Grammians In Nonlinear Evolution Equations

by

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A thesis submitted to

the Faculty of Science

at the University of Glasgow

for the degree of

Doctor of Philosophy

August 7, 1998

Summary

This thesis is concerned with solutions to nonlinear evolution equations. In particular we examine two specific equations; the Davey-Stewartson (DS) equation and the three-dimensional three-wave resonant interaction equation. More precisely we are interested in the role that grammians play in determining new solutions to three-dimensional three-wave resonant interactions (3D3WR), through Hirota's bilinear method [42] and the binary Darboux transformation [70]. We also exploit the grammian structure to obtain rational solutions to the DS equation.

The thesis is organised as follows. Chapter one is an introduction to the concepts, ideas and constructions that will be used throughout this thesis. We discuss bilinear equations, Laplace expansions of determinants and grammians, all with a view to their role in obtaining solutions to nonlinear evolution equations. The chapter attempts to provide an overall framework for the work that follows and an outline of the connections between the chapters. We also try and consider the motivation for working with a grammian approach.

In chapter two we focus on the DS equation with non-zero background, and in particular rational solutions for it. After background material to the DS equation and its derivation, we look more closely at methods that already exist to obtain solutions. Our aim is to provide a simple way to calculate rational solutions to the DS equation. The example of the KP equation [66] [7], and Gilson and Nimmo's work [35] provides the approach we need. We verify a broad class of solutions all written in terms of a grammian and from these we obtain singular rational solutions by exploiting the "long-wave limit". However, by relaxing the necessary reality conditions we may obtain rational solutions from a general grammian. By then verifying when these are solutions to the DS equation we obtain a wider class of rational solutions.

This mirrors the approach of Ablowitz and Satsuma [89]. It leads us to determine a class of non-singular rational solutions which describe multiple collisions of lumps. These lumps correspond to the ones found by Ablowitz and Satsuma but the grammian method is simpler and the solutions more "fully" rational.

In chapter three we consider 3D3WR using a bilinear approach to investigate a broad class of solutions. The solutions to 3D3WR described originally by Kaup [52] [51], can easily be recast in terms of grammians. This approach arises naturally by considering the Painlevé analysis for 3D3WR [31], through which we recover Kaup's Bäcklund transformation and the bilinear form. Kaup's solutions are generalised to give the n-lump solution, and then we prove a general grammian solution by using a Jacobi identity. Finally in chapter three we examine some specific examples of the lump solutions and provide some idea of what the solutions look like. The work in this chapter constitutes [37].

We stay with 3D3WR in chapter four. By focusing on its scattering problem and using the method developed by Nimmo [78] we derive Darboux transformations (DT) and binary Darboux transformations (BDT). It turns out that only the BDT preserves the structure that we need for a solution to 3D3WR and these are written in a grammian format. By determining a closed form of the solution to the iterated BDT we see that it corresponds to the lump solutions of chapter three. This provides a link between the Bäcklund transformation of Kaup [51] and the BDT. We look briefly at obtaining a discrete version of 3D3WR from the BDT.

Chapter five seeks to bring together the results of the various chapters and again identify the common theme of the grammian. We also discuss some open questions that arise from the work presented.

To God

Without whom this thesis
would not have been possible

Contents

List of Figures

Acknowledgements

I would like to express my deepest thanks and gratitude to my supervisor, Dr. Claire Gilson, for her guidance, inspiration and encouragement throughout the three years of this research. Particularly her patience, kindness and willingness to help.

I would like to acknowledge the Engineering and Physical Sciences Research Council (EPSRC) for their financial support through a studentship, as well as the Department of Mathematics, University of Glasgow for providing funding to attend conferences.

I would like to thank the "boys"; Iain, Mark, Mohammed, Andrew and Rob, without whom, this period of research would not have been nearly as stimulating, enjoyable and fun.

Finally special thanks to my family whose love and support have made this thesis possible.

Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

Chapter one introduces some of the concepts used throughout the thesis and provides some motivation. The beginning of each chapter, and section 3.2.1, provide introduction and background. Section 2.2 provides an example. The proofs in sections 4.3 and 4.4 are due to Gilson and Nimmo. References are given throughout these sections.

The remainder of the work is the original work of the author. The work presented in chapter three appears in [37].

Chapter 1

Introduction

1.1 Preliminaries

In this section we introduce some of the basic concepts that are used throughout the thesis.

1.1.1 Bilinear Equations

Throughout this thesis, especially in chapters two and three we will make use of bilinear equations, which are obtained from nonlinear evolution equations via dependent variable transformations. The dependent variable transformation was originally derived by Hirota [41] in 1971 as a clever way of obtaining the multi-soliton solution to the KdV equation. However this method gave insight into the connection between Bäcklund transformations, the inverse scattering transform [86], and conservation laws. It was further found to be applicable to a large class of nonlinear evolution equations [44].

We provide a brief introduction to this subject by using the KdV equation as an illustration. For a fuller explanation of the bilinear transformation method see the book by Matsuno [68] and the summary by Hirota [42].

We start with the KdV equation, first derived by Korteweg and de Vries as a model for shallow-water waves [57], in the form

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

with $u = u(x,t)$ and the boundary condition $u \to 0$ as $|x| \to \infty$. Equation (1.1) has the solution

$$u(x,t) = \frac{p^2}{2}\operatorname{sech}^2\frac{\eta}{2} \tag{1.2}$$

where $\eta = px - p^3t + \eta_0$ and p and η_0 are arbitrary constants. This solution can be rewritten as

$$u(x,t) = 2p^2(e^{\eta/2} + e^{-\eta/2})^{-2} = 2\left(\log(1+e^{\eta})\right)_{xx}, \tag{1.3}$$

which suggests the following dependent variable transformation

$$u(x,t) = 2\left(\log f(x,t)\right)_{xx}. \tag{1.4}$$

Substituting (1.4) into (1.1) and integrating twice with respect to x (setting integration constants to zero) we obtain

$$f_{xt}f - f_xf_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0 \tag{1.5}$$

which is the bilinearized form of the KdV equation. At this point we introduce the Hirota operators defined as

$$D_x^lD_y^mD_t^n f\cdot f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,y,t)f(x',y',t')\Big|_{\substack{x=x'\\y=y'\\t=t'}} \tag{1.6}$$

with l, m, n non-negative integers, so that (1.5) can be written

$$D_x(D_t + D_x^3)f\cdot f = 0. \tag{1.7}$$

This process of transforming the nonlinear evolution equation (1.1) into the form (1.7) is known as bilinearization. In the bilinear form it is very convenient to find exact solutions, in particular soliton solutions [42]. Essentially the method involves expanding f in powers of an arbitrary parameter, ϵ say, and then after substitution into (1.7) equating coefficients of ϵ^n to zero. This gives rise to a whole series of equations which must all be satisfied. The n-soliton is then given in bilinear form as

$$f = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^{n}\mu_j\eta_j + \sum_{j>k}^{(n)}\mu_j\mu_kA_{jk}\right] \tag{1.8}$$

where

$$\eta_j = p_j x + \Omega_j t + \eta_{0j} \qquad e^{A_{jk}} = \frac{(p_j - p_k)^2}{(p_j + p_k)^2} \tag{1.9}$$

with $\Omega_j = -p_j^3$ for $j = 1, 2, \dots, n$, $\sum_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_1 = 0, 1,\ \mu_2 = 0, 1, \dots, \mu_n = 0, 1$ and $\sum_{j>k}^{(n)}$ means the summation over all possible combinations of n elements under the condition $j > k$.

1.1.2 Laplacian Expansion of Determinants

We consider the $n \times n$ matrix A, and choose any m rows of $|A|$, (without any loss of generality we may choose the first m rows). From these m rows we can form $\binom{n}{m}$ minors ($|A|$ with various columns supressed) of order m. By multiplying each of these minors by its corresponding cofactor (signed minor) of order $(n - m)$, we obtain terms in $|A|$. Counting up all the terms we have altogther

$$\binom{n}{m} m!(n-m)! = n! \tag{1.10}$$

which is equivalent to the number of terms in $|A|$. This form of expansion for a determinant was first given by Laplace in 1772 [10].

As an example we consider

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} . \tag{1.11}$$

To illustrate the method we choose $m = 2$ ($m = n/2$ mirrors the procedure necessary for wronskian solutions). So we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \hline a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix}$$
$$+ \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix}$$
$$- \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}\begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} + \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}\begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}, \quad (1.12)$$

where the sign of the cofactor arises from the number of row and columns that are interchanged, such that the minor is brought into leading position.

1.1.3 Grammians

The n-soliton solutions in the bilinear form have been given in a variety of different forms. In 1971, Hirota [41] originally used a determinant of a matrix, which could be expanded as the sum of exponential polynomials. The proof that this is a solution relies on induction.

In the wronskian method, often related to the inverse scattering transform [88], the solutions are written as a $n \times n$ wronskian determinant. This type of solution is proved by Laplacian expansions (see previous section).

Sato theory encompasses a variety of methods which can be exploited to give soliton solutions. These include the determinants of infinite matrices [87], vertex operators [24] or Fermion operators [23]. The bilinear form is shown to be the Plücker relation with respect to the τ function.

Grammian solutions are the determinants of a matrix whose elements are in an integral form. For example in the case of the bilinear KP equation

$$\left[D_x(D_x^3 - D_t) + 3D_y^2\right] f \cdot f = 0, \quad (1.13)$$

solutions can be written in the form :

$$f = \det(h_{ij})_{1 \le i,j \le n} \tag{1.14}$$

with

$$h_{ij} = c_{ij} + \int^{x} dx' \psi_i(x', y, t) \varphi_j(x', y, t) \tag{1.15}$$

where

$$\left[\partial_x^3 - \partial_t\right] \psi_i(x,y,t) = 0, \tag{1.16}$$

$$\left[\partial_x^3 - \partial_t\right] \varphi_j(x,y,t) = 0, \tag{1.17}$$

$$\left[\partial_x^2 + \partial_y\right] \varphi_j(x,y,t) = 0, \tag{1.18}$$

$$\left[\partial_x^2 - \partial_y\right] \psi_i(x,y,t) = 0, \tag{1.19}$$

and c_{ij} arbitrary constants. Depending on the choice of the ψ_i and φ_j, we obtain different types of solutions e.g. solitons or explode-decay solitons. This sort of solution can be inferred from inverse scattering theory [100] [92]. Using direct methods we can avoid the use of the Gelfand-Levitan-Marchenko (GLM) integral equation, and instead make use of the Jacobi identity. Essentially we make an ansatz of a broad class of solutions, and carry out a direct verification that these satisfy the bilinear equations, which turn out to be a Jacobi identity. The basic Jacobi identity [10] is given as follows : consider an $n \times n$ matrix A, we write $A^{i,\cdots,j}_{k,\cdots,l}$ for the minor obtained by omitting the ith, $\cdots$,jth rows and the kth,$\cdots$,lth columns. In this notation the Jacobi identity is

$$|A|\, A^{i,j}_{k,l} = \begin{vmatrix} A^i_k & A^j_k \\ A^i_l & A^j_l \end{vmatrix}. \tag{1.20}$$

In practice we are normally required to check certain conditions for the grammian to be a solution, such as reality. Due to the nature of the entries in the grammian determinant, this is much simpler to do than in the case of the wronskian determinants.

Nakamura [75] was the first to explicitly propose such solutions. In 1989 he determined a bilinear n-soliton solution for the KP equation, which also had the advantage that periodic waves of explode-decay type could easily be derived. He showed that the method was applicable to the KdV equation, cylindrical KP equation, Boussinesq and Toda equations. This work was generalized by Miyake, Ohta and Satsuma [71] to include a grammian approach to all the equations in the KP hierachy, via a Plücker relation. Further in this paper the explicit link between grammian solutions and the GLM equation was demonstrated. The grammian approach was used further with the nonlinear Schrödinger, the Heisenberg spin and cylindrical Heisenberg spin equations by Nakamura [76].

The grammian method and wronskian method are both based on a determinant, however whereas nth-order wronskian solution requires entries with $(n-1)$ differentiations, the nth-order grammian has entries with only one simple integration. Simple soliton solutions, such as the sech type solutions, correspond to a bilinear form that is composed of exponentials. Hence whether we calculate derivatives of these exponentials for wronskian solutions, or integrals for grammian solutions, there is little to choose between the two methods. However with explode-decay type solitons (Airy functions) the derivatives are difficult to calculate, so the grammian approach has a distinct advantage over wronskian solutions. In fact it was whilst attempting to generalise cylindrical solitons for the periodic Toda equation, that Nakamura [74] was led to propose the grammian approach for continous systems. Gilson and Nimmo [35] also showed the advantages of the method in their paper of 1991. They used the grammian method obtain dromion solutions, and carried out a detailed asymptotic analysis.

1.2 Outline and Motivation

1.2.1 Outline

This thesis is organised as follows. The remainder of this chapter provides some motivation as to why a grammian method should be applied to nonlinear evolution equations, and to the connections that exist between grammians, Darboux transformations (DT) and rational solutions.

Chapter two concerns the Davey-Stewartson (DS) equation and the rational solutions which correspond to the non-singular lump solutions of Ablowitz and Satsuma. We start by briefly describing the physical background to the DS equation and its derivation. We further outline the role of grammians and wronskians to obtain the different types of solutions that exist; dromions, solitoffs and solitons.

Section 2.2 is an extended example, which seeks to use the KP equation to motivate the approach that is adopted for the DS equation. In particular the way the "long-wave" limit can be adapted to determinants to give rational solutions. This avoids the rather complicated summation process of Satsuma and Ablowitz.

In section 2.3 we verify that grammians are solutions to the DS equation and examine the various possibilities that exist for the choice of F. The reality and conjugacy conditions are proved.

We move on from these grammian solutions in section 2.4 to calculate rational solutions by taking the "long-wave" limit. This gives in general a class of singular solutions.

Section 2.5 compares and contrasts the equations and method of Ablowitz and Satsuma with the grammian method. We look in detail at the equations and transformations and then attempt to obtain a wider class of solutions. By relaxing the reality and conjugacy conditions, and again taking the "long-wave" limit we determine new rational solutions. By then verifying when these are solutions to the DS equation we construct non-singular rational solutions, which describe multiple collisions of lumps. Finally these solutions are compared with the ones of Ablowitz and Satsuma.

In chapter three we investigate features of this three dimensional three-wave interaction (3D3WR) problem from the point of view of the bilinear method and grammians. The equations and solutions described originally by Kaup can be easily recast in terms of a bilinear formulation. The equations correspond to lowest weight equations in the KP three-component hierachy.

In section 3.2 we recall the Painlevé test and the way it connects with Hirota's method. The singularity analysis is carried out, and the Laurent expansion allows us to generate the Bäcklund transformations which leads to the bilinear form. The connection between this and Kaup's Bäcklund transformation is noted.

In section 3.3 we consider the one and two-lump solutions obtained by Kaup via the Bäcklund transformation [51]. We re-write these in a grammian form, which we can then generalise to obtain an n-lump solution.

Section 3.4 gives a direct proof of solutions in the grammian form. This leads us to examine, in section 3.5 a set of more general solutions.

Finally in section 3.6 we shall look at some explicit examples and get an idea of what they look like.

In chapter four we continue to consider 3D3WR. Here the focus is on the Darboux transformation (DT) and binary Darboux transformation (BDT) and their role in obtaining new solutions. The introductory material provides a definition of the DT and the link to nonlinear evolution equations via the Lax pair. We consider how the inadequacy of the DT leads us to consider the BDT, and demonstrate this through the example of the KP equation.

In section 4.2 we start from the scattering problem for 3D3WR and re-write it as an appropriate matrix Lax pair. This allows us in section 4.3 to calculate and define the DT for 3D3WR. We find that we can iterate our DT n times and obtain closed form expressions for the new eigenfunctions. However the 3D3WR places certain constraints on our new solutions and we find the DT does not preserve these.

This leads us in section 4.4 to derive the composite mapping; the BDT. This has the advantage that the new solutions obtained from it, preserve the necessary structure and are given in a grammian form. By iterating the BDT we build up a hierachy of solutions which can be written in grammian form.

The solutions obtained in section 4.4 are the same as the lump solutions discussed in section 3.3 and we demonstrate the connections in section 4.5. We also discuss the role of the Bäcklund transformation compared with that of the BDT.

Finally, in section 4.5.5 we look at how we may consider applications of the BDT to a solution of the 3D3WR as giving rise to a discrete version of the equation.

Chapter five seeks to provide conclusions to each of the chapters and a summary of the results that we have obtained. We also discuss the open questions that are prompted by this thesis, and propose various areas in which the work may be extended to new areas.

1.2.2 Motivation

We give three pointers as to why the grammian method is to be preferred over other approaches to nonlinear evolution equations. Firstly the grammian method provides a compact and simple way of obtaining a broad class of solutions. As stated above, whilst the solutions could be inferred from inverse scattering, the great advantage lies in the fact that we avoid either the GLM equation or the need for difficult summation procedure. The Jacobi identity gives an elegant and beautiful way of proving a very general solution.
Secondly the integral nature of the entries. This gives rise to two specific advantages. Firstly that unlike the wronskian approach, in the nth-order solution we avoid the necessity of entries with $(n-1)$ derivatives. Then similarly, when derivatives are difficult to calculate (for example Airy functions), grammians give us new solutions. Finally the grammian method gives a unified way of looking at nonlinear evolution equations, as we attempt to find new solutions. For instance the relationship between rational solutions for the KdV equation and DT is spelt out in [70]. This relationship applies more generally to other nonlinear evolution equations. Grammians also appear naturally when the BDT is calculated.

Chapter 2

Davey-Stewartson Equation

2.1 Background

Physical Background

The Davey-Stewartson (DS) equation was derived by Davey and Stewartson in 1974 [25] to describe the evolution of three-dimensional surface waves. It arises in a wide range of physical applications, such as water waves, plasma physics and nonlinear optics. Whitham [95] had originally looked at the evolution of waves with slowly varying amplitude, moving under gravity in water of finite depth. However it was difficult to understand in what sense the amplitude was varying slowly. To improve on this, Davey and Stewartson made use of the multiple scales method, where a small parameter ϵ say is built into the expansion.

Davey and Stewartson introduced the following velocity potential $\phi(x, y, z, t)$ satisfying

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{2.1}$$

with appropriate boundary conditions. Writing

$$\phi(x, y, z, t) = \sum_{n=-\infty}^{\infty} \phi_n \exp[in(kx - \omega t)], \tag{2.2}$$

with the parameter

$$\phi_n = \sum_{j=n}^{\infty} \epsilon^j \phi_{nj} \tag{2.3}$$

and appropriately scaling dependent variables we have the general DS system of equations

$$iu_t + \frac{1}{2}(\epsilon_1 u_{xx} + u_{yy}) - u(v + \epsilon_2|u|^2) = 0, \tag{2.4}$$

$$\delta_1 v_{xx} + v_{yy} - \delta_2|u|^2_{xx} = 0. \tag{2.5}$$

δ_i and ϵ_i with $i = 1, 2$ are real constants, with $\epsilon_i = \pm 1$. We note that if we consider the waves as purely one-dimensional (in the y direction, so $u_x = 0$) then v can be taken as zero and we have the cubic nonlinear Schrödinger equation (NLS)

$$iu_t + \frac{u_{yy}}{2} = \epsilon_2|u|^2 u. \tag{2.6}$$

The DS system has a similar relationship to the NLS equation, as the KP equation has with the KdV equation, in that they provide a two-dimensional generalisation. Equations (2.4) and (2.5) have two basic cases, the first with $\epsilon_i = 1$ and $\delta_i = -1$ gives the DSI (hyperbolic case), the second $\epsilon_1 = -1$ and $\epsilon_2 = \delta_i = 1$ is DSII (elliptic case). Both DSI and DSII are completely integrable [100] [3].

For a fuller discussion on the derivation and physical applications of the DS equation, (particularly inverse scattering and the role of boundary conditions) see [1] and the references contained within.

Rational Solutions

The integrability of the DS system means that the n-soliton solution can be obtained [11]. Satsuma and Ablowitz [89] used the n-soliton solution to obtain rational solutions to the DS equation with non-zero boundary conditions, by taking a "long wave" limit. If the parameters are chosen carefully then this solution describes a non-singular lump, decaying in all directions.

Rational solutions are of interest both physically and theoretically. They were first discovered for the KdV equation by Airault, McKean and Moser [9] in 1977, with further work by Adler and Moser [8]. Since then a wide variety of methods have been developed to obtain rational solutions. Our aim is to use a "long wave" limit on the DS equation with non-zero asymptotic state, combined with grammians so avoiding the complex sum of exponentials.

Recent work on rational solutions for the DS equation was carried out by Pelinovsky [84]. He obtained rational solutions for the DSII equation, by making use of the general dressing method. The work by Mañas and Santini [65], again concentrates on the DSII equation.

Grammians and the DS system

Gilson and Nimmo [35] were the first to apply the grammian approach of Nakamura [75] to the DS equation, however this was with zero asymptotic state. We aim to show that the bilinear form of the DSI equation with non-zero asymptotic state has grammian solutions. It is from these that we attempt to calculate rational solutions. The grammian approach of Gilson and Nimmo gives rise to a class of solutions called dromions. The DS system was the first equation to be found which had this wider class of solutions. Originally they were discovered by use of a Bäcklund transformation [13], and then a wide variety of generalisations have been obtained, including the inverse scattering method of Fokas and Santini [28]. (It was Fokas and Santini who coined the term "dromion", from "dromos" for track). Essentially the DS system was found to posses exponentially localized coherent structures, which unlike the lump solutions had non-trivial interaction properties. In the simple case, the solution is localized in the complex u field, whilst the v field consists of a pair of perpendicular quasi-one-dimensional solitons. For a good summary of recent work see the review article by Boiti, Martina and Pempinelli [14].
The grammian approach has also been exploited by Gilson [34] to obtain solutions termed "solitoffs". These are somewhere in between solitons and dromions, and are localized in every direction except one, so that they end to a non-zero value in only one direction. The difference between solitons, solitoffs and dromions is down to the choice of grammian solution. If we represent our solution by a grammian determinant which contains the spectral parameters and a Hermitian matrix H, then it is the choice of H that is significant in deciding the type of solution.

Wronskian Solutions

Wronskian solutions were first derived for the KdV and modified KdV equation by Satsuma in 1979 [88]. However the bulk of the work was carried out by Freeman and Nimmo, for example see reference [30]. In particular they used a wronskian form to obtain rational solutions to the KdV equation [81]. This work was extended to include the DS system by Freeman [29] and then Hirota, Ohta and Satsuma [43]. We define a wronskian solution for the DS equation

$$F = W(\varphi_1, \varphi_2, \dots, \varphi_n) \tag{2.7}$$

$$= \begin{vmatrix} \varphi_1 & \cdots & \varphi_1^{(n-1)} \\ \vdots & & \vdots \\ \varphi_n & \cdots & \varphi_n^{(n-1)} \end{vmatrix} \tag{2.8}$$

$$= \left|\boldsymbol{\varphi}, \dots, \boldsymbol{\varphi}^{(n-1)}\right| \tag{2.9}$$

with

$$\varphi_j^{(i)} = \frac{\partial^i \varphi_j}{\partial x^i} \qquad\qquad \boldsymbol{\varphi}^i = \frac{\partial^i \boldsymbol{\varphi}}{\partial x^i} \tag{2.10}$$

and $\varphi_j = \varphi_j(x, y, t)$ satisfying the equations

$$\varphi_{j,t} + i\left[\varphi_{j,xx} - \varphi_{j,yy}\right] = 0 \tag{2.11}$$

$$\varphi_{j,y} + \rho_0^2 \int \varphi_j \, dx = 0, \tag{2.12}$$

and $\boldsymbol{\varphi} = [\varphi_1, \varphi_2, \dots, \varphi_n]^T$. We adopt the notation of Freeman [29] and omit reference to the function $\boldsymbol{\varphi}$, so that (2.9) becomes

$$F = \left(\widehat{n-1}\right), \tag{2.13}$$

where $\widehat{n-1}$ denotes that n columns are indicated. More generally we will write

$$\left(\widehat{n-i}, n-k_1, n-k_2, \dots, n-k_{i-1}\right) \tag{2.14}$$

for a general $n \times n$ determinant where $\widehat{n-i}$ denotes $n-i+1$ consecutive columns $0, 1, 2, \dots, n-i$ and the $n-k_j$ with $j = 1, \dots, i-1$, are individual columns in which $\boldsymbol{\varphi}$ is differentiated $n-k_j$ times, k_j is one of the integers $0, \dots, i$.

The n-soliton format originally obtained by Anker and Freeman [11], has a determinant with sums of exponentials, here differentiation leads to a sum of n determinants. In contrast the wronskian only has a contribution from the last column (differentiation of most of the columns leads to a later column and zero contribution). Hence the derivative of a wronskian is a single determinant. For example differentiating (2.13) by x gives

$$F_x = (\widehat{n-2}, n). \tag{2.15}$$

Higher derivatives leads to a sum of determinants, but it depends on the number of differentiations not on n. The proof of the n-soliton is easily done by application of a Laplace expansion.

Hietarinta and Hirota [40] [39] used this method to obtain dromion solutions to the DS system. The method however is complex, as in the bilinear form the reality of F is difficult to check. So in that sense the grammian approach of Gilson and Nimmo is much easier to work with as the reality condition is simple to show.

2.2 Rational Solutions for the KP Equation

As an example which motivates the work to be presented in this chapter, consider the Kadomstev and Petviashvili equation [47] (KP, a two-dimensional generalisation of the Korteweg-de Vries). It describes the evolution of weakly nonlinear, weakly dispersive and weakly two-dimensional waves (all three effects being of equivalent order), and can be used as a model for surface waves and internal waves in straits or channels of varying depth and width. It is written

$$(u_t + 6uu_x + u_{xxx})_x + 3\alpha u_{yy} = 0 \tag{2.16}$$

where $u = u(x, y, t)$ and $\alpha = \pm 1$, a parameter depending on the dispersive property of the system. $\alpha = 1$ is the KPII equation, with $\alpha = -1$ giving KPI.

The idea is to construct rational solutions in the independent variables, in a simple and elegant way by exploiting the phase constants in the n-soliton solution. By considering the dependent variable transformation

$$u = 2(\log f)_{xx} \tag{2.17}$$

where $f = f(x, y, t)$ we obtain the Hirota form of the KP equation [42]

$$\left[D_x(D_x^3 + D_t) + 3\alpha D_y^2\right] f \cdot f = 0 \tag{2.18}$$

with D_x, D_y and D_t defined as (1.6). The n-soliton solution of (2.18) can be ascertained from inverse scattering theory [66], [29] and is given by (for $\alpha = 1$)

$$f = \left|\delta_{ij} + \frac{a_i}{p_i + q_j} e^{\theta_i + \psi_i}\right| \tag{2.19}$$

where $\theta_i = p_i x + p_i^2 y - 4p_i^3 t$ and $\psi_i = q_i x - q_i^2 y - 4q_i^3 t$ with p_i, q_i and a_i complex constants for $i = 1, \dots, n$. If we use the substitution $y \to iy$ we obtain the solution of (2.18) with $\alpha = -1$.

We exploit the arbitrary nature of the a_i to allow us to calculate rational solutions. We adapt the method of Ablowitz and Satsuma [7], [89] and take a "long-wave limit" of the n-soliton solution.

We rewrite the n-soliton solution in the following way

$$\begin{aligned} f &= \left| e^{\theta_i + \psi_i} \left(\delta_{ij} e^{-\theta_i - \psi_i} + \frac{a_i}{p_i + q_j} \right) \right| & (2.20) \\ &= \prod_{i=1}^{n} e^{\theta_i + \psi_i} \left| \delta_{ij} e^{-\theta_i - \psi_i} + \frac{a_i}{p_i + q_j} \right|. & (2.21) \end{aligned}$$

Choosing every $a_i = -(p_i + q_i)$ in (2.21), then (2.21) has the following terms on the diagonal

$$e^{-\theta_i - \psi_i} - 1 = (1 - (\theta_i + \psi_i) + (\theta_i + \psi_i)^2/2! - \dots) - 1. \tag{2.22}$$

On the off-diagonal we have $-\frac{p_i + q_i}{p_i + q_j}$. We define $k_i = p_i + q_i$ and $l_i = p_i - q_i$ for $i = 1, \dots, n$. Putting these together we can express (2.21) as

$$f = \prod_{i=1}^{n} e^{\theta_i + \psi_i} \begin{vmatrix} -k_1\alpha_1 + O(k_1^2) & \frac{-2k_1}{l_1 - l_2 + k_1 + k_2} & \cdots & \frac{-2k_1}{l_1 - l_n + k_1 + k_n} \\ \frac{-2k_2}{l_2 - l_1 + k_2 + k_1} & -k_2\alpha_2 + O(k_2^2) & \cdots & \frac{-2k_2}{l_2 - l_n + k_2 + k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2k_n}{l_n - l_1 + +k_n + k_1} & \frac{-2k_n}{l_n - l_2 + k_n + k_2} & \cdots & -k_n\alpha_n + O(k_n^2) \end{vmatrix} \tag{2.23}$$

simplifying

$$f = (-1)^n \prod_{i=1}^{n} k_i e^{\theta_i+\psi_i} \begin{vmatrix} \alpha_1 + O(k_1) & \frac{2}{l_1-l_2+k_1+k_2} & \cdots & \frac{2}{l_1-l_n+k_1+k_n} \\ \frac{2}{l_2-l_1+k_2+k_1} & \alpha_2 + O(k_2) & \cdots & \frac{2}{l_2-l_n+k_2+k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{l_n-l_1+k_n+k_1} & \frac{2}{l_n-l_2+k_n+k_2} & \cdots & \alpha_n + O(k_n) \end{vmatrix}, \quad (2.24)$$

with $\alpha_i = [x + l_i y - (k_i^2 + 3l_i^2)t]$. We recall that u is invariant under transformations of the form $f \to af$, with a either a constant or exponential function of x, y, t. By taking the limit $k_i \to 0$ for $i = 1, \ldots, n$, with $l_i = O(1)$, we write f as

$$f = \begin{vmatrix} \eta_1 & \frac{2}{l_1-l_2} & \cdots & \frac{2}{l_1-l_n} \\ \frac{2}{l_2-l_1} & \eta_2 & \cdots & \frac{2}{l_2-l_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{l_n-l_1} & \frac{2}{l_n-l_2} & \cdots & \eta_n \end{vmatrix}, \quad (2.25)$$

and $\eta_i = (x + l_i y - 3l_i^2 t)$. This is our general rational solution, it may be compared directly to Ablowitz and Satsuma's determinant [89]

$$f_n = \begin{vmatrix} \theta_1 & \sqrt{B_{12}} & \cdots & \sqrt{B_{1n}} \\ -\sqrt{B_{12}} & \theta_2 & \cdots & \sqrt{B_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\sqrt{B_{1n}} & -\sqrt{B_{2n}} & \cdots & \theta_n \end{vmatrix}, \quad (2.26)$$

where $\theta_i = (x + p_i y - \alpha p_i^2 t)$ and $B_{ij} = 12/[\alpha(p_i - p_j)^2]$. If we set $\alpha = 3$ and $p_i = l_i$ then (2.26) is equal to (2.25). So we see that we may obtain the rational solutions of Ablowitz and Satsuma via a "long-wave" limit on a grammian determinant. This has the advantage of avoiding the difficult summation process.

