RINGS OF ENDOmorphisms

by

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SUMMARY

This dissertation reviews some results about rings of endomorphisms of modules, mainly in the form "if a module has the property $P$ then its ring of endomorphisms has the property $Q$".

After an introductory Chapter 0, Chapter 1 is devoted to develop some concepts that will be necessary later on; a detailed study of the uniform (Goldie) dimension of a module is carried out and, in this vein, some original results of the author, which will appear elsewhere, are included in Section 4.

In Chapter 2 we present the endomorphism ring of a module as well as a general technique for its study (Sections 5 and 6). The modules whose rings of endomorphisms have been reviewed are detailed next.

In Section 7, injective and quasi-injective modules are considered; it is shown that the factor ring of their endomorphism ring modulo its radical is a regular and (right) self-injective ring.

In Section 8, projective modules are discussed; the Morita Theorem is recollected and some properties of a ring which are inherited by the endomorphism rings of its finitely generated projective modules are stated; also, a study of the projective modules with local endomorphism rings is done.

In Section 9, we consider finite dimensional modules. First they are assumed to be also injective and, after dropping this hypothesis, we study the nilpotency of the nil subrings of their rings of endomorphisms; we also answer some questions about the quotient ring of the endomorphism ring of a finite dimensional nonsingular module.

Finally, in Section 10, we look at what happens when the module is assumed to satisfy some chain conditions, in general at a first stage and under the hypothesis of quasi-injectivity or quasi-projectivity in the final paragraph of the dissertation.
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This dissertation consists of two quite different chapters. The first of them intends to provide the basic tools of Module Theory that shall be used in the second, and that are usually beyond the scope of a very elementary course in Rings and Modules. There has been assumed some knowledge about direct sums and products, homomorphisms, isomorphism theorems, exact sequences, injectivity, projectivity and chain conditions, while concepts like essential submodules and extensions, complement submodules, singular and nonsingular modules, injective hulls or quasi-injective modules are introduced in some detail.

Particular emphasis is made in the topic of finite (Goldie) dimension, leading to a Section 4 in which the dimension of a sum of finite dimensional modules is studied. This section should be viewed as an Appendix to Chapter 1 and may be omitted without consequences for Chapter 2.

The second chapter deals with the topic announced in the title: the ring of endomorphisms (or endomorphism ring) of a module. It is assumed that the reader is familiar with concepts such as local, simple, semisimple, Artinian or Noetherian rings, Jacobson radical, idempotent elements, nilpotent ideals and subrings, factor rings or lifting of idempotents. Other ideas are introduced here although at times, in order to keep our attention in the main subject of this work, some strong results of Ring Theory are just quoted, e.g. some facts about perfect and semiperfect rings in Section 8, Goldie’s theorems in Section 9 or the Hopkins-Levitzki Theorem in Section 10.

The rings of endomorphisms are introduced along with some easy results which show how the structure of a module determines that of its ring of endomorphisms. This is the central idea of this dissertation, namely the search for theorems of the form "if a module has property \( P \) then its endomorphism ring has property \( Q \)." In fact, what is done here is a review of the results of that kind which already existed in the literature in case the module is injective, quasi-injective,
projective or finite dimensional, or satisfy some chain conditions.

A very general technique to find results of that type (the "correspondence theorems") is then presented; some of the results here will prove very helpful in the following sections.

Next, the endomorphism ring of a quasi-injective module, or rather its factor ring modulo the radical, is studied. This is followed by a quick review of the Morita Theorem, and some classes of projective modules whose rings of endomorphisms have nice properties are briefly introduced.

Injective modules are then revisited, now taking into account their dimension; later, nil subrings and quotient rings of the endomorphism ring of a finite dimensional module are studied. Finally, we state some results about Artinian, Noetherian and finite-length modules, and also about quasi-injective and quasi-projective modules with certain chain conditions.

Throughout the dissertation, an effort to pay tribute to the parents of the ideas which appear there has been made. The references do not necessarily mean that we have followed the proofs given in the quoted paper but that, to the best of the author's knowledge, that is the first time such a result appeared in the literature. Some papers which are not referred to in the text, but contain material which aided in the preparation of this work, have been included after the main text under the common label of List of References. Also, an index with the concepts assumed and defined through the dissertation is provided.

Finally, the author would like to express his gratitude to his supervisor, Professor P.F. Smith, for his guidance and for suggesting the topic of the dissertation, as well as to the British Council and the Caja de Ahorros del Mediterráneo (Spain) for their efficient collaboration in preparing and financially supporting him in this last study's year.

CHAPTER 0

NOTATION AND CONVENTIONS

Throughout this dissertation, by a ring $R$ we will mean an associative ring with identity $1_R$ (or 1 if there is no risk of confusion about the ring), and all modules will be unitary (i.e. the product of an element $x$ of the module by the identity of the ring equals $x$). The following right-sided conventions will also stand in their left-sided form.

The category of all (unitary) right $R$-modules will be denoted by $\text{Mod}_R$ ($\text{Mod}_L$ for left $R$-modules). $M=\text{MR}$ will mean that $M$ is an object of $\text{Mod}_R$. Given $M=\text{MR}$ and $N=\text{NR}$, the notation $f:MR\to NR$ will imply that $f$ is a morphism in $\text{Mod}_R$ (i.e., a right $R$-homomorphism), while $f:M\to N$ shall be viewed as a set theoretical map, unless otherwise specified. All morphisms in the categories $\text{Mod}_R$ and $\text{Mod}_L$ will be written in the side opposite to the scalars (i.e., given $f:MR\to NR$ and $g:NL\to NK$, the images of $x\in M$ and $y\in L$ will be $f(x)$ and $(y)g$, or more often $fx$ and $yg$). In the same way, the image of a submodule $PM\in \text{MR}$ will be written $f(P)$ or $fP$.

Given $M=\text{MR}$ and $N=\text{NR}$ we will denote by $\text{Hom}_R(M,N)$, or by $\text{Hom}(\text{MR},\text{NR})$ if we want to emphasize the side, the set of all right $R$-homomorphisms from $M$ into $N$. If $f,g\in \text{Hom}_R(M,N)$ then the map $f+g:M\to N$ defined via $(f+g)x=fx+gx$ for all $x\in M$, is actually in $\text{Hom}_R(M,N)$, and this 'sum of homomorphisms' provides $\text{Hom}_R(M,N)$ with the structure of Abelian group (with the zero map as zero element) which shall be assumed in the sequel. In case $M=N$, we call an element of $\text{Hom}_R(M,M)$ an endomorphism of $M$, and write $\text{End}_R(M)$ or $\text{End}_R(M)$ for $\text{Hom}_R(M,M)$.

Given two rings $T,R$, a $(T,R)$-bimodule is an Abelian group $M$ which is both a left $T$-module and a right $R$-module in such a way that, for all $t\in T$, $r\in R$ and $x\in M$, the equality $(tx)r=t(xr)$ is satisfied. We denote this situation by $M=\text{MR}$, and the category of $(T,R)$-bimodules by $\text{Mod}_{T,R}$.

For bimodules $\text{OMR}$ and $\text{NR}$, it is well known that $\text{Hom}(\text{MR},\text{NR})$ is an object of $\text{OM}_R$. Similarly, for $R\text{AOM}_{T}$ and $R\text{BR}$, $\text{Hom}(R\text{AOM}_{T},R\text{BR})$ is in $\text{OM}_{R}$. 
According to our notation, we will write $\text{Hom}(M_r, N_r)_Q$ and $\text{Hom}(R A, R B)$, where $Z$ is the ring of rational integers; therefore, given e.g. $Q M_r$ and $N_r$, we have $\text{Hom}(M_r, N_r)_Q$, and so on.

The symbols $\leq$ and $\subset$ will mean inclusion and strict inclusion, respectively. If $M=M_r$, the fact that $N$ is an $R$-submodule of $M$ will be abbreviated as $N \leq M_r$, while $N \subset M_r$ will be viewed as a set inclusion. Therefore, $\alpha \leq R_r$ will mean that $\alpha$ is a right ideal of $R$. By 'a is an ideal of R' (without further specification) we will understand 'a is a two-sided ideal of $R$'.

For a module $M_r$, the lattice of submodules of $M$ ordered by inclusion will be denoted by $\text{Lat}(M_r)$. Then $\text{Lat}(R R)$ ($\text{Lat}(R R))$ will stand for the lattice of right (left) ideals of $R$. Many times, we will speak about chain conditions in a subset $\Omega$ of $\text{Lat}(M_r)$ or $\text{Lat}(R R)$, as for example when we say ' granted has the descending chain condition (always abbreviated DCC) on complements'; this will mean that the subset $\Omega$ of $\text{Lat}(M_r)$ consisting of all complement submodules in $M_r$ satisfies the minimum condition: i.e., every nonempty subset of $\Omega$ contains a minimal element or, equivalently, every strictly descending chain of elements of $\Omega$ must be finite. Of course, a similar convention stands for the ascending chain condition, or ACC.

Recall that, for a module $M_r$, the lattice $\text{Lat}(M_r)$ is modular, i.e. $N+(L \cap K) = (N+L) \cap K$ whenever $N, L, K$ are submodules of $M_r$ such that $N \leq K$. This Modular Law will be used without further reference.

Given $N \leq M_r$ and a nonempty subset $X$ of $M$, we write $(N:X)$ for the right ideal $\{r \in R : x \in N \text{ for all } x \in X\}$ of $R$. If $X$ is a singleton $X=\{x\}$, then we write $(N:x)$. If, for example, $M$ is a bimodule $sM_r$, we avoid any confusion by writing $(N_r:X)$ or $(N_s:X)$. If $N$ is the zero submodule then $(0_r:X)$ is usually called the (right) annihilator of $X$ in $R$ and written $r_r(X)$; similarly, $(0_s:X)$ is called the (left) annihilator of $X$ in $S$, and we write $l_s(X)$.

If $M=R_r$ then the annihilator ideals of nonempty subsets $V$ of $R$ will be simply called the right annihilator ideals of $R$, and we shall write $R(V)$ for $r_r(V)$. Similarly $l(V)$ will stand for the left annihilator ideal $l_r(V)$. 
Also, for a bimodule $sM_r$, the annihilators in $M$ of nonempty subsets $V$ of $R$ and $W$ of $S$ will be considered, and our notation will be $l^*_M(V) = \{ m \in M : mr = 0 \text{ for all } r \in V \}$ and $r^*_M(W) = \{ m \in M : sm = 0 \text{ for all } s \in W \}$.

If $N \leq S M_r$ is a direct summand of $M$ (i.e. if there exists $L \leq S M_r$ such that $M = N \oplus L$) then we write $N \leq d M$. If $M$ can be decomposed as $M = \bigoplus_{i \in I} M_i$ and $j \in I$ then $f : \bigoplus_{i \in I} M_i \rightarrow M_j$ will mean that $f$ is the canonical projection of $M$ on $M_j$ for the given decomposition (i.e., if $x = \sum_{i \in F} x_i$, for some finite subset $F$ of $I$ and some $0 \neq x_i \in M_i$, is the unique expression of $x$ in $\bigoplus_{i \in I} M_i$, then $f(x) = x_j$ if $j \in F$ and $f(x) = 0$ otherwise). Similarly, if $N \leq S M_r$, then $p : M \rightarrow M/N$ should be read as ' $p$ is the natural epimorphism of $M$ onto $M/N$ (i.e., $p(x) = x + N$ for all $x \in M$).

Finally, a family $\{ M_i : i \in I \}$ of submodules of a module $M_r$ will be said to be independent if the sum $\bigoplus_{i \in I} M_i$ is direct, i.e. if, for any two nonempty finite subsets $J$ and $K$ of $I$, we have $(\sum_{j \in J} M_j) \cap (\sum_{k \in K} M_k) = 0$.
CHAPTER 1
GOLDIE DIMENSION

SECTION 1: ESSENTIAL EXTENSIONS AND COMPLEMENT SUBMODULES

In this first section, we introduce several concepts which will be used throughout this dissertation, and establish their first properties. The basic concepts are the essential extensions and complement submodules of a module and the nonsingular modules.

Essential Submodules and Essential Extensions

The concept of essentiality was introduced by R.E. Johnson [26; p.891] in the early fifties, although the terminology is due to B.Eckmann and A.Schopf [12]. Given two right R-modules \(N \leq M_r\), \(N\) is said to be an essential submodule of \(M\) if \(N\) has nonzero intersection with each nonzero submodule of \(M\) (A.W.Goldie's terminology [19] was \(N\) meets all submodules of \(M\)). We denote this situation by \(N \leq_e M\). If \(N \leq_e M\) and \(N \subseteq M\), then we write \(N \leq_e M\) and say that \(N\) is a proper essential submodule of \(M\). If \(M = R_r\) we call an essential submodule of \(R_r\) an essential right ideal of \(R\).

Clearly, if the zero submodule is an essential submodule of \(M_r\) then \(M\) itself must be zero. Also, if \(N\) is an essential direct summand of \(M_r\) then \(N = M\).

If \(N \leq_e M\) then we also say that \(M\) is an essential extension of \(N\) though, as C.Faith points out in [FA73; p.168], it might well be called an inessential extension. In Section 3 we shall slightly generalize our concept of essential extension, but for the other sections the definition given here will suffice.

Next, we give the following useful characterization of essential extensions:
LEMMA 1.1  N is an essential submodule of \( M \) if and only if, for all \( x \in M \setminus N \), there exists \( r \in R \) such that \( 0 \neq x \cdot r \in N \). In this case, for all \( x \in M \), \( (N:x) \subseteq R N \).

**Proof:** If \( N \subseteq M \) and \( x \in M \setminus N \) then, in particular, \( x \neq 0 \), whence \( x \cdot r \neq 0 \) and thus \( x \cdot r \cdot N \neq 0 \). Conversely, if there exists \( 0 \neq L \subseteq M \) such that \( N \cdot L = 0 \) then, for any \( x \in L \setminus N \neq 0 \), \( x \cdot r \cdot N = 0 \).

Assume now that \( N \subseteq M \), and let \( r \in R \setminus (N:x) \); then \( x \cdot r \cdot N \) and hence there exists \( s \in R \) such that \( 0 \neq (x \cdot r) \cdot s \subseteq N \); then \( 0 \neq s \subseteq (N:x) \) and thus, by the first part, \( (N:x) \subseteq R N \).

Then, for example, it is clear that \( \mathbb{Z} \subseteq \mathbb{Q} \) (or, more generally, every commutative domain is essential in its field of fractions).

The following lemma states, among some other useful properties of essential extensions, that the set of essential submodules of \( M \) is a filter in the lattice \( \text{Lat}(M) \).

**PROPOSITION 1.2** Let \( M \) be a module with submodules \( N, L; N_1, \ldots, N_n; L_1, \ldots, L_n \); and let \( f : K \to M \) be any homomorphism. Then
a) if \( N \subseteq L \), then \( N \subseteq M \) if and only if \( N \subseteq L \) and \( L \subseteq M \);
b) if \( N \subseteq N_i \) and \( L \subseteq L_i \), then \( N \cdot L \subseteq N_i \cdot L_i \); in particular, if \( N \subseteq M \) and \( L \subseteq M \) then \( N \cdot L \subseteq M \);
c) if \( N \subseteq L \) and \( (L/N) \subseteq (M/N) \), then \( L \subseteq M \);
d) if \( N \subseteq M \) then \( f^{-1}(N) \subseteq K \);
e) if \( N \subseteq L_1 \) for \( i = 1, \ldots, n \) and the sum \( \sum N_i \) is direct, then \( \sum L_1 \) is also direct and \( (\bigoplus N_i) \subseteq (\bigoplus L_1) \).

**Proof:**  
a) Assume \( N \subseteq M \); then for all \( 0 \neq A \subseteq M \), \( L \cap A \subseteq N \cdot A \neq 0 \), whence \( L \subseteq M \); and for all \( 0 \neq A \subseteq L \), \( N \cdot A \neq 0 \), whence \( N \subseteq L \). Conversely, if \( N \subseteq L \), \( L \subseteq M \) and \( A \subseteq M \) verifies \( A \cdot N = 0 \), then \( N \cdot (A \cap L) = 0 \), whence \( A \cdot L = 0 \) and thus \( A = 0 \); therefore \( N \subseteq M \).

b) Let \( 0 \neq A \subseteq N \); since \( N \subseteq N_i \), \( 0 \neq N_i \) and then, since \( L \subseteq L_1 \), \( L \cap (N \cdot A) = (L \cap N) \cdot A \neq 0 \); therefore \( L \cap N \subseteq N \cdot L \).

c) Assume \( (L/N) \subseteq (M/N) \) and let \( 0 \neq A \subseteq M \); if \( A \subseteq N \) then \( A \cdot L = A \neq 0 \); if \( A \notin N \) then \( A \cdot L = 0 \), whence \( A \cdot L = 0 \), i.e. \( N \subseteq (A + N) \cdot L = (A \cdot L) + N \), and thus \( A \cdot L = 0 \); therefore \( L \subseteq M \).

d) This is very easily proved using (1.1); however, in order to obtain
a dual proof for the next result, we proceed as follows: Assume first that \( f \) is monic; then if \( A \in \text{Ker} f \) is such that \( f^{-1}NnA = 0 \), we get
\[
0 = f(f^{-1}NnA) = fNnA = (NaK)nA = NaA,
\]
whence \( fA = 0 \) and thus \( f^{-1}N \subseteq \text{Ker} f \).

In general, since \( g: (\text{Ker} f) \to \text{M} \) given by \( g(k + \text{Ker} f) = fk \) is monic, we get
\[
g^{-1}N\subseteq K \subseteq \text{Ker} f \quad \text{and hence, by c), } f^{-1}N \subseteq \text{Ker} f.
\]
e) Since the case \( n=2 \) is easily extended to any finite number of submodules, we prove that \( N \subseteq L, N' \subseteq L' \) and \( N \cap N' = 0 \) implies \( L \cap L' = 0 \) and \( N \cap N' \subseteq L \cap L' \); by b), \( 0 = N \cap N' \subseteq L \cap L' \), whence \( L \cap L' = 0 \); consider now the projections \( \pi: L \cap L' \to L \) and \( \rho: L \cap L' \to L' \); by d), \( \pi^{-1}N = N \cap L \subseteq \text{Ker} f \) and \( \rho^{-1}N' = N \cap L' \subseteq \text{Ker} f \), and thus b) gives \( N \subseteq N' \subseteq \text{Ker} f \).

REMARK: (1.2.e) also holds for infinite direct sums [G; Prop.1.4].

There is a dual concept for essentiality which is convenient to introduce here, though it will not be used until the second chapter. A submodule \( N \) of \( \text{M} \) is said to be small or superfluous in \( \text{M} \) if the only submodule \( L \) of \( \text{M} \) which verifies \( N + L = \text{M} \) is \( \text{M} \) itself. Our notation for this situation is \( N \subseteq \text{M} \). Dual to (1.2) we have:

**PROPOSITION 1.3** Let \( \text{M} \) be a module with submodules \( N, L; N_1, \ldots, N_n; L_1, \ldots, L_n \); and let \( f: \text{M} \to \text{K} \) be any homomorphism. Then
a) if \( N \subseteq L \subseteq \text{M} \) if and only if \( N \subseteq \text{M} \) and \( (L/N) \subseteq (\text{M}/N) \);
b) if \( N \subseteq \text{M} \) and \( L \subseteq \text{M} \) then \( N + L \subseteq \text{M} \);
c) if \( N \subseteq L \) and \( N \subseteq L, \) then \( N \subseteq \text{M} \);
d) if \( N \subseteq \text{M} \) then \( f(N) \subseteq \text{K} \);
e) if \( N_i \subseteq L_i \) for \( i=1, \ldots, n \) and the sum \( \sum L_i \) is direct then \( \cap N_i \subseteq \text{K} \).

There are modules which possess submodules which are at the same time essential and superfluous, for example any nontrivial subgroup of the quasi-cyclic group \( \mathbb{Z}(p^\infty) \) (for any prime integer \( p \)), or the (two-sided) maximal ideal of any local ring which is not a division ring.

On the other hand, some modules \( \text{M} \) have no essential (resp. superfluous) submodules other than \( \text{M} \) (resp. \( \text{O} \)). These are precisely the semisimple modules (resp. the modules with zero radical). This
will follow immediately from Proposition 1.5, but before proving it we need to introduce the concept of relative complement.

**Complement Submodules**

Given a module $M$ and a submodule $N \subseteq M$, the set $\Omega = \{ L \subseteq M : L \cap N = 0 \}$ is clearly inductive and nonempty ($0 \in \Omega$). Any maximal element of $\Omega$ is said to be a *relative complement* for $N$ in $M$. This concept is reminiscent of the set-theoretical concept of complement subset and has an obvious generalization to arbitrary lattices.

Note that, if $N, K \subseteq M$ verify $N \cap K = 0$, then $\Omega' = \{ L \subseteq M : K \subseteq L \text{ and } L \cap N = 0 \}$ is also inductive and nonempty, so that we can take a relative complement for $N$ which contains $K$. The following result implies the remarkable fact, essentially proved by R.E. Johnson [26], that every submodule of a module $M$ is a direct summand of an essential submodule of $M$.

**PROPOSITION 1.4** Let $N \subseteq M$; if $L$ is a relative complement for $N$ in $M$ then $N \oplus L$ is an essential submodule of $M$.

**PROOF:** Since $N \cap L = 0$, the sum $N + L$ is direct. Suppose now that $K \subseteq M$ is such that $(N \oplus L) \cap K = 0$; then $(N \oplus L) + K = N \oplus L \oplus K$, whence $N \cap (L \oplus K) = 0$ and then, by maximality of $L$, we get $L \oplus K = L$, i.e. $K = 0$. ■

**PROPOSITION 1.5** For any module $M$, the socle of $M$ is the intersection of all essential submodules of $M$, and the radical of $M$ is the sum of the superfluous submodules of $M$.

**PROOF:** Write $\text{Soc} M$ and $\text{Rad} M$ for the socle and the radical of $M$, (i.e. the sum of all simple submodules of $M$ and the intersection of all maximal submodules of $M$, respectively); and set

\[ A = \cap \{ N : N \subseteq M \} \quad \text{B} = \Sigma \{ N : N \subseteq M \}. \]

For any simple submodule $S$ of $M$, and for any $N \subseteq M$, we have $0 \neq N \cap S S S$, whence $N \cap S = S$, i.e. $S \subseteq N$; therefore $\text{Soc} M \subseteq A$. On the other hand, let $L \subseteq A$, and let $L'$ be a relative complement for $L$ in $M$; then, by (1.4), $L \oplus L' \subseteq M$ and hence $A \oplus L \subseteq L'$; by modularity, $L \cap (A \cap L') = A \cap (L \cap L') = A$, which proves that every submodule of $A$ is a direct summand of $A$, i.e. $A$ is
semisimple and hence $A = \text{Soc} M$. Therefore $A = \text{Soc} M$, as desired.

For any maximal submodule $L$ of $M$, and for any $N \subseteq M$, we have $NSN + L \subseteq M$, whence $N = N + L$, i.e. $LSN$; therefore $B \subseteq \text{Rad} M$. To see that $\text{Rad} M \subseteq B$, we prove that $xR$ is superfluous for all $x \in \text{Rad} M$; if $xR$ is not superfluous and $N \subseteq M$ is such that $N + xR = M$, then clearly $x \in N$ and thus, by Zorn's Lemma, $\Delta = \{K \subseteq M : N \n S K$ and $x \notin K\}$ has a maximal element $K_0$, which is in turn a maximal submodule of $M$, since

$$\text{K}_0 \n S L \subseteq M \Rightarrow x \in L \Rightarrow M = N + xR \subseteq K_0 + xR \subseteq L \Rightarrow M = L;$$

therefore, since $x \notin K_0$, $x \notin \text{Rad} M$. This completes the proof.

By a complement in $M$ we will mean any submodule $N$ of $M$ which is a relative complement in $M$ for some submodule of $M$. In this case we write $N \subseteq M$. If $M = Rr$ then we call a complement submodule of $Rr$ a right complement in $R$. For example, every direct summand $N$ of $M$ is a complement in $M$ (if $M = N \oplus L$, then $N$ is a relative complement for $L$ in $M$). There exists a close relationship between the concepts of complement and essentiality, as the next result shows (in fact, some authors call complements closed or essentially closed submodules because of the equivalence $a) \Rightarrow b)$).

**Proposition 1.6** Let $N \subseteq M$. Then the following are equivalent:

a) $N$ is a complement in $M$;

b) $N$ does not admit proper essential extensions within $M$;

c) for any $L \subseteq M$ such that $N \subseteq L \subseteq M$, $(L/N) \subseteq (M/N)$.

**Proof:**

a) $\Rightarrow$ b) Assume that $N$ is a relative complement for some $K \subseteq M$, and suppose $N \subseteq L \subseteq M$. Since $(L \cap K) \cap N = K \cap N = 0$, we get $L \cap K = 0$ and then, by maximality of $N$, it must be $N = L$; this proves b).

b) $\Rightarrow$ c) Assume $N \subseteq L \subseteq M$, and suppose that $K$ is such that $N \subseteq K \subseteq M$ and $(L/N) \cap (K/N) = 0$; then $N = L \cap K \subseteq M \cap K = K$ (1.2.b), whence $N = K$ by b), i.e. $K/N = 0$; therefore $(L/N) \subseteq (M/N)$.

c) $\Rightarrow$ a) Let $K$ be a relative complement for $N$ in $M$; we prove that $N$ is a relative complement for $K$ in $M$. Since $N \cap K = 0$, we can find a complement $N'$ for $K$ in $M$ with $N \subseteq N'$; then, by modularity, $(N \cap K) \cap N' = N \cap (K \cap N') = N$;

since $N \subseteq N'$ by (1.4), the hypothesis gives $N = K \subseteq M$, but the previous argument gives $N' \subseteq N$, so that $N = N'$.
COROLLARY 1.7  Let $M_r$ be any module, and let $N$ be a complement in $M_r$. Then, for any $K \subseteq M_r$ such that $N \subseteq K$ and $(K/N) \subseteq (M/N)$, we have $K \subseteq M$.

PROOF: If $K \subseteq L \subseteq M$ then, by (1.6.c), $\frac{K}{N} \subseteq \frac{L}{N},$ whence $\frac{K}{N} \subseteq \frac{L}{N}$, i.e. $K=L$. ■

From (1.6.b), it is clear that, if $N \subseteq M$ and $L \subseteq M_r$ is such that $N \subseteq L$, then $N \subseteq L$. On the other hand, we have the following 'transitive' property of complements.

PROPOSITION 1.8  Let $L \subseteq N$ be submodules of $M_r$ such that $L \subseteq N$ and $N \subseteq M$. Then $L \subseteq M$.

PROOF [10; Theo. 2.2]: By hypothesis, $L$ is a relative complement in $N$ for some $L' \subseteq N$, and $N$ is a relative complement in $M$ for some $N' \subseteq M$. Then $L \cap (L' + N') = 0$ since, for $x \in L$, $y \in L'$ and $z \in N'$, we have $x = y + z \Rightarrow z = x - y$ for $K \subseteq L$, whence $K \subseteq N$, i.e. $K = L$. ■

Given $NSM_r$ we can take first a relative complement $K$ for $N$ in $M$, and then a relative complement $N'$ for $K$ in $M$ containing $N$. $N'$ is then a complement and also an essential extension of $N$. For, if $L \subseteq N'$ verifies $L \cap N = 0$ then we have $K \subseteq L$ and $(K \cap L) \cap N = 0$, since $K + x = 0$ (for $K \subseteq L$, $n \in N$) implies $K = n - x$ for $K \subseteq L$, i.e. $L = 0$, proving the claim. We shall call an e-closure (for essential closure) of $N$ in $M$ every complement in $M$ that is an essential extension of $N$. As we have just seen, e-closures do exist for any submodule of any module $M_r$, and by (1.6) and (1.2.a) the set of e-closures of $N$ in $M$ coincide with $\mathcal{F} = \{L \subseteq M_r : N \subseteq L\}$. The next proposition gives another description or the same set.
PROPOSITION 1.9 Let $N \subseteq M$. Then the e-closures of $N$ in $M$ are the minimal elements of $\mathcal{E} = \{K \subseteq M : K \subseteq cM \text{ and } N \subseteq K\}$. 

PROOF: If $N'$ is an e-closure for $N$ in $M$ then clearly $N' \in \mathcal{E}$; if $K \in \mathcal{E}$ is such that $K \subseteq N'$, then (1.2.a) $K \subseteq cN'$ and, since $K \subseteq cM$, $K = N'$; therefore $N'$ is minimal in $\mathcal{E}$.

On the other hand, if $K$ is a minimal element of $\mathcal{E}$, we have to prove that $N \subseteq K$; for, let $N'$ be an e-closure for $N$ in $K$; then $N' \subseteq M$ (1.8) and therefore $N' \in \mathcal{E}$, whence $K = N'$ and thus $N \subseteq K$. ■

An e-closure for $N$ in $M$ need not be unique: For example, if $R = \mathbb{Z}$ and $M = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, then $N = (0, 2)\mathbb{Z}$ has two e-closures in $M$, namely $(0, 1)\mathbb{Z}$ and $(1, 1)\mathbb{Z}$.

We close this paragraph with an application of the concept of e-closure, which characterizes the essential extensions within a module $M$.

PROPOSITION 1.10 Let $N, L \subseteq M$. Then the following conditions are equivalent

a) $N \subseteq L$;

b) $N$ and $L$ have a common relative complement in $M$;

c) $N$ and $L$ have a common e-closure in $M$.

PROOF: a) $\Rightarrow$ b). Let $L'$ be a relative complement for $L$ in $M$; then $N \cap L' \subseteq L \cap L' = 0$, and if $K \subseteq M$ is such that $L' \subseteq K$ and $N \cap K = 0$ then (1.2.b) $0 = N \cap K \subseteq L \cap K$, i.e. $0 = L \cap K$; by maximality of $L'$, this gives $K = L'$. Therefore $L'$ is also a relative complement for $N$ in $M$.

b) $\Rightarrow$ c). If $K$ is a common relative complement for $N$ and $L$, take a relative complement $K'$ for $K$ containing $L$ (and hence $N$); $K'$ is then the desired common e-closure.

c) $\Rightarrow$ a). This follows immediately from (1.2.a). ■
The concepts of singular ideal of a ring and nonsingular ring were introduced by R.E. Johnson [26; p.894], and extended some years later to modules by himself [27; p.537]. These concepts are closely related to those of essential and complement submodules and have proved to be very helpful in many different areas of Ring Theory, particularly in the study of quotient rings. They will be used frequently throughout this dissertation, and we shall compile here their definition and first properties.

**LEMMA 1.11** Let $M_r$ be any module. The set $Z(M_r) = \{x \in M : r_r(x) \leq RR = R\}$ is a submodule of $M_r$.

**Proof:** First note that, with the help of (1.2.a), it is clear that both sets in the statement of the lemma are actually equal. Since $r_r(0) = R \leq RR$, we have $0 \in Z(M_r)$. Let $x, y \in Z(M_r)$ and write $a = r_r(x)$, $b = r_r(y)$; then $(x-y)(a \cap b) = 0$ and $a \cap b \leq RR$, whence $x-y \in Z(M_r)$. Finally, if $r \in R$ then, by (1.1), $(a:r) \leq RR$ and $xr(a:r) \leq xa = 0$, whence $x \in Z(M_r)$. Therefore $Z(M_r)$ is a submodule of $M_r$. ■

$Z(M_r)$ is called the **singular submodule** of $M_r$. The module $M_r$ is called **nonsingular** (resp. **singular**) if $Z(M_r) = 0$ (resp. $Z(M_r) = M$). The **right singular ideal** of a ring $R$ is $Z_r(R) = Z(Rr)$, and $R$ is a **right nonsingular ring** if $Z_r(R) = 0$. The left singular ideal and left nonsingular rings are defined similarly. Since all our rings have an identity, it is easy to show that $Z_r(R) \neq R$ for any ring $R$ in which $1 \neq 0$, i.e. there do not exist 'singular rings'.

