REWRITING SYSTEMS, FINITENESS CONDITIONS AND SECOND ORDER DEHN FUNCTIONS OF MONOIDS

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Contents

Statement

Chapter 1 covers some basic material such as rewriting systems, monoid presentations, group presentations, graphs and 2-complexes, group pictures, monoid pictures, first and second order Dehn functions of groups and monoids. This material can be found, for example, in [12], [46], [47], [60], [67], [68], [69], [74], [76], [81].

Chapter 2, 3, 4 are the original work of the author, with the exception of instances indicated within the text, as well as the main results in §2.5, §2.6, §4.5, §4.6 which are joint work with S.J. Pride ([71], [72], [73]).

The main results in §3.3 ([79]) were published in the Theoretical Computer Science 191 (1998) 219-228, and the main results in §2.3, §4.1 ([80]) were published in the Journal of Algebra 204 (1998) 493-503. Results in §2.5, §2.6 ([72]) have been submitted for publication.

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Abstract

The main work of this thesis starts with Chapter 2.

In Chapter 2, we first give some basic definitions and results about rewriting systems, then we consider finite complete rewriting systems for small extensions of monoids and for semi-direct products of monoids. After introducing the notion of directed 2-complex and some results about it, we consider subgroups of finite index in groups with finite complete rewriting systems.

In Chapter 3, we first give some basic definitions and results about monoids of finite derivation type (FDT) and finite homological type (FHT), and their associated second order Dehn functions. Then we consider these properties for semi-direct products of monoids. We get that the class of FDT monoids and the class of FHT monoids are closed under semi-direct products. We also get some general bounds for second order Dehn functions of direct products of monoids.

In Chapter 4, we continue to consider FDT, FHT and second order Dehn functions for some monoid constructions, such as small extensions and relative monoids. We get that the class of FDT monoids and the class of FHT monoids are closed under small extensions. Let S be a monoid, and let S_0 be a submonoid of S such that $S \backslash S_0$ is an ideal of S. If S is FDT (respectively, FHT), then so is S_0, and we have $\tilde{\gamma}^{(2)}_{S_0} \preceq \tilde{\gamma}^{(2)}_{S}$ (respectively, $\gamma^{(2)}_{S_0} \preceq \gamma^{(2)}_{S}$). For a relative monoid $S = S(\mathcal{R})$ with a coefficient group H, if $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle-free, then S is FDT (respectively, FHT) if and only if H is. We also get some relations between the second order Dehn functions of S and H.

Notations

$\mathbb{Z}$	the set of all integers
$\mathbb{N}$	the set of all natural numbers
$\mathbb{Z}^+$	the set of all non-negative integers
$\mathbb{R}^+$	the set of all non-negative real numbers
X^*	the free monoid on X
$\varnothing$	empty word
$l(W)$	the length of a word W
$d_R(W)$	the disorder of a word W
$st(W)$	the stretch of a word W
$W_1 \equiv W_2$	two words W_1 and W_2 are identical
$\mathcal{P} = [X; R]$	monoid presentation
$\longrightarrow_R$	the single-step reduction relation induced by R
$\longrightarrow_R^*$	the reduction relation induced by R
$\longleftrightarrow_R^*$	the Thue congruence generated by R
$S(\mathcal{P})$	the monoid defined by $\mathcal{P}$
$[W]_R$	the element of $S(\mathcal{P})$ represented by W
$A \ltimes_Q B$	the semi-direct product of A and B relative to Q
Γ	graph or directed graph
$\mathcal{D} = \langle \Gamma; R \rangle$	2-complex with underlying graph Γ
$\pi_1(\mathcal{D}, v)$	the fundamental group of $\mathcal{D}$ at v
$\mathcal{K} = [\Gamma; R]$	directed 2-complex with underlying directed graph Γ
$\pi_1^+(\mathcal{K}, v)$	the fundamental monoid of $\mathcal{K}$ at v

$\hat{\mathcal{P}} = \langle X; R \rangle$	group presentation
$G(\hat{\mathcal{P}})$	the group defined by $\hat{\mathcal{P}}$
$\pi_2(\hat{\mathcal{P}})$	the second homotopy module of $\hat{\mathcal{P}}$
$\delta_G^{(1)}$	the first order Dehn function of a group G
$\delta_G^{(2)}$	the second order Dehn function of a group G
$\gamma_S^{(1)}$	the first order Dehn function of a monoid S
$\Gamma(\mathcal{P})$	the graph associated to $\mathcal{P}$
$\mathcal{D}(\mathcal{P})$	the Squier complex associated to $\mathcal{P}$
$\mathbb{Z}S$	the integral monoid ring
$A(p)$	the area of a monoid picture p
$\pi_2^{(b)}(\mathcal{P})$	the second homology bimodule of $\mathcal{P}$
$\tilde{V}_{\mathcal{P},\boldsymbol{X}}(p)$	the homotopical volume of p
$V_{\mathcal{P},c}(p)$	the homological volume of p
$\tilde{\gamma}_S^{(2)}$	the second order homotopical Dehn function of a monoid S
$\gamma_S^{(2)}$	the second order homological Dehn function of a monoid S
$\mathcal{R} = [H, \boldsymbol{y}; \boldsymbol{u}]$	relative monoid presentation
$LG(\mathcal{R})$	the left graph of $\mathcal{R}$
$RG(\mathcal{R})$	the right graph of $\mathcal{R}$

Introduction

In 1912, Max Dehn ([29]) formulated three fundamental decision problems for groups defined by group presentations. These problems are important for presentation theory and its applications. The word problem is the first of the three decision problems. It can be stated as follows. Let $\hat{\mathcal{P}} = \langle X; R\rangle$ be a group presentation. Given a word W on $X \cup X^{-1}$ decide in a finite number of steps whether W represents the identity of the group presented by $\hat{\mathcal{P}}$.

Having solvable word problem is an invariant property of finitely presented groups. The word problem has been solved successfully for many classes of groups, e.g., one relator groups, small cancellation groups and automatic groups. However, Novikov ([62], 1955) and Boone ([13], 1959) proved, independently, that the word problem is unsolvable in general. (See also Britton ([14], 1963).)

The word problem for groups can be extended to monoids as follows. Let $\mathcal{P} = [X; R]$ be a monoid presentation. Given two words U, V on X decide in a finite number of steps whether U and V represent the same element of the monoid presented by $\mathcal{P}$. Having solvable word problem is also an invariant property of finitely presented monoids. So we can speak of monoids that have solvable word problem. Of course, the word problem for monoids is unsolvable in general.

String rewriting systems have played a major role in the development of theoretical computer science. They are also used in combinatorial semigroup and group theory to present monoids and groups. In general, these rewriting systems do not give much information about the monoids or groups presented. However, if a monoid can be presented by a finite complete rewriting system, then the word problem for the monoid can be solved effectively through the normal form algorithm. Actually, if R is a finite complete rewriting system on X, given two words U, V on X, they can be reduced in a finite number of steps to their normal forms (the irreducible words) U_0 and V_0, respectively. Then U and V represent the same element of the monoid presented by $[X; R]$ if and only if U_0 and V_0 are identical.

So it is natural to ask whether finite complete rewriting systems provide a way to solve all word problems for finitely presented monoids, that is, can every finitely presented monoid with a solvable word problem be presented by a finite complete rewriting system? This important question remained open for many years. In 1987 Squier ([77]) answered this question in the negative. He showed that if a monoid can be presented by a finite complete rewriting system, then it must satisfy the homological finiteness condition FP_3. Then he showed by giving infinitely many examples that there exist finitely presented monoids with solvable word problems which can not be presented by finite complete rewriting systems.

It was later realised that results of Anick ([6], 1986) contained, in a different language and with some conditions which can be weakened, the stronger result that if a monoid can be presented by a finite complete rewriting system then it satisfies the homological finiteness condition FP_∞. Kobayashi ([53], 1990), Groves ([39], 1991) and Brown ([16], 1992) also proved this result.

Kapar and Narendran ([48]) showed that the rewriting system $\{(aba, bab)\}$ on the alphabet $\{a, b\}$ has no equivalent finite complete rewriting system. However the monoid it presents can also be presented by the finite complete rewriting system

$$\{(ab, c), (ca, bc), (bcb, cc), (ccb, acc)\}$$

on the alphabet $\{a, b, c\}$. It is therefore of interest to obtain conditions that a monoid must satisfy if it can be presented by a finite complete rewriting system.

In 1987 Squier wrote another paper which has been published posthumously in 1994 ([78]). In this paper Squier introduced the combinatoral finiteness property of having finite derivation type (FDT for short) for finitely presented monoids. He proved that the property of having finite derivation type is independent of the finite presentation of the monoid, that is, if one finite presentation of a monoid has finite derivation type, then so does every finite presentation of the monoid. He also proved that if a monoid can be presented by a finite complete rewriting system, then it has FDT. Moreover, he proved that a particular monoid S_1 does not have FDT, so S_1 can not be presented by a finite complete rewriting system. However, S_1 is a finitely presented monoid with a solvable word problem and it satisfies the homological finiteness condition FP_∞. Thus the condition FP_∞ is not sufficient for a finitely presented monoid with a solvable word problem to admit a presentation through a finite complete rewriting system.

Cremanns and Otto ([27]), Lafont ([54]) and Pride ([68], [69]) showed, independently, that for finitely presented monoids the property FDT implies the homological finiteness condition FP_3. Note that Squier's monoid S_1 shows that for finitely presented monoids with solvable word problems, the property FDT is strictly stronger than the property FP_3. Cremanns and Otto ([28]) also proved that for groups the two properties FDT and FP_3 are equivalent. Since there exist finitely presented groups with solvable word problems which are FP_3 but not FP_∞ (see [8]), the condition FDT is not sufficient for a finitely presented monoid with a solvable word problem to admit a presentation through a finite complete rewriting system.

Groves and Smith ([40], [41]) considered the property of having finite complete rewriting systems for various group-theoretic constructions (extending from subgroups, forming amalgamated free products, HNN-extensions and wreath products, etc.). They also proved the following two results.

Let K be a normal subgroup of the group G. If both K and G/K have finite complete rewriting systems, then so does G. In particular, a semi-direct product of groups with finite complete rewriting systems has a finite complete rewriting system.

Let H be a subgroup of the group G, and let H have finite index in G. If H has a finite complete rewriting system, then so does G. However, the converse of this result still remains open.

Many results about the properties FP_n and FP_∞ for various group constructions can be found in [8].

Another finiteness condition introduced by Pride and X. Wang is the finite homological type (FHT for short) for finite monoid presentations [74]. They proved that FHT is an invariant property of finitely presented monoids, that is, if one finite presentation of a monoid is FHT, then so is every finite presentation of the monoid. If a monoid is FDT, then it is FHT. They also proved that for groups FDT and FHT are equivalent. The question whether FHT implies FDT is open in general.

The first order Dehn functions (i.e., the minimum isoperimetric functions) of groups are important for discussing the complexity of the word problems for finitely presented groups. A finitely presented group G has a solvable word problem if and only if the first order Dehn function $\delta_G^{(1)}$ is recursive (see [36]). Gromov ([38]) proved that a finitely presented group G is hyperbolic if and only if the first order Dehn function of G is linear. The automatic groups satisfy the quadratic isoperimetric inequality (see [34]).

The first order Dehn function $\gamma_S^{(1)}$ of a monoid S has been discussed in [57], [69] (see also [9]). If G is a finitely presented group, then the two first order Dehn functions $\delta_G^{(1)}$ and $\gamma_G^{(1)}$ are equivalent (see [74]). The second order Dehn function $\delta_G^{(2)}$ of an FDT group G has been discussed in [4], [5].

Pride and X. Wang ([74]) introduced the second order Dehn functions $\tilde{\gamma}_S^{(2)}$ and $\gamma_S^{(2)}$ for an FDT and FHT monoid S, respectively. If S is FDT then $\tilde{\gamma}_S^{(2)} \succeq \gamma_S^{(2)}$. It is unknown (even for FDT groups) whether $\tilde{\gamma}_S^{(2)}$ and $\gamma_S^{(2)}$ are equivalent. If G is an FDT group, then the second order Dehn functions $\delta_G^{(2)}$ and $\gamma_G^{(2)}$ are equivalent ([74]).

In this thesis, we investigate how the properties of having finite complete rewriting systems, FDT and FHT behave under various monoid constructions. We also get some general bounds for the second order Dehn functions of monoids for some monoid constructions. The main work of this thesis starts with Chapter 2.

Chapter 2 contains our results about finite complete rewriting systems. In Sections 2.1 and 2.2 we give the basic definitions and some known results concerning finite complete rewriting systems. In Section 2.3 we prove (Theorem 2.3.4) that if a monoid T can be presented by a finite complete rewriting system, then every small extension S of T also can be presented by a finite complete rewriting system. (Let S be a monoid and let T be a submonoid of S. If the set $S \backslash T$ is finite, then S is called a *small extension* of T). This result appears in the paper [80]. In Section 2.4 we prove (Theorem 2.4.1) that the semi-direct product of monoids with finite complete rewriting systems also has a finite complete rewriting system. This result generalizes a result in [41] from groups to monoids. In Section 2.5 we introduce the notion of directed 2-complex and get some general results about 2-complexes. In Section 2.6 we use the results in Section 2.5 to get the following result (Theorem 2.6.1)([72]).

Let H be a subgroup of finite index n in a group G. If G has a finite complete rewriting system, then so does the free product $H * F_{n-1}$, where F_{n-1} is the free group of rank $n-1$.

Note that this result provides a link between the following two open questions. Question 1 ([41], [23]): Let H be a subgroup of finite index in a group G. If G has a finite complete rewriting system, does H have a finite complete rewriting system? Question 2 ([65]): Let A and B be finitely presented monoids. If the free product $A * B$ has a finite complete rewriting system, do A and B have finite complete rewriting systems?

From Chapter 3 we start to consider FDT, FHT and second order Dehn functions of monoids for various monoid constructions. In Sections 3.1 and 3.2 we give the basic definitions and some known results about FDT, FHT and second order Dehn functions of monoids. In Section 3.3 we prove (Theorem 3.3.1) that the class of FDT monoids is closed under semi-direct products. This result appears in [79]. The converse of this result is false. In Section 3.4 we prove (Theorem 3.4.1) that the class of FHT monoids is also closed under semi-direct products. In Section 3.5 we get the following general bounds for second order Dehn functions of direct products of monoids (Theorems 3.5.8 and 3.5.9).

If A and B are FDT, then

$$max\{\tilde{\gamma}_A^{(2)},\ \tilde{\gamma}_B^{(2)}\} \preceq \tilde{\gamma}_{A\times B}^{(2)} \preceq \overline{max\{\tilde{\gamma}_A^{(2)},\ \tilde{\gamma}_B^{(2)}\}} + n^2.$$

If A and B are FHT, then

$$max\{\gamma_A^{(2)},\ \gamma_B^{(2)}\} \preceq \gamma_{A\times B}^{(2)} \preceq \overline{max\{\gamma_A^{(2)},\ \gamma_B^{(2)}\}} + n^2.$$

This result generalizes a result in [4] from groups to monoids.

In Section 4.1 we prove (Theorem 4.1.8) that if a monoid T is FDT, then every small extension S of T is also FDT. This result appears in the paper [80]. We also get a similar result for FHT (Theorem 4.1.9). In Section 4.2 we consider FDT, FHT and second order Dehn functions for submonoids with ideal complements. Let S be a monoid, and let S_0 be a submonoid of S such that $S\backslash S_0$ is an ideal of S. If S is FDT (respectively, FHT), then so is S_0, and we have $\tilde{\gamma}_{S_0}^{(2)} \preceq \tilde{\gamma}_S^{(2)}$ (respectively, $\gamma_{S_0}^{(2)} \preceq \gamma_S^{(2)}$) (Propositions 4.2.3 and 4.2.5). In Section 4.3 we get some results concerning FDT, FHT and relative asphericity.

In [50] Kilgour introduced the notion of a relative monoid presentation $\mathcal{R}$. This is a triple $[H, \mathbf{y}; \mathbf{u}]$ where H is a group (the coefficient group), $\mathbf{y}$ is a set, and each element $U \in \mathbf{u}$ is an ordered pair (U_{+1}, U_{-1}) of elements of the free product $H * \mathbf{y}^*$ such that neither U_{+1} nor U_{-1} lies in H. The monoid $S(\mathcal{R})$ defined by $\mathcal{R}$ is the quotient of $H * \mathbf{y}^*$ by the congruence generated by $\mathbf{u}$. Kilgour associated with $\mathcal{R}$ two graphs $LG(\mathcal{R})$ (the left graph) and $RG(\mathcal{R})$ (the right graph) whose edges are labelled by elements of H. These graphs are said to be cycle-free if there is no non-empty reduced closed path such that the product of the labels of edges making up the path is equal to 1 in H. Using geometric methods, Kilgour proved that if $LG(\mathcal{R})$ (respectively, $RG(\mathcal{R})$) is cycle-free then S is left (respectively, right) cancellative, and if $LG(\mathcal{R})$ and $RG(\mathcal{R})$ are both cycle-free then S is embeddable in a group. (These results generalize results of Adian ([1], [2]) which deal with the case $H = 1$.)

In Section 4.4 we give some definitions about relative monoids. In Section 4.5 we prove (Theorem 4.5.2) that for a relative monoid $S = S(\mathcal{R})$ with a coefficient group H, if $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle-free, then S is FDT (respectively, FHT) if and only if H is. In Section 4.6 we get the following result (Theorem 4.6.1).

Let $S = S(\mathcal{R})$ be a relative monoid with the coefficient group H, and suppose that $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle-free. If H is FDT, then so is S, and we also have

$$\tilde{\gamma}_H^{(2)} \preceq \tilde{\gamma}_S^{(2)} \preceq \overline{\tilde{\gamma}}_H^{(2)},$$

$$\gamma_H^{(2)} \preceq \gamma_S^{(2)} \preceq \overline{\gamma}_H^{(2)},$$

$$(\delta_H^{(2)} \preceq \gamma_S^{(2)} \preceq \overline{\delta}_H^{(2)}).$$

Chapter 1

Preliminaries

1.1 Monoid presentations

1.1.1 Monoids and congruences

A *semigroup* is a set with an associative binary operation on it. A *monoid* is a semigroup with an identity.

Let M be a monoid. A relation ρ on the set M is called *compatible* if whenever $(a, b) \in \rho$ then $(cac', cbc') \in \rho$ for any $c, c' \in M$. A compatible equivalence relation on M is called a *congruence*.

If ρ is a congruence on a monoid M then we can define a binary operation on the quotient set M/ρ in a natural way as follows

$$[a]_\rho [b]_\rho = [ab]_\rho,$$

where $[c]_\rho$ denotes the congruence class containing c for any $c \in M$. It is easy to check that this operation is well-defined and M/ρ is a monoid with respect to this operation. The natural mapping ϕ_ρ from M onto M/ρ given by

$$\phi_\rho : \ M \longrightarrow M/\rho, \quad a \longmapsto [a]_\rho$$

is a homomorphism.

Theorem 1.1.1. *If $\psi : M \longrightarrow M'$ is a homomorphism of monoids, then the relation*

$$Ker\psi = \{(a,b) \in M \times M : \psi(a) = \psi(b)\}$$

is a congruence on M and there is a monomorphism $\alpha : M/Ker\psi \longrightarrow M'$ such that $Im\alpha = Im\psi$ and $\psi = \alpha \circ \phi_{Ker\psi}$, where $\phi_{Ker\psi} : M \longrightarrow M/Ker\psi$ is the natural homomorphism.

1.1.2 Free monoids

For a non-empty set X, let X^* denote the set of all finite *words* $x_1x_2\cdots x_m$ on X. A binary operation is defined on X^* by concatenation:

$$(x_1x_2\cdots x_m)(x_1'x_2'\cdots x_n') = x_1x_2\cdots x_mx_1'x_2'\cdots x_n'.$$

With respect to this operation X^* is a monoid, called the *free monoid* on X. The identity of X^* is the *empty word*, denoted by $\varnothing$. The *length* of a word $W = x_1x_2\cdots x_m$ is the number m of the letters involved in W, denoted $l(W)$. Note that $l(\varnothing) = 0$.

Theorem 1.1.2. *Let X be a non-empty set and let S be a monoid. If $\phi : X \longrightarrow S$ is an arbitrary mapping, then there exists a unique homomorphism $\psi : X^* \longrightarrow S$ such that $\psi|X = \phi$.*

Corollary 1.1.3. *Every monoid is isomorphic to a quotient of a free monoid by a congruence.*

1.1.3 Rewriting systems and monoid presentations

A *rewriting system* R on X is a subset of $X^* \times X^*$. Its elements are refered to as rewriting rules, and they are often written in the form $r : \; r_{+1} = r_{-1}$ for $(r_{+1}, r_{-1}) \in R$. We will assume that r_{+1} and r_{-1} are distinct words on X. The single-step reduction relation $\longrightarrow_R$ is the following relation on X^*:

$$U \longrightarrow_R V \quad \text{iff} \quad U \equiv W_1r_{+1}W_2 \text{ and } V \equiv W_1r_{-1}W_2$$
$$\text{for some } (r_{+1}, r_{-1}) \in R \text{ and } W_1, W_2 \in X^*.$$

Its reflexive, transitive closure $\longrightarrow_R^*$ is the reduction relation induced by R, and its reflexive, symmetric and transitive closure $\longleftrightarrow_R^*$ is the Thue congruence generated by R. A word U is called *reducible* if there is a word V such that $U \longrightarrow_R V$; if there is no such V we call U *irreducible*. For any $W \in X^*$, let $[W]_R$ denote the congruence class $\{U \in X^* | U \longleftrightarrow_R^* W\}$.

A *monoid presentation* $\mathcal{P}$ is a pair $[X; R]$, where R is a rewriting system on X. The quotient monoid $X^*/\longleftrightarrow_R^*$ will be called the monoid *presented* (or *defined*) by $\mathcal{P}$, denoted by $S(\mathcal{P})$. If a monoid M is isomorphic to $S(\mathcal{P})$, then we say that $\mathcal{P} = [X; R]$ is a monoid presentation of M with generators X and defining relations R.

Lemma 1.1.4. *Suppose we are given a monoid presentation $\mathcal{P} = [X; R]$, a monoid M and a mapping $\phi : X \longrightarrow M$. For any $W = x_1 x_2 \cdots x_m \in X^*$, let $\phi(W) = \phi(x_1)\phi(x_2)\cdots\phi(x_m)$. If $\phi(r_{+1}) = \phi(r_{-1})$ for every $r \in R$, then there is an induced homomorphism*

$$\Phi : \quad S(\mathcal{P}) \longrightarrow M, \qquad [W]_R \longmapsto \phi(W).$$

1.1.4 Tietze transformations

Let $\mathcal{P} = [X; R]$ be a monoid presentation. We consider some operations on $\mathcal{P}$.

(T1) : Let $U \longleftrightarrow_R^* V$, where U and V are words on X. Then replace $\mathcal{P}$ by $\mathcal{P}_1 = [X;\ R, U = V]$.

(T2) : Let y be a symbol not in X, and let W be a word on X. Then replace $\mathcal{P}$ by $\mathcal{P}_2 = [X, y;\ R, y = W]$.

The operations (T1), (T2) and their inverses are called *Tietze transformations.* Two finite monoid presentations are said to be *Tietze equivalent* if one can be obtained from the other by a finite number of Tietze transformations.

Theorem 1.1.5. *Two finite monoid presentations $\mathcal{P}$ and $\mathcal{P}'$ are Tietze equivalent if and only if the monoids presented by $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic.*

1.2 Semi-direct products of monoids

Let A, B be monoids, and let Q be a monoid homomorphism

$$Q: \ A \longrightarrow End(B), \quad a \longmapsto Q_a, \ 1 \longmapsto id,$$

where $End(B)$ denotes the collection of endomorphisms of B (this is itself a monoid with identity $id : B \longrightarrow B$). We write elements $\phi \in End(B)$ on the *right* of the argument (i.e., we write $b\phi$ rather than $\phi(b)$ for $b \in B$). Then we can construct a monoid $M = A \ltimes_Q B$ with elements (a, b) $(a \in A, b \in B)$ and product $(a, b)(a', b') = (aa', (bQ_{a'})b')$. The monoid M is called the *semi-direct product* of A and B relative to Q.

If we pick monoid presentations $\mathcal{P}_A = [X; R]$ and $\mathcal{P}_B = [Y; S]$ for A and B respectively, then we want to construct a monoid presentation for $M = A \ltimes_Q B$.

For any word W on X (or Y), let $[W]_A$ (or $[W]_B$) denote the element of A (or B) represented by W. For any $x \in X$, $y \in Y$, we use yQ_x to denote a fixed word on Y such that $[yQ_x]_B = [y]_B Q_{[x]_A}$. Note that yQ_x is unique modulo S. Let

$$\mathcal{P} = [X \cup Y; R, S, yx = x(yQ_x) \ \ (x \in X, y \in Y)]$$

and let $\overline{M} = S(\mathcal{P})$ be the monoid defined by $\mathcal{P}$. We will show that $\overline{M} \cong M$.

Consider the mapping

$$\phi : X \cup Y \longrightarrow M, \quad x \longmapsto ([x]_A, 1), \ y \longmapsto (1, [y]_B),$$

where $x \in X, \ y \in Y$. For any $r : r_{+1} = r_{-1} \in R, \ s : s_{+1} = s_{-1} \in S, \ x \in X$ and $y \in Y$, we have

$$\phi(r_{+1}) = ([r_{+1}]_A, 1) = ([r_{-1}]_A, 1) = \phi(r_{-1}),$$

$$\phi(s_{+1}) = (1, [s_{+1}]_B) = (1, [s_{-1}]_B) = \phi(s_{-1}),$$

$$\begin{aligned} \phi(yx) &= (1, [y]_B)([x]_A, 1) = ([x]_A, [y]_B Q_{[x]_A}) \\ &= ([x]_A, [yQ_x]_B) = ([x]_A, 1)(1, [yQ_x]_B) = \phi(x(yQ_x)). \end{aligned}$$

So by Lemma 1.1.4 we can get an induced homomorphism

$$\Phi : \quad \overline{M} \longrightarrow M, \qquad [W]_{\overline{M}} \longmapsto \phi(W)$$

where $W \in (X \cup Y)^*$. It is clear that Φ is surjective. It is not difficult to show that for each $W \in (X \cup Y)^*$, there exist $W_1 \in X^*$ and $W_2 \in Y^*$ such that $[W]_{\overline{M}} = [W_1 W_2]_{\overline{M}}$.

Let $U, V \in (X \cup Y)^*$ such that $\Phi([U]_{\overline{M}}) = \Phi([V]_{\overline{M}})$. Then there exist $U_1, V_1 \in X^*$ and $U_2, V_2 \in Y^*$ such that $[U]_{\overline{M}} = [U_1 U_2]_{\overline{M}}$ and $[V]_{\overline{M}} = [V_1 V_2]_{\overline{M}}$. Hence

$$([U_1]_A, [U_2]_B) = \phi(U_1 U_2) = \Phi([U_1 U_2]_{\overline{M}}) = \Phi([V_1 V_2]_{\overline{M}}) = \phi(V_1 V_2) = ([V_1]_A, [V_2]_B).$$

So $[U_1]_A = [V_1]_A$ and $[U_2]_B = [V_2]_B$. Thus $[U_1]_{\overline{M}} = [V_1]_{\overline{M}}$, $[U_2]_{\overline{M}} = [V_2]_{\overline{M}}$, and hence $[U]_{\overline{M}} = [U_1][U_2]_{\overline{M}} = [V_1][V_2]_{\overline{M}} = [V]_{\overline{M}}$. So Φ is injective. Thus $\overline{M} \cong M$. Therefore $\mathcal{P}$ is a monoid presentation of $M = A \ltimes_Q B$.

1.3 Group presentations

A group presentation $\hat{\mathcal{P}}$ is a pair $\langle X; R \rangle$ where X is a set (the *generators*) and R is a set of non-empty words on $X \cup X^{-1}$ (the *relators*), where X^{-1} is a set in one-to-one correspondence with X, $x \leftrightarrow x^{-1}$ $(x \in X)$ such that X and X^{-1} are disjoint. We say that $\hat{\mathcal{P}}$ is *finite*, if both X and R are finite.

Note that we will use angular brackets $\langle \cdots \rangle$ for *group* presentations and square brackets $[\cdots]$ for *monoid* presentations.

Let $F(X)$ be the free group on X, and let N_R be the normal closure of the set R in $F(X)$ (that is, the normal subgroup of $F(X)$ generated by R, where we consider a word W on $X \cup X^{-1}$ to be the same as the element of $F(X)$ represented by W). Then the quotient group $F(X)/N_R$ is called the group *presented* (or *defined*) by $\hat{\mathcal{P}}$, denoted by $G(\hat{\mathcal{P}})$. If a group G is isomorphic to $F(X)/N_R$, then we say that $\hat{\mathcal{P}} = \langle X; R \rangle$ is a group presentation of G.

For a group G, if $\hat{\mathcal{P}} = \langle X; R \rangle$ is a group presentation of G, then

$$\mathcal{P} = [X \cup X^{-1};\ xx^{-1} = \varnothing,\ x^{-1}x = \varnothing,\ r = \varnothing\ (x \in X,\ r \in R)]$$

is a monoid presentation of G.

1.4 Equivalence and subnegativity of number functions

Given two increasing functions $f_1 : \mathbb{N} \longrightarrow \mathbb{R}^+$ and $f_2 : \mathbb{N} \longrightarrow \mathbb{R}^+$, we write $f_1 \preceq f_2$ if there is an integer $A > 0$ such that

$$f_1(n) \leq Af_2(An) + An \quad (n \in \mathbb{N}).$$

We say that f_1 is *equivalent* to f_2, denoted $f_1 \sim f_2$, if $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

A function $f : \mathbb{N} \longrightarrow \mathbb{R}^+$ is called *subnegative* (some authors use *superadditive*) if

$$f(n_1) + f(n_2) \leq f(n_1 + n_2)$$

for all $n_1, n_2 \in \mathbb{N}$. For each function $f : \mathbb{N} \longrightarrow \mathbb{R}^+$ there exists a least subnegative function $\overline{f}$, called the *subnegative closure* of f, such that $f \preceq \overline{f}$. This is defined by

$$\overline{f}(n) = max\{\sum_{i=1}^{r} f(n_i) : n_1 + n_2 + \cdots + n_r = n,\ n_i \in \mathbb{N}\ (1 \leq i \leq r)\}.$$

If $f_1 \preceq f_2$ then $\overline{f_1} \preceq \overline{f_2}$, and so if $f_1 \sim f_2$ then $\overline{f_1} \sim \overline{f_2}$.

For any two functions $f, g : \mathbb{N} \longrightarrow \mathbb{R}^+$, we have a function $max\{f, g\} : \mathbb{N} \longrightarrow \mathbb{R}^+$ defined by

$$max\{f, g\}(n) = max\{f(n), g(n)\}, \quad n \in \mathbb{N}.$$

Note that if $f \preceq f'$ and $g \preceq g'$ then $max\{f, g\} \preceq max\{f', g'\}$. So if $f \sim f'$ and $g \sim g'$ then $max\{f, g\} \sim max\{f', g'\}$ and $\overline{max\{f, g\}} \sim \overline{max\{f', g'\}}$.

1.5 Graphs and 2-complexes

1.5.1 Graphs

In this thesis we need two main concepts of "graph", one a special case of the other. The more general graph is a quadruple $\Gamma = (V, E, \iota, \tau)$, where $V = V(\Gamma)$ and $E = E(\Gamma)$ are disjoint sets (the vertex and edge sets respectively), and

$$\iota : E \longrightarrow V, \quad \tau : E \longrightarrow V$$

are functions (the initial and terminal functions respectively). A subclass of this class of graphs contains those with an inverse function, that is a function

$$^{-1} : E \longrightarrow E$$

such that $\iota(e) = \tau(e^{-1})$, $(e^{-1})^{-1} = e$ and $e^{-1} \neq e$ for all $e \in E$. These are the graphs considered by Serre in [76], and are those most used in combinatorial group theory. *We will refer to a graph with an inverse function as simply a* **graph**. *The more general graph described above (which will be used in Section 2.5) will be refered to as a* **directed graph**.

Let $\Gamma = (V, E, \iota, \tau)$ be a directed graph. A non-empty *path* p of Γ is a finite sequence of edges written in the form $p = e_1 e_2 \cdots e_n$ $(n > 0,\ e_i \in E)$ such that $\tau(e_i) = \iota(e_{i+1})$, $1 \leq i \leq n-1$. The *initial vertex* $\iota(p)$ of p is defined to be $\iota(e_1)$, and the *terminal vertex* $\tau(p)$ of p is defined to be $\tau(e_n)$. The *length* of p is n, denoted $l(p) = n$. If $\iota(p) = \tau(p)$, then p is called a *closed path*. For each $v \in V$ we introduce the *empty path* 1_v at v which has no edges. We have $\iota(1_v) = \tau(1_v) = v$ and $l(1_v) = 0$. We let $P(\Gamma)$ denote the set of all paths in Γ, and let

$$P^{(2)}(\Gamma) := \{(p, q) \ : \ p, q \in P(\Gamma),\ \iota(p) = \iota(q), \tau(p) = \tau(q)\}.$$

If p and q are two paths of Γ with $\tau(p) = \iota(q)$, then the *product* pq of p and q is defined to be the path starting with p followed by q.

If Γ has an inverse function $^{-1}$, then the *inverse* p^{-1} of $p = e_1 e_2 \cdots e_n$ is defined to be $e_n^{-1} \cdots e_2^{-1} e_1^{-1}$ which is also a path in Γ. The inverse of empty path 1_v is defined to be 1_v for every $v \in V$. If $e_i^{-1} \neq e_{i+1}$ for all $i = 1, 2, \cdots, n-1$ then we say that p is a *reduced path*.

Let $V_0 \subseteq V$ and $E_0 \subseteq E$. If $\iota(e), \tau(e) \in V_0$ (and $e^{-1} \in E_0$ if Γ is a graph) for each $e \in E_0$, then we say that $\Gamma_0 = (V_0, E_0, \iota, \tau)$ is a *directed subgraph* of the directed graph Γ (respectively, *subgraph* of the graph Γ).

A (directed) graph Γ is *connected* if for any two vertices of Γ there is a path in Γ joining them. A maximal connected (directed) subgraph of Γ is called a *component* of Γ.

We will also briefly need one other type of graph in Section 2.2. A **simplicial graph** is a pair (V, E), where V is a nonempty set and E is a set of 2-element subsets of V.

1.5.2 2-complexes

A (combinatorial) *2-complex* $\mathcal{D}$ is a pair

$$\mathcal{D} = \langle \Gamma; R \rangle$$

where Γ is a graph and R is a set of closed paths (called *defining paths*) of $\mathcal{D}$. We will discuss the more general case (that is, directed 2-complexes) later in Section 2.5. We say $\mathcal{D}$ is *finite* if $V(\Gamma)$, $E(\Gamma)$ and R are finite. We say $\mathcal{D}$ is *connected* if Γ is.

Example 1.5.1. Let Γ be the graph in Figure 1.1, and let

$$\mathcal{D} = \langle \Gamma;\ e_1 e_2^{-1} e_3,\ e_4^3,\ e_5^{-1} e_4^2 e_3^{-1} e_2 \rangle.$$

Then $\mathcal{D}$ is a finite connected 2-complex.

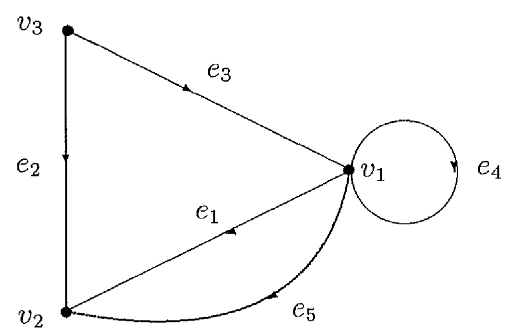


Figure 1.1

If Γ' is a subgraph of Γ and R' is a subset of R such that R' is a set of closed paths of Γ', then the 2-complex $\mathcal{D}' = \langle \Gamma'; R' \rangle$ is called a *subcomplex* of $\mathcal{D}$.

For any 2-complex $\mathcal{D} = \langle \Gamma; R \rangle$ there are two types of *elementary operations* on the paths of $\mathcal{D}$ introduced as follows.

(I) Insertion or deletion of an inverse pair ee^{-1} of two successive edges.

(II) Insertion or deletion of a subpath p with $p \in R \cup R^{-1}$.

Let $\boldsymbol{X}$ be another set of closed paths of Γ. We can get a new 2-complex $\mathcal{D}^{\boldsymbol{X}} = \langle \Gamma; R \cup \boldsymbol{X} \rangle$. Then $\mathcal{D}$ is a subcomplex of $\mathcal{D}^{\boldsymbol{X}}$. So in $\mathcal{D}^{\boldsymbol{X}}$ there is another type of operations as follows.

(III) Insertion or deletion of a subpath p with $p \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$.

Any two paths p and q in $\mathcal{D}$ are said to be *equivalent*, denoted by $p \sim q$, if one can be obtained from the other by a finite sequence of applications of operations (I) and (II). They are said to be *equivalent relative to* $\boldsymbol{X}$, denoted by $p \sim_{\boldsymbol{X}} q$, if one can be obtained from the other by a finite sequence of applications of operations (I), (II) and (III). Note that if two paths of $\mathcal{D}$ are equivalent relative to $\boldsymbol{X}$ then they are equivalent in $\mathcal{D}^{\boldsymbol{X}}$. A closed path of $\mathcal{D}$ which is equivalent to an empty path is said to be *contractible* in $\mathcal{D}$.

Let p be a path in $\mathcal{D}$ which is contractible in $\mathcal{D}^{\boldsymbol{X}}$. The *volume* of p with respect to $\boldsymbol{X}$, denoted by $V_{\mathcal{D},\boldsymbol{X}}(p)$, is the smallest number of operations of type (III) used in any transformation of p to an empty path.

Lemma 1.5.2. *Let p be a closed path at a vertex v in $\mathcal{D}$. If p is contractible in $\mathcal{D}^{\boldsymbol{X}}$ with $V_{\mathcal{D},\boldsymbol{X}}(p) = n$, then p is equivalent in $\mathcal{D}$ to a path of the form*

$$(p_1 q_1 p_1^{-1})(p_2 q_2 p_2^{-1}) \cdots (p_n q_n p_n^{-1}),$$

where $q_i \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$, p_i is a path in $\mathcal{D}$, $1 \le i \le n$. Conversely, if p is equivalent in $\mathcal{D}$ to a path of the above form, then $V_{\mathcal{D},\boldsymbol{X}}(p)$ is the smallest number n in the above form which p is equivalent to in $\mathcal{D}$.

Proof. We just prove the first part of the lemma. The second part is clear.

If $n = 0$, then p is contractible in $\mathcal{D}$ and hence $p \sim 1_v$. Now let $n > 0$. Since p is contractible in $\mathcal{D}^{\boldsymbol{X}}$ with $V_{\mathcal{D},\boldsymbol{X}}(p) = n$, there is a finite sequence of paths

$$p = p'_1,\ p'_2, \cdots, p'_{k-1},\ p'_k = 1_v$$

such that for each i $(1 \le i \le k-1)$, p'_{i+1} is obtained from p'_i by an application of operation (I), or (II), or (III), and there are precisely n applications of operation (III).

Let m be the first number for which an operation of type (III) is applied. Thus $p \sim p'_m$ and $V_{\mathcal{D},\boldsymbol{X}}(p'_{m+1}) = n-1$. By induction hypothesis we have

$$p'_{m+1} \sim (p_1 q_1 p_1^{-1})(p_2 q_2 p_2^{-1}) \cdots (p_{n-1} q_{n-1} p_{n-1}^{-1})$$

in $\mathcal{D}$, where $q_i \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$, p_i is a path in $\mathcal{D}$, $1 \le i \le n-1$. We just consider the case that p'_{m+1} is obtained from p'_m by an insertion of a subpath in $\boldsymbol{X} \cup \boldsymbol{X}^{-1}$ (the other case can be proved in a similar way) . We can assume that $p'_m = q' p_n^{-1}$ and $p'_{m+1} = q' t p_n^{-1}$ for some paths q', p_n in $\mathcal{D}$ and some $t \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$. So we have

$$\begin{aligned} p \sim p'_m &= q' p_n^{-1} \sim q' t p_n^{-1} p_n t^{-1} p_n^{-1} \\ &= p'_{m+1} p_n t^{-1} p_n^{-1} \\ &\sim (p_1 q_1 p_1^{-1})(p_2 q_2 p_2^{-1}) \cdots (p_{n-1} q_{n-1} p_{n-1}^{-1})(p_n q_n p_n^{-1}) \end{aligned}$$

where $q_n = t^{-1} \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$. This completes the proof. □

Lemma 1.5.3. *(1). If two closed paths p, p' contractible in $\mathcal{D}^{\boldsymbol{X}}$ are equivalent in $\mathcal{D}$, then $V_{\mathcal{D},\boldsymbol{X}}(p) = V_{\mathcal{D},\boldsymbol{X}}(p')$.*

(2). If p is contractible in $\mathcal{D}^{\boldsymbol{X}}$ and q is a path of $\mathcal{D}$ with $\tau(q) = \iota(p)$, then $V_{\mathcal{D},\boldsymbol{X}}(qpq^{-1}) = V_{\mathcal{D},\boldsymbol{X}}(p)$.

1.5.3 Fundamental groups of 2-complexes

For each path p in $\mathcal{D}$, let $[p] = [p]_{\mathcal{D}}$ denote the equivalence class consisting of all paths equivalent to p in $\mathcal{D}$. For each $v \in V$, let $\pi_1(\mathcal{D}, v)$ be the set of all equivalence classes $[p]$ with p a closed path in $\mathcal{D}$ at v. A multiplication can be defined on $\pi_1(\mathcal{D}, v)$ by

$$[p_1][p_2] = [p_1 p_2].$$

It is easy to check that this multiplication is well-defined. And $\pi_1(\mathcal{D}, v)$ is a group under this multiplication, called the *fundamental group* of $\mathcal{D}$ at v. If v and v' are in the same component of $\mathcal{D}$, then $\pi_1(\mathcal{D}, v) \cong \pi_1(\mathcal{D}, v')$. Hence, if $\mathcal{D}$ is connected, then all fundamental groups of $\mathcal{D}$ are isomorphic.

A group presentation $\hat{\mathcal{P}} = \langle X; R\rangle$ can be considered as a 2-complex $\mathcal{D} = \langle \Gamma; R\rangle$, where Γ is a graph with one vertex o and an edge pair x, x^{-1} ($\iota(x) = \tau(x) = o$) for each $x \in X$, and a word on $X \cup X^{-1}$ is considered as a path in Γ. It is clear that the group presented by $\hat{\mathcal{P}}$ is isomorphic to the fundamental group of $\mathcal{D}$ at o.

Suppose that $\boldsymbol{X}$ is a set of closed paths in $\mathcal{D}$ such that every closed path in $\mathcal{D}$ is equivalent relative to $\boldsymbol{X}$ to the empty path. Then all fundamental groups of $\mathcal{D}^{\boldsymbol{X}}$ are trivial. We then say that $\boldsymbol{X}$ *trivializes* $\mathcal{D}$ (and then $\boldsymbol{X}$ is a *trivializer* of $\mathcal{D}$).

The fundamental monoids of directed 2-complexes will be considered later in Section 2.5.

1.6 Pictures over group presentations

1.6.1 Pictures over group presentations

Let $\hat{\mathcal{P}} = \langle X; R\rangle$ be a group presentation. A *(group) picture* $\mathbb{P}$ over $\hat{\mathcal{P}}$ is a geometric configuration consisting of the following.

