Nonlinear stability analyses for variable viscosity and compressible convection problems

by

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A thesis submitted to the Faculty of Science, University of Glasgow, for the degree of Doctor of Philosophy

Department of Mathematics,
University of Glasgow
May, 1993

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Preface

This thesis is submitted to the University of Glasgow in accordance with the requirements for the degree of Doctor of Philosophy.

I would like to express my gratitude to my supervisor Professor B. Straughan for his guidance, assistance and encouragement throughout the period of my research.

My sincere thanks also go to Professor R. W. Ogden for the help and advice he has given me during my time in the Department of Mathematics.

I would also like to thank Professors J.D. Gibbon, R.N. Hills, L.E. Payne, S. Rionero and P.H. Roberts for various suggestions, help and advice.

I also thank the Science and Engineering Council of Great Britain for financing this research through a three year award.

Finally, I am very grateful to all my family for their patience and support during all my time at University.

Summary

In this thesis we present nonlinear energy stability analyses of a variety of convection problems, specifically concentrating on compressible convection and convection where the viscosity of the fluid depends on the temperature.

To begin we introduce the energy method, which we shall employ in later chapters in order to derive nonlinear stability criteria, and illustrate its effectiveness with a simple one-dimensional example. We then study the stability of a system containing a generalized incompressible fluid, employing linear theory, the energy method and an asymptotic analysis.

Next we study a variable viscosity fluid, first establishing continuous dependence of the solution to the Oberbeck-Boussinesq equations, both forward and backward in time, on the viscosity. We then look at convection where the viscosity is first a linear function of the temperature and then a quadratic one. In both cases we employ a generalized energy in order to derive a nonlinear stability boundary. To conclude our analysis of a variable viscosity fluid we also introduce a temperature dependent conductivity and carry out both linear and heuristic nonlinear analyses.

We then turn our attention to the phenomenon of convection within a porous medium, concentrating first on penetrative convection. We examine the stability of convection in a porous medium containing an internal heat source and a salt field, using linear theory, the energy method and a weighted energy. Then, returning to the topic of variable viscosity, we demonstrate that, in a porous medium, the solution to the equations of motion depends continuously on the viscosity both forward and backward in time. Finally we study a porous medium saturated with a variable viscosity fluid, employing the Brinkman equation, coupled with the energy method, in order to derive a nonlinear stability boundary.

CHAPTER ONE

INTRODUCTION.

1.1 The energy method

The energy method is an integral inequality technique which was originally based on the kinetic energy of the fluid motion. However subsequent work has, for a variety of reasons, introduced variations on this classical energy. These new functionals have much in common with the Lyapunov method which is used in solving partial differential equations and are generally referred to as "generalized energy methods". The present interest in the energy method owes much to the work of Serrin (1959a, 1959b) and Joseph (1965, 1966) whilst Rionero (1967, 1968) was the first to show that there existed a maximizing solution to the maximum problem of energy stability. In this thesis we will use the energy method in order to prove nonlinear stability results for a variety of convection problems. It allows us to obtain sufficient conditions for the nonlinear stability of some base solution, a condition which usually takes the form of a critical Rayleigh number; for Rayleigh numbers below this we know that the system is stable, i.e. any perturbation to the base solution will decay in time.

We will also establish continuous dependence results for two improperly posed problems. The definition of a well-posed problem is one for which a unique solution exists which depends continuously on the data. In order to establish the continuous dependence results we will use a technique known as the logarithmic convexity method cf. Payne (1975).

The first energy stability problem we study is that of convection in a generalized incompressible fluid. Here generalized incompressibility is taken in the sense defined by Hills and Roberts (1991), i.e. the density of the fluid, whilst being resistant to change by compression, can be changed by varying the pressure or the temperature. This model is important in that it represents a first approximation to compressible convection, a previously largely unforgiving area of research. We derive both linear instability and nonlinear stability results, the latter requiring the use of a generalized energy, and we see that the nonlinear stability boundary is very close to the linear instability one.

In Chapters 3–6 we consider a fluid whose viscosity is temperature dependent. First, in Chapter 3, we show that the solution to the Oberbeck-Boussinesq equations depends continuously on the viscosity both forward and backward in time. The backward in time problem is an improperly posed one and requires the use of a logarithmic convexity argument (cf. Payne (1975)). Having derived this result we then study, first in Chapter 4, convection when the viscosity is a linear function of temperature, and then, in Chapter 5, convection when the viscosity is a *quadratic* function of temperature. In both cases it is found that the nonlinear stability boundary coincides with the linear instability boundary, thus giving an optimum result.

Chapter 6 is devoted to the study of a variable viscosity fluid which also has a temperature dependent conductivity. The presence of these two competing effects complicates the analysis but we are able to deal with the linear problem and derive a linear instability boundary. For the nonlinear problem, however, the boundary conditions prove intransigent to analysis and we are forced to make dynamically inadmissible assumptions in order to derive the nonlinear stability results. Therefore we treat the problem heuristically, and, in this sense, are able to derive some interesting and useful results.

From Chapter 7 onwards we study convection within a fluid permeated porous medium. This type of model is important when studying various geophysical problems such as thawing subsea permafrost (cf. Payne *et al* (1988)) or patterned ground formation (cf. Gleason (1984)). In Chapter 7 we consider a porous medium containing an internal heat source and subject to a destabilizing salt field coupled with a nonlinear density. This gives rise to penetrative convection. For this problem we prove both conditional and unconditional nonlinear stability results, in the first case by using a generalized energy argument and in the second by employing a weighted energy.

In Chapter 8 we establish that, for a fluid-permeated porous medium, the solution to the equations of motion depends continuously on the viscosity. Again, as in Chapter 3, we consider both the forward in time and improperly posed backward in time problems, the latter once more necessitating the use of the logarithmic convexity method.

In Chapter 9 we prove that convection in a porous medium is, conditionally, nonlinearly stable. In order to prove the stability results we use the Brinkman equation, rather than the more common Darcy equation, when describing the velocity of the fluid. As in Chapters 4 and 5 we find that the nonlinear stability boundary coincides with the linear instability boundary and so we have an optimum result.

Throughout we will refer to various inequalities which are hereby defined.

1). Cauchy-Schwarz inequality.

$$< ab > \le < a^2 >^{1/2} < b^2 >^{1/2} = \|a\|\|b\|$$

where $a, b \in L^2(\Omega)$, $< . >$ means integration over Ω and $\| * \|$ refers to the $L^2(\Omega)$ norm. In most cases Ω will be represented by the period cell V.

2). Arithmetic-Geometric mean inequality.

$$uv \le \frac{1}{2\alpha}u^2 + \frac{\alpha}{2}v^2$$

where α is a positive constant.

Standard notation is employed throughout. Partial derivatives are denoted in the usual way and we employ indicial notation, e.g. $u_{i,i} = \Sigma_{i=1}^3 u_{i,i}$ and $u_{i,t} = \partial u_i/\partial t$. Also, Δ will mean the three-dimensional Laplacian.

In order to illustrate the techniques used throughout this thesis we now consider a simple example.

Let u be a solution to the one-dimensional equation

$$u_t + uu_x = \frac{1}{R}u_{xx} + u, \tag{1.1}$$

where $x \in (0,1)$ and $u(0,t) = u(1,t) = 0, \quad \forall t > 0$.

The zero solution $u \equiv 0$ is a solution to (1.1) and we will now investigate the stability of this solution by adding in a perturbation and considering the behaviour of this perturbation as $t \to \infty$. First we carry out a linear analysis. Upon linearizing (1.1) we have

$$u_t = \frac{1}{R}u_{xx} + u. \tag{1.2}$$

We write the perturbation in the form

$$u(x,t) = e^{\sigma t}\sin kx$$

where σ and k are constants. Note that for this to satisfy the boundary conditions we must have

$$k = n\pi, \qquad n = 1, 2, ...$$

When substituted into (1.2) we now have the equation

$$\sigma = -\frac{k^2}{R} + 1.$$

For linear stability we require $\sigma < 0$, as then the perturbation will decay with time. Hence we require $R < k^2$. As k^2(min)$= \pi^2$ our linear instability boundary is given by

$$R = \pi^2, \tag{1.3}$$

i.e. for $R > \pi^2$ the zero solution is unstable.

To study the *stability* of the solution to (1.1) we use an energy argument. First, multiply the equation by u and then integrate over $(0, 1)$ to find

$$\int_0^1 uu_t dx + \int_0^1 u^2 u_x dx = \frac{1}{R}\int_0^1 uu_{xx} dx + \int_0^1 u^2 dx. \tag{1.4}$$

If we write $\|u\|^2 = \int_0^1 u^2 dx$ and let $E(t) = \frac{1}{2}\|u\|^2$ then, after integration by parts, (1.4) becomes

$$\frac{dE}{dt} = -\frac{1}{R}\|u_x\|^2 + \|u\|^2. \tag{1.5}$$

For;

$$\int_0^1 uu_t dx = \int_0^1 \frac{1}{2}(u^2)_t dx = \frac{d}{dt}\frac{1}{2}\int_0^1 u^2 dx = \frac{d}{dt}\frac{1}{2}\|u\|^2,$$

$$\int_0^1 u^2 u_x dx = \int_0^1 \frac{1}{3}(u^3)_x dx = \frac{1}{3}[u^3(1) - u^3(0)] = 0,$$

$$\int_0^1 uu_{xx} dx = uu_x\Big|_0^1 - \int_0^1 u_x u_x dx = -\|u_x\|^2.$$

We now define

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{\|u\|^2}{\|u_x\|^2},$$

where $\mathcal{H}$ is the space of admissible functions, i.e. $\mathcal{H} = \{u \in C^2(0,1) \mid u = 0$ when $x = 0, 1\}$.

Then (1.5) can be written

$$\begin{aligned}\frac{dE}{dt} &= -\|u_x\|^2\Big[\frac{1}{R} - \frac{\|u\|^2}{\|u_x\|^2}\Big] \\ &\le -\|u_x\|^2\Big[\frac{1}{R} - \max_{\mathcal{H}}\frac{\|u\|^2}{\|u_x\|^2}\Big] \\ &= -\|u_x\|^2\Big(\frac{1}{R} - \frac{1}{R_E}\Big).\end{aligned}$$

If $R < R_E$ then $1/R - 1/R_E > 0$, say $1/R - 1/R_E = a$.
Hence

$$\frac{dE}{dt} \le -a\|u_x\|^2.$$

We may now use the Poincaré inequality $\|u_x\|^2 \ge \pi^2\|u\|^2$ to deduce

$$\frac{dE}{dt} \le -2a\pi^2 E$$

and this equation may be integrated to show

$$E(t) \le E(0)e^{-2a\pi^2 t}.$$

Thus we have shown that if $R < R_E$ then

$$E(t) = \frac{1}{2}\|u\|^2 \to 0 \quad \text{as} \quad t \to \infty$$

with the decay at least exponential in time. That is, we have proved that the perturbation is nonlinearly stable.

The problem remains to find R_E where, as defined earlier

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{\|u\|^2}{\|u_x\|^2}. \tag{1.6}$$

Let $I_1 = \|u\|^2$, $I_2 = \|u_x\|^2$. Then the Euler-Lagrange equations for (1.6) are derived as follows.

$$\begin{aligned}\frac{d}{d\epsilon}\frac{I_1(u+\epsilon\eta)}{I_2(u_x+\epsilon\eta_x)}\bigg|_{\epsilon=0} =&\delta\left(\frac{I_1}{I_2}\right) = \frac{I_2\delta I_1 - I_1\delta I_2}{I_2^2}\\ =&\frac{1}{I_2}\left(\delta I_1 - \frac{I_1}{I_2}\bigg|_{\max}\delta I_2\right)\\ =&\frac{1}{I_2}\left(\delta I_1 - \frac{1}{R_E}\delta I_2\right) = 0.\end{aligned}$$

Hence

$$\delta I_1 - \frac{1}{R_E}\delta I_2 = 0 \tag{1.7}$$

where

$$\begin{aligned}\delta I_1 =&\frac{d}{d\epsilon}\int_0^1 (u+\epsilon\eta)^2\,dx\bigg|_{\epsilon=0}\\ \delta I_2 =&\frac{d}{d\epsilon}\int_0^1 (u_x+\epsilon\eta_x)^2\,dx\bigg|_{\epsilon=0}\end{aligned}$$

and η is an arbitrary $C^2(0,1)$ function with $\eta(0) = \eta(1) = 0$.

Then (1.7) leads to

$$\int_0^1 (u\eta - \frac{1}{R_E}\eta_x u_x)dx = 0.$$

Upon integrating the second term by parts we can see that this becomes

$$\int_0^1 \eta(u_{xx} + R_E u)dx = 0$$

and, since η is an arbitrary function, we must have

$$u_{xx} + R_E u = 0 \qquad u(0) = u(1) = 0. \tag{1.8}$$

Equation (1.8) is the Euler problem which gives an eigenvalue problem for R_E.

The general solution to (1.8) is

$$u = A \sin \sqrt{R_E}x + B \cos \sqrt{R_E}x$$

where A and B are constants. The boundary condition $u(0) = 0$ shows that $B = 0$ and the condition $u(1) = 0$ shows that

$$\sqrt{R_E} = n\pi, \qquad n = \pm 1, \pm 2, \ldots.$$

For stability we need $R < R_E(\min)$ and $R_E(\min) = \pi^2$. Therefore,

$$R = \pi^2$$

is our nonlinear stability boundary. This criterion has been derived by using energy inequalities and is the same as the one found from the linear problem.

To complete our example we will now study the continuous dependence of the solution to the backward in time problem on the initial conditions. We will, for the moment, omit the convective term, as its inclusion leads to a problem whose complexity outweighs its usefulness as an illustrative example. However this term appears in the problem studied in Chapter 3, and the technique necessary to deal with it is explained there.

The equation governing the backward in time problem, omitting the nonlinear term and taking $R = 1$, is

$$v_t = -v_{xx} - v. \tag{1.9}$$

Let $\Omega \subseteq \mathbf{R}$ be a bounded domain. Here we take $\Omega = (0,1)$. Then equation (1.9) is defined on the domain $\Omega \times [0,T)$ and the boundary and initial conditions are

$$v(0,t) = v(1,t) = 0, \tag{1.10}$$
$$v(x,0) = v_0. \tag{1.11}$$

Suppose that v and v^* are two solutions to (1.9) which satisfy the same boundary conditions (1.10) but have different initial conditions v_0 and v_0^*.
Now define the difference quotient

$$u = v^* - v.$$

Then we can see that u satisfies

$$\begin{aligned} u_t &= -u_{xx} - u \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= v_0^* - v_0 = u_0. \end{aligned} \tag{1.12}$$

We now define the function

$$F(t) = \|u\|^2. \tag{1.13}$$

Differentiating (1.13), substituting for u_t using $(1.12)_1$ and then integrating by parts we can see that

$$\begin{aligned} F'(t) &= 2 < uu_t > \\ &= 2\|u_x\|^2 - 2\|u\|^2. \end{aligned} \tag{1.14}$$

Then, differentiating again, integrating by parts and substituting for u_t and u_{xx}, again using $(1.12)_1$, we may derive

$$\begin{aligned} F''(t) &= 4 < u_x u_{xt} > -4 < uu_t > \\ &= -4 < u_t u_{xx} > -4 < uu_t > \\ &= -4\|u_x\|^2 + 4\|u\|^2 + 4\|\dot{u}\|^2 + 4 < uu_t > \\ &= 4\|\dot{u}\|^2 + 4 < uu_t > -2F', \end{aligned} \tag{1.15}$$

where $\dot{u} \equiv u_t$. As the problem is an improperly posed one we require a suitable technique with which to solve it, one such is the logarithmic convexity method, cf. Payne (1975).

We form the expression $FF'' - (F')^2$ and can see, using (1.13), (1.14) and (1.15) that

$$\begin{aligned} FF'' - (F')^2 &= 4\|u\|^2\|\dot{u}\|^2 - 4 < uu_t >^2 + 2FF' - 2FF' \\ &= 4S^2 \\ &\geq 0, \end{aligned} \tag{1.16}$$

where $S^2 = \|u\|^2\|\dot{u}\|^2 - < uu_t >^2$ is obviously non-negative due to the Cauchy-Schwarz inequality.
Note that the LHS of (1.16) can be written $(\log F)''$. Then we now have

$$(\log F)'' \geq 0$$

and this expression can be integrated cf. Payne (1975) to yield

$$F(t) \leq \Big[F(\mathcal{T})\Big]^{t/\mathcal{T}} \Big[F(0)\Big]^{1-t/\mathcal{T}}$$

i.e.

$$\|u(t)\|^2 \leq \|u(\mathcal{T})\|^{2t/\mathcal{T}} \|u_0\|^{2(1-t/\mathcal{T})}. \tag{1.17}$$

Provided we restrict attention to the class of solutions such that

$$\|u(\mathcal{T})\| \leq K,$$

for some constant K, we may use (1.17) to establish continuous dependence of the solution u on the initial conditions u_0 on compact intervals of $[0, \mathcal{T})$.

Finally, we include a brief note about the various supremum inequalities used. In Chapter 2 we use

$$\sup_V |\theta| \leq c\|\Delta\theta\|^{3/4}\|\theta\|^{1/4} \tag{1.18}$$

and in Chapters 4,5 and 6

$$\sup_V |\theta| \leq c_1\|\Delta\theta\| \tag{1.19}$$

for c_1, c some constants.

These inequalities are, however, derived from the same source. The optimum result is given by Xie (1991) and is

$$\sup_V |\theta| \leq c_2\|\Delta\theta\|^{1/2}\|\nabla\theta\|^{1/2} \tag{1.20}$$

with $c_2 = 1/\sqrt{2\pi}$.

As explained in Chapter 2, by using integration by parts and the Cauchy-Schwarz inequality on the $\|\nabla\theta\|$ term we can derive (1.18). To derive (1.19), we note that Poincaré's inequality yields

$$\|\nabla\theta\| \leq k\|\Delta\theta\|$$

for some constant k. Then, using (1.20) and the above inequality we have

$$\begin{aligned} \sup_V |\theta| &\leq c_2\|\Delta\theta\|^{1/2}\|\nabla\theta\|^{1/2} \\ &\leq c_2\sqrt{k}\|\Delta\theta\|^{1/2}\|\Delta\theta\|^{1/2} \\ &= c_2\sqrt{k}\|\Delta\theta\|, \end{aligned}$$

which is (1.19).

CHAPTER TWO

A NONLINEAR STABILITY ANALYSIS OF CONVECTION IN A GENERALIZED INCOMPRESSIBLE FLUID.

2.1 Introduction

Problems in fluid mechanics have, in the past, generally concentrated on the case where the fluid is incompressible, as this is by far the more tractable case. In the context of thermally driven convection a few papers e.g. Joseph (1976) have sought to derive the Oberbeck-Boussinesq approximation from the full thermo-mechanical equations but these have in general required involved analysis. The Oberbeck-Boussinesq equations are, of course, a first approximation to compressible convection. A recent paper by Hills and Roberts (1991) however, derives the O-B approximation as a particular case of a new continuum model, the generalized incompressible fluid. Here generalized incompressibility is defined to mean that, although the density of the fluid cannot be changed by compression, there remains the possibility that ρ can be changed by varying the pressure or the temperature.

Hills and Roberts (1991) successfully derive the equations of motion for a linear, viscous, generalized incompressible fluid and these are the equations which we shall use. They contain, as we shall see, the O-B approximation but are more general, retaining some vestiges of compressibility. The beauty of this approach to compressible flow is its simplicity, for, although both linear and nonlinear analyses of compressible convection have been carried out in the past, their use of the full thermo-mechanical equations has required difficult and, at times, impenetrable analysis. Linear results have been known for many years (see Spiegel (1965)) and a more recent paper by Coscia and Padula (1990) derived a nonlinear stability result. However, in both cases, the analysis involved is necessarily very intricate and, in the case of Coscia and Padula, demands the use of a very complicated generalized energy function. Our approach, using as it does Hills and Roberts' model, is much clearer, involving a more straightforward generalized energy and fewer nonlinearities, and yet still allowing the effects of compressibility to be felt.

We shall derive linear instability and nonlinear energy stability results for thermally driven convection using both a numerical approach and an asymptotic analysis. In both cases it will be seen that the area of possible subcritical instabilities is very small. The conditional nonlinear stability results which we establish are an extension of those first derived by Prodi (1962) for the Navier-Stokes equations and subsequently developed by Sattinger (1970). However as our equations differ from the Navier-Stokes equations due to the inclusion of an incompressibility parameter, ε, the analysis is necessarily somewhat different. The proof used to establish conditional stability is somewhat reminiscent of those of Ladyzhenskaya (1969) Chapter 6, in the mathematical theory of the incompressible Navier-Stokes equations. Her conditions involved small initial data or a sufficiently large viscosity while ours depend on the initial data being smaller than some constant.

This chapter has essentially appeared in Richardson (1992a).

2.2 Equations for convection in a generalized incompressible fluid

The equations of motion for a generalized incompressible, linear, viscous fluid as derived in Hills and Roberts (1991) are

$$\rho_r v_{i,t} + \rho_r v_j v_{i,j} = - p_{,i} - \rho_r g \alpha_r (T - T_r) k_i + \mu_r \Delta v_i, \tag{2.1}$$

$$(\rho c_p)_r \dot{T} - \rho_r g \alpha_r T w = \rho_r Q + \kappa_r \Delta T + 2\mu_r D_{ij} D_{ij}, \tag{2.2}$$

$$v_{i,i} = 0. \tag{2.3}$$

Here Q is the heat source which we set equal to zero, p is the pressure, T the temperature, $\mathbf{v} = (u, v, w)$ the velocity, $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$, c_p is the specific heat at constant pressure, α is the thermal expansion coefficient, κ and μ are material coefficients and the density has the form

$$\rho = \rho_r[1 - \alpha_r(T - T_r) + \cdots].$$

Quantities with the subscript r denote the reference state about which the thermodynamic variables and material parameters were expanded.

Following the example of Hills and Roberts (1991) we measure z in the downward direction, along with gravity g. We suppose a layer of heat conducting viscous fluid is contained between the planes $z = 0$ and $z = -d$. The temperature at the boundaries is fixed with

$$T = T_0 \quad z = 0, \qquad T = T_1 \quad z = -d, \qquad (T_0 > T_1),$$

i.e the fluid is heated from below.

To investigate the problem of Bénard convection according to the Hills-Roberts theory we consider a steady solution $(\bar{v}_i, \bar{T}, \bar{p})$ with $\bar{v}_i \equiv 0$ and $\bar{T}$ a function of z. Then (2.1)–(2.3) give

$$0 = \kappa_r \bar{T}'', \tag{2.4}$$
$$0 = -\bar{p}_{,i} - \rho_r g \alpha_r (\bar{T} - T_r) k_i, \tag{2.5}$$
$$0 = \bar{v}_{i,i}. \tag{2.6}$$

From (2.4) and the boundary conditions we see that $\bar{T} = \beta z + T_0$ where β is a measure of the temperature gradient, with $\beta = (T_0 - T_1)/d$ (> 0). Also $\bar{p}_{,i} = -\rho_r g \alpha_r (\bar{T} - T_r) k_i$ so $\bar{p}$ is a function of z also and explicitly,

$$\bar{p} = -\rho_r g \alpha_r (\beta z^2/2 + T_0 z - T_r z) + p_o$$

where p_o is a constant.

In order to study the stability of this steady solution we introduce perturbations (u_i, θ, π) via

$$v_i = \bar{v}_i + u_i,$$
$$T = \bar{T} + \theta,$$
$$p = \bar{p} + \pi.$$

When substituted in (2.1)–(2.3) the resulting perturbation equations are

$$u_{i,t} + u_j u_{i,j} = -\frac{1}{\rho_r}\pi_{,i} - g\alpha_r \theta k_i + \frac{\mu_r}{\rho_r}\Delta u_i, \tag{2.7}$$
$$(\rho c_p)_r(\theta_{,t} + u_i \theta_{,i}) = \rho_r g \alpha_r (\beta z + T_0) w + \rho_r g \alpha_r \theta w - (\rho c_p)_r w \beta + \kappa_r \Delta\theta + 2\mu_r d_{ij} d_{ij}, \tag{2.8}$$
$$u_{i,i} = 0. \tag{2.9}$$

Here $d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

We now non-dimensionalize (2.7)–(2.9) using the following scales

$$x_i = Lx_i^*, \qquad t = \frac{L^2}{\nu_r}t^*, \qquad u_i = \frac{\nu_r}{L}u_i^*, \qquad P = \frac{\rho_r \nu_r^2}{L^2},$$
$$T_0 = \beta L Pr T_0^*, \qquad \theta = T^\sharp \theta^*, \qquad k_r = \frac{\kappa_r}{(\rho c_p)_r},$$
$$T^\sharp = \frac{\beta L \nu_r}{k_r}, \qquad R^2 = \frac{g\alpha_r \beta L^4}{\nu_r k_r}, \qquad \varepsilon = \frac{g\alpha_r L}{(c_p)_r}, \qquad Pr = \frac{\nu_r}{k_r},$$
$$\pi = Pp^*, \qquad \nu_r = \frac{\mu_r}{\rho_r}, \qquad \Gamma = z + PrT_0.$$

Here ε is a dissipation parameter and, if we assume that the layer is adiabatically thin in the sense that $\beta_{ad} \ll \beta$, then equivalently $\varepsilon \to 0$. The parameter ε is essentially a measure of the compressibility of the fluid layer. R^2 is the Rayleigh number, Pr is the Prandtl number and L is a typical length scale.

Equations (2.7)–(2.9) *non-dimensionalized* become, dropping all asterisks,

$$u_{i,t} + u_j u_{i,j} = -p_{,i} + \Delta u_i - R^2 \theta k_i, \tag{2.10}$$

$$Pr(\theta_{,t} + u_i \theta_{,i}) = Pr\varepsilon\theta w + \varepsilon\Gamma w - w + \Delta\theta + \frac{2Pr\varepsilon}{R^2} d_{ij} d_{ij}, \tag{2.11}$$

$$u_{i,i} = 0. \tag{2.12}$$

In order to proceed we let $R^2\theta = R\hat{\theta}$ in the above equations and ignore the $(\hat{\ })$. Equations (2.10)–(2.12) now become

$$u_{i,t} + u_j u_{i,j} = -p_{,i} + \Delta u_i - R\theta k_i, \tag{2.13}$$

$$Pr(\theta_{,t} + u_i \theta_{,i}) = Pr\varepsilon\theta w - R(1 - \varepsilon\Gamma) w + \Delta\theta + \frac{2Pr\varepsilon}{R} d_{ij} d_{ij}, \tag{2.14}$$

$$u_{i,i} = 0, \tag{2.15}$$

and these are the equations which will be used henceforth. It is important to realise that, due to the appearance of ε on the RHS of (2.14), these equations still retain some traces of compressibility.

The boundary conditions we impose are those for two free surfaces. By employing the method of Chandrasekhar (1981) we may show that the appropriate boundary conditions on $z = 0$ and $z = 1$ are

$$\begin{aligned} & w = \frac{\partial^2 w}{\partial z^2} = \Delta w = 0, \\ & \theta = 0, \\ & \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0. \end{aligned} \tag{2.16}$$

2.3 Linear theory

The first thing we do is investigate the linear instability of the system. Here we consider stationary convection, i.e we put

$$u_i(\mathbf{x},t) = u_i(\mathbf{x})e^{\sigma t}$$
$$\theta(\mathbf{x},t) = \theta(\mathbf{x})e^{\sigma t}$$

in the linearized version of (2.13)–(2.15) and then take the growth rate, $\sigma = 0$. We expect the linear and nonlinear Rayleigh numbers to be very close and so there will be little likelihood of sub-critical instabilities arising.
The linearized version of equations (2.13)–(2.14) for stationary convection are

$$0 = -p_{,i} + \Delta u_i - R\theta k_i, \tag{2.17}$$
$$0 = -R(1-\varepsilon\Gamma)w + \Delta\theta, \tag{2.18}$$

u_i again being solenoidal. If we take (curlcurl) of (2.17), take the third component and introduce normal modes

$$w = W(z)e^{i(mx+ny)},$$
$$\theta = \Theta(z)e^{i(mx+ny)},$$

then (2.17)–(2.18) becomes

$$(D^2-a^2)^2 W = -Ra^2\Theta, \tag{2.19}$$
$$(D^2-a^2)\Theta = R(1-\varepsilon\Gamma)W, \tag{2.20}$$

where $D^2 \equiv d^2/dz^2$ and $a^2 = m^2+n^2$ is a wave number.

We wish now to find, from equations (2.19)–(2.20), the critical Rayleigh number of stationary convection

$$Ra_L = \min_{a^2} R^2. \tag{2.21}$$

In order to do this we employ the compound matrix method (cf. Drazin and Reid (1981)) to find the eigenvalue R^2 and golden section search (cf. Cheney and Kincaid (1985)) in the minimisation. The numerical results are presented in §2.5.

2.4 Nonlinear theory

To study the stability of the nonlinear equations we use a generalized energy method, cf. Straughan (1992).

Let V denote a period cell for the solution. To commence, multiply (2.13) by u_i, (2.14) by θ and integrate over V. After some integration by parts this results in

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|^2 = -\|\nabla\mathbf{u}\|^2 - R < \theta w >, \tag{2.22}$$

$$\frac{d}{dt}\frac{1}{2}Pr\|\theta\|^2 = Pr\varepsilon < \theta^2 w > -R < (1-\varepsilon\Gamma)\theta w > -\|\nabla\theta\|^2 + \frac{2Pr\varepsilon}{R} < \theta d_{ij}d_{ij} > . \tag{2.23}$$

Here $\|\cdot\|$ denotes the $L^2(V)$ norm and $< f >= \int_V f\, dV$.

If we now define an energy

$$E(t) = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{\lambda Pr}{2}\|\theta\|^2,$$

where $\lambda(>0)$ is a coupling parameter to be chosen to the best advantage, then we see from (2.22) and (2.23) that

$$\frac{dE}{dt} = RI - D + Pr\varepsilon\lambda < \theta^2 w > + \frac{2Pr\varepsilon\lambda}{R} < \theta d_{ij}d_{ij} >, \tag{2.24}$$

where

$$I = - < \theta w(1+\lambda(1-\varepsilon\Gamma)) >,$$
$$D = \|\nabla\mathbf{u}\|^2 + \lambda\|\nabla\theta\|^2.$$

The motivation behind the following analysis is now briefly explained. We wish to be able to write (2.24) in such a way that

$$\frac{dE}{dt} \leq -D + AD^\alpha E^\beta,$$

where A, α and β are constants with $\alpha \leq 1$ and $\alpha + \beta \neq 1$. We will then be able to integrate the equation and show that, subject to conditions, $E(t) \to 0$ as $t \to \infty$. As E is composed of the perturbations we will have demonstrated that the nonlinear system is stable.

For this particular case, and indeed most non-trivial systems, the energy which we have already defined is not strong enough to control the nonlinearities and so we must introduce extra components to form a "generalized" energy. Straughan (1992) describes the logic and history of generalized energy methods, although the technique necessary here is of necessity new and hence developed afresh.

Considering our present problem we see that although the $< \theta^2 w >$ term would not be difficult to bound in terms of $DE^{1/2}$ by using a Sobolev inequality, the manipulation of $<\theta d_{ij}d_{ij}>$ is evidently novel and I do not see a way in which to proceed directly. Instead, we introduce another "piece" to our generalized energy.

For this section we need the Sobolev inequalities

$$\sup_V |\mathbf{u}| \leq c\|\Delta\mathbf{u}\|^{3/4}\|\mathbf{u}\|^{1/4}, \tag{2.25}$$

$$\Big(\int_V \theta^4 dV\Big)^{\frac{1}{2}} \leq c_1\|\theta\|^{1/2}\|\nabla\theta\|^{3/2}. \tag{2.26}$$

A proof of the second inequality can be found in Adams (1975); a proof for the first is now given.

The proof of this inequality follows from the work done by Gagliardo and Nirenberg in 1959/60, see e.g. Nirenberg (1959). An optimal constant was provided by Xie (1991) who proved that for Ω, an arbitrary open set in $\mathbf{R}^3$, $\mathbf{u} \in H_0^1$ and $\Delta\mathbf{u} \in L_2$ then

$$\sup_\Omega |\mathbf{u}| \leq \frac{1}{\sqrt{2\pi}}\|\nabla\mathbf{u}\|_2^{1/2}\|\Delta\mathbf{u}\|_2^{1/2}.$$

If we now use integration by parts and the Cauchy-Schwarz inequality on the $\|\nabla\mathbf{u}\|$ term, we can show that

$$\sup_\Omega |\mathbf{u}| \leq \frac{1}{\sqrt{2\pi}}\|\mathbf{u}\|_2^{1/4}\|\Delta\mathbf{u}\|_2^{3/4},$$

the desired result, with $c = 1/\sqrt{2\pi}$.

Note that it is possible to derive a slightly sharper result by using the Xie inequality as it stands, rather than the one derived from it.

