Fermion Masses and Partially Conserved Chiral Symmetries

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Gerard John Lowe

Department of Physics & Astronomy
University of Glasgow

To my mum, dad & sister.

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Declaration

With the exception of Chapter 1, the work presented in this thesis was performed by myself with the help of my supervisor, Dr. C.D. Froggatt, and Prof. H.B. Nielsen of the Niels Bohr Institute in Copenhagen. Part of Chapters 3 and 4 has been published in Phys. Lett. B311 (1993) 163, while the work of Chapter 5 is to be published in Nucl. Phys. B.

"*I'm drawn between the light and dark*
Where others see their targets, divine symmetry...
I'm sinking in the quicksand of my thought..."

David Bowie
Quicksand

Abstract

This thesis addresses one of the outstanding questions of modern theoretical particle physics: what underlies the fermion mass and quark mixing hierarchies? All fermions (except the top quark) have masses which are suppressed to varying degrees relative to their natural scale, which is set by electroweak symmetry breaking. Most models, including the Standard Model (SM), attribute this to a hierarchy amongst fundamental Yukawa couplings. Here, the view is taken that partially conserved chiral symmetries provide a much more satisfactory rationale for suppressed fermion masses, and the fermion mass sector of promising models is analysed.

Firstly, Chapter 1 discusses relevant aspects of the SM and some popular Standard Model Group (SMG) extensions.

In Chapter 2, a classification scheme is introduced for SMG extensions which possess no non-SM fermions in the low energy regime. This classification is based on the manner in which fermion irreducible representations (IRs) of the SM are collected to form IRs of these extensions. It is argued that the class of extensions whose members' IRs are identical to those of the SM show most promise of naturally generating the fermion mass hierarchy. The SMG is seen to be embedded within these extensions as a diagonal subgroup and, consequently, the non-abelian part of each extension must be gauged. Assuming that all abelian symmetries are also gauged, the anomaly-free members of this favoured class are discussed. They are seen to be closely related to the anti-grand unified group $SMG \times SMG \times SMG$. For this group, corresponding Weyl fermions in different generations belong to inequivalent IRs.

Chapter 3 begins by taking some time out to emphasise that the whole approach to fermion masses and quark mixing angles in this thesis is geared towards accounting for them order of magnitude-wise. It then becomes more quantitative in specifying how

various approximately conserved symmetries suppress fermion mass matrix elements, and several plausible ansätze for constructing these elements are introduced. Finally, the existence of the large inter-generation mass gaps points towards particular candidate symmetries before the intra-generation gaps are seen to lead in two quite different directions.

One of these directions is examined in Chapter 4, where the candidate symmetries of the previous chapter are extended to include a partially conserved and gauged abelian flavour symmetry. This is done in order to directly account for mass splitting within the generations, assuming that the heavy generation mass eigenstates are approximately equal to the corresponding gauge eigenstates (a well-defined concept for the type of groups under consideration). Unfortunately, the full symmetry group cannot then include $SMG \times SMG \times SMG$, but only a subgroup of it. Several anomaly-free flavour charge sets are found for each model, subject to some basic constraints. The resulting models are then analysed using the ansätze of Chapter 3, following a general discussion of how the ansätze parameters are chosen to fit the known data on fermion masses and quark mixing angles.

Finally, Chapter 5 examines the alternative method of obtaining intra-generation mass splitting. This is based on the hope that the abelian subgroup of the anti-grand unified group $SMG \times SMG \times SMG$ might itself be responsible for such splitting, providing the assumption of Chapter 4 regarding the heavy generation mass eigenstates is explicitly violated. The mass gaps thus generated turn out to be unrealistic, however, and again a gauged abelian flavour symmetry is introduced in an effort to rescue the anti-grand unified model. The resulting $SMG \times SMG \times SMG \times U(1)$ model is then analysed in exactly the same way as the models of Chapter 4.

Contents

"*Can you see it in the sky*
That the landscape is too high?"

David Bowie
Red Money

Chapter 1

Introduction

1.1 Shortcomings of the Standard Model

At present, the Standard Model [1, 2] is almost universally accepted as the "correct" physical theory of fundamental particles and the forces between them, down to a scale of tenths of millifermis. The success of the Standard Model (SM) in describing all known physics up to energies of over 100 GeV is certainly remarkable, and is seen to be more so every time precision measurements are performed in experiments at CERN and elsewhere. Nevertheless, few theorists (if any) believe that it is Nature's final word. That it is not a complete or final theory is evidenced by several unanswered questions, among which are:

- why are there 3 generations of fermions?
- why is electric charge quantized?
- why do the gauge coupling constants of the Standard Model Group almost intersect at a point at some incredibly high mass scale $M_X \sim 10^{15}$ GeV?
- what underlies the fermion mass hierarchy?

This thesis will focus on the last question, but first it is useful to review some basic features of the SM.

1.2 Construction of the Standard Model

For some time it has generally been thought that any description of Nature at a fundamental level must be formulated as a relativistic quantum field theory. In particular, *gauge* theories are much favoured: all known viable physical theories (quantum electrodynamics, quantum chromodynamics, the electroweak theory) are of this type, as are almost all plausible extensions and would-be successors. A gauge theory is one which is symmetric under transformations in some internal space which vary from point to point in space-time. The Standard Model Group (SMG), the group of internal transformations under which the SM Lagrangian is symmetric, is:

$$SMG \equiv SU(3)_c \times SU(2)_L \times U(1)_Y \tag{1.1}$$

The SM contains 45 Weyl fermions forming 3 generations which are simply copies of each other as far as the SMG is concerned. Suppressing colour degrees of freedom for the moment, these are usually grouped as:

Generation 1:	u_L	d_L	u_R	d_R	e_L	ν_{eL}	e_R
Generation 2:	c_L	s_L	c_R	s_R	μ_L	$\nu_{\mu L}$	μ_R
Generation 3:	t_L	b_L	t_R	b_R	τ_L	$\nu_{\tau L}$	τ_R

Within any generation each chiral fermion behaves quite differently; a brief discussion of this divides quite naturally into 2 sectors.

1.2.1 $SU(3)_c$

That part of the SM governed by the symmetry group $SU(3)_c$ is called quantum chromodynamics (QCD), a description of the strong force. Under $SU(3)_c$ transformations, all quark states are triplets **3** (each degree of freedom being labelled by a "colour"), while all leptons are singlets **1**. The quarks interact through the exchange of massless gluons, and some of the most interesting elements of particle physics feature in this sector *e.g.* asymptotic freedom and confinement.

Confinement poses an immediate question of relevance to this thesis: given that quarks do not exist as free particles, how then does one define the mass of a quark? One answer is the "constituent mass" used in phenomenological nonrelativistic models. Here, for example, the masses of the u and d quarks are taken to be approximately

1/3 the mass of the proton. A common alternative, used in this thesis, is the "current mass". This is the Lagrangian parameter corresponding to the quark mass used in current algebra which emerges from the commutation rules involving the quark currents and the energy momentum tensor. The light quark masses (u, d, s) are then estimated from chiral symmetry breaking calculations and QCD spectral sum rules, while the heavier masses (c, b) can be extracted, for example, from e^+e^- data using QCD sum rules [3].

1.2.2 $SU(2)_L \times U(1)_Y$

The SM sector governed by the symmetry group $SU(2)_L \times U(1)_Y$ is known as the electroweak theory, a partially unified description of the weak and electromagnetic forces. Under $SU(2)_L$ transformations, all right-handed fermions are singlets while the left-handed fermions pair off to form doublets **2** (this is the root of the famous parity violation observed in weak interactions). The doublet partners are:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \begin{pmatrix} c \\ s \end{pmatrix}_L \begin{pmatrix} t \\ b \end{pmatrix}_L \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

Under $U(1)_Y$ phase rotations, the weak hypercharges are (with the unconventional normalisation $Q = I_3 + Y/6$):

$$\begin{array}{lll} Y((u\ d)_L) = 1 & Y(u_R) = 4 & Y(d_R) = -2 \\ Y((\nu_e\ e)_L) = -3 & Y(e_R) = -6 & \end{array} \tag{1.2}$$

and these quantum numbers are repeated for the other two generations. The fermions interact through the exchange of the $W^\pm$ and Z^0 bosons and the photon γ. When the Higgs doublet and spontaneous symmetry breaking are added to the theory, the $W^\pm$ and Z^0 become massive and the well-known quantum electrodynamics emerges, governed by the symmetry group $U(1)_{\text{em}}$.

1.3 Summary of Fermion Representations

Discussion of these fermion representations will occur repeatedly throughout the text, so it is worthwhile to summarise them here. Denoting irreducible representations (IRs) of the SMG by:

$$(\text{IR of } SU(3)_c, \text{ IR of } SU(2)_L)_Y$$

Chiral Fermions			IR of	Triality	Duality
Gen 1	Gen 2	Gen 3	the SMG	t	d
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	$(\mathbf{3},\mathbf{2})_1$	1	1
u_R	c_R	t_R	$(\mathbf{3},\mathbf{1})_4$	1	0
d_R	s_R	b_R	$(\mathbf{3},\mathbf{1})_{-2}$	1	0
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	$(\mathbf{1},\mathbf{2})_{-3}$	0	1
e_R	μ_R	τ_R	$(\mathbf{1},\mathbf{1})_{-6}$	0	0

Table 1.1: The chiral fermions and their SMG representations.

all of the chiral fermions and their transformation properties are listed in Table 1.1. Although seemingly random, this set of IRs is in fact very special, as will be seen shortly when anomalies are discussed.

It is interesting to follow Michel [4] and O'Raifeartaigh [5] in giving a physical meaning to the Lie *group* as well as to the Lie algebra. This is done by requiring that all particle multiplets of the theory must form genuine (*i.e.* single-valued) representations of the Lie group. Of course, the group $SU(3) \times SU(2) \times U(1)$ is represented in this way, but requiring the smallest such group leads us to consider:

$$S(U(2) \times U(3)) \equiv \left\{ \mathbf{U} = \begin{pmatrix} \multicolumn{2}{c}{U_2} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & U_3 \end{pmatrix} : U_2 \in U(2),\ U_3 \in U(3),\ \det \mathbf{U} = 1 \right\} \tag{1.3}$$

This group is a subgroup of $SU(5)$ and has the same Lie algebra as $SU(3) \times SU(2) \times U(1)$. In fact:

$$S(U(2) \times U(3)) = SU(3) \times SU(2) \times U(1)/D_6 \tag{1.4}$$

where the discrete subgroup divided out is:

$$D_6 \equiv \{h^n : h = (\mathrm{e}^{\mathrm{i}2\pi/3}\mathrm{I}_3, \mathrm{e}^{\mathrm{i}\pi}\mathrm{I}_2, \mathrm{e}^{\mathrm{i}\pi/3}),\ \mathrm{n} \in \mathrm{Z}_6\} \tag{1.5}$$

It is representable on precisely those multiplets which satisfy:

$$\frac{t}{3} + \frac{d}{2} + \frac{y}{6} \equiv 0 \pmod 1 \tag{1.6}$$

where:

$$\begin{aligned} t &\equiv \text{triality} = \begin{cases} 1 & \text{for } SU(3) \text{ triplets} \\ 0 & \text{for } SU(3) \text{ singlets} \\ -1 & \text{for } SU(3) \text{ anti-triplets} \end{cases} \\ d &\equiv \text{duality} = \begin{cases} 1 & \text{for } SU(2) \text{ doublets} \\ 0 & \text{for } SU(2) \text{ singlets} \end{cases} \\ y &\equiv \text{weak hypercharge, normalised as in Table 1.1} \end{aligned} \tag{1.7}$$

The triality and duality values of the known chiral fermions are shown in Table 1.1, and (1.6) is seen to be satisfied for all of these multiplets.

It is in fact possible to argue for this $S(U(2) \times U(3))$ group [6, 7] on the basis that Nature "chooses" the group with the highest degree of skewness *i.e.* the fewest outer automorphisms.

1.4 Spontaneous Symmetry Breaking and Fermion Masses

At this point it is reasonable to consider a first attempt at an $SU(2)_L \times U(1)_Y$ invariant electroweak Lagrangian density:

$$\mathcal{L}_{\mathrm{EW}} = -\frac{1}{4}G^2 - \frac{1}{4}B^2 + i\sum_j (\overline{\psi^j_L}\gamma_\mu D^\mu \psi^j_L + \overline{\psi^j_R}\gamma_\mu D^\mu \psi^j_R) \tag{1.8}$$

where: G and B are the kinetic terms for the $SU(2)_L$ and $U(1)_Y$ gauge fields respectively (the physical $W^\pm$, Z^0 and γ are linear combinations of these fields); D^μ denotes the covariant derivative; and ψ^j are the Weyl fermions of Table 1.1. There is a glaring problem: the $W^\pm$, Z^0 and fermions have no mass, in gross contradiction with experiment (except for the neutrinos). Adding mass terms by hand like:

$$m_W^2 W^\dagger_\mu W^\mu + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \tag{1.9}$$

or:

$$-\, m_e \overline{\psi^e_L}\psi^e_R + h.c. \tag{1.10}$$

is out of the question because they violate the gauge symmetry and lead to horrible renormalization problems. The solution is spontaneous symmetry breaking (SSB).

A complex scalar doublet Φ is added to the theory (the Higgs doublet) and more $SU(2)_L \times U(1)_Y$ symmetric terms can appear in $\mathcal{L}_{\mathrm{EW}}$, describing the interactions of Φ with itself and with the other fields. The vacuum is assumed not to respect the $SU(2)_L \times U(1)_Y$ symmetry, Φ acquiring a vacuum expectation value (VEV):

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ \langle\phi\rangle_{\mathrm{ws}} \end{pmatrix} \tag{1.11}$$

which breaks the symmetry down to $U(1)_{\mathrm{em}}$. The gauge bosons (except γ) then receive a mass from the term:

$$|D_\mu \Phi|^2 \tag{1.12}$$

The fermions receive masses from the terms:

$$\mathcal{L}_Y = -\sum_{j,k}(M_l^\dagger)_{jk}\overline{l_L^j}\Phi l_R^k - \sum_{j,k}(M_D^\dagger)_{jk}\overline{q_L^j}\Phi D_R^k - i\sum_{j,k}(M_U^\dagger)_{jk}\overline{q_L^j}\sigma_2\Phi^* U_R^k + h.c. \tag{1.13}$$

after the substitution $\Phi = \langle\Phi\rangle + \ldots$ where:

$$l_L = \begin{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \\ \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \end{pmatrix} \qquad q_L = \begin{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}_L \\ \begin{pmatrix} c \\ s \end{pmatrix}_L \\ \begin{pmatrix} t \\ b \end{pmatrix}_L \end{pmatrix} \tag{1.14}$$

$$l_R = \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} \qquad D_R = \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} \qquad U_R = \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}$$

Note that this convention for $M_{U,D,l}$, where the rows and columns are indexed by right- and left-handed fermions respectively, will be employed throughout this thesis.

The leptonic mass matrix M_l is easily diagonalised by redefining the leptons appropriately, utilising the fact that there are no right-handed neutrinos in the SM. The situation regarding the quark mass matrices (M_U and M_D) is more tricky. They are diagonalised by non-trivial biunitary transformations. It is possible to find unitary matrices R_U, S_U, R_D and S_D which satisfy:

$$\begin{aligned} S_U^\dagger M_U R_U &= \mathrm{diag}(m_u, m_c, m_t) \\ S_D^\dagger M_D R_D &= \mathrm{diag}(m_d, m_s, m_b) \end{aligned} \tag{1.15}$$

yielding the quark masses. What has happened is that the gauge eigenstates $U_{L,R}$ and $D_{L,R}$ have been rotated to form mass eigenstates $U'_{L,R}$ and $D'_{L,R}$:

$$\begin{aligned} U'_L &= R_U^\dagger U_L, & U'_R &= S_U^\dagger U_R, \\ D'_L &= R_D^\dagger D_L, & D'_R &= S_D^\dagger D_R \end{aligned} \tag{1.16}$$

This rotation leads to another interesting feature of weak interactions, that they do not conserve flavour. The charged current interaction of quarks is given by (1.8) to be:

$$\begin{aligned} \mathcal{L}_{cc} &= \overline{U_L}\gamma^\mu W_\mu D_L + h.c. \\ &= \overline{U'_L}\gamma^\mu W_\mu [R_U^\dagger R_D] D'_L + h.c. \\ &\equiv \overline{U'_L}\gamma^\mu W_\mu [V_{\mathrm{CKM}}] D'_L + h.c. \end{aligned} \tag{1.17}$$

Here:

$$W_\mu \equiv \frac{g_2}{2}(W_{1\mu} - iW_{2\mu}) \tag{1.18}$$

where g_2 is the weak gauge coupling constant and (W_1, W_2, W_3) are the weak gauge fields. So this interaction is not in general diagonal in the mass eigenstate basis, since V_{CKM} does not have to be the unit matrix. This matrix:

$$V_{\mathrm{CKM}} \equiv R_U^\dagger R_D \tag{1.19}$$

is called the Cabibbo-Kobayashi-Maskawa matrix [8, 9] and it parameterises the mixing of quark flavours in weak charged current interactions. The KM representation employs 3 mixing angles θ_i (i=1,2,3) and a complex phase δ, and is given by:

$$V_{\mathrm{CKM}} = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix} \tag{1.20}$$

where $s_i \equiv \sin\theta_i$ and $c_i \equiv \cos\theta_i$ (i=1,2,3). The presence of a complex phase is highly significant because it signals the existence of CP violation in the theory. Throughout this thesis, interest is focussed on the matrix $\mathcal{V}$ whose elements are $\mathcal{V}(i,j) \equiv$

$|V_{\rm CKM}(i,j)|$, a matrix which we write as:

$$\mathcal{V} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1.21}$$

It is obviously described in terms of only 3 parameters.

Note that up to this point, the term "generation" has been used to describe a collection of *gauge* eigenstates; it is more conventionally applied to the *mass* eigenstates. It is more precise then to use the term "proto-generation" when referring to the gauge eigenstates, but this is cumbersome. In this work, the term "generation" shall be freely used to describe both gauge and mass eigenstates and it will always be clear from the context exactly which type of generation is being discussed.

1.5 Mass-Protection of Fermions

The SM fermion IRs and the generation of fermion mass via SSB illustrate a concept crucial to this thesis, that of "mass-protection". A fermion ψ is said to be mass-protected by a symmetry group G if ψ_L and ψ_R belong to inequivalent IRs of G (G is then called a *chiral* symmetry), for then a (Dirac) mass term:

$$-m\overline{\psi_R}\psi_L + h.c. \tag{1.22}$$

cannot be inserted in the Lagrangian without violating G-symmetry. G thus "protects" the fermion from gaining a mass. Note that this is exactly the situation for all the SM fermions (except the three ν_L which have no right-handed partners at all), which are mass-protected by $SU(2)_L \times U(1)_Y$ (but not by $SU(3)_c$).

Such a fermion can gain a mass when G is spontaneously broken. The procedure would be analogous to the Higgs mechanism in the SM: a scalar which carries G-quantum numbers, and couples to the left- and right-handed components of the fermion, gains a VEV which gives mass to the fermion.

Note that a Weyl fermion (ψ_L say) which is a singlet under G is immediately *not* mass-protected as it can always form a Majorana mass term:

$$-\frac{1}{2}m\psi_L^T C\psi_L + h.c. \tag{1.23}$$

where C is the charge conjugation operator.

1.6 The Fermion Mass Problem & Approximately Conserved Chiral Symmetry

From (1.13), the mass of any fermion can be written (after diagonalisation of $M_{U,D,l}$) as:

$$m_i = y_i \langle \phi \rangle_{\rm ws} \tag{1.24}$$

where i=e, μ, τ, d, s, b, u, c or t and the y_i are known as Yukawa coupling constants. The Higgs VEV is $\langle \phi \rangle_{\rm ws}$ = 174 GeV and the y_i are free parameters in the SM. The fermion mass hierarchy is thus not predicted in the SM, but is merely accommodated through an appropriate choice of the various y_i. Similarly, the parameters of $V_{\rm CKM}$ in (1.19) are not predicted, merely measured experimentally.

A very pronounced hierarchical structure is found to exist amongst the Yukawa couplings and mixing angles. They have orders of magnitude:

$$\begin{gathered} y_t \sim 10^0, \quad V_{us} \sim 10^{-1}, \quad y_c \sim y_b \sim y_\tau \sim V_{cb} \sim 10^{-2}, \\ y_s \sim y_\mu \sim V_{ub} \sim 10^{-3}, \quad y_d \sim y_u \sim 10^{-4}, \quad y_e \sim 10^{-5} \end{gathered} \tag{1.25}$$

As these are completely free parameters in the SM, it is very distasteful to find them ranging over 5 orders of magnitude. All fermion masses are naively expected to be of order $\langle \phi \rangle_{\rm ws}$, (from (1.24)) but patently most are much lighter than this. Why are all the Yukawa couplings not of $\mathcal{O}(1)$? Indeed, the oft-repeated question "Why is the top so heavy?" is seen to be somewhat misleading; a better question is "Why are the others so light?". Really, two questions spring to mind:

- **Question 1:**
  Why are the masses of all non-top fermions suppressed?

- **Question 2:**
  Why are they suppressed to such differing degrees?

This is the fermion mass problem, to which no satisfactory solution has been found; this thesis tries to answer these questions.

The suppression of masses relative to their natural scale surely constitutes a most (if not *the* most) promising window on physics at higher energy scales. Here, as is universally done in fundamental physics when certain quantities are observed to be much smaller than expected, symmetry will be invoked as the underlying explanation. The view is taken that the SMG is only a low energy remnant of some larger group G, and that the fermion mass and mixing hierarchies are consequences of the spontaneous breaking of G to the SMG. Suppose that some subgroup of the full chiral symmetry group G mass-protects some (or all) of the SM fermions, in addition to the mass-protection they receive from the SMG itself. Suppose further that such a subgroup is "approximately conserved" *i.e.* only weakly broken (the component symmetries of the relevant subgroup are "partially conserved chiral symmetries" or PCCSs). Fermion mass matrix elements will consequently be suppressed, quantitative details depending on precisely which symmetries are partially conserved, how the fermions behave *w.r.t.* these symmetries and on the symmetry-breaking mechanism itself. In this way, the basic structure of the fermion mass hierarchy does *not* depend on small input parameters in the Lagrangian. All *fundamental* Yukawa couplings in the "true" underlying theory may be of $\mathcal{O}(1)$ (a much more satisfactory scenario than (1.25) when the SM viewed is as a fundamental theory), while the *effective* Yukawa couplings of the fermions to the Weinberg-Salam Higgs in the familiar low energy theory can naturally be small.

1.7 The Renormalization Group

In a renormalizable theory (such as the SM), physically measurable quantities can be written as functions of couplings which are renormalized at some arbitrary scale μ. Physical quantities calculated in the theory must be independent of μ. For example, denoting some physical quantity by Q where:

$$Q = f(g_3(\mu), \mu) \tag{1.26}$$

and g_3 is the gauge coupling constant of QCD, it must be true that:

$$\mu\frac{d}{d\mu}f(g_3(\mu), \mu) = 0 \tag{1.27}$$

which is called the renormalization group equation (RGE). It represents the fact that a change in the renormalization scale must be compensated for by a modification of the coupling constants in order to leave physical quantities invariant. Equations such as (1.27) have proven very powerful indeed, for example in probing the asymptotic nature of theories.

Some analysis in this thesis makes use of the fact that the parameters of the Lagrangian, in particular the y_i of (1.24), are consequently viewed as functions of scale $t \equiv \ln \mu$. The 1 loop RGEs are [10]:

$$\frac{d}{dt} Y_{U,D,l} = \frac{1}{16\pi^2} \beta_{U,D,l} \, Y_{U,D,l} \tag{1.28}$$

where $Y_{U,D,l}$ are the matrices of Yukawa couplings and:

$$\begin{aligned} \beta_U &= \frac{3}{2}(Y_U^\dagger Y_U - Y_D^\dagger Y_D) + S - (\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2) \\ \beta_D &= \frac{3}{2}(Y_D^\dagger Y_D - Y_U^\dagger Y_U) + S - (\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2) \\ \beta_l &= \frac{3}{2} Y_l^\dagger Y_l + S - \frac{9}{4}(g_1^2 + g_2^2) \end{aligned} \tag{1.29}$$

In these equations $g_{1,2,3}$ are the gauge coupling constants of $U(1)_Y$, $SU(2)_L$ and $SU(3)_c$ respectively and:

$$S \equiv \mathrm{Tr}\{3Y_U^\dagger Y_U + 3Y_D^\dagger Y_D + Y_e^\dagger Y_e\} \tag{1.30}$$

These equations describe how the Yukawa couplings (*i.e.* the fermion masses) evolve with changes in the energy scale.

1.8 Anomalies

This thesis will consider some extensions of the SMG, and indeed such model-building has long been a favourite occupation of theorists. Perhaps the biggest constraint on the model-builder's imagination is due to the existence of anomalies. Anomalies arise when the symmetries of a classical field theory are broken by the quantum fluctuations inherent in the corresponding quantum field theory [11]. Three different kinds of chiral gauge anomaly have been identified:

1. the triangular gauge anomaly, which threatens the renormalizability of a theory;

2. the global $SU(2)$ anomaly, which threatens the mathematical consistency of a theory;

3. and the mixed gauge-gravitational anomaly, which again threatens the renormalizability of a theory.

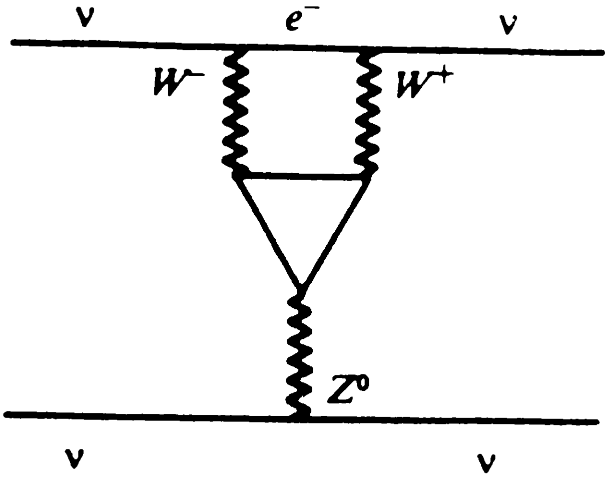

Figure 1.1: A contribution to ν–ν scattering in the Standard Model, involving the triangle diagram.

1.8.1 Triangular Gauge Anomaly

The quantum fluctuations represented by triangular fermion loop corrections in the renormalization process (such as those of Figure 1.1 which shows a triangular fermion loop in a diagram contributing to ν–ν scattering) break down classical chirality invariance and lead to the triangular gauge anomalies. These anomalies in the axial vector current part of chiral fermion interactions with the gauge fields eliminate the axial vector Ward-Takahashi identities and hence destroy the renormalizability of the quantum field theory. Anomalies also materialise in higher order graphs, but the triangular gauge anomlay is basic and its absence implies the absence of all other anomalous diagrams [12].

The only way of saving renormalizibility is to ensure that the *total* contribution of all triangle graphs is zero. This is a condition on the fermion content of a theory. A crucial point is that these anomalies are independent of the mass of the fermion circulating in the loop; it is possible to formulate general conditions for anomaly cancellation exclusively in terms of the properties of the transformation matrices at the three vertices of the loop. The general cancellation condition is:

$$\mathrm{Tr}\ [\{T_L^a, T_L^b\}T_L^c] = \mathrm{Tr}\ [\{T_R^a, T_R^b\}T_R^c] \tag{1.31}$$

where the trace is taken over all fermions which can circulate in the loop and $T_{L,R}$ are the transformation matrices of left- and right-handed fermions at each vertex. Such a cancellation guarantees gauge invariance, hence rescuing the renormalizibility of the theory. We shall also adopt a notation in which all fermions are left-handed, whence (1.31) becomes:

$$\text{Tr}\ [\{T_L^a, T_L^b\}T_L^c] = 0 \tag{1.32}$$

1.8.2 Global $SU(2)$ Anomaly

In 1982, Witten [13] showed that any $SU(2)$ gauge theory with an odd number of (left-handed) Weyl doublets is mathematically inconsistent. He showed that the fermion path integral, taken over Weyl fermions, changes sign (due to the properties of the chirality operator γ_5) under a topologically non-trivial $SU(2)$ gauge transformation. So although the global $SU(2)$ anomaly occurs for the same basic reason as the triangular anomaly (both effects depend crucially on the properties of the γ_5 operator), it differs from the triangular (perturbative) anomaly in requiring the full exploration of the gauge field space via these topologically non-trivial transformations. This $SU(2)$ anomaly introduces ambiguities in the evaluation of expectation values of quantum field operators and leads to a mathematically inconsistent theory. Witten showed that the only remedy was to insist on an even number of $SU(2)$ Weyl doublets.

1.8.3 Mixed Gauge-Gravitational Anomaly

The mixed gauge-gravitational anomaly was derived [14] in close analogy to the triangular gauge anomaly by replacing the two vector currents by two symmetric tensor (gravitational) currents at the two vertices of the triangular fermion loop. However, it was only realised much later [15] that the existence of a "long wavelength" limit (as well as a "short wavelength" limit) justified taking it seriously at the electroweak level. The basic reason for requiring cancellation of this anomaly is to simultaneously maintain gauge invariance and general covariance of the theory. It is certainly true that in the Planckian short wavelength limit, where gravitational fluctuations are as significant as quantum ones, then the unknown effects of quantum gravity have to be considered. But if only quantum fluctuations at the electroweak level are considered then quantum gravity is unimportant in this long wavelength limit of a mixed gauge-gravitational anomaly. Thus, any lack of knowledge about quantum gravity should not alter the need for anomaly cancellation at the electroweak scale. It was shown in [15] that a necessary condition for this cancellation is:

$$\operatorname{Tr} Y = 0 \tag{1.33}$$

(in a notation where all fermions are left-handed) where again the trace is taken over all fermions which can circulate in the loop.

1.8.4 The Standard Model is Anomaly-Free

Remarkably the SM, with its seemingly haphazard collection of chiral fermion representations, is completely free of anomalies as is necessary if it is to make sense. Note that all anomalies cancel *within* each generation. Probably the strongest theoretical argument for the existence of the t quark is the need for the 3rd generation of quarks and leptons to satisfy the anomaly-free constraints. So perhaps these representations are not as random as they seem; exactly how "special" is the SM collection of representations?

1.9 Anomaly-Free Constraints & Uniqueness of the Weyl Representations of the SMG

In order to illustrate just how strong the requirements of anomaly-freedom are, the question of anomaly cancellation within the SM can be turned around. If the gauge group is taken to be:

$$G_{321} \equiv S(U(2) \times U(3)) \tag{1.34}$$

and if all anomalies have to be cancelled, then what sets of fermion representations are possible? This question was discussed in [7]. It was also investigated in [16] and further elucidated in [17], but with $G_{321} \equiv SU(3) \times SU(2) \times U(1)$. The set of IRs shown in Table 1.1 for one generation is derivable and unique providing:

1. all anomalies vanish;
2. all fermions are mass-protected (*i.e.* no Dirac or Majorana masses are possible without SSB) by G_{321};
3. and the principle of minimality is adopted so that only the most economical solution (*i.e.* the smallest number of Weyl fermions) is accepted, although any number of identical copies (generations) is allowed.

$SU(3)$	$SU(2)$	$U(1)$	t	d
3	**2**	Q_i (i=1,2,...,j)	1	1
3	**1**	Q'_i (i=1,2,...,k)	1	0
$\overline{\mathbf{3}}$	**1**	$\overline{Q_i}$ (i=1,2,...,l)	-1	0
$\overline{\mathbf{3}}$	**2**	$\overline{Q'_i}$ (i=1,2,...,m)	-1	1
1	**2**	q_i (i=1,2,...,n)	0	1
1	**1**	$\overline{q_i}$ (i=1,2,...,p)	0	0

Table 1.2: Possible Weyl fermion representations of the G_{321} group.

Note that since the gauge group is $S(U(2) \times U(3))$, the condition:

$$\frac{t}{3} + \frac{d}{2} + \frac{y}{6} \equiv 0 \pmod 1 \tag{1.35}$$

must be satisfied. This would not have been the case if the gauge group was taken to be $SU(3) \times SU(2) \times U(1)$ as in [16, 17].

The derivation begins by assuming an arbitrary number of each of the (left-handed) G_{321} representations shown in Table 1.2. Of course larger IRs are possible, but minimality dictates the consideration of these possibilities first.

The freedom from triangular gauge anomalies requires (from (1.31)):

$$\begin{aligned}
\mathrm{Tr}\,[SU(3)^3] &= \sum_{i=1}^{j} 2 + \sum_{i=1}^{k} 1 - \sum_{i=1}^{l} 1 - \sum_{i=1}^{m} 2 = 0 && (1.36)\\
\mathrm{Tr}\,[SU(3)^2U(1)] &= 2\sum_{i=1}^{j} Q_i + \sum_{i=1}^{k} Q'_i + \sum_{i=1}^{l} \overline{Q_i} + 2\sum_{i=1}^{m} \overline{Q'_i} = 0 && (1.37)\\
\mathrm{Tr}\,[SU(2)^2U(1)] &= 3\sum_{i=1}^{j} Q_i + 3\sum_{i=1}^{m} \overline{Q'_i} + \sum_{i=1}^{n} q_i = 0 && (1.38)\\
\mathrm{Tr}\,[U(1)^3] &= 6\sum_{i=1}^{j} Q_i^3 + 3\sum_{i=1}^{k} Q_i'^3 + 3\sum_{i=1}^{l} \overline{Q_i}^3 + 6\sum_{i=1}^{m} \overline{Q'_i}^3 \\
&\quad + 2\sum_{i=1}^{n} q_i^3 + \sum_{i=1}^{p} \overline{q_i}^3 = 0 && (1.39)
\end{aligned}$$

The global $SU(2)$ anomaly-free condition is:

$$3j + 3m + n = 2N \tag{1.40}$$

for some integer N. Finally, the mixed anomaly-free condition is (from (1.33) with the help of (1.37)):

$$\mathrm{Tr}\, Y = 2\sum_{i=1}^{n} q_i + \sum_{i=1}^{p} \overline{q_i} = 0 \tag{1.41}$$

A reasonable first attempt at a minimal solution, looking at (1.40), is to take $j = m = 0$ and $n = 2$, and this leads to 10 Weyl fermions in the IRs:

$$(\mathbf{1},\mathbf{2})_q \quad (\mathbf{1},\mathbf{2})_{-q} \quad (\mathbf{3},\mathbf{1})_Q \quad (\overline{\mathbf{3}},\mathbf{1})_{-Q} \tag{1.42}$$

but then the first pair and the last pair can combine to generate Dirac fermion masses without SSB.

Going on to consider $j = n = 1$, minimality suggests $m = 0$ and so $l = 2$ and $k = p = 0$. This yields a solution with 14 Weyl fermions in the G_{321} representations:

$$(\mathbf{3},\mathbf{2})_0 \quad (\overline{\mathbf{3}},\mathbf{1})_{\overline{Q}} \quad (\overline{\mathbf{3}},\mathbf{1})_{-\overline{Q}} \quad (\mathbf{1},\mathbf{2})_0 \tag{1.43}$$

The problem here is that the IRs $(\mathbf{3},\mathbf{2})_0$ and $(\mathbf{1},\mathbf{2})_0$ do not satisfy (1.35). [Since [16, 17] do not use (1.35), they must eliminate this solution on different grounds. They discard it because: the $(\mathbf{1},\mathbf{2})_0$ doublet cannot acquire a Dirac mass term even after the SSB of $SU(3) \times SU(2) \times U(1)$; and because it "trivializes" the mixed anomaly-free condition (1.41). Neither of these reasons is completely satisfactory: in the SM set of IRs, the left-handed neutrino does not acquire a mass-term even after SSB, yet this solution is not discarded; and zero charges are as good a way to cancel anomalies as any other. We therefore prefer the argument presented in this section].

So, still accepting minimality, the next step is to consider $p = 1$ *i.e.* to add the state $(\mathbf{1},\mathbf{1})_{\overline{q}}$. The surviving anomaly-free constraints (corresponding to $j = 1$, $k = 0$, $l = 2$, $m = 0$, $n = 1$, $p = 1$) are:

$$\begin{aligned} 2Q + \overline{Q_1} + \overline{Q_2} &= 0 \\ 3Q + q &= 0 \\ 6Q^3 + 3\overline{Q_1}^3 + 3\overline{Q_2}^3 + 2q^3 + \overline{q}^3 &= 0 \\ 2q + \overline{q} &= 0 \end{aligned} \tag{1.44}$$

These equations possess two solutions:

1. the SM solution

$$(Q, \overline{Q_1}, \overline{Q_2}, q, \overline{q}) = (1, -4, 2, -3, 6) \times \text{constant} \tag{1.45}$$

2. and the so-called "bizarre" solution

$$(Q, \overline{Q_1}, \overline{Q_2}, q, \overline{q}) = (0, 1, -1, 0, 0) \times \text{constant} \tag{1.46}$$

The bizarre solution is discarded on the grounds that the state $(\mathbf{1}, \mathbf{1})_0$ can acquire a Majorana mass by combining with itself, and that the IRs $(\mathbf{3}, \mathbf{2})_0$ and $(\mathbf{1}, \mathbf{2})_0$ again do not satisfy (1.35).

Thus, the application of all anomaly-free conditions, the requirement of mass-protection and the concept of minimality yield a unique set of Weyl fermion representations that coincides with that of the observed quarks and leptons of one generation. Obviously no light is shed on the number of generations, but anomaly cancellation is nevertheless seen to be a very strong constraint on model-building. Much use will be made of this constraint. We might also say that this exercise illustrates Nature's preference for minimal solutions to the no-anomaly equations.

1.10 Standard Model Group Extensions & Fermion Masses

As has already been stated, the SM is generally viewed as an incomplete theory and this has led over the last 20 years to the investment of much effort in extending the SMG in order to remove some of its shortcomings. Some relevant extensions, and how they deal with fermion masses, are briefly discussed here.

1.10.1 Grand Unification

The classic Grand Unified Theory (GUT) postulates the existence of a simple group G_{GUT} of which the SMG is necessarily a subgroup. The Lagrangian is taken to be G_{GUT}-symmetric, and G_{GUT} is spontaneously broken to the SMG:

$$G_{\mathrm{GUT}} \xrightarrow{M_X} SMG \tag{1.47}$$

at some very high energy scale M_X, typically around 10^{15} GeV. The rationale of a GUT is that there exists in Nature only one force (gravity is, as usual, discounted) and the apparent existence of three forces is a low energy artifact of how G_{GUT} is broken. Popular GUTs include $SU(5)$ [18], $SO(10)$ [19] and the Pati-Salam partial GUT [20] where lepton number is treated as a fourth colour.

In a typical GUT, some IRs of the SMG are grouped together to form IRs of the unified group G_{GUT}. For example, in the standard $SU(5)$ model the SM fermions are grouped into the SU(5) IRs $\overline{\mathbf{5}}$ and $\mathbf{10}$:

$$\overline{\mathbf{5}} \ : \ \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu_e \end{pmatrix}_L$$

$$\mathbf{10} \ : \ \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & e^+ & 0 \end{pmatrix}_L \tag{1.48}$$

where a colour subscript has been introduced for the quarks. This structure is repeated for each of the 3 generations. Sometimes, "new" fermion states are postulated in order to furnish complete IRs of G_{GUT} in conjunction with the SM Weyl fermions. For example, a right-handed neutrino is added to the 15 SM Weyl states in the $SO(10)$

model in order to complete the $SO(10)$ IR **16**:

$$\mathbf{16}: \begin{pmatrix} \nu_e \\ u_1 \\ u_2 \\ u_3 \\ e^- \\ d_1 \\ d_2 \\ d_3 \\ -d_3^c \\ -d_2^c \\ d_1^c \\ -e^+ \\ u_3^c \\ -u_2^c \\ -u_1^c \\ \nu_e^c \end{pmatrix}_L \tag{1.49}$$

and again this structure is repeated for each generation. Note that the IRs of GUTs consist of non-isomorphic IRs of the SMG: isomorphic IRs such as $(u \;\; d)_L$ and $(c \;\; s)_L$, or e_R and μ_R, do not appear in the same IR of G_{GUT}.

Nowadays, supersymmetric (SUSY) GUTs [21] are the focus of much attention as the classic GUT predictions (for example, $\sin^2\theta_W$ and the intersection of the three SM gauge couplings in the course of their renormalisation group flow) need SUSY in order to remain compatible with experimental data of ever-increasing accuracy [22].

Unfortunately, the general GUT treatment of fermion masses can only be described as unsatisfactory. Although there are often some seductive predictions such as:

$$m_b = m_\tau \quad (\text{at } M_X) \tag{1.50}$$

(as well as some bad ones), generally speaking the mass hierarchy is merely incorporated, not explained. This is done by adopting some mass matrix ansatz, usually of a generalised Fritzsch [23] or Georgi-Jarlskog [24] stucture (see *e.g.* [25]), which essentially has the hierarchy built in via hugely differing fundamental Yukawa couplings.

Often these ansätze invoke ad-hoc discrete symmetries in order to eliminate several mass matrix elements, another unsatisfactory feature of their whole treatment of the masses.

1.10.2 Horizontal Symmetry

An alternative SMG extension, constructed with fermion masses and mixings as more of a central theme than in GUTs, is the perhaps less familiar horizontal symmetry group. Horizontal symmetries studied have included $U(1)_H$ [26], $SU(2)_H$ [27] and $SU(3)_H$ [28]. Whereas a GUT symmetry can be said to act "vertically" *i.e. within* a generation, a horizontal symmetry G_H acts *between* the generations.

$$G_{\mathrm{GUT}} \left\{ \underbrace{\begin{array}{ccc} \begin{pmatrix} u \\ d \end{pmatrix}_L & \begin{pmatrix} c \\ s \end{pmatrix}_L & \begin{pmatrix} t \\ b \end{pmatrix}_L \\ u_R & c_R & t_R \\ d_R & s_R & b_R \\ \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L & \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \\ e_R & \mu_R & \tau_R \end{array}}_{G_H} \right. \tag{1.51}$$

Isomorphic IRs of the SMG (*i.e.* corresponding IRs from different generations) are collected to form IRs of G_H. For any particular G_H, this can be done in a large number of ways (the possibilities for $SU(2)_H$, for example, are catalogued in [29]).

Typically, the requirement that the Lagrangian should respect G_H constrains the Yukawa sector of the theory and allows (for example) mixing angles to be expressed in terms of fermion masses (see *e.g.* [27]). Indeed, some very basic features of the mass and mixing hierarchies (the heaviness of the 3rd generation compared to the other two, the smallness of the mixing angles) might be explained by a horizontal symmetry [29], but generally the situation is still unsatisfactory. Unnatural features such as a hierarchy in fundamental Yukawa couplings, or a finely-tuned cancellation between two VEVs (see *e.g.* [27]), are still relied upon to provide the observed fermion masses.

It is convenient to note here that the term "horizontal" symmetry will hereafter only be used to describe non-abelian symmetries. An abelian symmetry has only 1d representations and so there does not exist a generator which connects Weyl states from different generations. This will be of crucial importance in what follows. Instead, any abelian symmetry different from the $U(1)_Y$ of the SMG will be called a "flavour" symmetry.

1.11 Abelian Flavour Symmetry

As this type of symmetry is being distinguished from horizontal symmetry and as it will be of great importance in this thesis, it is useful to give some further discussion, especially regarding its relevance to the fermion mass problem.