Considering $n = 2$, (2.25) gives

$$f_2 = \eta_1\eta_2 + \frac{4}{(l_1 - l_2)^2}. \quad (2.27)$$

To obtain nonsingular solutions we need $\eta_2 = \eta_1^*$, i.e. $l_2 = l_1^*$ (with * being complex conjugation). With this condition we have

$$f_2 = \eta_1\eta_1^* + \frac{4}{(l_1 - l_1^*)^2} \quad (2.28)$$

$$= \eta_1\eta_1^* - \frac{1}{l_I^2} \qquad l_1 = l_R + il_I, \quad (2.29)$$

which gives rise to singular u. However if we consider KPI ($\alpha = -1$), with (2.25) and $y \to iy$, then under $l_2 = -l_1^*$, f_2 is written

$$f_2 = \eta_1\eta_1^* + \frac{4}{(l_1 + l_1^*)^2} \tag{2.30}$$

$$= \eta_1\eta_1^* + \frac{1}{l_R^2}, \tag{2.31}$$

using $l_1 = l_R + il_I$. We note further

$$f_2 = \left[x - 3t(l_R^2 + l_I^2)\right]^2 + (l_R^2 + l_I^2)\left[y - 6l_I t\right]^2 - 2l_I\left[x - 3t(l_R^2 + l_I^2)\right]\left[y - 6l_I t\right] + 1/l_R^2, \tag{2.32}$$

which gives

$$f_2 = (X - l_I Y)^2 + l_R^2 Y^2 + 1/l_R^2 > 0, \tag{2.33}$$

where

$$X = x - 3t(l_R^2 + l_I^2) \qquad \text{and} \qquad Y = y - 6l_I t. \tag{2.34}$$

Inserting (2.33) into (2.17), we obtain

$$u_2 = 2\left(\log[(X - l_I Y)^2 + l_R^2 Y^2 + 1/l_R^2]\right)_{xx} \tag{2.35}$$

$$= \frac{4\left[-(X - l_I Y)^2 + l_R^2 Y^2 + 1/l_R^2\right]}{\left[(X - l_I Y)^2 + l_R^2 Y^2 + 1/l_R^2\right]^2}. \tag{2.36}$$

The rational solution is a real nonsingular function decaying with order $O(1/x^2, 1/y^2)$ for $|x|$, $|y| \to \infty$, and is known as a lump solution. It moves with velocity $v_x = 3(l_R^2 + l_I^2)$ and $v_y = 6l_I$ (obtained from (2.34)).

In general (2.25) gives nonsingular solutions provided we choose

$$n = 2M, \qquad \text{and} \qquad l_{M+i} = -l_i^* \tag{2.37}$$

for $i = 1, 2, \ldots, M$. So then f has the following structure

$$f_{2M} = \begin{vmatrix} C & A \\ -A^* & (C^*)^T \end{vmatrix}, \tag{2.38}$$

where C and A are $M \times M$ matrices defined by

$$C = \begin{bmatrix} \eta_1 & \frac{2}{l_1-l_2} & \cdots & \frac{2}{l_1-l_M} \\ \frac{2}{l_2-l_1} & \eta_2 & \cdots & \frac{2}{l_2-l_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{l_M-l_1} & \frac{2}{l_M-l_2} & \cdots & \eta_M \end{bmatrix}, \tag{2.39}$$

and

$$A = \begin{bmatrix} \frac{2}{l_1+l_1^*} & \frac{2}{l_1+l_2^*} & \cdots & \frac{2}{l_1+l_M^*} \\ \frac{2}{l_2+l_1^*} & \frac{2}{l_2+l_2^*} & \cdots & \frac{2}{l_2+l_M^*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{l_M+l_1^*} & \frac{2}{l_M+l_2^*} & \cdots & \frac{2}{l_M+l_M^*} \end{bmatrix}. \tag{2.40}$$

In this case the determinant (2.38) is always positive [66], so (2.17) does not become singular at any point in x or y. This gives the general n-lump solution to the KP equation. We note for future comparisons the solution describes the interaction of M lumps ($2M = n$). Making use of an expansion of (2.25) we may rewrite it as [89]

$$f_n = \prod_{i=1}^{n} \eta_i + \frac{1}{2} \sum_{i,j}^{(n)} D_{ij} \prod_{l \neq i,j}^{n} \eta_l + \cdots \\ + \frac{1}{L!2^L} \sum_{i,j,\dots,m,o}^{(n)} \overbrace{D_{ij} D_{kl} \dots D_{mo}}^{L} \prod_{p \neq i,j,\dots,m,o}^{n} \eta_p + \cdots \tag{2.41}$$

where $D_{ij} = \frac{4}{(l_i - l_j)^2}$ and $\sum_{i,j,\dots,m,o}^{(n)}$ denotes the summation over all possible combinations of $i, j, \dots, m, o$ which are taken from $1, 2, \dots, n$ and all different. We fix ourselves on one particular phase, η_L say (i.e. We travel with that particular phase) , so that $|\eta_L|^2$ is a constant. Then in the limit $t \to \pm\infty$, (with the condition (2.37))

$$|\eta_1|, |\eta_2|, \dots, |\eta_{L-1}|, |\eta_{L+1}|, \dots, |\eta_M| = O(t) \tag{2.42}$$

and f_{2M} has the following asymptotic state

$$f_{2M} \sim |\eta_1|^2 |\eta_2^2| \cdots |\eta_M|^2 + D_{L,M+L} |\eta_1|^2 \cdots |\eta_{L-1}|^2 |\eta_{L+1}|^2 \cdots |\eta_M|^2, \tag{2.43}$$

where we have used the fact that $|\eta_L|$ is a constant. Using the invariance of u (2.43) is equivalent to

$$f_{2M} \to |\eta_L|^2 + D_{L,M+L}, \tag{2.44}$$

which is a lump with phase η_L. This is true for all the possible M lumps, as they all have different velocities. So (2.25) describes the collision of M lumps. We notice that (2.44) is the same at $t = \pm\infty$, so there is no phase shift due to the interaction. We make use of similar determinant identities for the Davey-Stewartson equation to obtain rational solutions.

2.3 Set-up

The DSI equation can be expressed as [34]

$$iu_t + \Delta u + vu = 0 \tag{2.45}$$

$$v_{xy} - 2\Delta|u|^2 = 0 \tag{2.46}$$

where $\Delta = \partial_x^2 + \partial_y^2$, with u and v functions of x, y and t. Introducing the new dependent variables; F which is real, and G which is complex

$$u = \frac{G}{F} \tag{2.47}$$

$$v = 2\Delta \log F + a(x,t) + b(y,t), \tag{2.48}$$

the DSI equation is written in Hirota form

$$\left[iD_t + D_x^2 + D_y^2\right] G \cdot F - 2\left[a(x,t) + b(y,t)\right] GF = 0 \tag{2.49}$$

$$D_x D_y F \cdot F - 2\left[|G|^2 - \rho_0^2 F^2\right] = 0 \tag{2.50}$$

with D_x, D_y and D_t the Hirota derivatives defined in the previous section. The arbitary functions $a(x,t)$ and $b(y,t)$ correspond to the non-trivial boundary conditions on v, for a discussion of boundary conditions see [28]. For simplicity we consider the case $a(x,t) = b(y,t) = 0$. We have restricted ourselves to the boundary condition of non-zero asymptotic state i.e. $|u|^2 \to \rho_0^2$ as $|x| \to \infty$.

The non-zero background leads to restrictions in the type of solutions that we can consider. We verify a broad class of grammian solutions; depending on arbitary functions. These solutions are contrasted with the solutions obtained by [89].

2.3.1 Grammian Type Solutions

This type of solution is expressed in terms of grammian type determinants and is verified using identities from Jacobi's formula [10]. Consider a function F of the form

$$F = |\mathfrak{F}| = |I + H\Phi| \tag{2.51}$$

where Φ is a $n \times n$ matrix of the following form

$$\Phi = \left[\int_{-\infty}^{x} \varphi_i \overline{\psi}_j \, dx\right], \tag{2.52}$$

and H is a constant Hermitian matrix, with $\overline{\psi_j}$ indicating complex conjugation of ψ_j. The φ_i, ψ_j are functions of x, y, and t that satisfy

$$\varphi_{j,t} + i\left[\varphi_{j,xx} - \varphi_{j,yy}\right] = 0 \tag{2.53}$$

$$\varphi_{j,y} + \rho_0^2 \int \varphi_j \, dx = 0, \tag{2.54}$$

and $i, j \in \{1, \ldots, n\}$. We show the necessary conditions for F to be real later on. To prove F and G satisfy (2.49) and (2.50) we require various derivatives. These derivatives can be expressed as bordered determinants. In general for an $n \times n$ matrix A whose entries a_{ij} are such that the derivatives $a'_{ij} = \alpha_i \beta_j$, the derivative of its determinant can be written as [35], [10]

$$|A|' = \sum_{i,j=1}^{n} (-1)^{i+j} \alpha_i \beta_j A_{ij} = -\begin{vmatrix} 0 & \beta_1 & \cdots & \beta_n \\ \alpha_1 & & & \\ \vdots & & A & \\ \alpha_n & & & \end{vmatrix} \tag{2.55}$$

where A_{ij} is the (i,j)th minor of A. We recall that differentiating an $n \times n$ determinant leads to the sum of n determinants, obtained from differentiating each row in term. So for F we obtain

$$F_x = -\begin{vmatrix} 0 & \overline{l}^T \\ Hr & \mathfrak{F} \end{vmatrix}, \tag{2.56}$$

$$F_y = \begin{vmatrix} 0 & \overline{k}^T \\ Hs & \mathfrak{F} \end{vmatrix}, \tag{2.57}$$

where

$$r = (\varphi_1, \ldots, \varphi_n)^T, \tag{2.58}$$

$$s = \rho_0 \left(\int \varphi_1 \, dx, \ldots, \int \varphi_n \, dx \right)^T, \tag{2.59}$$

$$l = (\psi_1, \ldots, \psi_n)^T, \tag{2.60}$$

$$k = \rho_0 \left(\int \psi_1 \, dx, \ldots, \int \psi_n \, dx \right)^T. \tag{2.61}$$

The second derivatives of F are obtained from the bordered form of the first derivatives; for the xx and yy derivatives the only effect being to differentiate the terms in the borders.

$$F_{xx} = -\begin{vmatrix} 0 & \overline{l}^T \\ Hr_x & \mathfrak{F} \end{vmatrix} - \begin{vmatrix} 0 & \overline{l}_x^T \\ Hr & \mathfrak{F} \end{vmatrix} \tag{2.62}$$

$$F_{yy} = \begin{vmatrix} 0 & \overline{k}^T \\ Hs_y & \mathfrak{F} \end{vmatrix} + \begin{vmatrix} 0 & \overline{k}_y^T \\ Hs & \mathfrak{F} \end{vmatrix}. \tag{2.63}$$

The xy-derivative also has a contribution where the border has two rows and two columns,

$$F_{xy} = \rho_0 \begin{vmatrix} 0 & \overline{k}^T \\ Hr & \mathfrak{F} \end{vmatrix} + \rho_0 \begin{vmatrix} 0 & \overline{l}^T \\ Hs & \mathfrak{F} \end{vmatrix} - \begin{vmatrix} 0 & 0 & \overline{l}^T \\ 0 & 0 & \overline{k}^T \\ Hr & Hs & \mathfrak{F} \end{vmatrix}. \tag{2.64}$$

We have noted that $s_x = \rho_0 r$ and $k_x = \rho_0 l$. The t-derivative is calculated by considering (2.53) so that

$$\partial_t \int_{-\infty}^{x} \varphi_i \overline{\psi}_j \, dx = -i \int_{-\infty}^{x} \left[\left(\varphi_{i,xx} - \varphi_{i,yy} \right) \overline{\psi}_j - \varphi_i \left(\overline{\psi}_{j,xx} - \overline{\psi}_{j,yy} \right) \right] dx, \tag{2.65}$$

$$= i \left[-\varphi_{i,x} \overline{\psi}_j + \varphi_i \overline{\psi}_{j,x} - \frac{1}{\rho_0^2} \varphi_{i,yy} \overline{\psi}_{j,y} + \frac{1}{\rho_0^2} \varphi_{i,y} \overline{\psi}_{j,yy} \right], \tag{2.66}$$

by parts. Using (2.66) we obtain

$$F_t = -i \left[\begin{vmatrix} 0 & \overline{l}_x^T \\ Hr & \mathfrak{F} \end{vmatrix} - \begin{vmatrix} 0 & \overline{l}^T \\ Hr_x & \mathfrak{F} \end{vmatrix} - \begin{vmatrix} 0 & \overline{k}^T \\ Hs_y & \mathfrak{F} \end{vmatrix} + \begin{vmatrix} 0 & \overline{k}_y^T \\ Hs & \mathfrak{F} \end{vmatrix} \right]. \tag{2.67}$$

We wish to determine the expression corresponding to F given by (2.51). In order to do this we show that the Hirota equation (2.50) corresponds to a Jacobi-type identity. (2.50) is written

$$\frac{1}{2}D_xD_yF.F-|G|^2+\rho_0^2F^2=\rho_0|\mathfrak{F}|\left[\begin{vmatrix}0 & \overline{k}^T\\ Hr & \mathfrak{F}\end{vmatrix}+\begin{vmatrix}0 & \overline{l}^T\\ Hs & \mathfrak{F}\end{vmatrix}\right]$$

$$-|\mathfrak{F}|\begin{vmatrix}0 & 0 & \overline{l}^T\\ 0 & 0 & \overline{k}^T\\ Hr & Hs & \mathfrak{F}\end{vmatrix}+\begin{vmatrix}0 & \overline{k}^T\\ Hr & \mathfrak{F}\end{vmatrix}\begin{vmatrix}0 & \overline{k}^T\\ Hr & \mathfrak{F}\end{vmatrix}+\rho_0^2|\mathfrak{F}||\mathfrak{F}|-G\overline{G} \quad (2.68)$$

which is simplified by the Jacobi identity to

$$\rho_0|\mathfrak{F}|\left[\begin{vmatrix}0 & \overline{k}^T\\ Hr & \mathfrak{F}\end{vmatrix}+\begin{vmatrix}0 & \overline{l}^T\\ Hs & \mathfrak{F}\end{vmatrix}+\rho_0|\mathfrak{F}|\right]+\begin{vmatrix}0 & \overline{l}^T\\ Hs & \mathfrak{F}\end{vmatrix}\begin{vmatrix}0 & \overline{k}^T\\ Hr & \mathfrak{F}\end{vmatrix}-G\overline{G}. \quad (2.69)$$

So the second of the Hirota equations i.e. (2.50), is satisfied provided

$$G = \rho_0|\mathfrak{F}|+\begin{vmatrix}0 & \overline{l}^T\\ Hs & \mathfrak{F}\end{vmatrix} \quad (2.70)$$

$$\overline{G} = \rho_0|\mathfrak{F}|+\begin{vmatrix}0 & \overline{k}^T\\ Hr & \mathfrak{F}\end{vmatrix}. \quad (2.71)$$

That (2.71) is the conjugate of (2.70) will be shown later. As with F we are able to calculate the derivatives of G in terms of bordered determinants,

$$G_x = \begin{vmatrix}0 & \overline{l}_x^T\\ Hs & \mathfrak{F}\end{vmatrix} \quad (2.72)$$

$$G_y = \begin{vmatrix}0 & \overline{l}^T\\ Hs_y & \mathfrak{F}\end{vmatrix}. \quad (2.73)$$

The second derivatives take the form

$$G_{xx} = \rho_0\begin{vmatrix}0 & \overline{l}_x^T\\ Hr & \mathfrak{F}\end{vmatrix}+\begin{vmatrix}0 & \overline{l}_{xx}^T\\ Hs & \mathfrak{F}\end{vmatrix}-\begin{vmatrix}0 & 0 & \overline{l}_x^T\\ 0 & 0 & \overline{l}^T\\ Hs & Hr & \mathfrak{F}\end{vmatrix} \quad (2.74)$$

$$G_{yy} = \begin{vmatrix}0 & \overline{l}^T\\ Hs_{yy} & \mathfrak{F}\end{vmatrix}-\rho_0\begin{vmatrix}0 & \overline{k}^T\\ Hs_y & \mathfrak{F}\end{vmatrix}+\begin{vmatrix}0 & 0 & \overline{l}^T\\ 0 & 0 & \overline{k}^T\\ Hs_y & Hs & \mathfrak{F}\end{vmatrix}. \quad (2.75)$$

The t-derivative takes the form

$$G_t = \rho_0 |\mathfrak{F}|_t + i \left[\begin{vmatrix} 0 & 0 & \bar{l}^T \\ 0 & 0 & \bar{k}^T \\ Hs & Hs_y & \mathfrak{F} \end{vmatrix} - \begin{vmatrix} 0 & 0 & \bar{l}^T \\ 0 & 0 & \bar{l}_x^T \\ Hs & Hr & \mathfrak{F} \end{vmatrix} \right]$$
$$+ i \left[\begin{vmatrix} 0 & \bar{l}^T \\ Hs_{yy} & \mathfrak{F} \end{vmatrix} - \rho_0 \begin{vmatrix} 0 & \bar{l}^T \\ Hr_x & \mathfrak{F} \end{vmatrix} + \begin{vmatrix} 0 & \bar{l}_{xx}^T \\ Hs & \mathfrak{F} \end{vmatrix} + \rho_0 \begin{vmatrix} 0 & \bar{k}_y^T \\ Hs & \mathfrak{F} \end{vmatrix} \right]. \tag{2.76}$$

Subsituting these derivatives into the LHS of (2.49) we get

$$|\mathfrak{F}| \left[-2 \begin{vmatrix} 0 & 0 & \bar{l}_x^T \\ 0 & 0 & \bar{l}^T \\ Hs & Hr & \mathfrak{F} \end{vmatrix} + 2 \begin{vmatrix} 0 & 0 & \bar{l}^T \\ 0 & 0 & \bar{k}^T \\ Hs_y & Hr & \mathfrak{F} \end{vmatrix} \right] + 2 \begin{vmatrix} 0 & \bar{l}^T \\ Hr & \mathfrak{F} \end{vmatrix} \begin{vmatrix} 0 & \bar{l}_x^T \\ Hs & \mathfrak{F} \end{vmatrix}$$
$$-2 \begin{vmatrix} 0 & \bar{k}^T \\ Hs & \mathfrak{F} \end{vmatrix} \begin{vmatrix} 0 & \bar{l}^T \\ Hs_y & \mathfrak{F} \end{vmatrix} - 2 \begin{vmatrix} 0 & \bar{l}^T \\ Hs & \mathfrak{F} \end{vmatrix} \begin{vmatrix} 0 & \bar{l}_x^T \\ Hr & \mathfrak{F} \end{vmatrix} + 2 \begin{vmatrix} 0 & \bar{l}^T \\ Hs & \mathfrak{F} \end{vmatrix} \begin{vmatrix} 0 & \bar{k}^T \\ Hs_y & \mathfrak{F} \end{vmatrix}, \tag{2.77}$$

this can be seen to be identically zero by using a pair of Jacobi identities, thus completing verification of the solution.

Discussion on Conditions

We have 2 conditions to check. Firstly that F is real, and then that (2.71) is the conjugate of (2.70). F may be written

$$F = |I + H\Phi| \tag{2.78}$$
$$= |H| \left| H^{-1} + \Phi \right| \tag{2.79}$$

so

$$\overline{F} = \overline{|H||H^{-1} + \Phi|} \tag{2.80}$$
$$= \left| H^T \right| \left| \overline{H}^{-1} + \overline{\Phi} \right| \tag{2.81}$$
$$= F, \tag{2.82}$$

providing

$$\varphi_i = \psi_i, \tag{2.83}$$

and where we have used the fact that H is Hermitian and that taking the transpose does not alter the value of the determinant. So F is real. Checking the condition on G amounts to demonstrating that

$$\overline{\begin{vmatrix} 0 & \overline{l}^T \\ Hs & \mathfrak{F} \end{vmatrix}} = \begin{vmatrix} 0 & \overline{k}^T \\ Hr & \mathfrak{F} \end{vmatrix}. \tag{2.84}$$

We define A as

$$A = \begin{vmatrix} 0 & \overline{l}^T \\ Hs & \mathfrak{F} \end{vmatrix}. \tag{2.85}$$

With the reality condition (2.83), $r = l$, $s = k$ and $\overline{\Phi}^T = \Phi$, so

$$\overline{A} = \begin{vmatrix} I & 0 \\ 0 & H^T \end{vmatrix} \begin{vmatrix} 0 & l^T \\ \overline{s} & \overline{H}^{-1} + \overline{\Phi} \end{vmatrix} \tag{2.86}$$

$$= \begin{vmatrix} I & 0 \\ 0 & H \end{vmatrix} \begin{vmatrix} 0 & \overline{s}^T \\ l & H^{-1} + \overline{\Phi}^T \end{vmatrix} \tag{2.87}$$

$$= \begin{vmatrix} I & 0 \\ 0 & H \end{vmatrix} \begin{vmatrix} 0 & \overline{k}^T \\ r & H^{-1} + \Phi \end{vmatrix} \tag{2.88}$$

$$= \begin{vmatrix} 0 & \overline{k}^T \\ Hr & \mathfrak{F} \end{vmatrix}. \tag{2.89}$$

This is clearly the RHS of (2.84). So we have proved necessary conditions for a solution. However it turns out that it is not possible to exploit the structure of F, to calculate rational solutions, with a general H. There is no simple way to take a limit, as the grammian structure is complex, and the exponential factors interconnected. However simple choices of H do work and we will take H as a constant diagonal matrix.

2.3.2 Alternative Grammian Solutions

The reason behind the choice of F in this format (2.51) can be seen by considering the difference between this and the F chosen by Gilson and Nimmo [35]. The F chosen by them takes the form

$$F = |\mathfrak{F}| = |I + H\Phi| \tag{2.90}$$

where Φ is a $(M+n)\times(M+n)$ matrix with block structure

$$\Phi = \begin{bmatrix} \int_{-\infty}^{x} \varphi_i \overline{\varphi}_j \, dx & 0 \\ 0 & \int_{y}^{\infty} \psi_k \overline{\psi}_l \, dy \end{bmatrix}. \tag{2.91}$$

The φ_i are functions of x and t and the ψ_k of y and t. We have $i,j \in \{1,\dots,M\}$ and $k,l \in \{1,\dots,n\}$. H is a constant Hermitian matrix of the same dimension as Φ. The non-zero boundary conditions, namely the ρ_0^2 term in (2.50), means that Gilson and Nimmo's method for calculation of G and $\overline{G}$, and hence verification of a solution breaks down. For this choice of F

$$F_x = -\begin{vmatrix} 0 & \overline{l}^T \\ Hl & \mathfrak{F} \end{vmatrix} \tag{2.92}$$

$$F_y = \begin{vmatrix} 0 & \overline{m}^T \\ Hm & \mathfrak{F} \end{vmatrix} \tag{2.93}$$

$$F_{xy} = -\begin{vmatrix} 0 & 0 & \overline{l}^T \\ 0 & 0 & \overline{m}^T \\ Hl & Hm & \mathfrak{F} \end{vmatrix} \tag{2.94}$$

where

$$l = (\varphi_1,\dots,\varphi_M;0,\dots,0)^T, \qquad m = (0,\dots,0;\psi_1,\dots,\psi_n)^T, \tag{2.95}$$

are both column vectors. Equation (2.50) implies

$$G\overline{G} = |\mathfrak{F}|\,|\mathfrak{F}| + \begin{vmatrix} 0 & \overline{l}^T \\ Hm & \mathfrak{F} \end{vmatrix} \begin{vmatrix} 0 & \overline{m}^T \\ Hl & \mathfrak{F} \end{vmatrix} \tag{2.96}$$

So it is not possible to choose a G such that we can obtain a simple solution. Hence the motivation behind choosing F in the form we do (2.51).

2.4 Obtaining Rational Solutions

2.4.1 Method

To obtain the rational solutions we start from the n-soliton solution in grammian form (2.51), (2.70). In this case we choose

$$\varphi_i = \exp\left[\rho_0\left[p_i x - \frac{y}{p_i} - i\rho_0 t\left(p_i^2 - \frac{1}{p_i^2}\right)\right]\right] \tag{2.97}$$

where $\varphi_i \to 0$ as $x \to \infty$, $H = \text{Diag}[c_1, c_2, \ldots, c_n]$, the reality condition (2.83) holds and the p_i are complex constants. Rewriting (2.51)

$$\begin{aligned} F &= \left|\delta_{ij} + c_i \int_{-\infty}^{x} \varphi_i \overline{\varphi}_j \, dx\right| && (2.98) \\ &= \left|\delta_{ij} + \frac{c_i \varphi_i \overline{\varphi}_j}{\rho_0 (p_i + \overline{p}_j)}\right| && (2.99) \\ &= \prod_{i=1}^{n} \varphi_i \overline{\varphi}_i \left|\delta_{ij} \varphi_i^{-1} \overline{\varphi}_i^{-1} + \frac{c_i}{\rho_0 (p_i + \overline{p}_j)}\right|. && (2.100) \end{aligned}$$

The method to obtain rational solutions from soliton solutions relies on the arbitariness of the constants c_i. We adapt the method of Ablowitz and Satsuma [7], [89] of obtaining rational solutions by taking a "long wave" limit. We choose

$$c_i = -\rho_0 P_i \tag{2.101}$$

for $i = 1, \ldots, n$, with $P_i = p_i + \overline{p}_i$ (note that this ensures that H is Hermitian). Writing

$$\begin{aligned} \varphi_i \overline{\varphi}_i &= \exp\left[\rho_0 P_i \left[x - \frac{y}{p_i \overline{p}_i} - i \rho_0 t \,(p_i - \overline{p}_i) \left(1 + \frac{1}{p_i^2 \overline{p}_i^2}\right)\right]\right] && (2.102) \\ &= \exp\left[\rho_0 P_i \alpha_{ii}\right] && (2.103) \end{aligned}$$

with

$$\alpha_{ii} = x - \frac{y}{p_i \overline{p}_i} - i \rho_0 t \,(p_i - \overline{p}_i) \left(1 + \frac{1}{p_i^2 \overline{p}_i^2}\right), \tag{2.104}$$

a real function. This means (2.100) has the following terms on the diagonal

$$\begin{aligned} \varphi_i^{-1} \overline{\varphi}_i^{-1} + \frac{c_i}{\rho_0 P_i} &= \left[1 - \rho_0 P_i \alpha_{ii} + \frac{\rho_0^2}{2} P_i^2 \alpha_{ii}^2 - \ldots\right] - 1 && (2.105) \\ &= -\rho_0 P_i \alpha_{ii} + \frac{\rho_0^2}{2} P_i^2 \alpha_{ii}^2 - \ldots && (2.106) \end{aligned}$$

Off the diagonal we have

$$\frac{c_i}{\rho_0 \,(p_i + \overline{p}_j)} = -\frac{P_i}{p_i + \overline{p}_j}. \tag{2.107}$$

Putting these together we can express (2.100) as

$$F = \prod_{i=1}^{n} \varphi_i \overline{\varphi}_i (-1)^n$$

$$\begin{vmatrix} \rho_0 P_1 \alpha_{11} - O(P_1^2) & \frac{P_1}{p_1+\overline{p}_2} & \cdots & \frac{P_1}{p_1+\overline{p}_n} \\ \frac{P_2}{p_2+\overline{p}_1} & \rho_0 P_2 \alpha_{22} - O(P_2^2) & \cdots & \frac{P_2}{p_2+\overline{p}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{P_n}{p_n+\overline{p}_1} & \frac{P_n}{p_n+\overline{p}_2} & \cdots & \rho_0 P_n \alpha_{nn} - O(P_n^2) \end{vmatrix}. \tag{2.108}$$

We may simplify this by removing factors of P_i from rows for $i = 1, \dots, n$, so that F becomes

$$F = \prod_{i=1}^{n} P_i \varphi_i \overline{\varphi}_i (-1)^n \begin{vmatrix} \rho_0 \alpha_{11} - O(P_1) & \frac{1}{p_1+\overline{p}_2} & \cdots & \frac{1}{p_1+\overline{p}_n} \\ \frac{1}{p_2+\overline{p}_1} & \rho_0 \alpha_{22} - O(P_2) & \cdots & \frac{1}{p_2+\overline{p}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_n+\overline{p}_1} & \frac{1}{p_n+\overline{p}_2} & \cdots & \rho_0 \alpha_{nn} - O(P_n) \end{vmatrix}. \tag{2.109}$$

We recall that v ($v = 2\Delta \log F$) is invariant under transformations of the form $F \to aF$ with a a constant or exponential function of x, y, t, so taking the limit $P_i \to 0$ for $i = 1, \dots, n$, with $p_i \overline{p}_j$ and $p_i - \overline{p}_j$ of order 1, we write F as

$$F = \begin{vmatrix} \rho_0 \alpha_{11} & \frac{1}{p_1+\overline{p}_2} & \cdots & \frac{1}{p_1+\overline{p}_n} \\ \frac{1}{p_2+\overline{p}_1} & \rho_0 \alpha_{22} & \cdots & \frac{1}{p_2+\overline{p}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_n+\overline{p}_1} & \frac{1}{p_n+\overline{p}_2} & \cdots & \rho_0 \alpha_{nn} \end{vmatrix}. \tag{2.110}$$

Hence we have obtained a rational determinant form for F. We use a similar procedure to obtain a rational form for G. From section 2.3 we have

$$A = \begin{vmatrix} 0 & \overline{l}^T \\ Hs & \mathfrak{F} \end{vmatrix} \tag{2.111}$$

$$= \prod_{i=1}^{n} \varphi_i \overline{\varphi}_j \begin{vmatrix} 0 & \overline{l'}^T \\ s' & \delta_{ij} \varphi_i^{-1} \overline{\varphi}_i^{-1} + \frac{a_i b_j}{\rho_0 (p_i + \overline{p}_j)} \end{vmatrix} \tag{2.112}$$

with $s' = (c_1/p_1, \dots, c_n/p_n)^T$ and $l' = (1, \dots, 1)^T$. Again we expand $\varphi_i^{-1}\overline{\varphi}_i^{-1}$ and choose c_i, as (2.101). In this case A is written

$$A = \prod_{i=1}^{n} \varphi_i\overline{\varphi}_i(-1)^n \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ \frac{\rho_0 P_1}{p_1} & \rho_0 P_1 \alpha_{11} - O(P_1^2) & \frac{P_1}{p_1+\overline{p}_2} & \cdots & \frac{P_1}{p_1+\overline{p}_n} \\ \frac{\rho_0 P_2}{p_2} & \frac{P_2}{p_2+\overline{p}_1} & \rho_0 P_2 \alpha_{22} - O(P_2^2) & \cdots & \frac{P_2}{p_2+\overline{p}_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_0 P_n}{p_n} & \frac{P_n}{p_n+\overline{p}_1} & \frac{P_n}{p_n+\overline{p}_2} & \cdots & \rho_0 P_n \alpha_{nn} - O(P_n^2) \end{vmatrix}. \tag{2.113}$$

Removing factors of P_i from the rows gives

$$A = \prod_{i=1}^{n} P_i\varphi_i\overline{\varphi}_i(-1)^n \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ \frac{\rho_0}{p_1} & \rho_0 \alpha_{11} - O(P_1) & \frac{1}{p_1+\overline{p}_2} & \cdots & \frac{1}{p_1+\overline{p}_n} \\ \frac{\rho_0}{p_n} & \frac{1}{p_2+\overline{p}_1} & \rho_0 \alpha_{22} - O(P_2) & \cdots & \frac{1}{p_2+\overline{p}_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_0}{p_n} & \frac{1}{p_n+\overline{p}_1} & \frac{1}{p_n+\overline{p}_2} & \cdots & \rho_0 \alpha_{nn} - O(P_n) \end{vmatrix}. \tag{2.114}$$

Now u (2.47) is defined as

$$u = \frac{G}{F} \tag{2.115}$$

$$= \frac{\rho_0 F + A}{F} \tag{2.116}$$

$$= \rho_0 + \frac{A}{F} \tag{2.117}$$

By comparing (2.109) and (2.114), we see that we may write A in the following way

$$A = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ \frac{\rho_0}{p_1} & \rho_0 \alpha_{11} & \frac{1}{p_1+\overline{p}_2} & \cdots & \frac{1}{p_1+\overline{p}_n} \\ \frac{\rho_0}{p_2} & \frac{1}{p_2+\overline{p}_1} & \rho_0 \alpha_{22} & \cdots & \frac{1}{p_2+\overline{p}_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_0}{p_n} & \frac{1}{p_n+\overline{p}_1} & \frac{1}{p_n+\overline{p}_2} & \cdots & \rho_0 \alpha_{nn} \end{vmatrix} \tag{2.118}$$

where we have used the limit $P_i \to 0$. Therefore the rational version of G is written

$$G = \rho_0 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{p_1} & \rho_0\alpha_{11} & \frac{1}{p_1+\overline{p}_2} & \cdots & \frac{1}{p_1+\overline{p}_n} \\ \frac{1}{p_2} & \frac{1}{p_2+\overline{p}_1} & \rho_0\alpha_{22} & \cdots & \frac{1}{p_2+\overline{p}_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_n} & \frac{1}{p_n+\overline{p}_1} & \frac{1}{p_n+\overline{p}_2} & \cdots & \rho_0\alpha_{nn} \end{vmatrix}. \tag{2.119}$$

2.4.2 Examples

$n = 1$

For $n = 1$ we see from (2.110), (2.119) that F and G have the following form

$$F = \rho_0\alpha_{11} \tag{2.120}$$

$$G = \rho_0^2\alpha_{11} - \frac{\rho_0}{p_1}, \tag{2.121}$$

thus we obtain the singular solution

$$u = \rho_0 - \frac{1}{p_1\alpha_{11}}. \tag{2.122}$$

$n = 2$

We take $n = 2$

$$F = \rho_0^2\alpha_{11}\alpha_{22} - \frac{1}{(p_1+\overline{p}_2)(p_2+\overline{p}_1)} \tag{2.123}$$

$$G = \rho_0 F - \rho_0^2\left[\frac{\alpha_{11}}{p_2} + \frac{\alpha_{22}}{p_1}\right] + \rho_0\left[\frac{1}{p_2(p_1+\overline{p}_2)} + \frac{1}{p_1(p_2+\overline{p}_1)}\right]. \tag{2.124}$$

We recall that we have taken the limit

$$p_i + \overline{p}_i \to 0 \tag{2.125}$$

i.e. the p_i's are completely imaginary. We replace $\overline{p}_i$ by $-p_i$ so that

$$F = \frac{\rho_0^2(p_1-p_2)^2\alpha_{11}\alpha_{22}+1}{(p_1-p_2)^2} \tag{2.126}$$

$$G = \rho_0 F + \frac{\rho_0}{p_1p_2}\left[1 - \rho_0(p_2\alpha_{22} + p_1\alpha_{11})\right]. \tag{2.127}$$

Thus (2.126), (2.127) yield the solution

$$u = \rho_0 + \frac{\rho_0(p_1 - p_2)^2 \left[1 - \rho_0(p_2\alpha_{22} + p_1\alpha_{11})\right]}{p_1 p_2 \left[\rho_0^2(p_1 - p_2)^2\alpha_{11}\alpha_{22} + 1\right]}. \tag{2.128}$$

Now if we define $p_i = iq_i$ for $q_i \in \mathbb{R}$, we can write (2.128) as

$$u = \rho_0 + \frac{\rho_0(q_1 - q_2)^2 \left[1 - i\rho_0(q_2\alpha_{22} + q_1\alpha_{11})\right]}{q_1 q_2 \left[-\rho_0^2(q_1 - q_2)^2\alpha_{11}\alpha_{22} + 1\right]}. \tag{2.129}$$

It is clear in general we have a singular solution.

