New invariants for groups

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To A. L.

Abstract

The principal part of this thesis starts with Chapter 2, Chapter 1 containing preliminary material.

In Chapter 2, we give an exposition of the classical Alexander ideals of a group presentation whose set of generators is finite. These Alexander ideals are a group invariant; the chain of ideals calculated from presentations for isomorphic groups being equivalent. We also consider some classes of presentations whose groups cannot be distinguished by their Alexander ideals.

In Chapter 3, we define a chain of ideals, the B-ideals, which are calculated from a 3-presentation with finite set of generators and relators. We show that these too are a group invariant and, moreover, that they can distinguish groups which the Alexander ideals cannot.

In Chapter 4, we define for the class of groups of type FP_n another new group invariant, the E_n-ideals. These are calculated from a free resolution of type FP_n for the group. We show that these generalise the Alexander and B-ideals. The E_n-ideals of a group are actually a special case of an invariant for group modules of type FP_n. In the remainder of Chapter 4, we derive some properties of these module invariants and their equivalents for groups, including the connexion of these new invariants with the integral homology of a group.

In Chapter 5, we consider the classes of modules and of groups whose E_n-ideals are simple in a certain sense, the E-trivial modules and groups. In particular, we show that projective modules are E-trivial and, consequently, that groups of type FP are E-trivial. We consider how this relates to a question of Serre's concerning groups of type FP and of type FL. We then consider a larger class of groups, the E-linked groups, whose E_n-ideals are linked in adjacent dimensions in a certain sense.

For a subclass of these groups we define an Euler characteristic, which extends the definition of Euler characteristics of Serre, Chiswell and Brown. We then study the closure properties of these classes of groups and the behaviour of the new Euler characteristic when graphs of these groups are constructed. Extensions of certain E-trivial groups are considered next, and we then demonstrate that, for every $n > 1$, the E_n-ideals can distinguish groups which have the same E_i-ideals for $i < n$ and the same integral homology.

In Chapter 6, we extend the definition of these new invariants to monoids and their modules, distinguishing a right- and a left-hand version. We consider some of the properties of the monoid invariant, in particular, showing how the E_n-ideals of certain groups can be obtained from those of a submonoid. Finally, the E_n-ideals of monoids with a zero element are studied and we consider further the question of Serre.

Statement

Chapter 1 covers some basic material, such as presentations of groups and monoids, pictures over a group presentation, Tietze transformations of presentations, the Fox and picture derivatives, elementary ideals of matrices, resolutions of modules, ranks of projective modules, Euler characteristics of groups, graphs, Coxeter groups and graphs of groups. With the exception of §1.3.2, which covers the picture derivative, this material can be found elsewhere, such as in [16], [22], [23], [25], [33], [36], [41], [47], [52], [58], [66], [71], [85].

Chapter 2 includes material on Alexander ideals which can be found, for example, in [33] or [41]. Theorem 2.6 is an unpublished result of S. J. Pride. In Chapter 3, the definition of the B-ideals of a 3-presentation and Theorem 3.1 is unpublished work of S. J. Pride. The remainder of these chapters is the author's own work.

Chapters 4, 5 and 6 are the original work of the author, with the exception of instances indicated in the text as well as §5.1, §5.4 and Theorem 5.32, which are joint work with S. J. Pride.

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Introduction

Given two groups, a natural question to ask is whether or not they are isomorphic. This problem was raised as early as 1908 by Tietze [83] and was famously formulated by Dehn as the *isomorphism problem*:

> Two groups are given. To decide whether they are isomorphic or not (and also whether a given correspondence is an isomorphism). [34] (Translated in [28].)

A group could be given by a set of matrices or permutations, or by a presentation. We will consider groups given by presentations. Thus, we want to be able to determine whether two presentations define isomorphic groups. Now, while solutions to this problem have been found for some classes of groups (see, for example, [61], [63] and [75]), it is known to be unsolvable in general; for finitely presented groups, this was shown by Adyan [1] and, independently, Rabin [72].

In the absence of a single method for distinguishing groups, we must instead rely on a variety of *invariants*. Suppose that an object or quantity $f(\mathcal{P})$ can be calculated from a presentation $\mathcal{P}$ for a group G. For $f(\mathcal{P})$ to be a group invariant, if $\mathcal{P}_0$ is a presentation for a group G_0 which is isomorphic to G, then the object $f(\mathcal{P}_0)$ calculated from $\mathcal{P}_0$ must be equal to $f(\mathcal{P})$, or equivalent in some sense.

The integral homology $H_*(G)$ of a group G is such an invariant. This takes the form of an infinite sequence of abelian groups and can be calculated from any projective resolution for G.

Various Euler characteristics have been defined for certain classes of groups. These can be calculated in a variety of ways, and usually take values in $\mathbb{Z}$ or $\mathbb{Q}$. See, for example, [22], [29], [76] and [79].

Another example is the chain of Alexander ideals, defined by Fox [41]. These are calculated from a matrix which can be readily obtained from a presentation $\mathcal{P}$, and take the form of an ascending chain, $A(\mathcal{P})$, of ideals $A_\lambda(\mathcal{P})$ $(\lambda \in \mathbb{Z})$ in a commutative ring.

In this thesis, we define a family of new group invariants which are calculated in a similar way to the Alexander ideals.

Firstly, in Chapter 2, we review the definition of the Alexander ideals and we give the usual proof, using Tietze transformations of presentations, that they are a group invariant (Theorem 2.1). We show how these invariants can be used to distinguish non-isomorphic groups and also consider groups which cannot be distinguished in this way (Lemma 2.5 and Theorem 2.6).

In Chapter 3, we define a new group invariant, the B-ideals; if $\mathcal{T}$ is a 3-presentation with finite sets of generators and relators, we can readily obtain a matrix from which an ascending chain, $B(\mathcal{T})$, of ideals $B_\lambda(\mathcal{T})$ $(\lambda \in \mathbb{Z})$ in $\mathbb{Z}G(\mathcal{T})^{ab}$ can be calculated. We prove, using Tietze transformations on 3-presentations, that, if $\mathcal{T}$ and $\mathcal{T}_0$ are 3-presentations for isomorphic groups, then the chains $B(\mathcal{T})$ and $B(\mathcal{T}_0)$ are equivalent (Theorem 3.1). The B-ideals are thus a group invariant. We also show that the B-ideals are sometimes able to distinguish groups which the Alexander ideals cannot.

In Chapter 4, we define, for a group G of type FP_n, an ascending chain, $E_n(G)$, of ideals $E_{n,\lambda}(G)$ $(\lambda \in \mathbb{Z})$ in $\mathbb{Z}G^{ab}$. These are calculated from a matrix associated with the $(n+1)$-st boundary map of a free resolution of type FP_n for G. These generalise the Alexander and B-ideals, since $A(\mathcal{P}) = E_1(G(\mathcal{P}))$ and $B(\mathcal{T}) = E_2(G(\mathcal{T}))$. We also define $E_n^{tf}(G)$ and $E_n^{triv}(G)$ to be the images of the chain $E_n(G)$ in the rings $\mathbb{Z}G^{tf}$ and $\mathbb{Z}$ respectively, where G^{tf} is the largest torsion-free quotient of G^{ab}.

In order to prove that the E_n-ideals of a group are well-defined and a group invariant, we define, for a $\mathbb{Z}G$-module M of type FP_n, a chain, $E_n(M)$, of ideals $E_{n,\lambda}(M)$ $(\lambda \in \mathbb{Z})$ in $\mathbb{Z}G^{ab}$, which is calculated from a free resolution of M. When M is the trivial module $\mathbb{Z}$, $E_n(\mathbb{Z}) = E_n(G)$. By using an analogue of Tietze transformations, we show that the choice of free resolution is immaterial, and so the E_n-ideals of a module are well-defined (Theorem 4.7). We then consider some properties of the E_n-ideals of a group module, such as their behaviour when the co-efficient ring, $\mathbb{Z}G$, changes. As a consequence, we prove that, if G and G_0 are isomorphic groups of type FP_n, then $E_n(G)$ and $E_n(G_0)$ are equivalent, and so the E_n-ideals are, indeed, a group invariant (Theorem 4.1).

We also define associated invariants,

$$\nu_n(M) = \min\{\lambda \in \mathbb{Z} : E_{n,\lambda}(M) = \mathbb{Z}G^{ab}\},$$
$$\delta_n(M) = \min\{\lambda \in \mathbb{Z} : E_{n,\lambda}(M) \neq 0\}$$

and, similarly, we define $\nu_n(G)$, $\delta_n(G)$, ν_n^{tf}, ν_n^{triv}, etc. We then consider how the invariants of modules in a short exact sequence are related and how the integral homology, $H_*(G)$, is related to $E_n^{triv}(G)$ and $\delta_n^{triv}(G)$ (Theorem 4.24).

In Chapter 5, we first consider $E[m,n]$-trivial groups and modules; a module M of type FP_n is $E[m,n]$-trivial if $\nu_i(M) = \delta_i(M)$ for $i = m, \ldots, n$ and if $\delta_i(M) + \delta_{i+1}(M) = 0$ for $m \leq i < n$. A group G is $E[m,n]$-trivial when the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is. We show that finitely generated projective $\mathbb{Z}G$-modules are $E[0,\infty]$-trivial (Theorem 5.5) and, consequently, that groups of type FP are $E[\operatorname{cd} G, \infty]$-trivial (Theorem 5.10). However, we also show that, if we consider $\mathbb{Q}$-coefficients rather than $\mathbb{Z}$-coefficients, as we may, then this need not be the case. In particular, we show that, although finitely presented CA groups are of type FP over $\mathbb{Q}$ (Lemma 5.12), a large family of them are not of type FL over $\mathbb{Q}$ (Proposition 5.13). We then consider how this relates to a question of Serre's [76].

We next consider those groups and modules whose E_n^{tf}-ideals are linked in adjacent dimensions; if, for $m \leq i < n$, $\delta_i^{tf}(M) + \delta_{i+1}^{tf}(M) = 0$, M is said to be $E^{tf}[m,n]$-linked. A group is said to be $E^{tf}[m,n]$-linked if the module $\mathbb{Z}$ is. We consider how the invariant δ_n^{tf} behaves for short exact sequences of such modules (Proposition 5.14).

If a group G is $E^{tf}[l,\infty]$-linked for some l, we define a new Euler characteristic, $\delta^{tf}(G) = (-1)^l \delta_l^{tf}(G)$. This extends Euler characteristics of Serre [76] and Brown [21]. We then consider how E-linked and E-trivial groups behave when a graph of groups is constructed from them. We also look at the behaviour of δ^{tf} under this construction and extend a result of Chiswell's [29].

Finally, in order to show that these new invariants are useful, we look at the E_n-ideals of extensions of $E[0, n]$-trivial groups. For every $n > 1$, we are then able to construct an infinite family of pairwise non-isomorphic groups which can be distinguished neither by their integral homology, nor by their E_i-ideals for $i < n$, but which have distinct E_n-ideals (Theorem 5.33).

In the final chapter, the definition of the E_n-ideals is extended to monoids. We must, however, distinguish a left- and a right-hand case. If S is a monoid of type $FP_n^{(l)}$, we define a chain, $E_n^{(l)}(S)$, of ideals in $\mathbb{Z}S^{ab}$ and, if S is a monoid of type $FP_n^{(r)}$, we define a chain, $E_n^{(r)}(S)$, of ideals in $\mathbb{Z}S^{ab}$. We consider to what extent the properties of the E_n-ideals for groups extend to monoids and are able to show how the E_n-ideals of certain groups can be obtained from the $E_n^{(l)}$-ideals of a submonoid (Proposition 6.8) .

The invariants of a monoid which contains a zero element are considered next, in particular, the $E^{(l)}$- and $E^{(r)}$-ideals of a monoid to which a zero has been added (Proposition 6.9 and Theorem 6.12). Finally, we return briefly to the question of Serre.

Chapter 1

Preliminaries

1.1 General

All rings are assumed to have an identity element, 1, which all ring homomorphisms respect.

Although we will treat both left and right modules, we predominantly consider left modules. Thus, except where otherwise stated, *we assume that all modules are left modules.*

If $\{c_1, c_2, \dots\}$ is a set of elements of a ring C, then we write $(c_1, c_2, \dots)$ for the (two-sided) ideal of C generated by this set, $C.(c_1, c_2, \dots)$ for the left ideal and $(c_1, c_2, \dots).C$ for the right ideal. We will make frequent use of the fact that if, for some $c \in C$, $1 - c$ is a generator of a two-sided ideal of C, then we may substitute 1 for any further instances of c in the other generators. For instance, if $1 - c$ and $1 + c + \cdots + c^{p-1}$ are both generators of an ideal, we may replace them with the pair $1 - c, p$.

Let C be a ring with the *invariance of rank property*, that is, the rank of a free C-module is well-defined. If F is a finitely generated free C-module, we denote its rank by $\mathrm{rk}_C(F)$. If $r = \mathrm{rk}_C(F)$, then $F \cong C^r$, where

$$C^r = \underbrace{C \oplus C \oplus \cdots \oplus C}_{r \text{ times}}.$$

If C is a principal ideal domain (pid) and if M is any finitely generated C-module, then it can be written uniquely in the form

$$M \cong C^q \oplus \left(\oplus_{j=1}^{p} C/(c_j)\right) \tag{1.1}$$

for some integers $p, q \geq 0$ and some non-zero, non-unit $c_j \in C$ (well-defined up to multiplication by a unit) such that $c_j | c_{j+1}$. We define $\mathrm{rk}_C(M)$ to be the rank, q, of the free part of M.

The *group algebra*, KG, of a group G with coefficients in a commutative ring K consists of all formal sums $\sum_{g\in G} k_g g$, where $k_g \in K$ and k_g is non-zero for only finitely many $g \in G$. If $\alpha : G \to U(C)$ is a group homomorphism from a group G to the group of units, $U(C)$, of a K-algebra C, then it extends uniquely to a ring homomorphism

$$\alpha : KG \to C; \sum_{g\in G} k_g g \mapsto \sum_{g\in G} k_g \alpha(g)$$

(the use of α to denote the induced map is a convenient abuse of notation). In this way, any group homomorphism $\alpha : G \to G_0$ extends to a ring homomorphism $\alpha : KG \to KG_0$. For instance, the trivial group homomorphism $G \to 1; g \mapsto 1$ induces a ring homomorphism

$$aug : KG \to K; \sum_{g\in G} k_g g \mapsto \sum_{g\in G} k_g,$$

the *augmentation map*. When $K = \mathbb{Z}$, we call the kernel of this map the *augmentation ideal* of G and denote it IG. We then have a short exact sequence

$$0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0. \tag{1.2}$$

As a $\mathbb{Z}$-module, IG is freely generated by the set $\{1 - g : g \in G\}$.

The preceding paragraph applies also to monoids and monoid algebras and we have a monoid version of the short exact sequence (1.2).

If H is a subgroup of a group G, then KG is a free KH-module of rank $[G : H]$, the *index* of H in G. If M is any KH-module, we have the *induced* KG-module

$$M \uparrow_H^G = KG \otimes_{KH} M.$$

For any group G, the *abelianisation* of G, G^{ab}, is the quotient G/G' of G by its derived subgroup $G' = [G, G]$.

If G is a finitely generated group, then we denote by $d(G)$ the minimum number of generators of G.

If $\varepsilon \in \{1, -1\}$, then we define

$$\varepsilon^* = \frac{\varepsilon - 1}{2} = \begin{cases} 0 & \varepsilon = 1 \\ -1 & \varepsilon = -1 \end{cases}.$$

1.2 Presentations and pictures

A *word* on a set $\mathbf{x}$ consists of a finite sequence of symbols from the set $\mathbf{x}$.

If $\mathbf{x}$ is a non-empty set, then the *free monoid* $[\mathbf{x}]$ on $\mathbf{x}$ is the set of words on $\mathbf{x}$, with multiplication given by concatenation. The identity of $[\mathbf{x}]$ is the empty word, denoted 1.

If $\mathbf{x}$ is a non-empty set, we define $\mathbf{x}^{-1}$ to be a set $\{x^{-1} : x \in \mathbf{x}\}$ in one-one correspondence with $\mathbf{x}$. The *free group* $\langle \mathbf{x} \rangle$ on the set $\mathbf{x}$ consists of words on $\mathbf{x} \cup \mathbf{x}^{-1}$, subject to the relations $x^\varepsilon x^{-\varepsilon} = 1$ ($x \in \mathbf{x}$, $\varepsilon = \pm 1$). Although $\langle \mathbf{x} \rangle$ is thus the set of equivalence classes of words modulo these relations, we identify an element of the free group with any word which represents it.

Two words on $\mathbf{x} \cup \mathbf{x}^{-1}$ will be said to be *freely equal* if they represent the same element of $\langle \mathbf{x} \rangle$. A word on $\mathbf{x} \cup \mathbf{x}^{-1}$ will be said to be *reduced* if it does not contain a subword of the form $x^\varepsilon x^{-\varepsilon}$. Every word on $\mathbf{x} \cup \mathbf{x}^{-1}$ is freely equal to a unique reduced word [58]. A word on $\mathbf{x} \cup \mathbf{x}^{-1}$ will be said to be *cyclically reduced* if it is reduced and if its last symbol is distinct from the inverse of its first.

1.2.1 Presentations of groups

A *group presentation*

$$\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle \tag{1.3}$$

consists of a set of *generating symbols*, $\mathbf{x}$, the *generating set*, together with a set $\mathbf{r}$ of non-empty, cyclically reduced words on $\mathbf{x} \cup \mathbf{x}^{-1}$, the *defining relators*. The group $G(\mathcal{P})$ defined by $\mathcal{P}$ is then the quotient of the free group $\langle \mathbf{x} \rangle$ on $\mathbf{x}$ by the normal closure $\ll \mathbf{r} \gg$ of $\mathbf{r}$ in $\langle \mathbf{x} \rangle$. We have the natural quotient map

$$\gamma_{\mathcal{P}} : \langle \mathbf{x} \rangle \to G(\mathcal{P}); W \mapsto W \ll \mathbf{r} \gg.$$

Notation. When the context is clear, we will often write $\gamma_{\mathcal{P}}(W)$ as $\overline{W}$ or simply as W. We will also denote the image of any $\xi \in \mathbb{Z}\langle \mathbf{x} \rangle$ in $\mathbb{Z}G(\mathcal{P})$ as $\overline{\xi}$ or simply as ξ.

If $G \cong G(\mathcal{P})$, then we say that $\mathcal{P}$ is a presentation for G. Note, however, that G and $G(\mathcal{P})$ are distinct groups.

Instead of defining relators, we sometimes use *defining relations* of the form $U = V$ for some distinct words U, V on $\mathbf{x} \cup \mathbf{x}^{-1}$ which start and end with distinct symbols. This relation is equivalent to the (non-empty, cyclically reduced) relator UV^{-1}.

We will say that a word W on $\mathbf{x} \cup \mathbf{x}^{-1}$ is a *consequence* of $\mathbf{r}$ if $W \in \ll \mathbf{r} \gg = \ker \gamma_{\mathcal{P}}$.

A presentation is said to be *finite* if the sets $\mathbf{x}$ and $\mathbf{r}$ are both finite.

When $\mathbf{x}$ is a finite set, we define

$$\chi_1(\mathcal{P}) = |\mathbf{x}| - 1$$

and, when $\mathcal{P}$ is finite, we define

$$\chi_2(\mathcal{P}) = |\mathbf{r}| - |\mathbf{x}| + 1.$$

If it is clear which presentation we are referring to, we abbreviate these to χ_1, χ_2, respectively.

Associated with a presentation $\mathcal{P}$ is a $\mathbb{Z}G(\mathcal{P})$-module, $M(\mathcal{P})$, the *relation module* of $\mathcal{P}$, defined to be the abelianisation, $\ll \mathbf{r} \gg^{ab}$, of $\ll \mathbf{r} \gg$, written additively. The action of $G(\mathcal{P})$ is given by

$$\overline{W}.U \ll \mathbf{r} \gg' = WUW^{-1} \ll \mathbf{r} \gg',$$

for $W \in \langle \mathbf{x} \rangle$, $U \in \ll \mathbf{r} \gg$. If $U \in \ll \mathbf{r} \gg$, then it may be written as

$$U = \prod_i W_i R_i^{\varepsilon_i} W_i^{-1}$$

for some $W_i \in \langle \mathbf{x} \rangle$, $R_i \in \mathbf{r}$ and $\varepsilon_i = \pm 1$. Thus

$$U \ll \mathbf{r} \gg' = \left(\prod_i W_i R_i^{\varepsilon_i} W_i^{-1} \right) \ll \mathbf{r} \gg' = \sum_i \varepsilon_i \overline{W_i}.R_i \ll \mathbf{r} \gg',$$

so $M(\mathcal{P})$ is generated as a $\mathbb{Z}G(\mathcal{P})$-module by the set

$$\{R\ll \mathbf{r} \gg' : R \in \mathbf{r}\}.$$

The module $M(\mathcal{P})$ embeds in a free $\mathbb{Z}G$-module of rank $|\mathbf{x}|$, with cokernel IG, giving an exact sequence [25]

$$0 \to M(\mathcal{P}) \to \oplus_{x\in\mathbf{x}}\mathbb{Z}G \to IG \to 0 \tag{1.4}$$

(we define the maps of this sequence in §1.6.3, below).

1.2.2 Presentations of monoids

A *monoid presentation* (or *rewriting system*)

$$\mathcal{P} = [\mathbf{x}; \mathbf{r}]$$

consists of a set $\mathbf{x}$ of *generating symbols*, together with a set $\mathbf{r}$ of ordered pairs (R^+, R^-) of words on $\mathbf{x}$, the *relations.* We often write $R^+ = R^-$ instead of (R^+, R^-).

The monoid $S(\mathcal{P})$ defined by $\mathcal{P}$ is then the set of equivalence classes $\{\overline{W} : W \in [\mathbf{x}]\}$ of the congruence on $[\mathbf{x}]$ induced by $\mathbf{r}$. More explicitly, if W is a word on $\mathbf{x}$ which is of the form UR^+V for some $U, V \in [\mathbf{x}]$ and some $(R^+, R^-) \in \mathbf{r}$, then an *elementary transformation* on W replaces R^+ by R^-, giving the word UR^-V. The inverse of an elementary transformation replaces a word of the form UR^-V with the word UR^+V. Two words on $\mathbf{x}$ will be said to be *equivalent (relative to* $\mathbf{r}$*)* if one can be obtained from the other by a finite number of elementary transformations and their inverses. Multiplication is given by $\overline{W}.\overline{W'} = \overline{WW'}$ and is well-defined, so we have a surjective monoid homomorphism

$$\gamma_{\mathcal{P}} : [\mathbf{x}] \to S(\mathcal{P}); W \mapsto \overline{W}.$$

Notation. If it is clear that we are considering elements of $S(\mathcal{P})$ rather than $[\mathbf{x}]$, we will write W rather than $\overline{W}$.

Associated with any monoid S we have the *opposite monoid,* S^{opp}, which has the same underlying set as S and where the product $s.s'$ for $s, s' \in S^{opp}$ is defined to be the product $s's \in S$. There is a set map

$$opp : S \to S^{opp}; s \mapsto s,$$

such that $opp(ss') = s's$ $(s, s' \in S)$. If $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is a presentation for S, then $\mathcal{P}^{opp} = [\mathbf{x}; \mathbf{r}^{opp}]$ is a presentation for S^{opp}, where $\mathbf{r}^{opp}$ is obtained from $\mathbf{r}$ by reversing the words in each $(R^+, R^-) \in \mathbf{r}$. For example, if

$$\mathcal{P} = [a, b; (a^5, 1), (ba, a^2b)],$$

then

$$\mathcal{P}^{opp} = [a, b; (a^5, 1), (ab, ba^2)]$$

is a presentation for $S(\mathcal{P})^{opp}$.

1.2.3 Pictures, π_2 and 3-presentations of groups

A *picture* $\mathbb{P}$ over a presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ consists of the following:

1) An ambient disc D, with a basepoint $0_{\mathbb{P}}$ on its boundary.

2) Discs $\Delta_1, \ldots, \Delta_m$ in the interior of D, each of which has a basepoint 0_{Δ_i} on its boundary. To avoid confusion, when we refer to the discs of $\mathbb{P}$ we mean the discs $\Delta_1, \ldots, \Delta_m$, not the ambient disc D.

3) A finite number of disjoint arcs in the closure of $D \setminus \bigcup_i \Delta_i$, each of which is either a simple closed curve or a simple curve joining two distinct points on $\partial D \bigcup_i \partial \Delta_i$, neither of which is a basepoint. Each of the arcs has an orientation, indicated by a transverse arrow, and is labelled by a symbol from $\mathbf{x}$. A closed arc which encircles no discs or arcs is a *floating circle.*

4) For a disc Δ of $\mathbb{P}$, if we travel around $\partial\Delta$ in a clockwise direction, then we can read off a word from the successive labels of the arcs we cross; if we cross an arc labelled x in the direction of its orientation, we read the symbol x and, if we cross in the direction opposite to its orientation, we read x^{-1}. The word thus produced must be $R_\Delta^{\varepsilon_\Delta}$ for some $R_\Delta \in \mathbf{r}$ and $\varepsilon_\Delta = \pm 1$. We call R_Δ the *label* of Δ and we say that Δ has *positive orientation* if $\varepsilon_\Delta = +1$ and that Δ has *negative orientation* if $\varepsilon_\Delta = -1$.

We define $\partial\mathbb{P}$ to be ∂D. If we read clockwise around $\partial\mathbb{P}$ from $0_{\mathbb{P}}$, then we obtain a word $W_{\mathbb{P}}$ on $\mathbf{x} \cup \mathbf{x}^{-1}$, which is termed the *(boundary) label* of $\mathbb{P}$.

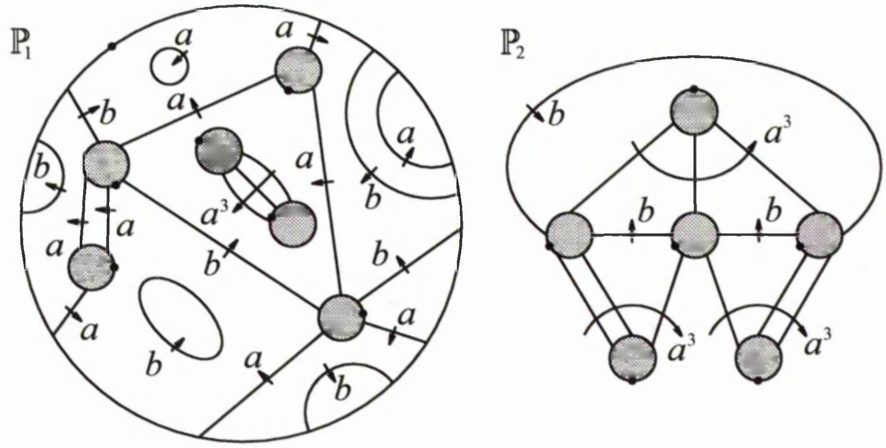

Figure 1.1: Some pictures over $\langle a, b; a^3, bab^{-1}a^{-2} \rangle$

A picture is said to be *spherical* if no arc intersects $\partial\mathbb{P}$. If a picture is spherical, we often omit $\partial\mathbb{P}$ and $0_{\mathbb{P}}$.

Some examples of pictures over the presentation $\mathcal{P} = \langle a, b; a^3, bab^{-1}a^{-2} \rangle$ are given in Figure 1.1. Note that, in places, a number of arrows have been amalgamated into one, whose label is the word consisting of the labels of the individual arrows. This will be common practice. The picture $\mathbb{P}_2$ is spherical and *connected*, that is, it contains only one connected component, while $\mathbb{P}_1$ contains a number of components, one of which is a spherical subpicture. Notice that $W_{\mathbb{P}_1} = ab^{-1}aa^{-1}bb^{-1}abb^{-1}aa^{-1}bb^{-1}b \in \ker \gamma_{\mathcal{P}}$.

Theorem 1.1 (Van Kampen's Lemma [66]). *Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and let W be a word on $\mathbf{x} \cup \mathbf{x}^{-1}$. Then $\overline{W} = 1 \in G(\mathcal{P})$ if, and only if, there is a picture $\mathbb{P}$ over $\mathcal{P}$ with $W_{\mathbb{P}} = W$.*

We now consider spherical pictures. Consider the following operations on a spherical picture:

(S1) Make a *bridge move* on two adjacent arcs with the same label, but opposing orientations, as illustrated in Figure 1.2.

(S2) Insert or delete a *cancelling pair*, that is, a spherical subpicture with exactly two discs, each of which has the same label, but which have opposite orien-

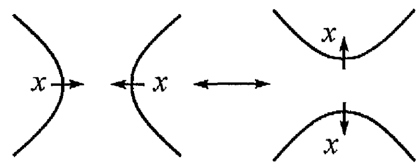

Figure 1.2: A bridge move

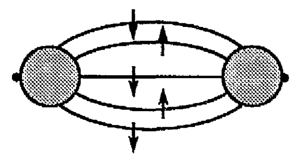

Figure 1.3: A cancelling pair

tations, and whose basepoints are in the same region, as illustrated in Figure 1.3.

(S3) Insert or delete a floating circle.

