USING PICTURES IN COMBINATORIAL GROUP AND SEMIGROUP THEORY

CALUM WALLACE KILGOUR

A thesis presented to the

University of Glasgow

Faculty of Science

for the degree of

Doctor of Philosophy

June 1995

To mum, Murdoch, Brian, Gary, Hope, Karl and Katharine.

Many encourage and inspire

but I strive each day to find myself in You

Contents

Statement iv

Acknowlegement v

Abstract vi

Notation vii

1 Preliminaries 1

1.1 Words 1

1.2 Monoid presentations 2

1.3 Pictures 4

1.4 Monoid pictures 6

1.5 Group presentations 10

1.6 Pictures over group presentations 12

1.7 Annular pictures 20

1.8 Conjugacy 22

1.9 Graphs 27

1.10 Detailed description of the thesis 28

2 Embeddability of monoids 31

2.1 Some background 31

2.2 Embeddability 34

2.3 Pictures and pieces 38

2.4 Tessellations of the sphere with one distinguished vertex 40

2.5 Proof of Theorem 2.2.4 45

3 Conjugacy in monoids **57**

3.1 Some background 57

3.2 Definition of conjugacy 59

3.3 Annular pictures and tessellations 61

3.4 Tessellations of the sphere with two distinguished vertices 67

3.5 Proof of Theorem 3.2.1 75

4 Relative monoid presentations I **84**

4.1 Some definitions 84

4.2 Mixed monoid presentations 87

4.3 Relating $\mathcal{R}$ and $\mathcal{M}$ 89

4.4 Pictures over mixed monoid presentations 94

4.5 Operations on monoid pictures 106

4.6 $\boldsymbol{t}$-circles 108

4.7 Asphericity of mixed monoid presentations 110

4.8 Cancellation properties of $S(\mathcal{R})$ 117

4.9 Examples 118

5 Relative monoid presentations II: Embeddability of $S(\mathcal{R})$ **119**

5.1 $\widehat{\mathcal{M}}$ and $G(\widehat{\mathcal{M}})$ 119

5.2 Words in $G(\widehat{\mathcal{M}})$ 123

5.3 An embedding theorem 125

5.4 Transverse paths and $\tilde{\boldsymbol{r}}$-discs 128

Bibliography **134**

STATEMENT

This thesis is submitted in accordance with the degree of Doctor of Philosophy in the University of Glasgow. It presents the results of research undertaken by the author between October 1991 and September 1994.

Chapter 1 covers some basic material concerning monoid and group presentations, and the use of pictures in studying the structures defined by the presentations. Similar material can be found in [10, 23, 33, 38, 42, 44, 45, 46, 50].

Chapters 2–5 are my own work, with the exception of §2.1 and §3.1, as well as the other instances indicated within the text.

ACKNOWLEDGEMENT

I would like to express my deepest thanks to my supervisor, Prof. S. J. Pride, for suggesting the topics studied in this thesis, and for his inspiration, assistance, guidance and encouragement during my time as a graduate student.

I would also like to thank the Science and Engineering Research Council for financing my research through an SERC Studentship Award, as well as Prof. R.R. Burnside for employing me as a Research Assistant, 1994–1995.

Finally, I would like to thank my family and friends for their faith and encouragement over the last few years.

ABSTRACT

Pictures over group presentations are the duals of van Kampen diagrams, known widely in geometric group theory. They have proved to be an effective tool in obtaining results concerning groups. Pictures over semigroup and monoid presentations have recently been introduced and show promise in yielding algebraic information.

In Chapter 1 we review existing theory concerning monoid and group presentations, and the concept of pictures over these. We remark that all the monoid presentations considered in this thesis have relations of the form $A = B$, where A and B are non-empty words. Therefore we refer to the monoid S or semigroup S_0 defined by a monoid presentation.

Related to any monoid presentation for a monoid S or semigroup S_0, there is a group presentation defining a group G. It is of interest to ascertain exactly how the monoid or semigroup structures are related to the group structure.

In Chapter 2 we prove a result which gives sufficient conditions on the group presentation for the embeddability of S in G. We prove that these conditions are implied by the conditions recently given by E.V. Kashintsev. Furthermore, we give an example of a group presentation which satisfies our embeddability conditions but has a corresponding monoid presentation which does not belong to the class of presentations defining embeddable semigroups, studied recently by Guba. Chapter 3 is concerned with the relationship between conjugacy in S and conjugacy in G. We prove under Kashintsev's embeddablity conditions, and Goldstein and Teymouri's definition of conjugacy in S, that two elements of S which are conjugate in G, are conjugate in S in an 'elementary' way.

Chapters 4 and 5 are concerned with relative monoid presentations. Generalising work by Adjan, we introduce the notions of left and right graphs for these presentations. We prove an asphericity result for mixed monoid presentations (for which there exists a natural notion of picture), as well as a cancellation result and an embeddability result for monoids given by relative monoid presentations.

NOTATION

$\mathcal{S} = [\boldsymbol{x}; \boldsymbol{r}]$	monoid presentation
$\mathcal{G} =< \boldsymbol{y}; \boldsymbol{s} >$	group presentation
$\mathcal{M} = [\boldsymbol{a}; \boldsymbol{t} \vert \boldsymbol{s}; \tilde{\boldsymbol{r}}]$	mixed monoid presentation
$\mathcal{R} = [H, \boldsymbol{t}; \boldsymbol{r}]$	relative monoid presentation
$S_0(\mathcal{S})$	semigroup defined by $\mathcal{S}$
$S(\mathcal{S})$	monoid defined by $\mathcal{S}$
$G(\mathcal{G})$	group defined by $\mathcal{G}$
$F(\boldsymbol{x})$	free group on $\boldsymbol{x}$
$\sim_{\mathcal{S}}$	equivalence of words with respect to $\mathcal{S}$
$\sim_{\mathcal{G}}$	equivalence of words with respect to $\mathcal{G}$
$=_S$	equality in S
$=_G$	equality in G
$\equiv$	identically equal
$\simeq$	freely equivalent
$\approx_G$	conjugate in G
$\approx_S^{(e)}$	elementarily conjugate in S
$\approx_S^{(ei)}$	elementary inversely conjugate in S
$\approx_S^{(se)}$	sequentially elementarily conjugate in S
$\approx_S$	conjugate in S

Let $\mathbb{P}$ be a picture with basepoints $O_{\mathbb{P}}, O'_{\mathbb{P}}$:

$\partial\mathbb{P}$	boundary of $\mathbb{P}$
$\partial^+\mathbb{P}$	section of $\partial\mathbb{P}$ travelling clockwise from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$
$\partial^-\mathbb{P}$	section of $\partial\mathbb{P}$ travelling anti-clockwise from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$
$W(\mathbb{P})$	boundary label of $\mathbb{P}$
$\iota(\mathbb{P})$	label on $\partial^+\mathbb{P}$
$\tau(\mathbb{P})$	label on $\partial^-\mathbb{P}$
$\mathbb{P}^{(\gamma)}$	picture with stratification γ

Let $\mathbb{P}$ be an annular picture with basepoint $O_{\mathbb{P}}$ ($O'_{\mathbb{P}}$) on the outer (respectively inner) boundary:

$\partial^+\mathbb{P}$	component of $\partial\mathbb{P}$ reading clockwise from $O_{\mathbb{P}}$
$\partial^-\mathbb{P}$	component of $\partial\mathbb{P}$ reading clockwise from $O'_{\mathbb{P}}$
$\iota(\mathbb{P})$	label on outer boundary of annulus
$\tau(\mathbb{P})$	label on inner boundary of annulus

Let Δ be a disc with basepoints O, O':

$\partial\Delta$	boundary of Δ
$\partial^+\Delta$	boundary of Δ travelling clockwise from O to O'
$\partial^-\Delta$	boundary of Δ travelling anticlockwise from O to O'
$i(\Delta)$	number of arcs incident with Δ which do not intersect with the boundary of the picture
$ai(\Delta)$	number of arcs incident with Δ which do not intersect with the outer or inner boundary of the annular picture

$L(W)$	length of word W
$deg(W)$	degree of W
$deg(W, W')$	degree of (W, W')
$conjdeg(W, W')$	conjugacy degree of (W, W')
K_p^q	class of monoid presentations satisfying $C_S(p)$ and $D(q)$

Let $\Gamma = (V, E)$ be a graph:

V	vertex set
E	edge set
E^+	an orientation of E
$\iota(e)$	initial vertex of edge e
$\tau(e)$	terminal vertex of edge e
$\phi(e)$	label of edge e
$LG(\ )$	left graph
$RG(\ )$	right graph

Let γ be a transverse path:

$\iota(\gamma)$	initial point of γ
$\tau(\gamma)$	terminal point of γ

D^2	disc
S^2	sphere
A	annulus
$\partial D^2, \partial A$	boundary of disc, annulus
$\cong$	isomorphic
$a \in A$	a belongs to A
$A \cup B$	union of sets A and B
$A \cap B$	intersection of sets A and B
$A - B$	set difference
$A \subseteq B$	A is a subset of B
$\ker \theta$	kernel of θ

Chapter 1

Preliminaries

This thesis is concerned with using the geometry of pictures to find algebraic results for monoids and groups given by presentations.

Pictures have recently proved to be an effective tool in the study of groups [3, 4, 5, 6, 28, 34, 42, 43]. They are essentially the duals of the more widely known van Kampen diagrams [26, 33] but are in many situations easier to work with. A calculus for the manipulation of pictures over group presentations of various group constructions is well developed (see [6] for a survey of the work carried out). However it is only recently that semigroups and monoids have been studied using pictures [22, 44, 45, 46] and diagrams [13, 21, 22, 23, 25, 29, 48].

In this first chapter we recall some background theory concerning monoid and group presentations. We describe the notions of pictures over monoid and group presentations, and prove some basic results concerning them.

1.1 Words

Let $\boldsymbol{x}$ be a non-empty set. We define $\boldsymbol{x}^{-1}$ to be a set in one-to-one correspondence with $\boldsymbol{x}$, $x \leftrightarrow x^{-1}$ $(x \in \boldsymbol{x})$, and let $\boldsymbol{x}^{\pm 1} = \boldsymbol{x} \cup \boldsymbol{x}^{-1}$. The elements of $\boldsymbol{x}^{\pm 1}$ are called *letters.* A *word* W *(on* $\boldsymbol{x}$*)* is an expression

$$x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$$

where $n \geq 0$, $x_i \in \boldsymbol{x}$, $\epsilon = \pm 1$ and $1 \leq i \leq n$. When $n = 0$ we have the *empty word* which we denote by 1. A *positive* word (on $\boldsymbol{x}$) is a word for which $\epsilon_i = 1$ $(1 \leq i \leq n)$, while a *negative* word (on $\boldsymbol{x}$) is a word for which $\epsilon_i = -1$ $(1 \leq i \leq n)$. If $\boldsymbol{x}' \subseteq \boldsymbol{x}$ then an *$\boldsymbol{x}'$-positive* (*$\boldsymbol{x}'$-negative*) word (on $\boldsymbol{x}$) is a word such that if $x_i \in \boldsymbol{x}'$ for any $1 \leq i \leq n$, then $\epsilon_i = 1$ ($\epsilon_i = -1$ respectively). We also define a word to be *trivially* $\boldsymbol{x}'$-positive ($\boldsymbol{x}'$-negative) if $x_i \notin \boldsymbol{x}'$ for all $1 \leq i \leq n$.

The *inverse of* W, denoted W^{-1}, is the word

$$x_n^{-\epsilon_n} x_{n-1}^{-\epsilon_{n-1}} \dots x_1^{-\epsilon_1}.$$

The *length* of a word W, denoted $L(W)$, is the number of letters in W.

Two words U and V on $\boldsymbol{x}$ are *identically equal*, written $U \equiv V$, if they are identical as sequences of letters of $\boldsymbol{x}^{\pm 1}$. The *product of* U *and* V, denoted UV, is the word consisting of the letters of U followed by the letters of V. It is clear that $(UV)W \equiv U(VW)$ and $1U \equiv U1 \equiv U$ for words U, V and W.

If $W \equiv XYZ$ (X, Y and Z words on $\boldsymbol{x}$) then Y is a *subword of* W. We say that a word on $\boldsymbol{x}$ is *reduced* if it does not contain any subwords $x^\epsilon x^{-\epsilon}$ $(x \in \boldsymbol{x}, \epsilon = \pm 1)$. Finally, $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$ $(x_i \in \boldsymbol{x}, \epsilon_i = \pm 1, 1 \leq i \leq n)$ is *cyclically reduced* if it is reduced and $x_1^{\epsilon_1} \neq x_n^{-\epsilon_n}$.

1.2 Monoid presentations

A *monoid presentation* $\boldsymbol{S}$ is a pair[1] $[\boldsymbol{x}; \boldsymbol{r}]$ where $\boldsymbol{x}$ is a set (the *generating symbols*) and each $R \in \boldsymbol{r}$ (a *relation*) is an ordered pair (R_{+1}, R_{-1}), where R_{+1} and R_{-1} are distinct, positive words on $\boldsymbol{x}$. We remark that one of R_{+1}, R_{-1} may be the empty word. We usually write $R : R_{+1} = R_{-1}$ and define $R^{-1} : R_{-1} = R_{+1}$. Also, in some situations we will be concerned with lists $R_1, R_2, \dots, R_k$ of elements of $\boldsymbol{r}$. We write each element as $R_i : R_{+1,i} = R_{-1,i}$ $(1 \leq i \leq k)$.

In order to define a monoid associated with $\boldsymbol{S}$ we introduce the following elementary operation on positive words on $\boldsymbol{x}$. Let W be a positive word on $\boldsymbol{x}$.

[1] We will use square brackets to denote a monoid presentation; angular brackets to denote a group presentation (see below).

$(A):$ If W contains a subword R_ϵ, where $\epsilon = \pm 1, R_{+1} = R_{-1} \in \boldsymbol{r}$, then replace it by $R_{-\epsilon}$.

Two positive words W_1, W_2 on $\boldsymbol{x}$ are *equivalent (relative to $\boldsymbol{\mathcal{S}}$)*, denoted $W_1 \sim_{\boldsymbol{\mathcal{S}}} W_2$, if there is a chain of elementary operations of type (A) leading from W_1 to W_2. This is obviously an equivalence relation on the set of all positive words on $\boldsymbol{x}$, and we denote the equivalence class containing W by $[W]_{\boldsymbol{\mathcal{S}}}$. We define a multiplication on equivalence classes by $[W_1]_{\boldsymbol{\mathcal{S}}}.[W_2]_{\boldsymbol{\mathcal{S}}} = [W_1W_2]_{\boldsymbol{\mathcal{S}}}$.

Lemma 1.2.1 *The multiplication defined above is well defined.*

Proof. Suppose that $U \sim_{\boldsymbol{\mathcal{S}}} U'$ and $V \sim_{\boldsymbol{\mathcal{S}}} V'$ for some positive words U, U', V, V' on $\boldsymbol{x}$. Then there exist chains of elementary operations of type (A):

$$U \equiv U_0, U_1, \ldots, U_n \equiv U' \text{ and } V \equiv V_0, V_1, \ldots\ V_m \equiv V'.$$

Now $UV \equiv U_0V, U_1V, \ldots, U_nV \equiv U'V$ is a chain taking UV to $U'V$ so $[UV]_{\boldsymbol{\mathcal{S}}} = [U'V]_{\boldsymbol{\mathcal{S}}}$. Also $U'V \equiv U'V_0, U'V_1, \ldots, U'V_m \equiv U'V'$ is a chain taking $U'V$ to $U'V'$, thus $[U'V]_{\boldsymbol{\mathcal{S}}} = [U'V']_{\boldsymbol{\mathcal{S}}}$. Therefore $[UV]_{\boldsymbol{\mathcal{S}}} = [U'V']_{\boldsymbol{\mathcal{S}}}$. □

The set of all equivalence classes together with this multiplication form a monoid, the *monoid defined by $\boldsymbol{\mathcal{S}}$*, denoted $S(\boldsymbol{\mathcal{S}})$. The identity in $S(\boldsymbol{\mathcal{S}})$ is $[1]_{\boldsymbol{\mathcal{S}}}$.

If $\boldsymbol{\mathcal{S}}$ is understood to be a monoid presentation for $S(\boldsymbol{\mathcal{S}})$ then we usually refer to simply S. We also dispense with the equivalence class notation and write W for $[W]_{\boldsymbol{\mathcal{S}}}$, and $W =_S W'$ if $[W]_{\boldsymbol{\mathcal{S}}} = [W']_{\boldsymbol{\mathcal{S}}}$.

We draw the readers attention to the fact that all the relations $R \in \boldsymbol{r}$ in the monoid presentations considered in this thesis, have both R_{+1} and R_{-1} non-empty.

In view of this, the set $S(\boldsymbol{\mathcal{S}}) - \{1\}$ together with the above multiplication form a semigroup, the *semigroup $S_0(\boldsymbol{\mathcal{S}})$ defined by $\boldsymbol{\mathcal{S}}$*, denoted $S_0(\boldsymbol{\mathcal{S}})$ or simply S_0.

1.3 Pictures

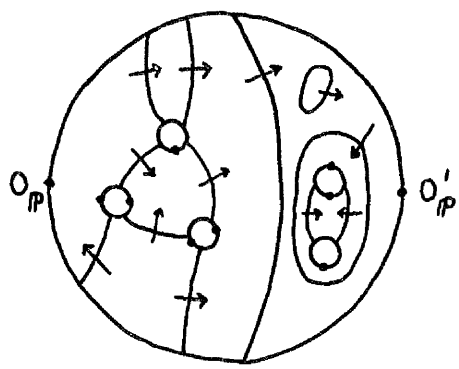
A *picture* $\mathbb{P}$ is a geometric configuration consisting of the following:

(a) A disc D^2 with two (not necessarily distinct) basepoints $O_{\mathbb{P}}$ and $O'_{\mathbb{P}}$ on ∂D^2;

(b) Disjoint discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ in the interior of D^2, where each disc Δ_i has two (not necessarily distinct) basepoints O_i and O'_i $(1 \leq i \leq n)$ on $\partial\Delta_i$;

(c) A finite number of disjoint arcs. Each arc lies in the closure of $D^2 - \{\cup_{i=1}^n \Delta_i\}$ and is either a simple closed curve having empty intersection with $\partial D^2 \cup \{\cup_{i=1}^n \partial\Delta_i\}$, or is a simple non-closed curve joining two points of $\partial D^2 \cup \{\cup_{i=1}^n \partial\Delta_i\}$, neither point being a basepoint. Each arc has a normal orientation indicated by a short arrow meeting the arc transversely.

Example 1:

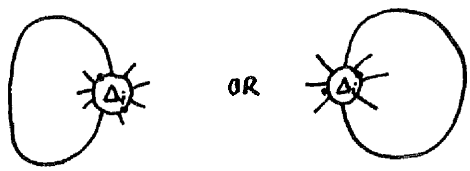


The *empty picture* is the picture which contains no arcs or discs.

We define $\partial\mathbb{P}$ to be ∂D^2. If $\mathbb{P}$ has two distinct basepoints $O_{\mathbb{P}}$ and $O'_{\mathbb{P}}$, then we define $\partial^+\mathbb{P}$ $(\partial^-\mathbb{P})$ to be the section of $\partial\mathbb{P}$ travelling in a clockwise (respectively anticlockwise) direction from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$. Note that when we refer to the 'discs of $\mathbb{P}$' we mean the discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ and not the ambient disc D^2. If a disc Δ_i has two distinct basepoints O_i and O'_i, then we define $\partial^+\Delta_i$ $(\partial^-\Delta_i)$ to be the section of $\partial\Delta_i$ travelling in a clockwise (respectively anticlockwise) direction from O_i to O'_i.

A picture $\mathbb{P}$ is *connected* if $\{\cup_{i=1}^n \Delta_i\} \cup \{\text{arcs}\}$ is connected. The connected components of $\{\cup_{i=1}^n \Delta_i\} \cup \{\text{arcs}\}$ are called the *components of* $\mathbb{P}$. We say that $\mathbb{P}$ is *spherically connected* if every component intersects with $\partial\mathbb{P}$.

If a disc Δ_i has an incident arc which intersects $\partial\mathbb{P}$ then Δ_i is a *boundary disc.* All discs which are not boundary discs are *interior discs.* A *self identified disc* Δ_i is a disc which has an arc which starts and ends on $\partial\Delta_i$.

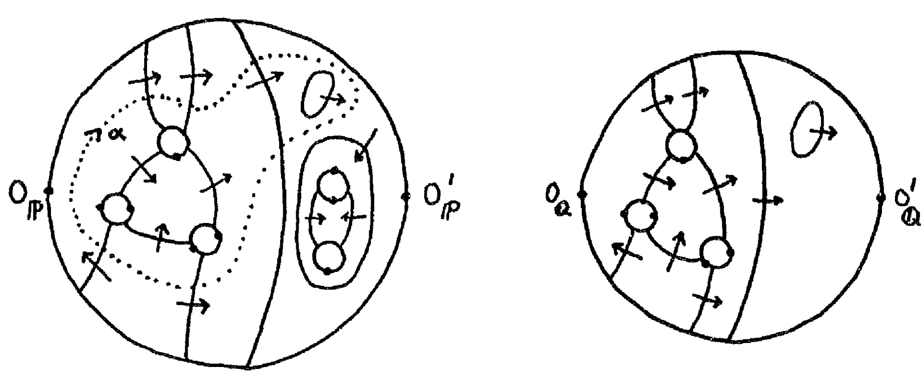


The closures of the connected components of $D^2 - \{\cup_{i=1}^{n}\Delta_i\} \cup \{\text{arcs}\}$ are called the *areas* of $\mathbb{P}$. An area of $\mathbb{P}$ which is incident with $\partial\mathbb{P}$ is called a *boundary area*, while all other areas are called *interior areas.*

A *transverse path* in $\mathbb{P}$ is a simple path in the closure of $D^2 - \{\cup_{i=1}^{n}\Delta_i\}$ which intersects the arcs of $\mathbb{P}$ finitely many times (note that if the transverse path intersects an arc then it crosses it and does not just touch it). The starting point of a transverse path γ is denoted by $\iota(\gamma)$, while the end point is denoted by $\tau(\gamma)$. We remark that whenever we refer to a *path* in a picture, we will always mean a transverse path. A *positive (negative)* transverse path is a transverse path which only crosses arcs in the direction (respectively, against the direction) of their orientation.

If $\mathbb{P}$ is a picture and α is a transverse path in $\mathbb{P}$ which does not intersect any discs, then the part of $\mathbb{P}$ enclosed by α is a *subpicture* $\mathbb{Q}$ of $\mathbb{P}$, and α is the *boundary of* $\mathbb{Q}$ (denoted $\partial\mathbb{Q}$). We can arbitrarily pick basepoints $O_{\mathbb{Q}}$ and $O'_{\mathbb{Q}}$ on α to be the basepoints of $\mathbb{Q}$, so that $\mathbb{Q}$ is a picture.

Example 2: Transverse path α in the picture on the left encloses subpicture $\mathbb{Q}$ shown on the right.



1.4 Monoid pictures

A picture $\mathbb{P}$ *over* $\boldsymbol{S} = [\boldsymbol{x}; \boldsymbol{r}]$ is a picture such that:

(a) Disc D^2 has two distinct basepoints $O_{\mathbb{P}}$ and $O'_{\mathbb{P}}$;

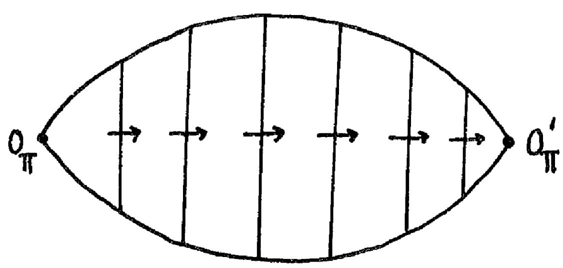
(b) Each arc is labelled by an element of $\boldsymbol{x}$;

(c) Each disc Δ_i $(1 \leq i \leq n)$ has two distinct basepoints O_i and O'_i, such that travelling around $\partial^+\Delta_i$ we read[2] $R_{\epsilon_i,i}$, while travelling around $\partial^-\Delta_i$ we read $R_{-\epsilon_i,i}$ $(\epsilon_i = \pm 1, R_i \in \boldsymbol{r})$.

See Example 3 below for an example of a picture over a monoid presentation.

We define $\iota(\mathbb{P})$ $(\tau(\mathbb{P}))$ to be the word read while travelling along $\partial^+\mathbb{P}$ (respectively $\partial^-\mathbb{P}$). If picture $\mathbb{P}$ over $\boldsymbol{S}$ has $\iota(\mathbb{P}) \equiv \tau(\mathbb{P})$ then $\mathbb{P}$ is a *spherical* picture.

A *trivial picture* $\mathbb{T}$ *over* $\boldsymbol{S}$ is a spherically connected picture over $\boldsymbol{S}$ which contains no discs and is such that $\iota(\mathbb{T}) \equiv \tau(\mathbb{T})$, where $\iota(\mathbb{T})$ and $\tau(\mathbb{T})$ are positive words on $\boldsymbol{x}$.



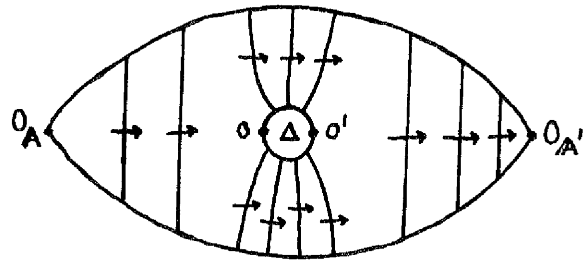
An *atomic picture* $\mathbb{A}$ *over* $\boldsymbol{S}$ is a spherically connected picture over $\boldsymbol{S}$ which has the following properties:

(a) $\mathbb{A}$ contains a single disc Δ;

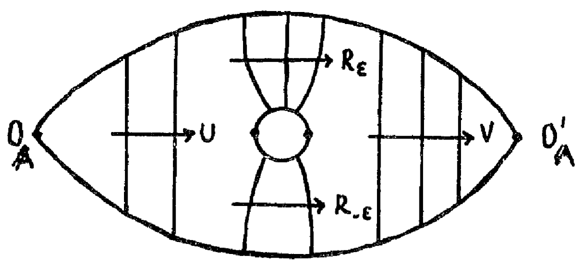
(b) Each arc intersecting $\partial^+\Delta$ $(\partial^-\Delta)$ also intersects $\partial^+\mathbb{A}$ $(\partial^-\mathbb{A}$ respectively);

(c) $\iota(\mathbb{A})$ and $\tau(\mathbb{A})$ are positive words on $\boldsymbol{x}$.

[2]When reading off labels on arcs we use the convention that if we cross an arc labelled by x $(x \in \boldsymbol{x})$ in the direction of its orientation we read x, whereas if we cross it against its orientation we read x^{-1}. Throughout this thesis we appeal to a similar convention for reading the labels on arcs in all other types of picture.



Suppose that we have two positive words W and W' on $\boldsymbol{x}$ such that $W \equiv UR_\epsilon V$ and $W' \equiv UR_{-\epsilon}V$ (U, V positive words on $\boldsymbol{x}$, $\epsilon = \pm 1$, $R_{+1} = R_{-1} \in \boldsymbol{r}$). Then $W =_S W'$ (since W' can be obtained from W by applying a single elementary operation of type (A)) and we can represent this fact geometrically by means of an atomic picture $\mathbb{A}$ over $\boldsymbol{S}$ with $\iota(\mathbb{A}) \equiv W$ and $\tau(\mathbb{A}) \equiv W'$, as shown below.



We have words associated with transverse paths in pictures over $\boldsymbol{S}$, namely the labels read off whilst travelling along the transverse paths and crossing the arcs in the manner described. Note that positive transverse paths are labelled by positive words on $\boldsymbol{x}$, while negative transverse paths are labelled by negative words on $\boldsymbol{x}$.

A *stratification* γ for a picture $\mathbb{P}$ over $\boldsymbol{S}$ is a sequence of positive transverse paths $\gamma_0, \gamma_1, \ldots, \gamma_n$, each starting at $O_{\mathbb{P}}$ and ending at $O_{\mathbb{P}}'$, and satisfying the following:

(i) γ_0 is $\partial^+\mathbb{P}$, while γ_n is $\partial^-\mathbb{P}$;

(ii) γ_λ and γ_μ intersect only at $O_{\mathbb{P}}$ and $O_{\mathbb{P}}'$ ($0 \leq \lambda < \mu \leq n$);

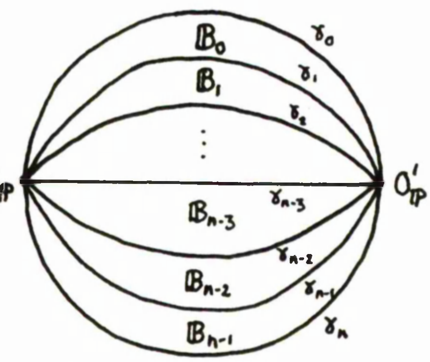
(iii) travelling in a clockwise (anticlockwise) direction around a small neighbourhood of $O_{\mathbb{P}}$ ($O_{\mathbb{P}}'$ respectively), we encounter the paths in the order $\gamma_0, \gamma_1, \ldots, \gamma_n$;

(iv) the subpicture of $\mathbb{P}$ with boundary $\gamma_i\gamma_{i+1}^{-1}$ ($0 \leq i \leq n-1$) is either an atomic picture or a trivial picture over $\boldsymbol{S}$.

A picture $\mathbb{P}$ over $\mathcal{S}$ together with a stratification γ for $\mathbb{P}$, is a *monoid picture* $\mathbb{P}^{(\gamma)}$ over $\mathcal{S}$. In general we will be unconcerned with the particular stratification associated with $\mathbb{P}$, and will usually refer simply to monoid picture $\mathbb{P}$, unless we are specifically concerned with the stratification of $\mathbb{P}$.

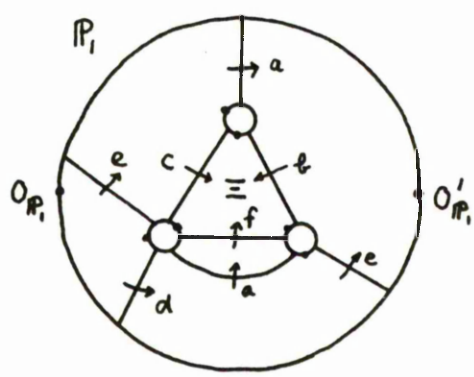
Stratifications will be indicated in our figures using bold lines.

For a monoid picture $\mathbb{P}^{(\gamma)}$ over $\mathcal{S}$ we will often write $\mathbb{P}^{(\gamma)} = \mathbb{B}_0\mathbb{B}_1 \ldots \mathbb{B}_{n-1}$, where $\mathbb{B}_i$ is the atomic or trivial picture over $\mathcal{S}$ with boundary $\gamma_i\gamma_{i+1}^{-1}$ $(0 \leq i \leq n-1)$.



Intuitively, we can think of any monoid picture over $\mathcal{S}$ as being constructed from atomic and trivial pictures over $\mathcal{S}$. Consequently, it is clear that no monoid picture over $\mathcal{S}$ contains a self identified disc.

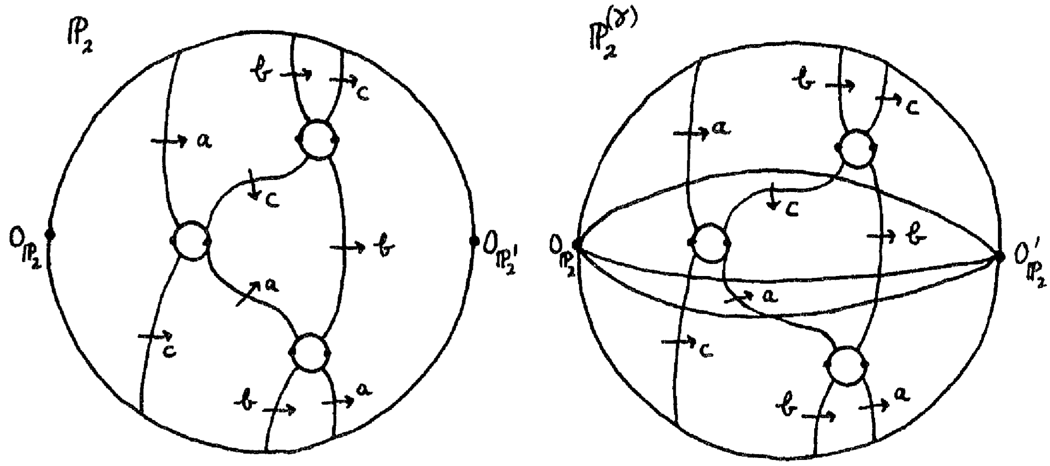
Example 3: Let $\mathcal{S}_1 = [a, b, c, d, e, f; ab = c, daf = ec, eb = af]$. Picture $\mathbb{P}_1$ below is a picture over $\mathcal{S}_1$.



However it is not possible to find a stratification for $\mathbb{P}_1$ due to the orientation of the arcs bounding interior area Ξ.

Let $\mathcal{S}_2 = [a, b, c; ab = ba, ac = ca, bc = cb]$. Picture $\mathbb{P}_2$ below (left) is a picture over $\mathcal{S}_2$. Furthermore, it is possible to find a stratification γ for $\mathbb{P}_2$. Monoid picture

$\mathbb{P}_2^{(\gamma)}$ is shown on the right.



Monoid pictures have been studied by S.J. Pride [44, 45]. They are the duals of monoid diagrams (see[3] for instance [23, pp. 73–80]). Work on semigroup and monoid diagrams has recently been carried out by Kilibarda [29], and jointly by Guba and Sapir [22]. Connected monoid diagrams and spherically connected monoid pictures are in one-to-one correspondence. The following result is the dual of Theorem 1.7.2 in [23].

Theorem 1.4.1 *Let U and V be positive words on $\boldsymbol{x}$. Then $U =_{\mathcal{S}} V$ if and only if there exists a monoid picture $\mathbb{P}$ over $\boldsymbol{\mathcal{S}}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.*

Proof. Suppose that $U =_S V$. Then there exists a sequence of positive words on $\boldsymbol{x}$

$$U \equiv U_0, U_1, \ldots, U_n \equiv V$$

where U_{i+1} is obtained from U_i $(0 \leq i \leq n-1)$ by applying a single elementary operation of type (A). Hence there exist atomic pictures $\mathbb{A}_0, \mathbb{A}_1, \ldots, \mathbb{A}_{n-1}$ over $\boldsymbol{\mathcal{S}}$ with $\iota(\mathbb{A}_i) \equiv U_i$, $\tau(\mathbb{A}_i) \equiv U_{i+1}$ $(0 \leq i \leq n-1)$ and $\tau(\mathbb{A}_j) \equiv \iota(\mathbb{A}_{j+1})$ $(0 \leq j \leq n-2)$. Let $\mathbb{P}$ be the picture over $\boldsymbol{\mathcal{S}}$ obtained by identifying $\partial^-\mathbb{A}_j$ with $\partial^+\mathbb{A}_{j+1}$ $(0 \leq j \leq n-2)$ and connecting the arcs in the obvious way. Then $\mathbb{P} = \mathbb{A}_0\mathbb{A}_1\ldots\mathbb{A}_{n-1}$ is a monoid picture with stratification $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$, where $\gamma_0 = \partial^+\mathbb{A}_0$, $\gamma_k = \partial^-\mathbb{A}_{k-1} = \partial^+\mathbb{A}_k$ $(1 \leq k \leq n-1)$ and $\gamma_n = \partial^-\mathbb{A}_{n-1}$. Also $\iota(\mathbb{P}) \equiv U$ while $\tau(\mathbb{P}) \equiv V$.

[3]Only semigroup diagrams are dealt with here, but a similar theory for monoid diagrams can be developed.

Conversely, suppose that $\mathbb{P}^{(\gamma)} = \mathbb{B}_0\mathbb{B}_1 \ldots \mathbb{B}_{n-1}$ ($\mathbb{B}_i$ an atomic or trivial picture over $\boldsymbol{\mathcal{S}}$ $(0 \leq i \leq n-1)$), where $\iota(\mathbb{P}) \equiv U$, $\tau(\mathbb{P}) \equiv V$ and $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$. Let U_i be the label read while travelling along γ_i $(0 \leq i \leq n)$ from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$. Now $U_0 \equiv U$ and $U_n \equiv V$. Furthermore, if $\mathbb{B}_i$ is trivial $(0 \leq i \leq n-1)$ then $U_i \equiv U_{i+1}$, while if $\mathbb{B}_i$ is atomic then, as noted above, U_{i+1} is obtained from U_i by an elementary operation of type (A) and so $U_i =_S U_{i+1}$. Consequently

$$U \equiv U_0 =_S U_1 =_S \ldots =_S U_n \equiv V.$$

$\square$

Example 4: Monoid picture $\mathbb{P}_2^{(\gamma)}$ in Example 3 demonstrates that $abc =_{S(\boldsymbol{\mathcal{S}}_2)} cba$.

A monoid picture $\mathbb{P}$ is said to be *minimal* if there is no other monoid picture $\mathbb{P}'$ over $\boldsymbol{\mathcal{S}}$ which contains fewer discs than $\mathbb{P}$, and has $\iota(\mathbb{P}) \equiv \iota(\mathbb{P}')$, $\tau(\mathbb{P}) \equiv \tau(\mathbb{P}')$.

1.5 Group presentations

A *group presentation* $\boldsymbol{\mathcal{G}}$ is a pair $< \boldsymbol{y}; \boldsymbol{s} >$ where $\boldsymbol{y}$ is a set (the *generating symbols*) and $\boldsymbol{s}$ is a set of non-empty, cyclically reduced words on $\boldsymbol{y}$ (the *relators*).

In order to define a group associated with $\boldsymbol{\mathcal{G}}$, we introduce the following elementary operations on words on $\boldsymbol{y}$. Let W be a word on $\boldsymbol{y}$.

(B_1) : If W contains a subword $y^\epsilon y^{-\epsilon}$ $(y \in \boldsymbol{y}, \epsilon = \pm 1)$ then delete it.
$(B_1)^{-1}$: Insert a word $y^\epsilon y^{-\epsilon}$ $(y \in \boldsymbol{y}, \epsilon = \pm 1)$ at any position in W.
(B_2) : If W contains a subword S^ϵ $(S \in \boldsymbol{s}, \epsilon = \pm 1)$ then delete it.
$(B_2)^{-1}$: Insert S^ϵ $(S \in \boldsymbol{s}, \epsilon = \pm 1)$ at any position in W.

Two words W_1, W_2 on $\boldsymbol{y}$ are *equivalent (relative to $\boldsymbol{\mathcal{G}}$)*, denoted $W_1 \sim_{\boldsymbol{\mathcal{G}}} W_2$, if there is a chain of elementary operations of types $(B_1)^{\pm 1}, (B_2)^{\pm 1}$ leading from W_1 to W_2. Now $\sim_{\boldsymbol{\mathcal{G}}}$ is an equivalence relation on the set of all words on $\boldsymbol{y}$. Let $[W]_{\boldsymbol{\mathcal{G}}}$ denote the equivalence class containing W. A multiplication can be defined on equivalence classes by $[W_1]_{\boldsymbol{\mathcal{G}}}.[W_2]_{\boldsymbol{\mathcal{G}}} = [W_1W_2]_{\boldsymbol{\mathcal{G}}}$, and this multiplication is easily checked to be well defined by an argument similar to that used in the proof of Lemma 1.2.1. The set of all equivalence classes together with this multiplication form a group, the *group defined by $\mathcal{G}$*, denoted $G(\mathcal{G})$. The identity in $G(\mathcal{G})$ is $[1]_{\mathcal{G}}$.

If $\mathcal{G}$ is understood to be a group presentation for $G(\mathcal{G})$ then we refer to simply G. We also usually dispense with the equivalence class notation and write W for $[W]_{\mathcal{G}}$, and $W =_G W'$ if $[W]_{\mathcal{G}} = [W']_{\mathcal{G}}$.

Two words W_1, W_2 on $\boldsymbol{y}$ are said to be *equivalent (relative to $\boldsymbol{y}$)*, written $W_1 \sim_{\boldsymbol{y}} W_2$, if there is a chain of elementary operations of types $(B_1)^{\pm 1}$ from W_1 to W_2. Now $\sim_{\boldsymbol{y}}$ is an equivalence relation on the set of all words on $\boldsymbol{y}$ and we denote the equivalence class containing W by $[W]_{\boldsymbol{y}}$. A (well defined) multiplication can be defined on these equivalence classes by $[W_1]_{\boldsymbol{y}}[W_2]_{\boldsymbol{y}} = [W_1 W_2]_{\boldsymbol{y}}$, and the set of all equivalence classes together with this multiplication form a group, the *free group on $\boldsymbol{y}$*, denoted $F(\boldsymbol{y})$. Once again, we usually dispense with equivalence class notation and write W for $[W]_{\boldsymbol{y}}$. If $[W]_{\boldsymbol{y}} = [W']_{\boldsymbol{y}}$ then we write $W \simeq W'$ and say that W and W' are *freely equivalent.*

Let W be a word on $\boldsymbol{y}$ and suppose that $W =_G 1$. Consider a chain of elementary operations from W to 1 which makes use of the smallest number (n say) of elementary operations of types $(B_2)^{\pm 1}$. We define the *degree of* W, denoted $deg(W)$, to be n. Note that $deg(W) = 0$ if and only if W is freely equivalent to 1.

Proposition 1.5.1 *Let W be a word on $\boldsymbol{y}$ which is not freely equivalent to* 1. *Then $W =_G 1$ if and only if $W \simeq \prod_{i=1}^{k} Y_i S_i^{\epsilon_i} Y_i^{-1}$, where $k \geq 1, S_i \in \boldsymbol{s}, \epsilon_i = \pm 1$ and Y_i is a word on $\boldsymbol{y}$ $(1 \leq i \leq k)$.*

Proof. We appeal to the equivalence class notation for clarity.

Let N be the normal closure in $F(\boldsymbol{y})$ of the set $\{[S]_{\boldsymbol{y}} : S \in \boldsymbol{s}\}$. We first show that $\frac{F(\boldsymbol{y})}{N}$ is isomorphic to $G(\mathcal{G})$.

Consider the mapping ψ from $\frac{F(\boldsymbol{y})}{N}$ to $G(\mathcal{G})$ defined by $[W]_{\boldsymbol{y}}N \mapsto [W]_{\mathcal{G}}$ (W a word on $\boldsymbol{y}$). Suppose that $[W_1]_{\boldsymbol{y}}N, [W_2]_{\boldsymbol{y}}N \in \frac{F(\boldsymbol{y})}{N}$ (W_1, W_2 words on $\boldsymbol{y}$). Then

$$\begin{aligned} & [W_1]_{\boldsymbol{y}}N = [W_2]_{\boldsymbol{y}}N \\ \Rightarrow \quad & [W_2^{-1}W_1]_{\boldsymbol{y}} \in N \end{aligned}$$

$$\Rightarrow \quad W_2^{-1}W_1 \simeq \prod_{i=1}^{k} U_i S_i^{\epsilon_i} U_i^{-1}$$

$$(U_i \text{ words on } \boldsymbol{y},\ \epsilon_i = \pm 1,\ S \in \boldsymbol{s})$$

$$\Rightarrow \quad [W_2^{-1}W_1]_{\mathcal{G}} = [1]_{\mathcal{G}}$$

$$\Rightarrow \quad [W_1]_{\mathcal{G}} = [W_2]_{\mathcal{G}}.$$

Hence ψ is well defined.

It is easily checked that ψ is an epimorphism. Hence we need only check that ψ is injective.

Let $[W]_{\boldsymbol{y}}N, [W']_{\boldsymbol{y}}N \in \frac{F(\boldsymbol{y})}{N}$ (W, W' words on $\boldsymbol{y}$) and suppose that $[W]_{\mathcal{G}} = [W']_{\mathcal{G}}$. Then there exists a sequence of elementary operations from W to W':

$$W \equiv W_1, W_2, \ldots, W_n \equiv W'$$

where W_{i+1} is obtained from W_i $(1 \leq i \leq n-1)$ by applying a single elementary operation of type $(B_1)^{\pm 1}$ or $(B_2)^{\pm 1}$. We will prove that $[W_i]_{\mathcal{G}} = [W_{i+1}]_{\mathcal{G}}$ implies that $[W_i]_{\boldsymbol{y}}N = [W_{i+1}]_{\boldsymbol{y}}N$, from which we can deduce that $[W]_{\boldsymbol{y}}N = [W']_{\boldsymbol{y}}N$.

Suppose that W_i, W_{i+1} differ by an elementary operation of type $(B_1)^{\pm 1}$. Then $[W_i]_{\boldsymbol{y}} = [W_{i+1}]_{\boldsymbol{y}}$ and so $[W_i]_{\boldsymbol{y}}N = [W_{i+1}]_{\boldsymbol{y}}N$.

Suppose that one of W_i, W_{i+1} is identically equal to $US^{\epsilon}V$ while the other is identically equal to UV (U, V words on $\boldsymbol{y}$, $\epsilon = \pm 1$, $S \in \boldsymbol{s}$). Then

$$[US^{\epsilon}V]_{\boldsymbol{y}}N = [UV(V^{-1}S^{\epsilon}V)]_{\boldsymbol{y}}N = [UV]_{\boldsymbol{y}}N.$$

Hence ψ is injective and we have proved that ψ is an isomorphism.

Let W be a word on $\boldsymbol{y}$ which is not freely equivalent to 1. Then $[W]_{\mathcal{G}} = [1]_{\mathcal{G}}$ if and only if $[W]_{\boldsymbol{y}} \in N$, and the result follows. □

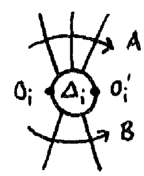
1.6 Pictures over group presentations

A picture $\mathbb{P}$ *over* $\mathcal{G} =< \boldsymbol{y}; \boldsymbol{s} >$ is a picture such that:

(a) Each arc is labelled by an element of $\boldsymbol{y}$;

(b) For each disc Δ_i in $\mathbb{P}$ $(1 \leq i \leq n)$, travelling clockwise around $\partial\Delta_i$ from O_i we read $S_i^{\epsilon_i}$ ($\epsilon_i = \pm 1, S_i \in \boldsymbol{s}$). We say that Δ_i is *labelled by* S_i^{ϵ}.

In the literature (see for instance [42]) it is normally assumed that every disc Δ_i in a picture over a group presentation has only one basepoint O_i, satisfying (b) above. We remark that we allow discs in our pictures over group presentations to have two basepoints O_i and O_i'. A disc Δ_i with two such basepoints will always be labelled by a relator of the form AB^{-1} (A, B words on $\boldsymbol{y}$), where travelling around $\partial^+\Delta$ ($\partial^-\Delta$) we read A (B respectively).



However the purpose of the second basepoint O_i' is merely to indicate the divide between the arcs which are labelled by A and those which are labelled by B. Hence, from a group theoretical perspective, such a Δ_i can effectively be viewed as having only one 'true' basepoint O_i if necessary.

