FINITE GROUP THEORY:

ODD PRIMES AND CC-SUBGROUPS

by

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A thesis submitted for the degree
of Master of Science at the
University of Glasgow

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University of Glasgow
September 1985

ACKNOWLEDGEMENTS

The work involved in preparing this thesis was done between October 1984 and September 1985 under the supervision of Dr Neil. K. Dickson, for whose patience and seemingly endless enthusiasm I am extremely grateful.

I would also like to thank Professor W. D. Munn, Head of Department, and the secretaries of the Mathematics Department, for giving me free use of the departmental word processor in producing the final versions of this thesis.

For its contribution to my financial support, I thank the Colonel Alexander MacLean Educational Trust.

Lastly, I thank my parents.

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SUMMARY

The majority of this thesis comprises a survey of existing knowledge.

Let G be a finite group and suppose that M is a subgroup of G such that (i) $1<M<G$, and

(ii) for all $x \in M^{\#}$, $C_G(x) \leq M$.

Such a subgroup M is called a CC-subgroup, the concept to which this thesis is predominantly devoted.

Following a brief introduction, Chapter II consists of a survey of the known results on the odd prime structure of finite groups. This survey is split into three sections as follows. The first gives an account of the development of a unified theory for characterising groups with CC-subgroups of order divisible by three. Section II introduces the twin ideas of closure and homogeneity, concluding with a theorem which has important applications later in Chapter V. Section III consists of a straightforward listing of the remaining odd prime structure results.

Chapter III is the theoretical base of the thesis, contributing all the major results which are required before proceeding. Section II of this chapter is itself an integral part of the survey, being a systematic exposition of basic CC-subgroup theory.

Chapter IV is a discussion on the various techniques and proofs involved in Chapter II, giving a readable yet rigorous explanation of the theory.

Chapter V highlights more recent, and more general, results involving CC-subgroups, giving detailed proofs, and sets the scene for the final chapter.

Chapter VI consists of two sections. Section I is given over entirely to the statement and proof of a single theorem which completely classifies groups containing CC-subgroups, a simple corollary of which initiates Section II, an outline of the search for CC-subgroups of the finite simple groups. This section, and the thesis, ends with four tables that give as complete a list as possible of the information currently available on the CC-subgroups of the simple groups.

CHAPTER I: INTRODUCTION

Late in 1980, the complete classification of the finite simple groups was achieved after some quarter century of continuous development. In effect, the result states that the only finite simple groups are:

(i) the cyclic groups of prime order (the only abelian simple groups),

(ii) the alternating groups of degree at least 5,

(iii) the simple groups of Lie type, and

(iv) the 26 sporadic simple groups.

Even a glance at the main theorems in specific characterisations of finite simple groups reveals that the classification was achieved by studying various aspects of the prime 2, for example properties of 2-elements and subgroups defined in terms of 2-elements, which are called 2-structure. Of the many good reasons for the fundamental importance of the even prime we shall consider two.

Firstly we have the Brauer and Fowler result of 1954, one form of which is the following.

Theorem 1.1[13]. If G is a finite simple group of even order and t is an involution in G (that is, an element of G of order 2) then

$$|G|\le(|C_G(t)|^2)!$$

Although the bound is not particularly useful in itself, its existence clearly implies that there are only finitely many simple groups with a particular centraliser of involution. (In fact, it is rare for more than two or three simple groups to have some centraliser of involution in common). Brauer realised the possibilities of this and proposed the idea of a classification scheme.

Secondly we have the fundamental result of Walter Feit and John Thompson, published in 1963[24], one form of which is that every finite non-abelian simple group has even order, thus bringing every such group within the scope of Brauer's scheme.

From the publication of these two results until the final papers which settled the classification (see for example, Griess[38] in which was proved the existence and uniqueness of the largest sporadic group, F_1), many mathematicians have been involved in developing new techniques with which to attack the problem.

First we have the centraliser of involution method in which was supposed that a centraliser of some involution (of no specific group) had a certain form, usually taken from already existing configurations. With this limited information, delicate character theory of the type pioneered by Brauer and Suzuki (given in detail by Higman[47]) was used to produce a set of possible orders of a hypothetical simple group containing the given centraliser. It was then necessary to either refer to some previous descriptive paper proving the existence and uniqueness of a simple group with such an order, or if such a paper did not exist, to seek an existence theorem from the information known.

Researchers were sometimes then faced with the fact that there was no known simple group of a particular order and the possibility arose of the existence of a new simple group. In this way several of the sporadic groups were first discovered and characterised, for example the four Janko groups, the Lyons and O'Nan groups, and the Held group which arose from Held's attempt to characterise the Matthieu group M_{24} and PSL(5,2) which contain isomorphic centralisers of involution. He found that one possible configuration could not be eliminated in any other way than by supposing the existence of a new group.

Another technique, which has dominated the more recent studies, is the investigation of groups via the structure of local subgroups, that is, normalisers of p-subgroups where p is a prime, called local analysis. There is now a very powerful theory of local analysis, based originally on the group-theoretic results of Thompson in the Odd Order paper and refined by Glauberman and Bender among many others. Here we also mention Thompson's results on N-groups, that is, insoluble groups all of whose local subgroups are soluble. In this work, which eventually stretched to over 400 pages of published research, Thompson in particular classified all the minimal simple insoluble groups, those with no soluble subgroups.

Many authors have also given general classifications on finite groups with restrictions on the form of their Sylow 2-subgroups, for example, to abelian[65], dihedral[36], [37] or quaternion[14], and also with restrictions on centralisers of arbitrary subgroups, for example, abelian[58], which extends to nilpotent[22].

However the initial interest of this thesis lay not in these "even" approaches, but in the odd prime structure, and with this restriction there are fewer publications. One of the earliest and most important of these is "Odd characterisations of finite simple groups" by G. Higman[47], which was the first comprehensive survey of such results, and is a basis for more recent researches on the subject. Nevertheless, the greatest interest in odd prime structure was comparatively short-lived. There are several reasons for this. Firstly, the progress of the even classification in the 1970's was so rapid that few felt the need or desire to waste valuable time exploring areas not directly connected with the main theory. Secondly, as was the view of many, it simply did not look as if odd prime structure would produce any significant breakthroughs.

Be that as it may, there does exist a body of work of some importance, almost exclusively concerned with the prime 3, on odd prime structure.

Chapter II consists of a survey of all the known results concerning odd primes and finite groups, in particular introducing us to the CC-subgroup. After listing in Chapter III the important results necessary for continuing, Chapter IV gives a detailed discussion on the techniques involved in proving the results of Chapter II.

Later chapters will expand and generalise the idea of the CC-subgroup, and the thesis will culminate in a classification theorem which will be used to list the CC-subgroups of simple groups.

Before proceeding, it is convenient to give the notation which will be met soon. As much as possible, this notation follows that of [Gorenstein]. All groups considered will be finite and all simple groups non-abelian unless otherwise stated. G always denotes a group, p always a prime and q a prime power, with π a set of primes. $|x|$ will denote the order of the element $x \in G$ and $\pi(n)$ and $\pi(G)$ denote the set of primes dividing respectively the positive integer n and the group G. The cyclic group of order n will be denoted by Z_n, and finally, G−H denotes the set of elements of G which are not in H. References are in square brackets as usual, with textbooks having the author's name within the reference brackets.

CHAPTER II: A MATHEMATICAL SURVEY

Section I: Groups with CC-subgroups of order divisible by 3

Felt and Thompson commenced the studies of the 3-structure of finite groups with a paper published in 1962. Subsequently both of these authors concentrated on 2-structure, but nevertheless their result, which is given now, is of fundamental importance.

Theorem 2.1 (Feit and Thompson[23]). Let G be a group with a self-centralising subgroup of order 3. Then one of the following is true.

(i) There exists a normal nilpotent subgroup N of G such that

$G/N \cong A_3$ or S_3.

(ii) There exists a normal 2-subgroup N of G such that

$G/N \cong A_5 \cong PSL(2,4)$.

(iii) $G \cong PSL(2,7)$.

We can see clearly in the above that restricting the group G to being a simple group provides us with an odd characterisation of A_5 and $PSL(2,7)$.

Now, retaining the hypothesis of Theorem 2.1, let us denote the self-centralising subgroup of order 3 by M. Clearly, M has the property that for all $x \in M^{\#}$, $C_G(x) = M$. This led to the following generalisation.

Definition 2.2. Let M be a subgroup of a group G. Then M is said to be a CC-subgroup of G, written $M \leq_{CC} G$, if and only if

(i) $1 < M < G$, and

(ii) for all $x \in M^{\#}$, $C_G(x) \leq M$.

Condition (i) is included to remove trivial cases.

Note: Clearly in condition (ii) of the definition, equality holds if and only if M is abelian.

It is easily proved that the following is true. (see, for example[43: Theorem 2.3]).

Theorem 2.3. Let $M \leq_{CC} G$. Then

(i) M is a Hall-subgroup of G, and

(ii) If $N_G(M) \neq M$ then $N_G(M)$ is a Frobenius group with kernel M.

Note: By Thompson[63] the kernel of a Frobenius group is nilpotent.

As a consequence of Theorem 2.3(i) we have the following definition.

Definition 2.4. Let $M \leq_{CC} G$ and $\pi = \pi(M)$. Then G is said to be a πCC-group with πCC-subgroup M.

The subsequent theory of the odd prime structure of finite groups dealt almost exclusively with πCC-groups, specifically when $3 \in \pi$. Hence the title of this section.

Note: We can now restate the hypothesis of Theorem 2.1 in terms of the CC-property, as: Let G be a 3CC-group with Sylow 3-subgroup of order 3.

In the 1960's, Marcel Herzog investigated the CC-property subject to the restriction that the CC-subgroup was cyclic[44], [45] or elementary abelian[43], clearly the simplest cases of nilpotency (see the note following Theorem 2.3). Although Herzog's work on the latter case proved fruitful, and will be given later in part (Theorems 2.8 to 2.11), his results for the cyclic case–an immediate generalisation of Theorem 2.1–were weighed down by unwanted restrictions and will not be given here. In this case, however, the initial breakthrough was made by W. B. Stewart.

Theorem 2.5 (Stewart[56]). Let $M \leq_{CC} G$ and suppose that M is abelian, 3 divides $|M|$ and $|N_G(M)/M| \leq 2$. Then one of the following is true.

(i) There exists a nilpotent normal subgroup N of G such that

$$G/N \cong N_G(M).$$

(ii) There exists a normal elementary abelian 2-subgroup of G such that $G/N \cong PSL(2, 2^n)$ for some $n \geq 2$.

(iii) $G \cong PSL(2, q)$ for some odd q.

A short argument, given in Chapter IV, shows that if $M \leq_{CC} G$, 3 divides $|M|$ and M is cyclic then indeed $|N_G(M)/M| \leq 2$, and so Theorem 2.5 can be applied.

Later, Herzog himself extended the result.

Theorem 2.6 (Herzog[46]). Let $M \leq_{CC} G$, and suppose that M is abelian, 3 divides $|M|$ and $|N_G(M)/M| = 2^s = d$ for some $s \geq 0$. Then one of the following is true.

(i) There exists a normal nilpotent subgroup N of G such that

$$G/N \cong N_G(M).$$

(ii) $d=8$ and if $P \in Syl_3(M)$ then $|\Omega_1(P)| = 9$.

(iii) $d=4$, $|M|=9$ and $G \cong PSL(2, 9)$.

(iv) $d=2$, M is cyclic and G has a normal elementary abelian

2-subgroup N such that $G/N \cong PSL(2,2^n)$ for some $n \geq 2$.

(v) $d=2$ and $G \cong PSL(2,q)$ for some odd q.

Theorem 2.7 (Herzog[46; Theorem 2]). Let $M \leq_{CC} G$ and suppose that M is abelian, 3 divides $|M|$ and $|N_G(M)/M| \leq 8$. Then $|N_G(M)/M|$ is a power of 2.

Both of these theorems were proved using the following results of Herzog on the case where a CC-subgroup is elementary abelian, Theorems 2.8 to 2.11. Throughout them, M will denote a CC-subgroup of a group G, $m=|M|$, and $d=|N_G(M)/M|$.

Theorem 2.8[43; Theorem 3.1]. Suppose that $d \neq 1$ or $m-1$, $M \ntrianglelefteq G$ and M is not a nonabelian p-group with $|M/M'| < 4d^2$. Then either

(i) $d < \sqrt{(m-1)}$, or

(ii) $d > \sqrt{m-1}$ and M is elementary abelian.

Theorem 2.9[43; Theorem 4.2, Corollaries 4.2 and 4.6]. Suppose that the hypothesis of Theorem 2.8 holds, and also that 3 divides $|M|$. Then all of the following is true.

(i) M has a normal Sylow 3-subgroup P with $P \leq Z(M)$.

(ii) M is abelian of odd order.

(iii) If d is odd then part (ii) of Theorem 2.8 holds.

Theorem 2.10[43; Theorem 4.4]. Suppose that M is a noncyclic elementary abelian 3-group. Then one of the following holds.

(i) $M \trianglelefteq G$ and so G is Frobenius with kernel M.

(ii) $d=(m-1)/2$ and G is simple.

(iii) $d=m-1$ and G is simple.

(iv) $d=m-1$ and G contains a simple normal subgroup of index 2.

Theorem 2.11[43; Theorem 5.1]. Suppose that the hypothesis of Theorem 2.8 is satisfied, 3 divides $|M|$ and d is odd. Then $m=3^n$, $d=(m-1)/2$ and $G \cong PSL(2,3^n)$, where $n>1$ and n is odd.

Before proceeding, it is necessary to introduce a concept first studied by G. Higman[47]. He considered groups in which centralisers of 3-elements are 3-groups, but we will work with a more general condition.

Definition 2.12. A <u>$C\pi\pi$-group</u> is a group in which centralisers of all non-trivial π-elements are π-groups. (By convention we suppose such a group is not a π-group).

The importance of this idea is made obvious by the following two results.

Proposition 2.13. A πCC-group is a $C\pi\pi$-group.

Proposition 2.14. A Cpp-group with abelian Sylow p-subgroups is a pCC-group.

Although Higman, and later Fletcher, studied $C\pi\pi$-groups in their own right, we use the idea purely for convenience, as some important theorems on πCC-groups were proved in the context of $C\pi\pi$-groups. In particular we have the following.

Theorem 2.15(Fletcher[31; Corollary]). A simple C33-group has abelian Sylow 3-subgroups.

We also have the following two results which though of interest to the survey, were obtained at a later date than the present results and do not take any direct part in the development of the CC-theory as given in this section(though the latter will be used eventually in Chapter VI).

Theorem 2.16(Arad and Herzog[11]). C33-groups of even order are 3CC-groups.

Theorem 2.17(Williams[66]). Suppose that $2\notin\pi$ and let G be a $C\pi\pi$-group such that $O^{\pi}(G)=G$ and G has a nilpotent Hall π-subgroup. Then G is a πCC-group.

Fletcher proceeded to give the following specific case using Theorems 2.6, 2.10 and 2.11, and also incorporating some of Higman's results.

Theorem 2.18(Fletcher[32]). Let G be a C33-group with Sylow 3-subgroup M of order 9. Then one of the following is true.

(a) M is cyclic and $G/O_{3'}(G)$ is isomorphic to either

(i) M,

(ii) $N_G(M)$, a dihedral group of order 18,

(iii) PSL(2,8) with $O_{3'}(G)$ being an elementary abelian 2-group,

or (iv) PSL(2,17) or PSL(2,19) with $O_{3'}(G)=1$.

(b) M is elementary abelian and either

(i) $M\trianglelefteq G$,

(ii) $|N_G(M)/M|=4$ and $G\cong PSL(2,9)$,

(iii) $|N_G(M)/M|=8$ and $G\cong PSL(3,4)$,

or (iv) $|N_G(M)/M|=8$ and G has a simple normal subgroup of index 2, isomorphic to PSL(2,9).

At this point we have considered 3CC-groups with Sylow 3-subgroup M given $|M|=3$(Theorem 2.1) and $|M|=9$(Theorem 2.18). It was therefore at this point desirable to seek a generalisation to arbitrary powers of 3. On this line, Pamela Ferguson proved the following result.

__Theorem 2.19__(Ferguson[25]). Let G be a 3CC-group with Sylow 3-subgroup M, and suppose that $N_G(M)/M$ is a non-trivial group of odd order. If M is nonabelian then M is normal in G.

This combines with Herzog's results, Theorems 2.8 to 2.11.

__Theorem 2.20__(Ferguson[25;Theorem 2]). Let $M \leq_{CC} G$, 3 divide $|M|$, M be noncyclic and suppose that $N_G(M)/M$ is a non-trivial group, of odd order not equal to $|M|-1$. Then either

(i) $M \trianglelefteq G$, or

(ii) M is an elementary abelian 3-subgroup of G of order 3^n, and $G \cong PSL(2,3^n)$.

Little further work on the subject has been published without the extensive involvement of Zvi Arad. In 1976, he produced the following important extension of Theorem 2.19.

__Theorem 2.21__(Arad[2;Lemma 2]). Let $M \leq_{CC} G$ and suppose that M is a Sylow 3-subgroup of G. Then either

(i) $M \trianglelefteq G$,

(ii) M is cyclic, or

(iii) M is elementary abelian.

Using this result, it was a short step to the complete classification of 3CC-groups.

__Theorem 2.22__(Arad[2;Theorem A]). Let G be a 3CC-group with Sylow 3-subgroup M. Then one of the following is true.

(i) $M \trianglelefteq G$ and G is Frobenius with kernel M.

(ii) There exists a normal nilpotent subgroup N of G such that

$G/N \cong N_G(M)$. M is cyclic.

(iii) There exists a normal elementary abelian 2-subgroup N of G such that $G/N \cong PSL(2,2^n)$ for some $n \geq 2$.

(iv) $G \cong PSL(2,q)$ for some odd $q \geq 5$.

(v) $G \cong PSL(3,4)$.

(vi) G has a simple normal subgroup of index 2, isomorphic to

$PSL(2,3^n)$ for some $n \geq 2$.

In three papers published in 1976 and 1977, the last two with the aid of Herzog, Arad completed the classification of all finite groups with a CC-subgroup of order divisible by 3.

Theorem 2.23(Arad[3]). Let $M \leq_{CC} G$ and suppose that 3 divides $|M|$. Then one of the following is true.

(i) $M \trianglelefteq G$ and G is Frobenius with kernel M.

(ii) $N_G(M)=M$.

(iii) M is an elementary abelian Sylow 3-subgroup of G.

(iv) M is a cyclic subgroup of G of odd order.

As groups arising in the cases (iii) and (iv) had already been classified, by Theorems 2.22 and 2.6 respectively, it only remained to classify groups arising from case (ii).

Theorem 2.24(Arad and Herzog[9]). Let $M \leq_{CC} G$, 3 divide $|M|$, and suppose that $N_G(M)=M$. Then either

(i) G is Frobenius with complement M, or

(ii) $G \cong PSL(2,q)$ for some $q \geq 5$.

In particular Arad listed all simple groups satisfying the conditions of Theorem 2.23.

Theorem 2.25(Arad[3;Theorem B]). Let G be a simple group with a CC-subgroup of order divisible by 3. Then G is isomorphic to either

(i) $PSL(3,4)$,

(ii) $PSL(2,2^n)$ for some $n \geq 2$,

(iii) $PSL(2,3^n)$ for some $n \geq 2$, or

(iv) $PSL(2,p^n)$ for some prime $p>3$ such that $p^n \equiv \pm 5 \bmod 12$.

Conversely, all the simple groups listed contain a CC-subgroup of order divisible by 3.

The various cases of Theorem 2.23 are not necessarily mutually exclusive. We can rectify this by restating the theorem as follows.

Theorem 2.26. Let G be a finite group with a CC-subgroup M of order divisible by 3. Then one and only one of the following is true.

(i) $M \trianglelefteq G$ and G is Frobenius with kernel M.

(ii) $N_G(M)=M$ and G is Frobenius with complement M.

(iii) $N_G(M)=M$ and $G \cong PSL(2,q)$ for some $q \geq 5$.

(iv) $M<N_G(M)<G$ and M is a noncyclic elementary abelian Sylow 3-subgroup of G.

(v) $M<N_G(M)<G$ and M is a cyclic subgroup of G of odd order.

Remark: The corresponding result in the case when 2 divides the order of the CC-subgroup was completely settled in the early 1960's by Suzuki. We state it here for completeness.

Theorem 2.27(Suzuki[59],[60]). Let $M \leqslant_{CC} G$ with 2 dividing $|M|$. Then one of the following is true.

(i) G is Frobenius with M as kernel or complement.

(ii) G is isomorphic to $PSL(2,2^n)$ or $Sz(2^{2n+1})$ for some n, and M is either a Sylow 2-subgroup or a Sylow 2-normaliser.

Section II: Closure and Homogeneity

Zvi Arad and Pamela Ferguson, who figured prominently in the previous section, were also involved in developing a theory to accompany the related concepts of closure and homogeneity, ideas introduced by Reinholt Baer[12] in which he gave the basic theorems that are required for further development. These results will follow immediately after the definitions.

Definition 2.28. A group G is said to be π-closed if the set of all its π-elements forms a π-subgroup.

In this case it is easy to see that the subgroup so formed is a normal Hall π-subgroup, and hence also characteristic. In particular, a π'-group is π-closed as the set of all its π-elements consists of the identity only, which is a normal π-subgroup. Clearly any primes in π not dividing the order of G do not affect the situation, and so it can generally be assumed that π is a subset of $\pi(G)$.

Definition 2.29. A group G is said to be π-homogeneous if, for all π-subgroups H of G, $N_G(H)/C_G(H)$ is a π-group.

It is clear, as in the above remarks, that we can suppose that π is a subset of $\pi(G)$.

We can now state Baer's results.

Theorem 2.30(Baer[12; Lemma 2.1]). If G is a π-closed group then G is π'-homogeneous.

Theorem 2.31[12; Lemma 2.3]. Subgroups, direct products and factor groups of π-homogeneous groups are π-homogeneous.

