On the Complex Cobordism of Flag Varieties Associated to Loop Groups

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Summary

This work is about the algebraic topology of LG/T, in particular, the complex cobordism of LG/T where G is a compact semi-simple Lie group. The loop group LG is the group of smooth parametrized loops in G, i.e. the group of smooth maps from the circle $\mathbb{S}^1$ into G. Its multiplication is pointwise multiplication of loops. Loop groups turn out to behave like compact Lie groups to a quite remarkable extent. They have Lie algebras which are related to affine Kač-Moody algebras. The details can be found in [**98**] and [**64**].

The class of cohomology theories which we study here are the complex orientable theories. These are theories with a reasonable theory of characteristic classes for complex vector bundles. Complex cobordism is the universal complex orientable theory. This theory has two descriptions. These are homotopy theoretic and geometric. The geometric description only holds for smooth manifolds.

Some comments about the structure of this thesis are in order. It is written for a reader with a first course in algebraic topology and some understanding of the structure of compact semi-simple Lie groups and their representations, plus some Hilbert space theory and some mathematical maturity. Some good general references are Kač [**60**] for Kač-Moody algebra theory, Pressley-Segal [**84**] for loop groups and their representations, Young [**96**] for Hilbert space theory, Adams [**3**] for complex orientable theories, Husemoller [**56**] and Switzer [**93**] for fiber bundle theory and topology, Ravenel [**87**] for Morava K-theories, Lang [**74**] for the differential topology of infinite dimensional manifolds, Conway [**27**] for Fredholm operator theory.

The organization of this thesis is as follows.

Chapter 1 includes all details about Schubert calculus and cohomology of the flag space G/B for Kač-Moody group G. We examine the finite type flag space in section 1. In the section 2, we give some facts and results about Kač-Moody Lie algebras and associated groups and the construction of dual Schubert cocycles on the flag spaces

by using the relative Lie algebra cohomology tools. The rest of chapter includes cup product formulas and facts about nil-Hecke rings.

Chapter 2 includes the general theory of loop groups. Stratifications and a cell decomposition of Grassmann manifolds and the homogeneous spaces of loop groups are given.

In chapter 3, we discuss the calculation of cohomology rings of LG/T. First we describe the root system and Weyl group of LG, then we give some homotopy equivalences between loop groups and homogeneous spaces, and investigate the cohomology ring structures of LSU_2/T and ΩSU_2. Also we prove that BGG-type operators correspond to partial derivation operators on the divided power algebras.

In chapter 4, we investigate the topological construction of BGG-type operators, giving details about complex orientable theories, Becker-Gottlieb transfer and a formula of Brumfiel-Madsen.

In chapter 5, we develop a version of Quillen's geometric cobordism theory for infinite dimensional separable Hilbert manifolds. For a separable Hilbert manifold X, we prove that this cobordism theory has a graded-group structure under the topological union operation and this theory has push-forward maps. In section 2 of this chapter, we discuss transversal approximations and products, and the contravariant property of this cobordism theory. In section 3, we discuss transversality for finite dimensional fiber bundle. In section 4, we define the Euler class of a finite dimensional complex vector bundle in this cobordism theory and we generalize Bressler-Evens's work on LG/T. In section 6, we prove that strata given in chapter 2 are cobordism classes of infinite dimensional homogeneous spaces. In section 7, we give some examples showing that in certain cases our infinite dimensional theory maps surjectively to complex cobordism.

CHAPTER 1

The Schubert Calculus and Cohomology of the Flag Space G/B for a Kač-Moody Group G

1. Schubert cells and integral cohomology of K/T for a compact Lie group K.

The general reference for this section is [**7**]. Let K be a compact semi-simple simply-connected Lie group. We fix a maximal torus $T \subseteq K$. The complexified Lie algebras of K and T will be denoted by $\mathbf{g}$ and $\mathbf{h}$ respectively. Let $\mathbf{b}$ be the Borel subalgebra of $\mathbf{g}$. The compact group K can be embedded into a complex Lie group G with Lie algebra $\mathbf{g}$. We choose a Borel subgroup B containing T. The analytic complexification $K/T \to G/B$ induces a complex structure on the flag space $K/T \cong G/B$. The flag space K/T will be denoted by X. In this section, the root system will be denoted by Δ, and the simple root system will be denoted by Σ. Δ_+ is the set of positive roots. From [**7**],

Theorem 1.1. *The finite dimensional flag space X is a non-singular complex projective variety.*

We give at this point two descriptions of the homology of X. The first of these makes use of the decomposition of X into cells, while the second involves the realization of two-dimensional cohomology classes as the Chern classes of one-dimensional holomorphic bundles.

Definition 1.2. *Let W be the Weyl group of G. Then the* length *of an element $w \in W$ is the least number of factors in the decomposition relative to the set of the simple reflections r_α, is denoted by $\ell(w)$.*

We know from [**12**] that $N_w = wN^-w^{-1} \cap N$ is a unipotent subgroup of G of (complex) dimension $\ell(w)$, where N is the unipotent radical of B and N^- is the opposite nilpotent subgroup of G. From [**21**], we have

Theorem 1.3. *Let G be a complex reductive Lie group. Then*

$$G = \bigsqcup_{w \in W} BwB.$$

In addition, there is an isomorphism of algebraic varieties

$$N_w \to BwB/B$$

given by $n \to nwB/B$.

Corollary 1.4.

$$X = \bigsqcup_{w \in W} BwB/B.$$

The cells $C_w = BwB/B$ are open and closed varieties in the Zariski topology. Let $\overline{C}_w$ be the closure of C_w in X respect to the usual topology, we have from [**81**],

Theorem 1.5. *Let Y be a projective variety and let Y° be the interior of Y with respect to the Zariski topology. Then the closure of Y° in the usual topology is Y.*

Since $C_w = BwB/B$ is a Zariski-open set, by Theorem 1.5, the closure of C_w coincides with the Zariski closure. $[\overline{C}_w] \in H_{2\ell(w)}(\overline{C}_w, \mathbb{Z})$ is the fundamental cycle of the complex algebraic variety $\overline{C}_w$. Let $s_w \in H_{2\ell(w)}(X, \mathbb{Z})$ be the image of $[\overline{C}_w]$ under the mapping induced by the embedding $\overline{C}_w \hookrightarrow X$.

Proposition 1.6. *The elements s_w form a basis of the free $\mathbb{Z}$-module $H_*(X, \mathbb{Z})$.*

Definition 1.7. *A group W is a* Coxeter group *if there is a subset S of W such that W has the presentation*

$$\langle s \in S : (ss')^{m_{ss'}} = 1 \rangle$$

where $m_{ss'} \in \{2, 3, \dots, \infty\}$ is the order of ss', $s \neq s'$ and $m_{ss} = 1$. The pair (W, S) is called a Coxeter system.

Theorem 1.8. [**50**] *The Weyl group W is a Coxeter group.*

Definition 1.9. *Let $w_1, w_2 \in W$, $\gamma \in \Delta_+$. Then we write $w_1 \xrightarrow{\gamma} w_2$ when $r_\gamma w_1 = w_2$ and $\ell(w_2) = \ell(w_1) + 1$. We put $w < w'$ if there is a chain*

$$w = w_1 \to w_2 \to \cdots \to w_k = w'.$$

This order is called the Bruhat order *on the Weyl group* W.

Here are some properties of this ordering.

Lemma 1.10. *Let* $w = r_{\alpha_1} \cdots r_{\alpha_l}$ *be a reduced decomposition of an element* $w \in W$. *We put* $\gamma_i = r_{\alpha_1} \cdots r_{\alpha_{i-1}}(\alpha_i)$. *Then the roots* $\gamma_1, \ldots, \gamma_l$ *are distinct and the set* $\{\gamma_1, \ldots, \gamma_l\}$ *coincides with* $\Delta_+ \cap w\Delta_-$.

Lemma 1.11. *Let* $w, w' \in W$ *and let* α *be a simple root. Assume that* $w < w'$. *Then, either* $r_\alpha w \leqslant w'$ *or* $r_\alpha w < r_\alpha w'$, *either* $w \leqslant r_\alpha w'$ *or* $r_\alpha w < r_\alpha w'$.

The properties in lemma 1.11 characterize the ordering $<$. From [**7**], we have

Proposition 1.12. *The Bruhat ordering* $<$ *on* W *is a partial order relation.*

Proposition 1.13. *Let* $w \in W$ *and let* $w = r_{\alpha_1} \cdots r_{\alpha_l}$ *be the reduced decomposition of* w. *If*

$$w' = r_{\alpha_{i_1}} \ldots r_{\alpha_{i_k}} \tag{1.1}$$

for $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l$, *then* $w' \leqslant w$. *If* $w' < w$, *then* w' *can be represented in the form 1.1 for some indexing set* $\{i_j\}$. *If* $w' \to w$, *then there is a unique index* i *satisfying* $1 \leqslant i \leqslant l$ *and such that*

$$w' = r_{\alpha_1} \cdots r_{\alpha_{i-1}} r_{\alpha_{i+1}} \cdots r_{\alpha_l}.$$

Proposition 1.13 yields an alternative definition of the ordering on W in [**92**]. The geometrical interpretation of this partial ordering is very interesting and useful in what follows.

Theorem 1.14. *Let* V *be a finite-dimensional irreducible representation of the Lie algebra* $\mathfrak{g}$ *with highest weight* λ *and let* $\mathfrak{n}$ *be the nilpotent part of* $\mathfrak{g}$. *Assume that all the weights* $w\lambda$, $w \in W$, *are distinct and select for each* w *a non-zero* $f_w \in V$ *of weight* $w\lambda$. *Then*

$$w' \leqslant w \iff f_{w'} \in U(\mathfrak{n}) f_w$$

where $U(\mathfrak{n})$ *is the enveloping algebra of Lie algebra* $\mathfrak{n}$.

We use Theorem 1.14 to describe the mutual disposition of the Schubert cells. From [**92**], we have

Theorem 1.15. *Let $w \in W$, $C_w \subseteq X$ a Schubert cell, and $\overline{C}_w$ its closure. Then*

$$C_{w'} \subseteq \overline{C}_w \iff w' \leqslant w.$$

We turn to the other approach to the description of the cohomology of X. For this purpose, we introduce in $\mathbf{h}$ the coroot lattice

$$Q^\vee = \bigoplus_i \mathbb{Z} h_i$$

where h_i is the coroot to dual to $\alpha_i \in \Delta$. We have the weight lattice

$$P = \{\chi \in \mathbf{h}^* : \chi(h_i) \in \mathbb{Z} \text{ for all } \alpha_i \in \Delta\}$$

dual to $Q^\vee$. We set $P_\mathbb{Q} = P \otimes_\mathbb{Z} \mathbb{Q}$. We denote by $\mathbf{h}_\mathbb{Q} \subseteq \mathbf{h}$ the vector space over $\mathbb{Q}$ spanned by the h_i. Let $R = S(P_\mathbb{Q})$ be the graded algebra of polynomial functions on $\mathbf{h}_\mathbb{Q}$ over rational coefficients where the graduation is given by the degree of polynomials. The Weyl group W acts on $\mathbf{h}^*$ by the rule

$$r_{\alpha_i}(\chi) = \chi - \chi(h_i)\alpha_i \quad \text{for } \alpha \in \Sigma \text{ and } \chi \in \mathbf{h}^*.$$

We can extend the action of the Weyl group W on $\mathbf{h}^*$ to R by the rule $wf(h) = f(w^{-1}h)$ for $f \in R$. We denote by R^W the subring of W-invariant elements in R and set

$$R_+^W = \{f \in R^W : f(0) = 0\},$$
$$J = R_+^W R.$$

We want to construct a ring homomorphism $\psi_\mathbb{Q} : R \longrightarrow H^*(X, \mathbb{Q})$ in the following way. First let $\chi \in P$. Since G is simply-connected, there is a character $\theta \in \mathrm{Hom}(B, \mathbb{C}^*)$ such that $\theta(\exp b) = \exp \chi(b)$, for $b \in \mathbf{b}$, where $\exp : \mathbf{b} \to B$ is an exponential map which is a locally diffeomorphism. Since $G \to X$ is a principal bundle with structure group B, this θ defines a one-dimensional complex holomorphic line bundle

$$L_\chi = \{[g, \zeta] : [g \exp(b), \exp \chi(b)\zeta] = [g, \zeta] \text{ for } b \in \mathbf{b},\ g \in G \text{ and } \zeta \in \mathbb{C}\}$$

on X. We set $\psi(\chi) = c_\chi$, where $c_\chi \in H^2(X, \mathbb{Z})$ is the first Chern class of L_χ. Then ψ is a group homomorphism of P into $H^2(X, \mathbb{Z})$, which extends naturally to a homomorphism of graded rings

$$\psi_\mathbb{Q} : R \to H^*(X, \mathbb{Q}).$$

From [**10**, **4**], we have

Proposition 1.16. *The homomorphism $\psi_{\mathbb{Q}}$ commutes with the action of W on R and $H^*(X,\mathbb{Q})$. $\ker\psi_{\mathbb{Q}} = J$ and the natural mapping $\overline{\psi_{\mathbb{Q}}} : R/J \to H^*(X,\mathbb{Q})$ is an isomorphism.*

We now study the rings R and $\overline{R} = R/J$. For each $w \in W$, we define an element $P_w \in \overline{R}$ and a functional D_w on R and investigate their properties. In the next section, we shall show that the D_w correspond to Schubert cell and that the P_w yield a basis, dual to the Schubert cell basis, for the rational cohomology of X. Let $\gamma \in \Delta$. We specify an operator A_γ on R by the rule

$$A_\gamma f = \frac{f - r_\gamma f}{\gamma}.$$

$A_\gamma f$ lies in R, since $f - r_\gamma f = 0$ on the hyperplane $\gamma = 0$ in $\mathbf{h}_{\mathbb{Q}}$. A_i will be called the *Bernstein-Gelfand-Gelfand operator* and it will be briefly indicated by *BGG-operator.* The properties of the A_γ are described in the following lemma.

Proposition 1.17. *For $\gamma \in \Delta$ and $w \in W$, we have*

$$A_{-\gamma} = -A_\gamma,$$

$$A_\gamma^2 = 0,$$

$$wA_\gamma w^{-1} = A_{w\gamma},$$

$$r_\gamma A_\gamma = -A_\gamma r_\gamma = A_\gamma,$$

$$r_\gamma = -\gamma A_\gamma + 1 = A_\gamma \gamma - 1,$$

$$A_\gamma f = 0 \iff r_\gamma f = f,$$

$$A_\gamma J \subseteq J.$$

Proposition 1.18. *Let $\chi \in \mathbf{h}_{\mathbb{Q}}^*$. Then the commutator of A_γ with the operator of multiplication by χ has the form $[A_\gamma, \chi] = \chi(h_\gamma) r_\gamma$.*

The following property of the BGG-operator A_γ is fundamental in what follows. From [**7**], we have

Theorem 1.19. *Let $\alpha_1, \dots, \alpha_l \in \Sigma$. We put $w = r_{\alpha_1} \cdots r_{\alpha_l}$ and $A_{(\alpha_1, \cdots, \alpha_l)} = A_{\alpha_1} \cdots A_{\alpha_l}$. If $\ell(w) < l$, then $A_{(\alpha_1, \cdots, \alpha_l)} = 0$. If $\ell(w) = l$, then $A_{(\alpha_1, \cdots, \alpha_l)}$ depends only on w and not on the set $\alpha_1, \cdots, \alpha_l$; in this case we put $A_w = A_{(\alpha_1, \cdots, \alpha_l)}$.*

Proposition 1.20. *The operators A_w satisfy the following commutator relation:*

$$[w^{-1}A_w, \chi] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)w^{-1}A_{w'},$$

where h_γ is a coroot.

We put $S_i = R_i^*$, where $R_i \subseteq R$ is the space of homogeneous polynomials of degree i and R_i^* is the dual space of R_i and $S = \bigoplus_i S_i$. We denote by (,) the natural pairing $S \times R \to \mathbb{Q}$. Then W acts naturally on the graded ring S.

Definition 1.21. *For any $\chi \in \mathbf{h}_{\mathbb{Q}}^*$, we let χ^* denote the transformation of S adjoint to the operator of multiplication by χ in R. We denote by $F_\gamma : S \to S$ the linear transformation adjoint to $A_\gamma : R \to R$.*

The next lemma gives an explicit description of the F_γ.

Lemma 1.22. *Let $\gamma \in \Delta$. For any $D \in S$ there is a $\widetilde{D} \in S$ such that $\chi^*(\widetilde{D}) = D$. If $\widetilde{D}$ is any such operator, then $\widetilde{D} - r_\gamma\widetilde{D} = F_\gamma(D)$, in particular, the left-hand side of this equation does not depend on the choice of $\widetilde{D}$.*

It is often convenient to interpret S as a ring of differential operators on $\mathbf{h}$ with constant rational coefficients. Then the pairing (,) is given by the formula $(D, f) = (Df)(0)$, $D \in S$, $f \in R$. Also, it is easy to check that $\chi^*(D) = [D, \chi]$, where $\chi \in \mathbf{h}_{\mathbb{Q}}$ and $D \in S$ are regarded as operators on R .

Theorem 1.23. *Let $\alpha_1, \dots, \alpha_l \in \Sigma$. We put $w = r_{\alpha_1} \cdots r_{\alpha_l}$. If $\ell(w) < l$, then $F_{\alpha_l} \cdots F_{\alpha_1} = 0$. If $\ell(w) = l$, then $F_{\alpha_l} \cdots F_{\alpha_1}$ depends only on w and not on $\alpha_1 \dots \alpha_l$ and in this case we write $F_w = F_{\alpha_l} \cdots F_{\alpha_1}$. Also, $F_w = A_w^*$ and*

$$[\chi^*, F_w w] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)F_{w'}w,$$

where h_γ is a coroot.

We set $D_w = F_w(1)$. As we shall show in the next section, the functionals D_w correspond to the Schubert cells in $H_*(X, \mathbb{Q})$ in the sense that $(D_w, f) = \langle s_w, \psi_{\mathbb{Q}}(f)\rangle$ for all $f \in R$. The properties of the D_w are listed in the following theorem.

Theorem 1.24. *Let $w \in W$, let α be a simple root and $\chi, \chi_1, \ldots, \chi_l \in \mathbf{h}^*_{\mathbb{Q}}$ and $D_w \in S_{\ell(w)}$. Then*

$$F_\alpha D_w = \begin{cases} 0 & \text{if } \ell(wr_\alpha) < \ell(w), \\ D_{wr_\alpha} & \text{if } \ell(wr_\alpha) \geqslant \ell(w). \end{cases}$$

$$\chi^*(D_w) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma) D_{w'}.$$

$$r_\alpha D_w = \begin{cases} -D_w & \text{if } \ell(wr_\alpha) < \ell(w), \\ -D_w + \displaystyle\sum_{w' \xrightarrow{\gamma} wr_\alpha} w'\alpha(h_\gamma) D_{w'} & \text{if } \ell(wr_\alpha) \geqslant \ell(w). \end{cases}$$

$$(D_w, \chi_1 \cdots \chi_l) = \sum \chi_1(h_{\gamma_1}) \cdots \chi_l(h_{\gamma_l}),$$

where the summation extends over all chains

$$e \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_1} w_2 \rightarrow \cdots \xrightarrow{\gamma_l} w_l = w^{-1}.$$

Let $J^\perp$ be the subspace of S orthogonal to the ideal $J \subseteq R$. It follows from lemma 1.18 that $J^\perp$ is invariant with respect to all the F_γ. It is also clear that $1 \in J^\perp$. Thus, $D_w \in J^\perp$ for all $w \in W$. From **[7]**, we have

Theorem 1.25. *For $w \in W$, the functionals D_w form a basis for $J^\perp$.*

The form (,) gives rise to a non-degenerate pairing between $\overline{R}$ and $J^\perp$. Let $\{P_w\}_{w \in W}$ be the basis of $\overline{R}$ dual to $\{D_w\}_{w \in W}$. The following properties of the P_w are immediate consequences of Theorem 1.24

Theorem 1.26. *Let $w \in W$, let α be a simple root and $\chi \in \mathbf{h}^*_{\mathbb{Q}}$. Then*

$$A_\alpha P_w = \begin{cases} 0 & \text{if } \ell(wr_\alpha) > \ell(w), \\ P_{wr_\alpha} & \text{if } \ell(wr_\alpha) \leqslant \ell(w). \end{cases}$$

$$\chi P_w = \sum_{w \xrightarrow{\gamma} w'} w\chi(h_\gamma) P_{w'}.$$

$$r_\alpha P_w = \begin{cases} P_w & \text{if } \ell(wr_\alpha) > \ell(w), \\ P_w - \sum_{wr_\alpha \xrightarrow{\gamma} w'} w\alpha(h_\gamma) P_{w'} & \text{if } \ell(wr_\alpha) \leqslant \ell(w). \end{cases}$$

From Theorem 1.26, it is clear that all the P_w can be expressed in terms of P_s, where $s \in W$ is the unique element of maximal length, $r = \ell(s)$. More precisely, let $w = r_{\alpha_1} \cdots r_{\alpha_l}$, $\ell(w) = l$. Then

$$P_w = A_{\alpha_l} \dots A_{\alpha_1} P_s.$$

To find an explicit form for the P_w it therefore suffices to determine the $P_s \in \overline{R}$. From [7], we have

Theorem 1.27. $P_s = \frac{1}{|W|} \prod_{\gamma \in \Delta_+} \gamma \pmod{J}$.

We now give some results on products of the P_w in $\overline{R}$.

Theorem 1.28. *Let α be a simple root and let $w \in W$. Then*

$$P_{r_\alpha} P_w = \sum_{w \xrightarrow{\gamma} w'} \chi_\alpha(h_{w^{-1}\gamma}) P_{w'},$$

where $\chi_\alpha \in \mathbf{h}^*_{\mathbb{Z}}$ is the fundamental weight corresponding to the root α. Let $w_1, w_2 \in W$ and satisfying $\ell(w_1) + \ell(w_2) = r$. If $w_2 \neq w_1 s$, and $P_{w_1} P_{w_1 s} = P_s$, then $P_{w_1} P_{w_2} = 0$. Let $w \in W$ and $f \in \overline{R}$. Then

$$f P_w = \sum_{w' \geqslant w} c_{w'} P_{w'}$$

If $w_1 \nleqslant w_2 s$, then $P_{w_1} P_{w_2} = 0$

We define the operator $\mathcal{P} : \overline{R} \to J^\perp$ of Poincaré duality by the formula

$$(\mathcal{P} f)(g) = D_s(fg),\ f, g \in \overline{R},\ \mathcal{P} f \in J^\perp.$$

Corollary 1.29. $\mathcal{P} P_w = D_w$.

We will show that the functionals $D_w, w \in W$ introduced in last section correspond to Schubert cells $s_w, w \in W$ and give the cap-product formula in the cohomology of flag variety K/T. Let $s_w \in H_*(X, \mathbb{Q})$ be a Schubert cell. It gives rise to a linear functional on $H^*(X, \mathbb{Q})$, which, by means of the ring homomorphism $\psi_{\mathbb{Q}} : R \to H^*(X, \mathbb{Q})$, can be regarded as a linear functional on R. This functional takes the value 0 on all homogeneous components P_k with $k \neq \ell(w)$, and thus determines an element $\widehat{D}_w \in S_{\ell(w)}$. From [7], we have

Theorem 1.30. $\widehat{D}_w = D_w$

This theorem is a natural consequence of the next two propositions.

Proposition 1.31. $\widehat{D}_e = 1$, *and for any* $\chi \in \mathbf{h}_{\mathbb{Z}}^*$

$$\chi^*(\widehat{D}_w) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)\widehat{D}_w.$$

Proposition 1.32. *Suppose that for each* $w \in W$ *we are given an element* $\widehat{D}_w \in S_{\ell(w)}$*, with* $\widehat{D}_e = 1$*, for which Proposition 1.31 holds for any* $\chi \in \mathbf{h}_{\mathbb{Z}}^*$*. Then* $\widehat{D}_w = D_w$*.*

For any topological space Y there is a bilinear mapping called the cap-product

$$\cap : H^i(Y, \mathbb{Q}) \times H_j(Y, \mathbb{Q}) \to H_{j-i}(Y, \mathbb{Q}).$$

such that

$$\langle c \cap y, z \rangle = \langle y, c.z \rangle$$

for all $y \in H_j(Y, \mathbb{Q})$, $z \in H^{j-i}(Y, \mathbb{Q})$, $c \in H^i(Y, \mathbb{Q})$. If $f : Y_1 \to Y_2$ is a continuous mapping, then

$$f_*(f^*c \cap y) = c \cap f_*y$$

for all $y \in H_*(Y_1, \mathbb{Q})$, $c \in H^*(Y_2, \mathbb{Q})$. Then, we have for any $\chi \in \mathbf{h}_{\mathbb{Z}}^*$, $f \in R$

$$(\chi^*(\widehat{D}_w), f) = (\widehat{D}_w, \chi f) = \langle s_w, \psi(\chi)\psi_{\mathbb{Q}}(f) \rangle = \langle s_w \cap \psi(\chi), \psi_{\mathbb{Q}}(f) \rangle.$$

Therefore, Proposition 1.31 is equivalent to the following geometrical fact.

Proposition 1.33. *For all* $\chi \in \mathbf{h}_{\mathbb{Z}}^*$

$$s_w \cap \psi(\chi) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)s_{w'}.$$

We restrict the one complex dimensional holomorphic line bundle L_χ to $\overline{C}_w \subseteq X$ and let $c_\chi \in H^2(\overline{C}_w, \mathbb{Q})$ be the first Chern class of L_χ. Then, it is sufficient to prove that

$$s_w \cap c_\chi = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma) s_{w'} \tag{1.2}$$

in $H_{2\ell(w)-2}(\overline{C}_w, \mathbb{Q})$. To prove Equation 1.2, we use the following lemma, which can be verified by standard arguments involving relative Poincaré duality, see [**43**].

Lemma 1.34. *Let Y be a compact complex analytic space of dimension n, such that the codimension of the space of singularities of Y is greater than 1. Let E be an analytic vector bundle on Y, and $c \in H^2(Y, \mathbb{Q})$ the first Chern class of E. Let μ be a non-zero analytic section of E and $\sum_i m_i Y_i = \mathrm{div}\mu$ the divisor of μ. Then $[Y] \cap c = \sum_i m_i [Y_i] \in H_{2n-2}(Y, \mathbb{Q})$ where $[Y]$ and $[Y_i]$ are the fundamental classes of Y and Y_i.*

Let $w \in W$, and let $C_w \subseteq X$ be the corresponding the Schubert cell. Then,

Proposition 1.35. *Let $w' \xrightarrow{\gamma} w$. Then $\overline{C}_w$ is non-singular at points $x \in C_{w'}$.*

We now give another proposition to prove Proposition 1.33

Proposition 1.36. *There is a section μ of the fibering E_χ over $\overline{C}_w$ such that*

$$\mathrm{div}\mu = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma)\overline{X}_{w'}.$$

2. Differential operators, Lie algebra cohomology and generalized Schubert cocycles.

First, we will give some facts about Kač-Moody Lie algebras and associated groups which will be used in this section. The general reference is [**60**] of V. Kač.

Definition 1.37. *Let $A = \{a_{ij}\}_{n\times n}$ be a complex matrix of rank l. A* realization *of A is a triple (h, π, π^V), where h is a complex vector space of dimension $n +$ corank A, $\pi = \{\alpha_i\}_{1 \leqslant i \leqslant n} \subseteq \mathrm{h}^*$ and $\pi^V = \{h_i\}_{1\leqslant i \leqslant n} \subseteq \mathrm{h}$ are free indexed sets satisfying $\alpha_j(h_i) = a_{ij}$.*

Definition 1.38. *Two realizations* $(\mathbf{h}, \pi, \pi^V)$ *and* $(\mathbf{h}_1, \pi_i, \pi_1^V)$ *are called* isomorphic *if there exists a vector space isomorphism* $\varphi : \mathbf{h} \to \mathbf{h}_1$ *such that* $\varphi(\pi^V) = \pi_1^V$ *and* $\varphi^*(\pi_1) = \pi$.

From **[60]**, we have

Theorem 1.39. *There exists a unique up to isomorphism realization for every* $n \times n$ *matrix.*

Definition 1.40. *A* generalized Cartan matrix $A = \{a_{ij}\}_{n\times n}$ *is a matrix of integers satisfying* $a_{ii} = 2$ *for all* i *and* $a_{ij} \leqslant 0$ *if* $i \neq j$, $a_{ij} = 0$ *implies* $a_{ji} = 0$.

Definition 1.41. *Given a realization* $(\mathbf{h}, \pi, \pi^V)$ *of a* $n \times n$ *generalized Cartan matrix* A*, the* Kač-Moody algebra $\mathbf{g} = \mathbf{g}(A)$ *is the Lie algebra over* $\mathbb{C}$*, generated by* $\mathbf{h}$ *and the elements* e_i *and* f_i *for* $1 \leqslant i \leqslant l$ *such that this basis elements satisfy the following relations:*

$$[\mathbf{h}, \mathbf{h}] = 0,\ [h, e_i] = \alpha_i(h)e_i,\ [h, f_i] = -\alpha_i(h)f_i$$

for $h \in \mathbf{h}$ *and all* $1 \leqslant i \leqslant l$*;* $[e_i, f_i] = \delta_{ij}h_j$ *for all* $1 \leqslant i, j \leqslant n$*;*

$$(\text{ad } e_i)^{1-a_{ij}}(e_j) = 0 = (\text{ad } f_i)^{1-a_{ij}}(f_j)$$

for all $1 \leqslant i \neq j \leqslant n$*. The elements* h_i, e_i, f_i *are called* Chevalley generators *and the subalgebra* $\mathbf{h}$ *of* $\mathbf{g}(A)$ *is called the* Cartan subalgebra.

The Kač-Moody algebra $\mathbf{g} = \mathbf{g}(A)$ has a root space decomposition.

Theorem 1.42. *For* $0 \neq \alpha$*, the root space* $\mathbf{g}_\alpha = \{x \in \mathbf{g} : [h, x] = \alpha(h)x, \forall\, h \in \mathbf{h}\}$ *is finite dimensional and there is a root space decomposition*

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\alpha\in\Delta} \mathbf{g}_\alpha = \mathbf{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathbf{g}_\alpha \oplus \bigoplus_{\alpha\in\Delta^-} \mathbf{g}_\alpha$$

where Δ^+ *(resp.* Δ^-*) is the positive (resp. negative) root system.*

We define *fundamental reflections* $r_i \in \text{Aut}_{\mathbb{C}}(\mathbf{h})$, $1 \leqslant i \leqslant n$, by $r_i(h) = h - \alpha_i(h)h_i$. They generate the *Weyl group* W, which is a Coxeter group on $\{r_i\}_{1\leqslant i\leqslant n}$.

We define the following Lie algebras.

$$\mathbf{n} = \bigoplus_{\alpha\in\Delta^+} \mathbf{g}_\alpha, \quad \mathbf{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathbf{g}_\alpha.$$

Then, $\mathbf{g} = \mathbf{h} \oplus \mathbf{n} \oplus \mathbf{n}^-$, where $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}$ is called the *Borel algebra*. We have a unique complex linear involution ω of $\mathbf{g}$ defined by $\omega(f_i) = e_i$ for all $1 \leqslant i \leqslant l$ and $\omega(h) = -h$ for $h \in \mathbf{h}$. The involution ω leaves the real points of $\mathbf{g}$ stable. ω is called the Chevalley involution. Also, we have a unique conjugate linear involution ω_0 which agrees with ω on the real points of $\mathbf{g}$. We can define a nondegenerate $\mathbf{g}$-invariant, symmetric complex bilinear form σ on $\mathbf{h}^*$ such that $\sigma(\alpha_i, \alpha_j) = \langle h_{\alpha_i}, h_{\alpha_j} \rangle$ where $\langle \, , \rangle$ is the standard complex inner product on $\mathbf{g}$. This form is called the *Killing form*. This gives a Hermitian form $\{ \, , \}$ on $\mathbf{g}$ defined by $\{x, y\} = -\langle x, \omega_0(y) \rangle$ for $x, y \in \mathbf{g}$.

Now, we will mention the highest weight module category of a Kač-Moody algebra $\mathbf{g}$. The fundamental reference is [**60**] of V. Kač. Let V be a $\mathbf{g}$-module, $\lambda \in \mathbf{h}^*$. We define

$$V_\lambda = \{x \in V : h \cdot x = \lambda(h) x \text{ for } \forall h \in \mathbf{h}\}.$$

Then, V_λ is a subspace of V. If $V_\lambda \neq 0$, λ is called a *weight* of the $\mathbf{g}$-module V, V_λ the *weight space* corresponding to λ, and $\dim V_\lambda$ the *multiplicity* of λ. If λ is a weight of V, then any non-zero vector of V_λ is called a *weight vector* of λ. We denote by

$$P(V) = \{\lambda \in \mathbf{h}^* : V_\lambda \neq 0\},$$

the set of weights of the $\mathbf{g}$-module V.

Lemma 1.43. *For any $\alpha \in \Delta \cup \{0\}$ and $\lambda \in \mathbf{h}^*$, we have*

$$\mathbf{g}_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}.$$

We set

$$D(\lambda) = \{\lambda - \alpha : \alpha \in Q_+\},$$

where $Q_+ = \bigoplus_i \mathbb{Z}_+ \alpha_i$. For any subset $F \subseteq \mathbf{h}^*$, we define

$$D(F) = \bigcup_{\lambda \in F} D(\lambda).$$

We can define a partial ordering $\geqslant$ on $\mathbf{h}^*$ by

$$\lambda \geqslant \mu \quad \Longleftrightarrow \quad \lambda - \mu \in Q_+ \quad \Longleftrightarrow \quad \mu \in D(\lambda).$$

We will give the definition of category $\mathcal{O}$ of $\mathbf{g}$-modules.

Definition 1.44. *The objects of $\mathcal{O}$ are* $\mathbf{g}$*-modules V which satisfy the conditions*

1. *V is* $\mathbf{h}$*-diagonalizable, i.e.,*

$$V = \bigoplus_{\lambda \in \mathbf{h}^*} V_\lambda,$$

2. $\dim V_\lambda < \infty$ *for all* $\lambda \in \mathbf{h}^*$,

3. *there exists a finite set $F \subseteq \mathbf{h}^*$ such that $P(V) \subseteq D(F)$, and whose morphisms are* $\mathbf{g}$*-module homomorphisms.*

By the property 3 of the definition of $\mathcal{O}$, we have

Proposition 1.45. *Every non-zero* $\mathbf{g}$*-module in $\mathcal{O}$ has at least one maximal weight.*

Definition 1.46. *A* $\mathbf{g}$*-module V is called a* highest weight module, *if V has a unique maximal weight Λ and V is generated by some weight vector $v_\Lambda \in V_\Lambda$.*

Theorem 1.47. *Let V be a highest weight module with maximal weight Λ. Then,*

$$V = U(\mathbf{g}) \cdot v_\Lambda = U(\mathbf{n}^-) \cdot v_\Lambda$$

for any weight vector v_Λ of Λ; $V \in \mathcal{O}$, $\dim V_\Lambda = 1$ *and $P(V) \subseteq D(\Lambda)$; $\mathbf{g}_\alpha \cdot V_\Lambda = 0$ for any $\alpha \in \Delta_+$; V has a unique maximal submodule, hence a unique quotient simple module; the homomorphic image of V is also a highest weight module with maximal weight Λ.*

Since $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}^+$, we can regard V_Λ as a $\mathbf{b}$-module with $\mathbf{n}^+$ acting on it trivially. We define

$$M(\Lambda) = U(\mathbf{g}) \otimes_{U(\mathbf{b})} V_\Lambda$$

where V_Λ is 1-dimensional weight space corresponding to the weight Λ. The g-module $M(\Lambda)$ is called *Verma module* corresponding to the weight Λ.

Theorem 1.48. *The Verma module $M(\Lambda)$ is a highest weight module with highest weight Λ. Any highest weight module with highest weight Λ is a homomorphic image of the Verma module $M(\Lambda)$. $M(\Lambda)$ has a unique maximal submodule with simple quotient $L(\Lambda)$. Any irreducible highest weight module with highest weight Λ is isomorphic to $L(\Lambda)$.*

Now, we will give the definition of lowest weight module. Let $L(\Lambda)$ be an irreducible highest weight module with the highest weight Λ. Let $L(\Lambda)^*$ be the $\mathbf{g}$-module contragredient to $L(\Lambda)$. Then

$$L(\Lambda)^* = \prod_{\lambda \in \mathbf{h}^*} L(\Lambda)^*_\lambda.$$

The subspace

$$L^*(\Lambda) = \bigoplus_{\lambda \in \mathbf{h}^*} L(\Lambda)^*_\lambda$$

is a submodule of the $\mathbf{g}$-module $L(\Lambda)^*$. The module $L^*(\Lambda)$ is irreducible and for $v \in L(\Lambda)^*_\lambda$, we have

$$\mathbf{n}_- \cdot v = 0 \quad \text{and} \quad h \cdot v = -\Lambda(h)v \,\text{for}\; h \in \mathbf{h}.$$

The module $L^*(\Lambda)$ is called an *irreducible module with lowest weight* $-\Lambda$.

Theorem 1.49. *There is a bijection between* $\mathbf{h}^*$ *and irreducible lowest weight modules given by* $\Lambda \to L^*(-\Lambda)$.

We denote by π_Λ the action of $\mathbf{g}$ on $L(\Lambda)$. We give a new action π^*_Λ on the space $L(\Lambda)$ by

$$\pi^*_\Lambda(g)v = \pi_\Lambda(\omega(g))v,$$

where ω is the Chevalley involution of $\mathbf{g}$. $(L(\Lambda), \pi^*_\Lambda)$ is an irreducible $\mathbf{g}$-module with lowest weight $-\Lambda$. By the uniqueness of irreducible lowest weight modules with the lowest weight $-\Lambda$, this module can be identified with $L^*(\Lambda)$.

Definition 1.50. *A* $\mathbf{g}$*-module* L *is called* quasi-simple *if it is a highest weight module with highest weight vector* x_0 *such that there exists* $n \in \mathbb{Z}_+$ *with* $f_i{}^n(x_0) = 0$ *for all* $1 \leqslant i \leqslant i$.

From **[41]**, we have

Proposition 1.51. *The quasi-simple* $\mathbf{g}$*-modules are indexed by the positive integral weights.*

We will denote by $L(\lambda)$ the quasi-simple module with highest weight λ. We will denote the derived algebra $[\mathbf{g}, \mathbf{g}]$ by $\mathbf{g}'$. From **[98]**, we have

Theorem 1.52. g' *is the subalgebra of* g *generated by the Chevalley generators* e_i *anf* f_i *for* $1 \leqslant i \leqslant l$ *and we have the decomposition*

$$\mathrm{g}' = \mathrm{h}' \oplus \mathrm{n} \oplus \mathrm{n}^-,$$

where $\mathrm{h}' = \mathrm{g}' \cap \mathrm{h}$.