See Example 5 below for a picture over a group presentation.

We define $W(\mathbb{P})$ to be the word read while travelling in a clockwise direction around $\partial\mathbb{P}$ from $O_\mathbb{P}$. If $\mathbb{P}$ has two distinct basepoints $O_\mathbb{P}$ and $O_\mathbb{P}'$ (in which case we say that $\mathbb{P}$ is a *two sided* picture over $\mathcal{G}$), then we also define $\iota(\mathbb{P})$ ($\tau(\mathbb{P})$) to be the word read while travelling along $\partial^+\mathbb{P}$ (respectively $\partial^-\mathbb{P}$).

We remark that it is normally assumed in the literature that pictures over group presentations have only one basepoint $O_\mathbb{P}$. However the second basepoint $O_\mathbb{P}'$ in a two sided picture (as defined above) merely indicates that we are considering $W(\mathbb{P})$ as a product of two words $W_1W_2^{-1}$, where $\iota(\mathbb{P}) \equiv W_1$, $\tau(\mathbb{P}) \equiv W_2$ and W_1, W_2 are words on $\boldsymbol{y}$. Hence we informally adhere to the convention by assuming that any picture over a group presentation has only one 'true' basepoint $O_\mathbb{P}$.

Note that positive transverse paths in a picture over $\mathcal{G}$ are labelled by positive words on $\boldsymbol{y}$, while negative transverse paths are labelled by negative words on $\boldsymbol{y}$.

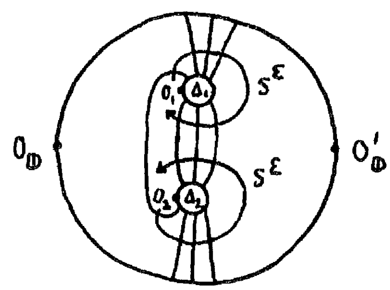
Picture $\mathbb{P}$ over $\mathcal{G}$ is said to be *spherical* if no arcs of $\mathbb{P}$ meet $\partial\mathbb{P}$. We remark that a calculus for the manipulation of spherical pictures has been developed, and that they have been used to study the second homotopy modules of group presentations (see [6] and the associated references).

A *dipole* $\mathbb{D}$ is a connected picture over $\mathcal{G}$ which contains exactly two discs Δ_1 and Δ_2 (with basepoints O_1 and O_2 respectively), and satisfies the following:

(i) There exists $S \in \boldsymbol{s}$ and $\epsilon = \pm 1$ such that reading clockwise (anticlockwise) around $\partial\Delta_1$ ($\partial\Delta_2$ respectively) from O_1 (O_2 respectively) we read S^ϵ.

(ii) There exists at least one arc intersecting both $\partial\Delta_1$ and $\partial\Delta_2$.

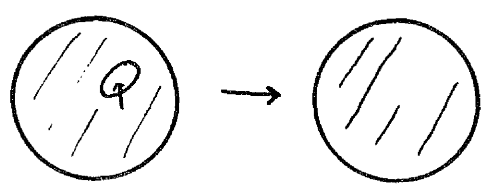
(iii) Basepoints O_1 and O_2 lie in the same area of $\mathbb{P}$.



A *cancelling pair* is a spherical picture which is also a dipole.

We define some operations on pictures over $\mathcal{G}$. Let $\mathbb{P}$ be a picture over $\mathcal{G}$.

- Deletion of a closed arc which encircles no discs or arcs of $\mathbb{P}$ (such a closed arc is called a *floating circle*)[4].

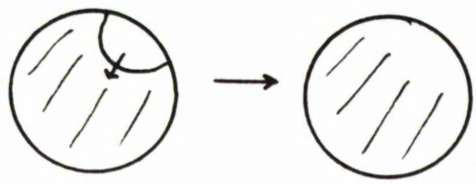


- Insertion of a floating circle into some area of $\mathbb{P}$.

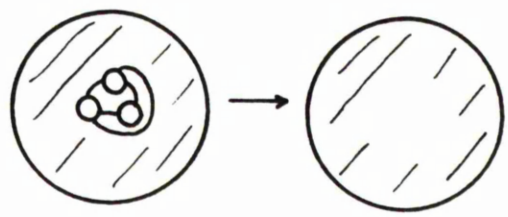
- Deletion of an arc of $\mathbb{P}$ which starts and ends on $\partial\mathbb{P}$ and encloses a boundary area which contains no discs, arcs or either of $O_{\mathbb{P}}$, $O'_{\mathbb{P}}$ (such an arc is called a

[4]In general we will assume that all floating circles have been deleted from our pictures.

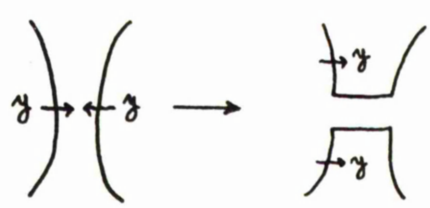
floating semicircle).



- Insertion of a floating semicircle into some boundary area of $\mathbb{P}$.
- If $\mathbb{P}$ contains a subpicture $\mathbb{Q}$ which is a spherical picture over $\mathcal{G}$, then delete $\mathbb{Q}$ from the picture.



- Insert a spherical picture over $\mathcal{G}$ into some area of $\mathbb{P}$.
- <u>Bridge move:</u>

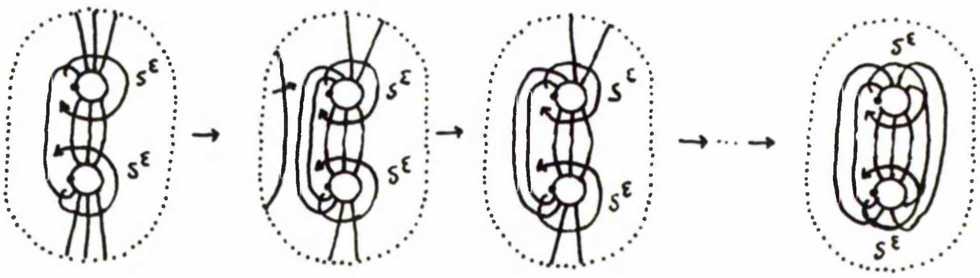


Two pictures over $\mathcal{G}$ are said to be *equivalent* if one can be obtained from the other by a finite number of the above elementary operations.

We remark that our definition of equivalence of pictures differs slightly from that already established in the literature (see for instance [42]). Here we allow any spherical pictures over $\mathcal{G}$, which arise as subpictures, to be deleted and inserted, whereas in the standard definition of equivalence of pictures, cancelling pairs are the only spherical pictures (containing discs) which can be deleted and inserted. However, the established definition of equivalence is normally extended to the relation *equivalent (relative to X)*, where X is some set of spherical pictures over $\mathcal{G}$, and the deletion and insertion of spherical pictures from X is allowed. Thus our definition of equivalence coincides with the established definition of equivalence (relative to X), where X is taken to be the set of all spherical pictures over $\mathcal{G}$.

Proposition 1.6.1 *Let $\mathbb{P}$ be a picture over $\mathcal{G}$ which contains a subpicture $\mathbb{D}$ which is also a dipole. Then $\mathbb{P}$ is equivalent to a picture $\mathbb{P}'$ which has $W(\mathbb{P}) \equiv W(\mathbb{P}')$ and contains two fewer discs than $\mathbb{P}$.*

Proof. We perform bridge moves on $\mathbb{D}$ and deform $\partial\mathbb{D}$, as shown below:



to obtain a cancelling pair. It is clear that this has no effect on the boundary label of $\mathbb{P}$. This cancelling pair can then be deleted from the picture to give a picture $\mathbb{P}'$ as required. □

A picture $\mathbb{P}$ over $\mathcal{G}$ is said to be *reduced* if it is not possible to apply bridge moves to $\mathbb{P}$ and obtain a subpicture which is a dipole.

A non-spherical picture $\mathbb{P}$ over $\mathcal{G}$ is said to be *minimal* if there is no other picture $\mathbb{P}'$ over $\mathcal{G}$ which contains fewer discs than $\mathbb{P}$, and has $W(\mathbb{P}) \equiv W(\mathbb{P}')$.

The next result follows immediately from the previous Proposition.

Proposition 1.6.2 *Every minimal picture over $\mathcal{G}$ is reduced.*

We also have the following.

Proposition 1.6.3 *(a) Every minimal picture over $\mathcal{G}$ is spherically connected.*

(b) Every subpicture of a minimal picture over $\mathcal{G}$ is itself minimal.

Proof. (a) Let $\mathbb{P}$ be a minimal picture over $\mathcal{G}$. Recall that we assume that all floating circles have been deleted from $\mathbb{P}$. Furthermore, since $\mathbb{P}$ contains the smallest number of discs possible it contains no spherical subpictures. Hence every component of $\mathbb{P}$ must intersect $\partial\mathbb{P}$.

(b) We prove this result by contradication.

Suppose subpicture $\mathbb{Q}$ of a minimal picture $\mathbb{P}$ over $\mathcal{G}$ was not minimal. Then there exists a picture $\mathbb{Q}'$ over $\mathbb{Q}$ which contains less discs than $\mathbb{Q}$ and has $W(\mathbb{Q}) \equiv W(\mathbb{Q}')$. Informally, $\mathbb{Q}'$ could be glued in place of $\mathbb{Q}$ in $\mathbb{P}$ to obtain a picture $\mathbb{P}'$ over $\mathcal{G}$ which contains less discs than $\mathbb{P}$ and has $W(\mathbb{P}) \equiv W(\mathbb{P}')$. This contradicts the minimality of $\mathbb{P}$. $\square$

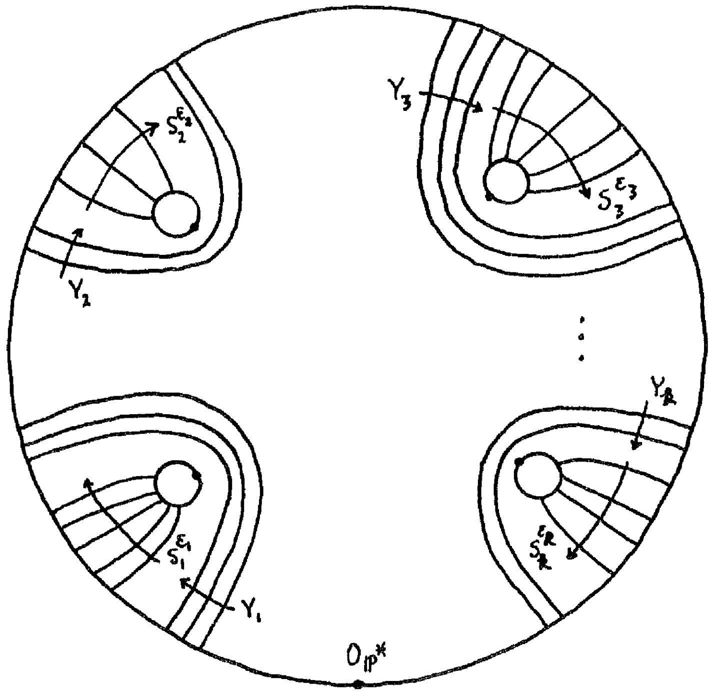
Group pictures are the duals of van Kampen diagrams (see for instance [33, pp. 235–240]). Connected van Kampen diagrams and spherically connected pictures are in one-to-one correspondence. The next result is the dual of Theorem V.1.1 and Lemma V.1.2 in [33].

Theorem 1.6.4 *Let W be a word on $\boldsymbol{y}$. Then $W =_G 1$ if and only if there exists a spherically connected picture $\mathbb{P}$ over $\mathcal{G}$ with $W(\mathbb{P}) \equiv W$.*

Proof. Suppose that $W =_G 1$. We can assume that W is a reduced word (floating semicircles can be added to the final picture we obtain below if W was not originally a reduced word). By Proposition 1.5.1

$$W \simeq \prod_{i=1}^{k} Y_i S_i^{\epsilon_i} Y_i^{-1}$$

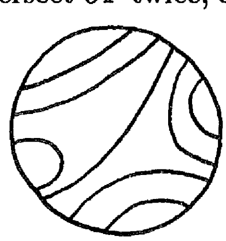
where $S_i \in \boldsymbol{s}, \epsilon = \pm 1$ and Y_i is a word on $\boldsymbol{y}$. We construct a spherically connected picture $\mathbb{P}^*$ over $\mathcal{G}$ with boundary label $\prod_{i=1}^{k} Y_i S_i^{\epsilon_i} Y_i^{-1}$, as shown below.



Performing bridge moves on the arcs which intersect $\partial\mathbb{P}^*$ and deleting floating semicircles which arise, we obtain a picture with boundary label W. Any spherical subpictures which arise can also be deleted, giving a picture $\mathbb{P}$ which is spherically connected and has $W(\mathbb{P}) \equiv W$.

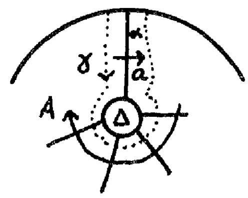
Suppose that $\mathbb{P}$ is a spherically connected picture over $\mathcal{G}$ with $W(\mathbb{P}) \equiv W$. We prove the result by induction on the number of discs, n say, in $\mathbb{P}$.

If $n = 0$ then since $\mathbb{P}$ is spherically connected it is either the empty picture, or contains only arcs which intersect $\partial\mathbb{P}$ twice, as shown below.



If $\mathbb{P}$ is the empty picture then $W \equiv 1$. In the latter case, W is freely equivalent to 1. Therefore the result is true if $n = 0$.

Suppose that $\mathbb{P}$ contains $n \geq 1$ discs. Since $\mathbb{P}$ is spherically connected it must contain a disc Δ which has an incident arc α which intersects $\partial\mathbb{P}$. Let γ be a transverse path in $\mathbb{P}$ which has $\iota(\gamma)$ on $\partial\mathbb{P}$, enters $\mathbb{P}$ in a small neighbourhood of $\alpha \cap \partial\mathbb{P}$, travels parallel to α in a small neighbourhood of α, travels around the remaining arcs which are incident with Δ, then leaves $\mathbb{P}$ after travelling parallel to α in a small neighbourhood of α (see Figure below).



Suppose that α is labelled by $a \in \boldsymbol{y}$ and that travelling around Δ in the direction of the orientation of α, we read a word A on $\boldsymbol{y}$. Then aA is a cyclic permutation of some S^ϵ ($\epsilon = \pm 1, S \in \boldsymbol{s}$), and therefore $aA \equiv YS^\epsilon Y^{-1}$ for some word Y on $\boldsymbol{y}$.

Suppose that $W \equiv W_1 a W_2$ (W_1, W_2 words on $\boldsymbol{y}$).

Consider cutting along γ to obtain a picture $\mathbb{P}'$ with $W(\mathbb{P}') \equiv W_1 A^{-1} W_2$. Note that $\mathbb{P}'$ is spherically connected and contains less discs than $\mathbb{P}$. Hence by the inductive hypothesis $W_1 A^{-1} W_2 =_G 1$. Therefore by Proposition 1.5.1

$$W_1 A^{-1} W_2 \simeq \prod_{i=1}^{k} Y_i S_i^{\epsilon_i} Y_i^{-1}$$

($k \geq 1, \epsilon_i = \pm 1, S_i \in \boldsymbol{s}$, Y_i words on $\boldsymbol{y}$, $1 \leq i \leq k$).

Now

$$\begin{aligned} W &\equiv W_1 a W_2 \\ &\simeq W_1 a A W_1^{-1}(W_1 A^{-1} W_2) \\ &\simeq W_1 a A W_1^{-1}\left(\prod_{i=1}^{k} Y_i S_i^{\epsilon_i} Y_i^{-1}\right) \text{ by the note above} \\ &\equiv (W_1 Y) S^\epsilon (W_1 Y)^{-1}\left(\prod_{i=1}^{k} Y_i S_i^{\epsilon_i} Y_i^{-1}\right). \end{aligned}$$

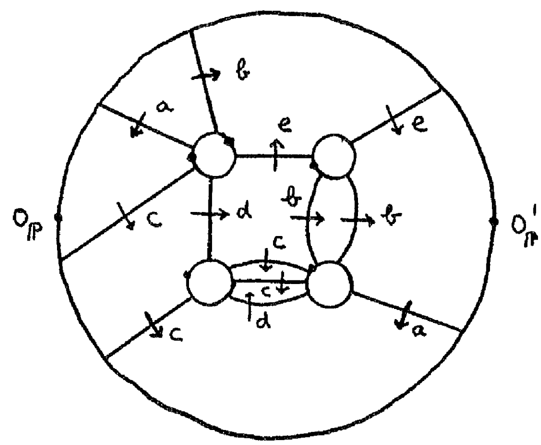
Hence by Proposition 1.5.1, $W =_G 1$. □

Corollary 1.6.5 *Let U and V be words on $\boldsymbol{y}$. Then $U =_G V$ if and only if there exists a spherically connected, two sided, picture $\mathbb{P}$ over $\mathcal{G}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.*

Proof. If $U =_G V$ then $UV^{-1} =_G 1$ and so by the previous Theorem there exists a spherically connected picture $\mathbb{P}$ over $\mathcal{G}$ with $W(\mathbb{P}) \equiv UV^{-1}$. By choosing a second basepoint $O'_{\mathbb{P}}$ on $\partial\mathbb{P}$ so that $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$, we obtain a two sided picture with the required properties.

Conversely, if $\mathbb{P}$ is a picture as stated then $W(\mathbb{P}) \equiv UV^{-1}$. By the previous Theorem $UV^{-1} =_G 1$ and so $U =_G V$. $\square$

Example 5: Let $\mathcal{G} =< a, b, c, d, e; a^{-1}be^{-1}d^{-1}c^{-1}, b^2e^{-2}, b^2adc^{-2}, dc^2d^{-1}c^{-1} >$. The following picture is a (two sided) picture over $\mathcal{G}$ showing that $U =_G V$, where $U \equiv a^{-1}be$ and $V \equiv c^2a^{-1}$.



1.7 Annular pictures

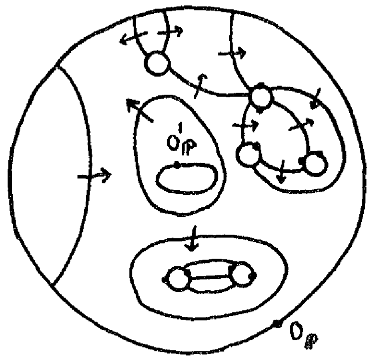
An *annular picture* $\mathbb{P}$ is a geometric configuration consisting of the following:

(a) An annulus A with outer boundary α and inner boundary β, with basepoint $O_{\mathbb{P}}$ on α and $O'_{\mathbb{P}}$ on β. We define $\partial A = \alpha \cup \beta$.

(b) Disjoint discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ in the interior of A, where each disc Δ_i has two (not necessarily distinct) basepoints O_i and O'_i $(1 \leq i \leq n)$;

(c) A finite number of disjoint arcs. Each arc lies in the closure of $A - \{\cup_{i=1}^{n}\Delta_i\}$ and is either a simple closed curve having empty intersection with $\partial A \cup \{\cup_{i=1}^{n}\partial\Delta_i\}$, or is a simple non-closed curve joining two points of $\partial A \cup \{\cup_{i=1}^{n}\partial\Delta_i\}$, neither point being a basepoint. Each arc has a normal orientation indicated by a short arrow meeting the arc transversely.

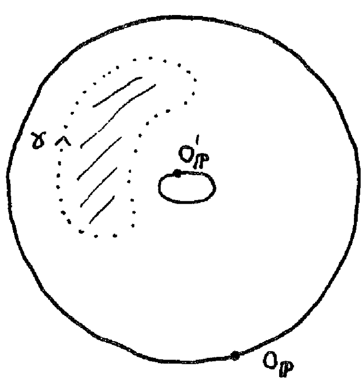
Example 6:

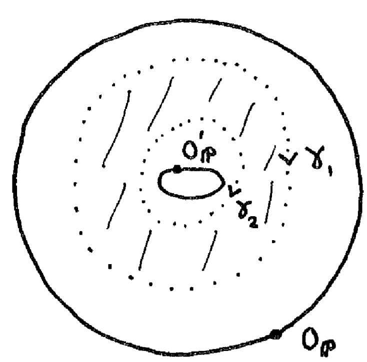


An annular picture $\mathbb{P}$ is *connected* if $\{\cup_{i=1}^{n}\Delta_i\} \cup \{\text{arcs}\}$ is connected, and *spherically connected* if $\partial A \cup \{\cup_{i=1}^{n}\Delta_i\} \cup \{\text{arcs}\}$ is connected.

We define $\partial\mathbb{P}$ to be ∂A, and $\partial^+\mathbb{P}$ ($\partial^-\mathbb{P}$) to be the component of $\partial\mathbb{P}$ travelling in a clockwise direction from $O_{\mathbb{P}}$ ($O'_{\mathbb{P}}$ respectively) around α (β respectively). We define $\partial^+\Delta$ and $\partial^-\Delta$ for a disc Δ, as for pictures (see section 1.3). We also define the concepts of *boundary disc, interior disc, boundary area, interior area* and *(positive / negative) transverse path* in an analogous way to which they were defined for pictures.

Suppose $\mathbb{P}$ is an annular picture and γ is a transverse path in $\mathbb{P}$ which does not intersect any discs. If γ encloses a simply connected component of A then we call this component a *subpicture of* $\mathbb{P}$, as in the Figure on the left below. If $\mathbb{P}$ has two transverse paths γ_1 and γ_2 which do not intersect any discs and encircle β (γ_1 may be α, γ_2 may be β), as shown in the Figure on the right below, then we call the component of A lying between γ_1 and γ_2, an *annular subpicture of* $\mathbb{P}$.





1.8 Conjugacy

If g_1 and g_2 are two elements of a group G, then g_1 and g_2 are *conjugate (in G)*, denoted $g_1 \approx_G g_2$, if there exists $g \in G$ such that

$$g_1 =_G g g_2 g^{-1}.$$

If g_1 and g_2 are represented by words W_1 and W_2 respectively, then we will often write $W_1 \approx_G W_2$.

An annular picture $\mathbb{P}$ *over* $\mathcal{G} =< \boldsymbol{y}; \boldsymbol{s} >$ is an annular picture such that:

(a) Each arc is labelled by an element of $\boldsymbol{y}$;

(b) For each disc Δ_i in $\mathbb{P}$ $(1 \leq i \leq n)$, travelling around $\partial\Delta_i$ from O_i we read $S_i^{\epsilon_i}$ $(\epsilon_i = \pm 1, S_i \in \boldsymbol{s})$.

We informally adhere to the established convention (as with pictures over group presentations) that every disc Δ_i with basepoints O_i and O_i' has only one 'true' basepoint O_i.

We remark that a subpicture of an annular picture over $\mathcal{G}$ is a picture over $\mathcal{G}$, while an annular subpicture of an annular picture over $\mathcal{G}$ is itself an annular picture over $\mathcal{G}$.

We define $\iota(\mathbb{P})$ $(\tau(\mathbb{P}))$ to be the word read while travelling around $\partial^+\mathbb{P}$ ($\partial^-\mathbb{P}$ respectively).

An annular picture $\mathbb{P}$ over $\mathcal{G}$ is *spherical* if no arcs of $\mathbb{P}$ meet $\partial\mathbb{P}$. A non-spherical annular picture is *reduced* if it is not possible to apply bridge moves to $\mathbb{P}$ and obtain a subpicture which is a dipole. An annular picture $\mathbb{P}$ over $\mathcal{G}$ is said to be *minimal* if there is no other annular picture $\mathbb{P}'$ over $\mathcal{G}$ which contains fewer discs than $\mathbb{P}$, and has $\iota(\mathbb{P}) \equiv \iota(\mathbb{P}')$, $\tau(\mathbb{P}) \equiv \tau(\mathbb{P}')$.

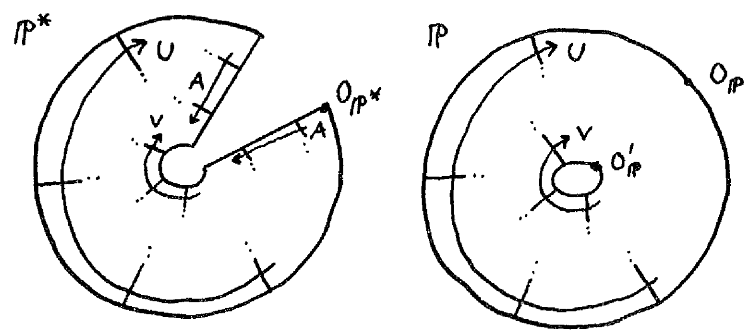
Annular pictures are the analogue of annular diagrams as described in [33, pp. 252–259]. The following result is the annular picture analogue of Lemmas V.5.1 and V.5.2 in [33].

Theorem 1.8.1 *Let U and V be words on $\boldsymbol{y}$. Then $U \approx_G V$ if and only if there exists an annular picture $\mathbb{P}$ over $\mathcal{G}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.*

Proof. Note the following:

$$
\begin{aligned}
& U \approx_G V \\
\Leftrightarrow\ & U =_G AVA^{-1} \text{ for some } A \in G \\
\Leftrightarrow\ & UAV^{-1}A^{-1} =_G 1 \\
\Leftrightarrow\ & \text{there exists (by Theorem 1.6.4) a picture } \mathbb{P}^* \text{ over } \mathcal{G} \text{ with } W(\mathbb{P}^*) \equiv UAV^{-1}A^{-1}
\end{aligned}
$$

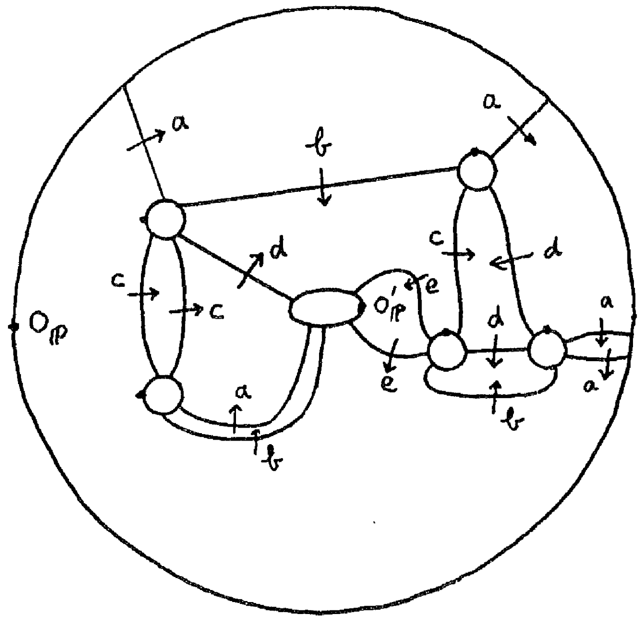
Identifying $\partial\mathbb{P}^*$ so that the arcs labelled by A are joined (as shown below), we obtain an annular picture $\mathbb{P}$ over $\mathcal{G}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.



Conversely, suppose that $\mathbb{P}$ is an annular picture over $\mathcal{G}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$. Let α be a transverse path in $\mathbb{P}$ from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$ and suppose that α is labelled by the word A on $\boldsymbol{y}$, reading from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$. Then cutting the annular picture along α we obtain a picture over $\mathcal{G}$ with boundary label $UAV^{-1}A^{-1}$, and the result follows by the above note. □

Example 7: Let $\mathcal{G} =< a, b, c, d, e; abd^{-1}c^{-2}, c^2a^{-1}b^{-1}, bcd^{-1}a^{-1}, e^2bd^{-1}c^{-1}, a^2bd^{-2} >$.

The following picture shows that $U \approx_G V$, where $U \equiv a^4$ and $V \equiv ebade$.



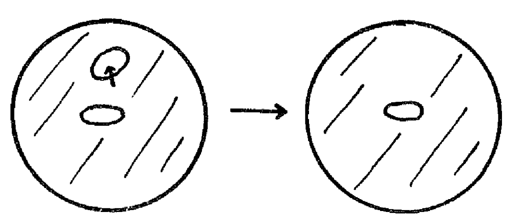
Using arguments similar to those used in the proofs of Propositions 1.6.2 and 1.6.3, we obtain the following.

Proposition 1.8.2 *(a) Every minimal annular picture is reduced.*

(b) Every subpicture or annular subpicture of a minimal annular picture is itself minimal.

We now define some operations on annular pictures over $\mathcal{G}$. Let $\mathbb{P}$ be an annular picture over $\mathcal{G}$.

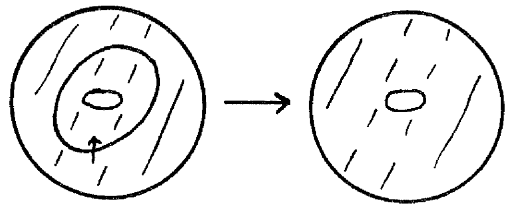
- Deletion of a closed arc which encircles no discs of $\mathbb{P}$, arcs of $\mathbb{P}$, or β (such a closed arc is called a *floating circle*)[5].



- Insertion of a floating circle into some area of $\mathbb{P}$.

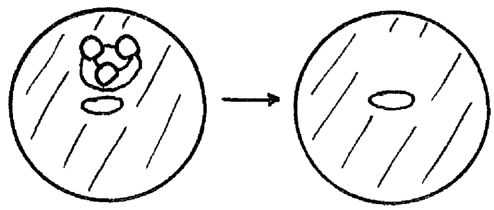
[5] In general we will assume that all floating circles have been deleted from our annular pictures.

- Deletion of a closed arc which encircles β (such a closed arc is called an *annular floating circle*)[6].



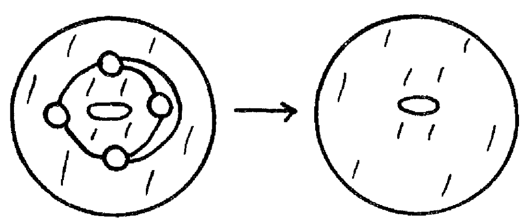
- Insertion of an annular floating circle into some area of $\mathbb{P}$.

- If $\mathbb{P}$ contains a spherical subpicture $\mathbb{Q}$ over $\mathcal{G}$, then delete $\mathbb{Q}$ from the picture.



- Insert a spherical picture over $\mathcal{G}$ into some area of $\mathbb{P}$.

- If $\mathbb{P}$ contains a spherical annular subpicture $\mathbb{Q}$ over $\mathcal{G}$, then delete $\mathbb{Q}$ from the picture.



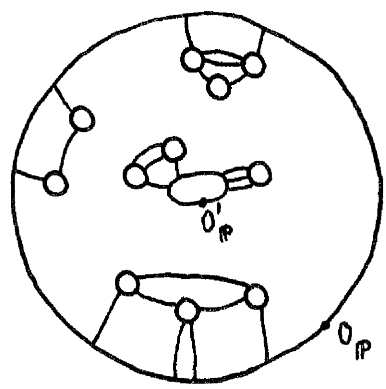
- Insert a spherical annular subpicture over $\mathcal{G}$ into some area of $\mathbb{P}$.

- Bridge move.

Proposition 1.8.3 *Let $\mathbb{P}$ be a minimal annular picture over $\mathcal{G}$ from which all floating circles and annular floating circles have been deleted. Furthermore, suppose that it is not possible to insert an annular floating circle into any area of $\mathbb{P}$. Then $\mathbb{P}$ is spherically connected.*

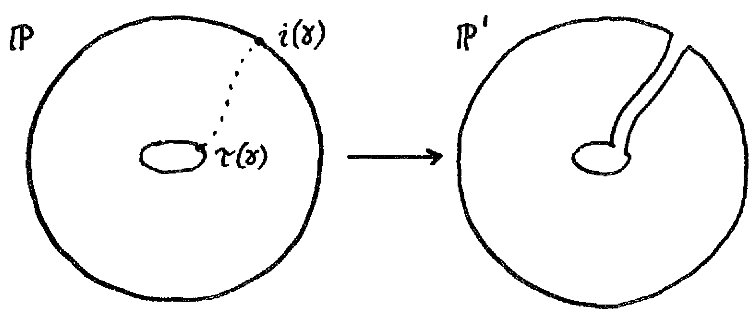
[6]Once again, we will assume that all annular floating circles have been deleted from our annular pictures.

Proof. Since $\mathbb{P}$ is minimal, $\mathbb{P}$ has no spherical subpictures or spherical annular subpictures which contain discs. Also, since it is not possible to insert an annular floating circle into $\mathbb{P}$ it cannot have two connected components (one intersecting α, the other intersecting β), as shown below:



Thus $\mathbb{P}$ is spherically connected. □

Proposition 1.8.4 *Let $\mathbb{P}$ be a minimal annular picture over $\mathcal{G}$. Let γ be a transverse path in $\mathbb{P}$ which intersects no discs and has $\iota(\gamma)$ on α and $\tau(\gamma)$ on β. If $\mathbb{P}$ is cut along γ to obtain a picture $\mathbb{P}'$ over $\mathcal{G}$, then $\mathbb{P}'$ is a minimal picture.*



Proof. We prove the result by contradiction.

Suppose that $\mathbb{P}'$ is not minimal. Then there exists a picture $\mathbb{P}''$ over $\mathcal{G}$ which contains fewer discs than $\mathbb{P}'$ and has $W(\mathbb{P}'') \equiv W(\mathbb{P}')$. However we can identify sections of $\partial\mathbb{P}''$ to obtain an annular picture $\mathbb{P}'''$ over $\mathcal{G}$ with $\iota(\mathbb{P}''') \equiv \iota(\mathbb{P})$ and $\tau(\mathbb{P}''') \equiv \tau(\mathbb{P})$. Since $\mathbb{P}'''$ contains fewer discs than $\mathbb{P}$, this contradicts the fact that $\mathbb{P}$ was minimal. □

1.9 Graphs

A *graph*[7] $\Gamma = (V, E)$ consists of two disjoint sets V (*vertex set*), E (*edge set*), and three functions

$$\iota : E \rightarrow V,\ \tau : E \rightarrow V,\ ^{-1} : E \rightarrow E$$

where

(i) $\iota(e^{-1}) = \tau(e)$;

(ii) $(e^{-1})^{-1} = e$;

(iii) $e^{-1} \neq e$

for all $e \in E$. We call $\iota(e)$ and $\tau(e)$ the *initial* and *terminal* vertices of e, respectively. Edge e^{-1} is the *inverse edge* of e.

A *subgraph* $\Gamma' = (V', E')$ of graph Γ consists of subsets V' of V and E' of E, such that for any $e^{\pm 1} \in E'$ we have $\iota(e^{\pm 1}), \tau(e^{\pm 1}) \in V'$. A *labelled graph* is a graph which has an additional function $\phi : E \rightarrow G$, for some given group G, such that $\phi(e^{-1}) = \phi(e)^{-1}$ $(e \in E)$.

A *path* γ *of length* $\ell > 0$ is a collection of vertices $v_1, v_2, \ldots, v_\ell$ from V and edges $e_1, e_2, \ldots, e_\ell$ from E such that $\iota(e_i) = v_i$ and $\tau(e_i) = v_{i+1}$ $(1 \leq i \leq \ell - 1)$. We usually denote the path by $\gamma = e_1 e_2 \ldots e_\ell$. We also define $\iota(\gamma) = v_1$ and $\tau(\gamma) = v_\ell$. The *inverse path of* γ, denoted γ^{-1}, is the path $e_\ell^{-1} e_{\ell-1}^{-1} \ldots e_2^{-1} e_1^{-1}$. A path consisting of one vertex is called an *empty path*. An empty path coincides with its inverse.

A graph is *connected* if every two vertices can be joined by a path. A *closed* path (of length ℓ) is a path for which $v_\ell = v_1$, while a *cycle* (of length ℓ) is a closed path (of length ℓ) for which all the vertices are distinct and it is not of the form ee^{-1} or $e^{-1}e$ $(e \in E)$.

In general when drawing graphs we draw only one edge from edge pair $\{e, e^{-1}\}$, and omit the other. More generally, an *orientation* E^+ for E is a choice of element from each edge pair $\{e, e^{-1}\}$. A graph with an orientation for E is called an *oriented*

[7]This definition is often attributed to Serre [50].

graph. When drawing such graphs we draw only the edges in E^+, putting an arrow on each edge pointing away from $\iota(e)$, towards $\tau(e)$ ($e \in E^+$).

$$\iota(e) \xrightarrow{e} \tau(e)$$

If $e \in E^+$ then e^{-1} is thought of as the edge travelled along from $\tau(e)$ to $\iota(e)$, against the orientation of e.

A *reduced* path in an oriented graph is a path which contains no consecutive edges ee^{-1} or $e^{-1}e$ ($e \in E^+$).

1.10 Detailed description of the thesis

Let $\mathcal{P}$ be a monoid presentation for a monoid S, or semigroup S_0. Then there is a related group presentation $\widehat{\mathcal{P}}$ defining a group G. It is of interest to ascertain exactly how (if at all) the structures of S (S_0) and G are related. For instance if G has some group theoretical property, does S (S_0) share the analogous monoid (semigroup) theoretical property? In Chapter 2 we use pictures to study the embeddability of monoids in groups, while in Chapter 3 we use pictures and annular pictures to study the connections between conjugacy in monoids and groups given by related presentations.

Monoid S (semigroup S_0) is said to be *embeddable* if it is embeddable in a group, and is therefore isomorphic to a subset of the group. It can be shown that S (S_0) is embeddable if and only if S (S_0) is embeddable in G [47]. Much work has been done in studying the embeddability of semigroups. It is known that any abelian semigroup is embeddable if and only if it is cancellative [32, 35]. In general however being a cancellative semigroup is not a sufficient condition for embeddability, as was shown by Malcev [35]. Malcev [36] also found necessary and sufficient conditions on a semigroup for it to be embeddable, in the form of a countably infinite set of implications. He later showed [37] that no finite subset of these conditions would suffice. Lambek [32] gave equivalent geometric conditions to Malcev's conditions involving polyhedra, while related (unpublished) work has been carried out by Krstić

[30]. Sufficient conditions for embeddability were also given by Ore [39], Dubreil-Jacotin [17] and Doss [16]. However it was Adjan [1] who was the first to give sufficient conditions on $\mathcal{P}$ for the embeddability of the semigroup S_0. Kashintsev [27] expanded on Adjan's result by describing a class $(K_3^3 \cup K_4^2)$ of presentations which gives rise to embeddable semigroups, while Guba [21] has shown recently that the class K_3^2 (which properly contains $K_3^3 \cup K_4^2$) gives rise to embeddable semigroups. This work appealed to small cancellation theory for semigroup presentations. However small cancellation theory for group presentations is further developed and better understood (see for instance [33, Chapter V]). It therefore seemed natural to ask whether it was possible to give sufficient conditions for embeddability in terms of the group presentation $\widehat{\mathcal{P}}$ for G. In Theorem 2.2.4 we do precisely this, in the more general setting of monoids. We also show in Theorem 2.2.5 that Kashintsev's conditions for embeddability, generalised to the monoid case, imply our conditions for embeddability. Furthermore we give an example of a group presentation which satisfies our embeddability conditions, but has a corresponding monoid presentation which does not belong to K_3^2.

The notion of conjugacy in semigroups, and in particular monoids, is of interest to both computer scientists and mathematicians. Computer scientists are interested in conjugacy in connection with the theory of string rewriting (see [11] for a survey of this area). However the actual definition of conjugacy in semigroups and monoids, as yet remains to be agreed upon. Lallement [31] gave two (natural) definitions of conjugacy for free monoids and showed that they are equivalent. However while these notions lead to an equivalence relation in free monoids, neither in general gives an equivalence relation for an arbitrary monoid. Further definitions have been proposed and discussed by Otto [40], Zhang [54], Choffrut [12], Silva [51] and Dauns [15]. Most recently Goldstein and Teymouri [20] have proposed a definition for conjugacy in semigroups. They have shown that under Adjan's conditions for the embeddability of S_0, that two elements of S_0 which are conjugate in G are conjugate in S_0 (with respect to their definition). In the main result of Chapter 3 (Theorem 3.2.1) we prove under Kashintsev's monoid embeddability conditions and Goldstein and Teymouri's definition of conjugacy, that two elements of S which are conjugate in G, are conjugate in S in an 'elementary' way. It is hoped that the techniques developed in this chapter can be used in addressing the conjugacy question for monoids with presentations which satisfy Guba's embeddability conditions (generalised to monoid presentations).

Both Chapters 2 and 3 require some theory on tessellations which we develop in sections 2.4 and 3.4 respectively.

The following group theoretical situation has received much attention (see for instance [5, 7, 24]). Let H be a group and $\boldsymbol{t}$ a set. Adjoin $\boldsymbol{t}$ to H as a set of generators and factor $H * F(\boldsymbol{t})$ by the normal closure of a set $\boldsymbol{r}$ of cyclically reduced elements of $\{H * F(\boldsymbol{t})\} - H$, giving a group G. Bogley and Pride [5] describe G as being defined by the relative presentation $\mathcal{P} =< H, \boldsymbol{t}; \boldsymbol{r} >$.

In Chapter 4 we define the concept of a relative monoid presentation $\mathcal{R}$ for a monoid $S(\mathcal{R})$. We introduce the concepts of left and right graphs for $\mathcal{R}$, generalising the analogous concepts defined by Adjan [1] for semigroup presentations. We show how to lift any relative presentation $\mathcal{R}$ to a mixed monoid presentation $\mathcal{M}$ for $S(\mathcal{R})$, and define a concept of picture over $\mathcal{M}$. We also develop the concepts of left and right graphs for $\mathcal{M}$. Our first major result is an asphericity result (Theorem 4.7.3) for mixed monoid presentations. We then go on to prove (Theorems 4.8.1, 4.8.2 and 4.8.3.) some results concerning the cancellation properties of monoids given by relative monoid presentations.

We continue our discussion of relative monoid presentations in Chapter 5 where we prove some embeddability results (Theorem 5.3.3 and Corollary 5.3.4) concerning monoids given by mixed monoid and relative monoid presentations. We remark that Adjan's Theorem (Theorem 2.1.1), generalised to the monoid setting, can be obtained as a special case of Theorem 5.3.3 (by taking H to be the trivial group).

Chapter 2

Embeddability of monoids

2.1 Some background

Let $\mathcal{P} = [\boldsymbol{x}; \boldsymbol{r}]$ be a monoid presentation for monoid $S(= S(\mathcal{P}))$, or semigroup $S_0(= S_0(\mathcal{P}))$. For each $R \in \boldsymbol{r}$ we assume that

($\star$) R_{+1} and R_{-1} are distinct, non-empty, positive words on $\boldsymbol{x}$, and that for any other relation $R' : R'_{+1} = R'_{-1} \in \boldsymbol{r}$, $R_\epsilon \not\equiv R'_\delta$ $(\epsilon, \delta = \pm 1)$.

Let $\hat{\boldsymbol{r}} = \{R_{+1}R_{-1}^{-1} : R \in \boldsymbol{r}\}$, then we have a corresponding group presentation $\widehat{\mathcal{P}} =< \boldsymbol{x}; \hat{\boldsymbol{r}} >$, for group $G(= G(\widehat{\mathcal{P}}))$. Let $\boldsymbol{r}^*$ be the symmetrised closure of $\hat{\boldsymbol{r}}$, that is, the set of all cyclic permutations of the words in $\hat{\boldsymbol{r}}$, and their inverses. We assume that words in $\hat{\boldsymbol{r}}$ are cyclically reduced, and therefore that the words in $\boldsymbol{r}^*$ are cyclically reduced. Finally, let π be the natural homomorphism from S (or S_0) to G defined by $[X]_{\mathcal{P}} \mapsto [X]_{\widehat{\mathcal{P}}}$ (X a word on $\boldsymbol{x}$). As usual we dispense with equivalence class notation and write $X \mapsto X$.

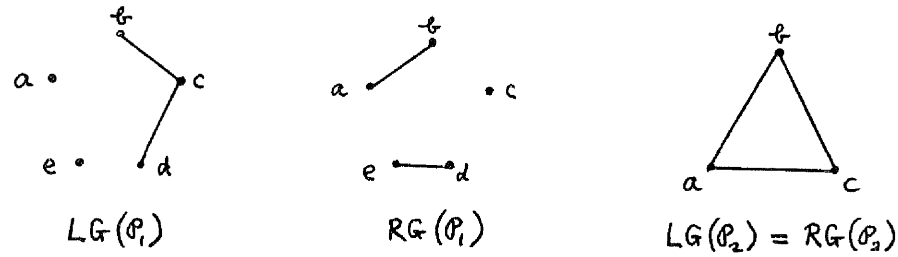
The problem of giving sufficient conditions which guarantee that a semigroup is embeddable in a group, is one that has received much attention [1, 8, 9, 13, 16, 17, 21, 27, 30, 32, 35, 48, 52]. We will consider the more general question concerning the embeddability of monoids. Now it can be shown (see for instance [47, Lemma 2.5]) that S (or S_0) is embeddable in a group if and only if π is injective, and so our task is reduced to finding sufficient conditions which guarantee that S (or S_0) is embeddable in G. We will say that S (S_0) is *embeddable* if it is embeddable in G.

Adjan [1] proved one of the most comprehensive embeddability results, introducing the concepts of the *left graph of* $\mathcal{P}$ ($LG(\mathcal{P})$) and the *right graph of* $\mathcal{P}$ ($RG(\mathcal{P})$). These unoriented graphs both have vertex set $\boldsymbol{x}$, and for each $R \in \boldsymbol{r}$ there corresponds an edge in $LG(\mathcal{P})$ (respectively $RG(\mathcal{P})$) joining the first (respectively last) letter of R_{+1} to the first (respectively last) letter of R_{-1}. If both $LG(\mathcal{P})$ and $RG(\mathcal{P})$ have no cycles then $\mathcal{P}$ is said to be *cycle free.*

Theorem 2.1.1 (Adjan [1]) *If* $\mathcal{P}$ *is cycle free then* $S_0(\mathcal{P})$ *is embeddable.*

It can be shown (as a special case of Theorem 5.3.3 proved in Chapter 5, see also [13]) that if $\mathcal{P}$ is cycle free then $S(\mathcal{P})$ is embeddable.

Example 1: Let $\mathcal{P}_1 = [a, b, c, d, e; cba = b, cde = d]$ and $\mathcal{P}_2 = [a, b, c; ab = ba, ac = ca, bc = cb]$. Then $\mathcal{P}_1$ is cycle free while $\mathcal{P}_2$ is not:

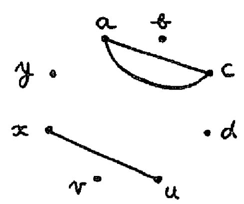


We conclude that $S(\mathcal{P}_1)$ is embeddable. Note however that $S(\mathcal{P}_2)$ is actually embeddable, demonstrating that whilst Adjan's conditions for embeddability are sufficient, they are not necessary.

