Bernstein Modes in a Weakly Relativistic e^-e^+ Plasma

David Arthur Keston

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Department of Physics & Astronomy,
Kelvin Building,
University of Glasgow,
Glasgow, G12 8QQ

ProQuest LLC.
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106 – 1346

Abstract

This thesis examines the behaviour of a homogeneous and quasineutral, equal-mass (electron-positron) plasma in the presence of a constant magnetic field. Two sets of comparisons are made: between e^-e^+ and e^--ion plasmas and between treatments of a progressively more relativistic nature.

Fundamental plasma quantities: plasma frequency, ω_p; cyclotron frequency, Ω; and Debye length, λ_D, are briefly introduced in Chapter 1.

The peculiar features of electron-positron plasmas are illustrated in Chapter 2. Quantities defined in Chapter 1 are redefined for e^-e^+ plasmas. Examples of physical situations where e^-e^+ plasmas may occur are given. The predictions of cold plasma theory for e^-e^+ plasmas are summarized at the end of this chapter.

Chapter 3 embraces the kinetic theory upon which the remainder of the thesis relies. After explaining the need for a kinetic theory, the development of that theory is reviewed. The first part of the chapter shows the microscopic (Klimontovich) description. Next the necessary concepts from Gibbs' and Boltzmann's statistical theory are presented. These ideas are married in the BBGKY hierarchy. The lowest order expansion of the hierarchy gives the Vlasov equation. A dielectric treatment can then be carried out using the Vlasov equation and Maxwell's equations. At this point the equilibrium conditions are stated. After some analysis general dispersion relations are found for both e^--ion and e^-e^+ plasmas. These expressions are general in the sense that they permit a choice of momentum distribution.

In Chapter 4 the momentum distribution function is the familiar Maxwellian distribution function. The dispersion relations are then used to derive the (electrostatic) Bernstein

modes. Bernstein modes propagate perpendicular to the magnetic field and resonate at electron cyclotron harmonics ($\omega \approx n\Omega_e$). In an electron-ion plasma, there are seen to be gaps in the frequency spectrum away from $n\Omega_e$ where these modes may not propagate. e^-e^+ plasmas are different: the theory leading to frequency gaps is exact and not a consequence of approximation. These relations are then illustrated.

Original work begins in Chapter 5. In this thesis, interest in a weakly relativistic plasma stems from the wish to observe the transition between the existing non-relativistic and fully relativistic kinetic treatments. The relativistic nature of these treatments is governed by the parameter $a = m_0c^2/kT$: $10 \leq a \leq 100$ corresponds to weakly relativistic conditions. Chapter 5 is concerned with weakly relativistic e^-e^+ plasmas. A novel combination of non-relativistic distribution function and otherwise fully relativistic dispersion relation leads to the dispersion relation for weakly relativistic e^-e^+ plasmas. This expression is then prepared for inclusion in computer code: it is restated in dimensionless units and rearranged so that double quadrature becomes single quadrature with a special function.

The design of the computer code is discussed in Chapter 6. The resulting dispersion curves are shown and described in Chapter 7. It is demonstrated that, as for e^--ion plasmas, the introduction of a weakly (or fully) relativistic treatment sees a broadening of the frequencies at which resonance occurs and a downshift in those frequencies. These results have been described briefly in conference proceedings [1, 2]. This chapter goes on to consider the possibility of an approximate analytical approach and suggests the direction future work will take.

There are three appendices. They deal respectively with: the properties of certain special functions; contour integration; and a listing of the code which is described in Chapter 6.

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Chapter 1

Introduction

> When a man gives himself up to the government of a ruling passion, — or, in other words, when his HOBBY-HORSE grows headstrong, — farewel cool reason and fair discretion!
>
> *Tristram Shandy*, Laurence Sterne

While it is often remarked that plasma is the fourth state of matter, it is certainly true that it is the least well-defined state. The consensus is that a plasma is an ionized gas in which *collective behaviour* dominates. Electrons and ions exchange energy and momentum with each other through their effects on, and reaction to, electromagnetic fields. The fields can be external or the result of the motions of the charged particles themselves, the so-called self-consistent field: often both types of field are involved.

This self-consistency is very important. Single particle interactions are treated as collisions: while the dynamics of a plasma are pictured as the superposition of two clouds, interacting only through the field to which both are coupled. If the coupling force is extreme enough to keep the clouds more or less together, their relative velocity is proportional to the electric current and the entire plasma is treated as a single, current-carrying fluid. The study of such plasmas is known as magnetohydrodynamics (MHD). Langmuir (or plasma) waves represent the other extreme: dynamics are so fast that the ions are essentially stationary – all that oscillates is the electron cloud and with it the electric

field.

Plasmas often possess significant thermal energy. Relativistic particle velocities are likely and complicate any kinetic description. This is especially true of low mass particles such as electrons which reach relativistic velocities at lower energies. Nevertheless in this investigation of electron-positron plasmas, we expect to reveal simplifications due to the nature of the plasma.

Throughout this work various assumptions have been made about the conditions under which the plasma exists. Namely, it is assumed that the plasma is locally neutral, homogeneous and hot. Note that under these conditions collisional terms may be neglected, see Section 3.2.

The propagation of waves in a *weakly* relativistic e^-e^+ plasma *perpendicular* to a uniform equilibrium magnetic field is of particular interest. Kinetic theory seems the most appropriate approach to plasmas in this domain. The remainder of this chapter defines the basic terms used throughout this work, no assumption is made about the species present in plasmas.

1.1 Plasma Frequency

At a simplistic level one may describe plasmas in the cold approximation. One takes a neutral assembly of negative and positive particles with a distribution which is initially uniform throughout. In addition one assumes that there is *no thermal motion.* Under such circumstances the effect of displacing a group of particles, say electrons, is to make the plasma non-neutral around the displacement. The displaced particles store potential energy and have a tendency to return to their equilibrium positions. Once they reach the equilibrium position however the potential energy has converted to kinetic energy and the particles overshoot. Away from equilibrium again the motion of the particles induces (and is opposed by) an electric field. In a cold plasma there is no loss mechanism and once a perturbation has been introduced the oscillations in the plasma will carry on indefinitely. The root mean square of the frequencies of these fundamental plasma oscillations for each

species, s, present is known as the plasma frequency and will be denoted ω_p.

$$\omega_p^2 = \sum_s \omega_{ps}^2 = \sum_s \frac{n_{s0} q_s^2}{\epsilon_0 m_{s0}} \tag{1.1}$$

In many electron-ion plasmas the ions may be treated as a stationary background to the extent that 'plasma frequency' often refers to the electron plasma frequency, ω_{pe}.

Note that ω_p depends on both the properties of the particles themselves, q_s and m_{s0}, and the parameters of the plasma, n_{s0}.

1.2 Debye length

$$\lambda_D = \sqrt{\frac{\epsilon_0 kT}{ne^2}} \tag{1.2}$$

The Debye length, λ_D, can be pictured as the effective range of the Coulomb force in a plasma. It is calculated by coupling an electrostatic formulation of Poisson's equation and the Boltzmann equation $n(E) = n_0 e^{-E/\kappa T}$ self-consistently through the charge density. A test particle at a distance greater than λ_D is effectively screened from the field due to the source by intervening ambient particles.

The dominance of collective effects may be estimated from the typical number of particles in a Debye sphere (a sphere of radius λ_D). If the characteristic interparticle distance, $n^{-1/3}$, is small in comparison to λ_D many particles are close enough to one another that Coulomb forces are not shielded, the plasma must behave collectively. One criterion for a plasma, then, is a large N_D, where

$$N_D = n\lambda_D{}^3$$

Coulomb collisions scale with the Debye length while large-angle particle collisions scale with interparticle distance. The former (long-range) force dominates in plasmas.

To behave as a plasma it is just as important that the Debye length be much less than the characteristic length of the plasma. That way the plasma may be considered as a continuum and effects may be treated locally.

1.3 Cyclotron Frequency

In the presence of a uniform magnetic field a single charged particle is accelerated in a circular path perpendicular to the field. This circular motion has a frequency, the cyclotron frequency, given by

$$\Omega_s = \frac{q_s B_0}{m_s} \tag{1.3}$$

Unlike the plasma frequency, the cyclotron frequency depends on particle parameters alone. Using this frequency one can create another useful quantity the characteristic cyclotron length or Larmor radius, r_L. This is the radius at which the force due to the magnetic field is in balance with the centripetal force.

$$r_L = \frac{v_\perp}{\Omega_s} \tag{1.4}$$

Chapter 2

Equal mass Plasmas

In an equal mass plasma the masses of the two species are equal while the charges are opposite. The study of equal mass plasmas is prompted by the possibility that mass equality will lead to substantially different phenomena. This in turn might permit insight into the physical behaviour of plasmas in general.

Mathematically there is a certain amount of cancellation in the calculation of dispersion relations.

For an exact equal mass plasma the only possibility is that of a particle-antiparticle plasma (Alfvén calls this *ambiplasma* [3]). The ambiplasma with the lightest component species is the electron-positron plasma.

In reference [4] basic e^-e^+ pair processes are discussed. In particular Lightman gives a review of the various possible creation and annihilation reactions [5]: whilst Takahara explores the realm of magnetized, relativistic e^-e^+ plasmas [6].

2.1 Electron-Positron Plasmas

The main positron annihilation path for positron temperatures above 100 eV is direct two-body collision annihilation (see [7]).

When discussing e^-e^+ plasmas it will be seen that particles will generally have a great deal of thermal energy, so that relativistic particle velocities are likely. One might expect

relativistic considerations to complicate the theory but by investigating electron-positron plasmas, one hopes that the symmetries inherent to an 'equal-mass, opposite charge' plasma will permit simplification.

Over a large range of conditions, a system of electrons and positrons is definitely a plasma. One important criterion is the number of particles in the Debye sphere, N_{D}. This is introduced on page 12. In the context of non-relativistic electrons and positrons the Debye length has a slightly different definition:

$$\lambda_{\mathrm{D}} = \sqrt{\frac{\epsilon_0 kT}{2ne^2}} \tag{2.1}$$

For a typical weakly relativistic temperature of around 10^8 K and electron (positron) density of say $10^{16}\,\mathrm{m}^{-3}$, $\lambda_{\mathrm{D}} \approx 8\,\mathrm{mm}$ and $N_{\mathrm{D}} \approx 10^{10}$. Densities up to $10^{30}\,\mathrm{m}^{-3}$ still fulfill the N_{D} criterion.

At this point the special properties of the e^-e^+ plasma may be illustrated by reviewing the definitions of fundamental plasma frequencies: the cyclotron frequency and plasma frequency. In the subscripts used e denotes *electron* and $\bar{e}$ represents *positron* quantities.

By definition (1.3) the electron cyclotron frequency, Ω_e, is identical to the positron cyclotron frequency but has opposite sign.

$$\Omega_e = \frac{(-e)B_0}{m_e} = -\frac{(+e)B_0}{m_{\bar{e}}} = -\Omega_{\bar{e}} \tag{2.2}$$

The plasma frequency for the pure neutral e^-e^+ plasma under consideration is given by:

$$\omega_p = \sqrt{\omega_{pe}^2 + \omega_{p\bar{e}}^2} = \sqrt{\frac{2n_{e0}e^2}{\epsilon_0 m_{e0}}} = \sqrt{2}\omega_{pe} \tag{2.3}$$

In contrast to electron-ion plasmas the positrons certainly may not be treated as a stationary background. The positron contribution is identical to that of the electron.

Electron-positron plasmas may be found in many places: in the atmospheres of pulsars; in extra-galactic jets; in the early universe; and recently, there has been an upsurge in interest in creating them in the laboratory, now that trapping techniques have been vastly improved.

2.2 Pulsars

2.2.1 What is a pulsar?

The prevailing model of pulsars, the lighthouse model, assumes that a pulsar is a rapidly rotating neutron star. Its magnetic field is exceptionally strong. Often the magnetic and rotational axes do not coincide. Charged material is channelled along the tightly packed magnetic field lines. Matter streaming from the pulsar at the poles collides violently with matter streaming into the pulsar. Large amounts of radiation are produced and this radiation in turn escapes from the vicinity of the pulsar. When observed from Earth the polar radiation will appear as a pulse, once for each revolution. This is illustrated in figure 2.1. The whole topic of pulsar magnetospheres is reviewed in many places including articles by Michel [8, 9] and Lominadze [10].

Crucial here is the nature of the matter which streams from the pulsar; it consists of electrons and positrons. This streams through a dense plasma magnetosphere, also electron-positron (the presence of which was suggested by Sturrock as far back as 1971 [11]). The interaction between streaming particles and the plasma is at the centre of the debate over the source of the exceptionally bright emission spectra of pulsars.

2.2.2 Current theories

The presence of e^-e^+ pairs in pulsars was predicted in the earliest models. Reviews may be found in [8] and in more detail in articles by Arons [12] and Ray [13]. In outline the mechanism of pair creation is as follows: the strong magnetic field separates particles of opposite charge and leaves a layer immediately above the surface of the pulsar which is relatively free of particles. This charge separation leads to a strong electric field along which electrons are accelerated to very high energies. The electrons also experience the dipolar magnetic field of the pulsar and because they are forced to move along these highly curved field lines they must radiate. This radiation in turn encounters the gap region and because of the electric field in that region produces pairs.

One of the major topics around pulsars is the mechanism of the radio emission observed

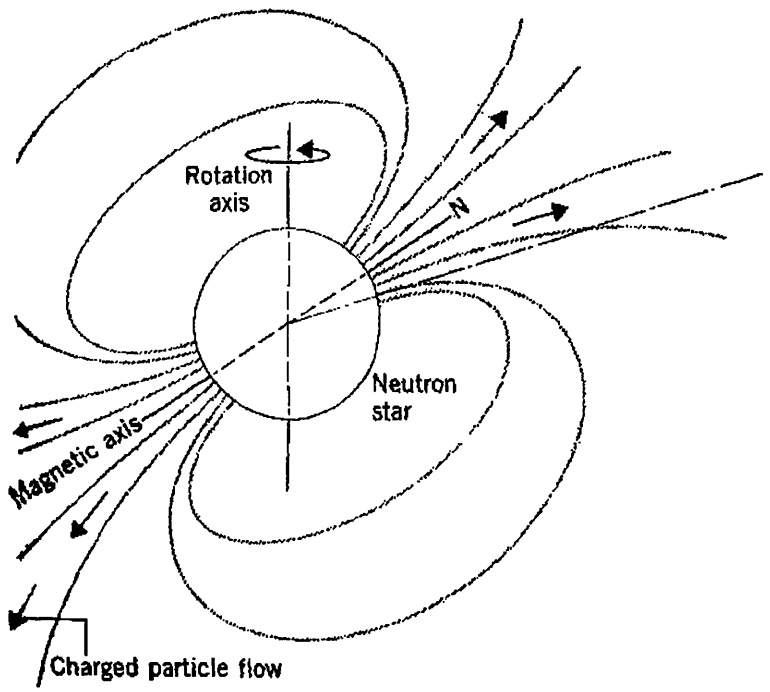


Figure 2.1: A diagram of the 'lighthouse' model of pulsars

from pulsars. In the seminal article by Arons [14] the theory of wave propagation in pulsars, *i.e.* in superstrong magnetic fields, is proposed. Many authors have worked on emission mechanisms: one good candidate is the two-stream instability, see for instance [15]. There are two regions in which researchers have considered this instability to take place: in the high polar gaps and in the pair production gaps. The latter case is ruled out by Egorenkov [16]. Usov [17] describes the creation of conditions for a two-stream instability in the polar gaps. The production of pairs is strongly non-stationary – thus electrons (and positrons) appear in clouds. These clouds disperse because of their velocity distribution and eventually, the author estimates at around 10^6m, the fastest particles from one cloud catch up with the slowest of the previous cloud. Thus there is the two-humped distribution necessary to cause the two-stream instability.

Lyubarskii too deals with the problem of the generation of pulsar radio emission [18]. He describes how emission due to the two-stream instability produces observable radio

waves: induced scattering converts the original subluminal modes into superluminal waves. These superluminal waves then transform easily into the observed electromagnetic waves.

2.3 Extragalactic Jets and Galactic Nuclei

Shrader [19] and Purcell [20] report the results from the Compton Gamma Ray Observatory in which galactic sources of annihilation line radiation are mapped for the first time. There is evidence for a central galactic emission bulge.

This is part of a body of evidence supporting the existence of e^-e^+ plasmas in the nucleus of the galaxy [21, 22]. The topic is reviewed in reference [4] in the contribution of MacCallum *et al.* [23].

Beyond this galaxy, observations of extragalactic jets suggest the presence of an e^-e^+ pair plasma [24]. The nuclei of other galaxies also appear to have annihilation line sources [25].

Some current theories of black-holes include a surrounding medium which consists of electrons and positrons. Observations of black-hole candidates agree [26, 27, 28].

2.4 Laboratory e^-e^+ Plasmas

As early as 1978, there was interest in observing the properties of an e^-e^+ plasma in the laboratory, see reference [29]. Here too can be found the first mention of the characteristic absence of Faraday rotation and thus whistler modes.

Since then the experimental work of groups specializing on positrons has made great advances. In particular the Greaves/Surko group (at the University of California, San Diego) have produced one-component positron plasmas through a scheme of positron trapping and cooling [30, 31, 32, 33, 34]. They collect and trap $\approx 10^8$ positrons and expose them to a beam of a similar number of electrons: this situation is however unstable.

At present laboratory equal mass plasmas are imperfect due to the presence of residual electrons. It is expected that the investigation of the properties of equal mass plasmas will be vastly improved with the creation of e^-e^+ plasmas. As of 1997 [32], no true e^-e^+

plasmas have been created in the laboratory.

That the creation of e^-e^+ plasmas in the laboratory is even considered is due to the recent improvements in trapping devices. The only trapping mechanism which might be able to trap both electrons and positrons simultaneously for long confinement times is the Paul trap (reviewed in [35]).

2.5 On Cold e^-e^+ Plasmas

In broad terms, a plasma may be treated as a superposition of two fluids - by making a *cold* assumption. This gives the simplest picture of the behaviour of waves in a magnetized plasma. The well-known CMA diagram is useful in illustrating this behaviour. This is general in that it does illustrate the way a mode would propagate at any angle to the equilibrium magnetic field.

The specific case of cold electron-positron plasmas has already been described by Stewart & Laing [36]. Using this research, one can plot the (now substantially simplified) CMA diagram (figure 2.2). For comparison, see the CMA diagram for a cold electron-ion plasma (figure 2.3). This simplification means that in e^-e^+ plasmas, Faraday rotation does not exist and consequently nor does the whistler mode [29, 36]. These phenomena occur as features of the difference in mass between plasma species. Any textbook will describe the theory of cold plasmas, for example [37, 38, 39]

Figure 2.2 illustrates distinct regions of parameter space. Here the ordinate is proportional to the magnetic field strength, B_0, and the abscissa is proportional to the electron density, n_{e0}. The marked regions correspond to

I High Frequency Electromagnetic Region

II Transition Region

III Highly Magnetized Region

IV Alfvén Wave Region

V Stop Region

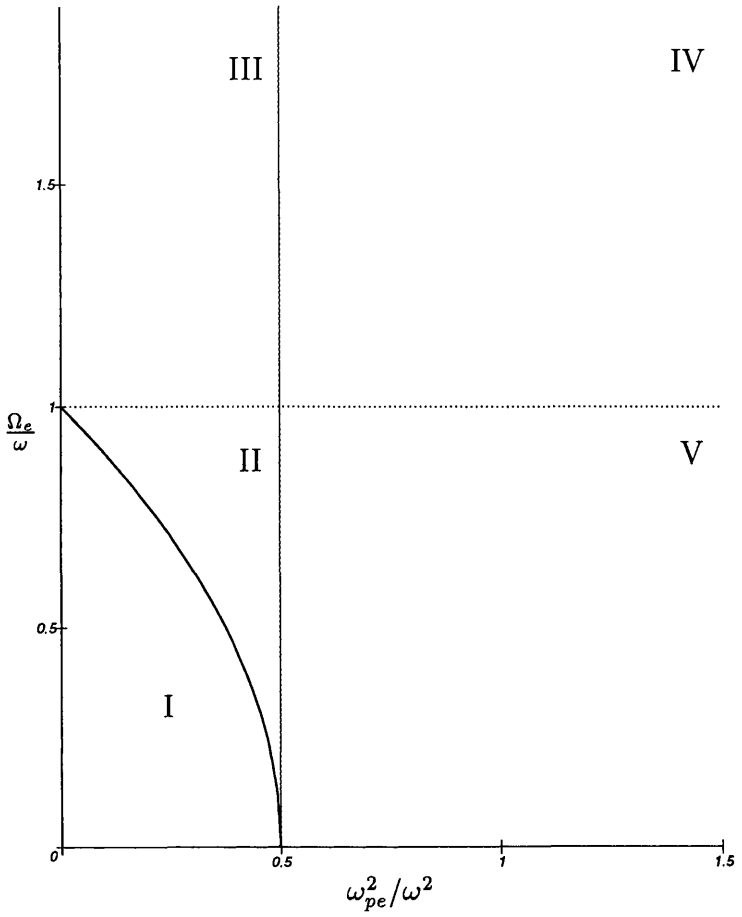


Figure 2.2: The CMA Diagram for a cold electron-positron plasma

The Stop Region is so-called because no propagation may take place in this region of parameter space.

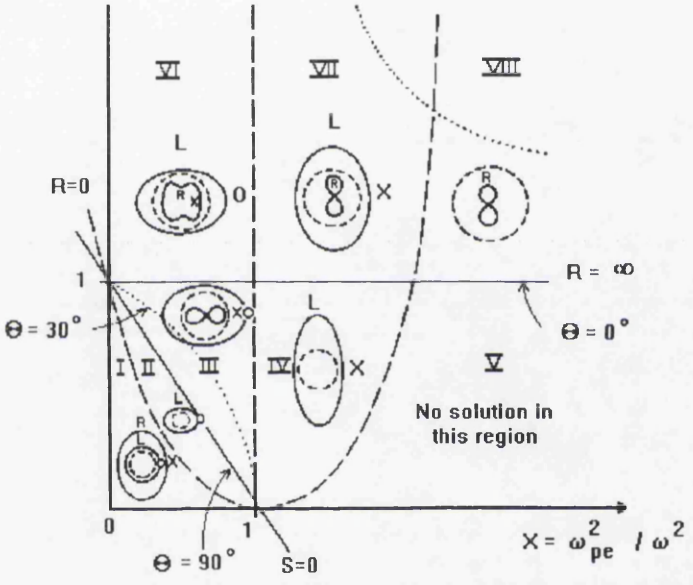


Figure 2.3: The CMA Diagram for a cold electron-ion plasma

Chapter 3

Mathematical Formulation

Seh ich die Werke der Meister an,
So seh ich das, was sie getan;
Betracht ich meine Siebensache,
Seh ich, was ich hätt' sollen machen.

Johann Wolfgang von Goethe

3.1 A Brief Introduction to Kinetic Theory

The principal characteristic of hot plasmas which sets them apart from other plasmas is the comparative length of their mean free path, λ_{mfp}. Other scale distances, such as the dimensions of plasma devices or the wavelength of the mode to be studied, are significantly smaller.

From the outset, this research was expected to deal with high frequency wave effects; partly because of the short characteristic time of pair annihilation and creation and partly because the main focus of study was the field of Bernstein modes. These conditions mean that the fluid approximation is not appropriate. Instead we require a more fundamental theory which accounts for combined distributions of all species, ion and electron, in both configuration and velocity space; the kinetic theory. In this common approach to waves in plasmas, plasmas are treated as a special class of gases. Just like a gas, a plasma is a

collection of particles, each with its own velocity and position at a particular time.

[40] gives an example. Consider the characteristic speed of one gas particle, say $100\,\mathrm{ms}^{-1}$, and the corresponding characteristic interaction distance, $10^{-10}\,\mathrm{m}$. This gives a time scale of about $10^{-12}\,\mathrm{s}$. In this time the velocity and position of a particle will change abruptly. Now treat each collision between each particle in the gas and the exact solution for all particles has become effectively impossible. This topic is introduced in many textbooks. The notation here follows that of [41]; the treatment from [40, 41, 42, 43].

3.1.1 Microscopic Description

Consider a gas of N particles with coordinates $\mathbf{X}_i = (\mathbf{x}_i, \mathbf{v}_i)$, with $i = 1...N$. The microscopic number density of gas particles may be represented intuitively by

$$N(\mathbf{X}, t) = \sum_{i=1}^{N} \delta(\mathbf{X} - \mathbf{X}_i(t)) \qquad \text{where} \qquad \mathbf{X} = (\mathbf{x}, \mathbf{v}) \tag{3.1}$$

In this way, $N(\mathbf{X}, t)$ will be 1 if there is a particle at $(\mathbf{X}, t)$ and zero otherwise. Integrating across all of six-dimensional μ-space will thus give the number of particles present in the volume. The *microscopic number density* is thus given by $\int N(\mathbf{X}, t) d\mathbf{v}$.

In addition an equation expressing the time dependence of this distribution can be introduced.

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \dot{\mathbf{X}} \cdot \frac{\partial N}{\partial \mathbf{X}} = 0 \tag{3.2}$$

which is a shorthand for

$$\frac{\partial N}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial N}{\partial \mathbf{x}} + \dot{\mathbf{v}} \cdot \frac{\partial N}{\partial \mathbf{v}} = 0 \tag{3.3}$$

3.2 Plasma Conditions

Plasma particles are charged. This single property ensures that plasmas behave rather differently to an ordinary gas. Commonly a plasma is modelled as an assembly of *charged* particles which is everywhere locally neutral. Non-neutral plasmas do exist but that subject lies outwith the ambit of this thesis.

Moving randomly, plasma particles interact with each other through (and indeed generate) *electromagnetic* forces. External electromagnetic disturbances give rise to corresponding particle motion. This superposition of fields around plasma particles is expressed in the equations:

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_{ext}(\mathbf{x},t) + \mathbf{e}(\mathbf{x},t) \tag{3.4}$$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}_{ext}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) \tag{3.5}$$

The microscopic electric and magnetic fields $(\mathbf{e}, \mathbf{b})$ due to the particles themselves may be written out explicitly using Maxwell's equations.

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t} \tag{3.6}$$

$$\nabla \times \mathbf{b} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{e}}{\partial t} \tag{3.7}$$

$$\nabla \cdot \mathbf{e} = \frac{\rho}{\epsilon_0} \tag{3.8}$$

$$\nabla \cdot \mathbf{b} = 0 \tag{3.9}$$

where charge and current density are given respectively by:

$$\rho = \sum_s q_s \int N(\mathbf{X},t)\, d\mathbf{v}$$

$$\mathbf{J} = \sum_s q_s \int N(\mathbf{X},t)\mathbf{v}\, d\mathbf{v}$$

The $\dot{\mathbf{x}}$ and $\dot{\mathbf{v}}$ together define the motion of a particle. Assuming that acceleration of a plasma particle is due solely to electromagnetic forces, the equations of motion of that particle are given by

$$\dot{\mathbf{x}} = \mathbf{v} \tag{3.10}$$

$$\dot{\mathbf{v}} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3.11}$$

Assuming the particular case of a two-component locally neutral plasma, the Klimontovich equation (3.3) with acceleration due to electric and magnetic fields can be written, as proposed by Dupree, and described in [42]:

$$\frac{\partial N}{\partial t} + \mathbf{v} \cdot \frac{\partial N}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial N}{\partial \mathbf{v}} = 0 \tag{3.12}$$

For future work it is useful to separate this equation into single particle and multiple particle expression

$$\left[\frac{\partial}{\partial t} + \mathcal{L}(\mathbf{X}) - \int d\mathbf{X}' \mathcal{V}(\mathbf{X}, \mathbf{X}') N(\mathbf{X}')\right] N(\mathbf{X}) = 0 \tag{3.13}$$

Here $\mathcal{L}(\mathbf{X})$ and $\mathcal{V}(\mathbf{X}, \mathbf{X}')$ are introduced. $\mathcal{L}(\mathbf{X})$ is the single particle operator:

$$\mathcal{L}(\mathbf{X}) = \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E}_{ext} + \mathbf{v} \times \mathbf{B}_{ext}) \cdot \frac{\partial}{\partial \mathbf{v}} \tag{3.14}$$

$\mathcal{V}(\mathbf{X}, \mathbf{X}')$ expresses the two particle Coulomb interaction.

$$\mathcal{V}(\mathbf{X}, \mathbf{X}') = \frac{q_s}{m_s}(\mathbf{e} + \mathbf{v} \times \mathbf{b}) \cdot \frac{\partial}{\partial \mathbf{v}} \tag{3.15}$$

3.2.1 The Statistical Approach of Boltzmann and Gibbs

The statistical properties of a system are completely determined by the distribution of its particles. Central to this approach is the concept of an average across equally probable and similar arrangements of the N particles under the same macroscopic conditions (*replicas*). This is Gibbs' *ensemble*. Replacing the single, impossibly random, $6N$ dimensional Γ-space point there is now a *swarm* of such points which is characterized by an average point.

In his treatment Boltzmann divides 6 dimensional configuration-velocity space (μ-space) into a number of small, finite cells of size ω. The N particles are then distributed amongst these cells so that there are n_1 in ω_1, n_2 in ω_2 and so on. The smallness of the cells is still large enough that the n_is are large numbers. The number of ways, N, of distributing the particles amongst the cells to give the same macroscopic conditions is

$$\mathcal{N} = \frac{N!}{n_1! n_2! \ldots}$$

provided : $\sum_i n_i = N$ and $\sum_i n_i \epsilon_i = E$

Finding the maximum of $\mathcal{N}$ will give the most probable distribution function. For large n_i this maximum is very sharp. We now identify this most probable distribution function with the equilibrium state. The identification is reasonable since almost the whole phase space volume belongs to that state.