(1) A disc D^2 with a basepoint o on the boundary ∂D^2 of D^2.

(2) Disjoint discs $d_1, d_2, \cdots, d_n$ in the interior of D^2. Each disc d_i has a basepoint o_i on the boundary ∂d_i of d_i.

(3) A finite number of disjoint arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$. Each arc lies in the closure of $D^2 - \cup_{i=1}^{n} d_i$ and is either a simple closed curve having trivial intersection with $\partial D^2 \cup (\cup_{i=1}^{n} d_i)$, or is a simple non-closed curve which joins two points of $\partial D^2 \cup (\cup_{i=1}^{n} \partial d_i)$, neither point being a basepoint. Each arc has a normal orientation indicated by a short arrow meeting the arc transversely, and is labelled by an element in X which is called the label of the arc.

(4) Reading off the labels on the arcs encountered while travelling once around ∂d_i ($1 \leq i \leq n$) in the clockwise direction from o_i to o_i gives a word which belongs to $R \cup R^{-1}$. We call this word the label of d_i. If $R_0 \subset R$ and a disc d_i is labelled by a word which belongs to $R_0 \cup R_0^{-1}$, then we say that d_i is an R_0-disc.

We define the *boundary* $\partial\mathbb{P}$ of $\mathbb{P}$ to be ∂D^2. The *boundary label* of $\mathbb{P}$ is the word read off by travelling around $\partial\mathbb{P}$ once in the clockwise direction starting from o. When we refer to the discs of $\mathbb{P}$ we mean the discs in the interior of the ambient disc D^2 not D^2 itself. A *region* of $\mathbb{P}$ is the closure of a component of $D^2 - ((\cup_{i=1}^{n} d_i) \cup (\cup_{j=1}^{m} \alpha_j))$. The *area* $A(\mathbb{P})$ of $\mathbb{P}$ is the disc number in $\mathbb{P}$. We say that $\mathbb{P}$ is *spherical* if no arc meets $\partial\mathbb{P}$. Usually, we would drop off the boundary of the ambient disc D^2 of a spherical picture.

Example 1.6.1. Let $\hat{\mathcal{P}} = \langle x, y, z; xyz^{-1}y, y^3, zyz^{-1}y^{-1}\rangle$. Then $\mathbb{P}$ in Figure 1.2 is a picture over $\hat{\mathcal{P}}$. The disc d_1 is a y^3- disc; the discs d_2 and d_3 are $xyz^{-1}y$-discs; the disc d_4 is a $yzy^{-1}z^{-1}$-disc. The boundary label of $\mathbb{P}$ is $xy^{-1}xyy^{-1}z^{-1}yz^{-1}yxzz^{-1}x^{-1}$.

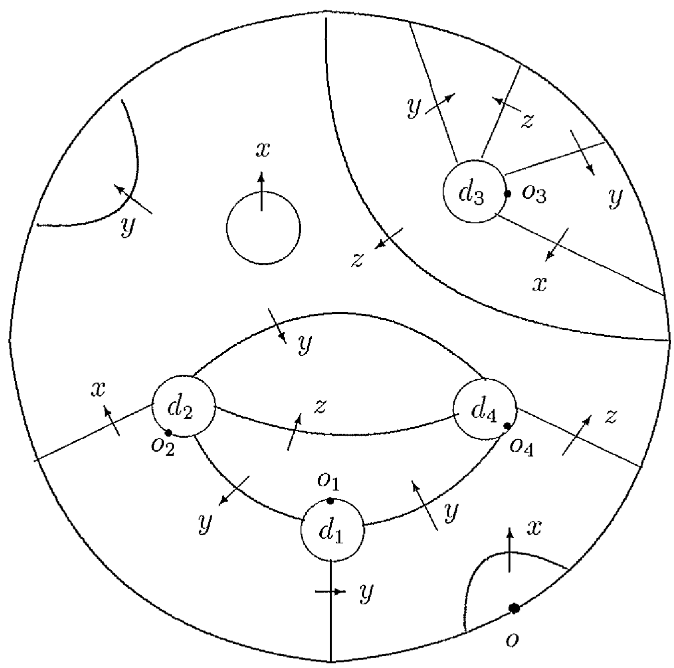


Figure 1.2

Example 1.6.2. Let $\hat{\mathcal{P}} = \langle a, b; a^3, b^{-1}aba^{-2}\rangle$. Then $\mathbb{P}$ in Figure 1.3 is a spherical picture over $\hat{\mathcal{P}}$.

1.6.2 First order Dehn functions of groups

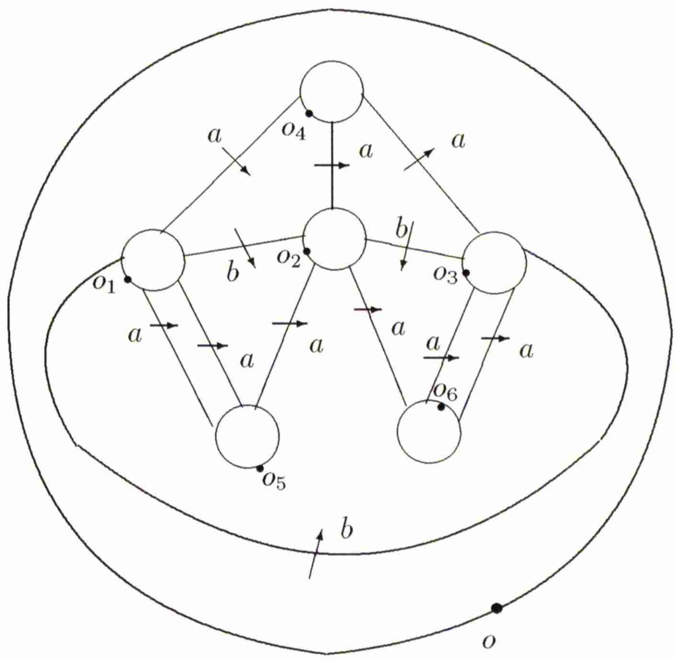


Figure 1.3

Theorem 1.6.3 (van Kampen). *Let $\hat{\mathcal{P}} = \langle X; R \rangle$ be a group presentation, and let $G = G(\hat{\mathcal{P}})$. Let W be a word on $X \cup X^{-1}$. Then $[W]_G = 1$ (that is, W represents the identity 1 in G) if and only if there exists a picture $\mathbb{P}$ over $\hat{\mathcal{P}}$ with boundary label $\partial\mathbb{P} = W$.*

For each word W on $X \cup X^{-1}$ representing the identity of $G = G(\hat{\mathcal{P}})$, we can define

$$A(W) = min\{A(\mathbb{P}) : \partial\mathbb{P} = W\}.$$

Suppose the presentation $\hat{\mathcal{P}}$ is finite. The *first order Dehn function* $\delta^{(1)}_{\hat{\mathcal{P}}} : \mathbb{N} \longrightarrow \mathbb{Z}^+$ of $\mathcal{P}$ is given by

$$\delta^{(1)}_{\hat{\mathcal{P}}}(n) = max\{A(W) : [W]_G = 1,\ l(W) \leq n\}.$$

This function is essentially an invariant of the group defined by $\hat{\mathcal{P}}$. If G is a finitely presented group and $\hat{\mathcal{P}}_1$, $\hat{\mathcal{P}}_2$ are any two finite group presentations of G, then $\delta^{(1)}_{\hat{\mathcal{P}}_1} \sim \delta^{(1)}_{\hat{\mathcal{P}}_2}$ (see [3]). Thus , up to equivalence, we can talk about the *first order Dehn function* $\delta^{(1)}_G$ of G.

1.6.3 Second homotopy modules of group presentations

Let $\hat{\mathcal{P}}$ be a group presentation, and let $\mathbb{P}$ be a group picture over $\hat{\mathcal{P}}$. A *floating circle* of $\mathbb{P}$ is a closed arc which encloses no discs or arcs of $\mathbb{P}$. In Example 1.6.1, the circle labelled by x in Figure 1.2 is a floating circle. A *cancelling pair* is a spherical picture with exactly two discs whose basepoints lie in the same region (see Figure 1.4).

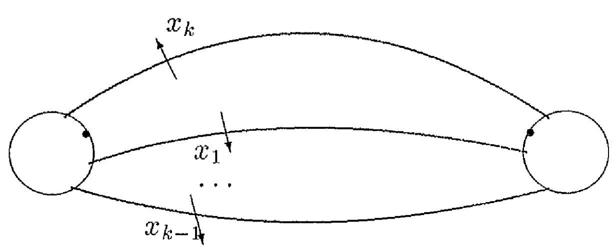


Figure 1.4

Note that in Figure 1.5, the first picture is a cancelling pair, but the second one is not.

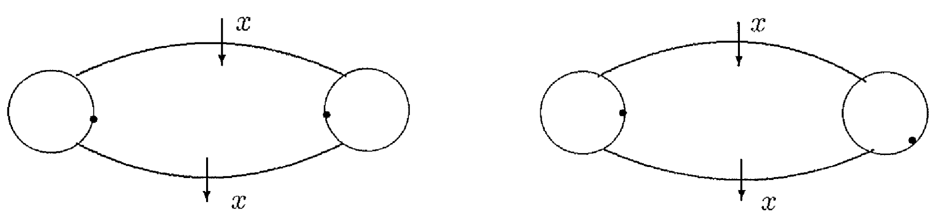


Figure 1.5

We now introduce some elementary operations on spherical pictures over $\hat{\mathcal{P}}$ as follows.

(A) Deletion of a floating circle.

$(A)^{-1}$ Insertion of a floating circle.

(B) Deletion of a cancelling pair.

$(B)^{-1}$ Insertion of a cancelling pair.

(C) Bridge move (see Figure 1.6).

Two spherical pictures over $\hat{\mathcal{P}}$ are *equivalent* if one can be obtained from the other by a finite number of operations (A), $(A)^{-1}$, (B), $(B)^{-1}$, (C).

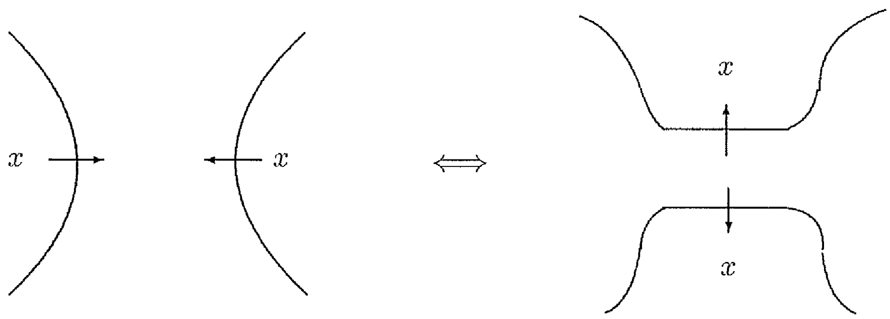


Figure 1.6

The *mirror image* of a picture $\mathbb{P}$ over $\hat{\mathcal{P}}$, denoted $-\mathbb{P}$, is also a picture over $\hat{\mathcal{P}}$. We form the *sum* of any two pictures $\mathbb{P}_1, \mathbb{P}_2$ over $\hat{\mathcal{P}}$ in the obvious way (see Figure 1.7).

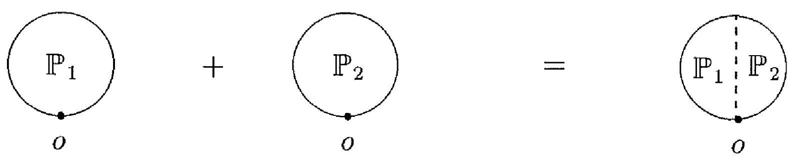


Figure 1.7

We will write $\mathbb{P}_1 - \mathbb{P}_2$ for $\mathbb{P}_1 + (-\mathbb{P}_2)$. For any picture $\mathbb{P}$ over $\hat{\mathcal{P}}$, it is clear that $\mathbb{P} - \mathbb{P}$ is equivalent to the empty picture. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are both spherical then $\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{P}_2 + \mathbb{P}_1$.

Let $\mathbb{P}$ be any spherical picture over $\hat{\mathcal{P}}$. We denote $\langle \mathbb{P} \rangle$ the equivalence class of spherical pictures over $\hat{\mathcal{P}}$ containing $\mathbb{P}$. We say that $\mathbb{P}$ is *minimal* if $A(\mathbb{P}) = \min\{A(\mathbb{Q}) | \mathbb{Q} \in \langle \mathbb{P} \rangle\}$. The set of all equivalence classes of spherical pictures over $\hat{\mathcal{P}}$ forms an abelian group, denoted $\pi_2(\hat{\mathcal{P}})$, under the following binary operation:

$$\langle \mathbb{P}_1 \rangle + \langle \mathbb{P}_2 \rangle = \langle \mathbb{P}_1 + \mathbb{P}_2 \rangle.$$

Let $\mathbb{P}$ be a spherical picture over $\hat{\mathcal{P}}$. For any $x \in X$, we can form new spherical pictures $\mathbb{P}^x$ and $\mathbb{P}^{x^{-1}}$ as in Figure 1.8. Let W be a word on $X \cup X^{-1}$. Inductively, we define $\mathbb{P}^W = (\mathbb{P}^U)^{x^\varepsilon}$, where $W = Ux^\varepsilon$ and $\varepsilon = \pm 1$.

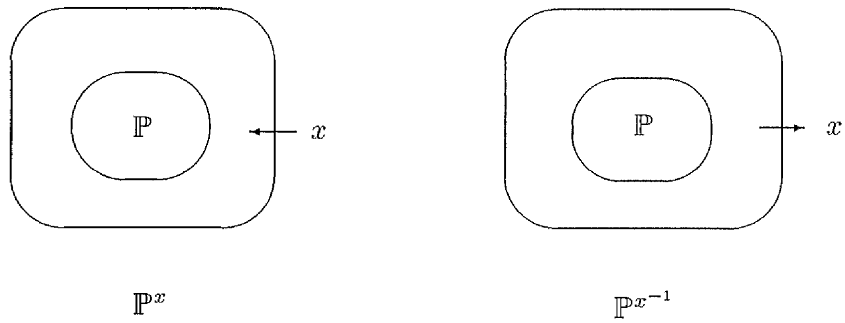


Figure 1.8

Let $G = G(\hat{\mathcal{P}})$. The G-action on $\pi_2(\hat{\mathcal{P}})$ given by

$$[W]_G \cdot \langle \mathbb{P} \rangle = \langle \mathbb{P}^W \rangle,$$

is well-defined. Thus $\pi_2(\hat{\mathcal{P}})$ is a left $\mathbb{Z}G$-module, called the *second homotopy module* of $\hat{\mathcal{P}}$.

1.6.4 Second order Dehn functions of groups

Let $\hat{\mathcal{P}} = \langle X; R \rangle$ be a group presentation for a group G. A collection $\hat{\boldsymbol{X}}$ of spherical pictures over $\hat{\mathcal{P}}$ will be called a set of *generating pictures* if the equivalence classes of elements in $\hat{\boldsymbol{X}}$ constitute a set of generators for the left $\mathbb{Z}G$-module $\pi_2(\hat{\mathcal{P}})$. Then every element $\hat{\xi} \in \pi_2(\hat{\mathcal{P}})$ can be expressed in the form

$$\sum_{i=1}^{m} \varepsilon_i g_i \langle \mathbb{P}_i \rangle,$$

where $\varepsilon_i = \pm 1, g_i \in G, \mathbb{P}_i \in \hat{\boldsymbol{X}}, i = 1, \cdots, m$. The *volume* of $\hat{\xi}$ with respect to $\hat{\boldsymbol{X}}$, denoted $V_{\hat{\boldsymbol{X}}}(\hat{\xi})$, is the minimal value of m over all expressions of the above form equal to $\hat{\xi}$. Let $A(\hat{\xi})$ denote the minimum of the areas of group pictures representing $\hat{\xi}$.

Now if we assume $\hat{\mathcal{P}}$ is a finite group presentation then the set

$$\chi_n = \{V_{\hat{\boldsymbol{X}}}(\hat{\xi}) : \; A(\hat{\xi}) \leq n\} \quad (n \in \mathbb{N})$$

is finite (although in general the set $\{\hat{\xi} : \; A(\hat{\xi}) \leq n\}$ is not) (See [74]). We can define the *second order Dehn function* of $\hat{\mathcal{P}}$ with respect to $\hat{\boldsymbol{X}}$ by

$$\delta^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}} : \mathbb{N} \longrightarrow \mathbb{Z}^+, \ n \longmapsto \max \chi_n.$$

A group G is said to be of *type* F_3 if it is finitely presented and each finite group presentation of G has finitely generated second homotopy module.

Theorem 1.6.4. ([5], [74], [82]) *Let G be a group of type F_3. Let $\hat{\mathcal{P}}_1$ and $\hat{\mathcal{P}}_2$ be any two finite group presentations for G, and let $\hat{\boldsymbol{X}}_1$ and $\hat{\boldsymbol{X}}_2$ be finite generating sets of pictures for $\pi_2(\hat{\mathcal{P}}_1)$ and $\pi_2(\hat{\mathcal{P}}_2)$ respectively. Then*

$$\delta^{(2)}_{\hat{\mathcal{P}}_1,\hat{\boldsymbol{X}}_1} \sim \delta^{(2)}_{\hat{\mathcal{P}}_2,\hat{\boldsymbol{X}}_2}.$$

By this theorem we see that for any group G of type F_3 we can define (up to equivalence) the *second order Dehn function* $\delta^{(2)}_G$ of G.

1.7 Monoid pictures

1.7.1 Associated graphs and first order Dehn functions of monoids

Let $\mathcal{P} = [X; R]$ be a monoid presentation, where a typical element $r \in R$ has the form $r_{+1} = r_{-1}$. Let X^* denote the free monoid on X.

As in [68], [69], for any $r \in R$, $U, V \in X^*$, $\varepsilon = \pm 1$, we can define a geometric object called an *atomic monoid picture* $e = (U, r, \varepsilon, V)$ over $\mathcal{P}$ as depicted in Figure 1.9.

Here each arc in the rectangle is transversely orientated from left to right and labelled by an element of X; the disc represents the relation $r : r_{+1} = r_{-1}$, with upper label r_ε and lower label $r_{-\varepsilon}$. The word $Ur_\varepsilon V$ we read off by travelling along the top of the picture e from left to right is called the *upper boundary label* of e, denoted $\iota(e)$. The word $Ur_{-\varepsilon}V$ we read off along the bottom is called the *lower boundary label* of e, denoted $\tau(e)$. We define the *inverse atomic picture* e^{-1} to be $(U, r, -\varepsilon, V)$ (see Figure 1.10).

We then have the graph $\Gamma = \Gamma(\mathcal{P})$ associated to $\mathcal{P} = [X; R]$, where the vertices are the elements of X^*, the edges are the atomic monoid pictures, and the initial, terminal and inversion functions ι, τ, $^{-1}$ are as above. The edge $e = (U, r, \varepsilon, V)$ is called *positive*

$e = (U, r, \varepsilon, V)$

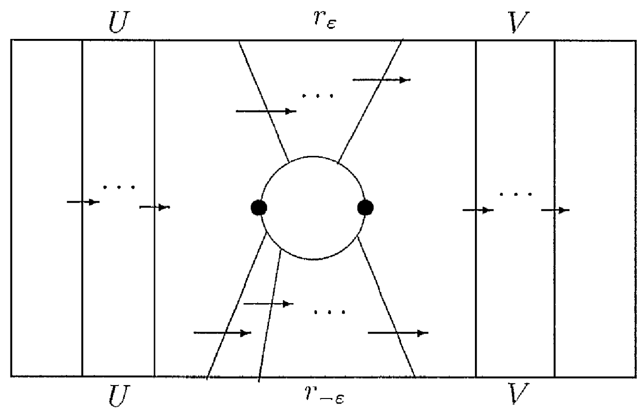


Figure 1.9

$e = (U, r, -\varepsilon, V)$

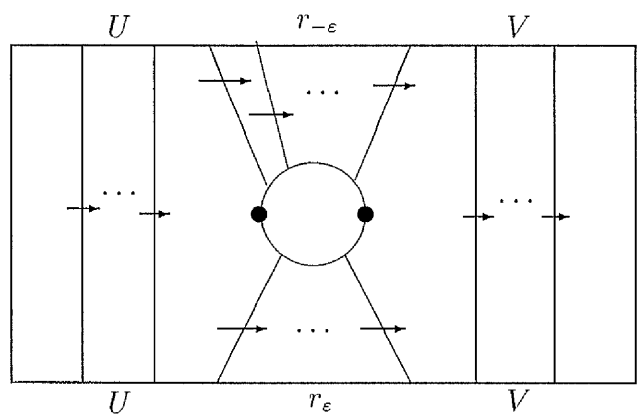


Figure 1.10

if $\varepsilon = +1$. Paths in Γ are called *monoid pictures* over $\mathcal{P}$. The empty path at a vertex V in Γ is called a *trivial picture* (see Figure 1.11), denoted 1_V.

Closed paths in Γ are called *spherical monoid pictures* over $\mathcal{P}$. An *arc* of a monoid picture consists of all those atomic picture arcs which are labelled by the same element of X and can be geometrically connected one by one (see Example 1.7.1).

Let p be a monoid picture over $\mathcal{P}$. The *area* $A(p)$ of p is the number of discs in p (this is the same as the length of p as a path in Γ).

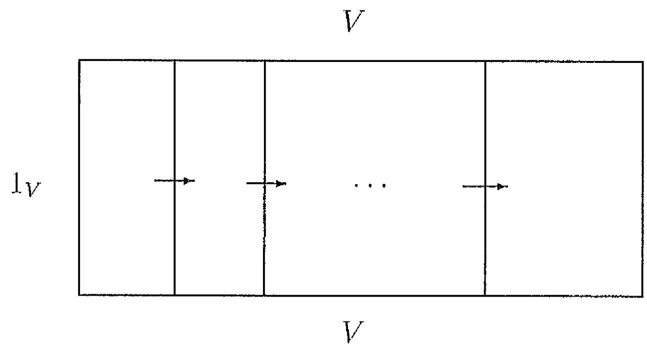


Figure 1.11

Example 1.7.1. Let $\mathcal{P} = [a, b, c; ab = c, ca = bc, bcb = cc, ccb = acc]$, and let

$$e_1 = (\varnothing, ca = bc, -1, bcb),\ \ e_2 = (c, ab = c, +1, cb),$$

$$e_3 = (c, ccb = acc, +1, \varnothing),\ \ e_4 = (\varnothing, ca = bc, +1, cc),$$

$$e_5 = (bc, bcb = cc, -1, \varnothing).$$

Then $\tau(e_i) = \iota(e_{i+1}), i = 1, 2, 3, 4$ and $\iota(e_1) = bcbcb = \tau(e_5)$. Thus $p = e_1e_2e_3e_4e_5$ is a closed path in $\Gamma(\mathcal{P})$, where, for example, the line joining the lower boundary of the disc in e_1 to the upper boundary of the disc in e_4 labelled by c is an arc of p. (See Figure 1.12.)

Usually we will omit the broken lines in monoid pictures, if no confusion arises (see Figure 1.13).

There is a two-sided action of X^* on Γ as follows. If $W, W' \in X^*$, then for any vertex V of Γ, $W \cdot V \cdot W' = WVW'$ (product in X^*), and for each edge $e = (U, r, \varepsilon, V)$ of Γ, $W \cdot e \cdot W' = (WU, r, \varepsilon, VW')$. This action can be extended to the paths in Γ: $W \cdot 1_V \cdot W' = 1_{WVW'}$; if $p = e_1e_2\cdots e_m$ is a non-empty path in Γ with e_i an edge of Γ $(1 \leq i \leq m)$ then for any $W, W' \in X^*$,

$$W \cdot p \cdot W' = (W \cdot e_1 \cdot W')(W \cdot e_2 \cdot W') \cdots (W \cdot e_m \cdot W').$$

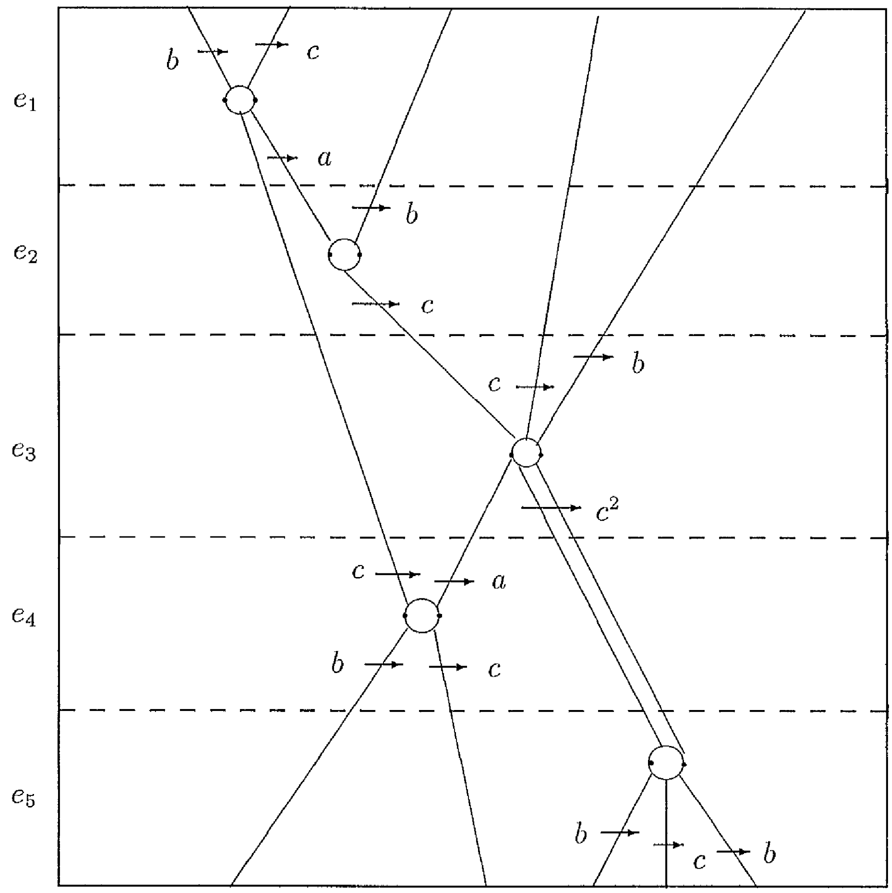


Figure 1.12

Theorem 1.7.2. *Let $\mathcal{P} = [X; R]$ be a monoid presentation, and let $\Gamma = \Gamma(\mathcal{P})$ be the associated graph. Let U and V be two words on X. Then $U \longleftrightarrow_R^* V$ if and only if there exists a path p in Γ with $\iota(p) = U$ and $\tau(p) = V$.*

Let U and V be two words on X such that $U \longleftrightarrow_R^* V$. Then we can define

$$A(U,V) = min\{A(p) : p \in P(\Gamma),\ \iota(p) = U,\ \tau(p) = V\}.$$

Assuming $\mathcal{P}$ is finite we can then define the *first order Dehn function* of $\mathcal{P}$ by

$$\gamma_{\mathcal{P}}^{(1)}(n) = max\{A(U,V) : U \longleftrightarrow_R^* V,\ l(U) + l(V) \leq n\}.$$

Again, up to equivalence this function is an invariant of the monoid [57] (see also [69]),

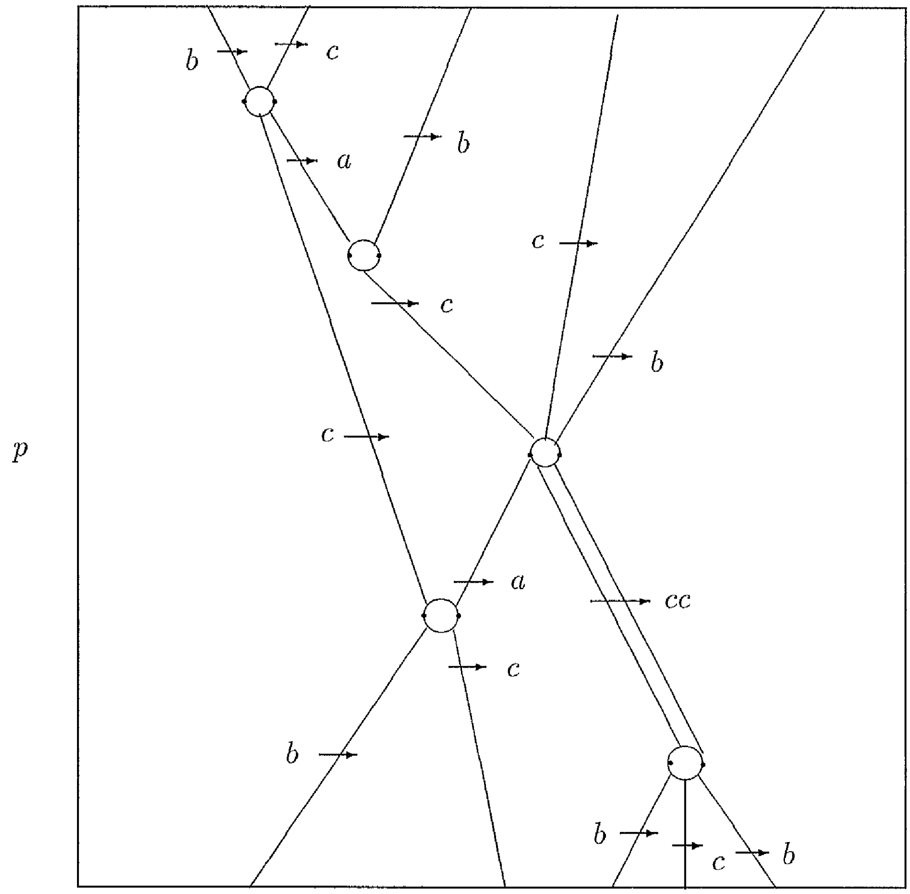


Figure 1.13

and so for a finitely presented monoid S we can talk about the *first order Dehn function* $\gamma_S^{(1)}$ of S.

Theorem 1.7.3. ([74]) *If G is a finitely presented group then*

$$\gamma_G^{(1)} \sim \delta_G^{(1)}.$$

We will give the definition of second order Dehn functions of monoids later in Chapter 3.

1.7.2 Squier complexes

Let $\mathcal{P} = [X; R]$ be a monoid presentation. We want to associate with $\mathcal{P}$ a certain 2-complex $\mathcal{D} = \mathcal{D}(\mathcal{P})$ where the underlying graph is the associated graph $\Gamma = \Gamma(\mathcal{P})$.

Let e_1, e_2 be two atomic monoid pictures over $\mathcal{P}$. We let $[e_1, e_2]$ denote the closed path

$$(e_1 \cdot \iota(e_2))(\tau(e_1) \cdot e_2)(e_1^{-1} \cdot \tau(e_2))(\iota(e_1) \cdot e_2^{-1})$$

in $\Gamma = \Gamma(\mathcal{P})$ (see Figure 1.14).

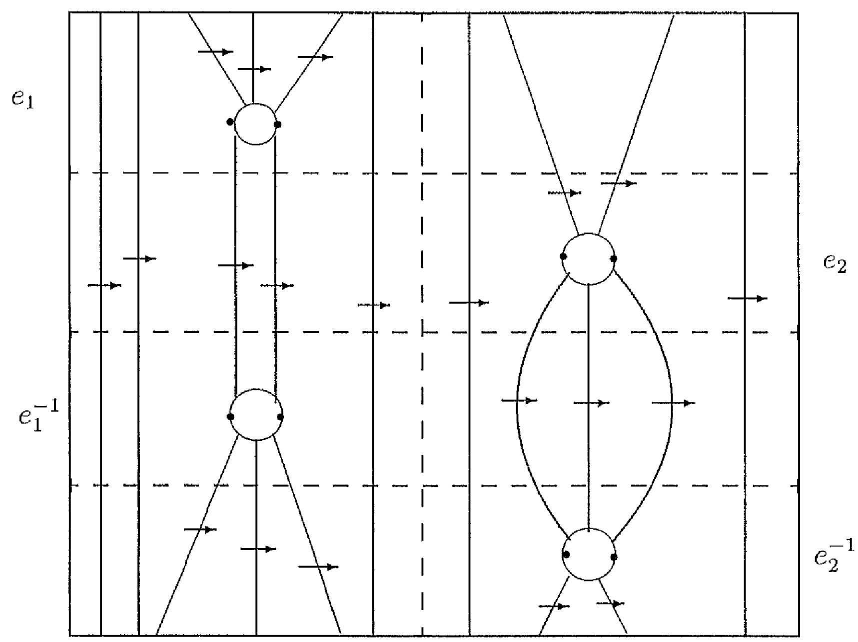


Figure 1.14

We let $\mathcal{D} = \mathcal{D}(\mathcal{P})$ denote the 2-complex with underlying graph Γ and where the defining paths are all the paths $[e_1, e_2]$. This 2-complex is called the *Squier complex* of $\mathcal{P}$. In fact, Squier [78] introduced the underlying graph of $\mathcal{D}(\mathcal{P})$ and a homotopy relation on the set of paths which coincides with the natural homotopy relation induced by $\mathcal{D}(\mathcal{P})$. The definition of $\mathcal{D}(\mathcal{P})$ presented here was introduced by Pride [68]. A similar definition was given independently by Kilibarda [51] (see also Guba and Sapir [42]).

It is clear that the following two paths in $\mathcal{D}$ are equivalent:

$$(e_1 \cdot \iota(e_2))(\tau(e_1) \cdot e_2) \sim (\iota(e_1) \cdot e_2)(e_1 \cdot \tau(e_2))$$

for any edges e_1 and e_2 in Γ. By induction we can get the following lemma.

Lemma 1.7.4 (Pull-down and push-up). *Let p_1, p_2 be any two paths in $\mathcal{D}(\mathcal{P})$. Then*

$$(p_1 \cdot \iota(p_2))(\tau(p_1) \cdot p_2) \sim (\iota(p_1) \cdot p_2)(p_1 \cdot \tau(p_2)).$$

Note that the set of defining paths of $\mathcal{D}$ is fixed under the two-sided action of X^* on Γ: for any $W, W' \in X^*$ and $[e_1, e_2]$ a defining path of $\mathcal{D}$ we have

$$W \cdot [e_1, e_2] \cdot W' = [W \cdot e_1, e_2 \cdot W'].$$

A spherical monoid picture over $\mathcal{P}$ is called *skewed* if (up to equivalence in $\mathcal{D}$) it has the form $(p_1 \cdot \iota(p_2))(\tau(p_1) \cdot p_2)$, where p_1 and p_2 are non-spherical monoid pictures over $\mathcal{P}$ (see Figure 1.15).

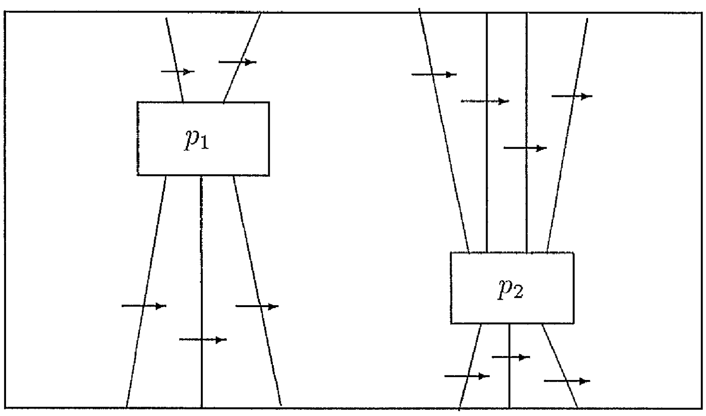


Figure 1.15

A non-skewed spherical monoid picture p has the form as in Figure 1.16 (up to equivalence in $\mathcal{D}$), where p_i is a spherical monoid picture with the property that every arc meets a disc ($i = 1, 2, \cdots, m$) and $\sum_{i=1}^{m} A(p_i) = A(p)$.

Lemma 1.7.5. ([74]) *If p is a skewed spherical monoid picture over $\mathcal{P}$, then there is a monoid picture q such that qpq^{-1} is equivalent to a non-skewed picture p' in $\mathcal{D}$ where $A(p') = A(p)$.*

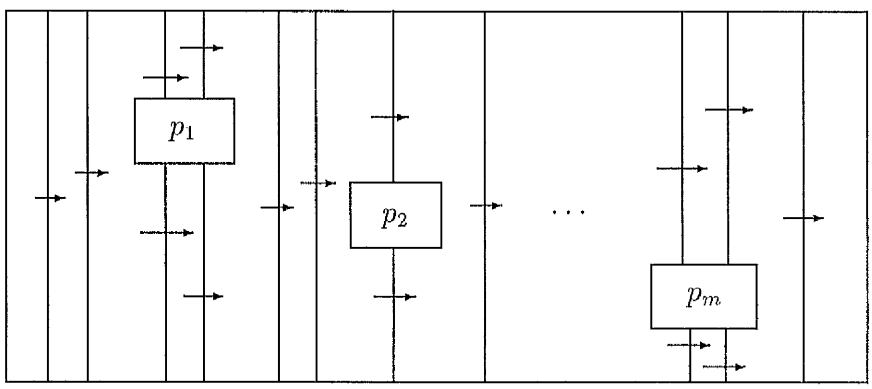


Figure 1.16

Chapter 2

Finite Complete Rewriting Systems

Abstract

In this chapter, we first give some basic definitions and results about rewriting systems, then we consider finite complete rewriting systems for small extensions of monoids and for semi-direct products of monoids. After introducing the notion of directed 2-complex and some results about it, we consider subgroups of finite index in groups with finite complete rewriting systems.

2.1 Preliminaries

2.1.1 Noetherian induction

Let X be a set and let $\longrightarrow$ be a relation on X. The relation $\longrightarrow$ is called *Noetherian* provided that there does not exist any infinite sequence

$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow \cdots$$

with $x_i \in X$.

Proposition 2.1.1 (Principle of Noetherian induction). *Let X be a set, let $\longrightarrow$ be a Noetherian relation on X and let P be a predicate on X. Suppose that whenever $x \in X$*

has the property that if $P(y)$ holds for every $y \in X$ with $x \longrightarrow .y$, then this implies that $P(x)$ holds. Then $P(x)$ holds for every $x \in X$.

Proof. Suppose that $P(x)$ does not hold for some $x \in X$. Then there exists some $x_1 \in X$ with $x \longrightarrow x_1$ and $P(x_1)$ does not hold. Repeating this argument for x_1 gives some $x_2 \in X$ with $x_1 \longrightarrow x_2$ and $P(x_2)$ does not hold. In this way we can get an infinite sequence $x \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots$ with $x_i \in X$, contradicting the fact that $\longrightarrow$ is a Noetherian relation on X. $\square$

2.1.2 Notation and basic definitions

Let R be a rewriting system on X. We say that R is *Noetherian* if the relation $\longrightarrow_R$ on X^* is Noetherian, that is, there is no infinite reduction sequence

$$W_1 \longrightarrow_R W_2 \longrightarrow_R W_3 \longrightarrow_R \cdots .$$

We say that R is *locally confluent* if whenever $U \longrightarrow_R V$ and $U \longrightarrow_R V'$ then there is a word $W \in X^*$ with $V \longrightarrow_R^* W$ and $V' \longrightarrow_R^* W$. Also, R is called *confluent* if whenever $U \longrightarrow_R^* V$ and $U \longrightarrow_R^* V'$ then there is a word $W \in X^*$ with $V \longrightarrow_R^* W$ and $V' \longrightarrow_R^* W$. If R is both Noetherian and confluent, we say that R is *complete*. It is easy to see that R is complete if and only if R is Noetherian and for each word $U \in X^*$ there is a unique irreducible word $W \in X^*$ such that $U \longrightarrow_R^* W$.

Proposition 2.1.2. *If R is Noetherian and locally confluent, then R is complete.*

Proof. We just need to show that for each word $U \in X^*$ there is a unique irreducible word $W \in X^*$ such that $U \longrightarrow_R^* W$. We use Noetherian induction to prove it. Since R is Noetherian, the existence of W is easy. If U is irreducible, the uniqueness is clear. In general, suppose that $U \longrightarrow_R V_1 \longrightarrow_R^* W_1$ and $U \longrightarrow_R V_2 \longrightarrow_R^* W_2$ with W_1, W_2 irreducible. Since R is locally confluent, there is a word $V \in X^*$ such that $V_1 \longrightarrow_R^* V$ and $V_2 \longrightarrow_R^* V$. Choose an irreducible word $W \in X^*$ such that $V \longrightarrow_R^* W$. Applying the inductive hypothesis to V_1 and V_2 respectively, we can get that $W_1 \equiv W$ and $W_2 \equiv W$. Thus $W_1 \equiv W_2$ as required. $\square$

Lemma 2.1.3. *Let R be a finite Noetherian rewriting system on X. Given a word $W \in X^*$, there is a bound on the lengths of all reduction sequences*

$$W \longrightarrow_R W_1 \longrightarrow_R W_2 \longrightarrow_R \cdots \longrightarrow_R W_n,$$

where the length of the above sequence is defined to be n. There is also a bound on the lengths of all of the words which appear in any of these reduction sequences.

Proof. Since there are only finitely many subwords of W, and each of these subwords may occur as the left hand side of at most finitely many rewrite rules in R, there are only finitely many ways to do a single-step reduction $W \longrightarrow_R W'$. If there is no bound on the lengths of all reduction sequences starting with W, then for some word W_1 with $W \longrightarrow_R W_1$, there must be no bound on the lengths of all reduction sequences starting with W_1. Repeating this argument gives a word W_2 with $W_1 \longrightarrow_R W_2$ and no bound on the lengths of all reduction sequences starting with W_2. In this way we can produce an infinite sequence $W \longrightarrow_R W_1 \longrightarrow_R W_2 \longrightarrow_R \cdots$, contradicting the fact that R is Noetherian. Now, since there is a bound on the lengths of all reduction sequences starting with W, and at each stage there are only finitely many ways to do a single-step reduction to a word, there are only finitely many words which appear in any of these reduction sequences. □

Let R be a finite complete rewriting system on X. Given a word $W \in X^*$, the *disorder* of W, denoted by $d_R(W)$, is the maximum of the lengths of all of the reduction sequences starting with W. If W is irreducible then $d_R(W) = 0$. The *stretch* of W, denoted by $st(W)$, is the maximum of the lengths of all of the words which appear in any of these sequences.

It is clear that if $U \longrightarrow_R V$ then $d_R(U) > d_R(V)$ and $st(U) \geq st(V)$.

2.1.3 Knuth-Bendix completion procedure

Let R be a rewriting system on X. For any word $W \in X^*$, let $l(W)$ denote the length of W. We call R *length-reducing* if $l(r_{+1}) > l(r_{-1})$ for all $(r_{+1}, r_{-1}) \in R$. We call R *length-plus-lexicographic* if $l(r_{+1}) \geq l(r_{-1})$ for all $(r_{+1}, r_{-1}) \in R$ and, further, if $l(r_{+1}) = l(r_{-1})$

then r_{-1} precedes r_{+1} in the lexicographic order (from the left) induced by some total ordering on X. An ordering $>$ on a set Y is called *well-founded* if there is no infinite chain $y_1 > y_2 > y_3 > \cdots$ with $y_i \in Y$. One usually establishes the Noetherian condition by imposing a well-founded ordering $>$ on X^*, which is compatible with concatenation (that is, if $W > W'$, then $UWV > UW'V$ for any $U, V \in X^*$), and then checking that if $(r_{+1}, r_{-1}) \in R$ then $r_{+1} > r_{-1}$. As a special case, if R is length-reducing or length-plus-lexicographic then R is Noetherian.