**PROPOSITION 1.12** Let $M_r$ be any module and let $N \leq M_r$. Then

a) $Z(Nr) = NrZ(Mr)$;

b) $Z(Mr)$ is a singular module;

c) if $M_r$ is singular (resp. nonsingular), then so is $Nr$;

d) if $Nr$ is nonsingular and $N \leq M$, then $M_r$ is nonsingular;

e) if $NS \leq M$ then $M/N$ is singular;

f) if $M/N$ is nonsingular then $NS \leq M$.

**Proof:** a) is clear from the definition.
b) follows by applying a) to the case $N=Z(M_R)$.

c) follows directly from a).

d) Since $N_R$ is nonsingular, $0=Z(N_R)=N\cap Z(M_R)$, and since $N\subseteq M$ this implies $Z(M_R)=0$, i.e., $M_R$ is nonsingular.

e) For any $x\in M$, $(N:x)$ is an essential right ideal of $R$ such that $x(N:x)\subseteq N$, i.e. $(x+N)(N:x)=0$ in $M/N$, and therefore $x+N\in Z(M/N)$, whence $Z(M/N)=M/N$.

f) If $M/N$ is nonsingular and $N\subseteq K\subseteq M$, then $K/N$ is singular by e) and nonsingular by c), since $K/N\subseteq M/N$. But obviously the only module which is both singular and nonsingular is the zero module, so that $K/N=0$, i.e. $N=K$, and hence $N\subseteq M$.

**PROPOSITION 1.13** Let $R$ be any ring and let $M_R$ be a nonsingular module. Then

a) for any right ideal $a$ of $R$, $a\subseteq R_R \Leftrightarrow R/a$ is singular as a right $R$-module; in this case $aM_R \subseteq M$;

b) for any $N\subseteq M_R$, $N\subseteq M \Leftrightarrow M/N$ is singular;

c) for any $N\subseteq M$, $N\subseteq M \Leftrightarrow M/N$ is nonsingular.

**PROOF:**

a) If $a\subseteq R_R$ and $(R/a)_R$ is singular, then there exists $b\subseteq R_R$ such that $(1+a)b=0$, i.e. $b\subseteq a$, whence $a\subseteq R_R$; the converse follows from (1.12. e); if $a\subseteq R_R$ then for all $0\neq x\in M$ we have, by nonsingularity of $M_R$, $0\neq xa$; thus there exists $r\in a$ such that $0\neq xreM$, and therefore $aM_R \subseteq M$.

b) Assume that $M_R$ is nonsingular and $M/N$ is singular; then, for all $x\in M\setminus N$, $(N:x)=R \Leftrightarrow (x+N)$ is an essential right ideal of $R$, and hence $x(N:x)\neq 0$; thus there exists $r\in R$ such that $0\neq xreN$, and therefore $N\subseteq M$. The converse is (1.12.e).

c) Assume that $M_R$ is nonsingular and $N\subseteq M$; let $K$ be such that $N\subseteq K\subseteq M$ and $K/N=Z(M/N)$; then, since $K$ is nonsingular and $K/N$ is singular, b) gives $N\subseteq K$ and hence $N=K$, i.e. $Z(M/N)=K/N=0$. The converse is (1.12.f).
SECTION 2: THE UNIFORM DIMENSION OF A MODULE

A module $M_r$ is called finite dimensional (abbreviated f.d.) if all direct sums of nonzero submodules of $M$ have a finite number of summands. Thus e.g. all Artinian or Noetherian modules are f.d. We shall show that, if $M$ is f.d., there is a least upper bound for the set $D(M) = \{ n \in \mathbb{Z} : \text{there is a direct sum of nonzero submodules of } M \text{ with } n \text{ summands} \}$. This fact will allow us to define a 'dimension' for finite dimensional modules which generalize the concept of dimension of a vector space.

Both concepts, finite dimensional modules and the dimension of a module, were introduced by A.W. Goldie. In [18] he concerned himself with ideals of a ring, but most of the proofs given there go through with minor changes when extending these concepts to modules, as Goldie did in [19].

Next, we give a first characterization of finite dimensional modules which sharpen our observation that either chain condition implies finite dimensionality.

**PROPOSITION 2.1** For any module $M_r$, the following are equivalent:

a) $M_r$ is finite dimensional;

b) $M_r$ satisfies the ACC on complement submodules;

c) $M_r$ satisfies the DCC on complement submodules.

**PROOF:**
a)$\Rightarrow$b) If $N_1 \supset N_2 \supset \cdots$ is an infinite chain of complements in $M$, then we construct an infinite direct sum of nonzero submodules of $M$ as follows: since each extension $N_i \supset N_{i+1}$ is not essential, choose $0 \neq L_i \subseteq N_{i+1}$ such that $L_i \cap N_i = 0$; then $N_1 \oplus L_1 \oplus L_2 \oplus \cdots$ is the announced sum.

b)$\Rightarrow$c) If $N_1 \supset N_2 \supset \cdots$ is an infinite chain of complements in $M$, then set $L_0 = 0$ and let $L_i$ (i=1,2,...) be a relative complement for $N_i$ in $M$ containing $L_{i-1}$; then $L_1 \subseteq L_2 \subseteq \cdots$ is an infinite ascending chain of complements in $M$ in which the inclusions are strict by (1.10).

c)$\Rightarrow$a). If $M$ contains an infinite direct sum $N_1 \oplus N_2 \oplus \cdots$ of nonzero submodules and $K_i$ is an e-closure for $N_i \oplus N_{i+1} \oplus \cdots$ in $K_{i-1}$ (where
K_0 \cong M), then K_1 \supset K_2 \supset \cdots \text{ is an infinite chain of complements in } M \text{ by } (1.8) \text{ and } (1.10). \text{■}

Clearly, every submodule of a f.d. module is f.d. Some other properties of stability for f.d. modules are listed below.

**PROPOSITION 2.2** Let M be any module and let N, N_1, \ldots, N_r be submodules of M. Then

a) if N is f.d. and N \subseteq M, then M is f.d.;

b) if M is f.d. and N \subseteq M, then M/N is f.d.;

c) if N and M/N are both f.d., then M is f.d.;

d) if each N_i is f.d. and the sum \sum N_i is direct, then \oplus N_i is f.d.

**PROOF:**

a) Clearly, a direct sum \bigoplus_{i \in I} M_i of nonzero submodules of M provides a direct sum \bigoplus_{i \in I} N_i of nonzero submodules of N, where N_i = N \cap M_i; therefore the index set I must be finite and hence M is f.d.

b) By (1.7), an infinite strictly ascending chain of complements in M/N would provide an infinite strictly ascending chain of complements in M, which is impossible by (2.1); thus, also by (2.1), M/N is f.d.

c) Assume that N and M/N are both f.d., and let M_1 \oplus M_2 \oplus \cdots be an infinite direct sum of submodules of M; set T_k = M_k \oplus M_{k+1} \oplus \cdots; we claim that N \cap T_k = 0 for some k.

Suppose not; since N \cap T_1 = 0, there exists r_1 \geq 1 such that N_1 = N \cap (M_1 \oplus \cdots \oplus M_{r_1}) \neq 0; but also N \cap T_{r_1+1} \neq 0 and hence we have, for some r_2 > r_1, N_2 = N \cap (M_{r_1+1} \oplus \cdots \oplus M_{r_2}) \neq 0. In this way, we produce an infinite independent set \{N_1, N_2, \ldots\} of nonzero submodules of N, which contradicts the hypothesis and hence proves the claim.

Let now p : M \to M/N; since \ker p \cap T_k = N \cap T_k = 0, we may view T_k as a submodule of the f.d. module M/N, and then all but a finite number of the M_r (for r \geq k) must be zero, proving that M is f.d.

d) By induction: If r=1 then there is nothing to prove; if r>1 then N_i and (\oplus N_i)/N_i \cong \bigoplus N_i are f.d., and hence so is \bigoplus N_i by c.).

Note that (2.2.b) fails if N is not a complement in M. For example, consider \mathbb{Q} as a \mathbb{Z}-module; since any two nonzero elements of \mathbb{Q} have a common nonzero multiple, any two nonzero submodules of \mathbb{Q} have nonzero
intersection and then $Q$ is finite dimensional; but $Q/Z$ may be expressed as the direct sum of all its $p$-primary components, which are infinitely many and all nonzero, and thus it is not finite dimensional.

The stated property about the submodules of $Q/Z$ is of interest in itself, and will be key in order to define the dimension of a finite dimensional module. A module $U$ is uniform if $U \neq 0$ and every two nonzero submodules of $U$ have nonzero intersection; or, equivalently, if $U \neq 0$ and every nonzero submodule of $U$ is essential in $U$. Note that an essential extension of a uniform module is uniform. For a uniform module $U$ it is clear that the least upper bound of $D(U)$ is 1.

**Lemma 2.3** Every nonzero finite dimensional module $M$ contains a uniform submodule.

**Proof** ([18; Lemma 1.2]): Suppose not. Then $M$ itself is not uniform and so there exist $0 \neq M_1, L_1 \leq M$ with $M_1 \cap L_1 = 0$. But again $L_1$ is not uniform, and we can find $0 \neq M_2, L_2 \leq L_1$ with $M_2 \cap L_2 = 0$. This process leads to an infinite direct sum $M_1 \oplus M_2 \oplus \cdots$ of nonzero submodules of $M$, a contradiction. 

**Proposition 2.4** If $0 \neq M$ is finite dimensional, then there exist uniform submodules $U_1, \cdots, U_n$ of $M$ such that the sum $U_1 + \cdots + U_n$ is direct and $U_1 \oplus \cdots \oplus U_n \subseteq M$.

**Proof:** Let $U_i$ be a uniform submodule of $M$, and let $K_i$ be a complement for $U_i$ in $M$. If $U_i$ is not essential in $M$ then $K_i \neq 0$ and $K_i$ is f.d., so that $K_i$ contains a uniform submodule $U_2$. If $U_i \oplus U_2$ is not essential in $M$ then it has a nonzero complement $K_3$ which contains a uniform submodule $U_3$ with $(U_i \oplus U_2) \cap U_3 \subseteq (U_i \oplus U_2) \cap K_2 = 0$. Since $M$ is f.d., this process must stop at some step $n$, and then $U_1, \cdots, U_n$ have the desired property.

**Theorem 2.5** Let $M$ be a module, and suppose that $M$ contains an essential submodule of the form $U_1 \oplus \cdots \oplus U_n$ where the $U_i$'s are uniform. Then any direct sum of nonzero submodules of $M$ has at most $n$ summands.
PROOF [18; Theo. 6]: Let $V_1, \ldots, V_k$ be an independent family of nonzero submodules of $M$ and suppose $k > n$. Assume also $n \geq 2$ (if $n = 1$ then $M$ is uniform or zero, and thus the result is clear).

If $N \subseteq M$ is not essential in $M$, then $N$ has zero intersection with some $U_i$. For, suppose $N \cap U_i \neq 0$ ($i = 1, \ldots, n$); then $N \cap U_i \subseteq U_i$, whence

$$\text{Hom}(N \cap U_i) \subseteq \text{Hom}(U_i) \subseteq M$$

by (1.2.e). Then, since $\text{Hom}(N \cap U_i) \subseteq N$, we have $N \subseteq M$ (1.2.a), a contradiction.

Let $\hat{V}_1 = V_2 \oplus \cdots \oplus V_k$; $\hat{V}_1$ is not essential in $M$, and so we can assume e.g. $\hat{V}_1 \cap U_i = 0$. Therefore the sum $U_1 \oplus V_2 \oplus \cdots \oplus V_k$ is direct.

Let $\hat{V}_2 = U_1 \oplus V_2 \oplus \cdots \oplus V_k$ which, as above, has zero intersection with some $U_i$, and not actually with $U_i$. Assume then $\hat{V}_2 \cap U_2 = 0$; therefore the sum $U_1 \oplus U_2 \oplus V_3 \oplus \cdots \oplus V_k$ is direct.

Since $n < k$ we can, by repeating this argument, give rise to a direct sum $(U_1 \oplus \cdots \oplus U_n) \oplus (V_{n+1} \oplus \cdots \oplus V_k)$ with the second parenthesis nonzero, but this contradicts the essentiality of $U_1 \oplus \cdots \oplus U_n$. Therefore it must be that $k \leq n$, as desired.■

COROLLARY 2.6 A module $M$ is finite dimensional if and only if it contains a finite direct sum of uniform submodules which is an essential submodule of $M$. In this case, the number of summands in such a sum is an invariant $n$ of $M$ which equals the least upper bound of $D(M) = \{k \in \mathbb{Z} : M \text{ contains } k \text{ independent nonzero submodules}\}$.

PROOF: The first statement follows immediately from (2.4) and (2.5). If $U_1 \oplus \cdots \oplus U_n$ and $V_1 \oplus \cdots \oplus V_k$ are essential submodules of $M$ with each $U_i$ and each $V_i$ uniform, then apply (2.5) twice to obtain $n \leq k$ and $k \leq n$, whence $n = k$. A new application of the previous theorem proves the last statement.■

If $M$ is finite dimensional then the integer $n$ of Corollary 2.6 is called the uniform or Goldie dimension of $M$. Our notation will be $u(M) = n$. If $M$ is not finite dimensional then we write $u(M) = \infty$. A ring $R$ is said to be right (left) finite dimensional if so is the regular module $R_R$ ($R_R$).

For example, a semisimple module $M$ is f.d. if and only if it is
finitely generated, if and only if it is of finite length, and in this case \( u(M) = \text{length}(M) \), the composition length of \( M \). In particular, this shows that our concept of dimension generalizes the usual one for vector spaces.

In (2.1) we showed that the finite dimensionality of \( M_r \) depends on the set of complement submodules of \( M_r \); in fact, also the dimension of \( M_r \) may be described in terms of its chains of complements. In the next proposition, for an strict chain \( K_0 < K_1 < \cdots < K_n \) of complements in \( M \), call \( n \) the length of the chain.

**Proposition 2.7**  
Let \( M_r \) be a finite dimensional module, and let \( n = u(M_r) \). Then \( n \) is the maximum of the lengths of all chains of complements in \( M \). Moreover, a chain \( 0 = K_0 < K_1 < \cdots < K_r = M \) of complements in \( M \) has length \( r = n \) if and only if \( (K_{i+1}/K_i) \) is uniform for \( i = 0, \ldots, n-1 \).

**Proof** [19; Lemma 1.4]: The construction methods used in (2.1) prove the first part: if \( K_0 < K_1 < \cdots < K_r \) is a chain of complements, then we get a direct sum of \( r \) nonzero submodules of \( M_r \) as in 'a)\( \Rightarrow \)b)'; also, if \( M_0 < \cdots < M_r \) is a direct sum of nonzero submodules, then we get a chain of complements with \( r \) strict inclusions as in 'c)\( \Rightarrow \)a)'.

Now, suppose that \( 0 = K_0 < K_1 < \cdots < K_n = M \) is a chain of complements and fix an \( i \in \{0, \ldots, n-1\} \). By the first part, we cannot insert any complement between \( K_i \) and \( K_{i+1} \); hence, for any \( N \in M_r \), with \( K_i < N \), \( K_{i+1} \) must be an \( e \)-closure of \( N \) by (1.9) and thus, by (1.6.c), \( (N/K_i)e(K_{i+1}/K_i) \); therefore \( K_{i+1}/K_i \) is uniform.

Conversely, let \( 0 = K_0 < K_1 < \cdots < K_r = M \) be a chain of complements with each \( K_{i+1}/K_i \) uniform. For \( i = 2, \ldots, r \) let \( L_i(\neq 0) \) be a relative complement of \( K_{i-1} \) in \( K_i \); then clearly the sum \( K_0 \oplus L_2 \oplus \cdots \oplus L_r \) is direct; moreover, since each \( L_i \) embeds in \( K_i/K_{i-1} \), all the summands are uniform; also, we have \( K_{i-1} \oplus L_i \subseteq K_i \) (1.4); then, repeated applications of (1.2.e) give

\[
K_0 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r \subseteq K_2 \oplus L_3 \oplus \cdots \oplus L_r \subseteq \cdots \subseteq K_{r-1} \oplus L_r \subseteq K_r = M,
\]

and hence, by (2.6), \( n = r \).

As we have already remarked, Artinian modules are f.d.; more can be said in this case since, if \( M_r \) is Artinian and \( U_1, \ldots, U_n \) are uniform submodules of \( M \) with \( \oplus U_i \subseteq M \), then we can find a simple submodule \( S \)
inside each $U_i$, and we get $\oplus S_i \subseteq M$, whence $\text{Soc}(M) \subseteq M$ and then $u(M) = u(\text{Soc}M) = \text{length}(\text{Soc}M)$, as a consequence of part a) of the next result.

**PROPOSITION 2.8** Let $M$ be any module, and let $N, N_1, \ldots, N_r$ be submodules of $M$. Then

a) if $N \subseteq M$ then $u(N) = u(M)$; if $M$ is f.d. then the converse holds;

b) if $N \subseteq M$ then $u(M) = u(N) + u(M/N)$;

c) if $K$ is an e-closure for $N$ in $M$ then $u(M) + u(K/N) = u(N) + u(M/N)$;

d) if $N_1, \ldots, N_r$ are independent then $u(\oplus N_i) = \sum u(N_i)$.

**Proof:**

a) If $u(N) = \infty$ then also $u(M) = \infty$; if $u(N) = n$ and $N \subseteq M$, then any direct sum $\bigoplus U_i$ of uniform submodules of $N$ which is essential in $N$ is also essential in $M$, whence $u(M) = n$. If $u(M) = u(N) = n$ and $\oplus U_i$ is as before, then $\oplus U_i$ must be essential in $M$, because otherwise it could be extended to a direct sum with more than $n$ terms, so that $N \subseteq M$ (1.2.a).

b) Note first that, if some term is not finite, then the formula holds by (2.2.b) and (2.2.c). Suppose then they are all finite; let $0 = N \subseteq N_1 \subseteq \cdots \subseteq N_r = N$ be a chain of complements in $N$ (and hence in $M$) with each $N_{i+1}/N_i$ uniform; and let $0 = (K_0/N) \subseteq (K_1/N) \subseteq \cdots \subseteq (K_s/N) = (M/N)$ be a chain of complements in $M/N$ (whence each $K_i \subseteq M$ by (1.7)) with each $(K_{i+1}/N)/(K_i/N)$ uniform (and then so is $K_{i+1}/K_i$). Thus

$$0 = N \subseteq N_1 \subseteq \cdots \subseteq N_r = N \subseteq K_0 \subseteq K_1 \subseteq \cdots \subseteq K_s = M$$

is a chain of complements in $M$ with each factor uniform, so that, by (2.7), $u(M) = r + s = u(N) + u(M/N)$.

c) As in b), if some summand is not finite, then the formula holds. Assume then they are all finite; by (1.2.c), $(K/N) \subseteq (M/N)$ and thus, applying b) twice, we get

$$u(M) = u(K) + u(M/K) = u(K) + u(M/N) - u(K/N),$$

and since $u(N) = u(K)$, the result follows.

d) This follows easily by induction from b) (recall that every direct summand of $M$ is a complement in $M$).
This section start with a proposition which shows how the injectivity of a module depends on its essential extensions; this is one of the ways in which the concept of injective hull of a module appears naturally (as a maximal essential extension of the module). The fact that every submodule of an injective module $E$ admits an injective hull within $E$ will be used to characterize the finite dimensional injective modules. This characterization will be used later when studying the endomorphism ring of $E$. At the end of the section, we define quasi-injective modules and prove some results which will be used later.

In this paragraph we shall change slightly our concept of essential extension. By a (proper) essential extension of $M$ we shall henceforth mean a monomorphism $f$ from $M$ to any module $N$ such that $(f(M)\neq N$ and) $f(M)\subseteq N$.

Recall that an injective module is a direct summand of any module containing it (in fact this is also a sufficient condition for the module to be injective). Next, we give two more characterizations of injective modules.

**Proposition 3.1** A module $E$ is injective if and only if it does not admit proper essential extensions.

**Proof (12; 4.2):** If $E$ is injective and $f:E\to N$ is an essential extension of $E$, then $f(E)=N$ is injective and hence a direct summand of $N$, but since $fE\subseteq N$ this implies $fN$ and therefore $f$ is not proper. Conversely, if $E$ is not injective, then there exists a module $L$ containing $E$ such that $E$ is not a direct summand of $L$; for a relative complement $E'$ of $E$ in $L$, $E\oplus E'$ is a proper essential submodule of $L$ by (1.4), and hence $E\oplus E'\subseteq L$ by (1.6.c); since $E\cap E'=0$, the natural map
**f:E→L→L/E'** is a monomorphism with image \( \frac{E\oplus E'}{E'} \), and hence a proper essential extension of \( E \).

Using (1.6), we get at once:

**COROLLARY 3.2** A module is injective if and only if it is a complement submodule in any module containing it.

The following lemma states that an injective module \( E_r \) containing \( M_r \) also contains an isomorphic copy of each essential extension of \( M_r \), so that \( E_r \) may be viewed as an 'upper bound' for the essential extensions of \( M_r \).

**LEMMA 3.3** Let \( E_r \) be injective and let \( f:M→E \) be a monomorphism. For any essential extension \( g:M_r→N_r \) of \( M_r \) there exists a monomorphism \( h:N→E \) such that \( f=hg \).

**Proof:** By injectivity of \( E \), there exists \( h:N→E_r \) with \( f=hg \), and we have \( gM\subseteq N \) and \( gM\cap\ker h=0 \) (since \( \ker f=0 \)), so that \( \ker h=0 \).

Let \( M\subseteq N_r \); if \( M \) is injective then \( M\subseteq N \), and if \( M\subseteq N \) then \( M\subseteq N_r \). When \( N_r \) is injective we get both converses.

**PROPOSITION 3.4** Let \( E_r \) be injective. For any \( M\subseteq E_r \) the following are equivalent:

a) \( M \) is injective;

b) \( M \) is a direct summand of \( E \);

c) \( M \) is a complement in \( E \).

**Proof:** We need to prove c)⇒a). Assume c), let \( g:M→N \) be an essential extension of \( M_r \) and consider the inclusion \( u:M→E \); by (3.3) there exists a monomorphism \( h:N→E \) with \( u=gh \); then \( h:N→hN \) is an isomorphism which carries \( gM \) onto \( hgM=uhM=M \), whence \( M\subseteq hNSE \) (1.2.d); by assumption, we get \( M=\ker h \) and hence \( gM=\ker h=\ker h^{-1}M=N \). Thus \( M_r \) does not admit proper essential extensions and then it is injective by (3.1).

Recall that any module is a submodule of an injective module [A-F; Prop.18.6]. We are now ready to prove the existence of minimal
injective extensions and of maximal essential extensions for any module, and also to show that both coincide. The first of these facts was essentially proved by R. Baer [3], and the rest is due to B. Eckmann and A. Schopf [12].

**THEOREM 3.5** Given any module $M_r$, there exists a module $E_r$ containing $M$ such that

a) $M \subseteq E$ and, for any essential extension $g: M \rightarrow N$, there exists a monomorphism $h: N \rightarrow E$ such that $hg$ is the inclusion map;

b) $E$ is injective and any monomorphism $f: M \rightarrow E'$ with $E'$ injective extends to a monomorphism $h: E \rightarrow E'$.

**PROOF** [12; 4.1.4 & 4.3]: Let $F_r$ be an injective module containing $M$, and let $E$ be an $e$-closure of $M$ in $F$. Then $E_r$ is injective by (3.4) and $M \subseteq E$, so that we already have the first parts of a) and b).

Since $E$ is injective, the second part of a) follows by taking $f$ in (3.3) as the inclusion map. Also the second part of b) follows from (3.3), taking $g$ as the inclusion $M \subseteq E$.

A module $E_r$ satisfying the conditions of (3.5) is called an **injective hull** for $M_r$. The injective hull of a module is not unique; in fact, (3.4) and the proof of (3.5) show that, if $M \subseteq F_r$ and $F_r$ is injective, the injective hulls of $M$ inside $F$ coincide with the $e$-closures of $M$ in $F$. However, we have the following unicity theorem, which will allow us to speak about 'the' injective hull of $M_r$ when any of the (isomorphic) injective hulls of $M_r$ serves our purposes.

**THEOREM 3.6** If $E$ and $E'$ are injective hulls of $M_r$, then there exists an isomorphism $f: E \rightarrow E'$ which is the identity over $M$.

**PROOF:** Since $E_r$ is injective, the inclusion $M \subseteq E$ extends to some $f: E \rightarrow E'$ which, by (3.3), is a monomorphism. Moreover, $M = fM \subseteq fE \subseteq E'$ together with $M \subseteq E'$ imply $fE \subseteq E'$, whence $f$ is an essential extension of $E$ and hence, by (3.1), an isomorphism.
Finite Dimensional Injective Modules

Finite dimensional injective modules admit a well behaved decomposition theory, which in turn serves to characterize all finite dimensional modules, as follows.

**Proposition 3.7** A nonzero module is uniform if and only if its injective hull is indecomposable.

*Proof:* Let $0 \neq M$ be any module, $E$ an injective hull for $M$. Since $M \subseteq E$, if $M$ is uniform then so is $E$, and every uniform module is indecomposable. Conversely, if $M$ is not uniform and $0 \neq K, L \subseteq M$ are such that $L \cap K = 0$, then, taking e-closures $L'$ and $K'$ for $L$ and $K$ in $E$, we know that $L'$ and $K'$ are injective (3.4) and that their sum is direct (1.2.e), so that $L' \oplus K'$ is injective and hence a direct summand of $E$; therefore $E$ is not indecomposable.

**Proposition 3.8** Let $M$ be a module which has a finite decomposition $M = \bigoplus_{i=1}^{n} M_i$ and let $E$ be an injective hull of $M$. Also, for each $i=1, \ldots, n$, let $E_i$ be an injective hull of $M_i$ within $E$; then $E = \bigoplus E_i$.

*Proof:* By (1.2.e), the sum $\sum E_i$ is direct, and hence it is an essential injective submodule of $E$; then (3.1) gives the result.

**Theorem 3.9** A module $M$ is finite dimensional if and only if its injective hull $E$ is a direct sum of finitely many nonzero indecomposable modules $E_1, \ldots, E_n$. In this case $u(M) = n$.

*Proof:* If $u(M) = n$ then there exist uniform submodules $U_1, \ldots, U_n$ of $M$ with $\bigoplus U_i \subseteq M$ (2.6). If $E_i$ is an $e$-closure of $U_i$ in $E$ (for $i=1, \ldots, n$) then $E = \bigoplus E_i$ by (3.8), and each $E_i$ is indecomposable by (3.7). The converse follows from (3.7), (2.6) and (2.8.a).

Quasi-Injective Modules

Quasi-injective modules are a generalization of injective modules introduced by R.E. Johnson and E.T. Wong [65; p.260]. Their endomorphism
rings have some nice properties that we shall study in Section 7. Here, we introduce them and prove their first properties.

A module \( M \) is said to be quasi-injective if, for every submodule \( N \) of \( M \) and every homomorphism \( f: M_R \to M \), there exists \( g: M_R \to M \) such that \( g|_N = f \). Obviously, every injective and every semisimple module is quasi-injective. Also, by Baer's Criterion [A-F; p.205], the regular module \( R_R \) (for any ring \( R \)) is injective if and only if it is quasi-injective. In this case \( R \) is called a right self-injective ring.

The following result characterizes quasi-injective modules in terms of their relationship with their injective hull.

**Proposition 3.10** Let \( M_R \) be a module and let \( E_R \) be an injective hull for \( M_R \). Then \( M_R \) is quasi-injective if and only if, for every endomorphism \( f \) of \( E_R \), \( fM \subseteq M \).

**Proof** [65; Theo.1.11]: Assume that \( M_R \) is quasi-injective and let \( f: E_R \to E_R \) be an endomorphism. Then \( N = Mnf^{-1}M \) is a submodule of \( M_R \) such that \( fN \subseteq M \); thus there exists \( g: M_R \to M_R \) with \( g|_N = f|_N \). Let \( u: M_R \to E_R \) be the inclusion map; then, by injectivity of \( E_R \), \( ug \) extends to some \( h: E_R \to E_R \), for which we have \( hM = huM = ugM = gM \subseteq M \), and hence \( M \n (h-f)^{-1}Mf^{-1}M \); on the other hand, since \( h \), \( g \) and \( f \) coincide over \( N \), we have \( N \subseteq \text{Ker}(h-f) \) and therefore

\[
M \n (h-f)^{-1}M \subseteq Mnf^{-1}M = N \subseteq \text{Ker}(h-f),
\]

whence \( (h-f)M = 0 \). Since \( M \subseteq E \), this implies \( (h-f)M = 0 \) and hence \( h \) and \( f \) coincide over \( M \), whence \( fM = hM \subseteq M \).

Conversely, if \( fM \subseteq M \) for all \( f: E_R \to E_R \) and we are given a submodule \( N \subseteq M \) and a homomorphism \( g: N_R \to M_R \) then, by injectivity of \( E_R \), \( ug \) extends to some \( h: E_R \to E_R \) for which \( huM = hM \subseteq M \), i.e. \( hu \) is an endomorphism of \( M_R \) which extends \( g \), and therefore \( M_R \) is quasi-injective.

**Corollary 3.11** If \( M_R \) is quasi-injective and \( E_R \) is an injective hull of \( M_R \), then any decomposition \( E = \bigoplus E_i \) yields a decomposition \( M = \bigoplus (M \cap E_i) \).

**Proof:** For each \( j \in I \), let \( f_j : \bigoplus E_i \to E_j \); since \( f_j \) may be viewed as an endomorphism of \( E_R \), \( f_j \subseteq M \). Thus, if \( x \in M \) is expressed in \( E = \bigoplus E_i \) as \( x = \sum_{K} x_k \) for some finite subset \( K \) of \( I \) and some \( x_k \in E_k \), then each \( x_k = f_{i_k} x_k \in M \cap E_k \); hence \( M \subseteq (M \cap E_i) \). That the sum is direct and contained in \( M \) is clear.
COROLLARY 3.12 Let Mr be quasi-injective with finite dimension n. Then M is the direct sum of n uniform submodules.

PROOF: If an injective hull Er of M is written as $E = \bigoplus_{i=1}^{n} E_i$ with each $E_i$ uniform (3.9), then $M = \bigoplus_{i=1}^{n} (M \cap E_i)$ with each $M \cap E_i$ uniform.

Analogously to (3.4), we have:

PROPOSITION 3.13 Let Mr be quasi-injective. Then every complement submodule of Mr is a direct summand of Mr, and every direct summand of Mr is quasi-injective.