Extending the SMG to $SMG \times U(1)_f$ is perhaps the simplest SMG extension imaginable. One extra generator is added, whose action on the SM states must be defined. Then the $U(1)_f$ must be chosen to be gauged (in which case anomaly cancellation must be addressed) or global. Finally, the $U(1)_f$ must be spontaneously broken; if it is an exact symmetry then it must be trivial as far as the quarks are concerned (*i.e.* all quark states must have identical charge) otherwise there exist vanishing mixing angles [30].

One of the first examples of $SMG \times U(1)_f$ was seen in [26]. There, the $U(1)_f$ was gauged and acted non-trivially on only two generations. The anomalies were largely cancelled by choosing the flavour charges Q_f of corresponding states in these two generations to be equal and opposite in sign. The work reproduced successful phenomenological relations such as:

$$\theta_C \simeq \sqrt{\frac{m_d}{m_s}} \tag{1.52}$$

where θ_C is the Cabibbo angle, but the flavour symmetry was not used to account for the fermion mass hierarchy; in particular, it was not taken to be partially conserved.

Much closer to the spirit of this thesis was [31]. Here the $U(1)_f$ of an $SMG \times U(1)_f$ model was assumed to be partially conserved and the suppression of fermion masses assumed to be a result of the weakly broken chiral charges Q_f. No precise charges were specified; instead the charges of left- and right-handed states were assumed to be positive and negative respectively, and some average distribution of charge differences was assumed. It was noted that large quark mass ratios and relations similar to (1.52) emerge naturally from such a scenario, but no detailed numerical results were given.

This was then developed in [32] which considered the SMG extension:

$$SMG \times \underbrace{U(1)_f \times U(1)'_f \times \dots}_{\text{N flavour symmetries}} \tag{1.53}$$

with N some arbitrary integer. Again the flavour symmetries were taken to be approximately conserved in the hope of accounting for the mass and mixing hierarchies. Flavour charges were randomly distributed and numerical results showed that the generation mass gaps could be satisfactorily accounted for, but not the observed structure *within* the generations. In both [31] and [32] it was suggested that the flavour symmetries are gauged, but no effort was made to ensure that the models were explicitly anomaly-free among the SM particles alone: rather it was assumed that there was a cancellation with unspecified very heavy fermions, vector-like under the SMG.

A slightly different approach to fermion masses was taken in the $SO(10) \times U(1)_f$ model of [33]. With very specific superheavy fermion and $U(1)_f$-breaking scalar sectors, fermion mass matrices with the Fritzsch texture arose naturally. In common with [31, 32], the $U(1)_f$ charge differences indicated the degree of suppression of mass matrix elements, but the predicted top masses are, in retrospect, too low. A more systematic approach to SUSY $SO(10) \times U(1)_f$ models (with a global flavour symmetry) was taken in [34], with good results for some models.

Closer to the approach of [31, 32] was the $SMG \times U(1)_f$ model of [35]. Again, the partially conserved $U(1)_f$ was supposed to account for the broad features of the mass and mixing hierarchies. Flavour charges were explicitly assigned to the Weyl fermions and, while moderately successful results were obtained for freely chosen charges, no promising anomaly-free charge set was found *i.e.* the $U(1)_f$ could not be taken to be a gauge symmetry.

Recently [30], the SMG extension:

$$SMG \times Z_3 \times Z_5 \subset SMG \times U(1)_f \times U(1)'_f \tag{1.54}$$

has been analysed in a SUSY context, with the gauged flavour symmetries partially conserved. The results for fermion masses and mixings are fairly successful, but the situation regarding cancellation of the anomalies involving the flavour symmetries is unclear. Similar models have been considered in [36], but without SUSY. The results are again fairly good, but the origin of the discrete symmetries is not specified and so anomalies are not addressed.

This thesis will take a much wider view of extending the SM to include PCCSs than any of these works. Particularly relevant will be the idea of anti-grand unification.

1.12 Anti-Grand Unification

Central to grand unification is the existence of a single gauge coupling constant; "anti-grand unification", as the name suggests, is the anithesis of this. The purest example is the gauge group [7, 37]:

$$G_{\mathrm{anti}} = SMG^3 \equiv SMG_1 \times SMG_2 \times SMG_3 \tag{1.55}$$

where SMG_i (i=1,2,3) behaves just like the SMG as far as the i^{th} generation is concerned, but acts trivially on the other 2 generations *i.e.* G_{anti} has three times as many generators as the SMG, each fermion generation effectively having its "own" SMG. This fundamental Planck scale ($M_P \equiv 1.22 \times 10^{19}$ GeV) group has 9 gauge coupling constants. It is assumed that all these couplings approach a multicritical point in the corresponding lattice gauge theory, where G_{anti} spontaneously breaks down just below M_P to its diagonal subgroup which is to be identified with the usual low energy SMG:

$$G_{\mathrm{anti}} \rightarrow G_{\mathrm{diag}} \equiv SMG \tag{1.56}$$

Consequently, the running fine structure constants of the SM, $\alpha_i(\mu)$ (i=1,2,3), are predicted to take on values:

$$\alpha_i(M_P) = \frac{\alpha_i^{\mathrm{crit}}}{N_{\mathrm{gen}}} = \frac{\alpha_i^{\mathrm{crit}}}{3} \tag{1.57}$$

where the critical couplings α_i^{crit} have been estimated using lattice gauge theory Monte Carlo results. Good agreement of (1.57) with the experimental couplings (extrapolated to M_P using the SM renormalisation group equations) is obtained for the two non-abelian groups $SU(2)_L$ and $SU(3)_c$ [38]; but there are as yet unresolved problems in estimating α_1^{crit} for the group $U(1)_Y$ due to the infinite number of invariant subgroups and corresponding phases expected for the U(1) groups in (1.55).

This anti-grand unified model provides much of the motivation underlying the models analysed in this thesis. It has been suggested [7] that the broken chiral gauge quantum numbers of the quarks and leptons under the symmetry groups SMG_i (i=1,2,3) could be responsible for the fermion mass hierarchy. Obviously, all of the SM fermions are mass-protected by SMG^3 in much the same way as they are by the SMG. If in the process of the breaking described by (1.56) some symmetries are only weakly broken, then a hierarchy in the fermion masses would appear in a most natural manner. Indeed, the bulk of this thesis is taken up with analysis of models like SMG^3 within which the SMG is embedded as a diagonal subgroup.

"He laughed at accidental sirens
That broke the evening gloom
The police had warned of repercussions
They followed none too soon
A trickle of strangers were all that were left alive"

David Bowie
Panic in Detroit

Chapter 2

SMG Extensions, Fermion Masses & the SMG as a Diagonal Subgroup

2.1 Introduction

Starting from the premise that partially conserved chiral symmetries are responsible for the broad features of the fermion mass and quark mixing hierarchies, this chapter sets out to determine which Standard Model Group extensions can most naturally produce these hierarchies. A classification of SMG extensions based on their Weyl fermion IRs is suggested and it is argued that one particular class offers most promise: that class whose members' IRs are identical to the IRs of the SMG. The embedding of the SMG as a diagonal subgroup within members of this class is then discussed, followed by an argument for gauging these members.

Some notational comments are necessary. The subscripts "c", "L" and "Y" on the SMG components are now dropped. A component with no subscript is understood to be exactly as in the Standard Model (*e.g.* $SU(2)$ is understood to be $SU(2)_L$). Other subscripts of Chapter 1 are maintained so that, for example, $SU(2)_H$ may be distinguished from $SU(2)$ and $U(1)_f$ from $U(1)$.

2.2 Rationale for Mass Suppression

The basic philosophy underlying the whole of this thesis is: all order of magnitude features of the fermion mass and quark mixing hierarchies should be accounted for by approximately conserved chiral symmetries. The appearance of such a striking range of masses and mixing angles should be potentially explainable from the dynamics of any particular model; in particular, *no* order of magnitude feature should result merely from a hierarchy in fundamental Yukawa couplings. Indeed, we shall take all such couplings to be $\mathcal{O}(1)$ (we shall return to this point later) so that they are responsible only for fine structure.

The goal, then, is to explain the broad hierarchy features and pay little attention to the fine details. This is not as modest as it perhaps first sounds. The fact is that there does not presently exist a model in which these gross features emerge in a completely natural manner *i.e.* without the insertion by hand of small parameters into the Lagrangian. Given this lack of understanding, it is only reasonable to ignore questions such as:

- Why does $m_e = 0.5$ MeV instead of 1.0 MeV?
- Why does $m_\mu = 105$ MeV instead of 50 MeV?

and focus instead on the much more basic questions:

- Why does $m_e/m_\mu \simeq \mathcal{O}(200)$?
- Why are they both so much lighter than $\langle\phi\rangle_{\mathrm{ws}}$?

Once these have been answered, we might then worry about how the fundamental couplings provide the necessary factors of 2 and 3 *etc* to precisely pin down the fermion masses. Perhaps Nature's fundamental couplings can never be calculated and it would then not be possible to calculate the quark or lepton mass ratios. Nevertheless, it should still be possible to understand why these ratios are small numbers.

2.3 *SMG* Extensions

We seek the *SMG* extensions most naturally suggested by the known fermion masses and quark mixing angles, and we assume that there exist no mass-protected fermions (other than those of the three generation SM) which provide a net non-zero contribution to any chiral gauge anomalies. In the case of "light" fermions *i.e.* those mass-protected by the SMG itself (so that their natural mass scale is $\langle\phi\rangle_{\text{ws}}$) this has a very good chance of being true due to the familiar LEP result that their are only 3 light neutrinos, and the fact that if "new" light fermions existed we might hope to have seen at least indirect signs of them. But even for fermions which are vector-like under the SMG and are only mass-protected by a symmetry G broken at some high energy scale $M \gg \langle\phi\rangle_{\text{ws}}$ (so that the fermions would naturally have a mass of order M and be effectively unobservable at presently obtainable energies), this is still a desirable assumption. We do not wish to become embroiled in the open-ended process of postulating what the fermion spectrum might look like at the GUT or Planck scales. Moreover, we do not wish our models to depend critically on specific sets of heavy fermions with specific quantum numbers. The thrust of our assumption is that any anomalies in a model must be cancelled amongst the SM fermions *alone*; there will be no fudging of this issue. It should perhaps be stated here that we make no *a priori* assumption that any extended symmetries must necessarily be gauged; this will emerge partly as a conclusion of our chain of argument.

So, we assume that only the 45 Weyl fermions of the SM exist in the low energy regime. Furthermore, we are only interested in that part of any extension which acts non-trivially on at least one of these 45 states (we now call this non-trivial group G) *i.e.* letting $\hat{G}$ denote the subgroup of the true fundamental gauge group which transforms these 45 states amongst themselves, we are only interested in:

$$G = \hat{G}/L \tag{2.1}$$

where L is the invariant subgroup of $\hat{G}$ consisting of those elements which act trivially on the known 45 Weyl fields. The elements of G can thus be distinguished by their action on the 45 states, and so G is naturally identified with a subgroup of $U(45)$, the unitary group in whose fundamental representation would sit all the known Weyl fermions. That is:

$$SMG \subseteq G \subseteq U(45) \tag{2.2}$$

and there essentially exists only a finite number of algebras corresponding to such G.

However, this will be an extremely large number and some classification of the many G obeying (2.2) would obviously be useful. Clearly, the 45 Weyl states fall into IRs of

G each of which must include a number of the IRs of the SMG. It is thus possible to classify the extensions G according to the manner in which the IRs of the SMG fit into the IRs of G. (Note that if G is fully gauged, then many groups obeying (2.2) would lead to gauge or mixed anomalies, or to the global anomaly, and so would not be acceptable as genuine models). We introduce our classification of groups G by first classifying collections of IRs of the SMG, which (as already outlined in Chapter 1) has 5 different IRs of type $(u\ d)_L$, u_R, d_R, $(\nu_e\ e)_L$ and e_R each occuring 3 times, once for each generation (we reserve the right to flip freely between u_R and u_L^c *etc* where:

$$\psi_L^c \equiv (\psi_L)^c = (\psi^c)_R \tag{2.3}$$

and:

$$\psi^c \equiv C\overline{\psi}^T \tag{2.4}$$

where C is the charge conjugation operator). The different types of collection are:

1. A collection consisting only of *isomorphic* IRs of the SMG (this might be called a "horizontal" collection) *e.g.* $\{u_R, c_R, t_R\}$ or $\{(u\ d)_L, (t\ b)_L\}$. It is this type of collection which forms the IRs of horizontal symmetry groups.
2. A collection consisting only of *non-isomorphic* IRs of the SMG (this might be called a "vertical" collection) *e.g.* $\{d_L^c, (\nu_e\ e)_L\}$ or $\{(u\ d)_L, u_L^c, e_L^c\}$. It is this type of collection which forms the IRs of GUTs.
3. A collection consisting of *both* isomorphic and non-isomorphic IRs of the SMG (this might be called a "mixed" collection) *e.g.* $\{d_L^c, (\nu_e\ e)_L, s_L^c, (\nu_\mu\ \mu)_L\}$.

We may now classify the groups G according to whether its IRs number among them collections of these 3 types (see Table 2.1) where, in our notation, numerical subscripts on a group component indicate which generations are transformed non-trivially by an element of that group component. For example, $SU_1(5)$ acts non-trivially (in the usual manner) only on the 1st generation while $SU_{23}(3)$ acts non-trivially (in the usual manner) only on the 2nd and 3rd generations. So a group like $SU_1(5)\times SU_2(5)\times SU_3(5)$ of Category (6) has all the fermions grouped into the usual $\overline{\mathbf{5}}$ and $\mathbf{10}$ $SU(5)$ representations but of *different* $SU(5)$ components *e.g.* $(d_L^c\ e_L\ \ -\nu_{eL})$ would transform like $(\overline{\mathbf{5}}, \mathbf{1}, \mathbf{1})$ whereas $(s_L^c\ \mu_L\ -\nu_{\mu L})$ would transform like $(\mathbf{1}, \overline{\mathbf{5}}, \mathbf{1})$ *etc* . A more complicated

Type of Collection:			Anomaly-Free Examples (involving only 3 generations where possible)	
Horiz-ontal	Vert-ical	Mixed		
$\surd$	$\surd$	$\surd$	(1) *e.g.*	Need ≥ 5 generations $SU_1(5) \times \{SMG \times SU(2)_H\}_{23} \times \{SU(5) \times SU(2)_H\}_{45}$
		$\times$	(2)	$SU_1(5) \times \{SMG \times SU(2)_H\}_{23}$
	$\times$	$\surd$	(3) *e.g.*	Need ≥ 4 generations $\{SMG \times SU(2)_H\}_{12} \times \{SU(5) \times SU(2)_H\}_{34}$
		$\times$	(4)	$SMG \times SU(2)_H$ $SU(3) \times SU(2) \times U_1(1) \times U_2(1) \times U_3(1) \times SU(2)_H$
$\times$	$\surd$	$\surd$	(5)	$SU_1(5) \times \{SU(5) \times SU(2)_H\}_{23}$
		$\times$	(6)	$SU(5)$ $SU_1(5) \times SU_2(5) \times SU_3(5)$
	$\times$	$\surd$	(7)	$SU(5) \times SU(2)_H$ $SMG_1 \times \{SU(5) \times SU(2)_H\}_{23}$
		$\times$	(8)	SMG $SMG_1 \times SMG_2 \times SMG_3$

Table 2.1: Classification of SMG extensions according to the manner in which IRs of the SMG fit into IRs of G. Subscript "H" means "horizontal" while subscript "ij" means that the relevant group component acts only on generations "i" and "j". Some examples of each category are given, and these are gauge and mixed anomaly free so that they may be considered valid *gauge* groups. Note that abelian flavour symmetries may be added on to any group, although the corresponding charges are constrained if anomalies have to be cancelled.

example is $SU_1(5) \times SMG_{23} \times SU_{23}(2)_H$ which appears in Category (2) and has: the 1st generation states grouped into the usual $SU(5)$ IRs $\overline{\mathbf{5}}$ and $\mathbf{10}$ and transformed only by the $SU_1(5)$ component; and the 2nd and 3rd generation states behaving exactly as in the SM under SMG_{23} but grouped horizontally under $SU_{23}(2)_H$ (in one of the scenarios catalogued in [29]).

Table 2.1 displays all combinations of the 3 different types of collection and gives one or two anomaly-free examples in each case. Note that any group in any of these categories may be extended by the addition of abelian flavour symmetries. Such symmetries are not restricted to groups whose IRs constitute horizontal collections, and this is why we made the distinction between horizontal symmetry and abelian flavour symmetry in Chapter 1. The usual GUT models which have no "extra" fermions appear in Category (6) - this really only means $SU(5)$, but this can be distorted to *e.g.* $SU_1(5) \times SU_2(5) \times SU_3(5)$. The horizontal symmetry models (*e.g.* $SMG \times SU(2)_H$, $SMG \times SU(3)_H$) appear in Category (4), but again distortions are possible: the SMG part might be widened to any larger subgroup of SMG^3, or the horizontal part might apply to only two of the three generations. Most of the other categories involve some combination of the different types of collection (and some even require more than three generations of fermions in order that no group components act trivially). The exception is the category our pedagogical arguments will shortly lead us to favour: Category (8).

Several of the non-trivial extensions in Category (8) display most clearly a feature which is prominent throughout the table *viz.* that different subgroups of the extension transform different generations with the corresponding part of the SMG as the "diagonal" subgroup: the subgroup whose elements correspond to identical transformations on each generation. (The "purest" such extension, the anti-grand unified gauge group SMG^3 seen here in Category (8), has been mentioned already in Chapter 1). The diagonal subgroup of a cross-product $H^3 \equiv H \times H \times H$ is defined to be:

$$H_{\text{diag}} \equiv \{(h,h,h) : h \in H\} \tag{2.5}$$

A diagonal subgroup of isomorphic non-abelian factors is thus clearly defined in terms of its generators: if, for example, $SU_1(3) \times SU_2(3) \times SU_3(3)$ is generated by $\lambda_a^{(i)}$, (i=1,2,3, a=1,...,8), then the diagonal subgroup $SU(3)_{\text{diag}}$ is generated by $\lambda_a^{(1)} + \lambda_a^{(2)} + \lambda_a^{(3)}$ (a=1,...,8). (As will be seen shortly, this unambiguous construction collapses when we consider abelian factors). It is possible to construct a generalised diagonal subgroup for cases where the non-abelian cross-product factors are not actually isomorphic, but contains only subgroups which are isomorphic. For example, Category (6) contains the group $SU_1(5) \times SMG_{23}$; the SMG is embedded in this as a generalised diagonal subgroup in the sense that it appears as:

$$SMG = \{(T(h), h) : h \in SMG\} \tag{2.6}$$

where $T(h)$ is the element of $SU(5)$ corresponding to the element h of the SMG when the SMG is embedded in $SU(5)$ in the standard Georgi-Glashow manner. If $SU_1(5)$ is generated by $L_a^{(1)}$ (a=1,...,24) and SMG_{23} by $\lambda_b^{(23)}$ (b=1,...,12), then the SMG is generated by $L_a^{(1)}+\lambda_a^{(23)}$ (a=1,...,12) where the $L_a^{(i)}$ (a=1,...,12) generate the subgroup SMG_1 of $SU_1(5)$.

We emphasise that the defining characteristic of the groups in Category (8) is not that they have the SMG embedded within them as some kind of diagonal subgroup, but is that their IRs are coincident with those of the SMG.

2.4 Which Categories Are Favoured?

The prominence of the diagonal subgroup idea throughout the table renders slightly curious the fact that such types of extension are the least popular in the literature. Obviously, the goal of GUTs to obtain a simple group G completely precludes consideration of these models, but even attempts to extend the SMG in an effort to address the fermion mass or generation problems have largely ignored them too, concentrating instead on the horizontal models of Category (4).

This relative lack of interest is even more surprising since we may intuitively argue that Nature in some sense favours Category (8) models. When examining the SM in an attempt to glean clues regarding its extension, the most remarkable feature is arguably not the near-coincidence of the running gauge couplings at some high energy scale [22] but is the presence of three generations of fermions with large mass gaps between them. The two outstanding features of this hierarchy are:

- all fermions except the top quark are light compared to the electroweak scale $\langle\phi\rangle_{\text{WS}}$;

- the average mass of each generation gets successively larger, by a factor of order 60.

No pair of corresponding particles from different generations is degenerate, even order of magnitude wise, and consequently it is reasonable to expect that Nature has chosen a model in which the chance of such degeneracy is very low.

This argument would seem to disfavour the use of non-abelian horizontal symmetry groups to account for the generation gaps. For example, the positioning of corresponding pairs of states such as $\{e_R, \mu_R\}$ into doublets of $SU(2)_H$ (in the $SMG \times SU(2)_H$ model), or even $\{u_R, c_R, t_R\}$ into triplets of $SU(2)_H$, creates the obvious risk of these states forming degenerate particles. Given that without more than three generations the horizontal group must have IRs which are triplets, doublets or singlets, it is extremely difficult to avoid degeneracies among the particles formed by the states in these IRs (without appealing to a hierarchy in the fundamental Yukawa couplings or a finely-tuned cancellation between scalar VEVs - see [29] for example). Nevertheless, further examination of such models is required. To avoid inviting these degeneracies the extended group G should thus have no IRs which constitute horizontal or mixed collections of IRs of the SMG. This argument weighs heavily against the extensions of Categories (1)–(5) and (7), leaving only the GUTs and anti-unified groups.

Really, in order for G to account for the large generation gaps, there must be some difference in the quantum numbers of particles in different generations (unlike the standard $SU(5)$ GUT, for example, where the generations are simply copies of each other as far as their symmetry properties are concerned) but without grouping IRs of the SMG horizontally *i.e.* corresponding particles from different generations should belong to inequivalent IRs of G. The crudest and most obvious way of achieving this is to extend the SMG to $SMG \times U(1)_f$ and arrange the new charges Q_f to vary from generation to generation. For example, a set of charges satisfying:

$$|Q_f(u_L) - Q_f(u_R)| > |Q_f(c_L) - Q_f(c_R)| > |Q_f(t_L) - Q_f(t_R)| \tag{2.7}$$

might naturally account for:

$$m_u \ll m_c \ll m_t \tag{2.8}$$

if the $U(1)_f$ was partially conserved. A more subtle method of achieving the same goal would be to extend the SU(2) sector of the SMG to $SU_1(2) \times SU_2(2) \times SU_3(2)$ so that

$(u\ d)_L$ transforms as $(\mathbf{2},\mathbf{1},\mathbf{1})$, $(c\ s)_L$ as $(\mathbf{1},\mathbf{2},\mathbf{1})$ and $(t\ b)_L$ as $(\mathbf{1},\mathbf{1},\mathbf{2})$ while u_R, c_R and t_R would transform as $(\mathbf{1},\mathbf{1},\mathbf{1})$. Then taking $SU_i(2)$ to be better conserved with decreasing i (so that the mass matrix element connecting u_L and u_R is more strongly suppressed than that connecting c_L and c_R *etc*) again might naturally produce (2.8).

This argument is enough to rule out the standard $SU(5)$ model where the large generation gaps are obtained via the fundamental Yukawa couplings, just as in the SM; this is exactly what we have set out to avoid. But it is not sufficient to dismiss all Category (6) models. For example, the group $SU_1(5)\times SU_2(5)\times SU_3(5)$ might naturally account for the generation gaps in much the same way as the $SU_1(2)\times SU_2(2)\times SU_3(2)$ mentioned above. Even the consequent standard $SU(5)$ predictions:

$$m_b = m_\tau; \qquad m_s = m_\mu; \qquad m_d = m_e \tag{2.9}$$

are perhaps acceptable order of magnitude relations. (They are far from being numerically exact - the Georgi-Jarlskog relations [24], obtained by complicating the scalar sector, are better. However, even the long-standing "success" $m_b = m_\tau$ is no longer thought to be numerically exact [39] without SUSY).

However, if we also consider the quark mass gaps *within* generations as "big" (*i.e.* not naturally obtainable via fundamental Yukawa couplings) then we should beware of collecting more than single IRs of the SMG to form IRs of G. That is, we should be very cautious about using the vertical collections which feature in *all* Category (6) models. Even in the $SU_1(5)\times SU_2(5)\times SU_3(5)$ model, the suppression of m_b *w.r.t.* m_t and of m_s *w.r.t.* m_c is obtained via the Yukawa couplings, just as in the usual $SU(5)$ model. But the predictions:

$$m_s \simeq m_c; \qquad m_b \simeq m_t \tag{2.10}$$

obtained if (following our philosophy) the fundamental Yukawa couplings are all of the same order of magnitude, might be avoided by use of an abelian flavour symmetry. Category (6) models extended by abelian flavour symmetries are therefore not ruled out *a priori*, although in [35] no suitable $SU(5)\times U(1)_f$ model was found and [40] rules out models such as $SU(5)\times U(1)_f\times U(1)'_f\times\ldots$, and $SU_1(5)\times SU_2(5)\times SU_3(5)\times U(1)_f$. So Category (6) models are not studied further in this thesis.

We are thus left in the position of having only Category (8) models "favoured" by Nature, at least as far as generating the fermion mass hierarchy is concerned.

2.5 Embedding the SMG and Gauging G

In this section we look more closely at the groups of Category (8), demonstrating that the SMG is embedded within each of them as a diagonal subgroup (despite a threat posed by the left-handed quarks) and arguing that their non-abelian parts should consequently be gauged.

Consider, then, one IR Ψ of a Category (8) group, whose IRs are no bigger than those already present in the SM. Leaving aside for the moment the left-handed quarks, we find that all possible unitary transformations of the Weyl components forming Ψ are in fact "already" performed by some SMG gauge transformation. For example, a right-handed quark with electric charge $+2/3$ (*e.g.* $\Psi = u_R$) is a 3-dimensional IR and the set of all unitary transformations of Ψ consequently forms a $U(3)$ group. The $SU(3)$ subgroup of these transformations can *as far as the transformation of* Ψ *is concerned* be identified with the familiar colour $SU(3)$ transformations, and the $U(1)$ subgroup (the overall phase) can similarly be identified with the weak hypercharge phase rotation. If we next look at $\Psi' = c_R$ (say), we can obviously make the same claim but Ψ and Ψ' could have their $SU(3)$ (or $U(1)$) transformations performed independently. Then the overall transformations of $\Psi + \Psi'$ would form the group $SU^{u_R}(3) \times SU^{c_R}(3)$ (in this suggestive notation the superscript indexes which representation is non-trivially transformed by the group component in question). So, taking account of all Ψ except the left-handed quarks, the transformations forming a Category (8) group can be described by some subgroup of a group G which looks like:

$$G = \prod_i SU^i(3) \times \prod_j SU^j(2) \times \prod_k U^k(1) \tag{2.11}$$

where i runs over right-handed quarks, j over left-handed leptons and k over right-handed quarks and all leptons. Each component acts just like its SM counterpart, but on one IR only.

Note that this group is considerably more fragmented than the examples given in Table 2.1 (*e.g.* $SMG_1 \times SMG_2 \times SMG_3$) where each *generation* had its own group component - here, each *irreducible representation* has its own group component.

The case of Ψ being a 6-dimensional IR of left-handed quarks, however, suggests the possibility of unitary transformations (*viz.* $U(6)$ or $SU(6)$) which are *not* identifiable with SM gauge transformations (restricted to Ψ). For example, for the IR $(u\ d)_L$ the

$SU(3) \times SU(2)$ gauge transformations of the SM can (schematically):

- change the colour of the u and d quarks simultaneously *e.g.*

$$\begin{pmatrix} \begin{pmatrix} u^r \\ u^b \\ u^g \end{pmatrix}_L \\ \begin{pmatrix} d^r \\ d^b \\ d^g \end{pmatrix}_L \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{pmatrix} u^g \\ u^r \\ u^b \end{pmatrix}_L \\ \begin{pmatrix} d^g \\ d^r \\ d^b \end{pmatrix}_L \end{pmatrix} \tag{2.12}$$

- change the weak isospin of all colours simultaneously *e.g.*

$$\begin{pmatrix} \begin{pmatrix} u^r \\ u^b \\ u^g \end{pmatrix}_L \\ \begin{pmatrix} d^r \\ d^b \\ d^g \end{pmatrix}_L \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{pmatrix} d^r \\ d^b \\ d^g \end{pmatrix}_L \\ \begin{pmatrix} u^r \\ u^b \\ u^g \end{pmatrix}_L \end{pmatrix} \tag{2.13}$$

- or some combination of these *e.g.*

$$\begin{pmatrix} \begin{pmatrix} u^r \\ u^b \\ u^g \end{pmatrix}_L \\ \begin{pmatrix} d^r \\ d^b \\ d^g \end{pmatrix}_L \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{pmatrix} d^g \\ d^r \\ d^b \end{pmatrix}_L \\ \begin{pmatrix} u^g \\ u^r \\ u^b \end{pmatrix}_L \end{pmatrix} \tag{2.14}$$

But they *cannot*, for example, change the weak isospin of some colours while leaving others untouched *e.g.*

$$\begin{pmatrix} \begin{pmatrix} u^r \\ u^b \\ u^g \end{pmatrix}_L \\ \begin{pmatrix} d^r \\ d^b \\ d^g \end{pmatrix}_L \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{pmatrix} d^r \\ d^b \\ u^g \end{pmatrix}_L \\ \begin{pmatrix} u^r \\ u^b \\ d^g \end{pmatrix}_L \end{pmatrix} \tag{2.15}$$

which an $SU(6)$ transformation would in general be able to do. As far as our Category (8) group is concerned, though, all we know is that $(u\ d)_L$ must form an IR and so *a priori* we should allow the full $SU(6)$ transformations. The largest Category (8) group (still neglecting abelian flavour symmetries) is then:

$$J_1 \times J_2 \times J_3 \times J_4 \tag{2.16}$$

where:

$$\begin{aligned}
J_1 &= SU^{d_L}(6) \times SU^{s_L}(6) \times SU^{b_L}(6) \\
J_2 &= SU^{u_R}(3) \times SU^{d_R}(3) \times SU^{c_R}(3) \times SU^{s_R}(3) \times SU^{t_R}(3) \times SU^{b_R}(3) \\
J_3 &= SU^{e_L}(2) \times SU^{\mu_L}(2) \times SU^{\tau_L}(2) \\
J_4 &= \prod_i U^i(1) \text{ where } i \text{ runs over all 15 IRs of the SM.}
\end{aligned} \tag{2.17}$$

and the left-handed doublets are described by their $I_3 = -1/2$ members. Consider then an algebra of globally conserved $SU(6)$ generators $\mathcal{A}_6^{\text{global}}$ possessing a locally conserved $SU(3) \times SU(2) \times U(1)$ subalgebra of generators $\mathcal{A}_{321}^{\text{local}}$. Closing this algebra by commutation yields a locally conserved $SU(6)$ algebra:

$$[\mathcal{A}_6^{\text{global}}, \mathcal{A}_{321}^{\text{local}}] = \mathcal{A}_6^{\text{local}} \tag{2.18}$$

because $SU(6)$ is a simple group and so possesses no non-trivial invariant subalgebras. This means that the J_1 component of (2.17), which we originally thought might merely be a *global* symmetry, is in fact a *gauge* symmetry. But this scenario is forbidden by the requirement that a viable model should have no net gauge anomalies. The now-gauged J_1 is obviously anomalous: each of its components possesses only one non-trivial fermion IR with nothing to cancel against it. Reducing J_1 to its diagonal subgroup:

$$J_1 \longrightarrow J_1^{\text{diag}} = SU(6) \tag{2.19}$$

under which $(u\ d)_L$, $(c\ s)_L$ and $(t\ b)_L$ all transform as **6**, does not help either. The $SU(3) \times SU(2)$ part of the SMG has to be embedded in J_1^{diag} in a manner independent of which IR we consider, and so if we have one left-handed quark representation Ψ transforming as a **6** under J_1^{diag} it is not possible to have another Ψ' transforming as a $\overline{\mathbf{6}}$ in an effort to cancel anomalies. So all we can really do in Category (8) groups for left-handed quarks is to retain the usual $SU(3) \times SU(2)$ transformations but, as in (2.11), allow different "copies" for different IRs.

Overall, then, Category (8) groups G are essentially subgroups of:

$$G' = \prod_i SU^i(3) \times \prod_j SU^j(2) \times \prod_k U^k(1) \tag{2.20}$$

where i runs over all quark representations, j over all left-handed representations and k over all 15 representations in the SM. Again, each component acts just like the corresponding SMG component but on one IR only. We do not let i and j run over all 15 representations because we wish to maintain consistency with (2.1) where we stated that we want no part of our group to act trivially on all known states and this is exactly what components like $SU^{e_R}(3)$ and $SU^{s_R}(2)$ would do. Nevertheless, allowing for this technicality, the SMG is identifiable for each IR with one $SU(3) \times SU(2) \times U(1)$ copy, so that the SMG is realised as a subgroup of G' in which the group elements corresponding to each copy are identical. But this is nothing other than the diagonal subgroup discussed earlier and we can thus claim that the 3 factors $U(1)$, $SU(2)$ and $SU(3)$ of the SMG occur as diagonal subgroups of the corresponding parts of G **for all G in Category (8)**. Note that some of these factors may be embedded in G as trivial diagonal subgroups *e.g.* for a group like:

$$G = SU(3) \times SU^{d_L}(2) \times SU^{s_L}(2) \times SU^{b_L}(2) \times SU^{e_L}(2) \times SU^{\mu_L}(2) \times SU^{\tau_L}(2) \times U(1) \tag{2.21}$$

the $SU(2)$ part of the SMG is embedded as a true diagonal subgroup while the embedding of the $SU(3)$ and $U(1)$ parts is trivial.

Note that nothing we have said actually prevents us from tagging abelian flavour symmetries onto the G' of (2.20) and the concept of a diagonal subgroup is not clean for abelian groups. Which subgroup of a product $U(1)^n$ is termed the "diagonal" subgroup depends on which linear combination of generators is chosen to form a basis. For example, if the group $SU_a(2) \times SU_b(2)$ has generators I_i^a and I_j^b $(i, j = 1, 2, 3)$ then the diagonal subgroup $SU(2)_{\text{diag}}$ is unambiguously defined to be generated by $I_i^a + I_i^b$ $(i = 1, 2, 3)$. But the same cannot be said of the group $U_a(1) \times U_b(1)$ generated by Y_a and Y_b. If we switch basis *e.g.* to Y_1 and Y_2 where:

$$\begin{aligned} Y_1 &= Y_a + Y_b \\ Y_2 &= Y_a - Y_b \end{aligned} \tag{2.22}$$

then (following the non-abelian construction) we have two equally valid possibilities for the generator of our supposed diagonal subgroup $U(1)_{\text{diag}}$ *viz.* $Y_a + Y_b$ and $Y_1 + Y_2$

which generate entirely different subgroups. Thus the concept is basis dependent and so use of the expression "diagonal" subgroup is a notational matter devoid of physical content.

Nevertheless, a strong consequence of concluding that the non-abelian part of the SMG (SMG_{na}) is embedded in the non-abelian part of G' (G'_{na}) as a diagonal subgroup is that G'_{na} must be gauged. Consider the algebra of globally conserved G'_{na} generators, $\mathcal{A}^{\mathrm{global}}_{G'_{\mathrm{na}}}$, possessing a locally conserved SMG_{na} subalgebra, $\mathcal{A}^{\mathrm{local}}_{SMG_{\mathrm{na}}}$. Again, closing the algebra by commutation yields a locally conserved G'_{na} algebra:

$$[\mathcal{A}^{\mathrm{global}}_{G'_{\mathrm{na}}}, \mathcal{A}^{\mathrm{local}}_{SMG_{\mathrm{na}}}] = \mathcal{A}^{\mathrm{local}}_{G'_{\mathrm{na}}} \tag{2.23}$$

because the subalgebras corresponding to the diagonal subgroups are not invariant *i.e.* do not close on themselves under commutation with the full algebra. (The only non-trivial proper invariant subgroups of $H \times H$, where H is simple, are:

$$\begin{aligned} H_1 &= \{(h,1):\ h \in H\} \text{ and} \\ H_2 &= \{(1,h):\ h \in H\} \end{aligned} \tag{2.24}$$

In particular:

$$H_{\mathrm{diag}} = \{(h,h):\ h \in H\} \tag{2.25}$$

is *not* invariant).

This line of argument for gauging is obviously not applicable to any abelian factors in G' (because the abelian generators irritatingly commute with all other generators). However, as the SM weak hypercharge symmetry and all of the non-abelian symmetries are gauged, we consider it reasonable to gauge all of the abelian factors too. Anyway, use of fundamental global charges should perhaps be avoided due to arguments relating to wormholes [7, 41].

2.6 Anomaly-Free Members of Category (8)

As it stands in (2.20), then, the now fully gauged G' is riddled with anomalies. Its biggest anomaly-free subgroup is:

$$G'' = SMG_1 \times SMG_2 \times SMG_3 \tag{2.26}$$

because the anomalies cannot be made to cancel on a smaller subset of states than that forming one generation. But, as previously stated, we can still add abelian flavour symmetries. The anomaly cancellation conditions for a $U(1)_f$ tagged onto G'' are (denoting the $U(1)_f$ charges by $Q_f(t_R) \equiv t_R$ *etc* , and still describing doublets by their $I_3 = -1/2$ members):

$$A_1 = \mathrm{Tr}\,[\mathrm{SU}_1(3)^2\mathrm{U}(1)_\mathrm{f}] = 2d_L - u_R - d_R = 0 \tag{2.27}$$

$$A_2 = \mathrm{Tr}\,[\mathrm{SU}_2(3)^2\mathrm{U}(1)_\mathrm{f}] = 2s_L - c_R - s_R = 0 \tag{2.28}$$

$$A_3 = \mathrm{Tr}\,[\mathrm{SU}_3(3)^2\mathrm{U}(1)_\mathrm{f}] = 2b_L - t_R - b_R = 0 \tag{2.29}$$

$$A_4 = \mathrm{Tr}\,[\mathrm{SU}_1(2)^2\mathrm{U}(1)_\mathrm{f}] = 3d_L + e_L = 0 \tag{2.30}$$

$$A_5 = \mathrm{Tr}\,[\mathrm{SU}_2(2)^2\mathrm{U}(1)_\mathrm{f}] = 3s_L + \mu_L = 0 \tag{2.31}$$

$$A_6 = \mathrm{Tr}\,[\mathrm{SU}_3(2)^2\mathrm{U}(1)_\mathrm{f}] = 3b_L + \tau_L = 0 \tag{2.32}$$

$$A_7 = \mathrm{Tr}\,[\mathrm{U}_1(1)^2\mathrm{U}(1)_\mathrm{f}] = d_L - 8u_R - 2d_R + 3e_L - 6e_R = 0 \tag{2.33}$$

$$A_8 = \mathrm{Tr}\,[\mathrm{U}_2(1)^2\mathrm{U}(1)_\mathrm{f}] = s_L - 8c_R - 2s_R + 3\mu_L - 6\mu_R = 0 \tag{2.34}$$

$$A_9 = \mathrm{Tr}\,[\mathrm{U}_3(1)^2\mathrm{U}(1)_\mathrm{f}] = b_L - 8t_R - 2b_R + 3\tau_L - 6\tau_L = 0 \tag{2.35}$$

$$A_{10} = \mathrm{Tr}\,[\mathrm{U}_1(1)\mathrm{U}(1)_\mathrm{f}^2] = d_L^2 - 2u_R^2 + d_R^2 - e_L^2 + e_R^2 = 0 \tag{2.36}$$

$$A_{11} = \mathrm{Tr}\,[\mathrm{U}_2(1)\mathrm{U}(1)_\mathrm{f}^2] = s_L^2 - 2c_R^2 + s_R^2 - \mu_L^2 + \mu_R^2 = 0 \tag{2.37}$$

$$A_{12} = \mathrm{Tr}\,[\mathrm{U}_3(1)\mathrm{U}(1)_\mathrm{f}^2] = b_L^2 - 2t_R^2 + b_R^2 - \tau_L^2 + \tau_R^2 = 0 \tag{2.38}$$

$$\begin{aligned} A_{13} = \mathrm{Tr}\,[\mathrm{U}(1)_\mathrm{f}^3] &= 6d_L^3 - 3u_R^3 - 3d_R^3 + 2e_L^3 - e_R^3 \\ &+6s_L^3 - 3c_R^3 - 3s_R^3 + 2\mu_L^3 - \mu_R^3 \\ &+6b_L^3 - 3t_R^3 - 3b_R^3 + 2\tau_L^3 - \tau_R^3 = 0 \end{aligned} \tag{2.39}$$

$$\begin{aligned} A_{14} = \mathrm{Tr}\,[(\mathrm{graviton})^2\mathrm{U}(1)_\mathrm{f}] &= 6d_L - 3u_R - 3d_R + 2e_L - e_R \\ &+6s_L - 3c_R - 3s_R + 2\mu_L - \mu_R \\ &+6b_L - 3t_R - 3b_R + 2\tau_L - \tau_R = 0 \end{aligned} \tag{2.40}$$

The quadratic equations are all trivially satisfied by virtue of the appropriate linear equations: (2.36) by virtue of (2.27), (2.30) and (2.33); (2.37) by virtue of (2.28), (2.31) and (2.34); and (2.38) by virtue of (2.29), (2.32) and (2.35). After a little algebra and use of the linear equations, (2.39) can be written:

$$A_{13} = 3(4d_L - u_R)(4s_L - c_R)(4b_L - t_R) = 0 \tag{2.41}$$

That is:

$$
\begin{aligned}
4d_L - u_R &= 0 \\
\text{or} \quad 4s_L - c_R &= 0 \\
\text{or} \quad 4b_L - t_R &= 0
\end{aligned}
\tag{2.42}
$$

Up to an overall scale factor, these equations permit 4 linearly independent solutions. Regardless of which of these last constraints of (2.42) holds, the following 3 charge sets are always anomaly-free:

$$
Q_f \equiv \begin{pmatrix} d_L & u_R & d_R & e_L & e_R \\ s_L & c_R & s_R & \mu_L & \mu_R \\ b_L & t_R & b_R & \tau_L & \tau_R \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 4 & -2 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -2 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -2 & -3 & -6 \end{pmatrix} \end{cases} \tag{2.43}
$$

because they each satisfy all of the equations (2.42). Note that these are simply copies of $U_a(1)$ (a=1,2,3). The fourth charge set is:

$$
Q_f = \begin{cases} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix} & \text{if } (2.42)_1 \text{ holds} \\ \\ \begin{pmatrix} 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix} & \text{if } (2.42)_2 \text{ holds} \\ \\ \begin{pmatrix} 0 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \text{if } (2.42)_3 \text{ holds} \end{cases} \tag{2.44}
$$

Note the resemblance of these charge sets to the "bizarre" hypercharge solution of the SMG no-anomaly equations given in Chapter 1. A general anomaly-free charge set is then a linear combination of the three solutions in (2.43) and any **one** of the solutions in (2.44).

The solutions in (2.43) (or any linear combination of them) are really superfluous. For example, if we have a $U(1)_f$ generated by Y_f which is an exact copy of $U_3(1)$ generated by Y_3, then we can change the basis of the $U_3(1) \times U(1)_f$ space to:

$$\begin{aligned} Y &= Y_3 + Y_f \\ Y' &= Y_3 - Y_f \end{aligned} \tag{2.45}$$

This would leave: one $U(1)$ (generated by Y) which is identical to $U_3(1)$ and $U(1)_f$; and a second $U(1)$ (generated by Y') which acts trivially on all SM Weyl states and so can be dropped as in (2.1). Thus, an abelian flavour symmetry only contains new information if one of the solutions in (2.44) is involved.