If $(UV, r_1) \in R$, $(VW, r_2) \in R$ and U, V, W are non-empty words, then the word UVW is called an *overlap ambiguity* of R. If $(V, r_1) \in R$ and $(UVW, r_2) \in R$ (if both U and W are empty then $r_1 \not\equiv r_2$), then the word UVW is called an *inclusion ambiguity* of R. The corresponding pair of words (r_1W, Ur_2) or (Ur_1W, r_2), respectively, is called a *critical pair* of R (see Figure 2.1). A critical pair (W_1, W_2) of R is *resolved* if there is a word $W' \in X^*$ such that $W_1 \longrightarrow_R^* W'$ and $W_2 \longrightarrow_R^* W'$.

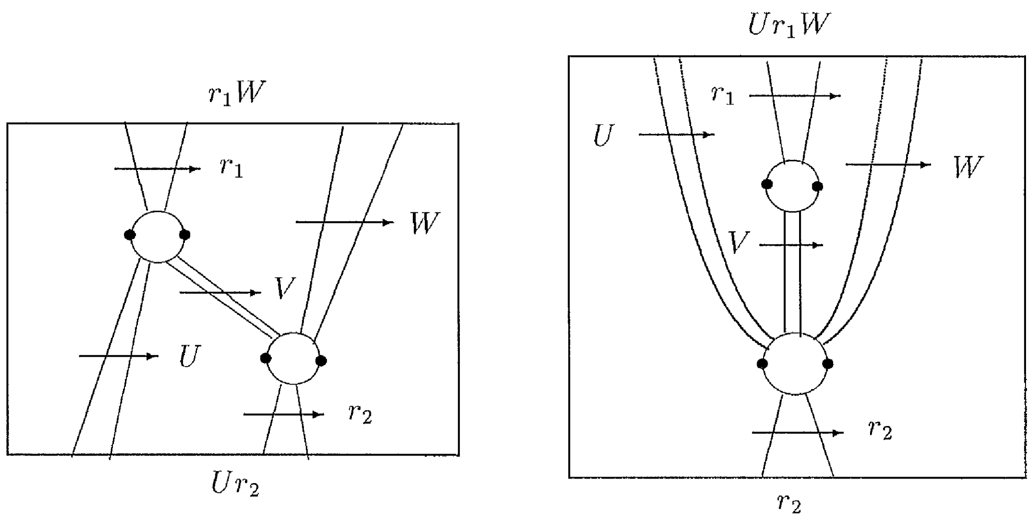


Figure 2.1

Proposition 2.1.4. *A rewriting system R is locally confluent if and only if all critical pairs of R are resolved.*

Proof. For each critical pair (W_1, W_2) of R, there is a word W such that $W \longrightarrow_R W_1$ and $W \longrightarrow_R W_2$. If R is locally confluent, then all critical pairs of R are resolved.

Suppose that all critical pairs of R are resolved. Let $U \longrightarrow_R V$ and $U \longrightarrow_R V'$, where a rewriting rule, say r, is applied to U to get V, and a rule, say r', is applied to U to get V'. If r and r' are disjoint, then we can assume that $U \equiv W_1 r_{+1} W_2 r'_{+1} W_3$, $V \equiv W_1 r_{-1} W_2 r'_{+1} W_3$ and $V' \equiv W_1 r_{+1} W_2 r'_{-1} W_3$. So $V \longrightarrow_R W_1 r_{-1} W_2 r'_{-1} W_3$ and $V' \longrightarrow_R W_1 r_{-1} W_2 r'_{-1} W_3$. If r and r' are not disjoint, then there is a critical pair (W_1, W_2) of R such that $V \equiv U_1 W_1 U_2$ and $V' \equiv U_1 W_2 U_2$. By assumption, there exists W' such that $W_1 \longrightarrow_R^* W'$ and $W_2 \longrightarrow_R^* W'$. Then $V \longrightarrow_R^* U_1 W' U_2$ and $V' \longrightarrow_R^* U_1 W' U_2$. Thus R is locally confluent. □

Two rewriting systems R_1 and R_2 on X are called *equivalent* if $\longleftrightarrow_{R_1}^*$ is the same as $\longleftrightarrow_{R_2}^*$. This condition is stronger than saying that the monoids presented by $[X; R_1]$ and $[X; R_2]$ are isomorphic. In fact, it can be shown that two rewriting systems R_1 and R_2 on X are equivalent if and only if $[X; R_2]$ can be obtained from $[X; R_1]$ by a finite number of Tietze transformations of type (T1) and its inverse. (On the other hand, see Theorem 1.1.5, the monoids presented by $[X; R_1]$ and $[X; R_2]$ are isomorphic if and only if $[X; R_2]$ can be obtained from $[X; R_1]$ by a finite number of Tietze transformations of types (T1), (T2) and their inverses.)

Every rewriting system is equivalent to a complete one, as is shown by the Knuth-Bendix completion procedure ([52]) given below.

Let R be a length-plus-lexicographic rewriting system. For each word W choose an irreducible word $I(W)$ such that $W \longrightarrow_R^* I(W)$. This can be done arbitrarily, but if we wish to perform an algorithmic process we could require $I(W)$ to be obtained by always rewriting the leftmost subword possible.

Let R' be obtained from R by considering all critical pairs (W_1, W_2) of R such that $I(W_1) \not\equiv I(W_2)$ and adding to R for such a critical pair either $(I(W_1), I(W_2))$ or $(I(W_2), I(W_1))$; the choice of pairs to add is to be made so that R' remains length-plus-lexicographic. (In practice, it is not always necessary to add all pairs $(I(W_1), I(W_2))$ or $(I(W_2), I(W_1))$ to R for all critical pairs (W_1, W_2) of R with $I(W_1) \not\equiv I(W_2)$, see Example 2.1.5). Evidently R is complete if and only if $R' = R$. Since $I(W_1) \longleftrightarrow_R^* I(W_2)$ for each critical pair (W_1, W_2), we have that R' is equivalent to R.

Now let R be an arbitrary rewriting system. Let R_0 be obtained from R by replacing some of the rewriting rules $(r_1, r_2) \in R$ by (r_2, r_1) in such a way that R_0 is length-plus-lexicographic. Inductively, define R_n for all n by $R_{n+1} = (R_n)'$, and let $R_\infty = \cup_n R_n$. It is easy to check that R_∞ is length-plus-lexicographic and equivalent to R. Consider a critical pair (W_1, W_2) of R_∞. Then (W_1, W_2) is a critical pair of R_n for some n. Letting $I_n(W_1)$ and $I_n(W_2)$ be the chosen irreducible words for W_1 and W_2 corresponding to R_n, we know that either $I_n(W_1) \equiv I_n(W_2)$ or one of $(I_n(W_1), I_n(W_2))$ and $(I_n(W_2), I_n(W_1))$ is in R_{n+1}. So every critical pair (W_1, W_2) of R_∞ is resolved. Thus R_∞ is complete.

When R is finite then each R_n is finite, but R_∞ may be infinite. If $R_{n+1} = R_n$ for some n, then R_n is complete and $R_\infty = R_n$. It follows that R is equivalent to the finite complete rewriting system R_n. Conversely, if R is equivalent to some finite complete rewriting system then there is some n such that R is equivalent to R_n (see [55]).

Example 2.1.5. Let $X = \{a, b\}$, and let $R = \{(b^2, a^3), (ba, a^2b)\}$.

Defining a total ordering on X by $b > a$, then $R_0 = \{(a^3, b^2), (a^2b, ba)\}$ is length-plus-lexicographic.

To get $R_1 = (R_0)'$, we need to consider all critical pairs of R_0. From the overlap ambiguity a^4 we get a critical pair (b^2a, ab^2). Since $I_0(b^2a) = b^2a$ and $I_0(ab^2) = ab^2$, we need to add a new rewriting rule (b^2a, ab^2) to R_0. From the overlap ambiguity a^3b we get a critical pair (b^3, aba). Since $I_0(b^3) = b^3$ and $I_0(aba) = aba$, we need to add another new rewriting rule (b^3, aba) to R_0.

Note that from the overlap ambiguity a^5 we can get a critical pair (b^2a^2, a^2b^2) and we should also add a new rewriting rule (b^2a^2, a^2b^2) to R_0, but it is superfluous after adding (b^2a, ab^2) to R_0. Similarly, we do not need to add any more new rewriting rules obtained from other critical pairs of R_0. Thus we may take

$$R_1 = \{(a^3, b^2), (a^2b, ba), (b^2a, ab^2), (b^3, aba)\}.$$

Similarly, we can get

$$R_2 = (R_1)' = \{(a^3, b^2), (a^2b, ba), (b^2a, ab^2), (b^3, aba), (baba, abab), (bab^2, aba^2)\},$$

and $R_3 = (R_2)' = R_2$. Thus R_2 is complete and R is equivalent to R_2.

2.2 Some known results and examples

Let M be a monoid, and let $\mathbb{Z}M$ denote the integral monoid ring of M. Consider the left $\mathbb{Z}M$-module $\mathbb{Z}$ where M acts trivially (that is, $m \cdot \lambda = \lambda$ for all $m \in M$, $\lambda \in \mathbb{Z}$). A *left free resolution* of $\mathbb{Z}$ is an exact sequence

$$\cdots P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

of free left $\mathbb{Z}M$-modules P_i and $\mathbb{Z}M$-module homomorphisms ∂_i. Such a resolution always exists (see [15]).

The monoid M is said to satisty the *left homological finiteness condition* FP_k for some integer $k \geq 0$ (or M is left FP_k for short) if there exists a left free resolution of $\mathbb{Z}$ such that P_i is finitely generated for all $i \leq k$. The monoid M is left FP_∞ if there is a left free resolution of $\mathbb{Z}$ with P_i finitely generated for all i. Similarly, we can define the *right homological finiteness condition* FP_k. For groups, left FP_k and right FP_k are equivalent. However, Cohen ([22]) gave an example of monoid which is right FP_∞ but not left FP_1 (and vice versa). It is known that one-relator groups and automatic groups are FP_∞.

Theorem 2.2.1. *Let M be a finitely presented monoid.*

(1) ([77]) *If M can be presented by a finite complete rewriting system, then M is (both left and right) FP_3.*

(2) ([6], [53], [39], [16]) *If M can be presented by a finite complete rewriting system, then M is (both left and right) FP_∞.*

As mentioned in the Introduction, if a monoid can be presented by a finite complete rewriting system, then it has a solvable word problem, and it has a finite derivation type. If the finite complete rewriting system has additional properties, then further properties of the monoid can be obtained. So far most such results are about groups.

A rewriting system R on X is called *special* if $R \subseteq (X^*\backslash\{\varnothing\}) \times \{\varnothing\}$. It is *monadic* if R is length-reducing and $r_{-1} \in X \cup \{\varnothing\}$ for all $r \in R$.

Theorem 2.2.2. ([64]) *Suppose a monoid M has a finite presentation $[X; R]$, where R is a length-reducing and confluent rewriting system on X. Then the problems of deciding* *whether M is a free monoid and whether it is a group are decidable.*

Theorem 2.2.3. ([63]) *Suppose a monoid M has a finite presentation $[X; R]$, where R is a monadic and confluent rewriting system on X. Then the problems of deciding whether M is torsion-free and whether it is a free group are decidable.*

Theorem 2.2.4. ([61]) *If a monoid M has a finite presentation $[X; R]$, where R is length-reducing and confluent on X, then the conjugacy problem for M is solvable.*

Theorem 2.2.5. ([20]) *A group G has a finite monoid presentation $[X; R]$, where R is a special and confluent rewriting system on X, if and only if G is the free product of finitely many cyclic groups.*

A group G is called a *plain group* if it is isomorphic to the free product of finitely many finite groups and a finitely generated free group.

Theorem 2.2.6. ([7]) *A group G has a finite monoid presentation $[X; R]$, where R is a monadic and confluent rewriting system on X that provides inverses of length 1 for all letters in X, if and only if G is a plain group.*

A function $g : X \longrightarrow \mathbb{Z}$ satisfying $g(x) > 0$ for all $x \in X$ is called a *weight-function.* It can be uniquely extended to a homomorphism from X^* into $\mathbb{Z}$, which will also be denoted by g. Define a binary relation $>_g$ on X^* as follows: $U >_g V$ if and only if $g(U) > g(V)$. Then $>_g$ is a well-founded partial ordering on X^*. A rewriting system R on X is called *weight-reducing* if there exists a weight function g such that $r_{+1} >_g r_{-1}$ holds for each $(r_{+1}, r_{-1}) \in R$. Weight-reducing systems are a generalization of length-reducing systems. It is clear that weight-reducing systems are Noetherian.

Theorem 2.2.7. ([33], [58]) *Let G be a group that has a finite monoid presentation $[X; R]$, where R is a weight-reducing and confluent rewriting system on X.*

(a) Every abelian subgroup of G is either finite or isomorphic to the free group of rank 1.

(b) If the centre of G is nontrivial, then G is either finite or isomorphic to the free group of rank 1.

(c) If G contains a nontrivial normal subgroup that is finite, then G itself is finite.

(d) Every factor group of G modulo a finitely generated nontrivial normal subgroup is finite.

Groves and Smith ([40]) investigate how the property of being presented by a finite complete rewriting system behaves under various group-theoretic constructions (extending from subgroups, forming amalgamated free products, HNN-extensions and wreath products, etc.). Complete rewriting systems (maybe infinite) for presenting various interesting groups are given in [55]. Here we mention some groups which can be presented by finite complete rewriting systems.

Given a finite simplicial graph with a group (or monoid) attached to each vertex , the associated *graph product* is the group (monoid) generated by all of the vertex groups (monoids) with the added relations that elements of distinct adjacent vertex groups commute. *Graph groups* are graph products of groups in which each vertex group is a free group of rank 1.

Theorem 2.2.8. ([44]) *The graph product of finitely many groups (or monoids) which can be presented by finite complete rewriting systems can be presented by a finite complete rewriting system.*

Example 2.2.9. All graph groups (with finite underlying graph) can be presented by finite complete rewriting systems.

A group is said to be *constructible* if it can be obtained from the trivial group by a finite sequence of the following operations:

(1) forming a finite extension of a previously constructed group;

(2) forming an HNN-extension in which both the base group and the associated subgroups have been previously constructed;

(3) forming an amalgamated free product in which the factors and the amalgamated subgroup have been previously constructed.

Example 2.2.10. ([41]) Each soluble constructible group can be presented by a finite complete rewriting system.

The *braid group* B_n on n strings is the group presented by the group presentation

$$\langle x_1, x_2, \cdots, x_{n-1}; x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1}\ (1 \le i \le n-2),\ x_i x_j x_i^{-1} x_j^{-1}\ (|i-j| \ge 2)\rangle.$$

Example 2.2.11. ([45]) All braid groups can be presented by finite complete rewriting systems.

A *Coxeter group* is a group with a group presentation of the form

$$\langle s_1, s_2, \cdots, s_n; s_i^2\ (1 \le i \le n), (s_i s_j)^{m_{ij}}\ (1 \le i < j \le n)\rangle,$$

where $2 \le m_{ij} \le \infty$, and $m_{ij} = \infty$ denotes that there is no relation involving s_i and s_j.

Example 2.2.12. ([43]) Let G be a Coxeter group. Suppose G satisfies one of the following two properties:

(1) G has three or fewer generators;

(2) G does not contain a special subgroup of the form

$$\langle s_i, s_j, s_k; s_i^2, s_j^2, s_k^2, (s_i s_j)^2, (s_i s_k)^m, (s_j s_k)^l\rangle$$

with m and l both finite and not both equal to two.

Then G can be presented by a finite complete rewriting system.

The fundamental group of a closed orientable surface of genus n has a group presentation

$$\langle a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n; a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}\rangle.$$

Example 2.2.13. ([55]) The fundamental group of a closed orientable surface of genus n can be presented by a finite complete rewriting system.

In general, it still remains open whether all one-relator groups can be presented by finite complete rewriting systems or not. However, there are some special one-relator groups which can be presented by finite complete rewriting systems.

Example 2.2.14. ([30]) Let G be the one-relator group defined by a group presentation

$$\langle X; aba^{-1}b^{-1}W\rangle,$$

where X is a finite set, $a, b \in X$, W is a word on $(X \cup X^{-1})\backslash\{a, a^{-1}, b, b^{-1}\}$. (Note that the fundamental group of a closed orientable surface of genus n is a special case of G). Then G can be presented by a finite complete rewriting system.

The technique of rewriting (giving complete rewriting systems) can be used for some normal form theorems. For example, Dekov [31] gave a simple proof of Bourbaki's normal form theorem for free products with amalgamation of monoids by using the technique of rewriting.

2.3 Finite Complete Rewriting Systems for Small Extensions of Monoids

Let S be a semigroup and let T be a subsemigroup of S. If T has finite index in S (that is, the set $S\backslash T$ is finite), then S is called a *small extension* of T, and T is called a *large subsemigroup* of S.

Example 2.3.1. Let $\mathcal{P} = [X; R]$ be a finite monoid presentation such that $l(r_{+1}) = l(r_{-1})$ for every $r \in R$, and let $S = S(\mathcal{P})$. For any integer $n > 0$, let

$$T_n = \{[W]_R : W \in X^*,\ l(W) > n\}.$$

Then it is easy to see that T_n is a subsemigroup of S, and $S\backslash T_n$ is finite. Thus S is a small extension of T_n.

Example 2.3.2. Let A be a finite semigroup, and let B be any semigroup which is disjoint from A. Let $S = A \cup B$ with a product defined as follows. For any $a_1, a_2 \in A$ and $b_1, b_2 \in B$, let a_1a_2 and b_1b_2 be the same as the products in A and B, respectively. Let $ab = ba = b$ for any $a \in A$ and $b \in B$. It is easy to check that S is a semigroup under this product, and B can be considered as a subsemigroup of S. Because $S\backslash B = A$ is finite, S is a small extension of B.

Some finiteness conditions, such as being finitely presented and having a solvable word problem, are preserved by forming small extensions or large semigroups (see [17], [18],

[19] and [75]). In this section we consider finite complete rewriting systems for small extension of monoids (it is similar for the semigroup case). We solve part of the Open Problem11.1 posed by Ruškuc ([75]).

Suppose S is a monoid and let T be a submonoid of finite index in S. Let $\mathcal{P}_T = [A; R_1]$ be a finite presentation for T. Then by [75], S has a finite presentation

$$\mathcal{P}_S = [A, S\backslash T; R_1, R_2],$$

where $R_2 = \{(sa,\ \rho(s,a)),\ (as,\ \lambda(a,s)),\ (ss',\ \pi(s,s')) : s, s' \in S\backslash T,\ a \in A\}$ for certain elements $\rho(s,a), \lambda(a,s), \pi(s,s') \in A^* \cup (S\backslash T)$.

Let $K = \max\{1,\ l(\rho(s,a)), l(\lambda(a,s)), l(\pi(s,s')) : s, s' \in S\backslash T, a \in A\}$. For any $V \in (A \cup (S\backslash T))^*$, let $V \equiv V_1 s_1 V_2 s_2 \cdots V_n s_n V_{n+1}$, where $V_i \in A^*,\ s_i \in S\backslash T$. We define

$$\alpha(V) = Kn + l(V_1) + l(V_2) + \cdots + l(V_n) + l(V_{n+1}). \tag{2.1}$$

It is clear that $\alpha(VV') = \alpha(V) + \alpha(V')$ for any $V, V' \in (A \cup (S\backslash T))^*$. For any $r \in R_2$, we have $\alpha(r_{+1}) \geq K + 1 > K \geq \alpha(r_{-1})$. So

$$\alpha(Ur_{+1}V) = \alpha(U) + \alpha(r_{+1}) + \alpha(V) > \alpha(U) + \alpha(r_{-1}) + \alpha(V) = \alpha(Ur_{-1}V).$$

Thus $\alpha(W) > \alpha(W')$ if $W \longrightarrow_{R_2} W'$.

Lemma 2.3.3. *For any word $U \in (A \cup (S\backslash T))^*$, there is a word $U' \in A^* \cup (S\backslash T)$ such that $U \longrightarrow^*_{R_2} U'$.*

Proof. We prove the result by induction on the non-negative integer $\alpha(U)$.

If $\alpha(U) = 0$, the result is clear. Suppose that the result is true for the case $\alpha(U) < n$. We consider the case of $\alpha(U) = n$.

If $U \in A^* \cup (S\backslash T)$, then choose $U' \equiv U$. If $U \notin A^* \cup (S\backslash T)$, then $U \longrightarrow_{R_2} U_1$ for some $U_1 \in (A \cup (S\backslash T))^*$. So $\alpha(U_1) < \alpha(U) = n$. By the induction hypothesis, there is a word $U' \in A^* \cup (S\backslash T)$ such that $U_1 \longrightarrow^*_{R_2} U'$. Thus $U \longrightarrow^*_{R_2} U'$. □

Theorem 2.3.4. *Let S be a small extension of T. If T can be presented by a finite complete rewriting system, then so can S.*

This theorem appears in the paper [80].

We remark that the converse of Theorem 2.3.4 (also raised as part of the Open Problem11.1 in [75]) remains open.

Proof. Suppose R_1 is a finite complete rewriting system on A. Let $R = R_1 \cup R_2$. We want to show that R is a finite complete rewriting system on $A \cup (S \backslash T)$.

First we show that R is confluent.

Let $U \longrightarrow_R^* V_1$ and $U \longrightarrow_R^* V_2$, where $U, V_1, V_2 \in (A \cup (S \backslash T))^*$. By Lemma 2.3.3, there are words $V_1', V_2' \in A^* \cup (S \backslash T)$ such that $V_1 \longrightarrow_R^* V_1'$ and $V_2 \longrightarrow_R^* V_2'$. Since $V_1' = V_2'$ holds in S, we either have $V_1' \equiv V_2' \in S \backslash T$ or we have $V_1', V_2' \in A^*$ and $V_1' = V_2'$ holds in T. In the second case we have $[V_1']_{R_1} = [V_2']_{R_1}$. Because R_1 is a finite complete rewriting system on A, there is an irreducible word $W \in A^*$ such that $V_1' \longrightarrow_{R_1}^* W$ and $V_2' \longrightarrow_{R_1}^* W$. Thus $V_1 \longrightarrow_R^* W$ and $V_2 \longrightarrow_R^* W$. Hence R is confluent.

To show that R is Noetherian, it suffices to define a well-founded partial order $>$ on $(A \cup (S \backslash T))^*$ such that if $W \longrightarrow_R V$ then $W > V$.

For any $W \in (A \cup (S \backslash T))^*$, let $W \equiv W_1 s_1 W_2 s_2 \cdots W_n s_n W_{n+1}$, where $W_1, \cdots, W_{n+1} \in A^*$, $s_1, s_2, \ldots, s_n \in S \backslash T$. We define functions $\phi_1, \phi_2, \cdots$ from $(A \cup (S \backslash T))^*$ to non-negative integers by

$$\begin{aligned} \phi_1(W) &= n, \\ \phi_{2i}(W) &= d_{R_1}(W_i) \text{ (the disorder of } W_i), \\ \phi_{2i+1}(W) &= l(W_i) \text{ (the length of } W_i), \\ \phi_j(W) &= 0, \end{aligned}$$

where $1 \leq i \leq n+1,\ j > 2n+3$.

For two words $W, U \in (A \cup (S \backslash T))^*$, we define $W > U$ if $\phi_k(W) > \phi_k(U)$ for some k and $\phi_j(W) = \phi_j(U)$ for all $j < k$.

First we show this ordering is well-founded. Suppose there is an infinite chain

$$U_1 > U_2 > U_3 > \cdots .$$

Since $\phi_1(U_1) \geq \phi_1(U_2) \geq \phi_1(U_3) \geq \cdots$, there must be some integer k_1 such that

$$\phi_1(U_{k_1}) = \phi_1(U_{k_1+1}) = \phi_1(U_{k_1+2}) = \cdots .$$

Then we have

$$\phi_2(U_{k_1}) \geq \phi_2(U_{k_1+1}) \geq \phi_2(U_{k_1+2}) \geq \cdots .$$

There must be some integer $k_2 \geq k_1$ such that

$$\phi_2(U_{k_2}) = \phi_2(U_{k_2+1}) = \phi_2(U_{k_2+2}) = \cdots .$$

So we have

$$\phi_3(U_{k_2}) \geq \phi_3(U_{k_2+1}) \geq \phi_3(U_{k_2+2}) \geq \cdots .$$

Continuing in this way, we can get some integers $k_m \geq k_{m-1} \geq \cdots \geq k_2 \geq k_1$, such that

$$\phi_j(U_{k_j}) = \phi_j(U_{k_j+1}) = \phi_j(U_{k_j+2}) = \cdots , \ (j = 1, 2, \cdots, m).$$

So $\phi_j(U_{k_m}) = \phi_j(U_{k_m+1})$ for $j = 1, 2, \cdots, m$.

Since $\phi_1(U_i) \leq \phi_1(U_1)$ for all i, we have $\phi_j(U_i) = 0$ for all i if $j > 2\phi_1(U_1) + 3$. Let $m = 2\phi_1(U_1) + 3$. Then we have $\phi_j(U_{k_m}) = \phi_j(U_{k_m+1})$ for all j. This is impossible for $U_{k_m} > U_{k_m+1}$. Thus there is no infinite chain $U_1 > U_2 > U_3 > \cdots$.

Now we show that if $W \longrightarrow_R V$ then $W > V$.

Let $W \equiv W_1 s_1 W_2 s_2 \cdots W_n s_n W_{n+1}$, where $W_i \in A^*$, $s_j \in S \backslash T$. Suppose a rule $r \in R$ is applied to W to get V. If $r \in R_1$, it must be applied to a subword W_i. So we have $\phi_j(W) = \phi_j(V)$ $(1 \leq j < 2i)$ and $\phi_{2i}(W) > \phi_{2i}(V)$. So $W > V$. If $r = (s_i s_{i+1}, \pi(s_i, s_{i+1}))$, or $r = (s_i a, \rho(s_i, a))$ with $\rho(s_i, a) \in A^*$, or $r = (a s_i, \lambda(a, s_i))$ with $\lambda(a, s_i) \in A^*$, then $\phi_1(W) > \phi_1(V)$, so $W > V$.

If $r = (s_i a, s_i')$ with $s_i' \in S \backslash T$, then

$$W \equiv W_1 s_1 \cdots W_i s_i a W_{i+1}' s_{i+1} \cdots W_{n+1},$$

$$V \equiv W_1 s_1 \cdots W_i s_i' W_{i+1}' s_{i+1} \cdots W_{n+1},$$

where $W_{i+1} \equiv a W_{i+1}'$. So $\phi_j(W) = \phi_j(V)$ $(1 \leq j < 2i + 2)$, $\phi_{2i+2}(W) = d_{R_1}(a W_{i+1}') \geq d_{R_1}(W_{i+1}') = \phi_{2i+2}(V)$, and $\phi_{2i+3}(W) = l(a W_{i+1}') > l(W_{i+1}') = \phi_{2i+3}(V)$. Thus $W > V$.

Similarly, if $r = (a s_i, s_i')$ with $s_i' \in S \backslash T$, we can also get $W > V$. $\square$

2.4 Finite complete rewriting systems for semi-direct products of monoids

Let A, B be monoids, and let Q be a monoid homomorphism

$$Q : A \longrightarrow End(B),\ a \longmapsto Q_a,\ 1 \longmapsto id.$$

Then we have the semi-direct product $M = A \ltimes_Q B$ of A and B relative to Q (see Section 1.2).

The main result in this section is the following theorem.

Theorem 2.4.1. *If A and B can be presented by finite complete rewriting systems, then so can $A \ltimes_Q B$.*

This theorem is mentioned in the paper [79]. The proof is based on the ideas of Groves and Smith [40].

Proof. Let $\mathcal{P}_A = [X; R_1]$ and $\mathcal{P}_B = [Y; R_2]$ be finite complete presentations for A and B respectively. If W is a word on X (or Y), then $[W]_A$ (or $[W]_B$) denotes the element of A (or B) represented by W. For any $x \in X$ and $y \in Y$, we use yQ_x to denote the unique irreducible word on Y such that $[yQ_x]_B = [y]_B Q_{[x]_A}$. Let R_3 denote the set of relations $yx = x(yQ_x)$ $(x \in X, y \in Y)$. Then

$$\mathcal{P} = [X \cup Y; R_1, R_2, R_3]$$

is a finite presentation for $M = A \ltimes_Q B$.

We want to show that $R = R_1 \cup R_2 \cup R_3$ is a complete rewriting system on $X \cup Y$.

To show that R is Noetherian, it suffices to define a well-founded partial order $>$ on $(X \cup Y)^*$ such that if $W \longrightarrow_R V$ then $W > V$.

For every word $W \in (X \cup Y)^*$, let

$$W \equiv W_1 x_1 W_2 x_2 \cdots W_n x_n W_{n+1},$$

where $W_1, W_2, \cdots, W_{n+1} \in Y^*$, $x_1, x_2, \ldots, x_n \in X$, and let $W' \equiv x_1 x_2 \cdots x_n$. We define functions $\psi_0, \psi_1, \cdots$ from $(X \cup Y)^*$ to non-negative integers by

$$\begin{aligned}
\psi_0(W) &= st(W') \text{ (the stretch of } W'),\\
\psi_1(W) &= d_{R_1}(W') \text{ (the disorder of } W'),\\
\psi_{2i}(W) &= d_{R_2}(W_i) \text{ (the disorder of } W_i),\\
\psi_{2i+1}(W) &= l(W_i) \text{ (the length of } W_i),\\
\psi_j(W) &= 0,
\end{aligned}$$

where $1 \le i \le n+1,\ j > 2n+3$. Note that $\psi_0(W) = st(W') \ge l(W') = n$.

For two words $W, U \in (X \cup Y)^*$, we define $W > U$ if $\psi_k(W) > \psi_k(U)$ for some k and $\psi_j(W) = \psi_j(U)$ for all $j < k$.

First we show this ordering is well-founded. Suppose there is an infinite chain

$$U_1 > U_2 > U_3 > \cdots .$$

Since $\psi_0(U_1) \ge \psi_0(U_2) \ge \psi_0(U_3) \ge \cdots$, there must be some integer k_0 such that

$$\psi_0(U_{k_0}) = \psi_0(U_{k_0+1}) = \psi_0(U_{k_0+2}) = \cdots .$$

Then we have

$$\psi_1(U_{k_0}) \ge \psi_1(U_{k_0+1}) \ge \psi_1(U_{k_0+2}) \ge \cdots .$$

There must be some integer $k_1 \ge k_0$ such that

$$\psi_1(U_{k_1}) = \psi_1(U_{k_1+1}) = \psi_1(U_{k_1+2}) = \cdots .$$

So we have

$$\psi_2(U_{k_1}) \ge \psi_2(U_{k_1+1}) \ge \psi_2(U_{k_1+2}) \ge \cdots .$$

Continuing in this way, we can get some integers $k_m \ge k_{m-1} \ge \cdots \ge k_1 \ge k_0$, such that

$$\psi_j(U_{k_j}) = \psi_j(U_{k_j+1}) = \psi_j(U_{k_j+2}) = \cdots,\ (j = 0, 1, 2, \cdots, m).$$

So $\psi_j(U_{k_m}) = \psi_j(U_{k_m+1})$ for $j = 0, 1, 2, \cdots, m$.

Since $\psi_0(U_i) \leq \psi_0(U_1)$ for all i, we have $\psi_j(U_i) = 0$ for all i if $j > 2\psi_0(U_1) + 3$. Let $m = 2\psi_0(U_1) + 3$. Then we have $\psi_j(U_{k_m}) = \psi_j(U_{k_m+1})$ for all j. This is impossible for $U_{k_m} > U_{k_m+1}$. Thus there is no infinite chain $U_1 > U_2 > U_3 > \cdots$.

Now we show that if $W \longrightarrow_R V$ then $W > V$.

Let $W \equiv W_1 x_1 W_2 x_2 \cdots W_n x_n W_{n+1}$, where $W_1, W_2, \cdots, W_{n+1} \in Y^*$, $x_1, x_2, \cdots, x_n \in X$. Suppose a rule $r \in R$ is applied to W to get V. If $r \in R_1$, it must be applied to a subword of W'. So $W' \longrightarrow_{R_1} V'$. Thus we have $\psi_0(W) = st(W') \geq st(V') = \psi_0(V)$ and $\psi_1(W) = d_{R_1}(W') > d_{R_1}(V') = \psi_1(V)$. So $W > V$. If $r \in R_2$, it must be applied to a subword W_k. In this case we have $W' \equiv V'$, $W_j \equiv V_j$ for $j < k$ and $W_k \longrightarrow_{R_2} V_k$. So $\psi_{2k}(W) > \psi_{2k}(V)$ and $\psi_j(W) = \psi_j(V)$ for all $j < 2k$. Thus $W > V$.

If $r \in R_3$, then the rule is applied to a subword of $W_k x_k$ for some k. Let $W_k \equiv V_k y$, where $y \in Y$, $V_k \in Y^*$. Then

$$V \equiv W_1 x_1 \cdots W_{k-1} x_{k-1} V_k x_k (y Q_{x_k}) W_{k+1} x_{i+1} \cdots W_{n+1}.$$

So $\psi_j(W) = \psi_j(V)$ $(1 \leq j < 2k)$, $\psi_{2k}(W) = d_{R_2}(W_k) = d_{R_2}(V_k y) \geq d_{R_2}(V_k) = \psi_{2k}(V)$, and $\psi_{2k+1}(W) = l(W_k) = l(V_k y) > l(V_k) = \psi_{2k+1}(V)$. Thus $W > V$.

Having proved that R is Noetherian, we just need to show that for each word $U \in (X \cup Y)^*$ there is a unique irreducible word $W \in (X \cup Y)^*$ such that $U \longrightarrow_R^* W$. It is clear that the irreducible words in $(X \cup Y)^*$ with respect to R are exactly the words of the form $V_1 V_2$, where V_1 is an irreducible word in X^* with respect to R_1 and V_2 is an irreducible word in Y^* with respect to R_2.

For each word $U \in (X \cup Y)^*$, it is easy to see that there exist words $U_1 \in X^*$ and $U_2 \in Y^*$ such that $U \longrightarrow_{R_3}^* U_1 U_2$. Let V_1 be the irreducible word in X^* with respect to R_1 such that $U_1 \longrightarrow_{R_1}^* V_1$, and let V_2 be the irreducible word in Y^* with respect to R_2 such that $U_2 \longrightarrow_{R_2}^* V_2$. Then $V_1 V_2$ is an irreducible word in $(X \cup Y)^*$ with respect to R such that $U \longrightarrow_R^* V_1 V_2$. Suppose that we also have $U \longrightarrow_R^* V_1' V_2'$ for some irreducible words $V_1' \in X^*$ and $V_2' \in Y^*$. Then

$$[V_1]_M [V_2]_M = [V_1 V_2]_M = [U]_M = [V_1' V_2']_M = [V_1']_M [V_2']_M.$$

Since A and B can be considered as submonoids of M, we have $[W_1]_A = [W_1]_M$ if $W_1 \in X^*$, and $[W_2]_B = [W_2]_M$ if $W_2 \in Y^*$. So $[V_1]_A[V_2]_B = [V_1']_A[V_2']_B$. Thus $[V_1]_A = [V_1']_A$ and $[V_2]_B = [V_2']_B$. Therefore $V_1 = V_1'$ and $V_2 = V_2'$.

Thus R is complete on $X \cup Y$. □

Note that the converse of Theorem 2.4.1 is false.

Example 2.4.2. Let $F_1 = \langle x \rangle$ be the free group of rank 1, and let F_∞ be the free group on generators $\{a_k : k \in \mathbb{Z}\}$. We define a right F_1-action on F_∞ by $a_k \cdot x = a_{k+1}$ for all k. Then the semi-direct product $F_1 \ltimes F_\infty$ can be presented by a finite complete rewriting system, but the factor F_∞ can not .

First we show that $F_1 \ltimes F_\infty \cong F_2$, where F_2 denotes the free group of rank 2.

A group presentation for $F_1 \ltimes F_\infty$ is

$$\langle x, a_k\ (k \in \mathbb{Z}); x^{-1}a_k x a_{k+1}^{-1}\ (k \in \mathbb{Z})\rangle.$$

Let $F_2 = \langle y, z;\ \rangle$, and let ϕ be the homomorphism from the free group $\langle x, a_k\ (k \in \mathbb{Z});\ \rangle$ to F_2 defined by $\phi(x) = y,\ \phi(a_k) = y^{-k}zy^k\ (k \in \mathbb{Z})$. Since

$$\phi(x^{-1}a_k x a_{k+1}^{-1}) = y^{-1}y^{-k}zy^k y(y^{-k-1}zy^{k+1})^{-1} = 1 \text{ for all } k \in \mathbb{Z},$$

we can extend ϕ to a homomorphism from $F_1 \ltimes F_\infty$ to F_2. Let ψ be the homomorphism from F_2 to $F_1 \ltimes F_\infty$ defined by $\psi(y) = x,\ \psi(z) = a_0$. Since $\phi\psi(y) = \phi(x) = y$ and $\phi\psi(z) = \phi(a_0) = z$, we have $\phi\psi = id$. Since $\psi\phi(x) = \psi(y) = x$ and $\psi\phi(a_k) = \psi(y^{-k}zy^k) = x^{-k}a_0x^k = a_k$, we have $\psi\phi = id$. Thus $F_1 \ltimes F_\infty \cong F_2$.

Since every free group of finite rank can be presented by a finite complete rewriting system, we have $F_1 \ltimes F_\infty$ can be presented by a finite complete rewriting system. Since F_∞ is not finitely generated, it can not be presented by any finite complete rewriting system.

2.5 Directed 2-complexes and rewriting systems on directed graphs

The work in this section and the next section is joint with S.J.Pride ([72]).

A *directed 2-complex* is a pair

$$\mathcal{K} = [\Gamma; R]$$

where $\Gamma = (V, E, \iota, \tau)$ is a directed graph and R is a subset of $P^{(2)}(\Gamma)$. We also say that R is a *rewriting system* on Γ. The elements of R are then called rewriting rules, and they are written sometimes in the form $r: \ r_{+1} = r_{-1}$ for $(r_{+1}, r_{-1}) \in R$.

Example 2.5.1. Let Γ be the directed graph in Figure 2.2. Let

$$R = \{e_1e_2e_3 = 1_{v_1},\ e_4e_1 = e_1e_2e_5,\ e_2e_3e_4 = e_6e_3,\ e_3e_1e_6 = e_5e_2\}.$$

Then R is a rewriting system on Γ.

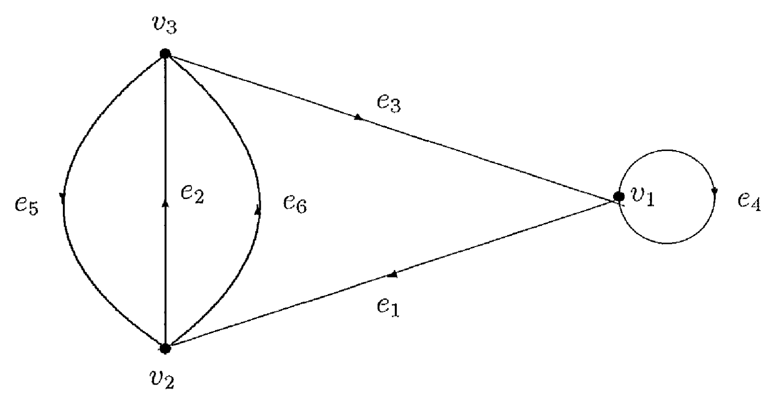


Figure 2.2

The single-step reduction relation $\longrightarrow_R$ is the following relation on $P(\Gamma)$: $p \longrightarrow_R q$ if and only if $p = p_1r_{+1}p_2$ and $q = p_1r_{-1}p_2$ for some $(r_{+1}, r_{-1}) \in R$ and $p_1, p_2 \in P(\Gamma)$. Its reflexive, transitive closure is denoted by $\longrightarrow_R^*$, and its reflexive, symmetric and transitive closure is denoted by $\longleftrightarrow_R^*$. For any $p \in P(\Gamma)$, let $[p]_R$ denote the equivalence class $\{q \in P(\Gamma) : q \longleftrightarrow_R^* p\}$. The empty path at v will be denoted by 1_v for any $v \in V$.

It is clear that if $p \longleftrightarrow_R^* p'$ and $q \longleftrightarrow_R^* q'$ then $pq \longleftrightarrow_R^* p'q'$ provided $\tau(p) = \iota(q)$. This enables us to define a partial multiplication of equivalence classes by

$$[p]_R[q]_R = [pq]_R \quad (\text{if } \tau(p) = \iota(q)).$$

In particular, if we fix a vertex v and consider the set $\{[p]_R : p \in P(\Gamma),\ \iota(p) = \tau(p) = v\}$, then we have a multiplication on this set, and it is a monoid (with identity $[1_v]$) under this multiplication. We call the above monoid the *fundamental monoid* of $\mathcal{K}$ at v, denoting it by $\pi_1^+(\mathcal{K}, v)$.

We say that R is *Noetherian* if there is no infinite reduction sequence

$$p_1 \longrightarrow_R p_2 \longrightarrow_R p_3 \longrightarrow_R \cdots .$$

We say that R is *locally confluent* if whenever $p \longrightarrow_R q_1$ and $p \longrightarrow_R q_2$ then there is a $q \in P(\Gamma)$ with $q_1 \longrightarrow_R^* q$ and $q_2 \longrightarrow_R^* q$. Also, R is called *confluent* if whenever $p \longrightarrow_R^* q_1$ and $p \longrightarrow_R^* q_2$ then there is a $q \in P(\Gamma)$ with $q_1 \longrightarrow_R^* q$ and $q_2 \longrightarrow_R^* q$. If R is both Noetherian and confluent, we say that R is *complete.* It is easy to prove (by Noetherian induction) that if R is Noetherian and locally confluent then R is complete.

A monoid presentation $\mathcal{P} = [X; R]$ can be considered as a directed 2-complex $\mathcal{K} = [\Gamma; R]$, where Γ is a directed graph with one vertex o and an edge x ($\iota(x) = \tau(x) = o$) for each $x \in X$, and a word on X is considered as a path in Γ. It is clear that the monoid presented by $\mathcal{P}$ is isomorphic to the fundamental monoid of $\mathcal{K}$ at o. Then R is a complete rewriting system on X if and only if R is a complete rewriting system on Γ.

Let Γ' be another directed graph with vertex set V' and edge set E'. A mapping

$$\phi : \Gamma \longrightarrow \Gamma'$$

is a function from $V \cup E$ to $V' \cup E'$ with $\phi(V) \subseteq V'$, $\phi(E) \subseteq E'$ and such that

$$\phi(\iota(e)) = \iota(\phi(e)),\ \phi(\tau(e)) = \tau(\phi(e))$$

for all $e \in E$. If $p = e_1 e_2 \cdots e_n$ is a path in Γ, then $\phi(e_1)\phi(e_2)\cdots\phi(e_n)$ is a path in Γ'. We will denote this path by $\phi(p)$. For empty paths 1_v ($v \in V$), we define $\phi(1_v) = 1_{\phi(v)}$.

For any $v \in V$, we let

$$Star(v) := \{e \in E : \iota(e) = v\}.$$

Clearly $\phi(Star(v)) \subseteq Star(\phi(v))$ for any $v \in V$. We say that ϕ is *locally bijective* if

$$\phi|_{Star(v)} : Star(v) \longrightarrow Star(\phi(v))$$

is bijective for every $v \in V$.

Lemma 2.5.2. *Let $\phi : \Gamma \longrightarrow \Gamma'$ be a locally bijective mapping of directed graphs. For any path p' in Γ', if $\iota(p') = \phi(v)$ for some vertex v in Γ, then there is a unique path p in Γ such that $\iota(p) = v$ and $\phi(p) = p'$.* (We will call p the *lift* of p' at v).