PROOF [38; Prop. 4.3]: Let $K \subseteq M$; let $E_r$ be an injective hull of $M_r$ and let $F$ be an $e$-closure of $K$ in $E$ (hence an injective hull of $K$); since $K \subseteq M \cap F \subseteq M$, i.e. $K = M \cap F$; if $G$ is such that $E = F \oplus G$ then we get (3.11) $M = (M \cap F) \oplus (M \cap G) = K \oplus (M \cap G)$, whence $K \subseteq M$.

Now, suppose that $M = N \oplus K$; let $E, F$ be as above, and let $G$ be an $e$-closure of $N$, so that $E = F \oplus G$. Let $h : G \to G$ be any endomorphism; if $u : G \to E$ and $p : E \to G$ are the canonical injection and projection of $E = F \oplus G$, then $uhp$ is an endomorphism of $E_r$ and therefore $uhpM \subseteq M$ by (3.10); hence $hN = hpN = uhpN \subseteq uhpM \subseteq M$ and $hNG \subseteq M$ imply, by the modular law, $hNG = (N \cap G) \cap G = N + (K \cap G) = N$, i.e. $hNG$ for all endomorphism $h$ of the injective hull $G$ of $N$, and therefore $N$ is quasi-injective.
Section 4: The Dimension Formula

Although the uniform dimension of a module is a generalization of the dimension of a vector space, in general the formula

\[ u(A+B) = u(A) + u(B) - u(A \cap B), \]

that we shall call the dimension formula (for \( A+B \)) does not hold for submodules \( A \) and \( B \) of an arbitrary module \( M_r \). Moreover, taking \( A,B \) of dimension 1 (i.e. uniform), \( u(A+B) \) may be any positive integer \( k \) or even \( \omega \). The following are two easy examples, one of each case:

**Example 1:** Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \) where \( n \) is a product of \( k-1 \) distinct primes (with \( n=1 \) if \( k=1 \)), and let \( A = (1,0) \mathbb{Z} \), \( B = (1,1+n\mathbb{Z}) \mathbb{Z} \). Then \( A \cap B = \mathbb{Z} \), whence \( u(A) = u(B) = 1 \), but \( u(A+B) = u(M) = k \).

**Example 2:** Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Q} \oplus (\mathbb{Q}/\mathbb{Z}) \). Let \( A = \{(q,0) : q \in \mathbb{Q} \} \) and \( B = \{(q,q+\mathbb{Z}) : q \in \mathbb{Q} \} \). Now \( A \cap B = \mathbb{Q} \) and \( M = A+B \), whence \( u(A) = u(B) = 1 \) and \( u(M) = \omega \) (\( \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}_p \omega \), where the sum runs over all prime integers).

Our purpose in this Section, which contains several results of the author [58], is to impose conditions on a module under which the dimension formula holds for any pair of submodules, as well as to give some alternative general formulae for \( u(A+B) \). Another task arises from Example 2: that of characterizing the modules \( M_r \) such that the sum of any two finite dimensional submodules of \( M_r \) is still finite dimensional; we shall give a partial answer to this in the last part of the section.

A Characterization of Modules which Satisfy the Dimension Formula

We shall say that a module \( M_r \) satisfies the dimension formula if any sum of two submodules of \( M_r \) does. It is obvious that the dimension formula for \( A+B \) holds if either \( u(A) = \omega \) or \( u(B) = \omega \), so that \( M_r \) satisfies the dimension formula if any sum of two f.d. submodules does. Here, we shall prove that this is equivalent to the apparently weaker condition
that any sum of two (f.d.) complements in \( M_r \) satisfies the dimension formula, and this happens when and only when any finite dimensional submodule of \( M_r \) has a unique e-closure.

Modules with the property that all their submodules (not only the f.d. ones) have a unique e-closure were studied by G. Renault in his doctoral thesis [R67; p.42], where they are shown to be exactly those modules such that the intersection of any two complement submodules is again a complement submodule. This will be proved in one of the following preparatory lemmas for our Theorem 4.4.

**Lemma 4.1** For a module \( M_r \), the following statements are equivalent:

a) \( M_r \) satisfies the dimension formula;

b) if \( A \subseteq A' \), \( B \subseteq B' \) are f.d. submodules of \( M_r \) then \( A + B \subseteq A' + B' \).

**Proof:**

a) \( \Rightarrow \) b) By (1.2.b) we get \( A \cap B \subseteq A' \cap B' \), and hence, using (2.8.a),

\[
u(A' + B') = \nu(A') + \nu(B') - \nu(A' \cap B') = \nu(A) + \nu(B) - \nu(A \cap B) = \nu(A + B)
\]

and all terms are finite, whence \( A + B \subseteq A' + B' \).

b) \( \Rightarrow \) a) Assume b) and note first that if \( N \) and \( K \) are complements in \( M \), then so is \( N \cap K \). For, let \( N \cap K \subseteq L \subseteq M \); then, by b), \( N + (N \cap K) \subseteq N + L \), i.e. \( N \subseteq N + L \), whence \( N = N + L \), i.e. \( L = N \); similarly \( L \subseteq K \) and thus \( N \cap K = L \). Therefore \( N \cap K \) is a complement in \( M \).

Now, let \( A \) and \( B \) be arbitrary finite dimensional submodules of \( M_r \), and take e-closures \( A' \) and \( B' \) for them in \( M \); then we get \( A \cap B \subseteq A' \cap B' \), \( A' \subseteq A' + B' \) and \( A' \cap B' \subseteq M \), whence \( A' \cap B' \subseteq B' \); thus, by (2.8.a & b)

\[
u(A + B) = \nu(A' + B') = \nu(A') + \nu((A' + B') \cap A') = \nu(A') + \nu(B'/(A' \cap B'))
\]

\[
= \nu(A') + \nu(B') - \nu(A' \cap B') = \nu(A) + \nu(B) - \nu(A \cap B).
\]

Therefore, by the remark preceding this lemma, \( M_r \) satisfies the dimension formula.

We use now (4.1) to show that the dimension formula holds in all nonsingular modules (more proofs of this fact will come later).

**Corollary 4.2** If \( M_r \) is nonsingular then it satisfies the dimension formula.
**Proof:** Suppose \( A \subseteq A' \subseteq \mathcal{M} \) and \( B \subseteq B' \subseteq \mathcal{M} \). Then \( A'/A \) and \( B'/B \) are singular (1.12.e) and hence so is \( (A'+B')/(A+B) \). For, let \( a \in A' \) and \( b \in B' \); then there exist essential right ideals \( e \) and \( f \) of \( R \) such that \( ae \subseteq A \) and \( bf \subseteq B \); thus \( ef \) is an essential right ideal of \( R \) with \( (a+b)(ef) \subseteq A+B \). This proves that \( (A'+B')/(A+B) \) is singular. Since \( M \) is nonsingular, this implies \( A+B \subseteq A'+B' \) (1.13.c) and therefore Lemma 4.1 applies.

**Lemma 4.3** For a module \( M \), the following statements are equivalent:

a) the dimension formula holds for all complements \( A, B \) in \( M \);

b) if \( A, B \) are finite dimensional complements in \( M \) then so is \( A \cap B \);

c) each f.d. submodule of \( M \) has a unique \( e \)-closure in \( M \).

**Proof:**

a) \( \Rightarrow \) b): Let \( A, B \) be f.d. complements in \( M \); then \( A \subseteq A+B \), whence \[ u(A+B) = u(A) + u((A+B)/A) = u(A) + u(B/(A \cap B)). \]

Then the dimension formula holds if and only if \( u(B/(A \cap B)) = u(B) - u(A \cap B) \), i.e. if and only if \( A \cap B \subseteq B \) (2.8.c), but since \( B \subseteq M \) this is equivalent to saying that \( A \cap B \subseteq M \).

b) \( \Rightarrow \) c) Let \( L \) be a f.d. submodule of \( M \), and suppose that \( A \) and \( B \) are \( e \)-closures of \( L \) in \( M \); then b) implies that \( A \cap B \) is a complement in \( M \) containing \( L \). By minimality of \( A \) and \( B \) (1.9) one gets \( A=B \).

c) \( \Rightarrow \) b) Let \( A, B \) be f.d. complements in \( M \). Since \( A \cap B \) is f.d. we can take its (unique) \( e \)-closure \( K \) in \( M \) but then, since \( A \) and \( B \) are complements in \( M \) containing \( A \cap B \), both must contain \( K \) (1.9), and hence \( A \cap B = K \), which is a complement in \( M \).

**Remark:** The equivalence of b) and c) without the hypotheses of finite dimension follows exactly in the same way as above.

**Theorem 4.4** The following statements about a module \( M \) are equivalent:

a) \( M \) satisfies the dimension formula;

b) the dimension formula holds for all complements \( A, B \) in \( M \);

c) if \( A \subseteq A' \), \( B \subseteq B' \) are f.d. submodules of \( M \) then \( A+B \subseteq A'+B' \);

d) if \( A, B \) are f.d. complements in \( M \) then so is \( A \cap B \);

e) each f.d. submodule of \( M \) has a unique \( e \)-closure in \( M \).

**Proof:** In view of the previous lemmas, and since a) \( \Rightarrow \) b)
is clear, it suffices to show that b), d) and e) together imply c).
Suppose then \( A \subseteq A' \subseteq M \), \( B \subseteq B' \subseteq M \) with \( u(A) < \infty \), \( u(B) < \infty \), and let us prove that \( A + B \subseteq A' + B' \). First, it is clear that we can assume \( A' \) and \( B' \) to be the e-closures of \( A \) and \( B \) in \( M \) (if not, take e-closures for them and apply (1.2.a)); then b) ensures that \( A' + B' \) is f.d. (and thus so is \( A + B \)); hence we can take their respective unique e-closures \( K' \) and \( K \).
Since \( K' \) is a complement in \( M \) containing \( A + B \), the uniqueness of \( K \) as minimal complement over \( A + B \) (1.9) implies \( K \subseteq K' \). The same argument applied to the inclusions \( A \subseteq K \) and \( B \subseteq K \), whence \( A' + B' \subseteq K \) and this implies \( K' \subseteq K \). Therefore \( K = K' \) and thus, by (1.10), we get \( A + B \subseteq A' + B' \). This proves c).\( \qed \)

Since, in a nonsingular module \( M_r \), every submodule \( N \) has a unique e-closure \( \bar{N} = \{ x \in M : x \in N \text{ for some } e \in eRR \} \) [FA67; p. 61]; and since every submodule of a semisimple module is a complement, we infer that the dimension formula holds in any nonsingular or semisimple module.

Next, we make use of (4.2) to determine which Abelian groups satisfy the dimension formula: Let \( M \) be an abelian group. If \( M \) contains an element of infinite order and a nonzero element of order \( n \), then \( M \) contains a copy of \( \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \), and therefore it does not satisfy the formula (see Example 1). If \( M \) is torsion and the primary component of \( M \) for some prime \( p \) is neither semisimple nor uniform, then \( M \) contains a copy of \( (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z}) \). This copy does not satisfy the dimension formula: let \( A = (0,1)\mathbb{Z}, B = (1,1)\mathbb{Z} \).

Therefore a necessary condition for \( M \) to satisfy the dimension formula is: \( M \) is either torsion-free, or torsion with each primary component either semisimple or uniform.

This condition is also sufficient. For, if \( M \) is torsion-free then (4.2) ensure that the dimension formula holds in \( M \). On the other hand, if \( M \) is torsion and \( M_p \) denotes the \( p \)-primary component of \( M \), then, if \( A, B \) are submodules of \( M \), we have clearly \( (A \cap B)_p = A_p \cap B_p \), and we claim that \( A_p + B_p \subseteq (A + B)_p \).

To prove it, let \( a \in A \), \( b \in B \) be such that \( o(a+b) = (\text{the order of } a+b) = p^n \) for some \( n \geq 1 \); we have to prove that some nonzero multiple of \( a+b \) lies in \( A_p + B_p \). If \( a \in A_p \), say \( o(a) = p^r \), then let \( m = \max\{n, r\} \) and note that
0 = p^m(a+b) = p^mb, i.e. b ∈ B_p and hence we are done. If a ∉ A_p then b ∈ B_p by the previous argument; in this case let q, t, r, s ∈ N be such that o(a) = qr, o(b) = tr, q > 1, t > 1, p | q, p | t.

Since p is prime, p^n | qt and hence qt(a+b) = qt(a) + qt(b) is a nonzero multiple of a+b with p^r(qt(a)) = 0 and p^s(qt(b)) = 0, i.e. qa ∈ A_p and qtb ∈ B_p, proving our claim.

Hence, if the dimension formula holds in each M_p, then we get
\[ u(A+B) = \sum_{p} u((A+B)_p) = \sum_{p} (u(A_p) + u(B_p)) = \sum_{p} (u(A_p) + u(B_p) - u(A_p \cap B_p)) \]
\[ = \sum_{p} (u(A_p) + u(B_p) - u(A_p \cap B_p)) = u(A) + u(B) - u(A \cap B) \]
(where the sums run over all prime integers p), and this proves that the stated condition is also sufficient.

**Some General Formulae for u(A+B)**

Suppose we are given submodules A, B of an arbitrary module M. Take e-closures A' for A in A+B, and C' for C = A ∩ B in B. By (2.8.c),
\[ u(A+B) + u(A'/A) = u(A) + u((A+B)/A) = u(A) + u(B/C) \]
\[ = u(A) + u(B) - u(C) + u(C'/C). \]

Note that, by modularity, A + (A' ∩ B) = A' and therefore
\[ A'/A = [(A+(A' \cap B))/A] \equiv (A' \cap B)/(A \cap B). \]

But A ∩ A' implies A ∩ B ∈ A' ∩ B and therefore, without loss of generality, we could have taken C' such that A' ∩ B ∈ C' ⊆ B, whence u(A'/A) ≤ u(C'/C). Thus we can restate our formula as
\[ u(A+B) = u(A) + u(B) - u(A \cap B) + [u(C'/C) - u(A'/A)]. \]

This shows at once that in general we have
\[ u(A+B) ≥ u(A) + u(B) - u(A \cap B), \]
and as a consequence we get the implication b) ⇒ a) of (4.4). For, given A and B, take e-closures A', B' for them; then b) implies
\[ u(A+B) ≤ u(A'+B') = u(A') + u(B') - u(A' \cap B') = u(A) + u(B) - u(A \cap B) ≤ u(A+B). \]

Therefore all terms are equal, so the dimension formula holds.
Consider now the short exact sequence \( 0 \rightarrow A \cap B \rightarrow A \oplus B \rightarrow A + B \rightarrow 0 \), where \( \alpha(x) = (x, x) \) and \( \beta(a, b) = a - b \), and let \( C = \text{Im} (\alpha) \). If \( C' \) is an e-closure for \( C \) in \( A \oplus B \) then by (2.8.c & d) we get
\[
u(A + B) = \nu(A) + \nu(B) - \nu(A \cap B) + \nu(C' / C)\].

In [8] Camillo and Zelmanowitz established, for submodules \( A, B \) of a module \( M \), the formula
\[
u(A + B) = \nu(A) + \nu(B) - \nu(D) + \nu(D / C),
\] where \( C = A \cap B \) and \( D \supseteq C \) is a submodule of \( A \) maximal with respect to the property of being the domain of a monic extension into \( B \) of the identity in \( C \). Note that (II) and (2.8.c) give (I).

We finally use (II) to give another proof of (4.2).

**Corollary 4.2'** If \( M_r \) is nonsingular then it satisfies the dimension formula.

**Proof:** Given \( A, B \in M \) let \( C, D \) be as in (II) and let \( f : D \rightarrow B \) be a monic extension of the identity in \( C \). We claim that \( D / C \) is nonsingular. For, let \( d \in D \) and suppose there exists an essential right ideal \( e \) of \( R \) such that \( de \subseteq C \); then for all \( e \in e \), \( de = f(de) = f(d)e \) so that \( (d - f(d))e = 0 \), but since \( M \) is nonsingular this implies \( d = f(d) \) and hence \( d \in C \). That is, \( d + C = 0 \) in \( D / C \), which proves that \( D / C \) is nonsingular. Thus (1.12.f) \( C \subseteq D \) and hence \( \nu(C) = \nu(D) - \nu(D / C) \). Therefore (II) takes the form of the dimension formula for \( A + B \), as desired.

**Finiteness of \( \nu(A + B) \).**

We give now a partial answer to the following question: If \( A, B \) are finite dimensional submodules of a module \( M_r \), when is \( A + B \) also finite dimensional?

Following [6], we say that a module \( M_r \) is *quotient finite dimensional* (or *q.f.d.* for short) if \( M / N \) is f.d. for all submodules \( N \) of \( M_r \). The following lemma is straightforward:
LEMMA 4.5 Let $M_r$ be any module and $N$ any submodule of $M_r$. Then $M$ is q.f.d. if and only if both $N$ and $M/N$ are q.f.d. In particular, a finite direct sum is q.f.d. if and only if each summand is q.f.d.

LEMMA 4.6 For a ring $R$ the following statements are equivalent:

a) $A+B$ is f.d. for all f.d. submodules $A, B$ of any right $R$-module $M$;

b) every f.d. right $M$-module is q.f.d.;

c) every f.d. injective right $R$-module is q.f.d.

Proof: a)$\Rightarrow$b) Suppose there exist modules $MSM_r$ such that $u(M)=n<\infty$ but $u(M/N)=\infty$. Then we can take submodules of $M\oplus(M/N)$ as in Example 2, namely $A=\{(x,0+N):x\in M\}$ and $B=\{(x,x+N):x\in N\}$, such that $A$ and $B$ are f.d. (they are isomorphic to $M$) but $A+B=M\oplus(M/N)$ is not f.d., contradicting a).

b)$\Rightarrow$a) Note that $u(A)<\infty$ and $u(B)<\infty$ imply $u(A\oplus B)<\infty$ (here $A\oplus B$ is an 'external' direct sum) and that $A+B$ is a quotient of $A\oplus B$. Since $A\oplus B$ is q.f.d. by hypothesis, $A+B$ is finite dimensional.

b)$\Leftrightarrow$c) This is clear since every finite dimensional module is contained in a finite dimensional injective module (3.9) and every submodule of a q.f.d. is q.f.d.

Next we make use of (4.6) to study in some detail the commutative case. Recall that a module $M_r$ is finitely embedded if its injective hull is a finite direct sum of injective hulls of simple modules (see [59]). The following proposition shows that examples in [7] cannot be extended to the infinite case:

PROPOSITION 4.7 Let $R$ be a commutative Artinian ring. Then every sum of finite dimensional $R$-modules is finite dimensional.

Proof [58; Prop. 8]: By (4.6), we just have to prove that every f.d. injective $R$-module is q.f.d. By [59; Theo. 1], every f.d. injective $R$-module $E_r$ is finitely embedded. Then [34; Prop. 3] shows that $E_r$ is Artinian and therefore it is q.f.d.

For the Noetherian case we have the following:
PROPOSITION 4.8 Let R be a commutative integrally closed Noetherian domain with field of fractions K. Then the following statements are equivalent:

a) If A, B are f.d. submodules of an R-module M then A + B is f.d.;

b) every f.d. (injective) R-module is q.f.d.;

c) K/R is f.d.;

d) R is a semilocal principal ideal domain.

PROOF [58; Prop. 9]: a)⇒b) This is just (4.6).

b)⇒c) Since R is a domain, Kr is uniform and injective. Apply b).

c)⇒d) This is just [52; Prop. 4.7]. Note that there the hypothesis 'K has Krull dimension' is only used to get 'K/R is f.d.'

d)⇒b) Since R is commutative Noetherian, every f.d. injective R-module E is a finite direct sum of injective hulls E(R/P) of R/P for some prime ideals P of R [33; Theorem 2.5, Prop. 3.1]. But d) implies that each P is either 0 (in which case E(R/P)≡K by [33; Theorem 3.4]) or maximal (and then E(R/P) is Artinian by [34; Prop. 3]), and therefore E takes the form E=K⊕⋯⊕K⊕A where A is Artinian and thus q.f.d. Hence by (4.5) it suffices to show that K is q.f.d. as R-module or, since R is Noetherian, that K/R is q.f.d.. In fact, we prove next that K/R is Artinian.

Let P1=Rp1,..., Pn=Rpn be the maximal ideals of R and denote by P1 the R-submodule of K generated by 1/p1. Since P1/R is annihilated by P1 and nonzero cyclic, it is simple. Moreover, every simple R-submodule of K/R is one of the Pi/R. For, let S/R be simple; then it is annihilated by some Pi, i.e. Sp1⊂R, whence R⊂Sp1 and thus S=Pi. Therefore the sum ∑(Pi/R) (which is indeed direct, see [52]) is the socle of K/R. But every torsion module over a principal ideal domain has essential socle, whence K/R is finitely embedded and thus Artinian by [34; Prop. 3].
CHAPTER 2
THE RING OF ENDMORPHISMS OF A MODULE

SECTION 5: ENDMORPHISM RINGS; FIRST RESULTS.

Given a module \( M = \mathbb{M} \), call \( S = \text{End}_R(M) = \text{Hom}_R(M, M) \). We can view the binary operation 'composition of maps' as a 'product' in \( S \), and easy computations show that the Abelian group \( S \) (see Chapter 0) becomes then a ring with identity \( 1_s = 1_M \) (the identity map in \( M \)). This ring \( S \) is called the ring of endomorphisms of \( M \) or the endomorphism ring of \( M \).

For example, consider the regular module \( \mathbb{R} \). For each \( r \in \mathbb{R} \), the map \( \lambda_r : \mathbb{R} \to \mathbb{R} \) defined by \( \lambda_r(t) = rt \) ('left multiplication by \( r \)') is an endomorphism of \( \mathbb{R} \), and \( \lambda_s \neq \lambda_r \) whenever \( r \neq s \). On the other hand, if \( f \in \text{End}(\mathbb{R}) \) and \( f(1) = r \), then \( f(t) = f(1t) = f(1)t = rt \) for all \( t \in \mathbb{R} \), so that \( f = \lambda_r \). In fact, the map \( r \mapsto \lambda_r \) defines a ring isomorphism \( \lambda : \mathbb{R} \to \text{End}(\mathbb{R}) \) with inverse given by \( f \mapsto f(1) \).

In the general situation, \( M \) becomes a left \( S \)-module if we define the product of an element \( x \) of \( M \) by an element \( f \) of \( S \) as \( fx = f(x) \) (and, fortunately, this agrees with our convention about the notation). Then \( M \) is clearly faithful as an \( S \)-module and it is an \( (S, R) \)-bimodule: \( M = \mathbb{M} \) (see Chapter 0).

Given a submodule \( N \) of \( M \) and a nonempty subset \( W \) of \( S \), we will write \( WN \) for the product submodule \( \Sigma \{ tN : t \in W \} \).

An \( S \)-submodule \( N \) of \( M \) need not be an \( R \)-submodule. For example consider the regular module \( \mathbb{R} \); by the previous example, the \( \text{End}(\mathbb{R}) \)-submodules of \( R \) are the left ideals of \( R \), which need not be submodules of \( \mathbb{R} \).

The \( (S, R) \)-submodules of \( M \) are usually called fully invariant submodules of \( M \), since they are precisely the \( R \)-submodules \( N \) of \( M \) whose image under any endomorphism of \( M \) remains inside \( N \). Therefore, if \( N \subseteq M_R \) is fully invariant, we can define the "restriction map" \( \text{End}(M_R) \to \text{End}(N_R) \) in the obvious way, and it is easy to check that it
is a ring homomorphism (and it is a ring epimorphism whenever arbitrary endomorphisms of N can be extended to End(MR), e.g. if NG\subseteq M or if MR is quasi-injective).

For example, we can restate (3.10) as 'A module MR is quasi-injective if and only if it is a fully invariant submodule of its injective hull'. Other examples of fully invariant submodules are the ideals of R (when M=RM); the annihilators in MR of left ideals of R; the socle of M (in fact, every sum of homogeneous components of Soc(M)); and the radical of M [FA67; p.179]. Clearly, since Lat(sMR) is a sublattice of Lat(MR), arbitrary sums and intersections of fully invariant submodules are still fully invariant.

The relationships between properties of M and properties of S have been widely studied, and the next sections are devoted to a review of the main results obtained in this area for certain classes of modules. Sometimes we shall make use of the fact that End(RR)=R and rewrite these results in the specific case M=RR, obtaining as corollaries some well-known theorems about rings.

With no further background we can already get some easy properties of the endomorphism rings of well-behaved modules, such as vector spaces, simple, semisimple or free modules. S will always stand for End_R(M).

**Proposition 5.1** If MR is a simple module then S is a division ring.

**Proof:** If f\in S is not the zero homomorphism then Ker(f)\subseteq MR and 0=Im(f)\subseteq MR; since M has no nontrivial submodules we have Ker(f)=0 and Im(f)=M, whence f is invertible in S. Therefore S is a division ring. ■

If MR admits a finite decomposition as M=\bigoplus \limits_j M_j and we write M_{ij} for Hom_R(M_j,M_i), then S may be identified with the ring of n-square matrices [f_{ij}] with each f_{ij} in M_{ij}. Specifically, if u_j: M_j\rightarrow M and p_i: M\rightarrow M_i are the injections and projections (respectively) of the coproduct \bigoplus M_i, then the map f\mapsto [f_{ij}], where f_{ij}=p_if_ju_j, provides the desired ring isomorphism.

In particular, if M^{(n)} represents the direct sum of n copies of M, then End_R(M^{(n)}) is isomorphic to the full ring of n by n matrices with entries in S=End_R(M). Using this facts, we can prove the next result.
PROPOSITION 5.2 If \( M_r \) is semisimple and finitely generated then \( S \) is a finite product of matrix rings over division rings (and hence \( S \) is a semisimple Artinian ring).

**Proof:** We can write \( M = \bigoplus_{i=1}^{B} M_i \) where the \( M_i \)'s are the homogeneous components of \( M \), each of which is a finite direct sum of copies of a simple module, whence each \( \text{End}_R(M_i) \) is a matrix ring over a division ring (5.1).

On the other hand, it is clear that any homomorphic image of a semisimple homogeneous module is again homogeneous of the same type, and then for the given decomposition of \( M \) we have \( \text{Hom}_R(M_j,M_i)=0 \) whenever \( i \neq j \). Therefore it is clear that \( S \) is the ring product of the \( \text{End}_R(M_i)'s \), which completes the proof. \( \Box \)

PROPOSITION 5.3 If \( M_r \) is free then \( S \) is a row-finite matrix ring.

**Proof:** Let \( \{x_i : i \in I\} \) be a basis for \( M_r \). Denote by \( \mathcal{RFM}_I(R) \) the ring of row-finite \( I \)-square matrices with entries in \( R \). Then the map \( \phi:S \rightarrow \mathcal{RFM}_I(R) \) given by \( \phi(f) = [r_{ij}] \) where the \( r_{ij} \)'s are such that \( f(x_i) = \sum_{j \in I} x_i r_{ij} \) is a ring isomorphism. \( \Box \)

**Idempotents**

The behavior of the idempotent elements in a ring of endomorphisms is very important, since they are closely related with the direct summands of the module, as the following lemma shows. Before stating it, we recall that a set \( \{t_i : i \in I\} \) of idempotents of a ring is said to be orthogonal if, for all \( i \neq j \) in \( I \), we have \( t_i t_j = 0 \). An idempotent \( t \) is primitive if it cannot be expressed as \( t = t_1 + \cdots + t_n \), with \( \{t_1, \ldots, t_n\} \) a family of orthogonal idempotents and \( n > 1 \). Finally, a finite orthogonal set \( \{t_1, \ldots, t_n\} \) of idempotents is said to be complete if \( t_1 + \cdots + t_n = 1 \).

**Lemma 5.4** Let \( M_r \) be any module and let \( S = \text{End}(M_r) \). Then:

a) if \( NS \subseteq M_r \), then \( NS \subseteq M \) if and only if there exist an idempotent \( t \) of \( S \) such that \( N = tM \); in this case \( M = tM \oplus (1-t)M \), and \( N \) is indecomposable if and only if \( t \) is primitive;
b) \( M \) is indecomposable if and only if 0 and 1 are the unique idempotents of \( S \), if and only if 1 is primitive in \( S \);

c) if \( \{ t_i : i \in I \} \) is a family of orthogonal idempotents of \( S \) then the sum \( \sum_{i} t_i M \) is direct; if \( I \) is finite and \( \sum_{i} t_i = 1 \), then \( M = \bigoplus_{i} M_i \);

d) if \( M = \bigoplus_{i \in I} M_i \) then there exists a family \( \{ t_i : i \in I \} \) of orthogonal idempotents in \( S \) with \( M_i = t_i M \) and \( \text{Ker}(t_i) = \bigoplus_{j \neq i} M_j \) for each \( i \in I \). If \( I \) is finite then \( \sum_{i} t_i = 1 \).

PROOF:  a) If \( M = N \oplus L \) then the canonical projection \( t : N \oplus L \rightarrow N \) is the desired idempotent. Conversely, it is clear that, for \( t^2 = t \in S \), we have \( M = tM \oplus (1 - t)M \). Of course, this \( t \) is not unique in general.

If \( N \) is the direct sum \( N = N_1 \oplus N_2 \) of two nonzero submodules and we define \( t_i : N_i \oplus N_2 \rightarrow N_i \) for \( i = 1, 2 \), then \( t_1, t_2 \) are nonzero orthogonal idempotents with \( t = t_1 + t_2 \), and thus \( t \) is not primitive. Conversely, if \( t = t_1 + t_2 \) where \( t_1 \) and \( t_2 \) are nonzero orthogonal idempotents of \( S \), then it is easy to see that \( N = t_1 M \oplus t_2 M \), and hence \( N \) is not indecomposable.

b) is clear from a).

c) To see that the sum \( \sum_{i} t_i M \) is direct, let \( \sum_{j} t_j x_j = 0 \) for some finite subset \( J \) of \( I \) and for some \( x_j \in M \); then, for all \( k \in J \), we have \( 0 = t_k (\sum_{j} t_j x_j) = t_k x_k \). Therefore \( \sum_{i} t_i M \) is direct. If \( I \) is finite and \( \sum_{i} t_i = 1 \), then, for all \( x \in M \), \( x = 1(x) = (\sum_{i} t_i) x = \sum_{i} t_i x = \sum_{i} t_i M_i \), so that \( M = \bigoplus_{i} M_i \).

d) If \( M = \bigoplus_{i \in I} M_i \) and we define \( t_j : M = \bigoplus_{i} M_i \rightarrow M_j \) then \( \{ t_j : j \in I \} \) is the desired family of orthogonal idempotents.

COROLLARY 5.5 \( M_\alpha \) admits a finite indecomposable decomposition if and only if \( S \) possesses a complete family of primitive idempotents.

COROLLARY 5.6 If \( S = \text{End}(M_\alpha) \) is a local ring then \( M_\alpha \) is an indecomposable module.

PROOF: Since in any ring the only invertible idempotent is the identity, and since in a local ring the non-invertible elements form an ideal, 1 is a primitive idempotent of \( S \), and therefore \( M_\alpha \) is indecomposable by (5.4.b).

PROPOSITION 5.7 Let \( M_\alpha \) be any module and let \( S = \text{End}(M_\alpha) \); for every idempotent \( t \) of \( S \) there is a ring isomorphism \( \phi \) between \( tS_\alpha t \) and \( \text{End}_R(tM) \) given by \( \phi(tf_t)(tx) = tf_tx \) for all \( f \in S \) and \( x \in M \).
proof: It is clear that the given map is a ring homomorphism and
that it is injective. If \( g \in \text{End}_R(tM) \) then it is easily checked that
\( g = \phi(t \gamma t) \), and hence \( \phi \) is a ring isomorphism. 

