak{g}$ is primitive in A° but in general its embedding $\Pi^\circ(f)$ is not primitive in H° (even though the functionals in $\mathfrak{g}$ act on $\overline{H}^*$ as derivations). This will become clearer in section 4.4 where we compute $W(H)$ for most of the examples of commutative-by-finite Hopf algebras introduced in §2.2.

3. When k has characteristic zero, we also know that $\ker\ \iota^\circ|_{W(H)} = (\overline{H}^*)^+ W(H)$.

   Proof. We know from Lemma 4.1.4(3) that

$$(\overline{H}^*)^+ W(H) \subseteq \ker\ \iota^\circ|_{W(H)} . \tag{4.10}$$

   Since k has characteristic zero, $A' \cong U(\mathfrak{g})$ for the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. It is clear from (4.7) that

$$W(H)/(\overline{H}^*)^+ W(H) \cong U(\mathfrak{g})$$

   and $W(H)/\ker\ \iota^\circ|_{W(H)} \cong \operatorname{im}\ \iota^\circ|_{W(H)} = U(\mathfrak{g})$ by (1). Since A is affine, $\mathfrak{g}$ is finite dimensional, and hence $U(\mathfrak{g})$ is a noetherian algebra [64, Corollary 1.7.4]. In particular $U(\mathfrak{g})$ cannot be isomorphic to a proper factor of itself, and hence (4.10) must be an equality, proving (4). $\square$

We now study the second important subspace of the dual of commutative-by-finite Hopf algebras, $\widehat{kG}$.

4.3 The character component $\widehat{kG}$

Throughout this section we assume that k is an algebraically closed field and A is reduced, so that $A = \mathcal{O}(G)$ is the coordinate ring of the algebraic group

$$G = \operatorname{Maxspec}(A) \cong \operatorname{Alg}(A, k) = G(A^\circ).$$

For each $g \in G$, we denote the corresponding maximal ideal and character of A respectively by $\mathbf{m}_g \in \operatorname{Maxspec}(A)$ and $\chi_g : A \to k$. In particular, notice that the maximal ideal associated to the identity 1_G is $\mathbf{m}_{1_G} = A^+$, the augmentation ideal of A, and the corresponding character is $\chi_{1_G} = \epsilon_A$, the counit of A.

Moreover, recall from the beginning of this chapter that A° decomposes into the smash product $A'\#kG$ as in (4.1). For each $g \in G$ the one-dimensional subspace $k\chi_g$ of the Hopf subalgebra kG can be thought of as the set of functionals that vanish on the maximal ideal $\mathbf{m}_g$. Therefore, the Hopf subalgebra kG of A° can be described as

$$kG = \{f \in A^\circ : f(\mathbf{m}_1 \cap \ldots \cap \mathbf{m}_r) = 0, r \geq 1, \mathbf{m}_i \in \operatorname{Maxspec}(A)\}. \tag{4.11}$$

In this section we extend the construction of the Hopf subalgebra kG of A° to the dual of H, by defining a subcoalgebra $\widehat{kG}$ of H°, the *character component*. We study the properties of $\widehat{kG}$; in particular, the nontrivial question of when it is a Hopf subalgebra of H°. And, similarly to what was done for $W(H)$, we prove some results on its decomposition. In §4.4 we compute the character component for many of the examples from §2.2.

4.3.1 Definition and basic properties

In order to define the character component $\widehat{kG}$ of the dual of commutative-by-finite Hopf algebras H, we make use here of the notions and results from section §3.1. Recall the notion of $\overline{H}$-core of an ideal of A, the equivalence relation $\sim^{(\overline{H})}$ on $\operatorname{Maxspec}(A)$ and the corresponding orbits.

As in §3.1 we denote the $\overline{H}$-core of a maximal ideal $\mathbf{m}_g$ of A by $\mathbf{m}_g^{(\overline{H})}$, this being the largest ideal contained in $\mathbf{m}_g$ which is invariant under the adjoint actions of H. Recall from Lemma 3.1.10 that, since we are assuming A is reduced, stability under left and right adjoint actions is equivalent, and by the same result $\mathbf{m}_g^{(\overline{H})}H = H\mathbf{m}_g^{(\overline{H})}$ is an ideal of H.

As we pointed out above, G can be thought of as $\operatorname{Maxspec}(A)$, so for simplicity we will consider $\sim^{(\overline{H})}$ to be an equivalence relation in G, where two elements $g \sim^{(\overline{H})} g'$ are equivalent if their corresponding maximal ideals have the same $\overline{H}$-core, that is $\mathbf{m}_g^{(\overline{H})} = \mathbf{m}_{g'}^{(\overline{H})}$. Recall the notion of $\overline{H}$-orbit associated to it as well. By Proposition 3.1.4(2),(3) each $\mathbf{m}_g^{(\overline{H})}$ has finite codimension in A and each orbit has the description $\mathcal{O}_{\mathbf{m}_g} = \{\mathbf{m}_h \in \operatorname{Maxspec}(A) : \mathbf{m}_g^{(\overline{H})} \subseteq \mathbf{m}_h\}$. Since H is a finitely generated A-module, the ideal $H\mathbf{m}_g^{(\overline{H})}$ also has finite codimension in H. This ensures that the following definition concerns subspaces of H°.

Definition 4.3.1. Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative semiprime Hopf subalgebra $A = \mathcal{O}(G)$.

1. The *character component* of H° is

$$\widehat{kG} := \left\{ f \in H^\circ : f\left(H\mathbf{m}_{g_1}^{(\overline{H})} \cap \ldots \cap H\mathbf{m}_{g_r}^{(\overline{H})}\right) = 0, \text{ for some } g_i \in G \right\}.$$

2. For each $g \in G$, we define the following subspace of $\widehat{kG}$:

$$\widehat{g} := \{f \in H^\circ : f(H\mathbf{m}_g^{(\overline{H})}) = 0\} = \left(H/H\mathbf{m}_g^{(\overline{H})}\right)^*.$$

Note that this is a natural generalization of the Hopf subalgebra kG of A° described in (4.11). Here are some basic properties of these spaces. Recall the injective map of right A°-comodules $\Pi^\circ : A^\circ \hookrightarrow H^\circ$ from Lemma 4.1.4.

Proposition 4.3.2. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative reduced Hopf subalgebra $A = \mathcal{O}(G)$. Then,*

1. *If $g, h \in G$ with $g \sim^{(\overline{H})} h$, then $\widehat{g} = \widehat{h}$.*
2. *$\widehat{kG} = \bigoplus_{g \in G/\sim^{(\overline{H})}} \widehat{g}$.*
3. *$\widehat{kG}$ is a subcoalgebra of H°.*
4. *For $g \in G$, $\widehat{g}$ is a subcoalgebra of $\widehat{kG}$.*
5. *For all $g \in G$,*

$$\widehat{g} \supseteq \bigoplus_{h \sim^{(\overline{H})} g} (H/H\mathbf{m}_h)^*,$$

and

$$(H/H\mathbf{m}_g)^* \supseteq \overline{H}^* \Pi^\circ(\chi_g) + \Pi^\circ(\chi_g)\overline{H}^*.$$

6. *For $g \in G$, $S^\circ(\widehat{g}) = \widehat{g^{-1}}$.*
7. *$\widehat{kG} \cap W(H) = \widehat{1_G} = \overline{H}^*$.*
8. *Assume that A is a domain. For $g \in G$,*

$$\dim_k(H/H\mathbf{m}_g) = \dim_k(H/\mathbf{m}_g H) = \dim_k(\overline{H}), \tag{4.12}$$

and

$$\dim_k(\widehat{g}) = \dim_k(A/\mathbf{m}_g^{(\overline{H})}) \dim_k(\overline{H}) \leq \left(\dim_k(\overline{H})\right)^2. \tag{4.13}$$

Proof. (1) This follows from the definition of $\sim^{(\overline{H})}$.

(2) If $g, h \in G$ with $g \nsim^{(\overline{H})} h$, then the ideals $\mathbf{m}_g^{(\overline{H})}$ and $\mathbf{m}_h^{(\overline{H})}$ of A are comaximal: by Proposition 3.1.4(3), the maximal ideals containing $\mathbf{m}_g^{(\overline{H})}$ are in the orbit $\mathcal{O}_{\mathbf{m}_g}$ and the same holds for h, but since $\mathcal{O}_{\mathbf{m}_g} \cap \mathcal{O}_{\mathbf{m}_h} = \varnothing$ no maximal ideal contains both $\mathbf{m}_g^{(\overline{H})}$ and $\mathbf{m}_h^{(\overline{H})}$, thus $\mathbf{m}_g^{(\overline{H})} + \mathbf{m}_h^{(\overline{H})} = A$.

Therefore, the ideals $H\mathbf{m}_g^{(\overline{H})}$ and $H\mathbf{m}_h^{(\overline{H})}$ of H are also comaximal and distinct subspaces $\widehat{g}$ intersect trivially. If $g_1, \ldots, g_r \in G$ are representatives of distinct $\sim^{(\overline{H})}$-orbits, the Chinese Remainder Theorem shows that

$$H \Big/ \bigcap_{i=1}^{r} H\mathbf{m}_{g_i}^{(\overline{H})} \cong \bigoplus_{i=1}^{r} H/H\mathbf{m}_{g_i}^{(\overline{H})}.$$

Taking duals on both sides yields

$$\left(H\Big/\bigcap_{i=1}^{r} H\mathbf{m}_{g_i}^{(\overline{H})}\right)^* \cong \left(\bigoplus_{i=1}^{r} H/H\mathbf{m}_{g_i}^{(\overline{H})}\right)^* \cong \bigoplus_{i=1}^{r}\left(H/H\mathbf{m}_{g_i}^{(\overline{H})}\right)^* = \bigoplus_{i=1}^{r}\widehat{g_i}.$$

Thus, any functional in $\widehat{kG}$ belongs to $\bigoplus_{g\in G/\sim^{(\overline{H})}}\widehat{g}$ and the reverse inclusion is clear.

(3),(4) For each $g \in G$, since $H\mathbf{m}_g^{(\overline{H})}$ is an ideal of H, $\widehat{g}$ is a subcoalgebra of H° by Lemma 1.3.13(3). By (2), $\widehat{kG}$ is also a subcoalgebra of H°.

(5) Let $g \in G$. By Proposition 3.1.4(3) $\mathbf{m}_g^{(\overline{H})} \subseteq \mathbf{m}_h$ for any $h \sim^{(\overline{H})} g$. Therefore, in terms of the dual, the functionals that vanish in $\bigcap_{h\sim^{(\overline{H})}g} H\mathbf{m}_h$ also vanish in $H\mathbf{m}_g^{(\overline{H})}$, that is,

$$\widehat{g} = \left(H/H\mathbf{m}_g^{(\overline{H})}\right)^* \supseteq \left(H\Big/\bigcap_{h\sim^{(\overline{H})}g} H\mathbf{m}_h\right)^* \cong \bigoplus_{h\sim^{(\overline{H})}g}(H/H\mathbf{m}_h)^*,$$

since as in the proof of (2) the left ideals $H\mathbf{m}_h$ of H are comaximal.

Aiming to prove the second inclusion, we first claim that for any $g, h \in G$

$$\Delta(\mathbf{m}_{gh}) \subseteq \mathbf{m}_g \otimes A + A \otimes \mathbf{m}_h. \tag{4.14}$$

We can decompose $A \otimes A$ as a vector space into $(\mathbf{m}_g \otimes A + A \otimes \mathbf{m}_h) \oplus k1_A \otimes 1_A$. Let $a \in \mathbf{m}_{gh}$ and write $\Delta(a) = \sum a_1 \otimes a_2 + \lambda 1 \otimes 1$ in terms of the previous decomposition. Since each functional $\chi_g : A \to k$ vanishes at the corresponding $\mathbf{m}_g$ and $\chi_g\chi_h = \chi_{gh}$ by the equivalence $G \cong \mathrm{Alg}(A, k)$, then we have

$$0 = \chi_{gh}(a) = \chi_g\chi_h(a) = \sum \chi_g(a_1)\chi_h(a_2) + \lambda = \lambda.$$

Since a is an arbitrary element of $\mathbf{m}_{gh}$, the claim follows.

Therefore,

$$\Delta(H\mathbf{m}_g) = \Delta(H)\Delta(\mathbf{m}_g) \subseteq H\mathbf{m}_{1_G} \otimes H + H \otimes H\mathbf{m}_g. \tag{4.15}$$

Let $f \in \overline{H}^*$, that is, f vanishes at $A^+H = HA^+$. Then, using (4.15) and the fact that $\mathbf{m}_{1_G} = A^+$,

$$(f\Pi^\circ(\chi_g))(H\mathbf{m}_g) \subseteq f(H\mathbf{m}_{1_G})\Pi^\circ(\chi_g)(H) + f(H)\Pi^\circ(\chi_g)(H\mathbf{m}_g) = 0.$$

This proves $\overline{H}^*\Pi^\circ(\chi_g) \subseteq (H/H\mathbf{m}_g)^*$. The proof for $\Pi^\circ(\chi_g)\overline{H}^*$ is analogous.

(6) Let $g \in G$. To prove that $S^\circ(\widehat{g}) = \widehat{g^{-1}}$ it suffices to show that

$$S\left(\mathbf{m}_g^{(\overline{H})}\right) = \mathbf{m}_{g^{-1}}^{(\overline{H})}. \tag{4.16}$$

For, if $f \in \widehat{g}$ then by stability of $\mathbf{m}_g^{(\overline{H})}$ we have

$$S^\circ(f)\left(H\mathbf{m}_{g^{-1}}^{(\overline{H})}\right) = f\left(S(\mathbf{m}_{g^{-1}}^{(\overline{H})})S(H)\right) = f\left(\mathbf{m}_g^{(\overline{H})}H\right) = f\left(H\mathbf{m}_g^{(\overline{H})}\right) = 0.$$

And, conversely, any functional $f \in \widehat{g^{-1}}$ can clearly be written as $f = S^\circ((S^{-1})^\circ(f))$ and as before $(S^{-1})^\circ(f)$ vanishes at $S^{-1}(\mathbf{m}_{g^{-1}}^{(\overline{H})}H) = \mathbf{m}_g^{(\overline{H})}H$, that is, $f \in S^\circ(\widehat{g})$.

To prove (4.16), we first note that it follows easily from the equivalence between G and $\mathrm{Maxspec}(A)$ that $S(\mathbf{m}_g) = \mathbf{m}_{g^{-1}}$. Also, as pointed out at the beginning of §4.3.1, left and right $\overline{H}$-cores coincide, that is, ${}^{(\overline{H})}\mathbf{m}_g = \mathbf{m}_g^{(\overline{H})}$. Then, by Lemma 3.1.12(1)

$$S\left(\mathbf{m}_g^{(\overline{H})}\right) = S(\mathbf{m}_g)^{(\overline{H})} = \mathbf{m}_{g^{-1}}^{(\overline{H})}.$$

(7) Since $\mathbf{m}_{1_G} = A^+$, we have $\widehat{1_G} = (H/A^+H)^* = \overline{H}^*$. This also shows that $\overline{H}^* \subseteq \widehat{kG}$ and, by Lemma 4.2.2(2), $\overline{H}^* \subseteq W(H)$.

For the reverse inclusion, suppose $f \in W(H) \cap \widehat{kG}$, that is f vanishes on $(A^+H)^n = (A^+)^nH$ and on $I := \bigcap_{i=1}^r \mathbf{m}_{g_i}^{(\overline{H})}H$, for some $n, r \geq 1$ and distinct $g_1, \ldots, g_r \in G/\sim^{(\overline{H})}$. We claim that

$$A^+H \subseteq I + (A^+)^nH. \tag{4.17}$$

On one hand, if $g_i \neq 1_G$ for all $i = 1, \ldots, r$, then I and $(A^+)^nH$ are comaximal by the discussion in the proof of (2), so that (4.17) is clear.

Suppose on the other hand that, say, $g_1 = 1_G$. If either $r = 1$ or $n = 1$, the inclusion (4.17) follows immediately, so we may assume now $n, r \geq 2$. We know that $I + (A^+)^nH \subseteq A^+H$ and the left A-module $A^+H/(I + (A^+)^nH)$ is annihilated by both of the ideals $(A^+)^{n-1}$ and by $\bigcap_{i=2}^r \mathbf{m}_{g_i}^{(\overline{H})}$. Again, as in the proof of (2) these ideals are comaximal, meaning $A^+H/(I + (A^+)^nH)$ is annihilated by A. Thus, $A^+H/(I + (A^+)^nH) = \{0\}$ and we have equality in (4.17).

Thus, f vanishes at A^+H, that is $f \in \overline{H}^*$, as required.

(8) Assume that A is a domain. Localize A at the maximal ideal $\mathbf{m}_g$, and consider the $A_{\mathbf{m}_g}$-module $H_{\mathbf{m}_g} := A_{\mathbf{m}_g} \otimes_A H$. Since H is a finitely generated projective A-module by Theorem 2.1.9(3), H is a locally free A-module with constant rank

$$r = \dim_k(\overline{H}),$$

by the same argument as in the proof of Corollary 3.3.2: $H_{\mathbf{m}_g}$ is a free $A_{\mathbf{m}_g}$-module by [77, Theorem 4.44 and following comments] with finite constant rank equal to $r = \dim_Q(Q \otimes_A H)$, where Q denotes the quotient field of A, and by Nakayama's lemma $r = \dim_{A_{\mathbf{m}_g}}(A_{\mathbf{m}_g} \otimes_A H) = \dim_{A_{A^+}}(A_{A^+} \otimes_A H) = \dim_k(\overline{H})$.

Now first, as left $A/\mathbf{m}_g$-modules we have

$$\begin{aligned} H_{\mathbf{m}_g}/\mathbf{m}_gH_{\mathbf{m}_g} &\cong (A_{\mathbf{m}_g}/\mathbf{m}_gA_{\mathbf{m}_g}) \otimes_{A_{\mathbf{m}_g}} H_{\mathbf{m}_g} = (A_{\mathbf{m}_g}/\mathbf{m}_gA_{\mathbf{m}_g}) \otimes_{A_{\mathbf{m}_g}} A_{\mathbf{m}_g} \otimes_A H \\ &\cong (A/\mathbf{m}_g) \otimes_A H \cong H/\mathbf{m}_gH. \end{aligned}$$

And by Nakayama's lemma [78, Remark 8.25] a k-basis of $H_{\mathbf{m}_g}/\mathbf{m}_g H_{\mathbf{m}_g}$ lifts to a minimal spanning set (hence a basis) of $H_{\mathbf{m}_g}$ over $A_{\mathbf{m}_g}$. Therefore,

$$\dim_k(H/\mathbf{m}_g H) = \dim_k(H_{\mathbf{m}_g}/\mathbf{m}_g H_{\mathbf{m}_g}) = \dim_{A_{\mathbf{m}_g}}(H_{\mathbf{m}_g}) = r.$$

Working with H as a right A-module yields the same conclusion for $\dim_k(H/H\mathbf{m}_g)$, proving (4.12).

Moreover, as above we have an isomorphism $H_{\mathbf{m}_g}/\mathbf{m}_g^{(H)} H_{\mathbf{m}_g} \cong H/\mathbf{m}_g^{(H)} H$ of $A/\mathbf{m}_g^{(H)}$-modules. But since $H_{\mathbf{m}_g} \cong (A_{\mathbf{m}_g})^{\oplus r}$, then $H_{\mathbf{m}_g}/\mathbf{m}_g^{(H)} H_{\mathbf{m}_g} \cong (A/\mathbf{m}_g^{(H)})^{\oplus r}$. Thus,

$$\dim_k(\widehat{g}) = \dim_k(H/\mathbf{m}_g^{(H)} H) = \dim_k(H_{\mathbf{m}_g}/\mathbf{m}_g^{(H)} H_{\mathbf{m}_g}) = r \dim_k(A/\mathbf{m}_g^{(H)}).$$

The inequality in (4.13) now follows from Corollary 3.3.2. $\square$

One aspect of $\widehat{kG}$ we do not discuss in the previous result is its algebra structure. This turns out to be more complex than one might think at first and we devote the next subsection to its discussion.

4.3.2 Subalgebras and subcoalgebras of $\widehat{kG}$

In this subsection we consider the effect on the structure of $\widehat{kG}$ of imposing the following additional hypotheses:

(i) the kernel X of Π can be chosen to be a coideal of H (see Remark 4.1.6(5));

(ii) $A \subseteq H$ is orbitally semisimple (see §3.1.2 and §3.1.3).

We show that these hypotheses have significant effect on the algebra structure of $\widehat{kG}$. In particular, under condition (i) we gain more information on the set of units of H° and the structure of $(H/H\mathbf{m}_g)^*$, while condition (ii) implies that $\widehat{kG}$ is closed under multiplication (hence a Hopf subalgebra of H°) with a nice formula for the product of the subcoalgebras $\widehat{g}$.

As we have pointed out before, we know of no examples where these conditions are not satisfied; see Remark 4.1.6(6) and Proposition 3.1.13. In particular, they are satisfied by every Hopf algebra in section 4.4, and may even hold for all affine commutative-by-finite Hopf algebras.

Recall the injection $\Pi^\circ : A^\circ \to H^\circ$ of right A°-comodules introduced in Lemma 4.1.4, and which is an algebra map when X can be chosen to be a coideal, as in Theorem 4.1.5(1). Also recall from Lemma 4.1.2 the Hopf surjection $\iota^\circ : H^\circ \to A^\circ$, given by restriction of functionals on H to A.

Lemma 4.3.3. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative reduced Hopf subalgebra $A = \mathcal{O}(G)$. Assume that the decomposition $H = A \oplus X$ of right A-modules can be achieved with X a coideal in H. Set $\widetilde{G} := \{\Pi^\circ(\chi_g) : g \in G\} \subseteq H^\circ$.*

1. *$\widetilde{G}$ is a subgroup of the group of units of H°, and the algebra injection $\Pi^\circ : A^\circ \to H^\circ$ restricts to an isomorphism of groups from the grouplikes G of A° to $\widetilde{G}$.*

2. *The subalgebra of H° generated by $\widetilde{G}$ is the group algebra $k\widetilde{G}$, and Π° restricts to an algebra isomorphism from kG to $k\widetilde{G}$.*

3. *For all $g \in G$, $\overline{H}^*\Pi^\circ(\chi_g) = \Pi^\circ(\chi_g)\overline{H}^*$ is free of rank 1 as an $\overline{H}^*$-bimodule.*

4. *Suppose that A is a domain. Then, for all $g \in G$*

$$(H/H\mathbf{m}_g)^* = \overline{H}^*\Pi^\circ(\chi_g),$$

hence

$$\widehat{g} \supseteq \bigoplus_{h \sim^{(\overline{H})} g} (H/H\mathfrak{m}_h)^* = \bigoplus_{h \sim^{(\overline{H})} g} \overline{H}^*\Pi^\circ(\chi_h).$$

Proof. (1),(2) Note that X being a coideal of H means the projection Π along X is a coalgebra map, hence Π° is an algebra homomorphism and it is an embedding by Lemma 4.1.4. Since the functionals χ_g are units of A° (each χ_g with inverse $\chi_{g^{-1}}$), then Π° maps G isomorphically to a subgroup $\widetilde{G}$ of the group of units of H°, and

$$k\widetilde{G} = \Pi^\circ(kG) \cong kG.$$

(3) By Remark 4.1.6(3), the action of $kG \subseteq A^\circ$ on $\overline{H}^*$ is given by

$$\chi_g \cdot f = \Pi^\circ(\chi_g) f \Pi^\circ(\chi_{g^{-1}}) \in \overline{H}^*,$$

for any $f \in \overline{H}^*, g \in G$. Hence $\Pi^\circ(\chi_g)f \in \overline{H}^*\Pi^\circ(\chi_g)$, proving $\Pi^\circ(\chi_g)\overline{H}^* \subseteq \overline{H}^*\Pi^\circ(\chi_g)$ and the converse is proved similarly.

Suppose that $f \in \overline{H}^*$ is such that $f\Pi^\circ(\chi_g) = 0$, where the multiplication takes place in H°. Then,

$$f\Pi^\circ(\chi_g)\Pi^\circ(\chi_{g^{-1}}) = 0$$

and hence, by part (1) of the lemma, $f = f\epsilon_H = 0$. This proves that $\overline{H}^*\Pi^\circ(\chi_g)$ is a free $\overline{H}^*$-bimodule of rank 1 with generator $\Pi^\circ(\chi_g)$, proving (3).

(4) By Proposition 4.3.2(5) $\overline{H}^*\Pi^\circ(\chi_g) \subseteq (H/H\mathbf{m}_g)^*$. Suppose A is a domain. Then, comparing dimensions using (4.12) in Proposition 4.3.2(8) and part (3) above yields $(H/H\mathbf{m}_g)^* = \overline{H}^*\Pi^\circ(\chi_g)$. The last statement is given by Proposition 4.3.2(5). □

Recall the notion of orbital semisimplicity from Definition 3.1.5 and the results and comments regarding it in subsections 3.1.2 and 3.1.3. In the following result we show the effect of this condition on $\widehat{kG}$.

Proposition 4.3.4. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative reduced Hopf subalgebra $A = \mathcal{O}(G)$. Suppose that $A \subseteq H$ is orbitally semisimple. Then,*

1. *The subspace $\widehat{kG}$ is a Hopf subalgebra of H°.*

2. *For $g, h \in G$,*

$$\widehat{g}\widehat{h} \subseteq \sum_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} \widehat{g'h'}. \tag{4.18}$$

3. *For $g \in G$, $\widehat{g}$ is a left and right $\overline{H}^*$-module.*

4. *Assume further that A is a domain. Then, for any $g \in G$,*

$$\dim_k(\widehat{g}) = \left|\mathcal{O}_{\mathbf{m}_g}\right| \dim_k(\overline{H})$$

and

$$\widehat{g} = \bigoplus_{h \sim^{(\overline{H})} g} (H/H\mathbf{m}_h)^* = \bigoplus_{h \sim^{(\overline{H})} g} (H/\mathbf{m}_h H)^*.$$

Proof. By Proposition 4.3.2, (1) will follow if $\widehat{kG}$ is closed under multiplication. Moreover, by Proposition 4.3.2(2), closure under multiplication will follow at once from (2). Note that (3) is a special case of (2), since $\overline{H}^* = \widehat{1_G}$. We prove (2) now.

We first claim that for distinct $g_1, \ldots, g_n \in G$ we have

$$\bigcap_{i=1}^{n}(H\mathbf{m}_{g_i}) = H\left(\bigcap_{i=1}^{n}\mathbf{m}_{g_i}\right).$$

For, since $\mathbf{m}_{g_1}, \ldots, \mathbf{m}_{g_n}$ are comaximal, the Chinese Remainder theorem yields the short exact sequence $0 \to \bigcap_{i=1}^n \mathbf{m}_{g_i} \to A \to \bigoplus_{i=1}^n A/\mathbf{m}_{g_i} \to 0$ and, when applying $H \otimes_A -$, the flatness of H_A afforded by Theorem 2.1.9(1) yields the exact sequence

$$0 \to H\left(\bigcap_{i=1}^{n}\mathbf{m}_{g_i}\right) \to H \to H \otimes_A \left(\bigoplus_{i=1}^{n} A/\mathbf{m}_{g_i}\right) \cong \bigoplus_{i=1}^{n} H/H\mathbf{m}_{g_i} \to 0.$$

But the kernel of the map $H \to \bigoplus_{i=1}^n H/H\mathbf{m}_{g_i}$ is $\bigcap_{i=1}^n(H\mathbf{m}_{g_i})$ and exactness of the previous sequence proves the claim.

We now claim that proving (4.18) amounts to showing that

$$\Delta\left(\bigcap_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} \mathbf{m}_{g'h'}\right) \subseteq \mathbf{m}_g^{(\overline{H})} \otimes A + A \otimes \mathbf{m}_h^{(\overline{H})} =: J. \tag{4.19}$$

For, given (4.19), the first claim gives

$$\Delta\left(\bigcap_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} H\mathbf{m}_{g'h'}^{(\overline{H})}\right) \subseteq H\mathbf{m}_g^{(\overline{H})} \otimes H + H \otimes H\mathbf{m}_h^{(\overline{H})}.$$

Consider now $\beta\gamma$ with $\beta \in \widehat{g}$ and $\gamma \in \widehat{h}$. It follows that

$$\beta\gamma \left(\bigcap_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} H\mathbf{m}_{g'h'}^{(\overline{H})} \right) \subseteq \beta \left(H\mathbf{m}_g^{(\overline{H})} \right) \gamma(H) + \beta(H)\gamma \left(H\mathbf{m}_h^{(\overline{H})} \right) = 0,$$

hence

$$\beta\gamma \in \left(H \Big/ \bigcap_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} H\mathbf{m}_{g'h'}^{(\overline{H})} \right)^* \cong \sum_{\substack{g' \sim^{(\overline{H})} g, \\ h' \sim^{(\overline{H})} h}} \left(H/H\mathbf{m}_{g'h'}^{(\overline{H})} \right)^* = \sum_{\substack{g' \sim^{(\overline{H})} g, \\ h' \sim^{(\overline{H})} h}} \widehat{g'h'}.$$

Let us prove (4.19). By (4.14) in the proof of Proposition 4.3.2,

$$\begin{aligned} \Delta \left(\bigcap_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} \mathbf{m}_{g'h'} \right) &\subseteq \bigcap_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} \Delta\left(\mathbf{m}_{g'h'}\right) \\ &\subseteq \bigcap_{g' \sim^{(\overline{H})} g, h' \sim^{(\overline{H})} h} \left(\mathbf{m}_{g'} \otimes A + A \otimes \mathbf{m}_{h'}\right) =: I. \end{aligned}$$

Thus, (4.19) will follow if it can be shown that $I = J$. Since A is orbitally semisimple, $\mathbf{m}_g^{(\overline{H})} = \bigcap_{g' \sim^{(\overline{H})} g} \mathbf{m}_{g'}$ for all $g \in G$ and so $J \subseteq I$. To prove equality, let $r = |\mathcal{O}_{\mathbf{m}_g}|$ and $s = |\mathcal{O}_{\mathbf{m}_h}|$. Then, on one hand, by the Chinese Remainder theorem

$$(A \otimes A)/I \cong \bigoplus_{\substack{g' \sim^{(\overline{H})} g, \\ h' \sim^{(\overline{H})} h}} (A \otimes A)/(\mathbf{m}_{g'} \otimes A + A \otimes \mathbf{m}_{h'}) \cong \bigoplus_{\substack{g' \sim^{(\overline{H})} g, \\ h' \sim^{(\overline{H})} h}} (A/\mathbf{m}_{g'}) \otimes (A/\mathbf{m}_{h'}),$$

so $\dim_k((A \otimes A)/I) = rs \dim_k(A/\mathbf{m}_{g'}) \dim_k(A/\mathbf{m}_{h'}) = rs$. And, on the other hand, by orbital semisimplicity we have

$$\dim_k((A \otimes A)/J) = \dim_k \left(A/\mathbf{m}_g^{(\overline{H})} \right) \dim_k \left(A/\mathbf{m}_h^{(\overline{H})} \right) = rs.$$

This completes the proof of (4.19).

(4) Since A is orbitally semisimple, $\left|\mathcal{O}_{\mathbf{m}_g}\right| = \dim_k(A/\mathbf{m}_g^{(\overline{H})})$, hence the first statement follows from (4.13), and $H/H\mathbf{m}_g^{(\overline{H})} = \bigoplus_{h \sim^{(\overline{H})} g} H/H\mathbf{m}_h$, so the second statement follows from the definition of $\widehat{g}$. □

The formula (4.18) is as good a formula for the product of the subcoalgebras $\widehat{g}$ as we can hope, as the following example illustrates.

Example 4.3.5. Consider $H = kD$ the group algebra of the dihedral group as in §2.2.5. Its commutative normal Hopf subalgebra is $A = k[b^{\pm 1}]$, which is reduced and whose corresponding group of characters is $G = k^\times$. The Hopf quotient $\overline{H}$ is kC_2.

Consider $\mathbf{m}_g = A(b - g)$ for some $g \in k^\times$. The $\overline{H}$-action on $\mathbf{m}_g$ is determined by $a(b - g)a = b^{-1} - g = -b^{-1}g(b - g^{-1})$, hence $a \cdot \mathbf{m}_g = \mathbf{m}_{g^{-1}}$. Thus, the $\overline{H}$-orbits of G are determined by $g \sim^{(\overline{H})} g^{-1}$.

Recall from Proposition 3.1.13 that $A \subseteq H$ is orbitally semisimple and from section 2.2.5 that $H = kD$ decomposes as the smash product $A\#\overline{H}$; in particular, it satisfies the hypothesis of Lemma 4.3.3 (as explained in Theorem 4.1.5). By Proposition 4.3.4(4) and Lemma 4.3.3(4),

$$\widehat{g} = \bigoplus_{h \sim^{(\overline{H})} g} (H/H\mathbf{m}_h)^* = \bigoplus_{h \sim^{(\overline{H})} g} \overline{H}^* \Pi^\circ(\chi_h).$$

Since $\overline{H}^* \cong kC_2$, $\widehat{1} = \overline{H}^* = kC_2$ and $\widehat{-1} = kC_2 \otimes k\Pi^\circ(\chi_{-1})$ and, for $g \in k^\times \setminus \{\pm 1\}$,

$$\widehat{g} = kC_2 \otimes (k\Pi^\circ(\chi_g) + k\Pi^\circ(\chi_{g^{-1}})).$$

Hence, taking $g, h \in k^\times \setminus \{\pm 1\}$ we have

$$\widehat{g}\widehat{h} = kC_2 \otimes [k\Pi^\circ(\chi_{gh}) + k\Pi^\circ(\chi_{g^{-1}h}) + k\Pi^\circ(\chi_{gh^{-1}}) + k\Pi^\circ(\chi_{g^{-1}h^{-1}})] = \widehat{gh} + \widehat{g^{-1}h},$$

that is in this example we obtain equality in (4.18).

In a similar fashion to what we did for $W(H)$, we now turn our attention to the problem of decomposing $\widehat{kG}$.

4.3.3 Decomposition of $\widehat{kG}$

In this subsection we suppose $A \subseteq H$ is orbitally semisimple and discuss the decomposition of the Hopf subalgebra $\widehat{kG}$. We first prove a few technical statements in an auxiliary lemma and move to the most important result of section 4.3, where we display a decomposition of $\widehat{kG}$ into a crossed product of the Hopf subalgebra $\overline{H}^*$ of H° and the Hopf subalgebra kG of A°.

Recall from Lemma 4.1.4 the right A°-comodule embedding $\Pi^\circ : A^\circ \hookrightarrow H^\circ$ and from Lemma 4.1.2 the Hopf surjection $\iota^\circ : H^\circ \twoheadrightarrow A^\circ$, so that H° is a right A°-comodule algebra via the map $\rho := (\mathrm{id} \otimes \iota^\circ) \circ \Delta$.

Lemma 4.3.6. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative reduced Hopf subalgebra $A = \mathcal{O}(G)$. Suppose that $A \subseteq H$ is orbitally semisimple. The following hold for each $g \in G$:*

1. $\iota^\circ(\widehat{g}) = \bigoplus_{h \sim^{(\overline{H})} g} k\chi_h$.
2. $\widehat{g} \supseteq \{f \in H^\circ : \rho(f) \in f \otimes \sum_{h \sim^{(\overline{H})} g} k\chi_h\}$*; and for* $g = 1_G$ *we have* $\overline{H}^* = \widehat{1_G} = \{f \in H^\circ : \rho(f) = f \otimes 1\}$.

Proof. (1) Let $g \in G$ and take $f \in \widehat{g}$, so $f|_A(\mathbf{m}_g^{(\overline{H})}) = 0$. Since $A \subseteq H$ is orbitally semisimple, by the Chinese remainder theorem

$$\iota^\circ(f) = f|_A \in \left(A/\mathbf{m}_g^{(\overline{H})}\right)^* \cong \bigoplus_{h \sim^{(\overline{H})} g} (A/\mathbf{m}_h)^* = \bigoplus_{h \sim^{(\overline{H})} g} k\chi_h.$$

Conversely, let $\alpha = \sum_{h \sim^{(\overline{H})} g} \lambda_h \chi_h \in A^\circ$ for some $\lambda_h \in k$. Then, $\alpha = \iota^\circ(\Pi^\circ(\alpha))$ by Lemma 4.1.4(2). Since $H = A \oplus X$ as right A-modules,

$$\Pi^\circ(\alpha)\left(H\mathbf{m}_g^{(\overline{H})}\right) = \Pi^\circ(\alpha)\left(\mathbf{m}_g^{(\overline{H})} \oplus X\mathbf{m}_g^{(\overline{H})}\right) = \alpha(\mathbf{m}_g^{(\overline{H})}) = \sum_{h \sim^{(\overline{H})} g} \lambda_h \chi_h(\mathbf{m}_g^{(\overline{H})}) = 0,$$

as $\mathbf{m}_g^{(\overline{H})} \subseteq \mathbf{m}_h$ for every $h \sim^{(\overline{H})} g$. Thus, $\Pi^\circ(\alpha) \in \widehat{g}$ and $\alpha \in \iota^\circ(\widehat{g})$, proving (1).