Note: Analogously to Theorem 2.31, it is obvious that subgroups and factor groups of π-closed groups are π-closed.

Theorem 2.32[12; Lemma 2.4]. Suppose that G is a π'-homogeneous group and $K \trianglelefteq G$. If both K and G/K are π-closed then G is π-closed.

The following definition is made for convenience.

Definition 2.33. A group is said to be π-separable if its composition factors are either π-groups or π'-groups.

Clearly π-separability and π'-separability are the same concept.

We can now give the first important partial converse to Theorem 2.30.

Theorem 2.34[12;Lemma 2.5]. A group G is π-closed if and only if G is both π'-homogeneous and π'-separable.

In particular we have the following.

Corollary 2.35. Suppose G is soluble. Then G is π-closed if and only if G is π'-homogeneous.

A theorem of Frobenius(see, for example[Rose; 10.47]) can now be given in terms of the above definitions.

Theorem 2.36. A group G is p-closed if and only if G is p'-homogeneous.

The problem, then, was to generalise Theorem 2.36.

Remark: It is necessary to stress the significance of the position of the "′", as interchange will usually cause theorems such as Theorem 2.36 to break down, as can be seen from the following counterexamples.

Example 2.37. A_5 is 5′-homogeneous but not 5-closed. [A_5 has only six conjugacy classes of 5′-subgroups(of orders 1, 2, 3, 4, 6 and 12) and it is easily shown that each of these satisfy the criterion for 5′-homogeneity. Clearly, as $5 \in \pi(A_5)$ and A_5 is simple, A_5 is not 5-closed].

Example 2.38. The simple groups $PSL(2, 2^{2n+1})$ where $n \geq 1$ are 3′-homogeneous but not 3-closed(see, for example[2]).

The best results obtained by Baer, Theorems 2.39 and 2.41, use Theorem 2.36 in their proofs.

Theorem 2.39[12;Theorem 5.2]. A group G is both p-homogeneous and p'-homogeneous if and only if G is the direct product of a p-group and a p'-group.

This result led Baer to make the following conjecture.

Conjecture 2.40. A group G is both π-homogeneous and π'-homogeneous if and only if G is the direct product of a π-group and a π'-group.

We shall investigate this conjecture later in this section, and also in Section II of Chapter IV.

Theorem 2.41[12;Theorem 5.3]. A group G is π-closed if and only if G is π'-homogeneous and G satisfies the following condition.

(*) If R is a maximal π-subgroup of G and P is a Sylow p-subgroup of G for some $p \notin \pi$ then $\langle R, P\rangle$ is a (π, p)-group.

Condition (*) is not very helpful. Much later, after an hiatus of fifteen years, Arad, in his doctoral thesis extracts published in 1974, improved slightly on Baer's results. The first theorem is in relation to the prime 2.

Theorem 2.42(Arad[1;Lemma 2.6]). A group G is 2-closed if and only if G is 2'-homogeneous.

Thus, taking Theorem 2.36 into account, the prime 2 has a certain symmetry which odd primes do not.

Arad then extended Frobenius' theorem.

Theorem 2.43(1;Theorem A]). Suppose that $2\in\pi$ and that all π-subgroups of a group G are 2-closed. Then G is π'-closed if and only if G is π-homogeneous.

He then attacked the general problem of equating π'-closure with π-homogeneity.

Theorem 2.44[1;Theorem B]. Suppose that $2\in\pi$. Then a group G is π'-closed if and only if G is π-homogeneous and any one of the following conditions holds.

(i) $3\notin\pi(G)$.

(ii) The π'-subgroups of G are soluble.

(iii) G has abelian or dihedral Sylow 2-subgroups.

(iv) Every chain of subgroups in G has length at most 7.

(Note that a chain of subgroups is an ordered list of subgroups, each of which is maximal in the next).

Clearly, the aim was to extend this list, or, alternatively, to seek more general statements. The latter view held, as will be seen after the following corollary of Theorem 2.44.

Corollary 2.45[1;Theorem C]. Suppose G is a group such that $|\pi(G)|=4$. Then G is π'-closed if and only if G is π-homogeneous.

It is convenient now to make the following definition.

Definition 2.46. A group G is said to be a D_π-group if G has a Hall π-subgroup, all Hall π-subgroups are conjugate, and every π-subgroup is contained in a Hall π-subgroup.

The last theorem of Arad from 1974 can now be given.

Theorem 2.47[1;Theorem D]. Let G be a π-homogeneous D_π-group. Then G is π'-closed if and only if one of the following holds.

(i) $3\notin\pi(G)$.

(ii) The proper subgroups of G are π'-closed.

Ferguson then gave an important result relating to 3′-homogeneity.

<u>Theorem 2.48</u>(Ferguson[26]). The only finite nonabelian simple 3′-homogeneous groups are $Sz(2^{2n+1})$ and $PSL(2,2^{2n+1})$ for some $n\geq 1$.

(Note that the Suzuki groups are obviously 3′-homogeneous as their orders are prime to 3, and the linear groups are 3′-homogeneous by Example 2.38).

Extending Theorem 2.48, Arad showed that in a sense the above linear groups are the only groups to violate the equivalence of 3′-homogeneity and 3-closure.

<u>Theorem 2.49</u>(Arad[2;Theorem B]). Let G be a 3′-homogeneous group and suppose that $PSL(2,2^{2n+1})$ is not involved in G for any $n\geq 1$. Then G is 3-closed.

(A group H is said to be involved in a group G if H is a factor group of a subgroup of G).

In the same paper, Arad gave the following.

<u>Theorem 2.50</u>[2;Theorem C]. Suppose that $2\notin\pi$ and $3\in\pi$. Then π-homogeneity is equivalent to π'-closure.

One year later, in 1977, Ferguson gave an important improvement over both Theorems 2.36 and 2.47.

<u>Theorem 2.51</u>(Ferguson[28]). Let G be a π-homogeneous D_π-group. Then G is π'-closed.

This result was proved without reference to any of the previous theorems except Baer's basic results, Theorems 2.30, 2.31, 2.32 and 2.34, by showing that the group G satisfies the hypothesis of a certain character-theoretic result of Brauer and Suzuki(Theorem 3.35) which immediately implies π'-closure.

There followed a period of seven years when there was no published work on the subject, until in 1984, Arad and Chillag gave the following theorem, the proof of which uses to good effect the complete classification of the finite simple groups.

<u>Theorem 2.52</u>(Arad and Chillag[8]). Suppose that $2\notin\pi$. Then a group G is π'-closed if and only if G is π-homogeneous.

In later chapters we shall see extensive use of this fundamental theorem to prove results of major significance. However we shall end this section by noting that Theorem 2.52 can be used to prove the truth of Conjecture 2.40, as will be seen in Chapter IV.

Theorem 2.53. A group G is both π-homogeneous and π'-homogeneous if and only if G is the direct product of a π-group and a π'-group.

Section III: Further odd characterisations of simple groups

For completeness, there now follows a listing of the remaining important theorems concerning odd characterisations of simple groups. As their proofs are so diverse in technique it is not possible in this thesis to give an adequate description and so no discussion of them will be given.

The first two theorems are contained in G. Higman's Odd Characterisation Lecture Notes.

Theorem 2.54(Higman[47; Theorem 11]). Let G be a simple group with elements x and y of orders 3 and 5 respectively, such that $C_G(x)=\langle x\rangle\times\langle y\rangle$ and $C_G(y)$ is insoluble. Then $G\cong PSU(3,4)$.

Theorem 2.55[47; Theorem 12]. Let G be a simple group with a subgroup isomorphic to $D_6\times D_{10}$ which is the normaliser in G of both the subgroup of order 3 and the subgroup of order 5 contained in it. Then $G\cong J_1$.

We now have an alternative characterisation involving PSL(2,7) to that of Theorem 2.1.

Theorem 2.56(Collins[17; Theorem A]). Let G be a simple group such that whenever K is a subgroup of odd order which is either abelian or a p-group, then $|N_G(K)/C_G(K)|\leq 3$. Then G is isomorphic to either PSL(2,7) or $PSL(2,2^n)$ for some $n\geq 2$.

The next theorem combines results from two papers to give a characterisation of certain symplectic groups.

Theorem 2.57(Hayden[40],[41]). Let G be a simple group containing a 3-central element x of order 3 such that $C_G(x)\cong C_{Psp(4,3^m)}(\alpha)$ for some 3-central element α of order 3 in $Psp(4,3^m)$ for some $m>1$, and that for all elements z in the centre of $C_G(x)$, $C_G(z)\cong C_G(x)$, and that not all 3-central elements of G belong to the same conjugacy class of G. Then $G\cong Psp(4,3^m)$.

Note: The corresponding result for Psp(4,3) given by Hayden[39] was shown to be wrong, and corrected, by Prince in Theorem 2.69 below.

We now have an extension of Theorem 2.54.

Theorem 2.58(Stewart[57]). Let G be a simple group containing an element x of order 3 such that $C_G(x)\cong Z_{15}$. Then G is isomorphic to either PSL(2,16), PSL(2,29), PSL(2,31) or PSU(3,4).

The next theorem, as with many of the other results in this section, requires 2-local analysis in its proof, although it contains no 2-structure in its hypothesis.

Theorem 2.59(O'Nan[51]). Let G be a simple group with an element x of order 3 such that $C_G(x)/\langle x\rangle$ is isomorphic to either PSL(2,q), PGL(2,q) or PΓL(2,q). Suppose also that there exists an elementary abelian 3-subgroup P of G, of order 9, such that all non-identity elements of P are conjugate in G. Then G is isomorphic to one of the following:

PSU(3,5), PSL(3,7), M_{22}, M_{23}, M_{24}, HS, Ru.

Later O'Nan improved this result as follows.

Theorem 2.60(O'Nan[52]). Let G and P satisfy the hypothesis of Theorem 2.59 minus the condition that the non-identity elements of P are conjugate. Then the only possibilities for G other than those listed in Theorem 2.59 are PSL(5,2), J_2 and Psp(4,4).

A result related to C33-groups is the following.

Theorem 2.61(Yoshida[69]). Let G be a simple group such that for each element x of order 3, $C_G(x)$ is isomorphic to either $Z_3 x Z_3$ or $Z_3 x A_4$. Then G is isomorphic to one of the following:

A_6, A_7, PSL(3,4), PSL(3,7), PSU(3,5), M_{22}.

Arad and Chillag improved Theorem 2.61.

Theorem 2.62(Arad and Chillag[5]). Let G be a simple group such that for each element x of order 3, $C_G(x)$ is either a 3-group or is isomorphic to $Z_3 x A_4$. Then G is isomorphic to one of the following:

PSL(2,q) where $q=2.3^n\pm1$ for some $n\geq1$,

PSL(2,3^n) for some $n\geq2$,

PSL(2,8), PSL(3,4), PSL(3,7), A_7, PSU(3,5), M_{22}.

The next theorems concern the presence or absence of elements of order 6.

Theorem 2.63(Hayden and Winter[42]). Let G be a simple group with a self-centralising element of order 6 such that G has only one conjugacy class of involution. Then G is isomorphic to one of the following:

M_{11}, J_1, PSL(3,3), PSL(2,11), PSL(2,13).

Theorem 2.64(Frohardt[34]). Let G be a simple group with an element x of order 3 such that $C_G(x)$ is abelian of odd order, no non-trivial 2-subgroup of G is normalised by $\langle x\rangle$ and G has a cyclic Sylow 3-subgroup.

Then $G\cong PSL(2,2^{2n+1})$ for some $n\geq 1$.

Theorem 2.65(Fletcher, Stewart and Stellmacher[33]). Let G be a simple group with no element of order 6, such that some non-trivial 2-subgroup of G is normalised by an element of order 3. Then G is isomorphic to one of the following:

$PSL(2,3^n)$, $PSL(2,p^n)$ for some $p^n\equiv\pm 5 \bmod 12$,

$PSL(2,2^{2n})$, $PSU(3,2^n)$, $PSL(3,2^n)$ for some $2^n\not\equiv 1 \bmod 9$.

see also L M Gordon Bull Austral MS

Theorem 2.66(Higman[48]). Let G be a simple group with an element x of order 3 and cyclic Sylow 3-subgroup such that $N_G(\langle x\rangle)=\langle H,\tau\rangle$ where H is an abelian subgroup of G of odd order and $(h\tau)^2=1$ for all $h\in H$. Then G is isomorphic to one of the following:

$PSL(2,2^n)$ for some $n\geq 2$,

$PSL(2,q)$ for some odd $q>5$ and $q\equiv\pm 5 \bmod 12$.

This result was improved by Dickson.

Theorem 2.67(Dickson[19]). Let G be as in the hypothesis of Theorem 2.66 except that now H is an abelian subgroup of order not divisible by 4. Then G is isomorphic to one of the following:

$PSL(2,2^n)$ for some $n\geq 2$,

$PSL(2,q)$ for some $q\equiv\pm 5, \pm 7$ or $\pm 11 \bmod 24$.

A result on CC-subgroups of order divisible by 3 is the following alternative characterisation of $PSL(2,3^n)$ which was published after Arad's classification(Theorem 2.23).

Theorem 2.68(Ferguson and Smith[29]). Let G be a simple 3CC-group with Sylow 3-subgroup M such that M is abelian and G has at least two conjugacy classes of elements of order 3. Then G is isomorphic to $PSL(2,3^n)$ for some $n\geq 2$.

The next result corrects a mistake in Hayden[39] as promised in the note following Theorem 2.57. First, it is convenient to state an hypothesis.

Hypothesis A. G is a group with an element x of order 3 such that

(i) $C_G(x)$ is isomorphic to the centraliser of a 3-central element of order 3 in $Psp(4,3)$, and

(ii) $C_G(x)$ contains an elementary abelian subgroup of order 3^3 which does not normalise any non-trivial 3′-subgroup of G.

<u>Theorem 2.69</u>(Prince[53]). Let G be a simple group satisfying Hypothesis A.

(i) If x is <u>not</u> conjugate to its inverse then $G \cong Psp(4,3)$.

(ii) If x is conjugate to its inverse then $G \cong Psp(6,2)$.

In connection with Theorem 2.69, we make the following definition.

<u>Definition 2.70</u>. We will say that two groups have the same 3-centraliser structure if they have the same number of conjugacy classes of elements of order 3 and the corresponding centralisers are isomorphic.

An extension of Theorem 2.69 can now be given.

<u>Theorem 2.71</u>[53].

(i) Let G be a simple group with the same 3-centraliser structure as $Psp(4,3)$. Then $G \cong Psp(4,3)$.

(ii) Let G be a simple group with the same 3-centraliser structure as $Psp(6,2)$. Then $G \cong Psp(6,2)$.

Prince later applied the same ideas to the group ${}^2D_4(2)$.

<u>Theorem 2.72</u>(Prince[54]). Let G be a simple group with the same 3-centraliser structure as ${}^2D_4(2)$. Then $G \cong {}^2D_4(2)$.

The next result requires two definitions.

<u>Definition 2.73</u>. A group G is said to be a <u>perfect central extension</u> of another group H if $G/Z(G) \cong H$ and $G' = G$.

<u>Definition 2.74</u>. Let G be a group with Sylow p-subgroup P. A subgroup W of P is said to be <u>weakly closed</u> in P with respect to G if whenever $W^g \le P$ for some $g \in G$ then $W^g = W$.

<u>Theorem 2.75</u>(Stafford[55]). Let G be a simple group with an element x of order 3 such that $C_G(x)$ is a perfect central extension of M_{22}, and that for some Sylow 3-subgroup P of G containing x, $\langle x \rangle$ is <u>not</u> weakly closed in P with respect to G. Then $G \cong J_4$.

Lastly we have another result condensed from more than paper.

<u>Theorem 2.76</u>(Thomas[62]). Let G be a simple group satisfying the following three conditions.

(i) The centraliser of each element of order 3 has a Sylow

2-subgroup which is either cyclic, dihedral, semidihedral or generalised quaternion.

(ii) The centraliser of some involution of G has a non-cyclic Sylow 3-subgroup.

(iii) All 2-local subgroups are 2-constrained.

Then G is isomorphic to either PSU(4,3), Psp(4,3) or $G_2(3)$.

Note: A group H is p-constrained if whenever P is a Sylow p-subgroup of $O_{p',p}(H)$ then $C_H(P) \leq O_{p',p}(H)$. In particular, every soluble group is p-constrained for every prime p.

CHAPTER III: RESULTS REQUIRED IN THIS THESIS

Section I: General Group Theory

This section consists of a listing of results from general group theory which are useful in later chapters whether for convenience or necessity. However we start with some unnumbered definitions.

First we redefine some terms appearing in the previous chapter.

Definition. Let M be a subgroup of a group G. Then M is said to be a CC-subgroup of G, denoted $M \leq_{CC} G$, if and only if it is proper, non-trivial and for all $x \in M^{\#}$, $C_G(x) \leq M$.

Now, although we have not yet proved that a CC-subgroup is a Hall subgroup (Theorem 3.36), we define a πCC group.

Definition. Suppose that $M \leq_{CC} G$, and let $\pi = \pi(M)$. Then we call G a πCC-group, with πCC-subgroup M.

Definition. A group G is said to be a $C\pi\pi$-group for some set π if and only if the centraliser of every non-trivial π-element is a π-group.

Definition. A group G is said to be a D_π-group if G has a Hall π-subgroup, all Hall π-subgroups of G are conjugate and every π-subgroup of G is contained in a Hall π-subgroup.

Definition. A group G is π-separable if the composition factors of G are either π-groups or π'-groups.

Definition. A group G is π-soluble if the composition factors of G are either π'-groups or p-groups for $p \in \pi$.

A useful idea encountered frequently in later chapters is that of a T.I-set.

Definition. A subgroup H of a group G is said to be a T.I-set in G if for all $g \in G - N_G(H)$, $H \cap H^g = 1$.

We now give the standard definitions of Frobenius and Zassenhaus groups.

Definition. Suppose that a group G has a subgroup H such that H is a T.I-set in G and $N_G(H) = H$. Then G is called a Frobenius group with complement H.

Note: It is obvious that any conjugate of H is also a complement.

Definition. A group G is a Zassenhaus group of degree n if it satisfies the following three conditions.

(i) G is doubly transitive on n symbols.

(ii) Any non-identity element of G has at most two fixed points.

(iii) G has no regular normal subgroup.

Lastly we have the following.

Definition. $P\Gamma L(2,q) = \mathrm{Aut}PSL(2,q)$.

$P\Sigma L = \mathrm{Aut}\, PSL$
$P\Gamma L \cdot \mathrm{Aut}\, \mathbb{P}_1(\mathbb{F}_q)$

We are now in a position to list the required results.

In this list there is no special significance in the ordering. However there is a general progression from elementary results to much deeper ones. It should be noted that proofs will only appear if there is no explicit reference available. In any particular proof, results in this section needed will precede that being proved.

Proposition 3.1 [Rose; Lemma 4.36]. Suppose that $H \trianglelefteq G$. Then $C_G(H) \trianglelefteq N_G(H)$ and $N_G(H)/C_G(H)$ can be embedded in AutH.

Proposition 3.2 [Rose; Lemma 4.38]. If G is a cyclic group then AutG is abelian.

Proposition 3.3 [Rose; Lemma 9.15]. If G is a cyclic group of prime order p then AutG is cyclic of order $p-1$.

Proposition 3.4 [Rose; Lemmas 11.8 and 11.9]. If P is a p-group then $P/\Phi(P)$ is elementary abelian.

Proposition 3.5 [Rose; Theorem 7.4]. Suppose $K \trianglelefteq G$. If K and G/K are both soluble then G is soluble.

Proposition 3.6. If G is a non-trivial soluble group then $F(G) \neq 1$.

Proof. Since G is soluble, every chief factor of G is elementary abelian. In particular G has a non-trivial nilpotent normal subgroup.

Proposition 3.7. Suppose that $K \trianglelefteq G$, H is a Hall π-subgroup of K and all Hall π-subgroups of K are conjugate in K. Then $G = N_G(H)K$.

Proof. $H \leq K$, and so $H^g \leq K^g = K$ for any $g \in G$. Since H^g is a Hall

π-subgroup of K, $H^g=H^k$ for some $k\in K$. Thus $gk^{-1}\in N_G(H)$ and so $g\in N_G(H)K$ as required.

<u>**Theorem 3.8**</u>[Rose; Theorem 11.18]. If G is a soluble group then G is a D_π-group for any π.

<u>**Theorem 3.9**</u>[Gorenstein; Theorem 6.3.6]. If G is π-soluble or π'-soluble then G is a D_π-group.

<u>**Theorem 3.10**</u>[Scott; 9.3.14]. If G has a nilpotent Hall π-subgroup then G is a D_π-group.

<u>**Proposition 3.11**</u>. Suppose N is a minimal normal subgroup of a group G. Then either N is elementary abelian or N is the direct product of isomorphic simple groups.

<u>Proof</u>. Clearly N is characteristically simple. The result follows by [Rose; Theorem 8.10].

<u>**Proposition 3.12**</u>. [Rose; Lemma 10.20]. Let P be a Sylow p-subgroup of a group G. Then any two elements of $Z(P)$ which are conjugate in G are already conjugate in $N_G(P)$.

<u>**Theorem 3.13**</u>[Rose; Theorem 10.21]. Let P be a Sylow p-subgroup of a group G. If $P\leq Z(N_G(P))$ then G has a normal p-complement.

<u>**Theorem 3.14**</u>[Gorenstein; Theorem 5.1.4]. Let ψ be a p'-automorphism of a p-group P which induces the identity on $P/\Phi(P)$. Then ψ is the identity automorphism on P.

<u>**Theorem 3.15**</u>[Gorenstein; Theorem 3.8.2]. Let K be a conjugacy class of p-elements of a group G such that every pair of elements of K generates a p-group. Then K lies in a normal p-subgroup of G.