Proof. We prove the result by induction on the length $l(p')$ of p'. If $l(p') = 0$, then $p' = 1_{\phi(v)}$. So we can take $p = 1_v$ and it is unique.

Suppose $p' = p'_1 e'$, where $l(p'_1) = n-1$ and $e' \in E'$. By inductive hypothesis, there is a unique path p_1 in Γ such that $\iota(p_1) = v$ and $\phi(p_1) = p'_1$. Since $\iota(e') = \tau(p'_1) = \tau(\phi(p_1)) = \phi(\tau(p_1))$ and $\phi|_{Star(\tau(p_1))} : Star(\tau(p_1)) \longrightarrow Star(\phi(\tau(p_1)))$ is bijective, there is a unique edge $e \in E$ such that $\iota(e) = \tau(p_1)$ and $\phi(e) = e'$. Let $p = p_1 e$. Then $\iota(p) = v$ and $\phi(p) = \phi(p_1)\phi(e) = p'_1 e' = p'$. If there is another path $\bar{p}$ such that $\iota(\bar{p}) = v$ and $\phi(\bar{p}) = p'$, then we can assume that $\bar{p} = \bar{p}_1 \bar{e}$ such that $\phi(\bar{p}_1) = p'_1$ and $\phi(\bar{e}) = e'$. By the uniqueness of p_1 we have $\bar{p}_1 = p_1$. Because $\phi(\bar{e}) = e'$ and $\iota(\bar{e}) = \tau(\bar{p}_1) = \tau(p_1)$, by the uniqeness of e we have $\bar{e} = e$. So $\bar{p} = p$. □

Let $\mathcal{K} = [\Gamma; R]$, $\mathcal{K}' = [\Gamma'; R']$ be directed 2-complexes. A mapping from $\mathcal{K}$ to $\mathcal{K}'$ is a mapping ϕ of directed graphs from Γ to Γ' such that $(\phi(r_{+1}), \phi(r_{-1})) \in R'$ for each $r \in R$. It is clear that if $p \longleftrightarrow^*_R q$ then $\phi(p) \longleftrightarrow^*_{R'} \phi(q)$. Thus we get an induced homomorphism

$$\phi_* : \pi_1^+(\mathcal{K}, v) \longrightarrow \pi_1^+(\mathcal{K}', \phi(v)),\ \phi_*([p]_R) = [\phi(p)]_{R'}.$$

A mapping ϕ from $\mathcal{K}$ to $\mathcal{K}'$ will be called *locally bijective* if the underlying mapping of directed graphs is locally bijective and if $\phi^{-1}(R') = R$ (that is, if $r' = (r'_{+1}, r'_{-1}) \in R'$ and r_{+1}, r_{-1} are the unique lifts of r'_{+1}, r'_{-1} at some vertex v of $\mathcal{K}$ then $\tau(r_{+1}) = \tau(r_{-1})$ and $(r_{+1}, r_{-1}) \in R$).

Lemma 2.5.3. *Let $\phi : \mathcal{K} \longrightarrow \mathcal{K}'$ be locally bijective, and let p, q be paths in $\mathcal{K}$ with $\iota(p) = \iota(q)$. Then $\phi(p) \longrightarrow^*_{R'} \phi(q)$ if and only if $p \longrightarrow^*_R q$, and $\phi(p) \longleftrightarrow^*_{R'} \phi(q)$ if and only if $p \longleftrightarrow^*_R q$. In particular, for any vertex v of $\mathcal{K}$ the induced homomorphism*

$$\phi_* : \pi_1^+(\mathcal{K}, v) \longrightarrow \pi_1^+(\mathcal{K}', \phi(v))$$

is injective.

Proof. We just prove that $\phi(p) \longrightarrow_{R'} \phi(q)$ if and only if $p \longrightarrow_R q$. It is clear that if $p \longrightarrow_R q$ then $\phi(p) \longrightarrow_{R'} \phi(q)$. Conversely, if $\phi(p) \longrightarrow_{R'} \phi(q)$, then $\phi(p) = p_1' r_{+1}' p_2'$ and $\phi(q) = p_1' r_{-1}' p_2'$ for some $r' \in R'$ and some paths p_1', p_2' in $\mathcal{K}'$. Let p_1 be the lift of p_1' at $\iota(p)$ (note that $\iota(p_1') = \iota(\phi(p)) = \phi(\iota(p))$). Since $\iota(r_{+1}') = \tau(p_1') = \tau(\phi(p_1)) = \phi(\tau(p_1))$, there exists the lift r_{+1} of r_{+1}' at $\tau(p_1)$. Since $\iota(p_2') = \tau(r_{+1}') = \tau(\phi(r_{+1})) = \phi(\tau(r_{+1}))$, there exists the lift p_2 of p_2' at $\tau(r_{+1})$. Thus $p_1 r_{+1} p_2$ is the lift of $p_1' r_{+1}' p_2'$ at $\iota(p)$. By the uniqueness of the lift at a vertex (Lemma 2.5.2), we have $p = p_1 r_{+1} p_2$. Similarly, we have $q = p_1 r_{-1} p_2$ and $(r_{+1}, r_{-1}) \in R$. Thus $p \longrightarrow_R q$. □

Lemma 2.5.4. *Let $\phi : \mathcal{K} \longrightarrow \mathcal{K}'$ be locally bijective. Suppose R' is a complete rewriting system on Γ'. Then R is a complete rewriting system on Γ.*

Proof. There can be no infinite reduction sequence in $\mathcal{K}$, for otherwise, applying ϕ would give an infinite reduction sequence in $\mathcal{K}'$. Thus R is Noetherian.

To show confluence of R, suppose that $p \longrightarrow_R^* q_1$ and $p \longrightarrow_R^* q_2$. Then $\phi(p) \longrightarrow_{R'}^* \phi(q_1)$ and $\phi(p) \longrightarrow_{R'}^* \phi(q_2)$. Since R' is confluent, there exists a path q' in $\mathcal{K}'$ such that $\phi(q_1) \longrightarrow_{R'}^* q'$ and $\phi(q_2) \longrightarrow_{R'}^* q'$. Let q be the lift of q' at $\iota(q_1)$ (note that $\iota(q') = \iota(\phi(q_1)) = \phi(\iota(q_1))$. Then $\phi(q_1) \longrightarrow_{R'}^* \phi(q)$ and $\phi(q_2) \longrightarrow_{R'}^* \phi(q)$. So $q_1 \longrightarrow_R^* q$ and $q_2 \longrightarrow_R^* q$ (by Lemma 2.5.3). □

Lemma 2.5.5. *Let Γ be a directed graph, and let o be a vertex of Γ. Suppose that there is a path in Γ from o to v for every vertex v in Γ. Then there exists a subgraph T of Γ, such that $V(T) = V(\Gamma)$ and there is a unique path in T from o to v for every $v \in V(\Gamma)$.*

We will call T a *maximal tree* of Γ at o.

Proof. Let $\mathcal{Y}$ be the collection of all subgraphs Y of Γ containing o such that there is a unique path in Y from o to v for each vertex v in Y, and partially order $\mathcal{Y}$ by inclusion. By Zorn's lemma, there is a maximal element T of $\mathcal{Y}$. Suppose there is a vertex v in Γ but not in T. Since there is a path p in Γ from $o \in T$ to $v \notin T$, there exists an edge e in p such that $\iota(e) \in T$ but $\tau(e) \notin T$. Then $T \cup \{\tau(e), e\} \in \mathcal{Y}$ contradicts the maximality of T. □

Let $\mathcal{K} = [\Gamma; R]$ be a directed 2-complex. Let T be a maximal tree of Γ at o, and for each vertex v let γ_v be the unique path in T from o to v. Let us say that T is *complemented* in $\mathcal{K}$ if for each vertex v there is a path γ'_v in Γ from v to o such that $\gamma_v\gamma'_v \longleftrightarrow^*_R 1_o$ and $\gamma'_v\gamma_v \longleftrightarrow^*_R 1_v$.

Every path p in Γ can be considered as a word on the edge set E.

Theorem 2.5.6. *If $\mathcal{K}$ has a complemented maximal tree T at o, then*

$$\mathcal{P}_{\mathcal{K},T} = [E; R \cup \{(e, \varnothing) : e \in E \cap T\}]$$

is a monoid presentation for $\pi_1^+(\mathcal{K}, o)$, where $\varnothing$ denotes the empty word on E.

Proof. For any closed path $p = e_1e_2 \cdots e_l$ in Γ at o, we have

$$\begin{aligned} [p]_R &= [1_o e_1 \gamma'_{\tau(e_1)}\gamma_{\tau(e_1)} e_2 \gamma'_{\tau(e_2)}\gamma_{\tau(e_2)} \cdots \gamma'_{\tau(e_{l-1})}\gamma_{\tau(e_{l-1})} e_l 1_o]_R \\ &= [\gamma_{\iota(e_1)} e_1 \gamma'_{\tau(e_1)}]_R [\gamma_{\iota(e_2)} e_2 \gamma'_{\tau(e_2)}]_R \cdots [\gamma_{\iota(e_l)} e_l \gamma'_{\tau(e_l)}]_R. \end{aligned}$$

Thus $\pi_1^+(\mathcal{K}, o)$ can be generated by $[\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R$ $(e \in E)$. If $e \in T$, then $\gamma_{\iota(e)} e = \gamma_{\tau(e)}$ (by uniqueness), so $[\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R = [\gamma_{\tau(e)}\gamma'_{\tau(e)}]_R = 1$.

Let M be the monoid defined by $\mathcal{P}_{\mathcal{K},T}$. Let ϕ_1 be the homomorphism from the free monoid on E to $\pi_1^+(\mathcal{K}, o)$ defined by

$$e \longmapsto [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R \quad (e \in E).$$

If $e \in T$, then $\phi_1(e) = 1 = \phi_1(\varnothing)$. Also, if $r \in R$ with $r_{+1} = e_1e_2 \cdots e_m$, $r_{-1} = e'_1e'_2 \cdots e'_l$ $(e_i, e'_j \in E)$ say, then

$$\phi_1(r_{+1}) = [\gamma_{\iota(e_1)} e_1 \gamma'_{\tau(e_1)} \gamma_{\iota(e_2)} e_2 \gamma'_{\tau(e_2)} \cdots \gamma_{\iota(e_m)} e_m \gamma'_{\tau(e_m)}]_R = [\gamma_{\iota(r_{+1})} r_{+1} \gamma'_{\tau(r_{+1})}]_R.$$

Similarly, $\phi_1(r_{-1}) = [\gamma_{\iota(r_{-1})} r_{-1} \gamma'_{\tau(r_{-1})}]_R$. So $\phi_1(r_{+1}) = \phi_1(r_{-1})$. Thus we have an induced homomorphism

$$\phi_{1*} : M \longrightarrow \pi_1^+(\mathcal{K}, o),\ [e]_{R'} \longmapsto [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R,$$

where $R' = R \cup \{(e, \varnothing) : e \in E \cap T\}$.

Now regard $\mathcal{P}_{\mathcal{K},T}$ as a directed 2-complex with one vertex c, and consider the following mapping of directed 2-complexes

$$\phi_2 : \mathcal{K} \longrightarrow \mathcal{P}_{\mathcal{K},T}, \ v \longmapsto c, e \longmapsto e \quad (v \in V, e \in E).$$

We have an induced homomorphism

$$\phi_{2*} : \pi_1^+(\mathcal{K}, o) \longrightarrow M, \ \phi_{2*}([p]_R) = [\phi_2(p)]_{R'} = [p]_{R'}.$$

Since $[\gamma_v]_{R'} = 1$ and $[\gamma_v \gamma'_v]_R = 1$, we have $[\gamma'_v]_{R'} = [\gamma_v]_{R'}[\gamma'_v]_{R'} = [\gamma_v\gamma'_v]_{R'} = 1$ for any $v \in V$. Thus

$$\phi_{2*}\phi_{1*}([e]_{R'}) = \phi_{2*}([\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R) = [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_{R'} = [e]_{R'},$$

$$\phi_{1*}\phi_{2*}([\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R = \phi_{1*}([e]_{R'}) = [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R.$$

So $M \cong \pi_1^+(\mathcal{K}, o)$. □

Example 2.5.7. Let Γ be the directed graph in Figure 2.3. Let

$$\mathcal{K} = [\Gamma; \ e_1e_2e_3 = 1_{v_1}, \ e_3e_1e_2 = 1_{v_2}]$$

be a directed 2-complex. Then $\mathcal{K}$ has a complemented maximal tree $T = \{v_1, v_2, e_3\}$ at v_2. So

$$\mathcal{P} = [e_1, e_2, e_3; \ e_1e_2e_3 = \varnothing, \ e_3e_1e_2 = \varnothing, \ e_3 = \varnothing]$$

is a monoid presentation of $\pi_1^+(\mathcal{K}, v_2)$. Note that $\mathcal{P}$ is Tietze equivalent to

$$\mathcal{P}' = [e_1, e_2; \ e_1e_2 = \varnothing].$$

Thus $\mathcal{P}'$ is a monoid presentation of $\pi_1^+(\mathcal{K}, v_2)$.

Let $\mathcal{K}^{cone}$ be the directed 2-complex obtained from $\mathcal{K}$ by adjoining a new vertex a, adjoining edges y_v, y_v^{-1} $(v \in V)$ with $\iota(y_v) = \tau(y_v^{-1}) = a, \tau(y_v) = \iota(y_v^{-1}) = v$, and adjoining additional rewriting rules $(y_v y_v^{-1}, 1_a), (y_v^{-1} y_v, 1_v)$.

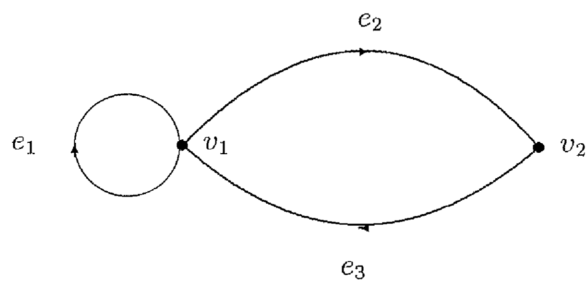


Figure 2.3

Theorem 2.5.8. *If $\mathcal{K}$ has a complemented maximal tree T at o, then*

$$\pi_1^+(\mathcal{K}^{cone}, o) \cong \pi_1^+(\mathcal{K}, o) * F_{n-1},$$

where $n = |V|$, F_{n-1} is the free group of rank $n-1$. Also, $\mathcal{P} = [E; R]$ is a monoid presentation of $\pi_1^+(\mathcal{K}^{cone}, o)$.

Proof. Let $\mathcal{K}^{cone} = [\Gamma'; R']$, where $\Gamma' = (V', E', \iota, \tau)$. Then $V' = V \cup \{a\}$, $E' = E \cup \{y_v, y_v^{-1} : v \in V\}$, and $R' = R \cup \{(y_v y_v^{-1}, 1_a), (y_v^{-1} y_v, 1_v) : v \in V\}$. It is clear that $\mathcal{K}^{cone}$ has a complemented maximal tree $T_1 = T \cup \{y_o^{-1}\} \cup \{a\}$ at o. By Theorem 2.5.6, $\pi_1^+(\mathcal{K}, o)$ has a monoid presentation $\mathcal{P}_{\mathcal{K},T}$, and $\pi_1^+(\mathcal{K}^{cone}, o)$ has a monoid presentation

$$[E \cup \{y_v, y_v^{-1} : v \in V\} \,;\, R \cup \{(y_v y_v^{-1}, \varnothing), (y_v^{-1} y_v, \varnothing) : v \in V\} \cup \{(e, \varnothing), (y_o^{-1}, \varnothing) : e \in E \cap T\}].$$

It is clear that the monoid defined by

$$[y_v, y_v^{-1}\ (v \in V) \,;\, (y_v y_v^{-1}, \varnothing), (y_v^{-1} y_v, \varnothing), (y_o^{-1}, \varnothing)\ (v \in V)]$$

is the free group F_{n-1} of rank $n-1$. Thus $\pi_1^+(\mathcal{K}^{cone}, o) \cong \pi_1^+(\mathcal{K}, o) * F_{n-1}$.

It is clear that $\mathcal{K}^{cone}$ also has a complemented maximal tree $T_2 = \{y_v : v \in V\} \cup V \cup \{a\}$ at a. So by Theorem 2.5.6, $\pi_1^+(\mathcal{K}^{cone}, a)$ has a monoid presentation

$$[E \cup \{y_v, y_v^{-1} : v \in V\} \,;\, R \cup \{(y_v y_v^{-1}, \varnothing), (y_v^{-1} y_v, \varnothing) : v \in V\} \cup \{(y_v, \varnothing) : v \in V\}].$$

The above presentation is Tietze equivalent to the presentation $[E; R]$. It is easy to check that the mapping

$$\pi_1^+(\mathcal{K}^{cone}, a) \longrightarrow \pi_1^+(\mathcal{K}^{cone}, o),\ [p]_{R'} \longmapsto [y_o^{-1} p y_o]_{R'}$$

is an isomorphism. Thus $\mathcal{P} = [E; R]$ is a monoid presentation of $\pi_1^+(\mathcal{K}^{cone}, o)$. □

Theorem 2.5.9. *Let $\mathcal{K} = [\Gamma; R]$ be a finite directed 2-complex and let $n = |V|$. If R is complete on Γ and $\mathcal{K}$ has a complemented maximal tree T at o, then $\pi_1^+(\mathcal{K}, o) * F_{n-1}$ can be presented by a finite complete rewriting system.*

Proof. By Theorem 2.5.8, $\pi_1^+(\mathcal{K}, o) * F_{n-1}$ has a monoid presentation $\mathcal{P} = [E; R]$. We just need to show that R is a complete rewriting system on E.

Any word w on E can be written as $w = p_1 p_2 \cdots p_k$, where $p_i \in P(\Gamma)$ and $\tau(p_i) \neq \iota(p_{i+1})$. It is clear that $w \longrightarrow_R w'$ in $\mathcal{P}$ for some word w' on E if and only if $w' = p_1 \cdots p_{i-1} q_i p_{i+1} \cdots p_k$ and $p_i \longrightarrow_R q_i$ in $\mathcal{K}$. Since R is complete on Γ, it is easy to prove that R is Noetherian and locally confluent on E. So R is a finite complete rewriting system on E. □

2.6 Subgroups of finite index in groups with finite complete rewriting systems

Groves and Smith ([40], [41]) have proved that if a group G has a subgroup of finite index with a finite complete rewriting system, then G also has such a system. However, the converse of this result (raised in [41] and [23]) still remains open. We will prove the following result.

Theorem 2.6.1. *Let H be a subgroup of finite index n in a group G. If G has a finite complete rewriting system, then so does the free product $H * F_{n-1}$, where F_{n-1} is the free group of rank $n-1$.*

Remark. This theorem provides a link with another open question ([65]): Let A and B be finitely presented groups (or monoids). If the free product $A * B$ has a finite complete rewriting system, do A and B also have finite complete rewriting systems?

Proof. Let G have a finite complete monoid presentation, which we can regard as a directed 2-complex $\mathcal{K}_1 = [\Gamma_1; R_1]$, where Γ_1 has a single vertex c and a finite edge set X, and where R_1 is a finite complete rewriting system on Γ_1. We will construct a finite directed 2-complex $\mathcal{K} = [\Gamma; R]$ satisfying the conditions of Theorem 2.5.9, with $\pi_1^+(\mathcal{K}, o) \cong H$.

The vertex set V of Γ will be the set of right cosets Hg ($g \in G$) of H in G. The edge set E of Γ will consist of all ordered pairs (Hg, x) ($g \in G, x \in X$) with $\iota((Hg, x)) = Hg$, $\tau((Hg, x)) = Hg[x]_{R_1}$. For any $v = Hg \in V$ and $r \in R_1$, with $r_{+1} = x_1 x_2 \cdots x_k$, $r_{-1} = x'_1 x'_2 \cdots x'_l$ say, let

$$r_{+1}^{(v)} = (Hg, x_1)(Hg[x_1]_{R_1}, x_2) \cdots (Hg[x_1 x_2 \cdots x_{k-1}]_{R_1}, x_k),$$

$$r_{-1}^{(v)} = (Hg, x'_1)(Hg[x'_1]_{R_1}, x'_2) \cdots (Hg[x'_1 x'_2 \cdots x'_{l-1}]_{R_1}, x'_l).$$

Then $\iota(r_{+1}^{(v)}) = Hg = \iota(r_{-1}^{(v)})$ and $\tau(r_{+1}^{(v)}) = Hg[r_{+1}]_{R_1} = Hg[r_{-1}]_{R_1} = \tau(r_{-1}^{(v)})$. So we have $r^{(v)} := (r_{+1}^{(v)}, r_{-1}^{(v)}) \in P^{(2)}(\Gamma)$. We let

$$R = \{r^{(v)} : v \in V, r \in R_1\} \subset P^{(2)}(\Gamma).$$

We define a mapping

$$\phi : \mathcal{K} \longrightarrow \mathcal{K}_1,$$

by

$$Hg \longmapsto c, \ (Hg, x) \longmapsto x \ (g \in G, x \in X).$$

Clearly ϕ is locally bijective. Hence by Lemma 2.5.4, R is complete on Γ.

Let $v = Hg \in V$, with $g = [x_1 x_2 \cdots x_m]_{R_1}$ ($x_i \in X$). Then

$$(H1, x_1)(H[x_1]_{R_1}, x_2) \cdots (H[x_1 x_2 \cdots x_{m-1}]_{R_1}, x_m)$$

is a path in Γ from $o = H1$ to v. Thus by Lemma 2.5.5, there exists a maximal tree T of Γ at o. Let γ_v denote the unique path in T from o to v. Let $\phi(\gamma_v) = w$ and $g_1 = [w]_{R_1}$. Then $Hg_1 = H[\phi(\gamma_v)]_{R_1} = \tau(\gamma_v) = Hg$. Let $g_1^{-1} = [w']_{R_1}$, and let $w' = x'_1 x'_2 \cdots x'_k$ ($x'_i \in X$). Since $[w]_{R_1}[w']_{R_1} = g_1 g_1^{-1} = 1$, there is a path $\gamma'_v = (Hg_1, x'_1)(Hg_1[x'_1]_{R_1}, x'_2) \cdots (Hg_1[x'_1 x'_2 \cdots x'_{k-1}]_{R_1}, x'_k)$ in Γ from v to o. Because

$$\phi(\gamma_v \gamma'_v) = \phi(\gamma_v)\phi(\gamma'_v) = ww' \longleftrightarrow^*_{R_1} \varnothing = \phi(1_o),$$

by Lemma 2.5.3 we have $\gamma_v \gamma'_v \longleftrightarrow^*_R 1_o$. Similarly, we have $\gamma'_v \gamma_v \longleftrightarrow^*_R 1_v$. Thus T is complemented.

Let

$$\phi_* : \pi_1^+(\mathcal{K}, o) \longrightarrow \pi_1^+(\mathcal{K}_1, c) = G$$

be the homomorphism induced by ϕ. By Lemma 2.5.3, ϕ_* is injective. Let $[p]_R \in \pi_1^+(\mathcal{K}, o)$ (p a closed path in Γ at o). Then $H1 = \tau(p) = H[\phi(p)]_{R_1}$, so we have $\phi_*([p]_R) = [\phi(p)]_{R_1} \in H$. Conversely, if $[p']_{R_1} \in H$, and if p is the lift of p' at o, then $\tau(p) = H[p']_{R_1} = H1 = o$, so $\phi_*([p]_R) = [\phi(p)]_{R_1} = [p']_{R_1}$ with $[p]_R \in \pi_1^+(\mathcal{K}, o)$. Thus $\phi_* : \pi_1^+(\mathcal{K}, o) \longrightarrow H$ is an isomorphism. □

Chapter 3

FDT, FHT and second order Dehn functions of monoids I: Semi-direct products of monoids

Abstract

In this chapter, we first give some basic definitions and results about FDT, FHT and second order Dehn functions of monoids. Then we consider these properties for semi-direct products of monoids. We get that the class of FDT monoids and the class of FHT monoids are closed under semi-direct products. We also get some general bounds for second order Dehn functions of direct products of monoids.

3.1 Preliminaries

3.1.1 FDT and the associated second order (homotopical) Dehn functions

Let $\mathcal{P} = [X; R]$ be a finite monoid presentation. Let $\mathcal{D} = \mathcal{D}(\mathcal{P})$ be the associated Squier complex, with underlying graph Γ.

Definition 3.1.1. The presentation $\mathcal{P}$ is said to have *finite derivation type* (FDT) if there is a finite set $\boldsymbol{X}$ of spherical monoid pictures over $\mathcal{P}$ such that $\overline{\boldsymbol{X}} = X^* \cdot \boldsymbol{X} \cdot X^*$ trivializes $\boldsymbol{\mathcal{D}}$ (see Section 1.5.3). A monoid is FDT if it has a finite presentation which is FDT.

We will often abuse notation and write $\boldsymbol{\mathcal{D}}^{\boldsymbol{X}}$ rather than $\boldsymbol{\mathcal{D}}^{\overline{\boldsymbol{X}}}$, and we will refer to $\boldsymbol{X}$ (rather than $\overline{\boldsymbol{X}}$) as a trivializer. If p is a closed path in $\boldsymbol{\mathcal{D}}$ then we will write $\tilde{V}_{\mathcal{P},\boldsymbol{X}}(p)$ for $V_{\boldsymbol{\mathcal{D}},\overline{\boldsymbol{X}}}(p)$ (see Section 1.5.2). Thus (Lemma 1.5.2) $\tilde{V}_{\mathcal{P},\boldsymbol{X}}(p)$ is the smallest number m such that p is equivalent to a path of the form

$$(*) \qquad p_1(U_1 \cdot q_1^{\varepsilon_1} \cdot V_1)p_1^{-1}p_2(U_2 \cdot q_2^{\varepsilon_2} \cdot V_2)p_2^{-1} \cdots p_m(U_m \cdot q_m^{\varepsilon_m} \cdot V_m)p_m^{-1},$$

where p_i is a path in $\boldsymbol{\mathcal{D}}$, $U_i, V_i \in X^*$, $q_i \in \boldsymbol{X}$, $\varepsilon_i = \pm 1$ $(1 \leq i \leq m)$.

Note (by Lemma 1.5.3) that if q is any path with $\tau(q) = \iota(p)$, then

$$\tilde{V}_{\mathcal{P},\boldsymbol{X}}(qpq^{-1}) = \tilde{V}_{\mathcal{P},\boldsymbol{X}}(p).$$

We define the *second order (homotopical) Dehn function* $\tilde{\gamma}^{(2)}_{\mathcal{P},\boldsymbol{X}}$ of $\mathcal{P}$ with respect to $\boldsymbol{X}$ as follows. For $n \in \mathbb{N}$ let

$$\Upsilon_n = \{p : \ p \text{ a closed path in } \boldsymbol{\mathcal{D}}, \ A(p) \leq n\},$$

$$\tilde{\lambda}_n = \{\tilde{V}_{\mathcal{P},\boldsymbol{X}}(p) : \ p \in \Upsilon_n\}.$$

In general, Υ_n is not finite. However, $\tilde{\lambda}_n$ is finite (see [74]). Then we define

$$\tilde{\gamma}^{(2)}_{\mathcal{P},\boldsymbol{X}} : \ \mathbb{N} \longrightarrow \mathbb{Z}^+, \ n \longmapsto \max \tilde{\lambda}_n.$$

The concept of finite derivation type was introduced by Squier in [78]. The above is not his definition (it is due to Pride [68]), but is equivalent to it. Squier's original definition is as follows.

An equivalence relation $\simeq$ on $P(\Gamma)$ is called a *homotopy relation* if $\simeq \subseteq P^{(2)}(\Gamma)$ and it satisfies the following conditions:

(a) If e_1, e_2 are edges of Γ, then

$$(e_1 \cdot \iota(e_2)(\tau(e_1) \cdot e_2) \simeq (\iota(e_1) \cdot e_2)(e_1 \cdot \tau(e_2)).$$

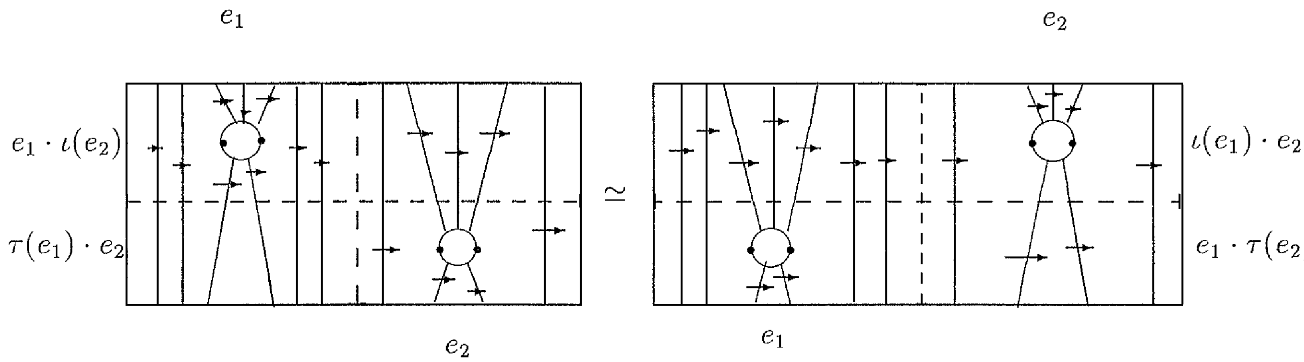


Figure 3.1

(See Figure 3.1.)

(b) If $p \simeq q$, then $W \cdot p \cdot W' \simeq W \cdot q \cdot W'$ for all $W, W' \in X^*$.

(c) If $p, q_1, q_2, r \in P(\Gamma)$ satisfy $\tau(p) = \iota(q_1) = \iota(q_2)$, $\tau(q_1) = \tau(q_2) = \iota(r)$, and $q_1 \simeq q_2$, then $pq_1r \simeq pq_2r$.

(d) If $p \in P(\Gamma)$ with $\iota(p) = W$, then $pp^{-1} \simeq 1_W$.

It is easily seen that the collection of all homotopy relations on $P(\Gamma)$ is closed under arbitrary intersection, and that $P^{(2)}(\Gamma)$ itself is a homotopy relation. Thus, if $B \subseteq P^{(2)}(\Gamma)$, then there is a unique smallest homotopy relation $\simeq_B$ on $P(\Gamma)$ that contains B. This homotopy relation will be called the *homotopy relation generated by B*.

Definition 3.1.2. The presentation $\mathcal{P}$ is FDT if there is a finite subset $B \subseteq P^{(2)}(\Gamma)$ which generates $P^{(2)}(\Gamma)$ as a homotopy relation, that is, $\simeq_B \; = P^{(2)}(\Gamma)$. A finitely presented monoid S is FDT if some finite presentation of S has FDT.

Proposition 3.1.3. *Definitions 3.1.1 and 3.1.2 are equivalent.*

Proof. First we prove that if $\mathcal{D}$ has a finite trivializer $\boldsymbol{X}$ then $\simeq_B \; = P^{(2)}(\Gamma)$, where B is the finite set $\{(p, 1_{\iota(p)}) : p \in \boldsymbol{X}\}$.

We need to show that the equivalence relation $\sim_{\overline{\boldsymbol{X}}}$ on $P(\Gamma)$ is the same as the homotopy relation $\simeq_B$ on $P(\Gamma)$, where $\overline{\boldsymbol{X}} = X^* \cdot \boldsymbol{X} \cdot X^*$. Suppose $p_1 \sim_{\overline{\boldsymbol{X}}} p_2$. To prove $p_1 \simeq_B p_2$, we just consider the case that p_1 is obtained fom p_2 by one operation of type (I) or (II) or (III) (see Section 1.5.2) (the more general case can be proved by induction).

For the operation of type (I), by the conditions (c) and (d) in the definition of homotopy relation we have $p_1 \simeq_B p_2$. For the operation of type (II), by the conditions (a), (c) and (d) in the definition of homotopy relation we have $p_1 \simeq_B p_2$. For the operation of type (III), by the conditions (b), (c), (d) in the definition of homotopy relation and note that $p \simeq_B 1$ for every $p \in \boldsymbol{X}$, we have $p_1 \simeq_B p_2$. Thus $\sim_{\overline{\boldsymbol{X}}} \subseteq \simeq_B$. Conversely, to prove that $\sim_{\overline{\boldsymbol{X}}} \supseteq \simeq_B$, we just need to prove that $\sim_{\overline{\boldsymbol{X}}}$ is a homotopy relation on $P(\Gamma)$ which contains B. It is clear that $\sim_{\overline{\boldsymbol{X}}}$ is an equivalence relation on $P(\Gamma)$ and $B \subseteq \sim_{\overline{\boldsymbol{X}}} \subseteq P^{(2)}(\Gamma)$. Since $\mathcal{D} = \mathcal{D}(\mathcal{P})$, it is clear that $\sim_{\overline{\boldsymbol{X}}}$ satisfies all the conditions (a), (b), (c) and (d) in the definition of homotopy relation. So we have proved that $\sim_{\overline{\boldsymbol{X}}} = \simeq_B$.

For every $(p, p') \in P^{(2)}(\Gamma)$, we have a closed path $p'p^{-1}$ in $\mathcal{D}$. Since $\boldsymbol{X}$ trivializes $\mathcal{D}$, we have $p'p^{-1} \sim_{\overline{\boldsymbol{X}}} 1_{\iota(p)}$. Thus $p'p^{-1} \simeq_B 1_{\iota(p)}$, and hence $p \simeq_B p'$. So $\simeq_B = P^{(2)}(\Gamma)$.

Next we prove that if there is a finite subset B of $P^{(2)}(\Gamma)$ such that $\simeq_B = P^{(2)}(\Gamma)$ then $\boldsymbol{X} = \{p_1 p_2^{-1} : (p_1, p_2) \in B\}$ is a finite trivializer of $\mathcal{D}$.

By a similar way as above, we can prove that $\sim_{\overline{\boldsymbol{X}}} = \simeq_B$. For every closed path p in $\mathcal{D}$, we have $p \simeq_B 1_{\iota(p)}$. So $p \sim_{\overline{\boldsymbol{X}}} 1_{\iota(p)}$. Thus $\boldsymbol{X}$ is a finite trivializer of $\mathcal{D}$. $\square$

Theorem 3.1.4. *Let S be a monoid, and let $\mathcal{P}_1, \mathcal{P}_2$ be any two finite monoid presentations for S.*

(1) ([78]) *if $\mathcal{P}_1$ is FDT then so is $\mathcal{P}_2$;*

(2) ([74]) *if $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are finite trivializers of $\mathcal{D}(\mathcal{P}_1)$ and $\mathcal{D}(\mathcal{P}_2)$ respectively, then*

$$\tilde{\gamma}^{(2)}_{\mathcal{P}_1, \boldsymbol{X}_1} \sim \tilde{\gamma}^{(2)}_{\mathcal{P}_2, \boldsymbol{X}_2}.$$

The first part of the theorem shows that having FDT is an invariant property of finitely presented monoids. The second part shows that, up to equivalence we can talk about the *second order (homotopical) Dehn function* $\tilde{\gamma}^{(2)}_S$ of an FDT monoid S.

3.1.2 FHT and the associated second order (homological) Dehn functions

Let $\mathcal{P} = [X; R]$ be a monoid presentation and let $S = S(\mathcal{P})$ be the monoid defined by $\mathcal{P}$. Let $\mathcal{D} = \mathcal{D}(\mathcal{P})$ be the associated Squier complex. Then we have the chain complex

$$0 \longrightarrow C_2(\mathcal{D}) \xrightarrow{\partial_2} C_1(\mathcal{D}) \xrightarrow{\partial_1} C_0(\mathcal{D}) \longrightarrow 0$$

of $\mathcal{D}$. Here $C_0(\mathcal{D})$ is the free abelian group with basis X^*, $C_1(\mathcal{D})$ is the free abelian group with basis the set of positive edges in $\mathcal{D}$, and $C_2(\mathcal{D})$ is the free abelian group on the set of defining paths $[e_1, e_2]$. Notice that these chain groups are $(\mathbb{Z}X^*, \mathbb{Z}X^*)$-bimodules via the two-sided action of X^* on the bases of the chain groups. The boundary maps are given by:

$$\partial_2[e_1^{\varepsilon_1}, e_2^{\varepsilon_2}] = \varepsilon_1\varepsilon_2 e_1 \cdot (\iota(e_2) - \tau(e_2)) - \varepsilon_1\varepsilon_2(\iota(e_1) - \tau(e_1)) \cdot e_2,$$

$$\partial_1(e) = \iota(e) - \tau(e),$$

where e_1, e_2 and e are positive edges in $\mathcal{D}$, $\varepsilon_1, \varepsilon_2 = \pm 1$.

For each path p in $\mathcal{D}$, say $p = e_1^{\varepsilon_1} \cdots e_k^{\varepsilon_k}$ (e_i a positive edge, $\varepsilon_i = \pm 1, i = 1, \cdots, k$), we let

$$z_p = \sum_{i=1}^{k} \varepsilon_i e_i \ \in C_1(\mathcal{D}).$$

It was shown in [68] that the first homology group $Ker\partial_1 / Im\partial_2$ is a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule under the action induced from the action of $\mathbb{Z}X^*$ (that is, $[W] \cdot (z_p + Im\partial_2) \cdot [W'] = z_{W \cdot p \cdot W'} + Im\partial_2$ for any $W, W' \in X^*$ and any closed path p in $\mathcal{D}$). This action is well-defined. As in [74], we will denote this bimodule by $\pi_2^{(b)}(\mathcal{P})$. For any closed path p in $\mathcal{D}$, we will write ξ_p for the element $z_p + Im\partial_2$ of $\pi_2^{(b)}(\mathcal{P})$. Note that if $\boldsymbol{X}$ trivialises $\mathcal{D}(\mathcal{P})$ then the elements ξ_p ($p \in \boldsymbol{X}$) generate the bimodule $\pi_2^{(b)}(\mathcal{P})$ (see [68]).

Let P be the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule with basis $\{t_r : r \in R\}$ in one-to-one correspondence with R. Then there is a $(\mathbb{Z}S, \mathbb{Z}S)$-homomorphism

$$\rho : \pi_2^{(b)}(\mathcal{P}) \longrightarrow P$$

defined as follows. Let

$$\alpha = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_n e_n$$

be an element of $Ker\partial_1$, where $e_i = (U_i, r_i, +1, V_i)$ say, $\varepsilon_i = \pm 1$, $(i = 1, 2, \cdots, n)$. Then

$$\rho(\alpha + Im\partial_2) = \sum_{i=1}^{n} \varepsilon_i [U_i] t_{r_i} [V_i].$$

The following lemma follows from results in [42]. See also [74].

Lemma 3.1.5. *The above homomorphism ρ is injective.*

Following Pride and X. Wang [74] we will say that $\mathcal{P}$ is of *finite homological type* (FHT for short) if $\mathcal{P}$ is finite and $\pi_2^{(b)}(\mathcal{P})$ is finitely generated as a bimodule. Note that $\mathcal{P}$ is FHT if and only if there is a finite set $\boldsymbol{X}$ of spherical monoid pictures such that the elements ξ_p $(p \in \boldsymbol{X})$ generate $\pi_2^{(b)}(\mathcal{P})$. In particular, if $\mathcal{P}$ is FDT then $\mathcal{P}$ is FHT. The converse is an open question, that is, if $\mathcal{P}$ is FHT we do not know whether it is FDT.

Let $\boldsymbol{c}$ be a generating set for $\pi_2^{(b)}(\mathcal{P})$. Then for any closed path p in $\mathcal{D}$, ξ_p can be expressed as a sum

$$\varepsilon_1 s_1 \xi_1 s_1' + \varepsilon_2 s_2 \xi_2 s_2' + \cdots + \varepsilon_m s_m \xi_m s_m'$$

where $\varepsilon_i = \pm 1$, $s_i, s_i' \in S$, $\xi_i \in \boldsymbol{c}$, $i = 1, 2, \cdots, m$. We let $V_{\mathcal{P},c}(p)$, the *homological volume* of p with respect to $\boldsymbol{c}$, be the minimum value of m over all expressions of the above form which are equal to ξ_p. Note that if q is any path with $\tau(q) = \iota(p)$ then $\xi_{qpq^{-1}} = \xi_p$, so

$$V_{\mathcal{P},c}(qpq^{-1}) = V_{\mathcal{P},c}(p).$$

Now suppose $\mathcal{P} = [X; R]$ is finite, and let $\boldsymbol{c}$ be a generating set for $\pi_2^{(b)}(\mathcal{P})$. For $n \in \mathbb{N}$ let

$$\Upsilon_n = \{p : \ p \text{ a closed path in } \mathcal{D}(\mathcal{P}), \ A(p) \leq n\},$$

$$\lambda_n = \{V_{\mathcal{P},c}(p) : \ p \in \Upsilon_n\}.$$

In general, Υ_n is not finite. However, λ_n is finite (see [74]). The *second order (homological) Dehn function* $\gamma^{(2)}_{\mathcal{P},c}$ of $\mathcal{P}$ with respect to $\boldsymbol{c}$ is defined by

$$\gamma^{(2)}_{\mathcal{P},c} : \ \mathbb{N} \longrightarrow \mathbb{Z}^+, \ n \longmapsto \max \lambda_n.$$

Theorem 3.1.6. ([74]) *Let S be a monoid, and let $\mathcal{P}_1, \mathcal{P}_2$ be any two finite monoid presentations for S.*

(1) if $\mathcal{P}_1$ is FHT then so is $\mathcal{P}_2$;

(2) if $\mathbf{c}_1$ and $\mathbf{c}_2$ are finite generating sets of $\pi_2^{(b)}(\mathcal{P}_1)$ and $\pi_2^{(b)}(\mathcal{P}_2)$ respectively, then

$$\gamma^{(2)}_{\mathcal{P}_{1},\mathbf{c}_1} \sim \gamma^{(2)}_{\mathcal{P}_{2},\mathbf{c}_2}.$$

Up to equivalence, this theorem allows us to talk about the *second order (homological) Dehn function* $\gamma_S^{(2)}$ of an FHT monoid S.

It is easy to see that for an FDT monoid S we have

$$\tilde{\gamma}_S^{(2)} \succeq \gamma_S^{(2)}.$$

It is an open question whether they are equivalent.

3.2 Some known results

Theorem 3.2.1. ([78]) *If a monoid M can be presented by a finite complete rewriting system, then M is FDT.*

If a monoid M has a (maybe infinite) complete presentation $\mathcal{P} = [X; R]$, then we can get a trivializer $\boldsymbol{X}$ of the associated Squier complex $\mathcal{D}(\mathcal{P})$ by the method given by Squier as follows.

Let Γ be the graph associated to $\mathcal{P} = [X; R]$. Let $P_+(\Gamma)$ denote the set of all those paths in Γ that only contain positive edges. It is clear that every path $p = e_1 e_2 \cdots e_n \in P_+(\Gamma)$ corresponds to a reduction sequence

$$\iota(p) \longrightarrow_R W_1 \longrightarrow_R W_2 \longrightarrow_R \cdots \longrightarrow_R W_{n-1} \longrightarrow_R \tau(p)$$

with $W_i = \tau(e_i)$, and vice versa.

An ordered pair (e_1, e_2) of positive edges in Γ is called a *critical pair of edges* if $\iota(e_1) = \iota(e_2)$ and $(\tau(e_1), \tau(e_2))$ is a critical pair of R. An ordered pair (p_1, p_2) of paths $p_1, p_2 \in P_+(\Gamma)$ is called a *resolution* of a critical pair of edges (e_1, e_2) if $\iota(p_1) = \tau(e_1), \iota(p_2) = \tau(e_2)$ and $\tau(p_1) = \tau(p_2)$.