2.5 Comparison of Solutions

In this section we look more closely at the equations and solutions obtained by Satsuma and Ablowitz in their paper of 1979. These solutions are compared and contrasted with the ones outlined in this chapter. Then we propose a method of obtaining a wider class of rational solutions, that follows a similar construction to that of Satsuma and Ablowitz.

2.5.1 Equations and Transformations

Rational solutions for the DS equation were obtained by Satsuma and Ablowitz [89]. These were constructed by taking the "long wave" limit in the n-soliton solution by careful choice of the phase factors. These solutions were in general singular, however if the parameters are chosen carefully, the method gives rise to non-singular lump solutions.

Satsuma and Ablowitz start with the DS equations written as

$$iA_t - \sigma_1 A_{XX} + A_{YY} - \sigma_2 A|A|^2 - 2\sigma_1\sigma_2 QA = 0 \tag{2.130}$$

$$\sigma_1 Q_{XX} + Q_{YY} + |A|^2_{XX} = 0 \tag{2.131}$$

where $\sigma_1, \sigma_2 = \pm 1$. We may transform (2.130), (2.131) to (2.45), (2.46) by use of the following

$$x = \frac{1}{\sqrt{2}}(X+Y) \quad (2.132)$$
$$y = \frac{1}{\sqrt{2}}(X-Y) \quad (2.133)$$
$$A = 2u \quad (2.134)$$
$$Q = \frac{v+4|u|^2}{2} \quad (2.135)$$

where we have chosen $\sigma_1 = -1$ and $\sigma_2 = 1$. It is worth noting that Hirota's bilinear form considered by Satsuma and Ablowitz (with $|A|^2 \to \mu_0^2$ as $|x| \to \infty$)

$$\left[iD_t - \sigma_1 D_X^2 + D_Y^2 - \sigma_2\mu_0^2\right] g' \cdot f' = 0 \quad (2.136)$$
$$\left[\sigma_1 D_X^2 + D_Y^2 - \sigma_2\mu_0^2\right] f' \cdot f' + \sigma_2 g'\overline{g}' = 0 \quad (2.137)$$

takes the following form when the independent variables are transformed as above

$$\left(iD_t + D_x^2 + D_y^2 - \mu_0^2\right) g \cdot f = 0 \quad (2.138)$$
$$\left[D_x D_y + \frac{\mu_0^2}{4}\right] f \cdot f - 2|g|^2 = 0 \quad (2.139)$$

where $g' = g$, $f' = 2f$. We may further transform these to the bilinear form used in this thesis (2.49), (2.50) by

$$G = \gamma g \quad (2.140)$$
$$F = \gamma f \quad (2.141)$$
$$\gamma = \exp[-\frac{\mu_0^2}{8}(x^2+y^2)] \quad (2.142)$$
$$\mu_0 = 2\rho_0. \quad (2.143)$$

this allows for a much simpler choice of the independent variable F. Combining these transformations allows us to move from a solution given by Satsuma and Ablowitz to one that satisfies (2.45) and (2.46).

2.5.2 A Wider Class of Solutions

Satsuma and Ablowitz first obtain rational solutions from the general n-soliton solution, by use of the "long wave" limit and then check that the F obtained is real.

This is true only in a limited number of cases, one of which gives rise to the non-singular lump solutions.

In contrast here, the grammian is first proved to be a solution and then we construct rational solutions. However if we relax the reality condition (2.83) ($\varphi_i = \psi_i$) and allow H to be a general complex constant diagonal matrix, we obtain a wider class of rational solutions. We then show that F is real, and that (2.71) is the conjugate of (2.70). This allows us to calculate non-singular rational solutions.

We follow the method outlined in section 2.4.1. We start from the n-soliton solution given in grammian form, (2.51), (2.70). In this case we choose

$$\varphi_i = \exp\left[\rho_0\left[p_i x - \frac{y}{p_i} - i\rho_0 t\left(p_i^2 - \frac{1}{p_i^2}\right)\right]\right] \tag{2.144}$$

$$\psi_j = \exp\left[\rho_0\left[q_j x - \frac{y}{q_j} - i\rho_0 t\left(q_j^2 - \frac{1}{q_j^2}\right)\right]\right] \tag{2.145}$$

where the p_i, q_j are complex constants. Rewriting (2.51)

$$F = \left|\delta_{ij} + c_i\int_{-\infty}^{x}\varphi_i\overline{\psi}_j\,dx\right| \tag{2.146}$$

$$= \left|\delta_{ij} + \frac{c_i\varphi_i\overline{\psi}_j}{\rho_0(p_i+\overline{q}_j)}\right| \tag{2.147}$$

$$= \prod_{i=1}^{n}\varphi_i\overline{\psi}_i\left|\delta_{ij}\varphi_i^{-1}\overline{\psi}_i^{-1} + \frac{c_i}{\rho_0(p_i+\overline{q}_j)}\right|. \tag{2.148}$$

Again we choose

$$c_i = -\rho_0 Q_i \tag{2.149}$$

for $i = 1, \ldots, n$, with $Q_i = p_i + \overline{q}_i$. Writing

$$\varphi_i\overline{\psi}_i = \exp\left[\rho_0 Q_i\left[x - \frac{y}{p_i\overline{q}_i} - i\rho_0 t\,(p_i - \overline{q}_i)\left(1 + \frac{1}{p_i^2\overline{q}_i^2}\right)\right]\right] \tag{2.150}$$

$$= \exp\left[\rho_0 Q_i\beta_{ii}\right] \tag{2.151}$$

with

$$\beta_{ii} = x - \frac{y}{p_i\overline{q}_i} - i\rho_0 t\,(p_i - \overline{q}_i)\left(1 + \frac{1}{p_i^2\overline{q}_i^2}\right), \tag{2.152}$$

gives the following terms on the diagonal

$$\varphi_i^{-1}\overline{\psi}_i^{-1} + \frac{c_i}{\rho_0 Q_i} = \left[1 - \rho_0 Q_i\beta_{ii} + \frac{\rho_0^2}{2}Q_i^2\beta_{ii}^2 - \ldots\right] - 1 \tag{2.153}$$

$$= -\rho_0 Q_i\beta_{ii} + \frac{\rho_0^2}{2}Q_i^2\beta_{ii}^2 - \ldots \tag{2.154}$$

Off the diagonal we have

$$\frac{c_i}{\rho_0\left(p_i+\overline{q}_j\right)}=-\frac{Q_i}{p_i+\overline{q}_j}. \tag{2.155}$$

Removing factors of Q_i from rows and making use of the invariance of v, we obtain a rational determinant form for F

$$F=\begin{vmatrix} \rho_0\beta_{11} & \frac{1}{p_1+\overline{q}_2} & \cdots & \frac{1}{p_1+\overline{q}_n} \\ \frac{1}{p_2+\overline{q}_1} & \rho_0\beta_{22} & \cdots & \frac{1}{p_2+\overline{q}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_n+\overline{q}_1} & \frac{1}{p_n+\overline{q}_2} & \cdots & \rho_0\beta_{nn} \end{vmatrix} \tag{2.156}$$

where we have taken the limit $Q_i \to 0$ for $i=1,\dots,n$, with $p_i\overline{q}_j$ and $p_i-\overline{q}_j$ of order 1. Using the same proceedure for A we obtain the following rational version of G

$$G=\rho_0\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{p_1} & \rho_0\beta_{11} & \frac{1}{p_1+\overline{q}_2} & \cdots & \frac{1}{p_1+\overline{q}_n} \\ \frac{1}{p_2} & \frac{1}{p_2+\overline{q}_1} & \rho_0\beta_{22} & \cdots & \frac{1}{p_2+\overline{q}_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_n} & \frac{1}{p_n+\overline{q}_1} & \frac{1}{p_n+\overline{q}_2} & \cdots & \rho_0\beta_{nn} \end{vmatrix}. \tag{2.157}$$

2.5.3 Examples

$n = 1$

For $n=1$ we see from (2.156), (2.157) that F and G take the following form

$$F = \rho_0\beta_{11} \tag{2.158}$$

$$G = \rho_0^2\beta_{11}-\frac{\rho_0}{p_1}. \tag{2.159}$$

Thus as before, we obtain the singular solution

$$u=\rho_0-\frac{1}{p_1\beta_{11}}. \tag{2.160}$$

$n = 2$

Next we take $n=2$

$$F = \rho_0^2\beta_{11}\beta_{22}-\frac{1}{(p_1+\overline{q}_2)(p_2+\overline{q}_1)} \tag{2.161}$$

$$G = \rho_0F-\rho_0^2\left[\frac{\beta_{11}}{p_2}+\frac{\beta_{22}}{p_1}\right]+\rho_0\left[\frac{1}{p_2(p_1+\overline{q}_2)}+\frac{1}{p_1(p_2+\overline{q}_1)}\right]. \tag{2.162}$$

The aim is to write

$$\beta_{22} = \overline{\beta}_{11} \tag{2.163}$$

so as to make F positive definite. For this to be true we need

$$p_2 = -\overline{p}_1 \tag{2.164}$$

$$q_2 = -\overline{q}_1 \tag{2.165}$$

so that F becomes

$$F = \rho_0^2\beta_{11}\overline{\beta}_{11} - \frac{1}{(p_1-q_1)(-\overline{p}_1+\overline{q}_1)} \tag{2.166}$$

$$= \rho_0^2\beta_{11}\overline{\beta}_{11} + \frac{1}{(p_1-q_1)(\overline{p}_1-\overline{q}_1)} \tag{2.167}$$

$$= \frac{\rho_0^2 R_1\overline{R}_1\beta_{11}\overline{\beta}_{11}+1}{R_1\overline{R}_1} \tag{2.168}$$

where $R_1 = p_1 - q_1$. In this case G is equivalent to

$$G = \rho_0 F - \frac{\rho_0^2}{p_1\overline{p}_1}\left[\overline{p}_1\overline{\beta}_{11} - p_1\beta_{11}\right] - \rho_0\left[\frac{p_1\overline{R}_1 + \overline{p}_1 R_1}{p_1\overline{p}_1 R_1\overline{R}_1}\right], \tag{2.169}$$

thus we have the solution

$$u = \rho_0 - \frac{\rho_0^2 R_1\overline{R}_1\left[\overline{p}_1\overline{\beta}_{11} - p_1\beta_{11}\right]}{p_1\overline{p}_1\left[\rho_0^2 R_1\overline{R}_1\beta_{11}\overline{\beta}_{11}+1\right]} - \frac{\rho_0\left[p_1\overline{R}_1 + \overline{p}_1 R_1\right]}{p_1\overline{p}_1\left[\rho_0^2 R_1\overline{R}_1\beta_{11}\overline{\beta}_{11}+1\right]}. \tag{2.170}$$

In the limit $Q_i \to 0$, $R_1 = \overline{R}_1 = p_1 + \overline{p}_1$ i.e. R_1 is real, we label it r_1. Hence we have the non-singular solution

$$u = \rho_0 - \frac{\rho_0^2 r_1^2\left[\overline{p}_1\overline{\beta}_{11} - p_1\beta_{11}\right]}{p_1\overline{p}_1\left[\rho_0^2 r_1^2\beta_{11}\overline{\beta}_{11}+1\right]} - \frac{\rho_0 r_1^2}{p_1\overline{p}_1\left[\rho_0^2 r_1^2\beta_{11}\overline{\beta}_{11}+1\right]} \tag{2.171}$$

$$= \rho_0 - \frac{\rho_0 r_1^2\left[1+\rho_0\left(\overline{p}_1\overline{\beta}_{11} - p_1\beta_{11}\right)\right]}{p_1\overline{p}_1\left[\rho_0^2 r_1^2\beta_{11}\overline{\beta}_{11}+1\right]}. \tag{2.172}$$

Clearly F is real and (2.71) is the conjugate of (2.70). So we have obtained a non-singular rational solution. The envelope of the solution is rational, and is written as

$$|u|^2 = \rho_0^2\left[1 - \frac{2r_1^2}{(p_1\overline{p}_1)\left[\rho_0^2 r_1^2\beta_{11}\overline{\beta}_{11}+1\right]} + \frac{r_1^4\left[1-\rho_0^2\left(\overline{p}_1\overline{\beta}_{11} - p_1\beta_{11}\right)^2\right]}{(p_1\overline{p}_1)^2\left[\rho_0^2 r_1^2\beta_{11}\overline{\beta}_{11}+1\right]^2}\right]. \tag{2.173}$$

To see the shape of the solution we rewrite (2.173) as

$$|u|^2 = \rho_0^2 \left[1 - \frac{8p_R^2}{\kappa} + \frac{16p_R^4}{\kappa^2}\left[1 + 4\rho_0^2\left[p_I X + (R_R p_I + R_I p_R)Y\right]^2\right]\right], \tag{2.174}$$

where

$$\kappa = (p_R^2 + p_I^2)\left[1 + 4\rho_0^2 p_R^2\left[R_I^2 Y^2 + (X + R_R Y)^2\right]\right], \tag{2.175}$$

and

$$X = x + \frac{(S_R R_I - S_I R_R)}{R_I}t, \qquad Y = y + \frac{S_I}{R_I}t, \tag{2.176}$$

with $p_1 = p_R + ip_I$ (so that $r_1 = R_1 = 2p_R$). We have redefined β_{11} as

$$\begin{aligned} \beta_{11} &= x + \frac{y}{p_1^2} - 2ip_1\rho_0 t\left(1 + \frac{1}{p_1^4}\right) &(2.177)\\ &= x + (R_R + iR_I)y + t(S_R + iS_I) &(2.178)\end{aligned}$$

so that

$$\begin{aligned} R_R &= \frac{p_R^2 - p_I^2}{(p_R^2 + p_I^2)^2}, &(2.179)\\ R_I &= \frac{-2p_R p_I}{(p_R^2 + p_I^2)^2}, &(2.180)\\ S_R &= 4\rho_0 p_R R_R R_I + 2\rho_0 p_I\left[1 + R_R^2 - R_I^2\right], &(2.181)\\ S_I &= 4\rho_0 p_I R_R R_I - 2\rho_0 p_R\left[1 + R_R^2 - R_I^2\right]. &(2.182)\end{aligned}$$

(2.174) decays as $O(1/x^2, 1/y^2)$ for $|x|, |y| \to \infty$ i.e. (2.174) expresses a permanent lump solution, with velocity

$$\begin{aligned} v_x &= \frac{R_R S_I - R_I S_R}{R_I} &(2.183)\\ v_y &= \frac{-S_I}{R_I}. &(2.184)\end{aligned}$$

By use of the substitution

$$T = 2\rho_0 p_R R_I Y \qquad Z = 2\rho_0 p_R (X + R_R Y) \tag{2.185}$$

we re-write (2.174) as

$$|u|^2 = \rho_0^2\left[1 - \frac{8p_R^2}{\kappa} + \frac{16p_R^2}{\kappa^2}\left[p_R^2 + (p_I Z + p_R T)\right]\right], \tag{2.186}$$

with

$$\kappa = (p_R^2 + p_I^2)\left[1 + Z^2 + T^2\right]. \tag{2.187}$$

This substitution allows us to determine more easily maxima and minima for $|u|^2$, these enable us to determine the exact shape of our envelope solution. Calculating $|u|_Z^2 = 0$ and $|u|_T^2 = 0$ we have five solutions

$$Z = 0 \qquad T = 0, \tag{2.188}$$

$$Z = \pm\frac{p_I\sqrt{(3p_I^2 - p_R^2)}}{(p_R^2 + p_I^2)} \qquad T = \pm\frac{p_R\sqrt{(3p_I^2 - p_R^2)}}{(p_R^2 + p_I^2)}, \tag{2.189}$$

$$Z = \pm\frac{p_R\sqrt{(3p_R^2 - p_I^2)}}{(p_R^2 + p_I^2)} \qquad T = \mp\frac{p_I\sqrt{(3p_R^2 - p_I^2)}}{(p_R^2 + p_I^2)}, \tag{2.190}$$

respectively. These solutions occur along the lines $T = \frac{p_R}{p_I}Z$, $T = -\frac{p_I}{p_R}Z$ i.e. at right angles to one another. By examining $|u|_{ZZ}^2$ and the Hessian, we can distinguish maximum, minimum and saddle points. We have three different cases, depending on the values of our parameters p_R and p_I. Firstly if

$$\frac{p_I^2}{3} \leq p_R^2 \leq 3p_I^2 \tag{2.191}$$

we have that (2.186) has two maximums at (2.189), two minimum values at (2.190) and a saddle at (2.188), see figure 2.1.

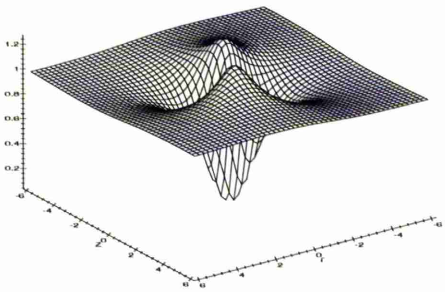


Figure 2.1: Envelope lump solution of (2.186) as seen in two dimensions at a fixed time. With $p_R = 1$, $p_I = 2$, $\rho_0^2 = 1$.

If

$$p_R^2 \leq 3p_I^2 \qquad \text{and} \qquad p_R^2 \not\geq \frac{p_I^2}{3} \tag{2.192}$$

then we have no solutions from (2.190) and then two maximums at (2.189) and a minimum at (2.188), see figure 2.2.

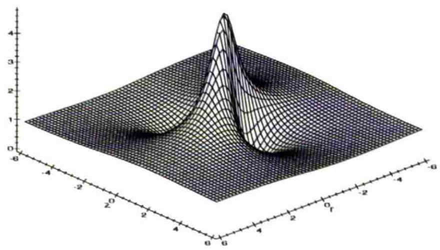


Figure 2.2: Envelope lump solution of (2.186) as seen in two dimensions at a fixed time. With $p_R = 2$, $p_I = 1$, $\rho_0^2 = 1$.

Finally if

$$p_R^2 \geq \frac{p_I^2}{3} \qquad \text{and} \qquad p_R^2 \not\leq 3p_I^2 \tag{2.193}$$

then we have no solutions at (2.189) and two minimums at (2.190) and a maximum at (2.188), figure 2.3. Thus (2.173) expresses a permanent lump, which may have either two maximum values and a minimum (figure 2.1), a maximum and two minimums (figure 2.2) or two maximums and minimums with a saddle point (figure 2.3). It has velocity given by (2.183) and (2.184).

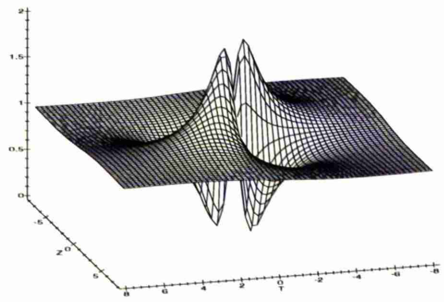


Figure 2.3: Envelope lump solution of (2.186) as seen in two dimensions at a fixed time. With $p_R = 1$, $p_I = 1$, $\rho_0^2 = 1$.

$n = 2M$

Finally we consider $n = 2M$, in this case we may express F and G as

$$F = \begin{vmatrix} \rho_0\beta_{11} & \frac{1}{p_1+\overline{q}_2} & \cdots & \frac{1}{p_1+\overline{q}_{2M}} \\ \frac{1}{p_2+\overline{q}_1} & \rho_0\beta_{22} & \cdots & \frac{1}{p_2+\overline{q}_{2M}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_{2M}+\overline{q}_1} & \frac{1}{p_{2M}+\overline{q}_2} & \cdots & \rho_0\beta_{2M2M} \end{vmatrix} \tag{2.194}$$

$$= \begin{vmatrix} \rho_0\beta_{11} & \frac{1}{p_1-p_2} & \cdots & \frac{1}{p_1-p_{2M}} \\ \frac{1}{p_2-p_1} & \rho_0\beta_{22} & \cdots & \frac{1}{p_2-p_{2M}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_{2M}-p_1} & \frac{1}{p_{2M}-p_2} & \cdots & \rho_0\beta_{2M2M} \end{vmatrix} \tag{2.195}$$

$$G = \rho_0 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{p_1} & \rho_0\beta_{11} & \frac{1}{p_1-p_2} & \cdots & \frac{1}{p_1-p_{2M}} \\ \frac{1}{p_2} & \frac{1}{p_2-p_1} & \rho_0\beta_{22} & \cdots & \frac{1}{p_2-p_{2M}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_{2M}} & \frac{1}{p_{2M}-p_1} & \frac{1}{p_{2M}-p_2} & \cdots & \rho_0\beta_{2M2M} \end{vmatrix}. \tag{2.196}$$

where we have used the limit $p_i + \overline{q}_i \to 0$ and $\beta_{ii} = x + \frac{y}{p_i^2} - 2i\rho_0 t p_i \left(1 + \frac{1}{p_i^4}\right)$. As with the KP equation we may write F in an expanded form

$$F = \prod_{i=1}^{2M} \rho_0^i \beta_{ii} + \frac{1}{2} \sum_{i,j}^{(2M)} D_{ij} \prod_{l \neq i,j}^{2M} \rho_0^l \beta_{ll} + \cdots$$
$$+ \frac{1}{L!2^L} \sum_{i,j,\dots,p,q}^{(2M)} \overbrace{D_{ij} D_{kl} \dots D_{pq}}^{L} \prod_{r \neq i,j,\dots,p,q}^{2M} \rho_0^r \beta_{rr} + \cdots \quad (2.197)$$

where $D_{ij} = \frac{1}{(p_i - p_j)^2}$ and $\sum_{i,j,\dots,p,q}^{(2M)}$ denotes the summation over all possible combinations of $i, j, \dots, p, q$ which are taken from $1, 2, \dots, 2M$ and all different. Generally this gives a singular solution solution, however if we choose

$$p_{M+i} = -\overline{p}_i \quad (2.198)$$

for $i = 1, \dots, M$ we obtain a class of non-singular rational solutions. F is rewritten

$$F = \begin{vmatrix} C & A \\ -\overline{A} & \overline{C}^T \end{vmatrix} \quad (2.199)$$

where C and A are $M \times M$ matrices defined by

$$C = \begin{bmatrix} \rho_0 \beta_{11} & \frac{1}{p_1 - p_2} & \cdots & \frac{1}{p_1 - p_M} \\ \frac{-1}{p_1 - p_2} & \rho_0 \beta_{22} & \cdots & \frac{1}{p_2 - p_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{p_1 - p_M} & \frac{-1}{p_2 - p_M} & \cdots & \rho_0 \beta_{MM} \end{bmatrix} \quad (2.200)$$

$$A = \begin{bmatrix} \frac{1}{p_1 + \overline{p}_1} & \cdots & \frac{1}{p_1 + \overline{p}_M} \\ \vdots & & \vdots \\ \frac{1}{p_M + \overline{p}_1} & \cdots & \frac{1}{p_M + \overline{p}_M} \end{bmatrix}. \quad (2.201)$$

Since the determinant

$$F = \begin{vmatrix} C & A \\ -\overline{A} & \overline{C}^T \end{vmatrix} \quad (2.202)$$

is always positive [66], then u does not become infinite at any position, hence we have non-singular solution. We calculate the conjugate of F

$$\overline{F} = \begin{vmatrix} \overline{C} & \overline{A} \\ -A & C^T \end{vmatrix} \quad (2.203)$$
$$= F \quad (2.204)$$

by interchanging rows and columns and taking the transpose. Hence the reality of F is guaranteed. The conjugacy condition (2.71) on G is satisfied for this choice of parameters. So we have a nonsingular solution to the Davey-Stewartson equation whose envelope $|u|^2$ is rational.

Since the form of (2.195) is the same as (2.25) we can apply the same asymptotic argument. In this case, we fix ourselves on the phase $|\beta_{LL}|$, such that $|\beta_{LL}|^2$ is a constant. Then in the limit of $t \to \pm\infty$.

$$|\beta_{11}|,\ |\beta_{22}|, \cdots, |\beta_{(L-1)(L-1)}|,\ |\beta_{(L+1)(L+1)}|, \cdots, |\beta_{MM}| = O(t), \tag{2.205}$$

and using (2.197) F has the following asymptotic state

$$F \sim \rho_0^{2M}|\beta_{11}|^2|\beta_{22}|^2\cdots|\beta_{MM}|^2 \\ + \rho_0^{2M-2}D_{L,M+L}|\beta_{11}|^2\cdots|\beta_{(L-1)(L-1)}|^2|\beta_{(L+1)(L+1)}|^2\cdots|\beta_{MM}|^2, \tag{2.206}$$

and G takes the form

$$G \sim \left[\rho_0^2|\beta_{LL}|^2 + D_{L,M+L} - \rho_0\left(\frac{\overline{\beta}_{LL}}{p_L} - \frac{\beta_{LL}}{\overline{p}_L}\right) - \frac{1}{\overline{p}_L(p_L+\overline{p}_L)} - \frac{1}{p_L(p_L+\overline{p}_L)}\right] \\ \times \rho_0^{2M-1}|\beta_{11}|^2\cdots|\beta_{(L-1)(L-1)}|^2|\beta_{(L+1)(L+1)}|^2\cdots|\beta_{MM}|^2, \tag{2.207}$$

where we have used the the nonsingular condition (2.198). Therefore u is written

$$u \to \frac{\rho_0\left[\rho_0^2|\beta_{LL}|^2 + D_{L,M+L} - \rho_0\left(\frac{\overline{\beta}_{LL}}{p_L} - \frac{\beta_{LL}}{\overline{p}_L}\right) - \frac{1}{\overline{p}_L(p_L+\overline{p}_L)} - \frac{1}{p_L(p_L+\overline{p}_L)}\right]}{[\rho_0^2|\beta_{LL}|^2 + D_{L,M+L}]}, \tag{2.208}$$

which is a lump, which has a rational envelope of the same functional form as (2.173). Hence we see that the nonsingular solution describes multiple collisions of M lumps, each of which has the functional form (2.174) and which have no phase shift when they collide with each other.

2.5.4 Discussion of Solutions

The solutions obtained (2.156) (2.157), are in many ways similar to those obtained by Satsuma and Ablowitz. They both have a class of lump solutions, and the same basic structure. For example Satsuma and Ablowitz have for their 1 and 2-soliton solution (after the "long wave" limit has been taken)

$$A_1 = \mu_0 e^{i\xi}\left(1+\frac{2iB_1}{\theta_1}\right), \tag{2.209}$$

$$A_2 = \frac{\mu_0 e^{i\xi}\left[\left(\theta_1+2iB_1\right)\left(\theta_2+2iB_2\right)+\alpha_{12}\right]}{\left(\theta_1\theta_2+\alpha_{12}\right)}, \tag{2.210}$$

where

$$\xi = kX+lY-\omega t+\xi^{(0)}, \tag{2.211}$$

$$\omega = -\sigma_1 k^2+l^2+\sigma_2\mu_0^2, \tag{2.212}$$

$$\theta_i = X+R_iY-t\left[-\sigma_1 k+2lR_i+(\sigma_1-R_i^2)/B_i\right], \tag{2.213}$$

and μ_0, B_i, R_i, α_{12} are all constants. $\alpha_{12}>0$ and $\sigma_1\sigma_2=-1$ for nonsingular solutions. We can see that by comparing these with (2.160) and (2.172), that θ_i corresponds to β_{ii} and that the order of the decay is the same. However to transform completely between them we must make use of the transformation derived in section 2.5.1. When we consider the envelope solution of Ablowitz and Satsuma, and the one derived in this thesis (2.174), we see that both describe multiple collisions of M lumps. These lumps have a similar functional form and experience no phase shift when they collide with each other. However, whereas the lumps of Satsuma and Ablowitz have two humps, the exact shape of the lumps described in this chapter depends on the choice of parameters (see section 2.5.3).

In contrast, the exponential factor in Ablowitz and Satsuma's solutions means that they are "less rational" than the solutions derived in this thesis. This factor cannot be easily removed. If we set $k=l=0$, the form of θ_i and β_{ii} compare more closely, and the exponential factor is simplified to

$$\exp\left[i\left(\xi^{(0)}-\sigma_2\mu_0^2 t\right)\right] \tag{2.214}$$

Rewriting

$$A = \exp\left[-i\sigma_2\mu_0^2 t\right]S, \tag{2.215}$$

$$Q = Q'+\frac{\sigma_1\mu_0^2}{2}, \tag{2.216}$$

where we set $\xi^{(0)}$ to zero, $S=S(X,Y,t)$ and $Q'=Q'(X,Y,t)$, the resulting equations in S and Q' satisfy the DS equations (2.130) (2.131). The variable S when transformed to a function of x, y, t, is analogous to the variable u used in this paper. In this notation equations (2.209) (2.210) become

$$A_1 = \exp[-i\sigma_2\mu_0^2 t]S_1 \qquad \text{and} \qquad A_2 = \exp[-i\sigma_2\mu_0^2 t]S_2. \tag{2.217}$$

So we obtain a correspondence between the dependent variables S and u. Therefore whilst we can say that the rational solutions obtained in this chapter are more "fully rational", than those of Ablowitz and Satsuma, when we consider the envelope solutions we see obvious similarities in their nature.