<u>**Theorem 3.16**</u>[Isaacs; Theorem 7.2]. Let G be a Frobenius group with complement H. Then there exists a normal subgroup K of G, called the kernel, such that $G=HK$ and $H\cap K=1$.

<u>**Theorem 3.17**</u>[Gorenstein; Theorem 5.4.3]. Let P denote either the dihedral group D_2m or the generalised quaternion group Q_2m for

D_{2^n} Q_{2^n}

some $m\geq 3$. Then the following hold.

(i) $|Z(P)|=2$ and $P/Z(P)\cong D_{2^{m-1}}$.

(ii) If $P=D_{2^m}$ then subgroups of P are cyclic or dihedral.

(iii) If $P=Q_{2^m}$ then subgroups of P are cyclic or generalised quaternion.

__Theorem 3.18__(Thompson[63]). If G admits a fixed-point-free automorphism of prime order then G is nilpotent.

__Theorem 3.19__[Gorenstein; Theorem 10.3.1][Huppert I; V 8.18]. Let G be a Frobenius group with kernel K and complement H. Then the following hold.

(i) $N_G(H)=H$.

(ii) H is a T.I-set in G.

(iii) $K\trianglelefteq G$.

(iv) $G=HK$, $H\cap K=1$.

(v) $|H|$ divides $|K|-1$.

(vi) H can be embedded in $\mathrm{Aut}K$.

(vii) K is nilpotent.

(viii) If p is odd then Sylow p-subgroups of H are cyclic.

(ix) Sylow 2-subgroups of H are cyclic or generalised quaternion.

(x) $Z(H)\neq 1$.

(xi) If $|H|$ is odd then H is metacyclic.

(xii) If $|H|$ is even then $|Z(H)|$ is even and K is abelian.

(xiii) If H is abelian then H is cyclic.

__Theorem 3.20__[Feit and Thompson]. Groups of odd order are soluble.

__Theorem 3.21__. Groups with non-trivial cyclic Sylow 2-subgroups are soluble.

__Proof__. Let G be a group of even order with a cyclic Sylow 2-subgroup, P. Let g be an element of odd order in $N_G(P)$. Then g acts as an automorphism of P by conjugation and so induces an automorphism on $P/\Phi(P)$. But $P/\Phi(P)\cong Z_2$ and $\mathrm{Aut}Z_2\cong 1$. Thus g induces the trivial automorphism on $P/\Phi(P)$ and so by Burnside's Theorem 3.14, g acts trivially on P, that is, g centralises P. Since we also have $P\leq C_G(P)$ it follows that $N_G(P)\leq C_G(P)$, that is, $N_G(P)=C_G(P)$. Thus $P\leq Z(N_G(P))$ and we use Theorem 3.13 to obtain that G has a normal 2-complement, K, say. Now K is soluble by Theorem 3.20, and since $G/K \cong P$, Proposition 3.5 implies that G is soluble.

<u>Proposition 3.22</u>. Suppose $K \leq H \leq G$ and that H is a T.I-set in G. Then $N_G(K) \leq N_G(H)$.

<u>Proof</u>. Let $g \in N_G(K)$. Then $K^g = K$. Thus $H \cap H^g \neq 1$. Then since H is a T.I-set in G, $g \in N_G(H)$, as required.

<u>Proposition 3.23</u>[Rose; 7.3]. Suppose that H, K, and L are subgroups of a group G such that $K \leq H$. Then $H \cap (KL) = K(H \cap L)$.

<u>Proposition 3.24</u>. Let H be a Hall π-subgroup of G and suppose N is a normal subgroup of G such that $(|H|, |N|) \neq 1$. Then $H \cap N \neq 1$.

<u>Proof</u>. Let p divide $(|H|, |N|)$ and let x be a p-element of N. Since $p \in \pi$ and H is a Hall π-subgroup, $x^g \in H$ for some $g \in G$. But $N \trianglelefteq G$ and so $x^g \in N$. Hence result.

<u>Theorem 3.25</u>[Huppert III; XI Lemmas 3.1 and 3.10 and Theorem 3.9]. Let G be the Suzuki group $Sz(q)$ where $q = 2^{2n+1}$ for some $n \geq 1$, and let $r = 2^n$. Then the following are true.

(i) If p is odd then Sylow p-subgroups of G are cyclic.

(ii) G has cyclic CC-subgroups of orders $q \pm 2r + 1$, denoted by H_i for $i = 1, 2$ such that $|N_G(H_i)/H_i| = 4$ for $i = 1, 2$.

(iii) G has a Frobenius CC-subgroup F of order $q^2(q-1)$ with a cyclic complement of order $q-1$.

<u>Theorem 3.26</u>(Suzuki[59]). A Zassenhaus group of odd degree is simple and is isomorphic to either $PSL(2, 2^n)$ or $Sz(2^{2n+1})$ for some $n \geq 1$.

<u>Theorem 3.27</u>(Suzuki[60]). Let $M \leq_{CC} G$ such that 2 divides $|M|$. Then one of the following is true.

(i) G is Frobenius with M as kernel or complement.

(ii) G is a Zassenhaus group of odd degree and M is either a Sylow 2-subgroup or a Sylow 2-normaliser.

<u>Theorem 3.28</u>(Gorenstein and Walter[36], [37]). Let G be a group with dihedral Sylow 2-subgroups. Then $G/O_{2'}(G)$ is isomorphic to one of the following.

(i) A Sylow 2-subgroup of G.

(ii) The alternating group A_7.

(iii) A subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ for some odd q.

__Theorem 3.29__(Brauer and Suzuki[14]). Let G be a group with generalised quaternion Sylow 2-subgroups. Then $|Z(G/O_{2'}(G))|=2$ and denoting $G/O_{2'}(G)$ by G^*, $G^*/Z(G^*)\cong H$ is a group with dihedral Sylow 2-subgroups and $O_{2'}(H)=1$.

__Proof__. G^* has generalised quaternion Sylow 2-subgroups and so factoring out the centre of such a group, of order 2, produces a group with dihedral Sylow 2-subgroups. Now the inverse image of $O_{2'}(H)$ in G^* is Z_2x(a group of odd order) and hence equals Z_2 since $O_{2'}(G^*)=1$. Hence $O_{2'}(H)=1$.

__Theorem 3.30__. Let G be a non-soluble Frobenius group with kernel K and complement H. Then the following are true.

(i) H has even order.

(ii) K is abelian of odd order.

(iii) G has generalised quaternion Sylow 2-subgroups.

(iv) $Z(G/O_{2'}(G))\cong Z_2$.

(v) $G/O_{2'}(G)/Z(G/O_{2'}(G))\cong PSL(2,p)$ for some p.

__Proof__. (i) Suppose that H had odd order. Then H would be soluble and hence so also would G, a contradiction.

(ii) It is immediate that K has odd order. By [Isaacs; Lemma 7.21], K is abelian.

(iii) A Sylow 2-subgroup of G is contained in H. By Theorem 3.19, H has cyclic or generalised quaternion Sylow 2-subgroups. If H had cyclic Sylow 2-subgroups then by Theorem 3.21, H would be soluble and hence G would be soluble, a contradiction.

(iv) Follows immediately from (iii) and Theorem 3.29.

(v) By Theorem 3.17, factoring out the centre of a group with generalised quaternion Sylow 2-subgroups leaves a group with dihedral Sylow 2-subgroups. Thus $L=G/O_{2'}(G)/Z(G/O_{2'}(G))$ has dihedral Sylow 2-subgroups. Also, $O_{2'}(L)=1$ as in Theorem 3.29 and so L has one of the three forms listed in Theorem 3.28.

Now if L was a 2-group then $G/O_{2'}(G)$ and hence G would be soluble, a contradiction. Also it is obvious that A_7 has a noncyclic elementary abelian 3-subgroup and so L, being a factor group of the complement H, would have a noncyclic 3-subgroup, contradicting Theorem 3.19. Thus L is a subgroup of $P\Gamma L(2,q)$ for some q, containing $PSL(2,q)$. But $PSL(2,q)$ has elementary abelian subgroups of order q, a contradiction unless $q=p$, a prime. In this case, $P\Gamma L(2,p)=PGL(2,p)$. Hence either $L=PGL(2,p)$ or $PSL(2,p)$. But $PGL(2,p)$ does not have dihedral Sylow 2-subgroups and so $L=PSL(2,p)$.

<u>Theorem 3.31</u>(Thompson, unpublished). Let G be a simple 3'-group. Then $G\cong Sz(2,2^{2n+1})$ for some $n\geq1$.

<u>Theorem 3.32</u>(Glauberman, unpublished). If G is an S_4-free group then G contains a strongly-closed abelian 2-subgroup in a Sylow 2-subgroup T with respect to G.

<u>Theorem 3.33</u>(see, for example[2]). Let G be a group. Then G is S_4-free if and only if $N_G(H)/C_G(H)$ is S_3-free for all 2-subgroups H of G.

<u>Theorem 3.34</u>(Goldschmidt[35]). Let G be a simple group with a strongly-closed abelian 2-subgroup in a Sylow 2-subgroup with respect to G. Then G is isomorphic to one of the following.

(i) $PSL(2,q)$ where $q=2^n$ or $q\equiv3$ or 5mod8.

(ii) $PSU(3,2^{2n})$ for some $n\geq1$.

(iii) $Sz(2^{2n+1})$ for some $n\geq1$.

(iv) J_1.

(v) ${}^2G_2(3^{2n+1})$ for some $n\geq1$.

<u>Theorem 3.35</u>[Isaacs; Theorem 8.22]. Let G be a group with a Hall subgroup H such that whenever two elements of H are conjugate in G they are already conjugate in H. Suppose also that for every elementary subgroup E in G(that is a direct product of a cyclic group and a p-group), if $|E|$ divides $|H|$ then E is conjugate to a subgroup of H. Then there exists a normal complement to H in G.

Section II: The basic theory of groups with CC-subgroups

This section consists of a systematic exposition of the basic results involving groups with CC-subgroups. Though in reality it is used extensively as a reference section, it is hoped that this part of the thesis will be seen to exist also as a self-contained whole. As such, all proofs are from first principles or at least starting from well known results. Some proofs are trivially simple but are given for completeness and several of the results have never actually been stated explicitly before.

Proposition 3.36. Let $M\leq_{CC}G$. Then M is a Hall subgroup.

Proof. Let p divide $|M|$ and suppose that P is a Sylow p-subgroup of M contained in a Sylow p-subgroup S of G. Then $Z(S)\leq C_G(P)\leq M$ and so $S\leq C_G(Z(S))\leq M$. Thus $P=S$, as required.

Proposition 3.37. Let $M\leq_{CC}G$. Then $M^g\leq_{CC}G$ for all $g\in G$.

Proof. Let $x\in M^{\#}$. Then $C_G(x^g)=(C_G(x))^g\leq M^g$, as required.

Proposition 3.38. Let $M,N\leq_{CC}G$. If $M\cap N\neq 1$ then $M\cap N\leq_{CC}G$.

Proof. Let $x\in (M\cap N)^{\#}$. Then $C_G(x)\leq M$ and $C_G(x)\leq N$. Hence $C_G(x)\leq M\cap N$, as required.

Proposition 3.39. Let $M\leq_{CC}G$ and suppose $Z(M)\neq 1$. Then M is a T.I-set in G.

Proof. Suppose $M\cap M^g\neq 1$ and let $x\in (M\cap M^g)^{\#}$. Clearly $Z(M)\cup Z(M^g)\leq C_G(x)\leq M\cap M^g$. Let $w\in (Z(M^g))^{\#}$. Then $w\in M$ and so $M^g=C_G(w)\leq M$. Thus $M^g=M$, as required.

Proposition 3.40. Let $M\leq_{CC}G$ and $H\leq G$. Then either $M\cap H=1$, $M\cap H=H$ or $M\cap H\leq_{CC}H$.

Proof. Suppose $M\cap H\neq 1$ or H and let $x\in (M\cap H)^{\#}$. Then $C_G(x)\leq M$ and so $C_H(x)\leq M\cap H$, as required.

Proposition 3.41. Let $K\leq_{CC}M\leq_{CC}G$. Then $K\leq_{CC}G$.

Proof. Let $k\in K^{\#}$. Then $C_G(k)\leq M$ and so $C_G(k)=C_M(k)\leq K$, as required.

Proposition 3.42. Let $M\leq_{CC}G$, $\pi=\pi(M)$ and let N be a normal π-subgroup of G. Then $N\leq M$.

<u>Proof</u>. MN is a π-subgroup of G containing M and so MN=M as M is Hall. Thus $N \leq M$.

<u>**Proposition 3.43**</u>. Let $M \leq_{CC} G$. Then $Z(G)=1$.

<u>Proof</u>. Suppose $Z(G)\neq 1$ and let $z \in (Z(G))^{\#}$. If $z \in M$ then $G=C_G(z) \leq M$, a contradiction, and if $z \notin M$ then $z \in C_G(M) \leq M$, a contradiction.

<u>**Proposition 3.44**</u>. Let $M \leq_{CC} G$ and suppose that $G=A \times B$. Then either $A=1$ or $B=1$.

<u>Proof</u>. Suppose neither A nor B is 1. Let p divide $|M|$ and suppose, without loss of generality, that p divides $|A|$. Let x be an element of order p in A. Then $x \in M^g$ for some $g \in G$. Thus $B \leq C_G(A) \leq C_G(x) \leq M^g$ and so $A \leq C_G(B) \leq M^g$. Hence $G \leq M^g$, a contradiction.

<u>**Proposition 3.45**</u>. Let $M \leq_{CC} G$ and suppose that $M \trianglelefteq G$. Then G is Frobenius with kernel M.

<u>Proof</u>. M is a normal Hall subgroup of G and so by the Schur–Zassenhaus Theorem G splits over M. That is, there exists $H \leq G$ such that $G=HM$ and $H \cap M=1$.

Let $g \in G-H$ and put $g=xm$ for some $x \in H$ and some $m \in M^{\#}$. Let $y \in H \cap H^g$. Then $y \in H^m$ and so $y=h^m$ for some $h \in H$. Since $y \in H$ we have $[h,m]=h^{-1}y \in H$. Since $M \trianglelefteq G$, $[h,m] \in M$ and so as $M \cap H=1$, $[h,m]=1$. Thus $h \in C_G(M) \leq M$. Hence $h=1$ and so $y=1$. Thus $H \cap H^g=1$. Then by the definition of a Frobenius group, G is Frobenius with complement H and so by Frobenius' Theorem 3.16, G has kernel M.

<u>**Proposition 3.46**</u>. Let G be a Frobenius group with kernel K and complement H. Then both K and H are CC-subgroups of G.

<u>Proof</u>. Let $h \in H^{\#}$ and $g \in C_G(h)$. Then $h \in H \cap H^g$ and so since H is a T.I-set in G, $H=H^g$. Thus $g \in N_G(H)=H$. Hence $H \leq_{CC} G$.

Let $k \in K^{\#}$, $x \in C_G(k)$ and suppose that $x \notin K$. Then $x \in H^g$ for some $g \in G$. Now $k \in C_G(x) \leq H^g$ by the first paragraph, a contradiction. Hence $K \leq_{CC} G$.

<u>**Proposition 3.47**</u>. Let $M \leq_{CC} G$ and suppose that G is Frobenius with kernel K and complement H. Then either $M=K$ or $M=H^g$ for some $g \in G$.

<u>Proof</u>. Suppose that $M \cap K=1$. Then $M \cap H^g \neq 1$ for some conjugate H^g of H. Since $Z(H)\neq 1$, Propositions 3.40 and 3.43 imply that

$H^g \not\leq M$. Suppose that $H^g < M$. Then since $G=H^gK$, $M\cap K\neq 1$, a contradiction. Hence $H^g=M$. Thus we can assume that $M\cap K\neq 1$. As before, since $Z(K)\neq 1$, we have that $K\leq M$. If $K<M$ then by the above argument $M\cap H^g\neq 1$ for some complement H^g of G and so $H^g\leq M$. Thus $G=H^gK\leq M$, a contradiction. Hence $K=M$.

<u>Proposition 3.48</u>. Let H be both a Frobenius complement and a Frobenius kernel. Then H is cyclic of odd order.

<u>Proof</u>. Since H is nilpotent, Sylow subgroups of H are normal in H, and a Sylow 2-subgroup, P, is either cyclic or generalised quaternion by Theorem 3.19. Suppose that $P\neq 1$. Then in either case for P, P contains a characteristic subgroup of order 2 (the latter case by Theorem 3.17). Thus H has a normal subgroup of order 2, N say. Now let G be a Frobenius group with kernel H and complement L. By Proposition 3.1, $N_G(N)/C_G(N)$ can be embedded in $\mathrm{Aut}N\cong\mathrm{Aut}Z_2\cong 1$. Now $C_G(N)\leq H$ as $H\leq_{cc}G$, and $H\trianglelefteq N_G(N)$. Thus $C_G(N)=N_G(N)=H$. Hence L normalises and so centralises N, contradicting the fact that $H\leq_{cc}G$. Hence 2 does not divide $|H|$. Thus H is the direct product of cyclic groups of relatively prime orders and therefore cyclic.

<u>Proposition 3.49</u>. Let G be a Frobenius group with kernel K and suppose that $N\trianglelefteq G$. Then either $K\leq N$ or $N\leq K$.

<u>Proof</u>. Suppose that $N\nleq K$. Then we can select $x\in N-K$ and without loss of generality suppose that x is of prime order p where p does not divide $|K|$ as K is a Hall subgroup. Thus $\langle x\rangle$ acts as a q'-group of automorphisms on each Sylow q-subgroup Q of K for any q dividing $|K|$. Thus by [Gorenstein; Theorem 5.3.5], $Q=[\langle x\rangle,Q]C_Q(x)$. Thus $Q=[\langle x\rangle,Q]$ since $C_Q(x)=1$ as $K\leq_{cc}G$. Thus $Q\leq N$ as $[\langle x\rangle,Q]\leq N$. Hence $K\leq N$, as required.

<u>Proposition 3.50</u>. Let $M\leq_{cc}G$ and N be a minimal normal subgroup of G such that $M\cap N\neq 1$. If $N\nleq M$ then N is simple.

<u>Proof</u>. By Proposition 3.11, N is either elementary abelian, in which case, by Propositions 3.40 and 3.43, $N\leq M$, or N is a direct product of isomorphic simple groups. In the latter case, if the product has length greater than one, then by Propositions 3.40 and 3.44, $N\leq M$. Thus in order that $N\nleq M$, N must be a simple group.

Proposition 3.51. Suppose that $1<K\leq M\leq_{cc}G$ and $H\leq G$ such that $M\cap H=1$. Then the following are true.

(i) If $K\trianglelefteq G$ and $H\ntrianglelefteq G$ then HK is Frobenius with kernel K and complement H.

(ii) If $H\trianglelefteq G$ and $K\ntrianglelefteq G$ then HK is Frobenius with kernel H and complement K.

Proof. (i) $C_G(k)\leq M$ for all $k\in K^{\#}$.

Thus $C_H(k)=1$ for all $k\in K^{\#}$.

Thus K is a normal CC-subgroup of HK and so HK is Frobenius with kernel K by Proposition 3.45. Similarly, $C_K(h)=1$ for all $h\in H^{\#}$ and so $H\leq_{cc}HK$. Thus by Proposition 3.47, and since $H\neq K$, H is a complement for HK.

The proof of (ii) is analogous.

We now require to introduce some character theory.

Proposition 3.52. Let $\#_{xyz}$ denote the number of different ways of expressing z as the product of a conjugate of x and a conjugate of y and let δ_{xy} denote the function that has value 1 if x and y are conjugate and 0 otherwise. Now suppose that G is a Frobenius group with kernel K and complement H and let $k\in K^{\#}, u, v\in G-K$.

(i) $\#_{uv^{-1}k}=\delta_{uv}|G|/|C_G(u)|$

(ii) $K\leq G'$ and $K=G'$ if and only if H is cyclic.

Proof. (i) A well known order formula is the following (see for example [Gorenstein; Theorem 4.2.12]).

$$\#_{xyz}=\frac{|G|}{|C_G(x)||C_G(y)|}\sum_{\chi\in \mathrm{Irr}G}\frac{\chi(x)\chi(y)\chi(z^{-1})}{\chi(1)}.$$

We neglect the preceding constant for the moment and consider the sum, which can be split into two as follows.

$$\sum_{\substack{\chi\in \mathrm{Irr}G\\ K\nleq \ker\chi}} + \sum_{\substack{\chi\in \mathrm{Irr}G\\ K\leq \ker\chi}}$$

But by [Gorenstein; Theorem 4.5.3] the first sum is identically zero (since we take $x=u\notin K$) and the second sum can be considered as the sum over the irreducible characters of $G/K\cong H$. Thus

$$\#_{uv^{-1}k}=\frac{|G|}{|C_G(u)||C_G(v)|}\sum_{\chi\in \mathrm{Irr}H}\chi(u)\chi(v^{-1})$$

By the second orthogonality relation, this sum is just $\delta_{uv}|C_H(v)|$, and the result follows since $H\leq_{cc}G$.

(ii) Suppose u and v are conjugate. Then $\#_{uv^{-1}k}\neq 0$ and it follows that $k=[w,z]$ for some $w,z\in G$ with w and z conjugate to u. Thus $K\leq G'$. Now $K=G'$ if and only if $G/K\cong H$ is abelian. That is, by Theorem 3.19(xiii), if and only if H is cyclic.

<u>Proposition 3.53</u>(Herzog[43; Theorem 2.3]). Let $M\leq_{cc}G$ and suppose M is a T.I-set in G. Then the following hold.

(i) If $N_G(M)$ has no normal complement in G and N is a normal subgroup of G such that $M\cap N\neq 1$ then $M\leq N$.

(ii) If $N_G(M)\neq M$ then $M\leq G'$.

(iii) If $N_G(M)$ has no normal complement in G then $M<G'$ and G is insoluble.

(iv) If $N_G(M)\neq M$ and N is a non-trivial normal subgroup of G such that $M\cap N=1$ then M is cyclic of odd order, N is nilpotent and $N_G(M)$ is metacyclic.

