PLANE VIBRATIONS AND STABILITY OF ELASTIC PLATES

by

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Preface.

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Abstract.

We consider plane harmonic incremental displacements superimposed on a finite deformation, corresponding to pure homogeneous strain, of a plate with rectangular cross–section, for which motions normal to the plane of this cross–section are neglected. We assume that the plate is isotropic and hyperelastic, with the underlying finite deformation satisfying the strong–ellipticity condition, and consider both incompressible and compressible materials. Having set up the governing equations of finite and incremental elasticity we consider the incompressible case and study the problem with mixed traction–displacement boundary conditions defined on two opposing faces and pure traction boundary conditions defined on the other sides. It is found that, in general, nine distinct cases may occur and frequency equations are derived for each case in turn, with two separate modes possible in most cases. Necessary conditions for the existence of nontrivial solutions are obtained for both the static and the dynamic problems and, in order to study the frequency equations numerically, we consider the restriction to equibiaxial underlying deformations as well as looking at the full problem for two particular strain–energy functions. The corresponding problem for compressible materials is then considered and frequency equations, similar to those in the incompressible case, are derived. Unfortunately, due to the more complicated algebra, fewer explicit results can be obtained in this case, as compared to the incompressible case, but some necessary existence conditions are derived. Numerical results are obtained in the static case for three particular strain–energy functions and one of these stain–energy functions is then used to illustrate the full dynamic problem. Finally we consider the related problem of an infinite layer, of finite thickness, with traction boundary conditions applied on the surfaces, and show how the results for the finite plate can be applied to this problem.

Chapter 1 – Introduction.

This thesis is concerned mainly with the derivation and study of frequency and bifurcation equations for incrementally linear, plane harmonic displacements which are superimposed on a nonlinearly deformed configuration of an isotropic hyperelastic material. We start off by considering vibrational modes of 'thick' plates, with rectangular cross–section, which are subjected to mixed traction–displacement boundary conditions on two opposing faces, such that no shear forces can be supported and no normal displacement is possible at these bounding surfaces. On the remaining two surfaces, pure traction boundary conditions are prescribed and displacements normal to the plane of this cross–section are neglected. Both incompressible and compressible materials are studied, but we restrict our attention to consider only underlying finite deformations that satisfy the strong–ellipticity condition. It is found that, when the dynamical aspects are considered, the controlling equations may also be hyperbolic or parabolic and in general nine distinct cases are possible. In the final chapter we then show how the results obtained for this bounded plate can be applied to the problem of an infinite elastic layer of finite thickness.

The static equivalent of the finite block problem studied here, with mixed boundary conditions on two surfaces, has been studied by many people, especially when compared to the similar problem with pure displacement boundary conditions applied at these surfaces, for both incompressible and compressible elastic materials. The reason for this interest is that the mixed boundary conditions admit solutions that are harmonic between the surfaces where they are defined, whereas the pure displacement boundary conditions do not. Early results in this field were obtained by Biot (1963, 1965), Levinson(1968), Nowinski (1969) and Wu and Widera (1969) for incompressible materials and Burgess and Levinson (1972) for compressible materials. In most of these papers the authors chose particular forms for the strain–energy functions at the outset and considered solutions with a hyperbolic dependence on the transverse variable of the plate. In the incompressible case, Sawyers and Rivlin (1974), together with Sawyers (1977), for a plate with traction–free lateral surfaces, obtained general bifurcation equations for arbitrary strain–energy functions in the elliptic domain. The authors also obtained necessary conditions for the existence of bifurcation points and deduced that bifurcation could only occur if the block was compressed along its length, while Sawyers and Rivlin (1982) studied the stability of these bifurcation modes for the neo–Hookean material. In the tensile case Hill and Hutchinson (1975) also considered both hyperbolic and parabolic, as well as the elliptic, regimes for a class of two–parameter strain–energy functions, while Young (1976) dealt with the corresponding compressive case. Again for incompressible materials, Ogden (1984) gave a comprehensive discussion of the possible bifurcation modes for deformations subject to the strong–ellipticity condition as well as deriving necessary conditions for their existence. Ogden (1984) also considered the corresponding compressible case and again gave a comprehensive list of the bifurcation modes possible, in or on the boundary of, the strongly–elliptic domain; however, existence criteria were not discussed for these materials. For compressible materials subject to compression Davies (1989), by considering stationary points of the energy, was able to show that for any block, with its lateral surfaces traction–free, there exists a value of the compression ratio such that bifurcation can occur.

So far all of the above papers have been dealing with the static case, but on turning our attention to the corresponding dynamic case we find that very little has been published on the subject, although Mindlin (1960) was able to show that, for the all–round traction–free problem, vibrational modes can only occur for certain discrete values of the length–to–width ratio of the plate. In contrast there has been a lot more work done on the related problem of waves propagating along an infinite layer, which can be thought of as a finite plate with one length allowed to increase to infinity. This problem was first studied by Rayleigh and Lamb in the late nineteenth century whereby they derived frequency equations for plane harmonic waves propagating along the layer. However, it was not until Mindlin (1960) that the complicated nature of the solutions, for such apparently simple equations, was finally understood for real and complex values of the wavenumber. Texts such as Graff (1975) and Achenbach (1984) further discussed the behaviour of the solutions to these classical equations, called the Rayleigh–Lamb equations, which, for real wavenumbers, correspond to the propagation of so called Lamb waves. Generalizations of the Rayleigh–Lamb equations were obtained by Biot (1965) for an arbitrary isotropic incompressible material which had been subjected to an initial compression along the length of the layer. Green (1982) obtained frequency equations for bending modes of a transversely isotropic layer, with the axis of transverse isotropy lying along the length of the layer, for waves propagating at any angle relative to this axis of isotropy. Willson (1977), for a compressible restricted Hadamard material, obtained frequency equations for waves propagating in a layer which had been subjected to a pure homogeneous strain, having studied the incompressible case in an earlier paper, Willson (1973).

If the thickness of the layer considered above is taken to be large in comparison with the wavelengths of the propagating waves, then it may be viewed as representing a half–space on which Rayleigh surface waves may occur. The classical secular equations for Rayleigh waves can be found in Ewing, Jardetzky and Press (1957), while Dowaikh and Ogden (1990) and (1991), dealing with incompressible and compressible materials respectively, obtained the secular equations of Rayleigh waves propagating on a homogeneously deformed isotropic half–space, for an arbitrary strain–energy function.

We now discuss each of the chapters of this thesis in some detail. In Chapter 2, following the approach of standard texts such as Truesdell and Noll (1965) and Ogden (1984), we introduce the notation that will be used throughout this thesis as well as setting up the equations of motion and constitutive relations used in finite elasticity, for both incompressible and compressible materials. We then introduce the corresponding equations for incremental deformations superimposed on a nonlinearly deformed state, as well as discussing the strong–ellipticity condition and, in the two–dimensional case, giving necessary and sufficient conditions for it to hold with respect to plane strain and plane displacements.

In Chapter 3 we study plane harmonic vibrations of an incompressible material, which, after being subjected to a pure homogeneous strain, is bounded by the lines $x_1 = \pm l_1$ and $x_2 = \pm l_2$, with mixed boundary conditions prescribed on the surfaces $x_1 = \pm l_1$ and pure traction boundary conditions on $x_2 = \pm l_2$. We then restrict our attention to finite deformations which satisfy the strong–ellipticity condition. It is found that on taking a time–dependent displacement, $\mathbf{u}$, of the form

$$\mathbf{u} = \mathbf{A}\exp(spx_2 + ipx_1 - i\omega t) , \tag{1.1}$$

the incremental equations of motion reduce to

$$c's^4 - 2b's^2 + a' = 0 , \tag{1.2}$$

where a', b' and c' are independent of s, so that the nature of the solution (1.1) depends on the relative values and signs of these terms. It is shown that, even while the strong–ellipticity condition holds, nine distinct cases can arise. The boundary conditions on the sides $x_1 = \pm l_1$ force p to be of the form $p = n\pi/2l_1$, for some integer n, while the boundary conditions on the other sides $x_2 = \pm l_2$ can be used to determine frequency equations for possible bifurcation modes. It is found that two separate modes can exist, one of which has displacements that are symmetric about the x_1–axis and the other has antisymmetric displacements about this axis. Frequency equations are then derived for each of the nine cases in turn. On taking the frequency ω to be zero, we obtain the static forms described in Sawyers and Rivlin (1974) and Ogden (1984) as well as generalizing slightly the existence criteria contained therein. The full dynamic problem is then reintroduced and necesssary existence conditions are found for those cases that do not involve trigonometric terms, together with some asymptotic results for 'thick' and 'thin' plates. For underlying deformations that are equibiaxial we find that, since the material is incompressible, the frequency equations simplify to a form which permits a numerical investigation. In order to obtain a quantitative idea of the nature of the solutions to the general frequency equations, we choose two particular strain–energy functions of the form

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{m}(\lambda_1^m + \lambda_2^m + \lambda_3^m - 3) \ , \qquad m \in \{1, 2\} \ , \tag{1.3}$$

which are shown to cover all nine possible cases and then we solve the resulting frequency equations numerically.

Chapter 4 deals with the same problem as Chapter 3 except that we now consider a compressible rather than an incompressible material. Equations (1.1) and (1.2) can again be shown to hold, with the terms a', b' and c' suitably modified, and, as for the incompressible material, nine distinct cases can arise for underlying deformations which satisfy the strong–ellipticity condition, with both symmetric and antisymmetric modes again possible. Frequency equations are then obtained for each of these nine cases but, when compared to the incompressible case, because of the more complicated algebra few explicit results can be deduced from them, although some necessary conditions for the existence of solutions and some limiting forms are given. For compressible materials, in the static limit, only three distinct cases can arise within the strong–ellipticity domain and we introduce three particular strain–energy functions to illustrate each of these possible cases. We then choose one of these strain–energy functions, namely a compressible neo–Hookean material of the form

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\ln(\lambda_1\lambda_2\lambda_3) - 3\right) \ , \tag{1.4}$$

as a basis for a numerical discussion of the frequency equations for the full dynamic problem.

Finally, in Chapter 5 we look at how the results obtained in the previous two chapters can be applied to the problem of plane harmonic waves propagating along an elastic layer. We briefly introduce the equations of linear elasticity and use them to derive the classical Rayleigh–Lamb equations for Lamb waves propagating along a waveguide. The incompressible limit of these equations is taken and dispersion curves are plotted for various values of the Poisson ratio. We then discuss how the results of Chapters 3 and 4 may be applied to this problem and use them to recover previously known results for both Lamb and Rayleigh waves.

Chapter 2 - The Basic Equations.

2.1 Notation and introduction to finite elasticity.

In this section we introduce the basic equations of elasticity, following the approach of standard works such as Truesdell and Noll (1965), Wang and Truesdell (1973) and Ogden (1984), as well as introducing the notation that will be used throughout this thesis.

2.1.1 Kinematics.

A homogeneous elastic body when unstressed is chosen to occupy the region B_0 in a three–dimensional Euclidean space. The points in this reference configuration are denoted by the position vector $\mathbf{X}$, relative to an arbitrarily chosen origin. The body is deformed isothermally into a new configuration, B_t, say in which the material point $\mathbf{X}$ moves to position $\mathbf{x}$ according to

$$\mathbf{x}(\mathbf{X}, t) = \chi_t(\mathbf{X}) , \qquad \mathbf{X} \in B_0 , \tag{2.1}$$

where the one–to–one mapping $\chi_t : B_0 \to B_t$ is the **deformation** of B_t relative to B_0 and it is endowed with sufficient regularity as the subsequent analysis requires. Cartesian bases $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$ ($i \in \{1, 2, 3\}$) are chosen to represent the reference and current configurations respectively.

The **deformation gradient tensor**, denoted by $\mathbf{A}$, is defined by

$$\mathbf{A} = \mathrm{Grad}\boldsymbol{\chi}_t , \tag{2.2}$$

where Grad is the gradient operator in B_0. In general, $\mathbf{A}$ depends on $\mathbf{X}$; however in the special case where $\mathbf{A}$ is independent of $\mathbf{X}$ the deformation is said to be **homogeneous.** The assumed regularity conditions ensure that $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$, which requires that $\det\mathbf{A} \neq 0$.

For a fixed time t, if we consider the differentials $\mathbf{dX}$ and $\mathbf{dx}$ in B_0 and B_t respectively, then they are related by

$$\mathbf{dx} = \mathbf{A}\mathbf{dX} , \tag{2.3}$$

which describes how line elements are transformed. Area elements are transformed according to **Nanson's formula** as

$$\mathbf{n}da = (\det\mathbf{A})\mathbf{A}^{-T}\mathbf{N}dA \ , \tag{2.4}$$

where $\mathbf{N}dA$ and $\mathbf{n}da$ represent the surface elements in B_0 and B_t with unit outward normals $\mathbf{n}$ and $\mathbf{N}$ respectively and $\mathbf{A}^{-T} = (\mathbf{A}^{-1})^T$ denotes the transpose of $\mathbf{A}^{-1}$. Similarly, volume elements are related by

$$dv = (\det\mathbf{A})dV \ , \tag{2.5}$$

for dv and dV in B_t and B_0 respectively. From (2.5), it can be seen that since finite volumes which, by definition are taken to be positive, cannot be annihilated

$$J = \det\mathbf{A} > 0 \ , \tag{2.6}$$

for all physically reasonable deformations.

If we assume $J = \det\mathbf{A} = 1$ as a constraint on the allowable deformations, then from (2.5) we have $dv = dV$, which means that the volume of any region in B_0 is unchanged by the deformation. Materials which satisfy this constraint are called **incompressible.**

By (2.6), the **Polar Decompostion Theorem** can be applied to $\mathbf{A}$ and gives

$$\mathbf{A} = \mathbf{RU} = \mathbf{VR} \ , \tag{2.7}$$

where $\mathbf{R}$ is proper orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are the symmetric, positive definite **right** and **left stretch tensors** respectively, defined by

$$\mathbf{U}^2 = \mathbf{A}^T\mathbf{A}, \qquad \mathbf{V}^2 = \mathbf{A}\mathbf{A}^T \ .$$

Since $\mathbf{U}$ is symmetric and positive definite, there exists positive numbers λ_i ($i \in \{1, 2, 3\}$) and vectors $\mathbf{u}^{(i)}$ ($i \in \{1, 2, 3\}$) such that

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} \ , \tag{2.8}$$

where λ_i are the **principal stretches** and $\mathbf{u}^{(i)}$ are the **Lagrangian principal axes** of the deformation.

Similarly $\mathbf{V}$ can be written as

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)} \ , \tag{2.9}$$

where from (2.7) the **Eulerian principal axes,** $\mathbf{v}^{(i)}$ ($i \in \{1,\ 2,\ 3\}$), are given by

$$\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)} \ . \tag{2.10}$$

2.1.2 <u>Analysis of stress and constitutive relations.</u>

The traction, $\mathbf{t}$, per unit area of a surface in the current configuration, B_t , can be written as

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \ , \tag{2.11}$$

where $\mathbf{n}$ is the unit outward normal to the surface in B_t and $\boldsymbol{\sigma}$ is a second order tensor called the **Cauchy stress tensor.** Likewise the traction per unit area of the reference configuration can be expressed as

$$\mathbf{t} = \mathbf{S}^T\mathbf{N} \ , \tag{2.12}$$

where $\mathbf{N}$ is the unit outward normal of the surface in B_0 and $\mathbf{S}$ is the **nominal stress tensor.** From Nanson's formula (2.4) we have

$$\mathbf{S} = J\mathbf{A}^{-1}\boldsymbol{\sigma} \ . \tag{2.13}$$

The balance of angular momentum forces $\boldsymbol{\sigma}$, and thus by (2.13) the tensor $\mathbf{AS}$, to be symmetric, so that in general $\mathbf{S} \neq \mathbf{S}^T$.

If ρ is the density per unit volume in the current configuration and ρ_0 is the density per unit volume in the reference configuration then by considering mass conservation and (2.5), they are related by

$$\rho = J^{-1}\rho_0 \ . \tag{2.14}$$

Note that for incompressible materials $\rho = \rho_0$ throughout the body, for all allowable deformations.

The (Eulerian) equation of motion is given by

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{x}_{,tt} \ , \tag{2.15}$$

where div is the divergence operator in the current configuration, $\mathbf{b}$ represents the body force per unit mass and $(\ )_{,t}$ represents differentiation with respect to time. This can be written in component form, with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho x_{i,tt} \ . \tag{2.16}$$

(Note that, unless otherwise stated, the summation convention is assumed.)

The corresponding Lagrangian formulation of the equation of motion is

$$\operatorname{Div} \mathbf{S} + \rho_0 \mathbf{b} = \rho_0 \mathbf{x}_{,tt} \ , \tag{2.17}$$

where Div is the divergence operator in the reference configuration.

For elastic materials deformed isothermally, the stress depends only on the deformation gradient, namely

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{A}) \ . \tag{2.18}$$

We require that this constitutive relation be objective under rigid body motions, that is, equation (2.18) is invariant under subsequent superimposed rotations. This can be written as

$$\boldsymbol{\sigma}(\mathbf{QA}) = \mathbf{Q}\boldsymbol{\sigma}(\mathbf{A})\mathbf{Q}^T \ , \tag{2.19}$$

where $\mathbf{Q}$ represents a general rotation.

If we now assume that the material is **isotropic**, that is , the material has no preferred directions, then we must have

$$\boldsymbol{\sigma}(\mathbf{A}) = \boldsymbol{\sigma}(\mathbf{AQ}) \ , \tag{2.20}$$

for all rotations $\mathbf{Q}$, which implies that $\boldsymbol{\sigma}$ can be defined solely in terms of the principal stretches (alternatively, the principal invariants) of $\mathbf{V}$. On taking $\mathbf{Q} = \mathbf{R}^T$, where $\mathbf{R}$ is obtained from the polar decomposition (2.7), then

$$\boldsymbol{\sigma}(\mathbf{A}) = \boldsymbol{\sigma}(\mathbf{AR}^T) = \boldsymbol{\sigma}(\mathbf{VRR}^T) = \boldsymbol{\sigma}(\mathbf{V}) \ ,$$

which means that $\boldsymbol{\sigma}$ and $\mathbf{V}$ must be coaxial, since both are symmetric. This means that $\boldsymbol{\sigma}$ can be written in the form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \, \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)} \ , \tag{2.21}$$

where $\sigma_i = \sigma_i(\lambda_1, \ \lambda_2, \ \lambda_3)$ $(i \in \{1, \ 2, \ 3\})$ are the principal values of the Cauchy stress.

The nominal stress tensor can be similarly expressed, in the form

$$\mathbf{S} = \sum_{i=1}^{3} t_i \, \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)} \ , \tag{2.22}$$

where $t_i = t_i(\lambda_1, \ \lambda_2, \ \lambda_3)$ $(i \in \{1, \ 2, \ 3\})$ are the principal Biot stresses, corresponding to the Biot stress tensor $\hat{\mathbf{T}}$=$\mathbf{SR}$ which, for an isotropic material, is symmetric.

An elastic material is said to be **hyperelastic** if there exists a scalar function $W(\mathbf{A})$ such that

$$\frac{d}{dt} W(\mathbf{A}) = J \mathrm{tr} \left(\boldsymbol{\sigma} \, \frac{d\mathbf{A}}{dt} \right) \ , \tag{2.23}$$

such a function is called the **strain–energy** (or **stored–energy) function** per unit reference volume and corresponds to the internal potential energy of the body.

For an unconstrained hyperelastic material the Cauchy stress tensor can be written in terms of the strain–energy function as

$$J\boldsymbol{\sigma} = \mathbf{A} \frac{\partial W}{\partial \mathbf{A}} \ , \tag{2.24}$$

where, in components, $(\partial W / \partial \mathbf{A})_{ij} \doteq \partial W / \partial \mathbf{A}_{ji}$. For the nominal stress, we get the simpler relation

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} \ , \tag{2.25}$$

so that, $\mathbf{S}$ and $\mathbf{A}$ can be viewed as conjugate variables.

If $W(\mathbf{A})$ is taken to be objective and isotropic, so that

$$W(\mathbf{A}) = W(\mathbf{QA}) = W(\mathbf{AQ}) \ ,$$

for all rotations $\mathbf{Q}$, then it can be shown that W depends solely on the principal stretches, that is

$$W = W(\lambda_1,\ \lambda_2,\ \lambda_3)\ , \tag{2.26}$$

where the order of the λ_1, λ_2 and λ_3 terms is immaterial. This means that for a homogeneous, objective, isotropic hyperelastic material equations (2.24) and (2.25), when evaluated along the principal axes, can be written as

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}\ , \qquad i \in \{1,\ 2,\ 3\}\ , \tag{2.27}$$

where no summation is implied, and

$$t_i = \frac{\partial W}{\partial \lambda_i}\ , \qquad i \in \{1,\ 2,\ 3\}\ , \tag{2.28}$$

respectively.

In the corresponding case for an incompressible material the constraint

$$\det\mathbf{A} = \lambda_1 \lambda_2 \lambda_3 \equiv 1\ ,$$

applies, which implies that the principal stretches are no longer independent. To circumvent this problem we introduce a Lagrange multiplier, p, with which the constrained form, for an incompressible material, of (2.24) is

$$\sigma = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}} - p\mathbf{I}\ . \tag{2.29}$$

It can be seen from (2.29) that the Lagrange multiplier, in effect, behaves as a hydrostatic stress acting in B_t. The corresponding form of (2.25) for the nominal stress in an incompressible material is given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} - p\mathbf{A}^{-1}\ . \tag{2.30}$$

For an isotropic, incompressible hyperelastic material (2.29) and (2.30) have component forms

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p\ , \qquad i \in \{1,\ 2,\ 3\}\ , \tag{2.31}$$

where again no summation is implied, and

$$t_i = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1}\ , \qquad i \in \{1,\ 2,\ 3\}\ , \tag{2.32}$$

when evaluated on the principal axes.

2.2 The equations of incremental elasticity.

2.2.1 Incremental constitutive relations.

An isotropic, homogeneous body is finitely deformed quasi–statically into the configuration B relative to the unstressed configuration B_0, so that the results of Section 2.1 apply. A small time–dependent displacement $\delta\mathbf{x}(\mathbf{X}, t)$ is now superimposed on this finite deformation, where small implies that terms of order $(\delta\mathbf{x})^2$ can be ignored, the change in the deformation gradient corresponding to this displacement is

$$\delta\mathbf{A} = \delta(\mathrm{Grad}\mathbf{x}) = \mathrm{Grad}(\mathbf{x} + \delta\mathbf{x}) - \mathrm{Grad}\mathbf{x} = \mathrm{Grad}(\delta\mathbf{x}) \ ,$$

which is an exact equation. The corresponding increment in det $\mathbf{A}$, however is not exact, and is given by

$$\delta J = J\mathrm{tr}\left((\delta\mathbf{A})\mathbf{A}^{-1}\right) \ , \tag{2.33}$$

to the first order, where tr denotes the trace of the tensor.

For an unconstrained material, the increment in the nominal stress to the first order is

$$\delta\mathbf{S} = \boldsymbol{\mathcal{A}}\,\delta\mathbf{A} \ , \tag{2.34}$$

where $\boldsymbol{\mathcal{A}} = \partial\mathbf{S}/\partial\mathbf{A}$ is the fourth–order tensor of instantaneous elastic moduli associated with the conjugate variables $(\mathbf{S}, \mathbf{A})$ and referred to B_0 . If we now let a superposed dot represent the increment in a quantity, so that $\dot{\mathbf{A}} = \delta\mathbf{A}$ for example, then (2.34) can be written in component form

$$\dot{\mathrm{S}}_{ij} = \mathcal{A}_{ijkl}\dot{x}_{l,k} \ . \tag{2.35}$$

For incompressible materials, since $J \equiv 1$ then (2.33) becomes

$$\mathrm{tr}\left((\dot{\mathbf{A}})\mathbf{A}^{-1}\right) = 0 \ , \tag{2.36}$$

whereas from (2.30) the corresponding form of (2.34) is

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{A}} - \dot{p}\mathbf{A}^{-1} + p\mathbf{A}^{-1}(\dot{\mathbf{A}})\mathbf{A}^{-1} \ . \tag{2.37}$$

If the reference configuration is updated to the quasi–statically deformed configuration, then relative to this new reference state we have

$$\dot{\mathbf{A}}_0 = \mathrm{grad}(\delta\mathbf{x}) = \dot{\mathbf{A}}\mathbf{A}^{-1} ,$$

together with

$$\dot{\mathbf{S}}_0 = J^{-1}\mathbf{A}\dot{\mathbf{S}} ,$$

and corresponding to (2.34), for unconstrained materials, we write

$$\dot{\mathbf{S}}_0 = \boldsymbol{\mathcal{A}}_0\,\dot{\mathbf{A}}_0 , \tag{2.38}$$

where $\boldsymbol{\mathcal{A}}_0$ is the fourth order tensor of elastic moduli referred to B, and the subscript zero represents a quantity evaluated relative to B. In component form, by comparing (2.34) and (2.38), we can see that $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}_0$ are related by

$$\mathcal{A}_{0ijkl} = J^{-1}\mathrm{A}_{is}\,\mathrm{A}_{kt}\,\mathcal{A}_{sjtl} .$$

For an incompressible material, when the reference configuration is updated to the deformed state, equations (2.36) and (2.37) simplify to

$$\mathrm{div}\,\dot{\mathbf{x}} = 0 , \tag{2.39}$$

and

$$\dot{\mathbf{S}}_0 = \boldsymbol{\mathcal{A}}_0\,\dot{\mathbf{A}}_0 - \dot{p}\mathbf{I} + p\dot{\mathbf{A}}_0 , \tag{2.40}$$

respectively.

The incremental constitutive relations (2.38) and (2.40) have component forms

$$\dot{\mathrm{S}}_{0ij} = \mathcal{A}_{0ijkl}\dot{x}_{l,k} , \tag{2.41}$$

and

$$\dot{\mathrm{S}}_{0ij} = \mathcal{A}_{0ijkl}\dot{x}_{l,k} + p\dot{x}_{i,j} - \dot{p}\delta_{ij} , \tag{2.42}$$

where $i,\ j,\ k,\ l \in \{1,\ 2,\ 3\}$, δ_{ij} is the Kronecker delta and $(\ )_{,j}$ represents $\partial/\partial x_j$.

If the material is hyperelastic then $\mathbf{S} = \partial W/\partial \mathrm{A}$, and so $\boldsymbol{\mathcal{A}}$ can be written as

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \mathrm{A}\, \partial \mathrm{A}} \, ,$$

or, in component form

$$\boldsymbol{\mathcal{A}}_{ijkl} = \frac{\partial^2 W}{\partial x_{j,i} \partial x_{l,k}} \, .$$

For isotropic, hyperelastic materials the nonzero components of $\boldsymbol{\mathcal{A}}_0$, relative to the principal axes of the quasi–static deformation, for an unconstrained material are given by

$$J\mathcal{A}_{0iijj} = \lambda_i \lambda_j W_{ij} \, , \tag{2.43}$$

$$J\mathcal{A}_{0ijij} = \begin{cases} \dfrac{(\lambda_i W_i - \lambda_j W_j)\lambda_i^2}{\lambda_i^2 - \lambda_j^2} \, , & i \neq j \, , \quad \lambda_i \neq \lambda_j \, , \\ \dfrac{1}{2}\left(J\mathcal{A}_{0iiii} - J\mathcal{A}_{0iijj} + \lambda_i W_i\right) \, , & i \neq j \, , \quad \lambda_i = \lambda_j \, , \end{cases} \tag{2.44}$$

$$J\mathcal{A}_{0ijji} = J\mathcal{A}_{0jiij} = J\mathcal{A}_{0ijij} - \lambda_i W_i \, , \quad i \neq j \, , \tag{2.45}$$

with $i, \ j \in \{1, \ 2, \ 3\}$, where $W_i = \partial W/\partial \lambda_i$ and $W_{ij} = \partial^2 W/\partial \lambda_i \, \partial \lambda_j$, using the form of W given in (2.26); note that the summation convention is not implied here.

If the quasi–static deformation is homogeneous the tensor $\boldsymbol{\mathcal{A}}_0$ is seen to be constant and in particular when the body is in an unstressed configuration, corresponding to classical linear theory, for an unconstrained material (2.43)–(2.45) become

$$\begin{gathered} \mathcal{A}_{0iiii} = \lambda + 2\mu \, , \qquad \mathcal{A}_{0iijj} = \lambda \, , \\ \mathcal{A}_{0ijij} = \mathcal{A}_{0ijji} = \mu \, , \end{gathered} \tag{2.46}$$

for $i \neq j$, where λ and μ are the classical Lamé moduli.

For incompressible materials, the components of $\boldsymbol{\mathcal{A}}_0$ are given by (2.43)–(2.45) with $J = 1$, but this time in the unstressed configuration, $\boldsymbol{\mathcal{A}}_0$ reduces to

$$\left.\begin{aligned} \mathcal{A}_{0iiii} = \mathcal{A}_{0ijij} = \mu \\ \mathcal{A}_{0iijj} = \mathcal{A}_{0ijji} = 0 \end{aligned}\right\} \quad i \neq j \, , \tag{2.47}$$

where $\mu > 0$ is the shear modulus.

2.2.2 Incremental dynamics.

In the absence of body forces, the incremental form of the equation of motion (2.17) is

$$\mathrm{Div}\,\dot{\mathrm{S}} = \rho_0 \dot{\mathbf{x}}_{,tt} \ , \tag{2.48}$$

and on updating the reference configuration to be the quasi–statically deformed state, this becomes

$$\mathrm{div}\,\dot{\mathrm{S}}_0 = \rho \dot{\mathbf{x}}_{,tt} \ . \tag{2.49}$$

For an unconstrained, compressible material (2.49) can be written in component form as

$$\frac{\partial}{\partial x_j}\left(\mathcal{A}_{0jikl}\,\dot{x}_{l,k}\right) = \rho \dot{x}_{i,tt} \tag{2.50}$$

for $i,\ j,\ k,\ l \in \{1,\ 2,\ 3\}$, which simplifies to

$$\mathcal{A}_{0jikl}\,\dot{x}_{l,jk} = \rho \dot{x}_{i,tt} \tag{2.51}$$

if the underlying deformation is homogeneous.

For an incompressible material, the component form of (2.49) is

$$\frac{\partial}{\partial x_j}\left(\mathcal{A}_{0jikl}\,\dot{x}_{l,k}\right) + p_{,j}\dot{x}_{j,i} - \dot{p}_{,i} = \rho \dot{x}_{i,tt} \ , \tag{2.52}$$

and, if the underlying deformation is homogeneous, this simplifies to

$$\mathcal{A}_{0jikl}\,\dot{x}_{l,jk} - \dot{p}_{,i} = \rho \dot{x}_{i,tt} \ , \tag{2.53}$$

for $i,\ j,\ k,\ l \in \{1,\ 2,\ 3\}$. In component form the incompressibility condition (2.39) can be written as

$$\dot{x}_{i,i} = 0 \ . \tag{2.54}$$

Note that, the incremental traction $\dot{\mathrm{S}}_0^T\mathbf{n}$ per unit area on a surface in B, with unit outward normal $\mathbf{n}$, has component form

$$\dot{\mathrm{S}}_{0ji} n_j = \mathcal{A}_{0jikl}\dot{x}_{l,k} n_j \ , \tag{2.55}$$

for compressible materials, and

$$\dot{\mathrm{S}}_{0ji} n_j = (\mathcal{A}_{0jikl} + p\delta_{jl}\,\delta_{ik})\,\dot{x}_{l,k} n_j - \dot{p} n_i \ , \tag{2.56}$$

for incompressible materials.

2.2.3 The strong–ellipticity condition.

The strong–ellipticity condition states that

$$\mathrm{tr}\,[(\boldsymbol{\mathcal{A}}(\mathbf{m}\otimes\mathbf{N}))(\mathbf{m}\otimes\mathbf{N})] > 0 \ , \tag{2.57}$$

for all rank one tensors $\mathbf{m}\otimes\mathbf{N} \neq \mathbf{0}$, where $\mathbf{m}$ is an Eulerian vector and $\mathbf{N}$ is a Lagrangian vector. If equality is permitted in (2.57) then this becomes the Legendre–Hadamard condition. When the strong–ellipticity condition holds for a given deformation, the equilibrium equations form an elliptic system and so rule out certain discontinuities in the solution, for example the shear–band solutions discussed by Knowles and Sternberg (1977) cannot occur when the strong–ellipticity condition holds.

The strong–ellipticity condition, first studied by Legendre and Hadamard, was introduced to study wave propagation in linear elastic materials, by means of the acoustic tensor $\mathbf{P}(\mathrm{N})$ defined, in component form, by

$$\mathrm{P}_{ij} = \mathcal{A}_{sitj}\, N_s\, N_t \ . \tag{2.58}$$

It was shown by Marsden and Hughes (1983) that the strong–ellipticity condition was a necessary and sufficient condition for the existence of travelling waves with real wavespeeds.

In terms of the acoustic tensor, necessary and sufficient conditions for strong–ellipticity to hold are

$$\begin{aligned} \mathrm{P}_{ii}(\mathrm{N}) > 0 \ , &\qquad i \in \{1,\,2,\,3\} \\ \mathrm{P}_{ii}(\mathrm{N})\mathrm{P}_{jj}(\mathrm{N}) - \mathrm{P}_{ij}^2(\mathrm{N}) > 0 \ , &\qquad j \neq i \in \{1,\,2,\,3\} \\ \det\mathbf{P}(\mathrm{N}) > 0 \ . & \end{aligned} \tag{2.59}$$

for all $\mathrm{N} \neq \mathbf{0}$.

Corresponding expressions for (2.59) in terms of the components of $\boldsymbol{\mathcal{A}}$ are in general very complicated; however, if attention is restricted to the two–dimensional case then simpler forms can be found. Dowaikh and Ogden (1991) noted that necessary and sufficient conditions for strong–ellipticity to hold in a compressible material are

$$\begin{gathered}
\mathcal{A}_{01111} > 0\ , \quad \mathcal{A}_{02222} > 0\ , \quad \mathcal{A}_{01212} > 0\ , \quad \mathcal{A}_{02121} > 0\ , \\
(\mathcal{A}_{01111}\,\mathcal{A}_{02222})^{\frac{1}{2}} + (\mathcal{A}_{01212}\,\mathcal{A}_{02121})^{\frac{1}{2}} \pm \delta > 0\ ,
\end{gathered} \tag{2.60}$$

with $\delta = \mathcal{A}_{01122} + \mathcal{A}_{02112}$, whereas for an incompressible material the constraint $\mathbf{m}.\mathrm{N} = 0$ must also apply and in two–dimensions, necessary and sufficient conditions for strong–ellipticity to hold are

$$\begin{gathered}
\mathcal{A}_{01212} > 0\ , \qquad \mathcal{A}_{02121} > 0\ , \\
\mathcal{A}_{01111} + \mathcal{A}_{02222} + (\mathcal{A}_{01212}\,\mathcal{A}_{02121})^{\frac{1}{2}} - 2\delta > 0\ .
\end{gathered} \tag{2.61}$$

Chapter 3 – Plane Vibrations of an Incompressible Elastic Plate.

3.1 Formulation of the problem.

We now consider the situation in which the unstressed configuration of a homogeneous, isotropic body, corresponding to the rectangular region defined by

$$-L_i \le X_i \le L_i \ , \qquad i \in \{1,\, 2,\, 3\}$$

is chosen to be the reference configuration, B_0. The body is then subjected to a pure homogeneous strain

$$x_1 = \lambda_1 X_1 \ , \ x_2 = \lambda_2 X_2 \ , \ x_3 = \lambda_3 X_3 \ ,$$

and is deformed into the configuration, B, defined by $-l_i \le x_i \le l_i$, where $l_i = \lambda_i L_i$, $i \in \{1,\, 2,\, 3\}$. A small time–dependent displacement $\mathbf{u} = (u_1,\, u_2,\, u_3)$ is now superimposed on this finite deformation and we restrict our attention to two–dimensional motions with $u_3 = 0$ and u_1 and u_2 independent of x_3, so that the equation of motion (2.53) reduces to

$$\begin{aligned} \mathcal{A}_{01111}u_{1,11} + \mathcal{A}_{02121}u_{1,22} + (\mathcal{A}_{01122} + \mathcal{A}_{02112})\, u_{2,12} - \dot{p}_{,1} &= \rho u_{1,tt} \ , \\ \mathcal{A}_{01212}u_{2,11} + \mathcal{A}_{02222}u_{2,22} + (\mathcal{A}_{01122} + \mathcal{A}_{02112})\, u_{1,12} - \dot{p}_{,2} &= \rho u_{2,tt} \ . \end{aligned} \tag{3.1}$$

The incompressibility condition (2.54) now becomes

$$u_{1,1} + u_{2,2} = 0 \ , \tag{3.2}$$

and from this we deduce the existence of a function ψ of x_1, x_2, t such that

$$u_1 = \psi_{,2} \ , \qquad u_2 = -\psi_{,1} \ . \tag{3.3}$$

On substituting from (3.3) into (3.1) and eliminating the terms in $\dot{p}$ between the two equations, the equation of motion can be written as

$$a\psi_{,1111} + 2b\psi_{,1122} + c\psi_{,2222} = \rho\,(\psi_{,11tt} + \psi_{,22tt}) \ , \tag{3.4}$$

where a, b and c are defined by

$$
\begin{aligned}
a = \mathcal{A}_{01212}\ , \qquad c = \mathcal{A}_{02121}\ ,\\
2b = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{02112}\ ,
\end{aligned}
\tag{3.5}
$$

which reduce to

$$
a = b = c = \mu\ , \tag{3.6}
$$

when the material is undeformed relative to B_0.

By substituting the definitions (3.5) into (2.61), we find that necessary and sufficient conditions for strong–ellipticity to hold are

$$
\begin{aligned}
a > 0\ , \qquad c > 0\ ,\\
b > -\sqrt{ac}\ ,
\end{aligned}
\tag{3.7}
$$

the first two inequalities being equivalent here.

We choose the boundary conditions so that the increment in the tractions $\dot{\mathrm{S}}_{0\,21}$ and $\dot{\mathrm{S}}_{0\,22}$ is zero on the sides $x_2 = \pm l_2$, while on the sides $x_1 = \pm l_1$ we take the displacement, u_1, in the x_1–direction to be zero, with the increment in the shear traction, $\dot{\mathrm{S}}_{0\,12}$, also zero. For values of $\lambda_1 < 1$ this could correspond to the problem of a block being compressed between two parallel, rigid, greased plates.

From (2.56) these boundary conditions can be written as

$$
\begin{aligned}
u_1 = 0\ ,\\
\dot{\mathrm{S}}_{0\,12} \equiv \mathcal{A}_{01212} u_{2,1} + \mathcal{A}_{01221} u_{1,2} = 0\ ,
\end{aligned}
\tag{3.8}
$$

when $x_1 = \pm l_1$, together with

$$
\begin{aligned}
\dot{\mathrm{S}}_{0\,21} \equiv \mathcal{A}_{02121} u_{1,2} + \mathcal{A}_{02112} u_{2,1} = 0\ ,\\
\dot{\mathrm{S}}_{0\,22} \equiv \mathcal{A}_{01122} u_{1,1} + (\mathcal{A}_{02222} + p)\, u_{2,2} - \dot{p} = 0\ ,
\end{aligned}
\tag{3.9}
$$

when $x_2 = \pm l_2$.

After using (3.3) and eliminating the term in $\dot{p}$ from (3.9) by use of $(3.1)_1$, (3.8) and (3.9) become

$$
\left.
\begin{aligned}
\psi_{,2} = 0\\
\psi_{,11} = 0
\end{aligned}
\right\}
\quad \text{on } x_1 = \pm l_1\ , \tag{3.10}
$$

and

$$\left.\begin{aligned}(2b+c-\sigma_2)\psi_{,112}+c\psi_{,222}-\rho\psi_{,2tt}=0\\ c\psi_{,22}+(\sigma_2-c)\psi_{,11}=0\end{aligned}\right\}\quad \text{on } x_2=\pm l_2\ , \tag{3.11}$$

so that by (3.4), (3.10) and (3.11), we have that both the equation of motion and the boundary conditions are linear in the scalar function $\psi = \psi(x_1, x_2, t)$, with coefficients depending only on a, b, c and σ_2.

3.2 Derivation of the frequency equations.

We consider time–harmonic vibrations of frequency ω and seek solutions for ψ of the form

$$\psi = A\exp(spx_2 + ipx_1 - i\omega t)\ , \tag{3.12}$$

where A is an arbitrary constant and s and p are to be determined. The boundary conditions (3.10) can be satisfied by taking linear combinations of expressions like (3.12) to give

$$\psi = \left\{\begin{array}{l}A\exp(spx_2 - i\omega t)\sin px_1\\ A\exp(spx_2 - i\omega t)\cos px_1\end{array}\right\}\quad \text{with } p=\frac{n\pi}{2l_1}\ ,\ n=\left\{\begin{array}{l}2,4,6,....\\ 1,3,5,....\end{array}\right\}\ . \tag{3.13}$$

Substitution of (3.13) into (3.4) leads to

$$p^2(a - 2bs^2 + cs^4) = \rho\omega^2(1 - s^2)\ ,$$

which can be rearranged as

$$c's^4 - 2b's^2 + a' = 0\ , \tag{3.14}$$

where

$$a' = a - \Omega^2\ ,\ 2b' = 2b - \Omega^2\ ,\ c' = c\ , \tag{3.15}$$

and

$$\Omega^2 = \rho\omega^2/p^2\ . \tag{3.16}$$

Equation (3.14) yields four solutions for s (which depend on Ω), and hence the general solution for ψ can be constructed. It then remains to satisfy the boundary conditions (3.11). Several different cases arise depending on the nature of the coefficients in the quadratic (3.14) for s^2, and we deal with these separately. Throughout, we assume that $a > 0$ in accordance with the strong–ellipticity condition (3.7), and hence $c' = c > 0$.

If s_1^2 and s_2^2 are the roots of (3.14) then the different cases are defined by:

If $b' > -(a'c')^{1/2}$ and $a' > 0$ we can have

case 1 — s_1^2 and s_2^2 are distinct and positive,

case 2 — $s_1^2 = s_2^2$ with both positive,

case 3 — s_1^2 and s_2^2 are complex conjugates.

If $b' \leq -(a'c')^{1/2}$ and $a' > 0$ we can have

case 4 — $s_1^2 = s_2^2$ with both negative.

case 5 — s_1^2 and s_2^2 are distinct and both negative,

If $a' < 0$ we have

case 6 — one of s_1^2 and s_2^2 is positive and the other is negative.

Or if $a' = 0$ we can have

case 7 — $s_1^2 = 0$ and $s_2^2 = (2b - \Omega^2)/c > 0$,

case 8 — $s_1^2 = 0$ and $s_2^2 = (2b - \Omega^2)/c < 0$.

case 9 — $s_1^2 = s_2^2 = 0$ when $b' = 0$ also.

We now look at each of these cases in turn,

<u>Case 1</u> : $a' > 0$, $b' > \sqrt{a'c}$.

In this case the roots, s_1^2 and s_2^2 say, of the quadratic (3.14) are distinct and positive, and given by

$$s_1^2 = \frac{\left(b' + \sqrt{b'^2 - a'c}\right)}{c} , \qquad s_2^2 = \frac{\left(b' - \sqrt{b'^2 - a'c}\right)}{c} , \tag{3.17}$$

with s_1 and s_2 chosen to be the positive square roots of these expressions.

The general solution for ψ can be written in the form

$$\psi = \phi(x_2) \left\{ \begin{array}{c} \sin px_1 \\ \\ \cos px_1 \end{array} \right\} e^{-i\omega t} , \tag{3.18}$$

where

$$\phi(x_2) = A\cosh(s_1px_2) + B\sinh(s_1px_2) + C\cosh(s_2px_2) + D\sinh(s_2px_2)\ , \tag{3.19}$$

A, B, C and D being constants.

Substitution of (3.18) with (3.19) into the boundary conditions (3.11) leads, after rearrangement and use of the formula $c(s_1^2 + s_2^2) = 2b'$, to

$$(cs_1^2 + c - \sigma_2)\cosh(s_1\eta)A + (cs_2^2 + c - \sigma_2)\cosh(s_2\eta)C = 0\ , \tag{3.20}$$

$$(cs_1^2 + c - \sigma_2)\sinh(s_1\eta)B + (cs_2^2 + c - \sigma_2)\sinh(s_2\eta)D = 0\ , \tag{3.21}$$

$$s_1(cs_2^2 + c - \sigma_2)\sinh(s_1\eta)A + s_2(cs_1^2 + c - \sigma_2)\sinh(s_2\eta)C = 0\ , \tag{3.22}$$

$$s_1(cs_2^2 + c - \sigma_2)\cosh(s_1\eta)B + s_2(cs_1^2 + c - \sigma_2)\cosh(s_2\eta)D = 0\ , \tag{3.23}$$

where the notation

$$\eta = pl_2\ , \tag{3.24}$$

has been introduced.