Now define the Liouville distribution function, $D(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_N, t)$, in $6N$ dimensions, with the normalization

$$\int \ldots \int D d\mathbf{q}_1 d\mathbf{q}_2 \ldots d\mathbf{q}_N d\mathbf{p}_1 d\mathbf{p}_2 \ldots d\mathbf{p}_N = 1$$

This describes the distribution of replica Γ-space phase points.

We now need only know that a system is in a small phase space volume, $\delta\Gamma$. The initial microscopic state of an assembly, at t_0, is no longer given by a phase point. Now

$$D(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_N, t) = \begin{cases} \text{const} & : \text{ within } \delta\Gamma(t_0) \\ 0 & : \text{ outside } \delta\Gamma(t_0) \end{cases} \tag{3.16}$$

In other words since we are treating particles of any one species as identical it suffices to know the number of particles which have velocities close to $\mathbf{v}$ at positions close to $\mathbf{x}$. The properties of any one such particle are assumed to be identical to those of any other 'nearby' particle.

It is then hypothesized that the macroscopic properties of an assembly of particles are given by the average value across the *ensemble.*

$$\langle Q \rangle = \int \ldots \int Q\, D d\mathbf{q}_1 d\mathbf{q}_2 \ldots d\mathbf{q}_N d\mathbf{p}_1 d\mathbf{p}_2 \ldots d\mathbf{p}_N \tag{3.17}$$

By definition the distribution function and macroscopic properties are constant with respect to time when the system is in equilibrium. Away from equilibrium phase points do not interact with each other and in fact the replica phase points behave much like an ideal gas. The time dependence of D is given by Liouville's (continuity) equation:

$$\frac{\partial D}{\partial t} + \sum_{j=1}^{N} \left(\frac{\partial D}{\partial \mathbf{q}_j} \dot{\mathbf{q}}_j + \frac{\partial D}{\partial \mathbf{p}_j} \dot{\mathbf{p}}_j \right) = 0 \tag{3.18}$$

Note that (3.18) is time reversible - some further work is called for to add in, or at least argue for, an *arrow of time.* A solution to this problem is the *coarse-graining* treatment of Boltzmann and Gibbs. By arguing that while replica systems not at equilibrium will evolve freely from any one state to any other, the fact that a great majority of these states are equilibrium states means that it virtually never happens that a phase point evolves from equilibrium to a particular (rare) state. The net result is the apparent preference of systems for relaxation to equilibrium.

3.2.2 Reduced Distributions

Now that both the Klimontovich and Gibbs approaches have been introduced the next logical step is to combine them so that there is a balance between microscopic detail and statistical practicality. *Reduced distributions* do just that. Averaging the products of successively more microscopic number densities gives a series of reduced distribution functions with far fewer variables than D. Of course they also contain less information.

Now we may adopt the statistical approach of (3.17) to the most basic microscopic property - the number density .

$$\langle N(\mathbf{X},t)\rangle = \int \ldots \int N(\mathbf{X},t)\, D d\mathbf{q}_1 d\mathbf{q}_2 \ldots d\mathbf{q}_N d\mathbf{p}_1 d\mathbf{p}_2 \ldots d\mathbf{p}_N = \mathcal{F}_1(\mathbf{X},t) \tag{3.19}$$

where we introduce the single particle distribution function $\mathcal{F}_1$. This by definition satisfies

$$\int \mathcal{F}_1(\mathbf{X}_1,t) d\mathbf{X}_1 = N \tag{3.20}$$

Thus $\mathcal{F}_1(\mathbf{X}_1,t)d\mathbf{X}_1$ describes how many single particles there are in the μ-space volume element $d\mathbf{X}_1$.

Likewise the average of the product of the microscopic number densities at two points in μ-space is given by

$$\left\langle N(\mathbf{X},t)N(\mathbf{X}',t)\right\rangle = \mathcal{F}_2(\mathbf{X},\mathbf{X}',t) + \delta(\mathbf{X}-\mathbf{X}')\mathcal{F}_1(\mathbf{X},t) \tag{3.21}$$

The second term appears only if the two μ-space points are one and the same.

Now for number conservation to hold $\mathcal{F}_2$ must satisfy:

$$\int \mathcal{F}_2(\mathbf{X}_1,\mathbf{X}_2,t) d\mathbf{X}_2 = (N-1)\mathcal{F}_1(\mathbf{X}_1,t) \tag{3.22}$$

Thus $\mathcal{F}_2(\mathbf{X}_1,\mathbf{X}_2,t)$ allows us to calculate how many pairs of particles there are such that $\mathbf{X}_1$ is in the μ-space volume element $d\mathbf{X}_1$ and $\mathbf{X}_2$ is in the μ-space volume $d\mathbf{X}_2$. This then is the two particle distribution function which governs the effects of two particle collisions.

This process could go on indefinitely for the products of successively more number density expressions.

Now if we take the Liouville average of the (microscopic) Klimontovich equation (3.13) the averaged time-dependence of the number density is

$$\left[\frac{\partial}{\partial t} + \mathcal{L}(\mathbf{X}_1)\right] \mathcal{F}_1(\mathbf{X}_1) = \int d\mathbf{X}_2 \mathcal{V}(\mathbf{X}_1, \mathbf{X}_2) \mathcal{F}_2(\mathbf{X}_1, \mathbf{X}_2) \tag{3.23}$$

Had we chosen, a longer equation of the same type as the Klimontovich equation (3.13) could have been used:

$$\left[\frac{\partial}{\partial t} + \mathcal{L}(\mathbf{X}) + \mathcal{L}(\mathbf{X}')\right] N(\mathbf{X}) N(\mathbf{X}') = \int d\mathbf{X}'' \left(\mathcal{V}(\mathbf{X}, \mathbf{X}'') + \mathcal{V}(\mathbf{X}', \mathbf{X}'')\right) N(\mathbf{X}) N(\mathbf{X}') N(\mathbf{X}'') \tag{3.24}$$

This would translate to:

$$\left[\frac{\partial}{\partial t} + \mathcal{L}(\mathbf{X}_1) + \mathcal{L}(\mathbf{X}_2) - (\mathcal{V}(\mathbf{X}_1, \mathbf{X}_2) + \mathcal{V}(\mathbf{X}_2, \mathbf{X}_1))\right] \mathcal{F}_2(\mathbf{X}_1, \mathbf{X}_2) = \int d\mathbf{X}_2 \left(\mathcal{V}(\mathbf{X}_1, \mathbf{X}_3) + \mathcal{V}(\mathbf{X}_2, \mathbf{X}_3)\right) \mathcal{F}_3(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \tag{3.25}$$

Taking this idea to an arbitrary number of such Klimontovich type equations will lead to a series of equations known as the *BBGKY hierarchy.*

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{s} \mathcal{L}(\mathbf{X}_i) - \sum_{i \neq j}^{s} \mathcal{V}(\mathbf{X}_i, \mathbf{X}_j)\right] \mathcal{F}_s(\mathbf{X}_1, \ldots, \mathbf{X}_s) = \sum_{i=1}^{s} \int d\mathbf{X}_{s+1} \mathcal{V}(\mathbf{X}_i, \mathbf{X}_{s+1}) \mathcal{F}_{s+1}(\mathbf{X}_1, \ldots, \mathbf{X}_s, \mathbf{X}_{s+1}) \tag{3.26}$$

Here again $\mathcal{L}(\mathbf{X}_i)$ is the single particle operator:

$$\mathcal{L}(\mathbf{X}_i) = \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} + \frac{q_s}{m_s}(\mathbf{E}_{ext} + \mathbf{v}_i \times \mathbf{B}_{ext}) \cdot \frac{\partial}{\partial \mathbf{v}_i} \tag{3.27}$$

And $\mathcal{V}(\mathbf{X}_i, \mathbf{X}_j)$ expresses the two particle Coulomb interaction.

$$\mathcal{V}(\mathbf{X}_i, \mathbf{X}_j) = \frac{q_s}{m_s}(\mathbf{e} + \mathbf{v}_i \times \mathbf{b}) \cdot \frac{\partial}{\partial \mathbf{v}_i} \tag{3.28}$$

3.3 The Vlasov approximation

In an ordinary gas, the dominant effect is *collision.* This is still true for plasmas, however the dominant *collision* effect is fundamentally different. Whereas the forces between gas

particles fall off with separation as r^{-7} or r^{-8}, in plasmas the inter-particle (Coulomb) force falls much less steeply (as r^{-2}). As a result plasma particles have a far greater chance of interacting with a large number of other particles at longer range. Collision in plasmas means Coulomb collision.

Arguments which hold when applied to gases lose their persuasiveness in the case of plasmas. The BBGKY hierarchy (3.26) is an attempt to ground these equations more firmly.

In hot plasmas, thermal motion may be so energetic that even the effects of Coulomb collisions are negligible. In this case the most appropriate equation of motion has no collision term at all! This is the *Vlasov equation.* It contains information only about the thermal behaviour of the plasma and the collective effect of external and self-consistent fields.

As mentioned in the introduction on page 12, a large number of particles in the Debye sphere is an essential criterion for the system to be considered a plasma. In the Vlasov approximation the number of particles in the Debye sphere, N_{D}, is very large (in the continuum limit where particles are 'smoothed out', $N_{\mathrm{D}} \to \infty$). Given the typical value of N_{D} found on page 15, the Vlasov approximation is entirely appropriate for a typical $e^- e^+$ plasma.

3.4 Dynamic processes

Rather than describe the macroscopic properties of a plasma, (pressure, velocity, density, *etc.*) we wish to investigate its (dynamical) microscopic properties. To do this appropriately we produce distribution functions which are closely related to the probability functions mentioned before. The requirement is to *approximately describe how many particles are in the phase space in the neighbourhood of a phase point* $\mathbf{X}$.

Relate the single particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ with the reduced single particle distribution function $\mathcal{F}_1$ in (3.19) by

$$\mathcal{F}_1 = n_0 f \tag{3.29}$$

In general we must describe the distributions of more than one species of particle. To that end we introduce the distribution function $f_s(\mathbf{x}, \mathbf{v}, t)$, where s is the label of the particle species. f_s is such that $n_{s0} f_s d\mathbf{x} d\mathbf{v}$ is the probable number of particles of species s in the configuration-velocity space volume element $d\mathbf{x}d\mathbf{v}$ at time t, where n_{s0} is the average particle density.

The usual fluid variables, density and fluid velocity *etc.*, can be written as velocity moments:

$$n_s(\mathbf{x}, t) = n_{s0} \int f_s(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} \tag{3.30}$$

$$\mathbf{u}_s(\mathbf{x}, t) = \int \mathbf{v} f_s(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} \tag{3.31}$$

Furthermore for a homogeneous system this becomes

$$\int f_s(\mathbf{v}) d\mathbf{v} = 1$$

making it equivalent to a probability distribution function for velocity. In later chapters a different normalization is more appropriate:

$$\int F_s(\mathbf{p}) d\mathbf{p} = 1 \tag{3.32}$$

An informal approach to this problem is to state that particles in a system are conserved, even in a $e^- e^+$ plasma this is true at the short timescales of plasma waves. In other words the distribution obeys a six-dimensional analogue of the fluid equation for the conservation of mass.

$$\frac{\partial}{\partial t} f_s + \dot{\mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{x}} (f_s) + \dot{\mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} (f_s) = 0$$

Now $\dot{\mathbf{r}} = \mathbf{v}$ and $\dot{\mathbf{v}} = \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$$\frac{\partial}{\partial t} f_s + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} (f_s) + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{v}} (f_s) = 0 \tag{3.33}$$

the Vlasov equation for the velocity distribution function describing species s.

3.5 Base Equations

The earlier sections of this chapter have illustrated the argument which leads to a Vlasov equation expressed in configuration-velocity space (3.33). This form of the equation is ideal for the development of a broad range of nonrelativistic theories. When casting a relativistic theory, the most tractable form of the Vlasov equation is one where the distribution is normalized in terms of momentum:

$$\frac{\partial F_s}{\partial t} + \mathbf{v} \cdot \frac{\partial F_s}{\partial \mathbf{x}} + q_s(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial F_s}{\partial \mathbf{p}} = 0 \tag{3.34}$$

where F_s is the distribution in configuration-momentum space, $F_s = F_s(\mathbf{p}, \mathbf{x}, t)$.

Since we now wish all our expressions to be valid in a relativistic regime we must be wary of our notation. From this point on, we must distinguish between m_s and the rest mass, m_{s0}. $m_s = \gamma m_{s0}$ and therefore quantities which involve mass, such as the cyclotron frequency, Ω_s, have associated rest values, Ω_{s0}.

$$\Omega_s = \frac{q_s B_0}{m_s} = \frac{q_s B_0}{\gamma m_{s0}} = \frac{\Omega_{s0}}{\gamma}$$

Our notation is now appropriate to the task. The first stage in shaping this relativistic theory is to consider our plasma to be at equilibrium with a small perturbation. This means linearizing Maxwell's equations and (3.34) and gives a set of equations for each species coupled through self-consistent electric and magnetic fields.

Choose the equilibrium conditions to be:

$$\mathbf{E} = \mathbf{0} \tag{3.35}$$

$$\mathbf{B} = \mathbf{B}_0 \tag{3.36}$$

$$F_s = F_{s0}(\mathbf{p}) \tag{3.37}$$

By convention the coordinate system is characterized by the direction of the equilibrium $\mathbf{B}$ field, *i.e.* $\mathbf{B}_0 = \hat{\mathbf{e}}_{||} B_0$. Throughout this work the directions *parallel* and *perpendicular* are relative to the direction of $\mathbf{B}_0$.

At equilibrium, (3.34) becomes

$$(\mathbf{p} \times \mathbf{B}) \cdot \frac{\partial F_{s0}}{\partial \mathbf{p}} = 0 \tag{3.38}$$

By expressing $\mathbf{p}$ in cylindrical polar coordinates $(p_\perp, p_{||}, \phi)$ we find that the equilibrium conditions require that F_{s0} be independent of ϕ *i.e.* $\frac{\partial F_{s0}}{\partial \phi} = 0$. The Maxwellian distribution certainly fulfills this criterion.

Allowing small harmonic perturbations

$$\mathbf{E} = \mathbf{E}_1 = \mathbf{E}_{\mathbf{k}\omega} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{3.39}$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 = \mathbf{B}_0 + \mathbf{B}_{\mathbf{k}\omega} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{3.40}$$

$$F_s = F_{s0}(\mathbf{p}) + F_{s1}(\mathbf{p}, \mathbf{x}, t) = F_{s0}(\mathbf{p}) + F_{s\mathbf{k}\omega} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{3.41}$$

Since a Fourier transform is implicit in the choice of expression for F_s, a further assumption is necessary to match the physical 'arrow of time'. We must assume that there was *no* perturbation $F_{s1}(t)$ before $t = 0$. None of this is necessary if the time-dependence is treated with the Laplace transforms. Implicitly a Laplace transform deals only with $t > 0$. For the treatment given here this difference of rigour does not affect the outcome.

In (3.34) this gives to first order

$$\frac{\partial F_{s1}}{\partial t} + \mathbf{v} \cdot \frac{\partial F_{s1}}{\partial \mathbf{x}} + q_s(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) \cdot \frac{\partial F_{s0}}{\partial \mathbf{p}} + q_s(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial F_{s1}}{\partial \mathbf{p}} = 0 \tag{3.42}$$

$$-i\omega F_{s\mathbf{k}\omega} + \mathbf{v} \cdot (i\mathbf{k}) F_{s\mathbf{k}\omega} + q_s(\mathbf{E}_{\mathbf{k}\omega} + \mathbf{v} \times \mathbf{B}_{\mathbf{k}\omega}) \cdot \frac{\partial F_{s0}}{\partial \mathbf{p}} + q_s(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial F_{s\mathbf{k}\omega}}{\partial \mathbf{p}} = 0 \tag{3.43}$$

The next step is to deal with $\frac{\partial F}{\partial \mathbf{p}}$ expressions. It is useful to switch from a linear (p_x, p_y, p_z) to a cylindrical coordinate system $(p_\perp, p_{||}, \phi)$. Now

$$\frac{\partial F}{\partial p_{x,y,z}} = \frac{\partial p_\perp}{\partial p_{x,y,z}} \frac{\partial F}{\partial p_\perp} + \frac{\partial p_{||}}{\partial p_{x,y,z}} \frac{\partial F}{\partial p_{||}} + \frac{\partial \phi}{\partial p_{x,y,z}} \frac{\partial F}{\partial \phi} \tag{3.44}$$

The switch in coordinate systems explicitly gives:

$$\frac{\partial F}{\partial p_x} = \cos\phi \frac{\partial F}{\partial p_\perp} - \frac{\sin\phi}{p_\perp}\frac{\partial F}{\partial \phi} \tag{3.45}$$

$$\frac{\partial F}{\partial p_y} = \sin\phi \frac{\partial F}{\partial p_\perp} + \frac{\cos\phi}{p_\perp}\frac{\partial F}{\partial \phi} \tag{3.46}$$

$$\frac{\partial F}{\partial p_z} = \frac{\partial F}{\partial p_{||}} \tag{3.47}$$

Now due to (3.38) when $F = F_{s0}$ this becomes

$$\frac{\partial F}{\partial p_x} = \cos\phi \frac{\partial F}{\partial p_\perp} \tag{3.48}$$

$$\frac{\partial F}{\partial p_y} = \sin\phi \frac{\partial F}{\partial p_\perp} \tag{3.49}$$

$$\frac{\partial F}{\partial p_z} = \frac{\partial F}{\partial p_{||}} \tag{3.50}$$

On the other hand for $F = F_{s\mathbf{k}\omega}$ the full expression must be used. This means that the final expression on the left hand side of 3.43 can be rewritten:

$$\begin{aligned} q_s(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial F_{s\mathbf{k}\omega}}{\partial \mathbf{p}} &= \frac{q_s}{m_s}(p_\perp \sin\phi\, B_0\hat{\mathbf{x}} - p_\perp\cos\phi\, B_0\hat{\mathbf{y}})\cdot\frac{\partial F_{s\mathbf{k}\omega}}{\partial \mathbf{p}} \\ &= -\frac{q_s}{m_s}B_0\frac{\partial F_{s\mathbf{k}\omega}}{\partial\phi} \end{aligned}$$

Now return to Maxwell's equations at equilibrium:

$$\frac{\partial}{\partial \mathbf{x}}\times\mathbf{E}_0 = \mathbf{0} = -\frac{\partial \mathbf{B}_0}{\partial t} \tag{3.51}$$

$$\frac{\partial}{\partial \mathbf{x}}\times\mathbf{B}_0 = \mu_0\sum_s q_s n_{s0}\int F_{s0}\mathbf{v}\,d\mathbf{p} \tag{3.52}$$

with the perturbed quantities this is

$$\frac{\partial}{\partial \mathbf{x}}\times\mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t} \tag{3.53}$$

$$\frac{\partial}{\partial \mathbf{x}}\times\mathbf{B}_1 = \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t} + \mu_0\sum_s q_s n_{s0}\int F_{s1}\mathbf{v}\,d\mathbf{p} \tag{3.54}$$

so that:

$$i\mathbf{k}\times\mathbf{E}_{\mathbf{k}\omega} = i\omega\mathbf{B}_{\mathbf{k}\omega} \tag{3.55}$$

$$i\mathbf{k}\times\mathbf{B}_{\mathbf{k}\omega} = -\frac{i\omega}{c^2}\mathbf{E}_{\mathbf{k}\omega} + \mu_0\sum_s q_s n_{s0}\int F_{s\mathbf{k}\omega}\mathbf{v}\,d\mathbf{p} \tag{3.56}$$

(3.54) may be restated by expressing the perturbed $\mathbf{B}$ and F_s in terms of $\mathbf{E}$. This is achieved by eliminating $\mathbf{B}_{\mathbf{k}\omega}$ using (3.55)

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) = -\frac{\omega^2}{c^2}\mathbf{E} + \frac{\omega}{i}\mu_0\sum_s \frac{q_s}{m_s}n_{s0}\int F_{s\mathbf{k}\omega}\mathbf{p}\,d\mathbf{p} \tag{3.57}$$

Broadly this approach views (3.57) as a dielectric equation:

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) + \frac{\omega^2}{c^2}\epsilon\cdot\mathbf{E} = \mathbf{0} \tag{3.58}$$

where ϵ is the *dielectric tensor* can be reduced even further to the terse statement

$$\mathcal{R}\cdot\mathbf{E} = \mathbf{0} \tag{3.59}$$

where the tensor $\mathcal{R}$ contains all the physical information in the system. This becomes a useful formalism in chapters 4 and 5.

To find $F_{s\mathbf{k}\omega}$ one must likewise eliminate $\mathbf{B}_{\mathbf{k}\omega}$ in (3.43)

$$-i\omega F_{s\mathbf{k}\omega} + i\mathbf{k}\cdot\mathbf{v}F_{s\mathbf{k}\omega} + q_s(\mathbf{E}_{\mathbf{k}\omega} + \frac{1}{\omega}\mathbf{v}\times(\mathbf{k}\times\mathbf{E}_{\mathbf{k}\omega}))\cdot\frac{\partial F_{s0}}{\partial\mathbf{p}} - \frac{q_s}{m_s}B_0\frac{\partial F_{s\mathbf{k}\omega}}{\partial\phi} = 0 \tag{3.60}$$

$$(i(\omega-\mathbf{k}\cdot\mathbf{v}) + \Omega_s\frac{\partial}{\partial\phi})F_{s\mathbf{k}\omega} = q_s(\mathbf{E}_{\mathbf{k}\omega} + \frac{1}{\omega}\mathbf{v}\times(\mathbf{k}\times\mathbf{E}_{\mathbf{k}\omega}))\cdot\frac{\partial F_{s0}}{\partial\mathbf{p}} \tag{3.61}$$

where the *relativistic* cyclotron frequency for species s, Ω_s, has been substituted for brevity. This quantity is introduced more formally on page 13.

In essence (3.61) is the first order partial differential equation:

$$\frac{\partial}{\partial\phi}F_{s\mathbf{k}\omega} + XF_{s\mathbf{k}\omega} = Y \tag{3.62}$$

where

$$X = \frac{i(\omega-\mathbf{k}\cdot\mathbf{v})}{\Omega_s} \tag{3.63}$$

and

$$Y = \frac{q_s}{\Omega_s}(\mathbf{E}_{\mathbf{k}\omega} + \frac{1}{\omega}\mathbf{v}\times(\mathbf{k}\times\mathbf{E}_{\mathbf{k}\omega}))\cdot\frac{\partial F_{s0}}{\partial\mathbf{p}} \tag{3.64}$$

3.5.1 Integrating Factor

The solution to (3.62) is found through an integrating factor $e^{\int^\phi X d\phi}$

$$\frac{\partial}{\partial\phi}(e^{\int^\phi X d\phi} F_{s\mathbf{k}\omega}) = Y e^{\int^\phi X d\phi} \tag{3.65}$$

$$F_{s\mathbf{k}\omega} = e^{-\int^\phi X d\phi} \int^\phi Y e^{\int^\phi X d\phi} \tag{3.66}$$

Since γ appears in Ω_s ($\Omega_s = \frac{\Omega_{s0}}{\gamma}$) and $\gamma = \gamma(p_\perp, p_{||})$ the integrating factor is calculated as follows:

$$X = \frac{i}{\Omega_s}(\omega - k_{||}v_{||} - k_\perp v_\perp \cos\phi) \tag{3.67}$$

thus

$$e^{\pm\int^\phi X d\phi} = e^{\pm\frac{i}{\Omega_s}(\omega - k_{||}v_{||})\phi} e^{\mp i\frac{k_\perp v_\perp}{\Omega_s}\sin\phi} \tag{3.68}$$

To clean up slightly introduce $a_s = k_\perp v_\perp/\Omega_s$ and a very useful identity:

$$e^{\mp i a_s \sin\phi} = \sum_{n=-\infty}^{\infty} J_n(a_s) e^{\mp in\phi} \tag{3.69}$$

So

$$e^{\pm\int^\phi X d\phi} = e^{\pm\frac{i}{\Omega_s}(\omega - k_{||}v_{||})\phi} \sum_{n=-\infty}^{\infty} J_n(a_s) e^{\mp in\phi} \tag{3.70}$$

3.5.2 The remaining part of the integrand

The term Y must similarly be dealt with:

$$Y = \frac{q_s}{\Omega_s}\left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + \begin{pmatrix} \frac{v_\perp}{\omega}\cos\phi \\ \frac{v_\perp}{\omega}\sin\phi \\ \frac{v_{||}}{\omega} \end{pmatrix} \times \left(\begin{pmatrix} k_\perp \\ 0 \\ k_{||} \end{pmatrix} \times \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}\right)\right] \cdot \begin{pmatrix} \frac{\partial}{\partial p_x}F_{s0} \\ \frac{\partial}{\partial p_y}F_{s0} \\ \frac{\partial}{\partial p_z}F_{s0} \end{pmatrix} \tag{3.71}$$

The expression in the square brackets is

$$\left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + \begin{pmatrix} \frac{v_\perp}{\omega}\cos\phi \\ \frac{v_\perp}{\omega}\sin\phi \\ \frac{v_\parallel}{\omega} \end{pmatrix} \times \begin{pmatrix} -k_\parallel E_x \\ -k_\perp E_z + k_\parallel E_x \\ k_\perp E_y \end{pmatrix}\right] \tag{3.72}$$

which leads to

$$\left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + \begin{pmatrix} -\frac{k_\parallel v_\parallel}{\omega}E_x + \frac{k_\perp v_\perp}{\omega}\sin\phi E_y + \frac{k_\perp v_\perp}{\omega}E_z \\ (-\frac{k_\perp v_\perp}{\omega}\cos\phi - \frac{k_\parallel v_\parallel}{\omega})E_y \\ \frac{k_\parallel v_\perp}{\omega}\cos\phi E_x + \frac{k_\parallel v_\perp}{\omega}\sin\phi E_y - \frac{k_\perp v_\perp}{\omega}\cos\phi E_z \end{pmatrix}\right] \tag{3.73}$$

Or

$$\frac{1}{\omega}\begin{bmatrix} (\omega - k_\parallel v_\parallel)E_x + (k_\perp v_\perp \sin\phi)E_y + (k_\perp v_\perp)E_z \\ (\omega - k_\perp v_\perp\cos\phi - k_\parallel v_\parallel)E_y \\ (k_\parallel v_\perp\cos\phi)E_x + (k_\parallel v_\perp\sin\phi)E_y + (\omega - k_\perp v_\perp\cos\phi)E_z \end{bmatrix} \tag{3.74}$$

Y now looks like this:

$$\frac{q_s}{\Omega_s\omega}\begin{bmatrix} (\omega - k_\parallel v_\parallel)E_x + (k_\perp v_\perp \sin\phi)E_y + (k_\perp v_\perp)E_z \\ (\omega - k_\perp v_\perp\cos\phi - k_\parallel v_\parallel)E_y \\ (k_\parallel v_\perp\cos\phi)E_x + (k_\parallel v_\perp\sin\phi)E_y + (\omega - k_\perp v_\perp\cos\phi)E_z \end{bmatrix} \cdot \begin{pmatrix} \cos\phi\frac{\partial F_{s0}}{\partial p_\perp} \\ \sin\phi\frac{\partial F_{s0}}{\partial p_\perp} \\ \frac{\partial F_{s0}}{\partial p_\parallel} \end{pmatrix} \tag{3.75}$$

And gathering terms in the components of $\mathbf{E}_{\mathbf{k}\omega}$

$$\begin{aligned} Y \;=\; & \frac{q_s}{\Omega_s\omega}\left\{\left[(\omega - k_\parallel v_\parallel)\frac{\partial F_{s0}}{\partial p_\perp} + k_\parallel v_\perp\frac{\partial F_{s0}}{\partial p_\parallel}\right]\cos\phi E_x\right. \\ & + \left[(\omega - k_\parallel v_\parallel)\frac{\partial F_{s0}}{\partial p_\perp} + k_\parallel v_\perp\frac{\partial F_{s0}}{\partial p_\parallel}\right]\sin\phi E_y \\ & \left. + \left[(k_\perp v_\parallel\cos\phi)\frac{\partial F_{s0}}{\partial p_\perp} + (\omega - k_\perp v_\perp\cos\phi)\frac{\partial F_{s0}}{\partial p_\parallel}\right]E_z\right\} \end{aligned} \tag{3.76}$$

That is

$$Y = \frac{q_s}{\Omega_s\omega}\left\{U\cos\phi E_x + U\sin\phi E_y + WE_z\right\} \tag{3.77}$$

where

$$U = (\omega - k_{||}v_{||})\frac{\partial F_{s0}}{\partial p_\perp} + k_{||}v_\perp \frac{\partial F_{s0}}{\partial p_{||}} \tag{3.78}$$

and

$$W = (k_\perp v_{||}\cos\phi)\frac{\partial F_{s0}}{\partial p_\perp} + (\omega - k_\perp v_\perp \cos\phi)\frac{\partial F_{s0}}{\partial p_{||}} \tag{3.79}$$

3.5.3 Substitutions

We have used the identity (A.4) on page 35. As may be seen in Appendix A this leads to two more identities:

$$\sum_{n=-\infty}^{\infty} \sin\phi J_n(a_s)e^{\mp in\phi} = \sum_{n=-\infty}^{\infty} (-i)J_n'(a_s)e^{\mp in\phi} \tag{3.80}$$

$$\sum_{n=-\infty}^{\infty} \frac{n}{a_s} J_n(a_s)e^{in\phi} = \sum_{n=-\infty}^{\infty} (\cos\phi)J_n(a_s)e^{in\phi} \tag{3.81}$$