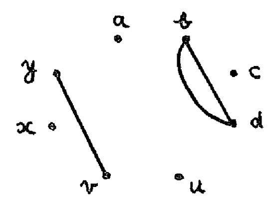
An alternative proof of Theorem 2.1.1 was given by Remmers [48] using diagrams (see [23] for an overview of this work), and later by Stallings [52] via the use of a graph theoretical lemma. In Chapters 4 and 5 we will see that these graphs can be generalised to *relative monoid presentations*, and various results on cancellability, asphericity and embeddability can be obtained. However our discussion in this chapter is motivated by work carried out by E.V. Kashintsev [27] on the embeddability of semigroups. He studied cases where $LG(\mathcal{P})$ and $RG(\mathcal{P})$ have cycles of various lengths, and $\mathcal{P}$ satisfies certain small cancellation conditions. As we shall see, his result generalises to give an embeddability result for monoids. We begin by recalling some definitions.

A monoid presentation $\mathcal{P}$ is said to satisfy condition $D(q)$ if there are no cycles in $LG(\mathcal{P})$ or $RG(\mathcal{P})$ of length less than q. Since we assume that the elements of $\hat{\boldsymbol{r}}$ are cyclically reduced, it is obvious that $\mathcal{P}$ satisfies $D(2)$. We define $\mathcal{C} = \{R_{+1}, R_{-1} : R_{+1} = R_{-1} \in \boldsymbol{r}\}$. A positive word X on $\boldsymbol{x}$ is an *S-piece* if there exists $C_j, C_k \in \mathcal{C}$ (where j, k are not necessarily distinct) such that $C_j \equiv UXV$, $C_k \equiv PXQ$ (U, V, P, Q positive words on $\boldsymbol{x}$), and either $U \not\equiv P$ or $V \not\equiv Q$. Monoid presentation $\mathcal{P}$ is said to satisfy condition $C_S(p)$ if no element of $\mathcal{C}$ can be written as a product of less than p S-pieces. Finally, we say that $\mathcal{P}$ belongs to class K_p^q if $\mathcal{P}$ satisfies $C_S(p)$ and $D(q)$.

Example 2: Let $\mathcal{P} = [a, b, c, d, u, v, x, y; ab = cd, av = cy, ub = xd]$. Then we have:



LG(P)



RG(P)

and $\mathcal{P} \in K_2^2$.

It is obvious that if S is embeddable then it is a cancellative monoid, and therefore that all non-cancellative monoids are not embeddable. Thus the search for monoids which *are* embeddable, must be carried out within the setting of cancellative monoids.

In the above language, Adjan's result says that if $\mathcal{P} \in K_1^\infty$ then $S(\mathcal{P})$ is embeddable. Bush [9] gave an example of an embeddable semigroup with presentation in K_2^2. However Malcev [35] in considering the presentation in Example 2, gave the first example of a cancellative semigroup which is not embeddable. Recall that this presentation also belongs to class K_2^2. More recently Kashintsev [27] has developed a scheme for constructing a non-cancellative (and therefore non-embeddable) semigroup, whose presentation belongs to K_2^q, for any $q > 2$. On the other hand he has also shown that if $\mathcal{P} \in K_3^3 \cup K_4^2$ then $S_0(\mathcal{P})$ is embeddable. Finally, Guba [21] has shown that if $\mathcal{P} \in K_3^2$ then $S_0(\mathcal{P})$ is embeddable, and that for any $q > 2$ there exists a cancellative semigroup with presentation in class K_2^q which is not embeddable. Note that class $K_3^3 \cup K_4^2$ is properly contained in K_3^2. Kashintsev's work makes use of small cancellation theory for groups while Guba uses techniques involving auxilliary graphs, developed by Ol'Shanskii [38].

Now the embeddability results mentioned in the previous paragraph all involve small cancellation conditions on $\mathcal{P}$. However, given that small cancellation theory is well developed for group presentations (see [33, Chapter V] and the references cited there), it seems natural to ask whether we can give small cancellation conditions on $\widehat{\mathcal{P}}$ which imply that $S(\mathcal{P})$ is embeddable. Therefore, in this chapter we give sufficient small cancellation conditions on $\widehat{\mathcal{P}}$ which guarantee that monoid $S(\mathcal{P})$ is embeddable. We will also show that whilst Kashintsev's conditions for embeddability imply that our conditions for embeddability hold (generalised to the monoid case), there exists a group presentation which satisfies our conditions but has a corresponding monoid presentation which does not belong to K_3^2, the class (properly containing $K_3^3 \cup K_4^2$) studied by Guba.

2.2 Embeddability

We assume from this point onwards that $\mathcal{P} = [\boldsymbol{x}; \boldsymbol{r}]$ is a monoid presentation for monoid S.

We begin by recalling some basic notions from small cancellation theory for group presentations $\widehat{\mathcal{P}} =< \boldsymbol{x}; \hat{\boldsymbol{r}} >$, and refer the reader to Lyndon and Schupp [33, Chapter V] for more details.

Suppose that $R_j \equiv XU$ and $R_k \equiv XV$ are distinct elements of $\boldsymbol{r}^*$ (X, U, V words on $\boldsymbol{x}$ where X is non-empty and at least one of U, V are non-empty), then X is said to be a *piece (relative to $\boldsymbol{r}^*$)*. Group presentation $\widehat{\mathcal{P}}$ is said to satisfy condition $C(p)$ if no element of $\boldsymbol{r}^*$ is the product of less than p pieces. The following results explain some of the relationships between pieces, S-pieces and the $D(q)$ condition.

Lemma 2.2.1 *Let X, Y be non-empty positive words on $\boldsymbol{x}$. If XY^{-1} or $X^{-1}Y$ is a piece then X and Y are S-pieces.*

Proof. Let $W \equiv XY^{-1}B^{-1}A$ and $W' \equiv XY^{-1}D^{-1}C$ be distinct elements of $\boldsymbol{r}^*$, where A, B, C and D are positive words on $\boldsymbol{x}$. Then there exist $R, R' \in \boldsymbol{r}$, $\epsilon, \epsilon' \in \{-1, 1\}$

such that

$$R_\epsilon \equiv AX \ , \ R_{-\epsilon} \equiv BY$$
$$R'_{\epsilon'} \equiv CX \ , \ R'_{-\epsilon'} \equiv DY.$$

Since W and W' are distinct elements of $\boldsymbol{r}^*$, either $A \not\equiv C$ or $B \not\equiv D$. However, we must actually have that $A \not\equiv C$ *and* $B \not\equiv D$, by assumption $(\star)$. Thus X and Y are S-pieces by definition. $\square$

Kashintsev gave a simple proof of the following fact.

Lemma 2.2.2 ([27], Lemma 3.2) *Word X or X^{-1} (X a positive word on $\boldsymbol{x}$) is a piece if and only if X is an S-piece.*

A piece is said to be *homogeneous* if it is either a positive word on $\boldsymbol{x}$ or a negative word on $\boldsymbol{x}$. Pieces which are not homogeneous are described as being *inhomogeneous.*

Lemma 2.2.3 [1] *Monoid presentation $\mathcal{P}$ satisfies $D(3)$ if and only if $\widehat{\mathcal{P}}$ has only homogeneous pieces.*

Proof.

XY^{-1} is a piece (X, Y non-empty positive words on $\boldsymbol{x}$)

$\Leftrightarrow$ there exist distinct elements $XY^{-1}A^{-1}B$, $XY^{-1}C^{-1}D \in \boldsymbol{r}^*$

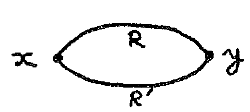
(A, B, C, D positive words on $\boldsymbol{x}$)

$\Leftrightarrow$ there exist (in view of assumption $(\star)$) distinct

elements $R, R' \in \boldsymbol{r}$, $\epsilon, \epsilon' \in \{-1, 1\}$ with

$R_\epsilon \equiv BX \ , \ R_{-\epsilon} \equiv AY$ and $R'_{\epsilon'} \equiv DX \ , \ R'_{-\epsilon'} \equiv CY$

$\Leftrightarrow$ there exist distinct edges



in $RG(\mathcal{P})$, where x is the last letter of X and y is the last letter of Y.

[1] Kashintsev proved the 'if' part of this result in Lemma 3.5 of his paper. However the proof given here is much simpler.

Similarly, $X^{-1}Y$ is a piece if and only if there are two distinct edges in $LG(\mathcal{P})$ which connect the first letter of X to the first letter of Y. $\square$

The following concept will be fundamental to our discussion.

Ω-Condition: *Group presentation $\widehat{\mathcal{P}}$ is said to satisfy the Ω-Condition if no subword of an element of $\boldsymbol{r}^*$ of the form $X^{-1}R_\epsilon Z^{-1}$ (where X and Z are positive words on $\boldsymbol{x}$ at least one of which is non-empty, $\epsilon = \pm 1, R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$) is a product of one, two or three pieces.*

Our main aim in this chapter will be the proof of the following Theorem.

Theorem 2.2.4 *If $\widehat{\mathcal{P}}$ satisfies $C(6)$ and the Ω-Condition then S is embeddable.*

In the light of the previous Theorem and the next Theorem, we observe that Kashintsev's embeddability conditions for $S_0(\mathcal{P})$ generalise to give an embeddability condition for $S(\mathcal{P})$.

Theorem 2.2.5 *If $\mathcal{P} \in K_3^3 \cup K_4^2$ then $\widehat{\mathcal{P}}$ satisfies $C(6)$ and the Ω-Condition.*

Proof. Kashintsev shows in Lemma 3.6 of his paper [27] that if $\mathcal{P} \in K_3^3 \cup K_4^2$ then $\widehat{\mathcal{P}}$ satisfies $C(6)$. Let $X^{-1}R_\epsilon Z^{-1}$ be a subword of an element of $\boldsymbol{r}^*$, where X and Z are positive words on $\boldsymbol{x}$ at least one of which is non-empty, $\epsilon = \pm 1$, $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$. Furthermore, we assume that $X^{-1}R_\epsilon Z^{-1}$ can be written as a product of pieces.

Suppose that $\mathcal{P} \in K_3^3$. Then R_ϵ is a product of at least three S-pieces. Also, by Lemma 2.2.3, $\widehat{\mathcal{P}}$ has homogeneous pieces. Therefore R_ϵ is a product of at least three pieces and so $X^{-1}R_\epsilon Z^{-1}$ is a product of at least four pieces, since at least one of X, Z is non-empty.

Suppose that $\mathcal{P} \in K_4^2$. If $X^{-1}R_\epsilon Z^{-1}$ can only be expressed as a product of homogeneous pieces then it is a product of at least five pieces, since R_ϵ is a product of at least four S-pieces (which are also pieces) and at least one of X, Z is non-empty. Thus it can be assumed that our factorisation of $X^{-1}R_\epsilon Z^{-1}$ involves inhomogeneous pieces. We note that the factorisation can involve at most two such inhomogenous pieces.

Case (I): Factorisation involves one inhomogeneous piece: Suppose $X \equiv X_1X_2$, $R_\epsilon \equiv Y_1Y_2$ and that $X_1^{-1}Y_1$ is a piece (X_1, X_2, Y_1, Y_2 positive words on $\boldsymbol{x}$ with at least X_1, Y_1 non-empty). By Lemma 2.2.1, Y_1 is an S-piece. Since $\mathcal{P}$ satisfies $C_S(4)$, Y_2 must be a product of at least three S-pieces, which are also pieces. Thus $X^{-1}R_\epsilon$ and therefore $X^{-1}R_\epsilon Z^{-1}$, is a product of at least four pieces.

A symmetrical argument gives the same conclusion if the factorisation involves a single inhomogeneous piece straddling R_ϵ and Z^{-1}.

Case (II): Factorisation involves two inhomogeneous pieces: Suppose $X \equiv X_1X_2$, $R_\epsilon \equiv Y_1Y_2Y_3$, $Z \equiv Z_1Z_2$ and that $X_1^{-1}Y_1$ and $Y_3Z_2^{-1}$ are pieces ($X_1, X_2, Y_1, Y_2, Y_3, Z_1, Z_2$ positive words on $\boldsymbol{x}$ with at least X_1, Y_1, Y_3, Z_2 non-empty). By Lemma 2.2.1, Y_1 and Y_3 are S-pieces. Since $\mathcal{P}$ satisfies $C_S(4)$ we conclude that Y_2 must be a product of at least two S-pieces, which are also pieces. Thus $X^{-1}R_\epsilon Z^{-1}$ is a product of at least four pieces. □

The following example shows that the converse of this result is false. We conclude that our embeddability result applies to a class of group presentations which include as a proper set, those group presentations whose corresponding monoid presentations belong to $K_3^3 \cup K_4^2$. Furthermore, the monoid presentation considered below does not belong to K_3^2.

Example 3: Consider the monoid presentation

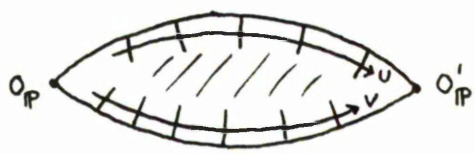
$$\mathcal{P} = [a, b, c, d, x, y, z, t; ab = zt, xacx = ycay, xbdx = ydby]$$

and let $\widehat{\mathcal{P}}$ be the corresponding group presentation.

Note that the only homogeneous pieces are the individual letters, except $z^{\pm 1}, t^{\pm 1}$, while the only inhomogeneous pieces are $(x^{-1}y)^{\pm 1}$ and $(xy^{-1})^{\pm 1}$. Also the relator $abt^{-1}z^{-1}$ is not factorisable into pieces, while the other relators of $\widehat{\mathcal{P}}$ together with their cyclic permutations and their inverses, are products of at least six pieces. We therefore conclude that $\widehat{\mathcal{P}}$ satisfies $C(6)$ and the Ω-Condition. However ab is a product of only two S-pieces and thus $\mathcal{P}$ does not belong to $K_3^3 \cup K_4^2$. Furthermore, $\mathcal{P}$ does not even belong to K_3^2.

2.3 Pictures and pieces

Recall from Corollary 1.6.5 that if words U and V on $\boldsymbol{x}$ are such that $U =_G V$, then there exists a spherically connected, two sided, picture $\mathbb{P}$ over $\widehat{\mathcal{P}} = <\boldsymbol{x}; \hat{\boldsymbol{r}}>$, with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.



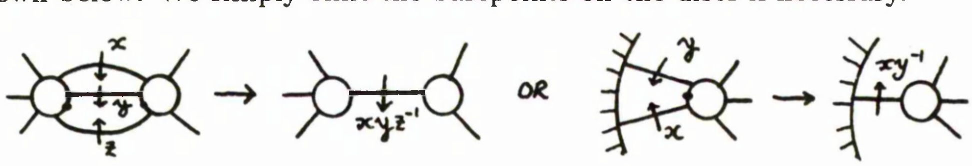
We also think of the discs in such a picture as having two 'sides'. If disc Δ_λ is labelled by $R_\epsilon R_{-\epsilon}^{-1}$ ($\epsilon = \pm 1$, $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$), then reading around $\partial^+\Delta_\lambda$ we read R_ϵ; reading around $\partial^-\Delta_\lambda$ we read $R_{-\epsilon}$.

We define

$$deg(U, V) = \min\{\text{number of discs in } \mathbb{P}\text{: } \mathbb{P} \text{ is a picture with boundary label } UV^{-1}\}$$

to be the *degree of* (U, V). Note that if the number of discs in $\mathbb{P}$ is equal to $deg(U, V)$ then $\mathbb{P}$ is minimal. Also, recall from Proposition 1.6.2 that if $\mathbb{P}$ is minimal then $\mathbb{P}$ is also reduced.

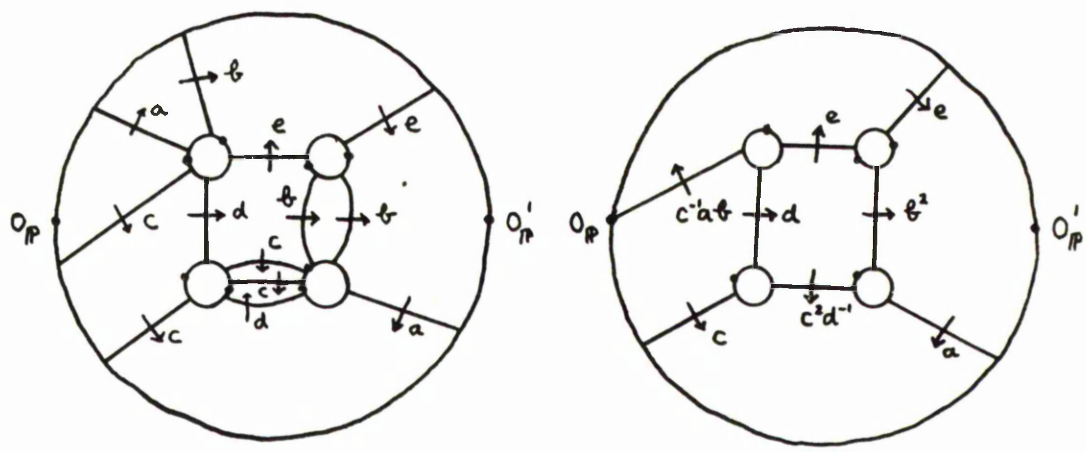
For the remainder of our work on embeddability, we will assume that pictures over $\widehat{\mathcal{P}}$ have been amended in the following way (unless otherwise indicated). If there are two or more parallel arcs connecting two discs, or connecting a disc to the boundary of the picture, then combine these arcs to form a single arc labelled by a word, as shown below. We simply omit the basepoints on the discs if necessary.



If the arcs which intersect the boundary also straddle $O_{\mathbb{P}}$ (or $O'_{\mathbb{P}}$), then we make the single arc intersect the boundary at $O_{\mathbb{P}}$ (or $O'_{\mathbb{P}}$, respectively).

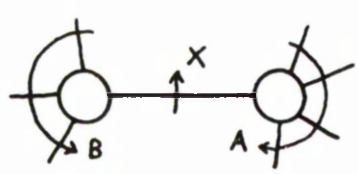
Note that since all the elements of $\boldsymbol{r}^*$ are cyclically reduced, any picture over $\widehat{\mathcal{P}}$ must have interior areas bounded by at least two arcs. However, the above amendment ensures that all interior areas are bounded by at least three arcs.

Example 4: Amending the picture on the left gives the picture shown on the right.



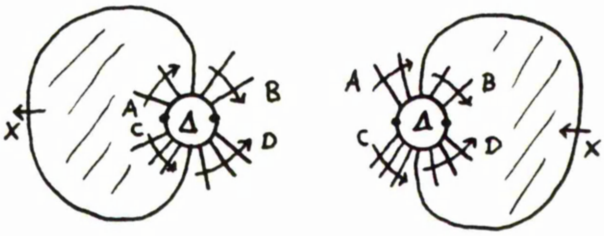
Lemma 2.3.1 *Let α be an arc in a reduced, amended picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$, which intersects at least one disc but does not intersect $\partial\mathbb{P}$. If α is labelled by a word X (on $\boldsymbol{x}$), then X is a piece (relative to $\boldsymbol{r}^*$).*

Proof. If α intersects two distinct discs, then there exist two elements $XA, XB \in \boldsymbol{r}^*$, where A and B are words on $\boldsymbol{x}$.



Since the picture is reduced $A \not\equiv B$, and X is a piece by definition.

If α intersects the same disc Δ twice, then Δ must be labelled by a relator of the form $AXBD^{-1}X^{-1}C^{-1}$, where A, B, C, D are positive words on $\boldsymbol{x}$ with at least one of A or C non-empty, and at least one of B or D non-empty.



Hence we have two possible elements of $\boldsymbol{r}^*$:

$$XBD^{-1}X^{-1}C^{-1}A \text{ and } X(BD^{-1})^{-1}X^{-1}(C^{-1}A)^{-1}$$

These could only be the same element of $\boldsymbol{r}^*$ if $BD^{-1} \equiv (BD^{-1})^{-1}$ and $C^{-1}A \equiv (C^{-1}A)^{-1}$. However, note that both BD^{-1} and $C^{-1}A$ are reduced words, and that the only reduced word which is identically equal to its own inverse is the empty word. Therefore we conclude that they are distinct elements of $\boldsymbol{r}^*$. Hence X is a piece. □

In view of this lemma, it is clear that if $\widehat{\mathcal{P}}$ satisfies $C(p)$ and $\mathbb{P}$ is a reduced, amended picture over $\widehat{\mathcal{P}}$, then any interior disc in $\mathbb{P}$ must have at least p arcs incident with its boundary.

In order to prove Theorem 2.2.4, we require to study some of the geometry underlying pictures over presentations which satisfy $C(6)$.

2.4 Tessellations of the sphere with one distinguished vertex

Any (amended) non-spherical, connected picture $\mathbb{P}$ gives rise to a tessellation T of the sphere S^2 with one distinguished vertex. We simply contract $\partial\mathbb{P}$ to a point (the distinguished vertex v_0 of the tessellation), contract discs to points (creating the remaining vertices of the tessellation), and consider arcs as unlabelled edges.

The vertices of T can be classified into one of the following types:

(i) v_0;

(ii) vertices connected to v_0 by a single edge;

(iii) vertices connected to v_0 by more than one edge;

(iv) vertices which are not connected to v_0 by any edges.

The closure of the connected components of $S^2 - T$ are called the *areas* of T. An *interior area* is an area which does not have v_0 as one of the vertices on its boundary. All areas which are not interior areas are called *distinguished areas.* The *valence* of a vertex v of T is the number of edges of T which intersect with v (where an edge is counted twice if it has both endpoints equal to v).

Let $\mathbb{P}$ be a non-spherical, amended, connected picture over a group presentation $\widehat{\mathcal{P}}$ which satisfies $C(6)$. Furthermore, suppose that $\mathbb{P}$ contains at least two discs, and that the intersection of any area of $\mathbb{P}$ with a disc is connected. Then $\mathbb{P}$ gives rise to a tessellation T of the sphere with at least two vertices distinct from v_0, and the following properties:

(A) every vertex of type (iv) has valence at least six;

(B) every interior area is bounded by at least three edges of the tessellation[2].

In this section we will be concerned with proving the following result.

Proposition 2.4.1 *Tessellation T has either*

(i) at least two vertices of type (ii) with valence exactly two, or

(ii) at least three vertices of type (ii) with valence between two and four.

We will require the following Lemma.

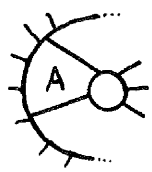
Lemma 2.4.2 *Every area of T is bounded by at least three edges of the tessellation.*

[2]In view of the comment preceeding Example 4, all tessellations arising from connected, amended pictures have this property.

Proof. Since every interior area of T is bounded by at least three edges, it is only necessary to consider distinguished areas.

Since $\mathbb{P}$ was connected, T has no edges with initial and terminal vertex v_0. Therefore T has no distinguished areas which are bounded by a single edge.

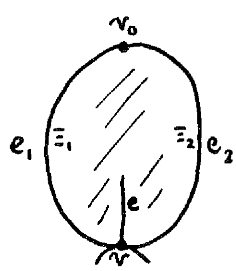
Since $\mathbb{P}$ was amended it had no boundary areas A as shown below.



It follows that T cannot have a distinguished area which is bounded by two edges of the tessellation. $\square$

Let M, P and N be the number of vertices in T of types (ii), (iii) and (iv), respectively. Also, let ℓ_1 and ℓ_2 be the average valence of vertices of type (ii) and (iii), respectively. Note that T must have at least one vertex of type (ii) or (iii) since $\mathbb{P}$ was non-spherical. We begin by proving that T has at least one vertex of type (ii).

Suppose that T has a vertex v of type (iii) with incident edges e_1 and e_2 which intersect with v_0.



Note that edges e_1 and e_2 must be separated by at least one edge, e say, of T, as shown. Let T_1 be the part of T enclosed by $e_1 \cup e_2$. We assume that v is chosen so that T_1 contains the smallest number of vertices possible.

Now suppose that T_1 contained a vertex v' of type (iii), with incident edges e_1' and e_2' which intersect with v_0. Let T_2 be the part of T_1 enclosed by $e_1' \cup e_2'$. Then T_2 contains less vertices than T_1 (since it does not contain v'), contradicting the choice of v. Hence we can assume that T_1 contains no vertices of type (iii).

Let Ξ_1 and Ξ_2 be the areas of T which are incident with e_1 and e_2 respectively, as indicated in the Figure above. Due to the fact that the picture from which T arose has no area Ξ and disc Δ such that $\partial\Xi \cap \partial\Delta$ is disconnected, it is clear that $\Xi_1 \neq \Xi_2$. Hence there must exist a sequence of edges $f_1, f_2, \ldots, f_k$ of T such that $\iota(f_1) = v$, $\tau(f_i) = \iota(f_{i+1})$ $(1 \leq i \leq k-1)$ and $\tau(f_k) = v_0$. Thus $\iota(f_k)$ is a vertex of type (ii), completing the proof that T contains at least one vertex of type (ii).

Since $\mathbb{P}$ was connected, no vertex of type (ii) has valence one, and therefore $\ell_1 \geq 2$. Let β be the average number of edges which are incident with vertices of type (iii) and intersect v_0. Note that P may be zero (in which case ℓ_2 and β are zero), but that if P is non-zero then $\beta \geq 2$.

Suppose that P is non-zero and that v is a vertex of type (iii) in T. Let $e_1, e_2, \ldots, e_m$ be the edges of T which are incident with both v and v_0, taken in a clockwise order around v. Then between e_i and e_{i+1} $(1 \leq i \leq m-1)$ there must exist at least one edge of T which is incident with v but not v_0. Furthermore, since $\mathbb{P}$ was connected, there must exist at least one edge between e_1 and e_m which intersects with v and some other vertex of T lying outside the part of T enclosed by $e_1 \cup e_m$ (otherwise e_1 and e_m would be have been identified when $\mathbb{P}$ was amended). Thus the valence of v is at least $2m$, and in view of this we see that $\ell_2 \geq 2\beta$.

Let E be the number of edges of this tessellation, V the number of vertices and Φ the number of areas. Note that $V = M + P + N + 1$. Also $\Phi - E + V = 2$, since the Euler Characteristic of the sphere is two.

Summing the valencies of each vertex (including the distinguished vertex), each edge is counted twice. Therefore:

$$2E \geq M + \beta P + \ell_1 M + \ell_2 P + 6N$$

Rearranging gives:

$$0 \geq -2E + (1 + \ell_1)M + (\beta + \ell_2)P + 6N \tag{2.1}$$

Also each edge is incident with two areas, and each area is bounded by at least three edges (by Lemma 2.4.2). Thus $2E \geq 3\Phi$ or,

$$0 \geq \Phi - \frac{2}{3}E \tag{2.2}$$

Now from (2.1):

$$\begin{aligned}
0 &\geq V - \frac{E}{3} + (\frac{1+\ell_1}{6})M + (\frac{\beta+\ell_2}{6})P + N - V \\
\Rightarrow 0 &\geq V - \frac{E}{3} + (\frac{1+\ell_1}{6})M + (\frac{\beta+\ell_2}{6})P + N - M - P - N - 1 \\
\Rightarrow 0 &\geq V - \frac{E}{3} + (\frac{\ell_1-5}{6})M + (\frac{\beta+\ell_2-6}{6})P - 1
\end{aligned}$$

Adding (2.2) gives:

$$\begin{aligned}
0 &\geq \Phi + V - E + (\frac{\ell-5}{6})M + (\frac{\beta+\ell_2-6}{6})P - 1 \\
\Rightarrow 0 &\geq (\frac{\ell_1-5}{6})M + (\frac{\beta+\ell_2-6}{6})P + 1
\end{aligned}$$

If $P = 0$ then we conclude that

$$0 \geq (\frac{\ell_1-5}{6})M + 1. \tag{2.3}$$

If $P \neq 0$ then $\beta \geq 2$ and $\ell_2 \geq 2\beta$. Therefore:

$$\frac{\beta+\ell_2-6}{6} \geq \frac{3\beta-6}{6} \geq 0.$$

Thus:

$$0 \leq (\frac{\beta+\ell_2-6}{6})P \leq -1 - (\frac{\ell_1-5}{6})M$$

and so equation (2.3) also holds in this case.

Since $\ell_1 \geq 2$ we conclude that $2 \leq \ell_1 \leq 4$, and that there exists at least one vertex of type (ii) in T with valence between two and four.

Suppose T has only one vertex of type (ii) with valence between two and four. Then all the other vertices of type (ii) in T have valence ≥ 5. Hence:

$$\ell_1 \geq \frac{2+5(M-1)}{M} \tag{2.4}$$

Substituting (2.4) into (2.3) gives:

$$\begin{aligned}
0 &\geq (\frac{\frac{2+5(M-1)}{M} - 5}{6})M + 1 \\
\Rightarrow 0 &\geq \frac{2+5M-5-5M}{6} + 1 \\
\Rightarrow \frac{1}{2} &\geq 1
\end{aligned}$$

We have a contradiction and so conclude that T has at least two vertices of type (ii) with valence between two and four.

Suppose there only exists two vertices of type (ii) in T with valence between two and four, and that at least one of them has valence three or four. Then:

$$\ell_1 \geq \frac{2+3+5(M-2)}{M} \tag{2.5}$$

Substituting (2.5) into (2.3) gives:

$$\begin{aligned} 0 &\geq \left(\frac{\frac{5+5(M-2)}{M} - 5}{6}\right)M + 1 \\ \Rightarrow 0 &\geq \frac{5+5M-10-5M}{6} + 1 \\ \Rightarrow \frac{5}{6} &\geq 1 \end{aligned}$$

In view of this contradiction, we conclude that either there exist at least two vertices of type (ii) with valence two in T, or there exist at least three vertices of type (ii) with valence between two and four in T. This completes the proof of Proposition 2.4.1. □

2.5 Proof of Theorem 2.2.4

Let Δ be a disc in a picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$ and define $i(\Delta)$ to be the number of arcs which are incident with Δ and do not intersect the boundary of $\mathbb{P}$. We say that Δ is a *simple boundary disc* if exactly one arc from Δ intersects the boundary of $\mathbb{P}$.

We now translate the result of Proposition 2.4.1 into pictures.

Lemma 2.5.1 *Let $\widehat{\mathcal{P}}$ be a presentation which satisfies $C(6)$. Suppose that $\mathbb{P}$ is a non-spherical, connected, amended picture over $\widehat{\mathcal{P}}$ which contains at least two discs, and is such that the intersection of any area of $\mathbb{P}$ with a disc is connected. Then $\mathbb{P}$ has either*

(i) at least two simple boundary discs with $i(\Delta) = 1$, or

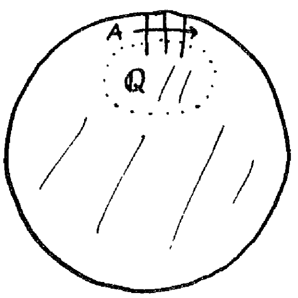
(ii) at least three simple boundary discs with $1 \leq i(\Delta) \leq 3$.

In proving Theorem 2.2.4 we will be concerned with minimal pictures. Recall from Proposition 1.6.3 that all minimal pictures are spherically connected, but not necessarily connected. Our next aim is therefore to extend the result of Lemma 2.5.1 to spherically connected pictures.

Let $\mathbb{P}$ be a spherically connected picture over $\widehat{\mathcal{P}}$. An *extremal component* of $\mathbb{P}$ is a subpicture, $\mathbb{Q}$ say, of $\mathbb{P}$ such that:

(i) reading once clockwise around $\partial\mathbb{Q}$ from some point on $\partial\mathbb{Q}$ we read a word A, and

(ii) A is a subword of a word read whilst travelling once clockwise around $\partial\mathbb{P}$ from some point on $\partial\mathbb{P}$.



The following result is the picture analogue of Lemma V.4.2 in [33].

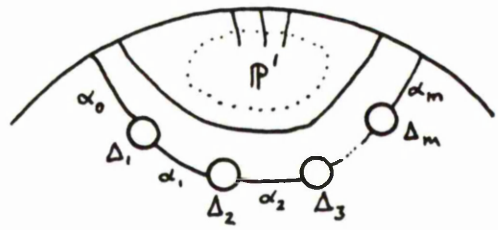
Lemma 2.5.2 *Let $\mathbb{P}$ be a spherically connected, amended picture over $\widehat{\mathcal{P}}$ which contains no floating semicircles. If $\mathbb{P}$ is not connected then $\mathbb{P}$ has at least two extremal components.*

Proof. We prove the result by induction on the number of discs in $\mathbb{P}$.

If $\mathbb{P}$ contains a single disc then $\mathbb{P}$ must be connected. Hence we can assume that $\mathbb{P}$ contains at least two discs. We can also assume that $\mathbb{P}$ contains at least two components.

If all the components (considered as subpictures) of $\mathbb{P}$ are extremal components then since $\mathbb{P}$ has no floating semicircles, the result follows. Hence we can assume that $\mathbb{P}$ has at least one component which is not an extremal component.

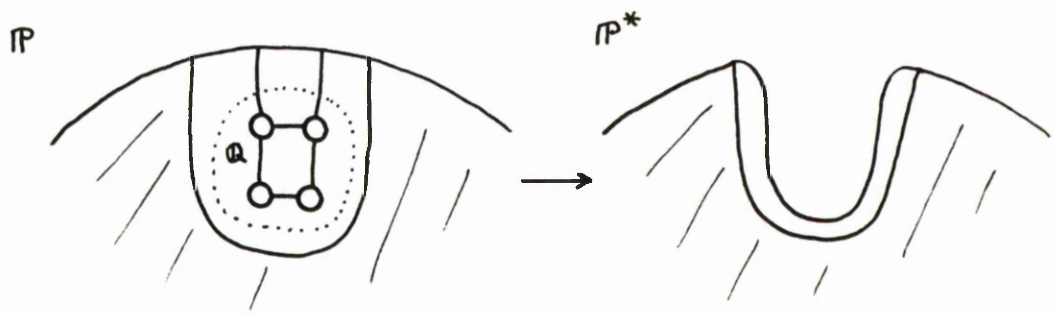
Consider a component of $\mathbb{P}$ which is not an extremal component. Then there must exist a sequence of arcs $\alpha_0, \alpha_1, \ldots, \alpha_m$ and discs $\Delta_1, \ldots, \Delta_m$ ($m \geq 1$) which belong to the component and are such that only α_0 and α_m intersect $\partial\mathbb{P}$, while α_i ($1 \leq i \leq m-1$) intersects Δ_i and Δ_{i+1}. Also there must exist a subpicture $\mathbb{P}'$ of $\mathbb{P}$, contained in the part of $\mathbb{P}$ enclosed by $\alpha_0 \cup \ldots \cup \alpha_m \cup \Delta_1 \cup \ldots \cup \Delta_m \cup \partial\mathbb{P}$, which contains at least one disc (since $\mathbb{P}$ contains no floating semicircles), has at least one arc which intersects $\partial\mathbb{P}$, and has no arcs which intersect $\Delta_1, \ldots, \Delta_m$.



Subpicture $\mathbb{P}'$ can be chosen so that it does not contain any floating semicircles. Also, note that $\mathbb{P}'$ is amended. Furthermore, we suppose that the original component is chosen so that $\mathbb{P}'$ contains the smallest number of discs possible.

Now suppose that $\mathbb{P}'$ contained a component which was not an extremal component. Then applying the above argument we could obtain a subpicture $\mathbb{P}''$ (as above) which contains less discs than $\mathbb{P}'$. However, this contradicts the original choice of component. Thus all the components of $\mathbb{P}'$ are extremal components, and we conclude that $\mathbb{P}$ has at least one extremal component, $\mathbb{Q}$ say.

Consider the picture $\mathbb{P}^*$ obtained from $\mathbb{P}$ by cutting $\mathbb{Q}$ out of $\mathbb{P}$.



We amend $\mathbb{P}^*$ in the usual way. Also $\mathbb{P}^*$ may contain floating semicircles, but these can be removed, one at a time, so that we can assume that $\mathbb{P}^*$ contains no floating semicircles. If $\mathbb{P}^*$ is connected then $\mathbb{P}^*$ (considered as a subpicture of $\mathbb{P}$) is an extremal component of $\mathbb{P}$ and we are finished. Otherwise, by the inductive hypothesis, $\mathbb{P}^*$ contains two extremal components, and at least one of these (considered as a subpicture of $\mathbb{P}$) must be an extremal component of $\mathbb{P}$. $\square$

Lemma 2.5.3 *Let $\widehat{\mathcal{P}}$ be a presentation which satisfies $C(6)$. Suppose that $\mathbb{P}$ is a minimal, amended picture over $\widehat{\mathcal{P}}$ which contains at least two discs, is such that the intersection of any area of $\mathbb{P}$ with a disc is connected, and has none of the following:*

(a) floating semicircles, or

(b) simple boundary discs which are incident with just one arc.

Then $\mathbb{P}$ has either

(i) at least two simple boundary discs with $i(\Delta) = 1$, or

(ii) at least three simple boundary discs with $1 \leq i(\Delta) \leq 3$.

Proof. By Proposition 1.6.3, $\mathbb{P}$ is spherically connected. If $\mathbb{P}$ is connected then the result follows immediately by Lemma 2.5.1. Otherwise, by Lemma 2.5.2, $\mathbb{P}$ has at least two extremal components.

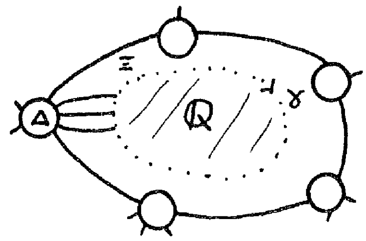
Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be extremal components of $\mathbb{P}$. Since $\mathbb{P}$ contains no simple boundary discs which are incident with just one arc, both $\mathbb{Q}_1$ and $\mathbb{Q}_2$ must contain at least two discs each. Considering $\mathbb{Q}_1$ and $\mathbb{Q}_2$ as pictures over $\widehat{\mathcal{P}}$, we amend them in the usual way and note that at most one disc in $\mathbb{Q}_i$ $(i = 1, 2)$ can be a simple boundary disc in $\mathbb{Q}_i$ but not $\mathbb{P}$. Hence applying Lemma 2.5.1 to $\mathbb{Q}_1$ and $\mathbb{Q}_2$, and summing the contributions of the relevant simple boundary discs from each, gives the result. □

We can now eliminate from our argument the case of a picture arising which has an area Ξ and disc Δ, such that $\partial\Xi \cap \partial\Delta$ is disconnected. The following result is in fact the picture analogue of a special case $(p = 6, q = 3)$ of Lemma V.4.1 in [33].

Lemma 2.5.4 *Let $\widehat{\mathcal{P}}$ be a group presentation which satisfies $C(6)$ and $\mathbb{P}$ a minimal, amended picture over $\widehat{\mathcal{P}}$. Then the intersection of any area of $\mathbb{P}$ with a disc is connected.*

Proof. The result is proved by contradiction.

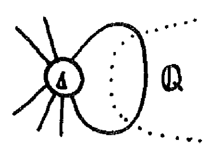
Let Δ be a disc in $\mathbb{P}$ and Ξ an area such that $\partial\Delta \cap \partial\Xi$ is disconnected. Then there exists a transverse path γ which encloses a subpicture $\mathbb{Q}$ of $\mathbb{P}$, as shown below. By Proposition 1.6.3, $\mathbb{Q}$ is minimal.



Let Δ and Ξ be chosen so that $\mathbb{Q}$ contains the smallest number of discs possible.

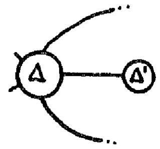
Suppose that $\mathbb{Q}$ contained a disc Δ' and area Ξ' such that $\partial\Delta' \cap \partial\Xi'$ is disconnected. Then there exists a transverse path γ' which encloses a subpicture $\mathbb{Q}'$ of $\mathbb{Q}$ (similar to above). However $\mathbb{Q}'$ contains a smaller number of discs than $\mathbb{Q}$ (since it does not contain Δ'), and so we have a contradiction to our original choice of Δ and Ξ. Thus we can assume that $\mathbb{Q}$ has no disc Δ' and area Ξ' such that $\partial\Delta' \cap \partial\Xi'$ is disconnected.

Now $\mathbb{Q}$ must contain at least one disc, otherwise Δ would have an arc which intersected $\partial\Delta$ twice, and Δ could not be labelled by a cyclically reduced word.



For the same reason $\mathbb{Q}$ cannot contain any floating semicircles.

Since $\widehat{\mathcal{P}}$ satisfies $C(6)$ and every disc in $\mathbb{Q}$ is an interior disc in $\mathbb{P}$, every disc in $\mathbb{Q}$ has at least six incident arcs. It follows that $\mathbb{Q}$ cannot contain a single disc Δ', for then Δ' would be forced to have a single arc which intersected Δ.



Thus we can assume that $\mathbb{Q}$ contains at least two discs.

Also $\mathbb{Q}$ cannot have a simple boundary disc Δ' with a single incident arc intersecting $\partial\mathbb{Q}$, since Δ' is an interior disc in $\mathbb{P}$ and must therefore have at least six incident arcs.

Hence by Lemma 2.5.3, there exist at least two simple boundary discs in $\mathbb{Q}$ with 1, 2 or 3 arcs which do not intersect $\partial\mathbb{Q}$. Let Δ^* be one of these discs. Since Δ^* has exactly one incident arc intersecting Δ and at most three other incident arcs, Δ^* has at most four incident arcs. However this contradicts the fact that Δ^* must have at least six incident arcs (since it is an interior disc of $\mathbb{P}$). $\square$

Combining the results of Lemma 2.5.3 and Lemma 2.5.4 gives the result we require.

Theorem 2.5.5 *Let $\widehat{\mathcal{P}}$ be a presentation which satisfies $C(6)$. Suppose that $\mathbb{P}$ is a minimal, amended picture over $\widehat{\mathcal{P}}$ which contains at least two discs and has none of the following:*

(a) floating semicircles, or

(b) simple boundary discs which are incident with just one arc.

Then $\mathbb{P}$ has either

(i) at least two simple boundary discs with $i(\Delta) = 1$, or

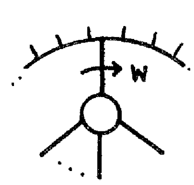
(ii) at least three simple boundary discs with $1 \leq i(\Delta) \leq 3$.

We now turn our attention to the geometric consequences for pictures over $\widehat{\mathcal{P}}$, given that $\widehat{\mathcal{P}}$ satisfies the Ω-Condition.

Theorem 2.5.6 *Suppose $\widehat{\mathcal{P}} =< \boldsymbol{x}; \hat{\boldsymbol{r}} >$ satisfies the Ω-Condition. Let Δ be a simple boundary disc in a reduced, amended picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$, which has its arc intersecting $\partial\mathbb{P}$ labelled by a positive or negative word W on $\boldsymbol{x}$. Then either*

(i) $W \equiv R_{-\epsilon}$, where $\epsilon = \pm 1$ and $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$, or

(ii) $i(\Delta) \geq 4$.



Proof. Let α be the unique arc between Δ and $\partial\mathbb{P}$. The remaining arcs are labelled by a word $X^{-1}R_\epsilon Z^{-1}$, where $\epsilon = \pm 1$, $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$ and X, Z are positive words on $\boldsymbol{x}$. Since $\widehat{\mathcal{P}}$ satisfies the Ω-Condition either

(i) both X and Z are the empty word, or

(ii) $X^{-1}R_\epsilon Z^{-1}$ is a product of at least four pieces.

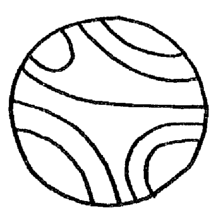
The result follows. □

Finally, before proving our main result, we require the following Lemma.

Lemma 2.5.7 *If $\widehat{\mathcal{P}}$ satisfies $C(6)$ and the Ω-Condition then there exist no non-empty, positive words U on $\boldsymbol{x}$ such that $U =_G 1$.*

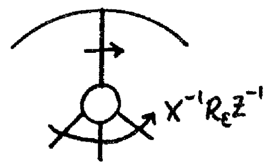
Proof. Suppose that U is a non-empty, positive word on $\boldsymbol{x}$ such that $U =_G 1$. By Theorem 1.6.4 there exists a spherically connected picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$ with boundary label U. We assume that $\mathbb{P}$ is minimal and amended. The result is proved by induction on $deg(U)$.

If $deg(U) = 0$ then $\mathbb{P}$ contains no discs and so must be the empty picture, otherwise it could not have boundary label a positive word on $\boldsymbol{x}$.



If $deg(U) = 1$ then $\mathbb{P}$ has a single disc Δ. Now all the arcs from Δ cannot intersect $\partial\mathbb{P}$ since U is a positive word on $\boldsymbol{x}$. Thus Δ must be a self identified disc and must not be labelled by a cyclically reduced word. This cannot happen and so we conclude that the result is true in this case.

Therefore we can assume that $deg(U) \geq 2$ and so $\mathbb{P}$ contains at least two discs. Since U is a positive word on $\boldsymbol{x}$, hypotheses (a) and (b) of Theorem 2.5.5 hold, and so there exists at least two simple boundary discs in $\mathbb{P}$ with $1 \leq i(\Delta) \leq 3$. Let Δ be any one of these discs. Note that the arc from Δ which intersects with $\partial\mathbb{P}$ is labelled by a positive word on $\boldsymbol{x}$. Hence the remaining arcs are labelled by a word $X^{-1}R_\epsilon Z^{-1}$ (X, Z positive words on $\boldsymbol{x}$, $\epsilon = \pm 1$ and $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$).



Now since $\widehat{\mathcal{P}}$ satisfies the Ω-Condition, by Theorem 2.5.6, X and Z are both the empty word. Cutting this disc out of $\mathbb{P}$ we obtain a minimal picture (by Proposition 1.6.3) with boundary label U', where U' is a non-empty, positive word on $\boldsymbol{x}$ such that $U' =_G 1$ and $deg(U') < deg(U)$. However, this is a contradiction to the inductive hypothesis. $\square$

Proof of Theorem 2.2.4: Let U and V be positive words on $\boldsymbol{x}$ such that $U =_G V$. We must show that $U =_S V$. By the previous lemma, we can assume that both U and V are non-empty.

The result will be proved by induction on

$$(deg(U, V), L(U) + L(V))$$

where ordered pairs are given the lexicographical order.

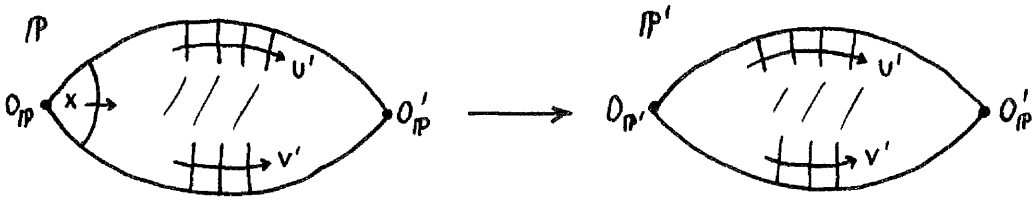
By Corollary 1.6.5 there exists a two sided, spherically connected picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$. We can assume that $\mathbb{P}$ is minimal and amended.