<u>Proof</u>. (i) Let $K=M\cap N$ and suppose that $N_N(K)=K$. Then by Frobenius' Theorem 3.16, K has a normal complement D in N such that $N=KD$ and $K\cap D=1$. D is characteristic in N and so normal in G, and $D\cap N_G(M)=1$. As K is a Hall subgroup of N by Proposition 3.36, we use Frattini's rule Proposition 3.7 and Proposition 3.22 to obtain that $G=N_G(K)N=N_G(M)D$, a contradiction. Hence we can assume that $N_N(K)\neq K$. Since $N_G(M)$ has no normal complement, $N_G(M)\neq M$, and we can choose $n\in N_G(K)\leq N_G(M)$ by Proposition 3.22, with $n\notin M$. Let $m\in M$. Then n and n^m are contained in $N_G(M)-M$ and so by Proposition 3.52, $\#_{n(n^m)^{-1}m}\neq 0$. Thus $m=n^g n^{mh}$ for some $g,h\in N_G(M)$ and so $m\in N$ as $N\trianglelefteq G$, as required.

(ii) By Proposition 3.52(ii), $M\leq (N_G(M))'\leq G'$. Hence result.

(iii) Again $N_G(M)\neq M$. If $M=G'$ then $N_G(M)$ would have normal complement 1, a contradiction. Now suppose that G is soluble and consider the derived series for G. If $N_{G'}(M)\neq M$ then we use Proposition 3.53(ii) to obtain that $M\leq G''$. By induction, then, either $N_{G^{(n)}}(M)=M$ for some n, or $M\leq 1$, and the latter is a contradiction. Thus we can assume that $N_{G^{(n)}}(M)=M$ for some n and thus that $G^{(n)}$ is Frobenius with complement M and kernel K say. K is a normal subgroup of G and if an asterisk denotes factoring by K, then $N_{G^*}(M^*)=G^*$. Thus $N_{G^*}(M^*)$ has normal complement 1 in G^* and so $N_G(M)$ has a normal complement in G, a contradiction. Hence G is insoluble.

(iv) $M\cap N=1$. Suppose that $N\cap N_G(M)\neq 1$ and let $n\in (N\cap N_G(M))^{\#}$. Since $n\notin M$, we can use Proposition 3.52(i) to obtain that $M\leq N$, a contradiction. Thus $N\cap N_G(M)=1$. Thus by Theorem 3.51, MN is

Frobenius with kernel N and complement M. But M is also a kernel and so by Proposition 3.48, M is cyclic of odd order. Since the complement of $N_G(M)$ is a group of automorphisms of M, it is abelian and so by Theorem 3.19(xiii), is cyclic. Thus $N_G(M)$ is metacyclic.

Proposition 3.54(Herzog[43]). Let $M \leq_{cc} G$, M a T.I-set of G, such that $N_G(M) \neq M$ or G, and M is not cyclic of odd order. Then the following hold.

(i) If there exists a proper non-trivial normal subgroup N of G then $M \leq N$ and G is insoluble.

(ii) G contains a simple normal subgroup containing M.

Proof. (i) By Proposition 3.53(iv), we have $M \cap N \neq 1$ and so by Proposition 3.53(i), $M \leq N$ and $N_G(M)$ has no normal complement in G. Thus by Proposition 3.53(iii), G is insoluble.

(ii) Let G^* be a minimal normal subgroup of G. If $M \cap G^* = 1$ then M is cyclic by Proposition 3.53(iv), a contradiction. Thus by Proposition 3.50, either $G^* \leq M$ or G^* is simple. But if $G^* \leq M$ then by Proposition 3.22, $M \trianglelefteq G$, a contradiction. Hence result.

Proposition 3.55(Herzog[43]). Let $M \leq_{cc} G$, M a T.I-set in G and suppose that G is soluble. Then one of the following holds.

(i) $M \trianglelefteq G$ and G is Frobenius with kernel M.

(ii) $N_G(M) = M$ and G is Frobenius with complement M.

(iii) $N_G(M) \neq M$ or G, $N_G(M)$ has a normal complement, M is cyclic of odd order and $N_G(M)$ is metacyclic.

Proof. If $N_G(M) = M$ or G, (ii) and (i) hold respectively, otherwise, by Proposition 3.53(iii), $N_G(M)$ has a normal complement and so by Proposition 3.53(iv), (iii) holds.

Proposition 3.56(Arad and Chillag). Let $H \trianglelefteq G$ and $x \in G$ such that $(|x|, |H|) = 1$, and denote by asterisk the image modulo H. Then $C_{G^*}(x^*) = C_G(x)H/H$.

Proof. $C_{G^*}(x^*) = \{g^* \in G^* : x^* g^* = g^* x^*\}$

$= \{g^* \in G^* : gxH = xgH\}$ where g is an inverse image of g^* in G. $= \{g^* \in G^* : [x, g] \in H\}$.

Clearly then, if $g \in C_G(x)$ then $g^* \in C_{G^*}(x^*)$.

Now let $c^* \in C_{G^*}(x^*)$ with inverse image c in G. We have $xcH = cxH$. That is, $x^c H = xH$, and so $\langle x^c \rangle H = \langle x \rangle H$.

Since $(|x|, |H|) = 1$ and $H \trianglelefteq G$, $H \leq \langle x \rangle H$ and we use the

Schur–Zassenhaus Theorem to obtain that $\langle x\rangle H$ splits over H, with the complements of H being conjugate. Thus $\langle x^c\rangle$ and $\langle x\rangle$ are conjugate in $\langle x\rangle H$. That is, there exists $h\in H$ such that $\langle x^c\rangle=\langle x\rangle^h$. Now let $n\in\mathbf{N}$ such that $x^c=(x^n)^h$. Then $x^nH=x^nhH=hx^cH=hxH$. Then as $H\trianglelefteq G$, $h^x\in H$ and so $x^{n-1}\in H$. Thus $x^{n-1}=1$. That is, $x^n=x$ and so $x^c=x^h$. Thus $ch^{-1}\in C_G(x)$, and so $c\in C_G(x)H$, as required.

Note: This result allows us to say that if $M\leq_{cc}G$ and $H\trianglelefteq G$ then either $M^*=1$, $M^*=G^*$ or $M^*\leq_{cc}G^*$.

Proposition 3.57. Let M be a Hall π-subgroup of G, $H\trianglelefteq G$ and suppose that either H is a π'-subgroup or MH is soluble. Then, denoting by an asterisk images modulo H, we have $N_{G^*}(M^*)=N_G(M)H/H$.

Proof. $N_{G^*}(M^*)=\{g^*\in G^*: g^*M^*=M^*g^*\}$

$=\{g^*\in G^*: (M^*)^{g^*}=M^*\}$

$=\{g^*\in G^*: (M^g)^*=M^*\}$.

Clearly if $g\in N_G(M)$ then $g^*\in N_{G^*}(M^*)$.

Now let $n^*\in N_{G^*}(M^*)$ with inverse image n in G. Then $(M^*)^{n^*}=M^*$, that is, $M^nH=MH$, and either by the Schur–Zassenhaus Theorem as in the previous proof, or by Hall's Theorem 3.8, there exists $h\in H$ such that $M^n=M^h$. Thus $nh^{-1}\in N_G(M)$ and so $n\in N_G(M)H$ as required.

Proposition 3.58. Let $M\leq_{cc}G$, $\pi=\pi(M)$ and suppose that $O_\pi(G)\neq1$. Then $O_\pi(G)=M$ and G is Frobenius with kernel M.

Proof. Let N be a non-trivial normal π-subgroup of G. Then by Proposition 3.42, $N\leq M$ and so $M_G\neq1$. If $M_G=M$ the result follows, and so we can assume that $M_G<M$. As M_G is a normal CC-subgroup of both G and M (by Propositions 3.37 and 3.38), M and G are Frobenius groups with kernel M_G. Let H be a complement to M_G in M. Then $H\leq_{cc}G$ and so by Proposition 3.56, $H\cong M/M_G\leq_{cc}G/M_G$. But G/M_G is isomorphic to a Frobenius complement and so has non-trivial centre, contradicting Proposition 3.43. Hence result.

Proposition 3.59. Let $M\leq_{cc}G$ and suppose that $(|F(G)|,|M|)\neq1$. Then $F(G)\leq M$.

Proof. Let p be a prime dividing both $|F(G)|$ and $|M|$ and let $\pi=\pi(M)$. Then $O_p(G)\leq M$ by Proposition 3.42. Now for all primes $q\neq p$, $O_p(G)\cap O_q(G)=1$ and so (by [Rose: Theorem 3.53]), $[O_p(G),O_q(G)]=1$.

Thus $O_q(G) \le C_G(O_p(G)) \le M$ as $M \le_{cc} G$. Thus $F(G) = \prod_{q \mid |G|} O_q(G) \le M$ as required.

<u>Proposition 3.60</u>. Let $M \le_{cc} G$ and suppose that $(|F(G)|, |M|) = 1$. Let asterisks denote images modulo $F(G)$.

(i) If $F(G^*) \ne 1$ then $F(G^*) \le M^*$.

(ii) If G is insoluble and $N_G(M) \ne M$ then $F(G^*) = 1$.

<u>Proof</u>. (i) Suppose that $F(G^*) \ne 1$ but that $(|F(G^*)|, |M^*|) = 1$. Consider $F_2(G)$, the inverse image of $F(G^*)$, so that $F_2(G)/F(G) = F(G^*)$. Since both $|F(G)|$ and $|F(G^*)|$ are coprime to $|M|$, $|F_2(G)|$ is coprime to $|M|$ and thus by Theorem 3.51, $MF_2(G)$ is Frobenius with kernel $F_2(G)$. But then $F_2(G)$ is a normal nilpotent subgroup of G strictly containing $F(G)$, a contradiction. Thus $(|F(G^*)|, |M^*|) \ne 1$. Now by Proposition 3.56, either $M^* \le_{cc} G^*$ or $M^* = G^*$, and thus by Proposition 3.59 applied to M^*, $F(G^*) \le M^*$.

(ii) Now suppose that G is insoluble and $N_G(M) \ne M$ but that $F(G^*) \ne 1$. Again applying Theorem 3.51, $MF(G)$ is a Frobenius group with complement M so that by Theorem 3.48, M is cyclic of odd order. Also, by (i), $F(G^*) \le M$ so that $F(G^*)$ is cyclic. We can thus choose a characteristic subgroup R^* of order p for some prime p. Then $R^* \trianglelefteq G^*$ and so by Propositions 3.1 and 3.3, $G^*/C_{G^*}(R^*)$ can be embedded in $\mathrm{Aut}Z_p \cong Z_{p-1}$. But G^* is insoluble since G is insoluble, and so $C_{G^*}(R^*)$ is insoluble. However, $C_{G^*}(R^*) \le M^*$ as $R^* \le M^*$, which is soluble since M is cyclic, a contradiction. The result follows.

CHAPTER IV: A MATHEMATICAL DISCUSSION

Section I: On groups with CC-subgroups of order divisible by 3

Most of the early results on CC-subgroups of order divisible by 3, up to and including Theorem 2.20(except for the two inserted results, Theorems 2.16 and 2.17) were obtained with the use of character theory, sometimes extensively. One reason for this is that, as CC-subgroups are in general trivial-intersection sets, a large body of well understood theory could be applied. In fact, before the concept of CC-subgroups existed on its own, the CC-property, along with the "T. I-ness", had been shown by Felt and others to be fundamentally involved with the theory of exceptional characters(see for example[Isaacs; Chapter 7]). Thus Herzog, whose researches stemmed from those of Felt, relied predominantly on the use of exceptional characters in proving Theorems 2.8 to 2.11, and also to a lesser extent in Theorem 2.6.

Another technique frequently invoked is the Brauer-Suzuki method, mentioned in the introduction, of constructing parts of character tables(usually blocks for ease of handling) to which can be applied order formulae involving centralisers and conjugacy classes. To show the applicability of this method to CC-subgroup theory we consider a group G. The method requires the existence of a subgroup H of G which has a subset of pairwise non-conjugate elements $\{h_1,\dots,h_n\}$, each of which satisfies $C_G(h_i)\leq H$(conjugacy classes in H of these elements are called the special classes of H). Clearly this condition is easily satisfied in the case that $H\leq_{CC}G$. An example of the order formulae is the well known "sharp" equation

$$\#(a\cdot b\cdot =c)=\frac{|G|}{|C_G(a)|\,|C_G(b)|}\sum_{\chi\in \mathrm{Irr}G}\frac{\chi(a)\chi(b)\chi(c^{-1})}{\chi(1)}\quad ,$$

where $\#(a\cdot b\cdot =c)$ denotes the number of different ways that c can be given as the product of a conjugate of a and a conjugate of b. Calculation of the left hand side of this equation in general requires knowledge of subgroups generated by conjugates of a and b, and so is restricted to elements of orders 2 and 3 (see for example[47; Theorem2]), whereas the right hand side can be made manageable provided at least one of a, b and c is in a special class, in that it then involves a fixed set of unknowns satisfying a set of equations previously obtained by the character theory.

In this way several of the preliminary results in the proof of Theorem 2.5 were obtained, and also the specific characterisations of simple groups by Higman and Fletcher in Theorem 2.18.

However as a policy the author of this thesis has chosen to concentrate on the group-theoretic arguments at the expense of any specific character theory. As a consequence, discussion on these early results will for the most part, only be qualitative.

The Feit and Thompson proof of Theorem 2.1, as with most of the following in this section, begins by considering a minimal counterexample, quickly seen to be simple, and proceeding to a contradiction via the existence of irreducible characters with particular properties. Extensive use is made of order formulae relating to conjugacy classes, and Sylow theory, to obtain three possible configurations for the counterexample, each of which is proved invalid, either by a false arithmetical relation or the existence of a simple group with an impossible order. However this proof was considered by Higman and Sa Stewart to be unsuitable for generalisation. Higman[47] proceeded to give two alternative proofs, both of which use the Brauer-Suzuki method, and can be generalised in particular to results on C33-groups. The first of these is standard in that it gives an arithmetical property of a minimal counterexample which is not compatible with the hypothesis. The second is more sophisticated and uses local group theory to show that no counterexample exists. It is this latter technique which Stewart extended to prove Theorem 2.5.

Stewart used much more local analysis, the Brauer-Suzuki method being used in a slightly different way from Higman to obtain a set of lemmas useful in the proof (for instance that a minimal counterexample has all its involutions conjugate).

As promised in the remarks made after the statement of Theorem 2.5, we now show that theorem's applicability to the cyclic case.

<u>Lemma 4.1</u>. Let $M \leq_{CC} G$, 3 divide $|M|$ and suppose that M is cyclic. Then $|N_G(M)/M| \leq 2$.

<u>Proof</u>. M is a Hall subgroup. Let P be a Sylow 3-subgroup of G contained in M. Since P is cyclic, $P \leq C_G(P)$ and so $N_G(P)/C_G(P)$ is a $3'$-group of automorphisms acting non-trivially on P. Thus by

Proposition 3.14, $N_G(P)/C_G(P)$ acts non-trivially on $P/\Phi(P)$. But $P/\Phi(P)$ is an elementary abelian cyclic 3-group and so has order 3. Thus $N_G(P)/C_G(P)$ can be embedded in $\mathrm{Aut}Z_3 \cong Z_2$.

Now, since clearly $C_G(P)=M$ and $N_G(M) \leq N_G(P)$, we have $|N_G(M)/M| \leq 2$, as required.

The proof of Herzog's further extension of Theorem 2.1, that is Theorem 2.6, proceeds again by minimal counterexample, G, say. A configuration is obtained for G which, on applying some further theory of exceptional characters, gives an upper and lower bound for the order of G in terms of polynomials in m, the order of the CC-subgroup M. These bounds are incompatible unless m=3, which by Theorem 2.5 implies that G satisfies the conclusion of the theorem, an immediate contradiction.

Theorem 2.7 is a consequence of Theorems 2.8 to 2.11(which are proved by exceptional characters and are not discussed here), and Theorem 2.6.

We now proceed to the Higman and Fletcher development of C33-group theory, and the generalisation introduced by Definition 2.12. In order that it may be seen how closely related this idea is to the study of CC-subgroups, we prove Propositions 2.13 and 2.14.

<u>Proof of Proposition 2.13</u>[πCC-groups are $C\pi\pi$-groups]. Let G be a πCC-group with πCC-subgroup M. Let x be a non-trivial π-element of G. Then for some power x^n of x, x^n is of prime order p for some $p \in \pi$. Since M contains a Sylow p-subgroup of G, x^n is conjugate to an element of M and hence $C_G(x^n)$ is a π-group. Since $C_G(x) \leq C_G(x^n)$, it follows that $C_G(x)$ is a π-group. Hence result.

<u>Proof of Proposition 2.14</u>[Cpp-groups with abelian Sylow p-subgroups are pCC-groups]. Let G be a Cpp-group with abelian Sylow p-subgroup P. Let $x \in P^{\#}$. Then $C_G(x)$ is a p-group containing P and so $C_G(x)=P$. Hence result.

Fletcher's proof of Theorem 2.15, that simple C33-groups have abelian Sylow 3-subgroups, uses standard group-theoretic arguments. However, though the result is required in later results, as this discussion concentrates on CC-subgroups, the proof will not be given here.

The important characterisation result Theorem 2.18 is for the most part an immediate consequence of previous results, by considering in case (a), Theorem 2.5 applied to the particular configuration, and in case (b), Theorem 2.10. The individual characterisations of the 3CC-groups PSL(2,9) and PSL(3,4) were given by Higman[47; Theorem 13.3] and Fletcher[32] respectively by using the Brauer-Suzuki method to find possible orders for groups satisfying the conditions of Theorem 2.10(ii) and (iii).

Lastly in the set of character theory dominated results is Theorem 2.19. The proof of this is extremely complicated, and as it was greatly simplified and extended by Arad without the use of characters in Theorem 2.21, we shall not dwell on it.

We now come to the body of work produced by Arad which culminated in the classification Theorem 2.23. Although this work relies on character theory inasmuch as it is based on many of the previous results, the actual proofs given by Arad contain no explicit use of characters whatsoever, since he made use of several advanced local group-theoretic results which had been made available in the intervening years, for example Glauberman's Theorem 3.32 and Goldschmidt's Theorem 3.34. Also used for the first time in the context of CC-subgroups of order divisible by 3 was Suzuki's result for the even case, Theorem 3.27. As these results would take several hundred pages to prove on their own, and involve theory far outwith the range of this thesis, they will not be discussed.

First we need a result of Fletcher which originated from his doctoral thesis.

<u>Theorem 4.2</u>(Fletcher[30]). Let G be a simple C33-group and suppose that one of the following is true.

(i) There exists an insoluble subgroup N of G such that 3 divides $|N|$ and $O_{3'}(N)\neq 1$.

(ii) Some non-trivial 2-subgroup of G is normalised by an element of order 3.

Then G is isomorphic to either PSL(3,4) or PSL(2,q) for some $q\geq 4$.

Unfortunately no proof of this result has been published to the best of this author's knowledge.

The following preliminary result is a convenience which Arad did not adopt.

Proposition 4.3. Let G be a simple $C\pi\pi$-group for some π such that $3\in\pi$ and $2\notin\pi$, and suppose that G is S_4-free. Then $G\cong PSL(2,q)$ for some q.

Proof. By Theorem 3.32, G(if it exists) must be one of the groups listed in Theorem 3.34. Clearly G is not a Suzuki group since 3 divides $|G|$. The possible sets π for J_1 and ${}^2G_2(3^{2n+1})$ are known(see Chapter VI) and none satisfy the necessary conditions. The possible sets π for PSU(3,4) are also known and none of these satisfy the necessary conditions either and since PSU(3,4) can be embedded in $PSU(3,4^n)$ for all $n\geq 1$, none of these groups is a $C\pi\pi$-group for a suitable π. Hence result.

Remark: This result holds in particular for πCC-groups where $3\in\pi$ and $2\notin\pi$.

The next result, which improves upon Theorem 4.2, is essential for further development. The proof is very representative of Arad's technique.

Proposition 4.4(Arad[2; Lemma 1]). Let G be a simple C33-group. Then G is isomorphic to either PSL(3,4) or PSL(2,q) for some q.

Proof. Let H be an arbitrary non-trivial 2-subgroup of G. If 3 divides $|N_G(H)|$ then the result follows by Theorem 4.2(ii). THus we can assume that 3 does not divide $|N_G(H)|$. Then by Theorem 3.33, G is S_4-free and so by Proposition 4.3, $G\cong PSL(2,q)$ for some q. Hence result.

Note: In particular, by Proposition 4.4, simple C33-groups have cyclic or elementary abelian Sylow 3-subgroups.

We can now give the following.

Proof of Theorem 2.21. Let G be a minimal counterexample and let N be a minimal normal subgroup of G. Suppose that 3 does not divide $|N|$. Then N admits a fixed-point-free automorphism of order 3 and so is nilpotent by Theorem 3.18. Thus by Theorem 3.11, N is a p-group for some $p\neq 3$. We can assume that $M\ntrianglelefteq G$ so that by Theorem 3.51, MN is a Frobenius group with complement M. Hence by Theorem 3.19(viii), M is cyclic, a contradiction.

Thus we can assume that 3 divides $|N|$. In this case $M\cap N\neq 1$ by Proposition 3.24, and so by Proposition 3.50, either $N\leq M$ or N is simple. If $N\leq M$ then by Proposition 3.22, $M\trianglelefteq G$, a contradiction. Hence

N is simple. If $N_G(M) \neq M$ then by Proposition 3.54, $M \triangleleft N$ and since G is a minimal counterexample, G=N. Thus by Proposition 4.4, M is cyclic or elementary abelian, a contradiction. Thus we can assume that $N_G(M)=M$, in which case M is a Frobenius complement of G and so by Theorem 3.19(viii), M is cyclic, a final contradiction.