Chapter 3

3D Three Wave Resonant Interaction

3.1 Introduction

3.1.1 Background

Of the nonlinear interactions possible in three dimensions, possibly the simplest is the three-wave resonant interaction (3WRI). This is found to occur in a variety of physical situations, including nonlinear optics, electronics and fluid mechanics. If we consider two colliding waves whose envelopes vary slowly compared to their central frequencies w_1, w_2 (wave vectors k_1 and k_2). It is possible for a third wave to exist which has a composite frequency $w_3 = w_1 \pm w_2$ ($k_3 = k_1 \pm k_2$). It may be viewed as the two original waves interacting (beating) with one another, giving rise to the third wave. Given the frequency of the third is a linear combination of the first two, the waves can "phase lock" allowing for growth and continuation of the new wave. Without this feature the third wave would quickly disperse after several oscillations. The reason that this kind of nonlinear interaction is so evident, is because the nonlinearity demonstrated is of the lowest possible, quadratic. The reverse of this process also occurs, where the interaction gives rise to the decay of a pump wave into two daughter waves.

We may also consider an interaction which satisfies $w_3 = -w_1 - w_2$ ($k_3 = -k_1 - k_2$), if we allow waves which have negative energy (e.g. plasma with a current). These will satisfy the 3WRI providing at least one of the w_i is negative. They describe the process of the "creation of three quanta from the vacuum", and are known as an explosive instability of the medium. A full review of these effects may be found in [18], [54].

The three-dimensional three-wave equations are conveniently expressed by characteristic coordinates as

$$\frac{\partial q_i}{\partial X_i} = \gamma_i q_j^* q_k^* \qquad \frac{\partial q_i^*}{\partial X_i} = \gamma_i q_j q_k \tag{3.1}$$

where i, j, k are cyclic and equal to $1, 2, 3$ and $*$ means complex conjugation. The X_i are characteristic coordinates, the γ's are coupling constants. In conservative systems, for example plasma's, the γ_i are real, or can be made so by rescaling of q_j. Solutions can then be obtained via inverse scattering or the Bäcklund transformation. However if we allow the γ_i to be complex these methods break down as energy conservation no longer happens. The non-conservative case is of physical interest, particular examples include hydrodynamic stability of viscous shear flow and plasma physics. It is interesting to note that the conservative system is really a special case of this and occurs physically only when more assumptions are made. For a fuller description of the non-conservative case, including derivation and particular solutions see [18], [19], [20]. However our focus will be on the conservative case.

The 3WRI has been extensively investigated. Originally this work was in one spatial dimension with the working towards the full three-dimensional problem. An inverse scattering transform for the homogenous-medium 3WRI in one space and time dimension was developed by Zakharov and Manakov [98], [99] and Kaup [48]. Case and Chiu [15] exploited the linear operators of the inverse scattering transform to obtain a Bäcklund transformation. For a full discussion of the evolution in time and one-spatial dimension,in a homogenous medium of the 3WRI see the paper by Kaup, Reiman and Bers [54]. The inhomogeous medium case was examined by Reiman [85], and recently by Kaup and Malomed using a Lagrangian formalism [53]. In the inhomogeneous case the γ_i need not be constants, so we may allow slowly varying functions or delta functions.

The above work led the way for the study of the full three-dimensional three-wave resonant interaction (3D3WR), which arises as the computability condition between two 3×3 differential systems. The inverse scattering problem was first formulated and a particular class of solutions found by Zakharov [97], and then a special case of these by Craik [17]. These solutions came to be known as "lumps".

Ablowitz and Haberman's work [2] work led Cornille [16] to reformulate the inverse scattering set-up with all three coordinates on the same footing. Zakharov's approach had been to have one coordinate as a variable and the other two as parameters. Kaup's work [50], [49], [52] was to use these characteristic coordinates to give explicitly the general inverse scattering solution and also an infinite set of conservation laws. We note that the $(2+1)$ and $(3+1)$-dimensional problems are represented by the same system. The only difference is that for the $(3+1)$ system we have a fourth characteristic coordinate defined, which appears in q as a parameter. The lump solutions obtained by Zakharov and later Craik are different to the soliton solutions found in the one-dimensional inverse scattering, however, they maintain some similarities. The one-dimensional soliton solutions are derived from separable kernels [54] and hence give rise to a closed form of solution. Similarly for lump solutions the kernels are also separable [49] and again we have a closed form of solutions. This is often used as a definition of lump solutions. Further in an analogous way to the construction of n-soliton solutions, one can construct n-lump solutions. Indeed this is possible from a Bäcklund transformation as shown by Kaup [51] (An alternative Bäcklund transformation was given by [63]).

However soliton solutions in one-dimensional inverse scattering theories always corresponds to a pure bound-state spectrum i.e. no continous spectrum, and there is a relationship between the amplitude and width so that only one is truly independent. For lump solutions this connection is lost and the profile may be quite arbitrary. Lump solutions may be thought of as having more freedom in their profile.

Work was carried out on the Lie point symmetries of the 3WRI by a variety of authors. Fokas et al. [27] showed that in one spatial and one temporal dimension, the 3WRI invariant solutions, satisfy a system of three first order nonlinear ODE's. These can further be reduced to a second order equation, which is a special case of an equation related to a transformation of the Painlevé VI (PVI) equation. Similarly the 3WRI in two dimensions is related to PIII, PIV and PVI [60], [55]. For a full discussion of the group analysis for the full three-dimensional three wave interaction and the new invariant solutions obtained see [67].

Recent work has focused on examining the non-symmetrical three wave resonant interaction. Normally when we consider the case of one of the waves linearly instable and two of them damped, we assume that the damped waves are symmetrical. However Hughes and Proctor considered the non-symmetric case where the damped waves are truly distinct [45]. We confine ourselves to considering symmetric cases only.

3.1.2 The System of Equations

The three-dimensional three-wave resonant equations (3D3WR) [50] take the form

$$\frac{\partial q_i}{\partial X_i} = \gamma_i q_j^* q_k^* \qquad \frac{\partial q_i^*}{\partial X_i} = \gamma_i q_j q_k \tag{3.2}$$

where i, j, k are cyclic and equal to $1, 2, 3$ and $*$ means complex conjugation. The X_i are characteristic coordinates, usually defined by

$$\frac{\partial}{\partial X_i} = -\partial_t - \underline{v}_i \cdot \nabla. \tag{3.3}$$

The γ's are the coupling constants and are scaled to unity in magnitude, i.e. $\gamma = \pm 1$, different choices for γ's will correspond to different classes of solution. By changing the signs of the fields q, one γ can be set equal to $+1$, it can then be seen that there are only two distinct cases $\gamma_1 = \gamma_2 = \gamma_3 = 1$ and $\gamma_1 = \gamma_2 = -\gamma_3 = 1$. However we may set γ_i (for $i = 1, 2, 3$) to one by absorbing it into the characteristic coordinates. We shall assume this for the rest of the thesis.

Although time, t, occurs in the system we shall generally be working with the characteristic coordinates, thus, we may think of this system as a stationary system in three-dimensional space represented by these characteristic coordinates. Alternatively, we can consider the solution of the system at a point in time to be a cross section through the three-dimensional space picture. As time progresses this cross section moves, $t \to -\infty$ corresponding to the characteristic coordinates $X_i \to +\infty$ and $t \to \infty$ corresponding to $X_i \to -\infty$.

The basic scattering problem [50] is given by

$$\frac{\partial \zeta_i}{\partial X_k} = q_j^* \zeta_k \qquad \frac{\partial \zeta_k}{\partial X_i} = q_j \zeta_i \tag{3.4}$$

where i, j, k take cyclic values over $1, 2, 3$. The integrability condition for (3.4) is the original non-linear system (3.2). Kaup points out [50] that this scattering problem is unusual in that there is no eigenvalue present, thus we have no bound states. This is because the envelope profiles in three dimensions are required to be integrable and square integrable, giving rise to only a continous spectrum. Thus solitons as understood for the one-dimensional system do not occur. However, localized solutions do occur, and are referred to as lumps rather than solitons.

3.2 Singularity Analysis

3.2.1 Background

We recall that the absence of moveable critical points for an ordinary differential equation is defined as the Painlevé property. Thus the 50 equations that Painlevé and Gambier [83] classified are the only rational second order equations that satisfy the Painlevé property.

Further this property lead Ablowitz, Ramani and Segur [4], [5], [6] to the Painlevé ordinary differential equation (ODE) test. This states that every ODE which arises as a similarity reduction of a completely integrable partial differential equation (PDE) is of Painlevé type perhaps after a transformation of variables. This conjecture acts as a practical test of integrability, however it is limited as we must test every similarity reduction.

In 1983 Weiss, Tabor and Carnevale [94] introduced the Painlevé property for PDE's as a method of applying the Painlevé test directly to a given PDE without having to reduce it to a given ODE. We outline the method.

We recall the idea of a Laurent expansion, and the idea of expanding around a singularity. However in several complex variables the singularities can't be isolated (in general) so we generalise in the following way.

Lemma : If $f = f(z_1, \dots, z_n)$ is a meromorphic function (analytic except for poles) of n complex variables, the singularities of f occur along manifolds of real dimension $2n-2$. They are determined by conditions of the form

$$\varphi(z_1, \dots, z_n) = 0 \tag{3.5}$$

with φ an analytic function of $z_1, \dots, z_n$ in a neighbourhood of the non-characteristic manifold (we may think of this locally as Euclidean space). Characteristic manifolds for PDE's are the analogue of fixed singularities in ODE's since they are determined by the PDE and not by the particular solution. So a non-characteristic manifold is essential because on a characteristic manifold any type of singularity may propagate. The PDE test may be summarised as :

Painlevé Test : A PDE is said to possess the Painlevé property when the solutions of the PDE are "single-valued" about the moveable singularity manifold. To be precise we say that a PDE is integrable if the singularity manifold is determined as above and if $u = u(z_1, \dots, z_n)$ is a solution of the PDE we assume

$$u = \sum_{j=0}^{\infty} u_j \varphi^{j-\alpha} \qquad \alpha \in \mathbb{Z} \tag{3.6}$$

where $u_j = u_j(z_1, \dots, z_n)$ and $\varphi = \varphi(z_1, \dots, z_n)$ are analytic functions in a neighbourhood of the manifold.

The PDE test is extremely successful in helping to decide whether a given PDE is integrable. For a discussion of it's limitations see [58], [59]. The PDE test is important however, as it allows other features of the PDE to be determined. These include the Lax pair [73], Bäcklund transformations [32], Schwarzian derivative [93], special and rational solutions [32].

There also exists a deep connection between Hirota's method and the PDE test. In 1985, Gibbon, Radmore, Tabor and Wood [33] demonstrated the link explicitly, and in doing so provided an explanation as to why Hirota's method works. We exploit this to obtain the bilinear transformation for the 3D3WR.

3.2.2 PDE Test for the 3D3WR

For this system the Painlevé analysis can be carried out. We recall that, informally, one says that a partial differential equation (PDE) possesses the Painlevé property when it's solutions are single-valued about the moveable singularity manifold [94]. In order to perform the Painlevé analysis we start from the system (3.2) (with γ_i's scaled to unity), and we make the ansatz that the variables q_i, q_i^* can be expanded about the singularity manifold $\varphi(X_1, X_2, X_3) = 0$ as

$$q_i = \sum_{m=0}^{\infty} u_{im}\varphi^{m+\alpha_i} \qquad (3.7)$$

$$q_i^* = \sum_{m=0}^{\infty} v_{im}\varphi^{m+\beta_i} \qquad (3.8)$$

where φ, u_{im} and v_{im} are all analytic functions of the X_i, in a neighborhood of $\varphi = 0$ and α_i, β_i are integers.

Inserting (3.7), (3.8) into equations (3.2) a leading order analysis provides

$$\alpha_i = \beta_i = -1 \qquad (3.9)$$

for all i, with

$$\frac{\partial\varphi}{\partial X_i} = \frac{-v_{j0}v_{k0}}{u_{i0}} \qquad (3.10)$$

$$\frac{\partial\varphi}{\partial X_i} = \frac{-u_{j0}u_{k0}}{v_{i0}}. \qquad (3.11)$$

From equations (3.10), (3.11) we may choose two of the u_{i0}, v_{i0} (u_{30} and v_{10} say) as arbitrary functions and then

$$u_{10} = \frac{1}{v_{10}}\frac{\partial\varphi}{\partial X_2}\frac{\partial\varphi}{\partial X_3} \qquad (3.12)$$

$$u_{20} = -\frac{v_{10}}{u_{30}}\frac{\partial\varphi}{\partial X_1} \qquad (3.13)$$

$$v_{20} = -\frac{u_{30}}{v_{10}}\frac{\partial\varphi}{\partial X_3} \qquad (3.14)$$

$$v_{30} = \frac{1}{u_{30}}\frac{\partial\varphi}{\partial X_2}\frac{\partial\varphi}{\partial X_1}. \qquad (3.15)$$

We obtain the resonances, that is values of m at which arbitrary functions enter into the series, by substituting (3.7), (3.8) into equations (3.2), retaining leading order terms only. As a result, we obtain the matrix equation

$$[M]\,[V]=0 \qquad [V]^T=[u_{1m},u_{2m},u_{3m},v_{1m},v_{2m},v_{3m}] \tag{3.16}$$

for the lowest-order coefficients, where

$$[M]=\begin{bmatrix} P_1 & 0 & 0 & 0 & -v_{30} & -v_{20} \\ 0 & P_2 & 0 & -v_{30} & 0 & -v_{10} \\ 0 & 0 & P_3 & -v_{20} & -v_{10} & 0 \\ 0 & -u_{30} & -u_{20} & P_1 & 0 & 0 \\ -u_{30} & 0 & -u_{10} & 0 & P_2 & 0 \\ -u_{20} & -u_{10} & 0 & 0 & 0 & P_3 \end{bmatrix} \tag{3.17}$$

with $P_i=(m-1)\frac{\partial\varphi}{\partial X_i}$ for $i=1,2,3$. The resonances are obtained when $\det M=0$, which yields the resonances $m=-1,0,2,3$ with 0 and 2 repeated twice. We recall that the resonance at $m=-1$ represents the arbitrariness of the singularity manifold $\varphi(X_1,X_2,X_3)=0$. While the "double" resonance at $m=0$ is associated with the arbitrary functions u_{30} and v_{10}.

In order to check the existence of arbitrary functions at the other resonances we use the Laurent expansion of (3.7), (3.8) in equations (3.2). Now collecting the coefficients of $(\varphi^{-1},\varphi^{-1},\varphi^{-1},\varphi^{-1},\varphi^{-1},\varphi^{-1})$ we obtain the set of equations

$$\frac{\partial u_{i0}}{\partial X_i}-[v_{j0}v_{k1}+v_{j1}v_{k0}] = 0 \tag{3.18}$$

$$\frac{\partial v_{i0}}{\partial X_i}-[u_{j0}u_{k1}+u_{j1}u_{k0}] = 0. \tag{3.19}$$

Similarly, collecting the coefficients of $(\varphi^0,\varphi^0,\varphi^0,\varphi^0,\varphi^0,\varphi^0)$, we obtain

$$\frac{\partial u_{i1}}{\partial X_i}+u_{i2}\frac{\partial\varphi}{\partial X_i}-[v_{j0}v_{k2}+v_{j2}v_{k0}+v_{j1}v_{k1}] = 0 \tag{3.20}$$

$$\frac{\partial v_{i1}}{\partial X_i}+v_{i2}\frac{\partial\varphi}{\partial X_i}-[u_{j0}u_{k2}+u_{j2}u_{k0}+u_{j1}u_{k1}] = 0. \tag{3.21}$$

Finally at $(\varphi^1,\varphi^1,\varphi^1,\varphi^1,\varphi^1,\varphi^1)$

$$\frac{\partial u_{i2}}{\partial X_i}+2u_{i3}\frac{\partial\varphi}{\partial X_i}-[v_{j0}v_{k3}+v_{j3}v_{k0}+v_{j1}v_{k2}+v_{j2}v_{k1}] = 0 \tag{3.22}$$

$$\frac{\partial v_{i2}}{\partial X_i}+2v_{i3}\frac{\partial\varphi}{\partial X_i}-[u_{j0}u_{k3}+u_{j3}u_{k0}+u_{j1}u_{k2}+u_{j2}u_{k1}] = 0. \tag{3.23}$$

To show the above equations have the required number of arbitrary functions becomes tedious for the general manifold. We adapt the Kruskal ansatz [90]. By assuming $\varphi(X_1, X_2, X_3) = X_1 + \psi(X_2, X_3)$ the calculations are somewhat simpler. Solving equations (3.18), (3.19) we can determine u_{i1} and v_{i1} uniquely, given u_{i0} and v_{i0} $(i = 1, 2, 3)$. Solving (3.20), (3.21) by repeated substitution we arrive at two arbitrary functions, u_{32} and v_{22} say, providing the following are satisfied

$$v_{10}\left[\gamma_2\frac{\partial v_{21}}{\partial X_2} + u_{30}\frac{\partial u_{11}}{\partial X_1} - u_{30}v_{21}v_{31} - u_{31}u_{11}\right] - u_{30}v_{20}\left[\frac{\partial u_{21}}{\partial X_2} + v_{30}\frac{\partial v_{11}}{\partial X_1} - u_{21}u_{31}v_{30} - v_{31}v_{11}\right] = 0, \quad (3.24)$$

$$u_{10}\left[\frac{\partial u_{31}}{\partial X_3} + v_{20}\frac{\partial v_{11}}{\partial X_1} - v_{20}u_{21}u_{31} - v_{11}v_{21}\right] - u_{30}v_{20}\left[\frac{\partial v_{31}}{\partial X_3} + u_{20}\frac{\partial u_{11}}{\partial X_1} - u_{20}v_{21}v_{31} - u_{11}u_{21}\right] = 0. \quad (3.25)$$

With a little algebra this is shown to be the case. So the"double" resonance at $m = 2$ corresponds to the arbitrariness of u_{32} and v_{22}, with u_{12}, u_{22}, v_{12} and v_{32} in terms of previously determined functions.

For (3.22), (3.23) we find that v_{33} is arbitrary providing the following is true

$$g_3 + \frac{u_{20}f_1}{2} + \frac{u_{20}v_{30}}{3\partial\varphi/\partial X_2}\left[g_2 + \frac{u_{30}f_1}{2\gamma_1}\right] + \frac{2u_{10}}{3\partial\varphi/\partial X_2}\left[f_2 + \frac{v_{30}g_1}{2}\right] - \frac{v_{20}}{3}\left[f_2 + \frac{v_{30}g_1}{2}\right] - \frac{\partial\varphi/\partial X_2}{u_{30}^2}\left[f_3 + \frac{v_{20}g_1}{2} + \frac{2v_{10}}{3\partial\varphi/\partial X_2}\left[g_2 + \frac{u_{30}f_1}{2}\right]\right] = 0 \quad (3.26)$$

with $f_i = \frac{\partial u_{i2}}{\partial X_i} - v_{j1}v_{k2} - v_{j2}v_{k1}$ and $g_i = \frac{\partial v_{i2}}{\partial X_i} - u_{j1}u_{k2} - u_{j2}u_{k1}$. Again this is shown to be true.

So the solution q_i, q_i^* of equations (3.7), (3.8) admits the required number of arbitrary functions without the introduction of moveable critical manifolds. Hence the Painlevé property is satisfied for (3.2).

3.2.3 Bäcklund Transformation

To generate the Bäcklund transformation of equations (3.2), we follow the same method as Ganesan and Lakshmanan [31], and truncate the Laurent series at the constant level term (see also [33]), that is

$$u_{ik} = v_{ik} = 0 \tag{3.27}$$

for $k \geq 2$. Thus from (3.7), (3.8) we have

$$q_i = \frac{u_{i0}}{\varphi} + u_{i1} \tag{3.28}$$

$$q_i^* = \frac{v_{i0}}{\varphi} + v_{i1} \tag{3.29}$$

where the u_{i1} and v_{i1} satisfy the original equations (3.2),

$$\frac{\partial u_{i1}}{\partial X_i} = v_{j1}v_{k1} \qquad \frac{\partial v_{i1}}{\partial X_i} = u_{j1}u_{k1} \tag{3.30}$$

and

$$u_{i0}\frac{\partial \varphi}{\partial X_i} = -v_{j0}v_{k0} \qquad v_{i0}\frac{\partial \varphi}{\partial X_i} = -u_{j0}u_{k0} \tag{3.31}$$

$$\frac{\partial u_{i0}}{\partial X_i} = (v_{j0}v_{k1} + v_{k0}v_{j1}) \qquad \frac{\partial v_{i0}}{\partial X_i} = (u_{j0}u_{k1} + u_{k0}u_{j1}) \tag{3.32}$$

for $i, j, k = 1, 2, 3$ and cyclic. Starting with a solution u_{i1}, v_{i1} we can generate another solution q_i, q_i^* via equations (3.28), (3.29), as long as equations (3.30) to (3.32) hold. Thus the above equations (3.28) to (3.32) represent a Bäcklund transformation for the system.

The Bäcklund transformation for this system has been previously been written down by Kaup [51]. Given a solution q_i to generate another solution $\bar{q}_i$ we use

$$\bar{q}_i = q_i + \frac{\zeta_k^* \zeta_j}{D}, \tag{3.33}$$

where the function D exists and

$$\partial_i D = \frac{\partial D}{\partial X_i} = -\zeta_i^* \zeta_i. \tag{3.34}$$

The ζ's come from the scattering problem for the original equation

$$\partial_k \zeta_i = q_j^* \zeta_k, \tag{3.35}$$

$$\partial_i \zeta_k = q_j \zeta_i, \tag{3.36}$$

where as before the i, j, k are cyclic. This Bäcklund transformation is indeed the same as the one generated by truncating the series expansion if we make the identifications

$$\begin{aligned} \varphi &\rightarrow D \\ u_{i0} &\rightarrow \zeta_j \zeta_k^* \\ v_{i0} &\rightarrow \zeta_k \zeta_j^* \end{aligned} \tag{3.37}$$

3.2.4 Bilinearization

We can use the Painlevé analysis as a guide to bilinearization, by considering the vacuum solution [31]

$$u_{i1} = v_{i1} = 0 \tag{3.38}$$

in the Bäcklund transformations (3.28), (3.29) to give

$$q_i = \frac{u_{i0}}{\varphi} \tag{3.39}$$

$$q_i^* = \frac{v_{i0}}{\varphi}. \tag{3.40}$$

Assuming that φ to be real and substituting equations (3.39), (3.40) into (3.2), and making use of the Hirota bilinear operators [42], we obtain the following Hirota bilinear form of the 3D3WR equation

$$D_{X_i} u_{i0} \cdot \varphi = v_{j0} v_{k0} \tag{3.41}$$

$$D_{X_i} v_{i0} \cdot \varphi = u_{j0} u_{k0}. \tag{3.42}$$

The D_{X_i} are standard Hirota bilinear operators defined in the introduction (1.6). Instead of using the Painlevé analysis as a guidance to the bilinearization we can directly bilinearize this system by choosing the following change of variables :

$$q_i = \frac{G_i}{F} \tag{3.43}$$

$$q_i^* = \frac{G_i^*}{F} \tag{3.44}$$

where F is understood to be real. This recasts our equations as

$$D_{X_i} F \cdot G_i = -G_j^* G_k^* \tag{3.45}$$

$$D_{X_i} F \cdot G_i^* = -G_j G_k \tag{3.46}$$

where i, j, k are cyclic permutations of $1, 2, 3$. It should perhaps be pointed out at this early stage that these equations come straight from the lowest equations in the three-component KP hierarchy [46], hence integrability is to be expected. The form of solution will be the τ-functions from this hierarchy. These in general will take the form of three component wronskians or three-component grammians. We will take an explicit look at this later.

3.3 Solutions

Let us consider the simplest type of solution to such a system. Consider the specific case where there is only one wave envelope present, q_1 say, with $q_2 = q_3 = 0$, here the equations reduce to

$$\frac{\partial q_1}{\partial X_1} = 0 \qquad\qquad \frac{\partial q_1^*}{\partial X_1} = 0 \tag{3.47}$$

this says that q_1 is independent of X_1 and has arbitrary dependence on X_2 and X_3, thus, to use the language of Kaup we can think of q_1 as being a "tube" extending in the X_1 direction with some profile in the X_2 and X_3 directions. In general however all three fields will be present in the solution, and they can be thought of as "tubes" lying in the three characteristic directions. In regions where there is an overlap of these "tubes" interactions will occur, however, as the $X_i \to \pm\infty$ the envelopes will separate and the solutions will cease from changing. Kaup looked at such solutions via inverse scattering methods [50], [49], [52].

Perhaps the simplest way to generate solutions is using the Bäcklund transformation [51]. Given a solution q_i we have a Bäcklund transformation to generate another solution

$$\bar{q_i} = q_i + \frac{\zeta_k^* \zeta_j}{D} \tag{3.48}$$

where the function D exists and

$$\partial_i D \equiv \frac{\partial D}{\partial X_i} = -\zeta_i^* \zeta_i. \tag{3.49}$$

The ζ's come from the scattering problem for the original equation

$$\partial_k \zeta_i = q_j^* \zeta_k \tag{3.50}$$

$$\partial_i \zeta_k = q_j \zeta_i \tag{3.51}$$

where as before the i, j, k are cyclic.

3.3.1 1-lump solution

Non-degenerate kernel case:

To generate the 1-lump solution we start with the trivial solution $q_i = 0, \forall i = 1, 2, 3$. We then use the Bäcklund transformation (BT) to generate another solution, solving first for the ζ's, then calculating D. Following Kaup's method [51] we have

$$\zeta_i = g_i(X_i) \tag{3.52}$$

$$D = \beta + \sum_{i=1}^{3} \Phi_i(X_i) \tag{3.53}$$

where β is a real constant and

$$\Phi_i(X_i) = \int_{X_i}^{\infty} g_i^*(u) g_i(u) du \tag{3.54}$$

the g_i's are arbitrary functions of the single variables X_i. Without loss of generality we may set $\beta = 1$, the 1-lump solution is then given by

$$\bar{q}_j = \frac{g_i^* g_k}{1 + \sum_{m=1}^{3} \Phi_m(X_m)}. \tag{3.55}$$

One of the aims of this chapter is to relate these solutions to grammians, in a similar way as Gilson and Nimmo's proceedure for the Davey-Stewartson equation [35]. The grammian approach being compact and direct allows a much wider class of solutions to be determined. Here we can write the solution in the following form

$$q_i = \frac{G_i}{F} \qquad q_i^* = \frac{G_i^*}{F} \tag{3.56}$$

with

$$F = 1 + \sum_{m=1}^{3} \Phi_m(X_m) = |I + H\Phi| = |\mathcal{F}| \tag{3.57}$$

where I is the (3×3) identity matrix, and

$$\Phi = \mathrm{diag}(\Phi_1, \Phi_2, \Phi_3) \tag{3.58}$$

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.59}$$

The G's can be expressed as a bordered determinants

$$G_1 = -\begin{vmatrix} 0 & 0 & 0 & g_3^* \\ g_2 & & & \\ g_2 & & \mathcal{F} & \\ g_2 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \underline{g_3^\dagger} \\ H\underline{g_2} & \mathcal{F} \end{vmatrix} \tag{3.60}$$

$$G_2 = -\begin{vmatrix} 0 & g_1^* & 0 & 0 \\ g_3 & & & \\ g_3 & & \mathcal{F} & \\ g_3 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \underline{g_1^\dagger} \\ H\underline{g_3} & \mathcal{F} \end{vmatrix} \tag{3.61}$$

$$G_3 = -\begin{vmatrix} 0 & 0 & g_2^* & 0 \\ g_1 & & & \\ g_1 & & \mathcal{F} & \\ g_1 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \underline{g_2^\dagger} \\ H\underline{g_1} & \mathcal{F} \end{vmatrix} \tag{3.62}$$

$$\tag{3.63}$$

similarly we have

$$G_i = -\begin{vmatrix} 0 & \underline{g_j^\dagger} \\ H\underline{g_k} & \mathcal{F} \end{vmatrix}. \tag{3.64}$$

for i, j, k cyclic. Here

$$\underline{g_1} = (g_1, 0, 0)^T \qquad \underline{g_2} = (0, g_2, 0)^T \qquad \underline{g_3} = (0, 0, g_3)^T \tag{3.65}$$

$$\underline{g_1^\dagger} = (g_1^*, 0, 0) \qquad \underline{g_2^\dagger} = (0, g_2^*, 0) \qquad \underline{g_3^\dagger} = (0, 0, g_3^*). \tag{3.66}$$

Degenerate kernel case:

In addition to this solution (the non-degenerate kernel case) there is, what is called the degenerate kernel case [50]. For the degenerate kernel case the solution is given by;

$$q_j = g_i^*(X_i)g_k(X_k)\frac{1-\Phi_j(X_j)}{D(X_i, X_j, X_k)} \tag{3.67}$$

where

$$\Phi_i(X_i) = \int_{X_i}^{\infty} g_i^*(u)g_i(u)du \tag{3.68}$$

and

$$D = 1 - \Phi_1\Phi_2 - \Phi_2\Phi_3 - \Phi_3\Phi_1 + 2\Phi_1\Phi_2\Phi_3. \tag{3.69}$$

D appears in the denominator of the solution, thus, if we want a nonsingular solution, it can be seen from the signs of the coefficients occuring in D that we cannot take the g's as exponentials. Cornille [16] has actually shown that these degenerate kernels can admit localised solutions as long as the q's are localised and the Φ's are always strictly less than 1.

Again this can be cast in terms of grammians as follows,

$$q_i = \frac{G_i}{F} \tag{3.70}$$

with

$$F = |I + H\Phi| = |\mathcal{F}| \tag{3.71}$$

where I is the (3×3) identity matrix and

$$\Phi = \operatorname{diag}(\Phi_1, \Phi_2, \Phi_3) \tag{3.72}$$

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3.73}$$

Thus the structure of the solution has not changed, only the form of the matrix H. The G_i's can be expressed as a bordered determinants, just as before, but now with the modified H. For example

$$G_1 = -\begin{vmatrix} 0 & 0 & 0 & g_3^* \\ g_2 & & & \\ 0 & & \mathcal{F} & \\ g_2 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \underline{g_3^\dagger} \\ H\underline{g_2} & \mathcal{F} \end{vmatrix} \tag{3.74}$$

i.e.

$$G_i = -\begin{vmatrix} 0 & \underline{g_k^\dagger} \\ H\underline{g_j} & \mathcal{F} \end{vmatrix} \qquad G_i^* = -\begin{vmatrix} 0 & \underline{g_j^\dagger} \\ H\underline{g_k} & \mathcal{F} \end{vmatrix}. \tag{3.75}$$

3.3.2 2-lump solution

Again using the Bäcklund transformation a 2-lump solution can be generated from the 1-lump solution. For convenience, especially later on in the paper we shall adapt notation somewhat different from Kaup's. Lets rewrite the 1-lump solution:

$$g_1 = \phi(X_1), \qquad g_2 = \psi(X_2), \qquad g_3 = \sigma(X_3) \tag{3.76}$$

$$\Phi_{11} = \int_{X_1}^{\infty} \phi^*\phi(u)du, \quad \Psi_{11} = \int_{X_2}^{\infty} \psi^*\psi(u)du, \quad \Sigma_{11} = \int_{X_3}^{\infty} \sigma^*\sigma(u)du, \tag{3.77}$$

with

$$D_{11} = 1 + \Phi_{11}(X_1) + \Psi_{11}(X_2) + \Sigma_{11}(X_3). \tag{3.78}$$

The 1-lump solution is then given by

$$q_1 = \frac{\psi\sigma^*}{D_{11}} \tag{3.79}$$

$$q_2 = \frac{\sigma\phi^*}{D_{11}} \tag{3.80}$$

$$q_3 = \frac{\phi\psi^*}{D_{11}} \tag{3.81}$$

with q_i^* in the obvious way, noting that D_{11} is real.