We now derive the auxiliary equations necessary for our generalized energy. First, multiply (2.13) by $-\Delta u_i$ and integrate over V. The resulting equation is

$$\frac{d}{dt}\frac{1}{2}\|\nabla\mathbf{u}\|^2 = < u_j u_{i,j}\Delta u_i > - \|\Delta\mathbf{u}\|^2 + R < \theta\Delta w > . \tag{2.27}$$

We now add this to the existing energy E, introducing another coupling parameter μ, and define our generalized energy $\mathcal{E}(t)$ by

$$\mathcal{E}(t) = E(t) + \frac{\mu}{2}\|\nabla\mathbf{u}\|^2 = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{Pr\lambda}{2}\|\theta\|^2 + \frac{\mu}{2}\|\nabla\mathbf{u}\|^2. \tag{2.28}$$

Thus, we see from (2.24), (2.27) and (2.28) that

$$\begin{aligned}\frac{d\mathcal{E}}{dt} =& RI - D + Pr\varepsilon\lambda < \theta^2 w > + \mu < u_j u_{i,j}\Delta u_i > + \frac{2Pr\varepsilon\lambda}{R} < \theta d_{ij}d_{ij} > \\ & - \mu\|\Delta\mathbf{u}\|^2 + R\mu < \theta\Delta w > .\end{aligned} \tag{2.29}$$

If we now define

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D}, \tag{2.30}$$

where $\mathcal{H}$ is the space of admissible functions, let $a = (R_E - R)/R_E$ and assume $R < R_E$, then, from (2.29), we may obtain

$$\begin{aligned}\frac{d\mathcal{E}}{dt} \leq & - aD + Pr\varepsilon\lambda < \theta^2 w > + \frac{2Pr\varepsilon\lambda}{R} < \theta d_{ij}d_{ij} > -\mu\|\Delta\mathbf{u}\|^2 \\ & + \mu < u_j u_{i,j}\Delta u_i > + R\mu < \theta\Delta w > .\end{aligned} \tag{2.31}$$

Now let $\mathcal{D} = aD + \mu\|\Delta\mathbf{u}\|^2 = a\|\nabla\mathbf{u}\|^2 + a\lambda\|\nabla\theta\|^2 + \mu\|\Delta\mathbf{u}\|^2$. The idea is to bound the cubic nonlinearities in (2.31) in terms of $\mathcal{D}^{7/8}\mathcal{E}^{5/8}$. Taking each one in turn we have

$$< \theta^2 w > \leq \|w\|\Big(\int_V \theta^4 dV\Big)^{\frac{1}{2}} \leq c_1\|w\|\|\nabla\theta\|^{3/2}\|\theta\|^{1/2} \tag{2.32}$$

$$\begin{aligned}< \theta d_{ij}d_{ij} > &= \frac{1}{2} < \theta u_{i,j}u_{i,j} > + \frac{1}{2} < \theta u_{j,i}u_{i,j} > \\ &= -\frac{1}{2} < u_i\theta_{,j}u_{i,j} > - \frac{1}{2} < u_i\theta\Delta u_i > - \frac{1}{2} < \theta_{,i}u_{i,j}u_j > \\ &\leq \frac{1}{2}\sup_V|\mathbf{u}|\ | < \theta_{,j}u_{i,j} > | + \frac{1}{2}\sup_V|\mathbf{u}|\ | < \theta\Delta u_i > | \\ &\qquad + \frac{1}{2}\sup_V|\mathbf{u}|\ | < \theta_{,i}u_{i,j} > | \\ &\leq c\|\Delta\mathbf{u}\|^{3/4}\|\nabla\theta\|\|\nabla\mathbf{u}\|\|\mathbf{u}\|^{1/4} + \frac{c}{2}\|\Delta\mathbf{u}\|^{7/4}\|\theta\|\|\mathbf{u}\|^{1/4}\end{aligned} \tag{2.33}$$

$$\begin{aligned}< u_j u_{i,j}\Delta u_i > &\leq \sup_V|\mathbf{u}|\ | < u_{i,j}\Delta u_i > | \\ &\leq c\|\Delta\mathbf{u}\|^{7/4}\|\mathbf{u}\|^{1/4}\|\nabla\mathbf{u}\|,\end{aligned} \tag{2.34}$$

where we have integrated by parts and used the Cauchy-Schwarz inequality, (2.25) and (2.26). The last term in (2.31) is quadratic and we handle this as follows.

$$\begin{aligned}R\mu < \theta\Delta w > &\leq \frac{\mu}{2\alpha}\|\theta\|^2 + \frac{R^2\mu\alpha}{2}\|\Delta w\|^2 \\ &\leq \frac{\mu}{2\alpha\pi^2}\|\nabla\theta\|^2 + \frac{R^2\mu\alpha}{2}\|\Delta w\|^2.\end{aligned} \tag{2.35}$$

Here we have used the arithmetic-geometric mean inequality which gives rise to the positive constant α, plus Poincaré's inequality on the $\|\theta\|^2$ term, i.e $\|\theta\|^2 \leq \pi^{-2}\|\nabla\theta\|^2$.

Using (2.32)–(2.35) in (2.31) we may obtain

$$\begin{aligned}\frac{d\mathcal{E}}{dt} \leq & -\mathcal{D} + \frac{c_1 2^{3/4} Pr^{3/4}\varepsilon}{a^{3/4}\xi^{1/8}}\mathcal{D}^{7/8}\mathcal{E}^{5/8} + \frac{(Pr\lambda)^{1/2}\varepsilon c 2^{5/8}}{\mu^{7/8}R}\mathcal{D}^{7/8}\mathcal{E}^{5/8} \\ & + \frac{\varepsilon c Pr\lambda^{1/2}2^{13/8}}{R\mu^{7/8}a^{1/2}}\mathcal{D}^{7/8}\mathcal{E}^{5/8} + \frac{c2^{5/8}}{\mu^{3/8}}\mathcal{D}^{7/8}\mathcal{E}^{5/8} \\ & + \frac{\mu}{2\alpha\pi^2}\|\nabla\theta\|^2 + \frac{R^2\mu\alpha}{2}\|\Delta w\|^2. \end{aligned} \tag{2.36}$$

It should be noted that, due to the definition of $\mathcal{D}$ and $\mathcal{E}$, there exists a constant ξ by Poincaré's inequality such that

$$\mathcal{D} \geq \xi\mathcal{E}. \tag{2.37}$$

We have used this result in deriving (2.36) and it will be crucial in proving the nonlinear stability result. We now choose the constant $\alpha = 1/R^2$ and the coupling parameter $\mu = \alpha\pi^2 a\lambda$. In this way λ is also defined. Then (2.36) reduces to

$$\frac{d\mathcal{E}}{dt} \leq -\mathcal{D} + A\mathcal{D}^{7/8}\mathcal{E}^{5/8} + \frac{\mathcal{D}}{2},$$

where the constant A is given by

$$A = \frac{c_1(2Pr)^{3/4}\varepsilon}{a^{3/4}\xi^{1/8}} + \frac{(Pr\lambda)^{1/2}\varepsilon c2^{5/8}}{\mu^{7/8}R} + \frac{\varepsilon c Pr\lambda^{1/2}2^{13/8}}{R\mu^{7/8}a^{1/2}} + \frac{c2^{5/8}}{\mu^{3/8}}.$$

In other words our energy inequality is

$$\frac{d\mathcal{E}}{dt} \leq -\frac{\mathcal{D}}{2} + A\mathcal{D}^{7/8}\mathcal{E}^{5/8}. \tag{2.38}$$

Recalling (2.37) we see that (2.38) can be written

$$\frac{d\mathcal{E}}{dt} \leq -\frac{\mathcal{D}}{2} + \frac{A}{\xi^{1/8}}\mathcal{D}\mathcal{E}^{1/2}, \tag{2.39}$$

and (2.39) can be integrated to show that, provided

$$(A)\ R < R_E \qquad \text{and} \qquad (B)\ \mathcal{E}^{1/2}(0) < \frac{\xi^{1/8}}{2A},$$

then $\mathcal{E}(t) \to 0$ as $t \to \infty$.

The proof is as follows:

If $\mathcal{E}^{1/2}(0) < \xi^{1/8}/2A$ then

$$1)\ \mathcal{E}^{1/2}(t) < \frac{\xi^{1/8}}{2A}, \qquad \forall t > 0$$

or

$$2)\ \exists\, \eta < \infty \quad \text{s.t.} \quad \mathcal{E}^{1/2}(\eta) = \frac{\xi^{1/8}}{2A},$$
$$\mathcal{E}^{1/2}(t) < \frac{\xi^{1/8}}{2A}, \qquad 0 \le t < \eta.$$

Suppose (2) holds.

Then, for $0 \le t < \eta$, $\frac{1}{2} - \frac{A}{\xi^{1/8}}\mathcal{E}^{1/2}(t) > 0$ so (2.39) shows that

$$\frac{d\mathcal{E}}{dt} \le -\mathcal{D}\Big(\frac{1}{2} - \frac{A}{\xi^{1/8}}\mathcal{E}^{1/2}(t)\Big) < 0.$$

Hence

$$\frac{d\mathcal{E}}{dt} \le 0$$

and so

$$\mathcal{E}(t) \le \mathcal{E}(0).$$

In particular,

$$\mathcal{E}^{1/2}(t) \le \mathcal{E}^{1/2}(0) < \frac{\xi^{1/8}}{2A}.$$

Since $\mathcal{E}(t)$ is assumed continuous in t this means that $\mathcal{E}^{1/2}(\eta) \ne \xi^{1/8}/2A$—which is a contradiction.

Therefore (1) holds and, further,

$$\frac{d\mathcal{E}}{dt} \le 0, \qquad \forall t \ge 0,$$

Hence

$$\mathcal{E}(t) \le \mathcal{E}(0), \qquad \forall t \ge 0,$$

and

$$\frac{d\mathcal{E}}{dt} \le -\mathcal{D}\Big(\frac{1}{2} - \frac{A}{\xi^{1/8}}\mathcal{E}^{1/2}(t)\Big) \le -\mathcal{D}\Big(\frac{1}{2} - \frac{A}{\xi^{1/8}}\mathcal{E}^{1/2}(0)\Big).$$

Let $K = \frac{1}{2} - A\mathcal{E}^{1/2}(0)/\xi^{1/8}$, (a constant). Then

$$\frac{d\mathcal{E}}{dt} \le -K\mathcal{D} \le -K\xi\mathcal{E},$$

(using (2.37)) and this equation can be integrated to show that

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-K\xi t}.$$

Thus we have demonstrated *conditional* nonlinear stability. For, as $t \to \infty$ then $\mathcal{E} \to 0$ at least exponentially, provided (A) and (B) hold.

It only remains now to resolve the maximum problem (2.30) for the nonlinear case.

Returning to (2.30), the Euler-Lagrange equations derived from this maximum problem are

$$2\Delta u_i - R_E M\theta k_i = \varpi_{,i}, \tag{2.40}$$

$$2\Delta\theta - R_E M w = 0, \tag{2.41}$$

where ϖ is a Lagrange multiplier introduced due to the solenoidal aspect of $\mathbf{u}$, we have put $\theta^* = \sqrt{\lambda}\theta$ and ignored the star and

$$M = \frac{1 + \lambda(1 - \varepsilon\Gamma)}{\sqrt{\lambda}}.$$

In order to investigate more thoroughly the relationship between R_E and the coupling parameter λ we use the technique of parametric differentiation cf. Joseph (1976).

Suppose that λ_1 and λ_2 are the first two eigenvalues and that with each one are associated corresponding values $R_E^1, R_E^2, M^1, M^2, u_i^1, u_i^2, \theta^1, \theta^2$. Take (2.40) with the first eigenvalue and multiply by u_i^2 before integrating over V. Then take (2.40) with the second eigenvalue and multiply by u_i^1 before again integrating. The process is repeated with (2.41), here multiplying by θ^2 and θ^1 respectively. The resulting equations are

$$2 < u_i^2 \Delta u_i^1 > - R_E^1 < M^1\theta^1 w^2 > = < \varpi_{,i} u_i^2 >,$$

$$2 < u_i^1 \Delta u_i^2 > - R_E^2 < M^2\theta^2 w^1 > = < \varpi_{,i} u_i^1 >,$$

$$2 < \theta^2 \Delta\theta^1 > - R_E^1 < M^1 w^1 \theta^2 > = 0,$$

$$2 < \theta^1 \Delta\theta^2 > - R_E^2 < M^2 w^2 \theta^1 > = 0.$$

Now use integration by parts to obtain

$$-2 < \nabla u_i^1 \nabla u_i^2 > - R_E^1 < M^1\theta^1 w^2 > = 0, \tag{i}$$

$$-2 < \nabla u_i^2 \nabla u_i^1 > - R_E^2 < M^2\theta^2 w^1 > = 0, \tag{ii}$$

$$-2 < \nabla\theta^1 \nabla\theta^2 > - R_E^1 < M^1 w^1 \theta^2 > = 0, \tag{iii}$$

$$-2 < \nabla\theta^1 \nabla\theta^2 > - R_E^2 < M^2 w^2 \theta^1 > = 0. \tag{iv}$$

Then, adding ((iv)-(iii))+((ii)-(i)) results in

$$< (w^2\theta^1 + w^1\theta^2)\Big[R_E^2\frac{(M^2 - M^1)}{\lambda_2 - \lambda_1} + M^1\frac{(R_E^2 - R_E^1)}{\lambda_2 - \lambda_1}\Big] >= 0,$$

where we have divided throughout by $\lambda_2 - \lambda_1$.
Now let $\lambda_2 \to \lambda_1 \equiv \lambda$. Then,

$$2 < w\theta(M\frac{\partial R_E}{\partial \lambda} + R_E\frac{\partial M}{\partial \lambda}) >= 0.$$

Since we require R_E to have a maximum in λ we need $\partial R_E/\partial\lambda = 0$. Thus

$$< w\theta R_E\frac{\partial M}{\partial \lambda} >= 0.$$

This suggests we integrate $\partial M/\partial\lambda$ from 0 to 1 with respect to z, set it equal to zero and solve for λ. This results in

$$\lambda(2 - \varepsilon - 2\varepsilon PrT_0) = 2,$$

and so when carrying out the numerical calculations this gives a useful guide as to where to begin a search.

It should also be pointed out that when $\varepsilon \to 0$, which is the limit we are considering, then $\lambda \to 1$, which is indeed the case when the Oberbeck-Boussinesq equations are used.

R_E also depends on Γ but parametric differentiation yields only

$$< w\theta R_E\frac{\partial M}{\partial \Gamma} >= 0$$

and here the technique of averaging $\partial M/\partial\Gamma$ does not give such a useful result.

In order to do the numerical calculations we again, as in the linear analysis, take (curlcurl) of (2.40), take the third component and normal modes and the resulting equations are

$$(D^2 - a^2)^2 W = -\frac{1}{2}R_E M a^2\Theta, \tag{2.42}$$

$$(D^2 - a^2)\Theta = \frac{1}{2}R_E M W, \tag{2.43}$$

where again $D^2 \equiv d^2/dz^2$ and a^2 is a wave number. It remains only to find

$$Ra_E = \max_{\lambda}\min_{a^2} R_E^2$$

and, as before, this is effected by the compound matrix method and golden section search. The numerical results are in the following section.

2.5 Numerical results and asymptotic analysis

Before presenting the numerical calculations we include an asymptotic analysis for small ε. Throughout this section we take the Prandtl number as 0.72, the value for air.

i) Linear Analysis

As derived in §2.3, the linear Euler-Lagrange equations are

$$(D^2 - a^2)^2 W = -Ra^2\Theta, \tag{2.44}$$

$$(D^2 - a^2)\Theta = R(1 - \varepsilon\Gamma)W, \tag{2.45}$$

together with the boundary conditions

$$W = D^2W = D^4W = 0 \text{ on } z = 0, 1. \tag{2.46}$$

If we now eliminate Θ in equation (2.44) using (2.45) then we obtain

$$(D^2 - a^2)^3 W = -R^2a^2(1 - \varepsilon\Gamma)W. \tag{2.47}$$

Now expand $Ra_L = R^2$ and W in terms of ε,

$$W = W_0 + \varepsilon W_1 + \cdots,$$

$$Ra_L = R_0 + \varepsilon R_1 + \cdots.$$

Then to $O(1)$, equation (2.47) looks like

$$(D^2 - a^2)^3 W_0 = -R_0 a^2 W_0. \tag{2.48}$$

Also, the boundary conditions on W_0 are

$$W_0 = D^2W_0 = D^4W_0 = 0 \text{ on } z = 0, 1. \tag{2.49}$$

By expanding (2.48) and then differentiating again we can show that

$$D^{(2n)}W_0 = 0 \text{ on } z = 0, 1, \qquad n = 1, 2, \ldots. \tag{2.50}$$

Since (2.48)–(2.50) is the eigenvalue problem for the classical Bénard equations we write $W_0 = A\sin n\pi z$. Substituted in (2.47) this yields

$$R_0 = \frac{(n^2\pi^2 + a^2)^3}{a^2},$$

and R_0 minimized over a^2 results in

$$R_0 = \frac{27\pi^4}{4}, \qquad a_c^2 = \frac{\pi^2}{2}.$$

If we now return to (2.47), then at $O(\varepsilon)$ we have

$$(D^2 - a^2)^3 W_1 = -a^2(R_0 W_1 + R_1 W_0 - \Gamma R_0 W_0), \tag{2.51}$$

with W_1 satisfying the boundary conditions

$$W_1 = D^2 W_1 = D^4 W_1 = 0 \text{ on } z = 0, 1. \tag{2.52}$$

We now multiply (2.51) by W_0, integrate over $(0,1)$ and, after integrating by parts, the resulting equation is

$$\int_0^1 W_1(D^2 - a^2)^3 W_0 dz = -a^2 R_0 \int_0^1 W_0 W_1 dz - a^2 R_1 \int_0^1 W_0^2 dz + a^2 R_0 \int_0^1 \Gamma W_0^2 dz,$$

where we have utilized the boundary conditions (2.50) and (2.52). Using (2.48) and (2.49) we see that this reduces to

$$R_1 \int_0^1 W_0^2 dz = R_0 \int_0^1 \Gamma W_0^2 dz,$$

and, on carrying out the integration, we have

$$R_1 = R_0(\frac{1}{2} + PrT_0).$$

Hence we have shown that

$$Ra_L = \frac{27\pi^4}{4}[1 + \varepsilon(T_0 Pr + \frac{1}{2}) + \cdots]. \tag{2.53}$$

ii) Nonlinear Analysis.

Here the Euler-Lagrange equations are,

$$(D^2 - a^2)^2 W = -\frac{1}{2} R_E M a^2 \Theta, \tag{2.54}$$

$$(D^2 - a^2)\Theta = \frac{1}{2} R_E M W, \tag{2.55}$$

together with (2.46). Recall that $M = (1 + \lambda(1 - \varepsilon\Gamma))/\sqrt{\lambda}$. Again we eliminate Θ from (2.54) to obtain

$$(D^2 - a^2)^3 W = -\frac{1}{4} R_E^2 M^2 a^2 W. \tag{2.56}$$

We expand $Ra_E = R_E^2$ and W in terms of ε as before, i.e

$$Ra_E = R_0 + \varepsilon R_1 + \cdots,$$
$$W = W_0 + \varepsilon W_1 + \cdots,$$

and to $O(1)$ equation (2.56) is

$$(D^2 - a^2)^3 W_0 = -\frac{a^2}{4\lambda}(1 + \lambda)^2 R_0 W_0. \tag{2.57}$$

Once again we may show $D^{(2n)} W_0 = 0$ on $z = 0, 1 \quad n = 1, 2, \ldots,$ and so $W_0 = A \sin n\pi z$. Equation (2.57) then becomes, once we have minimized R_0 over a^2,

$$R_0 = \frac{27\lambda\pi^4}{(1 + \lambda)^2}. \tag{2.58}$$

If at this stage we seek R_0 maximized over λ then we find that $\lambda = 1$ and $R_0 = 27\pi^4/4$. Henceforth we take $\lambda = 1$. To $O(\varepsilon)$, (2.56) is

$$(D^2 - a^2)^3 W_1 = -a^2 \Big(R_0 W_1 + R_1 W_0 - \Gamma R_0 W_0 \Big). \tag{2.59}$$

In order to solve equation (2.59) we once again multiply both sides by W_0 and integrate over $(0, 1)$. Integration by parts and the use of (2.57) yields the equation

$$R_1 = R_0(T_0 Pr + \frac{1}{2}),$$

and so

$$Ra_E = \frac{27\pi^4}{4}\Big(1 + \varepsilon(T_0 Pr + \frac{1}{2}) + \cdots\Big). \tag{2.60}$$

Thus, to $O(\varepsilon)$ we have demonstrated that $Ra_L = Ra_E$.

The numerical results are now presented. Here Ra_L is the critical value of the linear Rayleigh number, Ra_E is the critical value of the energy Rayleigh number, a_L^2 and a_E^2 are the respective critical wave numbers and λ is the best value of the coupling parameter.

Table 1. Critical Rayleigh and wave numbers of linear and energy theory for $0 \leq T_0 \leq 2$
$\varepsilon = 10^{-3}$.

T_0	Ra_L	Ra_E	a_L^2	a_E^2	λ
0	657.841	657.841	4.935	4.935	1.001
1	658.315	658.315	4.935	4.935	1.001
2	658.790	658.790	4.935	4.935	1.002

Table 2. Critical Rayleigh and wave numbers of linear and energy theory for $0 \leq T_0 \leq 2$
$\varepsilon = 10^{-2}$.

T_0	Ra_L	Ra_E	a_L^2	a_E^2	λ
0	660.816	660.816	4.935	4.935	1.005
1	665.632	665.632	4.935	4.935	1.012
2	670.520	670.520	4.935	4.935	1.020

Table 3. Critical Rayleigh and wave numbers of linear and energy theory for $0 \leq T_0 \leq 2$
$\varepsilon = 10^{-1}$.

T_0	Ra_L	Ra_E	a_L^2	a_E^2	λ
0	692.108	692.084	4.935	4.935	1.053
1	748.862	748.832	4.935	4.935	1.139
2	815.756	815.717	4.935	4.935	1.241

When ε is small ($= 10^{-3}$), then the linear and nonlinear Rayleigh numbers are the same to the accuracy considered here, which is in agreement with the asymptotic analysis and they diverge only slightly as ε increases. When $\varepsilon = 0$ then we return to the standard Bénard problem. As ε increases both Ra_L and Ra_E also increase, thus making it more difficult for convection to take place, i.e. the greater the degree of compressibility, the harder it is for instabilities to arise. However, ε has no apparent effect on the variation of the wave numbers a_E^2 and a_L^2.

CHAPTER THREE

CONTINUOUS DEPENDENCE ON THE VISCOSITY FOR THE OBERBECK-BOUSSINESQ EQUATIONS.

3.1 Introduction

In this chapter we study the relationship between the solution to the equations of motion for an incompressible heat-conducting viscous fluid and the viscosity of the fluid. We will establish that the solution depends continuously on the viscosity both forward and backward in time. The backward-in-time problem is an improperly posed one and hence more complicated than the other case; this necessitates the utilisation of a technique capable of dealing with such a problem.

The first study of the improperly posed backward in time problem for the Navier-Stokes equations was by Serrin (1963) who established uniqueness of the solution. Serrin's proof was based on that of Lees and Protter (1961) who established backward uniqueness for a class of parabolic equations. That paper was, in turn, based on the space-time weighted energy technique, cf. Protter (1961).

Knops and Payne (1968) established continuous dependence on the final data for a solution to the Navier-Stokes equations on a bounded domain. There has subsequently been much interest in various aspects of improperly posed problems for fluid dynamic systems and, in particular, in studies of continuous dependence on coefficients in the problem, such as heat source, micropolar coefficients, nonlinearities, see eg. Payne (1971,1975,1987), Payne and Straughan (1989a,1989b), Straughan (1990,1991).

Here we show that a solution to the Navier-Stokes equations on a bounded domain, for the improperly posed backward in time problem, depends continuously on the viscosity. The proof we employ is based on a logarithmic convexity argument cf. Knops and Payne (1968), Payne (1975). In the latter reference Payne details several techniques for dealing with improperly posed problems, one of which is the logarithmic convexity method. The most common procedure used in studying the evolution forward in time of solutions of differential equations, Lyapunov's direct method, makes use of first order differential inequalities. The logarithmic convexity method may be viewed as a method of second order differential inequalities. Here we seek a function $F(t)$ such that $F(t) \geq 0$, $F(0) = 0 \iff \mathbf{u}, \theta = 0$ and $FF'' - (F')^2 \geq -AF^2 - BFF'$ where A and B are constants. The final inequality can then be integrated in order to give us the required continuous dependence result.

Another approach to this type of problem was developed by Song (1988) who established continuous dependence results of various types for well-posed problems in continuum physics, concentrating primarily on the Navier-Stokes equations of viscous fluid flow and the equations of thermoelasticity. We are, of course, interested mainly in his treatment of establishing continuous dependence on the viscosity, but, as the technique used is a general one and is employed in other continuous dependence problems, we describe here the basic method.

The aim of Song's analysis is to show, that for the Navier-Stokes equation for viscous flow, when two solutions are close at $t = 0$, then they remain close in subsequent times. To this end he defines an energy, based on the difference of the two solutions and proceeds to divide the problem into two cases, $N = 2$ and $N = 3$. This division is necessary due to his use of a Sobolev inequality, different in different dimensions. Then, by using this Sobolev inequality and various other techniques, he derives a first order inequality which implies the desired result.

For the specific problem; continuous dependence on viscosity, Song derives a result in $N = 2$ which is expressed purely in terms of the initial data. When $N = 3$ he is forced, in order to derive continuous dependence for all t, to ensure that the L^2 norm of the time derivative of one of the initial solutions is sufficiently small. If this is not, in fact, the case, then continuous dependence can be obtained for a finite interval of t. Here our analysis includes the temperature field equation but the final result is, in the forward in time case, qualitatively the same as that derived by Song. However for our treatment we are forced to bound the L^2 norms of both the initial solutions and the temperature.

This chapter is essentially to appear in Richardson (1993).

3.2 Forward in time problem

The equations of motion for a linear, viscous, heat-conducting fluid forward in time are

$$v_{i,t} + v_j v_{i,j} = -\frac{1}{\rho_0} p_{,i} + \nu \Delta v_i - g k_i (1 - \alpha(T - T_0)), \tag{3.1}$$

$$T_{,t} + v_i T_{,i} = \kappa \Delta T, \tag{3.2}$$

$$v_{i,i} = 0, \tag{3.3}$$

where v_i is the velocity, p the pressure, T the temperature and ν has been taken to be constant.

Let $\Omega \subseteq \mathbf{R}^3$ be a bounded domain. Equations (3.1)–(3.3) are defined on the domain $\Omega \times (0, \mathcal{T}\,]$: Ω is here the volume occupied by the fluid. The boundary of Ω is denoted by Γ, and Γ is assumed sufficiently smooth to allow applications of the divergence theorem.

The boundary conditions considered are

$$v_i(\mathbf{x}, t) = \overline{v_i}(\mathbf{x}, t) \text{ on } \Gamma \times [0, \mathcal{T}] \tag{3.4}$$

$$T(\mathbf{x}, t) = T_1(\mathbf{x}, t) \text{ on } \Gamma \times [0, \mathcal{T}]. \tag{3.5}$$

The initial conditions are

$$v_i(\mathbf{x}, 0) = v_i^o(\mathbf{x}) \qquad\qquad T(\mathbf{x}, 0) = T_0(\mathbf{x}) \tag{3.6}$$

where (3.6) holds on $\Omega \times \{0\}$ and $\overline{v}_i$, v_i^o, T_0, T_1 are prescribed functions of the indicated arguments. Let $\chi = (\mathbf{v}, T, p)$ and $\chi^* = (\mathbf{v}^*, T^*, p^*)$ be two classical solutions to the above boundary-initial value problem; the quantities without asterisks denote the base flow and the quantities with asterisks denote the perturbed flow. We assume that v_i, v_i^*, T and T^* are of class $C^2(\overline{\Omega} \times (0, \mathcal{T}])$ and further suppose that χ, χ^* satisfy (3.4)–(3.6) for the same data $\overline{v}_i$, T_1, T_0 and v_i^o but for different viscosities ν and ν^*. We are thus considering explicitly the continuous dependence on the viscosity only.

In order to study the dependence of the solutions on the viscosity we use an "energy" argument. Hence we define the difference quotients

$$u_i = v_i^* - v_i,$$

$$\theta = T^* - T,$$

$$\varepsilon = \nu^* - \nu,$$

$$\pi = \frac{1}{\rho_0}(p^* - p),$$

and let $\kappa = 1$ and $\alpha g k_i = b_i$. Then u_i, θ, π and ε satisfy the boundary initial value problem

$$\begin{aligned} u_{i,t} + v_j^* u_{i,j} + u_j v_{i,j} =& -\pi_{,i} + \varepsilon \Delta v_i^* + \nu \Delta u_i + b_i \theta, \quad (3.7)\\ \theta_{,t} + v_i^* \theta_{,i} + u_i T_{,i} =& \Delta \theta, \quad (3.8)\\ u_{i,i} =& 0, \quad (3.9)\end{aligned}$$

on $\Omega \times (0, \mathcal{T}]$,

$$u_i = \theta = 0 \quad \text{on} \quad \Gamma \times [0, \mathcal{T}] \quad (3.10)$$

and

$$u_i = \theta = 0 \quad \text{at} \quad t = 0. \quad (3.11)$$

In order to show explicitly the dependence of $\mathbf{u}$ and θ on the viscosity we define $F(t) = \|\mathbf{u}\|^2 + \|\theta\|^2$ and suppose that the solution satisfies the bounds

$$\begin{aligned} |\mathbf{v}^*| &\leq A, \quad (3.12)\\ |T^*| &\leq B, \end{aligned}$$

for some constants A and B. The notation is as before.

We also require a bound for $\|\nabla \mathbf{v}^*\|^2$ but this can be obtained from the equations as is now shown.

First note that

$$\frac{d}{dt}\|\nabla \mathbf{v}^*\|^2 = \frac{d}{dt} < v_{i,j}^* v_{i,j}^* > = -2 < v_{i,t}^* \Delta v_i^* > .$$

We then substitute for Δv_i^* using (3.1) and the equation becomes

$$\begin{aligned} \frac{d}{dt}\|\nabla \mathbf{v}^*\|^2 =& -\frac{2}{\nu} < v_{i,t}^* v_{i,t}^* > -\frac{2}{\nu} < v_{i,t}^* v_j^* v_{i,j}^* > \\ & -\frac{2}{\nu} g k_i < (1 - \alpha(T^* - T_0^*)) v_{i,t}^* > \\ \leq & -\frac{2}{\nu}\|\mathbf{v}_{,t}^*\|^2 + \frac{A^2}{\nu \alpha_1}\|\nabla \mathbf{v}^*\|^2 + \frac{\alpha_1}{\nu}\|\mathbf{v}_{,t}^*\|^2 \\ & + \frac{2 g k_i \alpha}{\nu} < T^* v_{i,t}^* > - \frac{2 g k_i}{\nu} < (1 + \alpha T_0^*) v_{i,t}^* > \\ \leq & -\frac{2}{\nu}\|\mathbf{v}_{,t}^*\|^2 + \frac{A^2}{\nu \alpha_1}\|\nabla \mathbf{v}^*\|^2 + \frac{\alpha_1}{\nu}\|\mathbf{v}_{,t}^*\|^2 + \frac{g k_i \alpha \beta}{\nu}\|\mathbf{v}_{,t}^*\|^2 \\ & + \frac{g k_i \alpha}{\beta \nu}\|T^*\|^2 + \frac{g k_i \gamma}{\nu}(1 + \alpha T_0^*)^2 + \frac{g k_i}{\gamma \nu}\|\mathbf{v}_{,t}^*\|^2, \end{aligned}$$

where we have integrated by parts, used the arithmetic-geometric mean inequality and the bound on $|\mathbf{v}^*|$. Note that α_1, γ and β are constants to be chosen.

Now choose $\alpha_1 + gk_i\alpha\beta + gk_i/\gamma = 2$, let $K = gk_i\gamma(1 + \alpha T_0^*)^2/\nu$, and let $\sigma = gk_i\alpha/\beta\nu$. Then

$$\frac{d}{dt}\|\nabla\mathbf{v}^*\|^2 \leq \frac{A^2}{\alpha_1\nu}\|\nabla\mathbf{v}^*\|^2 + K + \sigma\|T^*\|^2. \tag{3.13}$$

Using equation (3.2) we can show that $\|T^*\|^2 \leq \|T^*(0)\|^2 e^{-2\kappa\lambda t}$, where λ is a constant due to Poincaré's inequality. Then (3.13) becomes

$$\frac{d}{dt}\|\nabla\mathbf{v}^*\|^2 \leq \frac{A^2}{\alpha_1\nu}\|\nabla\mathbf{v}^*\|^2 + K + \sigma\|T^*(0)\|^2 e^{-2\kappa\lambda t}.$$

This equation can be integrated to give

$$\|\nabla\mathbf{v}^*\|^2 \leq V + We^{-2\kappa\lambda t}, \tag{3.14}$$

where $V = -(\alpha_1\nu K)/A^2$ and $W = -(\sigma\alpha_1\nu\|T^*(0)\|^2)/(2\kappa\lambda\alpha_1\nu + A^2)$. Hence we have derived a bound for $\|\nabla\mathbf{v}^*\|^2$ which we will use shortly.