Definition 1.53. *A* g'*-module* (V, π) *is called* integrable *if* $\pi(e)$ *is locally nilpotent whenever* $e \in \mathrm{g}_\alpha$ *for any real root.*

Let G^* be the free product of the additive groups $\{\mathrm{g}_\alpha\}_{\alpha \in \Delta_{\mathrm{re}}}$ with canonical inclusions $i_\alpha : \mathrm{g}_\alpha \to G^*$. For any integrable g'-module (V, π), we define a homomorphism $\pi^* : G^* \to \mathrm{Aut}_{\mathbb{C}}(V)$ by $\pi^*(i_\alpha(e)) = \exp(\pi(e))$ for $e \in \mathrm{g}_\alpha$. Let N^* be the intersection of all $\ker \pi^*$ and let $q : G^* \to G^*/N^*$ be the canonical homomorphism. We put $G = G^*/N^*$. The next result comes from [**63**].

Proposition 1.54. *G is an algebraic group in the sense of Safarevič.*

We call G the *group associated to the Kač-Moody Lie algebra* g. G may be of three different types: finite, affine and wild. The finite type Kač-Moody groups are simply-connected semi-simple finite dimensional algebraic groups as introduced in the Section 1. The affine type Kač-Moody groups are the circle group extension of the group of polynomial maps from $\mathbb{S}^1$ to a group of finite type, or a twisted analogue. There is no concrete realization of the wild type groups. Now, we will introduce some subgroups of the Kač-Moody group G. For $e \in \mathrm{g}_\alpha$, we put $\exp(e) = q(i_\alpha(e))$ so that $U_\alpha = \exp \mathrm{g}_\alpha$ is an additive one parameter subgroup of G. We denote by U (resp. U^-) the subgroup of G generated by the U_α (resp. $U_{-\alpha}$) for $\alpha \in \Delta_+$. For $1 \leqslant i \leqslant l$, there exists a unique homomorphism $\varphi_i : SL_2(\mathbb{C}) \to G$, satisfying $\varphi \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \exp(ze_i)$ and $\varphi \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \exp(zf_i)$ for all $z \in \mathbb{C}$. We define

$$H_i = \left\{ \varphi \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^* \right\};$$

$G_i = \varphi(SL_2(\mathbb{C}))$. Let N_i be the normalizer of H_i in G_i, H the subgroup of G generated by all H_i and N the subgroup of G generated by all N_i. There is an isomorphism

$W \to N/H$. We put $B = HU$. B is called *standard Borel subgroup* of G. Also, we can define the negative Borel subgroup B^- as $B^- = HU^-$. G has Bruhat and Birkhoff decompositions. Details can be found in [**62**]. The conjugate linear involution ω_0 of g gives to an involution $\tilde{\omega_0}$ on G. Let K denote the set of fixed points of this involution. K is called the *standard real form* of G. Also, this involution preserves the subgroups G_i, H_i and H; we denote by K_i, T_i and T respectively the corresponding fixed point subgroups. Then, $K_i = \varphi(SU_2)$ and

$$T_i = \left\{ \varphi \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} : |u| = 1 \right\}$$

is a maximal torus of K_i and $T = \prod T_i$ is a maximal torus in K.

Now, we will give some facts about the topology of K. Let D (resp. D°) be the unit disk (resp. its interior) in $\mathbb{C}$ and let $\mathbb{S}^1$ be the unit circle. Given $u \in D$, let

$$z(u) = \begin{pmatrix} u & (1-|u|^2)^{1/2} \\ -(1-|u|^2)^{1/2} & \overline{u} \end{pmatrix} \in SU_2,$$

and $z_i(u) = \varphi_i(z(u))$. We also set

$$Y_i = \{z_i(u) : u \in D^\circ\} \subseteq K_i.$$

Let $w = r_{i_1} \cdots r_{i_n}$ be a reduced expression of $w \in W$. We put $Y_w = Y_{i_1} \cdots Y_{i_n}$. We have a fibration $\pi : K \to K/T$. The topological space K/T is called the *flag variety* of the K and G. Already we have given the topology of finite dimensional flag variety in the Section 1 of this chapter. Now, we will give the topological structure in the infinite dimensional case. We define $C_w = \pi(Y_w)$. From [**61**], we have

Proposition 1.55. *The decomposition*

$$K/T = \coprod_{w \in W} C_w$$

defines a CW structure on K/T.

The closure of C_w is given by

$$\overline{C}_w = \coprod_{w' \leqslant w} C_{w'}$$

The closures $\overline{C}_w$ are called Schubert varieties and they are finite dimensional complex spaces. The infinite type flag variety K/T is the inductive limit of these spaces and by Iwasawa decomposition in [**62**], we have a homeomorphism $K/T \to G/B$. From [**61**], we have

Proposition 1.56. *The flag variety K/T is an infinite dimensional complex projective variety.*

Proposition 1.57. *The elements $\overline{C}_w$ are a basis form of free $\mathbb{Z}$-module $H_*(K/T, \mathbb{Z})$.*

Now we will give the construction of the dual Schubert cocycles on the flag variety by using the relative Lie-algebra cohomology tools. This construction was done by B. Kostant in [**67**] for finite type and extended by S. Kumar in [**72**] for the Kač-Moody case.

First, we will introduce some notations for this section. Reference for the notations is [**53**]. By $\Lambda(V)$, we denote the exterior algebra on a $\mathbf{g}$-module V. For a Lie-algebra pair $(\mathbf{g}, \mathbf{h})$ and a left $\mathbf{g}$-module M, let $\Lambda(\mathbf{g}, \mathbf{h}, M^t)$ denote the standard chain complex with coefficients in the right module M^t, where M^t is the right $\mathbf{g}$-module, whose underlying space is M and on which $\mathbf{g}$ acts by the rule $m \cdot g = -g \cdot m$ for all $g \in \mathbf{g}$ and $m \in M$, and let $C(\mathbf{g}, \mathbf{h}, M)$ denote the standard cochain complex with coefficients in M.

Let L_λ be the quasi-simple $\mathbf{g}$-module with highest weight λ. Then, there is an invariant positive definite Hermitian form $\{,\}$ on $\Lambda(\mathbf{n}^-) \otimes L_\lambda$ due to V. Kač and D. Peterson [**62**]. Let $\partial : \Lambda(\mathbf{n}^-) \otimes L_\lambda \to \Lambda(\mathbf{n}^-) \otimes L_\lambda$ be the differential of degree -1 of the chain complex $\Lambda(\mathbf{n}^-, {L_\lambda}^t)$. We denote the adjoint of ∂ respect to $\{,\}$ by ∂^*. $\Lambda(\mathbf{n}^-) \otimes L_\lambda$ is a $\mathbf{h}$-module and ∂ is a $\mathbf{h}$-module map, as is ∂^*. We define the *Laplacian* by $\nabla = \partial\partial^* + \partial^*\partial$. We know from [**41**] that $\Lambda(\mathbf{n}^-) \otimes L_\lambda$ decomposes as a direct sum of finite dimensional irreducible $\mathbf{h}$-modules. From [**72**], we have

Theorem 1.58. *Let $\mathbf{g}$ be the Kač-Moody Lie algebra and let L_λ be the quasi-simple $\mathbf{g}$-module with the highest weight λ. Then the action of ∇ on $\Lambda(\mathbf{n}^-, {L_\lambda}^t)$ is as follows. Let S_β be an irreducible $\mathbf{h}$-submodule of $\Lambda(\mathbf{n}^-, {L_\lambda}^t)$ with highest weight β, then ∇ reduces to a scalar operator on S_β and the scalar is*

$$\frac{1}{2}[\sigma(\lambda+\rho, \lambda+\rho) - \sigma(\beta+\rho, \beta+\rho)]$$

where ρ is the sum of all simple roots.

From [**41**], we have

Theorem 1.59. *With the notations as in Theorem 1.58, the $2j$th homology space $H_{2j}(\mathrm{n}^-, L_\lambda{}^t)$ is finite dimensional and it is isomorphic as an* h*-module to the direct sum*

$$\bigoplus_{\ell(w)=j} M_{(w(\lambda+\rho)-\rho)}$$

of non-isomorphic irreducible h*-modules.*

$\mathcal{C}(\mathbf{g},\mathbf{h})$ denotes the standard cochain complex with differential d associated to the Lie algebra pair $(\mathbf{g},\mathbf{h})$ with trivial coefficients. That is, $\mathcal{C}(\mathbf{g},\mathbf{h})$ is defined to be $\sum_{s\geqslant 0} \mathrm{Hom}_{\mathbf{h}}(\Lambda^s(\mathrm{g}/\mathrm{h}),\mathbb{C})$ such that $\mathbf{h}$ acts trivially on $\mathbb{C}$. We define

$$\widetilde{\mathcal{C}} = \sum_{s\geqslant 0} \widetilde{\mathcal{C}}^s$$

where $\widetilde{\mathcal{C}}^s = \mathrm{Hom}_{\mathbb{C}}(\Lambda^s(\mathrm{g}/\mathrm{h}),\mathbb{C})$. We put the topology of pointwise convergence on $\widetilde{\mathcal{C}}^s$, i.e., $f_n \to f$ in $\widetilde{\mathcal{C}}^s$ if and only if $f_n(x) \to f(x)$ in $\mathbb{C}$ with usual topology, for all $x \in \Lambda(\mathrm{g},\mathrm{h})$. From [**26**], we have

Theorem 1.60. $\widetilde{\mathcal{C}}^s$ *is a complete, Hausdorff, topological vector space with respect to the pointwise topology.*

In [**72**], a continous map $\tilde{\partial} : \widetilde{\mathcal{C}}^s \to \widetilde{\mathcal{C}}^{s-1}$, and a cochain map of $\tilde{b}$ on $\widetilde{\mathcal{C}}$ are defined. We define ∂, b to be the restrictions of $\tilde{\partial}$ and $\tilde{b}$ to the subspace $\mathcal{C}(\mathbf{g},\mathbf{h})$. We define the following operators on $\mathcal{C}(\mathbf{g},\mathbf{h})$: $S = d\partial + \partial d$ and $L = b\partial + \partial b$. From [**72**], we have

Proposition 1.61. $\ker S \oplus \operatorname{im} S = \mathcal{C}$.

Theorem 1.62. *d and ∂ on $\mathcal{C}(\mathbf{g},\mathbf{h})$ are disjoint.*

Proposition 1.63 (Hodge type decomposition). *Let V be any vector space and $d, \partial : V \to V$ be two disjoint operators such that $d^2 = \partial^2 = 0$. Further, assume that $\ker S \oplus \operatorname{im} S = V$ where $S = d\partial + \partial d$. Then, $\ker S \to \ker d/\operatorname{im} d$ and $\ker S \to \ker \partial/\operatorname{im} \partial$ are both isomorphisms.*

By the Hodge type decomposition and Proposition 1.61, we have

Theorem 1.64. *The canonical maps* $\psi_{d,S} : \ker S \to H(\mathcal{C}, d)$ *and* $\psi_{\partial,S} : \ker S \to H(\mathcal{C}, \partial)$ *are both isomorphisms.*

Now, we describe a basis for $\ker L$. We fix $w \in W$ of length s. We define $\Phi_w = w\Delta_- \cap \Delta_+$. Φ_w consists of real roots $\{\gamma_1, \cdots, \gamma_s\}$. We pick $y_{\gamma_i} \in \mathbf{g}_{-\gamma_i}$ of unit norm with respect to the form $\{,\}$ and let $x_{\gamma_i} = -\omega_0(y_{\gamma_i})$. Let $M_{(w\rho-\rho)}$ be the irreducible $\mathbf{h}$-submodule with highest weight $(w\rho - \rho)$. By Proposition 2.5 in [**41**], the corresponding highest weight vector is $y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s}$. There exists a unique element $\overline{h^w} \in [M_{(w\rho-\rho)} \otimes \Lambda^s(\mathbf{n})]$ such that $\overline{h^w} = (2i)^s(y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s} \wedge x_{\gamma_1} \wedge \cdots \wedge x_{\gamma_s}) \mod P_w \otimes \Lambda^s(\mathbf{n})$, where P_w is the orthogonal complement of $y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s}$ in $M_{(w\rho-\rho)}$. Using the nondegenerate bilinear form $\langle\,\rangle$ on $\mathbf{g}$, we have embedding

$$e : \bigoplus_{k \geqslant 0} \Lambda^s(\mathbf{n} \oplus \mathbf{n}^-) \to \bigoplus_{k \geqslant 0} [\Lambda^s(\mathbf{n} \oplus \mathbf{n}^-)]^*.$$

Then $h_w = e(\overline{h_w}) \in \ker L$. These elements $\{h_w\}_{w \in W}$ is a $\mathbb{C}$-basis of $\ker L$. Then, we can define $s^w = {\psi_{\partial,S}}^{-1}([h^w]) \in H(\mathcal{C}, \partial)$. From [**72**], [**68**], we have

Theorem 1.65. *Let* $\mathbf{g}$ *be the Kač-Moody Lie algebra and let* G *be the group associated to the Kač-Moody algebra* $\mathbf{g}$ *and* B *be standard Borel subgroup of* G. *Then*

$$\int_{C_{w'}} s^w = \begin{cases} 0 & \text{if } w \neq w', \\ (4\pi)^{2\ell(w)} \displaystyle\prod_{\nu \in w^{-1}\Delta \cap \Delta_+} \sigma(\rho, \nu)^{-1} & \text{if } w = w'. \end{cases}$$

This gives the expression for the d, ∂ harmonic forms $s_0^w = \dfrac{s^w}{d_w}$ which are dual to the Schubert cells where $d_w = \displaystyle\int_{C_w} s^w$.

Theorem 1.66. *(see* [**73**]*)*

$$H(\int) : H^*(\mathbf{g}, \mathbf{h}) \to H^*(G/B, \mathbb{C})$$

is a graded algebra isomorphism.

Let ε^w denote the image of ${s_0}^w$ by the integral map in last theorem. These cohomology classes are dual to the closure of the Schubert cells, hence we have

Theorem 1.67. *The elements ε^w, $w \in W$, form a basis of the $\mathbb{Z}$-module $H^*(G/B,\mathbb{Z})$.*

For a finite type flag variety G/B, ε^w denotes $P_{w^{-1}}$ in the notation of Section 1. Now, we give a cup product formula in the cohomology of any type flag variety G/B. From [**68**],

Theorem 1.68. *Let χ_i be the fundamental weight of G for $1 \leqslant i \leqslant l$. For any simple reflection r_i and any element $w \in W$, and a coroot $\gamma^\vee$,*

$$\varepsilon^{r_i} \cdot \varepsilon^w = \sum_{w \xrightarrow{\gamma} w'} \chi_i(\gamma^\vee)\varepsilon^{w'}.$$

As an analogy of cohomology theory of the finite type flag space G/B, the cohomology of affine type flag space G/B and some operators will be introduced in this section. The fundamental reference is [**61**] of V. Kač.

Let $Q^\vee = \bigoplus_i \mathbb{Z}h_i$, where h_i is coroot, be the coroot lattice and let

$$P = \{\lambda \in \mathbf{h}'^* : \lambda(h_i) \in \mathbb{Z}\}$$

be the weight lattice dual to $Q^\vee$. Let $S(P) = \bigoplus_{j\geqslant 0} S^j(P)$ be the integral symmetric algebra over the lattice P, and $S(P)^+ = \bigoplus_{j>0} S^j(P)$ the augmentation ideal. Given a commutative ring $\mathbb{F}$ with unit, we denote $S(P)_\mathbb{F} = S(P)_\mathbb{F} \otimes_\mathbb{Z} \mathbb{F}$. We define the *characteristic homomorphism* $\psi : S(P) \to H^*(G/B,\mathbb{Z})$ as follows: given $\lambda \in P$, we have the corresponding character of B and the associated line bundle L_λ on G/B. We put $\psi(\lambda) \in H^2(G/B,\mathbb{Z})$ equal to the Chern class of L_λ and we extend this multiplicativity to the whole $S(P)$. We denote by $\psi_\mathbb{F}$ the extension of ψ by linearity to $S(P)_\mathbb{F}$. In order to describe the properties of $\psi_\mathbb{F}$, we define BGG-operator Δ_i for $1 \leqslant i \leqslant l$ on $S(P)$ by

$$\Delta_i(f) = \frac{f - r_i(f)}{\alpha_i}$$

and we extend this by linearity to $S(P)_\mathbb{F}$. We define

$$I_\mathbb{F} = \{f \in S(P)^+_\mathbb{F} : \Delta_{i_1} \cdots \Delta_{i_n}(f) \in S(P)^+_\mathbb{F} \ \forall \text{ sequence } (i_1, \cdots .i_n)\}.$$

Theorem 1.69. *We have $\ker \psi_\mathbb{F} = I_\mathbb{F}$ and $H^*(G/B,\mathbb{F})$ is a free module over $\operatorname{im} \psi_\mathbb{F}$.*

We will introduce certain operators on cohomology of the flag space G/B which are basic tools in the study of this theory. These operators are extension of action of the

BGG-operators Δ_i from the image of ψ to the whole cohomology operators. We know that the Weyl group W acts by right multiplication on K/T and this action induces an action of W on homology and cohomology of flag space. On the other hand, we have a fibration $p_i : K/T \to K/K_iT$ with fibre K_i/T_i. Since the odd degree cohomologies of K_i/T_i and K/K_iT are trivial, then the Leray-Serre spectral sequence of the fibration degenerates after the second term. So, $H^*(K/T, \mathbb{Z})$ is generated by $\operatorname{im} p_i^*$, which is r_i invariant and the element $\psi(\chi_i)$ where χ_i is fundamental weight. We define a $\mathbb{Z}$-linear operator A^i on $H^*(K/T, \mathbb{Z})$ lowering the degree by 2 such that r_i leaves the image of A^i invariant and

$$x - r_i(x) = A^i(x) \cup \psi(\alpha_i)$$

for $x \in H^*(K/T, \mathbb{Z})$. Similarly, we can define homology operators A_i on $H_*(K/T, \mathbb{Z})$ raising the degree by 2 such that $r_i(A_i(v)) = -A_i(v)$ and

$$v + r_i(v) = A_i(v) \cap \psi(\alpha_i)$$

for $v \in H_*(K/T, \mathbb{Z})$. The properties of the actions of the operators A^i (resp. A_i) on the cup product (resp. cap product) in the cohomology (resp. homology) can be found in [**61**]. Now, we will give the geometric interpretation of A_i. Given $w \in W$, we choose a reduced expression $w = r_{i_1} \cdots r_{i_s}$ and and define a map $\tau_w : D \to K/T$ given by $\tau_w(u_1, \cdots, u_s) = z_{i_1} \cdots z_{i_s} T$ where D is the unit disk in the complex space $\mathbb{C}^s$ and z_i has been defined in the previous section. By Proposition 1.55, the relative homology map τ_{w*} gives us an element $s_w \in H_{2\ell(w)}(K/T, \mathbb{Z})$. By Proposition 1.57, these elements are a basis of $H_*(K/T, \mathbb{Z})$; let ε^w be the dual basis of $H^*(K/T, \mathbb{Z})$.

Proposition 1.70.

$$r_i(\varepsilon^w) = \begin{cases} \varepsilon^w & \text{if } \ell(wr_i) > \ell(w), \\ \varepsilon^w - \displaystyle\sum_{wr_i \xrightarrow{\gamma} w'} \langle \alpha_i, \gamma \rangle \varepsilon^{w'} & \text{otherwise.} \end{cases}$$

Similarly, we can give the reflection action on Schubert cycles s_w.

Proposition 1.71.

$$A^i(\varepsilon^w) = \begin{cases} \varepsilon^{wr_i} & \text{if } \ell(wr_i) < \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

$$A_i(s_w) = \begin{cases} s_{wr_i} & \text{if } \ell(wr_i) > \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.72.

$$c_1(L_\lambda) \cap s_w = \sum_{w' \xrightarrow{\gamma} w} \langle \lambda, \gamma \rangle s_{w'}$$

The set of all functions from the Weyl group W to $\mathbb{C}$ will be denoted by $\mathbb{C}\{W\}$. $\mathbb{C}\{W\}$ is an algebra under pointwise addition and multiplication. Now, we will give the relation between $\mathbb{C}\{W\}$ and $\mathrm{End}_{\mathfrak{h}} H^*(\mathfrak{n}^-, \mathbb{C})$. From **[68]**, we have

Theorem 1.73. *Let $(\mathcal{A}, d)$ be a differential graded algebra over $\mathbb{C}$ and let δ be the derivation in $\mathrm{End}(\mathcal{A}, d)$ induced by d such that*

$$\delta\xi = d\xi - (-1)^i \xi d \text{ for } \xi \in \mathrm{End}^i(\mathcal{A}).$$

Then $\iota : H(\mathrm{End}(\mathcal{A}, d), \delta) \to \mathrm{End}\, H(\mathcal{A}, d)$ is an isomorphism graded algebras.

Theorem 1.74. *The standard cochain complexes $C(\mathfrak{g}, \mathfrak{h})$ and $C(\mathfrak{n}^-)$ with the topology of pointwise convergence are both differential graded algebras over $\mathbb{C}$.*

Also, we can put the topology of pointwise convergence on $\mathrm{End}\, C(\mathfrak{g}, \mathfrak{h})$ and $\mathrm{End}\, C(\mathfrak{n}^-)$. Then, the derivation map $\delta : \mathrm{End}\, C(\mathfrak{n}^-) \to \mathrm{End}\, C(\mathfrak{n}^-)$ is continuous under this topology and it commutes with the action of $\mathfrak{h}$ on $\mathrm{End}\, C(\mathfrak{n}^-)$. We denote by δ_0, the restriction of δ to $\mathrm{End}_{\mathfrak{h}}\, C(\mathfrak{n}^-)$. From **[68]**, we have

Proposition 1.75. *There exists a unique injective continuous map $\eta : C(\mathfrak{g}, \mathfrak{h}) \to \mathrm{End}_{\mathfrak{h}}\, C(\mathfrak{n}^-)$.*

Lemma 1.76. *We have $\eta(\ker S) \subseteq \ker \delta_0$.*

The map η induces a map $\tilde{\eta} : \ker S \to H(\mathrm{End}_{\mathbf{h}}\, \mathcal{C}(\mathbf{n}^-), \delta_0)$ Also, ι induces a map $\iota_0 : H(\mathrm{End}_{\mathbf{h}}\, \mathcal{C}(\mathbf{n}^-), \delta_0) \to \mathrm{End}_{\mathbf{h}}\, H^*(\mathbf{n}^-, \mathbb{C})$. By Theorem 1.59, as an $\mathbf{h}$-module, $H^{2j}(\mathbf{n}^-, \mathbb{C})$ is isomorphic to the direct sum

$$\bigoplus_{\ell(w)=j} M_{(w\rho-\rho)}$$

of non-isomorphic irreducible $\mathbf{h}$-submodules. By a property of the Hom functor, we have

$$\mathrm{End}_{\mathbf{h}}\, H^*(\mathbf{n}^-, \mathbb{C}) \cong \prod_{i\geqslant 0} \mathrm{End}_{\mathbf{h}}\, H^i(\mathbf{n}^-, \mathbb{C}) \cong \prod_{i\geqslant 0} \prod_{\ell(w)=i} \mathrm{End}_{\mathbf{h}}\, M_{(w\rho-\rho)}.$$

Since $M_{(w\rho-\rho)}$ is irreducible, $\mathrm{End}_{\mathbf{h}}\, M_{(w\rho-\rho)}$ is 1-dimensional with a canonical generator 1_w which is the identity map of $M_{(w\rho-\rho)}$. This identifies $\mathrm{End}_{\mathbf{h}}\, H^*(\mathbf{n}^-, \mathbb{C})$ with $\prod_{w\in W} \mathbb{C}1_w$. The space $\prod_{w\in W} \mathbb{C}1_w$ is the vector space $\mathbb{C}\{W\}$ of all functions from W to $\mathbb{C}$. Let $\overline{\eta}$ be the composite map

$$\ker S \xrightarrow{\tilde{\eta}} H(\mathrm{End}_{\mathbf{h}}\, \mathcal{C}(\mathbf{n}^-), \delta_0) \xrightarrow{\iota_0} \mathrm{End}_{\mathbf{h}}\, H^*(\mathbf{n}^-, \mathbb{C}) \cong \mathbb{C}\{W\}.$$

Now, we will give filtrations of $\mathcal{C}(\mathbf{g}, \mathbf{h})$ and $\mathbb{C}\{W\}$. We define a decrasing filtration $\mathcal{G} = (\mathcal{G}_p)_{p\in\mathbb{Z}^-}$ by $\mathcal{G}_p = \sum_{0\geqslant k+q\geqslant p} \mathcal{C}^{q,k}(\mathbf{g}, \mathbf{h})$ where $\mathcal{C}^{q,k}(\mathbf{g}, \mathbf{h}) = \mathrm{Hom}_{\mathbf{h}}(\Lambda^q(\mathbf{n}) \otimes \Lambda^k(\mathbf{n}^-))$. This gives rise to a filtration $\mathcal{F} = (\mathcal{F}_p)_{p\in\mathbb{Z}^-}$ of $\mathrm{End}_{\mathbf{h}}\, \mathcal{C}(\mathbf{n})$ by defining $\mathcal{F}_p = \eta(\mathcal{G}_p)$. By ι_0, we have filtration $\mathcal{H} = (\mathcal{H}_p)_{p\in\mathbb{Z}^+}$ of $\mathbb{C}\{W\}$. $\mathrm{Gr}\, \mathbb{C}\{W\}$ will denote the associated graded algebra with respect to the filtration of $\mathbb{C}\{W\}$. That is, $\mathrm{Gr}\, \mathbb{C}\{W\} = \sum_{p\geqslant 0} \mathrm{Gr}^p$, where $\mathrm{Gr}^p = \mathcal{H}_p/\mathcal{H}_{p+1}$. From [**68**], we have

Theorem 1.77. *Let* $\mathbf{g}$ *be a symmetrizable Kač-Moody Lie algebra. Let* $\mathbf{h}$ *be the Cartan subalgebra. Then,* $H^*(\mathbf{g}, \mathbf{h}) \to \mathrm{Gr}\, \mathbb{C}\{W\}$ *is a graded algebra isomorphism.*

By Theorem 1.66, we can give the following corollary.

Corollary 1.78. $H^*(G/B, \mathbb{C}) \to \mathrm{Gr}\, \mathbb{C}\{W\}$ *is a graded algebra isomorphism.*

Let $\mathbf{g}$ be an arbitrary Kač-Moody algebra associated to a generalized Cartan matrix A, with its Cartan subalgebra $\mathbf{h}$ and Weyl group W. Let $Q = Q(\mathbf{h}^*)$ be the field of the rational functions on $\mathbf{h}$. The Weyl group W acts as a group of automorphisms on the field Q. Let Q_W be the smash product of Q with the group algebra $\mathbb{C}[W]$, i.e., Q_W is a right Q-module with a basis $\{\delta_w\}_{w\in W}$ and the multiplication is given by

$$(\delta_v q_v)\cdot(\delta_w q_w) = \delta_{vw}(w^{-1}q_v)q_w$$

for $v, w \in W$ and $q_v, q_w \in Q$. The module Q_W admits an involutary anti-automorphism t, defined by $(\delta_w q)^t = \delta_{w^{-1}}(wq)$ for $w \in W$ and $q \in Q$. We define

$$x_i = -(\delta_{r_i} + \delta_e)\frac{1}{\alpha_i} = \frac{1}{\alpha_i}(\delta_{r_i} - \delta_e) \in Q_W$$

where $r_i \in W$ is a simple reflection and α_i is the simple root.

Proposition 1.79. *Let $w \in W$ and let $w = r_{i_1}\cdots r_{i_n}$ be a reduced expression. Then the element $x_{i_1}\cdots x_{i_n} \in Q_W$ does not depend upon the choice of reduced expression of w.*

The element $x_{i_1}\cdots x_{i_n} \in Q_W$ will be denoted by x_w and $(x_{w^{-1}})^t$ denoted by $\overline{x}_w$.

Proposition 1.80.

$$x_v \cdot x_w = \begin{cases} x_{vw} & \textit{if } \ell(vw) = \ell(v) + \ell(w), \\ 0 & \textit{otherwise.} \end{cases}$$

We know that Q_W is a right Q-module. Also, Q has a left Q_W-module structure defined by $(\delta_w q)q' = w(qq')$ for $w \in W$ and q, q'. We define subring $\mathcal{R} \subseteq Q_W$ given by

$$\mathcal{R} = \{x \in Q_W : x \cdot S \subseteq S\}$$

where $S = S(\mathbf{h}^*)$ is the polynomial algebra on $\mathbf{h}$. Let S_W be the smash product of S with the group algebra $\mathbb{C}[W]$. Obviously $S_W \subseteq \mathcal{R}$ since S has left S_W-module structure.

Theorem 1.81. *$\mathcal{R}$ is a free right S-module with basis $\{x_w\}_{w\in W}$ In particular, any $x \in \mathcal{R}$ can be uniquely written as $x = \sum_{w\in W} x_w p_w$ some $p_w \in S$.*

R will be referred as a *nil-Hecke ring.* Now, we will give the coproduct structure on Q_W. Let $Q_W \otimes_Q Q_W$ be the tensor product, considering both the copies of Q_W as right Q-modules. We define the diagonal map $\Delta : Q_W \to Q_W \otimes_Q Q_W$ by

$$\Delta(\delta_w q) = \delta_w q \otimes \delta_w = \delta_w \otimes \delta_w q$$

for $w \in W$ and $q \in Q$. Δ is right Q-linear.

Theorem 1.82. *For any $w \in W$, we have*

$$\Delta(\overline{x}_w) = \sum_{u,v \leqslant w} \overline{x}_u \otimes \overline{x}_v p_{u,v}^w$$

for some homogeneous polynomials $p_{u,v}^w \in S$ of degree $\ell(u) + \ell(v) - \ell(w)$. In particular, $p_{u,v}^w = 0$ unless $\ell(u) + \ell(v) \geqslant \ell(w)$.

Now, we will introduce some dual objects. Let $\Xi = \mathrm{Hom}_Q(Q_W, Q)$. Since any $\xi \in \Xi$ is determined by its restriction to the Q-basis $\{\delta_w\}_{w \in W}$, we can regard Ξ as the Q-module of all the functions $W \to Q$ with pointwise addition and scalar multiplication defined by the structure $(q\xi)w = q \cdot \xi(w)$ for $q \in Q$, $\xi \in \Xi$ and $w \in W$. Ξ has a commutative Q-algebra structure with the product as pointwise multiplication of functions on W. Also, Ξ has a left Q_W module structure defined by $(x \cdot \xi)y = \xi(x^t \cdot y)$ for $x, y \in Q_W$ and $\xi \in \Xi$. We have the Weyl group action as well as the *Hecke-type* operators A_w on Ξ defined by $w\xi = \delta_w \cdot \xi$ and $A_w \xi = x_w \cdot \xi$ for $w \in W$ and $\xi \in \Xi$. We define the important subring $\Lambda \subseteq \Xi$ as follows:

$$\Lambda = \{\xi \in \Xi : \xi(\mathcal{R}^t) \subseteq S \text{ and } \xi(\overline{x}_w) = 0 \text{ for all but a finite number of } w \in W\}$$

Proposition 1.83. *λ is a S-subalgebra of Ξ. $\{\xi^w\}_{w \in W}$ is a S-basis of Λ where ξ^w is dual to $\overline{x}_w$ for $w \in W$.*

Proposition 1.84. *$A_i \xi^w = \xi^{r_i w}$ if $r_i w < w$, 0 otherwise.*

Proposition 1.85. *$\xi^{r_i}(w) = \chi_i - w^{-1}\chi_i$ where χ_i is the fundamental weight dual to the coroot h_i corresponding to simple root α_i.*

Now, we will give the important formula equivalent to the cup product formula in the cohomology of G/B where G is a Kač-Moody group.

Proposition 1.86.

$$\xi^u \cdot \xi^v = \sum_{u,v \leqslant w} p_{u,v}^w \xi^w,$$

where $p_{u,v}^w$ is a homogeneous polynomial of degree $\ell(u) + \ell(v) - \ell(w)$.

Proposition 1.87.

$$r_i\xi^w = \begin{cases} \xi^w & \textit{if } r_i w > w, \\ -(w^{-1}\alpha_i)\xi^{r_i w} + \xi^w - \sum\limits_{r_i w \xrightarrow{\gamma} w'} \alpha_i(\gamma^\vee)\xi^{w'} & \textit{otherwise.} \end{cases}$$

Theorem 1.88. *Let $u, v \in W$. We write $w^{-1} = r_{i_1} \cdots r_{i_n}$ as a reduced expression.*

$$p^w_{u,v} = \sum_{\substack{j_1 < \cdots < j_m \\ r_{j_1} \cdots r_{j_m} = v^{-1}}} A_{i_1} \circ \cdots \circ \hat{A}_{i_{j_1}} \circ \cdots \circ \hat{A}_{i_{j_m}} \circ \cdots \circ A_{i_n}(\xi^u)(e)$$

where $m = \ell(v)$ and the notation $\hat{A}_i$ means that the operator A_i is replaced by the Weyl group action r_i.

Let $\mathbb{C}_0 = S/S^+$ be the S-module where S^+ is the augmentation ideal of S. It is 1-dimensional as $\mathbb{C}$-vector space. Since Λ is a S-module, we can define $\mathbb{C}_0 \otimes_S \Lambda$. It is an algebra and the action of $\mathcal{R}$ on Λ gives an action of $\mathcal{R}$ on $\mathbb{C}_0 \otimes_S \Lambda$. The elements $\sigma^w = 1 \otimes \xi^w \in \mathbb{C}_0 \otimes_S \Lambda$ is a $\mathbb{C}$-basis form of $\mathbb{C}_0 \otimes_S \Lambda$.

Proposition 1.89. *$\mathbb{C}_0 \otimes_S \Lambda$ is a graded algebra associated with the filtration of length of the element of the Weyl group W.*

Proposition 1.90. *The complex linear map $f : \mathbb{C}_0 \otimes_S \Lambda \to \mathrm{Gr}\,\mathbb{C}\{W\}$ is a graded algebra homomorphism.*

Theorem 1.91. *Let K be the standard real form of the group G associated to a symmetrizable Kač-Moody Lie algebra $\mathfrak{g}$ and let T denote the maximal torus of K. Then the map*

$$\theta : H^*(K/T, \mathbb{C}) \to \mathbb{C}_0 \otimes_S \Lambda$$

defined by $\theta(\varepsilon^w) = \sigma^w$ for any $w \in W$ is a graded algebra isomorphism. Moreover, the action of $w \in W$ and A^w on $H^*(K/T, \mathbb{C})$ corresponds respectively to that δ_w and $x_w \in \mathcal{R}$ on $\mathbb{C}_0 \otimes_S \Lambda$.

Corollary 1.92. *The operators A^i on $H^*(K/T, \mathbb{C})$ generate the nil-Hecke algebra.*

Corollary 1.93. *We can use Proposition 1.86 and Theorem 1.88 to determine the cup product $\varepsilon^u\varepsilon^v$ in terms of the Schubert basis $\{\varepsilon^w\}_{w\in W}$ of $H^*(K/T, \mathbb{Z})$.*

CHAPTER 2

Stratification of the homogeneous space LG/T, Birkhoff-Bruhat Decompositions and Grassmannian models

For this chapter, the general reference is [**84**] of A. Pressley & G. Segal.

1. Introduction.

Let G be a compact simply-connected semi-simple Lie group.

Definition 2.1. *The* loop group *LG is the set of all smooth maps from the circle $\mathbb{S}^1$ to the compact, connected and simply connected group G.*

Since the compact group G is simply-connected, the loop group LG is connected. It has the compact-open topology structure (usual map topology) and also the pointwise group multiplication given by the multiplication in G.

Theorem 2.2. *Let G be a compact simply-connected semi-simple Lie group. The loop group LG is an infinite dimensional Lie group modelled on separable Hilbert space.*

We want to mention some subgroups of the loop group LG. These are real-analytic and polynomial loop groups. Since G can be embedded in a unitary group U_n by the unitary representation of G on $\mathbb{C}^n$, a loop η in G is a matrix-valued function and can be expanded in a Fourier series

$$\eta(z) = \sum_{s=-\infty}^{\infty} \eta_s z^s. \tag{2.1}$$

The real-analytic loops are these such that the Fourier series converges in some annulus $r \leqslant z \leqslant r^{-1}$. The polynomial loop group $L_{\mathrm{pol}}G$ consists of these such that the matrix entries are finite Laurent polynomials in z and z^{-1}, i.e. loops of the form 2.1 where only finitely many of the matrices η_s are non-zero. This group is the union of the subsets $L_{\mathrm{pol},N}G$ consisting of the loops (2.1) for which $\eta_s = 0$ for $|s| > N$. Each of these subsets is a compact space, we can give $L_{\mathrm{pol}}G$ the direct limit topology. The polynomial loop group $L_{\mathrm{pol}}G$ has the complexification $L_{\mathrm{pol}}G_{\mathbb{C}}$ which is just the points of G with values in $\mathbb{C}[z, z^{-1}]$ in the sense of algebraic geometry. The polynomial loop group $L_{\text{pol}}G_{\mathbb{C}}$ has a central extension $\widehat{L}_{\text{pol}}G_{\mathbb{C}}$ by the circle group $\mathbb{T}$. From **[63]**, we have

Theorem 2.3. *$\widehat{L}_{\text{pol}}G_{\mathbb{C}}$ is an infinite dimensional algebraic group in the sense of Safarevič.*

Theorem 2.4. *Let G be a compact simply-connected semi-simple Lie group. Then $L_{\text{pol}}G$ is dense in LG.*

Definition 2.5. *The* based loop group *ΩG is the set of all based smooth maps from the circle $\mathbb{S}^1$ to the compact simply-connected G, i.e. the smooth loops which map the base point of the circle $\mathbb{S}^1$ to 1. Similarly, the complexified based loop group $\Omega G_{\mathbb{C}}$ is defined. Also, the polynomial based loop groups are defined as $\Omega_{\text{pol}}G = \Omega G \cap L_{\text{pol}}G$.*

Theorem 2.6. *The Lie group LG is the semidirect product of the subgroup G of constant loops and the normal subgroup ΩG of based loops γ such that the compact group G acts on ΩG by conjugation. In particular, $LG = G \times \Omega G$ as a manifold; and the homogeneous space LG/G can be identified with ΩG. Then the based loop group ΩG can be thought as a homogeneous space of LG, since the action of $\gamma \in LG$ on ΩG is $\omega \to \tilde{\omega}$, where*

$$\tilde{\omega}(z) = \gamma(z)\omega(z)\gamma(1)^{-1}.$$

2. The Grassmannian model of Hilbert space.

First, we will give the natural embedding of the smooth loop group $LGL_n(\mathbb{C})$ into the restricted general linear group of a complex separable Hilbert space H. Let $H^{(n)}$ denote the Hilbert space $L^2(\mathbb{S}^1; \mathbb{C}^n)$ of square-summable $\mathbb{C}^n$-valued functions on the circle. The group $L_{\text{cts}}GL_n(\mathbb{C})$ of continuous maps $\mathbb{S}^1 \to GL_n(\mathbb{C})$ acts on the separable Hilbert space $H^{(n)}$ by multiplication operators: if γ is a matrix valued fuction on the circle, we denote the corresponding multiplication operator by M_γ. The norm $||M_\gamma||$ is defined by

$$||M_\gamma|| = \sup\{|\gamma(\theta)| : \theta \in \mathbb{S}^1\}.$$

Then the mapping $\gamma \to M_\gamma$ embeds the Banach Lie group $L_{\text{cts}}GL_n(\mathbb{C})$ as a closed subgroup of the Banach Lie group $GL(H^{(n)})$ of all invertible bounded operators in

$H^{(n)}$, with operator-norm topology, which is

$$||T||_\infty = \sup\left\{\frac{||Tx||}{||x||} : x \neq 0\right\}$$

for $T \in GL(H^{(n)})$.