These two new identities (3.80) & (3.81) are immediately useful. Placing the new expressions for the integrating factor (3.70) and Y (3.77) into (3.66) the expression for $F_{s\mathbf{k}\omega}$ becomes

$$F_{s\mathbf{k}\omega} = e^{-\frac{i}{\Omega_s}(\omega - k_{||}v_{||})\phi} e^{ia_s \sin\phi} \cdot \int^{\phi} \sum_{n=-\infty}^{\infty} J_n(a_s)\frac{q_s}{\Omega_s \omega}\{U\cos\phi E_x + U\sin\phi E_y + WE_z\} e^{\frac{i}{\Omega_s}(\omega - k_{||}v_{||} - n\Omega_s)\phi} d\phi \tag{3.82}$$

Look closer at the integral over ϕ. Applying (3.80) & (3.81), gives a new expression

for $\int^{\phi} \ldots$:

$$\int^{\phi} \sum_{n=-\infty}^{\infty} \frac{q_s}{\Omega_s \omega} \left\{ U \frac{n\Omega_s}{k_\perp v_\perp} J_n(a_s) E_x - iU J_n'(a_s) E_y + W J_n(a_s) E_z \right\} e^{\frac{i}{\Omega_s}(\omega - k_\parallel v_\parallel - n\Omega_s)\phi} d\phi$$

$$= \sum_{n=-\infty}^{\infty} \frac{q_s}{\Omega_s \omega} \left\{ U \frac{n\Omega_s}{k_\perp v_\perp} J_n(a_s) E_x - iU J_n'(a_s) E_y + W J_n(a_s) E_z \right\} \int^{\phi} e^{\frac{i}{\Omega_s}(\omega - k_\parallel v_\parallel - n\Omega_s)\phi} d\phi$$

$$= \sum_{n=-\infty}^{\infty} \frac{q_s}{\Omega_s \omega} \begin{pmatrix} U \frac{n\Omega_s}{k_\perp v_\perp} J_n(a_s) \\ -iU J_n'(a_s) \\ W J_n(a_s) \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \frac{\Omega_s}{i(\omega - k_\parallel v_\parallel - n\Omega_s)} e^{\frac{i}{\Omega_s}(\omega - k_\parallel v_\parallel - n\Omega_s)\phi}$$

$$= \frac{q_s}{i\omega} e^{\frac{i}{\Omega_s}(\omega - k_\parallel v_\parallel)\phi} \sum_{n=-\infty}^{\infty} \begin{pmatrix} U \frac{n\Omega_s}{k_\perp v_\perp} J_n(a_s) \\ -iU J_n'(a_s) \\ W J_n(a_s) \end{pmatrix} \cdot \mathbf{E}_{\mathbf{k}\omega} \frac{e^{-in\phi}}{(\omega - k_\parallel v_\parallel - n\Omega_s)} \tag{3.83}$$

Now substitute (3.83) into (3.82)

$$F_{s\mathbf{k}\omega} = e^{ia_s \sin\phi} \frac{q_s}{i\omega} \sum_{n=-\infty}^{\infty} \begin{pmatrix} U \frac{n\Omega_s}{k_\perp v_\perp} J_n(a_s) \\ -iU J_n'(a_s) \\ W J_n(a_s) \end{pmatrix} \cdot \mathbf{E}_{\mathbf{k}\omega} \frac{e^{-in\phi}}{(\omega - k_\parallel v_\parallel - n\Omega_s)} \tag{3.84}$$

Thus (3.57) is written:

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_{\mathbf{k}\omega}) = -\frac{\omega^2}{c^2} \mathbf{E}_{\mathbf{k}\omega}$$

$$+ \frac{\omega}{i} \mu_0 \sum_s \frac{q_s}{m_s} n_{s0} \int e^{ia_s \sin\phi} \frac{q_s}{i\omega} \sum_{n=-\infty}^{\infty} \begin{pmatrix} U \frac{n\Omega_s}{k_\perp v_\perp} J_n(a_e) \\ -iU J_n'(a_e) \\ W J_n(a_e) \end{pmatrix} \cdot \mathbf{E}_{\mathbf{k}\omega} \frac{e^{-in\phi}}{(\omega - k_\parallel v_\parallel - n\Omega_s)} \mathbf{p}\, d\mathbf{p}$$

(3.85)

Its last term can now be expressed through (3.84)

$$(\frac{\omega}{i})\mu_0 \sum_s \frac{q_s}{m_s} n_{s0} \int e^{ia_s \sin\phi} \frac{q_s}{i\omega} \sum_{n=-\infty}^{\infty} \begin{pmatrix} U\frac{n\Omega_s}{k_\perp v_\perp} J_n(a_s) \\ -iUJ_n'(a_s) \\ WJ_n(a_s) \end{pmatrix} \cdot \mathbf{E}_{\mathbf{k}\omega} \frac{e^{-in\phi}}{(\omega - k_{||}v_{||} - n\Omega_s)} \mathbf{p}\, d\mathbf{p}$$

$$= -\mu_0 \sum_s \frac{q_s^2}{m_s} n_{s0} \iiint \ldots p_\perp dp_\perp dp_{||} d\phi$$

The $\mathbf{p}$ integral is developed as follows:

$$\iiint e^{ia_s \sin\phi} \sum_{n=-\infty}^{\infty} \begin{pmatrix} U\frac{n\Omega_s}{k_\perp v_\perp} J_n \\ -iUJ_n' \\ WJ_n \end{pmatrix} \cdot \mathbf{E}_{\mathbf{k}\omega} \frac{e^{-in\phi}}{(\omega - k_{||}v_{||} - n\Omega_s)} \begin{pmatrix} p_\perp \cos\phi \\ p_\perp \sin\phi \\ p_{||} \end{pmatrix} p_\perp dp_\perp dp_{||} d\phi$$

$$= \iint \sum_{n,l=-\infty}^{\infty} \frac{dp_\perp dp_{||}}{\omega - k_{||}v_{||} - n\Omega_s} \begin{pmatrix} p_\perp \left(\frac{l\Omega_s}{k_\perp v_\perp}\right) J_l \\ ip_\perp J_l' \\ p_\perp p_{||} J_l \end{pmatrix} \left[\begin{pmatrix} U\frac{n\Omega_s}{k_\perp v_\perp} J_n \\ -iUJ_n' \\ WJ_n \end{pmatrix} \cdot \mathbf{E}_{\mathbf{k}\omega} \right] \int_0^{2\pi} e^{i(l-n)\phi}\, d\phi$$

Note that

$$\int_0^{2\pi} e^{i(l-n)\phi}\, d\phi = 2\pi\delta_{ln}$$

So the full expression for a general plasma

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_{\mathbf{k}\omega}) = -\frac{\omega^2}{c^2}\mathbf{E}_{\mathbf{k}\omega} - 2\pi\mu_0 \sum_s \frac{q_s^2}{m_s} n_{s0} \int_0^\infty \int_{-\infty}^\infty dp_\perp dp_{||} \sum_{n=-\infty}^{\infty} \frac{\mathcal{Q}_n \cdot \mathbf{E}_{\mathbf{k}\omega}}{\omega - k_{||}v_{||} - n\Omega_s} \tag{3.86}$$

where $\mathcal{Q}_n$ is the tensor

$$\begin{pmatrix} p_\perp^2 \left(\frac{n\Omega_s}{k_\perp v_\perp}\right)^2 UJ_n^2(a_s) & -ip_\perp^2 \left(\frac{n\Omega_s}{k_\perp v_\perp}\right) UJ_n(a_s)J_n'(a_s) & p_\perp^2 \left(\frac{n\Omega_s}{k_\perp v_\perp}\right) WJ_n^2(a_s) \\ ip_\perp^2 \left(\frac{n\Omega_s}{k_\perp v_\perp}\right) UJ_n(a_s)J_n'(a_s) & p_\perp^2 U(J_n'(a_s))^2 & ip_\perp^2 WJ_n'(a_s)J_n(a_s) \\ p_{||}p_\perp \left(\frac{n\Omega_s}{k_\perp v_\perp}\right) UJ_n^2(a_s) & -ip_{||}p_\perp UJ_n'(a_s)J_n(a_s) & p_{||}p_\perp WJ_n^2(a_s) \end{pmatrix} \tag{3.87}$$

3.6 Assumptions

If we are to make progress we must now characterize the plasma we wish to study. Without going as far as using a unique expression for the distribution of momentum one may choose to investigate plasmas with isotropic distribution.

3.6.1 Isotropy

Since subsequent sections deal with relativistic effects, the γ-factor is introduced in the usual way: $\mathbf{p} = \gamma m_{s0}\mathbf{v}$. One direct consequence of any isotropic distribution is that $p_{||}U = p_{\perp}W$. This can be shown as follows:

$$\begin{aligned} p_{||}U - p_{\perp}W &= \gamma m_{s0}(v_{||}U - v_{\perp}W) \\ &= \gamma m_{s0}\left[(\omega v_{||} - k_{||}v_{||}^2)\frac{\partial F_{s0}}{\partial p_{\perp}} + k_{||}v_{||}v_{\perp}\frac{\partial F_{s0}}{\partial p_{||}}\right. \\ &\qquad \left. + (-v_{||}n\Omega_s)\frac{\partial F_{s0}}{\partial p_{\perp}} + (-\omega v_{\perp} + v_{\perp}n\Omega_s)\frac{\partial F_{s0}}{\partial p_{||}}\right] \\ &= \gamma m_{s0}\left[(\omega - k_{||}v_{||} - n\Omega_s)\left[v_{||}\frac{\partial F_{s0}}{\partial p_{\perp}} - v_{\perp}\frac{\partial F_{s0}}{\partial p_{||}}\right]\right] \\ &= (\omega - k_{||}v_{||} - n\Omega_s)\left[p_{||}\frac{\partial F_{s0}}{\partial p_{\perp}} - p_{\perp}\frac{\partial F_{s0}}{\partial p_{||}}\right] \end{aligned} \tag{3.88}$$

With this *isotropic distribution*, $\mathcal{Q}_n$ may be written:

$$\begin{pmatrix} p_{\perp}^2\left(\frac{n\Omega_s}{k_{\perp}v_{\perp}}\right)^2 UJ_n^2(a_s) & -ip_{\perp}^2\left(\frac{n\Omega_s}{k_{\perp}v_{\perp}}\right)UJ_n(a_s)J_n'(a_s) & p_{\perp}p_{||}\left(\frac{n\Omega_s}{k_{\perp}v_{\perp}}\right)UJ_n^2(a_s) \\ ip_{\perp}^2\left(\frac{n\Omega_s}{k_{\perp}v_{\perp}}\right)UJ_n(a_s)J_n'(a_s) & p_{\perp}^2U(J_n'(a_s))^2 & ip_{\perp}p_{||}UJ_n'(a_s)J_n(a_s) \\ p_{||}p_{\perp}\left(\frac{n\Omega_s}{k_{\perp}v_{\perp}}\right)UJ_n^2(a_s) & -ip_{||}p_{\perp}UJ_n'(a_s)J_n(a_s) & p_{||}^2UJ_n^2(a_s) \end{pmatrix} \tag{3.89}$$

As stated in the introduction this is the case throughout this work. This expression for $\mathcal{Q}_n$ is substituted in (3.86). All work presented here entails an isotropic distribution. In addition the novel work assumes an e^-e^+ plasma.

3.7 Electron-positron plasma

In a plasma of electrons and positrons in equal number densities and temperatures, the mathematics may be simplified, as was pointed out in the introduction (see Section 2.1 on page 14 and references [36, 44, 45, 46]).

$F_{\bar{e}0} = F_{e0}$ therefore U (defined at (3.78)) is independent of species. $q_{\bar{e}} = |e| = -q_e$ and so $\Omega_{\bar{e}} = -\Omega_e$: similarly $a_{\bar{e}} = -a_e$. In consequence the following results are true:

$$J_n^2(a_{\bar{e}}) = J_n^2(a_e) \tag{3.90}$$

$$J_n(a_{\bar{e}})J_n'(a_{\bar{e}}) = -J_n(a_e)J_n'(a_e) \tag{3.91}$$

$$(J_n'(a_{\bar{e}}))^2 = (J_n'(a_e))^2 \tag{3.92}$$

Here the basic Bessel function identities (A.2) and (A.3) are employed.

All this feeds in to the expression for the last term of (3.86)

$$-2\pi\mu_0\frac{e^2}{m_e}n_{e0}\sum_{n=-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}dp_\perp dp_{||}\mathcal{K}_n\cdot\mathbf{E}_{\mathbf{k}\omega} \tag{3.93}$$

with $\mathcal{K}_n$:

$$\begin{pmatrix} p_\perp^2\left(\frac{n\Omega_e}{k_\perp v_\perp}\right)^2 UJ_n^2(a_e)\sigma & -ip_\perp^2\left(\frac{n\Omega_e}{k_\perp v_\perp}\right)UJ_n(a_e)J_n'(a_e)\sigma & p_\perp p_{||}\left(\frac{n\Omega_e}{k_\perp v_\perp}\right)UJ_n^2(a_e)\delta \\ ip_\perp^2\left(\frac{n\Omega_e}{k_\perp v_\perp}\right)UJ_n(a_e)J_n'(a_e)\sigma & p_\perp^2U(J_n'(a_e))^2\sigma & ip_\perp p_{||}UJ_n'(a_e)J_n(a_e)\delta \\ p_{||}p_\perp\left(\frac{n\Omega_e}{k_\perp v_\perp}\right)UJ_n^2(a_e)\delta & -ip_{||}p_\perp UJ_n'(a_e)J_n(a_e)\delta & p_{||}^2UJ_n^2(a_e)\sigma \end{pmatrix} \tag{3.94}$$

where

$$\begin{aligned}\sigma &= \frac{1}{\omega - k_{||}v_{||} - n\Omega_e} + \frac{1}{\omega - k_{||}v_{||} + n\Omega_e} \\ &= \frac{2(\omega - k_{||}v_{||})}{(\omega - k_{||}v_{||})^2 - (n\Omega_e)^2}\end{aligned} \tag{3.95}$$

$$\begin{aligned}\delta &= \frac{1}{\omega - k_{||}v_{||} - n\Omega_e} - \frac{1}{\omega - k_{||}v_{||} + n\Omega_e} \\ &= \frac{2n\Omega_e}{(\omega - k_{||}v_{||})^2 - (n\Omega_e)^2}\end{aligned} \tag{3.96}$$

3.7.1 Eliminating the off-diagonal elements of $\mathcal{K}_n$

The sum over the n's, $\sum_{n=-\infty}^{\infty}\ldots$, can be separated into $\sum_{n=1}^{\infty}\ldots$ and $\sum_{n=-1}^{-\infty}\ldots$ and the n=0 term. For the (xy,yx) and (xz,zx) elements of $\mathcal{K}_n$ the n=0 term is itself zero. The two remaining parts exactly cancel one another, due to (3.90) & (3.91).

Rewriting (3.86), multiplied through by c^2, now gives

$$c^2\,\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_{\mathbf{k}\omega})+\omega^2\,\mathbf{E}_{\mathbf{k}\omega}=-2\pi\frac{\mu_0c^2e^2n_{e0}}{m_e}\sum_{n=-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}dp_\perp dp_{||}\mathcal{K}_n\cdot\mathbf{E}_{\mathbf{k}\omega}\qquad(3.97)$$

The left hand side of this equation may be expanded:

$$c^2\,\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_{\mathbf{k}\omega})+\omega^2\,\mathbf{E}_{\mathbf{k}\omega}=\begin{pmatrix}\omega^2-c^2k_{||}^2 & 0 & c^2k_\perp k_{||}\\ 0 & \omega^2-c^2k^2 & 0\\ c^2k_\perp k_{||} & 0 & \omega^2-c^2k_\perp^2\end{pmatrix}\cdot\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}_{\mathbf{k}\omega}\qquad(3.98)$$

Some of the remaining elements of the tensors on both sides depend on $k_{||}$, this suggests that the case where $k_{||}\to 0$ would be informative. In this choice the focus of this research has become Bernstein modes. These are wavemodes which propagate perpendicular to the equilibrium magnetic field, $\mathbf{B}_0$.

3.7.2 Perpendicular propagation

Apart from the obvious disappearance of elements dependent solely on $k_{||}$, the expression for U found at (3.78) is made simpler by this choice:

$$U\to\omega\frac{\partial F_{e0}}{\partial p_\perp}\qquad(3.99)$$

Overall this means that (3.97) may be written in the form:

$$\begin{pmatrix}\omega^2 & 0 & 0\\ 0 & \omega^2-c^2k_\perp^2 & 0\\ 0 & 0 & \omega^2-c^2k_\perp^2\end{pmatrix}\cdot\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}_{\mathbf{k}\omega}=-2\pi\frac{\mu_0c^2e^2n_{e0}}{m_e}\sum_{n=-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}dp_\perp dp_{||}\mathcal{K}_n\cdot\mathbf{E}_{\mathbf{k}\omega}\qquad(3.100)$$

with $\mathcal{K}_n$ equivalent to:

$$\begin{pmatrix} p_\perp^2 \left(\frac{n\Omega_e}{k_\perp v_\perp}\right)^2 U J_n^2(a_e)\sigma & 0 & 0 \\ 0 & p_\perp^2 U (J_n'(a_e))^2 \sigma & i p_\perp p_{||} U J_n'(a_e) J_n(a_e)\delta \\ 0 & -i p_{||} p_\perp U J_n'(a_e) J_n(a_e)\delta & p_{||}^2 U J_n^2(a_e)\sigma \end{pmatrix} \tag{3.101}$$

while

$$\begin{aligned} \sigma &= \frac{2\omega}{\omega^2 - (n\Omega_e)^2} \\ \delta &= \frac{2n\Omega_e}{\omega^2 - (n\Omega_e)^2} \end{aligned} \tag{3.102}$$

To make further progress we must also supply an isotropic expression for the equilibrium momentum distribution.

3.7.3 Relativistic Distribution Functions

This section concentrates on the possible choices of distribution function. As mentioned in the preamble, the only plasmas which will be discussed here will have locally Maxwellian distributions. This still permits a degree of freedom of interpretation. The classical (homogeneous) Maxwellian momentum distribution function has the form:

$$F_s(\mathbf{p}) = v e^{\mathbf{p}^2/2m_s kT} \tag{3.103}$$

The value of v is defined by the choice of normalization. In this case equation (3.32) gives:

$$v = (2\pi m_{s0}\, kT)^{-\frac{3}{2}} \tag{3.104}$$

To facilitate discussion we define the dimensionless inverse temperature: $a = m_{e0}c^2/kT$. As an illustration an a of 100 is the equivalent of a temperature of 6×10^7 K; $a = 1$ is equivalent to $T = 6\times10^9$ K. Thus v could be written in terms of this parameter, a.

$$v = \left(\frac{a}{2\pi m_0^2 c^2}\right)^{\frac{3}{2}} \left(\frac{a}{m_0^2 c^2}\right) \tag{3.105}$$

The fully relativistic Maxwellian distribution is discussed in many textbooks including [43]. As they point out, for the same configuration-momentum space normalization, (3.32), the fully relativistic Maxwellian distribution is:

$$F_s(\mathbf{p}) = v' e^{-a\gamma} \tag{3.106}$$

v' is given by an expression involving the modified Bessel function of the second kind of order two ($K_2(a)$):

$$v' = \left(\frac{1}{4\pi m_{e0}^3 c^3}\right) \frac{a}{K_2(a)} \tag{3.107}$$

This distribution is also known as the Maxwell-Boltzmann-Jüttner distribution. As usual γ is defined by:

$$\gamma = \left(1 + \frac{p^2}{m_0^2 c^2}\right)^{\frac{1}{2}} \tag{3.108}$$

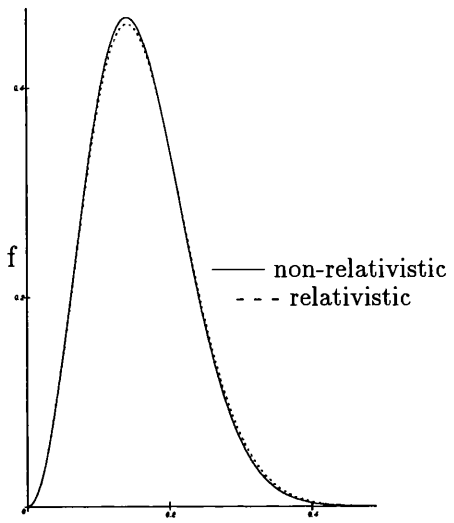
For a non-relativistic plasma $\gamma = 1$ and there is no momentum dependence at all; in which case the MBJ distribution is identical to the classical Maxwellian.

Rooney [44] used the MBJ distribution to produce a theory of fully relativistic $e^- e^+$ Bernstein modes. Rooney's theory treated plasmas for which $a < 10$. The present study of weakly relativistic plasmas (where $10 < a < 100$) is explained by the desire to bridge the gap between the unphysical classical Bernstein modes and Bernstein modes in more extremely relativistic plasmas.

Which distribution to adopt depends on the energy regime. The expression must accurately model the conditions and be easily incorporated in the analysis. In the weakly relativistic range there is little difference between the normalized classical and MBJ distribution functions. This can be seen in figures 3.1 and 3.2.

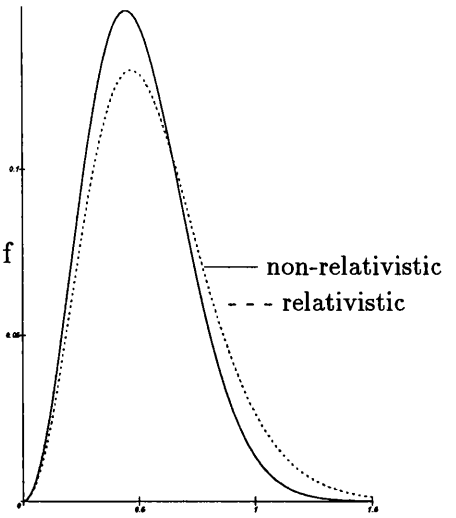
The present development accepts that the factors most strongly affected by relativistic factors are dependent on mass through the cyclotron frequency. This shows in the common denominator of (3.102).

$$\frac{1}{\omega^2 - (n\Omega_e)^2}$$



Momentum, **p**

Figure 3.1: $a = 100$



Momentum, **p**

Figure 3.2: $a = 10$

It is essential to note that Ω_e is not the *rest* cyclotron frequency. It can however be written in terms of the rest cyclotron frequency:

$$\Omega_e = \frac{\Omega_{e0}}{\gamma}$$

Furthermore note that because of this denominator the whole expression for any element of $\mathcal{R}$ resonates at frequencies close to the the *relativistic* cyclotron frequency and harmonics thereof. This becomes central in the appearance of modes such as the Bernstein modes.

One approach to the weakly relativistic regime, that taken by Robinson [47, 48] is to make an approximation for γ in an MBJ distribution. This method invokes the binomial expansion of γ,

$$\gamma \approx 1 + \frac{p^2}{2\, m_0^2 c^2} \tag{3.109}$$

Other approximations are made which result in a dispersion relation which is classical at frequencies away from harmonics, $\omega \neq n\Omega_{e0}$, and finite at harmonics, $\omega = n\Omega_{e0}$, where a classical theory would place discontinuities (n, an integer).

However the present treatment takes a different approach and retains just the non-relativistic Maxwellian, for reasons spelt out above. The strong relativistic effects are

retained in treatment of the denominator of the $p_{||}$ integrals in $\mathcal{R}$. Rather than make an approximation, this treatment retains the full relativistic expression for γ.

It is useful to register that no matter which expression is used for relativistic γ, it is always an *even* function of momentum: this feature stems from the isotropic condition laid down in the early stages.

3.7.4 A New Approach

Up to this point we have developed a general theory and left certain decisions unmade. To advance in the theory of weakly relativistic wave propagation we must specify an additional set of conditions:

1. $k_{||} \to 0$ (*i.e.* $U \to \omega \frac{\partial F_{e0}}{\partial p_\perp}$)
2. F_{e0} is a *non-relativistic* Maxwellian distribution
3. the denominator in (3.102) is expressed with a fully relativistic γ

This is an approach which has never been made before. These conditions immediately mean that:

$$U = -\omega v p_\perp e^{-\frac{1}{2}\frac{ap^2}{m_0^2c^2}} \tag{3.110}$$

where, as before

$$v = \left(\frac{a}{2\pi m_0^2 c^2}\right)^{\frac{3}{2}} \left(\frac{a}{m_0^2 c^2}\right) \tag{3.111}$$

Now look at the yz element of $\mathcal{K}_n$, (3.101).

$$ip_\perp p_{||} U J_n'(a_e) J_n(a_e) \delta$$

The zy element has the same expression except for a change in sign. Recall now that each of these elements is under a double integral. In each case the integrand is odd in $p_{||}$ and the range of the $p_{||}$ integral is $(-\infty, \infty)$. Splitting the range at zero the contributions

of the two parts of the integral exactly cancel. The $\mathcal{K}_n$ of (3.101) is therefore equivalent to a simpler form with no off-diagonal elements.

In summary (3.100) is

$$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 - c^2k_\perp^2 & 0 \\ 0 & 0 & \omega^2 - c^2k_\perp^2 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{\mathbf{k}\omega} = -2\pi \frac{\mu_0 c^2 e^2 n_{e0}}{m_e} \sum_{n=-\infty}^{\infty} \int_0^\infty \int_{-\infty}^\infty dp_\perp dp_{||} \mathcal{K}_n \cdot \mathbf{E}_{\mathbf{k}\omega} \tag{3.112}$$

where $\mathcal{K}_n$ is equivalent to the expression in (3.101) with only the diagonal elements:

$$-ve^{-\frac{ap^2}{2m_0^2c^2}} \frac{2\omega^2}{\omega^2 - (n\Omega_e)^2} \begin{pmatrix} \left(\frac{n\Omega_e}{k_\perp v_\perp}\right)^2 J_n^2(a_e) p_\perp^3 & 0 & 0 \\ 0 & (J_n'(a_e))^2 p_\perp^3 & 0 \\ 0 & 0 & J_n^2(a_e) p_{||}^2 p_\perp \end{pmatrix}$$

3.7.5 Diagonalized $\mathcal{R}$

Rearranging (3.112) into dielectric form (as in (3.59)) shows that the only non-zero elements of $\mathcal{R}$ are the *diagonal* elements. For perturbations of the electric field in the x-, y- and z-directions these correspond to three equations where the diagonal elements of $\mathcal{R}$ are zero. When $R_{xx} = 0$ the dispersion relation which results corresponds to the Bernstein modes. Likewise $R_{yy} = 0$ to the Extraordinary mode and $R_{zz} = 0$ to the Ordinary mode.

Written out these are:

$$R_{xx} = \omega^2 - 4\pi\omega_{pe}^2 \upsilon \sum_{n=-\infty}^{\infty} n^2 \int_0^{\infty}\int_{-\infty}^{\infty} dp_\perp dp_{||} p_\perp^3 \left(\frac{\Omega_e}{k_\perp v_\perp}\right)^2 J_n^2(a_e) e^{-\frac{ap^2}{2m_{e0}^2 c^2}} \frac{\omega^2}{\omega^2 - (n\Omega_e)^2} = 0 \tag{3.113}$$

$$R_{yy} = \omega^2 - c^2 k_\perp^2 - 4\pi\omega_{pe}^2 \upsilon \sum_{n=-\infty}^{\infty} \int_0^{\infty}\int_{-\infty}^{\infty} dp_\perp dp_{||} p_\perp^3 J_n'(a_e)^2 e^{-\frac{ap^2}{2m_{e0}^2 c^2}} \frac{\omega^2}{\omega^2 - (n\Omega_e)^2} = 0 \tag{3.114}$$

$$R_{zz} = \omega^2 - c^2 k_\perp^2 - 4\pi\omega_{pe}^2 \upsilon \sum_{n=-\infty}^{\infty} \int_0^{\infty}\int_{-\infty}^{\infty} dp_\perp dp_{||} p_\perp p_{||}^2 J_n^2(a_e) e^{-\frac{ap^2}{2m_{e0}^2 c^2}} \frac{\omega^2}{\omega^2 - (n\Omega_e)^2} = 0 \tag{3.115}$$

3.7.6 Introducing Dimensionless Variables

To simplify the expressions which follow (and the computer code which will eventually calculate the solutions) let us adopt dimensionless variables.

$$\begin{aligned} \hat{\omega} &= \frac{\omega}{\Omega_{e0}} & \hat{p}_\perp &= \frac{p_\perp}{m_e c} & \hat{k}_\perp &= \frac{c k_\perp}{\Omega_{e0}} \\ \hat{\omega}_{pe} &= \frac{\omega_{pe}}{\Omega_{e0}} & \hat{p}_{||} &= \frac{p_{||}}{m_e c} & \hat{k}_{||} &= 0 \\ & & \gamma^2 &= 1 + \hat{p}_{||}^2 + \hat{p}_\perp^2 & & \end{aligned} \tag{3.116}$$

The argument of the Bessel functions is

$$a_e = \frac{k_\perp v_\perp}{\Omega_e} = \frac{k_\perp c\ \gamma v_\perp m_{e0}}{\Omega_{e0}\ m_{e0} c} = \hat{k}_\perp \hat{p}_\perp \tag{3.117}$$

It is convenient too to define a dimensionless element of $\mathcal{R}$, $\hat{R}_{ii} = R_{ii}/\Omega_{e0}^2$.