If $deg(U, V) = 0$ then $\mathbb{P}$ is a trivial picture over $\mathcal{P}$ and so $U \equiv V$. If $deg(U, V) = 1$ then $\mathbb{P}$ is an atomic picture over $\mathcal{P}$ and so $U =_S V$.

Thus we can assume that $deg(U, V) \geq 2$, and therefore that $\mathbb{P}$ contains at least two discs.

Suppose that $\mathbb{P}$ has a floating semicircle α. Then α must intersect both $\partial^+\mathbb{P}$ and $\partial^-\mathbb{P}$. Suppose that α is labelled by a positive word X on $\boldsymbol{x}$ and that $U \equiv XU', V \equiv XV'$ for some positive words U' and V' on $\boldsymbol{x}$. (The case where $U \equiv U'X$ and $V \equiv V'X$

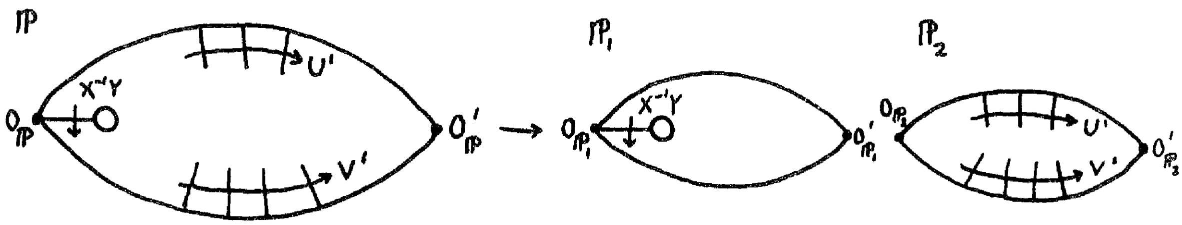
follows by a symmetrical argument.)



Consider the subpicture $\mathbb{P}'$ of $\mathbb{P}$ with boundary label $U'V'^{-1}$. By Proposition 1.6.3, $\mathbb{P}'$ is minimal. Now while $deg(U',V') = deg(U,V)$, we have $L(U') + L(V') < L(U) + L(V)$. Thus by the inductive hypothesis $U' =_S V'$. Consequently

$$U \equiv XU' =_S XV' \equiv V.$$

Suppose that $\mathbb{P}$ has a simple boundary disc Δ with just one incident arc which intersects $O_{\mathbb{P}}$ (the case where the arc intersects $O'_{\mathbb{P}}$ follows by a symmetrical argument). Suppose that the arc is labelled by the word $X^{-1}Y$ (X, Y non-empty positive words on $\boldsymbol{x}$) and that $U \equiv XU'$, while $V \equiv YV'$ (U', V' positive words on $\boldsymbol{x}$).



Consider the subpictures $\mathbb{P}_1$ and $\mathbb{P}_2$ of $\mathbb{P}$ with boundary labels $X^{-1}Y$ and $U'V'^{-1}$ respectively. Since $deg(X,Y) < deg(U,V)$ and $deg(U',V') < deg(U,V)$ we have by the inductive hypothesis that $X =_S Y$ and $U' =_S V'$. Consequently

$$U \equiv XU' =_S YU' =_S YV' \equiv V.$$

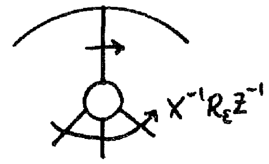
Thus we can assume that $\mathbb{P}$ satisfies the hypotheses of Theorem 2.5.5. Hence either

(i) $\mathbb{P}$ has at least two simple boundary discs with $i(\Delta) = 1$, or

(ii) $\mathbb{P}$ has at least three simple boundary discs with $1 \leq i(\Delta) \leq 3$.

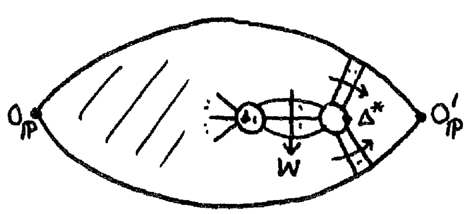
We deal with the second case first.

Case (ii): Since there exists at least three simple boundary discs with $1 \leq i(\Delta) \leq 3$, at least one of these discs must have its arc which intersects with $\partial\mathbb{P}$, labelled by either a positive or negative word. The remaining arcs are labelled by a word of the form $X^{-1}R_\epsilon Z^{-1}$ (X, Z positive words on $\boldsymbol{x}$, $\epsilon = \pm 1$, $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$).



Since $\widehat{\mathcal{P}}$ satisfies the Ω-Condition, by Theorem 2.5.6, X and Z must both be the empty word. Suppose (without loss of generality) that $U \equiv U_1 R_{-\epsilon} U_2$ (U_1, U_2 positive words on $\boldsymbol{x}$). Cutting this disc out of $\mathbb{P}$ we obtain a new minimal picture (by Proposition 1.6.3) with fewer discs and boundary label $U'V^{-1}$, where $U' \equiv U_1 R_\epsilon U_2$ and $U =_S U'$. Then $deg(U', V) < deg(U, V)$ and so $U' =_S V$. Consequently $U =_S V$.
Case (i): Let Δ^* be one of the simple boundary discs with $i(\Delta^*) = 1$. If the arc from Δ^* which intersects $\partial\mathbb{P}$ is labelled by a positive or negative word, then the interior arc of Δ^* will be labelled by a word of the form $X^{-1}R_\epsilon Z^{-1}$ (X, Z positive words on $\boldsymbol{x}$, $\epsilon = \pm 1$, $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$). The argument used in case (ii) above can now be repeated, giving $U =_S V$.

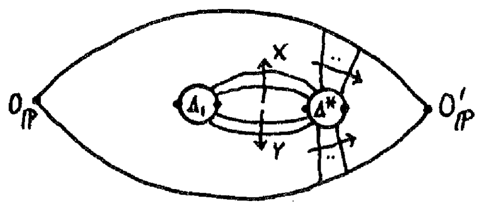
Hence we can assume that the arc from Δ^* which intersects $\partial\mathbb{P}$ actually intersects $O_\mathbb{P}$ or $O'_\mathbb{P}$. Without loss of generality we will assume that it intersects $O'_\mathbb{P}$ (a symmetrical argument holds if it intersects $O_\mathbb{P}$). For clarity, we will consider pictures in their unamended forms for the remainder of the proof. Suppose that Δ^* is connected to Δ_1 by arcs labelled by a word W on $\boldsymbol{x}$, as shown below.



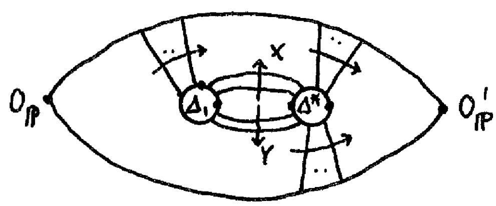
If W is a positive word on $\boldsymbol{x}$ then Δ^* can be cut out of the picture giving a new picture with fewer discs. By Proposition 1.6.3, this picture is minimal. Applying the usual argument on degree gives the result. A symmetrical argument follows if W is a negative word on $\boldsymbol{x}$.

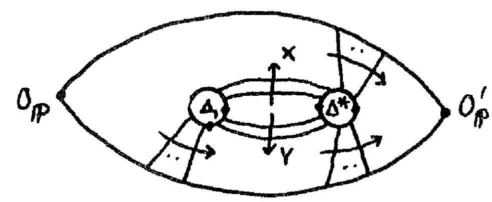
Hence we can assume that $W \equiv X^{-1}Y$, where X and Y are non-empty, positive words on x. We first prove that $\mathbb{P}$ must in fact contain three discs by examining all the possible situations involving two discs.

The following situation cannot arise since $\widehat{\mathcal{P}}$ satisfies $C(6)$.

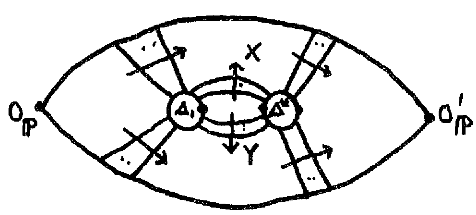


The two situations shown below cannot arise since $\widehat{\mathcal{P}}$ satisfies the Ω-Condition.





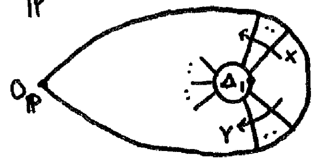
The only other possible situation is the following:



which cannot arise since Δ_1 is not a two sided disc.

Hence we can assume that $\mathbb{P}$ contains at least three discs. Let $\mathbb{P}'$ be the picture over $\widehat{\mathcal{P}}$, obtained from $\mathbb{P}$, by cutting out Δ^*. By Proposition 1.6.3, $\mathbb{P}'$ is minimal.

$\mathbb{P}'$



Now $\mathbb{P}'$ satisfies the hypotheses of Theorem 2.5.5, so that there exists at least two simple boundary discs in $\mathbb{P}'$ with $1 \leq i(\Delta) \leq 3$. If Δ_1 is a simple boundary disc in $\mathbb{P}'$ then $i(\Delta_1) \geq 5$, thus Δ_1 is not one of these discs. Hence at least one of the simple boundary discs in question must have its arc which intersects with $\partial\mathbb{P}$ labelled by either a positive or negative word. The remaining arcs of the disc must be labelled by a word of the form $X^{-1}R_\epsilon Z^{-1}$ (X, Z positive words on $\boldsymbol{x}$, $\epsilon = \pm 1$, $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$). Since $\widehat{\mathcal{P}}$ satisfies the Ω-Condition, by Theorem 2.5.6, X and Z must both be the empty word. We can therefore cut this disc out of the picture, apply the usual argument on degree, and conclude that $U =_S V$. $\square$

Chapter 3

Conjugacy in monoids

3.1 Some background

Let $\mathcal{P} = [\boldsymbol{x}; \boldsymbol{r}]$ be a monoid presentation for monoid $S(= S(\mathcal{P}))$, or semigroup $S_0(= S_0(\mathcal{P}))$. As in Chapter 2, we assume for each $R \in \boldsymbol{r}$ that R_{+1} and R_{-1} are distinct, non-empty positive words on $\boldsymbol{x}$, and that for any other relation $R' : R'_{+1} = R'_{-1} \in \boldsymbol{r}$, $R_\epsilon \not\equiv R'_\delta$ $(\epsilon, \delta = \pm 1)$. Also, let $\mathcal{C}$, $\hat{\boldsymbol{r}}$, $\widehat{\mathcal{P}} =< \boldsymbol{x}; \hat{\boldsymbol{r}} >$, $G(= G(\widehat{\mathcal{P}}))$ and π be defined as in Chapter 2.

In this chapter we will be concerned with the relationship between elements which are conjugate in S or S_0 and those which are conjugate in G, and vice versa. In particular it seems natural to ask the following questions:

1. If two elements of S (S_0) are conjugate in S (S_0), are their images under π conjugate in G?

2. If π is injective and two elements of S (S_0) have conjugate images under π in G, are they conjugate in S (S_0)?

If the first question can be answered positively we say that π *preserves* conjugacy. If the second question can be answered positively we say that π *reflects* conjugacy.

The major problem associated with posing such questions is that there seems to be no standard definition of conjugacy in an arbitrary monoid or semigroup (see for instance [12, 15, 31, 40, 54]). It would seem natural that for any reasonable concept of conjugacy we would have that π preserves conjugacy.

Goldstein and Teymouri [20] have modified the definition given in [31] to arrive at a definition of conjugacy in S_0 which not only gives π preserving conjugacy, but in addition, if $\mathcal{P}$ satisfies Adjan's conditions for S_0 to be embeddable in G, then π also reflects conjugacy.

Now conjugacy is reflected by $\pi : S_0(\mathcal{P}) \to G$ if for any two non-empty positive words U, V on $\boldsymbol{x}$ the following is true:

$$\pi(U) \approx_G \pi(V) \Rightarrow U \text{ is conjugate to } V \text{ in } S_0(\mathcal{P}).$$

Suppose that it is known that conjugacy *is* reflected by $\pi : S_0(\mathcal{P}) \to G$ and we wish to consider whether it is reflected by $\pi : S(\mathcal{P}) \to G$. It is clear that we need only consider the question of whether for any positive word U on $\boldsymbol{x}$ the following holds:

$$\pi(U) \approx_G 1 \Rightarrow \ U \text{ is conjugate to } 1 \text{ in } S(\mathcal{P}).$$

However $\pi(U) \approx_G 1$ if and only if $\pi(U) =_G 1$. Hence if we know that there exist no non-empty positive words W on $\boldsymbol{x}$ such that $W =_G 1$, then we have proved that conjugacy is reflected by $\pi : S(\mathcal{P}) \to G$.

It is known that this condition is satisfied for monoid presentations which satisfy Adjan's embeddability conditions (this can be deduced from Theorem 5.2.1 and Theorem 1.6.4, see also [13]). Thus Goldstein and Teymouri's result extends to $\pi : S(\mathcal{P}) \to G$.

Our task in this chapter will be to show that if monoid presentation $\mathcal{P}$ satisfies Kashintsev's conditions for $S(\mathcal{P})$ to be embeddable in G (that is $\mathcal{P} \in K_3^3 \cup K_4^2$), then π reflects conjugacy (with respect to Goldstein and Teymouri's definition) in an elementary way. Throughout we will appeal to the notation established in Chapter 2. We will also make use of the fact that Kashintsev's embeddability conditions imply the embeddability conditions given in Chapter 2 (Theorem 2.2.5).

We will assume from this point onwards that $\mathcal{P}$=[$\boldsymbol{x}$;$\boldsymbol{r}$] is a monoid presentation for monoid S.

3.2 Definition of conjugacy

Let $g, g' \in G$. Also, let s, s' be two distinct elements of S which are represented by positive words X and X' respectively.

We begin by recalling Goldstein and Teymouri's definition of conjugacy. This requires two preliminary definitions.

We say that s is *elementarily conjugate* to s', written $s \approx_S^{(e)} s'$, if there exists an element a in S such that either

$$sa =_S as' \text{ or } as =_S s'a.$$

We will often write $X \approx_S^{(e)} X'$.

Notes. (i) It is clear that if $s \approx_S^{(e)} s'$ then $\pi(s) \approx_G \pi(s')$.
(ii) If $X \equiv X_1X_2$ and $X' \equiv X_2X_1$ (X_1, X_2 positive words on $\boldsymbol{x}$) then $X \approx_S^{(e)} X'$ since

$$X_2.X_1X_2 =_S X_2X_1.X_2.$$

Hence if X is a cyclic permutation of X' then $X \approx_S^{(e)} X'$. We will often make use of this fact.
(iii) Elementary conjugacy is clearly a reflexive and symmetric relation, however it is not in general a transitive relation.

We say that s is *elementary inversely conjugate* to s', written $s \approx_S^{(ei)} s'$, if there exists an element a in S such that either

$$sas' =_S a \text{ or } s'as =_S a.$$

We will often write $X \approx_S^{(ei)} X'$.

Note. It is clear that if s is elementary inversely conjugate to s' then $\pi(s) \approx_G \pi(s')^{-1}$.

Finally, we say that s is *conjugate* to s' in S, written $s \approx_S s'$, if there exists a sequence of elements $s_1, s_2, \ldots s_n$ of S such that

(i) $s = s_1$ and $s' = s_n$;

(ii) s_{j+1} is either elementarily conjugate or elementary inversely conjugate to s_j $(1 \leq j \leq n-1)$;

(iii) the number of elementary inverse conjugations is even.

We will often write $X \approx_S X'$.

Consider the following new definition.

We say that s is *sequentially elementarily conjugate* to s', written $s \approx_S^{(se)} s'$, if there exists a sequence $s_1, s_2 \ldots, s_n$ of elements of S such that

(i) $s = s_1$ and $s' = s_n$;

(ii) s_{j+1} is elementarily conjugate to s_j $(1 \leq j \leq n-1)$.

We will often write $X \approx_S^{(se)} X'$.

Teymouri [53] considered the presentation

$$\mathcal{P} = [a, b, c, d, e; cba = b, cde = d].$$

This presentation is cycle free and $a \approx_G e$. However, Teymouri showed that a is not sequentially elementary conjugate to e. On the other hand, Goldstein and Teymouri [20] have shown that conjugacy (with respect to the above definition) is reflected by cycle free presentations. Thus $a \approx_S e$.

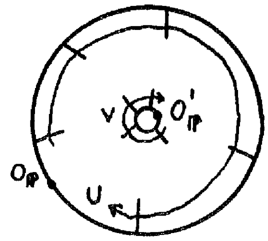
In contrast, we will prove the following result.

Theorem 3.2.1 *Let $\mathcal{P} \in K_3^3 \cup K_4^2$. If s and s' are elements of S such that $\pi(s) \approx_G \pi(s')$, then $s \approx_S^{(se)} s'$.*

Hence, under Kashintsev's embeddability conditions, $\pi : S \to G$ reflects sequential elementary conjugacy. In order to prove this result we must first understand some of the geometric concepts underlying conjugacy.

3.3 Annular pictures and tessellations

Recall, from Theorem 1.8.1, that if two words U and V on $\boldsymbol{x}$ are such that $U \approx_G V$, then there exists an annular picture $\mathbb{P}$ over $\widehat{\mathcal{P}} =< \boldsymbol{x}; \hat{\boldsymbol{r}} >$ such that $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.



Since we are assuming that each element of $\hat{\boldsymbol{r}}$ is of the form $R_\epsilon R_{-\epsilon}^{-1}$ ($\epsilon = \pm 1, R_{+1} = R_{-1} \in \boldsymbol{r}$) and both $R_\epsilon, R_{-\epsilon}$ are non-empty words on $\boldsymbol{x}$, we can think of the discs in $\mathbb{P}$ as having two 'sides', as with ordinary pictures over $\widehat{\mathcal{P}}$ (see section 2.3).

We define

$$conjdeg(U,V) = \min \left\{ \begin{array}{l} \text{number of discs in } \mathbb{P} : \mathbb{P} \text{ is an annular picture with} \\ \text{outer boundary label } U \text{ and inner boundary label } V \end{array} \right\}$$

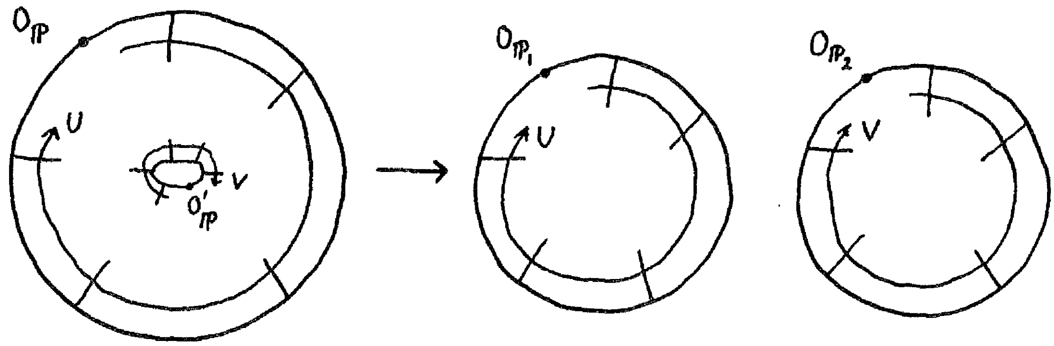
to be the *conjugacy degree of* (U, V).

Note that if the number of discs in $\mathbb{P}$ is equal to $conjdeg(U, V)$ then $\mathbb{P}$ is a minimal annular picture. Recall from Proposition 1.8.2 that if $\mathbb{P}$ is minimal, then $\mathbb{P}$ is reduced.

We make the following observation. Theorem 2.2.5 says that if $\mathcal{P} \in K_3^3 \cup K_4^2$ then $\widehat{\mathcal{P}}$ satisfies $C(6)$ and the Ω-Condition. Furthermore, Lemma 2.5.7 gives that if $\widehat{\mathcal{P}}$ satisfies $C(6)$ and the Ω-Condition, then there exist no non-empty positive words U on $\boldsymbol{x}$ such that $U =_G 1$. Thus if $\mathcal{P} \in K_3^3 \cup K_4^2$ then there exist no positive words U on $\boldsymbol{x}$ such that $U \approx_G 1$ (since this is equivalent to saying that $U =_G 1$).

In proving our main result we will be concerned with positive words U, V on $\boldsymbol{x}$ such that $U \approx_G V$. *In the light of the above observation, we can assume from this point onwards that U and V are non-empty, positive words on $\boldsymbol{x}$.*

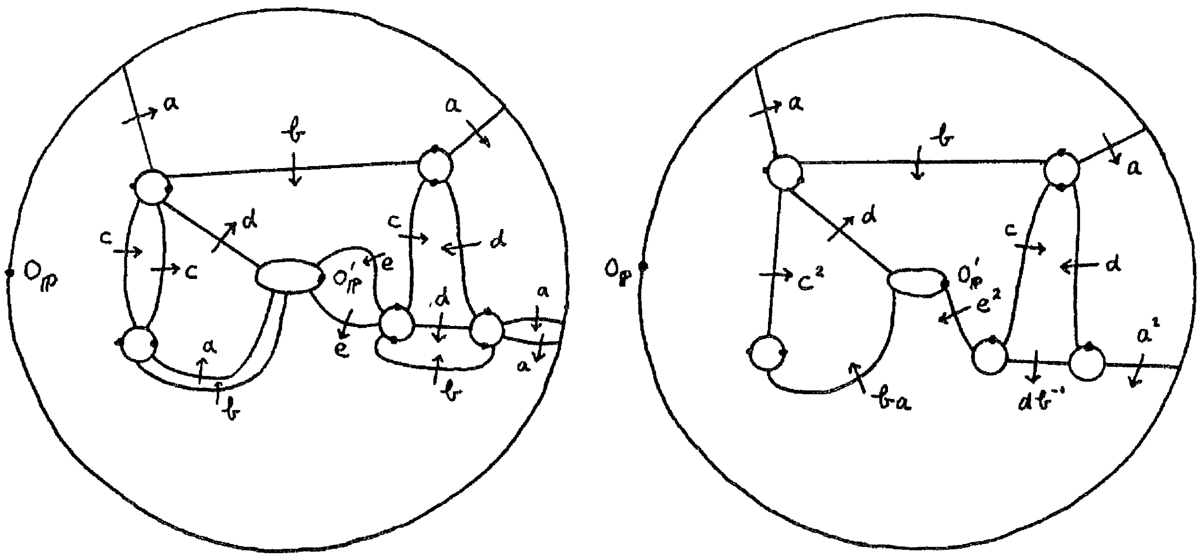
Recall that we assume that all floating circles and annular floating circles have been deleted from our annular pictures (see §1.8). Suppose that $\mathbb{P}$ is an annular picture over $\widehat{\mathcal{P}}$, where $\mathcal{P} \in K_3^3 \cup K_4^2$, and $\iota(\mathbb{P}) \equiv U, \tau(\mathbb{P}) \equiv V$, for some non-empty, positive words U, V on $\boldsymbol{x}$. Note that if we can insert an annular floating circle into an area of $\mathbb{P}$, then we can obtain two pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ over $\widehat{\mathcal{P}}$ with $W(\mathbb{P}_1) \equiv U$, $W(\mathbb{P}_2) \equiv V$.



By Theorem 1.6.4, $U =_G 1$ and $V =_G 1$. However, by the observation above we conclude that this is not possible. *Hence we can assume that all the minimal annular pictures considered in this Chapter satisfy the conditions of Proposition 1.8.3, and are therefore spherically connected.*

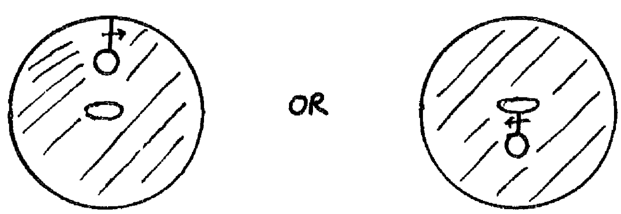
Throughout the rest of this chapter (unless otherwise indicated), we will amend each annular picture $\mathbb{P}$ in a manner similar to the way that ordinary pictures were amended in Chapter 2. Thus, if there are two or more parallel arcs connecting two discs, or connecting a disc to the outer (inner) boundary, then we combine these arcs to form a single arc labelled by a word. We simply omit basepoints on the discs if necessary. Also if the arcs which intersect the outer (inner) boundary also straddle $O_{\mathbb{P}}$ ($O'_{\mathbb{P}}$ respectively), then we make the single arc intersect the boundary at $O_{\mathbb{P}}$ (or $O'_{\mathbb{P}}$ respectively). Note that unlike in the ordinary picture case, all the single arcs which intersect with $\partial\mathbb{P}$ are labelled by a positive word.

Example 1: Amending the picture on the left gives the picture on the right.

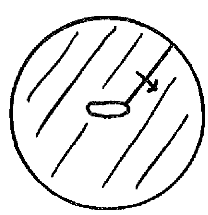


We remark that any annular (amended) picture $\mathbb{P}$ has the following properties:

- There are no discs in $\mathbb{P}$ which have only one arc which intersects the outer boundary, or intersects the inner boundary.



- If an arc[1] does not intersect any discs then it must intersect both the outer and inner boundaries of $\mathbb{P}$.



As with ordinary pictures, note that every interior area of an amended annular picture is bounded by at least three arcs.

[1]Recall that we assume that all floating circles and annular floating circles have been deleted from our annular pictures.

The following Lemma follows by a similar proof to Lemma 2.3.1.

Lemma 3.3.1 *Let α be an arc in a reduced, amended, annular picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$ which intersects at least one disc but does not intersect $\partial\mathbb{P}$. If α is labelled by a word X (on $\boldsymbol{x}$) then X is a piece (relative to $\boldsymbol{r}^*$).*

Hence if $\widehat{\mathcal{P}}$ satisfies $C(p)$ then all the interior discs in a reduced, amended, annular picture have at least p arcs incident with them.

Let Δ be a disc of $\mathbb{P}$ and define $ai(\Delta)$ to be the number of arcs incident with Δ which do not intersect the inner or outer boundary of $\mathbb{P}$. We say that Δ is a *simple boundary disc* in amended, annular picture $\mathbb{P}$, if exactly one arc from Δ intersects $\partial\mathbb{P}$.

Theorem 3.3.2 *Suppose $\widehat{\mathcal{P}}$ satisfies the Ω-Condition. Let Δ be a simple boundary disc in an amended, annular picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$, which has its arc intersecting $\partial\mathbb{P}$ labelled by W. Then either*

(i) $W \equiv R_\epsilon$, *where* $\epsilon = \pm 1$ *and* $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$, *or*

(ii) $ai(\Delta) \geq 4$.

Proof. This is proved in exactly the same way as Theorem 2.5.6, replacing the quantity $i(\ )$ by $ai(\ )$. □

In order to prove Theorem 3.2.1, we must study minimal, amended, annular pictures over $\widehat{\mathcal{P}}$, where $\mathcal{P} \in K_3^3 \cup K_4^2$. Annular pictures which have at least one of the following properties, are easily dealt with.

(I) An arc which intersects no discs, or

(II) A disc Δ which is labelled by $R_\epsilon R_{-\epsilon}^{-1}$ ($\epsilon = \pm 1, R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$), and has an arc labelled by R_ϵ which intersects the outer boundary, or intersects the inner boundary.

Note that by condition $D(2)$ every relator in $\widehat{\mathcal{P}}$ is labelled by a cyclically reduced word, and so every amended, annular picture with one disc satisfies (II).

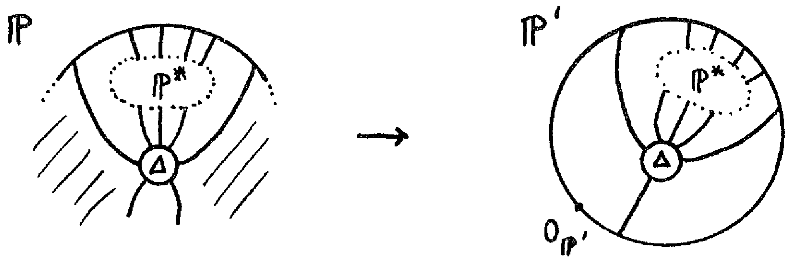
Let $\mathbb{P}$ be a minimal, amended, annular picture over $\widehat{\mathcal{P}}$, where $\mathcal{P} \in K_3^3 \cup K_4^2$. We will make use of that the fact that if $\mathcal{P} \in K_3^3 \cup K_4^2$, then $\widehat{\mathcal{P}}$ satisfies $C(6)$ and the Ω-Condition (Theorem 2.2.5).

Lemma 3.3.3 *Let $\widehat{\mathcal{P}}$ satisfy $C(6)$ and the Ω-Condition and suppose that $\mathbb{P}$ contains no discs as in (II) above. Then every disc in $\mathbb{P}$ has at most one arc which intersects the outer boundary of $\mathbb{P}$, and at most one arc which intersects the inner boundary of $\mathbb{P}$.*

Proof. Suppose $\mathbb{P}$ has a disc Δ which has two arcs intersecting the outer boundary of $\mathbb{P}$. Then within the area bounded by these two arcs, Δ, and the outer boundary of $\mathbb{P}$, there exists a picture $\mathbb{P}^*$ over $\widehat{\mathcal{P}}$ which contains at least one disc.

Let Δ and $\mathbb{P}^*$ be chosen so that $\mathbb{P}^*$ has the smallest number of discs possible. By this minimal choice all the discs in $\mathbb{P}^*$ have exactly one arc intersecting $\partial\mathbb{P}^*$.

Let $\mathbb{P}'$ be the picture over $\widehat{\mathcal{P}}$ consisting of disc Δ together with its incident arcs, and picture $\mathbb{P}^*$.



We will assume that $\mathbb{P}'$ has been amended in the usual way. By Proposition 1.8.2, $\mathbb{P}'$ is minimal. Also $\mathbb{P}'$ contains at least two discs and none of the following:

(a) floating semicircles, or

(b) simple boundary discs which are incident with just one arc.

Hence by Theorem 2.5.5 there exist at least two simple boundary discs Δ_j with $1 \leq i(\Delta_j) \leq 3$ $(j = 1, 2)$. Now Δ could be one of the Δ_j's. However each of the discs in $\mathbb{P}^*$ have at most one arc intersecting the boundary of $\mathbb{P}'$ (since they have at most one arc intersecting the boundary of $\mathbb{P}$) and this arc cannot be one side of a disc. Therefore by Theorem 2.5.6, $i(\Delta_k) \geq 4$ for each of the simple boundary discs Δ_k in $\mathbb{P}'$, and we have a contradiction.

A symmetrical argument holds if $\mathbb{P}$ has a disc which has two arcs intersecting the inner boundary of $\mathbb{P}$. $\square$

Now suppose that $\mathbb{P}$ satisfies neither (I) nor (II) above. Note in particular that $\mathbb{P}$ must contain at least two discs (since (II) is not satisfied). Then $\mathbb{P}$ gives rise to a graph T on the sphere S^2, with two distinguished vertices and at least two other vertices. We simply contract outer and inner boundaries to points (the *distinguished vertices* v_0 and v_1 of T, respectively), contract discs to points (creating the remaining vertices of the graph), and consider arcs as unlabelled edges.

Recall from §3.3 that we can assume that U and V are non-empty words. Therefore v_0 and v_1 have at least one incident edge each. Furthermore, we can also assume that $\mathbb{P}$ was spherically connected. In view of these observations T is a connected graph and is therefore a tessellation of the sphere.

In view of Lemma 3.3.3, the vertices of T can be classified into one of the following types:

(i) v_0;

(ii) v_1;

(iii) Those which have an incident edge intersecting v_0 but not v_1;

(iv) Those which have an incident edge intersecting v_1 but not v_0;

(v) Those which have an incident edge intersecting v_0 and an incident edge intersecting v_1;

(vi) Those which have no incident edges intersecting either v_0 or v_1.

The closure of the connected components of $S^2 - T$ are called the *areas* of T. An *interior area* is an area which does not have v_0 or v_1 as one of the vertices on its boundary. All areas which are not interior areas are called *distinguished areas.* Finally, we define the *valency* of a vertex v to be the number of edges of T which intersect with v (where an edge is counted twice if it has both endpoints equal to v).

3.4 Tessellations of the sphere with two distinguished vertices

Consider a tessellation T of the sphere with two distinguished vertices, as described in the previous section. Note that T has at least two vertices other than v_0 and v_1, as well as the following features:

(A) Every vertex has at most one incident edge which intersects with v_0 and at most one incident edge which intersects with v_1[2];

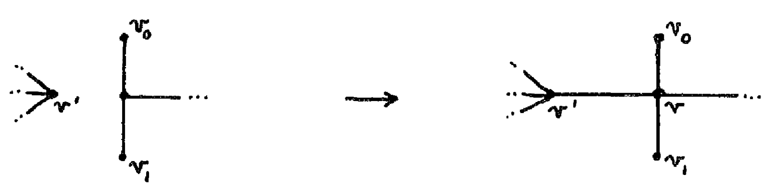
(B) Every vertex of type (iii) or (iv) has valence at least five, so that in particular every such vertex has at least four edges which do not intersect with v_0 or v_1, respectively;[3]

(C) Every vertex of type (vi) has valence at least six;

(D) Every interior area is bounded by at least three edges of the tessellation.

Let M, N, L and I be the number of vertices in T of types (iii), (iv), (v) and (vi), respectively. Also let ℓ be the average valence of vertices of type (v).

In the case where $L \neq 0$ we amend T in the following way. If T has a vertex v of type (v) with valency three, then we add an edge to T with initial vertex v and terminal vertex v', where v' is some vertex of T other than v, v_0 or v_1, in a neighbourhood of v, as shown below.



Hence we have that:

(E) If $L \neq 0$ then $\ell \geq 4$.

[2]This follows from Lemma 3.3.3.

[3]This follows from Theorem 3.3.2, using the fact that the picture from which T arises does not satisfy (II).

When $L = 0$ we define ℓ to be 4, then in both cases ℓL is equal to the sum of the valencies of vertices of type (v).

For clarity in figures we will draw T with v_0 expanded to a disc so that T can be visualised as a tessellation of the disc with centre v_1.

Proposition 3.4.1 *Tessellation T has the following properties:*

- *every vertex of type (iii) or (iv) has valency exactly five;*
- *every vertex of type (vi) has valency exactly six;*
- *every area of T is bounded by exactly three edges;*
- *either $L = 0$, or if $L \neq 0$ then every vertex of type (v) has valency exactly four.*

Proof. Let E be the number of edges of T, V the number of vertices and Φ the number of areas. Note that $V = M + N + L + I + 2$. We will also make use of the fact that the Euler Characteristic of the sphere is two, so that $\Phi - E + V = 2$.

Summing the valencies of each vertex (including v_0 and v_1), each edge is counted twice. Therefore

$$2E \geq (M + L) + (N + L) + 5(M + N) + \ell L + 6I.$$

Rearranging gives

$$0 \geq -2E + 6(M + N) + (\ell + 2)L + 6I. \tag{3.1}$$

Also each edge is incident with two areas and each area is bounded by at least three edges (for interior areas this is property (D); for distinguished areas this follows from properties (A) and (B)). Thus $2E \geq 3\Phi$, or

$$0 \geq \Phi - \frac{2}{3}E \tag{3.2}$$

Now from (3.1):

$$\begin{aligned} 0 &\geq V - \frac{E}{3} + (M + N) + (\frac{\ell + 2}{6})L + I - V \\ \Rightarrow 0 &\geq V - \frac{E}{3} + (M + N) + (\frac{\ell + 2}{6})L + I - M - N - L - I - 2 \\ \Rightarrow 0 &\geq V - \frac{E}{3} + (\frac{\ell - 4}{6})L - 2 \end{aligned}$$

Adding (3.2) gives:

$$\begin{aligned} 0 &\geq \Phi - E + V + (\frac{\ell - 4}{6})L - 2 \\ \Rightarrow 0 &\geq (\frac{\ell - 4}{6})L \end{aligned}$$

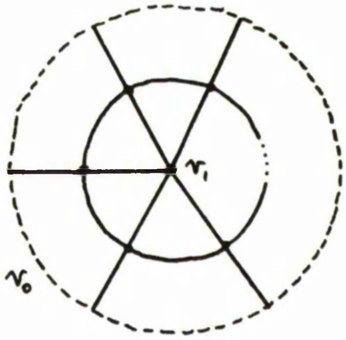
Now $\frac{\ell-4}{6}$ and L are both non-negative quantities. Hence we must actually have that

$$0 = (\frac{\ell - 4}{6})L$$

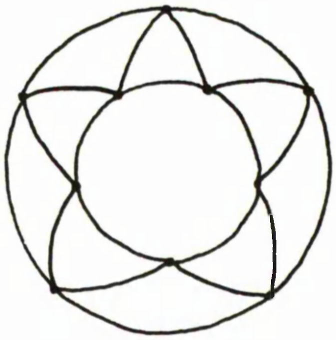
and thus either $L = 0$ or $\ell = 4$. Furthermore, all the inequalities in the above manipulation must be equalities. The first two properties in the Proposition follow from equation (3.1), while the third follows from (3.2). Recall that if $L \neq 0$ then $\ell \geq 4$. Hence if $\ell = 4$ then every vertex of type (v) must have valence exactly four. □

Our goal in the next two propositions is to describe the geometry of the tessellation T in the cases where $L \neq 0$ (Proposition 3.4.2) and $L = 0$ (Proposition 3.4.3).

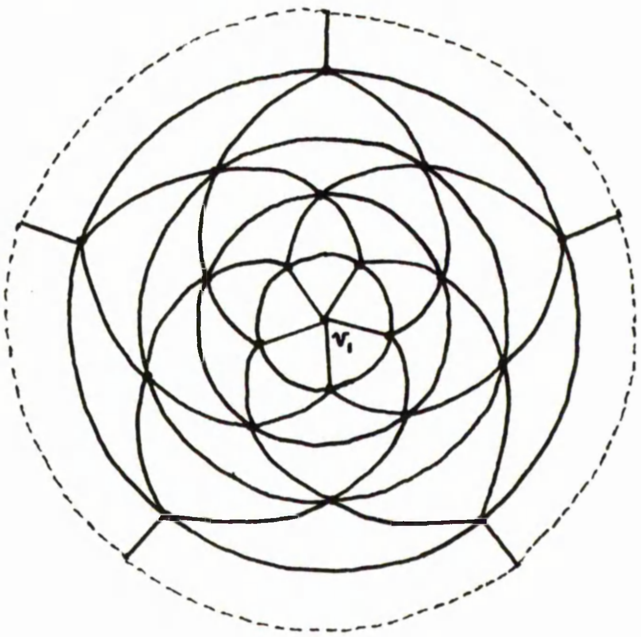
If $L \neq 0$ then we shall see that every vertex of T, except v_0 and v_1, is of type (v) so that every area of T is a distinguished area and has exactly three sides, as shown below:



In the case where $L = 0$ we shall see that T consists of a sequence of layers (where each layer has the same number of areas) of the form:

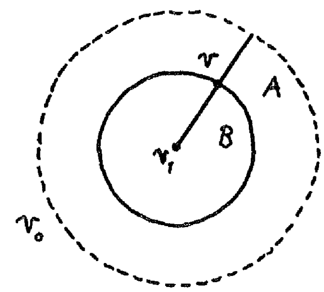


fitted together as shown below:



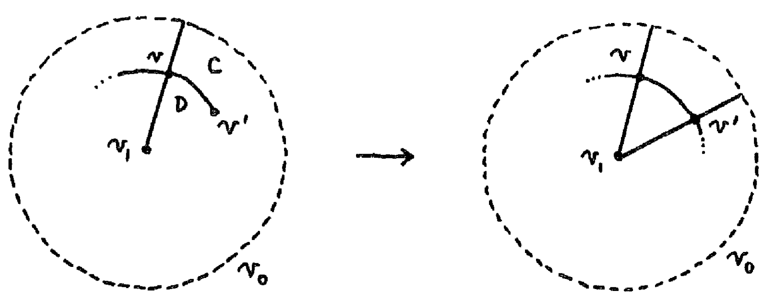
Proposition 3.4.2 *If $L \neq 0$ then all the vertices of T, except v_0 and v_1, are of type (v) and every area of T is a distinguished area with exactly three sides.*

Proof. Suppose that T has a vertex v of type (v). We first show that v must be connected by an edge of T to another vertex in T. Consider the following situation.



Since areas A and B (at this point of the argument) have three sides, there cannot exist any other vertices of T in areas A and B. However T has two vertices other than v_0 and v_1, and so we conclude that this situation is not possible.

Hence we can assume that v is connected to some other vertex v' (distinct from v_0 and v_1).



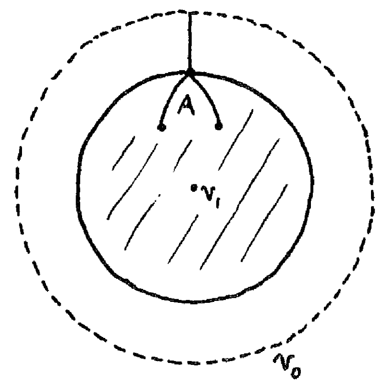
Since areas C and D have exactly three sides, it is clear that v' is forced to be a vertex of type (v), as shown in the second Figure above. Now either v' is connected to v by another edge of T, or the above argument can be repeated again, and so on. Hence T has the required form. □

Proposition 3.4.3 *If $L = 0$ then T has vertices of type (vi) with valence exactly six, and vertices of type (iii) and (iv) with valency exactly five.*

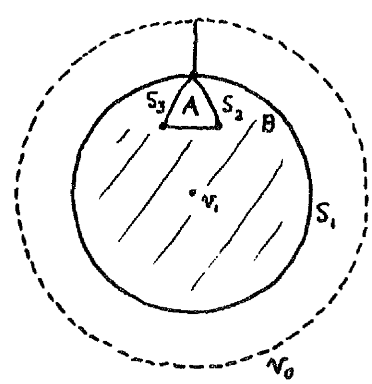
Proof. Suppose that T has no vertices of type (v). We will make use of the first three properties of T mentioned in Proposition 3.4.1.

Suppose T has a vertex of type (iii). Recall from Proposition 3.4.1 that all vertices of type (iii) have valency exactly five. We first show that it is not possible for T to have a single vertex of type (iii).

Suppose that T had a single vertex of type (iii). Since the distinguished area must be bounded by three sides of T we must have:

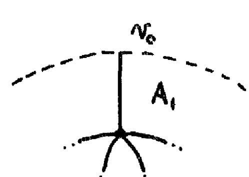


Now area A must be bounded by three sides of T. We must therefore have:

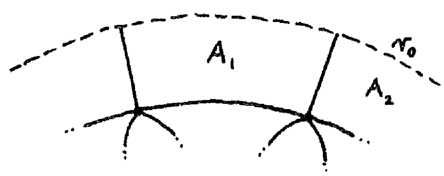


Now consider area B. At this point in the construction, it is bounded by three sides of T, namely s_1, s_2 and s_3. It is clear that B must be bounded by more than three sides, which gives a contradiction. We conclude that T must have at least two vertices of type (iii).

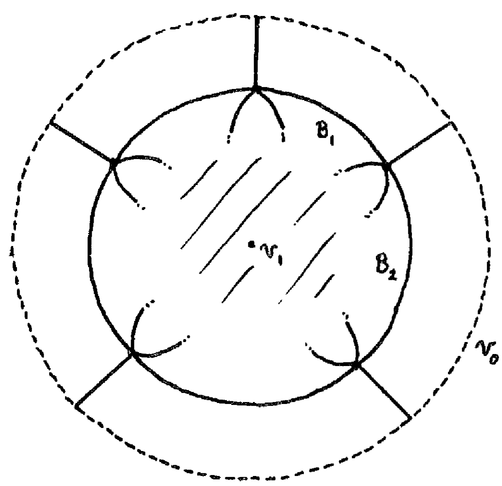
Consider any vertex of type (iii) in T.



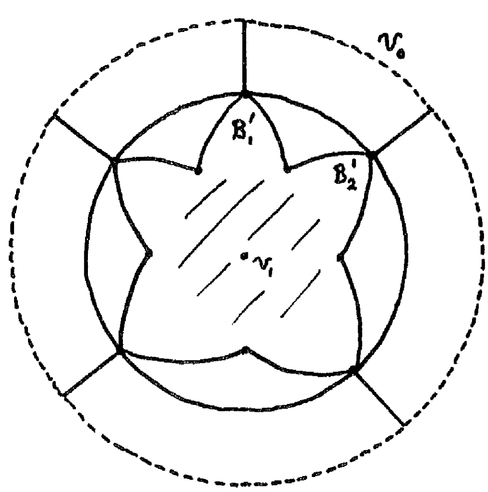
Now distinguished area A_1 must be bounded by three sides of T. Hence there must exist another vertex of type (iii) to the immediate right.



Applying the same argument to A_2, and so on, we conclude that T must take the form[4]:

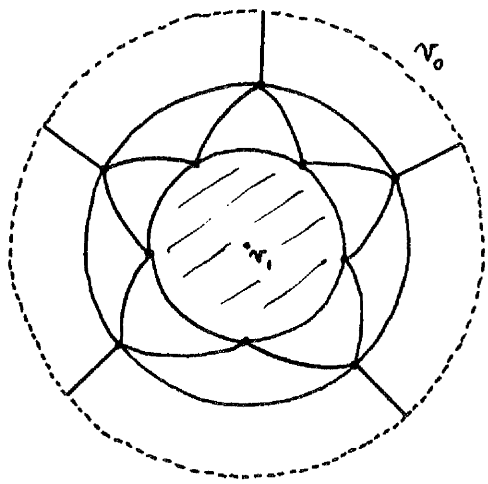


Now areas $B_1, B_2, \ldots$ must be bounded by three sides of T giving:

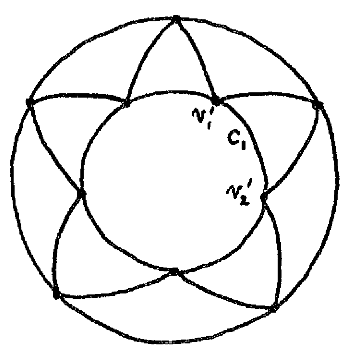


[4]The figures in this proof show the case where T has five vertices of type (iii).