Definition 2.7. *A linear operator $T : H_1 \to H_2$ between Hilbert spaces is* Hilbert-Schmidt *if for every complete orthonormal sequence $\{e_i\}$ in H_1, the series $\sum_i ||Te_i||^2$ converges. The* Hilbert-Schmidt norm *of T is defined by $||T||_2 = (\sum_i ||Te_i||^2)^{\frac{1}{2}}$.*

Theorem 2.8. *The set $\mathcal{F}_2(H_1; H_2)$ of Hilbert-Schmidt operators $H_1 \to H_2$ is a Hilbert space under the Hilbert-Schmidt norm $|| \, ||_2$.*

Definition 2.9. *A linear operator $T : H_1 \to H_2$ is called* compact *if for every bounded sequence (x_n) in H_1, the sequence (Ax_n) has a convergent subsequence in H_2.*

Theorem 2.10. *(see* [96]*) Every Hilbert-Schmidt operator is a compact operator.*

Now, we will introduce notion of restricted general linear group. It is defined for an infinite dimensional Hilbert space which is equipped with a *polarization*, i.e. a decomposition $H = H_- \oplus H_+$ as the orthogonal sum of two closed infinite dimensional subspaces.

Definition 2.11. *The* restricted linear group $GL_{\text{res}}(H)$ *is the subgroup of general linear group $GL(H)$ consisting of operators A such that the commutator $[J, A] = JAJ^{-1}A^{-1}$ is a Hilbert-Schmidt operator where $J : H \to H$ is a unitary operator given by*

$$J(h) = \begin{cases} h & \text{for } h \in H_+, \\ -h & \text{for } h \in H_-. \end{cases}$$

Definition 2.12. *Let H_1 and H_2 be Hilbert spaces. The linear operator $A : H_1 \to H_2$ is called* Fredholm *if* $\dim\ker A$ *and* $\dim\operatorname{coker} A$ *are both finite. Then the* index *of A is*

$$\operatorname{index} A = \dim\ker A - \dim\operatorname{coker} A.$$

Proposition 2.13. *(see* [27]*) If the operator $A : H_1 \to H_2$ is invertible modulo compact operators, it is a Fredholm operator.*

Proposition 2.14. *(see* [**27**]*) The set* $\mathrm{Fred}(H_1, H_2)$ *of the Fredholm operators is an open subset of the norm space* $\mathcal{L}(H_1; H_2)$ *of bounded operators in the sup-norm topology. The index function* index *from the space* $\mathrm{Fred}(H_1, H_2)$ *of Fredholm operators to* $\mathbb{Z}$ *is locally constant and hence it is continuous.*

The following proposition gives us a characterization of the restricted general linear group $GL_{\mathrm{res}}(H)$.

Proposition 2.15. *Let* $A \in GL(H)$ *be written as a* 2×2 *matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with respect to the decomposition* $H = H_- \oplus H_+$. *Then,* $A \in GL_{\mathrm{res}}(H)$ *if and only if* b *and* c *are both Hilbert-Schmidt operators. In particular, if* $A \in GL_{\mathrm{res}}(H)$, a *and* d *are Fredholm operators.*

Proposition 2.16. *The map* $GL_{\mathrm{res}}(H) \to \mathrm{Fred}(H_+)$ *given by* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to a$, *is a homotopy equivalance.*

We have seen that the continuous loops in $GL_n(\mathbb{C})$ can be regarded as a subgroup of $GL(H^{(n)})$. The smooth loops in $GL_n(\mathbb{C})$ are contained in $GL_{\mathrm{res}}(H^{(n)})$. This fact is proven by the next proposition.

Proposition 2.17. *If* $\gamma : \mathbb{S}^1 \to GL_n(\mathbb{C})$ *is continuously differentiable, the multiplication operator* M_γ *is in* $GL_{\mathrm{res}}(H^{(n)})$.

Definition 2.18. $\mathrm{Gr}(H)$ *is the set of all closed subspaces* W *of* H *such that the orthogonal projection* $\mathrm{pr}_+ : W \to H_+$ *is a Fredholm operator and the orthogonal projection* $\mathrm{pr}_- : W \to H_-$ *is a Hilbert-Schmidt operator.*

The definition of the Grassmannian of a Hilbert space can be given another way: $W \in \mathrm{Gr}(H)$ if it is image of an operator $w : H_+ \to H$ such that $\mathrm{pr}_+ \circ w$ is Fredholm and $\mathrm{pr}_- \circ w$ is Hilbert-Schmidt. The restricted general linear group $GL_{\mathrm{res}}(H)$ acts smoothly on the space $\mathrm{Gr}(H)$. We have

Proposition 2.19. *The restricted unitary subgroup* $U_{\mathrm{res}}(H)$ *acts transitively on* $\mathrm{Gr}(H)$, *and the stabilizer of* H_+ *is* $U(H_+) \times U(H_-)$.

Definition 2.20. *If $T : H_1 \to H_2$ is an operator, the* graph *of T is the set*

$$\{h \oplus Th \in H_1 \oplus H_2 : h \in \operatorname{dom} T\}.$$

Lemma 2.21. *The sum of a Fredholm operator and a Hilbert-Schmidt operator is a Fredholm operator.*

By Lemma 2.21, we have

Proposition 2.22. *If $W \in \mathrm{Gr}(H)$, so does the graph of every Hilbert-Schmidt operator $W \to W^{\perp}$.*

These graphs form the subset U_W of $\mathrm{Gr}(H)$ consisting of all W' for which the orthogonal projection $W' \to W$ is an isomorphism: it is in one-to-one correspondence with the Hilbert space $\mathcal{F}_2(W; W^{\perp})$ of Hilbert-Schmidt operators $W \to W^{\perp}$. Then, we have from [**84**]

Theorem 2.23. $\mathrm{Gr}(H)$ *is a separable Hilbert manifold modelled on $\mathcal{F}_2(H_+; H_-)$.*

We define the virtual dimension for an arbitrary $W \in \mathrm{Gr}(H)$.

Definition 2.24. *The* virtual dimension *of an arbitrary $W \in \mathrm{Gr}(H)$ is the index of the orthogonal projection $\mathrm{pr}_+ : W \to H_+$, i.e.*

$$\text{virt-dim}\, W = \dim \ker \mathrm{pr}_+ - \dim \operatorname{coker} \mathrm{pr}_+ .$$

Equivalently,

$$\text{virt-dim}\, W = \dim W \cap H_- - \dim W^{\perp} \cap H_+.$$

Now, we will introduce an orthonormal basis in the separable Hilbert space H. We know that $H^{(n)} = L^2(\mathbb{S}^1; \mathbb{C}^n)$ can be decomposed into the positive and negative eigenspaces of the infinitesimal rotation operator $-id/d\theta$ as follows:

$$H_+^{(n)} = \{f \in H^{(n)} : f(\theta) = \sum_{k \geqslant 0} f_k \, \mathrm{e}^{ik\theta}\}$$

and

$$H_-^{(n)} = \{f \in H^{(n)} : f(\theta) = \sum_{k < 0} f_k \, \mathrm{e}^{ik\theta}\}$$

where $f_k \in \mathbb{C}^n$. If we identify H with the space $L^2(\mathbb{S}^1; \mathbb{C})$ with natural basis $\{z^k\}_{k\in\mathbb{Z}}$, we have the collection of points $\{H_S\}$ in $\mathrm{Gr}(H)$ such that H_S is just the closed subspace spanned by z^s for $s \in S$, where S is a subset of $\mathbb{Z}$ which has finite difference from the positive integers $\mathbb{N}$. We will denote $\mathcal{S}$ for the family of such sets S. We see that

$$\text{virt-dim}\, H_S = \mathrm{card}(S - \mathbb{N}) - \mathrm{card}(\mathbb{N} - S).$$

This number will be called the *virtual cardinal* of S.

Proposition 2.25. *For any $W \in \mathrm{Gr}(H)$, there is a set $S \in \mathcal{S}$ such that the perpendicular projection $W \to H_S$ is an isomorphism.*

Then, by Proposition 2.25, the sets $\{U_S\}_{S\in\mathcal{S}}$, where $U_S = U_{H_S}$, form an open covering of $\mathrm{Gr}(H)$. We can have coordinate charts on separable Hilbert manifold $\mathrm{Gr}(H)$, indexed by $\mathcal{S}$. A point of U_S is the graph of a Hilbert-Schmidt operator $H_S \to {H_S}^{\perp}$, and is represented by an $\tilde{S} \times S$ matrix, where $\tilde{S} = \mathbb{Z} - S$. The transition functions are given by $T_1 = (c + dT_0)(a + bT_0)^{-1}$ where T_i is Hilbert-Schmidt operator from H_{S_i} to ${H_{S_i}}^{\perp}$ for $i \in \{0, 1\}$. Now, we will define an important dense submanifold of $\mathrm{Gr}(H)$ in terms of the coordinate charts U_{H_S}.

Definition 2.26. $\mathrm{Gr}_0(H)$ *is the set of all subspaces W such that $z^k H_+ \subseteq W \subseteq z^{-k} H_+$ for some k.*

$\mathrm{Gr}_0(H)$ is the union of the finite dimensional classical Grassmannians $\mathrm{Gr}(H_{-k,k})$ where $H_{-k,k} = z^{-k}H_+/z^k H_+$. In terms of coordinate charts, $\mathrm{Gr}_0(H)$ consists of the graphs of operators $H_S \to {H_S}^{\perp}$ with only finitely many non-zero matrix entries. These graphs are dense in the space $\mathcal{F}_2(H_S; {H_S}^{\perp})$ of Hilbert-Schmidt operators.

3. The stratification of $\mathrm{Gr}(H)$ and the Plücker embedding.

The stratification of the Grassmannian $\mathrm{Gr}(H)$ is analogous to that of finite dimensional Grassmannians. A stratification of $\mathrm{Gr}(H)$ may be by the dimension of the intersection $W \cap H_-$, which is necessarily finite. However, we can give a finer stratification, which records the dimension of $W \cap z^k H_-$ for every k. For this finer stratification, we will introduce some notions.

Definition 2.27. *An element f of the Hilbert space $H = L^2(\mathbb{S}^1;\mathbb{C})$ is of* finite order s *if it is the form* $\sum_{l=-\infty}^{s} f_l z^l$ *with* $f_s \neq 0$. *In other words, f is the boundary value of a function which is meromorphic in the hemisphere $|z| > 1$ with only a pole of order s at $z = \infty$. For any $W \in \mathrm{Gr}(H)$, the set of elements of finite order in W will be denoted by W^{fin}.*

Since the set of elements of finite order is dense in H_S for $S \in \mathcal{S}$, by Proposition 2.25, we have

Proposition 2.28. *W^{fin} is dense in W.*

For given W, we define

$$S_W = \{s \in \mathbb{Z} : W \text{contains an element of order } s\}.$$

Definition 2.29. *For given $S \in \mathcal{S}$, the set*

$$\Sigma_S = \{W \in \mathrm{Gr}(H) : S_W = S\}$$

is called the stratum *of* $\mathrm{Gr}(H)$ *corresponding to S.*

In other words, Σ_S consists of all W such that $\dim W_k = d_k(S)$ for all k, where $W_k = W \cap z^{k+1}H_-$ and $d_k(S)$ is the number of elements of S which are $\leqslant k$. An indexing set S of virtual cardinal d can be written

$$S = \{s_{-d}, s_{-d+1}, \dots\},$$

with $s_{-d} < s_{-d+1} < \dots$ and $s_l = l$ for large l. We will *order* the sets of the same virtual cardinal by

$$\begin{aligned} S \leqslant S' \quad &\Longleftrightarrow \quad s_l \geqslant s_l{}' \text{for all } l \\ &\Longleftrightarrow \quad d_k(S) \leqslant d_k(S') \text{for all } k \end{aligned}$$

Also, we define the *length* $\ell(S)$ of S by

$$\ell(S) = \sum_{l \geqslant 0} (l - s_l).$$

Note that $S < S'$ implies $\ell(S) < \ell(S')$.

Theorem 2.30. *The stratum Σ_S is a contractible closed submanifold of the open set U_S, of codimension $\ell(S)$. The stratum Σ_S is the orbit of H_S by $\mathcal{N}_-$ where $\mathcal{N}_-$ is* strictly lower triangular *subgroup of GL_{res} consisting of all elements A such that $A(z^k H_-) = z^k H_-$ for all k. The closure of Σ_S is the union of the strata $\Sigma_{S'}$ with $S' \geqslant S$.*

We know that the Grassmannian of a finite dimensional vector space has a Schubert cell decomposition. This details can be found in the [**43**]. By definition, $\mathrm{Gr}_0(H)$ is the union of the finite dimensional Grassmannians $\mathrm{Gr}(H_{-k,k})$. It too can be decomposed into Schubert cells. This decomposition is dual to the stratification of $\mathrm{Gr}(H)$ described in the last section in the following sense: the same set $\mathcal{S}$ indexes the cells C_S and the strata Σ_S; $\dim C_S = \operatorname{codim} \Sigma_S$; C_S meets Σ_S transversally in a single point. For the description of C_S, we will introduce some notation.

Definition 2.31. *The* co-order *of a polynomial element $f = \sum_{k=-N}^{N} f_k z^k$ of H is the smallest k such that $f_k \neq 0$.*

Then for $W \in \mathrm{Gr}_0(H)$, we define

$$S^W = \{s \in \mathbb{Z} : W \text{ contains an element of co-order } s\}$$

For $S \in \mathcal{S}$, we define

$$C_S = \{W \in \mathrm{Gr}_0(H) : S^W = S\}.$$

Theorem 2.32. *C_S is a closed submanifold of the open set U_S of $\mathrm{Gr}(H)$ and it is diffeomorphic to $\mathbb{C}^{\ell(S)}$. C_S is the orbit H_S under the* strictly upper triangular *subgroup $\mathcal{N}_+$ of GL_{res} which consists of all A such that $A(z^k H_+) = z^k H_+$ for all k. The closure of C_S is the union of the cells $C_{S'}$ with $S' \leqslant S$. C_S intersects $\Sigma_{S'}$ if and only if $S \geqslant S'$ and C_S intersects Σ_S transversally in the single point H_S.*

We will give the relation between rotation action of the circle group $\mathbb{T}$ on $\mathrm{Gr}(H)$ and the stratification of $\mathrm{Gr}(H)$. The circle group $\mathbb{T}$ acts on $H = L^2(\mathbb{S}^1; \mathbb{C})$ by rotating $\mathbb{S}^1$ and this action preserves the decomposition $H = H_+ \oplus H_-$, hence $\mathbb{T}$ also acts on the Grassmannian $\mathrm{Gr}(H)$. This action is continuous but it is not differentiable. The rotation action of $\mathbb{T}$ extends to an action

$$\mathbb{C}_{\leqslant 1}{}^{\times} \times \mathrm{Gr}(H) \to \mathrm{Gr}(H)$$

of the semigroup $\mathbb{C}_{\leqslant 1}{}^{\times}$ of non-zero complex numbers of modulus $\leqslant 1$. This action is holomorphic on the open set $\mathbb{C}_{<1}{}^{\times} \times \mathrm{Gr}(H)$. The rotation action of the circle $\mathbb{T}$ on the submanifold $\mathrm{Gr}_0(H)$ extends to a holomorphic action of the whole group $\mathbb{C}^{\times}$.

Proposition 2.33. *Σ_S consists of the points $W \in \mathrm{Gr}(H)$ such that $R_u W$ tends to H_S as $u \to 0$.*
C_S consists of the points $W \in \mathrm{Gr}_0(H)$ such that $R_u W$ tends to H_S as $u \to \infty$.

We know that a finite dimensional Grassmannian space can be embedded into a projective space and points of the Grassmannian can be written in coordinates of the projective space. Details can be found in [**43**]. We can do exactly the same with $\mathrm{Gr}(H)$. For $s \in S_W$, let w_s be an element of W of the form $\sum_{k=-\infty}^{s} f_k z^k$ with $f_s = 1$. Then $\{w_s\}$ is a basis of W^{fin} in the algebraic sense. We can choose w_s uniquely such that it projects to z^s; we will call this *canonical basis* of W. We will introduce the notion of admissible basis for W. Suppose that W has virtual dimension d.

Definition 2.34. *A sequence $\{w_k\}_{k \geqslant -d}$ in W is called an* admissible basis *for W if the linear map $w : z^{-d}H_+ \to W$ which takes z^k to w_k is a continuous isomorphism and the composite* $\mathrm{pr} \circ w$*, where* $\mathrm{pr} : W \to z^{-d}H_+$ *is orthogonal projection, is an operator with a determinant. The canonical basis for W is admissible: the composite map* $\mathrm{pr} \circ w$ *differs from the identity by an operator of finite rank.*

Definition 2.35. *If w is an admissible basis for W, $S \in \mathcal{S}$ is a set of virtual cardinal d, and $\mathrm{pr}_S : W \to H_S$ is the projection, we define* Plücker coordinate $\pi_S(w)$ *of the the basis w as the determinant* $\mathrm{Det}(\mathrm{pr}_S \circ w)$.

Proposition 2.36. *The Plücker coordinates $\{\pi_S\}_{S \in \mathcal{S}}$ define a holomorphic embedding*

$$\pi : \mathrm{Gr}(H) \to \mathbb{P}(\mathcal{H})$$

into the projective space of the separable Hilbert space $\mathcal{H} = \ell^2(\mathcal{S})$ *of the sequences* $s_i : S \to \mathbb{C}$. *such that* $\sum_{i \in S} |s_i|^2 < \infty$ *and* $s_i = 0$ *for all except a countable number of* $i \in S$.

Proposition 2.37. Det : $L \to \mathrm{Gr}(H)$ *is the holomorphic line bundle with fibre* $\mathrm{Det}(W)$ *at* $W \in \mathrm{Gr}(H)$ *which is to be thought of as the top exterior power of* W. *The line bundle* Det *is the pull-back of the tautological line bundle on* $\mathbb{P}(\mathcal{H})$.

Using the Plücker embedding, we have

Proposition 2.38. $\mathrm{Gr}(H)$ *is a Kähler manifold with the closed 2-form* ω *which represents the Chern class of the line bundle* Det *on* $\mathrm{Gr}(H)$.

4. The Birkhoff - Bruhat factorization theorems.

Definition 2.39. *The subgroup* $L^+GL_n(\mathbb{C})$ *of* $LGL_n(\mathbb{C})$ *is the set of loops* γ *which are the boundary values of holomorphic maps*

$$\gamma : \{z \in \mathbb{C} : |z| < 1\} \to GL_n(\mathbb{C}).$$

Theorem 2.40. *Any loop* $\gamma \in LGL_n(\mathbb{C})$ *can be factorized uniquely*

$$\gamma = \gamma_u \cdot \gamma_+$$

where γ_u *is an element of the based loop space* ΩU_n *and* $\gamma_+ \in L^+GL_n(\mathbb{C})$. *The product map*

$$\Omega U_n \times L^+GL_n(\mathbb{C}) \to LGL_n(\mathbb{C})$$

is a diffeomorphism.

Definition 2.41. *The subgroup* $L^-GL_n(\mathbb{C})$ *is set of loops* $\gamma \in LGL_n(\mathbb{C})$ *which are the boundary values of holomorphic maps*

$$\gamma : \{z \in \mathbb{C} \cup \infty : |z| > 1\} \to GL_n(\mathbb{C})$$

Theorem 2.42. *Any loop* $\gamma \in LGL_n(\mathbb{C})$ *can be factorized*

$$\gamma = \gamma_- \cdot \lambda \cdot \gamma_+,$$

where $\gamma_- \in L^-GL_n(\mathbb{C})$, $\gamma_+ \in L^+GL_n(\mathbb{C})$ and $\lambda \in \check{T}$ is a loop which is a homomorphism from the circle group $\mathbb{T}$ into the maximal torus T in $GL_n(\mathbb{C})$, i.e. λ is of the form

$$z^{\mathrm{a}} = \begin{pmatrix} z^{a_1} & & & & & \\ & z^{a_2} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & z^{a_n} \end{pmatrix}.$$

The product map

$$L_1{}^- \times L^+ \to LGL_n(\mathbb{C})$$

is a diffeomorphism where

$$L_1{}^- = \{\gamma_- \in L^- : \gamma_-(\infty) = 1\}.$$

In Theorem 2.40 and 2.42, $LGL_n(\mathbb{C})$ can be replaced by $LG_{\mathbb{C}}$ for any compact semi-simple Lie group G. Both theorems are referred as *Birkhoff's factorization theorems* and have exact analogues for the groups of rational and polynomial loops. Now, we will give the *Bruhat factorization theorem.*

Theorem 2.43. *Any polynomial loop $\gamma \in L_{\mathrm{pol}}GL_n(\mathbb{C})$ can be factorized*

$$\gamma = u \cdot \lambda \cdot v,$$

where u and v both belong to $L^+GL_n(\mathbb{C})$ and λ is a homomorphism from the circle group $\mathbb{T}$ into the maximal torus T.

5. The Grassmannian model for the based loop space ΩU_n.

We know that the group $LGL_n(\mathbb{C})$ acts by matrix multiplication on the separable Hilbert space $H^{(n)} = L^2(\mathbb{S}^1; \mathbb{C}^n)$, and by Proposition 2.17, on the Grassmannian $\mathrm{Gr}(H)$. Since the action of $\gamma \in LGL_n(\mathbb{C})$ commutes with multiplication by the function z, the subspaces W of the form γH_+ for $\gamma \in LGL_n(\mathbb{C})$ have the property $zW \subseteq W$.

Definition 2.44. $\mathrm{Gr}^{(n)}$ *denotes the closed subset of $\mathrm{Gr}(H^{(n)})$ which consists of subspaces W such that $zW \subseteq W$.*

We will show that $\mathrm{Gr}^{(n)}$ is gives a Grassmannian model of the loop space.

Definition 2.45. *$L_{\frac{1}{2}}GL_n(\mathbb{C})$ is set of the commutant elements of the multiplication operator M_z in $GL_{\mathrm{res}}(H^{(n)})$.*

Proposition 2.46. *The loop group $L_{\frac{1}{2}}U_n = L_{\frac{1}{2}}GL_n(\mathbb{C}) \cap LU_n$ acts transitively on $\mathrm{Gr}^{(n)}$, and the isotropy group of H_+ is the group U_n of constant loops.*

Then, by Proposition 2.46, $\Omega_{\frac{1}{2}}U_n = L_{\frac{1}{2}}U_n/U_n$ can be identified with $\mathrm{Gr}^{(n)}$ as a set.

Proposition 2.47. *In the correspondence $\mathrm{Gr}^{(n)} \leftrightarrow \Omega_{\frac{1}{2}}U_n$, $\mathrm{Gr}_0^{(n)}$ and $\mathrm{Gr}_\infty^{(n)}$ correspond to polynomial based loop space $\Omega_{\mathrm{pol}}U_n$ and ΩU_n respectively.*

The complexified group $L_{\frac{1}{2}}GL_n(\mathbb{C})$ acts on $\mathrm{Gr}^{(n)}$ as well as $L_{\frac{1}{2}}U_n$, and the stabilizer of H_+ in $L_{\frac{1}{2}}GL_n(\mathbb{C})$ is the closed subgroup $L_{\frac{1}{2}}{}^+GL_n(\mathbb{C})$. Thus, this gives the the proof of Theorem 2.40.

6. The stratification of $\mathrm{Gr}_\infty^{(n)}$.

In this section, we will drop the subscript ∞ for smooth manifolds $\mathrm{Gr}_\infty(H)$ and $\mathrm{Gr}_\infty^{(n)}$. Since $\mathrm{Gr}^{(n)}$ can be identified with $LGL_n(\mathbb{C})/L^+GL_n(\mathbb{C})$ by Theorem 2.40, Theorem 2.42 asserts that each L^- orbit contains a point of the form $z^a H_+{}^{(n)}$, unique up to the order of $\{a_1, a_2, \ldots, a_n\}$.

Definition 2.48. *N^- is the set of loops γ such that $\gamma(\infty)$ is upper triangular with 1's on the diagonal.*

Proposition 2.49. *Each orbit of N^- on $\mathrm{Gr}^{(n)}$ contains a unique point of the form $z^{\mathrm{a}}H_+{}^{(n)}$ and the orbits of N^- are the intersections of $\mathrm{Gr}^{(n)}$ with the strata of $\mathrm{Gr}(H)$.*

The fixed points of the rotation R_u action of $\mathbb{T}$ on ΩU_n given by $(R_u\omega)(\theta) = \omega(\theta - u)\omega(-u)^{-1}$ for $\omega \in \Omega U_n$ are the homomorphisms $\lambda : \mathbb{T} \to U_n$, corresponding to the subspaces $\lambda \cdot H_+$ in $\mathrm{Gr}^{(n)}$. The action of the circle group $\mathbb{T}$ extends to an action of the semigroup $\mathbb{C}_{\leqslant 1}{}^\times$ and for any $W \in \mathrm{Gr}^{(n)}$, the point $R_u W$ tends to $\lambda \cdot H_+$ as $u \to 0$. The stratification of $\mathrm{Gr}(H)$ was defined for $H = L^2(\mathbb{S}^1; \mathbb{C})$, whereas in this section, we are concerned with $H^{(n)} = L^2(\mathbb{S}^1; \mathbb{C}^n)$. Since all infinite dimensional separable Hilbert spaces are isomorphic, all that we need is a Hilbert space with an orthonormal basis indexed by the integers. Then, we can define an isomorphism between the separable Hilbert spaces: if $\{\varepsilon_i : 1 \leqslant i \leqslant n\}$ is the standard basis of $\mathbb{C}^n$, we let $\varepsilon_i z^k$ correspond to $z^{nk+i-1} \in H$. Given a vector valued fuction with components $(f_1, f_2, \ldots, f_n) \in H^{(n)}$, we have the scalar-valued $\tilde{f} \in H$ given by

$$\tilde{f} = f_1(\zeta^n) + \zeta f_2(\zeta^n) + \ldots + \zeta^{n-1} f_n(\zeta^n).$$

Conversely, given $\tilde{f} \in H$, we obtain $\{f_i\}_{1 \leqslant i \leqslant n} \in H^{(n)}$ by

$$f_{i+1}(z) = \frac{1}{n} \sum_{\zeta} \zeta^{-i} \tilde{f}(\zeta),$$

where ζ runs through the nth roots of z. The isomorphism is an isometry. Then, $H = H^{(n)}$ has the orthonormal bases $\{\zeta^k\}_{k \in \mathbb{Z}}$. We can define $\mathrm{Gr}^{(n)}$ in terms of ζ^k. Then, the definition of $\mathrm{Gr}^{(n)}$ can be rewritten

$$\mathrm{Gr}^{(n)} = \{W \in \mathrm{Gr}(H) : \zeta^n W \subseteq W\}.$$

We know from the definition of the strata Σ_S of $\mathrm{Gr}(H)$ that W belongs Σ_S if S is the set of integers s such that W contains an element of order s. If $W \in \mathrm{Gr}^{(n)}$ belongs to Σ_S, then $S + n \subseteq S$. Sets $S \in \mathcal{S}$ satisfying this condition are completely determined by the complement S^* of $S + n$ in S, which must consist of n elements, one in each congruence class modulo n. They corresponds to the homomorphisms from the circle $\mathbb{T}$ into the maximal torus. For the homomorphism z^a, there corresponds the set S_a such that ${S_a}^*$ is

$$\{na_1, na_2 + 1, \ldots, na_n + n - 1\}.$$

Thus the strata of $\mathrm{Gr}(H)$ which meet $\mathrm{Gr}^{(n)}$ can be indexed by the homomorphism z^a. We will write Σ_a for $\Sigma_{S_a} \cap \mathrm{Gr}^{(n)}$, and H_a for H_{S_a}.

Proposition 2.50. *The orbit of H_{a} under N^- is Σ_{a}. It can be identified with the subgroup ${L_{\mathrm{a}}}^-$ of N^-, where ${L_{\mathrm{a}}}^- = N^- \cap z^{\mathrm{a}} {L_1}^- z^{-\mathrm{a}}$.*

Proposition 2.51. $\mathrm{Gr}^{(n)}$ *is smooth submanifold of* $\mathrm{Gr}(H)$.

Theorem 2.52. *The map $\gamma \to \gamma H_{\mathrm{a}}$ defines a diffeomorphism between $z^{\mathrm{a}}L_1{}^{-}z^{-\mathrm{a}}$ and a contractible open neighbourhood U_{a} of H_{a} in $\mathrm{Gr}^{(n)}$. The stratum Σ_{a} is a contractible closed submanifold of U_{a}, of complex codimension*

$$d(\mathrm{a}) = \sum_{i<j} |a_i - a_j| - \nu(\mathrm{a}),$$

where $\nu(\mathrm{a})$ is the number of pairs i, j with $i < j$ but $a_i > a_j$. The orbit of H_{a} under N^+ is a complex cell C_{a} of complex dimension $d(\mathrm{a})$, which meets Σ_{a} in the single point H_{a}. The union of the cells C_{a} is $\mathrm{Gr}_0{}^{(n)}$ and C_{a} is the intersection of $\mathrm{Gr}^{(n)}$ with the cell $C_{S_{\mathrm{a}}}$ of Gr_0.

7. The Grassmannian model for ΩG where G is a compact semi-simple Lie group.

Since LG is a Lie group, by its adjoint representation, LG acts on the Hilbert space $H^{\mathfrak{g}} = L^2(\mathbb{S}^1; \mathfrak{g}_{\mathbb{C}})$ where $\mathfrak{g}_{\mathbb{C}}$ is the complexified Lie algebra of the compact Lie group G. If $\dim G = n$, we can identify $H^{\mathfrak{g}}$ with $H^{(n)}$. By the unitary representation of G on $\mathbb{C}^n$, LG is a subgroup of LU_n. Then, the based loop space ΩG is a submanifold of ΩU_n, which can be identified with a submanifold $\mathrm{Gr}(H^{\mathfrak{g}})$. W^{sm} is the subspace of smooth functions in W. It is dense in W.

Definition 2.53. $\mathrm{Gr}^{\mathfrak{g}}$ *is the subset of* $\mathrm{Gr}(H^{\mathfrak{g}})$ *consisting of subspaces W such that $zW \subseteq W$, $\overline{W}^{\perp} = zW$, where $\overline{W}$ is the conjugate space of W and W^{sm} is a Lie algebra under the bracket operation for $\mathfrak{g}_{\mathbb{C}}$-valued smooth functions in W.*

Theorem 2.54. *The action of $LG_{\mathbb{C}}$ on $\mathrm{Gr}(H^{\mathfrak{g}})$ preserves $\mathrm{Gr}^{\mathfrak{g}}$ and $\gamma \to \gamma H_+$ defines a diffeomorphism $\Omega G \to \mathrm{Gr}^{\mathfrak{g}}$.*

If we choose a maximal torus T of G and a positive root system Δ^+, then we can define the nilpotent subgroups N^+ and N^- of $G_{\mathbb{C}}$ whose Lie algebras are spanned by the positive (respectively negative) root vectors of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the positive (resp. negative) roots. Also, we can define nilpotent subgroups $\widetilde{N}^+$ and $\widetilde{N}^-$ of the loop group $LG_{\mathbb{C}}$: $\widetilde{N}^+$ is the set of the loops $\gamma \in L^+G_{\mathbb{C}}$ such that $\gamma(0) \in N^+$; $\widetilde{N}^-$ is the set of the loops $\gamma \in L^-G_{\mathbb{C}}$ such that $\gamma(\infty) \in N^-$ and $L_1^{\pm}G_{\mathbb{C}} \subseteq \widetilde{N}^{\pm} \subseteq L^{\pm}G_{\mathbb{C}}$.

Theorem 2.55. *$\mathrm{Gr}^{\mathfrak{g}} \cong \Omega G$ is the union of strata Σ_λ indexed by the coweight lattice $\check{T}$ of homomorphisms $\lambda : \mathbb{T} \to T$ where the strata Σ_λ is the orbit of $\lambda \cdot H_+$ under $\widetilde{N}^-$ and it is locally closed submanifold of finite codimension d_λ in $\mathrm{Gr}^{\mathfrak{g}}$. The complex cell C_λ of dimension d_λ is the orbit of $\lambda \cdot H_+$ under $\widetilde{N}^+$ and it meets the strata Σ_λ transversally in the single point $\lambda \cdot H_+$. The union of the cells C_λ is ${\mathrm{Gr}_0}^{\mathfrak{g}} \cong \Omega_{\mathrm{pol}} G$.*

Now, we will mention the homotopy equivalance between the based loop group ΩG and the polynomial based loop group $\Omega_{\mathrm{pol}} G$. We know that the cells C_λ of ${\mathrm{Gr}_0}^{\mathfrak{g}}$ have even dimension, thus the fundamental group $\pi_1({\mathrm{Gr}_0}^{\mathfrak{g}})$ must be trivial. Also, the fundamental group of ΩG is the second homotopy group $\pi_2(G)$. Then, we have

Proposition 2.56. *The inclusion $\Omega_{\mathrm{pol}} G \to \Omega G$ is a homotopy equivalance.*

Corollary 2.57. *The homotopy group $\pi_2(G)$ is trivial for any compact semi-simple Lie group G.*

8. The homogeneous space LG/T.

One of the most important homogeneous spaces of a compact group Lie group G is G/T, where T is a maximal torus of G. The analogue of G/T for a loop group LG is LG/T rather than the homogeneous space ΩG. LG/T is a complex manifold, because it is diffeomorphic to $LG_{\mathbb{C}}/\widetilde{B}^+$ where

$$\widetilde{B}^+ = \left\{ \sum_{k=0}^{\infty} \gamma_k z^k \in L^+ G_{\mathbb{C}} : \gamma_0 \in B \right\}.$$

LG/T is stratified by the orbits of $\widetilde{N}^-$, and the strata are indexed by the affine Weyl group $\widetilde{W}$. We know that the affine Weyl group $\widetilde{W}$ is the semi-direct product $W \ltimes \check{T}$ where W is the Weyl group of G and $\check{T}$ is the co-weight lattice of G. Since $\widetilde{W} = (N(T) \cdot \check{T})/T$ where $N(T)$ is the normalizer group of T in G, $\widetilde{W}$ is a subset of LG/T

Proposition 2.58. *The set of fixed points of the action of the circle group $\mathbb{T}$ on LG/T is the affine Weyl group $\widetilde{W}$.*

Theorem 2.59. *The stratum Σ_w is the orbit of w under $\widetilde{N}^-$, where Σ_w is a locally closed contractible complex submanifold of LG/T whose codimension is the length $\ell(w)$ of w. The stratum Σ_w is a closed subset of the open subset U_w of LG/T, where $U_w =$*

$w\dot{\Sigma}_e$. *The union of the strata* Σ_w *indexed by the affine Weyl group* $\widetilde{W}$ *is the complex manifold* $LG_{\mathbb{C}}/\widetilde{B}^+ \cong LG/T$. *A complex cell* C_w *of dimension* $\ell(w)$ *is the orbit of* w *under* A_w *where* $A_w = \widetilde{N}^+ \cap w\widetilde{N}^- w^{-1}$. *It intersects the strata* Σ_w *transversally at* w. *The union of the cells* C_w *is* $L_{\text{pol}}G/T$. *If* $\ell(w') = \ell(w) + 1$, *then* $\Sigma_{w'}$ *is contained in the closure of* Σ_w *if and only if* $w' = wr_{\mathbf{a}}$ *where* $r_{\mathbf{a}}$ *is the reflection corresponding to a simple affine root* $\mathbf{a}$.

If $\mathbf{a}$ is a simple affine root of LG, then there is a homomorphism $i_{\mathbf{a}} : SL_2(\mathbb{C}) \to LG_{\mathbb{C}}$ which maps the Borel group B of $SL_2(\mathbb{C})$ to $\widetilde{B}^+$. This gives us a map $\tilde{i}_{\mathbf{a}} : \mathbb{C}P^1 \to LG/T$ where $\mathbb{C}P^1$ is the two dimensional complex projective space which is diffeomorphic to the sphere $\mathbb{S}^2 = \mathbb{C} \cup \infty$. If $w' = wr_{\mathbf{a}}$ and $\ell(w') = \ell(w) + 1$, the map $z \to \tilde{i}_{\mathbf{a}} \cdot w$ from $\mathbb{S}^2$ to $LG_{\mathbb{C}}/\widetilde{B}^+$ defines a holomorphic projective curve in LG/T linking w to w'. This projective curve lies in Σ_w except for $\tilde{i}_{\mathbf{a}}(\infty) \cdot w = w'$, so the closure of Σ_w contains $\Sigma_{w'}$.

Now, we will describe about the Bott-Samelson resolution of the closure of the complex cell C_w.

Theorem 2.60. *The closure of* C_w *is a compact complex algebraic variety with singularities.*

If the element w of the affine Weyl group $\widetilde{W}$ is written as a product $w = r_{\mathbf{a}_1} r_{\mathbf{a}_2} \cdots r_{\mathbf{a}_k}$ of reflections corresponding to simple root $\mathbf{a}_i$ of LG, then the closed cell $\overline{C}_w$ is the image of the map

$$SU_2 \times \cdots \times SU_2 \to LG/T$$

given by $(g_1, \ldots, g_k) \to i_{\mathbf{a}_1} \cdots i_{\mathbf{a}_k} \cdot T$.

Proposition 2.61. $Z_w = SU_2 \times_{\mathbb{T}} SU_2 \times_{\mathbb{T}} \cdots \times_{\mathbb{T}} SU_2/\mathbb{T}$ *which is an iterated smooth projective bundle over* $\mathbb{C}P^1$ *is diffeomorphic to the complex manifold* $P_{\mathbf{a}_1} \times_{\widetilde{B}^+} P_{\mathbf{a}_2} \times_{\widetilde{B}^+} \cdots \times_{\widetilde{B}^+} P_{\mathbf{a}_k}/\widetilde{B}^+$, *where* $P_{\mathbf{a}_i} = i_{\mathbf{a}_i}(SL_2(\mathbb{C})) \cdot \widetilde{B}^+ \subseteq LG_{\mathbb{C}}$, *i.e.,* $P_{\mathbf{a}_i}$ *is a minimal parabolic subgroup which is containing* $\widetilde{B}^+$.