Now look at the integrand of R_{xx}

$$p_\perp^3 \left(\frac{\Omega_e}{k_\perp v_\perp}\right)^2 J_n^2(a_e) e^{-\frac{ap^2}{2m_{e0}^2 c^2}} \frac{\omega^2}{\omega^2 - (n\Omega_e)^2} \tag{3.118}$$

In particular

$$p_\perp^3 \left(\frac{\Omega_e}{k_\perp v_\perp}\right)^2 = p_\perp \left(\frac{\Omega_{e0}}{\gamma k_\perp} \frac{\gamma m_{e0} v_\perp}{v_\perp}\right)^2 = p_\perp \left(\frac{\Omega_{e0} m_{e0}}{k_\perp}\right)^2 = p_\perp \left(\frac{m_{e0}^2 c^2}{\hat{k}_\perp^2}\right) \tag{3.119}$$

and

$$\frac{\omega^2}{\omega^2-(n\Omega_e)^2}=\frac{1}{1-\left(\frac{n\Omega_{e0}}{\gamma\omega}\right)^2}=\frac{\gamma^2}{\left(\gamma^2-\frac{n^2}{\hat{\omega}^2}\right)} \tag{3.120}$$

This leaves expressions for the three decoupled modes which correspond to $\hat{R}_{xx}=0$, $\hat{R}_{yy}=0$ and $\hat{R}_{zz}=0$.

$$\hat{R}_{xx}=\hat{\omega}^2-\mathcal{C}\sum_{n=-\infty}^{\infty}n^2\int_0^{\infty}\int_{-\infty}^{\infty}dp_\perp dp_{||}(J_n(a_e))^2e^{-\frac{ap^2}{2m^2c^2}}\frac{\gamma^2p_\perp}{\left(\gamma^2-\frac{n^2}{\hat{\omega}^2}\right)} \tag{3.121}$$

where

$$\mathcal{C}=\frac{4\pi}{\Omega_{e0}^2}\omega_{pe}^2\upsilon\left(\frac{m_{e0}^2c^2}{\hat{k}_\perp^2}\right)=\frac{4\pi}{\Omega_{e0}^2}\omega_{pe}^2\left(\frac{a}{2\pi}\right)^{\frac{3}{2}}\frac{1}{m_{e0}^3c^3}\left(\frac{a}{\hat{k}_\perp^2}\right) \tag{3.122}$$

$$\hat{R}_{yy}=\hat{\omega}^2-\hat{k}_\perp^2-4\pi\hat{\omega}_{pe}^2\upsilon\sum_{n=-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}d\hat{p}_\perp d\hat{p}_{||}\hat{p}_\perp^3J_n'(a_e)^2e^{-\frac{a\hat{p}^2}{2}}\frac{\gamma^2}{\gamma^2-\left(\frac{n}{\omega}\right)^2} \tag{3.123}$$

$$\hat{R}_{zz}=\hat{\omega}^2-\hat{k}_\perp^2-4\pi\hat{\omega}_{pe}^2\upsilon\sum_{n=-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}d\hat{p}_\perp d\hat{p}_{||}\hat{p}_\perp\hat{p}_{||}^2J_n^2(a_e)e^{-\frac{a\hat{p}^2}{2}}\frac{\gamma^2}{\gamma^2-\left(\frac{n}{\omega}\right)^2} \tag{3.124}$$

Chapter 4

Classical Bernstein Modes

The crucial equation in the following chapters is (3.59):

$$\mathcal{R} \cdot \mathbf{E} = \mathbf{0}$$

As seen in Section 3.7.5 three independent modes of propagation are available: $E_x \neq 0$, $E_y \neq 0$, and $E_z \neq 0$. The first of these means, since propagation is perpendicular: $\mathbf{E} \perp \mathbf{B}$ and $\mathbf{k} \parallel \mathbf{E}$ – a *longitudinal* mode. Similarly $E_y \neq 0$ implies that $\mathbf{E} \perp \mathbf{B}$ and $\mathbf{k} \perp \mathbf{E}$ – a *transverse* mode. Finally $E_z \neq 0$ will give $\mathbf{E} \parallel \mathbf{B}$ and $\mathbf{k} \perp \mathbf{E}$ - likewise, a *transverse* mode. The longitudinal mode corresponds to a purely electrostatic mode, whose dispersion relation is given by $R_{xx} = 0$, the *Bernstein mode.* The two transverse modes are respectively $R_{yy} = 0$, the *Extraordinary*, and $R_{zz} = 0$, the *Ordinary Modes.*

4.1 Classical Bernstein Mode

The Bernstein mode propagates perpendicular to the magnetic field and resonates at electron cyclotron harmonics ($\omega \approx n\Omega_e$, with n an integer). This mode was first described by Bernstein [49] and is treated in most textbooks, however all these textbooks endeavour to treat only an electron-ion plasma. The usual static ion assumption is invoked early in the treatment, see [37, 38, 39, 42, 50, 51].

There is a whole body of research into Bernstein modes in electron-ion plasmas. As will be seen the e^-e^+ modes are very similar in form to electron Bernstein modes. The

equation for the latter is usually derived with the assumption of stationary ions. If this is not done, a low frequency phenomenon due to the ions must be accounted for - the ion Bernstein modes. Since some schemes of plasma heating in tokamaks rely on mode conversion into ion Bernstein modes, these modes are an important research topic, see for instance [52]. They are reviewed in a recent paper [53].

The early experimental work on Bernstein modes is described in reviews by Crawford [54] and Stix [55]. Recent experiments on NASA's Space Shuttle show electron Bernstein modes very clearly in the Earth's ionosphere [56].

4.2 Dispersion Relations

4.2.1 Electron-ion plasma

The electron-ion result may be derived here from the general dispersion equation (3.86) with an isotropic distribution function — thus (3.89) holds.

If the ion contribution to the sum across species is neglected (the static ion assumption) the three modes approximately decouple for perpendicular propagation. Put another way the resistivity tensor is essentially diagonalized. This approximation is handled in textbooks.

For the present purpose note that the R_{xx} element is decoupled from the other modes so it is sufficient to examine this element in isolation. Assuming the isotropic momentum distribution is the non-relativistic Maxwellian, U is defined at (3.110). Thus equation (3.86) gives the following expression for R_{xx}:

$$\omega^2 - c^2k_{||}^2 - 2\pi c^2\mu_0 \sum_s \frac{q_s^2}{m_s} n_{s0} \int_0^\infty \int_{-\infty}^\infty dp_\perp dp_{||} \sum_{n=-\infty}^{\infty} \frac{p_\perp^3 \omega v}{\omega - k_{||}v_{||} - n\Omega_s} \left(\frac{n\Omega_s}{k_\perp v_\perp}\right)^2 J_n^2(a_s) e^{-\frac{ap^2}{2m_0^2c^2}} \tag{4.1}$$

where as before v is defined as

$$v = \left(\frac{a}{2\pi m_0^2 c^2}\right)^{\frac{3}{2}} \left(\frac{a}{m_0^2 c^2}\right) \tag{4.2}$$

Now if low frequencies are avoided, the fact that $m_i \gg m_e$ means that to a good approximation, the ion contribution to the sum over species is zero. Mathematically that

is

$$\omega^2 - c^2k_{||}^2 - 2\pi\frac{e^2n_{e0}}{m_{e0}\epsilon_0}\int_0^\infty\int_{-\infty}^\infty dp_\perp dp_{||}\sum_{n=-\infty}^{\infty}\frac{p_\perp^3\omega v}{\omega - k_{||}v_{||} - n\Omega_e}\left(\frac{n\Omega_e}{k_\perp v_\perp}\right)^2 J_n^2(a_e)e^{-\frac{ap^2}{2m_{e0}^2c^2}} = 0 \tag{4.3}$$

For perpendicular propagation (*i.e.* $k_{||}\to 0$) the last term from the equation above becomes:

$$-2\pi\omega_{pe}^2\sum_{n=-\infty}^{\infty}\frac{m_{e0}^2\omega v}{\omega - n\Omega_e}\left(\frac{n\Omega_e}{k_\perp}\right)^2\int_0^\infty\int_{-\infty}^\infty dp_\perp dp_{||}p_\perp J_n^2(a_e)e^{-\frac{ap^2}{2m_{e0}^2c^2}} \tag{4.4}$$

The non-relativistic assumption has permitted the separation of the double integral. Taking each integral in turn:

$$\int_{-\infty}^\infty dp_{||}e^{-\frac{ap_{||}^2}{2m_{e0}^2c^2}} = \sqrt{\frac{\pi}{\left(\frac{a}{2m_{e0}^2c^2}\right)}} = \sqrt{\frac{2\pi}{a}}m_{e0}c \tag{4.5}$$

The very convenient identity mentioned in Appendix A applies directly to the $p_\perp$ integral — (A.10).

$$\begin{aligned}\int_0^\infty dp_\perp p_\perp J_n^2\left(\frac{k_\perp p_\perp}{\Omega_{e0}m_{e0}}\right)e^{-\frac{ap_\perp^2}{2m_{e0}^2c^2}} &= \frac{1}{2\left(\frac{a}{2m_{e0}^2c^2}\right)}\exp\left(-\frac{2\left(\frac{k_\perp}{m_{e0}\Omega_{e0}}\right)^2}{4\left(\frac{a}{2m_{e0}^2c^2}\right)}\right)I_n\left(\frac{\left(\frac{k_\perp}{m_{e0}\Omega_{e0}}\right)^2}{2\left(\frac{a}{2m_{e0}^2c^2}\right)}\right)\\ &= \frac{m_{e0}^2c^2}{a}\exp\left(-\frac{k_\perp^2c^2}{a\Omega_{e0}^2}\right)I_n\left(\frac{k_\perp^2c^2}{a\Omega_{e0}^2}\right)\end{aligned} \tag{4.6}$$

So that (4.3) is

$$\omega^2 - 2\pi\omega_{pe}^2\sum_{n=-\infty}^{\infty}\frac{m_{e0}^2\omega v}{\omega - n\Omega_e}\left(\frac{n\Omega_e}{k_\perp}\right)^2\sqrt{\frac{2\pi}{a}}\frac{m_{e0}^3c^3}{a}\exp\left(-\frac{k_\perp^2c^2}{a\Omega_{e0}^2}\right)I_n\left(\frac{k_\perp^2c^2}{a\Omega_{e0}^2}\right) = 0 \tag{4.7}$$

Using dimensionless parameters defined in (3.116) (so that for instance $\hat{k}_\perp = k_\perp c/\Omega_{e0}$) and introducing $\Lambda = \hat{k}_\perp^2/a$, one can write out the whole expression for $\hat{R}_{xx} = R_{xx}/\Omega_{e0}^2 = 0$. Note that there has been a great deal of cancellation.

$$\hat{\omega}^2 = \hat{\omega}_{pe}^2\frac{e^{-\Lambda}}{\Lambda}\sum_{n=-\infty}^{\infty}\frac{n^2I_n(\Lambda)}{1-\frac{n}{\hat{\omega}}} \tag{4.8}$$

Figure 4.1 illustrates one example of the dispersion curves which result (here $\hat{\omega}_{pe} = 3$).

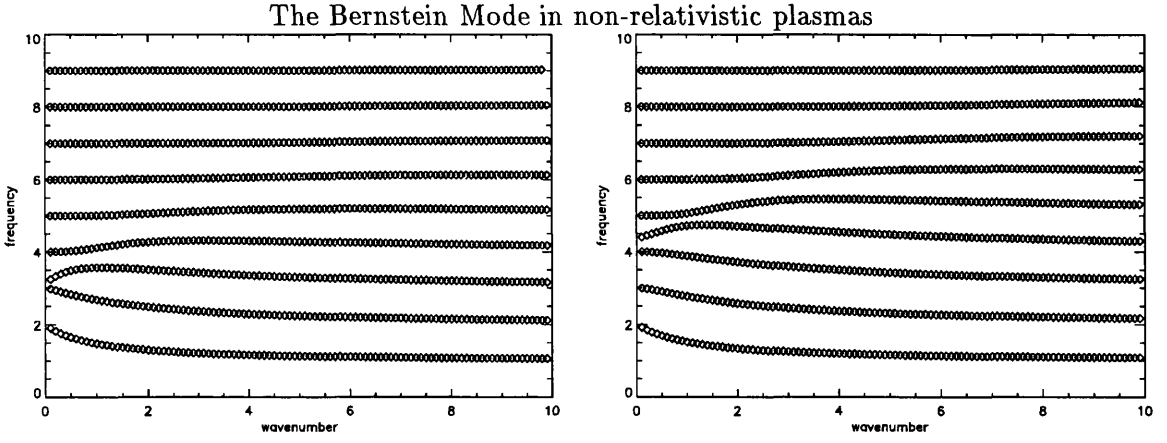


Figure 4.1: e^--ion plasma

Figure 4.2: e^-e^+ plasma

Note too that

$$\sum_{n=-\infty}^{\infty} \frac{n^2}{1-\left(\frac{n}{\hat{\omega}}\right)} I_n(\Lambda) = \sum_{n=-\infty}^{\infty} \frac{n^2}{1-\left(\frac{n}{\hat{\omega}}\right)^2} I_n(\Lambda) \tag{4.9}$$

So that (4.8) may also be written

$$\hat{\omega}^2 = \hat{\omega}_{pe}^2 \frac{e^{-\Lambda}}{\Lambda} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(\Lambda)}{1-\left(\frac{n}{\hat{\omega}}\right)^2} \tag{4.10}$$

4.2.2 Electron-positron plasma

For the e^-e^+ case on the other hand the static ion assumption is not valid, however the equal mass assumption *is*. To construct the non-relativistic equations one must return to the equation for the R_{xx} (3.113). Recall that this equation comes after the adoption of the non-relativistic Maxwellian distribution and take the non-relativistic limit, *i.e.* $\gamma = 1$, throughout.

$$R_{xx} = \omega^2 - 4\pi\omega_{pe}^2 v \sum_{n=-\infty}^{\infty} n^2 \int_0^{\infty}\int_{-\infty}^{\infty} dp_\perp dp_\parallel p_\perp^3 \left(\frac{\Omega_e}{k_\perp v_\perp}\right)^2 J_n^2(a_e) e^{-\frac{ap^2}{2m_{e0}^2 c^2}} \frac{\omega^2}{\omega^2 - (n\Omega_e)^2}$$

where v is defined in (4.2).

Now rearranged this is:

$$R_{xx} = \omega^2 - 4\pi\omega_{pe}^2 v \left(\frac{\Omega_{e0} m_{e0}}{k_\perp}\right)^2 \sum_{n=-\infty}^{\infty} \frac{\omega^2 n^2}{\omega^2 - (n\Omega_{e0})^2} \int_0^\infty \int_{-\infty}^\infty dp_\perp dp_{||} p_\perp J_n^2(a_e) e^{-\frac{ap^2}{2m_{e0}^2 c^2}} \tag{4.11}$$

Note that the double integral here is exactly that found for the electron-ion case (4.4) and so may be replaced by the expression

$$\sqrt{\frac{2\pi}{a}} \frac{m_{e0}^3 c^3}{a} \exp\left(-\frac{k_\perp^2 c^2}{a\Omega_{e0}^2}\right) I_n\left(\frac{k_\perp^2 c^2}{a\Omega_{e0}^2}\right) \tag{4.12}$$

Switch to dimensionless parameters again (see (3.116)). The expression for the complete term involving the double integral now reads

$$-4\pi\omega_{pe}^2 v \left(\frac{1}{\hat{k}_\perp^2}\right) \sum_{n=-\infty}^{\infty} \frac{n^2}{1 - \left(\frac{n}{\hat{\omega}}\right)^2} \sqrt{\frac{2\pi}{a}} \frac{m_0^5 c^5}{a} \exp\left(-\frac{\hat{k}_\perp^2}{a}\right) I_n\left(\frac{\hat{k}_\perp^2}{a}\right) \tag{4.13}$$

Now for clarity introduce $\Lambda = \hat{k}_\perp^2/a$ and write out the whole expression for $\hat{R}_{xx} = R_{xx}/\Omega_{e0}^2$

$$\hat{R}_{xx} = \hat{\omega}^2 - 4\pi\hat{\omega}_{pe}^2 v \sum_{n=-\infty}^{\infty} \left(\frac{1}{\Lambda}\right) \frac{n^2}{1 - \left(\frac{n}{\hat{\omega}}\right)^2} \sqrt{\frac{2\pi}{a}} \frac{m_0^5 c^5}{a^2} e^{-\Lambda} I_n(\Lambda) \tag{4.14}$$

Gathering all the constants in the rightmost term above gives some reassuring cancellation.

$$4\pi\hat{\omega}_{pe}^2 v \sum_{n=-\infty}^{\infty} \left(\frac{1}{\Lambda}\right) \frac{n^2}{1 - \left(\frac{n}{\hat{\omega}}\right)^2} \sqrt{\frac{2\pi}{a}} \frac{m_0^5 c^5}{a^2} e^{-\Lambda} I_n(\Lambda) = 2\hat{\omega}_{pe}^2 \frac{e^{-\Lambda}}{\Lambda} \sum_{n=-\infty}^{\infty} \frac{n^2}{1 - \left(\frac{n}{\hat{\omega}}\right)^2} I_n(\Lambda)$$

When $R_{xx} = 0$ for perpendicular propagation in an e^-e^+ plasma, the dispersion relation for Bernstein Modes is given by

$$\hat{\omega}^2 = \hat{\omega}_{pe}^2 \frac{2e^{-\Lambda}}{\Lambda} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(\Lambda)}{1 - \frac{n^2}{\hat{\omega}^2}} \tag{4.15}$$

This relation is illustrated in figure 4.2 and again in the diagrams of Section 4.2.

4.2.3 Comparison of e^--ion and e^-e^+ dispersion relations

Comparing the dispersion relations for each case (4.10) & (4.15) shows that they are almost identical - the sole difference is a factor of two in (4.15).

A point in a dispersion curve corresponds to an oscillation at a permitted pair of frequency and wavenumber. When these points lie on curves in frequency-wavenumber parameter space, as they do here, each curve defines a mode.

For both types of plasma there is a critical frequency above which the shape of the Bernstein modes changes. Modes above this frequency display gaps in the frequency spectrum away from $n\Omega_e$; below this, the frequency spectrum is continuous. The special symmetry involved in the theoretical treatment of the e^-e^+ plasma still allows such frequency gaps to occur, though in this case the theory leading to these gaps is exact and not a consequence of approximation. The only visible difference is the 'doubling' of the critical frequency. As can be seen by comparing the figures 4.1 and 4.2 for the same plasma frequency.

It is instructive to treat the low wavenumber/nonrelativistic limit: note that for $a \to \infty$ or $k_\perp \to 0$ (*i.e.* $\Lambda \to 0$) — $I_{\pm 1}(\Lambda)/\Lambda \to 1/2$ and $I_n(\Lambda)/\Lambda \to 0$ (for $n \neq \pm 1, 0$), (4.10) and (4.15) respectively reduce to simple equations:

$$\hat{\omega}^2 = \hat{\omega}_{pe}^2 + 1 \tag{4.16}$$

and

$$\hat{\omega}^2 = 2\hat{\omega}_{pe}^2 + 1 \tag{4.17}$$

In both cases the result is the hybrid frequency, $\omega = \sqrt{\omega_p^2 + \Omega_e^2}$.

The hybrid frequency and the critical frequency are one and the same.

4.2.4 Dispersion curves

The diagrams which follow illustrate classical Bernstein modes in e^-e^+ plasmas, as calculated in (4.15). They show curves for different values of plasma frequency.

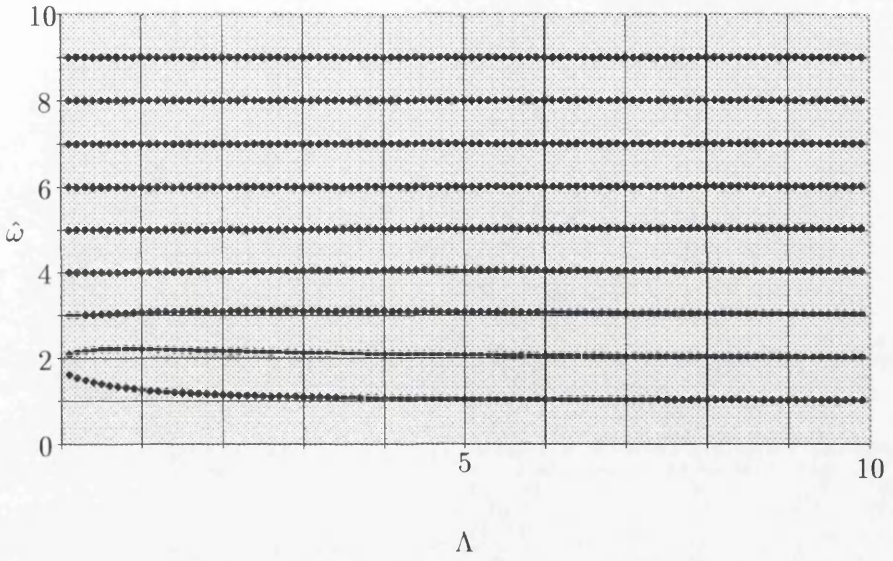


Figure 4.3: A plot of $\hat{\omega}_{ep} = 1$

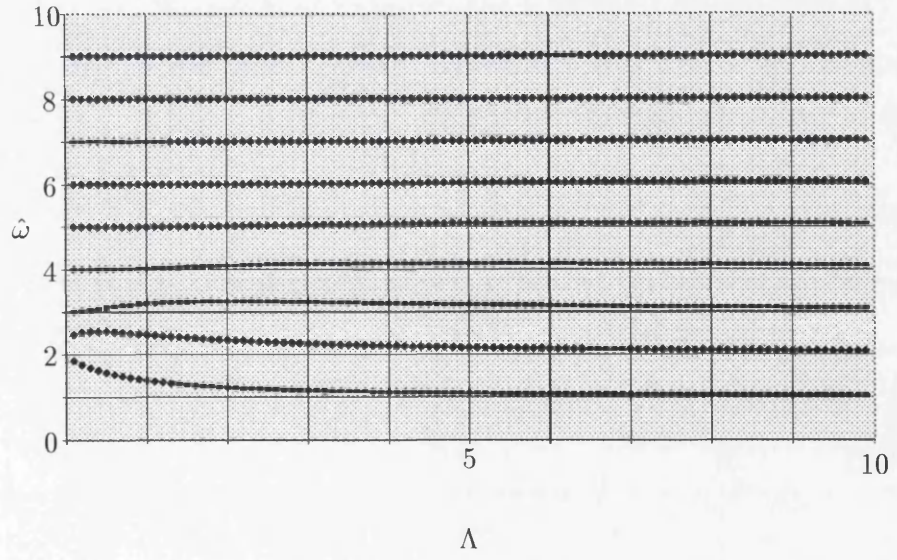


Figure 4.4: A plot of $\hat{\omega}_{ep} = 1.5$

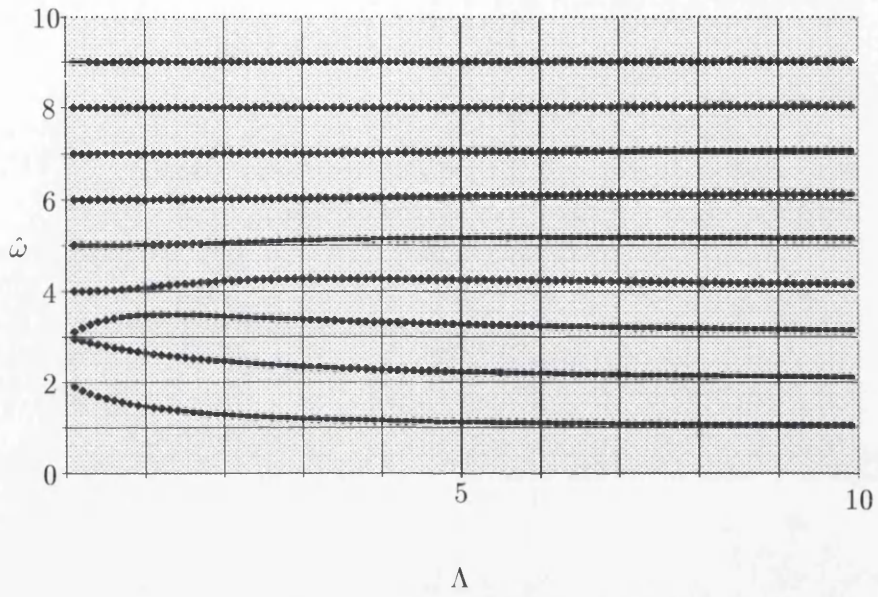


Figure 4.5: A plot of $\hat{\omega}_{ep} = 2$

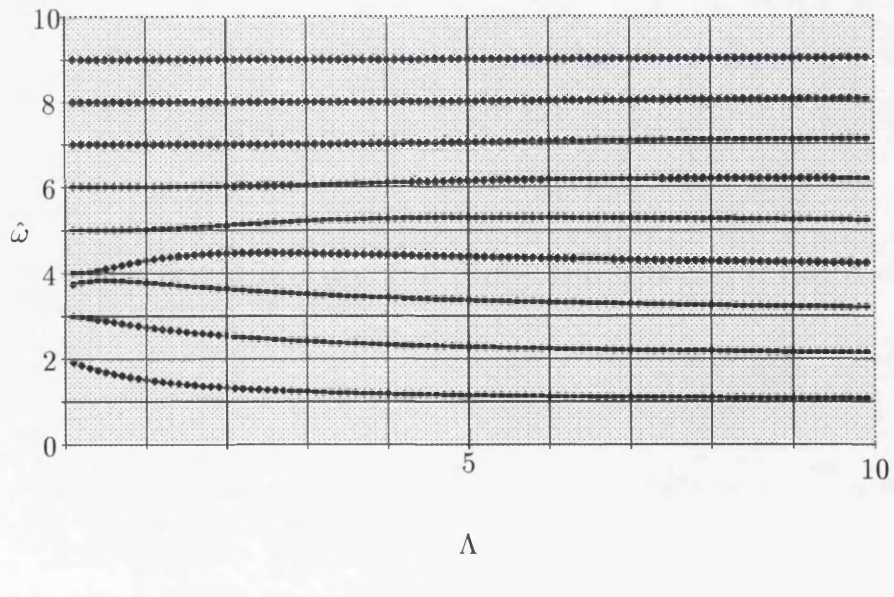


Figure 4.6: A plot of $\hat{\omega}_{ep} = 2.5$

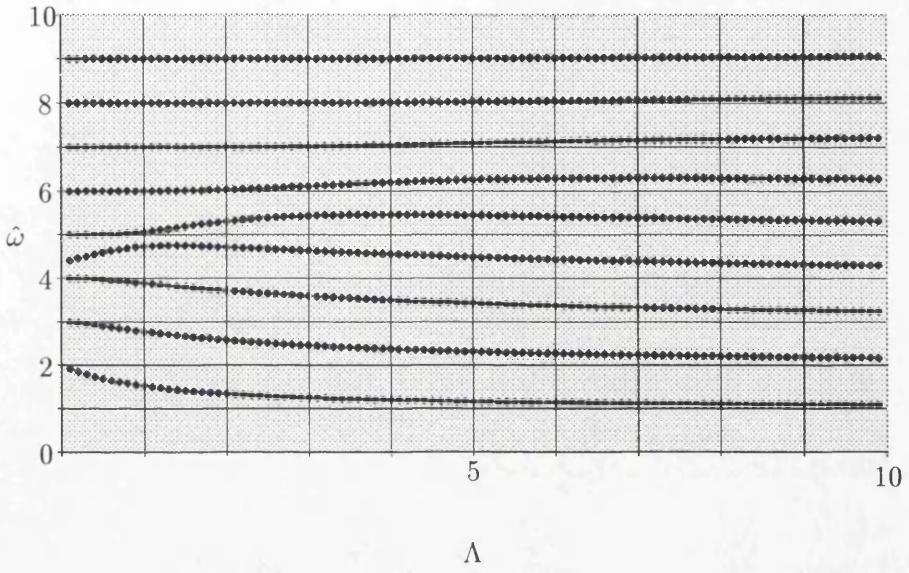


Figure 4.7: A plot of $\hat{\omega}_{ep} = 3$

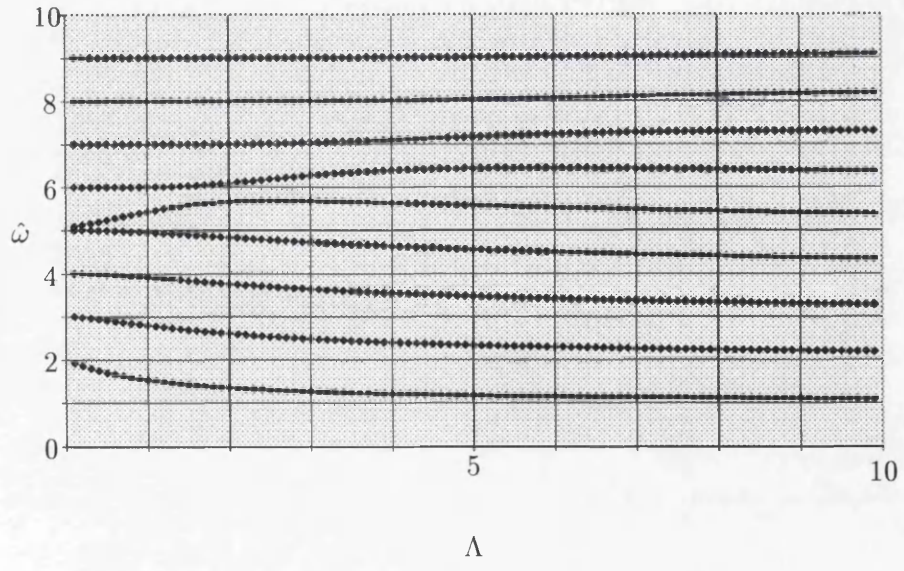


Figure 4.8: A plot of $\hat{\omega}_{ep} = 3.5$

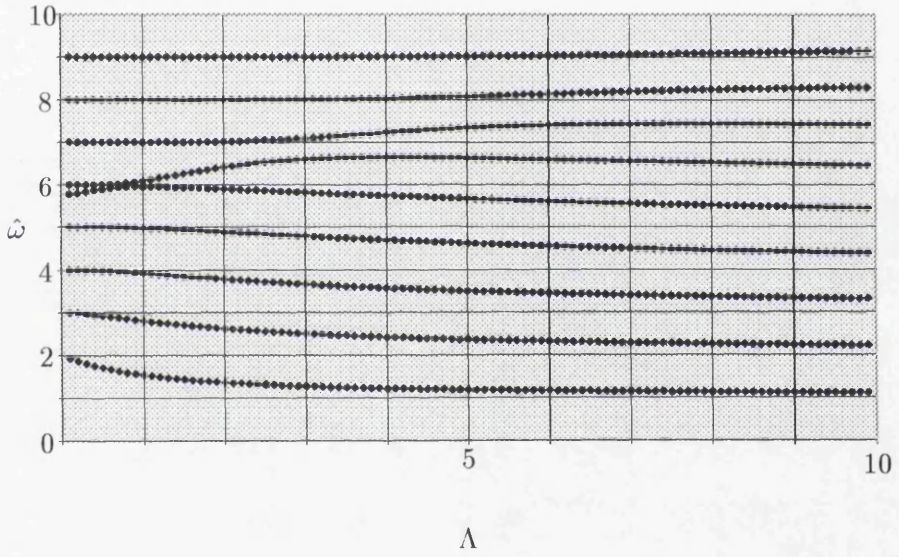


Figure 4.9: A plot of $\hat{\omega}_{ep} = 4$

Chapter 5

Relativistic Bernstein Modes

All sorts of funny thoughts
Run around my head:
"It isn't really
Anywhere!
It's somewhere else
Instead!"

in *Halfway down* by *A.A. Milne*

For e^--ion plasmas, the introduction of a weakly (or fully) relativistic treatment sees a broadening of the frequencies at which resonance occurs, and a downshift in those frequencies as thermal velocities become more relativistic. Waves at resonant frequencies are damped as a direct result of modifying the classical static ion dispersion relation to accommodate relativistic corrections [47, 48].