Also we will introduce

$$D_{ij} = \beta_{ij} + \Phi_{ij}(X_1) + \Psi_{ij}(X_2) + \Sigma_{ij}(X_3) \tag{3.82}$$

where β_{ij} are constants and

$$\Phi_{ij} = \int_{X_1}^{\infty} \phi_i^* \phi_j(u) du, \quad \Psi_{ij} = \int_{X_2}^{\infty} \psi_i^* \psi_j(u) du, \quad \Sigma_{ij} = \int_{X_3}^{\infty} \sigma_i^* \sigma_j(u) du. \tag{3.83}$$

For the 2-lump solution we let $i, j = 1, 2$ and can without loss of generality set $\beta_{11} = \beta_{22} = 1$. Then

$$q_i = \frac{G_i}{F} \tag{3.84}$$

with

$$F = D_{11}D_{22} - D_{12}D_{21} \tag{3.85}$$

$$G_1 = D_{22}\psi_1\sigma_1^* - D_{12}\psi_1\sigma_2^* - D_{21}\psi_2\sigma_1^* + D_{11}\psi_2\sigma_2^* \tag{3.86}$$

$$G_2 = D_{22}\sigma_1\phi_1^* - D_{12}\sigma_1\phi_2^* - D_{21}\sigma_2\phi_1^* + D_{11}\sigma_2\phi_2^* \tag{3.87}$$

$$G_3 = D_{22}\phi_1\psi_1^* - D_{12}\phi_1\psi_2^* - D_{21}\phi_2\psi_1^* + D_{11}\phi_2\psi_2^*. \tag{3.88}$$

This solution can also be recast as a grammian, this will now be a 6×6 determinant.

$$q_i = \frac{G_i}{F} \tag{3.89}$$

with

$$F = |I + H\Phi| = |\mathcal{F}| \tag{3.90}$$

where I is the (6×6) identity matrix, Φ is a matrix with on-diagonal (2×2) blocks and zeros elsewhere, H is a block matrix with nine identical (2×2) blocks.

$$\Phi = \begin{pmatrix} \Phi_{ij} & & 0 \\ & \Psi_{ij} & \\ 0 & & \Sigma_{ij} \end{pmatrix} \tag{3.91}$$

$$H = \begin{pmatrix} B & B & B \\ B & B & B \\ B & B & B \end{pmatrix} \tag{3.92}$$

with

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}, \tag{3.93}$$

and where we have redefined

$$\Phi_{ij} \int_{X_1}^{\infty} \phi_i \phi_j^*(u) du, \quad \Psi_{ij} \int_{X_2}^{\infty} \psi_i \psi_j^*(u) du, \quad \Sigma_{ij} \int_{X_3}^{\infty} \sigma_i \sigma_j^*(u) du. \tag{3.94}$$

The G_i's can be expressed as a bordered determinants

$$G_1 = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \mathcal{F} \end{vmatrix} \tag{3.95}$$

$$G_2 = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\sigma} & \mathcal{F} \end{vmatrix} \tag{3.96}$$

$$G_3 = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix} \tag{3.97}$$

where

$$\underline{\phi} = (\phi_1, \phi_2, 0, 0, 0, 0)^T, \quad \underline{\psi} = (0, 0, \psi_1, \psi_2, 0, 0)^T, \quad \underline{\sigma} = (0, 0, 0, 0, \sigma_1, \sigma_2)^T, \tag{3.98}$$

$$\underline{\phi}^\dagger = (\phi_1^*, \phi_2^*, 0, 0, 0, 0), \quad \underline{\psi}^\dagger = (0, 0, \psi_1^*, \psi_2^*, 0, 0), \quad \underline{\sigma}^\dagger = (0, 0, 0, 0, \sigma_1^*, \sigma_2^*). \tag{3.99}$$

We shall investigate the structure and what the one- and two-lump solutions look like, in section 3.6.

3.3.3 n-lump solution

The reason that we have taken time to change the notation is that now it is possible to postulate a simple form for the n-lump solution which has the structure of a three-component grammian.

$$q_i = \frac{G_i}{F} \tag{3.100}$$

with

$$F = |I + H\Phi| = |\mathcal{F}| \tag{3.101}$$

where I is the $(3n \times 3n)$ identity matrix, Φ is a matrix with on-diagonal $(n \times n)$ blocks and zeros elsewhere, H is a block matrix with nine identical $(n \times n)$ blocks.

$$\Phi = \begin{pmatrix} \Phi_{ij} & & 0 \\ & \Psi_{ij} & \\ 0 & & \Sigma_{ij} \end{pmatrix} \tag{3.102}$$

$$H = \begin{pmatrix} B & B & B \\ B & B & B \\ B & B & B \end{pmatrix} \tag{3.103}$$

with

$$B_{ij} = \beta_{ij}, \quad i,j = 1.....n. \tag{3.104}$$

The G's can be expressed as a bordered determinants

$$G_1 = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \mathcal{F} \end{vmatrix} \tag{3.105}$$

$$G_2 = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\sigma} & \mathcal{F} \end{vmatrix} \tag{3.106}$$

$$G_3 = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix} \tag{3.107}$$

where

$$\underline{\phi} = (\phi_1, \dots, \phi_n; 0, \dots, 0; 0, \dots, 0)^T \tag{3.108}$$

$$\underline{\psi} = (0, \dots, 0; \psi_1, \dots, \psi_n; 0, \dots, 0)^T \tag{3.109}$$

$$\underline{\sigma} = (0, \dots, 0; 0, \dots, 0; \sigma_1, \dots, \sigma_n)^T \tag{3.110}$$

$$\underline{\phi}^\dagger = (\phi_1^*, \dots, \phi_n^*; 0, \dots, 0; 0, \dots, 0) \tag{3.111}$$

$$\underline{\psi}^\dagger = (0, \dots, 0; \psi_1^*, \dots, \psi_n^*; 0, \dots, 0) \tag{3.112}$$

$$\underline{\sigma}^\dagger = (0, \dots, 0; 0, \dots, 0; \sigma_1^*, \dots, \sigma_n^*). \tag{3.113}$$

3.4 Direct Proof of Solution

For a direct proof of the solution we use a Jacobi identity (see the introduction). Here we need the following form of the identity:

$$|\mathcal{F}|\begin{vmatrix} 0 & 0 & \underline{\phi}^\dagger \\ 0 & 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & H\underline{\psi} & \mathcal{F} \end{vmatrix} = \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix}\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \mathcal{F} \end{vmatrix} - \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\psi} & \mathcal{F} \end{vmatrix}\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix}. \quad (3.114)$$

We can show that the derivatives we require can all be expressed in terms of bordered determinants. These expressions arise because, in general, for an $N \times N$ matrix A whose entries a_{ij} are such that

$$\frac{\partial a_{ij}}{\partial X} = \alpha_i \beta_j, \quad (3.115)$$

the derivative of the determinant can be written as

$$\frac{\partial |A|}{\partial X} = \sum_{i,j=1}^{N} (-1)^{i+j} \alpha_i \beta_j A^i_j = -\begin{vmatrix} 0 & \beta_1 & \dots & \beta_N \\ \alpha_1 & & & \\ \vdots & & A & \\ \alpha_N & & & \end{vmatrix}. \quad (3.116)$$

Thus, we can show that

$$\frac{\partial F}{\partial X_1} = \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix} \quad (3.117)$$

and

$$\frac{\partial G_1}{\partial X_1} = -\begin{vmatrix} 0 & 0 & \underline{\phi}^\dagger \\ 0 & 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & H\underline{\psi} & \mathcal{F} \end{vmatrix}. \quad (3.118)$$

Thus (3.114) is actually

$$-F\frac{\partial G_1}{\partial X_1} = -\frac{\partial F}{\partial X_1} G_1 - G_2^* G_3^* \quad (3.119)$$

This is precisely the bilinear equation

$$D_{X_1} G_1 \cdot F = G_2^* G_3^*. \quad (3.120)$$

Similarily if we choose

$$|A| = \frac{\partial G_i}{\partial X_i} \quad (3.121)$$

then equivalent identities to (3.114), give us the other bilinear equations.

3.5 A More General Solution

The proof that the n-lump solutions satisfy the equations, actually applies more generally. The proof will still hold for any constant hermitian matrix you may wish to take for H. In general, the form $F = |I + H\Phi|$, where Φ could be a block matrix, where each block may not necessarily be of the same size will still be a solution. Thus

$$q_i = \frac{G_i}{F}, \qquad q_i^* = \frac{G_i^*}{F} \tag{3.122}$$

will be a solution of the system, with

$$F = |I + H\Phi| = |\mathcal{F}| \tag{3.123}$$

where I is the $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ identity matrix, H is an hermitian matrix, Φ is an $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ matrix with non-zero on-diagonal blocks and zeros elsewhere

$$\Phi = \begin{pmatrix} \Phi_{ij} & & 0 \\ & \Psi_{kl} & \\ 0 & & \Sigma_{mn} \end{pmatrix}, \qquad \begin{array}{rcl} i,j & = & 1......n_1, \\ k,l & = & 1......n_2, \\ m,n & = & 1......n_3. \end{array} \tag{3.124}$$

and

$$\Phi_{ij} = \int_{X_1}^{\infty} \phi_i \phi_j(u)^* du, \quad \Psi_{ij} = \int_{X_2}^{\infty} \psi_i \psi_j(u)^* du, \quad \Sigma_{ij} = \int_{X_3}^{\infty} \sigma_i \sigma_j(u)^* du. \tag{3.125}$$

The G's can be expressed as a bordered determinants

$$G_1 = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \mathcal{F} \end{vmatrix}, \qquad G_1^* = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\sigma} & \mathcal{F} \end{vmatrix} \tag{3.126}$$

$$G_2 = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\sigma} & \mathcal{F} \end{vmatrix}, \qquad G_2^* = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix} \tag{3.127}$$

$$G_3 = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix}, \qquad G_3^* = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\psi} & \mathcal{F} \end{vmatrix}. \tag{3.128}$$

where

$$\begin{aligned}
\underline{\phi} &= (\phi_1, \dots, \phi_{n_1}; 0, \dots, 0; 0, \dots, 0)^T && (3.129)\\
\underline{\psi} &= (0, \dots, 0; \psi_1, \dots, \psi_{n_2}; 0, \dots, 0)^T && (3.130)\\
\underline{\sigma} &= (0, \dots, 0; 0, \dots, 0; \sigma_1, \dots, \sigma_{n_3})^T && (3.131)\\
\underline{\phi}^\dagger &= (\phi_1^*, \dots, \phi_{n_1}^*; 0, \dots, 0; 0, \dots, 0) && (3.132)\\
\underline{\psi}^\dagger &= (0, \dots, 0; \psi_1^*, \dots, \psi_{n_2}^*; 0, \dots, 0) && (3.133)\\
\underline{\sigma}^\dagger &= (0, \dots, 0; 0, \dots, 0; \sigma_1^*, \dots, \sigma_{n_3}^*). && (3.134)
\end{aligned}$$

This appears to be quite a broad class of solutions. However, we have to keep in mind that the H we actually want to take, should be non-zero everywhere, otherwise the solutions will contain singularities.

3.6 Examples

The essential difference between soliton solutions and lump solutions is the absence of (discrete) spectral parameters in the inverse scattering theory. This difference manifests itself in the functions ϕ_i, ψ_i and σ_i. In the case of related theories with solitons these functions contain the spectral parameters and are always restricted to obey some linear equation relating derivatives of the different independent variables together, here there is no such restriction. In this section we shall look at solutions to some simple cases.

3.6.1 The $(1,1,1)$ Case

Evaluating F in (3.123) gives

$$F = 1 + h_{11}\Phi_{11} + h_{22}\Psi_{11} + h_{33}\Sigma_{11} + \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \Phi_{11}\Psi_{11}$$

$$+ \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix} \Phi_{11}\Sigma_{11} + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} \Psi_{11}\Sigma_{11} + \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} \Phi_{11}\Psi_{11}\Sigma_{11} \quad (3.135)$$

and from (3.126)

$$G_1 = \psi_1 \sigma_1^* \left(h_{32} + \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix} \Phi_{11} \right) \tag{3.136}$$

with the Φ_{11}, Ψ_{11}, Σ_{11} given by (3.125), $n_1 = n_2 = n_3 = 1$ and H as

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}. \tag{3.137}$$

Localised Profiles

If ϕ_1, ψ_1 and σ_1 are taken to be localised functions, for instance Lorentzians, or Gaussians, their asymptotic behaviour is simple to describe. As

$$\begin{array}{llll} X_1 \to +\infty & \Phi_{11} \to 0, & \phi_1 \to 0, & \phi_1^* \to 0. \\ X_1 \to -\infty & \Phi_{11} \to \Phi_{-\infty}, & \phi_1 \to 0, & \phi_1^* \to 0. \end{array} \tag{3.138}$$

where $\Phi_{-\infty}$ is a constant. Similar sort of behaviour occurs for Ψ_{11} and Σ_{11} as $X_2, X_3 \to \pm\infty$. As $X_1 \to \infty$

$$F \to 1 + h_{22}\Psi_{11} + h_{33}\Sigma_{11} + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} \Psi_{11}\Sigma_{11} \tag{3.139}$$

$$G_1 \to h_{32}\psi_1\sigma_1^* \tag{3.140}$$

as $X_1 \to -\infty$

$$F \to a + b\Psi_{11} + c\Sigma_{11} + d\Psi_{11}\Sigma_{11} \tag{3.141}$$

$$G_1 \to e\psi_1\sigma_1^* \tag{3.142}$$

where a, b, c, d, e are constants

$$a = 1 + h_{11}\Phi_{-\infty}, \tag{3.143}$$

$$b = h_{22} + \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \Phi_{-\infty} \qquad c = h_{33} + \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix} \Phi_{-\infty} \tag{3.144}$$

$$d = \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} \Phi_{-\infty} \tag{3.145}$$

$$e = h_{32} + \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix} \Phi_{-\infty}. \tag{3.146}$$

We can choose ϕ, ψ, σ for instance as,

$$\phi_1 = c_1 e^{-(X_1 - X_1^0)^2}, \qquad \psi_1 = c_2 e^{-(X_2 - X_2^0)^2}, \qquad \sigma_1 = c_3 e^{-(X_3 - X_3^0)^2}, \tag{3.147}$$

where $c_1, c_2, c_3, X_1^0, X_2^0, X_3^0$ are real constants. The q_1–field ($q_i = G_i/F$) will take the form of a filled in "tube" in three-dimensional space in the direction of the X_1-axis, asymptotically for large X_1 the cross section profile will be determined by the functions ψ, σ. In this particular case we obtain a lump. Similarly at X_1 large and negative the profile will again be a lump of different size. The "interaction" region can be considered to be the region near the intersection of the coordinate axes. In figure 3.1 we show q_1 plotted in the X_2X_3-plane for fixed X_1, with

$$H = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}. \tag{3.148}$$

As in general the cross sectional profile of a $(1,1,1)$-solution is arbitrary, by choosing the ϕ_1, ψ_1 and σ_1's differently this simple solution could have a much more interesting appearence, however the basic character of the "tube" structure in the X_1 direction won't change. This behaviour is similar to the lumps as described by Kaup [51].

Non-localized Profiles

As an alternative to choosing our functions ϕ_1, ψ_1, σ_1 as above we could choose functions which don't decay as the $X_i \to \pm\infty$, we shall look at exponentials, we would expect the behaviour here to mimic more closely that of solitons. Choose

$$\phi_1 = e^{-p(X_1 - p)}, \qquad \psi_1 = e^{-q(X_2 - q)}, \qquad \sigma_1 = e^{-r(X_3 - r)}. \tag{3.149}$$

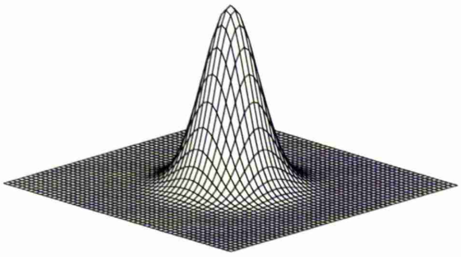


Figure 3.1: The q_1-field, plotted in the X_2X_3-plane for fixed $X_1 = -5$. $\phi_1 = \exp(-X_1^2)/2$, $\psi_1 = \exp(-X_2^2)/2$, $\sigma_1 = \exp(-X_3^2)/2$, $h_{ii} = 1$ and $h_{ij} = 1/2$ for $i \neq j$ $(i, j = 1, 2, 3)$.

With the way our boundary conditions have been set up earlier it is necessary to choose $Re(p), Re(q), Re(r) > 0$ (however with some reconsideration of the boundary conditions these can be taken negative). Now the choice of H in our solution is more critical, we can obtain solutions that are localised or are "ridge" like. For this exponential case the asymptotics is

$$\begin{array}{llll} X_1 \to +\infty & \Phi_{11} \to 0, & \phi_1 \to 0, & \phi_1^* \to 0. \\ X_1 \to -\infty & \Phi_{11} \to \infty, & \phi_1 \to \infty, & \phi_1^* \to \infty. \end{array} \tag{3.150}$$

thus as $X_1 \to +\infty$

$$F \to 1 + h_{22}\Psi_{11} + h_{33}\Sigma_{11} + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} \Psi_{11}\Sigma_{11} \tag{3.151}$$

$$G_1 \to h_{32}\psi_1\sigma_1^* \tag{3.152}$$

as $X_1 \to -\infty$

$$F \to a + b\Psi_{11} + c\Sigma_{11} + d\Psi_{11}\Sigma_{11} \tag{3.153}$$

$$G_1 \to e\psi_1\sigma_1^* \tag{3.154}$$

where a, b, c, d, e are constants

$$a = h_{11}, \qquad b = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}, \qquad c = \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix}, \tag{3.155}$$

$$d = \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix}, \qquad e = \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix}. \tag{3.156}$$

The basic structure of the solutions is as follows. If the coefficients of the Ψ_{11}, Σ_{11}, $\Psi_{11}\Sigma_{11}$ and the constant term are all present in F the solution will look like a lump in the X_2X_3-plane,for fixed X_1. This lump will change only in the interaction region (around $X_1 = 0$). This would be obtained with an H such as

$$H = \begin{pmatrix} 1 & \alpha & \beta \\ \alpha^* & 1 & \eta \\ \beta^* & \eta^* & 1 \end{pmatrix} \tag{3.157}$$

where $|\alpha|, |\beta|, |\gamma|$ are all strictly less than 1 (see figure 3.2). In figure 3.3 a surface of constant density of q_1-field has been plotted in three-dimensional space, this looks like a "tube" parallel to the X_1-axis.

In the case of the original lumps described by Kaup ($h_{ij} = 1, \forall ij = 1,2,3$) some of the coefficients of Ψ_{11} etc will be missing, this will have the effect of producing a ridge (see figure 3.4) as $X_1 \to +\infty$. At $X_1 = -\infty$ this ridge will disappear since

$$e = \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0. \tag{3.158}$$

3.6.2 The $(2, 1, 1)$ Case

The $(2, 1, 1)$ case is formed by introducing an extra arbitrary function $\phi_2(X_1)$ into the solution. This does not appear to correspond to the solutions discussed by Kaup via inverse scattering/Bäcklund transformations. The introduction of the function introduces extra features into the solutions. The structure will be reminiscent of say the (2,1) dromion solution in the Davey Stewartson equation. The function F will now be a 4×4 determinant. As before, the precise terms present will determine the shape of the solution.

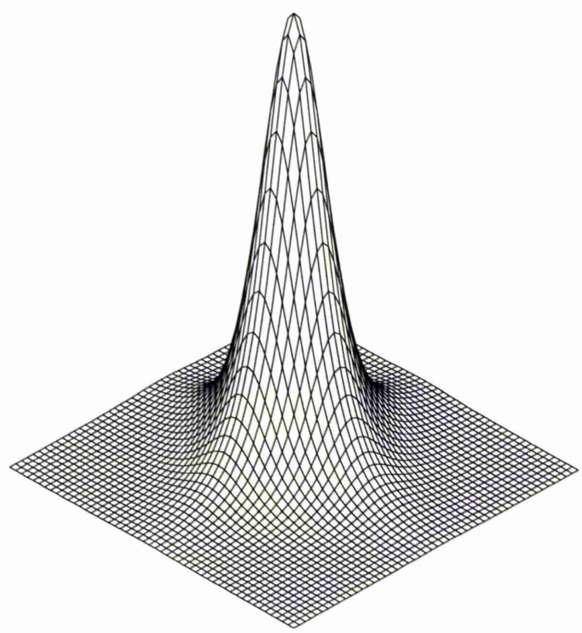


Figure 3.2: The q_1-field, plotted in the X_2X_3-plane for fixed $X_1 = 10$. $\phi_1 = \exp(-X_1)$, $\psi_1 = \exp(-X_2)$, $\sigma_1 = \exp(-X_3)$, $h_{11} = h_{22} = h_{33} = 1$, $h_{12} = h_{23} = h_{31} = 1/2$.

Localized Profiles

Choosing localised functions

$$\phi_1 = e^{-(X_1-\alpha)^2}, \quad \phi_2 = e^{-(X_1-\beta)^2}, \quad \psi_1 = e^{-(X_2-X_2^0)^2}, \quad \sigma_1 = e^{-(X_3-X_3^0)^2}, \qquad (3.159)$$

where $\alpha, \beta, X_2^0, X_3^0$ are constants. Looking at the q_2-field we observe, in three-dimensional space, two "tubes" centered on $(X_1, X_3) = (\alpha, X_3^0)$ and (β, X_3^0) lying in the direction of the X_2 axis. The q_3-field is similar with the "tubes lying in the direction of the X_3 axis. The q_1-field has just one "tube", however there are "inter-

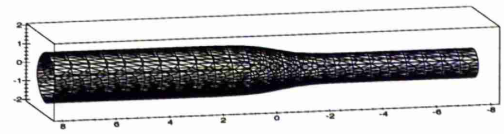


Figure 3.3: The q_1-field, here we have plotted the surface $|q_1| = 0.13$ in $X_1X_2X_3$-space. $\phi_1 = \exp(-X_1)$, $\psi_1 = \exp(-X_2)$, $\sigma_1 = \exp(-X_3)$, $h_{11} = h_{22} = h_{33} = 1$, $h_{12} = h_{23} = h_{31} = 1/2$.

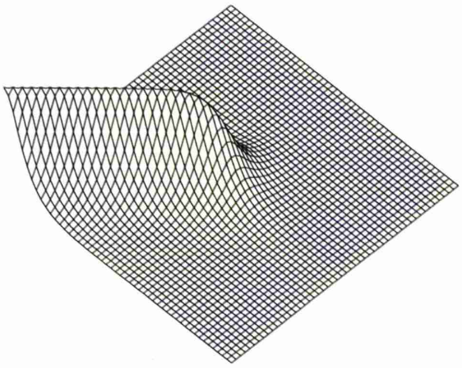


Figure 3.4: The q_1-field, plotted in the X_2X_3-plane for fixed $X_1 = 10$. $\phi_1 = \exp(-X_1)$, $\psi_1 = \exp(-X_2)$, $\sigma_1 = \exp(-X_3)$, $h_{ij} = 1$ for $i, j = 1, 2, 3$.

actions" around $X_1 = \alpha$ and $X_1 = \beta$. These solutions are shown in figures 3.5 and 3.6.

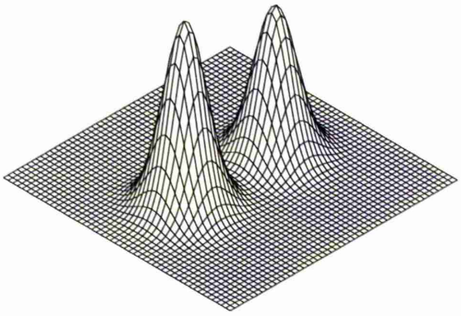


Figure 3.5: The q_2-field, plotted in the X_1X_3-plane for fixed $X_2 = 10$. $\phi_1 = \exp(-(X_1-2)^2)$, $\phi_2 = \exp(-(X_1+2)^2)$, $\psi_1 = \exp(-X_2^2)$, $\sigma_1 = \exp(-(X_3)^2)$, $h_{ij} = 1$ for $i, j = 1, 2, 3, 4$.

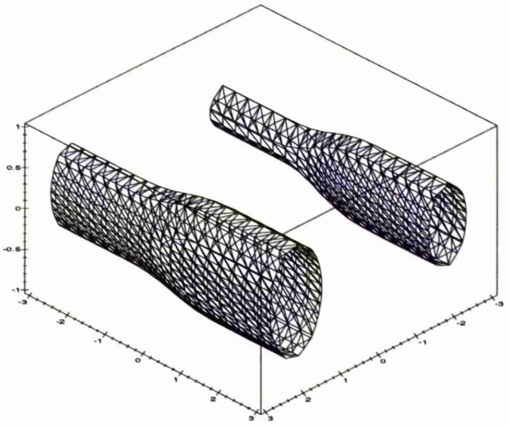


Figure 3.6: q_2-field, surface $q_2 = 0.12$ plotted in three-dimensional space. $\phi_1 = \exp(-(X_1-2)^2)$, $\phi_2 = \exp(-(X_1+2)^2)$, $\psi_1 = \exp(-X_2^2)$, $\sigma_1 = \exp(-X_3^2)$, $h_{ij} = 1$ for $i, j = 1, 2, 3, 4$.

Non-localized Profiles

Here as before the values of h_{ij} will determine the kind of solution. Taking exponentials

$$\phi_1 = e^{-p_1(X_1-p_1)}, \quad \phi_2 = e^{-p_2(X_1-p_2)}, \quad \psi = e^{-q(X_2-q)}, \quad \sigma = e^{-r(X_3-r)} \qquad (3.160)$$

for $p_1, p_2, q, r > 0$. Plotting for instance q_2 we can obtain a solution that appears to be two lumps in the X_1X_3-plane, i.e. two "tubes" parallel to the X_2 axes. Plotting q_1 the solution here is one "tube" parallel to the X_1 axis. This is similar to the localized profiles case. However, the value of the q_2-field can be negative, so that we may have a lump above and below the constant value of the field (see figure 3.7). Again by chosing H such that several minors vanish we can also get ridge type solutions (see figure 3.8).

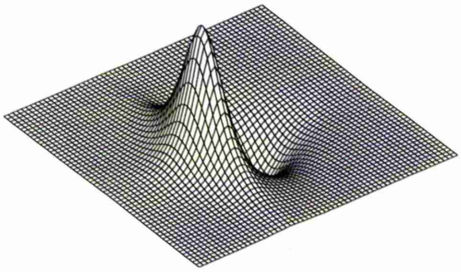


Figure 3.7: The q_2-field, plotted in the X_1X_3-plane for fixed $X_2 = 10$. $\phi_1 = \exp(-(X_1 - 1))$, $\phi_2 = \exp(-2(X_1 - 2))$, $\psi_1 = \exp(-(X_2 - 1))$, $\sigma_1 = \exp(-(X_3 - 1))$, $h_{ij} = 1$ for $i = j$ and $h_{ij} = 1/2$ for $i \neq j$, with $i, j = 1, 2, 3, 4$.

3.6.3 The (l, m, n) Case

In general these solutions will take the form of multiple lumps or ridges. The number of lumps will depend on the values of l, m, n. For instance in the least degenerate cases the q_1-field will have $m \times n$ lumps, if viewed in $X_1 =$ constant plane, with "interaction" regions occuring at l different values of X_1.

For example the $(3, 3, 3)$ case gives nine lumps, in whichever field is held constant, with three interaction regions for that variable (figure 3.9).

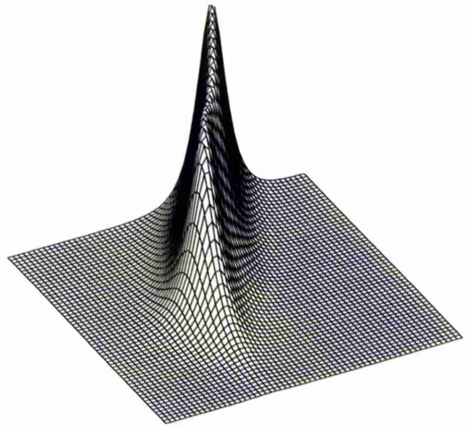


Figure 3.8: The q_2-field, plotted in the X_1X_3-plane for fixed $X_2 = 10$. $\phi_1 = \exp(-(X_1-1))$, $\phi_2 = \exp(-2(X_1-2))$, $\psi_1 = \exp(-(X_2-1))$, $\sigma_1 = \exp(-(X_3-1))$, $h_{11} = h_{13} = h_{22} = h_{24} = h_{31} = h_{33} = h_{42} = h_{44} = 1$ the other h's=0.

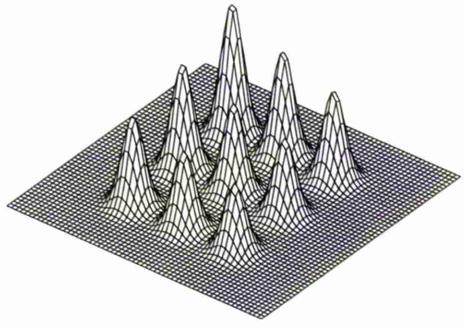


Figure 3.9: The q_1-field, plotted in the X_2X_3-plane for fixed $X_1 = 5$. $\phi_i = \exp(-X_1^2)$, for $i = 1,2,3$, $\psi_1 = \exp(-(X_2+4)^2)/2$, $\psi_2 = \exp(-X_2^2)/2$, $\psi_3 = \exp(-(X_2-4)^2)/2$, $\sigma_1 = \exp(-(X_3+5)^2)/2$, $\sigma_2 = \exp(-X_3^2)/2$, $\sigma_3 = \exp(-(X_3-5)^2)/2$, with $h_{ij} = 1$ for $i,j = 1,\dots,9$.

Chapter 4

Darboux Transformations for 3D3WR

4.1 Introduction

Darboux transformations (here after DT) define a mapping between the solutions of linear differential equations and a similar equation with new coefficients. DT arose originally in the work by Darboux [22] on the Sturm-Liouville problem,

$$y_{xx} + [\lambda - u(x)]\, y = 0. \tag{4.1}$$

He showed that (4.1) is covariant with respect to the transformation

$$y \to \tilde{y} = y_x - \sigma y \qquad\qquad u \to \tilde{u} = u - 2\sigma_x \tag{4.2}$$

with $\sigma = \theta_x/\theta$, where θ is a fixed solution of (4.1), with a particular eigenvalue

$$\theta_{xx} + [\lambda_1 - u(x)]\, \theta = 0. \tag{4.3}$$

Alternatively we can think of the DT as a mapping between two copies of Schrödingers equation, with different potentials i.e. from (4.1) to

$$\tilde{y}_{xx} + [\lambda - \tilde{u}]\, \tilde{y} = 0, \tag{4.4}$$

where we have a new potential $\tilde{u}$. Varying y in (4.2) allows us to recover new solutions.

DT can be conveniently considered by looking at the compatibility condition of a pair of linear operators; the Lax pair, as this gives rise to nonlinear evolution equations [78]. Using a DT we may build up a whole hierachy of solutions by iterating the transformation. Typically these solutions are exact, for example multi-soliton solutions.