If we next try to extend the group from $SMG^3 \times U(1)_f$ to $SMG^3 \times U(1)_f \times U(1)'_f$ then anomaly cancellation can only occur if the $U(1)'_f$ is a linear combination of the 3 charge sets in (2.43) and whichever charge set of (2.44) was used to obtain $U(1)_f$. This would again mean that the basis of the abelian subalgebra could be changed to yield a generator which acted trivially on all SM Weyl states. We can thus conclude that the biggest anomaly-free Category (8) group which does not possess trivial generators is:

$$G''' = SMG^3 \times U(1)_f \tag{2.46}$$

Can we say that all G in Category (8) are subgroups of G'''? In fact we cannot because the no-anomaly conditions which have to be satisfied by the flavour charges vary for different choices of G (so that different sets of anomaly-free flavour charges are possible). For example, consider the Category (8) groups $K_1 \times U(1)_f$ and $K_2 \times U(1)'_f$ where:

$$SMG \subseteq K_{1,2} \subseteq SMG^3 \text{ and } \mathcal{A}_{K_1} \subset \mathcal{A}_{K_2} \tag{2.47}$$

and $\mathcal{A}_{K_{1,2}}$ are the algebras of $K_{1,2}$ respectively. Then the anomaly-free charge sets for $U(1)'_f$ simply form a subset of those for $U(1)_f$, because the no-anomaly constraints for $U(1)_f$ form a subset of those for $U(1)'_f$. Taking $K_2 = SMG^3$, we see that it is thus *not* generally true that:

$$K_1 \times U(1)_f \subset K_2 \times U(1)'_f = G''' \tag{2.48}$$

but it *is* true that:

$$\mathcal{A}_{K_1 \times U(1)_f} \subset \mathcal{A}_{K_2 \times U(1)'_f} = \mathcal{A}_{G'''} \tag{2.49}$$

where $\mathcal{A}_{G'''}$ is the algebra of of G'''.

Can we then say that the algebras, $\mathcal{A}_G$, of all G in Category (8) are subalgebras of $\mathcal{A}_{G'''}$? In fact we cannot make this claim either. While it is certainly true for SMG^3 that only one non-trivial abelian flavour symmetry can be tagged on (as explained above, all further abelian symmetries must then essentially be copies of those already present), this is not true for subgroups of SMG^3. Consider Category (8) groups $J_{1,2}$ such that:

$$SMG \subseteq J_1 \subset J_2 \subseteq SMG^3 \tag{2.50}$$

Then J_2 can be expanded at least to $J_2 \times U(1)_f$ (*e.g.* with the $U(1)_f$ charges of one of the sets in (2.44)); whereas J_1 can be expanded at least to $J_1 \times U(1)_f \times U(1)'_f$ with the same $U(1)_f$ charge set (say), but with a $U(1)'_f$ charge set which is not a copy/linear combination of the other abelian charges already present. Taking $J_2 = SMG^3$, we see that it is thus *not* generally true that:

$$\mathcal{A}_{J_1 \times U(1)_f \times U(1)'_f} \subset \mathcal{A}_{J_2 \times U(1)_f} = \mathcal{A}_{G'''} \tag{2.51}$$

So we come to the overall conclusion:

1. discounting abelian flavour symmetries, all anomaly-free Category (8) groups G satisfy
$$SMG \subseteq G \subseteq SMG^3; \tag{2.52}$$
2. including such symmetries, the biggest anomaly-free Category (8) group (with no trivial generators) is $G''' \equiv SMG^3 \times U(1)_f$;
3. the algebra, $\mathcal{A}_G$, corresponding to a general Category (8) group G can be written as $\mathcal{A}_G = \mathcal{A} + \mathcal{B}_f$ where $\mathcal{A} \subseteq \mathcal{A}_{G'''}$ and $\mathcal{B}_f$ is a set of abelian flavour generators.

Finally, as we have already mentioned, it is certainly possible that there is only one copy of some SMG factor (*e.g.* $SU(3)$) in G *i.e.* the corresponding part of the SMG is embedded in G as a trivial diagonal subgroup. However, if we wish to obtain mass splitting between generations then we cannot have the completely trivial case of the

SMG as a diagonal subgroup of a cross product with only one factor. One or more charges, in addition to those in the SMG proper, have to exist and be approximately conserved in order to generate such splitting. This is simply a restatement of the fact that the SMG alone does not provide any real explanation for the origin of the fermion mass hierarchy.

"The vacuum created by the arrival of freedom
And the possibilities it seems to offer..."

David Bowie
Up the Hill Backwards

Chapter 3

Matrix Algebra, Matrix Element Ansätze & Model Selection

3.1 Introduction

Having argued that the fermion mass hierarchy seems to demand the consideration of a particular class of SMG extension, we now focus more closely on the mass matrices themselves. After discussing how these matrices are algebraically treated in our order of magnitude framework, we postulate different ansätze for the construction of the mass matrix elements from particular sets of partially conserved chiral symmetries (PCCSs). We finally decide which specific PCCSs look most promising as regards generation of the fermion mass hierarchy.

Since the basic premise of our analysis is that partially conserved symmetries are responsible for the observed masses, it is appropriate here to be a little more precise about how we envisage SM mass matrix elements being affected by such symmetries. A general SM fermion mass matrix element can be written:

$$M_{U,D,l}(i,j) = y_{ij}^{U,D,l} \langle\phi\rangle_{\mathrm{ws}} \tag{3.1}$$

where y_{ij} is a dimensionless effective Yukawa coupling and $\langle\phi\rangle_{\mathrm{ws}} = 174$ GeV is the usual Weinberg-Salam Higgs VEV. Throughout our algebra we will absorb $\langle\phi\rangle_{\mathrm{ws}}^{-1}$ into $M(i,j)$ *i.e.* masses will be algebraically specified in units of $\langle\phi\rangle_{\mathrm{ws}}$. The crux of our approach is that we assume the complex y_{ij} can be written as:

$$y_{ij}^{U,D,l} = \gamma_{ij}^{U,D,l} \; a_{ij}^{U,D,l} \tag{3.2}$$

where γ_{ij} is a complex number whose magnitude is of $\mathcal{O}(1)$ while a_{ij} is real and:

$$a_{ij} \begin{cases} = 1 & \text{if } M(i,j) \text{ is } \textit{not} \text{ mass-protected by any} \\ & \text{higher symmetry than that of the } SMG \\ \ll 1 & \text{if } M(i,j) \textit{ is} \text{ mass-protected by a higher} \\ & \text{symmetry than the } SMG \end{cases} \tag{3.3}$$

We take the view that the a_{ij} (which will obviously be responsible for the order of magnitude features of the fermion mass hierarchy) should be directly obtainable from the PCCSs *i.e.* from the chiral quantum numbers of the Weyl fermions which couple to form $M(i,j)$. As will shortly be seen, each a_{ij} will be related to one or more symmetry breaking parameters which may be thought of [31, 35] as the ratio of the symmetry breaking scale to the fundamental scale of the theory. For example in a simple $SMG \times U(1)_f$ model such as [30, 31, 35] with $U(1)_f$ approximately conserved, the order of magnitude of a suppressed matrix element is naturally given by a power of:

$$\epsilon \simeq \Lambda_f / M \ll 1 \tag{3.4}$$

where Λ_f is the scale at which the $U(1)_f$ is spontaneously broken (*e.g.* by the VEV of a scalar S which is an SMG singlet but has non-zero $U(1)_f$ charge) and M is the fundamental scale of the $SMG \times U(1)_f$ model.

Returning to (3.2), the $\mathcal{O}(1)$ γ_{ij} are unknown and maintained in algebraic analysis but dropped for numerical analysis. We can then only specify any $M(i,j)$ up to an unknown $\mathcal{O}(1)$ factor, but this is consistent with our aim of accounting only for the order of magnitude features of the mass and mixing hierarchies; the complex γ_{ij} are assumed to be responsible only for fine structure within these hierarchies.

3.2 Matrix Algebra

3.2.1 Matrix Diagonalisation

Our ignorance of these γ_{ij} poses certain obvious difficulties for the diagonalisation of the mass matrices. These difficulties are attacked by using prior knowledge of the fermion masses to state that any postulated mass matrix must be able to accommodate a definite hierarchy in its eigenvalues otherwise it is phenomenologically unacceptable.

Some clarification is necessary here. Given a fermion mass matrix M_a ($a = U, D, l$), and denoting the eigenvalues of $M_a^\dagger M_a$ as m_{ai}^2 (i=1,2,3), we demand that:

$$m_{a1}^2 \ll m_{a2}^2 \ll m_{a3}^2 \tag{3.5}$$

and actively use this in the algebraic diagonalisation process.

A basic assumption underlying this process is that for two arbitrary complex numbers α and β:

$$\mathcal{O}(\alpha+\beta) = \begin{cases} 0 & \text{if } \alpha = -\beta \\ max(\mathcal{O}(\alpha), \mathcal{O}(\beta)) & \text{otherwise} \end{cases} \tag{3.6}$$

Thus each step in the diagonalisation process is performed to leading order, with allowance made for *exact* algebraic cancellation (we will never rely on a finely-tuned cancellation in order to provide sufficiently small numbers). Further simplifying assumptions, justified by (3.5), are:

$$\begin{aligned} m_{a3}^2 &\simeq \text{leading term in } M_a^\dagger M_a \\ m_{a2}^2 m_{a3}^2 &\simeq \text{leading term in minors of } M_a^\dagger M_a \\ m_{a1}^2 m_{a2}^2 m_{a3}^2 &\simeq \text{leading term in } \det(M_a^\dagger M_a) \end{aligned} \tag{3.7}$$

for $a = U, D, l$. Having obtained leading order expressions for these masses, the eigenvalue equations are then solved for the quark mass matrices to yield leading order expressions for the eigenvectors. These form the matrices R_U and R_D of (1.15), for:

$$\begin{aligned} R_U^\dagger (M_U^\dagger M_U) R_U &= \text{diag}(m_u^2, m_c^2, m_t^2) \\ R_D^\dagger (M_D^\dagger M_D) R_D &= \text{diag}(m_d^2, m_s^2, m_b^2) \end{aligned} \tag{3.8}$$

so that the CKM matrix:

$$V_{\text{CKM}} \equiv R_U^\dagger R_D \tag{3.9}$$

can then be found to leading order.

3.2.2 A Diagonalisation Example

We now elucidate further on this treatment by explicitly diagonalising two typical quark mass matrices. One of the models which will be encountered in Chapter 4 features the matrices:

$$
\begin{aligned}
M_D &= \begin{pmatrix} \gamma^D_{11}x^3 & \gamma^D_{12}x^2yz & \gamma^D_{13}x^2z \\ \gamma^D_{21}xz^3 & \gamma^D_{22}yz^2 & \gamma^D_{23}z^4 \\ \gamma^D_{31}xz^3 & \gamma^D_{32}yz^4 & \gamma^D_{33}z^2 \end{pmatrix} \\
M_U &= \begin{pmatrix} \gamma^U_{11}x^3 & \gamma^U_{12}x^4yz & \gamma^U_{13}x^4z \\ \gamma^U_{21}xz & \gamma^U_{22}y & \gamma^U_{23}z^2 \\ \gamma^U_{31}xz & \gamma^U_{32}yz^2 & \gamma^U_{33} \end{pmatrix}
\end{aligned} \tag{3.10}
$$

where $x, y, z \ll 1$ correspond to symmetry breaking parameters of the type mentioned at the beginning of this chapter. Then, assuming $x^2 \ll z^3$, we have:

$$
\begin{aligned}
M_D^\dagger M_D &= \begin{pmatrix} (\gamma^{D2}_{21} + \gamma^{D2}_{31})x^2z^6 + \gamma^{D2}_{11}x^6 & \gamma^{D*}_{21}\gamma^D_{22}xyz^5 & \gamma^{D*}_{31}\gamma^D_{33}xz^5 \\ \gamma^D_{21}\gamma^{D*}_{22}xyz^5 & \gamma^{D2}_{22}y^2z^4 & (\gamma^{D*}_{22}\gamma^D_{23} + \gamma^{D*}_{32}\gamma^D_{33})yz^6 \\ \gamma^D_{31}\gamma^{D*}_{33}xz^5 & (\gamma^D_{22}\gamma^{D*}_{23} + \gamma^D_{32}\gamma^{D*}_{33})yz^6 & \gamma^{D2}_{33}z^4 \end{pmatrix} \\
M_U^\dagger M_U &= \begin{pmatrix} (\gamma^{U2}_{21} + \gamma^{U2}_{31})x^2z^2 + \gamma^{U2}_{11}x^6 & \gamma^{U*}_{21}\gamma^U_{22}xyz & \gamma^{U*}_{31}\gamma^U_{33}xz \\ \gamma^U_{21}\gamma^{U*}_{22}xyz & \gamma^{U2}_{22}y^2 & (\gamma^{U*}_{22}\gamma^U_{23} + \gamma^{U*}_{32}\gamma^U_{33})yz^2 \\ \gamma^U_{31}\gamma^{U*}_{33}xz & (\gamma^U_{22}\gamma^{U*}_{23} + \gamma^U_{32}\gamma^{U*}_{33})yz^2 & \gamma^{U2}_{33} \end{pmatrix}
\end{aligned} \tag{3.11}
$$

where $\gamma^2_{ij} \equiv \gamma^*_{ij}\gamma_{ij}$. The $\gamma^*_{11}\gamma_{11}$ terms in the (1,1) elements of both $M_U^\dagger M_U$ and $M_D^\dagger M_D$ have been retained in order to ensure that the leading order terms in $\det(M_U^\dagger M_U)$ and $\det(M_D^\dagger M_D)$ are seen to be:

$$(\gamma^{U*}_{11}\gamma^U_{11}\gamma^{U*}_{22}\gamma^U_{22}\gamma^{U*}_{33}\gamma^U_{33})x^6y^2$$

and:

$$(\gamma^{D*}_{11}\gamma^D_{11}\gamma^{D*}_{22}\gamma^D_{22}\gamma^{D*}_{33}\gamma^D_{33})x^6y^2z^8$$

respectively.

Now assuming $xz < y$ and using (3.7) gives:

$$
\begin{aligned}
m_t^2 &\simeq \gamma^{U*}_{33}\gamma^U_{33} & m_b^2 &\simeq \gamma^{D*}_{33}\gamma^D_{33}z^4 \\
m_c^2 &\simeq \gamma^{U*}_{22}\gamma^U_{22}y^2 & m_s^2 &\simeq \gamma^{D*}_{22}\gamma^D_{22}y^2z^4 \\
m_u^2 &\simeq \gamma^{U*}_{11}\gamma^U_{11}x^6 & m_d^2 &\simeq \gamma^{D*}_{11}\gamma^D_{11}x^6
\end{aligned} \tag{3.12}
$$

And calculating the eigenvectors $v_{t,c,u,b,s,d}$ corresponding to these eigenvalues by solving the equations:

$$(M_U^\dagger M_U)v_{t,c,u} = m^2_{t,c,u}\; v_{t,c,u}$$

$$(M_D^\dagger M_D)v_{b,s,d} \quad = \quad m_{b,s,d}^2\, v_{b,s,d} \tag{3.13}$$

gives:

$$\begin{aligned}
R_U &\simeq \begin{pmatrix} 1 & \frac{\gamma_{21}^{U*}}{\gamma_{22}^{U*}}\frac{xz}{y} & \frac{\gamma_{31}^{U*}}{\gamma_{33}^{U*}}xz \\ -R_U^*(1,2) & 1 & \frac{\gamma_{22}^{U*}\gamma_{23}^{U}+\gamma_{32}^{U*}\gamma_{33}^{U}}{\gamma_{33}^{U2}}yz^2 \\ -R_U^*(1,3) & -R_U^*(2,3)-R_U(1,2)R_U^*(1,3) & 1 \end{pmatrix} \\
R_D &\simeq \begin{pmatrix} 1 & \frac{\gamma_{21}^{D*}}{\gamma_{22}^{D*}}\frac{xz}{y} & \frac{\gamma_{31}^{D*}}{\gamma_{33}^{D*}}xz \\ -R_D^*(1,2) & 1 & \frac{\gamma_{22}^{D*}\gamma_{23}^{D}+\gamma_{32}^{D*}\gamma_{33}^{D}}{\gamma_{33}^{D2}}yz^2 \\ -R_D^*(1,3) & -R_D^*(2,3)-R_D(1,2)R_D^*(1,3) & 1 \end{pmatrix}
\end{aligned} \tag{3.14}$$

Note that it is not algebraically clear which of the two terms in $R_{U,D}(3,2)$ is dominant; this point will be discussed in Chapter 4. Strictly speaking, these two matrices should be orthonormalised, but the expressions given here are good to leading order. The CKM matrix is then given by (3.9) to be:

$$V_{\mathrm{CKM}} \simeq \begin{pmatrix} 1 & -R_U(1,2)+R_D(1,2) & -R_U(1,3)+R_D(1,3) \\ R_U^*(1,2)-R_D^*(1,2) & 1 & \begin{array}{l} -R_U(2,3)+R_D(2,3) \\ -R_U^*(1,2)R_U(1,3) \\ +R_U^*(1,2)R_D(1,3) \end{array} \\ R_U^*(1,3)-R_D^*(1,3) & \begin{array}{l} R_U^*(2,3)-R_D^*(2,3) \\ -R_D(1,2)R_D^*(1,3) \\ +R_U^*(1,3)R_D(1,2) \end{array} & 1 \end{pmatrix} \tag{3.15}$$

and again this should (strictly speaking) be orthonormalised.

Finally, as the $\gamma_{ij}^{U,D}$ are unknown but assumed to be of $\mathcal{O}(1)$, we really only have information on $\mathcal{V} \equiv |V_{\mathrm{CKM}}|$ which we choose to parameterise by V_{us}, V_{ub} and V_{cb}. So our order of magnitude predictions are:

$$\begin{array}{lll} m_u \simeq x^3 & m_c \simeq y & m_t \simeq 1 \\ m_d \simeq x^3 & m_s \simeq yz^2 & m_b \simeq z^2 \\ V_{us} \simeq \frac{xz}{y} & V_{ub} \simeq xz & V_{cb} \simeq \max(yz^2, \frac{x^2z^2}{y}) \end{array} \tag{3.16}$$

The assignation of numerical values to our model parameters such as x, y and z will be discussed in due course.

3.2.3 Typical Mass Matrix Textures

Having discussed how we diagonalise a typical mass matrix, we now enumerate the matrix textures which appear in this thesis. Let the square roots of the order of magnitude eigenvalues of $(M^{\dagger}_{U,D,l}M_{U,D,l})$ be $m_i^{U,D,l}$ (i=1,2,3) (we shall often loosely refer to these as the "eigenvalues" of the mass matrices themselves). Let $a^{U,D,l}$, $b^{U,D,l}$ and $c^{U,D,l}$ be real numbers with $0 < a, b, c \leq 1$. Then all mass matrices encountered in this thesis have one of the following textures (order of magnitude wise):

$$
\begin{array}{ccc}
\textbf{Texture 1:} & \textbf{Texture 2:} & \\
\begin{pmatrix} m_1 & \times & \times \\ a & m_2 & \times \\ b & c & m_3 \end{pmatrix} &
\begin{pmatrix} \times & \frac{m_1 m_2}{a} & \times \\ a & m_2 & \times \\ b & c & m_3 \end{pmatrix} & \\
\textbf{Texture 3:} & \textbf{Texture 4:} & \textbf{Texture 5:} \\
\begin{pmatrix} \times & m_1 & \times \\ m_2 & a & \times \\ b & c & m_3 \end{pmatrix} &
\begin{pmatrix} \frac{m_1 m_2}{a} & \times & \times \\ m_2 & a & \times \\ b & c & m_3 \end{pmatrix} &
\begin{pmatrix} \times & \times & \frac{m_1 m_2 m_3}{ab} \\ m_2 & a & \times \\ b & c & m_3 \end{pmatrix} \\
\textbf{Texture 6:} & \textbf{Texture 7:} & \textbf{Texture 8:} \\
\begin{pmatrix} m_1 & \times & \times \\ a & \times & \frac{m_2 m_3}{c} \\ b & c & m_3 \end{pmatrix} &
\begin{pmatrix} \times & \frac{m_1 m_2}{a} & \times \\ a & \times & \frac{m_2 m_3}{c} \\ b & c & m_3 \end{pmatrix} &
\begin{pmatrix} \times & \times & \frac{m_1 m_2 m_3}{ac} \\ a & \times & \frac{m_2 m_3}{c} \\ b & c & m_3 \end{pmatrix} \\
\textbf{Texture 9:} & & \\
\begin{pmatrix} \frac{m_1 m_2}{a} & m_2 & \times \\ \times & a & \times \\ \times & \times & m_3 \end{pmatrix} & &
\end{array}
\tag{3.17}
$$

where we have suppressed all superscripts. The entries denoted "$\times$" can assume any values between 0 and 1 provided the matrix structure remains compatible with:

$$m_3^2 \simeq \text{leading term in } M^{\dagger}M$$

Combination	M_U	M_D	$\mathcal{V}$
Type 1	Texture 1 or 2	Texture 1 or 2	$\mathcal{V}_1$
Type 2	Texture 3 or 4 or 5	Texture 3 or 4 or 5	$\mathcal{V}_2$
Type 3	Texture 1 or 2	Texture 6 or 7 or 8	$\mathcal{V}_3$

Table 3.1: Combination types of quark mass matrix textures and their corresponding mixing matrices.

$$
\begin{aligned}
m_2^2 m_3^2 &\simeq \text{leading term in minors of } M^\dagger M \\
m_1^2 m_2^2 m_3^2 &\simeq \text{leading term in } \det(M^\dagger M)
\end{aligned} \tag{3.18}
$$

All lepton matrices will turn out to be of Texture 1, and so are particularly simple. The promising quark matrices will assume Textures 1–8 which are distinguised by the origin of their eigenvalues. Different combinations of M_U and M_D textures will give different CKM matrices. In fact, only 3 different types of combination of these 8 textures occur for the models featured in this thesis. These 3 types are shown in Table 3.1 and the 3 corresponding mixing matrices are (order of magnitude wise):

$$
\mathcal{V}_1 = \begin{pmatrix}
1 & max\ (\frac{a^U}{m_2^U}, \frac{a^D}{m_2^D}) & max\ (\frac{b^U}{m_3^U}, \frac{b^D}{m_3^D}, \frac{a^U c^U}{m_2^U m_3^U}, \frac{a^U c^D}{m_2^U m_3^D}) \\
max\ (\frac{a^U}{m_2^U}, \frac{a^D}{m_2^D}) & 1 & max\ (\frac{c^U}{m_3^U}, \frac{c^D}{m_3^D}, \frac{a^U b^U}{m_2^U m_3^U}, \frac{a^U b^D}{m_2^U m_3^D}) \\
max\ (\frac{b^U}{m_3^U}, \frac{b^D}{m_3^D}, \frac{a^D c^D}{m_2^D m_3^D}, \frac{a^D c^U}{m_2^D m_3^U}) & max\ (\frac{c^U}{m_3^U}, \frac{c^D}{m_3^D}, \frac{a^D b^D}{m_2^D m_3^D}, \frac{a^D b^U}{m_2^D m_3^U}) & 1
\end{pmatrix}
$$

$$
\mathcal{V}_2 = \begin{pmatrix}
1 & max\ (\frac{a^U}{m_2^U}, \frac{a^D}{m_2^D}) & max\ (\frac{c^U}{m_3^U}, \frac{c^D}{m_3^D}, \frac{a^U b^U}{m_2^U m_3^U}, \frac{a^U b^D}{m_2^U m_3^D}) \\
max\ (\frac{a^U}{m_2^U}, \frac{a^D}{m_2^D}) & 1 & max\ (\frac{b^U}{m_3^U}, \frac{b^D}{m_3^D}, \frac{a^U c^U}{m_2^U m_3^U}, \frac{a^U c^D}{m_2^U m_3^D}) \\
max\ (\frac{c^U}{m_3^U}, \frac{c^D}{m_3^D}, \frac{a^D c^D}{m_2^D m_3^D}, \frac{a^D c^U}{m_2^D m_3^U}) & max\ (\frac{b^U}{m_3^U}, \frac{b^D}{m_3^D}, \frac{a^D b^D}{m_2^D m_3^D}, \frac{a^D b^U}{m_2^D m_3^U}) & 1
\end{pmatrix}
$$

$$
\mathcal{V}_3 = \begin{pmatrix}
1 & max\ (\frac{a^U}{m_2^U}, \frac{b^D}{c^D}) & max\ (\frac{b^U}{m_3^U}, \frac{b^D}{m_3^D}, \frac{a^U c^U}{m_2^U m_3^U}, \frac{a^U c^D}{m_2^U m_3^D}) \\
max\ (\frac{a^U}{m_2^U}, \frac{b^D}{c^D}) & 1 & max\ (\frac{c^U}{m_3^U}, \frac{c^D}{m_3^D}, \frac{a^U b^U}{m_2^U m_3^U}, \frac{a^U b^D}{m_2^U m_3^D}) \\
max\ (\frac{a^D c^D}{m_2^D m_3^D}, \frac{b^U}{m_3^U}, \frac{b^D c^U}{c^D m_3^U}) & max\ (\frac{c^U}{m_3^U}, \frac{c^D}{m_3^D}, \frac{b^D b^U}{c^D m_3^U}) & 1
\end{pmatrix} \tag{3.19}
$$

Note that there are generally several competing contributions to each element of the mixing matrix and, consistent with (3.6), the order of magnitude of such an element is taken to be the order of magnitude of the biggest contribution. The expressions shown here for the various mixing angles are obtained in exactly the same way as those of (3.15). The simple ratios (*e.g.* a^U/m_2^U and b^D/m_3^D) come wholly from R_U or R_D in a straightforward manner. The complex ratios such as $a^U b^D/m_2^U m_3^D$ come about only after forming the product $R_U^\dagger R_D$; whereas the other complex ratios such as $a^U b^U/m_2^U m_3^U$ and $a^D c^D/m_2^D m_3^D$ are also present wholly in R_U or R_D ((3.15) illustrates these points well).

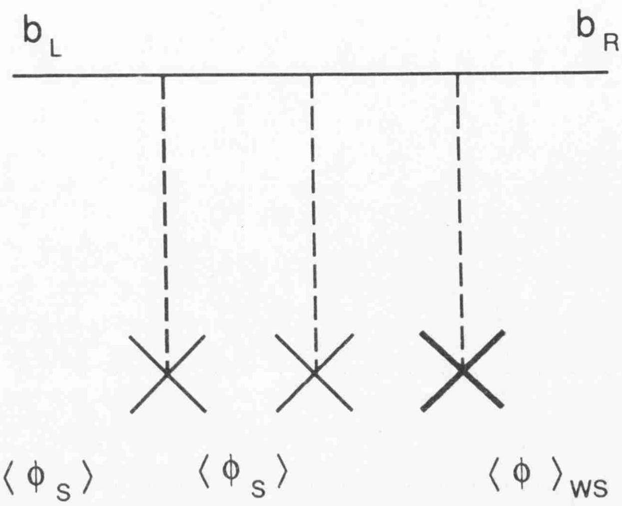

Figure 3.1: Tree-level contribution to the b mass via interaction of ϕ_S with superheavy fermions.

3.3 Ansätze for Mass Matrix Elements

3.3.1 Some Remarks on Symmetry Breaking

In order to be completely precise about the form of the fermion mass matrix elements, some mechanism for the spontaneous breaking:

$$G \rightarrow SMG \tag{3.20}$$

must be specified. There are different ways of achieving such a breaking.

Consider, for example, an $SMG \times U(1)_f$ model whose fundamental mass scale is M, broken to the SMG by the VEV of a scalar field ϕ_S where $\langle \phi_S \rangle < M$ and $Q_f(\phi_S) = 1$. Suppose further that $Q_f(b_L) = 0$ and $Q_f(b_R) = 2$. Then it is natural to expect the generation of a b mass of order:

$$\left(\frac{\langle \phi_S \rangle}{M} \right)^2 \langle \phi \rangle_{\mathrm{ws}} \tag{3.21}$$

via a Feynman diagram such as that of Figure 3.1. In this figure, the intermediate states are appropriately charged vector-like superheavy fermions (*i.e.* of mass M) (see [30, 31]).

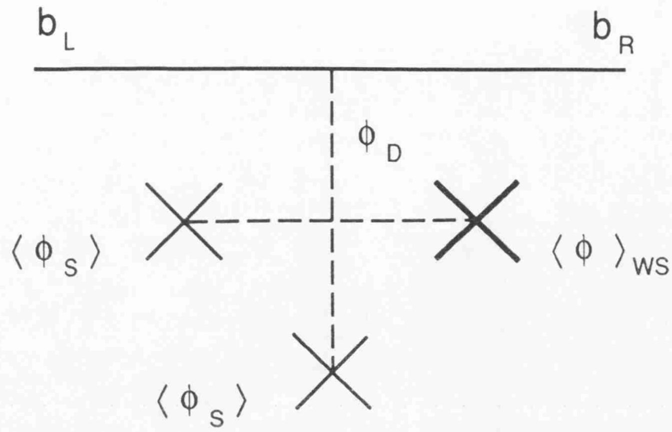

Figure 3.2: Tree-level contribution to the b mass via interaction of ϕ_S with a superheavy scalar doublet.

Alternatively, as in [35], a further scalar doublet Φ_D (in addition to the familiar doublet Φ_{ws}) can perform the job of the superheavy fermions. Suppose that its $I_3 = -1/2$ component, ϕ_D, has mass M and that $Q_f(\phi_D) = 2$. Then the Feynman diagram of Figure 3.2 naturally generates a b mass of the same order of magnitude as (3.21). In other words, ϕ_D acquires a VEV of order:

$$\langle \phi_D \rangle \sim \left(\frac{\langle \phi_S \rangle}{M} \right)^2 \langle \phi \rangle_{\text{ws}} \tag{3.22}$$

In either case, the existence of the appropriate spectrum of heavy fermions and scalars yields suppressed mass terms for the SM fermions. However, in this thesis we do not wish to become involved in the particulars of different symmetry breaking mechanisms. Instead, we will postulate some intuitive ansätze which will effectively parameterise our ignorance of such mechanisms. For example, in the above models the fact that:

$$|Q_f(b_L) - Q_f(b_R)| = 2 \tag{3.23}$$

might have prompted us to naturally suppose that:

$$m_b \simeq \epsilon^2 \langle \phi \rangle_{\text{ws}} \tag{3.24}$$

where $\epsilon < 1$, without being specific about how to break the $U(1)_f$ symmetry. Implicitly assumed in this approach, of course, is that there always exists an appropriate spectrum of superheavy fermions/scalars which can mediate all of the symmetry breaking transitions parameterised by the ansätze we are about to postulate. In particular, we will not assume the *absence* of appropriate superheavy states in order obtain texture zeroes in the mass matrices.

3.3.2 Component Symmetries of Category (8) Groups

The smallest non-trivial Category (8) group is $SMG \times U(1)_f$ while the largest is $SMG^3 \times U(1)_f$. The four building blocks of a group in this category are $SU_a(3)$, $SU_a(2)$, $U_a(1)$ ($a = 1, 2, 3$) and $U(1)_f$ and we look at each in turn.

(i) $SU_a(3)$

Under an $SU_a(3)$ all states are singlets except the quarks of the a^{th} generation which are triplets. So there are 3 types of matrix element:

1. an element linking singlets which will obviously remain unsuppressed because there is no quantum number difference between the left- and right-handed states which form it (*e.g.* $M_U(3,2)$ for $SU_1(3)$);
2. an element linking triplets which will be unsuppressed for the same reason (*e.g.* $M_U(1,1)$ for $SU_1(3)$);
3. and an element linking a triplet and a singlet. This will be suppressed if the $SU_a(3)$ is partially conserved (*e.g.* $M_U(2,1)$ for $SU_1(3)$); all such elements are naturally suppressed by the same amount.

Suppose, then, that some Category (8) group has a partially conserved $SU_1(3)$ with symmetry breaking parameter β_1. The mass matrices would be given by:

$$M_U \simeq M_D \simeq \begin{pmatrix} 1 & \beta_1 & \beta_1 \\ \beta_1 & 1 & 1 \\ \beta_1 & 1 & 1 \end{pmatrix}, \quad M_l \simeq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.25}$$

and obviously all eigenvalues of $M_i^\dagger M_i$ (*i.e.* masses) are of $\mathcal{O}(1)$ (we emphasise that the equalities in (3.25) are only approximate due to our lack of knowledge of the γ_{ij} in (3.2)).

When there is more than one partially conserved $SU_a(3)$ we assume that each is broken independently so that the overall symmetry breaking parameter is the product of all the individual ones. For example, if the group has $SU_1(3) \times SU_2(3) \times SU_3(3)$ as PCCSs and the symmetry breaking parameter for each $SU_a(3)$ is β_a, then the mass matrices are given by:

$$M_U \simeq M_D \simeq \begin{pmatrix} 1 & \beta_1\beta_2 & \beta_1\beta_3 \\ \beta_1\beta_2 & 1 & \beta_2\beta_3 \\ \beta_1\beta_3 & \beta_2\beta_3 & 1 \end{pmatrix}, \quad M_l \simeq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.26}$$

A smaller set of PCCSs is obtained by setting a particular $\beta_a = 1$ *i.e.* taking some $SU_a(3)$ to be so strongly broken as to be irrelevant for mass suppression. From (3.26) it is clear that a mass hierarchy is unobtainable without the help of other PCCSs as all masses are still of $\mathcal{O}(1)$.

(ii) $SU_a(2)$

Under an $SU_a(2)$ all states are singlets except the left-handed quarks and leptons of the a^{th} generation which form doublets in the usual manner. There are thus 2 types of matrix element:

1. an element linking singlets which is unsuppressed as usual (*e.g.* $M_U(2,3)$ for $SU_1(2)$);
2. and an element linking a doublet and a singlet which will be suppressed if the $SU_a(2)$ is approximately conserved (*e.g.* $M_U(2,1)$ for $SU_1(2)$). Again, all such elements are naturally suppressed to the same degree.

(The reader is reminded that in our notation the rows and columns of a mass matrix are indexed by right- and left-handed Weyl fermions respectively). There are no elements linking doublets because only left-handed states are doublets and any $M(i,j)$ links a left- and a right-handed state.

So, if some Category (8) group has $SU_1(2) \times SU_2(2) \times SU_3(2)$ as PCCSs and the symmetry breaking parameters are ϵ_1, ϵ_2 and ϵ_3 respectively, then the mass matrices are:

$$M_U \simeq M_D \simeq M_l \simeq \begin{pmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix} \tag{3.27}$$

Note that, unlike the $SU_1(3) \times SU_2(3) \times SU_3(3)$ case, each matrix element is affected by no more than one suppression factor. This is because the right-handed Weyl states are all singlets. The natural order of magnitude mass predictions are:

$$\begin{aligned} m_u \simeq m_d \simeq m_e \simeq \epsilon_1 \\ m_c \simeq m_s \simeq m_\mu \simeq \epsilon_2 \\ m_t \simeq m_b \simeq m_\tau \simeq \epsilon_3 \end{aligned} \tag{3.28}$$

and so it is easy to account for the inter-generation gaps by choosing:

$$\epsilon_1 \ll \epsilon_2 \ll \epsilon_3 \tag{3.29}$$

but mass-splitting within generations cannot be naturally obtained. Again, a smaller set of PCCSs is obtained simply by setting some $\epsilon_a = 1$ (a=1,2,3).

(iii) $U_a(1)$

To begin with, consider only one approximately conserved abelian symmetry, $U_1(1)$, with charges Q_1. All 2nd and 3rd generation states have $Q_1 = 0$ while the 1st generation states have Q_1 equal to the usual weak hypercharge which we normalise to assume integer values. That is, with $Q_1(u_R) \equiv u_R$ *etc* , the $U_1(1)$ charges are (with, as always, doublets denoted by their $I_3 = -1/2$ components):

$$d_L = 1, \quad u_R = 4, \quad d_R = -2, \quad e_L = -3, \quad e_R = -6 \tag{3.30}$$

If the symmetry breaking parameter of $U_1(1)$ is λ_1 then we take the matrix elements to be given by:

$$M(i,j) \simeq \lambda_1^{|Q_{1i}-Q_{1j}|} \tag{3.31}$$

where i runs over appropriate right-handed states and j over appropriate left-handed states. In this notation we have for example:

$$M_D = \begin{pmatrix} \lambda_1^{|d_R-d_L|} & \lambda_1^{|d_R-s_L|} & \lambda_1^{|d_R-b_L|} \\ \lambda_1^{|s_R-d_L|} & \lambda_1^{|s_R-s_L|} & \lambda_1^{|s_R-b_L|} \\ \lambda_1^{|b_R-d_L|} & \lambda_1^{|b_R-s_L|} & \lambda_1^{|b_R-b_L|} \end{pmatrix} \tag{3.32}$$

with similar expressions for M_U and M_l. We thus obtain:

$$M_U \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^4 & \lambda_1^4 \\ \lambda_1 & 1 & 1 \\ \lambda_1 & 1 & 1 \end{pmatrix}, \quad M_D \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^2 & \lambda_1^2 \\ \lambda_1 & 1 & 1 \\ \lambda_1 & 1 & 1 \end{pmatrix}, \quad M_l \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^6 & \lambda_1^6 \\ \lambda_1^3 & 1 & 1 \\ \lambda_1^3 & 1 & 1 \end{pmatrix} \tag{3.33}$$

The natural mass predictions are then:

$$m_u \simeq m_d \simeq m_e \simeq \lambda_1^3 \tag{3.34}$$

while all other masses are of $\mathcal{O}(1)$.

If we extend the PCCSs to $U_1(1) \times U_2(1) \times U_3(1)$ (with charges Q_1, Q_2 and Q_3 respectively) then we can combine the different symmetry breaking parameters λ_1, λ_2 and λ_3 in the same way as we did for the β_a in the $SU_1(3) \times SU_2(3) \times SU_3(3)$ case. That is, we can take:

$$M(i,j) \simeq \prod_{a=1}^{3} \lambda_a^{|Q_{ai}-Q_{aj}|} \tag{3.35}$$

which again means that each component symmetry is broken independently. The mass matrices would then be given by:

$$\begin{aligned} M_U &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^4\lambda_2 & \lambda_1^4\lambda_3 \\ \lambda_1\lambda_2^4 & \lambda_2^3 & \lambda_2^4\lambda_3 \\ \lambda_1\lambda_3^4 & \lambda_2\lambda_3^4 & \lambda_3^3 \end{pmatrix} \\ M_D &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^2\lambda_2 & \lambda_1^2\lambda_3 \\ \lambda_1\lambda_2^2 & \lambda_2^3 & \lambda_2^2\lambda_3 \\ \lambda_1\lambda_3^2 & \lambda_2\lambda_3^2 & \lambda_3^3 \end{pmatrix} \\ M_l &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^6\lambda_2^3 & \lambda_1^6\lambda_3^3 \\ \lambda_1^3\lambda_2^6 & \lambda_2^3 & \lambda_2^6\lambda_3^3 \\ \lambda_1^3\lambda_3^6 & \lambda_2^3\lambda_3^6 & \lambda_3^3 \end{pmatrix} \end{aligned} \tag{3.36}$$

with order of magnitude mass predictions:

$$\begin{aligned} m_u &\simeq m_d \simeq m_e \simeq \lambda_1^3 \\ m_c &\simeq m_s \simeq m_\mu \simeq \lambda_2^3 \\ m_t &\simeq m_b \simeq m_\tau \simeq \lambda_3^3 \end{aligned} \tag{3.37}$$

This is similar to the $SU_1(2) \times SU_2(2) \times SU_3(2)$ case with the generation gaps easily accounted for by taking:

$$\lambda_1^3 \ll \lambda_2^3 \ll \lambda_3^3 \tag{3.38}$$

but intra-generation splitting remaining unexplained.

The expression for $M(i,j)$ in (3.35), however, is dependent on which linear combinations of the 3 generators Y_1, Y_2, and Y_3 we choose to span the $U_1(1) \times U_2(1) \times U_3(1)$ space (it is implicitly assumed in (3.36) that the charges Q_a correspond to the generators Y_a ($a = 1,2,3$)). This is not completely satisfactory as we have no physical principle governing such a choice. What we really want instead of (3.35) is an expression for $M(i,j)$ which is invariant under any basis changes. Such an expression is provided courtesy of a general metric $\mathcal{G}$ in the charge space which gives:

$$\begin{aligned} M(i,j) &\simeq \exp(-\sqrt{(\mathbf{Q_i} - \mathbf{Q_j})\, \mathcal{G}\, (\mathbf{Q_i} - \mathbf{Q_j})}) \\ &\equiv \exp(-\sqrt{(Q_{ai} - Q_{aj})\, g_{ab}\, (Q_{bi} - Q_{bj})}) \end{aligned} \tag{3.39}$$

where $\mathbf{Q_i} = (Q_{1i}, Q_{2i}, Q_{3i})$ with Q_{1i} defined just as for (3.31) *etc* . This obviously still encompasses the motivation of (3.31) that the charge difference should be indicative of the strength of the suppression. The mass matrices generated by (3.39) are:

$$\begin{aligned} M_l &\simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & e^{-\sqrt{36g_{11}+9g_{22}-36g_{12}}} & e^{-\sqrt{36g_{11}+9g_{33}-36g_{13}}} \\ e^{-\sqrt{9g_{11}+36g_{22}-36g_{12}}} & e^{-\sqrt{9g_{22}}} & e^{-\sqrt{36g_{22}+9g_{33}-36g_{23}}} \\ e^{-\sqrt{9g_{11}+36g_{33}-36g_{13}}} & e^{-\sqrt{9g_{22}+36g_{33}-36g_{23}}} & e^{-\sqrt{9g_{33}}} \end{pmatrix} \\ M_U &\simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & e^{-\sqrt{16g_{11}+g_{22}-8g_{12}}} & e^{-\sqrt{16g_{11}+g_{33}-8g_{13}}} \\ e^{-\sqrt{g_{11}+16g_{22}-8g_{12}}} & e^{-\sqrt{9g_{22}}} & e^{-\sqrt{16g_{22}+g_{33}-8g_{23}}} \\ e^{-\sqrt{g_{11}+16g_{33}-8g_{13}}} & e^{-\sqrt{g_{22}+16g_{33}-8g_{23}}} & e^{-\sqrt{9g_{33}}} \end{pmatrix} \\ M_D &\simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & e^{-\sqrt{4g_{11}+g_{22}+4g_{12}}} & e^{-\sqrt{4g_{11}+g_{33}+4g_{13}}} \\ e^{-\sqrt{g_{11}+4g_{22}+4g_{12}}} & e^{-\sqrt{9g_{22}}} & e^{-\sqrt{4g_{22}+g_{33}+4g_{23}}} \\ e^{-\sqrt{g_{11}+4g_{33}+4g_{13}}} & e^{-\sqrt{g_{22}+4g_{33}+4g_{23}}} & e^{-\sqrt{9g_{33}}} \end{pmatrix} \end{aligned} \tag{3.40}$$

and should be compared with (3.36). In a large region of parameter space it is still natural to obtain expressions for the masses which are effectively the same as (3.37):

$$\begin{aligned} m_u &\simeq m_d \simeq m_e \simeq e^{-\sqrt{9g_{11}}} \\ m_c &\simeq m_s \simeq m_\mu \simeq e^{-\sqrt{9g_{22}}} \\ m_t &\simeq m_b \simeq m_\tau \simeq e^{-\sqrt{9g_{33}}} \end{aligned} \tag{3.41}$$

and again the generation gaps are easily explained by taking:

$$g_{11} > g_{22} > g_{33} \tag{3.42}$$

but splitting within generations is not accounted for. Of course, there are textures obtainable from (3.40) other than those leading to (3.41), and these will be explored in due course.