These equations decouple as two pairs of equations for the constants $(A,\ C)$ and $(B,\ D)$, which can therefore be treated independently. With $B = D = 0$, ψ becomes an even function of x_2, u_1 an odd function of x_2 and u_2 an even function of x_2. This gives rise to an **antisymmetric** (or flexural) mode. For equations (3.20) and (3.22) to yield non–trivial solutions for $(A,\ C)$, the determinant of coefficients must vanish. This leads to the **frequency equation for antisymmetric modes**, namely

$$\frac{\tanh(\eta s_1)}{\tanh(\eta s_2)} = \frac{s_2(cs_1^2 + c - \sigma_2)^2}{s_1(cs_2^2 + c - \sigma_2)^2}\ . \tag{3.25}$$

On the other hand, with $A = C = 0$, ψ corresponds to **symmetric** (or barreling) modes, and the **frequency equation for symmetric modes** is

$$\frac{\tanh(\eta s_1)}{\tanh(\eta s_2)} = \frac{s_1(cs_2^2 + c - \sigma_2)^2}{s_2(cs_1^2 + c - \sigma_2)^2}\ . \tag{3.26}$$

It is useful to note that (3.25) and (3.26) can be written jointly in the form

$$\begin{aligned} &\left[2\sqrt{a'c}(b'+c-\sigma_2)-a'c+(c-\sigma_2)^2\right]\frac{\sinh[\eta(s_1-s_2)]}{(s_1-s_2)} \\ &\qquad = \pm\left[2\sqrt{a'c}(b'+c-\sigma_2)+a'c-(c-\sigma_2)^2\right]\frac{\sinh[\eta(s_1+s_2)]}{(s_1+s_2)} \ , \end{aligned} \tag{3.27}$$

where the plus and minus signs correspond to (3.25) and (3.26) respectively. We also note that

$$c(s_1-s_2)^2 = 2(b'-\sqrt{a'c}) \ , \qquad c(s_1+s_2)^2 = 2(b'+\sqrt{a'c}) \ . \tag{3.28}$$

Case 2 : $a' > 0$, $b' = \sqrt{a'c}$.

Here $s_1^2 = s_2^2 > 0$ and we take $s_1 = s_2 = s = (a'/c)^{\frac{1}{4}}$. The general solution for ψ again has the form (3.18), but with $\phi(x_2)$ now given by

$$\begin{aligned} \phi(x_2) = A\cosh(spx_2) + B\sinh(spx_2) &+ Cspx_2\sinh(spx_2) \\ &+ Dspx_2\cosh(spx_2) \ . \end{aligned} \tag{3.29}$$

In this case the boundary conditions (3.11) can be cast in the form

$$\begin{aligned} Ae\sinh(s\eta) + C\left[(e-2b')\sinh(s\eta) + es\eta\cosh(s\eta)\right] = 0 \ , \\ Ae\cosh(s\eta) + C\left[2b'\cosh(s\eta) + es\eta\sinh(s\eta)\right] = 0 \ , \end{aligned} \tag{3.30}$$

$$\begin{aligned} Be\cosh(s\eta) + D\left[(e-2b')\cosh(s\eta) + es\eta\sinh(s\eta)\right] = 0 \ , \\ Be\sinh(s\eta) + D\left[2b'\sinh(s\eta) + es\eta\cosh(s\eta)\right] = 0 \ , \end{aligned} \tag{3.31}$$

where we have introduced the notation

$$e = b' + c - \sigma_2 \ . \tag{3.32}$$

Two possibilities arise :

(a) $e = 0$, in which case it follows that $C = D = 0$ (provided $\sigma_2 \neq c$; if $\sigma_2 = c$ then $a' = b' = 0$, which possibility is covered in Case 9). This allows both antisymmetric and symmetric modes to occur simultaneously.

Since $b' = \sigma_2 - c = \sqrt{a'c}$ we conclude that $\sigma_2 > c$ and

$$\Omega^2 = a - \frac{(\sigma_2-c)^2}{c} = 2(b+c-\sigma_2) \ . \tag{3.33}$$

The latter equation in (3.33) imposes restrictions on the state of stress and deformation for which this special case is attainable.

(b) $e \neq 0$, in which case the equations for (A, C) and (B, D) decouple. For $B = D = 0$ we obtain the frequency equation for antisymmetric modes, and for $A = C = 0$ the corresponding equation for symmetric modes. Jointly these equations are written

$$\frac{\sinh(2s\eta)}{2s\eta} = \pm\frac{(b' + c - \sigma_2)}{(3b' - c + \sigma_2)} , \tag{3.34}$$

the + (−) sign corresponding to antisymmetric (symmetric) modes. Again, with $b' = \sqrt{a'c}$, (3.34) yield restrictions on the state of stress and deformation. It is worth noting that (3.34) can be obtained directly from (3.27) by taking the limit $s_1 \to s_2 = s$ with $b' = \sqrt{a'c}$.

<u>Case 3</u> : $a' > 0$, $-\sqrt{a'c} < b' < \sqrt{a'c}$.

Here, s_1^2 and s_2^2 are complex conjugates and we write

$$s_1 = \gamma + i\delta , \qquad s_2 = \gamma - i\delta , \tag{3.35}$$

where

$$\gamma = \left(\frac{b' + \sqrt{a'c}}{2c}\right)^{\frac{1}{2}} , \qquad \delta = \left(\frac{\sqrt{a'c} - b'}{2c}\right)^{\frac{1}{2}} . \tag{3.36}$$

The general solution for ψ can again be written in the form (3.18) with (3.19) but with complex arguments; when expressed in real form it involves both trigonometric and hyperbolic functions, but we do not write it explicitly here. The resulting frequency equations are obtained directly from (3.27) by using (3.35) to give

$$\begin{aligned} &\left[2\sqrt{a'c}(b' + c - \sigma_2) - a'c + (c - \sigma_2)^2\right] \frac{\sin(2\eta\delta)}{\delta} \\ &\qquad = \pm\left[2\sqrt{a'c}(b' + c - \sigma_2) + a'c - (c - \sigma_2)^2\right] \frac{\sinh(2\eta\gamma)}{\gamma} , \end{aligned} \tag{3.37}$$

with the + (−) sign corresponding to antisymmetric (symmetric) modes.

Case 4 : $a' > 0\ ,\ b' = -\sqrt{a'c}.$

Here $s_1^2 = s_2^2 < 0$ and we write $s_1 = s_2 = is^*$, where s^* is real and positive with $s^{*2} = \sqrt{a'/c}$. This time we have

$$\begin{aligned}\phi(x_2) = A\cos(s^*px_2) + B\sin(s^*px_2) &+ Cs^*px_2\sin(s^*px_2) \\ &+ Ds^*px_2\cos(s^*px_2)\ ,\end{aligned} \tag{3.38}$$

and, as in Case 2, two possibilities can arise:

(a) $e = 0$, in which case, antisymmetric and symmetric modes can occur together with (3.33) again holding but this time we must have

$$\sigma_2 < c\ ,$$

(b) $e \neq 0$, the frequency equations are given by

$$\frac{\sin(2s^*\eta)}{2s^*\eta} = \pm\frac{(b' + c - \sigma_2)}{(3b' - c + \sigma_2)}\ , \tag{3.39}$$

where + (–) corresponds to the antisymmetric (symmetric) mode.

We note that the inequality

$$a > 2b\ ,$$

must be satisfied in this case.

Case 5 : $a' > 0\ ,\ b' < -\sqrt{a'c}.$

In this case s_1^2 and s_2^2 are distinct and negative and we write

$$s_1 = is_1^*\ , \qquad s_2 = is_2^*\ , \tag{3.40}$$

where s_1^* and s_2^* are real and positive. The expression (3.19) is now replaced by

$$\phi(x_2) = A\cos(s_1^*px_2) + B\sin(s_1^*px_2) + C\cos(s_2^*px_2) + D\sin(s_2^*px_2)\ , \tag{3.41}$$

and the frequency equations (3.25) and (3.26) become

$$\frac{\tan(\eta s_1^*)}{\tan(\eta s_2^*)} = \left[\frac{s_2^*(c - cs_1^{*2} - \sigma_2)^2}{s_1^*(c - cs_2^{*2} - \sigma_2)^2}\right]^{\pm 1}\ , \tag{3.42}$$

with the + (–) sign corresponding to the antisymmetric (symmetric) mode. The frequency equations can also be written, corresponding to (3.27), as

$$\begin{aligned}\left[2\sqrt{a'c}(b'+c-\sigma_2)-a'c+(c-\sigma_2)^2\right]\frac{\sin[\eta(s_1^*-s_2^*)]}{(s_1^*-s_2^*)}\\ =\pm\left[2\sqrt{a'c}(b'+c-\sigma_2)+a'c-(c-\sigma_2)^2\right]\frac{\sin[\eta(s_1^*+s_2^*)]}{(s_1^*+s_2^*)}\ .\end{aligned} \tag{3.43}$$

Note that, we again must have

$$a > 2b\ .$$

Case 6 : $a' < 0$.

With the definition (3.17) it follows that $s_1^2 > 0$ and $s_2^2 < 0$, so we write $s_2 = is_2^*$ with s_2^* positive. The expression for $\phi(x_2)$ now has the form

$$\phi(x_2) = A\cosh(s_1 p x_2) + B\sinh(s_1 p x_2) + C\cos(s_2^* p x_2) + D\sin(s_2^* p x_2)\ , \tag{3.44}$$

where A, B, C, D are real constants, and the frequency equations (3.25) and (3.26) become

$$\frac{\tanh(\eta s_1)}{\tan(\eta s_2^*)} = -\frac{s_2^*(cs_1^2+c-\sigma_2)^2}{s_1(c-\sigma_2-cs_2^{*2})^2}\ , \tag{3.45}$$

and

$$\frac{\tanh(\eta s_1)}{\tan(\eta s_2^*)} = \frac{s_1(c-\sigma_2-cs_2^{*2})^2}{s_2^*(cs_1^2+c-\sigma_2)^2}\ , \tag{3.46}$$

corresponding to antisymmetric and symmetric modes respectively.

Note that for this case, the frequency equations cannot be written in a form similar to (3.27).

Case 7 : $a' = 0\ ,\ b' > 0$.

Here, $s_1^2 = 2b'/c > 0$ and $s_2^2 = 0$ so we take

$$\phi(x_2) = A\cosh(s_1 p x_2) + B\sinh(s_1 p x_2) + C + Dx_2\ . \tag{3.47}$$

The boundary conditions (3.11) become

$$\begin{aligned}A(c-\sigma_2)\sinh(s_1\eta) = 0\ ,\\ A(2b'+c-\sigma_2)\cosh(s_1\eta) + C(c-\sigma_2) = 0\ ,\end{aligned} \tag{3.48}$$

$$
\begin{aligned}
&Bs_1p(c-\sigma_2)\cosh(s_1\eta)+D(2b'+c-\sigma_2)=0\ ,\\
&B(2b'+c-\sigma_2)\sinh(s_1\eta)+Dl_2(c-\sigma_2)=0\ .
\end{aligned}
\tag{3.49}
$$

For antisymmetric modes, (3.48) yields no nontrivial solutions for $\sigma_2 \neq c$; however, if $\sigma_2 = c$ then (3.48) gives $A = 0$, so that we can have a solution with

$$
\psi = C\left\{\begin{array}{c}\cos px_1\\ \\ \sin px_1\end{array}\right\}e^{-i\omega t}\ ,
\tag{3.50}
$$

which corresponds to a displacement

$$
\begin{aligned}
&u_1 = 0\ ,\\
&u_2 = Cp\left\{\begin{array}{c}\sin px_1\\ \\ -\cos px_1\end{array}\right\}e^{-i\omega t}\ .
\end{aligned}
\tag{3.51}
$$

For symmetric modes equations (3.49) lead to

$$
\frac{\tanh(s_1\eta)}{s_1\eta}=\frac{(c-\sigma_2)^2}{(2b'+c-\sigma_2)^2}\ ,
\tag{3.52}
$$

which can also be obtained directly from (3.26) by taking the limit $s_2 \to 0$. Note that no solution is possible when $\sigma_2 = c$. Also, the inequality

$$
a < 2b\ ,
$$

must be satisfied in this case.

Case 8 : $\quad a' = 0\ ,\ b' < 0.$

This time, $s_1^2 = 2b'/c < 0$ and $s_2^2 = 0$, so we set $s_1 = is_1^*$, where s_1^* is positive and take

$$
\phi(x_2) = A\cos(s_1^* p x_2) + B\sin(s_1^* p x_2) + C + Dx_2\ .
\tag{3.53}
$$

As in Case 7, no antisymmetric modes occur when $\sigma_2 \neq c$ but the solution represented by (3.51) can again occur for $\sigma_2 = c$.

For symmetric modes the frequency equation is

$$
\frac{\tan(s_1^*\eta)}{s_1^*\eta}=\frac{(c-\sigma_2)^2}{(2b'+c-\sigma_2)^2}\ .
\tag{3.54}
$$

Note that this time when $\sigma_2 = c$, equation (3.54) yields

$$s_1^* \eta = k\pi \ , \quad \text{for some integer } k. \tag{3.55}$$

The inequality

$$a > 2b \ ,$$

must hold here.

Case 9 : $\quad a' = b' = 0$.

In this case $s_1^2 = s_2^2 = 0$, and so $\phi(x_2)$ must be of the form

$$\phi(x_2) = A + Bx_2 + Cx_2^2 + Dx_2^3 \ . \tag{3.56}$$

The boundary conditions (3.11) give

$$\begin{aligned} (\sigma_2 - c)C &= 0 \ , \\ p^2(c - \sigma_2)A + \left[2c + \eta^2(c - \sigma_2)\right] C &= 0 \ , \end{aligned} \tag{3.57}$$

and

$$\begin{aligned} p^2(\sigma_2 - c)B + \left[3(\sigma_2 - c)\eta^2 + 6c\right] D &= 0 \ , \\ p^2(\sigma_2 - c)B + \left[(\sigma_2 - c)\eta^2 - 6c\right] D &= 0 \ . \end{aligned} \tag{3.58}$$

For antisymmetric modes if $\sigma_2 \neq c$ then equations (3.57) do not permit any nontrivial solutions, but when $\sigma_2 = c$ solutions of the form (3.51) can again occur.

This time for symmetric modes, the situation depends on whether $\sigma_2 = c$ or not. If $\sigma_2 = c$ then equations (3.58) reduce to $D = 0$, so that solutions of the form

$$\psi = Bx_2 \begin{Bmatrix} \cos px_1 \\ \sin px_1 \end{Bmatrix} e^{-i\omega t} \ , \tag{3.59}$$

exist for this case. In terms of the displacement $\mathbf{u}$, (3.59) can be written as

$$\begin{aligned} u_1 &= B \begin{Bmatrix} \cos px_1 \\ \sin px_1 \end{Bmatrix} e^{-i\omega t} \ , \\ u_2 &= Bpx_2 \begin{Bmatrix} \sin px_1 \\ -\cos px_1 \end{Bmatrix} e^{-i\omega t} \ . \end{aligned} \tag{3.60}$$

Otherwise, when $\sigma_2 \neq c$, the equations (3.58) give

$$Bp^2 + 2D\eta^2 = 0 \ , \tag{3.61}$$

together with

$$\eta^2(c - \sigma_2) = 6c \ , \tag{3.62}$$

and hence we require $\sigma_2 < c$ for these symmetric modes to exist.

Note that this time we must have

$$a = 2b \ .$$

3.3 The static case with $\omega = 0$.

3.3.1 Bifurcation criteria for quasi–static modes.

When $\omega = 0$ the solution in (3.13) corresponds to a quasi–static incremental mode of deformation. Then, if the strong–ellipticity condition (3.7) holds only Cases 1–3 in Section 3.2 can arise, while Case 4 corresponds to the transitional situation in which strong–ellipticity just fails. The frequency equations for these cases become **bifurcation equations** which describe the states of deformation and stress in which the incremental modes of deformation can appear on a path of pure homogeneous deformation from the natural configuration. Some special cases of these equations have been examined by Ogden (1984) and, for convenience, we summarize the relevant results here for reference and also generalize those results slightly.

Case 1 : $\quad a > 0 \ , \ b > \sqrt{ac}.$

Here the bifurcation equations are

$$\frac{\tanh(\eta s_1)}{\tanh(\eta s_2)} = \left[\frac{s_2(cs_1^2 + c - \sigma_2)^2}{s_1(cs_2^2 + c - \sigma_2)^2}\right]^{\pm 1} , \tag{3.63}$$

where the exponent $+1$ (-1) corresponds to antisymmetric (symmetric) modes. These are as in (3.25) and (3.26) respectively but now s_1 and s_2 are given by

$$s_1^2 = \frac{b + \sqrt{b^2 - ac}}{c} , \qquad s_2^2 = \frac{b - \sqrt{b^2 - ac}}{c} . \tag{3.64}$$

As in (3.27) the equations (3.63) can be rearranged in the form

$$\begin{aligned} &\left[2\sqrt{ac}(b + c - \sigma_2) - ac + (c - \sigma_2)^2\right] \frac{\sinh[\eta(s_1 - s_2)]}{(s_1 - s_2)} \\ &\quad = \pm \left[2\sqrt{ac}(b + c - \sigma_2) + ac - (c - \sigma_2)^2\right] \frac{\sinh[\eta(s_1 + s_2)]}{(s_1 + s_2)} . \end{aligned} \tag{3.65}$$

Equations (3.65) generalize the corresponding equations given by Ogden (1984, equation (6.3.138)) for $\sigma_2 = 0$. Contact is now made with the results in Ogden (1984) by introducing the notation

$$\lambda = \lambda_1 \lambda_3^{1/2} , \qquad \lambda^{-1} = \lambda_2 \lambda_3^{1/2} , \tag{3.66}$$

and writing the strain–energy as a function of λ and λ_3. Thus, we define

$$\hat{W}(\lambda, \lambda_3) = W(\lambda\lambda_3^{-1/2},\ \lambda^{-1}\lambda_3^{-1/2},\ \lambda_3) , \tag{3.67}$$

so that

$$\sigma_1 - \sigma_2 = \lambda \hat{W}_\lambda , \tag{3.68}$$

$$a = \lambda^4 c = \lambda^5 \hat{W}_\lambda / (\lambda^4 - 1) , \qquad 2(b + c) = \lambda^2 \hat{W}_{\lambda\lambda} . \tag{3.69}$$

We note that in Ogden (1984) the dependence of $\hat{W}$ on λ_3 was suppressed. Here, the subscript λ denotes partial differentiation with respect to λ.

In terms of $\hat{W}$ the strong–ellipticity inequalities (3.7) take the form

$$\frac{\lambda \hat{W}_\lambda}{\lambda^2 - 1} > 0 , \qquad \lambda^2 \hat{W}_{\lambda\lambda} + \frac{2\lambda \hat{W}_\lambda}{\lambda^2 + 1} > 0 . \tag{3.70}$$

For Case 1, which we are dealing with here, we have

$$\lambda^2 \hat{W}_{\lambda\lambda} > \frac{2\lambda \hat{W}_\lambda}{\lambda^2 - 1} > 0 , \tag{3.71}$$

from which it follows that (3.70) are automatically satisfied.

In Ogden (1984) it was shown that neither antisymmetric nor symmetric modes can occur in tension ($\lambda > 1$) when $\sigma_2 = 0$. In compression, on the other hand, again for $\sigma_2 = 0$, it was shown that for $\lambda_c < \lambda < 1$ only antisymmetric modes can occur and for $0 < \lambda < \lambda_c$ only symmetric modes can occur, where λ_c is the first (and possibly only) value of λ for which $\lambda^3 \hat{W}_{\lambda\lambda} + \hat{W}_\lambda$ vanishes on a path of deformation on which λ decreases from unity.

Turning now to states of stress with $\sigma_2 \neq 0$ we see that, since $s_1 > s_2$ and tanh is a monotonic increasing function of its argument, a necessary condition for antisymmetric modes to occur is that the right–hand side of (3.63) with the positive exponent should be greater than unity. Use of (3.64) enables this condition to be cast in the form

$$2\sqrt{ac}(b+c) + c(a-c) + 2(c-\sqrt{ac})\sigma_2 - \sigma_2^2 > 0 , \tag{3.72}$$

or, equivalently, in terms of $\hat{W}$,

$$\frac{\lambda^2 \hat{W}_\lambda}{(\lambda^4 - 1)} \left(\lambda^3 \hat{W}_{\lambda\lambda} + \hat{W}_\lambda\right) - \frac{2\lambda \hat{W}_\lambda}{(\lambda^2+1)}\sigma_2 - \sigma_2^2 > 0 . \tag{3.73}$$

On the other hand a necessary condition for the existence of symmetric modes is provided by reversing the inequality in (3.72) and (3.73). Clearly, the two sets of conditions are mutually exclusive.

Further, by considering (3.65) and the monotonicity of $\sinh x/x$ we deduce that

$$(c - \sigma_2)^2 > ac , \tag{3.74}$$

and

$$\sigma_2 < b + c , \tag{3.75}$$

are both necessary for either antisymmetric or symmetric modes.

Noting the chain of inequalities

$$c - \sqrt{ac} - \left[2\sqrt{ac}(b+\sqrt{ac})\right]^{\frac{1}{2}} < c - \sqrt{ac} < c + \sqrt{ac}$$
$$< c - \sqrt{ac} + \left[2\sqrt{ac}(b+\sqrt{ac})\right]^{\frac{1}{2}} < b + c ,$$

we conclude from (3.72), (3.74) and (3.75) that for antisymmetric modes to exist either

$$c-\sqrt{ac}-\left[2\sqrt{ac}(b+\sqrt{ac})\right]^{\frac{1}{2}}<\sigma_2<c-\sqrt{ac}\ , \tag{3.76}$$

or

$$c+\sqrt{ac}<\sigma_2<c-\sqrt{ac}+\left[2\sqrt{ac}(b+\sqrt{ac})\right]^{\frac{1}{2}}\ , \tag{3.77}$$

must hold.

Similarly, for symmetric modes to exist it is necessary that either

$$\sigma_2<c-\sqrt{ac}-\left[2\sqrt{ac}(b+\sqrt{ac})\right]^{\frac{1}{2}}\ , \tag{3.78}$$

or

$$c-\sqrt{ac}+\left[2\sqrt{ac}(b+\sqrt{ac})\right]^{\frac{1}{2}}<\sigma_2<b+c\ . \tag{3.79}$$

We also note that neither mode is possible if

$$c-\sqrt{ac}<\sigma_2<c+\sqrt{ac}\ . \tag{3.80}$$

Clearly, all the above inequalities can be expressed in terms of λ, $\hat{W}_\lambda$ and $\hat{W}_{\lambda\lambda}$, but we do not give details here.

Similar necessary conditions were first obtained by Sawyers and Rivlin (1974) for the case $\sigma_2=0$. On setting $\sigma_2=0$ we recover the results given in Ogden (1984). In particular, (3.77) cannot hold, while for (3.76) to hold it is necessary that $c>a$, and hence that $\hat{W}_\lambda<0$ and $\lambda<1$ (corresponding to plane strain compression). Similarly, (3.79) cannot hold, and (3.78) again requires that $c>a$. In tension, on the other hand, $c<a$ and (3.80) holds for $\sigma_2=0$, i.e. neither mode is possible in this case.

We note that, with reference to (3.65),

$$2\sqrt{ac}(b+c-\sigma_2)-ac+(c-\sigma_2)^2=\left[\sigma_2-(c-\sqrt{ac})\right]^2+2\sqrt{ac}(b-\sqrt{ac})\ ,$$

which is positive, and

$$2\sqrt{ac}(b+c-\sigma_2)+ac-(c-\sigma_2)^2=-\left[\sigma_2-(c-\sqrt{ac})\right]^2+2\sqrt{ac}(b+\sqrt{ac})\ ,$$

which again can take either sign. In the reference configuration (with $\lambda = \lambda_3 = 1$, $\sigma_2 = 0$) the latter expression is positive. Thus, by continuity, it will remain positive on a path of deformation and stress from the reference configuration until it vanishes; thereafter it can become negative. It follows from (3.65) that antisymmetric modes will always appear before symmetric modes in quasi–static loading from the reference configuration. This generalizes the result given in Ogden (1984) for the case $\sigma_2 = 0$.

Case 2 : $a > 0$, $b = \sqrt{ac}$.

Here $s = \lambda$ and (3.32) becomes

$$e = b + c - \sigma_2 \equiv (\lambda^2 + 1)c - \sigma_2 \ . \tag{3.81}$$

Then, either

(a) $e = 0$, which allows both antisymmetric and symmetric modes to occur together,

or

(b) $e \neq 0$, in which case (3.34) reduces to

$$\frac{\sinh(2\lambda\eta)}{2\lambda\eta} = \pm\frac{(\lambda^2 + 1)c - \sigma_2}{(3\lambda^2 - 1)c + \sigma_2} \ . \tag{3.82}$$

From (3.82) we deduce that necessary conditions for antisymmetric and symmetric modes to occur are

$$-(3\lambda^2 - 1)c < \sigma_2 < -(\lambda^2 - 1)c \ , \tag{3.83}$$

and

$$\sigma_2 < -(3\lambda^2 - 1)c \ , \tag{3.84}$$

respectively, coupled with the requirement $b = \lambda^2 c$. We note that the isolated value $\sigma_2 = (\lambda^2 + 1)c$ corresponding to (a) above lies outside the ranges of values in (3.83) and (3.84).

As for Case 1, neither mode can occur for $\lambda > 1$ when $\sigma_2 = 0$, in which case $\lambda^2 < 1/3$ is a necessary condition for symmetric modes to occur, as shown in Ogden (1984).

Case 2 applies when $\lambda = 1$ ($\lambda_1 = \lambda_2$) and then

$$a = b = c = \frac{1}{4}\hat{W}_{\lambda\lambda}(1, \lambda_3) \,. \tag{3.85}$$

Here $e = 0$ corresponds to $\sigma_2 = 2c$ and when $e \neq 0$ the bifurcation criterion (3.82) reduces to

$$\frac{\sinh(2\eta)}{2\eta} = \pm\frac{(2c - \sigma_2)}{(2c + \sigma_2)} \,, \tag{3.86}$$

while the inequalities (3.83) and (3.84) respectively become

$$-2c < \sigma_2 < 0 \,, \tag{3.87}$$

and

$$\sigma_2 < -2c \,. \tag{3.88}$$

In different notation the results (3.86)–(3.88) were given in Ogden (1984).

<u>Case 3</u> : $a > 0$, $-\sqrt{ac} < b < \sqrt{ac}$.

From(3.36) we have

$$\gamma = \left(\frac{b + \sqrt{ac}}{2c}\right)^{\frac{1}{2}} \,, \qquad \delta = \left(\frac{\sqrt{ac} - b}{2c}\right)^{\frac{1}{2}} \,, \tag{3.89}$$

and (3.37) reduces to

$$\begin{aligned} &\left[2\sqrt{ac}(b + c - \sigma_2) - ac + (c - \sigma_2)^2\right] \frac{\sin(2\eta\delta)}{\delta} \\ &\quad = \pm\left[2\sqrt{ac}(b + c - \sigma_2) + ac - (c - \sigma_2)^2\right] \frac{\sinh(2\eta\gamma)}{\gamma} \,. \end{aligned} \tag{3.90}$$

Because of the occurrence of the trigonometric function in (3.90) it is not possible to seek general necessary conditions for the existence of antisymmetric and symmetric modes in this case; however, if

$$2\sqrt{ac}(b + c - \sigma_2) + ac - (c - \sigma_2)^2 = 0,$$

and

$$2\eta\delta = n\pi \,, \qquad n = 1, 2, \ldots. \,,$$

then (3.90) is satisfied for both antisymmetric and symmetric modes. Sawyers (1977) and Ogden (1984), with $\sigma_2 = 0$, display the complex nature of the solution branches of (3.90).

<u>Case 4</u> : $a > 0$, $b = -\sqrt{ac}$.

The inequality $b = -\sqrt{ac}$ corresponds to loss of ellipticity, and we have $s^* = \lambda$ with

$$e = b + c - \sigma_2 \equiv -(\lambda^2 - 1)c - \sigma_2 \ . \tag{3.91}$$

As in Case 2 both modes are possible when $e = 0$ while if $e \neq 0$ the bifurcation criterion is

$$\frac{\sin(2\lambda\eta)}{2\lambda\eta} = \pm\frac{[(1-\lambda^2)c - \sigma_2]}{[\sigma_2 - (3\lambda^2 + 1)c]} \ . \tag{3.92}$$

Bifurcation criteria for Cases 5–9 may also be written down formally but they are of limited interest since strong–ellipticity does not hold.

Depending on the form of the strain–energy function, either Case 1 or Case 3 will arise on a path of deformation and stress from the reference configuration (in which the conditions for Case 2 hold). Case 4 may arise after passage through Case 3.

We now make direct contact with the results of Sawyers and Rivlin (1974) and Sawyers (1977), for the case with $\sigma_2 = 0$, by considering the parameter A given by

$$\mathrm{A} + 1 = \frac{2c(b - \sqrt{ac})}{(c - \sqrt{ac})^2} \ .$$

Sawyers and Rivlin (1974) noted that the nature of the bifurcation equations depend on the size of this parameter and studied the cases with $\mathrm{A} > -1$, corresponding to Case 1 here, and $\mathrm{A} = -1$, corresponding to Case 2, while Sawyers (1977) considered the situation, corresponding to Case 3, where A lay in the range

$$-\left(\frac{\lambda^2 + 1}{\lambda^2 - 1}\right)^2 < \mathrm{A} < -1 \ .$$

Note that the deformation parameter λ used by Sawyers and Rivlin is different to the definition, (3.66), given here.

3.3.2 Some limiting cases.

Here we consider the specialization of the results in Section 3.3.1 to the situations in which the aspect ratio η is very small or very large.

(a) $\eta \to 0$

For Case 1 it can be shown from either (3.63) or (3.65) that for antisymmetric modes the asymptotic form of the bifurcation equation, to the first order in η^2, is either

$$\sigma_2 = c + \sqrt{ac} + \frac{1}{3}(b - \sqrt{ac})\lambda^2\eta^2 , \tag{3.93}$$

or

$$\sigma_2 = c - \sqrt{ac} - \frac{1}{3}(b + \sqrt{ac})\lambda^2\eta^2 . \tag{3.94}$$

For symmetric modes, on the other hand, we obtain

$$\sigma_2 = b + c - \frac{1}{6}\frac{(b^2 - ac)}{c}\eta^2 . \tag{3.95}$$

together with the $O(\eta^{-2})$ solution

$$\sigma_2 = -6c\eta^{-2} ; \tag{3.96}$$

these formulae are consistent, respectively, with the inequalities (3.77), (3.76), (3.79) and (3.78).

In Case 2 (b) the asymptotic form of (3.82) is

$$\sigma_2 = -(\lambda^2 - 1)c - \frac{2}{3}\lambda^4 c\eta^2 , \tag{3.97}$$

for antisymmetric modes and $\sigma_2 = -6c\eta^{-2}$ for symmetric modes. Note that (3.97) is consistent with (3.83).

In Case 3 the asymptotic results obtained from (3.90) are again (3.93)–(3.96), as for Case 1, but the required chain of inequalities

$$b + c < c - \sqrt{ac} + \left[2\sqrt{ac}(b + \sqrt{ac})\right]^{\frac{1}{2}} < c + \sqrt{ac} ,$$

is the reverse of that applying in Case 1.

Finally, in Case 4, equation (3.92) yields

$$\sigma_2 = (\lambda^2 + 1)c - \frac{2}{3}\lambda^4 c\eta^2 , \tag{3.98}$$

for antisymmetric modes and $\sigma_2 = -6c\eta^{-2}$ for symmetric modes.

(b) $\eta \to \infty$

Asymptotic results for large η can be obtained in a similar way to those above for small η, but they involve exponentials. Here we give the limiting results for $\eta \to \infty$.

In Case 1 the equations (3.63) for antisymmetric and symmetric modes coincide in this limit and both yield

$$2\sqrt{ac}(b + c - \sigma_2) + ac - (c - \sigma_2)^2 = 0 . \tag{3.99}$$

On setting $b = \sqrt{ac}$ in (3.99) the results for Case 2 are recovered and yield either

$$\sigma_2 = (\lambda^2 + 1)c , \tag{3.100}$$

or

$$\sigma_2 = -(3\lambda^2 - 1)c . \tag{3.101}$$

These can also be obtained directly from (3.81) and (3.82) respectively. In Case 3 the limiting equation is again (3.99), while in Case 4 the limiting result

$$\sigma_2 = -(\lambda^2 - 1)c , \tag{3.102}$$

obtained from (3.92) coincides with the special case $e = 0$.

It is worth noting that (3.99), or, equivalently,

$$\frac{\lambda^2 \hat{W}_\lambda}{(\lambda^4 - 1)} \left(\lambda^3 \hat{W}_{\lambda\lambda} + \hat{W}_\lambda\right) - \frac{2\lambda \hat{W}_\lambda}{(\lambda^2 + 1)}\sigma_2 - \sigma_2^2 = 0 ,$$

is the bifurcation criterion for surface deformations on a half-space subject to the same deformation as considered here (Dowaikh and Ogden, 1990). In the present context the inequality (3.73) is sufficient to exclude symmetric modes of deformation, while for a half–space it provides an exclusion condition for surface deformations. This point will be discussed further in Chapter 5.

3.4 Discussion of the incompressible frequency equations.

We now consider the full dynamic problem and try to obtain necessary conditions for the existence of vibrational modes, noting that the quasi–static underlying deformation must satisfy the stability criteria discussed in Section 3.3.

3.4.1 General existence criteria.

Case 1: $a' > 0,\ b' > \sqrt{a'c}$.

Here the frequency equations for the antisymmetric and symmetric modes are given by (3.25) and (3.26) respectively, or alternatively (3.27) describes both modes.

In the antisymmetric case, since we chose $s_1 > s_2$, the left–hand side of (3.25) is strictly greater than unity, using the monotonicity of tanh, therefore by applying this restriction to the right–hand side and using the definition (3.17), we obtain the condition

$$2\sqrt{a'c}(b' + c) + c(a' - c) + 2(c - \sqrt{a'c})\sigma_2 - \sigma_2^2 > 0 \ , \qquad (3.103)$$

which is necessary for the existence of antisymmetric modes. Note the similarity of (3.103) to the corresponding result (3.72) in the static case; the frequency dependence of (3.103) entering solely through the terms

$$a' = a - \Omega^2 \quad \text{and} \quad 2b' = 2b - \Omega^2 \ .$$

A necessary condition for the existence of symmetric modes is given by (3.103) with the inequality reversed, so that for a given state of deformation and stress these modes are mutually exclusive.

By considering (3.27) and using the monotonicity of $\sinh x/x$ it is found that

$$(c - \sigma_2)^2 > a'c \ , \qquad (3.104)$$

and

$$\sigma_2 < b' + c\,, \tag{3.105}$$

are necessary for both antisymmetric and symmetric modes to occur. As in the static case we have a chain of inequalities

$$c - \sqrt{a'c} - \left[2\sqrt{a'c}(b' + \sqrt{a'c})\right]^{\frac{1}{2}} < c - \sqrt{a'c} < c + \sqrt{a'c}$$
$$< c - \sqrt{a'c} + \left[2\sqrt{a'c}(b' + \sqrt{a'c})\right]^{\frac{1}{2}} < b' + c\,,$$

which can be used to give existence criteria in terms of the stress σ_2.

From (3.103), (3.104) and (3.105) either

$$c - \sqrt{a'c} - \left[2\sqrt{a'c}(b' + \sqrt{a'c})\right]^{\frac{1}{2}} < \sigma_2 < c - \sqrt{a'c}\,, \tag{3.106}$$

or

$$c + \sqrt{a'c} < \sigma_2 < c - \sqrt{a'c} + \left[2\sqrt{a'c}(b' + \sqrt{a'c})\right]^{\frac{1}{2}}\,, \tag{3.107}$$

must hold for antisymmetric modes to exist. Whereas for symmetric modes to exist, either

$$\sigma_2 < c - \sqrt{a'c} - \left[2\sqrt{a'c}(b' + \sqrt{a'c})\right]^{\frac{1}{2}}\,, \tag{3.108}$$

or

$$c - \sqrt{a'c} + \left[2\sqrt{a'c}(b' + \sqrt{a'c})\right]^{\frac{1}{2}} < \sigma_2 < b' + c\,, \tag{3.109}$$

must be satisfied.

Note that neither mode can occur if

$$c - \sqrt{a'c} < \sigma_2 < c + \sqrt{a'c}\,. \tag{3.110}$$

Case 2: $a' > 0,\ b' = \sqrt{a'c}$.

We have the two possibilities:

(a) $e = 0$, which gives

$$\sigma_2 = b' + c\,, \tag{3.111}$$

so that antisymmetric and symmetric modes can occur together.

(b) $e \neq 0$, in which case the frequency equations for antisymmetric and symmetric modes are jointly given by (3.34). Using the result that $\sinh x/x > 1$ for all positive x, we find that for antisymmetric modes to occur we require

$$c - 3b' < \sigma_2 < c - b' , \tag{3.112}$$

whereas for symmetric modes to occur, the inequality

$$\sigma_2 < c - 3b' , \tag{3.113}$$

must hold. However, if $\sigma_2 > c - b'$ then neither mode can occur.

Note that, since $b' > 0$, the isolated value given by (3.111) lies outside the ranges indicated in (3.112) and (3.113).

Cases 3, 4, 5, 6 and 8 all contain trigonometric terms, and so necessary conditions for the existence of antisymmetric and symmetric modes cannot be obtained in these cases.

<u>Case 7</u>: $a' = 0, \; b' > 0$.

If $\sigma_2 = c$ then only the solution represented by (3.51) can occur, but otherwise the frequency equation for the symmetric mode is given by (3.52). By noting that $\tanh x/x < 1$ for all positive x, a necessary condition for the existence of symmetric modes, with $\sigma_2 \neq c$, is given by

$$\sigma_2 < b' + c . \tag{3.114}$$

Note that antisymmetric modes, with $\bar{\sigma} \neq c$, cannot occur for this case.

For Case 9, as was noted in Section 3.2, a necessary (and sufficient) condition for symmetric modes to exist is $\sigma_2 < c$, while if $\sigma_2 = c$ then the solutions represented by (3.51) and (3.60) occur.

3.4.2 Some limiting cases.

(a) $\eta \to 0$.

As in the static case, asymptotic forms, for small η, can be found from the frequency equations obtained in Section 3.2.

In Case 1, for antisymmetric modes, the asymptotic form to the first order in η^2 of the frequency equation (3.25) (or alternatively (3.27)), is either

$$\sigma_2 = c + \sqrt{a'c} + \frac{\sqrt{a'c}}{3c}(b' - \sqrt{a'c})\eta^2 > c\ , \tag{3.115}$$

or

$$\sigma_2 = c - \sqrt{a'c} - \frac{\sqrt{a'c}}{3c}(b' + \sqrt{a'c})\eta^2 < c\ , \tag{3.116}$$

whereas for symmetric modes the only $O(1)$ solution is given by

$$\sigma_2 = b' + c - \frac{1}{6c}(b'^2 - a'c)\eta^2 < b' + c\ . \tag{3.117}$$

Note that, (3.115)–(3.117) are consistent with the inequalities (3.107), (3.106) and (3.109) respectively. In the symmetric case, we also have the asymptotic form

$$\sigma_2 = -6c\eta^{-2}\ , \tag{3.118}$$

which obviously satisfies the inequality (3.108) when η is small. Equations (3.115)–(3.117) can be rearranged to give the asymptotic behaviour of Ω in terms of small values of η, but we do not list the results in this case.

The asymptotic forms represented by (3.115)–(3.118) are again obtained for Cases 3 and 5, while for Case 6 (3.115)–(3.118) still hold but with $\sqrt{a'c}$ replaced by $\sqrt{-(a'c)}$, since $a' < 0$ in this case.

For Case 2(b), the asymptotic form of (3.34) corresponding to antisymmetric modes is

$$\sigma_2 = c - \sqrt{a'c} - \frac{2}{3}a'\eta^2\ , \tag{3.119}$$

while for symmetric modes we obtain the asymptotic form given in (3.118). Note that (3.119) and (3.118) are consistent with the inequalities (3.112) and (3.113) in Section 3.4.1. The equations (3.119) and (3.118) also hold in Case 4(b), for antisymmetric and symmetric modes respectively. On rearranging (3.119), we find that Ω can be expressed as

$$\Omega = \sqrt{a - c(1 - \sigma_2/c)^2}\left[1 + \frac{2(c - \sigma_2)^3}{3c\left(ac - (c - \sigma_2)^2\right)}\eta^2\right] . \tag{3.120}$$

In Case 7, provided $\sigma_2 \neq c$, only symmetric modes can occur and the asymptotic form of (3.52), to the first order in η^2, is

$$\sigma_2 = b' + c - \frac{b'^2}{6c}\eta^2 , \tag{3.121}$$

which can also be obtained from (3.117) by setting $a' = 0$. The solution represented by (3.118) can again occur, and both (3.118) and (3.121) satisfy the inequality (3.114), while (3.121) can be rearranged to give

$$\Omega = \sqrt{2(b + c - \sigma_2)}\left[1 - \frac{(c - \sigma_2)^2}{12c(b + c - \sigma_2)}\eta^2\right] . \tag{3.122}$$

Case 8 gives rise to the same results as obtained for Case 7, while for Case 9, we quickly see from (3.62) that only the solution given by (3.118) can occur for $\sigma_2 < c$.

(b) $\underline{\eta \to \infty.}$

For large values of η, we ignore the asymptotic results, which involve exponentials, and consider only the limiting results as $\eta \to \infty$.

In Case 1, as $\eta \to \infty$, both antisymmetric and symmetric modes have the same limiting form, namely

$$2\sqrt{a'c}(b' + c) + c(a' - c) + 2\left(c - \sqrt{a'c}\right)\sigma_2 - \sigma_2^2 = 0 . \tag{3.123}$$

Equation (3.123) was derived, in another notation, in Dowaikh and Ogden (1990) as the secular equation for the existence of Rayleigh surface waves on an infinite half-space. This link between Rayleigh waves and vibrations of finite blocks will be discussed further in Chapter 5.

In Case 2, when $e = 0$, (3.33) gives

$$\sigma_2 = b' + c \ , \tag{3.124}$$

while if $e \neq 0$ then (3.34) has the limit

$$\sigma_2 = c - 3b' \ , \tag{3.125}$$

as $\eta \to \infty$. From (3.37), Case 3 has limiting form (3.123) as $\eta \to \infty$, while for Case 4 only (3.124) can arise, regardless of the value of e. Cases 5, 6 and 8 contain the trigonometeric term tan, and limits cannot be found in these cases. In Case 7, equation (3.52) has limit $\sigma_2 = c$ as does equation (3.62) of Case 9.

3.5 Equibiaxial deformations.

We now consider the special case in which the underlying deformation is equibiaxial with $\lambda_1 = \lambda_2$. Then, from (3.66), by incompressibility, $\lambda = \lambda_1 \lambda_3^{1/2} = 1$, with $\sigma_1 = \sigma_2 = \sigma$, say and, as in (3.85), $a = b = c$. On putting this into (3.17), we find that

$$s_1 = 1 \ , \qquad s_2 = \sqrt{1 - \bar{\Omega}^2} \ , \tag{3.126}$$

where

$$\bar{\Omega}^2 = \Omega^2 / a \ , \tag{3.127}$$

and Ω^2 is given by (3.16). For real, non–zero Ω only Cases 1, 6 and 7 arise, and we examine these separately.

3.5.1 Discussion of the frequency equations.

Case 1 : $0 < \bar{\Omega}^2 < 1$.

Equations (3.25) and (3.26) together specialize to

$$\frac{\tanh \eta}{\tanh \left(\eta \sqrt{1 - \bar{\Omega}^2} \right)} = \left[\frac{\sqrt{1 - \bar{\Omega}^2}(2 - \bar{\sigma})^2}{(2 - \bar{\sigma} - \bar{\Omega}^2)^2} \right]^{\pm 1} \ , \tag{3.128}$$

where the + (−) corresponds to antisymmetric (symmetric) modes, and $\bar{\sigma} = \sigma / a$. It can be shown that a necessary condition for the existence of antisymmetric modes is either

$$1+\sqrt{1-\bar{\Omega}^2} < \bar{\sigma} < 1-\sqrt{1-\bar{\Omega}^2}+(1-\bar{\Omega}^2)^{\frac{1}{4}}(1+\sqrt{1-\bar{\Omega}^2}) , \tag{3.129}$$

or

$$1-\sqrt{1-\bar{\Omega}^2}-(1-\bar{\Omega}^2)^{\frac{1}{4}}(1+\sqrt{1-\bar{\Omega}^2}) < \bar{\sigma} < 1-\sqrt{1-\bar{\Omega}^2} . \tag{3.130}$$

The corresponding necessary condition for symmetric modes is either

$$1-\sqrt{1-\bar{\Omega}^2}+(1-\bar{\Omega}^2)^{\frac{1}{4}}(1+\sqrt{1-\bar{\Omega}^2}) < \bar{\sigma} < 2-\frac{1}{2}\bar{\Omega}^2 , \tag{3.131}$$

or

$$\bar{\sigma} < 1-\sqrt{1-\bar{\Omega}^2}-(1-\bar{\Omega}^2)^{\frac{1}{4}}(1+\sqrt{1-\bar{\Omega}^2}) . \tag{3.132}$$

This time when η is small we obtain the asymptotic results, that for antisymmetric modes to occur, either

$$\bar{\sigma} = 1+\sqrt{1-\bar{\Omega}^2}+\frac{1}{6}\sqrt{1-\bar{\Omega}^2}\left(1-\sqrt{1-\bar{\Omega}^2}\right)^2\eta^2 , \tag{3.133}$$

or

$$\bar{\sigma} = 1-\sqrt{1-\bar{\Omega}^2}-\frac{1}{6}\sqrt{1-\bar{\Omega}^2}\left(1+\sqrt{1-\bar{\Omega}^2}\right)^2\eta^2 , \tag{3.134}$$

and we note that in (3.133) we must have $\bar{\sigma} \in (1,\, 2+\eta^2/6)$, while in (3.134) we have $\bar{\sigma} \in (-\frac{2\eta^2}{3},\, 1)$, with η small. Correspondingly, for symmetric modes, we must have either

$$\bar{\sigma} = 2-\bar{\Omega}^2/2-\bar{\Omega}^4\eta^2/24 , \tag{3.135}$$

or

$$\bar{\sigma} = -6\eta^{-2} , \tag{3.136}$$

where from (3.135) $\bar{\sigma}$ must lie in the interval $(\frac{3}{2}-\frac{\eta^2}{24},\, 2)$. Note that (3.133)–(3.136) are consistent with (3.129)–(3.132) above.