Since R is a complete rewriting system on X, every critical pair of R is resolved. If (e_1, e_2) is a critical pair of edges, then the critical pair $(\tau(e_1), \tau(e_2))$ of R is resolved. So there is a word $W \in X^*$ such that $\tau(e_1) \longrightarrow_R^* W$ and $\tau(e_2) \longrightarrow_R^* W$. Let p_1 and p_2 be the paths in $P_+(\Gamma)$ that correspond to the reduction sequences $\tau(e_1) \longrightarrow_R^* W$ and $\tau(e_2) \longrightarrow_R^* W$ respectively. Then (p_1, p_2) is a resolution of the critical pair of edges (e_1, e_2). Thus, a resolution exists for each critical pair of edges.

Let B be the following subset of $P^{(2)}(\Gamma)$:

$$B := \{(e_1p_1, e_2p_2) : (e_1, e_2) \text{ is a critical pair of edges}\},$$

where (p_1, p_2) is a fixed resolution of (e_1, e_2). Then we have the following theorem.

Theorem 3.2.2. ([78]) *Let $\mathcal{P} = [X; R]$ be a complete monoid presentation, let Γ be the associated graph, and let B be defined as above. Then $\simeq_B = P^{(2)}(\Gamma)$.*

It is clear that B is a finite set if R is finite. Let

$$\boldsymbol{X} := \{e_1p_1p_2^{-1}e_2^{-1} : (e_1p_1, e_2p_2) \in B\}.$$

Then $\boldsymbol{X}$ is a trivializer of the associated Squier complex $\mathcal{D}(\mathcal{P})$ (see the proof of Proposition 3.1.3). If R is finite, then $\boldsymbol{X}$ is a finite trivializer of $\mathcal{D}(\mathcal{P})$.

Example 3.2.3. Let $\mathcal{P} = [X; R]$, where $X = \{a, b, c\}$ and

$$R = \{ab = c, ca = bc, bcb = cc, ccb = acc\}.$$

It is easy to see that R is length-plus-lexicographic. So R is Noetherian. It is easy to check that (ccb, acc), (bcb, cc), $(cccb, bccc)$ and $(acccb, cccc)$ are all the critical pairs of R. Since all critical pairs of R are resolved, R is locally confluent. Thus R is a finite complete rewriting system on X. Let

$$e_1 = (\varnothing, ab = c, +1, cb),\ e_1' = (a, bcb = cc, +1, \varnothing),$$

$$e_2 = (\varnothing, ca = bc, +1, b),\ e_2' = (c, ab = c, +1, \varnothing),$$

$$e_3 = (\varnothing, bcb = cc, +1, cb),\ e_3' = (bc, bcb = cc, +1, \varnothing),$$

$$e_4 = (\varnothing, ccb = acc, +1, cb),\ \ e'_4 = (cc, bcb = cc, +1, \varnothing).$$

Then $(e_1, e'_1), (e_2, e'_2), (e_3, e'_3)$ and (e_4, e'_4) are all critical pairs of edges. Let

$$p_1 = (\varnothing, ccb = acc, +1, \varnothing),\ \ p_2 = (\varnothing, bcb = cc, +1, \varnothing),$$

$$p_3 = (c, ccb = acc, +1, \varnothing)(\varnothing, ca = bc, +1, cc),$$

$$p_4 = (ac, ccb = acc, +1, \varnothing)(a, ca = bc, +1, cc)(\varnothing, ab = c, +1, ccc),$$

and let $p'_i = 1$ (empty path), $i = 1, 2, 3, 4$. Then (p_i, p'_i) is a resolution of (e_i, e'_i), $i = 1, 2, 3, 4$. Thus

$$\boldsymbol{X} = \{e_i p_i {p'}_i^{-1} {e'}_i^{-1} :\ i = 1, 2, 3, 4\}$$

is a finite trivializer of the associated 2-complex $\mathcal{D}(\mathcal{P})$. The paths in $\boldsymbol{X}$ are as in Figures 3.2, 3.3, 3.4.

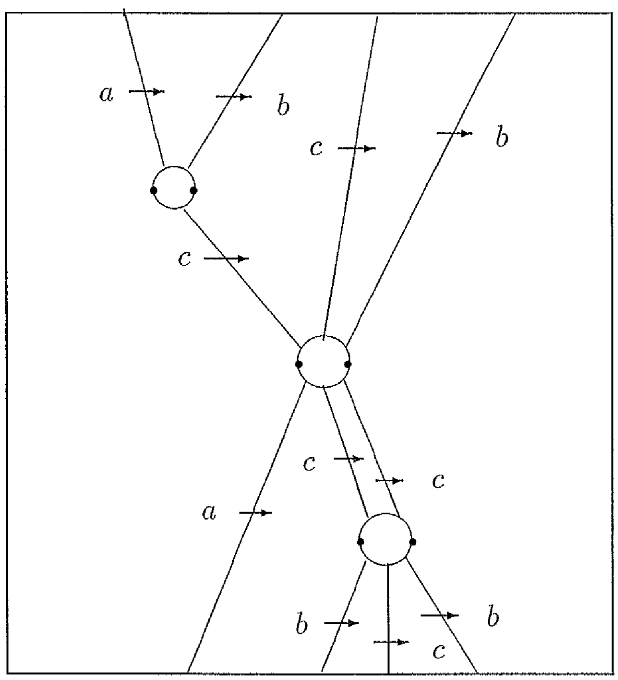


Figure 3.2

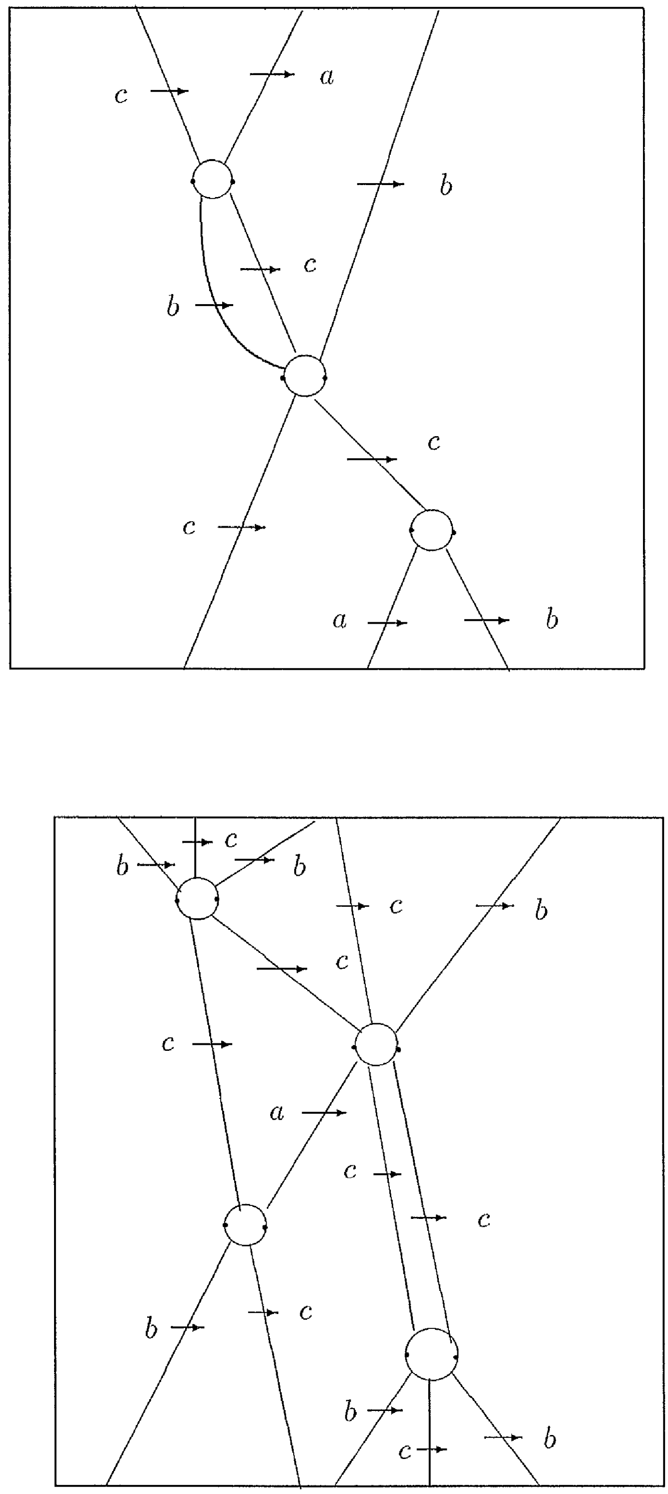


Figure 3.3

Theorem 3.2.4. ([27], [54], [68], [69]) *Let M be a finitely presented monoid. If M is*

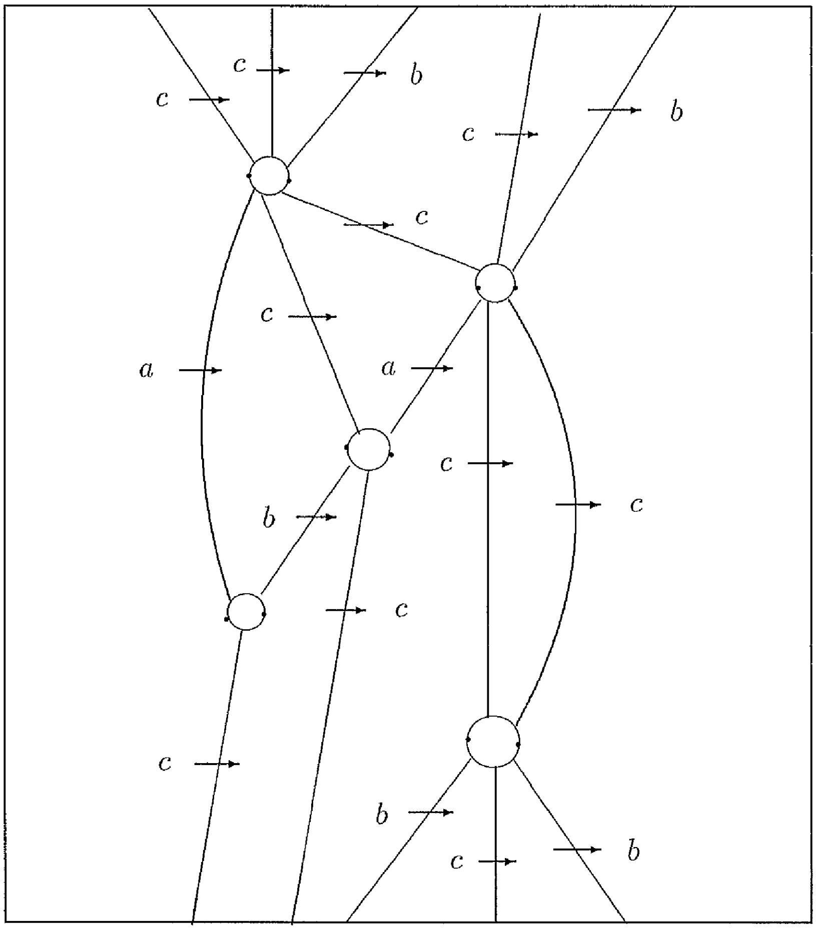


Figure 3.4

FDT, then M satisfies the (left and right) homological finiteness condition FP_3.

Theorem 3.2.5. ([28]) *Let G be a finitely presented group. Then G is FP_3 if and only if G is FDT.*

See also [37], [70] .

Example 3.2.6. ([78]) Let S_1 be the monoid presented by

$$\mathcal{P} = [a, b, t, x, y;\ ab = \varnothing, xa = atx, xt = tx, xb = bx, xy = \varnothing].$$

Then

(1) S_1 has a decidable word problem,

(2) S_1 satisfies the homological finiteness condition FP_∞,

(3) but S_1 is not FDT.

Example 3.2.7. ([49]) Let $X = \{a, b_i, c_i, d_i : i = 1, 2, 3\}$, and let

$$R = \{(b_i a, ab_i),\ (c_j b_j, c_1 b_1),\ (b_j d_j, b_1 d_1) : i = 1, 2, 3.\ j = 2, 3\}.$$

Let M be the monoid presented by $[X; R]$. Then

(1) M has a word problem solvable in linear time,

(2) M is FP_∞,

(3) M is FDT,

(4) but M can not be presented by a finite complete rewriting system.

A monoid S_0 is a *retract* of a monoid S if there are morphisms $\psi : S \longrightarrow S_0$, $\theta : S_0 \longrightarrow S$ such that $\psi\theta = id_{S_0}$ (ψ is then called a *retraction*).

Theorem 3.2.8. ([74]) *Let S_0 and S be two finitely presented monoids. Suppose that S_0 is a retract of S, and S is FDT (or FHT). Then S_0 is also FDT (or FHT) and*

$$\tilde{\gamma}^{(2)}_{S_0} \preceq \tilde{\gamma}^{(2)}_{S} \ (or\ \gamma^{(2)}_{S_0} \preceq \gamma^{(2)}_{S}).$$

Let G be a finitely presented group. Choose a finite group presentation $\hat{\mathcal{P}} = \langle X; R\rangle$ for G. This gives rise to a monoid presentation for G:

$$\mathcal{P} = [X \cup X^{-1}; xx^{-1} = \varnothing, x^{-1}x = \varnothing\ (x \in X),\ r = \varnothing\ (r \in R)].$$

Theorem 3.2.9. ([28], [74]) *Let G be a finitely presented group, and let $\hat{\mathcal{P}}$ and $\mathcal{P}$ be finite group and monoid presentations for G as above. Then the following are equivalent.*

(1) $\mathcal{P}$ is FDT;

(2) $\mathcal{P}$ is FHT;

(3) $\pi_2(\hat{\mathcal{P}})$ is finitely generated as a left $\mathbb{Z}G$-module.

Theorem 3.2.10. ([74]) *Let G be an FDT group. Then*

$$\gamma^{(2)}_{G} \sim \delta^{(2)}_{G}.$$

3.3 FDT for semi-direct products of monoids

The main result in this section is the following theorem.

Theorem 3.3.1. *Let A, B be monoids, and let $Q: A \longrightarrow End(B)$ be a monoid homomorphism. If A and B are FDT, then so is the semi-direct product $A \ltimes_Q B$.*

This theorem appears in [79].

Note that the converse of this theorem is false.

Example 3.3.2. (see Example 2.4.2) Let $F_1 = \langle x \rangle$ be the free group of rank 1, and let F_∞ be the free group on generators $\{a_k : k \in \mathbb{Z}\}$. We define a right F_1-action on F_∞ by $a_k \cdot x = a_{k+1}$ for all k. Then the semi-direct product $F_1 \ltimes F_\infty$ is FDT (in Example 2.4.2 we have proved that $F_1 \ltimes F_\infty \cong F_2$, where F_2 denotes the free group of rank 2). But the factor F_∞ is not FDT.

The proof of Theorem 3.3.1 will follow several lemmas.

If we pick monoid presentations $\mathcal{P}_A = [X; R]$ and $\mathcal{P}_B = [Y; S]$ for A and B respectively, then

$$\mathcal{P} = [X \cup Y; R, S, T]$$

is a monoid presentation for $M =: A \ltimes_Q B$, where T denote the set of relations $yx = x(yQ_x)$ $(x \in X, y \in Y)$.

If $W \equiv y_1 y_2 \cdots y_m$ is a word on Y, then for any $x \in X$ we denote the word $(y_1 Q_x)(y_2 Q_x) \cdots (y_m Q_x)$ by WQ_x. If $U \equiv x_1 x_2 \cdots x_n$ is a word on X, then for any $y \in Y$ we denote $(\cdots((yQ_{x_1})Q_{x_2})Q_{x_3} \cdots)Q_{x_n}$ by yQ_U.

Let Γ_A, Γ_B and Γ denote the graphs associated to $\mathcal{P}_A, \mathcal{P}_B$ and $\mathcal{P}$ respectively (Γ_A and Γ_B can be considered as subgraphs of Γ). Let $\overline{\Gamma}$ denote the subgraph of Γ which has the same set of vertices as Γ, but which contains only those edges (U, t, ε, V) of Γ with $t \in T$, $U, V \in (X \cup Y)^*$, $\varepsilon = \pm 1$. By $P_+(\overline{\Gamma})$ (respectively, $P_-(\overline{\Gamma})$) we denote the set of those paths in $\overline{\Gamma}$ that only contain edges of the form $(U, t, +1, V)$ (respectively, $(U, t, -1, V)$) . Let $\simeq$ denote an arbitrary homotopy relation on $P(\Gamma)$.

Lemma 3.3.3. *Let $p \in P(\overline{\Gamma})$. Then there exist paths $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$ such that $p \simeq p_+p_-$.*

Proof. Let $p = e_1e_2 \cdots e_m$, where $e_1, \cdots, e_m$ are edges of $\overline{\Gamma}$. Suppose there is an index i, such that $e_i \in P_-(\overline{\Gamma})$, $e_{i+1} \in P_+(\overline{\Gamma})$. Choose i minimal, and let

$$e_i = (U_i, t_i, -1, V_i), \quad t_i : y_i x_i = x_i(y_i Q_{x_i}),$$

$$e_{i+1} = (U_{i+1}, t_{i+1}, +1, V_{i+1}), \quad t_{i+1} : y_{i+1}x_{i+1} = x_{i+1}(y_{i+1}Q_{x_{i+1}}).$$

If $U_i \equiv U_{i+1}$, then $y_i \equiv y_{i+1}$, $x_i \equiv x_{i+1}$, $V_i \equiv V_{i+1}$. So $e_{i+1} = e_i^{-1}$, and hence,

$$p \simeq e_1 \cdots e_{i-1}e_{i+2} \cdots e_m.$$

If $U_i \not\equiv U_{i+1}$, then $U_iy_ix_iV_i \equiv U_{i+1}y_{i+1}x_{i+1}V_{i+1}$ implies that these edges involve disjoint applications of relations. In fact, if $U_i \equiv U_{i+1}y_{i+1}x_{i+1}W_{i+1}$, $V_{i+1} \equiv W_{i+1}y_ix_iV_i$, then

$$\begin{aligned} e_ie_{i+1} &= (U_{i+1}y_{i+1}x_{i+1}W_{i+1}, t_i, -1, V_i)(U_{i+1}, t_{i+1}, +1, W_{i+1}y_ix_iV_i) \\ &\simeq (U_{i+1}, t_{i+1}, +1, W_{i+1}x_i(y_iQ_{x_i})V_i)(U_{i+1}x_{i+1}(y_{i+1}Q_{x_{i+1}})W_{i+1}, t_i, -1, V_i) \\ &=: e_i'e_{i+1}', \end{aligned}$$

where $e_i' \in P_+(\overline{\Gamma})$, $e_{i+1}' \in P_-(\overline{\Gamma})$ (see Figure 3.5).

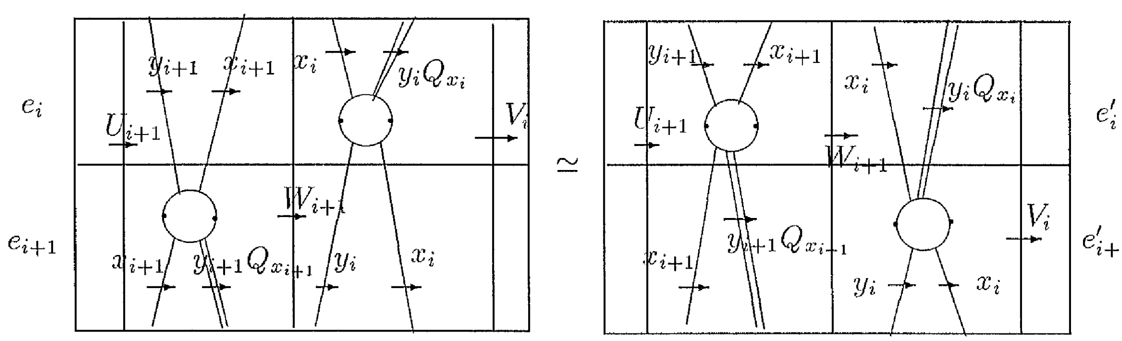


Figure 3.5

Hence, $p \simeq e_1 \cdots e_{i-1}e_i'e_{i+1}'e_{i+2} \cdots e_m$. By repeated use of this, p can be transformed into a path of required form. $\square$

Remark. A result similar to Lemma 3.3.3 holds for the paths in the graph associated to any monoid presentation without critical pairs. (Note that T above is a rewriting system without critical pairs. In paticular, it is confluent.)

Lemma 3.3.4. *Let* $p \in P(\overline{\Gamma})$. *If* $\iota(p) = UV$, $\tau(p) = U'V'$, *where* $U, U' \in X^*$, $V, V' \in Y^*$, *then* $U \equiv U'$, $V \equiv V'$ *and* $p \simeq 1$.

Proof. By Lemma 3.3.3, there exist paths $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$ such that $p \simeq p_+p_-$. Since $\iota(p_+) = \iota(p) = UV$, we have $p_+ = 1$. Since $\tau(p_-) = \tau(p) = U'V'$, we have $p_- = 1$. Hence, $p \simeq 1$ and $U \equiv U'$, $V \equiv V'$. □

We define a homomorphism $f : (X \cup Y)^* \longrightarrow X^*$ by $f(x) = x$ $(x \in X)$, $f(y) = 1$ $(y \in Y)$.

Lemma 3.3.5. *Let* $W \in (X \cup Y)^*$. *Then there is a path* $p_W \in P_+(\overline{\Gamma})$ *from* W *to* $f(W)V$ *for some* $V \in Y^*$. *If* $p \in P_+(\overline{\Gamma})$ *is a path from* W *to* $f(W)V'$ *for some* $V' \in Y^*$, *then* $V \equiv V'$ *and* $p_W \simeq p$.

Proof. Let $W \equiv W_0x_1W_1x_2\cdots x_mW_m$, where $x_1, x_2, \cdots, x_m \in X$, $W_0, W_1, \cdots, W_m \in Y^*$. Then $f(W) = x_1x_2\cdots x_m$. Let $W_0 \equiv y_1y_2\cdots y_k$ $(y_i \in Y)$, $t_i : y_ix_1 = x_1(y_iQ_{x_1})$ $(i = 1, 2, \cdots, k)$. Let $W' \equiv W_1x_2W_2x_3\cdots x_mW_m$. Then

$$(y_1y_2\cdots y_{k-1}, t_k, +1, W')(y_1y_2 \cdots y_{k-2}, t_{k-1}, +1, (y_kQ_{x_1})W') \cdots (1, t_1, +1, (y_2y_3\cdots y_k)Q_{x_1}W')$$

is a path in $P_+(\overline{\Gamma})$ from $W \equiv W_0x_1W'$ to $x_1(W_0Q_{x_1})W'$ (see Figure 3.6).

Continuing in this way, we can get a path $p_W \in P_+(\overline{\Gamma})$ from W to $f(W)V$ for some $V \in Y^*$. If $p \in P_+(\overline{\Gamma})$ is a path from W to $f(W)V'$ for some $V' \in Y^*$, then $p^{-1}p_W \in P(\overline{\Gamma})$ is a path from $f(W)V'$ to $f(W)V$. By Lemma 3.3.4, $p^{-1}p_W \simeq 1$, so $p_W \simeq p$ and $V \equiv V'$. □

By Lemma 3.3.5, we can define a map

$$h : (X \cup Y)^* \longrightarrow Y^*, \quad W \longrightarrow V \quad (W, V \text{ are as in Lemma 3.3.5}).$$

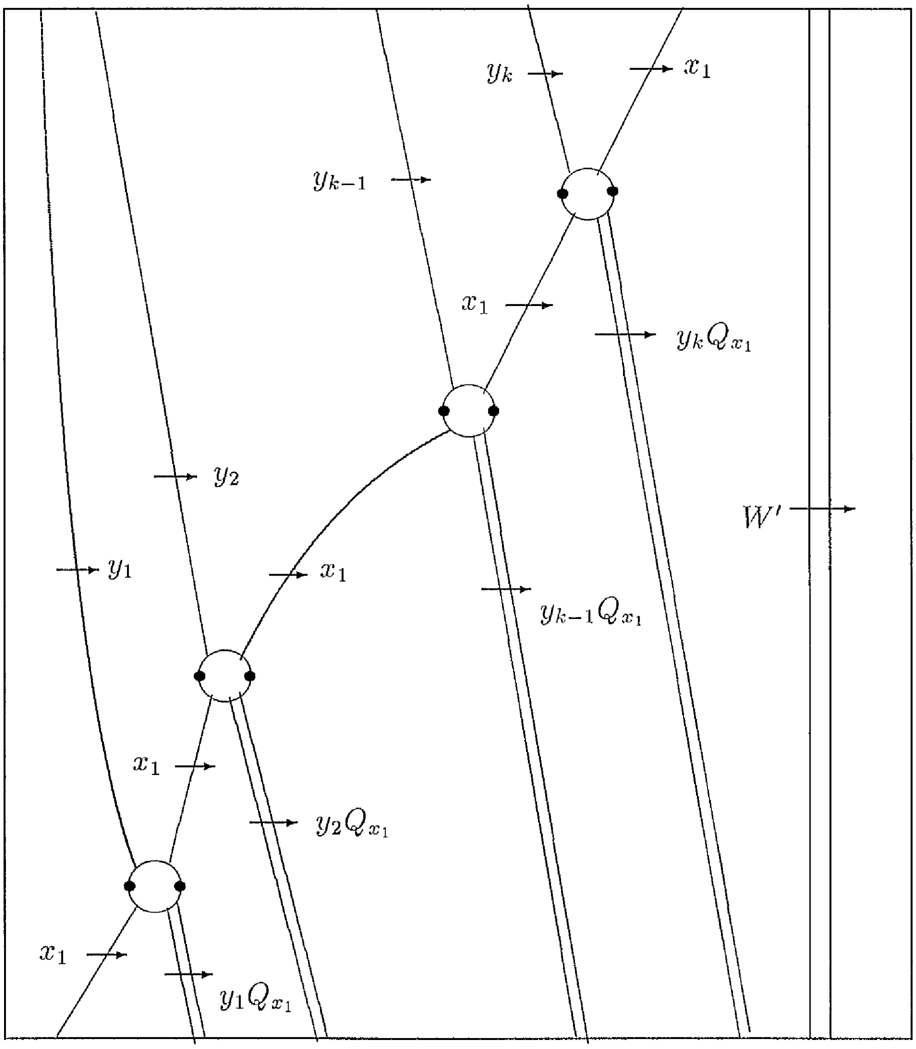


Figure 3.6

For each $s \in S$, where $s : s_{+1} = s_{-1}$, and each $x \in X$, by Lemma 3.3.5, there is a (homotopically unique) path $p_+ \in P_+(\overline{\Gamma})$ from $s_{+1}x$ to $x(s_{+1}Q_x)$, and a path $p_- \in P_-(\overline{\Gamma})$ from $x(s_{-1}Q_x)$ to $s_{-1}x$. Since $[s_{+1}]_B = [s_{-1}]_B$, we have $[s_{+1}Q_x]_B = [s_{-1}Q_x]_B$, so there is a path $p_{s,x} \in P(\Gamma_B)$ from $s_{+1}Q_x$ to $s_{-1}Q_x$. Hence, we have a path

$$q_{s,x} =: p_+(x \cdot p_{s,x})p_- \in P(\Gamma)$$

from $s_{+1}x$ to $s_{-1}x$. Let

$$C_1 = \{((1, s, +1, x), q_{s,x}) : \ s \in S, \ x \in X\} \subset P^{(2)}(\Gamma). \tag{3.1}$$

For each $r \in R$, where $r: \; r_{+1} = r_{-1}$, and each $y \in Y$, by Lemma 3.3.5, there is a (homotopically unique) path $p'_+ \in P_+(\overline{\Gamma})$ from yr_{+1} to $r_{+1}(yQ_{r_{+1}})$, and a path $p'_- \in P_-(\overline{\Gamma})$ from $r_{-1}(yQ_{r_{-1}})$ to yr_{-1}. For any $W \in Y^*, x \in X$, we have $[WQ_x]_B = [W]_B Q_{[x]_A}$. So $[yQ_{r_{+1}}]_B = [y]_B Q_{[r_{+1}]_A}$. Since $[r_{+1}]_A = [r_{-1}]_A$, we have $Q_{[r_{+1}]_A} = Q_{[r_{-1}]_A}$, and hence, $[yQ_{r_{+1}}]_B = [yQ_{r_{-1}}]_B$. So there is a path $p'_{r,y} \in P(\Gamma_B)$ from $yQ_{r_{+1}}$ to $yQ_{r_{-1}}$. Thus

$$q'_{r,y} =: p'_+(r_{+1} \cdot p'_{r,y})((1, r, +1, 1) \cdot yQ_{r_{-1}})p'_- \in P(\Gamma)$$

is a path from yr_{+1} to yr_{-1}. Note for further use that

$$q'_{r,y} \simeq p'_+((1, r, +1, 1) \cdot yQ_{r_{+1}})(r_{-1} \cdot p'_{r,y})p'_-.$$

Let

$$C_2 = \{((y, r, +1, 1), q'_{r,y}) : \; r \in R, y \in Y\} \subset P^{(2)}(\Gamma). \tag{3.2}$$

Let $p, q \in P(\Gamma)$ and let $\simeq$ be a homotopy relation on $P(\Gamma)$. If $p \simeq p_+ q p_-$ for some $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$, then we write $p \rightsquigarrow q$. (If $\simeq = \simeq_{C'}$ for some $C' \subset P^{(2)}(\Gamma)$ then we will write $p \rightsquigarrow_{C'} q$.) Note that $\rightsquigarrow$ is transitive and that this relation is compatible with the two-sided action of $(X \cup Y)^*$.

Lemma 3.3.6. *Let $p \in P_+(\overline{\Gamma})$, $q \in P(\Gamma)$. Then*

$$\iota(p) \cdot q \rightsquigarrow \tau(p) \cdot q,$$

$$q \cdot \iota(p) \rightsquigarrow q \cdot \tau(p).$$

Proof. By use of Pull-down and push-up we have (see Figure 3.7)

$$(p^{-1} \cdot \iota(q))(\iota(p) \cdot q) \simeq (\tau(p) \cdot q)(p^{-1} \cdot \tau(q)).$$

Thus we have

$$\iota(p) \cdot q \simeq (p \cdot \iota(q))(\tau(p) \cdot q)(p^{-1} \cdot \tau(q)),$$

and so (since $p \cdot \iota(q) \in P_+(\overline{\Gamma})$, $p^{-1} \cdot \tau(q) \in P_-(\overline{\Gamma})$)

$$\iota(p) \cdot q \rightsquigarrow \tau(p) \cdot q.$$

The other relation is similar. □

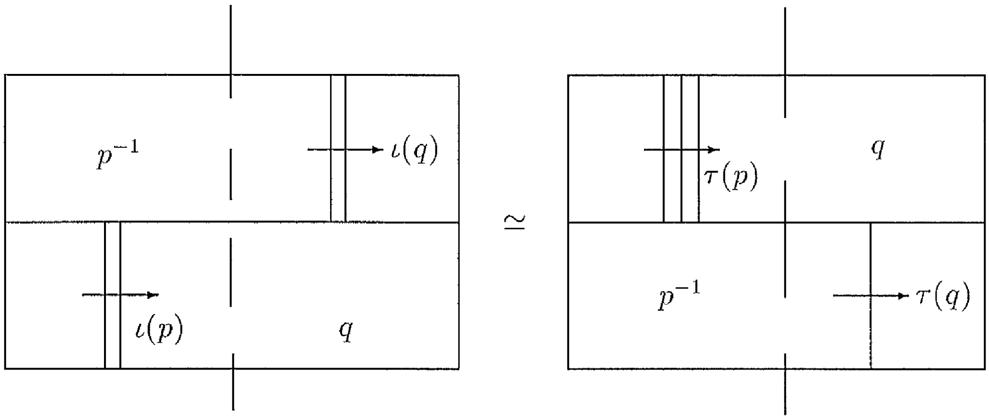


Figure 3.7

Lemma 3.3.7. *Let p, q be paths in Γ with $\tau(p) = \iota(q)$. If $p \rightsquigarrow p'$, $q \rightsquigarrow q'$ and $\tau(p'), \iota(q') \in X^*Y^*$, then $\tau(p') = \iota(q')$ and $pq \rightsquigarrow p'q'$.*

Proof. We have $p \simeq p_+p'p_-$, $q \simeq q_+q'q_-$, where $p_+, q_+ \in P_+(\overline{\Gamma})$, $p_-, q_- \in P_-(\overline{\Gamma})$. Then $pq \simeq p_+p'p_-q_+q'q_-$. By Lemma 3.3.4 $p_-q_+ \simeq 1$. Thus $\tau(p') = \iota(q')$ and $pq \rightsquigarrow p'q'$. □

Lemma 3.3.8. *Let e be an edge of Γ_B. Then for any $x \in X$ there is a path q in Γ_B such that $e \cdot x \rightsquigarrow_{C_1} x \cdot q$.*

Proof. Let $e = (U, s, \varepsilon, V)$, where $U, V \in Y^*$, $s \in S$, $\varepsilon = \pm 1$. By Lemma 3.3.5, there is a path in $P_+(\overline{\Gamma})$ from Vx to $xh(Vx)$. So by Lemma 3.3.6 we have

$$e \cdot x \rightsquigarrow_{C_1} (U, s, \varepsilon, xh(Vx)).$$

By the definition of C_1, we have

$$(U, s, \varepsilon, xh(Vx)) \rightsquigarrow_{C_1} Ux \cdot q_1,$$

where $q_1 = p^{\varepsilon}_{s,x} \cdot h(Vx) \in P(\Gamma_B)$. Now there is a path in $P_+(\overline{\Gamma})$ from Ux to $xh(Ux)$. By Lemma 3.3.6 we have

$$Ux \cdot q_1 \rightsquigarrow_{C_1} x \cdot (h(Ux) \cdot q_1).$$

The result follows. □

Lemma 3.3.9. *Let p be any non-empty path in Γ_B. Then for any $W \in X^*$ there exists a path q in Γ_B such that $p \cdot W \rightsquigarrow_{C_1} W \cdot q$.*

Proof. We will prove the result for W consisting of a single letter $x \in X$. The general result will follow from this by induction on the length of W. Let $p = e_1 e_2 \cdots e_m$. Then by Lemma 3.3.8, there exist $q_i \in P(\Gamma_B)$, such that $e_i \cdot x \rightsquigarrow_{C_1} x \cdot q_i$, $i = 1, 2, \cdots, m$. Thus by Lemma 3.3.7 we have $p \cdot x \rightsquigarrow_{C_1} x \cdot (q_1 q_2 \cdots q_m)$. □

Lemma 3.3.10. *Let (U, s, ε, V) be an edge in Γ, where $U, V \in (X \cup Y)^*$, $s \in S$, $\varepsilon = \pm 1$. Then there is a path q in Γ_B such that*

$$(U, s, \varepsilon, V) \rightsquigarrow_{C_1} f(UV) \cdot q.$$

Proof. By Lemmas 3.3.5 and 3.3.6 we have

$$(U, s, \varepsilon, V) \rightsquigarrow_{C_1} (U, s, \varepsilon, f(V)h(V)).$$

By Lemma 3.3.9 we can get that

$$(U, s, \varepsilon, f(V)h(V)) \rightsquigarrow_{C_1} Uf(V) \cdot q_1$$

for some $q_1 \in P(\Gamma_B)$. Now by Lemmas 3.3.5 and 3.3.6

$$Uf(V) \cdot q_1 \rightsquigarrow_{C_1} f(UV)h(Uf(V)) \cdot q_1.$$

Hence $(U, s, \varepsilon, V) \rightsquigarrow_{C_1} f(UV) \cdot q$, where $q = h(Uf(V)) \cdot q_1 \in P(\Gamma_B)$. □

Lemma 3.3.11. *Let $r \in R$, $W \in Y^*$. Then there is a path q in Γ_B such that*

$$W \cdot (1, r, \varepsilon, 1) \rightsquigarrow_{C_2} (r_\varepsilon \cdot q)((1, r, \varepsilon, 1) \cdot W')$$

for some $W' \in Y^*$.

Proof. For any $U \in (X \cup Y)^*$, $p \in P(\Gamma_B)$ we have (by Lemmas 3.3.5 and 3.3.6)

$$U \cdot p \rightsquigarrow f(U) \cdot p', \tag{1}$$

where $p' \in P(\Gamma_B)$. Also, for each $y \in Y$, by the definition of C_2, we have

$$y \cdot (1, r, \varepsilon, 1) \rightsquigarrow_{C_2} (r_\varepsilon \cdot q')((1, r, \varepsilon, 1) \cdot W_1), \tag{2}$$

where $q' \in P(\Gamma_B)$, $W_1 \in Y^*$. By repeated use of (1), (2) and Lemma 3.3.7, we have

$$W \cdot (1, r, \varepsilon, 1) \rightsquigarrow_{C_2} (r_\varepsilon \cdot q)((1, r, \varepsilon, 1) \cdot W'),$$

where $q \in P(\Gamma_B)$, $W' \in Y^*$. $\square$

Lemma 3.3.12. *Let (U, r, ε, V) be an edge in Γ, where $U, V \in (X \cup Y)^*$, $r \in R$, $\varepsilon = \pm 1$. Then there is a path $q \in P(\Gamma_B)$, such that*

$$(U, r, \varepsilon, V) \rightsquigarrow_{C_1 \cup C_2} (f(Ur_\varepsilon V) \cdot q)((f(U), r, \varepsilon, f(V)) \cdot h(Ur_{-\varepsilon} V)).$$

Proof. By Lemmas 3.3.5 and 3.3.6 we have

$$(U, r, \varepsilon, V) \rightsquigarrow (f(U)h(U), r, \varepsilon, f(V)h(V)).$$

By Lemma 3.3.11 we have

$$(f(U)h(U), r, \varepsilon, f(V)h(V)) \rightsquigarrow_{C_2} (f(U)r_\varepsilon \cdot q_1 \cdot f(V)h(V))((f(U), r, \varepsilon, 1) \cdot W_1 f(V)h(V))$$

for some $q_1 \in P(\Gamma_B)$ and $W_1 \in Y^*$. By Lemma 3.3.9 we have

$$f(U)r_\varepsilon \cdot q_1 \cdot f(V)h(V) \rightsquigarrow_{C_1} f(Ur_\varepsilon V) \cdot q$$

for some $q \in P(\Gamma_B)$. By Lemmas 3.3.5 and 3.3.6

$$(f(U), r, \varepsilon, 1) \cdot W_1 f(V)h(V) \rightsquigarrow (f(U), r, \varepsilon, f(V)) \cdot W$$

for some $W \in Y^*$. Using Lemma 3.3.7 and the above results we can get that

$$(U, r, \varepsilon, V) \rightsquigarrow_{C_1 \cup C_2} (f(Ur_\varepsilon V) \cdot q)((f(U), r, \varepsilon, f(V)) \cdot W).$$

Thus there is a path $p_- \in P_-(\overline{\Gamma})$ from $f(Ur_{-\varepsilon}V)W$ to $Ur_{-\varepsilon}V$. By Lemma 3.3.5 and the definition of h, we have $W = h(Ur_{-\varepsilon}V)$. $\square$

Proposition 3.3.13. *Let $p \in P(\Gamma)$. Then there exist paths $p_+ \in P_+(\overline{\Gamma})$, $p_- \in P_-(\overline{\Gamma})$, $q = f(\iota(p)) \cdot q'$ with $q' \in P(\Gamma_B)$, $\bar{q} = \bar{q}' \cdot h(\tau(p))$ with $\bar{q}' \in P(\Gamma_A)$, such that*

$$p \simeq_{C_1 \cup C_2} p_+ q \bar{q} p_-,$$

where $\tau(p_+) = f(\iota(p))h(\iota(p))$, $\iota(p_-) = f(\tau(p))h(\tau(p))$.

Proof. We will prove the result by induction on the length of p. If p consists of a single edge (U, r, ε, V), then the result is an immediate consequence of Lemma 3.3.12 (if $r \in R$), Lemma 3.3.10 (if $r \in S$), or Lemma 3.3.5 (if $r \in T$). Let $p = p_1 e$, where e is an edge, p_1 has length less than that of p. Then by the inductive hypothesis we have

$$p_1 \rightsquigarrow_{C_1 \cup C_2} (f(\iota(p)) \cdot q_1')(\bar{q}_1' \cdot h(\tau(p_1))),$$

$$e \rightsquigarrow_{C_1 \cup C_2} (f(\iota(e)) \cdot q_2')(\bar{q}_2' \cdot h(\tau(p))),$$

where $q_1'q_2' \in P(\Gamma_B)$, $\bar{q}_1', \bar{q}_2' \in P(\Gamma_A)$, and $\iota(q_1') = h(\iota(p)), \tau(\bar{q}_2') = f(\tau(p))$, $\tau(\bar{q}_1') = f(\tau(p_1)), \iota(q_2') = h(\tau(p_1))$. By Lemma 3.3.7,

$$p \rightsquigarrow_{C_1 \cup C_2} (f(\iota(p)) \cdot q_1')(\bar{q}_1' \cdot h(\tau(p_1)))(f(\iota(e)) \cdot q_2')(\bar{q}_2' \cdot h(\tau(p))).$$

Since the relations used in the path $\bar{q}_1' \cdot h(\tau(p_1))$ and the relations used in the path $f(\iota(e)) \cdot q_2'$ are disjoint, we can get (see Figure 3.8)

$$(\bar{q}_1' \cdot h(\tau(p_1)))(f(\iota(e)) \cdot q_2') \simeq (f(\iota(p)) \cdot q_2')(\bar{q}_1' \cdot h(\tau(p))).$$

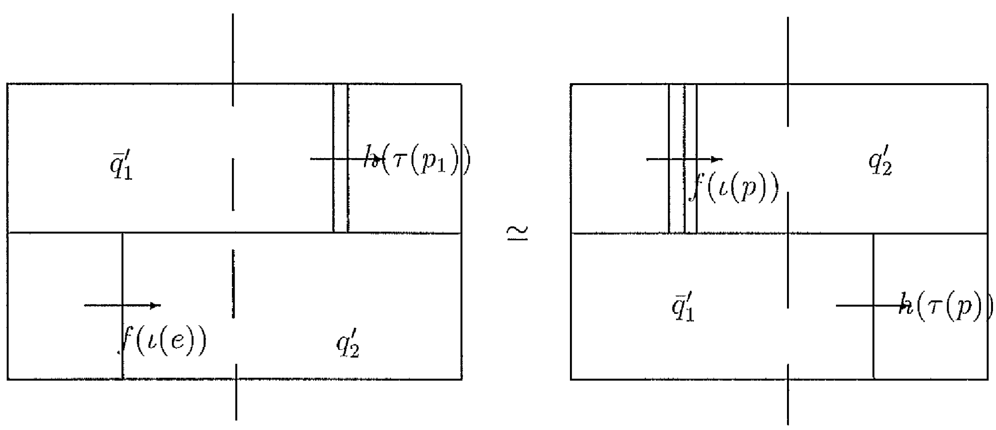


Figure 3.8

Let $q' = q_1'q_2', \bar{q}' = \bar{q}_1'\bar{q}_2'$. Then

$$p \rightsquigarrow_{C_1 \cup C_2} (f(\iota(p) \cdot q')(\bar{q}' \cdot h(\tau(p))),$$

where $\iota(q') = h(\iota(p)), \tau(\bar{q}') = f(\tau(p))$. □

Corollary 3.3.14. *If p is a closed path in Γ, then there exist paths $p_+ \in P_+(\overline{\Gamma})$, $q = f(\iota(p)) \cdot q'$ with q' a closed path in Γ_B, $\bar{q} = \bar{q}' \cdot h(\iota(p))$ with $\bar{q}'$ a closed path in Γ_A, such that*

$$p \simeq_{C_1 \cup C_2} p_+ q \bar{q} p_+^{-1},$$

where $\tau(p_+) = f(\iota(p))h(\iota(p))$.