We will say that two spherical pictures over a presentation $\mathcal{P}$ are *equivalent* if we can transform one to the other by a finite number of the above three operations. We denote the equivalence class of a spherical picture $\mathbb{P}$ under this equivalence by $[\mathbb{P}]$. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are two spherical pictures over $\mathcal{P}$, then $\mathbb{P}_1 + \mathbb{P}_2$ is defined to be the picture consisting of $\mathbb{P}_1$ and $\mathbb{P}_2$ side by side. We then obtain a well-defined addition on the set of equivalence classes given by $[\mathbb{P}_1] + [\mathbb{P}_2] = [\mathbb{P}_1 + \mathbb{P}_2]$. If we let $-\mathbb{P}$ denote the mirror image of $\mathbb{P}$, then $[\mathbb{P}_1] + [-\mathbb{P}_1] = [\mathbb{P}_1 + (-\mathbb{P}_1)] = [0]$, the equivalence class of the empty picture. We write $[-\mathbb{P}]$ as $-[\mathbb{P}]$ and $\mathbb{P}_1 + (-\mathbb{P}_2)$ as $\mathbb{P}_1 - \mathbb{P}_2$. The set of equivalence classes of pictures over $\mathcal{P}$ with this addition thus forms an abelian group.

There is an action of $\langle \mathbf{x} \rangle$ on this group; for $W \in \langle \mathbf{x} \rangle$, $W.[\mathbb{P}] = [\mathbb{P}^W]$, where $\mathbb{P}^W$ is obtained from $\mathbb{P}$ by surrounding it by a number of simple closed arcs whose total label, read from the outside in, is W. See Figure 1.4, where the spherical picture $\mathbb{P}$ is contained within the dotted line. This action is well-defined, since $x^{\varepsilon}x^{-\varepsilon}$ ($x \in \mathbf{x}$, $\varepsilon = \pm 1$) acts trivially modulo (S1) and (S3). Indeed, the elements of $\mathbf{r}$ act trivially modulo (S1), (S2), (S3) [66], and so we obtain a $\mathbb{Z}G(\mathcal{P})$-module, $\pi_2(\mathcal{P})$, the *second*

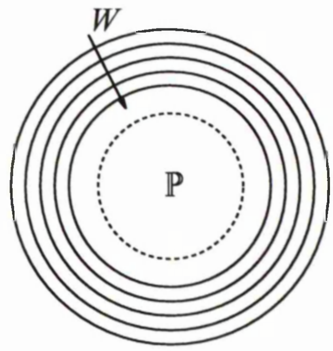

Figure 1.4: The action of the word W on the spherical picture $\mathbb{P}$

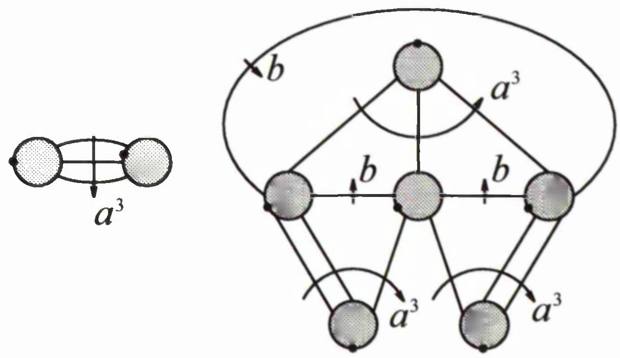

Figure 1.5: A generating set of pictures for $\langle a, b; a^3, bab^{-1}a^{-2}\rangle$

homotopy module of $\mathcal{P}$.

We will say that a set $\mathbf{d}$ of spherical pictures over a presentation $\mathcal{P}$ is a *generating set of pictures* for $\mathcal{P}$ if the set $\{[\mathbb{D}] : \mathbb{D} \in \mathbf{d}\}$ generates $\pi_2(\mathcal{P})$ as a $\mathbb{Z}G(\mathcal{P})$-module. Alternatively [66], $\mathbf{d}$ is a generating set if every picture over $\mathcal{P}$ can be transformed to the trivial picture by operations (S1), (S2) and (S3) along with the operation:

(S4) Insert or delete a subpicture $\varepsilon\mathbb{D}$ for $\varepsilon = \pm 1$, $\mathbb{D} \in \mathbf{d}$.

For example, it can be shown that the spherical pictures in Figure 1.5 constitute a generating set for $\langle a, b; a^3, bab^{-1}a^{-2}\rangle$ [9].

If $\mathbf{d}$ is a generating set of pictures for a presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}\rangle$, then we call the triple

$$\mathcal{T} = \langle \mathcal{P}; \mathbf{d}\rangle = \langle \mathbf{x}; \mathbf{r}; \mathbf{d}\rangle \tag{1.5}$$

a 3-*presentation* for $G(\mathcal{P})$. We will also write $G(\mathcal{T}) = G(\mathcal{P})$ and set $\gamma_{\mathcal{T}} = \gamma_{\mathcal{P}}$, $\chi_1(\mathcal{T}) = \chi_1(\mathcal{P})$ and $\chi_2(\mathcal{T}) = \chi_2(\mathcal{P})$ (when defined). We call $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ the *underlying presentation* of $\mathcal{T}$. When $\mathbf{x}$, $\mathbf{r}$ and $\mathbf{d}$ are finite, we will say that $\mathcal{T}$ is *finite.*

The $\mathbb{Z}G$-module $\pi_2(\mathcal{P})$ embeds in a free $\mathbb{Z}G$-module of rank $|\mathbf{r}|$, the cokernel of the embedding being isomorphic to the relation module $M(\mathcal{P})$. We therefore have a short exact sequence [66]

$$0 \to \pi_2(\mathcal{P}) \to \oplus_{R \in \mathbf{r}} \mathbb{Z}G \to M(\mathcal{P}) \to 0 \tag{1.6}$$

(we return to describe the maps of this sequence in §1.6.3, below).

Much work has been devoted to calculating generating sets of pictures for presentations. For example, in [9] the authors show how a 3-presentation for a group extension can be obtained from 3-presentations for the normal subgroup and the quotient group. In a similar vein, (generalised) graphs of groups (see §1.8.3, below) are considered in [10] and [20], where it is shown how a 3-presentation can be obtained from 3-presentations for the constituent groups.

The second homotopy module of a presentation in which each relator involves at most two generators is considered in [64], [67], [65] and [70].

When an extra generator and an extra relator are added to a presentation, giving a so-called relative presentation, the additional generating pictures required are determined for certain cases in [8], [19], [38] and [49].

See also [48] and [66] for interesting alternative definitions of $\pi_2(\mathcal{P})$. We remark that there is also a concept of a picture over a monoid presentation and that the definition of $\pi_2(\mathcal{P})$ can be extended to monoid presentations [68], [69].

1.2.4 Asphericity and combinatorial asphericity

Certain presentations have particularly simple generating sets of pictures.

A presentation $\mathcal{P}$ for which $\pi_2(\mathcal{P}) = 0$ is said to be *aspherical.* For example, a one relator presentation $\langle \mathbf{x}; R \rangle$ is aspherical if the word R is not a proper power.

In general, each relator $R \in \mathbf{r}$ of a presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ can be written as $R = R_0^{p_R}$ for some integer $p_R > 0$ and some word R_0 which is not a proper power. We call R_0 the *root* of R and p_R the *period.* If $p_R > 1$, then there are associated

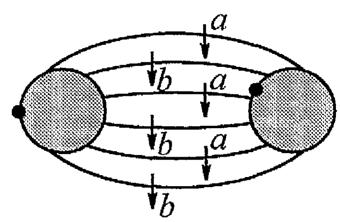

Figure 1.6: The dipole $\mathbb{D}_{(ab)^3}$

spherical pictures, called *dipoles*, which have two discs labelled R, with opposite orientations, whose basepoints are in different regions. There are then $p_R(p_R - 1)$ dipoles for each relator R, corresponding to the different choices of basepoints. We define $\mathbb{D}_R$ to be the dipole in which the basepoint of the positively oriented disc is in the outer region and the path from that basepoint to the basepoint of the other disc has label R_0. Such a dipole for the relator $(ab)^3$ is shown in Figure 1.6. Modulo the operations (S1), (S2), (S3) and inserting or deleting $\mathbb{D}_R$, all other dipoles are trivial.

For any presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$, we set

$$\mathbf{r}' = \{R \in \mathbf{r} : p_R > 1\}.$$

The presentation $\mathcal{P}$ is said to be *combinatorially aspherical* (CA) if the set of dipoles $\{\mathbb{D}_R : R \in \mathbf{r}'\}$ is a generating set of pictures. A group will be said to be CA if it has a CA presentation. For example, a finite cyclic group is CA by virtue of the CA presentation $\mathcal{P} = \langle x; x^p \rangle$, as are all groups with a one relator presentation. CA presentations have been extensively studied [25], [31], [51], [57], [66]. See [20] for some tests to determine whether a presentation is CA.

The following result describes fully the torsion elements of a CA group.

Theorem 1.2 ([51]). *Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ be a CA presentation and let $G = G(\mathcal{P})$. An element $g \in G$ is of finite order if, and only if,*

$$g = \overline{U R_0^q U^{-1}}$$

for some $R \in \mathbf{r}$, some word U on $\mathbf{x} \cup \mathbf{x}^{-1}$ and some integer q such that $0 < q < p_R$.

1.2.5 Tietze transformations

Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ be a group presentation as in (1.3). We define the following *Tietze transformations* on $\mathcal{P}$:

(T1) $\mathcal{P} \mapsto \mathcal{P}_0 = \langle \mathbf{x}, \mathbf{y}; \mathbf{r}, y = W_y (y \in \mathbf{y}) \rangle$ for some set $\mathbf{y}$ with $\mathbf{y} \cap \mathbf{x} = \emptyset$ and some $W_y \in \langle \mathbf{x} \rangle$.

(T2) $\mathcal{P} \mapsto \mathcal{P}_0 = \langle \mathbf{x}; \mathbf{r}, \mathbf{s} \rangle$ for some set $\mathbf{s}$ of cyclically reduced consequences of $\mathbf{r}$.

If the set $\mathbf{y}$ or the set $\mathbf{s}$ is finite, then the corresponding transformation is termed *finitary*.

In each case the group $G_0 = G(\mathcal{P}_0)$ is isomorphic to $G = G(\mathcal{P})$. Indeed, in the case of the transformation (T2) the groups G and G_0 are identical, since $G_0 = \langle \mathbf{x} \rangle / \ll \mathbf{r}, \mathbf{s} \gg$ and $\ll \mathbf{r}, \mathbf{s} \gg = \ll \mathbf{r} \gg$. If $\mathcal{P}_0$ is obtained from $\mathcal{P}$ by a transformation (T1), then there is an induced isomorphism $\alpha_{\mathcal{P}} : G \to G_0$ given by $x \mapsto x$ (for $x \in \mathbf{x}$), which has inverse given by $x \mapsto x$ ($x \in \mathbf{x}$), $y \mapsto W_y$ ($y \in \mathbf{y}$).

We will say that two presentations are *Tietze equivalent* if one can be obtained from the other by a finite number of Tietze transformations and their inverses. This is an equivalence relation. If two presentations are Tietze equivalent, then they define isomorphic groups. Conversely:

Lemma 1.3 ([58]). *Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and $\mathcal{Q} = \langle \mathbf{y}; \mathbf{s} \rangle$ be two group presentations. If there is an isomorphism $\alpha : G(\mathcal{P}) \to G(\mathcal{Q})$, then $\mathcal{P}$ and $\mathcal{Q}$ are Tietze equivalent and α is the isomorphism induced by the Tietze transformations. If both $\mathcal{P}$ and $\mathcal{Q}$ are finite presentations, then they are Tietze equivalent by finitary Tietze transformations. If $\mathbf{x}$ and $\mathbf{y}$ are finite, then any transformation (T1) in the equivalence of $\mathcal{P}$ and $\mathcal{Q}$ can be taken to be finitary.*

Proof. Initially, we assume that $\mathbf{x}$ and $\mathbf{y}$ are distinct sets. For each $x \in \mathbf{x}$, let W_x be a word on $\mathbf{y} \cup \mathbf{y}^{-1}$ such that $\alpha(x) = W_x$ and, for each $y \in \mathbf{y}$, let U_y be a word on $\mathbf{x} \cup \mathbf{x}^{-1}$ such that $\alpha(U_y) = y$. We can then apply Tietze transformations of type (T1) to $\mathcal{P}$ and $\mathcal{Q}$ to give presentations

$$\mathcal{P}_0 = \langle \mathbf{x}, \mathbf{y}; \mathbf{r}, y = U_y (y \in \mathbf{y}) \rangle, \quad \mathcal{Q}_0 = \langle \mathbf{x}, \mathbf{y}; \mathbf{s}, x = W_x (x \in \mathbf{x}) \rangle.$$

The isomorphism $\alpha_0 : G(\mathcal{P}_0) \to G(\mathcal{Q}_0)$ induced by α is then given by $x \mapsto x$ ($x \in \mathbf{x}$), $y \mapsto y$ ($y \in \mathbf{y}$) and so it is clear that

$$\ll \mathbf{r}, yU_y^{-1}(y \in \mathbf{y}) \gg = \ll \mathbf{s}, xW_x^{-1}(x \in \mathbf{x}) \gg \leq \langle \mathbf{x}, \mathbf{y} \rangle.$$

Thus, we may apply a Tietze transformations of type (T2) to each of $\mathcal{P}_0$ and $\mathcal{Q}_0$ to give the presentation

$$\mathcal{R} = \langle \mathbf{x}, \mathbf{y}; \mathbf{r}, y = U_y(y \in \mathbf{y}), \mathbf{s}, x = W_x(x \in \mathbf{x}) \rangle.$$

Hence, the presentations $\mathcal{P}$ and $\mathcal{Q}$ are Tietze equivalent.

The isomorphism induced by these Tietze transformations is the composition

$$G(\mathcal{P}) \xrightarrow{\alpha_{\mathcal{P}}} G(\mathcal{P}_0) \xrightarrow{\mathrm{Id}_{\mathcal{P}_0}} G(\mathcal{R}) \xrightarrow{\mathrm{Id}_{\mathcal{Q}_0}^{-1}} G(\mathcal{Q}_0) \xrightarrow{\alpha_{\mathcal{Q}}^{-1}} G(\mathcal{Q}).$$

Now, for $x \in \mathbf{x}$,

$$\begin{aligned} \alpha_{\mathcal{Q}}^{-1}\mathrm{Id}_{\mathcal{Q}_0}^{-1}\mathrm{Id}_{\mathcal{P}_0}\alpha_{\mathcal{P}}(x) &= \alpha_{\mathcal{Q}}^{-1}\mathrm{Id}_{\mathcal{Q}_0}^{-1}\mathrm{Id}_{\mathcal{P}_0}(x) \\ &= \alpha_{\mathcal{Q}}^{-1}(x) \\ &= \alpha_{\mathcal{Q}}^{-1}(W_x) \\ &= W_x \\ &= \alpha(x), \end{aligned}$$

so $\alpha = \alpha_{\mathcal{Q}}^{-1}\mathrm{Id}_{\mathcal{Q}_0}^{-1}\mathrm{Id}_{\mathcal{P}_0}\alpha_{\mathcal{P}}$, as required.

If $\mathbf{x} \cap \mathbf{y} \neq \emptyset$, then let $\mathbf{y}'$ be a set, distinct from $\mathbf{x} \cup \mathbf{y}$, in one-one correspondence with $\mathbf{y}$. This correspondence induces an isomorphism $\beta : \langle \mathbf{y} \rangle \to \langle \mathbf{y}' \rangle$. Let $\mathcal{Q}' = \langle \mathbf{y}'; \{\beta(S) : S \in \mathbf{s}\} \rangle$. The groups $G(\mathcal{Q})$ and $G(\mathcal{Q}')$ are clearly isomorphic. We then apply the first part of the proof twice; first to show that $\mathcal{Q}$ and $\mathcal{Q}'$ are Tietze equivalent and then to show that $\mathcal{P}$ and $\mathcal{Q}'$ are.

If the sets $\mathbf{x}$, $\mathbf{y}$ are finite, then each of the transformations (T1) above are finitary and, if the sets $\mathbf{r}$, $\mathbf{s}$ are also finite, then all of the Tietze transformations undertaken above are finitary. □

We wish to extend the definition of Tietze transformations to 3-presentations. In order to do this, we must first consider the effect on a generating set of pictures of a Tietze transformation on a presentation.

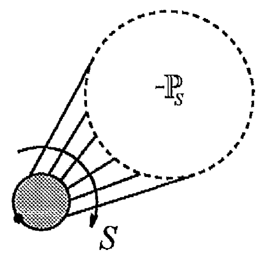

Figure 1.7: The picture $\mathbb{Q}_S$

If $\mathcal{P}_0$ is obtained from $\mathcal{P}$ by a transformation (T1) as above, then any spherical picture over $\mathcal{P}_0$ is equivalent to a spherical picture $\mathbb{P}$ over $\mathcal{P}$; if the picture includes a disc Δ with label yW_y^{-1} for some $y \in \mathbf{y}$, then, since yW_y^{-1} is the only relator of $\mathcal{P}_0$ which includes the symbol y, $\mathbb{P}$ must include a second disc labelled yW_y^{-1}, with opposite orientation, joined to Δ by an arc labelled y. It is then an easy matter to use bridge moves to create a cancelling pair with these two discs and remove them. Thus, any generating set of pictures for $\mathcal{P}$ is also a generating set of pictures for $\mathcal{P}_0$.

Now suppose that $\mathcal{P}_0$ is obtained from $\mathcal{P}$ by a transformation (T2) as above. Since each $S \in \mathbf{s}$ is a consequence of $\mathbf{r}$, by Theorem 1.1 there is a picture $\mathbb{P}_S$ over $\mathcal{P}$ with boundary label S. We can therefore construct a spherical picture $\mathbb{Q}_S$ over $\mathcal{P}_0$ by joining the arcs meeting the boundary of $-\mathbb{P}_S$ to a disc labelled S, as in Figure 1.7. Any picture over $\mathcal{P}_0$ is equivalent modulo the set $\{\mathbb{Q}_S : S \in \mathbf{s}\}$ to a picture over $\mathcal{P}$; if a picture includes a disc Δ with label S, insert a picture $-\varepsilon_\Delta \mathbb{Q}_S$ in the same region as the basepoint of Δ, use bridge moves to make a cancelling pair of the discs labelled S and then delete them. Thus, if $\mathbf{d}$ is a generating set of pictures for $\mathcal{P}$, then $\mathbf{d} \cup \{\mathbb{Q}_S : S \in \mathbf{s}\}$ is a generating set of pictures for $\mathcal{P}_0$.

Let $\mathcal{T} = \langle \mathbf{x}; \mathbf{r}; \mathbf{d} \rangle$ be a 3-presentation as in (1.5). We define *Tietze transformations* of $\mathcal{T}$ as follows:

(T1) $\mathcal{T} \mapsto \mathcal{T}_0 = \langle \mathbf{x}, \mathbf{y}; \mathbf{r}, y = W_y (y \in \mathbf{y}); \mathbf{d} \rangle$	for some set $\mathbf{y}$ with $\mathbf{y} \cap \mathbf{x} = \emptyset$ and some $W_y \in \langle \mathbf{x} \rangle$.
(T2) $\mathcal{T} \mapsto \mathcal{T}_0 = \langle \mathbf{x}; \mathbf{r}, \mathbf{s}; \mathbf{d}, \mathbb{Q}_S (S \in \mathbf{s}) \rangle$	for some set $\mathbf{s}$ of cyclically reduced consequences of $\mathbf{r}$.
(T3) $\mathcal{T} \mapsto \mathcal{T}_0 = \langle \mathbf{x}; \mathbf{r}; \mathbf{d}, \mathbf{e} \rangle$	for some set $\mathbf{e}$ of spherical pictures

over $\langle \mathbf{x}; \mathbf{r} \rangle$.

Each of these induces an isomorphism $\alpha_{\mathcal{T}} : G(\mathcal{T}) \to G(\mathcal{T}_0)$, which is the identity in the case of (T2) and (T3). We will say that a Tietze transformation on a 3-presentation is *finitary* if, in the case of (T1) and (T2), the underlying transformation of presentations is finitary or, in the case of (T3), if $\mathbf{e}$ is finite.

We will say that two 3-presentations are *Tietze equivalent* if one can be obtained from the other by a finite number of Tietze transformations and their inverses. Tietze equivalent 3-presentations define isomorphic groups and, conversely:

Lemma 1.4. *Let $\mathcal{T} = \langle \mathbf{x}; \mathbf{r}; \mathbf{d} \rangle$ and $\mathcal{S} = \langle \mathbf{y}; \mathbf{s}; \mathbf{e} \rangle$ be two 3-presentations. If there is an isomorphism $\alpha : G(\mathcal{T}) \to G(\mathcal{S})$, then $\mathcal{T}$ and $\mathcal{S}$ are Tietze equivalent and α is the isomorphism induced by the Tietze transformations. If $\mathbf{x}$ and $\mathbf{y}$ are finite, then any transformation (T1) in this equivalence can be taken to be finitary. If, in addition, $\mathbf{r}$ and $\mathbf{s}$ are finite, any transformation (T2) may also be taken to be finitary. If $\mathcal{T}$ and $\mathcal{S}$ are finite, then all the Tietze transformations in the equivalence can be taken to be finitary.*

Proof. By Lemma 1.3, $\langle \mathbf{x}; \mathbf{r} \rangle$ and $\langle \mathbf{y}; \mathbf{s} \rangle$ are Tietze equivalent and the equivalence induces α. We therefore need only show that, if $\mathbf{d}$, $\mathbf{d}_0$ are two generating sets of pictures for a presentation $\langle \mathbf{x}; \mathbf{r} \rangle$, then the 3-presentations $\langle \mathbf{x}; \mathbf{r}; \mathbf{d} \rangle$ and $\langle \mathbf{x}; \mathbf{r}; \mathbf{d}_0 \rangle$ are Tietze equivalent. But these are both equivalent by a single transformation (T3) to $\langle \mathbf{x}; \mathbf{r}; \mathbf{d}, \mathbf{d}_0 \rangle$, and the result follows. □

1.3 Derivatives

1.3.1 Derivations and the Fox derivative

If G is a group and M a KG-module, then a *derivation* from KG to M is a K-module homomorphism $d : KG \to M$ satisfying the additional property that, for $g, g_0 \in G$,

$$d(gg_0) = d(g) + g.d(g_0). \tag{1.7}$$

The following properties of derivations are immediate from the definition.

Lemma 1.5 ([40]). *If $d : KG \to M$ is a derivation, then*

i) for $k \in K$, $d(k) = 0$;

ii) for $g \in G$, $d(g^{-1}) = -g^{-1}d(g)$; and, more generally,

iii) for $g \in G$, $n \in \mathbb{Z}$,

$$d(g^n) = \begin{cases} (1 + g + \cdots + g^{n-1})d(g) & n > 0 \\ 0 & n = 0 \\ -(g^n + g^{n+1} + \cdots + g^{-1})d(g) & n < 0 \end{cases} .$$

Lemma 1.6 ([40]). *A derivation $d : KG \to M$ is uniquely determined by its values on any generating subset of G.*

Example 1.1. Take M to be the $\mathbb{Z}G$-module IG and $d : \mathbb{Z}G \to IG$ to be the $\mathbb{Z}$-linear map induced by $d(g) = g - 1$. Since, for $g, g_0 \in G$,

$$\begin{aligned} d(gg_0) &= gg_0 - 1 \\ &= g - 1 + g(g_0 - 1) \\ &= d(g) + gd(g_0), \end{aligned}$$

this defines a derivation, which extends to $d(\xi) = \xi - aug(\xi)$, for $\xi \in \mathbb{Z}G$. ◊

We now consider derivations from the group ring $\mathbb{Z}\langle \mathbf{x} \rangle$ of the free group on a set $\mathbf{x}$ to the $\mathbb{Z}\langle \mathbf{x} \rangle$-module $\mathbb{Z}\langle \mathbf{x} \rangle$. By Lemma 1.6, to define a derivation $d : \mathbb{Z}\langle \mathbf{x} \rangle \to \mathbb{Z}\langle \mathbf{x} \rangle$ we need only specify $d(x)$ for each $x \in \mathbf{x}$ and ensure that (1.7) holds. Indeed, the following result shows that any choice of $d(x)$ determines a derivation.

Lemma 1.7 ([40]). *For any set $\{\xi_x : x \in \mathbf{x}\} \subset \mathbb{Z}\langle \mathbf{x} \rangle$ there is a unique derivation $d : \mathbb{Z}\langle \mathbf{x} \rangle \to \mathbb{Z}\langle \mathbf{x} \rangle$ with $d(x) = \xi_x$ for $x \in \mathbf{x}$.*

We now define a family of derivations, the *Fox derivatives*, with a certain universal property. For $x \in \mathbf{x}$, let $\frac{\partial}{\partial x} : \mathbb{Z}\langle \mathbf{x} \rangle \to \mathbb{Z}\langle \mathbf{x} \rangle$ be the unique derivation with

$$\frac{\partial x_0}{\partial x} = \begin{cases} 1 & x_0 = x \\ 0 & \text{otherwise} \end{cases} .$$

This definition extends to $\langle \mathbf{x} \rangle$; Lemma 1.5(ii) gives

$$\frac{\partial x_0^{-1}}{\partial x} = -x_0 \frac{\partial x_0}{\partial x} = \begin{cases} -x^{-1} & x_0 = x \\ 0 & \text{otherwise} \end{cases}.$$

If W is a word on $\mathbf{x} \cup \mathbf{x}^{-1}$, we can write this word as

$$W = W_0 x^{\varepsilon_0} W_1 x^{\varepsilon_1} W_2 \dots W_n x^{\varepsilon_n} W_{n+1},$$

where each W_i, $0 \leq i \leq n+1$, is a (possibly trivial) word on $\mathbf{x} \cup \mathbf{x}^{-1} \setminus \{x, x^{-1}\}$ and $\varepsilon_i = \pm 1$. Then, using (1.7), the *Fox derivative of W with respect to x* is

$$\begin{aligned} \frac{\partial W}{\partial x} &= W_0 \frac{\partial x^{\varepsilon_0}}{\partial x} + W_0 x^{\varepsilon_0} W_1 \frac{\partial x^{\varepsilon_1}}{\partial x} + \dots + (W_0 x^{\varepsilon_0} W_1 \dots W_n) \frac{\partial x^{\varepsilon_n}}{\partial x} \\ &= \varepsilon_0 W_0 x^{\varepsilon_0^*} + \varepsilon_1 W_0 x^{\varepsilon_0} W_1 x^{\varepsilon_1^*} + \dots + \varepsilon_n (W_0 x^{\varepsilon_0} W_1 \dots W_n) x^{\varepsilon_n^*} \end{aligned}$$

This defines $\frac{\partial}{\partial x}$ for words on $\mathbf{x} \cup \mathbf{x}^{-1}$. To demonstrate that this is well-defined on $\langle \mathbf{x} \rangle$, we show that the derivatives of freely equal words are equal. Let U, V be words on $\mathbf{x} \cup \mathbf{x}^{-1}$ and let $x, x_0 \in \mathbf{x}$, $\varepsilon \in \{1, -1\}$. Then, since

$$\frac{\partial}{\partial x} x_0^{\varepsilon} x_0^{-\varepsilon} = \begin{cases} \frac{\partial x^{\varepsilon}}{\partial x} + x^{\varepsilon} \frac{\partial x^{-\varepsilon}}{\partial x} = \frac{\partial x^{\varepsilon}}{\partial x} - x^{\varepsilon} x^{-\varepsilon} \frac{\partial x^{\varepsilon}}{\partial x} = 0 & x_0 = x \\ 0 & x_0 \neq x \end{cases},$$

we have

$$\begin{aligned} \frac{\partial}{\partial x} U x_0^{\varepsilon} x_0^{-\varepsilon} V &= \frac{\partial U}{\partial x} + U \frac{\partial x_0^{\varepsilon} x_0^{-\varepsilon}}{\partial x} + U x_0^{\varepsilon} x_0^{-\varepsilon} \frac{\partial V}{\partial x} \\ &= \frac{\partial U}{\partial x} + U \frac{\partial V}{\partial x} \\ &= \frac{\partial}{\partial x} UV. \end{aligned}$$

For example, if $\mathbf{x} = \{x, y\}$ and $W = x^2 y x^{-2} y^{-1}$, then

$$\begin{aligned} \frac{\partial W}{\partial x} &= 1 + x - x^2 y x^{-1} - x^2 y x^{-2}, \\ \frac{\partial W}{\partial y} &= x^2 - x^2 y x^{-2} y^{-1}. \end{aligned}$$

These Fox derivatives are the universal derivations from $\mathbb{Z}\langle \mathbf{x} \rangle$ to $\mathbb{Z}\langle \mathbf{x} \rangle$ in the following sense.