Proposition 2.62. *The surjection* $Z_w \to \overline{C}_w$ *is a birational equivalence of algebraic varieties.*

Now we will describe the Borel-Weil theory for the loop group LG, which is analogous to that for compact Lie groups. Given a central extension $\widehat{L}G$ of LG by the circle group $\mathbb{T}$, we know that $LG/T = \widehat{L}G/\widehat{T} = \widehat{L}G_{\mathbb{C}}/\widehat{\widetilde{B}^+}$. Every character λ of $\widehat{T}$ extends canonically to a holomorphic homomorphism $\lambda : \widehat{\widetilde{B}^+} \to \mathbb{C}^\times$. Then, we can define a holomorphic line bundle $L_\lambda = \widehat{L}G_{\mathbb{C}} \times_{\widehat{\widetilde{B}^+}} \mathbb{C}$ on LG/T by acting $\widehat{\widetilde{B}^+}$ on $\mathbb{C}$ via $b \cdot \eta = \lambda(b)\eta$ for $b \in \widehat{\widetilde{B}^+}$ and $\eta \in \mathbb{C}$.

Proposition 2.63. *The line bundle*

$$L_\lambda = \left\{ [\gamma, \eta] \in \widehat{L}G_{\mathbb{C}} \times_{\widehat{\widetilde{B}^+}} \mathbb{C} : [\gamma, \eta] = [\gamma b, b \cdot \eta] \, \forall \, b \in \widehat{\widetilde{B}^+} \right\}$$

has an equivariant structure.

Proposition 2.64. *$\widehat{L}G$ acts on the holomorphic line bundle L_λ and this action is compatible with the action on LG/T.*

Proposition 2.65. *The rotation action on LG/T of the circle group $\mathbb{T}$ is covered by an action on L_λ.*

Now, we will introduce some useful notions and terminology. By a representation of a topological group G, we mean a complete locally convex complex topological vector space V on which G acts linearly and continuously in the sense that $(g, \eta) \to g \cdot \eta$ is a continuous map $G \times V \to V$.

Definition 2.66. *A representation is* irreducible *if it has no closed invariant subspace.*

When V is a representation of a Lie group G, then a vector $\eta \in V$ is called *smooth* if the action given by $g \to g \cdot \eta$ is smooth. The set of all such smooth vectors of V will be denoted by V_{sm}.

Definition 2.67. *The representation V is* smooth *if V_{sm} is dense in V.*

After we give some terminology, we return to the holomorphic line bundle L_λ on LG/T. We will denote the space of holomorphic sections of the line bundle L_λ by Γ_λ. It is a complete vector space with the compact-open topology. By Proposition 2.64, $\widehat{L}G$ acts on the holomorphic section space Γ_λ. Then Γ_λ is a holomorphic representation of $\widehat{L}G$ but it may be zero.

Definition 2.68. *The weight λ is called* antidominant *if $\lambda(h_{\mathrm{a}}) \leqslant 0$ for every positive coroot h_{a} of LG.*

Now, we will give an important theorem in this section.

Theorem 2.69. *If the representation Γ_λ is non-zero, then λ is antidominant and the representation is irreducible with lowest weight λ where λ is a character of $\mathbb{T} \times \widehat{T}$ which is trivial on $\mathbb{T}$. Furthermore, if μ is any other weight of Γ_λ, then $\mu - \lambda$ is a sum of positive roots LG.*

Definition 2.70. *Representations V and V' of a group G are* essentially equivalent *if there is an injective G-equivarant continuous linear map $V \to V'$ which has dense image.*

Warning: this is not an equivalence relation. Indeed, a reducible representation may be essential equivalent to an irreducible representation. If V and V' are finite dimensional topological vector spaces, then essential equivalance agrees with the notion of G-equivarant linear isomorphism.

Theorem 2.71. *Every irreducible representation of $\widehat{L}G$ is essentially equivalent to some Γ_λ.*

Proposition 2.72. *The line bundle L_λ on LG/T has non-vanishing holomorphic sections if and only if the weight λ is antidominant.*

We know that a character χ is a class function on G. When G is a compact Lie group, each element of G is conjugate to an element of the maximal torus T, so the character χ is described by the restriction to T. Now, we will be interested to the character theory of representations of loop groups. We consider the representations of $\mathbb{T} \ltimes \widehat{L}G$, where $\widehat{L}G$ is an arbitrary central extension of LG by the circle group $\mathbb{T}$. In the last section, we have seen that every irreducible Γ_λ contains a unique lowest weight vector, up to scalar multiple, transforming according to a character λ of $\mathbb{T} \times \widehat{T}$ which determines Γ_λ. Then,

Theorem 2.73. *Every representation of $\mathbb{T} \times \widehat{L}G$ is determined up to equivalence by its restriction to $\mathbb{T} \times \widehat{T}$.*

Definition 2.74. *The* character *of the representation V of $\mathbb{T} \times \widehat{L}G$ is the sum*

$$\chi_V = \sum_{\lambda} d_\lambda \cdot \mathrm{e}^{i\lambda}$$

where d_λ is the number of multiplicity of a character λ of $\mathbb{T} \times \widehat{T}$.

We will state an important result due to V. Kač [**59**] which is an exact analogue of the Weyl character formula for compact groups.

Theorem 2.75. **(The Kač character formula)**

$$\chi_\lambda = \prod_{\alpha} (1 - \mathrm{e}^{i\alpha})^{-1} \sum_{w \in \widetilde{W}} (-1)^{\ell(w)} \, \mathrm{e}^{i(w\lambda + s(w))},$$

where $s(w)$ is the sum of all positive roots α of LG for which $w^{-1}\alpha$ is negative.

Now, we will give an application of lowest weight representations of the affine type Kač-Moody algebras. This gives the algebraic proof of the character formula. The fundamental reference is W. Zhe-Xian [**98**]. First, we will give some notations. The universal enveloping algebra $U(\widehat{L_{\mathrm{pol}}\mathfrak{g}_{\mathbb{C}}})$ will be denoted by $\mathcal{U}$, $U(\widehat{\mathbf{b}^-_{\mathrm{pol}}})$ (resp. $U(\mathbf{n}^+_{\mathrm{pol}})$) will be denoted by $\mathcal{U}_-$. (resp. $\mathcal{U}_+$) By the Poincaré-Birkhoff-Witt Theorem, the multiplication map

$$\mathcal{U}^+ \otimes \mathcal{U}^- \to \mathcal{U}$$

is an isomorphism of complex vector spaces.

Definition 2.76. *A representation space M_λ of $\mathbb{T} \times \widehat{L_{\mathrm{pol}}}G$ which is generated by a vector ζ_λ, is called a* Verma module *if ζ_λ is annihilated by $\mathbf{n}^-_{\mathrm{pol}}$, ζ_λ is an eigenvector of $\mathbb{T} \times \widehat{T}$ correponding to the character λ and the map $\mathcal{U} \otimes_{\mathcal{U}_-} \mathbb{C} \to M_\lambda$ given by $a \otimes x \to xa \cdot \zeta_\lambda$ is an isomorphism.*

Corollary 2.77. *The Verma module M_λ is isomorphic to $\mathbb{C}_\lambda \otimes \mathrm{Sym}(\mathbf{n}^+_{\mathrm{pol}})$ as a representation of $\mathbb{T} \otimes \widehat{T}$, where $\mathbb{C}_\lambda$ denotes $\mathbb{C}$ with the action of $\mathbb{T} \otimes \widehat{T}$ given by the character λ.*

Proposition 2.78. *If a representation space V is a direct sum of one-dimensional irreducible representations ρ_i of a compact group K, then the character of the exterior algebra $\Lambda(V)$ is $\prod(1 + \rho_i)$, while that of* $\mathrm{Sym}(V)$ *is $\prod(1 + \rho_i)^{-1}$.*

Corollary 2.79. *The character φ_λ of the Verma module M_λ is*

$$\varphi_\lambda = \mathrm{e}^{i\lambda} \cdot \prod_{\alpha>0}(1-\mathrm{e}^{i\alpha})^{-1}.$$

Proposition 2.80. *The Verma module M_λ is a lowest weight module with lowest weight λ of the multiplicity 1. Any lowest weight module with lowest weight λ is homomorphic image of M_λ. M_λ has a unique maximal submodule with simple quotient L_λ. If V is an irreducible lowest weight module with lowest weight λ, then V is isomorphic to L_λ.*

Proposition 2.81. *The character χ_λ of the irreducible representation Γ_λ is a countable sum of the form $\sum_\mu n_\mu \varphi_\mu$, where $n_\mu \in \mathbb{Z}$ and μ runs through weights such that $\mu-\lambda$ is a sum of positive roots of $\widehat{L}G$ and $\sigma(\mu-\rho,\mu-\rho)=\sigma(\lambda-\rho,\lambda-\rho)$. Here, ρ denotes $(0,\rho,-c)$ where ρ is the half of sum of all positive roots of G and c is $\rho(h_{\alpha_0})+1$.*

Lemma 2.82. *For any weight λ and any $w \in \widetilde{W}$, we have*

$$w \cdot \varphi_\lambda = (-1)^{\ell(w)} \varphi_{w(\lambda-\rho)+\rho}.$$

Lemma 2.83. *Let λ be an antidominant weight, and let μ be a weight which satisfies the following conditions.*

1. *$\mu-\lambda$ is a sum of all positive roots of $\widehat{L}G$,*
2. *$\mu-\rho$ is an antidominant weight,*
3. *$\sigma(\mu-\rho,\mu-\rho)=\sigma(\lambda-\rho,\lambda-\rho)$.*

Then, $\lambda=\mu$.

Theorem 2.84.

$$\chi_\lambda = \sum_{w\in\widetilde{W}} (-1)^{\ell(w)} \varphi_{w(\lambda-\rho)+\rho}.$$

By Proposition 2.80 and Theorem 2.84, we have

Theorem 2.85. *There exists a resolution*

$$0 \leftarrow \Gamma_\lambda \leftarrow M_\lambda \leftarrow \bigoplus_{\ell(w)=1} M_{w(\lambda-\rho)+\rho} \leftarrow \bigoplus_{\ell(w)=2} M_{w(\lambda-\rho)+\rho} \leftarrow \cdots$$

of the irreducible representation Γ_λ by Verma modules.

This is called the *Bernstein-Gelfand-Gelfand resolution.* This resolution expresses the stratification of the basic homogeneous space LG/T. When Γ_λ is realized as the space of holomorphic sections of the line bundle L_λ on homogeneous space LG/T, we can identify the antidual of the canonical surjection $M_\lambda \to \Gamma_\lambda$ with the restriction map of sections $\Gamma(L_\lambda) \to \Gamma(L_\lambda|U)$, where U is the dense open stratum of LG/T.

Theorem 2.86. *If Γ_λ is the irreducible representation of $\widehat{L}G$ with lowest weight, then*

$$H^{2q}(\mathrm{n}^-;\Gamma_\lambda) \cong \bigoplus_{\ell(w)=q} \mathbb{C}_{w(\lambda-\rho)+\rho}$$

as a representation of $\mathbb{T}\times\widehat{T}$, where $\mathbb{C}_{w(\lambda-\rho)+\rho}$ denotes $\mathbb{C}$ with the action of $\mathbb{T}\times\widehat{T}$ given by the character $w(\lambda-\rho)+\rho$.

Theorem 2.87. *The map $H(\int) : H^*(L\mathrm{g},\mathrm{h};\mathbb{C}) \to H^*(LG/T;\mathbb{C})$ is a graded-algebra isomorphism.*

We already know that the $\mathbb{Z}$-cohomology of the homogeneous space LG/T is the free abelian group generated by the strata of complex codimension p , which are indexed by the elements of length p in the affine Weyl group.

CHAPTER 3

The Cohomology Ring of the Infinite Flag Manifold LG/T

1. Introduction.

In [**72**], Kumar described the Schubert classes which are the dual to the closures of the Bruhat cells in the flag varieties of the Kač-Moody groups associated to the infinite dimensional Kač-Moody algebras. These classes are indexed by affine Weyl groups and can be choosen as elements of integral cohomologies of the homogeneous space $\widehat{L}_{\text{pol}}G_{\mathbb{C}}/\widehat{B}$ for any compact simply connected semi-simple Lie group G. Later, S. Kumar and B. Kostant gave explicit cup product formulas of these classes in the cohomology algebras by using the relation between the invariant-theoretic relative Lie algebra cohomology theory (using the representation module of the nilpotent part) with the purely nil-Hecke rings [**68**]. These explicit product formulas involve some BGG-type operators A^i and reflections. Using some homotopy equivalances, we determine cohomology ring structures of LG/T where LG is the smooth loop space on G. Here, as an example we calculate the products and explicit ring structure of LSU_2/T using these ideas.

2. The root system, Weyl group and Cartan matrix of the loop group LG.

We know from compact simply-connected semi-simple Lie theory that the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of the compact Lie group G has a decomposition under the adjoint action of the maximal torus T of G. Then, from [**54**], we have

Theorem 3.1. *There is a decomposition*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha} \mathfrak{g}_{\alpha},$$

where $\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$ *is the complexified Lie algebra of* T *and*

$$\mathfrak{g}_{\alpha} = \{\xi \in \mathfrak{g}_{\mathbb{C}} : t \cdot \xi = \alpha(t)\xi \; \forall t \in T\}.$$

The homomorphisms $\alpha : T \to \mathbb{T}$ for which $\mathfrak{g}_\alpha \neq 0$ are called the *roots* of G. They form a finite subset of the lattice $\check{T} = \mathrm{Hom}(T, \mathbb{T})$. By analogy, the complexified Lie algebra $L\mathfrak{g}_{\mathbb{C}}$ of the loop group LG has a decomposition

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}} \cdot z^k$$

where $\mathfrak{g}_{\mathbb{C}}$ is the complexified Lie algebra of G. This is the decomposition into eigenspaces of the rotation action of the circle group $\mathbb{T}$ on the loops. The rotation action commutes with the adjoint action of the constant loops G, and from **[84]**, we have

Theorem 3.2. *There is a decomposition of $L\mathfrak{g}_{\mathbb{C}}$ under the action of the maximal torus T of G,*

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_0 \cdot z^k \oplus \bigoplus_{(k,\alpha)} \mathfrak{g}_\alpha \cdot z^k.$$

The pieces in this decomposition are indexed by homomorphisms

$$(k, \alpha) : \mathbb{T} \times T \to \mathbb{T}.$$

The homomorphisms $(k, \alpha) \in \mathbb{Z} \times \check{T}$ which occur in the decomposition are called the *roots* of LG.

Definition 3.3. *The set of roots is called the* root system *of LG and denoted by $\widehat{\Delta}$.*

Let δ be $(0, 1)$. Then

$$\widehat{\Delta} = \bigcup_{k\in\mathbb{Z}} (\Delta \cup \{0\} + k\delta) = \Delta \cup \{0\} + \mathbb{Z}\delta,$$

where Δ is the root system of G. The root system $\widehat{\Delta}$ is the union of real roots and imaginary roots:

$$\widehat{\Delta} = \widehat{\Delta}_{\mathrm{re}} \cup \widehat{\Delta}_{\mathrm{im}},$$

where

$$\widehat{\Delta}_{\mathrm{re}} = \{(\alpha, n) : \alpha \in \Delta,\ n \in \mathbb{Z}\}$$

$$\widehat{\Delta}_{\mathrm{im}} = \{(0, r) : r \in \mathbb{Z}\}.$$

Definition 3.4. *Let the rank of G be l. Then, the set of simple roots of LG is*

$$\{(\alpha_i, 0) : \alpha_i \in \Sigma \, for \; 1 \leqslant i \leqslant l\} \cup \{(-\alpha_{l+1}, 1)\},$$

where α_{l+1} is the highest weight of the adjoint representation of G.

The root system $\widehat{\Delta}$ can be divided into three parts as the positive and the negative and 0:

$$\widehat{\Delta} = \widehat{\Delta}^+ \cup \{0\} \cup \widehat{\Delta}^-$$

where

$$\widehat{\Delta}^+ = \widehat{\Delta}^+_{\text{re}} \cup \widehat{\Delta}^+_{\text{im}},$$
$$\widehat{\Delta}^- = \widehat{\Delta}^-_{\text{re}} \cup \widehat{\Delta}^-_{\text{im}},$$

where

$$\widehat{\Delta}^+_{\text{re}} = \{(\alpha, n) \in \widehat{\Delta}_{\text{re}} : n > 0\} \cup \{(\alpha, 0) : \alpha \in \Delta^+\},$$
$$\widehat{\Delta}^+_{\text{im}} = \{n\delta : n > 0\}$$

and

$$\widehat{\Delta}^-_{\text{re}} = -\widehat{\Delta}^+_{\text{re}},$$
$$\widehat{\Delta}^-_{\text{im}} = -\widehat{\Delta}^+_{\text{im}}.$$

Now, we will give some examples. First, we will discuss the case of SU_2. The root system $\widehat{\Delta}$ of the loop group $LSU(2)$ has two basis elements $\mathbf{a}_0 = (-\alpha, 1)$ and $\mathbf{a}_1 = (\alpha, 0)$ where α is the simple root of SU_2. All roots of LSU_2 can be written as a sum of the simple roots $\mathbf{a}_0$ and $\mathbf{a}_1$.

Proposition 3.5. *The set of roots of LSU_2 is given by $\widehat{\Delta} = \widehat{\Delta}_{\text{re}} \cup \widehat{\Delta}_{\text{im}}$ where*

$$\widehat{\Delta}_{\text{re}} = \{k\mathbf{a}_0 + l\mathbf{a}_1 : |k - l| = 1, k \in \mathbb{Z}\},$$
$$\widehat{\Delta}_{\text{im}} = \{k\mathbf{a}_0 + k\mathbf{a}_1 : k \in \mathbb{Z}\}.$$

Corollary 3.6. *The set of positive roots of LSU_2 is given by $\widehat{\Delta}^+ = \widehat{\Delta}^+_{\text{re}} \cup \widehat{\Delta}^+_{\text{im}}$ where*

$$\widehat{\Delta}^+_{\text{re}} = \{k\mathbf{a}_0 + l\mathbf{a}_1 : |k - l| = 1, k \in \mathbb{Z}^+\} = \{(\alpha, r), (-\alpha, s) : r \geqslant 0, s > 0\},$$
$$\widehat{\Delta}^+_{\text{im}} = \{k\mathbf{a}_0 + k\mathbf{a}_1 : k \in \mathbb{Z}^+\} \qquad .$$

In the case of LSU_n, for $n \geqslant 3$, the root system $\widehat{\Delta}$ of the loop group LSU_n has basis elements $\mathbf{a}_0 = (-\alpha_0, 1)$ and $\mathbf{a}_i = (\alpha_i, 0)$, $1 \leqslant i \leqslant n-1$ where α_i is the simple root of SU_n and $\alpha_0 = \sum_{i=1}^{n-1} \alpha_i$. All roots of LSU_n can be written as a sum of the simple roots $\mathbf{a}_i$.

Theorem 3.7. *(see [60]) The set of roots of LSU_n, for $n \geqslant 3$, is*

$$\widehat{\Delta} = \{k \sum_{r=0}^{i-1} \mathbf{a}_r + l \sum_{r=i}^{j-1} \mathbf{a}_r + k \sum_{r=j}^{n-1} \mathbf{a}_r : |k-l| = 1, k \in \mathbb{Z} \; and 0 \leqslant i \leqslant j \leqslant n\}.$$

Corollary 3.8. *The set of positive roots of LSU_n, for $n \geqslant 3$, is*

$$\widehat{\Delta}^+ = \{k \sum_{r=0}^{i-1} \mathbf{a}_r + l \sum_{r=i}^{j-1} \mathbf{a}_r + k \sum_{r=j}^{n-1} \mathbf{a}_r : |k-l| = 1, k \in \mathbb{Z}^+ \; and 0 \leqslant i \leqslant j \leqslant n\}.$$

Now, we will discuss the Weyl group of the loop group LG. In order to define this group, we need a larger group structure. We define the semi-direct product $\mathbb{T} \ltimes LG$ of $\mathbb{T}$ and LG in which $\mathbb{T}$ acts on LG by the rotation. From [84], we have

Theorem 3.9. *$\mathbb{T} \times T$ is a maximal abelian subgroup of $\mathbb{T} \ltimes LG$.*

Theorem 3.10. *The complexified Lie algebra of $\mathbb{T} \ltimes LG$ has a decomposition*

$$(\mathbb{C} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \left(\bigoplus_{k \neq 0} \mathfrak{t}_{\mathbb{C}} \cdot z^k \oplus \bigoplus_{(k,\alpha)} \mathfrak{g}_\alpha \cdot z^k, \right)$$

according to the characters of $\mathbb{T} \times T$.

We know from chapter 1 that the roots of G are permuted by the Weyl group W. This is the group of automorphisms of the maximal torus T which arise from conjugation in G, i.e. $W = N(T)/T$, where

$$N(T) = \{n \in G : nTn^{-1} = T\}$$

is the normalizer of T in G. In exactly same way, the infinite set of roots of LG is permuted by the Weyl group $\widetilde{W} = N(\mathbb{T} \times T)/(\mathbb{T} \times T)$, where $N(\mathbb{T} \times T)$ is the normalizer in $\mathbb{T} \ltimes LG$. The Weyl group $\widetilde{W}$ which was defined above is called the *affine Weyl group.*

Proposition 3.11. *The affine Weyl group $\widetilde{W}$ is the semidirect product of the coweight lattice $T^\vee = \mathrm{Hom}(\mathbb{T}, T)$ by the Weyl group W of G.*

We know from chapter 1, the Weyl group W of G acts on the Lie algebra of the maximal torus T, it is a finite group of isometries of the Lie algebra $\mathbf{t}$ of the maximal torus T. It preserves the coweight lattice $T^{\vee}$. For each simple root α, the Weyl group W contains an element r_α of order two represented by $\exp\left(\frac{\pi}{2}(e_\alpha + e_{-\alpha})\right)$ in $N(T)$. Since the roots α can be considered as the linear functionals on the Lie algebra $\mathbf{t}$ of the maximal torus T, the action of r_α on $\mathbf{t}$ is given by

$$r_\alpha(\xi) = \xi - \alpha(\xi)h_\alpha \, \text{for} \; \xi \in \mathbf{t},$$

where h_α is the coroot in $\mathbf{t}$ corresponding to simple root α. Also, we can give the action of r_α on the roots by

$$r_\alpha(\beta) = \beta - \alpha(h_\beta)\alpha \, \text{for} \; \alpha, \beta \in \mathbf{t}^*,$$

where $\mathbf{t}^*$ is the dual vector space of $\mathbf{t}$. The element r_α is the reflection in the hyperplane H_α of $\mathbf{t}$ whose equation is $\alpha(\xi) = 0$. These reflections r_α generate the Weyl group W. For the special unitary matrix group SU_2, we have only one simple root α with corresponding reflection r_α which generates the Weyl group of SU_2 and $W \cong \mathbb{Z}/2$. More generally, we have from [**55**].

Theorem 3.12. *The Weyl group of SU_n is the symmetric group S_n.*

Now, we want to describe the Weyl group structure of LG. By analogy with $\mathbb{R}$ for real form, the roots of the loop group LG can be considered as linear forms on the Lie algebra $\mathbb{R} \times \mathbf{t}$ of the maximal abelian group $\mathbb{T} \times T$. The Weyl group $\widetilde{W}$ acts linearly on $\mathbb{R} \times \mathbf{t}$, the action of W is an obvious reflection in the affine hyperplane $1 \times \mathbf{t}$ and the action of $\lambda \in T^{\vee}$ is given by

$$\lambda \cdot (x, \xi) = (x, \xi + x\lambda).$$

Thus, the Weyl group $\widetilde{W}$ preserves the hyperplane $1 \times \mathbf{h}$, and $\lambda \in \check{T}$ acts on it by translation by the vector $\lambda \in T^{\vee} \subseteq \mathbf{t}$. If $\alpha \neq 0$, the affine hyperplane $H_{\alpha,k}$ can be defined as follows. For each root (α, k),

$$H_{\alpha,k} = \{\xi \in \mathbf{t} : \alpha(\xi) = -k\}.$$

We know that the Weyl group W of G is generated by the reflections r_α in the hyperplanes H_α for the simple roots α. A corresponding statement holds for the affine Weyl group $\widetilde{W}$.

Proposition 3.13. *Let G be a simply-connected semi-simple compact Lie group. Then the Weyl group $\widetilde{W}$ of the loop group LG is generated by the reflections in the hyperplanes $H_{\alpha,k}$. The affine Weyl group $\widetilde{W}$ acts on the root system $\widehat{\Delta}$ by*

$$r_{(\alpha,k)}(\gamma, m) = (r_\alpha(\gamma), m - \alpha(h_\gamma)k) \, for \, (\alpha, k), (\gamma, m) \in \widehat{\Delta}.$$

Proposition 3.14. *The Weyl group $\widetilde{W}$ of LSU_2 is*

$$\widetilde{W} = \{(r_{\mathrm{a}_0} r_{\mathrm{a}_1})^k,\ (r_{\mathrm{a}_0} r_{\mathrm{a}_1})^k r_{\mathrm{a}_0},\ (r_{\mathrm{a}_1} r_{\mathrm{a}_0})^k,\ (r_{\mathrm{a}_1} r_{\mathrm{a}_0})^k r_{\mathrm{a}_1} : k \geqslant 0,\ r_{\mathrm{a}_0}^2 = r_{\mathrm{a}_1}^2 = \mathrm{Id}\}.$$

Proposition 3.15. *The Weyl group of LSU_n is the semi-direct product $S_n \ltimes \mathbb{Z}^{n-1}$ where S_n acts by permutation action on coordinates of $\mathbb{Z}^{n-1}$.*

Actually the symmetric group S_n acts on $\mathbb{Z}^n$ by the permutation action. $\mathbb{Z}^{n-1}$ is the fixed subgroup which corresponds to the eigen-value action. From [**50**], we have

Theorem 3.16. *The affine Weyl group $\widetilde{W}$ of LG is a Coxeter group.*

We will give some properties of the affine Weyl group $\widetilde{W}$.

Definition 3.17. *The* length *of an element $w \in \widetilde{W}$ is the least number of factors in the decomposition relative to the set of the reflections $\{r_{\mathrm{a}_i}\}$, is denoted by $\ell(w)$.*

Definition 3.18. *Let $w_1, w_2 \in \widetilde{W}, \gamma \in \Delta^+_{\mathrm{re}}$. Then $w_1 \xrightarrow{\gamma} w_2$ indicates the fact that*

$$r_\gamma w_1 = w_2,$$
$$\ell(w_2) = \ell(w_1) + 1.$$

We put $w \leqslant w'$ if there is a chain

$$w = w_1 \to w_2 \to \cdots \to w_k = w'.$$

The relation $\leqslant$ is called the Bruhat order *on the affine Weyl group $\widetilde{W}$.*

Proposition 3.19. *Let $w \in \widetilde{W}$ and let $w = r_{\mathrm{a}_1} r_{\mathrm{a}_2} \cdots r_{\mathrm{a}_l}$ be the reduced decomposition of w. If $1 \leqslant i_1 < \ldots < i_k \leqslant l$ and $w' = r_{\mathrm{a}_{i_1}} r_{\mathrm{a}_{i_2}} \cdots r_{\mathrm{a}_{i_k}}$, then $w' \leqslant w$. If $w' \leqslant w$, then w' can be represented as above for some indexing set $\{i_\xi\}$. If $w' \to w$, then there is a unique index i, $1 \leqslant i \leqslant l$ such that*

$$w' = r_{\mathrm{a}_1} \cdots r_{\mathrm{a}_{i-1}} r_{\mathrm{a}_{i+1}}.$$

The last proposition gives an alternative definition of the Bruhat ordering on $\widetilde{W}$. Now we will define the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ which will be used in the text later. We know that the Weyl group $\widetilde{W}$ of the loop group LG is a split extension $T^\vee \to \widetilde{W} \to W$, where W is the Weyl group of the compact group Lie group G. Since the Weyl group W is a sub-Coxeter system of the affine Weyl group $\widetilde{W}$, we can define the set of cosets $\widetilde{W}/W$.

Lemma 3.20. *The subgroup of $\widetilde{W}$ fixing 0 is the Weyl group W.*

Corollary 3.21. *Let $w, w' \in \widetilde{W}$. Then, $w(0) = w'(0)$ if and only if $wW = w'W$ in $\widetilde{W}/W$.*

By the last corollary, the map $\widetilde{W}/W \to T^\vee$ given by $wW \to w(0)$ is well-defined and has inverse map given by $\chi_i \to r_{\alpha_i} W$, so the coset set $\widetilde{W}/W$ is identified to $T^\vee$ as set. We have from [**18**],

Theorem 3.22. *Each coset in $\widetilde{W}/W$ has a unique element of the minimal length.*

We will write $\overline{\ell(w)}$ for the minimal length element occuring in the coset wW, for $w \in \widetilde{W}$. We see that each coset $wW, w \in \widetilde{W}$ has two distinguished representatives which are not in the general the same. Let the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ be the set of the minimal representative elements $\overline{\ell(w)}$ in the coset wW for each $w \in \widetilde{W}$. The subset $\widehat{W}$ has the Bruhat order since it identitifies the set of the minimal representative elements $\overline{\ell(w)}$. As a example, we calculate the subset $\widehat{W}$ of the Weyl group of LSU_2. Our aim is to find the minimal representative elements $\overline{\ell(w)}$ in the right coset wW for each the element $w \in \widetilde{W}$, where

$$\widetilde{W} = \{(r_{\mathrm{a}_0} r_{\mathrm{a}_1})^k,\ (r_{\mathrm{a}_0} r_{\mathrm{a}_1})^l r_{\mathrm{a}_0},\ (r_{\mathrm{a}_1} r_{\mathrm{a}_0})^m,\ (r_{\mathrm{a}_1} r_{\mathrm{a}_0})^n r_{\mathrm{a}_1} : k, l, m, n \geqslant 0,\ r_{\mathrm{a}_0}^2 = r_{\mathrm{a}_1}^2 = \mathrm{id}\},$$

and $W = \langle r_{\mathrm{a}_1};\, r_{\mathrm{a}_1}^2 = \mathrm{id}\rangle$. We have the minimal representative elements $\overline{\ell(w)}$ for each coset wW, $w \in \widetilde{W}$ as follows

$$\begin{aligned}
\overline{l((r_{\mathrm{a}_0} r_{\mathrm{a}_1})^k)} &= (r_{\mathrm{a}_0} r_{\mathrm{a}_1})^k \quad \text{for } k \geqslant 0 \\
\overline{l((r_{\mathrm{a}_0} r_{\mathrm{a}_1})^l r_{\mathrm{a}_0})} &= (r_{\mathrm{a}_0} r_{\mathrm{a}_1})^l r_{\mathrm{a}_0} \quad \text{for } l \geqslant 0 \\
\overline{l((r_{\mathrm{a}_1} r_{\mathrm{a}_0})^n r_{\mathrm{a}_1})} &= (r_{\mathrm{a}_0} r_{\mathrm{a}_1})^n \quad \text{for } n \geqslant 0
\end{aligned}$$

and

$$\overline{l((r_{\mathrm{a}_1} r_{\mathrm{a}_0})^m)} = \begin{cases} \mathrm{Id} & \text{for m} = 0 \\ (r_{\mathrm{a}_0} r_{\mathrm{a}_1})^{m-1} r_{\mathrm{a}_0} & \text{for } m > 0 \end{cases}$$

By the transformations $m-1, l$ and $k \to n$, we have the subset

$$\widehat{W} = \{\overline{\ell(w)} : w \in \widetilde{W}\} = \{(r_{\mathrm{a}_0} r_{\mathrm{a}_1})^n,\, (r_{\mathrm{a}_0} r_{\mathrm{a}_1})^n r_{\mathrm{a}_0} : n \geqslant 0\}.$$

Now we will describe the Lie algebra $L_{\mathrm{pol}}\mathrm{g}_{\mathbb{C}}$ and its universal central extension in terms of generators and relations. For a finite dimensional semi-simple Lie algebra $\mathrm{g}_{\mathbb{C}}$, we can choose a non-zero element e_α in g_α for each root α. From **[54]**, we have

Theorem 3.23. *$\mathrm{g}_{\mathbb{C}}$ is a Kač-Moody Lie algebra generated by $e_i = e_{\alpha_i}$ and $f_i = e_{-\alpha_i}$ for $i = 1, \ldots, l$ where the α_i are the simple roots and l is the rank of $\mathrm{g}_{\mathbb{C}}$ only if G is semi-simple.*

Let us choose generators e_j and f_j of $L\mathrm{g}_{\mathbb{C}}$ corresponding to simple affine roots. Since $\mathrm{g}_{\mathbb{C}} \subseteq L\mathrm{g}_{\mathbb{C}}$, we can take

$$e_j = \begin{cases} z e_{-\alpha_0} & \text{for } j = 0, \\ e_i & \text{for } 1 \leqslant j \leqslant l \end{cases}$$

and

$$f_j = \begin{cases} z^{-1} e_{\alpha_0} & \text{for } j = 0, \\ f_i & \text{for } 1 \leqslant j \leqslant l \end{cases}$$

where α_0 is the highest root of the adjoint representation. From **[84]**,

Theorem 3.24. *Let $\mathrm{g}_{\mathbb{C}}$ be a semi-simple Lie algebra. Then, $L_{\mathrm{pol}}\mathrm{g}_{\mathbb{C}}$ is generated by the elements e_j and f_j corresponding to simple affine roots.*

The Cartan matrix $A_{(l+1)\times(l+1)}$ of $L\mathfrak{g}_{\mathbb{C}}$ has the Cartan integers $a_{ij} = \mathbf{a}_j(h_{\mathbf{a}_i})$ as the entries where $\mathbf{a}_0 = -\alpha_0$, and $\mathbf{a}_j = \alpha_j$ if $1 \leqslant j \leqslant l$. As an example,

Proposition 3.25. *Let $G = SU_2$. The Cartan matrix $A_{2\times 2}$ of $L\mathfrak{g}_{\mathbb{C}}$ is the symmetric matrix* $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

Although the relations of the Kač-Moody algebra hold in $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$, they do not define it. By a theorem of Gabber and Kač in [**40**], the relations define the universal central extension $\widehat{L}_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ of $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$ which is described by the cocycle ω_K given by

$$\omega_K(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(\xi(\theta), \eta'(\theta))d\theta.$$

As a vector space $\widehat{L}_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ is $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}$ and the bracket is given by

$$[(\xi, \lambda), (\eta, \mu)] = ([\xi, \eta], \omega_K(\xi, \eta)).$$

Theorem 3.26. *$\widehat{L}\mathfrak{g}_{\mathbb{C}}$ is an affine Kač-Moody algebra.*

3. Some homotopy equivalences for the loop group LG and its homogeneous spaces.

From [**42**], we have

Theorem 3.27. *The compact group G is a deformation retract of $G_{\mathbb{C}}$ and so, the loop space LG is homotopic to the complexified loop space $LG_{\mathbb{C}}$.*

Now, we want to give a major result from [**84**]

Theorem 3.28. *The inclusion*

$$\iota : L_{\text{pol}}G_{\mathbb{C}} \to LG_{\mathbb{C}}$$

is a homotopy equivalence.

Now we will give some useful notations. The parabolic subgroup P of $L_{\text{pol}}G_{\mathbb{C}}$ is the set of maps $\mathbb{C} \to G_{\mathbb{C}}$ which have non-negative Laurent series expansions. Then $P = G_{\mathbb{C}}[z]$. The minimal parabolic subgroup B is the Iwahori subgroup

$$\{f \in P : f(0) \in \overline{B}\},$$

where $\overline{B}$ is the finite-dimensional Borel subgroup of G. Note also that the minimal parabolic subgroup B corresponds to the positive roots, the parabolic subgroup P to the roots (α, n) with $n \geqslant 0$. From [**42**],

Theorem 3.29. *The evaluation at zero map $e_0 : P \to G_{\mathbb{C}}$ is a homotopy equivalence with the homotopy inverse the inclusion of $G_{\mathbb{C}}$ as the constant loops.*

The following fact follows from the local rigidity of the trivial bundle on the projective line. From [**44**], we have

Proposition 3.30. *The projection*

$$L_{\text{pol}}G_{\mathbb{C}} \to L_{\text{pol}}G_{\mathbb{C}}/P$$

is a principal bundle with fiber P.

Now, as a consequence of Theorem 3.28, Remark 2.6, Proposition 3.30 and Theorem 3.29, we have

Theorem 3.31. *$\Omega G_{\mathbb{C}}$ is homotopy equivalent to $L_{\text{pol}}G_{\mathbb{C}}/P$.*

Theorem 3.32. *(see [**79**]) The homogeneous space*

$$L_{\text{pol}}G_{\mathbb{C}}/P = \coprod_{w \in \widetilde{W}/W} BwP/P.$$

Corollary 3.33. *The homogeneous space*

$$L_{\text{pol}}G_{\mathbb{C}}/B = \coprod_{w \in \widetilde{W}} BwB/B.$$

By Theorem 2.87 of chapter 2, we have an isomorphism

Theorem 3.34.

$$H^*(LG/T; \mathbb{C}) \cong H^*(L\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}; \mathbb{C}) \cong H^*(\widehat{L}\mathfrak{g}_{\mathbb{C}}, \widehat{\mathfrak{t}}_{\mathbb{C}}; \mathbb{C}) \cong H^*(\widehat{L}_{\text{pol}}G_{\mathbb{C}}/\widehat{B}; \mathbb{C}).$$

By Theorem 3.34, the $\mathbb{Z}$-cohomology ring of LG/T generated by the strata can be calculated using Corollary 1.93 of chapter 1. In the next section, we will work at an example.

4. Cohomology rings of the homogeneous spaces ΩSU_2 and LSU_2/T.

In order to determine the integral cohomology ring of LSU_2/T, we need some calculations in the integral cohomology of LSU_2/T.