The Dual Module and the Trace Ideal of a Module

Next, we introduce the concepts of the dual module of \( M_R \) and the trace ideal of \( M \) in \( R \), whose first properties we state here. These concepts
will be used in later sections.

For any module \( M_R \), we already know that \( M \) is an \((S,R)\)-bimodule, where
\( S = \text{End}(M_R) \). Moreover, for each module \( N = N_R \), the Abelian group \( \text{Hom}_R(M,N) \)
is, in a natural way (see Chapter 0), a right \( S \)-module. In case \( N = R_R \), we write \( \hat{M} \) for \( \text{Hom}(M_R,R_R) \), and call it the dual module of \( M_R \); since \( R \)
is an \((R,R)\)-bimodule, \( \hat{M} \) has a natural \((R,S)\)-bimodule structure (see
again Chapter 0), \( \hat{M} = R \hat{M} S \).

For a module \( N = N_R \), it may be of interest to know which elements of \( N \)
appear as images of elements of \( M \) under an homomorphism of \( \text{Hom}(M_R,N_R) \).
The trace of \( M_R \) in \( N_R \), written \( t_N(M) \), is the submodule of \( N \) generated
by these images, i.e.
\[
t_N(M) = \{ \varphi \in \text{Hom}(M_R,N_R) \}.
\]

In case \( N = R_R \), the trace of \( M \) in \( R \), i.e. \( t_R(M) = \{ \varphi \in \text{Hom}(M_R,N_R) \} \), is usually
called the trace ideal of \( M_R \); it is a two-sided ideal of \( R \) since \( \hat{M} \) is
a left \( R \)-module.

Now, consider the bimodules \( sM_R \) and \( \hat{M} S \); the tensor products \( \hat{M} \otimes_S M \) and
\( M \otimes_R \hat{M} \) are, respectively, \((R,R)\)- and \((S,S)\)-bimodules. Given \( x \in M \) and \( \varphi \in \hat{M} \),
let \( (\varphi, x) \) represent the image of \( x \) under \( \varphi \), i.e. \( (\varphi, x) = \varphi x \), and let
\( [x, \varphi] \) be the map \( M \rightarrow M \) defined by \( [x, \varphi] y = x(\varphi, y) \) for all \( y \in M \). It is
easy to check that \( [x, \varphi] \) is an endomorphism of \( M_R \) and that the maps
\((,): \hat{M} \otimes_M R \) and \([,]: M \otimes_R \hat{M} \rightarrow S \) are bilinear, so that they extend to
\(Z\)-homomorphisms \((,): \hat{M} \otimes M \rightarrow R \) and \([,]: M \otimes R \rightarrow S \). Easy computations show
that, in fact, \((,)\) is an homomorphism of \((R,R)\)-bimodules and \([,]\) is an
homomorphism of \((S,S)\)-bimodules. We rewrite here their action on
generators for easy reference:
Note that the image of $(,) \in \mathcal{H} \otimes \mathcal{M}$ is precisely $t(M)$, the trace ideal of $M_R S$. Note also that, for any $x \in M_R S$, $\varphi \in \mathcal{H}$, $r \in R$ and $f \in S$, we have

$$\begin{align*}
[xr, \varphi] &= [x, r\varphi] \\
(r, x) &= (r\varphi, x) \\
(\varphi, x) &= (\varphi, f) \\
([x, \varphi]) &= [f, x, \varphi] \\
[x, \varphi]f &= [x, \varphi f].
\end{align*}$$

All these relations will be assumed in the sequel, and we shall use them without further reference.
A very natural approach to the study of the relationships between properties of a module $M_R$ and properties of the ring $S=\text{End}(M_R)$ consists in seeking out bijections between the lattice $\mathcal{L}=\text{Lat}(M_R)$ and either of the lattices $\mathcal{L}_1=\text{Lat}(sS)$ or $\mathcal{L}_r=\text{Lat}(S_S)$.

The concept of "Galois connection", that we shall introduce shortly, provides a general source to get lattice (anti-)isomorphisms, and some examples of these connections will fit perfectly our purposes. For any module $M_R$ we shall find, in a natural way, two Galois connections: $\mathcal{G}_1$ between $\mathcal{L}$ and $\mathcal{L}_1$ and $\mathcal{G}_2$ between $\mathcal{L}^{\text{op}}$ (the opposite lattice of $\mathcal{L}$) and $\mathcal{L}_r$. For each one of these, we shall get the corresponding sets of "closed" elements in $\mathcal{L}$, $\mathcal{L}_1$ or $\mathcal{L}_r$, as well as lattice (anti-)isomorphisms between them.

This general setting seems to have been first introduced by Baer [4], and has been widely employed (see K.Wolfson, G.Tsukerman, and S.Khuri [63, 54, 28, 29, 30, 31]); most of the proofs in this section are to be found in [30] and [31].

If we want to study a property of $S$ which may be stated in terms of a class $\mathcal{C}$ of, say, right ideals, our two tasks will be: First, to check that the right ideals in $\mathcal{C}$ are closed objects of $\mathcal{L}_r$ for $\mathcal{G}_2$ (or, if in general they are not, to find conditions on $M_R$ under which they actually are); and second, to identify the images of the elements of $\mathcal{C}$ via the corresponding isomorphism. In this way, we obtain a bijection involving the ideals we are interested in and a certain class of submodules of $M_R$ (we shall call that a correspondence theorem), and from this bijection we can deduce necessary and sufficient conditions on $M$ in order to get the desired property on $S$.

Fortunately, these bijections will not only preserve (or reverse) the inclusions, but also we will be able to prove that every member of the domain $\mathcal{C}$ is a direct summand of $S$ if and only if every member of the image is a direct summand of $M_R$ (6.2), and this will enlarge the range of the applications of our correspondence theorems.

Let us first introduce the concepts of closure operator in a lattice.
and Galois connection between two lattices, and state their first properties (for details see e.g. [S; Chap.III, Sec.7 & 8]).

**Closure Operators and Galois Connections**

Let \((L, \leq)\) be a complete lattice. A closure operator in \(L\) is a map \(c : L \to L\) (we shall represent the image of \(a \in L\) under \(c\) by \(a^c\)) which satisfies:

\[
\begin{align*}
    a &\leq a^c, \quad \text{for all } a \in L; \\
    a \leq b &\Rightarrow a^c \leq b^c, \quad \text{for all } a, b \in L; \\
    (a^c)^c &\leq a^c, \quad \text{for all } a \in L.
\end{align*}
\]

For example, if \(M_R\) is a module in which every submodule has a unique e-closure, e.g. a nonsingular module or a f.d. module in which the 'dimension formula' holds (see Section 4), then 'taking e-closures' is a closure operator in \(\text{Lat}(M_R)\). In a ring \(R\), the most common closure operator acting on \(\text{Lat}(R_R)\) is given by \(a \mapsto \mathcal{R}(a)\).

An element \(a\) of \(L\) is said to be \(c\)-closed if \(a^c = a\); the \(c\)-closed elements of \(L\) are precisely the images under \(c\) of elements of \(L\); we denote the set of \(c\)-closed elements in \(L\) by \(L^c = \{a \in L : a^c = a\} = \{a^c : a \in L\}\).

In the previous examples, the closed elements were, respectively, the complement submodules of \(M_R\) and the right annihilator ideals of \(R\). The set \(L^c\) with the order inherited from \(L\) forms a complete lattice, (however, it is not in general a sublattice of \(L\)).

Let \(L_1\) and \(L_2\) be complete lattices (since the risk of confusion is small, we shall use the same symbol \(\leq\) for the partial orders in \(L_1\) and \(L_2\)). A Galois connection between \(L_1\) and \(L_2\) consists of a pair \(G=\{\tau, \sigma\}\) of mappings \(\tau : L_1 \to L_2, \sigma : L_2 \to L_1\) satisfying

\[
\begin{align*}
    a \leq b &\Rightarrow \tau(a) \leq \tau(b) \quad \text{for all } a, b \in L_1; \\
    x \leq y &\Rightarrow \sigma(x) \leq \sigma(y) \quad \text{for all } x, y \in L_2; \\
    a \leq \sigma(\tau(a)) &\quad \text{and} \quad x \leq \tau(\sigma(x)) \quad \text{for all } a \in L_1, x \in L_2.
\end{align*}
\]

For example, if \(R\) is any ring, then the annihilator operators \(\mathcal{Z} : \text{Lat}(R_R) \to \text{Lat}(R_R)\) and \(\mathcal{R} : \text{Lat}(R_R) \to \text{Lat}(R_R)\) form a Galois connection.

If \(G=\{\tau, \sigma\}\) is a Galois connection between \(L_1\) and \(L_2\), then the
composition maps \( \sigma: L_1 \to L_1 \) and \( \tau: L_2 \to L_2 \) are closure operators. Let \( \hat{L}_1 \) (resp. \( \hat{L}_2 \)) represent the lattice of \( \sigma \)-closed (resp. \( \tau \)-closed) elements of \( L_1 \) (resp. \( L_2 \)). It is easy to prove that \( \hat{L}_1 = \{ \sigma(x) : x \in L_2 \} \) and \( \hat{L}_2 = \{ \tau(a) : a \in L_1 \} \), and that the restrictions \( \tau: \hat{L}_1 \to \hat{L}_2 \) and \( \sigma: \hat{L}_2 \to \hat{L}_1 \) are inverse lattice anti-isomorphisms. The elements of \( \hat{L}_i \) are often called the Galois objects of \( L_i \) (for \( i=1,2 \)) with respect to \( G \).

For a lattice \( L \), let \( L^\text{op} \) stand for the opposite lattice of \( L \), i.e. the lattice consisting of the same underlying set with the opposite order. A Galois connection \( G = \{ \tau, \sigma \} \) between \( L^\text{op} \) and \( L_2 \) must then verify:

\[
\begin{align*}
& \text{a} \leq \text{b} \Rightarrow \tau(a) \leq \tau(b) \text{ for all } a, b \in L_1; \\
& x \leq y \Rightarrow \sigma(x) \leq \sigma(y) \text{ for all } x, y \in L_2; \\
& a \leq \sigma \tau(a) \text{ and } x \leq \tau \sigma(x) \text{ for all } a \in L_1, x \in L_2;
\end{align*}
\]

(where \( \leq \) always denotes the order in the original lattices \( L_1 \) and \( L_2 \)); in this case the restrictions \( \tau: \hat{L}_1 \to \hat{L}_2 \) and \( \sigma: \hat{L}_2 \to \hat{L}_1 \) are inverse lattice isomorphisms.

**The Galois Connections \( G_1 \) and \( G_2 \)**

Let \( M_\mathbb{R} \) be any module and let \( S = \text{End}(M_\mathbb{R}) \). Write \( \mathcal{L} = \text{Lat}(M_\mathbb{R}) \), \( \mathcal{L}_1 = \text{Lat}(sS) \) and \( \mathcal{L}_2 = \text{Lat}(sS) \). As always, let \( \mathcal{L} \) and \( \mathcal{R} \) denote the annihilator operators in \( S \), and let \( \mathcal{L}_S \) and \( \mathcal{R}_M \) denote the annihilators in \( S \) of subsets of \( M \) and in \( M \) of subsets of \( S \), respectively. Specifically, for any nonempty subsets \( W \) of \( S \) and \( X \) of \( M \),

\[
\begin{align*}
\mathcal{L}(W) &= \{ f \in S : fg = 0 \text{ for all } g \in W \}, & \mathcal{L}_S(X) &= \{ f \in S : fx = 0 \text{ for all } x \in X \}, \\
\mathcal{R}(W) &= \{ f \in S : gf = 0 \text{ for all } g \in W \}, & \mathcal{R}_M(W) &= \{ x \in M : gx = 0 \text{ for all } g \in W \} = \bigcap_{g \in W} \text{Ker} g
\end{align*}
\]

Also let, for any subset \( W \) of \( S \) and any submodule \( N \) of \( M_\mathbb{R} \),

\[
\begin{align*}
\tau_S(N) &= \{ f \in S : fM \subseteq N \}, & \sigma_M(W) &= \{ g \in S : g \in W \}.
\end{align*}
\]

With the notation of Chapter 0, we could have written \( (N; M) \) for \( \tau_S(N) \); note also that \( \tau_S(N) \) may be identified with \( \text{Hom}(M, N) \) (since \( N \) is a submodule of \( M \)), and it is a right ideal of \( S \). On the other hand, \( \sigma_M(W) \) is just the ‘product’ \( WM \), which is an \( R \)-submodule of \( M \).

It is easy, although a bit tedious, to check that

1) the mappings \( \mathcal{L}_S \) and \( \mathcal{R}_M \) form a Galois connection \( G_1 \) between \( \mathcal{L} \) and
the closed elements of $\mathcal{L}$ and $\mathcal{L}_1$ for $G_1$ will be called, respectively, $a$-closed (annihilator-closed) submodules and $a$-closed (left) ideals, and we shall write $M_a$ and $\mathcal{Y}_a$ for the sets of $a$-closed elements of $\mathcal{L}$ and $\mathcal{L}_1$, i.e. $M_a=\{r^*_M(W):W\subseteq S\}$ and $\mathcal{Y}_a=\{1^*_M(N):N\subseteq M\}$; if $N$ is a submodule of $M$, we shall sometimes write $N^a$ for $r^*_M(N)$.

2) the mappings $\tau_S^M$ and $\sigma_M^S$ form a Galois connection $G_2$ between $\mathcal{L}^\text{op}$ and $\mathcal{L}_r$; the closed elements of $\mathcal{L}$ for $G_2$ will be called, following [30], $M$-cotorsionless submodules of $M$; the Galois objects of $\mathcal{L}_r$ for $G_2$ will be simply called $\tau\sigma$-closed (right) ideals. We shall write $M_{\sigma T}^S=\{\sigma_M^S(W):W\subseteq S\}$ and $\mathcal{Y}_{\tau\sigma}^r=\{\tau_S^M(N):N\subseteq M\}$.

3) the mappings $\mathcal{K}$ and $\mathcal{L}$ (as we have already remarked) form a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_r$, whose Galois objects are respectively $\mathcal{A}_1$ and $\mathcal{D}_r$, where we write $\mathcal{A}_1$ (resp. $\mathcal{D}_r$) for the set of left (resp. right) annihilator ideals of $S$. In particular, from the existence of a lattice anti-isomorphism between $\mathcal{A}_1$ and $\mathcal{D}_r$ we infer the well known fact that, for any ring, ACC (DCC) on right annihilators is equivalent to DCC (ACC) on left annihilators.

We wish to notice here the importance in what follows of the sets $\mathcal{Y}_a$ and $\mathcal{Y}_{\tau\sigma}^r$, since for any class $\mathcal{C}$ of ideals of $S$ contained in either of these sets we will get a correspondence theorem involving $\mathcal{C}$.

Apart from the inclusion relations which are inherent to the fact that the above are Galois connections, some other relations always occur, as it is easily verified. They are listed below, and will be used throughout this section without further reference.

**Lemma 6.1** With the above notation we have, for all nonempty subsets $W$ of $S$, for all submodules $N$ of $M$ and for all $f, t \in S$ with $t^2=t$:

a) $\tau^*_S r^*_M(W)=\mathcal{R}(W)$; $\mathcal{L}(W)=\mathcal{L}(W)$;

b) $\mathcal{R}(Sf)=\mathcal{R}(f)$; $\mathcal{L}(fs)=\mathcal{L}(f)$; $\mathcal{R}(t)=(1-t)S$; $\mathcal{L}(t)=S(1-t)$

c) $r^*_M(Sf)=\mathcal{R}(f)=\text{Ker} f$; $r^*_M(St)=(1-t)M$;

\[1^*_S(fM)=\mathcal{L}(f); \quad 1^*_S(tM)=S(1-t) \]

d) $\sigma_M^S(Sf)=fM$; $\sigma_M^S(tM)=tS$;

**Proof:** We only prove the second half of d), since the rest of the proofs are mechanical. So let $t^2=t \in S$, then $t\sigma^*_S(tM)$ and hence $tS\tau^*_S(tM)$; if $g\sigma^*_S(tM)$ then, for all $x \in M$, there exists $y \in M$ with $gx=ty$, 


whence $tgx = t^2y = ty = gx$, i.e. $g = tgetS$, which completes the proof. ■

In particular, by (6.1.c,d), for each $f \in S$, Ker$f$ is $\alpha$-closed and $fM$ is $M$-cotorsionless. Then, using (5.4.a), we deduce that every direct summand of $Ma$ is $\alpha$-closed and $M$-cotorsionless.

We now intend to show some examples of situations in which these Galois connections are particularly useful. For example, as a consequence of (6.1.a), every member of $A_1$ (resp. $A_r$) is an $\alpha$-closed ideal (resp. a $\tau\alpha$-closed ideal), so that the 'first step' outlined at the beginning of the section is already done, and this will be helpful when studying conditions in $S$ which depend on its annihilator ideals, such as being a Baer ring or a ring with chain conditions on annihilator ideals.

In the same way we shall study conditions in $M$ under which the right complements of $S$ will be Galois objects of $G_2$; we will make further use of these conditions in Section 9.

We close this section with a brief study of the principal and finitely generated left or right ideals of $S$. This study will be carried on in Section 10, where we shall characterize the quasi-injective and quasi-projective modules whose endomorphism rings are Noetherian, semiprimary or Artinian.

In what follows, a bijection between two partially ordered sets which is order-preserving (resp. order-reversing) will be called a projectivity (resp. a duality). Note that, if $\varphi : L_1 \rightarrow L_2$ is a projectivity (e.g. a lattice isomorphism) and $K_1$ is a subset of $L_1$ ($i=1,2$), then $\varphi : K_1 \rightarrow K_2$ is a projectivity if and only if $\varphi(K_1)=K_2$, if and only if $\varphi^{-1}(K_2) \subseteq K_1$. Of course, a similar remark holds for dualities.

The following lemma, announced at the beginning of the section, will be used in the proof of most applications of our correspondence theorems:

**Lemma 6.2** a) Assume that $l_s$ and $r_m$ determine a duality between certain subsets $U$ of $Ma$ and $V$ of $Fa$. Then every member of $U$ is a direct summand of $M\alpha$ if and only if every member of $V$ is a direct
summand of sS.

b) Assume that \( \tau \) and \( \sigma \) determine a projectivity between the subsets \( U \) of \( M \) and \( V \) of \( S \). Then every member of \( U \) is a direct summand of \( M \) if and only if every member of \( V \) is a direct summand of \( S \).

**Proof:**

a) Let \( U, V \) be as stated, and assume \( U \subseteq D = \{ N \subseteq M : N \subseteq aM \} \); then, for all \( \mathcal{A} \in V \), there exists \( t^2 = t \in S \) such that \( r^\mathcal{M}_S(\mathcal{A}) = tM \); also, since \( V \subseteq \mathcal{Y} \), we have \( \mathcal{A} \subseteq l_S r^\mathcal{M}_S(\mathcal{A}) \), whence \( \mathcal{A} = l_S(tM) = S(1-t) \), which is a direct summand of \( sS \). Conversely, if every member of \( V \) is a direct summand of \( sS \) and \( N \subseteq \mathcal{U} \), then there exists \( t^2 = t \in S \) with \( l_S(N) = St \) and hence, since \( N \subseteq \mathcal{U} \subseteq M \), \( N = r^\mathcal{M}_S(N) = r^\mathcal{M}_S(St) = (1-t)M \) is a direct summand of \( M \).

b) is proved similarly.

**Correspondence Theorems for Annihilators**

As we have already remarked, the class \( A \) of left annihilator ideals of \( S \) is included in \( M \) so that, in order to obtain a correspondence theorem for left annihilators, all we have to do is to identify the a-closed submodules of \( M \) which correspond to the ideals in \( A \).

**Theorem 6.3**

For any module \( M \), set \( M_1 = \{ N \subseteq M : N = [\sigma \tau(N)]^a \} \). Then the maps \( 1 : M_1 \rightarrow A \) and \( r : A \rightarrow M_1 \) are inverse dualities.

**Proof:**

Since every element of \( M_1 \) is an a-closed submodule and every element of \( A \) is an a-closed ideal, the only things we have to check are that \( 1(M_1) \subseteq A \) and that \( r(A_1) \subseteq M_1 \).

If \( N \in M_1 \), then (6.1.a) gives \( l_S(N) = l_S \sigma \tau(N) = \tau(N) \in A \), i.e. \( 1(M_1) \subseteq A \). On the other hand, again using (6.1.a), if \( \mathcal{A} = \mathcal{Y}(\mathcal{A}) \) then \( r^\mathcal{M}_S(1_S \sigma \tau(N)) = r^\mathcal{M}_S(N) = \mathcal{Y}(\mathcal{A}) = r(\mathcal{A}) \), i.e. \( r(A) \subseteq M_1 \).

A **Baer ring** is a ring in which every left (or right) annihilator ideal is generated by an idempotent. From (6.2) and (6.3) we get

**Corollary 6.4**

a) \( S \) has ACC (DCC) on left annihilators if and only if \( M \) has DCC (ACC) on \( M_1 \).

b) \( S \) is a Baer ring if and only every member of \( M_1 \) is a direct summand of \( M \).
A module $M_a$ is a **self-generator** if $\tau_{N}(M)=N$ for all $N \subseteq M_a$ (see Section 5 and compare with the definition of generator in Section 8), i.e. if all its submodules are $M$-cotorsionless. For a self-generator $M_a$ it is clear that $M_1=M_a$; however, there exist modules which are not self-generators but verify $M_1=M_a$ [30;p.395]. A module for which $M_1=M_a$, i.e. a module $M_a$ such that, for each $N \subseteq M_a$, we have $N=[\sigma_{M_a}(N)]^a$, is called an $a$-**self-generator**. In this case we get not only a duality between $M_a$ and $A_1$, but also a projectivity between $M_a$ and $A_1$:

**THEOREM 6.5** For a module $M_a$ the following are equivalent:

a) $M_a$ is an $a$-self-generator;

b) the maps $l_s:M_a \rightarrow A_1$ and $r_M:A_1 \rightarrow M_a$ are inverse lattice anti-isomorphisms;

c) the maps $\tau_s:M_a \rightarrow A_1$ and $\tau_s:A_1 \rightarrow [\sigma_{M_a}(A)]^a$ from $A_1$ to $M_a$ are inverse lattice isomorphisms.

**PROOF:**

a)$\Rightarrow$b) Since $M_a=M_1$ by hypothesis, (6.3) gives the result.

b)$\Rightarrow$c) Since $\tau_{S}(M_a) \subseteq A_1$ (6.1.a) and $[\sigma_{M_a}(A)]^a \subseteq M_a$ for all $A \subseteq A_1$, we just have to prove that both mappings in c) are inverse of each other:

If $A \subseteq A_1$ then $\tau_s[A]^a = \tau_s \tau_s[A] = A_1 \subseteq A_1 = A_1$ (6.1.a).

If $N \subseteq M_a$ then, by b), $l_s(N) \subseteq A_1$, i.e. $l_s l_s(N) = l_s(N)$, and hence $[\sigma_{M_a}(N)]^a = \tau_s l_s(\tau_s(N)) = \tau_s l_s(N) = l_s(N) = N$.

c)$\Rightarrow$a) This is clear from the definition of $a$-self-generator.

**COROLLARY 6.6** Let $M_a$ be an $a$-self-generator. Then

a) $S$ has ACC (DCC) on left annihilators if and only if $M_a$ has DCC (ACC) on $a$-closed submodules;

b) $S$ is a Baer ring if and only if every $a$-closed submodule of $M_a$ is a direct summand of $M$.

**PROOF:** By Corollary 6.4.

A module $M_a$ in which every complement submodule is a direct summand is called a CS-module. For example, every quasi-injective module is a CS-module (3.13). If $M_a$ is a module for which the $a$-closed submodules coincide with the complements in $M$, then we can rewrite (6.6) in terms of the module being a CS-module or finite dimensional.
As we have remarked, for a nonsingular module $M_r$, 'taking e-closures' is a closure operator in $\text{Lat}(M_r)$ and hence, for any $N \subseteq M_r$, we can write $N^e$ for the (unique) e-closure of $N$ in $M$. For such a module (and by abuse of language for all modules) we shall write $M_e$ for the set of essentially closed (i.e. complement) submodules of $M_r$.

**Proposition 6.7**

a) If $M_r$ is nonsingular then $M_e \subseteq M_a$.

b) If $M_r$ is a CS-module then $M_e \subseteq M_a$.

**Proof:**

a) Assume that $M_r$ is nonsingular, and let $N \in M_a$. Let $K = N^e$; then $N \subseteq K$ and hence $1^e_s(K) \subseteq 1^e_s(N)$; on the other hand, if $f \in 1^e_s(N)$ and $x \in K$, then $(N:x) \subseteq RR$ and $fx(N:x) \leq fN = 0$, whence $fx = 0$ by nonsingularity; this means that $f \in 1^e_s(K)$ and hence $1^e_s(K) = 1^e_s(N)$. Therefore $K \subseteq r^e_s(N) = r^e_s(N) = N$ and thus $N = K$, so that $N \in M_e$.

b) If $M$ is a CS-module then every element of $M_e$ is a direct summand of $M_r$ and hence is a-closed, whence $M_e \subseteq M_a$.

**Corollary 6.8**

Let $M_r$ be a nonsingular $a$-self-generator CS-module. Then $S$ is a Baer ring. If, in addition, $M_r$ is finite dimensional, then $S$ has ACC and DCC on left (and right) annihilator ideals.

**Proof:** By (6.7), we have $M_a = M_e$. Then (6.6.b) gives the first part, while (6.6.a) and (2.1) yield the second.

**Correspondence Theorems for Right Complements**

In the next lemma, we shall make use of the concepts of the trace ideal $T$ of a module $M_r$ and of the dual module $\hat{M}$ of $M_r$, as well as of the maps $(,): \hat{M} \otimes M \to R$ and $[,] : M \otimes \hat{M} \to S$, which were defined in Section 5.

**Lemma 6.9**

Let $M_r$ be a module with trace ideal $T$. The following statements are equivalent:

a) $xT \neq 0$ for all $0 \neq x \in M$;

b) $[x, \hat{M}] \neq 0$ for all $0 \neq x \in M$;

c) $NT \subseteq N$ for all $N \subseteq M_r$.

**Proof:**

a)$\Rightarrow$b) If $[x, \hat{M}] = 0$ then $0 = [x, \hat{M}] M = x(\hat{M}, M) = xT$ and thus $x = 0$.

b)$\Rightarrow$c) Let $N \subseteq M_r$; if $N = 0$ there is nothing to prove; otherwise, for each
\(0 \neq x \in \mathbb{N}\), we have \([x, \hat{M}] \neq 0\), i.e., there exists \(\varphi \in \hat{M}\) such that \([x, \varphi] \neq 0\); thus there exists \(y \in \hat{M}\) with \(0 \neq [x, \varphi] y = x(\varphi, y) \in x \mathbb{N} \mathbb{M}\), what proves that \(N \mathbb{M} \subseteq \mathbb{N}\).

c) \(\Rightarrow\) a) Let \(0 \neq x \in \mathbb{M}\); thus we get \(x \mathbb{R} \neq 0\) and \(x \mathbb{T} = (x \mathbb{R}) \mathbb{T} \subseteq x \mathbb{R}\), whence \(x \mathbb{T} \neq 0\). 

A module which satisfies the equivalent conditions of (6.9) is called a non-degenerate module. If \(M_{\mathbb{R}}\) is a generator of \(\text{Mod}_{\mathbb{R}}\) (i.e., if \(t_{\mathbb{R}}(M) = \mathbb{N}\) for all \(N\) in \(\text{Mod}_{\mathbb{R}}\), see Section 8) then \(T = t_{\mathbb{R}}(M) = \mathbb{R}\) and hence \(M_{\mathbb{R}}\) is non-degenerate (and, as we already remarked, self-generator). However, none of these conditions implies that \(M_{\mathbb{R}}\) is a generator [30; p.387].

Let \(\mathfrak{C}_{\mathbb{R}}\) stand for the set of right complement ideals of \(\mathbb{R}\). Part d) of the following proposition, namely that every right complement in \(\mathbb{R}\) is \(\tau\)-closed (i.e., \(\mathfrak{C}_{\mathbb{R}} \subseteq \mathfrak{C}_{\mathbb{R}}\)), whenever \(M_{\mathbb{R}}\) is non-degenerate, suggests that non-degeneracy is a suitable condition under which we will be able to obtain correspondence theorems for right complements.

**Proposition 6.10** Let \(M_{\mathbb{R}}\) be non-degenerate. Then

a) for any \(0 \neq N \subseteq M_{\mathbb{R}}\) we have \(\tau_{\mathbb{S}}(N) \neq 0\);

b) if \(\mathfrak{A} \subseteq \mathfrak{B}\) are right ideals of \(\mathbb{S}\) then \(\mathfrak{A} \subseteq \mathfrak{B} \Leftrightarrow \sigma_{\mathbb{H}}(\mathfrak{A}) \subseteq \sigma_{\mathbb{H}}(\mathfrak{B})\);

c) for any right ideal \(\mathfrak{A}\) of \(\mathbb{S}\), \(\mathfrak{A} \subseteq \tau_{\mathbb{S}} \sigma_{\mathbb{H}}(\mathfrak{A})\);

d) every right complement in \(\mathbb{S}\) is \(\tau\)-closed;

e) for all \(N \subseteq M_{\mathbb{R}}\) we have \(\sigma_{\mathbb{H}}(\tau_{\mathbb{S}}(N)) \subseteq N_{\mathbb{S}}\);

f) if \(N \subseteq K\) are submodules of \(M_{\mathbb{R}}\) then \(N \subseteq K \Leftrightarrow \tau_{\mathbb{S}}(N) \subseteq \tau_{\mathbb{S}}(K)\);

g) \(M_{\mathbb{R}}\) is an \(a\)-self-generator.

**Proof:** a) Let \(0 \neq N \subseteq M_{\mathbb{R}}\); then (6.9) gives \(0 = [N, \hat{M}]\) and hence \(0 = \tau_{\mathbb{S}}(N)\), since clearly \([N, \hat{M}] \subseteq \tau_{\mathbb{S}}(N)\).

b) Assume first that \(\mathfrak{A} \subseteq \mathfrak{B}\) (as right \(\mathbb{S}\)-modules); for all \(0 \neq x \in \sigma_{\mathbb{H}}(\mathfrak{B})\) (which has the form \(x = \sum_{i} f_{i} [x_{i}, \hat{M}] \subseteq \mathbb{B}\), for some finite set \(I\) and some \(f_{i} \in \mathfrak{B}\), \(x_{i} \in \mathbb{M}\)), we have \(0 = [x, \hat{M}] = \sum_{i} f_{i} [x_{i}, \hat{M}] \subseteq \mathbb{B}\), and hence \(0 = \mathfrak{A} \cap [x, \hat{M}]\); thus

\[0 = ([\mathfrak{A} \cap [x, \hat{M}]]_{\mathbb{M}})_{\mathbb{M}} \subseteq \mathfrak{A} \cap [x, \hat{M}]_{\mathbb{M}} = \mathfrak{A} \mathbb{M} x \mathbb{R} \subseteq \mathfrak{A} \mathbb{M} x \mathbb{R},\]

whence \(\mathfrak{A} \mathbb{M} = \sigma_{\mathbb{H}}(\mathfrak{A}) \subseteq \sigma_{\mathbb{H}}(\mathfrak{B})\).