4.1.1 Example

We consider the example of the KP equation, discussed in chapter two. The KPII equation has the following Lax pair

$$\begin{aligned} L &= \partial_y + \partial_x^2 + u && (4.5)\\ M &= \partial_t + 4\partial_x^3 + 6\partial_x u - 3w, && (4.6)\end{aligned}$$

which are compatible, providing u and w satisfy

$$\begin{aligned} w_x - u_y - u_{xx} &= 0 && (4.7)\\ u_t + 6uu_x + u_{xxx} + 3w_y - 3u_{xy} &= 0, && (4.8)\end{aligned}$$

which gives

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \qquad (4.9)$$

We define S as the set of common eigenfunctions of L and M (non-trivial solutions of $L(\psi) = M(\psi) = 0$), and choose $\theta \in S$ such that $L(\theta) = M(\theta) = 0$. Then a DT (first given by Matveev [69]) is defined by the operator

$$G_\theta = \theta\partial_x\theta^{-1} = \partial_x - \theta_x\theta^{-1}, \qquad (4.10)$$

so that

$$L(\psi) = M(\psi) = 0 \qquad \Rightarrow \qquad \tilde{L}(\tilde{\psi}) = \tilde{M}(\tilde{\psi}) = 0, \qquad (4.11)$$

with $\tilde{\psi} = G_\theta(\psi)$ (our new eigenfunction) and $\tilde{L}$, $\tilde{M}$ the operators obtained from L and M by transforming u to our new potential,

$$\tilde{u} = u + 2(\log\theta)_{xx}. \qquad (4.12)$$

The result is easily checked by noting

$$\tilde{L}G_\theta = G_\theta L \qquad \tilde{M}G_\theta = G_\theta M. \tag{4.13}$$

The DT is dependent on our choice of θ, which is referred to as the gauge, so we may refer to the DT as a gauge transformation. (4.10) defines a mapping between solutions u and new solution $\tilde{u}$. So in this way we can build up a hierachy of new solutions, usually starting from a simple solution such as the vacuum solution. Notice also that

$$[L, M] = 0 \qquad \Rightarrow \qquad \left[\tilde{L}, \tilde{M}\right] = 0, \tag{4.14}$$

as

$$\left[\tilde{L}, \tilde{M}\right] = \tilde{L}\tilde{M} - \tilde{M}\tilde{L} \tag{4.15}$$

$$= G_\theta L M G_\theta^{-1} - G_\theta M L G_\theta^{-1} \tag{4.16}$$

$$= G_\theta (LM - ML) G_\theta^{-1} \tag{4.17}$$

$$= 0. \tag{4.18}$$

We see therefore that the DT induces an auto-Bäcklund transformation.
An alternative form of the KP equation, known as the KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{4.19}$$

has the Lax pair

$$L = -i\partial_y + \partial_x^2 + u \tag{4.20}$$

$$M = \partial_t + 4\partial_x^3 + 6\partial_x u - 3w, \tag{4.21}$$

with u and w satisfying

$$w_x + iu_y - u_{xx} = 0 \tag{4.22}$$

$$u_t + 6uu_x + u_{xxx} - 3iw_y + 3iu_{xy} = 0. \tag{4.23}$$

Defining the DT in the same way, we see that since θ is a solution to complex equation ($L(\theta) = M(\theta) = 0$), that in general $\tilde{u}$ (4.12) will not be real. For $\tilde{u}$ to be a solution we need it to be real, so we do not have solutions to the KPI equation. At the heart of this problem is that whilst L is self-adjoint, $\tilde{L}$ is not. This leads us to derive a composite or binary transformation, known as the binary Darboux transformation (here after BDT), which preserves the self-adjointness of $\tilde{L}$ [79]. The BDT also has the advantage of giving access to a class of solutions not available from the DT.

4.1.2 Classes of Solutions

Considering the effect of the DT on the Schrödinger equation, Crum [21] showed that the DT adds an eigenvalue to the spectrum of the operator. This result was related to nonlinear evolution equations by Wadati, Sanuki and Konno [91]. They realised that the n-soliton solution to the KdV equation is a reflectionless, n-eigenvalue potential of the Schrödinger equation. Hence adding an eigenvalue to the spectrum of the operator via the DT, leads to the addition of a soliton at the evolution equation level (this is also true for the BDT).

It turns out that n iterations of the DT gives the new solution in terms of a wronskian determinant, composed of n eigenfunctions. This was first proved by Crum [21] for the Schrödinger equation, and was found to be true in a wide variety of cases (see for example [70] for the KP equation or [78] for the $n \times n$ matrix spectral problem). In contrast, iteration of the BDT leads to grammian type solutions [79]. This result again generalises so that we find that the BDT gives rise to grammian type determinants for general $n \times n$ matrix Lax pairs [78] [77]. Recent work by Willox, Loris and Gilson has demonstrated a relationship between wronskian and grammian solutions [96].

Zhou [101] determined the Darboux transformation for the (1+2)-dimensional three-wave equation. Using this he calculated explicit localized soliton solutions, and examined their asymptotic behaviour. Work on the full three-dimensional three-wave resonant equation and the DT was carried out by Guil and Mãnas [38]. They obtained solutions written as grammian type determinants of vector solutions to the Lax pair.

4.1.3 Discrete Equations

Recent work [82] [80] on the DT has seen it used to obtain discrete versions of equations, by considering the superposition principle associated with it. The work is closely related to the reinterpretation of the Bäcklund transformation as giving rise to integrable discrete equations (see for example [61] [56]). This is perhaps unsurprising as the close connection between Bäcklund transformations has long been established [62].

Nimmo and Schief made use of the Moutard transformation with the $(2+1)$-dimensional sine-Gordon equation (the original DT was a specialization of the transformation first given by Moutard [72], and discussed recently in [12]). This transformation was used to construct new solutions, whose superposition principle may be reinterpreted as a discrete version of the $(2+1)$-dimensional sine-Gordon equation (discrete BKP equation). Nimmo's work on the two-dimensional Toda lattice, made use of DT to give the discrete KP equation. In both cases the DT is calculated for the discrete equation, and hence we have the possibilities of new solutions for these equations. The method relies on finding a suitable expression for the iteration of solutions to the DT or Moutard transformation.

4.2 Lax Pair and Scattering Problem

The basic scattering problem for the three-dimensional three-wave resonant equation (3D3WR), is given as [51]

$$\frac{\partial \zeta_i}{\partial X_k} = q_j^* \zeta_k \tag{4.24}$$

$$\frac{\partial \zeta_k}{\partial X_i} = q_j \zeta_i \tag{4.25}$$

where i, j, k take cyclic values over 1, 2, 3. The integrability condition gives

$$\frac{\partial q_i}{\partial X_i} = q_j^* q_k^* \tag{4.26}$$

and the complex conjugate. We recall here that there is no eigenvalue present, thus no bound states and hence no solitons [51]. However localised solutions do occur, and are referred to as lumps.

We re-write equations (4.24), (4.25) in a matrix form

$$L = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & 0 & \partial_y \end{bmatrix} + \begin{bmatrix} 0 & 0 & -q_2^* \\ -q_3^* & 0 & 0 \\ 0 & -q_1^* & 0 \end{bmatrix} \tag{4.27}$$

$$M = \begin{bmatrix} \partial_y & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_x \end{bmatrix} + \begin{bmatrix} 0 & -q_3 & 0 \\ 0 & 0 & -q_1 \\ -q_2 & 0 & 0 \end{bmatrix} \tag{4.28}$$

with $X_1 = x$, $X_2 = y$, $X_3 = z$. The scattering problem is equivalent to

$$L\zeta = 0 \tag{4.29}$$

$$M\zeta = 0 \tag{4.30}$$

with $\zeta = [\zeta_1, \zeta_2, \zeta_3]^T$. We define the following transformation of independent variables

$$X = a_1 x + b_1 y + c_1 z \tag{4.31}$$

$$Y = a_2 x + b_2 y + c_2 z \tag{4.32}$$

$$Z = a_3 x + b_3 y + c_3 z \tag{4.33}$$

which gives the new L and M as

$$L \to A_1 \partial_X + A_2 \partial_Y + A_3 \partial_Z + Q \tag{4.34}$$

$$M \to B_1 \partial_X + B_2 \partial_Y + B_3 \partial_Z + R \tag{4.35}$$

where we have $A_i = \text{Diag}[c_i, a_i, b_i]$, $B_i = \text{Diag}[b_i, c_i, a_i]$, and Q, R as

$$Q = \begin{bmatrix} 0 & 0 & -q_2^* \\ -q_3^* & 0 & 0 \\ 0 & -q_1^* & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & -q_3 & 0 \\ 0 & 0 & -q_1 \\ -q_2 & 0 & 0 \end{bmatrix}. \tag{4.36}$$

We consider the new operators L_1, M_1, composed of linear combinations of L and M

$$L_1 = (A_3^{-1}A_2 - B_3^{-1}B_2)^{-1}(A_3^{-1}L - B_3^{-1}M) \quad (4.37)$$

$$= \partial_Y + F_1\partial_X + Q' \quad (4.38)$$

$$M_1 = (A_2^{-1}A_3 - B_2^{-1}B_3)^{-1}(A_2^{-1}L - B_2^{-1}M) \quad (4.39)$$

$$= \partial_Z + F_2\partial_X + R' \quad (4.40)$$

$$(4.41)$$

with F_1, F_2 as

$$F_1 = \mathrm{Diag}[F_{11}, F_{12}, F_{13}] = \begin{bmatrix} \frac{c_1b_3-b_1c_3}{c_2b_3-c_3b_2} & 0 & 0 \\ 0 & \frac{c_1a_3-a_1c_3}{a_3c_2-a_2c_3} & 0 \\ 0 & 0 & \frac{b_1a_3-a_1b_3}{b_2a_3-b_3a_2} \end{bmatrix} \quad (4.42)$$

$$F_2 = \mathrm{Diag}[F_{21}, F_{22}, F_{23}] = \begin{bmatrix} \frac{c_2b_1-b_2c_1}{c_2b_3-c_3b_2} & 0 & 0 \\ 0 & \frac{c_2a_1-a_2c_1}{a_3c_2-a_2c_3} & 0 \\ 0 & 0 & \frac{b_2a_1-a_2b_1}{b_2a_3-b_3a_2} \end{bmatrix}, \quad (4.43)$$

and

$$Q' = \begin{bmatrix} 0 & \frac{q_3c_3}{c_2b_3-b_2c_3} & \frac{-q_2^*b_3}{c_2b_3-b_2c_3} \\ \frac{-q_3^*c_3}{a_2c_3-c_2a_3} & 0 & \frac{q_1a_3}{a_2c_3-c_2a_3} \\ \frac{q_2b_3}{b_2a_3-b_3a_2} & \frac{-q_1^*a_3}{b_2a_3-b_3a_2} & 0 \end{bmatrix} \quad (4.44)$$

$$R' = \begin{bmatrix} 0 & \frac{-q_3c_2}{c_2b_3-b_2c_3} & \frac{q_2^*b_2}{c_2b_3-b_2c_3} \\ \frac{q_3^*c_2}{a_2c_3-c_2a_3} & 0 & \frac{-q_1a_2}{a_2c_3-c_2a_3} \\ \frac{-q_2b_2}{b_2a_3-b_3a_2} & \frac{q_1^*a_2}{b_2a_3-b_3a_2} & 0 \end{bmatrix}. \quad (4.45)$$

L_1 and M_1 give us our new Lax pair, in a form appropriate for calculating the DT. We note that as we have only taken linear combinations of our original operators (4.34), (4.35), up to a change of variables, the compatability of L_1 and M_1 gives rise to 3D3WR.

4.3 Darboux Transformations

4.3.1 Set-Up

We mirror the approach of Nimmo [79], [78] and consider the 3×3 linear matrix operator

$$L = \partial_Y + F_1 \partial_X + Q \tag{4.46}$$

where F_1 is diagonal and constant (4.42), and $Q = Q(X, Y, Z)$ is off-diagonal (4.44). We have dropped the subscript on L and superscript on Q for convenience. We work solely with the operator L, however given the form of M_1 (4.40), the calculations all carry through. We return to the exact nature of the solutions determined from M_1 in section four. The aim is to calculate the Darboux and binary Darboux transformations.

Let S be the set of non-singular 3×3 solution matrices of $L(\psi) = 0$ (ψ the eigenfunction), and define $\tilde{S}$, $S^\dagger$ for operators $\tilde{L}$ and $L^\dagger$ respecively. Taking $\theta \in S$ we define the Darboux transformation as

$$\begin{aligned} G_\theta &= \theta \partial_X \theta^{-1} & (4.47) \\ &= \partial_X - \theta_X \theta^{-1}. & (4.48) \end{aligned}$$

This transformation, defines a mapping

$$G_\theta : S \to \tilde{S} \tag{4.49}$$

with

$$\tilde{L} = G_\theta L G_\theta^{-1}. \tag{4.50}$$

By considering the adjoint operator $G_\theta^\dagger$, we have

$$\tilde{L}^\dagger = G_\theta^{\dagger^{-1}} L^\dagger G_\theta^\dagger \tag{4.51}$$

so that

$$G_\theta^\dagger : \tilde{S}^\dagger \to S^\dagger. \tag{4.52}$$

We use the formal adjoint $\dagger$, in the following way; it is the linear operation defined by

$$(a\partial^i)^\dagger = (-1)^i \partial^i a^\dagger, \tag{4.53}$$

Explicitly for a matrix a, $a^\dagger$ is the Hermitian conjugate of a. We can show that

$$\tilde{L}^\dagger (\theta^{-1})^\dagger = 0 \tag{4.54}$$

and we use this to help us determine the exact form of $\tilde{Q}$, under the Darboux transformation. We have $\tilde{L} = \partial_Y + F_1\partial_X + \tilde{Q}$, so that

$$\begin{aligned}
\left(\tilde{L}^\dagger(\theta^{-1})^\dagger\right)^\dagger &= \left((-\partial_Y - \partial_X F_1 + (\tilde{Q})^\dagger)(\theta^{-1})^\dagger\right)^\dagger && (4.55)\\
&= \theta^{-1}\theta_Y\theta^{-1} + \theta^{-1}\theta_X\theta^{-1}F_1 + \theta^{-1}\tilde{Q} && (4.56)\\
&= \theta^{-1}\left((L\theta)\theta^{-1} + \tilde{Q} - Q - \left[F_1, \theta_X\theta^{-1}\right]\right) && (4.57)\\
&= 0 \qquad \Leftrightarrow \qquad \tilde{Q} = Q + \left[F_1, \theta_X\theta^{-1}\right]. && (4.58)
\end{aligned}$$

Expression (4.58) is consistent with (4.50) in the following sense. Consider $L = \partial_Y + F_1\partial_X + Q$ and $\tilde{L} = \partial_Y + F_1\partial_X + \tilde{Q}$ and G_θ as above, then

$$\tilde{L} = G_\theta L G_\theta^{-1}, \tag{4.59}$$

is equivalent to

$$\begin{aligned}
&\tilde{L}G_\theta - G_\theta L = \\
&\quad \left[\partial_Y + F_1\partial_X + \tilde{Q}\right]\left[\partial_X - \theta_X\theta^{-1}\right] - \left[\partial_X - \theta_X\theta^{-1}\right]\left[\partial_Y + F_1\partial_X + Q\right] = \\
&\quad - \theta_{XY}\theta^{-1} + \theta_X\theta^{-1}\theta_Y\theta^{-1} - F_1(\theta_{XX} - \theta_X\theta^{-1}\theta_X)\theta^{-1} - \tilde{Q}\theta_X\theta^{-1} \\
&\quad - Q_X - \theta_X\theta^{-1}Q + \left[\tilde{Q} - Q - \left[F_1, \theta_X\theta^{-1}\right]\right]\partial_X. \qquad (4.60)
\end{aligned}$$

If we make use of (4.58), then the operator coefficient vanishes and we obtain

$$\begin{aligned}
\tilde{L}G_\theta - G_\theta L = -\theta_{XY}\theta^{-1} + \theta_X\theta^{-1}\theta_Y\theta^{-1} - F_1\theta_{XX}\theta^{-1} - \left[Q, \theta_X\theta^{-1}\right] \\
- Q_X + \theta_X\theta^{-1}F_1\theta_X\theta^{-1} = -(G_\theta L\theta)\theta^{-1}. \qquad (4.61)
\end{aligned}$$

This vanishes as θ is an eigenvalue, so

$$\tilde{L}G_\theta - G_\theta L = 0. \tag{4.62}$$

Hence the definitions are consistent, and we have a new solution

$$\tilde{Q} = Q + \left[F_1, \theta_X \theta^{-1}\right] \tag{4.63}$$

and a new eigenfunction $\tilde{\psi} = G_\theta(\psi)$.

4.3.2 Constraints

As F_1 is diagonal we have the following representation for (4.58)

$$\tilde{Q}_{ij} = Q_{ij} + (F_{1i} - F_{1j})\frac{|\theta^{ij}|}{|\theta|} \tag{4.64}$$

where θ^{ij} is θ with jth row (θ_j) replaced with derivative of ith (we write this $\theta_{i,X}$, with , to separate the index and the partial derivative). With this explicit form of $\tilde{Q}$ we can check whether the DT preserves the correct structure of solutions to 3D3WR.

By considering a new set of solutions $\tilde{q}_i$, (for $i = 1, 2, 3$) and their conjugates to (4.26), and using the construction for Q in section 2, we obtain

$$\tilde{Q} = \begin{bmatrix} 0 & \frac{\tilde{q}_3 c_3}{c_2 b_3 - b_2 c_3} & \frac{-\tilde{q}_2^* b_3}{c_2 b_3 - b_2 c_3} \\ \frac{-\tilde{q}_3^* c_3}{a_2 c_3 - c_2 a_3} & 0 & \frac{\tilde{q}_1 a_3}{a_2 c_3 - c_2 a_3} \\ \frac{\tilde{q}_2 b_3}{b_2 a_3 - b_3 a_2} & \frac{-\tilde{q}_1^* a_3}{b_2 a_3 - b_3 a_2} & 0 \end{bmatrix}. \tag{4.65}$$

Comparing each of the entries with (4.64) we have

$$\tilde{q}_1 = q_1 + (F_{12} - F_{13})\frac{(a_2c_3 - c_2a_3)|\theta^{23}|}{a_3|\theta|} \tag{4.66}$$

$$\tilde{q}_1^* = q_1^* + (F_{12} - F_{13})\frac{(b_2a_3 - b_3a_2)|\theta^{32}|}{a_3|\theta|} \tag{4.67}$$

$$\tilde{q}_2 = q_2 + (F_{13} - F_{11})\frac{(b_2a_3 - a_2b_3)|\theta^{31}|}{b_3|\theta|} \tag{4.68}$$

$$\tilde{q}_2^* = q_2^* + (F_{13} - F_{11})\frac{(b_3c_2 - c_3b_2)|\theta^{13}|}{b_3|\theta|} \tag{4.69}$$

$$\tilde{q}_3 = q_3 + (F_{11} - F_{12})\frac{(b_3c_2 - c_3b_2)|\theta^{12}|}{c_3|\theta|} \tag{4.70}$$

$$\tilde{q}_3^* = q_3^* + (F_{11} - F_{12})\frac{(a_2c_3 - c_2a_3)|\theta^{21}|}{c_3|\theta|}. \tag{4.71}$$

It is clear from (4.65) that we have a constraint arising from complex conjugation. However from (4.66) to (4.71) it is obvious that other than in trivial cases this constraint is not satisfied. Hence the DT does not in general preserve the structure of the 3D3WR.

We note at this point that we have a similar closed expression for $\tilde{\psi}$, as the one obtained above for $\tilde{Q}$ (4.64). We have that $\tilde{\psi} = G_\theta(\psi)$, and make use of the following theorem

Theorem 4.3.1. *For any $m \times m$ matrices, A, B, C, with B invertible*

$$(AB^{-1}C)_{ij} = -\frac{1}{|B|}\begin{vmatrix} B & C_{.j} \\ A_{i.} & 0 \end{vmatrix}. \tag{4.72}$$

Proof. This follows immediately from two facts : the expansion of the (larger) determinant on the right by its last row and column is

$$-\sum_{k,l=1}^{m} A_{ik} B_{\hat{l}\hat{k}} C_{lj} \tag{4.73}$$

and

$$(B^{-1})_{ij} = \frac{1}{|B|} B_{\hat{j}\hat{i}}. \tag{4.74}$$

□

Then

$$\tilde{\psi}_{ij} = \psi_{ij,X} - \frac{1}{|\theta|}\begin{vmatrix} \theta & \psi_{.j} \\ \theta_{i.,X} & 0 \end{vmatrix} \tag{4.75}$$

$$= \frac{1}{|\theta|}\begin{vmatrix} \theta & \psi_j \\ \theta_{i,X} & \psi_{ij,X} \end{vmatrix}. \tag{4.76}$$

Where for a matrix A, $A_{i.}$ ($A_{.j}$) denotes its ith row, (jth column) and A_{ij} the (ij)th entry. $A_{\hat{i}.}$ ($A_{.\hat{j}}$) denotes the matrix obtained by removing its ith row (jth column) and $A_{\hat{i}\hat{j}}$ denotes the (ij)th minor. In this notation we write our new solution (4.64), as

$$\tilde{Q}_{1,ij} = Q_{1,ij} + (F_{1i} - F_{1j})\frac{\begin{vmatrix} \theta_{\hat{j}.} \\ \theta_{i.,X} \end{vmatrix}}{|\theta|}. \tag{4.77}$$

4.3.3 Iteration

This transformation can be iterated to give a whole hierachy of solutions [78], we give the details as the construction aids our understanding of the iterated BDT. However it must be stressed that DT does not preserve the structure of the 3D3WR and so the iterated form of Q are not solutions.

We start by assuming that we have r solutions $\theta_i \in S$ for $i = 1, ..., r$. We relabel S as S^0 to indicate our starting point. The aim is to determine the form of Q and ψ after r iterations of the Darboux transformation, we re-write these as Q^0, ψ^0 to indicate that these are the original functions, so that the aim is to find Q^r and ψ^r. We illustrate this pictorially as :

$$S^0 \xrightarrow{G_{\theta 0}} S^1 \xrightarrow{G_{\theta 1}} S^2 \longrightarrow \cdots \longrightarrow S^{r-1} \xrightarrow{G_{\theta r-1}} S^r$$

$$\theta_1 \longmapsto \theta^1$$

where S^1 corresponds to $\tilde{S}$, i.e. the set of solutions of non-singular 3×3 matrices of $\tilde{L}(\tilde{\psi}) = 0$. In the same way we have in general, S^i the set of non-singular 3×3 solution matrices of $L^i(\psi^i) = 0$, where L^i has the form,

$$L^i = \partial_Y + F_1 \partial_X + Q^i. \tag{4.78}$$

We obtain θ^i for $i = 0, ..., r-1$ by applying successive Darboux transformations

$$\theta^0 = \theta_0, \qquad \theta^1 = G_{\theta^0}\theta_1, \qquad \theta^2 = G_{\theta^1}G_{\theta^0}\theta_2, ... \tag{4.79}$$

so for example

$$\begin{aligned} \theta^1 &= G_{\theta^0}\theta_1 & (4.80)\\ &= [\partial_X - \theta^0_X(\theta^0)^{-1}]\theta_1 & (4.81)\\ &= \theta_{1,X} - \theta^0_X(\theta^0)^{-1}\theta_1. & (4.82)\end{aligned}$$

In general the DT $G_{\theta^r} = \theta^r \partial_X (\theta^r)^{-1}$, where $\theta^r \in S^r$ is invertible, is a mapping from S^r to S^{r+1}. This allows us to construct our ψ^r by repeated use of transformations, so

$$\psi^1 = G_{\theta^0}\psi^0, \qquad \psi^2 = G_{\theta^1}\psi^1 = G_{\theta^1}G_{\theta^0}\psi^0, ... \tag{4.83}$$

Q^r_{ij} is written in terms of Q^{r-1}_{ij} and θ^{r-1}, this allows us to build up the expressions for Q^r_{ij}. For example

$$Q^2_{ij} = Q^1_{ij} + (F_{1i} - F_{1j})\frac{|\theta^{1\,ij}|}{|\theta^1|} \tag{4.84}$$

$$= Q^0_{ij} + (F_{1i} - F_{1j})\left[\frac{|\theta^{1\,ij}|}{|\theta^1|} + \frac{|\theta^{0\,ij}|}{|\theta^0|}\right], \tag{4.85}$$

where $\theta^{r\,ij}$ is θ^r with jth row (θ^r_j) replaced with derivative of ith $(\theta^r_{i,X})$, for $r = 1, \dots, n$.

4.3.4 Closed Form Expressions

As with the single DT, it would be convienent to have an explicit and closed form for ψ^r and Q^r_{ij}. To motivate our determination of these, we consider the situation where the θ_i are all scalars, corresponding to a 1×1 matrix spectral problem. Here θ^1 is written

$$\theta^1 = \frac{\theta_{1,X}\theta^0 - \theta^0_X\theta_1}{\theta^0} = \frac{|\Theta|}{\theta^0}, \tag{4.86}$$

with $\Theta = \theta_{1,X}\theta^0 - \theta^0_X\theta_1$, which gives rise to

$$\psi^2 = G_{\theta^1}G_{\theta^0}\psi^0 \tag{4.87}$$

$$= \left[\partial_X - \frac{\theta^1_X}{\theta^1}\right]\left[\psi^0_X - \frac{\theta^0_X\psi^0}{\theta^0}\right] \tag{4.88}$$

$$= \psi^0_{XX} - \frac{\theta^1_X\psi^0_X}{\theta^1} - \left(\frac{\theta^0_X\psi^0}{\theta^0}\right)_X + \frac{\theta^1_X\theta^0_X\psi^0}{\theta^1\theta^0} \tag{4.89}$$

$$= \psi^0_{XX} + \frac{\psi^0}{|\Theta|}\left[\theta^0_X\theta_{1,XX} - \theta^0_{XX}\theta_{1,X}\right] - \frac{\psi^0_X}{|\Theta|}\left[\theta^0\theta_{1,XX} - \theta_1\theta^0_{XX}\right]. \tag{4.90}$$

This may be reformulated as

$$\psi^2 = \frac{\begin{vmatrix} \theta_0 & \theta_1 & \psi^0 \\ \theta_{0,X} & \theta_{1,X} & \psi^0_X \\ \theta_{0,XX} & \theta_{1,XX} & \psi^0_{XX} \end{vmatrix}}{|\Theta|} \tag{4.91}$$

$$= \frac{\begin{vmatrix} \boldsymbol{\theta} & \psi^0 \\ \boldsymbol{\theta}_X & \psi^0_X \\ \boldsymbol{\theta}_{XX} & \psi^0_{XX} \end{vmatrix}}{|\Theta|}, \tag{4.92}$$

where $\boldsymbol{\theta} = [\theta_0, \theta_1]$ and $\theta_0 = \theta^0$. This idea can be generalized to give an explicit form of ψ^r and Q^r_{ij} in the 3×3 case, after we have carried out r iterations. The new eigenfunction is represented in a wronksian form, as

$$\psi^r_{ij} = \frac{\begin{vmatrix} \boldsymbol{\theta}^0 & \psi^0_{.j} \\ \vdots & \vdots \\ \boldsymbol{\theta}^{r-1} & \psi^{r-1}_{.j} \\ \boldsymbol{\theta}^r_{i.} & \psi^r_{ij} \end{vmatrix}}{\begin{vmatrix} \boldsymbol{\theta}^0 \\ \vdots \\ \boldsymbol{\theta}^{r-1} \end{vmatrix}}, \tag{4.93}$$

and

$$Q^r_{ij} = Q^0_{ij} + (F_{1i} - F_{1j}) \frac{\begin{vmatrix} \boldsymbol{\theta}^0 \\ \vdots \\ \boldsymbol{\theta}^{r-1}_{j.} \\ \boldsymbol{\theta}^r_{i.} \end{vmatrix}}{\begin{vmatrix} \boldsymbol{\theta}^0 \\ \vdots \\ \boldsymbol{\theta}^{r-1} \end{vmatrix}}, \tag{4.94}$$

where we have written $\boldsymbol{\theta}^r$, for a $3 \times 3r$ matrix formed by concatenating invertible elements of S_0. $\boldsymbol{\theta}^{r,k}$ denotes the kth derivative with respect to X, and if the value of r is unimportant or clear from the context it will be omitted. ψ^r are arbitrary matrices in S^r acted upon by a DT.

The proof of (4.93) and (4.94) is due to Gilson and Nimmo [36], and we make use of their notation. We require the following lemma :

Lemma 4.3.2. *For each $r \geq 1$ define 3×3 matrices g^r, h^r, $\hat{g}^r$, $\hat{h}^r$ and G^r, and scalar F^r by*

$$g^r_{ij} = \begin{vmatrix} \boldsymbol{\theta}^0 & \psi^0_{\cdot j} \\ \vdots & \vdots \\ \boldsymbol{\theta}^{r-1} & \psi^{r-1}_{\cdot j} \\ \boldsymbol{\theta}^r_{i\cdot} & \psi^r_{ij} \end{vmatrix}, \qquad \hat{g}^r_{ij} = \begin{vmatrix} \boldsymbol{\theta}^0 & \psi^0_{\cdot j} \\ \vdots & \vdots \\ \boldsymbol{\theta}^{r-1} & \psi^{r-1}_{\cdot j} \\ \boldsymbol{\theta}^{r+1}_{i\cdot} & \psi^{r+1}_{ij} \end{vmatrix}, \tag{4.95}$$

$$h^r_{ij} = \begin{vmatrix} \boldsymbol{\theta}^0 & \theta^0_{\cdot j} \\ \vdots & \vdots \\ \boldsymbol{\theta}^{r-1} & \theta^{r-1}_{\cdot j} \\ \boldsymbol{\theta}^r_{i\cdot} & \theta^r_{ij} \end{vmatrix}, \qquad \hat{h}^r_{ij} = \begin{vmatrix} \boldsymbol{\theta}^0 & \theta^0_{\cdot j} \\ \vdots & \vdots \\ \boldsymbol{\theta}^{r-1} & \theta^{r-1}_{\cdot j} \\ \boldsymbol{\theta}^{r+1}_{i\cdot} & \theta^{r+1}_{ij} \end{vmatrix}, \tag{4.96}$$

$$G^r_{ij} = (-1)^{j-1} \begin{vmatrix} \boldsymbol{\theta}^0 \\ \vdots \\ \boldsymbol{\theta}^{r-1}_{\hat{j}\cdot} \\ \boldsymbol{\theta}^r_{i\cdot} \end{vmatrix}, \qquad F^r = \begin{vmatrix} \boldsymbol{\theta}^0 \\ \vdots \\ \boldsymbol{\theta}^{r-1} \end{vmatrix}. \tag{4.97}$$

If the matrices g^{r+1}, G^{r+1} and scalar F^{r+1} are given by

$$g^{r+1}_{ij} = \begin{vmatrix} \boldsymbol{\theta}^0 & \theta^0 & \psi^0_{\cdot j} \\ \vdots & \vdots & \vdots \\ \boldsymbol{\theta}^r & \theta^r & \psi^r_{\cdot j} \\ \boldsymbol{\theta}^{r+1}_{i\cdot} & \theta^{r+1}_{i\cdot} & \psi^{r+1}_{ij} \end{vmatrix},$$

$$G^{r+1}_{ij} = (-1)^{j-1} \begin{vmatrix} \boldsymbol{\theta}^0 & \theta^0 \\ \vdots & \vdots \\ \boldsymbol{\theta}^r_{\hat{j}\cdot} & \theta^r_{\hat{j}\cdot} \\ \boldsymbol{\theta}^{r+1}_{i\cdot} & \theta^{r+1}_{i\cdot} \end{vmatrix}, \qquad F^{r+1} = \begin{vmatrix} \boldsymbol{\theta}^0 & \theta^0 \\ \vdots & \vdots \\ \boldsymbol{\theta}^r & \theta^r \end{vmatrix}, \tag{4.98}$$

then

$$D_X g^r \cdot F^r + G^r g^r - \hat{g}^r F^r = 0, \tag{4.99}$$

$$D_X h^r \cdot F^r + G^r h^r - \hat{h}^r F^r = 0, \tag{4.100}$$

$$g^{r+1} F^r - \hat{g}^r F^{r+1} + G^{r+1} g^r = 0, \tag{4.101}$$

$$G^{r+1} h^r - \hat{h}^r F^{r+1} = 0. \tag{4.102}$$

The θ^r are 3×3 matrices, not already included in $\boldsymbol{\theta}^r$ i.e. A new solution. We note that $\psi^r = g^r/F^r$, $Q^r_{ij} = Q^0_{ij} + (F_{1i} - F_{1j})(-1)^{j-1}G_{ij}/F$.