For antisymmetric modes, equations (3.133) and (3.134) can be rearranged, in terms of $\bar{\Omega}$, and can be written jointly in the form

$$\bar{\Omega}^2 = (2-\bar{\sigma})\left[\bar{\sigma} + \frac{(2-\bar{\sigma})(\bar{\sigma}-1)^2}{3}\eta^2\right] , \tag{3.137}$$

where $\bar{\sigma} \in [0, 2)$, with $\bar{\sigma} \neq 1$, for real values of $\bar{\Omega} < 1$. Equation (3.135), in the symmetric case, can similarly be rearranged to give

$$\bar{\Omega} = \sqrt{2(2-\bar{\sigma})}\left(1-(2-\bar{\sigma})\eta^2/12\right) , \tag{3.138}$$

for values of $\bar{\sigma} \in (\frac{3}{2} - \frac{\eta^2}{24}, 2)$, with η small.

<u>Case 6</u> : $\quad \bar{\Omega}^2 > 1$.

In this case the frequency equation is

$$\frac{\tanh\eta}{\tan\left(\eta\sqrt{\bar{\Omega}^2-1}\right)} = \mp\left[\frac{\sqrt{\bar{\Omega}^2-1}(2-\bar{\sigma})^2}{(2-\bar{\sigma}-\bar{\Omega}^2)^2}\right]^{\pm 1} , \tag{3.139}$$

where the upper (lower) sign corresponds to antisymmetric (symmetric) modes.

A special feature here occurs when either $\bar{\sigma} = 2$ or $\bar{\sigma} + \bar{\Omega}^2 = 2$. For antisymmetric modes, for example, when $\bar{\sigma} = 2$ the following frequencies are possible:

$$\bar{\Omega}^2 = 1 + \left[\frac{(2k-1)\pi}{2\eta}\right]^2 , \qquad k = 1, 2, \ldots . \tag{3.140}$$

Similarly, if $\bar{\sigma} + \bar{\Omega}^2 = 2$ then

$$\bar{\sigma} = 1 - \left(\frac{k\pi}{\eta}\right)^2 , \qquad \bar{\Omega}^2 = 1 + \left(\frac{k\pi}{\eta}\right)^2 , \qquad k = 1, 2, \ldots . \tag{3.141}$$

For symmetric modes, the roles of k and $(2k-1)/2$ are reversed in the above.

For other values of $\bar{\sigma}$ each equation in (3.139) yields an infinite family of solutions which may be plotted as curves in the $(\eta,\ \bar{\Omega})$–plane. Note, however, that the chosen values of $\bar{\sigma}$ should be consistent with the requirements of stability deduced from Case 2 in Section 3.3.1. In particular, the configuration is stable for

$0 \le \bar{\sigma} < 2$, but stability fails in an antisymmetric mode for some value of $\bar{\sigma}$ in the interval $(-2,\ 0)$ depending on η. Specifically, for

$$\bar{\sigma} = \frac{2(2\eta - \sinh 2\eta)}{(2\eta + \sinh 2\eta)} \ . \tag{3.142}$$

<u>Case 7</u> : $\bar{\Omega}^2 = 1$.

In this transitional case antisymmetric modes are not possible, except for $\bar{\sigma} = 1$, as shown in Section 3.2 in a more general context. For symmetric modes the equation

$$\frac{\tanh \eta}{\eta} = \frac{(1-\bar{\sigma})^2}{(2-\bar{\sigma})^2} \ , \tag{3.143}$$

must be satisfied, and a necessary condition for this is $\bar{\sigma} < 3/2$, with $\bar{\sigma} \neq 1$. Equation (3.143) can be rearranged to give

$$\bar{\sigma} = \begin{cases} \dfrac{1 - 2(\tanh\eta/\eta)^{\frac{1}{2}}}{1 - (\tanh\eta/\eta)^{\frac{1}{2}}} & \bar{\sigma} < 1 \ , \\[2ex] \dfrac{1 + 2(\tanh\eta/\eta)^{\frac{1}{2}}}{1 + (\tanh\eta/\eta)^{\frac{1}{2}}} & 1 < \bar{\sigma} < \dfrac{3}{2} \ . \end{cases} \tag{3.144}$$

3.5.2 <u>Numerical results for equibiaxial deformations.</u>

In Figures 3.1–3.10 the line $\bar{\Omega} = 1$ corresponds to Case 7 and so divides the regions corresponding to Case 1 (below this line) and Case 6 (above it). From (3.143) we see that only symmetric modes may occur when $\bar{\Omega} = 1$ and so no antisymmetric branch may cross this line. Also, in this equibiaxial case, we see that the solution branches of the two different modes do not cross at any point.

In Figures 3.1–3.5 we plot the modified frequency $\bar{\Omega}$ against the mode number η for various values of the stress $\bar{\sigma} = \sigma/a$. By considering the intersections with the η–axis the inequalities (3.87) and (3.88) derived for the static case are shown to hold in that, when $\bar{\sigma} = -3$, a symmetric bifurcation mode may arise but when $\bar{\sigma} = -1$ an antisymmetric bifucation mode can occur while the underlying deformation is stable for $0 \le \bar{\sigma} < 2$, with stability failing in Case 2(a) when $\bar{\sigma} = 2$.

In Figure 3.4, with $\bar{\sigma} = 1$, the lowest antisymmetric mode lies along the line $\bar{\Omega} = 1$, corresponding to Case 7, and so represents the displacement given by (3.51).

In Figures 3.6–3.10 we plot the modified frequency $\bar{\Omega}$ against the stress $\bar{\sigma}$ for a range of successively decreasing values of the mode number η and it can be seen that although the spacing of the solution branches varies, the underlying character of the graphs is unchanged; with the point on the $\bar{\sigma}$–axis at $\bar{\sigma} = 2$ corresponding to Case 2(a) bifurcation in each of the graphs. On taking the limit as η tends to zero in (3.137), for the antisymmetric mode, it reduces to the upper half–circle

$$\bar{\Omega}^2 + (\bar{\sigma} - 1)^2 = 1 \ , \qquad \bar{\Omega} > 0 \ , \tag{3.145}$$

and we see from Figure 3.10 that this is almost the case even when η is as large as 0.2. Alternatively, the limit of (3.128) as $\eta \to \infty$ for both antisymmetric and symmetric modes is

$$(2 - \bar{\sigma} - \bar{\Omega}^2)^2 = \sqrt{1 - \bar{\Omega}^2}(2 - \bar{\sigma})^2 \ , \qquad \bar{\Omega} > 0 \ ,$$

which cuts the axis at $\bar{\sigma} = \pm 2$ and we see from Figure 3.6 that this is almost the case when $\eta = 5$.

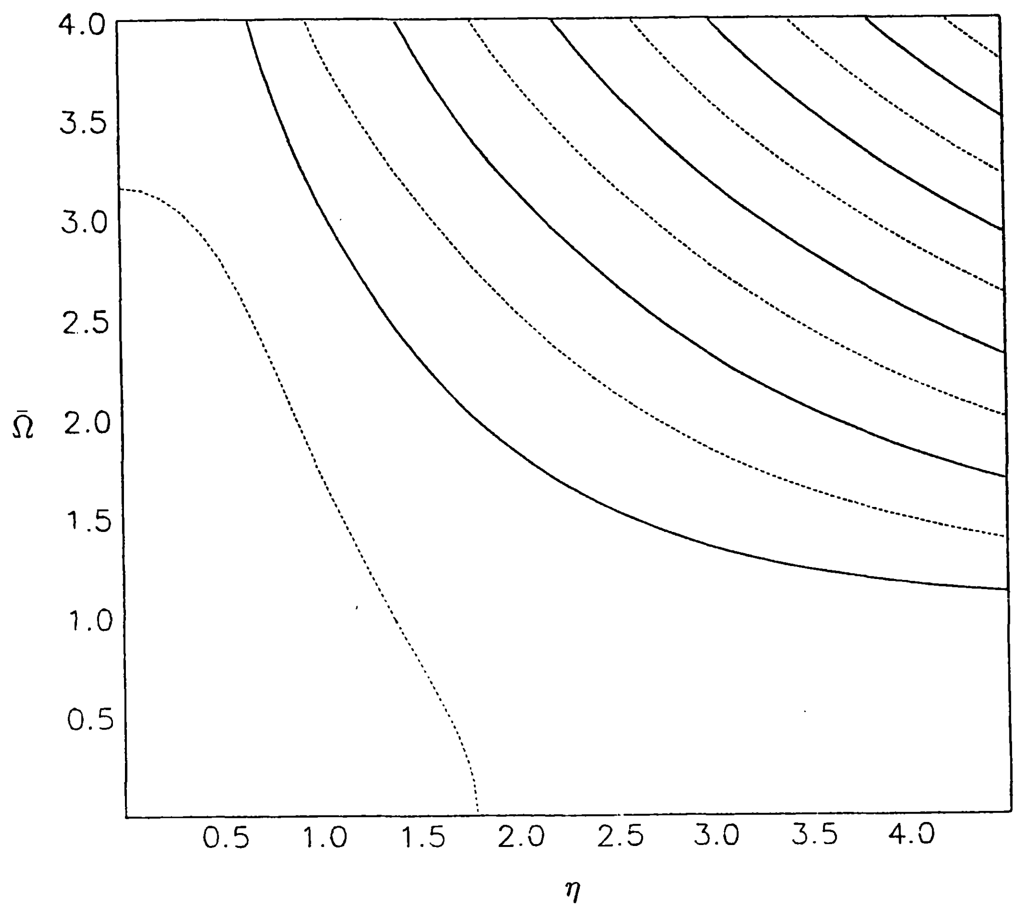

Figure 3.1: Dispersion spectrum for an equibiaxially deformed incompressible material subject to a compressive stress $\bar{\sigma} = -3$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

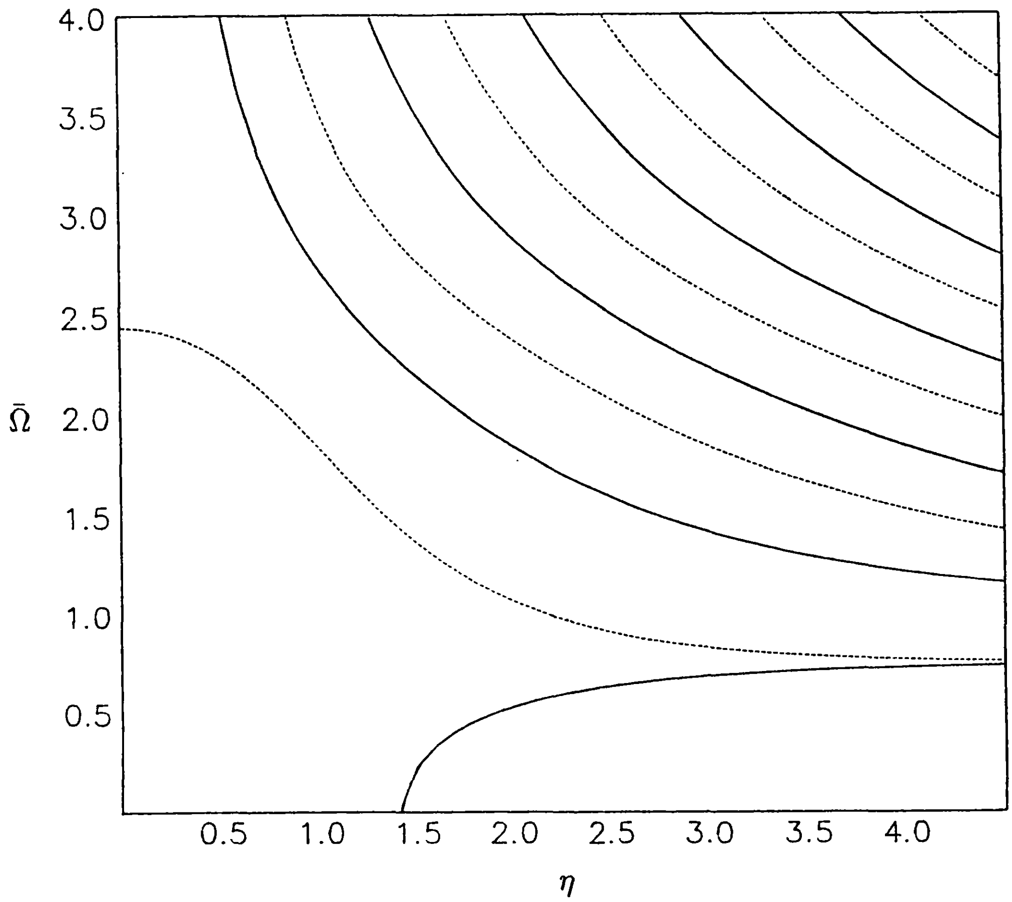

Figure 3.2: Dispersion spectrum for an equibiaxially deformed incompressible material subject to a compressive stress $\bar{\sigma} = -1$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

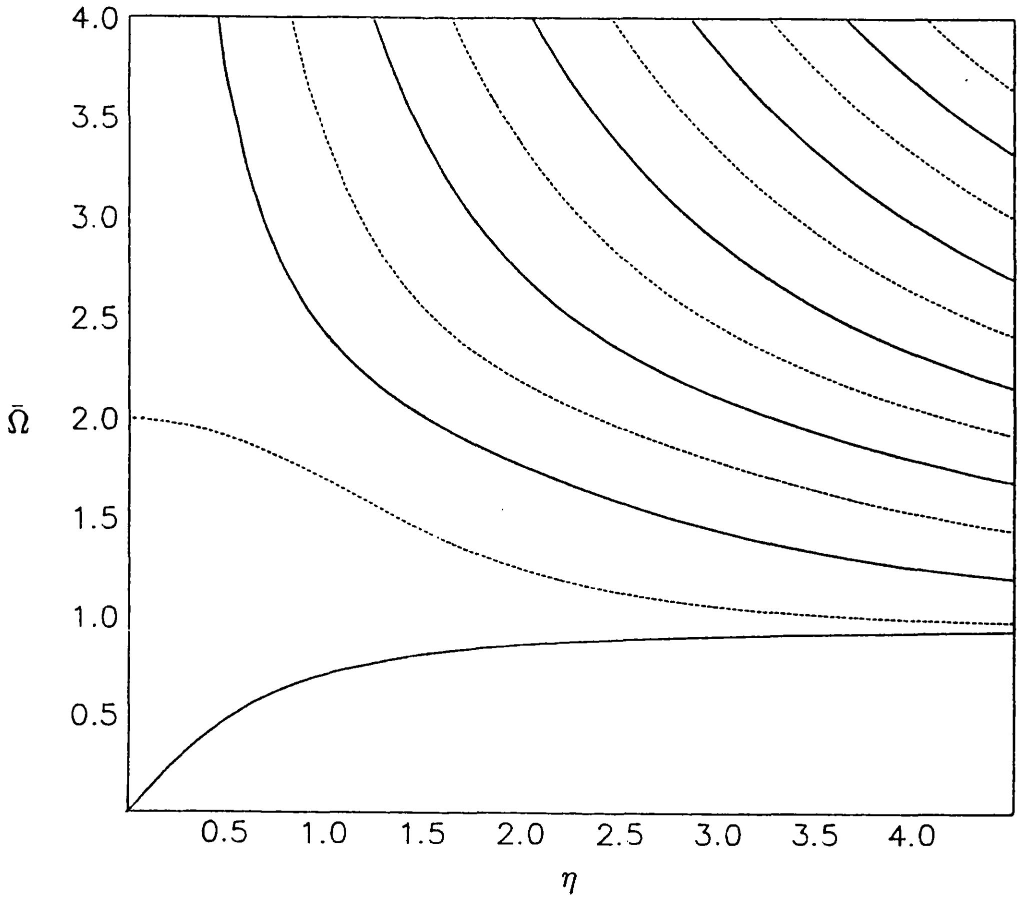

Figure 3.3: Dispersion spectrum for an unstressed incompressible body ($\bar{\sigma} = 0$), where the solid (broken) curves are taken to represent the antisymmetric (symmetric) solution branches.

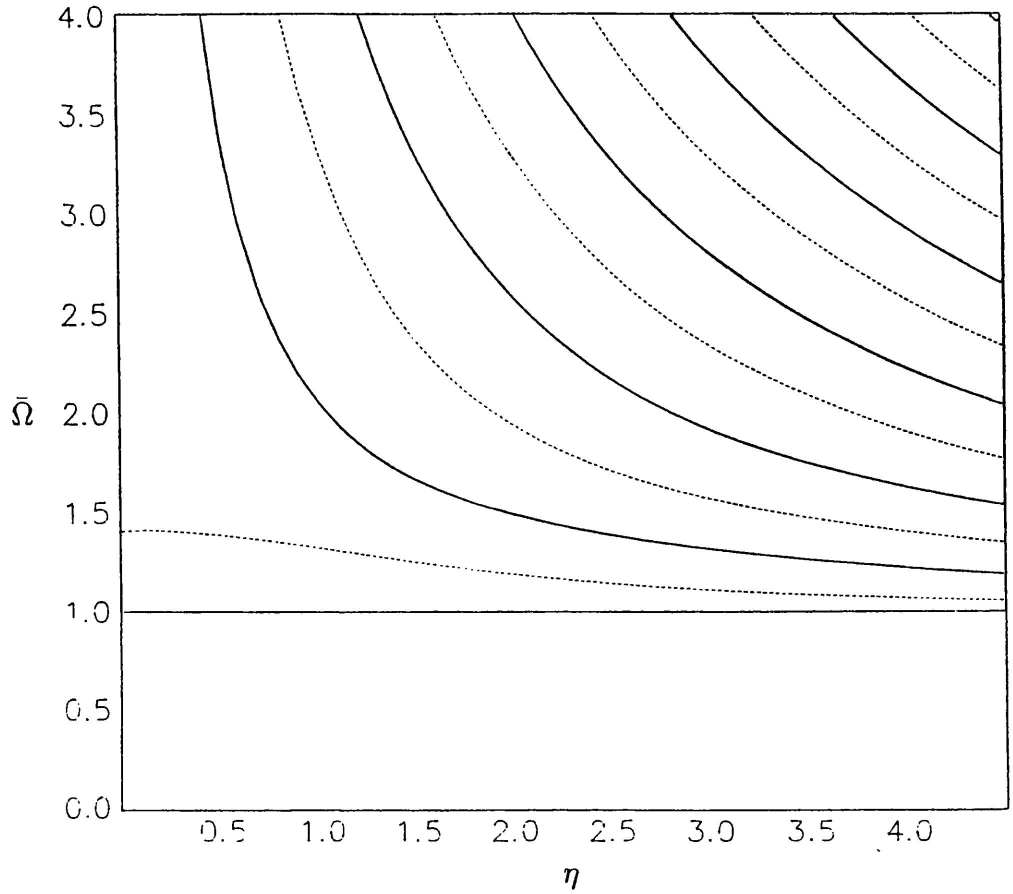

Figure 3.4: Dispersion spectrum for an equibiaxially deformed incompressible material subject to a tensile stress $\bar{\sigma} = 1$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

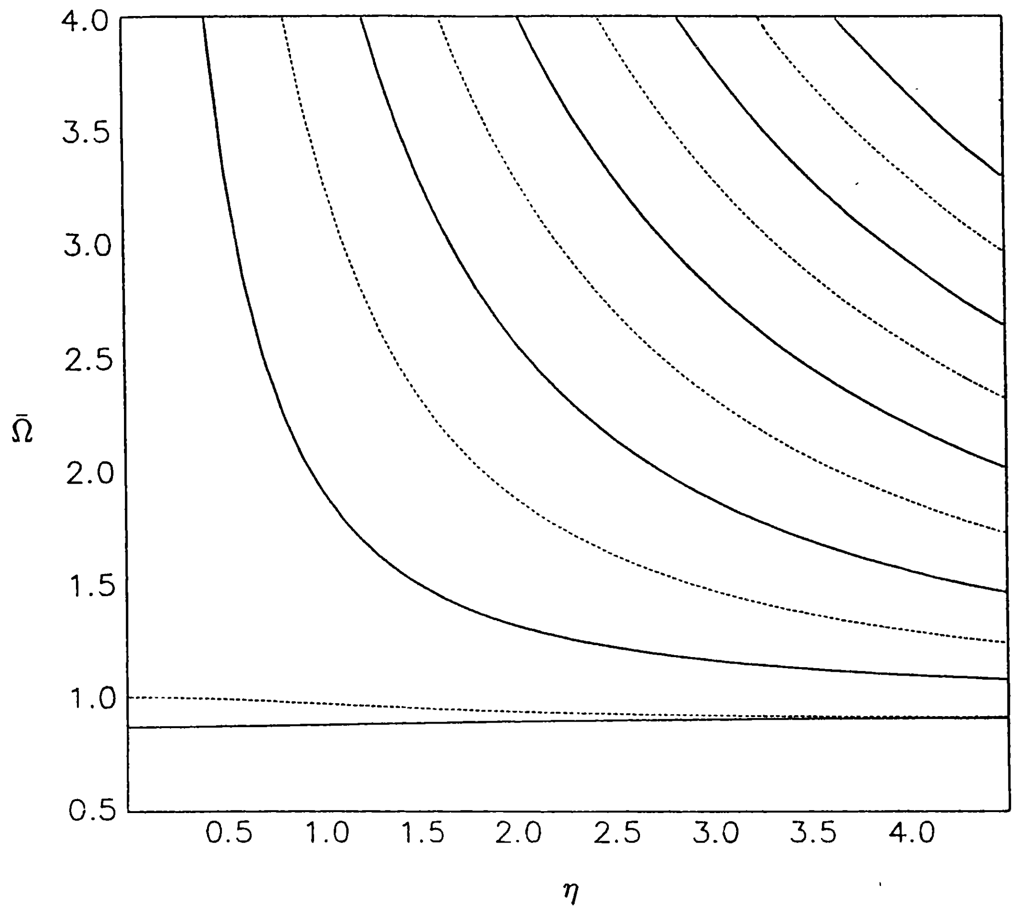

Figure 3.5: Dispersion spectrum for an equibiaxially deformed incompressible material subject to a tensile stress $\bar{\sigma} = 1.5$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

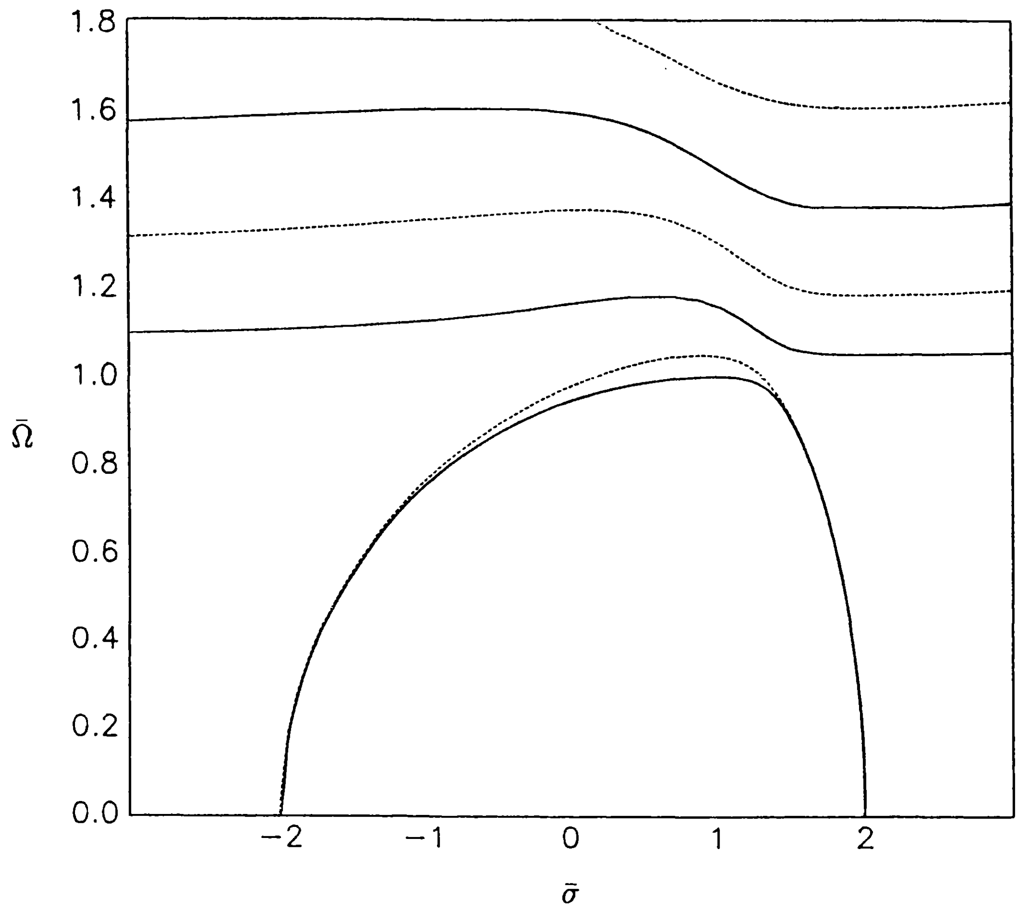

Figure 3.6: Frequency–stress plot, with mode number $\eta = 5$, for an equibiaxially deformed incompressible material . The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

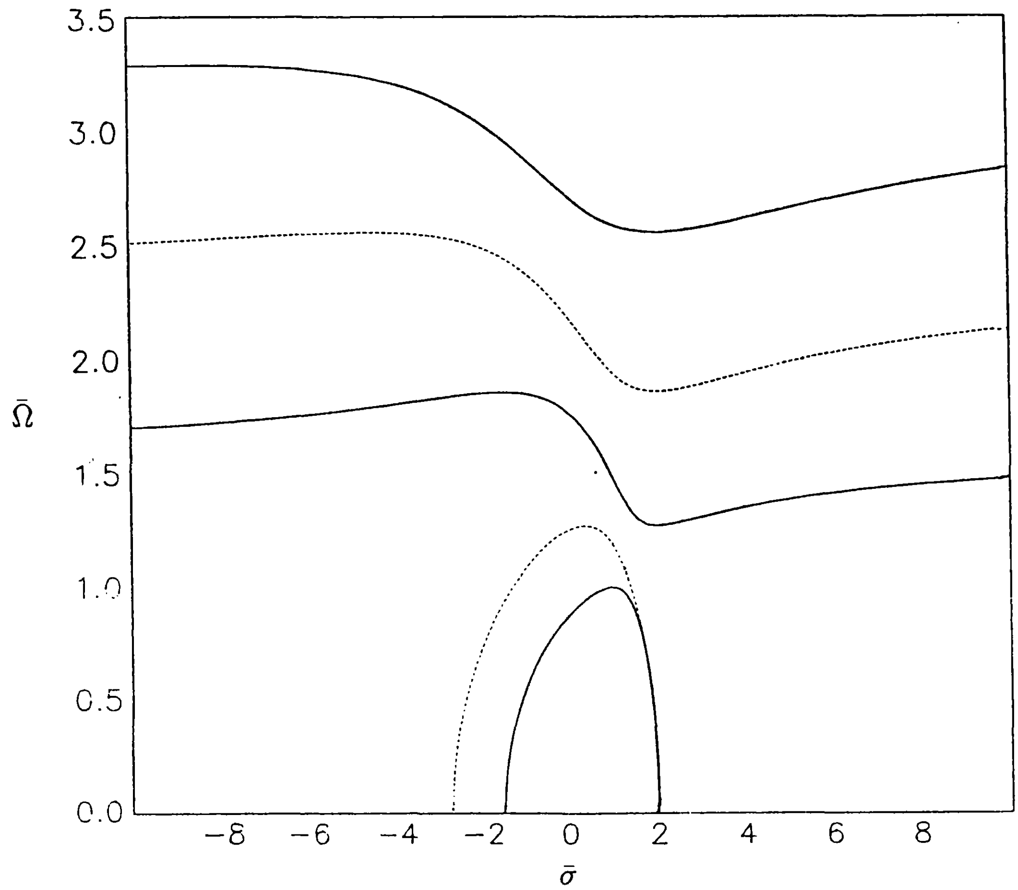

Figure 3.7: Frequency–stress plot, with mode number $\eta = 2$, for an equibiaxially deformed incompressible material . The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

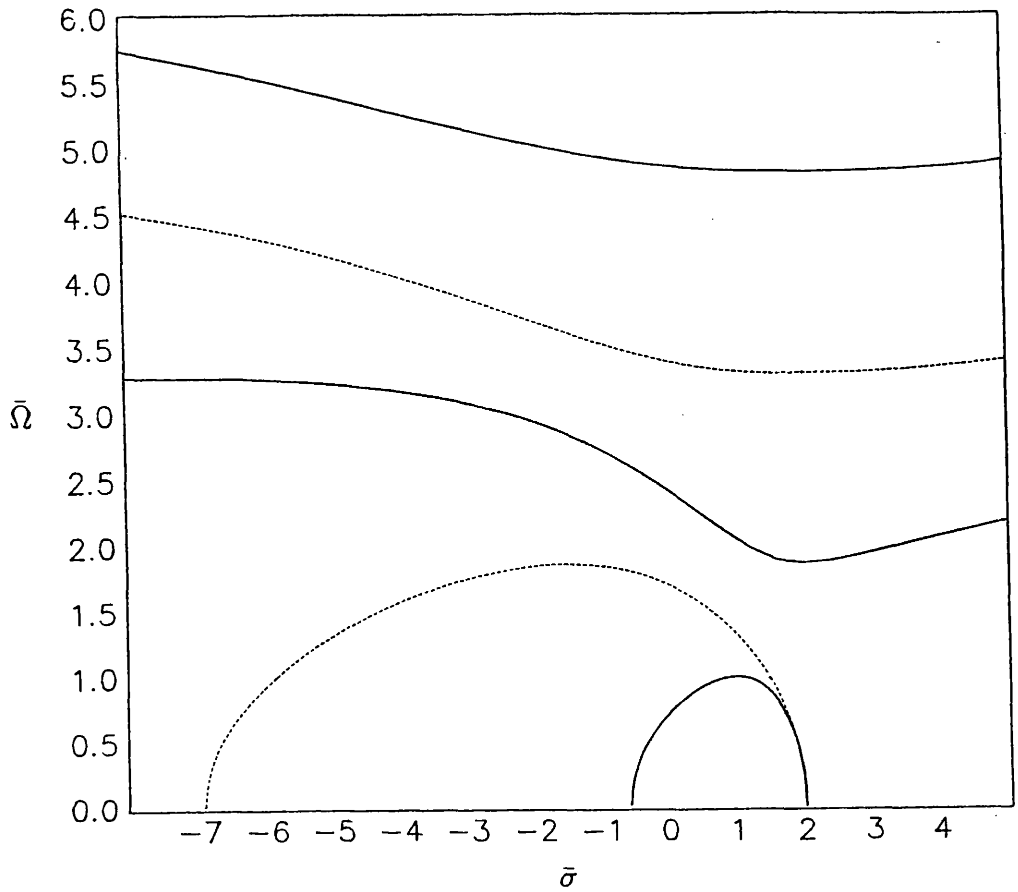

Figure 3.8: Frequency–stress plot, with mode number $\eta = 1$, for an equibiaxially deformed incompressible material . The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

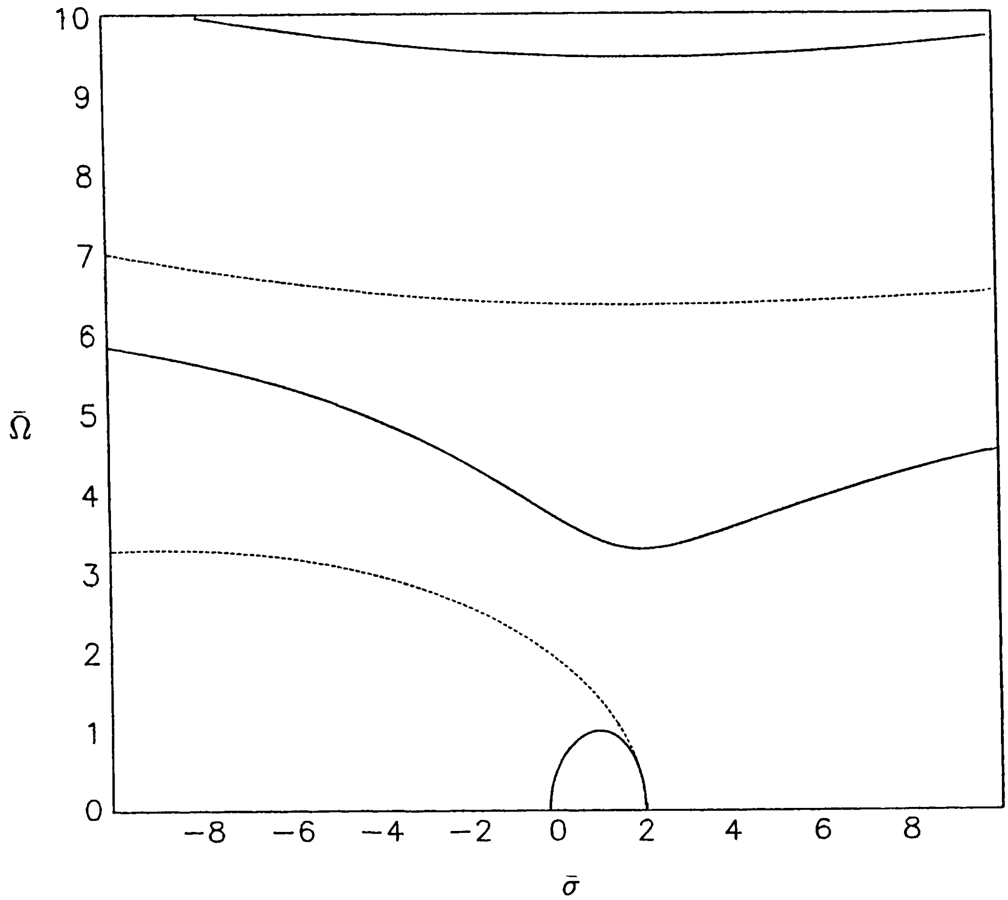

Figure 3.9: Frequency–stress plot, with mode number $\eta = 0.5$, for an equibiaxially deformed incompressible material . The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

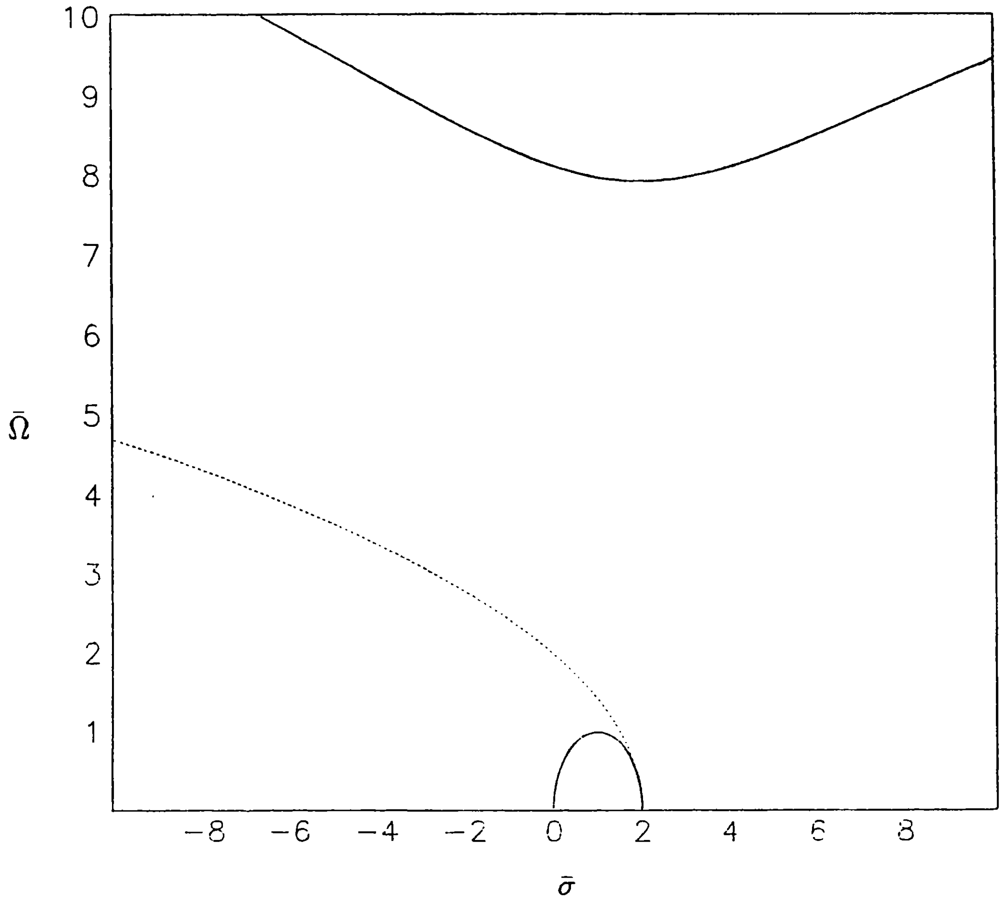

Figure 3.10: Frequency–stress plot, with mode number $\eta = 0.2$, for an equibiaxially deformed incompressible material . The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

3.6 Results for particular strain–energy functions.

In order to illustrate the results of the previous sections and to provide a basis for numerical calculations we now consider the class of strain–energy functions given by

$$W = \frac{\mu}{m}(\lambda_1^m + \lambda_2^m + \lambda_3^m - 3) \ ,$$

where μ and m are constants. It follows from (3.67) that

$$\hat{W}(\lambda,\ \lambda_3) = \frac{\mu}{m}\left(\lambda^m \lambda_3^{-m/2} + \lambda^{-m}\lambda_3^{-m/2} + \lambda_3^m - 3\right) \ , \tag{3.146}$$

and hence from (3.69) that

$$\begin{gathered} c = \mu\lambda_3^{-m/2}(\lambda^m - \lambda^{-m})/(\lambda^4 - 1) \ , \quad a = \lambda^4 c \ , \\ 2b = \mu\lambda_3^{-m/2}\left\{(m-1)\lambda^{m+4} - (m+1)\lambda^m + (m+1)\lambda^{4-m} \right. \\ \left. -(m-1)\lambda^{-m}\right\}/(\lambda^4 - 1) \ . \end{gathered} \tag{3.147}$$

For $c > 0$ we require that

$$\mu m > 0 \ . \tag{3.148}$$

From (3.147) we obtain

$$\begin{aligned} b - \sqrt{ac} = \frac{1}{2}\mu\lambda_3^{-m/2}\{&(m-1)\lambda^{m+2} - (m+1)\lambda^m \\ &+(m+1)\lambda^{2-m} - (m-1)\lambda^{-m}\}/(\lambda^2 - 1) \ . \end{aligned} \tag{3.149}$$

Without loss of generality we take $\mu > 0$ and $m > 0$ for consistency with (3.148) since, as noted in Ogden (1984), the properties of the expression on the right–hand side of (3.149) are unaffected by replacing (μ, m) by $(-\mu, -m)$. It then follows that

$$b - \sqrt{ac}\begin{cases} > 0 & \text{for } m > 1,\ \lambda \neq 1 \ , \\ \equiv 0 & \text{for } m = 1 \ , \\ < 0 & \text{for } 0 < m < 1,\ \lambda \neq 1 \ , \end{cases} \tag{3.150}$$

and $b = \sqrt{ac}$ when $\lambda = 1$ for any m.

For convenience we introduce the notation

$$\hat{\mu} = \mu\lambda_3^{-m/2} \ . \tag{3.151}$$

We now consider two particular values of m, namely $m = 2$ and $m = 1$, corresponding to the neo–Hookean and Varga strain–energy functions respectively.

3.6.1 The neo–Hookean strain–energy function.

Here we have

$$a = \hat{\mu}\lambda^2 \ , \quad c = \hat{\mu}\lambda^{-2} \ , \quad b = \frac{1}{2}(a + c) \ , \tag{3.152}$$

and the inequality $2b > a$ is satisfied. We also have

$$s_1 = 1 \quad , \quad s_2 = \lambda^2\sqrt{1 - \bar{\Omega}^2} \ , \tag{3.153}$$

where $\bar{\Omega}^2$ is defined by (3.127).

Recalling the definitions of p, Ω and η from (3.13), (3.16) and (3.24) respectively, we define corresponding quantities p_0, Ω_0, η_0 which are independent of the deformation by

$$p_0 = \frac{n\pi}{2L_1} \ , \quad \Omega_0^2 = \frac{\rho\omega^2}{p_0^2} \ , \quad \eta_0 = p_0 L_2 \ , \tag{3.154}$$

so that

$$p_0 = (\lambda\lambda_3^{1/2})p \ , \quad \Omega = (\lambda\lambda_3^{1/2})\Omega_0 \ , \quad \eta_0 = \lambda^2\eta \ . \tag{3.155}$$

It follows that, for the neo–Hookean strain–energy function, $\bar{\Omega}^2 = \Omega_0^2/\mu$ is also independent of the deformation.

Cases 1, 2, 6 and 7 are the only ones which arise for this material.

Case 1 : the inequalities $a' > 0$, $b' > \sqrt{a'c'}$ are equivalent to $\bar{\Omega}^2 < 1$ and $\bar{\Omega}^2 \neq 1 - \lambda^{-4}$ respectively, and the frequency equations (3.25) and (3.26) become

$$\frac{\tanh(\eta_0\lambda^{-2})}{\tanh(\eta_0\sqrt{1 - \bar{\Omega}^2})} = \left[\frac{\lambda^2\sqrt{1 - \bar{\Omega}^2}(2 - \lambda^2\bar{\sigma}_2)^2}{(\lambda^4 - \lambda^4\bar{\Omega}^2 + 1 - \lambda^2\bar{\sigma}_2)^2}\right]^{\pm 1} \ , \tag{3.156}$$

where $\bar{\sigma}_2 = \sigma_2/\hat{\mu}$ is the only term depending on λ_3.

When $\Omega = 0$ and $\sigma_2 = 0$ equations (3.156) reduce to

$$\frac{\tanh(\eta_0\lambda^{-2})}{\tanh\eta_0} = \left[\frac{4\lambda^2}{(\lambda^4 + 1)^2}\right]^{\pm 1} \ , \tag{3.157}$$

and this can be shown to agree with the results of Sawyers and Rivlin (1974) for the neo–Hookean material.

Case 2 : in this transitional case we have

$$\bar{\Omega}^2 = 1 - \lambda^{-4} , \tag{3.158}$$

and $s = 1$. Clearly, for real $\bar{\Omega} \neq 0$ we require $\lambda > 1$.

Subcase 2(a) in Section 3.2 corresponds to $\sigma_2 = 2c$, while for subcase 2(b) equation (3.34) becomes

$$\frac{\sinh 2\eta}{2\eta} = \pm\frac{(2c - \sigma_2)}{(2c + \sigma_2)} . \tag{3.159}$$

These are the same equations as apply for $\lambda = 1$ in the quasi–static situation for a general strain–energy function (equations (3.86)), but, of course, the value of c is different here. When $\Omega = 0$ equation (3.158) forces λ to be unity.

Case 6 : here we have $\bar{\Omega}^2 > 1$ and

$$s_1 = 1 \quad s_2^* = \lambda^2\sqrt{\bar{\Omega}^2 - 1} ,$$

and the frequency equations (3.45) and (3.46) become

$$\frac{\tanh(\eta_0\lambda^{-2})}{\tan(\eta_0\sqrt{\bar{\Omega}^2 - 1})} = \mp\left[\frac{\lambda^2\sqrt{\bar{\Omega}^2 - 1}(2 - \lambda^2\bar{\sigma}_2)^2}{(1 + \lambda^4 - \lambda^4\bar{\Omega}^2 - \lambda^2\bar{\sigma}_2)^2}\right]^{\pm 1} , \tag{3.160}$$

with the upper (lower) signs corresponding to antisymmetric (symmetric) modes. This reduces to (3.139) when $\lambda = 1$.

Case 7 : $\bar{\Omega}^2 = 1$ with $s_1 = 1,\ s_2 = 0$.