(2) Let $f \in H^\circ$ and suppose $\rho(f) = f \otimes \sum_{h \sim^{(\overline{H})} g} \lambda_h \chi_h$ for some $\lambda_h \in k$. Then,

$$\begin{aligned} f(H\mathbf{m}_g^{(\overline{H})}) &= \sum f_1(H) f_2(\mathbf{m}_g^{(\overline{H})}) = \sum f_1(H)\ f_2|_A(\mathbf{m}_g^{(\overline{H})}) \\ &= \rho(f)(H \otimes \mathbf{m}_g^{(\overline{H})}) = f(H)\left(\sum_{h \sim^{(\overline{H})} g} \lambda_h \chi_h(\mathbf{m}_g^{(\overline{H})})\right) = 0, \end{aligned}$$

as $\mathbf{m}_g^{(\overline{H})} \subseteq \mathbf{m}_h$ for every $h \sim^{(\overline{H})} g$. Hence $f \in \widehat{g}$.

Now let $f \in \overline{H}^* = \widehat{1_G}$. By part (1) and the fact that 1_G constitutes its own $\sim^{(\overline{H})}$-coset,

$$\rho(f) \in \rho(\overline{H}^*) \subseteq (\mathrm{id} \otimes \iota^\circ)(\overline{H}^* \otimes \overline{H}^*) = \overline{H}^* \otimes k\epsilon_A,$$

say $\rho(f) = f' \otimes \epsilon_A$ for some $f' \in \overline{H}^*$. By the counit axiom of right comodules,

$$f = f' \epsilon_{A^\circ}(\epsilon_A) = \epsilon_A(1_A) f' = f',$$

hence $\rho(f) = f \otimes 1_{A^\circ}$. □

We can now show that, when $A \subseteq H$ is orbitally semisimple, $\widehat{kG}$ decomposes as a crossed or smash product whenever H° does, generalizing the presence of the group algebra kG in the dual of a commutative semiprime Hopf algebra. Recall that sufficient conditions for H° to decompose as a crossed product are given in Theorem 4.1.5.

Theorem 4.3.7. *Let H be an affine commutative-by-finite Hopf algebra, finite over the normal commutative reduced Hopf subalgebra $A = \mathcal{O}(G)$. Suppose $A \subseteq H$ is orbitally semisimple.*

1. *Suppose that $H^\circ \cong \overline{H}^* \#_\sigma A^\circ$ decomposes as a crossed product for some cocycle σ and action of A° on $\overline{H}^*$. Then,*

$$\widehat{kG} \cong \overline{H}^* \#_{\sigma|_{kG \otimes kG}} kG \tag{4.20}$$

as algebras, left $\overline{H}^*$-modules and right kG-comodules.

2. *Suppose X can be chosen to be a coideal in the decomposition $H = A \oplus X$ as right A-modules. Then, (4.20) is a decomposition of $\widehat{kG}$ as the smash product*

$$\widehat{kG} \cong \overline{H}^* \# kG. \tag{4.21}$$

The action of kG on $\overline{H}^*$ is given by

$$\chi_g \cdot \varphi = \Pi^\circ(\chi_g)\varphi\Pi^\circ(\chi_{g^{-1}}),$$

for any $g \in G, \varphi \in \overline{H}^*$, that is as an algebra

$$\widehat{kG} \cong \overline{H}^* * G$$

is a skew group ring over $\overline{H}^*$.

3. *Conversely, if $\widehat{kG} \cong \overline{H}^* \#_\tau kG$ decomposes as a crossed product with a cleaving map $\gamma : kG \to H^\circ$ that can be extended to a right A°-comodule map $\gamma' : A^\circ \to H^\circ$, then H° decomposes as a crossed product $\overline{H}^* \#_\sigma A^\circ$ as algebras, left $\overline{H}^*$-modules and right A°-comodules such that the cocycles $\sigma_{|kG\otimes kG}$ and τ coincide.*

Proof. (1) Suppose that $H^\circ \cong \overline{H}^* \#_\sigma A^\circ$. By Theorem 1.1.17 there exists a convolution invertible right A°-comodule map $\gamma : A^\circ \to H^\circ$ such that the isomorphism is given by $f\#\varphi \mapsto f\gamma(\varphi)$. Also recall from Theorem 1.3.7 that $A^\circ \cong A'\#kG$. The core of this proof is showing that

$$\widehat{g} = \bigoplus_{h \sim^{(\overline{H})} g} \overline{H}^* \gamma(\chi_h). \tag{4.22}$$

We first claim that $\gamma(\chi_h) \in \widehat{g}$. Since $\gamma : A^\circ \to H^\circ$ is a right A°-comodule map (with H° and A° being A°-comodules respectively by ρ and Δ) and χ_h is a grouplike element of A°, then

$$\rho\gamma(\chi_h) = (\gamma \otimes \mathrm{id})\Delta(\chi_h) = \gamma(\chi_h) \otimes \chi_h$$

and the claim follows from Lemma 4.3.6(2). Therefore, by Proposition 4.3.4(3)

$$\overline{H}^* \gamma(\chi_h) \subseteq \overline{H}^* \widehat{g} \subseteq \widehat{g},$$

proving the inclusion ($\supseteq$) of (4.22).

For the reverse inclusion, let $f \in \widehat{g}$. By the first sentence of the proof, $f = \sum_{i=1}^n f_i\gamma(u_i\#\chi_{g_i})$, where $f_i \in \overline{H}^*, u_i \in A', g_i \in G$. Since ρ is an algebra homomorphism and $\gamma : A^\circ \to H^\circ$ is a right A°-comodule map, then

$$\begin{aligned} \rho(f) &= \sum_{i=1}^n \rho(f_i)\rho\gamma(u_i\#\chi_{g_i}) = \sum_{i=1}^n \rho(f_i)(\gamma\otimes \mathrm{id})\Delta(u_i\#\chi_{g_i}) \\ &= \sum_{i=1}^n \sum_{(u_i)} f_i\gamma(u_{i,1}\#\chi_{g_i}) \otimes (u_{i,2}\#\chi_{g_i}), \end{aligned}$$

the last equality holding because $\chi_{g_i} \in A^\circ$ is grouplike and $\rho(f_i) = f_i \otimes 1$ by Lemma 4.3.6(2). But since $f \in \widehat{g}$ and by Proposition 4.3.2(4) $\widehat{g}$ is a subcoalgebra of H°, then by Lemma 4.3.6(1) we have

$$\rho(f) = (\mathrm{id} \otimes \iota^\circ)\Delta(f) \in \widehat{g} \otimes \iota^\circ(\widehat{g}) = \bigoplus_{h \sim^{(\overline{H})} g} \widehat{g} \otimes \chi_h.$$

Moreover, since each u_i is a polynomial on primitive elements, we must have $u_i = 1$ and $g_i \sim^{(\overline{H})} g$ for every i. Hence $f = \sum_{i=1}^n f_i \gamma(\chi_{g_i})$ has the required form, and (4.22) follows.

Therefore, by Proposition 4.3.2(2) and (4.22)

$$\widehat{kG} = \bigoplus_{g \in G/\sim^{(\overline{H})}} \widehat{g} = \bigoplus_{g \in G} \overline{H}^* \gamma(\chi_g) = \overline{H}^* \gamma(kG).$$

By the calculations above $\rho(\widehat{kG}) \subseteq \widehat{kG} \otimes kG$, that is the A°-comodule structure of H° restricts to a kG-comodule structure for $\widehat{kG}$. In particular, by Lemma 4.1.2(2) and Proposition 4.3.2(7)

$$\widehat{kG}^{\mathrm{co}\, kG} = \widehat{kG} \cap (H^\circ)^{\mathrm{co}\, A^\circ} = \widehat{kG} \cap \overline{H}^* = \overline{H}^*.$$

Also, since $\gamma : A^\circ \to H^\circ$ is convolution invertible, then so is its restriction to the coradical of $A^\circ = A' \# kG$, [96, Lemma 14]. Since A' is connected, its coradical is trivial (as in the proof of Theorem 4.2.3), so the coradical of A° is kG and $\gamma|_{kG} : kG \to \widehat{kG}$ is convolution invertible. Therefore, by Theorem 1.1.17

$$\widehat{kG} \cong \overline{H}^* \#_\tau kG$$

and by inspecting the formulae for the cocycles σ and τ one easily sees that $\tau = \sigma|_{kG \otimes kG}$.

(2) If X can be chosen to be a coideal, then $H^\circ \cong \overline{H}^* \# A^\circ$ by Theorem 4.1.5(1), hence the cocycle σ is trivial and $\widehat{kG}$ decomposes as a smash product. The action of kG on $\overline{H}^*$ was justified in Remark 4.1.6(3).

(3) Suppose that $\widehat{kG}$ decomposes as stated. Since the coradical of A° is kG and the restriction $\gamma'|_{kG} = \gamma$ is convolution invertible, then γ' is also convolution invertible by [67, Lemma 5.2.10]. By Theorem 1.1.17

$$H^\circ \cong \overline{H}^* \#_\sigma A^\circ$$

and, again by inspecting their formulae, the cocycles $\sigma_{|kG \otimes kG}$ and τ coincide. □

Remark 4.3.8. Note that, as we mentioned back in Remark 4.1.6(2), the isomorphism (4.21) is not of coalgebras for in general the functionals $\Pi^\circ(\chi_g)$ indexed by the group G will not be grouplike in $\widehat{kG}$ (even though they act by conjugation on $\overline{H}^*$). This is illustrated in the next section, where we compute the character component of most of the commutative-by-finite Hopf algebras introduced in section 2.2.

4.4 Examples

In this section we compute the duals of most of the examples of commutative-by-finite Hopf algebras introduced in section 2.2, as well as the tangential component $W(H)$ and the character component $\widehat{kG}$ which we studied in sections 4.2 and 4.3. It is important to point out that the decompositions shown below are as algebras and in general not as coalgebras; in some cases, we explicitly present the coalgebra structure of H°. Here k continues to denote an algebraically closed field.

Quantized enveloping algebras of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$

We begin by computing the dual of two quantized enveloping algebras, which were introduced in §2.2.1. Note that the dual of $U_\epsilon(\mathfrak{sl}_2(k))$ has already been computed by Astrid Jahn, [44, §5.4]. Recall the restricted quantized coordinate rings $o_\epsilon(G)$ from subsection 2.2.2.

Corollary 4.4.1. *Let k be an algebraically closed field of characteristic zero and ϵ be a primitive lth root of unity of k, with l odd. Then,*

1. *The dual of the quantized enveloping algebra of $\mathfrak{sl}_2(k)$ is*

$$U_\epsilon(\mathfrak{sl}_2)^\circ \cong o_\epsilon(SL_2)\# \left(U(\mathfrak{g})\# k\left((k,+)^2 \rtimes k^\times\right)\right),$$

where $\mathfrak{g}$ is the Lie algebra with basis $\{a,b,c\}$ and brackets $[a,b]=0, [a,c]=a, [b,c]=b$. Moreover, this dual contains the Hopf subalgebras

$$W(U_\epsilon(\mathfrak{sl}_2)) \cong o_\epsilon(SL_2)\# U(\mathfrak{g})$$

and

$$\widehat{kG} \cong o_\epsilon(SL_2)\# k\left((k,+)^2 \rtimes k^\times\right).$$

2. *The dual of the quantized enveloping algebra of $\mathfrak{sl}_3(k)$ is*

$$U_\epsilon(\mathfrak{sl}_3)^\circ \cong o_\epsilon(SL_3)\# \left(U(\mathfrak{h})\# k\left((k,+)^6 \rtimes (k^\times)^2\right)\right),$$

where $\mathfrak{h}$ is the Lie algebra with basis $\{e_1,e_2,e_3,f_1,f_2,f_3,k_1,k_2\}$ and brackets

$$[e_1,e_2]=e_3, \quad [f_1,f_2]=-f_3, \quad [e_1,k_1]=-e_1, \quad [e_2,k_2]=-e_2,$$

$$[e_3,k_i]=-e_3, \quad [f_1,k_1]=-f_1, \quad [f_2,k_2]=-f_2, \quad [f_3,k_i]=-f_3$$

and zero elsewhere. In addition, it contains the Hopf subalgebras

$$W(U_\epsilon(\mathfrak{sl}_3)) \cong o_\epsilon(SL_3)\# U(\mathfrak{h})$$

and

$$\widehat{kG} \cong o_\epsilon(SL_3)\#k\left((k,+)^6 \rtimes (k^\times)^2\right).$$

The isomorphisms are of algebras, left $\overline{H}^*$-modules and right A°-comodules.

Proof. Let $E_i, F_i, K_i^{\pm 1}$ (with $1 \le i \le n-1$) be the generators of $H = U_\epsilon(\mathfrak{sl}_n(k))$. We denote all roots (simple and nonsimple) by E_{β_j}, F_{β_j} (with $1 \le j \le N := (n-1)n/2$). It is known that $A = k[K_i^{\pm l}, E_{\beta_j}^l, F_{\beta_j}^l]$ is a central Hopf subalgebra, over which $H = U_\epsilon(\mathfrak{sl}_n(k))$ is a free module with basis of monomials

$$F_{\beta_N}^{r_N}\dots F_{\beta_1}^{r_1}K_1^{s_1}\dots K_{n-1}^{s_{n-1}}E_{\beta_1}^{t_1}\dots E_{\beta_N}^{t_N},$$

where $0 \le r_i, s_i, t_i < l$; see [13, §III.6.2], [26, §19.1], [21, Proposition 9.2.7]. Note that A is obviously reduced, being in fact a domain.

The structure of this commutative Hopf subalgebra A is well-known. By [13, III.6.5], the corresponding affine algebraic group is the semidirect product

$$G = ((k,+)^N \times (k,+)^N) \rtimes (k^\times)^{n-1},$$

where $(k^\times)^{n-1}$ acts on $(k,+)^{2N}$ by multiplication. Therefore, by Theorem 1.3.7 we have

$$A^\circ \cong U(\mathfrak{g})\#k((k,+)^{2N} \rtimes (k^\times)^{n-1}),$$

where the Lie algebra is $\mathfrak{g} = \operatorname{Lie} G = (A^+/(A^+)^2)^*$ and its Lie brackets can be computed by commutators on A°.

Furthermore, recall from subsection 2.2.1 that the Hopf quotient $\overline{H}$ is the restricted quantized enveloping algebra $u_\epsilon(\mathfrak{sl}_n)$, whose dual is

$$u_\epsilon(\mathfrak{sl}_n)^* = o_\epsilon(SL_n),$$

the restricted quantized coordinate ring of SL_n; see for example [13, III.7.10].

We now focus on the case $n = 2$. In her thesis Astrid Jahn showed that we can decompose $U_\epsilon(\mathfrak{sl}_2(k)) = A \oplus X$ where the right A-submodule X has generators

$$\{F^rK^sE^t : 0 \le r,s,t < l \text{ and } r,t \text{ not both zero}\} \cup \{K^s - 1 : 1 \le s < l\}$$

and X is a coideal of $U_\epsilon(\mathfrak{sl}_2(k))$; see [44, §5.4]. Then, by Theorem 4.1.5(1) we have

$$U_\epsilon(\mathfrak{sl}_2(k))^\circ \cong \overline{H}^*\#A^\circ = o_\epsilon(SL_2)\#(U(\mathfrak{g})\#k((k,+)^2 \rtimes k^\times)).$$

The computations of the Lie brackets of $\mathfrak{g}$ can also be found in [44, §5.4].

Now consider quantized enveloping algebra $U_\epsilon(\mathfrak{sl}_3(k))$. A few more computations will show that there is a right A-module coalgebra projection defined, for $a \in A$, by

$$\begin{array}{rccll}
\Pi : & U_\epsilon(\mathfrak{sl}_3) & \to & A & \\
& aK_1^{s_1}K_2^{s_2} & \mapsto & a, & 0 \le s_i < l \\
& aF_3^{r_3}F_2^{l-r_3}F_1^{l-r_3}K_1^{s_1}K_2^{s_2} & \mapsto & \mu a F_3^l, & 1 \le r_3 < l \\
& aK_1^{s_1}K_2^{s_2}E_1^{l-t_3}E_2^{l-t_3}E_3^{t_3} & \mapsto & \xi a E_3^l, & 1 \le t_3 < l \\
& aF_3^{r_3}F_2^{l-r_3}F_1^{l-r_3}K_1^{s_1}K_2^{s_2}E_1^{l-t_3}E_2^{l-t_3}E_3^{t_3} & \mapsto & \mu\xi a F_3^l E_3^l, & 1 \le r_3, t_3 < l \\
& \text{elsewhere} & \mapsto & 0, &
\end{array}$$

where $\mu = \epsilon^{-r_3(r_3-1)/2}(1-\epsilon^{-2})^{r_3-l}$ and $\xi = \epsilon^{t_3(t_3-1)/2}(\epsilon-\epsilon^{-1})^{t_3-l}$. These calculations can be found in the Appendix, §A.1. It follows from Theorem 4.1.5(1) that

$$H^\circ \cong \overline{H}^* \# A^\circ = o_\epsilon(SL_3)\#(U(\mathfrak{h})\#k((k,+)^6 \rtimes (k^\times)^2)).$$

It remains to compute the Lie algebra $\mathfrak{h}$. As mentioned above, we know $\mathfrak{h} = (A^+/(A^+)^2)^*$. Since A^+ is generated as an ideal of A by the 8 elements $E_i^l, F_i^l, K_1^l - 1, K_2^l - 1$ ($i = 1,2,3$), their images under the quotient map $A^+ \twoheadrightarrow A^+/(A^+)^2$ must generate $A^+/(A^+)^2$. But $\dim_k(A/(A^+)^2) = \dim_k \mathfrak{h} = \dim_k G = 8$, hence those elements must be a basis. Let $\{e_i, f_i, k_1, k_2 : i = 1,2,3\}$ be the dual basis of $(A^+/(A^+)^2)^*$; by setting them to be zero at $(A^+)^2$ and 1_A they embed into A° as follows: for $\alpha = E_1^{t_1 l}E_2^{t_2 l}E_3^{t_3 l}F_1^{r_1 l}F_2^{r_2 l}F_3^{r_3 l}K_1^{s_1 l}K_2^{s_2 l}$, we have

$$e_i(\alpha) = \begin{cases} 1, & \text{for } \alpha = E_i^l K_1^{s_1 l}K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases}, \qquad f_i(\alpha) = \begin{cases} 1, & \text{for } \alpha = F_i^l K_1^{s_1 l}K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases},$$

$$k_i(\alpha) = \begin{cases} s_i, & \text{for } \alpha = K_1^{s_1 l}K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases}.$$

The Lie brackets of $\mathfrak{h} = \operatorname{Lie} G$ may be computed as the commutator in A°. Some calculations (see Appendix, §A.1) will yield the Lie brackets as follows

$$[e_1, e_2] = (\epsilon - \epsilon^{-1})^l e_3, \quad [f_1, f_2] = -(1-\epsilon^{-2})^l f_3, \quad [e_1, k_1] = -e_1, \quad [e_2, k_2] = -e_2,$$

$$[e_3, k_i] = -e_3, \quad [f_1, k_1] = -f_1, \quad [f_2, k_2] = -f_2, \quad [f_3, k_i] = -f_3$$

and zero elsewhere. Upon replacing e_i by $e_i/(\epsilon-\epsilon^{-1})^l$ and f_i by $f_i/(1-\epsilon^{-2})^l$, we get the Lie brackets as in the statement.

Lastly, for both Hopf algebras $U_\epsilon(\mathfrak{sl}_2)$ and $U_\epsilon(\mathfrak{sl}_3)$, the tangential component $W(H)$ decomposes as in the statement by Theorem 4.2.3(4). And, since in both cases $A \subseteq H$ is orbitally semisimple by Proposition 3.1.13, the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2). □

Remark 4.4.2. At the moment we do not know whether the action of A° on $\overline{H}^*$ is nontrivial in the two examples described in the previous result. And we also do not know whether quantized enveloping algebras are cleft extensions, therefore, unlike what will happen in the following subsection for quantized coordinate rings at roots of unity, we cannot apply part (2) of Theorem 4.1.5.

Quantized coordinate rings

Recall these Hopf algebras from subsection 2.2.2.

Corollary 4.4.3. *Let k be an algebraically closed field of characteristic zero and ϵ be a primitive lth root of unity of k. Then,*

$$\mathcal{O}_\epsilon(G)^\circ \cong u_\epsilon(\mathfrak{g}) \otimes (U(\mathfrak{g})\#kG),$$

where $\mathfrak{g} := \operatorname{Lie} G$. The isomorphism is of algebras, left $\overline{H}^*$-modules and right A°-comodules. Moreover, it contains the Hopf subalgebras

$$W(\mathcal{O}_\epsilon(G)) \cong u_\epsilon(\mathfrak{g}) \otimes U(\mathfrak{g})$$

and

$$\widehat{kG} \cong u_\epsilon(\mathfrak{g}) \otimes kG.$$

Proof. As we had mentioned already in §2.2.2 it is known that $\mathcal{O}(G) \subseteq \mathcal{O}_\epsilon(G)$ is a cleft extension; see the proof of [4, Proposition 2.8] for an explicit construction of a coalgebra cleaving map. Since $A = \mathcal{O}(G)$ is central in $H = \mathcal{O}_\epsilon(G)$, by Theorem 4.1.5(2) we have

$$H^\circ \cong \overline{H}^* \otimes A^\circ \cong u_\epsilon(\operatorname{Lie} G) \otimes (U(\operatorname{Lie} G) * kG).$$

The dual of $A = \mathcal{O}(G)$ follows from Theorem 1.3.7 and $\overline{H} := H/A^+H = o_\epsilon(G) = u_\epsilon(\operatorname{Lie} G)^*$ and, being finite dimensional, its dual is $u_\epsilon(\operatorname{Lie} G)$.

In addition, $A = \mathcal{O}(G)$ is clearly reduced, hence the decomposition of $W(\mathcal{O}_\epsilon(G))$ follows from Theorem 4.2.3(4) and the fact that the action of A° on $\overline{H}^*$ is trivial. By Proposition 3.1.13 $A \subseteq H$ is orbitally semisimple, hence the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2). □

Enveloping algebras of Lie algebras in positive characteristic

Recall this example from subsection 2.2.3.

Corollary 4.4.4. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra of dimension m over a field k with positive characteristic p. Then,*

$$U(\mathfrak{g})^\circ \cong u^{[p]}(\mathfrak{g})^* \otimes \left(k[f_1^{(n)}, \ldots, f_m^{(n)} : n \geq 0] \otimes k(k,+)^m\right),$$

as algebras, left $\overline{H}^*$-modules and right A°-comodules. Here $k[f_1^{(n)}, \dots, f_m^{(n)} : n \geq 0]$ denotes a divided power algebra on m variables as per Remark 1.3.9. In addition, it contains the two Hopf subalgebras

$$W(U(\mathfrak{g})) \cong u^{[p]}(\mathfrak{g})^* \otimes k[f_1^{(n)}, \dots, f_m^{(n)} : n \geq 0]$$

and

$$\widehat{kG} \cong u^{[p]}(\mathfrak{g})^* \otimes k(k,+)^m.$$

Proof. Let $\{x_1, \dots, x_m\}$ be a basis of $\mathfrak{g}$. As mentioned in §2.2.3, the enveloping algebra $U(\mathfrak{g})$ contains a central Hopf subalgebra $A = k\langle y_1, \dots, y_m\rangle$, where each y_i is a central p-polynomial on x_i. Moreover, $U(\mathfrak{g})$ is a free A-module with basis $\{x_1^{i_1} \dots x_m^{i_m} : 0 \leq i_j < d_j\}$ where $d_j = \deg(y_j)$. Thus, $H = U(\mathfrak{g})$ decomposes into $A \oplus X$ as A-modules, where X is the A-module generated by

$$\{x_1^{i_1} \dots x_m^{i_m} : 0 \leq i_j < d_j \text{ not all zero}\}.$$

Since each x_i is primitive, it is easy to prove that X is a coideal of H.

Since A is the polynomial algebra on the m primitive elements $y_1, \dots, y_m$, $A^\circ \cong k[f_1^{(n)}, \dots, f_m^{(n)} : n \geq 0] \otimes k(k,+)^m$ by Remark 1.3.9. Moreover, $\overline{H} = u^{[p]}(\mathfrak{g})$ and, since $U(\mathfrak{g})$ is cocommutative, $U(\mathfrak{g})^\circ$ is commutative and the action of A° on $\overline{H}^*$ is trivial. The statement now follows from Theorem 4.1.5(2).

Note that $A = k[y_1, \dots, y_m]$ is a domain, hence obviously reduced. The decomposition of $W(U(\mathfrak{g}))$ follows from Theorem 4.2.3(4) and the fact that $U(\mathfrak{g})^\circ$ is commutative. Moreover, since we have orbital semisimplicity by Proposition 3.1.13, the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2). □

Group algebras of abelian-by-finite groups

Recall this example from §2.2.4.

Corollary 4.4.5. *Let k be an algebraically closed field of characteristic zero, G a finitely generated abelian-by-finite group and N a free abelian normal subgroup of finite index. Let m denote the rank of N. Then,*

$$(kG)^\circ \cong (k(G/N))^* \otimes (kN)^\circ \cong (k(G/N))^* \otimes (k[f_1, \dots, f_m] \otimes k(k^\times)^m).$$

The isomorphism is of algebras, left $\overline{H}^*$-modules and right A°-comodules. Additionally, we have the Hopf subalgebras

$$W(kG) \cong (k(G/N))^* \otimes k[f_1, \dots, f_m]$$

and

$$\widehat{kG} \cong (k(G/N))^* \otimes k(k^\times)^m.$$

Proof. As in §2.2.4, kG decomposes into the crossed product $kN\#_\sigma k(G/N)$, whose cleaving map γ is defined by choosing a set of coset representatives for G/N in G, hence it is clearly a splitting coalgebra map. Note that such choice can be made so that $\gamma(g^{-1}N) = \gamma(gN)^{-1}$ for every $g \in G$. Moreover, since N is a free abelian group of rank m, kN is a free commutative Hopf algebra with m grouplike generators, that is $kN = k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, which is a domain (hence reduced). Thus, Theorem 4.1.5(2) gives

$$(kG)^\circ \cong (k(G/N))^* \otimes (kN)^\circ.$$

The dual of kN follows as in Example 1.3.10.

The decomposition of $W(kG)$ follows from Theorem 4.2.3(4) and the fact that $(kG)^\circ$ is commutative. Moreover, $kN \subseteq kG$ is orbitally semisimple by Proposition 3.1.13. Therefore, the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2). □

Prime regular affine Hopf algebras with GK-dimension 1

Recall the classification of prime regular affine Hopf algebras with Gelfand-Kirillov dimension one in §2.2.5. I should mention that the duals of the Hopf algebras (I)-(IV) in this classification have already been computed by Astrid Jahn, during her PhD; see [44, §6].

Corollary 4.4.6. *The duals of the affine prime regular Hopf algebras of GK-dimension 1 over an algebraically closed field of characteristic 0 read as follows. The isomorphisms are of algebras, left $\overline{H}^*$-modules and right A°-comodules.*

(I) We have

$$k[x]^\circ \cong k[f] \otimes k(k, +),$$

where $f(x^i) = \delta_{i,1}$ and for each $\lambda \in k$ the corresponding character is given by $\chi_\lambda(x^i) = \lambda^i$. Its tangential and character components are the Hopf subalgebras $W(k[x]) \cong k[f]$ and $\widehat{kG} \cong k(k, +)$, where f is primitive and $(k, +)$ consists of grouplike elements.

And

$$k[x^{\pm 1}]^\circ \cong k[f'] \otimes k(k^\times),$$

where $f'(x^i) = i$ and for each $\lambda \in k^\times$ the corresponding character is $\chi_\lambda(x^i) = \lambda^i$. As above, its Hopf subalgebras are $W(k[x^{\pm 1}]) \cong k[f']$ and $\widehat{kG} \cong k(k^\times)$, where f' is primitive and the elements of $(k^\times, \cdot)$ are grouplike.

(II) We have

$$(kD)^\circ \cong kC_2 \otimes (k[f] \otimes k(k^\times)),$$

where $f(a^ib^j) = j$*, each* $\lambda \in k^\times$ *corresponds to the functional* z_λ *defined by* $z_\lambda(a^ib^j) = \lambda^j$*, and the generator* α *of* C_2 *is defined by* $\alpha(a^ib^j) = (-1)^i$*.*

As for the coalgebra structure of $(kD)^\circ$*,* α *is grouplike,* f *is* $(\alpha, 1)$*-primitive and*

$$\Delta(z_\lambda) = \frac{1}{2}\left(1 \otimes (1+\alpha) + z_{\lambda^{-2}} \otimes (1-\alpha)\right)(z_\lambda \otimes z_\lambda).$$

Its tangential and character components are the Hopf subalgebras

$$W(kD) \cong kC_2 \otimes k[f] \cong \mathcal{O}\left((k,+) \rtimes C_2\right)$$

and

$$\widehat{kG} \cong kC_2 \otimes k(k^\times) = k\left(C_2 \times (k^\times, \cdot)\right).$$

(III) We have

$$T(n,t,q)^\circ \cong (T_f(n',t',q^d) \otimes kC_d) \otimes (k[f] \otimes k(k,+)),$$

where $d = (n,t), n' = n/d, t' = t/d$*. The functionals are as follows: the invertible generator* G *of* T_f *is given by* $G(x^ig^j) = \delta_{i,0}q^{-t'^{-1}dk}$ *and the nilpotent generator* X *is given by* $X(x^ig^j) = \delta_{i,1}$*, where* t'^{-1} *is the inverse of* t' *modulo* n' *and* $j = dk + r$ *with* $0 \leq r < d$*; the generator* α *of* C_d *is defined by* $\alpha(x^ig^j) = \delta_{i,0}q^{n'j}$*;* f *is given by* $f(x^ig^j) = \delta_{i,n'}q^{n'j}$ *and the functional* z_λ *corresponding to each* $\lambda \in k$ *is defined by* $z_\lambda(x^ig^j) = \delta_{n'|i}q^{ij}\lambda^{i/n'}$*.*

The tangential and character components are the Hopf subalgebras

$$W(T(n,t,q)) \cong (T_f(n',t',q^d) \otimes kC_d) \otimes k[f]$$

and

$$\widehat{kG} \cong (T_f(n',t',q^d) \otimes kC_d) \otimes k(k,+).$$

The coalgebra structure of $W(T(n,t,q))$ *is the following:* α *is grouplike,*

$$\Delta(X) = X \otimes 1 + \left(\sum \lambda_k \alpha^k\right) G^{t'} \otimes X,$$

$$\Delta(G) = \left(1 \otimes 1 + (q^{-t'^{-1}d} - 1) \sum_{\substack{l,m=1 \\ l+m \geq d}}^{d-1} \left(\sum_s \mu_s^l \alpha^s\right) \otimes \left(\sum_t \mu_t^m \alpha^t\right)\right)(G \otimes G),$$

and

$$\Delta(f) = f \otimes \alpha + 1 \otimes f + (\alpha \otimes \alpha)\left(\sum_{s=1}^{n'-1} X_{n'-s}\left[(\sum_k \lambda_k \alpha^k) G^{t'}\right]^s \otimes X_s\right),$$

where, for each $1 \leq s < n'$*,* X_s *is the functional defined by* $X_s(x^ig^j) = \delta_{i,s}$*, and the scalars* λ_k *and* μ_s^l *are such that* $\sum_k \lambda_k (q^{n'r})^k = q^{-r}$ *and* $\sum_s \mu_s^l (q^{n'r})^s = \delta_{r,l}$

for any $0 \leq r < d$.

(IV) *We have*

$$B(n,w,q)^\circ \cong T_f(n,1,q)\#(k[f] \otimes k(k^\times)),$$

where the grouplike generator of T_f *is given by* $G(y^i g^j x^k) = \delta_{i,0} q^{-j}$*, its skew-primitive generator is given by* $Y(y^i g^j x^k) = \delta_{i,1}$*;* f *is defined by* $f(y^i g^j x^k) = \delta_{i,0} k$ *and the functional corresponding to* $\lambda \in k^\times$ *is given by* $z_\lambda(y^i g^j x^k) = \delta_{i,0}\lambda^k$.

The coalgebra structure is: G *is grouplike,* Y *is* $(1,G)$*-primitive,*

$$\Delta(f) = f \otimes 1 + 1 \otimes f + w \sum_{\substack{l,m=1\\ l+m\geq n}}^{n-1} \left(\sum_r \mu_r^l G^r\right) \otimes \left(\sum_s \mu_s^m G^s\right) - w \sum_{l=1}^{n-1} Y_{n-l} G^l \otimes Y_l$$

and for each $\lambda \in k^\times$

$$\Delta(z_\lambda) = \left(1 \otimes 1 + (1-\lambda^w) \sum_{l=1}^{n-1} (Y_{n-l}G^l \otimes Y_l)\right)$$
$$\left(1 \otimes 1 + (\lambda^w - 1) \sum_{\substack{l,m=1\\ l+m\geq n}}^{n-1} \left(\sum_r \mu_r^l G^r\right) \otimes \left(\sum_s \mu_s^m G^s\right)\right)(z_\lambda \otimes z_\lambda),$$

where, for each $0 \leq l < n$, Y_l *is the functional given by* $Y_l(y^i g^j x^k) = \delta_{i,l}$ *and the scalars* μ_r^l *satisfy* $\sum_r \mu_r^l q^{-rj} = \delta_{j,l}$ *for all* j.

The tangential component is the Hopf subalgebra

$$W(B(n,w,q)) \cong T_f(n,1,q)\#k[f] = T_f(n,1,q)[f;\delta],$$

a skew polynomial algebra with coefficient ring $T_f(n,1,q)$ *and derivation given by* $\delta(G) = 0$ *and* $\delta(Y) = wY(\sum_r \mu_r^{n-1} G^r)$.

The character component is the Hopf subalgebra

$$\widehat{kG} \cong T_f(n,1,q)\#k(k^\times) = T_f(n,1,q) * k^\times,$$

a skew group ring with coefficient ring $T_f(n,1,q)$ *and multiplication determined by* $z_\lambda G = G z_\lambda$ *and* $z_\lambda Y = Y(\sum_r \eta_r G^r) z_\lambda$*, where* η_r *are such that* $\sum_r \eta_r q^{-rj} = 1 + (\lambda^w - 1)\sum_r \mu_r^{n-1} q^{-rj} = \lambda^{w\delta_{j,n-1}}$ *for all* j.

(V) *We have*

$$D(m,d,q)^\circ \cong (kC_2 \#_\tau T_f(m,1,q^2))\#(k[f] \otimes k(k^\times)).$$

The functionals are defined as follows: f *is given by* $f(y^i g^j x^l) = \delta_{i,0} l$, $f(g^j u_k x^l) = \delta_{k,0} l$ *and for each* $\lambda \in k^\times$ *the corresponding functional is* $z_\lambda(y^i g^j x^l) = \delta_{i,0}\lambda^l$, $z_\lambda(g^j u_k x^l) = \delta_{k,0}\lambda^l$*; the generator of* C_2 *is* $\alpha(y^i g^j x^l) = \delta_{i,0}$, $\alpha(g^j u_k x^l) = -\delta_{k,0}$, *and the generators of* $T_f(m,1,q^2)$ *are* $G(y^ig^jx^l)=q^{-2j}\delta_{i,0}$*,* $G(g^ju_kx^l)=q^{-2j}\delta_{k,0}$ *and* $Y(y^ig^jx^l)=\delta_{i,1}$*,* $Y(g^ju_kx^l)=(1-q^{-2})^{-1}q^{-2j}\delta_{k,1}$*.*

The tangential and character components are respectively the Hopf subalgebras

$$W(D(m,d,q))\cong(kC_2\#_\tau T_f(m,1,q^2))\#k[f]$$

and

$$\widehat{kG}\cong(kC_2\#_\tau T_f(m,1,q^2))\#k(k^\times).$$

Proof. We analyze each case separately. Some calculations, including the ones justifying the formulae of the functionals, can be found in section A.2 of the Appendix.