Proposition 1.8 **([40]).** *If $d : \mathbb{Z}\langle \mathbf{x} \rangle \to \mathbb{Z}\langle \mathbf{x} \rangle$ is a derivation, then, for $\xi \in \mathbb{Z}\langle \mathbf{x} \rangle$,*

$$d(\xi) = \sum_{x \in \mathbf{x}} \frac{\partial \xi}{\partial x} d(x).$$

We can apply this to the derivation $d : \mathbb{Z}\langle \mathbf{x} \rangle \to I\langle \mathbf{x} \rangle \leq \mathbb{Z}\langle \mathbf{x} \rangle$ given in Example 1.1:

Corollary 1.9. *For $\xi \in \mathbb{Z}\langle \mathbf{x} \rangle$,*

$$\xi - aug(\xi) = \sum_{x \in \mathbf{x}} \frac{\partial \xi}{\partial x}(x - 1).$$

For a word W on $\mathbf{x} \cup \mathbf{x}^{-1}$ and for $x \in \mathbf{x}$, the *exponent sum* of x in W, $\exp_x(W)$, is defined to be the number of occurrences of x in W minus the number of occurrences of x^{-1}. From the definition of the Fox derivative we then have

$$aug\left(\frac{\partial W}{\partial x}\right) = \exp_x(W). \tag{1.8}$$

In due course, we will have cause to consider the Fox derivatives of the relators of a presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and their image in $G = G(\mathcal{P})$, that is, the composition

$$\overline{\frac{\partial}{\partial x}} : \mathbb{Z}\langle \mathbf{x} \rangle \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}\langle \mathbf{x} \rangle \xrightarrow{\bar{\ }} \mathbb{Z}G.$$

If $\mathcal{P}$ is given by defining relations rather that relators, then these can still be used to find the derivative of the corresponding relator:

Lemma 1.10. *If $U = V$ is a defining relation of $\mathcal{P}$, then*

$$\overline{\frac{\partial UV^{-1}}{\partial x}} = \overline{\frac{\partial U}{\partial x}} - \overline{\frac{\partial V}{\partial x}}.$$

Proof. By (1.7) and Lemma 1.5,

$$\begin{aligned} \frac{\partial UV^{-1}}{\partial x} &= \frac{\partial U}{\partial x} + U \frac{\partial V^{-1}}{\partial x} \\ &= \frac{\partial U}{\partial x} - UV^{-1} \frac{\partial V}{\partial x}, \end{aligned}$$

and $\overline{UV^{-1}} = 1$. □

When S is a consequence of $\mathbf{r}$, then, applying Corollary 1.9, we have

$$\sum_{x \in \mathbf{x}} \overline{\frac{\partial S}{\partial x}}(\overline{x} - 1) = 0. \tag{1.9}$$

Also, $\overline{\frac{\partial S}{\partial x}}$ is a $\mathbb{Z}G$-linear combination of the elements $\overline{\frac{\partial R}{\partial x}}$ $(R \in \mathbf{r})$, that is:

Lemma 1.11. *If* $S = \prod_{i=1}^{n} W_i R_i^{\varepsilon_i} W_i^{-1} \in \ll \mathbf{r} \gg$, *for some* $W_i \in \langle \mathbf{x} \rangle$, *some* $R_i \in \mathbf{r}$ *and some* $\varepsilon_i \in \{1, -1\}$, *then*

$$\overline{\frac{\partial S}{\partial x}} = \sum_{i=1}^{n} \varepsilon_i \overline{W_i} \overline{\frac{\partial R_i}{\partial x}}.$$

Proof. By (1.7),

$$\frac{\partial S}{\partial x} = \sum_{i=1}^{n} S_i \left(\frac{\partial W_i}{\partial x} + \varepsilon_i W_i R_i^{\varepsilon_i^*} \frac{\partial R_i}{\partial x} - W_i R_i^{\varepsilon_i} W_i^{-1} \frac{\partial W_i}{\partial x} \right),$$

where $S_i = \prod_{j=1}^{i-1} W_j R_j^{\varepsilon_j} W_j^{-1}$. Since $\overline{S_i} = \overline{R_i} = 1$, the result follows. □

We consider one final property of the Fox derivative.

Proposition 1.12 (Chain rule. [40]). *If* $\alpha : \langle \mathbf{y} \rangle \to \langle \mathbf{x} \rangle$ *is a homomorphism, then, for* $W \in \langle \mathbf{y} \rangle$,

$$\frac{\partial \alpha(W)}{\partial x} = \sum_{y \in \mathbf{y}} \alpha \left(\frac{\partial W}{\partial y} \right) \frac{\partial \alpha(y)}{\partial x}.$$

1.3.2 The picture derivative

Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and $G = G(\mathcal{P})$. For each $R \in \mathbf{r}$, we define a map $\frac{\partial}{\partial R}$ from the set of pictures over $\mathcal{P}$ to $\mathbb{Z}G$. While this is certainly not a derivation, it shares many of the properties of the Fox derivatives, so we will call it the *picture derivative.*

Let $\mathbb{P}$ be a picture over $\mathcal{P}$, with ambient disc D and discs $\Delta_1, \ldots, \Delta_m$. For each disc Δ of $\mathbb{P}$, choose a simple path β_Δ in the closure of $D \setminus \cup_i \Delta_i$ from $0_{\mathbb{P}}$ to 0_Δ which intersects the arcs of $\mathbb{P}$ only finitely many times and does so by crossing the arc, not just touching it. By reading along this path, we obtain a word $W(\beta_\Delta)$ on $\mathbf{x} \cup \mathbf{x}^{-1}$. We define

$$\frac{\partial \mathbb{P}}{\partial R} = \sum_{\Delta : R_\Delta = R} \varepsilon_\Delta \overline{W(\beta_\Delta)},$$

the sum over those discs of $\mathbb{P}$ whose label is R.

Lemma 1.13. *The map* $\frac{\partial}{\partial R}$ *is well-defined.*

Proof. If we choose a different path β'_Δ from $0_{\mathbb{P}}$ to 0_Δ, then we obtain a possibly different word $W(\beta'_\Delta)$. However, $W(\beta_\Delta)$ and $W(\beta'_\Delta)$ represent the same element of G, as we now show.

If β_Δ and β'_Δ intersect only in their endpoints, then the region of $\mathbb{P}$ enclosed by β_Δ and β'_Δ is a picture over $\mathcal{P}$ whose boundary label is

$$(W(\beta_\Delta)W(\beta'_\Delta)^{-1})^\varepsilon$$

for some $\varepsilon \in \{-1,1\}$. Thus, by Theorem 1.1,

$$\overline{W(\beta_\Delta)} = \overline{W(\beta'_\Delta)}.$$

If β_Δ, β'_Δ intersect, then, assuming (as we may) that they do not intersect on an arc, the regions of $\mathbb{P}$ enclosed by β_Δ and β'_Δ are a number of pictures $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n$ over $\mathcal{P}$, with boundary labels $U_1V_1^{-1}, V_2U_2^{-1}, U_3V_3^{-1}, \ldots$, respectively, where

$$W(\beta_\Delta) = U_1U_2\ldots U_n,$$
$$W(\beta'_\Delta) = V_1V_2\ldots V_n$$

(or vice versa). Thus each $\overline{U_i} = \overline{V_i}$, and so $\overline{W(\beta_\Delta)} = \overline{W(\beta'_\Delta)}$. □

For example, consider the pictures over $\langle a, b; a^3, bab^{-1}a^{-2}\rangle$ of Figure 1.1. Denoting the relators a^3 and $bab^{-1}a^{-2}$ by R and S respectively, we have

$$\begin{aligned}
\frac{\partial \mathbb{P}_1}{\partial R} &= \overline{a}^{-1} + \overline{a}^{-1} - \overline{a}^{-2} - \overline{a}^{-1}\overline{b}^{-1} \\
&= 2\overline{a}^2 - \overline{a} - \overline{a}^2\overline{b}^{-1}, \\
\frac{\partial \mathbb{P}_1}{\partial S} &= -\overline{a}^{-1}\overline{b}^{-1} - \overline{a}\overline{b}^{-1} \\
&= -\overline{a}^2\overline{b}^{-1} - \overline{a}\overline{b}^{-1} \\
\frac{\partial \mathbb{P}_2}{\partial R} &= 1 + 1 - \overline{b} \\
&= 2 - \overline{b}, \\
\frac{\partial \mathbb{P}_2}{\partial S} &= 1 + \overline{a}^2 + \overline{a}^4 \\
&= 1 + \overline{a} + \overline{a}^2.
\end{aligned}$$

Now suppose that $\mathbb{P}$ is a spherical picture. If we make a bridge move on $\mathbb{P}$, or insert or delete a cancelling pair or a floating circle, giving an equivalent picture $\mathbb{P}'$, then it is easy to see that

$$\frac{\partial \mathbb{P}'}{\partial R} = \frac{\partial \mathbb{P}}{\partial R}.$$

Also, if $W \in \langle \mathbf{x} \rangle$, then

$$\frac{\partial \mathbb{P}^W}{\partial R} = \overline{W}\frac{\partial \mathbb{P}}{\partial R}$$

and, if $\mathbb{Q}$ is another spherical picture,

$$\frac{\partial(\mathbb{P} + \mathbb{Q})}{\partial R} = \frac{\partial \mathbb{P}}{\partial R} + \frac{\partial \mathbb{Q}}{\partial R}.$$

Thus, we have a well-defined $\mathbb{Z}G$-homomorphism, also denoted $\frac{\partial}{\partial R}$,

$$\frac{\partial}{\partial R} : \pi_2(\mathcal{P}) \to \mathbb{Z}G; [\mathbb{P}] \mapsto \frac{\partial \mathbb{P}}{\partial R}.$$

We consider some properties of the maps $\frac{\partial}{\partial R}$, the first of which could be thought of as an analogue of Corollary 1.9.

Proposition 1.14. *For a picture* $\mathbb{P}$ *over* $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ *and for* $x \in \mathbf{x}$,

$$\overline{\frac{\partial W_{\mathbb{P}}}{\partial x}} = \sum_{R \in \mathbf{r}} \frac{\partial \mathbb{P}}{\partial R} \overline{\frac{\partial R}{\partial x}}.$$

Proof. We use induction on the number of discs in $\mathbb{P}$. If $\mathbb{P}$ has no discs, then

$$\frac{\partial \mathbb{P}}{\partial R} = 0$$

for each R and, since $W_{\mathbb{P}}$ is freely equal to 1,

$$\frac{\partial W_{\mathbb{P}}}{\partial x} = 0$$

for each x. The proposition then holds in this case.

Now suppose that the proposition holds for pictures with m discs (for some integer $m \geq 0$) and that $\mathbb{P}$ is a picture with $m+1$ discs. Choose a disc Δ of $\mathbb{P}$. The picture $\mathbb{P}$ is then of the form illustrated in Figure 1.8, where we have divided $\mathbb{P}$ into two subpictures, one including Δ, a small area around Δ and a small region around a path from $0_{\mathbb{P}}$ to 0_{Δ} and the other, $\mathbb{Q}$, including the remaining m discs (note that the dotted line indicates the boundary between the two subpictures and that $\mathbb{P}$ and $\mathbb{Q}$ share a basepoint). Now, $W_{\mathbb{Q}} = W_{\mathbb{P}} U S^{-\varepsilon} U^{-1}$, so

$$\begin{aligned}\overline{\frac{\partial W_{\mathbb{Q}}}{\partial x}} &= \overline{\frac{\partial W_{\mathbb{P}}}{\partial x}} + \overline{W_{\mathbb{P}}}\overline{\frac{\partial U S^{-\varepsilon} U^{-1}}{\partial x}} \\ &= \overline{\frac{\partial W_{\mathbb{P}}}{\partial x}} - \varepsilon \overline{U}\overline{\frac{\partial S}{\partial x}},\end{aligned}$$

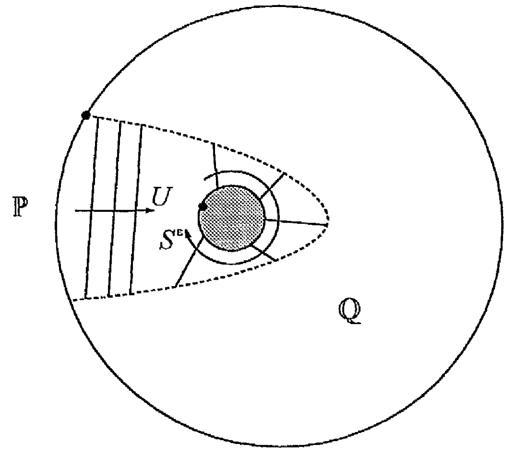

Figure 1.8: A picture $\mathbb{P}$ with $m+1$ discs

by Lemma 1.11 and since $\overline{W_{\mathbb{P}}} = 1$. On the other hand, since

$$\frac{\partial \mathbb{P}}{\partial R} = \begin{cases} \frac{\partial \mathbb{Q}}{\partial R} & R \neq S \\ \frac{\partial \mathbb{Q}}{\partial S} + \varepsilon \overline{U} & R = S \end{cases},$$

we have

$$\begin{aligned} \sum_{R \in \mathbf{r}} \frac{\partial \mathbb{P}}{\partial R} \overline{\frac{\partial R}{\partial x}} &= \sum_{R \in \mathbf{r}} \frac{\partial \mathbb{Q}}{\partial R} \overline{\frac{\partial R}{\partial x}} + \varepsilon \overline{U} \overline{\frac{\partial S}{\partial x}} \\ &= \overline{\frac{\partial W_{\mathbb{Q}}}{\partial x}} + \varepsilon \overline{U} \overline{\frac{\partial S}{\partial x}}, \end{aligned}$$

which gives the required result. □

In particular, if $\mathbb{P}$ is a spherical picture, then

$$\sum_{R \in \mathbf{r}} \frac{\partial \mathbb{P}}{\partial R} \overline{\frac{\partial R}{\partial x}} = 0. \tag{1.10}$$

For a picture $\mathbb{P}$ over $\mathcal{P}$ and for $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, $\exp_R(\mathbb{P})$, is defined to be the number of discs of $\mathbb{P}$ labelled R with positive orientation minus the number of discs of $\mathbb{P}$ labelled R with negative orientation. It is then easy to see that

$$aug\left(\frac{\partial \mathbb{P}}{\partial R}\right) = \exp_R(\mathbb{P})$$

(cf. (1.8)).

As with Lemma 1.11, we will find the following useful.

Lemma 1.15. *If* $\mathbf{d}$ *is a generating set of pictures for* $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ *and if* $\mathbb{P}$ *is a spherical picture over* $\mathcal{P}$*, then, for* $R \in \mathbf{r}$*,* $\frac{\partial \mathbb{P}}{\partial R}$ *is a* $\mathbb{Z}G$*-linear combination of the elements* $\frac{\partial \mathbb{D}}{\partial R}$ *(*$\mathbb{D} \in \mathbf{d}$*), that is, if*

$$[\mathbb{P}] = \sum_{\mathbb{D} \in \mathbf{d}} \xi_{\mathbb{D}} [\mathbb{D}]$$

for some $\xi_{\mathbb{D}} \in \mathbb{Z}G$*, then*

$$\frac{\partial \mathbb{P}}{\partial R} = \sum_{\mathbb{D} \in \mathbf{d}} \xi_{\mathbb{D}} \frac{\partial \mathbb{D}}{\partial R}.$$

Proof. The map $\frac{\partial}{\partial R} : \pi_2(\mathcal{P}) \to \mathbb{Z}G$ is a $\mathbb{Z}G$-homomorphism, and $\frac{\partial \mathbb{P}}{\partial R} = \frac{\partial [\mathbb{P}]}{\partial R}$. □

Proposition 1.12, the so-called chain rule, has a picture analogue. Let $\mathcal{Q} = \langle \mathbf{y}; \mathbf{s} \rangle$ and $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$. If $\alpha : G(\mathcal{Q}) \to G(\mathcal{P})$ is a group homomorphism, then it lifts to a homomorphism $\widehat{\alpha} : [\mathbf{y}, \mathbf{y}^{-1}] \to [\mathbf{x}, \mathbf{x}^{-1}]$ of free monoids such that, for $W \in [\mathbf{y}, \mathbf{y}^{-1}]$, $\overline{\widehat{\alpha}(W)} = \alpha(\overline{W})$. There is also an induced map from the set of pictures over $\mathcal{Q}$ to the set of pictures over $\mathcal{P}$, also denoted α, defined as follows: since, for every $S \in \mathbf{s}$, $\overline{\widehat{\alpha}(S)} = 1$, by Theorem 1.1, there is a picture $\mathbb{P}_S$ over $\mathcal{P}$ with boundary label $\widehat{\alpha}(S)$. Let $\mathbb{Q}$ be a picture over $\mathcal{Q}$. For each arc of $\mathbb{Q}$, if it is labelled by $y \in \mathbf{y}$, then replace it by a number of arcs with total label $\widehat{\alpha}(y)$ and for each disc Δ_0 of $\mathbb{Q}$, if it is labelled by $S \in \mathbf{s}$, then replace it with $\varepsilon_{\Delta_0} \mathbb{P}_S$, putting $0_{\mathbb{P}_S}$ where 0_{Δ_0} was, and matching up the arcs on the boundaries, as we may.

Proposition 1.16 (Chain Rule). *Let* $\mathcal{Q}$*,* $\mathcal{P}$*,* α *and* $\widehat{\alpha}$ *be as above. Then, for* $R \in \mathbf{r}$ *and a picture* $\mathbb{Q}$ *over* $\mathcal{Q}$*,*

$$\frac{\partial \alpha(\mathbb{Q})}{\partial R} = \sum_{S \in \mathbf{s}} \alpha \left(\frac{\partial \mathbb{Q}}{\partial S} \right) \frac{\partial \mathbb{P}_S}{\partial R}.$$

Proof. Suppose that Δ_0 is a disc of $\mathbb{Q}$ with label S and that β_{Δ_0} is a path from $0_{\mathbb{Q}}$ to 0_{Δ_0}. Let $\beta^{\alpha}_{\Delta_0}$ be the path in $\alpha(\mathbb{Q})$ which takes the same course as β_{Δ_0} from $0_{\alpha(\mathbb{Q})}$ to $0_{\mathbb{P}_S}$. Thus the label on this path is

$$W(\beta^{\alpha}_{\Delta_0}) = \widehat{\alpha}(W(\beta_{\Delta_0})).$$

For each disc Δ of $\varepsilon_{\Delta_0} \mathbb{P}_S$, let β_{Δ} be a path from $0_{\mathbb{P}_S}$ to 0_{Δ}. By composing $\beta^{\alpha}_{\Delta_0}$ with β_{Δ}, we obtain a path $\beta^{\alpha}_{\Delta_0} \beta_{\Delta}$ from $0_{\alpha(\mathbb{Q})}$ to 0_{Δ} with label

$$W(\beta^{\alpha}_{\Delta_0} \beta_{\Delta}) = \widehat{\alpha}(W(\beta_{\Delta_0})) W(\beta_{\Delta}).$$

Since the orientation of Δ in $\alpha(\mathbb{Q})$ is $\varepsilon_{\Delta_0}\varepsilon_{\Delta}$, if we apply this to each disc Δ of $\alpha(\mathbb{Q})$, we obtain the result. $\square$

Remark. Note that the map α of pictures depends on the choice of pictures $\mathbb{P}_S$ ($S \in \mathbf{s}$). Note also that if $\mathbb{Q}$ is spherical, then so is $\alpha(\mathbb{Q})$ and so α induces a map of second homotopy modules $\alpha : \pi_2(\mathcal{Q}) \to \pi_2(\mathcal{P})$.

1.4 Matrices, chains of ideals and elementary ideals of matrices

1.4.1 Matrices

If X is a matrix over a ring C, whose rows are indexed by an ordered set $\mathbf{u}$ and whose columns are indexed by an ordered set $\mathbf{v}$, then we write

$$X = \Big[c_{uv}\Big]_{\substack{u\in\mathbf{u}\\ v\in\mathbf{v}}}$$

or $X = [c_{uv}]_{\mathbf{u},\mathbf{v}}$, where $c_{uv} \in C$ is the (u,v)-th entry of X. We say that X is a $|\mathbf{u}| \times |\mathbf{v}|$ matrix. If $|\mathbf{u}| = |\mathbf{v}|$, then we say that X is *square*.

If $\mathbf{u} = \{1,\dots,m\}$, $\mathbf{v} = \{1,\dots,n\}$, then we write

$$X = \Big[c_{ij}\Big]_{\substack{1\le i\le m\\ 1\le j\le n}}$$

or, simply, $X = [c_{ij}]_{i,j}$. Here, we allow m or n to be infinite.

A choice of subsets $\mathbf{u}_0 \subseteq \mathbf{u}$, $\mathbf{v}_0 \subseteq \mathbf{v}$ defines a $|\mathbf{u}_0| \times |\mathbf{v}_0|$ *submatrix*

$$\Big[c_{uv}\Big]_{\substack{u\in\mathbf{u}_0\\ v\in\mathbf{v}_0}}$$

of X. If $X = [c_{ij}]_{i,j}$, then we write

$$\Big[c_{i_1 j_1} \quad c_{i_2 j_2} \quad \cdots\Big]$$

for the square submatrix of X with rows $i_1, i_2, \dots$ and columns $j_1, j_2, \dots$ of X.

A *diagonal matrix* $[d_{ij}]_{i,j}$ is a square matrix for which $d_{ij} = 0$ when $i \neq j$. The entries d_{ii}, $i = 1, 2, \dots$, are the *diagonal entries*. If $\{d_1, \dots, d_n\}$ is a set of elements of C, then we write $\mathrm{Diag}_n(d_1, \dots, d_n)$ for the $n \times n$ diagonal matrix with diagonal entries $d_1, \ldots, d_n$. If $d_1 = d_2 = \cdots = d_n = d$, then we will write $\mathrm{Diag}_n(d)$ for $\mathrm{Diag}_n(d, \ldots, d)$.

The $n \times n$ *identity matrix* over C is the diagonal matrix

$$I_n = \mathrm{Diag}_n(1).$$

If $\{d_u : u \in \mathbf{u}\}$ is a set of elements of C indexed by an ordered set $\mathbf{u}$, we write $\mathrm{Diag}_{u\in\mathbf{u}}(d_u)$ for the diagonal matrix with diagonal entries d_u ($u \in \mathbf{u}$). If, for $u \in \mathbf{u}$, X_u is an $m_u \times n_u$ over C, then $\mathrm{Diag}_{u\in\mathbf{u}}(X_u)$ is the diagonal matrix of matrices, thought of as a $\left(\sum_{u\in\mathbf{u}} m_u\right) \times \left(\sum_{u\in\mathbf{u}} n_u\right)$ matrix over C.

If $X = [c_{uv}]_{\mathbf{u},\mathbf{v}}$, then the *transpose* of X is the $|\mathbf{v}| \times |\mathbf{u}|$ matrix

$$X^t = \left[d_{vu}\right]_{\substack{v\in\mathbf{v} \\ u\in\mathbf{u}}},$$

where $d_{vu} = c_{uv}$.

If $\alpha : C \to C_0$ is a ring homomorphism and if $X = [c_{uv}]_{\mathbf{u},\mathbf{v}}$ is a matrix over C, then

$$X^\alpha = \left[\alpha(c_{uv})\right]_{\mathbf{u},\mathbf{v}}$$

is the matrix over C_0 obtained by applying α to each entry of X.

1.4.2 Chains of ideals

Let K be a commutative ring.

An *ascending chain of ideals* in K is a set of ideals $I = \{I_\kappa\}_{\kappa\in\mathbb{Z}}$, indexed by $\mathbb{Z}$, such that, for $\kappa \in \mathbb{Z}$,

$$I_\kappa \subseteq I_{\kappa+1}.$$

A *descending chain of ideals* in K is a set of ideals $I = \{I_\kappa\}_{\kappa\in\mathbb{Z}}$ such that, for $\kappa \in \mathbb{Z}$,

$$I_\kappa \supseteq I_{\kappa+1}.$$

We will say that a chain of ideals I is *trivial* if $I_\kappa = 0$ or K for every $\kappa \in \mathbb{Z}$.

If $I = \{I_\kappa\}_{\kappa\in\mathbb{Z}}$, $J = \{J_\kappa\}_{\kappa\in\mathbb{Z}}$ are two chains of ideals in K (either both ascending or both descending) for which $I_\kappa \subseteq J_\kappa$ for each $\kappa \in \mathbb{Z}$, then we will write $I \subseteq J$.

If $I = \{I_\kappa\}_{\kappa\in\mathbb{Z}}$ is a chain of ideals in K and if $\alpha : K \to K_0$ is a ring homomorphism to another commutative ring K_0, then we will write (αI) for the chain $\{(\alpha I_\kappa)\}_{\kappa\in\mathbb{Z}}$

of ideals in K_0 generated by the sets αI_κ. If α is onto, then $(\alpha I_\kappa) = \alpha I_\kappa$, so we will denote the chain (αI) simply by αI.

If I is a chain of ideals in K and J a chain of ideals in K_0, we will write $I \cong J$ if there is an isomorphism $\alpha : K \to K_0$ such that $J = \alpha I$.

Notation. In the special case where $K = K'G$, $K_0 = K'G_0$ for some commutative ring K' and abelian groups G, G_0, we will write $I \cong^{(0)} J$ if, in addition, the ring isomorphism $\alpha : K'G \to K'G_0$ is induced by a group isomorphism $\alpha : G \to G_0$. The same notation will also apply for monoid algebras.

If $I = \{I_\kappa\}_{\kappa\in\mathbb{Z}}$, $J = \{J_\kappa\}_{\kappa\in\mathbb{Z}}$ are two chains of ideals in K (either both ascending or both descending), then, for $\lambda \in \mathbb{Z}$, the *convolution of I and J, suspended by λ*, is the collection of ideals

$$\sum_{j\in\mathbb{Z}} I_{\lambda+j} J_{\kappa-j} \quad (\kappa \in \mathbb{Z}).$$

This collection is a chain, since, if I, J are both ascending, $J_{\kappa-j} \subseteq J_{\kappa+1-j}$ for each $j \in \mathbb{Z}$, so

$$I_{\lambda+j} J_{\kappa-j} \subseteq I_{\lambda+j} J_{\kappa+1-j}.$$

The case when I, J are both descending chains is similar. We denote this chain by $I *^{(\lambda)} J$. If $\lambda = 0$, then we omit the superscript $^{(\lambda)}$.

If, for $\kappa \in \mathbb{Z}$, I_κ is generated by the set $\{k_u : u \in \mathbf{u}_\kappa\} \subseteq K$ and J_κ by the set $\{k'_v : v \in \mathbf{v}_\kappa\} \subseteq K$, then $\left(I *^{(\lambda)} J\right)_\kappa$ is generated by the set

$$\left\{k_u k'_v : j \in \mathbb{Z}, u \in \mathbf{u}_{\lambda+j}, v \in \mathbf{v}_{\kappa-j}\right\}.$$

We can, in fact, form the convolution of more than two chains of ideal. If, for $u \in \mathbf{u}$, I_u is a chain of ideals in K, all of which are ascending or all of which are descending, then, for $\lambda \in \mathbb{Z}$, $*^{(\lambda)}_{u\in\mathbf{u}} I_u$ is the chain of ideals in K whose κ-th ideal is

$$\left(*^{(\lambda)}_{u\in\mathbf{u}} I_u\right)_\kappa = \sum_{\substack{j_u\in\mathbb{Z}(u\in\mathbf{u})\\ \sum_{\mathbf{u}} j_u=\kappa+\lambda}} \prod_{u\in\mathbf{u}} I_{u,j_u}.$$

Again, when $\lambda = 0$, we omit it. We could even have $|\mathbf{u}| = 1$, in which case $\left(*^{(\lambda)}_{u\in\mathbf{u}} I\right)_\kappa = I_{u,\kappa+\lambda}$.

1.4.3 Elementary ideals of matrices

Let X be an $m \times n$ matrix over a commutative ring K. We permit one of m, n to be infinite, but not both. For $\kappa \in \mathbb{Z}$, the *κ-th elementary ideal* of X is

$$J_\kappa(X) = \begin{cases} 0 & \kappa > \min\{m,n\} \\ \text{the ideal of } K \text{ generated} & \\ \text{by the determinants of all} & 0 < \kappa \le \min\{m,n\} \\ \kappa \times \kappa \text{ submatrices of } X & \\ K & \kappa \le 0 \end{cases} .$$

If X_0 is a $\kappa \times \kappa$ submatrix of X, then its determinant, $\det(X_0)$, is a K-linear combination of $(\kappa-1)\times(\kappa-1)$ submatrices of X_0. Thus $J_\kappa(X) \subseteq J_{\kappa-1}(X)$, and so $J(X) = \{J_\kappa(X)\}_{\kappa\in\mathbb{Z}}$ is a descending chain of ideals.

We consider some properties of the elementary ideals. Throughout, X is an $m\times n$ matrix over K.

Lemma 1.17. *If $\alpha : K \to K_0$ is a ring homomorphism, then*

$$J(X^\alpha) = (\alpha J(X)) .$$

If α is surjective, then

$$J(X^\alpha) = \alpha J(X).$$

Proof. Every $\kappa \times \kappa$ submatrix X_0 of X defines a $\kappa \times \kappa$ submatrix X_0^α of X^α, so $\alpha J_\kappa(X) \subseteq J_\kappa(X^\alpha)$. On the other hand, if X_0 is a $\kappa \times \kappa$ submatrix of X^α, then $X_0 = \widetilde{X}_0^\alpha$ for some $\kappa\times\kappa$ submatrix $\widetilde{X}_0$ of X and $\det(X_0) = \alpha(\det(\widetilde{X}_0))$, so $J_\kappa(X^\alpha) \subseteq (\alpha J_\kappa(X))$. □

Lemma 1.18. $J(X^t) = J(X)$.