Conversely, if \(\sigma_{\mathbb{H}}(\mathfrak{A}) \subseteq \sigma_{\mathbb{H}}(\mathfrak{B})\) then, for all \(0 \neq f \in \mathfrak{B}\), \(f_{\mathbb{M}}\) is nonzero, and hence \(f \mathbb{M} \sigma_{\mathbb{H}}(\mathfrak{A}) \neq 0\) whence, by non-degeneracy,

\[0 \neq [f \mathbb{M} \sigma_{\mathbb{H}}(\mathfrak{A}), \hat{M}] \subseteq [f \mathbb{M}, \hat{M}] \cap [\mathfrak{A} \mathbb{M}, \hat{M}] \subseteq f \mathfrak{A} \mathfrak{B} .\]

Therefore \(\mathfrak{A} \subseteq \mathfrak{B}\), and this finishes the proof of b).
c) Since, for all \( \mathcal{A} \subseteq \mathcal{S} \), we have \( \mathcal{A} \subseteq \sigma_s(\mathcal{A}) \) and \( \sigma_s(\mathcal{A}) = \sigma_s(\mathcal{A}) \), b) gives \( \mathcal{A} \subseteq \sigma_s(\mathcal{A}) \).

d) If \( \mathcal{A} \) is a complement then c) implies \( \mathcal{A} = \sigma_s(\mathcal{A}) \), as required.

e) Let \( \mathcal{A} \neq \{0\} \) and \( \mathcal{B} \neq \{0\} \); then, by a), \( \tau_s(\mathcal{A}) \neq \{0\} \), and for any \( \mathcal{A} \neq \{0\} \)
we have \( 0 \neq f \mathcal{M} \subseteq x\mathcal{N} \subseteq \tau_s(\mathcal{M}) \subseteq \mathcal{M} \), whence \( \tau_s(\mathcal{M}) \subseteq \mathcal{N} \). If \( \mathcal{N} = 0 \)
then the result is obvious.

f) The cases \( \mathcal{N} \neq 0 \) or \( \mathcal{K} = 0 \) are trivial. Assume then \( \mathcal{N} \neq 0 \subseteq \mathcal{K} \) whence, by e),
\( \sigma_s(\mathcal{N}) \subseteq \sigma_s(\mathcal{K}) \subseteq \mathcal{K} \) and \( \sigma_s(\mathcal{N}) \subseteq \mathcal{K} \); thus (1.2.a) \( \sigma_s(\mathcal{N}) \subseteq \sigma_s(\mathcal{K}) \)
and hence, by b), \( \tau_s(\mathcal{N}) \subseteq \tau_s(\mathcal{K}) \). Conversely, if \( \tau_s(\mathcal{N}) \neq \tau_s(\mathcal{K}) \) then, by
b) and e), \( \sigma_s(\mathcal{N}) \neq \sigma_s(\mathcal{K}) \); but \( \sigma_s(\mathcal{N}) \neq \mathcal{K} \), and therefore \( \mathcal{N} \neq \mathcal{K} \) by
(1.2.a).

g) Let \( \mathcal{N} \neq \mathcal{M} \); we have to prove that \( \mathcal{N} = [\sigma_s(\mathcal{N})]^{\mathcal{A}} \), and for this it will
suffice to see that \( \mathcal{L}_s(\mathcal{N}) = 1 \sigma_s(\mathcal{N}) \), since then the action of \( \mathcal{L}_s(\mathcal{N}) \)
on both sides will yield the desired equality. Also, since \( \sigma_s(\mathcal{N}) \subseteq \mathcal{N} \), it
will suffice to prove that \( \mathcal{L}_s(\mathcal{N}) \subseteq \mathcal{N} \).
Let then \( f \in 1 \sigma_s(\mathcal{N}) \); thus \( f(\sigma_s(\mathcal{N})) = 0 \), which clearly implies
\( f\tau_s(\mathcal{N}) = 0 \); now, since \( [\mathcal{N}, \mathcal{M}] \subseteq \tau_s(\mathcal{N}) \), we get \( [f\mathcal{N}, \mathcal{M}] = f[\mathcal{N}, \mathcal{M}] \subseteq f\tau_s(\mathcal{N}) = 0 \)
which, by non-degeneracy, implies \( f\mathcal{N} = 0 \), i.e. \( f \in 1_s(\mathcal{N}) \), as required.

Now, we are ready to prove the following corollary theorem:

**THEOREM 6.11** Let \( \mathcal{M}_r \) be non-degenerate and let \( \mathcal{M}_2 = \{ \mathcal{N} \in \mathcal{M}_r : \mathcal{N} \) is
\( \mathcal{M} \)-cotorsionless and \( \tau_s(\mathcal{N}) \subseteq \mathcal{N} \}. \) Then \( \tau_s : \mathcal{M}_2 \rightarrow \mathcal{E}_r \) and \( \sigma_s : \mathcal{E}_r \rightarrow \mathcal{M}_2 \) are
inverse projectivities.

**PROOF:** Since \( \mathcal{M}_2 \subseteq \mathcal{M}_r \) by definition and \( \mathcal{E}_r \subseteq \mathcal{E}_r \) (6.10), we just have to
prove that \( \tau_s(\mathcal{M}_2) \subseteq \mathcal{E}_r \) and that \( \sigma_s(\mathcal{E}_r) \subseteq \mathcal{M}_2 \). The first inclusion follows
directly from the definition of \( \mathcal{M}_2 \), and if \( \mathcal{A} \in \mathcal{E}_r \) then \( \mathcal{A} = \tau_s(\mathcal{A}) \) and
hence \( \sigma_s(\mathcal{A}) \in \mathcal{M}_2 \).}

A ring \( R \) is a **right (left) CS-ring** if \( R \mathcal{M} \) (\( \mathcal{M} R \)) is a CS-module. And \( R \) is
said to be a right Goldie ring if it is a **right finite dimensional ring** (i.e. \( R \mathcal{M} \) is f.d.) with ACC on right annihilators.

**COROLLARY 6.12** Let \( \mathcal{M}_r \) be non-degenerate. Then

a) \( S \) is a right CS-ring if and only if every \( \mathcal{N} \in \mathcal{M}_2 \) is a direct
b) if $M_r$ has ACC on $M_{\sigma r}$ then $S$ is a right Goldie ring.

**Proof:**
\hspace{1cm}a) follows from (6.11) and (6.2).

b) Assume that $M_r$ has ACC on $M$-cotorless submodules and let $\mathcal{R}(W_1)\subseteq \mathcal{R}(W_2)\subseteq \cdots$ be an ascending chain in $\mathcal{A}$; then, by hypothesis, $\sigma^*_\mathcal{H}(W_1)\subseteq \sigma^*_\mathcal{H}(W_2)\subseteq \cdots$ gets stationary at some step $n$ and then, for $k\leq n$, we get $\sigma^*_\mathcal{H}(W_k)=\mathcal{R}(W_k)$, and hence $\mathcal{R}(W_k)$ is ACC on $M_r$ for all $k\leq n$. Therefore $S$ has ACC on right annihilator ideals (note that we can prove in the same way that DCC on $M$-cotorless submodules implies DCC on right annihilators).

To see that $S$ is right Goldie, it remains to show that it is right f.d., but ACC on $M$-cotorless submodules implies ACC on $M_2$, which in turn implies ACC on $\mathcal{A}$ (6.11) and hence (2.1) $S$ is right f.d.

In the previous paragraph, we had to introduce the notion of a-self-generator in order to get $M_\sigma=M_1$. Now, it would be of interest to get conditions under which $M_2$ coincides with the set of complement submodules of $M_r$. Two of the concepts already introduced will suffice to get $M_2=M_e$, though in this case these conditions are not necessary.

**Theorem 6.13** Let $M_r$ be non-degenerate. If $M_r$ is a self-generator or a CS-module, then $\tau:S\mathcal{H} \rightarrow \mathcal{A}$ and $\sigma:S\mathcal{A} \rightarrow \mathcal{H}$ are inverse projectivities.

**Proof:** By (6.11), it will suffice to prove that, under the stated conditions, $M_2=M_e$.

Let $NeM_2$ and let $K$ be an e-closure for $N$ in $M$. Since $N\in M$, $\tau(N)\subseteq \sigma\mathcal{S}(K)$ by (6.10.f), but we have $\tau(N)\subseteq \mathcal{N}$, whence $\tau(N)=\tau(S)$ and $N=\sigma\mathcal{S}(N)=\sigma\mathcal{S}(K)$. If $M_r$ is a self-generator then $K$ is $M$-cotorless, and if $M_r$ is a CS-module then $K\subseteq M$; in any case $K=\sigma\mathcal{S}(K)=N$ and therefore $N$ is a complement in $M$.

For the converse inclusion, let $K\subseteq M$; then, as above, $K$ is $M$-cotorless, and hence it remains to show that $\tau(S)(K)=N$. Let $\mathcal{A}$ be an e-closure for $\tau(S)(K)$ in $S\mathcal{A}$; then $\tau(S)(K)\subseteq \sigma\mathcal{A}$ and, by (6.10.b), $K=\sigma\mathcal{S}(K)\subseteq \sigma\mathcal{S}(\mathcal{A})$, which implies $K=\sigma\mathcal{S}(\mathcal{A})$ and therefore $\mathcal{A}\subseteq \sigma\mathcal{S}(\mathcal{A})=\tau(S)(K)$, i.e. $\tau(S)(K)=\mathcal{A}\subseteq \mathcal{A}$ and hence $K\subseteq M_2$.■
COROLLARY 6.14 \ Let \( M_r \) be non-degenerate. Then
a) if \( M_r \) is a self-generator or a CS-module, then \( u(M_r) = u(S_S) \);
b) if \( M_r \) is a self-generator then \( M_r \) is a CS-module if and only if \( S \) is a right CS-ring;
c) if \( M_r \) is a CS-module then \( S \) is a right CS-ring.

PROOF: \( a) \) follows from (6.13) and (2.7); \( b) \) and \( c) \) follow from (6.12.a), using the fact that \( M_M = M_e \).

The conditions imposed on the non-degenerate module \( M_r \) in (6.13) are not the only ones under which \( M_e \) and \( S_r \) are isomorphic. The next result makes further use of the uniqueness of the e-closures in a nonsingular module to show that also nonsingularity of \( M_r \) implies the existence of such an isomorphism, although in this case we have to change slightly the definition of our maps.

THEOREM 6.15 \ Let \( M_r \) be nonsingular and non-degenerate. Then the maps \( \tau_S : M_M \rightarrow S_r \) and \( \lambda \rightarrow [\sigma_M(\lambda)]^e \) from \( S_r \) to \( M_M \) are inverse projectivities.

PROOF: First, we see that \( \tau_S(M_M) \leq S_r \); if \( K \leq M_M \) and \( \mathfrak{A} \) is an e-closure for \( \tau_S(K) \) in \( S_S \), then (6.10.b) \( \sigma_M \tau_S(K) \leq \sigma_M(\mathfrak{A}) \); but (6.10.e) \( K \) is the e-closure of \( \sigma_M \tau_S(K) \) and hence, by nonsingularity, also of \( \sigma_M(\mathfrak{A}) \), i.e. \( K = [\sigma_M(\mathfrak{A})]^e \); in particular, \( \sigma_M(\mathfrak{A}) \leq K \) and hence \( \mathfrak{A} \leq \sigma_M(\mathfrak{A}) \leq \sigma_S(K) \), i.e. \( \tau_S(K) = \mathfrak{A} \leq S_r \).

Since the image of \( \mathfrak{A} \) under \( \mathfrak{A} \rightarrow [\sigma_M(\mathfrak{A})]^e \) is in \( M_M \), it only remains to show that both maps are inverse of each other.

As we have mentioned above, \( K = [\sigma_M \tau_S(K)]^e \) for all \( K \leq M_M \); on the other hand, if \( \mathfrak{A} \leq S_r \), then \( \mathfrak{A} = \tau_S \sigma_M(\mathfrak{A}) \) by (6.10.d); since \( \sigma_M(\mathfrak{A}) \leq [\sigma_M(\mathfrak{A})]^e \), (6.10.f) yields \( \mathfrak{A} \leq \sigma_S([\sigma_M(\mathfrak{A})]^e) \) and hence \( \mathfrak{A} = \tau_S([\sigma_M(\mathfrak{A})]^e) \), showing that both maps are inverse of each other.

COROLLARY 6.16 \ Let \( M_r \) be nonsingular and non-degenerate. Then
a) \( u(M_r) = u(S_S) \);
b) \( S \) is a right CS-ring if and only if \( M_r \) is a CS-module.

PROOF: \( a) \) follows from (6.15) and (2.7), and \( b) \) follows from (6.15) and (6.2).
Correspondence Theorems for Principal and Finitely Generated Ideals

Write \( P_l \) and \( P_r \) for the sets of left and right principal ideals of \( S \):

\[
P_l = \{ Sf : f \in S \}, \quad P_r = \{ fS : f \in S \}.
\]

Also, let \( F_l \) and \( F_r \) represent the sets of finitely generated left and right ideals of \( S \), that is

\[
F_l = \{ \sum_{i=1}^{n} Sf_i : f_i \in S \}, \quad F_r = \{ \sum_{i=1}^{n} f_i S : f_i \in S \}.
\]

Recall that \( r^l(Sf) = \ker f \) and \( \sigma^l(fS) = \text{im} f \) for all \( f \in S \); hence, if we write

\[
K = \{ \ker f : f \in S \}, \quad J = \{ \text{im} f : f \in S \},
\]

then \( r^l(P_l) = K \) and \( \sigma^l(P_r) = F_r \). Further, if we set

\[
K_f = \{ \sum_{i=1}^{n} \ker f_i : f_i \in S \}, \quad J_f = \{ \sum_{i=1}^{n} f_i S : f_i \in S \},
\]

then \( r^l(F_l) = K_f \) and \( \sigma^l(F_r) = J_f \).

Since, on the other hand, \( K \) and \( K_f \) are always included in \( M \), while \( J \) and \( J_f \) always lie in \( M_{\tau T} \), we get the following equivalences:

**THEOREM 6.17**  With the above notation,

a) \( \tau_M, r_M \) determine a duality between \( K \) and \( P_1 \sim P_1 \leq \mathcal{A} \);  
b) \( \tau_S, \sigma_M \) determine a projectivity between \( J \) and \( P_r \sim P_r \leq \mathcal{J} \);  
c) \( \tau_M, r_M \) determine a duality between \( K_f \) and \( F_1 \sim F_1 \leq \mathcal{A} \);  
d) \( \tau_S, \sigma_M \) determine a projectivity between \( J_f \) and \( F_r \sim F_r \leq \mathcal{J} \).

Recall that a ring is **regular** if every principal (left or right) ideal (equivalently, every f.g. left or right ideal) is generated by an idempotent; and that a ring is **left (right) perfect** if its principal (equivalently, f.g.) right (left) ideals satisfy the DCC [5]. Therefore we get, from (6.17) and (6.2),

**COROLLARY 6.18**  With the above notation, and if \( D = \{ N \leq \mathcal{M} : N \leq M \} \),

a) if \( P_1 \leq \mathcal{A} \) then: \( S \) is regular \( \sim \) \( K \leq D \);  
a') if \( P_1 \leq \mathcal{A} \) then: \( S \) is right perfect \( \sim \) \( M \) has ACC on \( K \);  
b) if \( P_r \leq \mathcal{J} \) then: \( S \) is regular \( \sim \) \( J \leq D \);  
b') if \( P_r \leq \mathcal{J} \) then: \( S \) is left perfect \( \sim \) \( M \) has DCC on \( J \);
c) if $\mathcal{F} \subseteq \mathcal{G}'$ then: $S$ is regular $\iff \mathcal{K} \subseteq \mathcal{D}$;  
c') if $\mathcal{F} \subseteq \mathcal{G}'$ then: $S$ is right perfect $\iff M$ has ACC on $\mathcal{K}$;  
d) if $\mathcal{F} \subseteq \mathcal{G}'$ then: $S$ is regular $\iff \mathcal{I} \subseteq \mathcal{D}$;  
d') if $\mathcal{F} \subseteq \mathcal{G}'$ then: $S$ is left perfect $\iff M$ has DCC on $\mathcal{I}$.

Now, it is of interest to seek for conditions on $M_r$ under which one of the equivalent conditions of (6.17) holds. In fact, conditions a) and c) (resp. b) and d)) hold in any quasi-injective module (resp. quasi-projective module), but we shall postpone the proof of this until Section 10 since then we will have introduced the concept of T-nilpotency, which will be needed when applying these facts.

Nevertheless, we can prove now the following result, which gives as a corollary the fact that $S$ is regular if and only if $\mathcal{K} \subseteq \mathcal{D}$ and $\mathcal{I} \subseteq \mathcal{D}$, with the notation of (6.18).

**Proposition 6.19**

a) If $\mathcal{I} \subseteq \mathcal{D}$ then $\mathcal{P} \subseteq \mathcal{G}'$;  
b) if $\mathcal{K} \subseteq \mathcal{D}$ then $\mathcal{P} \subseteq \mathcal{G}'$.

**Proof:**
a) Assume $\mathcal{I} \subseteq \mathcal{D}$ and let $f \in S$; we need to prove that $Sf = \mathcal{R}_r(Sf)$ or, equivalently, that $\mathcal{R}_r(Sf) \subseteq Sf$. Let $g \in \mathcal{R}_r(Sf)$; then $\text{Ker}f \subseteq \text{Ker}g$ and hence $h_1 : \mathcal{M} \to g\mathcal{M}$ given by $h_1(fx) = gx$ is a well-defined $\mathcal{R}$-homomorphism. Now, if $M = f\mathcal{M} \oplus N$ and we define $h \in S$ by $h|_{f\mathcal{M}} = h_1$ and $h|_N = 0$, then $hf = g$ and therefore $g \in Sf$, as required.

d) Assume $\mathcal{K} \subseteq \mathcal{D}$ and let $f \in S$; we have to prove that $\tau \mathcal{G}_r(fS) \subseteq fS$. Let $g \in \mathcal{G}_r(fS)$, i.e. $g \in \mathcal{M}$, and let $t, q$ be idempotents of $S$ such that $\text{Ker}f = \text{Ker}t$ and $\text{Ker}g = \text{Ker}q$ (5.4.a).

For all $x \in \mathcal{M}$, there exists $y \in \mathcal{M}$ with $gx = fy$; we claim that the map $h_1 : \mathcal{M} \to \mathcal{M}$ given by $h_1(qx) = ty$ is well-defined (and hence it is clearly an $\mathcal{R}$-homomorphism); for, if $qx = qx'$ then $x - x' \in \text{Ker}q = \text{Ker}t$, so that $gx = gx'$, and similarly $fy = fy'$ implies $ty = ty'$.

Define then $h \in S$ by $h|_{q \mathcal{M}} = h_1$ and $h|_{\text{Ker}q} = 0$; we shall prove that $g = hf$, which will finish the proof. First note that, for any $z \in \mathcal{M}$, $z = tz + (1-t)z$ with $(1-t)z \in \text{Ker}t = \text{Ker}q$, so that $fz = ftz$. Now, for any $x \in \mathcal{M}$, $x = qx + (1-q)x$ and, if $y \in \mathcal{M}$ is such that $gx = fy$, then $hx = hq x = h_1 qx = ty$ and hence $f hx = fty = fy = gx$, so that $g = fh$.

**Corollary 6.20**

For any module $M_r$, $S$ is a regular ring if and only if the kernel and the image of every endomorphism of $M_r$ are direct summands of $M_r$.  

---  

A. del Valle; MSc, 1992; Rings of Endomorphisms; Sec.6
PROOF: Assume that $S$ is regular; then for every $f \in S$ there exist idempotents $t, q$ of $S$ such that $fS = tS$ and $Sf =Sq$, and hence we get

$$ fM = fSM = tSM = tM \in D \quad \text{and} \quad \text{Ker} f = r_H^*(Sf) = r_H^*(Sq) = \text{Ker} q \in D. $$

Conversely, if $K \in D$ and $J \in D$ then either (6.19.a) and (6.18.a) or (6.19.b) and (6.18.b) imply that $S$ is regular.

Finally, we prove a lemma which will allow us to apply our previous results to nonsingular continuous modules. A module $M$ is said to be continuous if it is a CS-module such that every submodule of $M$ is isomorphic to a direct summand of $M$.

**Lemma 6.21** Let $M$ be a module for which $K \in D$. Then $J \in D$ if and only if, for all $N \in D$ and for every monomorphism $h : N \to M$, we have $hN \in D$.

**Proof:** The 'only if' part does not need the hypothesis $K \in D$: if $J \in D$ and $N$, $h : N \to M$ are as stated, take $K \in M$ such that $M = N \oplus K$ and extend $h$ to $f \in S$ by requiring $f|_K = 0$; then $hN = fN$ which is in $D$ by assumption. Conversely, assume that $K \in D$ and that $hN \in D$ for all $N \in D$ and every monomorphism $h : N \to M$, and let $f \in S$; thus $M = \text{Ker} f \oplus K$ for some $K \in M$; hence $h = f|_K : K \to M$ is monic and thus $fM = hK \in D$. Therefore $J \in D$.

**Corollary 6.22** a) If $M$ is continuous then $S$ is regular $\iff K \in D$;

b) if $M$ is a nonsingular CS-module then $S$ is regular $\iff J \in D$;

c) if $M$ is nonsingular and continuous then $S$ is a regular ring.

**Proof:** a) The 'only if' part follows from (6.20). If $M$ is continuous and $K \in D$ then (6.21) implies $J \in D$ and hence $S$ is regular by (6.20).

b) If $M$ is nonsingular and $C$ then $M = \text{M} \oplus \text{M}$ (6.7.a) and $\text{M} = D$, whence $K \subseteq \text{M} \subseteq \text{M} = D$. Therefore $S$ is regular if and only if $J \in D$ by (6.20).

c) If $M$ is nonsingular and continuous then, as above, $K \in D$ and hence $S$ is regular by a).
**Section 7: The Endomorphism Ring of a Quasi-Injective Module**

Throughout this section we shall study the ring $S$ of endomorphisms of a quasi-injective module $M$. $J=J(S)$ will denote the Jacobson radical of $S$. The main results in this area concern, rather than $S$ itself, the factor ring $S/J$, which is sometimes called the associated ring of $M$. They are due to B. Osofsky [41], J. Roos [45] and G. Rénault [R75 & 44], among others, who followed techniques introduced by Y. Utumi [57], E. Wong and R. Johnson [64] to show that $S/J$ is a regular and right self-injective ring (7.11).

Further results on the endomorphism ring of quasi-injective or injective modules with some finiteness conditions (such as chain conditions or finite dimension) will be proved in Sections 9 and 10.

Let us denote the factor ring $S/J$ by $\overline{S}$ and, for any $f \in S$, let $\overline{f}$ be its image in $\overline{S}$. The ideal $\Gamma=\Gamma(S)$, introduced in the following lemma, will be of key importance in what follows.

**Lemma 7.1** Let $M$ be any module and let $S=\text{End}(M)$. Then the set $\Gamma(S)=\{f \in S : \ker f \subseteq E\}$ is an ideal of $S$.

**Proof:** Let $f, g \in \Gamma(S)$ and $h \in S$. Then $\ker f \subseteq M$ and $\ker g \subseteq M$, whence $\ker f \cap \ker g \subseteq M$. Since $\ker f \cap \ker g \subseteq \ker (f+g)$, we have $f+g \in \Gamma(S)$; and since $\ker f \subseteq \ker (hf)$, we get $hf \in \Gamma(S)$. Note that $\ker (fh) = h^{-1}(\ker f)$; then (1.2.d) gives $\ker (fh) \subseteq M$, whence $fh \in \Gamma(S)$. Therefore $\Gamma(S)$ is a two-sided ideal of $S$. ■

The next proposition was first proved, for $M$ injective, by Y. Utumi [55; Lemma 8].

**Proposition 7.2** If $M$ is quasi-injective then $J(S)=\Gamma(S)$ and $\overline{S}$ is a (von Neumann) regular ring.

**Proof** [13; Theo.3.1.a]: Write $J=J(S)$ and $\Gamma=\Gamma(S)$. First we prove that $\Gamma \subseteq J$: Let $f \in \Gamma$; $\ker f \cap \ker (1-f)=0$ implies $\ker (1-f)=0$, whence $(1-f):M \rightarrow (1-f)M$ is an isomorphism whose inverse $g:(1-f)M \rightarrow M$ extends to some $h \in S$, for which $h(1-f)=1$; thus every element of $\Gamma$ is left
quasi-regular and hence $T \subseteq J$ \cite[A-F; Theo.15.3].

Next we see that $S/\Gamma$ is a regular ring: Given $f \in S$, set $K = \ker f$ and take a relative complement $N$ for $K$ in $M$; then $f|_N$ is monic with inverse $g: fN \to N$. Since $M$ is quasi-injective, $g$ can be lifted to $h \in S$, and then $K \oplus NS \subseteq \ker (f - fhf)$: for, if $k \in K$ and $n \in N$, then

$$fhf(k + n) = fhf(n) = fgf(n) = f(n) = f(k + n).$$

Now, since $K \oplus N \subseteq M$, $f - fhf \in \Gamma$, i.e. $f$ and $fhf$ have the same image in $S/\Gamma$, and this shows that $S/\Gamma$ is regular.

Finally, we show that $J \subseteq \Gamma$, which will complete the proof. If $f \in J$, choose $h \in S$ such that $g = f - fhf \in \Gamma$; since $1 - fh$ has an inverse we get $f = (1 - fh)^{-1}g \in \Gamma$. Therefore $J \subseteq \Gamma$. $\blacksquare$

Our purpose now is to show that $\tilde{S}$ is a right self-injective ring. At a first stage, we will prove this for the endomorphism ring of an injective module, but at the end of the section we will see that the result also holds for quasi-injectives. We need some technical lemmas about the lifting of idempotents from $\tilde{S}$ to $S$; the proofs given here are due to G. Rénault \cite{44 or R75}.

**PROPOSITION 7.3** If $E_r$ is injective and $S = \text{End}(E_r)$, then every idempotent of $\tilde{S}$ can be lifted to an idempotent of $S$.

**Proof** \cite{13}: Suppose $f^2 = f \in \tilde{S}$; then $h = f^2 - f \in J$. If $f' = f|_{\ker h}$ then it is easy to see that $\ker h = \ker f \oplus \text{Im} f'$. Now, since $\ker h \subseteq E$ \cite[(7.2)], if $E_1$ and $E_2$ are injective hulls in $E$ of $\ker f$ and $\text{Im} f'$, respectively, then $E = E_1 \oplus E_2$ \cite[(3.8)]. Let $t: E_1 \oplus E_2 \to E_2$; then $\ker h \subseteq \ker(t - f)$ for if $x \in \ker h$ is written as $x = y + fz$ (with $y \in \ker f$, $z \in \ker h$) then

$$tx = fz = (f^2 - h)z = f^2z = f(fz + y) = fx.$$ 

This shows that $t - f \in J$ (because $h \in J$) and hence $f = t^2$ with $t^2 = t \in S$. $\blacksquare$

**Remark** By \cite[p.186-7]{S}, (7.3) shows that any countable family of orthogonal idempotents of $\tilde{S}$ lifts to a family of orthogonal idempotents of $S$. The same remark applies to the following corollary.

**COROLLARY 7.4** Let $R$ be a right self-injective ring, and let $J(R)$ be
the Jacobson radical of $R$. Then $J(R) = Z_r(R)$, $R/J(R)$ is a regular ring and idempotents can be lifted modulo $J(R)$.

proof [57; Theo. 4.6 & Cor. 4.10]: The last two statements are direct consequences of (7.2) and (7.3), using the ring isomorphism $S = \text{End}(R_R) \cong R$ (see Section 5).

To see that $J(R) = Z_r(R)$, note that the ring isomorphism $\varphi: R \to S$ carries $r \in R$ to the endomorphism "left multiplication by $r"$, and then $\text{Ker}(\varphi(r)) = R(r)$ for all $r \in R$. Therefore

$$r \in J(R) \Leftrightarrow \varphi(r) \in J(S) \Leftrightarrow \text{Ker}(\varphi(r)) \subseteq R(r) \Rightarrow r \in Z_r(R).$$

Hence $J(R) = Z_r(R)$.

**Lemma 7.5** Suppose $E_\alpha$ is injective and $S = \text{End}(E_\alpha)$. Let $\{e_I : i \in I\}$ be a family of idempotents of $S$ such that the sum $\sum_{i \in I} e_i$ is direct. Then the sum $\sum_{i \in I} t_i e_i$ is direct and, if $I$ is finite, then there exists $t^2 = t \in S$ such that $\sum_{i \in I} t_i e_i = t S$.

**proof**: Since in general a sum $\sum_{A_i}$ is direct if and only if so is $\sum_{F} A_i$ for every finite subset $F$ of $I$, we can assume $I$ to be finite, say $I = \{1, \ldots, n\}$.

Set $E_i = t_i E$ for $i = 1, \ldots, n$. If we prove that the $E_i$ are independent then so are the $t_i e_i$: for, suppose e.g. that $t_{I} f_{I} = t_{I} f_{I} + \cdots + t_{I} f_{I}$ is an element of $(t_S)(\sum_{I} t_i e_i)$; then for all $x \in E$ we get

$$t_{I} f_{I}(x) = \sum_{I} t_{I} f_{I}(x) \in E(\sum_{I} t_i e_i) = 0;$$

hence $t_{I} f_{I} = 0$.

Let us now prove that $\sum_{I} t_i e_i \in E_i$ by induction in $n$. Suppose the sum $F = \sum_{I} t_i e_i$ is direct and $G = F \cap E_i \neq 0$; let $H$ be an injective hull of $G$ contained in $E_i$ and let $e^2 = e \in S$ be such that $H = eE$. Then clearly $e = t_1 e$.

Since $F$ is injective, the inclusion map $G \to F$ lifts to some $h: E \to F$ for which we have $h = \sum_{I} t_i h$. For, if $x \in E$ then $hx = \sum_{I} t_i y_i$ for some $y_i \in E$, whence

$$\left( \sum_{I} t_i h \right) x = \sum_{I} \left( \sum_{I} t_i t_i y_i \right) = \sum_{I} t_2 y_i = \sum_{I} t_i y_i = hx.$$

Now, since $G \subseteq H$ and $e - H e \subseteq \text{Ker}(e)$, we get $G \cap \text{Ker}(e) \subseteq e E$; but $e$ and he coincide in $G$ (in fact they are both the identity in $G$), and thus so do they in $G \cap \text{Ker}(e)$, whence $e = \bar{e}$. 


Thus, we get $\bar{t}e = e = \bar{t}e = \sum_{i=1}^{n} \bar{t}_i \bar{e}$ with $e = 0$ (Ker($e)$) $\cap$ $H = 0$ with $H \neq 0$), contradicting the hypothesis that the sum $\sum_{i=1}^{n} \bar{t}_i S$ is direct. Therefore $\sum_{i=1}^{n} t_i S$ is direct, and then so is $\sum_{i=1}^{n} t_i S$.