(I) We have described the dual of $k[x]$ and $k[x^{\pm 1}]$ back in Examples 1.3.8 and 1.3.10. Moreover, their Hopf subalgebras $W(H)$ and $\widehat{kG}$ are respectively just the enveloping algebra and the group algebra in the dual.

(II) The dihedral group $D=\langle a,b : a^2=1, aba=b^{-1}\rangle$ is abelian-by-finite, with normal abelian subgroup $N=\langle b\rangle$ of index 2, hence by Corollary 4.4.5 and Examples 1.3.3 and 1.3.10

$$(kD)^\circ\cong(kC_2)^*\otimes(k[b^{\pm 1}])^\circ\cong kC_2\otimes(k[f]\otimes k(k^\times)).$$

The decompositions of $W(kD)$ and $\widehat{kG}$ were also already discussed in Corollary 4.4.5. The calculations sustaining both the formulae of the functionals of $(kD)^\circ$ and their coproduct can be found in §A.2.

Moreover, since $W:=W(kD)$ is an affine commutative Hopf algebra, it is the coordinate ring of the algebraic group $\mathrm{Maxspec}(W)\cong\mathrm{Alg}_k(W,k)$. The maximal ideals of W have the form $(f-\lambda,\alpha\pm 1)$ for some $\lambda\in k$, hence the corresponding characters $\phi_\lambda,\psi_\lambda$ of W are given by $\phi_\lambda(f)=\psi_\lambda(f)=\lambda$ and $\phi_\lambda(f)=1,\psi_\lambda(f)=-1$. Easy computations (which can be found in §A.2) yield the following convolution product:

$$\phi_\lambda\phi_\mu=\phi_{\lambda+\mu},\quad \phi_\lambda\psi_\mu=\psi_{-\lambda+\mu},\quad \psi_\lambda\phi_\mu=\psi_{\lambda+\mu},\quad \psi_\lambda\psi_\mu=\phi_{-\lambda+\mu},$$

for any $\lambda,\mu\in k$. Therefore, the algebraic group $\mathrm{Maxspec}(W)$ is the (nontrivial) semi-direct product $(k,+)\rtimes C_2$ and $W\cong\mathcal{O}(k\rtimes C_2)$.

(III) Consider the Taft algebra $H=T(n,t,q)$, which was defined in Example 1.1.4. Its commutative normal Hopf subalgebra is $A=k[x^{n'}]$, where $d=(n,t)$ and $n'=n/d$.

As we saw in Example 1.1.21, H decomposes into $A\#_\sigma\overline{H}$, where the cleaving map γ is a splitting coalgebra map and one can easily check that $S_H\circ\gamma=\gamma\circ S_{\overline{H}}$.

Therefore, by Theorem 4.1.5(2) we have

$$T(n,t,q)^\circ \cong \overline{H}^* \otimes k[x^{n'}]^\circ.$$

The dual of $A = k[x^{n'}]$ is well-known, since A is a commutative Hopf algebra; see Example 1.3.8. Moreover, we proved in Example 1.1.21 that the Hopf quotient $\overline{H} = H/x^{n'}H$ decomposes as an algebra into $T_f(n',t',q^d)\#_\tau kC_d$, where $t' = t/d$, and it decomposes into $T_f(n',t',q^d)\otimes kC_d$ as a coalgebra. Therefore, as an algebra $\overline{H}^* \cong T_f(n',t',q^d)^* \otimes (kC_d)^*$ and, since $(n',t') = 1$, $T_f(n',t',q^d)$ is self-dual, as is kC_d; see Examples 1.3.4 and 1.3.3. The computations of the formulae of the functionals and their coproduct can be found in §A.2.

The decomposition of $W(T(n,t,q))$ follows from Theorem 4.2.3(4) and the fact that the action of A° on $\overline{H}^*$ is trivial. By Proposition 3.1.13 $A \subseteq H$ is orbitally semisimple, therefore the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2).

(IV) Now consider the generalized Liu algebra $H = B(n,w,q)$. It is a free module over its central Hopf subalgebra $A = k[x^{\pm 1}]$,

$$B(n,w,q) = \bigoplus_{0\le i,j<n} y^i g^j A.$$

So we take the direct complement X of A to be the right A-module generated by

$$\{y^i g^j : 1 \le i < n, 0 \le j < n\} \cup \{g^j - 1 : 1 \le j < n\}.$$

Let us show that X is a coideal of H. It is easy to see $\epsilon(X) = 0$ and, since X is a right A-module, it suffices to check the coproduct of each of its generators belongs to $H\otimes X + X \otimes H$. For all $1 \le j < n$, $g^j - 1$ is $(g^j,1)$-primitive and for any $1 \le i < n, 0 \le j < n$,

$$\Delta(y^i g^j) = \sum_{k=0}^{i} \binom{i}{k}_{q^{-1}} y^{i-k} g^{k+j} \otimes y^k g^j,$$

where the summand corresponding to $k = 0$ is $y^i g^j \otimes g^j \in X \otimes H$, while all other summands clearly belong to $H \otimes X$. This proves X is a coideal.

Therefore, by Theorem 4.1.5(1) we have

$$B(n,w,q)^\circ \cong \overline{H}^* \# A^\circ.$$

The dual of $A = k[x^{\pm 1}]$ is described in Example 1.3.10, and the Hopf quotient is $\overline{H} = T_f(n,1,q)$, which is self-dual as in Example 1.3.4. The calculations that justify the formulae of the functionals are presented in §A.2, as are the calculations on their coproduct and on the action of A° on $\overline{H}^*$.

Moreover, by Theorem 4.2.3(4) $W(B(n,w,q))$ decomposes as in the statement. And $A \subseteq H$ is orbitally semisimple by Proposition 3.1.13, therefore the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2).

(V) Now consider the Hopf algebra $H = D(m,d,q)$. Recall from section 2.2 it is a free module over its commutative normal Hopf subalgebra $A = k[x^{\pm 1}]$,

$$H = \left(\bigoplus_{i,j=0}^{m-1} y^i g^j A\right) \oplus \left(\bigoplus_{j,k=0}^{m-1} g^j u_k A\right).$$

So we take the direct complement X of A to be the right A-module

$$\bigoplus_{\substack{1\le i<m \\ 0\le j<m}} y^i g^j A \oplus \bigoplus_{1\le j<m} (g^j-1)A \oplus \bigoplus_{\substack{0\le j<m \\ 1\le k<m}} g^j u_k A \oplus \bigoplus_{0\le j<m} g^j(u_0-1)A.$$

First one easily sees that $\epsilon(X)=0$. Recall that the subalgebra of H generated by x,g,y is the generalized Liu algebra $B(m,w,q^2)$, and the computations in (IV) show that both $\Delta(g^i y^j)$ and $\Delta(g^i-1)$ belong to $H\otimes X + X\otimes H$. Moreover,

$$\Delta(g^j u_k) = \sum_{l=0}^{m-1} \gamma^{l(k-l)} x^{-ld} g^{j+l} u_{k-l} \otimes g^j u_l,$$

in which the tensor corresponding to $l=0$ is $g^j u_k \otimes g^j u_0 \in X\otimes H$, while all other terms belong to $H\otimes X$. And the coproduct of the last generators is

$$\begin{aligned}
& \Delta(g^j(u_0-1)) \\
= \;& (g^j\otimes g^j)\left(\sum_{l=0}^{m-1} \gamma^{-l^2} x^{-ld} g^l u_{m-l}\otimes u_l - 1\otimes 1\right) \\
= \;& (g^j\otimes g^j)\left(u_0\otimes u_0 - 1\otimes 1 + \sum_{l=1}^{m-1} \gamma^{-l^2} x^{-ld} g^l u_{m-l}\otimes u_l\right) \\
= \;& g^j(u_0-1)\otimes g^j u_0 + g^j\otimes g^j(u_0-1) + \sum_{l=1}^{m-1} \gamma^{-l^2} x^{-ld} g^{l+j} u_{m-l}\otimes g^j u_l,
\end{aligned}$$

which clearly belongs to $H\otimes X + X\otimes H$, proving X is a coideal of H.

Therefore, Theorem 4.1.5(1) applies and gives

$$D(m,d,q)^\circ \cong \overline{H}^* \# A^\circ.$$

The dual of $A = k[x^{\pm 1}]$ was explained back in Example 1.3.10.

The dual of $\overline{H}$, however, requires a few more calculations. Lemma 2.2.3(3) yields the crossed product decomposition $\overline{H} \cong T_f(m,1,\gamma)\#_\sigma kC_2$, hence as a vector space $\overline{H}^*$ is $T_f(m,1,\gamma)^* \otimes (kC_2)^*$. The Hopf surjection $\pi : \overline{H} \twoheadrightarrow kC_2$ and the

Hopf embedding $\iota : T_f \hookrightarrow \overline{H}$ induce a Hopf embedding $\pi^* : (kC_2)^* \hookrightarrow \overline{H}^*$ and a Hopf surjection $\iota^* : \overline{H}^* \twoheadrightarrow T_f^*$, which provides $\overline{H}^*$ with a right T_f^*-comodule algebra structure. Moreover, we have a map $\Pi : \overline{H} \twoheadrightarrow T_f$ given by $tu_0^k \mapsto t$ for any $t \in T_f$, which dualizes to a right T_f^*-comodule embedding $\Pi^* : T_f^* \to \overline{H}^*$. Let $T_f^* = k\langle G, Y\rangle$. The map

$$\begin{array}{rccl} \gamma : & T_f^* & \to & \overline{H}^* \\ & G^r Y^s & \mapsto & \Pi^*(G)^r \Pi^*(Y)^s, \quad \text{for } 0 \leq r, s < m \end{array}$$

is a cleaving map, with convolution inverse given by

$$\gamma^{-1}(G^r Y^s) = (-1)^s \gamma^{\binom{s}{2}} \Pi^*(Y)^s \Pi^*(G)^{-(r+s \mod m)}.$$

Therefore, Theorem 1.1.17 gives

$$\overline{H}^* = (\overline{H}^*)^{\mathrm{co}\, T_f^*} \#_\tau T_f^*.$$

Furthermore, it is easy to see that the elements of $(kC_2)^*$ are coinvariants of the right T_f^*-coaction and, using a dimension argument, $(\overline{H}^*)^{\mathrm{co}\, T_f^*} = (kC_2)^*$. Therefore, by Examples 1.3.3 and 1.3.4 we have

$$\overline{H}^* \cong (kC_2)^* \#_\tau T_f(m, 1, \gamma)^* \cong kC_2 \#_\tau T_f(m, 1, \gamma).$$

The formulae of the functionals of H° is explained in §A.2.

Lastly, the decomposition of $W(D(m, d, q))$ is obtained from Theorem 4.2.3(4). As we saw in Proposition 3.1.13, $A \subseteq H$ is orbitally semisimple. Therefore, the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2).

□

Remarks 4.4.7.

1. In the dual of the infinite dimensional Taft algebras, family (III) in the previous result, we were not able to compute the coproduct of the functionals z_λ indexed by $\lambda \in k$. When $d := (n, t) = 1$, this coproduct is

$$\Delta(z_\lambda) = (z_\lambda \otimes z_\lambda) \left(1 \otimes 1 + \lambda \sum_{k=1}^{n-1} X_{n-k} G^{tk} \otimes X_k \right),$$

as was computed by Astrid Jahn; see [44, Remark 6.12].

2. With respect to family (V) in the previous result, at the moment we do not know whether the actions of $k[f]$ and of $k(k^\times)$ on $\overline{H}^* = kC_2 \#_\tau T_f(m, 1, q^2)$ or even whether the cocycle τ are nontrivial.

PI noetherian Hopf domains with GK-dimension 2

Recall the classification of the PI Hopf domains with Gelfand-Kirillov dimension two in §2.2.6.

Corollary 4.4.8. *The duals of the PI noetherian Hopf domains of GK-dimension 2 over an algebraically closed field of characteristic 0 are as follows. The isomorphisms are of algebras, left $\overline{H}^*$-modules and right A°-comodules.*

(I) We have

$$(k\mathbb{Z}^2)^\circ \cong k[f, f'] \otimes k(k^\times)^2.$$

The functionals are defined as $f(x^i y^j) = i$, $f'(x^i y^j) = j$ and $\chi_{\lambda,\mu}(x^i y^j) = \lambda^i \mu^j$ for each $(\lambda, \mu) \in (k^\times)^2$, where x, y are the generators of $\mathbb{Z}^2$. Its Hopf subalgebras are $W(k\mathbb{Z}^2) \cong k[f, f']$ and $\widehat{kG} \cong k(k^\times)^2$, where f, f' are primitive and $(k^\times)^2$ consists of grouplike elements.

We also have

$$(k(\mathbb{Z} \rtimes \mathbb{Z}))^\circ \cong kC_2 \otimes k[f, f'] \otimes k(k^\times)^2,$$

in which the functionals are: the generator α of C_2 is $\alpha(a^i b^j) = (-1)^i$, $f(a^i b^j) = k$, $f'(a^i b^j) = j$ and for each $(\lambda, \mu) \in (k^\times)^2$ we have $z_{\lambda,\mu}(a^i b^j) = \lambda^k \mu^j$ where $i = 2k + r$ and $r \in \{0, 1\}$. The components are the Hopf subalgebras

$$W(k(\mathbb{Z} \rtimes \mathbb{Z})) \cong kC_2 \otimes k[f, f'] = \mathcal{O}\left((k, +)^2 \rtimes C_2\right)$$

and

$$\widehat{kG} \cong kC_2 \otimes k(k^\times)^2 = k(C_2 \times \left(k^\times, \cdot\right)^2).$$

The coalgebra structure is as follows: α is grouplike, f' is $(\alpha, 1)$-primitive,

$$\Delta(f) = f \otimes 1 + 1 \otimes f + \frac{1}{4}(1 - \alpha) \otimes (1 - \alpha)$$

and

$$\Delta(z_{\lambda,\mu}) = \frac{1}{2}\left[1 \otimes (1 + \alpha) + \left(\frac{1+\lambda}{2} + \frac{1-\lambda}{2}\alpha\right) z_{1,\mu^{-2}} \otimes (1 - \alpha)\right](z_{\lambda,\mu} \otimes z_{\lambda,\mu}).$$

(II) We have

$$k[x, y]^\circ \cong k[f, f'] \otimes k(k, +)^2,$$

where $f(x^i y^j) = \delta_{i,1}\delta_{j,0}$, $f'(x^i y^j) = \delta_{i,0}\delta_{j,1}$ and $\chi_{\lambda,\mu}(x^i y^j) = \lambda^i \mu^j$ for each $(\lambda, \mu) \in (k, +)^2$. It contains the Hopf subalgebras $W(k[x, y]) \cong k[f, f']$ and $\widehat{kG} \cong k(k, +)^2$, with f, f' primitive and $(k, +)^2$ consisting of grouplike elements.

(III) For a primitive lth root of unity q,

$$A(l, n, q)^\circ \cong (T_f(l', n', q^{-d}) \otimes kC_d)\#(k[f, f'] \otimes k((k, +) \times k^\times)),$$

with $d := (n,l)$, $l' := l/d$, $n' := n/d$. *It contains the Hopf subalgebras*

$$W(A(l,n,q)) \cong (T_f(l',n',q^{-d}) \otimes kC_d) \# k[f,f']$$

and

$$\widehat{kG} \cong (T_f(l',n',q^{-d}) \otimes kC_d) \# k((k,+) \times k^\times).$$

The functionals are defined as follows: f *is given by* $f(y^i x^j) = \delta_{i,l'} q^{-l'j}$, f' *by* $f'(y^i x^j) = k\delta_{i,0}$ *and for each* $(\lambda,\mu) \in k \times k^\times$ *the corresponding character is* $z_{\lambda,\mu}(y^i x^j) = \delta_{l'|i} q^{-ij} \lambda^{i/l'} \mu^k$, *where* $j = lk + r$ *with* $0 \leq r < l$*; the generator* α *of* C_d *is given by* $\alpha(y^i x^j) = \delta_{i,0} q^{l'j}$, *the nilpotent generator* Y *of* T_f *is given by* $Y(y^i x^j) = \delta_{i,1}$ *and the invertible generator* X *of* T_f *is given by* $X(y^i x^j) = \delta_{i,0} q^{n'^{-1} dm}$, *where* $j = dm + s$ *with* $0 \leq s < d$ *and* n'^{-1} *is the inverse of* n' *modulo* l'.

(IV) *For a primitive* $(n/p_0)p_1 \dots p_s$*th root of unity* q,

$$B(n,p_0,\dots,p_s,q)^\circ \cong \overline{H}^* \# (k[f,f'] \otimes k((k,+) \times k^\times)),$$

where as algebras

$$\overline{H}^* \cong T_f(p_1,p_0,\xi_1) \otimes \dots \otimes T_f(p_s, p_0 p_1 \dots p_{s-1}, \xi_s) \otimes kC_{n/p_0},$$

for primitive p_i*th roots of unity* $\xi_i = q^{-(n/p_0)m_i}$. *Moreover, it contains the Hopf subalgebras*

$$W(B(n,p_0,\dots,p_s,q)) \cong \overline{H}^* \# k[f,f']$$

and

$$\widehat{kG} \cong \overline{H}^* \# k((k,+) \times k^\times).$$

The functionals are defined as follows: the generator α *of* C_{n/p_0} *is given by* $\alpha(y_1^{i_1} \dots y_s^{i_s} x^j) = \delta_{i_1,0} \dots \delta_{i_s,0} q^{p_1 \dots p_s r_0}$*; for each* $1 \leq k \leq s$*, the invertible generator* X_k *of* $T_f(p_k, p_0 \dots p_{k-1}, \xi_k)$ *is* $X_k(y_1^{i_1} \dots y_s^{i_s} x^j) = \delta_{i_1,0} \dots \delta_{i_s,0} \xi_k^{-(p_0 \dots p_{k-1})^{-1} r_k}$ *and the nilpotent generator* Y_k *of* $T_f(p_k, p_0 \dots p_{k-1}, \xi_k)$ *is defined by* $Y_k(y_1^{i_1} \dots y_s^{i_s} x^j) = \delta_{i_k,1}(\Pi_{l \neq k} \delta_{i_l,0}) q^{-(n/p_0)m_k(r_1 + \dots + r_{k-1})}$*, where the integers* r_k *are given by the unique decomposition* $j = r_0 + \frac{n}{p_0} r_s + \frac{n}{p_0} p_s r_{s-1} + \dots + \frac{n}{p_0} p_s \dots p_2 r_1$*; the generators of* $k[f,f']$ *are given by* $f(y_1^{i_1} \dots y_s^{i_s} x^j) = \sum_{k=1}^s \delta_{i_k,p_k} (\Pi_{l \neq k} \delta_{i_l,0}) q^{-p_1 \dots p_s j}$ *and* $f'(y_1^{i_1} \dots y_s^{i_s} x^j) = k\delta_{i_1,0} \dots \delta_{i_s,0}$*, and for each* $(\lambda,\mu) \in k \times k^\times$ *the corresponding functional is given by* $z_{\lambda,\mu}(y_1^{i_1} \dots y_s^{i_s} x^j) = \lambda^{i_1/p_1 + \dots + i_s/p_s} \mu^k$*, where* $j = lk + r$ *with* $0 \leq r < l$.

Proof. We compute each example separately. Some calculations, in particular the ones justifying the definitions of the functionals, can be found in §A.3.

(I) The group algebra of $\mathbb{Z}^2$ is $k\mathbb{Z}^2 \cong k[x^{\pm 1}, y^{\pm 1}]$, similarly to what was done in Example 1.1.8. Being a commutative Hopf algebra, its dual is $(k\mathbb{Z}^2)^\circ \cong k[f,f'] \otimes$

$k(k^\times)^2$ as in Example 1.3.10. More specifically, f, f' are given by $f(x^iy^j) = i$ and $f'(x^iy^j) = j$, and for each $(\lambda, \mu) \in (k^\times)^2$ the corresponding character is $\chi_{\lambda,\mu}(x^iy^j) = \lambda^i\mu^j$.

Moreover, the Hopf subalgebras $W(k\mathbb{Z}^2)$ and $\widehat{kG}$ are respectively just the enveloping algebra and group algebra in $(k\mathbb{Z}^2)^\circ$.

As for the second Hopf algebra, $\mathbb{Z} \rtimes \mathbb{Z} = \langle a, b : aba^{-1} = b^{-1}\rangle$ is an abelian-by-finite group with abelian normal subgroup $\mathbb{Z} \times 2\mathbb{Z} = \langle a^2, b\rangle$ of index 2, hence Corollary 4.4.5 gives

$$(k(\mathbb{Z} \rtimes \mathbb{Z}))^\circ \cong (kC_2)^* \otimes (k(\mathbb{Z} \times 2\mathbb{Z}))^\circ = kC_2 \otimes k[f, f'] \otimes k(k^\times, \cdot)^2.$$

The formulae of the functionals of $(k(\mathbb{Z} \rtimes \mathbb{Z}))^\circ$ are explained in §A.3, as is their coproduct.

The decomposition of $W(k(\mathbb{Z} \rtimes \mathbb{Z}))$ and $\widehat{kG}$ have already been discussed in Corollary 4.4.5.

The affine commutative Hopf algebra $W(k(\mathbb{Z} \rtimes \mathbb{Z}))$ is the coordinate ring of the algebraic group $\mathrm{Maxspec}(W) \cong \mathrm{Alg}_k(W, k)$. The maximal ideals of W have the form $(f - \lambda, f' - \mu, \alpha \pm 1)$ for some $\lambda, \mu \in k$, and the corresponding characters $\phi_{\lambda,\mu}, \psi_{\lambda,\mu}$ are given by $\phi_{\lambda,\mu}(f) = \psi_{\lambda,\mu}(f) = \lambda$, $\phi_{\lambda,\mu}(f') = \psi_{\lambda,\mu}(f') = \mu$ and $\phi_{\lambda,\mu}(\alpha) = 1$, $\psi_{\lambda,\mu}(\alpha) = -1$. Easy computations (which can be found in §A.3) give the following convolution products:

$$\phi_{\lambda,\mu}\phi_{\nu,\xi} = \phi_{\lambda+\nu,\mu+\xi}, \qquad \phi_{\lambda,\mu}\psi_{\nu,\xi} = \psi_{\lambda+\nu,-\mu+\xi},$$

$$\psi_{\lambda,\mu}\phi_{\nu,\xi} = \psi_{\lambda+\nu,\mu+\xi}, \qquad \psi_{\lambda,\mu}\psi_{\nu,\xi} = \phi_{\lambda+\nu+1,-\mu+\xi}.$$

Therefore, $\mathrm{Maxspec}(W)$ is a semidirect product $(k, +)^2 \rtimes C_2$.

(II) The dual of polynomial algebras had already been computed in Example 1.3.8. For them, the Hopf subalgebras $W(H)$ and $\widehat{kG}$ are respectively the enveloping algebra and group algebra contained in their dual.

(III) Consider the Hopf algebra $H = A(l, n, q)$. Let $d = (l, n), l' = l/d, n' = n/d$. Recall that H is a free module of finite rank over the normal commutative Hopf subalgebra $A = k[y^{l'}, (x^l)^{\pm 1}]$, with

$$H = \bigoplus_{\substack{0 \le i < l' \\ 0 \le j < l}} y^i x^j A.$$

So we take X to be the following right A-module

$$\bigoplus_{\substack{1 \le i < l' \\ 0 \le j < l}} y^i x^j A \oplus \bigoplus_{1 \le j < l} (x^j - 1)A.$$

And X is a coideal by calculations similar to the ones for the generalized Liu algebras in the proof of Corollary 4.4.6.

Therefore, Theorem 4.1.5(1) gives

$$A(l,n,q)^\circ \cong \overline{H}^* \# A^\circ.$$

The Hopf subalgebra A is $k[y^{l'}, (x^l)^{\pm 1}] = \mathcal{O}(k \times k^\times)$, whose dual can easily be computed from Theorem 1.3.7 as

$$A^\circ = k[f, f'] \otimes k(k \times k^\times).$$

And, as we saw in section 2.2.6, as a coalgebra the Hopf quotient decomposes into $\overline{H} \cong T_f(l', n', q^{-d}) \otimes kC_d$, hence as an algebra

$$\overline{H}^* \cong T_f(l', n', q^{-d})^* \otimes (kC_d)^*$$

and, since $(l', n') = 1$, both tensorands are self-dual by Examples 1.3.4 and 1.3.3. The formulae for the functionals of H° can be found in §A.3.

The decomposition of $W(A(l,n,q))$ follows from Theorem 4.2.3(4). As we saw in Proposition 3.1.13, $A \subseteq H$ is orbitally semisimple. Therefore, the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2).

(IV) Consider now the Hopf algebra $H = B(n, p_0, \ldots, p_s, q)$. Its commutative normal Hopf subalgebra is $A = k\langle y_1^{p_1}, \ldots, y_s^{p_s}, (x^l)^{\pm 1}\rangle = k[y^{p_1 \cdots p_s}, x^{\pm l}]$, over which H is a free module with A-basis

$$H = \bigoplus_{\substack{0 \le i_k < p_k \\ 0 \le j < l}} y_1^{i_1} \ldots y_s^{i_s} x^j A.$$

Then, we take X to be the right A-module

$$X = \bigoplus_{\substack{0 \le i_k < p_k \\ 0 \le j < l \\ \text{some } i_k \ge 1}} y_1^{i_1} \ldots y_s^{i_s} x^j A \oplus \bigoplus_{1 \le j < l} (x^j - 1)A.$$

It is easy to see that $\epsilon(X) = 0$ and $x^j - 1$ is $(x^j, 1)$-primitive. Moreover, the coproduct $\Delta(y_1^{i_1} \ldots y_s^{i_s} x^j)$ is

$$\sum_{k_l=0}^{i_l} \binom{i_1}{k_1}_{q^{nm_{i_1}}} \cdots \binom{i_s}{k_s}_{q^{nm_{i_s}}} (y_1^{i_1 - k_1} x^{nm_{i_1} k_1}) \ldots (y_s^{i_s - k_s} x^{nm_{i_s} k_s}) x^j \otimes y_1^{k_1} \ldots y_s^{k_s} x^j;$$

all 2-tensors for which some $k_l \ge 1$ belong to $H \otimes X$ and the tensor for which all k_l are zero is $y_1^{i_1} \ldots y_s^{i_s} x^j \otimes x^j \in X \otimes H$, proving X is a coideal.

Therefore, by Theorem 4.1.5(1) we have

$$B(n, p_0, \dots, p_s, q)^\circ \cong \overline{H}^* \# A^\circ.$$

The dual of A can be calculated just as in (III),

$$A^\circ = k[f, f'] \otimes k(k \times k^\times).$$

By Lemma 2.2.5 as a coalgebra the Hopf quotient $\overline{H}$ is

$$\overline{H} \cong T_f(p_1, p_0, \xi_1) \otimes \ldots \otimes T_f(p_s, p_0 p_1 \ldots p_{s-1}, \xi_s) \otimes kC_{n/p_0},$$

for some primitive p_ith roots of unity ξ_i, hence as an algebra

$$\overline{H}^* \cong T_f(p_1, p_0, \xi_1)^* \otimes \ldots \otimes T_f(p_s, p_0 p_1 \ldots p_{s-1}, \xi_s)^* \otimes (kC_{n/p_0})^*.$$

The group algebra of the cyclic group is self-dual as in Example 1.3.3 and, since the p_i's are pairwise coprime, the finite dimensional Taft algebras are also self-dual as in Example 1.3.4. The definition of the functionals of H° can be found in §A.3.

The decomposition of $W(B(n, p_0, \dots, p_s, q))$ follows from Theorem 4.2.3(4). And, by Proposition 3.1.13 $A \subseteq H$ is orbitally semisimple. Therefore, the character component $\widehat{kG}$ is a Hopf subalgebra by Proposition 4.3.4(1) and it decomposes as in the statement by Theorem 4.3.7(2).

□

Remark 4.4.9. At the moment we have not researched the coalgebra structure of the dual of families (III) and (IV), nor do we know whether the action of A° on $\overline{H}^*$ in their duals is nontrivial.

Remark 4.4.10. It is important to reiterate the fact that a substantial amount of work in this chapter is a natural generalization of work due to Astrid Jahn [44]. More specifically, she studied the dual of affine Hopf algebras H that are finitely generated modules over some central Hopf subalgebra A over an algebraically closed field of characteristic zero, or in short central-by-finite Hopf algebras [44, chapter 5]. Our setting is a natural generalization of such a family of Hopf algebras.

Thus, Theorem 4.1.5, which decomposes the dual of H into a smash or crossed product of the duals of A and $\overline{H}$, generalizes one of her results [44, Theorem 5.8]. Moreover, she also studied two Hopf subalgebras, $W(H)$ and $\widehat{kG}$, contained in the dual of such Hopf algebras. Theorems 4.2.3 and 4.3.7 also generalize her results [44, Theorems 5.19, 5.20, 5.21]. However, in our setting the structure of the coalgebra $\widehat{kG}$ is severely more intricate, in particular the question of when it possesses an algebra structure. Therefore, most of the results in subsections 4.3.1 and 4.3.2 are original and due to the author and his supervisor.

Lastly, some of the calculations in section 4.4 are also due to Astrid Jahn. More specifically, she had already computed the dual of the quantized enveloping algebra $U_\epsilon(\mathfrak{sl}_2)$ [44, Corollary 5.26] and the duals of families (I)-(IV) in Corollary 4.4.6 [44, Theorem 6.5]. All other computations in this section are original and due to the author and his supervisor.

Chapter 5

Their antipode and their Drinfeld double

This chapter concerns two main subjects: the antipode and the Drinfeld double of commutative-by-finite Hopf algebras. As before, k continues to denote an algebraically closed field.

We start section 5.1 by recalling a well-known formula due to Radford on the fourth power of the antipode of finite dimensional Hopf algebras. This was extended by Brown and Zhang for noetherian AS-Gorenstein Hopf algebras in [18, Corollary 4.6], which gives a formula for S^4 as the following composition

$$S^4 = \gamma \circ \tau^r_\chi \circ \tau^l_{\chi^{-1}}.$$

Just as in the finite dimensional case, τ^r_χ and $\tau^l_{\chi^{-1}}$ are winding automorphisms and γ is some inner automorphism of H; see Theorem 5.1.6. But it is not known what the map γ is, as so far there is no formula or recipe for computing it; see Question 5.1.7. This map is well-understood in the finite dimensional case and we propose a description of γ for affine commutative-by-finite Hopf algebras as follows:

Conjecture 5.0.1. *Let H be an affine commutative-by-finite Hopf algebra and the inner automorphism γ as above. Let $W(H)$ be the tangential component of H° defined in §4.2. Then,*

1. *$W(H)$ is AS-Gorenstein.*

Let α be the character of $W(H)$ corresponding to the right action of $W(H)$ on its left integrals. Then,

2. *$\alpha = ev_g$, evaluation at some unique grouplike g of H;*

3. *γ is conjugation by g, that is $\gamma(h) = g^{-1}hg$ for any $h \in H$, and therefore*

$$S^4(h) = \sum g^{-1}\chi^{-1}(h_1)h_2\chi(h_3)g.$$

This conjecture is supported by some computational examples, most of which are from section 2.2.5, and we prove some results aimed towards proving this conjecture.

Section 5.2 concerns the Drinfeld double of infinite dimensional Hopf algebras. Recall the finite dimensional case from section 1.4. We illustrate the double with some examples, again mostly from section 2.2.5. We discuss the unimodularity of the double in subsection 5.2.1 and propose the following conjecture, which is sustained by some examples.

Conjecture 5.0.2. *Let H be an affine commutative-by-finite Hopf algebra. Let $W(H)$ be the tangential component of H°. Then,*

1. *$W(H)^{cop} \bowtie H$ is AS-Gorenstein;*
2. *$W(H)^{cop} \bowtie H$ is unimodular;*
3. *If A is central in H, then $(\overline{H}^*)^{cop} \bowtie H$ is also unimodular.*

We finish this last chapter by slightly delving into the connection between these two subjects in subsection 5.2.2, especially how closely related these two conjectures are and the consequences they have on other questions. We also leave yet more questions that can possibly provide more future work.

Some results mentioned throughout this chapter are not original and they are referenced accordingly. However, the conjectures and supporting examples are attributed to the author. The unfinished work in this chapter provides material for future work.

5.1 A formula on the antipode

We start by reviewing a very important formula proved by Radford on the fourth power of the antipode of finite dimensional Hopf algebras.

First we need the definition of distinguished elements. Recall the notion of left and right integrals from section 1.1.2. The subspace of left integrals $\int_H^l$ of a finite dimensional Hopf algebra H is a one-dimensional 2-sided ideal of H, where H acts trivially on the left and possibly non-trivially on the right. Thus, the right action of H on $\int_H^l$ is given by some functional $\chi \in H^*$, that is $\chi : H \to k$ is such that for all $h \in H$

$$\int_H^l h = \int_H^l \chi(h).$$

It is easy to see that χ is a character of H, that is a grouplike element of H^*; and it is usually called the *distinguished grouplike* of H^*, [67, Definition 2.2.3].

We may also apply the same reasoning as before but starting with H^* rather than H. In this case, we obtain a character α of H^*, that is a grouplike element of $(H^*)^*$. But since $(H^*)^* \cong H$ are isomorphic as Hopf algebras by Lemma 1.3.2(4), under this isomorphism α corresponds to a grouplike element g of H, which is named the distinguished grouplike of H. More concretely, $g \in G(H)$ is such that for all $f \in H^*$

$$\int_{H^*}^{l} f = \int_{H^*}^{l} f(g).$$

Theorem 5.1.1 (Radford)**.** *Let H be a finite dimensional Hopf algebra. Recall the left $\rightharpoonup$ and right $\leftharpoonup$ H^*-actions on H from section 1.4. Then,*

$$S^4 = (\operatorname{ad} g) \circ (\operatorname{ad} \chi), \tag{5.1}$$

that is for all $h \in H$

$$S^4(h) = g^{-1}(\chi \rightharpoonup h \leftharpoonup \chi^{-1})g = \sum g^{-1}\chi^{-1}(h_1)h_2\chi(h_3)g,$$

where χ is the distinguished grouplike of H^* and g is the distinguished grouplike of H.

Example 5.1.2. Consider a finite dimensional Taft algebra $H = T_f(n, t, q)$ with $(n, t) = 1$. Recall from Example 1.1.4 that $S(g) = g^{-1}$ and $S(x) = -g^{-t}x$. Thus, $S^4(g) = g$ and $S^4(x) = q^{2t}x$. By Example 1.1.13 $\int_H^l = k(\sum_i g^i)x^{n-1}$, hence the distinguished grouplike χ of H^* is given by

$$\chi(x) = 0, \qquad \chi(g) = q^{-1}.$$

By Example 1.3.4, H is self-dual, so let G and X respectively denote the grouplike and skew-primitive generators of $T_f(n, t, q)^*$, which are given by $G(x^i g^j) = \delta_{i,0} q^{-t^{-1}j}$ and $X(x^i g^j) = \delta_{i,1}$. The left integrals of H^* are $\int_{H^*}^l = k(\sum_i G^i)X^{n-1}$. The right H^*-action on $\int_{H^*}^l$ is given by $\alpha(G) = q^{-1}$ and $\alpha(X) = 0$, which corresponds to evaluation at g^t, that is, under the canonical isomorphism $(H^*)^* \cong H$, α corresponds to g^t - the distinguished grouplike of H. Therefore,

$$S^4(h) = g^{-t} \sum \chi^{-1}(h_1)h_2\chi(h_3)g^t.$$

This formula can easily be checked for the generators g, x.

In [58] Lu, Wu and Zhang extended the notion of left (and right) integrals to noetherian AS-Gorenstein Hopf algebras as follows. Recall the definition of AS-Gorenstein Hopf algebras from section 1.2.

Definition 5.1.3 (Lu-Wu-Zhang, [58, Definition 1.1])**.** Let H be an AS-Gorenstein Hopf algebra, with injective dimension d. A left homological integral is an element of the H-bimodule $\operatorname{Ext}_H^d({}_Hk, {}_HH)$, and we write $\int_H^l = \operatorname{Ext}_H^d({}_Hk, {}_HH)$. An element of $\operatorname{Ext}_H^d(k_H, H_H)$ is a right homological integral, and we write $\int_H^r = \operatorname{Ext}_H^d(k_H, H_H)$. The Hopf algebra H is said to be unimodular if $\int_H^l$ and $\int_H^r$ are isomorphic as H-bimodules.

We point out that this definition of integrals extends the one given in subsection 1.1.2 for finite dimensional Hopf algebras [58, comments after Definition 1.1]. And by the AS-Gorenstein property, notice that both H-bimodules of left and right homological integrals are one-dimensional spaces.