Returning to our functional $F(t)$ we see that differentiating with respect to t results in

$$F'(t) = 2 < u_i u_{i,t} > +2 < \theta\theta_{,t} > .$$

We now substitute for $u_{i,t}$ and $\theta_{,t}$ from (3.7) and (3.8) and the equation becomes

$$\begin{aligned}
F'(t) &= 2 < u_i u_{i,t} > +2 < \theta\theta_{,t} > \\
&= 2 < u_i(-v_j^* u_{i,j} - u_j v_{i,j} - \pi_{,i} + \varepsilon\Delta v_i^* + \nu\Delta u_i + b_i\theta) > \\
&\quad + 2 < \theta(-v_i^*\theta_{,i} - u_i T_{,i} + \Delta\theta) > \\
&= 2 < u_{i,j} u_j v_i > -2\varepsilon < u_{i,j} v_{i,j}^* > -2\nu\|\nabla\mathbf{u}\|^2 \\
&\quad + 2 < w\theta > +2 < \theta_{,i} u_i T > -2\|\nabla\theta\|^2 \\
&= 2 < u_{i,j} u_j v_i^* > -2\varepsilon < u_{i,j} v_{i,j}^* > -2\nu\|\nabla\mathbf{u}\|^2 + 2 < w\theta > \\
&\quad - 2\|\nabla\theta\|^2 + 2 < \theta_{,i} u_i T^* > \\
&\leq \frac{A^2}{\alpha}\|\mathbf{u}\|^2 + \alpha\|\nabla\mathbf{u}\|^2 + \frac{\varepsilon^2 V}{\delta} + \frac{\varepsilon^2 W}{\delta}e^{-2\kappa\lambda t} + \delta\|\nabla\mathbf{u}\|^2 - 2\nu\|\nabla\mathbf{u}\|^2 \\
&\quad + \frac{1}{\beta}\|w\|^2 + \beta\|\theta\|^2 + \frac{B^2}{\mu}\|\mathbf{u}\|^2 + \mu\|\nabla\theta\|^2 - 2\|\nabla\theta\|^2
\end{aligned}$$

where we have integrated by parts, used the bounds (3.12) and (3.14) and the arithmetic-geometric mean inequality. Also we have substituted for v_i and T using the difference quotients defined earlier. Here α, β, δ and μ are arbitrary positive constants and $w = b_i u_i$. Pick $\mu = 2$, $\delta + \alpha = 2\nu$, say $\delta = \alpha = \nu$, and choose β such that $\beta = A^2/\alpha + |b_i|^2/\beta + B^2/\mu = a$, say. Also define $b = (V + We^{-2\kappa\lambda t})/\delta$. Then the above inequality yields

$$\frac{d}{dt}(\|\mathbf{u}\|^2 + \|\theta\|^2) \le a(\|\mathbf{u}\|^2 + \|\theta\|^2) + b\varepsilon^2,$$

or

$$F'(t) - aF \le b\varepsilon^2.$$

Since $F(0) = 0$ an integration shows that

$$F(t) \le \frac{\varepsilon^2 W}{\delta(a + 2\kappa\lambda)}[e^{at} - e^{-2\kappa\lambda t}] + \frac{\varepsilon^2 V}{\delta a}[e^{at} - 1]. \tag{3.15}$$

Hence on any bounded interval $(0, T]$, set $G = [e^{at} - e^{-2\kappa\lambda t}]W/(\delta(a + 2\kappa\lambda)) + V[e^{at} - 1]/\delta a$ and then from (3.15) we find

$$\|\mathbf{u}\|^2 + \|\theta\|^2 \le G\varepsilon^2.$$

So, by making ε arbitrarily small, we may likewise make $\mathbf{u}$ and θ arbitrarily small. Hence we have shown that, in the forward in time case, the solution depends continuously on the viscosity.

3.3 Backward in time problem

We now consider exactly the same problem as before but look at it backward in time. However as this problem is an improperly posed one it will necessitate the use of a different technique.

For this case u_i, θ, π and ε satisfy the boundary initial value problem

$$u_{i,t} = v_j^* u_{i,j} + u_j v_{i,j} + \pi_{,i} - \varepsilon \Delta v_i^* - \nu \Delta u_i - b_i \theta, \tag{3.16}$$

$$u_{i,i} = 0, \tag{3.17}$$

$$\theta_{,t} = v_i^* \theta_{,i} + u_i T_{,i} - \Delta\theta, \tag{3.18}$$

together with equations (3.10) and (3.11).

Now assume that the following bounds hold,

$$\sup_{\Omega\times[0,\mathcal{T}]}|v_m^* v_m^*|\leq b^2,\qquad \sup_{\Omega\times[0,\mathcal{T}]}|T_{,s}|\leq b,\qquad \sup_{\Omega\times[0,\mathcal{T}]}|v_{[i,k]}v_{[i,k]}|\leq b^2,$$
$$\sup_{\Omega\times[0,\mathcal{T}]}|v_{i,t}|\leq b,\qquad \sup_{\Omega\times[0,\mathcal{T}]}(|b_i|+|\nabla T|)\leq n^2,\qquad \|v_{i,j}^*\|\leq A,\tag{3.19}$$
$$\int_0^t\|\Delta v_i^*\|^2ds\leq K^2,\qquad \sup_{\Omega\times[0,\mathcal{T}]}|v_i|\leq A,\qquad \int_0^t\|\nabla v_{i,s}^*\|^2ds\leq M^2,$$
$$\sup_{\Omega\times[0,\mathcal{T}]}|b_i|\leq A,\qquad \sup_{\Omega\times[0,\mathcal{T}]}|T|\leq A,$$

where $v_{[i,k]}=\frac{1}{2}(v_{i,k}-v_{k,i})$ and, due to the definition of b_i, obviously $A=|\alpha g|$. For this case define

$$F(t)=\int_0^t(\|\mathbf{u}\|^2+\|\theta\|^2)ds+Q(\mathcal{T})\tag{3.20}$$

where $Q(\mathcal{T})$ is a data term to be defined later. Then

$$F'(t)=2\int_0^t(<u_iu_{i,s}>+<\theta\theta_{,s}>)ds\tag{3.21}$$

and

$$F''(t)=2\int_0^t(\|\dot{\mathbf{u}}\|^2+\|\dot{\theta}\|^2)ds+2\int_0^t<u_iu_{i,ss}>ds+2\int_0^t<\theta\theta_{,ss}>ds.$$

Note that we can establish an estimate for $F'(t)$ by the following method.

Multiplying (3.16) by u_i, (3.18) by θ and integrating with respect to t results in

$$\int_0^t<u_iu_{i,s}>ds=-\int_0^t<u_{i,j}u_jv_i>ds+\varepsilon\int_0^t<u_{i,j}v_{i,j}^*>ds$$
$$+\nu\int_0^t\|\nabla\mathbf{u}\|^2ds-b_i\int_0^t<u_i\theta>ds,$$
$$\int_0^t<\theta\theta_{,s}>ds=\int_0^t<\theta u_iT_{,i}>ds+\int_0^t\|\nabla\theta\|^2ds.$$

Then, by using (3.19) and the arithmetic geometric mean inequality we find

$$-\nu\int_0^t\|\nabla\mathbf{u}\|^2ds-\int_0^t\|\nabla\theta\|^2ds\geq -F'-2\Big(\frac{A^2}{\nu}+A^2+\frac{1}{2}\Big)F-\frac{A^2\mathcal{T}}{\nu}\varepsilon^2.\tag{3.22}$$

This estimate will be useful later.

Returning to our expression for F'' and substituting for $u_{i,s}$ and $\theta_{,s}$ from (3.16) and (3.18) we find

$$\begin{aligned}F''(t) =&2\int_0^t (\|\dot{\mathbf{u}}\|^2 + \|\dot{\theta}\|^2)ds\\ &+2\int_0^t (< u_i v^*_{j,s} u_{i,j} > + < u_i v^*_j u_{i,js} > + < u_i u_{j,s} v_{i,j} > + < u_i u_j v_{i,js} >\\ &+ < u_i \pi_{,is} > -\varepsilon < \Delta v^*_{i,s} u_i > -\nu < u_i \Delta u_{i,s} > -b_i < u_i \theta_{,s} >)ds\\ &+2\int_0^t (< \theta v^*_{i,s}\theta_{,i} > + < \theta v^*_i \theta_{,is} > + < \theta u_i T_{,is} >\\ &\quad - < \theta \Delta \theta_{,s} > + < \theta u_{i,s} T_{,i} >)ds.\end{aligned}$$

Now integrate by parts and substitute for Δu_i and $\Delta\theta$ from (3.16) and (3.18). This yields

$$\begin{aligned}F''(t) =&4\int_0^t (\|\dot{\mathbf{u}}\|^2 + \|\dot{\theta}\|^2)ds - 4\int_0^t < v^*_j (u_{i,j} u_{i,s} + \theta_{,j}\theta_{,s}) > ds\\ &- 2\int_0^t < (u_i \theta_{,s} - \theta u_{i,s})(b_i + T_{,i}) > ds\\ &- 4\int_0^t < u_{i,s} u_j v_{[i,j]} > ds - 2\int_0^t < u_j (v_{i,s} u_{i,j} + T_{,s}\theta_{,j}) > ds\\ &+ 2\varepsilon\int_0^t (< u_{i,j} v^*_{i,js} > + < u_{i,s}\Delta v^*_i >)ds. \end{aligned} \tag{3.23}$$

The next step is to form the expression $FF'' - (F')^2$ as outlined in §3.1. First it is helpful to define (cf. Payne (1975))

$$\begin{aligned}\chi_i =&u_{i,t} - \frac{1}{2}v^*_k u_{i,k} - \frac{1}{2}u_k v_{[i,k]},\\ \Phi =&\theta_{,t} - \frac{1}{2}v^*_m \theta_{,m}.\end{aligned}$$

Then, using (3.21) and (3.23), we find that

$$FF'' - (F')^2 = 4S^2 + \sum_{i=1}^{8} I_i + 4Q\int_0^t (\|\chi\|^2 + \|\Phi\|^2)ds, \tag{3.24}$$

where

$$S^2 = \int_0^t (\|\mathbf{u}\|^2 + \|\theta\|^2)ds \int_0^t (\|\chi\|^2 + \|\Phi\|^2)ds - \Big(\int_0^t (< u_i\chi_i + \theta\Phi >)ds\Big)^2, \tag{3.25}$$

and the quantities I_α are given below;

$$
\begin{aligned}
I_1 =& -F\int_0^t < (v_k^* u_{i,k} + u_k v_{[i,k]})(v_m^* u_{i,m} + u_m v_{[i,m]}) > ds,\\
I_2 =& -F\int_0^t < v_m^* v_r^* \theta_{,m}\theta_{,r} > ds,\\
I_3 =& -2F\int_0^t < u_j u_{i,j} v_{i,s} > ds,\\
I_4 =& -2F\int_0^t < T_{,s} u_j \theta_{,j} > ds,\\
I_5 =& -2F\int_0^t < (b_i + T_{,i}) u_i \theta_{,s} > ds,\\
I_6 =& 2F\int_0^t < (b_i + T_{,i}) u_{i,s}\theta > ds,\\
I_7 =& 2\varepsilon F\int_0^t < u_{i,s}\Delta v_i^* > ds,\\
I_8 =& 2\varepsilon F\int_0^t < u_{i,j} v_{i,js}^* > ds.
\end{aligned}
$$

$I_1, \ldots, I_8$ are estimated with the use of the arithmetic geometric mean inequality, the Cauchy-Schwarz inequality, (3.19), (3.20), (3.22) and (3.25). Then we have

$$
\begin{aligned}
I_1 \geq& -\frac{2b^2}{\nu}FF' - \frac{4b^2A_1}{\nu}F^2 - \frac{2b^2A^2\mathcal{T}}{\nu^2}\varepsilon^2 F - 2b^2F\int_0^t \|\mathbf{u}\|^2 ds,\\
I_2 \geq& -b^2FF' - 2A_1b^2F^2 - \frac{b^2A^2\mathcal{T}}{\nu}\varepsilon^2 F,\\
I_3 \geq& -Fb\int_0^t \|\mathbf{u}\|^2 ds - \frac{b}{\nu}FF' - \frac{2bA_1}{\nu}F^2 - \frac{bA^2\mathcal{T}}{\nu^2}\varepsilon^2 F,\\
I_4 \geq& -Fb\int_0^t \|\mathbf{u}\|^2 ds - bFF' - 2bA_1F^2 - \frac{A^2b\mathcal{T}}{\nu}\varepsilon^2 F,\\
I_5 \geq& -2n^2SF - n^2FF' - \frac{n^2b}{2}F\int_0^t \|\mathbf{u}\|^2 ds - \frac{n^2b}{2}FF'\\
& - n^2bA_1F^2 - \frac{A^2n^2b\mathcal{T}}{2\nu}\varepsilon^2 F,\\
I_6 \geq& -2n^2SF - n^2FF' - n^2bF\int_0^t \|\theta\|^2 ds - \frac{n^2b}{2}F\int_0^t \|\mathbf{u}\|^2 ds\\
& - \frac{n^2b}{2\nu}FF' - \frac{n^2bA_1}{\nu}F^2 - \frac{n^2bA^2\mathcal{T}}{2\nu^2}\varepsilon^2 F,
\end{aligned}
$$

$$I_7 \geq -2\varepsilon F K\Big(\int_0^t \|\chi\|^2 ds\Big)^2 - \frac{b}{2}F\int_0^t \|\mathbf{u}\|^2 ds - \frac{b}{2\nu}FF'$$
$$-\frac{bA_1}{\nu}F^2 - \frac{bA^2\mathcal{T}}{2\nu^2}\varepsilon^2,$$
$$I_8 \geq -\frac{M^2}{\nu}\varepsilon^2 F - FF' - 2A_1F^2 - \frac{A^2\mathcal{T}}{\nu}\varepsilon^2 F.$$

Here $A_1 = (A^2/\nu) + A^2 + (1/2)$. These bounds are inserted into (3.24) to find

$$FF'' - (F')^2 \geq 4S^2 - k_2\varepsilon^2 F - k_3FF' - c_1F^2 - c_2F\int_0^t \|\mathbf{u}\|^2 ds$$
$$- c_3F\int_0^t \|\theta\|^2 ds - 4n^2SF,$$

where

$$c_1 = \Big(\frac{4b^2}{\nu} + 2b^2 + 2b + \frac{3b}{\nu} + \frac{n^2b}{\nu} + 2 + n^2b\Big)A_1,$$
$$c_2 = 2b^2 + n^2b + \frac{5b}{2},$$
$$c_3 = n^2b,$$
$$k_2 = \frac{2b^2A^2\mathcal{T}}{\nu^2} + \frac{b^2A\mathcal{T}}{\nu} + \frac{A^2b\mathcal{T}}{\nu} + \frac{3bA^2\mathcal{T}}{2\nu^2} + \frac{A^2n^2b\mathcal{T}}{2\nu^2}$$
$$+ \frac{M^2}{\nu} + \frac{A^2\mathcal{T}}{\nu} + bK^2 + \frac{A^2\mathcal{T}n^2b}{2\nu},$$
$$k_3 = \frac{2b^2}{\nu} + b^2 + b + \frac{3b}{2\nu} + 2n^2 + 1 + \frac{n^2b}{2\nu} + \frac{n^2b}{2}.$$

Let $c_4 = \max\{c_2, c_3\}$, $\quad c_5 = c_1 + c_4$.

Then, using (3.22),

$$FF'' - (F')^2 \geq 4S^2 - 4n^2SF - c_5F^2 - k_2\varepsilon^2F - k_3FF'.$$

The S terms on the right-hand side are removed by completing the square and we find

$$FF'' - (F')^2 \geq -k_2\varepsilon^2 F - k_3FF' - k_1F^2 \tag{3.26}$$

where $k_1 = n^4 + c_5$. Now let $Q = \varepsilon^2$ in (3.20). Then (3.26) becomes

$$FF'' - (F')^2 \geq -k_4F^2 - k_3FF', \tag{3.27}$$

where $k_4 = k_2 + k_1$. This equation can be integrated using the following technique; cf. Payne (1975).

$$\begin{aligned} & FF'' - (F')^2 + k_4 F^2 + k_3 FF' \geq 0, \\ \iff & F^2(\log F)'' + k_4 F^2 + k_3 F^2 (\log F)' \geq 0, \\ \iff & (\log F)'' + k_4 + k_3 (\log F)' \geq 0, \\ \iff & [e^{k_3 t}(\log F)']' + k_4 e^{k_3 t} \geq 0. \end{aligned} \tag{3.28}$$

Let $\sigma = e^{-k_3 t}$. Then

$$\frac{d}{dt} = \frac{d}{d\sigma}\frac{d\sigma}{dt} = -k_3 \sigma \frac{d}{d\sigma}.$$

So (3.28) can be written

$$\begin{aligned} & -k_3 \sigma \frac{d}{d\sigma}[e^{k_3 t}(-k_3)\sigma \frac{d}{d\sigma}(\log F(\sigma))] + \frac{k_4}{\sigma} \geq 0, \\ \iff & \frac{d^2}{d\sigma^2}(\log F(\sigma)) + \frac{k_4}{(k_3\sigma)^2} \geq 0, \\ \iff & \frac{d^2}{d\sigma^2}[\log(F(\sigma)\sigma^{-k_4/k_3^2})] \geq 0. \end{aligned}$$

Now let $\mu = \frac{k_4}{k_3}$. Then

$$\frac{d^2}{d\sigma^2}[\log(F(\sigma)e^{\mu t})] \geq 0. \tag{3.29}$$

We now use the result that if $G''(x) \geq 0$, with $x \in [x_1, x_2]$, then

$$G(x) \leq G(x_2)\frac{(x - x_1)}{(x_2 - x_1)} + G(x_1)\frac{(x_2 - x)}{(x_2 - x_1)}.$$

Applied to (3.29) with $G(x) \equiv \log\big[F(\sigma)e^{\mu t}\big]$ and remembering that $t \in [0, \mathcal{T}]$ this gives

$$F(t) \leq [F(0)]^{\frac{(\sigma-\sigma_1)}{(1-\sigma_1)}}[F(\mathcal{T})e^{\mu\mathcal{T}}]^{\frac{(1-\sigma)}{(1-\sigma_1)}}e^{-\mu t} \tag{3.30}$$

where $\sigma_1 = e^{-k_3\mathcal{T}}$. Now define $K(\mathcal{T})$ by

$$K(\mathcal{T}) = [F(\mathcal{T})e^{\mu\mathcal{T}}]^{\frac{(1-\sigma)}{(1-\sigma_1)}}.$$

Then (3.30) leads directly to the desired continuous dependence estimate: for we now have

$$\int_0^t (\|\mathbf{u}\|^2 + \|\theta\|^2)ds + \varepsilon^2 \leq K(\mathcal{T})e^{-\mu t}[\varepsilon^2]^{\frac{(\sigma-\sigma_1)}{(1-\sigma_1)}}. \tag{3.31}$$

Inequality (3.31) establishes a continuous dependence of the solution $(\mathbf{u}, \theta)$ on the viscosity on compact intervals of $[0, \mathcal{T})$.

CHAPTER FOUR

A NONLINEAR ENERGY STABILITY ANALYSIS OF CONVECTION WITH TEMPERATURE DEPENDENT VISCOSITY.

I. ν(T).

4.1 Introduction

Having shown that the solutions to the Oberbeck-Boussinesq equations do depend continuously on the viscosity we now go on to prove a nonlinear stability result for the equations when the viscosity is a linear function of the temperature.

It was recognised long ago that changes of viscosity due to variations in temperature could have a pronounced effect on thermally driven convection, and the stability of flow of variable viscosity liquids has attracted the attention of many prominent writers, see e.g. Booker (1976), Joseph (1964), Palm (1960), Palm *et al.* (1967), Richter (1978), Tippelskirch (1956), Torrance & Turcotte (1971) and the references therein. Indeed, Tippelskirch (1956) demonstrated experimentally that if the viscosity decreases with increasing temperature then the cellular convection cells which arise are such that fluid rises in the centre of the cell, whereas if the viscosity increases when the temperature increases, the fluid descends at the centre of the cell and the circulation pattern is reversed. The working fluid of Tippelskirch's (1956) experiments was liquid sulphur which possesses the curious behaviour that at $153°C$ the viscosity changes the characteristic of its dependence on temperature. Torrance & Turcotte (1971) suggested a model where the viscosity varies and an energy stability analysis for one of their models which has the viscosity depending on the vertical coordinate has already been presented by Straughan (1986). However, the more general problem where the viscosity is a function of the temperature is more difficult, and is addressed here.

Tippelskirch (1956) suggests two possible equations for the viscosity, $\nu(T)$, namely

$$\nu(T) = \frac{\nu_0}{1 + \alpha T + \beta T^2}, \tag{4.1}$$

as being appropriate for a fluid, and

$$\nu(T) = \nu_0 \sqrt{\frac{T}{T_0}} \left(\frac{1 + c/T_0}{1 + c/T} \right), \tag{4.2}$$

as appropriate to gases, where $\nu_0, \alpha, \beta, c, T_0$ are constants. To analyse mathematically the effect of variable viscosity, Palm *et al.* (1967) choose a linear relation (i.e. the leading terms in a binomial expansion of (4.1)), such that

$$\nu(T) = \nu_0 \big(1 - \gamma(T - T_0)\big), \tag{4.3}$$

where ν_0 and γ are constants, taken positive, and T_0 is a reference temperature. (Palm *et al.* (1967) use $+\gamma$ in (4.3).) Equation (4.3) is the viscosity-temperature relation which we will adopt in the following analysis.

This chapter is adapted from Richardson and Straughan (1993a).

4.2 The governing equations

We suppose a layer of heat conducting incompressible viscous fluid is contained between the planes $z = 0$ and $z = d$. The equations governing the convective fluid motion, with relation (4.3) are then

$$\begin{aligned} v_{i,t} + v_j v_{i,j} =& -\frac{1}{\rho} p_{,i} + 2\nu_0 \Big\{ \big(1 - \gamma[T - T_0]\big) D_{ij} \Big\}_{,j} \\ & - g k_i \big(1 - \alpha(T - T_0)\big), \end{aligned} \tag{4.4}$$

$$v_{i,i} = 0, \tag{4.5}$$

$$T_{,t} + v_i T_{,i} = \kappa \Delta T, \tag{4.6}$$

where g, ρ, α, κ are gravity, (constant) density, thermal expansion coefficient and thermal diffusivity, respectively, p is pressure, v_i velocity and T is the temperature. The tensor D_{ij} is the symmetric part of the velocity gradient, i.e.

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

The boundaries $z = 0, d$ are kept at fixed, constant, temperatures;

$$T = T_1, \quad z = 0, \qquad T = T_2, \quad z = d, \qquad (T_1 > T_2),$$

and then the steady solution whose stability we investigate is

$$\bar{v}_i \equiv 0, \qquad \bar{T} = -\beta z + T_1, \tag{4.7}$$

where β is the temperature gradient given by

$$\beta = \frac{T_1 - T_2}{d},$$

and the steady pressure $\bar{p}(z)$ is found from (4.4).

Perturbations (u_i, θ, π) to (v_i, T, p) are introduced via,

$$v_i = \bar{v}_i + u_i, \quad T = \bar{T} + \theta, \quad p = \bar{p} + \pi,$$

and we then employ the non-dimensionalization

$$t = t^* \frac{d^2}{\nu_0}, \qquad \pi = \pi^* P, \qquad P = \frac{U \nu_0 \rho}{d}, \qquad Pr = \frac{\nu_0}{\kappa},$$

$$u_i = u_i^* U, \qquad \theta = \theta^* T^\sharp, \qquad T^\sharp = U \sqrt{\frac{\beta \nu_0}{\kappa g \alpha}}, \qquad R = \sqrt{\frac{\alpha g \beta d^4}{\kappa \nu_0}},$$

$$x_i = x_i^* d, \qquad U = \frac{\nu_0}{d}, \qquad \Gamma = \gamma \beta d, \qquad \xi = \frac{T_1 - T_0}{\beta d}.$$

The *non-dimensional* perturbation equations become (dropping all stars):

$$\begin{aligned} u_{i,t} + u_j u_{i,j} =& - \pi_{,i} + 2\Big\{ \big[1 + \Gamma(z - \xi)\big] d_{ij} \Big\}_{,j} + R\theta k_i \\ & - 2\frac{\Gamma Pr}{R} (\theta d_{ij})_{,j}, \qquad (4.8) \\ u_{i,i} =& 0, \qquad (4.9) \\ Pr(\theta_{,t} + u_i \theta_{,i}) =& Rw + \Delta\theta, \qquad (4.10) \end{aligned}$$

where $d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and the spatial domain is now the three-dimensional strip $z \in (0, 1)$.

We here employ boundary conditions appropriate to stress free surfaces, i.e on $z = 0, 1$

$$\begin{aligned} & w = \frac{\partial^2 w}{\partial z^2} = \Delta w = 0, \\ & \theta = \Delta\theta = 0, \qquad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \end{aligned} \qquad (4.11)$$

cf. Chapter 2.

4.3 Nonlinear stability

For this section we need the results (cf. Adams (1975), Gilbarg & Trudinger (1977)), that there exists a constant c such that

$$\sup_V |\theta| \leq c\|\Delta\theta\|, \qquad \sup_V |\mathbf{u}| \leq c\|\Delta\mathbf{u}\|, \tag{4.12}$$

where we observe that the proof of the latter inequality also uses the Wirtinger inequality due to the fact that only (4.11) hold, see Galdi & Straughan (1985). Inequalities (4.12) are important in the following analysis.

To establish a nonlinear stability result we commence by multiplying (4.8) by u_i, (4.10) by θ, and we then integrate over V (defined as in Chapter 2) to find:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 = -\|\nabla\mathbf{u}\|^2 - 2\Gamma < (z-\xi)d_{ij}d_{ij} > + R < \theta w > + 2\frac{\Gamma Pr}{R} < \theta d_{ij}d_{ij} >, \tag{4.13}$$

$$\frac{1}{2}Pr\frac{d}{dt}\|\theta\|^2 = R < w\theta > - \|\nabla\theta\|^2. \tag{4.14}$$

To employ a coupling parameter analysis we now add (4.13)+ λ(4.14) for a positive constant λ at our disposal. The equation which results is

$$\frac{dE}{dt} = RI - D + 2\frac{\Gamma Pr}{R} < \theta d_{ij}d_{ij} >, \tag{4.15}$$

where

$$E = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}\lambda Pr\|\theta\|^2, \quad I = (1+\lambda) < \theta w >, \quad D = \|\nabla\mathbf{u}\|^2 + 2\Gamma < (z-\xi)d_{ij}d_{ij} > + \lambda\|\nabla\theta\|^2. \tag{4.16}$$

In order to follow the standard energy method we now define

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D}, \tag{4.17}$$

where $\mathcal{H}$ is the set of admissible functions, and then derive from (4.15)

$$\frac{dE}{dt} \leq -D\Big(\frac{R_E - R}{R_E}\Big) + \frac{2\Gamma Pr}{R} < \theta d_{ij}d_{ij} > . \tag{4.18}$$

The Euler equations resulting from (4.17) are

$$2\Delta u_i - 4\Gamma(\mu d_{ij})_{,j} + R_E M k_i \phi = \varpi_{,i}, \tag{4.19}$$

$$2\Delta\phi + R_E M w = 0, \tag{4.20}$$

where we have put $\phi = \sqrt{\lambda}\theta$, ϖ is a Lagrange multiplier, $\mu = z - \xi$ and $M = \frac{1+\lambda}{\sqrt{\lambda}}$. The numerical results derived from these equations are in §4.4.

We now use the technique of parametric differentiation cf. Chapter 2 on these equations in order to first of all determine the best value of λ, and secondly in order to find $\partial R_E/\partial\Gamma$. On carrying out the analysis we find

$$< w\phi > \left[R_E \frac{\partial M}{\partial\lambda} + M\frac{\partial R_E}{\partial\lambda}\right] = 0. \tag{4.21}$$

For R_E to achieve a maximum in λ, we require $\frac{\partial R_E}{\partial\lambda} = 0$ and so (4.21) indicates that $\frac{\partial M}{\partial\lambda} = 0$, i.e $\lambda = 1$. Henceforth we take $\lambda = 1$.

If we repeat the process for eigenvalues Γ^1 and Γ^2 the the resulting equation is

$$\frac{\partial R_E}{\partial\Gamma} = \frac{< (z-\xi)d_{ij}d_{ij} >}{< w\phi >}. \tag{4.22}$$

However (4.20) gives us $< w\phi >= \frac{1}{R_E}\|\nabla\phi\|^2 \geq 0$. Hence from (4.22)

$$\frac{\partial R_E}{\partial\Gamma} = \frac{< (z-\xi)d_{ij}d_{ij} > R_E}{\|\nabla\phi\|^2}, \tag{4.23}$$

and so the sign of $\frac{\partial R_E}{\partial\Gamma}$ depends on the sign of $z - \xi$. The eigenvalue problem arising from (4.19) and (4.20) is the *same* as the one for the *linearized* version of (4.8)–(4.10) due to the fact that $\lambda = 1$ and so if we can demonstrate decay of E from (4.18) then we have an optimum result since the nonlinear stability boundary (4.17) is the same as the linear one. Unfortunately, a direct approach to the nonlinearity in (4.18) does not seem possible, since the d_{ij} terms are of the same order of derivatives as the quadratic dissipation term. Instead, we must develop a generalized energy analysis, as in Chapter 2.

We begin with an estimation of the nonlinearity in (4.18). To this end, using (4.12)

$$\begin{aligned} < \theta d_{ij}d_{ij} > &\leq \sup_V |\theta| \, \|d_{ij}\|^2 \\ &\leq \frac{1}{2}c\|\Delta\theta\|\|\nabla\mathbf{u}\|^2. \end{aligned} \tag{4.24}$$

Inequality (4.24) suggests that a suitable generalized energy might involve $\|\Delta\theta\|^2$. It transpires that we also need a combination of $\|\nabla\mathbf{u}\|^2$ in our energy, and so we derive the necessary auxiliary identities.

To commence, take the Laplacian of (4.10). This yields

$$Pr\Delta\theta_{,t} + Pr(\Delta u_i\theta_{,i} + 2u_{i,j}\theta_{,ij} + u_i\Delta\theta_{,i}) = R\Delta w + \Delta^2\theta. \tag{4.25}$$

From (4.8) and (4.25), we may now derive the following equations, with the aid of (4.11):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|^2 = -\|\Delta \mathbf{u}\|^2 - 2\Gamma < \big[(z-\xi)d_{ij}\big]_{,j}\Delta u_i > \\ - R < \theta\Delta w > + < \Delta u_i u_j u_{i,j} > \\ + 2\frac{\Gamma Pr}{R} < \Delta u_i(\theta d_{ij})_{,j} >,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\frac{1}{2}Pr\frac{d}{dt}\|\Delta\theta\|^2 = -\|\nabla\Delta\theta\|^2 + R < \Delta w\Delta\theta > \\ - Pr\big\{< \Delta\theta\Delta u_i\theta_{,i} > \\ + 2 < \Delta\theta u_{i,j}\theta_{,ij} > + < \Delta\theta u_i\Delta\theta_{,i} >\big\}.\end{aligned} \tag{4.27}$$

Equation (4.27) is immediately simplified by noting that by use of (4.11),

$$< u_i\Delta\theta\Delta\theta_{,i} >= 0.$$

Furthermore, to progress, we need an embedding inequality of the type:

$$\sup_V |\nabla\theta| \le c\|\nabla\Delta\theta\|. \tag{4.28}$$

To prove (4.28) we note that from Adams (1975), p. 95, the space

$$C_B^j(V) = \big\{u \in C^j(V)\big| D^\alpha u \text{ bounded on } V \text{ for } |\alpha| \le j\big\}$$

is a Banach space under the norm

$$\|u\|_{C_B^j(V)} = \max_{0\le|\alpha|\le j}\sup_{x\in V}|D^\alpha u(x)|.$$

The situation of relevance here is case C of Adams (1975), p. 97, where we take $m = p = 2$, $n = 3$, and then

$$\|u\|_{C_B^1(V)} \le c\|u\|_{W^{3,2}(V)}, \tag{4.29}$$

for a computable constant c. Inequality (4.29) is

$$\max\{\sup_V|u|, \sup_V|\nabla u|\} \le c\Big(\int_V u^2 dV + \int_V u_{,i}u_{,i}dV + \int_V u_{,ij}u_{,ij}dV + \int_V u_{,ijk}u_{,ijk}dV\Big)^{1/2}.$$

Interpreted in our context we hence deduce by using Poincaré's inequality,

$$\|\theta\|^2 \le \pi^{-2}\|\nabla\theta\|^2,$$

that (4.29) yields:

$$\sup_V |\nabla\theta| \le c\Big\{(1+\pi^{-2})\|\nabla\theta\|^2 + <\theta_{,ij}\theta_{,ij}> + <\theta_{,ijk}\theta_{,ijk}>\Big\}^{1/2}. \qquad (4.30)$$

Using the boundary conditions (4.11) and integrating by parts, we may now show

$$<\theta_{,ijk}\theta_{,ijk}> = \|\nabla\Delta\theta\|^2, \qquad (4.31)$$

$$<\theta_{,ij}\theta_{,ij}> = \|\Delta\theta\|^2. \qquad (4.32)$$

Also, since $\Delta\theta = 0$ on $z = 0, 1$, use of Poincaré's inequality shows

$$\|\Delta\theta\|^2 \le \pi^{-2}\|\nabla\Delta\theta\|^2,$$

and hence from (4.32),

$$<\theta_{,ij}\theta_{,ij}> \le \frac{1}{\pi^2}\|\nabla\Delta\theta\|^2. \qquad (4.33)$$

Also, integrating by parts, and using the Cauchy-Schwarz and Poincaré inequalities,

$$\begin{aligned}\|\nabla\theta\|^2 &= -<\theta\Delta\theta> \\ &\le \|\theta\|\|\Delta\theta\| \\ &\le \pi^{-1}\|\nabla\theta\|\|\Delta\theta\|,\end{aligned}$$

and hence

$$\|\nabla\theta\|^2 \le \pi^{-2}\|\Delta\theta\|^2 \le \pi^{-4}\|\nabla\Delta\theta\|^2. \qquad (4.34)$$

We thus use (4.31)–(4.34) in (4.30) to conclude

$$\sup_V |\nabla\theta| \le c\big[(1+\pi^{-2})\pi^{-4} + \pi^{-2} + 1\big]^{1/2}\|\nabla\Delta\theta\|, \qquad (4.35)$$

which is (4.28) for c suitably interpreted. A value for c in (4.35) may be found from the method given by Adams (1975), p. 99 and on.