Here equation (3.52), for symmetric modes, reduces to

$$\frac{\tanh \eta}{\eta} = \frac{(c - \sigma_2)^2}{(2c - \sigma_2)^2} , \tag{3.161}$$

which, by identifying σ_2/c with $\bar{\sigma}$, can be seen to have the same form as (3.143). The equations (3.144) for $\bar{\sigma}$ therefore apply equally to σ_2/c, which, unlike $\bar{\sigma}$, depends on λ. Antisymmetric modes are possible if $\sigma_2 = c$.

3.6.1 The Varga strain–energy function.

In this case $m = 1$,

$$a = \frac{\hat{\mu}\lambda^3}{(\lambda^2+1)} = \lambda^2 b = \lambda^4 c \, , \tag{3.162}$$

and

$$\begin{aligned} s_1^2 &= \lambda^2 - \hat{\Omega}^2/2 + \sqrt{\hat{\Omega}^2\left(\hat{\Omega}^2/4 - \lambda^2 + 1\right)} \, , \\ s_2^2 &= \lambda^2 - \hat{\Omega}^2/2 - \sqrt{\hat{\Omega}^2\left(\hat{\Omega}^2/4 - \lambda^2 + 1\right)} \, , \end{aligned} \tag{3.163}$$

where

$$\hat{\Omega}^2 = \Omega^2/c \, . \tag{3.164}$$

Although it is possible to express the results in terms of (3.154), which are independent of the deformation, this introduces an explicit dependence on λ_3 into the frequency equations and so we do not consider this here.

For this strain–energy function all nine of the cases described in Section 3.2 can occur, and arise as follows:

If $\lambda < 1$ then $s_1^2 > 0$, for all $\hat{\Omega}^2$, and so only Cases 1, 6 and 7 can arise, depending on the sign of s_2^2, which is the same as the sign of $(\lambda^4 - \hat{\Omega}^2)$. When $\lambda = 1$ we have $s_1^2 = 1$ and $s_2^2 = 1 - \hat{\Omega}^2$, so that the results of Section 3.5 hold. However when $\lambda > 1$ we have 3 possible solution paths depending on whether $\lambda^2 < 2$, $\lambda^2 = 2$ or $\lambda^2 > 2$,

i) $1 < \lambda^2 < 2$, here when $\hat{\Omega}^2$ is such that

$$\hat{\Omega}^2 < 4(\lambda^2 - 1) \, , \tag{3.165}$$

the expression (3.163) has complex roots (Case 3) but when equality holds in (3.165), we have $s_1^2 = s_2^2 > 0$ (Case 2). For values of $\hat{\Omega}^2$ such that

$$4(\lambda^2 - 1) < \hat{\Omega}^2 < \lambda^4 \, , \tag{3.166}$$

both roots of (3.14) will be distinct and positive (Case 1), with s_2^2 changing sign as we go through the value $\hat{\Omega}^2 = \lambda^4$ (Case 7 when $\hat{\Omega}^2 = \lambda^4$ and Case 6 for $\hat{\Omega}^2 > \lambda^4$).

ii) $\lambda^2 > 2$, again we have complex roots (Case 3) when (3.165) holds, but this time we have a negative double root when $\hat{\Omega}^2 = 4(\lambda^2 - 1)$ (Case 4). Both roots remain negative (Case 5) when (3.166) holds with s_1^2 changing sign at $\hat{\Omega}^2 = \lambda^4$ (Case 8 then Case 6).

iii) $\lambda^2 = 2$, we again have complex roots while (3.165) holds, but when $\hat{\Omega}^2 = 4(\lambda^2 - 1) = \lambda^4$ we have $s_1^2 = s_2^2 = 0$ (Case 9) with $s_1^2 > 0$ and $s_2^2 < 0$ (Case 6) for all values of $\hat{\Omega}^2$ greater than this.

We note, however, that since, by (3.162), $b = \sqrt{ac}$, only Case 2 is compatible with $\Omega = 0$.

Case 1 : $\quad 4(\lambda^2 - 1) < \hat{\Omega}^2 < \lambda^4 < 2\lambda^2$.

The frequency equations are given by (3.25) and (3.26) with (3.162)–(3.164).

Case 2 : $\quad$ either $\Omega = 0$ or $\hat{\Omega}^2 = 4(\lambda^2 - 1) < \lambda^4 < 2\lambda^2$.

Here we have $s = \sqrt{2 - \lambda^2}$ and we observe that $1 \leq \lambda^2 < 2$. Subcase 2(a) in Section 3.2 corresponds to

$$\hat{\sigma}_2 = \begin{cases} 3 - \lambda^2 & \text{if } \Omega \neq 0\ , \\ \lambda^2 + 1 & \text{if } \Omega = 0\ , \end{cases} \tag{3.167}$$

where $\hat{\sigma}_2 = \sigma_2/c$ and we note that these coincide if $\lambda = 1$.

For subcase 2(b), equation (3.34) gives

$$\frac{\sinh(2\eta\sqrt{2 - \lambda^2})}{2\eta\sqrt{2 - \lambda^2}} = \pm\frac{(3 - \lambda^2) - \hat{\sigma}_2}{(5 - 3\lambda^2) + \hat{\sigma}_2}\ , \tag{3.168}$$

if $\Omega \neq 0$ while (3.82) applies if $\Omega = 0$.

Case 3 : $\quad \hat{\Omega}^2 < 4(\lambda^2 - 1) \leq \lambda^4$.

Here equations (3.36) and (3.37) apply with (3.162), and we note that a necessary condition for this case to hold is $\lambda > 1$.

Case 4 : $2\lambda^2 < \hat{\Omega}^2 = 4(\lambda^2 - 1) < \lambda^4$.

This case applies for $\lambda^2 > 2$; subcase 4(a) corresponds to

$$\hat{\sigma}_2 = 3 - \lambda^2 \ , \tag{3.169}$$

while subcase 4(b) yields

$$\frac{\sin(2\eta\sqrt{\lambda^2 - 2})}{2\eta\sqrt{\lambda^2 - 2}} = \pm\frac{(3 - \lambda^2) - \hat{\sigma}_2}{(5 - 3\lambda^2) + \hat{\sigma}_2} \ . \tag{3.170}$$

Case 5 : $2\lambda^2 < 4(\lambda^2 - 1) < \hat{\Omega}^2 < \lambda^4$.

The frequency equations are given by (3.43) appropriately specialized.

Case 6 : $\hat{\Omega}^2 > \lambda^4$.

The frequency equations are (3.45) and (3.46), again specialized with use of (3.162)–(3.164).

Case 7 : $\hat{\Omega}^2 = \lambda^4 < 2\lambda^2$.

Antisymmetric modes are not possible unless $\sigma_2 = c$, while for $\sigma_2 \neq c$ the frequency equation (3.52) for symmetric modes specializes to

$$\frac{\tanh(\eta\lambda\sqrt{2 - \lambda^2})}{\eta\lambda\sqrt{2 - \lambda^2}} = \frac{(1 - \hat{\sigma}_2)^2}{[\lambda^2(2 - \lambda^2) + 1 - \hat{\sigma}_2]^2} \ . \tag{3.171}$$

Case 8 : $\hat{\Omega}^2 = \lambda^4 > 2\lambda^2$.

Here (3.171) is replaced by

$$\frac{\tan(\eta\lambda\sqrt{\lambda^2 - 2})}{\eta\lambda\sqrt{\lambda^2 - 2}} = \frac{(1 - \hat{\sigma}_2)^2}{[\lambda^2(2 - \lambda^2) + 1 - \hat{\sigma}_2]^2} \ , \tag{3.172}$$

with this time symmetric modes, as well as antisymmetric modes, also possible when $\sigma_2 = c$.

Case 9 : $\hat{\Omega}^2 = \lambda^4 = 2\lambda^2$.

If $\sigma_2 = c$ the antisymmetric and symmetric modes given by (3.51) and (3.60), respectively, can occur, while for $\sigma_2 \neq c$ only symmetric modes can arise, with

$$\hat{\sigma}_2 = (1 - 6\eta^{-2}) \ . \tag{3.173}$$

3.6.3 Numerical Results.

The solutions to the frequency equations obtained in Section 3.6.1 for the neo–Hookean material are shown in Figures 3.11–3.19, while those found in Section 3.6.2 for the Varga material are represented in Figures 3.20–3.28.

In Figures 3.11–3.17, for the incompressible neo–Hookean material, we plot dispersion spectra, that is we plot the modified frequency $\bar{\Omega}$ against the mode number η_0, for several values of the stretch $\lambda = \lambda_1\lambda_3^{1/2}$ and stress $\bar{\sigma}_2$, and we note that these graphs may be compared with Figures 3.1–3.5 in Section 3.5, for which $\lambda = 1$, with the line $\bar{\Omega} = 1$ again corresponding to Case 7 and so dividing the two regions with Case 1 lying below the line and Case 6 above it. It can be seen from these graphs that increasing (decreasing) the stretch λ has a stabilizing (destabilizing) effect on the underlying deformation for a given value of $\bar{\sigma}_2$. By specializing the results of Section 3.3.1 to the neo–Hookean material, we see that the underlying deformation is stable provided

$$\lambda^{-2} - 1 < \bar{\sigma}_2 < \lambda^{-2} + 1 \ ,$$

so that there is only a small range of values of $\bar{\sigma}_2$ where stability is assured. From Figures 3.3, 3.12 and 3.15, with $\bar{\sigma}_2 = 0$, we see in the static limit that the results obtained here agree with Ogden (1984), in that bifurcation modes only occur for $\lambda < 1$. However, if nonzero values of $\bar{\sigma}_2$ are permitted, as Figure 3.14 shows, bifurcation modes may occur for $\lambda > 1$.

Figures 3.17 and 3.18, together with Figure 3.8, show frequency–stress plots for several values of the stretch and we note that varying the mode number does not affect the nature of the graphs obtained but merely alters the spacing of the solution branches, with the spacing decreasing as the mode number increases, much as in the equibiaxial case. In Figure 3.19 we plot the frequency spectrum for $\bar{\Omega}$ against λ and we again note that the nature of this graph does not depend markedly on the parameters η_0 and $\bar{\sigma}_2$, although we do not show the results here.

The results shown in Figures 3.20–3.22 for the Varga material can be compared with Figures 3.12, 3.14 and 3.15 respectively in the neo–Hookean case and the behaviour of both materials is seen to be very similar, even though the variables have a slightly different meaning in each case; although Figure 3.23, for $\lambda = 1.2$ and $\bar{\sigma}_2 = 2$, has the lowest symmetric mode lying beneath the lowest antisymmetric mode, a feature which does not occur for the neo–Hookean material. However, if larger values of the stretch λ are considered the behaviour differs greatly; while the neo–Hookean material behaves much as in Figures 3.16 and 3.17, in the Varga case with $\lambda > \sqrt{2}$, as Figures 3.24 and 3.25 show, there is a layer corresponding to Case 5 in which the solution branches representing the two different modes cross over one another, so that for some values of η the lowest symmetric mode can lie beneath the lowest antisymmetric mode.

These changes in behaviour are also present in the frequency–stress plots; while graphs for $\lambda < 1$, not shown here, behave as those for the neo–Hookean material, when $1 < \lambda < \sqrt{2}$, see Figure 3.26 with $\lambda = 1.2$, the lowest antisymmetric and symmetric solution branches cross–over at a point, with $\hat{\sigma}_2 = 1.56$ and $\hat{\Omega} \approx 1.33$, corresponding to Case 2(a), but again the character of the graph does not change as η is varied. For values of $\lambda > \sqrt{2}$, as Figures 3.27 and 3.28 show for $\lambda = 1.8$, the character of the graph changes markedly depending on the value of the mode number, in that some of the higher modes can also cross over each other for certain values of η.

Figure 3.29 shows how the $(\lambda, \hat{\Omega})$–plane is divided up into the nine cases that arise for the Varga material; the parabola, $\hat{\Omega} = \lambda^2$, (on which Cases 7 and 8 occur) and the hyperbola, $\hat{\Omega} = 2\sqrt{\lambda^2 - 1}$, (on which Cases 2 and 4 occur) divide the four regions corresponding to Cases 1, 3, 5 and 6, with Case 9 arising at the single point where they meet. Figure 3.30 then shows the frequency spectrum obtained for the Varga material, with $\hat{\sigma}_2 = 0$ and $\eta = 1$, and we note that although varying the values of η and $\hat{\sigma}_2$ alters where the solution branches cross, they do not greatly change the overall character of the graph obtained.

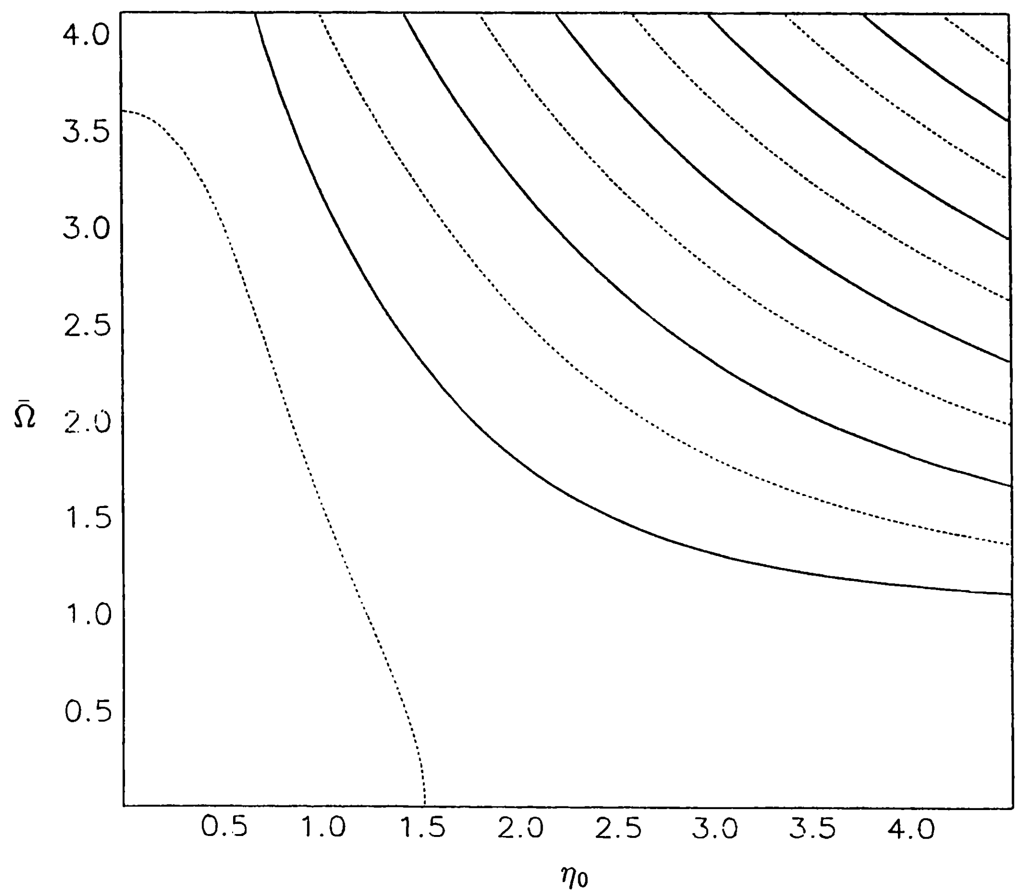

Figure 3.11: The dispersion spectrum for an incompressible neo–Hookean material subjected to a stretch $\lambda = 0.9$ and a stress $\bar{\sigma}_2 = -3$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

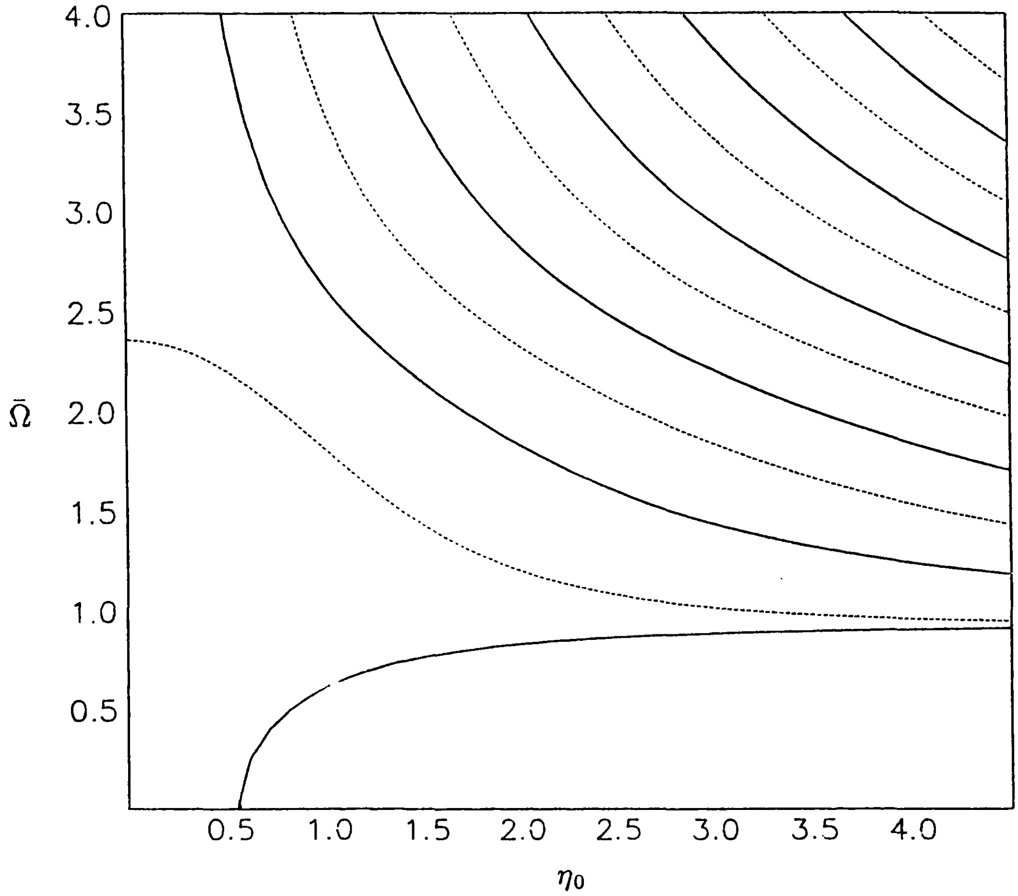

Figure 3.12: The dispersion spectrum for an incompressible neo–Hookean material subjected to a stretch $\lambda = 0.9$ and a stress $\bar{\sigma}_2 = 0$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

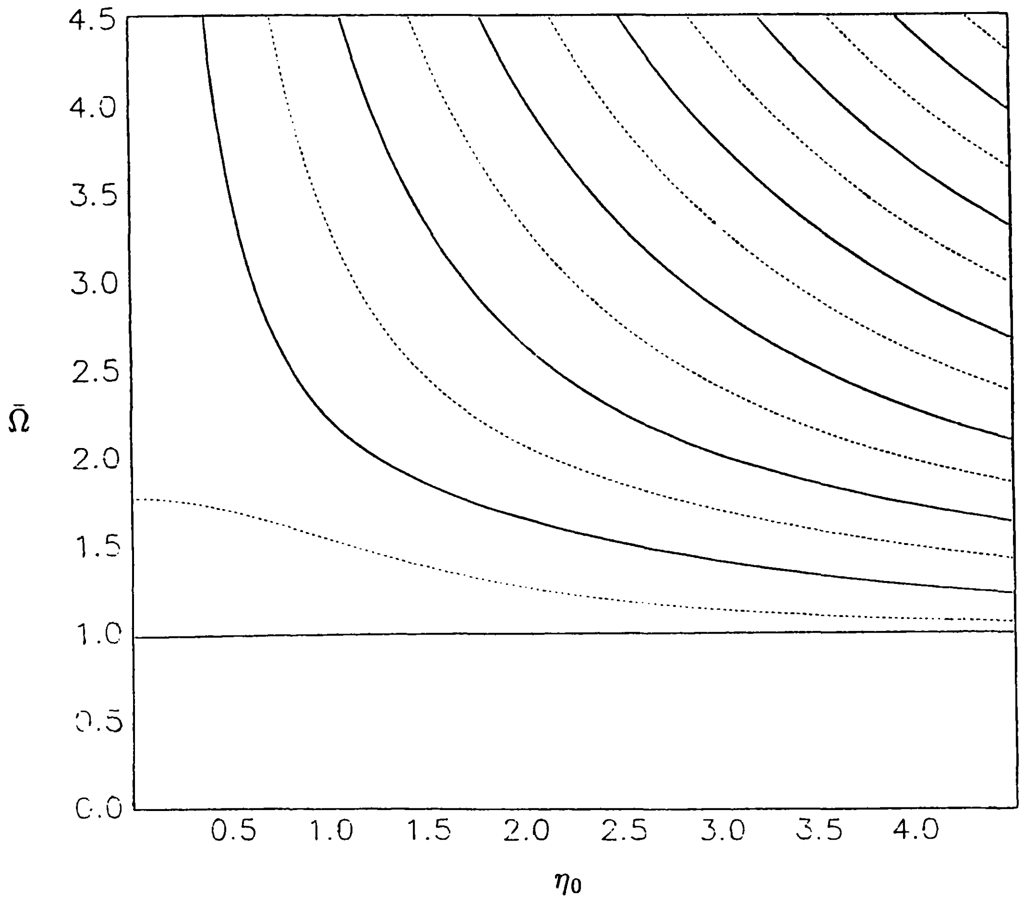

Figure 3.13: The dispersion spectrum for an incompressible neo–Hookean material subjected to a stretch $\lambda = 0.9$ and a stress $\bar{\sigma}_2 = 1$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

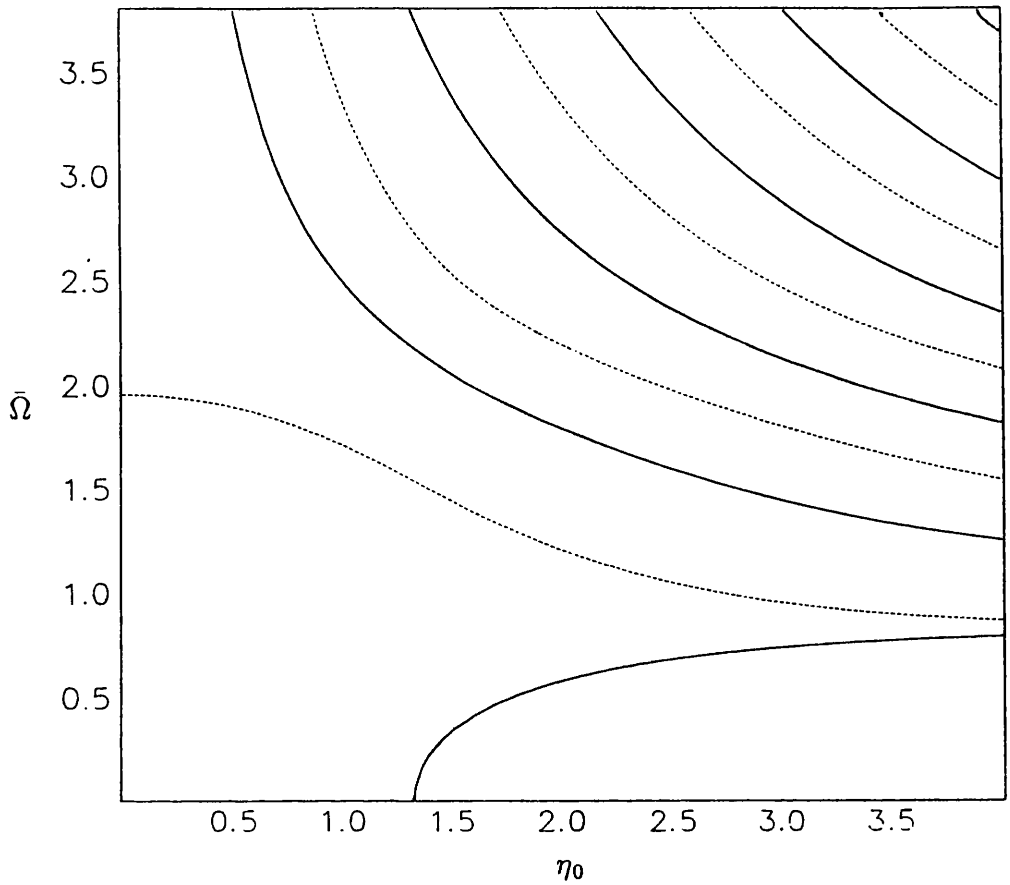

Figure 3.14: The dispersion spectrum for an incompressible neo–Hookean material subjected to a stretch $\lambda = 1.2$ and a stress $\bar{\sigma}_2 = -1$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

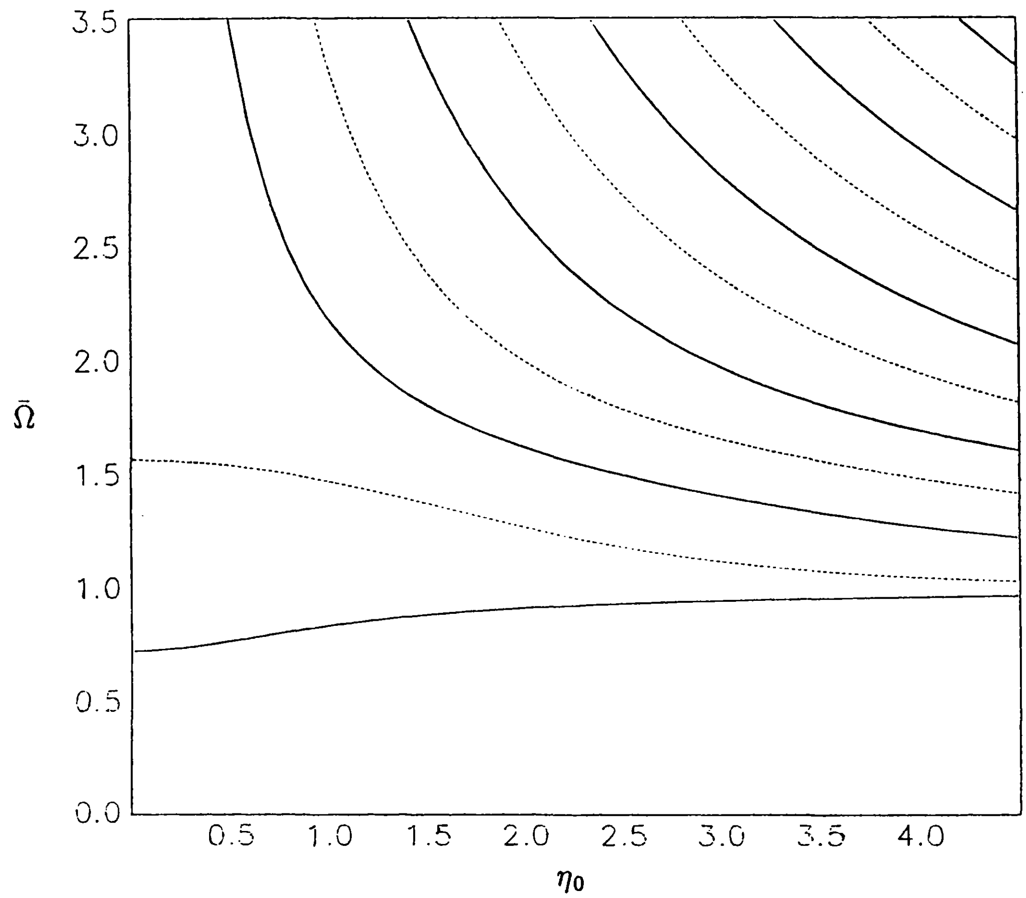

Figure 3.15: The dispersion spectrum for an incompressible neo–Hookean material subjected to a stretch $\lambda = 1.2$ and a stress $\bar{\sigma}_2 = 0$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

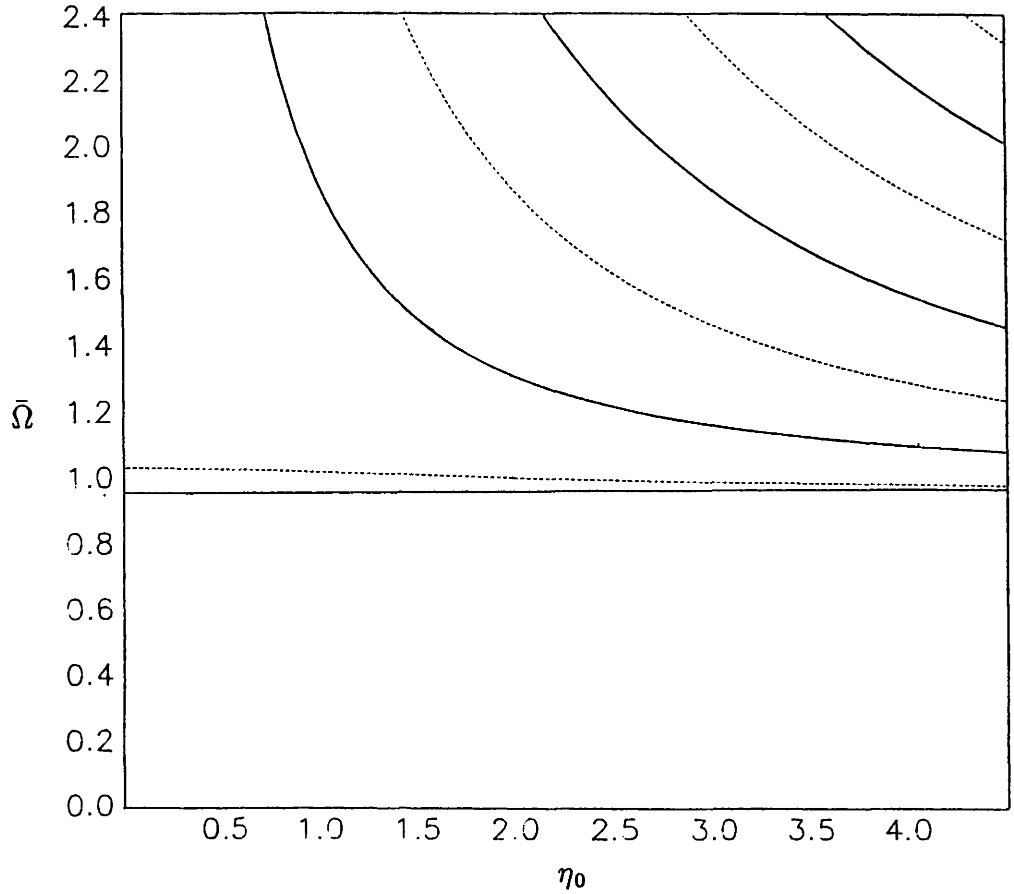

Figure 3.16: The dispersion spectrum for an incompressible neo–Hookean material subjected to a stretch $\lambda = 1.2$ and a stress $\bar{\sigma}_2 = 1$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

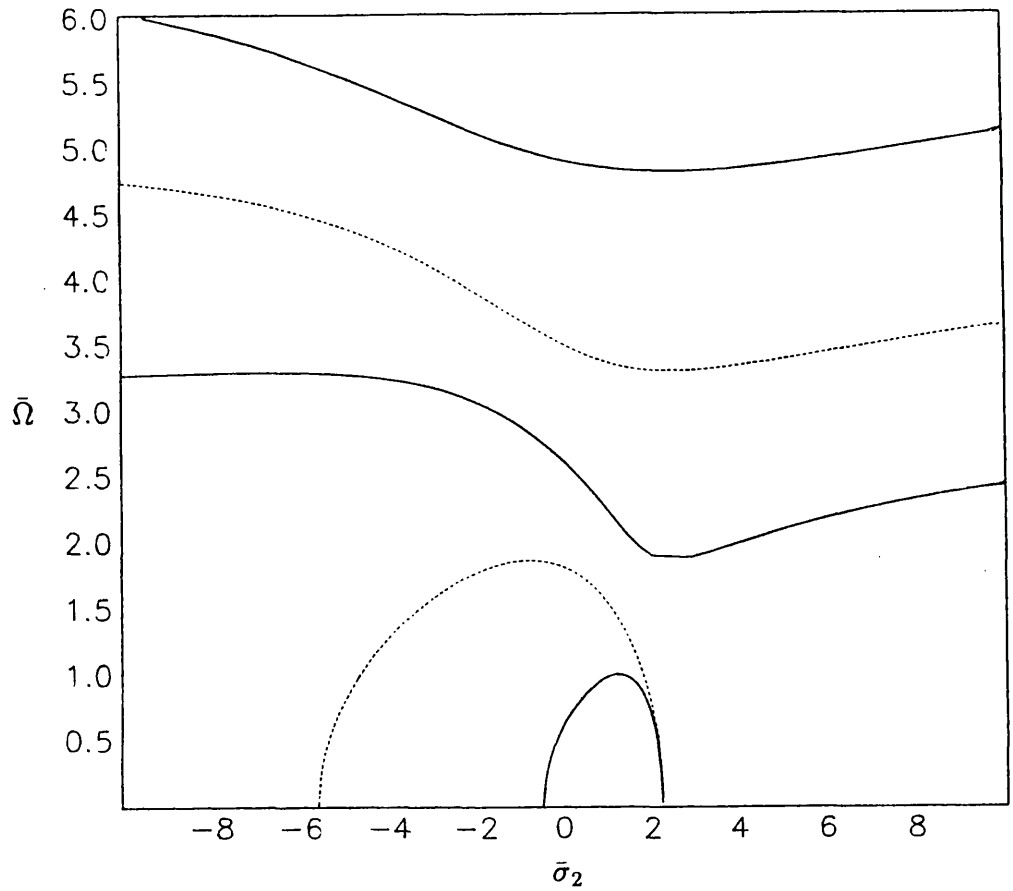

Figure 3.17: Frequency–stress plot for an incompressible neo–Hookean material, with mode number $\eta_0 = 1$, which has been subjected to a stretch $\lambda = 0.9$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

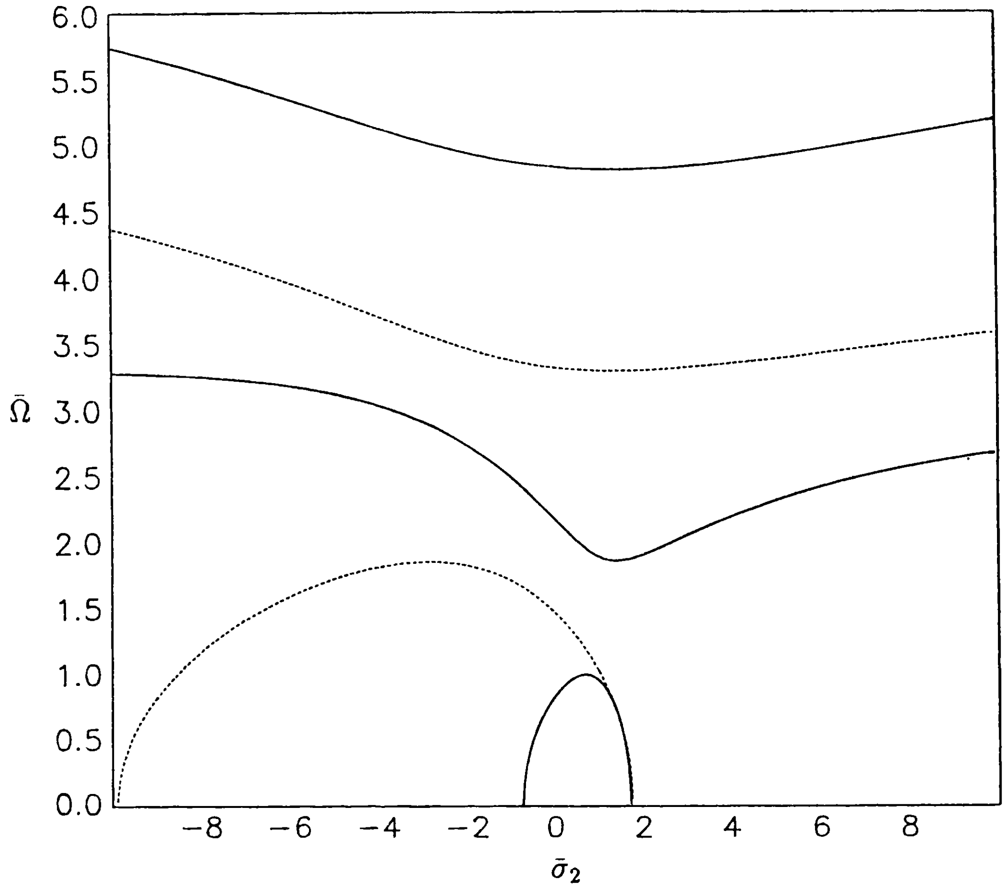

Figure 3.18: Frequency–stress plot for an incompressible neo–Hookean material, with mode number $\eta_0 = 1$, which has been subjected to a stretch $\lambda = 1.2$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

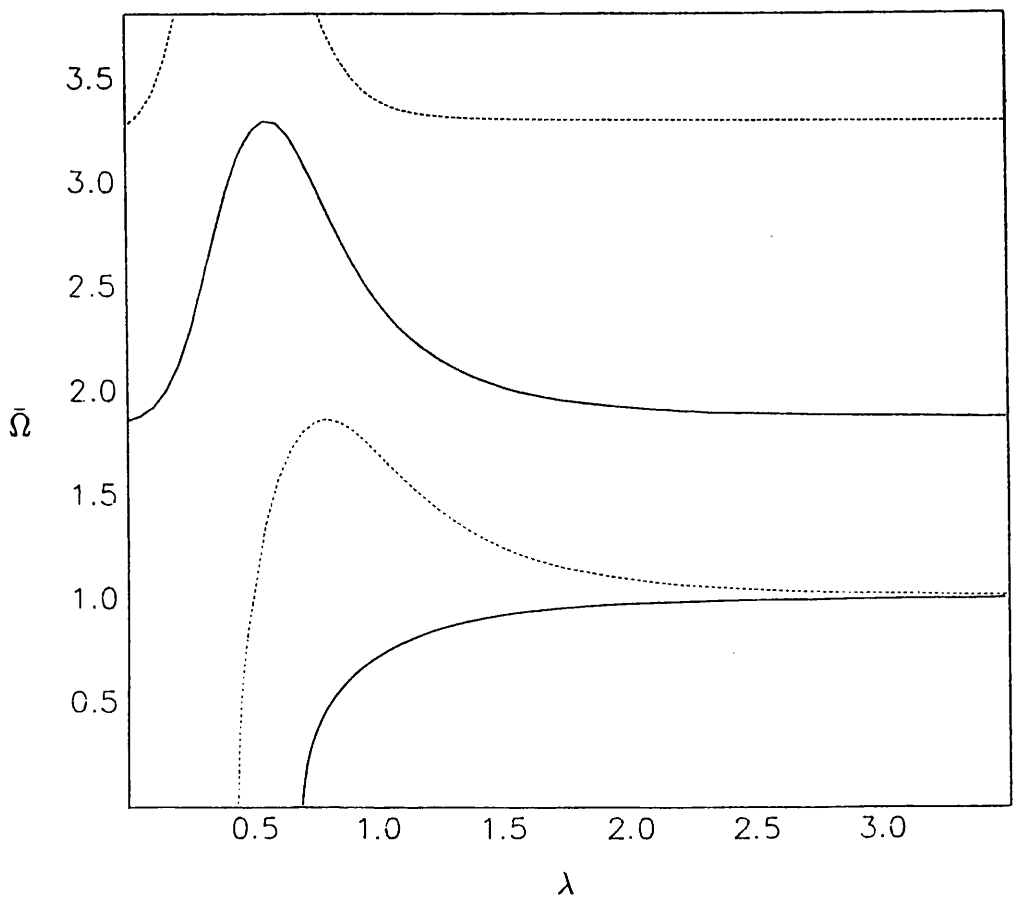

Figure 3.19: The frequency spectrum for an incompressible neo–Hookean material with $\bar{\sigma}_2 = 0$ and $\eta_0 = 1$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

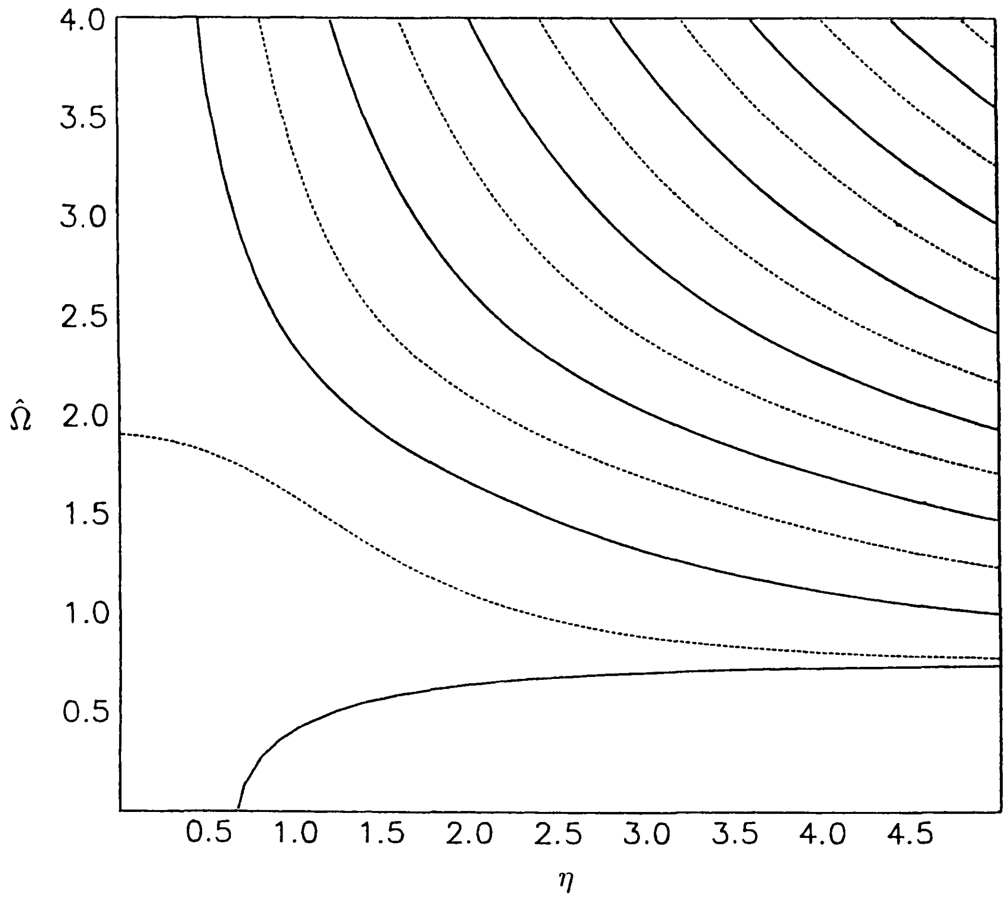

Figure 3.20: The dispersion spectrum for an incompressible Varga material subjected to a stretch $\lambda = 0.9$ with the stress $\hat{\sigma}_2 = \sigma_2/c = 0$, where the solid (broken) curves represent the antisymmetric (symmetric) modes.

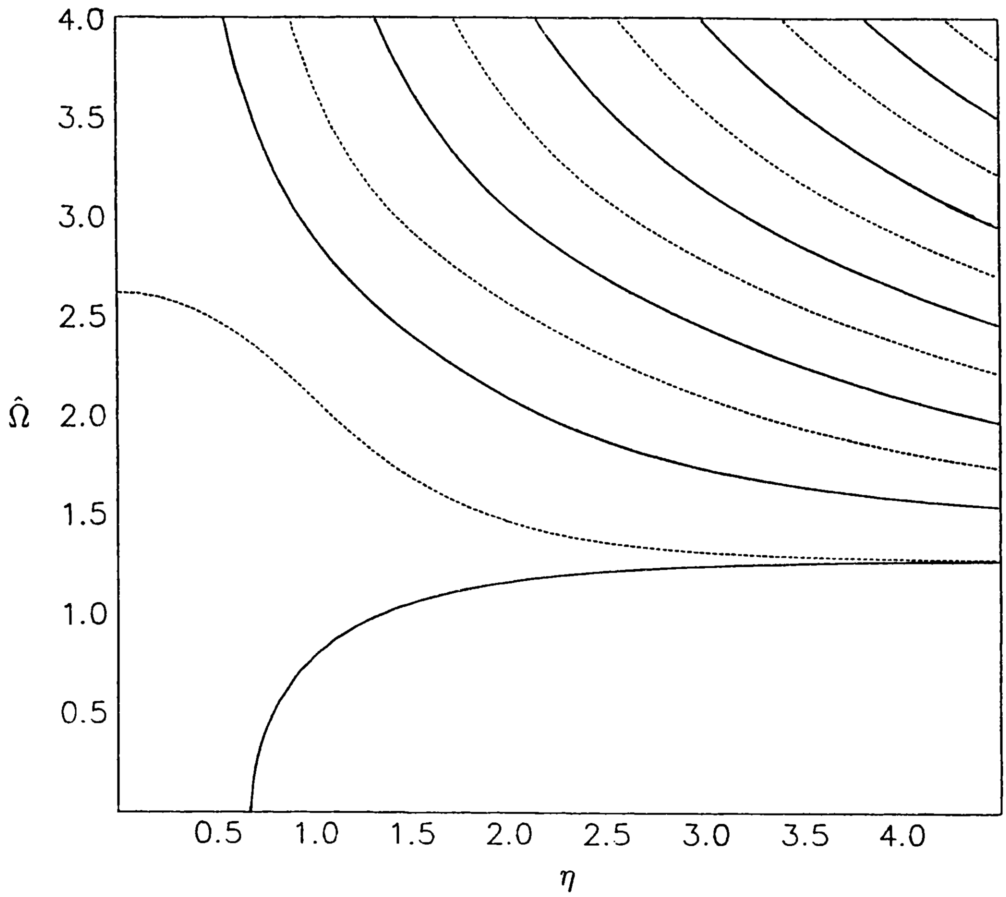

Figure 3.21: The dispersion spectrum for an incompressible Varga material subjected to a stretch $\lambda = 1.2$ and a stress $\hat{\sigma}_2 = -1$, where the solid (broken) curves represent the antisymmetric (symmetric) modes.