My computations of examples will often require the following two lemmas, the first of which is due to Lu, Wu and Zhang. An element x of a ring H is called *normal* if there exists an algebra automorphism τ such that $xh = \tau(h)x$ for all $h \in H$.

Lemma 5.1.4 (Lu-Wu-Zhang, [58, Lemma 2.6]). *Let H be an AS-Gorenstein Hopf algebra with injective dimension d and let x be a normal regular element of H such that (x) is a Hopf ideal of H. Let τ be the algebra automorphism such that $xh = \tau(h)x$ for all $h \in H$.*

1. *$H' := H/(x)$ is an AS-Gorenstein Hopf algebra with injective dimension $d-1$;*

2. *$\int_H^l \cong {}^1(\int_{H'}^l)^{\tau^{-1}}$ as H-bimodules, that is the left H-action is trivial and the right H-action on $\int_{H'}^l$ is twisted by τ^{-1}.*

Lemma 5.1.5. *Let H be an affine noetherian Hopf algebra of GK-dimension 1. Then, H is AS-Gorenstein with injective dimension 1 and it has a classical ring of fractions Q. Its set of left integrals is*

$$\int_H^l \cong \{q \in Q : H^+q \subseteq H\}/H = (kp + H)/H \subseteq Q/H,$$

for some $p \in Q \setminus H$ such that $H^+p \subseteq H$.

Proof. First, a noetherian ring of GK-dimension 1 satisfies a polynomial identity [90, Theorem]. By [102, Theorems 0.1, 0.2] H is AS-Gorenstein and GK-Cohen-Macaulay with $\operatorname{injdim}(H) = \operatorname{GKdim}(H) = 1$, and its classical ring of fractions Q of H is quasi-Frobenius artinian.

We claim that Q is the injective hull of ${}_HH$, that is, in simple terms, a minimal injective module that contains H; see [37, Chapter 5]. Clearly Q is an essential extension of H, meaning that H has nontrivial intersection with any nonzero H-submodule of Q: since for any nonzero fraction $x^{-1}h$ in Q, with $h, x \in H$ and x regular, we have $xx^{-1}h = h \in H$. And, since Q is quasi-Frobenius, that is self-injective, then it is an injective H-module by [37, Corollary 10.14] and the claim follows.

So $0 \to H \to Q$ is the start of an injective resolution of ${}_HH$ but, since $\operatorname{injdim} H = 1$, the injective resolution must have length 1, hence

$$0 \to H \to Q \xrightarrow{\pi} Q/H \to 0$$

is a minimal injective resolution as left (and right) H-modules.

Applying the functor $\operatorname{Hom}_H(k, -)$, we get the complex

$$0 \to \operatorname{Hom}_H(k, H) \to \operatorname{Hom}_H(k, Q) \xrightarrow{\pi^*} \operatorname{Hom}_H(k, Q/H) \to 0.$$

We claim that $\operatorname{im}\pi^* = 0$. For, any map $f \in \operatorname{Hom}_H(k, Q)$ induces a copy of ${}_Hk$ into Q but, since Q is an essential extension of H, such copy of ${}_Hk$ must be contained in H, hence it vanishes in Q/H. Therefore, $\pi \circ f = 0$.

By definition of an Ext-group, we have

$$\int_H^l := \operatorname{Ext}^1_H({}_Hk, {}_HH) = \operatorname{Hom}_H({}_Hk, {}_HQ/H)/\operatorname{im}\pi^* = \operatorname{Hom}_H({}_Hk, {}_HQ/H).$$

Each H-linear map $f : k \to Q/H$ is determined by the element $q + H := f(1)$, which satisfies $\epsilon(h)q + H = hq + H$ in Q/H for all $h \in H$, that is $H^+q \subseteq H$. Thus, the set of left integrals of H is the H-submodule $\{q \in Q : H^+q \subseteq H\}/H$ of Q/H. Given that $\int_H^l$ is one-dimensional, we must have

$$\int_H^l \cong (kp + H)/H$$

for some p as stated. □

Using the notion of homological integrals, Brown and Zhang proved the following generalization of Theorem 5.1.1, extending formula (5.1) on the antipode.

Theorem 5.1.6 (Brown-Zhang, [18, Corollary 4.6]). *Suppose that H is a noetherian AS-Gorenstein Hopf algebra with bijective antipode. Then,*

$$S^4 = \gamma \circ \tau^r_\chi \circ \tau^l_{\chi^{-1}}, \tag{5.2}$$

where χ is determined by the right H-module structure of $\int_H^l$, $\tau^r_{\chi^{-1}}$ and τ^l_χ are winding automorphisms, that is

$$\tau^l_{\chi^{-1}}(h) = h \leftharpoonup \chi^{-1} = \sum \chi^{-1}(h_1)h_2 \quad \textit{and} \quad \tau^r_\chi(h) = \chi \rightharpoonup h = \sum h_1\chi(h_2),$$

and γ is some inner automorphism of H.

Note that more recently it was proved that a noetherian AS-Gorenstein Hopf algebra has bijective antipode [57, Corollary 0.3].

The following simple question was asked in [18, Question 4.6] but so far has not been answered.

Question 5.1.7. What is the map γ? Is it conjugation by a (canonical) grouplike of H?

We conjecture below an answer to this question for affine commutative-by-finite Hopf algebras but first we look at the answer in the finite dimensional case.

Following Radford's formula, in the finite dimensional case γ is given by conjugation by the character of H^* which corresponds to the right H^*-action of $\int_{H^*}^l$ and, under the canonical isomorphism $(H^*)^* \cong H$, γ corresponds to conjugation by a grouplike of H - the distinguished grouplike of H.

When extending this to the infinite dimensional case, one immediate problem lies in the fact that in general H° will not be AS-Gorenstein, since it is often not even a noetherian algebra, as is demonstrated by the computations done in §4.4. Moreover, $(H^\circ)^\circ$ and H are also in general no longer isomorphic. Therefore, we aim to replace H° by a suitable Hopf subalgebra of H°.

Question 5.1.8. Let H be a noetherian AS-Gorenstein Hopf algebra. Can we find an AS-Gorenstein Hopf subalgebra U of H° such that γ is obtained from the right action of U on the left integrals $\int_U^l$?

For the class of affine commutative-by-finite Hopf algebras studied in this thesis, first notice that these are AS-Gorenstein by Theorem 2.1.8(1), hence Theorem 5.1.6 above applies. As for Question 5.1.8, here the obvious candidates for U are the finite dimensional $\overline{H}^*$, the tangential component $W(H)$ from §4.2 and, provided A is reduced and $A \subseteq H$ is orbitally semisimple, the character component $\widehat{kG}$ from §4.3. It is clear from the examples in section 4.4 that $\widehat{kG}$ is not a good guess, as quite often it is not even a noetherian algebra (hence obviously not AS-Gorenstein). As we will see in Example 5.1.18 below, $\overline{H}^*$ is also not a good choice - see Remark 5.1.19. Therefore, we propose the following conjecture.

Conjecture 5.1.9. *Let H be an affine commutative-by-finite Hopf algebra and the inner automorphism γ as in (5.2). Let $W(H)$ be the tangential component of H°. Then,*

1. *$W(H)$ is AS-Gorenstein.*

Let α be the character of $W(H)$ corresponding to the right action of $W(H)$ on its left integrals. Then,

2. *$\alpha = ev_g$, evaluation at some unique grouplike g of H;*
3. *γ is conjugation by g, that is $\gamma(h) = g^{-1}hg$ for any $h \in H$, and therefore*

$$S^4(h) = \sum g^{-1}\chi^{-1}(h_1)h_2\chi(h_3)g.$$

Remark 5.1.10. If H is a finite dimensional Hopf algebra, we can choose $A = k1$ to be the trivial Hopf subalgebra, and in this case the previous conjecture holds. For, the dual of A is trivial and $W(H) = H^*$ is the whole dual of H. Therefore, the previous conjecture proposes the same construction of the distinguished grouplikes χ of H^* and g of H as explained at the beginning of this section, and the same formula for S_H^4 as given by Radford (Theorem 5.1.1).

The first part of the previous conjecture is proved in the following result for affine commutative-by-finite Hopf algebras that decompose nicely. As we saw in section 4.4 this encompasses most of the examples we studied in this thesis.

Proposition 5.1.11. *Let H be an affine commutative-by-finite Hopf algebra. Suppose that k has characteristic 0 and $H = A \oplus X$ decomposes as right A-modules with X a coideal. Then, $W(H)$ is AS-Gorenstein, Auslander-Gorenstein and GK-Cohen-Macaulay.*

Proof. By Theorem 4.2.3(4) we immediately have

$$W(H) \cong \overline{H}^* \# U(\mathfrak{g}).$$

Let $\{f_1, \dots, f_n\}$ be a k-basis of $\mathfrak{g}$ and consider the following filtration of $W := W(H)$. Let $W_0 = \overline{H}^*$ and $W_j = W_{j-1} + \sum_{i_1,\dots,i_j=1}^{n} \overline{H}^* f_{i_1} \dots f_{i_j}$. Then, the associated graded ring

$$\operatorname{gr} W := W_0 \oplus \bigoplus_{i=1}^{\infty} W_i/W_{i-1} \cong \overline{H}^* \otimes k[\overline{f_1}, \dots, \overline{f_n}]$$

is the usual polynomial ring on $\overline{f_1}, \dots, \overline{f_n}$ with coefficient ring $\overline{H}^*$. For, since $f_i f_j = f_j f_i + [f_i, f_j]$ in $U(\mathfrak{g}) \subseteq W$, their images $\overline{f_i}, \overline{f_j}$ commute in $\operatorname{gr} W$, and by the formula of the action of $U(\mathfrak{g})$ on $\overline{H}^*$ from Theorem 4.2.3(4) $f_i \varphi = \varphi f_i + f_i \cdot \varphi$ in W for any $\varphi \in \overline{H}^*$, hence $\overline{f_i}$ commutes with any functional from $\overline{H}^*$ in $\operatorname{gr} W$. Moreover, by construction $\operatorname{gr} W/(\overline{H}^*)^+ \operatorname{gr} W$ is the associated graded ring of $U(\mathfrak{g})$ with respect to the standard filtration, so it is $k[\overline{f_1}, \dots, \overline{f_n}]$ by the PBW theorem (Theorem 1.1.3). Thus, $\operatorname{gr} W$ is the polynomial algebra over $\overline{H}^*$.

First, we prove W is Auslander-Gorenstein. Any finite dimensional Hopf algebra is Auslander-Gorenstein of injective dimension 0, as we mentioned in section 1.2, hence $\overline{H}^*$ is Auslander-Gorenstein. And every Ore extension of an Auslander-Gorenstein ring is itself Auslander-Gorenstein [32, Theorem 4.2], hence $\operatorname{gr} W$ is Auslander-Gorenstein. Since the filtration of W is Zariskian [42, I.3.3 Remark 5, II.2.2 Proposition 1], W is Auslander-Gorenstein by [7, Theorem 3.9].

Second, the graded ring $\operatorname{gr} W$ is GK-Cohen-Macaulay. Since $\operatorname{gr} W$ is affine, noetherian by [37, Corollary 1.5] and a PI ring by [64, Corollary 13.1.13(iii)], it is GK-Cohen-Macaulay by [102, Theorem 0.1, 0.2].

Third, W is GK-Cohen-Macaulay. Let M be a noetherian right W-module. By [49, Proposition 6.6] $\operatorname{GKdim}(\operatorname{gr} M_{\operatorname{gr} W}) = \operatorname{GKdim}(M_W)$ and $\operatorname{GKdim}(\operatorname{gr} W) = \operatorname{GKdim}(W)$. Moreover, by [7, proof of Theorem 3.9] the grades $j(\operatorname{gr} M_{\operatorname{gr} W})$ and $j(M_W)$ coincide. Since $\operatorname{gr} W$ is GK-Cohen-Macaulay, then so is W.

Lastly, since the Hopf algebra W is both Auslander-Gorenstein and GK-Cohen-Macaulay, by [18, Lemma 6.1] W is AS-Gorenstein. $\square$

The following lemma is a first step towards proving Conjecture 5.1.9(2). More specifically, it yields a group embedding of the group $G(H)$ of grouplike elements of H into the group of characters of $W(H)$, provided A is a domain. This implies that, in this case, there will be at most one grouplike of H determining the right H-module structure of the left integrals of $W(H)$.

Lemma 5.1.12. *Let H be an affine commutative-by-finite Hopf algebra. Suppose A is a domain. Then, we have an embedding*

$$\begin{array}{rcl} \phi : G(H) & \hookrightarrow & W(H)^\circ \\ g & \mapsto & ev_g \end{array},$$

where ev_g is evaluation at g, that is $ev_g(w) = w(g)$ for all $w \in W(H)$.

Proof. First, since each $g \in G(H)$ is grouplike, the evaluation functional ev_g is a character of $W(H)$ (that is, an algebra map $W(H) \to k$), hence $ev_g \in W(H)^\circ$ and ϕ is a well defined group homomorphism. We now prove injectivity.

Suppose that $g \neq 1_G$. Since A is a domain, flatness of H over A from Theorem 2.1.9(1) and Krull's Intersection theorem [31, Corollary 5.4] yield

$$\bigcap_n (A^+H)^n = \bigcap_n (A^+)^n H = \left(\bigcap_n (A^+)^n\right) H = 0.$$

Thus, $g - 1 \notin (A^+H)^n$ for some $n > 0$. We define the functional w that satisfies $w(g-1) = 1$, $w((A^+H)^n) = 0$ and that vanishes on a complement of $k(g-1) \oplus (A^+H)^n$. Since $(A^+H)^n$ has finite codimension, $w \in W(H)^\circ$. And $w(g) \neq w(1)$, therefore $ev_g \neq ev_{1_G}$. □

This lemma fails if we consider the Hopf subalgebra $\overline{H}^*$ instead of $W(H)$, as we explain in Remark 5.1.19.

We now look at a few examples, all of which support conjecture 5.1.9. We start with commutative examples.

Example 5.1.13 (Polynomial algebra and Laurent polynomial algebra)**.** Consider the polynomial ring $H = k[x]$ over a field of characteristic 0. Being commutative, $S^2 = \mathrm{id}$ and H is unimodular, which implies the winding automorphisms, and hence γ, are the identity. The tangential component is simply $W(H) = k[f]$. It is clearly AS-Gorenstein and unimodular, hence the right action of $W(H)$ on its left integrals is given by the counit $\epsilon_{W(H)}$, that is evaluation at 1_H. Therefore, the conjecture holds.

Similarly, the conjecture holds for the Laurent polynomial ring and for finitely many indeterminates.

We now consider a more interesting commutative noncocommutative example.

Example 5.1.14. Let k be a field of characteristic 0 and recall from Example 1.3.11 the coordinate ring of $G = (k, +) \rtimes k^\times$, that is $H = \mathcal{O}(G) = k[x, y^{\pm 1}]$ where x is $(1, y)$-primitive and y is grouplike. Again, since H is commutative, we must have $\gamma = \mathrm{id}$.

As in Example 1.3.11, the tangential component is $W(H) = U(\mathfrak{g})$, the enveloping algebra of the 2-dimensional (nonabelian) solvable Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$; that is $\mathfrak{g}$ has basis $\{f, f'\}$ with brackets $[f', f] = f$. More specifically, these functionals are defined by $f(x^i y^j) = \delta_{i,1}$ and $f'(x^i y^j) = \delta_{i,0} j$.

By Proposition 5.1.11, $W := W(H)$ is AS-Gorenstein. Let us calculate its left integrals. Since $ff' = f'f - f = (f'-1)f$ in W, f is normal and it is primitive and, since W is a domain, it is also regular. Thus, Lemma 5.1.4 yields

$$\int_W^l = \left(\int_{W/(f)}^l\right)^{\tau^{-1}},$$

where $\tau(f') = f'-1$. But $W/(f) \cong k[\overline{f'}]$ is commutative, hence unimodular. Therefore, as right W-modules we have

$$\int_W^l = (W/(f,f'))^{\tau^{-1}} = W/(f, f'+1).$$

The character of W corresponding to the right W-action on its left integrals is then given by $\alpha(f) = 0, \alpha(f') = -1$. Upon analyzing the definitions of these maps given above, we conclude this is evaluation at the grouplike y^{-1}. And, H being commutative, γ can be seen as conjugation by any grouplike of H.

We now look at three noncommutative examples arriving from section 2.2.5.

Example 5.1.15. Consider once again the group algebra of the dihedral group $H = kD = k\langle a, b : a^2 = 1, aba = b^{-1}\rangle$ introduced in §2.2.5. Being cocommutative, $S^2 = \mathrm{id}$, hence $S^4 = \mathrm{id}$.

We start by calculating the left integrals of H. First, note that $k\langle b\rangle \setminus \{0\}$ consists of regular elements of H, since H is a free $k\langle b\rangle$-module and $k\langle b\rangle$ is a domain. In particular, the (b, b^{-1})-primitive element $b - b^{-1}$ is regular and normal, because $(b-b^{-1})a = -a(b-b^{-1})$. Hence, by Lemma 5.1.4 we have

$$\int_H^l \cong \left(\int_{H/(b-b^{-1})}^l\right)^{\tau^{-1}}$$

as right H-modules, where $\tau(b) = b, \tau(a) = -a$. Since $H/(b-b^{-1}) = k(C_2 \times C_2)$, by Example 1.1.12 its space of (left) integrals is the subspace

$$\int_{k(C_2\times C_2)} = k(1 + \overline{a} + \overline{b} + \overline{ab}).$$

Thus, as right H-modules

$$\int_H^l \cong \left(\int_{k(C_2\times C_2)}\right)^{\tau^{-1}} = (H/(a-1, b-1))^{\tau^{-1}} = H/(a+1, b-1).$$

The character χ is therefore given by $\chi(a) = -1, \chi(b) = 1$. Since $\chi^{-1} = \chi\circ S = \chi$, the composition of winding automorphisms is given by

$$\tau^r_\chi \circ \tau^l_{\chi^{-1}} : \begin{array}{rcl} a & \mapsto & \chi^{-1}(a)a\chi(a) = a \\ b & \mapsto & \chi^{-1}(b)b\chi(b) = b \end{array},$$

so by (5.2) $\gamma = \mathrm{id}$, or in other words conjugation by 1. Let us try to see this in terms of the dual.

By Corollary 4.4.6 $H^\circ = kC_2 \otimes k[f] \otimes k(k^\times, \cdot)$ and the tangential component is

$$W(H) = kC_2 \otimes k[f] = \mathcal{O}(k \rtimes C_2).$$

This Hopf algebra is AS-Gorenstein by Proposition 5.1.11 (or simply by observing that $W(H)$ is commutative), hence it has integrals and is in fact unimodular. Therefore, $W(H)$ acts trivially (that is, by $\epsilon_{W(H)}$) on its left integrals, which is exactly evaluation at 1. The same holds if we take $\overline{H}^*$ instead of $W(H)$.

This example is a particular case of the following result. Recall the group algebras of abelian-by-finite groups from §2.2.4.

Lemma 5.1.16. *Conjecture 5.1.9 holds for the group algebras of finitely generated abelian-by-finite groups over a field of characteristic 0.*

Proof. Let $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated group with abelian normal subgroup N of finite index. Since $H = kG$ is cocommutative, $S^2 = \mathrm{id}$. Moreover, the character χ that determines the right H-structure of its left integrals depends on $\lambda_i := \chi(g_i) \in k^\times$ for $1 \leq i \leq n$. And then $\chi^{-1}(g_i) = \lambda_i^{-1}$, yielding $\tau^r_\chi \circ \tau^l_{\chi^{-1}} = \mathrm{id}$. By (5.2) $\gamma = \mathrm{id}$.

Since H is cocommutative, $W(H)$ (and $\overline{H}^*$) is commutative. By Proposition 5.1.11 (whose hypothesis follow as in the proof of Corollary 4.4.5), $W(H)$ is AS-Gorenstein. Hence, $W(H)$ is unimodular, that is the right structure of its left integrals is given by $\epsilon_{W(H)}$, which is precisely evaluation at $1_H = 1_G$. $\square$

Example 5.1.17. Consider once more the infinite dimensional Taft algebra $H = T(n, t, q) = k\langle g, x : g^n = 1, xg = qgx \rangle$, where g is grouplike and x is $(1, g^t)$-primitive. Since $S(g) = g^{-1}$ and $S(x) = -g^{-t}x$, we have $S^4(g) = g$ and $S^2(x) = -S(x)S(g^{-t}) = g^{-t}xg^t = q^t x$ so $S^4(x) = q^{2t}x$.

A computation of the left integrals of these Hopf algebras can be found in [58, Example 2.7]: since x is a normal regular element of H and $H/(x) \cong kC_n$, Lemma 5.1.4 and Example 1.1.12 yield that as right H-modules we have

$$\int_H^l \cong \left(\int_{H/(x)}^l \right)^{\tau^{-1}} = \left(k \sum_i \overline{g}^i \right)^{\tau^{-1}},$$

where $\tau(x) = x, \tau(g) = qg$. Thus, as right H-modules $\int_H^l \cong H/(x, g - q^{-1})$, and χ is defined by $\chi(g) = q^{-1}, \chi(x) = 0$. Since $\chi^{-1} = \chi \circ S$, we have $\chi^{-1}(g) = \chi(g^{-1}) = q, \chi^{-1}(x) = \chi(-g^{-t}x) = 0$, so the composition of the winding automorphisms is

$$\begin{array}{rcl} \tau^r_\chi \circ \tau^l_{\chi^{-1}} : g & \mapsto & \chi^{-1}(g) g \chi(g) = g \\ x & \mapsto & \chi^{-1}(x) + \chi^{-1}(g^t)x + \chi^{-1}(g^t)g^t\chi(x) = q^t x \end{array}.$$

By (5.2) we must have

$$\begin{array}{rcl}\gamma : g & \mapsto & g \\ x & \mapsto & q^t x\end{array}.$$

In other words, γ is conjugation by g^t. We now investigate the left integrals of $W(H)$.

By Corollary 4.4.6, the tangential component is

$$W(H) = kC_d \otimes T_f(n', t', q^d) \otimes k[f]$$

as an algebra, where $d = (n,t), n' = n/d, t' = t/d$. The generator of C_d is defined as $\alpha(x^i g^j) = \delta_{i,0} q^{n'j}$, the generators of T_f are given by $X(x^i g^j) = \delta_{i,1}$ and $G(x^i g^j) = \delta_{i,0} q^{-t'^{-1}dk}$ and f is given by $f(x^i g^j) = \delta_{i,n'} q^{n'j}$ (where t'^{-1} is the inverse of t' modulo n' and $j = dk + r$ where $0 \le r < d$).

First, note that any nonzero element of $k[f]$ is regular in $W(H)$, since $W(H)$ is a free module over the domain $k[f]$. Second, $W(H)$ is AS-Gorenstein by Proposition 5.1.11 and we now calculate its left integrals. Since f is central and regular in $W(H)$, by Lemma 5.1.4 the left integrals of $W(H)$ are

$$\int^l_{W(H)} \cong \int^l_{W(H)/(f)} = \int^l_{kC_d \otimes T_f} = k\left(\sum_{i=0}^{n'-1} \overline{G}^i\right)\overline{X}^{n'-1}\left(\sum_{i=0}^{d-1}\overline{\alpha}^i\right),$$

so the character for the right action is $\beta(\alpha) = 1, \beta(G) = (q^d)^{n'-1} = q^{-d}, \beta(X) = 0, \beta(f) = 0$. This is precisely the evaluation at g^t:

$$\alpha(g^t) = q^{n't} = q^{nt'} = 1, \quad X(g^t) = 0, \quad G(g^t) = q^{-t'^{-1}dt'} = q^{-d}, \quad f(g^t) = 0.$$

If we consider $\overline{H}^*$ instead of $W(H)$, then

$$\int^l_{\overline{H}^*} = k\left(\sum_{i=0}^{n'-1} G^i\right) X^{n'-1}\left(\sum_{i=0}^{d-1}\alpha^i\right),$$

so the character for the right $\overline{H}^*$-action is also evaluation at the grouplike g^t.

Example 5.1.18. Recall from §2.2.5 the generalized Liu algebras given by

$$H = B(n, w, q) = k\langle x^{\pm 1}, g^{\pm 1}, y : x \text{ central}, yg = qgy, g^n = x^w = 1 - y^n\rangle,$$

where g, x are grouplike and y is $(1, g)$-primitive. Since $S(y) = -g^{-1}y$, we have $S^2(y) = g^{-1}yg = qy$ and

$$S^4(x) = x, \qquad S^4(g) = g, \qquad S^4(y) = q^2 y. \tag{5.3}$$

As in the previous examples, we start by computing the left integrals of H and the map γ. Following the same argument as before, any nonzero element of $k[x^{\pm 1}]$ is regular in H. In particular, $x - 1$ is a central regular element of H, so Lemma 5.1.4

yields that the space of left integrals is

$$\int_H^l \cong \int_{H/(x-1)}^l = \int_{T_f}^l = k\left(\sum_i \overline{g}^i\right)\overline{y}^{n-1},$$

since $H/(x-1) = T_f(n,1,q)$ is the Taft algebra generated by $\overline{g}$ and $\overline{y}$, these being the images of g and y in $H/(x-1)$. Thus, as right H-modules $\int_H^l \cong H/(x-1, g-q^{-1}, y)$, hence the associated character is $\chi(x) = 1, \chi(y) = 0, \chi(g) = q^{-1}$. Its convolution inverse is $\chi^{-1}(x) = 1, \chi^{-1}(y) = 0, \chi^{-1}(g) = q$, so the winding automorphisms give

$$\begin{array}{rcl} \tau_\chi^r \circ \tau_{\chi^{-1}}^l : x & \mapsto & \chi^{-1}(x)x\chi(x) = x \\ g & \mapsto & \chi^{-1}(g)g\chi(g) = g \\ y & \mapsto & \chi^{-1}(y) + \chi^{-1}(g)y + \chi^{-1}(g)g\chi(y) = qy \end{array}.$$

By (5.2) and (5.3), we must have $\gamma(x) = x$, $\gamma(g) = g$ and $\gamma(y) = qy$. That is, γ is given by conjugation by g. Let us see this now in terms of characters of $W := W(H)$.

By Corollary 4.4.6,

$$W \cong T_f(n,1,q)\#k[f] = T_f(n,1,q)[f;\delta]$$

is a differential operator algebra, where the derivation is given by $\delta(G) = 0$ and $\delta(Y) = wY\sum_i \mu_i G^i$ in which w is the parameter of $H = B(n,w,q)$ and the scalars μ_i are such that $\sum_i \mu_i q^{-ij} = \delta_{j,n-1}$.

Clearly W is an affine Hopf algebra, noetherian by [37, Theorem 2.6], and, since W is a finitely generated $k[f]$-module, it has GK-dimension 1 by [49, Proposition 5.5]. Thus, by Lemma 5.1.5 W is AS-Gorenstein and its space of left integrals is

$$\int_W^l \cong \{r \in Q : W^+r \subseteq W\}/W \cong (kp+W)/W,$$

where $Q := Q(W)$ is the classical ring of fractions of W and $p \in Q \setminus W$ is an element such that $W^+p \subseteq W$.

First, note that any nonzero polynomial of $k[f]$ is regular in W, since $W = T_f(n,1,q)[f;\delta]$ is a free module over the domain $k[f]$. In particular, f is invertible in Q and, considering $p' = (\sum_i G^i)Y^{n-1}$, a left integral of T_f, we define

$$p := p'f^{-1} = \left(\sum_i G^i\right)Y^{n-1}f^{-1}.$$

Since p' is a left integral of T_f, G and Y act trivially on the left on p, that is $Gp = p$ and $Yp = 0$. As for the left action of f, it follows by induction that for each $0 \leq k < n$

$$fY^k = Y^k\left[f + w\sum_i \mu_i \left(\sum_{j=0}^{k-1} q^{-ij}\right) G^i\right],$$

where the scalars μ_i are such that $\sum_i \mu_i q^{-ij} = \delta_{j,n-1}$. Since q is a primitive nth root of unity, we have $\sum_{j=0}^{n-1}(q^{-i})^j = 0$, so $\sum_{j=0}^{n-2} q^{-ij} = -q^{-i(n-1)} = -q^i$, thus

$$\begin{aligned} fp &= \left(\sum_i G^i\right) fY^{n-1}f^{-1} = \left(\sum_i G^i\right) Y^{n-1}\left(f - w\sum_i q^i\mu_i G^i\right) f^{-1} \\ &= \left(\sum_i G^i\right) Y^{n-1} - w\left(\sum_i G^i\right) Y^{n-1}\left(\sum_i q^i\mu_i G^i\right) f^{-1}. \end{aligned}$$

But, since $\sum_i \mu_i = 0$, we have

$$\begin{aligned} \left(\sum_i G^i\right) Y^{n-1}\left(\sum_i q^i\mu_i G^i\right) &= \left(\sum_j G^j\right)\left(\sum_i \mu_i G^i\right) Y^{n-1} = \left(\sum_{i,j} \mu_i G^{i+j}\right) Y^{n-1} \\ &= \left(\sum_i \mu_i\right)\left(\sum_k G^k\right) Y^{n-1} = 0, \end{aligned}$$

so it follows that $fp = (\sum_i G^i)Y^{n-1}$. Therefore, p is such that $W^+p \subseteq W$, and therefore it determines the space of left integrals of W.

We now investigate the right action of W on $\int_W^l$. Since f and G commute in W, f^{-1} and G commute in Q, hence

$$pG = \left(\sum_i G^i\right) Y^{n-1}Gf^{-1} = q^{-1}\left(\sum_i G^{i+1}\right) Y^{n-1}f^{-1} = q^{-1}p.$$

Moreover, $pf = (\sum_i G^i)Y^{n-1}$, therefore the right action of f on $\int_W^l = (kp + W)/W$ is trivial, that is given by multiplication by $\epsilon(f) = 0$.

The right action of Y on $\int_W^l$ would require rather more calculations, so we use another argument instead. Note that $(kp + W)/W = \{r \in Q : W^+r \subseteq W\}/W$ and $W^+(rw) = (W^+r)w \subseteq Ww \subseteq W$ for any $w \in W$. Thus, $(kp+W)/W$ is a W-bimodule and, being $\int_W^l$, it is also one-dimensional. In particular, $(kp+W)/W$ is a simple right W-module, hence it must be annihilated by the nilradical of W. But Y is a normal element of W, for $YG = qGY$ and $fY = Y(f + w\sum \mu_i G^i)$, hence YW is a nilpotent ideal of W and must therefore annihilate $\int_W^l$. In particular, $pY = 0$.

Therefore, the character associated to the right action of W on $\int_W^l$ is $\alpha(G) = q^{-1}, \alpha(Y) = 0, \alpha(f) = 0$. Recalling from Corollary 4.4.6 that the functionals are defined as $G(y^ig^jx^k) = q^{-j}\delta_{i,0}, Y(y^ig^jx^k) = \delta_{i,1}, f(y^ig^jx^k) = k\delta_{i,0}$, α is precisely evaluation at g. Therefore, Conjecture 5.1.9 holds in this example.

Remark 5.1.19. Recall that Lemma 5.1.12 implies that, when A is a domain, there is at most one grouplike of H that determines the right H-module structure of the left integrals of $W(H)$. And throughout the previous examples such grouplike exists, and so it is unique. However, such uniqueness fails if we consider $\overline{H}^*$ instead of $W(H)$. In

Example 5.1.18, we have

$$\int_{\overline{H}^*}^{l} = k\left(\sum_i G^i\right) Y^{n-1},$$

so the associated character is $\beta(G) = q^{-1}, \beta(Y) = 0$, which is evaluation at any grouplike of the form $x^i g^{n-1}$. So here we do not have a canonical choice for the grouplike associated to γ conjectured in 5.1.9, which suggests that $\overline{H}^*$ may not be a good choice for the "suitable" Hopf subalgebra of H° to answer question 5.1.8.

5.2 The Drinfeld double in infinite dimension

In this section we extend the concept of the Drinfeld double of a finite dimensional Hopf algebra from section 1.4 to the infinite dimensional case. We mention a few basic properties that still hold for infinite dimensional Hopf algebras and illustrate it with a few examples, most of which are from §2.2.5. Recalling the important property of unimodularity of the double of finite dimensional Hopf algebras (Theorem 1.4.7), we conjecture that in the commutative-by-finite infinite dimensional case a particular Hopf subalgebra of the double, namely $W(H)^{cop} \bowtie H$, is unimodular. Once again, we support this hypothesis with a few examples, mostly from §2.2.5. In subsection 5.2.2 we briefly explain how the double relates to Conjecture 5.1.9 and the formula of the fourth power of the antipode, while also looking at some consequences of these conjectures and enumerating a few other questions that may provide future work.

We have introduced the Drinfeld double for finite dimensional Hopf algebras back in section 1.4. When trying to extend this notion to Hopf algebras of infinite dimension, the only problem that arises is the fact that the antipode may not be bijective. Thus, throughout this section we assume that

all Hopf algebras have bijective antipode.

This is not a very stringent assumption, since the antipode is quite often bijective. Such is the case for every noetherian AS-Gorenstein Hopf algebra by [57, Corollary 0.3] and for every affine noetherian PI Hopf algebra by [85, Corollary 2]. In particular, this assumption holds for the Hopf algebras in which we are most interested, commutative-by-finite Hopf algebras; see Theorem 2.1.4.

Taking this into consideration, the Drinfeld double $D(H)$ is defined similarly to the finite dimensional case, using H° rather than H^*. More specifically, as a coalgebra

$$D(H) = (H^\circ)^{cop} \otimes H$$

and the product and the antipode are respectively given by

$$hf = \sum (h_1 \rightharpoonup f \leftharpoonup S^{-1}h_3)h_2 \qquad \text{and} \qquad S_{D(H)}(fh) = S(h)(S^{-1})^*(f),$$

for any $h \in H, f \in H^\circ$. Recall from Lemma 1.4.2 an alternative formula for the product in $D(H)$ and an explicit formula of the antipode $S_{D(H)}$.

Proposition 5.2.1. *Let H be a Hopf algebra with bijective antipode. Then,*

1. *$D(H)$ is a Hopf algebra with the above structure.*
2. *H and $(H^\circ)^{cop}$ are Hopf subalgebras of $D(H)$.*

Suppose further that H is affine commutative-by-finite. Recall the tangential and character components, $W(H)$ and $\widehat{kG}$, respectively from sections 4.2 and 4.3. Then,

3. *$(\overline{H}^*)^{cop} \bowtie H$ and $W(H)^{cop} \bowtie H$ are Hopf subalgebras of $D(H)$.*
4. *If A is reduced and $A \subseteq H$ is orbitally semisimple, $\widehat{kG}^{cop} \bowtie H$ is also a Hopf subalgebra of $D(H)$.*

Proof. (1),(2) These follow just as in the finite dimensional case.

(3) By Lemma 4.2.2 and (2), $W(H)^{cop}$ is a Hopf subalgebra of $D(H)$. Thus, $W(H)^{cop} \bowtie H$ is a subcoalgebra of $D(H)$ and it suffices to prove that $W(H)^{cop} \bowtie H$ is closed under multiplication. By the formula above, it suffices to see that for any $f \in W(H), h, h' \in H$ we have

$$h \rightharpoonup f \leftharpoonup h' \in W(H).$$

But $f((A^+H)^n) = 0$ for some $n > 0$, hence closure follows from the fact that each $(A^+H)^n$ is an ideal of H. The proof is similar for $(\overline{H}^*)^{cop} \bowtie H$.

(4) $\widehat{kG}$ is defined, since A is reduced, and, under orbital semisimplicity, $\widehat{kG}$ is a Hopf subalgebra of H° by Proposition 4.3.4(1). By (2) $\widehat{kG}^{cop}$ is a Hopf subalgebra of $D(H)$. Now (4) follows by an argument similar to (3), which uses the fact that $H\mathbf{m}_{g_1}^{(\overline{H})} \cap \ldots \cap H\mathbf{m}_{g_r}^{(\overline{H})}$ is an ideal of H for any $g_i \in G$. ☐

Moreover, Proposition 1.4.6 on the commutativity of the Drinfeld double, can be extended infinite dimension as follows.

Lemma 5.2.2. *Let H be a Hopf algebra with bijective antipode. The following are equivalent:*

1. *H is commutative and cocommutative;*
2. *$D(H)$ is commutative and cocommutative.*

And, in this case $D(H) = H^\circ \otimes H$ as Hopf algebras.

Proof. This follows as in Proposition 1.4.6. ☐

Let us compute the Drinfeld double for some examples. Note that the computations of these examples can be found in the appendix, §A.2 and §A.4.