If we return to (4.26), then the following relations may be proved, using (4.12) and (4.28):

$$-2\Gamma < \big[(z-\xi)d_{ij}\big]_{,j}\Delta u_i > = -\Gamma < (z-\xi)|\Delta\mathbf{u}|^2 > -2\Gamma < d_{i3}\Delta u_i >, \qquad (4.36)$$

$$\begin{aligned}<\Delta u_i u_j u_{i,j}> &\le \sup_V |\mathbf{u}|\,\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\| \\ &\le c\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|^2, \qquad (4.37)\end{aligned}$$

$$\begin{aligned}2<\Delta u_i(\theta d_{ij})_{,j}> &= <\theta\Delta u_i\Delta u_i> + <\Delta u_i\theta_{,j}(u_{i,j}+u_{j,i})> \\ &\le \sup_V |\theta|\,\|\Delta\mathbf{u}\|^2 + 2\sup_V |\nabla\theta|\,\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\| \\ &\le c\|\Delta\theta\|\|\Delta\mathbf{u}\|^2 + 2c\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\|\nabla\Delta\theta\|. \qquad (4.38)\end{aligned}$$

Hence, employing (4.36)–(4.38) in (4.26),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 \leq -\|\Delta\mathbf{u}\|^2 - \Gamma < (z-\xi)|\Delta\mathbf{u}|^2 > -2\Gamma < d_{i3}\Delta u_i > \\ - R < \theta\Delta w > + c\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|^2 \\ + c\frac{\Gamma Pr}{R}\|\Delta\theta\|\|\Delta\mathbf{u}\|^2 + 2\frac{\Gamma Prc}{R}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\|\nabla\Delta\theta\|.\end{aligned} \tag{4.39}$$

To estimate the nonlinearities in (4.27) we also employ (4.28):

$$\begin{aligned}< \Delta\theta\Delta u_i\theta_{,i} > &\leq \sup_V |\nabla\theta| \, \|\Delta\theta\|\|\Delta\mathbf{u}\| \\ &\leq c\|\Delta\mathbf{u}\|\|\Delta\theta\|\|\nabla\Delta\theta\|,\end{aligned} \tag{4.40}$$

$$\begin{aligned}< \Delta\theta u_{i,j}\theta_{,ij} > &= - < u_{i,j}\Delta\theta_{,i}\theta_{,j} > \\ &\leq \sup_V |\nabla\theta| \, \|\nabla\mathbf{u}\|\|\nabla\Delta\theta\| \\ &\leq c\|\nabla\mathbf{u}\|\|\nabla\Delta\theta\|^2.\end{aligned} \tag{4.41}$$

With the aid of (4.40) and (4.41) we thus deduce from (4.27),

$$\begin{aligned}\frac{1}{2}Pr\frac{d}{dt}\|\Delta\theta\|^2 \leq -\|\nabla\Delta\theta\|^2 + R < \Delta w\Delta\theta > \\ + cPr\big\{\|\Delta\mathbf{u}\|\|\Delta\theta\|\|\nabla\Delta\theta\| + 2\|\nabla\mathbf{u}\|\|\nabla\Delta\theta\|^2\big\}.\end{aligned} \tag{4.42}$$

To utilize (4.39) and (4.42) we first recollect (4.18) which, with the help of (4.24), yields

$$\frac{dE}{dt} \leq -D\left(\frac{R_E - R}{R_E}\right) + c\frac{\Gamma Pr}{R}\|\Delta\theta\|\|\nabla\mathbf{u}\|^2. \tag{4.43}$$

We derive a *generalized energy inequality* by forming (4.43)+η(4.39)+ ζ(4.42), for coupling parameters $\eta, \zeta (> 0)$ at our disposal. The writing of the generalized energy inequality is facilitated by putting

$$\mathcal{E}(t) = E(t) + \frac{1}{2}\eta\|\nabla\mathbf{u}\|^2 + \frac{1}{2}\zeta Pr\|\Delta\theta\|^2, \tag{4.44}$$

$$D_1 = \left(\frac{R_E - R}{R_E}\right)D + \zeta\|\nabla\Delta\theta\|^2 + \eta < \big[1 + \Gamma(z-\xi)\big]|\Delta\mathbf{u}|^2 >, \tag{4.45}$$

$$Q = -2\eta\Gamma < d_{i3}\Delta u_i > -R\eta < \theta\Delta w > + \zeta R < \Delta w\Delta\theta >, \tag{4.46}$$

and we may then show

$$\begin{aligned}\frac{d\mathcal{E}}{dt} \leq & -D_1 + Q + c\frac{\Gamma Pr}{R}\|\Delta\theta\|\|\nabla\mathbf{u}\|^2 \\ & + c\eta\frac{\Gamma Pr}{R}\|\Delta\theta\|\|\Delta\mathbf{u}\|^2 + c\eta\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|^2 \\ & + 2c\eta\frac{\Gamma Pr}{R}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\|\nabla\Delta\theta\| + c\zeta Pr\|\Delta\mathbf{u}\|\|\Delta\theta\|\|\nabla\Delta\theta\| \\ & + 2c\zeta Pr\|\nabla\mathbf{u}\|\|\nabla\Delta\theta\|^2.\end{aligned} \tag{4.47}$$

In what follows we take

$$|\Gamma(z-\xi)| < 1 \qquad \text{and} \qquad R < R_E.$$

Of course, the latter condition is simply the requirement that the number R be less than the nonlinear stability threshold. We now deduce $\exists\ k_1, k_2(> 0)$ such that

$$\begin{aligned}D_1 =& \left(\frac{R_E - R}{R_E}\right)(\|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2 + 2\Gamma < (z-\xi)d_{ij}d_{ij} >) \\ & + \eta < \left[1 + \Gamma(z-\xi)\right]|\Delta\mathbf{u}|^2 > + \zeta\|\nabla\Delta\theta\|^2 \\ \geq & \left(\frac{R_E - R}{R_E}\right)(\|\nabla\theta\|^2 + k_1\|\nabla\mathbf{u}\|^2) \\ & + \eta k_2\|\Delta\mathbf{u}\|^2 + \zeta\|\nabla\Delta\theta\|^2 \\ \stackrel{\text{def}}{=} & \mathcal{D},\end{aligned} \tag{4.48}$$

$\mathcal{D}$ being defined as indicated. Clearly, we may (and do) replace D_1 by $\mathcal{D}$ in (4.47).

The cubic terms in (4.47) are now bounded in terms of $\mathcal{D}\mathcal{E}^{1/2}$,

$$\eta c\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|^2 \leq \frac{c\sqrt{2}}{\sqrt{\eta}k_2}\mathcal{D}\mathcal{E}^{1/2}, \tag{4.49}$$

$$\frac{\Gamma c Pr}{R}\|\Delta\theta\|\|\nabla\mathbf{u}\|^2 \leq \frac{\sqrt{2}cPr\Gamma}{R\pi\sqrt{\zeta\eta a k_1}}\mathcal{D}\mathcal{E}^{1/2}, \tag{4.50}$$

$$\zeta c Pr\|\Delta\mathbf{u}\|\|\Delta\theta\|\|\nabla\Delta\theta\| \leq \frac{c\sqrt{2Pr}}{\sqrt{\eta k_2}}\mathcal{D}\mathcal{E}^{1/2}, \tag{4.51}$$

$$2\zeta c Pr\|\nabla\Delta\theta\|^2\|\nabla\mathbf{u}\| \leq \frac{2^{3/2}cPr}{\sqrt{\eta}}\mathcal{D}\mathcal{E}^{1/2}, \tag{4.52}$$

$$\frac{2\eta c Pr\Gamma}{R}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\|\nabla\Delta\theta\| \leq \frac{2^{3/2}cPr\Gamma}{R\sqrt{\zeta k_2}}\mathcal{D}\mathcal{E}^{1/2}, \tag{4.53}$$

$$\frac{\eta c Pr\Gamma}{R}\|\Delta\theta\|\|\Delta\mathbf{u}\|^2 \leq \frac{\sqrt{2Pr}c\Gamma}{Rk_2\sqrt{\zeta}}\mathcal{D}\mathcal{E}^{1/2}, \tag{4.54}$$

where Poincaré's inequality has also been used. We now derive from (4.47),

$$\frac{d\mathcal{E}}{dt} \leq -\mathcal{D} + Q + A\mathcal{E}^{1/2}\mathcal{D}, \tag{4.55}$$

where the constant A is given by,

$$A = \frac{\sqrt{2}\Gamma c Pr}{\pi R\sqrt{\zeta\eta a k_1}} + c\sqrt{\frac{2Pr}{\eta k_2}} + \frac{cPr2^{3/2}}{\sqrt{\eta}} + \frac{c}{k_2}\sqrt{\frac{2}{\eta}} + \frac{c\Gamma Pr 2^{3/2}}{R\sqrt{\zeta k_2}} + \frac{c\Gamma}{Rk_2}\sqrt{\frac{2Pr}{\zeta}}.$$

It remains to handle the Q term and to this end we use the arithmetic-geometric mean inequality on the individual components of Q as follows,

$$-2\eta\Gamma < d_{i3}\Delta u_i > = -\eta\Gamma < u_{i,z}\Delta u_i > \leq \frac{\eta k_2}{6}\|\Delta \mathbf{u}\|^2 + \frac{3\eta\Gamma^2}{2k_2}\|\nabla \mathbf{u}\|^2, \tag{4.56}$$

$$-R\eta < \theta\Delta w > \leq \frac{\eta k_2}{6}\|\Delta w\|^2 + \frac{3\eta R^2}{2k_2}\|\theta\|^2, \tag{4.57}$$

$$\zeta R < \Delta w \Delta\theta > \leq \frac{\eta k_2}{6}\|\Delta w\|^2 + \frac{3\zeta^2 R^2}{2\eta k_2}\|\Delta\theta\|^2. \tag{4.58}$$

We thus bound Q, and additionally use Poincaré's inequality on the last terms in (4.57) and (4.58), to find

$$Q \leq \frac{\eta k_2}{2}\|\Delta \mathbf{u}\|^2 + \frac{3\zeta^2 R^2}{2\pi^2\eta k_2}\|\nabla\Delta\theta\|^2 + \frac{3\Gamma^2\eta}{2k_2}\|\nabla \mathbf{u}\|^2 + \frac{3R^2\eta}{2\pi^2 k_2}\|\nabla\theta\|^2. \tag{4.59}$$

The coefficients ζ and η are at our disposal and we select these such that

$$\zeta = \frac{\eta\pi^2 k_2}{3R^2}, \qquad \eta = \left(\frac{R_E - R}{3R_E}\right)\min\Big\{\frac{\pi^2 k_2}{R^2}, \frac{k_1 k_2}{\Gamma^2}\Big\}.$$

With these choices in (4.59) it easily follows that

$$Q \leq \frac{1}{2}\mathcal{D},$$

and hence from the energy inequality (4.55) we deduce that

$$\frac{d\mathcal{E}}{dt} \leq -\frac{1}{2}\mathcal{D} + A\mathcal{E}^{1/2}\mathcal{D}. \tag{4.60}$$

Since we may use Poincaré's inequality to show $\exists$ a constant ξ such that $\mathcal{D} \geq \xi\mathcal{E}$, it is straightforward to conclude from (4.60), cf. Chapter 2, that provided

$$1)\ \mathcal{E}^{1/2}(0) < \frac{1}{2A} \qquad \text{and} \qquad 2)\ R < R_E,$$

then

$$\mathcal{E}(t) \leq e^{-K\xi t}$$

where $K = 1/2 - A\mathcal{E}^{1/2}(0)$ and is a positive constant. Hence $\mathcal{E}(t) \to 0$ as $t \to \infty$, and nonlinear stability is assured.

4.4 Numerical results

As shown in the previous section, the nonlinear stability threshold is the same as the linear instability one. The critical Rayleigh number, Ra, is calculated as follows.

We return to the Euler equations (4.19)–(4.20) given by the maximum problem (4.17);

$$\begin{aligned} 2\Delta u_i - 4\Gamma(\mu d_{ij})_{,j} + 2R_E k_i \phi &= \varpi_{,i}, \\ \Delta\phi + R_E w &= 0. \end{aligned}$$

Following the same procedure as in Chapter 2 we take the third component of the double curl of the first equation and then normal modes. This results in

$$\begin{aligned} (1-\mu\Gamma)(D^2-a^2)^2 W + 2\Gamma(D^2-a^2)DW - a^2 R_E \Phi &= 0, \\ (D^2-a^2)\Phi + R_E W &= 0, \end{aligned}$$

and we now solve this system for the minimum eigenvalue Ra subject to the boundary conditions

$$W = D^2 W = \Phi = 0 \quad \text{on} \quad z = 0, 1.$$

Again, as in Chapter 2, we use the compound matrix method to find the eigenvalue and golden section search for the minimization.

The numerical results are now presented.

Table 1. Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.

$\xi = 0.0$.

Ra	a^2	Γ
690.17	4.935	0.1
722.45	4.935	0.2
754.39	4.935	0.3
786.02	4.935	0.4
817.40	4.936	0.5
848.53	4.936	0.6
879.44	4.937	0.7
910.16	4.937	0.8
940.70	4.937	0.9
971.08	4.938	1.0

Table 2. Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.

$\Gamma = 0.1$.

Ra	a^2	ξ
624.40	4.935	1.0
558.62	4.935	2.0
492.84	4.935	3.0
427.04	4.935	4.0

Table 3. Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.

$\xi = 1.0$.

Ra	a^2	Γ
624.40	4.935	0.1
590.77	4.935	0.2
556.51	4.935	0.3
521.50	4.936	0.4
485.54	4.938	0.5
448.37	4.940	0.6
409.54	4.943	0.7
368.31	4.949	0.8
323.01	4.958	0.9
318.12	4.960	0.91
313.14	4.961	0.92
308.06	4.951	0.93
302.86	4.966	0.94
297.52	4.968	0.95

From these tables we can see that when $\xi = 0$, Ra varies with Γ, and when $\xi = 1.0$, Ra varies inversely with Γ. This reinforces the parametric differentiation results derived earlier which said that the sign of $\partial R_E/\partial\Gamma$ depended on the sign of $z - \xi$. When $\xi = 1$ which corresponds to the reference temperature being defined as the temperature at $z = 1$, then a large value of Γ, i.e. a strongly varying viscosity, gives rise to a low Rayleigh number; that is, convection is easier to initiate.

When $\xi = 0$, which corresponds to the reference temperature being the temperature on the lower boundary, then a large value of Γ gives rise to a large Rayleigh number; that is, convection is becoming harder to initiate.

CHAPTER FIVE

A NONLINEAR ENERGY STABILITY ANALYSIS OF CONVECTION WITH TEMPERATURE DEPENDENT VISCOSITY.
II. $\nu(\mathbf{T}^2)$.

5.1 Introduction

In the previous chapter we studied the problem of convection in a fluid with temperature dependent viscosity. The temperature-viscosity relation which we chose was a linear one,

$$\nu(T) = \nu_0(1 - \gamma(T - T_0)) \tag{5.1}$$

as used by Palm *et al* (1967). This type of relation is of course only an approximation but for many fluids eg. sodium and potassium it is a very useful one. For these fluids the viscosity-temperature relation is a decreasing exponential and, for small temperature differences, may be said to approximate to a straight line, see e.g. figures 1 and 2. However not all fluids have this type of relationship between the viscosity and the temperature. For example liquid sulphur and bismuth both have a viscosity maximum, in fact sulphur also has a minimum, as can be seen in figures 3 and 4. For these types of fluid a quadratic relation like

$$\nu(T) = \nu_0(1 - \gamma(T - T_0)^2) \tag{5.2}$$

is more useful in approximating their behaviour. Sulphur could actually be modelled by a relation like

$$\nu(T) = A + BT + CT^2 + DT^3, \tag{5.3}$$

where A, B, C and D are constants. This would cause few changes in the following analysis, merely adding a $\mathcal{D}\mathcal{E}^{3/2}$ term into the final energy inequality.

Having given the viscosity quadratic behaviour the question arises–what of the density? In this chapter we use the Boussinesq approximation

$$\rho = \rho_0(1 - \alpha(T - T_0)) \tag{5.4}$$

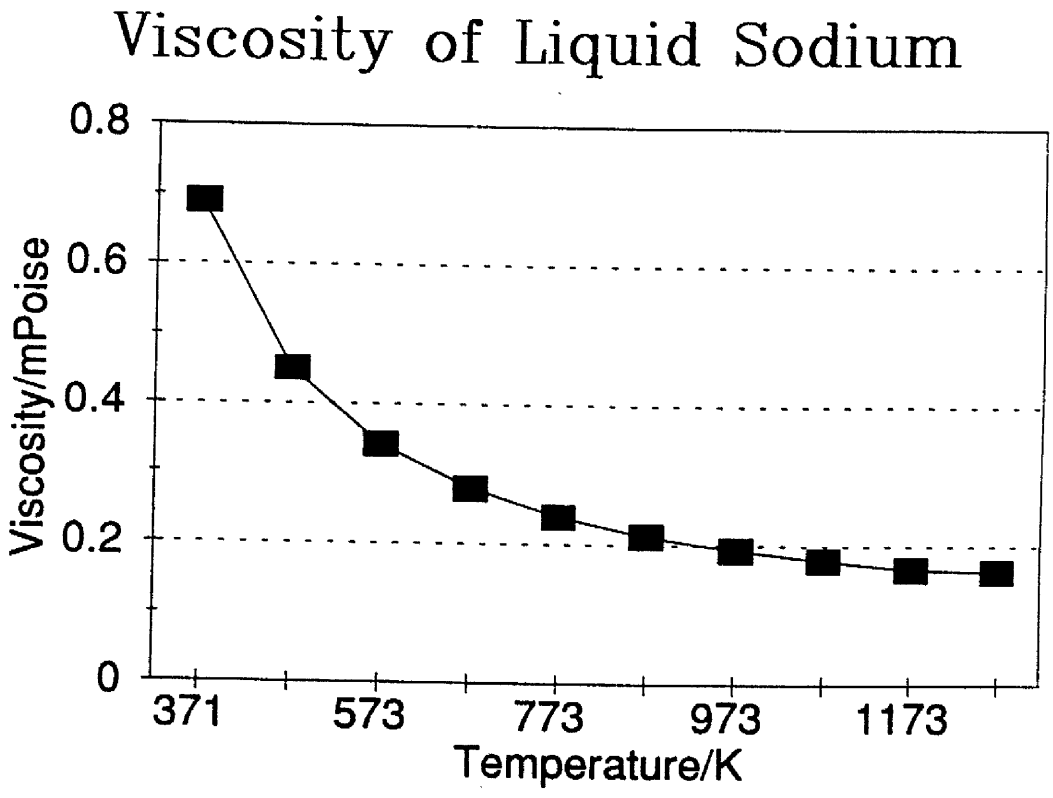
Viscosity of Liquid Sodium
0.8
0.6
0.4
0.2
0
Viscosity/mPoise
371
573
773
973
1173
Temperature/K

Figure 1.

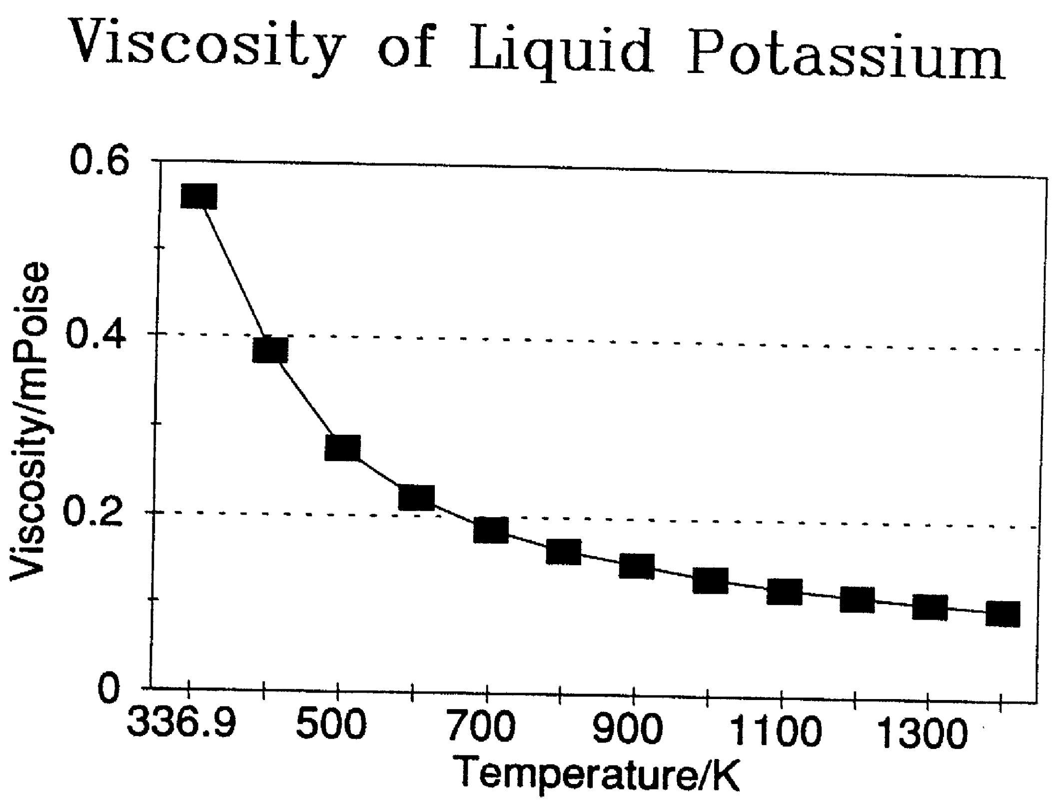
Viscosity of Liquid Potassium
0.6
0.4
0.2
0
Viscosity/mPoise
336.9
500
700
900
1100
1300
Temperature/K

Figure 2.

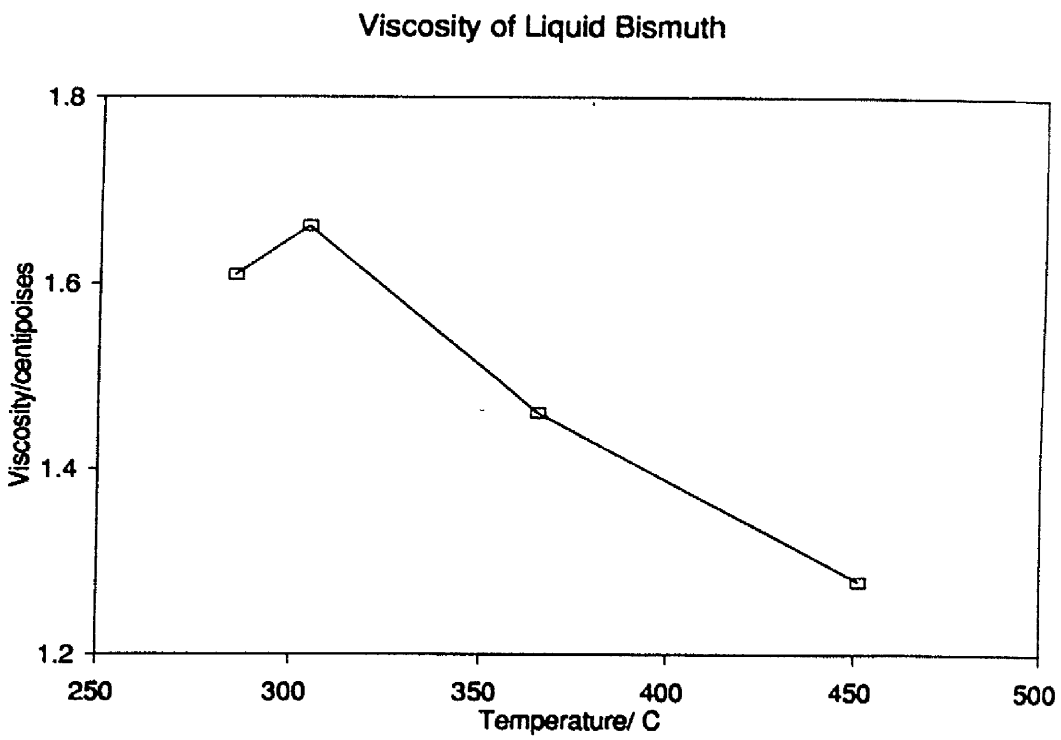
Viscosity of Liquid Bismuth
1.8
1.6
1.4
1.2
Viscosity/centipoises
250
300
350
400
450
500
Temperature/ C

Figure 3.

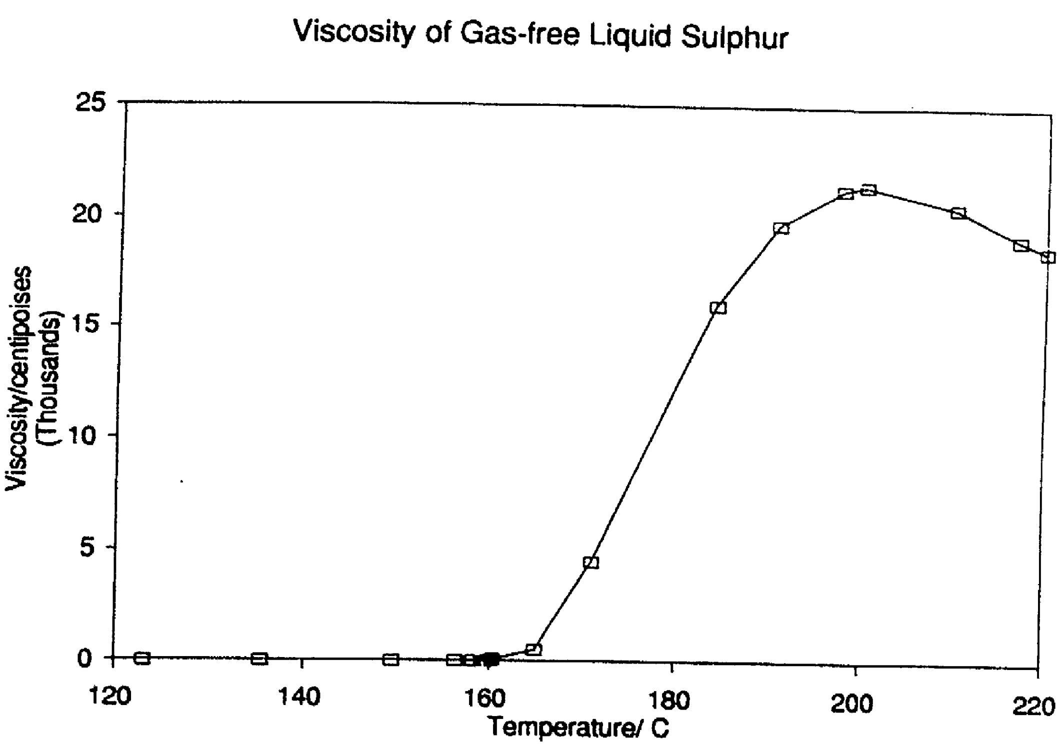
Viscosity of Gas-free Liquid Sulphur
Viscosity/centipoises (Thousands)
25
20
15
10
5
0
120
140
160
180
200
220
Temperature/ C

Figure 4a.

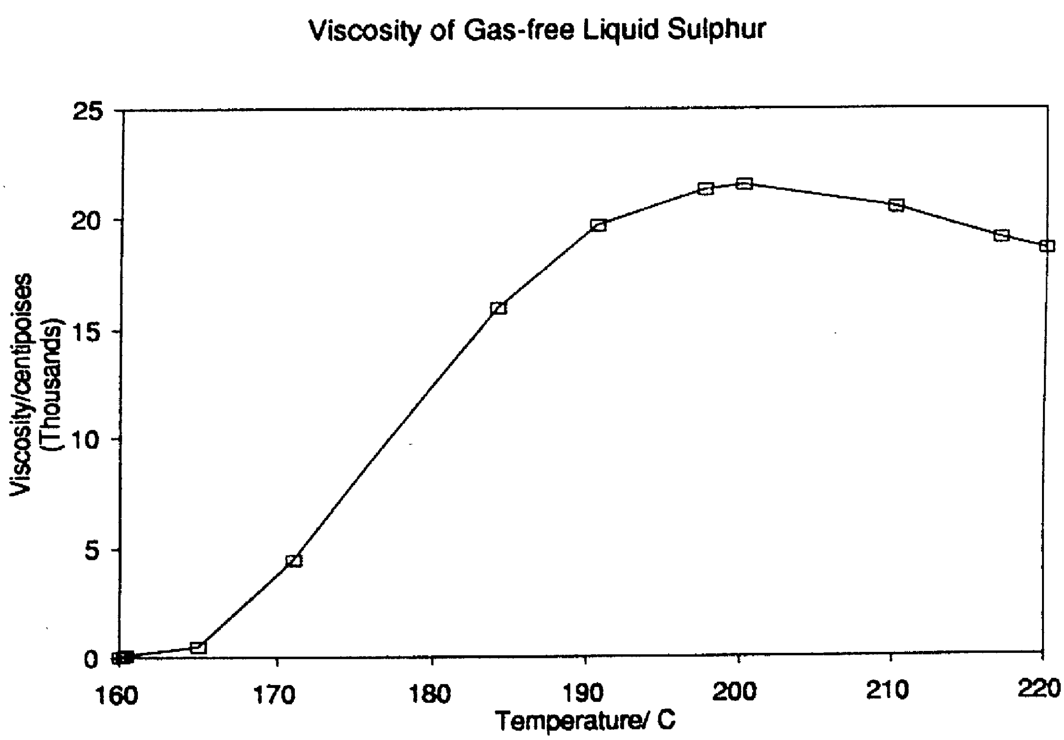
Viscosity of Gas-free Liquid Sulphur
Viscosity/centipoises (Thousands)
25
20
15
10
5
0
160
170
180
190
200
210
220
Temperature/ C

Figure 4b.

but a quadratic density would seem like the natural next step. Indeed when modelling certain problems in astrophysics, e.g. penetrative convection in clouds, then a quadratic density would be necessary.

So, for our problem we shall use the quadratic viscosity relation (5.2), the density given as in (5.4) and study the behaviour of an incompressible fluid under these conditions. Our model shall be an infinite plane layer $z \in (0, d)$, gravity pointing downwards and we shall assume two free surfaces at the boundaries with the layer being heated from below. We will show that the linear and nonlinear Rayleigh numbers are the same due to the symmetry of the problem and so, in a sense, we have an optimum result: however it must be remembered that the nonlinear stability result is a *conditional* one.

5.2 The governing equations

The equations for an incompressible Navier-Stokes fluid with a temperature dependent viscosity as in (5.2) are

$$\begin{aligned} v_{i,t} + v_j v_{i,j} =& -\frac{1}{\rho} p_{,i} + 2\nu_0[(1-\gamma(T-T_0)^2)D_{ij}]_{,j} - gk_i(1-\alpha(T-T_0)) && (5.5)\\ v_{i,i} =& 0, && (5.6)\\ T_{,t} + v_i T_{,i} =& \kappa \Delta T, && (5.7) \end{aligned}$$

where v_i is the velocity field, p is the pressure, g is gravity, ν is the viscosity, κ is the thermal diffusivity, α is the thermal expansion coefficient, T_0 is a reference temperature, ρ is the density, $\mathbf{k}$ is the unit vector in the z direction and T is the temperature field. The tensor D_{ij} is the symmetric part of the velocity gradient, i.e.

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

We prescribe the temperature on the boundaries to be $T = T_1$ on $z = 0$ and $T = T_2$ on $z = d$ with $T_1 > T_2$, (thus we are heating the layer from below).

We investigate the stability of the steady state $\bar{v}_i \equiv 0$, $\bar{T}(z)$, $\bar{p}$. Using equations (5.5)–(5.7) together with the boundary conditions we can show that $\bar{T}$ can be represented as follows,

$$\bar{T} = -\beta z + T_1, \tag{5.8}$$

where $\beta = (T_1 - T_2)/d$, and $\bar{p}$ is then found from

$$\frac{d\bar{p}}{dz} = -\rho g(1 - \alpha(T_1 - T_0 - \beta z)). \tag{5.9}$$

For the purposes of this chapter we prefer to take an average of the viscosity in our equations and we define this average, ν_m, as follows:

$$\begin{aligned}\nu_m &= \frac{1}{d}\int_0^d \nu(\bar{T})dz \\ &= \frac{1}{d}\int_0^d \nu_0(1-\gamma(\bar{T}-T_0)^2)dz \\ &= \frac{\nu_0}{3}(3-\gamma\beta^2 d^2),\end{aligned}$$

i.e $\nu_0 = 3\nu_m/(3-\gamma\beta^2 d^2)$, where we have taken the reference temperature T_0 to be T_1, the temperature on the lower boundary.

We now introduce perturbations (u_i, θ, π) to $(\bar{v}_i, \bar{T}, \bar{p})$ via

$$v_i = \bar{v}_i + u_i, \quad T = \bar{T} + \theta, \quad p = \bar{p} + \pi,$$

and then non-dimensionalize the resulting equations according to the scales,

$$\begin{aligned} &t = t^* \frac{d^2}{\nu_m}, \qquad \pi = \pi^* P, \qquad P = \frac{U\nu_m\rho}{d}, \qquad Pr = \frac{\nu_m}{\kappa}, \\ &u_i = u_i^* U, \qquad \theta = \theta^* T^\sharp, \qquad T^\sharp = U\sqrt{\frac{\beta\nu_m}{\kappa g\alpha}}, \qquad R = \sqrt{\frac{\alpha g\beta d^4}{\kappa\nu_m}}, \\ &x_i = x_i^* d, \qquad U = \frac{\nu_m}{d}, \qquad \Gamma = \gamma\beta^2 d^2. \end{aligned}$$

The equations now become, ignoring asterisks

$$\begin{aligned} u_{i,t} + u_j u_{i,j} =& -\pi_{,i} + Rk_i\theta + \frac{6}{(3-\Gamma)}\left[d_{ij}(1-\Gamma z^2)\right]_{,j} + \frac{12\Gamma Pr}{(3-\Gamma)R}\left[d_{ij}\theta z\right]_{,j} \\ & - \frac{6\Gamma Pr^2}{(3-\Gamma)R^2}\left[d_{ij}\theta^2\right]_{,j}, \end{aligned} \tag{5.10}$$

$$u_{i,i} = 0, \tag{5.11}$$

$$Pr(\theta_{,t} + u_i\theta_{,i}) = \Delta\theta + Rw. \tag{5.12}$$

The boundary conditions we employ are those for two free surfaces, i.e. on $z = 0, 1$

$$\begin{aligned} w =& \frac{\partial^2 w}{\partial z^2} = \Delta w = 0 \\ \theta =& \Delta\theta = 0, \qquad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0. \end{aligned} \tag{5.13}$$

cf. Chandrasekhar (1981) and Chapter 2.

5.3 Nonlinear analysis

Once more we employ the energy method in order to study the nonlinear stability of the system.

To begin, multiply (5.10) by u_i, (5.12) by θ and then integrate over the period cell, V.