Proof. First consider the $(6r+1) \times (6r+1)$ determinant

$$\left|\begin{array}{cccc|cccc|ccc} \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^{r-1}_{k.} & \theta^r_{k.} & \theta^r_{i.} \\ \psi^0_{.j} & \cdots & \psi^{r-2}_{.j} & \psi^{r-1}_{\hat{k}j} & \psi^0_{.j} & \cdots & \psi^{r-2}_{.j} & \psi^{r-1}_{\hat{k}j} & \psi^{r-1}_{kj} & \psi^r_{kj} & \psi^r_{ij} \\ \hline 0 & \cdots & 0 & 0 & \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^{r-1}_{k.} & \theta^r_{k.} & \theta^r_{i.} \end{array}\right| \quad (4.103)$$

where $i, j, k \in 1, 2, 3$. This may be evaluated by a Laplace expansion as it stands or after subtracting the final row block from the first. The second expansion gives 0 and so we get the identity

$$\begin{aligned} &\begin{vmatrix} \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^{r-1}_{k.} & \theta^r_{k.} \\ \psi^0_{.j} & \cdots & \psi^{r-2}_{.j} & \psi^{r-1}_{\hat{k}j} & \psi^{r-1}_{kj} & \psi^r_{kj} \end{vmatrix} \begin{vmatrix} \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^r_{i.} \end{vmatrix} \\ &- \begin{vmatrix} \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^{r-1}_{k.} & \theta^r_{i.} \\ \psi^0_{.j} & \cdots & \psi^{r-2}_{.j} & \psi^{r-1}_{\hat{k}j} & \psi^{r-1}_{kj} & \psi^r_{ij} \end{vmatrix} \begin{vmatrix} \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^r_{k.} \end{vmatrix} \\ &+ \begin{vmatrix} \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^r_{k.} & \theta^r_{i.} \\ \psi^0_{.j} & \cdots & \psi^{r-2}_{.j} & \psi^{r-1}_{\hat{k}j} & \psi^r_{kj} & \psi^r_{ij} \end{vmatrix} \begin{vmatrix} \theta^0 & \cdots & \theta^{r-2} & \theta^{r-1}_{\hat{k}.} & \theta^{r-1}_{k.} \end{vmatrix} = 0 \end{aligned} \quad (4.104)$$

Taking the sum over k gives

$$\left(\sum_{k=1}^{3} g^r_{kj} G^r_{ik}\right) - g^r_{ij} F^r_X + (g^r_{ij,X} - \hat{g}^r_{ij}) F^r = 0 \quad (4.105)$$

which is the ijth entry in (4.99). The identity (4.100) follows from (4.99) when we choose $\psi = \theta$.

Also consider the $(6r+4) \times (6r+4)$ determinant

$$\left|\begin{array}{ccc|ccc|cc} \theta^0 & \cdots & \theta^{r-1} & \theta^0 & \cdots & \theta^{r-1} & \theta^r & \theta^{r+1}_{i.} \\ \theta^0 & \cdots & \theta^{r-1} & \theta^0 & \cdots & \theta^{r-1} & \theta^r & \theta^{r+1}_{i.} \\ \hline 0 & \cdots & 0 & \theta^0 & \cdots & \theta^{r-1} & \theta^r & \theta^{r+1}_{i.} \\ 0 & \cdots & 0 & \psi^0_{.j} & \cdots & \psi^{r-1}_{.j} & \psi^r_{.j} & \psi^{r+1}_{ij} \end{array}\right| \quad (4.106)$$

By considering the third row block subtracted from the first it can be seen that we have $F^r g^{r+1}$. Then by carrying out the Laplace expansion we obtain (4.101). Finally choosing $\psi = \theta$, $g^{r+1} = 0$ and the above expansion gives (4.102).

□

These identities allow us to prove the following theorem, which gives a proof for (4.93).

Theorem 4.3.3. *For $n \geq 1$ let ψ^n be the solution obtained from n iterations of the DT. In the notation of Lemma 4.3.2 we have*

$$\psi^n = \frac{g^n}{F^n}, \tag{4.107}$$

with $F^n \neq 0$.

Proof. The proof is by induction on n. For $n = 1$, consider 3×3 solution matrices $\boldsymbol{\theta} = \theta$ and ψ. After an application of the DT we get

$$\psi^1 = G_\theta(\psi) = \psi_X - \theta_X \theta^{-1} \psi. \tag{4.108}$$

Re-writing this

$$(\theta_X \theta^{-1} \psi)_{ij} = -\frac{1}{|\theta|} \begin{vmatrix} \theta & \psi_{.j} \\ (\theta_{i.})_X & 0 \end{vmatrix}, \tag{4.109}$$

and hence

$$\psi^1_{ij} = \frac{1}{|\theta|} \begin{vmatrix} \theta & \psi_{.j} \\ (\theta_{i.})_X & (\psi_{ij})_X \end{vmatrix} = \frac{1}{|\boldsymbol{\theta}^0|} \begin{vmatrix} \boldsymbol{\theta}^0 & \psi^0_{.j} \\ \boldsymbol{\theta}^1_{i.} & \psi^1_{ij} \end{vmatrix} \tag{4.110}$$

and so $\psi^1 = g^1/F^1$ as required. Now assume that the result holds for $n = r \geq 1$, that is by applying r DTs we have obtained solutions $\psi^r = g^r/F^r$. We also have another solution $\theta^r = h^r/F^r$, a particular solution obtained from ψ^r by taking $\psi = \theta$. Then after a further DT we get a solution

$$\psi^{r+1} = G_{\theta^r}(\psi^r) = \psi^r_X - \theta^r_X (\theta^r)^{-1} \psi^r, \tag{4.111}$$

that is

$$\psi^{r+1} = \frac{(D_x g^r \cdot F^r) - (D_X h^r \cdot F^r)(h^r)^{-1} g^r}{(F^r)^2}. \tag{4.112}$$

Now using the identities in Lemma 4.3.2 we get

$$\begin{aligned} \psi^{r+1} &= \frac{(-G^r g^r + \hat{g}^r F^r) - (-G^r h^r + \hat{h}^r F^r)(h^r)^{-1} g^r}{(F^r)^2} && (4.113)\\ &= \frac{\hat{g}^r - \hat{h}^r (h^r)^{-1} g^r}{F^r} && (4.114)\\ &= \frac{g^{r+1} F^r + G^{r+1} g^r - G^{r+1} h^r (h^r)^{-1} g^r}{F^r F^{r+1}} && (4.115)\\ &= \frac{g^{r+1}}{F^{r+1}}, && (4.116) \end{aligned}$$

as required. □

We explicitly give the details for ψ in the case $r = 2$. We assume that we have 2, 3×3 solutions, θ_1 and θ_2 both in S_0, so that $\boldsymbol{\theta}$ is a 3×6 matrix, written

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}. \tag{4.117}$$

Thus

$$\psi^2_{ij} = \frac{\begin{vmatrix} \theta_1 & \theta_2 & \psi^0_{.j} \\ \theta_{1,X} & \theta_{2,X} & \psi^0_{.j,X} \\ \theta_{i.,XX} & \theta_{i.,XX} & \psi_{ij,XX} \end{vmatrix}}{\begin{vmatrix} \theta_1 & \theta_2 \\ \theta_{1,X} & \theta_{2,X} \end{vmatrix}}. \tag{4.118}$$

Theorem 4.3.4. *For $r \geq 1$, Q, the new solution, takes the following form after r iterations of the DT*

$$Q^r_{ij} = Q^0_{ij} + (F_{1j} - F_{1i})(-1)^j \frac{G^r_{ij}}{F^r}, \tag{4.119}$$

i.e. a 3×3 matrix, where F_{1i} are the diagonal entries in F_1 for $i = 1, 2, 3$.

Proof. By induction on r. For $r = 1$ we have

$$\begin{aligned} Q^1_{ij} &= Q^0_{ij} - (F_{1i} - F_{1j})(-1)^j \frac{G^1_{ij}}{F^1} && (4.120) \\ &= Q^0_{ij} + (F_{1i} - F_{1j}) \frac{\begin{vmatrix} \boldsymbol{\theta}^0_{j.} \\ \boldsymbol{\theta}^1_{i.} \end{vmatrix}}{|\boldsymbol{\theta}^0|} && (4.121) \end{aligned}$$

which is (4.77). Assume that the result (4.119) holds for r. We also have that

$$Q^{r+1} = Q^r + \left[F_1, \theta^r_X (\theta^r)^{-1}\right], \tag{4.122}$$

from (4.63), where we have generalised to the rth iteration. Substituting (4.119) into the (ij)th entry of (4.122) we have

$$\begin{aligned} Q^{r+1}_{ij} &= Q^0_{ij} + (F_{1j} - F_{1i}) \left[(-1)^j \frac{G^r_{ij}}{F^r} - (\theta^r_X (\theta^r)^{-1})_{ij}\right], && (4.123) \\ &= Q^0_{ij} + (F_{1j} - F_{1i}) \left[(-1)^j \frac{G^r_{ij}}{F^r} + (-1)^j \frac{(D_X(h^r \cdot F^r)(h^r)^{-1})_{ij}}{F^r}\right] && (4.124) \end{aligned}$$

by using $\theta^r = h^r/F^r$ (we note that the $(-1)^{j-1}$ comes from the derivatives). We use the identities in lemma 4.3.2 and obtain

$$Q_{ij}^{r+1} = Q_{ij}^0 + (F_{1j} - F_{1i})(-1)^j \left[\frac{G_{ij}}{F^r} + \frac{((-G^r h^r + \hat{h}^r F^r)(h^r)^{-1})_{ij}}{F^r} \right], \quad (4.125)$$

$$= Q_{ij}^0 + (F_{1j} - F_{1i})(-1)^j \frac{G_{ij}^{r+1}}{F^{r+1}}, \quad (4.126)$$

as required. □

4.4 Binary Darboux Transformations

4.4.1 Set-Up

We consider the set of 3×3 non-singular matrices $\hat{S}$ such that $\hat{L}(\psi) = 0$. With $\rho \in \hat{S}$ we define $G_\rho : \hat{S} \to \tilde{S}$, which gives rise to the composite mapping

$$G_\rho^{-1} G_\theta : S \to \hat{S}. \quad (4.127)$$

The aim of the BDT is to allow us to construct new solutions ($\hat{q}_i$, for $i = 1, 2, 3$) to the 3D3WR, in effect $\hat{Q}$, corresponding to the operator

$$\hat{L} = \partial_Y + F_1 \partial_X + \hat{Q}, \quad (4.128)$$

again we have dropped subscripts and superscripts. However the difficulty with this definition, is that we need ρ, one of the solutions we are trying to obtain. To overcome this we make use of the adjoint transformation.
Recall that we have the transformation $G_\theta^\dagger : \tilde{S}^\dagger \to S^\dagger$ and $(\theta^{-1})^\dagger \in \tilde{S}^\dagger$. In a similar way, we have $(\rho^{-1})^\dagger \in \tilde{S}^\dagger$. We now use the mapping $G_\theta^{\dagger^{-1}} : S^\dagger \to \tilde{S}^\dagger$, to obtain

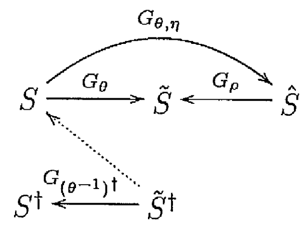
$$(\rho^{-1})^\dagger = G_\theta^{\dagger^{-1}} \eta \quad (4.129)$$

for $\eta \in S^\dagger$ i.e. an adjoint eigenfunction More generally we have $S^\dagger$ the set of non-singular 3×3 matrices such that $L^\dagger(\phi) = 0$. Noting that

$$G_{(\theta^{-1})^\dagger} = (\theta^{-1})^\dagger \partial_X (\theta)^\dagger \quad (4.130)$$

$$= -G_\theta^\dagger, \quad (4.131)$$

implies that the transformations they define are the same, so we work with $G_{(\theta^{-1})^\dagger}$ as it is simpler. We illustrate the compostite mapping in the diagram below.



$$\eta \longmapsto (\rho^{-1})^\dagger$$

This gives rise to the following definition of a BDT.

Definition : Consider an operator L and gauge operator G_θ, where $\theta \in S$. For each $\eta \in S^\dagger$, define

$$G_{\theta,\eta} = G_\rho^{-1} G_\theta \tag{4.132}$$

where $\rho = ((G^{-1}_{(\theta^{-1})^\dagger}\eta)^{-1})^\dagger$. Then

$$G_{\theta,\eta} : S \to \hat{S}. \tag{4.133}$$

where $\hat{L} = G_{\theta,\eta} L G^{-1}_{\theta,\eta} = \partial_Y + F_1\partial_X + \hat{Q}$, is called a binary Darboux transformation (BDT).

To calculate $G^{-1}_{(\theta^{-1})^\dagger}$ we recall

$$\tilde{\psi} = G_\theta(\psi) \qquad \Rightarrow \qquad \psi = G_\theta^{-1}(\tilde{\psi}), \tag{4.134}$$

or

$$\tilde{\psi} = \theta\partial_X\theta^{-1}(\psi) \qquad \Rightarrow \qquad \psi = \theta V, \tag{4.135}$$

where $V_X = \theta^{-1}\tilde{\psi}$. This gives

$$G_\theta^{-1}(\tilde{\psi}) = \theta\partial_X^{-1}(\theta^{-1}\tilde{\psi}), \tag{4.136}$$

and therefore

$$\rho = ((G^{-1}_{(\theta^{-1})^\dagger}\eta)^{-1})^\dagger \tag{4.137}$$

$$= \left(((\theta^{-1})^\dagger\partial_X^{-1}(\theta^\dagger\eta))^{-1}\right)^\dagger \tag{4.138}$$

$$= \left(((\theta^{-1})^\dagger\Omega^\dagger)^{-1}\right)^\dagger \tag{4.139}$$

$$= \theta\Omega^{-1} \tag{4.140}$$

where $\Omega = \partial_X^{-1}(\eta^\dagger\theta)$. Using this expression for ρ in our binary transformation we have our BDT

$$\begin{aligned} G_{\theta,\eta} &= \rho\partial_X^{-1}\rho^{-1}\theta\partial_X\theta^{-1} && (4.141)\\ &= \theta\Omega^{-1}\partial_X^{-1}\Omega\theta^{-1}\theta\partial_X\theta^{-1} && (4.142)\\ &= \theta\Omega^{-1}\left[\Omega\theta^{-1} - \partial_X^{-1}\Omega_X\theta^{-1}\right] \qquad \text{by parts} && (4.143)\\ &= I - \theta\Omega^{-1}\partial_X^{-1}\eta^\dagger\theta\theta^{-1} \qquad \text{as} \qquad \Omega_X = \eta^\dagger\theta && (4.144)\\ &= I - \theta\Omega^{-1}\partial_X^{-1}\eta^\dagger. && (4.145) \end{aligned}$$

Hence we have new Q, and new eigenfunctions $\hat{\psi} = G_{\theta,\eta}(\psi)$. We also have the adjoint BDT which is written $G_{\eta,\theta} : S^\dagger \to \hat{S}^\dagger$, and is calculated

$$\begin{aligned} G_{\eta,\theta} &= G_{(\rho^{-1})^\dagger}G^{-1}_{(\theta^{-1})^\dagger} && (4.146)\\ &= (\rho^{-1})^\dagger\partial_X\rho^\dagger(\theta^{-1})^\dagger\partial_X^{-1}\theta^\dagger && (4.147)\\ &= (\theta^{-1})^\dagger\Omega^\dagger\left[(\Omega^{-1})^\dagger\partial_X - (\Omega^{-1})^\dagger\theta^\dagger\eta(\Omega^{-1})^\dagger\right]\partial_X^{-1}\theta^\dagger && (4.148)\\ &= I - \eta(\Omega^{-1})^\dagger\partial_X^{-1}\theta^\dagger. && (4.149) \end{aligned}$$

So new adjoint eigenfunctions are given by $\hat{\phi} = G_{\eta,\theta}(\phi)$. We notice the way the eigenfunctions and adjoint eigenfunctions have interchanged role in the adjoint BDT.

4.4.2 Constraints

As with the DT, the idea is to examine the structure of the new solution ($\hat{Q}$), to see if the structure is preserved under the BDT. To do this we find closed form expressions for $\hat{Q}_{ij}$ and $\hat{\psi}_i$. Considering the two DT G_θ and G_ρ, we have two expressions for $\tilde{Q}$

$$\tilde{Q} = Q + [F_1, \theta_X\theta^{-1}], \qquad \tilde{Q} = \hat{Q} + [F_1, \rho_X\rho^{-1}]. \tag{4.150}$$

Given that $\rho = \theta\Omega^{-1} = \theta(\Omega(\eta,\theta))^{-1}$

$$\rho_X\rho^{-1} = \theta_X\theta^{-1} - \theta(\Omega(\eta,\theta))^{-1}\eta^\dagger, \tag{4.151}$$

we have

$$\hat{Q} = Q + [F_1, \theta(\Omega(\eta,\theta))^{-1}\eta^\dagger]. \tag{4.152}$$

Using theorem 4.3.1, this is re-written as

$$\hat{Q}_{ij} = Q_{ij} + (F_{1j} - F_{1i}) \frac{\begin{vmatrix} \Omega(\eta,\theta) & \eta^{\dagger}_{.j} \\ \theta_{i.} & 0 \end{vmatrix}}{|\Omega(\eta,\theta)|}, \tag{4.153}$$

with θ, η 3×3 matrices. The new eigenfunction arises by considering

$$\begin{aligned} \hat{\psi} &= G_{\theta,\eta}(\psi) && (4.154) \\ &= \left[I - \theta\Omega^{-1}\partial_X^{-1}\eta^{\dagger}\right](\psi) && (4.155) \\ &= \psi - \theta\Omega^{-1}\partial_X^{-1}(\eta^{\dagger}\psi). && (4.156) \end{aligned}$$

So in a compact form

$$\hat{\psi}_{ij} = \frac{\begin{vmatrix} \Omega(\eta,\theta) & \Omega(\eta,\psi)_{.j} \\ \theta_{i.} & \psi_{ij} \end{vmatrix}}{|\Omega(\eta,\theta)|}, \tag{4.157}$$

where we have defined a general Ω as

$$\partial_X \Omega(\sigma,\tau) = \sigma^{\dagger}\tau. \tag{4.158}$$

Similarly the new adoint eigenfunction is written

$$\hat{\phi}_{ij} = \frac{\begin{vmatrix} \Omega(\theta,\eta) & \Omega(\theta,\phi)_{.j} \\ \eta_{i.} & \phi_{ij} \end{vmatrix}}{|\Omega(\theta,\eta)|}. \tag{4.159}$$

From (4.153) we note that

$$\begin{aligned} \frac{\begin{vmatrix} \Omega(\eta,\theta) & \eta^{\dagger}_{.j} \\ \theta_{i.} & 0 \end{vmatrix}^{*}}{|\Omega(\eta,\theta)|^{*}} &= \frac{\begin{vmatrix} \Omega(\eta,\theta)^{*} & \eta^{T}_{.j} \\ \theta^{*}_{i.} & 0 \end{vmatrix}}{|\Omega(\eta,\theta)^{*}|} && (4.160) \\ &= \frac{\begin{vmatrix} \Omega(\eta,\theta)^{*T} & \theta^{\dagger}_{.i} \\ \eta_{j.} & 0 \end{vmatrix}}{|\Omega(\eta,\theta)^{*T}|} \quad \text{taking transpose} && (4.161) \\ &= \frac{\begin{vmatrix} \Omega(\eta,\theta) & \eta^{\dagger}_{.i} \\ \theta_{j.} & 0 \end{vmatrix}}{|\Omega(\eta,\theta)|}, && (4.162) \end{aligned}$$

if we choose

$$\eta = \theta, \qquad \Rightarrow \qquad \theta_{i.} = \eta_{i.}, \tag{4.163}$$

for $i = 1, 2, 3$. Once again, considering a new set of solutions $\hat{q}_i$, (for $i = 1, 2, 3$) and their conjugates to (4.26), and using the construction for Q in section 2, we obtain

$$\hat{Q} = \begin{bmatrix} 0 & \frac{\hat{q}_3 c_3}{c_2 b_3 - b_2 c_3} & \frac{-\hat{q}_2^* b_3}{c_2 b_3 - b_2 c_3} \\ \frac{-\hat{q}_3^* c_3}{a_2 c_3 - c_2 a_3} & 0 & \frac{\hat{q}_1 a_3}{a_2 c_3 - c_2 a_3} \\ \frac{\hat{q}_2 b_3}{b_2 a_3 - b_3 a_2} & \frac{-\hat{q}_1^* a_3}{b_2 a_3 - b_3 a_2} & 0 \end{bmatrix} \tag{4.164}$$

Using the explicit representation (4.153), and comparing this with (4.164), we see for example

$$\frac{\hat{q}_3 c_3}{c_2 b_3 - b_2 c_3} = \frac{q_3 c_3}{c_2 b_3 - b_2 c_3} + (F_{12} - F_{11}) \frac{\begin{vmatrix} \Omega(\eta, \theta) & \eta_{.2}^{*\,T} \\ \theta_{1.} & 0 \end{vmatrix}}{|\Omega(\eta, \theta)|} \tag{4.165}$$

$$\frac{-\hat{q}_3^* c_3}{a_2 c_3 - c_2 a_3} = \frac{-q_3^* c_3}{a_2 c_3 - c_2 a_3} + (F_{11} - F_{12}) \frac{\begin{vmatrix} \Omega(\eta, \theta) & \eta_{.1}^{*\,T} \\ \theta_{2.} & 0 \end{vmatrix}}{|\Omega(\eta, \theta)|}. \tag{4.166}$$

It is again obvious that up to a factor, (4.165) and (4.166) are related by complex conjugation. Hence we must show that starting with (4.165), that under complex conjugation we have (4.166). Noting (4.160) and the condition $\eta = \theta$, then (4.165) is equal to the complex conjugate of (4.166), providing

$$b_3 c_2 - b_2 c_3 = a_2 c_3 - a_3 c_2. \tag{4.167}$$

By examining the rest of (4.164) we see that the BDT preserves the structure of $\hat{Q}$, and hence provides new solutions to 3D3WR if

$$c_2 b_3 - c_3 b_2 = a_2 c_3 - a_3 c_2 = b_2 a_3 - b_3 a_2 \neq 0. \tag{4.168}$$

With this condition (4.168), (4.31) is invertible, providing

$$a_1 + b_1 + c_1 \neq 0. \tag{4.169}$$

Further we restrict ourselves to the situation where no pair of a_3, b_3, c_3 are zero and the equivalent term from a_2, b_2, c_2 is also non-zero. We note in passing that by considering any pair of (4.168) we have for example (for $a_2, c_2 \neq 0$)

$$b_3 = \frac{a_2c_3 - a_3c_2 + c_3b_2}{c_2} \qquad \text{or} \qquad b_3 = \frac{a_3b_2 - a_2c_3 + a_3c_2}{a_2} \tag{4.170}$$

and by eliminating b_3

$$\frac{(a_2 + b_2 + c_2)(a_2c_3 - a_3c_2)}{c_2a_2} = 0, \tag{4.171}$$

giving the relationship

$$a_2 + b_2 + c_2 = 0. \tag{4.172}$$

Similarly, considering for example b_2, we obtain

$$a_3 + b_3 + c_3 = 0. \tag{4.173}$$

We recall at this point that the operator M (where we have dropped the subscript from section two) has the same form as our operator L. Hence the definition (4.47) of the DT holds and we obtain the new operator

$$\tilde{M} = G_\theta M G_\theta^{-1} = \partial_Z + F_2\partial_X + \tilde{Q}_2', \tag{4.174}$$

where $\theta \in S$ the set of non-singular 3×3 solution matrices of $L(\psi) = M(\psi) = 0$. Once again $\tilde{R}$ is constrained (we have removed superscripts)

$$\tilde{R} = R + \left[F_2, \theta_X\theta^{-1}\right], \tag{4.175}$$

which gives us a defintion that is consistent. However we encounter the same problem for $\tilde{M}$ as we do with $\tilde{L}$, that is the complex conjugation condition is not satisfied. Hence we have to examine the new solution obtained from the BDT.

We consider the new solutions $\hat{R}$ corresponding to the operator

$$\hat{M} = G_{\theta,\eta} M G_{\theta,\eta}^{-1} = \partial_Z + F_2\partial_X + \hat{R}, \tag{4.176}$$

with $G_{\theta,\eta}$ defined as in (4.145). From (4.153) we see $\hat{R}$ takes the form

$$\hat{R}_{ij} = R_{ij} + (F_{2j} - F_{2i})\frac{\begin{vmatrix} \Omega(\eta,\theta) & \eta_{\cdot j}^{*T} \\ \theta_{i\cdot} & 0 \end{vmatrix}}{|\Omega(\eta,\theta)|}, \tag{4.177}$$

also by definition

$$\hat{R} = \begin{bmatrix} 0 & \frac{-\hat{q}_3 c_2}{c_2 b_3 - b_2 c_3} & \frac{\hat{q}_2^* b_2}{c_2 b_3 - b_2 c_3} \\ \frac{\hat{q}_3^* c_2}{a_2 c_3 - c_2 a_3} & 0 & \frac{-\hat{q}_1 a_2}{a_2 c_3 - c_2 a_3} \\ \frac{-\hat{q}_2 b_2}{b_2 a_3 - b_3 a_2} & \frac{\hat{q}_1^* a_2}{b_2 a_3 - b_3 a_2} & 0 \end{bmatrix}. \tag{4.178}$$

Equations (4.177) and (4.178) are consistent providing (4.168) holds. Under this condition we see that Q, $\hat{Q}$, R, and $\hat{R}$ are skew-hermitian (i.e. A is skew-hermitian if $A^T = -A^*$). Further we have the following new solutions from $\hat{Q}$ and $\hat{R}$ (where we have used (4.172) and (4.173) to guarantee equivalence)

$$\hat{q}_1 = q_1 + [a_1 + b_1 + c_1] \frac{\begin{vmatrix} \Omega(\eta, \theta) & \eta_{.3}^{*\,T} \\ \theta_{2.} & 0 \end{vmatrix}}{|\Omega(\eta, \theta)|}, \tag{4.179}$$

$$\hat{q}_2 = q_2 + [a_1 + b_1 + c_1] \frac{\begin{vmatrix} \Omega(\eta, \theta) & \eta_{.1}^{*\,T} \\ \theta_{3.} & 0 \end{vmatrix}}{|\Omega(\eta, \theta)|}, \tag{4.180}$$

$$\hat{q}_3 = q_3 + [a_1 + b_1 + c_1] \frac{\begin{vmatrix} \Omega(\eta, \theta) & \eta_{.2}^{*\,T} \\ \theta_{1.} & 0 \end{vmatrix}}{|\Omega(\eta, \theta)|}. \tag{4.181}$$

Choosing $b_1 = c_1 = 0$ and $a_1 = 1$ we obtain

$$\hat{q}_i = q_i + \frac{\begin{vmatrix} \Omega(\eta, \theta) & \eta_{.k}^{*\,T} \\ \theta_{j.} & 0 \end{vmatrix}}{|\Omega(\eta, \theta)|}, \tag{4.182}$$

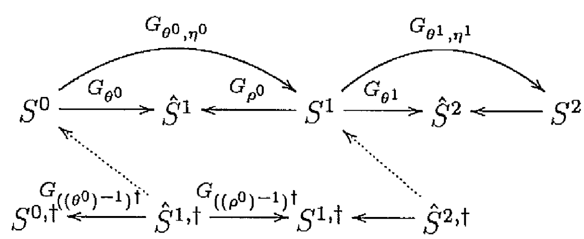
for $i, j, k = 1, 2, 3$. With these constraints, we may choose the following form for (4.31) to (4.33) ($A\boldsymbol{x} = \boldsymbol{X}$, with $\boldsymbol{x} = [x, y, z]$ and $\boldsymbol{X} = [X, Y, Z]$)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1-\alpha & \alpha & 1 \\ \alpha & -1-\alpha & 1 \end{bmatrix}, \tag{4.183}$$

with $\alpha = b_2 = a_3 \in \mathbb{R}$. To ensure that (4.168) is not violated we restrict α such that $\alpha \neq -1, 0, -1/2$.

4.4.3 Iteration

Once again we aim to iterate the BDT to obtain a whole hierachy of solutions. As the BDT preserves the correct structure of Q, then new solutions will be solutions of the 3D3WR. As in section 3 we relabel S and $S^\dagger$ as S^0 and $S^{0,\dagger}$ to indicate the base level. Once again the aim is to determine Q, ψ and ϕ after r iterations, of the BDT (we do not need to consider R, as Q gives us all the solutions). We have our general eigenfunction ψ^r, corresponding to $L^r(\psi^r) = (\partial_Y + F_1\partial_X + Q^r)(\psi^r) = 0$ and general adjoint eigenfunction ϕ^r, corresponding to $L^{r,\dagger}(\phi^r) = 0$. The original eigenfunction and adjoint eigenfunction will be labelled $\psi^0 = \psi$ and $\phi^0 = \phi$ respectively. Considering (4.157) and (4.159), we may regard ψ^r as function of ψ and ϕ^r as a function of ϕ. We illustrate the set-up with the case where $r = 2$. Assume that we have $\theta_i \in S^0$ and $\eta_i \in S^{0,\dagger}$ for $i = 0, 1$.



$$\eta_0 \longmapsto (\rho_0^{-1})^\dagger$$

We have distinguished between S^i corresponding to the DT by the use of $\hat{}$. The construction of the BDT (4.145), adjoint BDT (4.146) and the new solutions to the 3D3WR (4.153), relied on θ and η being 3×3 square matrices (i.e. invertible). However the final form (4.145) does not actually depend on their inverses. Thus we may consider more general eigenfunctions and adjoint eigenfunctions and the formulaes remain consistent. For example, considering θ and η as $n \times m$ matrices, gives Ω^{-1} as $m \times m$ and forces ψ and $\hat{\psi}$ to be $n \times p$, with $m, n, p \in \mathbb{N}$.

For simplicity we will consider the case $n = 3$, $m = 1$ i.e. θ and η are 3×1 column vectors. This implies that ψ and ϕ are 3×1 and hence so are $\hat{\psi}$ and ϕ. Explicitly the BDT is given by (4.145)

$$G_{\theta,\eta} = I - \theta\Omega^{-1}\partial_X^{-1}\eta^\dagger., \tag{4.184}$$

so with θ and η as above, we have

$$G_{\theta,\eta}(\psi) = \psi - \theta\Omega(\eta,\theta)^{-1}\Omega(\eta,\psi). \tag{4.185}$$

In the same way the adjoint BDT is

$$G_{\eta,\theta}(\phi) = \phi - \eta\Omega(\eta,\theta)^{-1}\Omega(\theta,\phi). \tag{4.186}$$

By taking r eigenfunctions θ_i, we may form the $3\times r$ matrix $\boldsymbol{\theta}$ by concatenating the elements. Similarly with η_i we form $\boldsymbol{\eta}$. These matrices help us to determine the effect of the BDT and adjoint BDT after r iterations. We form the $r+1$ eigenfunction (adjoint eigenfunction) by acting on the rth eigenfunction (adjoint eigenfunction) in the following way

$$\psi^{r+1} = G_{\theta^r,\eta^r}(\psi^r), \tag{4.187}$$
$$\phi^{r+1} = G_{\eta^r,\theta^r}(\phi^r). \tag{4.188}$$

4.4.4 Closed Form Expressions

(4.187), (4.188) allow us to consider the closed form of ψ^r and ϕ^r and then finally Q^r. We assume we have r eigenfunctions θ_i, and r adjoint eigenfunctions η_i $(i = 1, \dots, r)$, in S_0 and $S_0^\dagger$ respectively. Then we have the following theorems :

Theorem 4.4.1. *For* $r \geq 1$

$$\psi_i^r = \frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\psi) \\ \boldsymbol{\theta}_{i.} & \psi_i \end{vmatrix}}{|\Omega(\boldsymbol{\eta},\boldsymbol{\theta})|}, \quad \text{and} \quad \phi_i^r = \frac{\begin{vmatrix} \Omega(\boldsymbol{\theta},\boldsymbol{\eta}) & \Omega(\boldsymbol{\theta},\phi) \\ \boldsymbol{\eta}_{i.} & \phi_i \end{vmatrix}}{|\Omega(\boldsymbol{\theta},\boldsymbol{\eta})|}, \tag{4.189}$$

where $\boldsymbol{\theta}$ is an $3\times r$ matrix formed from the r eigenfunctions, θ_i and $\boldsymbol{\eta}$ similarly is $3\times r$, made up of the adjoint eigenfunctions η_i.