Proof. If X_0 is a $\kappa \times \kappa$ submatrix of X, then X_0^t is a $\kappa \times \kappa$ submatrix of X^t and $\det(X_0^t) = \det(X_0)$, so $J(X) \subseteq J(X^t)$. Also, $(X^t)^t = X$, giving the result. □

We now consider matrices of the form

$$\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix},$$

where X is an $m \times n$ matrix, Y and $m' \times n'$ matrix and Z an $m' \times n$ matrix over K and 0 represents an $m \times n'$ matrix of zeroes. For $0 < \kappa \leq \min\{m+m', n+n'\}$, every $\kappa \times \kappa$ submatrix of this matrix must be of the form

$$\begin{bmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{bmatrix},$$

where X_0 is a $\kappa_1 \times \kappa_2$ submatrix of X, Y_0 a $\kappa_3 \times \kappa_4$ submatrix of Y and Z_0 a $\kappa_3 \times \kappa_2$ submatrix of Z, with $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \geq 0$ and $\kappa_1 + \kappa_3 = \kappa_2 + \kappa_4 = \kappa$. The following observations prove useful when dealing with the determinants of such matrices:

i) If $\kappa_1 = \kappa_2$, then
$$\det\left(\begin{bmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{bmatrix}\right) = \det(X_0)\det(Y_0).$$
Note that $\kappa_1 = \kappa_2$ if, and only if, $\kappa_3 = \kappa_4$.

ii) If $\kappa_1 > \kappa_2$, then
$$\det\left(\begin{bmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{bmatrix}\right) = 0$$
Note that $\kappa_1 > \kappa_2$ if, and only if, $\kappa_4 > \kappa_3$.

These follow from the elementary properties of the determinant [2].

Proposition 1.19. *For X, Y, Z as above,*

$$J\left(\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix}\right) \supseteq J(X) * J(Y),$$

with equality when $Z = 0$.

Proof. Let $\kappa, j \in \mathbb{Z}$. When $j \leq 0$, $J_j(X) = K$ and $\kappa - j \geq \kappa$, so

$$J_j(X)J_{\kappa-j}(Y) = J_{\kappa-j}(Y) \subseteq J_\kappa(Y) \subseteq J_\kappa\left(\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix}\right),$$

since every submatrix of Y is certainly a submatrix of $\left[\begin{smallmatrix} X & 0 \\ Z & Y \end{smallmatrix}\right]$. Similarly, for $j \geq \kappa$, $J_{\kappa-j}(Y) = K$, so

$$J_j(X)J_{\kappa-j}(Y) = J_j(X) \subseteq J_\kappa(X) \subseteq J_\kappa\left(\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix}\right).$$

For $j < \kappa - \min\{m'n'\}$, $J_{\kappa-j}(Y) = 0$ and for $j > \min\{m,n\}$, $J_j(X) = 0$, so, in either of these cases,

$$0 = J_j(X)J_{\kappa-j}(Y) \subseteq J_\kappa\left(\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix}\right).$$

This leaves $\max\{1, \kappa - m', \kappa - n'\} \leq j \leq \min\{m, n, \kappa - 1\}$. For such a j, if X_0 is any $j \times j$ submatrix of X and Y_0 any $(\kappa - j) \times (\kappa - j)$ submatrix of Y, then there is a $(\kappa - j) \times j$ submatrix Z_0 of Z such that $\left[\begin{smallmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{smallmatrix}\right]$ is a $\kappa \times \kappa$ submatrix of $\left[\begin{smallmatrix} X & 0 \\ Z & Y \end{smallmatrix}\right]$. Since

$$\det\left(\begin{bmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{bmatrix}\right) = \det(X_0)\det(Y_0),$$

we must have $J_j(X)J_{\kappa-j}(Y) \subseteq J_\kappa\left(\left[\begin{smallmatrix} X & 0 \\ Z & Y \end{smallmatrix}\right]\right)$.

If $Z = 0$, then any submatrix of $\left[\begin{smallmatrix} X & 0 \\ Z & Y \end{smallmatrix}\right]$ with non-zero determinant must be of the form $\left[\begin{smallmatrix} X_0 & 0 \\ 0 & Y_0 \end{smallmatrix}\right]$, where X_0, Y_0 are square submatrices of X, Y respectively. Thus

$$J\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = J(X) * J(Y).$$

□

Corollary 1.20. *For matrices $X_1, X_2, \ldots, X_p$ over K,*

$$J\left(Diag_p(X_1, \ldots, X_p)\right) = *_{i=1}^{p} J(X_i).$$

Proposition 1.21. *Let X be an $m \times n$ matrix over K.*

i) If Y is an $m' \times n$ matrix over K, each of whose rows is a K-linear combination of the rows of X, then

$$J(Y) \subseteq J(X);$$

ii) If Z is an $m \times n'$ matrix over K, each of whose columns is a K-linear combination of the columns of X, then

$$J(Z) \subseteq J(X).$$

Proof. We prove (i), the other result following with an application of Lemma 1.18.

For $\kappa \leq 0$, $J_\kappa(Y) = J_\kappa(X) = K$. For $\kappa > 0$, if Y_0 is a $\kappa \times \kappa$ submatrix of Y, then $\det(Y_0)$ is a K-linear combination of determinants of $\kappa \times \kappa$ submatrices of X, and so $J_\kappa(Y) \subseteq J_\kappa(X)$ (note, in particular, that if $\kappa > \min\{m, n\}$, then $\det(Y_0) = 0$, since X has no $\kappa \times \kappa$ submatrices). □

Corollary 1.22. *If X is an $m \times n$ matrix and Y an $l \times m$ matrix over K, then $J(YX) \subseteq J(Y)$ and $J(YX) \subseteq J(X)$.*

Proof. The rows of YX are a K-linear combination of the rows of X and the columns a K-linear combination of the columns of Y. □

If Y is an $m' \times m$ matrix, then a *left inverse* of Y is an $m \times m'$ matrix $\overline{Y}$ such that $\overline{Y}Y = I_m$. We define *right inverses* similarly.

Corollary 1.23. *Let X be an $m \times n$ matrix over K. If Y is an $m' \times m$ matrix with a left inverse and Z an $n \times n'$ matrix with a right inverse, then*

$$J(YXZ) = J(X).$$

Proof. If $\overline{Y}$ is a left inverse for Y and $\overline{Z}$ a right inverse for Z, then

$$J(YXZ) \subseteq J(X) = J(\overline{Y}YXZ\overline{Z}) \subseteq J(YXZ). \qquad \square$$

We can also conclude that, if $X = [c_{uv}]_{\mathbf{u},\mathbf{v}}$, then the orderings of the sets $\mathbf{u}$, $\mathbf{v}$ do not affect the elementary ideals of X. More generally, consider the following *elementary row operations* on a matrix X over K:

(ERO1) swap two rows of X;

(ERO2) add a K-multiple of one row of X to another;

(ERO3) multiply each entry in a row of X by a unit of K,

and the following *elementary column operations*:

(ECO1) swap two columns of X;

(ECO2) add a K-multiple of one column of X to another;

(ECO3) multiply each entry in a column of X by a unit of K.

Lemma 1.24. *If X' is obtained from X by a finite number of elementary row and column operations, then $J(X') = J(X)$.*

Proof. Each of the above operations may be accomplished by multiplying by an invertible matrix.

For $1 \leq i, j \leq m$, let $I_{i,j}^{(m)}$ be the $m \times m$ matrix over K, all of whose entries are 0 with the exception of the (i,j)-th entry, which is 1. To swap rows i and j of X, multiply X on the left by the matrix $I_m - I_{i,i}^{(m)} - I_{j,j}^{(m)} + I_{i,j}^{(m)} + I_{j,i}^{(m)}$. For $k \in K$, to add k times row i to row j, multiply X on the left by $I_m + kI_{j,i}^{(m)}$. To multiply row i of X by a unit $k \in K$, multiply X on the left by $I_m + (k-1)I_{i,i}^{(m)}$. Since each of these matrices is invertible, the elementary ideals of X are unchanged.

Similarly, the elementary column operations can be equated with multiplication on the right by invertible matrices such as these. □

Corollary 1.25. *If X is an $m \times n$ matrix and Z an $l \times n$ matrix over K, then, for $\kappa \in \mathbb{Z}$,*

$$J_\kappa\left(\begin{bmatrix} X & 0 \\ Z & I_l \end{bmatrix}\right) = J_{\kappa-l}(X).$$

Proof. By elementary column operations, we can obtain the matrix $\left(\begin{smallmatrix} X & 0 \\ 0 & I_l \end{smallmatrix}\right)$ from $\left(\begin{smallmatrix} X & 0 \\ Z & I_l \end{smallmatrix}\right)$. Now, applying Proposition 1.19, since

$$J_\kappa(I_l) = \begin{cases} 0 & \kappa > l \\ K & \kappa \leq l \end{cases},$$

we have

$$\begin{aligned} J_\kappa\left(\begin{bmatrix} X & 0 \\ Z & I_l \end{bmatrix}\right) &= J_\kappa\left(\begin{bmatrix} X & 0 \\ 0 & I_l \end{bmatrix}\right) \\ &= \sum_{j \in \mathbb{Z}} J_j(X) J_{\kappa-j}(I_l) \\ &= \sum_{j \geq \kappa - l} J_j(X) \\ &= J_{\kappa-l}(X). \end{aligned}$$

□

1.5 Abelianising functors

We will be interested in certain (covariant) functors from the category of groups to the category of abelian groups.

We define an *abelianising functor*, ^{T}, *on groups* to be a natural transformation from groups to abelian groups which assigns to each group G an abelian group G^T together with a surjective group homomorphism $\tau_G^T : G \to G^T$ and which has the property that, if $\alpha : G \to G_0$ is a group homomorphism, then there is an induced homomorphism $\alpha^T : G^T \to G_0^T$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G_0 \\ {\scriptstyle \tau_G^T}\downarrow & & \downarrow{\scriptstyle \tau_{G_0}^T} \\ G^T & \xrightarrow{\alpha^T} & G_0^T \end{array}$$

commutes. Note that α, α^T, τ_G^T and $\tau_{G_0}^T$ will each induce homomorphisms of group rings such that $\tau_{G_0}^T \alpha = \alpha^T \tau_G^T$ still holds.

We now give some examples of abelianising functors on groups.

Example 1.2 (Trivialisation, ^{triv}). For any group G, let G^{triv} be the trivial group 1 and let τ_G^{triv} be the trivial map. For any homomorphism $\alpha : G \to G_0$, let $\alpha^{triv} : G^{triv} \to G_0^{triv}$ be the (necessarily) trivial map. Note that $\tau_G^{triv} : KG \to KG^{triv}$ is the augmentation map, *aug*. ◊

Example 1.3 (Abelianisation, ^{ab}). For any group G, we have the abelianisation $G^{ab} = G/G'$. Let τ_G^{ab} be the natural surjection. If $\alpha : G \to G_0$ is a group homomorphism, then α carries G' to G_0', so we have an induced homomorphism

$$\alpha^{ab} : G^{ab} \to G_0^{ab}; gG' \mapsto \alpha(g)G_0'.$$ ◊

Example 1.4 (n-abelianisation, $^{n-ab}$). For any group G, let G^{n-ab} be the quotient G/G^nG' and let τ_G^{n-ab} be the natural surjection. For any map $\alpha : G \to G_0$,

$$\alpha^{n-ab} : G/G^nG' \to G_0/G_0^nG_0'$$

is the induced homomorphism. ◊

Example 1.5 (Torsion-free abelianisation, ^{tf}). For any group G, let G^{tf} be the quotient of G^{ab} by its torsion subgroup. Thus, if G^{ab} is finitely generated, G^{tf} is (isomorphic to) the free part of G^{ab}. We let τ_G^{tf} be the composition of τ_G^{ab} with the quotient map $G^{ab} \to G^{tf}$. If $\alpha : G \to G_0$ is a group homomorphism, then α^{ab} will carry torsion elements to torsion elements, and so we have an induced homomorphism

$$\alpha^{tf} : G^{tf} \to G_0^{tf}. \qquad \diamond$$

Example 1.6 (Factoring out π-torsion, $^{\pi-tf}$). Let π be a set of primes. If, in the preceding example, we take the quotient of G^{ab} only by those torsion elements whose orders are π-numbers, then we obtain a functor $^{\pi-tf}$. $\diamond$

Of all possible abelianising functors, ^{ab} is universal in that G^{ab} is the largest abelian quotient of G. Thus, for any functor ^{T}, τ_G^T factors through G^{ab}, that is, $\tau_G^T = \beta\tau_G^{ab}$ for some $\beta : G^{ab} \to G^T$.

Notation. The abelianising functor ^{ab}, being universal in this way, will be the most frequently used. To avoid clutter, we will often omit ^{ab} from the terminology where the abelianising functor in use is usually specified. **Therefore, when it is clear that we are using an abelianising functor, but no functor is specified, then we assume that it is ^{ab}.**

The functor ^{tf} is universal in the same way amongst functors ^{T} where G^T is torsion-free. At the other end of the spectrum, τ_G^{triv} factors through G^T for every ^{T} and $\tau_G^{triv} = \tau_{G^T}^{triv}\tau_G^T$.

If X is a matrix over KG, we will write X^T for the matrix $X^{\tau_G^T}$ obtained by applying $\tau_G^T : KG \to KG^T$ to each entry.

Notation. Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and $W \in \langle \mathbf{x} \rangle$. If we wish to distinguish the image of W in $G(\mathcal{P})$ from the image of W in $G(\mathcal{P})^T$, we write $\overline{W}$ for the former and $\widehat{W}$ for the latter. Thus $\overline{W} = \gamma_{\mathcal{P}}(W)$ and $\widehat{W} = \tau_G^T(\overline{W}) = \tau_G^T\gamma_{\mathcal{P}}(W)$. **For simplicity, where the context permits no confusion, we will write W to represent its own image in whichever group we are considering.**

We can also define *abelianising functors on monoids* in an analogous manner to abelianising functors over groups. Then, any abelianising functor over monoids always restrict to give an abelianising functor over groups. The most useful such functors are ^{ab}, where S^{ab} is the largest abelian quotient of a monoid S, and ^{triv}, where S^{triv} is the trivial monoid.

Notation. An analogous convention to the one above will apply to monoids.

1.6 Resolutions and homology

1.6.1 Resolutions

Let C be a ring and let M be a (left) C-module. A *resolution* of M is an exact sequence of C-modules

$$\cdots \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0.$$

We call $\mathcal{F} = (F_i, \partial_i)$ a *resolution of* M. A *partial resolution* of M is a sequence

$$F_l \xrightarrow{\partial_l} F_{l-1} \xrightarrow{\partial_{l-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0,$$

which is exact at $M, F_0, \ldots, F_{l-1}$.

If each F_i is projective, then $\mathcal{F}$ is called a *projective resolution.* If, in addition, each F_i is free, $\mathcal{F}$ is called a *free resolution.* Every module has a free (and thus projective) resolution [22].

A projective resolution $\mathcal{F} = (F_i, \partial_i)$ of M is said to be of *type* FP_n if F_i is finitely generated for $i \leq n$. It is said to be of type FP_∞ if F_i is finitely generated for all i. If M has a resolution of type FP_n, then M is said to be of *type* FP_n.

Lemma 1.26. *If M is of type FP_n, then M has a free resolution of type FP_n.*

Proof. Let $\mathcal{P} = (P_i, \varepsilon_i)$ be a resolution of M of type FP_n. Since each P_i is projective, there is a projective module Q_i such that $\overline{P}_i = P_i \oplus Q_i$ is free. The composition of the projection map $\alpha_i : \overline{P}_i \to P_i$ with the inclusion map $\iota_i : P_i \to \overline{P}_i$ gives an idempotent endomorphism $\pi_i = \iota_i \circ \alpha_i : \overline{P}_i \to \overline{P}_i$. Similarly, the projection $\alpha'_i : \overline{P}_i \to Q_i$ and the inclusion $\iota'_i : Q_i \to \overline{P}_i$ give rise to an idempotent endomorphism $\pi'_i = \iota'_i \circ \alpha'_i : \overline{P}_i \to \overline{P}_i$.

We turn $\mathcal{P}$ into a free resolution as follows. Set $F_0 = \overline{P}_0$. The chain complex $\mathcal{P}^{(0)} = (P_i^{(0)}, \varepsilon_i^{(0)})$, where

$$P_i^{(0)} = \begin{cases} F_0 & i = 0 \\ P_1 \oplus Q_0 & i = 1 \\ P_i & i > 1 \end{cases}, \quad \varepsilon_i^{(0)} = \begin{cases} \varepsilon_0 \circ \alpha_0 & i = 0 \\ (\iota_0 \circ \varepsilon_1) \oplus \iota'_0 & i = 1 \\ \varepsilon_i & i > 1 \end{cases},$$

remains a projective resolution of M. Setting $F_1 = \overline{P}_1 \oplus \overline{P}_0 \cong (P_1 \oplus Q_0) \oplus (Q_1 \oplus P_0)$, we obtain a new resolution $\mathcal{P}^{(1)} = (P_i^{(1)}, \varepsilon_i^{(1)})$, where

$$P_i^{(1)} = \begin{cases} F_i & i = 0,1 \\ P_2 \oplus (P_0 \oplus Q_1) & i = 2 \\ P_i & i > 2 \end{cases}, \quad \varepsilon_i^{(1)} = \begin{cases} \varepsilon_0 \circ \alpha_0 & i = 0 \\ (\iota_0 \circ \varepsilon_1 \circ \alpha_1) \oplus \pi'_0 & i = 1 \\ (\iota_1 \circ \varepsilon_2) \oplus \iota'_1 \oplus \iota_0 & i = 2 \\ \varepsilon_i & i > 2 \end{cases}.$$

Continuing in this way, we obtain a free resolution $\mathcal{F} = (F_i, \partial_i)$ of M, where

$$F_i = \oplus_{j=0}^{i} \overline{P}_j,$$

$\partial_0 = \varepsilon_0 \circ \alpha_0$, and

$$\partial_i = (\iota_{i-1} \circ \varepsilon_i \circ \alpha_i) \oplus \pi'_{i-1} \oplus \pi_{i-2} \oplus \pi'_{i-3} \oplus \cdots$$

If P_i can be finitely generated for $i = 0, 1, \ldots, n$, then the $\overline{P}_i$ can be chosen to be of finite rank, and so $\mathcal{F}$ is of type FP_n. □

Notation. Throughout this thesis we will assume, without further comment, that any resolution of type FP_n is a free resolution.

This is an abuse of the notation in that we should more correctly refer to resolutions of type FL_n or to free resolutions of type FP_n.

We also require the following result, which is an extension of Schanuel's Lemma.

Lemma 1.27 ([22]). *If M is of type FP_n and $\mathcal{F} = (F_i, \partial_i)$ is a resolution of type FP_m of M, $m < n$, then* $\ker \partial_m$ *is of type FP_{n-m-1}.*

A resolution $\mathcal{F} = (F_i, \partial_i)$ of M is said to be of finite length if, for some $l \geq 0$, $F_i = 0$ for $i > l$. If $F_l \neq 0$, we call $l = l(\mathcal{F})$ the *length* of $\mathcal{F}$.

A module M is of *type FP* if it has a resolution $\mathcal{F} = (F_i, \partial_i)$ of finite length such that each F_i is a finitely generated projective module. Such a resolution is said to be of *type FP*. If, in addition, each F_i is free, then M and the resolution are said to be of type FL. In contrast with Lemma 1.26, not every module of type FP is of type FL (see, for instance, [55] or Proposition 5.13, below).

Suppose now that C has the invariance of rank property. If $\mathcal{F} = (F_i, \partial_i)$ is a free resolution of M of type FP_n, then we define the *n-th directed partial Euler characteristic* of $\mathcal{F}$ to be

$$\chi_n(\mathcal{F}) = \mathrm{rk}_C(F_n) - \mathrm{rk}_C(F_{n-1}) + \cdots + (-1)^n \mathrm{rk}_C(F_0),$$

abbreviating it to χ_n when the context allows.

For a free resolution $\mathcal{F} = (F_i, \partial_i)$, choose ordered bases $\mathbf{z}$ and $\mathbf{e}$ for the free modules F_n and F_{n+1} respectively, so $F_n \cong \oplus_{z \in \mathbf{z}} Cz$ and $F_{n+1} \cong \oplus_{e \in \mathbf{e}} Ce$. For each $e \in \mathbf{e}$,

$$\partial_{n+1}(e) = \sum_{z \in \mathbf{z}} c_{e,z} z,$$

for some $c_{e,z} \in C$. We can then associate with the map ∂_{n+1} a $\mathrm{rk}_C(F_{n+1}) \times \mathrm{rk}_C(F_n)$ matrix

$$D_n(\mathcal{F}) = \Big[c_{e,z} \Big]_{\substack{e \in \mathbf{e} \\ z \in \mathbf{z}}}$$

over C. Notice that $D_{n+1}(\mathcal{F}) D_n(\mathcal{F}) = 0$.

There are analogous definitions of *resolutions*, *projective/free resolution*, *types FP_n, FP_∞, FP, FL* and χ_n for right C-modules. Lemma 1.26 also holds for right C-modules.

Notation. We adopt the convention that, if $\mathcal{F} = (F_i, \partial_i)$ is a free resolution of a right C-module M, then $D_n(\mathcal{F})$ is the $\mathrm{rk}_C(F_n) \times \mathrm{rk}_C(F_{n+1})$ matrix over C associated with ∂_{n+1}.

Thus, in the right-hand case, $D_n(\mathcal{F})D_{n+1}(\mathcal{F}) = 0$.

An *anti-isomorphism* of rings C and C_0 is a map $* : C \to C_0; c \mapsto c^*$ which is an isomorphism of the underlying additive groups such that $(cc')^* = c'^*c^*$ for $c, c' \in C$. There is then an induced (covariant) functor, also denoted $*$, from the category of left C_0-modules, ${}_{C_0}\mathcal{M}$, to the category of right C-modules, $\mathcal{M}_C$; if $M \in {}_{C_0}\mathcal{M}$, then M^* is the right C-module with the same underlying abelian group as M and with right C-action given by

$$m.c = c^*m \quad (m \in M, c \in C)$$

and, if $\alpha : M \to M_0$ is a left C_0-homomorphism (that is, a morphism of ${}_{C_0}\mathcal{M}$), then $*$ gives the right C-homomorphism

$$\alpha^* : M^* \to M_0^*; m \mapsto \alpha(m) \quad (m \in M).$$

This is a right C-homomorphism, since, for $m \in M$, $c \in C$,

$$\alpha^*(m.c) = \alpha(c^*m) = c^*\alpha(m) = \alpha^*(m).c.$$

Similarly, there is a functor $* : \mathcal{M}_{C_0} \to {}_C\mathcal{M}$.

The inverse mapping, also denoted $*$, is also an anti-isomorphism and induces functors $* : \mathcal{M}_C \to {}_{C_0}\mathcal{M}$ and $* : {}_C\mathcal{M} \to \mathcal{M}_{C_0}$. For any C_0-module M and any C_0 homomorphism α, we have $M^{**} = M$ and $\alpha^{**} = \alpha$.

Lemma 1.28. *If M is of type FP_n or FL, then so is M^*. Indeed, if $\mathcal{F} = (F_i, \partial_i)$ is a free resolution of M of type FP_n, then $\mathcal{F}^* = (F_i^*, \partial_i^*)$ is a free resolution of type FP_n of M^* and $D_n(\mathcal{F}^*) = (D_n(\mathcal{F})^*)^t$ (for appropriate bases).*

Proof. The first part of the lemma is due to F_i^* being free of finite rank when F_i is and $\mathcal{F}^*$ remaining exact.

We prove the last part in the case when M is a left C-module. If $\mathbf{z}$ and $\mathbf{e}$ are ordered bases for the free left C-modules F_n and F_{n+1} respectively, then they are also bases for the free right C_0-modules F_n^* and F_{n+1}^*. If

$$\partial_{n+1}(e) = \sum_{z \in \mathbf{z}} c_{e,z} z$$

for some $c_{e,z} \in C$, then

$$\partial^*_{n+1}(e) = \sum_{z \in \mathbf{z}} c_{e,z} z = \sum_{z \in \mathbf{z}} z c^*_{e,z},$$

and so

$$\begin{aligned} D_n(\mathcal{F}^*) &= \left[c^*_{e,z}\right]_{\substack{z\in\mathbf{z} \\ e\in\mathbf{e}}} \\ &= \left(\left[c_{e,z}\right]^*_{\substack{e\in\mathbf{e} \\ z\in\mathbf{z}}}\right)^t \\ &= (D_n(\mathcal{F})^*)^t. \end{aligned}$$

□

We now consider the case when $C = KG$, the group ring of a group G with coefficients in the commutative ring K. The ring KG has the invariance of rank property. If M is a K-module, then we obtain a KG-module, ${}_GM$, upon which G acts trivially. More precisely, ${}_GM$ is obtained from M by restriction of scalars via the map $aug : KG \to K$. In particular, the K-module K induces a KG-module, ${}_GK$. If it is clear that we are considering K as a KG-module, then we omit the subscript G.

More generally, if S is a monoid, then any K-module M induces a left KS-module ${}_SM$ and a right KS-module M_S. In particular, we have the left KS-module ${}_SK$ and the right KS-module K_S.

We will say that a monoid S is of *type* $FP_n^{(l)}$ *over* K if the module ${}_SK$ is of type FP_n and that S is of *type* $FP_n^{(r)}$ *over* K if K_S is of type FP_n. We define monoids of *types* $FP_\infty^{(l)}$, $FP_\infty^{(r)}$, $FP^{(l)}$, $FP^{(r)}$, $FL^{(l)}$ *and* $FL^{(r)}$ *over* K in a similar fashion. A resolution of type $FP_n^{(l)}$ of ${}_SK$ is called a *resolution of type* $FP_n^{(l)}$ *over* K *for* S, and similarly for $FP_n^{(r)}$, $FP_\infty^{(l)}$, etc. When $K = \mathbb{Z}$, we often omit mention of it.

The opposite map $opp : S \to S^{opp}; s \mapsto s$ extends to an anti-isomorphism $opp : KS \to KS^{opp}$. Lemma 1.28 then tells us, for example, that a right KS-module M is of type FP_n if, and only if, the left KS^{opp}-module M^{opp} is. Thus, since $K_S^{opp} = {}_{S^{opp}}K$, S is of type $FP_n^{(r)}$ if, and only if, S^{opp} is of type $FP_n^{(l)}$.

Results of Cohen [32] and of Guba and Pride [45] show that monoids can have very different properties on the left and right. For example, Cohen [32] gives an example of a monoid which is of type $FP_\infty^{(l)}$, but not of type $FP_1^{(r)}$.

However, when the monoid is a group G, there ceases to be a distinction between left and right: the map $g \mapsto g^{-1}$ $(g \in G)$ extends linearly to an anti-automorphism $inv : KG \to KG$, and so, since $(K_G)^{inv} = {}_GK$, a group is of type $FP_n^{(r)}$ over K if, and only if, it is of type $FP_n^{(l)}$ over K. We therefore say that a group G is of *type* FP_n*,* FP_∞*,* FP *or* FL *over* K if the left KG-module ${}_GK$ is.

For some rings K, a group of type FP over K need not be of type FL over K. Lee and Park in [55] give a family of groups which are of type FP over $\mathbb{Q}$ but not of type FL over $\mathbb{Q}$, and we will give a larger such family below. However, it remains an open question, Serre's question [76], whether a group of type FP (over $\mathbb{Z}$) must also be of type FL.

1.6.2 Homology and the partial directed Euler characteristic of a group

If M is a right C-module and M' a left C-module, then, for $i \geq 0$,

$$\mathrm{Tor}_i^C(M, M') = H_i(M \otimes_C \mathcal{F}') = H_i(\mathcal{F} \otimes_C M'),$$

where $\mathcal{F}$ is any projective resolution of M and $\mathcal{F}'$ is any projective resolution of M'. The *homology of a group* G with coefficients in a right KG-module M is

$$H_i(G, M) = \mathrm{Tor}_i^{KG}(M, {}_GK).$$

If $M = K_G$, we write $H_i^K(G)$ for $H_i(G, K_G)$. If $K = \mathbb{Z}$, we omit it, writing $H_i(G)$ for the *integral homology* of G. We write $H_*^K(G)$ for the totality of the homology groups $H_i^K(G)$ $(i \geq 0)$. It is sufficient, in the following sense, to consider only the integral homology of G:

Theorem 1.29 (Universal Coefficients Theorem, [47]). *For any coefficient ring K and for $n > 0$,*

$$H_n^K(G) \cong (K \otimes_{\mathbb{Z}} H_n(G)) \oplus Tor_1^{\mathbb{Z}}(K, H_{n-1}(G)).$$

For every group G,

$$H_0(G) = \mathbb{Z}$$

and

$$H_1(G) \cong G^{ab}.$$

The *cohomological dimension*, $\mathrm{cd}\, G$, of a group G is defined to be the length of the shortest projective resolution for G.