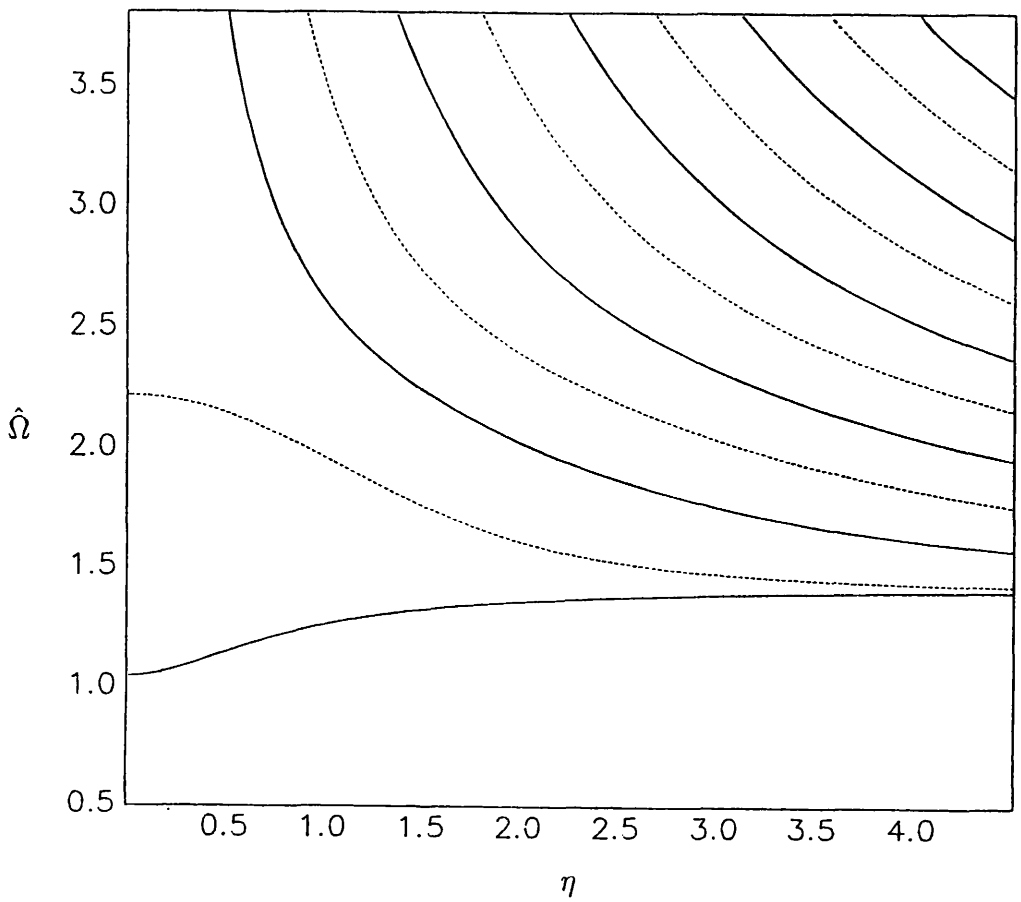

Figure 3.22: The dispersion spectrum for an incompressible Varga material subjected to a stretch $\lambda = 1.2$ with the stress $\hat{\sigma}_2 = 0$, where the solid (broken) curves represent the antisymmetric (symmetric) modes.

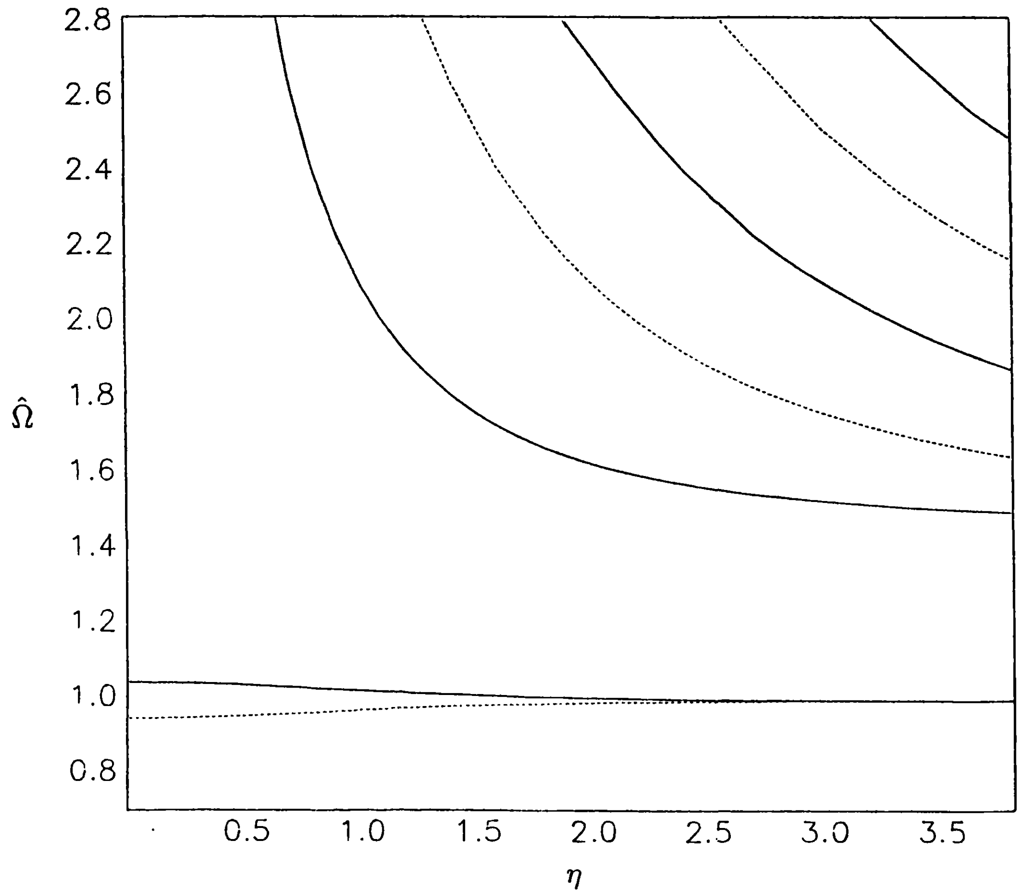

Figure 3.23: The dispersion spectrum for an incompressible Varga material subjected to a stretch $\lambda = 1.2$ with the stress $\hat{\sigma}_2 = 2$, where the solid (broken) curves represent the antisymmetric (symmetric) modes.

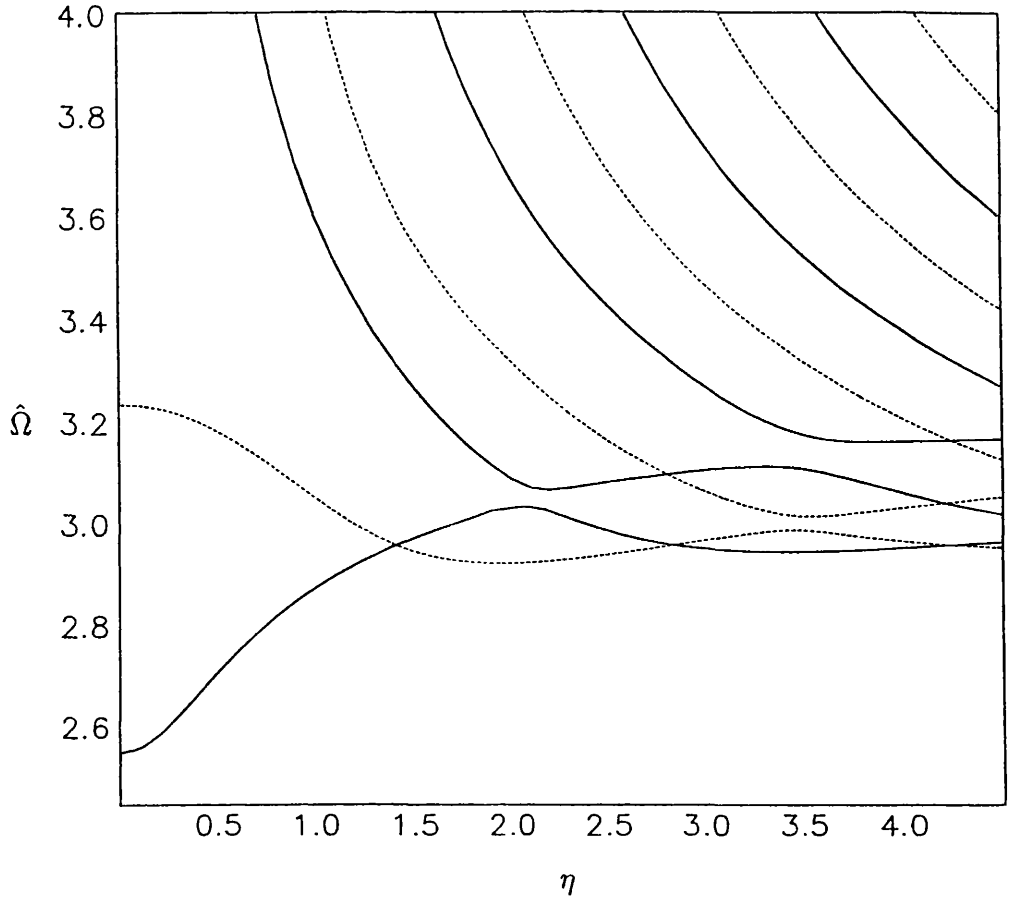

Figure 3.24: The dispersion spectrum for an incompressible Varga material subjected to a stretch $\lambda = 1.8$ and a stress $\hat{\sigma}_2 = -1$, where the solid (broken) curves represent the antisymmetric (symmetric) modes.

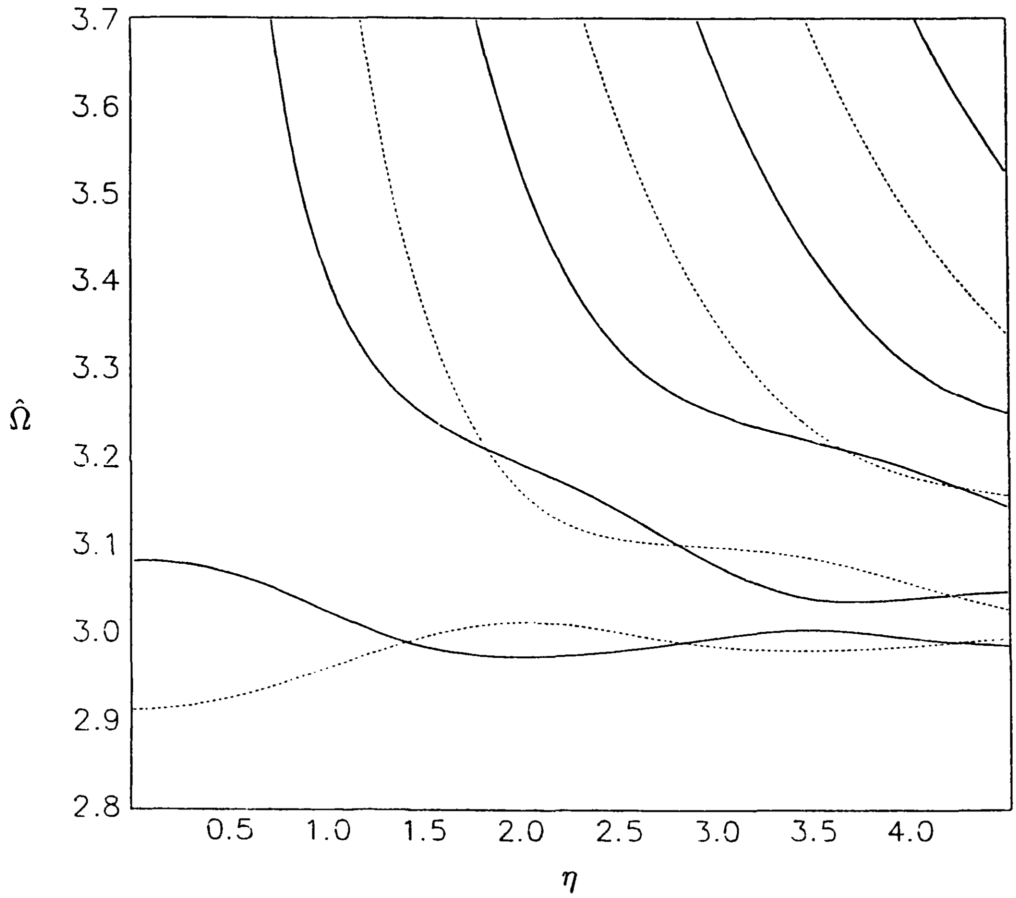

Figure 3.25: The dispersion spectrum for an incompressible Varga material subjected to a stretch $\lambda = 1.8$ with the stress $\hat{\sigma}_2 = 0$, where the solid (broken) curves represent the antisymmetric (symmetric) modes.

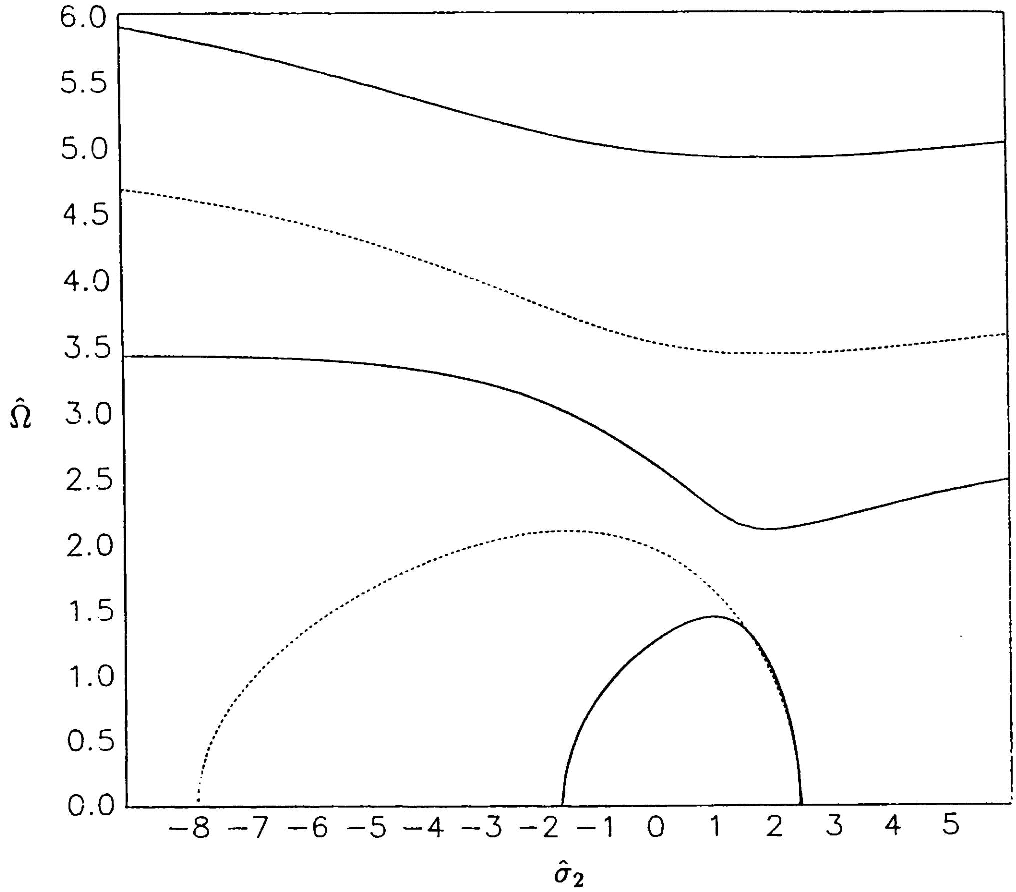

Figure 3.26: Frequency–stress plot for an incompressible Varga material, with mode number $\eta = 1$, which has been subjected to a stretch $\lambda = 1.2$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

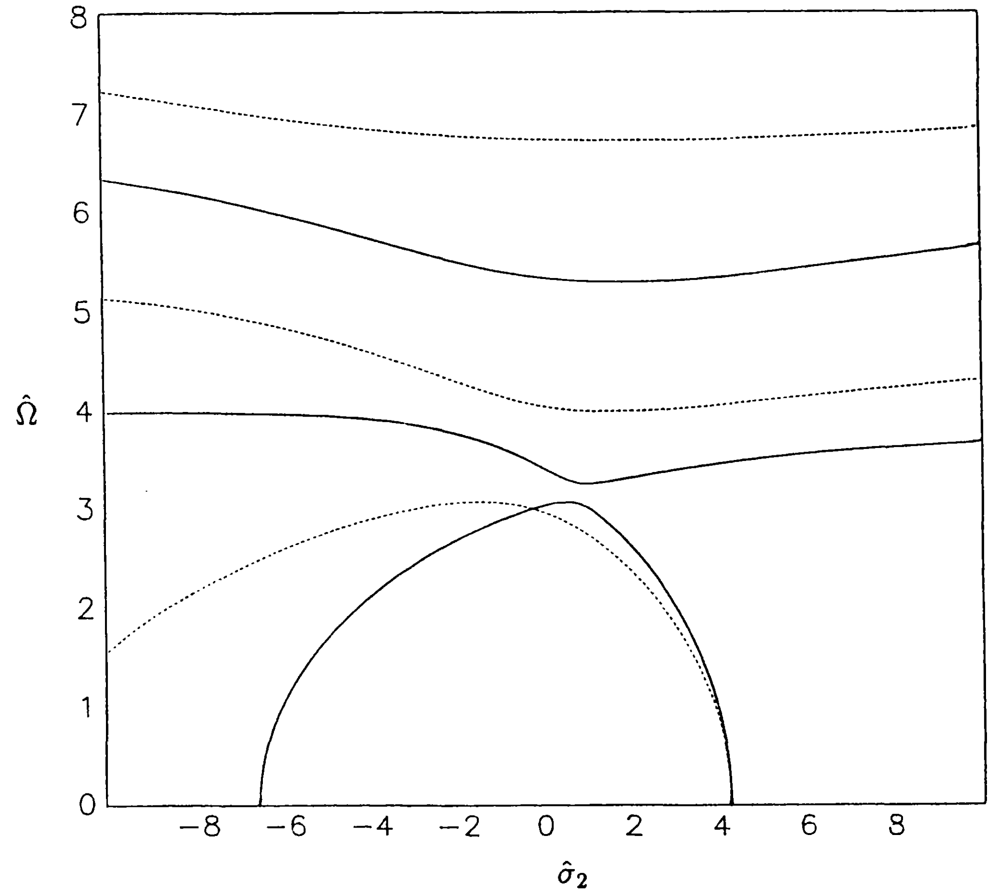

Figure 3.27: Frequency–stress plot for an incompressible Varga material, with mode number $\eta = 1$, which has been subjected to a stretch $\lambda = 1.8$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

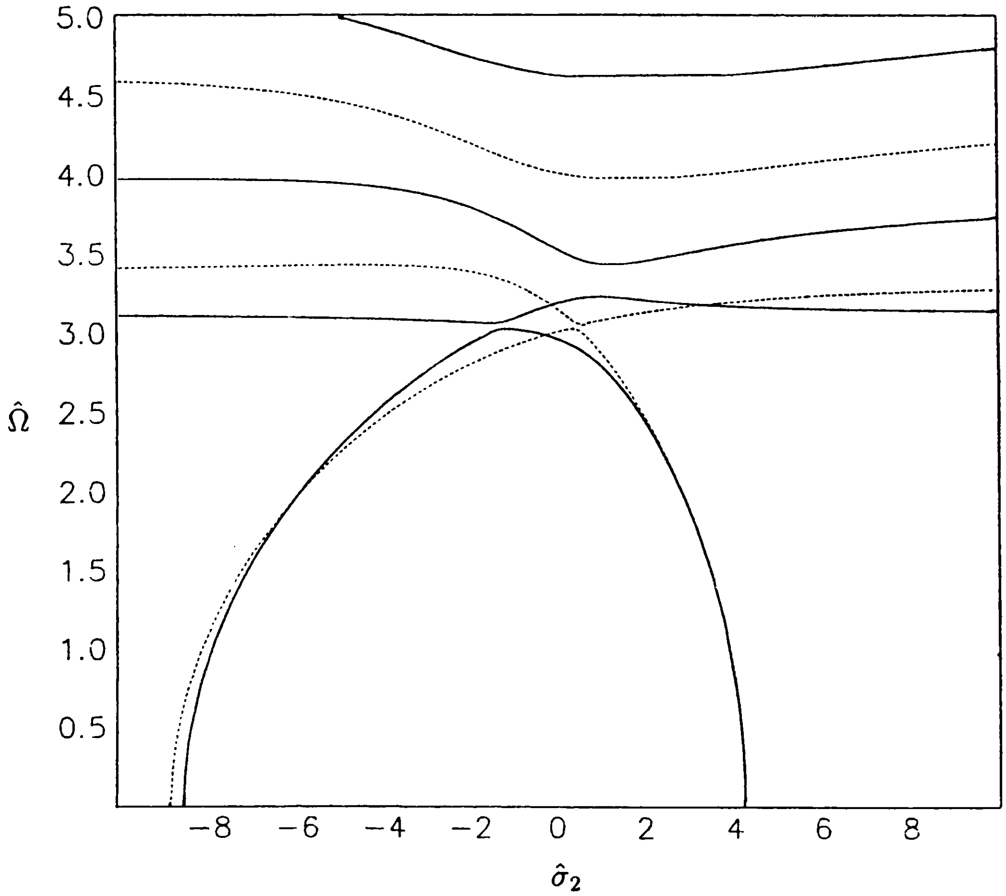

Figure 3.28: Frequency–stress plot for an incompressible Varga material, with mode number $\eta = 2$, which has been subjected to a stretch $\lambda = 1.8$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

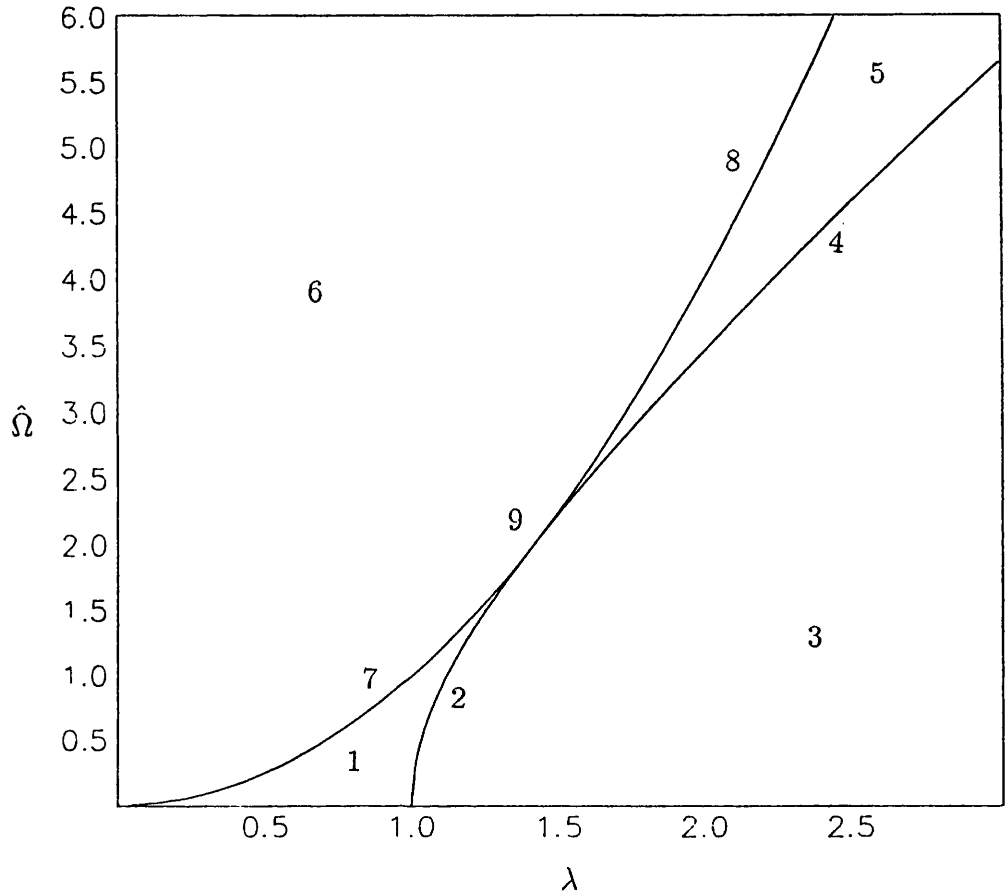

Figure 3.29: Disposal of the nine cases possible for the Varga strain-energy function.

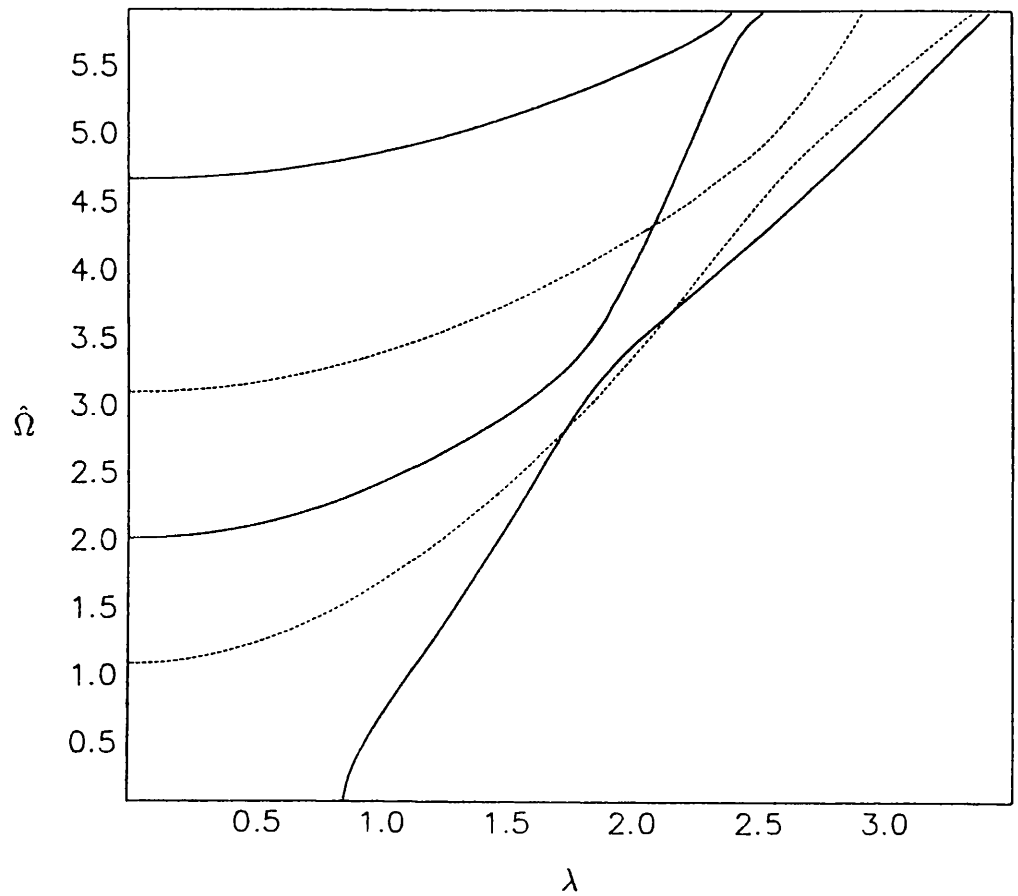

Figure 3.30: The frequency spectrum for an incompressible Varga material with stress $\hat{\sigma}_2 = 0$ and mode number $\eta = 1$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

Chapter 4 – Plane Vibrations of a Compressible Elastic Plate.

4.1 Formulation of the compressible problem.

As in the incompressible case, we assume that when unstressed the body corresponds to the rectangular region defined by

$$-L_i \le X_i \le L_i\ , \quad i \in \{1, 2, 3\}\ ,$$

which is subjected to a pure homogeneous strain

$$x_i = \lambda_i X_i\ , \quad i \in \{1, 2, 3\}\ ,$$

and is deformed into the configuration defined by $-l_i \le x_i \le l_i$, where $l_i = \lambda_i L_i$, ($i \in \{1, 2, 3\}$). We now consider a small time–dependent displacement $\mathbf{v}$ superimposed on this finite deformation and we again restrict our attention to two–dimensional motions with $v_3 = 0$ and v_1 and v_2 independent of x_3. With this the incremental equations of motion (2.51) reduce to

$$\begin{aligned} \mathcal{A}_{01111}v_{1,11} + \mathcal{A}_{02121}v_{1,22} + (\mathcal{A}_{01122} + \mathcal{A}_{02112})v_{2,12} &= \rho v_{1,tt}\ , \\ \mathcal{A}_{01212}v_{2,11} + \mathcal{A}_{02222}v_{2,22} + (\mathcal{A}_{01122} + \mathcal{A}_{02112})v_{1,12} &= \rho v_{2,tt}\ , \end{aligned} \tag{4.1}$$

and we introduce the notation

$$\begin{gathered} \alpha_{ij} = \mathcal{A}_{0iijj}\ , \quad (i \in \{1, 2\}) \\ \gamma_1 = \mathcal{A}_{01212}\ , \qquad \gamma_2 = \mathcal{A}_{02121}\ , \\ \mathcal{A}_{2112} = \mathcal{A}_{02112} = \mathcal{A}_{01221}\ , \\ \delta = \mathcal{A}_{02112} + \mathcal{A}_{01122}\ , \end{gathered} \tag{4.2}$$

and note that for a hyperelastic material $\alpha_{12} = \alpha_{21}$. The two equations of motion (4.1) can be combined to give

$$av_{i,1111} + 2bv_{i,1122} + cv_{i,2222} = \rho(\alpha_{11}+\gamma_1)v_{i,11tt} + \rho(\alpha_{22}+\gamma_2)v_{i,22tt} + \rho^2 v_{i,tttt}\ , \tag{4.3}$$

(with $i \in \{1, 2\}$), where

$$\begin{gathered} a = \alpha_{11}\gamma_1\ , \qquad c = \alpha_{22}\gamma_2\ , \\ 2b = \alpha_{11}\alpha_{22} + \gamma_1\gamma_2 - \delta^2\ . \end{gathered} \tag{4.4}$$

We again assume that the incremental tractions $\dot{S}_{0\,21}$ and $\dot{S}_{0\,22}$ vanish on the sides $x_2 = \pm l_2$, while on the boundaries $x_1 = \pm l_1$, we take $\dot{S}_{0\,12}$ and v_1 to be zero. On using (2.55), restricted to two dimensions, we see that these boundary conditions can be written as

$$\left.\begin{aligned} v_1 &= 0 \ , \\ v_{2,1} &= 0 \ , \end{aligned}\right\} \quad \text{on} \quad x_1 = \pm l_1 \ , \tag{4.5}$$

together with

$$\left.\begin{aligned} \gamma_2 v_{1,2} + \mathcal{A}_{2112} v_{2,1} &= 0 \ , \\ \alpha_{12} v_{1,1} + \alpha_{22} v_{2,2} &= 0 \ , \end{aligned}\right\} \quad \text{on} \quad x_2 = \pm l_2 \ . \tag{4.6}$$

On substituting the definitions (4.2) and (4.4) into the strong–ellipticity conditions (2.60), for a compressible material, it can be seen that necessary and sufficient conditions for strong–ellipticity to hold are

$$\alpha_{ii} > 0 \ , \quad \gamma_i > 0 \ (i \in \{1,\, 2\}), \quad b + \sqrt{ac} > 0 \ , \tag{4.7}$$

which give $a > 0$ and $c > 0$ as necessary conditions for strong–ellipticity to hold.

4.2 Derivation of the frequency equations.

We consider time–harmonic solutions of frequency ω, and look for incremental motions, $\mathbf{v}$, of the form

$$v_i = A_i \exp(spx_2 + ipx_1 - i\omega t) \ , \qquad i \in \{1,\, 2\} \ , \tag{4.8}$$

where A_i ($i \in \{1,\, 2\}$) are arbitrary constants and s and p are to be determined. It is quickly seen that the conditions (4.5), on the boundary $x_2 = \pm l_2$, can be satisfied by solutions of the form

$$\begin{aligned} v_1 &= \phi_1(x_2) \begin{Bmatrix} -\sin(px_1) \\ \cos(px_1) \end{Bmatrix} e^{-i\omega t} \ , \\ v_2 &= \phi_2(x_2) \begin{Bmatrix} \cos(px_1) \\ \sin(px_1) \end{Bmatrix} e^{-i\omega t} \ , \end{aligned} \qquad \text{where } p = \frac{n\pi}{2l_1}, \text{ and } n = \begin{cases} 0,\, 2,\, 4 \ldots \\ 1,\, 3,\, 5 \ldots \end{cases} \tag{4.9}$$

for arbitrary functions $\phi_1(x_2)$ and $\phi_2(x_2)$. On taking linear combinations of expressions like (4.8), so that they are of the form (4.9), and substituting these into the equations of motion (4.3), we obtain the polynomial

$$cp^4s^4 - 2bp^4s^2 + ap^4 = \rho(\alpha_{11} + \gamma_1)p^2\omega^2 - \rho(\alpha_{22} + \gamma_2)p^2\omega^2 - \rho^2\omega^4 \ , \tag{4.10}$$

which is a quadratic in s^2. On writing

$$\Omega^2 = \rho\omega^2/p^2 \ , \tag{4.11}$$

equation (4.10) can be expressed as

$$cs^4 - \left(2b - (\alpha_{22} + \gamma_2)\Omega^2\right)s^2 + (\alpha_{11} - \Omega^2)(\gamma_1 - \Omega^2) = 0 \ . \tag{4.12}$$

Equation (4.12) yields four solutions for s, the nature of which being determined by the underlying deformation and the frequency ω. The general solutions are then found by taking suitable linear combinations of (4.8) and then satisfying the boundary conditions (4.6) together with a second equation of motion from (4.1). Again, as in the incompressible case, nine possible cases arise for deformations within the strong–ellipticity domain, and they are labelled, as in Chapter 3, according to the nature of the solutions s_1^2 and s_2^2 of the quadratic (4.12).

It is worth noting that by writing $\bar{\alpha}_{11} = \alpha_{11} - \Omega^2$ and $\bar{\gamma}_1 = \gamma_1 - \Omega^2$ equation (4.12) can be written in the form

$$c's^4 - 2b's^2 + a' = 0 \ , \tag{4.13}$$

where

$$\begin{aligned} c' &= c = \alpha_{22}\gamma_2 \ , \\ 2b' &= \bar{\alpha}_{11}\alpha_{22} + \bar{\gamma}_1\gamma_2 - \delta^2 \ , \\ a' &= \bar{\alpha}_{11}\bar{\gamma}_1 \ , \end{aligned} \tag{4.14}$$

so that the frequency dependence of (4.13) occurs only through the terms $\bar{\alpha}_{11}$ and $\bar{\gamma}_1$. On making the associations $\bar{\alpha}_{11} \leftrightarrow \alpha_{11}$ and $\bar{\gamma}_1 \leftrightarrow \gamma_1$ we see that (4.13) and (4.14) reduce to the results in Ogden (1984, Section 6.3) for the static case, but this time, even when strong–ellipticity holds, we may have $\bar{\alpha}_{11} < 0$ or $\bar{\gamma}_1 < 0$ (or both).

Also note that

$$s_1^2 s_2^2 = \frac{a'}{c} = \frac{\bar{\alpha}_{11}\bar{\gamma}_1}{\alpha_{22}\gamma_2} \ , \tag{4.15}$$

and

$$s_1^2 + s_2^2 = \frac{2b'}{c} = \frac{(\bar{\alpha}_{11}\alpha_{22} + \bar{\gamma}_1\gamma_2 - \delta^2)}{(\alpha_{22}\gamma_2)} \ . \tag{4.16}$$

We now consider each of these cases in turn:

Case 1: $a' > 0, \quad b' > \sqrt{a'c}.$

In this case the roots s_1^2 and s_2^2 of (4.13) are distinct and positive, and are given by

$$s_1^2 = \frac{b' + \sqrt{b'^2 - a'c}}{c} \ , \qquad s_2^2 = \frac{b' - \sqrt{b'^2 - a'c}}{c} \ , \tag{4.17}$$

with s_1 and s_2 the positive square roots of these expressions. The general solution can be written in the form (4.9) with $\phi_1(x_2)$ and $\phi_2(x_2)$ defined by

$$\begin{aligned} \phi_1(x_2) &= A_1 \sinh(s_1 p x_2) + B_1 \cosh(s_1 p x_2) + C_1 \sinh(s_2 p x_2) \\ &\qquad + D_1 \cosh(s_2 p x_2) \\ \phi_2(x_2) &= A_2 \cosh(s_1 p x_2) + B_2 \sinh(s_1 p x_2) + C_2 \cosh(s_2 p x_2) \\ &\qquad + D_2 \sinh(s_2 p x_2) \end{aligned} \tag{4.18}$$

where the A_i, B_i, C_i and D_i ($i \in \{1, 2\}$) are constants. On substituting (4.18) and (4.9) into the equation of motion $(4.1)_1$, we obtain the identities

$$\begin{aligned} (\bar{\alpha}_{11} - \gamma_2 s_1^2) \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} &= \delta s_1 \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \ , \\ (\bar{\alpha}_{11} - \gamma_2 s_2^2) \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} &= \delta s_2 \begin{bmatrix} C_2 \\ D_2 \end{bmatrix} \ , \end{aligned} \tag{4.19}$$

while on substituting (4.18) and (4.9) into the remaining boundary conditions (4.6), on $x_2 = \pm l_2$, we get

$$\begin{aligned} (\gamma_2 s_1 A_1 + \mathcal{A}_{2112} A_2)\cosh(s_1\eta) + (\gamma_2 s_2 C_1 + \mathcal{A}_{2112} C_2)\cosh(s_2\eta) = 0 \ , \\ (\alpha_{12} A_1 - s_1 \alpha_{22} A_2)\sinh(s_1\eta) + (\alpha_{12} C_1 - s_2 \alpha_{22} C_2)\sinh(s_2\eta) = 0 \ , \end{aligned} \tag{4.20}$$

$$
\begin{aligned}
(\gamma_2 s_1 B_1 + \mathcal{A}_{2112} B_2)\sinh(s_1\eta) + (\gamma_2 s_2 D_1 + \mathcal{A}_{2112} D_2)\sinh(s_2\eta) = 0 \ , \\
(\alpha_{12} B_1 - s_1\alpha_{22} B_2)\cosh(s_1\eta) + (\alpha_{12} D_1 - s_2\alpha_{22} D_2)\cosh(s_2\eta) = 0 \ ,
\end{aligned} \tag{4.21}
$$

where $\eta = pl_2$. It can be seen from (4.19)–(4.21) that, as in the incompressible case, the equations decouple into two systems in $(A_i,\ C_i)$ and $(B_i,\ D_i)$ respectively and so we can treat them independently. Considering first the situation where $B_i = D_i = 0$ $(i \in \{1,\ 2\})$ we see from (4.9) and (4.18) that v_1 is an odd function of x_2 and v_2 is an even function of x_2, so that this gives rise to **antisymmetric** (or flexural) modes. Using the identities (4.19) in (4.20) and taking the determinant of the coefficients to be zero, we obtain the frequency equation for antisymmetric modes,

$$
\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \frac{s_1(\mathcal{A}_{2112}\bar{\alpha}_{11} + \alpha_{12}\gamma_2 s_2^2)(\alpha_{12}\delta - \bar{\alpha}_{11}\alpha_{22} + \gamma_2\alpha_{22}s_1^2)}{s_2(\mathcal{A}_{2112}\bar{\alpha}_{11} + \alpha_{12}\gamma_2 s_1^2)(\alpha_{12}\delta - \bar{\alpha}_{11}\alpha_{22} + \gamma_2\alpha_{22}s_2^2)} \ ,
$$

which, following the corresponding result in Ogden (1984) for the static case, can be simplified to

$$
\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \frac{s_2(\mathcal{A} - s_1^2)}{s_1(\mathcal{A} - s_2^2)} \ , \tag{4.22}
$$

where

$$
\mathcal{A} = \frac{\bar{\alpha}_{11}(\bar{\gamma}_1\gamma_2 - \mathcal{A}_{2112}^2)}{\gamma_2(\bar{\alpha}_{11}\alpha_{22} - \alpha_{12}^2)} \ . \tag{4.33}
$$

Alternatively, on considering the situation where $A_i = C_i = 0$ $(i \in \{1,\ 2\})$ we find, from (4.18) and (4.9), that v_1 is an even function of x_2 while v_2 is an odd function of x_2, which gives rise to **symmetric** (or barreling) modes. In this case, the frequency equation for symmetric modes is given by

$$
\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \frac{s_1(\mathcal{A} - s_2^2)}{s_2(\mathcal{A} - s_1^2)} \ . \tag{4.24}
$$

It is worth noting that (4.22) and (4.24) can be written jointly as

$$
\frac{\sinh(s_1 + s_2)\eta}{(s_1 + s_2)}[s_1 s_2 + \mathcal{A}] = \pm\frac{\sinh(s_1 - s_2)\eta}{(s_1 - s_2)}[s_1 s_2 - \mathcal{A}] \ , \tag{4.25}
$$

where the $+(-)$ corresponds to antisymmetric (symmetric) modes.

Case 2: $a' > 0, \quad b' = \sqrt{a'c}.$

Here (4.13) has a positive double root s^2 with

$$s^2 = b'/c = \sqrt{a'/c}\,, \tag{4.26}$$

with s taken to be the positive square root. The general solution is again of the form (4.9) but this time

$$\begin{aligned}\phi_1(x_2) &= A_1 \sinh(spx_2) + B_1 \cosh(spx_2) + C_1 spx_2 \cosh(spx_2) \\ &\qquad + D_1 spx_2 \sinh(spx_2)\,, \\ \phi_2(x_2) &= A_2 \cosh(spx_2) + B_2 \sinh(spx_2) + C_2 spx_2 \sinh(spx_2) \\ &\qquad + D_2 spx_2 \cosh(spx_2)\,.\end{aligned} \tag{4.27}$$

Substituting (4.27) with (4.9) into the equation of motion $(4.1)_1$ gives the identities

$$\begin{aligned}(\bar{\alpha}_{11} - \gamma_2 s^2)\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} &= \delta s \begin{bmatrix} C_2 \\ D_2 \end{bmatrix}, \\ (\bar{\alpha}_{11} - \gamma_2 s^2)\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} - (\bar{\alpha}_{11} + \gamma_2 s^2)\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} &= \delta s \begin{bmatrix} A_2 \\ B_2 \end{bmatrix},\end{aligned} \tag{4.28}$$

and using (4.28) together with (4.27) and (4.9), the boundary conditions on the sides, $x_2 = \pm l_2$, can be written as

$$\begin{aligned}m\cosh(s\eta)A_1 + [n\cosh(s\eta) + s\eta m\sinh(s\eta)]\,C_1 &= 0\,, \\ r\sinh(s\eta)A_1 + [t\sinh(s\eta) + s\eta r\cosh(s\eta)]\,C_1 &= 0\,,\end{aligned} \tag{4.29}$$

$$\begin{aligned}m\sinh(s\eta)B_1 + [n\sinh(s\eta) + s\eta m\cosh(s\eta)]\,D_1 &= 0\,, \\ r\cosh(s\eta)B_1 + [t\cosh(s\eta) + s\eta r\sinh(s\eta)]\,D_1 &= 0\,,\end{aligned} \tag{4.30}$$

where

$$m = \alpha_{12}\gamma_2 s^2 + \bar{\alpha}_{11}\mathcal{A}_{2112}\,, \qquad n = \alpha_{12}\gamma_2 s^2 - \bar{\alpha}_{11}\mathcal{A}_{2112}\,,$$

$$r = \bar{\alpha}_{11}\alpha_{22} - \alpha_{12}\delta - \gamma_2\alpha_{22}s^2\,, \qquad t = -2\gamma_2\alpha_{22}s^2\,.$$

After considerable algebraic manipulation, it can be shown from (4.29), with $B_i = D_i = 0$ $(i \in \{1, 2\})$, that the frequency equation for antisymmetric modes is

$$\frac{\sinh(2s\eta)}{2s\eta} = \frac{s^2 - \mathcal{A}}{s^2 + \mathcal{A}}\,, \tag{4.31}$$

where $\mathcal{A}$ is again defined by (4.23). While from (4.30) we find that the frequency equation for symmetric modes is

$$\frac{\sinh(2s\eta)}{2s\eta} = -\frac{s^2 - \mathcal{A}}{s^2 + \mathcal{A}} \,. \tag{4.32}$$

Note that (4.31) and (4.32) can be derived from (4.25) by taking the limit as $s_1 \to s_2$.