Previous work in e^-e^+ plasmas has handled the fully relativistic regime (where $a \leq 10$)[44] but not a weakly relativistic regime ($10 \leq a \leq 100$).

The purpose of this chapter is to develop the novel treatment for weakly relativistic e^-e^+ plasmas outlined earlier (see Section 3.7.4). Our goal is a treatment which results in figures which compare directly with those of chapter 4. At the risk of anticipating the theory, we present one such figure, figure 5.1.

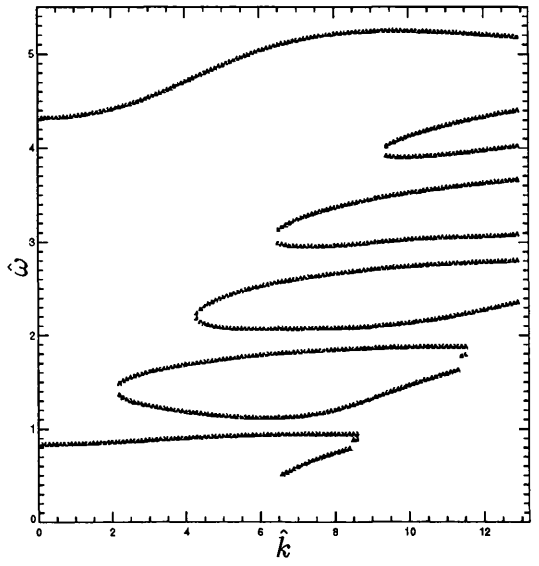


Figure 5.1: The Bernstein Mode in weakly relativistic e^-e^+ plasma ($a = 10$)

The effects noted in the weakly relativistic e^--ion plasma are mirrored in the e^-e^+ treatment. However no approximation dependent on the disparate masses of species is required and the damping is consequently not present. Furthermore in the ultrarelativistic limit ($a \to 0$) of the weakly relativistic dispersion relation for e^-e^+ plasmas, the Bernstein modes are not present and only the plasma oscillation remains perpendicular to the magnetic field, consistent with the predictions of the fully relativistic analysis of [44]. In the classical limit, the dispersion relation can be shown to relax to the non-relativistic form.

5.1 The Expression for $\hat{R}_{xx}$

5.1.1 Simplification

For now, let's narrow our interest to the expression for $\hat{R}_{xx}$ found at the end of Chapter 3 (3.121):

$$\hat{R}_{xx} = \hat{\omega}^2 - C \sum_{n=-\infty}^{\infty} n^2 \int_0^{\infty} \int_{-\infty}^{\infty} dp_\perp dp_{||} (J_n(a_e))^2 e^{-\frac{ap^2}{2m^2c^2}} \frac{\gamma^2 p_\perp}{\left(\gamma^2 - \frac{n^2}{\hat{\omega}^2}\right)}$$

where

$$C = \frac{4\pi}{\Omega_{e0}^2}\omega_{pe}^2 v\left(\frac{m_{e0}^2c^2}{\hat{k}_\perp^2}\right) = \frac{4\pi}{\Omega_{e0}^2}\omega_{pe}^2\left(\frac{a}{2\pi}\right)^{\frac{3}{2}}\frac{1}{m_{e0}^3c^3}\left(\frac{a}{\hat{k}_\perp^2}\right)$$

Rearrange the part of the integrand in (3.120)

$$\frac{\gamma^2}{\left(\gamma^2-\frac{n^2}{\hat{\omega}^2}\right)} = \frac{\left(\gamma^2-\frac{n^2}{\hat{\omega}^2}\right)+\frac{n^2}{\hat{\omega}^2}}{\left(\gamma^2-\frac{n^2}{\hat{\omega}^2}\right)} = 1+\frac{n^2}{\hat{\omega}^2}\left[\frac{1}{\gamma^2-\frac{n^2}{\hat{\omega}^2}}\right] \tag{5.1}$$

So that:

$$\hat{R}_{xx} = \hat{\omega}^2 - C\sum_{n=-\infty}^{\infty} n^2\int_0^\infty\int_{-\infty}^\infty \left(m_{e0}^3c^3\right)d\hat{p}_\perp d\hat{p}_{||}\,\hat{p}_\perp(J_n(a_e))^2e^{-\frac{a\hat{p}^2}{2}}\left(1+\frac{n^2}{\hat{\omega}^2}\left[\frac{1}{\gamma^2-\frac{n^2}{\hat{\omega}^2}}\right]\right) \tag{5.2}$$

5.1.2 Bessel Identities

To tidy up a few more items use the identity (A.1)

$$J_{-n}(a) = (-1)^nJ_n(a)$$

Now the integrals are summed over index n; with the range $(-\infty,\infty)$. Compare the parts of the sum $\sum_{n=1}^{\infty}\cdots$ and $\sum_{n=-1}^{-\infty}\cdots$ ($n=0$ gives zero). Since $J_{-n}^2(a) = (-1)^{2n}J_n^2(a) = J_n^2(a)$ and all the other occurrences of n are of *even* order both parts will have identical elements.

$$\sum_{n=-\infty}^{\infty}\cdots = 2\sum_{n=1}^{\infty}\cdots \tag{5.3}$$

Thus

$$\begin{aligned}\hat{R}_{xx} &= \hat{\omega}^2 - 2\left(m_{e0}^3c^3\right)C\sum_{n=1}^{\infty}\left[n^2\int_0^\infty\int_{-\infty}^\infty d\hat{p}_\perp d\hat{p}_{||}\,\hat{p}_\perp(J_n(a_e))^2e^{-\frac{a\hat{p}^2}{2}}\right.\\ &\left.+\frac{n^4}{\hat{\omega}^2}\int_0^\infty\int_{-\infty}^\infty d\hat{p}_\perp d\hat{p}_{||}\,\frac{\hat{p}_\perp}{\gamma^2-\frac{n^2}{\hat{\omega}^2}}(J_n(a_e))^2e^{-\frac{a\hat{p}^2}{2}}\right]\end{aligned}$$

Now $2\left(m_{e0}^3c^3\right)C = 4\hat{\omega}_{pe}^2a^{5/2}/\left(\sqrt{2\pi}\,\hat{k}_\perp^2\right)$, which leaves:

$$\begin{aligned}\hat{R}_{xx} &= \hat{\omega}^2 - 4\hat{\omega}_{pe}^2\frac{a^{\frac{5}{2}}}{\sqrt{2\pi}}\left(\frac{1}{\hat{k}_\perp^2}\right)\left[\sum_{n=1}^{\infty}n^2\int_0^\infty\int_{-\infty}^\infty d\hat{p}_\perp d\hat{p}_{||}\,\hat{p}_\perp(J_n(a_e))^2e^{-\frac{a\hat{p}^2}{2}}\right.\\ &\left.+\sum_{n=1}^{\infty}\frac{n^4}{\hat{\omega}^2}\int_0^\infty\int_{-\infty}^\infty d\hat{p}_\perp d\hat{p}_{||}\,\frac{\hat{p}_\perp}{\gamma^2-\frac{n^2}{\hat{\omega}^2}}(J_n(a_e))^2e^{-\frac{a\hat{p}^2}{2}}\right]\end{aligned}$$

Now examine the first sum:

$$-4\hat{\omega}_{pe}^2 \frac{a^{\frac{5}{2}}}{\sqrt{2\pi}} \left(\frac{1}{\hat{k}_\perp^2}\right) \sum_{n=1}^{\infty} \left[n^2 \int_0^\infty \int_{-\infty}^\infty d\hat{p}_\perp d\hat{p}_{||}\, \hat{p}_\perp (J_n(\hat{k}_\perp \hat{p}_\perp))^2 e^{-\frac{a\hat{p}^2}{2}} \right] \tag{5.4}$$

The double integral may be simplified here. First deal with the $\hat{p}_{||}$ integral by quoting the identity:

$$\int_{-\infty}^{\infty} d\hat{p}_{||}\, e^{-\frac{a\hat{p}_{||}^2}{2}} = \sqrt{\frac{2\pi}{a}} \tag{5.5}$$

The $\hat{p}_\perp$ integral requires a different identity:

$$\int_0^\infty d\hat{p}_\perp\, \hat{p}_\perp (J_n(\hat{k}_\perp \hat{p}_\perp))^2 e^{-\frac{a\hat{p}_\perp^2}{2}} = \frac{1}{a} e^{-\left(\frac{\hat{k}_\perp^2}{a}\right)} I_n\left(\frac{\hat{k}_\perp^2}{a}\right) \tag{5.6}$$

Thus the whole double integral expression is:

$$-4\hat{\omega}_{pe}^2 \frac{a^{\frac{5}{2}}}{\sqrt{2\pi}} \left(\frac{1}{\hat{k}_\perp^2}\right) \sum_{n=1}^{\infty} \left[n^2 \frac{\sqrt{2\pi}}{a^{\frac{3}{2}}} e^{-\left(\frac{\hat{k}_\perp^2}{a}\right)} I_n\left(\frac{\hat{k}_\perp^2}{a}\right) \right] \tag{5.7}$$

Now according to [57], there is a further identity

$$\sum_{n=1}^{\infty} n^2 \frac{e^{-\Lambda} I_n(\Lambda)}{\Lambda} = \frac{1}{2} \tag{5.8}$$

with $\Lambda = \hat{k}_\perp^2/a$. This means the first term of the sum becomes

$$-4\hat{\omega}_{pe}^2 \frac{a^{\frac{5}{2}}}{\sqrt{2\pi}} \left(\frac{1}{\hat{k}_\perp^2}\right) \frac{\sqrt{2\pi}}{a^{\frac{3}{2}}} \left(\frac{1}{2} \frac{\hat{k}_\perp^2}{a}\right) = -2\hat{\omega}_{pe}^2 \tag{5.9}$$

And so we have the expression

$$\hat{R}_{xx} = \hat{\omega}^2 - 2\hat{\omega}_{pe}^2 - \frac{4\,\hat{\omega}_{pe}^2}{\hat{k}_\perp^2} \left(\frac{a^{\frac{5}{2}}}{\sqrt{2\,\pi}}\right) \sum_{n=1}^{\infty} \frac{n^4}{\hat{\omega}^2} \int_0^\infty \hat{p}_\perp (J_n(a_e))^2 e^{-\frac{a\hat{p}_\perp^2}{2}} d\hat{p}_\perp \int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\left(\gamma^2 - \frac{n^2}{\hat{\omega}^2}\right)} d\hat{p}_{||} \tag{5.10}$$

5.2 Theoretical treatment

The integral part of the final term of (5.10) is a double integral in $\hat{p}_\perp$ and $\hat{p}_{||}$:

$$\int_0^\infty \hat{p}_\perp (J_n(\hat{k}_\perp \hat{p}_\perp))^2 e^{-\frac{a\hat{p}_\perp^2}{2}} d\hat{p}_\perp \int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\left(\gamma^2 - \frac{n^2}{\hat{\omega}^2}\right)} d\hat{p}_{||} \tag{5.11}$$

Bearing this in mind we now aim to replace one of the parts of this double integral with a special function. In the event one special function is not enough but the principle is the same: there are good routines for calculating special functions which can then be adopted by the code.

Now it can been seen that the $\hat{p}_{||}$ integral has a discontinuity when $\gamma^2 = \frac{n^2}{\hat{\omega}^2}$. The denominator takes the form:

$$1 + \hat{p}_{||}^2 + \hat{p}_{\perp}^2 - \frac{n^2}{\hat{\omega}^2} = \hat{p}_{||}^2 + \hat{p}_{\perp}^2 - b_n^2 = \hat{p}_{||}^2 + \hat{p}_{\perp}^2 + q_n^2 \tag{5.12}$$

Two related quantities are defined here:

$$b_n^2 = \frac{n^2}{\hat{\omega}^2} - 1 \quad \text{if} \quad \hat{\omega} \leq n \tag{5.13}$$

$$q_n^2 = 1 - \frac{n^2}{\hat{\omega}^2} \quad \text{if} \quad \hat{\omega} > n \tag{5.14}$$

Three situations may arise

Case I $\hat{\omega} \leq n$ and $\hat{p}_{\perp} \leq b_n$

Case II $\hat{\omega} \leq n$ and $\hat{p}_{\perp} > b_n$

Case III $\hat{\omega} > n$

The first two cases must be considered together. Here $\hat{\omega} \leq n$ and b_n is used.

For each n the $\hat{p}_{\perp}$ integral is split around b_n:

$$\int_0^{\infty} [\cdots] = \int_0^{b_n} [\cdots] + \int_{b_n}^{\infty} [\cdots]$$

In the case of $\int_0^{b_n}[\cdots]$, case one, the denominator (5.12) may become zero. Consequently evaluating the $\hat{p}_{||}$ integral must be done by contour integration so that poles are correctly handled. On the other hand, for $\int_{b_n}^{\infty}[\cdots]$ has no such problem, (5.12) may never be zero.

The third case is separate: again there are no singularities to worry about. q_n^2 is always greater than 0 by (5.14).

5.2.1 Case I: $\hat{\omega} \leq n$ and $\hat{p}_\perp \leq b_n$

$$\int_0^{b_n} \hat{p}_\perp (J_n(\hat{k}_\perp \hat{p}_\perp))^2 e^{-\frac{a\hat{p}_\perp^2}{2}} d\hat{p}_\perp \int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\left(\gamma^2 - \frac{n^2}{\hat{\omega}^2}\right)} d\hat{p}_{||} \tag{5.15}$$

Of primary interest here is the behaviour around the zeros of the denominator (5.12). Make a change of variable: $\beta^2 = {b_n}^2 - \hat{p}_\perp^2$ hence $\hat{p}_\perp d\hat{p}_\perp = -\beta d\beta$

The whole double integral is:

$$\int_0^{b_n} \beta \left(J_n(\hat{k}_\perp \left(b_n^2 - \beta^2\right)^{\frac{1}{2}})\right)^2 e^{-\frac{a\left(b_n^2 - \beta^2\right)}{2}} d\beta \int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2 - \beta^2} d\hat{p}_{||} \tag{5.16}$$

Now the $\hat{p}_{||}$ integral may be written

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2 - \beta^2} d\hat{p}_{||} \tag{5.17}$$

As stated previously there is a discontinuity for $\hat{p}_{||}^2 = \beta^2$ and contour integration is necessary.

The integral has two simple poles. Take a closed integration path as follows: a semicircular path in the upper half of the complex plane; return along the real axis making a small semicircular indentation into the UHP above each pole (either $\pm\beta$). The residues at the poles are:

$$\text{Res}_{p=\beta}\left(\frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2 - \beta^2}\right) = \frac{e^{-\frac{a\beta^2}{2}}}{2\beta}$$

$$\text{Res}_{p=-\beta}\left(\frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2 - \beta^2}\right) = \frac{e^{-\frac{a\beta^2}{2}}}{-2\beta}$$

Contour integration is described in more detail in Appendix B. Using the results described in that appendix for poles lying on the contour of integration means that the

corresponding closed path integral may be calculated

$$\begin{aligned}\oint_{-\infty}^{\infty}\frac{e^{-\frac{a\hat{p}^2}{2}}}{\hat{p}^2-\beta^2}d\hat{p} &= P\int_{-\infty}^{\infty}\frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2-\beta^2}d\hat{p}_{||} - i\pi\left\{\frac{e^{-\frac{a\beta^2}{2}}}{2\beta}+\frac{e^{-\frac{a\beta^2}{2}}}{-2\beta}\right\}\\ &= P\int_{-\infty}^{\infty}\frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2-\beta^2}d\hat{p}_{||} \qquad (5.18)\end{aligned}$$

The integral which corresponds to the integral in (5.17) is the principal part integral as defined in the appendix.

We intend to 'cancel out' the denominator of (5.17). To this end propose a new substitution, $\hat{p}_{||} = \beta u$. This leads to

$$\lim_{L\to\infty} P\int_{-L}^{L}\frac{e^{-\frac{a\beta^2u^2}{2}}}{\beta^2u^2-\beta^2}\beta du = \frac{1}{\beta}\lim_{L\to\infty} P\int_{-L}^{L}\frac{e^{-\frac{a\beta^2u^2}{2}}}{u^2-1}du = \frac{1}{\beta}A\left(\frac{a\beta^2}{2}\right) \qquad (5.19)$$

If $x = a\beta^2/2$, this manipulation defines the function $A(x)$, *i.e.*:

$$A(x) = \lim_{L\to\infty} P\int_{-L}^{L}\frac{e^{-xu^2}}{u^2-1}du \qquad (5.20)$$

The double integral is now written

$$\int_0^{b_n}\left(J_n(\hat{k}_\perp\left(b_n^2-\beta^2\right)^{\frac{1}{2}})\right)^2 e^{-\frac{a\left(b_n^2-\beta^2\right)}{2}}d\beta A\left(\frac{a\beta^2}{2}\right) \qquad (5.21)$$

Taking the derivative of $A(x)$ with respect to x gives

$$\frac{\partial A(x)}{\partial x} = \lim_{L\to\infty} P\int_{-L}^{L}-u^2\frac{e^{-xu^2}}{u^2-1}du \qquad (5.22)$$

so that

$$\begin{aligned}A+\frac{\partial A(x)}{\partial x} &= \lim_{L\to\infty} P\int_{-L}^{L}\left(1-u^2\right)\frac{e^{-xu^2}}{u^2-1}du\\ &= -\lim_{L\to\infty} P\int_{-L}^{L}e^{-xu^2}du = -\sqrt{\frac{\pi}{x}} \qquad (5.23)\end{aligned}$$

Now show that $A(0) = 0$.

$$A(0) = \lim_{L\to\infty} P\int_{-L}^{L}\frac{1}{u^2-1}du \qquad (5.24)$$

Split the denominator by partial fractions

$$\frac{1}{u^2-1} = \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right] \tag{5.25}$$

Thus

$$A(0) = \frac{1}{2}\left[\lim_{L\to\infty} P\int_{-L}^{L}\frac{1}{u-1}du - \lim_{L\to\infty} P\int_{-L}^{L}\frac{1}{u+1}du\right] \tag{5.26}$$

Now the second term in the expression on the right hand side is identical to the first under a sign change in the dummy variable. That is to say

$$-\lim_{L\to\infty} P\int_{u=-L}^{u=L}\frac{1}{u+1}du = -\lim_{L\to\infty} P\int_{u'=L}^{u'=-L}\frac{1}{-u'+1}(-du') = \lim_{L\to\infty} P\int_{u'=-L}^{u'=L}\frac{1}{u'-1}du' \tag{5.27}$$

Leaving just

$$A(0) = \lim_{L\to\infty} P\int_{-L}^{L}\frac{1}{u-1}du = 0 \tag{5.28}$$

This is true because we can make a substitution $\kappa = u - 1$ giving an *odd* integrand.

To solve (5.23) we propose an integrating factor. Try e^x.

$$\frac{d}{dx}(e^x A) = -e^x\sqrt{\frac{\pi}{x}} \tag{5.29}$$

thus

$$\begin{aligned} A &= e^{-x}\int_0^x (-\sqrt{\pi})\frac{e^{x'}}{\sqrt{x'}}dx' + C \\ &= -\sqrt{\pi}e^{-x}\left\{\int_0^x \frac{e^{x'}}{\sqrt{x'}}dx'\right\} \end{aligned}$$

Finally, substitute $x' = w^2$, and this becomes

$$\begin{aligned} A &= -\sqrt{\pi}e^{-x}\int_0^{\sqrt{x}}\frac{e^{w^2}}{w}(2wdw) \\ &= -2\sqrt{\pi}e^{-x}\int_0^{\sqrt{x}} e^{w^2}dw \end{aligned}$$

The full expression for the double integral is:

$$-2\sqrt{\pi}\int_0^{b_n}\left(J_n(\hat{k}_\perp\left(b_n^2-\beta^2\right)^{\frac{1}{2}})\right)^2 e^{-\frac{a\left(b_n^2-\beta^2\right)}{2}} D\left(\sqrt{\frac{a\beta^2}{2}}\right)d\beta \tag{5.30}$$

For computational purposes note that the integral function $D(\eta)$ is the Dawson integral. The definition is given along with an illustration in Appendix A

5.2.2 Case II: $\hat{\omega} \leq n$ and $\hat{p}_\perp > b_n$

$$\int_{b_n}^{\infty} \hat{p}_\perp (J_n(\hat{k}_\perp \hat{p}_\perp))^2 e^{-\frac{a\hat{p}_\perp^2}{2}} d\hat{p}_\perp \int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\left(\gamma^2 - \frac{n^2}{\hat{\omega}^2}\right)} d\hat{p}_{||} \tag{5.31}$$

Here as before the first stage is to change the variable $\hat{p}_\perp^2 = b_n^2 + \delta^2$ so that $\hat{p}_\perp d\hat{p}_\perp = \delta d\delta$

The $\hat{p}_{||}$ integral is now

$$P\int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2 + \delta^2} d\hat{p}_{||} = \lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-\frac{a\hat{p}_{||}^2}{2}}}{\hat{p}_{||}^2 + \delta^2} d\hat{p}_{||} \tag{5.32}$$

To simplify this another change of variables is called for: $\hat{p}_{||} = \delta u$ thus (5.32) becomes

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-\frac{a\delta^2 u^2}{2}}}{\delta^2 u^2 + \delta^2} d\hat{p}_{||} = \lim_{L\to\infty} \frac{1}{\delta} \int_{-L}^{L} \frac{e^{-\left(\frac{a\delta^2}{2}\right)u^2}}{u^2+1} du \tag{5.33}$$

Using these variable changes together means that the double integral is re-expressed as:

$$\lim_{L\to\infty} \int_0^{\infty} \delta (J_n(\hat{k}_\perp \left(b_n^2 + \delta^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(b_n^2+\delta^2\right)}{2}} d\delta \; \frac{1}{\delta} \int_{-L}^{L} \frac{e^{-\left(\frac{a\delta^2}{2}\right)u^2}}{u^2+1} du \tag{5.34}$$

which simplifies to:

$$\int_0^{\infty} (J_n(\hat{k}_\perp \left(b_n^2 + \delta^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(b_n^2+\delta^2\right)}{2}} d\delta \; B\left(\frac{a\delta^2}{2}\right) \tag{5.35}$$

where

$$B\left(\frac{a\delta^2}{2}\right) = \lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-\left(\frac{a\delta^2}{2}\right)u^2}}{u^2+1} du \tag{5.36}$$

Now let $y = \frac{a\delta^2}{2}$ and examine $B(y)$ on its own. What is $B(0)$?

$$\begin{aligned} B(0) &= \lim_{L\to\infty} \int_{-L}^{L} \frac{1}{u^2+1} du \\ &= 2 \lim_{L\to\infty} \int_0^{L} \frac{1}{u^2+1} du \\ &= 2 \lim_{L\to\infty} \left[\tan^{-1} L\right] \\ B(0) &= \pi \end{aligned}$$

Now take the derivative of B with respect to y:

$$\frac{dB(y)}{dy} = \lim_{L\to\infty} \int_{-L}^{L} (-u^2) \frac{e^{-yu^2}}{u^2+1} du \tag{5.37}$$

It follows that:

$$\frac{dB(y)}{dy} - B(y) = \lim_{L\to\infty} \int_{-L}^{L} (-u^2 - 1) \frac{e^{-yu^2}}{u^2+1} du = -\lim_{L\to\infty} \int_{-L}^{L} e^{-yu^2} du = -\sqrt{\frac{\pi}{y}} \tag{5.38}$$

Thus the problem is reduced to the differential equation:

$$\frac{dB}{dy} - B = -\sqrt{\frac{\pi}{y}} \tag{5.39}$$

As before the standard approach is to propose an integrating factor (here e^{-y}) that way:

$$\frac{d}{dy}\left(e^{-y}B\right) = -e^{-y}\sqrt{\frac{\pi}{y}} \tag{5.40}$$

thus

$$\begin{aligned} B &= e^{y} \int_{0}^{y} (-\sqrt{\pi}) \frac{e^{-y'}}{\sqrt{y'}} dy' + C'e^{y} \\ &= \pi e^{y} \left\{ -\frac{1}{\sqrt{\pi}} \int_{0}^{y} \frac{e^{-y'}}{\sqrt{y'}} dy' + 1 \right\} \end{aligned}$$

Make one last substitution, $y' = w^2$, and this becomes

$$\begin{aligned} B &= \pi e^{y} \left\{ 1 - \frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{y}} \frac{e^{-w^2}}{w} (2wdw) \right\} \\ &= \pi e^{y} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{y}} e^{-w^2} dw \right\} \\ &= \pi e^{y} \left(1 - \operatorname{erf}\left(\sqrt{y}\right)\right) \end{aligned}$$

Note that in the final stage of this algebra, the error function, erf, appears. For more information about this function see Section A.4.1 in the appendix.

Write out the full expression

$$\int_{0}^{\infty} (J_n(\hat{k}_\perp \left(b_n^2 + \delta^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(b_n^2+\delta^2\right)}{2}} d\delta \; \pi e^{\frac{a\delta^2}{2}} \left(1 - \operatorname{erf}\left(\sqrt{\frac{a\delta^2}{2}}\right)\right) \tag{5.41}$$

So that the final expression for the double integral is then

$$\pi e^{\frac{-ab_n^2}{2}} \int_{0}^{\infty} d\delta (J_n(\hat{k}_\perp \left(b_n^2 + \delta^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(b_n^2+\delta^2\right)}{2}} \left\{ 1 - \operatorname{erf}\left(\sqrt{\frac{a\delta^2}{2}}\right) \right\} \tag{5.42}$$

5.2.3 Case III: $\hat{\omega} > n$

The solution of this case is close to that of Case II. In this regime the quantity q_n defined at (5.14) is appropriate. The double integral is

$$\int_0^\infty \hat{p}_\perp (J_n(\hat{k}_\perp \hat{p}_\perp))^2 e^{-\frac{a\hat{p}_\perp^2}{2}} d\hat{p}_\perp \int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_\parallel^2}{2}}}{\left(\gamma^2 - \frac{n^2}{\hat{\omega}^2}\right)} d\hat{p}_\parallel \tag{5.43}$$

Here as before the first stage is to change the variable $\hat{p}_\perp^2 = \epsilon^2 - q_n^2$ hence $\hat{p}_\perp d\hat{p}_\perp = \epsilon d\epsilon$

So that the $\hat{p}_\parallel$ integral is now

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{a\hat{p}_\parallel^2}{2}}}{\hat{p}_\parallel^2 + \epsilon^2} d\hat{p}_\parallel = \lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-\frac{a\hat{p}_\parallel^2}{2}}}{\hat{p}_\parallel^2 + \epsilon^2} d\hat{p}_\parallel \tag{5.44}$$

To simplify this another change of variables is called for: $\hat{p}_\parallel = \epsilon u$ thus (5.44) becomes

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-\frac{a\epsilon^2 u^2}{2}}}{\epsilon^2 u^2 + \epsilon^2} d\hat{p}_\parallel = \lim_{L\to\infty} \frac{1}{\epsilon} \int_{-L}^{L} \frac{e^{-\left(\frac{a\epsilon^2}{2}\right)u^2}}{u^2 + 1} du \tag{5.45}$$

Using these variable changes together means that the double integral is re-expressed as:

$$\lim_{L\to\infty} \int_0^\infty \epsilon (J_n(\hat{k}_\perp \left(\epsilon^2 - q_n^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(\epsilon^2 - q_n^2\right)}{2}} d\epsilon \, \frac{1}{\epsilon} \int_{-L}^{L} \frac{e^{-\left(\frac{a\epsilon^2}{2}\right)u^2}}{u^2 + 1} du \tag{5.46}$$

which simplifies to:

$$\int_0^\infty (J_n(\hat{k}_\perp \left(q_n^2 + \epsilon^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(q_n^2 + \epsilon^2\right)}{2}} d\epsilon \, C\left(\frac{a\epsilon^2}{2}\right) \tag{5.47}$$

where

$$C\left(\frac{a\epsilon^2}{2}\right) = \lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-\left(\frac{a\epsilon^2}{2}\right)u^2}}{u^2 + 1} du \tag{5.48}$$

This C is identical to $B(y)$ at (5.36).

$$C(z) = \pi e^z \left(1 - \operatorname{erf}\left(\sqrt{z}\right)\right) \tag{5.49}$$

Write out the full expression

$$\int_{q_n}^\infty (J_n(\hat{k}_\perp \left(\epsilon^2 - q_n^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(\epsilon^2 - q_n^2\right)}{2}} d\epsilon \quad \pi e^{\frac{a\epsilon^2}{2}} \left(1 - \operatorname{erf}\left(\sqrt{\frac{a\epsilon^2}{2}}\right)\right) \tag{5.50}$$

So that the final expression for the double integral is then

$$\pi e^{\frac{aq_n^2}{2}} \int_{q_n}^{\infty} d\epsilon (J_n(\hat{k}_\perp (\epsilon^2 - q_n^2)^{\frac{1}{2}}))^2 \left\{ 1 - \text{erf}\left(\sqrt{\frac{a\epsilon^2}{2}} \right) \right\} \tag{5.51}$$

5.3 A Single Integral Form of $\hat{R}_{xx}$

Recall that the aim of the previous section was to reformulate the expression for $\hat{R}_{xx}$. Quadrature of a single integral with a special function is more efficient than quadrature of the original double integral in (5.10). The form of the dispersion relation used from this point on is:

$$\hat{R}_{xx} = \hat{\omega}^2 - 2\hat{\omega}_{pe}^2 - \frac{4\hat{\omega}_{pe}^2}{\hat{k}_\perp^2} \frac{a^{\frac{5}{2}}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{n^4}{\hat{\omega}^2} \{\text{Case I} + \text{Case II}\}_n \text{ or } \{\text{Case III}\}_n \tag{5.52}$$

where the cases refer to (5.30), (5.42), & (5.51) respectively. Cases I and II arose from either side of $\hat{p}_\perp = b_n$: as a result their contributions are summed. Which cases are involved is decided by the index value of the infinite sum, n, (*i.e.* whether it is greater than or less than the value of the frequency, $\hat{\omega}$).