The resulting equations are

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|^2 = R < \theta w > - \frac{6}{(3-\Gamma)} < d_{ij}d_{ij}(1-\Gamma z^2) > - \frac{12\Gamma Pr}{(3-\Gamma)R} < d_{ij}d_{ij}\theta z >$$
$$+ \frac{6\Gamma Pr^2}{(3-\Gamma)R^2} < d_{ij}d_{ij}\theta^2 >, \tag{5.14}$$

$$\frac{d}{dt}\frac{Pr}{2}\|\theta\|^2 = R < \theta w > - \|\nabla\theta\|^2. \tag{5.15}$$

To derive our energy equation we now add (5.14) + λ(5.15) where $\lambda > 0$ is a coupling parameter to be chosen to the best advantage. This yields

$$\frac{dE}{dt} = RI - D + N \tag{5.16}$$

where

$$I = (1+\lambda) < \theta w >,$$
$$D = \lambda\|\nabla\theta\|^2 + \frac{6}{(3-\Gamma)} < d_{ij}d_{ij}(1-\Gamma z^2) >,$$
$$E = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}Pr\lambda\|\theta\|^2,$$
$$N = \frac{6\Gamma Pr^2}{(3-\Gamma)R^2} < d_{ij}d_{ij}\theta^2 > - \frac{12\Gamma Pr}{(3-\Gamma)R} < d_{ij}d_{ij}\theta z > .$$

If we now define

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D}, \tag{5.17}$$

where $\mathcal{H}$ is the space of admissible functions and let $(R_E - R)/R_E = a$ then, if $R < R_E$, we have

$$\frac{dE}{dt} \leq -aD + N. \tag{5.18}$$

The number R_E will be our nonlinear critical Rayleigh number.

In order to bound the nonlinearities in N we will have to add extra terms to our energy but first we will concentrate on the maximum problem posed in (5.17).

The Euler-Lagrange equations associated with this problem are

$$2\Delta\theta + R_E M w = 0, \tag{5.19}$$

$$\frac{6}{(3-\Gamma)}\Delta u_i - \frac{12\Gamma}{(3-\Gamma)}(z^2 d_{ij})_{,j} + R_E M k_i \theta = \varpi_{,i}, \tag{5.20}$$

where we have put $\sqrt{\lambda}\theta = \theta^*$, ignored the star, and $M = (1+\lambda)/\sqrt{\lambda}$. As before ϖ is a Lagrange multiplier.

In order to find the relationship between R_E and λ we employ parametric differentiation, as in Chapters 2 and 4. This yields the equation

$$< w\theta > \left[R_E \frac{\partial M}{\partial \lambda} + M \frac{\partial R_E}{\partial \lambda} \right] = 0. \qquad (5.21)$$

We wish to choose a value for λ such that it maximises R_E, i.e. we need $\partial R_E/\partial\lambda = 0$. Hence (5.21) becomes

$$\frac{\partial M}{\partial \lambda} = 0$$

and so, recalling the definition of M we can see that $\bar{\lambda}$, the best value for λ, is $\lambda = 1$. Hence $M = 2$ and the Euler-Lagrange equations now become

$$\frac{3}{(3-\Gamma)}\Delta u_i - \frac{6\Gamma}{(3-\Gamma)}(z^2 d_{ij})_{,j} + R_E k_i \theta = \frac{1}{2}\varpi_{,i}, \qquad (5.22)$$

$$\Delta\theta + R_E w = 0. \qquad (5.23)$$

It should be pointed out that the eigenvalue problem arising from these equations is the same as that arising from the linearized problem, due to the fact that $\lambda = 1$. Hence we have an optimum result as the nonlinear stability boundary (5.17) is the same as the linear one.

In order to solve the system numerically for the eigenvalue R_E we take the double curl of (5.22) and then take the third component. Employing normal modes, i.e.

$$w = W(z)e^{i(mx+ny)},$$

$$\theta = \Theta(z)e^{i(mx+ny)},$$

with $D = d/dz$ and $a^2 = (m^2+n^2)$ the system becomes

$$D^4 W = -\frac{4\Gamma z}{(\Gamma z^2 - 1)}D^3 W - \frac{2(\Gamma + a^2(1-\Gamma z^2))}{(\Gamma z^2 - 1)}D^2 W + \frac{4a^2\Gamma z}{(\Gamma z^2 - 1)}DW$$
$$- \frac{a^2}{(\Gamma z^2 - 1)}(2\Gamma + a^2(\Gamma z^2 - 1))W + \frac{(\Gamma - 3)R_E a^2}{3(\Gamma z^2 - 1)}\Theta, \qquad (5.24)$$

$$D^2\Theta = a^2\Theta - R_E W, \qquad (5.25)$$

with $W = D^2 W = \Theta = 0$ on $z = 0, 1$. We wish to find

$$Ra = \min_{a^2} R_E^2.$$

To do this we utilise the compound matrix method and golden section search as before.

The numerical results are now presented.

Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.
$0 \leq \Gamma \leq 0.9$.

Ra	a^2	Γ
657.512	4.935	0.0
654.575	4.958	0.1
650.845	4.983	0.2
646.139	5.010	0.3
640.212	5.040	0.4
632.735	5.074	0.5
623.241	5.112	0.6
611.041	5.155	0.7
595.017	5.207	0.8
573.074	5.271	0.9

From this table we can see that Ra varies inversely with Γ. This is physically correct as increasing Γ means decreasing viscosity and we would then expect convective motion to be easier. When $\Gamma = 0$ then we have the case of constant viscosity and we see that indeed $Ra = 27\pi^4/4$, which is the value for Bénard convection.

5.4 Generalized energy and manipulation of nonlinearities

We now return to our energy equation (5.18),

$$\frac{dE}{dt} \leq -aD + N, \tag{5.26}$$

where

$$\begin{aligned} E =& \frac{1}{2}\|\mathbf{u}\|^2 + \frac{Pr}{2}\|\theta\|^2, \\ D =& \|\nabla\theta\|^2 + \frac{6}{3-\Gamma} < d_{ij}d_{ij}(1-\Gamma z^2) >, \\ N =& \frac{6\Gamma Pr^2}{(3-\Gamma)R^2} < d_{ij}d_{ij}\theta^2 > - \frac{12\Gamma Pr}{(3-\Gamma)R} < d_{ij}d_{ij}\theta z > . \end{aligned}$$

For this section we will need the inequalities

$$\sup_V |\theta| \leq c\|\Delta\theta\|, \tag{5.27}$$

$$\sup_V |\nabla\theta| \leq c_1\|\nabla\Delta\theta\|, \tag{5.28}$$

as used in Chapter 4. We wish to bound the nonlinearities in N by the energy. However in order to do this our energy will need to include $\|\Delta\theta\|^2$ and $\|\nabla\mathbf{u}\|^2$. We now derive the necessary auxiliary equations.

First, we take the Laplacian of (5.12). We then multiply the resulting equation by $\Delta\theta$ and integrate over V. This results in the equation

$$\begin{aligned}\frac{d}{dt}\frac{Pr}{2}\|\Delta\theta\|^2 &= -\|\nabla\Delta\theta\|^2 + R < \Delta w\Delta\theta > -Pr < \theta_{,i}\Delta\theta\Delta u_i > -2Pr < u_{i,j}\theta_{,ij}\Delta\theta > \\ &= -\|\nabla\Delta\theta\|^2 + R < \Delta w\Delta\theta > -Pr < \theta_{,i}\Delta\theta\Delta u_i > \\ &\quad + 2Pr < u_{i,j}\theta_{,j}\Delta\theta_{,i} >, \end{aligned} \tag{5.29}$$

where we have integrated the last term by parts.

Considering the last two terms of (5.29) we see that, by using (5.28) and the Cauchy-Schwarz inequality, they can be bounded as follows;

$$\begin{aligned} -Pr < \theta_{,i}\Delta\theta\Delta u_i > &\leq Pr \sup_V |\nabla\theta| \; |< \Delta\theta\Delta u_i >| \\ &\leq Prc_1\|\nabla\Delta\theta\|\|\Delta\theta\|\|\Delta\mathbf{u}\|, \\ 2Pr < u_{i,j}\theta_{,j}\Delta\theta_{,i} &\leq 2Pr \sup_V |\nabla\theta| \; |< u_{i,j}\Delta\theta_{,i} >| \\ &\leq 2Prc_1\|\nabla\Delta\theta\|^2\|\nabla\mathbf{u}\|. \end{aligned}$$

Substituting for these terms in (5.29) our equation becomes

$$\begin{aligned}\frac{d}{dt}\frac{Pr}{2}\|\Delta\theta\|^2 \leq &-\|\nabla\Delta\theta\|^2 + R < \Delta w\Delta\theta > +Prc_1\|\nabla\Delta\theta\|\|\Delta\theta\|\|\Delta\mathbf{u}\| \\ &+ 2Prc_1\|\nabla\Delta\theta\|^2\|\nabla\mathbf{u}\|. \end{aligned} \tag{5.30}$$

For our second auxiliary equation, we multiply (5.10) by Δu_i and again integrate over V to derive

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\nabla\mathbf{u}\|^2 = &\frac{12\Gamma}{(3-\Gamma)} < \Delta u_i d_{ij} z\delta_{j3} > - \frac{3}{(3-\Gamma)} < |\Delta u_i|^2(1-\Gamma z^2) > \\ &- R < \theta\Delta w > - \frac{6\Gamma Pr}{(3-\Gamma)R} < |\Delta u_i|^2\theta z > - \frac{12\Gamma Pr}{(3-\Gamma)R} < \Delta u_i d_{ij}\theta_{,j} z > \\ &- \frac{12\Gamma Pr}{(3-\Gamma)R} < \Delta u_i d_{ij}\theta\delta_{j3} > + \frac{3\Gamma Pr^2}{(3-\Gamma)R^2} < |\Delta u_i|^2\theta^2 > \\ &+ \frac{6\Gamma Pr^2}{(3-\Gamma)R^2} < \Delta u_i d_{ij}(\theta^2)_{,j} > . \end{aligned} \tag{5.31}$$

Here δ_{j3} is the Kronecker delta.

Again we estimate the nonlinear terms in (5.31). Here, as $z \in (0,1)$ we have taken $|z| \leq 1$, and used (5.27) and (5.28).

$$
\begin{aligned}
-\frac{6\Gamma Pr}{(3-\Gamma)R} < |\Delta u_i|^2 \theta z > &\leq \frac{6\Gamma Pr}{(3-\Gamma)R} \sup_V |\theta| < |\Delta u_i|^2 > \\
&\leq \frac{6\Gamma Prc}{(3-\Gamma)R} \|\Delta\theta\| \|\Delta \mathbf{u}\|^2, \\
-\frac{12\Gamma Pr}{(3-\Gamma)R} < \Delta u_i d_{ij} \theta_{,j} z > &\leq \frac{12\Gamma Pr}{(3-\Gamma)R} \sup_V |\nabla\theta| \; | < \Delta u_i d_{ij} > | \\
&\leq \frac{12\Gamma Prc_1}{(3-\Gamma)R} \|\nabla\Delta\theta\| \|\Delta \mathbf{u}\| \|\nabla \mathbf{u}\|, \\
-\frac{12\Gamma Pr}{(3-\Gamma)R} < \Delta u_i d_{ij} \theta \delta_{j3} > &\leq \frac{12\Gamma Pr}{(3-\Gamma)R} \sup_V |\theta| \; | < \Delta u_i d_{i3} > | \\
&\leq \frac{12\Gamma Prc}{(3-\Gamma)R} \|\Delta\theta\| \|\Delta \mathbf{u}\| \|\nabla \mathbf{u}\|, \\
\frac{3\Gamma Pr^2}{(3-\Gamma)R^2} < |\Delta u_i|^2 \theta^2 > &\leq \frac{3\Gamma Pr^2}{(3-\Gamma)R^2} \sup_V |\theta| < \theta |\Delta u_i|^2 > \\
&\leq \frac{3\Gamma Pr^2 c}{(3-\Gamma)R^2} \sup_V |\theta| \|\Delta\theta\| < |\Delta u_i|^2 > \\
&\leq \frac{3\Gamma Pr^2 c^2}{(3-\Gamma)R^2} \|\Delta\theta\|^2 \|\Delta \mathbf{u}\|^2, \\
\frac{6\Gamma Pr^2}{(3-\Gamma)R^2} < \Delta u_i d_{ij} (\theta^2)_{,j} > &= \frac{12\Gamma Pr^2}{(3-\Gamma)R^2} < \Delta u_i d_{ij} \theta \theta_{,j} > \\
&\leq \frac{12\Gamma Pr^2}{(3-\Gamma)R^2} \sup_V |\theta| \; | < \Delta u_i d_{ij} \theta_{,j} > | \\
&\leq \frac{12\Gamma Pr^2 c}{(3-\Gamma)R^2} \|\Delta\theta\| \; | < \Delta u_i d_{ij} \theta_{,j} > | \\
&\leq \frac{12\Gamma Pr^2 c}{(3-\Gamma)R^2} \|\Delta\theta\| \sup_V |\nabla\theta| \; | < \Delta u_i d_{ij} > | \\
&\leq \frac{12\Gamma Pr^2 cc_1}{(3-\Gamma)R^2} \|\Delta\theta\| \|\nabla\Delta\theta\| \|\Delta \mathbf{u}\| \|\nabla \mathbf{u}\|.
\end{aligned}
$$

And so (5.31) beomes

$$\frac{d}{dt}\frac{1}{2}\|\nabla \mathbf{u}\|^2 \leq \frac{12\Gamma}{(3-\Gamma)} < \Delta u_i d_{i3} z > -\frac{3}{3-\Gamma} < |\Delta u_i|^2(1-\Gamma z^2) > -R < \theta \Delta w >$$
$$+ \frac{6\Gamma Prc}{(3-\Gamma)R}\|\Delta \mathbf{u}\|^2\|\Delta\theta\| + \frac{12\Gamma Prc_1}{(3-\Gamma)R}\|\nabla\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\|$$
$$+ \frac{12\Gamma Prc}{(3-\Gamma)R}\|\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\| + \frac{3\Gamma Pr^2c^2}{(3-\Gamma)R^2}\|\Delta\theta\|^2\|\Delta\mathbf{u}\|^2$$
$$+ \frac{12\Gamma Pr^2cc_1}{(3-\Gamma)R^2}\|\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\Delta\theta\|\|\nabla\mathbf{u}\|. \tag{5.32}$$

We derive our generalized energy inequality by adding (5.26)+η(5.30)+μ(5.32) where η, $\mu(> 0)$ are coupling parameters. We now have

$$\frac{d\mathcal{E}}{dt} \leq - D_1 + Q + \frac{6\Gamma Pr^2}{(3-\Gamma)R^2} < d_{ij}d_{ij}\theta^2 > -\frac{12\Gamma Pr}{(3-\Gamma)R} < d_{ij}d_{ij}\theta z >$$
$$+ Prc_1\eta\|\nabla\Delta\theta\|\|\Delta\theta\|\|\Delta\mathbf{u}\| + 2Prc_1\eta\|\nabla\Delta\theta\|^2\|\nabla\mathbf{u}\| + \frac{6\Gamma Prc\mu}{(3-\Gamma)R}\|\Delta\mathbf{u}\|^2\|\Delta\theta\|$$
$$+ \frac{12\Gamma Prc_1\mu}{(3-\Gamma)R}\|\nabla\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\| + \frac{12\Gamma Prc\mu}{(3-\Gamma)R}\|\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\|$$
$$+ \frac{3\Gamma Pr^2c^2\mu}{(3-\Gamma)R^2}\|\Delta\theta\|^2\|\Delta\mathbf{u}\|^2 + \frac{12\Gamma Pr^2cc_1\mu}{(3-\Gamma)R^2}\|\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\Delta\theta\|\|\nabla\mathbf{u}\|, \tag{5.33}$$

where

$$\mathcal{E} = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{Pr}{2}\|\theta\|^2 + \frac{\eta Pr}{2}\|\Delta\theta\|^2 + \frac{\mu}{2}\|\nabla\mathbf{u}\|^2,$$
$$D_1 = a(\|\nabla\theta\|^2 + \frac{6}{3-\Gamma} < d_{ij}d_{ij}(1-\Gamma z^2) >) + \eta\|\nabla\Delta\theta\|^2$$
$$+ \frac{3\mu}{3-\Gamma} < |\Delta u_i|^2(1-\Gamma z^2) >,$$
$$Q = R\eta < \Delta w\Delta\theta > + \frac{12\Gamma\mu}{3-\Gamma} < \Delta u_i z d_{i3} > -R\mu < \theta\Delta w > .$$

Note that the third and fourth terms in (5.33) can be bounded as follows,

$$\frac{6\Gamma Pr^2}{(3-\Gamma)R^2} < d_{ij}d_{ij}\theta^2 > \leq \frac{6\Gamma Pr^2}{(3-\Gamma)R^2}\sup_V|\theta| < \theta d_{ij}d_{ij} >$$
$$\leq \frac{6\Gamma Pr^2c}{(3-\Gamma)R^2}\sup_V|\theta|\|\Delta\theta\| < d_{ij}d_{ij} >$$
$$\leq \frac{3\Gamma Pr^2c^2}{(3-\Gamma)R^2}\|\Delta\theta\|^2\|\nabla\mathbf{u}\|^2,$$
$$-\frac{12\Gamma Pr}{(3-\Gamma)R} < d_{ij}d_{ij}\theta z > \leq \frac{12\Gamma Pr}{(3-\Gamma)R}\sup_V|\theta|\ |< d_{ij}d_{ij} >|$$
$$\leq \frac{6\Gamma Prc}{(3-\Gamma)R}\|\Delta\theta\|\|\nabla\mathbf{u}\|^2.$$

Also, provided that $|\Gamma z^2| < 1$, there exist k_1, $k_2 (> 0)$ such that

$$\begin{aligned}
D_1 &= a(\|\nabla\theta\|^2 + \frac{6}{3-\Gamma} < d_{ij}d_{ij}(1-\Gamma z^2) >) + \eta\|\nabla\Delta\theta\|^2 \\
&\quad + \frac{3\mu}{3-\Gamma} < |\Delta u_i|^2(1-\Gamma z^2) > \\
&\geq a(\|\nabla\theta\|^2 + k_1\|\nabla\mathbf{u}\|^2) + \eta\|\nabla\Delta\theta\|^2 + \mu k_2\|\Delta\mathbf{u}\|^2 \\
&= \mathcal{D} \text{ say .}
\end{aligned}$$

Clearly we can replace D_1 with $\mathcal{D}$.

The cubic terms in (5.33) can now be bounded in terms of $\mathcal{D}\mathcal{E}^{1/2}$ and the quadratic terms by $\mathcal{D}\mathcal{E}$ as follows;

$$\begin{aligned}
\frac{6\Gamma Prc}{(3-\Gamma)R}\|\Delta\theta\|\|\nabla\mathbf{u}\|^2 &\leq \frac{6\Gamma c(2Pr)^{1/2}}{(3-\Gamma)Rak_1\sqrt{\eta}}\mathcal{D}\mathcal{E}^{1/2}, \\
\frac{3\Gamma Pr^2c^2}{(3-\Gamma)R^2}\|\Delta\theta\|^2\|\nabla\mathbf{u}\|^2 &\leq \frac{6\Gamma Prc^2}{(3-\Gamma)R^2\eta ak_1}\mathcal{D}\mathcal{E}, \\
Pr\eta c_1\|\nabla\Delta\theta\|\|\Delta\theta\|\|\Delta\mathbf{u}\| &\leq c_1\sqrt{\frac{2Pr}{\mu k_2}}\mathcal{D}\mathcal{E}^{1/2}, \\
2Pr\eta c_1\|\nabla\Delta\theta\|^2\|\nabla\mathbf{u}\| &\leq \frac{2\sqrt{2}Prc_1}{\sqrt{\mu}}\mathcal{D}\mathcal{E}^{1/2}, \\
\frac{6\Gamma Prc\mu}{(3-\Gamma)R}\|\Delta\mathbf{u}\|^2\|\Delta\theta\| &\leq \frac{6\Gamma c(2Pr)^{1/2}}{(3-\Gamma)Rk_2\sqrt{\eta}}\mathcal{D}\mathcal{E}^{1/2}, \\
\frac{12\Gamma Pr\mu c_1}{(3-\Gamma)R}\|\nabla\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\| &\leq \frac{12\sqrt{2}\Gamma Prc_1}{(3-\Gamma)R\sqrt{\eta k_2}}\mathcal{D}\mathcal{E}^{1/2}, \\
\frac{12\Gamma Pr\mu c}{(3-\Gamma)R}\|\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\| &\leq \frac{12\Gamma c}{(3-\Gamma)R}\sqrt{\frac{2Pr\mu}{\eta k_2 ak_1}}\mathcal{D}\mathcal{E}^{1/2}, \\
\frac{3\Gamma Pr^2\mu c^2}{(3-\Gamma)R^2}\|\Delta\theta\|^2\|\Delta\mathbf{u}\|^2 &\leq \frac{6\Gamma Prc^2}{(3-\Gamma)R^2k_2\eta}\mathcal{D}\mathcal{E}, \\
\frac{12\Gamma Pr^2cc_1\mu}{(3-\Gamma)R^2}\|\Delta\theta\|\|\nabla\Delta\theta\|\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\| &\leq \frac{24\Gamma Pr^{3/2}cc_1}{(3-\Gamma)R^2\eta\sqrt{k_2}}\mathcal{D}\mathcal{E}.
\end{aligned}$$

Hence (5.33) can now be written

$$\frac{d\mathcal{E}}{dt} \leq -\mathcal{D} + Q + A\mathcal{D}\mathcal{E}^{1/2} + B\mathcal{D}\mathcal{E},$$

where

$$A = \frac{6\Gamma c(2Pr)^{1/2}}{(3-\Gamma)Rk_2\sqrt{\eta}} + \frac{6\Gamma c(2Pr)^{1/2}}{(3-\Gamma)Rak_1\sqrt{\eta}} + c_1\sqrt{\frac{2Pr}{\mu k_2}} + 2Prc_1\sqrt{\frac{2}{\mu}}$$
$$+ \frac{12\sqrt{2}\Gamma Prc_1}{(3-\Gamma)R\sqrt{\eta k_2}} + \frac{12\Gamma c}{(3-\Gamma)R}\sqrt{\frac{2Pr\mu}{\eta k_2 ak_1}},$$
$$B = \frac{6\Gamma Prc^2}{(3-\Gamma)R^2k_2\eta} + \frac{6\Gamma Prc^2}{(3-\Gamma)R^2\eta ak_1} + \frac{24\Gamma Pr^{3/2}cc_1}{(3-\Gamma)R^2\eta\sqrt{k_2}}.$$

All that remains now is to estimate Q and this can be done as follows. Here we will use Poincaré's inequality, the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality. So, again taking each component of Q in turn,

$$\begin{aligned}
R\eta < \Delta w\Delta\theta > &\leq \frac{\mu k_2}{6}\|\Delta w\|^2 + \frac{3R^2\eta^2}{2\mu k_2}\|\Delta\theta\|^2 \\
&\leq \frac{\mu k_2}{6}\|\Delta w\|^2 + \frac{3R^2\eta^2}{2\mu k_2\pi^2}\|\nabla\Delta\theta\|^2, \\
\frac{12\Gamma\mu}{3-\Gamma} < \Delta u_i z d_{i3} > &\leq \frac{12\Gamma\mu}{3-\Gamma}| < \Delta u_i d_{i3} > | \\
&\leq \frac{\mu k_2}{6}\|\Delta \mathbf{u}\|^2 + \frac{216\Gamma^2\mu}{k_2(3-\Gamma)^2}\|\nabla\mathbf{u}\|^2, \\
-R\mu < \theta\Delta w > &\leq \frac{3R^2\mu}{2k_2}\|\theta\|^2 + \frac{\mu k_2}{6}\|\Delta w\|^2 \\
&\leq \frac{3R^2\mu}{2k_2\pi^2}\|\nabla\theta\|^2 + \frac{\mu k_2}{6}\|\Delta\mathbf{u}\|^2.
\end{aligned}$$

Hence

$$Q \leq \frac{\mu k_2}{2}\|\Delta\mathbf{u}\|^2 + \frac{3R^2\eta^2}{2\pi^2\mu k_2}\|\nabla\Delta\theta\|^2 + \frac{216\Gamma^2\mu}{k_2(3-\Gamma)^2}\|\nabla\mathbf{u}\|^2 + \frac{3R^2\mu}{2\pi^2 k_2}\|\nabla\theta\|^2.$$

As η and μ are constants which can be chosen at our discretion we can now take

$$\eta = \frac{\pi^2\mu k_2}{3R^2}, \qquad \mu = \frac{a}{2}\min\left\{\frac{k_1k_2(3-\Gamma)^2}{216\Gamma^2}, \frac{2\pi^2k_2}{3R^2}\right\},$$

and Q becomes

$$Q \leq \frac{\mu k_2}{2}\|\Delta\mathbf{u}\|^2 + \frac{\eta}{2}\|\nabla\Delta\theta\|^2 + \frac{ak_1}{2}\|\nabla\mathbf{u}\|^2 + \frac{a}{2}\|\nabla\theta\|^2 = \frac{\mathcal{D}}{2}.$$

Therefore our energy inequality can now be written

$$\frac{d\mathcal{E}}{dt} \leq -\frac{\mathcal{D}}{2} + A\mathcal{D}\mathcal{E}^{1/2} + B\mathcal{D}\mathcal{E},$$

and so, provided

$$(1) \quad R < R_E \qquad \text{and} \qquad (2) \quad \frac{1}{2} - A\mathcal{E}^{1/2}(0) - B\mathcal{E}(0) > 0$$

we can prove conditional nonlinear stability as in the previous chapters. For, if we define $K = 1/2 - A\mathcal{E}^{1/2}(0) - B\mathcal{E}(0)$, then we can show that

$$\frac{d\mathcal{E}}{dt} \leq -K\mathcal{D}$$

and so

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-K\xi t}$$

where ξ is a constant given by the result $\mathcal{D} \geq \xi\mathcal{E}$. Therefore $\mathcal{E}(t) \to 0$ as $t \to \infty$.

CHAPTER SIX

CONVECTION WITH TEMPERATURE DEPENDENT VISCOSITY AND CONDUCTIVITY.

6.1 Introduction

Amongst the many assumptions that we frequently make when studying convection problems in fluid dynamics is that the thermal conductivity is essentially a constant. In fact this is not the case as can be seen by studying tables of data in e.g. Weast (1988). From these it can be seen that the conductivity does in reality vary with temperature and that for many fluids may be approximated by a linear relation in which conductivity decreases with temperature, e.g.

$$\kappa(T) = \kappa_0 \Big[1 - \mu(T - T_0)\Big]. \tag{6.1}$$

This variation of conductivity with temperature is, however, much weaker than that of the viscosity, a case which we have already studied in Chapters 4 and 5. It is of obvious interest to study the competing effects of variable viscosity and conductivity on an incompressible fluid and this is what we now do.

From previous chapters we have a choice of two relationships between the viscosity and the temperature;

$$\nu(T) = \nu_0(1 - \gamma(T - T_0)) \tag{6.2}$$

or

$$\nu(T) = \nu_0(1 - \gamma(T - T_0)^2). \tag{6.3}$$

The case of variable conductivity with a *linear* viscosity-temperature relation has already been studied, cf. the manuscript of Mulone *et al* (1993) and so for our problem we choose a viscosity as in (6.3) (which is more accurate than (6.2)), together with the conductivity relation (6.1). The linear problem is solved completely and we are able to establish a linear instability boundary. However for the nonlinear problem the boundary conditions prove intransigent to analysis. We are forced to make dynamically inadmissible assumptions but, if treated heuristically, the analysis reveals interesting and useful results.

6.2 The equations

For our model we once more take an infinite plane layer $z \in (0, d)$, with gravity pointing downwards and assume that the fluid is heat-conducting and incompressible. Then the equations of motion are, remembering our viscosity and conductivity relations, (6.1) and (6.3),

$$\begin{aligned} v_{i,t} + v_j v_{i,j} =& -\frac{1}{\rho} p_{,i} + 2\nu_0 [D_{ij}(1 - \gamma(T - T_0)^2)]_{,j} \\ & - g k_i (1 - \alpha(T - T_0)), \qquad (6.4) \\ T_{,t} + v_i T_{,i} =& \kappa_0 [T_{,j}(1 - \mu(T - T_0))]_{,j}, \qquad (6.5) \\ v_{i,i} =& 0. \qquad (6.6) \end{aligned}$$

Here all quantities are as defined in the previous chapter.
The boundary conditions on the temperature are $T(0) = T_1$, $T(d) = T_2$ where $T_1 > T_2$, i.e. the layer is heated from below. Those for the velocity follow by assuming that the surfaces are free of tangential stress.

First we look at a stationary solution, $\bar{v}_i \equiv 0$, $\bar{T}(z)$. Then equations (6.4) and (6.5) yield

$$\begin{aligned} 0 =& -\frac{1}{\rho} \bar{p}_{,i} - g k_i (1 - \alpha(\bar{T} - T_0)), \\ 0 =& \kappa_0 [\bar{T}_{,j}(1 - \mu(\bar{T} - T_0))]_{,j}. \end{aligned}$$

The second of these two equations gives us an explicit expression for $\bar{T}$ for, after integration we can see that

$$\bar{T} = \frac{1}{\mu} + T_0 - \frac{1}{\mu}\sqrt{1 + \frac{z}{d}[\mu^2(\Delta T)^2 + 2\mu \Delta T]},$$

and

$$\frac{d\bar{T}}{dz} = -\frac{\mu(\Delta T)^2 + 2\Delta T}{2d\sqrt{1 + \frac{z}{d}[\mu^2(\Delta T)^2 + 2\mu\Delta T]}},$$

where $\Delta T = T_1 - T_2$. Let us also define β to be the absolute value of $d\bar{T}/dz$ evaluated at $z = 0$, i.e $\beta = (\mu(\Delta T)^2/2d) + (\Delta T/d)$. As can be seen $\bar{T}$ is no longer a linear function of z and this obviously complicates the analysis.

We now introduce perturbations:

$$v_i = \bar{v}_i + u_i, \qquad T = \bar{T} + \theta, \qquad p = \bar{p} + \pi,$$

and non-dimensionalize according to the scales

$$t = t^*\frac{d^2}{\nu_0}, \qquad \pi = \pi^* P, \qquad P = \frac{U\nu_0\rho}{d}, \qquad Pr = \frac{\nu_0}{\kappa_0},$$

$$u_i = u_i^* U, \qquad \theta = \theta^* T^\sharp, \qquad T^\sharp = U\sqrt{\frac{\beta\nu_0}{\kappa_0 g\alpha}}, \qquad R = \sqrt{\frac{\alpha g\beta d^4}{\kappa_0\nu_0}},$$

$$x_i = x_i^* d, \qquad U = \frac{\nu_0}{d}, \qquad \Gamma = \gamma\beta^2 d^2,$$

$$\xi = 2\mu\beta d, \qquad f = \sqrt{1+\xi z}.$$

The perturbation equations then become, ignoring asterisks,

$$u_{i,t} + u_j u_{i,j} = -\pi_{,i} + 2\big[d_{ij}(1-B)\big]_{,j} - \frac{8\Gamma Pr}{R\xi}\big[d_{ij}\theta(1-f)\big]_{,j} - \frac{2\Gamma Pr^2}{R^2}\big[d_{ij}\theta^2\big]_{,j} + Rk_i\theta, \tag{6.7}$$

$$u_{i,i} = 0, \tag{6.8}$$

$$Pr(\theta_{,t} + u_i\theta_{,i}) = R\frac{w}{f} + \frac{\xi}{2}\left[\frac{\theta}{f}\right]_{,z} - \frac{\xi Pr}{2R}\left[\theta\theta_{,j}\right]_{,j} + \left[f\theta_{,j}\right]_{,j}, \tag{6.9}$$

where B(z) is defined by

$$B = \frac{4\Gamma z^2}{(1+f)^2}. \tag{6.10}$$

The boundary conditions are, on $z = 0, 1$

$$w = \frac{\partial^2 w}{\partial z^2} = \Delta w = 0$$

$$\theta = 0, \qquad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0,$$

since we have assumed two stress free surfaces.

6.3 Linear analysis

We linearise (6.7) and (6.9) and then proceed as in Chapter 2, i.e. we write

$$u_i(\mathbf{x}, t) = u_i(\mathbf{x})e^{\sigma t},$$

$$\theta(\mathbf{x}, t) = \theta(\mathbf{x})e^{\sigma t},$$

and then take $\sigma = 0$. As before we do not expect subcritical instabilities to arise and this is indeed borne out by the numerical analysis. So (6.7) and (6.9) become

$$0 = -\pi_{,i} + \Delta u_i - 2(Bd_{ij})_{,j} + Rk_i\theta, \tag{6.11}$$

$$0 = R\frac{w}{f} + \frac{\xi}{f}\theta_z - \frac{\xi^2}{4f^3}\theta + f\Delta\theta. \tag{6.12}$$

In order to solve (6.11) and (6.12) for the eigenvalue R^2 we take the double curl of (6.11) and then the third component. Then, writing

$$w = W(z)e^{i(mx+ny)},$$
$$\theta = \Theta(z)e^{i(mx+ny)},$$

we have

$$(1-B)(D^2-a^2)^2 W = 2B''D^2W - B''(D^2-a^2)W + 2B'(D^2-a^2)DW + Ra^2\Theta, \tag{6.13}$$

$$f(D^2-a^2)\Theta = \frac{\xi^2}{4f^3}\Theta - \frac{\xi}{f}D\Theta - \frac{R}{f}W. \tag{6.14}$$

Here $D = d/dz$ and $a^2 = (m^2+n^2)$ is a wave number.

We now solve these equations for

$$Ra_L = \min_{a^2} R^2,$$

where R^2 is the lowest eigenvalue to (6.13) and (6.14), subject to the boundary conditions

$$W = \Theta = D^2W = 0 \text{ on } z = 0, 1.$$

As before we use the compound matrix method and golden section search. The numerical results appear in §6.5.