Case 3: $b'^2 - a'c < 0$.

In this case s_1^2 and s_2^2 are complex conjugates, and we write

$$s_1 = \gamma + i\epsilon \,, \qquad s_2 = \gamma - i\epsilon \,, \tag{4.33}$$

where

$$\gamma = \left(\frac{b' + \sqrt{a'c}}{2c}\right)^{1/2} \,, \qquad \epsilon = \left(\frac{\sqrt{a'c} - b'}{2c}\right)^{1/2} \,. \tag{4.34}$$

The general solution for this case can again be written in the form (4.9) with (4.18), but this time it will involve trigonometric as well as hyperbolic terms and so we do not list it here. However, the frequency equations can be found by substituting (4.34) into (4.22) and (4.24) (alternatively (4.25)) to give

$$\frac{\sinh(2\gamma\eta)}{\sin(2\epsilon\eta)} = \pm\frac{\gamma(\gamma^2 + \epsilon^2 - \mathcal{A})}{\epsilon(\gamma^2 + \epsilon^2 + \mathcal{A})} \,, \tag{4.35}$$

where the $+(-)$ corresponds to the antisymmetric (symmetric) modes.

Case 4: $a' > 0, \quad b' = -\sqrt{a'c}$.

In this case, we find that s^2 is the unique negative double root of (4.13) and so we set $s = is^*$, where s^* is positive and $s^{*2} = \sqrt{a'/c}$. This time we have

$$\begin{aligned} \phi_1(x_2) &= A_1 \sin(s^* p x_2) + B_1 \cos(s^* p x_2) + C_1 s^* p x_2 \cos(s^* p x_2) \\ &\qquad + D_1 s^* p x_2 \sin(s^* p x_2) \,, \\ \phi_2(x_2) &= A_2 \cos(s^* p x_2) + B_2 \sin(s^* p x_2) + C_2 s^* p x_2 \sin(s^* p x_2) \\ &\qquad + D_2 s^* p x_2 \cos(s^* p x_2) \,, \end{aligned} \tag{4.36}$$

and the frequency equations for this case can be found by putting $s = is^*$ into the results of Case 2, namely

$$\frac{\sin(2s^*\eta)}{2s^*\eta} = \pm\frac{s^{*2} + \mathcal{A}}{s^{*2} - \mathcal{A}} , \tag{4.37}$$

with the $+(-)$ sign corresponding to antisymmetric (symmetric) modes.

Case 5: $a' > 0, \quad b' < \sqrt{a'c}$.

This time s_1^2 and s_2^2 are distinct and negative, so we write

$$s_1 = is_1^* , \qquad s_2 = is_2^* , \tag{4.38}$$

where s_1^* and s_2^* are positive. The general solution is given by (4.9) with this time

$$\begin{aligned} \phi_1(x_2) &= A_1 \sin(s_1^* p x_2) + B_1 \cos(s_1^* p x_2) + C_1 \sin(s_2^* p x_2) \\ &\qquad + D_1 \cos(s_2^* p x_2) , \\ \phi_2(x_2) &= A_2 \cos(s_1^* p x_2) + B_2 \sin(s_1^* p x_2) + C_2 \cos(s_2^* p x_2) \\ &\qquad + D_2 \sin(s_2^* p x_2) , \end{aligned} \tag{4.39}$$

and on putting (4.38) into (4.22) and (4.24) the frequency equations become

$$\frac{\tan(s_1^*\eta)}{\tan(s_2^*\eta)} = \left[\frac{s_2^*(\mathcal{A} + s_1^{*2})}{s_1^*(\mathcal{A} + s_2^{*2})}\right]^{\pm 1} , \tag{4.40}$$

where $+(-)$ corresponds to antisymmetric (symmetric) modes. Note that corresponding to (4.25), the frequency equations can be written in the form

$$\frac{\sin(s_1^* + s_2^*)\eta}{(s_1^* + s_2^*)}[s_1^* s_2^* - \mathcal{A}] = \pm\frac{\sin(s_1^* - s_2^*)\eta}{(s_1^* - s_2^*)}[s_1^* s_2^* + \mathcal{A}] . \tag{4.41}$$

Case 6: $a' < 0$.

In this case one of s_1^2, s_2^2 is positive and the other is negative. We take $s_1^2 > 0$ and set $s_2 = is_2^*$, where s_2^* is taken to be positive, so that the general solution is given by (4.9) together with

$$\begin{aligned} \phi_1(x_2) &= A_1 \sinh(s_1 p x_2) + B_1 \cosh(s_1 p x_2) + C_1 \sin(s_2^* p x_2) \\ &\qquad + D_1 \cos(s_2^* p x_2) , \\ \phi_2(x_2) &= A_2 \cosh(s_1 p x_2) + B_2 \sinh(s_1 p x_2) + C_2 \cos(s_2^* p x_2) \\ &\qquad + D_2 \sin(s_2^* p x_2) . \end{aligned} \tag{4.42}$$

This time the frequency equation for antisymmetric modes is given by

$$\frac{\tanh(s_1\eta)}{\tan(s_2^*\eta)} = -\frac{s_2^*(\mathcal{A} - s_1^2)}{s_1(\mathcal{A} + s_2^{*2})} , \tag{4.43}$$

and the frequency equation for symmetric modes is

$$\frac{\tanh(s_1\eta)}{\tan(s_2^*\eta)} = \frac{s_1(\mathcal{A} + s_2^{*2})}{s_2^*(\mathcal{A} - s_1^2)} . \tag{4.44}$$

Note that in this case the frequency equations can not be written in a form similar to (4.25).

Case 7: $a' = 0, \quad b' > 0.$

In this case, one of the roots of (4.13) is zero with the other being positive and so we choose $s^2 = s_1^2 > 0$ and $s_2^2 = 0$, where

$$s^2 = \frac{2b'}{c} = \frac{\bar{\alpha}_{11}\alpha_{22} + \bar{\gamma}_1\gamma_2 - \delta^2}{\alpha_{22}\gamma_2} . \tag{4.45}$$

This time we take

$$\begin{aligned} \phi_1(x_2) &= B_1 + A_1 x_2 + D_1 \cosh(spx_2) + C_1 \sinh(spx_2) , \\ \phi_2(x_2) &= A_2 + B_2 x_2 + C_2 \cosh(spx_2) + D_2 \sinh(spx_2) , \end{aligned} \tag{4.46}$$

which can be substituted into the boundary conditions (4.6), with $x_2 = \pm l_2$, to give the paired equations

$$\begin{aligned} \gamma_2 A_1 + \mathcal{A}_{2112} p A_2 + \gamma_2 sp \cosh(s\eta) C_1 + \mathcal{A}_{2112} p \cosh(s\eta) C_2 = 0 , \\ -\alpha_{12} l_2 A_1 - \alpha_{12} \sinh(s\eta) C_1 + \alpha_{22} s \sinh(s\eta) C_2 = 0 , \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} \mathcal{A}_{2112} l_2 B_2 + \gamma_2 s \sinh(s\eta) D_1 + \mathcal{A}_{2112} \sinh(s\eta) D_2 = 0 , \\ -p\alpha_{12} B_1 + \alpha_{22} B_2 - p\alpha_{12} \cosh(s\eta) D_1 + \alpha_{22} sp \cosh(s\eta) D_2 = 0 . \end{aligned} \tag{4.48}$$

This case arises when $a' = 0$, which can happen only if either $\bar{\alpha}_{11} = 0$ or $\bar{\gamma}_1 = 0$ (both cannot hold together, since (4.45) would give $s^2 = -\delta^2/\alpha_{22}\gamma_2$, but $s^2 > 0$ by definition) and we look at each case separately.

(a) $\bar{\alpha}_{11} = 0 \quad (\alpha_{11} = \Omega^2)$.

On substituting (4.46), together with (4.9), into the equation of motion $(4.1)_2$, we obtain the identities

$$\begin{aligned} B_2 &= 0 \ , \\ -p\gamma_2 A_2 &= \delta A_1 \ , \\ (\alpha_{22}s^2 - \bar{\gamma}_1)\begin{bmatrix} C_2 \\ D_2 \end{bmatrix} &= \delta s \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} . \end{aligned} \tag{4.49}$$

If we consider first symmetric modes, with $A_i = C_i = 0$ $(i \in \{1, 2\})$, then (4.48) becomes

$$\begin{aligned} -\alpha_{12}\delta D_2 &= 0 \ , \\ -\alpha_{12}(B_1 + \cosh(s\eta)D_1) + \alpha_{22}s\cosh(s\eta)D_2 &= 0 \ , \end{aligned} \tag{4.50}$$

and we see that the solution depends on whether $\alpha_{12} = 0$ or not.

(i) If $\alpha_{12} = 0$ then (4.49) and (4.50) reduce to $B_2 = D_1 = D_2 = 0$, with B_1 undefined, so that we may have solutions of the form

$$\begin{aligned} v_1 &= B_1 \begin{Bmatrix} \cos px_1 \\ -\sin px_1 \end{Bmatrix} e^{-i\omega t} \ , \\ v_2 &= 0 \ , \end{aligned} \tag{4.51}$$

occurring for all values of η.

(ii) If $\alpha_{12} \neq 0$ then (4.49) and (4.50) give $B_i = D_i = 0$ $(i \in \{1, 2\})$, so that no nontrivial solutions are possible in this case.

Alternatively, if we take $B_i = D_i = 0$ $(i \in \{1, 2\})$, corresponding to antisymmetric modes, then on using (4.49) the boundary conditions (4.47) can be written as

$$\begin{aligned} (\bar{\gamma}_1\gamma_2 - \mathcal{A}_{2112}\delta)A_2 + \alpha_{12}\delta\cosh(s\eta)C_2 &= 0 \ , \\ \alpha_{12}\bar{\gamma}_1 s\eta A_2 + (\bar{\gamma}_1\alpha_{12} + \mathcal{A}_{2112}\alpha_{22}s^2)\sinh(s\eta)C_2 &= 0 \ , \end{aligned} \tag{4.52}$$

and again the nature of the solution depends on whether $\alpha_{12} = 0$ or not.

(i) If $\alpha_{12} = 0$ then on noting that $s^2 > 0$ and $\delta \neq 0$ (if $\delta = 0$ the equations of motion (4.1) decouple), we get that $A_i = C_i = 0$ $(i \in \{1, 2\})$ and so no nontrivial solutions are possible.

(ii) If $\alpha_{12} \neq 0$ then by taking the determinant of the coefficients of (A_2, C_2) in (4.52) to be zero, we obtain the frequency equation for antisymmetric modes

$$\frac{\tanh(s\eta)}{s\eta} = \frac{\alpha_{12}^2 \bar{\gamma}_1 \gamma_2}{(\bar{\gamma}_1 \gamma_2 - \mathcal{A}_{2112}\delta)^2} \, . \tag{4.53}$$

Since the left–hand side of (4.53) is strictly positive, a necessary condition for the existence of solutions in this case is clearly $\bar{\gamma}_1 > 0$, which can be written as

$$\alpha_{11} = \Omega^2 < \gamma_1 \, . \tag{4.54}$$

(b) $\bar{\gamma}_1 = 0 \quad (\gamma_1 = \Omega^2)$.

The results obtained in this case are essentially complementary to those obtained in part (a), but this time the nature of the solution depends on whether $\mathcal{A}_{2112} = 0$ or not.

(i) $\mathcal{A}_{2112} = 0$. Here no symmetric modes can arise but the solution given by

$$\begin{aligned} v_1 &= 0 \, , \\ v_2 &= A_2 \begin{Bmatrix} \sin(px_1) \\ \cos(px_1) \end{Bmatrix} e^{-i\omega t} \, , \end{aligned} \tag{4.55}$$

can occur in the antisymmetric case for all values of η.

(ii) $\mathcal{A}_{2112} \neq 0$. This time no antisymmetric modes may arise, but the frequency equation for symmetric modes is given by

$$\frac{\tanh(s\eta)}{s\eta} = \frac{\bar{\alpha}_{11}\alpha_{22}\mathcal{A}_{2112}^2}{(\bar{\alpha}_{11}\alpha_{22} - \alpha_{12}\delta)^2} \, , \tag{4.56}$$

which requires that

$$\gamma_1 = \Omega^2 < \alpha_{11} \, , \tag{4.57}$$

for solutions to exist.

Case 8: $a' = 0, \quad b' < 0$.

This time we have one root of (4.13) zero with the other root, s^2 say, negative and so we write $s = is^*$ with s^* taken to be positive. The results in this case can be obtained from those of Case 7, by setting $s = is^*$ and give:

(a) $\bar{\alpha}_{11} = 0$.

If $\alpha_{12} \neq 0$ then no symmetric modes can occur, while the frequency equation for antisymmetric modes is

$$\frac{\tan(s^*\eta)}{s^*\eta} = \frac{\alpha_{12}^2\bar{\gamma}_1\gamma_2}{(\bar{\gamma}_1\gamma_2 - \mathcal{A}_{2112}\delta)^2} . \tag{4.58}$$

If $\alpha_{12} = 0$ then the solution given by (4.51) can again occur, with no nontrivial antisymmetric solutions possible in this case either.

(b) $\bar{\gamma}_1 = 0$.

The frequency equation for symmetric modes, provided $\mathcal{A}_{2112} \neq 0$, is given by

$$\frac{\tan(s^*\eta)}{s^*\eta} = \frac{\bar{\alpha}_{11}\alpha_{22}\mathcal{A}_{2112}^2}{(\bar{\alpha}_{11}\alpha_{22} - \alpha_{12}\delta)^2} , \tag{4.59}$$

with only the solution represented by (4.55) occurring when $\mathcal{A}_{2112} = 0$.

Note that in this case, if $\bar{\alpha}_{11} = \bar{\gamma}_1 = 0$ then (4.58) (and (4.59)) have the solution

$$s^*\eta = k\pi , \quad \text{for some integer } k. \tag{4.60}$$

<u>Case 9</u>: $a' = b' = 0$.

Here zero is a double root of the quadratic (4.13), so that we must have

$$\bar{\alpha}_{11}\alpha_{22} + \bar{\gamma}_1\gamma_2 - \delta^2 = \bar{\alpha}_{11}\bar{\gamma}_1 = 0 , \tag{4.61}$$

and since we require $\delta \neq 0$, by (4.61), we cannot have $\bar{\alpha}_{11} = \bar{\gamma}_1 = 0$. The general solution is of the form (4.9) with

$$\begin{aligned} \phi_1(x_2) &= B_1 + A_1x_2 + D_1x_2^2 + C_1x_2^3 , \\ \phi_2(x_2) &= A_2 + B_2x_2 + C_2x_2^2 + D_2x_2^3 , \end{aligned} \tag{4.62}$$

and on substituting this into the boundary conditions (4.6), on $x_2 = \pm l_2$, they become

$$\begin{aligned} \gamma_2(A_1 + 3l_2^2C_1) + p\mathcal{A}_{2112}(A_2 + l_2^2C_2) = 0 , \\ -\alpha_{12}p(A_1 + l_2^2C_1) + 2\alpha_{22}C_2 = 0 , \end{aligned} \tag{4.63}$$

and

$$
\begin{aligned}
2\gamma_1 D_1 + p\mathcal{A}_{2112}(B_2 + l_2^2 D_2) = 0 \ , \\
-\alpha_{12} p(B_1 + l_2^2 D_1) + \alpha_{22}(B_2 + 3l_2^2 D_2) = 0 \ .
\end{aligned}
\tag{4.64}
$$

We again have two possibilities, namely, either $\bar{\alpha}_{11} = 0$ or $\bar{\gamma}_1 = 0$, and we look at each separately.

(a) $\bar{\alpha}_{11} = 0$.

In this case (4.61) becomes

$$
\bar{\gamma}_1 \gamma_2 = \delta^2 \ . \tag{4.65}
$$

On substituting (4.62) into the equation of motion $(4.1)_2$, we obtain the identities

$$
\begin{aligned}
D_2 &= 0 \ , \\
2\delta D_1 &= -p\bar{\gamma}_1 B_2 \ , \\
3\delta C_1 &= -p\bar{\gamma}_1 C_2 \ , \\
p\delta A_1 &= 2\alpha_{22} C_2 - p^2 \bar{\gamma}_1 A_2 \ .
\end{aligned}
\tag{4.66}
$$

Consider first the antisymmetric case with $B_i = D_i = 0$ ($i \in \{1, 2\}$), so that on substituting from (4.66) into (4.63), the boundary conditions can be written as

$$
\begin{aligned}
\alpha_{12}\delta p^2 A_2 + (\alpha_{12}\delta\eta^2 - 2\alpha_{22}\gamma_2) C_2 = 0 \ , \\
p^2 \bar{\gamma}_1 \alpha_{12} A_2 + (2\mathcal{A}_{2112}\alpha_{22} + \alpha_{12}\gamma_2\eta^2/3) C_2 = 0 \ ,
\end{aligned}
\tag{4.67}
$$

and so the nature of the solution again depends on the value of α_{12}.

If $\alpha_{12} = 0$ then (4.67) reduces to $C_2 = 0$, since $\alpha_{22} > 0$ by strong–ellipticity, and (4.66) then gives $C_1 = 0$. This permits solutions of the form

$$
\begin{aligned}
v_1 &= \frac{-p\bar{\gamma}_1}{\mathcal{A}_{2112}} A_2 x_2 \left\{ \begin{array}{c} \cos px_1 \\ -\sin px_1 \end{array} \right\} e^{-i\omega t} \ , \\
v_2 &= A_2 \left\{ \begin{array}{c} \sin px_1 \\ \cos px_1 \end{array} \right\} e^{-i\omega t} \ ,
\end{aligned}
\tag{4.68}
$$

which exist for all values of η.

However, if $\alpha_{12} \neq 0$ then we obtain from (4.67) the frequency equation for antisymmetric modes

$$\eta^2 = \frac{3c(\delta + \mathcal{A}_{2112})}{\alpha_{12}\delta^2} \, . \tag{4.69}$$

A necessary condition for (4.69) to give positive solutions for η^2 is

$$\frac{\mathcal{A}_{2112}}{\alpha_{12}} > -\frac{1}{2} \, . \tag{4.70}$$

For symmetric modes with $A_i = C_i = 0$ ($i \in \{1, 2\}$), on using (4.66), the boundary condition $(4.64)_1$ reduces to

$$(\bar{\gamma}_1\gamma_2 - \mathcal{A}_{2112}\delta)D_1 = 0 \, , \tag{4.71}$$

and from (4.65) we see that this requires either $\alpha_{12} = 0$ or $D_1 = 0$. If $\alpha_{12} \neq 0$ then no nontrivial solutions can occur in this case, however when $\alpha_{12} = 0$ the solution described by (4.51) can again occur.

(b) $\bar{\gamma}_1 = 0$.

This time we find that for antisymmetric modes, no nontrivial solution is possible when $\mathcal{A}_{2112} \neq 0$, but if $\mathcal{A}_{2112} = 0$ the solution represented by (4.55) can occur. For symmetric modes it is found that if $\mathcal{A}_{2112} = 0$ then the solution given by

$$\begin{aligned} v_1 &= B_1 \left\{ \begin{matrix} \cos px_1 \\ -\sin px_1 \end{matrix} \right\} e^{-i\omega t} \, , \\ v_2 &= \frac{p\bar{\alpha}_{11}}{\alpha_{12}} B_1 x_2 \left\{ \begin{matrix} \sin px_1 \\ \cos px_1 \end{matrix} \right\} e^{-i\omega t} \, , \end{aligned} \tag{4.72}$$

exists for all values of η, but otherwise the frequency equation is given by

$$\eta^2 = \frac{3c(\delta + \alpha_{12})}{\mathcal{A}_{2112}\delta^2} \, , \tag{4.73}$$

which requires that

$$\frac{\alpha_{12}}{\mathcal{A}_{2112}} > -\frac{1}{2} \, , \tag{4.74}$$

for positive values of η^2.

4.3 The compressible static case.

4.3.1 General bifurcation criteria.

When $\omega = 0$ the solutions found in Section 4.2 correspond to quasi–static incremental deformations and the frequency equations derived there can be viewed as bifurcation criteria. When the strong–ellipticity condition (4.7) holds, only Cases 1, 2 and 3 of Section 4.2, can arise and we restrict our attention to just these three cases. These equations, together with the results for the cases that occur on the boundary of the strongly–elliptic domain, were obtained by Ogden (1984), but a full discussion of their nature was omitted. Here we restate the relevant equations, giving a brief discussion of them, before studying them in more detail using particular strain–energy functions.

Case 1: $b > \sqrt{ac}$.

The bifurcation criteria are given by

$$\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \left[\frac{s_2(\mathcal{A} - s_1^2)}{s_1(\mathcal{A} - s_2^2)}\right]^{\pm 1} , \tag{4.75}$$

where the $+(-)$ sign corresponds to antisymmetric (symmetric) modes of deformation, and we have

$$s_1^2 = \frac{b + \sqrt{b^2 - ac}}{c} , \qquad s_2^2 = \frac{b - \sqrt{b^2 - ac}}{c} ,$$

with

$$\mathcal{A} = \frac{\alpha_{11}(\gamma_1\gamma_2 - \mathcal{A}_{2112}^2)}{\gamma_2(\alpha_{11}\alpha_{22} - \alpha_{12}^2)} . \tag{4.76}$$

As in (4.25), equations (4.75) can be rearranged as

$$\frac{\sinh(s_1 + s_2)\eta}{(s_1 + s_2)}[s_1 s_2 + \mathcal{A}] = \pm\frac{\sinh(s_1 - s_2)\eta}{(s_1 - s_2)}[s_1 s_2 - \mathcal{A}] . \tag{4.77}$$

By using the monotonicity of tanh, it can be seen from (4.75) that a necessary condition for the existence of antisymmetric modes is

$$s_1 s_2 > | \mathcal{A} | \,, \tag{4.78}$$

while for symmetric modes to exist we must have

$$s_1 s_2 < | \mathcal{A} | \ . \tag{4.79}$$

However, since $\sinh x/x$ is also strictly increasing (for nonzero x) (4.77) gives us that, for either mode to occur we must have

$$\mathcal{A} < 0 \,, \tag{4.80}$$

so that on combining (4.78)–(4.80) we find that a necessary condition for antisymmetric modes to exist is

$$-s_1 s_2 < \mathcal{A} < 0 \,, \tag{4.81}$$

while

$$\mathcal{A} < -s_1 s_2 < 0 \,, \tag{4.82}$$

is necessary for symmetric modes. In the unstressed configuration, from (4.76) and (2.46), we see that $\mathcal{A} = 0$ and $s_1 = s_2 = 1$, so that on a path of pure homogeneous deformation from this unstressed configuration, by continuity, we must pass through (4.81) before (4.82) can be satisfied, that is, **antisymmetric modes can always occur before symmetric modes.**

By noting that $s_1^2 s_2^2 = a/c$, (4.81) and (4.82) can be written as

$$\frac{\alpha_{12}^2}{(\alpha_{11}\alpha_{22})^{1/2}} + \frac{\mathcal{A}_{2112}^2}{(\gamma_1\gamma_2)^{1/2}} - \left((\gamma_1\gamma_2)^{1/2} + (\alpha_{11}\alpha_{22})^{1/2}\right) < 0 \,, \tag{4.83}$$

and

$$\frac{\alpha_{12}^2}{(\alpha_{11}\alpha_{22})^{1/2}} + \frac{\mathcal{A}_{2112}^2}{(\gamma_1\gamma_2)^{1/2}} - \left((\gamma_1\gamma_2)^{1/2} + (\alpha_{11}\alpha_{22})^{1/2}\right) > 0 \,, \tag{4.84}$$

respectively. In Dowaikh and Ogden (1991), equation (4.83), in a slightly different notation, was found to be an exclusion condition for the existence of quasi–static surface deformations on a half–space, whereas here it can viewed as an exclusion condition for the existence of symmetric modes.

Case 2: $b = \sqrt{ac}$.

Here, $s^2 = \sqrt{a/c}$ and the bifurcation equations are given by

$$\frac{\sinh(2s\eta)}{2s\eta} = \pm\frac{s^2 - \mathcal{A}}{s^2 + \mathcal{A}}\,, \tag{4.85}$$

with the $+(-)$ corresponding to antisymmetric (symmetric) modes. Again by noting the monotonicity of $\sinh x/x$, a necessary condition for antisymmetric modes to exist is

$$-s^2 < \mathcal{A} < 0\,, \tag{4.86}$$

while for symmetric modes to exist, the inequality

$$\mathcal{A} < -s^2 < 0\,, \tag{4.87}$$

must hold and so, as in Case 1, antisymmetric modes can always occur before symmetric modes. Note that (4.86) and (4.87) are consistent with (4.81) and (4.82), and that they can again be written in the forms (4.83) and (4.84) respectively.

Case 3: $b^2 < ac$.

Here we have complex roots and the bifurcation equations are given by

$$\frac{\sinh(2\gamma\eta)}{\sin(2\epsilon\eta)} = \pm\frac{\gamma(\gamma^2 + \epsilon^2 - \mathcal{A})}{\epsilon(\gamma^2 + \epsilon^2 + \mathcal{A})}\,, \tag{4.88}$$

where from (4.34), we have

$$\gamma = \left(\frac{b + \sqrt{ac}}{2c}\right)^{\frac{1}{2}}\,, \qquad \epsilon = \left(\frac{\sqrt{ac} - b}{2c}\right)^{\frac{1}{2}}\,, \tag{4.89}$$

and the $+(-)$ corresponds to antisymmetric (symmetric) modes. The occurrence of trigonometric as well as hyperbolic terms in (4.88) does not allow us to obtain necessary existence criteria as in the other two cases.

4.3.2 Results for particular strain–energy functions.

In order to obtain a further understanding of the results of the previous section we consider three particular strain–energy functions, one or more of which lie within each of the three cases that can arise while strong–ellipticity holds.

a) A compressible neo–Hookean material.

If we consider the strain–energy function

$$W = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\ln(\lambda_1\lambda_2\lambda_3)\right) , \tag{4.90}$$

and take $\lambda_3 = 1$ to be fixed, we get from (4.4) and (4.90) that

$$a = \frac{\mu^2}{\lambda_2^2}(\lambda_1^2+1) , \quad c = \frac{\mu^2}{\lambda_1^2}(\lambda_2^2+1) , \quad 2b = \mu^2(2+\lambda_1^{-2}+\lambda_2^{-2}) , \tag{4.91}$$

$$\mathcal{A} = \frac{(\lambda_1^2\lambda_2^2-1)}{\lambda_2^2(\lambda_2^2+1)} , \tag{4.92}$$

which satisfy

$$\begin{gathered} b > 0 , \\ b^2 - ac = \mu^4(\lambda_1^{-2} - \lambda_2^{-2})^2/4 \geq 0 , \end{gathered} \tag{4.93}$$

for all values (λ_1, λ_2). So, from (4.93), we see that if $\lambda_1 = \lambda_2$ then Case 2 holds, while Case 1 holds for all other values of λ_1 and λ_2.

(*i*) $\lambda_1 \neq \lambda_2$.

Here the solution lies wholly within Case 1 and on introducing the quantities p_0 and η_0, as in (3.154), which are independent of the deformation, we find that

$$p_0 = \lambda_1 p , \qquad \eta_0 = \frac{\lambda_1}{\lambda_2}\eta . \tag{4.94}$$

The bifurcation equations, from (4.75), can be written as

$$\frac{\tanh(\eta_0)}{\tanh\left[\left(\frac{1+\lambda_1^{-2}}{1+\lambda_2^{-2}}\right)^{\frac{1}{2}}\eta_0\right]} = \left[\frac{\lambda_2(\lambda_1^2+1)^{3/2}}{\lambda_1(\lambda_2^2+1)^{3/2}}\right]^{\pm 1} , \tag{4.95}$$

where the $+(-)$ sign corresponds to antisymmetric (symmetric) modes.

If we take the sides $x_2 = \pm l_2$ to be traction free, that is $\sigma_2 = 0$, then from (4.90) and (2.27) we must have $\lambda_2 = 1$, which on substituting into (4.95) gives

$$\frac{\tanh(\eta_0)}{\tanh\left[(1+\lambda_1^{-2})^{\frac{1}{2}}\eta_0/\sqrt{2}\right]} = \left[\frac{(\lambda_1^2+1)^{3/2}}{2\sqrt{2}\lambda_1}\right]^{\pm 1} , \tag{4.96}$$

and by considering the monotonicity of tanh, a necessary condition for antisymmetric modes to occur is

$$\lambda_1^2 > \sqrt{5} - 2 , \tag{4.97}$$

while for symmetric modes we require

$$\lambda_1^2 < \sqrt{5} - 2 . \tag{4.98}$$

From (4.92) and (4.80) we see that for either mode to occur, we must have $\lambda_1 < 1$, so that neither mode can arise for tensile values of σ_1, while in compression, as the body is deformed on a path of deformation and stress from the unstressed configuration, only antisymmetric modes may arise until $\lambda_1^2 = \sqrt{5} - 2$ and thereafter only symmetric modes may occur.

(*ii*) $\lambda_1 = \lambda_2$.

In this equibiaxial case, if $\lambda_1 = \lambda_2 = \lambda$, say, then

$$\begin{gathered} a = b = c = \mu^2(1+\lambda^{-2}) , \\ \mathcal{A} = (1-\lambda^{-2}) , \end{gathered} \tag{4.99}$$

so that $b = \sqrt{ac}$ and the solution thus lies within Case 2. The bifurcation equation, from (4.85), is

$$\frac{\sinh 2\eta_0}{2\eta_0} = \pm\frac{1}{2\lambda^2 - 1} , \tag{4.100}$$

which requires that $1/\sqrt{2} < \lambda < 1$ for antisymmetric modes to occur and $\lambda < 1/\sqrt{2}$ for symmetric modes.

b) Varga material.

Similar to the neo-Hookean material, we choose the strain–energy function

$$W = \mu\,(\lambda_1 + \lambda_2 + \lambda_3 - 3 - \ln(\lambda_1\lambda_2\lambda_3)) , \tag{4.101}$$

which gives, when $\lambda_3 = 1$,

$$a = \frac{\mu^2}{\lambda_2^2(\lambda_1 + \lambda_2)} = \frac{\lambda_1}{\lambda_2} b = \frac{\lambda_1^2}{\lambda_2^2} c \ , \tag{4.102}$$

which are all positive with $b^2 - ac = 0$ and thus Case 2 holds for all admissible deformations in this case. From (4.85), the bifurcation equations for antisymmetric (+) and symmetric (–) modes can be written as

$$\frac{\sinh 2\eta_0}{2\eta_0} = \pm \frac{\lambda_1^{-1} + \lambda_2^{-1} - 1}{3 - \lambda_1^{-1} - \lambda_2^{-1}} \ . \tag{4.103}$$

From (4.103) it can be seen that a necessary condition for antisymmetric modes to occur is

$$2 < \lambda_1^{-1} + \lambda_2^{-1} < 3 \ , \tag{4.104}$$

while for symmetric modes

$$\lambda_1^{-1} + \lambda_2^{-1} > 3 \ , \tag{4.105}$$

is a necessary condition.

c) Blatz–Ko material.

If we take the strain–energy function given by

$$W = \frac{\mu}{2}(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5) \ , \tag{4.106}$$

which for fixed $\lambda_3 = 1$ has

$$\begin{aligned} a = c &= \frac{3\mu^2}{\lambda_1^4\lambda_2^4} \ , \\ 2b &= \frac{\mu^2}{\lambda_1^6\lambda_2^6}(8\lambda_1^2\lambda_2^2 - \lambda_1^4 - \lambda_2^4) \ , \end{aligned} \tag{4.107}$$

and so gives

$$b - \sqrt{ac} = -\frac{\mu^2(\lambda_1^2 - \lambda_2^2)^2}{2\lambda_1^6\lambda_2^6} \leq 0 \ , \tag{4.108}$$

with

$$b + \sqrt{ac} = \frac{\mu^2}{2\lambda_1^6\lambda_2^6}(14\lambda_1^2\lambda_2^2 - \lambda_1^4 - \lambda_2^4) \ . \tag{4.109}$$

The strong–ellipticity condition requires that $b+\sqrt{ac}>0$, which from (4.109) can be seen to be

$$\lambda_1^4+\lambda_2^4-14\lambda_1^2\lambda_2^2<0\ , \tag{4.110}$$

and this corresponds to the region, in $(\lambda_1,\ \lambda_2)$–space, in the first quadrant between the two lines

$$\lambda_1=(2\pm\sqrt{3})\lambda_2\ . \tag{4.111}$$

When strong–ellipticity holds, we see from (4.108) and (4.110) that $b^2-ac\le 0$, with equality only when $\lambda_1=\lambda_2$. Therefore, for equibiaxial deformations, the solution lies within Case 2, but for all other admissible deformations Case 3 applies.

(i) $\lambda_1\neq\lambda_2$.

From (4.107) and (4.89), we see that $\gamma^2+\epsilon^2=1$, which can then be substituted into the bifurcation equation (4.88) together with

$$\begin{aligned}\gamma^2&=\frac{1}{12}(14-\xi^2-\xi^{-2})\ ,\\ \epsilon^2&=\frac{1}{12}(\xi-\xi^{-1})^2\ ,\\ \mathcal{A}&=\frac{3\left(1-(\xi+\xi^{-1}-\lambda_1^2\lambda_2^2)^2\right)}{(9-\lambda_1^4\lambda_2^4)}\ ,\end{aligned} \tag{4.112}$$

where $\xi=\lambda_2/\lambda_1$.

(ii) $\lambda_1=\lambda_2(=\lambda)$.

In this equibiaxial case we have

$$a=b=c=3\mu^2/\lambda^8\ ,$$

$$\mathcal{A}=\frac{3(\lambda^4-1)}{(\lambda^4+3)}\qquad(\text{provided }\lambda^4\neq 3)\ ,$$

and the bifurcation equation (4.85) becomes, for $\lambda^4\neq 3$,

$$\frac{\sinh 2\eta_0}{2\eta_0}=\pm\frac{3-\lambda^4}{2\lambda^4}\ , \tag{4.113}$$

where the $+(-)$ corresponds to antisymmetric (symmetric) modes. By considering (4.113), it can be seen that symmetric modes cannot arise in this case, while antisymmetric modes may only occur in compression.

4.3.3 Numerical results for the static case.

The bifurcation criteria derived in the previous section were studied numerically and the results are shown in Figures 4.1–4.8.

Figures 4.1–4.3 deal with the neo–Hookean strain–energy function and Figure 4.1 represents the solution to (4.100) in the equibiaxial case. Figure 4.2 shows plots of the stretch λ_1 against the mode number η_0 for various values of the stretch λ_2 and, as in the equibiaxial case, for each graph, there exists a critical value of λ_1, λ_c say, such that $\lambda_1 > \lambda_c$ corresponds to antisymmetric modes while $\lambda_1 < \lambda_c$ corresponds to symmetric modes and the value of λ_c decreases as the stretch λ_2 increases. Figure 4.3 shows a (λ_1, λ_2) phase plot of the bifurcation equations and, as the value of the mode number η_0 increases, the two solution branches can be shown to approach one another, although only the branch corresponding to the symmetric mode intersects the λ_1–axis.

Figures 4.4 and 4.5 dealing with the Varga material are seen to be very similar to Figures 4.2 and 4.3, respectively, for the neo–Hookean material; however, in Figure 4.5, as the value of η_0 is increased, both curves approach the hyperbola

$$(\lambda_1 - \frac{1}{3})(\lambda_2 - \frac{1}{3}) = \frac{1}{9}\ . \tag{4.114}$$

For the Blatz–Ko material, we see from Figure 4.6 that there are two regions in which bifurcation can occur and, for the case corresponding to larger values of λ_1, symmetric modes can occur before antisymmetric modes on a path of deformation from the reference configuration. The sole antisymmetric solution branch predicted from (4.113), in the equibiaxial case, is shown in Figure 4.7, with Figure 4.8 giving a phase plot for this Blatz–Ko material. The upper two solution branches in Figure 4.8 converge as they approach the line $\lambda_1 = \lambda_2$ with neither mode possible on this line, which is consistent with (4.113).

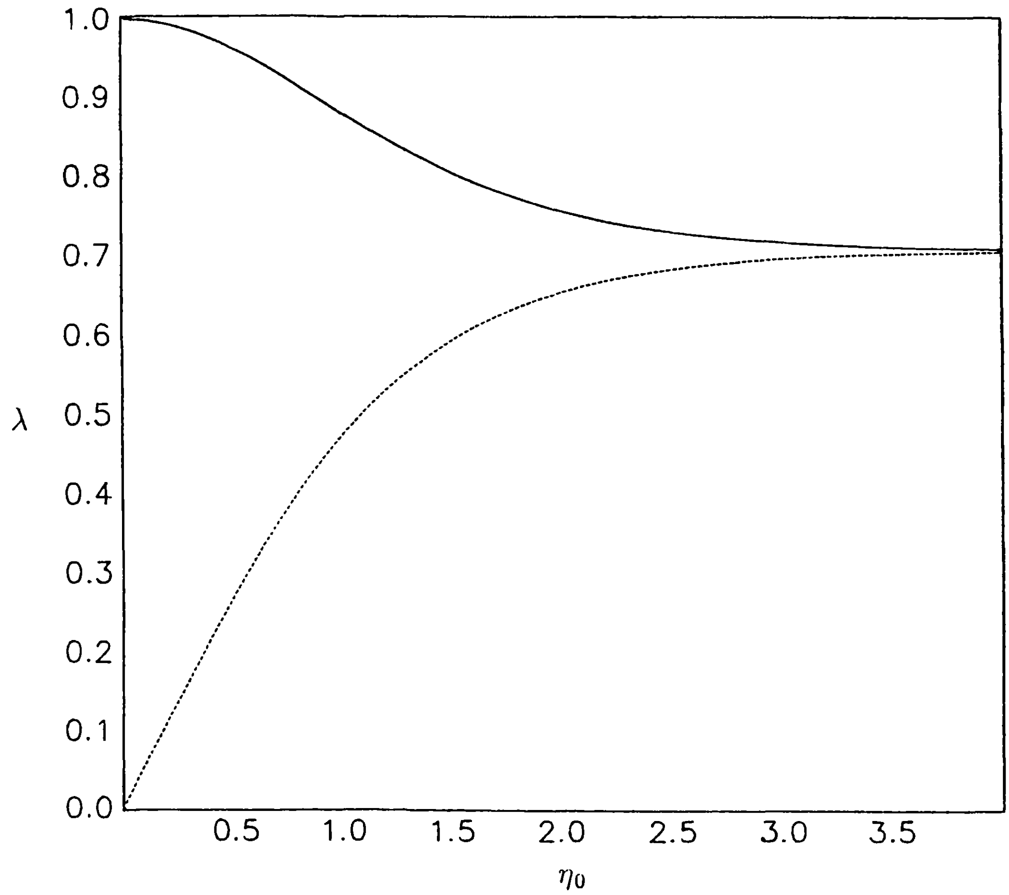

Figure 4.1: Bifurcation curves for a compressible neo–Hookean material subjected to an equibiaxial deformation with $\lambda_1 = \lambda_2 = \lambda$, where the solid (broken) lines correspond to antisymmetric (symmetric) modes.

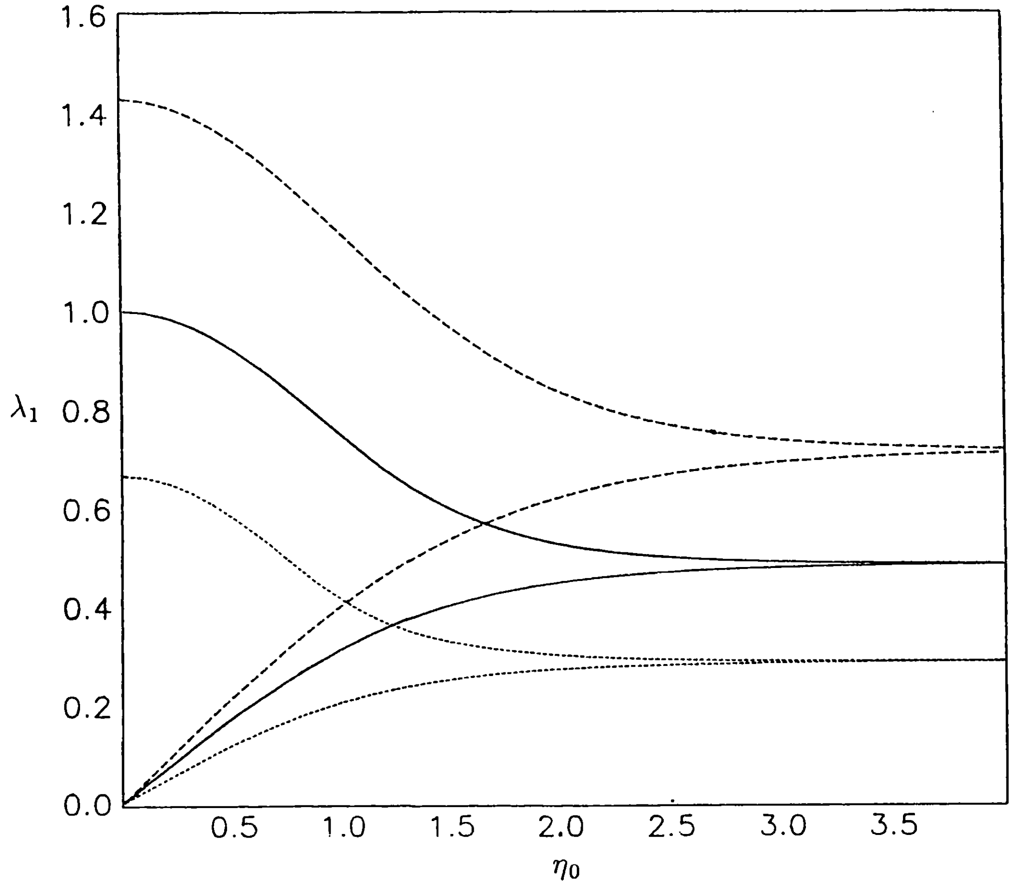

Figure 4.2: Bifurcation curves for a compressible neo–Hookean material for several values of the stretch λ_2; the solid line corresponds to $\lambda_2 = 1.0$, the dotted line corresponds to $\lambda_2 = 1.5$ and the dashed line corresponds to $\lambda_2 = 0.7$. On each graph there is a critical value of the stretch λ_1 such that antisymmetric modes occur above this value and symmetric modes occur below it.

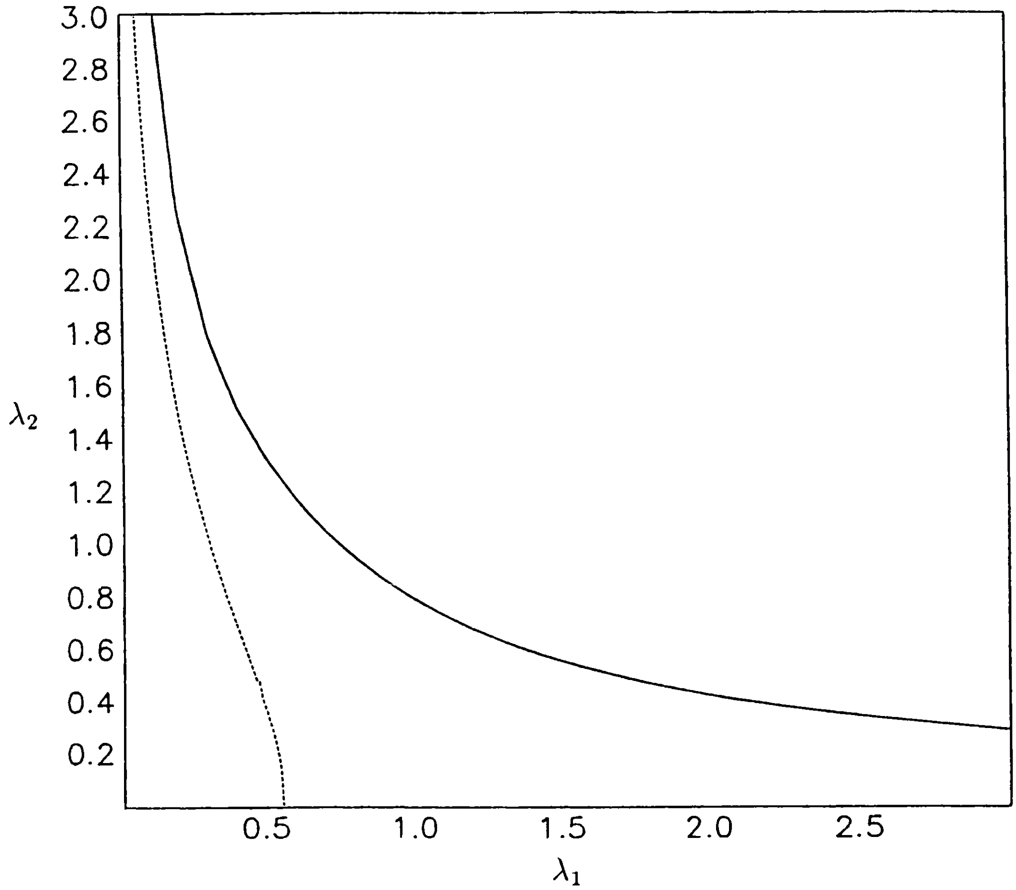

Figure 4.3: Phase plot for bifurcation modes of a compressible neo-Hookean material with mode number $\eta_0 = 1.0$, where the solid (broken) lines correspond to antisymmetric (symmetric) modes.