Proof. The proof due to Gilson and Nimmo [36] is by induction. For $r=1$ we have

$$\psi_i^1 = (G_{\theta,\eta}(\psi))_i \tag{4.190}$$
$$= \psi_i - \left(\frac{\theta\Omega(\eta,\psi)}{\Omega(\eta,\theta)}\right)_i \tag{4.191}$$
$$= \frac{\begin{vmatrix} \Omega(\eta,\theta) & \Omega(\eta,\psi) \\ \theta_i & \psi_i \end{vmatrix}}{|\Omega(\eta,\theta)|}, \tag{4.192}$$

and

$$\phi_i^1 = (G_{\eta,\theta}(\phi))_i \tag{4.193}$$
$$= \phi_i - \left(\frac{\eta\Omega(\theta,\phi)}{\Omega(\theta,\eta)}\right)_i \tag{4.194}$$
$$= \frac{\begin{vmatrix} \Omega(\theta,\eta) & \Omega(\theta,\phi) \\ \eta_i & \phi_i \end{vmatrix}}{|\Omega(\theta,\eta)|}. \tag{4.195}$$

Assume the result holds for r. First we prove

$$\Omega(\eta^r,\psi^r) = \frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\psi) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\psi) \end{vmatrix}}{|\Omega(\boldsymbol{\eta},\boldsymbol{\theta})|}, \tag{4.196}$$

where η is a new adjoint eigenfunction, not already included in $\boldsymbol{\eta}$. In the same way we have θ a new eigenfunction not included in $\boldsymbol{\theta}$. Taking the derivative of the right-hand side, we obtain

$$-\sum_{i=1}^{3} \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\psi) & \boldsymbol{\eta}^\dagger_{.i} \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\psi) & \eta^\dagger_i \\ \boldsymbol{\theta}_{i.} & \psi_i & 0 \end{vmatrix} \Big/ |\Omega(\boldsymbol{\eta},\boldsymbol{\theta})|$$
$$+\sum_{i=1}^{3} \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\psi) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\psi) \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger_{.i} \\ \boldsymbol{\theta}_{i.} & 0 \end{vmatrix} \Big/ |\Omega(\boldsymbol{\eta},\boldsymbol{\theta})|^2, \tag{4.197}$$

which using a Jacobi expansion can be written

$$\sum_{i=1}^{3} \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger_{.i} \\ \Omega(\eta,\boldsymbol{\theta}) & \eta^\dagger_i \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\psi) \\ \boldsymbol{\theta}_{i.} & \psi_i \end{vmatrix} \Big/ |\Omega(\boldsymbol{\eta},\boldsymbol{\theta})|^2 = (\eta^r)^\dagger \psi^r, \tag{4.198}$$

where we get η^r from ϕ^r by the specialization $\phi = \eta$. Hence we have (4.196). Consequently $\psi_i^{r+1} = G_{\theta^r,\eta^r}(\psi_i^r)$ is equivalent to

$$\frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\psi) \\ \boldsymbol{\theta}_{i.} & \psi_i \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger\theta \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\theta) \end{vmatrix} - \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\psi) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\psi) \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger\theta \\ \boldsymbol{\theta}_{i.} & \theta_i \end{vmatrix}}{|\Omega(\boldsymbol{\eta},\boldsymbol{\theta})| \begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\theta) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\theta) \end{vmatrix}}, \tag{4.199}$$

which using a Jacobi identity is

$$\frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\theta) & \Omega(\boldsymbol{\eta},\psi) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta^{\dagger},\theta) & \Omega(\eta,\psi) \\ \boldsymbol{\theta}_{i.} & \theta_i & \psi_i \end{vmatrix}}{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\theta) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega\eta,\theta) \end{vmatrix}} = \frac{\begin{vmatrix} \Omega((\boldsymbol{\eta},\eta),(\boldsymbol{\theta},\theta)) & \Omega((\boldsymbol{\eta},\eta),\psi) \\ (\boldsymbol{\theta},\theta)_{i.} & \psi_i \end{vmatrix}}{|\Omega((\boldsymbol{\eta},\eta),(\boldsymbol{\theta},\theta))|} = \psi_i^{r+1}, \tag{4.200}$$

as required, where (A, v) denotes the matrix formed by appending the vector v as the last column to matrix A. Similarly

$$\phi_i^{r+1} = \phi_i^r - \left(\eta^r \frac{\Omega(\theta^r,\phi^r)}{\Omega(\theta^r,\eta^r)}\right)_i, \tag{4.201}$$

which we write in full as

$$\left[\begin{vmatrix} \Omega(\boldsymbol{\theta},\boldsymbol{\eta}) & \Omega(\boldsymbol{\theta},\phi) \\ \boldsymbol{\eta}_{i.} & \phi_i \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\theta},\boldsymbol{\eta}) & \Omega(\boldsymbol{\theta},\eta) \\ \Omega(\theta,\boldsymbol{\eta}) & \Omega(\theta,\eta) \end{vmatrix} - \begin{vmatrix} \Omega(\boldsymbol{\theta},\boldsymbol{\eta}) & \Omega(\boldsymbol{\theta},\eta) \\ \boldsymbol{\eta}_{i.} & \eta_i \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\theta},\boldsymbol{\eta}) & \Omega(\boldsymbol{\theta},\phi) \\ \Omega(\theta,\boldsymbol{\eta}) & \Omega(\theta,\phi) \end{vmatrix}\right] \Big/ \begin{vmatrix} \Omega(\boldsymbol{\theta},\boldsymbol{\eta}) & \Omega(\boldsymbol{\theta},\eta) \\ \Omega(\theta,\boldsymbol{\eta}) & \Omega(\theta,\eta) \end{vmatrix} |\Omega(\boldsymbol{\theta},\boldsymbol{\eta})|, \tag{4.202}$$

where we note that

$$\Omega(\theta^r,\phi^r) = \frac{\begin{vmatrix} \Omega(\boldsymbol{\theta},\boldsymbol{\eta}) & \Omega(\boldsymbol{\theta},\phi) \\ \Omega(\theta,\boldsymbol{\eta}) & \Omega(\theta,\phi) \end{vmatrix}}{|\Omega(\boldsymbol{\theta},\boldsymbol{\eta})|}. \tag{4.203}$$

This is proved in the same way as (4.196). Simplifying using a Jacobi expansion we have

$$\frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\theta) & \boldsymbol{\eta}^{\dagger}_{.i} \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\theta) & \eta^{\dagger}_i \\ \Omega(\phi,\boldsymbol{\theta}) & \Omega(\phi,\theta) & \phi_i \end{vmatrix}}{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\theta) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\theta) \end{vmatrix}} = \frac{\begin{vmatrix} \Omega((\boldsymbol{\eta},\eta),(\boldsymbol{\theta},\theta)) & (\boldsymbol{\eta},\eta)^{\dagger}_{.i} \\ \Omega(\phi,(\boldsymbol{\theta},\theta)) & \phi_i \end{vmatrix}}{|\Omega((\boldsymbol{\eta},\eta),(\boldsymbol{\theta},\theta))|} = \phi_i^{r+1}, \tag{4.204}$$

as required. □

Theorem 4.4.2. *For $r \geq 1$ our new solutions Q^r are given by*

$$Q^r_{ij} = Q^0_{ij} + (F_{1j} - F_{1i}) \frac{\begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger_{.j} \\ \boldsymbol{\theta}_{i.} & 0 \end{vmatrix}}{|\Omega(\boldsymbol{\eta}, \boldsymbol{\theta})|}. \tag{4.205}$$

Proof. This is by induction on r. For $r = 1$ we have

$$Q^1_{ij} = Q^0_{ij} + (F_{1j} - F_{1i}) \frac{\begin{vmatrix} \Omega(\eta, \theta) & \eta^\dagger_{.j} \\ \theta_{i.} & 0 \end{vmatrix}}{|\Omega(\eta, \theta)|}. \tag{4.206}$$

which is equivalent to (4.153). Assume the result holds for r. From (4.152) we have

$$Q^{r+1} = Q^r + \left[F_1, \theta^r(\Omega(\eta^r, \theta^r))^{-1}(\eta^r)^\dagger\right], \tag{4.207}$$

where we have generalised to the rth iteration. Subsitituting for Q^r (4.205), and looking at the (ij)th entry we obtain

$$Q^{r+1}_{ij} = Q^0_{ij} + (F_{1j} - F_{1i}) \left[\frac{\begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger_{.j} \\ \boldsymbol{\theta}_{i.} & 0 \end{vmatrix}}{|\Omega(\boldsymbol{\eta}, \boldsymbol{\theta})|} - (\theta^r(\Omega(\eta^r, \theta^r))^{-1}(\eta^r)^\dagger)_{ij} \right]. \tag{4.208}$$

$\Omega(\eta^r, \theta^r)$ is scalar, so that the (ij)th entry of the second term, is equivalent to $(\theta^r_i(\eta^r_j)^\dagger)/\Omega(\eta^r, \theta^r)$. Using theorem 4.4.1 we have

$$Q^{r+1}_{ij} = Q^0_{ij} + (F_{1j} - F_{1i}) \left[\begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger_{.j} \\ \boldsymbol{\theta}_{i.} & 0 \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \Omega(\boldsymbol{\eta}, \theta) \\ \Omega(\eta, \boldsymbol{\theta}) & \Omega(\eta, \theta) \end{vmatrix} \right.$$
$$\left. - \begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \Omega(\boldsymbol{\eta}, \theta) \\ \boldsymbol{\theta}_{i.} & \theta_i \end{vmatrix} \begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger_{.j} \\ \Omega(\eta, \boldsymbol{\theta}) & \eta^\dagger_j \end{vmatrix} \right] / |\Omega(\boldsymbol{\eta}, \boldsymbol{\theta})| \begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \Omega(\boldsymbol{\eta}, \theta) \\ \Omega(\eta, \boldsymbol{\theta}) & \Omega(\eta' \theta) \end{vmatrix}, \tag{4.209}$$

which simplifies using a Jacobi identity

$$Q_{ij}^{r+1} = Q_{ij}^0 + (F_{1j} - F_{1i}) \frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\theta) & \boldsymbol{\eta}_{.j}^{\dagger} \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\theta) & \eta_j^{\dagger} \\ \boldsymbol{\theta}_{i.} & \theta_i & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \Omega(\boldsymbol{\eta},\theta) \\ \Omega(\eta,\boldsymbol{\theta}) & \Omega(\eta,\theta) \end{vmatrix}} \tag{4.210}$$

$$= Q_{ij}^0 + (F_{1j} - F_{1i}) \frac{\begin{vmatrix} \Omega((\boldsymbol{\eta},\eta),(\boldsymbol{\theta},\theta)) & (\boldsymbol{\eta},\eta)_{.j}^{\dagger} \\ (\boldsymbol{\theta},\theta)_{i.} & 0 \end{vmatrix}}{|\Omega((\boldsymbol{\eta},\eta),(\boldsymbol{\theta},\theta))|}, \tag{4.211}$$

as required. □

4.5 Solutions

We examine the solutions that we have obtained via the BDT and subsequent iterations. The aim is to show that the solutions in this chapter correspond to the ones in chapter three.

4.5.1 Method

Firstly we return to the exact nature of the integral (4.136) in section 4.4.1. We have $\Omega(\eta, \theta)$, such that

$$\partial_X(\Omega(\eta,\theta)) = \eta^{\dagger}\theta. \tag{4.212}$$

Recalling $\psi = \theta V$ and $V_X = \theta^{-1}\tilde{\psi}$ (4.135), then

$$L(\psi) = (\partial_Y + F_1\partial_X + Q)(\theta V) \tag{4.213}$$

$$= (L(\theta))V + \theta V_Y + F_1\tilde{\psi} \tag{4.214}$$

$$= 0 \quad \Leftrightarrow \quad V_Y = -\theta^{-1}F_1\tilde{\psi}, \tag{4.215}$$

and

$$M(\psi) = (\partial_Z + F_2\partial_X + R)(\theta V) \tag{4.216}$$

$$= 0 \quad \Leftrightarrow \quad V_Z = -\theta^{-1}F_2\tilde{\psi}. \tag{4.217}$$

Putting these expressions together we have

$$\Omega(\eta,\theta) = \int d\Omega \tag{4.218}$$

$$= \int_C \eta^\dagger \theta dX - \eta^\dagger F_1 \theta dY - \eta^\dagger F_2 \theta dZ, \tag{4.219}$$

along an appropriate curve C in three-dimensional space.

For the constraints to be satisfied and the BDT to preserve the correct structure of the solutions we had $\eta = \theta$ (4.163), specifically with θ and η any 3×3 matrices. We have equivalently in the general case (i.e. $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ $3\times r$) $\boldsymbol{\theta} = \boldsymbol{\eta}$.

We notice that with (4.183), we have

$$dX = dx, \tag{4.220}$$

$$dY = (-1-\alpha)dx + \alpha dy + dz, \tag{4.221}$$

$$dZ = \alpha dx + (-1-\alpha)dy + dz. \tag{4.222}$$

The action of the original Lax pair (4.29) (4.30), (with the vacuum solution i.e. Q^0 set to zero), on the eigenfunctions and adjoint eigenfunctions, gives

$$L(\theta_i) = 0 \qquad M(\theta_i) = 0, \tag{4.223}$$

$$L^\dagger(\eta_i) = 0 \qquad M^\dagger(\eta_i) = 0, \tag{4.224}$$

for $i = 1, \ldots, r$. This implies that if we label our eigenfunctions $\theta_i = [\theta_{i1}, \theta_{i2}, \theta_{i3}]^T$, and adjoint eigenfunctions $\eta_i = [\eta_{i1}, \eta_{i2}, \eta_{i3}]^T$, then

$$\theta_{i1} = \theta_{i1}(x) \qquad \eta_{i1} = \eta_{i1}(x), \tag{4.225}$$

$$\theta_{i2} = \theta_{i2}(y) \qquad \eta_{i2} = \eta_{i2}(y), \tag{4.226}$$

$$\theta_{i3} = \theta_{i3}(z) \qquad \eta_{i3} = \eta_{i3}(z). \tag{4.227}$$

We return to the closed form of our new solution, and we will consider the simplest type of BDT, the vacuum solution (if we choose Q^0 non-zero then we obtain a wider class of solutions). We have therefore

$$q_i^r = \frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \boldsymbol{\eta}^\dagger_{.k} \\ \boldsymbol{\theta}_{j.} & 0 \end{vmatrix}}{|\Omega(\boldsymbol{\eta},\boldsymbol{\theta})|}. \tag{4.228}$$

where we have generalised (4.182) to the r iterated case (same choice of A (4.183)).

4.5.2 $r = 1$ Case

With $r = 1$, we have one eigenfunction θ_1, and one adjoint eigenfunction η_1, so $\boldsymbol{\theta} = \theta_1$ and $\boldsymbol{\eta} = \eta_1$. From the constraint on solutions we have $\theta_1 = \eta_1$. Choosing $\theta_1 = [\theta_{11}, \theta_{12}, \theta_{13}]^T$ and $\eta_1 = [\eta_{11}, \eta_{12}, \eta_{13}]^T$, and using (4.219), the right hand side denominator of (4.228) takes the form

$$\begin{aligned} |\Omega(\eta_1, \theta_1)| &= \left| \sum_{i=1}^{3} \left[\int \eta_{1i}^* \theta_{1i} (dX - F_{1i} dY - F_{2i} dZ) \right] \right|, && (4.229) \\ &= \left| c + \int \eta_{11}^* \theta_{11} dx + \eta_{12}^* \theta_{12} dy + \eta_{13}^* \theta_{13} dz \right|, && (4.230) \end{aligned}$$

with c a constant. Rewriting (4.230) we have

$$\begin{aligned} |\Omega(\eta_1, \theta_1)| &= c \left| 1 + c^{-1} \left(\int \eta_{11}^* \theta_{11} dx + \eta_{12}^* \theta_{12} dy + \eta_{13}^* \theta_{13} dz \right) \right|, && (4.231) \\ &= c \begin{vmatrix} 1 + c^{-1} \int \eta_{11}^* \theta_{11} dx & c^{-1} \int \eta_{12}^* \theta_{12} dy & c^{-1} \int \eta_{13}^* \theta_{13} dz \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} && (4.232) \end{aligned}$$

Adding rows we get

$$c \begin{vmatrix} 1 + c^{-1} \int \eta_{11}^* \theta_{11} dx & c^{-1} \int \eta_{12}^* \theta_{12} dy & c^{-1} \int \eta_{13}^* \theta_{13} dz \\ c^{-1} \int \eta_{11}^* \theta_{11} dx & 1 + c^{-1} \int \eta_{12}^* \theta_{12} dy & c^{-1} \int \eta_{13}^* \theta_{13} dz \\ c^{-1} \int \eta_{11}^* \theta_{11} dx & c^{-1} \int \eta_{12}^* \theta_{12} dy & 1 + c^{-1} \int \eta_{13}^* \theta_{13} dz \end{vmatrix} = c\,|I + H\beta|, \qquad (4.233)$$

with

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \beta = \begin{bmatrix} c^{-1} \int \eta_{11}^* \theta_{11} dx & & 0 \\ & c^{-1} \int \eta_{12}^* \theta_{12} dy & \\ 0 & & c^{-1} \int \eta_{13}^* \theta_{13} dz \end{bmatrix}, \qquad (4.234)$$

and I the (3×3) identity matrix, which corresponds to (3.57). Thus we can write (4.228) as

$$q_i^1 = \frac{-\theta_{1j} \eta_{1k}^*}{c\,|I + H\beta|}, \qquad (4.235)$$

which is equivalent to (3.55) or (3.56) (up to a minus sign, this arises from our choice of A (4.183)). To make the correspondence exact we choose

$$c^{-1/2}\theta_{1i} = g_i, \tag{4.236}$$

and $X_1 = x$, $X_2 = y$ and $X_3 = z$, with the factor of c cancelling. So we see that with $r = 1$ in the BDT we have the 1-lump solution in the non-degenerate kernel case.

4.5.3 General n Case

This corresponds to the n-lump solution discussed in section 3.3.3. We have n eigenfunctions and adjoint eigenfunctions, each of which are separable as described above. $\boldsymbol{\eta} = \boldsymbol{\theta}$ to ensure we have solutions. Then

$$\boldsymbol{\eta}^\dagger \boldsymbol{\theta} = \sum_{i=1}^{3} \begin{bmatrix} \eta_{1i}^*\theta_{1i} & \cdots & \eta_{1i}^*\theta_{ni} \\ \vdots & & \vdots \\ \eta_{ni}^*\theta_{1i} & \cdots & \eta_{ni}^*\theta_{ni} \end{bmatrix} \tag{4.237}$$

so that

$$\begin{aligned} |\Omega(\boldsymbol{\eta}, \boldsymbol{\theta})| = |\, c + \int \begin{bmatrix} \eta_{11}^*\theta_{11} & \cdots & \eta_{11}^*\theta_{n1} \\ \vdots & & \vdots \\ \eta_{n1}^*\theta_{11} & \cdots & \eta_{n1}^*\theta_{n1} \end{bmatrix} dx \\ + \int \begin{bmatrix} \eta_{12}^*\theta_{12} & \cdots & \eta_{12}^*\theta_{n2} \\ \vdots & & \vdots \\ \eta_{n2}^*\theta_{12} & \cdots & \eta_{n2}^*\theta_{n2} \end{bmatrix} dy + \int \begin{bmatrix} \eta_{13}^*\theta_{13} & \cdots & \eta_{13}^*\theta_{n3} \\ \vdots & & \vdots \\ \eta_{n3}^*\theta_{13} & \cdots & \eta_{n3}^*\theta_{n3} \end{bmatrix} dz\,| \end{aligned} \tag{4.238}$$

where c is a constant $n \times n$ matrix. If we choose the following representation

$$\Phi = \begin{bmatrix} \int \eta_{11}^* \theta_{11} dx & \cdots & \int \eta_{11}^* \theta_{n1} dx \\ \vdots & & \vdots \\ \int \eta_{n1}^* \theta_{11} dx & \cdots & \int \eta_{n1}^* \theta_{n1} dx \end{bmatrix}, \tag{4.239}$$

$$\Psi = \begin{bmatrix} \int \eta_{12}^* \theta_{12} dy & \cdots & \int \eta_{12}^* \theta_{n2} dy \\ \vdots & & \vdots \\ \int \eta_{n2}^* \theta_{12} dy & \cdots & \int \eta_{n2}^* \theta_{n2} dy \end{bmatrix}, \tag{4.240}$$

$$\Sigma = \begin{bmatrix} \int \eta_{13}^* \theta_{13} dz & \cdots & \int \eta_{13}^* \theta_{n3} dz \\ \vdots & & \vdots \\ \int \eta_{n3}^* \theta_{13} dz & \cdots & \int \eta_{n3}^* \theta_{n3} dz \end{bmatrix}, \tag{4.241}$$

then (4.238) is equivalent to writing

$$|\Omega(\boldsymbol{\eta}, \boldsymbol{\theta})| = |c| \left| I + c^{-1}\Phi + c^{-1}\Psi + c^{-1}\Sigma \right|, \tag{4.242}$$

with I the $(n \times n)$ identity matrix. Following the same proceedure as the previous section of subtracting and adding rows and columns we write (4.242) as

$$|c| \begin{vmatrix} I + c^{-1}\Phi & c^{-1}\Psi & c^{-1}\Sigma \\ c^{-1}\Phi & I + c^{-1}\Psi & c^{-1}\Sigma \\ c^{-1}\Phi & c^{-1}\Psi & I + c^{-1}\Sigma \end{vmatrix} = |c| \left| \overline{I} + H\beta \right|, \tag{4.243}$$

with H the $(3n \times 3n)$ block matrix with nine identical $(n \times n)$ blocks, $\overline{I}$ the $(3n \times 3n)$ identity and β a matrix with on-diagonal $(n \times n)$ blocks

$$\beta = \begin{bmatrix} \Phi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Sigma \end{bmatrix}, \qquad H = \begin{bmatrix} c^{-1} & c^{-1} & c^{-1} \\ c^{-1} & c^{-1} & c^{-1} \\ c^{-1} & c^{-1} & c^{-1} \end{bmatrix}. \tag{4.244}$$

Carrying out the same procedure on

$$\begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \boldsymbol{\eta}_{.k}^{\dagger} \\ \boldsymbol{\theta}_{j.} & 0 \end{vmatrix}, \tag{4.245}$$

we find that we can write

$$q_1^r = \frac{\begin{vmatrix} \overline{I}+H\beta & H\underline{\sigma} \\ \underline{\psi}^\dagger & 0 \end{vmatrix}}{|\overline{I}+H\beta|}, \tag{4.246}$$

$$q_2^r = \frac{\begin{vmatrix} \overline{I}+H\beta & H\underline{\phi} \\ \underline{\sigma}^\dagger & 0 \end{vmatrix}}{|\overline{I}+H\beta|}, \tag{4.247}$$

$$q_3^r = \frac{\begin{vmatrix} \overline{I}+H\beta & H\underline{\psi} \\ \underline{\phi}^\dagger & 0 \end{vmatrix}}{|\overline{I}+H\beta|}, \tag{4.248}$$

where $\phi_i(X_1) = \theta_{i1}(x)$, $\psi_i(X_2) = \theta_{i2}(y)$, $\sigma_i(X_3) = \theta_{i3}(z)$, for $i = 1, \ldots, n$ and

$$\underline{\phi} = (\phi_1, \ldots, \phi_n; 0, \ldots, 0; 0, \ldots, 0)^T \tag{4.249}$$

$$\underline{\psi} = (0, \ldots, 0; \psi_1, \ldots, \psi_n; 0, \ldots, 0)^T \tag{4.250}$$

$$\underline{\sigma} = (0, \ldots, 0; 0, \ldots, 0; \sigma_1, \ldots, \sigma_n)^T \tag{4.251}$$

$$\underline{\phi}^\dagger = (\phi_1^*, \ldots, \phi_n^*; 0, \ldots, 0; 0, \ldots, 0) \tag{4.252}$$

$$\underline{\psi}^\dagger = (0, \ldots, 0; \psi_1^*, \ldots, \psi_n^*; 0, \ldots, 0) \tag{4.253}$$

$$\underline{\sigma}^\dagger = (0, \ldots, 0; 0, \ldots, 0; \sigma_1^*, \ldots, \sigma_n^*). \tag{4.254}$$

By swopping rows and columns, and with the correct boundary conditions, (4.246) to (4.248) are equivalent to q_1^*, q_2^* and q_3^* respectively, as defined in section 3.3.3, (up to a minus sign, again due to our choice of A). So we see that iterating the BDT n times, leads us to obtain the n-lump solution to 3D3WR.

4.5.4 Bäcklund Transformation

The BDT gives rise to the lump solutions discussed in chapter three. This is a specific case of the general solutions of section 3.5 with H made up of nine $(r \times r)$ identity matrices. An iteration of the BDT is therefore equivalent to the addition of a lump to the 3D3WR. We recall that the scattering problem contains no eigenvalue so that we have no bound states and hence no solitons. However in the case of the 3D3WR the lump solutions appear to be "equivalent" to soliton solutions. Whereas an iteration of the DT for the KP hierachy gives rise to a new eigenvalue (or addition of a soliton) [70] [21] [91], the 3D3WR after iteration of the BDT leads to the addition of a lump.

The Bäcklund transformation given by Kaup [51] is another way of generating lump solutions, which involves solving ζ for the scattering problem and then determining D (3.34). However in practice this is far from easy, and the grammian approach given in the previous chapter was a significant simplification. In fact the n-lump had not been written down explicitly before. The BDT provides a method of building up lump solutions, which avoids the difficulities associated with Kaup's Bäcklund transformation of having to solve for D.

We recall (4.14) finally that we may consider the DT or the BDT as an auto-Bäcklund transformation. With $\hat{L} = G_{\theta,\eta} L G_{\theta,\eta}^{-1}$ and $\hat{M}$ similarly, we have

$$[L, M] = 0 \qquad \Rightarrow \qquad [\hat{L}, \hat{M}] = 0. \tag{4.255}$$

So the link between the solutions obtained by Kaup and the Bäcklund transformation, and the ones from the BDT is to be expected.

4.5.5 Discrete Equation

In section 4.4.4, we established two concise superposition formulae. The first in theorem 4.4.2 expresses the form of a new solution after r iterations of the BDT, the second in theorem 4.4.1 gives the relationship between the new eigenfunctions (adjoint eigenfunctions) and the original eigenfunction (adjoint eigenfunction). The aim is to use these formulae to try and construct an integrable discrete system, and associated linear system, which in a particular form provide discretizations of 3D3WR and its scattering problem.

Recall from theorem 4.4.1 we have

$$\psi_i^r = \frac{\begin{vmatrix} \Omega(\boldsymbol{\eta}, \boldsymbol{\theta}) & \Omega(\boldsymbol{\eta}, \psi) \\ \boldsymbol{\theta}_{i.} & \psi_i \end{vmatrix}}{|\Omega(\boldsymbol{\eta}, \boldsymbol{\theta})|}, \qquad \phi_i^r = \frac{\begin{vmatrix} \Omega(\boldsymbol{\theta}, \boldsymbol{\eta}) & \Omega(\boldsymbol{\theta}, \phi) \\ \boldsymbol{\eta}_i & \phi_i \end{vmatrix}}{|\Omega(\boldsymbol{\theta}, \boldsymbol{\eta})|}, \tag{4.256}$$

and from theorem (4.4.2)

$$Q^r_{ij} = Q^0_{ij} + (F_{1j} - F_{1i})\frac{\begin{vmatrix} \Omega(\boldsymbol{\eta},\boldsymbol{\theta}) & \boldsymbol{\eta}^{\dagger}_{.j} \\ \boldsymbol{\theta}_{i.} & 0 \end{vmatrix}}{|\Omega(\boldsymbol{\eta},\boldsymbol{\theta})|}. \tag{4.257}$$

We write (4.257) as

$$Q^r_{ij} = Q^0_{ij} + (F_{1j} - F_{1i})\frac{G^r_{ij}}{F^r}, \tag{4.258}$$

so that

$$Q^{r+1}_{ij} = Q^0_{ij} + (F_{1j} - F_{1i})\frac{G^{r+1}_{ij}}{F^{r+1}}. \tag{4.259}$$

From (4.208) we have

$$Q^{r+1}_{ij} = Q^0_{ij} + (F_{1j} - F_{1i})\left[\frac{G^r_{ij}}{F^r} - \frac{\theta^r_i(\eta^r_j)^{\dagger}}{\Omega(\eta^r\theta^r)}\right]. \tag{4.260}$$

We notice that (4.203) becomes

$$\Omega(\eta^r, \theta^r) = \frac{F^{r+1}}{F^r}, \tag{4.261}$$

and using (4.256) we write θ^r and η^r in a more convenient form as (we mimic the notation of section 4.3.4)

$$\theta^r = \frac{h^r_{ij}}{F^r} \qquad\qquad \eta^r = \frac{\hat{h}^r_{ij}}{F^r}. \tag{4.262}$$

This gives the superposition formula

$$G^{r+1}_{ij}F^r = G^r_{ij}F^{r+1} - h^r_i\hat{h}^r_j. \tag{4.263}$$

The formulae obtained by Nimmo [80] [82] were particularly elegant and simple and allowed the identification of indices with increments in the associated discrete variables. However (4.263) contains too many different dependent variables. We further have the problem that $\boldsymbol{\eta} = \boldsymbol{\theta}$, restricting our freedom in choice of eigenfunctions and adjoint eigenfunctions. So whilst (4.263) could be interpreted as a giving rise to a discrete equation, it is of questionable use. We return briefly to the question of a discrete version of 3D3WR in chapter five.

Chapter 5

Conclusion

5.1 Summary

Grammians have proved to be a powerful tool in obtaining new solutions; in particular they have led us to replicate the rational solutions of Ablowitz and Satsuma to the DSI equation, whilst avoiding some of the complications involved in the summation process. The iterative Bäcklund transformation by Kaup for 3D3WR, has been considerably simplified by the use of the grammian approach, indeed the explicit n-lump can be written down and more general solutions calculated. The BDT for 3D3WR has led to the extensive use of grammian solutions, in providing a closed form for our new solutions and eigenfunctions. The connection between the solutions from the BDT and the lump solutions of chapter three has been established.
Chapter two has seen us derive a set of rational solutions for the non-zero background DSI equation. For a particular choice of the parameters we may obtain non-singular solutions, which are lumps, with no phase shift when they collide. They correspond to the lump solutions obtained by Satsuma and Ablowitz, but the final form has no exponential factor, so could be said to be more "fully" rational.
The aim of chapter three was to investigate a broad class of solutions to the 3D3WR interaction. The types of solutions can be broadly categorised into two classes, lump solutions and ridge solutions. The lump solutions arise from either choosing our basic functions as localized or by choosing our matrix H so as the coefficients of all the possible terms in F and G are present. The ridge solutions are essentially resonant

solutions where by choice of a specific H not all the coefficients are present. The class of solutions presented here includes the "lump" solutions of Kaup.

In chapter four the DT and BDT have been constructed for 3D3WR, and the BDT used to establish new solutions. These solutions include those obtained by Kaup via the Bäcklund transformation. Indeed we may view the BDT as providing explicit machinery to obtain new solutions. This approach is simpler than the one of Kaup and more general.

5.2 Open Questions

We propose two questions that could stimulate further work. Firstly, can we apply the wronskian approach of Freeman [29] to DSI with non-zero background conditions? The obstreperous step appears to be the reality condition on the proposed F and then the conjugacy condition on the corresponding G. Freemans work in 1984, gave a double wronskian solution to DSI but with zero background conditions. Again, Hietarinta and Hirota's work [39] [40], followed the approach of Hirota, Ohta and Satsuma, but only examined the zero background case, with the difficulty focusing on proving the reality conditions.

Secondly we would like to obtain a discrete integrable version of 3D3WR, and to connect it with the work carried out by Santini and Doliwa on lattices [64] [26]. In particular to make use of the method of reinterpreting the superposition formula as a discrete equation to give rise to a discrete 3D3WR.

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