6.4 Nonlinear analysis

To establish a nonlinear stability result we wish to again use the energy method as in the previous chapters. However, in order to carry out the analysis we require $\Delta\theta = 0$ on the boundaries which is unfortunately not the case when the conductivity is not a constant. In fact, on the boundaries, we can see from (6.9) that

$$\Delta\theta = \frac{\xi Pr}{2Rf}\theta_z^2 - \frac{\xi}{f^2}\theta_z.$$

Hence the following analysis only holds for a class of perturbations for which $\Delta\theta = 0$ on the boundaries. Admittedly this is not dynamically admissable but, if we treat the problem as essentially a heuristic one, the assumption may be justified. Indeed when the numerical analysis is carried out the nonlinear Rayleigh numbers are found to be very close to their linear counterparts.

So, in order to proceed with the nonlinear analysis we continue as follows, with the assumption that $\Delta\theta = 0$ on $z = 0, 1$.
Multiply (6.7) by u_i, (6.9) by θ and integrate over the period cell V. The resulting equations are, after integration by parts,

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|^2 =& -2 < d_{ij}d_{ij}(1-B) > + \frac{2\Gamma Pr^2}{R^2} < d_{ij}d_{ij}\theta^2 > + \frac{8\Gamma Pr}{R\xi} < d_{ij}d_{ij}\theta(1-f) > \\ & + R < \theta w >, \end{aligned} \tag{6.15}$$

$$\begin{aligned} \frac{d}{dt}\frac{Pr}{2}\|\theta\|^2 =& R < \frac{\theta w}{f} > - \frac{\xi^2}{8} < \theta^2 f^{-3} > + \frac{\xi Pr}{2R} < \theta|\nabla\theta|^2 > \\ & - < f|\nabla\theta|^2 > . \end{aligned} \tag{6.16}$$

Now add (6.15) + λ(6.16) where $\lambda(> 0)$ is a coupling parameter to be chosen to the best advantage. We can then write

$$\frac{dE}{dt} = RI - D + N, \tag{6.17}$$

where

$$\begin{aligned} E =& \frac{1}{2}\|\mathbf{u}\|^2 + \frac{\lambda Pr}{2}\|\theta\|^2, \\ I =& < \theta w > + \lambda < \frac{\theta w}{f} >, \\ D =& 2 < d_{ij}d_{ij}(1-B) > + \frac{\xi^2\lambda}{8} < \theta^2 f^{-3} > + \lambda < f|\nabla\theta|^2 >, \\ N =& \frac{2\Gamma Pr^2}{R^2} < d_{ij}d_{ij}\theta^2 > + \frac{8\Gamma Pr}{R\xi} < d_{ij}d_{ij}\theta(1-f) > + \frac{\xi Pr\lambda}{2R} < \theta|\nabla\theta|^2 > . \end{aligned}$$

If we now define

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D} \tag{6.18}$$

where $\mathcal{H}$ is defined as in previous chapters and let $R < R_E$, then (6.17) can be written

$$\frac{dE}{dt} \le -aD + N, \tag{6.19}$$

where $a = (R_E - R)/R_E$.

The Euler equations associated with (6.18) are

$$4(d_{ij}(1-B))_{,j} + R_E M k_i \phi = \varpi_{,i}, \tag{6.20}$$

$$2(f\phi_{,i})_{,i} - \frac{\xi^2}{4f^3}\phi + R_E M w = 0. \tag{6.21}$$

Here $\phi = \sqrt{\lambda}\theta$, $M = \lambda^{-1/2} + f^{-1}\lambda^{1/2}$ and ϖ is a Lagrange multiplier introduced due to the fact that u_i is solenoidal.

In order to solve these equations for the minimum eigenvalue R_E we follow standard procedure, i.e. we take the double curl of (6.20) and then the third component and write

$$\begin{aligned} w =& W(z)e^{i(mx+ny)}, \\ \theta =& \Theta(z)e^{i(mx+ny)}. \end{aligned}$$

Then the resulting equations are

$$\begin{aligned} D^4 W =& \frac{(2a^2(B-1) - B'')}{B-1} D^2 W - \frac{a^2(B'' + a^2(B-1))}{(B-1)} W + \frac{2a^2 B'}{B-1} DW \\ & - \frac{2B'}{B-1} D^3 W - \frac{R_E M a^2}{2(B-1)} \Phi, \qquad (6.22) \\ D^2 \Phi =& a^2 \Phi - \frac{\xi}{2f^2} D\Phi + \frac{\xi^2}{8f^4}\Phi - \frac{R_E M}{2f} W, \qquad (6.23) \end{aligned}$$

where again $D = d/dz$ and $a^2 = m^2 + n^2$. Equations (6.22) and (6.23) will be solved with the use of compound matrices and then optimised using golden section search in order to find

$$Ra_E = \max_{\lambda} \min_{a^2} R_E^2.$$

The numerical results are in §6.5.

We now return to the nonlinear stability analysis and equation (6.19),

$$\frac{dE}{dt} \leq -aD + N.$$

We wish to bound the nonlinearities by D but unfortunately the terms in D would not appear strong enough and so we must use a generalized energy method, as before.

Multiply (6.7) by Δu_i and integrate over V; then, after some integration by parts the equation becomes

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|\nabla \mathbf{u}\|^2 =& - \|\Delta \mathbf{u}\|^2 + 2 < B'\Delta u_i d_{i3} > + \frac{\Gamma Pr^2}{R^2} < \theta^2 |\Delta u_i|^2 > + < B|\Delta u_i|^2 > \\ & + \frac{4\Gamma Pr^2}{R^2} < \Delta u_i \theta \theta_{,j} d_{ij} > + \frac{4\Gamma Pr}{R\xi} < \theta |\Delta u_i|^2 > \\ & - \frac{4\Gamma Pr}{R\xi} < \theta f |\Delta u_i|^2 > + \frac{8\Gamma Pr}{R\xi} < \Delta u_i \theta_{,j} d_{ij} > \qquad (6.24) \\ & - \frac{8\Gamma Pr}{R\xi} < \Delta u_i \theta_{,j} d_{ij} f > - \frac{4\Gamma Pr}{R} < \theta d_{i3} \Delta u_i f^{-1} > \\ & - R < \theta \Delta w > . \end{aligned}$$

Now take the Laplacian of (6.9), then multiply by $\Delta\theta$ and again integrate over V. After integration by parts we can show that the equation becomes

$$\begin{aligned}\frac{d}{dt}\frac{Pr}{2}\|\Delta\theta\|^2 =& - Pr < \theta_{,i}\Delta\theta\Delta u_i > +2Pr < u_{i,j}\theta_{,j}\Delta\theta_{,i} > \\ &+ R < \Delta\theta\Delta(\frac{w}{f}) > +\frac{\xi}{2} < \Delta\theta\Delta(\frac{\theta}{f})' > \\ &- \frac{\xi Pr}{2R} < \Delta\theta\Delta(\theta\theta_{,i})_{,i} > + < \Delta\theta\Delta(f\theta_{,i})_{,i} > . \qquad (6.25)\end{aligned}$$

Considering the 4th and 6th terms in (6.25) we can show, after much integration by parts, that, provided

$$\xi^4 \leq \frac{8\pi^4}{87},$$

(a weak constraint), then the above equation can be manipulated to derive

$$\begin{aligned}\frac{d}{dt}\frac{Pr}{2}\|\Delta\theta\|^2 \leq& - Pr < \theta_{,i}\Delta\theta\Delta u_i > +2Pr < u_{i,j}\theta_{,j}\Delta\theta_{,i} > \\ &+ R < f^{-1}\Delta\theta\Delta w > -R\xi < f^{-3}w_{,z}\Delta\theta > \\ &+ \frac{3R\xi^2}{4} < f^{-5}w\Delta\theta > +\frac{\xi Pr}{2R} < \Delta\theta_{,i}\Delta(\theta\theta_{,i}) > \\ &- \frac{1}{2} < f|\nabla\Delta\theta|^2 > . \qquad (6.26)\end{aligned}$$

These results will be used along with (6.19) to build our generalized energy. We add (6.19) + k_1(6.24) + k_2(6.26) where k_1, k_2 are positive coupling parameters to be chosen to the best advantage. The outcome is the following inequality and definitions:

$$\frac{d\mathcal{E}}{dt} \leq - D_1 + Q + \mathcal{N} \qquad (6.27)$$

where

$$\begin{aligned}D_1 =& 2a < d_{ij}d_{ij}(1 - B) > +\frac{a\xi^2\lambda}{8} < \theta^2 f^{-3} > +a\lambda < f|\nabla\theta|^2 > +k_1\|\Delta\mathbf{u}\|^2 \\ &- k_1 < B|\Delta u_i|^2 > +\frac{1}{2}k_2 < f|\nabla\Delta\theta|^2 > \\ Q =& 2k_1 < B'\Delta u_i d_{i3} > -Rk_1 < \theta\Delta w > +Rk_2 < f^{-1}\Delta\theta\Delta w > \\ &- R\xi k_2 < f^{-3}w_{,z}\Delta\theta > +\frac{3R\xi^2 k_2}{4} < f^{-5}w\Delta\theta > \\ \mathcal{E} =& \frac{1}{2}\|\mathbf{u}\|^2 + \frac{\lambda Pr}{2}\|\theta\|^2 + \frac{k_1}{2}\|\nabla\mathbf{u}\|^2 + \frac{k_2 Pr}{2}\|\Delta\theta\|^2\end{aligned}$$

$$\begin{aligned}\mathcal{N} =&\frac{2\Gamma Pr^2}{R^2} < d_{ij}d_{ij}\theta^2 > + \frac{8\Gamma Pr}{R\xi} < d_{ij}d_{ij}\theta(1-f) > + \frac{\xi Pr\lambda}{2R} < \theta|\nabla\theta|^2 > \\ &+ \frac{\Gamma Pr^2 k_1}{R^2} < \theta^2|\Delta u_i|^2 > + \frac{4\Gamma Pr^2 k_1}{R^2} < \Delta u_i \theta\theta_{,j} d_{ij} > + \frac{4\Gamma Pr k_1}{R\xi} < \theta|\Delta u_i|^2 > \\ &- \frac{4\Gamma Pr k_1}{R\xi} < \theta f|\Delta u_i|^2 > + \frac{8\Gamma Pr k_1}{R\xi} < \Delta u_i \theta_{,j} d_{ij} > - \frac{8\Gamma Pr k_1}{R\xi} < f\Delta u_i d_{ij}\theta_{,j} > \\ &- \frac{4\Gamma Pr k_1}{R} < \theta d_{i3}\Delta u_i f^{-1} > - Pr k_2 < \theta_{,i}\Delta\theta\Delta u_i > + 2Pr k_2 < u_{i,j}\theta_{,j}\Delta\theta_{,i} > \\ &+ \frac{\xi Pr k_2}{2R} < \Delta\theta_{,i}\Delta(\theta\theta_{,i}) > .\end{aligned}$$

After some intricate analysis, cf. Chapters 4 and 5, (6.27) can be written

$$\frac{d\mathcal{E}}{dt} \le -\frac{\mathcal{D}}{2} + A\mathcal{D}\mathcal{E}^{1/2} + B\mathcal{D}\mathcal{E}, \tag{6.28}$$

where

$$\begin{aligned}\mathcal{D} =& ak_3\|\nabla\mathbf{u}\|^2 + \frac{\xi^2 a\lambda}{8} < \theta^2 f^{-3} > + a\lambda\|\nabla\theta\|^2 \\ &+ k_1k_3\|\Delta\mathbf{u}\|^2 + \frac{k_2}{2}\|\nabla\Delta\theta\|^2, \\ A =& \frac{4c\Gamma \sup|1-f|(2Pr)^{1/2}}{R\xi a k_3\sqrt{k_2}} + \frac{\xi c}{Ra}\sqrt{\frac{Pr}{2k_2}} + \frac{4\Gamma c \sup|1-f|}{R\xi k_3}\sqrt{\frac{2Pr}{k_2}} \\ &+ \frac{16\Gamma Prc\sup|1-f|}{R\xi\sqrt{k_2k_3}} + \frac{4\Gamma c(2Prk_1)^{1/2}}{Rk_3\sqrt{ak_2}} + 2c\sqrt{\frac{Pr}{k_1k_3}} \\ &+ \frac{4Prc2^{1/2}}{k_1^{1/2}} + \frac{4\xi Prc}{R\pi}\sqrt{\frac{2}{k_2}}, \\ B =& \frac{2\Gamma Prc^2}{R^2ak_2k_3} + \frac{2\Gamma Prc^2}{R^2k_2k_3} + \frac{8\Gamma Pr^{3/2}c^2 2^{1/2}}{R^2k_2\sqrt{k_3}},\end{aligned}$$

(k_3 being a positive constant). We can now show as in Chapters 2, 4 and 5, that, provided

$$(1)\ R < R_E \quad \text{and} \quad (2)\ \frac{1}{2} - A\mathcal{E}^{1/2}(0) - B\mathcal{E}(0) > 0$$

we have conditional nonlinear stability.

Let $K = 1/2 - A\mathcal{E}^{1/2}(0) - B\mathcal{E}(0)$. Then, remembering that there exists a constant ζ, such that $\mathcal{D} \ge \zeta\mathcal{E}$ we can see that

$$\frac{d\mathcal{E}}{dt} \le -K\zeta\mathcal{E}.$$

This equation can be integrated to give

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-K\varsigma t}$$

and so $\mathcal{E}(t) \to 0$ as $t \to \infty$.

6.5 Numerical results

We carried out a coupled parametric analysis of Γ, a measure of the viscosity, and ξ, a measure of the conductivity. We varied ξ between 0.1 and 0.5. It was found that increasing ξ meant that the Rayleigh number was increased. The net effect of adding in a variable viscosity was to decrease the Rayleigh number, that is,

$$Ra(\xi = 0.1, \Gamma = 0.1) > Ra(\xi = 0.1, \Gamma = 0.2).$$

Thankfully when Γ and ξ were both zero the Rayleigh numbers were equivalent and $= 27\pi^4/4$, the classical Bénard result. Note that in all cases the nonlinear stability threshold was very close to the linear instability threshold. Also λ seems to depend only on ξ, not on Γ, which is explained by the fact that when only the viscosity varies with temperature (Chapters 4 & 5) then $\lambda = 1$ and $Ra_L = Ra_E$.

Critical Rayleigh numbers Ra_L and Ra_E
with their respective critical wave numbers
a_L and a_E.

Table 1. $\Gamma = 0$. (variable conductivity only.)

ξ	Ra_L	Ra_E	a_L^2	a_E^2	λ
0.0	657.512	657.512	4.935	4.935	1.000
0.1	690.207	690.167	4.935	4.935	1.024
0.2	722.572	722.422	4.935	4.935	1.048
0.3	754.648	754.324	4.936	4.935	1.071
0.4	786.471	785.916	4.936	4.935	1.093
0.5	818.067	817.232	4.937	4.935	1.114

Table 2. $\Gamma = 0.1$.

ξ	Ra_L	Ra_E	a_L^2	a_E^2	λ
0.1	665.350	665.312	4.956	4.956	1.025
0.2	697.611	697.466	4.955	4.954	1.048
0.3	729.582	729.269	4.953	4.953	1.071
0.4	761.298	760.762	4.953	4.952	1.093
0.5	792.787	791.978	4.952	4.950	1.115

Table 3. $\Gamma = 0.2$.

ξ	Ra_L	Ra_E	a_L^2	a_E^2	λ
0.1	639.976	639.940	4.979	4.979	1.025
0.2	672.157	672.018	4.976	4.975	1.048
0.3	704.044	703.742	4.973	4.972	1.071
0.4	735.671	735.154	4.970	4.969	1.093
0.5	767.070	766.289	4.968	4.967	1.115

Table 4. $\Gamma = 0.4$.

ξ	Ra_L	Ra_E	a_L^2	a_E^2	λ
0.1	587.344	587.311	5.030	5.031	1.025
0.2	619.468	619.340	5.022	5.022	1.049
0.3	651.276	650.997	5.016	5.015	1.072
0.4	682.808	682.328	5.010	5.009	1.094
0.5	714.097	713.371	5.005	5.003	1.116

Table 5. $\Gamma = 0.5$.

ξ	Ra_L	Ra_E	a_L^2	a_E^2	λ
0.1	559.852	559.821	5.061	5.061	1.025
0.2	592.028	591.905	5.049	5.049	1.049
0.3	623.863	623.596	5.040	5.040	1.072
0.4	655.405	654.945	5.032	5.031	1.094
0.5	686.669	685.993	5.026	5.024	1.116

CHAPTER SEVEN

POROUS PENETRATIVE CONVECTION WITH A SALT FIELD AND INTERNAL HEAT SOURCE.

7.1 Introduction

Previously the problems studied have concerned a layer of fluid but another system with just as much physical interest is that of porous convection, i.e. when we have a layer of some porous material permeated by a fluid. This mechanism is important when considering certain geophysical situations such as patterned ground formation due to thawing permafrost (cf. Ray *et al* (1983)) and convection within the Earth's mantle. The remaining chapters deal with porous convection under various different conditions. Here we will look at porous convection in a layer containing an internal heat source with a salt field and impose a density with quadratic temperature dependence on the fluid.

Penetrative convection occurs when a layer of fluid, heated in some suitable manner, can be seperated into stable and unstable regions. When conditions are right the convection which occurs in the unstable region then penetrates into the stable region, hence the name. There are at least two possible ways in which to create these areas of instability within a stable layer. One is to include an internal heat source and the other is to use a nonlinear buoyancy law e.g. to adopt an equation of state in the buoyancy term like

$$\rho = \rho_0[1 - \alpha(T - T_R)^2],$$

where T is the temperature and T_R is a reference temperature. If we consider a layer whose lower boundary is maintained at a temperature lower than T_R and whose upper boundary is maintained at a temperature greater than T_R, then, due to the quadratic term, we can create a layer of gravitationally stable fluid above a layer of gravitationally unstable fluid. Convection will occur in the lower layer and penetrate into the stable upper layer.

On the other hand, if an internal heat source is introduced into any type of layer, then we can create an unstable area sandwiched between two stable areas.

Again, convection which occurs in the unstable region will penetrate into the stable regions.

Convection of this type is very useful when modelling certain geophysical problems. For example, it can be used when studying convection within the Earth's mantle. It is known that the forces which maintain the movement of the plates forming the Earth's crust are governed by the equations of fluid dynamics, although these equations are difficult to determine. One possible mechanism suggested to explain these forces is that of heat generated by radiation within the mantle, (see McKenzie, Roberts and Weiss (1974)). A known energy source within the mantle is the heat generated by the radioactive decay of $^{40}K, ^{235}U, ^{238}U, ^{232}Th$. The total energy released by this, when compared to the energy released by earthquakes, shows that the efficiency of the conversion of heat into seismic energy need only be $\leq 1\%$. Hence much work on this problem has used this approach, (see Hales (1936), Runcorn (1965), Vening Meinesz (1962) and Knopoff (1964)). Obviously a model which includes an internal heat source could be used to give an idea of the mechanics of convection within the Earth's mantle.

A nonlinear buoyancy law, however, is useful when investigating a fluid which exhibits a density maximum e.g. water, antimony, bismuth, gallium or liquid helium.

The phenomenon of penetrative convection is prevalent in stellar regions and both nonlinear bouyancy laws and internal heat sources are employed to model this situation, see e.g. Veronis (1963). Other geophysical problems where penetrative convection in a porous layer is involved include modelling thawing subsea permafrost (see e.g. Payne *et al* (1988)) or patterned ground formation (see e.g. Ray *et al* (1983), Gleason (1984), George *et al* (1989)).

There are many other areas of research in which penetrative convection is involved, Veronis (1963) cites many instances, and others are reviewed in the monograph of Straughan (1993), but those detailed above help to motivate our particular choice of conditions.

We study the combined effect of a salt field and an internal heat source on the motion of an incompressible heat-conducting viscous fluid in a porous layer. We use a nonlinear buoyancy law, specifically

$$\rho = \rho_0[1 - \alpha(T - T_R)^2 + \alpha_s(S - S_R)],$$

and choose to study water as our fluid with $T_R = 4°C$. If we consider a layer which has a lower boundary maintained at $0°C$ and an upper boundary whose temperature is greater than $4°C$ then, due to water having a maximum density at $4°C$, we have divided our layer into two parts, one stable and the other unstable, as described above. Hence two conditions for penetrative convection, a nonlinear buoyancy law and an internal heat source, are present. A destabilizing salt field is chosen and we take three different types of heat source. Our analysis involves calculation of both linear and nonlinear Rayleigh numbers. For the nonlinear analysis we calculate both conditional and unconditional numbers. For the unconditional nonlinear analysis we are required to use a "weighted" energy.

The boundary conditions for a fluid saturated porous medium are as given in Joseph (1976). We shall regard the normal velocity and the temperature as prescribed at the boundary. In a porous medium the fluid will stick to a solid wall but this effect is confined to a boundary layer whose size is measured in pore diameters. As the wall friction does not overtly influence the motion in the interior it is reasonable to replace the true wall with a free surface.

This chapter has essentially appeared in Richardson (1992b).

7.2 The governing equations

We study the problem of the onset of convection in a porous medium, permeated by some incompressible fluid, in this case water, containing a heat source in which a destabilizing salt field and a stabilizing temperature gradient are present. An infinite horizontal layer where $z \in (0, d)$ is considered and the density is chosen to reflect the temperature of the layer, namely

$$\rho = \rho_0[1 - \alpha(T - T_R)^2 + \alpha_s(S - S_R)],$$

where $\alpha \simeq 7.68\text{x}10^{-6}(°C^{-2})$ in the salt free situation as given by Veronis (1963) and $T_R = 4°C$. In the Oberbeck-Boussinesq equations of motion the Δu_i term is replaced by one involving only u_i; this is Darcy's Law (cf. Joseph (1976)) which is largely based on a generalization of empirical observations and is commonly used when modelling porous convection. Additionally omitting the inertia in the momentum equation, the relevant equations are then

$$\frac{1}{\rho_0}p_{,i} = -gk_i[1-\alpha(T-T_R)^2+\alpha_s(S-S_R)] - \frac{\mu}{k\rho_0}v_i, \quad (7.1)$$
$$T_{,t}+v_iT_{,i} = \kappa\Delta T + Q(z), \quad (7.2)$$
$$S_{,t}+\delta v_iS_{,i} = \kappa_s\Delta S, \quad (7.3)$$

together with the boundary conditions

$$\begin{aligned} T(0) &= T_l \qquad S(0) = S_l,\\ T(d) &= T_u \qquad S(d) = S_u,\\ W &= 0 \text{ at } z = 0, d, \end{aligned} \quad (7.4)$$

where $S_u > S_l$, $T_l \cong 0$ and $T_u > 4$.

By choosing the boundary temperatures in this way we are dividing the layer into a stable and an unstable region, due to the fact that below $4°C$ the density of water is a decreasing function of temperature. This situation raises the possibility of convection penetrating into the stable region.

Here ρ_0 is the density at reference temperature and salt concentration, p is the pressure field, α_s the salt expansion coefficient, μ, k the (constant) dynamic viscosity and permeability, κ, κ_s the thermal and salt diffusivities and S and T are the temperature and salt concentrations. We take $\mathbf{k} = (0,0,1)$ to be the unit vector in the z direction and t is $(time)\times(\rho_0 c_o)_f/(\rho_0 c_o)_m$. $Q(z)$ is the heat source and $\delta = (c_0\rho_0)_m/\phi(c_0\rho_0)_f$ where ϕ is the porosity which we assume constant, c_0 is the specific heat, f refers to the fluid and m to the matrix. Specifically

$$\phi = \frac{\text{fluid volume}}{\text{total volume}}.$$

Three heat supply functions are considered;

$$\begin{aligned} &\text{I.} & Q &= Q(\text{constant}),\\ &\text{II.} & Q &= Q(e^{z/d}-1),\\ &\text{III.} & Q &= Q\sin(2\pi z/d). \end{aligned} \quad (7.5)$$

Note that the third function heats the lower half of the layer whilst cooling the rest.

We consider a steady solution $(\bar{\mathbf{v}}, \bar{T}, \bar{S}, \bar{p})$ where $\bar{\mathbf{v}} \equiv 0$ and $\bar{T}$, $\bar{S}$ are functions of z. Then (7.1)–(7.3) become

$$\frac{1}{\rho_0}\bar{p}_{,i} = -\, g k_i[1 - \alpha(\bar{T} - T_R)^2 + \alpha_s(\bar{S} - S_R)], \tag{7.6}$$

$$\bar{T}'' = -\,\frac{1}{\kappa}Q, \tag{7.7}$$

$$\bar{S}'' = 0. \tag{7.8}$$

Then from (7.8) and (7.4) we see that $\bar{S} = \beta_s z + S_l$ where $\beta_s = (S_u - S_l)/d > 0$; (7.7) yields, utilising (7.4) and (7.5),

$$\begin{aligned} \text{I.} \quad \bar{T} =& \frac{Qd^2}{\kappa}\Big(\frac{z}{2d} - \frac{z^2}{2d^2}\Big) + \frac{z}{d}(T_u - T_l) + T_l, \\ \text{II.} \quad \bar{T} =& \frac{Qd^2}{\kappa}\Big(1 + \frac{z^2}{2d^2} - e^{z/d} + \frac{z}{d}(e - 3/2)\Big) \\ & + \frac{z}{d}(T_u - T_l) + T_l, \\ \text{III.} \quad \bar{T} =& \frac{Qd^2}{4\pi^2\kappa}\sin 2\pi z/d + \frac{z}{d}(T_u - T_l) + T_l, \end{aligned} \tag{7.9}$$

and from (7.6) we see that $\bar{p}$ is a function of z, specifically

$$\frac{d\bar{p}}{dz} = -g\rho_0[1 - \alpha(\bar{T} - \bar{T}_R)^2 + \alpha_s(\bar{S} - S_R)].$$

To investigate the stability of these solutions we introduce perturbations via

$$v_i = \bar{v}_i + u_i, \qquad T = \bar{T} + \theta, \qquad S = \bar{S} + \phi, \qquad p = \bar{p} + \pi.$$

The perturbation equations are then non-dimensionalized according to the scales

$$t = \frac{t^* d}{U}, \qquad \mathbf{x} = \mathbf{x}^* d, \qquad \theta = T^{\#}\theta^*, \qquad \mathbf{u} = \mathbf{u}^* U,$$

$$Pr = \frac{dU}{\kappa}, \qquad \pi = p^* P, \qquad P = \frac{\mu U d}{k}, \qquad Ps = \frac{dU}{\kappa_s},$$

$$\phi = S^{\#}\phi^*, \qquad Q = \frac{\kappa T_u}{d^2}Q^*, \qquad \xi = \frac{T_R}{T_u},$$

$$T^{\#} = U\sqrt{\frac{d\mu}{\kappa k \rho_0 \alpha g}}, \qquad S^{\#} = U\sqrt{\frac{\mu d^2 \delta \beta_s}{k\rho_0 g \alpha_s \kappa_s}},$$

$$R = T_u\sqrt{\frac{k\rho_0 g \alpha d}{\mu\kappa}}, \qquad R_s = \sqrt{\frac{k\rho_0 g \alpha_s d^2 \delta\beta_s}{\mu\kappa_s}}.$$

Here Pr is the Prandtl number, Ps the salt Prandtl number and R^2 and R_s^2 are the Rayleigh and salt Rayleigh numbers respectively. Omitting all stars the *non-dimensional* equations are

$$p_{,i} = 2k_i R\theta(g(z) + z - \xi) + Prk_i\theta^2 - R_s k_i \phi - u_i, \qquad (7.10)$$

$$Pr(\theta_{,t} + u_i\theta_{,i}) = \Delta\theta - Rw(1 + h(z)), \qquad (7.11)$$

$$Ps(\phi_{,t} + \delta u_i\phi_{,i}) = \Delta\phi - R_s w, \qquad (7.12)$$

where

$$g(z) = \begin{cases} Q(z - z^2)/2 & \text{I.} \\ Q(1 + \frac{z^2}{2} - e^z + ez - \frac{3}{2}z) & \text{II.} \\ (Q\sin 2\pi z)/4\pi^2 & \text{III.} \end{cases} \qquad (7.13)$$

and

$$h(z) = \begin{cases} Q(1 - 2z)/2 & \text{I.} \\ Q(z - e^z + e - \frac{3}{2}) & \text{II.} \\ (Q\cos 2\pi z)/2\pi & \text{III.} \end{cases} \qquad (7.14)$$

The boundary conditions which are applicable are a matter of contention in a porous medium. A clear account is provided in §§1.5, 1.6 of Nield & Bejan (1992). When there is no slip at the boundary we may take

$$u = v = w = 0 \qquad \text{on} \qquad z = 0, 1.$$

However the situation which we wish to consider is the one which arises when the porous medium is free at the boundary. Here the situation is not so clear. Nevertheless the usual stress free boundary conditions should hold, viz.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \qquad \text{on} \qquad z = 0, 1. \qquad (7.15a)$$

We shall also assume u, v, w, θ and p are periodic on the x, y boundaries of the convection cell V. The temperature and salt perturbations also satisfy

$$\phi = \theta = 0 \qquad \text{on} \qquad z = 0, 1. \qquad (7.15b)$$

7.3 Linear instability analysis

The study of linear instability of this system follows standard procedures. First we let $R_s = \alpha R$ so that we have only one set of eigenvalues to consider and then, on linearizing (7.10)–(7.12), substituting

$$\begin{aligned} u(\mathbf{x},t) =& u(\mathbf{x})e^{\sigma t}, \\ \theta(\mathbf{x},t) =& \theta(\mathbf{x})e^{\sigma t}, \\ \phi(\mathbf{x},t) =& \phi(\mathbf{x})e^{\sigma t}, \end{aligned}$$

and taking $\sigma = 0$ we have,

$$p_{,i} = 2k_i R\theta(g + z - \xi) - \alpha R k_i \phi - u_i, \tag{7.16}$$

$$0 = \Delta\theta - Rw(1 + h), \tag{7.17}$$

$$0 = \Delta\phi - \alpha R w. \tag{7.18}$$

Here we are considering stationary convection ($\sigma = 0$) as we expect the linear and nonlinear Rayleigh numbers to be close, thus minimising the region of possible subcritical instabilities. Our justification for this assumption is that exchange of stabilities holds for both simple penetrative convection cf. Veronis (1963) and for penetrative convection with a salt field cf. Payne *et al* (1988).

Now take (curlcurl) of equation (7.16), take the third component and introduce normal modes, i.e

$$\begin{aligned} w =& W(z)e^{i(mx+ny)}, \\ \theta =& \Theta(z)e^{i(mx+ny)}, \\ \phi =& \Phi(z)e^{i(mx+ny)}. \end{aligned}$$

The equations now become

$$0 = (D^2 - a^2)W - \alpha a^2 R\Phi + 2a^2 R(g + z - \xi)\Theta, \tag{7.19}$$

$$0 = (D^2 - a^2)\Theta - RW(1 + h), \tag{7.20}$$

$$0 = (D^2 - a^2)\Phi - \alpha R W, \tag{7.21}$$

together with the boundary conditions

$$W = \Theta = \Phi = 0 \text{ at } z = 0, 1.$$

Then the critical Rayleigh number of linear instability theory, Ra_L, is defined as

$$Ra_L = \min_{a^2} R^2$$

where R^2 was found numerically from (7.19)–(7.21) by employing the compound matrix method and the minimisation was effected using golden section search as in previous chapters. The numerical results are presented in §7.6.

7.4 Conditional nonlinear stability

In order to study the nonlinear stability of the system we once again employ a generalized energy argument.

To begin, multiply (7.10) by u_i, (7.11) by $\lambda\theta$ and (7.12) by $\mu\phi$, where $\lambda(>0)$ and $\mu(>0)$ are coupling parameters to be chosen to the best advantage. Next we integrate the resulting equations over the period cell V. This yields

$$\begin{aligned} 0 =& -\|\mathbf{u}\|^2 - \alpha R < \phi w > + 2R < (g + z - \xi)\theta w > \\ & + Pr < w\theta^2 >, \qquad (7.22)\\ \frac{d}{dt}\frac{1}{2}\lambda Pr\|\theta\|^2 =& -\lambda\|\nabla\theta\|^2 - R\lambda < \theta w(1+h) >, \qquad (7.23)\\ \frac{d}{dt}\frac{1}{2}\mu Ps\|\phi\|^2 =& -\mu\|\nabla\phi\|^2 - \alpha R\mu < \phi w >, \qquad (7.24)\end{aligned}$$

where again we have substituted $R_s = \alpha R$.

Adding equations (7.22)–(7.24) results in

$$\frac{dE}{dt} = -D + RI + Pr < w\theta^2 >, \qquad (7.25)$$

where

$$\begin{aligned} E =& \frac{1}{2}Pr\lambda\|\theta\|^2 + \frac{1}{2}Ps\mu\|\phi\|^2, \\ D =& \|\mathbf{u}\|^2 + \lambda\|\nabla\theta\|^2 + \mu\|\nabla\phi\|^2, \qquad (7.26)\\ I =& < (2(g + z - \xi) - \lambda(1+h))\theta w > -\alpha(1+\mu) < \phi w > . \end{aligned}$$

From (7.25) it now follows that

$$\frac{dE}{dt} \le -RD\Big(\frac{1}{R} - \frac{1}{R_E}\Big) + Pr < w\theta^2 >, \qquad (7.27)$$

where R_E is defined by

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D}, \qquad (7.28)$$

($\mathcal{H}$ being defined as before).

Set $a = (R_E - R)/R_E$ and suppose $R < R_E$. Then $a > 0$ and (7.27) becomes

$$\frac{dE}{dt} \leq -aD + Pr < w\theta^2 > . \tag{7.29}$$

We wish now to bound the $< \theta^2 w >$ term in (7.29) by the energy but it is not evident how to do this as $\|w\|$ does not appear in E. Therefore we try to utilise the θ^2 part in some way.