Example 5.2.3. Consider the polynomial algebra $k[x]$ from Example 1.1.9. Since $k[x]$ is commutative and cocommutative, by Lemma 5.2.2 and Example 1.3.8 we have

$$D(k[x]) = k[x]^\circ \otimes k[x] \cong k[f] \otimes k(k,+) \otimes k[x]$$

as Hopf algebras.

On a similar fashion, the algebra $k[x^{\pm 1}]$ of Laurent polynomials from Example 1.1.8 is commutative and cocommutative, hence Lemma 5.2.2 and Example 1.3.10 give

$$D(k[x^{\pm 1}]) = k[x^{\pm 1}]^\circ \otimes k[x^{\pm 1}] \cong k[f] \otimes k(k^\times, \cdot) \otimes k[x^{\pm 1}].$$

Example 5.2.4. Recall from Example 1.3.11 the coordinate ring $H = k[x, y^{\pm 1}]$ of the affine algebraic group $G = (k,+) \rtimes k^\times$, where x is $(1,y)$-primitive and y is grouplike. It follows from that example that its Drinfeld double is

$$D(H) = H^\circ \bowtie H = (U(\mathfrak{g}) * G) \bowtie k[x, y^{\pm 1}],$$

where $U(\mathfrak{g}) = k\langle f, f' : [f', f] = f\rangle$ is the enveloping algebra of the 2-dimensional nonabelian solvable Lie algebra.

With the notation as in Example 1.3.11, the product in $D(H)$ is given by the relations in H and H° as well as

$$xf = fx + (1-y), \quad xf' = (f'+1)x, \quad x\chi_{(\alpha,\beta)} = \chi_{(\alpha,\beta)}(\alpha(1-y) + \beta x), \quad y \text{ central}.$$

The calculations can be found in §A.4.

Example 5.2.5. Recall the group algebra of the dihedral group $D = \langle a, b : a^2 = 1, aba = b^{-1}\rangle$ from section 2.2.5. From Corollary 4.4.6 it follows that

$$D(kD) = (kD)^\circ \bowtie kD \cong (kC_2 \otimes k[f] \otimes k(k^\times, \cdot)) \bowtie kD.$$

As an algebra the double is in fact a skew group ring

$$D(kD) = (kD)^\circ * D,$$

where the action of D on $(kD)^\circ$ is determined by

$$\alpha \text{ central}, \qquad af = -fa, \qquad az_\lambda = z_{\lambda^{-1}}a,$$

$$bf = (f + 1 - \alpha)b, \qquad bz_\lambda = \frac{1}{2}\left((1+\lambda^2) + (1-\lambda^2)\alpha\right) z_\lambda b,$$

where the functional α is the generator of C_2 and z_λ is the functional associated to $\lambda \in k^\times$. The calculations can be found in §A.2.

This is a particular example of the following result. For the finite dimensional case, revisit Example 1.4.3.

Lemma 5.2.6. *For any group G, the Drinfeld double of kG is a skew group ring*

$$D(kG) = ((kG)^\circ)^{cop} * G.$$

Proof. It suffices to notice that for any $g \in G, f \in (kG)^\circ$ we have $gf = (g \rightharpoonup f \leftharpoonup g^{-1})g$ and that $g \rightharpoonup f \leftharpoonup g^{-1}$ yields a group action of G on $(kG)^\circ$. $\square$

Example 5.2.7. Let $H = T(n,t,q) = k\langle g, x : g^n = 1, xg = qgx\rangle$ be the infinite dimensional Taft algebra. The coalgebra structure is given by g grouplike and x $(1, g^t)$-primitive.

Let $d = (n,t), n' = n/d, t' = t/d$. It follows from Corollary 4.4.6 that

$$D(H) = H^\circ \bowtie H \cong \left[kC_d \otimes T_f(n', t', q^d) \otimes k[f] \otimes k(k,+)\right] \bowtie T(n,t,q).$$

With the notation as in Corollary 4.4.6(III), the product is determined by the relations in H° and H as well as

$$\alpha \text{ central}, \quad gG = Gg, \quad gX = qXg, \quad gf = q^{n'}fg, \quad gz_\lambda = z_{\lambda q^{n'}}g, \quad xG = q^{-d}Gx,$$

$$xX = Xx + \left(\sum_k \lambda_k \alpha^k\right) G^{t'} - g^t, \quad xf = fx + X_{n'-1}\alpha\left[\left(\sum_k \lambda_k\alpha^k\right)G^{t'} - q^{-t}g^t\right],$$

$$xz_\lambda = z_\lambda x + \lambda X_{n'-1}\alpha z_\lambda\left[\left(\sum_k \lambda_k\alpha^k\right)G^{t'} - q^{-t}g^t\right],$$

where $X_{n'-1}(x^ig^j) = \delta_{i,n'-1}$ (it is a scalar multiple of $X^{n'-1}$) and λ_k are scalars such that $\sum_k \lambda_k (q^{n'r})^k = q^{-r}$ (for $0 \le r < d$), which had already appeared in Corollary 4.4.6(III). Once again, the computations can be found in §A.2.

Example 5.2.8. Consider the generalized Liu algebras introduced in section 2.2.5 $H = B(n,w,q) = k\langle x^{\pm 1}, g, y : x \text{ central}, yg = qgy, g^n = x^w = 1 - y^n\rangle$, where x, g are grouplike and y is $(1,g)$-primitive. From Corollary 4.4.6, we have

$$D(H) = H^\circ \bowtie H \cong \left[T_f(n,1,q)\#(k[f] \otimes k(k^\times, \cdot))\right] \bowtie B(n,w,q).$$

With the notation as in Corollary 4.4.6(IV), the product is determined by the relations in H and H° as well as the following:

$$x \text{ central}, \qquad gG = Gg, \qquad gY = qYg, \qquad gf = fg, \qquad gz_\lambda = z_\lambda g,$$

$$yG = q^{-1}Gy, \quad yY = Yy + (G - qg), \quad yf = fy + w\left[\left(\sum_r \mu_r G^r\right)y + Y_{n-1}(g - G)\right],$$

$$yz_\lambda = \left(\sum_r \eta_r G^r\right) z_\lambda y + (1 - \lambda^w) Y_{n-1} z_\lambda (G - g),$$

where the scalars μ_r and η_r are such that $\sum_r \mu_r q^{-rj} = \delta_{j,n-1}$ and $\sum_r \eta_r q^{-rj} = \lambda^{w\delta_{j,n-1}}$, and Y_{n-1} is the functional given by $Y_{n-1}(y^i g^j x^k) = \delta_{i,n-1}$. Recall that we have already seen these scalars in Corollary 4.4.6(IV). The computations can be found in §A.2.

5.2.1 Unimodularity

As was pointed out in Theorem 1.4.7, the Drinfeld double of finite dimensional Hopf algebras is unimodular and its integral can be computed from the right integrals of H and the left integrals of H^*. We conjecture a generalization of that result for affine commutative-by-finite Hopf algebras as follows.

As we have discussed before, the dual of affine commutative-by-finite Hopf algebras is often not even a noetherian algebra, hence in general neither is the Drinfeld double. Therefore, it makes sense to try and extend the unimodularity property to interesting Hopf subalgebras of $D(H)$ instead.

Conjecture 5.2.9. *Let H be an affine commutative-by-finite Hopf algebra. Recall the tangential component $W(H)$ of H° from §4.2. Then,*

1. *$W(H)^{cop} \bowtie H$ is AS-Gorenstein;*
2. *$W(H)^{cop} \bowtie H$ is unimodular;*
3. *If A is central in H, then $(\overline{H}^*)^{cop} \bowtie H$ is AS-Gorenstein and unimodular.*

Remarks 5.2.10.

1. Note that, in terms of the coalgebra structure, only the counit affects the AS-Gorenstein property and unimodularity. But, it is clear from the counit axioms that a Hopf algebra H and its coproduct-twisted version H^{cop} have the same counit, hence we may consider $W(H)$ and $\overline{H}^*$ in the conjecture above rather than their coproduct-twisted versions.

2. The previous conjecture holds for finite dimensional Hopf algebras. If H is finite dimensional, let $A = k1$ be the trivial Hopf subalgebra and so $W(H) = H^*$ is the dual of H as already mentioned in Remark 5.1.10. Therefore, $W(H)^{cop} \bowtie H = D(H)$ is the Drinfeld double of H. Being a finite dimensional Hopf algebra, it is clearly AS-Gorenstein and it is unimodular as per Theorem 1.4.7.

This conjecture is supported by the following examples. It holds trivially for the Hopf algebras $k[x]$ and $k[x^{\pm 1}]$, since their doubles are commutative and, in particular, so is the Hopf subalgebra $W(H) \otimes H$, hence it is unimodular. We now consider a more interesting commutative noncocommutative example.

Example 5.2.11. Consider the Hopf algebra $H = \mathcal{O}((k, +) \rtimes k^\times) = k[x, y^{\pm 1}]$ and its Drinfeld double studied in Example 5.2.4. We investigate the unimodularity of the Hopf subalgebra

$$U := W(H)^{cop} \bowtie H = U(\mathfrak{g}) \bowtie k[x, y^{\pm 1}],$$

where y is central, $\mathfrak{g} = kf \oplus kf'$ is the Lie algebra with $[f', f] = f$ and we have the relations $xf = fx + (1 - y), xf' = (f' + 1)x$.

We start by proving that U is AS-Gorenstein. Notice that, given the above relations, U can be seen as an iterated differential operator ring $U = H[f;\delta_1][f';\delta_2]$, for some derivations δ_1 of H and δ_2 of $H[f;\delta_1]$. Since H is commutative, it is clearly Auslander-Gorenstein and applying [32, Theorem 4.2] twice yields that U is also Auslander-Gorenstein.

Furthermore, filtering U by degree of f', the associated graded ring is $\operatorname{gr} U = H[\overline{f};\overline{\delta_1}] \otimes k[\overline{f'}] = (H[\overline{f'}])[\overline{f};\overline{\delta_1}]$. And subsequently filtering $\operatorname{gr} U$ by degree of f, the associated graded ring is a polynomial algebra in 3 primitive generators and 1 grouplike generator. In particular, $\operatorname{gr}(\operatorname{gr} U)$ is GK-Cohen-Macaulay. Let M be a right noetherian U-module. Since both these gradings are clearly Zariskian [42, I.3.3 Remark 5, II.2.2 Proposition 1], the grades $j(M_U)$, $j(\operatorname{gr} M_{\operatorname{gr} U})$ and $j(\operatorname{gr}(\operatorname{gr} M)_{\operatorname{gr}(\operatorname{gr} U)})$ coincide by [7, proof of Theorem 3.9], and by [65, Theorem 1.3, Corollary 1.4] $\operatorname{GKdim}(\operatorname{gr}(\operatorname{gr} U)) = \operatorname{GKdim}(\operatorname{gr} U) = \operatorname{GKdim}(U)$ and $\operatorname{GKdim}(\operatorname{gr}(\operatorname{gr} M)) = \operatorname{GKdim}(\operatorname{gr} M) = \operatorname{GKdim}(M)$. Since $\operatorname{gr}(\operatorname{gr} U)$ is GK-Cohen-Macaulay, then so is U. And, therefore, by [18, Lemma 6.1] U is AS-Gorenstein.

We now calculate the left integrals of U. First, note that any nonzero element of $k\langle y\rangle$ is regular in U, since it is a free module over the domain $k\langle y\rangle$. Since y is central and grouplike and the image of x in $U/(y-1)$ is normal, applying Lemma 5.1.4 twice yields

$$\int_U^l \cong \int_{U/(y-1)}^l \cong \left(\int_{U/(x,y-1)}^l\right)^{\tau^{-1}},$$

where $\tau(f) = f, \tau(f') = f' + 1$. But $U/(x, y-1) \cong W(H)$ and we have calculated its left integrals back in Example 5.1.14. Thus, as right U-modules we have

$$\int_U^l \cong (U/(x, y-1, f, f'+1))^{\tau^{-1}} \cong U/(x, y-1, f, f').$$

Therefore, U is unimodular.

And now we verify the conjecture for the three noncommutative examples studied in this chapter.

Example 5.2.12. Consider the Hopf algebra $H = kD$ and its Drinfeld double from Example 5.2.5. We investigate the unimodularity of the Hopf subalgebra of the double

$$U := W(H)^{cop} \bowtie H = (kC_2 \otimes k[f]) \bowtie kD,$$

where the algebra structure is determined by the generator α of C_2 being central and the relations $\alpha^2 = 1 = a^2$, $ab = b^{-1}a$, $af = -fa$ and $bf = (f + 1 - \alpha)b$. The coalgebra structure of U is given by α, a, b being grouplike and f being $(1, \alpha)$-primitive.

We begin by proving that U is AS-Gorenstein. Considering the relations above, U can be regarded as an Ore extension $U = (kC_2 \otimes kD)\,[f;\sigma,\delta]$, for some automorphism

σ and derivation δ of $R := kC_2 \otimes kD$. Clearly, R is the group algebra of the abelian-by-finite group $C_2 \otimes D$, hence it is a commutative-by-finite Hopf algebra. In particular, R is Auslander-Gorenstein by Theorem 2.1.8(1) and then so is U by [7, Theorem 4.2].

Now filtering U by degree in f, the associated graded ring is a skew polynomial ring $\operatorname{gr} U = (kC_2 \otimes kD)\,[\overline{f}; \sigma']$. Being finitely generated over the commutative subalgebra $k[\overline{f}, \overline{b}]$, $\operatorname{gr} U$ is an affine noetherian PI Hopf algebra, hence it is GK-Cohen-Macaulay by [102, Theorem 0.1, 0.2]. Let M be a right noetherian U-module. Since this grading is clearly Zariskian [42, I.3.3 Remark 5, II.2.2 Proposition 1], the grades $j(M_U)$ and $j(\operatorname{gr} M_{\operatorname{gr} U})$ coincide by [7, proof of Theorem 3.9], and by [65, Theorem 1.3] $\operatorname{GKdim}(\operatorname{gr} U) = \operatorname{GKdim}(U)$ and $\operatorname{GKdim}(\operatorname{gr} M) = \operatorname{GKdim}(M)$. Since $\operatorname{gr} U$ is GK-Cohen-Macaulay, then so is U. And, therefore, by [18, Lemma 6.1] U is AS-Gorenstein.

We cannot calculate the left integral of U by factoring by $(\alpha - 1)$, since $\alpha - 1$ is not a regular element of U. Therefore, we take a more indirect approach. Consider the Hopf algebra

$$\widehat{U} := (k\mathbb{Z} \otimes k[f]) \bowtie kD$$

subject to the relations: the generator $\widehat{\alpha}$ of $\mathbb{Z}$ is central, $a^2 = 1$, $ab = b^{-1}a$, $af = -fa$ and $bf = (f + 1 - \widehat{\alpha})b$. Moreover, let $\widehat{\alpha}, a, b$ be grouplike and f be $(1, \widehat{\alpha})$-primitive. Note that $\widehat{U}$ is AS-Gorenstein by the same argument used for U. With a slight abuse of notation, clearly

$$U \cong \widehat{U}/(\widehat{\alpha}^2 - 1).$$

Since $\widehat{\alpha}^2 - 1$ is a central regular element of $\widehat{U}$, by Lemma 5.1.4

$$\int_U^l \cong \int_{\widehat{U}}^l$$

and, since $\widehat{\alpha} - 1$ is another central regular element of $\widehat{U}$, the same result yields

$$\int_U^l \cong \int_{\widehat{U}}^l \cong \int_{\widehat{U}/(\widehat{\alpha}-1)}^l,$$

where $\widehat{U}/(\widehat{\alpha}-1) \cong U/(\alpha-1)$. The image of f is a regular normal element of $U/(\alpha-1)$, so by the same result we have

$$\int_U^l \cong \int_{U/(\alpha-1)}^l \cong \left(\int_{U/(\alpha-1,f)}^l\right)^{\tau^{-1}} \cong \left(\int_{kD}^l\right)^{\tau^{-1}},$$

where $\tau(a) = -a, \tau(b) = b$. These integrals were calculated in Example 5.1.15, thus as right U-modules

$$\int_U^l \cong U/(\alpha - 1, f, a + 1, b - 1)^{\tau^{-1}} = U/(\alpha - 1, f, a - 1, b - 1).$$

Each generator of U acts trivially on $\int_U^l$ on the right, that is U is unimodular.

Remark 5.2.13. This example shows that in general the Hopf subalgebra $(\overline{H}^*)^{cop} \bowtie H$ of $D(H)$ is not unimodular. Proceeding as before, the left integrals of $U' = kC_2 \bowtie kD$ are

$$\int_{U'}^l \cong \int_{\widehat{U'}}^l \cong \int_{\widehat{U'}/(\widehat{\alpha}-1)}^l \cong \int_{U'/(\alpha-1)}^l \cong \int_{kD}^l \cong U'/(\alpha - 1, a+1, b-1)$$

as right U'-modules, where $\widehat{U'} = k\mathbb{Z} \bowtie kD$ with similar Hopf structure to U'. Clearly a acts on $\int_{U'}^l$ on the right as multiplication by -1. Therefore, U' is not unimodular.

Example 5.2.14. Consider the double of Taft algebras studied in Example 5.2.7 and consider the Hopf subalgebra

$$U := W(H)^{cop} \bowtie H = (kC_d \otimes T_f(n', t', q^d) \otimes k[f])^{cop} \bowtie T(n, t, q),$$

whose product is determined by the relations in $W(H)$ and H as well as

$$\alpha \text{ central}, \qquad gG = Gg, \qquad gX = qXg, \qquad gf = q^{n'} fg, \qquad xG = q^{-d}Gx,$$

$$xX = Xx + \left(\sum_k \lambda_k \alpha^k\right) G^{t'} - g^t, \qquad xf = fx + X_{n'-1}\alpha \left[\left(\sum_k \lambda_k \alpha^k\right) G^{t'} - q^{-t} g^t\right],$$

where $X_{n'-1}$ is a scalar multiple of $X^{n'-1}$, which is given by $X_{n'-1}(x^i g^j) = \delta_{i,n'-1}$, and λ_k are such that $\sum_k \lambda_k (q^{n'r})^k = q^{-r}$ (for any $0 \leq r < d$).

We start by showing that U is AS-Gorenstein. Considering the relations above, U can be regarded as an Ore extension $U = [(kC_d \otimes T_f \otimes k[f]) \bowtie k\langle g\rangle] [x; \sigma, \delta]$, for some automorphism σ and derivation δ of $R := (kC_d \otimes T_f \otimes k[f]) \bowtie k\langle g\rangle$. Clearly, R is affine and, being a finitely generated module over the commutative subalgebra $k[f]$, it is noetherian by [37, Corollary 1.5] and PI by [64, Corollary 13.1.13(iii)], hence the Hopf algebra R is Auslander-Gorenstein by [102, Theorems 0.1, 0.2] and then so is U by [7, Theorem 4.2].

Now filtering U by degree in x, the associated graded ring is a skew polynomial ring $\operatorname{gr} U = \left[(kC_d \otimes T_f \otimes k[\overline{f}]) \bowtie k\langle \overline{g}\rangle\right] [\overline{x}; \sigma']$. Being finitely generated over the commutative subalgebra $k[\overline{f}, \overline{x}]$, $\operatorname{gr} U$ is an affine noetherian PI Hopf algebra, hence it is GK-Cohen-Macaulay by [102, Theorem 0.1, 0.2]. Let M be a right noetherian U-module. Since this grading is clearly Zariskian [42, I.3.3 Remark 5, II.2.2 Proposition 1], the grades $j(M_U)$ and $j(\operatorname{gr} M_{\operatorname{gr} U})$ coincide by [7, proof of Theorem 3.9], and by [65, Theorem 1.3] $\operatorname{GKdim}(\operatorname{gr} U) = \operatorname{GKdim}(U)$ and $\operatorname{GKdim}(\operatorname{gr} M) = \operatorname{GKdim}(M)$. Since $\operatorname{gr} U$ is GK-Cohen-Macaulay, then so is U. And, therefore, by [18, Lemma 6.1] U is AS-Gorenstein.

We now compute the left integrals of U. Just as in Example 5.2.12, the element $\alpha - 1$ is central in $D(H)$ but it is not regular, so we start with a slightly indirect approach. Consider the Hopf algebra

$$\widehat{U} := (k\mathbb{Z} \otimes T_f(n', t', q^d) \otimes k[f])^{cop} \bowtie T(n, t, q),$$

subject to the same relations as above with the generator $\widehat{\alpha}$ of $\mathbb{Z}$ replacing α and excluding the relation $\alpha^d = 1$. Note that $\widehat{U}$ is AS-Gorenstein by the same argument as for U. Clearly $U \cong \widehat{U}/(\widehat{\alpha}^d - 1)$ and $\widehat{\alpha}^d - 1$ is a regular central element of $\widehat{U}$. Since $\widehat{\alpha} - 1$ is also regular and central in $\widehat{U}$, applying Lemma 5.1.4 twice yields

$$\int_U^l \cong \int_{\widehat{U}}^l \cong \int_{\widehat{U}/(\widehat{\alpha}-1)}^l \cong \int_{U/(\alpha-1)}^l .$$

Moreover, $x^{n'}$ is primitive in H, hence in $D(H)$ we have

$$x^{n'}\varphi = \varphi x^{n'} + (x^{n'} \rightharpoonup \varphi) - (\varphi \leftharpoonup x^{n'})$$

for any $\varphi \in H^\circ$. Clearly, $x^{n'}g = q^{n'}gx^{n'}$ and $x^{n'}$ commutes with α and G. For the functionals X and f, the calculations are as follows. Recall from Corollary 4.4.6 that $X(x^ig^j) = \delta_{i,1}$, $f(x^ig^j) = \delta_{i,n'}q^{n'j}$ and $\alpha(x^ig^j) = \delta_{i,0}q^{n'j}$. Then,

- $X(x^ig^jx^{n'}) = q^{-n'j}X(x^{i+n'}g^j) = 0$, so $x^{n'} \rightharpoonup X = 0$,
- $f(x^ig^jx^{n'}) = q^{-n'j}f(x^{i+n'}g^j) = q^{-n'j}q^{n'j}\delta_{i,0} = \delta_{i,0}$, so $x^{n'} \rightharpoonup f = 1_{H^\circ}$,
- $X(x^{i+n'}g^j) = 0$, so $X \leftharpoonup x^{n'} = 0$,
- $f(x^{i+n'}g^j) = \delta_{i,0}q^{n'j}$, so $f \leftharpoonup x^{n'} = \alpha$.

So, we have $x^{n'}X = Xx^{n'}$ and $x^{n'}f = fx^{n'} + (1-\alpha)$. Therefore, the image of $x^{n'}$ in $U/(\alpha-1)$ is primitive and *normal*, and Lemma 5.1.4 yields

$$\int_U^l \cong \int_{U/(\alpha-1)}^l \cong \left(\int_{U/(\alpha-1,x^{n'})}^l\right)^{\tau^{-1}}$$

as right U-modules, where $\tau(g) = q^{n'}g$, $\tau(\alpha) = \alpha$, $\tau(G) = G$, $\tau(X) = X$, $\tau(f) = f$.

Let

$$\overline{U} := U/(\alpha - 1, x^{n'}) \cong (T_f(n',t',q^d) \otimes k[f]) \bowtie T(n,t,q)/(x^{n'}),$$

where we slightly abuse the notation by writing for instance f instead of $\bar{f}$. This affine Hopf algebra is an Ore extension $(T_f(n',t',q^d) \bowtie T(n,t,q)/(x^{n'}))[f;\alpha,\delta]$ for some automorphism α and derivation δ, hence it is noetherian by [37, Theorem 2.6] and, being a finitely generated $k[f]$-module, $\overline{U}$ has GK-dimension 1 [49, Proposition 5.5]. By Lemma 5.1.5, the space of left integrals is

$$\int_{\overline{U}}^l \cong \{q \in Q : \overline{U}^+ q \subseteq \overline{U}\}/\overline{U} = (kp + \overline{U})/\overline{U},$$

where $Q := Q(\overline{U})$ is its classical ring of fractions and $p \in Q \setminus \overline{U}$ is such that $\overline{U}^+ p \subseteq \overline{U}$.

The same argument as in Example 5.1.17 yields that every nonzero element of $k[f]$

is regular in $\overline{U}$, hence invertible in Q. So, consider

$$p' = \left(\sum_{i=0}^{n'-1} G^i\right) X^{n'-1}x^{n'-1} \left(\sum_{i=0}^{n-1} g^i\right) \in \int^l_{T_f(n',t',q^d)} \int^r_{T(n,t,q)/(x^{n'})},$$

and let $p = p'f^{-1}$. Since $(\sum_i G^i)\, X^{n'-1}$ is a left integral of T_f, we have $Gp = p$ and $Xp = 0$. Moreover, g commutes with G and $X^{n'-1}x^{n'-1}$, so $gp = p$. It follows by induction that f commutes with $X^{n'-1}x^i$ for $0 \le i < n'$, hence

$$fp = \left(\sum_i G^i\right) X^{n'-1}x^{n'-1} f \left(\sum_i g^i\right) f^{-1} = \left(\sum_i G^i\right) X^{n'-1}x^{n'-1} \left(\sum_i q^{-n'i} g^i\right),$$

an element of $\overline{U}$. The left action of x requires some more calculations. It follows by induction that

$$xX^k = X^k x + \left[\left(\sum_{l=0}^{k-1} q^{lt}\right) G^{t'} - \left(\sum_{l=0}^{k-1} q^{-lt}\right) g^t\right] X^{k-1}.$$

Since q^t is a primitive n'th root of unity, $\sum_{l=0}^{n'-2} q^{lt} = \sum_{l=0}^{n'-1} q^{lt} - q^{(n'-1)t} = -q^{-t}$ and similarly $\sum_{l=0}^{n'-2} q^{-lt} = -q^t$, hence

$$xX^{n'-1} = X^{n'-1}x + (q^t g^t - q^{-t}G^{t'})X^{n'-2},$$

and so xp' is

$$\begin{aligned}
&(\sum_i q^{-di}G^i)xX^{n'-1}x^{n'-1}(\sum_i g^i) = (\sum_i q^{-di}G^i)(q^t g^t - q^{-t}G^{t'})X^{n'-2}x^{n'-1}(\sum_i g^i) \\
= \; &(\sum_i q^{-di}G^i)X^{n'-2}x^{n'-1}g^t(\sum_i g^i) - (\sum_i q^{-d(i+t')}G^{i+t'})X^{n'-2}x^{n'-1}(\sum_i g^i) \\
= \; &(\sum_i q^{-di}G^i)X^{n'-2}x^{n'-1}(\sum_i g^i) - (\sum_i q^{-di}G^i)X^{n'-2}x^{n'-1}(\sum_i g^i) = 0.
\end{aligned}$$

Thus $xp = 0$. Therefore, p is the required element and it determines the left integrals of $\overline{U}$.

We now look into the right action of $\overline{U}$ on its left integrals. Since $gf = q^{n'}fg$, we have $f^{-1}g = q^{n'}gf^{-1}$ and, since $x^{n'-1}(\sum_i g^i)$ is a right integral of $T(n,t,q)/(x^{n'})$, we have

$$pg = q^{n'} \left(\sum_i G^i\right) X^{n'-1}x^{n'-1} \left(\sum_i g^i\right) gf^{-1} = q^{n'}p.$$

And since G commutes with f, g and $X^{n'-1}x^{n'-1}$, we have $pG = p$. Also clearly $pf = p'$, hence the right action of f on $\int^l_{\overline{U}} = (kp + \overline{U})/\overline{U}$ is trivial. It follows by induction that

$$x^k X = X x^k + \left[\left(\sum_{l=0}^{k-1} q^{-lt}\right) G^{t'} - \left(\sum_{l=0}^{k-1} q^{lt}\right) g^t\right] x^{k-1},$$

so $x^{n'-1}X = Xx^{n'-1} + (q^{-t}g^t - q^tG^{t'})x^{n'-2}$. Since X and f commute, pX is given by

$$\begin{aligned}
& (\sum_i G^i)X^{n'-1}x^{n'-1}X(\sum_i q^ig^i)f^{-1} \\
= \; & (\sum_i G^i)X^{n'-1}(q^{-t}g^t - q^tG^{t'})x^{n'-2}(\sum_i q^ig^i)f^{-1} \\
= \; & [(\sum_i G^i)X^{n'-1}x^{n'-2}(\sum_i q^{i+t}g^{i+t}) - (\sum_i G^{i+t'})X^{n'-1}x^{n'-2}(\sum_i q^ig^i)]f^{-1} \\
= \; & [(\sum_i G^i)X^{n'-1}x^{n'-2}(\sum_i q^ig^i) - (\sum_i G^i)X^{n'-1}x^{n'-2}(\sum_i q^ig^i)]f^{-1} = 0.
\end{aligned}$$

And lastly the right action of x. Since $xf = fx + \phi$, where $\phi = X_{n'-1}(G^{t'} - q^{-t}g^t)$, and f commutes with ϕ, we have

$$xf^2 = (fx + \phi)f = fxf + f\phi = f(xf + \phi),$$

then $f^{-1}x = (xf + \phi)f^{-2} = xf^{-1} + \phi f^{-2}$. Since $x^{n'-1}(\sum_i g^i)$ is a right integral of $T(n,t,q)/(x^{n'})$ and $p'X = 0$ as above, we must have

$$px = p'xf^{-1} + p'X_{n'-1}(G^{t'} - q^{-t}g^t)f^{-2} = 0.$$

Therefore, as right U-modules we have

$$\int_U^l \cong (U/(\alpha - 1, G - 1, X, f, g - q^{n'}, x))^{\tau^{-1}} = U/(\alpha - 1, G - 1, X, f, g - 1, x),$$

hence $U = W(H)^{cop} \bowtie H$ is unimodular.

Remark 5.2.15. Note that, when $(n,t) = 1$, $A = k[x^n]$ is central in H and the Hopf subalgebra $(\overline{H}^*)^{cop} \bowtie H$ is also unimodular. In this case, as we calculated above, x^n is central and regular in $D(H)$ and, since $(\overline{H}^*)^{cop} \bowtie H/(x^n) \cong (T_f(n,t,q)^*)^{cop} \bowtie T_f(n,t,q) = D(T_f)$, Lemma 5.1.4 yields

$$\int^l_{(\overline{H}^*)^{cop} \bowtie H} \cong \int_{D(T_f)}.$$

Example 5.2.16. Recall the double of generalized Liu algebras studied in Example 5.2.8. We focus on the Hopf subalgebra

$$U = W(H)^{cop} \bowtie H = (T_f(n,1,q)\#k[f])^{cop} \bowtie B(n,w,q),$$

whose product is determined by the relations in H and $W(H)$ as well as the following:

$$x \text{ central}, \qquad gG = Gg, \qquad gY = qYg, \qquad gf = fg, \qquad yG = q^{-1}Gy,$$

$$yY = Yy + (G - qg), \qquad yf = fy + w\left[\left(\sum_r \mu_r G^r\right) y + Y_{n-1}(g - G)\right],$$

where the scalars μ_r are such that $\sum_r \mu_r q^{-rj} = \delta_{j,n-1}$ and Y_{n-1} is the functional given by $Y_{n-1}(y^i g^j x^k) = \delta_{i,n-1}$.

We start by showing that U is AS-Gorenstein. Considering the relations above, U can be regarded as the differential operator ring $U = (T_f \bowtie B(n,w,q))\,[f;\delta]$, for some derivation δ of $R := T_f \bowtie B(n,w,q)$. Clearly, R is affine and, being a finitely generated module over the commutative subalgebra $k\langle x\rangle$, it is noetherian by [37, Corollary 1.5] and PI by [64, Corollary 13.1.13(iii)], hence the Hopf algebra R is Auslander-Gorenstein by [102, Theorems 0.1, 0.2] and then so is U by [7, Theorem 4.2].

Now filtering U by degree in f, the associated graded ring is $\operatorname{gr} U = (T_f \bowtie B(n,w,q))[\overline{f}]$, which is an affine noetherian PI Hopf algebra, hence it is GK-Cohen-Macaulay by [102, Theorem 0.1, 0.2]. Let M be a right noetherian U-module. Since this grading is clearly Zariskian [42, I.3.3 Remark 5, II.2.2 Proposition 1], the grades $j(M_U)$ and $j(\operatorname{gr} M_{\operatorname{gr} U})$ coincide by [7, proof of Theorem 3.9], and by [65, Theorem 1.3] $\operatorname{GKdim}(\operatorname{gr} U) = \operatorname{GKdim}(U)$ and $\operatorname{GKdim}(\operatorname{gr} M) = \operatorname{GKdim}(M)$. Since $\operatorname{gr} U$ is GK-Cohen-Macaulay, then so is U. And, therefore, by [18, Lemma 6.1] U is AS-Gorenstein.

We now compute the left integrals of U. Since x is central and regular in U, Lemma 5.1.4 gives

$$\int_U^l \cong \int_{U/(x-1)}^l .$$

Let

$$\overline{U} = (W(H)^{cop} \bowtie H)/(x-1) \cong (T_f(n,1,q)^* \# k[f])^{cop} \bowtie T_f(n,1,q) = D(T_f)[f;\delta],$$

which is a differential operator ring with derivation given by $\delta(g) = \delta(G) = 0$, $\delta(y) = w[Y_{n-1}(G-g) - (\sum_i \mu_i G^i)y]$ and $\delta(Y) = wY(\sum_i \mu_i G^i)$. We note a slight abuse of notation here, by denoting for example f instead of $\bar{f}$.

The affine Hopf algebra $\overline{U}$ is noetherian by [37, Theorem 2.6] and, being a finitely generated $k[f]$-module, it has GK-dimension 1 [49, Proposition 5.5]. Thus, Lemma 5.1.5 gives

$$\int_{\overline{U}}^l = (kp + \overline{U})/\overline{U},$$

a submodule of $Q/\overline{U}$, where $Q := Q(\overline{U})$ is its classical ring of fractions and $p \in Q \setminus \overline{U}$ such that $\overline{U}^+ p \subseteq \overline{U}$.

By an argument similar to Example 5.1.18, any nonzero element of $k[f]$ is regular in $\overline{U}$; in particular, f is invertible in Q. Consider the integral of $D(T_f)$

$$p' = \left(\sum_i G^i\right) Y^{n-1} y^{n-1} \left(\sum_i g^i\right)$$

and let $p := p' f^{-1}$. Since p' is an integral of $D(T_f)$, clearly the left action of g, y, G and Y on p is trivial.

We now show that f also acts trivially. By induction we have

$$fY^k = Y^k \left[f + w \sum \mu_i \left(\sum_{j=0}^{k-1} q^{-ij} \right) G^i \right],$$

hence $fY^{n-1} = Y^{n-1}(f - w\sum_i \mu_i q^i G^i) = Y^{n-1}f - w(\sum_i \mu_i G^i)Y^{n-1}$ and

$$fY^{n-1}y^{n-1} = Y^{n-1}fy^{n-1} - w\left(\sum_i \mu_i G^i\right) Y^{n-1}y^{n-1}.$$

Again by induction we have

$$Y^k f y^k = Y^k y^k f - w \sum_i \mu_i \left(\sum_{j=1}^{k} q^{ij} \right) G^i Y^k y^k,$$

hence $Y^{n-1}fy^{n-1} = Y^{n-1}y^{n-1}f + w(\sum_i \mu_i G^i)Y^{n-1}y^{n-1}$ and

$$fY^{n-1}y^{n-1} = Y^{n-1}fy^{n-1} - w\left(\sum_i \mu_i G^i\right) Y^{n-1}y^{n-1} = Y^{n-1}y^{n-1}f,$$

that is f commutes with $Y^{n-1}y^{n-1}$. Since it also commutes with G and g, we have

$$fp = \left(\sum_i G^i\right) Y^{n-1}y^{n-1} \left(\sum_i g^i\right) \in \overline{U}.$$

Therefore, p determines the space of left integrals of $\overline{U}$.