Proof. By Proposition 3.3.13, we just need to show that q' and $\bar{q}'$ are closed paths and $p_- \simeq p_+^{-1}$.

Since $\iota(p) = \tau(p)$, we have

$$\iota(q') = h(\iota(p)) = h(\tau(p)) = \tau(q'),$$

and

$$\iota(\bar{q}') = f(\iota(p)) = f(\tau(p)) = \tau(\bar{q}').$$

Because $p_- p_+ \in P(\overline{\Gamma})$ and $\iota(p_- p_+) = f(\tau(p))h(\tau(p)) = f(\iota(p))h(\iota(p)) = \tau(p_- p_+)$, by Lemma 3.3.4 we have $p_- p_+ \simeq 1$. Thus $p_- \simeq p_+^{-1}$. □

Let $C_3 \subset P^{(2)}(\Gamma_A)$ and $C_4 \subset P^{(2)}(\Gamma_B)$ be such that $\simeq_{C_3} = P^{(2)}(\Gamma_A)$, $\simeq_{C_4} = P^{(2)}(\Gamma_B)$.

Corollary 3.3.15. *Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$. Then $\simeq_C = P^{(2)}(\Gamma)$.*

Proof. Let $(p_1, p_2) \in P^{(2)}(\Gamma)$. By Proposition 3.3.13, $p_1 \simeq_C p_+ q_1 \bar{q}_1 p_-$, $p_2 \simeq_C p'_+ q_2 \bar{q}_2 p'_-$, where $p_+, p'_+ \in P_+(\overline{\Gamma})$, $p_-, p'_- \in P_-(\overline{\Gamma})$, $q_i = f(\iota(p_i)) \cdot q'_i$ with $q'_i \in P(\Gamma_B)$, $\bar{q}_i = \bar{q}'_i \cdot h(\tau(p_i))$ with $\bar{q}'_i \in P(\Gamma_A)$ $(i = 1, 2)$. Since $\iota(p_1) = \iota(p_2)$ and $\tau(p_1) = \tau(p_2)$, we have

$$\tau(p_+) = f(\iota(p_1))h(\iota(p_1)) = f(\iota(p_2))h(\iota(p_2)) = \tau(p'_+),$$

$$\iota(p_-) = f(\tau(p_1))h(\tau(p_1)) = f(\tau(p_2))h(\tau(p_2)) = \iota(p'_-).$$

Hence $p_+ \simeq_C p'_+$, $p_- \simeq_C p'_-$ (by Lemma 3.3.5). It is easy to see that $\iota(q'_i) = h(\iota(p_i))$, $\tau(q'_i) = h(\tau(p_i))$ $(i = 1, 2)$. So $\iota(q'_1) = \iota(q'_2)$, $\tau(q'_1) = \tau(q'_2)$. Thus $(q'_1, q'_2) \in P^{(2)}(\Gamma_B)$. Since $\simeq_{C_4} = P^{(2)}(\Gamma_B)$, and $C_4 \subset C$, we have $q'_1 \simeq_C q'_2$ and hence $q_1 \simeq_C q_2$. Similarly, $\iota(\bar{q}'_1) = f(\iota(p_1)) = f(\iota(p_2)) = \iota(\bar{q}'_2)$ and $\tau(\bar{q}'_1) = f(\tau(p_1)) = f(\tau(p_2)) = \tau(\bar{q}'_2)$, so $(\bar{q}'_1, \bar{q}'_2) \in P^{(2)}(\Gamma_A)$. Since $\simeq_{C_3} = P^{(2)}(\Gamma_A)$, and $C_3 \subset C$, we have $\bar{q}'_1 \simeq_C \bar{q}'_2$, and hence, $\bar{q}_1 \simeq_C \bar{q}_2$. Thus $p_1 \simeq_C p_+ q_1 \bar{q}_1 p_- \simeq_C p'_+ q_2 \bar{q}_2 p'_- \simeq_C p_2$. Therefore $\simeq_C = P^{(2)}(\Gamma)$. □

Proof of the Theorem 3.3.1. If A and B are FDT, then we can assume $\mathcal{P}_A$ and $\mathcal{P}_B$ are finite presentations, and C_3, C_4 are finite set. So $\mathcal{P}$ is a finite presentation and C is a finite set. By Corollary 3.3.15, $\simeq_C = P^{(2)}(\Gamma)$. Hence $A \ltimes_Q B$ is FDT. □

3.4 *FHT* for semi-direct products of monoids

Theorem 3.4.1. *Let A, B be monoids, and let $Q: A \longrightarrow End(B)$ be a monoid homomorphism. If A and B are FHT, then so is the semi-direct product $A \ltimes_Q B$.*

Proof. Let $\mathcal{P}_A$, $\mathcal{P}_B$, $\mathcal{P}$, Γ_A, Γ_B and Γ be as in last section. Suppose $\mathcal{P}_A$ and $\mathcal{P}_B$ are finite, then so is $\mathcal{P}$. Let $\mathcal{D}_A = \mathcal{D}(\mathcal{P}_A)$, $\mathcal{D}_B = \mathcal{D}(\mathcal{P}_B)$ and $\mathcal{D} = \mathcal{D}(\mathcal{P})$. The inclusion map from $\mathcal{P}_A$ to $\mathcal{P}$ induces a mapping of 2-complexes $\phi_A : \mathcal{D}_A \longrightarrow \mathcal{D}$. Then the mapping ϕ_A induces a chain map from the chain complex

$$0 \longrightarrow C_2(\mathcal{D}_A) \xrightarrow{\partial_2^A} C_1(\mathcal{D}_A) \xrightarrow{\partial_1^A} C_0(\mathcal{D}_A) \longrightarrow 0$$

of $\mathcal{D}_A$ to the chain complex

$$0 \longrightarrow C_2(\mathcal{D}) \xrightarrow{\partial_2} C_1(\mathcal{D}) \xrightarrow{\partial_1} C_0(\mathcal{D}) \longrightarrow 0$$

of $\mathcal{D}$, and this chain map then induces a homomorphism

$$\phi_A^* : \pi_2^{(b)}(\mathcal{P}_A) \longrightarrow \pi_2^{(b)}(\mathcal{P}),\ z_p + Im\partial_2^A \longmapsto z_p + Im\partial_2,$$

where $p = e_1^{\varepsilon_1} \cdots e_k^{\varepsilon_k}$ is a closed path in $\mathcal{D}_A$ (e_i a positive edge, $\varepsilon_i = \pm 1,\ i = 1, \cdots, k$) and $z_p = \sum_{i=1}^k \varepsilon_i e_i$. Similarly, we have a homomorphism

$$\phi_B^* : \pi_2^{(b)}(\mathcal{P}_B) \longrightarrow \pi_2^{(b)}(\mathcal{P}),\ z_q + Im\partial_2^B \longmapsto z_q + Im\partial_2,$$

where q is a closed path in $\mathcal{D}_B$.

If A and B are FHT, then there exist a finite set $\boldsymbol{X}_1$ of spherical monoid pictures over $\mathcal{P}_A$ and a finite set $\boldsymbol{X}_2$ of spherical monoid pictures over $\mathcal{P}_B$ such that the elements $z_p + Im\partial_2^A$ ($p \in \boldsymbol{X}_1$) generate the bimodule $\pi_2^{(b)}(\mathcal{P}_A)$ and the elements $z_q + Im\partial_2^B$ ($q \in \boldsymbol{X}_2$) generate the bimodule $\pi_2^{(b)}(\mathcal{P}_B)$ (see §3.1.2). Let

$$\boldsymbol{X}_3 = \{pq^{-1} : (p, q) \in C_1 \cup C_2\},$$

where C_1 and C_2 are as in Section 3.3. Then $\boldsymbol{X}_3$ is a finite set of spherical monoid pictures over $\boldsymbol{\mathcal{P}}$. Let

$$\boldsymbol{X} = \boldsymbol{X}_1 \cup \boldsymbol{X}_2 \cup \boldsymbol{X}_3.$$

Then $\boldsymbol{X}$ is a finite set of spherical monoid pictures over $\boldsymbol{\mathcal{P}}$. We want to show that the elements $z_p + Im\partial_2$ $(p \in \boldsymbol{X})$ generate the bimodule $\pi_2^{(b)}(\boldsymbol{\mathcal{P}})$.

For any spherical monoid picture p over $\boldsymbol{\mathcal{P}}$, by Corollary 3.3.14 we have

$$p \simeq_{C_1 \cup C_2} p_+ q \bar{q} p_+^{-1},$$

where $p_+ \in P_+(\overline{\Gamma})$, $q = f(\iota(p)) \cdot q'$ with q' a closed path in $\boldsymbol{\mathcal{D}}_B$, $\bar{q} = \bar{q}' \cdot h(\iota(p))$ with $\bar{q}'$ a closed path in $\boldsymbol{\mathcal{D}}_A$. Thus

$$\begin{aligned} z_p + Im\partial_2 = (z_{p_+} + Im\partial_2) + (z_q + Im\partial_2) \quad &+ \quad (z_{\bar{q}} + Im\partial_2) - (z_{p_+} + Im\partial_2) \\ &+ \quad \sum_i \varepsilon_i d_i (z_i + Im\partial_2) d_i', \end{aligned}$$

where $\varepsilon_i = \pm 1$, $d_i, d_i' \in M = A \ltimes_Q B$ and $z_i = z_{p_i}$ for some $p_i \in \boldsymbol{X}_3$. Note that

$$z_q + Im\partial_2 = [f(\iota(p))]_M (z_{q'} + Im\partial_2),$$

$$\begin{aligned} z_{q'} + Im\partial_2 &= \phi_B^*(z_{q'} + Im\partial_2^B) \\ &= \phi_B^*\Big(\sum_j \varepsilon_j b_j (z_j + Im\partial_2^B) b_j'\Big) \\ &= \sum_j \varepsilon_j b_j (z_j + Im\partial_2) b_j', \end{aligned}$$

where $\varepsilon_j = \pm 1$, $b_j, b_j' \in B$, $z_j = z_{p_j}$ for some $p_j \in \boldsymbol{X}_2$. We also have

$$z_{\bar{q}} + Im\partial_2 = (z_{\bar{q}'} + Im\partial_2)[h(\iota(p))]_M,$$

$$\begin{aligned} z_{\bar{q}'} + Im\partial_2 &= \phi_A^*(z_{\bar{q}'} + Im\partial_2^A) \\ &= \phi_A^*\Big(\sum_k \varepsilon_k a_k (z_k + Im\partial_2^A) a_k'\Big) \\ &= \sum_k \varepsilon_k a_k (z_k + Im\partial_2) a_k', \end{aligned}$$

where $\varepsilon_k = \pm 1$, $a_k, a'_k \in A$ and $z_k = z_{p_k}$ for some $p_k \in \boldsymbol{X}_1$. So

$$z_p + Im\partial_2 = \sum_n \varepsilon_n d_n (z_n + Im\partial_2) d'_n,$$

where $\varepsilon_n = \pm 1$, $d_n, d'_n \in M$ and $z_n = z_{p_n}$ for some $p_n \in \boldsymbol{X}$. Since every element of $\pi_2^{(b)}(\mathcal{P})$ has the form of a finite sum $\sum(z_q + Im\partial_2)$ for some spherical pictures q over $\mathcal{P}$, the elements $z_p + Im\partial_2$ ($p \in \boldsymbol{X}$) generate the bimodule $\pi_2^{(b)}(\mathcal{P})$. Thus $M = A \ltimes_Q B$ is FHT. □

3.5 Second order Dehn functions for direct products of monoids

Let A, B be monoids. Let $\mathcal{P}_A = [X; R]$ and $\mathcal{P}_B = [Y; S]$ be finite monoid presentations for A and B respectively, where X and Y are disjoint. Then

$$\mathcal{P} = [X \cup Y; R, S, T]$$

is a monoid presentation for $M =: A \times B$, where T denote the set of relations $yx = xy$ ($x \in X, y \in Y$).

Let Γ_A, Γ_B and Γ denote the graphs associate to $\mathcal{P}_A, \mathcal{P}_B$ and $\mathcal{P}$ respectively(Γ_A and Γ_B can be considered as subgraphs of Γ). Let $\overline{\Gamma}$ denote the subgraph of Γ which has the same set of vertices as Γ, but which contains only those edges (U, t, ε, V) of Γ with $t \in T$, $U, V \in (X \cup Y)^*$, $\varepsilon = \pm 1$. By $P_+(\overline{\Gamma})$ (respectively, $P_-(\overline{\Gamma})$) we denote the set of those paths in $\overline{\Gamma}$ that only contain edges of the form $(U, t, +1, V)$ (respectively, $(U, t, -1, V)$) . Let $\mathcal{D} = \mathcal{D}(\mathcal{P})$ be the associated 2-complex.

We define two homomorphisms

$$f : (X \cup Y)^* \longrightarrow X^*, \ f(x) = x, f(y) = 1 \ (x \in X, y \in Y),$$

$$g : (X \cup Y)^* \longrightarrow Y^*, \ g(x) = 1, g(y) = y \ (x \in X, y \in Y).$$

Because the direct product of monoids is a special case of semi-direct products of monoids, by Lemmas 3.3.3, 3.3.4, 3.3.5 and 3.3.6 we have the following four lemmas.

Lemma 3.5.1. *Let $p \in P(\overline{\Gamma})$. Then there exist paths $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$ such that $p \sim p_+p_-$ in $\mathcal{D}$.* $\square$

Lemma 3.5.2. *Let $p \in P(\overline{\Gamma})$. If $\iota(p) = UV$, $\tau(p) = U'V'$, where $U, U' \in X^*$, $V, V' \in Y^*$, then $U \equiv U'$, $V \equiv V'$ and $p \sim 1$ in $\mathcal{D}$.* $\square$

Lemma 3.5.3. *Let $W \in (X \cup Y)^*$. Then there is a path $p_W \in P_+(\overline{\Gamma})$ from W to $f(W)g(W)$. If $p \in P_+(\overline{\Gamma})$ is a path from W to UV for some $U \in X^*$ and $V \in Y^*$, then $U \equiv f(W)$, $V \equiv g(W)$ and $p_W \sim p$ in $\mathcal{D}$.* $\square$

Lemma 3.5.4. *Let $p \in P_+(\overline{\Gamma})$, $q \in P(\Gamma)$. Then*

$$\iota(p) \cdot q \sim p_+(\tau(p) \cdot q)p_-,$$

$$q \cdot \iota(p) \sim p'_+(q \cdot \tau(p))p'_-,$$

in $\mathcal{D}$ for some $p_+, p'_+ \in P_+(\overline{\Gamma})$ and $p_-, p'_- \in P_-(\overline{\Gamma})$. $\square$

For each $r \in R$, where $r: \ r_{+1} = r_{-1}$, and each $y \in Y$, by Lemma 3.5.3, there is a path $q_{r,y} \in P_+(\overline{\Gamma})$ from yr_{+1} to $r_{+1}y$, and a path $q'_{r,y} \in P_-(\overline{\Gamma})$ from $r_{-1}y$ to yr_{-1}. Let

$$\boldsymbol{X}_1 = \{(y, r, -1, 1)q_{r,y}(1, r, +1, y)q'_{r,y} : r \in R, y \in Y\}.$$

Then $\boldsymbol{X}_1$ is a finite set of closed paths in $\mathcal{D}$.

For each $s \in S$, where $s : s_{+1} = s_{-1}$, and each $x \in X$, by Lemma 3.5.3, there is a path $p_{s,x} \in P_+(\overline{\Gamma})$ from $s_{+1}x$ to xs_{+1}, and a path $p'_{s,x} \in P_-(\overline{\Gamma})$ from xs_{-1} to $s_{-1}x$. Let

$$\boldsymbol{X}_2 = \{(1, s, -1, x)p_{s,x}(x, s, +1, 1)p'_{s,x} : s \in S, \ x \in X\}.$$

Then $\boldsymbol{X}_2$ is a finite set of closed paths in $\mathcal{D}$.

Lemma 3.5.5. *Let (U, r, ε, V) be an edge in Γ with $r \in R$. Then there are paths $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$ such that*

$$(U, r, \varepsilon, V) \sim_{\boldsymbol{X}_1} p_+(f(U), r, \varepsilon, f(V)g(UV))p_-$$

in $\mathcal{D}$, and $p_+(f(U), r, \varepsilon, f(V)g(UV))p_-$ can be obtained from (U, r, ε, V) by $l(g(U))$ operations of type (III) and some operations of type (I) and (II).

Proof. By Lemmas 3.5.3 and 3.5.4, we have

$$(U, r, \varepsilon, V) \sim p_1(f(U)g(U), r, \varepsilon, V)p_1'$$

in $\mathcal{D}$ for some $p_1 \in P_+(\overline{\Gamma})$ and $p_1' \in P_-(\overline{\Gamma})$. By the definition of $\boldsymbol{X}_1$ we have

$$(f(U)g(U), r, \varepsilon, V) \sim_{\boldsymbol{X}_1} p_2(f(U), r, \varepsilon, g(U)V)p_2'$$

in $\mathcal{D}$ for some $p_2 \in P_+(\overline{\Gamma})$ and $p_2' \in P_-(\overline{\Gamma})$, and $p_2(f(U), r, \varepsilon, g(U)V)p_2'$ can be obtained from $(f(U)g(U), r, \varepsilon, V)$ by $l(g(U))$ operations of type (III) and some operations of type (I) and (II). By Lemmas 3.5.3 and 3.5.4 we have

$$(f(U), r, \varepsilon, g(U)V) \sim p_3(f(U), r, \varepsilon, f(V)g(UV))p_3'$$

in $\mathcal{D}$ for some $p_3 \in P_+(\overline{\Gamma})$ and $p_3' \in P_-(\overline{\Gamma})$. Let $p_+ = p_1p_2p_3$ and $p_- = p_1'p_2'p_3'$. Then we can get the result. □

Lemma 3.5.6. *Let (U, s, ε, V) be an edge in Γ with $s \in S$. Then there are paths $q_+ \in P_+(\overline{\Gamma})$ and $q_- \in P_-(\overline{\Gamma})$ such that*

$$(U, s, \varepsilon, V) \sim_{\boldsymbol{X}_2} q_+(f(UV)g(U), s, \varepsilon, g(V))q_-$$

in $\mathcal{D}$, and $q_+(f(UV)g(U), s, \varepsilon, g(V))q_-$ can be obtained from (U, s, ε, V) by $l(f(V))$ operations of type (III) and some operations of type (I) and (II).

Proof. By Lemmas 3.5.3 and 3.5.4, we have

$$(U, s, \varepsilon, V) \sim q_1(U, s, \varepsilon, f(V)g(V))q_1'$$

in $\mathcal{D}$ for some $q_1 \in P_+(\overline{\Gamma})$ and $q_1' \in P_-(\overline{\Gamma})$. By the definition of $\boldsymbol{X}_2$ we have

$$(U, s, \varepsilon, f(V)g(V)) \sim_{\boldsymbol{X}_2} q_2(Uf(V), s, \varepsilon, g(V))q_2'$$

in $\mathcal{D}$ for some $q_2 \in P_+(\overline{\Gamma})$ and $q_2' \in P_-(\overline{\Gamma})$, and $q_2(Uf(V), s, \varepsilon, g(V))q_2'$ can be obtained from $(U, s, \varepsilon, f(V)g(V))$ by $l(f(V))$ operations of type (III) and some operations of type (I) and (II). By Lemmas 3.5.3 and 3.5.4, we have

$$(Uf(V), s, \varepsilon, g(V)) \sim q_3(f(UV)g(U), s, \varepsilon, g(V))q_3'$$

in $\mathcal{D}$ for some $q_3 \in P_+(\overline{\Gamma})$ and $q_3' \in P_-(\overline{\Gamma})$. Let $q_+ = q_1q_2q_3$ and $q_- = q_1'q_2'q_3'$. Then we can get the result. □

Lemma 3.5.7. *Let p be a path in Γ. Then there exist paths $p_+ \in P_+(\overline{\Gamma})$, $p_- \in P_-(\overline{\Gamma})$, $q_A \in P(\Gamma_A)$ and $q_B \in P(\Gamma_B)$ with $\iota(q_A) = f(\iota(p))$, $\tau(q_A) = f(\tau(p))$, $\iota(q_B) = g(\iota(p))$ and $\tau(q_B) = g(\tau(p))$ such that*

$$p \sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p_+(q_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)p_-$$

in $\mathcal{D}$, and $p_+(q_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)p_-$ can be obtained from p by at most mn operations of type (III) and some operations of type (I) and (II), and $A(p) \geq A(q_A) + A(q_B)$, where $n = A(p)$ and m is the maximum of $l(\iota(e))$ for all edges e in p.

Proof. If p consists of a single edge, then the result is an immediate consequence of Lemma 3.5.3 or Lemma 3.5.5 or Lemma 3.5.6. Now let $p = p_1 e$, where e is an edge of p and p_1 is a subpath of p. Inductively we have

$$p_1 \sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p_+(q_A \cdot g(\iota(p)))(f(\tau(p_1)) \cdot q_B)p_- \quad (1)$$

with $n - 1 \geq A(p_1) = A(q_A) + A(q_B)$, and

$$e \sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p'_+(q'_A \cdot g(\iota(e)))(f(\tau(p)) \cdot q'_B)p'_- \quad (2)$$

with $1 \geq A(q'_A) + A(q'_B)$. So we have

$$p \sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p_+(q_A \cdot g(\iota(p)))(f(\tau(p_1)) \cdot q_B)p_- p'_+(q'_A \cdot g(\iota(e)))(f(\tau(p)) \cdot q'_B)p'_-.$$

Since $p_- p'_+ \in P(\overline{\Gamma})$ and $\iota(p_- p'_+), \tau(p_- p'_+) \in X^*Y^*$, by Lemma 3.5.2 we have $p_- p'_+ \sim 1$ in $\mathcal{D}$. Note that $\iota(q'_A) = f(\iota(e)) = f(\tau(p_1))$, $\tau(q_B) = g(\tau(p_1)) = g(\iota(e))$, $\iota(q_B) = g(\iota(p_1)) = g(\iota(p))$, $\tau(q'_A) = f(\tau(e)) = f(\tau(p))$. By use of Pull -down and push-up, we have (see Figure 3.9)

$$(f(\tau(p_1)) \cdot q_B)(q'_A \cdot g(\iota(e))) \sim (q'_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)$$

in $\mathcal{D}$. Thus

$$\begin{aligned} p &\sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p_+(q_A \cdot g(\iota(p)))(q'_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)(f(\tau(p)) \cdot q'_B)p'_- \\ &\sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p_+((q_A q'_A) \cdot g(\iota(p)))(f(\tau(p)) \cdot (q_B q'_B))p'_- \end{aligned}$$

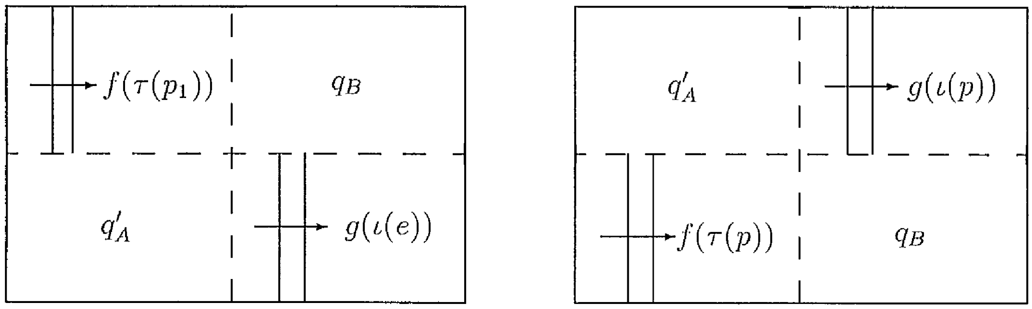


Figure 3.9

with $A(p) = n \geq A(q_A q'_A) + A(q_B q'_B)$. From the left side to the right side of (1) we need at most $m(n-1)$ operations of type (III), and from the left side to the right side of (2) we need at most m operations of type (III). Thus totally we need at most mn operations of type (III). □

Theorem 3.5.8. *If A and B are FDT, then*

$$max\{\tilde{\gamma}_A^{(2)},\ \tilde{\gamma}_B^{(2)}\} \preceq \tilde{\gamma}_{A\times B}^{(2)} \preceq \overline{max\{\tilde{\gamma}_A^{(2)},\ \tilde{\gamma}_B^{(2)}\} + n^2}.$$

Proof. Because A and B are retracts of $A \times B$, by Theorem 3.2.8 we have

$$\tilde{\gamma}_A^{(2)} \preceq \tilde{\gamma}_{A\times B}^{(2)}, \quad \tilde{\gamma}_B^{(2)} \preceq \tilde{\gamma}_{A\times B}^{(2)}.$$

Thus

$$max\{\tilde{\gamma}_A^{(2)},\ \tilde{\gamma}_B^{(2)}\} \preceq \tilde{\gamma}_{A\times B}^{(2)}.$$

Let $\boldsymbol{X}_3$ and $\boldsymbol{X}_4$ be finite trivializers of $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_B)$, respectively. Then by Lemma 3.5.7 we know that $\boldsymbol{X} = \boldsymbol{X}_1 \cup \boldsymbol{X}_2 \cup \boldsymbol{X}_3 \cup \boldsymbol{X}_4$ is a finite trivializer of $\mathcal{D} = \mathcal{D}(\mathcal{P})$.

Let $\boldsymbol{Y}_n$ be the set of all closed paths p in $\mathcal{D}$ with the property that every arc of p meets a disc and $A(p) \leq n$. Let

$$c = max\{l(r_\varepsilon), l(s_\varepsilon), 1 : r \in R, s \in S, \varepsilon = \pm 1\}.$$

For each $p \in \boldsymbol{Y}_n$, we have

$$max\{l(\iota(e)) : \text{ for all edges } e \text{ of } p\} \leq cn.$$

By Lemma 3.5.7 we have

$$p \sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p_+(q_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)p_-$$

in $\mathcal{D}$, and $A(q_A)+A(q_B) \leq A(p) \leq n$. Because $\iota(p_+) = \tau(p_-)$ and $\tau(p_+) = \iota(p_-) \in X^*Y^*$, by Lemma 3.5.2 we have $p_-p_+ \sim 1$. So $p_- \sim p_+^{-1}$. Thus

$$p \sim_{\boldsymbol{X}_1 \cup \boldsymbol{X}_2} p_+(q_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)p_+^{-1}$$

in $\mathcal{D}$. Since $p_+(q_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)p_+^{-1}$ can be obtained from p by at most $(cn)n$ operations of type (III) and some operations of type (I) and (II), we have

$$\begin{aligned}
\tilde{V}_{\mathcal{P},\boldsymbol{X}}(p) &\leq \tilde{V}_{\mathcal{P}_A,\boldsymbol{X}_3}(q_A) + \tilde{V}_{\mathcal{P}_B,\boldsymbol{X}_4}(q_B) + (cn)n \\
&\leq \tilde{\gamma}^{(2)}_{\mathcal{P}_A,\boldsymbol{X}_3}(A(q_A)) + \tilde{\gamma}^{(2)}_{\mathcal{P}_B,\boldsymbol{X}_4}(A(q_B)) + cn^2 \\
&\leq \overline{max\{\tilde{\gamma}^{(2)}_{\mathcal{P}_A,\boldsymbol{X}_3}, \tilde{\gamma}^{(2)}_{\mathcal{P}_B,\boldsymbol{X}_4}\}}(n) + cn^2.
\end{aligned}$$

Now let q be any closed path in $\mathcal{D}$ with $A(q) = n$. By Lemma 1.7.5 we know that q has a conjugate which is equivalent to a non-skewed spherical picture q' of area n in $\mathcal{D}$. Thus q' has the form as in Figure 1.16, where $p_i \in \boldsymbol{Y}_{n_i}$ $(i = 1, 2, \cdots, m)$ and $\sum_{i=1}^m n_i = n$. So

$$\begin{aligned}
\tilde{V}_{\mathcal{P},\boldsymbol{X}}(q) = \tilde{V}_{\mathcal{P},\boldsymbol{X}}(q') &\leq \sum_{i=1}^m \tilde{V}_{\mathcal{P},\boldsymbol{X}}(p_i) \\
&\leq \sum_{i=1}^m \left(\overline{max\{\tilde{\gamma}^{(2)}_{\mathcal{P}_A,\boldsymbol{X}_3}, \tilde{\gamma}^{(2)}_{\mathcal{P}_B,\boldsymbol{X}_4}\}}(n_i) + cn_i^2 \right) \\
&\leq \overline{max\{\tilde{\gamma}^{(2)}_{\mathcal{P}_A,\boldsymbol{X}_3}, \tilde{\gamma}^{(2)}_{\mathcal{P}_B,\boldsymbol{X}_4}\}}(n) + cn^2.
\end{aligned}$$

Hence $\tilde{\gamma}^{(2)}_{A\times B} \preceq \overline{max\{\tilde{\gamma}^{(2)}_A, \tilde{\gamma}^{(2)}_B\}} + n^2$. □

Theorem 3.5.9. *If A and B are FHT, then*

$$max\{\gamma^{(2)}_A,\ \gamma^{(2)}_B\} \preceq \gamma^{(2)}_{A\times B} \preceq \overline{max\{\gamma^{(2)}_A,\ \gamma^{(2)}_B\}} + n^2.$$

Remark. This theorem generalizes a result in [4] from groups to monoids. The n^2 term for groups needs some ingenuity, whereas it falls out naturally in our proof for monoids.

Proof. By Theorem 3.2.8 we have

$$\gamma_A^{(2)} \preceq \gamma_{A\times B}^{(2)}, \quad \gamma_B^{(2)} \preceq \gamma_{A\times B}^{(2)}.$$

Thus

$$max\{\gamma_A^{(2)},\ \gamma_B^{(2)}\} \preceq \gamma_{A\times B}^{(2)}.$$

Let $\boldsymbol{X}_3'$ and $\boldsymbol{X}_4'$ be finite sets of spherical pictures over $\boldsymbol{\mathcal{P}}_A$ and $\boldsymbol{\mathcal{P}}_B$ such that $\boldsymbol{c}_A := \{z_p + Im\partial_2^A : p \in \boldsymbol{X}_3'\}$ and $\boldsymbol{c}_B := \{z_q + Im\partial_2^B : q \in \boldsymbol{X}_4'\}$ are generating sets of $\pi_2^{(b)}(\boldsymbol{\mathcal{P}}_A)$ and $\pi_2^{(b)}(\boldsymbol{\mathcal{P}}_B)$, respectively. Let $\boldsymbol{c} = \{z_p + Im\partial_2 : p \in \boldsymbol{X}_1 \cup \boldsymbol{X}_2 \cup \boldsymbol{X}_3' \cup \boldsymbol{X}_4'\}$. Let $\boldsymbol{Y}_n$ and c be as in the proof of Theorem 3.5.8. For each $p \in \boldsymbol{Y}_n$, by Lemmas 3.5.7 and 3.5.2 we have

$$p \sim_{\boldsymbol{X}_1\cup\boldsymbol{X}_2} p_+(q_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)p_+^{-1}$$

in $\boldsymbol{\mathcal{D}}$, and $A(q_A) + A(q_B) \leq A(p) \leq n$. So

$$\begin{aligned} z_p + Im\partial_2 = (z_{q_A} + Im\partial_2)\,[g(\iota(p))]_M \quad &+ \quad [f(\tau(p))]_M\,(z_{q_B} + Im\partial_2) \\ &+ \quad \sum_{j=1}^{m_1} \varepsilon_j d_j(z_j + Im\partial_2)d_j', \end{aligned}$$

where $z_j = z_{p_j}$ for some $p_j \in \boldsymbol{X}_1 \cup \boldsymbol{X}_2$, $\varepsilon_j = \pm 1$, $d_j, d_j' \in M$, and $m_1 \leq (cn)n$ (for $p_+(q_A \cdot g(\iota(p)))(f(\tau(p)) \cdot q_B)p_+^{-1}$ can be obtained from p by at most $(cn)n$ operations of type (III) and some operations of type (I) and (II)). Let

$$z_{q_A} + Im\partial_2^A = \sum_{i=1}^{m_2} \varepsilon_i a_i(z_i + Im\partial_2^A)a_i',$$

$$z_{q_B} + Im\partial_2^B = \sum_{k=1}^{m_3} \varepsilon_k b_k(z_k + Im\partial_2^B)b_k',$$

where $\varepsilon_i = \pm 1$, $a_i, a_i' \in A$, $z_i = z_{p_i}$ for some $p_i \in \boldsymbol{X}_3'$, $\varepsilon_k = \pm 1$, $b_k, b_k' \in B$ and $z_k = z_{p_k}$ for some $p_k \in \boldsymbol{X}_4'$, and $m_2 = V_{\boldsymbol{\mathcal{P}}_{A},\boldsymbol{c}_A}(q_A)$, $m_3 = V_{\boldsymbol{\mathcal{P}}_{B},\boldsymbol{c}_B}(q_B)$. Then

$$\begin{aligned} z_{q_A} + Im\partial_2 &= \phi_A^*(z_{q_A} + Im\partial_2^A) \\ &= \phi_A^*\Big(\sum_{i=1}^{m_2} \varepsilon_i a_i(z_i + Im\partial_2^A)a_i'\Big) \\ &= \sum_{i=1}^{m_2} \varepsilon_i a_i(z_i + Im\partial_2)a_i', \end{aligned}$$

$$\begin{aligned} z_{q_B} + Im\partial_2 &= \phi_B^*(z_{q_B} + Im\partial_2^B) \\ &= \phi_B^*\Big(\sum_{k=1}^{m_3} \varepsilon_k b_k (z_k + Im\partial_2^B) b'_k\Big) \\ &= \sum_{k=1}^{m_3} \varepsilon_k b_k (z_k + Im\partial_2) b'_k. \end{aligned}$$

Thus

$$\begin{aligned} V_{\boldsymbol{\mathcal{P}},c}(p) \leq m_1 + m_2 + m_3 &\leq V_{\boldsymbol{\mathcal{P}}_{A,c_A}}(q_A) + V_{\boldsymbol{\mathcal{P}}_{B,c_B}}(q_B) + (cn)n \\ &\leq \gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{A,c_A}}(A(q_A)) + \gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{B,c_B}}(A(q_B)) + cn^2 \\ &\leq \overline{max\{\gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{A,c_A}}, \gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{B,c_B}}\}}(n) + cn^2. \end{aligned}$$

Now let q be any closed path in $\boldsymbol{\mathcal{D}}$ with $A(q) = n$. By Lemma 1.7.5 we know that q has a conjugate which is equivalent to a non-skewed spherical picture q' of area n in $\boldsymbol{\mathcal{D}}$. Thus q' has the form as in Figure 1.16, where $p_i \in \boldsymbol{Y}_{n_i}$ $(i = 1, 2, \cdots, m)$ and $\sum_{i=1}^m n_i = n$. So

$$\begin{aligned} V_{\boldsymbol{\mathcal{P}},c}(q) = V_{\boldsymbol{\mathcal{P}},c}(q') &\leq \sum_{i=1}^m V_{\boldsymbol{\mathcal{P}},c}(p_i) \\ &\leq \sum_{i=1}^m \Big(\overline{max\{\gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{A,c_A}}, \gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{B,c_B}}\}}(n_i) + cn_i^2 \Big) \\ &\leq \overline{max\{\gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{A,c_A}}, \gamma^{(2)}_{\boldsymbol{\mathcal{P}}_{B,c_B}}\}}(n) + cn^2. \end{aligned}$$

Hence $\gamma^{(2)}_{A \times B} \preceq \overline{max\{\gamma^{(2)}_A, \gamma^{(2)}_B\}} + n^2$. □

Chapter 4

FDT, FHT and second order Dehn functions of monoids II: Small extensions and relative monoids

Abstract

In this chapter we continue to consider FDT, FHT and second order Dehn functions for some monoid constructions, such as small extensions and relative monoids. We get that the class of FDT monoids and the class of FHT monoids are closed under small extensions. For a relative monoid $S = S(\mathcal{R})$ with a coefficient group H, if $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle-free, then S is FDT (respectively, FHT) if and only if H is. We also get some relations between the second order Dehn functions of S and H.

4.1 Small extensions of monoids

4.1.1 FDT for small extensions of monoids

Suppose a monoid S is a small extension of a monoid T, that is, T is a submonoid of finite index in S. Let $\mathcal{P}_T = [A; R_1]$ be a finite presentation for T. Then S has a finite presentation

$$\mathcal{P}_S = [A, S\backslash T; R_1, R_2],$$

where $R_2 = \{(sa,\ \rho(s,a)),\ (as,\ \lambda(a,s)),\ (ss',\ \pi(s,s')) : s, s' \in S\backslash T,\ a \in A\}$ for certain elements $\rho(s,a), \lambda(a,s), \pi(s,s') \in A^* \cup (S\backslash T)$. Let Γ_T and Γ_S denote the graphs associated with $\mathcal{P}_T$ and $\mathcal{P}_S$ respectively (Γ_T can be considered as a subgraph of Γ_S). Let Γ denote the subgraph of Γ_S which has the same vertex set as Γ_S, but which contains only those edges (U, r, ε, V) of Γ_S with $r \in R_2$, $U, V \in (A \cup (S\backslash T))^*$, $\varepsilon = \pm 1$. By $P_+(\Gamma)$ (respectively $P_-(\Gamma)$) we denote the set of those paths in Γ that only contain edges of the form $(U, r, +1, V)$ (respectively $(U, r, -1, V)$) .

Then Lemma 2.3.3 can be written as the following

Lemma 4.1.1. *For every $U \in (A \cup (S\backslash T))^*$, there is a path $p \in P_+(\Gamma)$ from U to some $U' \in A^* \cup (S\backslash T)$.*

For any $r \in R_1 \cup R_2$ and $s_1, s_2 \in (S\backslash T) \cup \{\varnothing\}$, by Lemma 4.1.1 we can choose a path $p_{s_1, r_{+1}, s_2} \in P_+(\Gamma)$ from $s_1 r_{+1} s_2$ to some $V \in A^* \cup (S\backslash T)$, and a path $p_{s_1, r_{-1}, s_2} \in P_+(\Gamma)$ from $s_1 r_{-1} s_2$ to some $V' \in A^* \cup (S\backslash T)$. Since $s_1 r_{+1} s_2 = s_1 r_{-1} s_2$ holds in S, we have $V = V'$ holds in S. Thus either $V \equiv V' \in S\backslash T$, or $V, V' \in A^*$ and $V = V'$ holds in T. So there is a path $q_{s_1, r, s_2} \in \overline{P(\Gamma_T)}$ from V to V', where

$$\overline{P(\Gamma_T)} := P(\Gamma_T) \cup \{1_s : s \in S\backslash T\}.$$

Let

$$C_1' = \{((s_1, r, +1, s_2), p_{s_1, r_{+1}, s_2} q_{s_1, r, s_2} p^{-1}_{s_1, r_{-1}, s_2}) : s_1, s_2 \in (S\backslash T) \cup \{\varnothing\}, r \in R_1 \cup R_2\}.$$

Then $C_1' \subset P^{(2)}(\Gamma_S)$.

For any $s \in S\backslash T$, $x, x' \in A \cup (S\backslash T)$, $s_1, s_2 \in (S\backslash T) \cup \{\varnothing\}$, we can choose a path $p_{s_1, \phi(x,s), x', s_2} \in P_+(\Gamma)$ from $s_1 \phi(x,s) x' s_2$ to some $V \in A^* \cup (S\backslash T)$, and a path $p_{s_1, x, \psi(s,x'), s_2} \in P_+(\Gamma)$ from $s_1 x \psi(s,x') s_2$ to some $V' \in A^* \cup (S\backslash T)$, where $\phi(x,s) = \lambda(x,s)$ if $x \in A$, or $\phi(x,s) = \pi(x,s)$ if $x \in S\backslash T$; $\psi(s,x') = \rho(s,x')$ if $x' \in A$, or $\psi(s,x') = \pi(s,x')$ if $x' \in S\backslash T$. Since $s_1 \phi(x,s) x' s_2 = s_1 x \psi(s,x') s_2 (= s_1 x s x' s_2)$ holds in S, we have

$V \equiv V' \in S\backslash T$, or $V, V' \in A^*$ and $V = V'$ holds in T. So there is a path $q_{s_1,x,s,x',s_2} \in \overline{P(\Gamma_T)}$ from V' to V. Let

$$C_2' = \{(p'_{s_1,x,s,x',s_2}, p''_{s_1,x,s,x',s_2}) : s \in S\backslash T, x, x' \in A \cup (S\backslash T), s_1, s_2 \in (S\backslash T) \cup \{\varnothing\}\},$$

where $p'_{s_1,x,s,x',s_2} = (s_1, (xs, \phi(x,s)), +1, x's_2)p_{s_1,\phi(x,s),x',s_2}$,

$p''_{s_1,x,s,x',s_2} = (s_1x, (sx', \psi(s,x')), +1, s_2)p_{s_1,x,\psi(s,x'),s_2}q_{s_1,x,s,x',s_2}$.

Then $C_2' \subset P^{(2)}(\Gamma_S)$.

For any $s, s' \in S\backslash T,\ a \in A,\ s_1, s_2 \in (S\backslash T) \cup \{\varnothing\}$, we can choose a path $p_{s_1,\rho(s,a),s',s_2} \in P_+(\Gamma)$ from $s_1\rho(s,a)s's_2$ to some $V \in A^* \cup (S\backslash T)$, and a path $p_{s_1,s,\lambda(a,s'),s_2} \in P_+(\Gamma)$ from $s_1s\lambda(a,s')s_2$ to some $V' \in A^* \cup (S\backslash T)$. Since $s_1\rho(s,a)s's_2 = s_1s\lambda(a,s')s_2 (= s_1sas's_2)$ holds in S, we have $V \equiv V' \in S\backslash T$, or $V, V' \in A^*$ and $V = V'$ holds in T. So there is a path $q_{s_1,s,a,s',s_2} \in \overline{P(\Gamma_T)}$ from V' to V. Let

$$C_3' = \{(q'_{s_1,s,a,s',s_2}, q''_{s_1,s,a,s',s_2}) : a \in A, s, s' \in S\backslash T, s_1, s_2 \in (S\backslash T) \cup \{\varnothing\}\} \subset P^{(2)}(\Gamma_S),$$

where $q'_{s_1,s,a,s',s_2} = (s_1, (sa, \rho(s,a)), +1, s's_2)p_{s_1,\rho(s,a),s',s_2}$,

$q''_{s_1,s,a,s',s_2} = (s_1s, (as', \lambda(a,s')), +1, s_2)p_{s_1,s,\lambda(a,s'),s_2}q_{s_1,s,a,s',s_2}$.

Let

$$C' = C_1' \cup C_2' \cup C_3' \subset P^{(2)}(\Gamma_S).$$

Lemma 4.1.2. *For any edge e in Γ_S, there exist paths $p_+ \in P_+(\Gamma)$, $p_- \in P_-(\Gamma)$ and $q \in \overline{P(\Gamma_T)}$, such that $\tau(p_+), \iota(p_-) \in A^* \cup (S\backslash T)$, and $e \simeq_{C_1'} p_+qp_-$.*

Proof. Let $e = (U, r, \varepsilon, V)$, where $U, V \in (A \cup (S\backslash T))^*$, $r \in R_1 \cup R_2$, $\varepsilon = \pm 1$. By Lemma 4.1.1, there are paths $p_1 \in P_+(\Gamma)$ from U to some $U' \in A^* \cup (S\backslash T)$, and $p_2 \in P_+(\Gamma)$ from V to some $V' \in A^* \cup (S\backslash T)$. By use of Pull-down and push-up, we have

$$e \simeq_{C_1'} (p_1 \cdot r_\varepsilon V)(U'r_\varepsilon \cdot p_2)(U', r, \varepsilon, V')(p_1^{-1} \cdot r_{-\varepsilon}V')(Ur_{-\varepsilon} \cdot p_2^{-1}).$$

(See Figure 4.1.)