Similarly $B_1', B_2', \ldots$ must be bounded by three sides of T:

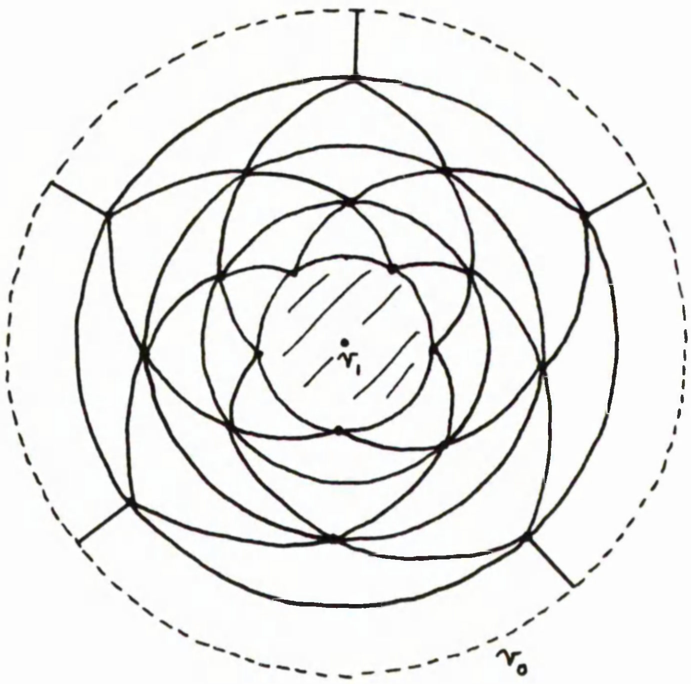


Observe that we have obtained a 'layer' of the form:



If all the vertices $v_1', v_2', \ldots$ are of type (vi), then using a similar argument to the above we can deduce that there is another layer, of the form shown, lying inside the previous layer.

We can repeat this argument until we obtain a layer where one of $v_1', v_2', \ldots$ is a vertex of type (iv). We illustrate below the case where T has three such layers:



Suppose that v_1' is of type (iv). Area C_1 must be bounded by at least three sides of T and so we deduce that v_2' must also be a vertex of type (iv). Applying this argument again we deduce that v_3' must be a vertex of type (iv), and so on. Hence T has the required form.

A symmetrical argument holds in the case where T is assumed to have at least one vertex of type (iv). Since T must have at least one vertex of type (iii) or (iv), the result is proved. □

Finally, we remark that M.S. El-Mosalamy and S.J. Pride [18] have used similar ideas involving tessellations of the sphere, to give simple solutions to the word and conjugacy problems for $T(6)$-groups.

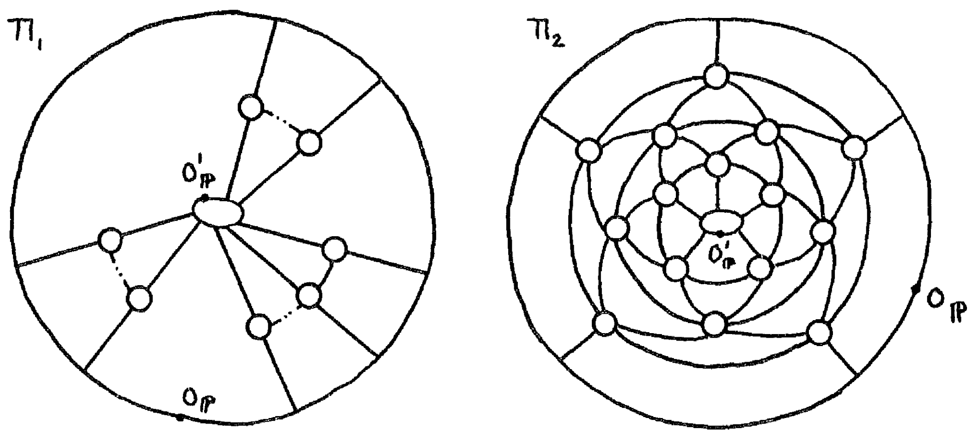
3.5 Proof of Theorem 3.2.1

We now translate the results of Propositions 3.4.2 and 3.4.3 into pictures.

Theorem 3.5.1 *Let $\mathcal{P} \in K_3^3 \cup K_4^2$ and suppose $\mathbb{P}$ is a minimal, amended, annular picture over $\widehat{\mathcal{P}}$ which does not satisfy properties (I) or (II) described in section 3.3. Then either*

(a) Every disc of $\mathbb{P}$ has an edge intersecting the outer boundary, an edge intersecting the inner boundary, and at least one other edge which intersects another disc (as in Π_1 below), or

(b) $\mathbb{P}$ has interior discs, each with exactly six incident arcs, and simple boundary discs with exactly five incident arcs (as in Π_2 below).



Proof. Let $\mathbb{P}$ be an annular picture as descibed. Then $\mathbb{P}$ gives rise to a tessellation T of the sphere with two distinguished vertices, as descibed in section 3.3. We amend vertices of type (v) (as previously described) so that T satisfies condition (E). Now T must have one of the forms shown in Propositions 3.4.2 and 3.4.3. Undoing the amendments made to T (so that we could assume condition (E) held), and translating this information back into annular picture form, we see that $\mathbb{P}$ takes one of the stated forms. Note that if $\mathbb{P}$ has form Π_1 then there exist no discs in $\mathbb{P}$ which only have arcs intersecting the boundary, otherwise condition (II) would be satisfied for $\mathbb{P}$. □

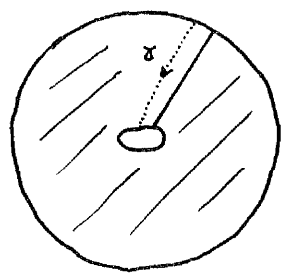
Proof of Theorem 3.2.1: In order to prove the result, we will prove that if $\mathcal{P} \in K_3^3 \cup K_4^2$ and U, V are distinct, non-empty, positive words on $\boldsymbol{x}$ such that $\pi(U) \approx_G \pi(V)$, then $U \approx_S^{(se)} V$.

As usual with regard to π (see §2.1), we will abuse notation in a natural way and assume that $\pi(U) \equiv U$, $\pi(V) \equiv V$. Since $U \approx_G V$, by Theorem 1.8.1 there exists an annular picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$. We can assume that $\mathbb{P}$ is minimal and has been amended. The result will be proved by induction on $conjdeg(U, V)$.

If $conjdeg(U, V) = 0$ then U is a cyclic permuation of V and so $U \approx_S^{(e)} V$.

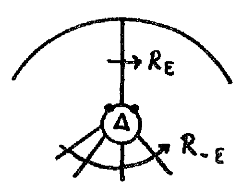
Let $conjdeg(U, V) \geq 1$.

Suppose $\mathbb{P}$ satisfies condition (I). Then it is possible to find a transverse path γ in $\mathbb{P}$ from the outer boundary to the inner boundary, which does not intersect any arcs or discs.



Suppose that U^* (V^*) is the word obtained by reading clockwise round the outer (respectively, inner) boundary from the initial (terminal) point of γ. Then by Proposition 1.8.4, cutting $\mathbb{P}$ along γ gives a minimal picture over $\widehat{\mathcal{P}}$ with boundary label U^*V^{*-1}. Thus by Corollary 1.6.5, $U^* =_G V^*$. However S is embeddable in G so that $U^* =_S V^*$. Since $U \approx_S^{(e)} U^*$ and $V \approx_S^{(e)} V^*$, we conclude that $U \approx_S^{(se)} V$.

Suppose $\mathbb{P}$ satisfies condition (II).

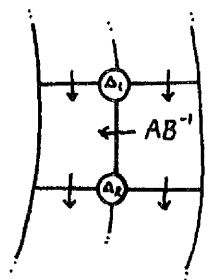


We deal with the case where Δ has the arc in question intersecting the outer boundary (the inner boundary case follows by a similar argument). Suppose that $U \equiv U_1 R_\epsilon U_2$ (U_1, U_2 positive words on $\boldsymbol{x}$, $\epsilon = \pm 1$, $R_{+1} R_{-1}^{-1} \in \hat{\boldsymbol{r}}$). Cutting this disc out of $\mathbb{P}$ we obtain a new minimal annular picture with outer boundary label U', where $U' \equiv U_1 R_{-\epsilon} U_2$ and $U =_S U'$. By Proposition 1.8.2, this annular picture is minimal. Furthermore, it contains less discs than $\mathbb{P}$ and so $conjdeg(U', V) < conjdeg(U, V)$. Thus, by the inductive hypothesis $U' \approx_S^{(se)} V$ and consequently $U \approx_S^{(se)} V$.

If $\mathbb{P}$ does not satisfy (I) or (II) then $conjdeg(U, V) \geq 2$, and we appeal to Theorem 3.5.1. Recall that $\mathbb{P}$ must take one of the forms Π_1 or Π_2.

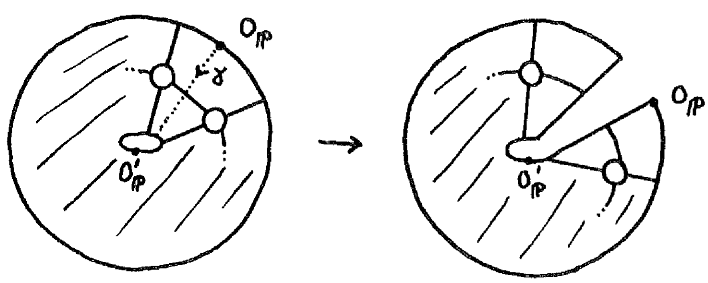
Suppose that $\mathbb{P}$ takes the form Π_1.

We note that if there were an arc in $\mathbb{P}$ between two discs Δ_1 and Δ_2, labelled by a word of the form AB^{-1} (A, B non-empty positive words on $\boldsymbol{x}$)



then it would not be possible for Δ_2 to be a two-sided disc. Similarly, if the arc were labelled by a word of the from $A^{-1}B$ then Δ_1 could not be a two-sided disc. Hence any arc between discs in $\mathbb{P}$ must be labelled by a positive or negative word.

Let γ be a transverse path from $O_{\mathbb{P}}$ to a point on the inner boundary of $\mathbb{P}$, which crosses at most one arc joining a pair of discs. Let W be the word read while travelling along γ. We cut along γ obtaining (by Proposition 1.8.4) a minimal picture over $\widehat{\mathcal{P}}$ with boundary label $UWV^{*-1}W^{-1}$, where V^* is some cyclic permutation of V.



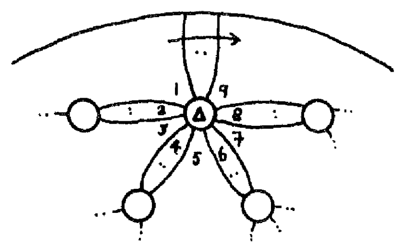
If W is a positive word on $\boldsymbol{x}$ then we deduce that $UW =_G WV^*$. However S is embeddable in G, so that $UW =_S WV^*$. Therefore $U \approx_S^{(e)} V^*$ and so $U \approx_S^{(se)} V$.

If W is a negative word on $\boldsymbol{x}$ then we deduce that $W^{-1}U =_G V^*W^{-1}$, and therefore that $W^{-1}U =_S V^*W^{-1}$. Thus $V^* \approx_S^{(e)} U$ and so $V \approx_S^{(se)} U$.

Now suppose that $\mathbb{P}$ takes form Π_2.

We will be concerned with the simple boundary discs of $\mathbb{P}$ which have an arc intersecting the outer boundary of $\mathbb{P}$. We call these discs the *outer layer* of $\mathbb{P}$.

Consider any disc Δ in the outer layer. For the moment we unamend the arcs of $\mathbb{P}$ so that each arc of $\mathbb{P}$ is labelled by an element of $\boldsymbol{x}$. Note that there are nine possible positions that the basepoints of Δ could occupy.



However, unless

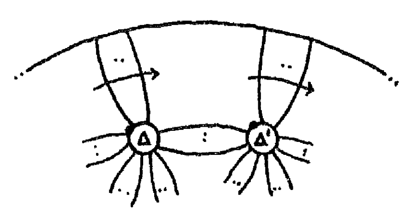
(i) Δ has a basepoint in position 1, or

(ii) Δ has a basepoint in position 9, or

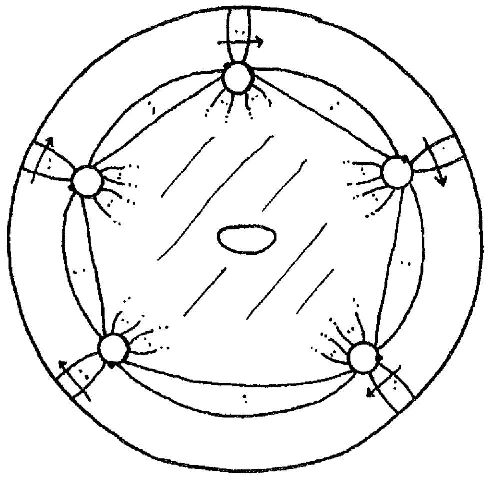
(iii) Δ has basepoints in positions 2 and 8

then it is possible to find a subword of an element of $\boldsymbol{r}^*$ of the form $X^{-1}R_\epsilon Z^{-1}$ (X and Z positive words on $\boldsymbol{x}$ with at least one of X, Z non-empty, $\epsilon = \pm 1$, $R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$) which is a product of one, two or three pieces. This is not possible since $\widehat{\mathcal{P}}$ satisfies the Ω-Condition (by Theorem 2.2.5).

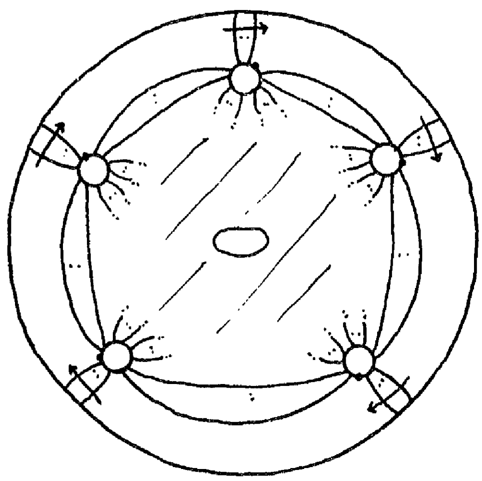
Suppose Δ has a basepoint in position 1 and that Δ' is the disc adjacent on the right to Δ.



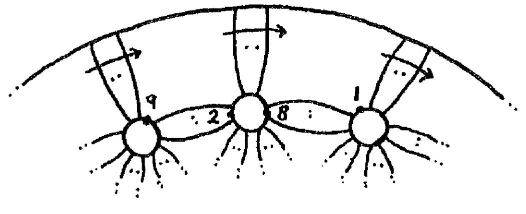
Since $\mathbb{P}$ does not satisfy condition (II), Δ cannot have its other basepoint in position 9 and so Δ' is forced to have a basepoint in position 1. Continuing this argument we see that $\mathbb{P}$ must take the form:



On the other hand if Δ has a basepoint in position 9, then by a similar argument we see that $\mathbb{P}$ must take the form:



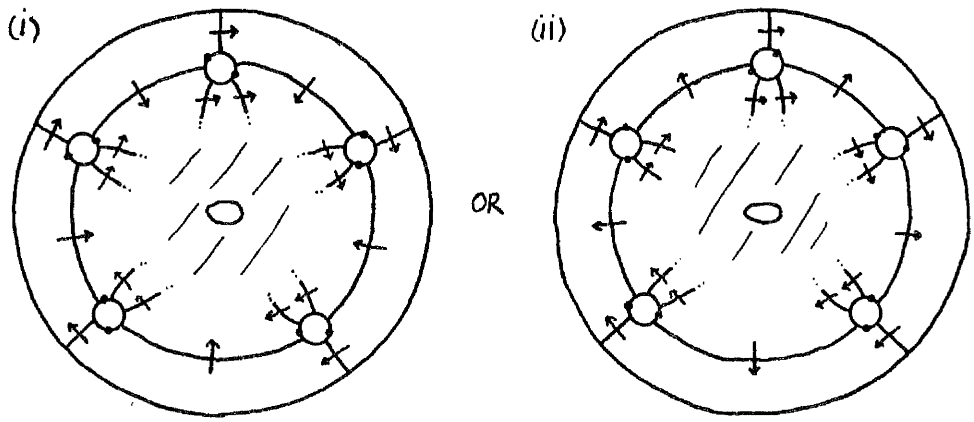
Suppose there exists a disc in the outer layer with basepoints in positions 2 and 8. Then the discs to the immediate left and right have basepoints in positions 9 and 1 respectively.



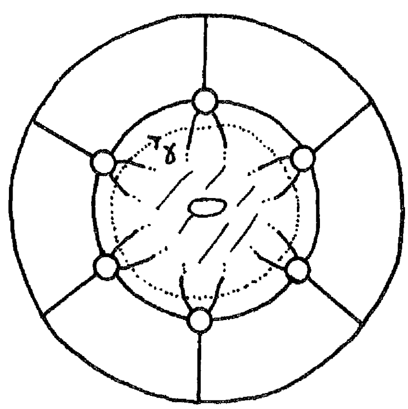
In view of the previous paragraphs we conclude that case (iii) is not possible.

We now split our argument into two cases:

<u>*Case* $\mathcal{P} \in K_3^3$:</u> By Lemma 2.2.3, $\widehat{\mathcal{P}}$ has homogeneous pieces. Hence the arcs in $\mathbb{P}$ are labelled by S-pieces. Since $\mathcal{P}$ satisfies $C_S(3)$, each side of a relation in $\boldsymbol{r}$ is the product of at least three pieces. Therefore $\mathbb{P}$ (in its amended state) must take one of the following forms:



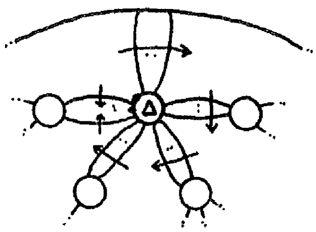
Consider the annular subpicture $\mathbb{P}'$ enclosed by transverse path γ in $\mathbb{P}$. Let W be the word read while travelling around γ from some point on the path.



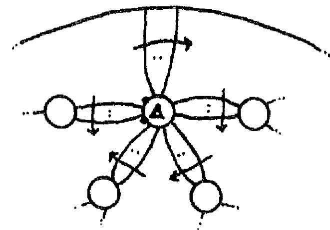
Now $\mathbb{P}'$ is minimal by Proposition 1.8.2, and $conjdeg(W, V) < conjdeg(U, V)$. Thus by the inductive hypothesis $W \approx_S^{(se)} V$. Also $conjdeg(U, W) < conjdeg(U, V)$ so that $U \approx_S^{(se)} W$. Consequently $U \approx_S^{(se)} V$.

<u>*Case* $\mathcal{P} \in K_4^2$:</u> Suppose that all the discs in the outer layer of $\mathbb{P}$ have basepoints in position 1. Let Δ be any such disc in the outer layer and consider where the second basepoint of Δ could be situated. Our claim is that it must be situated in position 8.

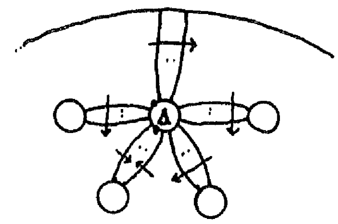
Suppose that the second basepoint is situated in one of positions 2–7 as shown below (respectively).



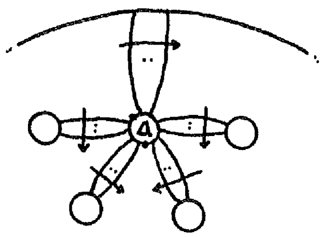
2



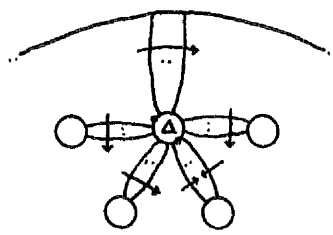
3



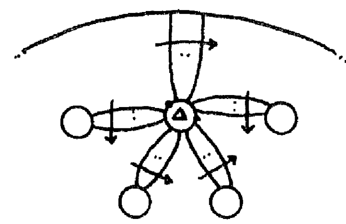
4



5



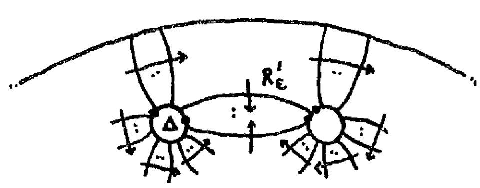
6



7

Recall that if XY^{-1} is a piece (X, Y positive words on $\boldsymbol{x}$) then X and Y are S-pieces (Lemma 2.2.1). Suppose Δ is labelled by $R_\epsilon R_{-\epsilon}^{-1}$ ($\epsilon = \pm 1, R : R_{+1}R_{-1}^{-1} \in \hat{\boldsymbol{r}}$). Note that each side of R is a product of at least four S-pieces. Consequently, cases (2) and (3) cannot occur for one side of R would be a single piece; cases (4) and (5) cannot occur for one side of R would be the product of two S-pieces; cases (6) and (7) cannot occur for one side of R would be the product of three S-pieces. Thus our claim is substantiated.

However if Δ has basepoints in positions 1 and 8, then the disc to the immediate right in the outer layer is labelled by an element $R'_\epsilon R'^{-1}_{-\epsilon} \in \hat{\boldsymbol{r}}$ ($\epsilon = \pm 1$) where R'_ϵ is a single S-piece.



We conclude that this case is not possible.

A symmetrical argument holds if the discs in the outer layer all have basepoints in position 9. Our final conclusion is that it is not possible to have $\mathcal{P} \in K_4^2$ and a picture $\mathbb{P}$ over $\widehat{\mathcal{P}}$ of form Π_2. $\square$

We have seen that if monoid presentation $\mathcal{P}$ satisfies Kashintsev's embeddability conditions then π reflects sequential elementary conjugacy. Also, recall that if $\mathcal{P}$ is cycle free (Adjan's embeddability condition) then π reflects conjugacy. Further research could be aimed at finding further 'classes' of injective natural maps which reflect sequential elementary conjugacy or conjugacy. In particular it is hoped that some of the ideas developed in this chapter can be used in studying conjugacy for monoid presentations in K_3^2, for which Guba, in the semigroup case, has shown that the natural maps are always injective.

Chapter 4

Relative monoid presentations I

In this chapter we introduce the concepts of *left* and *right graphs* for *relative monoid presentations.* They will be used to generalise results obtained by Adjan for semigroup presentations [1, 2]. Our tool in studying relative monoid presentations will be the concept of a *mixed monoid presentation*, and we will in passing obtain an asphericity result for mixed monoid presentations.

4.1 Some definitions

A *relative monoid presentation* $\mathcal{R}$ is a triple $[H, \boldsymbol{t}; \boldsymbol{r}]$ where H is a group, $\boldsymbol{t}$ is a set and $\boldsymbol{r}$ is a set of non-empty, distinct pairs of non-trivial $\boldsymbol{t}$-positive words (R_{+1}, R_{-1}) on $H \cup \boldsymbol{t}$. As usual we dispense with coordinate notation and write $R_{+1} = R_{-1}$. In some situations we will be dealing with lists $R_1, \ldots, R_k$ of elements of $\boldsymbol{r}$ and so we write each element as $R_i : R_{+1,i} = R_{-1,i}$ $(1 \leq i \leq k)$. Recall that we write $R_i^{-1} : R_{-1,i} = R_{+1,i}$.

Let W be a $\boldsymbol{t}$-positive word on $H \cup \boldsymbol{t}$. Consider the following elementary operations on the set of all $\boldsymbol{t}$-positive words on $H \cup \boldsymbol{t}$.

(A) If W contains a subword h_1h_2 then replace it by h, where $h = h_1h_2$ in H $(h, h_1, h_2 \in H)$;

$(A)^{-1}$ If W contains a subword h then replace it by h_1h_2, where $h_1h_2 = h$ in H $(h, h_1, h_2 \in H)$;

(B) Delete 1, the identity in H, from W;

$(B)^{-1}$ Insert 1 in W at any position;

(C) If W contains a subword R_ϵ $(\epsilon = \pm 1,\ R_{+1} = R_{-1} \in \boldsymbol{r})$, then replace it by $R_{-\epsilon}$.

Two $\boldsymbol{t}$-positive words W_1, W_2 on $H \cup \boldsymbol{t}$ are *equivalent (relative to $\mathcal{R}$)*, written $W_1 \sim_{\mathcal{R}} W_2$, if there is a chain of elementary operations leading from W_1 to W_2. This is obviously an equivalence relation on the set of all $\boldsymbol{t}$-positive words on $H \cup \boldsymbol{t}$, and we denote the equivalence class containing W by $[W]_{\mathcal{R}}$. The *monoid* $S(\mathcal{R})$ *defined by* $\mathcal{R}$ is the set of all equivalence classes under multiplication defined by $[W_1]_{\mathcal{R}}.[W_2]_{\mathcal{R}} = [W_1W_2]_{\mathcal{R}}$ (this multiplication is easily checked to be well defined by an argument similar to that used in the proof of Lemma 1.2.1).

We now introduce two labelled, oriented graphs which we associate with $\mathcal{R}$. These are generalisations of Adjan's left and right graphs, which we met in Chapters 2 and 3.

The *left graph of* $\mathcal{R}$ ($LG(\mathcal{R})$) and *right graph of* $\mathcal{R}$ ($RG(\mathcal{R})$) have vertex set $\boldsymbol{t}$, and edge set consisting of an edge pair $\{e_R, e_R^{-1}\}$ for each $R \in \boldsymbol{r}$. The graphs are oriented by choosing $E^+ = \{e_R : R \in \boldsymbol{r}\}$. The initial, terminal and labelling functions ι, τ and ϕ, are defined as follows. Let

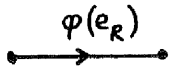
$$R : h_1t_1 \dots h_{r-1}t_{r-1}h_r = h'_1t'_1 \dots h'_{s-1}t'_{s-1}h'_s \in \boldsymbol{r}$$

where $h_i, h'_j \in H$, $t_i, t'_j \in \boldsymbol{t}$, $1 \le i \le r$, $1 \le j \le s$.

$LG(\mathcal{R})$: $\iota(e_R) = t_1$, $\tau(e_R) = t'_1$ and $\phi(e_R)$ is the element $h_1^{-1}h'_1$ of H, while $\phi(e_R^{-1})$ is the element $(h_1^{-1}h'_1)^{-1}$ of H.

$RG(\mathcal{R})$: $\iota(e_R) = t_{r-1}$, $\tau(e_R) = t'_{s-1}$ and $\phi(e_R)$ is the element $h_rh_s'^{-1}$ of H, while $\phi(e_R^{-1})$ is the element $(h_rh_s'^{-1})^{-1}$ of H.

When drawing these oriented graphs we will write the label of an edge adjacent to the edge.

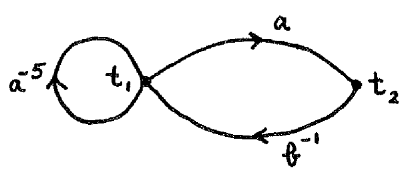


If $e_{R_1}^{\epsilon_1}e_{R_2}^{\epsilon_2}\dots e_{R_n}^{\epsilon_n}$ is a path in $LG(\mathcal{R})$ or $RG(\mathcal{R})$ ($R_i \in \boldsymbol{r}$, $\epsilon_i = \pm 1$, $1 \leq i \leq n$) then the path is labelled by the element $\phi(e_{R_1}^{\epsilon_1})\phi(e_{R_2}^{\epsilon_2})\dots\phi(e_{R_n}^{\epsilon_n})$ of H. We say that $LG(\mathcal{R})$ ($RG(\mathcal{R})$) is *cycle free* if there exist no non-empty, closed, reduced paths in $LG(\mathcal{R})$ ($RG(\mathcal{R})$ respectively) which are labelled by elements equal to the identity in H. Furthermore, $\mathcal{R}$ is *cycle free* if both $LG(\mathcal{R})$ and $RG(\mathcal{R})$ are cycle free.

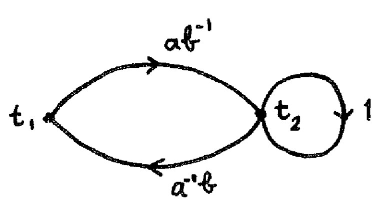
Example 1: Let H be the free abelian group of rank two, generated by a and b. Suppose

$$\mathcal{R} = [H, t_1, t_2; a^{-1}t_1a = t_2b, bt_2 = t_1b^{-2}t_1b^{-1}a, a^2t_1b^{-3}t_2ab = a^{-3}t_1t_2ba].$$

Then $LG(\mathcal{R})$ is:



while $RG(\mathcal{R})$ is:



Now while $LG(\mathcal{R})$ is cycle free, $RG(\mathcal{R})$ is not cycle free. Not only does $RG(\mathcal{R})$ have a cycle of length one which is labelled by the identity in H, but it also has a cycle of length two labelled by an element of H which equals the identity in H. Hence $\mathcal{R}$ is not cycle free.

Recall that a monoid S is *left cancellative* if for any $a, b, c \in S$, $ca = cb$ implies $a = b$; *right cancellative* if $ac = bc$ implies $a = b$; *cancellative* if it is both left and right cancellative. Our main aims in this chapter are the proofs of the following results:

- If $LG(\mathcal{R})$ is cycle free then $S(\mathcal{R})$ is left cancellative.
- If $RG(\mathcal{R})$ is cycle free then $S(\mathcal{R})$ is right cancellative.

4.2 Mixed monoid presentations

A *mixed monoid presentation* $\mathcal{M} = [\boldsymbol{a}; \boldsymbol{t} | \boldsymbol{s}; \tilde{\boldsymbol{r}}]$ consists of:

(i) two sets $\boldsymbol{a}$ and $\boldsymbol{t}$;

(ii) a set $\boldsymbol{s}$ of non-empty, cyclically reduced words on $\boldsymbol{a}$;

(iii) a set $\tilde{\boldsymbol{r}}$ of non-empty, distinct pairs of non-trivial $\boldsymbol{t}$-positive words $\tilde{R} = (\tilde{R}_{+1}, \tilde{R}_{-1})$ on $\boldsymbol{a} \cup \boldsymbol{t}$. As usual we write $\tilde{R} : \tilde{R}_{+1} = \tilde{R}_{-1}$.

Let W be a $\boldsymbol{t}$-positive word on $\boldsymbol{a} \cup \boldsymbol{t}$. Consider the following elementary operations on the set of all $\boldsymbol{t}$-positive words on $\boldsymbol{a} \cup \boldsymbol{t}$.

(D)	If W contains a subword $a^{\epsilon}a^{-\epsilon}$ $(a \in \boldsymbol{a}, \epsilon = \pm 1)$ then delete it;
$(D)^{-1}$	Insert a word $a^{\epsilon}a^{-\epsilon}$ $(a \in \boldsymbol{a}, \epsilon = \pm 1)$ at any position in W;
(E)	If W contains a subword S^{ϵ} $(S \in \boldsymbol{s}, \epsilon = \pm 1)$ then delete it;
$(E)^{-1}$	Insert S^{ϵ} $(S \in \boldsymbol{s}, \epsilon = \pm 1)$ at any position in W;
(F)	If W contains a subword $\tilde{R}_{\epsilon}$ $(\epsilon = \pm 1,\ \tilde{R}_{+1} = \tilde{R}_{-1} \in \tilde{\boldsymbol{r}})$, then replace it by $\tilde{R}_{-\epsilon}$.

Two $\boldsymbol{t}$-positive words W_1, W_2 on $\boldsymbol{a} \cup \boldsymbol{t}$ are *equivalent (relative to $\mathcal{M}$)*, written $W_1 \sim_{\mathcal{M}} W_2$, if there is a chain of elementary operations leading from W_1 to W_2. This is clearly an equivalence relation on the set of all $\boldsymbol{t}$-positive words on $\boldsymbol{a} \cup \boldsymbol{t}$, and we denote the equivalence class containing W by $[W]_{\mathcal{M}}$. The *monoid $S(\mathcal{M})$ defined by $\mathcal{M}$* is the set of all equivalence classes under multiplication defined by $[W_1]_{\mathcal{M}} . [W_2]_{\mathcal{M}} = [W_1 W_2]_{\mathcal{M}}$ (this multiplication is easily checked to be well defined by an argument similar to that used in the proof of Lemma 1.2.1).

Note. Mixed monoid presentation $\mathcal{M} = [\boldsymbol{a}; \boldsymbol{t} | \boldsymbol{s}; \tilde{\boldsymbol{r}}]$ is essentially shorthand notation for the ordinary monoid presentation:

$$[\boldsymbol{a}, \boldsymbol{a}^{-1}, \boldsymbol{t}; S = 1\,(S \in \boldsymbol{s}), \tilde{\boldsymbol{r}}, aa^{-1} = 1, a^{-1}a = 1\,(a \in \boldsymbol{a})]$$

which also defines $S(\mathcal{M})$.

Let H be the group defined by the group presentation $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$.

We now define the *left graph of* $\mathcal{M}$ ($LG(\mathcal{M})$) and the *right graph of* $\mathcal{M}$ ($RG(\mathcal{M})$). These oriented graphs have vertex set $\boldsymbol{t}$ and edge set consisting of an edge pair $\{e_{\tilde{R}}, e_{\tilde{R}}^{-1}\}$ for each $\tilde{R} \in \tilde{\boldsymbol{r}}$. They are oriented by choosing $E^+ = \{e_{\tilde{R}} : \tilde{R} \in \tilde{\boldsymbol{r}}\}$. The initial, terminal and labelling functions ι, τ and ϕ, are defined as follows. Let

$$\tilde{R} : \tilde{h}_1 t_1 \dots \tilde{h}_{r-1} t_{r-1} \tilde{h}_r = \tilde{h}'_1 t'_1 \dots \tilde{h}'_{s-1} t'_{s-1} \tilde{h}'_s \in \tilde{\boldsymbol{r}}$$

where $\tilde{h}_i, \tilde{h}'_j$ are words on $\boldsymbol{a}$, $t_i, t'_j \in \boldsymbol{t}$, $1 \le i \le r$, $1 \le j \le s$.

$LG(\mathcal{M})$: $\iota(e_{\tilde{R}}) = t_1$, $\tau(e_{\tilde{R}}) = t'_1$ and $\phi(e_{\tilde{R}})$ is the element of H represented by the word $\tilde{h}_1^{-1}\tilde{h}'_1$, while $\phi(e_{\tilde{R}}^{-1})$ is the element of H represented by the word $(\tilde{h}_1^{-1}\tilde{h}'_1)^{-1}$.

$RG(\mathcal{M})$: $\iota(e_{\tilde{R}}) = t_{r-1}$, $\tau(e_{\tilde{R}}) = t'_{s-1}$ and $\phi(e_{\tilde{R}})$ is the element of H represented by the word $\tilde{h}_r\tilde{h}'^{-1}_s$, while $\phi(e_{\tilde{R}}^{-1})$ is the element of H represented by the word $(\tilde{h}_r\tilde{h}'^{-1}_s)^{-1}$.

Once again when drawing these oriented graphs, we write the label associated with an edge adjacent to the edge.

Paths in these graphs take labels induced from the labels on the individual edges in the path, as for paths in $LG(\mathcal{R})$ or $RG(\mathcal{R})$ (see the previous section). We say that $LG(\mathcal{M})$ ($RG(\mathcal{M})$) is *cycle free* if there exist no closed, reduced paths in $LG(\mathcal{M})$ ($RG(\mathcal{M})$ respectively) which are labelled by words representing the identity in H. Also $\mathcal{M}$ is said to be *cycle free* if both $LG(\mathcal{M})$ and $RG(\mathcal{M})$ are cycle free.

4.3 Relating $\mathcal{R}$ and $\mathcal{M}$

Let $\mathcal{R} = [H, \boldsymbol{t}; \boldsymbol{r}]$ be a relative monoid presentation for $S(\mathcal{R})$. We will show how to construct a mixed monoid presentation for $S(\mathcal{R})$ when given a group presentation $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$, for group H.

Firstly one has an epimorphism θ mapping the free group on $\boldsymbol{a}$, $F(\boldsymbol{a})$, to H, where $\ker\theta$ is the normal closure of $\boldsymbol{s}$. For each element $h \in H$ choose a reduced word $\tilde{h}$ on $\boldsymbol{a}$ (the *lift of* h) such that $\theta(\tilde{h}) = h$. In particular we lift the identity element in H back to the empty word on $\boldsymbol{a}$. Now if $W \equiv h_1 t_1 \ldots h_{k-1} t_{k-1} h_k$ ($h_i \in H$, $t_i \in \boldsymbol{t}$, $1 \leq i \leq k$) is a $\boldsymbol{t}$-positive word on $H \cup \boldsymbol{t}$, we can lift it to the $\boldsymbol{t}$-positive word $\tilde{W} \equiv \tilde{h}_1 t_1 \tilde{h}_2 \ldots \tilde{h}_{k-1} t_{k-1} \tilde{h}_k$ on $\boldsymbol{a} \cup \boldsymbol{t}$ (the *lift of* W), where $\tilde{h}_i$ is the lift of h_i ($1 \leq i \leq k$).

Let $R : R_{+1} = R_{-1} \in \boldsymbol{r}$. Lifting the words on both sides of this relation gives a new relation $\tilde{R} : \tilde{R}_{+1} = \tilde{R}_{-1}$. We denote the set of all such relations by $\tilde{\boldsymbol{r}}$. Hence we have two sets $\boldsymbol{s}$ and $\tilde{\boldsymbol{r}}$, and therefore a mixed monoid presentation $\mathcal{M} = [\boldsymbol{a}; \boldsymbol{t} | \boldsymbol{s}; \tilde{\boldsymbol{r}}]$.

Lemma 4.3.1 *Let h_1, h_2 be distinct elements of H. Then*

$$\tilde{h}_1 \tilde{h}_2 \simeq \left(\prod_{i=1}^{k} A_i S_i^{\epsilon_i} A_i^{-1}\right) \widetilde{h_1 h_2}$$

where $S_i \in \boldsymbol{s}, \epsilon_i = \pm 1$ and the A_i's are words on $\boldsymbol{a}$.

Proof. Suppose that $h = h_1 h_2$ in H. Then $\theta(\tilde{h}) = h = h_1 h_2$ and so $\theta(\tilde{h}) = \theta(\tilde{h}_1)\theta(\tilde{h}_2)$. Therefore $\tilde{h}_1 \tilde{h}_2 \tilde{h}^{-1} \in \ker\theta$ and so using Proposition 1.5.1

$$\tilde{h}_1 \tilde{h}_2 \tilde{h}^{-1} \simeq \prod_{i=1}^{k} A_i S_i^{\epsilon_i} A_i^{-1}$$

for some $S_i \in \boldsymbol{s}, \epsilon_i = \pm 1$ and A_i's words on $\boldsymbol{a}$. □

Lemma 4.3.2 *Let h_1 and h_2 be distinct elements of H. Then*

$$[\widetilde{h_1 h_2}]_{\mathcal{M}} = [\tilde{h}_1 \tilde{h}_2]_{\mathcal{M}}.$$

Proof. Note that by Lemma 4.3.1

$$\tilde{h}_1 \tilde{h}_2 \simeq \left(\prod_{i=1}^{k} A_i S_i^{\epsilon_i} A_i^{-1}\right) \widetilde{h_1 h_2}$$

where $S_i \in \boldsymbol{s}, \epsilon_i = \pm 1$ and the A_i's are words on $\boldsymbol{a}$. Then

$$\begin{aligned}
[\widetilde{h_1 h_2}]_{\mathcal{M}} &= [(\prod_{i=1}^{k} A_i S_i^{\epsilon_i} A_i^{-1})\widetilde{h_1 h_2}]_{\mathcal{M}} \text{ using } (D)^{-1} \text{ and } (E)^{-1} \\
&= [\tilde{h}_1 \tilde{h}_2]_{\mathcal{M}} \text{ by the note and using } (D)^{\pm 1}.
\end{aligned}$$

□

Theorem 4.3.3 *$S(\mathcal{R}) \cong S(\mathcal{M})$ for any presentation $\mathcal{Q}$ of H.*

Proof. Using the lifting procedure induced by θ we define a map ψ from $S(\mathcal{R})$ to $S(\mathcal{M})$ by

$$[W]_{\mathcal{R}} \mapsto [\tilde{W}]_{\mathcal{M}}$$

where W is a $\boldsymbol{t}$-positive word on $H \cup \boldsymbol{t}$.

We begin by showing that this map is well defined.

Suppose that $[W_1]_{\mathcal{R}} = [W_2]_{\mathcal{R}}$ (W_1, W_2 $\boldsymbol{t}$-positive words on $H \cup \boldsymbol{t}$). Let

$$U \equiv m_1 t_1 \ldots m_{k-1} t_{k-1} m_k \quad \text{and} \quad V \equiv n_1 s_1 \ldots n_{l-1} s_{l-1} n_l$$

(m_i's and n_i's elements of H, t_i's and s_i's elements of $\boldsymbol{t}$). We can assume that W_1 and W_2 differ by one of the elementary operations $(A)^{\pm 1}, (B)^{\pm 1}$ or (C).

$(A^{\pm 1})$: Suppose $W_1 \equiv U h_1 h_2 V$ and $W_2 \equiv U h V$, where $h, h_1, h_2 \in H$ and $h_1 h_2 = h$ in H. Then

$$\begin{aligned}
\psi([W_1]_{\mathcal{R}}) &= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \widetilde{m_k h_1 h_2 n_1} s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \widetilde{h_1 h_2 n_1} s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \text{ by Lemma 4.3.2} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \widetilde{h_1 h_2} \tilde{n}_1 s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \text{ by Lemma 4.3.2} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \tilde{h} \tilde{n}_1 s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \text{ since } \tilde{h} = \widetilde{h_1 h_2} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \widetilde{h n_1} s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \text{ by Lemma 4.3.2} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \widetilde{m_k h n_1} s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \text{ by Lemma 4.3.2} \\
&= \psi([W_2]_{\mathcal{R}}).
\end{aligned}$$

$(B)^{\pm 1}$: Suppose $W_1 \equiv U1V$ and $W_2 \equiv UV$, where 1 is the identity in H. Then

$$\begin{aligned}
\psi([W_1]_{\mathcal{R}}) &= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \widetilde{m_k 1 n_1} s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \tilde{1} \tilde{n}_1 s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \\
&\quad \text{applying Lemma 4.3.2 twice} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \tilde{n}_1 s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \\
&\quad \text{since } \tilde{1} \text{ is the empty word} \\
&= [\tilde{m}_1 t_1 \tilde{m}_{k-1} t_{k-1} \widetilde{m_k n_1} s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \text{ by Lemma 4.3.2} \\
&= \psi([W_2]_{\mathcal{R}}).
\end{aligned}$$

(C): Suppose $R : h_1 \bar{t}_1 \ldots h_{r-1} \bar{t}_{r-1} h_r = h'_1 \bar{t}'_1 \ldots h'_{s-1} \bar{t}'_{s-1} h'_s \in \boldsymbol{r}$. Let $W_1 \equiv U R_{+1} V$ and $W_2 \equiv U R_{-1} V$. Then

$$\begin{aligned}
\psi([W_1]_{\mathcal{R}}) &= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \widetilde{m_k h_1} \bar{t}_1 \ldots \tilde{h}_{r-1} \bar{t}_{r-1} \widetilde{h_r n_1} s_1 \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \tilde{R}_{+1} \tilde{n}_1 s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \\
&\quad \text{using Lemma 4.3.2 twice} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \tilde{m}_k \tilde{R}_{-1} \tilde{n}_1 s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \text{ using operation (F)} \\
&= [\tilde{m}_1 t_1 \ldots \tilde{m}_{k-1} t_{k-1} \widetilde{m_k h'_1} \bar{t}'_1 \ldots \tilde{h}'_{s-1} \bar{t}'_{s-1} \widetilde{h'_s n_1} s_1 \ldots \tilde{n}_{l-1} s_{l-1} \tilde{n}_l]_{\mathcal{M}} \\
&\quad \text{using Lemma 4.3.2 twice} \\
&= \psi([W_2]_{\mathcal{R}}).
\end{aligned}$$

It is easily proved (using Lemma 4.3.2) that ψ is a monoid homomorphism. Also ψ is clearly surjective since the image of $S(\mathcal{R})$ contains a generating set for $S(\mathcal{M})$. We need only prove that ψ is injective.

Let $[W_1]_{\mathcal{R}}, [W_2]_{\mathcal{R}} \in S(\mathcal{R})$ and suppose that $\psi([W_1]_{\mathcal{R}}) = \psi([W_2]_{\mathcal{R}})$, that is $[\tilde{W}_1]_{\mathcal{M}} = [\tilde{W}_2]_{\mathcal{M}}$. Then there is a chain of elementary operations

$$\tilde{W}_1 \equiv X_0^*, X_1^*, \ldots, X_n^* \equiv \tilde{W}_2$$

where X_i^* $(0 \leq i \leq n)$ is a $\boldsymbol{t}$-positive word on $\boldsymbol{a} \cup \boldsymbol{t}$, and X_{j+1}^* can be obtained from X_j^* $(0 \leq j \leq n-1)$ by performing one of the elementary operations $(D), (D)^{-1}, (E), (E)^{-1}$ or (F).

We will show that the above sequence can be used to construct a chain of words

$$W_1 \equiv X_0, X_1, \ldots, X_n \equiv W_2$$

where $X_j \sim_{\mathcal{R}} X_{j+1}$ $(1 \leq j \leq n-1)$.

Let

$$U^* \equiv a_1 t_1 a_2 t_2 \ldots t_{k-1} a_k \quad \text{and} \quad V^* \equiv b_1 s_1 b_2 s_2 \ldots s_{l-1} b_l$$

be two arbitrary $\boldsymbol{t}$-positive words on $\boldsymbol{a} \cup \boldsymbol{t}$ (a_i's and b_i's are elements of $\boldsymbol{a}^{\pm 1}$, the t_i's and s_i's are elements of $\boldsymbol{t}$). Define

$$U \equiv \theta(a_1) t_1 \theta(a_2) t_2 \ldots t_{k-1} \theta(a_k) \quad \text{and} \quad V \equiv \theta(b_1) s_1 \theta(b_2) s_2 \ldots s_{l-1} \theta(b_l).$$

In the same way we can obtain a sequence $X_0, X_1, \ldots, X_n$ of $\boldsymbol{t}$-positive words on $H \cup \boldsymbol{t}$ from the sequence $X_0^*, X_1^*, \ldots, X_n^*$. Note that since $\tilde{W}_1$ is the lift of W_1 and $\tilde{W}_2$ is the lift of W_2, we have that $X_0 \equiv W_1$ and $X_n \equiv W_2$. We now show that $X_j \sim_{\mathcal{R}} X_{j+1}$ $(0 \leq j \leq n-1)$.

Consider the cases where X_j^* and X_{j+1}^* differ by one of the elementary operations $(D)^{\pm 1}, (E)^{\pm 1}$ or (F) respectively. Let U^* and V^* be $\boldsymbol{t}$-positive words on $\boldsymbol{a} \cup \boldsymbol{t}$, as defined above.