Define $\psi \equiv \theta^2$ and, by multiplying (7.11) by θ^3, it is possible to see that

$$\frac{d}{dt}\frac{1}{4}Pr\|\psi\|^2 = -\frac{3}{4}\|\nabla\psi\|^2 - R < \theta\psi w(1+h) > . \tag{7.30}$$

Let $E_1 = \frac{1}{4}Pr\|\psi\|^2$ and form the combination (7.29) + τ(7.30) for $\tau(> 0)$ a coupling parameter to be chosen.
Then

$$\frac{d\mathcal{E}}{dt} \leq -aD + Pr < \psi w > -\frac{3\tau}{4}\|\nabla\psi\|^2 - R\tau < \theta\psi w(1+h) >, \tag{7.31}$$

where $\mathcal{E}(t) = E(t) + \tau E_1 = \frac{1}{2}Pr\lambda\|\theta\|^2 + \frac{1}{2}Ps\mu\|\phi\|^2 + \frac{1}{2}Pr\tau\|\psi\|^2$ is our generalized energy.

To handle the last term on the right-hand side of (7.31) we require the Sobolev inequality

$$< \psi^4 >^{1/4} \leq c\|\nabla\psi\|^{3/4}\|\psi\|^{1/4},$$

as used in Chapter 2. Then, by repeatedly using the Cauchy-Schwarz inequality we have

$$\begin{aligned} < \theta w\psi(1+h) > &\leq \sup_V |1+h| \, | < \theta w\psi > |, \\ &\leq A\|w\|\Big(\int_V \theta^2\psi^2 dV\Big)^{1/2}, \\ &\leq A\|w\|\Big(< \theta^4 >^{1/2} < \psi^4 >^{1/2}\Big)^{1/2}, \\ &= A\|w\| < \psi^2 >^{1/4} < \psi^4 >^{1/4}, \\ &\leq Ac\|w\|\|\psi\|^{3/4}\|\nabla\psi\|^{3/4}, \end{aligned}$$

where $A = \sup_V |1+h|$.
Hence

$$R\tau < \theta w\psi(1+h) > \leq RA\tau c\|w\|\|\psi\|^{3/4}\|\nabla\psi\|^{3/4}. \tag{7.32}$$

Furthermore, use of the arithmetic-geometric mean inequality together with

Poincaré's inequality on the $<\psi w>$ term in (7.31) yields

$$\begin{aligned}<\psi w> &\le \frac{1}{2\alpha}\|\psi\|^2 + \frac{\alpha}{2}\|w\|^2 \\ &\le \frac{1}{2\alpha\lambda_1}\|\nabla\psi\|^2 + \frac{\alpha}{2}\|w\|^2,\end{aligned}$$

where α is a parameter to be chosen later and λ_1 is a constant arising from the use of Poincaré's inequality. Therefore (7.31) now becomes

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &\le -aD + \frac{Pr}{2\alpha\lambda_1}D(\psi) + \frac{Pr\alpha}{2}\|w\|^2 - \frac{3\tau}{4}\|\nabla\psi\|^2 - RA\tau c\|w\|\|\psi\|^{3/4}\|\nabla\psi\|^{3/4} \\ &\le -aD + \frac{Pr}{2\alpha\lambda_1}D(\psi) + \frac{Pr\alpha}{2}D - \frac{3\tau}{4}D(\psi) - RA\tau c\|w\|\|\psi\|^{3/4}\|\nabla\psi\|^{3/4} \quad (7.33)\end{aligned}$$

where $D(\psi) = \|\nabla\psi\|^2$. As the parameters α and τ are at our disposal we may now choose

$$\frac{Pr\alpha}{2} = \frac{a}{2} \quad \text{and } \tau = \frac{Pr}{\alpha\lambda_1}.$$

Then

$$\frac{d\mathcal{E}}{dt} \le -\frac{a}{2}D - \frac{Pr}{4\alpha\lambda_1}D(\psi) - RA\tau c\|w\|\|\psi\|^{3/4}\|\nabla\psi\|^{3/4}. \qquad (7.34)$$

Now define $\mathcal{D} = D + D(\psi) = \|\mathbf{u}\|^2 + \lambda\|\nabla\theta\|^2 + \mu\|\nabla\phi\|^2 + \|\nabla\psi\|^2$ and let

$$b = \min\left\{\frac{a}{2}, \frac{Pr}{4\alpha\lambda_1}\right\}.$$

We can now bound (7.34) in terms of $\mathcal{D}^{7/8}\mathcal{E}^{3/8}$ as follows,

$$\frac{d\mathcal{E}}{dt} \le -b\mathcal{D} - RAc\frac{2^{3/8}\tau^{5/8}}{Pr^{3/8}}\mathcal{D}^{7/8}\mathcal{E}^{3/8}, \qquad (7.35)$$

and if we take $c_2 = -RAc2^{3/8}\tau^{5/8}/Pr^{3/8}$, then

$$\frac{d\mathcal{E}}{dt} \le -b\mathcal{D} + c_2\mathcal{D}^{7/8}\mathcal{E}^{3/8}. \qquad (7.36)$$

Since we may use Poincaré's inequality to show $\exists$ a constant η s.t. $\mathcal{D} \ge \eta\mathcal{E}$, it is straightforward to conclude from (7.36), cf. Chapter 2 that, provided

$$(A)\ R < R_E, \qquad \text{and} \qquad (B)\ \mathcal{E}(0)^{1/4} < \left(\frac{b\eta^{1/8}}{c_2}\right), \qquad (7.37)$$

then $\mathcal{E}(t) \to 0$ at least exponentially as $t \to \infty$. For, as before, we can integrate (7.36) to produce

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{-K\eta t}$$

where $K = b - c_2\mathcal{E}^{1/4}(0)/\eta^{1/8}$.
Thus we have shown that the stationary solution is nonlinearly stable provided R satisfies the restriction (7.37)(A) and the initial energy satisfies (7.37)(B).

It remains however to determine R_E. To this end define

$$N(z;\lambda,\xi,Q) = \frac{2(g+z-\xi)-\lambda(1+h)}{\sqrt{\lambda}}, \qquad F(\mu,\alpha) = \frac{\alpha(1+\mu)}{\sqrt{\mu}}. \tag{7.38}$$

Set $\sqrt{\lambda}\theta = \theta^*$ and $\sqrt{\mu}\phi = \phi^*$ and the Euler-Lagrange equations derived from the maximum problem (7.28) are, (ignoring asterisks),

$$R_E k_i(N\theta - F\phi) - 2u_i = \varpi_{,i}, \tag{7.39}$$

$$2\Delta\theta + R_E N w = 0, \tag{7.40}$$

$$2\Delta\phi - R_E F w = 0, \tag{7.41}$$

where ϖ is a Lagrange multiplier introduced since $\mathbf{u}$ is solenoidal. We solve this system numerically for the first eigenvalue R_E, but before doing so we may obtain useful information by using the technique of parametric differentiation, cf. Chapter 2.

The eigenvalue R_E depends on the coupling parameters μ and λ and we form an expression for the derivative of each of these.
Firstly we find

$$< Nw\theta > \frac{\partial R_E}{\partial \mu} - < w\phi > \left[R_E \frac{\partial F}{\partial \mu} + F\frac{\partial R_E}{\partial \mu}\right] = 0. \tag{7.42}$$

We know that for R_E to obtain a maximum in μ, $\partial R_E/\partial\mu = 0$ and so (7.42) tells us that $\partial F/\partial\mu = 0$. Therefore we have that

$$\bar{\mu} = 1,$$

where $\bar{\mu}$ is the best value of μ. Hence $F = 2\alpha$ and we take this value henceforth.
Partial differentiation with respect to λ results in

$$2 < w\phi > \frac{\partial R_E}{\partial\lambda} = < w\theta(N\frac{\partial R_E}{\partial\lambda} + R_E\frac{\partial N}{\partial\lambda}) > . \tag{7.43}$$

Again we have that $\partial R_E/\partial\lambda = 0$ and so

$$< w\theta R_E \frac{\partial N}{\partial\lambda} >= 0. \tag{7.44}$$

This expression is useful as it gives us an idea as to where to search for λ. If we find $\partial N/\partial\lambda$, integrate it with respect to z from 0 to 1 and equate to zero we find that, for the three different heat sources,

$$\begin{aligned} \text{I.} \qquad \lambda &= 2\xi - 1 - \frac{Q}{6}, \\ \text{II.} \qquad \lambda &= 2\xi - 1 + Qe - \frac{17}{6}Q, \\ \text{III.} \qquad \lambda &= 2\xi - 1. \end{aligned} \tag{7.45}$$

These estimates are in fact remarkably accurate as can be seen by looking at the tables of numerical results.

To solve (7.39)–(7.41) numerically we take curlcurl of (7.39) and then take the third component of the result. Employing normal modes, as in §7.3, with $D = d/dz$ and $a^2 = (\text{wavenumber})^2$ we now wish to solve

$$\begin{aligned} (D^2 - a^2)W &= R_E a^2 \alpha \Phi - \frac{1}{2} R_E N a^2 \Theta, \\ (D^2 - a^2)\Theta &= -\frac{1}{2} R_E N W, \\ (D^2 - a^2)\Phi &= R_E \alpha W, \end{aligned}$$

for the eigenvalue R_E with $W = \Phi = \Theta = 0$ at $z = 0, 1$. We wish to find

$$Ra_E = \max_{\lambda > 0} \min_{a^2} R_E^2$$

numerically. To do this both the compound matrix method and golden section search are again used. The results are in §7.6.

7.5 Unconditional nonlinear stability

In §7.4 the conditional restriction (7.37)(B) is undesirable. By introducing a spatial weight into the thermal part of the energy, nonlinear stability may be recovered for arbitrarily sized initial perturbations, cf. Payne and Straughan (1987). Therefore our analysis is now continued along these lines.

Define

$$E_2(t) = \frac{Pr}{2} < (\eta - 2z)\theta^2 >$$

where $\eta > 2$ is a constant. Then, by direct calculation and use of (7.11), we have

$$\frac{dE_2}{dt} = - < |\nabla\theta|^2 (\eta - 2z) > - Pr < w\theta^2 > - R < w\theta(\eta - 2z)(1 + h) > . \tag{7.46}$$

Similarly, as in §7.4, multiply (7.10) by u_i, (7.12) by $\sigma\phi$, integrate over a period cell V and add, together with (7.46). This yields

$$\frac{d\mathcal{E}}{dt} = -\,[\|\mathbf{u}\|^2 + \|\nabla\phi\|^2 + < |\nabla\theta|^2(\eta - 2z) >] \tag{7.47}$$
$$+ R[2 < \theta w(g + z - \xi) > - < w\theta(\eta - 2z)(1 + h) > - \frac{\alpha(1+\sigma)}{\sqrt{\sigma}} < \phi w >],$$

where $\mathcal{E} = \frac{1}{2}\sigma Ps\|\phi\|^2 + E_2 = \frac{1}{2}\sigma Ps\|\phi\|^2 + \frac{1}{2}Pr < (\eta - 2z)\theta^2 >$ and we have put $\sqrt{\sigma}\phi = \phi^*$ and ignored the star. Again $\sigma(> 0)$ is a coupling parameter to be chosen to the best advantage.

Comparing (7.47) with the corresponding equation in §7.4 we can see that parametric differentiation will yield $\sigma = 1$, and so we take this value from now on. Therefore we have

$$\frac{d\mathcal{E}}{dt} = -\,D + IR, \tag{7.48}$$

where

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}Ps\|\phi\|^2 + \frac{1}{2}Pr < (\eta - 2z)\theta^2 >, \\ D &= \|\mathbf{u}\|^2 + \|\nabla\phi\|^2 + < |\nabla\theta|^2(\eta - 2z) >, \\ I &= < \theta w(2(g + z - \xi) - (\eta - 2z)(1 + h)) > -2\alpha < \phi w > . \end{aligned} \tag{7.49}$$

Let $\hat{a} = (R_W - R)/R_W$ where R_W is defined by

$$\frac{1}{R_W} = \max_{\mathcal{H}} \frac{I}{D}, \tag{7.50}$$

$\mathcal{H}$ being the set of admissible functions, and, provided $R < R_W$, (7.48) becomes

$$\frac{d\mathcal{E}}{dt} \leq -\hat{a}D. \tag{7.51}$$

Then, by the use of a weighted Poincaré inequality on the right-hand side of (7.51), we may deduce that

$$\frac{d}{dt}\left[\frac{1}{2}Ps\|\phi\|^2 + \frac{1}{2}Pr < (\eta - 2z)\theta^2 >\right] \leq -\,k[Ps\|\phi\|^2 + Pr < (\eta - 2z)\theta^2 > + \|\mathbf{u}\|^2], \tag{7.52}$$

where k is a constant. This establishes at least exponential decay of $\|\phi\|$ and $\|\theta\|$, for arbitrarily large initial disturbances.

It remains to find the weighted critical Rayleigh number R_W^2. For this the Euler-Lagrange equations derived from (7.50) are

$$R_W(m\theta - 2\alpha\phi)k_i - 2u_i = \pi_{,i}, \qquad (7.53)$$

$$2\Delta\phi - 2\alpha R_W w = 0, \qquad (7.54)$$

$$R_W m w + 2(n\theta_{,i})_{,i} = 0, \qquad (7.55)$$

where we have set $n = \eta - 2z$, $m = 2(g + z - \xi) - n(1 + h)$ and π is a Lagrange multiplier. To solve (7.53)–(7.55) for the eigenvalue R_W we once more take (curlcurl) of (7.53), take the third component and normal modes. It then remains to determine R_W from the system

$$(D^2 - a^2)W = -\frac{1}{2}R_W m a^2 \Theta + \alpha a^2 R_W \Phi, \qquad (7.56)$$

$$(D^2 - a^2)\Phi = \alpha R_W W, \qquad (7.57)$$

$$(D^2 - a^2)\Theta = \frac{2}{n}D\Theta - \frac{R_W m W}{2n}, \qquad (7.58)$$

where again $D \equiv d/dz$ and a is a wavenumber.

The boundary conditions are $W = \Theta = \Phi = 0$ at $z = 0, 1$, and we then determine

$$Ra_W = \max_{\eta>0} \min_{a^2} R_W^2.$$

The numerical procedure is as before, ie. we use the compound matrix method to find the eigenvalue and golden section search in the optimization. The results are in §7.6.

7.6 Numerical results

We calculated the linear instability Rayleigh number, Ra_L, the conditional energy stability Rayleigh number, Ra_E, and the unconditional energy stability Rayleigh number Ra_W, together with the best values of the corresponding parameters λ and η.

We took values for ξ between 1 and 0.5. Recalling that ξ is defined to be $4/T_u$, a value of 1 corresponds to the upper boundary being at a temperature of $4°C$, i.e. the whole layer is gravitationally unstable, and $\xi = 0.5$ corresponds to the upper boundary being at a temperature of $8°C$, i.e. there is a part of the layer which is gravitationally stable. As Q is just a number, indicative of the strength of the heat source, we took $Q = 1$ and, in some cases, $Q = 2$, in order to provide a measure of how the Rayleigh numbers varied with the heat source. $\alpha = Rs/R$ we took values for between 3 and 0.5, these being respectively the upper and lower limits for which results could be found.

Each calculation was then carried out for the three different types of heat source, I, II, III (see §7.2). The results of these calculations are now presented below. Here a_L^2, a_E^2 and a_W^2 are the respective wave numbers for the linear, conditional nonlinear and unconditional nonlinear cases. λ is the best value of the coupling parameter in §7.4 and η is the best value of the constant in the weighted energy of §7.5.

Case I.

Table 1. Critical Rayleigh and wave numbers of linear and energy theory for $0.6 \leq \alpha \leq 3$
$Q = 1, \xi = 1.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.0224	4.0199	3.9939	9.893	9.896	9.842	0.815	2.111
2.0	8.1891	8.1782	8.0703	9.932	9.939	9.830	0.820	2.110
1.0	21.440	21.360	20.627	10.18	10.19	9.894	0.838	2.105
0.9	23.852	23.751	22.848	10.24	10.26	9.923	0.841	2.104
0.8	26.506	26.379	25.270	10.32	10.34	9.960	0.844	2.102
0.7	29.371	29.214	27.862	10.42	10.44	10.01	0.848	2.101
0.6	32.383	32.190	30.559	10.53	10.55	10.06	0.852	2.099

Table 2. Critical Rayleigh and wave numbers of linear and energy theory for $0.8 \leq \alpha \leq 3$
$Q = 1, \xi = 0.9$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1064	4.1018	4.0681	9.892	9.898	9.841	0.628	2.040
2.0	8.5462	8.5256	8.3800	9.932	9.944	9.826	0.633	2.040
1.0	24.075	23.903	22.770	10.21	10.24	9.896	0.654	2.038
0.9	27.151	26.931	25.500	10.29	10.32	9.932	0.658	2.038
0.8	30.631	30.349	28.543	10.40	10.43	9.981	0.663	2.037

Table 3. Critical Rayleigh and wave numbers of linear and energy theory for $0.8 \leq \alpha \leq 3$
$Q = 1, \xi = 0.8$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1940	4.1850	4.1379	9.892	9.903	9.851	0.448	2.011
2.0	8.9358	8.8949	8.6826	9.932	9.954	9.845	0.454	2.012
1.0	27.449	27.060	25.149	10.25	10.30	9.959	0.478	2.012
0.9	31.509	30.998	28.512	10.36	10.41	10.01	0.484	2.012
0.8	36.268	35.594	32.354	10.51	10.56	10.08	0.490	2.012

Table 4. Critical Rayleigh and wave numbers of linear and energy theory for $0.9 \leq \alpha \leq 3$
$Q = 1, \xi = 0.7$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.2854	4.2657	4.1998	9.892	9.912	9.869	0.295	2.003
2.0	9.3628	9.2697	8.9613	9.931	9.972	9.881	0.300	2.003
1.0	31.925	30.918	27.663	10.32	10.39	10.07	0.322	2.003
0.9	37.530	36.168	31.783	10.47	10.53	10.15	0.327	2.003

Table 5. Critical Rayleigh and wave numbers of linear and energy theory for $2 \leq \alpha \leq 3$ $Q = 1, \xi = 0.6$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.3809	4.3283	4.2515	9.891	9.918	9.890	0.228	2.000
2.0	9.8326	9.5760	9.2026	9.931	9.982	9.924	0.229	2.000

Table 6. Critical Rayleigh and wave numbers of linear and energy theory for $2 \leq \alpha \leq 3$ $Q = 1, \xi = 0.5$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.4807	4.3558	4.2912	9.891	9.907	9.910	0.313	2.000
2.0	10.352	9.7148	9.3941	9.931	9.954	9.964	0.312	2.000

Table 7. Critical Rayleigh and wave numbers of linear and energy theory for $0.6 \leq \alpha \leq 3$ $Q = 2, \xi = 1.0$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1014	4.0994	4.0503	9.912	9.915	9.861	0.600	2.035
2.0	8.5178	8.5090	8.2957	9.983	9.989	9.880	0.604	2.034
1.0	23.660	23.584	21.942	10.46	10.48	10.16	0.621	2.030
0.9	26.574	26.478	24.415	10.61	10.62	10.26	0.624	2.028
0.8	29.826	29.703	27.122	10.79	10.80	10.37	0.628	2.027
0.7	33.381	33.224	30.021	10.02	11.03	10.52	0.633	2.026
0.6	37.152	36.956	33.034	11.30	11.31	10.70	0.638	2.025

Table 8. Critical Rayleigh and wave numbers of linear and energy theory for $0.7 \leq \alpha \leq 3$ $Q = 2, \xi = 0.9$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1892	4.1834	4.1201	9.911	9.918	9.872	0.426	2.010
2.0	8.9094	8.8827	8.5961	9.980	9.994	9.901	0.432	2.010
1.0	27.004	26.752	24.186	10.50	10.54	10.25	0.455	2.009
0.9	30.864	30.534	27.215	10.68	10.71	10.38	0.460	2.008
0.8	35.321	34.890	30.601	10.92	10.95	10.54	0.466	2.008
0.7	40.370	39.810	34.304	11.25	11.27	10.75	0.473	2.008

Table 9. Critical Rayleigh and wave numbers of linear and energy theory for $0.8 \leq \alpha \leq 3$ $Q = 2, \xi = 0.8$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.2809	4.2640	4.1819	9.910	9.926	9.891	0.280	2.002
2.0	9.3384	9.2585	8.8731	9.976	10.01	9.938	0.285	2.002
1.0	31.466	30.617	26.559	10.56	10.61	10.39	0.308	2.002
0.9	36.828	35.690	30.252	10.80	10.83	10.55	0.314	2.002
0.8	43.317	41.790	34.476	11.15	11.14	10.77	0.321	2.002

Case II.

Table 10. Critical Rayleigh and wave numbers of linear and energy theory for $0.5 \le \alpha \le 3$ $Q = 1, \xi = 1$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	3.9836	3.9796	3.9580	9.887	9.893	9.834	0.851	2.127
2.0	8.0310	8.0144	7.9270	9.918	9.930	9.812	0.857	2.127
1.0	20.425	20.387	19.755	10.11	10.13	9.824	0.876	2.124
0.9	22.612	22.466	21.792	10.16	10.18	9.838	0.880	2.123
0.8	24.998	24.817	23.999	10.21	10.24	9.857	0.884	2.123
0.7	27.551	27.328	26.344	10.28	10.31	9.882	0.887	2.122
0.6	30.212	29.940	28.767	10.36	10.39	9.913	0.892	2.121
0.5	32.882	32.557	31.181	10.44	10.48	9.949	0.896	2.120

Table 11. Critical Rayleigh and wave numbers of linear and energy theory for $0.8 \le \alpha \le 3$ $Q = 1, \xi = 0.9$.

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.0694	4.0633	4.0347	9.887	9.895	9.832	0.669	2.050
2.0	8.3881	8.3616	8.2409	9.919	9.936	9.806	0.676	2.050
1.0	22.895	22.684	21.814	10.14	10.18	9.813	0.698	2.049
0.9	25.672	25.404	24.320	10.20	10.24	9.831	0.702	2.049
0.8	28.780	28.440	27.090	10.28	10.32	9.855	0.707	2.049

Table 12. Critical Rayleigh and wave numbers of linear and energy theory for $0.9 \leq \alpha \leq 3$
$Q = 1, \xi = 0.8.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1590	4.1489	4.1079	9.887	9.900	9.839	0.494	2.016
2.0	8.7785	8.7331	8.5525	9.920	9.946	9.820	0.501	2.016
1.0	26.043	25.634	24.132	10.18	10.24	9.858	0.526	2.016
0.9	29.684	29.151	27.227	10.27	10.33	9.886	0.531	2.016

Table 13. Critical Rayleigh and wave numbers of linear and energy theory for $0.9 \leq \alpha \leq 3$
$Q = 1, \xi = 0.7.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.2526	4.2338	4.1741	9.887	9.908	9.856	0.336	2.004
2.0	9.2071	9.1194	8.8457	9.921	9.963	9.853	0.342	2.004
1.0	30.189	29.289	26.628	10.25	10.34	9.958	0.367	2.005
0.9	35.167	33.962	30.441	10.38	10.46	10.01	0.373	2.005

Table 14. Critical Rayleigh and wave numbers of linear and energy theory for $2 \leq \alpha \leq 3$
$Q = 1, \xi = 0.6.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.3506	4.3073	4.2308	9.886	9.918	9.878	0.236	2.001
2.0	9.6797	9.4721	9.1064	9.922	9.983	9.896	0.238	2.001

Table 15. Critical Rayleigh and wave numbers of linear and energy theory for $2 \leq \alpha \leq 3$
$Q = 1, \xi = 0.5.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.4531	4.3482	4.2762	9.886	9.912	9.899	0.269	2.000
2.0	10.204	9.6767	9.3221	9.923	9.965	9.940	0.268	2.000

Table 16. Critical Rayleigh and wave numbers of linear and energy theory for $0.7 \leq \alpha \leq 3$
$Q = 2, \xi = 1.0.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.0257	4.0218	3.9835	9.902	9.908	9.840	0.686	2.057
2.0	8.1987	8.1820	8.0235	9.958	9.969	9.832	0.692	2.056
1.0	21.416	21.290	20.238	10.30	10.32	9.941	0.713	2.053
0.9	23.801	23.643	22.355	10.39	10.42	9.981	0.716	2.052
0.8	26.415	26.218	24.646	10.50	10.53	10.03	0.721	2.051
0.7	29.224	28.978	27.076	10.63	10.66	10.10	0.725	2.050

Table 17. Critical Rayleigh and wave numbers of linear and energy theory for $0.7 \leq \alpha \leq 3$
$Q = 2, \xi = 0.9.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1173	4.1100	4.0596	9.902	9.911	9.846	0.518	2.019
2.0	8.5897	8.5573	8.3396	9.957	9.976	9.842	0.525	2.019
1.0	24.310	24.036	22.369	10.35	10.39	9.987	0.550	2.018
0.9	27.420	27.069	24.974	10.46	10.51	10.04	0.555	2.018
0.8	30.929	30.480	27.852	10.61	10.66	10.12	0.560	2.018
0.7	34.814	34.243	30.972	10.80	10.84	10.21	0.567	2.017

Table 18. Critical Rayleigh and wave numbers of linear and energy theory for $0.8 \le \alpha \le 3$
$Q = 2, \xi = 0.8.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.2132	4.1984	4.1297	9.901	9.918	9.863	0.364	2.005
2.0	9.0200	8.9523	8.6427	9.957	9.990	9.874	0.370	2.005
1.0	28.110	27.464	24.697	10.42	10.48	10.09	0.397	2.006
0.9	32.334	31.486	27.902	10.58	10.64	10.17	0.404	2.006
0.8	37.286	36.171	31.526	10.80	10.85	10.28	0.411	2.006

Case III.

Table 19. Critical Rayleigh and wave numbers of linear and energy theory for $0.5 \le \alpha \le 3$
$Q = 1, \xi = 1.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	3.9787	3.9746	3.9677	9.872	9.878	9.841	1.113	2.298
2.0	8.0169	7.9997	7.9716	9.881	9.893	9.818	1.118	2.299
1.0	20.448	20.328	20.142	9.947	9.972	9.786	1.134	2.299
0.9	22.665	22.516	22.287	9.965	9.991	9.786	1.136	2.299
0.8	25.097	24.911	24.630	9.987	10.02	9.789	1.139	2.299
0.7	27.715	27.487	27.143	10.01	10.04	9.794	1.142	2.299
0.6	30.465	30.186	29.769	10.04	10.08	9.802	1.146	2.299
0.5	33.250	32.913	32.416	10.08	10.11	9.812	1.149	2.298

Table 20. Critical Rayleigh and wave numbers of linear and energy theory for $0.7 \leq \alpha \leq 3$
$Q = 1, \xi = 0.9.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.0539	4.0487	4.0390	9.872	9.880	9.835	0.902	2.152
2.0	8.3280	8.3055	8.2646	9.882	9.896	9.806	0.907	2.152
1.0	22.590	22.415	22.112	9.963	9.996	9.755	0.924	2.153
0.9	25.323	25.101	24.720	9.988	10.02	9.753	0.928	2.153
0.8	28.390	28.109	27.629	10.02	10.06	9.755	0.931	2.153
0.7	31.778	31.422	30.822	10.06	10.10	9.760	0.936	2.153

Table 21. Critical Rayleigh and wave numbers of linear and energy theory for $0.9 \leq \alpha \leq 3$
$Q = 1, \xi = 0.8.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1320	4.1250	4.1098	9.872	9.882	9.832	0.694	2.056
2.0	8.6642	8.6329	8.5662	9.883	9.902	9.798	0.699	2.056
1.0	25.230	24.955	24.396	9.986	10.03	9.734	0.718	2.057
0.9	28.681	28.324	27.604	10.02	10.07	9.732	0.723	2.057

Table 22. Critical Rayleigh and wave numbers of linear and energy theory for $0.9 \leq \alpha \leq 3$
$Q = 1, \xi = 0.7.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.2132	4.2027	4.1765	9.873	9.886	9.839	0.493	2.016
2.0	9.0287	8.9806	8.8613	9.884	9.911	9.811	0.498	2.016
1.0	28.562	28.082	26.941	10.02	10.08	9.767	0.520	2.017
0.9	33.051	32.410	30.900	10.07	10.14	9.772	0.525	2.017

Table 23. Critical Rayleigh and wave numbers of linear and energy theory for $1 \leq \alpha \leq 3$
$Q = 1, \xi = 0.6.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.2976	4.2789	4.2354	9.873	9.894	9.854	0.316	2.004
2.0	9.4251	9.3365	9.1312	9.885	9.928	9.842	0.320	2.004
1.0	32.893	31.886	29.608	10.06	10.16	9.859	0.339	2.004

Table 24. Critical Rayleigh and wave numbers of linear and energy theory for $2 \leq \alpha \leq 3$
$Q = 1, \xi = 0.5.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.3854	4.3403	4.2838	9.873	9.902	9.873	0.224	2.000
2.0	9.8578	9.6365	9.3608	9.887	9.944	9.881	0.225	2.000

Table 25. Critical Rayleigh and wave numbers of linear and energy theory for $0.6 \leq \alpha \leq 3$
$Q = 2, \xi = 1.0.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.0110	4.0074	4.0021	9.872	9.877	9.849	1.209	2.375
2.0	8.1498	8.1346	8.1119	9.879	9.890	9.833	1.212	2.375
1.0	21.344	21.233	21.067	9.941	9.964	9.822	1.224	2.373
0.9	23.773	23.634	23.426	9.959	9.984	9.826	1.227	2.373
0.8	26.464	26.289	26.029	9.982	10.01	9.834	1.229	2.372
0.7	29.394	29.177	28.851	10.01	10.04	9.846	1.232	2.372
0.6	32.506	32.238	31.835	10.04	10.07	9.861	1.235	2.372

Table 26. Critical Rayleigh and wave numbers of linear and energy theory for $0.7 \leq \alpha \leq 3$
$Q = 2, \xi = 0.9.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.0808	4.0764	4.0690	9.872	9.878	9.843	0.977	2.200
2.0	8.4428	8.4240	8.3917	9.880	9.892	9.821	0.981	2.200
1.0	23.471	23.319	23.056	9.953	9.981	9.792	0.995	2.200
0.9	26.438	26.244	25.908	9.976	10.01	9.796	0.998	2.200
0.8	29.802	29.554	29.123	10.01	10.00	9.803	1.001	2.199
0.7	33.562	33.245	32.693	10.05	10.08	9.816	1.005	2.199

Table 27. Critical Rayleigh and wave numbers of linear and energy theory for $0.9 \leq \alpha \leq 3$
$Q = 2, \xi = 0.8.$

α	Ra_L	Ra_E	Ra_W	a_L^2	a_E^2	a_W^2	λ	η
3.0	4.1530	4.1474	4.1358	9.872	9.880	9.839	0.748	2.075
2.0	8.7577	8.7327	8.6807	9.880	9.896	9.812	0.752	2.076
1.0	26.065	25.839	25.367	9.970	10.01	9.766	0.769	2.076
0.9	29.770	29.474	28.856	10.00	10.04	9.768	0.772	2.076

It can be seen that for all cases $Ra_L > Ra_E > Ra_W$ and that all three numbers are very close, justifying our assumption of stationary convection in §7.3. Also Q increasing and ξ decreasing both force the Rayleigh numbers to increase, ie., convection is becoming harder to initiate. In most cases it is found that the wave numbers follow the pattern $a_W^2 < a_L^2 < a_E^2$. Note that the weighted unconditional Rayleigh number is very close to Ra_E, the conditional Rayleigh number. We include the conditional results because they are (marginally) larger than the unconditional results.

CHAPTER EIGHT

CONTINUOUS DEPENDENCE ON THE VISCOSITY FOR A POROUS MEDIUM.

8.1 Introduction and governing equations

In Chapter 3 we established that the solution to the Oberbeck-Boussinesq equations exhibited continuous dependence on the viscosity. We now continue the theme by studying continuous dependence on the viscosity within a porous medium. Again we will consider both the forward in time and improperly posed backward in time problems.

The main difference between this problem and the one studied in Chapter 3 lies in the equation governing the velocity of the fluid, as for a porous medium we use Darcy's law as explained in Chapter 7. Howevever the basic logic behind the analysis remains the same; we will strive to show that both the velocity and temperature fields depend continuously on the viscosity. In order to prove this result for the backward in time problem we again use a logarithmic convexity argument as in Chapter 3. We are also forced to impose bounds on the initial solutions but these are essentially weak.

The equations of motion for a porous medium containing an incompressible viscous heat-conducting fluid are (cf. Joseph (1976) and Chapter 7),

$$p_{,i} = -\frac{\mu}{k}v_i - gk_i\rho_0(1-\alpha(T-T_0)), \tag{8.1}$$

$$T_{,t} + v_iT_{,i} = \Delta T, \tag{8.2}$$

$$v_{i,i} = 0, \tag{8.3}$$

where v_i is the velocity of the fluid, T is the temperature field, T_0 is a reference temperature, μ is the dynamic viscosity, k is the permeability, ρ_0 is the density, p is the pressure, α is a constant, g is gravity (taken to be pointing downwards) and $\mathbf{k} = (0,0,1)$.

Let $\Omega \subseteq \mathbf{R}^3$ be a bounded domain. The above equations are defined on $\Omega \times (0, \mathcal{T}]$, where Ω is the volume of the porous medium. The boundary of Ω is denoted by Γ, and Γ is assumed sufficiently smooth to allow applications of the divergence theorem. The boundary conditions imposed are

$$\begin{aligned} n_i v_i(\mathbf{x}, t) &= n_i \overline{v_i}(\mathbf{x}, t) \text{ on } \Gamma \times [0, \mathcal{T}], \\ T(\mathbf{x}, t) &= \overline{T}(\mathbf{x}, t) \text{ on } \Gamma \times [0, \mathcal{T}], \end{aligned} \tag{8.4}$$

and the initial conditions are

$$\begin{aligned} v_i(\mathbf{x}, 0) &= v_i^o(\mathbf{x}), \\ T(\mathbf{x}, 0) &= T^o(\mathbf{x}), \end{aligned} \tag{8.5}$$

where (8.5) holds in Ω.

Let $(\mathbf{v}, T, p)$ and $(\mathbf{v}^*, T^*, p^*)$ be two classical solutions to the above boundary-initial value problem; the quantities without asterisks being the base flow and those with asterisks being the perturbed flow. We assume that v_i, v_i^*, T and T^* are of class $C^2(\overline{\Omega} \times (0, \mathcal{T}])$ and further suppose that the two solutions satisfy the same boundary and initial conditions but have different viscosities μ and μ^*. We are thus considering explicitly the continuous dependence on the viscosity only.