By the construction of C_1', it is easy to see that

$$(U', r, \varepsilon, V') \simeq_{C_1'} p_3qp_4^{-1},$$

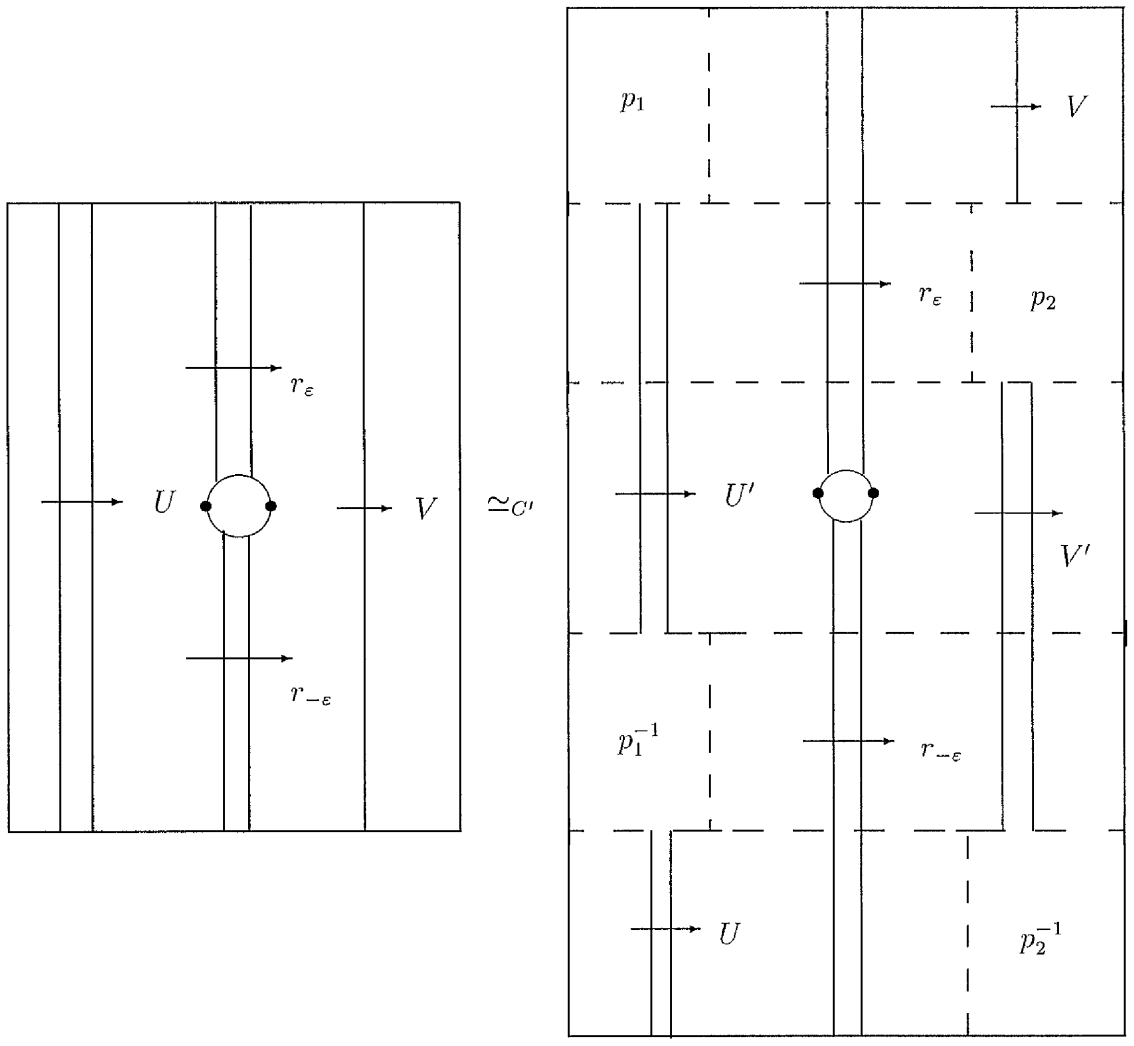


Figure 4.1

where $p_3, p_4 \in P_+(\Gamma)$, $q \in \overline{P(\Gamma_T)}$. Thus

$$e \simeq_{C_1'} p_+ q p_-,$$

where $p_+ = (p_1 \cdot r_\varepsilon V)(U' r_\varepsilon \cdot p_2)p_3 \in P_+(\Gamma)$, $p_- = p_4^{-1}(p_1^{-1} \cdot r_{-\varepsilon} V')(U r_{-\varepsilon} \cdot p_2^{-1}) \in P_-(\Gamma)$. $\square$

Lemma 4.1.3. *For any two edges e_1, e_2 in $P_+(\Gamma)$, if $\iota(e_1) = \iota(e_2)$, then there are paths $p, p' \in P_+(\Gamma)$, and $q \in \overline{P(\Gamma_T)}$, such that $\tau(p), \tau(p') \in A^* \cup (S \backslash T)$, and $e_1 p \simeq_{C'} e_2 p' q$.*

Proof. If $e_1 = e_2$, the result is true. Suppose $e_1 \neq e_2$. Since $\iota(e_1) = \iota(e_2)$, only the following three cases can occur.

1). $e_1 = (U_1, r_1, +1, U_2 r_{2,+1} U_3)$, $e_2 = (U_1 r_{1,+1} U_2, r_2, +1, U_3)$, where $U_1, U_2, U_3 \in (A \cup (S\backslash T))^*$, $r_1 = (r_{1,+1}, r_{1,-1}) \in R_2$ and $r_2 = (r_{2,+1}, r_{2,-1}) \in R_2$.

2). $e_1 = (U_1, (xs, \phi(x,s)), +1, x'U_2)$, $e_2 = (U_1 x, (sx', \psi(s,x'), +1, U_2)$, where $U_1, U_2 \in (A \cup (S\backslash T))^*$, $x, x' \in A \cup (S\backslash T)$, $s \in S\backslash T$, $\phi(x,s) = \lambda(x,s)$ if $x \in A$, or $\phi(x,s) = \pi(x,s)$ if $x \in S\backslash T$; $\psi(s,x') = \rho(s,x')$ if $x' \in A$, or $\psi(s,x') = \pi(s,x')$ if $x' \in S\backslash T$.

3). $e_1 = (U_1, (sa, \rho(s,a)), +1, s'U_2)$, $E_2 = (U_1 s, (as', \lambda(a,s')), +1, U_2)$, where $U_1, U_2 \in (A \cup (S\backslash T))^*$, $a \in A$, $s, s' \in S\backslash T$.

For the case 1), we have $e_1 e_1' \simeq_{C'} e_2 e_2'$, where $e_1' = (U_1 r_{1,-1} U_2, r_2, +1, U_3) \in P_+(\Gamma)$, $e_2' = (U_1, r_1, +1, U_2 r_{2,-1} U_3) \in P_+(\Gamma)$. By Lemma 4.1.1 there is a path $p_1 \in P_+(\Gamma)$ from $\tau(e_1') = \tau(e_2')$ to some $V \in A^* \cup (S\backslash T)$. Let $p = e_1' p_1 \in P_+(\Gamma)$, $p' = e_2' p_1 \in P_+(\Gamma)$. Then $\tau(p), \tau(p') \in A^* \cup (S\backslash T)$, and $e_1 p \simeq_{C'} e_2 p'$.

For the case 2), we can choose paths $p_1 \in P_+(\Gamma)$ from U_1 to some $V_1 \in A^* \cup (S\backslash T)$, and $p_2 \in P_+(\Gamma)$ from U_2 to some $V_2 \in A^* \cup (S\backslash T)$. Then we have

$$e_1(U_1\phi(x,s)x' \cdot p_2)(p_1 \cdot \phi(x,s)x'V_2) \simeq_{C'} p_3(V_1, (xs, \phi(x,s)), +1, x'V_2)$$

with $p_3 = (p_1 \cdot xsx'U_2)(V_1 xsx' \cdot p_2) \in P_+(\Gamma)$ (see Figure 4.2), and

$$e_2(U_1 x\psi(s,x') \cdot p_2)(p_1 \cdot x\psi(s,x')V_2) \simeq_{C'} p_3(V_1 x, (sx', \psi(s,x')), +1, V_2).$$

By the construction of C_2', it is easy to see that there exist paths $p_4, p_5 \in P_+(\Gamma)$ and $q \in \overline{P(\Gamma_T)}$, such that

$$(V_1, (xs, \phi(x,s)), +1, x'V_2)p_4 \simeq_{C'} (V_1 x, (sx', \psi(s,x')), +1, V_2)p_5 q.$$

Let

$$p = (U_1\phi(x,s)x' \cdot p_2)(p_1 \cdot \phi(x,s)x'V_2)p_4 \in P_+(\Gamma),$$

$$p' = (U_1 x\psi(s,x') \cdot p_2)(p_1 \cdot x\psi(s,x')V_2)p_5 \in P_+(\Gamma).$$

Then $e_1 p \simeq_{C'} e_2 p' q$, and $\tau(p), \tau(p') \in A^* \cup (S\backslash T)$.

For the case 3), by the construction of C_3', we can get the result (by an argument similar to that used in case 2)). $\square$

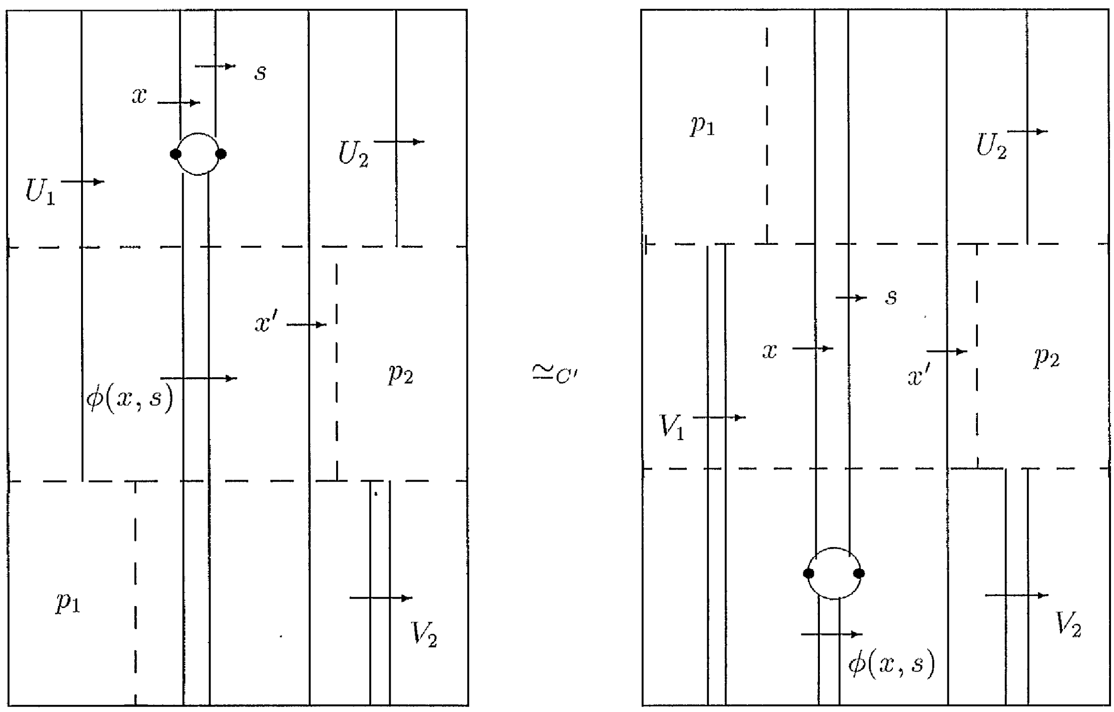


Figure 4.2

For any $V \in (A \cup (S\backslash T))^*$, let K and $\alpha(V)$ be defined as in Section 2.3.

For any edge $e = (U, r, +1, V)$ in $P_+(\Gamma)$, since $r \in R_2$, we have $\alpha(r_{+1}) \geq K+1 > K \geq \alpha(r_{-1})$. So $\alpha(Ur_{+1}V) = \alpha(U) + \alpha(r_{+1}) + \alpha(V) > \alpha(U) + \alpha(r_{-1}) + \alpha(V) = \alpha(Ur_{-1}V)$. Thus $\alpha(\iota(e)) > \alpha(\tau(e))$.

Lemma 4.1.4. *For any two paths $p_1, p_2 \in P_+(\Gamma)$, if $\iota(p_1) = \iota(p_2)$, then there are paths $p_1', p_2' \in P_+(\Gamma)$ and $q \in \overline{P(\Gamma_T)}$ such that $\tau(p_1'), \tau(p_2') \in A^* \cup (S\backslash T)$ and $p_1p_1' \simeq_{C'} p_2p_2'q$.*

Proof. We will prove the result by induction on the non-negative integer $\alpha(\iota(p_1))$.

If $\alpha(\iota(p_1)) = 0$, the result is true. Suppose the result is true for the case $\alpha(\iota(p_1)) < n$. We consider the case of $\alpha(\iota(p_1)) = n$.

If one of p_i is the empty path, the result is true. So let $p_1 = e_1\overline{p_1}$, $p_2 = e_2\overline{p_2}$, where e_i are deges in $P_+(\Gamma)$ and $\overline{p_i} \in P_+(\Gamma)$. By Lemma 4.1.3, there are paths $p', p'' \in P_+(\Gamma)$ and $q' \in \overline{P(\Gamma_T)}$ such that $\tau(p'), \tau(p'') \in A^* \cup (S\backslash T)$ and

$$(1) \qquad e_1 p' \simeq_{C'} e_2 p'' q'.$$

Now $\overline{p_1}, p' \in P_+(\Gamma)$, $\iota(\overline{p_1}) = \iota(p')$, and $\alpha(\iota(\overline{p_1})) = \alpha(\tau(e_1)) < \alpha(\iota(e_1)) = \alpha(\iota(p_1)) = n$, so by the induction hypothesis and the fact that $\tau(p') \in A^* \cup (S\backslash T)$, there are paths $p_1' \in P_+(\Gamma)$ and $q_1 \in \overline{P(\Gamma_T)}$ such that $\tau(p_1') \in A^* \cup (S\backslash T)$ and

$$(2) \qquad \overline{p_1} p_1' \simeq_{C'} p' q_1.$$

Similarly, there are paths $p_2' \in P_+(\Gamma)$ and $q \in \overline{P(\Gamma_T)}$ such that $\tau(p_2') \in A^* \cup (S\backslash T)$ and

$$(3) \qquad \overline{p_2} p_2' \simeq_{C'} p'' q_2.$$

Now by (1), (2) and (3) we have

$$p_1 p_1' = e_1 \overline{p_1} p_1' \simeq_{C'} e_1 p' q_1 \simeq_{C'} e_2 p'' q' q_1,$$

$$p_2 p_2' = e_2 \overline{p_2} p_2' \simeq_{C'} e_2 p'' q_2.$$

Let $q = q_2^{-1} q' q_1$. Then $q \in \overline{P(\Gamma_T)}$, and $p_1 p_1' \simeq_{C'} e_2 p'' q_2 q_2^{-1} q' q_1 \simeq_{C'} p_2 p_2' q$. □

Corollary 4.1.5. *For any two paths $p_1, p_2 \in P_+(\Gamma)$, if $\iota(p_1) = \iota(p_2)$ and $\tau(p_1), \tau(p_2) \in A^* \cup (S\backslash T)$, then there is a path $q \in \overline{P(\Gamma_T)}$ such that $p_1 \simeq_{C'} p_2 q$.*

Lemma 4.1.6. *For any path p in Γ_S, there exist paths $p_+ \in P_+(\Gamma)$, $p_- \in P_-(\Gamma)$ and $q \in \overline{P(\Gamma_T)}$ such that $\tau(p_+), \iota(p_-) \in A^* \cup (S\backslash T)$ and $p \simeq_{C'} p_+ q p_-$.*

Proof. Let $p = e_1 e_2 \cdots e_m$, where $e_1, e_2, \cdots, e_m$ are edges in Γ_S. By Lemma 4.1.2, there are paths $p_i, p_i' \in P_+(\Gamma)$ and $q_i \in \overline{P(\Gamma_T)}$, such that $\tau(p_i), \tau(p_i') \in A^* \cup (S\backslash T)$ and $e_i \simeq_{C'} p_i' q_i p_i^{-1}$ $(i = 1, 2, \cdots, m)$. Thus

$$p \simeq_{C'} p_1' q_1 p_1^{-1} p_2' q_2 p_2^{-1} \cdots p_m' q_m p_m^{-1}.$$

Since $\iota(p_i) = \iota(p_{i+1}')$ and $\tau(p_i), \tau(p_{i+1}') \in A^* \cup (S\backslash T)$, by the Corollary 4.1.5, there is a path $q_i' \in \overline{P(\Gamma_T)}$, such that $p_{i+1}' \simeq_{C'} p_i q_i'$, so $p_i^{-1} p_{i+1}' \simeq_{C'} q_i'$ $(i = 1, 2, \cdots, m)$. Thus

$$p \simeq_{C'} p_1' q p_m^{-1},$$

where $q = q_1q_1'q_2q_2' \cdots q_{m-1}'q_m \in \overline{P(\Gamma_T)}$. □

Suppose $\simeq_B = P^{(2)}(\Gamma_T)$ for some $B \subset P^{(2)}(\Gamma_T)$. Let $\overline{C} = B \cup C'$.

Lemma 4.1.7. $\simeq_{\overline{C}} = P^{(2)}(\Gamma_S)$.

Proof. For any $(p_1, p_2) \in P^{(2)}(\Gamma_S)$, by Lemma 4.1.6, there are paths $p_+, p_+' \in P_+(\Gamma)$, $p_-, p_-' \in P_-(\Gamma)$ and $q_1, q_2 \in \overline{P(\Gamma_T)}$ such that $\tau(p_+), \tau(p_+'), \iota(p_-), \iota(p_-') \in A^* \cup (S\backslash T)$, $p_1 \simeq_{\overline{C}} p_+q_1p_-$ and $p_2 \simeq_{\overline{C}} p_+'q_2p_-'$. If one of $\tau(p_+), \tau(p_+'), \iota(p_-), \iota(p_-')$ is in $S\backslash T$, then $\tau(p_+) = \tau(p_+') = \iota(p_-) = \iota(p_-') = s \in S\backslash T$, and $q_1 = q_2 = 1_s$ (for $\iota(p_+) = \iota(p_+'), \tau(p_-) = \tau(p_-')$). By the Corollary 4.1.5 we can get that $p_+ \simeq_{\overline{C}} p_+', p_- \simeq_{\overline{C}} p_-'$. Hence

$$p_1 \simeq_{\overline{C}} p_+p_- \simeq_{\overline{C}} p_+'p_-' \simeq_{\overline{C}} p_2.$$

Now suppose that $\tau(p_+), \tau(p_+'), \iota(p_-), \iota(p_-') \in A^*$. Then $q_1, q_2 \in P(\Gamma_T)$. By the Corollary 4.1.5, there are paths $q_3, q_4 \in P(\Gamma_T)$ such that $p_+ \simeq_{\overline{C}} p_+'q_3$ and $p_- \simeq_{\overline{C}} q_4^{-1}p_-'$. Since $(q_1, q_3^{-1}q_2q_4) \in P^{(2)}(\Gamma_T)$ and $\simeq_B = P^{(2)}(\Gamma_T)$, we have $q_1 \simeq_{\overline{C}} q_3^{-1}q_2q_4$. Thus

$$p_1 \simeq_{\overline{C}} p_+q_1p_- \simeq_{\overline{C}} p_+'q_3q_3^{-1}q_2q_4q_4^{-1}p_-' \simeq_{\overline{C}} p_+'q_2p_-' \simeq_{\overline{C}} p_2.$$

Therefore $\simeq_{\overline{C}} = P^{(2)}(\Gamma_S)$. □

Theorem 4.1.8. *Let S be a small extension of T. If T is FDT, then so is S.*

This theorem appears in the paper [80]. It answers part of the Open Problem 11.1 posed by Ruskuc ([75]).

We remark that the converse of this theorem remains open.

Proof. If T is FDT, we can choose B finite. Then $\overline{C}$ is finite and so since $\simeq_{\overline{C}} = P^{(2)}(\Gamma_S)$, S is FDT. □

4.1.2 FHT for small extensions of monoids

Theorem 4.1.9. *Let S be a small extension of T. If T is FHT, then so is S.*

Proof. Let $\mathcal{P}_T$, $\mathcal{P}_S$, Γ_T and Γ_S be as in Section 4.1.1. Suppose $\mathcal{P}_T$ is finite, then so is $\mathcal{P}_S$. Let $\mathcal{D}_T = \mathcal{D}(\mathcal{P}_T)$ and $\mathcal{D}_S = \mathcal{D}(\mathcal{P}_S)$. The inclusion map from $\mathcal{P}_T$ to $\mathcal{P}_S$ induces a mapping of 2-complexes $\phi : \mathcal{D}_T \longrightarrow \mathcal{D}_S$. Then the mapping ϕ induces a chain map from the chain complex

$$0 \longrightarrow C_2(\mathcal{D}_T) \xrightarrow{\partial_2^T} C_1(\mathcal{D}_T) \xrightarrow{\partial_1^T} C_0(\mathcal{D}_T) \longrightarrow 0$$

of $\mathcal{D}_T$ to the chain complex

$$0 \longrightarrow C_2(\mathcal{D}_S) \xrightarrow{\partial_2^S} C_1(\mathcal{D}_S) \xrightarrow{\partial_1^S} C_0(\mathcal{D}_S) \longrightarrow 0$$

of $\mathcal{D}_S$, and this chain map then induces a homomorphism

$$\phi^* : \pi_2^{(b)}(\mathcal{P}_T) \longrightarrow \pi_2^{(b)}(\mathcal{P}_S),\ z_p + Im\partial_2^T \longmapsto z_p + Im\partial_2^S,$$

where $p = e_1^{\varepsilon_1} \cdots e_k^{\varepsilon_k}$ is a closed path in $\mathcal{D}_T$ (e_i a positive edge, $\varepsilon_i = \pm 1,\ i = 1, \cdots, k$) and $z_p = \sum_{i=1}^k \varepsilon_i e_i$.

If T is FHT, then there exists a finite set $\boldsymbol{X}_1$ of spherical monoid pictures over $\mathcal{P}_T$ such that the elements $z_p + Im\partial_2^T$ ($p \in \boldsymbol{X}_1$) generate the bimodule $\pi_2^{(b)}(\mathcal{P}_T)$ (see Chapter 3). Let

$$\boldsymbol{X}_2 = \{pq^{-1} : (p, q) \in C'\},$$

where C' is as in Section 4.1.1. Then $\boldsymbol{X}_2$ is a finite set of spherical monoid pictures over $\mathcal{P}_S$. Let

$$\boldsymbol{X} = \boldsymbol{X}_1 \cup \boldsymbol{X}_2.$$

Then $\boldsymbol{X}$ is a finite set of spherical monoid pictures over $\mathcal{P}_S$. We want to show that the elements $z_p + Im\partial_2^S$ ($p \in \boldsymbol{X}$) generate the bimodule $\pi_2^{(b)}(\mathcal{P}_S)$.

For any spherical monoid picture p over $\mathcal{P}_S$, by Lemma 4.1.6 we have

$$p \simeq_{C'} p_+ q p_-,$$

where $p_+ \in P_+(\Gamma)$, $p_- \in P_-(\Gamma)$ and $q \in \overline{P(\Gamma_T)}$ with $\tau(p_+), \iota(p_-) \in A^* \cup (S\backslash T)$. Since $\iota(p_+) = \iota(p_-^{-1})$, $\tau(p_+), \tau(p_-^{-1}) \in A^* \cup (S\backslash T)$ and $p_+, p_-^{-1} \in P_+(\Gamma)$, by Corollary 4.1.5, there is a path $q' \in \overline{P(\Gamma_T)}$ such that $p_+ \simeq_{C'} p_-^{-1} q'$. Thus

$$p \simeq_{C'} p_-^{-1} q' q p_-,$$

and $q'q$ is a closed path in $\overline{P(\Gamma_T)} = P(\Gamma_T) \cup \{1_s : s \in S\backslash T\}$. So

$$\begin{aligned} z_p + Im\partial_2^S = -(z_{p_-} + Im\partial_2^S) &+ (z_{q'q} + Im\partial_2^S) + (z_{p_-} + Im\partial_2^S) \\ &+ \sum_i \varepsilon_i s_i (z_i + Im\partial_2^S) s_i', \end{aligned}$$

where $\varepsilon_i = \pm 1$, $s_i, s_i' \in S$ and $z_i = z_{p_i}$ for some $p_i \in \boldsymbol{X}_2$. We just need to consider the case that $q'q$ is a closed path in Γ_T. Let

$$z_{q'q} + Im\partial_2^T = \sum_j \varepsilon_j t_j (z_j + Im\partial_2^T) t_j',$$

where $\varepsilon_j = \pm 1$, $t_j, t_j' \in T$, $z_j = z_{p_j}$ for some $p_j \in \boldsymbol{X}_1$. Then we have

$$\begin{aligned} z_{q'q} + Im\partial_2^S &= \phi^*(z_{q'q} + Im\partial_2^T) \\ &= \phi^*\Big(\sum_j \varepsilon_j t_j (z_j + Im\partial_2^T) t_j'\Big) \\ &= \sum_j \varepsilon_j t_j (z_j + Im\partial_2^S) t_j'. \end{aligned}$$

Thus

$$z_p + Im\partial_2^S = \sum_j \varepsilon_j t_j (z_j + Im\partial_2^S) t_j' + \sum_i \varepsilon_i s_i (z_i + Im\partial_2^S) s_i'.$$

Since every element of $\pi_2^{(b)}(\boldsymbol{\mathcal{P}}_S)$ has the form of a finite sum $\sum(z_q + Im\partial_2^S)$ for some spherical pictures q over $\boldsymbol{\mathcal{P}}_S$, the elements $z_p + Im\partial_2^S$ $(p \in \boldsymbol{X})$ generate the bimodule $\pi_2^{(b)}(\boldsymbol{\mathcal{P}}_S)$. Thus S is FHT. □

4.2 Submonoids with ideal complements

Let S be a monoid, and let S_0 be a submonoid of S such that $S\backslash S_0$ is an ideal of S, that is, $s_1 s_2, s_2 s_1 \in S\backslash S_0$ for any $s_1 \in S\backslash S_0$ and $s_2 \in S$.

For each monoid presentation $\boldsymbol{\mathcal{P}} = [X; R]$ of S, let

$$X_0 = \{x \in X : [x] \in S_0\} \subseteq X,$$

$$R_0 = \{r \in R : r_{+1}, r_{-1} \in X_0^*\} \subseteq R.$$

We want to show that $\mathcal{P}_0 = [X_0; R_0]$ is a monoid presentation of S_0, and for any $r \in R\backslash R_0$, both r_{+1} and r_{-1} contain at least one letter in $X\backslash X_0$.

Let

$$\psi : X_0^* \longrightarrow S_0, \quad x_0 \longmapsto [x_0] \ (x_0 \in X_0).$$

Because $R_0 \subseteq R$, ψ can induce a homomorphism

$$\psi_* : X_0^*/\rho_{R_0} \longrightarrow S_0, \quad [x_0]_0 \longmapsto [x_0],$$

where ρ_{R_0} is the congruence relation generated by R_0. We want to show that ψ_* is an isomorphism.

For any $s_0 \in S_0$, let $s_0 = [x_1 x_2 \cdots x_n]$, where $x_i \in X^*$ $(i = 1, 2, \cdots, n)$. If $[x_i] \in S\backslash S_0$ for some i, then $s_0 = [x_1 \cdots x_{i-1}][x_i][x_{i+1} \cdots x_n] \in S\backslash S_0$, a contradiction. So $x_i \in X_0$, $i = 1, 2, \cdots, n$. Thus

$$\psi_*([x_1 x_2 \cdots x_n]_0) = [x_1 x_2 \cdots x_n] = s_0.$$

So ψ_* is surjective.

Let $r \in R$ with $r_\varepsilon \in X^*\backslash X_0^*$. Then $r_\varepsilon = UxV$ for some $x \in X\backslash X_0$, $U, V \in X^*$. So $[r_\varepsilon] = [U][x][V] \in S\backslash S_0$. If $r_{-\varepsilon} \in X_0^*$, then $[r_\varepsilon] = [r_{-\varepsilon}] \in S_0$, a contradiction. Thus for any $r \in R$, we have proved that if $r_\varepsilon \in X^*\backslash X_0^*$, then $r_{-\varepsilon} \in X^*\backslash X_0^*$. That is, for any $r \in R\backslash R_0$, both r_{+1} and r_{-1} contain at least one letter from $X\backslash X_0$.

If $\psi_*([W]_0) = \psi_*([W']_0)$ for some $W, W' \in X_0^*$, then $[W] = [W']$. So there is a sequence

$$W \equiv W_1, \ W_2, \ \cdots, \ W_m \equiv W'$$

such that $W_i \equiv U_i r_{i,\varepsilon_i} V_i$ and $W_{i+1} \equiv U_i r_{i,-\varepsilon_i} V_i$ for some $r_i = (r_{i,+1}, r_{i,-1}) \in R$ and $U_i, V_i \in X^*$ $(\varepsilon_i = \pm 1, \ i = 1, 2, \cdots, m-1)$. Since $W_1 \in X_0^*$, we have $U_1, V_1, r_{1,\varepsilon_1} \in X_0^*$. So $r_{1,-\varepsilon_1} \in X_0^*$, and hence $W_2 \in X_0^*$. Using induction on i we can get that $U_i, V_i \in X_0^*$ and $r_i \in R_0$ for all i. So $[W]_0 = [W']_0$. Thus ψ_* is injective. Hence $\mathcal{P}_0$ is a monoid presentation of S_0. Thus we have the following lemma.

Lemma 4.2.1. *Let S be a monoid, and let S_0 be a submonoid of S such that $S\backslash S_0$ is an ideal of S. Then for each monoid presentation $\mathcal{P} = [X; R]$ of S, we can get a monoid* *presentation $\mathcal{P}_0 = [X_0; R_0]$ of S_0 which is a subpresentation of $\mathcal{P}$ such that both r_{+1} and r_{-1} contain at least one letter from $X \backslash X_0$ for any $r \in R \backslash R_0$.* $\square$

Conversely, let $\mathcal{P} = [X; R]$ be a monoid presentation, and let $\mathcal{P}_0 = [X_0; R_0]$ be a subpresentation of $\mathcal{P}$ such that for any $r \in R \backslash R_0$, both r_{+1} and r_{-1} contain at least one letter from $X \backslash X_0$. The inclusion map $\phi : \mathcal{P}_0 \longrightarrow \mathcal{P}$ induces a homomorphism

$$\phi_* : S_0 \longrightarrow S,$$

where S_0 and S are the monoids defined by $\mathcal{P}_0$ and $\mathcal{P}$, respectively. A typical element of S_0 is an equivalence class $[W]_0$, where W is a word on X_0; and a typical element of S is an equivalence class $[W]$, where W is a word on X. For any $W_0 \in X_0^*$ and $W \in X^* \backslash X_0^*$, there is no path in $\mathcal{D}(\mathcal{P})$ from W_0 to W (using the assumption on $R \backslash R_0$). So if two words $W_0, W_0' \in X_0^*$ are equivalent in $\mathcal{P}$, then they are also equivalent in $\mathcal{P}_0$. Thus ϕ_* is injective. So we can consider S_0 as a submonoid of S. For any $s_1 \in S \backslash S_0$ and $s_2 \in S$, we have $s_1 = [W_1]$ and $s_2 = [W_2]$ for some $W_1 \in X^* \backslash X_0^*$ and $W_2 \in X^*$. Since $W_1 W_2 \in X^* \backslash X_0^*$ and $W_2 W_1 \in X^* \backslash X_0^*$, we have $s_1 s_2 = [W_1 W_2] \in S \backslash S_0$ and $s_2 s_1 = [W_2 W_1] \in S \backslash S_0$. Thus $S \backslash S_0$ is an ideal of S.

Lemma 4.2.2. *Let $\mathcal{P}_0 = [X_0; R_0]$ be a subpresentation of $\mathcal{P} = [X; R]$ such that both r_{+1} and r_{-1} contain at least one letter from $X \backslash X_0$ for any $r \in R \backslash R_0$. Let S_0 and S be the monoids defined by $\mathcal{P}_0$ and $\mathcal{P}$, respectively. Then we can consider S_0 as a submonoid of S, and $S \backslash S_0$ is an ideal of S.* $\square$

Proposition 4.2.3. *Let S be a monoid, and let S_0 be a submonoid of S such that $S \backslash S_0$ is an ideal of S. If S is FDT, then so is S_0, and*

$$\tilde{\gamma}_{S_0}^{(2)} \preceq \tilde{\gamma}_S^{(2)}.$$

Proof. Let $\mathcal{P} = [X; R]$ be a finite monoid presentation of S. Then we can get a finite monoid presentation $\mathcal{P}_0 = [X_0; R_0]$ of S_0 which is a subpresentation of $\mathcal{P}$ such that both r_{+1} and r_{-1} contain at least one letter from $X \backslash X_0$ for any $r \in R \backslash R_0$. Let $\mathcal{D}(\mathcal{P})$ and $\mathcal{D}(\mathcal{P}_0)$ be the associated 2-complexes. Let $\boldsymbol{X}$ be a finite set of spherical pictures over $\mathcal{P}$

which trivialize $\mathcal{D}(\mathcal{P})$, and let $\boldsymbol{X}_0$ be the subset of $\boldsymbol{X}$ which contains just those pictures over $\mathcal{P}_0$. We want to show that $\boldsymbol{X}_0$ is a finite trivializer of $\mathcal{D}(\mathcal{P}_0)$.

Given any spherical monoid picture p over $\mathcal{P}_0$, p is also a spherical monoid picture over $\mathcal{P}$. Thus p is equivalent to the empty path 1 in $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}}$, so there is a sequence of pictures

$$p = p_1, p_2, \cdots, p_{n-1}, p_n = 1,$$

such that p_{i+1} is obtained from p_i by one operation σ_i over $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}}$, where σ_i is inserting or deleting an inverse pair ee^{-1} or a subpicture of form $[e_1, e_2]$ or $W \cdot q^{\pm 1} \cdot V$ (e, e_1, e_2 are atomic pictures over $\mathcal{P}$, $q \in \boldsymbol{X}$, and W, V are words on X). Considering σ_1, since all vertices in the path p are words on X_0, we have that $\iota(e), \iota(e_1), \iota(e_2), \iota(q), W$ and V must be words on X_0. Thus e, e_1, e_2 and q contain only X_0-arcs and R_0-discs (using the assumption on $R \backslash R_0$). So e, e_1, e_2 and q are pictures over $\mathcal{P}_0$, and $q \in \boldsymbol{X}_0$. We can get the same results inductively for $\sigma_i, i = 2, 3, \cdots, n$. Hence p is equivalent to the empty path in $\mathcal{D}(\mathcal{P}_0)^{\boldsymbol{X}_0}$. Therefore, $\boldsymbol{X}_0$ is a finite trivializer of $\mathcal{D}(\mathcal{P}_0)$, and hence $\mathcal{P}_0$ is FDT.

Suppose $\tilde{V}_{\mathcal{P},\boldsymbol{X}}(p) = k$ and p is equivalent to

$$p' := p_1(U_1 \cdot q_1^{\varepsilon_1} \cdot V_1)p_1^{-1} p_2(U_2 \cdot q_2^{\varepsilon_2} \cdot V_2)p_2^{-1} \cdots p_k(U_k \cdot q_k^{\varepsilon_k} \cdot V_k)p_k^{-1}$$

in $\mathcal{D}(\mathcal{P})$, where p_i is a path in $\mathcal{D}(\mathcal{P})$, $U_i, V_i \in X^*$, $q_i \in \boldsymbol{X}$, $\varepsilon_i = \pm 1$ $(1 \leq i \leq k)$. Because p contains only X_0-arcs, using the assumption on $R \backslash R_0$ we can get that p_i is a path in $\mathcal{D}(\mathcal{P}_0)$, $U_i, V_i \in X_0^*$, $q_i \in \boldsymbol{X}_0$ $(1 \leq i \leq k)$, and p is equivalent to p' in $\mathcal{D}(\mathcal{P}_0)$. So

$$\tilde{V}_{\mathcal{P}_0,\boldsymbol{X}_0}(p) \leq k = \tilde{V}_{\mathcal{P},\boldsymbol{X}}(p).$$

Thus $\tilde{\gamma}_{S_0}^{(2)} \preceq \tilde{\gamma}_S^{(2)}$. □

Let $\mathcal{P}$ be a monoid presentation, and let $\mathcal{P}_0$ be a subpresentation of $\mathcal{P}$. Let $\mathcal{D}_0 = \mathcal{D}(\mathcal{P}_0)$ and $\mathcal{D} = \mathcal{D}(\mathcal{P})$. The inclusion map from $\mathcal{P}_0$ to $\mathcal{P}$ induces a mapping of 2-complexes $\phi : \mathcal{D}_0 \longrightarrow \mathcal{D}$. Then the mapping ϕ induces a chain map from the chain complex

$$0 \longrightarrow C_2(\mathcal{D}_0) \xrightarrow{\partial_2^0} C_1(\mathcal{D}_0) \xrightarrow{\partial_1^0} C_0(\mathcal{D}_0) \longrightarrow 0$$

of $\mathcal{D}_0$ to the chain complex

$$0 \longrightarrow C_2(\mathcal{D}) \xrightarrow{\partial_2} C_1(\mathcal{D}) \xrightarrow{\partial_1} C_0(\mathcal{D}) \longrightarrow 0$$

of $\mathcal{D}$, and this chain map then induces a homomorphism

$$\phi^* : \pi_2^{(b)}(\mathcal{P}_0) \longrightarrow \pi_2^{(b)}(\mathcal{P}),\ z_p + Im\partial_2^0 \longmapsto z_p + Im\partial_2,$$

where $p = e_1^{\varepsilon_1} \cdots e_k^{\varepsilon_k}$ is a closed path in $\mathcal{D}_0$ (e_i a positive edge, $\varepsilon_i = \pm 1,\ i = 1, \cdots, k$) and $z_p = \sum_{i=1}^k \varepsilon_i e_i$.

Lemma 4.2.4. *Let $\mathcal{P}_0 = [X_0; R_0]$ be a subpresentation of $\mathcal{P} = [X; R]$ such that both r_{+1} and r_{-1} contain at least one letter from $X \backslash X_0$ for each $r \in R \backslash R_0$. Then the above homomorphism ϕ^* is injective.*

Proof. For any $\xi \in Ker\phi^*$, we have

$$\xi = \sum_i s_i z_{p_i} s_i' + Im\partial_2^0,$$

where p_i are some spherical pictures in $\mathcal{D}_0$, $s_i, s_i' \in S_0$. Let $s_i = [U_i]_0$, $s_i' = [V_i]_0$ for some $U_i, V_i \in X_0^*$. Then

$$\xi = \sum_i z_{U_i p_i V_i} + Im\partial_2^0.$$

Because $\phi^*(\xi) = \sum_i z_{U_i p_i V_i} + Im\partial_2$, we have $\sum_i z_{U_i p_i V_i} \in Im\partial_2$. So

$$\begin{aligned} \sum_i z_{U_i p_i V_i} &= \partial_2\Big(\sum_j \varepsilon_j [a_j, b_j] + \sum_k \varepsilon_k [c_k, d_k]\Big) \\ &= \sum_j \varepsilon_j z_{[a_j, b_j]} + \sum_k \varepsilon_k z_{[c_k, d_k]}, \end{aligned}$$

where a_j, b_j are some edges in $\mathcal{D}_0$, and at least one of c_k and d_k is an edge in $\mathcal{D} \backslash \mathcal{D}_0$ for every k. So all edges in the path $[c_k, d_k]$ are in $\mathcal{D} \backslash \mathcal{D}_0$ (note that for every path p in $\mathcal{D}$, if one edge of p is in $\mathcal{D} \backslash \mathcal{D}_0$, then all edges of p are in $\mathcal{D} \backslash \mathcal{D}_0$). Thus we have

$$\sum_i z_{U_i p_i V_i} = \sum_l \varepsilon_l e_l,$$

$$\sum_j z_{[a_j, b_j]} = \sum_m \varepsilon_m' e_m',$$

$$\sum_k z_{[c_k,d_k]} = \sum_n \varepsilon''_n e''_n,$$

where $\varepsilon_l, \varepsilon'_m, \varepsilon''_n = \pm 1$, e_l, e'_m are some positive edges in $\mathcal{D}_0$ and e''_n are some positive edges in $\mathcal{D}\backslash\mathcal{D}_0$. Then

$$\sum_l \varepsilon_l e_l = \sum_m \varepsilon'_m e'_m + \sum_n \varepsilon''_n e''_n.$$

So $\sum_n \varepsilon''_n e''_n = 0$, that is $\sum_k z_{[c_k,d_k]} = 0$. Thus

$$\sum_i z_{U_i p_i V_i} = \partial_2\Big(\sum_j [a_j, b_j]\Big) = \partial_2^0\Big(\sum_j [a_j, b_j]\Big) \in Im\partial_2^0.$$

So $\xi = 0 + Im\partial_2^0$. Thus $Ker\phi^* = 0$. □

Proposition 4.2.5. *Let S be a monoid, and let S_0 be a submonoid of S such that $S\backslash S_0$ is an ideal of S. If S is FHT, then so is S_0, and*

$$\gamma_{S_0}^{(2)} \preceq \gamma_S^{(2)}.$$

Proof. Let $\mathcal{P} = [X; R]$ be a finite monoid presentation of S. Then we can get a finite monoid presentation $\mathcal{P}_0 = [X_0; R_0]$ of S_0 which is a subpresentation of $\mathcal{P}$ such that both r_{+1} and r_{-1} contain at least one letter from $X\backslash X_0$ for any $r \in R\backslash R_0$. If S is FHT, then there is a finite set $\boldsymbol{X}$ of spherical monoid pictures over $\mathcal{P}$ such that $\boldsymbol{c} = \{z_p + Im\partial_2 : p \in \boldsymbol{X}\}$ is a generating set for $\pi_2^{(b)}(\mathcal{P})$. Let

$$\boldsymbol{c}_0 = \{z_{p_0} + Im\partial_2^0 : p_0 \in \boldsymbol{X}_0\},$$

where $\boldsymbol{X}_0$ is the subset of $\boldsymbol{X}$ which contains just those pictures over $\mathcal{P}_0$. We want to show that $\boldsymbol{c}_0$ is a finite generating set for $\pi_2^{(b)}(\mathcal{P}_0)$.