<u>Proof of Theorem 2.22</u>. We use Theorem 2.21. If $M \ntrianglelefteq G$ then clearly (i) holds. If M is cyclic then by Theorem 2.5, one of (ii), (iii) and (iv) holds, and if M is elementary abelian then Theorem 2.10 and Proposition 4.4 cover the remaining possibilities.

To give a proof of Theorem 2.23 we require the following immediate corollary of Theorem 2.65.

<u>Proposition 4.5</u>. Let G be a simple group and suppose that the following hold.

(i) G has no element of order 6.

(ii) Some non-trivial 2-subgroup of G is normalised by an element of order 3.

Then CC-subgroups of G, of order divisible by 3, if they exist, are either cyclic of odd order or elementary abelian.

We can now proceed to the proof of the main result.

<u>Proof of Theorem 2.23</u>. Let G be a minimal counterexample. Then $N_G(M) \neq 1$ and so M is nilpotent. By Proposition 3.54, G contains a simple normal subgroup G^* containing M and satisfying $N_{G^*}(M) \neq M$. Thus, as G is a minimal counterexample, we can suppose that $G=G^*$. If 2 divides $|M|$ then Theorem 3.27 implies that $G \cong PSL(2,2^n)$ for some n, a contradiction. Thus we can assume that 2 does not divide $|M|$ and then that G has no element of order 6 (for such an element would centralise an element of M). Let H be an arbitrary non-trivial 2-subgroup of G. If 3 divides $|N_G(H)|$ then by Proposition 4.5, M is either cyclic of odd order or elementary abelian, a contradiction. Thus we can assume that 3 does not divide $|N_G(H)|$ in which case by Theorem 3.33 and Proposition 4.3, $G \cong PSL(2,q)$ for some q, a contradiction.

We prove Theorem 2.25 before Theorem 2.24 as the latter needs an additional preliminary result.

<u>Proof of Theorem 2.25</u>. Let G be a minimal counterexample. If 2 divides $|M|$ then $G \cong PSL(2,2^n)$ by Theorem 3.27, a contradiction. Thus we can assume that 2 does not divide $|M|$ and so G has no element of order 6. As in the proof of Theorem 2.23, if 3 divides $|N_G(H)|$ for some non-trivial 2-subgroup H of G, then either M is cyclic or elementary abelian and using Theorem 2.5 and Proposition 4.4, G is one of the groups listed, and if 3 does not divide $|N_G(H)|$ then by Theorem 3.33 and Proposition 4.3, $G \cong PSL(2,q)$ for some suitable q.

<u>Note</u>: Here we use the values for q as given in Theorem 2.65.

The last result to be proved from Chapter II, Section I, that is, Theorem 2.24, requires the following result. Here an inconsistency in the original proof by Arad and Herzog has been corrected.

<u>Proposition 4.6</u>(Arad and Herzog[9]). Let $M \leq_{cc} G$ and suppose that $N_G(M)=M$. Then one of the following is true.

(i) G is Frobenius with M as complement.

(ii) G has a simple section H/K and if an asterisk denotes an image modulo K we have

(a) $M \leq N_G(H) \cap N_G(K)$,

(b) $M^* \leq_{cc} (MH)^*$,

(c) $(H \cap M)^* \leq_{cc} H^*$, and

(d) $N_{H^*}((H \cap M)^*) = (H \cap M)^*$.

<u>Proof</u>. Let G be a minimal counterexample. M is a Hall π-subgroup of G where $\pi=\pi(M)$. Clearly G is neither simple nor Frobenius with complement M. Thus by Theorem 3.27, 2 does not divide $|M|$ and so M is soluble.

Suppose that $O_{\pi'}(G) \neq 1$. If $G/O_{\pi'}(G) \cong M$ then (i) holds, a contradiction. Hence $G/O_{\pi'}(G) \ncong M$. If we denote factoring modulo $O_{\pi'}(G)$ by a bar, it follows that $\overline{M} \leq_{cc} \overline{G}$ and $N_{\overline{G}}(\overline{M})=\overline{M}$. Hence, as G is a minimal counterexample, either (i) or (ii) holds for $\overline{G}$, and so also for G, a contradiction. Thus we can assume that $O_{\pi'}(G)=1$. Let N be a minimal normal subgroup of G. Then since $(|M|,|N|) \neq 1$, $M \cap N \neq 1$. Suppose $N \leq M$. Then $1 < M_G < M$ and so G is Frobenius with kernel M_G by Proposition 3.45. But then by Proposition 3.47, either $M=M_G$ or $M \cap M_G=1$, a contradiction. Hence by Proposition 3.50 we can assume that N is simple. Let $R=M \cap N$. Then $R \leq_{cc} N$. Suppose that $N_N(R) \neq R$. Then $M < N_G(R)$ and so by the minimality of G, either $N_G(R)$ is Frobenius with complement M, contradicting the fact that $R \trianglelefteq N_G(R)$, or (ii) holds for $N_G(R)$ and so also for G, a contradiction. Thus $N_N(R)=R$. But then G satisfies (ii) with H=N and K=1, a final contradiction.

This enables us to give the following.

<u>Proof of Theorem 2.24</u>. Let G be a minimal counterexample. Then condition (ii) of Proposition 4.6 holds. If 2 divides $|M|$ then $G\cong PSL(2,2^n)$, a contradiction. Thus we can assume that 2 does not divide $|M|$. Suppose that 3 divides $|(H\cap M)^*|$. Then by Theorem 2.23, H^* is one of a known list of simple groups, none of which except PSL(2, q) satisfy condition (ii)(d) of Proposition 4.6 such that 2 does not divide $|(H\cap M)^*|$, and so (ii) of this theorem holds. Thus we can assume that 3 does not divide $|(H\cap M)^*|$. Then 3 does not divide $|H^*|$ and so by Theorem 3.31, $H^*\cong Sz(2^{2n+1})$ for some $n\geq 1$. Let $m\in M$ be an element of order 3. Then by condition (ii)(a) of Proposition 4.6 and [Gorenstein: Theorem 6.2.2(i)], m^* normalises the centre of a Sylow 2-subgroup of H^*, which has order 2^{2n+1}. As 3 does not divide $2^{2n+1}-1$, m^* centralises an involution in H^*. Since 2 does not divide $|M|$ we have a final contradiction to condition (ii)(b) of Proposition 4.6.

Another proof of Theorem 2.24 was given later by Arad and Herzog[10] which is similar in technique but does not involve factor groups.

Section II: On closure and homogeneity

All the results in Section II of Chapter II were obtained group-theoretically, making use of several highly non-trivial results which, as in the previous section, lie far outwith the domain of this thesis. However, using these results, it is possible to prove every result in this part of the survey. Nevertheless, in cases where the proof involves lengthy computation or repetition of previous arguments, only an outline of the proof will be given.

The main reason for including the ideas of closure and homogeneity in the survey, apart from the obvious dependence on primes, is that the central body of results were produced by Arad and Ferguson simultaneously with their development of CC-subgroup theory, and many of the proofs in this section involve largely the same techniques. Indeed at a glance it is difficult to distinguish the proofs of Theorems 2.48 and 2.49 from those of Theorems 2.21, 2.23 and 2.25. Another reason is that certain important CC-results rely on the main theorem of this section, Theorem 2.52, as will be seen for instance at the end of Chapter V.

Important additions to the collections of useful results are the list of minimal simple groups obtained by Thompson, from the N-group paper[64], and, again by Thompson, the classification of all simple $3'$-groups, Theorem 3.31, which though used once in the previous section, is used extensively here and in Chapter V.

However we first have to consider the more elementary group-theoretically proved results of Baer.

<u>Proof of Theorem 2.30</u>[π-closure implies π'-homogeneity]. Let π be a set of primes and let G be a π-closed group. Then R, the set of π-elements of G, is a characteristic Hall π-subgroup of G. Suppose now that S is a π'-subgroup of G and let $g \in R \cap N_G(S)$. Then $[g,s] \in S$ for all $s \in S$ and also, since $R \trianglelefteq G$, $[g,s] \in R$. Hence $[g,s] \in R \cap S = 1$, and so $g \in C_G(S)$. Thus $R \cap N_G(S) \leq C_G(S)$. It follows that $N_G(S)/C_G(S)$ is a π'-group, as required.

We shall refrain from giving proofs of Theorems 2.31 and 2.32. They involve straightforward but lengthy arguments of Sylow theory and automorphism results. However, as Theorem 2.33 is a partial converse to Theorem 2.30(which is central motivation to the subsequent theory), we shall give its proof.

<u>Proof of Theorem 2.34</u>. Let G be a π-closed group. By Theorem 2.30, G is π'-homogeneous. Also, since subgroups and factor groups of π-closed groups are π-closed, the composition factors of G are π-closed simple groups(here, simple includes cyclic of prime order). But simple groups are π-closed if and only if they are either π-groups or π'-groups, and thus G is π'-separable.

Conversely, suppose that G is π'-homogeneous and π'-separable. We shall use induction. By definition of π'-separability, there exist subgroups $\{G_i: i=0..n\}$ of G for some n such that $G_0=1, G_n=G$ and $G_i \trianglelefteq G_{i+1}$ where G_{i+1}/G_i is either a π-group or a π'-group. In particular, G_{i+1}/G_i is π-closed for $i=0..n-1$. Now G_0 is π-closed and G_1/G_0 are π-closed. Also, G_1 is π'-homogeneous by Theorem 2.31. Hence by Theorem 2.32, G_1 is π-closed. Now suppose that G_i is π-closed. Then again by Theorem 2.32, G_{i+1} is π-closed. Hence by induction, $G=G_n$ is π-closed, as required.

We shall require Frobenius' Theorem 2.36 in the following.

<u>Proof of Theorem 2.39</u>. If G is the direct product of a p-group and a p'-group then clearly $G=O_p(G) \times O_{p'}(G)$ and obviously G is p-closed and p'-closed, hence p-homogeneous and p'-homogeneous by Theorem 2.30.

Conversely, suppose that G is both p-homogeneous and p'-homogeneous. Then G is p'-closed by Theorem 2.36 and so there exists a characteristic Hall p'-subgroup Q of G such that $|G/Q|$ is a power of p. If P is a Sylow p-subgroup of G then $G=PQ$ and $P \cap Q=1$. Since G is p'-homogeneous, $G/C_G(Q)=N_G(Q)/C_G(Q)$ is a p'-group and so $P \leq C_G(Q)$. Thus $G=P \times Q$ as required.

Theorem 2.41 is proved using similar techniques to those of the above results, but although it remained the best result available until 1974, its proof will not be given as it was superceded by what follows, and more importantly, was not used in the proofs of later theorems. (This cannot be said, for example, of the early results of Stewart and Herzog in the previous section).

The proof of Theorem 2.42 is deceptively simple.

<u>Proof of Theorem 2.42</u>[2'-homogeneity implies 2-closure]. Let G be a minimal counterexample. By Theorems 2.31 and 2.32, G must be a simple group. Let K be the conjugacy class of some involution $u \in G$.

Obviously $|K|>1$. Then by Baer's Theorem 3.15, there must exist $v \in K, v \neq u$, such that uv is not a 2-element. If $|uv|=2^k m$ where $m>1$ and is odd, then let $t=(uv)^{2^k}$. Then $|t|=m$. Now

$t^u=u(uv\ldots uv)u=(vu\ldots vu)=t^{-1}$ and so $N_G(\langle t\rangle)/C_G(t)$ is not a 2'-group. Hence G is not 2'-homogeneous, a contradiction.

The conciseness of the proof of Theorem 2.42 is only possible, however, by using Theorem 3.15, the proof of which in itself is a 3-to-4 page continuous flow of Sylow theory. As it is not directly relevant to this thesis, but only used as a means to an end, the proof is reluctantly omitted.

We now give a proof of Theorem 2.43.

Proof of Theorem 2.43. If G is a π'-closed group then G is π-homogeneous by Theorem 2.30.

Conversely, suppose that G is a minimal counterexample to the theorem. Then G is π-homogeneous. If $|G|$ is odd then G is soluble and so π-separable and thus by Theorem 2.34, G is π'-closed, a contradiction. Thus we can assume that $|G|$ is even. From the hypothesis, $2 \in \pi$. If $\pi=\{2\}$ then G is π'-closed by Theorem 2.36, a contradiction. Thus $|\pi| \geq 2$. Hence G has the following properties.

(i) G is π-homogeneous, $2 \in \pi(G)$, $2 \in \pi$ and $|\pi| \geq 2$.

(ii) π-subgroups of G are 2-closed (from the hypothesis).

(iii) G is not π'-closed.

(iv) Proper subgroups of G are π'-closed (by minimality of G).

Let H be a proper subgroup of G. Then H is an extension of a π'-group by a π-group and the π-group, by (ii) is itself an extension of a 2-group by a 2'-group. Thus since $2 \in \pi$, H is soluble by Theorems 3.5 and 3.20. (Note that it is this step which requires the additional hypothesis (ii) and is crucial to the remainder of the proof).

Now suppose that G is not simple and let N be a minimal normal subgroup of G. Then by the previous paragraph, N is soluble and so by Proposition 3.11, N is an elementary abelian p-group. Suppose that $p \in \pi$ and let K/N be a π-subgroup of G/N. Then K is a π-subgroup of G and so is 2-closed. Thus the π-subgroups of G/N are 2-closed, and by the mimlnality of G, G/N is π'-closed. Then by Theorem 2.32, G is π'-closed, a contradiction. Thus we can assume that $p \notin \pi$. Suppose again that K/N is a π-subgroup of G/N. Then by the Schur-Zassenhaus Theorem, $K=HN$ where H is a Hall π-subgroup of K. Thus $K/N \cong H$ has a normal Sylow 2-subgroup by condition (ii). Again by the minimality of G, G/N, and hence also G, is π'-closed, a contradiction. Hence G is simple.

Since all proper subgroups of G are soluble, G is now a minimal simple group and so by Thompson[64], G is one of a known list of simple groups whose subgroup structures are well known. Each of these is checked in turn and it happens that none of them satisfy all the conditions for a minimal counterexample to the theorem. The result follows.

The proof of Theorem 2.44 is in effect a set of independent proofs for each part. In actual fact Arad gave two further conditions involving chains of subgroups. However we will not prove the equivalence in all cases, but only in case (i) and partially in case (ii). The other parts are proved similarly.

<u>Proof of Theorem 2.44(i)</u>. The forward implication is obvious by Theorem 2.30. Let G be a minimal counterexample to the converse. Then G is π-homogeneous, $2 \notin \pi$ and $3 \notin \pi(G)$. If G were not simple then Theorems 2.31 and 2.32 would imply that G is π'-closed, a contradiction. Hence by Thompson's classification of simple $3'$-groups, Theorem 3.31, $G \cong Sz(q)$ for some q. Thus by [1; Lemma 2.1](which simply gives some properties of the Suzuki groups), G is not π-homogeneous, a contradiction.

<u>Proof of Theorem 2.44(ii)</u>. Let G be a minimal counterexample. Then G is π-homogeneous, $2 \notin \pi$ and π'-subgroups of G are soluble. Also, proper subgroups of G are π'-closed and so are soluble by Theorems 3.5 and 3.20. In an analogous way to the proof of Theorem 2.43, G is proved simple and so minimal simple. Thus, again using Thompson's list, we find that no group satisfies all the properties held by G. Hence result.

We have the following succinct proof of Theorem 2.45.

<u>Proof of Theorem 2.45</u>. If G is π-homogeneous and $|G|$ has exactly 4 distinct prime factors then by Frobenius' Theorem 2.36, Theorem 2.42, Theorem 2.44(ii), Burnside's $p^{\alpha}q^{\beta}$-theorem and the Odd Order Theorem, G is π'-closed.

From now on, Definition 2.46 of a D_π-group will be used for convenience.

We have the following sketch of a proof.

Proof of Theorem 2.47(i). Let G be a minimal counterexample. Then Theorems 2.31 and 2.32 together imply that G is simple. Theorem 3.31 then implies that $G \cong Sz(q)$ for some q. In addition, $2 \in \pi$ since otherwise Theorem 2.44(i) would imply that G is π'-closed, a contradiction. A reference to Suzuki's descriptive paper[59] then implies that G has more than one class of maximal π-subgroup and so is not a D_π-group, a contradiction.

Theorem 2.47(ii) is proved similarly.

Arad conjectured that in Theorem 2.47, neither of the two extra conditions was necessary, a result subsequently proved by Ferguson, Theorem 2.51, which we shall discuss shortly after the following.

Proof of Theorem 2.48. Let G be a simple 3'-homogeneous group. If 3 does not divide $|G|$ then $G \cong Sz(2^{2n+1})$ for some $n \geq 1$. Hence we can assume that 3 divides $|G|$. By Theorem 3.33, G is S_4-free if and only if whenever H is an arbitrary non-trivial 2-subgroup of G then S_3 is not involved in $N_G(H)/C_G(H)$. But since G is 3'-homogeneous, 3 does not divide $|N_G(H)/C_G(H)|$ for such an H. Thus by Glauberman's Theorem 3.32 and Goldschmidt's result, Theorem 3.34, G is one of a known list of simple groups. A check of this list shows that G is isomorphic to $PSL(2,2^{2n+1})$ for some $n \geq 1$. Hence result.

Proof of Theorem 2.49. Let G be a minimal counterexample. Clearly by Theorems 2.31 and 2.32, G is simple. As in the above proof of Theorem 2.48, we know that every 3'-homogeneous group is S_4-free and so by Theorems 3.32 and 3.34, G is one of a known list, from which it follows that G is isomorphic to $PSL(2,2^{2n+1})$ for some n, a contradiction.

The proof of Theorem 2.50 follows a similar pattern. A minimal counterexample is seen to be a simple group form Goldchmidt's list in Theorem 3.34, and checking of this list gives a contradiction.

The next result, Theorem 2.51, is proved using the standard techniques of group theory. However its length restricts us to giving a brief sketch.

__Sketch of the proof of Theorem 2.51__. To begin, G is supposed a minimal counterexample, that is, G is a π-homogeneous D_π-group which is not π'-closed. We let H be a Hall π-subgroup of G.

__Step 1__: Ferguson originally proved that if G is a D_π-group, then $G/O_\pi(G)$ is a D_π-group. (The proof of this however is easily extendable to the case where $O_\pi(G)$ is replaced by an arbitrary normal π-subgroup). It is then obvious that if $O_\pi(G)\neq 1$ then both $G/O_\pi(G)$ and $O_\pi(G)$ are π'-closed and so by Theorem 2.32, G is π'-closed, a contradiction. Hence $O_\pi(G)=1$.

__Step 2__: By far the longest stage, here are obtained certain fusion results, as follows.

Let S be a non-trivial p-subgroup of H for any $p\in\pi$. Then the following hold.

(i) $N_G(S)$ is π'-closed.

(ii) $N_G(S)=N_H(S)O_{\pi'}(C_G(S))$.

(iii) If $S\leq H^g$ for some $g\in G$ then $H^g=H^y$ for some $y\in O_{\pi'}(C_G(S))$.

By Step 1, $N_G(S)<G$. Thus by the minimality of G, in order to prove (i) it is sufficient to show that $N_G(S)$ is a D_π-group. Then proof is then split into two cases, according as S is a Sylow p-subgroup of H(and hence G) or not. Once these have been proved, (ii) and (iii) follow quickly in each case. The proofs of (i) proceed by elementary Sylow theory and fusion results, though in the latter, the profusion of conjugacy prompts the hope that a shorter argument might be found. It is from amidst the various conjugacy inequalities that parts (ii) and (iii) arise.

__Step 3__: Here it is proved that if $z\in H^{\#}$ then $C_G(z)$ is π'-closed, and also that if $z\in(H\cap H^g)^{\#}$ for some $g\in G$ then $H^g=H^y$ for some $y\in O_{\pi'}(C_G(z))$. The element z is supposed a counterexample of minimal order, and by Steps 2(i) and 2(ii) respectively for the two parts of the result, must be of composite order. The two results now follow easily by supposing that $z=z_1z_2$ where z_1 is of prime power order, and the fact that $C_G(z)\leq C_G(z_i)$ for i=1, 2, and again using Step 2.

__Step 4__: Any two elements of $H^{\#}$ conjugate in G are conjugate in H. This is easily proved by similar techniques to the above.

__Step 5__: The proof of the theorem follows immediately by applying Theorem 3.35.

Without a doubt, the single most important result in this survey is Theorem 2.52. (It is also the one with the most complex proof). Not only does it represent the final aim of the theory of closure and homogeneity, but it can be used to prove many other results which have important applications of their own. We shall see such results later (for example, Theorem 5.12). However we will now try to give a general flavour of the proof.

<u>Sketch of the proof of Theorem 2.52</u>. As in all the other results, π'-closure implies π-homogeneity, and thus in discussing the proof we shall consider a minimal counterexample G which is π-homogeneous but not π'-closed.

Theorems 2.31 and 2.32 immediately imply that G is simple. The first step is to obtain additional information about G and π. We can assume that $\pi \subseteq \pi(G)$, and since if $|\pi|=1$, Theorem 2.36 gives a contradiction, we can also assume that $2 \leq |\pi| < |\pi(G)|$.

Now since G is a minimal counterexample, any group properly involved in G is π'-closed. In particular, if H is a simple group properly involved in G then, as $|H|$ is even, H must be a π'-group. This point is critical to the proof.

Using the complete classification of the simple groups, each simple group is then checked, and it is shown that no simple group is π-homogeneous for any suitable π. There are three main techniques involved in the proof which sometimes overlap in use.

(a) We can show that if G is π-homogeneous then $|\pi|=1$, a contradiction. This is possible when the exact orders of the simple groups in question are known and a table of containment is available, as with the sporadics in [38]. The alternating groups are also handled in this way.

(b) We can attempt to locate a π-subgroup H of G for which $N_G(H)/C_G(H)$ contains an involution, a contradiction. This is particularly suitable when the subgroup structure of G is well known, as in the small cases of the groups of Lie type when the dimension or the field size is low, for example PSL(2, q) for any q.