$(D)^{\pm 1}$: Suppose $X_j^* \equiv U^* a^\epsilon a^{-\epsilon} V^*$ and $X_{j+1}^* \equiv U^* V^*$ where $a \in \boldsymbol{a}$, $\epsilon = \pm 1$ and $\theta(a^\epsilon) = h$ for some $h \in H$. Then

$$\begin{aligned}
X_j &\equiv \theta(a_1) t_1 \ldots t_{k-1} \theta(a_k a^\epsilon a^{-\epsilon} b_1) s_1 \theta(b_2) \ldots s_{l-1} \theta(b_l) \\
&\sim_{\mathcal{R}} \theta(a_1) t_1 \ldots t_{k-1} \theta(a_k) h h^{-1} \theta(b_1) s_1 \theta(b_2) \ldots s_{l-1} \theta(b_l) \\
&\qquad \text{using operation } (A)^{-1} \text{ and } \theta(a^\epsilon) = h \\
&\sim_{\mathcal{R}} \theta(a_1) t_1 \ldots t_{k-1} \theta(a_k) 1 \theta(b_1) s_1 \theta(b_2) \ldots s_{l-1} \theta(b_l) \\
&\qquad \text{using operation } (A) \\
&\sim_{\mathcal{R}} \theta(a_1) t_1 \ldots t_{k-1} \theta(a_k) \theta(b_1) s_1 \theta(b_2) \ldots s_{l-1} \theta(b_l) \\
&\qquad \text{using operation } (B) \\
&\sim_{\mathcal{R}} \theta(a_1) t_1 \ldots t_{k-1} \theta(a_k b_1) s_1 \theta(b_2) \ldots s_{l-1} \theta(b_l) \\
&\qquad \text{using operation } (A) \\
&\equiv X_{j+1}
\end{aligned}$$

$(E)^{\pm 1}$: Let $S \in \boldsymbol{s}$. Suppose $X_j^* \equiv U^* S^\epsilon V^*$ $(\epsilon = \pm 1)$ and $X_{j+1}^* \equiv U^* V^*$. Then

$$\begin{aligned}
X_j &\equiv \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k S^\epsilon b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\sim_{\mathcal{R}} \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k)1\theta(b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\qquad \text{using operation } (A)^{-1} \text{ and } \theta(S^\epsilon) = 1 \\
&\sim_{\mathcal{R}} \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k)\theta(b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\qquad \text{using operation } (B) \\
&\sim_{\mathcal{R}} \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\qquad \text{using operation } (A) \\
&\equiv X_{j+1}
\end{aligned}$$

(F): Let $\tilde{R} : \tilde{h}_1\bar{t}_1 \ldots \tilde{h}_{r-1}\bar{t}_{r-1}\tilde{h}_r = \tilde{h}'_1\bar{t}'_1 \ldots \tilde{h}'_{s-1}\bar{t}'_{s-1}\tilde{h}'_s \in \tilde{\boldsymbol{r}}$. Suppose that $X_j^* \equiv U^* \tilde{R}_{+1} V^*$ and $X_{j+1}^* \equiv U^* \tilde{R}_{-1} V^*$. Then

$$\begin{aligned}
X_j &\equiv \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k\tilde{h}_1)\bar{t}_1 \ldots \bar{t}_{r-1}\theta(\tilde{h}_r b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\sim_{\mathcal{R}} \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k)\theta(\tilde{h}_1)\bar{t}_1 \ldots \bar{t}_{r-1}\theta(\tilde{h}_r)\theta(b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\qquad \text{using } (A)^{-1} \\
&\equiv \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k)R_{+1}\theta(b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\sim_{\mathcal{R}} \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k)R_{-1}\theta(b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\qquad \text{using operation } (C) \\
&\equiv \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k)\theta(\tilde{h}'_1)\bar{t}'_1 \ldots \theta(\tilde{h}'_s)\theta(b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\sim_{\mathcal{R}} \theta(a_1)t_1 \ldots t_{k-1}\theta(a_k\tilde{h}'_1)\bar{t}'_1 \ldots \theta(\tilde{h}'_s b_1)s_1 \ldots s_{l-1}\theta(b_l) \\
&\qquad \text{using operation } (A) \\
&\equiv X_{j+1}
\end{aligned}$$

The result is proved. □

Now that we have developed all the necessary background theory concerning $S(\mathcal{R})$ and $S(\mathcal{M})$, we dispense with the equivalence class notation for the elements of these monoids. We identify $[W]_{\mathcal{R}} \in S(\mathcal{R})$ with word W and write $W_1 =_{S(\mathcal{R})} W_2$ if

$[W_1]_{\mathcal{R}} = [W_2]_{\mathcal{R}}$ ($[W_1]_{\mathcal{R}}, [W_2]_{\mathcal{R}} \in S(\mathcal{R})$). Similarly we identify $[V]_{\mathcal{M}} \in S(\mathcal{M})$ with word V and write $V_1 =_{S(\mathcal{M})} V_2$ if $[V_1]_{\mathcal{M}} = [V_2]_{\mathcal{M}}$ ($[V_1]_{\mathcal{M}}, [V_2]_{\mathcal{M}} \in S(\mathcal{M})$).

Let $\mathcal{M}$ be any mixed presentation for $S(\mathcal{R})$. Our strategy will be to prove results concerning $S(\mathcal{R})$ by proving results about $S(\mathcal{M})$.

The graphs we have defined for $\mathcal{R}$ and $\mathcal{M}$ are related in the following way.

Proposition 4.3.4 *$LG(\mathcal{R})$ ($RG(\mathcal{R})$) is identical to $LG(\mathcal{M})$ ($RG(\mathcal{M})$, respectively).*

Proof. Note that the underlying graphs of $LG(\mathcal{R})$ and $LG(\mathcal{M})$ are identical since there is a one-to-one correspondence between the relations of $\boldsymbol{r}$ and the relations of $\tilde{\boldsymbol{r}}$, namely (using the established notation):

$$\begin{aligned} R : h_1 t_1 \dots h_{r-1} t_{r-1} h_r &= h'_1 t'_1 \dots h'_{s-1} t'_{s-1} h'_s \in \boldsymbol{r} \\ &\leftrightarrow \tilde{R} : \tilde{h}_1 t_1 \dots \tilde{h}_{r-1} t_{r-1} \tilde{h}_r = \tilde{h}'_1 t'_1 \dots \tilde{h}'_{s-1} t'_{s-1} \tilde{h}'_s \in \tilde{\boldsymbol{r}}. \end{aligned}$$

Consider now the labels on the edges of these graphs. Let R be an edge of $LG(\mathcal{R})$, corresponding to relation $R \in \boldsymbol{r}$ above. It is labelled by the element $h_1^{-1} h'_1$ of H. However the corresponding edge $\tilde{R}$ of $LG(\mathcal{M})$ is labelled by the element of H represented by the word $\tilde{h}_1^{-1} \tilde{h}'_1$ on $\boldsymbol{a}$, namely $h_1^{-1} h'_1$.

A similar analysis gives the result for $RG(\mathcal{R})$ and $RG(\mathcal{M})$. □

For the remainder of this chapter we will assume that $\mathcal{R} = [H, \boldsymbol{t}; \boldsymbol{r}]$ is a relative monoid presentation for $S(\mathcal{R})$, and that $\mathcal{M} = [\boldsymbol{a}; \boldsymbol{t} | \boldsymbol{s}; \tilde{\boldsymbol{r}}]$ is any mixed monoid presentation for $S(\mathcal{R})$, where $\mathcal{Q} = < \boldsymbol{a}; \boldsymbol{s} >$ is a group presentation for H.

4.4 Pictures over mixed monoid presentations

One of the benefits of being equipped with mixed monoid presentations is that there is a natural notion of a picture over such presentations.

A *picture* $\mathbb{P}$ *over* $\mathcal{M} = [\boldsymbol{a}; \boldsymbol{t} | \boldsymbol{s}; \tilde{\boldsymbol{r}}]$ is a picture such that:

(a) Disc D^2 has two distinct basepoints $O_{\mathbb{P}}$ and $O'_{\mathbb{P}}$;

(b) Arcs are labelled by elements of $\boldsymbol{a}$ ($\boldsymbol{a}$-*arcs*) and elements of $\boldsymbol{t}$ ($\boldsymbol{t}$-*arcs*);

(c) Discs of $\mathbb{P}$ are of the following types:

- $\boldsymbol{s}$-*discs:* $\Delta_i^{S_i}$ $(1 \leq i \leq m)$ with one basepoint O_i. Travelling clockwise around $\partial\Delta_i^{S_i}$ from O_i we read the word $S_i^{\epsilon_i}$ $(\epsilon_i = \pm 1, S_i \in \boldsymbol{s})$.
- $\tilde{\boldsymbol{r}}$-*discs:* $\Delta_j^{\tilde{R}_j}$ $(1 \leq j \leq n)$ with two basepoints O_j, O_j'. Travelling around $\partial^+\Delta_j^{\tilde{R}_j}$ $(1 \leq j \leq n)$ we read $\tilde{R}_{\epsilon_j,j}$, while travelling around $\partial^-\Delta_j^{\tilde{R}_j}$ we read $\tilde{R}_{-\epsilon_j,j}$ $(\epsilon_j = \pm 1, \tilde{R}_{+1,j} = \tilde{R}_{-1,j} \in \tilde{\boldsymbol{r}})$.

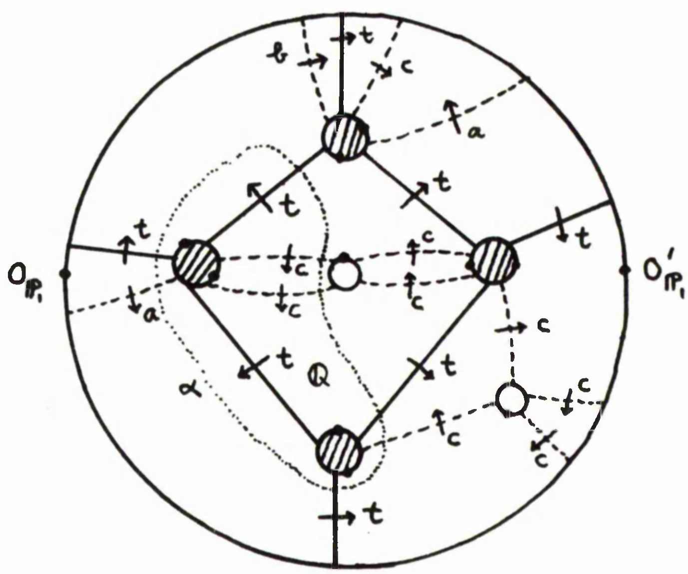
We define $\iota(\mathbb{P})$ $(\tau(\mathbb{P}))$ to be the words read when travelling along $\partial^+\mathbb{P}$ (respectively $\partial^-\mathbb{P}$). If picture $\mathbb{P}$ over $\mathcal{M}$ has $\iota(\mathbb{P}) \equiv \tau(\mathbb{P})$ then $\mathbb{P}$ is a *spherical picture.*

For clarity, when drawing pictures over $\mathcal{M}$ we will shade $\tilde{\boldsymbol{r}}$-discs and indicate $\boldsymbol{a}$-arcs using broken lines.

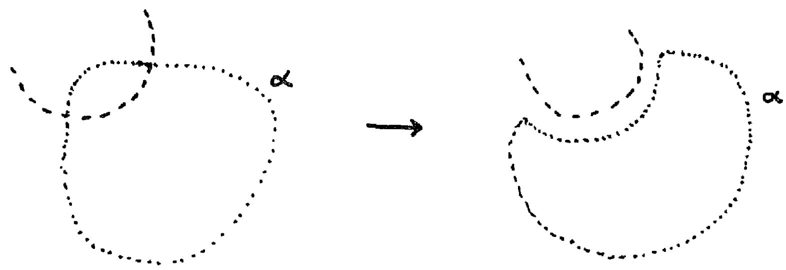
Example 2: Picture $\mathbb{P}_1$ below is a picture over

$$\mathcal{M} = [a, b, c; t | c^4; cta^{-1}t = c^{-1}t, tbtc = ta, ct^2 = c^{-1}tc, t^2 = tc^{-1}],$$

and α is a transverse path in $\mathbb{P}_1$ which encloses a subpicture $\mathbb{Q}$.



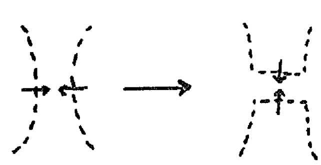
Suppose that a subpicture $\mathbb{Q}$ has an $\boldsymbol{a}$-arc in a small neighbourhood of $\partial\mathbb{Q}$, as shown below, then we can *deform* $\partial\mathbb{Q}$ so that $\mathbb{Q}$ does not contain this $\boldsymbol{a}$-arc.



Any subpicture of a picture $\mathbb{P}$ over $\mathcal{M}$, which does not contain any $\tilde{\boldsymbol{r}}$-discs or $\boldsymbol{t}$-arcs, can be thought of as a group picture over $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$. We can perform elementary group picture operations on such subpictures as follows (see §1.6).

(A) *Operations on* $\boldsymbol{s}$*-discs and* $\boldsymbol{a}$*-arcs:*

- Deletion of floating circles[1].
- Insertion of a floating circle into some area of $\mathbb{P}$.
- Deletion of floating semicircles.
- Insertion of a floating semicircle into some boundary area of $\mathbb{P}$.
- If $\mathbb{P}$ contains a subpicture $\mathbb{Q}$ which is a spherical picture over $\mathcal{Q}$, then delete $\mathbb{Q}$ from the picture.
- Insert a spherical picture into some area of $\mathbb{P}$.
- Bridge move:



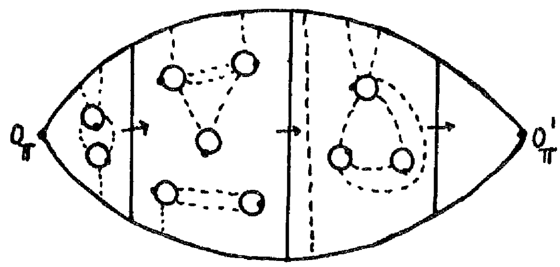
We will define an operation on $\tilde{\boldsymbol{r}}$-discs later in this chapter.

A $\boldsymbol{t}$*-positive* ($\boldsymbol{t}$*-negative*) transverse path in a picture $\mathbb{P}$ over $\mathcal{M}$ is a transverse path which only crosses $\boldsymbol{t}$-arcs in the direction (respectively, against the direction) of

[1]In general we will assume that all floating circles have been deleted from our pictures.

their orientation. As usual we have words associated with transverse paths. Note that $\boldsymbol{t}$-positive transverse paths are labelled by $\boldsymbol{t}$-positive words (on $\boldsymbol{a} \cup \boldsymbol{t}$), while $\boldsymbol{t}$-negative transverse paths are labelled by $\boldsymbol{t}$-negative words (on $\boldsymbol{a} \cup \boldsymbol{t}$).

A *trivial* picture $\mathbb{T}$ over $\mathcal{M}$ is a spherically connected picture over $\mathcal{M}$ which contains no $\tilde{\boldsymbol{r}}$-discs and is such that $\iota(\mathbb{T})$ and $\tau(\mathbb{T})$ are $\boldsymbol{t}$-positive words (on $\boldsymbol{a} \cup \boldsymbol{t}$).



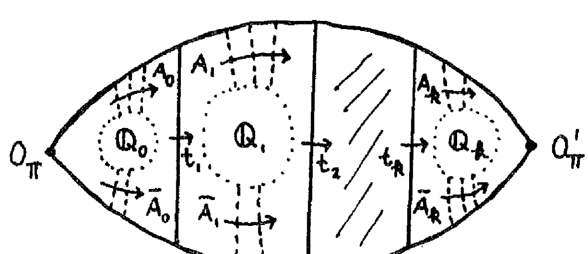
Lemma 4.4.1 *Let $\mathbb{T}$ be a trivial picture over $\mathcal{M}$ with $\iota(\mathbb{T}) \equiv U$ and $\tau(\mathbb{T}) \equiv V$, where U and V are $\boldsymbol{t}$-positive words. Then $U =_{S(\mathcal{M})} V$.*

Proof. We first note that U and V must take the forms

$$U \equiv A_0 t_1 A_1 t_2 \dots A_{k-1} t_k A_k \text{ and } V \equiv \bar{A}_0 t_1 \bar{A}_1 t_2 \dots \bar{A}_{k-1} t_k \bar{A}_k$$

where $A_i, \bar{A}_i$ $(0 \le i \le k)$ are words on $\boldsymbol{a}$ and $t_j \in \boldsymbol{t}$ $(1 \le j \le k)$.

Let $\mathbb{Q}_i$ $(0 \le i \le k)$ be the subpicture of $\mathbb{P}$ with boundary label $A_i \bar{A}_i^{-1}$, as shown below:



Then $\mathbb{Q}_i$ is a picture over group presentation $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$ for H. By Corollary 1.6.5, $A_i =_H \bar{A}_i$ $(0 \le i \le k)$. Since the elementary operations involved in transforming A_i

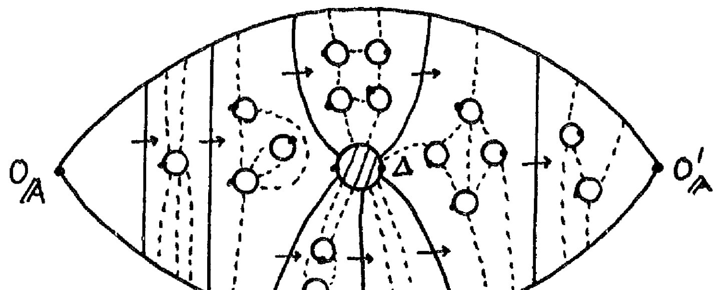
to $\bar{A}_i$ $(0 \leq i \leq k)$ are of types $(D)^{\pm 1}$ and/or $(E)^{\pm 1}$, it is clear that $A_i =_{S(\mathcal{M})} \bar{A}_i$ $(0 \leq i \leq k)$. It now follows that $U =_{S(\mathcal{M})} V$. □

An *atomic* picture $\mathbb{A}$ over $\mathcal{M}$ is a spherically connected picture over $\mathcal{M}$ which has the following properties:

(a) $\mathbb{A}$ contains a single $\tilde{\boldsymbol{r}}$-disc Δ;

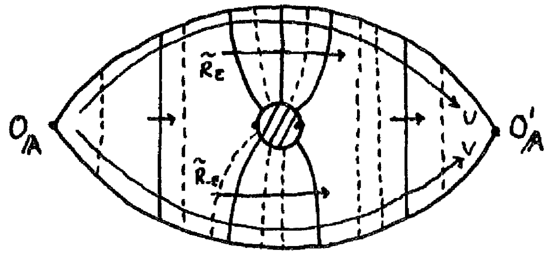
(b) Each $\boldsymbol{t}$-arc intersecting $\partial^+\Delta$ $(\partial^-\Delta)$ also intersects $\partial^+\mathbb{A}$ $(\partial^-\mathbb{A}$ respectively);

(c) $\iota(\mathbb{A})$ and $\tau(\mathbb{A})$ are $\boldsymbol{t}$-positive words.

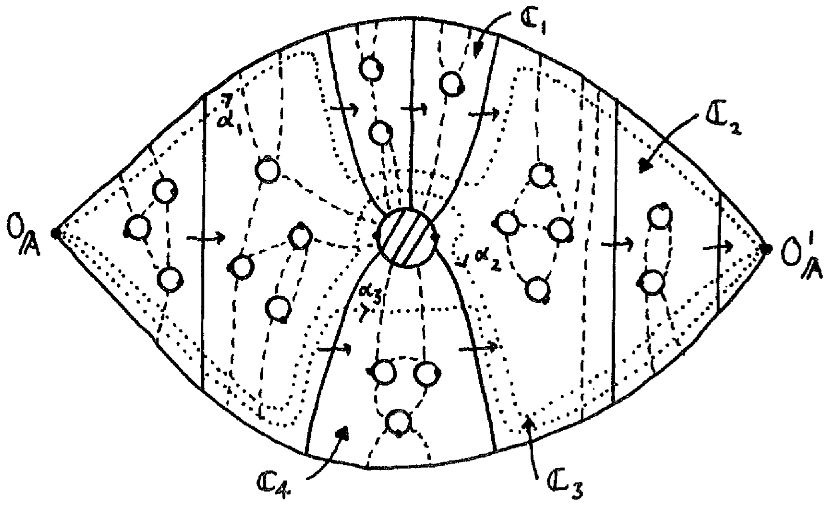


Lemma 4.4.2 *Let $\mathbb{A}$ be an atomic picture over $\mathcal{M}$ with $\iota(\mathbb{A}) \equiv U$ and $\tau(\mathbb{A}) \equiv V$, where U and V are **t**-positive words. Then $U =_{S(\mathcal{M})} V$.*

Proof. If $\mathbb{A}$ contains no $\boldsymbol{s}$-discs then $\mathbb{A}$ is a geometric representation of an elementary operation of type (F) which takes U to V $(\epsilon = \pm 1, \tilde{R}_{+1} = \tilde{R}_{-1} \in \tilde{\boldsymbol{r}})$.



Hence we can assume that $\mathbb{A}$ does contain $\boldsymbol{s}$-discs. We define three $\boldsymbol{t}$-positive transverse paths α_1, α_2 and α_3, as shown in the figure below, so that the subpictures $\mathbb{C}_1, \mathbb{C}_2$ and $\mathbb{C}_4$ with boundaries $\partial^+\mathbb{A}\alpha_1^{-1}$, $\alpha_1\alpha_2^{-1}$ and $\alpha_3(\partial^-\mathbb{A})^{-1}$ respectively, are trivial pictures over $\mathcal{M}$. Furthermore, subpicture $\mathbb{C}_3$ with boundary $\alpha_2\alpha_3^{-1}$ is an atomic picture over $\mathcal{M}$ which contains no $\boldsymbol{s}$-discs.



Suppose that α_1, α_2 and α_3 are labelled by the $\boldsymbol{t}$-positive words W_1, W_2 and W_3 respectively. By Lemma 4.4.1, $U =_{S(\mathcal{M})} W_1$, $W_1 =_{S(\mathcal{M})} W_2$ and $W_3 =_{S(\mathcal{M})} V$. Also by the observation at the start of the proof $W_2 =_{S(\mathcal{M})} W_3$. The result follows. $\square$

The following definition is the natural generalisation of our notion of stratification in the ordinary monoid picture case.

A *stratification* γ for a picture $\mathbb{P}$ over $\mathcal{M}$ is a sequence of $\boldsymbol{t}$-positive transverse paths $\gamma_0, \gamma_1, \ldots, \gamma_n$, each starting at $O_{\mathbb{P}}$ and ending at $O'_{\mathbb{P}}$, and satisfying the following:

(i) γ_0 is $\partial^+\mathbb{P}$, while γ_n is $\partial^-\mathbb{P}$;

(ii) γ_λ and γ_μ intersect only at $O_{\mathbb{P}}$ and $O'_{\mathbb{P}}$ $(0 \leq \lambda < \mu \leq n)$;

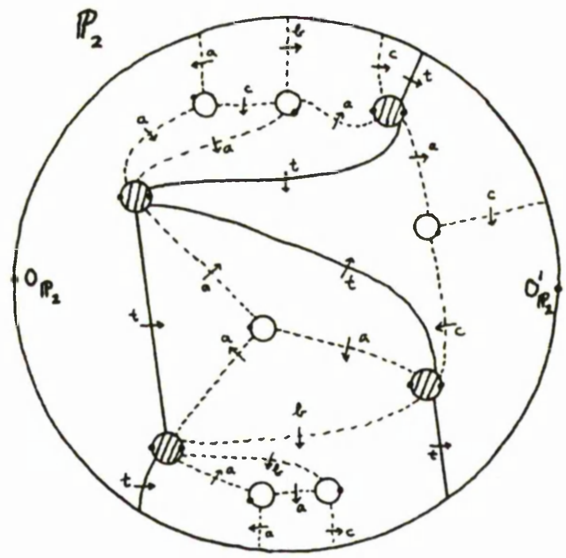
(iii) travelling in a clockwise (anticlockwise) direction around a small neighbourhood of $O_{\mathbb{P}}$ ($O'_{\mathbb{P}}$ respectively), we encounter the paths in the order $\gamma_0, \gamma_1, \ldots, \gamma_n$;

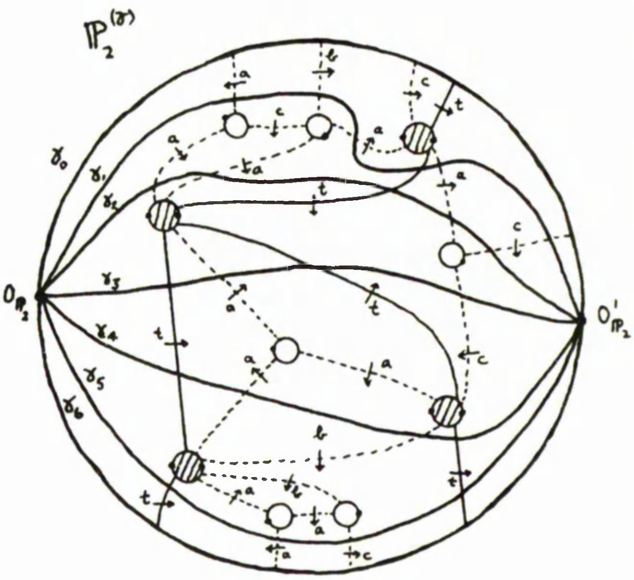
(iv) the subpicture of $\mathbb{P}$ with boundary $\gamma_i\gamma_{i+1}^{-1}$ $(0 \leq i \leq n-1)$ is either an atomic or trivial picture over $\mathcal{M}$.

A picture $\mathbb{P}$ over $\mathcal{M}$ together with a stratification γ for $\mathbb{P}$, is a *monoid picture* $\mathbb{P}^{(\gamma)}$ *over* $\mathcal{M}$. We will in most cases be unconcerned with the particular stratification associated with $\mathbb{P}$, and will usually refer to monoid picture $\mathbb{P}$ (over $\mathcal{M}$), unless we are specifically concerned with the stratification of $\mathbb{P}$.

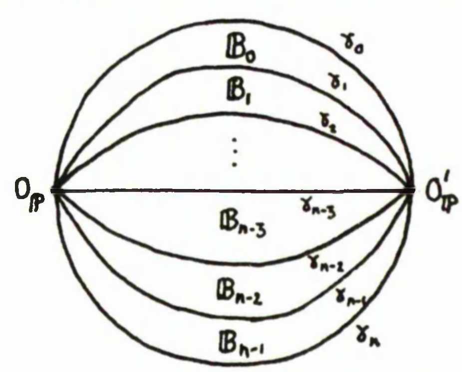
Stratifications will be indicated in our Figures using bold lines.

Example 3: Consider again picture $\mathbb{P}_1$ of Example 2. Due to the orientation of the $\boldsymbol{t}$-arcs which do not intersect with $\partial\mathbb{P}_1$, it is not possible to find a stratification for $\mathbb{P}_1$. Let $\mathbb{P}_2$ be the picture (over some suitable mixed monoid presentation $\mathcal{M}_2$) shown below on the left. It is possible to find a stratification γ for $\mathbb{P}_2$, as shown in the monoid picture $\mathbb{P}_2^{(\gamma)}$ on the right.



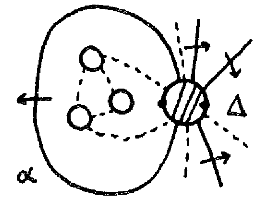


For a monoid picture $\mathbb{P}^{(\gamma)}$ over $\mathcal{M}$ we will often write $\mathbb{P}^{(\gamma)} = \mathbb{B}_0\mathbb{B}_1 \ldots \mathbb{B}_{n-1}$, where $\mathbb{B}_i$ is the atomic or trivial picture over $\mathcal{M}$ with boundary $\gamma_i\gamma_{i+1}^{-1}$ $(0 \leq i \leq n-1)$.

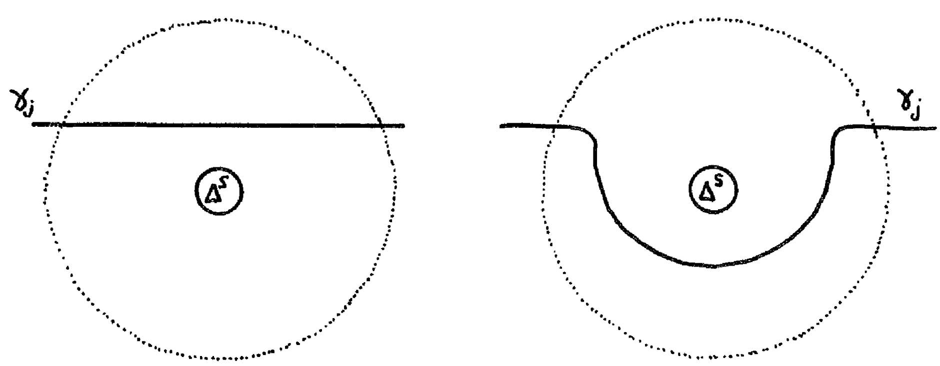


Intuitively, we can think of any monoid picture as being constructed from atomic and trivial pictures over $\mathcal{M}$. Consequently, it is clear that no monoid picture over $\mathcal{M}$

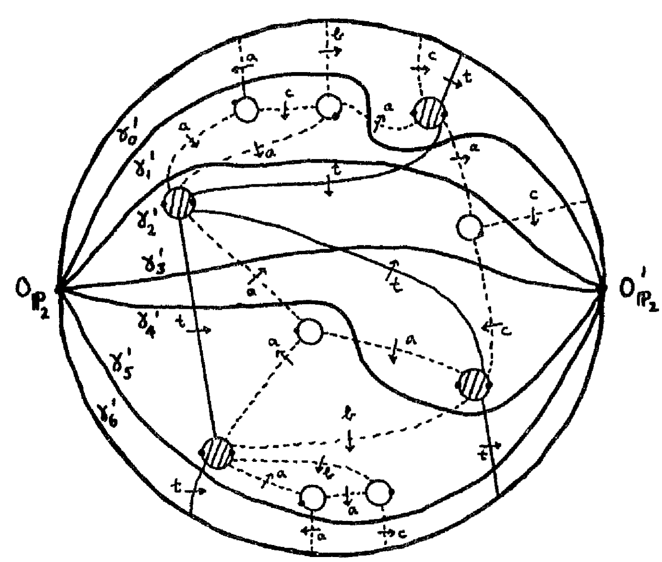
contains an $\tilde{r}$-disc Δ, with a single t-arc α which intersects both $\partial^+\Delta$ and $\partial^-\Delta$.



The stratification $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ of any monoid picture $\mathbb{P}^{(\gamma)}$, can be deformed in the following way. If there is a neighbourhood of an s-disc Δ^S ($S \in s$) which contains a subpath of some stratification path γ_j ($1 \leq j \leq n-1$) (and no other stratification paths), then we can deform γ_j by *passing it over* disc Δ^S, as shown in the Figure below.



Example 4: We can deform γ_4 in monoid picture $\mathbb{P}_2^{(\gamma)}$ of Example 3 to obtain the monoid picture $\mathbb{P}_2^{(\gamma')}$:



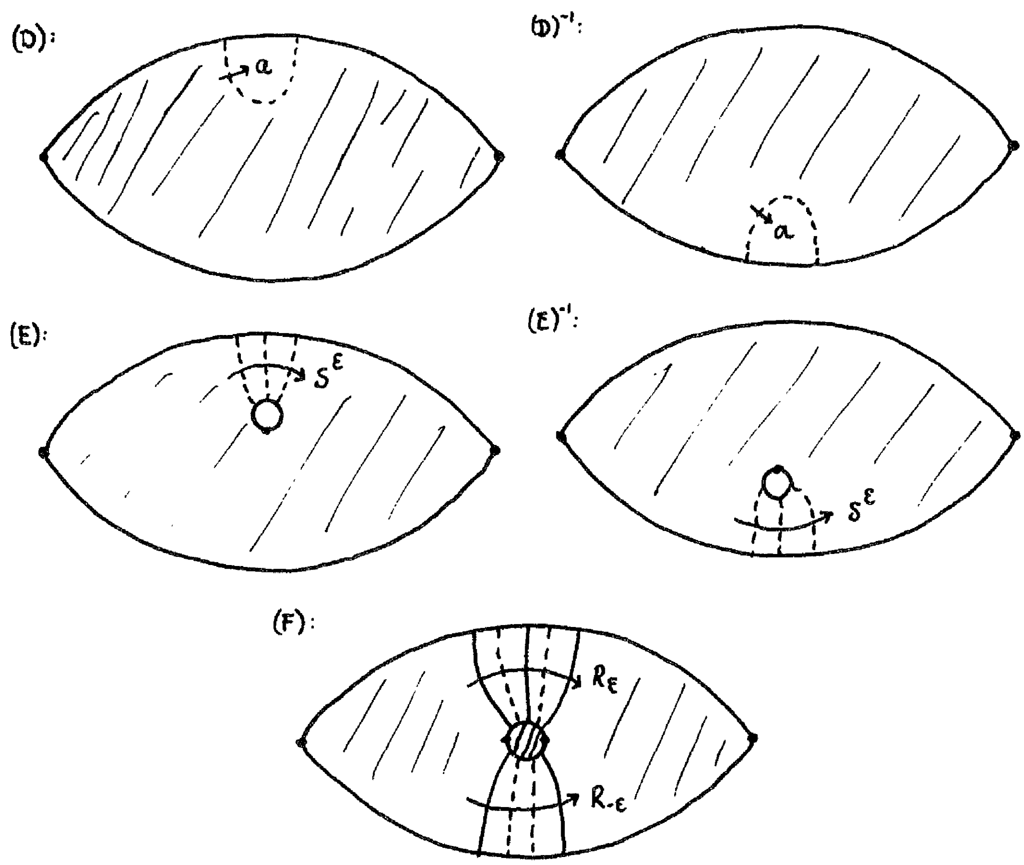
Theorem 4.4.3 *Two $\boldsymbol{t}$-positive words U and V are such that $U =_{S(\mathcal{M})} V$ if and only if there exists a monoid picture over $\mathcal{M}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.*

Proof. Suppose that $U =_{S(\mathcal{M})} V$. Then there exists a sequence of $\boldsymbol{t}$-positive words

$$U \equiv U_0, U_1, \ldots, U_n \equiv V$$

such that U_{i+1} can be obtained from U_i $(0 \leq i \leq n-1)$ by applying a single elementary operation of type $(D)^{\pm 1}, (E)^{\pm 1}$ or (F).

Now the elementary operations $(D)^{\pm 1}, (E)^{\pm 1}$ and (F) can be represented geometrically by means of the following trivial and atomic pictures over $\mathcal{M}$ (compare with the operations defined in §4.2).



Let $\mathbb{B}_i$ $(0 \le i \le n-1)$ be the relevant atomic or trivial picture over $\boldsymbol{\mathcal{M}}$ which corresponds to the elementary operation which takes U_i to U_{i+1}, so that $\iota(\mathbb{B}_i) \equiv U_i$, $\tau(\mathbb{B}_i) \equiv U_{i+1}$ and $\tau(\mathbb{B}_j) \equiv \iota(\mathbb{B}_{j+1})$ $(0 \le j \le n-2)$. Let $\mathbb{P}$ be the picture over $\boldsymbol{\mathcal{M}}$ obtained by identifying $\partial^-\mathbb{B}_j$ with $\partial^+\mathbb{B}_{j+1}$ $(0 \le j \le n-2)$ and connecting the arcs in the obvious way. Then $\mathbb{P} = \mathbb{B}_0\mathbb{B}_1\ldots\mathbb{B}_{n-1}$ is a monoid picture over $\boldsymbol{\mathcal{M}}$ with stratification $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$, where $\gamma_0 = \partial^+\mathbb{B}_0$, $\gamma_k = \partial^-\mathbb{B}_{k-1} = \partial^+\mathbb{B}_k$ $(1 \le k \le n-1)$ and $\gamma_n = \partial^-\mathbb{B}_{n-1}$. Also $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$.

Conversely, suppose that $\mathbb{P}^{(\gamma)} = \mathbb{B}_0\mathbb{B}_1\ldots\mathbb{B}_{n-1}$ is a monoid picture over $\boldsymbol{\mathcal{M}}$ ($\mathbb{B}_i$ an atomic or trivial picture over $\boldsymbol{\mathcal{M}}$ $(0 \le i \le n-1)$) where $\iota(\mathbb{B}_0) \equiv U$, $\tau(\mathbb{B}_{n-1}) \equiv V$ and $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$. Let $W_1, W_2, \ldots, W_{n-1}$ be the $\boldsymbol{t}$-positive words read while travelling from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$ along $\gamma_1, \ldots, \gamma_{n-1}$ respectively. By Lemma 4.4.1, if $\mathbb{B}_i$ $(0 \le i \le n-1)$ is a trivial picture over $\boldsymbol{\mathcal{M}}$ then $\iota(\mathbb{B}_i) =_{S(\boldsymbol{\mathcal{M}})} \tau(\mathbb{B}_i)$. Also, if $\mathbb{B}_i$ is an atomic picture over $\boldsymbol{\mathcal{M}}$ then by Lemma 4.4.2, $\iota(\mathbb{B}_i) =_{S(\boldsymbol{\mathcal{M}})} \tau(\mathbb{B}_i)$. Thus

$$U =_{S(\boldsymbol{\mathcal{M}})} W_1 =_{S(\boldsymbol{\mathcal{M}})} W_2 =_{S(\boldsymbol{\mathcal{M}})} \cdots =_{S(\boldsymbol{\mathcal{M}})} W_{n-1} =_{S(\boldsymbol{\mathcal{M}})} V.$$

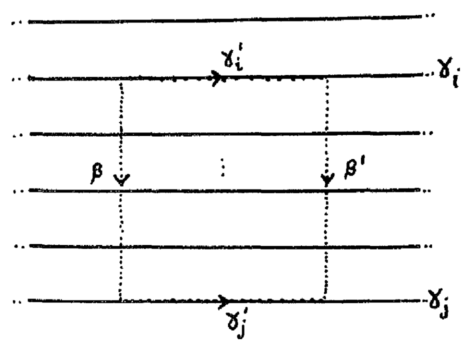
$\square$

Example 5: Picture $\mathbb{P}_2^{(\gamma)}$ in Example 3 shows that $a^{-1}bctc =_{S(\boldsymbol{\mathcal{M}}_2)} ta^{-1}ct$.

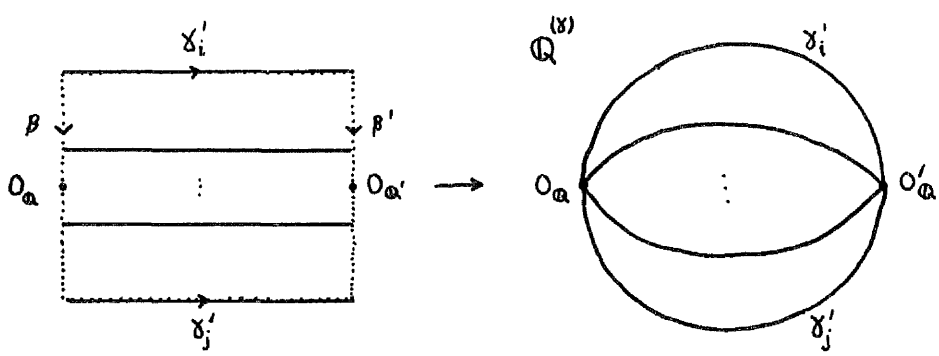
Let $\mathbb{P}$ be a monoid picture with stratification $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$. Suppose α is a simple closed path in the interior of $D^2 - \{\boldsymbol{s}\text{-discs } \cup \tilde{\boldsymbol{r}}\text{-discs}\}$ which consists of:

- a subpath γ_i' of γ_i and a subpath γ_j' of γ_j $(0 \le i < j \le n)$;
- a simple path β from the starting point of γ_i' to the starting point of γ_j', and a simple path β' from the end point of γ_i' to the endpoint of γ_j'. Furthermore β and β' intersect no $\boldsymbol{t}$-arcs.

Then the subpicture $\mathbb{Q}$ of $\mathbb{P}$ enclosed by $\alpha = \gamma_i'\beta'\gamma_j'^{-1}\beta^{-1}$ is said to be *compatible* with the stratification γ.

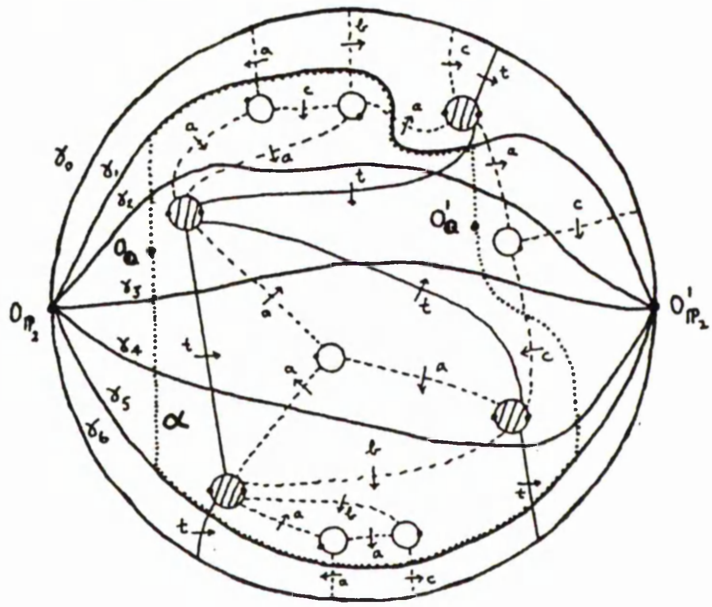


Suppose that we cut $\mathbb{Q}$ out of $\mathbb{P}$ and select two points $O_{\mathbb{Q}}$ and $O'_{\mathbb{Q}}$ on β and β' respectively. If we deform each stratification path within $\mathbb{Q}$ (without passing any such paths over $\boldsymbol{s}$-discs) to a path from $O_{\mathbb{Q}}$ to $O'_{\mathbb{Q}}$, and discard any segments of $\boldsymbol{a}$-arc which lie outside the picture with boundary $\gamma_i'\gamma_j'^{-1}$, then we obtain a monoid picture over $\mathcal{M}$ *with stratification induced from* γ, as shown below. We denote this picture by $\mathbb{Q}^{(\gamma)}$.

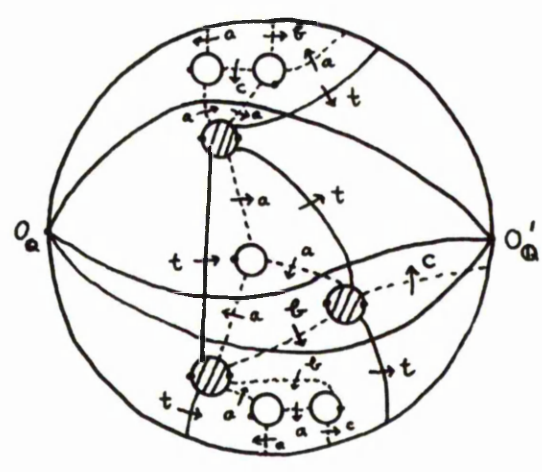


Example 6: Consider again picture $\mathbb{P}_2$ from Example 3. Path α, indicated below, is

the boundary of a subpicture $\mathbb{Q}$ which is compatible with the stratification γ of $\mathbb{P}_2$.



Monoid picture $\mathbb{Q}^{(\gamma)}$ is shown below.



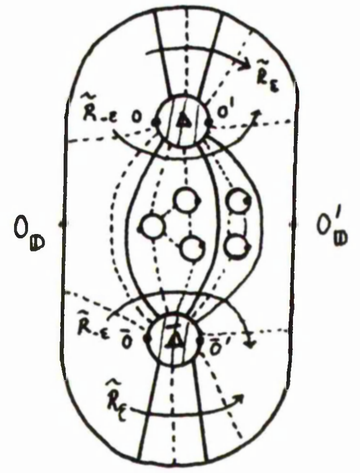
A spherically connected picture $\mathbb{D}$ over $\mathcal{M}$ is a *dipole* if it contains exactly two $\tilde{\boldsymbol{r}}$-discs Δ and $\bar{\Delta}$ (with basepoints O, O' and $\bar{O}, \bar{O}'$ respectively), together with a number of $\boldsymbol{s}$-discs (but possibly none), and satisfies the following:

(i) There exists $\tilde{R} \in \tilde{\boldsymbol{r}}, \epsilon = \pm 1$ such that the label on $\partial^+\Delta$ and $\partial^-\bar{\Delta}$ is $\tilde{R}_\epsilon$; the label on $\partial^-\Delta$ and $\partial^+\bar{\Delta}$ is $\tilde{R}_{-\epsilon}$.

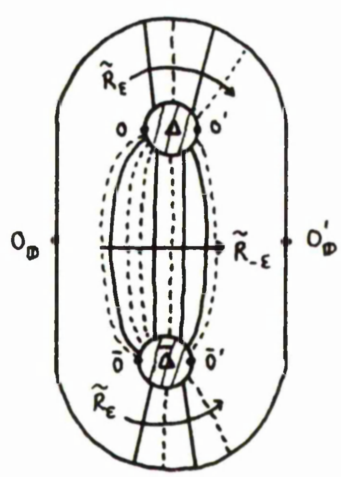
(ii) Each $\boldsymbol{t}$-arc intersecting $\partial^-\Delta$ also intersects $\partial^+\bar{\Delta}$.

(iii) The arcs intersecting $\partial^+\Delta$ and $\partial^-\bar{\Delta}$ intersect $\partial\mathbb{D}$.

(iv) Every s-disc in $\mathbb{D}$ is contained in a subpicture bounded by $\partial^-\Delta, \partial^+\bar{\Delta}$ and a pair of t-arcs which intersect with $\partial^-\Delta$ and $\partial^+\bar{\Delta}$.



A dipole which does not contain any s-discs and is such that any a-arc intersecting $\partial^-\Delta$ also intersects $\partial^+\bar{\Delta}$, is called a *cancelling pair of $\tilde{r}$-discs.*

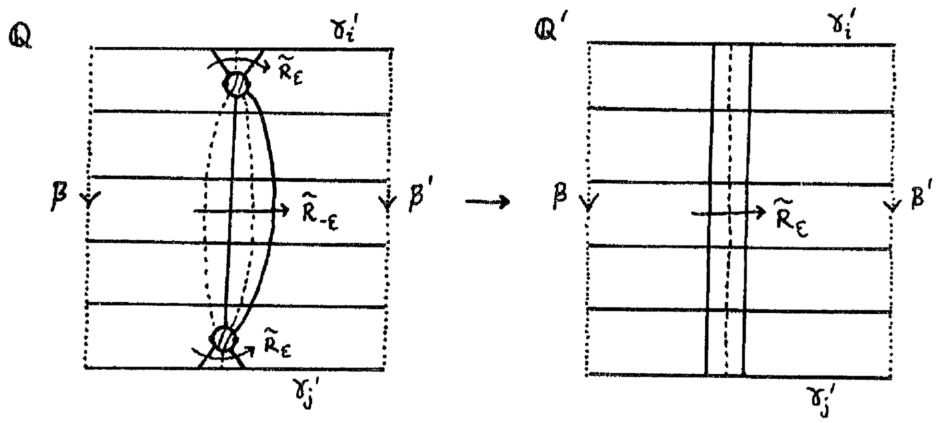


4.5 Operations on monoid pictures

In the previous section we defined type (A) elementary operations on pictures over $\mathcal{M}$. We now define one further elementary operation on monoid pictures over $\mathcal{M}$:

(B) *Operation on $\tilde{\boldsymbol{r}}$-discs:*

- Suppose a monoid picture contains a subpicture $\mathbb{Q}$ which is compatible with the stratification and is a cancelling pair of $\tilde{\boldsymbol{r}}$-discs. Then replace $\mathbb{Q}$ by a subpicture $\mathbb{Q}'$, which is compatible with the stratification and contains no discs, as shown below.