We now prove that $\overline{U}$ acts trivially on the right on $\int_{\overline{U}}^{l}$. Clearly f acts trivially. Since f commutes with g and G and p' is an integral of $D(T_f)$, we have $pg = p'gf^{-1} = p$ and similarly $pG = p$. As for y, since f commutes with g and $Y^{n-1}y^{n-1}$, so does f^{-1}, hence

$$py = \left(\sum_i G^i\right) f^{-1}Y^{n-1}y^{n-1} \left(\sum_i g^i\right) y = 0.$$

It remains to compute pY. We know that $fY = Y(f + w\sum_i \mu_i G^i)$ and, since f and G commute, we have

$$f\left(\sum_i G^i\right) Y = \left(\sum_i G^i\right) Y \left(f + w\sum_i \mu_i G^i\right) = Y\left(\sum_i q^{-i}G^i\right)\left(f + w\sum_i \mu_i G^i\right).$$

And, since $\sum_i \mu_i q^{-ij} = \delta_{j,n-1}$, we have

$$\left(\sum_i q^{-i}G^i\right)\left(\sum_j \mu_j G^j\right) = \sum_{i,j} q^{-i}\mu_j G^{i+j} = \left(\sum_j q^j \mu_j\right)\left(\sum_k q^{-k}G^k\right) = \sum q^{-k}G^k,$$

so $f(\sum_i G^i)Y = (\sum_i G^i)Y(f+w)$. Then,

$$\left(\sum_i G^i\right) Y(f+w)^{-1} = f^{-1}\left(\sum_i G^i\right) Y = \left(\sum_i G^i\right) f^{-1}Y.$$

Since G commutes with $Y^{n-1}y^{n-1}$ and g and p' is an integral of $D(T_f)$, it follows that

$$pY = p'f^{-1}Y = p'Y(f+w)^{-1} = 0.$$

Therefore, $\overline{U}$ is unimodular, hence so is $U = W(H)^{cop} \bowtie H$.

Remark 5.2.17. Note that $(\overline{H}^*)^{cop} \bowtie H$ is also unimodular. As above, x is central, regular and grouplike in $(\overline{H}^*)^{cop} \bowtie H$ and, since $(\overline{H}^*)^{cop} \bowtie H/(x) \cong (T_f(n,1,q)^*)^{cop} \bowtie T_f(n,1,q) = D(T_f(n,1,q))$,

$$\int^l_{(\overline{H}^*)^{cop}\bowtie H} \cong \int_{D(T_f)}.$$

This, together with Remark 5.2.15, motivates part (3) of conjecture 5.2.9.

5.2.2 The two conjectures

In this subsection we shed some light into the connection between the two subjects of this chapter, more specifically the proposed formula for the fourth power of the antipode of an affine commutative-by-finite Hopf algebra in Conjecture 5.1.9 and the unimodularity of a Hopf subalgebra of their Drinfeld double suggested in Conjecture 5.2.9. We also leave a few further questions for possible future work.

Let H be an affine commutative-by-finite Hopf algebra. Recall from Conjecture 5.1.9 that we propose the following formula for the fourth power of the antipode

$$S^4(h) = \sum g^{-1}\chi^{-1}(h_1)h_2\chi(h_3)g,$$

where χ is the distinguished grouplike of H° and g is associated to the right action of the tangential component $W(H)$ on its left integrals.

Looking at the character χ as an element of the Drinfeld double of H, which is grouplike, the above formula can be seen in the double $D(H)$ as conjugation by the grouplike $\chi^{-1}g$,

$$S^4 = ad(\chi^{-1}g).$$

Since χ is grouplike in $D(H)$ and $(S^*)^{-1}(\chi^{-1}) = \chi^{-1} \circ S^{-1} = \chi$, it follows by Lemma 1.4.2(1) that

$$\chi h \chi^{-1} = \chi\chi^{-1}((S^*)^{-1}(\chi^{-1}) \rightharpoonup h \leftharpoonup \chi^{-1}) = \chi \rightharpoonup h \leftharpoonup \chi^{-1} = \sum \chi^{-1}(h_1)h_2\chi(h_3),$$

hence

$$(\chi^{-1}g)^{-1}h(\chi^{-1}g) = g^{-1}\chi h\chi^{-1}g = \sum g^{-1}\chi^{-1}(h_1)h_2\chi(h_3)g = S^4(h).$$

We now consider the following formula on the fourth power of the antipode of the double of a finite dimensional Hopf algebra.

Lemma 5.2.18. *Let H be a finite dimensional Hopf algebra and consider its double $D(H)$ and the distinguished grouplikes χ of H^* and g of H. Then, the fourth power of the antipode of $D(H)$ is conjugation by $\chi^{-1}g$.*

Proof. First, since H is finite dimensional, Radford's formula (5.1) states $S_H^4(h) = \sum g^{-1}\chi^{-1}(h_1)h_2\chi(h_3)g$, so in the double S_H^4 is conjugation by $\chi^{-1}g$ by the argument above. Similarly, the same argument gives that $S_{H^*}^4$ is conjugation by $g^{-1}\chi$. However, the double is built from $(H^*)^{cop}$, whose antipode is $S_{H^*}^{-1}$, and so $S_{(H^*)^{cop}}^4 = ad(\chi^{-1}g)$.

Since $S_{D(H)}$ is an algebra anti-homomorphism, $S_{D(H)}^4$ is an algebra homomorphism. Then, for any $f \in H^*$ and $h \in H$,

$$S_{D(H)}^4(fh) = S_{(H^*)^{cop}}^4(f)S_H^4(h) = (g^{-1}\chi)f(\chi^{-1}g)(g^{-1}\chi)h(\chi^{-1}g) = (g^{-1}\chi)fh(\chi^{-1}g),$$

as required. □

The next obvious question is whether the previous statement can be extended for affine commutative-by-finite Hopf algebras.

Question 5.2.19. Let H be an affine commutative-by-finite Hopf algebra and consider the distinguished grouplike χ of H° and the grouplike g associated to the right action of the tangential component $W(H)$ on its left integrals as in Conjecture 5.1.9. Considering the Hopf subalgebra $W(H)^{cop} \bowtie H$ of the double of H, is the fourth power of its antipode given by conjugation by $\chi^{-1}g$?

We can say a bit more here. If Conjecture 5.2.9 holds, then $U := W(H)^{cop} \bowtie H$ is AS-Gorenstein. In particular, by Brown and Zhang's formula [18, Corollary 4.6]

$$S_U^4 = \gamma' \circ \tau_\alpha^r \circ \tau_{\alpha^{-1}}^l,$$

where γ' is some inner automorphism of U and α is the character for the right U-module structure of $\int_U^l$. But the same conjecture proposes that U is also unimodular, hence $\alpha = \epsilon_U$. Therefore, the fourth power of the antipode of U is inner and Conjecture 5.1.9 proposes a grouplike element that realizes this inner automorphism.

Time has not allowed us to carry on the research of this chapter any further but we still leave one other interesting question which also provides material for future work. Recall the notion of a quasi-triangular Hopf algebra and its properties from the end of section 1.4.

Question 5.2.20. Let H be an affine commutative-by-finite Hopf algebra. Is $D(H)$ a quasi-triangular Hopf algebra? What about the Hopf subalgebra $W(H)^{cop} \bowtie H$?

Conclusion

In this thesis we provided a comprehensive study into the class of commutative-by-finite Hopf algebras (chapters 2 and 3) and into the dual of these Hopf algebras (chapter 4), as well as a few questions and conjectures for the future (chapter 5).

Throughout this work we managed to better understand this quite large class of Hopf algebras, its homological properties, its nilradical, semiprimeness and primeness and even its representation theory. Moreover, we shed some light onto the action of the quotient Hopf algebra $\overline{H}$ on the spectrum of maximal ideals of the commutative Hopf algebra A, and we described quite thoroughly the subclass of commutative-by-(co)semisimple Hopf algebras.

Moreover, we studied intensively the dual of this class of Hopf algebras. We were able to decompose it into the smash product

$$H^\circ \cong \overline{H}^* \# A^\circ$$

under quite general hypotheses. We also studied the coalgebra structure of H° by describing important Hopf subalgebras that it contains, namely the tangential component $W(H)$ and the connected component $\widehat{kG}$. This allowed us to decompose and understand the duals of many Hopf algebras that had not been computed thus far, among them being the duals of the quantized enveloping algebra $U_\epsilon(\mathfrak{sl}_3)$ and quantized coordinate rings $\mathcal{O}_\epsilon(G)$ of connected semisimple Lie groups G at roots of unity ϵ.

At last, our work on the duals of commutative-by-finite Hopf algebras allowed us to tackle the important question of completely understanding the formula on the fourth power of the antipode

$$S^4 = \gamma \circ \tau^r_\chi \circ \tau^l_{\chi^{-1}}.$$

It also allowed us to start the research into the properties of the Drinfeld double of these Hopf algebras, providing a lot of material for future work.

Appendix A

Computations on duals and doubles

A.1 The quantized enveloping algebra $U_\epsilon(\mathfrak{sl}_3(k))$

Let ϵ be a primitive lth root of unity, with l odd. This quantum group is generated by E_i, F_i, K_i (for $i = 1, 2$) with relations

$$K_iK_j = K_jK_i, \qquad K_iE_j = \begin{cases} \epsilon^2 E_jK_i, & \text{if } i = j \\ \epsilon^{-1}E_jK_i, & \text{if } i \neq j \end{cases}, \quad K_iF_j = \begin{cases} \epsilon^{-2} F_jK_i, & \text{if } i = j \\ \epsilon F_jK_i, & \text{if } i \neq j \end{cases},$$

$$[E_i, F_j] = \delta_{i,j}\frac{K_i - K_i^{-1}}{\epsilon - \epsilon^{-1}}, \qquad E_i^2E_j - (\epsilon + \epsilon^{-1})E_iE_jE_i + E_jE_i^2 = 0 \quad (i \neq j),$$
$$F_i^2F_j - (\epsilon + \epsilon^{-1})F_iF_jF_i + F_jF_i^2 = 0 \quad (i \neq j).$$

Moreover, K_i is grouplike, E_i is $(1, K_i)$-primitive and F_i is $(K_i^{-1}, 1)$-primitive [13, I.6.2].

Introducing the nonsimple roots $E_3 = E_1E_2 - \epsilon^{-1}E_2E_1$ and $F_3 = F_1F_2 - \epsilon^{-1}F_2F_1$, the relations involving them are

$$K_iE_3 = \epsilon E_3K_i, \qquad E_1E_3 = \epsilon E_3E_1, \qquad E_2E_3 = \epsilon^{-1}E_3E_2,$$

$$F_1E_3 = E_3F_1 + \epsilon^{-1}K_1^{-1}E_2, \qquad F_2E_3 = E_3F_2 - K_2E_1,$$
$$K_iF_3 = \epsilon^{-1}F_3K_i, \qquad F_1F_3 = \epsilon F_3F_1, \qquad F_2F_3 = \epsilon^{-1}F_3F_2,$$
$$E_1F_3 = F_3E_1 + F_2K_1, \qquad E_2F_3 = F_3E_2 - \epsilon^{-1}F_1K_2^{-1},$$
$$E_3F_3 = F_3E_3 + \frac{1}{1-\epsilon^2}(K_1K_2 - K_1^{-1}K_2^{-1})$$

and their coproduct is

$$\Delta(E_3) = E_3 \otimes 1 + K_1K_2 \otimes E_3 + (\epsilon - \epsilon^{-1})K_2E_1 \otimes E_2$$

$$\Delta(F_3) = F_3 \otimes K_1^{-1}K_2^{-1} + 1 \otimes F_3 + (1 - \epsilon^{-2})F_2 \otimes F_1K_2^{-1}.$$

It is known that $H = U_\epsilon(\mathfrak{sl}_3)$ is a free module over the central Hopf subalgebra

$$A = k[K_1^{\pm l}, K_2^{\pm l}, E_1^l, E_2^l, E_3^l, F_1^l, F_2^l, F_3^l]$$

with basis $\{F_3^{r_3}F_2^{r_2}F_1^{r_1}K_1^{s_1}K_2^{s_2}E_1^{t_1}E_2^{t_2}E_3^{t_3} : 0 \le r_i, s_i, t_i < l\}$ [13, III.6.2].

X coideal

Lemma A.1.1. *There exists an A-module coideal X such that $H = A \oplus X$.*

Proof. Let X be the A-module with basis given by the union of the following sets:

1. $\{\alpha_0 := K_1^{s_1}K_2^{s_2} - 1 : 0 \le s_i < l$ not both zero$\}$;

2. $\{\alpha_1 := (F_3^{r_3}F_2^{l-r_3}F_1^{l-r_3} - \mu F_3^l)K_1^{s_1}K_2^{s_2} : 1 \le r_3 < l, 0 \le s_i < l\}$, where $\mu = \epsilon^{-r_3(r_3-1)/2}(1-\epsilon^{-2})^{r_3-l}$;

3. $\{\alpha_2 := K_1^{s_1}K_2^{s_2}(E_1^{l-t_3}E_2^{l-t_3}E_3^{t_3} - \xi E_3^l) : 1 \le t_3 < l, 0 \le s_i < l\}$, where $\xi = \epsilon^{t_3(t_3-1)/2}(\epsilon - \epsilon^{-1})^{t_3-l}$;

4. $\{\alpha_3 := F_3^{r_3}F_2^{l-r_3}F_1^{l-r_3}K_1^{s_1}K_2^{s_2}E_1^{l-t_3}E_2^{l-t_3}E_3^{t_3} - \mu\xi F_3^l K_1^{s_1}K_2^{s_2}E_3^l : 1 \le r_3, t_3 < l, 0 \le s_i < l\}$;

5. $\big\{\alpha_4 := F_3^{r_3}F_2^{r_2}F_1^{r_1}K_1^{s_1}K_2^{s_2}E_1^{t_1}E_2^{t_2}E_3^{t_3} : 0 \le r_i, s_i, t_i < l,$ not of the following types

$$(\text{all } r_i, t_i = 0) \vee (r_1 = r_2 = l - r_3, r_3 \ge 1, \text{ all } t_i = 0) \vee$$

$$(\text{all } r_i = 0, t_1 = t_2 = l - t_3, t_3 \ge 1) \vee (r_1 = r_2 = l - r_3, t_1 = t_2 = l - t_3, r_3, t_3 \ge 1)\big\}$$

We want to prove X is a coideal. It is easy to see all generators belong to H^+. Since K_1, K_2 are grouplike, clearly $\Delta(K_1^{s_1}K_2^{s_2} - 1) \in X \otimes H + H \otimes X$ but proving this for the other elements of the basis is more intricate. So we begin by computing the coproduct of a generic element $\alpha = F_3^{r_3}F_2^{r_2}F_1^{r_1}K_1^{s_1}K_2^{s_2}E_1^{t_1}E_2^{t_2}E_3^{t_3}$.

- $\Delta(E_i^{t_i}) = \sum_{q_i=0}^{t_i}\binom{t_i}{q_i}_{\epsilon^{-2}} K_i^{q_i}E_i^{t_i-q_i} \otimes E_i^{q_i}$, for $i = 1, 2$.

- $\Delta(E_3^{t_3}) = \sum_{q_3=0}^{t_3}\sum_{v=0}^{q_3}\binom{t_3}{q_3}_{\epsilon^{-2}}\binom{q_3}{v}_{\epsilon^2}(\epsilon - \epsilon^{-1})^{q_3-v}(K_2E_1 \otimes E_2)^{q_3-v}(K_1K_2 \otimes E_3)^v(E_3 \otimes 1)^{t_3-q_3} = \sum_{q_3,v} \dots K_1^v K_2^{q_3} E_1^{q_3-v}E_3^{t_3-q_3} \otimes E_2^{q_3-v}E_3^v$.

- $\Delta(F_i^{r_i}) = \sum_{p_i=0}^{r_i}\binom{r_i}{p_i}_{\epsilon^2} F_i^{p_i} \otimes F_i^{r_i-p_i}K_i^{-p_i}$, for $i = 1, 2$.

- $\Delta(F_3^{r_3}) = \sum_{p_3=0}^{r_3}\sum_{u=0}^{p_3}\binom{r_3}{p_3}_{\epsilon^2}\binom{p_3}{u}_{\epsilon^{-2}}(1-\epsilon^{-2})^{p_3-u}(1 \otimes F_3)^{r_3-p_3}(F_3 \otimes K_1^{-1}K_2^{-1})^u(F_2 \otimes F_1K_2^{-1})^{p_3-u} = \sum_{p_3,u} \dots F_3^u F_2^{p_3-u} \otimes F_3^{r_3-p_3}F_1^{p_3-u}K_1^{-u}K_2^{-p_3}$.

Therefore, the coproduct of α is

$$
\begin{aligned}
& \sum \dots F_3^u F_2^{p_3-u} F_2^{p_2} F_1^{p_1} K_1^{s_1} K_2^{s_2} K_1^{q_1} E_1^{t_1-q_1} K_2^{q_2} E_2^{t_2-q_2} K_1^v K_2^{q_3} E_1^{q_3-v} E_3^{t_3-q_3} \\
& \otimes F_3^{r_3-p_3} F_1^{p_3-u} K_1^{-u} K_2^{-p_3} F_2^{r_2-p_2} K_2^{-p_2} F_1^{r_1-p_1} K_1^{-p_1} K_1^{s_1} K_2^{s_2} E_1^{q_1} E_2^{q_2} E_2^{q_3-v} E_3^v \\
= & \sum \dots F_3^u F_2^{p_3-u+p_2} F_1^{p_1} K_1^{s_1+q_1+v} K_2^{s_2+q_2+q_3} E_1^{t_1-q_1} E_2^{t_2-q_2} E_1^{q_3-v} E_3^{t_3-q_3} \\
& \otimes F_3^{r_3-p_3} F_1^{p_3-u} F_2^{r_2-p_2} F_1^{r_1-p_1} K_1^{s_1-u-p_1} K_2^{s_2-p_2-p_3} E_1^{q_1} E_2^{q_2+q_3-v} E_3^v \\
= & \sum \dots F_3^u F_2^{p_3-u+p_2} F_1^{p_1} K_1^{s_1+q_1+v} K_2^{s_2+q_2+q_3} E_1^{t_1-q_1+q_3-v-w} E_2^{t_2-q_2-w} E_3^{t_3-q_3+w} \\
& \otimes F_3^{r_3-p_3+x} F_2^{r_2-p_2-x} F_1^{p_3-u-x+r_1-p_1} K_1^{s_1-u-p_1} K_2^{s_2-p_2-p_3} E_1^{q_1} E_2^{q_2+q_3-v} E_3^v,
\end{aligned}
$$

where the sum runs through $0 \le p_i \le r_i, 0 \le u \le p_3, 0 \le q_i \le t_i, 0 \le v \le q_3, 0 \le w \le \min(t_2-q_2, q_3-v), 0 \le x \le \min(r_2-p_2, p_3-u)$. We now investigate the coproduct of each α_i separately.

Part 1: $\Delta(\alpha_1)$

$$
\sum \dots F_3^u F_2^{p_3-u+p_2} F_1^{p_1} K_1^{s_1} K_2^{s_2} \otimes F_3^{r_3-p_3+x} F_2^{l-r_3-p_2-x} F_1^{p_3-u-x+l-r_3-p_1} K_1^{s_1-u-p_1} K_2^{s_2-p_2-p_3}
$$

$$
-\mu \left[F_3^l \otimes K_1^{-l} K_2^{-l} + 1 \otimes F_3^l + (1-\epsilon^{-2})^l F_2^l \otimes F_1^l K_2^{-l}\right] (K_1^{s_1} K_2^{s_2} \otimes K_1^{s_1} K_2^{s_2}),
$$

where the first sum runs through $0 \le p_1, p_2 \le l - r_3, 0 \le u \le p_3 \le r_3, 0 \le x \le \min(l - r_3 - p_2, p_3 - u)$.

- The subtraction of the term in the first sum with $p_1 = p_2 = l - r_3, u = p_3 = r_3$ (so $x = 0$) and the first term of the second sum yields $\alpha_1 \otimes K_1^{-l+s_1} K_2^{-l+s_2} \in X \otimes H$.
- Similarly, the subtraction of the term in the first sum with $p_1 = p_2 = p_3 = u = 0$ (so $x = 0$) and the second term of the second sum gives $K_1^{s_1} K_2^{s_2} \otimes \alpha_1 \in H \otimes X$.
- The term in the first sum with $p_1 = 0, p_2 = l - r_3, p_3 = r_3, u = 0$ (so $x = 0$) cancels out with the third term of the second sum.

All other terms (in the first sum) are of type α_4:

- The terms with $p_1 \ne 0$ or $u \ne 0$ are of the form $F_3^u F_2^{p_3-u+p_2} F_1^{p_1} \dots \otimes \dots \in X \otimes H$, otherwise $p_1 = p_2 = l - r_3$ and $u = p_3 = r_3$ (already dealt with).
- And the terms with $p_1 = u = 0$ are $F_2^{p_2+p_3} \dots \otimes \dots \in X \otimes H$, unless $p_2 + p_3 = 0$ in which case $p_2 = p_3 = 0$ (already dealt with) or $p_2 + p_3 = l$ in which case $p_2 = l - r_3, p_3 = r_3$ (already dealt with).

Part 2: $\Delta(\alpha_2)$ - very similar to part 1

$$
\sum \dots K_1^{s_1+q_1+v} K_2^{s_2+q_2+q_3} E_1^{(l-t_3)-q_1+q_3-v-w} E_2^{(l-t_3)-q_2-w} E_3^{t_3-q_3+w} \otimes K_1^{s_1} K_2^{s_2} E_1^{q_1} E_2^{q_2+q_3-v} E_3^v
$$

$$
-\xi (K_1^{s_1} K_2^{s_2} \otimes K_1^{s_1} K_2^{s_2}) \left[E_3^l \otimes 1 + K_1^l K_2^l \otimes E_3^l + (\epsilon - \epsilon^{-1})^l K_2^l E_1^l \otimes E_2^l\right],
$$

where the first sum runs through $0 \le q_1, q_2 \le l - t_3, 0 \le v \le q_3 \le t_3, 0 \le w \le \min(l - t_3 - q_2, q_3 - v)$. We can simplify this into

$$\alpha_2 \otimes K_1^{s_1} K_2^{s_2} + K_1^{s_1+l} K_2^{s_2+l} \otimes \alpha_2 + (\text{other terms}).$$

- The first term comes from $q_1 = q_2 = q_3 = v = 0$ in the first sum and first term of the sum; the second from $q_1 = q_2 = l - t_3, q_3 = v = t_3$ and second term of the second sum; and the term with $q_1 = v = w = 0, q_2 = l - t_3, q_3 = t_3$ cancels out with the third term of the second sum.

- Similarly to Part 1 all other terms are of type α_4. If $q_1 \neq 0$ or $v \neq 0$, $\ldots \otimes \ldots E_1^{q_1} E_2^{q_2+q_3-v} E_3^v \in H \otimes X$, otherwise $q_1 = q_2 = l - t_3, q_3 = v = t_3$ (which was already dealt with). When $q_1 = v = 0$, $\ldots \otimes \ldots E_2^{q_2+q_3} \in H \otimes X$, unless $q_2 + q_3 = 0$ (and so $q_2 = q_3 = 0$) or $q_2 + q_3 = l$ (hence $q_2 = l - t_3, q_3 = t_3$), both of which have already been dealt with.

Part 3: $\Delta(\alpha_3)$

$$\sum \ldots F_3^u F_2^{p_3-u+p_2} F_1^{p_1} K_1^{s_1+q_1+v} K_2^{s_2+q_2+q_3} E_1^{(l-t_3)-q_1+q_3-v-w} E_2^{(l-t_3)-q_2-w} E_3^{t_3-q_3+w}$$

$$\otimes F_3^{r_3-p_3+x} F_2^{(l-r_3)-p_2-x} F_1^{p_3-u-x+(l-r_3)-p_1} K_1^{s_1-u-p_1} K_2^{s_2-p_2-p_3} E_1^{q_1} E_2^{q_2+q_3-v} E_3^v$$
$$-\mu\xi(F_3^l \otimes K_1^{-l} K_2^{-l} + 1 \otimes F_3^l + (1-\epsilon^{-2})^l F_2^l \otimes F_1^l K_2^{-l})(K_1^{s_1} K_2^{s_2} \otimes K_1^{s_1} K_2^{s_2})$$
$$(E_3^l \otimes 1 + K_1^l K_2^l \otimes E_3^l + (\epsilon - \epsilon^{-1})^l K_2^l E_1^l \otimes E_2^l),$$

where the first sum runs through $0 \le p_1, p_2 \le l - r_3, 0 \le u \le p_3 \le r_3, 0 \le x \le \min(l - r_3 - p_2, p_3 - u), 0 \le q_1, q_2 \le l - t_3, 0 \le v \le q_3 \le t_3, 0 \le w \le \min(l - t_3 - q_2, q_3 - v)$. This simplifies to

$$\alpha_3 \otimes K_1^{s_1-l} K_2^{s_2-l} + F_3^{r_3} F_2^{l-r_3} F_1^{l-r_3} K_1^{s_1+l} K_2^{s_2+l} \otimes K_1^{-l} K_2^{-l} \alpha_2 + \xi \alpha_1 K_1^l K_2^l \otimes K_1^{s_1-l} K_2^{s_2-l} E_3^l$$

$$+\xi(\epsilon - \epsilon^{-1})^l \alpha_2 K_2^l \otimes K_1^{s_1-l} K_2^{s_2-l} E_2^l + \alpha_2 \otimes F_3^{r_3} F_2^{l-r_3} F_1^{l-r_3} K_1^{s_1} K_2^{s_2} + K_1^{s_1} K_2^{s_2} E_3^l \otimes \xi \alpha_1$$

$$+ K_1^{s_1+l} K_2^{s_2+l} \otimes \alpha_3 + (\epsilon - \epsilon^{-1})^l \xi K_1^{s_1} K_2^{s_2} E_1^l \otimes \alpha_1 E_2^l + \mu(1-\epsilon^{-2})^l F_2^l \alpha_2 \otimes F_1^l K_1^{s_1} K_2^{s_2-l}$$

$$+\mu(1-\epsilon^{-2})^l F_2^l K_1^{s_1+l} K_2^{s_2+l} \otimes F_1^l K_2^{-l} \alpha_3 + 0 + (\text{other terms})$$

- These explicit terms come from all possible combinations in the first sum between conditions $(p_1 = p_2 = l - r_3, p_3 = u = r_3), (p_1 = p_2 = p_3 = u = 0), (p_1 = u = 0, p_2 = l - r_3, p_3 = r_3)$ and $(q_1 = q_2 = q_3 = v = 0), (q_1 = q_2 = l - t_3, q_3 = v = t_3), (q_1 = v = 0, q_2 = l - t_3, q_3 = t_3)$ together with all 9 terms coming from the second product; note that the last combination cancels out with the last term from the product. More importantly, all these terms belong to $X \otimes H + H \otimes X$.

It remains to be shown that all other terms are also in $X \otimes H + H \otimes X$.

- If $u \neq 0$, we have $F_3^u F_2^{p_3-u+p_2} F_1^{p_1} \ldots \otimes \ldots \in X \otimes H$, unless $p_1 = l-u, p_3-u+p_2 = l-u \Leftrightarrow p_1 = p_2 = l-r_3, p_3 = u = r_3$, in which case $\ldots \otimes \ldots E_1^{q_1} E_2^{q_2+q_3-v} E_3^v$ must be in $H \otimes X$, for $v = q_1 = q_2 + q_3 - v = 0 \Leftrightarrow q_1 = q_2 = q_3 = v = 0$ (already dealt with) and $q_1 = l - v = q_2 + q_3 - v \Leftrightarrow q_1 = q_2 = l - t_3, q_3 = v = t_3$ (dealt with).

- If $u = 0$ and $p_1 \neq 0$ or $0 < p_2 + p_3 < l$ we have $F_2^{p_2+p_3} F_1^{p_1} \ldots \otimes \ldots \in X \otimes H$.

- If $p_1 = u = 0$ and $p_2 + p_3 = 0 \Leftrightarrow p_2 = p_3 = 0$ then $\ldots \otimes \ldots E_1^{q_1} E_2^{q_2+q_3-v} E_3^v$ must be in $H \otimes X$, by proceeding similarly to above. If instead $p_1 = u = 0$ and $p_2 + p_3 = l \Leftrightarrow p_2 = l - r_3, p_3 = r_3$ then again $\ldots \otimes \ldots E_1^{q_1} E_2^{q_2+q_3-v} E_3^v$ must be in $H \otimes X$.

Part 4: $\Delta(\alpha_4)$

$$\sum \ldots F_3^u F_2^{p_3-u+p_2} F_1^{p_1} K_1^{s_1+q_1+v} K_2^{s_2+q_2+q_3} E_1^{t_1-q_1+q_3-v-w} E_2^{t_2-q_2-w} E_3^{t_3-q_3+w}$$

$$\otimes F_3^{r_3-p_3+x} F_2^{r_2-p_2-x} F_1^{p_3-u-x+r_1-p_1} K_1^{s_1-u-p_1} K_2^{s_2-p_2-p_3} E_1^{q_1} E_2^{q_2+q_3-v} E_3^v,$$

where the sum runs through $0 \leq p_i \leq r_i, 0 \leq u \leq p_3, 0 \leq q_i \leq t_i, 0 \leq v \leq q_3, 0 \leq w \leq \min(t_2 - q_2, q_3 - v), 0 \leq x \leq \min(r_2 - p_2, p_3 - u)$. We will focus on the case where some $r_i \neq 0$; the case where some $t_i \neq 0$ is similar.

- We start with the case where $r_3 = 0$. The coproduct of these elements has the form $\sum \ldots F_2^{p_2} F_1^{p_1} \ldots \otimes F_2^{r_2-p_2} F_1^{r_1-p_1} \ldots$. If $r_2 \geq 1$, either $p_2 \neq r_2$ (so the summand belongs to $H \otimes X$) or $p_2 = r_2$ (and the summand is in $X \otimes H$); proceed similar for $r_1 \neq 0$.

We now focus on the case where $r_3 \geq 1$. Keep in mind that in α_4 we do *not* have $r_1 = r_2 = l - r_3$.

- The terms with $p_3 = 0$ are of the form $F_2^{p_2} F_1^{p_1} \ldots \otimes F_3^{r_3} F_2^{r_2-p_2} F_1^{r_1-p_1} \ldots$ and as above it belongs to $X \otimes H + H \otimes X$.

- If $p_3 \neq 0$, $u \neq 0$, we have $F_3^u F_2^{p_3-u+p_2} F_1^{p_1} \ldots \otimes F_3^{r_3-p_3+x} F_2^{r_2-p_2-x} F_1^{p_3-u-x+r_1-p_1} \ldots$. The first part of the summand is of type α_4 unless $p_1 = l - u$ and $p_2 = l - p_3$, in which case the second part is

$$F_3^{r_3-p_3+x} F_2^{r_2-(l-p_3)-x} F_1^{p_3-x+r_1-l} \ldots.$$

  This is of type α_1 if and only if $r_1 = r_2 = 2l - r_3 > l$ (which is not allowed) and it is of type α_0 if and only if $r_1 = r_2 = l - r_3$ (also not allowed). Considering the F_3-degree is strictly less than l, it is of type α_4 and belongs to X.

- If $p_1, p_3 \neq 0, u = 0$, we get $F_2^{p_3+p_2} F_1^{p_1} \ldots \otimes \ldots \in X \otimes H$.

- If $p_3 \neq 0, u = 0, p_1 = 0$, we have $F_2^{p_2+p_3} \ldots \otimes F_3^{r_3-p_3+x} F_2^{r_2-p_2-x} F_1^{p_3-x+r_1} \ldots$. If $p_3 < r_3$, the F_3-degree in the second part is between 1 and l; it is of type α_1 if and only if $r_1 = l - r_3, r_2 = (l - r_3) + (p_3 + p_2)$, hence $p_3 + p_2 \neq 0, l$ but in this case the first part of the tensor $F_2^{p_2+p_3} \ldots$ belongs to X.

- If $p_3 = r_3 \neq 0, u = 0, p_1 = 0$, we have $F_2^{p_2+r_3} \ldots \otimes F_3^{x} F_2^{r_2-p_2-x} F_1^{r_3-x+r_1} \ldots$.
  - For $x > 0$, the second part of the tensor cannot be of type α_1 (because $r_2 - p_2 - x + x < l$), so it belongs to X.
  - For $x = 0$, we get $F_2^{r_2+r_3} \ldots \otimes F_2^{r_2-p_2} F_1^{r_3+r_1} \ldots$. If $p_2 < r_2$, this belongs to $H \otimes X$; and if $p_2 = r_2$ we get $F_2^{r_2+r_3} \ldots \otimes F_1^{r_3+r_1} \ldots$ and since both powers are not allowed to be l, it belongs to $X \otimes H + H \otimes X$.

□

To sum up, we have an A-module coalgebra projection given by

$$
\begin{array}{llll}
\Pi: & H := U_\epsilon(\mathfrak{sl}_3) & \to & A := k[F_i^l, K_i^{\pm l}, E_i^l] \\
 & aK_1^{s_1}K_2^{s_2} & \mapsto & a, \qquad 0 \leq s_i < l \\
 & aF_3^{r_3}F_2^{l-r_3}F_1^{l-r_3}K_1^{s_1}K_2^{s_2} & \mapsto & \mu a F_3^l, \qquad 1 \leq r_3 < l \\
 & aK_1^{s_1}K_2^{s_2}E_1^{l-t_3}E_2^{l-t_3}E_3^{t_3} & \mapsto & \xi a E_3^l, \qquad 1 \leq t_3 < l \\
 & aF_3^{r_3}F_2^{l-r_3}F_1^{l-r_3}K_1^{s_1}K_2^{s_2}E_1^{l-t_3}E_2^{l-t_3}E_3^{t_3} & \mapsto & \mu\xi a F_3^l E_3^l, \qquad 1 \leq r_3, t_3 < l \\
 & \text{elsewhere} & \mapsto & 0,
\end{array}
$$

for any $a \in A$, where $\mu = \epsilon^{-r_3(r_3-1)/2}(1-\epsilon^{-2})^{r_3-l}$ and $\xi = \epsilon^{t_3(t_3-1)/2}(\epsilon - \epsilon^{-1})^{t_3-l}$.

The brackets of the Lie algebra $\mathfrak{h}$

As we saw in subsection 4.4, the Lie algebra $\mathfrak{h} = \operatorname{Lie} G$ of the algebraic group associated to A has dimension 8, with basis $\{e_1, e_2, e_3, f_1, f_2, f_3, k_1, k_2\}$. These functionals of A are defined as follows: for a generic element $\alpha = E_1^{t_1 l}E_2^{t_2 l}E_3^{t_3 l}F_1^{r_1 l}F_2^{r_2 l}F_3^{r_3 l}K_1^{s_1 l}K_2^{s_2 l}$ of A, we have

$$
e_i(\alpha) = \begin{cases} 1, & \text{for } \alpha = E_i^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases}, \qquad f_i(\alpha) = \begin{cases} 1, & \text{for } \alpha = F_i^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases},
$$

$$
k_i(\alpha) = \begin{cases} s_i, & \text{for } \alpha = K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases}.
$$

The Lie brackets of $\mathfrak{h}$ can be computed as commutators in A°. Recall the coproducts of the generators E_i^l, F_i^l, K_i^l of A: K_i^l is grouplike, E_i^l is $(1, K_i^l)$-primitive and F_i^l is $(K_i^{-l}, 1)$-primitive for $i = 1, 2$, and

$$
\Delta(E_3^l) = E_3^l \otimes 1 + K_1^l K_2^l \otimes E_3^l + (\epsilon - \epsilon^{-1})^l K_2^l E_1^l \otimes E_2^l
$$

and

$$\Delta(F_3^l) = F_3^l \otimes K_1^{-l}K_2^{-l} + 1 \otimes F_3^l + (1-\epsilon^{-2})^l F_2^l \otimes F_1^l K_2^{-l}.$$

We begin with $[e_1, e_2]$. Examining the definitions of e_1, e_2 and the coproducts above we easily see that

$$e_1e_2(\alpha) = \begin{cases} 1, & \text{for } \alpha = E_1^l E_2^l K_1^{s_1 l} K_2^{s_2 l} \\ (\epsilon - \epsilon^{-1})^l, & \text{for } \alpha = E_3^l K_1^{s_1^l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases}$$

and

$$e_2e_1(\alpha) = \begin{cases} 1, & \text{for } \alpha = E_1^l E_2^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases},$$

hence $[e_1, e_2] = e_1e_2 - e_2e_1 = (\epsilon - \epsilon^{-1})^l e_3$. Similarly, we have

$$e_1e_3(\alpha) = \begin{cases} 1, & \text{for } \alpha = E_1^l E_3^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases} = e_3e_1(\alpha)$$

and

$$e_2e_3(\alpha) = \begin{cases} 1, & \text{for } \alpha = E_2^l E_3^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases} = e_3e_2(\alpha),$$

hence $[e_1, e_3] = 0 = [e_2, e_3]$.