Theorem 3.35. *For $n \geqslant 0$, the action of affine Weyl group of LSU_2 on the real root system is given by*

$$(r_{a_0}r_{a_1})^n(-\alpha, s) = (-\alpha, s+2n); \tag{3.1}$$

$$(r_{a_0}r_{a_1})^n(\alpha, r) = (\alpha, r-2n), \tag{3.2}$$

$$(r_{a_0}r_{a_1})^n r_{a_0}(-\alpha, s) = (\alpha, s-2n-2); \tag{3.3}$$

$$(r_{a_0}r_{a_1})^n r_{a_0}(\alpha, r) = (-\alpha, r+2n+2), \tag{3.4}$$

$$(r_{a_1}r_{a_0})^n(-\alpha, s) = (-\alpha, s-2n); \tag{3.5}$$

$$(r_{a_1}r_{a_0})^n(\alpha, r) = (\alpha, r+2n), \tag{3.6}$$

$$(r_{a_1}r_{a_0})^n r_{a_1}(-\alpha, s) = (\alpha, s+2n); \tag{3.7}$$

$$(r_{a_1}r_{a_0})^n r_{a_1}(\alpha, r) = (-\alpha, r-2n). \tag{3.8}$$

PROOF. First, by induction on n, we shall show that

$$(r_{a_0}r_{a_1})^n(-\alpha, s) = (-\alpha, s+2n)$$
$$(r_{a_0}r_{a_1})^n(\alpha, r) = (\alpha, r-2n),$$

for $(-\alpha, s)$, $(\alpha, r) \in \widehat{\Delta}_{\mathrm{re}}$. The case $n = 0$ is trivially true.

Now, we assume that the equations (3.1) and (3.2) hold for $n = l$. Then,

$$\begin{aligned}
(r_{a_0}r_{a_1})^{l+1}(-\alpha, s) &= (r_{a_0}r_{a_1})(r_{a_0}r_{a_1})^l(-\alpha, s)\\
&= (r_{a_0}r_{a_1})(-\alpha, s+2l)\\
&= r_{a_0}(\alpha, s+2l)\\
&= (-\alpha, s+2(l+1)),
\end{aligned}$$

and

$$\begin{aligned}(r_{a_0}r_{a_1})^{l+1}(\alpha,r) &= (r_{a_0}r_{a_1})(r_{a_0}r_{a_1})^{l}(\alpha,r)\\ &= (r_{a_0}r_{a_1})(\alpha,r-2l)\\ &= r_{a_0}(-\alpha,r-2l)\\ &= (\alpha,r-2(l+1)).\end{aligned}$$

This means that Equations (3.1) and (3.2) hold for any $n \geqslant 0$.

Since $(r_{a_1}r_{a_0})^n r_{a_1} = r_{a_1}(r_{a_0}r_{a_1})^n$, we can find easily the action of the reflection $(r_{a_1}r_{a_0})^n r_{a_1}$ on the real root system. Then, we have Equation (3.7) and (3.8),

$$(r_{a_1}r_{a_0})^n r_{a_1}(-\alpha,s) = r_{a_1}(r_{a_0}r_{a_1})^n(-\alpha,s) = r_{a_1}(-\alpha,s+2n) = (\alpha,s+2n),$$

and

$$(r_{a_1}r_{a_0})^n r_{a_1}(\alpha,r) = r_{a_1}(r_{a_0}r_{a_1})^n(\alpha,r) = r_{a_1}(\alpha,r-2n) = (-\alpha,r-2n).$$

Since $(r_{a_1}r_{a_0})^n$ is inverse of $(r_{a_0}r_{a_1})^n$, the action of $(r_{a_1}r_{a_0})^n$ on the real root system is given by

$$\begin{aligned}(r_{a_1}r_{a_0})^n(\alpha,r) &= (\alpha,r+2n)\\ (r_{a_1}r_{a_0})^n(-\alpha,s) &= (-\alpha,s-2n).\end{aligned}$$

Also, since $(r_{a_0}r_{a_1})^n r_{a_0} = r_{a_0}(r_{a_1}r_{a_0})^n$, the action of $(r_{a_0}r_{a_1})^n r_{a_0}$ on the real root system is given by

$$(r_{a_0}r_{a_1})^n r_{a_0}(\alpha,r) = r_{a_0}(r_{a_1}r_{a_0})^n(\alpha,r) = r_{a_0}(\alpha,r+2n) = (-\alpha,r+2n+2),$$

and

$$(r_{a_0}r_{a_1})^n r_{a_0}(-\alpha,s) = r_{a_0}(r_{a_1}r_{a_0})^n(-\alpha,s) = r_{a_0}(-\alpha,s-2n) = (\alpha,s-2n-2).$$

□

Corollary 3.36. *Let (α, u) and $(-\alpha, v), u \geqslant 0, v > 0$, be real positive roots of LSU_2. For $n \geqslant 0$,*

$$r_{(\alpha,u)}(r_{a_0}r_{a_1})^n(-\alpha, s) = (\alpha, s+2n+2u); \tag{3.9}$$

$$r_{(\alpha,u)}(r_{a_0}r_{a_1})^n(\alpha, r) = (-\alpha, r-2n-2u), \tag{3.10}$$

$$r_{(\alpha,u)}(r_{a_0}r_{a_1})^n r_{a_0}(-\alpha, s) = (-\alpha, s-2n-2u-2); \tag{3.11}$$

$$r_{(\alpha,u)}(r_{a_0}r_{a_1})^n r_{a_0}(\alpha, r) = (\alpha, r+2n+2u+2), \tag{3.12}$$

$$r_{(\alpha,u)}(r_{a_1}r_{a_0})^n(-\alpha, s) = (\alpha, s-2n+2u); \tag{3.13}$$

$$r_{(\alpha,u)}(r_{a_1}r_{a_0})^n(\alpha, r) = (-\alpha, r+2n-2u), \tag{3.14}$$

$$r_{(\alpha,u)}(r_{a_1}r_{a_0})^n r_{a_1}(-\alpha, s) = (-\alpha, s+2n-2u); \tag{3.15}$$

$$r_{(\alpha,u)}(r_{a_1}r_{a_0})^n r_{a_1}(\alpha, r) = (\alpha, r-2n+2u), \tag{3.16}$$

$$r_{(-\alpha,v)}(r_{a_0}r_{a_1})^n(-\alpha, s) = (\alpha, s+2n-2v); \tag{3.17}$$

$$r_{(-\alpha,v)}(r_{a_0}r_{a_1})^n(\alpha, r) = (-\alpha, r-2n+2v), \tag{3.18}$$

$$r_{(-\alpha,v)}(r_{a_0}r_{a_1})^n r_{a_0}(-\alpha, s) = (-\alpha, s-2n+2v-2); \tag{3.19}$$

$$r_{(-\alpha,v)}(r_{a_0}r_{a_1})^n r_{a_0}(\alpha, r) = (\alpha, r+2n-2v+2), \tag{3.20}$$

$$r_{(-\alpha,v)}(r_{a_1}r_{a_0})^n(-\alpha, s) = (\alpha, s-2n-2v); \tag{3.21}$$

$$r_{(-\alpha,v)}(r_{a_1}r_{a_0})^n(\alpha, r) = (-\alpha, r+2n+2v), \tag{3.22}$$

$$r_{(-\alpha,v)}(r_{a_1}r_{a_0})^n r_{a_1}(-\alpha, s) = (-\alpha, s+2n+2v); \tag{3.23}$$

$$r_{(-\alpha,v)}(r_{a_1}r_{a_0})^n r_{a_1}(\alpha, r) = (\alpha, r-2n-2v). \tag{3.24}$$

Theorem 3.37. *For $k \geqslant 0$, the following equations hold in $H^*(LSU_2/T, \mathbb{Z})$.*

$$(\varepsilon^{r_{a_0}})^{2k} = (2k)!\,\varepsilon^{(r_{a_0}r_{a_1})^k}, \tag{3.25}$$

$$(\varepsilon^{r_{a_1}})^{2k} = (2k)!\,\varepsilon^{(r_{a_1}r_{a_0})^k}, \tag{3.26}$$

$$(\varepsilon^{r_{a_0}})^{2k+1} = (2k+1)!\,\varepsilon^{(r_{a_0}r_{a_1})^k r_{a_0}}, \tag{3.27}$$

$$(\varepsilon^{r_{a_1}})^{2k+1} = (2k+1)!\,\varepsilon^{(r_{a_1}r_{a_0})^k r_{a_1}} \tag{3.28}$$

PROOF. By induction on k, we will show that these equations hold in $H^*(LSU_2/T, \mathbb{Z})$. For $k=0$, these equations hold.

Now, we assume that these equations hold for $k = n$. Then, we have to show that they hold for $k = n + 1$. By assumption,

$$\begin{aligned}(\varepsilon^{r_{a_0}})^{2n+2} &= (\varepsilon^{r_{a_0}}) \cdot (\varepsilon^{r_{a_0}})^{2n+1} \\ &= (2n+1)!\, \varepsilon^{r_{a_0}} \cdot \varepsilon^{(r_{a_0} r_{a_1})^n r_{a_0}}.\end{aligned}$$

We have

$$(\varepsilon^{r_{a_0}})^{2n+2} = (2n+1)! \sum_{(r_{a_0} r_{a_1})^n r_{a_0} \xrightarrow{\gamma} w} \chi_0(h_\gamma) \varepsilon^w.$$

When we check the action of the reflections which have length $2n + 2$, by the action of $r_{(\alpha,u)}(r_{a_0} r_{a_1})^n r_{a_0}$ and $r_{(-\alpha,v)}(r_{a_0} r_{a_1})^n r_{a_0}$ on the real root system, we see that the sum in the right side of the last cup product equation holds the only for the positive root

$$(-\alpha, 2n+2) = (2n+2)\mathbf{a}_0 + (2n+1)\mathbf{a}_1.$$

Then,

$$(\varepsilon^{r_{a_0}})^{2n+2} = (2n+2)!\, \varepsilon^{r_{(-\alpha,2n+2)} (r_{a_0} r_{a_1})^n r_{a_0}}.$$

The composition of reflections $r_{(-\alpha,2n+2)}(r_{a_0} r_{a_1})^n r_{a_0}$ can be represented by the Weyl group element $(r_{a_0} r_{a_1})^{n+1}$, so

$$(\varepsilon^{r_{a_0}})^{2n+2} = (2n+2)!\, \varepsilon^{(r_{a_0} r_{a_1})^{n+1}}.$$

If we continue the induction for equation (3.27), by assumption,

$$\begin{aligned}(\varepsilon^{r_{a_0}})^{2n+3} &= (\varepsilon^{r_{a_0}}) \cdot (\varepsilon^{r_{a_0}})^{2n+2} \\ &= (2n+2)!\, \varepsilon^{r_{a_0}} \cdot \varepsilon^{(r_{a_0} r_{a_1})^{n+1}}.\end{aligned}$$

We have

$$(\varepsilon^{r_{a_0}})^{2n+3} = (2n+2)! \sum_{(r_{a_0} r_{a_1})^{n+1} \xrightarrow{\gamma} w} \chi_0(h_\gamma) \varepsilon^w.$$

When we check the action of the reflections which have length $2n + 3$, by the action of $r_{(\alpha,u)}(r_{a_0} r_{a_1})^{n+1}$ and $r_{(-\alpha,v)}(r_{a_0} r_{a_1})^{n+1}$ on the real root system, we see that the sum in the right side of the last cup product equation holds only for the positive root

$$(-\alpha, 2n+3) = (2n+3)\mathbf{a}_0 + (2n+2)\mathbf{a}_1.$$

Then,

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+3} = (2n+3)!\,\varepsilon^{r_{(-\alpha,2n+3)}(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{n+1}}.$$

The composition of reflections $r_{(-\alpha,2n+3)}(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{n+1}$ can be represented by the Weyl group element $(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{n+1}r_{\mathbf{a}_0}$, so

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+3} = (2n+3)!\,\varepsilon^{(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{n+1}r_{\mathbf{a}_0}}.$$

Thus, we have proved that the equations (3.25) and (3.27) hold in $H^*(LSU_2/T, \mathbb{Z})$.

Similarly, by assumption,

$$\begin{aligned}(\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} &= (\varepsilon^{r_{\mathbf{a}_1}}) \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{2n+1} \\ &= (2n+1)!\,\varepsilon^{r_{\mathbf{a}_1}} \cdot \varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}}.\end{aligned}$$

We have

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} = (2n+1)! \sum_{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1} \xrightarrow{\gamma} w} \chi_1(h_\gamma)\varepsilon^w.$$

When we check the action of the reflections which have length $2n+2$, by the action of $r_{(\alpha,u)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}$ on the real root system, we see that the sum in the right side of the last cup product equation holds the only for the positive root

$$(\alpha, 2n+1) = (2n+1)\mathbf{a}_0 + (2n+2)\mathbf{a}_1.$$

Then,

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} = (2n+2)!\,\varepsilon^{r_{(\alpha,2n+1)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}}.$$

The composition of reflections $r_{(\alpha,2n+1)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}$ can be represented by the Weyl group element $(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^{n+1}$, so

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} = (2n+2)!\,\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^{n+1}}.$$

If we continue the induction for equation (3.28),

$$\begin{aligned}(\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} &= (\varepsilon^{r_{\mathbf{a}_1}}) \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} \\ &= (2n+2)!\,\varepsilon^{r_{\mathbf{a}_1}} \cdot \varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^{n+1}}.\end{aligned}$$

We have

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} = (2n+2)! \sum_{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1} \xrightarrow{\gamma} w} \chi_1(h_\gamma)\varepsilon^w.$$

When we check the action of the reflections which have length $2n+3$, by the action of $r_{(\alpha,u)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}$ on the real root system, we see that the sum in the right side of the last cup product equation holds the only for the positive root

$$(\alpha, 2n+2) = (2n+2)\mathbf{a}_0 + (2n+3)\mathbf{a}_1.$$

Then,

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} = (2n+3)!\, \varepsilon^{r_{(\alpha,2n+2)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}}.$$

The composition of reflections $r_{(\alpha,2n+2)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}$ can be represented by the Weyl group element $(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1} r_{\mathbf{a}_1}$, so

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} = (2n+3)!\, \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1} r_{\mathbf{a}_1}}.$$

So, the induction is completed and we have proved that all equations hold in $H^*(LSU_2/T, \mathbb{Z})$. □

We will make another calculation in the integral cohomology algebra of LSU_2/T.

Theorem 3.38. *For $n, m \geqslant 0$, the following equation holds in $H^*(LSU_2/T, \mathbb{Z})$.*

$$(n+m)(\varepsilon^{r_{\mathbf{a}_0}})^n \cdot (\varepsilon^{r_{\mathbf{a}_1}})^m = n(\varepsilon^{r_{\mathbf{a}_0}})^{n+m} + m(\varepsilon^{r_{\mathbf{a}_1}})^{n+m}.$$

PROOF. By induction on m, we shall prove that the result holds in $H^*(LSU_2/T, \mathbb{Z})$. Since the integral cohomology ring of LSU_2/T is torsion-free, the integral cohomology ring can be embedded in the rational cohomology ring hence the calculations can be done in the rational cohomology. For $m = 0$, the equation obviously holds.

First, we will verify the equation for $m = 1$. For $m = 1$, the equation reduces to

$$(n+1)(\varepsilon^{r_{\mathbf{a}_0}})^n \cdot (\varepsilon^{r_{\mathbf{a}_1}}) = n(\varepsilon^{r_{\mathbf{a}_0}})^{n+1} + (\varepsilon^{r_{\mathbf{a}_1}})^{n+1}. \tag{3.29}$$

Now, we will use sub-induction with respect to n on the equation (3.29). The equation (3.29) obviously holds for $n = 0$.

Now, we assume that equation (3.29) holds for $n = k$. We verify that equation (3.29) holds for $n = k+1$. By the induction hypothesis, we have

$$\begin{aligned}
\varepsilon^{r_{a_1}} \cdot \left(\varepsilon^{r_{a_0}}\right)^{k+1} &= \left(\varepsilon^{r_{a_1}} \cdot \left(\varepsilon^{r_{a_0}}\right)^{k}\right) \cdot \varepsilon^{r_{a_0}} \\
&= \left(\frac{k}{k+1}\left(\varepsilon^{r_{a_0}}\right)^{k+1} + \frac{1}{k+1}\left(\varepsilon^{r_{a_1}}\right)^{k+1}\right) \cdot \varepsilon^{r_{a_0}} \\
&= \frac{k}{k+1}\left(\varepsilon^{r_{a_0}}\right)^{k+2} + \frac{1}{k+1}\left(\varepsilon^{r_{a_1}}\right)^{k+1} \cdot \varepsilon^{r_{a_0}}. \qquad (3.30)
\end{aligned}$$

Now, we calculate the cup product

$$\left(\varepsilon^{r_{a_1}}\right)^{k+1} \cdot \varepsilon^{r_{a_0}}$$

in the above equation. We now treat the case k odd or even separately. If $k = 2l-1$, by equation (3.26),

$$\varepsilon^{r_{a_0}} \cdot \left(\varepsilon^{r_{a_1}}\right)^{2l} = (2l)!\left(\varepsilon^{r_{a_0}} \cdot \left(\varepsilon^{(r_{a_1}r_{a_0})^l}\right)\right). \qquad (3.31)$$

By the cup product formula,

$$\varepsilon^{r_{a_0}} \cdot \varepsilon^{(r_{a_1}r_{a_0})^l} = \sum_{(r_{a_1}r_{a_0})^l \xrightarrow{\gamma} w} \chi_0(h_\gamma)\varepsilon^{w}.$$

When we check the action of reflections $r_{(\alpha,u)}(r_{a_1}r_{a_0})^{l1}$ and $r_{(-\alpha,v)}(r_{a_1}r_{a_0})^l$ by the action of the Weyl group elements $(r_{a_1}r_{a_0})^l r_{a_1}$ and $(r_{a_0}r_{a_1})^l r_{a_0}$ which has length $2l+1$, we see that the reflections $r_{(-\alpha,1)}(r_{a_1}r_{a_0})^l$ and $r_{(\alpha,2l)}(r_{a_1}r_{a_0})^l$ can be represented by the Weyl group elements $(r_{a_0}r_{a_1})^l r_{a_0}$ and $(r_{a_1}r_{a_0})^l r_{a_1}$ respectively. Using the positive root $(\alpha, 2l) = (2l)\,a_0 + (2l+1)\,a_1$ in the cup product formula,

$$\varepsilon^{r_{a_0}} \cdot \varepsilon^{(r_{a_1}r_{a_0})^l} = \varepsilon^{(r_{a_0}r_{a_1})^l r_{a_0}} + (2l)\,\varepsilon^{(r_{a_1}r_{a_0})^l r_{a_1}}. \qquad (3.32)$$

By equations (3.27) and (3.28),

$$\varepsilon^{r_{a_0}} \cdot \varepsilon^{(r_{a_1}r_{a_0})^l} = \frac{1}{(2l+1)!}\left(\varepsilon^{r_{a_0}}\right)^{2l+1} + \frac{2l}{(2l+1)!}\left(\varepsilon^{r_{a_1}}\right)^{2l+1}. \qquad (3.33)$$

When the last result is placed in the equation (3.31), we have

$$\begin{aligned}
\varepsilon^{r_{a_0}} \cdot \left(\varepsilon^{r_{a_1}}\right)^{2l} &= (2l)!\left(\varepsilon^{r_{a_0}} \cdot \left(\varepsilon^{(r_{a_1}r_{a_0})^l}\right)\right) \\
&= (2l)!\left(\frac{1}{(2l+1)!}\left(\varepsilon^{r_{a_0}}\right)^{2l+1} + \frac{2l}{(2l+1)!}\left(\varepsilon^{r_{a_1}}\right)^{2l+1}\right) \\
&= \frac{1}{2l+1}\left(\varepsilon^{r_{a_0}}\right)^{2l+1} + \frac{2l}{2l+1}\left(\varepsilon^{r_{a_1}}\right)^{2l+1}.
\end{aligned}$$

Using $k = 2l - 1$, we have

$$\varepsilon^{r_{a_0}} \cdot (\varepsilon^{r_{a_1}})^{k+1} = \frac{1}{k+2}(\varepsilon^{r_{a_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{a_1}})^{k+2}. \tag{3.34}$$

When the last result is placed in the equation (3.30), we have

$$\begin{aligned}
\varepsilon^{r_{a_1}} \cdot (\varepsilon^{r_{a_0}})^{k+1} &= \frac{k}{k+1}(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+1}(\varepsilon^{r_{a_1}})^{k+1} \cdot \varepsilon^{r_{a_0}} \\
&= \frac{k}{k+1}(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+1}\left(\frac{1}{k+2}(\varepsilon^{r_{a_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{a_1}})^{k+2}\right) \\
&= \left(\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}\right)(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+2}(\varepsilon^{r_{a_1}})^{k+2} \\
&= \frac{k+1}{k+2}(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+2}(\varepsilon^{r_{a_1}})^{k+2}.
\end{aligned}$$

If $k = 2l$, by the equation (3.28),

$$(\varepsilon^{r_{a_0}}) \cdot (\varepsilon^{r_{a_1}})^{2l+1} = (2l+1)!\left((\varepsilon^{r_{a_0}}) \cdot (\varepsilon^{(r_{a_1}r_{a_0})^l r_{a_1}})\right). \tag{3.35}$$

By the cup product formula,

$$(\varepsilon^{r_{a_0}}) \cdot (\varepsilon^{(r_{a_1}r_{a_0})^l r_{a_1}}) = \sum_{(r_{a_1}r_{a_0})^l r_{a_1} \xrightarrow{\gamma} w} \chi_0(h_\gamma)\varepsilon^w.$$

When we check the action of reflections $r_{(\alpha,u)}(r_{a_1}r_{a_0})^l r_{a_1}$ and $r_{(-\alpha,v)}(r_{a_1}r_{a_0})^l r_{a_1}$ by the action of the Weyl group elements $(r_{a_1}r_{a_0})^{l+1}$ and $(r_{a_0}r_{a_1})^{l+1}$, which has length $2l+2$, we see that the reflections $r_{(-\alpha,1)}(r_{a_1}r_{a_0})^l r_{a_1}$ and $r_{(\alpha,2l+1)}(r_{a_1}r_{a_0})^l r_{a_1}$ can be represented by the Weyl group elements $(r_{a_0}r_{a_1})^{l+1}$ and $(r_{a_1}r_{a_0})^{l+1}$ respectively. Using the positive root $(\alpha, 2l+1) = (2l+1)\,\mathbf{a}_0 + (2l+2)\,\mathbf{a}_1$, we have

$$(\varepsilon^{r_{a_0}}) \cdot (\varepsilon^{(r_{a_1}r_{a_0})^l r_{a_1}}) = \varepsilon^{(r_{a_0}r_{a_1})^{l+1}} + (2l+1)\,\varepsilon^{(r_{a_1}r_{a_0})^{l+1}}. \tag{3.36}$$

By equations (3.25) and (3.26),

$$\varepsilon^{r_{a_0}} \cdot \varepsilon^{(r_{a_1}r_{a_0})^l r_{a_1}} = \frac{1}{(2l+2)!}(\varepsilon^{r_{a_0}})^{2l+2} + \frac{2l+1}{(2l+2)!}(\varepsilon^{r_{a_1}})^{2l+2}. \tag{3.37}$$

When the last result is placed in the equation (3.35), we have

$$\begin{aligned}
\varepsilon^{r_{a_0}} \cdot (\varepsilon^{r_{a_1}})^{2l+1} &= (2l+1)!\left(\varepsilon^{r_{a_0}} \cdot (\varepsilon^{(r_{a_1}r_{a_0})^l r_{a_1}})\right) \\
&= (2l+1)!\left(\frac{1}{(2t+2)!}(\varepsilon^{r_{a_0}})^{2l+2} + \frac{2l+1}{(2l+2)!}(\varepsilon^{r_{a_1}})^{2l+2}\right) \\
&= \frac{1}{2l+2}(\varepsilon^{r_{a_0}})^{2l+2} + \frac{2l+1}{2l+2}(\varepsilon^{r_{a_1}})^{2l+2}.
\end{aligned}$$

Using $k = 2l$, we have

$$\varepsilon^{r_{a_0}} \cdot (\varepsilon^{r_{a_1}})^{k+1} = \frac{1}{k+2}(\varepsilon^{r_{a_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{a_1}})^{k+2}. \tag{3.38}$$

When the last result is placed in the equation (3.30), we have

$$\begin{aligned}
\varepsilon^{r_{a_1}} \cdot (\varepsilon^{r_{a_0}})^{k+1} &= \frac{k}{k+1}(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+1}(\varepsilon^{r_{a_1}})^{k+1} \cdot \varepsilon^{r_{a_0}} \\
&= \frac{k}{k+1}(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+1}\left(\frac{1}{k+2}(\varepsilon^{r_{a_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{a_1}})^{k+2}\right) \\
&= \left(\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}\right)(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+2}(\varepsilon^{r_{a_1}})^{k+2} \\
&= \frac{k+1}{k+2}(\varepsilon^{r_{a_0}})^{k+2} + \frac{1}{k+2}(\varepsilon^{r_{a_1}})^{k+2}.
\end{aligned}$$

The induction on n is completed. Thus, we proved that the equation holds for $m = 1$.

We assume that equation holds for $m = s$. Then, we will verify that it holds for $m = s + 1$. By assumption,

$$\begin{aligned}
(\varepsilon^{r_{a_0}})^n \cdot (\varepsilon^{r_{a_1}})^{s+1} &= \left(\frac{n}{n+s}(\varepsilon^{r_{a_0}})^{n+s} + \frac{s}{n+s}(\varepsilon^{r_{a_1}})^{n+s}\right) \cdot \varepsilon^{r_{a_1}} \\
&= \frac{n}{n+s}(\varepsilon^{r_{a_0}})^{n+s} \cdot \varepsilon^{r_{a_1}} + \frac{s}{n+s}(\varepsilon^{r_{a_1}})^{n+s+1} \\
&= \frac{n}{n+s}\left(\frac{n+s}{n+s+1}(\varepsilon^{r_{a_0}})^{n+s+1} + \frac{1}{n+s+1}(\varepsilon^{r_{a_1}})^{n+s+1}\right) + \\
&\quad \frac{s}{n+s}(\varepsilon^{r_{a_1}})^{n+s+1} \\
&= \frac{n}{n+s+1}(\varepsilon^{r_{a_0}})^{n+s+1} + \left(\frac{n}{(n+s).(n+s+1)} + \frac{s}{n+s}\right)(\varepsilon^{r_{a_1}})^{n+s+1} \\
&= \frac{n}{n+s+1}(\varepsilon^{r_{a_0}})^{n+s+1} + \frac{s^2+s(n+1)+n}{(n+s)(n+s+1)}(\varepsilon^{r_{a_1}})^{n+s+1} \\
&= \frac{n}{n+s+1}(\varepsilon^{r_{a_0}})^{n+s+1} + \frac{s+1}{(n+s+1)}(\varepsilon^{r_{a_1}})^{n+s+1}.
\end{aligned}$$

Thus, the induction is completed. □

Let R be a commutative ring with unit and let $\Gamma_R(x_0, x_1)$ be the divided power algebra over R, where $\deg x_0 = \deg x_1 = 2$.

Theorem 3.39. *Then, $H^*(LSU_2/T, R)$ is graded isomorphic to $\Gamma_R(x_0, x_1)/I_R$ where the ideal I_R is given by*

$$I_R = \left(x_0^{[n]}x_1^{[m]} - \binom{n+m-1}{m}x_0^{[n+m]} - \binom{n+m-1}{n}x_1^{[n+m]} : m, n \geqslant 1\right)$$

and which has the R-module basis $\{x_0^{[n]}, x_1^{[n]}\}$ in each degree $2n$ for $n \geqslant 1$.

PROOF. Since the odd dimensional cohomology is trivial, by the universal coefficient theorem, it suffices to prove this for $R = \mathbb{Z}$. The Schubert classes $\{\varepsilon^w\}_{w \in \widetilde{W}_{LSU(2)}}$ form a basis of the integral cohomology $H^*(LSU_2/T, \mathbb{Z})$ such that $\varepsilon^w \in H^{2\ell(w)}(LSU_2/T, \mathbb{Z})$. Since the cohomology module basis is indexed by the affine Weyl group $\widetilde{W}$, the Poincaré series over $\mathbb{Z}$ of cohomology of LSU_2/T is

$$P(t, \mathbb{Z}) = 1 + \sum_{k=1}^{\infty} 2t^{2k}.$$

Now we will show that the integral cohomology algebra $H^*(LSU_2/T, \mathbb{Z})$ is isomorphic to the quotient of divided power algebra $\Gamma_{\mathbb{Z}}(x_0, x_1)/I_{\mathbb{Z}}$. Then, we define a $\mathbb{Z}$-algebra homomorphism ψ from the divided power algebra $\Gamma_{\mathbb{Z}}(x_0, x_1)$ to the integral cohomology of LSU_2/T as follows.

For $U = \sum_{i=0}^{n} u_i \, x_0^{[i]} x_1^{[n-i]}$ with $u_i \in \mathbb{Z}$, let

$$\psi(U) = u_n X(n) + u_0 Y(n) + \sum_{i=1}^{n-1} \left[\binom{n-1}{n-i} X(n) + \binom{n-1}{i} Y(n) \right] u_i,$$

where

$$X(n) = \begin{cases} \varepsilon^{(r_{a_0} r_{a_1})^l} & \text{for } n = 2l \\ \varepsilon^{(r_{a_0} r_{a_1})^l r_{a_0}} & \text{for } n = 2l+1 \end{cases}$$

$$Y(n) = \begin{cases} \varepsilon^{(r_{a_1} r_{a_0})^l} & \text{for } n = 2l \\ \varepsilon^{(r_{a_1} r_{a_0})^l r_{a_1}} & \text{for } n = 2l+1. \end{cases}$$

We will show that ψ is a $\mathbb{Z}$-algebra homomorphism. Let

$$U = \sum_{i=0}^{n} u_i \, x_0^{[i]} x_1^{[n-i]} \quad V = \sum_{j=0}^{m} v_j \, x_0^{[j]} x_1^{[m-j]},$$

where $u_i, v_j \in \mathbb{Z}$. First, let us calculate

$$\psi(U)\cdot\psi(V) = \psi\left(\sum_{i=0}^{n} u_i\, x_0^{[i]} x_1^{[n-i]}\right)\cdot\psi\left(\sum_{j=0}^{m} v_j\, x_0^{[j]} x_1^{[m-j]}\right)$$

$$= \left(u_0 Y(n) + u_n X(n) + \sum_{i=1}^{n-1} u_i\left[\binom{n-1}{i-1}X(n) + \binom{n-1}{i}Y(n)\right]\right)\cdot$$

$$\left(v_0 Y(m) + v_m X(m) + \sum_{j=1}^{m-1} v_j\left[\binom{m-1}{j-1}X(m) + \binom{m-1}{j}Y(m)\right]\right)$$

$$= u_0v_0Y(n)Y(m) + u_0v_mY(n)X(m) + \sum_{j=1}^{m-1} u_0v_j\left[\binom{m-1}{j-1}Y(n)X(m) + \binom{m-1}{j}Y(n)Y(m)\right]$$

$$+u_nv_0X(n)Y(m) + u_nv_mX(n)X(m) + \sum_{j=1}^{m-1} u_nv_j\left[\binom{m-1}{j-1}X(n)X(m) + \binom{m-1}{j}X(n)Y(m)\right]$$

$$+\sum_{i=1}^{n-1} u_iv_0\left[\binom{n-1}{i-1}X(n)Y(m) + \binom{n-1}{i}Y(n)Y(m)\right] +$$

$$\sum_{i=1}^{n-1} u_iv_m\left[\binom{n-1}{i-1}X(n)X(m) + \binom{n-1}{i}Y(n)X(m)\right]$$

$$+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_iv_j\left[\binom{n-1}{i-1}\binom{m-1}{j-1}X(n)X(m) + \binom{n-1}{i-1}\binom{m-1}{j}X(n)Y(m)\right]$$

$$+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_iv_j\left[\binom{n-1}{i}\binom{m-1}{j-1}Y(n)X(m) + \binom{n-1}{i}\binom{m-1}{j}Y(n)Y(m)\right].$$

By equations (3.25), (3.26), (3.27), (3.28) and (3.38),

$$Y(n)Y(m) = \binom{n+m}{n}Y(n+m)$$
$$X(n)X(m) = \binom{n+m}{n}X(n+m),$$

$$X(n)Y(m) = \binom{n+m-1}{m}X(n+m) + \binom{n+m-1}{n}Y(n+m)$$

and

$$Y(n)X(m) = \binom{n+m-1}{n}X(n+m) + \binom{n+m-1}{m}Y(n+m).$$

If we put the last results in the equation, we have

$$\psi(U)\cdot\psi(V) = X(n+m)\Bigg\{u_0v_m\binom{m+n-1}{n} + \sum_{j=1}^{m-1} u_0v_j\binom{m-1}{j-1}\binom{m+n-1}{n} + u_nv_0$$

$$\binom{m+n-1}{m} + u_nv_m\binom{n+m}{n} + \sum_{j=1}^{m-1} u_nv_j\left[\binom{m-1}{j-1}\binom{n+m}{n} + \binom{m-1}{j}\binom{m+n-1}{m}\right] +$$

$$\sum_{i=1}^{n-1} u_iv_0\binom{n-1}{i-1}\binom{m+n-1}{m} + \sum_{i=1}^{n-1} u_iv_m\left[\binom{n-1}{i-1}\binom{n+m}{n} + \binom{n-1}{i}\binom{n+m-1}{n}\right] +$$

$$\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_iv_j\left[\binom{n-1}{i-1}\binom{m-1}{j-1}\binom{n+m}{n} + \binom{n-1}{i-1}\binom{m-1}{j}\binom{n+m-1}{m}\right] +$$

$$\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_iv_j\left[\binom{n-1}{i}\binom{m-1}{j-1}\binom{n+m-1}{n}\right]\Bigg\} +$$

$$Y(n+m)\Bigg\{u_0v_0\binom{n+m}{n} + u_0v_m\binom{n+m-1}{m} +$$

$$\sum_{j=1}^{m-1} u_0v_j\left[\binom{m-1}{j-1}\binom{m+n-1}{m} + \binom{m-1}{j}\binom{n+m}{n}\right] +$$

$$u_nv_0\binom{m+n-1}{n} + \sum_{j=1}^{m-1} u_nv_j\binom{m-1}{j}\binom{m+n-1}{n} +$$

$$\sum_{i=1}^{n-1} u_iv_0\left[\binom{n-1}{i-1}\binom{n+m-1}{n} + \binom{n-1}{i}\binom{n+m}{n}\right]$$

$$+\sum_{i=1}^{n-1} u_iv_m\binom{n-1}{i}\binom{n+m-1}{m} + \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_iv_j\binom{n-1}{i-1}\binom{m-1}{j}\binom{n+m-1}{n} +$$

$$\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_iv_j\left[\binom{n-1}{i}\binom{m-1}{j-1}\binom{n+m-1}{m} + \binom{n-1}{i}\binom{m-1}{j}\binom{n+m}{n}\right]\Bigg\}.$$

Now an expanding,

$$U\cdot V = u_0v_0\binom{n+m}{n}x_1^{[n+m]} + u_0v_mx_0^{[m]}x_1^{[n]} + \sum_{j=1}^{m-1}u_0v_j\binom{n+m-j}{n}x_0^{[j]}x_1^{[n+m-j]}$$
$$+ u_nv_0x_0^{[n]}x_1^{[m]} + u_nv_m\binom{n+m}{n}x_0^{[n+m]} + \sum_{j=1}^{m-1}u_nv_j\binom{n+j}{n}x_0^{[n+j]}x_1^{[m-j]}$$
$$+\sum_{i=1}^{n-1}u_iv_0\binom{n+m-i}{m}x_0^{[i]}x_1^{[n+m-i]} + \sum_{i=1}^{n-1}u_iv_m\binom{m+i}{i}x_0^{[m+i]}x_1^{[n-i]}$$
$$+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\binom{i+j}{i}\binom{(n+m)-(i+j)}{n-i}x_0^{[i+j]}x_1^{[(n+m)-(i+j)]}.$$

Hence,

$$\psi(U\cdot V) = X(n+m)\Bigg\{u_0v_m\binom{n+m-1}{n} + \sum_{j=1}^{m-1}u_0v_j\binom{n+m-j}{n}\binom{n+m-1}{j-1}+$$
$$u_nv_0\binom{n+m-1}{m} + u_nv_m\binom{n+m}{n} + \sum_{i=1}^{n-1}u_iv_m\binom{m+i}{i}\binom{m+n-1}{n-i}+$$
$$\sum_{j=1}^{m-1}u_nv_j\binom{n+j}{n}\binom{n+m-1}{m-j} + \sum_{i=1}^{n-1}u_iv_0\binom{n+m-i}{m}\binom{n+m-1}{i-1}$$
$$+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\binom{i+j}{i}\binom{(n+m)-(i+j)}{n-i}\binom{n+m-1}{i+j-1}\Bigg\}$$
$$+Y(n+m)\Bigg\{u_0v_0\binom{n+m}{n} + u_0v_m\binom{n+m-1}{m} + \sum_{j=1}^{m-1}u_0v_j\binom{n+m-j}{n}\binom{n+m-1}{j}+$$
$$u_nv_0\binom{n+m-1}{n}+\sum_{j=1}^{m-1}u_nv_j\binom{n+j}{n}\binom{n+m-1}{n+j}+\sum_{i=1}^{n-1}u_iv_0\binom{n+m-i}{m}\binom{n+m-1}{i}+$$
$$\sum_{i=1}^{n-1}u_iv_m\binom{i+m}{i}\binom{m+n-1}{m+i} + \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\binom{i+j}{i}\binom{(n+m)-(i+j)}{n-i}\binom{n+m-1}{i+j}\Bigg\}.$$

We show that $\psi(U\cdot V)=\psi(u)\cdot\psi(V)$ for all polynomials U,V. In order to verify this equation, we need the equality of the coefficients of u_iv_j in the both sides of this equation. We see that the coefficients of $u_iv_j, i=0,\dots,n$ and $j=0,\dots,n$ in the both sides of the equation are equal for $X(n+m)$ as well as $Y(n+m)$. Then ψ is a $\mathbb{Z}$-algebra homomorphism.

We will show that the $\mathbb{Z}$-algebra homomorphism ψ is surjective. Because, for every element $aX(n)+bY(n) \in H^{2n}(LSU_2/T,\mathbb{Z})$, we have $a\,x_0^{[n]}+b\,x_1^{[n]}$ such that $\psi(a\,x_0^{[n]}+b\,x_1^{[n]}) = aX(n)+bY(n)$, where $a,b \in \mathbb{Z}$.