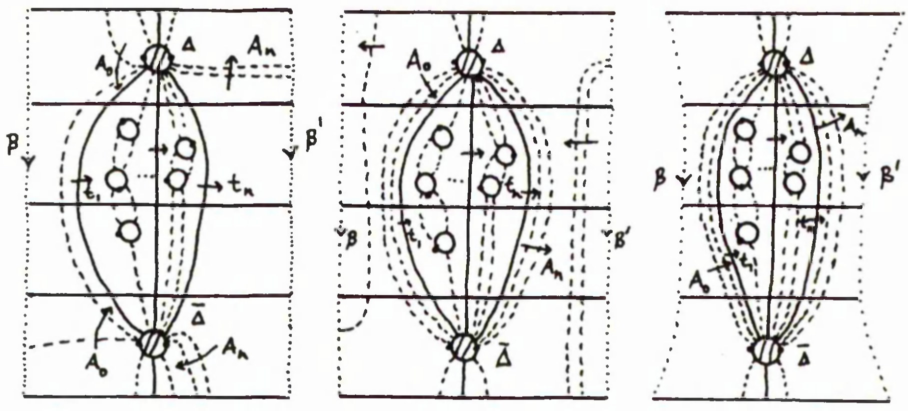
We say that two monoid pictures or subpictures are *equivalent* if one can be transformed into the other using a finite number of operations of type (A) or (B). In particular they are (A)-*equivalent* if one can be transformed into the other using a finite number of type (A) operations.

Lemma 4.5.1 *Let $\mathbb{P}^{(\gamma)}$ be a picture containing a subpicture $\mathbb{D}$ which is compatible with γ and is also a dipole. Then $\mathbb{D}$ is (A)-equivalent to a subpicture which is compatible with γ and is a cancelling pair of $\tilde{\boldsymbol{r}}$-discs.*

Proof. Let Δ and $\bar{\Delta}$ be the $\tilde{\boldsymbol{r}}$-discs of $\mathbb{D}$. We appeal to the notation used in the definition of a dipole. Also let $\mathbb{D}$ have boundary $\gamma_i'\beta'\gamma_j^{-1}\beta^{-1}$, as in the definition of a subpicture compatible with stratification γ.

Suppose that $\tilde{R}_{-\epsilon} \equiv A_0t_1A_1t_2A_2\ldots t_nA_n$ (A_i words on $\boldsymbol{a}$ $(0 \leq i \leq n)$, $t_j \in \boldsymbol{t}$ $(1 \leq j \leq n)$, $\epsilon = \pm 1, \tilde{R} \in \tilde{\boldsymbol{r}}$).

We proceed by performing bridge moves (as required) on the $\boldsymbol{a}$-arcs labelled by A_0 and A_n, until there are $\boldsymbol{a}$-arcs intersecting with $\partial^-\Delta$ and $\partial^+\bar{\Delta}$, which are labelled by A_0 and A_n (see Figs. (i) and (ii) below). Subpaths β and β' of $\partial\mathbb{D}$ can then be deformed to exclude the resulting semicircular $\boldsymbol{a}$-arcs (see Figs. (ii) and (iii)).



(i) (ii) (iii)

Now consider the region of $\mathbb{D}$ enclosed by Δ, $\bar{\Delta}$ and the arcs labelled by t_j, t_{j+1}, for each $1 \leq j \leq n-1$. This region contains a subpicture over $\mathcal{Q}$ with boundary label $A_j A_j^{-1}$. If this subpicture contains any $\boldsymbol{s}$-discs then they can be removed by first performing bridge moves on the $\boldsymbol{a}$-arcs, and then deleting the spherical picture over $\mathcal{Q}$ which arises. Consequently we eventually obtain a cancelling pair of $\tilde{\boldsymbol{r}}$-discs which is compatible with the stratification. $\square$

4.6 t-circles

Let $\mathbb{P}^{(\gamma)}$ be a monoid picture over $\mathcal{M}$, where $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$. A $\boldsymbol{t}$*-circle* in $\mathbb{P}^{(\gamma)}$ is either

(I) a $\boldsymbol{t}$-arc α which intersects γ_0 and γ_n, but no $\tilde{\boldsymbol{r}}$-discs;

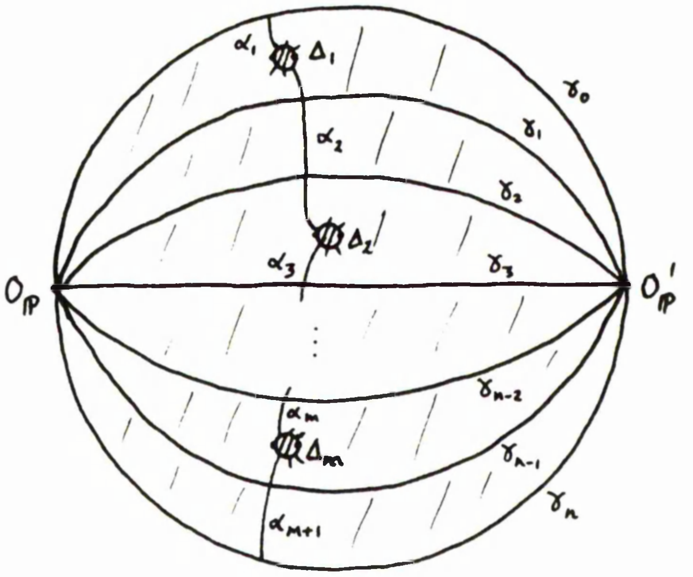
or

(II) a sequence of $\tilde{\boldsymbol{r}}$-discs $\Delta_1, \ldots, \Delta_m$ and $\boldsymbol{t}$-arcs $\alpha_1, \ldots, \alpha_{m+1}$, running from the top to the bottom of $\mathbb{P}^{(\gamma)}$, such that the following conditions are satisfied:

(i) α_1 intersects with γ_0 and Δ_1, while α_{m+1} intersects with γ_n and Δ_m;

(ii) α_j connects Δ_{j-1} to Δ_j $(2 \leq j \leq m)$;

(iii) α_1 and α_{m+1} are labelled by the same element of $\boldsymbol{t}$.



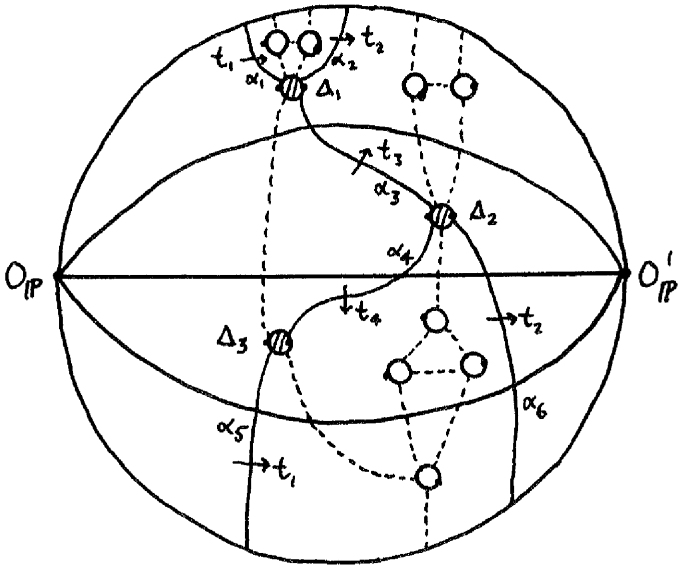
A ***t**-circle* of type (I) is called a *trivial **t**-circle.*

In the case where the first element of $\boldsymbol{t}$ which appears in $\iota(\mathbb{P})$ and $\tau(\mathbb{P})$ is the same, there exists a $\boldsymbol{t}$-circle in $\mathbb{P}$, defined as follows. Let α_1 be the first $\boldsymbol{t}$-arc which intersects γ_0 when travelling from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$ along γ_0. If α_1 is a trivial $\boldsymbol{t}$-circle then α_1 is called the *left **t**-circle of* $\mathbb{P}^{(\gamma)}$. On the other hand, suppose α_1 intersects disc Δ_1 (with basepoints O_1, O'_1). Let α_2 be the $\boldsymbol{t}$-arc which intersects with Δ_1 and is the first $\boldsymbol{t}$-arc encountered while travelling around $\partial\Delta_1$ from O_1 to O'_1 in an anticlockwise direction. Suppose α_2 intersects Δ_2 (with basepoints O_2, O'_2). Let α_3 be the $\boldsymbol{t}$-arc which intersects with Δ_2 and is the first $\boldsymbol{t}$-arc encountered while travelling around $\partial\Delta_2$ from O_2 to O'_2 in an anticlockwise direction. Continuing in this way we obtain the *left **t**-circle* of $\mathbb{P}^{(\gamma)}$. Note that since $\alpha_1, \alpha_2, \ldots$ are uniquely selected, the left $\boldsymbol{t}$-circle of $\mathbb{P}^{(\gamma)}$ is unique. In the former case we also often say that $\mathbb{P}^{(\gamma)}$ has a *trivial left **t**-circle.*

Intuitively we can think of the left $\boldsymbol{t}$-circle as the $\boldsymbol{t}$-circle of $\mathbb{P}^{(\gamma)}$ which lies 'nearest' to $O_{\mathbb{P}}$.

In the case where the last element of $\boldsymbol{t}$ which appears in $\iota(\mathbb{P})$ and $\tau(\mathbb{P})$ is the same, we can define (in a symmetrical way to the left $\boldsymbol{t}$-circle case) the *right **t**-circle* of $\mathbb{P}^{(\gamma)}$. We think of the right $\boldsymbol{t}$-circle as being the $\boldsymbol{t}$-circle of $\mathbb{P}^{(\gamma)}$ which lies 'nearest' to $O'_{\mathbb{P}}$. Clearly the right $\boldsymbol{t}$-circle of $\mathbb{P}^{(\gamma)}$ is also unique.

Example 7:



In this monoid picture discs Δ_1, Δ_2 and Δ_3, together with $\boldsymbol{t}$-arcs $\alpha_1, \alpha_3, \alpha_4$ and α_5 (α_2, α_3 and α_6) form the left (respectively, right) $\boldsymbol{t}$-circle.

4.7 Asphericity of mixed monoid presentations

The concept of asphericity for relative group presentations is one which is well developed (see [5] and the references cited there). In this section we develop the analogous concept of asphericity for a mixed monoid presentation $\mathcal{M}$, and give sufficient conditions for $\mathcal{M}$ to be aspherical.

In the group theory case, the work on asphericity leads to results concerning the (co)homology of groups given by aspherical relative presentations [5]. It is hoped that analogous monoid results can be obtained in the future.

A mixed monoid presentation $\mathcal{M}$ is said to be *aspherical* if every spherical monoid picture over $\mathcal{M}$ is equivalent to a monoid picture over $\mathcal{M}$ which contains no discs.

In order to prove our asphericity result, we require the following Lemmas.

Lemma 4.7.1 *Suppose that $LG(\mathcal{M})$ is cycle free. Let $\mathbb{P}$ be a monoid picture over $\mathcal{M}$ with $\iota(\mathbb{P}) \equiv At\ldots$ and $\tau(\mathbb{P}) \equiv At\ldots$, where A is a word on $\boldsymbol{a}$ and $t \in \boldsymbol{t}$. Let*

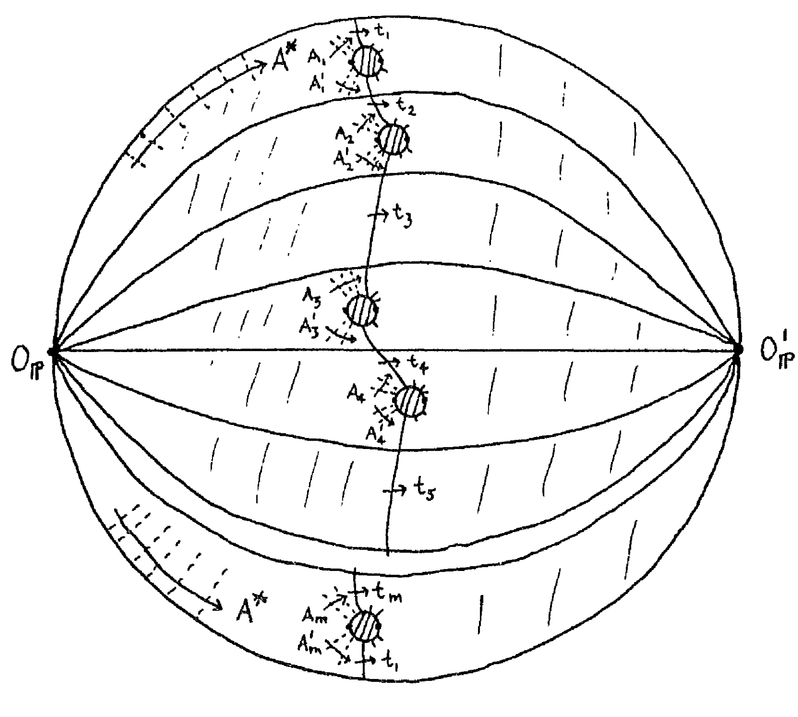
T be the left $\boldsymbol{t}$*-circle of* $\mathbb{P}$*. Then* $\mathbb{P}$ *is equivalent to a picture* $\mathbb{P}'$*, where* $\iota(\mathbb{P}) \equiv \iota(\mathbb{P}')$*,* $\tau(\mathbb{P}) \equiv \tau(\mathbb{P}')$ *and* $\mathbb{P}'$ *has a trivial left* $\boldsymbol{t}$*-circle. Moreover, if* T *is non-trivial then the number of* $\tilde{\boldsymbol{r}}$*-discs in* $\mathbb{P}'$ *is less than in* $\mathbb{P}$*.*

Proof. The key obeservation in this proof is the following. Consider the left $\boldsymbol{t}$-circle in some monoid picture $\mathbb{P}^*$ over $\mathcal{M}$ with $\iota(\mathbb{P}^*) \equiv A^*t_1\ldots$ and $\tau(\mathbb{P}^*) \equiv A^*t_1\ldots$ (A^* a word on $\boldsymbol{a}$, $t_1 \in \boldsymbol{t}$), and assume that it is non-trivial. Note that a non-trivial left $\boldsymbol{t}$-circle must involve at least two $\tilde{\boldsymbol{r}}$-discs, otherwise $LG(\mathcal{M})$ would have a cycle of length one which is labelled by the identity in H.

Suppose that the discs in the left $\boldsymbol{t}$-circle are labelled by the following relations, starting with the $\tilde{\boldsymbol{r}}$-disc at the top of the picture and working downwards:

$$\begin{aligned}
\tilde{R}_1^{\epsilon_1} : A_1t_1\ldots &= A_1't_2\ldots \\
\tilde{R}_2^{\epsilon_2} : A_2t_2\ldots &= A_2't_3\ldots \\
&\vdots \\
\tilde{R}_m^{\epsilon_m} : A_mt_m\ldots &= A_m't_1\ldots
\end{aligned}$$

(A_i, A_i' are words on $\boldsymbol{a}$, $t_i, t_i' \in \boldsymbol{t}$, $\epsilon_i = \pm 1$, $1 \le i \le m$).



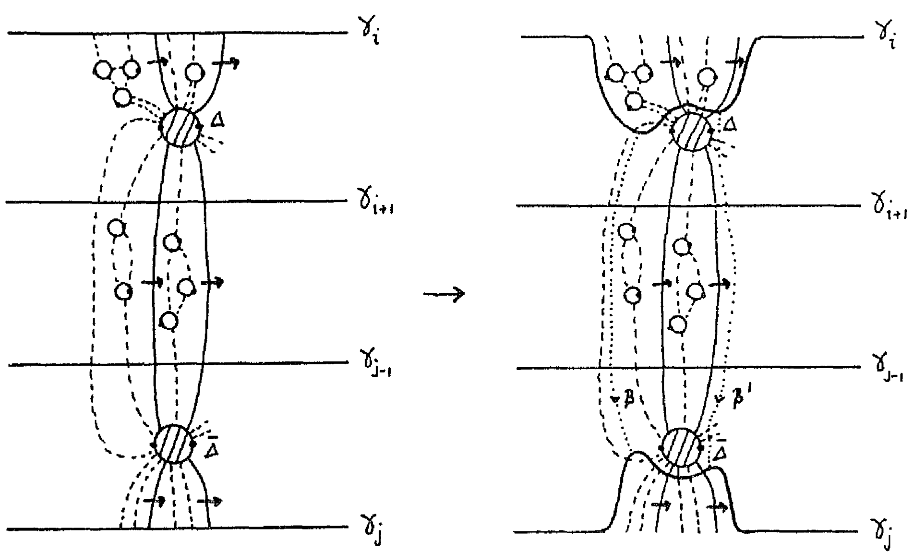
Then there is a closed path $e_{\tilde{R}_1}^{\epsilon_1} e_{\tilde{R}_2}^{\epsilon_2} \ldots e_{\tilde{R}_m}^{\epsilon_m}$ in $LG(\mathcal{M})$ at t_1 which is labelled by the element of H represented by $W \equiv A_1^{-1}A_1'A_2^{-1}A_2' \ldots A_m^{-1}A_m'$. Now $W' \equiv A^*WA^{*-1}$ represents the identity in H since it is the boundary label of the picture over $\mathcal{Q}$ lying to the left of the left $\boldsymbol{t}$-circle in question. Thus W also represents the identity in H. Since $LG(\mathcal{M})$ is cycle free we conclude that the closed path $e_{\tilde{R}_1}^{\epsilon_1}e_{\tilde{R}_2}^{\epsilon_2}\ldots e_{\tilde{R}_m}^{\epsilon_m}$ must not be reduced. Thus $e_{\tilde{R}_k}^{\epsilon_k} = e_{\tilde{R}_{k+1}}^{-\epsilon_{k+1}}$ and $\tilde{R}_k^{\epsilon_k} = \tilde{R}_{k+1}^{-\epsilon_{k+1}}$ for some $1 \le k \le m-1$.

We now turn to the proof proper. The result is proved by induction on the number of $\tilde{r}$-discs in $\mathbb{P}$. Suppose that $\mathbb{P}$ has stratification $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$.

If there are no $\tilde{\boldsymbol{r}}$-discs in $\mathbb{P}$ then any $\boldsymbol{t}$-circle in $\mathbb{P}$ must be trivial and there is nothing to prove. More generally, if T is trivial then there is nothing to prove. Hence we can suppose that $\mathbb{P}$ contains $\tilde{\boldsymbol{r}}$-discs and that T is non-trivial.

By the key observation there must exist two discs Δ and $\bar{\Delta}$ (with basepoints O, O' and $\bar{O}, \bar{O}'$ respectively) in T, $\tilde{R} \in \tilde{\boldsymbol{r}}$ and $\epsilon = \pm 1$, such that the label on $\partial^+\Delta$ and $\partial^-\bar{\Delta}$ is $\tilde{R}_\epsilon$; the label on $\partial^-\Delta$ and $\partial^+\bar{\Delta}$ is $\tilde{R}_{-\epsilon}$. Also the first $\boldsymbol{t}$-arc intersecting $\partial^-\Delta$ intersects $\partial^+\bar{\Delta}$.

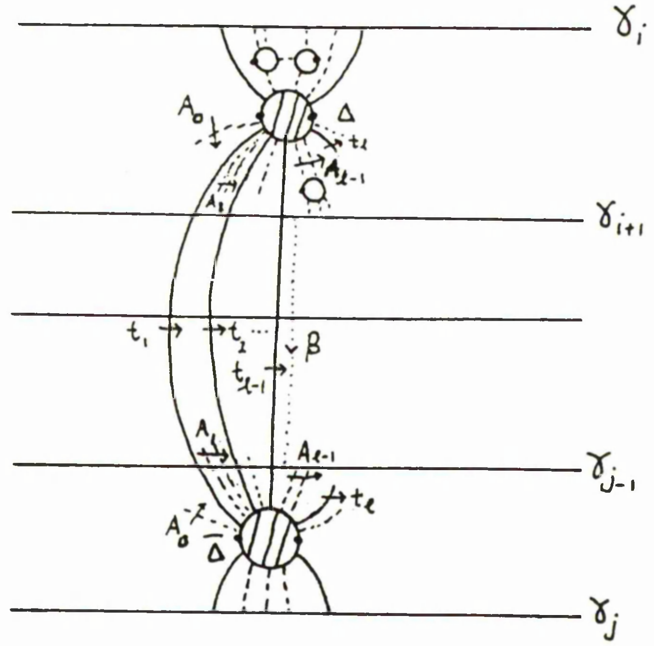
Suppose that Δ and $\bar{\Delta}$ lie in the subpictures with boundaries $\gamma_i\gamma_{i+1}^{-1}$ and $\gamma_{j-1}\gamma_j^{-1}$ respectively. If (after possibly deforming γ_i and γ_j, as shown below)



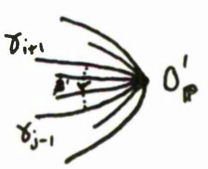
we can find a subpicture with boundary $\gamma_i'\beta'\gamma_j'^{-1}\beta^{-1}$ (γ_i' a subpath of γ_i, γ_j' a subpath of γ_j, β a simple path from $\iota(\gamma_i')$ to $\iota(\gamma_j')$, β' a simple path from $\tau(\gamma_i')$ to $\tau(\gamma_j')$), which is compatible with the stratification and is a dipole, then by Lemma 4.5.1 this dipole is (A)-equivalent to a subpicture which is compatible with γ and is a cancelling pair of $\tilde{\boldsymbol{r}}$-discs. This cancelling pair of $\tilde{\boldsymbol{r}}$-discs can be replaced by a subpicture which contains no $\tilde{\boldsymbol{r}}$-discs, using operation (B). The resulting picture $\mathbb{P}'$ has $\iota(\mathbb{P}) \equiv \iota(\mathbb{P}')$, $\tau(\mathbb{P}) \equiv \tau(\mathbb{P}')$ and contains less $\tilde{\boldsymbol{r}}$-discs than $\mathbb{P}$. Hence the result, in this case, follows by the inductive hypothesis.

If it not possible to find such a subpicture then there must exist an $\tilde{\boldsymbol{r}}$-disc in the subpicture with boundary $\gamma_{i+1}\gamma_{j-1}^{-1}$, which has a $\boldsymbol{t}$-arc intersecting Δ or $\bar{\Delta}$.

Suppose that $\tilde{R}_{-\epsilon} \equiv A_0 t_1 A_1 t_2 \ldots t_n A_n$ (A_i a word on $\boldsymbol{a}$ $(0 \leq i \leq n)$, $t_k \in \boldsymbol{t}$ $(1 \leq k \leq n)$). Suppose that the arcs labelled by $t_2, \ldots, t_{l-1}$ all intersect $\partial^-\Delta$ and $\partial^+\bar{\Delta}$, while t_l does not $(2 \leq l \leq n)$. Let β be a simple path from γ_{i+1} to γ_{j-1} which runs parallel to the $\boldsymbol{t}$-arc labelled by t_{l-1}, in a small neighbourhood to the left of the $\boldsymbol{t}$-arc, as shown below.



Also let β' be a simple path from γ_{i+1} to γ_{j-1} in a small neighbourhood of $O'_{\mathbb{P}}$.



Let γ'_{i+1} be the subpath of γ_{i+1} from the initial point of β to the initial point of β'. Let γ'_{j-1} be the subpath of γ_{j-1} from the terminal point of β to the terminal point of β'. We deform γ_{i+1} and γ_{j-1} (if necessary) so that travelling along γ'_{i+1} and γ'_{j-1} from

$\iota(\gamma'_{i+1})$ and $\iota(\gamma'_{j-1})$ respectively, we read the word $A_{\ell-1}t_\ell \ldots$. Let $\mathbb{Q}$ be the subpicture of $\mathbb{P}$ with boundary $\gamma'_{i+1}\beta'\gamma'^{-1}_{j-1}\beta^{-1}$. Note that $\mathbb{Q}$ is compatible with γ. Now choose points $O_{\mathbb{Q}}$ and $O'_{\mathbb{Q}}$ on β and β' respectively, and consider the monoid picture $\mathbb{Q}^{(\gamma)}$.

Now monoid picture $\mathbb{Q}^{(\gamma)}$ has a non-trivial left $\boldsymbol{t}$-circle, T' say, $\iota(\mathbb{Q}) \equiv A_{\ell-1}t_\ell \ldots$ and $\tau(\mathbb{Q}) \equiv A_{\ell-1}t_\ell \ldots$. By the inductive hypothesis this monoid picture is equivalent to one which has a trivial left $\boldsymbol{t}$-circle and contains less $\tilde{\boldsymbol{r}}$-discs. However the mechanics of the transformation in $\mathbb{Q}^{(\gamma)}$ can be repeated in $\mathbb{P}$, giving a monoid picture $\mathbb{P}'$ which contains less $\tilde{\boldsymbol{r}}$-discs than $\mathbb{P}$, and has $\iota(\mathbb{P}) \equiv \iota(\mathbb{P}')$, $\tau(\mathbb{P}) \equiv \tau(\mathbb{P}')$. Hence the result follows by the inductive hypothesis. □

A symmetrical argument involving right $\boldsymbol{t}$-circles gives the following result.

Lemma 4.7.2 *Suppose that $RG(\mathcal{M})$ is cycle free. Let $\mathbb{P}$ be a monoid picture over $\mathcal{M}$ with $\iota(\mathbb{P}) \equiv \ldots tA$ and $\tau(\mathbb{P}) \equiv \ldots tA$, where A is a word on $\boldsymbol{a}$ and $t \in \boldsymbol{t}$. Let T be the right $\boldsymbol{t}$-circle of $\mathbb{P}$. Then $\mathbb{P}$ is equivalent to a picture $\mathbb{P}'$, where $\iota(\mathbb{P}) \equiv \iota(\mathbb{P}')$, $\tau(\mathbb{P}) \equiv \tau(\mathbb{P}')$ and $\mathbb{P}'$ has a trivial right $\boldsymbol{t}$-circle. Moreover, if T is non-trivial then the number of $\tilde{\boldsymbol{r}}$-discs in $\mathbb{P}'$ is less than in $\mathbb{P}$.*

Theorem 4.7.3 *If $LG(\mathcal{M})$ or $RG(\mathcal{M})$ is cycle free then $\mathcal{M}$ is aspherical.*

Proof. Let $\mathbb{P}$ be a spherical monoid picture over $\mathcal{M}$, and suppose that $LG(\mathcal{M})$ is cycle free.

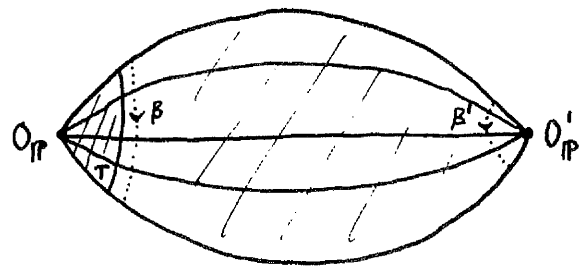
In order to prove the result, we will first show that $\mathbb{P}$ is equivalent to a spherical monoid picture over $\mathcal{M}$ which contains no $\tilde{\boldsymbol{r}}$-discs. This will be proved by induction on the number of $\tilde{\boldsymbol{r}}$-discs in $\mathbb{P}$.

If $\mathbb{P}$ contains no $\tilde{\boldsymbol{r}}$-discs then there is nothing to prove. If $\mathbb{P}$ contains at least one $\tilde{\boldsymbol{r}}$-disc then since each $\tilde{\boldsymbol{r}}$-disc has a $\boldsymbol{t}$-arc incident with each side of the disc and $\iota(\mathbb{P}) \equiv \tau(\mathbb{P})$, $\mathbb{P}$ must have a left $\boldsymbol{t}$-circle T.

Suppose that T is non-trivial. Since $\iota(\mathbb{P}) \equiv \tau(\mathbb{P})$, $\mathbb{P}$ is equivalent to a spherical monoid picture $\mathbb{P}'$ which has a trivial left $\boldsymbol{t}$-circle, by Lemma 4.7.1. Moreover $\mathbb{P}'$ contains fewer $\tilde{\boldsymbol{r}}$-discs than $\mathbb{P}$ and so the result follows by the inductive hypothesis.

Suppose that T is a trivial left $\boldsymbol{t}$-circle. Let β be a simple path from γ_0 to γ_n which runs parallel to T, in a small neighbourhood to the right of T as shown below. Let β'

be a simple path from γ_0 to γ_n in a small neighbourhood of $O'_{\mathbb{P}}$.



Let γ'_0 be the subpath of γ_0 from the initial point of β to the initial point of β'. Let γ'_n be the subpath of γ_n from the terminal point of β to the terminal point of β'. Let $\mathbb{Q}$ be the subpicture of $\mathbb{P}$ (compatible with the stratification) with boundary $\gamma'_0\beta'\gamma'^{-1}_n\beta^{-1}$. Choosing points $O_{\mathbb{Q}}$ and $O'_{\mathbb{Q}}$ on β and β' respectively, we consider the monoid picture $\mathbb{Q}^{(\gamma)}$, hereafter referred to as simply $\mathbb{Q}$. Note that $\mathbb{Q}$ is spherical.

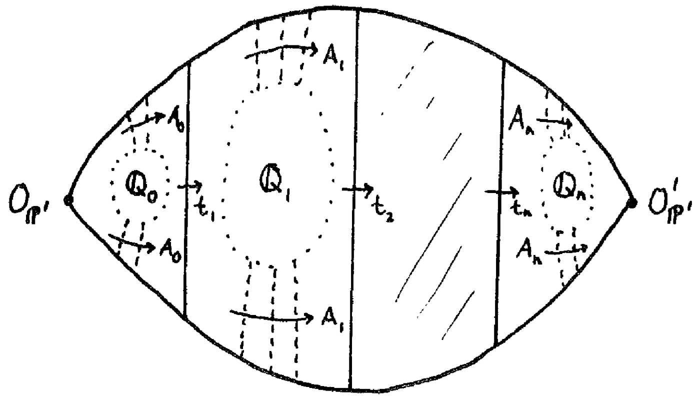
If $\mathbb{Q}$ has a non-trivial left $\boldsymbol{t}$-circle T', then since $\iota(\mathbb{Q}) \equiv \tau(\mathbb{Q})$, $\mathbb{Q}$ is equivalent to a spherical monoid picture $\mathbb{Q}'$ which has a trivial left $\boldsymbol{t}$-circle and less $\tilde{\boldsymbol{r}}$-discs than $\mathbb{Q}$, by Lemma 4.7.1. However the mechanics of this transformation can be performed in $\mathbb{P}$ to give a spherical monoid picture $\mathbb{P}'$ which has fewer $\tilde{\boldsymbol{r}}$-discs than $\mathbb{P}$. Hence the result follows by the inductive hypothesis.

If however T' is a trivial left $\boldsymbol{t}$-circle then we return to $\mathbb{P}$ and instead let β be a simple path which is parallel to T', in a small neighbourhood of T', and consider the resulting $\mathbb{Q}^{(\gamma)}$. Repeating (if necessary) this process a finite number of times we must eventually obtain a $\mathbb{Q}$ which has a non-trivial left $\boldsymbol{t}$-circle. We can then argue as above, and the result follows by the inductive hypothesis.

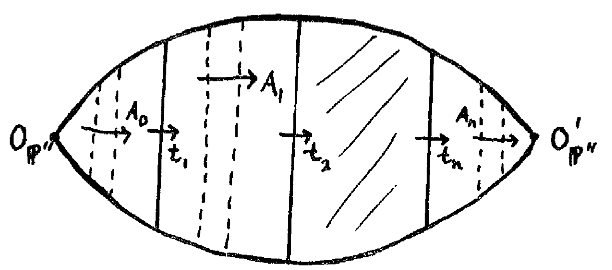
Thus we can assume that $\mathbb{P}$ is equivalent to a spherical monoid picture $\mathbb{P}'$ over $\boldsymbol{\mathcal{M}}$ which contains no $\tilde{\boldsymbol{r}}$-discs and has

$$\iota(\mathbb{P}') \equiv \tau(\mathbb{P}') \equiv A_0 t_1 A_1 t_2 \ldots t_n A_n$$

(A_i a word on $\boldsymbol{a}$ $(0 \leq i \leq n)$, $t_k \in \boldsymbol{t}$ $(1 \leq k \leq n)$). Let $\mathbb{Q}_i$ $(0 \leq i \leq n)$ be the subpicture of $\mathbb{P}'$ with boundary label $A_i A_i^{-1}$, as shown below.



We can apply bridge moves on the $\boldsymbol{a}$-arcs labelled by A_i $(0 \leq i \leq n)$ to obtain spherical pictures over $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$. These can be deleted to give a spherical picture $\mathbb{P}''$ over $\mathcal{M}$ which contains no discs, as required.



If $RG(\mathcal{M})$ is cycle free, then the result is proved by considering right $\boldsymbol{t}$-circles and applying Lemma 4.7.2, using a symmetrical argument to the above. $\square$

Corollary 4.7.4 *If $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle free and $\mathcal{M}$ is any mixed monoid presentation for $S(\mathcal{R})$ then $\mathcal{M}$ is aspherical.*

Proof. By Proposition 4.3.4, $LG(\mathcal{R})$ is identical to $LG(\mathcal{M})$, while $RG(\mathcal{R})$ is identical to $RG(\mathcal{M})$. Hence if $LG(\mathcal{R})$ $(RG(\mathcal{R}))$ is cycle free then so is $LG(\mathcal{M})$ $(RG(\mathcal{M})$ respectively). The result follows from Theorem 4.7.3 $\square$

4.8 Cancellation properties of $S(\mathcal{R})$

In this section we give sufficient conditions on $\mathcal{R}$ for $S(\mathcal{R})$ to be left and right cancellative. This work generalises Adjan's Theorem concerning cancellation in the ordinary semigroup presentation case [1].

Theorem 4.8.1 *If $LG(\mathcal{R})$ is cycle free then $S(\mathcal{R})$ is left cancellative.*

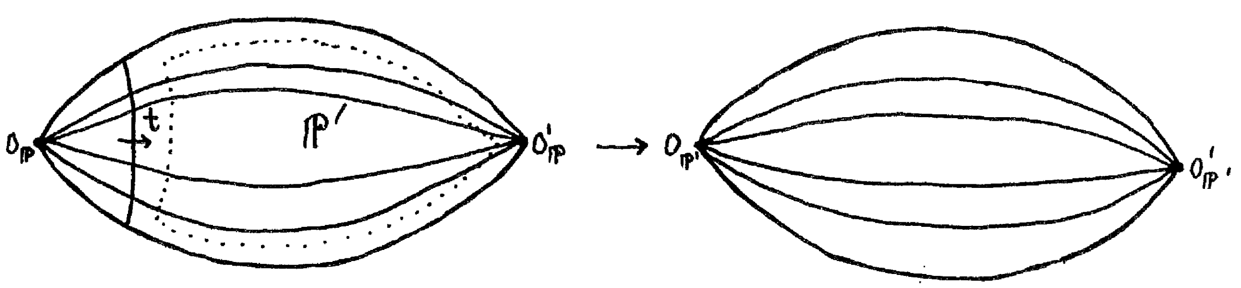
Proof. We prove the result by showing that $S(\mathcal{M})$ is left cancellative.

Let $U, V, W \in S(\mathcal{M})$ (U, V, W $\boldsymbol{t}$-positive words on $\boldsymbol{a} \cup \boldsymbol{t}$) and suppose that

$$UV =_{S(\mathcal{M})} UW \qquad (\star)$$

We require to show that $V =_{S(\mathcal{M})} W$. It is enough to consider the case where $U \equiv a \in \boldsymbol{a}^{\pm 1}$ or $U \equiv t \in \boldsymbol{t}$. If $U \equiv a$ then premultiplying both sides of $(\star)$ by a^{-1} and applying an operation of type (D), gives that $V =_{S(\mathcal{M})} W$.

Suppose that $U \equiv t$. By Theorem 4.4.3 there exists a monoid picture $\mathbb{P}$ over $\mathcal{M}$ with $\iota(\mathbb{P}) \equiv tV$ and $\tau(\mathbb{P}) \equiv tW$. By Lemma 4.7.1, we can assume that the left $\boldsymbol{t}$-circle of $\mathbb{P}$ is trivial (and of course labelled by t). This $\boldsymbol{t}$-circle can be deleted from $\mathbb{P}$ giving a monoid picture $\mathbb{P}'$ with $\iota(\mathbb{P}') \equiv V$ and $\tau(\mathbb{P}') \equiv W$.



Thus by Theorem 4.4.3, $V =_{S(\mathcal{M})} W$. □

A symmetrical argument gives the following result.

Theorem 4.8.2 *If $RG(\mathcal{R})$ is cycle free then $S(\mathcal{R})$ is right cancellative.*

Combining Theorems 4.8.1 and 4.8.2 gives the last of our cancellation results.

Theorem 4.8.3 *If $\mathcal{R}$ is cycle free then $S(\mathcal{R})$ is cancellative.*

4.9 Examples

Example 8. (i) Let H be a free group and suppose we choose two bases B_1, B_2 for $L \leq H$ of rank n:

$$B_1 = h_1, \ldots, h_n \quad B_2 = k_1, \ldots, k_n \ (h_i, k_i \in H,\ 1 \leq i \leq n)$$

Let $S(\mathcal{R})$ be the monoid defined by the relative monoid presentation:

$$\mathcal{R} = [H, t; h_i t = t k_i \ (1 \leq i \leq n)].$$

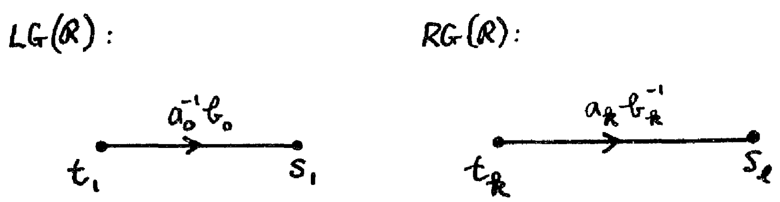
Note that since L is free, $\mathcal{R}$ is cycle free. Thus any mixed monoid presentation $\mathcal{M}$ for $S(\mathcal{R})$ is aspherical. Furthermore $S(\mathcal{R})$ is cancellative.

(ii) Let H be any group and consider the general one relator relative monoid presentation

$$\mathcal{R} = [H, \boldsymbol{t}; a_0 t_1 a_1 t_2 \ldots t_k a_k = b_0 s_1 b_1 s_2 \ldots s_l b_l]$$

where the a_i $(0 \leq i \leq k)$ and b_j $(0 \leq j \leq l)$ are elements of H, t_m $(1 \leq m \leq k)$ and s_n $(1 \leq n \leq l)$ are elements of $\boldsymbol{t}$.

Observe that $\mathcal{R}$ has the following left and right graphs:



Thus $\mathcal{R}$ is cycle free if and only if one of the following conditions hold:

(i) $t_1 \neq s_1$ and $t_k \neq s_l$.

(ii) $t_1 = s_1$, $t_k \neq s_l$ and $a_0^{-1} b_0 \neq_H 1$.

(iii) $t_1 \neq s_1$, $t_k = s_l$ and $a_k b_l^{-1} \neq_H 1$.

(iv) $t_1 = s_l$, $t_k = s_l$, $a_0^{-1} b_0 \neq_H 1$ and $a_k b_l^{-1} \neq_H 1$.

Chapter 5

Relative monoid presentations II: Embeddability of $S(\mathcal{R})$

5.1 $\widehat{\mathcal{M}}$ and $G(\widehat{\mathcal{M}})$

Throughout this chapter we appeal to the notation established in Chapter 4. Let $\mathcal{R} = [H, \boldsymbol{t}; \boldsymbol{r}]$ be a relative monoid presentation for $S(\mathcal{R})$ and let $\mathcal{M} = [\boldsymbol{a}; \boldsymbol{t}|\boldsymbol{s}; \tilde{\boldsymbol{r}}]$ be any mixed monoid presentation for $S(\mathcal{R})$, where $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$ is a group presentation for H. Our main aim in this chapter is to prove that $S(\mathcal{R})$ is embeddable in a group, provided that $\mathcal{R}$ is cycle free.

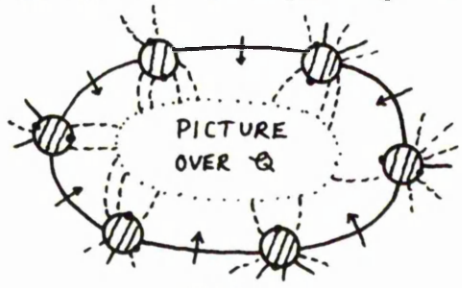
Let $\hat{\boldsymbol{r}} = \{\tilde{R}_{+1}\tilde{R}_{-1}^{-1} : \tilde{R} \in \tilde{\boldsymbol{r}}\}$ and $G(\widehat{\mathcal{M}})$ be the group defined by group presentation $\widehat{\mathcal{M}} =< \boldsymbol{a}, \boldsymbol{t}; \boldsymbol{s}, \hat{\boldsymbol{r}} >$. By Proposition 4.3.4, $\mathcal{R}$ is cycle free if and only if $\mathcal{M}$ is cycle free. Our result will be proved by showing that $S(\mathcal{M})$ is embeddable in $G(\widehat{\mathcal{M}})$, provided that $\mathcal{M}$ is cycle free.

We begin by introducing some concepts associated with pictures over $\widehat{\mathcal{M}}$, referring the reader to Chapter 1 for basic definitions and more details concerning pictures over group presentations. As with pictures over $\mathcal{M}$, we shade $\tilde{\boldsymbol{r}}$-discs and indicate $\boldsymbol{a}$-arcs using broken arcs.

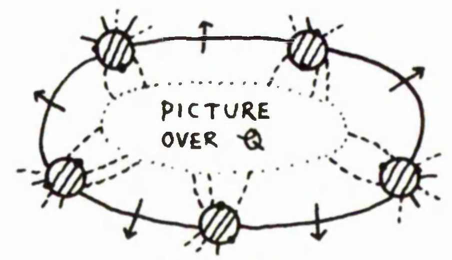
Let $\mathbb{P}$ be a picture over $\widehat{\mathcal{M}}$. The closures of the connected components of $\mathbb{P} - \{\tilde{\boldsymbol{r}}\text{-discs} \cup \boldsymbol{t}\text{-arcs}\}$ are called *$\tilde{\boldsymbol{r}}$-areas.* An *interior $\tilde{\boldsymbol{r}}$-area* is an $\tilde{\boldsymbol{r}}$-area which does not intersect with the boundary of $\mathbb{P}$. All $\tilde{\boldsymbol{r}}$-areas which are not interior $\tilde{\boldsymbol{r}}$-areas are called *boundary* $\tilde{\boldsymbol{r}}$*-areas.*

The connected components of $\{\tilde{\boldsymbol{r}}\text{-discs}\} \cup \{\boldsymbol{t}\text{-arcs}\}$ are called the ***t-components*** of $\mathbb{P}$. We say that $\mathbb{P}$ is *spherically* ***t****-connected* if every $\boldsymbol{t}$-component intersects $\partial\mathbb{P}$.

An $\tilde{\boldsymbol{r}}$-area Ξ of $\mathbb{P}$ is called a *sink* (*source*) if it is a simply connected, interior $\tilde{\boldsymbol{r}}$-area with all the $\boldsymbol{t}$-arcs in $\partial\Xi$ pointing into (respectively out from) Ξ.



SINK



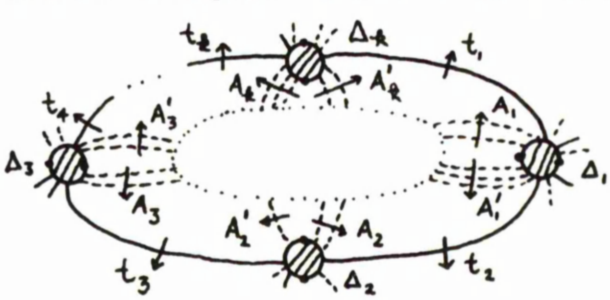
SOURCE

We assume that the transverse paths considered in this chapter cross any particular arc in a picture at most once. If γ is a transverse path then the label on γ will be denoted by $W(\gamma)$.

Adopting similar terminology to that used in Chapter 4, a ***t****-positive (****t****-negative)* transverse path in $\mathbb{P}$ is a transverse path which only crosses $\boldsymbol{t}$-arcs in the direction (respectively, against the direction) of their orientation. A *non-trivial* $\boldsymbol{t}$-positive ($\boldsymbol{t}$-negative) transverse path is a $\boldsymbol{t}$-positive ($\boldsymbol{t}$-negative) transverse path which crosses at least one $\boldsymbol{t}$-arc. For brevity we will refer to non-trivial $\boldsymbol{t}$-positive ($\boldsymbol{t}$-negative) transverse paths as $\boldsymbol{t}^+$*-paths* ($\boldsymbol{t}^-$*-paths* respectively). Note that $\boldsymbol{t}^+$-paths ($\boldsymbol{t}^-$-paths) are labelled by non-trivial $\boldsymbol{t}$-positive ($\boldsymbol{t}$-negative) words (on $\boldsymbol{a} \cup \boldsymbol{t}$).

Lemma 5.1.1 *If $LG(\mathcal{M})$ is cycle free, then any minimal picture over $\widehat{\mathcal{M}}$ has no sources.*

Proof. Let $\mathbb{P}$ be a minimal picture over $\widehat{\mathcal{M}}$ with a source Ξ. Note that each $\tilde{\boldsymbol{r}}$-disc on the boundary of Ξ has a basepoint which lies on the boundary of Ξ.