A smaller set of abelian PCCSs than $U_1(1) \times U_2(1) \times U_3(1)$ can be obtained by taking the appropriate g_{ab} to be zero.

(iv) $U(1)_f$

The fermion charges Q_f of the $U(1)_f$ must not lead to any gauge or mixed anomalies, but once an anomaly-free set of charges has been found the $U(1)_f$ should be treated exactly as the $U_a(1)$ discussed previously; it is merely another gauged abelian symmetry. In particular, the charge vectors $\mathbf{Q_i}$ of (3.39) are expanded in an obvious manner to $\mathbf{Q_i} = (Q_{1i}, Q_{2i}, Q_{3i}, Q_{fi})$ and the metric $\mathcal{G}$ is enlarged to encompass this extra component. Obviously no specific mass matrices can be discussed without an anomaly-free charge set so explicit examples are postponed until later.

3.3.3 Putting It All Together

We finally have to decide on some ansatz for $M(i,j)$ when several of these group components are all partially conserved. We shall in fact use 3 different ansätze. These are:

(1) Product Ansatz

Here we assume that all symmetry breaking parameters combine in product form so that:

$$M(i,j) = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 \tag{3.43}$$

where:

- the abelian symmetry breaking factor is, by a simple extension of (3.35),

$$\mathcal{M}_1 \simeq [\prod_{c=1}^{3} \lambda_c^{|Q_{ci}-Q_{cj}|}] \lambda_f^{|Q_{fi}-Q_{fj}|} \tag{3.44}$$

  and λ_f is the symmetry breaking parameter of $U(1)_f$;

- the $SU_1(2) \times SU_2(2) \times SU_3(2)$ symmetry breaking factor is, from (3.27),

$$\mathcal{M}_2 \simeq \epsilon_j \tag{3.45}$$

- while the $SU_1(3) \times SU_2(3) \times SU_3(3)$ symmetry breaking factor is, from (3.26),

$$\mathcal{M}_3 \simeq \begin{cases} 1 & \text{for quarks if } i = j \text{, or for leptons} \\ \beta_i \beta_j & \text{otherwise} \end{cases} \quad (3.46)$$

This expression (3.43) is relevant for the full $SMG^3 \times U(1)_f$ algebra, but we repeat that any smaller algebra can be obtained by setting particular symmetry breaking parameters equal to 1. Note that we will always take (Y_1, Y_2, Y_3, Y_f) as a basis for the $U_1(1) \times U_2(1) \times U_3(1) \times U(1)_f$ space, so that the charges (Q_1, Q_2, Q_3, Q_f) correspond to the generators in the obvious manner. All models are initially analysed using this ansatz.

(2) Mixed Ansatz

Here we use a general metric in the abelian charge space to get:

$$M(i,j) = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 \quad (3.47)$$

where the abelian symmetry breaking factor is, by a simple extension of (3.39):

$$\mathcal{M}_1 \simeq \exp(-\sqrt{(Q_{ai} - Q_{aj})\, g_{ab}\, (Q_{bi} - Q_{bj})}) \quad (3.48)$$

with $a, b = 1, 2, 3, f$ while $\mathcal{M}_{2,3}$ are as in (3.45) and (3.46). A smaller abelian algebra is obtained by setting particular metric elements g_{ab} to 0. All models are analysed using this ansatz.

(3) Metric Ansatz

This will only apply to models whose non-abelian PCCSs are subgroups of $SU_1(2) \times SU_2(2) \times SU_3(2)$. We would like to make a slight relaxation of the assumption that all non-abelian symmetries are broken separately and independently of the abelian ones; after all, the $SU(2)_L$ and $U(1)_Y$ of the familiar electroweak theory are simultaneously broken to $U(1)_{\mathrm{em}}$ by the Weinberg-Salam Higgs field. We do this by incorporating the $SU_a(2)$ symmetry breaking terms into the general metric structure which previously only applied to the abelian symmetries. Specifically, we take:

$$M(i,j) \simeq \exp(-\sqrt{(\mathbf{Q_i} - \mathbf{Q_j})\, \mathcal{G}\, (\mathbf{Q_i} - \mathbf{Q_j})}) \quad (3.49)$$

where $\mathbf{Q_i} = (2I^3_{1i}, 2I^3_{2i}, 2I^3_{3i}, Q_{1i}, Q_{2i}, Q_{3i}, Q_{fi})$ and I^3_a is the quantum number corresponding to the usual diagonal generator T^3_a of $SU_a(2)$.

3.3.4 Comments

Note that the mixed ansatz is generally less suppressive than the product ansatz (the same can be said of the metric ansatz compared to the mixed one). With the natural identification:

$$\begin{aligned} \lambda_c &\leftrightarrow e^{-\sqrt{g_{cc}}} \quad (c=1,2,3) \\ \lambda_f &\leftrightarrow e^{-\sqrt{g_{ff}}} \end{aligned} \tag{3.50}$$

any $M(i,j)$ defined by (3.47) is at least as big as the corresponding element defined by (3.43). We only expect that this easing of suppression might make a real difference when in (3.43) the abelian charge differences are large and/or two or more of the λ_i (i=1,2,3) are very small. Then two (or more) small suppression factors will not combine as restrictively as in the product ansatz.

Note further that, with this natural identification, each ansatz yields the same suppression for a matrix element which is affected by only one PCCS.

Finally, it should be said that (3.48) is not the most general parameterisation of the breaking of the abelian symmetries, as evidenced by the fact that the product ansatz is not a special case of the mixed ansatz. The completely general case would have extra parameters in the abelian sector.

3.4 Selection of Suitable Models

Having discussed how the mass matrices will be constructed, we now move on to consider which specific PCCSs merit a detailed examination. We demand that a model should first of all promise to provide the mass gaps *between* the generations, and secondly promise to produce structure *within* the generations.

3.4.1 Inter-Generation Splitting

The first priority in the search for a model is to find one which naturally accounts for the huge mass differences between the generations - this is the single most compelling feature of the mass hierarchy. As a first approximation, we take the mass scale of each generation (m_1, m_2, and m_3 respectively) to be set by its u-type quark; structure like $m_b \ll m_t$ and $m_s \ll m_c$ will be addressed later. We thus require that a set of PCCSs gives:

$$m_1 \ll m_2 \ll m_3 \sim 1 \tag{3.51}$$

where we are assuming that the top mass is unsuppressed order of magnitude wise *w.r.t.* $\langle\phi\rangle_{\text{ws}}$. This is consistent with its expected appearance in the approximate range 100–200 GeV [42, 43]. The above discussion of component symmetries of Category (8) groups immediately suggests 4 candidate sets of PCCSs which might achieve this required hierarchy in the values of m_1, m_2 and m_3:

$$\begin{aligned} G'_1 &= U_1(1) \times SU_2(2) \\ G'_2 &= U_1(1) \times U_2(1) \\ G'_3 &= SU_1(2) \times SU_2(2) \\ G'_4 &= SU_1(2) \times U_2(1) \end{aligned} \tag{3.52}$$

The full groups G_i (i=1,. . .,4) corresponding to these PCCSs must then satisfy:

$$\begin{array}{rcl} H_1 \equiv & SU(3) \times SU_2(2) \times SU_{13}(2) \times U_1(1) \times U_{23}(1) & \subseteq G_1 \subseteq SMG^3 \\ H_2 \equiv & SU(3) \times SU(2) \times U_1(1) \times U_2(1) \times U_3(1) & \subseteq G_2 \subseteq SMG^3 \\ H_3 \equiv & SU(3) \times SU_1(2) \times SU_2(2) \times SU_3(2) \times U(1) & \subseteq G_3 \subseteq SMG^3 \\ H_4 \equiv & SU(3) \times SU_1(2) \times SU_{23}(2) \times U_2(1) \times U_{13}(1) & \subseteq G_4 \subseteq SMG^3 \end{array} \tag{3.53}$$

The H_i (i=1,. . .,4) defined in (3.53) are the smallest groups containing both G'_i and the SMG as subgroups. The ansätze, however, care only for the G'_i and not for the G_i.

We make no mention of abelian flavour symmeteries at this point. Besides their omission being aesthetically more pleasing, this is because we were initially motivated to study SMG^3 and its subgroups for the reasons outlined during the discussion of anti-grand unification in Chapter 1. It is interesting to see how far we can go in our analysis without being forced to postulate the existence of such flavour symmetries. Anyway, in [35] the gauge group $SMG \times U(1)_f$ with a partially conserved $U(1)_f$ was dismissed as a candidate for the generation of a realistic mass hierarchy and, although the results in [30] are good for an $SMG \times U(1)_f \times U(1)'_f$ model with the abelian flavour symmetries approximately conserved (strictly speaking the symmetry group is $SMG \times Z_3 \times Z_5$), the situation regarding anomaly cancellation is unclear.

The product ansatz mass matrices $M_{U,D,l}$ corresponding to each candidate in (3.52) are easily seen from (3.43) to be as in Table 3.2. Natural order of magnitude mass predictions for the various possible textures of these matrices are shown in Table 3.3.

PCCSs	M_U	M_D	M_l
G_1'	$\begin{pmatrix} \lambda_1^3 & \epsilon_2\lambda_1^4 & \lambda_1^4 \\ \lambda_1 & \epsilon_2 & 1 \\ \lambda_1 & \epsilon_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \lambda_1^3 & \epsilon_2\lambda_1^2 & \lambda_1^2 \\ \lambda_1 & \epsilon_2 & 1 \\ \lambda_1 & \epsilon_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \lambda_1^3 & \epsilon_2\lambda_1^6 & \lambda_1^6 \\ \lambda_1^3 & \epsilon_2 & 1 \\ \lambda_1^3 & \epsilon_2 & 1 \end{pmatrix}$
G_2'	$\begin{pmatrix} \lambda_1^3 & \lambda_1^4\lambda_2 & \lambda_1^4 \\ \lambda_1\lambda_2^4 & \lambda_2^3 & \lambda_2^4 \\ \lambda_1 & \lambda_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \lambda_1^3 & \lambda_1^2\lambda_2 & \lambda_1^2 \\ \lambda_1\lambda_2^2 & \lambda_2^3 & \lambda_2^2 \\ \lambda_1 & \lambda_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \lambda_1^3 & \lambda_1^6\lambda_2^3 & \lambda_1^6 \\ \lambda_1^3\lambda_2^6 & \lambda_2^3 & \lambda_2^6 \\ \lambda_1^3 & \lambda_2^3 & 1 \end{pmatrix}$
G_3'	$\begin{pmatrix} \epsilon_1 & \epsilon_2 & 1 \\ \epsilon_1 & \epsilon_2 & 1 \\ \epsilon_1 & \epsilon_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon_1 & \epsilon_2 & 1 \\ \epsilon_1 & \epsilon_2 & 1 \\ \epsilon_1 & \epsilon_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon_1 & \epsilon_2 & 1 \\ \epsilon_1 & \epsilon_2 & 1 \\ \epsilon_1 & \epsilon_2 & 1 \end{pmatrix}$
G_4'	$\begin{pmatrix} \epsilon_1 & \lambda_2 & 1 \\ \epsilon_1\lambda_2^4 & \lambda_2^3 & \lambda_2^4 \\ \epsilon_1 & \lambda_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon_1 & \lambda_2 & 1 \\ \epsilon_1\lambda_2^2 & \lambda_2^3 & \lambda_2^2 \\ \epsilon_1 & \lambda_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon_1 & \lambda_2^3 & 1 \\ \epsilon_1\lambda_2^6 & \lambda_2^3 & \lambda_2^6 \\ \epsilon_1 & \lambda_2^3 & 1 \end{pmatrix}$

Table 3.2: Product ansatz mass matrices for G_i' (i=1,...,4).

PCCSs	M_U				M_D				M_l			
	Texture	m_t	m_c	m_u	Texture	m_b	m_s	m_d	Texture	m_τ	m_μ	m_e
G_1'	1	1	ϵ_2	λ_1^3	1,2, 6,7,8	1	ϵ_2	λ_1^3	1,6	1	ϵ_2	λ_1^3
	4	1	λ_1	$\epsilon_2\lambda_1^2$	3,4,5	1	λ_1	$\epsilon_2\lambda_1^2$				
G_2'	1	1	λ_2^3	λ_1^3	1,2, 6,7,8	1	λ_2^3	λ_1^3	1	1	λ_2^3	λ_1^3
G_3'	1,2	1	ϵ_2	ϵ_1	1,2, 6,7,8	1	ϵ_2	ϵ_1	1,2	1	ϵ_2	ϵ_1
G_4'	9	1	λ_2	$\epsilon_1\lambda_2^2$	9	1	λ_2	$\epsilon_1\lambda_2^2$	1	1	λ_2^3	ϵ_1

Table 3.3: Product ansatz mass predictions for G_i' (i=1,...,4).

The PCCSs G'_1, G'_2 and G'_3 can thus fulfill their promise to satisfactorily provide realistic generation gaps, whereas G'_4 features the very poor order of magnitude prediction $m_\mu \simeq m_{c,s}^3$ and is therefore discarded at this point. Note that this prediction is unchanged if we use a different ansatz to construct the matrices because the matrix elements involved are suppressed by only one PCCS.

3.4.2 Intra-Generation Splitting

Obviously none of the 3 remaining candidates has anything to say about mass-splitting *within* the generations. We are particularly interested in accounting for:

$$\begin{aligned} m_b &\ll m_t \\ m_s &\ll m_c \end{aligned} \tag{3.54}$$

as these are the most prominent order of magnitude features after (3.51). The situation concerning leptonic masses is complicated by their contrasting running behaviour as governed by the renormalisation group equations. In Chapter 4 we assume that our ansätze hold at some fundamental high energy scale which we take to be the Planck scale ($M_P \simeq 10^{19}$ GeV) and so our order of magnitude predictions are applicable only at this scale. In [44] the fermion masses are evolved from 1 GeV to M_P using the SM renormalisation group equations. The order of magnitude results of this analysis of particular interest to us are:

- lepton masses change very little (they get smaller by approximately 10%)
- non-top quark masses get smaller by a factor of $\mathcal{O}(5)$.

The renormalisation group thus naturally splits the leptons from the quarks as our ansätze are evolved back down to 1 GeV. We bear this in mind throughout our algebra, concentrating for the time being on (3.54).

In fact, requiring the natural appearance of a $t-b$ splitting leaves us at a crossroads. What kind of PCCSs can generate this mass gap? Non-abelian candidates ($SU_a(3)$ and $SU_a(2)$) are of no use because, as is obvious from (3.26) and (3.27), they have the same effect on both M_U and M_D. Abelian symmetries must therefore provide the solution, but there is an immediate problem. Any partially conserved subgroup of SMG^3 (using any ansatz) yields identical diagonal elements (order of magnitude wise) in the 3 fermion mass matrices:

$$M_U(i,i) \simeq M_D(i,i) \simeq M_l(i,i) \quad (i = 1,2,3) \tag{3.55}$$

Thus if m_t receives its dominant contribution from a diagonal element of M_U, m_b *cannot* naturally be made lighter than it. This problem can apparently be resolved in two quite different ways:

- **Method 1**

  Noting that abelian symmetries ($U_a(1)$) treat the off-diagonal quark matrix elements differently, expand the PCCSs to include $U_1(1) \times U_2(1) \times U_3(1)$ and arrange for some/all of the 3rd generation masses to receive their dominant contributions from off-diagonal matrix elements. A $t - b$ splitting might then be obtained.

- **Method 2**

  Simply introduce a partially conserved abelian flavour symmetry $U(1)_f$ and arrange that the fermion charges satisfy:

  $$|Q_f(b_L) - Q_f(b_R)| > |Q_f(t_L) - Q_f(t_R)| \tag{3.56}$$

  which might then naturally generate $m_b \ll m_t$.

Method 1 will be discussed in Chapter 5 while Method 2 will be dealt with now in Chapter 4.

"I still don't know what I was waiting for
And my time was running wild
A million dead-end streets
Every time I thought I'd got it made
It seemed the taste was not so sweet"

David Bowie
Changes

Chapter 4

Intra-Generation Structure from Abelian Flavour Symmetry

4.1 Introduction

In this chapter, we discuss Method 2 of generating intra-generation mass-splitting: the introduction of a gauged and partially conserved abelian flavour symmetry. For each candidate set of PCCSs, we find all anomaly-free flavour charge sets subject to certain constraints. After stating the data which we will attempt to fit, we explain the fitting procedure itself. We finally give results and discussion for all models in this Method 2 scenario.

Throughout this discussion of Method 2, we will assume that the 3rd generation masses m_t, m_b and m_τ receive their dominant contributions from $M_U(3,3)$, $M_D(3,3)$ and $M_l(3,3)$ respectively. That is, we assume that the 3rd generation mass eigenstates are approximately equal to the 3rd generation symmetry eigenstates (see (1.16)). This natural assumption is a feature of virtually all models of fermion mass matrices (*e.g.* the Fritzsch matrices [23]). It has been true of all mass matrices encountered here so far and, moreover, such an assumption is implicit in (3.56). It is sensible to insist on it here because a discussion of off-diagonal 3rd generation masses is tied up with Method 1. A useful consequence is that it allows us to make statements about the 3rd generation masses purely from knowledge of the symmetry transformation properties of the corresponding Weyl states.

4.2 The Flavour Charges

The remaining candidate sets of PCCSs are, from (3.52), but now including $U(1)_f$:

$$\begin{aligned}
G_1' \times U(1)_f &= U_1(1) \times SU_2(2) \times U(1)_f \\
G_2' \times U(1)_f &= U_1(1) \times U_2(1) \times U(1)_f \\
G_3' \times U(1)_f &= SU_1(2) \times SU_2(2) \times U(1)_f
\end{aligned} \tag{4.1}$$

The no-anomaly equations for triangle diagrams involving one or more $U(1)_f$ gauge bosons must be solved for each set of PCCSs in order to provide acceptable fermion charges (all other anomalies cancel because SMG^3 and its subgroups are anomaly-free in exactly the same manner as the SMG itself). To do this, the groups G_i (i=1,2,3) of (3.53) must be specified, whence the full groups are $G_i \times U(1)_f$. But how should we go about choosing the G_i?

The no-anomaly equations for $H_i \times U(1)_f$ (the H_i are defined in (3.53)) form a subset of those for $SMG^3 \times U(1)_f$. More generally, if we consider successively larger G_i in the chain of groups between H_i and SMG^3:

$$H_i \equiv G_i^{(1)} \subset G_i^{(2)} \subset G_i^{(3)} \subset \ldots \subset SMG^3 \tag{4.2}$$

we find that the set of no-anomaly equations for some $G_i^{(n)} \times U(1)_f$ (i=1,2,3) forms a subset of those for $G_i^{(n+1)} \times U(1)_f$. This is because each step down the chain involves taking the diagonal subgroup of cross product *i.e.* we perform one of:

$$\begin{aligned}
SU_a(3) \times SU_b(3) &\rightarrow SU_{ab}(3) \\
SU_a(2) \times SU_b(2) &\rightarrow SU_{ab}(2) \\
U_a(1) \times U_b(1) &\rightarrow U_{ab}(1)
\end{aligned} \tag{4.3}$$

where $a, b = 1, 2, 3$. Obviously there are many different chains of subgroups corresponding to the $G_i^{(n)}$ of (4.2), but this argument holds for each such chain. As far as anomaly cancellation is concerned, any such step means that we add together each corresponding pair of no-anomaly equations. For example, for the toy symmetry group $SU_a(3) \times SU_b(3) \times U(1)_f$ we have three anomalies to cancel:

$$\begin{aligned}
\mathrm{Tr}\,[SU_a(3)^2\, U(1)_f] &= 0 \\
\mathrm{Tr}\,[SU_b(3)^2\, U(1)_f] &= 0 \\
\mathrm{Tr}\,[U(1)_f^3] &= 0
\end{aligned} \tag{4.4}$$

while for $SU_{ab}(3) \times U(1)_f$ we have only two:

$$\begin{aligned} \mathrm{Tr}\,[SU_{ab}(3)^2\,U(1)_f] &= \mathrm{Tr}\,[SU_a(3)^2\,U(1)_f] + \mathrm{Tr}\,[SU_b(3)^2\,U(1)_f] = 0 \\ \mathrm{Tr}\,[U(1)_f^3] &= 0 \end{aligned} \tag{4.5}$$

The anomaly-free $U(1)_f$ charge sets for $G_i^{(n+1)} \times U(1)_f$ thus form a subset of those for $G_i^{(n)} \times U(1)_f$, as claimed. We will therefore assume that $G_i = H_i$ (i=1,2,3) and once we have an anomaly-free $U(1)_f$ charge set for $H_i \times U(1)_f$ it is easy to check if it also constitutes an anomaly-free set for some larger group $J_i \times U(1)_f$ where $H_i \subset J_i \subseteq SMG^3$ *i.e.* having assumed that the full symmetry group is $H_i \times U(1)_f$, it is simple to see if a particular $U(1)_f$ charge set is in fact compatible with a larger group. So we now go on to consider the gauge groups $H_i \times U(1)_f$ (i=1,2,3) with PCCSs $U_1(1) \times SU_2(2) \times U(1)_f$, $U_1(1) \times U_2(1) \times U(1)_f$ and $SU_1(2) \times SU_2(2) \times U(1)_f$ respectively.

Why do we only consider the addition of one abelian flavour symmetry? The sole purpose of such a symmetry is to provide structure within the generations, in particular the 2nd and 3rd generations (recall we have (3.54) in mind). It may seem reasonable to consider a model like $H_i \times U(1)_f \times U(1)'_f$ where $U(1)_f$ and $U(1)'_f$ act only on the 2nd and 3rd generations respectively, providing the required structure within each generation. However, it is straightforward to show from the anomaly cancellation conditions that any abelian symmetry which affects only the a^{th} generation must be a copy of the usual weak hypercharge (as far as that generation is concerned). That is, the appropriate fermion charges must be multiples of those given in (3.30) and the abelian symmetry is nothing other than $U_a(1)$, already dismissed as a candidate for explaining (3.54) as far as Method 2 is concerned. We would therefore be forced to extend the influence of both $U(1)_f$ and $U(1)'_f$ beyond a single generation. It is thus very awkward to have separate abelian symmetries neatly providing $m_b \ll m_t$ and $m_s \ll m_c$ without "interfering" with one another and/or disturbing already satisfactory mass relations of the partially conserved H_i (*e.g.* $m_u \simeq m_d$; see Table 3.3). So it seems simpler to maintain our original assumption and consider only a single abelian flavour symmetry, which will generate structure within both the 2nd and 3rd generations. In any case, more symmetries mean more parameters and less predictability.

What about the 1st generation states and their charges? Since the partially conserved H_i already account for:

$$m_{u,d,e} \ll \text{all other masses} \tag{4.6}$$

and the renormalisation group for:

$$m_e < m_{u,d} \tag{4.7}$$

the effect of an approximately conserved $U(1)_f$ is likely to be counter-productive as far as this generation is concerned. We thus take the charges Q_f of all 1st generation Weyl states to be zero *i.e.* with $Q_f(u_R) \equiv u_R$ *etc*, we take:

$$d_L = u_R = d_R = e_L = e_R = 0 \tag{4.8}$$

As already explained, the no-anomaly equations for any $H_i \times U(1)_f$ are easily obtained from the anomalies of the group $SMG^3 \times U(1)_f$, which were given in (2.27)-(2.40). These are restated here for convenience:

$$A_1 = \mathrm{Tr}\,[\mathrm{SU_1(3)^2U(1)_f}] = 2d_L - u_R - d_R = 0 \tag{4.9}$$

$$A_2 = \mathrm{Tr}\,[\mathrm{SU_2(3)^2U(1)_f}] = 2s_L - c_R - s_R = 0 \tag{4.10}$$

$$A_3 = \mathrm{Tr}\,[\mathrm{SU_3(3)^2U(1)_f}] = 2b_L - t_R - b_R = 0 \tag{4.11}$$

$$A_4 = \mathrm{Tr}\,[\mathrm{SU_1(2)^2U(1)_f}] = 3d_L + e_L = 0 \tag{4.12}$$

$$A_5 = \mathrm{Tr}\,[\mathrm{SU_2(2)^2U(1)_f}] = 3s_L + \mu_L = 0 \tag{4.13}$$

$$A_6 = \mathrm{Tr}\,[\mathrm{SU_3(2)^2U(1)_f}] = 3b_L + \tau_L = 0 \tag{4.14}$$

$$A_7 = \mathrm{Tr}\,[\mathrm{U_1(1)^2U(1)_f}] = d_L - 8u_R - 2d_R + 3e_L - 6e_R = 0 \tag{4.15}$$

$$A_8 = \mathrm{Tr}\,[\mathrm{U_2(1)^2U(1)_f}] = s_L - 8c_R - 2s_R + 3\mu_L - 6\mu_R = 0 \tag{4.16}$$

$$A_9 = \mathrm{Tr}\,[\mathrm{U_3(1)^2U(1)_f}] = b_L - 8t_R - 2b_R + 3\tau_L - 6\tau_L = 0 \tag{4.17}$$

$$A_{10} = \mathrm{Tr}\,[\mathrm{U_1(1)U(1)_f^2}] = d_L^2 - 2u_R^2 + d_R^2 - e_L^2 + e_R^2 = 0 \tag{4.18}$$

$$A_{11} = \mathrm{Tr}\,[\mathrm{U_2(1)U(1)_f^2}] = s_L^2 - 2c_R^2 + s_R^2 - \mu_L^2 + \mu_R^2 = 0 \tag{4.19}$$

$$A_{12} = \mathrm{Tr}\,[\mathrm{U_3(1)U(1)_f^2}] = b_L^2 - 2t_R^2 + b_R^2 - \tau_L^2 + \tau_R^2 = 0 \tag{4.20}$$

$$\begin{aligned} A_{13} = \mathrm{Tr}\,[\mathrm{U(1)_f^3}] &= 6d_L^3 - 3u_R^3 - 3d_R^3 + 2e_L^3 - e_R^3 \\ &\quad +6s_L^3 - 3c_R^3 - 3s_R^3 + 2\mu_L^3 - \mu_R^3 \\ &\quad +6b_L^3 - 3t_R^3 - 3b_R^3 + 2\tau_L^3 - \tau_R^3 = 0 \end{aligned} \tag{4.21}$$

$$A_{14} = \mathrm{Tr}\,[(\mathrm{graviton})^2\mathrm{U(1)_f}] = 6d_L - 3u_R - 3d_R + 2e_L - e_R$$

$$+6s_L - 3c_R - 3s_R + 2\mu_L - \mu_R$$
$$+6b_L - 3t_R - 3b_R + 2\tau_L - \tau_R = 0 \qquad (4.22)$$

Note that (4.8) immediately gives:

$$A_1 = A_4 = A_7 = A_{10} = 0 \qquad (4.23)$$

In order to avoid notational confusion of the charges in (4.9)-(4.22) with the corresponding Weyl states, from now on we denote the states with a tilde *e.g.* $\tilde{d_R}$.

Note that the requirement:

$$A_3 = 0 \qquad (4.24)$$

which must hold for any group of which $SU_3(3)$ is a subgroup, is very troublesome. It simply gives (since $t_L \equiv b_L$):

$$|t_L - t_R| = |b_L - b_R| \qquad (4.25)$$

thus destroying any hope of generating $m_b \ll m_t$ so long as we assume that the 3rd generation masses receive their dominant contributions from the appropriate $M_i(3,3)$. In particular, this means that the full gauge group *cannot* be $SMG^3 \times U(1)_f$ within this Method 2 framework; this is a blow to the original motivation behind this work which was to suppose that the fundamental gauge group was SMG^3 (or at least contained SMG^3 as a subgroup) and that the fermion masses could be explained by taking some subgroup of it to be partially conserved. (This assumption regarding the 3rd generation mass eigenstates will be dropped in Chapter 5, allowing us to recover $SMG^3(\times U(1)_f)$ as the fundamental gauge group).

The general method of selecting anomaly-free charge sets for each $H_i \times U(1)_f$ (i=1,2,3) is as follows. For each case the no-anomaly equations are constructed using (4.9)-(4.22). All integer solutions are then found, subject to certain conditions:

1. All fermion charges satisfy $|Q_f| \leq 15$. This number is arbitrary.
2. The 3rd generation charges should satisfy:

$$|b_L - b_R|, |\tau_L - \tau_R| \begin{cases} > 0 & \text{if } t_L = t_R \\ \geq 3|t_L - t_R| & \text{if } t_L \neq t_R \end{cases} \qquad (4.26)$$

Ideally we would like to assume:

$$t_L = t_R \qquad (4.27)$$

so that (as stated before) m_t is unsuppressed, order of magnitude wise, relative to $\langle\phi\rangle_{\mathrm{ws}}$ ($\tilde{t_L}$ and $\tilde{t_R}$ have identical transformation properties *w.r.t.* all other PCCSs under consideration here). We lend most credence to models which have this feature, but also consider those for which m_t is very slightly suppressed.

Certainly, for such models the condition (4.26) could be tightened. For if:

$$|b_L - b_R| = 3|t_L - t_R| \tag{4.28}$$

then:

$$\begin{aligned} m_b &= |\gamma_{33}^D|\epsilon^3\langle\phi\rangle_{\mathrm{ws}} \\ m_t &= |\gamma_{33}^U|\epsilon\langle\phi\rangle_{\mathrm{ws}} \end{aligned} \tag{4.29}$$

and:

$$\epsilon \sim \sqrt{\frac{m_b}{m_t}} \sim \frac{1}{5} \text{ or } \frac{1}{6} \tag{4.30}$$

We would then really have to rely on the "$\mathcal{O}(1)$" $\gamma_{33}^{U,D}$ attaining "nice" values in order to keep our whole approach consistent with $\langle\phi\rangle_{\mathrm{ws}} = 174$ GeV. Nevertheless, we judge it prudent to accept all solutions provided they satisfy (4.26), looking unfavourably upon them later should this problem become impossible to ignore.

3. The 3rd generation charges should also satisfy:

$$\frac{2}{3}(|b_L-b_R|-|t_L-t_R|) \leq (|\tau_L-\tau_R|-|t_L-t_R|) \leq \frac{3}{2}(|b_L-b_R|-|t_L-t_R|) \tag{4.31}$$

This effectively says:

$$\left(\frac{m_b}{m_t}\right)^{2/3} \leq \frac{m_\tau}{m_t} \leq \left(\frac{m_b}{m_t}\right)^{3/2} \tag{4.32}$$

and again this is not too restrictive; models can still be re-evaluated at the end of the day. The good $SU(5)$ GUT scale prediction $m_b = m_\tau$, though no longer thought to be numerically exact in the minimal GUT model [39], remains an excellent order of magnitude relation. We thus lend most credence to models satisfying:

$$|b_L - b_R| = |\tau_L - \tau_R| \tag{4.33}$$

Note that these first three constraints have the same consequences (as far as elimination of possible charge sets is concerned) regardless of which ansatz is under consideration, as they relate only to the elements $M_i(3,3)$ (i=U,D,l) which are affected by only one PCCS.

After all anomaly-free charge sets satisfying these constraints have been found, the fermion mass matrices are formed using the product ansatz. These display one of the textures of Chapter 3, and often a matrix is compatible with more than one texture (only the origin of the 3rd generation eigenvalues is determined *a priori*). Allowing for all possible origins of the 2nd generation masses, we then make further restrictions. If it is then observed to be unavoidable (*i.e.* an algebraic certainty) that:

4. the 2nd generation hierarchy is unsatisfactory *i.e.*

$$m_s \geq m_c \quad \text{or} \quad m_\mu \geq m_c \tag{4.34}$$

5. the relative $m_s - m_c$ and $m_b - m_t$ suppressions are unsatisfactory *i.e.*

$$\frac{m_s}{m_c} < \frac{m_b}{m_t} \quad \text{or} \quad \frac{m_b}{m_t} < \left(\frac{m_s}{m_c}\right)^3 \tag{4.35}$$

then the charge sets are discarded at this point. These are simple demands that, having already obtained a reasonable 3rd generation hierarchy, it should also be possible to obtain a reasonable 2nd generation hierarchy. Although these last two constraints are implemented only for the product ansatz, in virtually all cases a discarded model is not saved by choosing a different ansatz. In the very rare occasions where it might be saved, we choose to discard the model anyway. It is not clear whether one should trust a model whose mass hierarchy is terrible with one ansatz but acceptable with another.

4.2.1 $H_1 \times U(1)_f \equiv SU(3) \times SU_{13}(2) \times SU_2(2) \times U_1(1) \times U_{23}(1) \times U(1)_f$

Here the no-anomaly equations are:

$$\begin{aligned} A_2 + A_3 = 0, \quad A_5 = 0, \quad A_6 = 0, \quad A_8 + A_9 = 0, \\ A_{11} + A_{12} = 0, \quad A_{13} = 0, \quad A_{14} = 0 \end{aligned} \tag{4.36}$$

Table 4.1 shows how many integer solutions (with $|Q_f| \leq 15$) of these equations exist, and how many survive each of our restrictions. Tables 4.2 and 4.3 list the overall survivors and their prominent features.

Restriction	No. of surviving solutions with $\lvert Q_f\rvert \leq 15$
All anomalies cancel	514
$\lvert b_L - b_R\rvert \geq 3\lvert t_L - t_R\rvert$ and $\lvert b_L - b_R\rvert \neq 0$	119
$\lvert\tau_L - \tau_R\rvert \geq 3\lvert t_L - t_R\rvert$ and $\lvert\tau_L - \tau_R\rvert \neq 0$	84
$\frac{m_b}{m_t} \geq (\frac{m_\tau}{m_t})^{3/2}$	76
$\frac{m_\tau}{m_t} \geq (\frac{m_b}{m_t})^{3/2}$	69
$m_s < m_c$	50
$m_\mu < m_c$	49
$\frac{m_s}{m_c} \geq \frac{m_b}{m_t}$	42
$\frac{m_b}{m_t} \geq (\frac{m_s}{m_c})^3$	25
No. of overall survivors	25

Table 4.1: Selection process for anomaly-free charge sets of $H_1 \times U(1)_f$ model. Each restriction is implemented sequentially.

Note that in these $H_1 \times U(1)_f$ models the only symmetry which distinguishes the corresponding right-handed Weyl states in the 2nd and 3rd generations (*e.g.* $\tilde{s_R}$ and $\tilde{b_R}$) is $U(1)_f$ because *all* right-handed states, including those of the 2nd generation, are singlets under $SU_2(2)$. This means that the labelling of ($\tilde{s_R}$,$\tilde{b_R}$), ($\tilde{\mu_R}$,$\tilde{\tau_R}$) and ($\tilde{c_R}$,$\tilde{t_R}$) is essentially arbitrary, and there exist many solutions (not counted in Table 4.1) which are identical under the interchange of charges:

$$s_R \leftrightarrow b_R \quad \text{and/or} \quad \mu_R \leftrightarrow \tau_R \quad \text{and/or} \quad c_R \leftrightarrow t_R \tag{4.37}$$

We choose to label the states in a fashion consistent with the assumption that:

$$m_{t,b,\tau} \simeq M_{U,D,l}(3,3) \tag{4.38}$$

i.e. we take $\tilde{t_R}$ to be the state whose charge lies closest to that of $\tilde{t_L}$ *etc* .

4.2.2 $H_2 \times U(1)_f \equiv SU(3) \times SU(2) \times U_1(1) \times U_2(1) \times U_3(1) \times U(1)_f$

The no-anomaly equations for this model are:

$$\begin{array}{llll} A_2 + A_3 = 0, & A_5 + A_6 = 0, & A_8 = 0, & A_9 = 0, \\ A_{11} = 0, & A_{12} = 0, & A_{13} = 0, & A_{14} = 0 \end{array} \tag{4.39}$$

	$\begin{pmatrix} s_L & c_R & s_R & \mu_L & \mu_R \\ b_L & t_R & b_R & \tau_L & \tau_R \end{pmatrix}$	$t_L = t_R$?	$\lvert b_L - b_R \rvert = \lvert \tau_L - \tau_R \rvert$?
(1)	$\begin{pmatrix} 1 & 2 & 4 & -3 & 0 \\ -1 & -2 & -4 & 3 & 0 \end{pmatrix}$	×	√
(2)	$\begin{pmatrix} 1 & 1 & 3 & -3 & -1 \\ -1 & -1 & -3 & 3 & 1 \end{pmatrix}$	√	√
(3)	$\begin{pmatrix} 2 & 3 & 7 & -6 & -1 \\ -2 & -3 & -7 & 6 & 1 \end{pmatrix}$	×	√
(4)	$\begin{pmatrix} 2 & 1 & 5 & -6 & -3 \\ -2 & -1 & -5 & 6 & 3 \end{pmatrix}$	×	√
(5)	$\begin{pmatrix} 3 & 0 & 10 & -9 & -2 \\ -4 & -4 & -8 & 12 & 8 \end{pmatrix}$	√	√
(6)	$\begin{pmatrix} 3 & 5 & 11 & -9 & -1 \\ -3 & -5 & -11 & 9 & 1 \end{pmatrix}$	×	√
(7)	$\begin{pmatrix} 3 & 4 & 10 & -9 & -2 \\ -3 & -4 & -10 & 9 & 2 \end{pmatrix}$	×	√
(8)	$\begin{pmatrix} 3 & 2 & 8 & -9 & -4 \\ -3 & -2 & -8 & 9 & 4 \end{pmatrix}$	×	√
(9)	$\begin{pmatrix} 3 & 4 & 6 & -9 & -6 \\ -2 & 0 & -8 & 6 & 0 \end{pmatrix}$	×	√
(10)	$\begin{pmatrix} 4 & 1 & 13 & -12 & -3 \\ -5 & -5 & -11 & 15 & 9 \end{pmatrix}$	√	√
(11)	$\begin{pmatrix} 4 & 0 & 12 & -12 & -4 \\ -5 & -4 & -10 & 15 & 10 \end{pmatrix}$	×	√
(12)	$\begin{pmatrix} 4 & 7 & 15 & -12 & -1 \\ -4 & -7 & -15 & 12 & 1 \end{pmatrix}$	×	√
(13)	$\begin{pmatrix} 4 & 5 & 13 & -12 & -3 \\ -4 & -5 & -13 & 12 & 3 \end{pmatrix}$	×	√

Table 4.2: Surviving anomaly-free charge sets and their prominent features for the $H_1 \times U(1)_f$ model.

	$\begin{pmatrix} s_L & c_R & s_R & \mu_L & \mu_R \\ b_L & t_R & b_R & \tau_L & \tau_R \end{pmatrix}$	$t_L = t_R$?	$\lvert b_L - b_R \rvert = \lvert \tau_L - \tau_R \rvert$?
(14)	$\begin{pmatrix} 4 & 3 & 11 & -12 & -5 \\ -4 & -3 & -11 & 12 & 5 \end{pmatrix}$	×	√
(15)	$\begin{pmatrix} 4 & 9 & 13 & -12 & -3 \\ -3 & -5 & -15 & 9 & -3 \end{pmatrix}$	×	√
(16)	$\begin{pmatrix} 4 & 8 & 12 & -12 & -4 \\ -3 & -4 & -14 & 9 & -2 \end{pmatrix}$	×	√
(17)	$\begin{pmatrix} 4 & 7 & 11 & -12 & -5 \\ -3 & -3 & -13 & 9 & -1 \end{pmatrix}$	√	√
(18)	$\begin{pmatrix} 4 & 6 & 10 & -12 & -6 \\ -3 & -2 & -12 & 9 & 0 \end{pmatrix}$	×	√
(19)	$\begin{pmatrix} 4 & 5 & 9 & -12 & -7 \\ -3 & -1 & -11 & 9 & 1 \end{pmatrix}$	×	√
(20)	$\begin{pmatrix} 5 & 5 & 13 & -15 & -9 \\ -5 & -5 & -13 & 15 & 9 \end{pmatrix}$	√	×
(21)	$\begin{pmatrix} 5 & 4 & 14 & -15 & -6 \\ -5 & -4 & -14 & 15 & 6 \end{pmatrix}$	×	√
(22)	$\begin{pmatrix} 5 & 3 & 13 & -15 & -7 \\ -5 & -3 & -13 & 15 & 7 \end{pmatrix}$	×	√
(23)	$\begin{pmatrix} 5 & 7 & 13 & -15 & -7 \\ -4 & -3 & -15 & 12 & 1 \end{pmatrix}$	×	√
(24)	$\begin{pmatrix} 5 & 6 & 12 & -15 & -8 \\ -4 & -2 & -14 & 12 & 2 \end{pmatrix}$	×	√
(25)	$\begin{pmatrix} 5 & 5 & 11 & -15 & -9 \\ -4 & -1 & -13 & 12 & 3 \end{pmatrix}$	×	√

Table 4.3: Surviving anomaly-free charge sets and their prominent features for the $H_1 \times U(1)_f$ model (cont.).

Restriction	No. of surviving solutions with $|Q_f| \leq 15$
All anomalies cancel	1143
$|b_L - b_R| \geq 3|t_L - t_R|$ and $|b_L - b_R| \neq 0$	264
$|\tau_L - \tau_R| \geq 3|t_L - t_R|$ and $|\tau_L - \tau_R| \neq 0$	176
$\frac{m_b}{m_t} \geq (\frac{m_\tau}{m_t})^{3/2}$	134
$\frac{m_\tau}{m_t} \geq (\frac{m_b}{m_t})^{3/2}$	127
$m_s < m_c$	53
$m_\mu < m_c$	52
$\frac{m_s}{m_c} \geq \frac{m_b}{m_t}$	45
$\frac{m_b}{m_t} \geq (\frac{m_s}{m_c})^3$	6
No. of overall survivors	6

Table 4.4: Selection process for anomaly-free charge sets of $H_2 \times U(1)_f$ model. Each restriction is implemented sequentially.

and again Table 4.4 enumerates how many solutions are discarded with each successive restriction. Table 4.5 lists the survivors and their prominent features.

In this model the $U_2(1)$ symmetry adequately differentiates between all corresponding states of the 2nd and 3rd generations, so there is no ambiguity in the labelling.