Now we want to find the kernel of the homomorphism ψ. For $n,m \geqslant 1$, let

$$u_{n,m} = x_0^{[n]} \cdot x_1^{[m]} - \binom{n+m-1}{m} x_0^{[n+m]} - \binom{n+m-1}{n} x_1^{[n+m]}. \tag{3.39}$$

We claim that the kernel of the homomorphism ψ is equal to the following ideal $I_{\mathbb{Z}}$ generated by the elements $u_{n,m}$.

$$I_{\mathbb{Z}} = \sum_{k \geqslant 2} I_{\mathbb{Z}}^k,$$

where

$$I_{\mathbb{Z}}^k = \left\{ \sum_{0<r<k} t_r^k \left(x_0^{[r]} x_1^{[k-r]} - \binom{k-1}{k-r} x_0^{[k]} - \binom{k-1}{r} x_1^{[k]} \right) : t_r^k \in \Gamma_{\mathbb{Z}}(x_0,x_1) \right\}.$$

Now we will prove that our claim is true. Let $U \in I_{\mathbb{Z}}^k$. Then

$$\begin{aligned}
\psi(U) &= \psi\left(\sum_{0<r<k} t_r^k (x_0^{[r]} x_1^{[k-r]} - \binom{k-1}{k-r} x_0^{[k]} - \binom{k-1}{r} x_1^{[k]}) \right) \\
&= \sum_{0<r<k} \psi(t_r^k) \cdot \psi\left(x_0^{[r]} x_1^{[k-r]} - \binom{k-1}{k-r} x_0^{[k]} - \binom{k-1}{r} x_1^{[k]} \right).
\end{aligned}$$

Then $\psi(U)$ is equal to

$$\sum_{0<r<k} \psi(t_r^k) \left(\binom{k-1}{k-r} X(k) + \binom{k-1}{r} Y(k) - \binom{k-1}{k-r} X(k) - \binom{k-1}{r} Y(k) \right).$$

Then $\psi(U)=0$. So, $U \in \ker\psi$.

Conversely, let $U = \sum_{i=0}^{k} u_i x_0^{[i]} x_1^{[k-i]} \in \ker\psi$. Then,

$$\psi(U) = u_0 Y(k) + u_k X(k) + \sum_{i=1}^{k-1} u_i \left[\binom{k-1}{k-i} X(k) + \binom{k-1}{i} Y(k) \right] = 0,$$

So, we have to determine the solution of the homogeneous linear equations system $A \cdot v = 0$, where

$$A = \begin{pmatrix} 1 & k-1 & \dots & \binom{k-1}{i} & \dots & 1 & 0 \\ 0 & 1 & \dots & \binom{k-1}{k-i} & \dots & k-1 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ u_{k-1} \\ u_k \end{pmatrix}.$$

The rank of the matrix A is 2, so we have infinite solution vectors which have $k-1$ linear independent components and other two components depend these linear independent components. Then,

$$v = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ u_{k-1} \\ u_k \end{pmatrix} = \begin{pmatrix} -\sum_{i=1}^{k-1} t_i \binom{k-1}{i} \\ t_1 \\ \vdots \\ t_i \\ \vdots \\ t_{k-1} \\ -\sum_{i=1}^{k-1} t_i \binom{k-1}{k-i}, \end{pmatrix}$$

where $t_i \in \mathbb{Z}$ for $i = 1, \dots, k-1$. So, $U \in \ker \psi$ is given by

$$\begin{aligned} U &= -\sum_{i=1}^{k-1} t_i \binom{k-1}{i} x_1^{[k]} - \sum_{i=1}^{k-1} t_i \binom{k-1}{k-i} x_0^{[k]} + \sum_{i=1}^{k-1} t_i x_0^{[i]} x_1^{[k-i]} \\ &= \sum_{i=1}^{k-1} t_i \left(x_0^{[i]} x_1^{[k-i]} - \binom{k-1}{k-i} x_0^{[k]} - \binom{k-1}{i} x_1^{[k]} \right) \end{aligned}$$

for some $t_i \in \mathbb{Z}$. Thus, we have proved that $U \in I_{\mathbb{Z}}^k$. □

Theorem 3.40. *Under the isomorphism ψ, the $\mathbb{Z}$-module operator A^i of $H^*(LSU_2/T, \mathbb{Z})$ corresponds to the partial derivation operator*

$$\begin{cases} \frac{\partial}{\partial x_j} & \text{for degree } 4n \\ \frac{\partial}{\partial x_i} & \text{for degree } 4n+2 \end{cases}$$

for $i \neq j$, $i = 0, 1$.

Proof. We will prove that $\mathbb{Z}$-cohomology operator A^i corresponds to the partial derivation operators as stated. By definition of A^i, we have

$$\begin{aligned}
A^0 \varepsilon^{(r_{a_0} r_{a_1})^n} &= 0, \\
A^1 \varepsilon^{(r_{a_0} r_{a_1})^n} &= \varepsilon^{(r_{a_0} r_{a_1})^{n-1} r_{a_0}}, \\
A^0 \varepsilon^{(r_{a_0} r_{a_1})^n r_{a_0}} &= \varepsilon^{(r_{a_0} r_{a_1})^n}, \\
A^1 \varepsilon^{(r_{a_0} r_{a_1})^n r_{a_0}} &= 0, \\
A^0 \varepsilon^{(r_{a_1} r_{a_0})^n} &= \varepsilon^{(r_{a_1} r_{a_0})^{n-1} r_{a_1}}, \\
A^1 \varepsilon^{(r_{a_1} r_{a_0})^n} &= 0, \\
A^0 \varepsilon^{(r_{a_1} r_{a_0})^n r_{a_1}} &= 0, \\
A^1 \varepsilon^{(r_{a_1} r_{a_0})^n r_{a_1}} &= \varepsilon^{(r_{a_1} r_{a_0})^n}.
\end{aligned}$$

By ψ isomorphism, we have the following correspondences

$$\varepsilon^{(r_{a_0} r_{a_1})^n} \longleftrightarrow x_0^{[2n]}, \quad \varepsilon^{(r_{a_0} r_{a_1})^n r_{a_0}} \longleftrightarrow x_0^{[2n+1]},$$
$$\varepsilon^{(r_{a_1} r_{a_0})^n} \longleftrightarrow x_1^{[2n]}, \quad \varepsilon^{(r_{a_1} r_{a_0})^n r_{a_1}} \longleftrightarrow x_1^{[2n+1]}.$$

The last equations and correspondences verify our claim. □

Corollary 3.41. *The partial derivation operator $\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}$ on the divided power algebra induces a derivation on cohomology of LSU_2/T.*

Now we will discuss cohomology of ΩG respect to LG/T and G/T where G is a compact semi-simple Lie group. Since ΩG is homotopic to Ω_{pol}, the discussion can be restricted to the Kač-Moody groups and homogeneous spaces. The Lie algebras of $L_{\text{pol}} G_{\mathbb{C}}/B^+$, $L_{\text{pol}} G_{\mathbb{C}}/G_{\mathbb{C}}$ and $G_{\mathbb{C}}/B$ are $\mathbf{g}[t, t^{-1}]/\mathbf{b}^+$, $\mathbf{g}[t, t^{-1}]/\mathbf{g}$ and $\mathbf{g}/\mathbf{b}$ respectively.

There is a surjective homomorphism

$$\mathrm{ev}_{t=1} : \mathbf{g}[t, t^{-1}]/\mathbf{b}^{+} \to \mathbf{g}/\mathbf{b},$$

with $\ker \mathrm{ev}_{t=1} = \mathbf{g}[t, t^{-1}]/\mathbf{g}$. Since the odd cohomology groups of $\mathbf{g}[t, t^{-1}]/\mathbf{b}^{+}$ and $\mathbf{g}/\mathbf{b}$ are trivial, the second term E_2^{**} of the Leray-Serre spectral sequence collapses and hence we have

Theorem 3.42. *Let R is a commutative ring with unit. Then there exists an injective homomorphism $j : H^*(G/T, R) \to H^*(LG/T, R)$ and a surjective homomorphism $i : H^*(LG/T, R) \to H^*(\Omega G, R)$. In particular, $J = \mathrm{im}\, j^{+}$ is an ideal of $H^*(LG/T, R)$ and*

$$H^*(\Omega G, R) \cong H^*(LG/T, R)//J.$$

Corollary 3.43.

$$H^*(\Omega SU_2, R) \cong \Gamma_R(x, y)/\left(I_R, a(x^{[1]} - y^{[1]})\right) \cong \Gamma_R(x),$$

where $a \in R$.

Now we will give a different approach to determine the cohomology ring of based loop group ΩG using the Schubert calculus. For a compact simply-connected semi-simple Lie group G, we have from Theorem 2.40 of chapter 2

Theorem 3.44. *The natural map*

$$G \to LG \to LG/G \cong \Omega G,$$

is a split extension of Lie groups.

Theorem 3.45. *Let G be a compact simply-connected semi-simple Lie group and let T be a maximal torus of G. Then $\pi : LG/T \to LG/G$ is a fiber bundle with the fibre G/T.*

PROOF. Since $LG \to LG/G$ is a principal G-bundle and G/T is a left G-space by the action $g_1 \cdot g_2T = g_1g_2T$ for $g_1, g_2 \in G$, we have a fibration

$$G/T \to LG \times_G G/T \to \Omega G.$$

Therefore, we have to show that $LG \times_G G/T$ is diffeomorphic to LG/T. Since $LG \times_G G/T$ is equal to

$$\{[\gamma, gT] : [\gamma, gT] = [\gamma h, h^{-1}gT] \, \forall g, h \in G, \gamma \in LG\},$$

we define a smooth map $\tau : LG \times_G G/T \to LG/T$ given by $[\gamma, gT] \to \gamma gT$. It is well-defined because for $h \in G$,

$$\begin{aligned}\tau([\gamma h, h^{-1}gT]) &= \gamma h h^{-1} gT \\ &= \gamma gT \\ &= \tau([\gamma, gT]).\end{aligned}$$

For every γT, we can find an element $[\gamma, T] \in LG \times_G G/T$ such that $\tau([\gamma, T]) = \gamma T$. So, τ is a surjective map. Now, we will show that τ is an injective map. Let $[\gamma_1, g_1T], [\gamma_2, g_2T] \in LG \times_G G/T$ such that

$$\tau([\gamma_1, g_1T]) = \tau([\gamma_2, g_2T]). \tag{3.40}$$

The equation (3.40) gives

$$\gamma_1 g_1 T = \gamma_2 g_2 T.$$

So, $(\gamma_1 g_1)^{-1}(\gamma_2 g_2), (\gamma_2 g_2)^{-1}(\gamma_1 g_1) \in T$. Then,

$$\begin{aligned}[\gamma_1, g_1T] &= [\gamma_1 g_1, g_1^{-1} g_1 T] \\ &= [\gamma_1 g_1, T] \\ &= [(\gamma_1 g_1)(\gamma_1 g_1)^{-1}(\gamma_2 g_2), (\gamma_2 g_2)^{-1}(\gamma_1 g_1)T] \\ &= [\gamma_2 g_2, T] \\ &= [\gamma_2 g_2 g_2^{-1}, g_2 T] \\ &= [\gamma_2, g_2 T].\end{aligned}$$

Thus, we proved that τ is an injective map and it's inverse is given by $\gamma T \to [\gamma, T]$ which is smooth map. Then, $\pi : LG/T \to LG/G = \Omega G$ given by $\gamma T \to \gamma G$ is a fiber bundle map. $\square$

Since LG/T is a fiber bundle over ΩG with the fiber G/T, by the Leray-Serre spectral sequence of the fibration and Corollary (5.13) of Kostant and Kumar [**68**], $\theta : H^*(\Omega G, \mathbb{Z}) \to H^*(LG/T, \mathbb{Z})$ is injective and $\theta(H^*(\Omega G, \mathbb{Z}))$ is generated by the Schubert classes $\{\varepsilon^w\}_{w\in\widehat{W}}$ in the cohomology of $LG/T$and hence we can determine the cohomology ring of ΩG.

Let R be a commutative ring with unit and let $\Gamma_R(\gamma)$ be the divided power algebra with $\deg\gamma = 2$.

Theorem 3.46. *$H^*(\Omega SU(2), R)$ is isomorphic to $\Gamma_R(\gamma)$ with the R-module basis $\gamma^{[n]}$ in each degree $2n$ for $n \geqslant 1$.*

PROOF. Since the odd cohomology is trivial, by the universal coefficient theorem, it suffices to prove this for $R = \mathbb{Z}$. The integral cohomology of ΩSU_2 is generated by the Schubert classes indexed

$$\widehat{W} = \{\overline{\ell(w)} : w \in \widetilde{W}\} = \{(r_{a_0}r_{a_1})^n, (r_{a_0}r_{a_1})^n r_{a_0} : n \geqslant 0\}.$$

Then, we define a $\mathbb{Z}$-algebra homomorphism η from $\Gamma_{\mathbb{Z}}(\gamma)$ to $H^*(\Omega SU_2, \mathbb{Z})$ given as follows. For $n \geqslant 0, u_n \in \mathbb{Z}$, $\eta(u_n\gamma^{[n]}) = u_n X(n)$. Now, we will show that η is a $\mathbb{Z}$-algebra homomorphism. We have

$$\begin{aligned}\eta\left(\gamma^{[n]} \cdot \gamma^{[m]}\right) &= \eta\left(\binom{n+m}{n}\gamma^{[n+m]}\right)\\ &= \binom{n+m}{n} X(n+m).\end{aligned}$$

Let us calculate $\eta(\gamma^{[n]}) \cdot \eta(\gamma^{[m]}) = X(n) \cdot X(m)$. By equations (3.25) and (3.27), we have

$$X(n) \cdot X(m) = \binom{n+m}{n} X(n+m).$$

So,

$$\eta(\gamma^{[n]}) \cdot \eta(\gamma^{[m]}) = \binom{n+m}{n} X(m+n).$$

Then, we have shown that η is a $\mathbb{Z}$-algebra homomorphism.

Also, it is surjective and injective. Because, for every element $u_n X(n) \in H^*(\Omega SU_2, \mathbb{Z})$, we have $u_n \gamma^n$ such that $\eta(u_n \gamma^n) = u_n X(n)$ and

$$\begin{aligned} \ker \eta &= \{u_n \gamma^n : \eta(u_n \gamma^n) = u_n X(n) = 0\} \\ &= \{u_n \gamma^n : u_n = 0\} \\ &= 0. \end{aligned}$$

We have completed the proof. □

CHAPTER 4

The Generalized Cohomology Theories and Topological Construction of BGG-type Operators

In Chapter 1, we gave the homology and cohomology ring structure of the flag space G/B where G is a Kač-Moody group. In this section, we will discuss the generalized complex-oriented cohomology and homology theories of the flag space G/B, and the classical BGG and Kač operators will be constructed topologically using the transfer map for compact fibre bundles. In order to do this, first we will give some topological notations.

1. Topological preliminaries.

The reference for this section is [**3**].

1.1. Generalities on generalized cohomology. A generalized cohomology theory $h^*(\,)$ is a contravariant functor from topological spaces to graded abelian groups which satisfies all the Eilenberg-Steenrod axioms except the dimension axiom. That is, the coefficients $h^* = h^*(pt)$ need not be concentrated in a single degree. We will always assume that h^* is multiplicative, and that the associated ring structure is commutative in the graded sense. Then for a topological space X, $h^*(X)$ is a h^*-module. The first example is ordinary cohomology with coefficients in $\mathbb{Z}$.

We take $H^i(X) = H^i(X,\mathbb{Z}) = [X, K(\mathbb{Z},i)]$, where $K(\mathbb{Z},i)$ is an Eilenberg-Maclane space, and $[X,Y]$ denotes homotopy classes of based maps from X to Y for X and Y topological spaces with based points.

For a generalized theory $h^*(\,)$, there is a spectral sequence which computes $h^*(X)$ in terms of $H^*(X;h^*)$. This spectral sequence is called the *Atiyah-Hirzebruch spectral sequence*, and details can be found in [**3**].

Theorem 4.1. *There is a spectral sequence with E_2 term $H^p(X,h^q(pt)) \Longrightarrow h^{p+q}(X)$. The differential d_r is of bi-degree $(r, 1-r)$.*

Corollary 4.2. *Suppose that X has no odd dimensional cells and $h^q(pt) = 0$ for q odd. Then the Atiyah-Hirzebruch spectral sequence collapses at the E_2 term.*

Now we define reduced cohomology. Let $i : pt \to X$ be the inclusion of a point and $\pi : X \to pt$ be the collapsing map. Then $\pi \circ i = id$, so $i^* \circ \pi^* = id$ on $h^*(pt)$. Let $\tilde{h}^*(X) = \ker i^*$ be the reduced cohomology of X. Then, as a h^*-module,

$$h^*(X) = \tilde{h}^*(X) \oplus h^*.$$

1.2. Classifying spaces. In this section, we give some facts about the construction of universal bundles and classifying spaces of groups. The general reference for this section is [**56**]. Let G be a group. There is a universal space EG with a free right G-action and $\pi_i(EG) = 0$ for all $i > 0$. Moreover, EG is a limit of manifolds with the inductive limit topology. For example, for $G = U(n)$, the unitary group,

$$EU(n) = \lim_{m \to \infty} V_n(\mathbb{C}^{n+m}),$$

where

$$V_n(\mathbb{C}^{n+m}) = \frac{U(n+m)}{U(m)}.$$

is a Stiefel manifold. The classifying space BG is defined as EG/G. For $G = U(n)$,

$$BG \cong \lim_{m \to \infty} G_n(\mathbb{C}^{n+m}),$$

the Grassmannian manifold of n-planes.

We have the universal bundle (EG, p, BG), where $EG \xrightarrow{p} BG$ is the obvious projection map. Then BG has the following universal property.

Theorem 4.3. *Let $P \xrightarrow{\rho} B$ be a numerable right G-principal bundle. Then there exists a unique (up to homotopy) classifying map $f : B \to BG$ such that $f^*(EG) \cong P$ as G-principal bundles over B, where $\tilde{f} : P \to EG$ is the canonical morphism of the induced bundle given by $\rho^{-1}(b) \cong p^{-1}(f(b))$ for each $b \in B$.*

As a consequence,

Corollary 4.4. *BG is well-defined up to homotopy and classifies induced vector bundles.*

Let $P \xrightarrow{\rho} B$ be a right G-principal bundle. Then, if F is a finite dimensional representation of G, $E = P \times_G F$ is the associated vector bundle over B with structure group G, where

$$E = P \times_G F = P \times F / \sim$$

is the space obtained as the quotient of the product space $P \times F$ by the relation

$$(x, y) \sim (xt, t^{-1}y), \quad t \in G, x \in P, y \in F.$$

Theorem 4.5. *Let $E \to B$ be a vector bundle associated to the fibre F with structure group G. Then there exists $f : B \to BG$ with $f^*(EG \times_G F) \cong E$ as vector bundles over B.*

Consider the special case of the classifying space for a complex line bundle. The appropriate structure group is $U(1)$, so the appropriate classifying space is $BU(1)$. By the above construction,

$$BU(1) = \lim_{m \to \infty} \mathbb{C}P^m = \mathbb{C}P^\infty.$$

We know from [**56**] that

$$H^*(BU(1), \mathbb{Z}) = \mathbb{Z}[x],$$

where $\mathbb{Z}[x]$ is the graded ring of polynomials in one variable with coefficients in $\mathbb{Z}$ and $\deg x = 2$. Let

$$T = \prod_{i=1}^{l} U(1)$$

be a torus. Then,

$$BT = \prod_{i=1}^{l} BU(1),$$

and since $H^*(BU(1), \mathbb{Z})$ is torsion-free, by the Kunneth formula, we have

$$H^*(BT, \mathbb{Z}) \cong \bigotimes_{i=1}^{l} H^*(BU(1), \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_l],$$

where $\mathbb{Z}[x_1, \ldots, x_l]$ is the graded ring of polynomials in l variables with coefficients in the ring $\mathbb{Z}$.

1.3. Complex orientable cohomology theories. We follow [**3**] in this discussion.

Let $i : \mathbb{C}P^1 \to \mathbb{C}P^\infty = BU(1)$ be the inclusion.

Definition 4.6. *We say that the multiplicative cohomology theory h^* is* complex oriented *if there exists a class $x \in \tilde{h}^*(\mathbb{C}P^\infty)$ such that $i^*(x)$ is a generator of $\tilde{h}^*(\mathbb{C}P^1)$ over the ring $h^*(pt)$. Such a class x is called a* complex orientation.

$\tilde{h}^*(\mathbb{C}P^1) \cong \tilde{h}^*(\mathbb{S}^2)$ is generated by one element over $h^*(pt)$.

As an example, if $h^* = H^*$, then x can be taken as a ring generator of $H^*(\mathbb{C}P^\infty, \mathbb{Z})$, so $x \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$. $\mathbb{C}P^\infty$ has a universal line bundle L_λ given as follows. Let e^λ be the one-dimensional representation of $\mathbb{T} = U(1)$ given by

$$\mathrm{e}^\lambda(\mathrm{e}^{i\theta}).v = \mathrm{e}^{i\theta}\,.v,$$

where $\lambda \in \mathrm{Lie}(T)$ is a fundamental weight. Then, for a complex orientable theory h^* with orientation given by x, the first Chern class is given by $x = c_1(L_\lambda)$, where L_λ is the line bundle associated to e^λ. Let T be an l-dimensional torus.

Theorem 4.7. *With the above notation, we have isomorphisms of graded h^*-algebras*

$$h^*(\mathbb{C}P^\infty) \cong h^*(pt)[[x]],$$

$$h^*(BT) \cong h^*(pt)[[x_1, \dots, x_l]],$$

$$h^*(\mathbb{C}P^n) \cong h^*(pt)[[x]]/(x^{n+1}),$$

$$h^*(\prod_{i=1}^{l} \mathbb{C}P^{n_i}) \cong h^*(BT)/(x_1^{n_1+1}, \dots, x_l^{n_l+1}).$$

Now let $\pi : L \to X$ be a line bundle over X. Then L induces a classifying map $\theta : X \to \mathbb{C}P^\infty$. Then the first Chern class of L is $c_1(L) = \theta^*(x)$. Next we define the top Chern class of a vector bundle.

Definition 4.8. *Let $\pi : E \to X$ be a vector bundle. If there is a space Y and a map $f : Y \to X$ such that $f^* : h^*(X) \to h^*(Y)$ is injective and $f^*(E) \cong \bigoplus L_i$, where L_i are line bundles on Y, f is called a* splitting map *for π.*

From [**56**],

Theorem 4.9. *If $\pi : E \to X$ is a vector bundle, there exists a splitting map of π.*

Then,

Definition 4.10. *The top Chern class $c_n(E)$ where $\dim E = n$, which also will be referred as the Euler class $\chi(E)$, is defined by the formula*

$$f^*(c_n(E)) = \prod_i c_1(L_i),$$

where f is a splitting map for π.

1.4. Formal group laws. Let $\mathbb{F}$ be a commutative ring with unit.

Definition 4.11. *A* formal group law *over $\mathbb{F}$ is a power series $F(x,y)$ over $\mathbb{F}$ that satisfies the following conditions:*

1. $F(x,0) = F(0,x) = x$,
2. $F(x,y) = F(y,x)$,
3. $F(F(x,y),z) = F(x,F(y,z))$,
4. *there exists a series $i(x)$ such that $F(x,i(x)) = 0$.*

From [**87**], we have

Theorem 4.12. *In a complex oriented theory, for line bundles L, M, we have*

$$c_1(L \otimes M) = F(c_1(L), c_1(M))$$

where F is a formal group law over the coefficient ring h^*.

Now, we will explain this. A line bundle L over a space X is equivalent to a homotopy class of maps $f_L : X \to \mathbb{C}P^\infty$. Let L and M be two line bundles. Then we have

$$f_L \times f_M : X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty.$$

$\mathbb{C}P^\infty$ has an H-space structure $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. The homotopy class of $m \circ (f_L \times f_M)$ is then equivalent to the tensor product $L \otimes M$. There is an induced map $m^* : h^*(\mathbb{C}P^\infty) \to h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Since, $h^*(\mathbb{C}P^\infty) \cong h^*(pt)[[x]]$ and $h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong h^*(pt)[[x_1, x_2]]$, m^* has the form,

$$m^*(x) = \sum a_{ij} {x_1}^i {x_2}^j = F(x_1, x_2).$$

Then $c_1(L \otimes M) = F(c_1(L), c_1(M))$. As an example, if L and M are line bundles, we see in ordinary cohomology $H^*(\quad)$ that

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

1.5. Complex cobordism. Complex cobordism is the universal complex orientable theory. There are two different descriptions of complex cobordism, homotopy theoretic and geometric. The homotopy theoretic definition is valid for topological spaces but the geometric definition is valid only for manifolds.

We begin with the geometrical definition due to Quillen in [**85**]. By a manifold, we mean a smooth manifold, and maps of manifolds will always be smooth. The first step is to define a complex oriented map of manifolds $f : Z \to X$. This is a generalization of a weakly-complex structure on Z when X is a point.

Definition 4.13. *The* dimension *of f at z is defined to be*

$$\dim f_z = \dim Z_z - \dim X_{f(z)}.$$

If X and Z are connected, then

$$\dim f = \dim Z - \dim X.$$

Definition 4.14. *Suppose first that at each point z of Z, the dimension of f is even. Then, a* complex orientation *of f is an equivalance class of factorizations of f*

$$Z \xrightarrow{i} E \xrightarrow{p} X$$

where $p : E \to X$ is a complex vector bundle over X and where i an embedding endowed with a complex structure on its normal bundle ν_i. If the dimension of f is odd, a complex orientation for f is defined to be one for the map $(f, \varepsilon) : Z \to X \times \mathbb{R}$, where $\varepsilon(Z) = 0$. For a general map f, let $Z = Z' \sqcup Z''$, where

$$\dim Z'_z - \dim X_{f(z)}$$

is even, and

$$\dim Z''_z - \dim X_{f(z)}$$

is odd. Then f is complex oriented if it is complex oriented both pieces.

Two factorization $Z \xrightarrow{i} E \xrightarrow{p} X$, $Z \xrightarrow{i'} E' \xrightarrow{p'} X$ of f are equivalent if there is a finite dimensional complex vector bundle $Z \xrightarrow{i''} E'' \xrightarrow{p''} X$ such that i and i' are isotopic in E'' compatibly with the complex structure on the normal bundle.

Let $f : Z \to X$ be a complex orientable map and $g : Y \to X$ be a map transverse to f. Then the induced map g^* is defined by

$$g^*(f) : Y \prod_X Z \to Y$$

which has a complex orientation given by the pullback of the bundle defining the complex orientation on f. The transversality condition implies $Y \prod_X Z$ is a manifold.

Definition 4.15. *Let $f_1 : Z_1 \to X$, $f_2 : Z_2 \to X$ be two proper complex oriented maps. Let $\varepsilon_i : X \to X \times \mathbb{R}$ be given by $\varepsilon_i(x) = (x, i)$ for $i = 1, 2$. Then, we say that (Z_1, f_1) is* cobordant *to (Z_2, f_2) if there is a proper complex-oriented map $b : W \to X \times \mathbb{R}$ such that b is transversal to ε_i and the pull-back ${\varepsilon_i}^*(b)$ of b by ε_i is equivalent to f_i.*

Proposition 4.16. *Cobordism is an equivalence relation.*

By the Thom transversality theorem of [**94**], we can find a map $\tilde{g} : Y \to X$ which is homotopic to a map $g : Y \to X$ such that $\tilde{g}$ transverse to f. So, we can define $\tilde{g}^*(Z, f) \cong g^*(Z, f)$ in a homotopy invariant way.

Definition 4.17. *For a manifold, the cobordism $MU^n(X)$ is the set of cobordism classes of proper complex-oriented maps of dimension $-n$.*

$MU^*(X)$ has a ring structure given as follows. Let $[Z_1, f_1]$ and $[Z_2, f_2]$ be two cobordism classes of X. Then $[Z_1, f_1] + [Z_2, f_2]$ is the class of the well-defined map

$$f_1 \bigsqcup f_2 : Z_1 \bigsqcup Z_2 \to X.$$

The negative of the cobordism class of $f : Z \to X$ is the cobordism class of f endowed with the negative complex orientation which is defined as follows. Let the orientation of f be represented by a factorization $Z \xrightarrow{i} \mathbb{C}^n \times X \to X$ with complex structure on the normal bundle ν_i. Then the negative orientation is represented by the same factorization, with the same complex structure on ν_i, but with the new complex structure on $\mathbb{C}^n$ given by

$$i(z_1, \cdots, z_n) = (iz_1, \cdots, iz_{n-1}, -iz_n).$$

The class of empty set $\emptyset$ is the zero element.

The product structure on $MU^*(X)$ is given by external products of manifolds. If $[Z_1, f_1] \in MU^p(X)$ and $[Z_2, f_2] \in MU^r(X)$, then

$$f_1 \times f_2 : Z_1 \times Z_2 \to X \times X \in MU^{p+r}(X \times X).$$

Then the cup product is defined by

$$[Z_1, f_1] \cdot [Z_2, f_2] = \Delta^*((Z_1, Z_2), f_1 \times f_2)$$

where $\Delta : X \to X \times X$ is the diagonal embedding and $\Delta^* : MU^*(X \times X) \to MU^*(X)$ is induced by the map Δ. The unit element $1 \in MU^0(X)$ is given by the identity map $X \to X$.

Now we want to give the homotopy theoretic description of the complex cobordism theory. Let $EU(n) \to BU(n)$ be the universal n rank vector bundle. Let $D(EU(n))$ and $S(EU(n))$ be its disc and sphere bundles respectively. Then, $MU(n) = D(EU(n))/S(EU(n))$. $MU(n)$ is called the *universal Thom space*. $MU = \{MU(n)\}_n$ is a Σ-spectrum. The graded-group $[\Sigma^\infty X, MU]^*$ defines a cohomology theory **[56]**, where $[\Sigma^\infty X, MU]^*$ is the graded group of homotopy classes.

Theorem 4.18. *(see* [**85**]*) If X is a smooth manifold, $MU^n(X)$ is isomorphic to the group $[\Sigma^\infty X, MU]^n$.*

By Theorem 4.18, the complex cobordism functor $MU^*(\ )$ is a generalized cohomology theory functor. We will show geometrically that it has the contravariant property.

Let $g : Y \to X$ be a map of manifolds and let $f : Z \to X$ be a proper complex-oriented map. By the transversality theorem, g may be moved by a homotopy until it is transverse to f. The cobordism class of the pull-back map $g^*(f) : Y \times_X Z \to Y$ depends on the cobordism class of f, and this gives the map

$$g^* : MU^q(X) \to MU^q(Y)$$

defined by $g^*[Z, f] = [Y \times_X Z, g * (f)]$ for each q.

A proper complex-oriented map $g : X \to Y$ of dimension d induces a map

$$g_* : MU^q(X) \to MU^{q-d}(Y)$$

which sends the cobordism class of $f : Z \to X$ into the class of the composition $gf : Z \to Y$, since the composition of proper complex-oriented maps gf is a proper complex-oriented map, [**33**], see page 57. This g_* is called the *Gysin homomorphism.*

Definition 4.19. *The push-forward g_* is defined by*

$$g_*[Z, f] = [Z, gf].$$

From [**3**],

Theorem 4.20. *$MU^*(\quad)$ is a complex orientable theory.*

Proposition 4.21. *(see [**85**]) Given an element a of $h^*(pt)$, there is a unique morphism $\Theta : MU^* \to h^*$ of functors commuting with push-forwards such that $\Theta 1 = a$, where 1 is the cobordism class of the identity.*

Thus MU^* is the universal cohomology theory with respect to push-forwards. From [**3**],

Theorem 4.22. *The formal group law of MU^* is the Lazard's universal formal group law.*

2. The Becker-Gottlieb map and transfer.

The general reference for this section is [**5**].

Let $\pi : E \to B$ be a fiber bundle with the fiber F, which is a compact differentiable G-manifold for a compact Lie group G. For any cohomology theory h^* we have the induced map $\pi^* : h^*(B) \to h^*(E)$. A transfer map is a backwards map $h^*(E) \to h^*(B)$. We get a transfer map from a section of the bundle. However sections usually do not exist. Here, we will give a technique for producing a transfer map.

Definition 4.23. *Let $\xi \to B$ be a vector bundle. Let $D(\xi) = \{x \in \xi : |x| \leqslant 1\}$ and $S(\xi) = \{x \in \xi : |x| = 1\}$ be the disk and sphere bundles respectively. Then, $M\xi = D(\xi)/S(\xi)$ is called the* Thom space *of the vector bundle ξ.*

Now we give the useful propositions from [**56**],

Proposition 4.24. *If $\xi \to B$ is a trivial n dimensional vector bundle, then the Thom space $M\xi = \Sigma^n B^+$, where B^+ is the union of B with a point.*

Proposition 4.25. *If ξ and η are two vector bundles over B, then $M\xi \wedge M\eta = M(\xi \oplus \eta)$.*

We define transfer for the map from the fiber F to a point. We can embed F equivariantly into a real G-representation V of dimension r such that $r >> \dim F$. Let $\nu \to F$ be the normal bundle of the embedding. By the tubular neighbourhood theorem, we can identify the normal bundle N with a neighbourhood U of F by a diffeomorphism φ. There is an associated Pontryagin-Thom collapsing map $c : S^V \to M\nu$, where S^V is the one point compactification of V, defined by

$$c(x) = \begin{cases} \text{base point of } F^N & \text{if } x \notin U, \\ \varphi(x) & \text{if } x \in U. \end{cases}$$

Let $T(F)$ be the tangent bundle of F. Then we can identify $T(F) \oplus N$ with the trivial bundle $F \times V$. There is an inclusion $i : N \to N \oplus T(F) \cong F \times V$ and hence we have an inclusion of Thom spaces $i : M\nu \to S^V \wedge F^+$.

Definition 4.26. *The transfer τ to a point is the composition $\tau = i \circ c$.*

Let $\pi : E \to B$ be a fiber bundle associated to the principle G-bundle $p : P \to B$. Then the transfer to a point gives a map

$$\mathrm{Id} \times \tau : P \times_G S^V \to P \times_G (F \times V)^+.$$

When we collapse the section at ∞ to a point, which is equivalent to taking Thom spaces, we get a map $t : M\xi \to M\pi^*(\xi)$ where ξ is a vector bundle associated to the representation V. Then there is a map $t \wedge \mathrm{Id} : M\xi \wedge M\overline{\xi} \to M\pi^*(\xi) \wedge M\overline{\xi}$. If we restrict to the diagonal Δ in $B \times B$, we have transfer map

$$\tau(\pi) : \Sigma^m B^+ \to \Sigma^m E^+.$$

3. The Brumfiel-Madsen formula for transfer.

The general reference for this section is [**22**].

Let G be a compact connected semi-simple Lie group with maximal torus T. Let H be a closed connected subgroup of G containing the maximal torus T. Let W_G and W_H be the Weyl groups of G and H respectively. Suppose that $P \to B$ is a principal G-bundle. We have associated bundles

$$\pi_1 : E_1 = P \times_G G/T \to B$$

$$\pi_2 : E_2 = P \times_G G/H \to B.$$

Then there is a fibration $\pi : E_1 \to E_2$ with the fiber H/T. Since the Weyl group W_G acts on G/T, W_G also acts on E_1. The Weyl group W_H of H also acts on E_1 over E_2. Thus, cosets $w \in W_G/W_H$ define maps $\pi \circ w$ on E_1.

Theorem 4.27. *We have*

$$\pi_1{}^* \circ \tau(\pi_2)^* = \sum_{w \in W_G/W_H} w \circ \pi^*.$$

Corollary 4.28. *If we choose $H = T$, we get*

$$\pi_1{}^* \circ \tau(\pi_1)^* = \sum_{w \in W_G} w.$$

Although Brumfiel and Madsen were the first to assert that Theorem 4.27 is true, their proof was wrong. In [**37**] Feshbach, and in [**75**] Lewis, May and Steinberger have given different proofs of Theorem 4.27. Since EG is the universal space for G, we have the principle bundle $EG \to BG$.

Corollary 4.29. *Let $BT \to BG$ be the fiber bundle with the fiber G/T. Then*

$$\pi^* \circ \tau(\pi)^* = \sum_{w \in W_G} w.$$

For a compact semi-simple Lie group G, any root α defines a subgroup $M_\alpha = K_\alpha \cdot T$ such that the complexified Lie algebra $\mathbf{m}_\alpha$ contains the root spaces $\mathbf{g}_\alpha$ and $\mathbf{g}_{-\alpha}$ where K_α already has been defined in Chapter 1. The induced fiber bundle $\pi_i : BT \to BM_i$ has fiber $M_i/T \cong SU_2/T \cong \mathbb{C}P^1$. Then

Corollary 4.30.

$$\pi_i^* \circ \tau(\pi_i)^* = 1 + r_{\alpha_i},$$

if r_{α_i} is the reflection to corresponding to the simple root α_i.

Theorem 4.31. *(see [**38**]) Let B be a finite CW complex and $\pi : E \to B$ be a fiber bundle such that fiber F is a compact G-manifold. For any multiplicative complex oriented cohomology theory h^*,*

$$\tau(\pi)^* \circ \pi^*(x) = \chi(F)x + ux,$$

where $u \in h^0(B)$ is nilpotent and $\chi(F)$ is the Euler characteristic of F.

Corollary 4.32. *If $\chi(F)$ is a unit, then π^* is injective.*

Since $\chi(G/T) = |W_G|$, we have

Corollary 4.33. *If $|W_G|$ is a unit in $h^*(pt)$, then $\pi^* : h^*(BG) \to h^*(BT)$ is injective.*

4. The transfer and the Gysin homomorphism.

Let $\xi : E \to X$ be a vector bundle and h^* be the complex oriented theory. Then there is the associated Thom class $u \in h^*(M\xi)$. From [**32**], we have

Theorem 4.34. *The Thom map $\Phi : h^*(X) \to h^*(M\xi)$ given by $\Phi(x) = u \cdot \pi^*(x)$ is an isomorphism.*

Let $\pi : E \to B$ be a fiber bundle with compact smooth f-dimensional fiber F. Suppose that the tangent bundle $TF \to F$ is a complex vector bundle. Then we have the Gysin homomorphism $\pi_* : h^k(E) \to h^{k-f}(B)$. Since the tangent bundle $T(F)$ has complex structure, in the complex orientable theory h^* T_π has an Euler class, so $\chi(T_\pi) = c_n(T_\pi)$.