Lemma 1.30 ([22]). *A group G is of type FP if, and only if, it is of type FP_∞ and cdG is finite.*

Let G be a group of type FP_n (over $\mathbb{Z}$). In [82], Swan defines the *n-th directed partial Euler characteristic* of G to be

$$\chi_n(G) = \min\{\chi_n(\mathcal{F}) : \mathcal{F} \text{ is a resolution of type } FP_n \text{ for } G\}$$

and shows that it has a finite lower bound, since

$$d(H_n(G)) - \mathrm{rk}_{\mathbb{Z}}(H_{n-1}(G)) + \mathrm{rk}_{\mathbb{Z}}(H_{n-2}(G)) - \cdots \\ \cdots + (-1)^n \mathrm{rk}_{\mathbb{Z}}(H_0(G)) \leq \chi_n(G). \quad (1.11)$$

1.6.3 Efficiency and a partial resolution for groups

Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and $G = G(\mathcal{P})$. If $\mathbf{d}$ is a generating set of pictures for $\mathcal{P}$, let $\mathcal{T} = \langle \mathcal{P}; \mathbf{d} \rangle$.

If we compose

$$\oplus_{\mathbb{D}\in\mathbf{d}} \mathbb{Z}Ge_{\mathbb{D}} \xrightarrow{e_{\mathbb{D}} \mapsto [\mathbb{D}]} \pi_2(\mathcal{P}) \to 0$$

with the short exact sequences (1.2), (1.4) and (1.6), then we obtain a partial resolution $\mathcal{F}_{\mathcal{T}}$ of ${}_G\mathbb{Z}$ [25], [66]

$$\oplus_{\mathbb{D}\in\mathbf{d}} \mathbb{Z}Ge_{\mathbb{D}} \xrightarrow{\partial_3} \oplus_{R\in\mathbf{r}} \mathbb{Z}Ge_R \xrightarrow{\partial_2} \oplus_{x\in\mathbf{x}} \mathbb{Z}Ge_x \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} {}_G\mathbb{Z} \to 0, \quad (1.12)$$

where $\partial_0 = aug$,

$$\partial_1(e_x) = 1 - \overline{x} \quad (x \in \mathbf{x}),$$

$$\partial_2(e_R) = \sum_{x\in\mathbf{x}} \overline{\frac{\partial R}{\partial x}} e_x \quad (R \in \mathbf{r}),$$

$$\partial_3(e_{\mathbb{D}}) = \sum_{R\in\mathbf{r}} \frac{\partial \mathbb{D}}{\partial R} e_R \quad (\mathbb{D} \in \mathbf{d}).$$

The formulæ (1.9) and (1.10) then just say that $\partial_1\partial_2 = 0$ and $\partial_2\partial_3 = 0$.

Notice that $\chi_i(\mathcal{F}_\mathcal{T}) = \chi_i(\mathcal{T}) = \chi_i(\mathcal{P})$ for $i = 1, 2$.

Every group is thus of type FP_0. If a group can be finitely generated, then it is of type FP_1. In fact, a group is of type FP_1 if, and only if, it can be finitely generated (see [22], for example). If a group can be finitely presented, then it is of type FP_2. The converse of this last statement is, however, false; Bestvina and Brady [15] have found a group of type FP_2 which cannot be finitely presented.

There is an analogue of (1.12) for monoids [69].

By (1.11), if $\mathcal{P}$ is finite,

$$\begin{aligned}\chi_2(\mathcal{P}) &\geq d(H_2(G)) - \mathrm{rk}_\mathbb{Z}(H_1(G)) + \mathrm{rk}_\mathbb{Z}(H_0(G)) \\ &= d(H_2(G)) - \mathrm{rk}_\mathbb{Z}(G^{ab}) + 1.\end{aligned}$$

If $\mathcal{P}$ achieves this lower bound, then $\mathcal{P}$ is called an *efficient presentation* and G is said to be *efficient* [39].

If $\mathcal{P}$ is a finite presentation for G such that $\chi_2(\mathcal{P}) \leq \chi_2(\mathcal{Q})$ for all other finite presentations $\mathcal{Q}$ for G, then $\mathcal{P}$ is said to be a *minimal* presentation for G.

1.6.4 Resolutions for short exact sequences of modules

Let

$$0 \to M' \xrightarrow{\iota} M \xrightarrow{\alpha} M'' \to 0$$

be a short exact sequence of C-modules. In this section we show how, given free resolutions for any two of M', M, M'', we obtain a resolution for the third. In doing so, we prove the following result.

Lemma 1.31 ([16]). *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of C-modules.*

i) If M' and M'' are of type FP_n, then so is M.

ii) If M is of type FP_n $(n > 0)$ and M' is of type FP_{n-1}, then M'' is of type FP_n. If M is of type FP_0, then so is M''.

iii) If M is of type FP_n and M'' is of type FP_{n+1}, then M' is of type FP_n.

In what follows, we will make use of the so-called Snake Lemma.

Lemma 1.32 (Snake Lemma, [73]). *If*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

is a commuting diagram with exact rows, then there is an exact sequence

$$0 \to \ker\theta \to \ker\phi \to \ker\psi \to \mathit{coker}\theta \to \mathit{coker}\phi \to \mathit{coker}\psi \to 0.$$

The horseshoe construction

Let $\mathcal{F}' = (F_i', \partial_i')$, $\mathcal{F}'' = (F_i'', \partial_i'')$ be free resolutions of M', M'' respectively. We construct a free resolution of M (see [73, p187], for example, for fuller details).

First consider the "horseshoe"

$$\begin{array}{ccccccccc} & & F_0' & & & & F_0'' & & \\ & & \big\downarrow{\scriptstyle\partial_0'} & & & & \big\downarrow{\scriptstyle\partial_0''} & & \\ 0 & \longrightarrow & M' & \xrightarrow{\iota} & M & \xrightarrow{\alpha} & M'' & \longrightarrow & 0 \end{array}.$$

Since F_0'' is free and α is onto, there is a C-homomorphism $\phi_0 : F_0'' \to M$, such that $\alpha\phi_0 = \partial_0''$. Set $F_0 = F_0' \oplus F_0''$ and

$$\partial_0 : F_0 \to M; (f_0', f_0'') \mapsto \iota\partial_0(f_0') + \phi_0(f_0'').$$

This map is onto. Now, by the Snake Lemma, the bottom row of

$$\begin{array}{ccccccccc} & & F_1' & & & & F_1'' & & \\ & & \big\downarrow{\scriptstyle\partial_1'} & & & & \big\downarrow{\scriptstyle\partial_1''} & & \\ 0 & \longrightarrow & \ker\partial_0' & \longrightarrow & \ker\partial_0 & \longrightarrow & \ker\partial_0'' & \longrightarrow & 0 \end{array}$$

is exact. Proceeding as above, if we set $F_1 = F_1' \oplus F_1''$, we obtain a surjection $\partial_1 : F_1 \to \ker\partial_0$. Carrying on in this way, we construct a resolution $\mathcal{F} = (F_i, \partial_i)$ of M, where, for $n \geq 0$, $F_n = F_n' \oplus F_n''$ and

$$D_n(\mathcal{F}) = \begin{bmatrix} D_n(\mathcal{F}') & 0 \\ X_n & D_n(\mathcal{F}'') \end{bmatrix}$$

for some matrix X_n over C.

The mapping cylinder construction

Let $\mathcal{F}' = (F'_i, \partial'_i)$, $\mathcal{F} = (F_i, \partial_i)$ be free resolutions of M', M respectively. We construct a free resolution of M'' (see [76] for fuller details).

The map $\iota : M' \to M$ lifts to a chain map $\iota : \mathcal{F}' \to \mathcal{F}$. The *mapping cylinder* of this chain map is the complex $\mathcal{F}'' = (F''_i, \partial''_i)$, where

$$F''_i = \begin{cases} F_0 & i = 0 \\ F_i \oplus F'_{i-1} & i > 0 \end{cases},$$

and, for $f_i \in F_i$, $f'_i \in F'_i$,

$$\partial''_i(f_i, f'_{i-1}) = \begin{cases} \partial_1(f_1) + \iota_0(f'_0) & i = 1 \\ (\partial_i(f_i) + \iota_{i-1}(f'_{i-1}), -\partial'_{i-1}(f'_{i-1})) & i > 1 \end{cases}.$$

The complex $\mathcal{F}''$ is exact at F''_i for $i > 0$ and $\operatorname{coker} \partial''_1 \cong M''$, so $\mathcal{F}''$ becomes a free resolution of M'' via the map

$$\partial''_0 : F''_0 = F_0 \to M''; f_0 \to \alpha\partial_0(f_0).$$

If, for $n \geq 0$, X_n is the matrix of the map $\iota_n : F'_n \to F_n$, then

$$D_0(\mathcal{F}'') = \begin{bmatrix} D_0(\mathcal{F}) \\ X_0 \end{bmatrix}$$

and, for $n > 0$,

$$D_n(\mathcal{F}'') = \begin{bmatrix} D_n(\mathcal{F}) & 0 \\ X_n & -D_{n-1}(\mathcal{F}') \end{bmatrix}.$$

Proof of Lemma 1.31. (i) and (ii) follow from the above constructions.

(iii) If $\partial_0 : F_0 \to M$ is a surjection, with F_0 a free module of finite rank, then we have the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & F_0 & \xrightarrow{\text{Id}} & F_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \partial_0} & & \downarrow{\scriptstyle \alpha\partial_0} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \xrightarrow{\alpha} & M'' & \longrightarrow & 0 \end{array}$$

with exact rows. The Snake Lemma then gives the exact sequence

$$0 \to \ker \partial_0 \to \ker \alpha\partial_0 \xrightarrow{\alpha} M' \to 0. \tag{1.13}$$

Now, by Lemma 1.27, if M is of type FP_n ($n > 0$), then $\ker \partial_0$ is of type FP_{n-1} and if M'' is of type FP_{n+1}, then $\ker \alpha\partial_0$ is of type FP_n. Thus, by (ii), M' is of type FP_n. If M is of type FP_0 and M'' is of type FP_1, then $\ker \alpha\partial_0$ is of type FP_0, and so M' is of type FP_0 too. □

Remark. Part (iii) above is actually a slightly stronger result than that stated in [16].

1.6.5 Resolutions for extensions of groups

Let G be an extension of the normal subgroup H by its quotient G_0, so there is an exact sequence

$$1 \to H \to G \xrightarrow{\alpha} G_0 \to 1.$$

In his paper [85], Wall shows how, given a free resolution $\mathcal{Q} = (Q_i, \delta_i)$ for H and a free resolution $\mathcal{P} = (P_i, \varepsilon_i)$ for G_0, a free resolution $\mathcal{F} = (F_i, \partial_i)$ for G can be constructed.

Since H is a subgroup of G, $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module, and so $\mathbb{Z}G \otimes_{\mathbb{Z}H} \mathcal{Q}$ is a free resolution of the $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}H} {}_H\mathbb{Z} \cong \mathbb{Z}G_0$.

Let $r_i = \mathrm{rk}_{\mathbb{Z}G_0}(P_i)$. We define $\mathcal{C}_i$ to be the direct sum of r_i copies of $\mathbb{Z}G \otimes_{\mathbb{Z}H} \mathcal{Q}$, so $\mathcal{C}_i$ is a free resolution of r_i copies of $\mathbb{Z}G_0$, which we identify with P_i. If we let $C_{i,j}$ be the sum of r_i copies of $\mathbb{Z}G \otimes_{\mathbb{Z}H} Q_j$, then $\mathcal{C}_i = (C_{i,j}, \partial^{(0)}_{i,j})$, where $\partial^{(0)}_{i,j} : C_{i,j} \to C_{i,j-1}$ is the map induced by r_i copies of $\delta_j : Q_j \to Q_{j-1}$.

For $j \geq 0$, we let $F_j = \oplus_{i=0}^{j} C_{i,j-i}$.

By defining $\mathbb{Z}G$-homomorphisms $\partial^{(k)}_{i,j} : C_{i,j} \to C_{i-k,j+k-1}$ for each $k \geq 1$ and for each $i \geq k$, $j \geq 0$, we obtain maps $\partial_{j+1} = \sum_{i=0}^{j+1} \sum_{k=1}^{i} \partial^{(k)}_{i,j-i+1} : F_{j+1} \to F_j$.

For $k = 1$, let $\partial^{(1)}_{i,0}$ be a $\mathbb{Z}G$-homomorphism such that

$$\begin{array}{ccc} C_{i,0} & \xrightarrow{\partial^{(1)}_{i,0}} & C_{i-1,0} \\ {\scriptstyle \partial^{(0)}_{i,0}} \downarrow & & \downarrow {\scriptstyle \partial^{(0)}_{i-1,0}} \\ P_i & \xrightarrow{\varepsilon_i} & P_{i-1} \end{array}$$

commutes (such a map exists, since $C_{i,0}$ is free and $\partial^{(0)}_{i-1,0}$ is surjective). For $j > 0$, $\partial^{(1)}_{i,j}$ is defined by induction on j; if we know $\partial^{(1)}_{i,j-1}$, choose $\partial^{(1)}_{i,j}$ to satisfy

$$\partial^{(1)}_{i,j-1}\partial^{(0)}_{i,j} + \partial^{(0)}_{i-1,j}\partial^{(1)}_{i,j} = 0. \tag{1.14}$$

For $k > 1$, suppose that each $\partial_{i,j}^{(l)}$ is defined for $l < k$. The maps $\partial_{i,j}^{(k)}$ are defined by induction on j; for $j = 0$, choose $\partial_{i,0}^{(k)}$ such that

$$\partial_{i-1,0}^{(k-1)}\partial_{i,0}^{(1)} + \partial_{i-2,1}^{(k-2)}\partial_{i,0}^{(2)} + \cdots + \partial_{i-k+1,k-2}^{(1)}\partial_{i,0}^{(k-1)} + \partial_{i-k,k-1}^{(0)}\partial_{i,0}^{(k)} = 0. \tag{1.15}$$

For $j > 0$, if $\partial_{i,j-1}^{(k)}$ is defined, choose $\partial_{i,j}^{(k)}$ such that

$$\sum_{l=0}^{k} \partial_{i-l,j+l-1}^{(k-l)}\partial_{i,j}^{(l)} = 0. \tag{1.16}$$

The existence of such maps is proved in [85].

Theorem 1.33 ([85]). *$\mathcal{F} = (F_i, \partial_i)$ is a free resolution for G.*

Corollary 1.34. *If both H and G_0 are of type FP_n, then so is G.*

If we choose bases for each free module Q_i, then we can induce bases for the free modules $C_{i,j}$ and thus for each F_j. With respect to the bases for $C_{i,j}$, $C_{i-k,j+k-1}$ ($k \geq 0$) we can write the map $\partial_{i,j}^{(k)} : C_{i,j} \to C_{i-k,j+k-1}$ as an $r_i\mathrm{rk}_{\mathbb{Z}H}(Q_j) \times r_{i-k}\mathrm{rk}_{\mathbb{Z}H}(Q_{j+k-1})$ matrix $D_{i,j}^{(k)}$ over $\mathbb{Z}G$. With respect to the induced bases for F_j, F_{j-1}, the matrix $D_j(\mathcal{F})$ for ∂_{j+1} is of the form

$$\begin{bmatrix}
D_{0,j+1}^{(0)} & 0 & \cdots & & \cdots & 0 \\
D_{1,j}^{(1)} & D_{1,j}^{(0)} & 0 & \cdots & & \vdots \\
\vdots & \vdots & \ddots & & & \\
D_{i,j+1-i}^{(i)} & D_{i,j+1-i}^{(i-1)} & \cdots & D_{i,j+1-i}^{(0)} & 0 & \vdots \\
\vdots & \vdots & & & \ddots & 0 \\
D_{j,1}^{(j)} & D_{j,1}^{(j-1)} & \cdots & \cdots & D_{j,1}^{(1)} & D_{j,1}^{(0)} \\
D_{j+1,0}^{(j+1)} & D_{j+1,0}^{(j)} & \cdots & \cdots & D_{j+1,0}^{(2)} & D_{j+1,0}^{(1)}
\end{bmatrix}.$$

In these terms, the requirement that (1.14), (1.15) and (1.16) hold is equivalent to $D_{j+1}(\mathcal{F})D_j(\mathcal{F}) = 0$.

We note that, if $D_j(\mathcal{Q})$ is the matrix for $\delta_j : Q_j \to Q_{j-1}$ with respect to the choice of bases, then the matrix for $\partial_{i,j}^{(0)} : C_{i,j} \to C_{i,j-1}$ is the $r_i\mathrm{rk}_{\mathbb{Z}H}(Q_j) \times r_i\mathrm{rk}_{\mathbb{Z}H}(Q_{j-1})$ matrix $D_{i,j}^{(0)} = \mathrm{Diag}_{r_i}(D_j(\mathcal{Q}))$.

of bases, then the matrix for $\partial_{i,j}^{(0)} : C_{i,j} \to C_{i,j-1}$ is the $r_i\mathrm{rk}_{\mathbb{Z}H}(Q_j) \times r_i\mathrm{rk}_{\mathbb{Z}H}(Q_{j-1})$ matrix $D_{i,j}^{(0)} = \mathrm{Diag}_{r_i}(D_j(\mathcal{Q}))$.

Example 1.7. Let $H = \langle t \rangle$ be an infinite cyclic normal subgroup of G, generated by t, and suppose that G is a central extension of $\langle t \rangle$ by G_0 (so G_0 acts trivially on t). We choose the resolution $\mathcal{Q}$ for $\langle t \rangle$ to be

$$0 \to \mathbb{Z}\langle t \rangle \xrightarrow{1 \mapsto 1-t} \mathbb{Z}\langle t \rangle \xrightarrow{t \mapsto 1} {}_{\langle t \rangle}\mathbb{Z} \to 0.$$

If $\mathcal{P} = (P_i, \varepsilon_i)$ is a resolution for G_0, with $r_i = \mathrm{rk}_{\mathbb{Z}G_0}(P_i)$, Wall's method gives a resolution $\mathcal{F} = (F_i, \partial_i)$ for G with $\mathrm{rk}_{\mathbb{Z}G}(F_0) = r_0$ and $\mathrm{rk}_{\mathbb{Z}G}(F_j) = r_j + r_{j-1}$ $(j > 0)$. We find the maps $\partial_{i,1}^{(0)}$, $\partial_{i,0}^{(1)}$, $\partial_{i,1}^{(1)}$, $\partial_{i,0}^{(2)}$ (in that order) in terms of their matrices, all other maps being trivial.

As in the general case, $D_{i,1}^{(0)} = (1-t)I_{r_i}$ $(i \geq 0)$.

We now choose $\partial_{i,0}^{(1)}$ $(i > 0)$ such that $\partial_{i-1,0}^{(0)}\partial_{i,0}^{(1)} = \varepsilon_i \partial_{i,0}^{(0)}$. Let $\beta : G_0 \to G$ be a section (that is, β is a set map such that $\alpha\beta = \mathrm{Id}_{G_0}$) and extend it linearly to a map $\beta : \mathbb{Z}G_0 \to \mathbb{Z}G$. Since the map

$$\partial_{i,0}^{(0)} : (\mathbb{Z}G \otimes_{\mathbb{Z}\langle t \rangle} Q_0)^{r_i} \cong (\mathbb{Z}G)^{r_i} \to P_i \cong (\mathbb{Z}G_0)^{r_i}$$

is induced by (r_i copies of) α, we can choose $\partial_{i,0}^{(1)}$ such that $D_{i,0}^{(1)} = D_{i-1}(\mathcal{P})^\beta$.

For $\partial_{i,1}^{(1)}$, we require

$$D_{i,1}^{(0)} D_{i,0}^{(1)} + D_{i,1}^{(1)} D_{i-1,1}^{(0)} = 0.$$

But $D_{i,1}^{(0)} = (1-t)I_{r_i}$ and t is central, so $(1-t)D_{i,1}^{(1)} = -(1-t)D_{i-1}(\mathcal{P})^\beta$, and thus $D_{i,1}^{(1)} = -D_{i-1}(\mathcal{P})^\beta$.

Finally, we require matrices $D_{i,0}^{(2)}$ such that

$$D_{i,0}^{(1)} D_{i-1,0}^{(1)} + D_{i,0}^{(2)} D_{i-2,1}^{(0)} = 0.$$

Since $D_{i-1}(\mathcal{P})D_{i-2}(\mathcal{P}) = 0$, each entry of $D_{i,0}^{(1)}D_{i-1,0}^{(1)} = D_{i-1}(\mathcal{P})^\beta D_{i-2}(\mathcal{P})^\beta$ must be in $\ker \alpha = \mathbb{Z}G.(1-t)$. There is then an $r_i \times r_{i-2}$ matrix X_i over $\mathbb{Z}G$ such that $D_{i-1}(\mathcal{P})^\beta D_{i-2}(\mathcal{P})^\beta = X_i(1-t) = X_i D_{i-2,1}^{(0)}$, and we set $D_{i,0}^{(2)} = -X_i$. Thus, for $i \geq 1$,

$$D_i(\mathcal{F}) = \begin{bmatrix} -D_{i-1}(\mathcal{P})^\beta & (1-t)I_{r_i} \\ -X_{i+1} & D_i(\mathcal{P})^\beta \end{bmatrix}$$

and

$$D_0(\mathcal{F}) = \begin{bmatrix} (1-t)I_{r_0} \\ D_0(\mathcal{P})^\beta \end{bmatrix}. \qquad \Diamond$$

1.6.6 Resolutions for monoids with complete presentations

Let $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ be a monoid presentation. For $U, V \in [\mathbf{x}]$, we will write $U \to V$ if V can be obtained from U by a single elementary transformation (relative to $\mathbf{r}$), $U \xrightarrow{*} V$ if V can be obtained from U by a finite number of elementary transformations and $U \xleftrightarrow{*} V$ if U and V are equivalent relative to $\mathbf{r}$.

The presentation $\mathcal{P}$ is said to be *terminating* (or *Noetherian*) if there is no infinite sequence

$$W_1 \to W_2 \to W_3 \to \cdots .$$

The presentation $\mathcal{P}$ is said to be *confluent* if, for any $U, V_1, V_2 \in [\mathbf{x}]$ such that $U \xrightarrow{*} V_1$ and $U \xrightarrow{*} V_2$, there exists a word W such that $V_1 \xrightarrow{*} W$ and $V_2 \xrightarrow{*} W$. The presentation $\mathcal{P}$ is said to be *complete* if it is both terminating and confluent.

A word U on $\mathbf{x}$ is said to be *irreducible* (with respect to $\mathcal{P}$) if there is no word V such that $U \to V$. Otherwise, U is said to be *reducible.*

If $\mathcal{P}$ is complete, then for each word W on $\mathbf{x}$ there exists a unique irreducible word $\overline{W}$ such that $W \xrightarrow{*} \overline{W}$ (we can then identify the elements of $S(\mathcal{P})$ with the subset of $[\mathbf{x}]$ consisting of the irreducible words).

If a monoid presentation $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is terminating, then it is confluent if, and only if, the following two conditions are satisfied [35], [44], [60], [78]: for all $U_1, U_2, V, R_1^-, R_2^+ \in [\mathbf{x}]$,

(C1) if $(U_1V, R_1^-), (VU_2, R_2^-) \in \mathbf{r}$, then there is a word W on $\mathbf{x}$ such that $R_1^-U_2 \xrightarrow{*} W$ and $U_1R_2^- \xrightarrow{*} W$;

(C2) if $(U_1VU_2, R_1^-), (V, R_2^-) \in \mathbf{r}$, then there is a word W on $\mathbf{x}$ such that $R_1^- \xrightarrow{*} W$ and $U_1R_2^-U_2 \xrightarrow{*} W$.

Monoids which can be presented by a finite complete presentation are of types $FP_\infty^{(l)}$ and $FP_\infty^{(r)}$ [6], [23], [43], [54], [78]. We show how a resolution $\mathcal{F}^{(l)} = (F_i^{(l)}, \partial_i^{(l)})$ of type $FP_\infty^{(l)}$ and a resolution $\mathcal{F}^{(r)} = (F_i^{(r)}, \partial_i^{(r)})$ of type $FP_\infty^{(r)}$ for $S = S(\mathcal{P})$ can be obtained from a finite complete presentation $\mathcal{P}$.

Let $F_n^{(l)}$ be the free left $\mathbb{Z}S$-module with basis consisting of all ordered n-tuples $(W_1, \ldots, W_n)$ of irreducible words on $\mathbf{x}$ such that:

(L1) $W_n \in \mathbf{x}$;

(L2) $W_i W_{i+1}$ is reducible for $1 \leq i < n$;

(L3) any proper terminal subword of $W_i W_{i+1}$ is irreducible (i.e., there are no words U, V, with $U \neq 1$, $W_i = UV$ and VW_{i+1} reducible).

The second and third conditions imply that, for $1 \leq i < n$, W_i is an initial subword of R^+ for some $(R^+, R^-) \in \mathbf{r}$. Since $\mathbf{r}$ is finite, the set of all such n-tuples is finite, and so $F_n^{(l)}$ is of finite rank. The boundary map $\partial_n^{(l)} : F_n^{(l)} \to F_{n-1}^{(l)}$ is given by

$$\begin{aligned}\partial_n^{(l)}(W_1, \ldots, W_n) = \overline{W_1}(W_2, \ldots, W_n) - (\overline{W_1 W_2}, W_3, \ldots, W_n) + \cdots \\ \cdots + (-1)^i (W_1, W_2, \ldots, \overline{W_i W_{i+1}}, \ldots, W_n) + \cdots \\ \cdots + (-1)^n (W_1, W_2 \ldots, W_{n-2}, W_{n-1}).\end{aligned}$$

Now, it could be that one of the $(n-1)$-tuples in the above formula do not satisfy (L1)-(L3), and so is not a basis element of $F_{n-1}^{(l)}$. These, however, can be rewritten as a combination of basis elements, as described in [23]. Since we avoid this in the examples we encounter, we do not give details.

Similarly, the module $F_n^{(r)}$ is free on all ordered n-tuples $(W_1, \ldots, W_n)$ of irreducible words on $\mathbf{x}$ such that:

(R1) $W_1 \in \mathbf{x}$;

(R2) $W_i W_{i+1}$ is reducible for $1 \leq i < n$;

(R3) any proper initial subword of $W_i W_{i+1}$ is irreducible.

The boundary map $\partial_n^{(r)} : F_n^{(r)} \to F_{n-1}^{(r)}$ is given by

$$\begin{aligned}\partial_n^{(r)}(W_1, \ldots, W_n) = (W_2, \ldots, W_n) - (\overline{W_1 W_2}, W_3, \ldots, W_n) + \cdots \\ \cdots + (-1)^i (W_1, W_2, \ldots, \overline{W_i W_{i+1}}, \ldots, W_n) + \cdots \\ \cdots + (-1)^n (W_1, W_2 \ldots, W_{n-2}, W_{n-1})\overline{W_n}.\end{aligned}$$

Example 1.8. The presentation

$$\mathcal{P} = [x, \theta; (\theta x, \theta), (\theta\theta, \theta)]$$

is terminating, since the right-hand side of each relation is shorter than the left. To verify confluence, we note that there are two instances of (C1) (and none of (C2)), namely, the overlap of θx and $\theta\theta$ and the overlap of $\theta\theta$ with itself. Since

$$\theta(\theta x) \to \theta\theta \to \theta, \ (\theta\theta)x \to \theta x \to \theta$$

and

$$(\theta\theta)\theta \to \theta\theta, \ \theta(\theta\theta) \to \theta\theta,$$

the presentation is confluent. The monoid $S = S(\mathcal{P})$ is thus of type $FP_\infty^{(l)}$ and of type $FP_\infty^{(r)}$.

The left resolution $\mathcal{F}^{(l)}$ as above has $F_0^{(l)}$ free on the 0-tuple $e_0 = ()$ and $F_n^{(l)}$ $(n > 0)$ free on the n-tuples

$$e_n^x = (\theta, \ldots, \theta, x),$$
$$e_n^\theta = (\theta, \theta, \ldots, \theta).$$

The boundary maps are

$$\partial_1^{(l)}(e_1^x) = (x - 1)e_0,$$
$$\partial_1^{(l)}(e_1^\theta) = (\theta - 1)e_0$$

and, for $n > 1$,

$$\partial_n^{(l)}(e_n^x) = \theta e_{n-1}^x + \sum_{i=1}^{n-2}(-1)^i e_{n-1}^x + (-1)^{n-1}e_{n-1}^\theta + (-1)^n e_{n-1}^\theta$$
$$= \begin{cases} (\theta - 1)e_{n-1}^x & n \text{ odd} \\ \theta e_{n-1}^x & n \text{ even} \end{cases},$$
$$\partial_n^{(l)}(e_n^\theta) = \theta e_{n-1}^\theta + \sum_{i=1}^{n}(-1)^i e_{n-1}^\theta$$
$$= \begin{cases} (\theta - 1)e_{n-1}^\theta & n \text{ odd} \\ \theta e_{n-1}^\theta & n \text{ even} \end{cases}.$$

The right resolution $\mathcal{F}^{(r)}$ has $F_n^{(r)}$ free on the same basis as $F_n^{(l)}$ and, for $n > 0$,

$$\begin{aligned}
\partial_n^{(r)}(e_n^x) &= \sum_{i=0}^{n-2}(-1)^i e_{n-1}^x + (-1)^{n-1}e_{n-1}^\theta + (-1)^n e_{n-1}^\theta x \\
&= \begin{cases} e_{n-1}^\theta(1-x) & n \text{ odd} \\ e_{n-1}^x + e_{n-1}^\theta(x-1) & n \text{ even} \end{cases}, \\
\partial_n^{(r)}(e_n^\theta) &= \sum_{i=0}^{n-1}(-1)^i e_{n-1}^\theta + (-1)^n e_{n-1}^\theta \theta \\
&= \begin{cases} e_{n-1}^\theta(1-\theta) & n \text{ odd} \\ e_{n-1}^\theta \theta & n \text{ even} \end{cases}.
\end{aligned}$$

◊

1.7 Ranks and Euler characteristics

1.7.1 Ranks of projective modules

Throughout this section, P and Q are finitely generated projective modules.