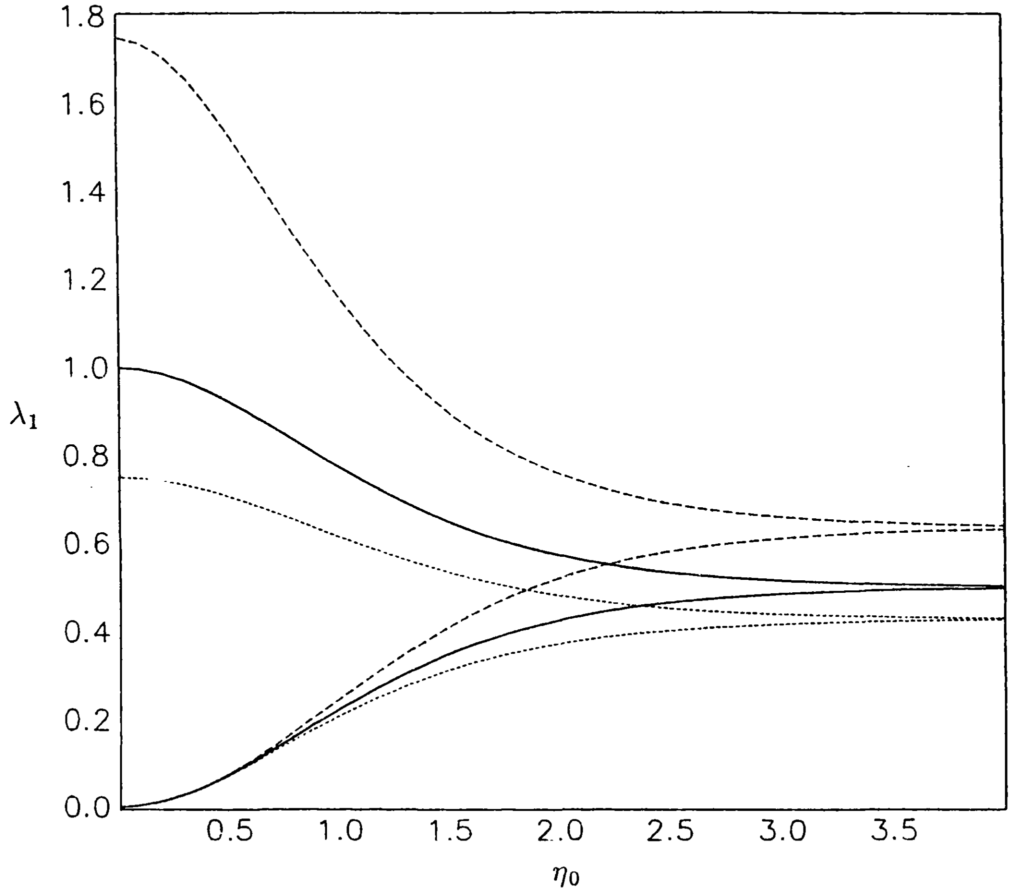

Figure 4.4: Bifurcation curves for a compressible Varga material for several values of the stretch λ_2; the solid line corresponds to $\lambda_2 = 1.0$, the dotted line corresponds to $\lambda_2 = 1.5$ and the dashed line corresponds to $\lambda_2 = 0.7$. On each graph there is a critical value of the stretch λ_1 such that antisymmetric modes occur above this value and symmetric modes occur below it.

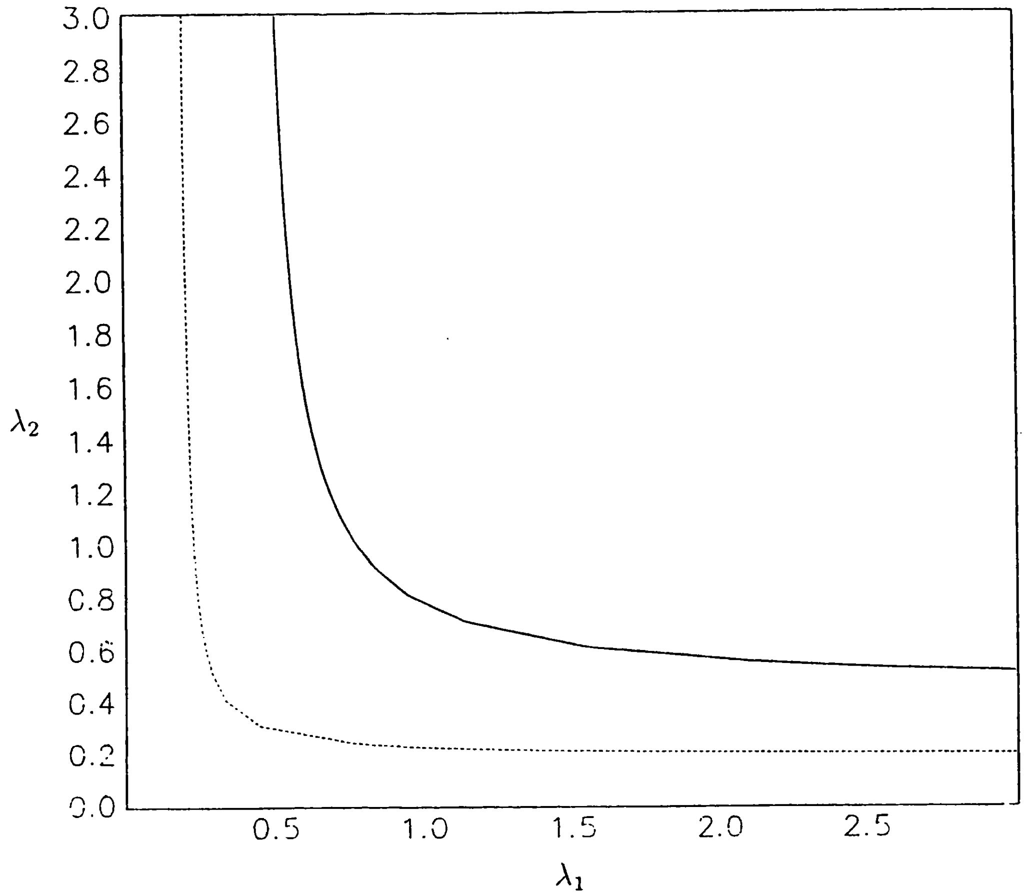

Figure 4.5: Phase plot for bifurcation modes of a compressible Varga material with mode number $\eta_0 = 1.0$, where the solid (broken) lines correspond to antisymmetric (symmetric) modes.

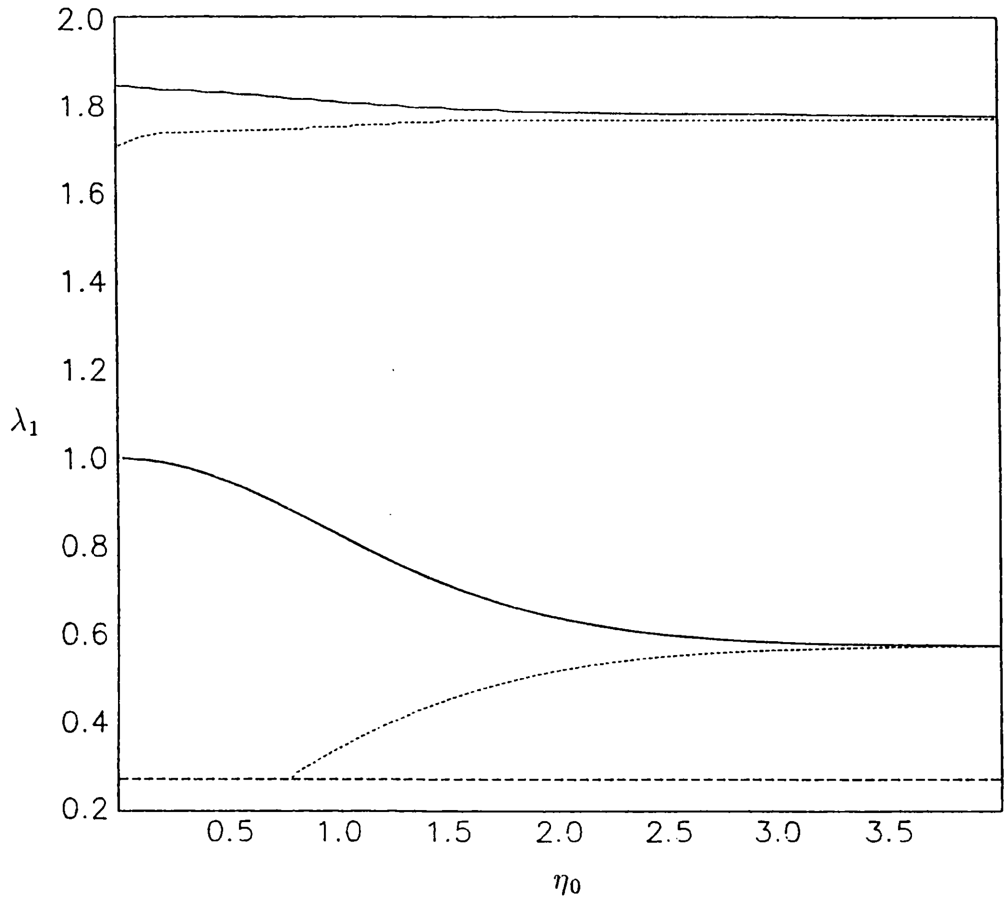

Figure 4.6: Bifurcation curves for a compressible Blatz–Ko material with stretch $\lambda_2 = 1.0$. The solid (dotted) curves represent antisymmetric (symmetric) solution branches while the dashed line represents the boundary of the strongly-elliptic domain.

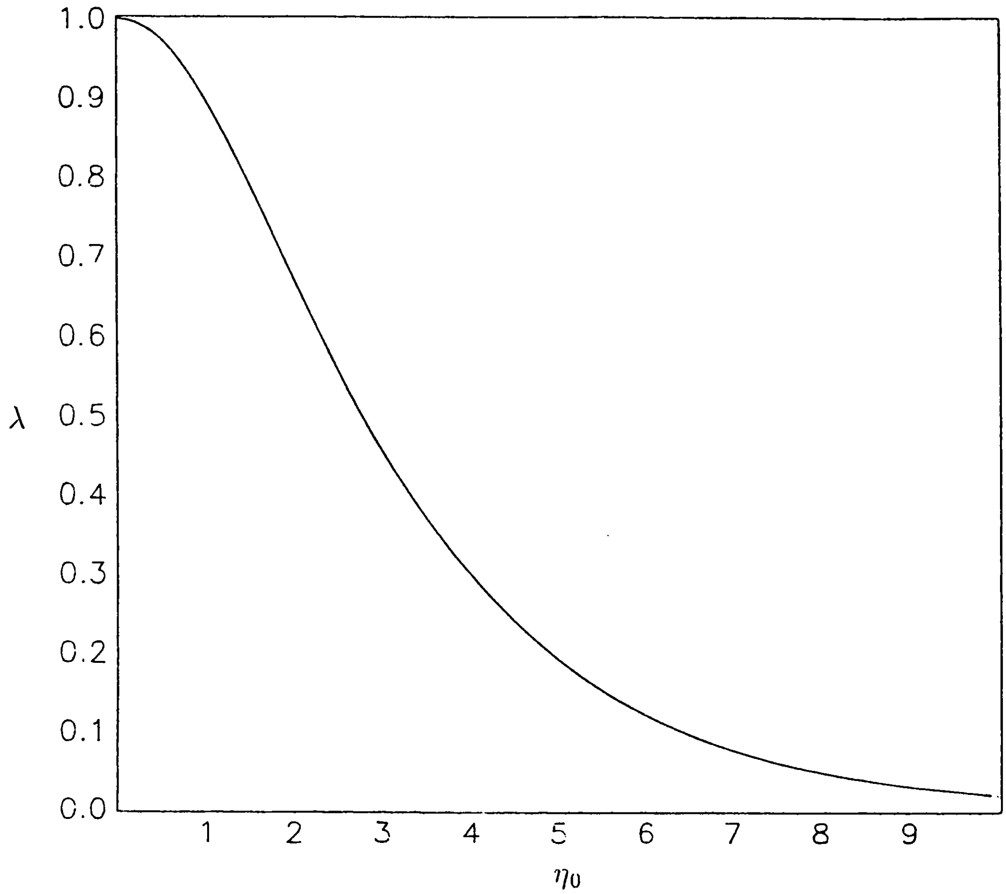

Figure 4.7: Bifurcation curve for the only, antisymmetric, mode that occurs for a compressible Blatz–Ko material subjected to an equi-biaxial deformation with $\lambda_1 = \lambda_2 = \lambda$.

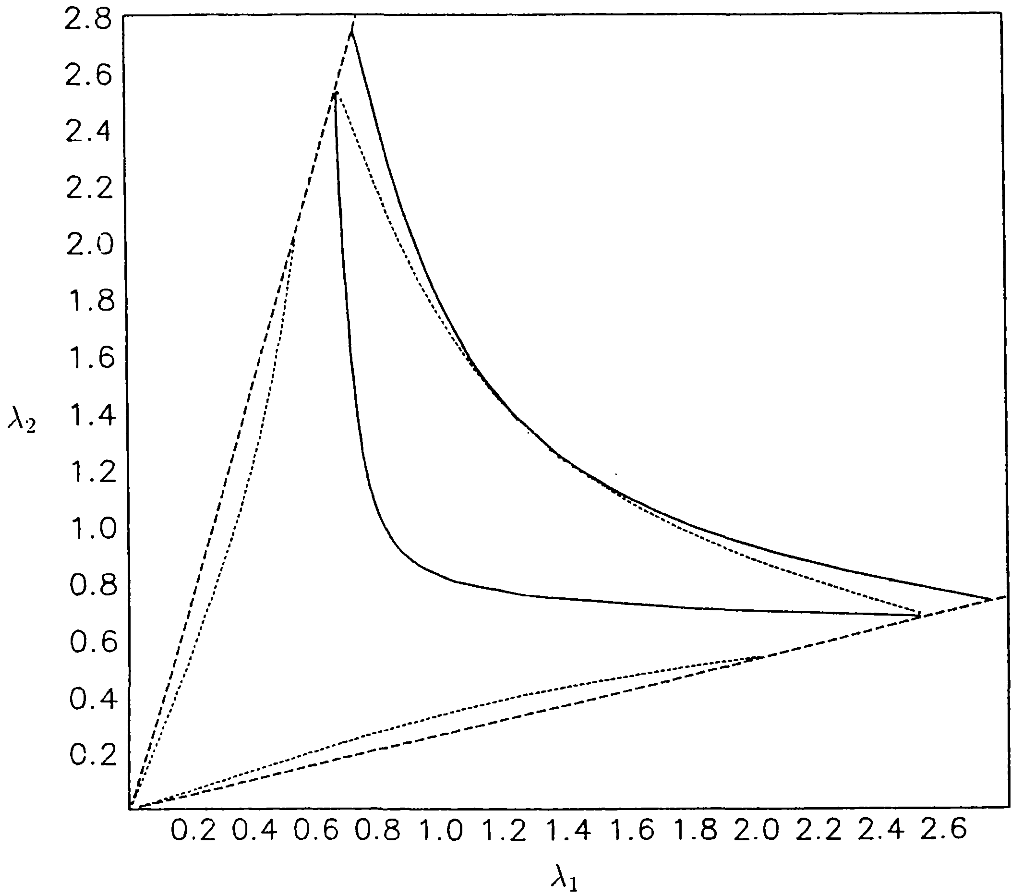

Figure 4.8: Phase plots for bifurcation modes of a compressible Blatz–Ko material with mode number $\eta_0 = 1.0$. The solid (dotted) lines represent antisymmetric (symmetric) solution branches while the dashed lines represent the boundary of the strongly–elliptic domain.

4.4 Results for the dynamic problem.

We now consider, for arbitrary ω, the frequency equations derived in Section 4.2. Unfortunately for compressible materials, few worthwhile results can be found for an arbitrary strain–energy function, so that here we specialize the results by restricting our attention to a neo–Hookean material.

4.4.1 Some general results.

Corresponding to the static case, existence criteria can be obtained for Cases 1 and 2, the other cases either containing trigonometric terms or having been discussed in Section 4.2.

For Case 1, we again find that (4.80)–(4.82) hold, although the terms involved have a different meaning, so that corresponding to (4.83) and (4.84), we get

$$\frac{\alpha_{12}^2}{(\bar{\alpha}_{11}\alpha_{22})^{1/2}} + \frac{\mathcal{A}_{2112}^2}{(\bar{\gamma}_1\gamma_2)^{1/2}} - \left((\bar{\alpha}_{11}\alpha_{22})^{1/2} + (\bar{\gamma}_1\gamma_2)^{1/2}\right) < 0 \ , \tag{4.115}$$

as a necessary condition for the existence of antisymmetric modes in the dynamic case, with

$$\frac{\alpha_{12}^2}{(\bar{\alpha}_{11}\alpha_{22})^{1/2}} + \frac{\mathcal{A}_{2112}^2}{(\bar{\gamma}_1\gamma_2)^{1/2}} - \left((\bar{\alpha}_{11}\alpha_{22})^{1/2} + (\bar{\gamma}_1\gamma_2)^{1/2}\right) > 0 \ , \tag{4.116}$$

necessary for symmetric modes. In Case 2, we again obtain the inequalities (4.115) and (4.116) for the existence of solutions, but for this case using the condition that $b' = \sqrt{a'c}$, we have in addition

$$\left[(\bar{\alpha}_{11}\alpha_{22})^{1/2} - (\bar{\gamma}_1\gamma_2)^{1/2}\right]^2 = \delta^2 \ . \tag{4.117}$$

Unlike the incompressible case, asymptotics for small and large values of η yield no constructive results and so here we only list results for the limits as $\eta \to 0$ and $\eta \to \infty$.

a) $\eta \to 0.$

On taking the limit $\eta \to 0$ in the frequency equations derived in Section 4.2, we find that for Cases 1–6, the frequency equations for symmetric modes have no limiting forms, while those for antisymmetric modes reduce to

$$\mathcal{A} = 0 \ , \tag{4.118}$$

which can be written as

$$\Omega^2 = \frac{\gamma_1\gamma_2 - \mathcal{A}_{2112}^2}{\gamma_2} \ . \tag{4.119}$$

Note that (4.118) holds for any material in its unstressed configuration, so that in this state the block, with $\eta \to 0$, can be unstable with regard to quasi-static antisymmetric modes.

In Cases 7 and 8, ignoring the solutions (4.51) and (4.55) which exist for all values of η, we find that on taking the limit $\eta \to 0$ in (4.53) (or (4.58)), with $\alpha_{11} = \Omega^2$, we can get either

$$\gamma_2(\gamma_1 - \alpha_{11}) = \delta^2 \quad \text{or} \quad \mathcal{A}_{2112}^2 \ , \tag{4.120}$$

which are distinct, since $\alpha_{12} \neq 0$. While if we take $\gamma_1 = \Omega^2$ then (4.56) (or (4.59)), in the limit $\eta \to 0$, becomes

$$\alpha_{22}(\alpha_{11} - \gamma_1) = \delta^2 \quad \text{or} \quad \alpha_{12}^2 \ , \tag{4.121}$$

with $\mathcal{A}_{2112} \neq 0$. Since the right-hand sides of (4.120) and (4.121) are both positive, only one or the other of these modes can arise, depending on the relative sizes of α_{11} or γ_1 (both must be positive by the strong-ellipticity condition).

For Case 9, taking the limit $\eta \to 0$ in (4.69) and (4.73) gives

$$\alpha_{12} + 2\mathcal{A}_{2112} = 0 \ , \tag{4.122}$$

with $\alpha_{11} = \Omega^2$ and

$$2\alpha_{12} + \mathcal{A}_{2112} = 0 \ , \tag{4.123}$$

with $\gamma_1 = \Omega^2$, respectively.

b) $\eta \to \infty$.

This time, on taking $\eta \to \infty$ we find that limits only exist for Cases 1–4 and the limits for the antisymmetric and symmetric modes coincide. For all four cases, the limit as $\eta \to \infty$, of the frequency equations, is

$$\mathcal{A} = -\sqrt{\frac{a'}{c}} \,, \tag{4.124}$$

which can be written as

$$\frac{\alpha_{12}^2}{(\bar{\alpha}_{11}\alpha_{22})^{1/2}} + \frac{\mathcal{A}_{2112}^2}{(\bar{\gamma}_1\gamma_2)^{1/2}} - \left[(\bar{\alpha}_{11}\alpha_{22})^{1/2} + (\bar{\gamma}_1\gamma_2)^{1/2}\right] = 0 \,. \tag{4.125}$$

In Dowaikh and Ogden (1991) equation (4.125), in a different notation and form, was found to be the secular equation for Rayleigh surface waves on a compressible half–space; the reasons behind this link will be discussed in Chapter 5.

4.4.2 The particular case of a compressible neo–Hookean material.

We again consider a strain–energy function of the form (4.90), namely

$$W = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\ln(\lambda_1\lambda_2\lambda_3)\right) \,,$$

although this time we also consider nonzero ω. We again take $\lambda_3 = 1$, so that a, b and c are given by (4.91) with, from (4.14),

$$\begin{aligned} a' &= a - \frac{\mu}{\lambda_1\lambda_2}(2\lambda_1^2 + 1)\Omega^2 + \Omega^4 \,, \\ 2b' &= 2b - \frac{\mu}{\lambda_1\lambda_2}(2\lambda_2^2 + 1)\Omega^2 \,, \\ c' &= c \,, \end{aligned} \tag{4.126}$$

and

$$\mathcal{A} = \frac{\lambda_1^2\lambda_2^2 - \lambda_1^3\lambda_2^3\Omega_0^2/\mu - 1}{\lambda_2^2(\lambda_2^2 + 1)} \,, \tag{4.127}$$

where Ω_0 is chosen to be independent of the deformation, as in (3.154), and is given by

$$\Omega = \lambda_1\Omega_0 \,. \tag{4.128}$$

On substituting (4.126) into (4.13) we find that

$$s_1^2 = \frac{\lambda_1^2 + 1 - \lambda_1^3\lambda_2\bar{\Omega}_0^2}{(\lambda_2^2+1)} \, ,$$
$$s_2^2 = \frac{\lambda_1^2}{\lambda_2^2}(1 - \lambda_1\lambda_2\bar{\Omega}_0^2) \, , \tag{4.129}$$

(or possibly vice–versa) where

$$\bar{\Omega}_0^2 = \frac{\Omega_0^2}{\mu} \, . \tag{4.130}$$

From (4.129) we see that both roots are positive for $\bar{\Omega}_0^2 < 1/\lambda_1\lambda_2$, one root positive and the other negative when $1/\lambda_1\lambda_2 < \bar{\Omega}_0^2 < (1+\lambda_1^{-2})/\lambda_1\lambda_2$ and both roots negative for $\bar{\Omega}_0^2 > (1+\lambda_1^{-2})/\lambda_1\lambda_2$.

<u>Case 1</u>: $\bar{\Omega}_0^2 < 1/\lambda_1\lambda_2$.

Substituting from (4.127) and (4.129) into the frequency equations (4.22) and (4.24), for antisymmetric and symmetric modes respectively, gives

$$\frac{\tanh\left(\frac{1+\lambda_1^{-2}-\lambda_1\lambda_2\bar{\Omega}_0^2}{1+\lambda_2^{-2}}\right)^{\frac{1}{2}}\eta_0}{\tanh\,(1-\lambda_1\lambda_2\bar{\Omega}_0^2)^{\frac{1}{2}}\eta_0} = \left[\frac{(1-\lambda_1\lambda_2\bar{\Omega}_0^2)^{1/2}(\lambda_2^2+1)^{3/2}}{\lambda_1^2\lambda_2(1+\lambda_1^{-2}-\lambda_1\lambda_2\bar{\Omega}_0^2)^{3/2}}\right]^{\pm 1} \, , \tag{4.131}$$

where the $+(-)$ sign corresponds to antisymmetric (symmetric) modes and η_0 is defined as in (4.94).

<u>Case 2</u>: $\bar{\Omega}_0^2 = \lambda_2(\lambda_2^{-2} - \lambda_1^{-2})/\lambda_1$.

Here we have $s_1^2 = s_2^2 = 1$, so that this case lies wholly within Case 1, and the frequency equations, from (4.31) and (4.32), can be written jointly as

$$\frac{\sinh(2\lambda_2\eta_0/\lambda_1)}{2\lambda_2\eta_0/\lambda_1} = \pm\frac{1}{2\lambda_2^2 - 1} \, , \tag{4.132}$$

where the $+(-)$ sign corresponds to antisymmetric (symmetric) modes.

Clearly for either mode to arise we require that $\lambda_1 > \lambda_2$, with antisymmetric modes possible when $1/\sqrt{2} < \lambda_2 < 1$ and symmetric modes possible when $0 < \lambda_2 < 1/\sqrt{2}$.

On considering (4.129) it is clear that Cases 3, 4 and 9 cannot occur for this material.

Case 5: $\bar{\Omega}_0^2 > (1+\lambda_1^{-2})/\lambda_1\lambda_2$.

Here s_1^2 and s_2^2 are both negative and the frequency equations can be written in the form

$$\frac{\tan\left(\frac{\lambda_1\lambda_2\bar{\Omega}_0^2-1-\lambda_1^{-2}}{1+\lambda_2^{-2}}\right)^{\frac{1}{2}}\eta_0}{\tan(\lambda_1\lambda_2\bar{\Omega}_0^2-1)^{\frac{1}{2}}\eta_0} = -\left[\frac{(\lambda_1\lambda_2\bar{\Omega}_0^2-1)^{1/2}(\lambda_2^2+1)^{3/2}}{\lambda_1^2\lambda_2(\lambda_1\lambda_2\bar{\Omega}_0^2-1-\lambda_1^{-2})^{3/2}}\right]^{\pm 1}, \tag{4.133}$$

where the $+(-)$ sign corresponds to antisymmetric (symmetric) modes.

Case 6: $1/\lambda_1\lambda_2 < \bar{\Omega}_0^2 < (1+\lambda_1^{-2})/\lambda_1\lambda_2$.

Here $s_1^2 > 0$ and $s_2^2 < 0$ and the frequency equations can be written similarly to (4.131) and (4.133), but this time in the forms (4.43) and (4.44) for antisymmetric and symmetric modes respectively.

Case 7: $\bar{\Omega}_0^2 = 1/\lambda_1\lambda_2$.

The condition $\bar{\Omega}_0^2 = 1/\lambda_1\lambda_2$ corresponds to $\bar{\gamma}_1 = 0$ and this, together with the fact that $\mathcal{A}_{2112} = \mu/\lambda_1\lambda_2 \neq 0$, implies that (4.56) holds in this case. This means that here only symmetric modes may arise, with frequency equation

$$\tanh\left[\frac{\lambda_2\eta_0}{\lambda_1(\lambda_2^2+1)^{1/2}}\right] = \frac{\lambda_2\eta_0}{\lambda_1(\lambda_2^2+1)^{3/2}}\ . \tag{4.134}$$

Case 8: $\bar{\Omega}_0^2 = (1+\lambda_1^{-2})/\lambda_1\lambda_2$.

This time the condition $\bar{\Omega}_0^2 = (1+\lambda_1^{-2})/\lambda_1\lambda_2$ corresponds to $\bar{\alpha}_{11} = 0$, and for this material we have $\alpha_{12} \equiv 0$, for all admissible deformations. This means that only the solution represented by (4.51), namely

$$v_1 = B_1\left\{\begin{matrix}\cos px_1\\ -\sin px_1\end{matrix}\right\}e^{-i\omega t}\ ,$$
$$v_2 = 0\ ,$$

may occur in this case, with

$$\omega^2 = \frac{\mu p_0^2}{\rho\lambda_1\lambda_2}(\lambda_1^{-2}+1) \; . \tag{4.135}$$

4.4.3 Numerical results for the neo–Hookean material.

For this strain–energy function we find that in the first quadrant of $(\lambda_1,\, \lambda_2,\, \bar{\Omega}_0)$–space the surface represented by $\bar{\Omega}_0^2 = 1/\lambda_1\lambda_2$, corresponding to Case 7, lies strictly below the surface $\bar{\Omega}_0^2 = (1+\lambda_1^{-2})/\lambda_1\lambda_2$, corresponding to Case 8, and so this quadrant is divided into three regions corresponding to Case 1 on the bottom, Case 5 on top and Case 6 lying between these two surfaces; the remaining Case 2 lies wholly within Case 1. In order to permit a comparison with the results for incompressible materials, it is found that on substituting the strain–energy function (4.90) into (2.31), the stress σ_2 can be written as

$$\bar{\sigma}_2 = \frac{(\lambda_2^2-1)}{\lambda_1\lambda_2} \; , \tag{4.136}$$

where $\bar{\sigma}_2 = \lambda_3\sigma_2/\mu$ is defined as in Section 3.6 for the incompressible neo–Hookean material.

In Figures 4.9–4.13 we show dispersion spectra, i.e. we plot $\bar{\Omega}_0$ against η_0, for the compressible neo–Hookean material proposed in the previous section. On comparison with the results for the corresponding incompressible neo–Hookean material the most immediately striking difference is that now the solution branches for the antisymmetric and symmetric modes can now cross over one another; all such cross–over points occur within Case 5, which cannot arise in the incompressible case, and correspond to values of λ_1, λ_2 and $\bar{\Omega}_0$ such that

$$s_1^* + s_2^* = \frac{k\pi}{\eta} \; ,$$

for some integer k, where s_1^* and s_2^* are defined as in (4.38).

Figures 4.9–4.11, with $\lambda_1 = 0.7$, $\lambda_2 = 0.7$, 1 and 1.5 which from (4.136) can be seen to correspond to $\bar{\sigma}_2 \approx -1.04$ in Figure 4.9, $\bar{\sigma}_2 = 0$ in Figure 4.10 and

$\bar{\sigma}_2 \approx 1.19$ in Figure 4.11, can be compared with Figures 3.11–3.13, respectively, in the incompressible case. The only major difference between the two cases, apart from the existence of cross–over points, occurs with the lowest symmetric mode, which in the incompressible case has a cut–off frequency, above which it cannot occur for any value of the mode number, but which has no such bound in the compressible case. On taking $\bar{\Omega}_0 = 0$ it is seen that similar stability arguments hold for both cases. Similarly Figures 4.10, 4.12 and 4.13, with $\lambda_2 = 1$, can be compared with Figures 3.12, 3.3 and 3.15, respectively, in the incompressible case.

In Figures 4.14 and 4.15 we plot the frequency spectrum, for $\bar{\Omega}_0$ against λ_1, corresponding to the cases with $\lambda_2 = 1.0$ and $\lambda_2 = 0.7$ respectively, and we note that the lowest antisymmetric mode in Figure 4.14 does not occur in Figure 4.15; on solving (4.132) for Figure 4.15 we find that $\lambda_1 \approx 0.22$, which is less than $\lambda_2 = 0.7$, so that Case 2 does not arise. Figure 4.14 can be compared with Figure 3.19, $\bar{\sigma}_2 = 0$ and $\eta_0 = 1$ in each, and, apart from the cross–over points, the only significant difference is that while all the incompressible modes occur within a finite range of values for $\bar{\Omega}_0$ only the lowest symmetric and antisymmetric modes are so bounded in the compressible case. Finally, in Figure 4.16 we plot a frequency spectrum for $\bar{\Omega}_0$ against λ_2, with a fixed value of λ_1, which can be compared with Figures 4.14 and 4.15 above for the reverse situation.

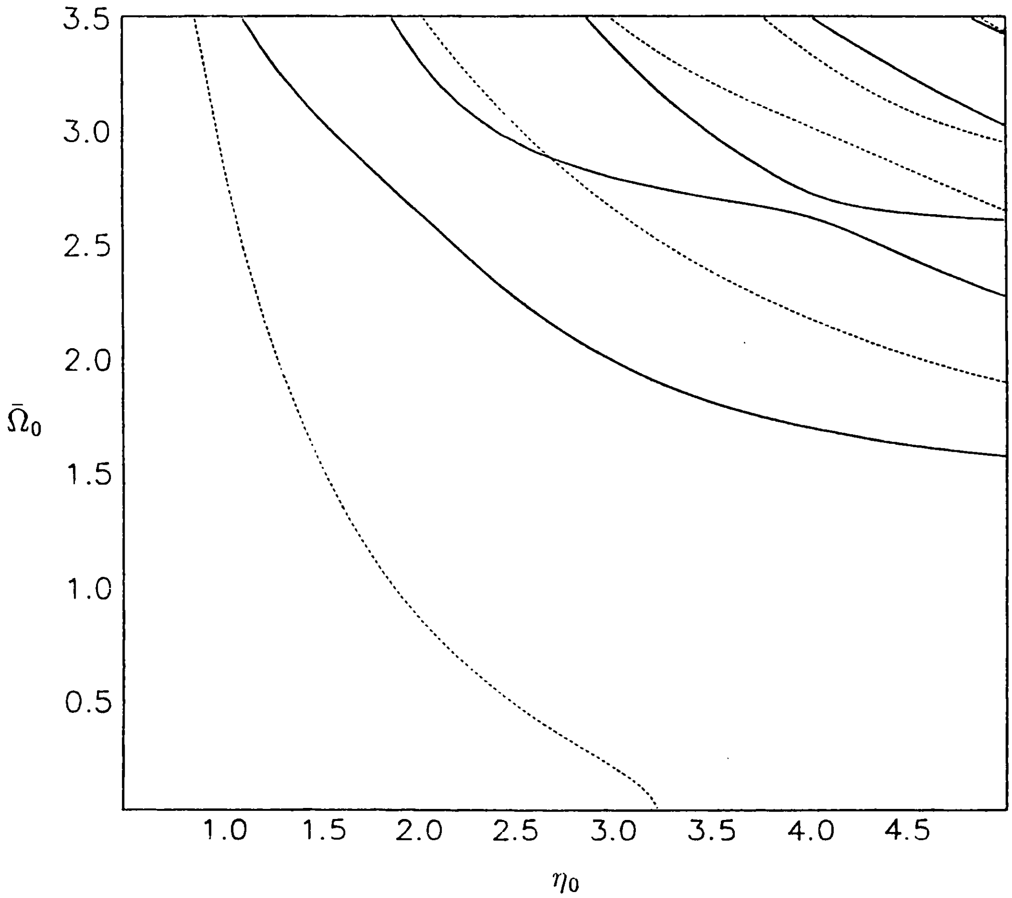

Figure 4.9: Dispersion spectrum for a compressible neo–Hookean material which has been subjected to an equibiaxial deformation with $\lambda_1 = \lambda_2 = 0.7$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

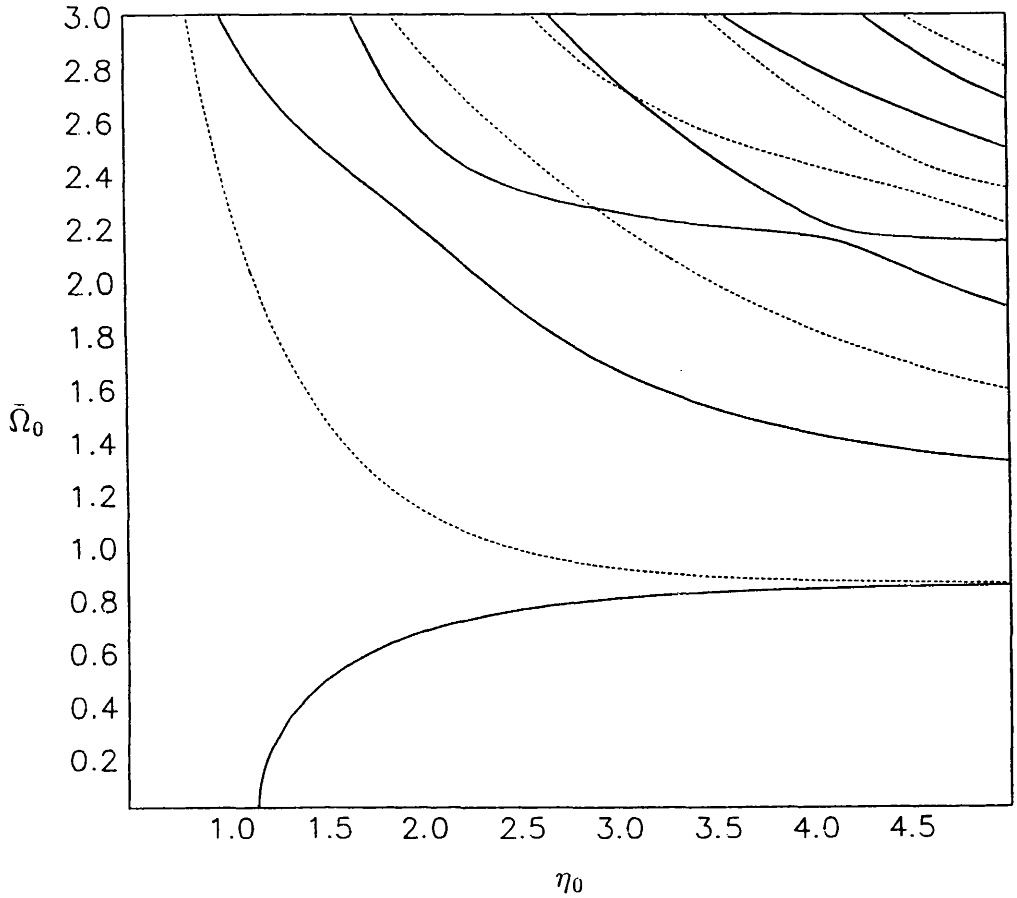

Figure 4.10: Dispersion spectrum for a compressible neo–Hookean material which has been subjected to a homogeneous deformation with $\lambda_1 = 0.7$ and $\lambda_2 = 1.0$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

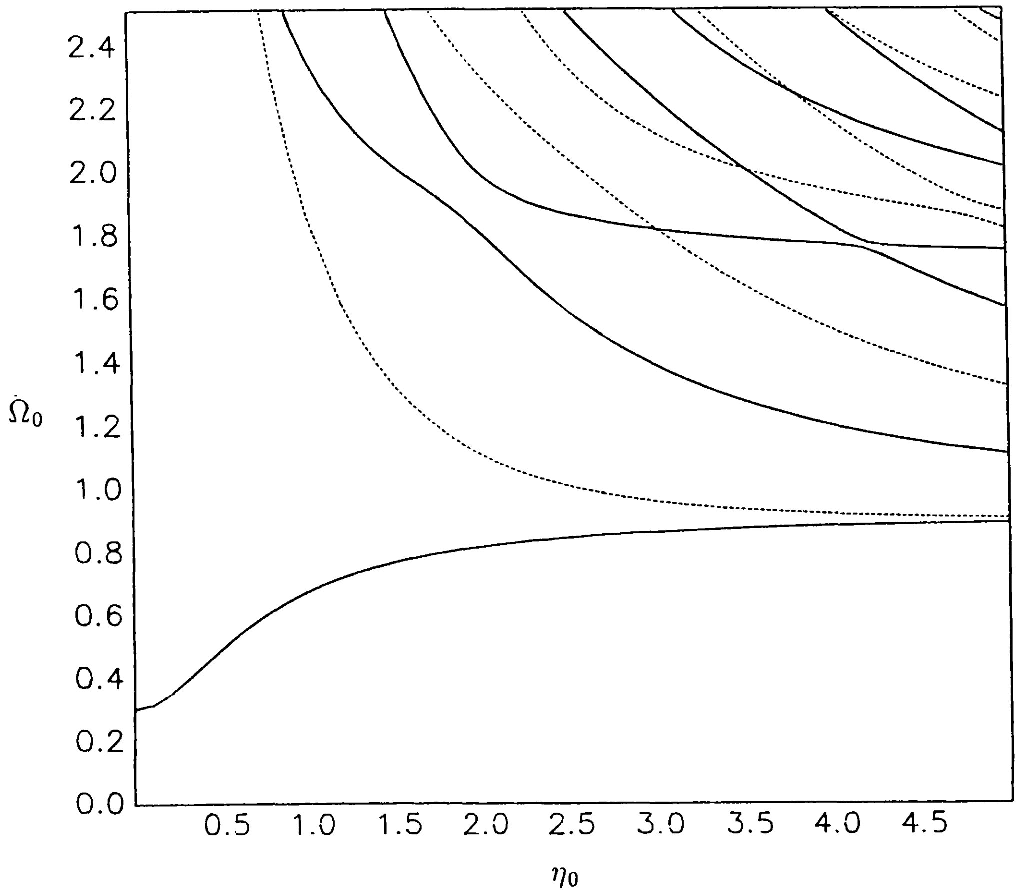

Figure 4.11: Dispersion spectrum for a compressible neo–Hookean material which has been subjected to a homogeneous deformation with $\lambda_1 = 0.7$ and $\lambda_2 = 1.5$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

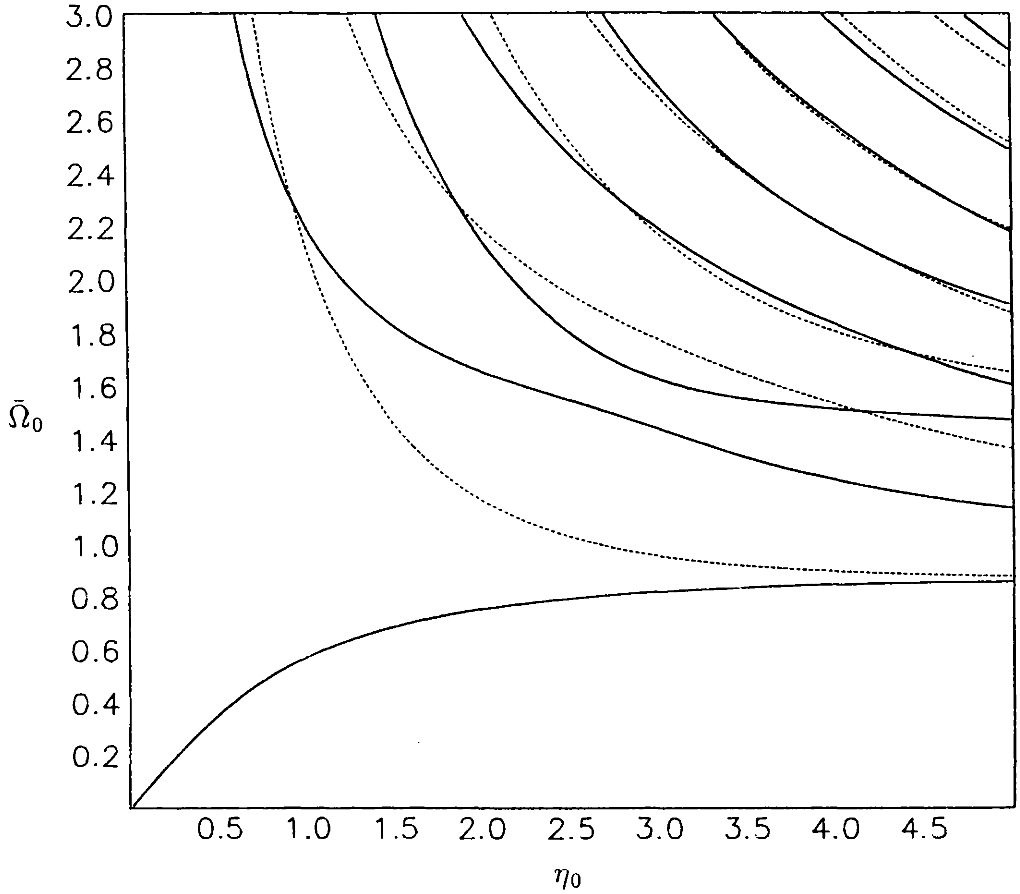

Figure 4.12: Dispersion spectrum for an undeformed compressible material, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

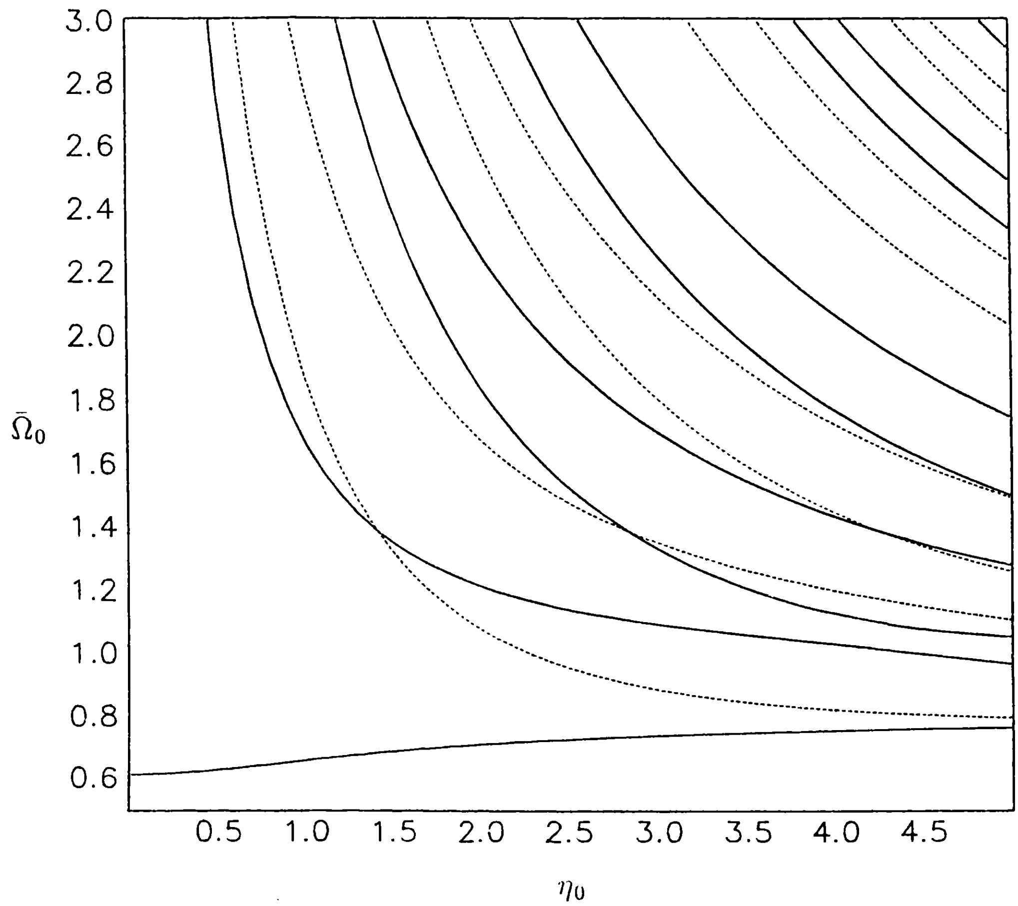

Figure 4.13: Dispersion spectrum for a compressible neo–Hookean material which has been subjected to a homogeneous deformation with $\lambda_1 = 1.5$ and $\lambda_2 = 1.0$. The solid (broken) curves represent the antisymmetric (symmetric) solution branches.