Now, each $E_i$ is injective and thus so is $\sum_{i=1}^{n} E_i$; hence there exists $E = \sum_{i=1}^{n} E_i$; let $q : \sum_{i=1}^{n} E_i \to E$, so that, for $i = 1, \ldots, n$, we have $q_i = q_i q_i$, and call $t = q_1 + \cdots + q_n$. Then, for all $f \in S$, we get

$$t f = \sum_{i=1}^{n} q_i f = \sum_{i=1}^{n} t_i f \in \sum_{i=1}^{n} t_i S,$$

and, for all $f_1, \ldots, f_n \in S$, we get $(\sum_{i=1}^{n} t_i f) E \subseteq \sum_{i=1}^{n} E_i$ and thus

$$(\sum_{i=1}^{n} t_i f) = t (\sum_{i=1}^{n} t_i f) \in t S.$$

Therefore $t S = \sum_{i=1}^{n} t_i S$, which proves the last statement of the lemma.

**Lemma 7.6** With the above notation, let $\{t_i, i \in I\}$ be a family of idempotents of $S$ such that the sum $\sum_{i=1}^{n} t_i S$ is direct, and let $B = \sum_{i=1}^{n} t_i S$.

Then every homomorphism $\phi : B \to S$ extends to an endomorphism of $S$.

**Proof:** Consider the right $R$-module $F = \sum_{i=1}^{n} E_i$ and define $g : F \to E$ as follows: If $z = \sum_{j=1}^{n} t_{j} f_{j} x_{j}$ for some finite set $J$ (where, for all $j \in J$, $f_j \in S$, $x_j \in E$ and the $t_j$'s are elements of $\{t_i, i \in I\}$, possibly repeated) then set $g(z) = \sum_{j=1}^{n} \phi(t_j f_j) x_j$.

If $g$ is well-defined, then it is clearly an $R$-homomorphism. To see that $g$ is actually single-valued, suppose another expression of $z$ is given and let $K$ be the (finite) set consisting of those members of $\{t_i, i \in I\}$ which appear in any of these two expression. Then, by (7.5), there exists an idempotent $t$ of $S$ such that $\sum_{k=1}^{K} t_k S = t S$, and then $t_j = t t_j$ for all $j \in J$; hence

$$g(z) = \sum_{j=1}^{n} \phi(t_j f_j) x_j = \sum_{j=1}^{n} \phi(t) t_j f_j x_j = \phi(t) z$$

independently of the representation of $z$.

Therefore $g$ extends to some $h \in S$; now, if $f \in B$ may be written as $f = \sum_{k=1}^{K} t_k f_k$ for some finite subset $K$ of $I$ and some $f_k \in S$, then, for all $x \in E$,

$$\phi(f) x = \phi(\sum_{k=1}^{K} t_k f_k x) = \sum_{k=1}^{K} \phi(t_k f_k x) = g(\sum_{k=1}^{K} t_k f_k x) = h(\sum_{k=1}^{K} t_k f_k x) = (hf) x.$$

Therefore $\phi(f) = hf$ for all $f \in B$ and thus $\phi$ can be extended to an endomorphism of $S$. ■
In view of (7.6), the following characterization of regular self-injective rings will clearly help us.

**Lemma 7.7** Let $R$ be a regular ring. Then $R$ is right self-injective if and only if for every right ideal $b$ of $R$ of the form $b = t_1 R$ (where $\{t_i : i \in I\}$ is a family of idempotents of $R$), every homomorphism $b R \to R^n$ can be extended to an endomorphism of $R^n$.

**Proof:** The necessity is clear. For the sufficiency let $a$ be any right ideal of $R$. By Zorn's Lemma there exists a maximal element $b$ in the family of all direct sums $\oplus t_i R \subseteq a$ where the $t_i$'s are idempotents of $R$, and since $R$ is regular this maximal element is an essential submodule of $a$ (recall that every principal right ideal of $R$ is generated by an idempotent).

Then for an arbitrary $f : a \to R$ let $g = f|_b$ and let $h \in \text{End}(R^n)$ be an extension of $g$. Then, for all $x \in a$, $b = (b : x)$ is an essential right ideal of $R$ (1.1) and $(h-f)x = 0$; since $R$ is right nonsingular [S; p.244], we get $hx = fx$ and therefore $h$ extends $f$, as desired.

**Theorem 7.8** Let $E_R$ be an injective module, $S$ its endomorphism ring and $J = \text{J}(S)$ the Jacobson radical of $S$. Then $S/J$ is a regular right self-injective ring.

**Proof** [44; Theor.3.2] or [R75;p.85]: By (7.2), $\tilde{S} = S/J$ is regular. Then, by (7.7), it suffices to show that, for each right ideal of $\tilde{S}$ of the form $\tilde{S} = \oplus t_i \tilde{S}$ (with $\{t_i : i \in I\}$ a family of idempotents of $\tilde{S}$ which, by (7.5), may be taken in such a way that every $t_i^2 = t_i$ in $S$), and for each homomorphism $\phi : \tilde{S} \to \tilde{S}$, there exists a right $\tilde{S}$-endomorphism of $\tilde{S}$ which extends $\phi$.

Given $\phi$, let $\tilde{f}_i = \phi(t_i)$; then $\tilde{f}_i = \phi(t_i^2) = \phi(t_i) t_i = \tilde{f}_i t_i$. By (7.5), the sum $\tilde{S} = \oplus t_i S$ is direct, so that the correspondence $t_i \mapsto \tilde{f}_i$ defines a right $S$-homomorphism $\phi : \tilde{S} \to S$. By (7.6), there exists $f_0 \in S$ with $\phi(t_i) = f_0 t_i$ for all $i \in I$. Then the right endomorphism of $\tilde{S}$ defined by left multiplication by $f_0$ extends $\phi$, and this completes the proof of the theorem.

Theorem 7.8 yields as a corollary the following result of Y. Utumi [57; Theo.8].
COROLLARY 7.9  If \( R \) is a right self-injective ring then \( R/J(R) \) is also right self-injective.

The following proposition allows us to extend our results to quasi-injective modules.

PROPOSITION 7.10  Let \( M_r \) be a quasi-injective module, \( E_r \) its injective hull, \( S=\text{End}(M_r) \) and \( H=\text{End}(E_r) \). Then \( H/J(H) \) and \( S/J(S) \) are ring isomorphic.

PROOF:  Since \( M_r \) is quasi-injective, (3.10) implies that the map \( \phi:H \rightarrow S \) given by \( \phi(f)=f|_M \) is well defined. Clearly, it is a ring homomorphism and, since \( E_r \) is injective, \( \phi \) is surjective. Composing with the natural ring epimorphism \( \pi:S \rightarrow S/J(S) \) we get \( S/J(S) \cong H/\ker(\pi\phi) \), and then all we have to check is that \( J(H) \) coincides with \( \ker(\pi\phi)=\phi^{-1}(\ker\pi)=\phi^{-1}J(S) \). For, let \( f \in H \); then we have

\[
 f \in \phi^{-1}(J(S)) \leftrightarrow \phi(f) \in J(S) \leftrightarrow \ker(f|_M) \subseteq \ker(f) \subseteq M \cap \ker f \subseteq M \cap \ker f \subseteq E \cap f \in J(H),
\]

whence effectively \( J(H)=\phi^{-1}J(S) \).

Thus we get as a corollary the announced result of B.Osofsky [41; Theo.12], G.Renault [44; Cor.3.5] and J.E.Roos [45; p.176].

THEOREM 7.11  If \( M_r \) is a quasi-injective module, \( S \) its endomorphism ring and \( J=J(S) \) the Jacobson radical of \( S \), then \( S/J \) is a regular right self-injective ring.

There is an important case, namely when \( M_r \) is not only quasi-injective but also nonsingular, in which case \( J(S)=0 \) and then we obtain a result which, together with (7.4) and (7.9), was the motivation for the study undertaken in this section. It appeared in the form given here in [64; Theo.5], although the proof of the self-injectivity of \( S \) is attributed to Y.Utumi.

THEOREM 7.12  If \( M_r \) is a nonsingular quasi-injective module then \( S=\text{End}(M_r) \) is a regular right self-injective ring.

PROOF:  After (7.11) and (7.2) it suffices to see that \( \Gamma=\Gamma(S)=0 \). For,
let $f \in \Gamma$ and $K = \text{Ker} f \subseteq M$. For any $x \in M$ we have $a = (K : x) \subseteq \text{Rr}$ (1.1) and $xa \subseteq K$; then $(fx)a = 0$, and the nonsingularity of $M$ gives $fx = 0$. Therefore $f = 0$ and thus $\Gamma = 0$. □

**REMARKS**

1) Nonsingularity is necessary in (7.12). For example, consider the Abelian group $\mathbb{Z}_p$ (the $p$-primary component of $\mathbb{Q}/\mathbb{Z}$, where $p$ is any prime integer), which is an injective $\mathbb{Z}$-module, but its endomorphism ring is the ring of $p$-adic integers $[F; p.211]$, which is not self-injective $[41; p.897]$.

2) Any semisimple module is quasi-injective with $\Gamma(S) = 0$, so that the endomorphism ring of a semisimple module is right self-injective.

3) B. Osofsky has investigated when $S/J$ is also left self-injective. In $[41]$ she proves, for $E$ quasi-injective and using results of Utumi $[56]$ that $S/J$ is left self-injective if, for every orthogonal set $\{t_i : i \in I\}$ of idempotents of $S$, the map $\phi : E \to \prod_{i \in I} t_i E$ given by $\phi(m) = \langle t_i m \rangle_{i \in I}$ is onto. In particular, for a right vector space $V_D$ over a division ring $D$, $S = \text{End}(V_D)$ is always right self-injective by the previous remark, but $S$ is left self-injective if and only if $V$ is finitely generated (i.e. finite dimensional) $[G; \text{Prop. 2.23}]$. This may also be proved for free modules over QF-rings $[S; p.278]$. 

Although there do not exist results for the endomorphism ring of an arbitrary projective module as strong as those given for (quasi-)injective modules in Section 7 (but see (8.5)), the literature about the subject is fairly wide. In particular, Morita’s Theorem characterizes finitely generated projective generators of $\text{Mod}_R$ as those modules $M_R$ such that there exists a category equivalence between $\text{Mod}_R$ and $\text{Mod}_S$, where $S=\text{End}(M_R)$. We begin this section by recalling Morita’s Theorem and drawing some consequences.

Next, we study the Jacobson radical of $S=\text{End}(M_R)$ when $M_R$ is projective and, as a consequence, we determine when $S$ is a local ring. This prompts us to a brief introductory discussion of the so-called local, regular, perfect and semiperfect (projective) modules, with which we close the section.

The Morita Theorem; Finitely Generated Projective Modules

In this paragraph we shall make use of the language of Category Theory, with which the reader will be assumed to be familiar. For the standard definitions we refer to [A–F].

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F:\mathcal{C} \to \mathcal{D}$ is said to be a category equivalence if there exists a functor $G:\mathcal{D} \to \mathcal{C}$ such that $GF$ (resp. $FG$) is naturally isomorphic to the identity functor in $\mathcal{C}$ (resp. in $\mathcal{D}$). This occurs if and only if $F$ is full and faithful and, for every object $D$ of $\mathcal{D}$, there exists an object $C$ of $\mathcal{C}$ such that $FC$ and $D$ are isomorphic. Therefore, a category equivalence between $\mathcal{C}$ and $\mathcal{D}$ preserves and reflects most of the categorical properties of the objects and the morphisms of $\mathcal{C}$ and $\mathcal{D}$.

Two rings $R$ and $S$ are (Morita) equivalent if there exists a category equivalence between $\text{Mod}_R$ and $\text{Mod}_S$ (it turns out that it happens if and only if there exists a category equivalence between $\text{Mod}_R$ and $\text{Mod}_S$). Therefore, all the properties of a ring which may be stated in
categorical terms are preserved by Morita equivalence. For example, if 
$R$ and $S$ are equivalent rings, then each one of the properties listed 
below hold in $R$ if and only if it holds in $S$ (such properties are 
called Morita invariants; see [FA73; p.220] for an extended list):

- (right or left) Noetherian
- (right or left) Artinian
- (von Neumann) regular
- (right or left) self-injective
- (right or left) hereditary
- (right or left) perfect
- (right or left) primitive

A generator for $\text{Mod}_R$ is a right $R$-module $M$ such that, for any module 
$N$, there exists a set $I$ and an epimorphism $M^{(I)} \rightarrow N$, where $M^{(I)}$ 
represent the direct sum of copies of $M$ indexed by $I$. If we set 
$H=\text{Hom}_R(M,N)$ and define a map $\pi: M^{(H)} \rightarrow N$ via $\pi(h) = \sum h(x)$, then 
clearly $\text{Im}(\pi) = t_N(M)$ (see Section 5), and from that it is easy to see 
that $M$ is a generator of $\text{Mod}_R$ if and only if $t_N(M)=N$ for all right 
$R$-modules $N$.

A progenerator of $\text{Mod}_R$ is just a finitely generated projective 
generator of $\text{Mod}_R$. A well known theorem of Morita [39] characterizes 
the rings which are equivalent to a given ring $R$ (see e.g. [A-F; §22]).

**Theorem** For two rings $R, S$ the following statements are equivalent:

a) $R$ and $S$ are Morita equivalent;

b) there exists a progenerator $M$ of $\text{Mod}_R$ with $S=\text{End}(M)$;

c) there exists a progenerator $M$ of $\text{Mod}_R$ with $S=\text{End}(M)$.

Therefore, the endomorphism ring of a progenerator of $\text{Mod}_R$ inherits 
many of the properties of $R$ (but not all, for example being a domain, 
a field, a commutative ring or an indecomposable ring are not Morita 
invariants [FA73; p.221]).

Sometimes, however, being a progenerator is too restrictive a 
condition in $M$ for $S=\text{End}(M)$ to preserve some Morita invariants. In 
fact, many times 'finitely generated (f.g.) projective' is a 
sufficient condition for that.
For example, note that a f.g. projective module $M_R$ is isomorphic to a direct summand of some finite direct sum of copies of $R$ and so, in particular, $M_R$ is a direct summand of a generator $P_R$ of $\text{Mod}_R$. If we write $S=\text{End}(M_R)$, $H=\text{End}(P_R)$, and if $e$ is an idempotent of $H$ such that $M=eP$, then it is easily checked that the assignment $f \mapsto efe$ defines a ring isomorphism between $S$ and $eHe$ with inverse $ehe \mapsto ehe^{-1}$. Thus, Morita invariants which do not vanish when one passes from a ring $Q$ to $qQq$ (for some $q^2=q\in Q$) are preserved for the endomorphism ring of f.g. projective modules. Specifically:

**THEOREM 8.1** Let $M_R$ be a finitely generated projective module. If $R$ has one of the properties below, then so does $S=\text{End}(M_R)$:

a) $R$ is a regular ring;

b) $R$ is a semiperfect ring;

c) $R$ is a right perfect ring.

**Proof:** All three properties are Morita invariants. We see that $eRe$ is regular whenever $R$ is and $e^2=eR$: given $r \in R$, let $x \in R$ be such that $(ere)x(ere)=ere$; then $(ere)(exe)(ere)=ere$, so that $eRe$ is regular. The same property for semiperfect rings follows from [A-F; Cor.27.7], and for perfect rings from [A-F; Theo.28.4.b] and Lemma 28.18.$\blacksquare$

In fact, we can add 'R is right Noetherian' and 'R is right Artinian' to the list in (8.1), and it will follow as a particular case of Theorem 8.3. Before stating it, we need a result which is more easily proved if we introduce the notion of smallness in $\text{Mod}_R$.

A module $M_R$ is said to be small in $\text{Mod}_R$ [M; p.74] if, for any direct sum $\bigoplus M_i$ of right $R$-modules and for any homomorphism $f:M \rightarrow \bigoplus M_i$, there exist a finite subset $J$ of $I$ and a homomorphism $\bar{f}:M \rightarrow \bigoplus M_j$ such that $f=u_{ij} \bar{f}_i$, where $u_{ij}$ is the canonical inclusion of $\bigoplus M_j$ in $\bigoplus M_i$.

For example, every f.g. module $M_R$ is small in $\text{Mod}_R$, since the images via $f:M \rightarrow \bigoplus M_i$ of a finite generating set of $M_R$ (and hence all of $fM$) lie in only finitely many of the $M_i$'s. In fact, for a projective module, finite generation and smallness are equivalent conditions.
PROPOSITION 8.2 If \( M_R \) is a finitely generated projective module then, for all right ideals \( A \) of \( S=\text{End}(M_R) \), we have \( A=\text{Hom}_R(M,AM) \) (i.e. every right ideal is \( \tau \)-closed).

PROOF [23;Lemma 2.6]: Note that, from the definitions, for any right ideal \( A \) of \( S \), we get \( \tau_A(\mathfrak{a})=\text{Hom}(M,AM) \) and \( A=\text{Hom}(M,AM) \). On the other hand, let \( f:M \to AM \) be any homomorphism and consider the coproduct \( M(\mathfrak{a}) \) with canonical inclusions \( \{u_h:heA\} \); the maps \( \{h:M \to AM: heA\} \) induce a homomorphism \( \pi:M(\mathfrak{a}) \to AM \) such that \( \pi u_h h \) for all \( heA \), and \( \pi \) is clearly an epimorphism.

Then, by projectivity of \( M_R \), there exists \( g:M \to M(\mathfrak{a}) \) such that \( f=\pi g \), and by smallness there exist a finite subset \( J \) of \( \mathfrak{a} \) and a homomorphism \( \bar{g}:M \to MJ \) such that \( (\text{writing } u \text{ for } u_{\mathfrak{a}}) \ g=u\bar{g} \).

Then, if \( \{e_h:heJ\} \) and \( \{p_h:heJ\} \) are the injections and projections, respectively, of the coproduct \( M' \), we get

\[
f = \pi g = \pi u\bar{g} = \pi(\sum e_h p_h)\bar{g} = \sum \pi(ue_h)p_h\bar{g} = \sum \pi u_h p_h\bar{g} = \sum h(p_h\bar{g}) \in A
\]

since \( J \) is finite, \( heA \) and, for each \( heJ \), \( p_h\bar{g} \in S \).

Therefore \( A=\text{Hom}_R(M,AM) \). ■

REMARK This proof may be slightly modified in order to obtain a similar result in which finite generation is required not in \( M_R \) but in \( A \). This will be done in (10.11) in a more general situation.

THEOREM 8.3 a) If \( M_R \) is projective and Noetherian then \( S=\text{End}(M_R) \) is right Noetherian. b) If \( M_R \) is finitely generated, projective and Artinian then \( S \) is right Artinian.

PROOF: Since in any case \( M_R \) is f.g., \( A=\text{Hom}_R(M,AM) \) for every right ideal \( A \) of \( S \), and therefore for any two right ideals \( A,B \) of \( S \) we have \( AM=BM \) if and only if \( A=B \). Thus, assuming a) (resp. b)), for a nonempty set \( \Sigma=\{A_i; i \in I\} \) in \( \text{Lat}(S) \), the set \( \{A_iM; i \in I\} \) has a maximal (resp. minimal) element \( A_iM \), and then \( A_i \) is maximal (resp. minimal) in \( \Sigma \). ■
REMARK: In fact, any projective Artinian module is f.g., so that the condition 'f.g.' in (8.3.b) is redundant (see the proof of [22; Theo. 2.8]). Moreover, even the endomorphism ring of a \( \Sigma \)-quasi-projective Artinian module is right Artinian [43; Theo. 7].

The Jacobson Radical of the Endomorphism Ring of a Projective Module; Local Endomorphism Rings.

The Jacobson radical of the endomorphism ring of a projective module admits a description that is dual to that given for an injective module, and that may be improved if, in addition, \( \text{Rad} M \) (the radical of \( M \), see (1.5)) is a superfluous submodule of \( M \).

It is known that, if \( M_r \) is projective and \( J=J(R) \) is the Jacobson radical of \( R \), then \( \text{Rad} M \subseteq M J \subseteq M \) [A-F; Prop. 17.10 & 17.14]. Also, if \( M_r \) is f.g. then every proper submodule \( L \) of \( M_r \) is included in a maximal submodule \( K \) and thus \( L+\text{Rad} M \subseteq K \subseteq M \); therefore \( \text{Rad} M \subseteq M \).

The proof of the next lemma is dual to (7.1).

**Lemma 8.4** Let \( M_r \) be any module and let \( S=\text{End}(M_r) \). Then the set \( \Delta(S)=\{ f \in S : f M \subseteq M \} \) is a two-sided ideal of \( S \).

**Proposition 8.5** Let \( M_r \) be a projective module, \( S=\text{End}(M_r) \) and \( N=\text{Rad} M \). Then

a) \( J(S) \subseteq \Delta(S) \subseteq \text{Hom}(M,N) \);

b) there exists a ring epimorphism \( S \rightarrow \text{End}(M/N) \) with kernel \( \text{Hom}(M,N) \).

c) if \( N \subseteq M \) (e.g. if \( M_r \) is finitely generated) then \( J(S) = \text{Hom}(M,N) \) and hence \( S/J(S) \) is ring isomorphic to \( \text{End}(M/N) \).

**Proof:**

a) \( \Delta(S) \subseteq J(S) \): Let \( f \in \Delta(S) \); since \( M=f M+(1-f) M \) and \( f M \subseteq M \), we get \( (1-f) M = M \). Then the short exact sequence \( 0 \rightarrow \text{Ker}(1-f) \rightarrow M \rightarrow M \rightarrow 0 \) splits (by projectivity of \( M \)) and hence \( \text{Ker}(1-f) \subseteq M \); but since \( \text{Ker}(1-f) \subseteq f M \), \( \text{Ker}(1-f) \) is a superfluous direct summand of \( M \) (1.3), i.e. \( \text{Ker}(1-f)=0 \). Hence, \( 1-f \) is invertible for all \( f \in \Delta(S) \), whence \( \Delta(S) \subseteq J(S) \).

b) \( J(S) \subseteq \Delta(S) \): Let \( f \in J(S) \) and suppose \( f M + N = M \) for some \( N \subseteq M_r \); let \( \pi : M \rightarrow M/N \), thus for any \( x \in M \) we have \( x=f y+z \) for some \( y \in M \), \( z \in N \), whence \( x+N = f y+N \); therefore \( \pi f : M \rightarrow M/N \) is epic and thus there exists \( g \in S \) such that \( \pi g = \pi \).

...
i.e. \(\pi(1-fg)=0\), which means that \((1-fg)M\subseteq N\); but \((1-fg)M=M\) since \(f\in J(S)\), whence \(N=M\). Therefore \(fM\subseteq M\), i.e. \(f\in \Delta(S)\).

\(J(S)\leq \text{Hom}(M,N)\): if \(f\in J(S)\) then \(fM\subseteq M\) and hence \(fM\subseteq N\) by (1.5).

b) Define \(\phi:S\to \text{End}_R(M/N)\) as follows: Let \(f\in S\); since \(N=\text{Rad}M\) is a fully invariant submodule of \(M\) we have \(fN\subseteq N\) and then \(\tilde{f}:x+N\to fx+N\) defines an endomorphism of \(M/N\). Let then \(\phi(f)=\tilde{f}\); this clearly defines a ring homomorphism, which is indeed an epimorphism by projectivity of \(M\). It is also clear that \(\tilde{f}=0\) if and only if \(fM\subseteq N\), so that \(\ker \phi=\text{Hom}(M,N)\).

c) If \(N\subseteq M\) and \(f\in \text{Hom}(\mathbb{M}, M)\) then (1.3.d) \(fM\subseteq M\), whence \(f\in \Delta(S)\). Then a) and b) yield c).

REMARKS

1) Further characterizations of \(J(S)\) may be found in [61].

2) A module \(M\) is said to be quasi-projective if for any module \(N=\mathbb{N}\), any epimorphism \(f:M\to N\) and any homomorphism \(g:M\to N\), there exists an endomorphism \(h\) of \(M\) such that \(g=fh\). A careful look at the proof of (8.5) reveals that it may be proved for \(M\) quasi-projective.

As a consequence of (8.5) we can characterize those projective modules which have a local endomorphism ring. Recall that a ring \(R\) is said to be local if its radical \(J(R)\) is a maximal right or left ideal or, equivalently, if \(J(R)=\{r\in R: r\text{ is not invertible}\}\).

Before stating the next theorem, a dual of which will be proved in Section 9, we need to introduce the dual concept of the injective hull, namely the projective cover: A projective cover for \(M\) is a projective module \(P\), together with an epimorphism \(\pi:P\to M\), such that \(\ker \pi\) is a superfluous submodule of \(P\).

Unlike injective hulls, projective covers for arbitrary modules seldom exist; e.g. if \(R\) is a ring with zero radical then only the projective \(R\)-modules possess a projective cover [A-F; Ex.17.14]. In particular, for \(R=\mathbb{Z}\), this implies that an Abelian group has a projective cover if and only if it is free. In fact, the only rings \(R\) for which every right \(R\)-module has a projective cover are the right perfect rings defined in Section 6 (see [5], [A-F; §28] or [FA67; §22]).
THEOREM 8.6 Let $M_R$ be a projective module, $N=\text{Rad}M$ and $S=\text{End}(M_R)$. The following statements are equivalent:

a) $S$ is a local ring;

b) $M_R$ is the projective cover of a simple right $R$-module;

c) $M_R$ contains a submodule which is both superfluous and maximal;

d) $N$ is a superfluous and maximal submodule of $M_R$;

e) $M_R$ has a unique maximal submodule (necessarily equal to $N$) which contains every proper submodule of $M_R$.

PROOF: a) $\Rightarrow$ b) Assume that $S$ is local. In particular, $S\neq 0$ and hence $M\neq 0$; therefore $NcM$, i.e. $M$ contains a maximal submodule $K$. Thus $M/K$ is a simple module, and if we prove $K\subseteq M$ then obviously $M_R$ will be a projective cover for $M/K$, proving b).

Suppose then $L\subseteq M_R$ is such that $K+L=M$; we have to prove $L=M$. Since $K\subseteq M$ and $M/K = (K+L)/K \cong L/(K\cap L)$, there exists a nonzero homomorphism $g:M \to L/(K\cap L)$, and by projectivity of $M_R$ there exists $f:M_R \to L$ such that $g=\pi f$, where $\pi:L \to L/(K\cap L)$. Now, since $g\neq 0$, $fM$ is not included in $K$, and hence $K+fM=M$ by maximality of $K$; therefore $fM$ is not superfluous in $M$, i.e. $f\notin J(S)$ (8.5). But, since $S$ is local, this implies that $f$ is invertible, so that $M=fMSL$, i.e. $L=M$, as required.

b) $\Rightarrow$ c) Note that b) just means that $M_R$ contains a superfluous submodule $K$ such that $M/K$ is simple, i.e. $K$ is also a maximal submodule of $M_R$.

c) $\Rightarrow$ d) Let $K \subseteq M_R$ be superfluous and maximal in $M_R$; then (1.5)

$$K \subseteq \Sigma \{L \subseteq M_R: L \subseteq M\} = N = \cap \{L \subseteq M_R: L \text{ is maximal in } M_R\} \subseteq K,$$

i.e. $N=K$.

d) $\Rightarrow$ e) Obviously, if $N=\text{Rad}M$ is maximal, then it is the only maximal submodule of $M_R$; now, if $L \subseteq M_R$ then, since $N \subseteq M$, $N \subseteq N+L \subseteq M$ and thus, by maximality of $N$, $N=N+L$, i.e. $L \subseteq N$.

e) $\Rightarrow$ a) Clearly, if $N$ contains every proper submodule of $M_R$, then $N \subseteq M$ and hence (8.5) $S/J(S) \cong \text{End}_R(M/N)$. Since $N$ is maximal by hypothesis, $M/N$ is simple and hence $S/J(S)$ is a division ring (5.1), i.e. $S$ is a local ring.

REMARKS: 1) Since every proper submodule of $M_R$, for $M_R$ satisfying the conditions of (8.6), is superfluous in $M_R$, we can add to these
equivalences 'Mr is a projective cover for all its nonzero quotient modules' (c.f. (9.2.c)).

2) In [60; §4], R.Ware remarks that the equivalence of the conditions which define a local ring depend largely on the fact that any ring is projective as a module over itself, and proves that most of these conditions remain equivalent when translated to an arbitrary projective module (e.g. b), c), d), e) of (8.6)).

He calls a module local if it is a projective module which satisfies these conditions, so that with this terminology (8.6) says that a projective module is local if and only if it has a local endomorphism ring [60; Theo.4.2].

3) Also, some of the conditions which define a regular, semiperfect or perfect ring remain equivalent when extended to projective modules. Thus, one can define, always within the classes of projective modules, regular, semiperfect or perfect modules. We state here, without proof, the properties of their endomorphism rings.

Regular modules are defined by R.Ware in [60; §§ 2 and 3], as those projective modules Mr with the property that every cyclic submodule of Mr is a direct summand of Mr (definitions of arbitrary regular modules which agree with this one in the projective case may be found in [14] and [66]). For the given definition, f.g. regular modules have regular endomorphism rings [60; Theo.3.6] and, over a commutative ring, projective modules whose ring of endomorphism is regular are regular [60; Theo.3.9] (finite generation and commutativity are necessary).

Perfect and semiperfect modules were introduced by E.Mares in [32]; a projective module Mr is semiperfect if every factor module of Mr has a projective cover or, equivalently, if RadMrM, M/RadM is semisimple and decompositions of M/RadM can be lifted to M [32]. A projective module has a semiperfect endomorphism ring if and only if it is finitely generated and semiperfect ([32; Theo.6.1] and [60; Prop.1.5]).

A projective module Mr is perfect if, for every set I and every factor module N of the direct sum Mr(I), N has a projective cover. A projective module has a perfect endomorphism ring if and only if it is finitely generated and perfect ([32; Theo.2.4 and Cor.7.5] and [60; Prop.5.2]).
SECTION 9: THE ENDOMORPHISM RING OF A FINITE DIMENSIONAL MODULE

We start this section by characterizing finite dimensional injective modules in terms of their endomorphism rings. In fact, an injective module $E\mathcal{R}$ will be f.d. if and only if $S=\text{End}(E\mathcal{R})$ is semiperfect (Theorem 9.5). This will allow us to embed the factor ring $S/T(S)$ of any f.d. module $\mathcal{M}\mathcal{R}$ in a semisimple ring, and as a consequence we will find conditions under which every nil subring of $S$ is nilpotent (Theorems 9.8 and 9.13). These latter results are due to R. Shock [50].

Later on, we shall look for situations in which not only finite dimensionality, but also the dimension of $\mathcal{M}\mathcal{R}$, is inherited by $S=\text{End}(\mathcal{M}\mathcal{R})$; this will be used, for example, to characterize some modules which have Goldie rings of endomorphisms (J. Hutchinson and J. Zelmanowitz [25]). In this area we will find some help in the results and techniques of Section 6.

Before that, we prove an easy but interesting result, which should be compared with (10.2).

**PROPOSITION 9.1** Let $\mathcal{M}\mathcal{R}$ be a finite dimensional module and let $f\in S=\text{End}(\mathcal{M}\mathcal{R})$. Then $f$ is invertible if and only if it is left (or right) invertible.

**PROOF:** We have to prove that, for all $f, g \in S$, $fg=1$ implies $gf=1$ (in fact, this condition is equivalent to $\mathcal{M}\mathcal{R}$ being directly finite, i.e. such that $\mathcal{M}\mathcal{R}$ is not isomorphic to a proper direct summand of itself, see [G; Lemma 6.9]).