We now present the calculations for f_i. We have

$$f_1f_2(\alpha) = \begin{cases} 1, & \text{for } \alpha = F_1^l F_2^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases}$$

and

$$f_2f_1(\alpha) = \begin{cases} 1, & \text{for } \alpha = F_1^l F_2^l K_1^{s_1 l} K_2^{s_2 l} \\ (1 - \epsilon^{-2})^l, & \text{for } \alpha = F_3^l K_1^{s_1^l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases},$$

hence $[f_1, f_2] = f_1f_2 - f_2f_1 = -(1-\epsilon^{-2})^l f_3$. Similarly, we have

$$f_1f_3(\alpha) = \begin{cases} 1, & \text{for } \alpha = F_1^l F_3^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases} = f_3f_1(\alpha)$$

and

$$f_2f_3(\alpha) = \begin{cases} 1, & \text{for } \alpha = F_2^l F_3^l K_1^{s_1 l} K_2^{s_2 l} \\ 0, & \text{elsewhere} \end{cases} = f_3f_2(\alpha),$$

hence $[f_1, f_3] = 0 = [f_2, f_3]$.

Moreover, $[k_1, k_2] = 0$ because

$$k_1k_2(\alpha) = \begin{cases} s_1s_2, & \text{for } \alpha = K_1^{s_1l}K_2^{s_2l} \\ 0, & \text{elsewhere} \end{cases} = k_2k_1(\alpha).$$

On a similar fashion, $[e_i, f_j] = 0$ for any $1 \leq i, j \leq 3$ because

$$e_if_j(\alpha) = \begin{cases} 1, & \text{for } \alpha = E_i^lF_j^lK_1^{s_1l}K_2^{s_2l} \\ 0, & \text{elsewhere} \end{cases} = f_je_i(\alpha).$$

Also, $[e_1, k_1] = -e_1, [e_2, k_2] = -e_2$ and $[e_1, k_2] = 0 = [e_2, k_1]$ because for any $1 \leq i, j \leq 2$

$$e_ik_j(\alpha) = \begin{cases} s_j, & \text{for } \alpha = E_i^lK_1^{s_1l}K_2^{s_2l} \\ 0, & \text{elsewhere} \end{cases},$$

while

$$k_je_i(\alpha) = \begin{cases} s_j + \delta_{i,j}, & \text{for } \alpha = E_i^lK_1^{s_1l}K_2^{s_2l} \\ 0, & \text{elsewhere} \end{cases}.$$

And $[e_3, k_1] = -e_3 = [e_3, k_2]$ because for $1 \leq i \leq 2$ we have

$$e_3k_i(\alpha) = \begin{cases} s_i, & \text{for } \alpha = E_3^lK_1^{s_1l}K_2^{s_2l} \\ 0, & \text{elsewhere} \end{cases}$$

and

$$k_ie_3(\alpha) = \begin{cases} s_i + 1, & \text{for } \alpha = E_3^lK_1^{s_1l}K_2^{s_2l} \\ 0, & \text{elsewhere} \end{cases}.$$

In a complete similar fashion, we get $[f_1, k_1] = -f_1, [f_2, k_2] = -f_2, [f_1, k_2] = 0 = [f_2, k_1], [f_3, k_1] = -f_3 = [f_3, k_2]$.

A.2 Prime regular affine Hopf algebras of GKdim 1

These are additional calculations for the proof of Corollary 4.4.6 and some examples in § 5.2.

Group algebra of the dihedral group D

Recall the dihedral group $D = \langle a, b : a^2 = 1, aba = b^{-1}\rangle$, which is abelian-by-finite with abelian group $N = \langle b\rangle$ of index 2.

The dual

By Corollary 4.4.6(II), its dual is

$$(kD)^\circ \cong kC_2 \otimes (k[f] \otimes k(k^\times)),$$

where the generator of C_2 is denoted by α and the functional indexed by $\lambda \in k^\times$ is denoted by z_λ.

First, we explain the formulas of the functionals. The Hopf quotient $\pi : H = kD \to \overline{H} \cong kC_2 = k\langle \bar{a} \rangle$ is given by $\pi(a^i b^j) = \bar{a}^i$. Moreover, $\overline{kD}^* = (kC_2)^*$ is self-dual, and by Example 1.3.3 its generator β is given by $\beta(\bar{a}^i) = (-1)^i$. Thus, $\alpha := \pi^\circ(\beta)$ is given by $\alpha(a^i b^j) = \beta\pi(a^i b^j) = \beta(\bar{a}^i) = (-1)^i$.

Furthermore, $A = kN = k[b^{\pm 1}]$ and $H = A \oplus (a-1)A$ is a right A-module decomposition in which $X = (a-1)A$ is a coideal. Thus, the corresponding projection $\Pi : H \to A$ is given by $\Pi(a^i b^j) = b^j$. Example 1.3.10 yields $A^\circ = k[b^{\pm 1}]^\circ \cong k[f'] \otimes k(k^\times)$, where $f'(b^j) = j$ and the character corresponding to each $\lambda \in k^\times$ is given by $\chi_\lambda(b^j) = \lambda^j$. Then, $f := \Pi^\circ(f')$ is given by $f(a^i b^j) = f'(b^j) = j$, and for each $\lambda \in k^\times$ $z_\lambda := \Pi^\circ(\chi_\lambda)$ is given by $z_\lambda(a^i b^j) = \chi_\lambda(b^j) = \lambda^j$.

Second, we explain the coproduct structure of H°. First, for any $i, m \in \{0, 1\}, j, n \in \mathbb{Z}$ we have $a^i b^j \cdot a^m b^n = a^{i+m} b^{(-1)^m j + n}$. Thus,

- $(\alpha \otimes \alpha)(a^i b^j \otimes a^m b^n) = (-1)^{i+m} = \alpha(a^i b^j \cdot a^m b^n)$;
- $(f \otimes \alpha + 1 \otimes f)(a^i b^j \otimes a^m b^n) = j(-1)^m + n = f(a^i b^j \cdot a^m b^n)$;
- $\frac{1}{2}\left(1 \otimes (1+\alpha) + z_{\lambda^{-2}} \otimes (1-\alpha)\right)(z_\lambda \otimes z_\lambda)(a^i b^j \otimes a^m b^n) = \frac{1}{2}(1 + (-1)^m + \lambda^{-2j}(1 - (-1)^m))\lambda^{j+n} = \lambda^{(-1)^m j + n} = z_\lambda(a^i b^j \cdot a^m b^n)$.

Third, we explain the convolution product of the characters $\phi_\lambda, \psi_\lambda$ of $W(kD) = kC_2 \otimes k[f]$, where $\lambda \in k$. Recall that $\phi_\lambda(f) = \psi_\lambda(f) = \lambda$ and $\phi_\lambda(\alpha) = 1, \psi_\lambda(\alpha) = -1$.

1. $(\phi_\lambda\phi_\mu)(f) = \phi_\lambda(f)\phi_\mu(\alpha) + \phi_\lambda(1)\phi_\mu(f) = \lambda + \mu$ and $(\phi_\lambda\phi_\mu)(\alpha) = \phi_\lambda(\alpha)\phi_\mu(\alpha) = 1$, so $\phi_\lambda\phi_\mu = \phi_{\lambda+\mu}$;
2. $(\phi_\lambda\psi_\mu)(f) = \phi_\lambda(f)\psi_\mu(\alpha) + \phi_\lambda(1)\psi_\mu(f) = -\lambda + \mu$ and $(\phi_\lambda\psi_\mu)(\alpha) = \phi_\lambda(\alpha)\psi_\mu(\alpha) = -1$, so $\phi_\lambda\psi_\mu = \psi_{-\lambda+\mu}$;
3. $(\psi_\lambda\phi_\mu)(f) = \psi_\lambda(f)\phi_\mu(\alpha) + \psi_\lambda(1)\phi_\mu(f) = \lambda + \mu$ and $(\psi_\lambda\phi_\mu)(\alpha) = \psi_\lambda(\alpha)\phi_\mu(\alpha) = -1$, so $\psi_\lambda\phi_\mu = \psi_{\lambda+\mu}$;
4. $(\psi_\lambda\psi_\mu)(f) = \psi_\lambda(f)\psi_\mu(\alpha) + \psi_\lambda(1)\psi_\mu(f) = -\lambda + \mu$ and $(\psi_\lambda\psi_\mu)(\alpha) = \psi_\lambda(\alpha)\psi_\mu(\alpha) = 1$, so $\psi_\lambda\psi_\mu = \phi_{-\lambda+\mu}$.

The double

We present here the calculations on the algebra relations of the double of kD. Recall from above that the functionals of $(kD)^\circ$ are given as follows: $f(a^i b^j) = j$, each $\lambda \in k^\times$ corresponds to the functional $z_\lambda(a^i b^j) = \lambda^j$, and the generator of C_2 is $\alpha(a^i b^j) = (-1)^i$.

First note that, since kD is cocommutative, it is involutory, that is $S^2 = \mathrm{id}$, and the product in the double is determined by

$$g\varphi = (g \rightharpoonup \varphi \leftharpoonup g^{-1})g$$

for all $g \in D, \varphi \in H^\circ$. Thus,

- $af = -fa$, because $(a \rightharpoonup f \leftharpoonup a)(a^i b^j) = f(aa^i b^j a) = f(a^i b^{-j}) = -j = -f(a^i b^j)$.
- $az_\lambda = z_{\lambda^{-1}} a$, since $(a \rightharpoonup z_\lambda \leftharpoonup a)(a^i b^j) = z_\lambda(a^i b^{-j}) = \lambda^{-j} = z_{\lambda^{-1}}(a^i b^j)$.
- $a\alpha = \alpha a$, since $(a \rightharpoonup \alpha \leftharpoonup a)(a^i b^j) = \alpha(a^i b^{-j}) = (-1)^i$.

Moreover, we have

$$b^{-1}a^i b^j b = \begin{cases} b^j, & \text{if } i = 0 \\ ab^{j+2}, & \text{if } i = 1 \end{cases},$$

so

- $f(b^{-1}a^i b^j b) = j + 2\delta_{i,1}$, so $b \rightharpoonup f \leftharpoonup b^{-1} = f + 1 - \alpha$.
- $z_\lambda(b^{-1}a^i b^j b) = \lambda^{j+2\delta_{i,1}}$, hence $b \rightharpoonup z_\lambda \leftharpoonup b^{-1} = \frac{1}{2}\left((1+\lambda^2) + (1-\lambda^2)\alpha\right) z_\lambda$.
- $\alpha(b^{-1}a^i b^j b) = (-1)^i$, so $b \rightharpoonup \alpha \leftharpoonup b^{-1} = \alpha$.

Thus,

$$bf = (f + 1 - \alpha)b, \quad bz_\lambda = \frac{1}{2}\left((1+\lambda^2) + (1-\lambda^2)\alpha\right) z_\lambda b, \quad b\alpha = \alpha b.$$

Taft algebras

Recall from Example 1.1.4 the infinite dimensional Taft algebra

$$H = T(n,t,q) = k\langle g, x : g^n = 1, xg = qgx\rangle,$$

where g is grouplike and x is $(1, g^t)$-primitive. Its corresponding normal commutative Hopf subalgebra is $A = k[x^{n'}]$, where $d = (n,t), n' = n/d$, and the Hopf quotient is $\overline{H} \cong T_f(n', t', q^d)\#_\sigma kC_d$ by Example 1.1.21.

The dual

By Corollary 4.4.6(III) its dual is

$$T(n,t,q)^\circ \cong \overline{H}^* \otimes A^\circ \cong (T_f(n',t',q^d) \otimes kC_d) \otimes (k[f] \otimes k(k,+)),$$

where $t' = t/d$. Let us compute the definition of the functionals of H°.

Let G and X respectively denote the invertible and the nilpotent generators of $T_f(n',t',q^d)^*$ as in Example 1.3.4 and α the generator of $(kC_d)^*$ as in Example 1.3.3.

First, we extend these to functionals of $\overline{H}^*$. By Example 1.1.21 as a vector space $\overline{H} \cong T_f(n', t', q^d) \otimes kC_d$, with isomorphism given by $\bar{x}^i\bar{g}^j \mapsto \bar{x}^i\bar{g}^{dk} \otimes \bar{g}^r$ where $j = dk + r$ for some $0 \le r < d$. Thus, by Examples 1.3.4 and 1.3.3 G, X, α extend to functionals on $\overline{H}$ and (with a slight abuse of notation) these are given by $X(\bar{x}^i\bar{g}^j) = \delta_{i,1}$, $G(\bar{x}^i\bar{g}^j) = \delta_{i,0}(q^d)^{-t'^{-1}k} = \delta_{i,0}q^{-t'^{-1}dk}$, where t'^{-1} is the inverse of t' modulo n', and $\alpha(\bar{x}^i\bar{g}^j) = \delta_{i,0}q^{n'r} = \delta_{i,0}q^{n'j}$, since $q^{n'}$ is a primitive dth root of unity.

And now we extend them to functionals on H. Since the Hopf quotient $\pi : H \to \overline{H}$ is given by $x^ig^j \mapsto \bar{x}^i\bar{g}^j$, (again with a slight abuse of notation) the extensions of the functionals X, G and α are given by:

$$X(x^ig^j) = \delta_{i,1}, \qquad G(x^ig^j) = \delta_{i,0}q^{-t'^{-1}dk} \quad \text{and} \quad \alpha(x^ig^j) = \delta_{i,0}q^{n'j}.$$

We now focus on the functionals coming from the dual of A. We know from Example 1.3.8 that $A^\circ \cong k[f] \otimes k(k, +)$. Moreover, as we saw in Example 1.1.21, H decomposes into a crossed product $A\#_\sigma\overline{H}$, where the cleaving map γ is a splitting coalgebra map that satisfies $S_H \circ \gamma = \gamma \circ S_{\overline{H}}$. Therefore, as in the proof of Theorem 4.1.5(2) H decomposes into $A \oplus X$ as right A-submodules where

$$X = \gamma(\overline{H}^+)A = \bigoplus_{\substack{1\le i<n' \\ 0\le j<n}} x^ig^jA \oplus \bigoplus_{1\le j<n} (g^j - 1)A$$

is a coideal. Therefore, the corresponding right A-module projection $\Pi : H \twoheadrightarrow A$ is given by

$$\Pi(x^ig^j) = \delta_{n'|i}q^{ij}x^i.$$

Then, with a slight abuse of notation, f extends through Π into the functional on H given by $f(x^ig^j) = \delta_{i,n'}q^{n'j}$, and for each $\lambda \in k$ the character χ_λ extends to the functional $z_\lambda = \Pi^\circ(\chi_\lambda)$ given by $z_\lambda(x^ig^j) = \delta_{n'|i}q^{ij}\lambda^{i/n'}$.

We now explain the coproduct of the functionals of $T(n, t, q)^\circ$. First, for any $i, l \ge 0$ and $0 \le j, m < n$, $x^ig^j \cdot x^lg^m = q^{-jl}x^{i+l}g^{j+m}$. Let λ_k be such that $\sum \lambda_k(q^{n'r})^k = q^{-r}$ for any $0 \le r < d$.

- Since $(\alpha \otimes \alpha)(x^ig^j \otimes x^lg^m) = \delta_{i,0}\delta_{l,0}q^{n'(j+m)} = \alpha(x^ig^j \cdot x^lg^m)$, α is grouplike.
- $[X \otimes 1 + (\sum_k \lambda_k\alpha^k)G^{t'} \otimes X](x^ig^j \otimes x^lg^m) = \delta_{i,1}\delta_{l,0} + q^{-j}\delta_{i,0}\delta_{l,1} = X(x^ig^j \cdot x^lg^m)$.

We now focus on G. Let $j = dk + r, m = dk' + r'$ and $j + m = dk'' + r''$ for some $0 \le r, r', r'' < d$. We know that $G(x^ig^j \cdot x^lg^m) = \delta_{i,0}\delta_{l,0}q^{-t'^{-1}dk''}$.

- $(G \otimes G)(x^ig^j \otimes x^lg^m) = \delta_{i,0}\delta_{l,0}q^{-t'^{-1}d(k+k')}$. This coincides with the above expression whenever $r + r' < d$.
- Let μ^l_s be scalars such that $\sum_s \mu^l_s(q^{n'r})^s = \delta_{r,l}$ for any $0 \le r < d$. Then, it is easy to see that $\sum_{l+m\ge d}[(\sum_s \mu^l_s\alpha^s) \otimes (\sum_t \mu^m_t\alpha^t)](G \otimes G)$ gives $\delta_{i,0}\delta_{l,0}q^{-t'^{-1}d(k+k')}$ when $r + r' \ge d$.

Thus,

$$\Delta(G) = \left(1 \otimes 1 + (q^{-t'^{-1}d} - 1) \sum_{\substack{l,m=1 \\ l+m \geq d}}^{d-1} (\sum_s \mu_s^l \alpha^s) \otimes (\sum_t \mu_t^m \alpha^t) \right) (G \otimes G).$$

As for f, we know that $f(x^i g^j \cdot x^l g^m) = \delta_{i+l,n'} q^{n'(j+m)} q^{-jl}$.

- $(f \otimes \alpha + 1 \otimes f)(x^i g^j \otimes x^l g^m) = \delta_{i,n'} \delta_{l,0} q^{n'j+n'm} + \delta_{i,0} \delta_{l,n'} q^{n'm}$;

  It remains to obtain the part of the coproduct responsible for $i + l = n'$ with $1 \leq i, l < n'$.

- $(\alpha \otimes \alpha) \sum_{s=1}^{n'-1} [X_{n'-s}[(\sum_k \lambda_k \alpha^k) G^{t'}]^s \otimes X_s](x^i g^j \otimes x^l g^m) = q^{n'j} q^{n'm} \delta_{n'-s,i} q^{-js} \delta_{s,l} = \delta_{i+l,n'} q^{n'(j+m)} q^{-jl}$.

Therefore,

$$\Delta(f) = f \otimes \alpha + 1 \otimes f + (\alpha \otimes \alpha) \left(\sum_{s=1}^{n'-1} X_{n'-s}[(\sum_k \lambda_k \alpha^k) G^{t'}]^s \otimes X_s \right).$$

The double

We present here the computations that justify the following relations on the double of $H = T(n, t, q)$:

$$\alpha \text{ central}, \qquad gG = Gg, \qquad gX = qXg, \qquad gf = q^{n'} fg, \qquad gz_\lambda = z_{\lambda q^{n'}} g$$

$$xG = q^{-d} Gx, \qquad xX = Xx + \left(\sum_k \lambda_k \alpha^k \right) G^{t'} - g^t,$$

$$xf = fx + X_{n'-1} \alpha \left[\left(\sum_k \lambda_k \alpha^k \right) G^{t'} - q^{-t} g^t \right],$$

$$xz_\lambda = z_\lambda x + \lambda X_{n'-1} \alpha z_\lambda \left[\left(\sum_k \lambda_k \alpha^k \right) G^{t'} - q^{-t} g^t \right],$$

where $X_{n'-1}(x^i g^j) = \delta_{i,n'-1}$ (it is a scalar multiple of $X^{n'-1}$) and λ_k are such that $\sum_k \lambda_k (q^{n'r})^k = q^{-r}$ (for $0 \leq r < d$).

Recall from above that the functionals of H° are defined as follows: the generator of C_d is the functional $\alpha(x^i g^j) = \delta_{i,0} q^{n'j}$, the generators of T_f are the functionals $G(x^i g^j) = \delta_{i,0} q^{-t'^{-1}dk}$ and $X(x^i g^j) = \delta_{i,1}$, f is given by $f(x^i g^j) = \delta_{i,n'} q^{n'j}$ and for each $\lambda \in k$ the corresponding functional is $z_\lambda(x^i g^j) = \delta_{n'|i} q^{ij} \lambda^{i/n'}$, where t'^{-1} is the inverse of t' modulo n' and $j = dk + r$ with $0 \leq r < d$.

We know that for every $\varphi \in H^\circ$

$$g\varphi = (g \rightharpoonup \varphi \leftharpoonup g^{-1}) g$$

and since $g^{-1}x^ig^jg = q^ix^ig^j$ we have

- $q^i\alpha(x^ig^j) = \delta_{i,0}q^{n'j}$, so $g\alpha = \alpha g$.
- $q^iG(x^ig^j) = \delta_{i,0}q^{-t'^{-1}dk}$, so $gG = Gg$.
- $q^iX(x^ig^j) = q\delta_{i,1}$, so $gX = qXg$.
- $q^if(x^ig^j) = q^{n'}q^{n'j}\delta_{i,n'}$, so $gf = q^{n'}fg$.
- $q^iz_\lambda(x^ig^j) = q^i\delta_{n'|i}q^{ij}\lambda^{i/n'} = \delta_{n'|i}q^{ij}(\lambda q^{n'})^{i/n'}$, so $gz_\lambda = z_{\lambda q^{n'}}g$.

Moreover, for any $\varphi \in H^\circ$ we have

$$x\varphi = (x \rightharpoonup \varphi) + (g^t \rightharpoonup \varphi)x - (g^t \rightharpoonup \varphi \leftharpoonup xg^{-t})g^t$$

- $(x \rightharpoonup \alpha)(x^ig^j) = q^{-j}\alpha(x^{i+1}g^j) = \delta_{i+1,0}q^{n'j-j} = 0$, so $x \rightharpoonup \alpha = 0$.
- $(g^t \rightharpoonup \alpha)(x^ig^j) = \alpha(x^ig^{j+t}) = \delta_{i,0}q^{n'j+n't} = \delta_{i,0}q^{n'j}$, so $g^t \rightharpoonup \alpha = \alpha$.
- $(g^t \rightharpoonup \alpha \leftharpoonup xg^{-t})(x^ig^j) = \alpha(xg^{-t}x^ig^jg^t) = q^{it}\alpha(x^{i+1}g^j) = q^{it}\delta_{i+1,0}q^{n'j} = 0$, so $g^t \rightharpoonup \alpha \leftharpoonup xg^{-t} = 0$, thus
$$x\alpha = \alpha x.$$
- Similarly, $x \rightharpoonup G = 0$ and $g^t \rightharpoonup G \leftharpoonup xg^{-t} = 0$ and $(g^t \rightharpoonup G)(x^ig^j) = G(x^ig^{j+t}) = \delta_{i,0}q^{-t'^{-1}dk-t'^{-1}dt'} = \delta_{i,0}q^{-t'^{-1}dk-d} = q^{-d}G(x^ig^j)$, hence
$$xG = q^{-d}Gx.$$
- $q^{-j}X(x^{i+1}g^j) = q^{-j}\delta_{i,0}$, so $x \rightharpoonup X = (\sum_k \lambda_k\alpha^k)G^{t'}$, where the scalars λ_k are such that $\sum_k \lambda_k(q^{n'r})^k = q^{-r}$ (for $0 \leq r < d$);
- $X(x^ig^{j+t}) = \delta_{i,1}$, so $g^t \rightharpoonup X = X$.
- $q^{it}X(x^{i+1}g^j) = q^{it}\delta_{i+1,1} = \delta_{i,0}$, so $g^t \rightharpoonup X \leftharpoonup xg^{-t} = 1$. Then,
$$xX = Xx + (\sum_k \lambda_k\alpha^k)G^{t'} - g^t.$$
- $q^{-j}f(x^{i+1}g^j) = \delta_{i,n'-1}q^{n'j}q^{-j}$, so $x \rightharpoonup f = X_{n'-1}\alpha(\sum_k \lambda_k\alpha^k)G^{t'}$, where the scalars λ_k are as above;
- $f(x^ig^{j+t}) = \delta_{i,n'}q^{n'(j+t)} = \delta_{i,n'}q^{n'j}$, so $g^t \rightharpoonup f = f$;
- $q^{it}f(x^{i+1}g^j) = q^{it}\delta_{i+1,n'}q^{n'j} = q^{-t}\delta_{i,n'-1}q^{n'j}$, so $g^t \rightharpoonup f \leftharpoonup xg^{-t} = q^{-t}X_{n'-1}\alpha$, hence
$$xf = fx + X_{n'-1}\alpha\left[(\sum_k \lambda\alpha^k)G^{t'} - q^{-t}g^t\right]$$

- $q^{-j} z_\lambda(x^{i+1}g^j) = \delta_{n'|i+1}\lambda^{(i+1)/n'}q^{(i+1)j}q^{-j} = \delta_{n'|i+1}\lambda^{(i+1)/n'}q^{ij}$, therefore $x \rightharpoonup z_\lambda = \lambda X_{n'-1}\alpha(\sum \lambda_k \alpha^k)G^{t'} z_\lambda$.

- $z_\lambda(x^i g^{j+t}) = \delta_{n'|i}\lambda^{i/n'}q^{i(j+t)} = \delta_{n'|i}\lambda^{i/n'}q^{ij}$ (since i is a multiple of n'), so $g^t \rightharpoonup z_\lambda = z_\lambda$.

- $q^{it} z_\lambda(x^{i+1}g^j) = q^{it}\delta_{n'|i+1}\lambda^{(i+1)/n'}q^{(i+1)j} = q^{-t}\delta_{n'|i+1}\lambda^{(i+1)/n'}q^{(i+1)j}$, so $g^t \rightharpoonup z_\lambda \leftharpoonup xg^{-t} = q^{-t}\lambda X_{n'-1}\alpha z_\lambda$, thus

$$xz_\lambda = z_\lambda x + \lambda X_{n'-1}\alpha z_\lambda \left[(\sum_k \lambda_k \alpha^k)G^{t'} - q^{-t}g^t \right].$$

Generalized Liu algebras

Recall from §2.2.5 the generalized Liu algebra

$$H = B(n,w,q) = k\langle x^{\pm 1}, g^{\pm 1}, y : x \text{ central}, yg = qgy, g^n = x^w = 1 - y^n \rangle,$$

where x, g are grouplike and y is $(1,g)$-primitive. It is a free module over the central Hopf subalgebra $A = k[x^{\pm 1}]$, with basis $\{y^i g^j : 0 \le i, j < n\}$, and the corresponding Hopf quotient is $\overline{H} = T_f(n,1,q)$.

The dual

As in Corollary 4.4.6(IV), the dual is

$$B(n,w,q)^\circ \cong \overline{H}^* \# A^\circ \cong T_f(n,1,q)\#(k[f] \otimes k(k^\times)).$$

We first compute the formulae for the functionals of $B(n,w,q)^\circ$. Let G and Y respectively denote the grouplike and skew-primitive generators of $T_f(n,1,q)^*$. By Example 1.3.4, these functionals embed into H° as follows: $G(y^i g^j x^k) = \delta_{i,0}q^{-j}$ and $Y(y^i g^j x^k) = \delta_{i,1}$.

Furthermore, as in the proof of Corollary 4.4.6, H decomposes as a right A-module into $A \oplus X$, where

$$X = \bigoplus_{\substack{1 \le i < n \\ 0 \le j < n}} y^i g^j A \oplus \bigoplus_{1 \le j < n} (g^j - 1)A$$

is a coideal of H. Therefore, the corresponding coalgebra projection $\Pi : H \to A$ is given by

$$\Pi(y^i g^j x^k) = \delta_{i,0}x^k.$$

Since $A^\circ \cong k[f'] \otimes k(k^\times)$ by Example 1.3.10, the functional $f = \Pi^\circ(f')$ is given by $f(y^i g^j x^k) = \delta_{i,0}k$ and, for each $\lambda \in k^\times$, the functional $z_\lambda = \Pi^\circ(\chi_\lambda)$ is given by $z_\lambda(y^i g^j x^k) = \delta_{i,0}\lambda^k$.

Let us compute the coproduct of the generators of the dual. First, since $\overline{H}^* \cong T_f(n,1,q)$ embeds into $B(n,w,q)^\circ$ as a Hopf subalgebra, G must be grouplike and Y

must be $(1,G)$-primitive. We now consider f. For any $0 \leq i,j,r,s < n, k,t \in \mathbb{Z}$ we have

$$f(y^i g^j x^k \cdot y^r g^s x^t) = \begin{cases} \delta_{i,0}\delta_{r,0}(k+t), & \text{if } i+r<n \wedge j+s<n \\ \delta_{i,0}\delta_{r,0}(k+t+w), & \text{if } i+r<n \wedge j+s \geq n \\ -w\delta_{i+r,n}q^{-rj}, & \text{if } i+r \geq n \end{cases}.$$

Let μ_r^l be scalars such that $\sum_r \mu_r^l q^{-rj} = \delta_{j,l}$ for all j. Then,

- $(f\otimes 1 + 1\otimes f)(y^i g^j x^k \otimes y^r g^s x^t) = \delta_{i,0}\delta_{r,0}(k+t)$;
- $\sum_{l+m\geq n}[(\sum_r \mu_r^l G^r)\otimes(\sum_s \mu_s^m G^s)](y^i g^j x^k \otimes y^r g^s x^t) = \delta_{i,0}\delta_{r,0}\sum_{l+m\geq n}\delta_{j,l}\delta_{s,m}$, hence it gives $\delta_{i,0}\delta_{r,0}$ when $j+s\geq n$;
- $\sum_{l=1}^{n-1}(Y_{n-l}G^l \otimes Y_l)(y^i g^j x^k \otimes y^r g^s x^t) = \sum_{l=1}^{n-1}\delta_{i,n-l}q^{-lj}\delta_{r,l} = \delta_{i+r,n}q^{-rj}$.

Therefore,

$$\Delta(f) = f\otimes 1 + 1\otimes f + w\sum_{\substack{l,m=1\\ l+m\geq n}}^{n-1}(\sum_r \mu_r^l G^r)\otimes(\sum_s \mu_s^m G^s) - w\sum_{l=1}^{n-1} Y_{n-l}G^l \otimes Y_l.$$

Let us now compute the coproduct of z_λ. For any $0 \leq i,j,r,s < n, k,t \in \mathbb{Z}$ we have

$$z_\lambda(y^i g^j x^k \cdot y^r g^s x^t) = \begin{cases} \delta_{i,0}\delta_{r,0}\lambda^{k+t}, & \text{if } i+r<n \wedge j+s<n \\ \delta_{i,0}\delta_{r,0}\lambda^{k+t+w}, & \text{if } i+r<n \wedge j+s\geq n \\ \delta_{i+r,n}q^{-rj}\lambda^{k+t}(1-\lambda^w), & \text{if } i+r\geq n \wedge j+s<n \\ \delta_{i+r,n}q^{-rj}\lambda^{k+t+w}(1-\lambda^w), & \text{if } i+r\geq n \wedge j+s\geq n \end{cases}.$$

- For the first case, $(z_\lambda \otimes z_\lambda)(y^i g^j x^k \otimes y^r g^s x^t) = \delta_{i,0}\delta_{r,0}\lambda^{k+t}$;
- For the second case, $\sum_{l+m\geq n}[(\sum_r \mu_r^l G^r)\otimes(\sum_s \mu_s^m G^s)](z_\lambda\otimes z_\lambda)(y^i g^j x^k \otimes y^r g^s x^t) = \delta_{i,0}\delta_{r,0}\lambda^{k+t}\sum_{l+m\geq n}\delta_{j,l}\delta_{s,m}$, hence it gives $\delta_{i,0}\delta_{r,0}\lambda^{k+t}$ when $j+s\geq n$;

Thus, the first two cases above are realized by

$$\left[1\otimes 1 + (\lambda^w - 1)\sum_{l+m\geq n}(\sum_r \mu_r^l G^r)\otimes(\sum_s \mu_s^m G^s)\right](z_\lambda\otimes z_\lambda).$$

- For the third case, $(\sum_{l=1}^{n-1} Y_{n-l}G^l\otimes Y_l)(z_\lambda\otimes z_\lambda)(y^i g^j x^k \otimes y^r g^s x^t) = \delta_{i+r,n}q^{-rj}\lambda^{k+t}$;
- For the last case, $(\sum_{l=1}^{n-1} Y_{n-l}G^l\otimes Y_l)\sum_{l+m\geq n}[(\sum_r \mu_r^l G^r)\otimes(\sum_s \mu_s^m G^s)](z_\lambda\otimes z_\lambda) = \delta_{i+r,n}q^{-rj}\lambda^{k+t}\sum_{l+m\geq n}\delta_{j,l}\delta_{s,m}$, that is it gives $\delta_{i+r,n}q^{-rj}\lambda^{k+t}$ when $j+s\geq n$.

Thus, the last two cases above are realized by

$$(1-\lambda^w)\left(\sum_{l=1}^{n-1} Y_{n-l}G^l\otimes Y_l\right)\left[1\otimes 1 + (\lambda^w - 1)\sum_{l+m\geq n}(\sum_r \mu_r^l G^r)\otimes(\sum_s \mu_s^m G^s)\right](z_\lambda\otimes z_\lambda).$$

Therefore,

$$\Delta(z_\lambda) = \left(1 \otimes 1 + (1-\lambda^w)\sum_{l=1}^{n-1}(Y_{n-l}G^l \otimes Y_l)\right)$$
$$\left(1 \otimes 1 + (\lambda^w - 1)\sum_{\substack{l,m=1 \\ l+m\geq n}}^{n-1}(\sum_r \mu_r^l G^r) \otimes (\sum_s \mu_s^m G^s)\right)(z_\lambda \otimes z_\lambda),$$

We now compute the action of A° on $\overline{H}^*$. Recall from Remark 4.1.6(3) that f acts on $\overline{H}^*$ as a derivation, while each z_λ acts by conjugation. First, for any $0 \leq i,j < n$, $k \in \mathbb{Z}$ we have

$$\Delta(y^i g^j x^k) = \sum_{l=0}^{i} \binom{i}{l}_{q^{-1}} y^{i-l}g^{l+j}x^k \otimes y^l g^j x^k.$$

Thus,

- $fG(y^ig^jx^k) = \sum_{l=0}^i \binom{i}{l}_{q^{-1}} f(y^{i-l}g^{l+j}x^k)G(y^lg^jx^k) = \delta_{i,0}q^{-j}k$ and one similarly obtains $Gf(y^ig^jx^k) = \delta_{i,0}q^{-j}k$, whence

$$fG = Gf.$$

- $fY(y^ig^jx^k) = \sum_{l=0}^i \binom{i}{l}_{q^{-1}} f(y^{i-l}g^{l+j}x^k)Y(y^lg^jx^k) = \delta_{i,1}f(g^{j+1}x^k)Y(yg^jx^k) = \delta_{i,1}(k + w\delta_{j,n-1})$;
- And $Yf(y^ig^jx^k) = \sum_{l=0}^i \binom{i}{l}_{q^{-1}} Y(y^{i-l}g^{l+j}x^k)f(y^lg^jx^k) = \delta_{i,1}k$.

  Thus, $\delta(Y) = fY - Yf$ is given by $\delta(Y)(y^ig^jx^k) = w\delta_{i,1}\delta_{j,n-1}$.

  Recalling that μ_r^{n-1} are scalars that satisfy $\sum_r \mu_r^{n-1}q^{-rj} = \delta_{j,n-1}$, then we have that $Y(\sum_r \mu_r^{n-1}G^r)(y^ig^jx^k) = \sum_{l=0}^i \binom{i}{l}_{q^{-1}} Y(y^{i-l}g^{l+j}x^k)(\sum_r \mu_r^{n-1}G^r)(y^lg^jx^k) = \delta_{i,1}(\sum_r \mu_r^{n-1}q^{-rj}) = \delta_{i,1}\delta_{j,n-1}$.

  Therefore,

$$\delta(Y) = fY - Yf = wY\left(\sum_r \mu_r^{n-1}G^r\right).$$

Moreover, for each $\lambda \in k^\times$ we have

- $z_\lambda G(y^ig^jx^k) = \sum_{l=0}^i \binom{i}{l}_{q^{-1}} z_\lambda(y^{i-l}g^{l+j}x^k)G(y^lg^jx^k) = \delta_{i,0}q^{-j}\lambda^k$ and one similarly obtains $Gz_\lambda(y^ig^jx^k) = \delta_{i,0}q^{-j}\lambda^k$, whence $z_\lambda G = Gz_\lambda$.
- $z_\lambda Y(y^ig^jx^k) = \sum_{l=0}^i \binom{i}{l}_{q^{-1}} z_\lambda(y^{i-l}g^{l+j}x^k)Y(y^lg^jx^k) = \delta_{i,1}z_\lambda(g^{j+1}x^k)Y(yg^jx^k) = \delta_{i,1}\lambda^{k+w\delta_{j,n-1}}$;

  and, provided η_r are scalars such that $\sum_r \eta_r q^{-rj} = \lambda^{w\delta_{j,n-1}}$,
- then $[Y(\sum_r \eta_r G^r)z_\lambda](y^ig^jx^k) = \sum_{l=0}^i \binom{i}{l}_{q^{-1}}[Y(\sum_r \eta_r G^r)](y^{i-l}g^{l+j}x^k)z_\lambda(y^lg^jx^k) = [Y(\sum_r \eta_r G^r)](y^ig^jx^k)\lambda^k = \sum_{l=0}^i \binom{i}{l}_{q^{-1}} Y(y^{i-l}g^{l+j}x^k)(\sum_r \eta_r G^r)(y^lg^jx^k)\lambda^k = \delta_{i,1}(\sum_r \eta_r q^{-rj})\lambda^k = \delta_{i,1}\lambda^{k+w\delta_{j,n-1}}$.