4.2.3 $H_3 \times U(1)_f \equiv SU(3) \times SU_1(2) \times SU_2(2) \times SU_3(2) \times U(1) \times U(1)_f$

Here, the no-anomaly equations are exactly the same as for the $H_1 \times U(1)_f$ model. However, the arbitrariness in the labelling of right-handed states is even greater for $H_3 \times U(1)_f$, extending to the 1st generation. This is because the $U(1)_f$ is the only symmetry which differentiates between *any* corresponding right-handed states (*e.g.* $\tilde{d}_R$, $\tilde{s}_R$ and $\tilde{b}_R$). Table 4.6 enumerates how many solutions are discarded with each successive restriction, and there are no survivors left over. This is because of the relabelling freedom, which ruins the structure of the mass matrices. Put slightly differently, the suppression of the elements $M_{U,D,l}(1,2)$ and $M_{U,D,l}(1,3)$ by the $U_1(1)$ symmetry in the $H_1 \times U(1)_f$ model (see Table 3.2) is absent here in the $H_3 \times U(1)_f$ model; these elements consequently become larger and get involved in contributing to the eigenvalues

	$\begin{pmatrix} s_L & c_R & s_R & \mu_L & \mu_R \\ b_L & t_R & b_R & \tau_L & \tau_R \end{pmatrix}$	$t_L = t_R$?	$\lvert b_L - b_R\rvert = \lvert\tau_L - \tau_R\rvert$?
(1)	$\begin{pmatrix} 1 & 1 & 13 & 13 & 1 \\ -1 & -1 & -13 & -13 & -1 \end{pmatrix}$	$\surd$	$\surd$
(2)	$\begin{pmatrix} 9 & 6 & 0 & 5 & -4 \\ -9 & -6 & 0 & -5 & 4 \end{pmatrix}$	$\times$	$\surd$
(3)	$\begin{pmatrix} 13 & 7 & 1 & 9 & -3 \\ -14 & -11 & 1 & -6 & 9 \end{pmatrix}$	$\times$	$\surd$
(4)	$\begin{pmatrix} 5 & 5 & -1 & 1 & -5 \\ -5 & -5 & 1 & -1 & 5 \end{pmatrix}$	$\surd$	$\surd$
(5)	$\begin{pmatrix} 1 & -1 & -15 & -15 & -1 \\ -1 & 1 & 15 & 15 & 1 \end{pmatrix}$	$\times$	$\times$
(6)	$\begin{pmatrix} 14 & 11 & -1 & 6 & -9 \\ -14 & -11 & 1 & -6 & 9 \end{pmatrix}$	$\times$	$\surd$

Table 4.5: Surviving anomaly-free charge sets and their prominent features for the $H_2 \times U(1)_f$ model.

Restriction	No. of surviving solutions with $\lvert Q_f\rvert \leq 15$
All anomalies cancel	514
$\lvert b_L - b_R\rvert \geq 3\lvert t_L - t_R\rvert$and $\lvert b_L - b_R\rvert \neq 0$	59
$\lvert \tau_L - \tau_R\rvert \geq 3\lvert t_L - t_R\rvert$ and $\lvert \tau_L - \tau_R\rvert \neq 0$	42
$\frac{m_b}{m_t} \geq (\frac{m_\tau}{m_t})^{3/2}$	41
$\frac{m_\tau}{m_t} \geq (\frac{m_b}{m_t})^{3/2}$	17
$m_s < m_c$	3
$m_\mu < m_c$	3
$\frac{m_s}{m_c} \geq \frac{m_b}{m_t}$	0
$\frac{m_b}{m_t} \geq (\frac{m_s}{m_c})^3$	0
No. of overall survivors	0

Table 4.6: Selection process for anomaly-free charge sets of $H_3 \times U(1)_f$ model. Each restriction is implemented sequentially.

and consequently ruin the hierarchy. For example, *before* any relabelling we have from (3.43):

$$M_U(1,2) \simeq \epsilon_2 \lambda_f^{|s_L|} \simeq M_D(1,2) \tag{4.40}$$

and since these elements often form m_c and m_s respectively, this can easily lead to $m_s \simeq m_c$.

4.2.4 Comment

For all 3 models, the most restrictive condition is seen to be:

$$|b_L - b_R| \begin{cases} \neq 0 \\ \geq 3|t_L - t_R| \end{cases} \tag{4.41}$$

(see Tables 4.1, 4.4 and 4.6). But for the $H_2 \times U(1)_f$ model, the restrictions relating to the 2nd generation masses cut the number of acceptable solutions from 127 to 6, whereas for the $H_1 \times U(1)_f$ model the drop is only from 69 to 25. The difference is because the quark matrices of the latter model are compatible with two different matrix textures (any one of Textures 1–2 *and* any one of Textures 3–5) whereas those of the former are really only compatible with one texture (any one of Textures 1–2 *or* any one of Textures 6–8). It is consequently harder to dismiss charge sets for the $H_1 \times U(1)_f$ model.

Numerical analysis will thus now concentrate on the twenty-five $H_1 \times U(1)_f$ models and the six $H_2 \times U(1)_f$ models.

4.3 The Full Symmetry Groups

Having found promising anomaly-free flavour charge sets for the groups $H_1 \times U(1)_f$ and $H_2 \times U(1)_f$, it is appropriate here to return to the question of whether any of these sets is compatible with a larger group. That is, for a given set of $U(1)_f$ charges, can we find some J_i such that:

$$H_i \subset J_i \subseteq SMG^3 \tag{4.42}$$

and $J_i \times U(1)_f$ remains free of anomalies (i=1,2). The maximal such J_i would yield the largest possible symmetry group compatible with each set of flavour charges, bearing in mind the proviso of (2.1) that we are not interested in symmetries which act trivially on all of the known 45 Weyl states.

Firstly, any anomalies involving $U(1)_f$ receive no contribution from the 1st generation states since all of the 1st generation flavour charges are zero. Hence the groups $H_i \times U(1)_f$ (i=1,2) can be enlarged to include $SU_1(3) \times SU_1(2) \times U_1(1)$ as a subgroup. That is:

$$H_1 \times U(1)_f \equiv SU(3) \times [SU_{13}(2) \times SU_2(2)] \times [U_1(1) \times U_{23}(1)] \times U(1)_f \tag{4.43}$$

can be enlarged to:

$$\begin{aligned} J_1 \times U(1)_f \;&\equiv\; [SU_1(3) \times SU_{23}(3)] \times [SU_1(2) \times SU_2(2) \times SU_3(2)] \\ &\quad \times [U_1(1) \times U_{23}(1)] \times U(1)_f \end{aligned} \tag{4.44}$$

and:

$$H_2 \times U(1)_f \equiv SU(3) \times SU(2) \times [U_1(1) \times U_2(1) \times U_3(1)] \times U(1)_f \tag{4.45}$$

can be enlarged to:

$$\begin{aligned} J_2 \times U(1)_f \;&\equiv\; [SU_1(3) \times SU_{23}(3)] \times [SU_1(2) \times SU_{23}(2)] \\ &\quad \times [U_1(1) \times U_2(1) \times U_3(1)] \times U(1)_f \end{aligned} \tag{4.46}$$

It is less trivial to find out if any further enlargement is possible. For each flavour charge set, the anomalies for all possible $J_i \times U(1)_f$ (with J_i satisfying (4.42)) are calculated (i=1,2). It is found that the largest anomaly-free $J_i \times U(1)_f$ are in fact given by (4.44) and (4.46) above.

Finally, it should be restated that it makes no difference to the results shortly to be presented here whether the full symmetry group is $H_i \times U(1)_f$ or $J_i \times U(1)_f$ (i=1,2). The results depend only on the PCCSs, which are $U_1(1) \times SU_2(2) \times U(1)_f$ and $U_1(1) \times U_2(1) \times U(1)_f$ respectively. The significant point of this brief interlude is that it re-emphasises the disappointing fact that in no Method 2 model can the full symmetry group contain SMG^3 as a subgroup.

4.4 Numerology

4.4.1 The Data to be Fitted

We now take a little time out to state the numerical values of the masses and mixing angles which we will attempt to fit with the parameters of our various ansätze.

For the quark and lepton masses, we follow [44]. At 1 GeV we take the running masses to be:

$$\begin{aligned}
&m_u = 5.2 \text{ MeV}, \quad m_c = 1.41 \text{ GeV}, \\
&m_d = 9.2 \text{ MeV}, \quad m_s = 194 \text{ MeV}, \quad m_b = 6.33 \text{ GeV} \\
&m_e = 0.5 \text{ MeV}, \quad m_\mu = 105 \text{ MeV}, \quad m_\tau = 1.78 \text{ GeV}
\end{aligned} \tag{4.47}$$

The running c and b masses shown here correspond to physical masses (*i.e.* "pole" masses) of:

$$m_c^{\text{phys}} = 1.53 \text{ GeV}, \quad m_b^{\text{phys}} = 4.89 \text{ GeV} \tag{4.48}$$

where the physical mass is defined as:

$$m_q^{\text{phys}} = m_q(m_q^{\text{phys}})[1 + \frac{4}{3\pi}\alpha_s(m_q^{\text{phys}})] \tag{4.49}$$

and m_q is the running mass. For completeness, we also give the b running mass value:

$$m_b(m_b) = 4.39 \text{ GeV} \tag{4.50}$$

Finally, we will assume that the t mass lies in the range [42, 43]:

$$100 \text{ GeV } \leq m_t^{\text{phys}} \leq 200 \text{ GeV} \tag{4.51}$$

The light quark masses (m_u, m_d, m_s) were estimated using chiral perturbation techniques and QCD spectral sum rules while the heavy quark masses (m_c, m_b) were estimated using the nonrelativistic bound state approximation and J/ψ and Υ sum rules [44]. The physical masses of the leptons are very well known [45] and the running masses are easily obtained from them.

For the mixing angles at 1 GeV we use the geometric mean of the ranges given by the Particle Data Group [45]:

$$\begin{aligned} \mathcal{V} &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} 0.9747-0.9759 & 0.218-0.224 & 0.002-0.007 \\ 0.218-0.224 & 0.9735-0.9751 & 0.032-0.054 \\ 0.003-0.018 & 0.030-0.054 & 0.9985-0.9995 \end{pmatrix} \end{aligned} \tag{4.52}$$

We are justified in assuming that these are the "known" values at 1 GeV (rather than some other scale) because it is shown in [44] that the running of the mixing angles is very flat, all the way up to M_P. The elements which we will fit are V_{us}, V_{cb} and V_{ub}. We will only require that the other predicted elements are compatible with the bounds shown in (4.52).

None of the results to be presented here depend critically on the precise values of the masses and mixings. Only the hierarchical features are important, and this is consistent with our whole approach.

4.4.2 Running Masses and the Top Mass

Our ansätze are taken to hold at the Planck scale, $M_P = 1.22 \times 10^{19}$ GeV. This is a legacy of our hopes for the anti-grand unified model of [37, 38] mentioned in Chapter 1 where the gauge group SMG^3 (a good symmetry at M_P) is spontaneously broken just below M_P to the SMG, its diagonal subgroup. Therefore in fitting the parameters of our ansätze to the above data, we must modify the masses and mixing angles so that they assume values appropriate to the scale M_P. In [44] the masses and mixings are evolved from 1 GeV to M_P using 2-loop SM renormalisation group equations (extensions of the 1-loop equations given in Chapter 1), and we use their results to extract values at M_P which we then take as input for our fits. The resulting 11 pieces of Planck scale data ($m_{u,c}$, $m_{d,s,b}$, $m_{e,\mu,\tau}$ and $V_{us,cb,ub}$) are hereafter denoted E_i (i=1,...,11) respectively.

The running behaviour of these masses is complicated by the fact that the top mass is "heavy" *i.e.* y_t is of comparable order of magnitude to g_3 (the strong gauge coupling constant). Equations (1.28), (1.29), (1.30) show that if *all* Yukawa couplings could be neglected compared to the gauge couplings (in particular, g_3), then the running of the masses would depend only on the gauge couplings and (1.28) could be easily solved. However, y_t cannot be so neglected and the running of all masses thus depends on the top mass. In [44] the running behaviour of all masses is shown for both $m_t^{\text{phys}} = 100$ GeV and $m_t^{\text{phys}} = 200$ GeV. The difference between these two possibilities, as far as the running of the non-top fermions is concerned is seen to be small (even for m_b, bearing in mind that we are only interested in order of magnitude features). We nevertheless perform fits for both of these m_t^{phys} values.

Furthermore, we perform a fit (for each model), where m_t^{phys} is allowed to assume any value between 100 and 200 GeV. Then, since the top mass is only evolved in [44] for these two limiting values, we use a crude interpolation to calculate any intermediate top masses. That is, if $m_t^{\text{phys}} = 100$ GeV corresponds to a running top mass $m_1(q)$ and $m_t^{\text{phys}} = 200$ GeV to a running top mass $m_2(q)$, then we take:

$$m_t(m_t) = m_1(m_1) + (m_t(M_P) - m_1(M_P)) \left[\frac{m_2(m_2) - m_1(m_1)}{m_2(M_P) - m_1(M_P)}\right] \tag{4.53}$$

Again, since we are only interested in order of magnitude features, the crudeness of this approximation is excusable. The valuable point of such an exercise is to determine whether the top mass naturally lies between 100 and 200 GeV in a given model (without worrying too much about its exact value) or whether this has to be "forced" on a model. The relevant 2-loop results [44] for the running of y_t are summarised in Table 4.7.

4.5 The Fitting Procedure

We now discuss how the parameters of our ansätze are chosen to provide a best fit to the data pieces E_i (i=1,...,11). The general fitting procedure consists of several steps:

$m_t^{\rm phys}$ (GeV)	100		200	
Scale (GeV)	10^2	M_P	10^2	M_P
y_t	0.54	0.16	1.14	0.65

Table 4.7: Running behaviour of y_t for different values of $m_t^{\rm phys}$.

1. Firstly, algebraic expressions for the 3 mass matrices are obtained *i.e.* collectively denoting the parameter set of a particular model by ξ, we calculate all matrix elements as functions of ξ, $M_{ij}(\xi)$, according to one of the ansätze of Chapter 3. Note that all M_{ij} are real and specified only up to factors of $\mathcal{O}(1)$.

2. For each mass matrix we calculate $M^\dagger(\xi)M(\xi)$ and algebraically diagonalise it to leading order in small quantities, as discussed and illustrated in Chapter 3. The diagonalisation process yields 3 eigenvalues from each matrix which give the quark/lepton masses as functions of ξ. For the quarks, the corresponding eigenvectors are calculated - these give the mixing matrix $\mathcal{V}$, again as discussed and illustrated in Chapter 3. At this point, then, we have 11 algebraic expressions for known quantities:

$$\left\{ \begin{array}{l} m_i(\xi) \quad i = u, c, d, s, b, e, \mu, \tau \\ V_{us}(\xi),\ V_{cb}(\xi),\ V_{ub}(\xi) \end{array} \right. \tag{4.54}$$

which we now respectively denote by $f_i(\xi)$ (i=1,...,11). We also have the expression (from (3.1) and (3.2)):

$$m_t \simeq |\gamma_{33}^U|\, a_{33}^U(\xi)\, \langle\phi\rangle_{\rm ws} \quad (a_{33}^U = 1 \ \text{if} \ t_L = t_R) \tag{4.55}$$

so that $|\gamma_{33}^U|$ is a free parameter in our analysis which we can:

(a) use to fix the top mass by hand to correspond to either $m_t^{\rm phys} = 100$ GeV or $m_t^{\rm phys} = 200$ GeV; or

(b) incorporate in our fitting procedure so that the "most favourable" top mass (corresponding to a physical mass between 100 and 200 GeV) is predicted for each model.

In any event, we absorb $|\gamma_{33}^U| a_{33}^U$ into $\langle\phi\rangle_{\rm ws}$ so that all masses are now predicted in units of m_t.

3. We now construct a penalty function, $\chi^2(\xi)$, whose minimum will indicate the best fit possible of a particular ansatz to the data. Firstly, as our procedure is only designed to fit the data up to numerical factors, we demand that a prediction which is inaccurate by a factor r receives the same penalty as one which is inaccurate by a factor of $1/r$, where r is some real number. In addition, we quantify our oft-repeated statement that our predictions are accurate only up to factors of $\mathcal{O}(1)$ by assuming that the error on each prediction f_i is a factor of e. This is numerically convenient, but it should be borne in mind that it is perhaps too strict since we are only interested in order of magnitude features. Should some predictions creep outside this assumed error, the model in question should not necessarily be dismissed - our models should live or die purely on the basis of their order of magnitude predictions (a factor of 5 is probably more realistic). Note further that this "theoretical" error on the f_i completely swamps the "experimental" error on the E_i. We choose the simplest function which meets our requirements:

$$\chi^2(\xi) = \sum_{i=1}^{11}[\log f_i(\xi) - \log E_i]^2 \tag{4.56}$$

4. Finally, we minimise this function for any particular model and ansatz using a routine from the NAG library (which implements a sequential quadratic programming method) and run our M_P results back down to 1 GeV for presentation here.

The minimisation procedure is complicated by the fact that it is often unclear how exactly the functions $f_i(\xi)$ should be specified *i.e.* which matrix elements should be fitted to a given mass or mixing angle. Often a given mass matrix is compatible with more than one of the textures of Chapter 3. This difficulty was somewhat glossed over in the diagonalisation example of Chapter 3, so we illustrate it now with Charge Set 2 of the $H_1 \times U(1)_f$ model (see Table 4.2), using the product ansatz (in fact, this *is* the example of Chapter 3).

We concentrate on the quark mass contributions to χ^2. Using Charge Set 2 from Table 4.2 and the product ansatz (3.43), the quark mass matrices are seen to be:

$$
\begin{aligned}
M_U &= \begin{pmatrix} \gamma_{11}^U \lambda_1^3 & \gamma_{12}^U \lambda_1^4 \epsilon_2 \lambda_f & \gamma_{13}^U \lambda_1^4 \lambda_f \\ \gamma_{21}^U \lambda_1 \lambda_f & \gamma_{22}^U \epsilon_2 & \gamma_{23}^U \lambda_f^2 \\ \gamma_{31}^U \lambda_1 \lambda_f & \gamma_{32}^U \epsilon_2 \lambda_f^2 & \gamma_{33}^U \end{pmatrix} \\
M_D &= \begin{pmatrix} \gamma_{11}^D \lambda_1^3 & \gamma_{12}^D \lambda_1^2 \epsilon_2 \lambda_f & \gamma_{13}^D \lambda_1^2 \lambda_f \\ \gamma_{21}^D \lambda_1 \lambda_f^3 & \gamma_{22}^D \epsilon_2 \lambda_f^2 & \gamma_{23}^D \lambda_f^4 \\ \gamma_{31}^D \lambda_1 \lambda_f^3 & \gamma_{32}^D \epsilon_2 \lambda_f^4 & \gamma_{33}^D \lambda_f^2 \end{pmatrix}
\end{aligned} \tag{4.57}
$$

where we maintain the $\gamma_{ij}^{U,D}$ for clarity. Then, bearing in mind the hierarchy we are trying to reproduce, we have two real choices in choosing the various $f_i(\lambda_1, \epsilon_2, \lambda_f)$. Recalling the approximations (3.7), we can take:

$$
m_b \simeq \gamma_{33}^D \lambda_f^2 \tag{4.58}
$$

and either:

$$
\begin{cases} m_c \simeq \gamma_{22}^U \epsilon_2, \ m_u \simeq \gamma_{11}^U \lambda_1^3 \\ m_s \simeq \gamma_{22}^D \epsilon_2 \lambda_f^2, \ m_d \simeq \gamma_{11}^D \lambda_1^3 \end{cases} \quad \text{if } \epsilon_2 \geq \lambda_1 \lambda_f \tag{4.59}
$$

or:

$$
\begin{cases} m_c \simeq \gamma_{21}^U \lambda_1 \lambda_f, \ m_u \simeq \frac{\gamma_{11}^U \gamma_{22}^U}{\gamma_{21}^U} \frac{\lambda_1^2 \epsilon_2}{\lambda_f} \\ m_s \simeq \gamma_{21}^D \lambda_1 \lambda_f^3, \ m_d \simeq \frac{\gamma_{11}^D \gamma_{22}^D}{\gamma_{21}^D} \frac{\lambda_1^2 \epsilon_2}{\lambda_f} \end{cases} \quad \text{if } \epsilon_2 < \lambda_1 \lambda_f \tag{4.60}
$$

We thus define the f_i (i=1,...,5) to be:

$$
\begin{aligned}
&\begin{cases} f_1 = \lambda_1^3, \ f_2 = \epsilon_2, \ f_3 = \lambda_1^3, \\ f_4 = \epsilon_2 \lambda_f^2, \ f_5 = \lambda_f^2 \end{cases} \quad \text{for } \epsilon_2 \geq \lambda_1 \lambda_f \\
&\begin{cases} f_1 = \frac{\lambda_1^2 \epsilon_2}{\lambda_f}, \ f_2 = \lambda_1 \lambda_f, \ f_3 = \frac{\lambda_1^2 \epsilon_2}{\lambda_f}, \\ f_4 = \lambda_1 \lambda_f^3, \ f_5 = \lambda_f^2 \end{cases} \quad \text{for } \epsilon_2 < \lambda_1 \lambda_f
\end{aligned} \tag{4.61}
$$

Thus, χ^2 is non-differentiable when $\epsilon_2 = \lambda_1 \lambda_f$ and this is a potential problem in the minimisation process. In practice, the χ^2 minimum almost invariably lies away from such regions of parameter space. Nevertheless, we take the precaution of making a preliminary analysis of each model by hand and then (in this illustrative model) minimising χ^2 using the first set of f_i (i=1,...,5), then separately minimising χ^2 for the second set of f_i and finally minimising the non-differentiable χ^2 (which has a choice of which expressions to assign to each f_i, depending on the relative sizes of ϵ_2 and

$\lambda_1\lambda_f$). Furthermore, a mesh is constructed in parameter space in order to provide a large number of initial solution estimates for the NAG routine.

For simplicity, we have chosen to illustrate this problem with a model where the quark mass origins are unclear. However, it is much more common to meet a closely related problem: the mixing angle origins are virtually always unclear, as a glance at $\mathcal{V}_{1,2,3}$ of Chapter 3 reveals. There are always competing contributions for each mixing angle, and we define f_i (i=9,10,11) to be whichever is the biggest contribution at any particular point in parameter space. This introduces more non-differentiable points, and these are dealt with as above.

4.6 $H_1 \times U(1)_f$: Results & Discussion

We analyse the $H_1 \times U(1)_f$ models using all three ansätze (recall that the PCCSs are $U_1(1) \times SU_2(2) \times U(1)_f$). Within each ansatz, many features of the χ^2 fits are common to virtually all of the 25 models (the flavour charges are given in Tables 4.2 and 4.3). We will therefore discuss these features quite generally while using one specific model for illustrative purposes *viz.* Charge Set 2. As will shortly be seen, this model provides neither the best nor the worst fits; it is fairly typical in this respect. We nevertheless favour this model not only because its charges satisfy:

$$\begin{aligned} t_L &= t_R \\ |b_L - b_R| &= |\tau_L - \tau_R| \end{aligned} \tag{4.62}$$

but perhaps most importantly because it has the smallest charges of all the 25 sets (its flavour charges satisfy $|Q_f| \leq 3$). In fact, demanding that (4.62) be satisfied and also that the fermion charges satisfy:

$$s_L = -b_L, \quad c_R = -t_R, \quad s_R = -b_R, \quad \mu_L = -\tau_L, \quad \mu_R = -\tau_R \tag{4.63}$$

yields Charge Set 2 as the unique solution to the no-anomaly equations for $H_1 \times U(1)_f$ (a candidate second solution is equivalent to (4.63) after a simple relabelling of the right-handed states). We call charges satisfying (4.63) DKW solutions, after [26]. As can be seen from Tables 4.2 and 4.3, (4.63) holds for many of the 25 charge sets; it is a very tidy way of cancelling many anomalies between two generations of fermions.

We restate that all results are given for both $m_t^{\rm phys} = 100$ GeV and $m_t^{\rm phys} = 200$ GeV. Furthermore, the physical top mass is allowed to vary between these two values, and corresponding results are shown if a physical top mass $m_t^{\rm free}$ satisfying:

$$100 < m_t^{\rm free}(\text{GeV}) < 200 \tag{4.64}$$

can provide a lower χ^2 than either of these two extremes.

4.6.1 Product Ansatz

Table 4.8 shows the χ^2 minima for all 25 charge sets, using the product ansatz. It also shows the corresponding values of a_{33}^U, the suppression factor on $M_U(3,3)$ (see (4.55)) and (where appropriate) results for $m_t^{\rm free}$.

Recall that in order to generate a $t-b$ splitting we demanded:

$$|b_L - b_R| \geq 3|t_L - t_R| \tag{4.65}$$

as opposed to the more stringent:

$$\begin{aligned} t_L &= t_R \\ b_L &\neq b_R \end{aligned} \tag{4.66}$$

In Table 4.8, those models with $t_L = t_R$ have $a_{33}^U = 1$ whereas the others do not. As mentioned already, the general top mass expression is:

$$m_t = \gamma_{33}^U \, a_{33}^U \, \langle\phi\rangle_{\rm ws} \tag{4.67}$$

so that any suppression ($a_{33}^U < 1$) must generally be compensated by γ_{33}^U. But the γ_{33}^U are assumed to be of $\mathcal{O}(1)$ which we might generously interpret to mean:

$$|\gamma_{33}^U| \leq 5 \tag{4.68}$$

Consistency with $\langle\phi\rangle_{\rm ws} = 174$ GeV therefore requires that the inequalities:

$$a_{33}^U \geq \begin{cases} 0.11 & \text{for } m_t^{\rm phys} = 100 \text{ GeV} \\ 0.23 & \text{for } m_t^{\rm phys} = 200 \text{ GeV} \end{cases} \tag{4.69}$$

be satisfied. Some doubt may thus be cast on Charge Sets 4, 9 and 25, at least for $m_t^{\rm phys} = 200$ GeV (see Table 4.8).

$H_1 \times U(1)_f$: Product Ansatz Results							
Charge	m_t^{free}	χ^2 for $m_t^{\text{phys}}(GeV) =$			a_{33}^U for $m_t^{\text{phys}}(GeV) =$		
Set	(GeV)	100	200	m_t^{free}	100	200	m_t^{free}
1	-	9.88	17.1	-	0.38	0.26	-
2	-	8.94	14.4	-	1.0	1.0	-
3	-	9.34	15.7	-	0.60	0.49	-
4	-	9.33	11.8	-	0.25	0.16	-
5	-	18.0	27.5	-	1.0	1.0	-
6	-	9.54	16.2	-	0.51	0.40	-
7	-	9.23	15.3	-	0.70	0.62	-
8	-	9.98	12.5	-	0.62	0.43	-
9	-	12.7	13.3	-	0.32	0.14	-
10	-	13.7	28.6	-	1.0	1.0	-
11	-	18.6	38.9	-	0.78	0.71	-
12	-	9.62	16.4	-	0.47	0.36	-
13	-	9.15	15.1	-	0.77	0.70	-
14	-	9.39	14.2	-	0.71	0.56	-
15	-	7.51	9.87	-	0.60	0.50	-
16	-	7.34	9.28	-	0.77	0.70	-
17	-	7.20	8.73	-	1.0	1.0	-
18	104	7.82	8.68	7.81	0.73	0.64	0.72
19	120	9.81	10.1	9.71	0.39	0.28	0.35
20	-	9.51	17.2	-	1.0	1.0	-
21	-	9.18	13.5	-	0.77	0.67	-
22	-	10.7	18.5	-	0.54	0.41	-
23	-	7.80	9.46	-	0.78	0.70	-
24	-	9.06	10.2	-	0.56	0.40	-
25	-	11.8	13.9	-	0.28	0.18	-

Table 4.8: χ^2 values for the $H_1 \times U(1)_f$ models (product ansatz). Also shown are: a_{33}^U, the suppression factor on $M_U(3,3)$; and $100 < m_t^{\text{free}}$ (GeV) < 200, if applicable.

Charge Set	m_t^{phys} (GeV)	m_e (MeV)	m_μ (MeV)	m_τ (GeV)	m_d (MeV)	m_s (MeV)	m_b^{phys} (GeV)	m_u (MeV)	m_c^{phys} (GeV)
1	100	1.6	29	4.6	5.0	220	15	5.0	1.6
	200	1.6	28	8.7	6.4	170	27	6.4	2.2
2	100	1.6	28	3.7	4.7	220	12	4.7	1.9
	200	1.5	26	6.3	5.9	160	20	5.9	2.8
3	100	1.6	28	4.1	4.9	220	14	4.9	1.7
	200	1.6	27	7.4	6.2	160	23	6.2	2.5
4	100	0.88	46	2.0	4.3	220	7.1	4.3	3.2
	200	0.99	32	3.0	4.8	160	10	4.8	5.1
5	100	1.8	67	13	8.7	330	30	1.5	4.8
	200	1.0	170	5.5	4.9	830	18	4.9	13
6	100	1.6	29	4.3	4.9	220	14	4.9	1.7
	200	1.6	27	7.9	6.3	170	24	6.3	2.4
7	100	1.6	28	4.0	4.8	220	13	4.8	1.8
	200	1.6	26	7.0	6.1	160	22	6.1	2.6
8	100	1.6	30	4.7	6.9	170	15	2.6	2.7
	200	1.2	30	4.2	5.7	140	14	5.7	3.7
9	100	0.71	65	2.3	3.4	310	8.0	3.4	1.8
	200	0.73	58	2.5	3.6	280	8.6	3.6	2.0
10	100	1.6	52	7.6	8.0	250	23	1.9	3.8
	200	1.9	60	16	9.4	290	45	2.1	7.8
11	100	1.8	71	12	8.9	340	35	1.2	5.5
	200	2.0	100	31	9.8	490	85	0.98	13
12	100	1.6	29	4.3	4.9	220	14	4.9	1.6
	200	1.6	27	8.1	6.3	170	25	6.3	2.4
13	100	1.6	28	3.9	4.8	220	13	4.8	1.8
	200	1.6	26	6.8	6.1	160	21	6.1	2.6

Table 4.9: Masses for the $H_1 \times U(1)_f$ models with the product ansatz. All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	m_t^{phys} (GeV)	m_e (MeV)	m_μ (MeV)	m_τ (GeV)	m_d (MeV)	m_s (MeV)	m_b^{phys} (GeV)	m_u (MeV)	m_c^{phys} (GeV)
14	100	1.6	28	4.2	6.2	170	14	3.1	2.4
	200	2.1	24	3.7	10	120	12	10	3.3
15	100	1.7	35	2.6	4.5	320	9.2	4.5	1.0
	200	1.8	33	3.8	5.6	250	13	5.6	1.2
16	100	1.7	35	2.5	4.4	320	8.7	4.4	1.1
	200	1.8	33	3.4	5.5	250	12	5.5	1.2
17	100	1.7	35	2.3	4.3	320	8.2	4.3	1.1
	200	1.8	33	3.1	5.3	255	11	5.3	1.2
18	100	1.6	30	2.5	4.0	290	8.8	4.0	1.2
	200	1.7	26	3.4	4.9	210	11	4.9	1.4
19	100	0.85	54	1.9	4.2	260	6.9	4.2	1.7
	200	0.96	38	2.6	4.7	180	9.2	4.7	2.2
20	100	1.8	36	6.4	5.0	180	12	5.0	1.6
	200	1.8	39	14	6.2	130	21	6.2	2.2
21	100	1.6	28	4.1	5.8	180	13	3.5	2.2
	200	1.2	30	5.0	6.1	140	16	6.1	3.1
22	100	1.6	31	5.1	7.5	160	17	2.2	3.0
	200	1.1	30	3.7	5.4	150	12	5.4	4.0
23	100	1.6	29	2.7	4.2	270	9.3	4.2	1.3
	200	1.6	25	3.8	5.2	190	13	5.2	1.6
24	100	1.5	26	3.1	3.8	250	11	3.8	1.5
	200	1.1	33	3.1	5.2	160	11	5.2	2.4
25	100	0.80	46	2.6	3.9	220	9.0	3.9	2.5
	200	0.88	30	4.0	4.3	150	13	4.3	3.9

Table 4.10: Masses for the $H_1 \times U(1)_f$ models with the product ansatz (cont.). All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	$m_t^{\rm phys}$ (GeV)					
	100			200		
	V_{us}	V_{cb}	V_{ub}	V_{us}	V_{cb}	V_{ub}
1	0.53	0.010	0.0063	0.62	0.0040	0.0032
2	0.53	0.012	0.0076	0.62	0.0052	0.0041
3	0.53	0.011	0.0069	0.62	0.0045	0.0036
4	0.62	0.015	0.019	0.71	0.011	0.011
5	0.71	0.012	0.012	0.54	0.0014	0.009
6	0.53	0.011	0.0067	0.62	0.0043	0.0035
7	0.53	0.011	0.0071	0.61	0.0047	0.0037
8	0.66	0.019	0.0064	0.69	0.0064	0.0068
9	0.39	0.028	0.068	0.37	0.027	0.067
10	0.71	0.011	0.011	0.71	0.0044	0.044
11	0.71	0.017	0.010	0.71	0.0080	0.0040
12	0.53	0.011	0.0066	0.62	0.0042	0.0034
13	0.53	0.012	0.0072	0.61	0.0048	0.0038
14	0.62	0.017	0.067	0.71	0.0065	0.0065
15	0.47	0.015	0.0080	0.54	0.0067	0.0043
16	0.47	0.016	0.0084	0.54	0.0073	0.0047
17	0.46	0.017	0.0089	0.53	0.0080	0.0050
18	0.45	0.024	0.012	0.51	0.013	0.0078
19	0.43	0.017	0.035	0.43	0.012	0.025
20	0.50	0.0098	0.0057	0.58	0.0039	0.0028
21	0.60	0.016	0.0069	0.68	0.0053	0.0056
22	0.68	0.022	0.0061	0.69	0.0075	0.0078
23	0.47	0.021	0.011	0.54	0.010	0.0067
24	0.45	0.030	0.015	0.48	0.0083	0.016
25	0.42	0.020	0.044	0.41	0.015	0.034

Table 4.11: Mixing angles for the $H_1 \times U(1)_f$ models with the product ansatz.

Charge Set	m_t^{phys} (GeV)									
	100					200				
	λ_1	ϵ_2	λ_f	Textures		λ_1	ϵ_2	λ_f	Textures	
				M_U	M_D				M_U	M_D
1	0.027	0.0063	0.38	4	4	0.015	0.0032	0.26	4	4
2	0.036	0.0076	0.34	4	4	0.023	0.0041	0.22	4	4
3	0.031	0.0069	0.60	4	4	0.018	0.0036	0.49	4	4
4	0.019	0.023	0.25	1	1	0.011	0.011	0.16	1	1
5	0.038	0.016	0.75	4	4	0.020	0.32	0.46	1	1
6	0.029	0.0067	0.71	4	4	0.017	0.0035	0.63	4	4
7	0.033	0.0071	0.70	4	4	0.020	0.0037	0.62	4	4
8	0.031	0.0064	0.62	4	4	0.016	0.0071	0.43	1	1
9	0.024	0.0018	0.57	4	4	0.094	0.0012	0.37	1	1
10	0.037	0.014	0.79	4	4	0.025	0.011	0.71	4	4
11	0.035	0.013	0.78	4	4	0.023	0.0090	0.71	4	4
12	0.029	0.0066	0.78	4	4	0.017	0.0034	0.71	4	4
13	0.033	0.0072	0.77	4	4	0.021	0.0038	0.70	4	4
14	0.033	0.0067	0.71	4	4	0.021	0.0065	0.56	1	1
15	0.032	0.0062	0.78	4	4	0.019	0.0031	0.70	4	4
16	0.034	0.0064	0.77	4	4	0.021	0.0032	0.70	4	4
17	0.037	0.0068	0.77	4	4	0.024	0.0035	0.69	4	4
18	0.033	0.0047	0.73	4	4	0.020	0.0020	0.64	4	4
19	0.021	0.0069	0.62	1	1	0.013	0.0021	0.53	1	1
20	0.038	0.0057	0.76	4	4	0.024	0.0028	0.69	1	1
21	0.034	0.0069	0.77	4	4	0.019	0.0060	0.67	1	1
22	0.030	0.0061	0.74	4	4	0.014	0.0082	0.56	1	1
23	0.034	0.0052	0.78	4	4	0.021	0.0024	0.70	4	4
24	0.030	0.0035	0.75	4	4	0.015	0.0027	0.63	1	1
25	0.019	0.0050	0.66	1	1	0.011	0.0014	0.56	1	1

Table 4.12: Fit parameters and the corresponding mass matrix textures for the $H_1 \times U(1)_f$ models with the product ansatz.

Note that only 2 Charge Sets (18 and 19) naturally feature a top mass in the required range. For all other models, χ^2 rises steadily from $m_t^{\rm phys} = 100$ GeV to $m_t^{\rm phys} = 200$ GeV.

Tables 4.9 and 4.10 show the quark and lepton masses corresponding to the χ^2 minima, while Table 4.11 shows the corresponding mixing angles. For completeness, the parameter values and corresponding quark matrix textures at the χ^2 minima are shown in Table 4.12.

The product ansatz expressions for the matrices of Charge Set 2 are (from (3.43)):

$$
\begin{aligned}
M_U &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^4\epsilon_2\lambda_f & \lambda_1^4\lambda_f \\ \lambda_1\lambda_f & \epsilon_2 & \lambda_f^2 \\ \lambda_1\lambda_f & \epsilon_2\lambda_f^2 & 1 \end{pmatrix} \\
M_D &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^2\epsilon_2\lambda_f & \lambda_1^2\lambda_f \\ \lambda_1\lambda_f^3 & \epsilon_2\lambda_f^2 & \lambda_f^4 \\ \lambda_1\lambda_f^3 & \epsilon_2\lambda_f^4 & \lambda_f^2 \end{pmatrix} \\
M_l &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^6\epsilon_2\lambda_f^3 & \lambda_1^6\lambda_f^3 \\ \lambda_1^3\lambda_f & \epsilon_2\lambda_f^2 & \lambda_f^4 \\ \lambda_1^3\lambda_f & \epsilon_2\lambda_f^4 & \lambda_f^2 \end{pmatrix}
\end{aligned}
\tag{4.70}
$$

At the χ^2 minima we have the M_P-scale predictions:

$$
\begin{array}{lll}
m_\tau \simeq M_l(3,3) \simeq \lambda_f^2, & m_\mu \simeq M_l(2,2) \simeq \epsilon_2\lambda_f^2, & m_e \simeq M_l(1,1) \simeq \lambda_1^3 \\
m_b \simeq M_D(3,3) \simeq \lambda_f^2, & m_s \simeq M_D(2,1) \simeq \lambda_1\lambda_f^3, & m_d \simeq \frac{M_D(1,1)M_D(2,2)}{M_D(2,1)} \simeq \frac{\lambda_1^2\epsilon_2}{\lambda_f} \\
m_t \simeq 1 & m_c \simeq M_U(2,1) \simeq \lambda_1\lambda_f, & m_u \simeq \frac{M_U(1,1)M_U(2,2)}{M_U(2,1)} \simeq \frac{\lambda_1^2\epsilon_2}{\lambda_f}
\end{array}
\tag{4.71}
$$

and:

$$
V_{us} \simeq \frac{M_U(2,2)}{M_U(2,1)} \simeq \frac{\epsilon_2}{\lambda_1\lambda_f}, \quad V_{cb} \simeq \frac{M_U(3,1)}{M_U(3,3)} \simeq \lambda_1\lambda_f, \quad V_{ub} \simeq V_{us}V_{cb} \simeq \epsilon_2 \tag{4.72}
$$

For $m_t^{\rm phys} = 100$ GeV the mass and mixing hierarchies are fairly good, but there is a marked deterioration when $m_t^{\rm phys} = 200$ GeV. Attention is drawn to the following features, which are in fact common to many of the charge sets here:

1. Large m_b.

   This is a result of the M_P-scale prediction:

$$
\frac{m_b}{m_t} \simeq \frac{m_s}{m_c} \tag{4.73}
$$

a relation which gets less realistic as m_t rises. For DKW charge sets this prediction is automatic for quark mass matrices having Textures 1 or 2. It is common for Textures 3, 4 and 5 because most DKW charge sets satisfy $s_R,\ c_R > s_L$ (s_R and c_R are limited only by $|s_R|, |c_R| \leq 15$ whereas s_L is limited to $|s_L| \leq 5$ due to the conditions $A_5 \equiv 3s_L + \mu_L = 0$ and $|\mu_L| \leq 15$) so that:

$$|s_R| - |c_R| = |s_L - s_R| - |s_L - c_R| \tag{4.74}$$

Then, since we have a DKW charge set in mind, we have:

$$|s_R| - |c_R| = |b_L - b_R| - |b_L - t_R| \tag{4.75}$$

which implies (4.73) for matrix Textures 3, 4 and 5.

This argument does not hold for non-DKW charge sets and indeed many of these sets have a better m_b; but some are still burdened by (4.73) or something close to it.

2. Large m_τ.

   This is due to the almost universal M_P scale relation:

$$m_\tau \simeq m_b \tag{4.76}$$

   and the large m_b.

3. Large m_c (especially at $m_t^{\text{phys}} = 200$ GeV).

   This is bound up with predictions like (4.73).

4. Small m_μ/large m_s/large V_{us}.

   For Textures 1 and 2 we often get the M_P-scale prediction:

$$m_s \simeq m_\mu \tag{4.77}$$

   whence the renormalisation group gives at 1 GeV [44]:

$$m_s \simeq 5m_\mu \tag{4.78}$$

   which should be compared with the desired relation:

$$m_s \simeq 2m_\mu \tag{4.79}$$

hence giving a high m_s/low m_μ (a low m_μ is the favoured option due to predictions like (4.73)). Also, the scale of V_{us} (usually originating from $M_D(2,1)/M_D(2,2)$) is set by:

$$\frac{\lambda_1\lambda_f}{\epsilon_2} \simeq \left(\frac{m_d}{m_t}\right)^{1/3}\frac{m_b}{m_s} \simeq \mathcal{O}(1) \tag{4.80}$$

Textures 3, 4 and 5 often predict at M_P (as here for Charge Set 2):

$$V_{us} \simeq \frac{m_\mu}{m_s} \tag{4.81}$$

Running this down to 1 GeV (V_{us} is practically constant, m_μ runs by about 10 % and m_s by a factor of 5–6 [44]) gives:

$$\frac{V_{us}m_s}{m_\mu} \simeq 5 \tag{4.82}$$

which should be compared with the desired relation:

$$\frac{V_{us}m_s}{m_\mu} \simeq \frac{1}{2} \tag{4.83}$$

The low m_μ and high V_{us} are then immediate, while m_s again cannot stray too high due to relations like (4.73).

5. Low m_d/high m_e.

   For Texture 1, this is due to the universal M_P-scale relation:

$$m_u \simeq m_d \simeq m_e \tag{4.84}$$

   while for Texture 4, the M_P-scale relation:

$$m_u \simeq m_d \tag{4.85}$$

   is still common, although there is scope for some splitting (Charge Sets 5, 8, 10, 11, 14, 21, 22). This is an unexpected bonus (recall we left the 1st generation states uncharged and so naively we could not expect splitting), but only a small one.

6. Small V_{cb} (especially at $m_t^{\text{phys}} = 200$ GeV).

   This is due to the common M_P-scale prediction:

$$V_{cb} \simeq \frac{m_c}{m_t} \tag{4.86}$$

seen most frequently for DKW charge sets with matrices of Texture 3, 4 or 5. Some models even feature:

$$V_{cb} < \frac{m_c}{m_t} \tag{4.87}$$

7. Good V_{ub}.

This is due to the M_P-scale prediction:

$$V_{ub} \simeq V_{us} V_{cb} \tag{4.88}$$

which is often true algebraically, or at least numerically, and the large V_{us} compensating the small V_{cb}.

Most of the problems mentioned here are relatively minor, the only serious one (*i.e.* which is unacceptable order of magnitude wise) being the low V_{cb} at $m_t^{\rm phys} = 200$ GeV.