Theorem 4.35. *(see [**5**]) The transfer $\tau(\pi)^* : h^k(E) \to h^k(B)$ is given by $\tau(\pi)^*(x) = \pi_*(x \cdot \chi(T_\pi))$.*

Let L_α be the line bundle on BT associated to the character e^α where α is a root. We want to determine when its characteristic classes are not zero divisors. We know that the characters e^α do not usually generate the representation ring $R(T)$. Let λ_i be the fundamental weight corresponding to the simple root α_i such that $\lambda_i(h_{\alpha_i}) = 1$, where h_{α_i} is the coroot. Then

Theorem 4.36. *(see [56]) These e^{λ_i} generate the representation ring $R(T)$.*

By Theorem 4.7,

$$h^*(BT) \cong h^*(pt)[[c_1(L_{\lambda_1}), \cdots, c_1(L_{\lambda_l})]]$$

where l is the rank of the compact Lie group G. Since $\chi(L_{\lambda_i})$ are generaters of $h^*(BT)$, the $\chi(L_{\lambda_i})$ are not zero-divisors in $h^*(BT)$. This implies that $\chi(L_{\lambda_i})$ is not nilpotent. We know that for any weight $\lambda \in \mathbf{h}^*$, λ can be written as

$$\lambda = \sum_{i=1}^{l} n_i \lambda_i,$$

where n_i is the multiplicity number. Using the formal group law in h^*, the Euler class $\chi(L_\lambda)$ of the line bundle L_λ in h^* is equal to

$$\sum_{i=1}^{l} n_i c_1(L_{\lambda_i}) + \text{higher order terms}.$$

If n_i is not a zero-divisor in $h^*(pt)$, then $\chi(L_\lambda)$ is not a zero-divisor in $h^*(BT)$. If the weight λ is a root corresponding to the adjoint representation, the multiplicity numbers n_i in the sum are the Cartan integers. By an examination of the Cartan matrices, we have

Proposition 4.37. *If $p \geqslant 3$ is a prime, there is some n_i such that p does not divide n_i.*

Corollary 4.38. *If $h^*(pt)$ has no 2-torsion, then the Euler class $\chi(L_{\alpha_i})$ is not a zero-divisor for any simple root α_i.*

Since every root is the image of a simple root by an element of the Weyl group W_G and the Weyl group acts by automorphism on $h^*(BT)$, we have

Corollary 4.39. *If $h^*(pt)$ has no 2-torsion, then the Euler class $\chi(L_\alpha)$ is not a zero-divisor for any root α.*

Now we want to give the Brumfiel-Madsen formula for the Gysin map of the fibration $\pi : BT \to BG$ with the fiber G/T. We need a complex structure on G/T. We know that the smooth manifold G/T is diffeomorphic to the complexified space $G_{\mathbb{C}}/B$ where B is a Borel group. Then we can determine the tangent bundle of the fiber $G_{\mathbb{C}}/B$. The tangent bundle $T(G_{\mathbb{C}}/B)$ is isomorphic to $G \times_T \mathbf{g}/\mathbf{b}$. Using the adjoint representation of T, we have

$$\mathbf{g} = \mathbf{b} \oplus \bigoplus_{\alpha \in \Delta^+} \mathbf{g}_{-\alpha},$$

where Δ^+ is the set of positive roots corresponding to B. Thus

$$\mathbf{g}/\mathbf{b} = \bigoplus_{\alpha \in \Delta^+} \mathbf{g}_{-\alpha}.$$

Therefore the tangent bundle along the fiber G/T is

$$T_\pi = EG \times_T \mathbf{g}/\mathbf{b} \cong \bigoplus_{\alpha \in \Delta^+} L_{-\alpha},$$

where $L_{-\alpha}$ is as above. We know that

$$\chi^n(T_\pi) = \prod_{\alpha \in \Delta^+} c_1(L_{-\alpha}),$$

where $\prod$ is the cup product in any complex orientable theory h^*. By Theorem 4.35, we have

$$\pi^* \circ \tau(\pi)^*(x) = \pi^* \circ \pi_*(x \cdot \chi(T_\pi))$$

for $x \in h^*(BT)$. Since $\chi(T_\pi)$ is a product of the non-zero divisors in $h^*(BT)$, we have

Theorem 4.40. *(see* [**19**]*) For $x \in h^*(BT)$,*

$$\pi^* \circ \pi_*(x) = \sum_{w \in W} w\left(\frac{x}{\prod \chi(L_{-\alpha})}\right),$$

here the right hand side is in a localization $h^*(BT)[\frac{1}{\prod \chi(L_{-\alpha})}]$.

But since the left hand side preserves the subring $h^*(BT)$, it may be regarded as an identity on $h^*(BT)$.

Corollary 4.41. *If $\chi(L_{-\alpha})$ is a non-zero divisor, for the fibration $\pi_i : BT \to BM_i$ with the fiber M_i/T,*

$$D_i(x) = \pi_i{}^* \circ \pi_{i*}(x) = (1 + r_i)\left(\frac{x}{\chi(L_{-\alpha})}\right).$$

Let h^* be the ordinary cohomology with complex coefficients. From chapter 1, we know that there is an isomorphism $\Theta : \mathbf{h}^* \to H^2(BT, \mathbb{C})$ given by $\lambda \to \chi(L_\lambda)$. Θ extends to an inclusion of the symmetric algebra $R = S(\mathbf{h}^*)$ into $H^*(BT, \mathbf{C})$. Then

$$H^*(BT, \mathbb{C}) \cong \mathbb{C}[\lambda_1, \cdots, \lambda_l]$$

under the identification $\chi(L_{\lambda_i}) = \lambda_i$. When $G = M_i$,

Corollary 4.42.

$$D_i = \frac{1}{\alpha}(r_i - 1)$$

is just the classical BGG operator which was defined in chapter 1.

If we apply Theorem 4.40 to K-theory, for $G = M_i$, the formula $D_i = \pi_i{}^* \circ \pi_{i*}$ in K-theory gives the Demazure operator.

Now, we will apply this result to BP-theory and Morava K-theory. In order to do this, we will give some definitions. Let F be a formal group law over commutative ring with unit R.

Definition 4.43. *For each n, the n-*series *$[n](x)$ of F is given by*

$$\begin{aligned} [1](x) &= x, \\ [n](x) &= F(x, [n-1](x)) \quad \textit{for } n > 1, \\ [-n](x) &= i([n](x)). \end{aligned}$$

Of particular interest is the p-series, where p is a prime. In characteristic p it always has leading term ax^q where $q = p^h$ for some integer h. This leads to the following.

Definition 4.44. *Let $F(x, y)$ be a formal group law over an $\mathbb{F}_p$-algebra. If $[p](x)$ has the form*

$$[p](x) = ax^{p^h} + \textit{higher terms}$$

with a invertible, then we say that F has height h *at* p. *If* $[p](x) = 0$ *then the height is infinity.*

Suppose that h^* is an $\mathbb{F}_p$-algebra and the formal group law F has the height h. Since the elements $x = \chi(L_{\lambda_i}) \in h^*(BT)$ are non-zero divisors, $[p](x)$ has the form

$$[p](x) = ax^{p^h} + \text{higher terms}, \quad (a \text{ is a unit.})$$

This lead us to mod p K-theory and the Morava K-theories. The reference for these cohomology theories is [**87**]. We generalize Corollary 4.38 and 4.39.

Theorem 4.45. *For any prime p, in $K(n)^*(BT)$, the Euler class $\chi(L_{\alpha_i})$ is not a zero divisor for any simple root α_i.*

Theorem 4.46. *For any prime p, in $K(n)^*(BT)$, the Euler class $\chi(L_\alpha)$ is not a zero divisor for any root α.*

Let $\pi : BT \to BG$ is a fiber bundle with the fiber G/T.

Theorem 4.47. *For $x \in K(n)^*(BT)$,*

$$\pi^* \circ \pi_*(x) = \sum_{w \in W} w\left(\frac{x}{\prod \chi(L_{-\alpha})}\right),$$

here the right hand side is in a localization $K(n)^*(BT)[\frac{1}{\prod \chi(L_{-\alpha})}]$.

Corollary 4.48. *Let $\pi_i : BT \to BM_i$ be a fiber bundle with the fiber M_i/T. For $x \in K(n)^*(BT)$,*

$$D_i(x) = {\pi_i}^* \circ \pi_{i*}(x) = (1 + r_i)\left(\frac{x}{\chi(L_{-\alpha})}\right).$$

Of course, these results can be generalized to $\mathbb{F}_p$-algebra h^* which has a formal group law F with the height n. In this section, so far we have concentrated our attention on BT. Now, we will give some interesting results about the flag variety G/T. Since the cohomology of G/T vanishes in odd degrees, Corollary 4.2 gives

Corollary 4.49. *Let h^* be any complex oriented cohomology theory. Then the Atiyah-Hirzebruch spectral sequence for G/T collapses at the E_2-term.*

Let $\pi_i : BT \to BM_i$. Since G/T is a T-principal bundle, there is a classifying map $\theta : G/T \to BT$. Similarly there is a classifying map $\theta_i : G/M_i \to BM_i$. The following diagram is a cartesian square.

$$\begin{tikzcd} G/T \arrow[r, "\theta"] \arrow[d, "p_i"] & BT \arrow[d, "\pi_i"] \\ G/M_i \arrow[r, "\theta_i"] & BM_i \end{tikzcd}$$

Let $C_i = {p_i}^* \circ {p_i}_*$. Then $\theta^* \circ D_i = C_i \circ \theta^*$. The following theorem gives a topological description of the operator C_i. From [**36**],

Theorem 4.50. *If $h^*(pt)$ contains $\mathbb{Z}[\frac{1}{|W_G|}]$, then θ^* is surjective.*

Definition 4.51. *For $i = 1, \dots, l$, let D_i be the linear operator associated to the simple root α_i. Then we say that D_i satisfy* braid relations *if*

$$(D_i D_j D_i)^{m_{ij}} = (D_j D_i D_j)^{m_{ij}},$$

where m_{ij} is the number of factors in each side for all pairs i and j.

For any Kač-Moody group, since T and M_i are still finite dimensional compact groups, we can define D_i operators on $h^*(BT)$.

Theorem 4.52. *Let G be a Kač-Moody group and let h^* be torsion-free. Then the operators D_i satisfy braid relations if and only if the formal group law is polynomial.*

PROOF. There are three cases to consider. These cases are when two non-orthogonal roots α_i and α_j have $m_{ij} = 3, 4$, or 6. In the finite dimensional case, the reference for case $m_{ij} = 3$ is [**20**] and for the remaining two cases the reference is [**45**]. In the affine case, it can be done similar way. □

This theorem tells that the ordinary cohomology and K-theory satisfy braid relations but cobordism and elliptic cohomology and another complex oriented cohomology theories do not satisfy.

By Theorem 4.50 and 4.52, we have

Theorem 4.53. *The operators C_i satisfy braid relations for ordinary cohomology and K-theory.*

Now we will give our result about the infinite dimensional flag variety. Let G be an affine Kač-Moody group and K be the unitary form of G. For every simple root α_i, let $M_i = K_i \cdot T$. We have a principal M_i-bundle $K \to K/M_i$, and the associated fiber bundle $K/T \to K/M_i$ with fiber M_i/T. M_i/T is diffeomorphic to complex projective space $\mathbb{C}P^1$.

Theorem 4.54. *Let $\pi_i : K/T \to K/M_i$ be the fiber bundle with the compact fiber $\mathbb{C}P^1$ and let $\mathbb{F}$ be a commutative ring with unit. For $x \in H^*(K/T, \mathbb{F})$,*

$$O_i(x) = \pi_i^* \circ \pi_{i*}(x) = -(1 + r_i)\left(\frac{x}{\varepsilon^{r_i}}\right),$$

here the right hand side is in the localization $H^*(BT)[\frac{1}{\prod \chi(L_{-\alpha})}]$. In fact O_i is the Kač operator which was introduced in chapter 1.

PROOF. By the Brumfiel-Madsen formula and Theorem 4.35, we have the following identity

$$\pi_i^* \circ \tau(\pi_i)^*(x) = \pi_i^* \circ \pi_{i*}(\psi(-\chi_i) \cdot x) = (1 + r_i)(x),$$

where r_i is the simple reflection associated to α_i and χ_i is the fundamental weight corresponding to the simple root α_i. Let $x \in H^*(K/T, \mathbb{F})$. We know from Chapter 1 that $\psi(\chi_i) = \varepsilon^{r_i}$ where $\psi : S(\mathrm{h}^*) \to H^*(K/T, \mathbb{F})$. In $H^*(K/T, \mathbb{F})$, we know that the element ε^{r_i} is a non zero-divisor, so we can define the local ring $H^*(K/T, \mathbb{F})[\frac{1}{\varepsilon^{r_i}}]$. Then, we have the following identity in the local ring $H^*(K/T, \mathbb{F})[\frac{1}{\varepsilon^{r_i}}]$,

$$\pi_i^* \circ \pi_{i*}(x) = -(1 + r_i)\left(\frac{x}{\varepsilon^{r_i}}\right).$$

Since the left hand side of the identity is an element of $H^*(K/T, \mathbb{F})$, we are done. $\square$

We know from [**61**] that the Kač operators satisfy braid relations for all affine Kač-Moody groups.

CHAPTER 5

Fredholm Maps and Cobordism of Separable Hilbert Manifolds

Introduction.

In [**85**], Quillen gave a geometric interpretation of cobordism groups which suggests a way of defining the cobordism of separable Hilbert manifolds equipped with suitable structure. In order that such a definition be sensible, it ought to reduce to his for finite dimensional manifolds and smooth maps of manifolds and be capable of supporting reasonable calculations for important types of infinite dimensional manifolds such as homogeneous spaces of free loop groups of finite dimensional Lie groups.

1. Cobordism of separable Hilbert manifolds.

By a manifold, we mean a smooth manifold modelled on a separable Hilbert space.

Definition 5.1. *Let X and Y be manifolds. Then the smooth map $f : X \to Y$ is called* proper *if the preimages of compact sets are compact.*

Definition 5.2. *Let U and V be normed vector spaces. The linear operator $A : U \to V$ is called* Fredholm *if* $\dim\ker A$ *and* $\dim\operatorname{coker} A$ *are both finite. Then the* index *of the operator A is*

$$\operatorname{index} A = \dim\ker A - \dim\operatorname{coker} A.$$

Proposition 5.3. *(see* [**27**]*) The set* $\operatorname{Fred}(U, V)$ *of Fredholm operators is open in the space of all bounded operators $L(U, V)$ in the norm topology. The index function* $\operatorname{index} : \operatorname{Fred}(U, V) \to \mathbb{Z}$ *is locally constant, hence continuous.*

Definition 5.4. *Let X and Y be as above. Then the map $f : X \to Y$ is called the* Fredholm *if for each $x \in X$, $df_x : T_xX \to T_{f(x)}Y$ is a Fredholm operator. Thus, at each point of X, we can define the* index *of f at x,*

$$\operatorname{index} f_x = \dim\ker df_x - \dim\operatorname{coker} df_x.$$

Proposition 5.5. *The function from X to $\mathbb{Z}$ given by $x \to \operatorname{index} df_x$ is locally constant, hence continuous.*

Definition 5.6. *Let S_i, and R be smooth manifolds and $f_i : S_i \to R$ be smooth maps for $i = 1, 2$. We say that f_1 and f_2 are* transversal *at $r \in R$ if*

$$df_1(T_{s_1}S_1) + df_2(T_{s_2}S_2) = T_rR$$

whenever $f_1(s_1) = f_2(s_2) = r$. The maps f_1 and f_2 are said to be transversal *if they are transversal everywhere.*

Lemma 5.7. *Let S_i, and R be smooth manifolds and $f_i : S_i \to R$ be smooth maps for $i = 1, 2$. Then f_1 is transversal to f_2 iff $f_1 \times f_2$ is transversal to the diagonal embedding*

$$\Delta : R \to R \times R.$$

Definition 5.8. *Let $f_i : S_i \to R$ be smooth maps between smooth Hilbert manifolds for $i = 1, 2$. If f_1 and f_2 are transversal, then* topological pullback

$$S_1 \coprod_R S_2 = \{(s_1, s_2) \in S_1 \times S_2 : f_1(s_1) = f_2(s_2)\}$$

is a submanifold of $S_1 \times S_2$ and the diagram

$$\begin{array}{ccc} S_1 \coprod_R S_2 & \xrightarrow{f_2^*(f_1)} & S_2 \\ \downarrow{\scriptstyle f_1^*(f_2)} & & \downarrow{\scriptstyle f_2} \\ S_1 & \xrightarrow{f_1} & R \end{array}$$

is commutative where the map $f_i^*(f_j)$ is pull-back of f_j by the map f_i.

Proposition 5.9. *Suppose that $A : U \to V$ is a linear operator, where U and V are finite dimensional vector spaces. Let L be any linear complement of $\ker A$, i.e. L is a vector subspace of U such that $U = \ker A \oplus L$. Then the restriction $A : L \to \operatorname{im} A$ is a vector space isomorphism. Hence* $\operatorname{codim} \ker A = \dim \operatorname{im} A$.

Proposition 5.10. *Let U and V be finite dimensional vector spaces. Then every linear operator $A : U \to V$ is Fredholm and*

$$\operatorname{index} A = \dim U - \dim V.$$

PROOF. By Proposition 5.9, we have the following equation,

$$\operatorname{codim}\ker A = \dim \operatorname{im} A.$$

Then,

$$\begin{aligned} \operatorname{index} A &= \dim\ker A - \dim\operatorname{coker} A \\ &= (\dim U - \operatorname{codim}\ker A) - (\dim V - \dim\operatorname{im} A) \\ &= (\dim U - \dim\operatorname{im} A) - (\dim V - \dim\operatorname{im} A) \\ &= \dim U - \dim V. \end{aligned}$$

□

Proposition 5.11. *(see [97]) Let $U \xrightarrow{A} V \xrightarrow{B} W$ be a sequence of Fredholm operators A and B, where U, V and W are vector spaces. Then the composite linear operator $U \xrightarrow{BA} W$ is also Fredholm and*

$$\operatorname{index} BA = \operatorname{index} B + \operatorname{index} A.$$

Proposition 5.12. *Let X and Y be finite dimensional connected smooth manifolds and let $f : X \to Y$ be a smooth map. Then f is a Fredholm map and, for any point $x \in X$,*

$$\operatorname{index} f_x = \dim X - \dim Y.$$

PROOF. For each $x \in X$ and $y \in Y$, since the manifolds X and Y are finite dimensional,

$$\dim T_x X = \dim X \quad \text{and} \quad \dim T_y Y = \dim Y.$$

Then the differential df_x of f at x is a linear operator from the tangent space of X at the point x to the tangent space of Y at the point $f(x)$. By Proposition 5.10 and the definition of Fredholm map, f is a Fredholm map and

$$\begin{aligned} \operatorname{index} f &= \dim\ker df_x - \dim\operatorname{coker} df_x \\ &= \dim T_x X - \dim T_{f(x)} Y \\ &= \dim X - \dim Y. \end{aligned}$$

□

By Proposition 5.11, we have

Proposition 5.13. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be a sequence of Fredholm maps* f *and* g*, where* X, Y *and* Z *are smooth manifolds. Then the composite smooth map* $X \xrightarrow{gf} Z$ *is also Fredholm and for* $x \in X$*,*

$$\operatorname{index}(gf)_x = \operatorname{index} g_{f(x)} + \operatorname{index} f_x.$$

Definition 5.14. *Suppose that* $f : X \to Y$ *is a proper Fredholm map with even index at each point. Then* f *is an* admissible complex-orientable map *if there is a smooth factorization*

$$f : X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y,$$

where $q : \xi \to Y$ *is a finite dimensional smooth complex vector bundle and* $\tilde{f}$ *is a smooth embedding endowed with a complex structure on its normal bundle* $\nu(\tilde{f})$.

A complex orientation for a Fredholm map f *of odd index is defined to be one for the map* $(f, \varepsilon) : X \to Y \times \mathbb{R}$ *given by* $(f, \varepsilon)(x) = (f(x), 0)$ *for every* $x \in X$*. At* $x \in X$*,* $\operatorname{index}(f, \varepsilon)_x = (\operatorname{index} f_x) - 1$*. Also the finite dimensional complex vector bundle* ξ *in the smooth factorization will be replaced by* $\xi \times \mathbb{R}$*.*

Suppose that f is an admissible complex orientable map. Then since the map f is the Fredholm and ξ is a finite dimensional vector bundle, we see $\tilde{f}$ is also a Fredholm map. By Proposition 5.13 and the surjectivity of q,

$$\operatorname{index} \tilde{f} = \operatorname{index} f - \dim \xi.$$

Before we give a notion of equivalence of such factorizations $\tilde{f}$ of f, we want to give some definitions.

Definition 5.15. *Let* X*,* Y *be the smooth separable Hilbert manifolds and* $F : X \times \mathbb{R} \to Y$ *a smooth map. Then we will say that* F *is an* isotopy *if it satisfies the following conditions.*

1. *For every* $t \in \mathbb{R}$*, the map* F_t *given by* $F_t(x) = F(x, t)$ *is an embedding.*
2. *There exist numbers* $t_0 < t_1$ *such that* $F_t = F_{t_0}$ *for all* $t \leqslant t_0$ *and* $F_t = F_{t_1}$ *for all* $t \geqslant t_1$*.*

The closed interval $[t_0, t_1]$ *is called a* proper domain *for the isotopy. We say that two embeddings* $f : X \to Y$ *and* $g : X \to Y$ *are* isotopic *if there exists an isotopy* $F_t : X \times \mathbb{R} \to Y$ *with proper domain* $[t_0, t_1]$ *such that* $f = F_{t_0}$ *and* $g = F_{t_1}$.

Proposition 5.16. *(see* [**74**]*) The relation of isotopy between smooth embeddings is an equivalence relation.*

Definition 5.17. *Two factorizations* $f : X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y$ *and* $f : X \xrightarrow{\tilde{f}'} \xi' \xrightarrow{q'} Y$ *are* equivalent *if* ξ *and* ξ' *can be embedded as subvector bundles of a vector bundle* $\xi'' \to Y$ *such that* $\tilde{f}$ *and* $\tilde{f}'$ *are isotopic in* ξ'' *and this isotopy is compatible with the complex structure on the normal bundle. That is, there is an isotopy* F *such that for all* $t \in [t_0, t_1]$, $F_t : X \to \xi''$ *is endowed with a complex structure on its normal bundle which matches that of* $\tilde{f}$ *and* $\tilde{f}'$ *in* ξ'' *at* t_0 *and* t_1 *respectively.*

By Proposition 5.16, we have

Proposition 5.18. *The relation of equivalence of admissible complex orientability of proper Fredholm maps between separable Hilbert manifolds is an equivalence relation.*

This definition is actually a generalization of Quillen's definition of the complex-orientable map for finite dimensional manifolds. Similarly we can define the notion of cobordism of two Fredholm maps $X_i \xrightarrow{f_i} Y$ for $i \in \{1, 2\}$. We will need the following proposition.

Proposition 5.19. *Let* $f : X \to Y$ *be an admissible complex orientable map and* $g : Z \to Y$ *be a smooth map transversal to* f*. Then the pull-back map*

$$g^*(f) : Z \prod_Y X \to Z$$

is an admissible complex orientable map with the finite dimensional pull-back vector bundle

$$g^*(\xi) = Z \prod_Y \xi = \{(z, (y, v)) \in Z \times \xi : g(z) = q((y, v))\}$$

in the factorization of $g^*(f)$ *where* $q : \xi \to Y$ *is the finite-dimensional complex vector bundle in the factorization of* f.

Now we will define a notion of cobordism for the admissible complex-orientable maps between separable Hilbert manifolds. This is also a generalization of Quillen's notation of complex cobordism for finite dimensional manifolds.

Definition 5.20. *Let $f_i : X_i \to Y$ be admissible complex-orientable maps for $i \in \{0,1\}$. Then, we say that f_0 is* cobordant *to f_1 if there is an admissible complex-orientable map $b : W \to Y \times \mathbb{R}$ such that $\varepsilon_i : Y \to Y \times \mathbb{R}$ given by $\varepsilon_i(y) = (y, i)$ for $i \in \{0,1\}$, is transversal to b and the pull-back map ${\varepsilon_i}^*(b)$ is equivalent to f_i. The cobordism class of $f : X \to Y$ will be denoted by $[X, f]$.*

Proposition 5.21. *Cobordism is an equivalence relation.*

Proof. **Reflexivity**: Let $f : X \to Y$ be an admissible complex orientable map. If we choose W as $X \times \mathbb{R}$, then we have the admissible complex orientable map $(f, \mathrm{Id}_{\mathbb{R}}) : X \times \mathbb{R} \to Y \times \mathbb{R}$ which is transverse to ε_i. Now we have to show that the pull-back map ${\varepsilon_0}^*(f, \mathrm{Id}_{\mathbb{R}})$ is equivalent to f. Firstly, we will prove that X, the domain of f, is diffeomorphic to the intersection manifold $Y \prod_{Y\times\mathbb{R}} X \times \mathbb{R}$, which is the domain of the pull-back map ${\varepsilon_0}^*(f, \mathrm{Id}_{\mathbb{R}})$. By the definition of the maps ε_0, $(f, \mathrm{Id}_{\mathbb{R}})$ and transversality, we get

$$\begin{aligned} Y \prod_{Y\times\mathbb{R}} X \times \mathbb{R} &= \{(y,(x,v)) : \varepsilon_0(y) = (f, \mathrm{Id}_{\mathbb{R}})(x,v)\} \\ &= \{(y,(x,v)) : (y,0) = (f(x), v)\} \\ &= \{(f(x),(x,0)) : \forall x \in X\} \end{aligned}$$

So, we have a bijective smooth map h from intersection manifold $Y \prod_{Y\times\mathbb{R}} X \times \mathbb{R}$ to X given by $h((f(x),(x,0))) = x, \forall x \in X$. Also, we have the smooth inverse h^{-1} given by $h^{-1}(x) = (f(x),(x,0))$. Thus $Y \prod_{Y\times\mathbb{R}} X \times \mathbb{R}$ is diffeomorphic to X. Since ${\varepsilon_0}^*((f, \mathrm{Id}_{\mathbb{R}}))(f(x),(x,0)) = f(x)$, $\forall x \in X$, the pull-back map ${\varepsilon_0}^*((f, \mathrm{Id}_{\mathbb{R}}))$ is equivalent to f. Similarly we can show that ${\varepsilon_1}^*(f, \mathrm{Id}_{\mathbb{R}})$ is equivalent to f. Therefore (X, f) is cobordant to itself.

Symmetry: Let $f_i : X_i \to Y$ be an admissible complex orientable map for $i \in \{0,1\}$. Suppose that (X_0, f_0) is cobordant to (X_1, f_1). By the definition of cobordant class, we have an admissible complex orientable map $b : W \to Y \times \mathbb{R}$ such that $\varepsilon_i : Y \to Y \times \mathbb{R}$, is given by $\varepsilon_i(y) = (y, i)$, is transversal to b and the pull-back map

${\varepsilon_i}^*(b)$ is equivalent to f_i. Let $b' : W \to Y \times \mathbb{R}$ be the negative orientation class of the admissible complex orientable map b be represented by the same factorization but with a new complex structure on the normal bundle of embedding. Since the admissible complex orientable map b is transverse to ε_i and the pull-back map ${\varepsilon_i}^*(B)$ is equivalent to f_i, the opposite orientable map b' is transverse to ε_{1-i} and ${\varepsilon_{1-i}}^*(b')$ is equivalent to f_{1-i}. Therefore (X_1, f_1) is cobordant to (X_0, f_0).

Transitivity: Let $f_i : X_i \to Y$ be admissible complex orientable maps for $i \in \{0, 1, 2\}$. Suppose that (X_0, f_0) is cobordant to (X_1, f_1), and (X_1, f_1) is cobordant to (X_2, f_2). Then we have to show that (X_0, f_0) is cobordant to (X_2, f_2). By the equivalence of $(X_0, f_0) \cong (X_1, f_1)$, we have an admissible complex orientable map $b_1 : W_1 \to Y \times \mathbb{R}$ such that $\varepsilon_i : Y \to Y \times \mathbb{R}$ is transverse to b_1 and ${\varepsilon_i}^*(b_1)$ is equivalent to f_i for $i \in \{0, 1\}$. By the equivalence of $(X_1, f_1) = (X_2, f_2)$, we have an admissible complex orientable map $b_2 : W_2 \to Y \times \mathbb{R}$ such that ε_i is transverse to b_2 and ${\varepsilon_i}^*(b_2)$ is equivalent to f_i for $i \in \{1, 2\}$. Since the b_1 and b_2 are transvere to ε_1, by the implicit mapping theorem, there exists a smooth submanifold $U_1 \subseteq W_1$ such that for $\varepsilon_1 > 0$, U_1 is diffeomorphic to $X_0 \times (-\varepsilon_1, \varepsilon_1)$ and similarly, there exists a smooth submanifold $U_2 \subseteq W_2$ such that for $\varepsilon_2 > 0$, U_2 is diffeomorphic to $X_0 \times (-\varepsilon_2, \varepsilon_2)$. By the gluing of U_1 and U_2, since ε_1 and ε_2 can be choosen as arbitrary small, we have a smooth map $\Theta : W_1 \coprod\limits_{W_1 \cap W_2} W_2 \to Y \times \mathbb{R}$ given by

$$\Theta(w) = \begin{cases} b_1 & \text{if } w \in W_1, \\ b_2 & \text{if } w \in W_2. \end{cases}$$

where $W_1 \coprod\limits_{W_1 \cap W_2} W_2$ is a gluing of manifolds W_1 and W_2 along the common submanifold $W_1 \cap W_2$. By definition, the smooth map Θ is an admissible complex orientable map such that ε_i is transversal to Θ and ${\varepsilon_i}^*(\Theta)$ is equivalent to f_i for $i \in \{0, 1, 2\}$. Therefore (X_0, f_0) is cobordant to (X_2, f_2). □

Definition 5.22. *For a separable Hilbert manifold Y, the cobordism set $\mathcal{U}^d(Y)$ is the set of cobordism classes of the admissible complex-orientable Fredholm maps of index $-d$.*

In the above definition, instead of proper maps, closed maps could be used for infinite dimensional Hilbert manifolds, because of the following proposition of Smale [**91**].

Theorem 5.23. *If $f : X \to Y$ is a closed Fredholm map where $\dim X = \infty$, then f is proper.*

Lemma 5.24. *If $f : X \to Y$ and $g : Y \to Z$ are proper, so is $g \circ f$.*

By Proposition 5.13 and Lemma 5.24, we have the following theorem.

Theorem 5.25. *If $f : X \to Y$ is an admissible complex orientable map with index d_1 and $g : Y \to Z$ is an admissible complex orientable map with index d_2, then $g \circ f : X \to Z$ is an admissible complex orientable map with index $d_1 + d_2$.*

PROOF. For the complex orientation of the composition map $g \circ f$, see [**33**]. $\square$

Let $g : Y \to Z$ be an admissible complex orientable map with index $-r$. By Theorem 5.25, we have push-forward, or Gysin map

$$g_* : \mathcal{U}^d(Y) \to \mathcal{U}^{d+r}(Z)$$

given by $g_*([X, f]) = ([X, g \circ f])$. We will show that it is well-defined. Suppose that $g : Y \to Z$ be an admissible complex orientable map and $f_0 : X_0 \to Y$ and $f_1 : X_1 \to Y$ are cobordant. Then, there is an admissible complex orientable map $b : W \to Y \times \mathbb{R}$ such that $\varepsilon_i : Y \to Y \times \mathbb{R}$ given by $\varepsilon_i(y) = (y, i)$ for $i \in \{0, 1\}$, is transverse to b and the pull-back map $\varepsilon_i{}^*(b)$ is equivalent to f_i. So,

$$W \coprod_{Y \times \mathbb{R}, \varepsilon_0} Y = \{(w, y) : b(w) = (y, 0)\} \cong X_0$$

and

$$W \coprod_{Y \times \mathbb{R}, \varepsilon_1} Y = \{(w, y) : b(w) = (y, 1)\} \cong X_1.$$

Since $g : Y \to g(Y)$ is an admissible complex orientable map,

$$h = (g, \mathrm{Id}) \circ b : W \to Y \times \mathbb{R} \to g(Y) \times \mathbb{R}$$

is an admissible complex orientable map transverse to $\varepsilon_i : g(Y) \to g(Y) \times \mathbb{R}$. Then

$$\begin{aligned} W \prod_{g(Y)\times\mathbb{R},\varepsilon_0} g(Y) &= \{(w, g(y)) : y \in Y, h(w) = (g(y), 0)\} \\ &= \{(w, g(y)) : y \in Y, (g, \mathrm{Id}) \circ b(w) = (g(y), 0)\} \\ &= \{(w, g(y)) : b(w) = (g^{-1}g(y), 0)\} \\ &= \{(w, y) : b(w) = (y, 0)\} \cong X_0, \end{aligned}$$

and

$$\begin{aligned} W \prod_{g(Y)\times\mathbb{R},\varepsilon_1} g(Y) &= \{(w, g(y)) : y \in Y, h(w) = (g(y), 1)\} \\ &= \{(w, g(y)) : y \in Y, (g, \mathrm{Id}) \circ b(w) = (g(y), 1)\} \\ &= \{(w, g(y)) : b(w) = (g^{-1}g(y), 1)\} \\ &= \{(w, y) : b(w) = (y, 1)\} \cong X_1. \end{aligned}$$

Thus we have verified that it is well-defined.

The graded cobordism set $\mathcal{U}^*(Y)$ of the separable Hilbert manifold Y has a group structure given as follows. Let $[X_1, f_1]$ and $[X_2, f_2]$ be cobordism classes, then

$$[X_1, f_1] + [X_2, f_2]$$

is the class of the map

$$f_1 \sqcup f_2 : X_1 \sqcup X_2 \to Y$$

where $X_1 \sqcup X_2$ is the topological sum (disjoint union) of manifolds X_1 and X_2. We will show that this is well-defined. Then, we have to show that if $[X, f] = [X', f']$ and $[Z, g] = [Z', g']$, then

$$[X, f] + [Z, g] = [X', f'] + [Z', g'].$$

There is an admissible complex orientable map $b : U \to Y \times \mathbb{R}$ such that the pull-back map $\varepsilon_0{}^*(b)$ is equivalent to f and $\varepsilon_1{}^*(b)$ is equivalent to f' where $\varepsilon_t : Y \to Y \times \mathbb{R}$ given by $\varepsilon_t(y) = (y, t)$ for $t = 0, 1$, is a map transverse to b. Similarly, since $[Z, g] = [Z', g']$, we have an admissible complex orientable map $c : W \to Y \times \mathbb{R}$ such that the pull-back map $\varepsilon_0{}^*(c)$ is equivalent to g and $\varepsilon_1{}^*(c)$ is equivalent to g', where $\varepsilon_t : Y \to Y \times \mathbb{R}$ given by $\varepsilon_t(y) = (y,t)$ for $t = 0, 1$, is a map transverse to c. Because the disjoint union of Fredholm maps is a Fredholm map too, there is an admissible complex orientable map $b \sqcup c : U \sqcup W \to Y \times \mathbb{R}$ such that it is transversal to ε_t and $\varepsilon_0{}^*(b \sqcup c)$ is equivalent to $f \sqcup g$, $\varepsilon_1{}^*(b \sqcup c)$ is equivalent to $f' \sqcup g'$. So, $[X, f] + [Z, g] = [X', f'] + [Z', g']$. We have shown that the addition on the graded cobordism set $\mathcal{U}^*(Y)$ is well-defined. As usual, the empty set $\emptyset$ is the zero element of the cobordism set and the negative of $[X, f]$ is itself with the opposite orientation on the normal bundle of the embedding $\tilde{f}$. Then we have

Proposition 5.26. *The graded cobordism set $\mathcal{U}^*(Y)$ of the admissible complex orientable maps of Y is a graded abelian group.*

If the cobordism functor $\mathcal{U}$ of admissible complex orientable maps is restricted to finite dimensional Hilbert manifolds, it agrees Quillen's complex cobordism functor MU^*.

Theorem 5.27. *For a finite dimensional separable Hilbert manifold Y, $\mathcal{U}^*(Y)$ is isomorphic to the Quillen's complex cobordism $MU^*(Y)$.*

2. Transversality, cup product and contravariant property.

We would like to define a product structure on the graded cobordism group $\mathcal{U}^*(Y)$. If $[X_1, f_1] \in \mathcal{U}^{d_1}(Y)$ and $[X_2, f_2] \in \mathcal{U}^{d_2}(Y)$ are two cobordism classes, then we have the external product

$$[(f_1, f_2) : X_1 \times X_2] \in \mathcal{U}^{d_1+d_2}(Y \times Y).$$

Although there is the external product in the category of cobordism of separable Hilbert manifolds, we can not define the internal product unless Y is a finite dimensional manifold. But, if admissible complex orientable maps f_1 and f_2 are transversal, then we have internal product

$$[X_1, f_1] \cup [X_2, f_2] = \Delta^*[(X_1, X_2), (f_1, f_2)]$$

where Δ is the diagonal embedding. The unit element 1 is given by the identity map $Y \to Y$ with index 0. If the separable Hilbert manifold Y is finite dimensional, by Thom's transversality theorem in [**94**], every complex orientable map to Y has a transversal approximation. Therefore the cobordism set $\mathcal{U}^*(Y)$ has a ring structure by the cup product $\cup$.

We want to give a useful theorem due to F. Quinn [**86**].

Theorem 5.28. *Let N be a smooth separable Hilbert manifold. Let $f : M \to N$ be a Fredholm map, $g : W \to N$ an inclusion of a finite-dimensional submanifold of N. Then there exists an approximation g' of g in $C^\infty(W, N)$ with the fine topology such that g' is transversal to f.*

Details about the fine topology and smooth maps space $C^\infty(W, N)$ can be found in [**77**]. Here, the derivatives of the difference function between the function g and its approximation g' are bounded. We would like to interprete this approximation in the fine topology. In order to make this, we need some notation.

Definition 5.29. *Let X and Y be smooth manifolds. A* k-jet from X to Y *is an equivalence class $[f, x]_k$ of pairs (f, x) where $f : X \to Y$ is a smooth mapping, $x \in X$. Two pairs (f, x) and (f', x') are equivalent if $x = x'$, f and f' have same Taylor expansion of order k at x in some pair of coordinate charts centered at x and $f(x)$ repectively. We write $[f, x]_k = J^k f(x)$ and call that the* k-jet of f *at x.*

Note that there is another definition of the equivalence relation: $[f, x]_k = [f', x']_k$ if $x = x'$ and $T_x^k f = T_x^k f'$ where T^k is the kth tangent mapping.

Definition 5.30. *Let X be a topological space. A covering of X is* locally finite *if every point has a neighborhood which intersects only finitely many elements of the covering.*

Now we interprete the approximation g' of g in the smooth fine topology as follows. Let $\{L_i\}_{i \in I}$ be a locally finite cover of W. For every open set L_i, there is a bounded continous map $\varepsilon_i : L_i \to [0, \infty)$ such that

$$||J^k g(x) - J^k g'(x)|| < \varepsilon_i(x),$$

for every $x \in W$ and $k > 0$.

By Theorem 5.28, a smooth map $g : Z \to Y$, where Z is finite dimensional manifold, can be moved by a homotopy until it is transversal to an admissible complex orientable map $f : X \to Y$. Then the cobordism functor is contravariant for any map from a finite dimensional manifold to a Hilbert manifold.

Theorem 5.31. *Let $f : X \to Y$ be an admissible complex oriented map and $g : Z \to Y$ be a smooth map from a finite dimensional manifold Z. The cobordism class of the pull-back $Z \prod_Y X \to Z$ depends only on the cobordism class of f, hence there is a map $g^* : \mathcal{U}^d(Y) \to \mathcal{U}^d(Z)$ given by*

$$g^*([X, f]) = [Z \prod_{Y} X, g'^*(f)],$$

where g' is an approximation of g such that it is transverse to f.