There are a number of ways in which the rank of a finitely generated projective module can be defined in order to generalise the rank of a free module. They all have the following properties: for an arbitrary ring C, a *rank* is a function ρ on finitely generated projective C-modules which takes values in an additive abelian group with a distinguished element 1, such that

a) $\rho(P \oplus Q) = \rho(P) + \rho(Q)$;

b) $\rho(C) = 1$.

When $C = \mathbb{Z}G$ for some group G, $\mathbb{Z} \otimes_{\mathbb{Z}G} P$ is a finitely generated projective $\mathbb{Z}$-module. Since all projective $\mathbb{Z}$-modules are $\mathbb{Z}$-free [47], we define

$$\widetilde{\rho}_G(P) = \mathrm{rk}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P).$$

If G is finite, then P is finitely generated and projective as a $\mathbb{Z}$-module, and [81]

$$\widetilde{\rho}_G(P) = \frac{\mathrm{rk}_{\mathbb{Z}}(P)}{|G|}.$$

For an arbitrary ring C, P is a direct summand of a free C-module F of finite rank. There is then the projection $\pi_P : F \to F$, with $\mathrm{im}\,\pi_P = P$. If we choose a basis for F, let $D(\pi_P)$ be the matrix of π_P. The *Hattori-Stallings rank* of P, $\rho_C(P)$, is defined to be the image of the trace of $D(\pi_P)$ in the abelian group $C/[C,C]$ (where $[C,C]$ is the additive subgroup of C generated by $\{cc' - c'c : c, c' \in C\}$). So, if $D(\pi_P) = [c_{ij}]_{i,j}$, then

$$\rho_C(P) = \sum_i c_{ii} + [C,C].$$

The rank $\rho_C(P)$ depends neither on the choice of F, nor on the choice of basis for F [79].

A useful property of the Hattori-Stallings rank is:

Lemma 1.35 ([29]). *If $\alpha : C \to C_0$ is a ring homomorphism, then there is an induced homomorphism $\widetilde{\alpha} : C/[C,C] \to C_0/[C_0,C_0]$. If P is a finitely generated projective C-module, then $C_0 \otimes_C P$ is a finitely generated projective C_0-module and*

$$\widetilde{\alpha}(\rho_C(P)) = \rho_{C_0}(C_0 \otimes_C P).$$

When $C = KG$, a group ring, $[KG,KG]$ is the K-submodule generated by

$$\begin{aligned} gg' - g'g &= g(g'g)g^{-1} - g'g \\ &= gg''g^{-1} - g'', \end{aligned}$$

where $g, g' \in G$ and $g'' = g'g$. Thus, $KG/[KG,KG]$ is K-free on the conjugacy classes of G. We can therefore think of $\rho_{KG}(P)$ as a K-valued function on G, with finite support, which is constant on each conjugacy class. For $g \in G$, we denote the coefficient of the conjugacy class $[g]$ of g by $\rho_{KG}(P)(g)$.

In [12], Bass conjectured that, if K is a subring of $\mathbb{C}$, the complex numbers, intersecting $\mathbb{Q}$ only in $\mathbb{Z}$, then

$$\rho_{KG}(P)(g) = 0 \text{ if } g \neq 1.$$

This conjecture has been found to be true for a large class of infinite torsion-free groups [12], [56], [74], [80]. It is also true for abelian groups [22, Chapter IX] and finite groups [81]. In fact, if G is a finite group, then [11]

$$\rho_{\mathbb{Z}G}(P) = \widetilde{\rho}_G(P) + [\mathbb{Z}G, \mathbb{Z}G].$$

More generally, we can apply Lemma 1.35 to the augmentation map $aug : \mathbb{Z}G \to \mathbb{Z}$ for any group G.

Lemma 1.36. *If P is a finitely generated projective $\mathbb{Z}G$-module, then*

$$\widetilde{aug}(\rho_{\mathbb{Z}G}(P)) = \widetilde{\rho}_G(P).$$

Here, $\widetilde{aug}$ will be the map from $\mathbb{Z}G/[\mathbb{Z}G,\mathbb{Z}G]$ to $\mathbb{Z}/[\mathbb{Z},\mathbb{Z}] = \mathbb{Z}$, which sends $\sum_{[g]} n_{[g]}[g]$ to $\sum_{[g]} n_{[g]}$.

When G is abelian, in giving a formula for determining the Hattori-Stallings rank of a projective $\mathbb{Z}G$-module P from the entries of the matrix $D(\pi_P)$, it is shown in [22, §IX.3] (see also [5]) that, if a_i is the coefficient of t^i in the polynomial $\det(I+tD(\pi_P))$, then

$$\sum_{j\geq i}(-1)^{j-i}\binom{j}{i}a_j = \begin{cases} 1 & i = \rho_{\mathbb{Z}G}(P) \\ 0 & \text{otherwise} \end{cases}. \tag{1.17}$$

Note that, if $D(\pi_P) = [\xi_{ij}]_{i,j}$, then [2]

$$a_i = \sum_{j_1<j_2<\cdots<j_i} \det\left(\begin{bmatrix} \xi_{j_1j_1} & \xi_{j_2j_2} & \dots & \xi_{j_ij_i}\end{bmatrix}\right).$$

Whenever we have a well-defined rank function ρ for finitely generated projective C-modules, we can extend it to a rank function for C-modules of type FP as follows: if M is a C-module of type FP, let $\mathcal{P} = (P_i, \varepsilon_i)$ be a resolution of M of type FP. We define

$$\rho(M) = \sum_{i\geq 0}(-1)^i\rho(P_i),$$

although we must ensure that this does not depend on the choice of $\mathcal{P}$. This is the case for ρ_C and for $\widetilde{\rho}_G$, so we duly extend their definition. These ranks have the following property.

Lemma 1.37 ([12]). *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of C-modules of type FP, then*

$$\rho_C(M) = \rho_C(M') + \rho_C(M'').$$

If $C = \mathbb{Z}G$, then

$$\widetilde{\rho}_G(M) = \widetilde{\rho}_G(M') + \widetilde{\rho}_G(M'').$$

1.7.2 Euler characteristics of groups

An *Euler characteristic* is a function on a class $\mathcal{C}$ of groups, usually taking values in $\mathbb{Q}$ or $\mathbb{Z}$. For example, $\chi_n(G)$ is defined for all groups G of type FP_n (see §1.6.2) and takes values in $\mathbb{Z}$.

For a class of groups $\mathcal{C}$, we will say that a group is *virtually in* $\mathcal{C}$, $v\mathcal{C}$, if it has a subgroup of finite index in $\mathcal{C}$. For instance, we will say that a group is of type vFL if it has a subgroup of finite index of type FL.

Let $\mathcal{C}$ be a class of groups for which $H \in \mathcal{C}$ whenever H is a subgroup of finite index of a group $G \in \mathcal{C}$. Suppose that $\chi_{\mathcal{C}}$ is an Euler characteristic on $\mathcal{C}$ with the property that

$$\chi_{\mathcal{C}}(H) = [G : H]\chi_{\mathcal{C}}(G) \tag{1.18}$$

whenever $G \in \mathcal{C}$ and H is a subgroup of finite index $[G : H]$ in G. Then $\chi_{\mathcal{C}}$ can be extended to the class $v\mathcal{C}$ by

$$\chi_{v\mathcal{C}}(G) = \frac{1}{[G : H]}\chi_{\mathcal{C}}(H)$$

when G is virtually in $\mathcal{C}$ and H is a subgroup of G of finite index which is in $\mathcal{C}$. This is well-defined [84], since, if H' is another subgroup of G of finite index which is in $\mathcal{C}$, then, setting $H_0 = H \cap H'$,

$$\begin{aligned} \frac{\chi_{\mathcal{C}}(H)}{[G : H]} &= \frac{\chi_{\mathcal{C}}(H_0)/[H : H_0]}{[G : H]} \\ &= \frac{\chi_{\mathcal{C}}(H_0)}{[G : H_0]} \end{aligned}$$

and, similarly,

$$\frac{\chi_{\mathcal{C}}(H')}{[G : H']} = \frac{\chi_{\mathcal{C}}(H_0)}{[G : H_0]}.$$

Note that χ_n cannot be extended in this way.

For groups G of type FL, Serre [76] defines

$$\chi_{FL}(G) = \sum_{i \geq 0}(-1)^i \mathrm{rk}_{\mathbb{Z}G}(F_i) \in \mathbb{Z},$$

where (F_i, ∂_i) is a resolution of type FL for G. For large enough n,

$$\chi_{FL}(G) = (-1)^n \chi_n(G).$$

Since (1.18) holds here, this can be extended to an Euler characteristic χ_{vFL} for groups of type vFL, taking values in $\mathbb{Q}$. Brown showed in [21] that, if G is torsion-free and of type vFL, then $\chi_{vFL}(G) \in \mathbb{Z}$.

Stallings [79] defines for a group G of type FP over K

$$\chi_{FP(K)}(G) = \rho_{KG}({}_GK) \in KG/[KG, KG].$$

When $K = \mathbb{Z}$, we omit it.

If G is torsion-free and of type vFL (and so of type FP), then

$$\chi_{FP}(G) = \chi_{vFL}(G).$$

In [29], Chiswell defines, for a group G of type FP over K, $\widetilde{\chi}_{FP(K)}(G) \in K$ to be the sum of the coefficients in $\chi_{FP(K)}(G)$. That is, if $\chi_{FP(K)}(G) = \sum_{[g]} k_{[g]}[g]$, then $\widetilde{\chi}_{FP(K)}(G) = \sum_{[g]} k_{[g]}$.

The property (1.18) does not hold for $\widetilde{\chi}_{FP(K)}$. For example, finite groups are of type FP over $\mathbb{Q}$ and, for any finite group G, $\chi_{FP(\mathbb{Q})}(G) = \frac{1}{|G|}\sum_{g\in G}[g]$, and so $\widetilde{\chi}_{FP(\mathbb{Q})}(G) = 1$.

When $K = \mathbb{Z}$, Lemma 1.36 gives

$$\widetilde{\chi}_{FP}(G) = \widetilde{\rho}_G({}_G\mathbb{Z})$$

for a group G of type FP. When K is a pid, there is an alternative definition of $\widetilde{\chi}_{FP(K)}$, viz.:

Lemma 1.38 ([29]). *If G is of type FP over a pid K, then*

$$\widetilde{\chi}_{FP(K)}(G) = \sum_{i\geq 0}(-1)^i rk_K(H_i^K(G)).$$

If G is torsion-free and of type vFL, then

$$\widetilde{\chi}_{FP}(G) = \chi_{FP}(G) = \chi_{vFL}(G).$$

We will say that a group G is of *type FR* if its rational homology, $H_*^{\mathbb{Q}}(G)$, is of finite rank, that is, if each $H_i^{\mathbb{Q}}(G)$ is of finite rank and is trivial for large i. Brown [21] defines, for a group G of type FR,

$$\widetilde{\chi}_{FR}(G) = \sum_{i\geq 0}(-1)^i \mathrm{rk}_{\mathbb{Q}}(H_i^{\mathbb{Q}}(G)).$$

If G is of type FP, then it is of type FR and, by Lemma 1.38,

$$\widetilde{\chi}_{FR}(G) = \widetilde{\chi}_{FP}(G).$$

Thus, if G is torsion-free and of type vFL, then

$$\widetilde{\chi}_{FR}(G) = \widetilde{\chi}_{FP}(G) = \chi_{FP}(G) = \chi_{vFL}(G).$$

This Euler characteristic satisfies (1.18) only under certain conditions (see, for instance, [22, §IX.5]).

1.8 Graphs and some associated groups

1.8.1 Graphs

A *graph* $\Gamma = \mathbf{v} \cup \mathbf{e}$ consists of two disjoint sets, the *vertices*, $\mathbf{v}$, and the *edges*, $\mathbf{e}$, along with three functions

$$\iota : \mathbf{e} \to \mathbf{v},$$
$$\tau : \mathbf{e} \to \mathbf{v},$$
$$^{-1} : \mathbf{e} \to \mathbf{e},$$

called, respectively, the *initial*, *terminal* and *inverse* functions, which satisfy the conditions

$$\iota(e^{-1}) = \tau(e),$$
$$\tau(e^{-1}) = \iota(e),$$
$$e^{-1} \neq e,$$

for $e \in \mathbf{e}$.

We will say that Γ is finite if both the sets $\mathbf{v}$ and $\mathbf{e}$ are.

An *orientation* $\mathbf{e}^+$ of Γ is a choice of one edge from each of the pairs $\{e, e^{-1}\}$ for $e \in \mathbf{e}$.

Notation. Since $e^{-1} \in \mathbf{e}$ whenever $e \in \mathbf{e}$, we often omit mention of the edges of $\mathbf{e} \setminus \mathbf{e}^+$ when describing or drawing a graph.

An edge e of Γ is called a *loop* if $\tau(e) = \iota(e)$. We will say that Γ has *multiple edges* if there are distinct edges $e, e' \in \mathbf{e}$ such that $\iota(e) = \iota(e')$ and $\tau(e) = \tau(e')$. When Γ has no multiple edges, if $\iota(e) = u$, $\tau(e) = v$, then we identify the edge e with the ordered pair (u, v).

A *non-empty path* β in Γ is a finite sequence of edges

$$e_1, e_2, \ldots, e_n,$$

where, for $1 \leq i < n$, $\iota(e_{i+1}) = \tau(e_i)$. We write $\iota(\beta) = \iota(e_1)$ and $\tau(\beta) = \tau(e_n)$. If, for $1 \leq i < n$, $e_{i+1} \neq e_i^{-1}$, then we will say that β is *reduced*. For every $v \in \mathbf{v}$, there is also the *empty path*, 1_v, which has no edges and has $\iota(1_v) = \tau(1_v) = v$.

A non-empty path β is called a *cycle* if $\tau(\beta) = \iota(\beta)$. If β is reduced and if, in addition, $e_1 \neq e_n^{-1}$, then β is said to be a *reduced cycle*.

A *subgraph* Γ_0 of Γ consists of a subset $\mathbf{v}_0$ of the vertices of Γ together with a subset $\mathbf{e}_0$ of the edges of Γ such that, if $e \in \mathbf{e}_0$, then $\iota(e), \tau(e) \in \mathbf{v}_0$ and $e^{-1} \in \mathbf{e}_0$.

The graph Γ is said to be *connected* if, for every two vertices $u, v \in \mathbf{v}$, there is a path β with $\iota(\beta) = u$ and $\tau(\beta) = v$. The *connected components* of Γ are the largest connected subgraphs of Γ.

A graph is called a *forest* if it has no non-empty, reduced cycles. A *tree* is a connected forest. A *subforest* of Γ is a subgraph which is a forest. A connected subforest is a *subtree*. A *spanning subtree* of Γ is a subtree which includes every vertex of Γ. If $\Gamma = \mathbf{v} \cup \mathbf{e}$ is a finite tree, then $|\mathbf{e}^+| = |\mathbf{v}| - 1$.

An *extremal edge* of Γ is an edge $e \in \mathbf{e}$ such that either there is no other edge e' of Γ with $\iota(e') = \iota(e)$, or there is no other edge e' with $\tau(e') = \tau(e)$. Thus, if e is extremal, then so is e^{-1}. A finite forest with non-empty edge set must have an extremal edge.

1.8.2 Coxeter groups

Let $\Gamma = \mathbf{v} \cup \mathbf{e}$ be a finite, connected graph, without loops or multiple edges and let $\psi : \mathbf{e} \to \mathbb{Z}^+$ be a map assigning to each edge $e \in \mathbf{e}$ an integer $\psi(e) > 1$, such that $\psi(e^{-1}) = \psi(e)$. We call $\psi(e)$ the *weight* of e. Associated with the pair Γ, ψ and with a choice of orientation $\mathbf{e}^+$ is a finite group presentation.

$$\mathcal{P}_{\Gamma,\psi} = \langle \mathbf{v}; v^2 (v \in \mathbf{v}), (uv)^{\psi(u,v)}((u,v) \in \mathbf{e}^+)\rangle. \tag{1.19}$$

The group $C_{\Gamma,\psi} = G(\mathcal{P}_{\Gamma,\psi})$ defined by this presentation is called a *Coxeter group* and the pair Γ, ψ a *Coxeter system.* Notice that the choice of orientation does not affect the resulting group, the two presentations arising from two choices of orientation differing only by two Tietze transformations (T2).

Consider the graph obtained from Γ by removing all the edges with an even weight. Let $\Gamma_1, \dots, \Gamma_{n_\Gamma}$ be the connected components of this graph. The abelianisation of $C_{\Gamma,\psi}$ is then the direct product of n_Γ cyclic groups of order two, that is, $C^{ab}_{\Gamma,\psi} = \langle \widehat{v_1} \rangle \times \cdots \times \langle \widehat{v_{n_\Gamma}} \rangle \cong \mathbb{Z}_2^{n_\Gamma}$, where $\widehat{v_i}$ is the image of each generator of $C_{\Gamma,\psi}$ corresponding to a vertex of Γ_i.

If, for each $e \in \mathbf{e}$, $\psi(e)$ is odd, then we will say that Γ, ψ is an *odd Coxeter system* and that $C_{\Gamma,\psi}$ is an *odd Coxeter group.* The abelianisation of an odd Coxeter group is then a cyclic group of order 2.

A Coxeter system Γ, ψ is said to be *aspherical* if the subgroup generated by the images in $C_{\Gamma,\psi}$ of any three distinct vertices is infinite [3], [71]. This is equivalent to the condition that, for any three edges e_1, e_2, e_3 of Γ which form a triangle,

$$\frac{1}{\psi(e_1)} + \frac{1}{\psi(e_2)} + \frac{1}{\psi(e_3)} \leq 1.$$

The group $C_{\Gamma,\psi}$ is called an *aspherical Coxeter group.* For such groups, a generating set, $\mathbf{d}_{\Gamma,\psi}$, of pictures for $\mathcal{P}_{\Gamma,\psi}$ is given in [71]. It consists of a dipole $\mathbb{D}_{v^2}$ for each $v \in \mathbf{v}$ and for each $e = (u,v) \in \mathbf{e}^+$ a dipole $\mathbb{D}_{(uv)^{\psi(e)}}$ together with a picture $\mathbb{D}_e$, as illustrated in Figure 1.9, with two positively orientated discs with label $(uv)^{\psi(e)}$ and $\psi(e)$ negatively oriented discs for each label u^2 and v^2.

More generally, we could also consider Coxeter systems Γ, ψ, where Γ is not a connected graph. However, $C_{\Gamma,\psi}$ is then the free product of the Coxeter groups corresponding to the connected components of Γ. For simplicity, we will treat free products separately, and consider only the connected case.

1.8.3 Graphs of groups

A *graph of groups* consists of:

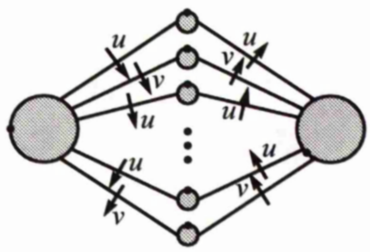

Figure 1.9: The picture $\mathbb{D}_e$ for $e = (u, v) \in \mathbf{e}^+$

1. a connected graph $\Gamma = \mathbf{v} \cup \mathbf{e}$, together with an orientation $\mathbf{e}^+$;
2. a group H_v for each $v \in \mathbf{v}$;
3. for each $e \in \mathbf{e}$, a subgroup H_e of $H_{\iota(e)}$; and
4. for each $e \in \mathbf{e}^+$, an isomorphism $\phi_e : H_e \to H_{e^{-1}}$.

The fundamental group G_Γ of this graph of groups is the quotient of the free product

$$(*_{v\in\mathbf{v}} H_v) * (*_{e\in\mathbf{e}^+}\langle t_e\rangle)$$

by the normal closure of the set

$$\{t_e^{-1} h t_e \phi_e(h)^{-1} : e \in \mathbf{e}^+, h \in H_e\} \cup \{t_e : e \in T \cap \mathbf{e}^+\},$$

where T is a choice of spanning subtree of Γ. A different choice of orientation or spanning subtree gives an isomorphic group. There are natural embeddings $\iota_\gamma : H_\gamma \to G_\Gamma$. See, for example, [36], [77]. There is also a more general concept of a graph of groups (see [10], [20]), which we do not consider here, but which admits the same treatment.

Two special cases of a graph of groups are free products with amalgamation and HNN extensions. If H is a subgroup of H_1 and if there is a monomorphism $\phi : H \to H_2$, then $G = H_1 *_H H_2$, the *free product of* H_1 *and* H_2 *amalgamating* H, arises as the graph of groups of the graph with two vertices $1, 2$ joined by an edge, with associated group H. An *HNN extension* $G = H_1 *_{H,\phi}$ arises as a graph of groups with one vertex, with associated group H_1, and a loop, with associated group $H \leq H_1$ and monomorphism $\phi : H \to H_1$.

Associated with a graph of groups G_Γ is a short exact sequence of $\mathbb{Z}G_\Gamma$-modules [30], [36]

$$0 \to \oplus_{e\in \mathbf{e}^+}\mathbb{Z}\uparrow_{H_e}^{G_\Gamma}\xrightarrow{\iota} \oplus_{v\in\mathbf{v}}\mathbb{Z}\uparrow_{H_v}^{G_\Gamma}\xrightarrow{\alpha} \mathbb{Z}\to 0. \qquad (1.20)$$

If we write $-\otimes_\gamma -$ for $-\otimes_{\mathbb{Z}H_\gamma}-$, then the maps are

$$\iota(1\otimes_e 1) = 1\otimes_{\iota(e)} 1 - t_e\otimes_{\tau(e)} 1$$

and

$$\alpha(1\otimes_v 1) = 1.$$

Applying Lemma 1.31(ii) to (1.20) gives:

Lemma 1.39 ([30]). *If Γ is a finite graph of groups and if each edge group is of type FP_{n-1} and each vertex group is of type FP_n, then G_Γ is of type FP_n. If each edge group and each vertex group is of type FL, then so is G_Γ.*

If $G = H_1 *_H H_2$ is a free product, as above, then this sequence becomes

$$0\to \mathbb{Z}\uparrow_H^G\xrightarrow{\iota}\mathbb{Z}\uparrow_{H_1}^G\oplus\mathbb{Z}\uparrow_{H_2}^G\xrightarrow{\alpha}\mathbb{Z}\to 0, \qquad (1.21)$$

where $\iota(1\otimes 1) = (1\otimes_1 1, -1\otimes_2 1)$ and $\alpha(1\otimes_1 1, 0) = 1 = \alpha(0, 1\otimes_2 1)$.

If $G = H_1 *_{H,\phi}$ is an HNN extension, as above, then 1.20 becomes

$$0\to \mathbb{Z}\uparrow_H^G\xrightarrow{\iota}\mathbb{Z}\uparrow_{H_1}^G\xrightarrow{\alpha}\mathbb{Z}\to 0, \qquad (1.22)$$

where $\iota(1\otimes 1) = (1-t)\otimes_1 1$ and $\alpha(1\otimes_1 1) = 1$.

Derived from (1.20) is a long exact Mayer-Vietoris sequence in homology [30], [47]

$$\begin{aligned}\cdots\to H_{n+1}(G_\Gamma)\to \oplus_{\mathbf{e}^+}H_n(H_e)\to\oplus_{\mathbf{v}}H_n(H_v)\to H_n(G_\Gamma)\to\cdots\\ \cdots\to H_2(G_\Gamma)\to\oplus_{\mathbf{e}^+}H_1(H_e)\to\oplus_{\mathbf{v}}H_1(H_v)\to H_1(G_\Gamma)\to\\ \oplus_{\mathbf{e}^+}\mathbb{Z}\to\oplus_{\mathbf{v}}\mathbb{Z}\to\mathbb{Z}\to 0.\end{aligned} \qquad (1.23)$$

Chapter 2

Alexander ideals: Group invariants from a presentation

Alexander [4] defined for knots an invariant which takes the form of a polynomial and is calculated from a matrix. This is known as the Alexander polynomial of a knot. Later, Fox [41] defined invariants for finitely generated groups, which are also calculated from a matrix obtained from a presentation. These take the form of an ascending chain of ideals in $\mathbb{Z}G^{ab}$, which are known as the Alexander ideals, and a sequence of Laurent polynomials, the Alexander polynomials. For a knot group, the Alexander polynomial of the knot occurs as the generator of one of these ideals and in the sequence of polynomials.

In this chapter, we review the definition of the Alexander ideals and polynomials. Note, however, that the chain of ideals and the sequence of polynomials defined here are indexed slightly differently to those of [41].

2.1 Definition of the A-ideals

Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ be a group presentation, as in (1.3), with a finite set of generators, $\mathbf{x}$, and let $G = G(\mathcal{P})$. Giving some order to the sets $\mathbf{x}$ and $\mathbf{r}$, let

$$D(\mathcal{P}) = \left[\overline{\frac{\partial R}{\partial x}}\right]_{\substack{R\in\mathbf{r}\\ x\in\mathbf{x}}},$$

an $|\mathbf{r}| \times |\mathbf{x}|$ matrix over $\mathbb{Z}G$. This matrix is called the *Jacobian matrix* of $\mathcal{P}$. When no confusion can arise, we abbreviate $D(\mathcal{P})$ to D.

Now choose an abelianising functor ^{T}. For $\lambda \in \mathbb{Z}$, we define the λ-th A^T*-ideal* of $\mathcal{P}$ to be

$$A_\lambda^T(\mathcal{P}) = J_{\chi_1(\mathcal{P})-\lambda}\left(D(\mathcal{P})^T\right).$$

The chain $A^T(\mathcal{P})$ of ideals $A_\lambda^T(\mathcal{P})$ $(\lambda \in \mathbb{Z})$ is then an ascending chain of ideals in the ring $\mathbb{Z}G^T$.

Throughout, if no ^{T} is specified, assume that it is ^{ab}. Thus, we denote $A^{ab}(\mathcal{P})$ simply by $A(\mathcal{P})$. We will write $\frac{\partial^T}{\partial x}$ for the composition of the maps

$$\mathbb{Z}\langle\mathbf{x}\rangle \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}\langle\mathbf{x}\rangle \xrightarrow{\gamma_{\mathcal{P}}} \mathbb{Z}G \xrightarrow{\tau_G^T} \mathbb{Z}G^T,$$

so

$$D(\mathcal{P})^T = \left[\frac{\partial^T R}{\partial x}\right]_{\substack{R\in\mathbf{r}\\ x\in\mathbf{x}}}.$$

The ideal $A_\lambda(\mathcal{P})$ is the $(\lambda+1)$-st Alexander ideal of $\mathcal{P}$ (see [4], [33], [41], [42], for example). The reason for our shift of indices should become apparent in Chapter 4.

By Lemma 1.24, the ideals are independent of the choice of order given to $\mathbf{x}$ and $\mathbf{r}$. In addition, they have the following invariance property.

Theorem 2.1 **([4], [33]).** *Let $\mathcal{P}$, $\mathcal{P}_0$ be two group presentations with finite generating sets. If there is an isomorphism $\alpha : G(\mathcal{P}) \to G(\mathcal{P}_0)$, then the induced isomorphism $\alpha^T : \mathbb{Z}G(\mathcal{P})^T \to \mathbb{Z}G(\mathcal{P}_0)^T$ carries $A_\lambda^T(\mathcal{P})$ onto $A_\lambda^T(\mathcal{P}_0)$ for each $\lambda \in \mathbb{Z}$. So, if $G(\mathcal{P}) \cong G(\mathcal{P}_0)$, then $A^T(\mathcal{P}) \cong^{(0)} A^T(\mathcal{P}_0)$.*

Proof. By Lemma 1.3, the presentations $\mathcal{P}$ and $\mathcal{P}_0$ are Tietze equivalent and α is induced by this equivalence. Indeed, we may transform $\mathcal{P}$ to $\mathcal{P}_0$ by a finite number of transformations (T2) and *finitary* transformations (T1). Thus, to prove the theorem, we need only consider the case when $\mathcal{P}_0$ is obtained from $\mathcal{P}$ by a single Tietze transformation and α is the induced isomorphism.