(c) We can use the involvement of simple groups in each other (see for example, Kantor[49]) to firstly reduce the possibilities for π and then use the existence of a special type of maximal abelian subgroup called a torus (see for example, Carter[16]) to obtain that a Sylow p-subgroup P for some $p \in \pi$ is centralised by such a subgroup (of known order) whose index is a π'-number. Thus, as G is

π'-homogeneous, $N_G(P)/C_G(P)=1$ and so by Burnside's Theorem 3.13, G is not simple, a contradiction. In this way most of the groups of Lie type are dealt with.

It can happen that because of the involvement in G of other simple groups, it is found that $\pi(G)$ is completely accounted for by the divisors of these simple groups so that G is a π'-group, an impossibility, and there is no need to proceed further. This happens for instance with $F_4(q)$ and $E_7(q)$.

It is also necessary in some cases to work some additional theory, but this is only used to bring the group in question to the point where one of the above three techniques can be applied.

In this way all the simple groups are shown not to be counterexamples to the theorem, establishing the result.

Finally we use Theorem 2.52 to prove Theorem 2.53.

Proof of Theorem 2.53. Clearly, as in the proof of Theorem 2.39, the backward implication holds. We thus suppose that G is π-homogeneous and π'-homogeneous. Then, without loss of generality, we can suppose that $2\notin\pi$ and so by Theorem 2.52, G is π'-closed, that is, G has a normal Hall π'-subgroup $H_{\pi'}$. By the Schur-Zassenhaus Theorem, G splits over $H_{\pi'}$. Thus $G=H_{\pi'}H_{\pi}$ where H_{π} is a Hall π-subgroup of G. Since G is π'-homogeneous, $N_G(H_{\pi'})/C_G(H_{\pi'})=G/C_G(H_{\pi'})$ is a π'-group and so $H_{\pi}\leq C_G(H_{\pi'})$. Thus $H_{\pi'}\leq C_G(H_{\pi})$. We then have that both H_{π} and $H_{\pi'}$ are contained in $N_G(H_{\pi})$. Hence $G=H_{\pi'}H_{\pi}\leq N_G(H_{\pi})$. Thus $H_{\pi}\trianglelefteq G$ and $G=H_{\pi'}\times H_{\pi}$.

CHAPTER V: THE RECENT THEORY OF CC-SUBGROUPS

In this chapter we develop certain Sylow-like properties of CC-subgroups(Theorems 5.8 and 5.12). In order to do this a collection of important structural results is given, which will also provide a base for further study, especially towards the classification of all groups with CC-subgroups in the next chapter. We also include an interesting result involving 3'-groups with non-cyclic CC-subgroups.

Extensive use will be made of Chapter III, in particular Wielandt's Theorem 3.10, and of course all of Section II.

This entire chapter is based on the published researches of Arad and Chillag, but many of the proofs have been altered for the sake of clarity and a better unity of technique.

The first theorem has an important bearing on simple groups.

<u>**Theorem 5.1**</u>(Arad and Chillag[6; Lemma 1.2(d)]). Let $M \leq_{CC} G$ and suppose that $N_G(M) \neq M$ and $F(G)=1$. Then the normal closure V of M in G is a simple minimal normal subgroup of G, and G=VH where H is a Frobenius complement of $N_G(M)$.

<u>Proof</u>. Let N be a minimal normal subgroup of G. If $M \cap N=1$ then by Theorem 3.51, MN is Frobenius with kernel N and so N is nilpotent, contradicting $F(G)=1$. Thus we can assume that $M \cap N \neq 1$. Suppose that $N_G(M)$ has a normal complement L in G. Then again by Theorem 3.51, since $M \cap L=1$, ML is Frobenius with kernel L, a contradiction. Thus, since M is nilpotent and so is a T.I-set in G, we can apply Proposition 3.53(i) to obtain that $M \leq N$. Since $M \neq N$ as $F(G)=1$, N is simple by Proposition 3.50, and $M \leq_{CC} N$. Let V be the normal closure of M in G. Then $V \leq N$ and so V=N since N is minimal normal.

Since M is nilpotent, we can apply Theorem 3.10 to obtain that all subgroups of V of order $|M|$ are conjugate in V. Thus by Frattini's Argument, Proposition 3.7, $G=VN_G(M)$. But $N_G(M)=MH$ where H is a Frobenius complement of $N_G(M)$. Hence G=VH as required.

Suppose that M is a CC-subgroup of a group G. An important property of M is that if $N_G(M) \neq M$ then M is <u>nilpotent</u>. Later, in Chapter VI, we will investigate this case further. The following result covers all non-nilpotent cases.

<u>Theorem 5.2</u>[6; Proposition 1.3]. Let $M \leq_{cc} G$. If $N_G(M)=M$ then one of the following is true.

(i) G is Frobenius with M as complement.

(ii) G is a simple group and M is Frobenius with nilpotent kernel K and cyclic complement H of odd order, where $N_G(K)=M$ and $N_G(H) \neq H$.

<u>Remark</u>: By Proposition 3.41, H and K are CC-subgroups of G.

<u>Proof of Theorem 5.2</u>. (Johnson and Bowshall). If M is a T.I-set in G then by definition, (i) holds. Thus we can assume that M is not a T.I-set in G. Now suppose that $F(G) \neq 1$. If $M \cap F(G)=1$ then by Theorem 3.51, M is a Frobenius complement of the group MF(G) and so by Theorem 3.19(x), $Z(M) \neq 1$, which implies by Proposition 3.39 that M is a T.I-set, a contradiction. Thus $M \cap F(G) \neq 1$. Then, since M is a Hall $\pi(M)$-subgroup for G, there exists a prime $p \in \pi(M)$ such that $1 < O_p(G) \leq M$. Hence by Proposition 3.58, G is Frobenius with M as kernel, a contradiction. Thus we can assume that $F(G)=1$.

Since M is not a T.I-set in G, there exists $g \in G$ such that $1 < M \cap M^g < M$. Then by Propositions 3.37 and 3.38, $M \cap M^g \leq_{cc} G$. Suppose $M_G \neq 1$. Then by Proposition 3.45, M_G is a Frobenius kernel and so nilpotent, contradicting $F(G)=1$. Thus $M_G=1$. Now since M_G is the intersection of all conjugates of M, there must exist some intersection H of conjugates of M such that $H \neq 1$ but $H \cap M^x=1$ for all $x \in G$ such that $H \nleq M^x$. We can assume that $H \leq M \cap M^g$ (otherwise we can replace g). Clearly H is a T.I-set in G and $H \leq_{cc} G$. If $N_G(H)=H$ then G is a Frobenius group, contradicting $F(G)=1$. Thus $N_G(H) \neq H$ and so H is nilpotent. Then by Proposition 3.43, H contains no CC-subgroup of G.

Now H and $H^{g^{-1}}$ are nilpotent Hall subgroups of M and so by Theorem 3.10, $H^{g^{-1}}=H^k$ for some $k \in M$. Then $kg \in N_G(H)$, and since $g \notin M$, $kg \notin M$. Hence $N_G(H) > N_M(H)$. It follows by Proposition 3.4, that $N_M(H) \leq_{cc} N_G(H)$ and since $N_G(H)$ is Frobenius, either $N_M(H)=H$ or $N_M(H)$ is a Frobenius complement of $N_G(H)$. But $H \leq N_M(H)$ and so $N_M(H)=H$. Since H is a T.I-set, M is Frobenius with complement H by definition, and kernel K, say. As $K \leq_{cc} M \leq_{cc} G$, $K \leq_{cc} G$. Now by Proposition 3.43, $Z(M)=1$ and so M is neither a kernel nor a complement of $N_G(K)$. Thus $M=N_G(K)$ by Proposition 3.47, as required.

Let V and W be the normal closures in G of K and H respectively. It follows by Theorem 5.1 (all conditions being fulfilled) that V and W are simple minimal normal subgroups of G and G=VH (since H is the

Frobenius complement of $N_G(K)=M$). Since $[K,H]\neq 1$ (as a result of the CC-property), we have $1<[K,H]\leq[V,W]\leq V\cap W$. Hence $V\cap W\neq 1$ and so since V and W are minimal normal, V=W. In particular, $H\leq V$ so that G=V, as required.

Finally, H is a Frobenius complement and kernel and so is cyclic by Proposition 3.48.

The next result is a simple generalisation of Proposition 3.55. There are several proofs but none quite as short as the one given here.

Theorem 5.3[6: Lemma 1.4]. Let G be a soluble group with a CC-subgroup M. Then one of the following is true.

(i) G is Frobenius with M as kernel or complement.

(ii) $N_G(M)\neq M$ or G, $N_G(M)$ has a normal nilpotent complement in G, M is cyclic of odd order and $N_G(M)$ is metacyclic.

Proof. If $N_G(M)=M$ then the result follows from Theorem 5.2, and if $N_G(M)\neq M$, it follows from Proposition 3.55.

To prove Theorem 5.6 it is convenient to give the next two lemmas.

Lemma 5.4 (Arad and Chillag[4; Lemma 2]). Let $M\leq_{CC}G$ with $Z(M)\neq 1$ and let N be a non-trivial proper normal subgroup of G. Then one of the following holds.

(i) G is Frobenius with M as kernel or complement.

(ii) $M\cap N=1$, M is cyclic of odd order and MN is Frobenius with kernel N and complement M.

(iii) $M\leq N$ and M has odd order.

(iv) $N_G(M)$ has a normal complement in G and M is cyclic of odd order.

Proof. If $|M|$ is even then by Theorem 3.27, (i) is true since G is not simple. Thus we can assume that $|M|$ is odd. If $N_G(M)=M$ then as M is a T.I-set in G, (i) holds by definition. Hence we can assume that $N_G(M)\neq M$. If $M\cap N\neq 1$ then either $N_G(M)$ has no normal complement in G so that by Proposition 3.53, (iii) holds, or $N_G(M)$ has a normal complement and by Theorem 3.51, M is a Frobenius complement so that since M is also a kernel, M is cyclic by Proposition 3.48, and (iv) holds. Thus we can assume that $M\cap N=1$. If $M\trianglelefteq G$, (i) holds. Thus we may suppose that $M\ntrianglelefteq G$. We can then apply Theorem 3.51 to obtain that MN is Frobenius with kernel N and complement M and so (ii) holds, again applying Proposition 3.48.

<u>Lemma 5.5</u>[4; Lemma 5]. Let $M \leq_{CC} G$ with $Z(M) \neq 1$ and let N be a minimal normal subgroup of G. Then one of the following holds.

(i) G is Frobenius with M as kernel or complement.

(ii) M is cyclic.

(iii) $M \leq N$ and N is simple.

<u>Proof</u>. Suppose that neither parts (i) nor (ii) hold. If $N=G$ then G is simple and (iii) holds. If $N<G$ then by Lemma 5.4, $M \leq N$. We now apply Proposition 3.50 to obtain that N is simple.

Before investigating the Sylow-like properties of groups with CC-subgroups, we can now prove a result which complements Theorem 2.23.

<u>**Theorem 5.6**</u>[4; Theorem 4]. Let $M \leq_{CC} G$, M being non-cyclic. If $(3, |G|)=1$ then one of the following holds.

(i) G is Frobenius with M as kernel or complement.

(ii) $G \cong Sz(q)$ for some q, and M is either a Sylow 2-subgroup or a Sylow 2-normaliser.

<u>Proof</u>. Let G be a minimal counterexample. By Suzuki's Theorem 3.27, $|M|$ is odd. Suppose that $N_G(M) \neq M$. Then M is nilpotent. Assume that G is not simple and let H be a minimal normal subgroup of G. By Lemma 5.5, $M \leq H$ and H is simple. Thus by Theorem 3.31, $H \cong Sz(q)$ for some q. Hence by Theorem 3.25, all Sylow p-subgroups of H are cyclic for odd p, and since M is nilpotent, M is cyclic, a contradiction. Thus G is simple and again by Theorem 3.31, $G \cong Sz(q)$ for some q, and similarly to the previous sentence, M is cyclic, a contradiction. Thus we can assume that $N_G(M)=M$.

Suppose that G is not simple and let H be a minimal normal subgroup of G.

Now suppose that H is an elementary abelian p-group for some p. Let $\pi=\pi(M)$ and suppose that $p \notin \pi$. Then, with an asterisk denoting factoring modulo H and using Propositions 3.56 and 3.57, we have $M^* \leq_{CC} G^*$ and $N_{G^*}(M^*)=M^*$, where $M^* \cong M$ since $M \cap H=1$. Hence by the minimality of G, M^* is a Frobenius complement of G^* since $|M|$ is odd and so part (ii) cannot hold. Thus there exists a normal Hall π'-subgroup R^* of G^* (that is, the kernel of G^*). The inverse image R of R^* in G is a normal Hall π'-subgroup of G. It follows by Theorem 3.51 that G is Frobenius with M as complement, a contradiction. We can assume then that $p\in\pi$. Thus by Proposition 3.42, $H\trianglelefteq M_G<M$. Since $M_G\neq 1$, G is Frobenius with kernel M_G and so by Proposition 3.47, either $M=M_G$ or $M\cap M_G=1$, both contradictions.

Thus we can suppose that H is the direct product of isomorphic simple groups $H_1,\ldots H_k$ for some integer k. Suppose that $M\cap H=1$. Then by Theorem 3.51, MH is Frobenius with kernel H, a contradiction. Thus $L=M\cap H\neq 1$. We have that $L\leq_{CC}H$. If $L\cap H_1=1$ then LH_1 is Frobenius with kernel H_1, a contradiction. Hence $L\cap H_1\neq 1$. Thus there exist elements of L centralised by $H_2\times\ldots\times H_k$. Hence $H_2\times\ldots H_k=1$ and H is simple. Thus $H\cong Sz(q)$ for some q and since $|L|$ is odd, L is cyclic. If $N_H(L)=L=C_H(L)$, then H is not simple by Theorem 3.13 and so $N_H(L)\neq L$, and also $M\leq N_G(L)$ since $L\trianglelefteq M$. If $M=N_G(L)$ then $L=N_H(L)$, a contradiction. Hence $M<N_G(L)<G$. Thus by the minimality of G, we can apply the theorem to $N_G(L)$. Since $|M|$ is odd, part (ii) does not hold. If $N_G(L)$ was Frobenius with kernel M then M would be nilpotent and so a T.I-set, making G Frobenius with complement M, a contradiction. Thus $N_G(L)$ is Frobenius with complement M, and kernel K, say. Since $K\trianglelefteq N_G(L)$ and $K\cap L=1$ we obtain that $K\leq C_G(L)\leq M$, a contradiction.

Hence G is simple and so $G\cong Sz(q)$ for some q. Thus as $|M|$ is odd, M is cyclic, a contradiction.

We now investigate the Sylow-like properties.

<u>Theorem 5.7</u>[6: Lemma 1.5]. Let M and N be CC-subgroups of a group G. If $|M|=|N|$ then M and N are conjugate.

<u>Proof</u>. If either $N_G(M)\neq M$ or $N_G(N)\neq N$ then the result follows immediately by Wielandt's Theorem 3.10. Thus we can assume that both $N_G(M)=M$ and $N_G(N)=N$. Applying Theorem 5.2 we can suppose that condition (ii) of that result holds for both M and N, since otherwise M and N would be Frobenius complements and so conjugate.

Thus $M=KH$ and $N=XY$ where K and X are the kernels and H and Y the complements of M and N respectively. Also G is simple and by the remark following Theorem 5.2, K and X are nilpotent CC-subgroups of G and H and Y are cyclic CC-subgroups of G of odd order.

If $|M|$ is even then by Theorem 3.27, M and N are Sylow 2-normalisers and thus conjugate. Thus we can assume that $|M|$ is odd so that M and N are soluble.

Let $\pi=\pi(K)$. Then N contains a subgroup of order $|K|$ by Theorem 3.8 and so by Theorem 3.10, N contains some conjugate K^l of K for some $l\in G$. Since K^l is a CC-subgroup of the Frobenius group N, either $K^l=X$ or K^l is a Frobenius complement for N by Proposition 3.47. If $K^l=X$ then $N_G(K)=M$ is conjugate to $N_G(X)=N$, as required. Thus we can assume that K^l is a Frobenius complement for N so that by Theorem 3.19(v), $|K|$ divides $|X|-1$. Similarly we can assume that M contains some conjugate X^g of X. If X^g is a Frobenius complement for M then $|X|$ divides $|K|-1$, which is impossible number-theoretically. Thus $X^g=K$ and $N_G(K)=M$ is conjugate to $N_G(X)=N$, as required.

The first main theorem of this chapter can now be given.

Theorem 5.8[6: Theorem 1.1]. Let $M\leq_{CC}G$ and $\pi=\pi(M)$. Then M is a Hall π-subgroup of G and all Hall π-subgroups of G are conjugate.

Proof. We already know that M is a Hall π-subgroup of G. Let N be any other Hall π-subgroup of G. In view of Theorem 5.7 it is enough to prove that N is a CC-subgroup of G.

By Theorem 3.10, if M is nilpotent then the theorem holds. Hence we can assume that $N_G(M)=M$ and then that part (ii) of Theorem 5.2 holds. If $|M|$ is even then by Theorem 3.27, M and N are both Sylow 2-normalisers and so conjugate. Hence we can assume that $|M|$ is odd and so N is soluble. Let M=KH as in Theorem 5.2(ii), and let $\pi_1=\pi(K)$ and $\pi_2=\pi(H)$. Since N is soluble, N contains Hall π_i-subgroups N_i for i=1,2. By Theorem 3.8, $N_1=K^{g_1}$ and $N_2=H^{g_2}$ for some $g_1,g_2\in G$. Since both K^{g_1} and H^{g_2} are CC-subgroups of G, we have that N_1 and N_2 are CC-subgroups of G. Now let $n\in N$ and suppose that $n=n_1n_2$ where n_i is a non-trivial π_i-element of N for i=1,2. Then $n_i\in C_G(n)$ and so $n\in C_G(n_i)$. But $n_i\in N_i^{l_i}$ for some $l_i\in N$ for i=1,2 and as $N_i\leq_{CC}G$ we have that $n\in N_i^{l_i}$ for i=1,2. Thus, since $(|N_1|,|N_2|)=1$, $n=1$. Thus every element of N is either a π_1-element or a π_2-element. Now let n be an arbitrary element of N and suppose that n is a π_i-element for i=1 or 2. Then $C_G(n)\leq N_i^g$ for some $g\in N$ and so $N\leq_{CC}G$, as required.

For the second main theorem we need another structural result.

Proposition 5.9[6: Lemma 1.8]. Let G be a soluble group containing two non-conjugate CC-subgroups M and L. Then G is Frobenius with M as kernel and L as complement, or vice-versa.

Proof. Let G be a counterexample. Then by Theorem 5.3, $N_G(M)$ has a nilpotent normal complement K. M is cyclic and also, since L can replace M. L is cyclic.

Suppose that $(|M|,|L|)\neq 1$ and let p divide $(|M|,|L|)$. M and L contain Sylow p-subgroups P and Q of G respectively. Since M and L are cyclic CC-subgroups, $M=C_G(P)$ is conjugate to $C_G(Q)=L$, a contradiction. Hence $(|M|,|L|)=1$.

Suppose that $(|K|,|L|)\neq 1$. Then by Proposition 3.59, $K\leq L$ and since L is a T.I-set in G, $L\trianglelefteq G$ by Proposition 3.22. Hence L is the Frobenius kernel of G, a contradiction. Thus $(|K|,|L|)=1$. We already know that $(|K|,|M|)=1$ and so since G is soluble, G has a Hall subgroup T of order $|M||L||K|$. Let V be a subgroup of T of order $|M||L|$. Then, as in the proof of Theorem 5.8, $V\leq_{CC}G$ and so $V\leq_{CC}T$. Hence, since $(|V|,|K|)=1$ and $K\trianglelefteq T$, K is a normal CC-subgroup of T. This implies that V is a Frobenius complement of the Frobenius group T and so by Theorem 3.19, $Z(V)\neq 1$, contradicting Proposition 3.43. Hence result.

We can now proceed to the second main theorem.

Theorem 5.10 (Arad and Chillag[7; Theorem B]). Let $M\leq_{CC}G$ and let $\pi=\pi(M)$. Then all of the following are true.

(i) M is a Hall π-subgroup of G and all Hall π-subgroups of G are conjugate.

(ii) G has at most two classes of soluble maximal π-subgroup.

(iii) If M is not nilpotent then either G is a Frobenius group with complement M or M is a Frobenius group with kernel K and complement L where $N_G(K)=M$ and $N_G(L)\neq L$, and L is cyclic of odd order.

(iv) If G has two classes of soluble maximal π-subgroup then

(a) G is simple,

(b) M is not nilpotent,

(c) the normaliser of every non-trivial soluble π-subgroup of G is a π-subgroup,

(d) M and $N_G(L)$ are representatives of the two classes of soluble maximal π-subgroups of G (where L is as in part (iii)).

Proof. Part (i) holds by Theorem 5.8. If $N_G(M)\neq M$ then M is nilpotent and so G is a D_π-group by Theorem 3.8, and the theorem holds. Thus we can assume that $N_G(M)=M$. Suppose that G is a Frobenius group with complement M. Then clearly G is π'-soluble and so by Theorem 3.9, G is again a D_π-group and the result holds.

Hence we may assume that G is not Frobenius with complement M. Then by Theorem 5.2, G is simple and M is a Frobenius group with kernel K and complement L, where $N_G(K)=M$ and $N_G(L)\neq L$ and L is cyclic of odd order. We also know that K and L are CC-subgroups of G, and T.I-sets in G. Let $\pi_1=\pi(K)$ and $\pi_2=\pi(L)$. Then by Theorem 3.8, G is a D_{π_1}-group and a D_{π_2}-group.

(*) Let T be a π_i-subgroup of G for i=1,2. Then either $T\leq K^g$ or $T\leq L^f$ for some $f,g\in G$. In particular, T is contained in a conjugate of M. Thus by Proposition 3.22, either $N_G(T)\leq N_G(K^g)=M^g$ or $N_G(T)\leq N_G(L^f)=N_G(L)^f$.