Chapter 6

The Computational Strategy

With expressions for the dispersion relations for e^-e^+ plasmas in both the non-relativistic and weakly relativistic cases, the next step is to plot the dispersion relations for a range of parameters. Naturally this job falls to the computer. Any code then must perform two tasks: it must calculate the value of an expression for R_{xx} on a grid in $(\hat{\omega}, \hat{k}_\perp)$ space; and it must locate the points at which that surface intersects the $R_{xx} = 0$ plane. This code should take as input a set of basic parameters: $\hat{\omega}_{pe}$ and if necessary $a = mc^2/kT$. The output should be a series of $(\hat{\omega}, \hat{k}_\perp)$ pairs for which $R_{xx} = 0$.

Such a program was eventually written in a modular form and can be found listed in Appendix C. The 'front end' is the module `bmodes.c`. The main function in `bmodes.c` calls functions in either of two function files `non.c` or `weakly.c` depending on which routine is needed. In each of `non.c` and `weakly.c` there is a routine which calculates the value of R_{xx} for a $(\hat{\omega}, \hat{k}_\perp)$ pair.

While a regularly spaced grid of values of R_{xx} is generated for pairs of parameters, the sign at each point on the grid is monitored. More precisely, for each $\hat{k}_\perp$ column of that grid, any change of sign of R_{xx} between sequential values of $\hat{\omega}$ denotes a potential intersect point. Provided R_{xx} is continuous we can be certain that these values of $\hat{\omega}$ bracket `root`, which corresponds to the true intersection, $R_{xx} = 0$.

Zero-finding is then performed by the routine `zbrent()` in `zbrent.c`. This routine is called in `bmodes.c`. The algorithm used in this routine, Brent's algorithm, is a hybrid of

one-dimensional methods: bisection and inverse quadratic interpolation. If the function behaves smoothly an inverse quadratic approach is very effective at zero-finding. As described in Numerical Recipes [58], the algorithm switches to a bisection step if the function is not smooth.

In addition to zero-finding this code makes use of a number of other Numerical Recipes routines, namely those for Bessel functions (standard and modified – first kind), for the Dawson integral and for quadrature. The interaction of the various functions is summarized in 6.1:

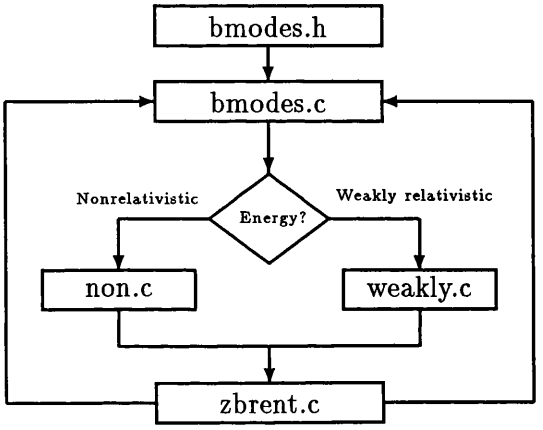


Figure 6.1: Modular structure

6.1 The Front End

The code in this file deals with the user options and ensures that the correct routine (non-relativistic or weakly relativistic) is adopted.

Using command line information the program then opens and names a data file; adds informative comments; loops through wavenumber and frequency; finds and records the $R_{xx} = 0$ contour in $(\hat{k}_\perp, \hat{\omega})$ space; and all the while, tells the user what it is doing.

Some basic errors are trapped: if the datafile is unclosed for whatever reason at the end of an execution a warning is given; or, if too few or too many arguments are given at the command line, the usage message is given.

On the very first pass through the $\hat{k}_\perp$ loop, comments are written to the data file. That way an immediate record is made of the parameters under which the subsequent data was produced.

Each first pass through the $\hat{\omega}$ loop skips all the commands in that loop due to the condition on `check`. On every other pass the values of two successive $R_{xx}(\hat{\omega})$ are compared. If they are of opposite sign the zero finding routine is started and a best value for $\hat{\omega}$, `root`, is found. Just in case this is a discontinuity (and not a zero) the value of R_{xx}(`root`) is squared and only if that number is still less than 1.0 is the ($\hat{k}_\perp$,`root`) pair written to the data file.

6.2 The Classical Routine

`non.c` contains a routine that numerically calculates R_{xx} for non-relativistic e^-e^+ plasmas.

This section of code simply calculates:

$$R_{xx} = 1 - 4\hat{\omega}_{pe}^2 \sum_{n=1}^{10} n^2 \frac{\exp(-\Lambda)}{\Lambda} \frac{I_n(\Lambda)}{\hat{\omega}^2 - n^2}$$

This is a version of (4.15) with the sum truncated at $n = 10$.

6.3 The Weakly Relativistic Routine

`weakly.c` contains a routine that numerically calculates R_{xx} for weakly relativistic e^-e^+ plasmas. In particular this routine calculates a value of R_{xx} for a given $\hat{\omega}$: the value of $\hat{k}_\perp$ is determined externally in the main routine.

The additional problem for the weakly relativistic routine is that of quadrature. The expression for R_{xx}, (5.52), contains three integrals which must themselves be performed. To solve the equations (5.30), (5.42), & (5.51), a practical scheme had to be devised.

This section of code calculates:

$$\hat{R}_{xx} = \hat{\omega}^2 - 2\hat{\omega}_{pe}^2 - \frac{4\hat{\omega}_{pe}^2}{\hat{k}_\perp^2} \frac{a^{\frac{5}{2}}}{\sqrt{2\pi}} \sum_{n=1}^{6} \frac{n^4}{\hat{\omega}^2} \{\text{Case I} + \text{Case II}\}_n \text{ or } \{\text{Case III}\}_n$$

This is simply a statement of (5.52) with the infinite sum truncated at $n = 6$. Case I refers to (5.30); Case II to (5.42); and Case III to (5.51).

6.4 Quadrature

The equations (5.30), (5.42) and (5.51) do not permit analytical solution in any obvious way. The next step is naturally to attempt a numerical approximation to the solution. The midpoint rule for integration was initially chosen because the limits of the integrals to be calculated were indefinite.

Quadrature is performed using the Midpoint routine (`midpnt()`) from Numerical Recipes [58].

6.5 Truncations

Our aim is to be able to plot the zeros of R_{xx} for frequencies as high as $10\Omega_{e0}$. We would rather not exclude *any* details of behaviour: neither do we wish to spend hours of computing time for details which are only marginally more accurate. A balance is struck between a swift program and an accurate one.

6.5.1 Truncating infinite sums

In both 6.2 and 6.3 the infinite sum must be truncated somewhere. The largest value for the index n is chosen as 10 in section 6.2. The factor $I_n/(\hat{\omega}^2 - n^2)$ is only significant when $(\hat{\omega}^2 - n^2) \approx 0$ so the choice of a maximum frequency of $10\Omega_{e0}$ also caps the indices which it will be necessary to sum over. Effects for larger index are only significant at frequencies greater than $10\Omega_{e0}$.

The upper limiting influence in section 6.3 is the processing time. Runs of the code have been made at different maximum indices: 4, 6, 7 and 8. At $n = 8$ the program took many hours to run (depending on the number of points on the grid). Comparing the results of code which used these different indices showed that very little detail was added in the jumps from 6 to 7 or 8 *in the frequency range below* $n = 6$: a great difference was apparent in the $n = 4$ case. A decision was made to cap the index for the weakly relativistic routine at 6 since runs took the order of half an hour to complete yet all significant detail was retained in the frequency range of interest.

6.5.2 Case I

The simplistic statement of the case I integral is:

$$-2\sqrt{\pi}\int_0^{b_n}\left(J_n(\hat{k}_\perp\left(b_n^2-\beta^2\right)^{\frac{1}{2}})\right)^2 e^{-\frac{a\left(b_n^2-\beta^2\right)}{2}}d\beta\, D\left(\sqrt{\frac{a\beta^2}{2}}\right) \tag{6.1}$$

Here $D(x)$ is the Dawson integral, (see Appendix A), defined as:

$$D(x) = e^{-x^2}\int_0^x e^{t^2}dt$$

There is a Numerical Recipes routine (in `dawson.c`) which swiftly calculates this integral. In the code this integral is treated as a special function.

The β integral, however, can only be evaluated numerically.

6.5.3 Case II

This case deals with an integrand which includes an erfc(x) function. This function is described in more detail in Appendix A. As illustrated there, for argument x greater than 2, this function may be approximated as 0. The infinite limit of the integral is misleading since the overwhelming contribution comes when argument x is less than 2.

Now the double integral (5.42) is:

$$\pi e^{\frac{-ab_n^2}{2}}\int_0^\infty d\delta(J_n(\hat{k}_\perp\left(b_n^2+\delta^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(b_n^2+\delta^2\right)}{2}}\left\{1-\text{erf}\left(\sqrt{\frac{a\delta^2}{2}}\right)\right\}$$

With virtually no loss of information this is now calculated as:

$$\pi e^{\frac{-ab_n^2}{2}}\int_0^{\text{cut}} d\delta(J_n(\hat{k}_\perp\left(b_n^2+\delta^2\right)^{\frac{1}{2}}))^2 e^{-\frac{a\left(b_n^2+\delta^2\right)}{2}}\left\{1-\text{erf}\left(\sqrt{\frac{a\delta^2}{2}}\right)\right\} \tag{6.2}$$

6.5.4 Case III

As in the previous case there is an erfc in the integrand. The difference lies in exactly where these cuts may be introduced.

Rather than take the upper limit of the integral as infinite one recalls that for arguments greater than 2, erfc(x) = 0. In this case the argument is $\sqrt{a/2}t$. erfc($\sqrt{a/2}t$) = 0 corresponds to a value of t greater than $\sqrt{8/a}$.

So in the code the integral which needs to be calculated is:

$$\pi e^{\frac{aq_n^2}{2}} \int_{q_n}^{\sqrt{8/a}} d\epsilon (J_n(\hat{k}_\perp (\epsilon^2 - q_n^2)^{\frac{1}{2}}))^2 \left\{ 1 - \mathrm{erf}\left(\sqrt{\frac{a\epsilon^2}{2}} \right) \right\} \tag{6.3}$$

6.5.5 Initial values for $\hat{\omega}$

In addition to the problems above there are computational limits. The initial value of the frequency, $\hat{\omega}$, is bounded below by the presence, in each case, of the exponential factor: $\mathcal{F} = e^{-ab_n^2/2}$. A quick calculation shows that if the computer can make no distinction between 10^{-40} and zero whenever $\mathcal{F} < 10^{-40}$ the resulting code will be returned a zero. To avoid this, the initial value of $\hat{\omega}$ is never set below 0.5.

The problem is most pronounced for index $n = 1$. $\exp(-ab_n^2/2) < 10^{-40}$ is equivalent to $ab_n^2/2 > 92$ and for $n = 1$ this is:

$$\begin{aligned} \frac{1}{\hat{\omega}^2} - 1 &> \frac{184}{a} \\ \hat{\omega}^2 &< \frac{a}{184 + a} \end{aligned}$$

This results in a less arbitrary lower limit for the initial value of $\hat{\omega}$. For $a = 10$, $\hat{\omega} \leq 0.23$ will effectively give $\mathcal{F} = 0$: the equivalent condition for $a = 100$ is $\hat{\omega} \leq 0.6$.

In fact to improve computing time conditions have been placed on all such exponential factors so that when their exponent passes below 14 the result is taken to be zero. This is reasonable since such small contributions will be swamped out by those contributions with exponents greater than 14.

Chapter 7

Results

Лбом стену не прошибёшь

You can't break through a wall with your forehead

Russian Proverb

This chapter ties in strongly with the preceding chapters. The diagrams shown here are chosen to match those produced for the nonrelativistic case in Section 4.2. These diagrams contain the output from the program described in the previous chapter.

7.1 Dispersion Curves

The notation remains as before, see (3.116): $\hat{\omega}$ is the frequency in units of rest electron cyclotron frequency, Ω_{e0}; $\hat{k}_\perp$ is the perpendicular wave number normalized so that $\hat{k}_\perp = ck_\perp/\Omega_{e0}$; and $a = mc^2/kT$. Notice that when the phase velocity is equal to the velocity of light in a vacuum (*i.e.* $\omega/k_\perp = c$) $\hat{\omega}/\hat{k}_\perp = 1$.

The first series of plots deals with the behaviour of the Bernstein modes for a fixed plasma frequency ($\hat{\omega}_{ep} = 3$). The inverse temperature is changed across the "weakly relativistic" regime between $a = 100$ and $a = 5$, figures 7.1 – 7.5.

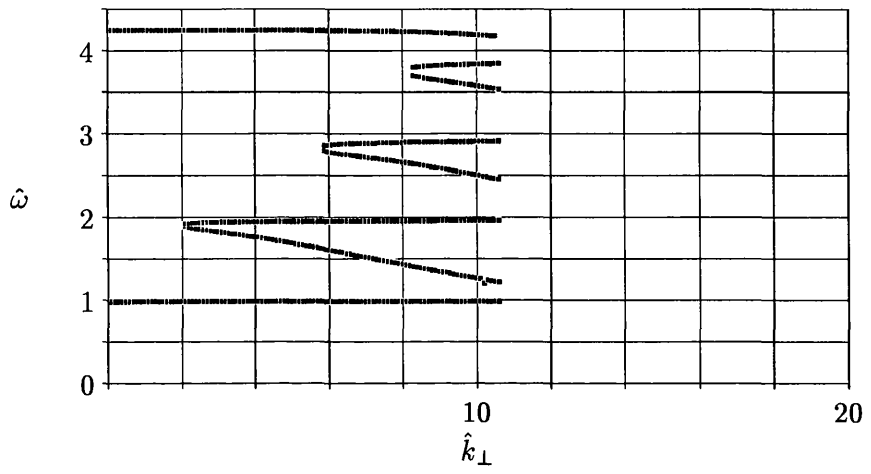


Figure 7.1: Bernstein modes for $a = 100$ and $\hat{\omega}_{ep} = 3$

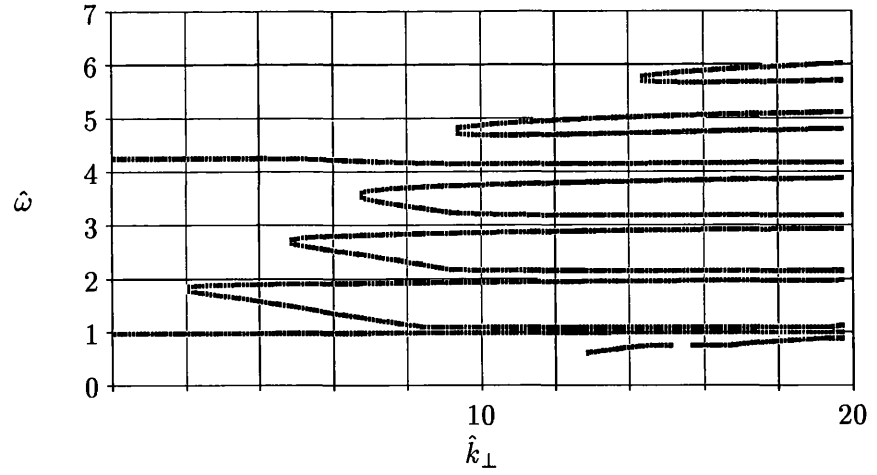


Figure 7.2: Bernstein modes for $a = 50$ and $\hat{\omega}_{ep} = 3$

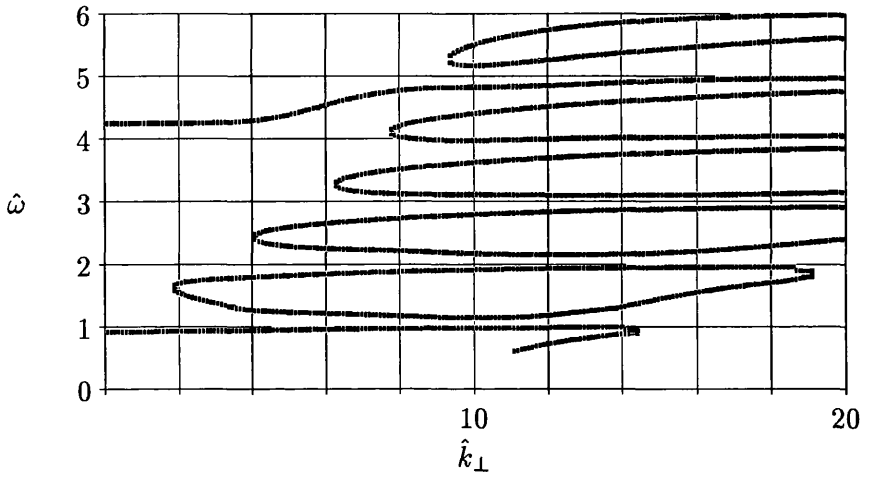


Figure 7.3: Bernstein modes for $a = 20$ and $\hat{\omega}_{ep} = 3$

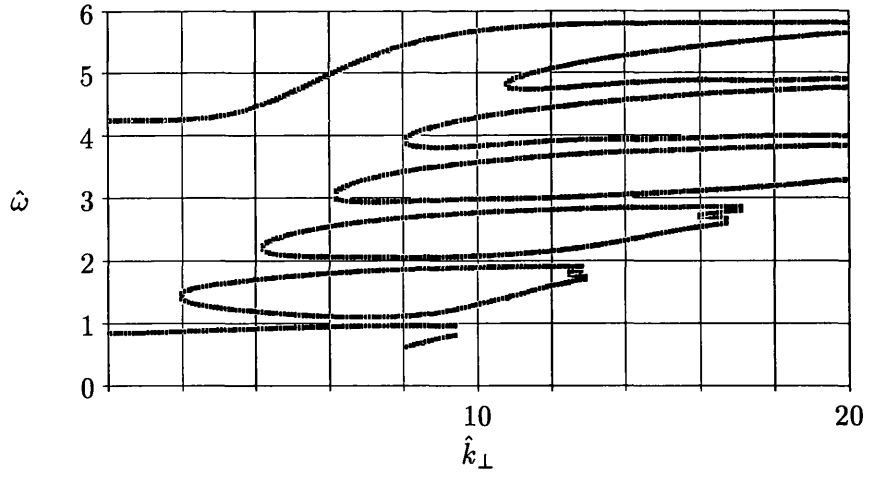


Figure 7.4: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 3$

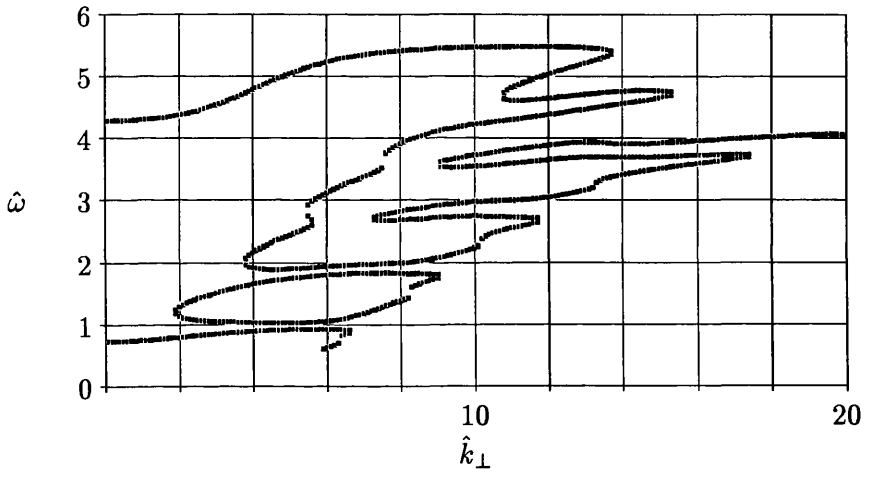


Figure 7.5: Bernstein modes for $a = 5$ and $\hat{\omega}_{ep} = 3$

It is also informative to observe the way plasma frequency affects behaviour. Holding $a = 10$ and varying the plasma frequency in the range $\hat{\omega}_{ep} = 1.5$ to 4.5 gives figures 7.6 – 7.12.

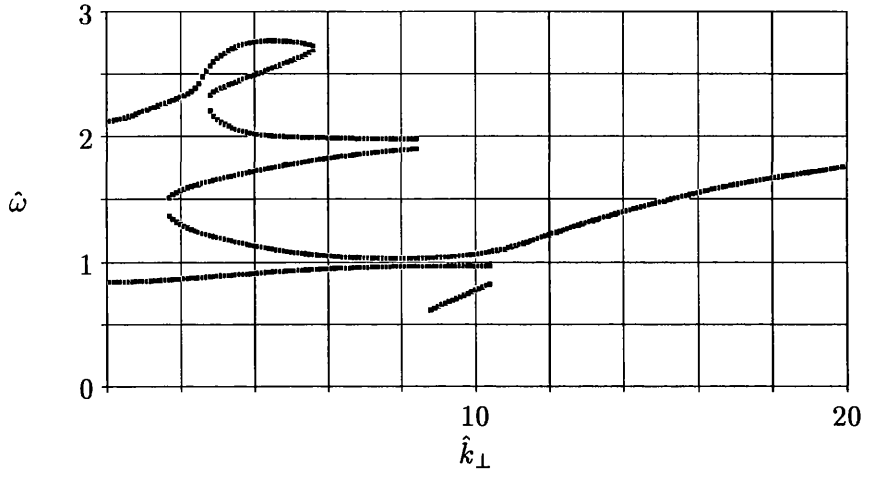


Figure 7.6: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 1.5$

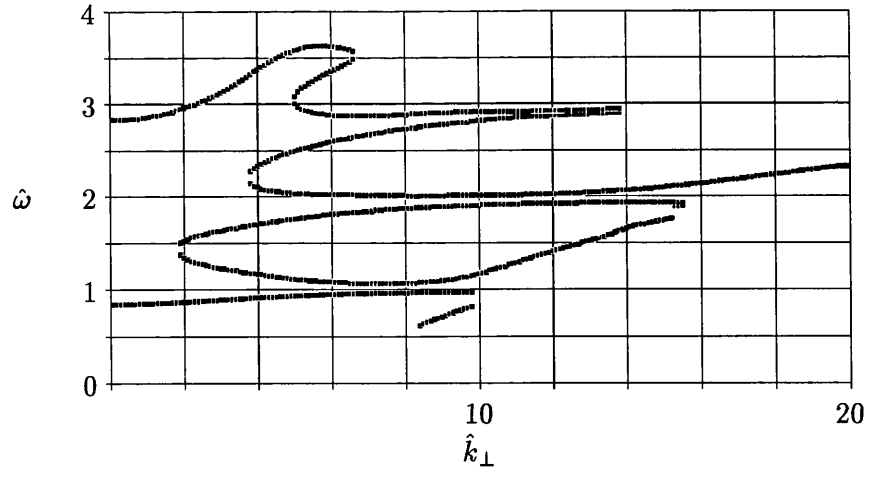


Figure 7.7: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 2$

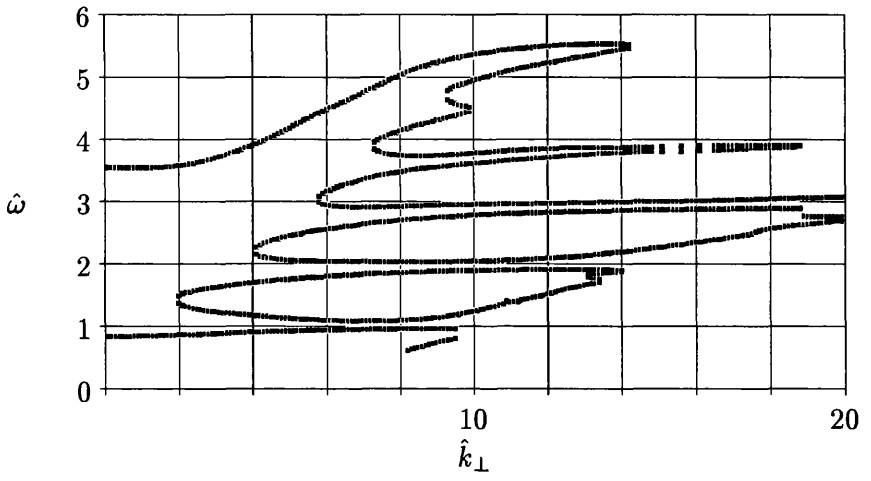


Figure 7.8: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 2.5$

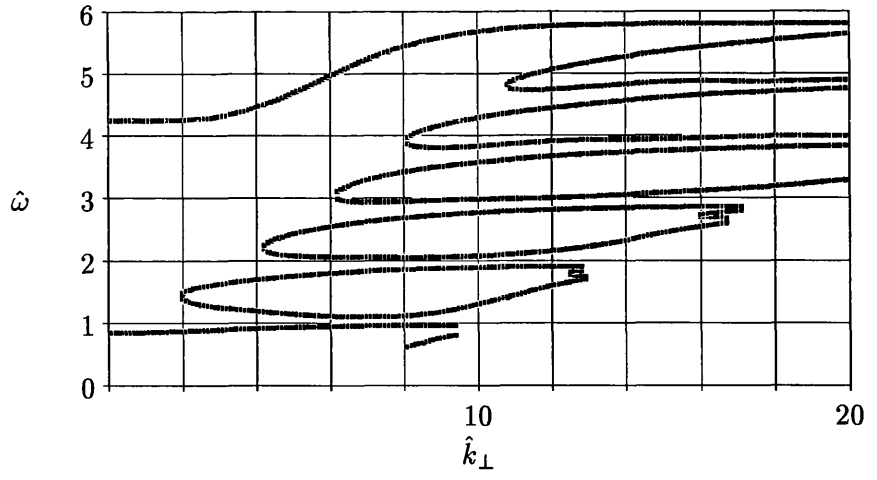


Figure 7.9: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 3$

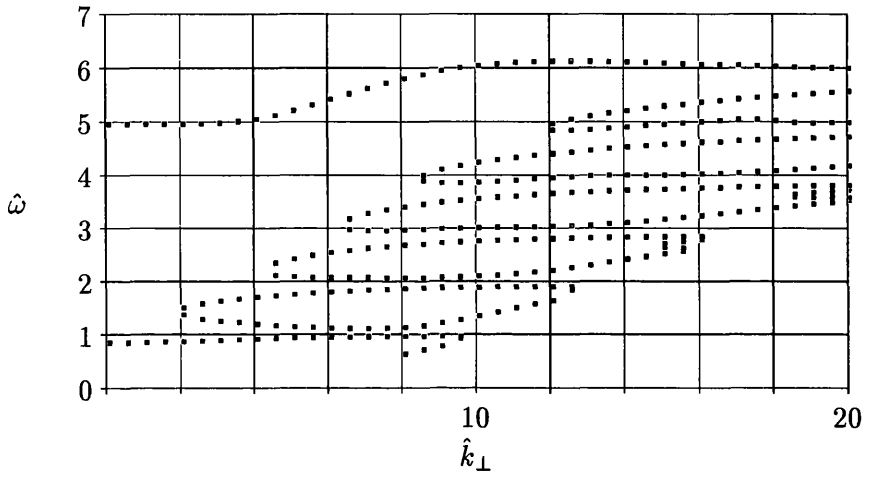


Figure 7.10: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 3.5$

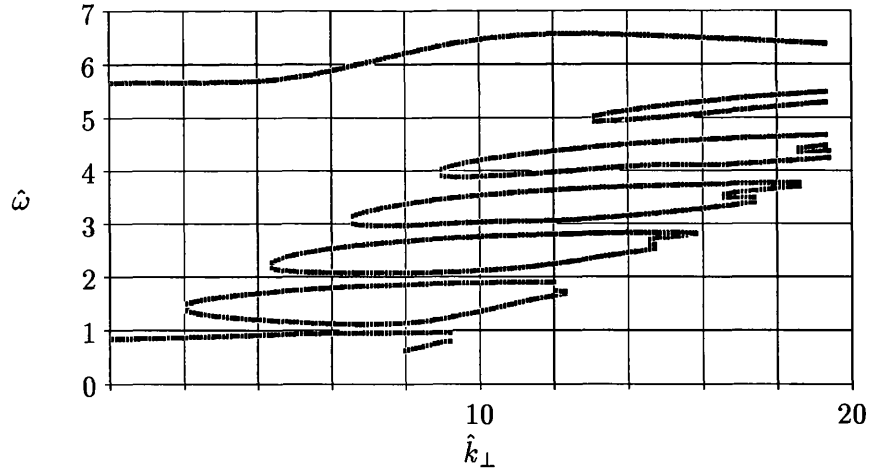


Figure 7.11: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 4$

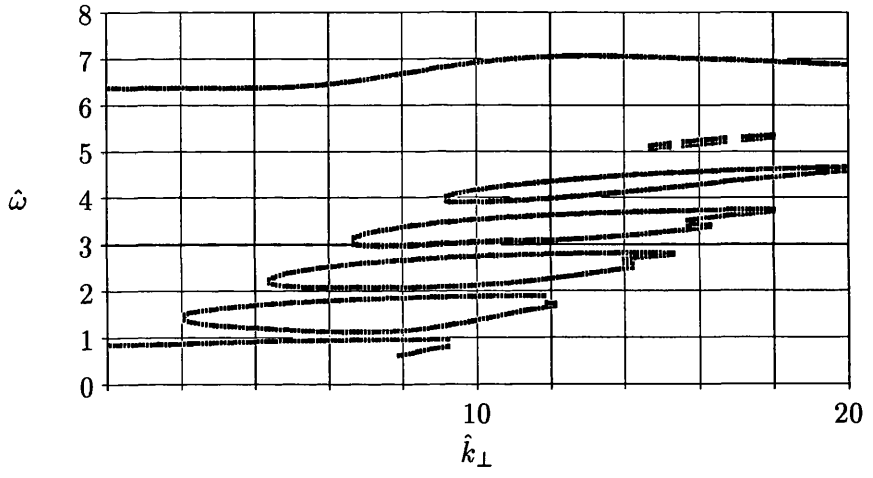


Figure 7.12: Bernstein modes for $a = 10$ and $\hat{\omega}_{ep} = 4.5$

7.2 Phenomenology

It will aid interpretation of the results to review the properties of the plasma. The initial conditions imposed go no further than a uniform **B** field, $\mathbf{B}_0$, and local neutrality.