Suppose that $\mathcal{P}_0 = \langle\mathbf{x}, \mathbf{y}; \mathbf{r}, y = W_y (y \in \mathbf{y})\rangle$ is obtained from $\mathcal{P} = \langle\mathbf{x}; \mathbf{r}\rangle$ by a finitary Tietze transformation of type (T1), so $\mathbf{y}$ is a finite set. Here $\alpha = \alpha_{\mathcal{P}}$, so, for

$x \in \mathbf{x}$, $\alpha\gamma_{\mathcal{P}}(x) = \gamma_{\mathcal{P}_0}(x)$. Giving orderings to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{r}$, we obtain

$$D(\mathcal{P}_0) = \begin{bmatrix} \left[\gamma_{\mathcal{P}_0}\left(\frac{\partial R}{\partial x}\right)\right]_{\substack{R\in\mathbf{r}\\x\in\mathbf{x}}} & 0 \\ \left[-\gamma_{\mathcal{P}_0}\left(\frac{\partial W_y}{\partial x}\right)\right]_{\substack{y\in\mathbf{y}\\x\in\mathbf{x}}} & I_{|\mathbf{y}|} \end{bmatrix} = \begin{bmatrix} D(\mathcal{P})^{\alpha} & 0 \\ X & I_{|\mathbf{y}|} \end{bmatrix},$$

where $X = \left[-\alpha\gamma_{\mathcal{P}}\left(\frac{\partial W_y}{\partial x}\right)\right]_{\mathbf{y},\mathbf{x}}$. So

$$\begin{aligned} J_\kappa(D(\mathcal{P}_0)^T) &= J_\kappa\left(\begin{bmatrix} D(\mathcal{P})^{\tau^T_{G_0}\alpha} & 0 \\ X^T & I_{|\mathbf{y}|} \end{bmatrix}\right) \\ &= J_{\kappa-|\mathbf{y}|}(D(\mathcal{P})^{\alpha^T\tau^T_G}) \\ &= \alpha^T J_{\kappa-|\mathbf{y}|}(D(\mathcal{P})^T), \end{aligned}$$

since $\tau^T_{G_0}\alpha = \alpha^T\tau^T_G$ and α^T is onto. Thus, since $\chi_1(\mathcal{P}_0) = \chi_1(\mathcal{P}) + |\mathbf{y}|$,

$$\begin{aligned} A^T_\lambda(\mathcal{P}_0) &= J_{\chi_1(\mathcal{P}_0)-\lambda}(D(\mathcal{P}_0)^T) \\ &= \alpha^T J_{\chi_1(\mathcal{P})-\lambda}(D(\mathcal{P})^T) \\ &= \alpha^T A^T_\lambda(\mathcal{P}). \end{aligned}$$

Now suppose that $\mathcal{P}_0 = \langle \mathbf{x}; \mathbf{r}, \mathbf{s}\rangle$ is obtained from $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}\rangle$ by a Tietze transformation of type (T2). Since each $S \in \mathbf{s}$ is a consequence of $\mathbf{r}$, by Lemma 1.11 the last $|\mathbf{s}|$ rows of

$$D(\mathcal{P}_0) = \begin{bmatrix} \left[\overline{\frac{\partial R}{\partial x}}\right]_{\substack{R\in\mathbf{r}\\x\in\mathbf{x}}} \\ \left[\overline{\frac{\partial S}{\partial x}}\right]_{\substack{S\in\mathbf{s}\\x\in\mathbf{x}}} \end{bmatrix}$$

are a linear combination of the first $|\mathbf{r}|$. Hence, by Proposition 1.21(i) and since $\chi_1(\mathcal{P}_0) = \chi_1(\mathcal{P})$,

$$\begin{aligned} A^T_\lambda(\mathcal{P}) &= J_{\chi_1(\mathcal{P}_0)-\lambda}(D(\mathcal{P}_0)^T) \\ &= J_{\chi_1(\mathcal{P})-\lambda}(D(\mathcal{P})^T) \\ &= A^T_\lambda(\mathcal{P}), \end{aligned}$$

and the result follows, α^T being the identity map here. □

Remarks. 1. In proving this theorem, we only require the property that $\tau_{G_0}^T \alpha = \alpha^T \tau_G^T$ when α is an *isomorphism.* It is when we come to compare the A^T-ideals of non-isomorphic groups that we require that this holds for all homomorphisms.

2. The second part of the above proof actually shows that, if $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and if $\mathbf{s}$ is an alternative set of normal generators for $\ll \mathbf{r} \gg = \ker \gamma_{\mathcal{P}}$, then $A^T(\mathcal{P}) = A^T(\mathcal{P}_0)$, where $\mathcal{P}_0 = \langle \mathbf{x}; \mathbf{s} \rangle$. Thus, for a given finite generating set $\mathbf{x}$, any choice of defining relators gives exactly the same chain of A-ideals.

We give an example of how the A-ideals may be used to distinguish groups.

Example 2.1. For $l = 1, 2, \ldots$, let

$$\mathcal{P}_l = \langle a, b, t; a^3, b^3, (ab)^7, t^l a t^{-l} a^{-1} \rangle$$

and let $G_l = G(\mathcal{P}_l)$. Then, for each l,

$$G_l^{ab} = \langle a \rangle \times \langle t \rangle \cong \mathbb{Z}_3 \times \mathbb{Z},$$

where $b = a^2$. The Jacobian matrix of $\mathcal{P}_l$ is

$$D = \begin{bmatrix} 1+a+a^2 & 0 & 0 \\ 0 & 1+b+b^2 & 0 \\ \sum_{i=0}^{6}(ab)^i & \sum_{i=0}^{6}(ab)^i a & 0 \\ t^l - 1 & 0 & \sum_{i=0}^{l-1} t^i (1-a) \end{bmatrix},$$

so

$$D^{ab} = \begin{bmatrix} 1+a+a^2 & 0 & 0 \\ 0 & 1+a^2+a & 0 \\ 7 & 7a & 0 \\ t^l - 1 & 0 & \sum_{i=0}^{l-1} t^i (1-a) \end{bmatrix}.$$

From this we obtain

$$J_\kappa(D^{ab}) = \begin{cases} 0 & \kappa > 2 \\ \big((1+a+a^2)^2, 7(1+a+a^2), (1+a+a^2)(t^l-1), & \\ \quad 7(t^l-1), 7\sum_{i=0}^{l-1} t^i(1-a)\big) & \kappa = 2 \\ \left(1+a+a^2, 7, t^l - 1, \sum_{i=0}^{l-1} t^i (1-a)\right) & \kappa = 1 \\ \mathbb{Z}(\langle a \rangle \times \langle t \rangle) & \kappa \leq 0 \end{cases},$$

which gives (after some simplification)

$$A_\lambda(\mathcal{P}_l) = \begin{cases} \mathbb{Z}(\langle a\rangle \times \langle t\rangle) & \lambda \geq 2 \\ \left(1+a+a^2, 7, \sum_{i=0}^{l-1} t^i\right) & \lambda = 1 \\ \left(1+a+a^2, 7(t^l-1), 7\sum_{i=0}^{l-1} t^i(1-a)\right) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

It proves more convenient to consider

$$A_\lambda^{tf}(\mathcal{P}_l) = \begin{cases} \mathbb{Z}\langle t\rangle & \lambda \geq 1 \\ (3, t^l - 1) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

Suppose $l \neq l'$. If $G_l \cong G_{l'}$, then there is an automorphism of $\mathbb{Z}G_l^{tf} = \mathbb{Z}G_{l'}^{tf} = \mathbb{Z}\langle t\rangle$, induced by an automorphism of $\langle t\rangle$, which carries $A_\lambda^{tf}(\mathcal{P}_l)$ to $A_\lambda^{tf}(\mathcal{P}_{l'})$ for each λ. However, neither of the possible automorphisms

$$\mathbb{Z}\langle t\rangle \to \mathbb{Z}\langle t\rangle; t \mapsto t^{\pm 1}$$

carries $A_0^{tf}(\mathcal{P}_l) = (3, t^l - 1)$ to $A_0^{tf}(\mathcal{P}_{l'}) = (3, t^{l'} - 1)$, so $G_l \ncong G_{l'}$. ◊

When $^T = {}^{tf}$, the ring $\mathbb{Z}G^{tf}$ is a ring of Laurent polynomials which, in particular, is a greatest common divisor (gcd) domain (that is, any set of elements of $\mathbb{Z}G^{tf}$ has a gcd [33]). Thus, every ideal of $\mathbb{Z}G^{tf}$ is contained in a smallest principal ideal. For $\lambda \in \mathbb{Z}$, we define the λ-th *a-polynomial* of $\mathcal{P}$, $a_\lambda(\mathcal{P})$, to be a generator of the smallest principal ideal containing $A_\lambda^{tf}(\mathcal{P})$. Such a generator is unique up to multiplication by an element of the unit group $\pm G^{tf}$ of $\mathbb{Z}G^{tf}$ and, since $A_\lambda^{tf}(\mathcal{P}) \subseteq A_{\lambda+1}^{tf}(\mathcal{P})$, $a_{\lambda+1}(\mathcal{P}) | a_\lambda(\mathcal{P})$. If $t_1, \ldots, t_n$ is a set of free generators for G^{tf}, then the a-polynomials are Laurent polynomials $a_\lambda(\mathcal{P})(t_1, \ldots, t_n)$ on $t_1, \ldots, t_n$. The polynomial $a_\lambda(\mathcal{P})$ is the $(\lambda+1)$-st Alexander polynomial of $\mathcal{P}$ ([4], [33], [42]).

That these polynomials are well-defined is a consequence of Theorem 2.1.

Corollary 2.2. *Let $\mathcal{P}$, $\mathcal{P}_0$, α be as in Theorem 2.1. For each $\lambda \in \mathbb{Z}$, $a_\lambda(\mathcal{P}_0)$ may be chosen such that*

$$\alpha^{tf}(a_\lambda(\mathcal{P})) = a_\lambda(\mathcal{P}_0).$$

A common use of Alexander ideals and polynomials has been to distinguish (tame) knots. The group of a knot in 3-dimensional space $\mathbb{R}^3$ is the fundamental group of its complement in $\mathbb{R}^3$. There are a number of ways of obtaining a presentation of the group of a knot, one of which, using a projection onto $\mathbb{R}^2$, gives a *Wirtinger presentation* [33] which has the same (finite) number of relators as generators. In fact, each of the relators of such a presentation is a consequence of the others, and so we can assume that a knot group is given by a presentation

$$\mathcal{Q} = \langle \mathbf{y}; \mathbf{s} \rangle \tag{2.1}$$

with $|\mathbf{s}| = |\mathbf{y}| - 1$. We call such a presentation a *deleted Wirtinger presentation.* These presentations have the additional property that $G(\mathcal{Q})^{ab}$ is infinite cyclic with a generator t which is the image of each of the generators $\mathbf{y}$.

The following result applies not only to knot groups.

Lemma 2.3. *If* $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$, ^{T} *are such that* $G(\mathcal{P})^T = \langle t \rangle \cong \mathbb{Z}$ *and* t *is the image of each* $x \in \mathbf{x}$, *then*

$$\sum_{x \in \mathbf{x}} \frac{\partial^T R}{\partial x} = 0,$$

for each $R \in \mathbf{r}$. *The columns of* $D(\mathcal{P})^T$ *thus sum to zero.*

Proof. By Corollary 1.9, $R - 1 = \sum_{x \in \mathbf{x}} \frac{\partial R}{\partial x}(x-1)$, so

$$\sum_{x \in \mathbf{x}} \frac{\partial^T R}{\partial x}(t-1) = 0.$$

But $\mathbb{Z}\langle t \rangle$ is an integral domain, and the result follows. □

So, if $\mathcal{Q}$ is a presentation of a knot group as above, $D(\mathcal{Q})^{ab}$ is a $(|\mathbf{y}| - 1) \times |\mathbf{y}|$ matrix over $\mathbb{Z}\langle t \rangle$ whose columns sum to zero. Thus $A_{-1}(\mathcal{Q}) = 0$ and $A_0(\mathcal{Q})$ is a principal ideal, generated by the determinant of any $(|\mathbf{y}| - 1) \times (|\mathbf{y}| - 1)$ submatrix of $D(\mathcal{Q})^{ab}$. This determinant can then be taken to be $a_0(\mathcal{Q})$. This polynomial, which is relatively easy to obtain, was the original invariant polynomial defined by Alexander [4] and is often called *the* Alexander polynomial of the knot.

The Alexander polynomial of a knot has some interesting properties:

Proposition 2.4 ([33]). *If* $a_0(\mathcal{Q})(t)$ *is the Alexander polynomial of a knot as above, then*

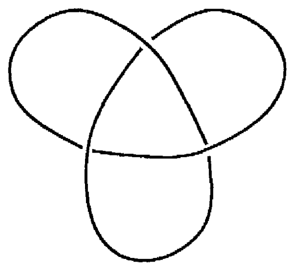

Figure 2.1: The clover-leaf knot

i) it is of even degree;

ii) it is reciprocal, that is, $a_0(\mathcal{Q})(t^{-1}) = t^n a_0(\mathcal{P})(t)$ *for some* $n \in \mathbb{Z}$*;*

iii) $a_0(\mathcal{Q})(1) = \pm 1$.

We show how (iii) can be proved in §4.4. Properties (ii) and (iii) actually characterise the Alexander polynomials of knots [33].

Example 2.2. a) The clover-leaf knot, as illustrated in Figure 2.1, has Wirtinger presentation

$$\langle x, y, z; x = yzy^{-1}, y = zxz^{-1}, z = xyx^{-1} \rangle.$$

Using Tietze transformations, this simplifies to

$$\mathcal{P} = \langle x, y; xyx = yxy \rangle.$$

The matrix

$$D^{ab} = \begin{bmatrix} 1 - t + t^2 & -1 + t - t^2 \end{bmatrix}$$

then gives $a_0(\mathcal{P}) = 1 - t + t^2$.

b) The figure-eight knot, as illustrated in Figure 2.2, has Wirtinger presentation

$$\langle x, y, z, w; x = z^{-1}wz, y = wxw^{-1}, z = x^{-1}yx, w = yzy^{-1} \rangle,$$

which simplifies to

$$\mathcal{P} = \langle x, y; yx^{-1}yxy^{-1} = x^{-1}yxy^{-1}x \rangle.$$

This gives

$$D^{ab} = \begin{bmatrix} t - 3 + t^{-1} & -t + 3 - t^{-1} \end{bmatrix}$$

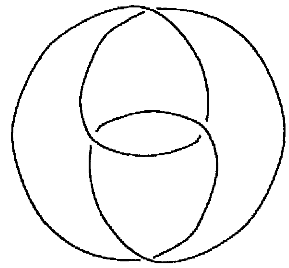

Figure 2.2: The figure-eight knot

and $a_0(\mathcal{P}) = t^2 - 3t + 1$.

Since no automorphism of $\mathbb{Z}\langle t\rangle$ induced by an automorphism of $\langle t\rangle$ takes the Alexander polynomial of the clover-leaf knot to that of the figure-eight knot, their respective knot groups are non-isomorphic. Hence, the two knots are distinct. $\diamond$

We now consider some circumstances where the A-ideals are unable to distinguish groups.

Lemma 2.5. *If G is a finitely generated perfect group, so $G^{ab} = 1$, then*

$$A_\lambda(\mathcal{P}) = \begin{cases} \mathbb{Z} & \lambda \geq -1 \\ 0 & \lambda < -1 \end{cases}$$

for every presentation $\mathcal{P}$ for G.

Proof. Every finitely generated perfect group G has a presentation of the form

$$\mathcal{P} = \langle x_1, \ldots, x_n; x_1R_1, \ldots, x_nR_n, R_{n+1}, \ldots\rangle,$$

where $\exp_{x_i}(R_j) = 0$ for each i, j, called a *preabelian presentation* [58]. Since $G^{ab} = G^{triv} = 1$,

$$D(\mathcal{P})^{ab} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

Thus

$$A_\lambda(\mathcal{P}) = J_{n-1-\lambda}(I_n) = \begin{cases} \mathbb{Z} & \lambda \geq -1 \\ 0 & \lambda < -1 \end{cases}.$$

If $\mathcal{P}_0$ is another presentation for G, then, since the only automorphism of $\mathbb{Z}$ induced by an isomorphism $G(\mathcal{P})^{triv} \to G(\mathcal{P}_0)^{triv}$ is the identity, $A(\mathcal{P}_0) = A(\mathcal{P})$. $\square$

The A-ideals, therefore, cannot distinguish perfect groups. Also, consider the following example.

Example 2.3. For $l = 1, 2, \ldots$, let

$$\mathcal{P}_l = \langle a, b, t; a^3, b^2, (ab)^7, t^l a t^{-l} a^{-1} \rangle$$

and let $G_l = G(\mathcal{P}_l)$. Notice that these presentations differ from those in Example 2.1 only by the power of b and that the subgroup generated by a and b is perfect.

For each l,

$$G_l^{ab} = \langle t \rangle \cong \mathbb{Z},$$

where $b = a = 1$. Now,

$$D(\mathcal{P}_l)^{ab} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 7 & 7 & 0 \\ t^l - 1 & 0 & 0 \end{bmatrix},$$

and so

$$A_\lambda(\mathcal{P}_l) = \begin{cases} \mathbb{Z}\langle t \rangle & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}.$$

for each l. ◊

This example is illustrative of the following situation: let $\mathcal{Q} = \langle \mathbf{y}; \mathbf{s} \rangle$ be a presentation for a perfect group $H = G(\mathcal{Q})$, let $\mathbf{x}$ be a set disjoint from H and choose a set $\mathbf{r}$ of cyclically reduced elements of $H * \langle \mathbf{x} \rangle \setminus H$. We set

$$G = (H * \langle \mathbf{x} \rangle) / \ll \mathbf{r} \gg.$$

A presentation for G is then given by

$$\mathcal{P} = \langle \mathbf{x}, \mathbf{y}; \widetilde{\mathbf{r}}, \mathbf{s} \rangle, \tag{2.2}$$

where $\widetilde{\mathbf{r}} = \{\widetilde{R} : R \in \mathbf{r}\} \subseteq \langle \mathbf{x}, \mathbf{y} \rangle$ is obtained by replacing each term from H of each $R \in \mathbf{r}$ by a word on $\mathbf{y} \cup \mathbf{y}^{-1}$ representing it.

Also, for each $R \in \mathbf{r}$, let R_0 be the word on $\mathbf{x} \cup \mathbf{x}^{-1}$ obtained from R by deleting each occurrence of a term from H and cyclically reducing. We set

$$\mathcal{P}_0 = \langle \mathbf{x}; R_0 (R \in \mathbf{r}) \rangle$$

and $G_0 = G(\mathcal{P}_0)$. Thus $G_0 \cong G/\ll \iota H \gg$, where $\iota : H \to G$ is the natural map $y \mapsto y$ ($y \in \mathbf{y}$).

If $\alpha : G \to G_0$ is the natural surjection, then

$$\alpha^{ab} : G^{ab} \to G_0^{ab}; gG' \mapsto \alpha(g)G_0'$$

is an isomorphism, since

$$\begin{aligned} G_0^{ab} &= G_0/G_0' \\ &\cong (G/\ll \iota H \gg)/(G/\ll \iota H \gg)' \\ &= (G/\ll \iota H \gg)/(G'/\ll \iota H \gg) \\ &\cong G/G' \\ &= G^{ab}. \end{aligned}$$

Theorem 2.6. *For $\mathcal{P}$, $\mathcal{P}_0$, α as above, with* $\mathbf{x}$, $\mathbf{y}$ *finite,*

$$A(\mathcal{P}_0) = \alpha^{ab} A(\mathcal{P}).$$

Proof. Note that $\widehat{y} = 1$ for each $y \in \mathbf{y}$, that $\alpha(\overline{x}) = \overline{x}$ for each $x \in \mathbf{x}$ and that, if $\widetilde{R} = \prod_i U_i V_i$, where each $U_i \in \langle \mathbf{x} \rangle$ and each $V_i \in \langle \mathbf{y} \rangle$, then

$$\begin{aligned} \frac{\partial^{ab} \widetilde{R}}{\partial x} &= \sum_i \prod_{j=1}^{i-1} \widehat{U_j} \widehat{V_j} \frac{\partial^{ab} U_i}{\partial x} \\ &= \sum_i \prod_{j=1}^{i-1} \widehat{U_j} \frac{\partial^{ab} U_i}{\partial x} \\ &= \frac{\partial^{ab} R_0}{\partial x}, \end{aligned}$$

so $\alpha^{ab}\left(\frac{\partial^{ab}\widetilde{R}}{\partial x}\right) = \frac{\partial^{ab} R_0}{\partial x}$. Thus

$$(D(\mathcal{P})^{ab})^{\alpha^{ab}} = \begin{bmatrix} \left[\exp_y(S)\right]_{\substack{S \in \mathbf{s} \\ y \in \mathbf{y}}} & 0 \\ \left[\alpha^{ab}\left(\frac{\partial^{ab}\widetilde{R}}{\partial y}\right)\right]_{\substack{R \in \mathbf{r} \\ y \in \mathbf{y}}} & D(\mathcal{P}_0)^{ab} \end{bmatrix}.$$

Now, by elementary row and column operations and by discarding any resulting rows of zeros, the matrix $\left[\exp_y(S)\right]$ can be transformed to the identity matrix $I_{|\mathbf{y}|}$.

Applying these operations to the first $|\mathbf{s}|$ rows and first $|\mathbf{y}|$ columns of the above matrix, we obtain the matrix

$$\begin{bmatrix} I_{|\mathbf{y}|} & 0 \\ X & D(\mathcal{P}_0)^{ab} \end{bmatrix}$$

for some matrix X over $\mathbb{Z}G_0^{ab}$. Thus, by Corollary 1.25,

$$\begin{aligned} \alpha^{ab} A_\lambda(\mathcal{P}) &= J_{\chi_1(\mathcal{P})-\lambda}((D(\mathcal{P})^{ab})^{\alpha^{ab}}) \\ &= J_{|\mathbf{x}|+|\mathbf{y}|-1-\lambda}\left(\begin{bmatrix} I_{|\mathbf{y}|} & 0 \\ X & D(\mathcal{P}_0)^{ab} \end{bmatrix}\right) \\ &= J_{\chi_1(\mathcal{P}_0)-\lambda}(D(\mathcal{P}_0)^{ab}) \\ &= A_\lambda(\mathcal{P}_0), \end{aligned}$$

as required. □

So, if we have two presentations $\mathcal{P}$, $\widehat{\mathcal{P}}$ arising from perfect groups H, $\widehat{H}$ respectively in the above manner, then, if the A-ideals cannot distinguish $G(\mathcal{P}_0)$ from $G(\widehat{\mathcal{P}}_0)$, they cannot distinguish $G(\mathcal{P})$ from $G(\widehat{\mathcal{P}})$. In the next chapter we define new group invariants which are (sometimes) able to do this. Before doing so, we look at some further computations of A-ideals.

2.2 Some examples

2.2.1 The A-ideals of the braid groups

For $n \geq 3$, the *braid group on n strings*, B_n, is given by the presentation

$$\begin{aligned} \mathcal{B}_n = \langle a_1, \ldots, a_{n-1};\, & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} (1 \leq i < n-1), \\ & a_i a_j = a_j a_i (1 \leq i < n-2, i+1 < j \leq n-1)\rangle. \end{aligned}$$

A full exposition of the properties of braid groups is given in [17]. In particular, the abelianisation of each braid group is infinite cyclic, with generator t the image of each a_i.

We calculate the A-ideals of the braid groups and show that:

Theorem 2.7. *For $n = 3, 4$,*

$$A_\lambda(\mathcal{B}_n) = \begin{cases} \mathbb{Z}\langle t\rangle & \lambda \geq 1 \\ (1 - t + t^2) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

and for $n \geq 5$

$$A_\lambda(\mathcal{B}_n) = \begin{cases} \mathbb{Z}\langle t\rangle & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}.$$

Proof. If, for $1 \leq i < j \leq n - 1$, we set

$$R_{i,j} = \begin{cases} a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1} & j = i + 1 \\ a_i a_j a_i^{-1} a_j^{-1} & j - i > 1 \end{cases},$$

then

$$\frac{\partial^{ab} R_{i,j}}{\partial a_k} = \begin{cases} 1 - t + t^2 & j = i + 1, k = i \\ -1 + t - t^2 & j = i + 1, k = j \\ 1 - t & j - i > 1, k = i \\ -1 + t & j - i > 1, k = j \\ 0 & \text{otherwise} \end{cases}.$$

Ordering the relators so that $R_{i,j}$ precedes $R_{k,l}$ if $j < l$ or if $j = l$ and $i < k$, we find that

$$D(\mathcal{B}_n)^{ab} = \begin{bmatrix} & D(\mathcal{B}_{n-1})^{ab} & & & 0 \\ 1 - t & 0 & \dots & 0 & t - 1 \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & 1 - t & 0 & t - 1 \\ 0 & \dots & 0 & t^2 - t + 1 & -1 + t - t^2 \end{bmatrix},$$

for $n > 3$, and

$$D(\mathcal{B}_3)^{ab} = \begin{bmatrix} 1 - t + t^2 & -1 + t - t^2 \end{bmatrix}.$$

By Lemma 2.3, the columns of $D(\mathcal{B}_n)^{ab}$ sum to zero.

Manual calculation gives

$$A_\lambda(\mathcal{B}_n) = \begin{cases} \mathbb{Z}\langle t\rangle & \lambda \geq 1 \\ (1-t+t^2) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

for $n = 3, 4$, and

$$A_\lambda(\mathcal{B}_5) = \begin{cases} \mathbb{Z}\langle t\rangle & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}.$$

We now show inductively that, for $n \geq 5$,

$$J_\kappa(D(\mathcal{B}_n)^{ab}) = \begin{cases} 0 & \kappa > \chi_1(\mathcal{B}_n) \\ \mathbb{Z}\langle t\rangle & \kappa \leq \chi_1(\mathcal{B}_n) \end{cases}.$$

Suppose that this holds for $n-1$. Since the columns of $D(\mathcal{B}_n)^{ab}$ sum to zero, $J_\kappa(D(\mathcal{B}_n)^{ab}) = 0$ for $\kappa > n-2$. For $\kappa \leq n-2$, by Proposition 1.19,

$$\begin{aligned} J_\kappa(D(\mathcal{B}_n)^{ab}) &\supseteq J_{\kappa-1}(D(\mathcal{B}_{n-1})^{ab}) J_1\left(\begin{bmatrix} t-1 \\ \vdots \\ t-1 \\ -1+t-t^2 \end{bmatrix}\right) \\ &= \mathbb{Z}\langle t\rangle, \end{aligned}$$

from our assumption, which gives the required result. □

2.2.2 The A-ideals of odd Coxeter groups

Let $\Gamma = \mathbf{v} \cup \mathbf{e}$, ψ be an odd Coxeter system. We use the presentation $\mathcal{P}_{\Gamma,\psi}$ (1.19) to calculate the A-ideals of an odd Coxeter group, showing that:

Theorem 2.8. *If Γ, ψ is an odd Coxeter system, then*

$$A_\lambda(\mathcal{P}_{\Gamma,\psi}) = \begin{cases} \mathbb{Z}\langle \widehat{v_1}\rangle & \lambda \geq |\mathbf{v}|-1 \\ (1+\widehat{v_1}, \phi_{|\mathbf{v}|-1-\lambda}) & -1 \leq \lambda < |\mathbf{v}|-1 \\ 0 & \lambda < -1 \end{cases},$$

where

$$\phi_\kappa = \gcd\left\{ \prod_{e\in\Phi\cap\mathbf{e}^+} \psi(e) : \begin{array}{c}\Phi \text{ is a subforest of} \\ \Gamma \text{ with } \kappa \text{ edges}\end{array}\right\}.$$

Proof. For $v \in \mathbf{v}$,

$$\frac{\partial v^2}{\partial v'} = \begin{cases} 1+v & v' = v \\ 0 & \text{otherwise} \end{cases}$$

and, for $e = (u, v) \in \mathbf{e}^+$,

$$\frac{\partial (uv)^{\psi(e)}}{\partial v'} = \begin{cases} 1 + uv + \cdots + (uv)^{\psi(e)-1} & v' = u \\ u + uvu + \cdots + (uv)^{\psi(e)-1}u & v' = v \\ 0 & \text{otherwise} \end{cases}.$$

Before giving the Jacobian matrix of $\mathcal{P}_{\Gamma,\psi}$ we must choose an ordering on $\mathbf{v}$ and on $\mathbf{e}^+$ (which will also induce an ordering on the set of relators). It proves convenient to show that:

Claim. For any $v_1 \in \mathbf{v}$ we can chose an ordering $<$ on $\mathbf{v}$ and an orientation $\mathbf{e}^+$ such that $u < v$ whenever $(u, v) \in \mathbf{e}^+$ and such that, for each $v \in \mathbf{v} \setminus \{v_1\}$, $v_1 < v$ and there is an edge $e \in \mathbf{e}^+$ with $\tau(e) = v$.