Proof. Suppose that $f : X \to Y$ be admissible complex orientable map. By Therorem 5.28, there exists an approximation $g_0 : Z \to Y$ of g such that it is transverse to f. We will show that $g_0^*[X, f]$ depends only on the cobordism class $[X, f]$. Assume that $f : X \to Y$ and $f' : X' \to Y$ are cobordant and another approximation $g_1 : Z \to Y$ of g is transverse to f'. Then there is an admissible complex orientable map $b : W \to Y \times \mathbb{R}$ such that $\varepsilon_i : Y \to Y \times \mathbb{R}$ given by $\varepsilon_i(y) = (y, i)$ for $i \in \{0, 1\}$, is transversal to b and the pull-back map $\varepsilon_0^*(b)$ is equivalent to f and $\varepsilon_1^*(b)$ is equivalent to f'. So,

$$W \prod_{Y \times \mathbb{R}} Y = \{(w, y) : b(w) = (y, 0)\} \cong X,$$

and

$$W \prod_{Y \times \mathbb{R}} Y = \{(w, y) : b(w) = (y, 1)\} \cong X'.$$

There is a smooth map $(g_0 \sqcup g_1, \mathrm{Id}_{\mathbb{R}}) : Z \times \mathbb{R} \to Y \times \mathbb{R}$ transverse to admissible complex orientable map $b : W \to Y \times \mathbb{R}$. By Proposition 5.19, the map $(g_0 \sqcup g_1, \mathrm{Id}_{\mathbb{R}})^*(b) : W \prod_{Y \times \mathbb{R}} Z \times \mathbb{R} \to Z \times \mathbb{R}$ is an admissible complex orientable map and it is transverse to $\varepsilon_i : Z \to Z \times \mathbb{R}, i = 0, 1$. By Proposition 5.19, we have induced map

$$\varepsilon_0^*(g_0 \sqcup g_1, \mathrm{Id}_{\mathbb{R}})^*(b) : \left(W \prod_{Y \times \mathbb{R}} Z \times \mathbb{R} \right) \prod_{Z \times \mathbb{R}} Z \to Z.$$

The product manifold $\left(W\prod_{Y\times\mathbb{R}} Z\times\mathbb{R}\right)\prod_{Z\times\mathbb{R}} Z$ is equal to

$$= \{(w,(z_1,t),z_2) : b(w) = (g_0(z_1),t) \,\text{or}\, b(w) = (g_1(z_1),t), (z_1,t) = (z_2,0)\}$$
$$= \{(w,(z_1,0) : b(w) = (g_0(z_1),0) \,\text{or}\, b(w) = (g_1(z_1),0)\} \cong Z\prod_Y X.$$

Similarly, we have induced map

$$\varepsilon_1^*(g_0\sqcup g_1, \mathrm{Id}_{\mathbb{R}})^* : \left(W\prod_{Y\times\mathbb{R}} Z\times\mathbb{R}\right)\prod_{Z\times\mathbb{R}} Z \to Z.$$

The product manifold $\left(W\prod_{Y\times\mathbb{R}} Z\times\mathbb{R}\right)\prod_{Z\times\mathbb{R}} Z$ is diffeomorphic to $Z\prod_Y X'$. The induced map $\varepsilon_0^*(g_0\sqcup g_1,\mathrm{Id}_{\mathbb{R}})^*(b)$ are equivalent to $g_0^*(f)$ and $\varepsilon_1^*(g_0\sqcup g_1,\mathrm{Id}_{\mathbb{R}})^*(b)$ are equivalent to ${g_1}^*(f')$. □

In the case that g is a smooth map between infinte dimensional separable Hilbert manifolds, F. Quinn did the best approach to solve this problem in [**86**].

Theorem 5.32. *Let U be an open set in separable infinite dimensional Hilbert space H and let $F : M \to N$ be a proper Fredholm map between separable infinite dimensional Hilbert manifolds M and N. Then the set of maps transversal to F is dense in the closure of Sard function space $\overline{S(U,N)}$ in the C^∞ fine topology.*

For the terminology of Sard functions, see Definition 5.41. We will require the open embedding theorem of Eells and Elworthy in [**34**].

Theorem 5.33. *Let X be a smooth manifold modelled on the separable infinite dimensional Hilbert space H. Then X is diffeomorphic to an open subset of H.*

Using the open embedding theorem, we have the transversal smooth approximation of Sard functions in the C^∞ fine topology. From [**34**], we have

Theorem 5.34. *Let X and Y be two smooth manifold modelled on the separable infinite dimensional Hilbert space H. If there is a homotopy equivalence $\varphi : X \to Y$, then φ is homotopic to a diffeomorphism of X onto Y.*

By Theorem 5.32, 5.33 and 5.34, we have the following theorem.

Theorem 5.35. *Let X, Y and Z be infinite dimensional smooth separable Hilbert manifold. Suppose that $f : X \to Y$ is an admissible complex orientable map and*

$g : Z \to Y$ is a Sard function. Then the cobordism class of the pull-back $Z \prod_Y X \to Z$ depends on the cobordism class of f hence there is a map $g^* : \mathcal{U}^d(Y) \to \mathcal{U}^d(Z)$ given by

$$g^*([X, f]) = [Z \times_Y X, g^*(f)].$$

Then, $\mathcal{U}^*$ is a contravariant functor for Sard functions between infinite dimensional separable Hilbert manifolds.

Of course, the question of whether it agrees with other cobordism functors such as representable cobordism is not so obviously answered. There is also no obvious dual bordism functor.

3. Finite dimensional smooth fiber bundles and transversality.

Definition 5.36. *A smooth fiber bundle ξ over a smooth manifold B (the* base*) consists of a smooth manifold E (the* total space*), a smooth map $\pi : E \to B$ (*projection map*). For each $b \in B$, $E_b = \pi^{-1}(b)$ is called the* fiber *over b. These must satisfy the following* local triviality condition. *There is an open covering $\{U_i\}$ of B and a finite dimensional smooth manifold V such that for $\forall i$, $\tau_i : \pi^{-1}(U_i) \to U_i \times V$ is a diffeomorphism commuting with the projection onto U_i and for $\forall\, x \in U_i$, the induced map on the fiber $\tau_{i_x} : \pi^{-1}(x) \to V$ is a diffeomorphism. If U_i and U_j are two members of the covering, then the map $g_{ij} : U_i \cap U_j \to \mathrm{Diff}(V)$ given by $x \to \tau_{j_x}{\tau_{i_x}}^{-1}$ is a smooth map which satisfies the identities*

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x) \quad \forall x \in U_i \cap U_j \cap U_k$$

$$g_{ii}(x) = 1.$$

g_{ij} is called the transition function *of the pair of smooth charts (τ_i, U_i) and (τ_j, U_j). A subgroup G of the diffeomorphism group $\mathrm{Diff}(V)$ is called a* structure group *of fiber bundle if $\operatorname{im} g_{ij} \subseteq G$. When the smooth manifold V is a vector space, (E, π, B, V) is called a* smooth vector bundle *B if the structure group G is a subgroup of $GL(V)$.*

Definition 5.37. *A smooth map $f : X \to Y$ is called a* submersion *at $x \in X$ if there exists a chart (U, φ) at x and a chart (V, ψ) at $f(x)$ such that φ gives an diffeomorphism of U on a product $U_1 \times U_2$ and the map $\psi f \varphi^{-1} : U_1 \times U_2 \to V$ is a projection where U_1, U_2 are open sets in the vector space V.*

The following proposition gives us the characterization of the notion of a submersion map between separable Hilbert manifolds.

Proposition 5.38. *Let $f : X \to Y$ be a smooth map between smooth separable Hilbert manifolds X and Y. Then f is a submersion at $x \in X$ if and only if the derivative map $d_x f$ is surjective and its kernel splits.*

By Proposition 5.38, we have

Proposition 5.39. *The smooth fiber bundle $\pi : E \to B$ is a submersion, hence it is transversal to any smooth map $f : X \to B$.*

By Proposition 5.39, the cobordism functor is contravariant for any finite dimensional smooth fiber bundle map π. Let $\pi : E \to Y$ be a smooth fiber bundle, $f : X \to Y$ be an admissible complex orientable map between separable Hilbert manifolds. Then the cobordism class of the pull-back $E \prod_Y X \to E$ depends on the cobordism class of f and we get the map $\pi^* : \mathcal{U}^d(Y) \to \mathcal{U}^d(E)$ given by

$$\pi^*([X, f]) = [E \times_Y X, \pi^*(f)].$$

4. The Euler class of a finite dimensional bundle.

In this section, we will introduce the Euler class of a vector bundle in complex cobordism for the separable Hilbert manifolds. In order to do this, we will give some definitions.

Definition 5.40. *Let E be a Banach space. We say that a collection $\mathcal{S}$ of smooth functions $\alpha : E \to \mathbb{R}$ is a* Sard class *if it satisfies the following conditions:*

1. *for $r \in \mathbb{R}$, $y \in E$ and $\alpha \in \mathcal{S}$, then the function $x \to \alpha(rx + y)$ is also in the class $\mathcal{S}$,*
2. *if $\alpha_n \in \mathcal{S}$, then the rank of differential $D_x(\alpha_1, \dots, \alpha_n)$ is constant for all x not in some closed finite dimensional submanifold of E.*

Definition 5.41. *Let $\mathcal{S}$ be a Sard class on E, U open in E, and M a smooth Banach manifold. We define $\mathcal{S}(U, M)$ to be the collection of* Sard functions *$f : U \to M$ such that for each $x \in U$ there is a neighbourhood V of x, functions $\alpha_1, \dots, \alpha_n \in \mathcal{S}$, and a* *smooth map $g : W \to M$, where W open in $\mathbb{R}^n$ contains $(\alpha_1, \ldots, \alpha_n)(V)$, all such that $f|V = g \circ (\alpha_1, \ldots, \alpha_n)|V$.*

Definition 5.42. *The* support *of a function $f : X \to \mathbb{R}$ is the closure of the set of points x such that $f(x) \neq 0$.*

From [**86**], we have

Theorem 5.43. *E admits a Sard class $\mathcal{S}$ if $\mathcal{S}(E, \mathbb{R})$ contains a function with bounded nonempty support. In particular, the separable Hilbert space admits Sard classes.*

Definition 5.44. *A* refinement *of a covering of X is a second covering, each element of which is contained in an element of the first covering.*

Definition 5.45. *A topological space is* paracompact *if it is Hausdorff, and every open covering has a locally finite open refirement.*

Definition 5.46. *A smooth* partition of unity *on a manifold X consists of an covering $\{U_i\}$ of X and a system of smooth functions $\psi_i : X \to \mathbb{R}$ satisfying the following conditions.*

1. *$\forall x \in X$, we have $\psi_i(x) \geqslant 0$;*

2. *the support of ψ_i is contained in U_i;*

3. *the covering is locally finite;*

4. *for each point $x \in X$, we have*

$$\sum_i \psi_i(x) = 1.$$

Definition 5.47. *A manifold X will be said to* admit partitions of unity *if it is paracompact, and if, given a locally finite open covering $\{U_i\}$, there exists a partition of unity $\{\psi_i\}$ such that the support of ψ_i is contained in some U_i.*

From [**74**], we have

Theorem 5.48. *Let X be a paracompact smooth manifold modelled on a separable Hilbert space H. Then X admits smooth partitions of unity.*

From [**35**],

Theorem 5.49. *On a separable Hilbert manifold the functions constructed using the partitions of unity form a Sard class.*

We know from [**58**] that global sections of a vector bundle on smooth separable Hilbert manifold can be constructed using the partitions of unity, then all sections are Sard. By Theorem 5.33 , the smooth separable Hilbert manifold B, which is the base space of a vector bundle, can be embedded into the separable Hilbert space H as a open subset and by Theorem 5.32, we have

Corollary 5.50. *Let $\pi : E \to B$ be a finite dimensional complex vector bundle over the separable Hilbert manifold B and let $i : B \to E$ be the zero-section of the vector bundle. Then, there is an approximation $\tilde{i}$ of i such that $\tilde{i}$ is transversal to i.*

By Theorem 5.34 and 5.35, we define the Euler class of a finite dimensional complex vector bundle on separable Hilbert manifolds. Note that Theorem 5.35 implies that Euler class is well-defined.

Definition 5.51. *Let $E \to B$ be a finite dimensional complex vector bundle on a separable Hilbert manifold B and let $i : B \to E$ be the zero-section of vector bundle. Then the element $i^*i_*(1)$ is called the* Euler class *of the vector bundle in the complex cobordism $\mathcal{U}^*(B)$ and denoted by $\chi(E)$.*

Now, we will give the projection formula of Gysin map for submersion maps.

Theorem 5.52. *Let $f : X \to Y$ be an admissible complex-orientable submersion map and let $\pi : E \to Y$ be a finite dimensional complex vector bundle. Then*

$$\chi(E) \cup [X, f] = f_*\chi(f^*E).$$

PROOF. Let ξ denote the cobordism class $[X, f]$. Then

$$\chi(E) = \{y \in Y : s(y) = (y, 0)\},$$

where s is the generic section of π. So,

$$\begin{aligned}\chi(E) \cup \xi &= \{(y,x) : y = f(x), s(y) = (y,0)\} \\ &= \{f(x) : s(f(x)) = (f(x),0)\}.\end{aligned}$$

Now, we determine the expression $f_*\chi(f^*E)$.

$$\begin{aligned}f^*(E) &= \{((y,v),x) : \pi(y,v) = y = f(x)\} \\ &= \{(x,v) : y = f(x), f^*\pi(x,v) = x\}.\end{aligned}$$

Then

$$\chi(f^*(E)) = \{x : y = f(x), s(y) = (y,0)\}.$$

So,

$$f_*\chi f^*(E) = \{f(x) : s(f(x)) = (f(x),0)\}.$$

Thus, we have completed the proof. $\square$

5. Complex cobordism of LG/T and cup product formula.

We will now give the construction of a collection of some elements in the negative part of the complex cobordism of the smooth Hilbert manifold LG/T. We know from the Chapter 2 that LG/T is a complex manifold by the diffeomorphism $LG/T \cong LG_{\mathbb{C}}/\widetilde{B}^+$.

Let $p_i : LG_{\mathbb{C}}/\widetilde{B}^+ \to LG_{\mathbb{C}}/P_i$ be the fiber bundle with the fiber $P_i/\widetilde{B}^+ \cong \mathbb{C}P^1$, where P_i is the parabolic subspace of $LG_{\mathbb{C}}$ corresponding to the simple affine root α_i. Let $1 \in U^0(LG/T)$ denote cobordism class of the identity map $\mathrm{Id} : LG_{\mathbb{C}}/\widetilde{B}^+ \to LG_{\mathbb{C}}/\widetilde{B}^+$. Then we have $p_{i*}(1) = [LG_{\mathbb{C}}/\widetilde{B}^+, p_i \circ \mathrm{Id}] \in \mathcal{U}^{-2}(LG_{\mathbb{C}}/P_i)$. Since the bundle map p_i is a submersion, it is transversal to any map, so

$$p_i{}^*(p_{i*}(1)) = [LG_{\mathbb{C}}/\widetilde{B}^+ \prod_{LG_{\mathbb{C}}/P_i} LG_{\mathbb{C}}/\widetilde{B}^+, p_i{}^*(p_i \circ \mathrm{Id})] \in \mathcal{U}^{-2}(LG/T).$$

By the definition of the product, the space $LG_{\mathbb{C}}/\widetilde{B}^+ \prod_{LG_{\mathbb{C}}/P_i} LG_{\mathbb{C}}/\widetilde{B}^+$ is equal to

$$\begin{aligned}&= \{(\xi_1\widetilde{B}^+, \xi_2\widetilde{B}^+) : \xi_1 P_i = \xi_2 P_i\} \\ &= \{(\xi_1\widetilde{B}^+, \xi_2\widetilde{B}^+) : \xi_2{}^{-1}\xi_1 \in P_i\}.\end{aligned}$$

The last space is diffemorphic to $LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i/\widetilde{B}^+$ given by smooth map $(\xi_1\widetilde{B}^+, \xi_2\widetilde{B}^+) \to [\xi_2, {\xi_2}^{-1}\xi_1\widetilde{B}^+]$ where $\widetilde{B}^+$ acts on $LG_{\mathbb{C}} \times P_i/\widetilde{B}^+$ by $b \cdot (\xi, x_i\widetilde{B}^+) = (\xi b, b^{-1}x_i\widetilde{B}^+)$. The smooth map from $LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i/\widetilde{B}^+$ to $LG_{\mathbb{C}}/\widetilde{B}^+ \prod_{LG_{\mathbb{C}}/P_i} LG_{\mathbb{C}}/\widetilde{B}^+$ is given by $[\xi, x_i\widetilde{B}^+] \to (\xi x_i\widetilde{B}^+, \xi\widetilde{B}^+)$. Hence the pull-back map ${p_i}^*(p_i \circ \mathrm{Id})$ from $LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i/\widetilde{B}^+$ to $LG_{\mathbb{C}}/\widetilde{B}^+$ is given by $[\xi, x_i\widetilde{B}^+] \to \xi \cdot x_i\widetilde{B}^+$.

For $i \neq j$, let $p_j : LG_{\mathbb{C}}/\widetilde{B}^+ \to LG_{\mathbb{C}}/P_j$ be the $\mathbb{C}P^1$ bundle associated to a different parabolic space P_j. Then ${p_j}^*{p_j}_*{p_i}^*{p_i}_*(1)$ is represented by the smooth map

$$s : LG_{\mathbb{C}}/\widetilde{B}^+ \prod_{LG_{\mathbb{C}}/P_j} LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i/\widetilde{B}^+ \to LG_{\mathbb{C}}/\widetilde{B}^+.$$

By the definition of product, the smooth submanifold $LG_{\mathbb{C}}/\widetilde{B}^+ \prod_{LG_{\mathbb{C}}/P_j} LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i/\widetilde{B}^+$ is equal to

$$\begin{aligned} &= \{(\xi_1\widetilde{B}^+, [\xi_2, x_i\widetilde{B}^+]) : \xi_1 P_j = \xi_2 \cdot x_i P_j\} \\ &= \{(\xi_1\widetilde{B}^+, [\xi_2, x_i\widetilde{B}^+]) : \xi_2 {x_i}^{-1}\xi_1 \in P_j\}. \end{aligned}$$

The space $LG_{\mathbb{C}}/\widetilde{B}^+ \prod_{LG_{\mathbb{C}}/P_j} LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i/\widetilde{B}^+$ is diffeomorphic to $LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i \times_{\widetilde{B}^+} P_j/\widetilde{B}^+$ given by the smooth map $(\xi_1\widetilde{B}^+, [\xi_2, x_i\widetilde{B}^+]) \to [\xi_2, x_i, \xi_2{x_i}^{-1}\xi_1\widetilde{B}^+]$. The smooth map

$$LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i \times_{\widetilde{B}^+} P_j/\widetilde{B}^+ \to LG_{\mathbb{C}}/\widetilde{B}^+ \prod_{LG_{\mathbb{C}}/P_j} LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i/\widetilde{B}^+$$

is given by $[\xi, x_i, x_j\widetilde{B}^+] \to (\xi x_i x_j\widetilde{B}^+, [\xi, x_i\widetilde{B}^+])$. Then the cobordism class of

$$s : LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_i \times_{\widetilde{B}^+} P_j/\widetilde{B}^+ \to LG_{\mathbb{C}}/\widetilde{B}^+.$$

is given by $[\xi, x_i, x_j\widetilde{B}^+] \to \xi \cdot x_i \cdot x_j\widetilde{B}^+$. Continuing in similar way, by induction, for any sequence $I = (i_1, i_2, \dots, i_n)$ such that $i_k \neq i_{k+1}$, we can construct a space

$$Z_I = LG_{\mathbb{C}} \times_{\widetilde{B}^+} P_{i_1} \times_{\widetilde{B}^+} \cdots \times_{\widetilde{B}^+} P_{i_n}/\widetilde{B}^+$$

together with a smooth map $z_I : Z_I \to LG_{\mathbb{C}}/\widetilde{B}^+$ given by $[\xi, x_{i_1}, \dots, x_{i_n}\widetilde{B}^+] \to \xi \cdot x_{i_1} \cdots x_{i_n}\widetilde{B}^+$. Here $\widetilde{B}^+$ acts by the multiplication on the right side of the each term in the sequence and by the inverse multiplication on the left side of each term for any $i \in I$.

Proposition 5.53. *For any sequence $I = (i_1, i_2, \dots, i_n)$ such that $i_k \neq i_{k+1}$, Z_I is a smooth complex manifold and $z_I : Z_I \to LG_{\mathbb{C}}/\widetilde{B}^+$ is a proper holomorphic map.*

Definition 5.54. *Let $r_{i_1}r_{i_2}\cdots r_{i_n}$ be the reduced decomposition of the element w of the affine Weyl group W. Then*

$$\widetilde{Z}_w = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_n}/B$$

in the Z_I is called Bott-Samelson variety *associated to w.*

Definition 5.55. *A* rational *map of a complex manifold M to projective space $\mathbb{C}P^n$ is a map*

$$f : z \to [1, f_1(z), \dots , f_n(z)]$$

given by n meromorphic functions on M. A rational map $f : M \to N$ to the algebraic variety $N \subseteq \mathbb{C}P^n$ is a rational map $f : M \to \mathbb{C}P^n$ whose image lies in N.

Definition 5.56. *A rational map $f : M \to N$ is* birational *if there exists a rational map $g : N \to M$ such that $f \circ g$ is the identity as a rational map. Two algebraic varieties are said to be* birationally isomorphic, *or simply* birational, *if there exists a birational map between them.*

Definition 5.57. *Let Y be a complex manifold with singularities and let $\Theta : X \to Y$ be a map. Then (X, Θ) is a* resolution of singularities *of Y if X is smooth and the map Θ is proper and birational.*

Theorem 5.58. *(see [28]) $z_w : \widetilde{Z}_w \to \overline{C_w}$ is a resolution of singularities of the closure of the cell C_w of $LG_{\mathbb{C}}/\widetilde{B}^+$ in the usual complex topology.*

Since the resolution Z_I is a complex manifold and the map z_I is a holomorphic map, naturally z_I has complex orientable structure. Then, $[Z_I, z_I]$ is a cobordism class of $\mathcal{U}^*(LG/T)$. We will denote this class by x_I for any sequence $I = (i_1, \dots , i_n)$.

Let $p_i : LG_{\mathbb{C}}/\widetilde{B}^+ \to LG_{\mathbb{C}}/P_i$ be a $\mathbb{CP}^1$ bundle associated the parabolic subgroup P_i. Then, the operator

$${p_i}^* {p_i}_* : \mathcal{U}^*(LG/T) \to \mathcal{U}^{*-2}(LG/T)$$

will be denoted A_i. This operator is analogous of Kač operator which has been introduced in the work about ordinary cohomology of flag spaces in the chapter 1.

Proposition 5.59. *We have cobordism classes $x_I = A_I(1)$ for $I = (i_1, \dots , i_n)$.*

Proposition 5.60. *Let $I = (i_1, \dots, i_n)$ and $J = (i_1, \dots, i_{n+1})$. Then, $A_{n+1}(x_I) = x_J$.*

Now we will describe a method for computing the products of the cobordism classes x_I with characteristic classes of line bundles on LG/T. Let L_λ be the line bundle on LG/T associated to a weight λ. We know that $i^* i_*(1)$ gives the Euler class in the $\mathcal{U}^2(LG/T)$ where i is the zero-section of the line bundle L_λ.

Theorem 5.61.

$$\chi(L_\lambda) \cup x_I = z_{I*}\chi(z_I^* L_\lambda).$$

PROOF. Since z_I is a complex orientable and submersion map, by Theorem 5.52, the equality is a direct consequence of the projection formula for the Gysin homomorphism. □

Given a index $I = (i_1, \dots, i_n)$, we define new indices $I^k, I_{>k}$ by

$$I^k = (i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n), \quad I_{>k} = (i_{k+1}, \dots, i_n).$$

A subindex J of I of length k is determined by a one-to-one order preserving map

$$\sigma : \{1, \dots, k\} \to \{1, \dots, n\}$$

by the rule $j_m = i_{\sigma(m)}$.

For the subindex J of I of length k there is a natural embedding

$$i_{J,I} : Z_J \to Z_I$$

defined by converting a $k+1$-tuple $(\xi, x_{j_1}, \dots, x_{j_k})$ to the $n+1$-tuple in the $i_{\sigma(m)}$th slot for $1 \leqslant m \leqslant k$ and the identity element elsewhere. A pair $x_J = [Z_J, i_{J,I}]$ represents an element of $\mathcal{U}^*(Z_I)$.

The classes of the form x_J are precisely the classes we use to obtain the expression for $\chi(L_\lambda)$.

A complex line bundle is determined up to isomorphism by its first Chern class $c_1(L)$ in integral cohomology. The Picard group of line bundles on Z_I denoted $\mathrm{Pic}(Z_I)$ is isomorphic to $H^2(Z_I, \mathbb{Z})$, which is free and has a basis of elements whose liftings to $\mathcal{U}^*(Z_I)$ can be chosen to be x_{I^k}

The first Chern class is given as

$$c_1 : \mathrm{Pic}(Z_I) \to H^2(Z_I, \mathbb{Z}).$$

$H^2(Z_I, \mathbb{Z})$ is free with basis consisting of classes x_{I^k} with $1 \leqslant k \leqslant n$. Therefore we can choose a basis for $\mathrm{Pic}(Z_I)$ consisting of line bundles L_k, where $1 \leqslant k \leqslant n$, satisfying

$$c_1(L_k) = x_{I^k}.$$

We take L_k to be the line bundle associated with the divisor Z_{I^k}. This means L_k has a section which intersects the zero section transversally on Z_{I^k} so that $\chi(L_k) = x_{I^k}$ in complex cobordism. This basis connects line bundles with a geometric basis for $\mathcal{U}^*(Z_I)$. In ordinary cohomology, the Euler class of a line bundle on Z_I is a linear combination of x_{I^k}.

Theorem 5.62. *Let λ be a weight. Let $I = (i_1, \dots, i_n)$ be a index and r_I denote the corresponding product of reflections. Then the line bundle L_λ on Z_I decomposes as*

$$L_\lambda = \bigotimes_{k=1}^{n} L_k^{-\langle r_{I_{>k}}\lambda, \mathrm{a}_{i_k}\rangle}.$$

PROOF. For the proof, see [20]. □

6. Examples of some infinite dimensional cobordism classes.

In this section, we will show that the stratas introduced in chapter 2 give some cobordism classes of separable Hilbert manifolds.

We know from Theorem 2.23 in chapter 2 that the Grassmannian space $\mathrm{Gr}\, H$ is a separable Hilbert manifold. By Theorem 2.30 of chapter 2, we know that the stratum Σ_S is a locally closed contractible complex submanifold of $\mathrm{Gr}(H)$ of codimension $\ell(S)$. The inclusion $i : \Sigma_S \to \mathrm{Gr}(H)$ is closed in the open subset U_S and it is a Fredholm map with index $-\ell(S)$. Since the strata Σ_S is an infinite dimensional manifold, by Theorem 5.23 of this chapter, the embedding $i : \Sigma_S \to \mathrm{Gr}(H)$ is a proper Fredholm map.

Therefore, we have

Theorem 5.63. *The strata $\Sigma_S \to \mathrm{Gr}(H)$, $S \in \mathcal{S}$, is an element of cobordism group $\mathcal{U}^{2\ell(S)}(\mathrm{Gr}(H))$.*

These stratas Σ_S are dual to the Schubert cells C_S in the following sense:

1. the dimension of C_S is the codimension of Σ_S and
2. C_S meets Σ_S transversally in a single point, and meets no other stratum the same codimension.

We know from chapter 2 that the based loop group ΩG of compact semi-simple Lie group G has a stratafication Σ_λ where $\lambda : \mathbb{T} \to T$ is a homomorphism. By Theorem 2.55 of chapter 2, we know that the strata Σ_λ is a locally closed contractible complex submanifold of ΩG of codimension d_λ where d_λ has been defined in chapter 2. Then, $i : \Sigma_\lambda \to \Omega G$ is a proper Fredholm complex orientable map. Then, we have

Theorem 5.64. *The stratas $\Sigma_\lambda \to \Omega G$ give elements of cobordism group $\mathcal{U}^{2d_\lambda}(\Omega G)$.*

Also these stratification holds for the homogeneous space LG/T where G is a compact semi-simple Lie group. Then, we have

Theorem 5.65. *The strata $\Sigma_w \to LG/T$, $w \in \widetilde{W}$, give elements of cobordism group $\mathcal{U}^{2\ell(w)}(LG/T)$.*

We don't know whether these stratifications are a basis form of cobordism groups of homogeneous spaces.

7. The relationship between $\mathcal{U}$-theory and MU-theory.

In this section we consider the relationship between $\mathcal{U}$-theory and MU-theory, in particular, for Grassmannians and LG/T.

First we give the general relationship between $\mathcal{U}^*(\ )$ and $MU^*(\ )$. Let X be a separable Hilbert manifold. Then for each smooth map $f : M \longrightarrow X$ where M is finite dimensional, there is a pullback homomorphism $f^* : \mathcal{U}^*(X) \longrightarrow \mathcal{U}^*(M) = MU^*(M)$. If we now consider all such maps into X, then there is a unique homomorphism $\mathcal{U}^*(X) \longrightarrow \lim\limits_{M \longrightarrow X} MU^*(M)$. In particular, the following seem reasonable.

Conjecture 5.66.
1. *The natural homomorphism $\mathcal{U}^*(X) \longrightarrow \lim\limits_{M \longrightarrow X} MU^*(M)$ is surjective.*
2. *If $\mathcal{U}^{\mathrm{ev}}(X) = 0$ or $\mathcal{U}^{\mathrm{odd}}(X) = 0$, the natural homomorphism $\mathcal{U}^*(X) \longrightarrow \lim\limits_{M \longrightarrow X} MU^*(M)$ is surjective.*
3. *If $MU^{\mathrm{ev}}(X) = 0$ or $MU^{\mathrm{odd}}(X) = 0$, the natural homomorphism $\mathcal{U}^*(X) \longrightarrow \lim\limits_{M \longrightarrow X} MU^*(M)$ is surjective.*

Now we discuss some important special cases. For a separable complex Hilbert space H, let H^n $(n \geqslant 1)$ denote an increasing sequence of finite dimensional subspaces with $\dim H^n = n$ with $H^\infty = \bigcup_n H^n$ dense in H.

From [**71**], we have Kuiper's theorem:

Theorem 5.67. *The unitary group $U(H)$ of a separable Hilbert space H is contractible.*

Let $\mathrm{Gr}_n(H)$ be the space of all n-dimensional subspaces of separable Hilbert space and let $\mathrm{Gr}_n(H^\infty) = \bigcup\limits_{k \geqslant n} \mathrm{Gr}_n(H^k)$. Then $\mathrm{Gr}_n(H^\infty)$ is subset of $\mathrm{Gr}_n(H)$ and it is dense.

Theorem 5.68. *The natural embedding $\mathrm{Gr}_n(H^\infty) \to \mathrm{Gr}_n(H)$ is a homotopy equivalence, and the natural n-plane bundle $\xi_n \to \mathrm{Gr}_n(H)$ is universal.*

PROOF. By a theorem of Pressley and Segal [**84**], the unitary group $U(H)$ acts on $\mathrm{Gr}(H)$ transitively and hence $U(H)$ acts on $\mathrm{Gr}_n(H)$ transitively. Let H^n be an n-dimensional subspace of infinite dimensional separable Hilbert space H and let H_1 be its orthogonal complement in H. The stabilizer group of H^n is $U(H^n) \times U(H_1)$ and this acts freely on the contractible space $U(H)$ and hence

$$\begin{aligned}
\mathrm{Gr}_n(H) &= \frac{U(H)}{U(H^n) \times U(H_1)} \\
&= B(U(H^n) \times U(H_1)) \\
&= BU(H^n) \times BU(H_1).
\end{aligned}$$

Since $U(H_1)$ is contractible by Theorem 5.67, then $BU(H_1)$ is contractible. Hence $\mathrm{Gr}_n(H) \simeq BU(H^n) = BU(n)$. On the other hand,

$$\mathrm{Gr}_n(H^\infty) = \bigcup_{k \geqslant n} \frac{U(H^k)}{U(H^n) \times U(H_2^{k-n})} \subseteq \mathrm{Gr}_n(H),$$

where H_2^{k-n} is the orthogonal complement of H^n in H^k.

By the construction, the natural n-plane bundle $\xi_n \to \mathrm{Gr}_n(H)$ is universal. Also, the natural bundle $\xi_n^\infty \to \mathrm{Gr}_n(H^\infty)$ is classified by the inclusion $\mathrm{Gr}_n(H^\infty) \to \mathrm{Gr}_n(H)$. Since the latter is universal, this inclusion is a homotopy equivalence. □

In particular, the inclusion $P(H^\infty) = \bigcup\limits_n P(H^n) \subseteq P(H)$ is a homotopy equivalence.

Theorem 5.69. *The natural map* $\mathcal{U}^*(P(H)) \longrightarrow \lim_n MU^*(P(H^n)) = MU^*(P(H^\infty))$ *is surjective.*

Proof. We will show by induction that for each n, $\mathcal{U}^*(P(H)) \xrightarrow{i_n^*} MU^*(P(H^{n+1}))$ is surjective. It will suffice to show that $x^i \in \operatorname{im} i_n^*$ for $i = 0, \dots, n$. For $n = 0$, this is trivial. Now we verify this for $n = 1$. By Theorem 5.68, since $\xi : L \to P(H) \simeq P(H^\infty)$ is a universal line bundle, the following diagram commutes $n \geqslant 1$

$$\begin{tikzcd} i_n^*(L) = \eta \arrow[r, "i_n^*"] \arrow[d, "i_n^*(\xi)"] & L \arrow[d, "\xi"] \\ \mathbb{C}P^n = P(H^{n+1}) \arrow[r, "i_n"] & P(H), \end{tikzcd}$$

where $i_n : \mathbb{C}P^n = P(H^{n+1}) \to P(H)$ is an inclusion map. Then for the generator $x = \chi(\eta) \in MU^*(P(H^{n+1})), n \geqslant 1$, by the compatibility of induced bundles, there exists an Euler class $\tilde{x} = \chi(L) \in \mathcal{U}^2(P(H))$ such that $i_n^*(\tilde{x}) = x$, where $i_n : P(H^{n+1}) \to P(H)$ is an inclusion map.

Assume that i_n^* is surjective. Then there exists $y_i \in \mathcal{U}^{2i}(P(H)), i = 0, \dots, n$ such that $i_n^* y_i = x^i \in MU^{2i}(P(H^{n+1}))$. Also, $i_{n+1}^* y_i = x^i + z_i x^{n+1} \in MU^{2i}(P(H^{n+2}))$ where $z_i \in MU_{2(n-i-+1)}$.

In particular, let $y_n = [W, f] \in \mathcal{U}^{2(n)}(P(H))$. Then the following diagram commutes

$$\begin{tikzcd} f^*(L) \arrow[r, "f^*"] \arrow[d, "f^*(\xi)"] & L \arrow[d, "\xi"] \\ W \arrow[r, "f"] & P(H) \end{tikzcd}$$

and there exists an Euler class $\chi(f^*(L)) = [W', g] \in \mathcal{U}^2(W)$. Now $y_{n+1} = f_* \chi(f^*(L)) \in \mathcal{U}^{2n+2}(P(H))$ satisfies

$$\begin{aligned} i_{n+1}^* y_{n+1} &= x^n \chi(\eta) \\ &= x^{n+1}. \end{aligned}$$

Hence, $\operatorname{im} i_{n+1}^*$ contains the MU^*-submodule generated by x^i, $i = 0, \dots, n$. Hence, i_{n+1}^* is surjective. This completes the induction. This shows that the map $\mathcal{U}^*(P(H)) \longrightarrow \lim_n MU^*(P(H^n)) = MU^*(P(H^\infty))$ is surjective. □

Now we need some geometry of Grassmannians. We know from chapter 2 that $\mathrm{Gr}_0(H)$ is the union of the finite dimensional Grassmannians $\mathrm{Gr}(H_{-k,k})$, where $H_{-k,k} =$

$z^{-k}H_+/z^kH_+$. The Grassmannian space Gr_0 is homotopic to the classfying space $BU \times \mathbb{Z}$ and it is dense in $\mathrm{Gr}(H)$.

Theorem 5.70. *The map* $\mathcal{U}^*(\mathrm{Gr}_n(H)) \to MU^*(Gr_n(H))$ *is surjective for* $n \geqslant 1$.

PROOF. For $k \geqslant n$, the inclusion $i : \mathrm{Gr}_n(H_{-k,k}) \to \mathrm{Gr}_n(H)$ induces a contravariant map

$$\mathcal{U}^*(Gr_n(H)) \to \mathcal{U}^*(\mathrm{Gr}_n(H_{-k,k}) = MU^*(\mathrm{Gr}_n(H_{-k,k})).$$

For $k \geqslant n$, since $C_S \subseteq \mathrm{Gr}_n(H_{-k,k})$ is transversal to Σ_S, by Theorem 2.30 of chapter 2 and section 6 of this chapter, there exists stratum $\Sigma_{S'}$ such that

$$\sigma_{S',k} = [\mathrm{Gr}_n(H_{-k,k}) \cap \Sigma_{S'} \to \mathrm{Gr}_n(H_{-k,k})] \in MU^*(\mathrm{Gr}_n(H_{-k,k}))$$

are the classical Schubert cells. From [**78**], these $\sigma_{S',k}$ generate $MU^*(\mathrm{Gr}_n(H_{-k,k}))$ as a MU^*-module. Then i^* is surjective. Since $MU^{\mathrm{odd}}(\mathrm{Gr}_n(H_{-k,k}))$ is trivial for every k, then

$$\begin{aligned}\mathcal{U}^*(\mathrm{Gr}_n(H)) \to \lim_k MU^*(\mathrm{Gr}_n(H_{-k,k}))\\ = MU^*(\mathrm{Gr}_n(H^\infty))\\ \cong MU^*(\mathrm{Gr}_n(H))\end{aligned}$$

is onto. □

Theorem 5.71. *Let G be a semi-simple compact Lie group. Then* $j^* : \mathcal{U}^*(LG/T) \to MU^*(LG/T)$ *is surjective.*

PROOF. If the following composition

$$\mathcal{U}^*(LG/T) \to MU^*(LG/T) \to H^*(LG/T, \mathbb{Z})$$

is onto, this implies that j^* is onto since the Atiyah- Hirzebruch spectral sequence collapses here (because LG/T has no odd cells). We know from chapter 3 that $H^*(LG/T, \mathbb{Z})$ is generated by the Schubert classes $\varepsilon^w, w \in W$ which are dual to the Schubert cells C_w. Since Σ_w is dual to C_w by Theorem 2.59 of chapter 2, the image of the strata Σ_w by composition map gives ε^w. Hence the composition map is onto. □

Similarly, by Theorem 2.55 of chapter 2, we have

Theorem 5.72. *Let G be a semi-simple compact Lie group. Then $j : \mathcal{U}^*(\Omega G) \to MU^*(\Omega G)$ is surjective.*

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