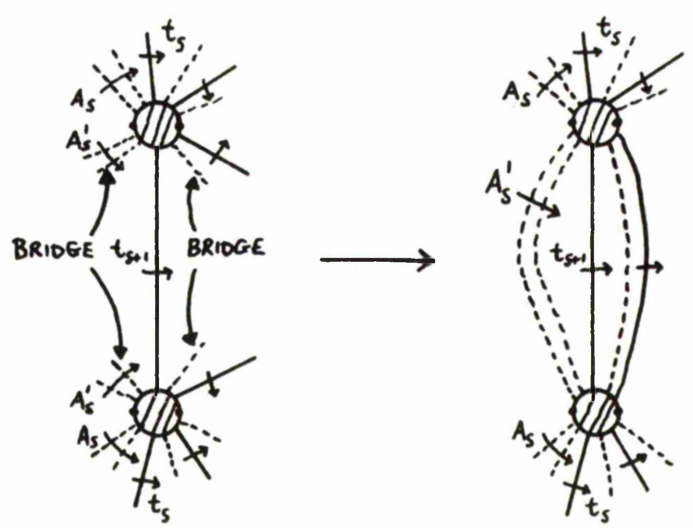


Suppose that the discs on the boundary of Ξ are labelled by the following relations.

$$\begin{aligned}
\Delta_1 \text{ by } \tilde{R}_1^{\epsilon_1} : A_1 t_1 \ldots &= A_1' t_2 \ldots \\
\Delta_2 \text{ by } \tilde{R}_2^{\epsilon_2} : A_2 t_2 \ldots &= A_2' t_3 \ldots \\
&\vdots \\
\Delta_k \text{ by } \tilde{R}_k^{\epsilon_k} : A_k t_k \ldots &= A_k' t_1 \ldots
\end{aligned}$$

where A_i, A_i' are words on $\boldsymbol{a}$, $t_i \in \boldsymbol{t}$, $\epsilon_i = \pm 1$, $\tilde{R}_i \in \tilde{\boldsymbol{r}}$ and $1 \leq i \leq k$. Travelling clockwise around the boundary of Ξ, we can find a closed transverse path γ which encloses all the $\boldsymbol{s}$-discs of Ξ and has $W(\gamma) \equiv A_1^{-1} A_1' A_2^{-1} A_2' \ldots A_k^{-1} A_k'$. Note that $W(\gamma)$ represents the identity in H by Theorem 1.6.4, being the boundary label of a picture over $\mathcal{Q}$.

Now γ corresponds to the closed path $e_{\tilde{R}_1}^{\epsilon_1} e_{\tilde{R}_2}^{\epsilon_2} \ldots e_{\tilde{R}_k}^{\epsilon_k}$ in $LG(\mathcal{M})$ which is labelled by the identity in H. By hypothesis this path must not be reduced. Suppose that $e_{\tilde{R}_s}^{\epsilon_s} = e_{\tilde{R}_{s+1}}^{-\epsilon_{s+1}}$ for some $1 \leq s \leq k-1$, and so $\tilde{R}_s^{\epsilon_s} = \tilde{R}_s^{-\epsilon_{s+1}}$. By performing bridge moves (as necessary) on the $\boldsymbol{a}$-arcs and $\boldsymbol{t}$-arcs, as shown below, we can obtain a subpicture which is a dipole[1].



By Proposition 1.6.1, $\mathbb{P}$ is equivalent to a picture $\mathbb{P}'$ which has two fewer discs than $\mathbb{P}$, and has the same boundary label. This contradicts the fact that $\mathbb{P}$ was minimal.

□

[1]Note that the term *dipole* is used in the sense of a picture over a *group* presentation (see §1.6).

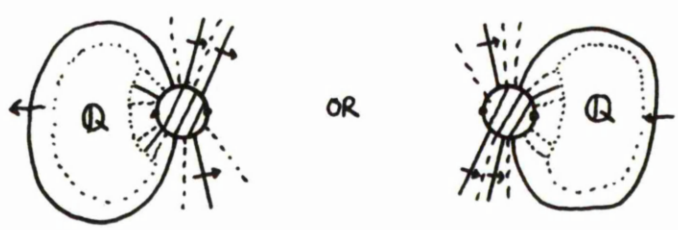
Lemma 5.1.2 *If $RG(\mathcal{M})$ is cycle free, then any minimal picture over $\widehat{\mathcal{M}}$ has no sinks.*

Proof. The proof is similar to that of the previous lemma. However in this case, travelling anticlockwise around the boundary of Ξ, we find a closed transverse path γ which is labelled by a word $W(\gamma)$ on $\boldsymbol{a}$, and represents the identity in H. This corresponds to a closed path in $RG(\mathcal{M})$ which is labelled by the identity in H. By hypothesis, this path must not be reduced. Proceeding as in the proof of Lemma 5.1.1, we obtain a contradiction to the minimality of $\mathbb{P}$. □

Combining the results of the previous Lemmas gives the following result.

Lemma 5.1.3 *If $\mathcal{M}$ is cycle free, then any minimal picture over $\widehat{\mathcal{M}}$ has no sources or sinks.*

In what follows, proving our results is made more complicated by the possible existence in a picture $\mathbb{P}$ over $\widehat{\mathcal{M}}$, of an $\tilde{\boldsymbol{r}}$-disc Δ which has a $\boldsymbol{t}$-arc which intersects both $\partial^+\Delta$ and $\partial^-\Delta$.



For brevity, we refer to such discs as *self identified $\tilde{\boldsymbol{r}}$-discs.* Let $\mathbb{Q}$ be the subpicture over $\widehat{\mathcal{M}}$ indicated in the Figures. Note that if $\mathbb{Q}$ does not contain any $\tilde{\boldsymbol{r}}$-discs, then $\mathbb{P}$ must have a source in the case illustrated on the left, a sink in the case on the right.

The following technical theorem deals with the existence of $\boldsymbol{t}^+$-paths in pictures over $\widehat{\mathcal{M}}$, and the pictures enclosed by them. We postpone the proof until the end of the chapter.

Theorem 5.1.4 *Let $\mathbb{P}$ be a spherically $\boldsymbol{t}$-connected picture over $\widehat{\mathcal{M}}$ with at least two $\tilde{\boldsymbol{r}}$-discs and no sources, sinks or self identified $\tilde{\boldsymbol{r}}$-discs. Suppose that travelling from $O_{\mathbb{P}}$*

to $O'_{\mathbb{P}}$ along $\partial^+\mathbb{P}$ ($\partial^-\mathbb{P}$) we read U (respectively V), where U, V are $\boldsymbol{t}$-positive words, with at least U non-empty. Then one of the following statements hold concerning $\mathbb{P}$:

(A) there exists a closed, $\mathbf{t}^+$-path γ either

(a) at $O_{\mathbb{P}}$ or $O'_{\mathbb{P}}$, or

(b) between interior $\tilde{\boldsymbol{r}}$-areas of $\mathbb{P}$

which encloses at least one $\tilde{\boldsymbol{r}}$-disc, but not all the $\tilde{\boldsymbol{r}}$-discs of $\mathbb{P}$;

(B) there exists a $\mathbf{t}^+$-path γ from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$, such that the subpictures $\mathbb{P}_1$ and $\mathbb{P}_2$ of $\mathbb{P}$, with boundaries $\partial^+\mathbb{P}\gamma^{-1}$ and $\partial^-\mathbb{P}\gamma^{-1}$ respectively, contain at least one $\tilde{\boldsymbol{r}}$-disc each.

We remark that J.H. Remmers [48] proved a result concerning group diagrams, which is analogous to the case where U and V are both non-trivial $\boldsymbol{t}$-positive words in the result above. He used this to give a geometric proof of Theorem 2.1.1 [Adjan]. Theorem 2.1.1 can in fact be obtained as a special case of Theorem 5.3.3, by taking H to be the trivial group.

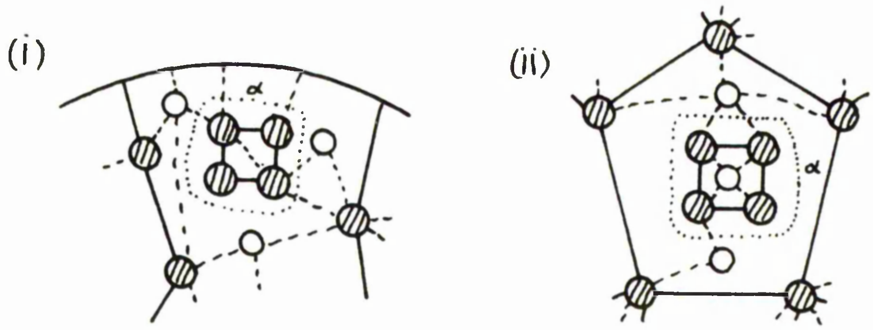
5.2 Words in $G(\widehat{\mathcal{M}})$

Theorem 5.2.1 *If $\mathcal{M}$ is cycle free, then no minimal picture over $\widehat{\mathcal{M}}$ which contains at least one $\tilde{\boldsymbol{r}}$-disc but no self identified $\tilde{\boldsymbol{r}}$-discs, can have boundary label which is a $\boldsymbol{t}$-positive word.*

Proof. We proceed by assuming that the result is false.

Consider the set of all minimal pictures over $\widehat{\mathcal{M}}$ which contain at least one $\tilde{\boldsymbol{r}}$-disc but no self identified $\tilde{\boldsymbol{r}}$-discs, and have boundary label a $\boldsymbol{t}$-positive word. By assumption this set is non-empty. Let $\mathbb{P}$ be an element from this set which contains the smallest number of $\tilde{\boldsymbol{r}}$-discs.

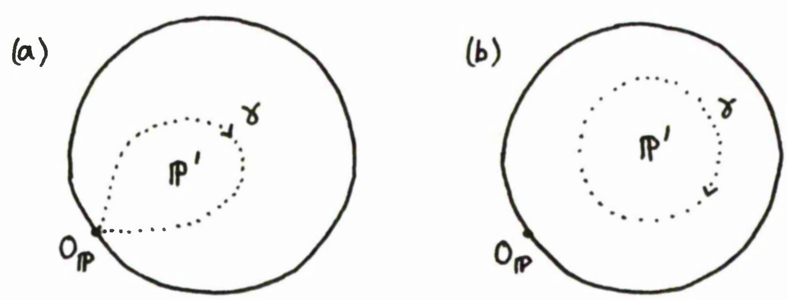
First note that $\mathbb{P}$ has no $\boldsymbol{t}$-arcs which do not intersect an $\tilde{\boldsymbol{r}}$-disc, otherwise the boundary label of $\mathbb{P}$ could not be a $\boldsymbol{t}$-positive word. Also $\mathbb{P}$ must be spherically $\boldsymbol{t}$-connected otherwise it would be possible to find a closed transverse path γ (as in (i) or (ii) below), such that $W(\gamma)$ is a $\boldsymbol{t}$-positive word (in fact just a word on $\boldsymbol{a}$), and the (sub)picture enclosed by γ (which is minimal by Proposition 1.6.3) contains less $\tilde{\boldsymbol{r}}$-discs than $\mathbb{P}$, contradicting the original choice of $\mathbb{P}$.



Suppose that $\mathbb{P}$ contains exactly one $\tilde{\boldsymbol{r}}$-disc Δ. Since the boundary label of $\mathbb{P}$ is a $\boldsymbol{t}$-positive word, not all the $\boldsymbol{t}$-arcs incident with Δ can intersect $\partial\mathbb{P}$. (Recall that there exists at least one $\boldsymbol{t}$-arc incident with $\partial^+\Delta$, and at least one incident with $\partial^-\Delta$.) Hence there must exist a $\boldsymbol{t}$-arc which intersects $\partial^+\Delta$ and $\partial^-\Delta$. We conclude that Δ is a self identified disc, contrary to assumption.

Hence we can assume that $\mathbb{P}$ contains two or more $\tilde{\boldsymbol{r}}$-discs.

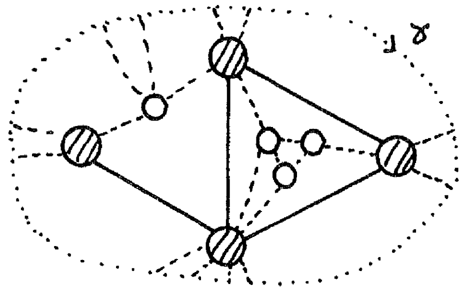
By Lemma 5.1.3, $\mathbb{P}$ has no sources or sinks, and so by Theorem 5.1.4 there exists a closed, $\boldsymbol{t}^+$-path γ, either (a) at $O_{\mathbb{P}} = O_{\mathbb{P}'}$, or (b) between interior $\tilde{\boldsymbol{r}}$-areas of $\mathbb{P}$. Furthermore γ encloses at least one $\tilde{\boldsymbol{r}}$-disc but not all the $\tilde{\boldsymbol{r}}$-discs of $\mathbb{P}$.



In either case, the boundary of the picture $\mathbb{P}'$ enclosed by γ is labelled by a $\boldsymbol{t}$-positive word $W(\gamma)$. Also, by Proposition 1.6.3, $\mathbb{P}'$ is minimal. However $\mathbb{P}'$ contains less $\tilde{\boldsymbol{r}}$-discs than $\mathbb{P}$, and so we have a contradiction to our original choice of $\mathbb{P}$. □

Corollary 5.2.2 *If $\mathcal{M}$ is cycle free then any minimal picture over $\widehat{\mathcal{M}}$ which does not contain any self identified $\tilde{\boldsymbol{r}}$-discs, is spherically $\boldsymbol{t}$-connected.*

Proof. Suppose that $\mathbb{P}$ is a minimal picture over $\widehat{\mathcal{M}}$ and that $\mathbb{P}$ has a $\boldsymbol{t}$-component which does not intersect with $\partial\mathbb{P}$. Then it must be possible to find a closed transverse path γ which encloses the $\boldsymbol{t}$-component as a subpicture $\mathbb{Q}$ of $\mathbb{P}$, and is labelled by a word $W(\gamma)$ on $\boldsymbol{a}$.



It is clear that $\mathbb{Q}$ must contain at least one $\tilde{\boldsymbol{r}}$-disc. Also, by Proposition 1.6.3, $\mathbb{Q}$ is a minimal picture. However this is not possible by Theorem 5.2.1. □

5.3 An embedding theorem

Before proving our main result, we deal with the technicalities arising from self identified $\tilde{\boldsymbol{r}}$-discs.

Theorem 5.3.1 *Suppose that $\mathcal{M}$ is cycle-free, then any minimal picture over $\widehat{\mathcal{M}}$ with boundary label UV^{-1} (U, V $\boldsymbol{t}$-positive words) and no self identified $\tilde{\boldsymbol{r}}$-discs, is a monoid picture over $\mathcal{M}$.*

Proof. Let $\mathbb{P}$ be a minimal picture over $\widehat{\mathcal{M}}$, as described in the statement of the Theorem. Then $\mathbb{P}$ is clearly a picture over $\mathcal{M}$. We require to show that we can find a stratification for $\mathbb{P}$.

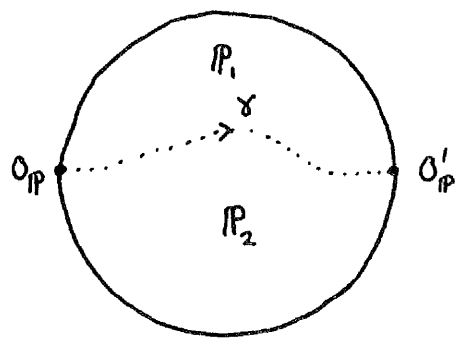
We prove the result by induction on the number of $\tilde{\boldsymbol{r}}$-discs in $\mathbb{P}$.

If $\mathbb{P}$ has no $\tilde{\boldsymbol{r}}$-discs then $\mathbb{P}$ is a trivial monoid picture over $\mathcal{M}$ and there is nothing to prove. Also, if $\mathbb{P}$ has a single $\tilde{\boldsymbol{r}}$-disc then $\mathbb{P}$ is an atomic monoid picture over $\mathcal{M}$ and there is nothing to prove.

Hence we can assume that $\mathbb{P}$ contains at least two $\tilde{\boldsymbol{r}}$-discs. By Corollary 5.2.2, $\mathbb{P}$ is spherically $\boldsymbol{t}$-connected, while by Lemma 5.1.3, $\mathbb{P}$ has no sources or sinks. Thus we can apply Theorem 5.1.4.

In case (A), by considering $W(\gamma)$ as the boundary of the picture over $\widehat{\mathcal{M}}$ enclosed by γ and using Proposition 1.6.3, we obtain a minimal picture over $\widehat{\mathcal{M}}$ with no self identified $\tilde{\boldsymbol{r}}$-discs and boundary label a $\boldsymbol{t}$-positive word. This contradicts Theorem 5.2.1.

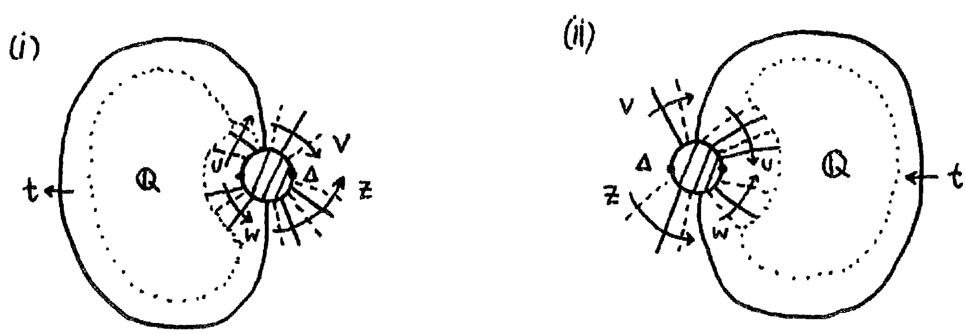
In case (B), in view of Proposition 1.6.3, we can apply the inductive hypothesis to the pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ over $\widehat{\mathcal{M}}$, with boundary labels $UW(\gamma)^{-1}$ and $W(\gamma)V^{-1}$ respectively.



The stratifications for $\mathbb{P}_1$ and $\mathbb{P}_2$, together with γ, give a stratification for $\mathbb{P}$. □

Corollary 5.3.2 *Suppose that $\mathcal{M}$ is cycle free, then no minimal picture $\mathbb{P}$ over $\widehat{\mathcal{M}}$ contains a self identified $\tilde{\boldsymbol{r}}$-disc.*

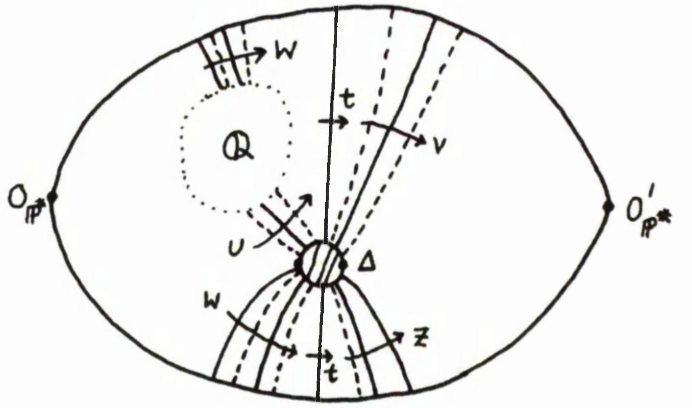
Proof. Suppose, by way of contradication, that $\mathbb{P}$ is a minimal picture over $\widehat{\mathcal{M}}$ with a self identified $\tilde{\boldsymbol{r}}$-disc Δ. Then $\mathbb{P}$ has a subpicture of one of the following forms:



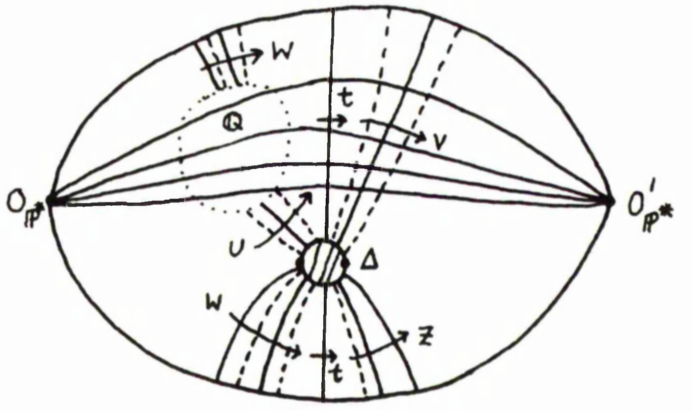
where U, V, W, Z are $\boldsymbol{t}$-positive words and $\mathbb{Q}$ is a picture over $\widehat{\mathcal{M}}$ which is minimal (by Proposition 1.6.3). We can assume that $\mathbb{Q}$ contains no self identified $\tilde{\boldsymbol{r}}$-discs by choosing Δ appropriately. Also $\mathbb{Q}$ must contain at least one $\tilde{\boldsymbol{r}}$-disc, otherwise $\mathbb{P}$ would have a source in case (i), a sink in case (ii). Furthermore, U and W must be non-trivial $\boldsymbol{t}$-positive words, otherwise $\mathbb{Q}$ contradicts Theorem 5.2.1.

We consider case (i) and remark that case (ii) follows by a symmetrical argument to what follows.

Consider the following picture $\mathbb{P}^*$ over $\widehat{\mathcal{M}}$.



Since $\mathbb{Q}$ is a minimal picture over $\widehat{\mathcal{M}}$, by Theorem 5.3.1, $\mathbb{Q}$ is a monoid picture over $\mathcal{M}$. Now the stratification for $\mathbb{Q}$ can be used to obtain a stratification for $\mathbb{P}^*$, as shown below.



Thus $\mathbb{P}^*$ is a monoid picture over $\mathcal{M}$. Also since $\iota(\mathbb{P}^*)$ and $\tau(\mathbb{P}^*)$ both start with the subword Wt, $\mathbb{P}^*$ has a non-trivial left $\boldsymbol{t}$-circle. By Lemma 4.7.1, $\mathbb{P}^*$ is equivalent to a picture $\mathbb{P}^{*'}$ with $\iota(\mathbb{P}^{*'}) \equiv \iota(\mathbb{P}^*)$, $\tau(\mathbb{P}^{*'}) \equiv \tau(\mathbb{P}^*)$ and trivial left $\boldsymbol{t}$-circle. Hence it must be possible to remove at least two $\tilde{\boldsymbol{r}}$-discs from $\mathbb{P}^*$, using the operations described in Chapter 4. However it must also be possible to remove the corresponding $\tilde{\boldsymbol{r}}$-discs from $\mathbb{P}$, using group picture operations, and obtain a picture with the same boundary label as $\mathbb{P}$ but fewer discs. This contradicts the fact that $\mathbb{P}$ was minimal. $\square$

We can now prove our main result.

Theorem 5.3.3 *If $\mathcal{M}$ is cycle free then $S(\mathcal{M})$ is embeddable in $G(\widehat{\mathcal{M}})$.*

Proof. Let U and V be $\boldsymbol{t}$-positive words such that $U =_{G(\widehat{\mathcal{M}})} V$. We require to show that $U =_{S(\mathcal{M})} V$.

By Corollary 1.6.5 there exists a picture over $\widehat{\mathcal{M}}$ with $\iota(\mathbb{P}) \equiv U$ and $\tau(\mathbb{P}) \equiv V$. We can assume that $\mathbb{P}$ is minimal.

By Corollary 5.3.2, $\mathbb{P}$ has no self identified $\tilde{\boldsymbol{r}}$-discs, and hence by Theorem 5.3.1, $\mathbb{P}$ is a monoid picture over $\mathcal{M}$. Thus by Theorem 4.4.3, $U =_{S(\mathcal{M})} V$. □

The following result is immediate.

Corollary 5.3.4 *If $\mathcal{R}$ is cycle free then $S(\mathcal{R})$ is embeddable in $G(\widehat{\mathcal{M}})$.*

5.4 Transverse paths and $\tilde{r}$-discs

In this section we give a proof of Theorem 5.1.4. However, we first require a Lemma.

Lemma 5.4.1 *Let $\mathbb{P}$ be a picture over $\widehat{\mathcal{M}}$ with boundary $\alpha\beta^{-1}$ (where β is possibly empty) and basepoints $O_{\mathbb{P}}$, $O'_{\mathbb{P}}$ such that $\partial^+\mathbb{P} = \alpha$ and $\partial^-\mathbb{P} = \beta$. Let γ be a $\boldsymbol{t}^+$-path in $\mathbb{P}$ which crosses at least one $\boldsymbol{t}$-arc connecting two distinct $\tilde{\boldsymbol{r}}$-discs in $\mathbb{P}$. Furthermore, suppose that γ is either*

(i) a closed path at $O_{\mathbb{P}}$ or $O'_{\mathbb{P}}$, or

(ii) a closed path between interior $\tilde{\boldsymbol{r}}$-areas of $\mathbb{P}$, or

(iii) a path from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$.

Then γ, or in case (iii) $\gamma\alpha^{-1}$, is a closed path which encloses at least one $\tilde{\boldsymbol{r}}$-disc, but not all the $\tilde{\boldsymbol{r}}$-discs of $\mathbb{P}$.

Proof. We first identify $\partial\mathbb{P}$ to a point C, obtaining a sphere S^2. Now in all cases γ is a closed transverse path on S^2. We define the *interior of* γ, $\mathrm{int}(\gamma)$, to be the part of S^2 enclosed by γ, which does not contain C. Let P be any point in $S^2 - \{\gamma \cup \mathrm{int}(\gamma) \cup C \cup$ all discs and arcs of $\mathbb{P}\}$. We enlarge this point to a circle and so obtain a disc D. Transverse path γ now exists as a closed path within D, and so by the Jordan Curve Theorem [38, Theorem 9.1] γ splits D into two connected components.

Suppose that γ crosses a $\boldsymbol{t}$-arc which intersects two distinct $\tilde{\boldsymbol{r}}$-discs, Δ_1 and Δ_2 say. Recall that our transverse paths cross any particular $\boldsymbol{t}$-arc (or $\boldsymbol{a}$-arc) at most once. Thus one of Δ_1, Δ_2 must lie in the component of D which intersects with ∂D, while the other must not. The result is proved. $\square$

Proof of Theorem 5.1.4. We begin by choosing one point in the closure of each $\tilde{\boldsymbol{r}}$-area, in the following way. For the boundary $\tilde{\boldsymbol{r}}$-areas containing $O_{\mathbb{P}}$ and $O'_{\mathbb{P}}$, we choose precisely $O_{\mathbb{P}}$ and $O'_{\mathbb{P}}$ respectively, while in all other $\tilde{\boldsymbol{r}}$-areas we arbitrarily choose a point which does not lie on the boundary of the $\tilde{\boldsymbol{r}}$-area.

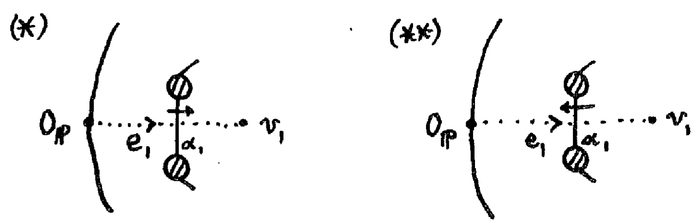
We first prove the Theorem under the assumption that $\mathbb{P}$ has at least one interior $\tilde{\boldsymbol{r}}$-area.

We will construct a $\boldsymbol{t}^+$-path or $\boldsymbol{t}^-$-path, edge by edge, with initial vertex $O_{\mathbb{P}}$.

Suppose that it is possible to join $O_{\mathbb{P}}$ to a vertex v_1 in some interior $\tilde{\boldsymbol{r}}$-area Ξ_1 of $\mathbb{P}$, by an edge e_1 which crosses a single $\boldsymbol{t}$-arc α_1. In that case there are two possibilities (see Figure below):

(*) e_1 crosses α_1 in the direction of its orientation, or

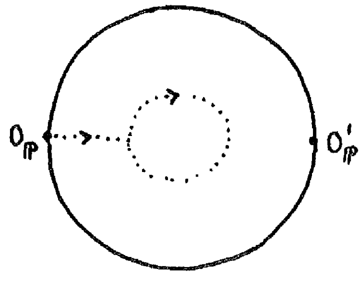
(**) e_1 crosses α_1 against the direction of its orientation.



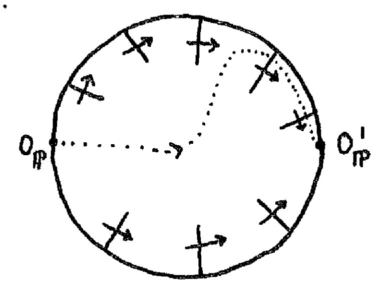
Note that since $\mathbb{P}$ is spherically $\boldsymbol{t}$-connected, every interior $\tilde{\boldsymbol{r}}$-area is simply connected.

Case (*): Since $\mathbb{P}$ has only simply connected interior $\tilde{\boldsymbol{r}}$-areas and no sources, sinks or self identified $\tilde{\boldsymbol{r}}$-discs, we can find an edge e_2 which crosses a single $\boldsymbol{t}$-arc α_2 in the direction of its orientation, and joins v_1 with some vertex v_2 in an interior or boundary $\tilde{\boldsymbol{r}}$-area, Ξ_2, of $\mathbb{P}$. If Ξ_2 is an interior $\tilde{\boldsymbol{r}}$-area, then repeating the argument we can find an edge e_3 which crosses a single $\boldsymbol{t}$-arc α_3 in the direction of its orientation, and joins v_2 with some vertex v_3 in an interior or boundary $\tilde{\boldsymbol{r}}$-area, Ξ_3, of $\mathbb{P}$. Repeating this argument a finite number of times, we must eventually re-enter an interior $\tilde{\boldsymbol{r}}$-area that we have been in before, or enter a boundary $\tilde{\boldsymbol{r}}$-area.

In the first case we let γ be the closed subpath of the $\boldsymbol{t}^+$-path we have constructed. It is clear that γ must have crossed at least one $\boldsymbol{t}$-arc which connects two distinct $\tilde{\boldsymbol{r}}$-discs of $\mathbb{P}$, and therefore the result follows by Lemma 5.4.1.



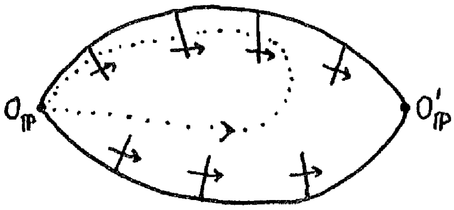
In the latter case, suppose that our $\boldsymbol{t}^+$-path enters a boundary $\tilde{\boldsymbol{r}}$-area Ξ_k with associated vertex v_k. If $v_k \neq O'_{\mathbb{P}}$, then we can find an edge e_{k+1} which crosses a single $\boldsymbol{t}$-arc α_{k+1} in the direction of its orientation, and joins v_k with some vertex v_{k+1} in an adjacent boundary $\tilde{\boldsymbol{r}}$-area to Ξ_k. Continuing in this way, we can extend our $\boldsymbol{t}^+$-path round the boundary $\tilde{\boldsymbol{r}}$-areas until it reaches $O'_{\mathbb{P}}$. We then take γ to be the $\boldsymbol{t}^+$-path that we have constructed.



It is clear that we can apply Lemma 5.4.1, and so the result is proved in this case.

Case (**): Since $\mathbb{P}$ has only simply connected interior $\tilde{\boldsymbol{r}}$-areas and no sources, sinks or self identified $\tilde{\boldsymbol{r}}$-discs, we can find an edge e_2 which crosses a single $\boldsymbol{t}$-arc α_2 against the direction of its orientation, and joins v_1 with some vertex v_2 in an interior $\tilde{\boldsymbol{r}}$-area, Ξ_2, of $\mathbb{P}$. If Ξ_2 is an interior $\tilde{\boldsymbol{r}}$-area, then repeating the argument we can find an edge e_3 which crosses a single $\boldsymbol{t}$-arc α_3 against the direction of its orientation, and joins v_2 with some vertex v_3 in an interior or boundary $\tilde{\boldsymbol{r}}$-area, Ξ_3, of $\mathbb{P}$. Repeating this argument a finite number of times, we must eventually re-enter an interior $\tilde{\boldsymbol{r}}$-area that we have been in before, or enter a boundary $\tilde{\boldsymbol{r}}$-area.

In the first case, as in Case (*), the $\boldsymbol{t}^-$-path we have constructed has a closed subpath between interior $\tilde{\boldsymbol{r}}$-areas of $\mathbb{P}$. We take γ to be the inverse path of this closed subpath (so that γ is a $\boldsymbol{t}^+$-path) and apply Lemma 5.4.1 to get the required result. In the latter case, we can assume that the $\boldsymbol{t}^-$-path we have constructed is a closed path at $O_{\mathbb{P}}$ by extending it (using a similar argument to that used in the latter part of Case (*)) round the boundary $\tilde{\boldsymbol{r}}$-areas until it returns to $O_{\mathbb{P}}$.

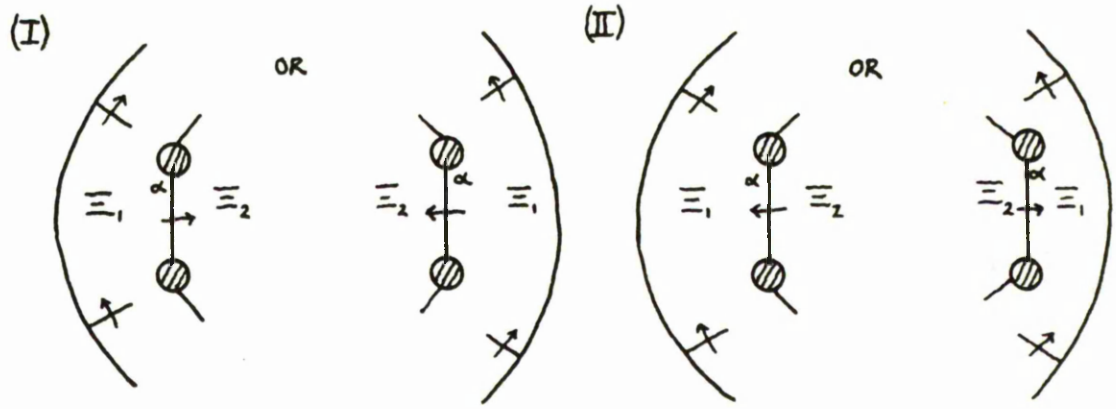


We now take γ to be the inverse of the $\boldsymbol{t}^-$-path that we have constructed. Applying Lemma 5.4.1 finishes the argument.

Now if UV^{-1} is a word on $\boldsymbol{a}$ then one of (*) or (**) must hold, since $\mathbb{P}$ contains at least one interior $\tilde{\boldsymbol{r}}$-area. Hence we can assume that UV^{-1} involves at least one element of $\boldsymbol{t}^{\pm 1}$, and therefore that there exists at least one $\boldsymbol{t}$-arc in $\mathbb{P}$ which intersects $\partial\mathbb{P}$.

Suppose that neither (*) nor (**) are attainable. Then the first edge of any $\boldsymbol{t}^+$-path or $\boldsymbol{t}^-$-path starting at $O_{\mathbb{P}}$ must cross over a $\boldsymbol{t}$-arc into another boundary $\tilde{\boldsymbol{r}}$-area.

Since $\mathbb{P}$ has an interior $\tilde{\boldsymbol{r}}$-area, there exists a $\boldsymbol{t}$-arc α which intersects two distinct $\tilde{\boldsymbol{r}}$-discs and lies on both the boundary of some boundary $\tilde{\boldsymbol{r}}$-area Ξ_1, as well as on the boundary of some interior $\tilde{\boldsymbol{r}}$-area Ξ_2, as in Figure (I) or (II) below.



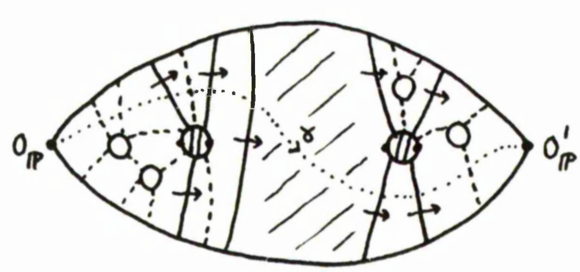
Case (I): Starting at $O_{\mathbb{P}}$ we construct a $\boldsymbol{t}^+$-path by travelling around the boundary $\tilde{\boldsymbol{r}}$-areas until Ξ_1 is reached, then crossing over α into Ξ_2. An analysis identical to that applicable if (*) held can then be applied.

Case (II): Starting at $O'_{\mathbb{P}}$ we construct a $\boldsymbol{t}^-$-path by travelling around the boundary areas until Ξ_1 is reached, then crossing over α into Ξ_2. An analysis identical to that applicable if (**) held can then be applied.

This completes the proof in the case where $\mathbb{P}$ has at least one interior $\tilde{\boldsymbol{r}}$-area.

Now suppose that $\mathbb{P}$ has no interior $\tilde{\boldsymbol{r}}$-areas so that all $\tilde{\boldsymbol{r}}$-areas are boundary $\tilde{\boldsymbol{r}}$-areas.

Assume first that all the $\boldsymbol{t}$-components involve at most one $\tilde{\boldsymbol{r}}$-disc. Since $\mathbb{P}$ has no self identified $\tilde{\boldsymbol{r}}$-discs, it must have the form:

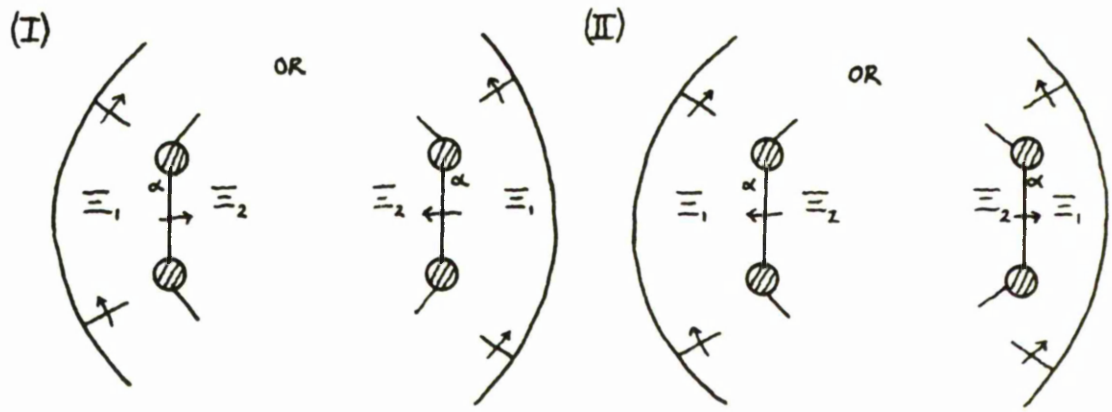


In this case it is obvious that we can find a $\boldsymbol{t}^+$-path γ from $O_{\mathbb{P}}$ to $O'_{\mathbb{P}}$ such that the subpictures with boundaries $\partial^+\mathbb{P}\gamma^{-1}$ and $\partial^-\mathbb{P}\gamma^{-1}$ each contains at least one $\tilde{\boldsymbol{r}}$-disc of

$\mathbb{P}$.

Hence we can assume that there exists a $\boldsymbol{t}$-component which involves at least two $\tilde{\boldsymbol{r}}$-discs, and thus that there exists a $\boldsymbol{t}$-arc α which intersects two distinct $\tilde{\boldsymbol{r}}$-discs.

Let Ξ_1 and Ξ_2 be boundary $\tilde{\boldsymbol{r}}$-areas such that arc α lies on $\partial\Xi_1$ and $\partial\Xi_2$, as shown below.



If the situation is of type (I) ((II)), starting at $O_{\mathbb{P}}$ ($O'_{\mathbb{P}}$, respectively) we can construct a $\boldsymbol{t}^+$-path ($\boldsymbol{t}^-$-path) by travelling around the boundary $\tilde{\boldsymbol{r}}$-areas until Ξ_1 is reached. After crossing over α into Ξ_2, we return to $O_{\mathbb{P}}$ ($O'_{\mathbb{P}}$) around the boundary $\tilde{\boldsymbol{r}}$-areas, giving a closed path at $O_{\mathbb{P}}$ ($O'_{\mathbb{P}}$). In case (I) we take γ to be the $\boldsymbol{t}^+$-path constructed, while in case (II) we take γ to be the inverse of the $\boldsymbol{t}^-$-path constructed. Finally, we apply Lemma 5.4.1. to get the result. $\square$

Bibliography

[1] S. I. Adjan, *On the embeddability of semigroups in groups*, Doklady Akad. Nauk SSSR **133** (1960), 255–257; English translation: Soviet Mathematics **1** (1960), Am. Math. Soc.

[2] S. I. Adjan, *Defining relations and algorithmic problems for groups and semigroups*, Trudy Mat. Inst. Steklov **85** (1966) (in Russian); Translations of the Am. Math. Soc **152** (1967).

[3] Y. G. Baik, J. Howie and S. J. Pride, *The identity problem for graph products of groups*, to appear in J. Algebra.

[4] Y. G. Baik and S. J. Pride, *Generators of the second homotopy module of presentations arising from group constructions*, preprint, University of Glasgow, 1992.

[5] W. A. Bogley and S. J. Pride, *Aspherical relative presentations*, Proc. Edinburgh Math. Soc. **35** (1992), 1–39.

[6] W. A. Bogley and S. J. Pride, *Calculating generators of π_2*, in *Two dimensional homotopy and combinatorial group theory* (C. Hog-Angeloni, W. Metzler, A. Sieradski, editors), CUP (1993), 157–188.

[7] S. G. Brick, *Normal-convexity and equations over groups*, Invent. Math. **94** (1988), 81–104.

[8] G. C. Bush, *The embedding theorems of Malcev and Lambek*, Canad. J. Math. **15** (1963), 49–58.

[9] G. C. Bush, *Note on an embedding theorem of Adjan*, Proc. Am. Math. Soc. **14** (1963), 597–599.

[10] D. E. Cohen, *Combinatorial group theory: a topological approach*, LMS Student Texts **14**, CUP (1989).

[11] D. E. Cohen, *String rewriting - a survey for group theorists*, in *Geometric group theory, Volume 1* (G.A. Niblo, M.A. Roller, editors), CUP (1993), 37–47.

[12] C. Choffrut, *Conjugacy in free inverse monoids*, in *Word equations and related topics* (H. Abdulrab, J-P. Pécuchet, editors), Lect. notes in Comp. Sci. **677** (1993), 6–22.

[13] J. R. Cho and S. J. Pride, *Embedding semigroups into groups and the asphericity of semigroups*, Internat. J. Alg. Comp. **3** (1993), 1–13.

[14] A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups, Vol. II*, Math. Surveys **7**, Amer. Math. Soc. (1967).

[15] J. Dauns, *Centers of semigroup rings and conjugacy classes*, Semigroup Forum **38** (1989), 355–364.

[16] M. R. Doss, *Sur l'immersion d'un semi-groupe dans un groupe*, Bull. Sci. Math. (2) **72** (1948), 139–150.

[17] M.-L. Dubreil-Jacotin, *Sur l'immersion d'un semigroupe dans un groupe*, C.R. Acad. Sci. Paris **225** (1947), 787–788.

[18] M.S. El-Mosalamy and S.J. Pride, *On T(6) groups*, Math. Proc. Camb. Phil. Soc. **102** (1987), 443–451.

[19] D. B. A. Epstein, *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford Ser. (2), **12** (1961), 205-212.

[20] R. Goldstein and J. Teymouri, *Adjan's theorem and conjugacy in semigroups*, Semigroup Forum **47** (1993), 299 –304.

[21] V.S. Guba, *On the embeddability of semigroups in groups*, to appear in Matematicheskiye Zametki (in Russian).

[22] V.S. Guba and M.V. Sapir, *Diagram groups*, preprint, University of Nebraska–Lincoln, 1995.

[23] P. M. Higgins, *Techniques of semigroup theory*, Oxford Univ. Press (1992).

[24] J. Howie, *How to generalise one-relator group theory*, in *Combinatorial group theory and topology*, Annals of Math. Studies **111** (S. Gersten and J.R. Stallings eds.), Princeton University Press, (1987), 53–78.

[25] S.V. Ivanov, *Relation modules and relation bimodules of groups, semigroups and associative algebras*, Internat. J. Alg. Comp. **1** (1991), 89–114.

[26] D. L. Johnson, *Presentations of groups*, London Math. Soc. Student Texts **15**, CUP (1990).

[27] E. V. Kashintsev, *Small Cancellation conditions and embeddability of semigroups in groups*, Internat. J. Alg. Comp. **2** (1992), 433–441.

[28] C.W. Kilgour and S.J. Pride, *Cockcroft presentations*, to appear in J. Pure and Applied Alg.

[29] V. Kilibarda, *On the algebra of semigroup diagrams*, Ph.D. Thesis, University of Nebraska–Lincoln, 1994.

[30] S. Krstić, *Embedding semigroups in groups — A geometrical approach*, preprint.

[31] G. Lallement, *Semigroups and combinatorial applications*, John Wiley & Sons, New York (1979).

[32] J. Lambek, *The immersibility of a semigroup into a group*, Can. J. Math. **3** (1951), 34–43.

[33] R.C. Lyndon and P.E. Schupp, *Combinatorial group theory*, Springer-Verlag (1977).

[34] I. Lysionok and S. J. Pride, *On the structure of the second homotopy module of presentations of split extensions*, preprint, University of Glasgow, 1993.

[35] A.I. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann. **133** (1937), 686–691.

[36] A.I. Malcev, *On the immersion of associative systems in groups*, Mat. Sbornik (N.S.) **6** (1939), 331–336.

[37] A.I. Malcev, *On the immersion of associative systems in groups II*, Mat. Sbornik (N.S.) **8** (1940), 251–264.

[38] A.Y. Ol'Shanskii, *The geometry of defining relations in groups*, Kluwer Academic Press (1991).

[39] O. Ore, *Linear equations in non-commutative fields*, Ann. of Math. (2) **32** (1931), 463–477.

[40] F. Otto, *Conjugacy in monoids with a special Church-Rosser presentation is decidable*, Semigroup Forum **29** (1984), 223–240.

[41] M. Petrich, *Introduction to semigroups*, C.E. Merril Pub. Co. (1973).

[42] S. J. Pride, *Identities among relations of group presentations*, in *Group theory from a geometrical viewpoint, Trieste 1990* (E. Ghys, A. Haefliger, A. Verjovsky, editors), World Scientific Publishing (1991), 687–717.

[43] S. J. Pride, *Examples of presentations which are minimally Cockcroft in several different ways*, J. Pure Appl. Algebra, **88** (1993), 199–204.

[44] S. J. Pride, *Low dimensional homotopy theory for monoids*, to appear in Intern. J. Alg. Comput. **5** (1995).

[45] S. J. Pride, *Geometric methods in combinatorial semigroup theory*, to appear in *Proceedings of international conference in groups, semigroups and formal languages, York 1993* (J. Fountain, editor), Kluwer Publications (1995).

[46] S. J. Pride, *Low dimensional homotopy for monoids II: Groups*, preprint, University of Glasgow, 1995.

[47] J.H. Remmers, *Some algebraic problems for semigroups: a geometric approach*, Ph.D. Thesis, University of Michigan (1971).

[48] J.H. Remmers, *On the geometry of semigroup presentations*, Adv. in Math. **36** (1980), 283–296.

[49] J. J. Rotman, *An introduction to the theory of groups*, 3rd Edition, Wm.C. Brown Pub. (1988).

[50] J.-P. Serre, *Trees*, Springer-Verlag (1980).

[51] P.V. Silva, *Conjugacy and transposition for inverse monoid presentations*, preprint, Universidade do Porto, 1995.

[52] J.R. Stallings, *A graph theoretic lemma and group embeddings*, in *Combinatorial group theory and topology*, Annals of Math. Studies **111** (S. Gersten and J.R. Stallings eds.), Princeton University Press, (1987), 145–155.

[53] J. Teymouri, *Geometric methods in group theory*, Ph.D. Thesis, SUNY Albany (1988).

[54] L. Zhang, *Conjugacy in special monoids*, J. Alg. **143** (1991), 487–497.