From now on, given a soluble subgroup X of G, we will denote by X_i, a Hall π_i-subgroup of X.

Let B be a soluble π-subgroup of G which is neither a π_1-group nor a π_2-group. Then $B=B_1B_2$ where $1<B_1\leq K^h$ and $1<B_2\leq L^d$ for some $h,d\in G$, by paragraph (*). Thus $1<B\cap K^h$ so that $B\cap K^h\leq_{cc}B$. Then $B\cap K^h$ is a Hall π_1-subgroup of B and so $B_1=B\cap K^h$ and $B_1\leq_{cc}G$. Similarly $B_2=B\cap L^d$ and $B_2\leq_{cc}B$. Proposition 5.9 now implies that B is Frobenius with B_i as kernel or complement for i=1 or 2. Suppose that B_j is the kernel. Then $B\leq N_G(B_j)$ and either $B\leq N_G(B_j)\leq N_G(K^h)=M^h$ or $B\leq N_G(L)^d$, as in paragraph (*), and part (ii) of the theorem holds. Also since B_j is characteristic in B (being a normal Hall subgroup), we have that $N_G(B)\leq N_G(B_j)$. Thus either $N_G(B)\leq M^h$ or $N_G(B)\leq N_G(L)^d$.

Thus every maximal soluble π-subgroup is contained in a conjugate of M or $N_G(L)$ and the normaliser of every non-trivial soluble π-subgroup is contained in a conjugate of M or $N_G(L)$. Hence in order to prove the remainder of the theorem it is enough to show that if G has more than one class of soluble maximal π-subgroup then $N_G(L)$ is a soluble π-subgroup.

Since $N_G(L)$ is Frobenius with a cyclic kernel L, the complement, which can be embedded in AutL by Theorem 3.19, is abelian by Proposition 3.2, and so cyclic by Proposition 3.19(xiii). Thus $N_G(L)$ is metacyclic and hence soluble. Let A be a soluble maximal π-subgroup not contained in a conjugate of M. Then $A=A_1A_2$ where $A_1\leq K^b$ and $A_2\leq L^c$ for some $b,c\in G$ (as with B above). If A is a π_i-group for i=1 or 2 then A would be contained in a conjugate of M, a contradiction. Thus, as with B, A is a Frobenius group with A_i as kernel or complement. If A_1 is the kernel then $A\leq N_G(A_1)\leq N_G(K^b)\leq M^b$ by paragraph (*), a contradiction. Hence A_2 is the kernel and $A \leq N_G(A_2) \leq N_G(L)^c$. Then, as $A_2 \leq L^c$, $A \leq A_1L^c$. Also, since $A_1L^c/L^c \cong A_1/A_1 \cap L^c$ and both L^c and A_1 are soluble, A_1L^c is a soluble π-subgroup. Hence $A=A_1L^c$. Let R be a Frobenius complement for $N_G(L)^c$. Then $|A_1|$ divides $|R|$. Hence R contains a p-element for some $p \in \pi_1$. Since no p-element of G commutes with a π'-element of G(as $M \leq_{CC} G$), and since by Proposition 3.19, $Z(R) \neq 1$, we have that R contains no π'-element. Thus $N_G(L)^c$ is a π-group and so $A=N_G(L)^c$. This proves the theorem.

In order to prove the last theorem in this chapter, we give a straightforward corollary of Theorem 5.10.

<u>Corollary 5.11</u>. Let $M \leq_{CC} G$, $\pi=\pi(M)$ and suppose that $|M|$ is odd. Then all of the following are true.

(i) M is a Hall π-subgroup of G and all Hall π-subgroups of G are conjugate.

(ii) G has at most two classes of maximal π-subgroup.

(iii) If M is not nilpotent then either G is a Frobenius group with M as complement or M is a Frobenius group with kernel K and complement L where $N_G(K)=M$ and $N_G(L) \neq L$ and L is cyclic of odd order.

(iv) If G has two classes of maximal π-subgroup then

(a) G is simple,

(b) M is not nilpotent,

(c) the normaliser of every non-trivial π-subgroup of G is a π-subgroup, and

(d) M and $N_G(L)$ are representatives of the two conjugacy classes of maximal π-subgroups of G(where L is as in part (iii)).

<u>Proof</u>. Immediate from Theorem 5.10 and the fact that groups of odd order are soluble.

We can now use Theorem 2.52 to give the last theorem of this chapter.

<u>Theorem 5.12</u>(Arad and Chillag[8; Corollary 3]). Let M be a CC-subgroup of a group G, $\pi=\pi(M)$ and suppose that $|M|$ is odd.

Then M is a Hall π-subgroup of G, all Hall π-subgroups of G are conjugate, and every π-subgroup of G is contained in a conjugate of M.

<u>Proof</u>. By Theorem 5.8, we already know that M is a Hall π-subgroup and that all Hall π-subgroups of G are conjugate. Suppose that not

every π-subgroup of G is contained in a conjugate of M. Then by Theorem 5.11(ii), G has two classes of maximal π-subgroup and so by Theorem 5.11(iv), G is π-homogeneous. Thus by Theorem 2.52, G is π'-closed. Then, since G is simple, G must be a π-group. But this is impossible since $2 \notin \pi$. Hence result.

<u>Note</u>: Theorem 5.12 would be false without the hypothesis of odd order. Arad and Chillag gave the specific example of PSL(2,16) with $\pi=\{2,3,5\}$ in Examples 1 and 2 of [7], for which there are at least three non-conjugate maximal π-subgroups at least one of which is a CC-subgroup.

CHAPTER VI: THE SEARCH FOR CC-SUBGROUPS

Section I: A classification theorem for groups with CC-subgroups

This section consists of a single result which completely covers every possible case for a group to have a CC-subgroup.

Theorem 6.1. Let M be a CC-subgroup of a group G. Then one of the following is true.

(i) G is a Frobenius group with M as kernel or complement.

(ii) G is soluble, $N_G(M)\neq M$ and $G=KML$ where M is cyclic of odd order, L is a cyclic Frobenius complement of $N_G(M)$ of odd order and K is a nilpotent normal complement to $N_G(M)$ in G.

(iii) G is simple, $N_G(M)=M$ and M is Frobenius with kernel K and complement L where K and L are CC-subgroups of G, $N_G(K)=M$, $N_G(L)\neq L$ and L is cyclic of odd order.

(iv) $(|F(G)|,|M|)=1$, $N_G(M)\neq M$ and if asterisks denote images modulo $F(G)$ then $M^*\not\leq_{cc}G^*$ and $G^*=V^*H^*$ where V^* is a simple normal subgroup of G^* containing M^* and H^* is a Frobenius complement of $N_{G^*}(M^*)$.

Proof. This will consist of a sequence of lemmas. We let G be a minimal counterexample to the theorem.

Lemma 6.1.1. G is insoluble.

Proof. Suppose G is soluble. Then by Theorem 5.3, G satisfies parts (i) or (ii) of the theorem, a contradiction.

Lemma 6.1.2. $|M|$ is odd.

Proof. Suppose $|M|$ is even. Then by Theorem 3.27, either G is simple and satisfies (iv) with $F(G)=1$ and $G^*=V^*$, or G satisfies (i), a contradiction.

Lemma 6.1.3. $N_G(M)\neq M$.

Proof. Suppose $N_G(M)=M$. Then by Theorem 5.2, either G satisfies (i) or G satisfies (iii), a contradiction.

Lemma 6.1.4. $M\ntrianglelefteq G$.

Proof. Suppose $M\trianglelefteq G$. Then by Proposition 3.45, (i) holds, a contradiction.

<u>**Lemma 6.1.5**</u>. $F(G)\neq 1$.

<u>Proof</u>. Suppose $F(G)=1$. Then by Lemma 6.1.3 and Theorem 5.1, G satisfies (iv), a contradiction.

<u>**Lemma 6.1.6**</u>. $F(G)\leq M$.

<u>Proof</u>. Suppose that $(|F(G)|,|M|)=1$ and let asterisks denote factors modulo $F(G)$. Then by Proposition 3.56, either $M^*=1^*$, $M^*=G^*$ or $M^*\leq_{cc}G^*$.

Suppose $M^*=1^*$. Then $M\leq F(G)$, a contradiction since their orders are coprime. Thus $M^*\neq 1^*$.

Suppose $M^*=G^*$. Then $G=MF(G)$. But since $|M|$ is odd by Lemma 6.1.2, M is soluble. As $F(G)$ is soluble we then have that G is soluble, a contradiction. Hence $M^*\neq G^*$.

Thus we can assume that $M^*\leq_{cc}G^*$. Now by Proposition 3.57, since $N_G(M)\neq M$, $N_{G^*}(M^*)\neq M^*$ and by Proposition 3.60(ii), $F(G^*)=1$. Since $|G^*|<|G|$ we can apply the theorem to G^* to obtain that G^* satisfies (iv) with $F(G^*)=1$, and hence G satisfies (iv), a contradiction.
Hence by Proposition 3.59, $F(G)\leq M$.

<u>**Lemma 6.1.7**</u>. G has generalised quaternion Sylow 2-subgroups.

<u>Proof</u>. Let T be a Sylow 2-subgroup of G. We can apply Theorem 3.51 to obtain that $F(G)T$ is a Frobenius group with kernel $F(G)$ and complement T. Then by Theorem 3.19, T is either cyclic or generalised quaternion. If T were cyclic then by Theorem 3.21, G would be soluble, a contradiction. Hence result.

<u>**Lemma 6.1.8**</u>. $F(G)=M$.

<u>Proof</u>. Let K be the inverse image in G of $Z(G/O_{2'}(G))$. Then by Lemma 6.1.7 and Theorem 3.29, $K/O_{2'}(G)\cong Z_2$. Hence K is soluble. Since also M is soluble (being of odd order), KM is a soluble subgroup of G. We have $M\leq_{cc}KM$ and thus as $|KM|<|G|$, we can apply the theorem by induction. Then either

(a) KM is Frobenius with kernel M,

(b) KM is Frobenius with kernel L say, and complement M, or

(c) KM has a normal nilpotent complement L say, to $N_{KM}(M)$.

Now in cases (b) and (c), $(|L|,|M|)=1$ and $L\trianglelefteq KM$ so that $[L,F(G)]\leq F(G)\cap L=1$ as $F(G)\leq M$ by Lemma 6.1.6. Thus $L\leq C_{KM}(F(G))\leq M$ as $M\leq_{cc}G$. Hence $L=1$.

Thus case (b) collapses and case (c) coincides with case (a). But now $K \trianglelefteq KM$ and $M \trianglelefteq KM$ and so by Proposition 3.49, either $M \leq K$ or $K \leq M$. As $|K|$ is even and $|M|$ is odd, the latter is impossible and so $M \leq K$. Therefore M is a nilpotent normal subgroup of K and so $M \leq F(K) \leq F(G)$. But $F(G) \leq M$ by Lemma 6.1.6. Hence result.

Now since Lemma 6.1.8 contradicts Lemma 6.1.4, G as a minimal counterexample does not exist and hence the theorem is true.

Section II: CC-subgroups of simple groups

In this section we locate the CC-subgroups for a "large" set of simple groups, which information is presented in Tables 1 to 4 as a suitable end to this thesis. We outline how the information is obtained and give results which may aid in the completion of the search.

Throughout this section, the complete classification of the finite simple groups is tacitly used.

To start, it can be seen that restricting the hypothesis of Theorem 6.1 to the case of a simple group provides the following result.

Corollary 6.2. Let G be a simple group with a CC-subgroup M. Then either

(i) M is nilpotent, or

(ii) $N_G(M)=M$ and M is a soluble Frobenius group with nilpotent kernel and cyclic complement of odd order, both of which are CC-subgroups of G.

Thus a simple group with a CC-subgroup necessarily contains a nilpotent CC-subgroup. We first engage in a search for such subgroups and then use the information obtained to find the remaining CC-subgroups.

In view of Theorem 3.27, which completely classifies the case when the CC-subgroup has even order, we can restrict the search to the odd case. It may also be convenient in some situations to use the results of Chapter II, Section I to further restrict the search to the case when the CC-subgroup has order prime to 3. Such a case arises now.

We consider first the alternating groups A_n of degree $n \geq 5$. Now $A_5 \cong PSL(2,4)$ and $A_6 \cong PSL(2,9)$ are both 3CC-groups (with Sylow 3-subgroups of orders 3 and 9 respectively, the latter being elementary abelian). Also, by Theorem 2.27, A_5 is a 2CC-group. Since no other alternating group is isomorphic to any of the groups listed in Theorem 2.27, we need only consider possible CC-subgroups of orders prime to 2 or 3.

Theorem 6.3. Let A_n be the alternating group of degree $n \geq 5$. Then A_n contains nilpotent CC-subgroups of orders prime to 2 or 3 if and only if $n=p, p+1$ or $p+2$ for some prime p, in which case A_n contains a CC-subgroup of order p.

Remark: Clearly if $p+2$ is also a prime then A_{p+2} has two classes of nilpotent CC-subgroup of order prime to 2 or 3, namely one each of orders p and $p+2$.

Proof of Theorem 6.3. For convenience we let $G\cong A_n$ and suppose that $M\leq_{CC}G$ such that $|M|$ is odd. Let p divide $|M|$. Then by conjugating if necessary, we can suppose that the element $x=(1\ 2\cdots p)$ of order p is contained in M. Now clearly $C_G(x)$ contains all the elements of G which act only on the letters $p+1,\cdots,n$. Thus $C_G(x)\geq A_{n-p}$. But A_{n-p} is not nilpotent of order prime to 2 or 3 unless $n-p=0,1$ or 2, in which case $n=p,p+1$ or $p+2$.

Conversely, we consider the three cases separately. We note that in the symmetric group, elements are conjugate if and only if they have the same cycle pattern. Thus the conjugates of x are precisely those of length p. However some of these conjugates may not belong to the corresponding alternating group. Thus we introduce the number ϵ where ϵ can take values 1 or 1/2 and which will be calculated in each case.

Denote by $cl(x)$ the conjugacy class containing x. Then $|C_G(x)|=|G|/|cl(x)|$. The values of $cl(x)$ are obtained combinatorially.

First, suppose that $n=p$. Then $|cl(x)|=\epsilon p!/p$. Thus

$$|C_G(x)|=(p!/2)/(\epsilon p!/p)=p/2\epsilon.$$

Since p is odd we must have $\epsilon=1/2$ and hence G is a Cpp-group with a cyclic Sylow p-subgroup. Thus by Proposition 2.14, G is a pCC-group.

Now suppose that $n=p+1$. Then $|cl(x)|=(p+1)\epsilon p!/p$. Thus

$$|C_G(x)|=((p+1)!/2)/((p+1)\epsilon p!/p)=p/2\epsilon.$$

As in the above case, $\epsilon=1/2$ and G is a pCC-group.

Lastly suppose that $n=p+2$. Then $|cl(x)|=((p+2)(p+1)/2)\epsilon p!/p$. Thus

$$|C_G(x)|=((p+2)!/2)/((p+2)(p+1)\epsilon p!/2p)=p/\epsilon.$$

Now consider $S_n\cong H$. Then $|C_H(x)|=(p+2)!/((p+2)(p+1)p!/2p)=2p$. But now the element $(p+1\ \ p+2)$ is contained in $C_H(x)$ but not in $C_G(x)$ as transpositions are odd. Hence $\epsilon=1$ and $|C_G(x)|=p$. As before then, G is a pCC-group.

Note: For convenience, the CC-subgroups of the alternating groups are given in Table 1.

We now give a result which has great practical use if, for example, the character table of a simple group is known.

<u>**Theorem 6.4**</u>. Let $M \leq_{CC} G$ where G is a simple group. Then M is nilpotent if and only if M is the centraliser of every p-central element of M for each p dividing $|M|$.

<u>Proof</u>. Suppose that M is nilpotent. Then M is the direct product of its Sylow subgroups, which are all then normal in M. Let p divide $|M|$. Then the Sylow p-subgroup of M centralises all p'-elements of M and a p-central element $x \in M$ centralises every p-element in M. Thus $M \leq C_G(x)$. Thus since $M \leq_{CC} G$, $C_G(x) = M$.

Conversely, if M is the centraliser of any p-central element of M for any p dividing $|M|$ then $Z(M) \neq 1$. By Corollary 6.2, M is either nilpotent or Frobenius with CC-subgroups of its own. In the latter case, by Proposition 3.43, $Z(M)=1$, which is not so. Hence M is nilpotent.

We use Theorem 6.4 as follows.

<u>**Proposition 6.5**</u>. If the orders of centralisers of conjugacy classes of all elements of a simple group G are known then the orders of every nilpotent CC-subgroup of odd order can be found.

<u>Proof</u>. It is easy, given such information, to locate the orders of all odd order centralisers of elements of G which are also Hall subgroups. Let such an order be N and let $\pi = \pi(N)$. Then for each $p \in \pi$, in order that the centraliser whose order is N is a CC-subgroup, it is necessary by Theorem 6.4 that there exists a p-central element whose centraliser has order N. When all such N are located (there are not many for a particular simple group), we use the fact that πCC-groups are $C\pi\pi$-groups to further restrict the possibilities by ensuring that for each N and $\pi = \pi(N)$, centralisers of π-elements are π-groups. We then apply Theorem 2.17 to obtain that for all the remaining N, N is the order of some nilpotent CC-subgroup of G.

We now account for the sporadic simple groups.

<u>**Theorem 6.6**</u>. Nilpotent CC-subgroups of sporadic simple groups are as given in Table 2 and are all cyclic of prime order.

<u>Proof</u>. Let G be a sporadic simple group. Then by Theorem 3.27, any possible CC-subgroup of G has odd order. Also, the orders of centralisers in G are available from [18], [38], [50], [61], [68] and hopefully, when it is published, [Atlas]. Applying the technique of Proposition 6.5, all the orders of nilpotent CC-subgroups are

obtained and it is seen that they are all prime as given in Table 2.

We need now only consider the simple groups of Lie type. The following approach which culminates in Theorem 6.13 is due to J. S. Williams[67].

<u>Definiton 6.7</u>. Let G be a group. The <u>prime graph</u> of G is a graph whose vertices are the primes dividing $|G|$ where two primes p and q are joined by an edge if and only if G has an element of order pq. The components of the prime graph of G(that is, the maximal connected subgraphs) are called the components of G and are denoted by π_i for $i=1..n$, where n is the number of components of G. If $|G|$ is even we suppose that $2\in\pi_1$.

<u>Lemma 6.8</u>. If π is a component of G then G is a $C\pi\pi$-group.

<u>Proof</u>. Let x be a π-element of G and suppose that $y\in C_G(x)$. Suppose that $p\in\pi(|x|)$ and $q\in(|y|)$ such that $p\neq q$. Then a power of x which is an element of order p centralises a power of y which is an element of order q, and so there exists an element of order pq. Thus $q\in\pi$ and y is a π-element as required.

We introduce here the idea of a torus, mentioned in the proof of Theorem 2.52.

<u>Definition 6.9</u>. Let G^* be a group of Lie type with simple factor G and let A_G be the corresponding algebraic group. A torus of A_G is defined to be an abelian subgroup of A_G isomorphic to a direct product of subgroups of A_G each of which is isomorphic to the multiplicative group of the base field of A_G. A torus of G^* is defined to be an intersection of a torus of A_G with G^*. A torus of G is defined to be the image in G of a torus of G^*. A maximal torus is simply a torus which is maximal in the set of torii.

<u>Note</u>: By their construction, maximal torii are Hall subgroups and every prime dividing the order of the group of Lie type which is not the characteristic of the group divides the order of some torus.

The significance of the torus is revealed in the following result.

<u>Theorem 6.10</u>(Williams[67; Lemma 5]). Let G be a simple group of Lie type and let G^* be the universal covering of G. Let T be a maximal torus of G. If $|T|$ is odd and $(|T|,|Z(G^*)|)=1$ then the following are equivalent.

(i) $\pi(T)$ is a component of G.

(ii) $T \leq_{CC} G$.

(iii) $(|T|, |C_G(t)|)=1$ for every involution $t \in G$.

Proof. (ii)=>(i). Let $\pi=\pi(T)$. Since T is abelian, π is contained in a single component. Suppose that there exists $p \notin \pi$ such that for some $q \in \pi$, G has an element x of order pq. Then x^p has order q and x^q has order p and $[x^p, x^q]=1$. Since $T \leq_{CC} G$, T is a Hall subgroup. Hence some conjugate $(x^p)^g$ of x^p lies in T. Then $(x^q)^g \in C_G((x^p)^g) \leq T$. Thus $|x^q|$ divides $|T|$, that is, $p \in \pi$, a contradiction. Hence π is a component of G.

(i)=>(iii). Let π^* be the set of primes dividing the orders of centralisers of involutions in G. Then π^* is contained in a component containing 2, that is, $\pi^* \subseteq \pi_1$. Since $\pi_1 \cap \pi = 0$ by (i), (iii) holds.

(iii)=>(ii). This part of the result is proved using algebraic group theory and is therefore omitted.

Theorem 6.10 allows us to give the following important step in the approach.

Proposition 6.11[67: Lemma 6]. If G is a simple group then the components of G consist of the following possibilities.

(i) π_1.

(ii) $\{p\}$, a single odd prime.

(iii) Primes dividing the order of some maximal torus of G satisfying the hypothesis of Theorem 6.10.

Proof. If G is an alternating or sporadic simple group then the result holds by Theorems 6.3 and 6.6. Thus we can suppose that G is of Lie type, of characteristic p say. Then G contains a number of maximal tori. Clearly the orders of these subgroups T are π_i-numbers for suitable components π_i of G. If $|T|$ is even then $\pi(T) \subseteq \pi_1$. On the other hand let π be a component of G not containing the prime 2 and let $q \in \pi$. Then either $q=p$ or q divides the order of some maximal torus T. If $(|T|, |Z(G^*)|)=1$ and q does not divide $|Z(G^*)|$ then $\pi=\pi(T)$ by Theorem 6.10. It remains to show that if q divides the order of $Z(G^*)$ then $q \in \pi_1$.