4.6.2 Mixed Ansatz

Table 4.13 shows the χ^2 minima, $m_t^{\rm free}$ and a_{33}^U for all 25 charge sets, using the mixed ansatz. Consistency of $m_t^{\rm phys}$ with both $\langle\phi\rangle_{\rm ws} = 174$ GeV and $\gamma_{33}^U = \mathcal{O}(1)$ casts doubt on Charge Sets 1, 9 and 25 (for $m_t^{\rm phys} = 200$ GeV). Several charge sets now feature top masses which lie naturally in the range 100–200 GeV, while the preference of the rest is divided between the two extremes (recall that with the product ansatz, the preference of such models was invariably for $m_t^{\rm phys} = 100$ GeV). A general deterioration in χ^2 is evident, although there are some notable exceptions (*e.g.* Charge Sets 5, 10 and 11).

Tables 4.14 and 4.15 show the fermion masses corresponding to the χ^2 minima, while Table 4.16 shows the mixing angles. Finally, Tables 4.17 and 4.18 show the parameter values and corresponding quark matrix textures for all 25 charge sets. The mixed ansatz expressions for the mass matrices of Charge Set 2 are, from (3.47):

$$M_U \simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & \epsilon_2 e^{-\sqrt{16g_{11}+g_{ff}-8g_{1f}}} & e^{-\sqrt{16g_{11}+g_{ff}+8g_{1f}}} \\ e^{-\sqrt{g_{11}+g_{ff}-2g_{1f}}} & \epsilon_2 & e^{-\sqrt{4g_{ff}}} \\ e^{-\sqrt{g_{11}+g_{ff}+2g_{1f}}} & \epsilon_2 e^{-\sqrt{4g_{ff}}} & 1 \end{pmatrix}$$

$$M_D \simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & \epsilon_2 e^{-\sqrt{4g_{11}+g_{ff}+4g_{1f}}} & e^{-\sqrt{4g_{11}+g_{ff}-4g_{1f}}} \\ e^{-\sqrt{g_{11}+9g_{ff}-6g_{1f}}} & \epsilon_2 e^{-\sqrt{4g_{ff}}} & e^{-\sqrt{16g_{ff}}} \\ e^{-\sqrt{g_{11}+9g_{ff}+6g_{1f}}} & \epsilon_2 e^{-\sqrt{16g_{ff}}} & e^{-\sqrt{4g_{ff}}} \end{pmatrix}$$

$H_1 \times U(1)_f$: Mixed Ansatz Results							
Charge	m_t^{free}	χ^2 for $m_t^{\text{phys}}(GeV) =$			a_{33}^U for $m_t^{\text{phys}}(GeV) =$		
Set	(GeV)	100	200	m_t^{free}	100	200	m_t^{free}
1	163	12.3	12.1	12.0	0.29	0.20	0.22
2	-	14.7	16.6	-	1.0	1.0	-
3	108	13.0	13.5	12.9	0.49	0.40	0.47
4	-	8.42	10.5	-	0.33	0.25	-
5	-	7.76	15.4	-	1.0	1.0	-
6	122	12.7	12.9	12.6	0.41	0.31	0.37
7	-	13.4	14.2	-	0.61	0.52	-
8	107	9.99	10.8	9.97	0.51	0.42	0.49
9	-	14.6	28.1	-	0.29	0.16	-
10	-	7.70	12.9	-	1.0	1.0	-
11	-	7.85	16.1	-	0.60	0.55	-
12	130	12.6	12.6	12.4	0.37	0.27	0.32
13	-	13.6	14.7	-	0.68	0.61	-
14	102	11.2	12.2	11.2	0.62	0.53	0.61
15	-	13.8	12.9	-	0.52	0.42	-
16	-	13.9	12.9	-	0.70	0.62	-
17	-	14.1	12.9	-	1.0	1.0	-
18	-	14.3	13.0	-	0.66	0.57	0.58
19	-	14.4	24.7	-	0.40	0.26	-
20	101	11.6	24.4	11.6	1.0	1.0	1.0
21	-	11.9	13.0	-	0.69	0.62	-
22	102	9.05	10.5	9.05	0.43	0.34	0.42
23	-	14.1	13.3	-	0.71	0.64	-
24	-	14.5	23.5	-	0.48	0.34	-
25	-	14.8	14.4	-	0.31	0.19	-

Table 4.13: χ^2 values for the $H_1 \times U(1)_f$ models (mixed ansatz). Also shown are: a_{33}^U, the suppression factor on $M_U(3,3)$; and $100 < m_t^{\text{free}}$ (GeV) < 200, if applicable.

Charge Set	m_t^{phys} (GeV)	m_e (MeV)	m_μ (MeV)	m_τ (GeV)	m_d (MeV)	m_s (MeV)	m_b^{phys} (GeV)	m_u (MeV)	m_c^{phys} (GeV)
1	100	1.1	35	0.81	4.4	1000	3.3	2.3	3.8
	200	1.1	29	0.98	4.5	710	3.9	2.6	5.4
2	100	1.3	26	1.2	2.2	880	4.7	4.0	4.5
	200	1.3	22	1.7	2.0	570	6.4	4.8	7.4
3	100	1.2	31	0.92	3.3	950	3.7	2.9	4.2
	200	1.2	26	1.2	3.2	640	4.7	3.4	6.4
4	100	1.2	53	1.2	3.7	450	4.7	2.5	4.4
	200	1.2	46	2.0	5.3	250	7.1	2.0	7.9
5	100	0.96	43	2.6	7.3	350	9.1	2.5	4.0
	200	0.86	53	6.1	8.5	270	19	1.9	8.4
6	100	1.1	33	0.87	3.7	980	3.6	2.7	4.0
	200	1.1	27	1.1	3.6	670	4.4	3.1	6.0
7	100	1.2	30	0.99	3.0	930	4.0	3.2	4.3
	200	1.2	24	1.3	2.8	620	5.1	3.8	6.7
8	100	1.3	36	1.1	3.3	460	4.2	3.3	4.3
	200	1.2	34	1.6	3.7	270	5.9	3.3	7.8
9	100	1.4	78	0.77	3.2	1700	3.2	2.6	3.1
	200	1.7	470	0.55	8.5	2300	2.4	8.5	9.8
10	100	1.0	36	1.9	6.0	320	6.9	2.7	4.4
	200	0.88	43	4.0	6.6	250	13	2.3	8.9
11	100	1.1	55	2.5	6.9	370	8.9	2.1	4.1
	200	1.1	68	6.3	7.6	330	20	1.5	8.9
12	100	1.1	33	0.85	3.9	990	3.5	2.6	4.0
	200	1.1	28	1.1	3.8	680	4.2	3.0	5.9
13	100	1.3	29	1.0	2.8	920	4.1	3.4	4.4
	200	1.2	24	1.4	2.6	610	5.3	4.0	6.9

Table 4.14: Masses for the $H_1 \times U(1)_f$ models with the mixed ansatz. All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	$m_t^{\rm phys}$ (GeV)	m_e (MeV)	m_μ (MeV)	m_τ (GeV)	m_d (MeV)	m_s (MeV)	$m_b^{\rm phys}$ (GeV)	m_u (MeV)	$m_c^{\rm phys}$ (GeV)
14	100	1.3	33	1.1	2.9	550	4.3	3.5	4.4
	200	1.4	26	1.5	3.3	260	5.5	4.1	7.2
15	100	1.2	63	0.67	3.8	1500	2.9	1.8	3.0
	200	1.2	56	0.67	3.7	1200	2.9	2.1	3.6
16	100	1.2	61	0.68	3.3	1500	2.9	2.0	3.1
	200	1.3	55	0.69	3.1	1200	2.9	2.3	3.8
17	100	1.3	60	0.70	2.9	1400	3.0	2.2	3.2
	200	1.3	54	0.73	3.5	1100	3.1	2.7	4.0
18	100	1.3	59	0.74	2.5	1400	3.1	2.5	3.3
	200	1.4	52	0.79	3.4	1100	3.3	3.1	4.4
19	100	1.4	57	0.81	2.4	1400	3.3	2.9	3.5
	200	1.8	250	0.58	14	1200	2.6	8.5	12
20	100	2.0	32	2.0	2.0	310	3.2	5.7	3.7
	200	1.9	180	3.1	9.1	250	3.7	9.1	22
21	100	1.3	31	1.1	2.7	600	4.4	3.6	4.4
	200	1.3	28	1.6	2.7	360	6.1	4.0	7.3
22	100	1.3	41	1.1	3.6	410	4.2	3.0	4.3
	200	1.2	38	1.7	4.3	260	6.3	2.7	7.7
23	100	1.3	50	0.78	2.4	1200	3.3	2.8	3.6
	200	1.4	43	0.89	2.8	900	3.6	3.4	4.9
24	100	1.4	48	0.85	2.1	1200	3.5	3.2	3.7
	200	1.7	160	0.54	11.7	760	2.4	8.4	14
25	100	1.5	45	0.96	2.0	1200	3.9	3.6	3.9
	200	1.7	29	0.90	2.4	620	3.7	4.7	5.3

Table 4.15: Masses for the $H_1 \times U(1)_f$ models with the mixed ansatz (cont.). All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	m_t^{phys} (GeV)					
	100			200		
	V_{us}	V_{cb}	V_{ub}	V_{us}	V_{cb}	V_{ub}
1	0.40	0.043	0.018	0.45	0.033	0.017
2	0.52	0.034	0.021	0.60	0.026	0.019
3	0.45	0.038	0.019	0.52	0.028	0.017
4	0.49	0.070	0.029	0.66	0.064	0.021
5	0.53	0.011	0.0058	0.70	0.0076	0.0035
6	0.43	0.039	0.019	0.50	0.030	0.017
7	0.47	0.036	0.020	0.55	0.027	0.018
8	0.45	0.041	0.021	0.52	0.031	0.017
9	0.34	0.12	0.045	0.40	0.041	0.14
10	0.48	0.016	0.0089	0.64	0.011	0.055
11	0.59	0.019	0.0076	0.71	0.013	0.0039
12	0.42	0.040	0.019	0.49	0.030	0.017
13	0.49	0.036	0.020	0.56	0.027	0.018
14	0.47	0.032	0.017	0.51	0.024	0.014
15	0.31	0.062	0.021	0.34	0.052	0.019
16	0.33	0.060	0.021	0.36	0.049	0.019
17	0.34	0.057	0.021	0.38	0.046	0.019
18	0.37	0.054	0.021	0.41	0.045	0.020
19	0.39	0.078	0.033	0.45	0.022	0.083
20	0.50	0.025	0.014	0.39	0.0028	0.018
21	0.48	0.031	0.017	0.54	0.024	0.015
22	0.44	0.051	0.025	0.58	0.041	0.019
23	0.40	0.049	0.021	0.45	0.040	0.020
24	0.42	0.058	0.027	0.44	0.014	0.059
25	0.44	0.10	0.052	0.38	0.025	0.14

Table 4.16: Mixing angles for the $H_1 \times U(1)_f$ models with the mixed ansatz.

Charge Set	$m_t^{\rm phys} = 100$ GeV					
	ϵ_2	g_{11}	g_{ff}	g_{1f}	Textures	
					M_U	M_D
1	0.044	11.8	1.51	1.34	4	3
2	0.022	11.3	2.70	1.45	4	3
3	0.034	11.6	0.506	0.738	4	3
4	0.044	11.5	1.20	0.719	4	3
5	0.11	12.1	0.396	0.711	4	3
6	0.037	11.7	0.204	0.479	4	3
7	0.030	11.5	0.248	0.500	4	3
8	0.034	11.4	0.465	0.439	4	3
9	0.016	11.1	0.388	0.387	4	3
10	0.079	11.9	0.224	0.435	4	3
11	0.099	11.8	0.258	0.510	4	3
12	0.039	11.7	0.109	0.353	4	3
13	0.028	11.5	0.147	0.376	4	3
14	0.030	11.3	0.234	0.346	4	3
15	0.036	11.7	0.105	0.300	4	3
16	0.032	11.6	0.123	0.317	4	3
17	0.027	11.5	0.147	0.330	4	3
18	0.023	11.3	0.177	0.336	4	5
19	0.018	11.2	0.213	0.325	4	5
20	0.016	10.4	0.218	0.316	4	4
21	0.028	11.3	0.140	0.283	4	3
22	0.038	11.4	0.181	0.258	4	3
23	0.023	11.3	0.114	0.280	4	3
24	0.019	11.2	0.132	0.273	4	5
25	0.015	11.1	0.153	0.250	4	5

Table 4.17: Fit parameters and the corresponding mass matrix textures for the $H_1 \times U(1)_f$ models with the mixed ansatz.

Charge Set	$m_t^{\text{phys}} = 200$ GeV					
	ϵ_2	g_{11}	g_{ff}	g_{1f}	Textures	
					M_U	M_D
1	0.030	15.2	2.61	1.42	4	3
2	0.012	14.6	4.57	1.21	4	3
3	0.021	14.9	0.861	0.728	4	3
4	0.023	14.7	1.91	0.195	4	4
5	0.084	15.7	0.569	0.638	4	3
6	0.024	15.0	0.348	0.487	4	3
7	0.018	14.8	0.421	0.474	4	3
8	0.022	14.7	0.767	0.275	4	4
9	0.056	13.9	0.815	0.177	1	2
10	0.059	15.6	0.328	0.392	4	3
11	0.065	15.2	0.357	0.417	4	3
12	0.026	15.1	0.187	0.364	4	3
13	0.017	14.8	0.248	0.347	4	3
14	0.018	14.4	0.405	0.154	4	4
15	0.023	14.8	0.190	0.318	4	3
16	0.019	14.7	0.223	0.321	4	5
17	0.016	14.6	0.264	0.313	4	5
18	0.012	14.4	0.316	0.285	4	5
19	0.057	13.9	0.450	0.135	1	2
20	0.057	13.7	0.377	0.109	1	1
21	0.017	14.7	0.232	0.177	4	3
22	0.026	14.7	0.289	0.132	4	4
23	0.013	14.5	0.202	0.238	4	5
24	0.057	13.9	0.297	0.0827	1	2
25	0.0063	13.9	0.300	0.130	4	5

Table 4.18: Fit parameters and the corresponding mass matrix textures for the $H_1 \times U(1)_f$ models with the mixed ansatz (cont.).

$$M_l \simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & \epsilon_2 e^{-\sqrt{36g_{11}+9g_{ff}-36g_{1f}}} & e^{-\sqrt{36g_{11}+9g_{ff}+36g_{1f}}} \\ e^{-\sqrt{9g_{11}+g_{ff}-6g_{1f}}} & \epsilon_2 e^{-\sqrt{4g_{ff}}} & e^{-\sqrt{16g_{ff}}} \\ e^{-\sqrt{9g_{11}+g_{ff}+6g_{1f}}} & \epsilon_2 e^{-\sqrt{16g_{ff}}} & e^{-\sqrt{4g_{ff}}} \end{pmatrix} \quad (4.89)$$

Then our algebraic M_P-scale predictions are (at the χ^2 minima):

$$\begin{array}{lll} m_\tau \simeq M_l(3,3), & m_\mu \simeq M_l(2,2), & m_e \simeq M_l(1,1) \\ m_b \simeq M_D(3,3), & m_s \simeq M_D(2,1), & m_d \simeq M_D(1,2) \\ m_t \simeq 1 & m_c \simeq M_U(2,1), & m_u \simeq \frac{M_U(1,1)M_U(2,2)}{M_U(2,1)} \end{array} \quad (4.90)$$

and:

$$V_{us} \simeq \frac{M_U(2,2)}{M_U(2,1)}, \quad V_{cb} \simeq \frac{M_U(3,1)}{M_U(3,3)}, \quad V_{ub} \simeq V_{us}V_{cb} \quad (4.91)$$

Note that several matrix elements are actually algebraically equivalent to the corresponding elements in (4.70) with the natural identification of (3.50). Those which contribute to χ^2 and are inequivalent are $M_{U,D}(2,1)$, $M_{U,D}(3,1)$ and $M_D(1,2)$.

Consider the down quark matrix. As before we have the algebraic predictions:

$$\begin{aligned} m_s &\simeq M_D(2,1) \\ V_{ub} &\simeq \left[\frac{V_{us}}{M_D(3,3)}\right] M_D(3,1) \end{aligned} \quad (4.92)$$

In the product ansatz case, m_s and V_{ub} were both good (although V_{ub} was a little high). But now the contrasting contributions of g_{1f} to $M_D(2,1)$ and $M_D(3,1)$ cause problems. For example, in order to numerically hold $M_D(2,1)$ as low as it was previously we would have to choose g_{1f} to lie on its lower limit:

$$g_{1f} = -\sqrt{g_{11}g_{ff}} \quad (4.93)$$

(constraints on the metric parameters are easily derived from the fact that $\mathcal{G}$ must be positive semi-definite). But then $M_D(3,1)$ would be much larger than it was previously, causing V_{ub} to be too large. Conversely, numerically holding $M_D(3,1)$ as low as its previous value requires that g_{1f} lies on its upper limit:

$$g_{1f} = \sqrt{g_{11}g_{ff}} \quad (4.94)$$

But then $M_D(2,1)$ (and hence m_s) becomes very large. The χ^2 minima shown in Table 4.13 represent a balance in this conflict, with m_s and V_{ub} both fairly high. A similar problem exists in M_U with the charm mass.

Indeed, since for virtually all models we have s_R opposite in sign to b_R and c_R opposite in sign to t_R (these are the only flavour charges affecting $M_{U,D}(2,1)$ and $M_{U,D}(3,1)$), this $m_s - V_{ub}$ conflict is present in most models.

Improvements to note are that the same effect (specifically, $M_D(3,1)$ getting numerically bigger) allows an improvement (*i.e.* rise) in $V_{cb}(\simeq M_D(3,1)/M_D(3,3))$, but this does not offset the $m_s - V_{ub}$ problem. Also, the problem of high m_b largely disappears: old M_P-scale predictions like (4.73) are not true of the mixed ansatz with matrix Textures 3, 4 and 5 because the 2nd generation eigenvalues (originating from $M_{U,D}(2,1)$) have greater numerical freedom (they are determined by 3 parameters as opposed to 2 in the product ansatz).

Some models fight the basic $m_s - V_{ub}$ problem by pushing the top mass up to 200 GeV, for then g_{ff} must be large in order to get a sufficiently low bottom mass and this partially holds down both $M_D(2,1)$ and $M_D(3,1)$. Other models actually manage to alleviate the problem altogether by maintaining a high m_b (*e.g.* Charge Sets 5, 10 and 11). There, the prediction:

$$V_{ub} \simeq V_{us}\left[\frac{M_D(3,1)}{M_D(3,3)}\right] \simeq [V_{us}M_D(3,1)]\frac{1}{m_b} \tag{4.95}$$

means that a high m_b can hold down the size of V_{ub} while g_{1f} is chosen to give a good m_s.

So the mixed ansatz results are broadly similar to those of the product ansatz, with some of the same minor problems (high V_{us}, low m_μ), except for a more serious problem in the $m_c - m_s - V_{ub}$ sector. This is the main reason for the deterioration in χ^2 in going from the product to the mixed ansatz.

4.6.3 Metric Ansatz

Table 4.19 shows the χ^2 minima, $m_t^{\rm free}$ and a_{33}^U for all 25 charge sets, using the metric ansatz. Consistency of $m_t^{\rm phys}$ with both $\langle\phi\rangle_{\rm ws} = 174$ GeV and $\gamma_{33}^U = \mathcal{O}(1)$ again casts doubt on Charge Sets 1, 9 and 25 (for $m_t^{\rm phys} = 200$ GeV). Again, several charge sets feature top masses which lie naturally in the range 100–200 GeV and the preference of the other sets is divided between these two extremes. This table shows a general improvement in χ^2 (for most models the results obtained are the best yet), although perhaps not as dramatic as the increase in the number of parameters might

$H_1 \times U(1)_f$: Metric Ansatz Results							
Charge Set	$m_t^{\rm free}$ (GeV)	χ^2 for $m_t^{\rm phys}(GeV) =$ 100	200	$m_t^{\rm free}$	a_{33}^U for $m_t^{\rm phys}(GeV) =$ 100	200	$m_t^{\rm free}$
1	-	9.46	8.85	-	0.29	0.20	-
2	104	8.02	8.80	8.01	1.0	1.0	1.0
3	141	8.73	8.63	8.51	0.48	0.39	0.43
4	-	8.08	12.0	-	0.35	0.26	-
5	-	8.79	17.8	-	1.0	1.0	-
6	162	8.98	8.68	8.65	0.40	0.30	0.32
7	125	8.48	8.62	8.36	0.60	0.51	0.56
8	-	7.85	10.0	-	0.52	0.43	-
9	-	11.6	10.7	-	0.28	0.17	-
10	-	8.20	15.0	-	1.0	1.0	-
11	-	9.16	18.2	-	0.61	0.54	-
12	176	9.10	8.72	8.70	0.37	0.27	0.28
13	119	8.35	8.64	8.27	0.67	0.60	0.65
14	-	7.84	9.53	-	0.62	0.54	-
15	-	11.9	10.8	-	0.53	0.42	-
16	-	11.5	10.5	-	0.71	0.62	-
17	-	11.1	10.1	-	1.0	1.0	-
18	-	10.7	9.74	-	0.65	0.57	-
19	-	10.2	9.35	-	0.38	0.28	-
20	-	5.83	7.34	-	1.0	1.0	-
21	-	7.85	9.31	-	0.69	0.62	-
22	-	7.91	10.6	-	0.44	0.36	-
23	-	9.97	9.14	-	0.71	0.63	-
24	-	9.56	8.89	-	0.47	0.37	-
25	197	9.13	8.67	8.67	0.29	0.19	0.20

Table 4.19: χ^2 values for the $H_1 \times U(1)_f$ models (metric ansatz). Also shown are: a_{33}^U, the suppression factor on $M_U(3,3)$; and $100 < m_t^{\rm free}$ (GeV) < 200, if applicable.

Charge Set	m_t^{phys} (GeV)	m_e (MeV)	m_μ (MeV)	m_τ (GeV)	m_d (MeV)	m_s (MeV)	m_b^{phys} (GeV)	m_u (MeV)	m_c^{phys} (GeV)
1	100	1.1	60	0.79	9.2	710	3.3	1.1	3.1
	200	1.1	49	0.94	9.2	500	3.8	1.3	4.4
2	100	1.1	56	0.98	9.2	510	4.0	1.3	3.7
	200	0.97	44	1.4	9.2	340	5.3	1.5	6.0
3	100	1.1	56	0.85	9.2	620	3.5	1.2	3.4
	200	1.0	45	1.1	9.2	430	4.3	1.4	5.1
4	100	1.2	72	1.4	9.2	420	5.1	1.1	4.1
	200	1.3	62	2.2	10	300	8.0	0.92	8.0
5	100	1.0	75	2.7	9.2	370	9.2	1.6	4.1
	200	0.97	89	5.9	12	430	19	1.6	8.7
6	100	1.1	57	0.82	9.2	660	3.4	1.2	3.3
	200	1.0	46	1.0	9.2	450	4.1	1.4	4.8
7	100	1.1	55	0.88	9.2	590	3.6	1.3	3.5
	200	0.99	44	1.2	9.2	400	4.6	1.5	5.4
8	100	1.1	63	1.2	9.2	440	4.6	1.2	4.0
	200	1.1	55	1.9	9.3	270	6.8	1.2	6.7
9	100	1.0	77	0.68	9.2	1200	2.9	1.0	2.5
	200	1.1	68	0.66	9.2	980	2.9	1.1	3.1
10	100	1.0	72	2.1	9.2	350	7.4	1.4	4.4
	200	1.1	57	3.9	8.6	280	13	1.3	9.6
11	100	1.2	86	2.7	9.3	420	9.5	1.5	4.2
	200	0.93	85	5.6	11	410	18	0.99	8.6
12	100	1.1	58	0.81	9.2	670	3.4	1.2	3.2
	200	1.0	47	1.0	9.2	460	4.0	1.4	4.8
13	100	1.1	55	0.90	9.2	570	3.7	1.2	3.5
	200	0.98	43	1.2	9.2	390	4.7	1.5	5.6

Table 4.20: Masses for the $H_1 \times U(1)_f$ models with the metric ansatz. All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	$m_t^{\rm phys}$ (GeV)	m_e (MeV)	m_μ (MeV)	m_τ (GeV)	m_d (MeV)	m_s (MeV)	$m_b^{\rm phys}$ (GeV)	m_u (MeV)	$m_c^{\rm phys}$ (GeV)
14	100	1.1	60	1.1	9.2	450	4.4	1.2	3.9
	200	1.0	52	1.7	9.2	280	6.2	1.3	6.5
15	100	1.1	75	0.70	9.2	1200	3.0	0.96	2.5
	200	1.1	67	0.68	9.2	940	2.9	1.1	3.0
16	100	1.1	73	0.69	9.2	1100	2.9	0.99	2.5
	200	1.1	65	0.69	9.2	880	2.9	1.1	3.1
17	100	1.1	72	0.69	9.2	1100	2.9	1.0	2.6
	200	1.1	63	0.71	9.2	820	3.0	1.1	2.3
18	100	1.1	69	0.70	9.2	1000	3.0	1.1	2.7
	200	1.1	60	0.73	9.2	760	3.1	1.2	3.5
19	100	1.1	67	0.71	9.2	950	3.0	1.1	2.8
	200	1.1	57	0.77	9.2	690	3.2	1.2	3.7
20	100	1.2	60	2.1	9.2	270	3.5	1.6	3.1
	200	1.1	52	4.0	9.2	160	4.9	1.9	4.4
21	100	1.1	59	1.1	9.2	460	4.3	1.3	3.9
	200	1.0	50	1.6	9.2	290	6.0	1.4	6.4
22	100	1.1	66	1.2	9.2	410	4.2	3.0	4.3
	200	1.1	58	2.1	9.3	280	7.7	1.0	7.1
23	100	1.1	65	0.72	9.3	1200	3.3	2.8	3.6
	200	1.1	55	0.81	9.2	630	3.3	1.3	3.9
24	100	1.0	63	0.74	9.2	820	3.1	1.2	3.0
	200	1.1	52	0.85	9.2	580	3.5	1.3	4.2
25	100	1.1	62	0.77	9.2	750	3.2	1.2	3.1
	200	1.0	51	0.92	9.2	520	3.7	1.4	4.5

Table 4.21: Masses for the $H_1 \times U(1)_f$ models with the metric ansatz (cont.). All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	$m_t^{\rm phys}$ (GeV)					
	100			200		
	V_{us}	V_{cb}	V_{ub}	V_{us}	V_{cb}	V_{ub}
1	0.38	0.056	0.012	0.43	0.042	0.011
2	0.47	0.038	0.0094	0.53	0.028	0.0086
3	0.40	0.046	0.011	0.46	0.034	0.0098
4	0.49	0.070	0.029	0.71	0.063	0.0095
5	0.71	0.012	0.0042	0.71	0.0051	0.0037
6	0.39	0.049	0.011	0.45	0.037	0.010
7	0.41	0.043	0.011	0.47	0.032	0.0095
8	0.58	0.042	0.0093	0.71	0.033	0.0076
9	0.29	0.11	0.023	0.32	0.12	0.025
10	0.71	0.018	0.0050	0.71	0.0095	0.0056
11	0.71	0.020	0.0051	0.71	0.0098	0.0036
12	0.38	0.051	0.012	0.44	0.038	0.010
13	0.43	0.042	0.010	0.47	0.030	0.0094
14	0.55	0.035	0.0080	0.67	0.027	0.0070
15	0.29	0.078	0.014	0.33	0.065	0.013
16	0.30	0.073	0.014	0.34	0.061	0.012
17	0.31	0.068	0.014	0.35	0.056	0.012
18	0.32	0.063	0.013	0.36	0.052	0.012
19	0.33	0.075	0.016	0.37	0.067	0.016
20	0.50	0.026	0.0075	0.54	0.019	0.0067
21	0.53	0.035	0.0083	0.64	0.027	0.0074
22	0.60	0.051	0.011	0.71	0.043	0.0079
23	0.34	0.056	0.012	0.39	0.045	0.011
24	0.35	0.056	0.013	0.40	0.046	0.012
25	0.38	0.10	0.023	0.43	0.098	0.026

Table 4.22: Mixing angles for the $H_1 \times U(1)_f$ models with the metric ansatz.

Charge Set	$m_t^{\rm phys} = 100$ GeV							
	g_{11}	g_{22}	g_{ff}	g_{12}	g_{1f}	g_{2f}	Textures	
							M_U	M_D
1	11.8	26.6	1.54	-3.96	0.899	-0.133	4	3
2	11.8	24.8	3.05	-4.34	1.06	0.872	4	3
3	11.9	25.9	0.530	-4.00	0.517	0.133	4	3
4	11.6	23.0	1.12	-5.06	0.444	0.704	4	3
5	11.9	17.1	0.391	-4.57	0.717	0.0328	4	3
6	11.8	26.1	0.211	-3.96	0.332	0.0361	4	3
7	11.9	25.6	0.264	-4.07	0.354	0.149	4	3
8	11.7	23.6	0.440	-4.81	0.332	0.428	4	3
9	11.9	31.7	0.415	-2.38	0.282	0.181	4	3
10	11.9	19.1	0.211	-4.76	0.399	0.0578	4	3
11	11.6	18.3	0.244	-4.79	0.481	0.0797	4	3
12	11.8	26.3	0.112	-3.96	0.243	0.00981	4	3
13	11.9	25.4	0.158	-4.12	0.267	0.137	4	3
14	11.8	23.9	0.232	-4.68	0.250	0.299	4	3
15	11.9	29.7	0.102	-3.11	0.202	-0.0689	4	3
16	11.9	30.0	0.122	-2.97	0.212	-0.0468	4	3
17	11.9	30.2	0.148	-2.86	0.222	-0.00377	4	3
18	11.9	30.2	0.182	-2.85	0.230	0.0826	4	3
19	11.9	29.8	0.228	-3.11	0.235	0.269	4	3
20	11.6	25.5	0.207	-4.59	0.243	0.549	4	3
21	11.8	24.1	0.143	-4.61	0.203	0.227	4	3
22	11.7	23.4	0.167	-4.92	0.187	0.269	4	3
23	11.9	28.9	0.120	-3.21	0.198	0.127	4	3
24	11.9	28.4	0.143	-3.53	0.200	0.260	4	3
25	11.8	27.7	0.173	-4.06	0.187	0.431	4	3

Table 4.23: Fit parameters and the corresponding mass matrix textures for the $H_1 \times U(1)_f$ models with the metric ansatz.

Charge Set	$m_t^{\text{phys}} = 200$ GeV							
	g_{11}	g_{22}	g_{ff}	g_{12}	g_{1f}	g_{2f}	Textures	
							M_U	M_D
1	15.1	35.2	2.66	-5.49	0.910	0.410	4	3
2	15.4	30.4	5.05	-7.40	0.763	3.19	4	3
3	15.3	33.9	0.896	-5.87	0.461	0.650	4	3
4	14.7	28.8	1.80	-8.39	0.175	2.15	4	3
5	15.4	16.5	0.581	-6.28	1.05	0.539	4	3
6	15.2	34.6	0.359	-5.66	0.307	0.304	4	3
7	15.3	33.0	0.444	-6.22	0.306	0.614	4	3
8	15.1	29.1	0.705	-8.63	0.0392	1.29	4	3
9	15.1	45.4	0.762	-3.36	0.0992	0.692	4	3
10	15.1	20.0	0.334	-9.18	0.475	0.674	4	3
11	15.5	18.5	0.384	-6.51	0.733	0.633	4	3
12	15.2	34.8	0.192	-5.59	0.230	0.189	4	3
13	15.4	32.3	0.264	-6.48	0.229	0.548	4	3
14	15.2	28.9	0.379	-8.39	0.0896	0.943	4	3
15	15.0	41.6	0.188	-3.87	0.190	-0.00388	4	3
16	15.0	42.0	0.223	-3.74	0.184	0.0563	4	3
17	15.0	42.4	0.267	-3.72	0.172	0.162	4	3
18	15.1	42.3	0.326	-3.94	0.150	0.357	4	3
19	15.1	41.4	0.405	-4.76	0.111	0.756	4	3
20	15.1	33.1	0.328	-8.15	0.105	1.31	4	3
21	15.2	29.2	0.234	-8.18	0.0906	0.733	4	3
22	15.0	30.4	0.258	-8.46	-0.00535	0.742	4	3
23	15.1	40.1	0.210	-4.66	0.129	0.391	4	3
24	15.2	38.8	0.248	-5.52	0.107	0.677	4	3
25	15.2	36.1	0.298	-7.27	0.0658	1.17	4	3

Table 4.24: Fit parameters and the corresponding mass matrix textures for the $H_1 \times U(1)_f$ models with the metric ansatz (cont.).

have merited (there are 6 parameters in the metric ansatz compared to only 3 in the product ansatz).

Tables 4.20 and 4.21 show the fermion masses corresponding to the χ^2 minima, while Table 4.22 shows the mixing angles. Finally, Tables 4.23 and 4.24 show the parameter values and corresponding matrix textures for all 25 charge sets.

To facilitate presentation of the mass matrices for Charge Set 2, we introduce 3 matrices N_U, N_D and N_l whose elements are the squares of the exponents of the matrix elements defined in (3.49). That is:

$$N_a(i,j) = (\log[M_a(i,j)])^2 \tag{4.96}$$

(a=U, D, l) and so, using equation (3.49), we have:

$$\begin{aligned}
N_l &\simeq \begin{pmatrix} 9g_{11} & \begin{array}{c}36g_{11} + g_{22} + 9g_{ff} \\ -12g_{12} - 36g_{1f} + 6g_{2f}\end{array} & 36g_{11} + 9g_{ff} + 36g_{1f} \\ 9g_{11} + g_{ff} - 6g_{1f} & g_{22} + 4g_{ff} + 4g_{2f} & 16g_{ff} \\ 9g_{11} + g_{ff} + 6g_{1f} & g_{22} + 16g_{ff} + 8g_{2f} & 4g_{ff} \end{pmatrix} \\
N_U &\simeq \begin{pmatrix} 9g_{11} & \begin{array}{c}16g_{11} + g_{22} + g_{ff} \\ -8g_{12} - 8g_{1f} + 2g_{2f}\end{array} & 16g_{11} + g_{ff} + 8g_{1f} \\ g_{11} + g_{ff} - 2g_{1f} & g_{22} & 4g_{ff} \\ g_{11} + g_{ff} + 2g_{1f} & g_{22} + 4g_{ff} + 4g_{2f} & 0 \end{pmatrix} \\
N_D &\simeq \begin{pmatrix} 9g_{11} & \begin{array}{c}4g_{11} + g_{22} + g_{ff} \\ -4g_{12} + 4g_{1f} - 2g_{2f}\end{array} & 4g_{11} + g_{ff} - 4g_{1f} \\ g_{11} + 9g_{ff} - 6g_{1f} & g_{22} + 4g_{ff} + 4g_{2f} & 16g_{ff} \\ g_{11} + 9g_{ff} + 6g_{1f} & g_{22} + 16g_{ff} - 8g_{2f} & 4g_{ff} \end{pmatrix}
\end{aligned} \tag{4.97}$$

With this ansatz, our algebraic predictions are (at the χ^2 minima):

$$\begin{array}{lll} m_\tau \simeq M_l(3,3), & m_\mu \simeq M_l(2,2), & m_e \simeq M_l(1,1) \\ m_b \simeq M_D(3,3), & m_s \simeq M_D(2,1), & m_d \simeq M_D(1,2) \\ m_t \simeq 1 & m_c \simeq M_U(2,1), & m_u \simeq \frac{M_U(1,1)M_U(2,2)}{M_U(2,1)} \end{array} \tag{4.98}$$

and:

$$V_{us} \simeq \tfrac{M_D(2,2)}{M_D(2,1)}, \quad V_{cb} \simeq \tfrac{M_D(3,1)}{M_D(3,3)}, \quad V_{ub} \simeq \tfrac{M_D(3,2)}{M_D(3,3)} \tag{4.99}$$

Some of the problems of the previous two ansätze still persist (*e.g.* the $m_s - m_\mu - V_{us}$ problem), but most other features are improved.

The general improvement in χ^2 from the product ansatz is largely because of:

1. The improved $m_d - m_u - m_e$.

   This is due to the fact that $\det M_D$ $(= m_d m_s m_b)$ is now bigger than its previous value of $M_D(1,1)M_D(2,2)M_D(3,3)$ (*i.e.* M_D now prefers Texture 4 to Textures 1 or 3), allowing m_d to assume a larger, more realistic value.

2. The rise in V_{cb}. A prediction for all ansätze has been:

   $$V_{cb} \simeq \frac{M_D(3,1)}{M_D(3,3)} \tag{4.100}$$

   and the metric ansatz caters for a larger $M_D(3,1)$ than the product ansatz.

The big general improvement from the mixed ansatz is due mainly to a resolution of the $m_s - V_{ub}$ problem seen there. Previously, M_P-scale predictions common to most models included:

$$\begin{aligned} V_{us} &\simeq \frac{M_U(2,2)}{M_U(2,1)} \\ m_s &\simeq M_D(2,1) \\ V_{ub} &\simeq \max\left(\frac{M_D(3,2)}{M_D(3,3)}, \frac{M_U(2,2)}{M_U(2,1)}\frac{M_D(3,1)}{M_D(3,3)}\right) \\ &= \frac{M_U(2,2)}{M_U(2,1)}\frac{M_D(3,1)}{M_D(3,3)} \end{aligned} \tag{4.101}$$

The $M_D(3,1) - M_D(2,1)$ conflict is evident (recall that g_{1f} contributes to the exponents of each of these matrix elements with opposite sign). It is not possible to drive down $M_U(2,2)/M_U(2,1)$ in order to hold down V_{ub} while g_{1f} is chosen to give a good m_s

because it forms the leading contribution to V_{us}. The above predictions contrast with those of the metric ansatz:

$$\begin{aligned}
V_{us} &\simeq \frac{M_D(2,2)}{M_D(2,1)} \\
m_s &\simeq M_D(2,1) \\
V_{ub} &\simeq \max\left(\frac{M_D(3,2)}{M_D(3,3)}, \frac{M_U(2,2)}{M_U(2,1)}\frac{M_D(3,1)}{M_D(3,3)}\right) = \frac{M_D(3,2)}{M_D(3,3)}
\end{aligned} \tag{4.102}$$

Here, then, $M_U(2,2)/M_U(2,1)$ *can* be driven down because the leading order contribution to V_{us} comes now from $M_D(2,2)/M_D(2,1)$. Thus g_{1f} can be chosen to fit m_s while its adverse effect on V_{ub} (via a large $M_D(3,1)$) can be negated. Indeed, the easing of suppression on $M_D(3,2)$ in going from the mixed to the metric ansatz means that $M_D(3,2)/M_D(3,3)$ now forms the leading contribution to V_{ub}.

4.6.4 Comments

Generally speaking, these fits are probably as good as could have been expected in our order of magnitude framework. With the crude 3-parameter product ansatz, most of the data was fitted to within a factor of 3 or 4. The biggest sticking point was V_{cb} for $m_t^{\rm phys} = 200$ GeV, which was usually an order of magnitude too small. There was a marked deterioration for the 4-parameter mixed ansatz. The V_{cb} problem disappeared only to be replaced by values of m_s and V_{ub} which were often a factor of 5 or 6 too high. However, there was a definite improvement for the 6-parameter metric ansatz, all the data being fitted within a factor of 3 or so for most charge sets. Judging these $H_1 \times U(1)_f$ models purely on their results, then, it is perhaps fair to say that while many of them are reasonable, none is outstanding.

4.7 $H_2 \times U(1)_f$: Results & Discussion

Recall here that the PCCSs are $U_1(1) \times U_2(1) \times U(1)_f$ and that the flavour charges are given in Table 4.5. The mixed and metric ansätze are equivalent for any $H_2 \times U(1)_f$ model because all PCCSs are abelian (there are no partially conserved $SU_a(2)$ symmetries). Each model is therefore analysed using only the product and mixed ansätze. As for the $H_1 \times U(1)_f$ case, many features of the fits are common to all the charge sets so we again use one particular model for illustration *viz.* Charge Set 1. We favour this model for similar reasons to before. It satisfies (4.62) and in fact demanding that the fermion charges satisfy this in conjuction with (4.63) yields Charge Sets 1 and 4 as the only solutions to the no-anomaly equations; but Charge Set 4 is manifestly bad as far as generating suitable fermion masses is concerned. The down quark mass matrices of Charge Sets 1 and 4 are of Textures 1 and 6 respectively. So for Charge Set 1:

$$m_s \simeq M_D(2,2) \tag{4.103}$$

which leads to the M_P-scale prediction:

$$\frac{m_b}{m_t} \simeq \frac{m_s}{m_c} \tag{4.104}$$

while for Charge Set 4:

$$m_s \simeq \frac{M_D(2,3)M_D(3,2)}{M_D(3,3)} \tag{4.105}$$

which leads to the less attractive M_P-scale prediction:

$$\frac{m_b}{m_t} \simeq \left(\frac{m_s}{m_c}\right)^3 \tag{4.106}$$

As can be seen from Table 4.5, five of the six charge sets to be analysed here are DKW solutions.

4.7.1 Product Ansatz

Table 4.25 shows the χ^2 minima for all 6 charge sets, as well as the corresponding values of $m_t^{\rm free}$ and a_{33}^U. Two Charge Sets (1 and 5) feature a top mass which sits naturally in the required range (another reason for favouring Charge Set 1) while the remaining models prefer $m_t^{\rm phys} = 100$ GeV. Note that none of these χ^2 values can really compete with those of typical $H_1 \times U(1)_f$ models, except perhaps those of Charge Set 1.

Tables 4.26 and 4.27 show the fermion masses and mixing angles corresponding to these χ^2 minima. Finally, Table 4.28 shows the parameter values and corresponding quark matrix textures for the 6 charge sets. The product ansatz expressions for the mass matrices of Charge Set 1 are, from (3.43):

$$M_U \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^4\lambda_2\lambda_f & \lambda_1^4\lambda_f \\ \lambda_1\lambda_2^4\lambda_f & \lambda_2^3 & \lambda_2^4\lambda_f^2 \\ \lambda_1\lambda_f & \lambda_2\lambda_f^2 & 1 \end{pmatrix}$$

$H_2 \times U(1)_f$: Product Ansatz Results							
Charge	$m_t^{\rm free}$	χ^2 for $m_t^{\rm phys}(GeV) =$			a_{33}^U for $m_t^{\rm phys}(GeV) =$		
Set	(GeV)	100	200	$m_t^{\rm free}$	100	200	$m_t^{\rm free}$
1	116	10.1	10.8	10.0	1.0	1.0	1.0
2	-	26.5	37.1	-	0.71	0.52	-
3	-	25.7	33.9	-	0.79	0.64	-
4	-	24.7	31.7	-	1.0	1.0	-
5	108	13.8	14.5	13.8	0.69	0.59	0.68
6	-	25.5	34.2	-	0.80	0.66	-

Table 4.25: χ^2 values for the $H_2 \times U(1)_f$ models (product ansatz). Also shown are: a_{33}^U, the suppression factor on $M_U(3,3)$; and $100 < m_t^{\rm free}$ (GeV) < 200, if applicable.