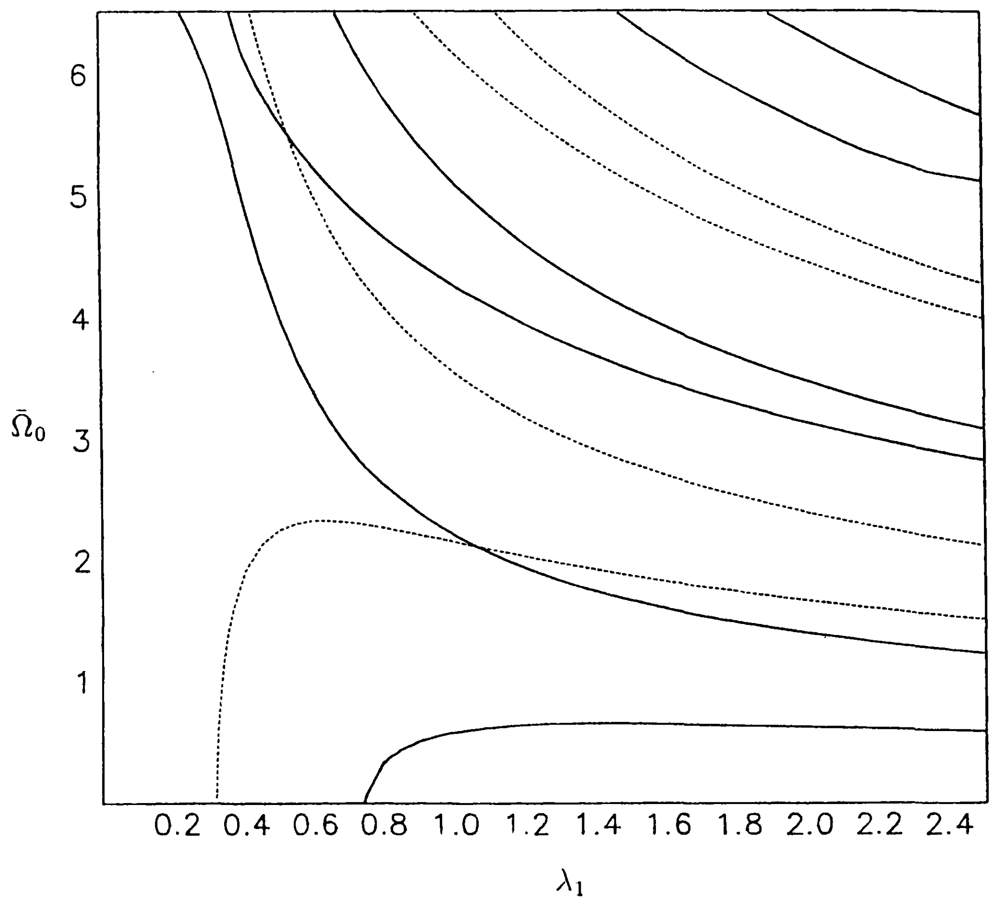

Figure 4.14: Frequency spectrum for a compressible neo–Hookean material, with stretch $\lambda_2 = 1.0$ and mode number $\eta_0 = 1$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

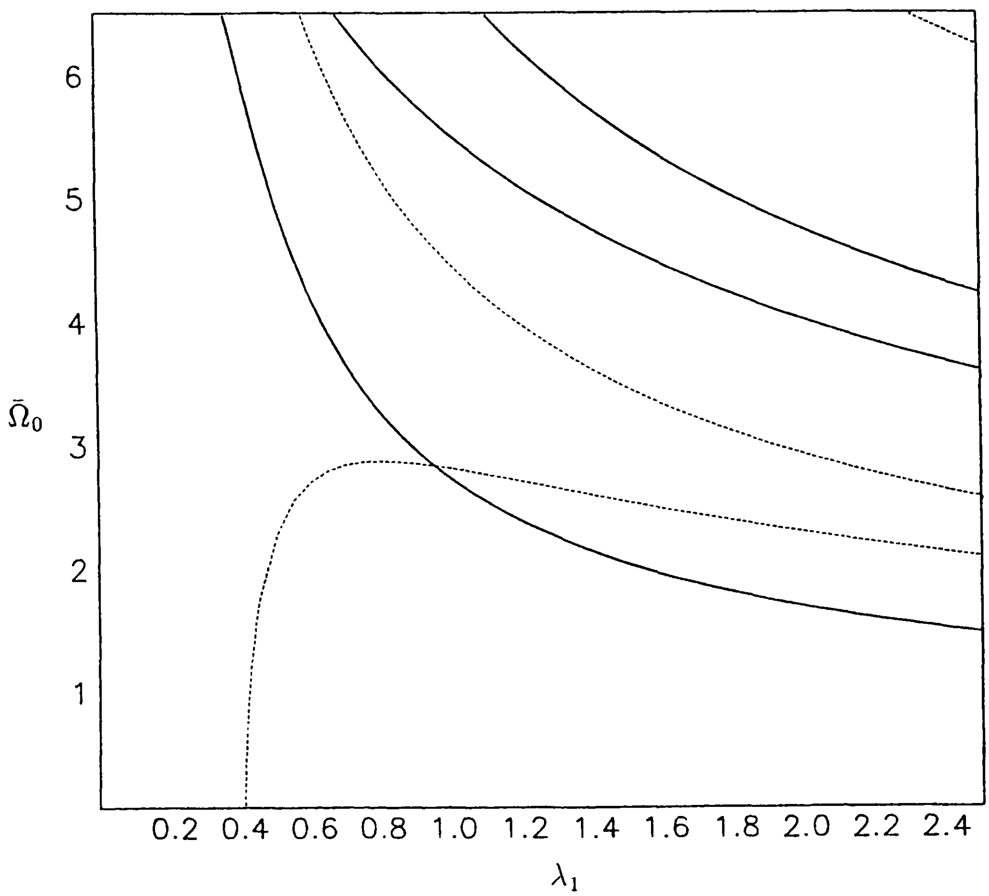

Figure 4.15: Frequency spectrum for a compressible neo–Hookean material, with stretch $\lambda_2 = 0.7$ and mode number $\eta_0 = 1$, where the solid (broken) curves represent the antisymmetric (symmetric) solution branches.

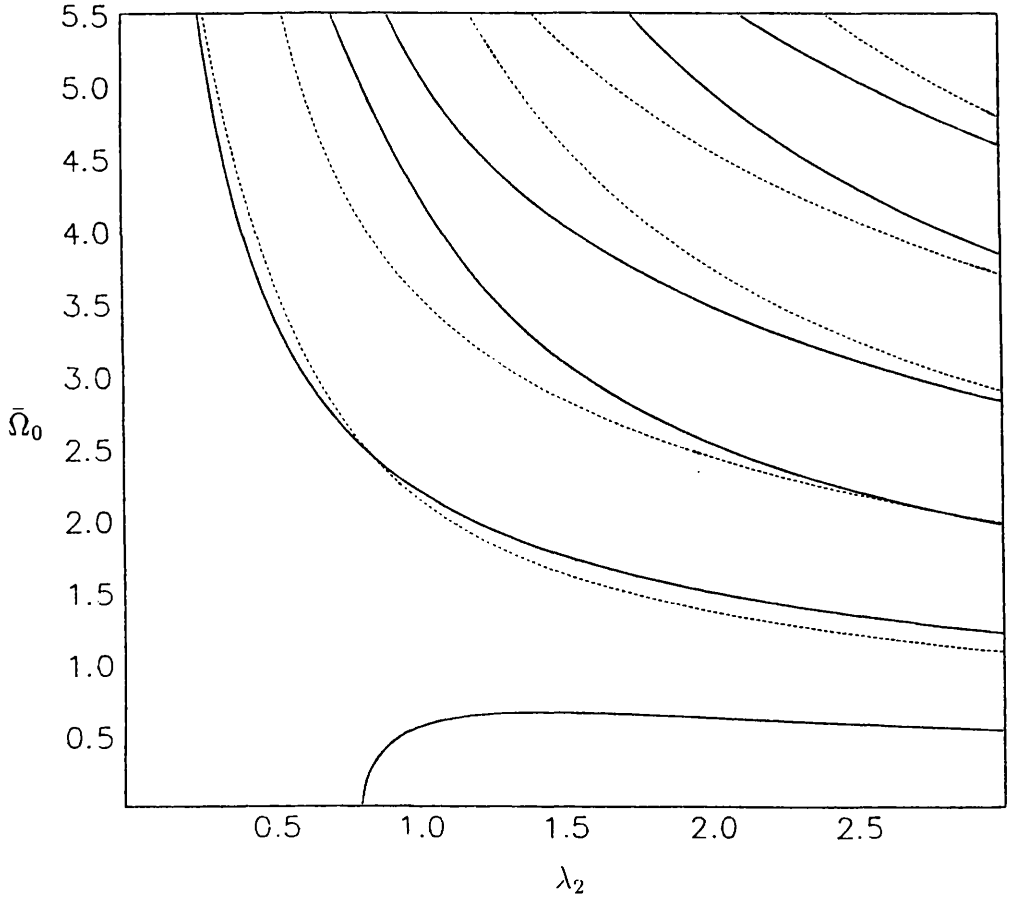

Figure 4.16: Frequency spectrum for a compressible neo–Hookean material, with stretch $\lambda_1 = 1.0$ and mode number $\eta = 1$, where the solid (broken) lines represent the antisymmetric (symmetric) solution branches.

Chapter 5 – Applications to Elastic Waveguides.

In this chapter we discuss briefly the theory of linear elasticity and with it derive the Rayleigh–Lamb frequency equations for plane waves in a linearly elastic waveguide. We then show how the results of Chapters 3 and 4 can be applied to the related problem of an infinite elastic layer of pre–stressed material and justify them by deriving known results for Lamb and Rayleigh waves.

5.1 Linear theory of elastic waveguides.

If $\mathbf{u}(\mathbf{x}, t)$ is a (small) displacement from a given unstressed reference configuration then the corresponding small–strain tensor, $\mathbf{E}$, is given in component form by

$$\mathrm{E}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) , \tag{5.1}$$

so that $\mathbf{E}$ is clearly symmetric. If $\mathbf{T}$ is the linear stress tensor then it is related to the strain $\mathbf{E}$ by the constitutive relation

$$\mathbf{T} = \boldsymbol{\mathcal{A}}_0 \, \mathbf{E} , \tag{5.2}$$

where $\boldsymbol{\mathcal{A}}_0$ has the component form given by (2.46) when evaluated along its principal axes, and with this $\mathbf{T}$ can be written in component form as

$$\mathrm{T}_{ij} = \lambda \mathrm{E}_{kk}\delta_{ij} + 2\mu \mathrm{E}_{ij} , \tag{5.3}$$

where λ and μ are the usual Lamé moduli.

On substituting from (5.1) and (5.3) into the equation of motion

$$\operatorname{div} \mathbf{T} = \rho \mathbf{u}_{,tt} , \tag{5.4}$$

we obtain the component form of (5.4), namely

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} = \rho u_{i,tt} . \tag{5.5}$$

In order to simplify the form of the equations of motion we can take the Helmholtz decomposition of the vector $\mathbf{u}$, namely

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\Psi} \ , \qquad \nabla . \boldsymbol{\Psi} = 0 \ , \tag{5.6}$$

where ϕ is an arbitrary scalar potential and $\boldsymbol{\Psi}$ is an arbitrary vector potential; the condition $\operatorname{div} \boldsymbol{\Psi} = 0$ ensures that the decomposition is unique. It can be shown, Achenbach (1984) for example, that in an unbounded medium, any solution of (5.5) can be written in the form (5.6); however when boundary conditions must be satisfied such a decomposition provides a restriction on possible solutions.

On substituting from (5.6) into (5.5) we obtain two uncoupled wave equations

$$\begin{aligned} \nabla^2\phi &= \frac{1}{c_L^2}\frac{\partial^2\phi}{\partial t^2} \ , \\ \nabla^2\boldsymbol{\Psi} &= \frac{1}{c_T^2}\frac{\partial^2\boldsymbol{\Psi}}{\partial t^2} \ , \end{aligned} \tag{5.7}$$

where c_L and c_T are constants given by

$$c_L^2 = \frac{\lambda + 2\mu}{\rho} \ , \qquad c_T^2 = \frac{\mu}{\rho} \ , \tag{5.8}$$

which represent the speed of propagation of P–waves and S–waves respectively; see, for example, Miklowitz (1966).

We now consider the specific problem of an infinite layer with $-l_2 \leq x_2 \leq l_2$, and restrict our attention to plane strain solutions in the (x_1, x_2)–plane, so that we have

$$u_3 \equiv 0 \ , \qquad \frac{\partial}{\partial x_3}(\,) \equiv 0 \ ,$$

and thus the equations (5.7) reduce to

$$\begin{aligned} \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} &= \frac{1}{c_L^2}\frac{\partial^2\phi}{\partial t^2} \ , \\ \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} &= \frac{1}{c_T^2}\frac{\partial^2\psi}{\partial t^2} \ , \end{aligned} \tag{5.9}$$

where ψ is a scalar potential function. We now look for plane–harmonic waves propagating in the x_1–direction and so we consider solutions of the form

$$\begin{aligned} \phi &= w_1(x_2)\exp\left(i(\omega t - kx_1)\right) \ , \\ \psi &= w_2(x_2)\exp\left(i(\omega t - kx_1)\right) \ , \end{aligned} \tag{5.10}$$

where ω is the frequency, k is the wavenumber and w_1 and w_2 are arbitrary functions of x_2. On putting (5.10) into (5.9), the equations reduce to

$$\begin{aligned} w_1'' &= \alpha^2 w_1 \ , \\ w_2'' &= \beta^2 w_2 \ , \end{aligned} \tag{5.11}$$

where

$$\alpha^2 = k^2 - \frac{\omega^2}{c_L^2} \ , \qquad \beta^2 = k^2 - \frac{\omega^2}{c_T^2} \ . \tag{5.12}$$

The equations (5.9) then have the general solution

$$\begin{aligned} \phi &= [A\sinh(\alpha x_2) + C\cosh(\alpha x_2)]\, e^{i(\omega t - k x_1)} \ , \\ \psi &= [B\cosh(\beta x_2) + D\sinh(\beta x_2)]\, e^{i(\omega t - k x_1)} \ . \end{aligned} \tag{5.13}$$

Note that α^2 and β^2 could also be negative, in which case we would get trigonometric, instead of hyperbolic, terms (isolated transitional cases also occur, but we do not consider them here).

If we now take the surfaces $x_2 = \pm l_2$ to be stress–free, so that

$$\mathrm{T}_{21} = \mathrm{T}_{22} = 0 \ , \quad \text{when } x_2 = \pm l_2 \ , \tag{5.14}$$

then on substituting from (5.13) into (5.3), (5.6) and (5.14), we find that two distinct modes are possible. On taking $B = D = 0$, which corresponds to antisymmetric (or flexural) modes, we obtain the frequency equation

$$\frac{\tanh(\alpha l_2)}{\tanh(\beta l_2)} = \frac{4k^2\alpha\beta}{(\beta^2 + k^2)^2} \ , \tag{5.15}$$

where α and β are chosen to be positive. On the other hand, if we take $A = C = 0$, which corresponds to symmetric (or barreling or longitudinal) modes, then the frequency equation is given by

$$\frac{\tanh(\alpha l_2)}{\tanh(\beta l_2)} = \frac{(\beta^2 + k^2)^2}{4k^2\alpha\beta} \ . \tag{5.16}$$

Equations (5.15) and (5.16) are the **Rayleigh–Lamb frequency equations** for antisymmetric and symmetric modes respectively (Rayleigh (1889), Lamb (1890)). For a material with positive bulk and shear moduli, we must have

$\lambda + \frac{2}{3}\mu > 0$ and $\mu > 0$ respectively, so that from (5.8) we must have $c_L^2 > c_T^2$ for all such materials. From (5.12) we now see that three distinct possibilities can arise, namely

(i) $\alpha^2 > 0$, $\beta^2 > 0$.

The frequency equations are given by (5.15) and (5.16) in this case.

(ii) $\alpha^2 > 0$, $\beta^2 < 0$.

On writing $\beta = i\bar{\beta}$, with $\bar{\beta}$ taken to be positive, the frequency equations can be written as

$$\frac{\tanh(\alpha l_2)}{\tan(\bar{\beta} l_2)} = -\frac{4k^2\alpha\bar{\beta}}{(k^2 - \bar{\beta}^2)^2} , \tag{5.17}$$

for antisymmetric modes, and

$$\frac{\tanh(\alpha l_2)}{\tan(\bar{\beta} l_2)} = \frac{(k^2 - \bar{\beta}^2)^2}{4k^2\alpha\bar{\beta}} , \tag{5.18}$$

for symmetric modes.

(iii) $\alpha^2 < 0$, $\beta^2 < 0$.

Here we take $\alpha = i\bar{\alpha}$ and $\beta = i\bar{\beta}$, with $\bar{\alpha}$ and $\bar{\beta}$ both taken to be positive, and with this the frequency equations can be written as

$$\frac{\tan(\bar{\alpha} l_2)}{\tan(\bar{\beta} l_2)} = \left[-\frac{4k^2\bar{\alpha}\bar{\beta}}{(k^2 - \bar{\beta}^2)^2}\right]^{\pm 1} , \tag{5.19}$$

where the +(–) sign corresponds to antisymmetric (symmetric) modes.

In Case (i), by making use of the monotonicity of tanh, it is possible to obtain necessary conditions for the existence of solutions, though we do not list them here; however, the situation in Cases (ii) and (iii) is much more complicated. It was not until Mindlin (1960) that the full frequency spectra, for such apparently simple equations, were understood for these two cases. Mindlin (1960) obtained bounding curves for each of the solution branches by considering the related problem with mixed boundary conditions, $\mathrm{T}_{12} = 0$ and $u_2 = 0$ on the surfaces $x_2 = \pm l_2$, instead of (5.14). In this case the corresponding frequency equations have a much simpler structure than the Rayleigh–Lamb equations and the solution branches can be described analytically. The use of modern computers has allowed the results of Mindlin to be easily verified, and in Figures 5.1–5.3 the dispersion curves of the Rayleigh–Lamb equations are shown for several values of Poisson's ratio, with Figure 5.3 corresponding to Figure 18 of Mindlin (1960) for $\nu = 1/3$.

Note that here we are only interested in propagating waves and so we restrict our attention, solely, to real values of the wavenumber.

So far in this section we have been dealing with compressible materials but we now consider the incompressible limit of these equations. For an incompressible material we must have the classical Lamé modulus $\lambda \to \infty$, which from (5.8) and (5.12) requires that $\alpha^2 \to k^2$, in this limit; this corresponds to the fact that longitudinal P–waves cannot propagate in an incompressible media. This means that the Rayleigh–Lamb frequency equation for an incompressible material, from (5.15) and (5.16), can be written as

$$\frac{\tanh(kl_2)}{\tanh(\beta l_2)} = \left[\frac{4k^3\beta}{(\beta^2 + k^2)^2}\right]^{\pm 1} , \tag{5.20}$$

where the $+(-)$ corresponds to antisymmetric (symmetric) modes, with a similar expression possible for Case (*ii*). The dispersion curves in this case are shown in Figure 5.4.

Note that since $\alpha^2 = k^2 > 0$, Case (*iii*) cannot occur for an incompressible material.

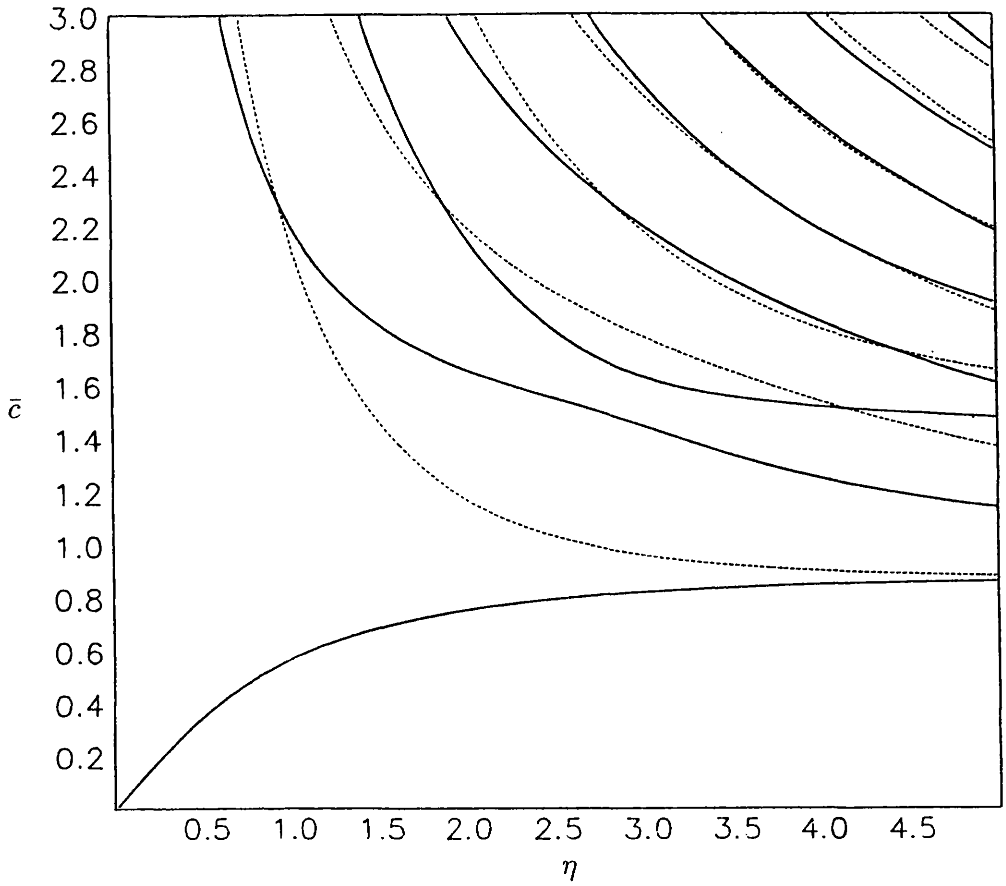

Figure 5.1: Dispersion curves for the compressible Rayleigh–Lamb equations with $\nu = \lambda = 0$, where $\bar{c} = c/c_T = \bar{\Omega}$ and $\eta = kl_2$. The solid lines represent the antisymmetric modes while the broken lines represent the symmetric modes.

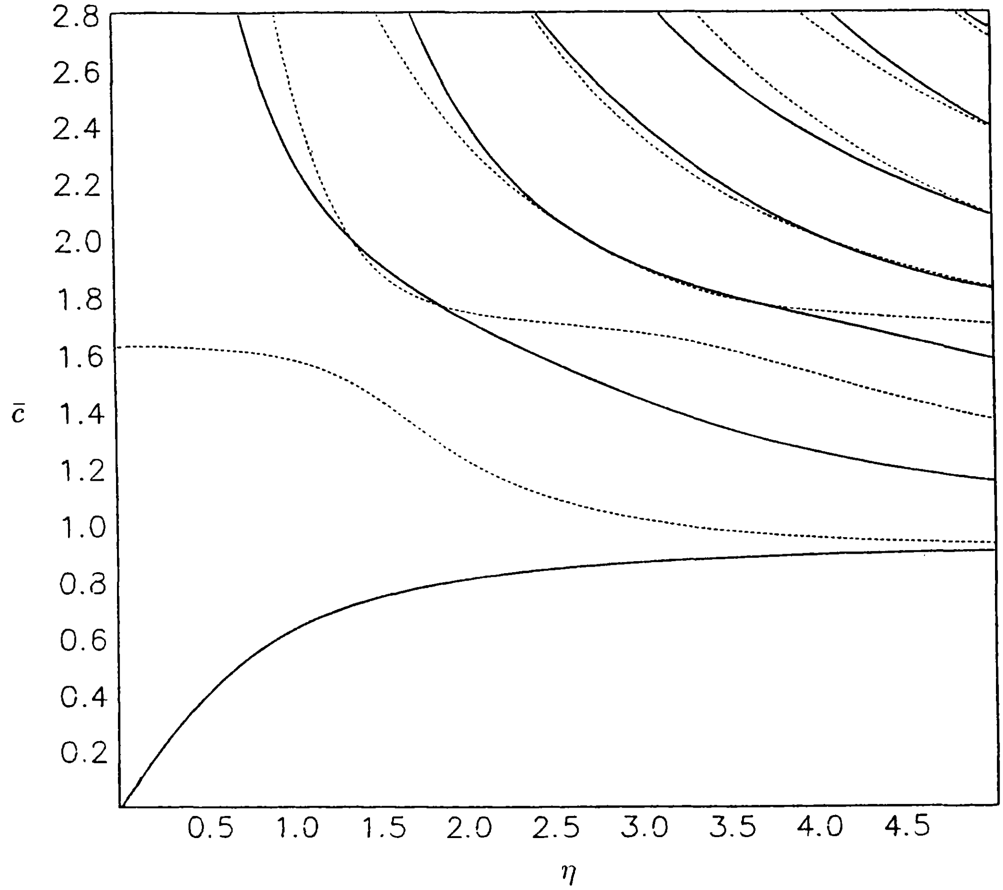

Figure 5.2: Dispersion curves for the compressible Rayleigh–Lamb equations with $\nu = 1/4$, where $\bar{c} = c/c_T = \bar{\Omega}$ and $\eta = kl_2$. The solid (broken) lines represent the antisymmetric (symmetric) modes respectively.

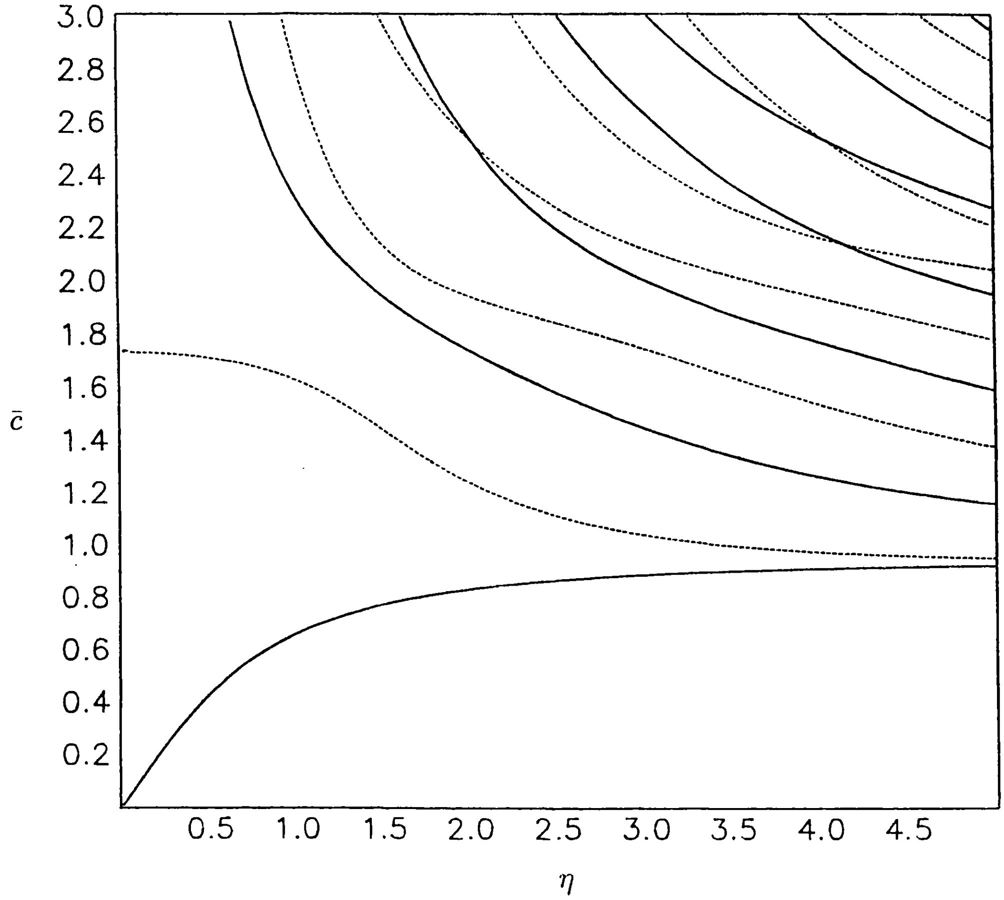

Figure 5.3: Dispersion curves for the compressible Rayleigh–Lamb equations with $\nu = 1/3$, where $\bar{c} = c/c_T = \bar{\Omega}$ and $\eta = kl_2$. The solid (broken) lines represent the antisymmetric (symmetric) modes respectively.

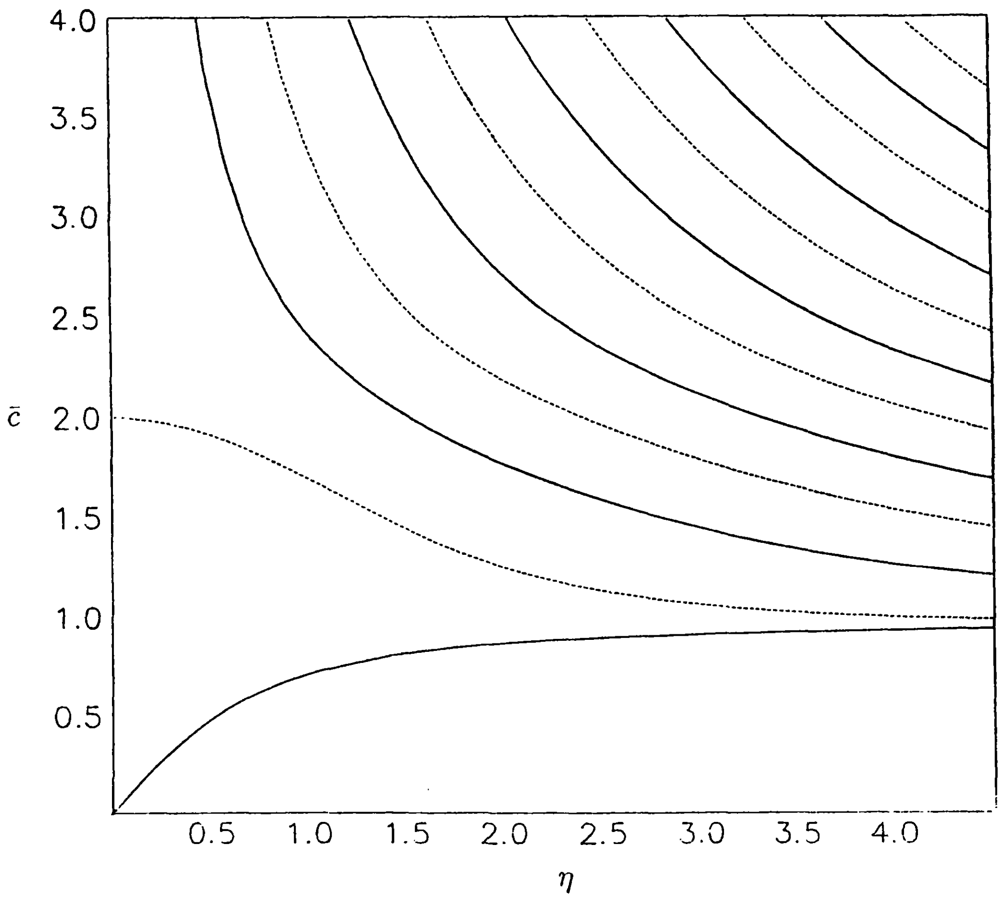

Figure 5.4: Dispersion curves for the incompressible Rayleigh–Lamb equations ($\nu = 1/2$), where $\bar{c} = c/c_T = \bar{\Omega}$ and $\eta = kl_2$. The solid (broken) lines represent the antisymmetric (symmetric) modes.

5.2 Lamb Waves.

If we allow $l_1 \to \infty$ for the finite block studied in Chapters 3 and 4 then, in this limit, it can be represented by an infinite layer in the x_1–direction of thickness $2l_2$. Now, if we again look for plane displacements $\mathbf{v}$, of the form (4.8), that is

$$\begin{aligned} v_i &= A_i \exp(spx_2 + ipx_1 - i\omega t)\ , \quad i \in \{1,\, 2\} \\ v_3 &= 0\ , \end{aligned} \tag{5.21}$$

then the expression (5.21) can be viewed as representing a plane wave in the $(x_1,\, x_2)$–plane. If p and ω are chosen to be real then we see that (5.21) represents a wave propagating in the x_1–direction with wavenumber p and phase velocity ω/p.

Note that unlike the results of Chapters 3 and 4, where the boundary conditions on $x_1 = \pm l_1$ force p to be of the form $n\pi/2l_1$, in this case no such boundary conditions apply and p is arbitrary.

The equations of motion and the boundary conditions on the top and bottom surfaces, $x_2 = \pm\, l_2$, of the layer, are identical to those of the finite plate in Chapter 4 (or Chapter 3, if the material is incompressible), which means that the results obtained in Chapters 3 and 4 also apply to this case. The frequency equations derived earlier can now be viewed as frequency equations for Lamb waves in an elastic waveguide.

5.2.1 Special cases for a compressible material.

From the results of Section 4.2, we see from equations (4.22) and (4.24), for Case 1, that the frequency equations for antisymmetric (+) and symmetric (–) Lamb waves can be written as

$$\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \left[\frac{s_2(\mathcal{A} - s_1^2)}{s_1(\mathcal{A} - s_2^2)}\right]^{\pm 1}\ , \tag{5.22}$$

where $\eta = pl_2$ and from (4.23)

$$\mathcal{A} = \frac{\bar{\alpha}_{11}(\bar{\gamma}_1\gamma_2 - \mathcal{A}_{2112}^2)}{\gamma_2(\bar{\alpha}_{11}\alpha_{22} - \alpha_{12}^2)}\ . \tag{5.23}$$

For an isotropic material, when the elastic layer is unstressed, the components of the tensor of elastic moduli $\mathcal{A}_0$ are given by (2.46), which on substituting into (4.13) gives us

$$
\begin{aligned}
s_1^2 &= 1 - \frac{\Omega^2}{2\mu + \lambda} = 1 - \frac{c^2}{c_L^2} \,, \\
s_2^2 &= 1 - \frac{\Omega^2}{\mu} = 1 - \frac{c^2}{c_T^2} \,,
\end{aligned}
\tag{5.24}
$$

where c_L^2 and c_T^2 are defined as in (5.8) and $c = \omega/p$. Further, by substituting from (2.46) into (5.23) and then into (5.22), the frequency equations can be simplified to

$$
\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \left[\frac{4s_1s_2}{(s_2^2+1)^2}\right]^{\pm 1} \,, \tag{5.25}
$$

where the $+(-)$ sign corresponds to antisymmetric (symmetric) modes. On writing $\alpha = s_1 p$ and $\beta = s_2 p$ we see that (5.25) becomes

$$
\frac{\tanh(\alpha l_2)}{\tanh(\beta l_2)} = \left[\frac{4p^2\alpha\beta}{(\beta^2+p^2)^2}\right]^{\pm 1} \,. \tag{5.26}
$$

We see from (5.15) and (5.16) that (5.26) are the Rayleigh–Lamb frequency equations for a compressible material. Correspondingly, the frequency equations (4.40), for Case 5 in Section 4.2, in the unstressed configuration reduce to (5.19), while for Case 6 the frequency equations (4.43) and (4.44) simplify to (5.17) and (5.18), respectively.

If we now take the material to be transversely isotropic, with the axis of isotropy lying along the x_1–axis of the layer, then the components of $\mathcal{A}_0$ can be written as

$$
\begin{aligned}
&\mathcal{A}_{01111} = \lambda + 4\mu_L - 2\mu_T \,, \qquad \mathcal{A}_{02222} = \lambda + 2\mu_T \,, \\
&\mathcal{A}_{01212} = \mathcal{A}_{02121} = \mathcal{A}_{02112} = \mu_L \,, \qquad \mathcal{A}_{01122} = \lambda \,,
\end{aligned}
\tag{5.27}
$$

where μ_L and μ_T are the the shear moduli along and transverse to the x_1–axis respectively. Note that when $\mu_L = \mu_T = \mu$, (5.27) reduce to the isotropic forms (2.46). The coefficients (5.27) can then be substituted into the equation of motion (4.13), but unfortunately, in this case, the roots s_1^2 and s_2^2 of this quadratic do not have simple forms.

On substituting from (5.27) into (5.22) and (5.23) we find that the frequency equations can be written in the form

$$\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \left[\frac{s_2[s_1^2(m_1m_2 - \lambda^2) + m_1\Omega^2]}{s_1[s_2^2(m_1m_2 - \lambda^2) + m_1\Omega^2]}\right]^{\pm 1} , \tag{5.28}$$

where the +(−) corresponds to antisymmetric (symmetric) modes and

$$m_1 = \lambda + 4\mu_L - 2\mu_T - \Omega^2 , \qquad m_2 = \lambda + 2\mu_T .$$

The antisymmetric form of (5.28) was obtained by Green (1982), in a different notation. On setting $\mu_L = \mu_T = \mu$, (5.28) reduces to the Rayleigh–Lamb equations.

5.2.2 Special cases for an incompressible material.

For a traction–free incompressible layer we see from the results of Section 3.2 that, when Case 1 holds, the frequency equations for Lamb waves are given by (3.25) and (3.26), namely

$$\frac{\tanh(s_1\eta)}{\tanh(s_2\eta)} = \left[\frac{s_2(s_1^2+1)^2}{s_1(s_2^2+1)^2}\right]^{\pm 1} , \tag{5.29}$$

where the +(−) sign corresponds to antisymmetric (symmetric) modes.

In the unstressed configuration, from (3.14) and (2.47), we find that

$$s_1^2 = 1 , \qquad s_2^2 = 1 - \frac{\Omega^2}{\mu} . \tag{5.30}$$

On writing $\beta^2 = p^2 s_2^2 = p^2 - \omega^2/c_T^2$, with c_T^2 given by (5.8), the frequency equations (5.29) can be written as

$$\frac{\tanh(pl_2)}{\tanh(\beta l_2)} = \left[\frac{4p^3\beta}{(\beta^2+p^2)^2}\right]^{\pm 1} , \tag{5.31}$$

which, on comparison with (5.20), can be seen to be the Rayleigh–Lamb equations for an incompressible material. From (5.30) we see that Case 6 modes can also occur here and correspond to Case (*ii*) of Section 5.1.

If we now consider applying a hydrostatic stress to the layer, so that $\sigma_1 = \sigma_2 = \sigma$ say, then since the material is incompressible we must have $\lambda_1 = \lambda_2 = 1$.

The results of Section 3.5 can now be applied to this problem and we note that the frequency equations (3.128), for Case 1, can also be written in the form

$$\frac{\tanh(pl_2)}{\tanh(\beta l_2)} = \left[\frac{(2-\bar{\sigma})^2 p^3 \beta}{(\beta^2 + p^2(1-\bar{\sigma}))^2}\right]^{\pm 1} , \tag{5.32}$$

where the $+(-)$ corresponds to antisymmetric (symmetric) modes, and $\bar{\sigma} = \sigma / \mu$. The numerical investigation of (5.32), and the corresponding equation when Case 6 holds, was carried out in Chapter 3 and we refer back to Figures 3.1–3.5 in Section 3.5 for the dispersion curves of the Lamb waves.

When the wavelength is taken to be large compared to the thickness of the layer, that is the product $\eta = pl_2$ is small, we see from equations (3.137) and (3.138) that Case 1 modes can only occur, in the limit, for $\bar{\sigma} \in [0, 2)$, with the phase velocity of the antisymmetric waves being given by given by

$$c^2 = c_T^2(2-\bar{\sigma})\left[\bar{\sigma} + \frac{(2-\bar{\sigma})(\bar{\sigma}-1)^2}{3}\eta^2\right] . \tag{5.33}$$

Whereas for symmetric waves we find that

$$c = c_T\sqrt{2(2-\bar{\sigma})}\left(1 - \frac{(2-\bar{\sigma})}{12}\eta^2\right) , \tag{5.34}$$

for $\bar{\sigma} \in (\frac{3}{2}, 2)$.

5.3 Rayleigh Waves.

If we now consider what happens in the short wavelength limit, which corresponds to the wavenumber being large, then the thickness of the layer is much greater than the wavelength. From (4.9) and (4.18), when Case 1 holds, the amplitude of the displacement has a hyperbolic dependence on the thickness variable x_2, so that on taking the limit $p \to \infty$, most of the displacement will take place at or near the surfaces $x_2 = \pm l_2$. This is the condition for Rayleigh surface waves to occur.

In Section 4.4.1 it was noted that, for Case 1, in the limit as $\eta = pl_2 \to \infty$, with the frequency ω suitably chosen so that in this limit the ratio $c = \omega/p$ remains finite, both the antisymmetric and the symmetric forms of the frequency equations have the same limit (4.125), namely

$$\frac{\alpha_{12}^2}{(\bar{\alpha}_{11}\alpha_{22})^{\frac{1}{2}}} + \frac{\mathcal{A}_{2112}^2}{(\bar{\gamma}_1\gamma_2)^{\frac{1}{2}}} - \left[(\bar{\alpha}_{11}\alpha_{22})^{\frac{1}{2}} + (\bar{\gamma}_1\gamma_2)^{\frac{1}{2}}\right] = 0\ , \tag{5.35}$$

which was shown in Dowaikh and Ogden (1991), in a different notation, to be the secular equation for Rayleigh surface waves, on a compressible half-space. When (5.35) is evaluated in the undeformed configuration, it can be written in the form

$$\xi^3 - 8\xi^2 + 8\left(3 - \frac{2\mu}{\lambda + 2\mu}\right)\xi - 16\left(1 - \frac{\mu}{\lambda + 2\mu}\right) = 0\ , \tag{5.36}$$

where $\xi = c^2/c_T^2$, which is the classical Rayleigh equation given in Ewing, Jardetzky and Press (1957).

In the incompressible case, if we again consider the short wavelength limit, the frequency equations (3.25) and (3.26), being viewed as holding for an infinite layer, again have a common limiting form given by (3.123), which in the notation of Chapter 4 can be written as

$$\sqrt{\bar{\gamma}_1\gamma_2}(\bar{\alpha}_{11} + \alpha_{22} - 2\delta + 2\gamma_2) + \gamma_2(\bar{\gamma}_1 - \gamma_2) + 2(\gamma_2 - \sqrt{\bar{\gamma}_1\gamma_2})\sigma_2 - \sigma_2^2 = 0\ , \tag{5.37}$$

which was shown in Dowaikh and Ogden (1990), in a different notation, to be the secular equation for Rayleigh waves on an incompressible half-space. On specializing (5.37) to the undeformed configuration it becomes

$$\xi^3 - 4(2 - \sigma_2)\xi^2 + 6(\sigma_2 - 2)^2\xi + (\sigma_2^2 - 4)(\sigma_2 - 2)^2 = 0\ , \tag{5.38}$$

where σ_2 can be viewed as a hydrostatic stress and $\xi = c^2/c_T^2$. Equation (5.38) was given in Dowaikh and Ogden (1990) and the corresponding classical form, with $\sigma_2 = 0$,

$$\xi^3 - 8\xi^2 + 24\xi - 16 = 0\ , \tag{5.39}$$

was given in Ewing, Jardetzky and Press (1957).

The static exclusion conditions obtained in Sections 3.3.2 and 4.3.1 for incompressible and compressible materials respectively can similarly be explained.

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