In order to study the dependence of the solutions on the viscosity we first define the difference quotients,

$$\begin{aligned} u_i &= v_i^* - v_i, \\ \theta &= T^* - T, \\ \varepsilon &= \frac{1}{k}(\mu^* - \mu), \\ \pi &= p^* - p, \end{aligned}$$

and, without loss of generality, let $\mu^*/k = 1$ and $\alpha g k_i \rho_0 = b_i$. Then u_i, θ, π and ε satisfy

$$\pi_{,i} = b_i \theta - \varepsilon v_i - u_i, \tag{8.6}$$

$$\theta_{,t} = - v_i^* \theta_{,i} - T_{,i} u_i + \Delta \theta, \tag{8.7}$$

$$u_{i,i} = 0, \tag{8.8}$$

where $u_i = \theta = 0$ on $\Gamma \times [0, \mathcal{T}]$, and $u_i = \theta = 0$ at $t = 0$.

8.2 Forward in time problem

This particular problem is satisfied by (8.6)–(8.8), together with the associated boundary conditions.

Let us assume that

$$\|\mathbf{v}\| \le M, \qquad |T| \le M, \tag{8.9}$$

for some constant M.

We can see that, by multiplying (8.6) by u_i and integrating over Ω we can derive

$$\|\mathbf{u}\|^2 = \varepsilon(\mathbf{v}, \mathbf{u}) + (b_i\theta, \mathbf{u}),$$

where $(*, *)$ denotes the inner product. Let $|b_i| \le 1$. Then we can show that

$$\|\mathbf{u}\| \le M\varepsilon + \|\theta\|, \tag{8.10}$$

by using the Cauchy-Schwarz inequality and (8.9). Hence we can write

$$\|\mathbf{u}\|^2 \le 2M^2\varepsilon^2 + 2\|\theta\|^2. \tag{8.11}$$

By multiplying (8.7) by θ and then integrating over Ω we can derive the equation

$$< \theta\theta_{,t} >= - < v_i^*\theta\theta_{,i} > - < \theta T_{,i}u_i > + < \theta\Delta\theta >,$$

which becomes, after integration by parts,

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = -\|\nabla\theta\|^2 + < u_i T\theta_{,i} > . \tag{8.12}$$

Then, by using the arithmetic-geometric mean inequality on the last term on the RHS of (8.12) together with (8.9) and then substituting for $\|\mathbf{u}\|^2$ using (8.11), the equation becomes

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 \le \Big(\frac{M\alpha}{2} - 1\Big)\|\nabla\theta\|^2 + \frac{M}{2\alpha}(2M^2\varepsilon^2 + 2\|\theta\|^2). \tag{8.13}$$

Here α is a positive constant and we can choose it so that it satisfies $\alpha = 2/M$. Then (8.13) can be written

$$\frac{d}{dt}\|\theta\|^2 \le M^4\varepsilon^2 + M^2\|\theta\|^2. \tag{8.14}$$

Equation (8.14) can be integrated to show that

$$\|\theta(t)\|^2 \le M^2\varepsilon^2(e^{M^2t} - 1), \tag{8.15}$$

for $t \in (0, T]$. Also, by substituting (8.15) into (8.11) we can see that

$$\|\mathbf{u}\|^2 \leq \left[2M^2 e^{M^2 t}\right] \varepsilon^2, \tag{8.16}$$

and hence we have proved that both $\|\mathbf{u}\|$ and $\|\theta\|$ depend continuously on the viscosity for the forward in time problem.

8.3 Backward in time problem

The backward in time problem is an improperly posed one and is satisfied by the following equations,

$$\pi_{,i} = b_i \theta - \varepsilon v_i - u_i, \tag{8.17}$$

$$\theta_{,t} = v_i^* \theta_{,i} + T_{,i} u_i - \Delta\theta, \tag{8.18}$$

$$u_{i,i} = 0, \tag{8.19}$$

together with $u_i = \theta = 0$ on $\Gamma \times [0, T]$, and $u_i = \theta = 0$ at $t = 0$. Note that (8.11) and (8.10) still hold.

To handle this problem we will need to impose the weak bounds

$$\|\mathbf{v}\| \leq M, \quad |\nabla T| \leq M, \quad |\mathbf{v}^*| \leq M, \quad |T| \leq M, \tag{8.20}$$

where M is a constant.

We also define the function $F(t)$ such that

$$F(t) = \int_0^t \|\theta\|^2 ds + \varepsilon^2. \tag{8.21}$$

As in Chapter 3 we will be looking at the relation $FF'' - (F')^2$ in order to use a logarithmic convexity argument. First we differentiate (8.21) and can then derive

$$\begin{aligned} F'(t) &= 2\int_0^t < \theta\theta_{,s} > ds \\ &= 2\int_0^t \|\nabla\theta\|^2 ds + 2\int_0^t < \theta u_i T_{,i} > ds \end{aligned} \tag{8.22}$$

where we have substituted for $\theta_{,s}$ using (8.18) and then integrated by parts.

Equation (8.22) can be used to provide an estimate for $\int_0^t \|\nabla\theta\|^2 ds$, as we now show.

From (8.22) we have

$$
\begin{aligned}
-\int_0^t \|\nabla\theta\|^2 ds &= -\frac{F'}{2} + \int_0^t < u_i T_{,i}\theta > ds \\
&\geq -\frac{F'}{2} - \frac{M}{2\alpha}\int_0^t \|\theta\|^2 ds - \frac{M\alpha}{2}\int_0^t \|\mathbf{u}\|^2 ds \\
&\geq -\frac{F'}{2} - \left(\frac{M}{2\alpha} + M\alpha\right)\int_0^t \|\theta\|^2 ds - M^3\alpha\varepsilon^2 T \\
&\geq -\frac{F'}{2} - cF, \qquad (8.23)
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality, the arithmetic-geometric mean inequality, (8.11), (8.20) and have then set $c = \max\{M/2\alpha + M\alpha, M^3\alpha T\}$. Here T is the length of the time interval. This estimate will be used later on.

Returning to our problem we see that differentiating (8.22) results in

$$
\begin{aligned}
F''(t) &= -4\int_0^t < \theta_{,s}\Delta\theta > ds + 2 < \theta u_i T_{,i} > \\
&= 4\int_0^t \|\dot{\theta}\| ds - 4\int_0^t < \theta_{,s} v_i^* \theta_{,i} > ds - 4\int_0^t < \theta_{,s} u_i T_{,i} > ds \\
&\quad + 2 < \theta u_i T_{,i} >, \qquad (8.24)
\end{aligned}
$$

where $\dot{\theta} \equiv \theta_{,s}$. Again we have integrated by parts and then substituted for $\Delta\theta$ from (8.18).

It is now helpful to define $\phi = \theta_{,t} - \frac{1}{2}v_i^*\theta_{,i}$, cf. Payne (1975). Then (8.22) can be written

$$
\begin{aligned}
F'' \geq 4\int_0^t \|\phi\|^2 ds - \int_0^t \|v_i^*\theta_{,i}\|^2 ds - 4\int_0^t < \phi u_i T_{,i} > ds \\
- 2\int_0^t < u_i T_{,i} v_i^* \theta_{,i} > ds - 2M\|\theta\|(M\varepsilon + \|\theta\|),
\end{aligned}
$$

where we have also used (8.20), the Cauchy-Schwarz inequality and (8.10). This equation can be further estimated with the use of (8.20) and the Cauchy-Schwarz inequality, becoming

$$
\begin{aligned}
F'' \geq 4\int_0^t \|\phi\|^2 ds - M^2\int_0^t \|\nabla\theta\|^2 ds - 4M\int_0^t \|\mathbf{u}\|\|\phi\| ds \\
- 2M^2\int_0^t \|\mathbf{u}\|\|\nabla\theta\| ds - 2M\|\theta\|^2 - 2M^2\varepsilon\|\theta\|.
\end{aligned}
$$

Next we make use of the arithmetic-geometric mean inequality and (8.11) once more; then

$$F'' \geq 4\int_0^t \|\phi\|^2 ds - \left(M^2 + \frac{M^2}{\gamma}\right)\int_0^t \|\nabla\theta\|^2 ds - 4M\int_0^t \|\mathbf{u}\|\|\phi\| ds$$
$$- \left(2M + \frac{M^2}{\beta}\right)\|\theta\|^2 - M^2\beta\varepsilon^2 - M^2\gamma\int_0^t (2M^2\varepsilon^2 + 2\|\theta\|^2) ds \quad (8.25)$$

where β and γ are positive constants.

Now note that, due to the definition of F we can write $F'(t) = \|\theta\|^2$. Hence, setting $k_1 = 2M + M^2/\beta$ and $k_2 = \max\{M^2\beta + 2M^4\gamma T, 2M^2\gamma\}$, enables us to write (8.25) as

$$F'' \geq 4\int_0^t \|\phi\|^2 ds - M^2\left(1 + \frac{1}{\gamma}\right)\int_0^t \|\nabla\theta\|^2 ds - 4M\int_0^t \|\mathbf{u}\|\|\phi\| ds - k_1 F' - k_2 F.$$

Equation (8.23) can now be used, and, if we let $k_3 = M^2 c + M^2 c/\gamma + k_2$ and $k_4 = M^2/2 + M^2/2\gamma + k_1$, then

$$F'' \geq 4\int_0^t \|\phi\|^2 ds - k_3 F - k_4 F' - 4M\int_0^t \|\mathbf{u}\|\|\phi\| ds. \quad (8.26)$$

Therefore

$$FF'' - (F')^2 \geq 4S^2 + 4\varepsilon^2\int_0^t \|\phi\|^2 ds - 4MF\int_0^t \|\mathbf{u}\|\|\phi\| ds$$
$$- k_3 F^2 - k_4 FF', \quad (8.27)$$

where

$$S^2 = \int_0^t \|\theta\|^2 ds \int_0^t \|\phi\|^2 ds - \left[\int_0^t < \theta\phi > ds\right]^2, \quad (8.28)$$

and is clearly non-negative thanks to the Cauchy-Schwarz inequality. Note that from (8.28) we can see that

$$\left(\int_0^t \|\theta\|^2 ds\right)^{1/2}\left(\int_0^t \|\phi\|^2 ds\right)^{1/2} = \sqrt{S^2 + \frac{(F')^2}{4}} \leq S + \frac{F'}{2}. \quad (8.29)$$

To handle the third term in (8.27) we proceed as follows,

$$
\begin{aligned}
-4MF\int_0^t \|\mathbf{u}\|\|\phi\|ds \geq & -4M^2\varepsilon F\int_0^t \|\phi\|ds - 4MF\int_0^t \|\theta\|\|\phi\|ds \\
\geq & -\frac{2M^2F^2}{\xi} - 2M^2\xi\varepsilon^2\Big(\int_0^t \|\phi\|ds\Big)^2 \\
& -4MF\Big(\int_0^t \|\theta\|^2 ds\Big)^{1/2}\Big(\int_0^t \|\phi\|^2 ds\Big)^{1/2} \\
\geq & -\frac{2M^2F^2}{\xi} - 2M^2\xi T\varepsilon^2\int_0^t \|\phi\|^2 ds \\
& -4MF\Big(\int_0^t \|\theta\|^2 ds\Big)^{1/2}\Big(\int_0^t \|\phi\|^2 ds\Big)^{1/2}.
\end{aligned}
$$

Here we have used (8.10), the arithmetic-geometric mean inequality (which gives rise to ξ) and the Cauchy-Schwarz inequality (twice). Let $\xi = 2/M^2T$ and then, on substituting this expression into (8.27), we have

$$
\begin{aligned}
FF'' - (F')^2 \geq 4S^2 - 4MF\Big(\int_0^t \|\theta\|^2 ds\Big)^{1/2}\Big(\int_0^t \|\phi\|^2 ds\Big)^{1/2} \\
- k_5F^2 - k_4FF',
\end{aligned}
\tag{8.30}
$$

where $k_5 = k_3 + M^4T$.

We now use (8.29) to handle the second term in (8.30) and the equation becomes

$$
\begin{aligned}
FF'' - (F')^2 &\geq 4S^2 - 4MSF - 2MFF' - k_5F^2 - k_4FF', \\
&= (2S - MF)^2 - M^2F^2 - 2MFF' - k_5F^2 - k_4FF', \\
&\geq -c_1F^2 - c_2FF',
\end{aligned}
\tag{8.31}
$$

where $c_1 = M^2 + k_5$ and $c_2 = 2M + k_4$.

Equation (8.31) can now be integrated in order to give us the required continuous dependence result. The analysis is similar to that in Chapter 3 but differs in that here we use (8.31) to establish that θ depends on ε and then utilise (8.11) to derive a result for $\mathbf{u}$.

As in Chapter 3, (8.31) can be written

$$
\Big[e^{c_2t}(\log F)'\Big]' + c_1e^{c_2t} \geq 0. \tag{8.32}
$$

We then substitute $\sigma = e^{-c_2t}$. Then (8.32) becomes

$$
\frac{d^2}{d\sigma^2}\Big(\log[F(\sigma)\sigma^{-c_1/c_2^2}]\Big) \geq 0,
$$

and, if we substitute $\mu = c_1/c_2$,

$$\frac{d^2}{d\sigma^2}\Big(\log[F(\sigma)e^{\mu t}]\Big) \geq 0.$$

Hence, arguing as in Chapter **3**,

$$F(t) \leq [F(0)]^{(\sigma-\sigma_1)/(1-\sigma_1)}[F(\mathcal{T})e^{\mu\mathcal{T}}]^{(1-\sigma)/(1-\sigma_1)}e^{-\mu t},$$

where $\sigma_1 = e^{-c_2\mathcal{T}}$. Therefore if we write

$$K(\mathcal{T}) = [F(\mathcal{T})e^{\mu\mathcal{T}}]^{(1-\sigma)/(1-\sigma_1)}$$

then we have

$$\int_0^t \|\theta\|^2 ds + \varepsilon^2 \leq [\varepsilon^2]^{(\sigma-\sigma_1)/(1-\sigma_1)}K(\mathcal{T})e^{-\mu t}. \tag{8.33}$$

Also, from (8.11) we have

$$\|\mathbf{u}\|^2 \leq 2M^2\varepsilon^2 + 2\|\theta\|^2$$

and therefore

$$\int_0^t \|\mathbf{u}\|^2 ds \leq 2M^2\mathcal{T}\varepsilon^2 + 2K(\mathcal{T})e^{-\mu t}[\varepsilon^2]^{(\sigma-\sigma_1)/(1-\sigma_1)} - 2\varepsilon^2. \tag{8.34}$$

The inequalities (8.33) and (8.34) establish continuous dependence of the solution $(\mathbf{u},\theta)$ on the viscosity on compact intervals of $[0,\mathcal{T})$.

CHAPTER NINE

POROUS CONVECTION WITH A TEMPERATURE DEPENDENT VISCOSITY.

9.1 Introduction

In the previous chapter we proved that a fluid in a porous medium exhibits continuous dependence on its viscosity. A natural next step, considering earlier chapters, is now to study the nonlinear stability problem of convection in a porous medium when the fluid has a temperature dependent viscosity. This type of convection is also highly important in geophysical and other contexts, see e.g. Or (1989), and the book of Nield & Bejan (1992), and the references therein.

Many studies of convection in porous media involve Darcy's law, see e.g. Nield & Bejan (1992). However, there have recently been several articles which advocate using instead the model of Brinkman (1947), see e.g. Hsu & Cheng (1985), Katto & Masuoka (1967), Nield (1983), Qin & Kaloni (1992), Rudraiah & Masuoka (1982), Vafai & Kim (1989), Vafai & Tien (1981), and Walker & Homsy (1977). A lucid discussion on the relative merits of the Brinkman and Darcy equations, along with other alternatives, is given in chapter 1 of Nield & Bejan (1992). In this chapter we also employ Brinkman's equation although our motivation is mainly a mathematical one in that we find this form necessary to implement the nonlinear energy stability analysis which follows: it may be possible that by some suitable selection of a generalized energy (or Lyapunov functional; such functionals are constructed in other contexts by e.g. Mulone & Rionero (1992), Straughan (1992)) one may find a way to proceed without the Brinkman term but at present we have not seen such an avenue.

The Brinkman equation has the form

$$p_{,i} = -\frac{\mu}{k}v_i - gk_i\rho_0(1-\alpha(T-T_0)) + \tilde{\mu}\Delta v_i. \tag{9.1}$$

Here μ is the viscosity of the fluid and $\tilde{\mu}$ is an effective viscosity. The reasoning behind the use of this equation is that to understand the onset of convection in a porous medium made up of a sparse distribution of particles the viscous shear must be taken into account no matter how small it may be in relation to the Darcy resistance.

For us, using the Brinkman model makes sense mathematically, as it enables us to prove an energy stability theorem. The reason for this is that the Δv_i term in the velocity equation is necessary in order to control the nonlinear term that arises in the energy analysis. Without it I do not see how to proceed in order to derive a result. The question remains— does the Brinkman equation make sense physically?

Nield (1983) deduced that the Brinkman equation could be successfully used for problems where the velocity of the fluid within the porous medium is constant except in regions close to the boundary. He also concluded that it was useful for porous media whose porosity was close to unity, which is indeed the case that it was designed to deal with. However he also states that the Brinkman equation is not generally applicable to flow in porous media. Notwithstanding this statement I feel that mathematically using the equation is justified and as there are some physical situations where it is relevant the analysis is not entirely without use.

We shall use an energy argument as before in order to prove conditional nonlinear stability. It will be seen that the linear and nonlinear problems coincide and so we have an optimum result. The numerical calculations appear in §9.4.

This chapter is essentially to appear in Richardson and Straughan (1993b).

9.2 The governing equations

For our viscosity-temperature relation we once more use a linear approximation as in Chapter 4. That is

$$\mu(T) = \mu_0(1 - \gamma(T - T_0)). \tag{9.2}$$

The model is a horizontal plane layer, $z \in (0, d)$ with gravity taken to be pointing downwards. The layer is a porous medium but we shall assume that the porosity is approaching unity, so that our use of Brinkman's equation is justified (according to Nield (1983)).

The governing equations of motion are, incorporating (9.2) into (9.1),

$$\begin{aligned} p_{,i} =& -\frac{\mu_0}{k}(1-\gamma(T-T_0))v_i + \tilde{\mu}\Delta v_i - gk_i\rho_0(1-\alpha(T-T_0)), && (9.3)\\ v_{i,i} =& 0, && (9.4)\\ T_{,t} + v_i T_{,i} =& \kappa \Delta T, && (9.5) \end{aligned}$$

where p is the pressure, ρ_0 is the density, v_i is the velocity, $\tilde{\mu}$ is the (constant) effective viscosity, g is gravity, $\mathbf{k} = (0, 0, 1)$, T is the temperature field, T_0 is a reference temperature and we have assumed that the fluid is incompressible.

We now consider a steady solution, $\bar{v}_i \equiv 0$, $\bar{T}(z)$, $\bar{p}$. Then equations (9.3) and (9.5) reveal

$$\begin{aligned} 0 =& \bar{T}'', \\ \bar{p}_{,i} =& -gk_i\rho_0(1-\alpha(\bar{T}-T_0)). \end{aligned}$$

If we then impose the boundary conditions $T(0) = T_1$ and $T(d) = T_2$ with $T_1 > T_2$, i.e. the layer is heated from below, then from above we can see that

$$\bar{T}(z) = -\beta z + T_1$$

where $\beta = (T_1 - T_2)/d$, and $\bar{p}$ is found from

$$\frac{d\bar{p}}{dz} = -\rho_0 g(1-\alpha(T_1 - \beta z - T_0)).$$

The next step is to introduce perturbations u_i, θ, π via

$$v_i = \bar{v}_i + u_i, \qquad T = \bar{T} + \theta, \qquad p = \bar{p} + \pi.$$

Substituting these into (9.3)–(9.5) and then non-dimensionalizing according to the scales

$$\begin{aligned} &t = t^*\frac{d}{\mu_0}, \quad \pi = \pi^* P, \quad P = \frac{U\mu_0 d}{k}, \quad Pr = \frac{\mu_0}{\kappa}, \\ &u_i = u_i^* U, \quad \theta = \theta^* T^\sharp, \quad T^\sharp = U\sqrt{\frac{\beta\mu_0 d^2}{\kappa g\alpha k\rho_0}}, \quad R = \sqrt{\frac{\alpha g\beta d^2 k\rho_0}{\kappa\mu_0}}, \\ &x_i = x_i^* d, \quad \xi = \frac{(T_1 - T_0)}{\beta d}, \quad \Gamma = \gamma\beta d, \quad \lambda = \frac{\tilde{\mu}k}{d^2\mu_0}, \quad U = \frac{\mu_0}{d}, \end{aligned}$$

the equations governing the system, (dropping all stars), become

$$\begin{aligned} \pi_{,i} =& -u_i(1+\Gamma(z-\xi)) + \frac{\Gamma Pr}{R}\theta u_i + \lambda\Delta u_i + Rk_i\theta, && (9.6)\\ Pr(\theta_{,t} + u_i\theta_{,i}) =& \Delta\theta + Rw, && (9.7)\\ u_{i,i} =& 0. && (9.8) \end{aligned}$$

Here Γ is a measure of the viscosity variation with temperature and λ is a measure of the effective viscosity.

The boundary conditions which we employ are those for two free surfaces as described in Chapter 7, i.e.

$$u_z = v_z = w = 0$$

$$\theta = 0$$

on $z = 0, 1$ with u, v, w, θ and p being periodic on the x, y boundaries of the convection cell V.

9.3 Nonlinear analysis

We once more employ an energy analysis in order to study the nonlinear system. We multiply (9.6) by u_i, (9.7) by $\eta\theta$ and integrate over the period cell V. Here η is a positive coupling parameter and, as always, $\|.\|$ is the $L^2(V)$ norm. Upon adding the resulting equations we may derive

$$\frac{dE}{dt} = -D + RI + \frac{\Gamma Pr}{R} < \theta u_i u_i >, \tag{9.9}$$

where

$$\begin{aligned} E =& \frac{Pr\eta}{2}\|\theta\|^2, \\ D =& \|\mathbf{u}\|^2 + \Gamma < u_i u_i(z-\xi) > + \lambda\|\nabla\mathbf{u}\|^2 + \eta\|\nabla\theta\|^2, \\ I =& (1+\eta) < \theta w > . \end{aligned}$$

If we now set

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D} \tag{9.10}$$

where $\mathcal{H}$ is the space of admissible solutions and argue as in previous chapters we may derive the inequality

$$\frac{dE}{dt} \leq -aD + \frac{\Gamma Pr}{R} < \theta u_i u_i >, \tag{9.11}$$

where $a = (R_E - R)/R$. The number R_E is the critical Rayleigh number of energy stability theory and we determine this number below. It is henceforth understood that we shall be working with Rayleigh numbers such that $a > 0$.

The Euler equations associated with (9.10) are

$$2\lambda\Delta u_i - 2u_i(1 + \Gamma\mu) + R_E M k_i \theta = \varpi_{,i}, \tag{9.12}$$

$$2\Delta\theta + R_E M w = 0, \tag{9.13}$$

where

$$M = \frac{1+\eta}{\sqrt{\eta}}, \qquad \mu = z - \xi,$$

and ϖ is a Lagrange multiplier.

In deriving these equations we have put $\sqrt{\eta}\theta = \theta^*$ and ignored the $*$. In order to determine the best value of η we use parametric differentiation as in Chapter 2. (The best value of η is that which maximises R_E). Upon carrying out the differentiation we find that

$$2 < w\theta > \left(R_E \frac{\partial M}{\partial \eta} + M \frac{\partial R_E}{\partial \eta} \right) = 0. \tag{9.14}$$

Since we require $\partial R_E / \partial \eta = 0$, (9.14) tells us that $\partial M / \partial \eta = 0$ i.e. $\eta = 1$ and $M = 2$. Therefore in the following analysis this is the value that we shall use. Note that the eigenvalue problem arising from (9.12) and (9.13) is now the same as that arising from the linear system, and so, if we can demonstrate decay of E from (9.11) we will have an optimum result. These circumstances arise due to the fact that $\eta = 1$.

We can also carry out a parametric differentiation with respect to λ. This yields

$$\frac{\partial R_E}{\partial \lambda} = \frac{\|\nabla \mathbf{u}\|^2}{2 < w\theta >}.$$

By multiplying (9.13) by θ and integrating over V we can derive an expression for $< w\theta >$ which is then substituted into the above equation to give

$$\frac{\partial R_E}{\partial \lambda} = \frac{R_E \|\nabla \mathbf{u}\|^2}{2\|\nabla \theta\|^2}.$$

Thus

$$\frac{\partial R_E}{\partial \lambda} \geq 0,$$

a result which is useful in the numerical calculations.

To solve (9.12) and (9.13) numerically for the eigenvalue R_E we first take the double curl of (9.12) and then the third component. Then we take normal modes, i.e.

$$\begin{aligned} w =& W(z) e^{i(mx+ny)}, \\ \theta =& \Theta(z) e^{i(mx+ny)}, \end{aligned}$$

and the system becomes

$$\lambda (D^2 - a^2)^2 W - (1 + \Gamma\mu)(D^2 - a^2) W - \Gamma D W - R_E a^2 \Theta = 0, \tag{9.15}$$

$$(D^2 - a^2)\Theta + R_E W = 0. \tag{9.16}$$

Here $a^2 = m^2 + n^2$ and $D \equiv d/dz$. As in the previous chapters we employ the compound matrix method and golden section search in order to find

$$Ra = \min_{a^2} R_E^2,$$

subject to the boundary conditions

$$W = D^2 W = \Theta = 0 \text{ on } z = 0, 1.$$

The numerical results appear in §9.4.

We now return to (9.11) and the energy stability analysis. Recalling that the energy equation was

$$\frac{dE}{dt} \le -aD + \frac{\Gamma Pr}{R} < \theta u_i u_i >$$

we see that a way must be found to control the nonlinear term. In order to do this we employ the Sobolev inequality (cf. Chapter 2)

$$\left(\int_V (u_i u_i)^2 dV \right)^{1/2} \le c \|\nabla \mathbf{u}\|^{3/2} \|\mathbf{u}\|^{1/2}.$$

We proceed as follows;

$$\begin{aligned} < \theta u_i u_i > &\le \left(\int_V \theta^2 dV \right)^{1/2} \left(\int_V (u_i u_i)^2 dV \right)^{1/2} \\ &\le c \|\theta\| \|\nabla \mathbf{u}\|^{3/2} \|\mathbf{u}\|^{1/2}. \end{aligned} \tag{9.17}$$

Note that there exists a positive constant $k_1 > 0$ such that

$$\begin{aligned} D &= \|\mathbf{u}\|^2 + \Gamma < u_i u_i (z - \xi) > + \lambda \|\nabla \mathbf{u}\|^2 + \|\nabla \theta\|^2 \\ &\ge k_1 \|\mathbf{u}\|^2 + \lambda \|\nabla \mathbf{u}\|^2 + \|\nabla \theta\|^2 \\ &= \mathcal{D}, \text{ say .} \end{aligned}$$

Then (9.11) can be written, incorporating (9.17),

$$\frac{dE}{dt} \le -a\mathcal{D} + \frac{\Gamma Prc}{R} \|\theta\| \|\nabla \mathbf{u}\|^{3/2} \|\mathbf{u}\|^{1/2}, \tag{9.18}$$

and (9.18) can be bound in terms of $\mathcal{D}$ and E. For,

$$c \|\theta\| \|\nabla \mathbf{u}\|^{3/2} \|\mathbf{u}\|^{1/2} \le \frac{\sqrt{2} c}{\sqrt{Pr} \lambda^{3/4} k_1^{1/4}} \mathcal{D} E^{1/2},$$

and so

$$\frac{dE}{dt} \leq -a\mathcal{D} + A\mathcal{D}E^{1/2}, \tag{9.19}$$

where

$$A = \frac{\Gamma c\sqrt{2Pr}}{R(\lambda^3 k_1)^{1/4}}.$$

Hence, provided

$$(1)\ R < R_E \qquad \text{and} \qquad (2)\ E^{1/2}(0) < \frac{a}{A}$$

then (9.19) can be integrated as in Chapter 2 to give

$$E(t) \leq E(0)e^{-K\zeta t}$$

where $K = a - AE^{1/2}(0)$, is a positive constant and ζ is given by the inequality $\mathcal{D} \geq \zeta E$. Thus we have conditional nonlinear stability. It is important to realise that the above analysis hinges on the presence of the Brinkman term as otherwise there is no $||\nabla \mathbf{u}||^2$ term in $\mathcal{D}$ to control the cubic term. The equivalent problem without the Brinkman term is unresolved.

9.4 Numerical results and asymptotic analysis

Before proceeding to the numerical results it is instructive to consider a special case, namely when $\Gamma = 0$, i.e. the viscosity is constant. Recall that the relevant equations are

$$\begin{aligned} \lambda(D^2 - a^2)^2 W - (1 + \Gamma\mu)(D^2 - a^2)W - \Gamma DW - R_E a^2 \Theta &= 0, \\ (D^2 - a^2)\Theta + R_E W &= 0, \end{aligned}$$

together with the boundary conditions

$$W = D^2 W = \Theta = 0 \text{ on } z = 0, 1.$$

For the case $\Gamma = 0$ we may solve the system exactly by firstly eliminating Θ to find

$$\lambda(D^2 - a^2)^3 W - (D^2 - a^2)^2 W + R_E^2 a^2 W = 0. \tag{9.20}$$

The boundary conditions together with (9.15) show that

$$D^{(2n)} W = 0 \qquad \text{on} \qquad z = 0, 1, \tag{9.21}$$

$n = 0, 1, 2, \dots$, and so $W = \sin \pi z$ which yields

$$R_E^2 = \frac{\lambda(\pi^2 + a^2)^3 + (\pi^2 + a^2)^2}{a^2}. \quad (9.22)$$

The minimum wavenumber is then found as

$$a_c^2 = \frac{-(1 + \lambda\pi^2) + \sqrt{(1 + \lambda\pi^2)^2 + 8\lambda\pi^2(1 + \lambda\pi^2)}}{4\lambda}. \quad (9.23)$$

It is further instructive to develop the analysis for $0 < \lambda << 1$. In the limit $\lambda \to 0$ we should recover the classical result for convection with Darcy's law and constant viscosity. Of course, in the classical case we must recognize that the equation (9.20) is fourth order and hence the boundary condition $D^2W = 0$ is not required. For λ small, use of the binomial theorem in (9.23) shows

$$a_c^2 \sim \pi^2 + O(\lambda), \quad (9.24)$$

and then putting this in (9.22) gives

$$R_E^2 \sim 4\pi^2 + O(\lambda). \quad (9.25)$$

Expressions (9.24) and (9.25) are in complete agreement with the classical results obtained via Darcy's law, as they should be.

The numerical results are now presented. For the purpose of the numerical calculations we have taken $\xi = 0$, i.e. the reference temperature is the same as the temperature on the lower boundary. It can be seen that increasing λ sharply increases the Rayleigh number as might be expected both intuitively and from the parametric differentiation. Physically this means that the greater the effective viscosity versus the fluid viscosity the more difficult it is for convection to commence.

Table 1. Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.

$\Gamma = 0.0$.

Ra	a^2	λ
108.573	6.111	0.1
174.948	5.621	0.2
240.970	5.420	0.3
306.873	5.310	0.4
372.722	5.241	0.5

Table 2. Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.
$\Gamma = 0.1$.

Ra	a^2	λ
110.658	6.153	0.1
177.079	5.649	0.2
243.124	5.441	0.3
309.041	5.327	0.4
374.899	5.255	0.5

Table 3. Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.
$\Gamma = 0.2$.

Ra	a^2	λ
112.733	6.193	0.1
179.203	5.672	0.2
245.273	5.462	0.3
311.204	5.344	0.4
377.072	5.269	0.5

Table 4. Critical Rayleigh numbers Ra, with their respective critical wavenumbers a^2.
$\Gamma = 0.5$.

Ra	a^2	λ
118.901	6.307	0.1
185.539	5.756	0.2
251.690	5.522	0.3
317.671	5.393	0.4
383.571	5.310	0.5

CONCLUDING REMARKS.

In the preceeding chapters we have seen that the energy method is a very powerful technique which yields sharp results. One area however in which it was not so useful was in Chapter 6, the case where we had both temperature dependent conductivity and viscosity. Recall that the relevant perturbation equations were

$$\begin{aligned} u_{i,t} + u_j u_{i,j} =& - \pi_{,i} + 2\big[d_{ij}(1-B)\big]_{,j} - \frac{8\Gamma Pr}{R\xi}\big[d_{ij}\theta(1-f)\big]_{,j} \\ & - \frac{2\Gamma Pr^2}{R^2}\big[d_{ij}\theta^2\big]_{,j} + Rk_i\theta, \\ Pr(\theta_{,t} + u_i\theta_{,i}) =& R\frac{w}{f} + \frac{\xi}{2}\Big[\frac{\theta}{f}\Big]_{,z} - \frac{\xi Pr}{2R}\Big[\theta\theta_{,j}\Big]_{,j} + \Big[f\theta_{,j}\Big]_{,j}, \end{aligned}$$

where B(z) is defined by

$$B = \frac{4\Gamma z^2}{(1+f)^2},$$

together with the boundary conditions

$$\begin{aligned} w =& \frac{\partial^2 w}{\partial z^2} = \Delta w = 0 \\ \theta =& 0, \qquad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \end{aligned}$$

on $z = 0, 1$.

Here, in order to establish a nonlinear stability result we require $\Delta\theta = 0$ on the boundaries, which is unfortunately not the case. In fact, on the boundaries, the temperature equation tells us

$$\Delta\theta = \frac{\xi Pr}{2Rf}\theta_z^2 - \frac{\xi}{f^2}\theta_z.$$

We treat the problem heuristically and, by assuming $\Delta\theta = 0$ on the boundaries, derive some useful results. However the problem incorporating the true boundary conditions still remains to be solved.

The computer routines used throughout to derive the numerical results are omitted from this thesis but are available on request.

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