Charge	$m_t^{\rm phys}$	m_e	m_μ	m_τ	m_d	m_s	$m_b^{\rm phys}$	m_u	$m_c^{\rm phys}$
Set	(GeV)	(MeV)	(MeV)	(GeV)	(MeV)	(MeV)	(GeV)	(MeV)	(GeV)
1	100	0.86	35	2.4	4.2	170	8.4	4.2	2.2
	200	0.94	26	3.5	4.6	130	12	4.6	3.8
2	100	0.65	78	16	3.2	380	47	3.2	0.91
	200	0.59	70	34	2.9	340	92	2.9	1.3
3	100	0.66	80	13	3.2	390	38	3.2	0.81
	200	0.59	74	21	2.9	360	58	2.9	1.1
4	100	0.68	49	11	3.3	500	32	3.3	0.91
	200	0.63	30	16	3.1	570	47	3.1	1.2
5	100	0.83	34	3.6	4.0	240	8.8	4.0	1.5
	200	0.90	23	5.3	4.4	190	11	4.4	2.4
6	100	0.66	64	13	3.2	420	38	3.2	0.96
	200	0.59	48	23	2.9	410	64	2.9	1.4

Table 4.26: Masses for the $H_2 \times U(1)_f$ models with the product ansatz. All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	m_t^{phys} (GeV)					
	100			200		
	V_{us}	V_{cb}	V_{ub}	V_{us}	V_{cb}	V_{ub}
1	0.097	0.16	0.024	0.074	0.11	0.014
2	0.37	0.17	0.068	0.59	0.13	0.094
3	0.33	0.21	0.074	0.50	0.21	0.12
4	0.28	0.24	0.070	0.36	0.24	0.094
5	0.10	0.29	0.031	0.077	0.27	0.021
6	0.33	0.20	0.073	0.53	0.17	0.10

Table 4.27: Mixing angles for the $H_2 \times U(1)_f$ models with the product ansatz.

Charge Set	m_t^{phys} (GeV)									
	100					200				
	λ_1	ϵ_2	λ_f	Textures		λ_1	ϵ_2	λ_f	Textures	
				M_U	M_D				M_U	M_D
1	0.030	0.25	0.80	1	1	0.020	0.20	0.74	1	1
2	0.024	0.17	0.89	1	1	0.014	0.13	0.81	1	1
3	0.025	0.17	0.93	1	1	0.014	0.13	0.86	1	1
4	0.028	0.17	0.83	1	6	0.017	0.12	0.71	1	6
5	0.026	0.21	0.83	1	6	0.016	0.16	0.77	1	6
6	0.025	0.17	0.93	1	6	0.015	0.13	0.87	1	6

Table 4.28: Fit parameters and the corresponding mass matrix textures for the $H_2 \times U(1)_f$ model with the product ansatz.

$$M_D \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^2\lambda_2\lambda_f & \lambda_1^2\lambda_f \\ \lambda_1\lambda_2^2\lambda_f^{13} & \lambda_2^3\lambda_f^{12} & \lambda_2^2\lambda_f^{14} \\ \lambda_1\lambda_f^{13} & \lambda_2\lambda_f^{14} & \lambda_f^{12} \end{pmatrix}$$

$$M_l \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^6\lambda_2^3\lambda_f^{13} & \lambda_1^6\lambda_f^{13} \\ \lambda_1^3\lambda_2^6\lambda_f & \lambda_2^3\lambda_f^{12} & \lambda_2^6\lambda_f^{14} \\ \lambda_1^3\lambda_f & \lambda_2^3\lambda_f^{14} & \lambda_f^{12} \end{pmatrix} \tag{4.107}$$

We then have the algebraic M_P-scale predictions (at the χ^2 minima):

$$\begin{array}{lll} m_\tau \simeq M_l(3,3) \simeq \lambda_f^{12}, & m_\mu \simeq M_l(2,2) \simeq \lambda_2^3\lambda_f^{12}, & m_e \simeq M_l(1,1) \simeq \lambda_1^3 \\ m_b \simeq M_D(3,3) \simeq \lambda_f^{12}, & m_s \simeq M_D(2,2) \simeq \lambda_2^3\lambda_f^{12}, & m_d \simeq M_D(1,1) \simeq \lambda_1^3 \\ m_t \simeq 1 & m_c \simeq M_U(2,2) \simeq \lambda_2^3, & m_u \simeq M_U(1,1) \simeq \lambda_1^3 \end{array} \tag{4.108}$$

and:

$$V_{us} \simeq \frac{M_D(2,1)}{M_D(2,2)} \simeq \frac{\lambda_1\lambda_f}{\lambda_2}, \quad V_{cb} \simeq \frac{M_D(3,2)}{M_D(3,3)} \simeq \lambda_2\lambda_f^2, \quad V_{ub} \simeq \frac{M_D(3,1)}{M_D(3,3)} \simeq \lambda_1\lambda_f \tag{4.109}$$

Some of the problems present in the $H_1 \times U(1)_f$ models are seen here too, for example:

1. High m_b.

   This is due to M_P-scale relations such as:

   $$\frac{m_b}{m_t} \simeq \frac{m_s}{m_c} \tag{4.110}$$

   (for *e.g.* Charge Set 1). In truth, the bottom masses are unacceptably high for all Charge Sets except 1 and 5 (and consequently so are the tau masses due to the M_P-scale prediction $m_\tau \simeq m_b$);

2. Low m_μ/high m_s.

   This is due to M_P-scale predictions such as:

   $$m_\mu \simeq m_s \tag{4.111}$$

   (*e.g.* for Charge Set 1), or even:

   $$m_\mu < m_s \tag{4.112}$$

   (*e.g.* for Charge Set 4) which lead to an $m_\mu - m_s$ splitting of the order of a factor of 5 or bigger.

$H_2 \times U(1)_f$: Mixed Ansatz Results							
Charge Set	$m_t^{\rm free}$ (GeV)	χ^2 for $m_t^{\rm phys}(GeV) =$ 100	200	$m_t^{\rm free}$	a_{33}^U for $m_t^{\rm phys}(GeV) =$ 100	200	$m_t^{\rm free}$
1	-	18.4	17.8	-	1.0	1.0	-
2	-	25.8	34.9	-	0.80	0.53	-
3	-	26.1	35.4	-	0.89	0.69	-
4	-	26.7	36.4	-	1.0	1.0	-
5	-	24.4	28.0	-	0.81	0.65	-
6	-	26.1	35.4	-	0.89	0.69	-

Table 4.29: χ^2 values for the $H_2 \times U(1)_f$ models (mixed ansatz). Also shown are: a_{33}^U, the suppression factor on $M_U(3,3)$; and $100 < m_t^{\rm free}$ (GeV) < 200, if applicable.

There is a more obvious problem with the mixing angles which was not present before: although the V_{us} values are fine, V_{cb} is generally predicted to be a factor of 5 or more too big, while V_{ub} is an order of magnitude too big. This can be traced to the lack of suppression of $M_{U,D}(3,2)$ compared to $M_{U,D}(2,2)$, a problem not present in any $H_1 \times U(1)_f$ model (contrast the suppressing effect of ϵ_2 on $M_{U,D}(3,2)$ relative to $M_{U,D}(2,2)$ in (4.70) with that of λ_2 on the same elements in (4.107)).

4.7.2 Mixed Ansatz

Table 4.29 shows the χ^2 minima, $m_t^{\rm free}$ and a_{33}^U for the 6 charge sets using the mixed ansatz. A general deterioration in χ^2 is obvious. Tables 4.30 and 4.31 show the corresponding masses and mixing angles respectively. Finally, Tables 4.32 and 4.33 show the parameter values and quark matrix textures.

Using (3.47), Charge Set 1 is seen to yield the mass matrices (again using the

Charge Set	m_t^{phys} (GeV)	m_e (MeV)	m_μ (MeV)	m_τ (GeV)	m_d (MeV)	m_s (MeV)	m_b^{phys} (GeV)	m_u (MeV)	m_c^{phys} (GeV)
1	100	0.33	35	2.4	31	620	8.4	1.6	0.76
	200	0.33	45	2.1	33	310	7.5	1.6	1.0
2	100	0.45	53	17	8.7	750	48	2.2	0.71
	200	0.36	49	18	9.2	1100	52	1.8	0.66
3	100	0.46	64	18	9.1	780	51	2.2	0.71
	200	0.38	45	20	9.9	1200	56	1.8	0.60
4	100	0.38	58	19	8.4	690	55	1.9	0.66
	200	0.46	48	27	8.8	1200	75	2.2	0.55
5	100	0.36	33	7.3	25	1000	19	1.7	0.55
	200	0.38	26	5.9	27	1400	13	1.8	0.55
6	100	0.46	64	18	9.1	780	51	2.2	0.71
	200	0.38	45	20	9.9	1200	56	1.8	0.60

Table 4.30: Masses for the $H_2 \times U(1)_f$ models with the mixed ansatz. All masses are running masses evaluated at 1 GeV unless otherwise stated.

Charge Set	m_t^{phys} (GeV)					
	100			200		
	V_{us}	V_{cb}	V_{ub}	V_{us}	V_{cb}	V_{ub}
1	0.18	0.24	0.046	0.19	0.27	0.056
2	0.28	0.16	0.046	0.37	0.23	0.094
3	0.26	0.16	0.043	0.29	0.25	0.079
4	0.21	0.17	0.037	0.21	0.27	0.057
5	0.15	0.27	0.042	0.10	0.43	0.048
6	0.26	0.16	0.043	0.29	0.25	0.079

Table 4.31: Mixing angles for the $H_2 \times U(1)_f$ models with the mixed ansatz.

Charge Set	$m_t^{\rm phys} = 100$ GeV							
	g_{11}	g_{22}	g_{ff}	g_{12}	g_{1f}	g_{2f}	Textures M_U	M_D
1	14.7	3.56	0.0466	2.25	0.363	0.111	1	7
2	13.9	3.90	0.00531	3.02	0.0478	0.104	1	7
3	13.9	3.96	0.00154	2.69	0.0509	0.0712	1	7
4	14.3	3.89	0.00711	3.26	-0.0102	0.130	1	7
5	14.5	4.18	0.0113	2.01	-0.221	-0.0911	1	7
6	13.9	3.83	0.00149	3.08	0.00412	0.0679	1	7

Table 4.32: Fit parameters and the corresponding mass matrix textures for the $H_2 \times U(1)_f$ models with the mixed ansatz.

Charge Set	$m_t^{\rm phys} = 200$ GeV							
	g_{11}	g_{22}	g_{ff}	g_{12}	g_{1f}	g_{2f}	Textures M_U	M_D
1	18.3	4.82	0.116	3.55	0.392	0.036	1	7
2	18.1	5.87	0.0454	5.96	0.380	0.0928	1	8
3	17.9	6.20	0.0148	5.43	-0.371	0.0692	1	7
4	17.4	6.24	0.0645	5.85	-0.714	0.0888	1	7
5	18.0	6.10	0.0473	2.73	-0.419	-0.0911	1	7
6	17.9	6.06	0.0148	5.77	-0.370	0.0552	1	7

Table 4.33: Fit parameters and the corresponding mass matrix textures for the $H_2 \times U(1)_f$ models with the mixed ansatz (cont.).

matrices $N_{U,D,l}$ of (4.96) to facilitate presentation):

$$N_l \simeq \begin{pmatrix} 9g_{11} & \begin{array}{c} 36g_{11}+9g_{22}+169g_{ff} \\ -36g_{12}+156g_{1f}-78g_{2f} \end{array} & 36g_{11}+169g_{ff}-156g_{1f} \\ \begin{array}{c} 9g_{11}+36g_{22}+g_{ff} \\ -36g_{12}+6g_{1f}-12g_{2f} \end{array} & 9g_{22}+144g_{ff}+72g_{2f} & 36g_{22}+196g_{ff}-168g_{2f} \\ 9g_{11}+g_{ff}-6g_{1f} & 9g_{22}+196g_{ff}-84g_{2f} & 144g_{ff} \end{pmatrix}$$

$$N_D \simeq \begin{pmatrix} 9g_{11} & \begin{array}{c} 4g_{11}+g_{22}+g_{ff} \\ +4g_{12}+4g_{1f}+2g_{2f} \end{array} & 4g_{11}+g_{ff}-4g_{1f} \\ \begin{array}{c} g_{11}+4g_{22}+169g_{ff} \\ +4g_{12}-26g_{1f}-52g_{2f} \end{array} & 9g_{22}+144g_{ff}-72g_{2f} & 4g_{22}+196g_{ff}-56g_{2f} \\ g_{11}+169g_{ff}+26g_{1f} & g_{22}+196g_{ff}+28g_{2f} & 144g_{ff} \end{pmatrix}$$

$$N_U \simeq \begin{pmatrix} 9g_{11} & \begin{array}{c} 16g_{11}+g_{22}+g_{ff} \\ -8g_{12}-8g_{1f}+2g_{2f} \end{array} & 16g_{11}+g_{ff}+8g_{1f} \\ \begin{array}{c} g_{11}+16g_{22}+g_{ff} \\ -8g_{12}-2g_{1f}+8g_{2f} \end{array} & 9g_{22} & 16g_{22}+4g_{ff}+16g_{2f} \\ g_{11}+g_{ff}+2g_{1f} & g_{22}+4g_{ff}+4g_{2f} & 0 \end{pmatrix} \tag{4.113}$$

Our algebraic predictions are then (at the χ^2 minima):

$$\begin{array}{lll} m_\tau \simeq M_l(3,3), & m_\mu \simeq M_l(2,2), & m_e \simeq M_l(1,1) \\ m_b \simeq M_D(3,3), & m_s \simeq \frac{M_D(2,3)M_D(3,2)}{M_D(3,3)}, & m_d \simeq \frac{M_D(1,2)M_D(2,1)M_D(3,3)}{M_D(2,3)M_D(3,2)} \\ & m_c \simeq M_U(2,2), & m_u \simeq M_U(1,1) \end{array} \tag{4.114}$$

and:

$$V_{us} \simeq \tfrac{M_D(3,1)}{M_D(3,2)}, \quad V_{cb} \simeq \tfrac{M_D(3,2)}{M_D(3,3)}, \quad V_{ub} \simeq V_{us}V_{cb} \tag{4.115}$$

As can be seen from Table 4.29, there is not much change in the χ^2 of Charge Sets 2, 3, 4 and 6 in going from the product to the mixed ansatz (there is a slight deterioration).

This is initially surprising since the increase in the number of parameters (3 to 6) might naively be expected to lead to an improvement in χ^2. But the mixed ansatz is less suppressive than the product ansatz, and V_{cb} and V_{ub} were already too big; the extra parameters merely perform the task of ensuring that this problem is not consequently exacerbated. In fact, for Charge Set 1 such an exacerbation cannot be avoided, hence the acute deterioration in χ^2.

Also seen for all models, but most markedly for Charge Set 5 (hence its poor performance with the mixed ansatz), is a large rise in m_s coupled with a drop in m_c (to such an extent that an $m_s - m_c$ splitting disappears altogether, or is even manifest as $m_s > m_c$). The rise in m_s is due to the easing of suppression of the matrix elements $M_D(2,3)$ and $M_D(3,2)$: all down quark matrices consequently adopt Textures 6, 7 or 8 (at the χ^2 minima), with a very large m_s to boot. The fall in m_c is largely due to the fact that g_{22} has to give a larger suppression here in the mixed ansatz than ϵ_2 gave in the product ansatz, again in order to try and hold down both m_s and V_{cb}.

4.7.3 Comments

Note that Charge Sets 3 and 6 give exactly the same fits with the mixed ansatz. This is because the abelian PCCSs are $U_1(1) \times U_2(1) \times U(1)_f$ and Charge Set 6 can be obtained simply by adding the $U_2(1)$ quantum numbers to Charge Set 3. That is, we merely make a change of basis, to which the mixed ansatz is insensitive: if the space of abelian PCCSs is generated by Y_1, Y_2, and Y_f for Charge Set 3, then that for Charge Set 6 is generated by Y_1, Y_2 and $Y_f' \equiv Y_f + Y_2$. There is no similar phenomenon amongst the 25 charge sets for $H_1 \times U(1)_f$ because there the space of abelian PCCSs is $U_1(1) \times U(1)_f$; obviously no set of flavour charges can be obtained from another by linearly combining with the $U_1(1)$ quantum numbers, since the flavour charges are non-trivial only for the 2nd and 3rd generations.

It must be said that the fits for the $H_2 \times U(1)_f$ models are poor, for both the product and mixed ansätze. Most of the ills can be traced to the fact that the $U_2(1)$ symmetry does not suppress the 2nd column of the quark mass matrices quite as well as the $SU_2(2)$ symmetry did for the $H_1 \times U(1)_f$ models. No case can be made for these $H_2 \times U(1)_f$ models.

"Andy, where's my fifteen minutes?"

David Bowie

I Can't Read

Chapter 5

Intra-Generation Structure from Off-Diagonal Third Generation Mass: the Anti-Grand Unified Model

5.1 Introduction

In this chapter we discuss Method 1 of generating intra-generation mass splitting, hoping to avoid the disappointment of Method 2 where the anti-grand unified model was ruled out as the fundamental gauge group. We also hope to avoid the introduction of abelian flavour symmetries. Method 1 is based on the observation that the $U_a(1)$ components (a=1,2,3) of SMG^3 affect the off-diagonal matrix elements of M_U and M_D differently, hinting at the possibility of a $t-b$ splitting. This promise is found to be fulfilled, but a serious problem is uncovered *viz.* the prediction $m_d m_s m_b \geq m_u m_c m_t$. Ironically, in order to circumvent this difficulty and rescue the anti-grand unified model, an abelian flavour symmetry must be introduced. Finally, results and discussion of this $SMG^3 \times U(1)_f$ model are given. The matrix diagonalisation and data fitting procedures are performed exactly as in Chapter 4.

5.2 Choice of PCCSs and Ansätze

In Chapter 4, the assumption that $m_{t,b,\tau} \simeq M_{U,D,l}(3,3)$ meant that the basic structure of the fermion mass matrices was to some extent clear from the outset. This in turn meant that the sets of PCCSs suggested towards the end of Chapter 3 as candidates for a suitable inter-generation hierarchy were easily postulated and then accepted or dismissed. However, such an approach is inappropriate here in this Method 1 scenario. The origin of all nine fermion masses (*i.e.* the correspondence between the mass matrix elements and the eigenvalues) is uncertain *a priori*, so an intuitive selection of PCCSs is not possible. Therefore, throughout this chapter we assume that *all* component symmetries of SMG^3 (or $SMG^3 \times U(1)_f$) are approximately conserved; this can easily be rectified at any point by dropping particular symmetry breaking parameters from the analysis *i.e.* by taking particular symmetries to be so badly broken as to be irrelevant for mass suppression. We emphasise, though, that in this chapter the anti-grand unified group SMG^3 will always form a subgroup of the full symmetry group, unlike the models of Chapter 4.

Furthermore, we will make use of only the product and the mixed ansätze, paying particular attention to the latter. The metric ansatz would involve a plethora of parameters, more than the number of data pieces we are trying to fit, and so we do not use it. Of the remaining two, we prefer the mixed ansatz because it treats the abelian symmetries without prejudice and it will be seen to be crucial that particular linear combinations of abelian generators are badly broken (*i.e.* irrelevant) while others are approximately conserved.

5.3 The Failure of SMG^3

For the purpose of clarity, we choose to separately display the abelian and non-abelian contributions to the fermion mass matrices, $M^{\mathrm{ab}}_{U,D,l}$ and $M^{\mathrm{non-ab}}_{U,D,l}$ respectively. It is understood that, for all i and j, the overall matrix elements are given by:

$$M(i,j) = M^{\mathrm{non-ab}}(i,j)\, M^{\mathrm{ab}}(i,j) \tag{5.1}$$

Dealing first with the non-abelian components of SMG^3, and taking the full non-abelian subgroup $SU(3)^3 \times SU(2)^3$ to be partially conserved, we find that for both the product and mixed ansätze (see (3.43) and (3.47)) the contribution of these non-abelian components to the mass matrices is:

$$M_U^{\mathrm{non-ab}} \simeq M_D^{\mathrm{non-ab}} \simeq \begin{pmatrix} \epsilon_1 & \beta_1\beta_2\epsilon_2 & \beta_1\beta_3\epsilon_3 \\ \beta_1\beta_2\epsilon_1 & \epsilon_2 & \beta_2\beta_3\epsilon_3 \\ \beta_1\beta_3\epsilon_1 & \beta_2\beta_3\epsilon_2 & \epsilon_3 \end{pmatrix}, \quad M_l^{\mathrm{non-ab}} \simeq \begin{pmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix} \tag{5.2}$$

These matrices re-emphasise the point that the production of intra-generation splitting depends crucially on the abelian PCCSs.

The contribution of the abelian components of SMG^3 to the mass matrices depends upon which ansatz we use. With the product ansatz (see (3.43)), taking the full $U(1)^3$ subgroup to be approximately conserved, we get:

$$\begin{aligned} M_U^{\mathrm{ab}} &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^4\lambda_2 & \lambda_1^4\lambda_3 \\ \lambda_1\lambda_2^4 & \lambda_2^3 & \lambda_2^4\lambda_3 \\ \lambda_1\lambda_3^4 & \lambda_2\lambda_3^4 & \lambda_3^3 \end{pmatrix} \\ M_D^{\mathrm{ab}} &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^2\lambda_2 & \lambda_1^2\lambda_3 \\ \lambda_1\lambda_2^2 & \lambda_2^3 & \lambda_2^2\lambda_3 \\ \lambda_1\lambda_3^2 & \lambda_2\lambda_3^2 & \lambda_3^3 \end{pmatrix} \\ M_l^{\mathrm{ab}} &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^6\lambda_2^3 & \lambda_1^6\lambda_3^3 \\ \lambda_1^3\lambda_2^6 & \lambda_2^3 & \lambda_2^6\lambda_3^3 \\ \lambda_1^3\lambda_3^6 & \lambda_2^3\lambda_3^6 & \lambda_3^3 \end{pmatrix} \end{aligned} \tag{5.3}$$

again assuming that the Q_a (a=1,2,3) involved in (3.43) correspond to the generators Y_a of $U_a(1)$. It is obvious from (5.3) that:

$$M_U^{\mathrm{ab}}(i,j) \leq M_D^{\mathrm{ab}}(i,j) \qquad \text{for all } i,j \tag{5.4}$$

so that from (5.1) and (5.2):

$$M_U(i,j) \leq M_D(i,j) \qquad \text{for all } i,j \tag{5.5}$$

This in turn means that:

$$m_t \leq m_b \tag{5.6}$$

since these masses come from the biggest elements of M_U and M_D respectively, and so the SMG^3 model with the product ansatz seems to run up against an insurmountable difficulty. However, a different choice of Q_a (*i.e.* a different choice of the basis for the $U(1)^3$ space) gives matrices different to those of (5.3), and we return to this point shortly. For now we simply state that with no physics to help us make such a choice, we prefer to use the mixed ansatz which is insensitive to such basis changes.

With the mixed ansatz (see (3.47)), things seem more promising. The contribution of the abelian components to the mass matrices becomes:

$$\begin{aligned}
M_U^{\mathrm{ab}} &\simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & e^{-\sqrt{16g_{11}+g_{22}-8g_{12}}} & e^{-\sqrt{16g_{11}+g_{33}-8g_{13}}} \\ e^{-\sqrt{g_{11}+16g_{22}-8g_{12}}} & e^{-\sqrt{9g_{22}}} & e^{-\sqrt{16g_{22}+g_{33}-8g_{23}}} \\ e^{-\sqrt{g_{11}+16g_{33}-8g_{13}}} & e^{-\sqrt{g_{22}+16g_{33}-8g_{23}}} & e^{-\sqrt{9g_{33}}} \end{pmatrix} \\
M_D^{\mathrm{ab}} &\simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & e^{-\sqrt{4g_{11}+g_{22}+4g_{12}}} & e^{-\sqrt{4g_{11}+g_{33}+4g_{13}}} \\ e^{-\sqrt{g_{11}+4g_{22}+4g_{12}}} & e^{-\sqrt{9g_{22}}} & e^{-\sqrt{4g_{22}+g_{33}+4g_{23}}} \\ e^{-\sqrt{g_{11}+4g_{33}+4g_{13}}} & e^{-\sqrt{g_{22}+4g_{33}+4g_{23}}} & e^{-\sqrt{9g_{33}}} \end{pmatrix} \\
M_l^{\mathrm{ab}} &\simeq \begin{pmatrix} e^{-\sqrt{9g_{11}}} & e^{-\sqrt{36g_{11}+9g_{22}-36g_{12}}} & e^{-\sqrt{36g_{11}+9g_{33}-36g_{13}}} \\ e^{-\sqrt{9g_{11}+36g_{22}-36g_{12}}} & e^{-\sqrt{9g_{22}}} & e^{-\sqrt{36g_{22}+9g_{33}-36g_{23}}} \\ e^{-\sqrt{9g_{11}+36g_{33}-36g_{13}}} & e^{-\sqrt{9g_{22}+36g_{33}-36g_{23}}} & e^{-\sqrt{9g_{33}}} \end{pmatrix}
\end{aligned} \tag{5.7}$$

The dependency of intra-generation mass splitting on some/all of the third generation masses originating from off-diagonal matrix elements is seen to be crucial. In particular, since the biggest element in each quark matrix will approximately give m_t and m_b, we do not want m_t to come from a diagonal element of M_U since this would almost certainly mean $m_b \simeq m_t$ (the corresponding element also being present in M_D). This feature, namely:

$$M_U(i,i) \simeq M_D(i,i) \simeq M_l(i,i) \qquad (i = 1,2,3) \tag{5.8}$$

is a constant thorn in the flesh of the SMG^3 model. We will therefore demand that m_t comes from an off-diagonal element and, due to the symmetry between such elements, we can without loss of generality choose:

$$m_t \simeq M_U(2,3) \tag{5.9}$$

Then, since we want m_t unsuppressed relative to $\langle\phi\rangle_{\mathrm{ws}}$, we take:

$$a_{23}^U = 1 \tag{5.10}$$

(recall that a_{ij}^U describes the suppression of $M_U(i,j)$ - see (3.2)). So to avoid suppression of $M_U(2,3)$ from the non-abelian PCCSs we take:

$$\beta_2 = \beta_3 = \epsilon_3 = 1 \tag{5.11}$$

which means that our PCCSs are reduced to:

$$SU_1(3) \times SU_1(2) \times SU_2(2) \times U_1(1) \times U_2(1) \times U_3(1) \tag{5.12}$$

Similarly, to avoid suppression of $M_U(2,3)$ from the abelian PCCSs we take:

$$16g_{22} + g_{33} - 8g_{23} = 0 \tag{5.13}$$

This last equation can be formulated more generally as:

$$(\mathbf{Q}_{t_L} - \mathbf{Q}_{c_R})^T \, \mathcal{G} \, (\mathbf{Q}_{t_L} - \mathbf{Q}_{c_R}) = 0 \tag{5.14}$$

in the notation of (3.47), where:

$$\begin{aligned} \mathbf{Q}_{t_L} &= (0,0,1) \\ \mathbf{Q}_{c_R} &= (0,4,0) \end{aligned} \tag{5.15}$$

But since $\mathcal{G}$ is a positive semi-definite metric, it is a simple matter to prove that this implies:

$$\mathcal{G} \, (\mathbf{Q}_{t_L} - \mathbf{Q}_{c_R}) = \mathbf{0} \tag{5.16}$$

which yields 3 linear equations:

$$g_{33} = 16g_{22}, \;\; g_{23} = 4g_{22}, \;\; g_{13} = 4g_{12} \tag{5.17}$$

Physically what we are saying in (5.16) is that the generator $4Y_2 - Y_3$ is not relevant for mass suppression *i.e.* it is very badly broken; only the $U(1)^2$ space spanned by two generators orthogonal[1] to this one is relevant. That is, our PCCSs are further reduced to:

$$SU_1(3) \times SU_1(2) \times SU_2(2) \times U_1(1) \times U_2(1) \times U_3(1)/U(1)' \tag{5.18}$$

where $U(1)'$ is generated by $4Y_2 - Y_3$.

[1]Assuming for convenience that Y_1, Y_2 and Y_3 form a Cartesian basis for the space of abelian generators.

So, using (5.11) and (5.17), the mass matrices become:

$$
\begin{aligned}
M_U &\simeq \begin{pmatrix} \epsilon_1 e^{-\sqrt{9g_{11}}} & \beta_1\epsilon_2 e^{-\sqrt{16g_{11}+g_{22}-8g_{12}}} & \beta_1 e^{-\sqrt{16g_{11}+16g_{22}-32g_{12}}} \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+16g_{22}-8g_{12}}} & \epsilon_2 e^{-\sqrt{9g_{22}}} & 1 \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+256g_{22}-32g_{12}}} & \epsilon_2 e^{-\sqrt{225g_{22}}} & e^{-\sqrt{144g_{22}}} \end{pmatrix} \\
M_D &\simeq \begin{pmatrix} \epsilon_1 e^{-\sqrt{9g_{11}}} & \beta_1\epsilon_2 e^{-\sqrt{4g_{11}+g_{22}+4g_{12}}} & \beta_1 e^{-\sqrt{4g_{11}+16g_{22}+16g_{12}}} \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+4g_{22}+4g_{12}}} & \epsilon_2 e^{-\sqrt{9g_{22}}} & e^{-\sqrt{36g_{22}}} \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+64g_{22}+16g_{12}}} & \epsilon_2 e^{-\sqrt{81g_{22}}} & e^{-\sqrt{144g_{22}}} \end{pmatrix} \\
M_l &\simeq \begin{pmatrix} \epsilon_1 e^{-\sqrt{9g_{11}}} & \epsilon_2 e^{-\sqrt{36g_{11}+9g_{22}-36g_{12}}} & e^{-\sqrt{36g_{11}+144g_{22}-144g_{12}}} \\ \epsilon_1 e^{-\sqrt{9g_{11}+36g_{22}-36g_{12}}} & \epsilon_2 e^{-\sqrt{9g_{22}}} & e^{-\sqrt{36g_{22}}} \\ \epsilon_1 e^{-\sqrt{9g_{11}+576g_{22}-144g_{12}}} & \epsilon_2 e^{-\sqrt{441g_{22}}} & e^{-\sqrt{144g_{22}}} \end{pmatrix}
\end{aligned}
\tag{5.19}
$$

At first glance this seems promising. Given that $m_t \simeq M_U(2,3) \simeq 1$ and that we want to obtain $m_\tau \simeq m_b$ and $\mathcal{V}(3,3) \equiv V_{tb} \simeq 1$ (so that we require m_b to come from the 3rd column of M_D), we should choose our parameters to give:

$$
\begin{aligned}
m_b &\simeq M_D(2,3) \simeq e^{-6\sqrt{g_{22}}} \\
m_\tau &\simeq M_l(2,3) \simeq e^{-6\sqrt{g_{22}}}
\end{aligned}
\tag{5.20}
$$

But things go drastically wrong when we turn to the 2nd generation masses. Likely predictions are:

$$
\begin{aligned}
m_c &\simeq \left[M_U(3,2) + \frac{M_U(2,2)M_U(3,3)}{M_U(2,3)}\right] \simeq \epsilon_2 e^{-15\sqrt{g_{22}}} \\
m_s &\simeq \left[M_D(3,2) + \frac{M_D(2,2)M_D(3,3)}{M_D(2,3)}\right] \simeq \epsilon_2 e^{-9\sqrt{g_{22}}} \\
m_\mu &\simeq \left[\frac{M_l(2,2)M_l(3,3)}{M_l(2,3)}\right] \simeq \epsilon_2 e^{-9\sqrt{g_{22}}}
\end{aligned}
\tag{5.21}
$$

and a terrible splitting is obtained ($m_c \ll m_s \simeq m_\mu$). This in fact points to a deeper problem, caused by the fact that:

$$M_U(2,3)M_U(3,2) \simeq M_U(2,2)M_U(3,3) \tag{5.22}$$

It is possible to prove (see Appendix A) for all β_1, ϵ_1 and ϵ_2 (< 1) that:

$$M_U(1,1)M_U(2,2)M_U(3,3) \simeq \det M_U \simeq \det M_l \leq \det M_D \tag{5.23}$$

using the fact that $\mathcal{G}$ is positive semi-definite. That is, the product of diagonal mass matrix elements dominates both $\det M_U$ and $\det M_l$, but $\det M_D$ can be even larger (it is possible to achieve $M_D(1,3)M_D(2,2)M_D(3,1) > M_D(1,1)M_D(2,2)M_D(3,3)$ for example). Then:

$$m_u m_c m_t \simeq m_e m_\mu m_\tau \leq m_d m_s m_b \tag{5.24}$$

and so although we can for example accommodate $m_b \ll m_t$ we cannot simultaneously get $m_s \ll m_c$ and $m_u \simeq m_d$. Moreover, this prediction (5.24) is so far from being satisfied that even a slight relaxation of $m_t \simeq \langle\phi\rangle_{\text{ws}}$ to $m_t \leq \langle\phi\rangle_{\text{ws}}$ (*i.e.* a relaxation of (5.11) and/or (5.16)) will not alter it radically enough to render it acceptable. With the mixed ansatz, the gauge group SMG^3 cannot, therefore, naturally accommodate the observed intra-generation fermion mass hierarchy without some new ingredient.

It should be said that there is nothing mysterious about choosing $4Y_2 - Y_3$ to be irrelevant for mass suppression. Quite simply, if we want to obtain $M_U(2,3) \simeq 1$ then we must assume that only generators orthogonal to this one are relevant, regardless of which ansatz we intend to use. Returning to the product ansatz, then, suppose we choose the generators Y_1 and $Y_2 + 4Y_3$ to span the space of abelian PCCSs *viz.* $U_1(1) \times U_2(1) \times U_3(1)/U(1)'$, where $U(1)'$ is again generated by $4Y_2 - Y_3$. The quantum numbers of all Weyl fermions corresponding to these generators are shown in Table 5.1. Then, using (3.43) with λ_1 and λ_2 corresponding to Y_1 and $Y_2 + 4Y_3$ respectively, the abelian contributions to the quark mass matrices are:

$$M_U^{\text{ab}} \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^4\lambda_2 & \lambda_1^4\lambda_2^4 \\ \lambda_1\lambda_2^4 & \lambda_2^3 & 1 \\ \lambda_1\lambda_2^{16} & \lambda_2^{15} & \lambda_2^{12} \end{pmatrix}, \quad M_D^{\text{ab}} \simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^2\lambda_2 & \lambda_1^2\lambda_2^4 \\ \lambda_1\lambda_2^2 & \lambda_2^3 & \lambda_2^6 \\ \lambda_1\lambda_2^8 & \lambda_2^9 & \lambda_2^{12} \end{pmatrix} \tag{5.25}$$

and $M_{U,D}^{\text{non-ab}}$ are still given by (5.2) with (5.11). While we no longer find that $M_U(i,j) \leq M_D(i,j)$, it is nevertheless true that the most likely predictions still give $m_s \gg m_c$, as for the mixed ansatz, since $\det M_U \simeq \det M_D \simeq \lambda_1^3\lambda_2^{15}$:

$$\begin{aligned} m_t &\simeq M_U(2,3) \simeq 1 \\ m_c &\simeq \left[M_U(3,2) + \frac{M_U(2,2)M_U(3,3)}{M_U(2,3)}\right] \simeq \epsilon_2\lambda_2^{15} \\ m_b &\simeq M_D(2,3) \simeq \lambda_2^6 \\ m_s &\simeq \left[M_D(3,2) + \frac{M_D(2,2)M_D(3,3)}{M_D(2,3)}\right] \simeq \epsilon_2\lambda_2^9 \end{aligned} \tag{5.26}$$

Weyl	Charge Corresponding to Generator:			
State	Y_1	Y_2	Y_3	$Y_2 + 4Y_3$
d_L	1	0	0	0
u_R	4	0	0	0
d_R	-2	0	0	0
e_L	-3	0	0	0
e_R	-6	0	0	0
s_L	0	1	0	1
c_R	0	4	0	4
s_R	0	-2	0	-2
μ_L	0	-3	0	-3
μ_R	0	-6	0	-6
b_L	0	0	1	4
t_R	0	0	4	16
b_R	0	0	-2	-8
τ_L	0	0	-3	-12
τ_R	0	0	-6	-24

Table 5.1: Charges of the Weyl fermions corresponding to particular generators of $U_1(1) \times U_2(1) \times U_3(1)$.

Different origins for m_s and m_c, compatible with both $m_s \ll m_c$ and $\mathcal{V} \simeq \mathrm{diag}(1,1,1)$, are not possible. Of course, we can try yet more basis changes but we always find that (see Appendix B):

$$M_U(2,3)M_U(3,2) \simeq M_U(2,2)M_U(3,3) \simeq M_D(2,2)M_D(3,3) \tag{5.27}$$

which is the basic problem here, because it makes it extremely hard to avoid getting:

$$m_c m_t \simeq m_s m_b \tag{5.28}$$

This whole line of analysis with the SMG^3 model is unpromising, to say the least.

5.4 The Rescue: $SMG^3 \times U(1)_f$

Strictly speaking, then, Method 1 of generating intra-generation mass structure has drawn a blank. But due to the success of the SMG^3 model in predicting the SM gauge couplings [38], we are reluctant to dismiss it so quickly. One possible way to "rescue" it is to assume that SMG^3 does not constitute the full gauge group, but that there exists at least one other symmetry which we can take to be partially conserved. Within the confines of the Category (8) groups of Chapter 2, this ironically means that we must consider an abelian flavour symmetry so that the gauge group is $SMG^3 \times U(1)_f$ (we have discussed in Chapter 2 why this is the only possible extension of SMG^3). The no-anomaly equations for the $U(1)_f$ charges were given in Chapter 2 where it was shown that the only solution carrying new information (*i.e.* which is not a linear combination of the $U_a(1)$ charges (a=1,2,3)) is essentially given by:

$$\begin{pmatrix} d_L & u_R & d_R & e_L & e_R \\ s_L & c_R & s_R & \mu_L & \mu_R \\ b_L & t_R & b_R & \tau_L & \tau_R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix} \tag{5.29}$$

There are another two solutions (shown in Chapter 2) corresponding to permutations of the non-zero charges shown in (5.29) among the generations, but they are equivalent to this one after simply relabelling the states. Furthermore, any linear combination of the charges of (5.29) with the $U_a(1)$ charges (a=1,2,3) is also an anomaly-free solution. But we restate that such basis changes are irrelevant for the mixed ansatz on which all of our numerical analysis will be based.

The contributions of the non-abelian components of $SMG^3 \times U(1)_f$ to the mass matrix elements is the same as in (5.2), again for both the product and mixed ansätze, but the abelian sector has now been expanded by the addition of the $U(1)_f$ of (5.29). With the product ansatz, the contribution of the abelian components to the mass matrices is, assuming as usual that the full $U_1(1) \times U_2(1) \times U_3(1) \times U(1)_f$ subgroup is partially conserved:

$$
\begin{aligned}
M_U^{\mathrm{ab}} &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^4\lambda_2 & \lambda_1^4\lambda_3 \\ \lambda_1\lambda_2^4\lambda_f & \lambda_2^3\lambda_f & \lambda_2^4\lambda_3\lambda_f \\ \lambda_1\lambda_3^4\lambda_f & \lambda_2\lambda_3^4\lambda_f & \lambda_3^3\lambda_f \end{pmatrix} \\
M_D^{\mathrm{ab}} &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^2\lambda_2 & \lambda_1^2\lambda_3 \\ \lambda_1\lambda_2^2\lambda_f & \lambda_2^3\lambda_f & \lambda_2^2\lambda_3\lambda_f \\ \lambda_1\lambda_3^2\lambda_f & \lambda_2\lambda_3^2\lambda_f & \lambda_3^3\lambda_f \end{pmatrix} \\
M_l^{\mathrm{ab}} &\simeq \begin{pmatrix} \lambda_1^3 & \lambda_1^6\lambda_2^3 & \lambda_1^6\lambda_3^3 \\ \lambda_1^3\lambda_2^6\lambda_f & \lambda_2^3\lambda_f & \lambda_2^6\lambda_3^3\lambda_f \\ \lambda_1^3\lambda_3^6\lambda_f & \lambda_2^3\lambda_3^6\lambda_f & \lambda_3^3\lambda_f \end{pmatrix}
\end{aligned}
\tag{5.30}
$$

Note that we have again assumed that the charges Q_a (a=1,2,3,f) involved in (3.43) correspond to the generators Y_1, Y_2, Y_3 and Y_f respectively. It is still true that:

$$M_U(i,j) \leq M_D(i,j) \quad \text{for all } i,j \tag{5.31}$$

and so no satisfactory hierarchy is obtainable.

With the mixed ansatz (see (3.47)), things are again more promising. $\mathcal{G}$ is expanded in an obvious manner to accommodate the abelian flavour symmetry. Introducing once more for display purposes the three matrices $N_{U,D,l}^{\mathrm{ab}}$ whose elements are defined by:

$$N_{U,D,l}^{\mathrm{ab}}(i,j) = (\log[M_{U,D,l}^{\mathrm{ab}}(i,j)])^2 \tag{5.32}$$

the contributions of the abelian components become:

$$N_l^{\rm ab} \simeq \begin{pmatrix} 9g_{11} & 36g_{11}+9g_{22}-36g_{12} & 36g_{11}+9g_{33}-36g_{13} \\ \\ 9g_{11}+36g_{22}+g_{ff}- & 9g_{22}+g_{ff}+6g_{2f} & 36g_{22}+9g_{33}+g_{ff}- \\ 36g_{12}-6g_{1f}+12g_{2f} & & 36g_{23}+12g_{2f}-6g_{3f} \\ \\ 9g_{11}+36g_{33}+g_{ff}- & 9g_{22}+36g_{33}+g_{ff}- & 9g_{33}+g_{ff}-6g_{3f} \\ 36g_{13}+6g_{1f}-12g_{3f} & 36g_{23}+6g_{2f}-12g_{3f} & \end{pmatrix}$$

$$N_U^{\rm ab} \simeq \begin{pmatrix} 9g_{11} & 16g_{11}+g_{22}-8g_{12} & 16g_{11}+g_{33}-8g_{13} \\ \\ g_{11}+16g_{22}+g_{ff}- & 9g_{22}+g_{ff}+6g_{2f} & 16g_{22}+g_{33}+g_{ff}- \\ 8g_{12}-2g_{1f}+8g_{2f} & & 8g_{23}+8g_{2f}-2g_{3f} \\ \\ g_{11}+16g_{33}+g_{ff}- & g_{22}+16g_{33}+g_{ff}- & 9g_{33}+g_{ff}-6g_{3f} \\ 8g_{13}+2g_{1f}-8g_{3f} & 8g_{23}+2g_{2f}-8g_{3f} & \end{pmatrix}$$

$$N_D^{\rm ab} \simeq \begin{pmatrix} 9g_{11} & 4g_{11}+g_{22}+4g_{12} & 4g_{11}+g_{33}+4g_{13} \\ \\ g_{11}+4g_{22}+g_{ff}+ & 9g_{22}+g_{ff}+6g_{2f} & 4g_{22}+g_{33}+g_{ff}+ \\ 4g_{12}+2g_{1f}+4g_{2f} & & 4g_{23}+4g_{2f}+2g_{3f} \\ \\ g_{11}+4g_{33}+g_{ff}+ & g_{22}+4g_{33}+g_{ff}+ & 9g_{33}+g_{ff}-6g_{3f} \\ 4g_{13}-2g_{1f}-4g_{3f} & 4g_{23}-2g_{2f}-4g_{3f} & \end{pmatrix} \quad (5.33)$$

Now, the diagonal elements are still identical in all 3 matrices:

$$M_U(i,i) \simeq M_D(i,i) \simeq M_l(i,i) \quad (i=1,2,3) \quad (5.34)$$

so m_t must again receive its dominant contribution from an off-diagonal element of M_U. Bearing in mind the need for:

$$m_t \gg m_b \simeq m_\tau \quad (5.35)$$

$$m_c \gg m_s \simeq m_\mu \quad (5.36)$$

$$m_u \simeq m_d \simeq m_e \quad (5.37)$$

$$\mathcal{V} \simeq \mathrm{diag}(1,1,1) \quad (5.38)$$

we might hope to aim, as a first approximation, for a parameter choice which yields matrices looking approximately like:

$$M_U \simeq \begin{pmatrix} m_u & \times & \times \\ \times & \times & m_t \\ \times & m_c & \times \end{pmatrix}, \quad M_D \simeq \begin{pmatrix} m_d & \times & \times \\ \times & m_s & \times \\ \times & \times & m_b \end{pmatrix}, \quad M_l \simeq \begin{pmatrix} m_e & \times & \times \\ \times & m_\mu & \times \\ \times & \times & m_\tau \end{pmatrix} \tag{5.39}$$

That is, we aim for all 3 matrices to be of Texture 1 (after switching the 2nd and 3rd rows of M_U).