Thus,

$$z_\lambda Y = Y(\sum_r \eta_r G^r)z_\lambda.$$

The double

The following are the calculations on the product of the double of $H = B(n, w, q)$:

$$x \text{ central}, \qquad gG = Gg, \qquad gY = qYg, \qquad gf = fg, \qquad gz_\lambda = z_\lambda g$$

$$yG = q^{-1}Gy, \qquad yY = Yy + (G - qg),$$

$$yf = fy + w\left[\left(\sum_r \mu_r G^r\right)y + Y_{n-1}(g - G)\right],$$

$$yz_\lambda = \left(\sum_r \eta_r G^r\right)z_\lambda y + (1 - \lambda^w)Y_{n-1}z_\lambda(G - g),$$

where the scalars μ_r and η_r are such that $\sum_r \mu_r q^{-rj} = \delta_{j,n-1}$ and $\sum_r \eta_r q^{-rj} = \lambda^{w\delta_{j,n-1}}$, and Y_{n-1} is the functional given by $Y_{n-1}(y^i g^j x^k) = \delta_{i,n-1}$.

Recall from above that the functionals are defined as follows: the generators of $T_f(n, 1, q)$ are $G(y^i g^j x^k) = q^{-j}\delta_{i,0}$ and $Y(y^i g^j x^k) = \delta_{i,1}$, $f(y^i g^j x^k) = \delta_{i,0}k$ and for each $\lambda \in k^\times$ we have $z_\lambda(y^i g^j x^k) = \delta_{i,0}\lambda^k$.

Since x is central and grouplike in H, we have $\varphi(x^{-1}hx) = \varphi(h)$ for any $h \in H$ and $\varphi \in H^\circ$, hence $x\varphi = (x \rightharpoonup \varphi \leftharpoonup x^{-1})x = \varphi x$ for all $\varphi \in H^\circ$, that is, x is central in $D(H)$.

For any $\varphi \in H^\circ$, $g\varphi = (g \rightharpoonup \varphi \leftharpoonup g^{-1})g$. Since $g^{-1}y^i g^j x^k g = q^i y^i g^j x^k$, we have

$$gG = Gg, \quad gY = qYg, \quad gf = fg, \quad gz_\lambda = z_\lambda g.$$

As for the relations involving y, we know that for any $\varphi \in H^\circ$

$$y\varphi = (y \rightharpoonup \varphi) + (g \rightharpoonup \varphi)y - (g \rightharpoonup \varphi \leftharpoonup g^{-1}y)g.$$

We start with $y \rightharpoonup \varphi$ for any $\varphi \in H^\circ$. Since

$$y^i g^j x^k y = q^{-j}y^{i+1}g^j x^k = \begin{cases} q^{-j}y^{i+1}g^j x^k, & \text{if } i < n-1 \\ q^{-j}(g^j x^k - g^j x^{k+w}), & \text{if } i = n-1 \end{cases},$$

then

- $(y \rightharpoonup G)(y^i g^j x^k) = q^{-j}(q^{-j} - q^{-j})\delta_{i,n-1} = 0$, hence $y \rightharpoonup G = 0$.
- $(y \rightharpoonup Y)(y^i g^j x^k) = q^{-j}\delta_{i,0}$, so $y \rightharpoonup Y = G$.
- $(y \rightharpoonup f)(y^i g^j x^k) = -wq^{-j}\delta_{i,n-1}$, so $y \rightharpoonup f = -wY_{n-1}G$, where Y_{n-1} is the functional given by $Y_{n-1}(y^i g^j x^k) = \delta_{i,n-1}$ (which is just a scalar multiple of Y^{n-1}).

- $(y \rightharpoonup z_\lambda)(y^i g^j x^k) = (1-\lambda^w)\lambda^k q^{-j}\delta_{i,n-1}$, so $y \rightharpoonup z_\lambda = (1-\lambda^w)Y_{n-1}Gz_\lambda$.

We now calculate $g \rightharpoonup \varphi$ for all $\varphi \in H^\circ$. We know

$$y^i g^j x^k g = y^i g^{j+1} x^k = \begin{cases} y^i g^{j+1} x^k, & \text{if } j < n-1 \\ y^i x^{k+w}, & \text{if } j = n-1 \end{cases},$$

hence

- $(g \rightharpoonup G)(y^i g^j x^k) = \delta_{i,0} q^{-(j+1)}$, so $g \rightharpoonup G = q^{-1}G$.
- $(g \rightharpoonup Y)(y^i g^j x^k) = \delta_{i,1}$, so $g \rightharpoonup Y = Y$.
- $(g \rightharpoonup f)(y^i g^j x^k) = \delta_{i,0}(k + w\delta_{j,n-1})$, so $g \rightharpoonup f = f + w(\sum_r \mu_r G^r)$, where the scalars μ_r are such that $\sum_r \mu_r q^{-rj} = \delta_{j,n-1}$.
- $(g \rightharpoonup z_\lambda)(y^i g^j x^k) = \delta_{i,0}\lambda^k \lambda^{w\delta_{j,n-1}}$, so $g \rightharpoonup z_\lambda = (\sum_r \eta_r G^r)z_\lambda$ where η_r are such that $\sum_r \eta_r q^{-rj} = \lambda^{w\delta_{j,n-1}}$.

And finally we compute $g \rightharpoonup \varphi \leftharpoonup g^{-1}y$ for all $\varphi \in H^\circ$. Since

$$g^{-1}yy^i g^j x^k g = q^{i+1}y^{i+1}g^j x^k = \begin{cases} q^{i+1}y^{i+1}g^j x^k, & \text{if } i < n-1, \\ g^j x^k - g^j x^{k+w}, & \text{if } i = n-1 \end{cases}$$

we have

- $(g \rightharpoonup G \leftharpoonup g^{-1}y)(y^i g^j x^k) = \delta_{i,n-1}(q^{-j} - q^{-j}) = 0$, so $g \rightharpoonup G \leftharpoonup g^{-1}y = 0$.
- $(g \rightharpoonup Y \leftharpoonup g^{-1}y)(y^i g^j x^k) = q\delta_{i,0}$, so $g \rightharpoonup Y \leftharpoonup g^{-1}y = q1_{H^\circ}$.
- $(g \rightharpoonup f \leftharpoonup g^{-1}y)(y^i g^j x^k) = -w\delta_{i,n-1}$, so $g \rightharpoonup f \leftharpoonup g^{-1}y = -wY_{n-1}$.
- $(g \rightharpoonup z_\lambda \leftharpoonup g^{-1}y)(y^i g^j x^k) = (1-\lambda^w)\lambda^k\delta_{i,n-1}$, so $g \rightharpoonup z_\lambda \leftharpoonup g^{-1}y = (1-\lambda^w)Y_{n-1}z_\lambda$.

Ding-Liu-Wu's examples

Recall this example from §2.2.5, whose normal commutative Hopf subalgebra is $A = k[x^{\pm 1}]$, and its dual

$$D(m,d,q)^\circ \cong \overline{H}^* \# A^\circ \cong (kC_2 \#_\tau T_f(m,1,q^2)) \# (k[f] \otimes k(k^\times))$$

from Corollary 4.4.6(V). We explain here the formulae of the functionals of H°.

Also recall that H can be decomposed as a right A-module into $A \oplus X$, where X is the coideal

$$X = \bigoplus_{\substack{1 \le i < m \\ 0 \le j < m}} y^i g^j A \oplus \bigoplus_{1 \le j < m} (g^j - 1)A \oplus \bigoplus_{\substack{0 \le j < m \\ 1 \le k < m}} g^j u_k A \oplus \bigoplus_{0 \le j < m} g^j(u_0 - 1)A.$$

Thus, the corresponding coalgebra projection $\Pi : H \to A$ is defined by

$$\Pi(y^i g^j x^l) = \delta_{i,0} x^l, \qquad \Pi(g^j u_k x^l) = \delta_{k,0} x^l.$$

We know that $A^\circ \cong k[f] \otimes k(k^\times)$ by Example 1.3.10, and it embeds into H° through Π°, hence f extends to a functional of H° defined by $f(y^i g^j x^l) = \delta_{i,0} l$ and $f(g^j u_k x^l) = \delta_{k,0} l$ and for each $\lambda \in k^\times$ the corresponding functional $z_\lambda = \Pi^\circ(\chi_\lambda)$ is given by $z_\lambda(y^i g^j x^l) = \delta_{i,0} \lambda^l$ and $z_\lambda(g^j u_k x^l) = \delta_{k,0} \lambda^l$.

Moreover, as in the proof of Corollary 4.4.6, $\overline{H}^* \cong (kC_2)^* \#_\tau T_f(m, 1, q^2)^*$. Let α be the generator of $(kC_2)^*$ and let G and Y respectively denote the invertible and the nilpotent generators of $T_f(m, 1, q^2)^*$. First, we extend these to functionals on $\overline{H}$. By Lemma 2.2.3 we have $\overline{H} \cong T_f(m, 1, q^2) \#_\sigma kC_2$. Thus, it follows from Example 1.3.3 that α extends to the functional given by $\alpha(\bar{y}^i \bar{g}^j \bar{u_0}^k) = \delta_{i,0}(-1)^k$, and by Example 1.3.4 G and Y extend to functionals defined by $Y(\bar{y}^i \bar{g}^j \bar{u_0}^k) = \delta_{i,1}$ and $G(\bar{y}^i \bar{g}^j \bar{u_0}^k) = \delta_{i,0} q^{-2j}$.

And now we extend them to functionals on H. The Hopf surjection $\pi : H \twoheadrightarrow \overline{H}$ is given by

$$\pi(y^i g^j x^l) = \bar{y}^i \bar{g}^j, \qquad \pi(g^j u_k x^l) = \alpha_k \gamma^{-jk} \bar{y}^k \bar{g}^j \bar{u}_0,$$

where $\alpha_k = \prod_{l=1}^{k} (1 - \gamma^{-l})^{-1}$. Therefore, α, G, Y extend to functionals in H° respectively given by $\alpha(y^i g^j x^l) = \delta_{i,0}$ and $\alpha(g^j u_k x^l) = -\delta_{k,0}$, $G(y^i g^j x^l) = q^{-2j} \delta_{i,0}$ and $G(g^j u_k x^l) = q^{-2j} \delta_{k,0}$ and $Y(y^i g^j x^l) = \delta_{i,1}$ and $Y(g^j u_k x^l) = (1 - \gamma^{-1})^{-1} \gamma^{-j} \delta_{k,1}$.

A.3 Noetherian PI Hopf domains of GKdim 2

This section contains the additional calculations for Corollary 4.4.8.

Group algebra of the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$

Recall the semidirect product $\mathbb{Z} \rtimes \mathbb{Z} = \langle a, b : aba^{-1} = b^{-1} \rangle$, which is an abelian-by-finite group with normal abelian subgroup $\mathbb{Z} \times 2\mathbb{Z} = \langle a^2, b \rangle$ of index 2. Let $H = k(\mathbb{Z} \rtimes \mathbb{Z})$ and $A = k(\mathbb{Z} \times 2\mathbb{Z})$. By Corollary 4.4.8(I) the dual is

$$H^\circ \cong \overline{H}^* \otimes A^\circ \cong kC_2 \otimes (k[f, f'] \otimes k(k^\times)^2).$$

We start by computing the formulae of the functionals of H°. Let α denote the generator of $(kC_2)^*$. The Hopf quotient $\pi : H \twoheadrightarrow \overline{H} = kC_2$ is given by $\pi(a^i b^j) = \bar{a}^r$ where $r \in \{0, 1\}$ such that $i = 2k + r$, so by Example 1.3.3 α extends to the functional of H° given by $\alpha(a^i b^j) = (-1)^r = (-1)^i$.

Moreover, H decomposes as a right A-module into $A \oplus (a-1)A$ and $(a-1)A$ is a coideal, so the corresponding projection is given by $\Pi(a^i b^j) = a^{2k} b^j$, where $i = 2k + r$ and $r \in \{0, 1\}$. Therefore, the functionals f, f' and $\chi_{\lambda,\mu}$ (for each $(\lambda, \mu) \in (k^\times)^2$) in $A^\circ \cong k[f, f'] \otimes k(k^\times)^2$ extend to H as follows: $\Pi^\circ(f)(a^i b^j) = k, \Pi^\circ(f')(a^i b^j) = j$ and for each $(\lambda, \mu) \in (k^\times)^2$ we have $\Pi^\circ(\chi_{\lambda,\mu})(a^i b^j) = \lambda^k \mu^j$.

We now explain the coalgebra structure of H°. First, note that $a^ib^j \cdot a^mb^n = a^{i+m}b^{(-1)^mj+n}$ for any $i, j, m, n \in \mathbb{Z}$. Let q, q', q'' be such that $i = 2q + r, m = 2q' + r'$ and $i + m = 2q'' + r''$ with $r, r', r'' \in \{0, 1\}$. Then,

- $(\alpha \otimes \alpha)(a^ib^j \otimes a^mb^n) = (-1)^{i+m} = \alpha(a^ib^j \cdot a^mb^n)$;
- $[f \otimes 1 + 1 \otimes f + \frac{1}{4}(1-\alpha) \otimes (1-\alpha)](a^ib^j \otimes a^mb^n) = q + q' + \frac{1}{4}(1-(-1)^i)(1-(-1)^m) = q'' = f(a^ib^j \cdot a^mb^n)$;
- $(f' \otimes \alpha + 1 \otimes f')(a^ib^j \otimes a^mb^n) = j(-1)^m + n = f'(a^ib^j \cdot a^mb^n)$;
- $\frac{1}{2}\left[1 \otimes (1+\alpha) + \left(\frac{1+\lambda}{2} + \frac{1-\lambda}{2}\alpha\right) z_{1,\mu^{-2}} \otimes (1-\alpha)\right] (z_{\lambda,\mu} \otimes z_{\lambda,\mu})(a^ib^j \otimes a^mb^n) = \frac{1}{2}[1 + (-1)^m + (\frac{1+\lambda}{2} + \frac{1-\lambda}{2}(-1)^i)\mu^{-2j}(1-(-1)^m)]\lambda^{q+q'}\mu^{j+n} = \lambda^{q''}\mu^{(-1)^mj+n} = z_{\lambda,\mu}(a^ib^j \cdot a^mb^n)$.

Lastly, we explain the convolution product of the characters $\phi_{\lambda,\mu}, \psi_{\lambda,\mu}$ of $W(H) = kC_2 \otimes k[f, f']$, where $\lambda, \mu \in k$. Recall that these are defined by $\phi_{\lambda,\mu}(f) = \psi_{\lambda,\mu}(f) = \lambda$, $\phi_{\lambda,\mu}(f') = \psi_{\lambda,\mu}(f') = \mu$ and $\phi_{\lambda,\mu}(\alpha) = 1$, $\psi_{\lambda,\mu}(\alpha) = -1$.

- $\phi_{\lambda,\mu}\phi_{\nu,\xi}(f) = \phi_{\lambda,\mu}(f)\phi_{\nu,\xi}(1) + \phi_{\lambda,\mu}(1)\phi_{\nu,\xi}(f) + \frac{1}{4}\phi_{\lambda,\mu}(1-\alpha)\phi_{\nu,\xi}(1-\alpha) = \lambda + \nu$;
  $\phi_{\lambda,\mu}\phi_{\nu,\xi}(f') = \phi_{\lambda,\mu}(f')\phi_{\nu,\xi}(\alpha) + \phi_{\lambda,\mu}(1)\phi_{\nu,\xi}(f') = \mu + \xi$;
  $\phi_{\lambda,\mu}\phi_{\nu,\xi}(\alpha) = \phi_{\lambda,\mu}(\alpha)\phi_{\nu,\xi}(\alpha) = 1$, so $\phi_{\lambda,\mu}\phi_{\nu,\xi} = \phi_{\lambda+\nu,\mu+\xi}$.
- $\phi_{\lambda,\mu}\psi_{\nu,\xi}(f) = \phi_{\lambda,\mu}(f)\psi_{\nu,\xi}(1) + \phi_{\lambda,\mu}(1)\psi_{\nu,\xi}(f) + \frac{1}{4}\phi_{\lambda,\mu}(1-\alpha)\psi_{\nu,\xi}(1-\alpha) = \lambda + \nu$;
  $\phi_{\lambda,\mu}\psi_{\nu,\xi}(f') = \phi_{\lambda,\mu}(f')\psi_{\nu,\xi}(\alpha) + \phi_{\lambda,\mu}(1)\psi_{\nu,\xi}(f') = -\mu + \xi$;
  $\phi_{\lambda,\mu}\psi_{\nu,\xi}(\alpha) = \phi_{\lambda,\mu}(\alpha)\psi_{\nu,\xi}(\alpha) = -1$, so $\phi_{\lambda,\mu}\psi_{\nu,\xi} = \psi_{\lambda+\nu,-\mu+\xi}$.
- $\psi_{\lambda,\mu}\phi_{\nu,\xi}(f) = \psi_{\lambda,\mu}(f)\phi_{\nu,\xi}(1) + \psi_{\lambda,\mu}(1)\phi_{\nu,\xi}(f) + \frac{1}{4}\psi_{\lambda,\mu}(1-\alpha)\phi_{\nu,\xi}(1-\alpha) = \lambda + \nu$;
  $\psi_{\lambda,\mu}\phi_{\nu,\xi}(f') = \psi_{\lambda,\mu}(f')\phi_{\nu,\xi}(\alpha) + \psi_{\lambda,\mu}(1)\phi_{\nu,\xi}(f') = \mu + \xi$;
  $\psi_{\lambda,\mu}\phi_{\nu,\xi}(\alpha) = \psi_{\lambda,\mu}(\alpha)\phi_{\nu,\xi}(\alpha) = -1$, so $\psi_{\lambda,\mu}\phi_{\nu,\xi} = \psi_{\lambda+\nu,\mu+\xi}$.
- $\psi_{\lambda,\mu}\psi_{\nu,\xi}(f) = \psi_{\lambda,\mu}(f)\psi_{\nu,\xi}(1) + \psi_{\lambda,\mu}(1)\psi_{\nu,\xi}(f) + \frac{1}{4}\psi_{\lambda,\mu}(1-\alpha)\psi_{\nu,\xi}(1-\alpha) = \lambda + \nu + 1$;
  $\psi_{\lambda,\mu}\psi_{\nu,\xi}(f') = \psi_{\lambda,\mu}(f')\psi_{\nu,\xi}(\alpha) + \psi_{\lambda,\mu}(1)\psi_{\nu,\xi}(f') = -\mu + \xi$;
  $\psi_{\lambda,\mu}\psi_{\nu,\xi}(\alpha) = \psi_{\lambda,\mu}(\alpha)\psi_{\nu,\xi}(\alpha) = 1$, so $\psi_{\lambda,\mu}\psi_{\nu,\xi} = \phi_{\lambda+\nu+1,-\mu+\xi}$.

Localized quantum plane $A(l, n, q)$

Recall from §2.2.6 the localized quantum plane

$$A(l, n, q) = k\langle x^{\pm 1}, y : xy = qyx\rangle,$$

where x is grouplike and y is $(1, x^n)$-primitive. Its corresponding normal commutative Hopf subalgebra is $A = k[(x^l)^{\pm 1}, y^{l'}]$ and the Hopf quotient is $\overline{H} = T_f(l', n', q^{-d})\#_\sigma kC_d$, where $d = (l, n), l' = l/d, n' = n/d$. From Corollary 4.4.8(III) we have

$$A(l, n, q)^\circ \cong \overline{H}^*\#A^\circ \cong (T_f(l', n', q^{-d}) \otimes kC_d)\#(k[f, f'] \otimes k(k \times k^\times)).$$

We investigate the definition of these functionals.

As in the proof of Corollary 4.4.8, H decomposes as a right A-module into $A \oplus X$, where

$$X = \bigoplus_{\substack{1 \le i < l' \\ 0 \le j < l}} y^i x^j A \oplus \bigoplus_{1 \le j < l} (x^j - 1)A$$

is a coideal. Thus, the associated A-module coalgebra projection $\Pi : H \twoheadrightarrow A$ is given by

$$\Pi(y^i x^j) = \delta_{l'|i} \Pi(y^i x^{lk+r}) = \delta_{l'|i} q^{-ir} \Pi(x^r y^i x^{lk}) = \delta_{l'|i} q^{-ij} y^i x^{lk},$$

where $j = lk + r$ with $0 \le r < l$.

Then, the functionals of A° embed into H° through Π and are defined as follows: f is given by $f(y^i x^j) = \delta_{i,l'} q^{-l'j}$, f' by $f'(y^i x^j) = k\delta_{i,0}$ and for each $(\lambda, \mu) \in k \times k^\times$ the corresponding character $z_{\lambda,\mu} = \Pi^\circ(\chi_{\lambda,\mu})$ is given by $z_{\lambda,\mu}(y^i x^j) = \delta_{l'|i} q^{-ij} \lambda^{i/l'} \mu^k$, where $j = lk + r$ with $0 \le r < l$.

We now look at the formulae for the functionals coming from $\overline{H}^*$. Let α be the generator of $(kC_d)^*$ and let X and Y respectively denote the invertible and the nilpotent generators of $T_f(l', n', q^{-d})^*$. First, the coalgebra decomposition of $\overline{H}$ into $T_f(l', n', q^{-d}) \otimes kC_d$ is given by $\bar{y}^i \bar{x}^j \mapsto \bar{y}^i (\bar{x}^d)^m \otimes \bar{x}^s$, where $j = dm + s$ with $0 \le s < d$. Thus, the functionals of $\overline{H}^*$ are defined as follows: from Example 1.3.3 we have $\alpha(\bar{y}^i \bar{x}^j) = \delta_{i,0} (q^{l'})^s = \delta_{i,0} q^{l'j}$, since $q^{l'}$ is a primitive dth root of unity; and by Example 1.3.4 X and Y are given by $X(\bar{y}^i \bar{x}^j) = \delta_{i,0} (q^{-d})^{n'^{-1} m} = \delta_{i,0} q^{-n'^{-1} dm}$ and $Y(\bar{y}^i \bar{x}^j) = \delta_{i,1}$, where n'^{-1} is the inverse of n' modulo l'.

Lastly, we extend these to functionals of H°. These embed into H° through the Hopf surjection $\pi : H \twoheadrightarrow \overline{H}$, which is given by $\pi(y^i x^j) = \delta_{i<l'} \bar{y}^i \bar{x}^r$, where $j = lk + r$ with $0 \le r < l$. Therefore, α, X and Y extend to the functionals given by $\alpha(y^i x^j) = \delta_{i,0} q^{l'r} = \delta_{i,0} q^{l'j}$, $X(y^i x^j) = \delta_{i,0} q^{n'^{-1} dm}$ and $Y(y^i x^j) = \delta_{i,1}$.

The family $B(n, p_0, \ldots, p_s, q)$

Recall the Hopf algebra $H = B(n, p_0, \ldots, p_s, q)$ from §2.2.6, whose normal commutative Hopf subalgebra is $A = k[y^{p_1 \cdots p_s}, x^{\pm l}] = \mathcal{O}(k \times k^\times)$. The corresponding Hopf quotient $\overline{H}$ is

$$T_f(p_1, p_0, \xi_1) \otimes \ldots \otimes T_f(p_s, p_0 p_1 \ldots p_{s-1}, \xi_s) \otimes kC_{n/p_0}$$

as a coalgebra by Lemma 2.2.5. Also recall its dual from Corollary 4.4.8(IV) as being

$$B(n, p_0, \ldots, p_s, q)^\circ \cong \overline{H}^* \# A^\circ.$$

We explain here the definition of the functionals of H°.

Recall from the proof of Corollary 4.4.8 that H decomposes as a right A-module into $A \oplus X$, where X is the coideal

$$X = \bigoplus_{\substack{0 \le i_k < p_k \\ 0 \le j < l \\ \text{some } i_k \ge 1}} y_1^{i_1} \dots y_s^{i_s} x^j A \oplus \bigoplus_{1 \le j < l} (x^j - 1)A.$$

Thus, the associated A-module coalgebra projection map $\Pi : H \twoheadrightarrow A = k[y^{p_1 \dots p_s}, x^{\pm l}]$ is

$$\begin{aligned} \Pi(y_1^{i_1} \dots y_s^{i_s} x^j) &= q^{-(m_1 i_1 + \dots + m_s i_s)j} \delta_{p_1|i_1} \dots \delta_{p_s|i_s} y_1^{i_1} \dots y_s^{i_s} x^{lk} \\ &= q^{-(m_1 i_1 + \dots + m_s i_s)j} \delta_{p_1|i_1} \dots \delta_{p_s|i_s} (y^{p_1 \dots p_s})^{i_1/p_1 + \dots + i_s/p_s} (x^l)^k, \end{aligned}$$

where $j = lk + r$ with $0 \le r < l$.

Thus, the functionals in $A^\circ \cong k[f, f'] \otimes k(k \times k^\times)$ extend through Π into functionals of H° as follows: the functional f is $f(y_1^{i_1} \dots y_s^{i_s} x^j) = \sum_{k=1}^s \delta_{i_k, p_k} (\Pi_{l \ne k} \delta_{i_l,0}) q^{-p_1 \dots p_s j}$, f' is given by $f'(y_1^{i_1} \dots y_s^{i_s} x^j) = k \delta_{i_1,0} \dots \delta_{i_s,0}$, and for each $(\lambda, \mu) \in k \times k^\times$ the corresponding functional is $z_{\lambda,\mu}(y_1^{i_1} \dots y_s^{i_s} x^j) = \lambda^{i_1/p_1 + \dots + i_s/p_s} \mu^k$, where $j = lk + r$ with $0 \le r < l$.

Let us investigate the formulae of the functionals arriving from $\overline{H}^*$. Given the decomposition of $\overline{H}$ into crossed products studied in Lemma 2.2.5 and the simple formulae of the associated cleaving maps, we may write a generic element of $\overline{H}$ as $\bar{y_1}^{i_1} \bar{x}^{r_1} \bar{y_2}^{i_2} \bar{x}^{r_2} \dots \bar{y_s}^{i_s} \bar{x}^{r_s} \bar{x}^{r_0}$, where $0 \le i_k, r_k < p_k$ for every $k = 1, \dots, s$ and $0 \le r_0 < n/p_0$.

Let α denote the generator of $(kC_{n/p_0})^*$; it extends to the functional of $\overline{H}^*$ given by $\alpha(\bar{y_1}^{i_1} \bar{x}^{r_1} \dots \bar{y_s}^{i_s} \bar{x}^{r_s} \bar{x}^{r_0}) = \delta_{i_1,0} \dots \delta_{i_s,0} q^{p_1 \dots p_s r_0}$, since $q^{p_1 \dots p_s}$ is a primitive (n/p_0)th root of unity. Let Y_k, X_k respectively denote the nilpotent and the invertible generators of $T_f(p_k, p_0 \dots p_{k-1}, \xi_k)^*$. By Example 1.3.4, these extend to functionals in $\overline{H}^*$ given by $Y_k(\bar{y_1}^{i_1} \bar{x}^{r_1} \dots \bar{y_s}^{i_s} \bar{x}^{r_s} \bar{x}^{r_0}) = \delta_{i_k,1} (\Pi_{l \ne k} \delta_{i_l,0})$ and $X_k(\bar{y_1}^{i_1} \bar{x}^{r_1} \dots \bar{y_s}^{i_s} \bar{x}^{r_s} \bar{x}^{r_0}) = \delta_{i_1,0} \dots \delta_{i_s,0} \xi_k^{-(p_0 \dots p_{k-1})^{-1} r_k}$, where the inverse is taken modulo p_k.

Before extending them to H°, we give the formulae of these functionals in terms of an easier basis of $\overline{H}^*$: $\alpha(\bar{y_1}^{i_1} \dots \bar{y_s}^{i_s} \bar{x}^j) = \delta_{i_1,0} \dots \delta_{i_s,0} q^{p_1 \dots p_s r_0}$, $Y_k(\bar{y_1}^{i_1} \dots \bar{y_s}^{i_s} \bar{x}^j) = \delta_{i_k,1} (\Pi_{l \ne k} \delta_{i_l,0}) q^{-(n/p_0) m_k (r_1 + \dots + r_{k-1})}$ and $X_k(\bar{y_1}^{i_1} \dots \bar{y_s}^{i_s} \bar{x}^j) = \delta_{i_1,0} \dots \delta_{i_s,0} \xi_k^{-(p_0 \dots p_{k-1})^{-1} r_k}$. Here the integers r_k are obtained by the unique decomposition

$$j = r_0 + \frac{n}{p_0} r_s + \frac{n}{p_0} p_s r_{s-1} + \dots + \frac{n}{p_0} p_s \dots p_2 r_1,$$

which can be obtained by successive integer divisions.

We now extend these functionals to H° through the Hopf quotient $\pi : H \twoheadrightarrow \overline{H}$, which is given by $\pi(y_1^{i_1} \dots y_s^{i_s} x^j) = \delta_{i_1 < p_1} \dots \delta_{i_s < p_s} \bar{y_1}^{i_1} \dots \bar{y_s}^{i_s} \bar{x}^r$, where $j = lk + r$ with $0 \le r < l$. Since $l = (n/p_0) p_1 \dots p_s$, the formulae of the extensions of these functionals are as follows: α is given by $\alpha(y_1^{i_1} \dots y_s^{i_s} x^j) = \delta_{i_1,0} \dots \delta_{i_s,0} q^{p_1 \dots p_s r_0}$, Y_k by $Y_k(y_1^{i_1} \dots y_s^{i_s} x^j) = \delta_{i_k,1} (\Pi_{l \ne k} \delta_{i_l,0}) q^{-(n/p_0) m_k (r_1 + \dots + r_{k-1})}$ and X_k by $X_k(y_1^{i_1} \dots y_s^{i_s} x^j) = \delta_{i_1,0} \dots \delta_{i_s,0} \xi_k^{-(p_0 \dots p_{k-1})^{-1} r_k}$, where the integers r_k are given as above.

A.4 The coordinate ring of $G = (k, +) \rtimes k^\times$

Let k be a field of characteristic 0 and $G = (k, +) \rtimes k^\times$ the affine algebraic group where $k^\times$ acts on $(k, +)$ by multiplication. Consider its coordinate ring $H = \mathcal{O}(G) = k[x, y^{\pm 1}]$. As in Example 3.2.2, x is $(1, y)$-primitive and y is grouplike. This section contains the calculations that support the formulas in Examples 1.3.11 and 5.2.4.

The dual

As in Example 1.3.11, its dual is

$$H^\circ \cong U(\mathfrak{g}) * G,$$

where $\mathfrak{g}$ is the 2-dimensional nonabelian solvable Lie algebra. More specifically, $\mathfrak{g}$ has basis $\{f, f'\}$ and these functionals are defined by $f(x^i y^j) = \delta_{i,1}$ and $f'(x^i y^j) = \delta_{i,0} j$, and for each $(\alpha, \beta) \in G$ the corresponding character is defined by $\chi_{(\alpha,\beta)} = \alpha^i \beta^j$.

The product in H° is then given by

$$f'f - ff' = f, \qquad \chi_{(\alpha,\beta)} f = \beta f \chi_{(\alpha,\beta)}, \qquad \chi_{(\alpha,\beta)} f' = f' \chi_{(\alpha,\beta)}.$$

We justify these relations with the following calculations:

- $(ff')(x^i y^j) = \sum_k \binom{i}{k} f(x^{i-k} y^{k+j}) f'(x^k y^j) = \delta_{i,1} j$;
- $(f'f)(x^i y^j) = \sum_k \binom{i}{k} f'(x^{i-k} y^{k+j}) f(x^k y^j) = \delta_{i,1}(j+1)$;

  It is easy to see that the inverse of (α, β) in G is $(-\alpha\beta^{-1}, \beta^{-1})$, hence $\chi^{-1}_{(\alpha,\beta)} = \chi_{(-\alpha\beta^{-1},\beta^{-1})}$.
- $\chi_{(\alpha,\beta)} f(x^i y^j) = \sum_k \binom{i}{k} \chi_{(\alpha,\beta)}(x^{i-k} y^{k+j}) f(x^k y^j) = \sum_k \binom{i}{k} \alpha^{i-k} \beta^{k+j} \delta_{k,1} = i\alpha^{i-1}\beta^{j+1}$;
- $\chi_{(\alpha,\beta)} f \chi_{(-\alpha\beta^{-1},\beta^{-1})}(x^i y^j) = \sum_k \binom{i}{k} \chi_{(\alpha,\beta)} f(x^{i-k} y^{k+j}) \chi_{(-\alpha\beta^{-1},\beta^{-1})}(x^k y^j) = \sum_k \binom{i}{k}(i-k)\alpha^{i-k-1}\beta^{k+j+1}(-\alpha\beta^{-1})^k \beta^{-j} = \left(\sum_k \binom{i}{k}(i-k)(-1)^k\right)\alpha^{i-1}\beta = \left(\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k\right) i\alpha^{i-1}\beta = \beta\delta_{i,1}$, since the alternating sum in parenthesis is zero if $i-1 \geq 1$.
- $\chi_{(\alpha,\beta)} f'(x^i y^j) = \sum_k \binom{i}{k} \chi_{(\alpha,\beta)}(x^{i-k} y^{k+j}) f'(x^k y^j) = \sum_k \binom{i}{k} \alpha^{i-k}\beta^{k+j}\delta_{k,0} j = \alpha^i \beta^j j$;
- $\chi_{(\alpha,\beta)} f' \chi_{(-\alpha\beta^{-1},\beta^{-1})}(x^i y^j) = \sum_k \binom{i}{k} \chi_{(\alpha,\beta)} f'(x^{i-k} y^{k+j}) \chi_{(-\alpha\beta^{-1},\beta^{-1})}(x^k y^j) = \sum_k \binom{i}{k} \alpha^{i-k}\beta^{k+j} j(-\alpha\beta^{-1})^k \beta^{-j} = \left(\sum_k \binom{i}{k}(-1)^k\right)\alpha^i j = \delta_{i,0} j$, since the alternating sum in parenthesis vanishes if $i \geq 1$.

The double

We explain here the formulae in Example 5.2.4:

$$xf = fx + (1-y), \quad xf' = (f'+1)x, \quad x\chi_{(\alpha,\beta)} = \chi_{(\alpha,\beta)}(\alpha(1-y) + \beta x), \quad y \text{ central}.$$

First, it is very easy to see that, since y is grouplike and central in H, then it must be central in the double: $y\varphi = (y \rightharpoonup \varphi \leftharpoonup y^{-1})y$ and $(y \rightharpoonup \varphi \leftharpoonup y^{-1})(h) = \varphi(y^{-1}hy) = \varphi(h)$ for every $\varphi \in H^\circ, h \in H$. Thus, we need only justify the relations involving x. Since x is $(1, y)$-primitive and H is commutative, for any $\varphi \in H^\circ$ we have

$$x\varphi = (x \rightharpoonup \varphi) + (y \rightharpoonup \varphi)x - (y \rightharpoonup \varphi \leftharpoonup y^{-1}x)y = (x \rightharpoonup \varphi)(1 - y) + (y \rightharpoonup \varphi)x.$$

- $(x \rightharpoonup f)(x^i y^j) = f(x^{i+1}y^j) = \delta_{i,0}$, hence $x \rightharpoonup f = 1$;
- $(y \rightharpoonup f)(x^i y^j) = f(x^i y^{j+1}) = \delta_{i,1}$, hence $y \rightharpoonup f = f$, which explains the first relation;
- $(x \rightharpoonup f')(x^i y^j) = f'(x^{i+1}y^j) = 0$;
- $(y \rightharpoonup f')(x^i y^j) = f'(x^i y^{j+1}) = \delta_{i,0}(j + 1)$, hence $y \rightharpoonup f' = f' + 1$, which explains the second relation;
- $(x \rightharpoonup \chi_{(\alpha,\beta)})(x^i y^j) = \chi_{(\alpha,\beta)}(x^{i+1}y^j) = \alpha^{i+1}\beta^j$, hence $x \rightharpoonup \chi_{(\alpha,\beta)} = \alpha\chi_{(\alpha,\beta)}$;
- $(y \rightharpoonup \chi_{(\alpha,\beta)})(x^i y^j) = \chi_{(\alpha,\beta)}(x^i y^{j+1}) = \alpha^i\beta^{j+1}$, thus $y \rightharpoonup \chi_{(\alpha,\beta)} = \beta\chi_{(\alpha,\beta)}$, giving the third relation.

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