Let P be the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule with basis $\{t_r : r \in R\}$ in one-to-one correspondence with R. Then there is a $(\mathbb{Z}S, \mathbb{Z}S)$-homomorphism

$$\rho : \pi_2^{(b)}(\mathcal{P}) \longrightarrow P$$

defined in Section 3.1.2. That is, if

$$\alpha = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_n e_n$$

is an element of $Ker\partial_1$, where $e_i = (U_i, r_i, +1, V_i)$ say, $\varepsilon_i = \pm 1$, $(i = 1, 2, \cdots, n)$, then

$$\rho(\alpha + Im\partial_2) = \sum_{i=1}^{n} \varepsilon_i [U_i] t_{r_i} [V_i].$$

For each spherical monoid picture p_0 over $\mathcal{P}_0$, p_0 is also a spherical monoid picture over $\mathcal{P}$. Suppose $V_{\mathcal{P},\boldsymbol{c}}(p_0) = m$, and

$$z_{p_0} + Im\partial_2 = (\varepsilon_1 s_1 \xi_1 s_1' + \cdots + \varepsilon_k s_k \xi_k s_k') + (\varepsilon_{k+1} s_{k+1} \xi_{k+1} s_{k+1}' + \cdots + \varepsilon_m s_m \xi_m s_m'),$$

where $0 \le k \le m$, $s_1, s_1', \cdots, s_k, s_k' \in S_0$, $\xi_1, \cdots, \xi_k \in \phi^*(\boldsymbol{c}_0)$, and $\xi_j \in \boldsymbol{c} \backslash \phi^*(\boldsymbol{c}_0)$, or, at least one of s_j, s_j' is in $S \backslash S_0$ $(k+1 \le j \le m)$. Let

$$\beta_1 = \varepsilon_1 s_1 \rho(\xi_1) s_1' + \cdots + \varepsilon_k s_k \rho(\xi_k) s_k',$$

$$\beta_2 = \varepsilon_{k+1} s_{k+1} \rho(\xi_{k+1}) s_{k+1}' + \cdots + \varepsilon_m s_m \rho(\xi_m) s_m'.$$

Then

$$\rho(z_{p_0} + Im\partial_2) = \beta_1 + \beta_2.$$

For each term $\varepsilon s t_r s'$ in $\rho(z_{p_0} + Im\partial_2)$ or in β_1, we have $r \in R_0$ and $s, s' \in S_0$. For each term $\varepsilon s t_r s'$ in β_2, we have $r \in R \backslash R_0$ or at least one of s, s' is in $S \backslash S_0$. Because P is the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule with basis $\{t_r : r \in R\}$, we have $\beta_2 = 0$ and $\rho(z_{p_0} + Im\partial_2) = \beta_1$. Since ρ is injective (Lemma 3.1.5), we have

$$z_{p_0} + Im\partial_2 = \varepsilon_1 s_1 \xi_1 s_1' + \cdots + \varepsilon_k s_k \xi_k s_k'.$$

Let $\xi_i = \phi^*(\xi_i')$ for some $\xi_i' \in \boldsymbol{c}_0$, $(i = 1, 2, \cdots, k)$. Then

$$\begin{aligned} \phi^*(z_{p_0} + Im\partial_2^0) &= \varepsilon_1 s_1 \phi^*(\xi_1') s_1' + \cdots + \varepsilon_k s_k \phi^*(\xi_k') s_k' \\ &= \phi^*(\varepsilon_1 s_1 \xi_1' s_1' + \cdots + \varepsilon_k s_k \xi_k' s_k'). \end{aligned}$$

Because ϕ^* is injective (Lemma 4.2.4), we have

$$z_{p_0} + Im\partial_2^0 = \varepsilon_1 s_1 \xi_1' s_1' + \cdots + \varepsilon_k s_k \xi_k' s_k'.$$

So $V_{\mathcal{P}_0, \boldsymbol{c}_0}(p_0) \le k \le m = V_{\mathcal{P},\boldsymbol{c}}(p_0)$. Thus $\gamma_{S_0}^{(2)} \preceq \gamma_S^{(2)}$. □

4.3 Relative asphericity

In group theory there is a notion of asphericity relative to a subpresentation (see [10]). Here we introduce and study a monoid analogue of this concept.

Let $\mathcal{P} = [X; R]$ be a finite monoid presentation, and let $\mathcal{P}_0 = [X_0; R_0]$ be a subpresentation of $\mathcal{P}$. We will say that $\mathcal{P}$ is *aspherical over* $\mathcal{P}_0$ if the set of all spherical monoid pictures over $\mathcal{P}_0$ trivializes $\mathcal{D}(\mathcal{P})$. Note that this is equivalent to asserting that if $\boldsymbol{X}_0$ is a set of spherical pictures over $\mathcal{P}_0$ which trivializes $\mathcal{D}(\mathcal{P}_0)$, then $\boldsymbol{X}_0$ trivializes $\mathcal{D}(\mathcal{P})$. Thus we have the following lemma.

Lemma 4.3.1. *Suppose $\mathcal{P}$ is aspherical over $\mathcal{P}_0$. If $\mathcal{P}_0$ is FDT then so is $\mathcal{P}$.* $\square$

Lemma 4.3.2. *Suppose $\mathcal{P}$ is aspherical over $\mathcal{P}_0$. If $\mathcal{P}_0$ is FHT then so is $\mathcal{P}$.*

Proof. Let $\boldsymbol{X}_0$ be a finite set of spherical monoid pictures over $\mathcal{P}_0$ such that $z_p + Im\partial_2^0$ $(p \in \boldsymbol{X}_0)$ generate the bimodule $\pi_2^{(b)}(\mathcal{P}_0)$. We want to show that $z_p + Im\partial_2$ $(p \in \boldsymbol{X}_0)$ generate the bimodule $\pi_2^{(b)}(\mathcal{P})$.

For each spherical monoid picture q over $\mathcal{P}$, since $\mathcal{P}$ is aspherical over $\mathcal{P}_0$, there exist finite number of spherical pictures q_i over $\mathcal{P}_0$ such that q is equivalent to

$$p_1(U_1 \cdot q_1 \cdot V_1)p_1^{-1}p_2(U_2 \cdot q_2 \cdot V_2)p_2^{-1} \cdots p_k(U_k \cdot q_k \cdot V_k)p_k^{-1}$$

in $\mathcal{D}(\mathcal{P})$, where p_i is a path in $\mathcal{D}(\mathcal{P})$, $U_i, V_i \in X^*$ $(1 \leq i \leq k)$. So

$$\begin{aligned} z_q + Im\partial_2 &= \sum_{i=1}^{k}[U_i]z_{q_i}[V_i] + Im\partial_2 \\ &= \sum_{i=1}^{k}[U_i](z_{q_i} + Im\partial_2)[V_i]. \end{aligned}$$

Let

$$z_{q_i} + Im\partial_2^0 = \sum_{j=1}^{m_i}[W_{ij}]_0(z_{p_{ij}} + Im\partial_2^0)[W'_{ij}]_0$$

for some $p_{ij} \in \boldsymbol{X}_0$ and $W_{ij}, W'_{ij} \in X_0^*$. Then

$$\begin{aligned} z_{q_i} + Im\partial_2 &= \phi^*(z_{q_i} + Im\partial_2^0) \\ &= \phi^*\Big(\sum_{j=1}^{m_i}[W_{ij}]_0(z_{p_{ij}} + Im\partial_2^0)[W'_{ij}]_0\Big) \\ &= \sum_{j=1}^{m_i}\phi^*([W_{ij}]_0(z_{p_{ij}} + Im\partial_2^0)[W'_{ij}]_0) \\ &= \sum_{j=1}^{m_i}[W_{ij}](z_{p_{ij}} + Im\partial_2)[W'_{ij}]. \end{aligned}$$

Thus

$$\begin{aligned} z_q + Im\partial_2 &= \sum_{i=1}^{k}[U_i]\Big(\sum_{j=1}^{m_i}[W_{ij}](z_{p_{ij}} + Im\partial_2)[W'_{ij}]\Big)[V_i] \\ &= \sum_{i=1}^{k}\sum_{j=1}^{m_i}[U_i][W_{ij}](z_{p_{ij}} + Im\partial_2)[W'_{ij}][V_i]. \end{aligned}$$

So $z_p + Im\partial_2$ ($p \in \boldsymbol{X}_0$) generate the bimodule $\pi_2^{(b)}(\mathcal{P})$. Thus $\mathcal{P}$ is FHT. □

4.4 Relative monoids

In [50], Kilgour introduced the notion of a *relative monoid presentation* $\mathcal{R}$. This is a triple $[H, \mathbf{y}; \mathbf{u}]$ where H is a group (the coefficient group), $\mathbf{y}$ is a set, and each element $U \in \mathbf{u}$ is an ordered pair (U_{+1}, U_{-1}) of words on $H \cup \mathbf{y}$, both involving at least one symbol from $\mathbf{y}$. We write $U : U_{+1} = U_{-1}$, and define $U^{-1} : U_{-1} = U_{+1}$. We can define operations on words W on $H \cup \mathbf{y}$ as follows:

(1) If W contains two successive terms $h, h' \in H$ then replace them by a single term which is their product in H.

(2) If W contains a term which is the identity of H then delete that term.

(3) If W contains a subword U_ε ($U : U_{+1} = U_{-1}$ in $\mathbf{u}$, $\varepsilon = \pm 1$) then replace it by $U_{-\varepsilon}$.

Two words are said to be *equivalent* (relative to $\mathcal{R}$) if one can be obtained from the other by a finite number of operations (1), $(1)^{-1}$, (2), $(2)^{-1}$, (3). Denote the equivalence class of W by $[W]$. These equivalences classes form a monoid $S(\mathcal{R}) = S$ (*the monoid defined by* $\mathcal{R}$) under the multiplication

$$[W_1][W_2] = [W_1W_2],$$

where W_1W_2 is the concatenation of W_1, W_2. There is a natural homomorphism

$$H \longrightarrow S \quad h \mapsto [h].$$

This is clearly injective, so we can identify H with its image in S. It is clear that the monoid $S(\mathcal{R})$ is isomorphic to the quotient of the free product $H * \mathbf{y}^*$ by the congruence generated by $\mathbf{u}$.

A *labelled graph* is a graph (Section 1.5.1) with vertex set V, edge set E, and initial, terminal and inversion function $\iota, \tau, -1$, say, together with a group G and a "labelling function"

$$\phi : E \longrightarrow G,$$

where $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E$. The label on a path is the product in G of the labels on the edges making up the path. A *cycle* in the labelled graph is a non-empty reduced closed path whose label is 1.

We associate with the relative monoid presentation $[H, \mathbf{y}; \mathbf{u}]$ two labelled graphs, the *left graph* $LG(\mathcal{R})$ and the *right graph* $RG(\mathcal{R})$, as follows. Both graphs have vertex set $\mathbf{y}$, and edge set $E = \{e_U, e_U^{-1} : U \in \mathbf{u}\}$. The graphs are oriented by choosing $E^+ = \{e_U : U \in \mathbf{u}\}$. The initial, terminal and labelling functions are as follows. Let

$$U : h_1y_1 \ldots h_{r-1}y_{r-1}h_r = h'_1y'_1 \ldots h'_{s-1}y'_{s-1}h'_s$$

be a relator in $\mathbf{u}$, where the h's belong to H and the y's belong to $\mathbf{y}$.

$$LG(\mathcal{R}): \quad \iota(e_U) = y_1,\ \tau(e_U) = y'_1,\ \phi(e_U) = h_1^{-1}h'_1$$
$$RG(\mathcal{R}): \quad \iota(e_U) = y_{r-1},\ \tau(e_U) = y'_{s-1},\ \phi(e_U) = h_rh_s'^{-1}.$$

Example 4.4.1. Let H be the free abelian group on $\{a, b\}$, and let

$$\mathcal{R} = [H, y_1, y_2;\ a^{-1}y_1a = y_2b, by_2 = y_1b^{-2}y_1b^{-1}a, a^2y_1b^{-3}y_2ab = a^{-3}y_1y_2ba].$$

Then $LG(\mathcal{R})$ and $RG(\mathcal{R})$ are as in Figure 4.3.

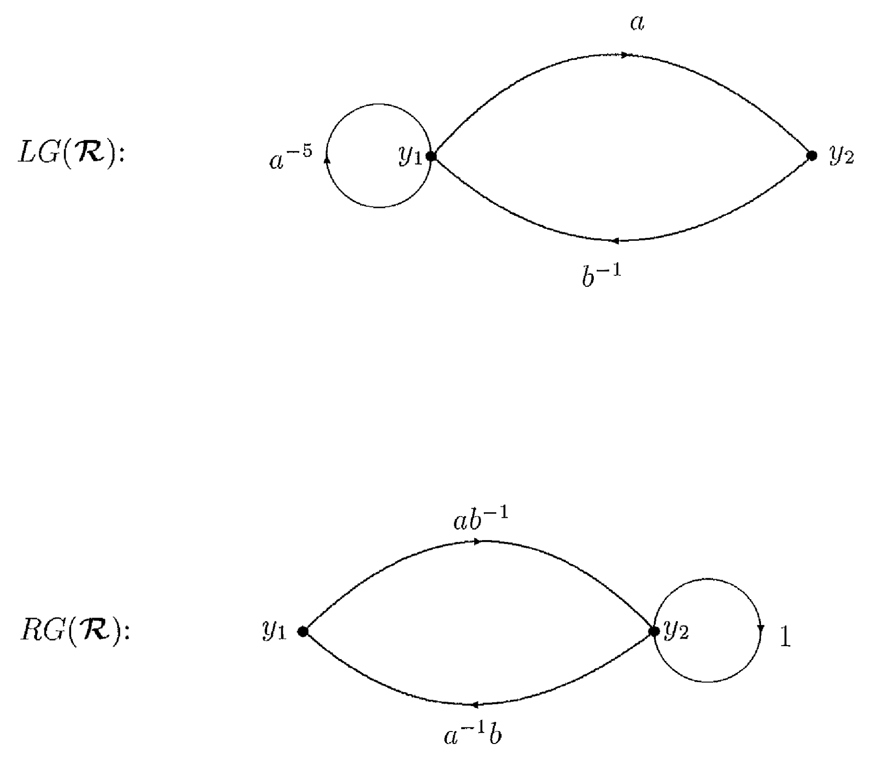


Figure 4.3

It is easy to check that in this example $LG(\mathcal{R})$ is cycle free, but $RG(\mathcal{R})$ is not cycle free. Note that not only does $RG(\mathcal{R})$ have a cycle of length one which is labelled by the identity 1 of H, but it also has cycles of length two labelled by elements of H which equal to the identity of H.

Using geometric methods, Kilgour ([50]) proved the following results.

Theorem 4.4.2. *(a) If $LG(\mathcal{R})$ (resp. $RG(\mathcal{R})$) is cycle-free then S is left (resp. right) cancellative.*

(b) If $LG(\mathcal{R})$ and $RG(\mathcal{R})$ are both cycle-free then S is embeddable in a group.

This generalizes results of Adian ([1], [2]) which deal with the case $H = 1$.

4.5 FDT and FHT for relative monoids

Let $\mathcal{R} = [H, \mathbf{y}; \mathbf{u}]$ be a finite relative monoid presentation, and let $S = S(\mathcal{R})$. Let $\langle \mathbf{a}; \mathbf{t} \rangle$ be a finite *group* presentation for H. Then we have a finite *monoid* presentation

$$\mathcal{P}_0 = \left[\mathbf{a}, \mathbf{a}^{-1}; T = 1 (T \in \mathbf{t}),\ a^{\varepsilon} a^{-\varepsilon} = 1\ (a \in \mathbf{a}, \varepsilon = \pm 1)\right]$$

for H, and then

$$\mathcal{P} = \left[\mathbf{a}, \mathbf{a}^{-1}, \mathbf{y}; T = 1 (T \in \mathbf{t}), a^{\varepsilon} a^{-\varepsilon} = 1 (a \in \mathbf{a}, \varepsilon = \pm 1), \mathbf{u}\right]$$

is a finite monoid presentation for S. By Lemma 4.2.2 we can get that $S \backslash H$ is an ideal of S.

By adapting arguments of Kilgour ([50]), we will prove the following.

Proposition 4.5.1. *If $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle-free then $\mathcal{P}$ is aspherical over $\mathcal{P}_0$.*

Combining Proposition 4.5.1 with Lemma 4.3.1, 4.3.2, Proposition 4.2.3 and 4.2.5 we can get our main result in this section.

Theorem 4.5.2. *Suppose $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle-free. Then S is FDT (respectively, FHT) if and only if H is.* □

For the proof of Proposition 4.5.1, we just consider the case that $LG(\mathcal{R})$ is cycle-free.

Let $\boldsymbol{X}_0$ be the set of all spherical monoid pictures over $\mathcal{P}_0$.

A *partial (left) dipole* in a monoid picture p over $\mathcal{P}$ consists of two discs Δ, Δ', one labelled by U and the other by U^{-1} for some $U \in \mathbf{u}$, and such that the first $\mathbf{y}$-arc meeting $\partial^- \Delta$ (reading left to right) coincides with the first $\mathbf{y}$-arc meeting $\partial^+ \Delta'$. A partial dipole will be called a *dipole* if *all* the $\mathbf{y}$-arc meeting $\partial^- \Delta$ also meet $\partial^+ \Delta'$, and will be called a *complete dipole* if *all the arcs* meeting $\partial^- \Delta$ also meet $\partial^+ \Delta'$. See Figure 4.4.

It is clear that if p contains a complete dipole then by swapping levels of discs if necessary (which is permissible modulo the 2-cells of $\mathcal{D}(\mathcal{P})$) we can convert the complete dipole into the discs of a cancelling pair ee^{-1} of edges in p which can be removed giving a picture p_1 equivalent to p with less $\mathbf{u}$-discs.

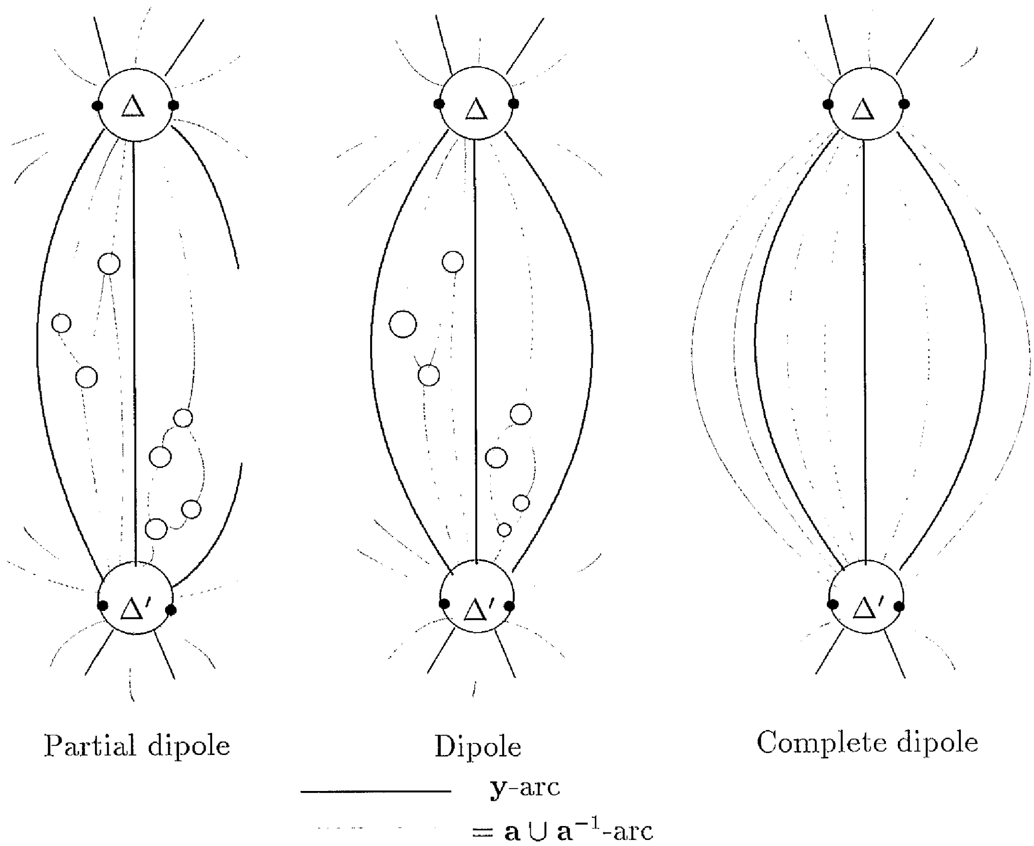


Figure 4.4

Now suppose that p contains a dipole. Let us show that p is equivalent in $\mathcal{D}(\mathcal{P})^{\mathbf{X}_0}$ to a picture with the same number of $\mathbf{u}$-discs as p but where the dipole has now been converted into a complete dipole.

Let Δ, Δ' be a dipole in p as above and let $\alpha_1, \alpha_2, \ldots \alpha_m$ be the $\mathbf{y}$-arcs meeting $\partial^-\Delta \cup \partial^+\Delta'$ (taken in order from left to right). Then the part of p between α_i and α_{i+1} $(1 \leq i < m)$ will be a spherical picture p_i over $\mathcal{P}_0$. After swapping levels of disjoint discs in p if necesary we can suppose that for certain words W_i, V_i, $W_i \cdot p_i \cdot V_i$ is a subpicture of p which, working in $\mathcal{D}(\mathcal{P})^{\mathbf{X}_0}$, we can remove. In this way we can convert p to a equivalent picture in $\mathcal{D}(\mathcal{P})^{\mathbf{X}_0}$, in which all the arcs ($\mathbf{a} \cup \mathbf{a}^{-1}$-arcs and $\mathbf{y}$-arcs) meeting $\partial^-\Delta$ from α_1 to α_m (reading left to right) also meet $\partial^+\Delta'$.

Now consider the $\mathbf{a} \cup \mathbf{a}^{-1}$-arcs meeting $\partial^- \Delta$ which lie to the left of α_1. We can connect these to the corresponding $\mathbf{a} \cup \mathbf{a}^{-1}$-arcs meeting $\partial^+ \Delta'$ to the left of α_1 by using the operation as depicted in Figure 4.5.

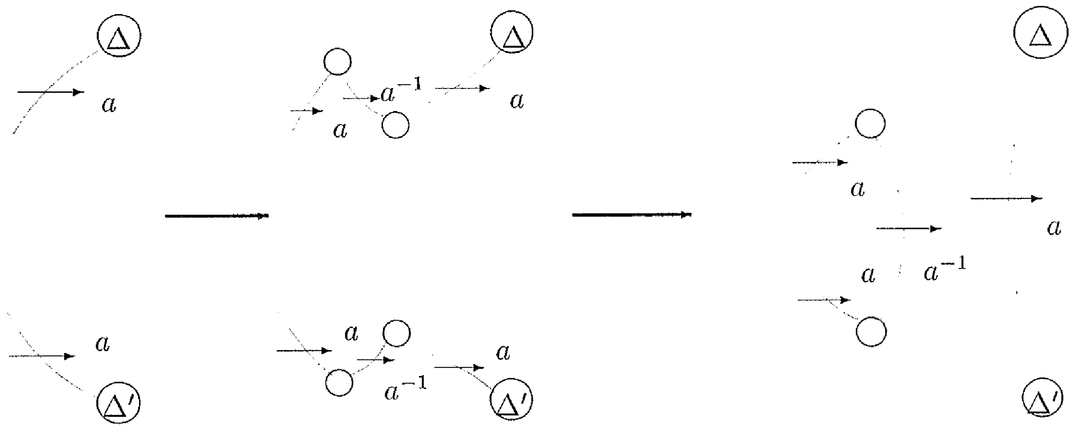


Figure 4.5

This process does not change the homotopy type of p in $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}_0}$. A similar process can be used to connect all the $\mathbf{a} \cup \mathbf{a}^{-1}$-arcs meeting $\partial^- \Delta$ to the right of α_m, to the corresponding arcs meeting $\partial^+ \Delta'$ to the right of α_m.

To prove the Proposition 4.5.1, we now see that it suffices to prove the following:

Lemma 4.5.3. *Every spherical picture containing a* $\mathbf{u}$*-disc contains a dipole.*

For by the above discussion it will then follow inductively that every spherical picture is equivalent in $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}_0}$ to a spherical picture without $\mathbf{u}$-discs; clearly such a picture is equivalent to the empty path in $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}_0}$.

To prove Lemma 4.5.3 we must consider pictures p_1 satisfying

(**) $\iota(p_1) = AyW, \tau(p_1) = AyV$ where A is a word on $\mathbf{a} \cup \mathbf{a}^{-1}, W, V$ are words on $\mathbf{a} \cup \mathbf{a}^{-1} \cup \mathbf{y}$, $y \in \mathbf{y}$.

Following [50] we define the *left* **y**-*circle* C of such a picture p_1. This is a sequence $\beta_0, \Delta_1, \beta_1, \Delta_2, \beta_2, \ldots, \Delta_n, \beta_n$ of **y**-arcs β_i and **u**-discs Δ_i, obtained as follows. Let β_0 be the left-most arc meeting $\partial^+ p_1$, labelled y. If β_0 also meets $\partial^- p_1$ then β_0 is C. Otherwise β_0 meets $\partial^+\Delta_1$ for some disc Δ_1 (and is, moreover, the left-most **y**-arc meeting $\partial^+\Delta_1$). Let β_1 be the left-most **y**-arc meeting $\partial^-\Delta_1$. If β_1 meets $\partial^- p_1$ then $\beta_0, \Delta_1, \beta_1$ is C. Otherwise, β_1 is the left-most **y**-arc meeting $\partial^+\Delta_2$ for some disc Δ_2. Let β_2 be the left-most **y**-arc meeting $\partial^-\Delta_2$. And so on. Continuing this process we eventually obtain C. We say that C is non-trivial if $n > 0$.

Lemma 4.5.4. *Suppose that p_1 satisfies $(**)$ and that its left* **y**-*circle is non-trivial. Then p_1 contains a dipole.*

Proof. The proof is by induction on the number of **u**-discs in p_1.

Consider the left **y**-circle C of p_1 as above, and suppose Δ_i is labelled by

$$U_i^{\varepsilon_i} : A_i y_i \ldots = A_i' y_i' \ldots$$

where A_i, A_i' are words on $\mathbf{a} \cup \mathbf{a}^{-1}$ and $y_i, y_i' \in \mathbf{y}$. Note that $y_1 = y_n' = y$. We thus have the closed path $\gamma = e_{U_1}^{\varepsilon_1} \ldots e_{U_n}^{\varepsilon_n}$ at y in $LG(\mathcal{R})$. This path is labelled by the element of H represented by the word

$$A_0 \equiv A_1^{-1} A_1' A_2^{-1} A_2' \ldots A_n^{-1} A_n'.$$

Now the part of p_1 to the left of C can be converted to a group picture over the group presentation $\langle \mathbf{a}; \mathbf{t} \rangle$ of H by contracting all discs labelled $a^{\pm 1} a^{\mp 1}$ $(a \in \mathbf{a})$ to points, and re-orienting all arcs labelled a^{-1} and relabelling them by a $(a \in \mathbf{a})$. This picture has boundary label AA_0A^{-1}. By Theorem 1.6.3 the boundary label of a picture over a group presentation defines the identity of the group presented, so AA_0A^{-1} (and hence A_0) represents the identity of H. Thus the label on γ is 1, so γ must not be reduced, as $LG(\mathcal{R})$ has no cycles. We conclude that for some $i, e_{U_i}^{\varepsilon_i} = e_{U_{i+1}}^{-\varepsilon_{i+1}}$. Thus the discs Δ_i, Δ_{i+1} constitute a partial dipole. If this partial dipole is a dipole then we are finished. Otherwise, we can single out a part of p_1 to the right of $\Delta_i, \beta_i, \Delta_{i+1}$ satisfying the assumptions of our lemma but with fewer **u**-discs (see Figure 4.6). We can then use the inductive hypothesis. $\square$

Proof of Lemma 4.5.3. If p is a spherical picture over p containing $\mathbf{u}$-discs then p will be equivalent in $\mathcal{D}(\mathcal{P})$ to a picture $(p_0 \cdot \iota(p_1))(\tau(p_0) \cdot p_1)$ where p_0 contains no $\mathbf{u}$-discs and p_1 satisfies the assumptions of Lemma 4.5.4. Thus, by Lemma 4.5.4, p_1 (and hence p) contains a dipole. □

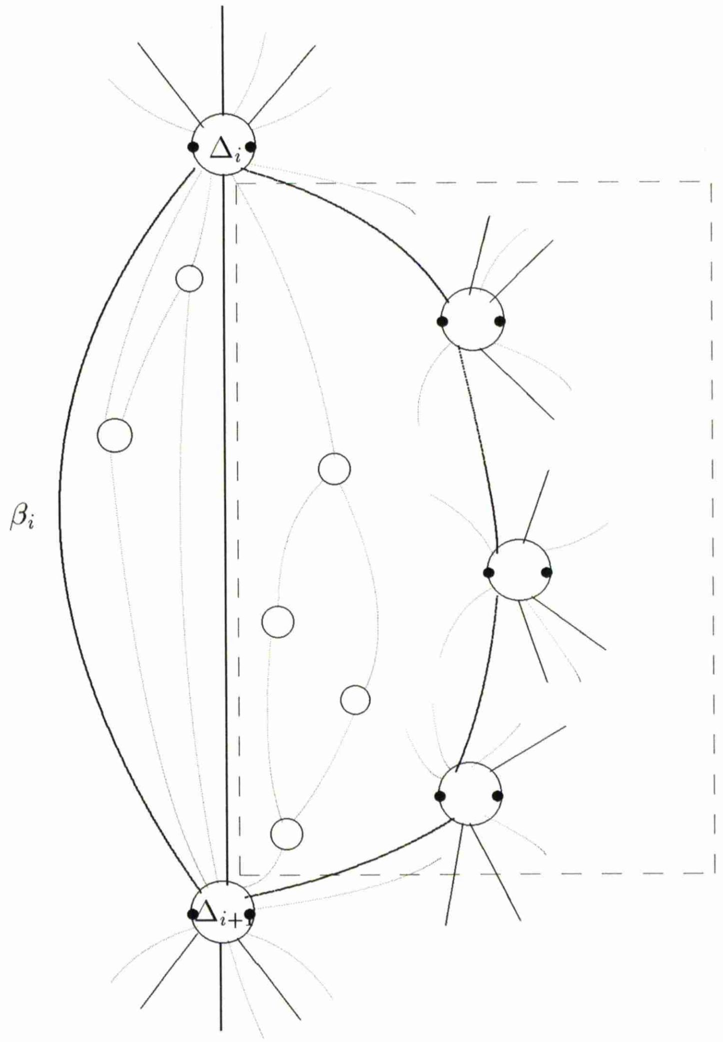
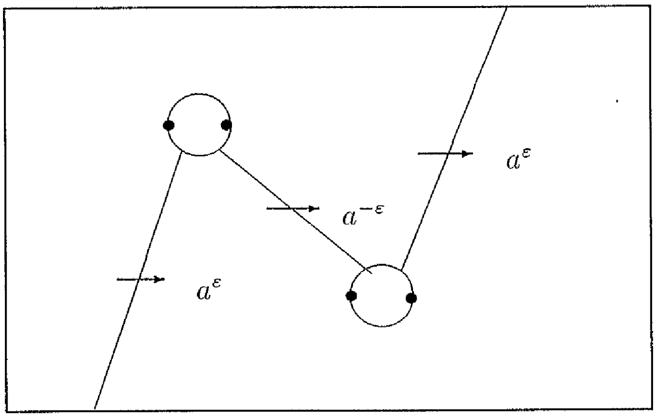


Figure 4.6

4.6 Second order Dehn functions for relative monoids

Let $\mathcal{R}$, H, S, $\mathcal{P}_0$ and $\mathcal{P}$ be as in Section 4.5. Suppose that $LG(\mathcal{R})$ is cycle free. If H is FDT, then so is S. Let $\boldsymbol{X}_0$ be a finite trivializer of $\mathcal{D}(\mathcal{P}_0)$ which contains all spherical pictures of the form in Figure 4.7. Then $\boldsymbol{X}_0$ is also a finite trivializer of $\mathcal{D}(\mathcal{P})$.



$(a \in \boldsymbol{a},\ \varepsilon = \pm 1)$

Figure 4.7

Let p be any spherical picture over $\mathcal{P}$ with area $A(p) = n$. Suppose p contains m $\boldsymbol{u}$-discs. If $m \geq 1$, then p contains a dipole (by Lemma 4.5.3). Let the two discs Δ and Δ' in the dipole are labelled by $U^{-\varepsilon}$ and U^{ε} for some $U \in \boldsymbol{u}$, respectively, where $\varepsilon = \pm 1$. Suppose that

$$U_\varepsilon \equiv W_0 y_1 W_1 y_2 \cdots W_{j-1} y_j W_j,$$

where $W_i \in (\boldsymbol{a} \cup \boldsymbol{a}^{-1})^*$, $y_i \in \boldsymbol{y}$. We can convert this dipole to a complete dipole by at most c_1 operations of type (III) and some operations of type (I) and (II), where

$$c_1 = 2l(W_0) + 2l(W_j) + \sum_{i=1}^{j-1} \tilde{\gamma}^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}(k_i)$$

and k_i is the number of discs between the $\boldsymbol{y}$-arcs labelled by y_i and y_{i+1} (see Figures 4.4 and 4.5). Then by swapping levels of discs if neccessary we can convert the complete dipole into the discs of a cancelling pair ee^{-1} of edges in p which can be removed giving a picture p_1 equivalent to p in $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}_0}$, such that p_1 contains $m-2$ $\boldsymbol{u}$-discs and at most

$n - m + 2(l(W_0) + l(W_j)) - n_1$ non $\boldsymbol{u}$-discs, where $n_1 = \sum_{i=1}^{j-1} k_i$. Note that from p to p_1 we did at most c_1 operations of type (III), and we have

$$c_1 \leq 2d + \bar{\tilde{\gamma}}^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}(n_1),$$

where $d = max\{l(U_\varepsilon) : U \in \boldsymbol{u},\ \varepsilon = \pm 1\}$.

If $m - 2 \geq 1$, we can repeat the above procedure to p_1, and so on, to get a picture $p_{\frac{m}{2}}$ equivalent to p in $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}_0}$, such that $p_{\frac{m}{2}}$ contains no $\boldsymbol{u}$-disc and $A(p_{\frac{m}{2}}) \leq n - m + md - \sum_{i=1}^{\frac{m}{2}} n_i$. From p to $p_{\frac{m}{2}}$ we did at most c operations of type (III), where

$$c = \sum_{i=1}^{\frac{m}{2}} c_i \leq md + \bar{\tilde{\gamma}}^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}(\sum_{i=1}^{\frac{m}{2}} n_i).$$

It is easy to see that

$$\tilde{V}_{\mathcal{P}, \boldsymbol{X}_0}(p_{\frac{m}{2}}) \leq \bar{\tilde{\gamma}}^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}(n - m + md - \sum_{i=1}^{\frac{m}{2}} n_i).$$

So we have

$$\tilde{V}_{\mathcal{P}, \boldsymbol{X}_0}(p) \leq \tilde{V}_{\mathcal{P}, \boldsymbol{X}_0}(p_{\frac{m}{2}}) + c \leq md + \bar{\tilde{\gamma}}^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}(n - m + md) \leq dn + \bar{\tilde{\gamma}}^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}(dn + n).$$

Thus $\tilde{\gamma}^{(2)}_{\mathcal{P}, \boldsymbol{X}_0} \preceq \bar{\tilde{\gamma}}^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}$, and hence

$$\tilde{\gamma}^{(2)}_S \preceq \bar{\tilde{\gamma}}^{(2)}_H. \qquad (1)$$

Let $\boldsymbol{c}_0 = \{\xi^0_p : p \in \boldsymbol{X}_0\}$ and let $\boldsymbol{c} = \{\xi_p : p \in \boldsymbol{X}_0\}$, where $\xi^0_p = z_p + Im\partial^0_2$ and $\xi_p = z_p + Im\partial_2$. Then $\boldsymbol{c}_0$ and $\boldsymbol{c}$ are finite generating sets for $\pi^{(b)}_2(\mathcal{P}_0)$ and $\pi^{(b)}_2(\mathcal{P})$, respectively. The inclusion map from $\mathcal{P}_0$ to $\mathcal{P}$ induces a mapping $\phi : \mathcal{D}(\mathcal{P}_0) \longrightarrow \mathcal{D}(\mathcal{P})$. Then we have a homomorphism

$$\phi^* : \pi^{(b)}_2(\mathcal{P}_0) \longrightarrow \pi^{(b)}_2(\mathcal{P}),\ z_p + Im\partial^0_2 \longmapsto z_p + Im\partial_2,$$

where p is a spherical picture over $\mathcal{P}_0$. It is easy to see that $V_{\mathcal{P}, \boldsymbol{c}}(\phi(p)) \leq V_{\mathcal{P}_0, \boldsymbol{c}_0}(p)$ for any spherical picture p over $\mathcal{P}_0$ (since $\phi^*(\xi^0_p) = \xi_{\phi(p)}$).

Now let p be a spherical picture over $\mathcal{P}$ with area $A(p) = n$, and let m be the number of $\boldsymbol{u}$-discs in p. If $m = 0$, then

$$p \sim (U_1 \cdot p_1 \cdot V_1)(U_2 \cdot p_2 \cdot V_2) \cdots (U_j \cdot p_j \cdot V_j),$$

where p_i is a spherical picture over $\mathcal{P}_0$, $U_i, V_i \in (\boldsymbol{a} \cup \boldsymbol{a}^{-1} \cup \boldsymbol{y})^*$, and $\sum_{i=1}^{j} A(p_i) = n$. So

$$\xi_p = \sum_{i=1}^{j} [U_i]\xi_{p_i}[V_i] = \sum_{i=1}^{j} [U_i]\phi^*(\xi^0_{p_i})[V_i].$$

$$V_{\mathcal{P},c}(p) \leq \sum_{i=1}^{j} V_{\mathcal{P},c}(\phi(p_i)) \leq \sum_{i=1}^{j} V_{\mathcal{P}_0,c_0}(p_i).$$

Thus

$$V_{\mathcal{P},c}(p) \leq \sum_{i=1}^{j} \gamma^{(2)}_{\mathcal{P}_0,c_0}(A(p_i)) \leq \overline{\gamma}^{(2)}_{\mathcal{P}_0,c_0}(n).$$

If $m \geq 1$, then p contains a dipole. We can convert this dipole to a dipole of the form in Figure 4.8 by at most $2d$ operations of type (III) and some operations of type (I) and (II), where $d = max\{l(U_\varepsilon) : U \in \boldsymbol{u},\ \varepsilon = \pm 1\}$.

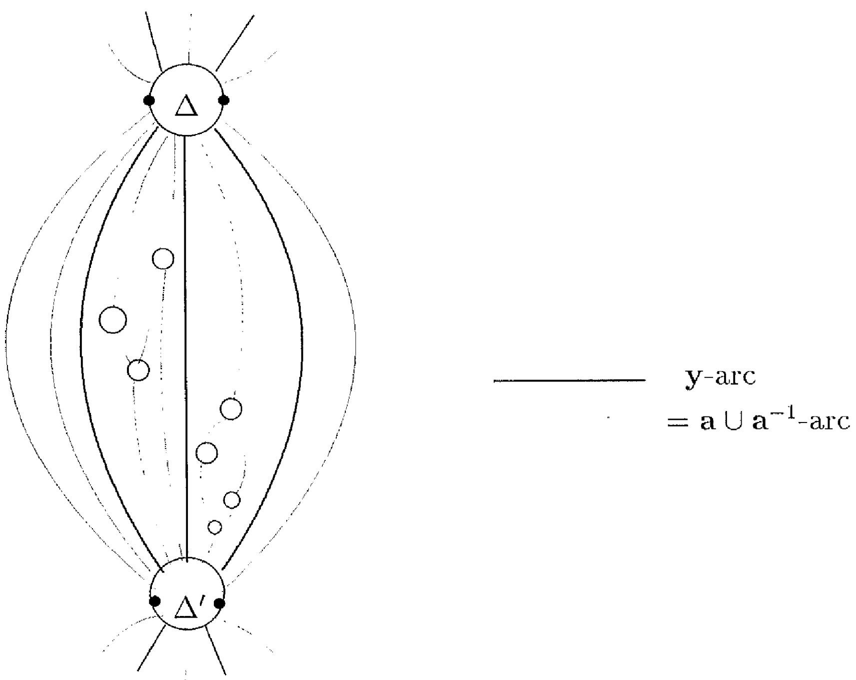


Figure 4.8

Then by swapping levels of discs if necessary we can get a picture $q_1 q_1' q_1''$ which is equivalent to p in $\mathcal{D}(\mathcal{P})^{\boldsymbol{X}_0}$, such that $q_1' = U \cdot p_1' \cdot V$, where p_1' is a spherical picture which consists of one dipole of the form in Figure 4.8, $U, V \in (\boldsymbol{a} \cup \boldsymbol{a}^{-1} \cup \boldsymbol{y})^*$. Note that $q_1 q_1''$ is a spherical picture with at most $m-2$ $\boldsymbol{u}$-discs, and $A(q_1 q_1'') \leq A(p) + 2d - A(q_1')$. Let $\bar{p}_1$ be the subpicture of p_1' obtained by removing the two $\boldsymbol{u}$-discs. So $\bar{p}_1$ is a spherical picture with no $\boldsymbol{u}$-disc. Since $A(\bar{p}_1) \leq A(p_1') = A(q_1')$, we have

$$V_{\mathcal{P},c}(\bar{p}_1) \leq \overline{\gamma}^{(2)}_{\mathcal{P}_0,c_0}(A(q_1')).$$

Because $V_{\mathcal{P},c}(p) \leq 2d + V_{\mathcal{P},c}(q_1') + V_{\mathcal{P},c}(q_1 q_1'')$ and $V_{\mathcal{P},c}(q_1') \leq V_{\mathcal{P},c}(\bar{p}_1)$, we have

$$V_{\mathcal{P},c}(p) \leq 2d + V_{\mathcal{P},c}(q_1 q_1'') + \overline{\gamma}^{(2)}_{\mathcal{P}_0,c_0}(A(q_1')).$$

If $m-2 \geq 1$, we can repeat the above procedure to $q_1 q_1''$, and so on, to get a spherical picture $q_{\frac{m}{2}} q''_{\frac{m}{2}}$ which contains no $\boldsymbol{u}$-disc such that

$$A(q_{\frac{m}{2}} q''_{\frac{m}{2}}) \leq n + md - \sum_{i=1}^{\frac{m}{2}} A(q_i'),$$

$$V_{\mathcal{P},c}(p) \leq md + \sum_{i=1}^{\frac{m}{2}} \overline{\gamma}^{(2)}_{\mathcal{P}_0,c_0}(A(q_i')) + \overline{\gamma}^{(2)}_{\mathcal{P}_0,c_0}(A(q_{\frac{m}{2}} q''_{\frac{m}{2}})).$$

Thus we have

$$V_{\mathcal{P},c}(p) \leq md + \overline{\gamma}_{\mathcal{P}_0,c_0}(n+md) \leq nd + \overline{\gamma}_{\mathcal{P}_0,c_0}(n+nd).$$

Thus $\gamma^{(2)}_{\mathcal{P},c} \preceq \overline{\gamma}^{(2)}_{\mathcal{P}_0,c_0}$, and hence

$$\gamma^{(2)}_S \preceq \overline{\gamma}^{(2)}_H. \qquad (2)$$

Combining above (1), (2) and Proposition 4.2.3 and 4.2.5 we have the following theorem.

Theorem 4.6.1. *Suppose that $LG(\mathcal{R})$ or $RG(\mathcal{R})$ is cycle-free. If H is FDT, then so is S, and we also have*

$$\tilde{\gamma}^{(2)}_H \preceq \tilde{\gamma}^{(2)}_S \preceq \overline{\gamma}^{(2)}_H,$$

$$\gamma^{(2)}_H \preceq \gamma^{(2)}_S \preceq \overline{\gamma}^{(2)}_H,$$

$$(\delta^{(2)}_H \preceq \gamma^{(2)}_S \preceq \overline{\delta}^{(2)}_H).$$

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