As before, we demand that $m_t \simeq M_U(2,3)$ be unsuppressed relative to $\langle\phi\rangle_{\mathrm{ws}}$ *i.e.* $a_{23}^U = 1$. This again gives:

$$\beta_2 = \beta_3 = \epsilon_3 = 1 \tag{5.40}$$

and:

$$\begin{aligned} 8g_{2f} &= -16g_{22} + g_{33} - g_{ff} \\ g_{13} &= 4g_{12} + g_{1f} \\ 2g_{3f} &= -16g_{22} + g_{33} + g_{ff} \\ 8g_{23} &= 16g_{22} + g_{33} - g_{ff} \end{aligned} \tag{5.41}$$

where this last set of equations follows from:

$$\mathcal{G}(\mathbf{Q}_{t_L} - \mathbf{Q}_{c_R}) = \mathbf{0} \tag{5.42}$$

with:

$$\begin{aligned} \mathbf{Q}_{t_L} &= (0,0,1,0) \\ \mathbf{Q}_{c_R} &= (0,4,0,1) \end{aligned} \tag{5.43}$$

This time we are saying physically that the generator $4Y_2 - Y_3 + Y_f$ is not relevant for mass suppression. So the PCCSs are now:

$$SU_1(3) \times SU_1(2) \times SU_2(2) \times U_1(1) \times U_2(1) \times U_3(1) \times U(1)_f/U(1)' \tag{5.44}$$

where $U(1)'$ is generated by $4Y_2 - Y_3 + Y_f$. We then have:

$$
\begin{aligned}
N_U^{\rm ab} &\simeq \begin{pmatrix} 9g_{11} & 16g_{11}+g_{22}-8g_{12} & 16g_{11}+g_{33}-32g_{12} \\ & & -8g_{1f} \\ g_{11}+g_{33}-8g_{12} & -3g_{22}+\frac{3}{4}g_{33}+\frac{1}{4}g_{ff} & 0 \\ -2g_{1f} & & \\ g_{11}+64g_{22}+12g_{33} & 45g_{22}+\frac{45}{4}g_{33}-\frac{9}{4}g_{ff} & 48g_{22}+6g_{33}-2g_{ff} \\ -3g_{ff}-32g_{12}-6g_{1f} & & \end{pmatrix} \\
N_D^{\rm ab} &\simeq \begin{pmatrix} 9g_{11} & 4g_{11}+g_{22}+4g_{12} & 4g_{11}+g_{33}+16g_{12} \\ & & +4g_{1f} \\ g_{11}-4g_{22}+\frac{1}{2}g_{33} & -3g_{22}+\frac{3}{4}g_{33}+\frac{1}{4}g_{ff} & -12g_{22}+3g_{33}+g_{ff} \\ +\frac{1}{2}g_{ff}+4g_{12}+2g_{1f} & & \\ g_{11}+32g_{22}+2g_{33} & 45g_{22}+\frac{9}{4}g_{33}-\frac{5}{4}g_{ff} & 48g_{22}+6g_{33}-2g_{ff} \\ -g_{ff}+16g_{12}+2g_{1f} & & \end{pmatrix} \\
N_l^{\rm ab} &\simeq \begin{pmatrix} 9g_{11} & 36g_{11}+9g_{22}-36g_{12} & 36g_{11}+9g_{33}-144g_{12} \\ & & -36g_{1f} \\ 9g_{11}+12g_{22}+\frac{3}{2}g_{33} & -3g_{22}+\frac{3}{4}g_{33}+\frac{1}{4}g_{ff} & -12g_{22}+3g_{33}+g_{ff} \\ -\frac{1}{2}g_{ff}-36g_{12}-6g_{1f} & & \\ 9g_{11}+96g_{22}+30g_{33} & 21g_{22}+\frac{105}{4}g_{33}-\frac{5}{4}g_{ff} & 48g_{22}+6g_{33}-2g_{ff} \\ -5g_{ff}-144g_{12}-30g_{1f} & & \end{pmatrix}
\end{aligned}
\tag{5.45}
$$

It seems perfectly possible that these matrices can adopt the textures of (5.39). In particular, we are no longer burdened by the prediction:

$$\det M_U \simeq M_U(1,1)M_U(2,2)M_U(3,3) \tag{5.46}$$

which caused the downfall of the SMG^3 model, since it is now possible to achieve:

$$M_U(2,3)M_U(3,2) > M_U(2,2)M_U(3,3) \tag{5.47}$$

After switching the 2nd and 3rd rows of M_U, all three matrices are presumed to be of Texture 1 at the χ^2 minima (self-consistency is of course confirmed when the minima have been found). The fit to the data is then carried out exactly as in Chapter 4 with the algebraic M_P-scale predictions (at the χ^2 minima):

$$
\begin{aligned}
m_t &\simeq M_U(2,3) \\
m_b &\simeq m_\tau \simeq M_D(3,3) \simeq \exp(-\sqrt{48g_{22}+6g_{33}-2g_{ff}}) \\
m_c &\simeq M_U(3,2) \simeq \epsilon_2 \exp(-\sqrt{45g_{22}+\frac{45}{4}g_{33}-\frac{9}{4}g_{ff}}) \\
m_s &\simeq m_\mu \simeq M_D(2,2) \simeq \epsilon_2 \exp(-\sqrt{-3g_{22}+\frac{3}{4}g_{33}+\frac{1}{4}g_{ff}}) \\
m_u &\simeq m_d \simeq m_e \simeq M_D(1,1) \simeq \epsilon_1 \exp(-\sqrt{9g_{11}}) \\
V_{us} &\simeq \frac{M_U(3,1)}{M_U(3,2)} \simeq \beta_1\epsilon_1 \frac{\exp(-\sqrt{g_{11}+64g_{22}+12g_{33}-3g_{ff}-32g_{12}-6g_{1f}})}{m_c/m_t} \\
V_{cb} &\simeq \frac{M_D(3,2)}{M_D(3,3)} \simeq \epsilon_2 \frac{\exp(-\sqrt{45g_{22}+\frac{9}{4}g_{33}-\frac{5}{4}g_{ff}})}{m_b/m_t} \\
V_{ub} &\simeq V_{us}V_{cb}
\end{aligned}
\tag{5.48}
$$

We have 9 parameters left after the imposition of (5.40) and (5.41) (*viz.* β_1, ϵ_1, ϵ_2, g_{11}, g_{22}, g_{33}, g_{ff}, g_{12} and g_{1f}) which we can use to fit the 11 data pieces, an unconvincingly large number. Fortunately, several of these parameters are redundant. The predictions of (5.48) reveal that only 6 matrix elements have to be fitted *viz.* $M_D(3,3)$, $M_U(3,2)$, $M_D(2,2)$, $M_D(1,1)$, $M_U(3,1)$ and $M_D(3,2)$. Note firstly that $M_D(3,3)$, $M_D(2,2)$ and $M_{U,D}(3,2)$ depend only on ϵ_2, g_{22}, g_{33} and g_{ff}. Of these, ϵ_2 is redundant: given the free choice $0 < \epsilon_2 \leq 1$, it is always true that at the χ^2 minimum we have $\epsilon_2 = 1$. This is because ϵ_2 works to suppress m_c and V_{cb} which from Table 5.2 (see later) are seen to be smaller than we would like already. The remaining two elements ($M_D(1,1)$ and $M_U(3,1)$) thus have an abundance of parameters with which to be fitted *viz.* β_1, ϵ_1, g_{11}, g_{12} and g_{1f} (crudely assuming that the other parameters are already pinned down). We can discard any of the following four combinations:

$$
\left.
\begin{array}{c}
(\beta_1, g_{12}) \\
(\beta_1, g_{1f}) \\
(\epsilon_1, g_{12}) \\
(\epsilon_1, g_{1f})
\end{array}
\right\} = (1,0)
\tag{5.49}
$$

and still remain on the χ^2 minimum. We arbitrarily choose to discard ϵ_1 and g_{1f}.

Obviously we would like to discard a third parameter, but unfortunately we cannot do so and yet remain on the χ^2 minimum. The resulting 6 parameter fit is shown in the first two columns of Table 5.2 for both $m_t^{\rm phys} = 100$ and 200 GeV (given a free choice, $m_t^{\rm phys}$ will lie at 100 GeV). In fact there is a curve of equal χ^2 at the minimum so that effectively only 5 (complicated) functions of these 6 parameters are used. We restate that this is because we only discarded 2 parameters in (5.49) leaving 3 parameters to fit 2 matrix elements; unfortunately we cannot make a further simple discard *e.g.* $\beta_1 = 1$ or $g_{12} = 0$.

Table 5.2 also shows the parameter values, masses and mixing angles at the χ^2 minima. The fit is generally very good, indeed it is much better than any of the fits in Chapter 4. Several points should be noted, some of which are by now familiar:

1. The good m_b and m_τ.

   This is a result of the $SU(5)$-like M_P-scale prediction:
$$m_b \simeq m_\tau \tag{5.50}$$

2. The slightly high m_s.

   This is similarly a result of the $SU(5)$-like M_P-scale prediction:
$$m_s \simeq m_\mu \tag{5.51}$$

3. The good mixing hierarchy *viz.* $V_{us} \gg V_{cb} \gg V_{ub}$.

   This is partly thanks to the M_P-scale prediction:
$$V_{ub} \simeq V_{us} V_{cb} \tag{5.52}$$

4. The failure to generate $m_s \ll m_c$.

   This is the only real fault in these fits, and it is especially apparent at high $m_t^{\rm phys}$. It is only partly caused by the slightly high m_s, explained above. There is in fact a conflict between getting V_{cb} to be as high as its experimental value and getting m_c to be higher than m_s. Using (5.48) we can write:
$$\begin{aligned}\frac{m_s}{m_c} &\simeq \frac{\exp(-\sqrt{-3g_{22} + \frac{3}{4}g_{33} + \frac{1}{4}g_{ff}})}{\exp(-\sqrt{45g_{22} + \frac{45}{4}g_{33} - \frac{9}{4}g_{ff}})} \\ V_{cb} &\simeq \frac{\exp(-\sqrt{45g_{22} + \frac{9}{4}g_{33} - \frac{5}{4}g_{ff}})}{\exp(-\sqrt{48g_{22} + 6g_{33} - 2g_{ff}})}\end{aligned} \tag{5.53}$$

Fit Results	$m_t^{\rm phys} = 100$ GeV	$m_t^{\rm phys} = 200$ GeV	
		unbiased	biased
χ^2	3.7	5.6	6.9
β_1	0.0910	0.0174	0.129
g_{11}	12.0	15.3	15.3
g_{22}	4.38	6.21	7.52
g_{33}	12.7	18.6	19.7
g_{ff}	138	195	230
g_{12}	0.263	0.940	0.231
m_e (MeV)	1.0	1.0	1.0
m_μ (MeV)	120	160	110
m_τ (GeV)	1.4	1.5	1.5
m_d (MeV)	4.9	4.9	4.9
m_s (MeV)	600	790	530
$m_b^{\rm phys}$ (GeV)	5.4	5.5	5.3
m_u (MeV)	4.9	4.9	4.9
m_c (GeV)	0.73	0.53	0.84
V_{us}	0.19	0.22	0.22
V_{cb}	0.016	0.012	0.0048
V_{ub}	0.0030	0.0027	0.0027

Table 5.2: Results of a χ^2-fit to fermion masses and mixing angles for $m_d \simeq m_u \simeq m_e$ at M_P. The active parameters are shown. All masses are running masses evaluated at 1 GeV unless otherwise stated.

The region of parameter space in which $m_s/m_c < 1$ (*i.e.* where the g_{ff} contribution to the exponents in the expression for m_s/m_c compensates for that of g_{22} and g_{33} which drives $m_s/m_c > 1$) favours a small V_{cb}. This is confirmed in the third column of Table 5.2 where we bias the fit by weighting V_{cb} to be less important than the other data pieces. We do this by minimising:

$$\chi'^2 = \sum_{i=1}^{11} n_i \left(\log \frac{f_i}{E_i}\right)^2 \tag{5.54}$$

in the notation of (4.56) where:

$$n_i = \begin{cases} \frac{1}{3} & \text{if } i = 10 \\ 1 & \text{otherwise} \end{cases} \tag{5.55}$$

The function value quoted in the table, however, is for the original χ^2 of (4.56). A (small) $m_s < m_c$ splitting is seen to be obtained at the expense of V_{cb} being almost an order of magnitude too small.

As a short coda to this section, we show that it is possible to make use of some of the previously redundant parameters. Throughout this thesis, we have been willing to accept the order of magnitude M_P-scale relation:

$$m_d \simeq m_u \simeq m_e \tag{5.56}$$

and have viewed the better M_P-scale relation:

$$m_d > m_u \simeq m_e \tag{5.57}$$

only as a fortuitous bonus whenever it occurred. But here we have some scope for deliberately obtaining (5.57) from the outset. This is done by aiming for a differently structured M_D *viz.* Texture 2 instead of Texture 1. Returning to the 9 parameters we had after the imposition of (5.40) and (5.41), the only discards we make are:

$$\beta_1 = \epsilon_2 = 1 \tag{5.58}$$

This means that the PCCSs are:

$$SU_1(2) \times U_1(1) \times U_2(1) \times U_3(1) \times U(1)_f / U(1)' \tag{5.59}$$

where $U(1)'$ is generated by $4Y_2 - Y_3 + Y_f$.

Fit Results	$m_t^{\rm phys} = 100$ GeV	$m_t^{\rm phys} = 200$ GeV unbiased	$m_t^{\rm phys} = 200$ GeV biased
χ^2	3.1	5.0	6.4
ϵ_1	0.257	0.236	0.515
g_{11}	9.69	12.5	14.4
g_{22}	4.38	6.21	7.52
g_{33}	12.7	18.6	19.7
g_{ff}	138	195	230
g_{12}	4.09	5.95	6.30
g_{1f}	-22.6	-32.2	-35.6
m_e (MeV)	0.73	0.73	0.73
m_μ (MeV)	120	160	110
m_τ (GeV)	1.4	1.5	1.5
m_d (MeV)	9.2	9.3	9.5
m_s (MeV)	600	790	530
$m_b^{\rm phys}$ (GeV)	5.4	5.5	5.3
m_u (MeV)	3.6	3.5	3.5
m_c (GeV)	0.73	0.53	0.84
V_{us}	0.19	0.22	0.22
V_{cb}	0.016	0.012	0.0048
V_{ub}	0.0030	0.0027	0.0026

Table 5.3: Results of a χ^2-fit to fermion masses and mixing angles for $m_d > m_u \simeq m_e$ at M_P. The active parameters are shown. All masses are running masses evaluated at 1 GeV unless otherwise stated.

The predictions of (5.48) still hold, except that for m_d which becomes:

$$m_d \simeq \frac{M_D(1,2)M_D(2,1)}{M_D(2,2)} \tag{5.60}$$

and results for the 7 parameter fit are shown in Table 5.3; this time there are no residual degrees of freedom. This table also shows the parameter values, masses and mixing angles corresponding to the χ^2 minima. The 15% improvement in χ^2 is entirely due to the improved 1st generation mass hierarchy; no other results are affected by the change of texture in M_D. In particular, the $m_s - m_c - V_{cb}$ problem remains, as can be seen in the second and third columns of Table 5.3 which respectively show (for $m_t^{\text{phys}} = 200$ GeV) a free fit and a fit biased against V_{cb} as in (5.54) and (5.55). It should however be said that the use of 2 extra degrees of freedom is far too high a price to pay merely to account for $m_d > m_{u,e}$; such a feature of the fermion mass hierarchy is obviously not as compelling as, for example, $m_b \ll m_t$.

5.5 Comments

The fit utilising 5 degrees of freedom (*i.e.* for $m_u \simeq m_d \simeq m_e$ at M_P) is very successful indeed for $m_t^{\text{phys}} = 100$ GeV. All data pieces are fitted to within a factor of 2 except m_s and V_{cb} which are fitted to within a factor of 3. These results are far superior to those of the last chapter. There is a slight niggle when $m_t^{\text{phys}} = 200$ GeV: m_s is a factor of 4 too high while m_c is a factor of 3 too low. While these are individually acceptable within our order of magnitude framework, they nevertheless combine to give the disappointing result:

$$m_s \simeq m_c \tag{5.61}$$

However, this is certainly not serious enough to kill off the $SMG^3 \times U(1)_f$ model.

"And the road is coming to its end
Now the damned have no time to make amends..."

David Bowie
Cygnet Committee

Chapter 6

Overview & Conclusions

Dissatisfied with the Standard Model (SM) treatment of fermion masses, where a vast hierarchy among fundamental Yukawa couplings is responsible for the observed range of masses, we suggested that partially conserved chiral symmetries (PCCSs) might provide a much more natural rationale for suppressed masses (and mixing angles). We assumed essentially that (below some very high energy scale) only the 45 Weyl fermions of the SM existed, so that all anomalies had to be cancelled amongst only these states. This enabled us to classify the possible extensions of the Standard Model Group (SMG) according to how their fermion irreducible representations (IRs) were composed of IRs of the SMG. We then argued that a natural generation of the fermion mass hierarchy pointed towards those extensions whose IRs were identical to those of the SMG. Finding that the SMG was embedded within each of these extensions as a diagonal subgroup, we further argued that the non-abelian part of any such extension should consequently be gauged. We then assumed that all abelian symmetries should also be gauged. All such "Category (8)" anomaly-free groups were seen to be closely related to the anti-grand unified group SMG^3.

We then moved on to examine in more detail how the PCCSs of the Category (8) groups suppress the fermion mass matrix elements. We volunteered different ansätze for constructing these elements, noting that we could only specify each element order of magnitude-wise. The large mass gaps between the generations suggested that we examine some particular PCCSs, before the intra-generation quark mass gaps (in particular, the $t-b$ splitting) offered two quite different paths.

The first (and crudest) path was simply to introduce a gauged and partially conserved abelian flavour symmetry and hope that the various flavour charges would provide mass splitting within the 2nd and 3rd generations, where it is most evident in Nature. With this approach we assumed that the 3rd generation mass eigenstates were approximately equal to the corresponding gauge eigenstates. We then noted immediately that, in order to generate the required splitting, the full symmetry group could *not* be:

$$SMG^3 \times U(1)_f \tag{6.1}$$

but only:

$$J \times U(1)_f \quad \text{where} \quad SMG \subseteq J \subset SMG^3 \tag{6.2}$$

Nevertheless, we found several promising anomaly-free charge sets for the candidate PCCSs mentioned above. Then we analysed all resulting models using our various ansätze after establishing a procedure for fitting the known mass and mixing data with the ansätze parameters. One symmetry group, J_1, satisfying:

$$K_1 \subseteq J_1 \subseteq K_1' \tag{6.3}$$

where:

$$\begin{aligned} K_1 &= [SU(3)] \times [SU_{13}(2) \times SU_2(2)] \times [U_1(1) \times U_{23}(1) \times U(1)_f] \\ K_1' &= [SU_1(3) \times SU_{23}(3)] \times [SU_1(2) \times SU_2(2) \times SU_3(2)] \\ &\quad \times [U_1(1) \times U_{23}(1) \times U(1)_f] \end{aligned} \tag{6.4}$$

and having PCCSs:

$$U_1(1) \times SU_2(2) \times U(1)_f \tag{6.5}$$

was found to give reasonable results for a number of $U(1)_f$ charge sets. But a second symmetry group, J_2, satisfying:

$$K_2 \subseteq J_2 \subseteq K_2' \tag{6.6}$$

where:

$$\begin{aligned} K_2 &= [SU(3)] \times [SU(2)] \times [U_1(1) \times U_2(1) \times U_3(1) \times U(1)_f] \\ K_2' &= [SU_1(3) \times SU_{23}(3)] \times [SU_1(2) \times SU_{23}(2)] \\ &\quad \times [U_1(1) \times U_2(1) \times U_3(1) \times U(1)_f] \end{aligned} \tag{6.7}$$

and having PCCSs:

$$U_1(1) \times U_2(1) \times U(1)_f \tag{6.8}$$

was found to give much poorer results. Even though the results for J_1 were not themselves discouraging, it has to be said that the existence of such a gauge group is almost aesthetically unpalatable. Breakthroughs in physics towards more fundamental theories are traditionally characterised by (among other things) beauty, and the group J_1 can only be described as very ugly.

The second path to intra-generation quark mass gaps was to suppose that the abelian sector of the SMG^3 model could generate these gaps without recourse to any abelian flavour symmetry. Splitting within the generations was indeed obtained, but it was shackled by the disasterous overall prediction:

$$m_d m_s m_b \geq m_u m_c m_t \simeq m_e m_\mu m_\tau \tag{6.9}$$

This anti-grand unified model, however, has a much better grounding outside of fermion mass issues than the groups shown above and we were therefore reluctant to dismiss it. Ironically, a gauged abelian flavour symmetry had to be introduced in order to effect its rescue. The resulting $SMG^3 \times U(1)_f$ model was analysed using the same fitting procedure as before. The fits were found to be very successful, much better than those of the J_1 model previously examined. Furthermore, we cannot level the same aesthetic complaints against the $SMG^3 \times U(1)_f$ model that we could against the $J_{1,2} \times U(1)_f$ models. Even its most unsatisfactory aspect, the abelian flavour symmetry, is to some extent unique (given our assumptions about the absence of non-SM fermions), having its fermion quantum numbers completely pinned down by the requirements of anomaly freedom.

The alert reader will, however, be wondering how to reconcile this $SMG^3 \times U(1)_f$ analysis with an earlier warning that we should not trust a model which gives excellent results with one mass matrix ansatz but poor results with another - an accusation which might be levelled at the $SMG^3 \times U(1)_f$ model with some justification. We say only that this touches on a part of our whole approach which any future work must address: the details of the symmetry breaking mechanism itself. Our ansätze should only be viewed as a parameterisation of our ignorance of such details. To give the model proper status, we must be much more specific about how we envisage the spontaneous breaking which occurs just below M_P.

It should also be said that what we have really done is drop the explanation of one hierarchy (the fermion mass hierarchy) on the doorstep of another (the gauge hierarchy). While the overall scale of the fermion masses is set by $\langle\phi\rangle_{\mathrm{ws}}$ which lies 17 orders of magnitude below M_P, the scale of the mass *hierarchy* is set by $\langle\phi\rangle / M_P$ where the symmetry breaking scale $\langle\phi\rangle$ is within only a few orders of magnitude of M_P. How do these scalars maintain such a disparity in their VEVs? This is the gauge hierarchy problem, a solution to which is beyond the scope of this thesis. However, a possible explanation was suggested in [46].

In conclusion, we make no grandiose claims to have uttered the last word on the fermion mass hierarchy. What we have done is taken the appealing idea that approximately conserved symmetry might be responsible for this hierarchy, and shown that such an approach can indeed bear fruit.

Appendix A

Proof that $\det M_U \simeq \det M_l \leq \det M_D$ in the SMG^3 Model with the Mixed Ansatz

We firstly reproduce the mass matrices of (5.19), which were formed using the mixed ansatz for the SMG^3 model assuming that:

$$M_U(2,3) \simeq 1 \tag{A.1}$$

These mass matrices are:

$$\begin{aligned}
M_U &\simeq \begin{pmatrix} \epsilon_1 e^{-\sqrt{9g_{11}}} & \beta_1\epsilon_2 e^{-\sqrt{16g_{11}+g_{22}-8g_{12}}} & \beta_1 e^{-\sqrt{16g_{11}+16g_{22}-32g_{12}}} \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+16g_{22}-8g_{12}}} & \epsilon_2 e^{-\sqrt{9g_{22}}} & 1 \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+256g_{22}-32g_{12}}} & \epsilon_2 e^{-\sqrt{225g_{22}}} & e^{-\sqrt{144g_{22}}} \end{pmatrix} \\
M_D &\simeq \begin{pmatrix} \epsilon_1 e^{-\sqrt{9g_{11}}} & \beta_1\epsilon_2 e^{-\sqrt{4g_{11}+g_{22}+4g_{12}}} & \beta_1 e^{-\sqrt{4g_{11}+16g_{22}+16g_{12}}} \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+4g_{22}+4g_{12}}} & \epsilon_2 e^{-\sqrt{9g_{22}}} & e^{-\sqrt{36g_{22}}} \\ \beta_1\epsilon_1 e^{-\sqrt{g_{11}+64g_{22}+16g_{12}}} & \epsilon_2 e^{-\sqrt{81g_{22}}} & e^{-\sqrt{144g_{22}}} \end{pmatrix} \\
M_l &\simeq \begin{pmatrix} \epsilon_1 e^{-\sqrt{9g_{11}}} & \epsilon_2 e^{-\sqrt{36g_{11}+9g_{22}-36g_{12}}} & e^{-\sqrt{36g_{11}+144g_{22}-144g_{12}}} \\ \epsilon_1 e^{-\sqrt{9g_{11}+36g_{22}-36g_{12}}} & \epsilon_2 e^{-\sqrt{9g_{22}}} & e^{-\sqrt{36g_{22}}} \\ \epsilon_1 e^{-\sqrt{9g_{11}+576g_{22}-144g_{12}}} & \epsilon_2 e^{-\sqrt{441g_{22}}} & e^{-\sqrt{144g_{22}}} \end{pmatrix}
\end{aligned} \tag{A.2}$$

Since ϵ_1 and ϵ_2 only multiply the 1st and 2nd columns repsectively of $M_{U,D,l}$, they will have no bearing on which terms dominate the determinants, so we take:

$$\epsilon_1 = \epsilon_2 = 1 \tag{A.3}$$

It is also clear that the best chance of avoiding arriving at:

$$\det M_U \simeq M_U(1,1)M_U(2,2)M_U(3,3) \tag{A.4}$$

is to take:

$$\beta_1 = 1 \tag{A.5}$$

We now look at the exponents of all terms contributing to $\det M_{U,D,l}$. We define:

$$\begin{aligned}
\det i_1 &= |\log\{M_i(1,1)M_i(2,2)M_i(3,3)\}| \\
\det i_2 &= |\log\{M_i(1,1)M_i(2,3)M_i(3,2)\}| \\
\det i_3 &= |\log\{M_i(1,2)M_i(2,1)M_i(3,3)\}| \\
\det i_4 &= |\log\{M_i(1,2)M_i(2,3)M_i(3,1)\}| \\
\det i_5 &= |\log\{M_i(1,3)M_i(2,1)M_i(3,2)\}| \\
\det i_6 &= |\log\{M_i(1,3)M_i(2,2)M_i(3,1)\}|
\end{aligned} \tag{A.6}$$

where $i = U, D, l$. We will make use of the fact that:

$$g_{12}^2 \leq g_{11}g_{22} \tag{A.7}$$

For M_U we have:

$$\begin{aligned}
\det U_1 \;\simeq\; \det U_2 &\simeq 3\sqrt{g_{11}} + 15\sqrt{g_{22}} \\
\det U_3 &\simeq \sqrt{16g_{11} + g_{22} - 8g_{12}} + \sqrt{g_{11} + 16g_{22} - 8g_{12}} + 12\sqrt{g_{22}} \\
&\geq |4\sqrt{g_{11}} - \sqrt{g_{22}}| + |\sqrt{g_{11}} - 4\sqrt{g_{22}}| + 12\sqrt{g_{22}} \\
&= \begin{cases} 5\sqrt{g_{11}} + 7\sqrt{g_{22}} & \text{if} \quad 4\sqrt{g_{22}} < \sqrt{g_{11}} \\ 3\sqrt{g_{11}} + 15\sqrt{g_{22}} & \text{if} \quad \frac{1}{4}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 4\sqrt{g_{22}} \\ -5\sqrt{g_{11}} + 17\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} < \frac{1}{4}\sqrt{g_{22}} \end{cases} \\
&= \begin{cases} \det U_1 + (2\sqrt{g_{11}} - 8\sqrt{g_{22}}) & \text{if} \quad 4\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det U_1 & \text{if} \quad \frac{1}{4}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 4\sqrt{g_{22}} \\ \det U_1 + (2\sqrt{g_{22}} - 8\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} < \frac{1}{4}\sqrt{g_{22}} \end{cases}
\end{aligned}$$

$$
\begin{aligned}
& \geq \det U_1 \\
\det U_4 & \simeq \sqrt{16g_{11} + g_{22} - 8g_{12}} + \sqrt{g_{11} + 256g_{22} - 32g_{12}} \\
& \geq |4\sqrt{g_{11}} - \sqrt{g_{22}}| + |\sqrt{g_{11}} - 16\sqrt{g_{22}}| \\
& = \begin{cases} 5\sqrt{g_{11}} - 17\sqrt{g_{22}} & \text{if} \quad 16\sqrt{g_{22}} < \sqrt{g_{11}} \\ 3\sqrt{g_{11}} + 15\sqrt{g_{22}} & \text{if} \quad \frac{1}{4}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 16\sqrt{g_{22}} \\ -5\sqrt{g_{11}} + 17\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} < \frac{1}{4}\sqrt{g_{22}} \end{cases} \\
& = \begin{cases} \det U_1 + (2\sqrt{g_{11}} - 32\sqrt{g_{22}}) & \text{if} \quad 16\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det U_1 & \text{if} \quad \frac{1}{4}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 16\sqrt{g_{22}} \\ \det U_1 + (2\sqrt{g_{22}} - 8\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} < \frac{1}{4}\sqrt{g_{22}} \end{cases} \\
& \geq \det U_1 \\
\det U_5 & \simeq \sqrt{16g_{11} + 16g_{22} - 32g_{12}} + \sqrt{g_{11} + 16g_{22} - 8g_{12}} + 15\sqrt{g_{22}} \\
& \geq |4\sqrt{g_{11}} - 4\sqrt{g_{22}}| + |\sqrt{g_{11}} - 4\sqrt{g_{22}}| + 15\sqrt{g_{22}} \\
& = \begin{cases} 5\sqrt{g_{11}} + 7\sqrt{g_{22}} & \text{if} \quad 4\sqrt{g_{22}} < \sqrt{g_{11}} \\ 3\sqrt{g_{11}} + 15\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 4\sqrt{g_{22}} \\ -5\sqrt{g_{11}} + 23\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} < \sqrt{g_{22}} \end{cases} \\
& = \begin{cases} \det U_1 + (2\sqrt{g_{11}} - 8\sqrt{g_{22}}) & \text{if} \quad 4\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det U_1 & \text{if} \quad \sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 4\sqrt{g_{22}} \\ \det U_1 + (8\sqrt{g_{22}} - 8\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} < \sqrt{g_{22}} \end{cases} \\
& \geq \det U_1 \\
\det U_6 & \simeq \sqrt{16g_{11} + 16g_{22} - 32g_{12}} + 3\sqrt{g_{22}} + \sqrt{g_{11} + 256g_{22} - 32g_{12}} \\
& \geq |4\sqrt{g_{11}} - 4\sqrt{g_{22}}| + 3\sqrt{g_{22}} + |\sqrt{g_{11}} - 16\sqrt{g_{22}}| \\
& = \begin{cases} 5\sqrt{g_{11}} - 17\sqrt{g_{22}} & \text{if} \quad 16\sqrt{g_{22}} < \sqrt{g_{11}} \\ 3\sqrt{g_{11}} + 15\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 16\sqrt{g_{22}} \\ -5\sqrt{g_{11}} + 23\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} < \sqrt{g_{22}} \end{cases} \\
& = \begin{cases} \det U_1 + (2\sqrt{g_{11}} - 32\sqrt{g_{22}}) & \text{if} \quad 16\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det U_1 & \text{if} \quad \sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 16\sqrt{g_{22}} \\ \det U_1 + (8\sqrt{g_{22}} - 8\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} < \sqrt{g_{22}} \end{cases} \\
& \geq \det U_1
\end{aligned}
\tag{A.8}
$$

So we have shown that:

$$\det U_j \geq \det U_1 \qquad (j = 2, \ldots, 6) \tag{A.9}$$

and it follows, then, that:

$$\det M_U \simeq M_U(1,1)M_U(2,2)M_U(3,3) \tag{A.10}$$

Similarly for M_l:

$$\begin{aligned}
\det l_1 &\simeq 3\sqrt{g_{11}} + 15\sqrt{g_{22}} \\
\det l_2 &\simeq 3\sqrt{g_{11}} + 27\sqrt{g_{22}} \\
&> \det l_1 \\
\det l_3 &\simeq 3\sqrt{4g_{11} + g_{22} - 4g_{12}} + 3\sqrt{g_{11} + 4g_{22} - 4g_{12}} + 12\sqrt{g_{22}} \\
&\geq 3|2\sqrt{g_{11}} - \sqrt{g_{22}}| + 3|\sqrt{g_{11}} - 2\sqrt{g_{22}}| + 12\sqrt{g_{22}} \\
&= \begin{cases} 9\sqrt{g_{11}} + 3\sqrt{g_{22}} & \text{if} \quad 2\sqrt{g_{22}} < \sqrt{g_{11}} \\ 3\sqrt{g_{11}} + 15\sqrt{g_{22}} & \text{if} \quad \frac{1}{2}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 2\sqrt{g_{22}} \\ -9\sqrt{g_{11}} + 21\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} < \frac{1}{2}\sqrt{g_{22}} \end{cases} \\
&= \begin{cases} \det l_1 + (6\sqrt{g_{11}} - 12\sqrt{g_{22}}) & \text{if} \quad 2\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det l_1 & \text{if} \quad \frac{1}{2}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 2\sqrt{g_{22}} \\ \det l_1 + (6\sqrt{g_{22}} - 12\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} < \frac{1}{2}\sqrt{g_{22}} \end{cases} \\
&\geq \det l_1 \\
\det l_4 &\simeq 3\sqrt{4g_{11} + g_{22} - 4g_{12}} + 6\sqrt{g_{22}} + 3\sqrt{g_{11} + 64g_{22} - 16g_{12}} \\
&\geq 3|2\sqrt{g_{11}} - \sqrt{g_{22}}| + 6\sqrt{g_{22}} + 3|\sqrt{g_{11}} - 8\sqrt{g_{22}}| \\
&= \begin{cases} 9\sqrt{g_{11}} - 21\sqrt{g_{22}} & \text{if} \quad 8\sqrt{g_{22}} < \sqrt{g_{11}} \\ 3\sqrt{g_{11}} + 27\sqrt{g_{22}} & \text{if} \quad \frac{1}{2}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 8\sqrt{g_{22}} \\ -9\sqrt{g_{11}} + 33\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} < \frac{1}{2}\sqrt{g_{22}} \end{cases} \\
&= \begin{cases} \det l_1 + (6\sqrt{g_{11}} - 36\sqrt{g_{22}}) & \text{if} \quad 8\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det l_1 + 12\sqrt{g_{22}} & \text{if} \quad \frac{1}{2}\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 8\sqrt{g_{22}} \\ \det l_1 + (18\sqrt{g_{22}} - 12\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} < \frac{1}{2}\sqrt{g_{22}} \end{cases} \\
&> \det l_1 \\
\det l_5 &\simeq 9\sqrt{g_{11} + 4g_{22} - 4g_{12}} + 21\sqrt{g_{22}} \\
&\geq 9|\sqrt{g_{11}} - 2\sqrt{g_{22}}| + 21\sqrt{g_{22}} \\
&= \begin{cases} 9\sqrt{g_{11}} + 3\sqrt{g_{22}} & \text{if} \quad 2\sqrt{g_{22}} < \sqrt{g_{11}} \\ -9\sqrt{g_{11}} + 39\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} \leq 2\sqrt{g_{22}} \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} \det l_1 + (6\sqrt{g_{11}} - 12\sqrt{g_{22}}) & \text{if} \quad 2\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det l_1 + (24\sqrt{g_{22}} - 12\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} \leq 2\sqrt{g_{22}} \end{cases} \\
&\geq \det l_1
\end{aligned}
$$

$$
\begin{aligned}
\det l_6 &\simeq 6\sqrt{g_{11} + 4g_{22} - 4g_{12}} + 3\sqrt{g_{22}} + 3\sqrt{g_{11} + 64g_{22} - 16g_{12}} \\
&\geq 6|\sqrt{g_{11}} - 2\sqrt{g_{22}}| + 3\sqrt{g_{22}} + 3|\sqrt{g_{11}} - 8\sqrt{g_{22}}| \\
&= \begin{cases} 9\sqrt{g_{11}} - 33\sqrt{g_{22}} & \text{if} \quad 8\sqrt{g_{22}} < \sqrt{g_{11}} \\ 3\sqrt{g_{11}} + 15\sqrt{g_{22}} & \text{if} \quad 2\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 8\sqrt{g_{22}} \\ -9\sqrt{g_{11}} + 39\sqrt{g_{22}} & \text{if} \quad \sqrt{g_{11}} < 2\sqrt{g_{22}} \end{cases} \\
&= \begin{cases} \det l_1 + (6\sqrt{g_{11}} - 48\sqrt{g_{22}}) & \text{if} \quad 8\sqrt{g_{22}} < \sqrt{g_{11}} \\ \det l_1 & \text{if} \quad 2\sqrt{g_{22}} \leq \sqrt{g_{11}} \leq 8\sqrt{g_{22}} \\ \det l_1 + (24\sqrt{g_{22}} - 12\sqrt{g_{11}}) & \text{if} \quad \sqrt{g_{11}} < 2\sqrt{g_{22}} \end{cases} \\
&\geq \det l_1 \qquad \text{(A.11)}
\end{aligned}
$$

So we have shown that:

$$
\det l_j \geq \det l_1 \qquad (j = 2, \ldots, 6) \qquad \text{(A.12)}
$$

and it follows that:

$$
\det M_l \simeq M_l(1,1)M_l(2,2)M_l(3,3) \qquad \text{(A.13)}
$$

But for M_D we have:

$$
\det D_1 \simeq 3\sqrt{g_{11}} + 15\sqrt{g_{22}} \qquad \text{(A.14)}
$$

and find that it is possible (for example) to get:

$$
\det D_6 < \det D_1 \qquad \text{(A.15)}
$$

so that:

$$
\begin{aligned}
\det M_D &\simeq M_D(1,3)M_D(2,2)M_D(3,1) \\
&> M_D(1,1)M_D(1,1)M_D(3,3) \qquad \text{(A.16)}
\end{aligned}
$$

For example, taking:

$$
\begin{aligned}
g_{11} &= 4g_{22} \\
g_{12} &= -\frac{1}{2}\sqrt{g_{11}g_{22}} = -g_{22} \qquad \text{(A.17)}
\end{aligned}
$$

gives, using (A.2) with $\epsilon_1 = \epsilon_2 = \beta_1 = 1$:

$$M_D \simeq \begin{pmatrix} e^{-6\sqrt{g_{22}}} & e^{-\sqrt{13g_{22}}} & e^{-4\sqrt{g_{22}}} \\ e^{-2\sqrt{g_{22}}} & e^{-3\sqrt{g_{22}}} & e^{-6\sqrt{g_{22}}} \\ e^{-\sqrt{52g_{22}}} & e^{-9\sqrt{g_{22}}} & e^{-12\sqrt{g_{22}}} \end{pmatrix} \tag{A.18}$$

whence:

$$\begin{aligned} \det M_D &\simeq M_D(1,3)M_D(2,2)M_D(3,1) \\ &> M_D(1,1)M_D(2,2)M_D(3,3) \end{aligned} \tag{A.19}$$

Overall, then, since the corresponding diagonal entries in each mass matrix are approximately equal, we have:

$$\det M_U \simeq \det M_l \leq \det M_D \tag{A.20}$$

as required.

Appendix B

Proof that $M_U(2,3)M_U(3,2) \simeq M_U(2,2)M_U(3,3) \simeq M_D(2,2)M_D(3,3)$ in the SMG^3 Model with the Product Ansatz

We are concerned with the product ansatz approach to the SMG^3 model. In order to obtain:

$$M_U(2,3) \simeq 1 \tag{B.1}$$

we must assume that the space of abelian PCCSs is:

$$U_1(1) \times U_2(1) \times U_3(1)/U(1)' \tag{B.2}$$

where $U(1)'$ is generated by:

$$4Y_2 - Y_3 \tag{B.3}$$

In Chapter 5, we have considered the case where this space is generated by Y_1 and $Y_2 + 4Y_3$; now we wish to consider the general case. We take the space to be generated by:

$$\begin{aligned} X_1 &\equiv Y_1 + \alpha(Y_2 + 4Y_3) \\ X_2 &\equiv \beta Y_1 + \gamma(Y_2 + 4Y_3) \end{aligned} \tag{B.4}$$

Weyl	Charge Corresponding to Generator:				
State	Y_1	Y_2	Y_3	X_1	X_2
d_L	1	0	0	1	β
u_R	4	0	0	4	4β
d_R	-2	0	0	-2	-2β
s_L	0	1	0	α	γ
c_R	0	4	0	4α	4γ
s_R	0	-2	0	-2α	-2γ
b_L	0	0	1	4α	4γ
t_R	0	0	4	16α	16γ
b_R	0	0	-2	-8α	-8γ

Table B.1: Charges of the quarks corresponding to particular generators of $U_1(1) \times U_2(1) \times U_3(1)$.

for some constants α, β and γ with $\gamma \neq \alpha\beta$. That is, we choose two linearly independent generators orthogonal[1] to $4Y_2 - Y_3$ but not necessarily orthogonal to each other. Then Table B.1 shows the quark charges corresponding to the generators X_1 and X_2. Using the product ansatz (see (3.43)) with $\lambda_{1,2}$ corresponding to $X_{1,2}$ respectively, we get:

$$
\begin{aligned}
M_U &\simeq \begin{pmatrix} \times & \times & \times \\ \times & \lambda_1^{|3\alpha|}\lambda_2^{|3\gamma|} & 1 \\ \times & \lambda_1^{|15\alpha|}\lambda_2^{|15\gamma|} & \lambda_1^{|12\alpha|}\lambda_2^{|12\gamma|} \end{pmatrix} \\
M_D &\simeq \begin{pmatrix} \times & \times & \times \\ \times & \lambda_1^{|3\alpha|}\lambda_2^{|3\gamma|} & \times \\ \times & \times & \lambda_1^{|12\alpha|}\lambda_2^{|12\gamma|} \end{pmatrix}
\end{aligned}
\tag{B.5}
$$

so that:

$$M_U(2,3)M_U(3,2) \simeq M_U(2,2)M_U(3,3) \simeq M_D(2,2)M_D(3,3) \tag{B.6}$$

for all α, β and γ, as required.

[1]Assuming for convenience that Y_1, Y_2 and Y_3 form a Cartesian basis for the space of abelian generators.

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"He who binds to himself a joy
Doth the winged life destroy
But he who kisses the joy as it flies
Lives in eternity's sunrise"

William Blake
Eternity

Thank-you Orla, for everything.