For most groups of Lie type, the prime 2 is easily seen to divide $|Z(G^*)|$ and so $q \in \pi_1$. The only remaining groups are either linear or unitary and are dealt with by showing that the odd primes dividing $|Z(G^*)|$ also divide the order of a centraliser of some involution and so again $q \in \pi_1$. Hence result.

We now come to the main theorem.

Theorem 6.12[67;Theorem 3]). Let G be a simple group and let π be a component of G not containing the prime 2. Then G contains a nilpotent πCC-subgroup.

Proof. By Proposition 6.11, π is either a single prime or the set of primes dividing some torus of G satisfying Theorem 6.10. In either case there exists a nilpotent Hall π-subgroup T, and also centralisers of π-elements are π-groups by Lemma 6.8. Hence by Theorem 2.17, $T \leq_{CC} G$.

The major application of Theorem 6.12 is the following.

Theorem 6.13. Let G be a simple group of Lie type of odd characteristic. Then the nilpotent CC-subgroups of G are as listed in Table 3.

Proof. We can discount $PSL(2,5) \cong PSL(2,4) \cong A_5$ as CC-subgroups of A_5 are already known. Thus by Theorem 3.27, any CC-subgroup of G has odd order. Hence by Theorem 6.12 it is sufficient to locate the odd components of G. Williams in [67] gives all the components of simple groups of Lie type of odd characteristic, in most cases actually presenting them as the sets of primes dividing given orders. For the odd components these orders are the orders of the corresponding CC-subgroups.

We sketch Williams' study.

Clearly the set π^* of primes dividing the orders of centralisers of involutions of G is contained in π_1. For simple groups of Lie type of odd characteristic, π^* is obtained from [15]. Now every prime dividing $|G|$ which is not the characteristic divides the order of some torus of G. Suppose that $p \notin \pi^*$. Then either p is the characteristic and $\{p\}$ is a component or by Theorem 6.10, p divides the order of some torus T of G such that $\pi(T)$ is an odd component of G. In either case $p \notin \pi_1$. Hence $\pi^* = \pi_1$ and any prime which is not a neighbour of 2 is not even connected to 2.

Thus after removing the orders of centralisers of involutions of G we look for maximal tori of G of odd orders such that their orders are coprime to π^* and these will be CC-subgroups. Williams proceeded to do this with the information in [15] and the orders of tori available from [16], together with certain number-theoretic lemmas which were used to find whether given orders are coprime or not.

We need now only consider the simple groups of Lie type of even characteristic. Shortly we will give some results pertaining to these, but at this point we mention the reasons why they have not yet been dealt with, and suggest some results which may help to complete the search for the CC-subgroups of simple groups.

William's results used in the above description of the proof of Theorem 6.13 rely heavily on the availability of orders of centralisers of involutions of simple groups. Unfortunately such information is incompletely available for simple groups of Lie type of even characteristic. It is hoped however that complete information will eventually be provided. The only other possible barrier in studying the groups of even characteristic is the number theory. As was mentioned in the sketch above, certain number-theoretic lemmas were used to establish coprimeness. We will give these now.

<u>Lemma 6.14</u>(Williams[67; Lemmas 7 to 10]). Suppose that q is an odd prime power. Then the following hold.

(i) $(q^k+1)/2$ is coprime to the set $\{q^i\pm1, i=1..k-1$, and $q^k-1\}$ if and only if k is a power of 2.

(ii) $(q^m-1)/((q-1)\gcd(m,q-1))$, where $m\geq3$, is coprime to the set $\{q^i-1, i=1..m-1\}$ if and only if m is an odd prime.

(iii) $(q^m+1)/((q+1)\gcd(m,q+1))$, where m is odd, is coprime to the set $\{q^i\pm1, i=1..m-1\}$ if and only if m is prime.

Parts (ii) and (iii) of Lemma 6.14 hold equally well when q is a power of 2. In fact the proofs in these cases are identical for (iii) with only a slight modification for part (ii). However the statement and proof of the analogous result to part (i) are substantially different and are as follows.

<u>Lemma 6.15</u>. Let q be a power of 2. Then the number q^k+1 is coprime to the set of numbers $\{q^i\pm1, i=1..k-1$, and $q^k-1\}$ if and only if k is a power of 2.

<u>Proof</u>. Suppose that k is not a power of 2. Then $k=2^nm$ for some m odd and $m\geq3$. Then $q^k+1=(q^{2^n}+1)t$ where $1\leq2^n<k$. Thus q^k+1 is not coprime to the given set.

Conversely, suppose that k is a power of 2. Obviously q^k+1 is coprime to q^k-1 as they are odd and differ by 2. Suppose that q^k+1 is not coprime to the given set and take the smallest i such that

$(q^k+1, q^l\pm1)\neq1$. Then either q^l-1 or q^l+1 is not coprime to q^k+1. In either case, $\gcd(q^{2k}-1, q^{2l}-1)\neq1$. Now $\gcd(q^{2k}-1, q^{2l}-1)=q^d-1$ where $d=\gcd(2k, 2l)$ and k is a power of 2 and so d is a power of 2. We have $\gcd(q^d-1, q^k+1)\neq1$. Now $q^d-1=(q^{(d/2)}-1)(q^{(d/2)}+1)$ and so either $\gcd(q^k+1, q^{(d/2)}-1)\neq1$ or $\gcd(q^k-1, q^{(d/2)}+1)\neq1$. This contradicts the minimality of l unless $d=2l$. Suppose that $d=2l$. Then $\gcd(q^k+1, q^{2l}-1)\neq1$. But $d=2l$ and so $2l$ divides $2k$, and since $l<k$ and k is a power of 2, $2l$ divides k. Thus $q^{2l}-1$ divides q^k-1, a contradiction, since $\gcd(q^k-1, q^k+1)=1$.

A result which may help in the search for CC-subgroups is the following, which relies on Theorem 5.12 and is therefore recent(and indeed unavailable to Williams).

<u>Theorem 6.16</u>. Let G be a πCC-group for some set of odd primes π. Then a subgroup H of G is either a π-group, a π'-group or a π_HCC-group where $\pi_H=\pi\cap\pi(H)$.

<u>Proof</u>. Let L be a maximal π_H-subgroup of H. Then L is contained in a maximal π-subgroup M of G. By the maximality of L, $L=M\cap H$. By Theorem 5.12, M is conjugate to a πCC-subgroup of G and is therefore itself a πCC-subgroup of G. We then apply Proposition 3.40 to obtain that either $L=H$, $L=1$ or $L\leq_{CC}H$, as required.

We give an example of the uses of Theorem 6.16.

Consider the simple group $PSL(n, q)$ where n is odd and $n\geq3$, and q is an odd prime power, and suppose that $PSL(n, q)$ is a πCC-group for some set of odd primes π. It is known that $SL(n-1, q)\leq PSL(n, q)$ and thus by Theorem 6.16, $SL(n-1, q)$ is either a π-group, a π'-group or a πCC-group. But $|Z(SL(n-1, q)|=\gcd(n-1, q-1)\neq1$ and so by Proposition 3.43, $SL(n-1, q)$ is not a πCC-group. Suppose that $SL(n-1, q)$ is a π-group. Then by Theorem 5.12, $SL(n-1, q)$ is contained in a CC-subgroup of a simple group and so by Corollary 6.2, $SL(n-1, q)$ is soluble, a contradiction. Thus $SL(n-1, q)$ is a π'-group and we can discount all the primes dividing its order when searching for a suitable π.

Thus Theorem 6.16 can be used to restrict the possibilities for π. Another result which allows us to do this is an extension of Proposition 3.44. First we need a definition.

<u>Definition 6.17</u>. A group G is said to be the <u>central product</u> of subgroups A and B if $G=AB$ and $[A,B]=1$.

Note that from $G=AB$ & $[A,B]=1$ we can deduce that A and B are normal subgroups of G and $A\cap B\leq Z(G)$. It is clear that direct products are special cases of central products corresponding to the case $A\cap B=1$.

<u>Theorem 6.18</u>. A non-trivial central product has no CC-subgroups.

<u>Proof</u>. Let $M\leq_{CC}G$, where G is the central product of A and B, both non-trivial. Let p divide $|M|$ and suppose without loss of generality that p divides $|A|$. Let x be an element of order p in A. Then $x\in M^g$ for some $g\in G$. Thus $B\leq C_G(A)\leq C_G(x)\leq M^g$ and so $A\leq C_G(B)\leq M^g$. Hence $G\leq M^g$, a contradiction.

Central products can be obtained from direct products in the following manner.

<u>Proposition 6.19</u>. Every homomorphic image of a direct product is a central product.

<u>Proof</u>. Let $G=A\times B$ and let asterisks denote images under the homomorphism. Then since $G=AB$ and $[A,B]=1$, we have $G^*=A^*B^*$ and $[A^*,B^*]=1$, as required.

We give an example of the possible uses of the above idea.

Consider the group $SL(n,q)$ where $PSL(n,q)$ is a simple group with $n\geq 4$. Clearly by simple matrix properties of linear groups, we have that $SL(n-2,q)\times SL(2,q)\leq SL(n,q)$. We wish to factor out the scalar matrices to obtain $PSL(n,q)$. Then the left hand side of the above equation, by Proposition 6.19, becomes a central product of $(SL(n-2,q))^*$ and $(SL(2,q))^*$. But by considering the possibilities for scalar matrices in this product, it is clear that $(SL(n-2,q))^*\cong SL(n-2,q)$ and $(SL(2,q))^*\cong SL(2,q)$. Hence $PSL(n,q)$ contains a central product of $SL(n-2,q)$ and $SL(n,q)$. Now suppose that $PSL(n,q)$ is a πCC-group. Then clearly by Theorems 6.16 and 6.18 and Corollary 6.2, the above central product is a π'-group.

This technique can be applied to all of the classical matrix groups to aid in restricting suitable π.

Returning to the simple groups of Lie type of even characteristic, we give some specific results.

__Proposition 6.20__. The Suzuki group $Sz(2^{2n+1})$ for some $n\geq 1$ has the following nilpotent CC-subgroups.

(i) Cyclic CC-subgroups of order $2^{2n+1}\pm 2^{n+1}+1$.

(ii) Cyclic CC-subgroups of order $2^{2n+1}-1$.

(iii) CC-subgroups of order 2^{4n+2}.

__Proof__. Theorem 3.25.

__Proposition 6.21__. The linear group $PSL(2,2^n)$ for some $n\geq 1$ has the following nilpotent CC-subgroups.

(i) Elementary abelian CC-subgroups of order 2^n.

(ii) Cyclic CC-subgroups of order $2^n\pm 1$.

__Proof__. [Huppert I;II Theorem 8.27] and Suzuki's characterisation in [60].

__Proposition 6.22__. The simple groups $PSL(3,4)$, $PSU(3,4)$ and $PSp(4,4)$ have nilpotent CC-subgroups as given in Table 4.

__Proof__. By Theorem 3.27, none of the above groups have CC-subgroups of even order. Also the orders of centralisers are available from [50]. Proposition 6.5 is then applied and the results are as given in Table 4.

__Proposition 6.23__. The simple group ${}^2F_4(2)'$ has only one class of nilpotent CC-subgroup and that is of order 13.

__Proof__. As Proposition 6.21 where the orders of centralisers are available in [68].

Now we consider the Frobenius CC-subgroups of the groups already dealt with. Clearly any simple group with only one class of nilpotent CC-subgroup cannot have a Frobenius CC-subgroup since such a subgroup is the product of two distinct nilpotent CC-subgroups by Corollary 6.2. Also, given the kernel K and complement H of such a Frobenius subgroup, it is necessary by Theorem 3.19 that $|H|$ divides $|K|-1$. This requirement rules out most possibilities.

__Theorem 6.24__. The Frobenius CC-subgroups of the sporadic simple groups are as given in Table 2.

__Proof__. By enforcing the above two criteria, the only possibilities remaining are for M_{11}, M_{22}, M_{23}, M_{24} and C_2. We consider M_{11} which has nilpotent CC-subgroups of orders 5 and 11. Since 5 divides

$11-1=10$. It is <u>possible</u> that the normaliser of a Sylow 11-subgroup is a Frobenius CC-subgroup. We denote Sylow p-subgroups by P_p. Now $N_G(P_{11})/P_{11}$ by Proposition 3.1 can be embedded in $\mathrm{Aut}P_{11}$ and so by Proposition 3.3, has possible orders 1, 2, 5 or 10. Let $|N_G(P_{11})/P_{11}|=a(11)$. Clearly $a(11)\neq 1$ since $N_G(P_{11})\neq P_{11}$ by Burnside's Transfer Theorem 3.13. From now on we use Theorem 3.12. Suppose that $a(11)=2$. Then an element of order 2 in $N_G(P_{11})$ permutes elements of order 11 into 5 cycles of length 2, resulting in 5 conjugacy classes of element of order 11, a contradiction (since there are only two such classes). Now suppose that $a(11)=10$. Then an element of order 10 in $N_G(P_{11})$ forces all elements of order 11 conjugate, a contradiction. The only possibility then is that $a(11)=5$ in which case an element of order 5 in $N_G(P_{11})$ produces 2 conjugacy classes each of 5 elements of order 11, which indeed happens. Thus $N_G(P_{11})$ is Frobenius with kernel P_{11} and complement P_5, and is a Frobenius CC-subgroup.

The other listed groups are considered similarly.

<u>Theorem 6.25</u>. The only alternating simple group which contains a Frobenius CC-subgroup is A_5 with a single class of such subgroups of order 12 with kernel of order 4.

<u>Proof</u>. Consider A_n with $n\geq 5$. Clearly if $n\geq 7$ then by Theorem 6.3 and the remarks pertaining to that result, for A_n to contain a Frobenius CC-subgroup it is necessary that both n and n-2 are prime and that, by Theorem 3.19, n-2 divides n-1, which is impossible. Similarly, for $n=6$, A_6 has two nilpotent CC-subgroups, of orders 3 and 5, which cannot produce a Frobenius group. For $n=5$, A_5 is 2CC, 3CC and 5CC with CC-subgroups of orders 4, 3 and 5 respectively. The only possibility then is a Frobenius subgroup such that the kernel has order 4 and complement has order 3, which is indeed the case by Theorem 3.27.

<u>Theorem 6.26</u>. The only groups of Lie type of odd characteristic which can possibly contain Frobenius CC-subgroups are PSL(2, q) for some $q\equiv 3 \bmod 4$.

<u>Proof</u>. Most of the groups considered here have only one nilpotent CC-subgroup and so cannot contain a Frobenius CC-subgroup. For those with more than one nilpotent CC-subgroup, the requirement of Theorem 3.19(v) excludes all but the PSL(2, q) with $q\equiv 3 \bmod 4$. Hence result.

Theorem 6.27. The Suzuki group $Sz(2^{2n+1})$ for $n\geq 1$ has one class of Frobenius CC-subgroup, and that is of order $q^2(q-1)$ with kernel of order q^2(where $q=2^{2n+1}$).

Theorem 6.28. The linear group $PSL(2,2^n)$ for some $n\geq 1$ has one class of Frobenius CC-subgroup, and that is of order $2^n(2^n-1)$ with kernel of order 2^n.

Theorem 6.29. None of the groups $PSL(3,4)$, $PSU(3,4)$, $PSp(4,4)$ and ${}^2F_4(2)'$ have Frobenius CC-subgroups.

Proof. In only the first case is there more than one nilpotent CC-subgroup, and $PSL(3,4)$ is excluded since Theorem 3.19(v) is not satisfied by any pair of nilpotent CC-subgroup orders.

This result ends the currently available information on CC-subgroups of simple groups. It is interesting to note that all the nilpotent CC-subgroups of odd order of simple groups found so far are actually abelian(that is, by Proposition 6.11, all pCC-subgroups are abelian for odd p). We conjecture that this is always true. We also conjecture on the basis of available information that there are no simple groups with Frobenius CC-subgroups apart from those listed in this chapter.

Apart from presenting the tables we have now come to the end of this thesis. As has been mentioned at various stages, there is still work to be done in the area of CC-subgroups of finite groups. It is reasonable to expect that a complete listing of the CC-subgroups of the simple groups will be published in the near future.

Table 1. Nilpotent CC-subgroups of the alternating groups $A_n, n \geq 5$.

n	orders of nilpotent CC-subgroups
5	4, 3 and 5
6	3 and 5
p where $p+2$ is not prime	p
p where $p+2$ is a prime	p and $p+2$

Table 2. CC-subgroups of the sporadic simple groups

Group	order of nilpotent CC-subgroups	order of Frobenius CC-subgroups
M_{11}	5, 11	55(kernel 11)
M_{12}	11	
M_{22}	5, 7, 11	55(kernel 11)
M_{23}	11, 23	253(kernel 23)
M_{24}	11, 23	253(kernel 23)
J_1	7, 11, 19	
J_2	7	
J_3	17, 19	
J_4	23, 29, 31, 37, 43	
C_3	23	
C_2	11, 23	253(kernel 23)
C_1	23	
HS	7, 11	
Suz	11, 13	
M^cL	11	
Ru	29	
He	17	
Ly	31, 37, 67	
O'N	11, 19, 31	
F_{22}	13	
F_{23}	17, 23	
F_{24}'	17, 23, 29	
F_5(HN)	19	
F_3(T)	19, 31	
F_2(BM)	31, 47	
F_1(M)	41, 59, 71	

Table 3. All nilpotent CC-subgroups of simple groups of Lie type of odd characteristic.

Group	restriction	orders of nilpotent CC-subgroups	
$A_{p-1}(q)$	$p\neq 2$	$(q^p-1)/((q-1)\gcd(p,q-1))$	
$A_p(q)$	$q-1$ divides $p+1$	$(q^p-1)/(q-1)$	
$B_{2^m}(q)$	$m\geq 2$	$q^2+1/2$	
$B_p(3)$		$3^p-1/2$	
$C_{2^m}(q)$	$m\geq 1$	$q^2+1/2$	
$C_p(3)$		$3^p-1/2$	
$D_p(3)$	$p\geq 5$	$3^p-1/2$	
$D_{p+1}(3)$	$p\geq 3$	$3^p-1/2$	
$D_p(5)$	$p\geq 5$	$5^p-1/4$	
$E_6(q)$	$q\equiv 1 \bmod 3$	$(q^6+q^3+1)/3$	
$E_6(q)$	$q\not\equiv 1 \bmod 3$	q^6+q^3+1	
$F_4(q)$		q^4-q^2+1	
$G_2(q)$	$q\equiv 1 \bmod 3$	q^2-q+1	
$G_2(q)$	$q\equiv -1 \bmod 3$	q^2+q+1	
${}^2A_{p-1}(q^2)$		$(q^p+1)/((q+1)\gcd(p,q+1))$	
${}^2A_p(q^2)$	$q+1$ divides $p+1$	$(q^p+1)/(q+1)$	
${}^2D_{2^n}(q^2)$		$q^2+1/2$	
${}^2D_p(3^2)$	$p\geq 5, p\neq 2^n+1$	$3^p+1/4$	
${}^3D_4(q^3)$		q^4-q^2+1	
${}^2E_6(q^2)$		$(q^6-q^3+1)/\gcd(3,q+1)$	
${}^2D_l(3^2)$	$l=2^n+1, l\neq p$	$3^{l-1}+1/2$	
$A_1(q)$	$q\equiv 1 \bmod 4, q\neq 5$	q	$q+1/2$
$A_1(q)$	$q\equiv 3 \bmod 4$	q	$q-1/2$
$E_7(3)$		757	1093
$G_2(q)$	$q\equiv 0 \bmod 3$	q^2+q+1	q^2-q+1
${}^2G_2(q)$	$q=3^{2m+1}, m\geq 1$	$q^2-\sqrt{(3q)}+1$	$q^2+\sqrt{(3q)}+1$
${}^2D_p(3^2)$	$p=2^n+1, n\geq 2$	$3^{p-1}+1/2$	$3^p+1/4$
$E_8(q)$	$q\equiv 2,3 \bmod 5$	$q^8+q^7-q^5-q^4-q^3+q+1$, $q^8-q^7+q^5-q^4+q^3-q+1$, q^8-q^4+1	
$E_8(q)$	$q\equiv 0,1,4 \bmod 5$	$q^8+q^7-q^5-q^4-q^3+q+1$, $q^8-q^7+q^5-q^4+q^3-q+1$, $q^8-q^6+q^4-q^2+1$, q^8-q^4+1.	

All other such groups have no CC-subgroups whatsoever, for example

$A_l(q)$, $l\geq 2$, $l\neq p$ or $p-1$, and if $l=p$ then $q-1$ does not divide $l+1$.

$B_l(q)$, $l\geq 3$, if $q\neq 3$ then $l\neq 2^m$ and if $q=3$ then $l\neq p$ or 2^m.

$C_l(q)$, $l\geq 2$, if $q\neq 3$ then $l\neq 2^m$ and if $q=3$ then $l\neq p$ or 2^m.

$D_l(q)$, $l\geq 4$, if $q\neq 3$ or 5 and if $q=3$ then $l\neq p, p+1$ and if $q=5$ then $l\neq p$.

$E_7(q)$, $q\neq 3$.

${}^2A_l(q)$, $l\geq 2$, if $l\neq p-1$ and if $l=p$ then $q+1$ does not divide $l+1$.

${}^2D_l(q)$, $l\geq 4$, if $q\neq 3$ then $l\neq 2^m$ and if $q=3$ then $l\neq p$ or 2^n+1.

Table 4. Some CC-subgroups of simple groups of Lie type of even characteristic

Group	order of nilpotent CC-subgroups
PSL(3,4)	9(elementary abelian), 5 and 7
PSU(3,4)	13
PSp(4,4)	17
${}^2F_4(2)'$	13
$Sz(2^{2n+1})$	$2^{2n+1}\pm 2^{n+1}+1$ $\quad 2^{2n+1}-1$ $\quad 2^{4n+2}$
$PSL(2,2^n)$	2^n $\quad 2^n\pm 1$

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