The theory we have developed handles the propagation of waves perpendicular to the magnetic field and in the case of the Bernstein modes longitudinal with respect to perturbations in the electric field. The plasma is e^-e^+ and thus particles of both charges experience these fields at equal magnitudes and in opposite senses. In developing the theory needed we made the assumption that the plasma was locally *isotropic*; further that F_{e0} is the *non-relativistic* Maxwellian distribution. Finally we did retain the relativistic γ where it made a significant contribution to the dispersion relation, *i.e.* where it appeared in the cyclotron frequency.

The main reason a kinetic approach was needed was the short timescale. The cyclotron timescale in this plasma is *not* small in comparison to the timescales of dynamical effects, such as the propagation of Bernstein modes. On the other hand the annihilation-creation timescale is considerably shorter: typically by at least three orders of magnitude.

As for any plasma the characteristic interparticle spacing $n^{-1/3}$ is much less than the Debye length, λ_D.

7.2.1 Features of Weakly Relativistic Dispersion Relations

Figures 7.1 - 7.12 share a few common features. At regular intervals of frequency a small gap appears between the lower branch of one pair of modes and the upper branch of the pair below it. These gaps are, it appears, associated with integer multiples of the relativistic cyclotron frequency. This might be predicted by comparison with the appearance of frequency gaps in the non-relativistic theory (see 4.2). The difference between weakly relativistic and non-relativistic gaps lies in the shape of the $R_{xx} = 0$ curves. The form of R_{xx} differs in the two regimes. The non-relativistic R_{xx} has an intersection associated with a discontinuity at each cyclotron harmonic and thus $R_{xx} = 0$ has only one branch between harmonics. In contrast the relativistic R_{xx} has a further intersection in the place of a discontinuity and for each harmonic, $R_{xx} = 0$ has a two branches, one on either side

of the frequency gap.

Another change in the form of the dispersion relation occurs as soon as any degree of relativity is considered. For any $(a, \hat{\omega}_{ep})$ pair there are just two superluminal modes (*i.e.* two modes for which the phase velocity is greater than the speed of light). When compared to the classical dispersion curves, the absence of most modes in the superluminal region and the absence of modes above the hybrid frequency mode are the most obvious new features.

There is, however, no damping of those modes which do remain: a result of the exact equal mass nature of the plasma. A classical Bernstein mode is undamped; due to mass effects relativistic Bernstein modes in e^--ion plasmas *are* damped; however in e^-e^+ plasmas even the mass effects are balanced and these modes are undamped.

7.3 Diagnostic Applications

As has been described earlier, there are features of the dispersion curves which will act as diagnostics for certain parameters of astrophysical objects (a and n_{e0}), which could not otherwise be measured directly. The gaps between pairs of modes change according to the inverse temperature, a; as do the gaps between consecutive lower branches. The mapping of these features, which are illustrated in Chapter 7, is a numerically intensive task. It would be convenient if some method could be developed to approximate one or two principal features of these diagrams. This would give us yet another way of recognizing distant equal mass plasmas. For plasmas which can be identified as equal mass by some unrelated observation (the presence of annihilation radiation for instance), the diagnostic formulae should give an estimate of the electron density, n_{e0} and so on.

One way to progress is to approach branches a pair at a time and approximate the dispersion curve of each pair. This is possible if one uses the full expression (5.10) on page 63.

7.3.1 Frequency range $0 < \hat{\omega} < 1$

Our first step is to consider the behaviour for $\hat{\omega} < 1$. While this condition holds the third case integral is never encountered (see the conditions laid out in the weakly relativistic chapter). The other two integrals are both called upon. The sum over the index n starts at $n = 1$.

Equation (5.30) is the final version of the expression for case I.

$$-2\sqrt{\pi}\int_0^{b_n}\left(J_n(\hat{k}_\perp\left(b_n^2-\beta^2\right)^{\frac{1}{2}})\right)^2 e^{-\frac{a\left(b_n^2-\beta^2\right)}{2}} D\left(\sqrt{\frac{a\beta^2}{2}}\right) d\beta \tag{7.1}$$

The Dawson integral notation, so helpful in the code, is less useful here. This integral may be rewritten as a relative of the error function. To show this requires a detour:

$$D(\eta) = e^{-\eta^2}\int_0^{\eta} e^{\tau^2} d\tau \tag{7.2}$$

and

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^{y} e^{-w^2} dw \tag{7.3}$$

From the latter it can be seen that:

$$\mathrm{erf}(-i\eta) = \frac{2}{\sqrt{\pi}}\int_0^{-i\eta} e^{-w^2} dw \tag{7.4}$$

and then substituting $\tau = iw$:

$$\begin{aligned}\mathrm{erf}(-i\eta) &= \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{\tau^2}(-id\tau) \\ &= -\frac{2i}{\sqrt{\pi}}\int_0^{\eta} e^{\tau^2} d\tau\end{aligned} \tag{7.5}$$

Thus $D(\eta)$ may be rewritten in terms of the error function:

$$D(\eta) = e^{-\eta^2}\left[i\frac{\sqrt{\pi}}{2}\right]\mathrm{erf}(-i\eta) \tag{7.6}$$

Thus the new form of the case I integral is:

$$-2\sqrt{\pi}\left[i\frac{\sqrt{\pi}}{2}\right]\int_0^{b_n}\left(J_n(\hat{k}_\perp\left(b_n^2-\beta^2\right)^{\frac{1}{2}})\right)^2 e^{-\frac{a\left(b_n^2-\beta^2\right)}{2}} e^{-\frac{a\beta^2}{2}}\,\mathrm{erf}(-i\sqrt{\frac{a}{2}}\beta)\, d\beta \tag{7.7}$$

$$-\pi i\, e^{-\frac{ab_n^2}{2}} \int_0^{b_n} \left(J_n(\hat{k}_\perp \left(b_n^2 - \beta^2\right)^{\frac{1}{2}})\right)^2 \operatorname{erf}(-i\sqrt{\frac{a}{2}}\beta)\, d\beta \tag{7.8}$$

Since $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ we finally have:

$$\pi i\, e^{-\frac{ab_n^2}{2}} \int_0^{b_n} \left(J_n(\hat{k}_\perp \left(b_n^2 - \beta^2\right)^{\frac{1}{2}})\right)^2 \operatorname{erf}(i\rho\beta)\, d\beta \tag{7.9}$$

where ρ is defined by $\rho^2 = \frac{a}{2}$.

7.3.2 Expanding the Bessel function

The series expansion of the square of a Bessel function $J_n(\xi)$ is

$$J_n^2(\xi) = \left(\frac{\xi}{2}\right)^{2n} \left(\frac{1}{n!}\right)^2 \left\{1 - \frac{\xi^2}{2(n+1)} + \frac{\xi^4}{16(n+2)(n+1)} + \frac{\xi^4}{16(n+1)^2} + \mathcal{O}(\xi^6)\right\} \tag{7.10}$$

For small argument, ξ, one can neglect terms in higher orders of ξ. The lowest order in ξ also depends on the index n. In the situation of an infinite sum of 'squares of Bessel functions', the requirement that terms are of low order in the argument will leave only a very few terms. Expansions for Bessel functions with the smallest indices naturally have terms of lowest order in the argument. With $n = 1$ the lowest order in ξ is ξ^2; with $n = 2$ the lowest order in ξ is ξ^4 and so on.

In the problem facing us, the argument is $\hat{k}_\perp \left(b_n^2 - \beta^2\right)^{\frac{1}{2}}$. To order $\hat{k}_\perp^2$, only the $n = 1$ expansion needs to be made. J_1^2 is:

$$\frac{\hat{k}_\perp^2}{4}\left(b_1^2 - \beta^2\right) + \mathcal{O}(\hat{k}_\perp^4) \tag{7.11}$$

giving an altogether more tractable expression for the integral:

$$\pi i\, e^{-\frac{ab_1^2}{2}} \frac{\hat{k}_\perp^2}{4} \int_0^{b_1} \left(b_1^2 - \beta^2\right) \operatorname{erf}(i\rho\beta)\, d\beta \tag{7.12}$$

A useful set of identities can be derived from one core identity:

$$\int_0^b \operatorname{erf}(\mu\, t) dt = \operatorname{erf}(\mu\, b) b + \frac{e^{-\mu^2 b^2}}{\sqrt{\pi}\mu} - \frac{1}{\sqrt{\pi}\mu}$$

by integrating by parts then applying the chain rule the integral of any even power of t times erf can be calculated. Here we need only go as far as t^2:

$$\int_0^b t^2 \mathrm{erf}(\mu\, t) dt = \frac{1}{3}\mathrm{erf}(\mu\, b) b^3 + e^{-\mu^2 b^2} \frac{\mu^2 b^2 + 1}{3\sqrt{\pi}\mu^3} - \frac{1}{3\sqrt{\pi}\mu^3}$$

thus the expression for the whole integral is

$$\pi i\, e^{-\frac{ab_1^2}{2}} \frac{\hat{k}_\perp^2}{4} \left\{ \frac{2}{3} b_1^3 \mathrm{erf}(i\rho b_1) + e^{\rho^2 b_1^2} \left(\frac{2b_1^2}{3\sqrt{\pi} i\rho} \right) - e^{\rho^2 b_1^2} \left(\frac{1}{3\sqrt{\pi} i^3 \rho^3} \right) - \frac{b_1^2}{\sqrt{\pi} i\rho} + \frac{1}{3\sqrt{\pi} i^3 \rho^3} \right\} \tag{7.13}$$

Leaving this for the moment, we must now include the contribution from the case II integral. This has a different initial form (5.42):

$$\pi e^{\frac{-ab_n^2}{2}} \int_0^\infty \left(J_n(\hat{k}_\perp \left(b_n^2 + \delta^2\right)^{\frac{1}{2}}) \right)^2 \mathrm{erfc}(\rho\delta)\, d\delta \tag{7.14}$$

On this occasion, J_1^2 is approximated by:

$$\frac{\hat{k}_\perp^2}{4} \left(b_1^2 + \delta^2\right) + \mathcal{O}(\hat{k}_\perp^4) \tag{7.15}$$

As before there are identities which simplify the procedure. They are:

$$\begin{aligned} \int_0^\infty \mathrm{erfc}(\mu\, t) dt &= \frac{1}{\sqrt{\pi}\mu} \\ \int_0^\infty t^2 \mathrm{erfc}(\mu\, t) dt &= \frac{1}{3\sqrt{\pi}\mu^3} \end{aligned}$$

The full case II expression for $n = 1$ is:

$$\pi e^{\frac{-ab_n^2}{2}} \frac{\hat{k}_\perp^2}{4} \left\{ \frac{1}{3\sqrt{\pi}\rho^3} + \frac{b_1^2}{\sqrt{\pi}\rho} \right\} \tag{7.16}$$

Note that there are terms in common between case I and case II. Indeed these terms cancel. This leaves one expression for the combined cases I and II for index $n = 1$

$$\pi i\, e^{-\frac{ab_1^2}{2}} \frac{\hat{k}_\perp^2}{4} \left\{ \frac{2}{3} b_1^3 \mathrm{erf}(i\rho b_1) \right\} + \sqrt{\pi} \frac{\hat{k}_\perp^2}{4} \left(\frac{2b_1^2}{3\rho} + \frac{1}{3\rho^3} \right) \tag{7.17}$$

which gives an approximation to the dispersion relation in the frequency range $0 < \hat{\omega} < 1$ when included in the $R_{xx} = 0$ equation (5.10). To make matters slightly easier one can

assume that for small values of b_1 the first term (the erf term) is negligible. The whole thing then looks like this:

$$\hat{\omega}^2 - 2\hat{\omega}_{pe}^2 - \frac{4\hat{\omega}_{pe}^2}{\hat{k}_\perp^2}\frac{a^{\frac{5}{2}}}{\sqrt{2\pi}}\frac{1}{\hat{\omega}^2}\left[\sqrt{\pi}\frac{\hat{k}_\perp^2}{4}\left(\frac{2\sqrt{2}}{3\sqrt{a}}\right)\left(b_1^2+\frac{1}{a}\right)\right] = 0$$

which simplifies to:

$$\hat{\omega}^2 - 2\hat{\omega}_{pe}^2 - \frac{\hat{\omega}_{pe}^2}{\hat{\omega}^2}\frac{2a^2}{3}\left(b_1^2+\frac{1}{a}\right) = 0 \tag{7.18}$$

Recall that $b_1^2 = 1/\hat{\omega}^2 - 1$ and multiply the resulting expression by $\hat{\omega}^6$ to give:

$$\hat{\omega}^6 - 2\hat{\omega}_{pe}^2\hat{\omega}^4 - \hat{\omega}_{pe}^2\frac{2a^2}{3}\left(1-\hat{\omega}^2+\frac{\hat{\omega}^2}{a}\right) = 0 \tag{7.19}$$

By judiciously substituting $\hat{\omega} = 1$ in the last expression (when $\hat{\omega}$ is raised to powers higher than 2)

$$1 - 2\hat{\omega}_{pe}^2 + \frac{2}{3}\hat{\omega}_{pe}^2 a^2\left[\hat{\omega}^2\left(1-\frac{1}{a}\right) - 1\right] = 0 \tag{7.20}$$

and requiring that a is in the weakly relativistic regime, thus $a \gg 1$, means that $1/a \to 0$.

$$1 - 2\hat{\omega}_{pe}^2 + \frac{2}{3}\hat{\omega}_{pe}^2 a^2\left[\hat{\omega}^2 - 1\right] = 0 \tag{7.21}$$

and thus

$$\left[\hat{\omega}^2 - 1\right] = \frac{2\hat{\omega}_{pe}^2 - 1}{\frac{2}{3}\hat{\omega}_{pe}^2 a^2} \tag{7.22}$$

Make the substitution $\hat{\omega} = 1 - \Delta\omega$ and truncate the binomial expansion of $\hat{\omega}^2$.

$$\Delta\omega \approx \frac{1 - 2\hat{\omega}_{pe}^2}{\frac{4}{3}\hat{\omega}_{pe}^2 a^2} \tag{7.23}$$

7.3.3 Frequency range $1 < \hat{\omega} < 2$

Now we must extend the thinking from the beginning of the section. Useful information about gaps on the dispersion diagrams can only be gleaned if the next frequency range is also treated. Here $1 < \hat{\omega} < 2$, and the first case I and II integral contributions have index $n = 2$, in addition the contribution of the case III integral for $n = 1$ must be taken into account.

However by demanding that the expression is second order (in $\hat{k}_\perp$), the same as for the branch in the previous frequency range (0,1), we reduce the expression to something very similar to the derivation in the previous section.

As before take cases I and II together. The expansion at the core of these integrals is that of $J_2(\ldots)^2$ (see equation (7.10)):

$$J_2(\ldots)^2 = \frac{\hat{k}_\perp^4}{64}(\ldots)^2 + \mathcal{O}(\hat{k}_\perp^6) \tag{7.24}$$

Immediately the expressions for cases I and II are seen to be at least fourth order in $\hat{k}_\perp$. By calling for expressions of second order both cases are implicitly neglected.

This leaves just the case III expression evaluated for index $n = 1$.

$$\pi\, e^{\frac{aq_1^2}{2}} \int_{q_1}^{\infty} \left(J_1(\hat{k}_\perp (\epsilon^2 - q_1^2)^{\frac{1}{2}})\right)^2 \operatorname{erfc}(\rho\epsilon) d\epsilon \tag{7.25}$$

where ρ is defined by $\rho^2 = \frac{a}{2}$.

Now

$$\left(J_1(\hat{k}_\perp (\epsilon^2 - q_1^2)^{\frac{1}{2}})\right)^2 = \frac{\hat{k}_\perp^2}{4}(\epsilon^2 - q_1^2) + \mathcal{O}(\hat{k}_\perp^4) \tag{7.26}$$

Following the same method as in the previous section, a familiar expression is found:

$$\frac{\sqrt{2\pi}}{\sqrt{a}} \left(\frac{\hat{k}_\perp^2}{6}\right) \left(-q_1^2 + \frac{1}{a}\right) \tag{7.27}$$

Since $-q_1^2 = b_1^2$, we need no further algebra to find the form of $R_{xx} = 0$: it is the same as equation (7.18).

7.4 Empirical Rules

By looking at the data set from which figures 7.1 – 7.12 derive one can pick out empirical rules. These rules in their turn can be used to elicit information about the plasma, given a certain set of features.

The frequency gap between branch pairs increases as the inverse temperature, a, decreases. This trend can be observed in figures 7.1 – 7.5. To assess the trend quantitatively,

the data from which the figures are plotted is treated in a computer application such as EXCEL. The difference at any particular wavenumber was plotted for each value of a.

The frequency gap between the upper branch of the pair below $\hat{\omega} = 1$ and the $\hat{\omega} = 1$ line (illustrated in figure 7.13) has the following rule:

$$\Delta\omega \propto a^{-0.7} \tag{7.28}$$

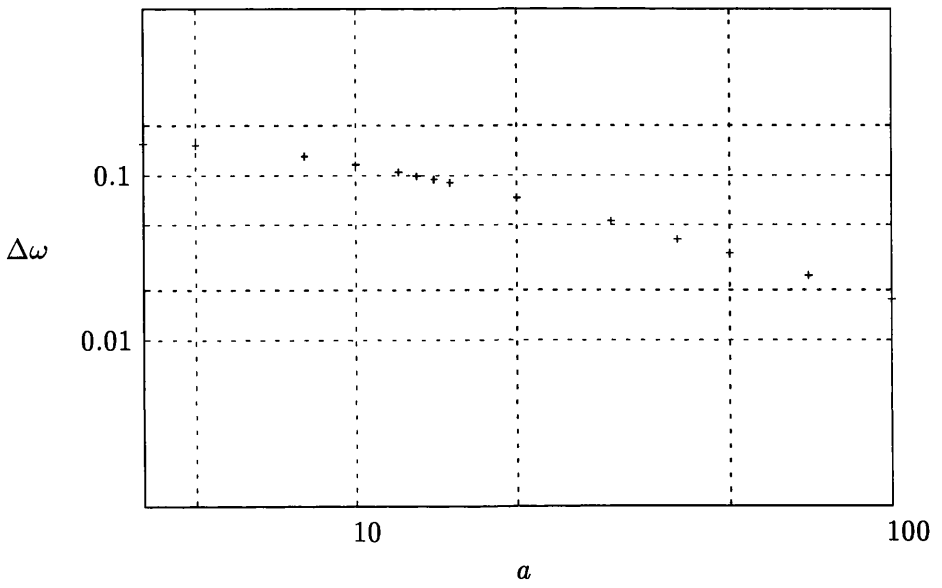


Figure 7.13: $\Delta\omega$ for a range of values of a ($\hat{\omega}_{ep} = 3$)

The gap between the $\hat{\omega} = 1$ line and the lower branch of the pair above it can be shown to follow a similar rule:

$$\Delta\omega \propto a^{-0.7} \tag{7.29}$$

These empirical rules restate the contents of the figures in Chapter 7: the previous section produced analytical rules which are substantially different. In the latter case, *i.e.* (7.23),

$$\Delta\omega \propto a^{-2} \tag{7.30}$$

Analytical solutions even to approximations of the problem need refinement. One way forward would be to introduce higher order terms in the approximate treatment. The additional error function terms which would then come into play may well be non-negligible.

7.5 The Next Step

The common goal of the previous sections was draw predictive rules from the raw mathematics. So in the case of Section 7.3 the equations governing the gaps between pairs of modes were derived from a series of approximations. Section 7.4 on the other hand is distilled from the same data that produced figures 7.1 – 7.12. Even at the first attempt both the empirical rule and the approximate analytical rule predict wider gaps between branch pairs as the inverse temperature decreases (though they disagree on the exact dependence).

Further steps could be taken to treat higher branches in the approximate treatment in Section 7.3. This would certainly improve the prediction of behaviour of the e^-e^+ Bernstein modes in general.

Such a predictive theory could be tested on spectra from astrophysical sources. These spectra may have harmonic features which are, at present, interpreted as cyclotron harmonic resonance peaks. A series of spectral peaks from a source which has a strong e^-e^+ component might be misinterpreted since this work suggests that an alternative conclusion should be drawn: that the peaks appear in pairs around the true (relativistically downshifted) cyclotron harmonics.

Now that the ground work for the theory of e^-e^+ plasmas in this weakly relativistic regime has been carried out there are many potential avenues of continuing research.

- The other two modes, extraordinary and ordinary, are touched upon but not developed in this work. The analytical work on these transverse modes is expected to be similar to that performed here; the code being closely related too.

- This treatment runs parallel to that for electron-ion plasmas and it is appropriate

to consider developments made in that field, such as the treatment of slightly off-perpendicular propagation.

- Relatively minor changes could be made to the present work to show its application to the study of other particle-antiparticle plasmas and of nearly equal mass plasmas.

- The research into wave propagation perpendicular to a uniform magnetic field in weakly and fully relativistic e^-e^+ plasmas leaves unanswered questions. So, for instance, as there appears to be a smooth transition from weakly relativistic to fully relativistic dispersion relations, one might wish to demonstrate this transition in a single analytical relativistic treatment. Can such a treatment be constructed?

The next step in e^-e^+ plasma research will certainly be an interesting one.

Appendix A

Special Functions

A.1 Basic Identities

The basic Bessel identities crop up repeatedly. A fuller description may be found in [59] and elsewhere. The identities used here are:

$$J_{-n}(a) = (-1)^n J_n(a) \tag{A.1}$$

and

$$J_n(-a) = (-1)^n J_n(a) \tag{A.2}$$

There is also an identity which covers the derivative of $J_n(-a)$ with respect to its argument:

$$J'_n(-a) = J'_n(a) \tag{A.3}$$

A.2 Infinite Sum Identities

In the kinetic treatment of hot plasmas infinite sums of Bessel functions crop up repeatedly. One identity is often invoked:

$$e^{\mp i a_s \sin\phi} = \sum_{n=-\infty}^{\infty} J_n(a_s) e^{\mp i n \phi} \tag{A.4}$$

This identity is first required on page 35. One may use it to derive some other results. Take the derivative with respect to a_s, $\frac{\partial}{\partial a_s}$

$$i \sin\phi e^{\mp i a_s \sin\phi} = \sum_{n=-\infty}^{\infty} J_n'(a_s) e^{\mp in\phi} \tag{A.5}$$

where $J_n'(a_s) = \frac{\partial J_n(a_s)}{\partial a_s}$ so that

$$\sum_{n=-\infty}^{\infty} \sin\phi J_n(a_s) e^{\mp in\phi} = \sum_{n=-\infty}^{\infty} (-i) J_n'(a_s) e^{\mp in\phi} \tag{A.6}$$

Also if we use the recurrence relation for Bessel functions

$$J_n'(a_s) = J_{n-1}(a_s) - \frac{n}{a_s} J_n(a_s) \tag{A.7}$$

we can gain a relation which deals with the $\cos\phi$:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{n}{a_s} J_n(a_s) e^{in\phi} &= \sum_{n=-\infty}^{\infty} \left[J_{n-1}(a_s) - J_n'(a_s) \right] e^{in\phi} \\
&= \sum_{n=-\infty}^{\infty} J_{n-1}(a_s) e^{in\phi} - \sum_{n=-\infty}^{\infty} J_n'(a_s) e^{in\phi} \\
&= \sum_{n'+1=-\infty}^{\infty} J_{n'}(a_s) e^{i(n'+1)\phi} - \sum_{n=-\infty}^{\infty} J_n'(a_s) e^{in\phi} \\
&= \sum_{n'=-\infty}^{\infty} e^{i\phi} J_{n'}(a_s) e^{in'\phi} - \sum_{n=-\infty}^{\infty} (i \sin\phi) J_n(a_s) e^{in\phi} \\
&= \sum_{n=-\infty}^{\infty} (e^{i\phi} - i \sin\phi) J_n(a_s) e^{in\phi}
\end{aligned} \tag{A.8}$$

Thus

$$\sum_{n=-\infty}^{\infty} \frac{n}{a_s} J_n(a_s) e^{in\phi} = \sum_{n=-\infty}^{\infty} (\cos\phi) J_n(a_s) e^{in\phi} \tag{A.9}$$

A.3 Integral Identities

In the later development of the kinetic theory, integrals of the squares of Bessel functions appear. When feasible these integrals are restated in the form:

$$\int_0^{\infty} dp\, p (J_n(kp))^2 e^{-\frac{ap^2}{2}} = \frac{1}{a} e^{-\left(\frac{k^2}{a}\right)} I_n\left(\frac{k^2}{a}\right) \tag{A.10}$$

This identity can be found in Watson [60]. Often an expression involving the modified Bessel functions, $I_n(\Lambda)$, is simpler to calculate.

A.4 Special functions

In the weakly relativistic treatment, both the Dawson integral and the error function appear. One stage in the numerical calculation of the Bernstein modes is the reduction of a double integral to a single integral containing these special functions. There are well known numerical routines for calculating such functions. It will be informative to illustrate the shapes of these integral functions as certain assumptions are made in the section describing the code.

A.4.1 Error function

The equation which defines the error function is:

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-w^2} dw \tag{A.11}$$

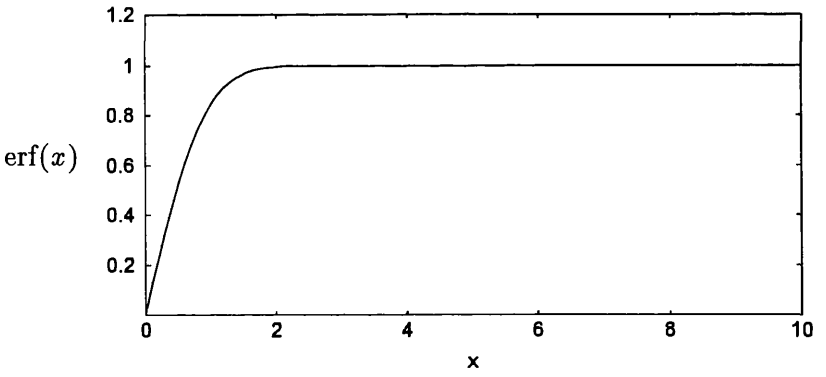


Figure A.1: The Error Function

As may be inferred from figure A.1 for argument greater than 2, $\mathrm{erf}(x) \approx 1$. Along with the error function one can define the related *complementary error function*, $\mathrm{erfc}(x)$, as $1 - \mathrm{erf}(x)$. Note that this complementary function tails off to zero for $x > 2$.

A.4.2 Dawson integral

The other special integral function is the Dawson integral. It has a similar definition to the error function:

$$D(\eta) = e^{-\eta^2} \int_0^{\eta} e^{\tau^2} d\tau \tag{A.12}$$

This integral is a special case of the familiar plasma dispersion function, which crops up in many different situations [61].