First, note that a f.d. module cannot be isomorphic to a proper direct summand of itself, because if $M=\mathbb{N}\mathcal{M}$ with $\mathbb{N}\mathcal{M}$ and $L\neq 0$ then $M=\mathbb{N}\mathcal{M}^n$ for all $n\in\mathbb{N}$, which is impossible.

Now suppose that $f, g \in S$ satisfy $fg=1$; then $t=1-gf$ is an idempotent of $S$ such that $tg=0$, and thus $M=\mathcal{M}tM$: if $gx=gy \in M$ then $ty=t^2y=tgx=0$, whence $gM\cap M=0$; and for all $x \in M$, $x=gfx+(x-gfx)eM+tM$.

Since $g$ is monic ($fg=1$), we get, by the above remark, that $gM=M$, and hence $tM=t(1-gf)M=0$, i.e. $gf=1$.■
Finite Dimensional Injective Modules

Here, the results of Section 3 will be used to show that an injective module is finite dimensional if and only if its endomorphism ring $S$ is semiperfect (i.e. idempotents lift modulo $J=J(S)$ and $S/J$ is semisimple [5]). As a first step towards this result, we characterize those injective modules which are uniform (compare with (8.6)).

**Proposition 9.2** Let $E_R$ be a nonzero injective module, and let $S=\text{End}(E_R)$. The following conditions are equivalent.

a) $E_R$ is uniform;

b) $E_R$ is indecomposable;

c) $E_R$ is the injective hull of all its nonzero submodules;

d) $S$ is a local ring.

**Proof:** a)$\Rightarrow$b) follows directly from (3.7).
a)$\Rightarrow$c) since every nonzero submodule of $E$ is essential in $E$.
c)$\Rightarrow$d). We show that the sum of any two noninvertible elements $f,g$ of $S$ is noninvertible. For, note that $\text{Ker}f\neq 0$, because otherwise $fE$ would be a nonzero injective submodule of $E$ and thus, by c), $E=fE$, whence $f$ would be an isomorphism. Similarly, $\text{Ker}g\neq 0$ and hence, by a), 0$\neq \text{Ker}f\cap \text{Ker}g \subseteq \text{Ker}(f+g)$, whence $f+g$ is not invertible.
d)$\Rightarrow$b) is (5.6). $\blacksquare$

**Theorem 9.3** Let $E_R$ be an injective module, $n$ a positive integer and write $S=\text{End}(E_R)$. The following conditions are equivalent.

a) $E_R$ is finite dimensional and $u(E_R)=n$;

b) $E_R$ is a finite direct sum of indecomposable injective modules, and any such decomposition of $E$ consists of exactly $n$ nonzero summands;

c) $S$ contains no infinite family of orthogonal idempotents, and $n$ is the maximum cardinality of all families of nonzero orthogonal idempotents of $S$.

**Proof:** a)$\Rightarrow$b) follows directly from (3.9).
b)$\Rightarrow$c). By (9.2) and by the Krull-Schmidt-Azumaya Theorem, every direct decomposition of $E$ has at most $n$ summands (and at least one of them has exactly $n$); then apply (5.4.c).
c)→a). Let \( \oplus_{i} M_{i} \) be any direct sum of nonzero submodules of \( E_{r} \); for each \( i \in I \), let \( E_{i} \) be an \( e \)-closure in \( E \) for \( M_{i} \); now, for each \( j \in I \), let \( p_{j}: E_{i} \to E_{j} \) and \( e_{j}: E_{j} \to E_{i} \) be the canonical projection and injection, and let \( u: \oplus_{i} E_{i} \to E \) be the inclusion map.

By injectivity of \( E_{j} \) (3.4), there exists \( f_{j}: E_{j} \to E_{j} \) (which may be viewed as an element of \( S \)) such that \( f_{j} u = p_{j} \); thus, for each \( x \in E \), \( f_{j} x e_{j} E_{j} \) and hence \( f_{j} x = p_{j} f_{j} x = u f_{j} x \). Now let \( j, k \in I \) and \( x \in E \); we get

\[
\delta_{jk} f_{j} x = f_{k} u f_{j} x = p_{k} p_{j} f_{j} x = \delta_{jk} f_{j} x
\]

(where \( \delta_{jk} \) is the 'Kronecker delta'), and therefore \( \{ f_{j}: j \in I \} \) is a family of nonzero orthogonal idempotents of \( S \).

Thus, for any direct sum of nonzero submodules of \( E_{r} \) we get a family of nonzero orthogonal idempotents of \( S \) with the same cardinality, and vice-versa (5.4.c), whence the implication c)→a) follows readily.

The following lemma is the key to prove our Theorem 9.6.

**Lemma 9.4** Let \( R \) be a regular ring. Then \( R \) is semisimple of length \( n \) (equivalently, of right or left dimension \( n \)) if and only if \( R \) contains a family of \( n \) nonzero orthogonal idempotents, and no set of nonzero orthogonal idempotents of \( R \) has more than \( n \) elements.

**Proof:** We prove that a regular ring \( R \) such that \( R \) does not contain infinite families of orthogonal idempotent elements is semisimple. The lemma follows then easily from the well-behaved decomposition theory of semisimple modules and rings.

From the hypothesis about the idempotents in \( R \), it is easy to see that \( 1 \in R \) may be written as a finite sum of primitive orthogonal idempotents of \( R \), and therefore \( R \) is a finite direct sum of indecomposable right (or left) ideals; but, in a regular ring, an indecomposable right ideal \( \alpha \neq 0 \) must be simple: for any \( 0 \neq \alpha a \), there exists \( e^{2} = e \in E \) with \( rR = eR \), whence \( rR \) is a direct summand of \( R \) and hence of \( \alpha \); since \( rR \neq 0 \) and \( \alpha \) is indecomposable, we get \( rR = \alpha \), and thus \( \alpha \) is simple. Therefore \( R \) is semisimple.

**Theorem 9.5** Let \( E_{r} \) be injective. Then \( E \) is finite dimensional if and only if \( S = \text{End}(E_{r}) \) is semiperfect. In this case \( u(E_{r}) = u(S/J(S)) \).
PROOF: Let J=J(S). Recall that S/J is regular (7.2) and idempotents may be lifted modulo J (7.3). Now (9.3) and (9.4) give:

\[ S \text{ is semiperfect with } u(S/J)=n \Leftrightarrow S/J \text{ is semisimple with } u(S/J)=n \]

the idempotents of S/J verify the conditions of (9.4) \( \Rightarrow \)

so do the idempotents of S \( \Rightarrow u(ER)=n. \]

COROLLARY 9.6 Let R be a right self-injective ring. The following conditions are equivalent.

a) R is semiperfect;

b) R has no infinite family of orthogonal idempotents;

c) R is right finite dimensional;

d) R is left finite dimensional. \[ \]

COROLLARY 9.7 The endomorphism ring of a f.d. injective nonsingular module is semisimple.

PROOF: Such a ring is semiperfect by (9.5) and has zero radical by (7.2) and the proof of (7.12); hence it is semisimple. \[ \]

Nil Subrings of the Endomorphism Ring of a Finite Dimensional Module

The fact that the injective hull \( E_R \) of a finite dimensional module \( M_R \) has the same dimension as \( M_R \), together with the previous study of finite dimensional injective modules and with the fact that \( S=\text{End}(M_R) \) may be embedded in \( \text{End}(E_R) \), will allow us to study in some detail the nilpotency of nil subrings of S. A sufficient condition for these to be nilpotent will be found in terms of the rationally closed submodules of \( M_R \), which we shall shortly introduce. The main results in this area are due to R.Shock [50].

THEOREM 9.8 Let \( M_R \) be any module (resp. a f.d. module), and let \( \Gamma=\Gamma(S) \). Then \( S/\Gamma \) is ring isomorphic to a subring of a regular (resp. semisimple) ring.

PROOF [50; Lemma 2]: Let \( E_R \) be an injective hull for \( M_R \), and let \( H=\text{End}(E_R) \); then \( J(H)=\Gamma(H) \) and \( H/J(H) \) is regular (semisimple if \( M_R \) is f.d.), and thus it suffices to embed \( S/\Gamma \) in \( H/\Gamma(H) \).
Let \( S' = \{ \text{heH} : h \in M \} \) and \( \Gamma' = \{ \text{heS'} : \ker h \subseteq E \} = \Gamma(H) \cap S' \). For \( f \in S \), let \( f' \) be an extension to \( E \) of \( f \); then it is easy to check that the correspondence \( f + \Gamma \mapsto f' + \Gamma' \) gives a ring isomorphism between \( S/\Gamma \) and \( S'/\Gamma' \), and \( S'/\Gamma' \) may be embedded in \( H/J(H) \) via \( h + \Gamma' \mapsto h + \Gamma(H) \).

We now turn our attention to the study of the nil subrings of \( S \). The starting point is the next result.

**Proposition 9.9** Let \( M \) be finite dimensional and \( S = \text{End}(M) \). Then

a) every nil subring of \( S/\Gamma(S) \) is nilpotent;

b) if \( \Gamma(S) \) is nilpotent then every nil subring of \( S \) is nilpotent;

c) a nil subring \( Q \) of \( S \) is nilpotent if and only if \( Q \cap \Gamma(S) \) is nilpotent.

**Proof:**

a) From a result of I. Hernstein and L. Small [24], in a ring with ACC on right and left annihilators every nil subring is nilpotent (see also [C-H; Theo.1.34]), so that, from (9.8) and by symmetry, it suffices to see that, in general, a subring of a semisimple (Artinian) ring has ACC on right annihilators.

Let \( B \) be a subring of the semisimple ring \( T \). Write \( R_B, L_B \) (resp. \( R_T, L_T \)) for the annihilator operators in \( B \) (resp. \( T \)) and note that, for a nonempty subset \( X \) of \( B \), \( R_B(X) = R_T(X) \cap B \). Now suppose \( X_1, X_2, \ldots \) are nonempty subsets of \( B \) such that \( R_B(X_1) \subseteq R_B(X_2) \subseteq \cdots \) and set \( Y_i = L_B R_B(X_i) \) for \( i = 1, 2, \ldots \), so that \( R_B(X_i) = R_B(Y_i) \) for all \( i \). If \( L_B \) acts in that chain then we get \( Y_1 \subseteq Y_2 \subseteq \cdots \) and hence \( R_T(Y_1) \subseteq R_T(Y_2) \subseteq \cdots \). By hypothesis there exists \( n \in \mathbb{N} \) such that, for all \( k \geq n \), \( R_T(Y_k) = R_T(Y_n) \) and hence

\[
R_B(X_k) = R_B(Y_k) = R_B(Y_n) \cap B = R_B(Y_n) \cap B = R_B(Y_n) = R_B(X_n),
\]

proving that \( B \) has ACC on right annihilators, as desired.

b) This follows then easily from a).

c) If \( Q \) is a nil subring of \( S \) and we write \( \Gamma \) for \( \Gamma(S) \), then \( (Q + \Gamma)/\Gamma \) is a nil subring of \( S/\Gamma \) and then \( (Q + \Gamma)/\Gamma = Q/(Q \cap \Gamma) \) is nilpotent; if \( Q \cap \Gamma \) is also nilpotent, then so is \( Q \), and this proves c).  

**Corollary 9.10** Let \( R \) be a right finite dimensional ring; then a nil subring \( Q \) of \( R \) is nilpotent if and only if so is \( Q \cap Z_r(R) \). If \( Z_r(R) \) is nilpotent then nil subrings of \( R \) are nilpotent.

**Proof:** As we already remarked in the proof of (7.4), \( \Gamma(R) = Z_r(R) \).
Now, we can look for f.d. modules $M_r$ for which $\Gamma(S)$ is nilpotent, and then use (9.9.b) to deduce that all nil subrings of $S$ are nilpotent. A sufficient condition both for $M_r$ to be f.d. and for $\Gamma(S)$ to be nilpotent will depend on the concept of $M$-rationally closed submodules of $M_r$, which we define next.

Let $L \subseteq N$ be submodules of $M_r$; we say that $N$ is an $M$-rational extension of $L$ provided $\text{Hom}_R(K/L,M)=0$ for all $K \subseteq M_r$ such that $L \subseteq K \subseteq N$. If $L \subseteq M_r$ has no proper $M$-rational extensions within $M$, we say that $L$ is $M$-rationally closed. A concept of rational closure ($r$-closure), similar to that of $e$-closure developed in Section 1, may be defined in terms of a class of right ideals of $R$ called $M$-dense ideals: $\alpha \subseteq R$ is $M$-dense if $\alpha x \neq 0$ for every nonzero element $x$ in an injective hull of $M_r$. Now, for $L \subseteq M_r$, let $L'=\{x \in M_r: (L:x) \text{ is } M\text{-dense}\}$; $L'$ is called the $r$-closure of $L$ in $M$, and it turns out that $L$ is rationally closed in $M$ if and only if $L=L'$ [50].

Before state the next theorem, we need two previous results. We shall again make use of the notation and results of Section 6.

**Lemma 9.11** If $S$ has DCC on a-closed ideals, then $\Gamma(S)$ is nilpotent.

**Proof [36]:** We know from (6.1.a) that every left annihilator ideal of $S$ is a-closed, so that $S$ has DCC on left annihilator ideals and hence ACC on right annihilator ideals, whence the chain $\mathcal{R}(\Gamma) \subseteq \mathcal{R}(\Gamma^2) \subseteq \cdots$ (where $\Gamma=\Gamma(S)$) stops, i.e. $\mathcal{R}(\Gamma^n)=\mathcal{R}(\Gamma^{n+1})$ for some $n \in \mathbb{N}$; now, we shall prove that $\Gamma \subseteq \mathcal{R}(\Gamma^n)$, which will imply $\Gamma^{n+1}=0$, proving the lemma.

Suppose there exists $f \in \Gamma \setminus \mathcal{R}(\Gamma^n)$; since $f \notin \mathcal{R}(\Gamma^{n+1})$, there exists $f_1 \in \Gamma$ such that $\Gamma^n f_1 f \neq 0$, i.e. $f_1 f \in \Gamma \setminus \mathcal{R}(\Gamma^n)$; in this way, we can construct an infinite sequence $f, f_1, f_2, \ldots$ of elements of $\Gamma$ such that $g_r=f_r \cdots f_1 f \neq 0$ for all $r \in \mathbb{N}$. Since $g_r \neq 0$ and $\text{Ker}(f_{r+1}) \subseteq M$, $g_r \cap \text{Ker}(f_{r+1}) \neq 0$, whence $\text{Ker}(g_r) \subset \text{Ker}(g_{r+1})$. Since the strictly ascending chain $\text{Ker}(g_1) \subset \text{Ker}(g_2) \subset \cdots$ consists of right annihilators in $M$ of subsets of $S$, also $1_S(\text{Ker}_1) \supset 1_S(\text{Ker}_2) \supset \cdots$ is strict, contradicting the hypothesis of the lemma and hence proving our claim. ■

**Lemma 9.12** Let $L \subseteq N$ be submodules of $M_r$; then $L' \subseteq N'$ if and only if there exists $x \in N \setminus L$ such that $(L:x)$ is not $M$-dense.
PROOF: Clearly, \( L' = N' \) if and only if \( N \subseteq L' \), and by definition this occurs if and only if \( (x : L) \) is \( M \)-dense for all \( x \in N \). Since \( (x : L) = R \) (which is \( M \)-dense) for all \( x \in L \), the lemma follows. ■

THEOREM 9.13 Let \( M_r \) be a module with \( \text{ACC} \) on rationally closed submodules, and let \( S = \text{End}(M_r) \). Then

a) \( M_r \) is finite dimensional;

b) \( S \) has \( \text{DCC} \) on \( a \)-closed left ideals;

c) every nil subring of \( S \) is nilpotent.

PROOF [50; Theo. 3.10]:

a) Let \( L, N \) be nonzero independent submodules of \( M_r \); for any \( 0 \neq x \in N \) we have \( x(L : x) \subseteq N \cap L = 0 \), so that \( (L : x) \) is not \( M \)-dense and hence \( L' \subseteq (N \cap L)' \) by the preceding lemma. Therefore, any infinite direct sum of nonzero submodules of \( M_r \) would force a strictly ascending chain of rationally closed submodules of \( M \), a contradiction which proves a).

b) Let \( \mathfrak{A}, \mathfrak{B} \) be \( a \)-closed left ideals of \( S \) with \( \mathfrak{A} \subseteq \mathfrak{B} \), and set \( E = r_M(\mathfrak{A}) \), \( F = r_M(\mathfrak{B}) \); thus \( l_S(E) = \mathfrak{A} \subseteq \mathfrak{B} = l_S(F) \). Take then \( f \in S \) such that \( fF = 0 \) and \( fE \neq 0 \), and pick \( z \in E \) with \( fz \neq 0 \); since \( fz(F : z) \subseteq fF = 0 \), \( (F : z) \) is not \( M \)-dense and thus \( F' \subseteq E' \) by (9.12). Therefore, for each strictly descending chain \( l_S(F) > l_S(E) > \cdots \) of left annihilators in \( S \) of subsets of \( M_r \), we find a strictly increasing chain \( E' \subseteq F' \subseteq \cdots \) of rationally closed submodules of \( M_r \), which must be finite by hypothesis.

c) This follows directly from a), b), (9.11) and (9.9.b). ■

COROLLARY 9.14 If \( M_r \) is injective with \( \text{ACC} \) on rationally closed submodules, then \( S \) is semiprimary.

PROOF: By (9.13.a) and (9.5), \( S \) is semiperfect, and by (7.2), (9.13.b) and (9.11), \( J(S) \) is nilpotent. Thus \( S \) is semiprimary. ■

Quotient Rings of the Endomorphism Ring of a Finite Dimensional Nonsingular Module

In Section 6 we saw that, for a non-degenerate module \( M_r \), we can obtain a satisfactory correspondence theorem for right complements of \( S = \text{End}(M_r) \). Next, we shall make use of this and other facts to get some
information about the endomorphism ring of a non-degenerate finite dimensional module. The results here are due to J. Hutchinson and J. Zelmanowitz [25].

Our first two results are of key importance in what follows, and they state that, for a non-degenerate module $M_r$, the dimension of $M_r$ and the right dimension of $S$ coincide, generalizing (6.14.a) and (6.16.a), and that $M_r$ is nonsingular if and only if $S$ is right nonsingular.

**Theorem 9.15** If $M_r$ is a non-degenerate module and $S=\text{End}(M_r)$, then $u(M_r)=u(S)$. 

**Proof** [1; Theo. 2]: Let $N_1 \oplus \cdots \oplus N_r$ be a direct sum of nonzero submodules of $M_r$; by hypothesis, the right ideals $[N_i, \hat{M}]$ $(i=1, \ldots, r)$ of $S$ are nonzero. We claim that they are independent, which will imply $u(M_r)=u(S)$. 

Suppose that $f_i \in [N_i, \hat{M}]$ $(i=1, \ldots, r)$ are such that $\Sigma f_i = 0$; for all $x \in M$ we get $0=(\Sigma f_i)x=\Sigma (f_i x)$ with each $f_i \in f_i M \subseteq [N_i, \hat{M}] M = N_i R(M) \subseteq N_i$, and then, by assumption, $f_i x = 0$ for each $i$; this shows that $f_i = 0$ for each $i$, so that the $[N_i, \hat{M}]$'s are independent. 

On the other hand, for each direct sum $M_1 \oplus \cdots \oplus M_t$ of nonzero right ideals of $S$, we claim that the nonzero submodules $M_1 M$ $(i=1, \ldots, t)$ of $M_r$ are independent, whence $u(S) \leq u(M_r)$ and hence the theorem is proved. 

Suppose then that $x_i \in M_1 M$ $(i=1, \ldots, t)$ are such that $\Sigma x_i = 0$; then, for any $\varphi \in \hat{M}$, $y \in M$, we have $(\Sigma x_i, \varphi)y=(\Sigma x_i)(y, \varphi)=0$, whence $\Sigma x_i, \varphi)=0$; but, for each $i=1, \ldots, t$, we have $[x_i, \varphi] \in [M_1 M, \hat{M}] = M_1 [M, \hat{M}] \subseteq M_1 S = M_1$ and thus, by assumption, $[x_i, \varphi]=0$; therefore $\Sigma x_i, \varphi)=0$ and hence, by hypothesis, $x_i = 0$, as required. 

**Theorem 9.16** Let $M_r$ be a non-degenerate module. Then $M_r$ is nonsingular if and only if $S$ is right nonsingular.

**Proof** [25; Prop. 2]: Assume first that $M_r$ is nonsingular, and let $f \in Z_r(S)$; then there exists $\mathfrak{A} \in \mathfrak{A}$ such that $fS=0$. Hence $\mathfrak{A} \subseteq \text{Ker} f$ and thus, since $\mathfrak{A} \subseteq \mathfrak{A}$ by (6.10.b), $\text{Ker} f \subseteq \mathfrak{A}$. Now, for any $0 \neq x \in M$, we have $e=(\text{Ker} f : x) \subseteq \mathfrak{A}$ (1.1) and $fx=0$; by assumption, this implies $fx=0$, whence $f=0$, i.e. $Z_r(S)=0$. 

Conversely, assume that $Z_r(S)=0$ and suppose there exists $0 \neq x \in Z_r(M_r)$. By
hypothesis, we can find \( \varphi \in \mathcal{M} \) such that \( f = [x, \varphi] \neq 0 \), and we claim that \( \text{Ker} f \subseteq \mathcal{M} \).

To prove so, let \( y \in \mathcal{M} \setminus \text{Ker} f \); then \( 0 \neq fy = (\varphi, y) \neq 0 \); now, since \( \varepsilon = r(x) \in \mathcal{R} \), we can choose \( r \in \mathcal{R} \) such that \( 0 \neq (\varphi, y)r \neq 0 \), i.e. \( fyr = (\varphi, y)r = 0 \); and then we have got \( 0 \neq yr \in \text{Ker} f \), proving the claim.

Thus, if we prove that \( [\text{Ker} f, \mathcal{M}] \subseteq \mathcal{S} \), since clearly \( [\text{Ker} f, \mathcal{M}] \subseteq \mathcal{R}(f) \), we will get \( 0 \neq f \in \mathcal{Z}(\mathcal{S}) \), a contradiction which will imply \( \mathcal{Z}(\mathcal{M}) = 0 \).

We prove in general that \( N \subseteq \mathcal{M} \) implies \( [N, \mathcal{M}] \subseteq \mathcal{S} \): let \( 0 \neq g \in \mathcal{S} \); then \( g \mathcal{M} \neq 0 \) and hence \( N \cap (g \mathcal{M}) = 0 \); thus, by hypothesis,

\[
0 \neq [N : (g \mathcal{M}), \mathcal{M}] \subseteq [N, \mathcal{M}] \cap (g \mathcal{M}, \mathcal{M}) \subseteq [N, \mathcal{M}] \cap (g \mathcal{M}, \mathcal{M}) \cap \mathcal{S};
\]

therefore \( [N, \mathcal{M}] \subseteq \mathcal{S} \).

With this information in hand, and making use of some results of Section 6, it is straightforward to characterize those non-degenerate modules which have a right Goldie endomorphism ring.

**COROLLARY 9.17** Let \( M_\mathcal{R} \) be a non-degenerate module. Then \( \mathcal{S} \) is a right Goldie ring if and only if \( M_\mathcal{R} \) is finite dimensional with ACC on \( \mathcal{M} \)-closed submodules (and both conditions hold if \( M_\mathcal{R} \) has ACC on \( \mathcal{M} \)-cotorsionless submodules).

**PROOF:** By (9.15), \( S \) is right f.d. if and only if \( M_\mathcal{R} \) is f.d. By (6.10.g), \( M_\mathcal{R} \) is an \( \mathcal{M} \)-self-generator and hence (6.6.a) \( S \) has ACC on right annihilators if and only if \( M_\mathcal{R} \) has ACC on \( M \)-closed submodules. Therefore the result follows, the statement in parenthesis being a direct consequence of (6.12.b).

**COROLLARY 9.18** Let \( M_\mathcal{R} \) be a non-degenerate nonsingular module. Then \( S \) is a right Goldie ring if and only if \( M_\mathcal{R} \) is f.d.

**PROOF:** Note that every \( M \)-closed submodule of \( M_\mathcal{R} \) is a complement in \( M \) (6.7.a), and therefore, if \( M_\mathcal{R} \) is f.d., then it has ACC on \( M \)-closed submodules. Thus (9.17) gives the result.

Now, we can study when \( \mathcal{S} \) has a semisimple classical or maximal right quotient ring. Rings with these properties may be characterized as follows:
(1) A ring \(R\) has a semisimple (simple) classical right quotient ring if and only if \(R\) is a semiprime (prime) right Goldie ring [19] (see e.g. [S; p.54] or [G; Theo.3.35]).

(2) A semiprime (prime) ring \(R\) has a semisimple (simple) right quotient ring if and only if \(R\) is a right nonsingular and right f.d. ring [G; Cor.3.32].

(3) A ring \(R\) has a semisimple maximal right quotient ring if and only if \(R\) is a right nonsingular and right f.d. ring [46; Theo.1.6].

Therefore we obtain at once, from (3), (9.15) and (9.16),

**COROLLARY 9.19** Let \(M_R\) be a non-degenerate module. Then \(S\) has a semisimple maximal right quotient ring if and only if \(M_R\) is f.d. and nonsingular.

For the 'classical' case, we need a lemma. Recall that a ring \(R\) is prime if \(aRb\) is nonzero for any nonzero elements \(a, b\) of \(R\); and that \(R\) is semiprime if and only if \(aRa^0\) for any \(0^*a\in R\).

**LEMMA 9.20** Let \(M_R\) be a nonzero nondegenerate module. Then \(S\) is a prime (semiprime) ring if and only if \(\overline{R}=R/_{R_R}(M)\) is prime (semiprime) and \([M,\hat{M}]x^0\) for all \(0^*x\in M\).

**PROOF** [25; Theo.11]: Assume that \(S\) is a prime ring and write \(a=r_{R_R}(M)\). To see that \(\overline{R}\) is prime we have to show that, for \(r, s\in R\setminus a\), \(rRs\notin a\). But in this case we get \(M_R\neq 0\), \(M_S\neq 0\), whence \([M,\hat{M}]\neq 0\), \([Ms,\hat{M}]\neq 0\). Thus, since \(S\) is prime,

\[
0 \neq [M,\hat{M}][Ms,\hat{M}] = [(M,\hat{M})Ms,\hat{M}] = [M(R\hat{M}),s,\hat{M}]
\]

whence \(M(R\hat{M})s\neq 0\) and thus, in particular, \(MrRs\neq 0\), i.e. \(rRs\notin a\).

Now, if \(x\in M\) is such that \([M,\hat{M}]x=0\), then \([M,\hat{M}]x,\hat{M}] = [(M,\hat{M})x,\hat{M}] = 0\) whence, by primeness of \(S\) and since \([M,\hat{M}]\neq 0\), \([x,\hat{M}]\neq 0\). Then, by hypothesis, \(x=0\).

Conversely, assume that \(\overline{R}\) is prime and that \([M,\hat{M}]x=0\) for \(x\in M\) implies \(x=0\). Let \(f, g\) be nonzero elements of \(S\); then \(fM\neq 0\) and \(gM\neq 0\) imply \(N(\hat{M},fM)=(M,\hat{M})fM\neq 0\) and \(M(\hat{M},gM)=(M,\hat{M})gM\neq 0\), i.e. \((\hat{M},fM)\notin a\) and \((\hat{M},gM)\notin a\) whence, by primeness of \(\overline{R}\), \((\hat{M},fM)(\hat{M},gM)\notin a\). But
\[(\hat{M}, fM)(\hat{M}, gM) = (\hat{M}, fM(\hat{M}, gM)) = (\hat{M}, f[M, \hat{M}]gM) = (\hat{M}, f[M, \hat{M}]gM),\]
whence \((\hat{M}, f[M, \hat{M}]gM)\in S\) and thus, in particular, \(f[M, \hat{M}]g\neq 0\) and \(fSg\neq 0\).
Therefore \(S\) is a prime ring.
The semiprime case follows by taking \(r=s\) and \(f=g\).

**COROLLARY 9.21** Let \(M_r\) be a nondegenerate module. Then the following conditions are equivalent:
a) \(S\) has a simple (semisimple) classical right quotient ring;
b) \(M_r\) is f.d. and nonsingular, \(R\sim R\cap (M)\) is a prime (semiprime) ring
and \([M, M]\neq 0\) for all \(0\neq x\in M\).

**PROOF:** This follows from (1), (2), (9.15), (9.16) and (9.20).

Finally, we go one step further in the study of the maximal quotient ring of \(S\); if \(S\) is right nonsingular (i.e. if \(M_r\) is nonsingular, always under the hypothesis of non-degeneracy) then it possesses a maximal right quotient ring. We shall describe this maximal quotient ring in the next proposition.

**PROPOSITION 9.22** Let \(M_r\) be a non-degenerate nonsingular module, let \(E_r\) be an injective hull for \(M_r\) and write \(S=\text{End}(M_r), H=\text{End}(E_r)\). Then \(H\)

**PROOF** [25;Prop.4]: By injectivity of \(E_r\), we may view \(S\) as a subring of \(H\); by (1.1.d) and (7.12), \(H\) is a regular right self-injective ring, and hence a right nonsingular ring by [G; Prop.1.27] or (9.16).
Suppose we prove that, for any \(0\neq h\in H\), \(S\cap hS\neq 0\); this is clearly equivalent to \(S\subseteq Hs\), and then \(Hs\) is nonsingular (1.12.d), whence it is a rational extension of \(Ss\) [G; Lemma 2.24], and thus \(H\) is a right quotient ring of \(S\). Now, \(Hn\) is injective, and then it has no proper rational extensions [G; Lemma 2.24], which implies that \(H\) is a maximal right quotient ring of \(S\) [G; Prop.2.28].
Let us then prove that \(0\neq h\in H\) implies \(0\neq S\cap hS\). Since \(M\subseteq E\), \(N=M\cap h^{-1}M\subseteq M\) (1.2.a & d) and hence \(hN\neq 0\), because \(\Gamma(H)=0\) (7.12). Therefore, by non-degeneracy of \(M_r\),

\(0 \neq [hN, \hat{M}] = h[N, \hat{M}] = h[Mh^{-1}M, \hat{M}] \subseteq h[M, \hat{M}] \cap h^{-1}M, \hat{M}] \subseteq hS\cap S\).
SECTION 10: THE ENDMORPHISM RING OF MODULES WITH CHAIN CONDITIONS

We start this section by proving the classical Fitting's Lemma, and obtain as a consequence the fact that every indecomposable module of finite length has a local endomorphism ring. Next, we prove a recent result of Camps and Dicks, whose characterization of semilocal rings in [9] gives as a corollary that the endomorphism ring of an Artinian module is semilocal.

Later on we introduce the concept of T-nilpotency and use it to prove that, in the endomorphism ring of a module which is either Noetherian or Artinian, every nil subring is nilpotent; and that every module of finite length has a semiprimary endomorphism ring. These results are due to Fisher and Small [16].

We close Section 10 proving that the correspondence theorems for finitely generated ideals of S studied in Section 6 work for quasi-injective or quasi-projective modules, and under these hypothesis we obtain necessary and sufficient conditions on Mr for S to be Noetherian, semiprimary, or Artinian. The main results in this area are due to M.Harada and T.Ishii ([22] and [23]), though our proofs of them make use of different techniques (those of Section 6).