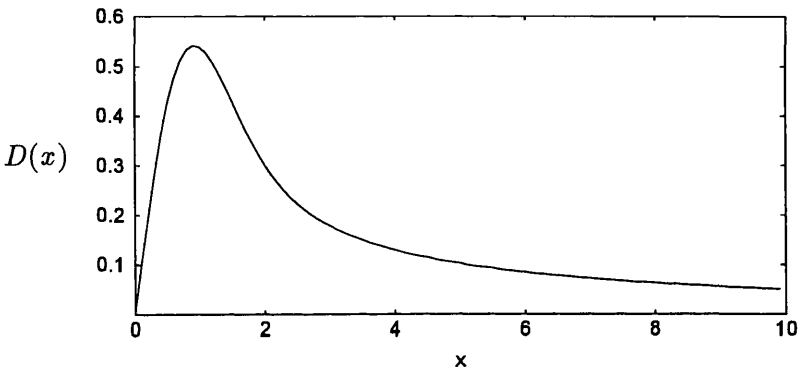


Figure A.2: The Dawson Integral

Appendix B

Contour Integration

By using the residue, a useful approximation to the behaviour of a complex function at a pole can be made.

$$f(z) \approx \frac{\lim_{z \to z_1} [(z - z_1) f(z)]}{z - z_1} = \frac{\mathrm{Res}_{z=z_1}(f(z))}{z - z_1} \tag{B.1}$$

The residue theorem states that if $f(z)$ (where z is complex) is analytic on and inside a closed contour C, as shown in figure B.1, except for a finite number of isolated singularities at $z = z_1, z_2, \ldots, z_n$, which are all located inside C, then

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z=z_k}(f(z)) \tag{B.2}$$

This theorem allows the evaluation of some special types of improper integrals. The background to the analysis which follows was obtained from Stewart [62] and Butkov [63]. If we can specify the three following conditions:

1. $f(z)$ is analytic in the upper half-plane, $\Im(z) \geq 0$ except for a finite number of singularities, none of which lie on the real axis
2. On the semicircle C_R, of radius R, $\{R \times$ maximum value of $|f(z)|$ on $C_R\}$ tends to 0 as $R \to \infty$
3. $\int_{-\infty}^{0} f(x)\, dx$ and $\int_{0}^{\infty} f(x)\, dx$ both exist

then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \{\text{residues at the poles with } \Im(z) \geq 0\} \tag{B.3}$$

For the purposes of this work, this analysis must be extended to include functions which have a *simple pole* on the real axis, as shown in figure B.2. As in the proof of the residue theorem this is achieved by making a semi-circular indentation in the contour into the upper half-plane. This semi-circle has radius r around the pole and removes the pole from the interior of the contour. For a pole at $z = z_1$ the full contour integration is now:

$$\oint_C f(z)\,dz = \int_{-R}^{z_1-r} f(x)\,dx + \int_{C_r} f(z)\,dz + \int_{z_1+r}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \tag{B.4}$$

As $r \to 0$ the two integrals along the real axis can be combined to give a definition for the *principal value* of the integral:

$$P\int_{-R}^{R} f(x)dx \equiv \lim_{r\to 0}\left\{\int_{-R}^{z_1-r} f(x)\,dx + \int_{z_1+r}^{R} f(x)\,dx\right\}. \tag{B.5}$$

If we assume condition 2 above is still valid, then the calculation is similar to the one for a pole in the upper half plane but with the additional contribution from the semi-circle C_r, which is equal to $-i\pi \mathrm{Res}_{z=z_1}(f(z))$. The negative sign arises since C_r is transversed in a clockwise, or negative, sense.

If on the other hand the alternative semi-circular contour is taken (*i.e.* below the real pole) the real pole contributes in a positive fashion but now there is a pole wholly within the closed contour. As might be wished the same contribution results. Taken together (as is the case in the analysis in Section 5.2.1) two real poles $z = a$ and $-a$ can cancel each other out.

So, in the limit $r \to 0$ and $R \to \infty$, the integral can thus be written as

$$\oint_C f(z)\,dz = P\int_{-\infty}^{\infty} f(x)\,dx - i\pi \sum \{\text{Residues at the simple poles lying on the real axis}\} \tag{B.6}$$

or, if we make use of the Residue Theorem (B.2), as

$$P\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum\{\text{Residues at poles in upper half-plane}\} + \pi i \sum\{\text{Residues at the simple poles lying on the real axis}\} \tag{B.7}$$

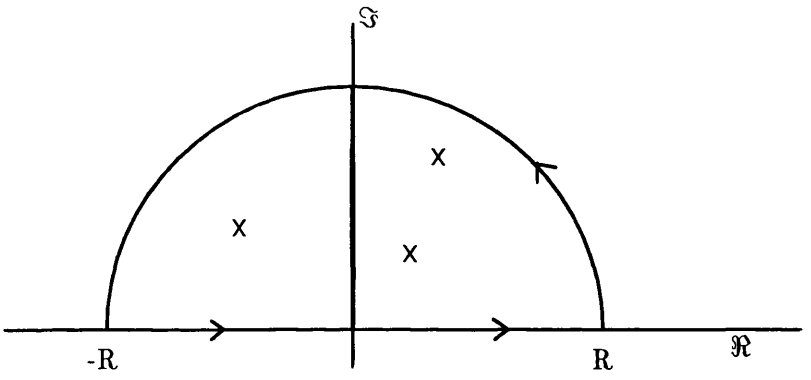


Figure B.1: A closed contour C over which the function f is integrated

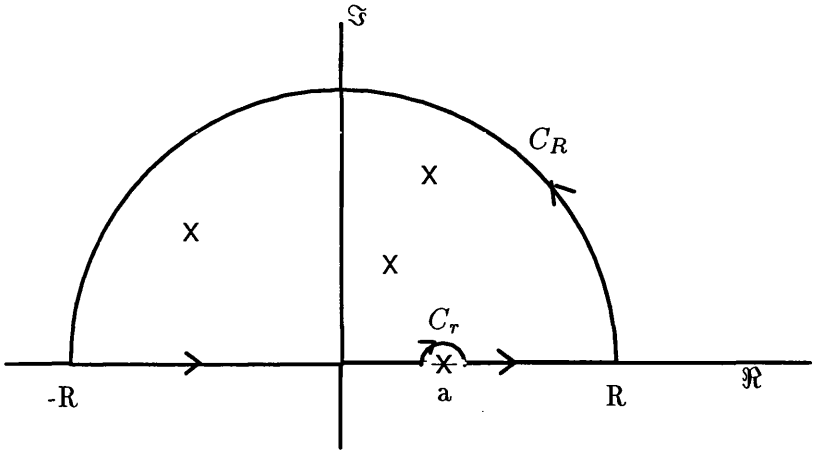


Figure B.2: A closed contour C with a simple pole lying on the real axis

Appendix C

The Code

There follows a listing of the code described in Chapter 6.

C.1 bmodes.c

```
#include <stdio.h>
#include <stdlib.h>  /*definition of atoi() */
#include <math.h>
#include <string.h>

#include "bmodes.h"


/*Declarations
These variables are declared here so that they are global.
*/

  /* plas - the plasma frequency */
  /* This is user-defined and constant through one execution */

float plas;

  /* plasma - the square of plas */

float plasma;

  /* howrel - the degree of relativity (called 'a' in notes) */
  /* This is user-defined and constant through one execution */

float howrel;

  /* Isrel - variable */
  /*         0  -  use non.c */
  /*         1  -  use weakly.c */
  /*         2  -  error */
```

```
  /* This is user-defined and constant through one execution */

int Isrel=2;

  /* execute - the name of the executable */
  /* This is defined in the makefile */

char execute[10];

  /* L - the Lambda or 'wave number' value */
  /* This is a true global variable, it varies */
  /* in steps in what_to_do() */
float L;


/*Prototypes*/

void usage(char exec[],int verbose);
void what_to_do(float (*choose)(float));
float Rxx(float);
float Rel(float);


int main(int argc, char *argv[ ])
{
extern float plas;
extern float howrel;

  /* Isrel, choice of weakly relativistic (1) or nonrelativistic (0) */
  /* default of 2 is chosen to be different again */
extern int Isrel;

  /* should the usage message be verbose? (0 - No; 1 - Yes) */
```

```
int verb=0;

  /* is there a good entry for the plasma frequency? (0 - No; 1 - Yes) */
int good=0;

extern char execute[10];

  /* execute is defined to be the name of the executable */
  /* which in turn was defined in the makefile */
strcpy(execute,argv[0]);

while ((argc >1) && (argv[1][0] == '-')){
  switch(argv[1][1])
    {
    case 'n':     /* nonrelativistic option is flagged */
      Isrel=0;
      break;

    case 'w':     /* weakly relativistic option is flagged */
      if((argv[1][2]>='0') && (argv[1][2]<='9')){
         /* degree of relativity, howrel (a=mc^2/kT) is set */
Isrel=1;
howrel=atof(&argv[1][2]);
      }
      else {
        usage(execute,1);
        exit(8);
      }
      break;

    case 'p':     /* plas, the plasma frequency is defined */
      if((argv[1][2]>='0') && (argv[1][2]<='9')){
++good;
```

```
        plas=atof(&argv[1][2]);
      }
      else {
        usage(execute,1);
        exit(8);
      }
      break;

    case 'h':     /* gives the verbose usage */
      verb=1;
      /* fall through */

    default:
      if(argv[1][1]!='h') printf("-%c is a bad option\n",argv[1][1]);
      usage(execute,verb);
      exit(8);
    }
  --argc;
  ++argv;
}

switch(Isrel){
 case 1:
    /* Based on the value of Isrel. This passes the name of the weakly */
    /* relativistic routine, Rxx to what_to_do() */
  if (good>0)
    printf("(a, plas) is (%g, %g)\n",howrel,plas);
    what_to_do(Rxx);
  break;

 case 0:
  /* If however the nonrelativistic option is requested this will pass */
  /* the name of the nonrelativistic routine, Rel to what_to_do() */
```

```
  if (good>0)
    printf("(plas) is (%g)\n",plas);
    what_to_do(Rel);
  break;

 case 2:
  printf("\n****** Please choose either -n or -w<howrel> flag ******\n\n");
  /* fall through */

 default:
  if (good==0) printf("\n****** Please give a -p<plas> flag ******\n\n");
  usage(execute,verb);
  exit(8);
}
return 0;

}

void usage(char exec[],int verbose)
{
printf("Usage: %s -n|-w<howrel> -p<plasma frequency> [-h]\n\n",exec);
if (verbose==1){
  printf("   either  -n or -w<howrel>\n\n");
  printf("           -n    nonrelativistic routine\n");
  printf("           -w    weakly relativistic routine\n");
  printf("                 immediately followed by the float <howrel>\n\n");
  printf("           -p    initialises the plasma frequency\n");
  printf("                 immediately followed by the float <plas>\n");
  printf("           -h    shows this message\n");
  }
}
```

```
void what_to_do(float (*choose)(float))
{
/*Global variables*/
extern float L;
extern float plasma;
extern int Isrel;
extern float plas;
extern float howrel;
extern char execute[10];


/*Local variables*/

  /* collect - a pointer to a FILE is declared this will be the data file */
FILE *collect;

  /* callname - the name of the data file */
char callname[40];

  /* x - the angular frequency */
float x;

  /* root - the value of x closest to a zero */
float root=0.0;

  /* val - the current value of Rxx(x) (or Rel(x)) */
float val=0.0;

  /* lastval - the value of Rxx(x) (or Rel(x)) on the last pass through */
  /* the x loop */
float lastval=0.0;

  /* f - the value of Rxx(root) (or Rel(root)) - should be close to zero */
```

```
  /* as long as there is no discontinuity */
float f;

  /* count - simply counts the number of passes through the L loop */
  /* check - counts the number of passes  through the x loop */
  /* flag - goes to 1 as soon as the first zero is found */
int count=0,check=0,flag=0;

  /* plasma - this is where the value of plasma is first declared */
plasma=plas*plas;

if (Isrel==0){
  printf("Processing for non-relativistic case .....\n");
  printf("\tplasma frequency is %g\n\n",plas);
  sprintf(callname,"non%.2f.dat",plas);
}
else {
  printf("Processing for weakly relativistic case ..... (a=%g)\n",howrel);
  printf("\tplasma frequency is %g\n\n",plas);
  sprintf(callname,"F%.1f_%.1f.dat",howrel,plas);
}

  /* This asks that a file named 'callname' be opened and, if */
  /* it already exists, appended to. */
collect=fopen(callname, "a");


for (L=0.1,count=1;L<=10.0;L+=0.1){
  if (count==1) {
    fprintf(collect,"# This data is from %s, with parameters: \n",execute);
    fprintf(collect,"#\t Howrel = %.1f\tplasma freq. = %e\n#\n",howrel,plas);
  }
  for (x=XINIT,check=0;x<XMAX;x+=XINC,check++){
```

```
    if (check!=0) {
      val=(*choose)(x);
      lastval=(*choose)(x-XINC);
      if ( ( (lastval>0) && (val<0) ) || ( (lastval<0) && (val>0) ) ) {
root=zbrent((*choose),x-XINC,x,1.0e-8);
f=(*choose)(root);
if ((f*f)<1.0) fprintf(collect,"%.1f\t%e\n",L,root);
flag=1;
      }
    }
  }
  if (flag==0) fprintf(collect,"\n"); /* No zero in the chosen range */
  fprintf(collect,"\n");
  if (count%5==0) printf("..... %4.1f .....\n",L); /* progress report */
  count++;
}
if (fclose(collect) != 0)
  printf("Error in closing file %s\n", callname);

}
```

C.2 non.c

```
/*********************************************************************
*    This will calculate Rxx for non-rel. case for a e+e- plasmas     *
*                                                                     *
*          DAK started this on the 19th of April, 1996               *
*********************************************************************/

/*For all parameters and common libraries */

#include "/usr/local/src/recipes_c-ansi/include/nr.h"
#include "/usr/local/src/recipes_c-ansi/include/nrutil.h"
#include "defs.h"

/*Needed for Bessel I part of program*/

#define ACC 40.0
#define BIGNO 1.0e10
#define BIGNI 1.0e-10

/*Prototypes of the main parts of the Integrand*/
float Rel(float);

float Rel(float omega)
{
extern float L;
extern float plasma;
extern int Isrel;

float Rval,term,expL;
int nu;
```

```
Isrel=0;

expL=exp(-L)/L;

term=expL*bessi1(L)/(omega*omega-1.0);
for (nu=2;nu<=10;nu++)
  {
    term+=nu*nu*expL*bessi(nu,L)/(omega*omega-nu*nu);
  }
Rval=1.0-4.0*plasma*term;

return Rval;
}



float bessi0(float x)
{
  float ax,ans;
  double y;

  if ((ax=fabs(x)) < 3.75) {
    y=x/3.75;
    y*=y;
    ans=1.0+y*(3.5156229+y*(3.0899424+y*(1.2067492
+y*(0.2659732+y*(0.360768e-1+y*0.45813e-2)))));
  } else {
    y=3.75/ax;
    ans=(exp(ax)/sqrt(ax))*(0.39894228+y*(0.1328592e-1
+y*(0.225319e-2+y*(-0.157565e-2+y*(0.916281e-2
+y*(-0.2057706e-1+y*(0.2635537e-1+y*(-0.1647633e-1
+y*0.392377e-2)))))));
  }
  return ans;
```

```
}

float bessi1(float x)
{
  float ax,ans;
  double y;

  if ((ax=fabs(x)) < 3.75) {
    y=x/3.75;
    y*=y;
    ans=ax*(0.5+y*(0.87890594+y*(0.51498869+y*(0.15084934
   +y*(0.2658733e-1+y*(0.301532e-2+y*0.32411e-3))))));
  } else {
    y=3.75/ax;
    ans=0.2282967e-1+y*(-0.2895312e-1+y*(0.1787654e-1
 -y*0.420059e-2));
    ans=0.39894228+y*(-0.3988024e-1+y*(-0.362018e-2
+y*(0.163801e-2+y*(-0.1031555e-1+y*ans))));
    ans *= (exp(ax)/sqrt(ax));
  }
  return x < 0.0 ? -ans : ans;
}


float bessi(int n, float x)
{
  float bessi0(float x);
  void nrerror(char error_text[]);
  int j;
  float bi,bim,bip,tox,ans;

  if (n < 2) nrerror("Index n less than 2 in bessi");
  if (x == 0.0)
```

```
    return 0.0;
  else {
    tox=2.0/fabs(x);
    bip=ans=0.0;
    bi=1.0;
    for (j=2*(n+(int) sqrt(ACC*n));j>0;j--) {
      bim=bip+j*tox*bi;
      bip=bi;
      bi=bim;
      if (fabs(bi) > BIGNO) {
ans *= BIGNI;
bi *= BIGNI;
bip *= BIGNI;
      }
      if (j == n) ans=bip;
    }
    ans *= bessi0(x)/bi;
    return x < 0.0 && (n & 1) ? -ans : ans;
  }
}
#undef ACC
#undef BIGNO
#undef BIGNI
```

C.3 weakly.c

```
/*************************************************************************
*    This will calculate Rxx for the weakly relativistic case           *
*    for e+e- plasmas                                                   *
*                                                                       *
*          DAK wrote this on the 24th of September, 1996                *
*************************************************************************/

/* In particular the function Rxx() calculates a value of Rxx for a
   given omega: the value of k_perp is determined externally from the
   main program                                                        */

/*For all parameters and common libraries */

#include "/usr/local/src/recipes_c-ansi/include/nr.h"
#include "/usr/local/src/recipes_c-ansi/include/nrutil.h"
#include "defs.h"
#include "constant.h"

/*Needed for Bessel J squared part of program*/

#define ACC 40.0
#define BIGNO 1.0e10
#define BIGNI 1.0e-10

/*Needed for Dawson Integral part of program*/

#define NRANSI
#define NMAX 6
#define H 0.4
#define A1 (2.0/3.0)
```

```
#define A2 0.4
#define A3 (2.0/7.0)

/*Needed for Romberg Integration driver*/

#define EPS 1.0e-6          /* fractional accuracy required */
#define JMAX 16             /* no. iterations before exit (default 14) */
#define JMAXP (JMAX+1)
#define K 5                 /* no. points used in the extrapolation */

/*Needed for Midpoint rule integration*/

#define FUNCT(x) ((*func)(x))

/*External variables*/

float howrel;
int NU;
float L;
float plasma;
float bb;

float integrandI(float);
float integrandII(float);

float Rxx(float omega)
{
  extern float howrel;
  extern int NU;
  extern float L;
  extern float plasma;
  extern int Isrel;
  extern float bb;
```

```
  float answerI,answerII,answerIII,cut;
  float q,bc,All,Rvalue,Kperp,marker,expon;

  Rvalue=0.0;
  Isrel=1;
  Kperp=L;
  q=bb=bc=0.0;
  All=0.0;
  cut=sqrt(8.0/howrel);

/* Change this for accuracy of erf(x) ~= 0  */

  for (NU=1;NU<7;NU++) {

    bb=NU*NU/(omega*omega)-1.0;
    answerI=answerII=answerIII=0.0;

    if (omega>NU){

      q=sqrt(-bb);

/*      printf("III\tomega: %.2f\tKperp: %.2f\tNU: %d\n",omega,Kperp,NU);*/

      if (q<cut) {
answerIII=qromo(integrandII,q,cut,midpnt);
      }
      else answerIII=0.0;
/*      printf("ansIII is: %e\n",answerIII);*/
    }

    else {
```

```
/* Omega is less than NU    */

      bc=sqrt(bb);

/* printf("I&II\tomega: %.2f\tKperp: %.2f\tNU: %d\n",omega,Kperp,NU);*/

/* Case I */

      answerI=qromo(integrandI,0.0,bc,midpnt);
/* printf("ansI is: %e\n",answerI);*/

/* Case II */

      answerII=qromo(integrandII,0.0,cut,midpnt);

/* printf("ansII is: %e\n",answerII);*/

    }
    if (0.5*howrel*bb<POW) expon=exp(-(0.5*howrel)*bb)*(answerII+answerIII);
    else expon=0.0;

    All+=NU*NU*NU*NU*(-answerI+0.5*sqrt(PI)*expon);

  }

/*  marker=omega*omega-2.0*plasma;
  printf("marker(%.2f,%.2f) is: %e\n",omega,Kperp,marker);

  printf("All is: %e\n",All);*/

  Rvalue=omega*omega-2.0*plasma-4.0*plasma*howrel*howrel*
      sqrt(2.0*howrel)*All/(omega*omega*Kperp*Kperp);
/*  printf("Rxx(%.2f,%.2f) is: %e\n",omega,Kperp,Rvalue);*/
```

```
  return Rvalue;
}

float integrandI(float alpha)
{
  extern float howrel;
  extern int NU;
  extern float bb;
  extern float L;

  float x,caseI,exponI,Kperp,arg,bessel;

  caseI=0.0;
  Kperp=L;

  x=sqrt(0.5*howrel*(alpha*alpha));

  arg=sqrt(bb-alpha*alpha);

  if (0.5*howrel*arg*arg<POW)
    {
      exponI=exp(-0.5*howrel*arg*arg);
      if (NU==1) bessel=bessj1(Kperp*arg);
      else bessel=bessj(NU,Kperp*arg);

      caseI=exponI*bessel*bessel*dawson(x);
    }
  else caseI=0.0;

  return caseI;
}

float integrandII(float delta)
```

```
{
  extern float howrel;
  extern int NU;
  extern float bb;
  extern float L;

  float y,caseII,Kperp,arg,bessel;

  caseII=0.0;
  Kperp=L;

  y=sqrt(howrel*(delta*delta)/2.0);

  arg=sqrt(delta*delta+bb);

  /*printf("y: %g\targ: %g\terfc: %g\t",y,arg,erfc(y));*/

  if (NU==1) bessel=bessj1(Kperp*arg);
  else bessel=bessj(NU,Kperp*arg);

  caseII=bessel*bessel*erfc(y);
  return caseII;
}


float bessj(int n, float x)
{
void nrerror(char error_text[]);
int j,jsum,m;
float ax,bj,bjm,bjp,sum,tox,ans;

if (n < 2) nrerror("Index n less than 2 in bessj");
ax=fabs(x);
```

```
if (ax == 0.0)
return 0.0;
else if (ax > (float) n) {
tox=2.0/ax;
bjm=bessj0(ax);
bj=bessj1(ax);
for (j=1;j<n;j++) {
bjp=j*tox*bj-bjm;
bjm=bj;
bj=bjp;
}
ans=bj;
} else {
tox=2.0/ax;
m=2*((n+(int) sqrt(ACC*n))/2);
jsum=0;
bjp=ans=sum=0.0;
bj=1.0;
for (j=m;j>0;j--) {
bjm=j*tox*bj-bjp;
bjp=bj;
bj=bjm;
if (fabs(bj) > BIGNO) {
bj *= BIGNI;
bjp *= BIGNI;
ans *= BIGNI;
sum *= BIGNI;
}
if (jsum) sum += bj;
jsum=!jsum;
if (j == n) ans=bjp;
}
sum=2.0*sum-bj;
```

```
ans /= sum;
}
return x < 0.0 && (n & 1) ? -ans : ans;
}


float bessj0(float x)
{
float ax,z;
double xx,y,ans,ans1,ans2;

if ((ax=fabs(x)) < 8.0) {
y=x*x;
ans1=57568490574.0+y*(-13362590354.0+y*(651619640.7
+y*(-11214424.18+y*(77392.33017+y*(-184.9052456)))));
ans2=57568490411.0+y*(1029532985.0+y*(9494680.718
+y*(59272.64853+y*(267.8532712+y*1.0))));
ans=ans1/ans2;
} else {
z=8.0/ax;
y=z*z;
xx=ax-0.785398164;
ans1=1.0+y*(-0.1098628627e-2+y*(0.2734510407e-4
+y*(-0.2073370639e-5+y*0.2093887211e-6)));
ans2 = -0.1562499995e-1+y*(0.1430488765e-3
+y*(-0.6911147651e-5+y*(0.7621095161e-6
-y*0.934935152e-7)));
ans=sqrt(0.636619772/ax)*(cos(xx)*ans1-z*sin(xx)*ans2);
}
return ans;
}

float bessj1(float x)
```

```
{
float ax,z;
double xx,y,ans,ans1,ans2;

if ((ax=fabs(x)) < 8.0) {
y=x*x;
ans1=x*(72362614232.0+y*(-7895059235.0+y*(242396853.1
+y*(-2972611.439+y*(15704.48260+y*(-30.16036606))))));
ans2=144725228442.0+y*(2300535178.0+y*(18583304.74
+y*(99447.43394+y*(376.9991397+y*1.0))));
ans=ans1/ans2;
} else {
z=8.0/ax;
y=z*z;
xx=ax-2.356194491;
ans1=1.0+y*(0.183105e-2+y*(-0.3516396496e-4
+y*(0.2457520174e-5+y*(-0.240337019e-6))));
ans2=0.04687499995+y*(-0.2002690873e-3
+y*(0.8449199096e-5+y*(-0.88228987e-6
+y*0.105787412e-6)));
ans=sqrt(0.636619772/ax)*(cos(xx)*ans1-z*sin(xx)*ans2);
if (x < 0.0) ans = -ans;
}
return ans;
}

float dawson(float x)
{
int i,n0;
float d1,d2,e1,e2,sum,x2,xp,xx,ans;
static float c[NMAX+1];
static int init = 0;
```

```
if (init == 0) {
init=1;
for (i=1;i<=NMAX;i++) c[i]=exp(-SQR((2.0*i-1.0)*H));
}
if (fabs(x) < 0.2) {
x2=x*x;
ans=x*(1.0-A1*x2*(1.0-A2*x2*(1.0-A3*x2)));
} else {
xx=fabs(x);
n0=2*(int)(0.5*xx/H+0.5);
xp=xx-n0*H;
e1=exp(2.0*xp*H);
e2=e1*e1;
d1=n0+1;
d2=d1-2.0;
sum=0.0;
for (i=1;i<=NMAX;i++,d1+=2.0,d2-=2.0,e1*=e2)
sum += c[i]*(e1/d1+1.0/(d2*e1));
ans=0.5641895835*SIGN(exp(-xp*xp),x)*sum;
}
return ans;
}


float qromo(float (*func)(float), float a, float b,
float (*choose)(float(*)(float), float, float, int))
{
  void polint(float xa[], float ya[], int n, float x, float *y, float *dy);
  void nrerror(char error_text[]);
  int j;
  int fix;
  float ss,dss,h[JMAXP+1],s[JMAXP+1];
```

```
This is the function file which deals with the weakly relativistic case
  h[1]=1.0;
  for (j=1;j<=JMAX;j++) {
    s[j]=(*choose)(func,a,b,j);
/*************************************************************************
These have the value of the integration (midpoint) between a and b,
*************************************************************************/

    if (j >= K) {
      polint(&h[j-K],&s[j-K],K,0.0,&ss,&dss);

/*************************************************************************
The call above to the polint function feeds the offset arrays
h[j-K+1..j-K+5]  and s[j-K+1..j-K+5] s.t. a 4th order polynomial P(h[i])
= s[i] is created, it returns a pair of values (by changing the contents
of addresses) the value of  P(0.0) = ss, and an estimate of error, dss.
*************************************************************************/

      if (fabs(dss) <= EPS*fabs(ss)) return ss;
    }
    s[j+1]=s[j];
    h[j+1]=h[j]/9.0;
      /* This 9.0 assumes step tripling and an even error series is used */
  }

  for (fix=1;fix<=JMAX;fix++){
    printf("%d.\t%e\n",fix,s[fix]);
/*************************************************************************
      This is a loop to print out consecutive values of s[fix]
*************************************************************************/
  }

  nrerror("Too many steps in routing qromo");
```

```
  return 0.0;
}

void polint(float xa[], float ya[], int n, float x, float *y, float *dy)
{

/*************************************************************************
A little explanation is in order - Given arrays xa[1..n] and ya[1..n],
and  given a value x, this routine returns the value y, and an error
estimate dy. If P(x) is a polynomial of degree N-1 s.t. P(xa[i])=ya[i],
i=1..n, then the returned value y = P(x).
*************************************************************************/

int i,m,ns=1;
float den,dif,dift,ho,hp,w;
float *c,*d;

dif=fabs(x-xa[1]);
c=vector(1,n);
d=vector(1,n);
for (i=1;i<=n;i++) {
if ( (dift=fabs(x-xa[i])) < dif) {
ns=i;
dif=dift;
}
c[i]=ya[i];
d[i]=ya[i];
}
*y=ya[ns--];
for (m=1;m<n;m++) {
for (i=1;i<=n-m;i++) {
ho=xa[i]-x;
hp=xa[i+m]-x;
```

```
w=c[i+1]-d[i];
if ( (den=ho-hp) == 0.0) nrerror("Error in routine polint");
den=w/den;
d[i]=hp*den;
c[i]=ho*den;
}
*y += (*dy=(2*ns < (n-m) ? c[ns+1] : d[ns--]));
}
free_vector(d,1,n);
free_vector(c,1,n);
}

float midpnt(float (*func)(float), float a, float b, int n)
{
float x,tnm,sum,del,ddel;
static float s;
int it,j;

/* printf("I got here\n");*/
if (n == 1) {
return (s=(b-a)*FUNCT(0.5*(a+b)));
} else {
for(it=1,j=1;j<n-1;j++) it *= 3;
tnm=it;
del=(b-a)/(3.0*tnm);
ddel=del+del;
x=a+0.5*del;
sum=0.0;
for (j=1;j<=it;j++) {
sum += FUNCT(x);
x += ddel;
sum += FUNCT(x);
x += del;
```

```
}
s=(s+(b-a)*sum/tnm)/3.0;
return s;
}
}

#undef FUNCT
#undef EPS
#undef JMAX
#undef JMAXP
#undef K
#undef ACC
#undef BIGNO
#undef BIGNI
#undef NRANSI
#undef NMAX
#undef H
#undef A1
#undef A2
#undef A3
```

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