Fitting's Lemma and Consequences

**LEMMA 10.1** (Fitting) If Mr is a module of finite length n and f is an endomorphism of M, then M=Imf^n@Kerf^n.

**Proof:** Let Kf=Kerf; the chain 0£K1£K2£⋯ becomes stationary at some step j, and for the least j with this property the inclusions K1-cK1 are strict for 1≤i≤j: Suppose not; then if x£K1+i, fx£K1=K1-1 and thus x£K1, i.e. K1=K1+i; by induction K1-1=Kj, against the minimality of j. This shows in particular that j≥n, whence Kn=K2n. Then, if x£Kerf^n@Imf^n, we get x=f^n(y) (y£M) and 0=f^n x=f^n y, i.e. yeK2n=Kn and thus x=0. Hence Kerf^n@Imf^n=0.

On the other hand, let M1=Imf; the chain M2M3⋯ stops at some
minimal j and then $M_i = M_{i+1}$ for $1 \leq i \leq j$: If not, for all $x \in M_i$ we have $x = f_i^{-1}y = f_i^{-1}f_1^{-1}y$ (yeM), and $f_i^{-1}y = f_1^{-1}y(zeM)$ and then $x = f_i^{-1}z \in M_{i+1}$; thus $M_i = M_{i+1}$ and by induction $M_{i-1} = M_j$, a contradiction. Therefore $j = sn$ and thus $M_n = M_{2n}$.

It remains to see that $M = \text{Ker}f^n + \text{Im}f^n$. For all $x \in M$, $y = f^n x \in M_{2n}$ and then there exists $z \in M$ such that $y = f^{2n}z$, whence $x - f^n z \in \text{Ker}f^n$ and thus $x = (x - f^n z) + f^n z \in \text{Ker}f^n + \text{Im}f^n$. This completes the proof.

**Corollary 10.2** Let $M_R$ be a Noetherian (resp. Artinian) module and let $f \in \text{End}(M_R)$, then $f$ is an epimorphism (resp. a monomorphism) if and only if it is an isomorphism.

**Proof:** Assume that $M_R$ is Noetherian. From the first part of the proof of Fitting's Lemma we know that, for some $n$, $\text{Ker}f^n \cap \text{Im}f^n = 0$; but if $f$ is epic then so is $f^n$, whence $\text{Ker}f^n = 0$ and thus $\text{Ker}f = 0$.

The proof when $M_R$ is Artinian follows by duality.

**Lemma 10.3** Let $R$ be a nonzero ring in which every element is either invertible or nilpotent. Then $R$ is a local ring.

**Proof:** First note that if $a \in R$ has no right inverse, then it has no left inverse: if $ba = 1$ and $n$ is the least integer such that $a^n = 0$, then $0 = ba^n = (ba)a^{n-1} = a^{n-1}$, a contradiction. This remark and its right-left symmetric show that, for all $a \in R$,

- $a$ is invertible $\iff$ $a$ is left invertible $\iff$ $a$ is right invertible.

Take now two non-invertible elements $a, b$ of $R$; to see that $R$ is local it suffices to show that $a + b$ is non-invertible. Suppose then that $(a + b)c = 1$ for some $c \in R$; since $bc$ is not invertible (b would then be right invertible), there exists $n \in \mathbb{N}$ with $(bc)^n = 0$; then $(1-bc)(1+bc+\cdots+(bc)^{n-1}) = 1-(bc)^n = 1$, but $1-bc-ac$ cannot have a right inverse (a would also have one). This contradiction implies that $a + b$ is non-invertible, and hence that $R$ is local.

**Theorem 10.4** If $M_R$ is a nonzero indecomposable module of finite length, then $S = \text{End}(M_R)$ is a local ring.

**Proof** ([17; Satz 3]): Let $n$ be the composition length of $M_R$; then for all $f \in S$ we get $M = \text{Ker}f^n \oplus \text{Im}f^n$ (10.1). But, since $M$ is indecomposable, it
must be that either $\text{Im} f^n = 0$ (whence $f^n = 0$), or $\text{Im} f^n = M$, $\text{Ker} f^n = 0$ (whence $\text{Im} f = M$, $\text{Ker} f = 0$ and thus $f$ is invertible). Therefore any element of $S$ is either invertible or nilpotent and then (10.3) applies.\[\square\]

**Remark** H. Fitting [17; Satz 8] also proved that, in the above situation, $J(S)$ is nilpotent of index at most the length of $M^n$ (see [FA73; Theo. 17.20]).

The Endomorphism Ring of an Artinian Module is Semilocal

In this paragraph we state one of the implications of a characterization of semilocal rings given by Camps and Dicks in [9], who used it to solve a conjecture made by their teacher P. Menal, namely that the endomorphism ring of an Artinian module is semilocal (i.e. semisimple modulo its radical). Let us first introduce the concept of maximum condition with respect to summands in a set of subgroups of an Abelian group.

Let $\Omega$ be a set of subgroups of an Abelian group $M$. Given $X, Y, Z$ in $\Omega$ with $X \oplus Y = Z$, we say that $X$ and $Y$ are $\Omega$-summands of $Z$; if $Y \neq 0$ then $X$ is said to be a proper $\Omega$-summand of $Z$. The set $\Omega$ satisfies the maximum condition with respect to summands if every nonempty subset $\Delta$ of $\Omega$ contains an element which is a proper $\Omega$-summand of no member of $\Delta$.

For example, given a module $M^n$, $\Omega = \{1_n^r : r \in R\}$ is a family of subsets of the Abelian group $M^n$. Suppose further that there exists a ring $Q$ such that $M = QM^n$ and $QM$ is finite dimensional; since, for each $r \in R$, $1_n^r$ is a submodule of $QM$, an easy argument shows that, in this case, $\Omega$ satisfies the maximum condition with respect to summands. This fact will be used in the proof of Theorem 10.6.

**Proposition 10.5** Let $R$ be a ring such that there exists an $R$-module $M^n$ satisfying the following two conditions:

1) the set $\Omega = \{1_n^r : r \in R\}$ satisfies the maximum condition with respect to summands;

2) if $r \in R$ is not invertible then $1_n^r \neq 0$.

Then $R$ is a semilocal ring.
In what follows, let \( U(R) \) be the set of all units in \( R \), \( J=J(R) \), \( \bar{R}=R/J \) and, for all \( r \in R \), write \( \bar{r}=r+J \). We have to show that \( \bar{R} \) is a semisimple ring.

First, we give a partial ordering in \( R \): For \( a,b \in R \), write \( a>\bar{b} \) if \( 1_{\bar{a}}(a) \) is a proper \( 1_{\bar{a}} \)-summand of \( 1_{\bar{b}}(b) \); then the relation \( \geq \) (defined in the obvious way) is a partial ordering in \( R \) which, by 1), satisfies the minimum condition. Henceforth, by minimal we shall mean minimal with respect to this ordering.

We recall that \( a \in J \) if and only if \( 1-ab \) (and \( 1-ba \)) belongs to \( U(R) \) for all \( b \in R \). This will be helpful after proving that

\[
3) \quad a,x \in R, \quad 1-ax \notin U(R) \Rightarrow a>\bar{a}-\bar{a}x\bar{a}.
\]

Since, by 2), \( 1_{\bar{a}}(1-ax)\neq 0 \), 3) will be proved if we show that \( 1_{\bar{a}}(1-ax)=1_{\bar{a}}(a)\oplus 1_{\bar{a}}(1-ax) \); clearly the sum \( 1_{\bar{a}}(a)+1_{\bar{a}}(1-ax) \) is direct and it is contained in \( 1_{\bar{a}}(1-ax) \); on the other hand, if \( m \in 1_{\bar{a}}(a-axa) \), then \( m=\max+m(1-ax) \) with \( \max \in 1_{\bar{a}}(1-ax) \) and \( m(1-ax)\in 1_{\bar{a}}(a) \).

Now let \( \mathcal{E}=\{aeR: \bar{a}^{2}=\bar{a} \text{ and } (\bar{I}-\bar{a})\bar{R} \text{ is semisimple in } \text{Mod}_R \} \); since \( \mathcal{E} \) is nonempty (1\( \in \mathcal{E} \)), it contains a minimal element, say \( a \); clearly, the proposition will be proved if we show that \( a \in J \).

Suppose then that \( a \notin J \); thus \( aR\setminus J \) is nonempty and hence there exists \( b \in B \) such that \( ab \) is minimal in \( aR\setminus J \). We claim that

\[
4) \quad x \in R, \quad 1-abx \notin U(R) \Rightarrow \bar{a}b\bar{x}\bar{a}b=\bar{a}b.
\]

By 3) we have \( \bar{a}b>\bar{a}b-abx\bar{a}b \), and since \( ab \) is minimal in \( aR\setminus J \), it follows that \( (ab-abx\bar{a}b)\in J \), proving 4). Now we can prove

\[
5) \quad \bar{a}b\bar{R} \text{ is a simple right } \bar{R}-\text{module}.
\]

We show that, for any \( x \in R \) such that \( \bar{a}b\neq 0 \), \( \bar{a}b \) generates \( \bar{a}b\bar{R} \); for, since \( ab \notin J \), there exists \( y \in R \) such that \( 1-abxy \notin U(R) \), whence \( \bar{a}b\bar{y}=\bar{a}b \) and thus \( \bar{a}b\bar{R} \孙 \bar{a}b\bar{R}=\bar{a}b\bar{y}\bar{a}b\bar{R} \孙 \bar{a}b\bar{R} \); hence \( \bar{a}b\bar{R}=\bar{a}b\bar{R} \).

Now, since \( ab \notin J \), there exists \( c \in R \) with \( 1-abc \notin U(R) \); for this \( c \) we claim that \( a-abca \notin \mathcal{E} \). By 4), \( \bar{a}b\bar{c}a=\bar{a}b \); this implies that \( \bar{a}b\bar{c}a=\bar{a}b \bar{R} \) (as in the proof of 5)) and that \( \bar{a}b\bar{c}a \) is idempotent, since

\[
(\bar{a}-\bar{a}c) = \bar{a}-\bar{a}c = \bar{a}-\bar{a}c \bar{a}b = \bar{a}-\bar{a}c = \bar{a}-\bar{a}c.
\]

It remains to check that \( (\bar{I}-\bar{a}+\bar{a}b\bar{c}a)\bar{R} \) is semisimple; now, if \( (\bar{I}-\bar{a})\bar{r}=\bar{a}b\bar{c}a \) for some \( r,s \in R \), then \( \bar{r}=0 \) and, since \( \bar{a}=\bar{a} \), this implies \( \bar{r}=\bar{a}r \), i.e. \( (\bar{I}-\bar{a})\bar{r}=0 \); thus the sum \( (\bar{I}-\bar{a})\bar{R} \bar{a}b\bar{c}a \bar{R} \) is direct and clearly it
contains \((1-a+abc\tilde{a})R\). Since \(ae\mathcal{E}\) and \(abc\tilde{a}R=abR\) is simple (5), 
\((1-a+abc\tilde{a})R\) is a submodule of a semisimple module and hence is semisimple itself, as required.

Finally, since \(1-abc\in\mathcal{U}(R)\), we can apply 3) to get \(a>a-abca\), but \(a-abca\not\in\mathcal{E}\) and \(a\) is minimal in \(\mathcal{E}\). This contradiction shows that \(ae\mathcal{J}\), as desired.

\textbf{THEOREM 10.6} \textit{If \(M_R\) is an Artinian module then \(S=\text{End}(M_R)\) is a semilocal ring.}

\textbf{Proof} [9, Theo. 6]: Consider the left \(S\)-module \(sM\); for any \(f\in S\), \(r_f(M)\) is precisely \(\ker f\), and thus (10.2) shows that \(r_f(M)\neq 0\) for all \(f\in S\setminus \mathcal{U}(S)\). On the other hand, \(M_R\) is finite dimensional and then the remark preceding (10.5) shows that \(\mathcal{E}=\{\ker f: f\in S\}\) satisfies the maximum condition with respect to summands. Now the left-right symmetric of (10.5) applies and therefore \(S\) is semilocal.

\textbf{Modules of Finite Length}

We intend to prove that an Artinian or Noetherian module has an endomorphism ring in which every nil subring is nilpotent (here we do not require that subrings contain the identity of the overring), and as a consequence the endomorphism ring of a module of finite length is \textit{semiprimary} (i.e. semilocal with nilpotent radical). The first part was first announced by A.Goldie and L.Small in [20] for Noetherian modules. Later on, J.Fisher [16] gave a proof for the Artinian case which was dualizable. The second part is here easily proved using our Theorem 10.6.

At this point, we need to introduce the concept of T-nilpotency (for transfinite nilpotency). A subset \(W\) of a ring \(R\) is said to be \textit{left} (resp. \textit{right}) \(T\)-\textit{nilpotent} if, for every infinite sequence \(w_1, w_2, \ldots\) of elements of \(W\), there exists an integer \(k\) such that \(w_1 \cdots w_k = 0\) (resp. \(w_k \cdots w_1 = 0\)). Every nilpotent subring of \(R\) is left (resp. right) \(T\)-\textit{nilpotent}, and every \(T\)-\textit{nilpotent} subset of \(R\) is nil. Counterexamples for both converses do exist (see [A-F; Ex.15.8]); however, \(T\)-\textit{nilpotency} does imply nilpotency in the following particular case.
LEMMA 10.7 Let $R$ be a ring with ACC on left (right) annihilators, and let $B$ be a subring of $R$. Then $B$ is nilpotent if and only if $B$ is left (right) T-nilpotent.

PROOF: As we have already remarked, if $B$ is nilpotent then it is (left and right) T-nilpotent. Assume then that $B$ is not nilpotent; by hypothesis, $\mathcal{L}(B)\subseteq\mathcal{L}(B^2)\subseteq\cdots$ gets stationary at some $n\in\mathbb{N}$, and by assumption $B^{n+1}\neq 0$; then there exists $b_1\in B$ with $b_1^B\neq 0$, i.e. $b_1\in B\setminus\mathcal{L}(B^n)$. Since $\mathcal{L}(B^n)=\mathcal{L}(B^{n+1})$, this implies $b_1^B^{n+1}\neq 0$; take then $b_2\in B$ such that $b_1b_2^B\neq 0$, so that $b_1b_2\in\mathcal{L}(B^n)$ and in particular $b_1b_2\neq 0$. In this way, we get a sequence $b_1, b_2, \ldots$ of elements of $B$ with $b_1 \cdots b_k \neq 0$ for all $k\in\mathbb{N}$, and hence $B$ is not left T-nilpotent.

THEOREM 10.8 If $M^r$ is Artinian or Noetherian, then each nil subring of $S=\text{End}(M^r)$ is nilpotent.

PROOF [16; Theo. 1.51] or [A-F; Theo. 29.2]: We assume that $M^r$ is Artinian; the proof if $M$ is Noetherian may follow dually to this one, but in fact is a direct consequence of (9.13).

Let $B$ be a nil subring of $S$. By (6.4.a) $S$ has ACC on left annihilator ideals, so that it suffices to see that $B$ is left T-nilpotent (10.7).

Let us first introduce the following two concepts: a sequence $\{b_n\}$ in $B$ is an $\omega$-chain if $b_1 \cdots b_n \neq 0$ for all $n$ (all subscripts will belong to $\mathbb{N}$); clearly every tail $\{b_n, b_{n+1}, \ldots\}$ of an $\omega$-chain is an $\omega$-chain. An element $b\in B$ has an $\omega$-chain if there exists an $\omega$-chain $\{b_n\}$ with $b_1=b$.

If $\{b_n\}$ is an $\omega$-chain and $i\leq j$, then the product $b_1 \cdots b_i \cdots b_j$ has an $\omega$-chain.

Suppose that $B$ is not left T-nilpotent; then $\mathcal{L}(\Omega)=\{b\in B: b$ has an $\omega$-chain$\}$ and thus, since $M^r$ is Artinian, there exists $b_1\in\Omega$ such that $b_1^B$ is minimal in $\{bM: b\in\Omega\}$. By induction, and using the previous remarks, we can construct the nonempty set $\Omega_n=\{b_1^B \cdots b_{n-1}^B b \in \Omega: b_1^B \cdots b_{n-1}^B b$ has an $\omega$-chain$\}$ and find $b_n\in\Omega_n$ such that $b_n^B$ is minimal in $\{bM: b\in\Omega_n\}$. It is clear that $\{b_n\}$ is then an $\omega$-chain.

Moreover, for each $i\leq j$, we have $(b_1 \cdots b_j)\subseteq bM$ and $(b_1 \cdots b_j)\in\Omega_n$ whence, by minimality of $b_1^B$, we get

\[ (b_1 \cdots b_j)M=b_1^B. \]

Now, for each $n$, call $f=b_1 \cdots b_n (\neq 0)$. By the last remark, $f^M=fM$.
M=(\text{Ker} f_n)+(b_{n+1} M).

To see that, let \( x \in M \) and take \( y \in M \) such that \( f(x) = f_{n+1} y \); then
\[
x = (x - b_{n+1} y) + b_{n+1} y \quad \text{with} \quad f(x - b_{n+1} y) = 0.
\]
Let us now prove that, for \( n = m \), \( f_{n+1} = 0 \). Suppose \( n = m \) and \( f_{n+1} = 0 \); for any \( k \geq m \) we have (1) \( b_{n+1} = b_{n+2} \cdot \cdot \cdot b_k \cdot M \) and thus \( f_{n+1} b_{n+2} \cdot \cdot \cdot b_k = 0 \), what means \( f_{n+1} b_{n+2} \cdot \cdot \cdot b_k = 0 \) for all \( k \geq m \), i.e. \( f_{n+1} \) has an \( \omega \)-chain; but
\[
f_{n+1} b_{n+2} \cdot \cdot \cdot b_{k+1}(b_{n+2} \cdot \cdot \cdot b_k) \quad \text{whence} \quad (b_{n+2} \cdot \cdot \cdot b_k) \in \gamma_m; \quad \text{then, by}
\]
minimality of \( b_{n+1} M \), we get
\[
0 \neq b_{n+1} M = (b_{n+2} \cdot \cdot \cdot b_k) \cdot b_{n+1} M = \cdot \cdot \cdot,
\]
contrary to the nilpotency of \( b_{n+2} \cdot \cdot \cdot b_k \in \beta).

This also proves that \( b_{n+1} \text{Ker} f_n \) for all \( n = m \), and thus
\[
(3) \quad b_{n+1} M \subseteq \bigcap_{n-1} \text{Ker} f_n \quad \text{and} \quad \sum_{i=1}^n b_i M \subseteq \text{Ker} f_n.
\]

Next, we show by induction that, for any \( n \),
\[
(4) \quad \sum_{k=1}^n \text{Ker} f_k + \sum_{j=2}^{n+1} b_j M = M.
\]
The case \( n=1 \) is covered by (2). Assume now that (4) holds for \( n-1 \); then using (2), (3) and the modular law we get
\[
\left( \sum_{k=1}^n \text{Ker} f_k \right) + \sum_{j=2}^{n+1} b_j M = \left[ \left( \sum_{k=1}^{n-1} \text{Ker} f_k \right) \cap \text{Ker} f_n \right] + \sum_{j=2}^{n+1} b_j M = \left[ \left( \sum_{k=1}^{n-1} \text{Ker} f_k \right) + \sum_{j=2}^{n+1} b_j M \right] \cap \text{Ker} f_n = \text{Ker} f + b_{n+1} M = M.
\]
Now, since \( M \) is Artinian, the sequence \( \left( \sum_{k=1}^n \text{Ker} f_k \right) \) stops at some \( n \), for which \( \sum_{k=1}^n \text{Ker} f_k \subseteq \text{Ker} f_n \). This, together with (3) and (4), shows that \( M = \text{Ker} f_{n+1} \), a contradiction since \( f_{n+1} \neq 0 \) for all \( i \). This contradiction proves the theorem. 

**COROLLARY 10.9** If \( M \) is Noetherian (resp. Artinian) then \( \Gamma(S) \) (resp. \( \Delta(S) \)) is nilpotent.

**PROOF**: Suppose \( M \) is Noetherian and recall that \( \Gamma = \Gamma(S) = \{ f \in S : \text{Ker} f \leq M \} \). By (10.8), it suffices to show that \( \Gamma \) is nil. Let \( f \in \Gamma \) and consider the ascending chain \( \text{Ker} f \leq \text{Ker} f^2 \leq \cdot \cdot \cdot \) of submodules of \( M \); by hypothesis there exists \( n \in \mathbb{N} \) such that \( \text{Ker} f^n = \text{Ker} f^{n+1} = \cdot \cdot \cdot \). Then it is easy to see that \( f^n M \cap \text{Ker} f = 0 \) and thus, since \( \text{Ker} f \leq M \), we get \( f^n = 0 \). Hence \( \Gamma \) is nil.

The result for \( \Delta(S) = \{ f \in S : f \leq M \} \) follows by duality. 


THEOREM 10.10 If $M^r$ has finite length, then $S=\text{End}(M^r)$ is semiprimary.

PROOF [A-F; Cor. 29.3]: We have to prove that $S/J(S)$ is semisimple and $J(S)$ is nilpotent. Since $M$ has finite length, it is in particular Artinian, and therefore (10.6) $S$ is semilocal.

Then, by (10.8), it suffices to see that $J=J(S)$ is nil. Let $f \in J$; by (10.1) $M=\ker f \oplus \text{im} f$ for decomposition length of $M$. Then $\text{im} f \cap \ker f = 0$ and $\text{im} f = \text{im} (f+1)$ (see the proof of (10.1)); therefore $f$ is an automorphism of $N=\text{im} f$; its inverse $g:N \rightarrow N$ can be extended to some $h \in S$ since $N \subseteq M$, and then $fh$ is the identity on $N$, what implies $N \subseteq \ker (1-fh)$. But $f \in J$ implies that $1-fh$ is invertible, so that $N=0$ and thus $f^m=0$, as required.

REMARK This last theorem may be proved starting with a weaker assumption on $M$, namely that $M^r$ is Artinian with finite homogeneous length, and the index of nilpotency of $J(S)$ (i.e. the Loewy length of $S$) may be bounded in terms of all simple submodules of $M^r$. For details see [48, 49, 51].

**Quasi-Injective and Quasi-Projective Modules with Noetherian, Semiprimary and Artinian Endomorphism Rings**

We extend now some of the results at the end of Section 6, and characterize the quasi-injective (resp. quasi-projective) modules $M^r$ which have semiprimary or left (resp. right) Noetherian or Artinian endomorphism rings in terms of chain conditions in their lattices of annihilator-closed (resp. $M$-cotorsionless) submodules. We shall keep the notation introduced in Section 6.

The key steps in what follows are the next two theorems:

THEOREM 10.11 If $M^r$ is quasi-injective then every finitely generated left ideal of $S$ is annihilator-closed.

PROOF [23; Lemma 1]: We prove that, if the left ideal $\mathcal{A}$ is a-closed and $f \in S$ then $\mathcal{A} + Sf$ is a-closed; thus, since the zero ideal is a-closed, the result follows by induction.

Assume then $\sum_{s \in M}^r (\mathcal{A}) = \mathcal{A}$ and $f \in S$; our task is reduced to show that
Note that, for arbitrary left ideals \(I, J\) of \(S\), we have \(r(I + J) = r(I) \cap r(J)\). Let then \(g \in \text{ker} r(I + J)\); we have at once \(r(I + J) \supseteq r(I) \cap r(J)\); consider the following diagram in which the rows are exact (\(r(h) = \text{Ker} h\) for any \(h \in S\)), \(i, j\) are inclusion maps (so that the square on the left commutes) and \(r = r:\)

\[
\begin{array}{cccc}
0 & \to & r(I) & \to r(J) & \to r(I + J) & \to 0 \\
& i & & j & & k \\
0 & \to & r(g) & \to M & \to gM & \to 0 \\
\end{array}
\]

An easy exercise of diagram chasing shows that there exists \(k : r(I) \to gM\) such that the resulting diagram commutes. From the quasi-injectivity of \(M\), \(k\) extends to some \(h \in S\) and then \(hf\) and \(g\) coincide in \(r(I);\) therefore \(g - hf \in \text{ker} r(I + J) = H\), whence \(g \in A + SF\), proving the theorem.\]

**THEOREM 10.12** If \(M_r\) is quasi-projective then every finitely generated right ideal of \(S\) is \(\tau\varepsilon\)-closed.

**PROOF** ([A-N; Prop. 4.9]): The proof is dual to that of (10.11), and consists of showing that \(A + FS\) is \(\tau\varepsilon\)-closed whenever the right ideal \(A\) of \(S\) is and \(feS\).

So, assume \(A = r(S)\), \(feS\), and let us prove that \(r(S) + FS\) is \(\tau\varepsilon\)-closed. It is clear that \(r(S) + FS \subseteq AM + FM\), so that if \(g \in r(S) + FS\) then \(gM \subseteq AM + FM\); write \(N = AM - r(S)\), \(p : M \to M/N\), \(f = pf\) and \(g = pg\), and note that

\[
gM = pgM = \frac{gM + N}{N} \subseteq \frac{fM + N}{N} = pfM = \bar{f}M.
\]

Thus, by quasi-projectivity of \(M\), there exists \(h \in S\) such that \(g = \bar{f}h\), i.e. \(pg = pfh\), or \(p(g - fh) = 0\), but that just means that \(g - fh \in N\) and hence \(gN + FS = r(S) + FS\), as required.\]

Thus we get the next two corollaries from (6.18):  

**COROLLARY 10.13** Let \(M_r\) be quasi-injective and consider the following statements:

a) \(S\) is a regular ring;

b) the kernel of every element of \(S\) is a direct summand of \(M_r\);

c) finite intersections of kernels of elements of \(S\) are direct summands of \(M_r\);
a') $S$ is a right perfect ring;

b') $M$ has ACC on $K=\{\text{Ker } f; f \in S\}$;

c') $M$ has ACC on $K_f=\{\bigoplus_{i=1}^n \text{Ker } f_i; f_1, \ldots, f_n \in S\}$.

Then we have $a') \Rightarrow b') \Rightarrow c')$.

**COROLLARY 10.14** Let $M_r$ be quasi-projective and consider the following statements:

a) $S$ is a regular ring;

b) the image of every element of $S$ is a direct summand of $M_r$;

c) finite sums of images of elements of $S$ are direct summands of $M_r$;

a') $S$ is a right perfect ring;

b') $M$ has ACC on $\mathcal{J} = \{\text{Im } f; f \in S\}$;

c') $M$ has ACC on $\mathcal{J}_f = \{\sum_{i=1}^n \text{Im } f_i; f_1, \ldots, f_n \in S\}$.

Then we have $a) \Rightarrow b) \Rightarrow c)$ and $a') \Rightarrow b') \Rightarrow c')$.

The following theorems, proved here for quasi-injective and quasi-projective modules, may be proved under the weaker assumptions $\mathcal{J} \subseteq M_a$ and $\mathcal{J}_r \subseteq M_{\text{ot}}$, respectively, and in fact the proofs given here use only these hypotheses.

**THEOREM 10.15** If $M_r$ is quasi-injective then the following assertions are equivalent:

a) $S$ is left Noetherian;

b) $M$ has DCC on $K_f$;

c) $M$ has DCC on $M_a$.

**PROOF** [31; Theo. 4.3]: Since $S$ is left Noetherian if and only if it satisfies ACC on finitely generated left ideals, i.e. on $\mathcal{J}_l$, the equivalence $a) \Rightarrow b)$ follows from (10.11) and (6.17.c).

The implications $a) \Rightarrow c) \Rightarrow b)$ follow without requiring $M_r$ to be quasi-injective: by definition, $K_r \subseteq M_a$, so that c) implies b). On the other hand, let $N_1 \supseteq N_2 \supseteq \cdots$ be a chain of $a$-closed submodules of $M$ and consider $1_S(N_1) \supseteq 1_S(N_2) \supseteq \cdots$; if $S$ is left Noetherian then there exists $n \in \mathbb{N}$ such that $1_S(N_n) = 1_S(N_k)$ for all $k \geq n$ and hence

$$N_n = r.1_S(N_n) = r.1_S(N_k) = N_k$$

for all $k \geq n$, and this proves that $a)$ implies $c)$.■
Dualizing the proof of (10.15) we get:

**THEOREM 10.16** If \( M_r \) is quasi-projective then the following statements are equivalent:

a) \( S \) is right Noetherian;

b) \( M \) has ACC on \( J_r \);

c) \( M \) has ACC on \( M_{\text{ct}} \). □

Now we turn to the question of when is \( S \) semiprimary; proofs of Theorems 10.16 and 10.17 which use different techniques may be found in [23; Theorem 1] and [22; Prop.2.4].

**THEOREM 10.17** If \( M_r \) is a quasi-injective module with ACC on a-closed submodules then \( S \) is a semiprimary ring.

**Proof** [31; Theo.4.5]: Since \( M_r \) is quasi-injective, (10.11) and (6.18.c') imply that \( S \) is a right perfect ring, i.e. \( S \) is semilocal and its radical \( J(S) \) is right T-nilpotent. Now, from (6.4.a) and the hypothesis, we know that \( S \) has DCC on left annihilators, i.e. ACC on right annihilators, and hence \( J(S) \) is indeed nilpotent by (10.7). Therefore \( S \) is semiprimary. □

Dually, we get

**THEOREM 10.18** If \( M_r \) is a quasi-projective module with DCC on \( M \)-cotorsionless submodules then \( S \) is a semiprimary ring.

Next, using the Hopkins-Levitzki Theorem (a ring is left Artinian if and only if it is left Noetherian and semiprimary [S; p.181]), we get:

**THEOREM 10.19** Let \( M_r \) be quasi-injective. Then \( S \) is left Artinian if and only if \( M \) satisfies ACC and DCC on \( M_a \).

**Proof** [31; Theo.4.6]: If \( M \) satisfies both chain conditions on \( M_a \) then it is left Noetherian (10.15) and semiprimary (10.17), i.e. \( S \) is left Artinian.

Conversely, if \( S \) is left Artinian then it is left Noetherian and hence \( M \) has DCC on \( M_a \) (10.15); moreover, if \( N_1 \subseteq N_2 \subseteq \cdots \) is a chain in \( M_a \) then
the chain \( l_s^1(N_1) \leq l_s^1(N_2) \leq \cdots \) stops by hypothesis and thus so does \( N_1 \leq N_2 \leq \cdots \), whence \( M \) has ACC on \( M_a \). Note that this 'only if' part may be proved without requiring \( M_a \) to be quasi-injective. ■

Dually,

**THEOREM 10.20** Let \( M_r \) be quasi-projective. Then \( S \) is right Artinian if and only if \( M \) satisfies ACC and DCC on \( M^{\text{op}} \).

**REMARK** Theorem 10.19 generalizes the well known fact that the endomorphism ring of a quasi-injective module of finite length is left Artinian (for an easy proof see [21]).

**COROLLARY 10.21** Let \( M_r \) be quasi-projective and quasi-injective; then

a) if \( M_r \) is Noetherian, then \( S \) is right Artinian.

b) if \( M_r \) is Artinian, then \( S \) is left Artinian.

**PROOF:** a) \( S \) is right Noetherian by (10.16) and semiprimary by (10.17); thus \( S \) is right Artinian.

b) This follows similarly from (10.15) and (10.18). ■
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The numbers following each entry refer to the Section where it was defined. Terms not defined in the text are followed by (#), and their definitions may be found e.g. in [A-F], [FA73], [FU], [L], [M] or [S].