Proof of Claim. We proceed by induction on the number of vertices in Γ. This is trivial when $|\mathbf{v}| = 1$. Suppose that $|\mathbf{v}| = n$ and that the claim holds when there are fewer than n vertices. Let $v_1 \in \mathbf{v}$. The graph obtained from Γ by removing v_1 along with all the edges incident at v_1 will be a graph each of whose connected components $\Gamma^{(i)}$ has fewer than n vertices. Each component $\Gamma^{(i)}$ will have a vertex $v_1^{(i)}$ connected to v_1 by an edge of Γ. Applying the inductive hypothesis, we obtain orderings on the vertices of $\Gamma^{(i)}$, with $v_1^{(i)}$ the first, and an orientation of the edges of $\Gamma^{(i)}$ such that for every vertex v of $\Gamma^{(i)}$ other than $v_1^{(i)}$ there is an edge e with $\tau(e) = v$. Giving an ordering to the components induces an ordering on $\mathbf{v} \setminus \{v_1\}$ and we then set $v_1 < v$ for each $v \in \mathbf{v} \setminus \{v_1\}$. We next replace the edges incident at v_1, orienting then appropriately and note that these include edges e with $\tau(e) = v_1^{(i)}$ for each i. This establishes the claim. □

Given such an ordering, we have

$$D^{ab} = \begin{bmatrix} \mathrm{Diag}_{|\mathbf{v}|}(1+\widehat{v_1}) \\ X \end{bmatrix},$$

where X is the $|\mathbf{e}^+| \times |\mathbf{v}|$ matrix with a row

$$\begin{bmatrix} \dots & 0 & \psi(e) & 0 & \dots & 0 & \psi(e)\widehat{v_1} & 0 & \dots \end{bmatrix}$$

for each $e \in \mathbf{e}^+$. We will call this the e-th row. If we call the column corresponding to the vertex v the v-th column, then the non-zero entries $\psi(e)$ and $\psi(e)\widehat{v_1}$ in the e-th row occur in the $\iota(e)$-th and $\tau(e)$-th column respectively.

We now show that

$$J_\kappa(D^{ab}) = \begin{cases} 0 & \kappa > |\mathbf{v}| \\ (1+\widehat{v_1}, \phi_\kappa) & 0 < \kappa \leq |\mathbf{v}| \\ \mathbb{Z}\langle \widehat{v_1} \rangle & \kappa \leq 0 \end{cases}.$$

Clearly, for $0 < \kappa \leq |\mathbf{v}|$,

$$\det(\mathrm{Diag}_\kappa(1+\widehat{v_1})) = (1+\widehat{v_1})^\kappa = 2^{\kappa-1}(1+\widehat{v_1}) \in J_\kappa(D^{ab}).$$

For a vertex v, let e_v be the first edge $e \in \mathbf{e}^+$ with $\tau(e) = v$. Let L_κ be the $(\kappa-1) \times (\kappa-1)$ submatrix of X consisting of columns $2, \dots, \kappa$, corresponding to vertices $v_2, \dots, v_\kappa$, and the e_{v_2}-th, $\dots$, e_{v_κ}-th rows. This is then a lower triangular matrix with diagonal entries $\psi(e_{v_2})\widehat{v_1}, \dots, \psi(e_{v_\kappa})\widehat{v_1}$. The $\kappa \times \kappa$ submatrix consisting of L_κ together with the first row and first column of D^{ab} is then a lower triangular matrix with determinant

$$\left(\prod_{i=2}^{\kappa} \psi(e_{v_i})\right)(1+\widehat{v_1}) \in J_\kappa(D^{ab}).$$

Since each $\psi(e)$ is odd, we then have

$$1+\widehat{v_1} \in J_\kappa(D^{ab})$$

for $0 < \kappa \leq |\mathbf{v}|$.

As the determinant of any submatrix which includes rows from $\mathrm{Diag}_{|\mathbf{v}|}(1+\widehat{v_1})$ will be divisible by $1+\widehat{v_1}$, we need only now consider submatrices of X.

Let Y be a $\kappa \times \kappa$ submatrix of X. The κ rows of Y give a set $\mathbf{e}_0$ of edges, so, if we set $\mathbf{v}_0 = \{\iota(e), \tau(e) : e \in \mathbf{e}_0\}$, then $\Gamma_0 = \mathbf{v}_0 \cup (\mathbf{e}_0 \cup \mathbf{e}_0^{-1})$ is a subgraph of Γ. Let Y_0 be the $\kappa \times |\mathbf{v}|$ submatrix of X consisting of the e-th row for each $e \in \mathbf{e}_0$. We distinguish two cases:

Case 1. Γ_0 has a non-empty, reduced cycle $e_1 e_2 \ldots e_n$. By rearranging the rows and columns of Y_0 and multiplying rows by $\widehat{v_1}$ where appropriate, we can obtain a matrix of the form

$$\begin{bmatrix} Y_0' & 0 \\ Y_0'' & Y_0''' \end{bmatrix},$$

where

$$Y_0' = \begin{bmatrix} \psi(e_1) & \psi(e_1)\widehat{v_1} & 0 & \ldots & & \\ 0 & \psi(e_2) & \psi(e_2)\widehat{v_1} & 0 & \ldots & \\ \vdots & 0 & \ddots & \ddots & & \\ 0 & \vdots & & \psi(e_{n-1}) & \psi(e_{n-1})\widehat{v_1} \\ \psi(e_n) & 0 & \ldots & 0 & \psi(e_n)\widehat{v_1} \end{bmatrix},$$

Y'' is some $(\kappa - n) \times n$ matrix and Y_0''' some $(\kappa - n) \times (\kappa - n)$ matrix. So, by an observation in §1.4.3, if Y doesn't include the v-th column of Y_0 for each vertex v of the cycle, $\det(Y) = 0$. On the other hand, if Y does include each of these columns, then $\det(Y_0') \,|\, \det(Y)$. But, by adding columns $2, \ldots, n$ of Y_0' to column 1, we see that $1 + \widehat{v_1} \,|\, \det(Y_0')$. Thus, if $\det(Y)$ is non-zero, it is a multiple of $1 + \widehat{v_1}$, and so adds nothing more to the generation of $J_\kappa(D^{ab})$.

Case 2. Γ_0 has no non-empty, reduced cycles, that is, Γ_0 is a forest. Since $\psi(e)$ is a common factor of each entry of the e-th row, we have

$$\prod_{e \in \mathbf{e}_0} \psi(e) \,|\, \det(Y).$$

We now show, by induction, that there is a $\kappa \times \kappa$ submatrix of Y_0 with determinant $\prod_{e \in \mathbf{e}_0} \psi(e)$ or $\left(\prod_{e \in \mathbf{e}_0} \psi(e)\right) \widehat{v_1}$. Once we have shown this, we have proven the theorem.

Clearly, when $\kappa = 1$, there is such a submatrix. Now let $\kappa > 1$ and suppose that, for every $(\kappa - 1) \times |\mathbf{v}|$ submatrix of X defining a subforest Φ of Γ, there is a submatrix with determinant $\prod_{e \in \Phi} \psi(e)$ or $\left(\prod_{e \in \mathbf{e}_0} \psi(e)\right) \widehat{v_1}$. Since Γ_0 is a forest, it has an extremal edge $e_0 \in \mathbf{e}_0$. Suppose v_0 is the end point of e_0 incident with no other edge, then the only non-zero entry in the v_0-th column is the entry $\psi(e_0)$ or $\psi(e_0)\widehat{v_1}$ in the e_0-th row. Now, removing e_0 from Γ_0 leaves a $\kappa-1$ edged subforest, so, by the inductive hypothesis, there is a $(\kappa-1)\times(\kappa-1)$ submatrix of Y_0, which doesn't include the e_0-th row, with determinant $\prod_{e\in\mathbf{e}_0\setminus\{e_0\}}\psi(e)$ or $\left(\prod_{e\in\mathbf{e}_0\setminus\{e_0\}}\psi(e)\right)\widehat{v_1}$. Since the v_0-th entry of the e-th row for each $e\in\mathbf{e}_0\setminus\{e_0\}$ is zero, this submatrix cannot include the v_0-th column. The $\kappa\times\kappa$ submatrix consisting of this smaller submatrix along with the e_0-th row and the v_0-th column will then have the required determinant.

This exhausts all possible submatrices of D^{ab}. □

Chapter 3

B-ideals: Group invariants from a 3-presentation

In the spirit of the Alexander ideals and polynomials, we now define new group invariants, a chain of ideals and a sequence of polynomials. Analogous to the way in which the A-ideals and a-polynomials are calculated from the Jacobian matrix obtained from a presentation, these new invariants are calculated from a matrix which is obtained from a 3-presentation.

3.1 Definition of the B-ideals

Let $\mathcal{T} = \langle \mathbf{x}; \mathbf{r}; \mathbf{d} \rangle$ be a 3-presentation with $\mathbf{x}$, $\mathbf{r}$ finite and let $G = G(\mathcal{T})$. Giving some order to the sets $\mathbf{r}$ and $\mathbf{d}$, let

$$D(\mathcal{T}) = \left[\frac{\partial \mathbb{D}}{\partial R}\right]_{\substack{\mathbb{D} \in \mathbf{d} \\ R \in \mathbf{r}}},$$

a $|\mathbf{d}| \times |\mathbf{r}|$ matrix over $\mathbb{Z}G$. We abbreviate $D(\mathcal{T})$ to D when the context is clear.

Now choose an abelianising functor ^{T}. For $\lambda \in \mathbb{Z}$, we define the λ-th B^T*-ideal* of $\mathcal{T}$ to be

$$B^T_\lambda(\mathcal{T}) = J_{\chi_2(\mathcal{T})-\lambda}\left(D(\mathcal{T})^T\right).$$

The chain $B^T(\mathcal{T})$ of ideals $B^T_\lambda(\mathcal{T})$ ($\lambda \in \mathbb{Z}$) is then an ascending chain of ideals in the ring $\mathbb{Z}G^T$. Again, if no functor ^{T} is specified, we assume that it is ^{ab}.

These B-ideals have the same invariance property as the A-ideals, viz.:

Theorem 3.1. *Let $\mathcal{T}$, $\mathcal{T}_0$ be two 3-presentations with finite underlying presentations. If there is an isomorphism $\alpha : G(\mathcal{T}) \to G(\mathcal{T}_0)$, then the induced isomorphism α^T : $\mathbb{Z}G(\mathcal{T})^T \to \mathbb{Z}G(\mathcal{T}_0)^T$ carries $B_\lambda^T(\mathcal{T})$ onto $B_\lambda^T(\mathcal{T}_0)$ for each $\lambda \in \mathbb{Z}$. So, if $G(\mathcal{T}) \cong G(\mathcal{T}_0)$, then $B^T(\mathcal{T}) \cong^{(0)} B^T(\mathcal{T}_0)$.*

Proof. By Lemma 1.4, we need only show that the Theorem holds when $\mathcal{T}_0$ is obtained from $\mathcal{T}$ by a single Tietze transformation (T1), (T2), (T3), finitary in the cases (T1) and (T2), and α is the induced isomorphism.

If $\mathcal{T}_0 = \langle \mathbf{x}, \mathbf{y}; \mathbf{r}, y = W_y (y \in \mathbf{y}); \mathbf{d}\rangle$ is obtained from $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}; \mathbf{d}\rangle$ by a finitary Tietze transformation of type (T1), then

$$\begin{aligned} D(\mathcal{T}_0) &= \begin{bmatrix} \left[\frac{\partial \mathbb{D}}{\partial R}\right]_{\substack{\mathbb{D}\in \mathbf{d} \\ R \in \mathbf{r}}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} D(\mathcal{T})^\alpha & 0 \end{bmatrix}, \end{aligned}$$

since $\gamma_{\mathcal{T}_0}|_{\langle \mathbf{x}\rangle} = \alpha\gamma_{\mathcal{T}}$. As $\chi_2(\mathcal{T}) = \chi_2(\mathcal{T}_0)$, this gives

$$\begin{aligned} B_\lambda^T(\mathcal{T}_0) &= J_{\chi_2(\mathcal{T}_0)-\lambda}(D(\mathcal{T}_0)^T) \\ &= \alpha^T J_{\chi_2(\mathcal{T})-\lambda}(D(\mathcal{T})^T) \\ &= \alpha^T B_\lambda(\mathcal{T}), \end{aligned}$$

as required.

Now suppose that $\mathcal{T}_0 = \langle \mathbf{x}; \mathbf{r}, \mathbf{s}; \mathbf{d}, \mathbb{Q}_S(S \in \mathbf{s})\rangle$ is obtained from $\mathcal{T} = \langle \mathbf{x}; \mathbf{r}; \mathbf{d}\rangle$ by a finitary Tietze transformation of type (T2). Since, for $S, S' \in \mathbf{s}$,

$$\frac{\partial \mathbb{Q}_S}{\partial S'} = \begin{cases} 1 & S' = S \\ 0 & \text{otherwise} \end{cases}$$

and since α is the identity here, we have

$$D(\mathcal{T}_0) = \begin{bmatrix} \left[\frac{\partial \mathbb{D}}{\partial R}\right]_{\substack{\mathbb{D}\in \mathbf{d} \\ R \in \mathbf{r}}} & 0 \\ X & I_{|\mathbf{s}|} \end{bmatrix} = \begin{bmatrix} D(\mathcal{T}) & 0 \\ X & I_{|\mathbf{s}|} \end{bmatrix}$$

for some $|\mathbf{s}| \times |\mathbf{r}|$ matrix X. Noting that $\chi_2(\mathcal{T}_0) = \chi_2(\mathcal{T}) + |\mathbf{s}|$, we obtain

$$\begin{aligned} B_\lambda^T(\mathcal{T}_0) &= J_{\chi_2(\mathcal{T}_0)-\lambda}(D(\mathcal{T}_0)^T) \\ &= J_{\chi_2(\mathcal{T}_0)-|\mathbf{s}|-\lambda}(D(\mathcal{T})^T) \\ &= B_\lambda(\mathcal{T}), \end{aligned}$$

as required.

Finally, suppose that $\mathcal{T}_0 = \langle \mathbf{x}; \mathbf{r}; \mathbf{d}, \mathbf{e} \rangle$ is obtained from $\mathcal{T} = \langle \mathbf{x}; \mathbf{r}; \mathbf{d} \rangle$ by a Tietze transformation of type (T3). Then, since $\{[\mathbb{D}] : \mathbb{D} \in \mathbf{d}\}$ generates $\pi_2(\langle \mathbf{x}; \mathbf{r} \rangle)$, by Lemma 1.15 the last $|\mathbf{e}|$ rows of

$$D(\mathcal{T}_0) = \begin{bmatrix} \left[\frac{\partial \mathbb{D}}{\partial R}\right]_{\substack{\mathbb{D}\in\mathbf{d}\\ R\in\mathbf{R}}} \\ \left[\frac{\partial \mathbb{E}}{\partial R}\right]_{\substack{\mathbb{E}\in\mathbf{e}\\ R\in\mathbf{r}}} \end{bmatrix}$$

are a linear combination of the first $|\mathbf{d}|$, which constitute $D(\mathcal{T})$. Noting that $\chi_2(\mathcal{T}_0) = \chi_2(\mathcal{T})$, the result follows. $\square$

Remarks. 1. As in Theorem 2.1, in proving this theorem we only require that $\tau_{G_0}^T \alpha = \alpha^T \tau_G^T$ when α is an *isomorphism.*

2. The last part of the above proof shows that, given a finite group presentation, any choice of generating set of pictures for that presentation will give exactly the same ideals. Thus, for a finite presentation $\mathcal{P}$, we define $B^T(\mathcal{P})$ to be $B^T(\langle \mathcal{P}; \mathbf{d} \rangle)$ for any generating set of pictures $\mathbf{d}$ for $\mathcal{P}$. In fact, the proof shows that, for a fixed generating set $\mathbf{x}$, any choice of normal generators for the kernel of the map $\langle \mathbf{x} \rangle \to G$ and any generating set of pictures for the corresponding presentation give exactly the same ideals.

For a 3-presentation $\mathcal{T}$ with finite underlying presentation we define, for $\lambda \in \mathbb{Z}$, the λ-th *b-polynomial* of $\mathcal{T}$, $b_\lambda(\mathcal{T})$, to be a generator of the smallest principal ideal containing $B_\lambda^{tf}(\mathcal{T})$. As with the a-polynomials, these are unique up to multiplication by a unit. We thus obtain a sequence of Laurent polynomials with $b_{\lambda+1}(\mathcal{T}) | b_\lambda(\mathcal{T})$.

Corollary 3.2. *Let $\mathcal{T}$, $\mathcal{T}_0$, α be as Theorem 3.1. For each $\lambda \in \mathbb{Z}$, $b_\lambda(\mathcal{T}_0)$ may be chosen such that*

$$\alpha^{tf}(b_\lambda(\mathcal{T})) = b_\lambda(\mathcal{T}_0).$$

Returning to Example 2.3, where the A-ideals failed to distinguish a family of groups, we find that these new B-ideals can be useful.

Example 3.1 (Example 2.3 continued). Recall that, for $l = 1, 2, \ldots$,

$$\mathcal{P}_l = \langle a, b, t; a^3, b^2, (ab)^7, t^l a t^{-l} a^{-1} \rangle,$$

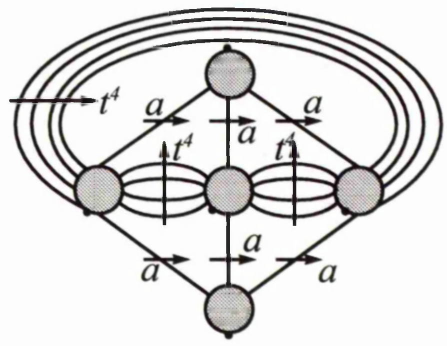

Figure 3.1: The picture $\mathbb{D}_4$ over $\mathcal{P}_4$

$G_l = G(\mathcal{P}_l)$ and $G_l^{ab} = \langle t \rangle \cong \mathbb{Z}$. A generating set $\mathbf{d}_l$ of pictures for $\mathcal{P}_l$, obtained using the methods of [10], consists of a dipole for each of the first three relators, together with an extra picture $\mathbb{D}_l$, as illustrated in Figure 3.1 for $l = 4$. (For general l, $\mathbb{D}_l$ differs from the illustrated picture by having l arcs labelled by t where $\mathbb{D}_4$ has 4.) Setting $\mathcal{T}_l = \langle \mathcal{P}_l; \mathbf{d}_l \rangle$ gives

$$D(\mathcal{T}_l) = \begin{bmatrix} 1-a & 0 & 0 \\ 0 & 1-b & 0 \\ 0 & 0 & 1-ab \\ 1-t^l & 0 & 1+a+a^2 \end{bmatrix},$$

whence

$$D(\mathcal{T}_l)^{ab} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-t^l & 0 & 3 \end{bmatrix},$$

and

$$B_\lambda(\mathcal{T}_l) = \begin{cases} \mathbb{Z}\langle t \rangle & \lambda \geq 2 \\ \left(1-t^l, 3\right) & \lambda = 1 \\ 0 & \lambda < 1 \end{cases}.$$

Thus, if $l \neq l'$, then $G_l \not\cong G_{l'}$, so, while the A-ideals cannot distinguish these groups, the B-ideals can. $\Diamond$

When we can find a generating set of pictures for a presentation, which is the case for an ever increasing number of groups, as discussed in §1.2.3, these B-ideals are just as readily calculated as the A-ideals. For example, we know that a CA presentation has a generating set of pictures consisting of dipoles. Recall from §1.2.4 that for a presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$, for $R \in \mathbf{r}$ we write $R = R_0^{p_R}$ and set $\mathbf{r}' = \{R \in \mathbf{r} : p_R > 1\}$.

Proposition 3.3. *If $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ is a finite CA presentation, then*

$$B_\lambda^T(\mathcal{P}) = \begin{cases} \mathbb{Z}G(\mathcal{P})^T & \lambda \geq \chi_2(\mathcal{P}) \\ \left(\prod_{R\in\mathbf{s}}(1-R_0) : \mathbf{s} \subseteq \mathbf{r}', |\mathbf{s}| = \chi_2 - \lambda\right) & \chi_2 - |\mathbf{r}'| \leq \lambda < \chi_2 \\ 0 & \lambda < \chi_2(\mathcal{P}) - |\mathbf{r}'| \end{cases}$$

Proof. Taking the set of dipoles $\{\mathbb{D}_R : R \in \mathbf{r}'\}$ as a generating set of pictures for $\mathcal{P}$, we have

$$\frac{\partial \mathbb{D}_R}{\partial R'} = \begin{cases} 1 - R_0 & R' = R \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$D = \begin{bmatrix} \mathrm{Diag}_{R\in\mathbf{r}'}(1-R_0) & 0 \end{bmatrix}$$

and the required result follows. □

Corollary 3.4. *If $\mathcal{P}$ is a finite aspherical presentation, then*

$$B_\lambda^T(\mathcal{P}) = \begin{cases} \mathbb{Z}G(\mathcal{P})^T & \lambda \geq \chi_2(\mathcal{P}) \\ 0 & \lambda < \chi_2(\mathcal{P}) \end{cases}.$$

For example, the deleted Wirtinger presentation (2.1) for a knot group is known to be aspherical [62], and so the B-ideals can give only limited information.

This last result then gives a test for whether a group has an aspherical presentation; if a presentation has a B-ideal distinct from 0 and the whole ring, then the group it defines can have no aspherical presentation. One form of Whitehead's conjecture [86] states that if $\mathcal{Q} = \langle \mathbf{y}; \mathbf{s} \rangle$ is a subpresentation of an aspherical presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ (that is, $\mathbf{y} \subseteq \mathbf{x}$ and $\mathbf{s} \subseteq \mathbf{r}$), then $\mathcal{Q}$ itself is aspherical. If $H = G(\mathcal{Q})$, $G = G(\mathcal{P})$, then the inclusion $\mathbf{y} \subseteq \mathbf{x}$ induces a homomorphism $\alpha : H \to G; y \mapsto y$.

Proposition 3.5. *If $\mathcal{Q}$ is a subpresentation of a finite aspherical presentation $\mathcal{P}$ as above, then*

$$\left(\alpha^T B_\lambda^T(\mathcal{Q})\right) = \begin{cases} \mathbb{Z}G^T & \lambda \geq \chi_2(\mathcal{Q}) \\ 0 & \lambda < \chi_2(\mathcal{Q}) \end{cases}.$$

In particular, $B^T_{\chi_2(\mathcal{Q})}(\mathcal{Q}) = \mathbb{Z}H^T$ *and* $B^T_{\chi_2(\mathcal{Q})-1}(\mathcal{Q}) \neq \mathbb{Z}H^T$ *for every* ^{T}.

Proof. Let $\mathbf{d}$ be a generating set of pictures for $\mathcal{Q}$, then, since $\pi_2(\mathcal{P}) = 0$, $\mathbf{d}$ also constitutes a generating set of pictures for $\mathcal{P}$. Now

$$D(\langle \mathcal{P}; \mathbf{d}\rangle) = \begin{bmatrix} D(\langle \mathcal{Q}; \mathbf{d}\rangle)^\alpha & 0 \end{bmatrix},$$

and so

$$\left(\alpha^T B^T_\lambda(\mathcal{Q})\right) = B^T_{\lambda+\chi_2(\mathcal{P})-\chi_2(\mathcal{Q})}(\mathcal{P}) = \begin{cases} \mathbb{Z}G^T & \lambda \geq \chi_2(\mathcal{Q}) \\ 0 & \lambda < \chi_2(\mathcal{Q}) \end{cases}.$$

By definition, $B^T_{\chi_2(\mathcal{Q})}(\mathcal{Q}) = J_0(D^T) = \mathbb{Z}H^T$. Also,

$$0 = B^T_{\chi_2(\mathcal{P})-1}(\mathcal{P}) = \left(\alpha^T B^T_{\chi_2(\mathcal{Q})-1}(\mathcal{Q})\right),$$

so $B^T_{\chi_2(\mathcal{Q})-1}(\mathcal{Q}) \subseteq \ker \alpha^T \subseteq IH^T \subset \mathbb{Z}H^T$. □

In the next chapter, we will derive many more properties of these new group invariants in a much wider context. Before moving on, however, we consider the B-ideals of two families of groups.

3.2 Some examples

3.2.1 The B-ideals of odd Coxeter groups

If $\Gamma = \mathbf{v} \cup \mathbf{e}$, ψ is an odd Coxeter system, then, since $\psi(e) \geq 3$ for each e, $C_{\Gamma,\psi}$ is aspherical. We use the 3-presentation $\mathcal{T}_{\Gamma,\psi} = \langle \mathcal{P}_{\Gamma,\psi}; \mathbf{d}_{\Gamma,\psi}\rangle$ of §1.8.2 for $C_{\Gamma,\psi}$ to calculate the B-ideals of an odd Coxeter group, showing that:

Theorem 3.6. *If* Γ, ψ *is an odd Coxeter system, then*

$$B_\lambda(\mathcal{T}_{\Gamma,\psi}) = \begin{cases} \mathbb{Z}\langle \widehat{v_1}\rangle & \lambda \geq |\mathbf{e}^+| - |\mathbf{v}| + 2 \\ (1-\widehat{v_1}, 1+\widehat{v_1}) & \lambda = |\mathbf{e}^+| - |\mathbf{v}| + 1 \\ \left(2^{|\mathbf{e}^+|-|\mathbf{v}|+1-\lambda}(1+\widehat{v_1})\right) & 1 \leq \lambda < |\mathbf{e}^+| - |\mathbf{v}| + 1 \\ 0 & \lambda < 1 \end{cases}$$

when $|\mathbf{e}^+| \geq |\mathbf{v}|$ *and*

$$B_\lambda(\mathcal{T}_{\Gamma,\psi}) = \begin{cases} \mathbb{Z}\langle \widehat{v_1}\rangle & \lambda \geq 1 \\ (1-\widehat{v_1}) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}$$

when Γ *is a tree.*

Proof. For $e = (u,v) \in \mathbf{e}^+$,

$$\frac{\partial \mathbb{D}_e}{\partial R} = \begin{cases} -(1 + \overline{uv} + \cdots + (\overline{uv})^{\psi(e)-1}) & R = u^2 \\ -(\overline{u} + \overline{uvu} + \cdots + (\overline{uv})^{\psi(e)-1}\overline{u} & R = v^2 \\ 1 + \overline{u} & R = (uv)^{\psi(e)} \\ 0 & \text{otherwise} \end{cases},$$

so, using the same orderings on $\mathbf{v}$ and $\mathbf{e}^+$ as in §2.2.2, we have

$$D^{ab} = \begin{bmatrix} \mathrm{Diag}_{|\mathbf{v}|}(1-\widehat{v_1}) & 0 \\ 0 & 0 \\ -X & \mathrm{Diag}_{|\mathbf{e}^+|}(1+\widehat{v_1}) \end{bmatrix},$$

where X is as in §2.2.2. We now show that, if $|\mathbf{e}^+| \geq |\mathbf{v}|$, then

$$J_\kappa(D^{ab}) = \begin{cases} 0 & \kappa > |\mathbf{e}^+| \\ (2^{\kappa-|\mathbf{v}|}(1+\widehat{v_1})) & |\mathbf{v}| < \kappa \leq |\mathbf{e}^+| \\ (1-\widehat{v_1}, 1+\widehat{v_1}) & \kappa = |\mathbf{v}| \\ \mathbb{Z}\langle \widehat{v_1}\rangle & \kappa < |\mathbf{v}| \end{cases},$$

and, if $|\mathbf{e}^+| = |\mathbf{v}| - 1$, then

$$J_\kappa(D^{ab}) = \begin{cases} 0 & \kappa > |\mathbf{v}| \\ (1-\widehat{v_1}) & \kappa = |\mathbf{v}| \\ \mathbb{Z}\langle \widehat{v_1}\rangle & \kappa < |\mathbf{v}| \end{cases}.$$

For $\kappa > \max\{|\mathbf{v}|, |\mathbf{e}^+|\}$, any $\kappa \times \kappa$ submatrix of D^{ab} without a column or a row of zeros must have a row whose only non-zero entry is $1 - \widehat{v_1}$ and a column whose only non-zero entry is $1 + \widehat{v_1}$. Thus, since $(1-\widehat{v_1})(1+\widehat{v_1}) = 0$, $J_\kappa(D^{ab}) = 0$.

As for the A-ideals, using the submatrix $-L_\kappa$ of $-X$ gives

$$1-\widehat{v_1}, 1+\widehat{v_1} \in J_\kappa(D^{ab})$$

for $0 < \kappa \leq \min\{|\mathbf{v}|, |\mathbf{e}^+|\}$. So $2 = (1-\widehat{v_1}) + (1+\widehat{v_1}) \in J_\kappa(D^{ab})$. But, for $0 < \kappa < \min\{|\mathbf{v}|, |\mathbf{e}^+|\}$, we also have

$$\det(L_{\kappa+1}) = \prod_{i=2}^{\kappa+1} \psi(e_{v_i}) \in J_\kappa(D^{ab}).$$

Since this is an odd number, we have

$$J_\kappa(D^{ab}) = \mathbb{Z}\langle\widehat{v_1}\rangle$$

for $\kappa < \min\{|\mathbf{v}|, |\mathbf{e}^+|\}$.

To determine $J_\kappa(D^{ab})$ for $\min\{|\mathbf{v}|, |\mathbf{e}^+|\} \leq \kappa \leq \max\{|\mathbf{v}|, |\mathbf{e}^+|\}$, we distinguish two cases:

Case 1. $|\mathbf{e}^+| \geq |\mathbf{v}|$. For $|\mathbf{v}| \leq \kappa \leq |\mathbf{e}^+|$, a $\kappa \times \kappa$ submatrix has non-zero determinant only when it either takes all its columns from the first $|\mathbf{v}|$ or all its rows from the last $|\mathbf{e}^+|$. When $\kappa = |\mathbf{v}|$, we already know that $1-\widehat{v_1}, 1+\widehat{v_1} \in J_{|\mathbf{v}|}(D^{ab})$, so we need only consider submatrices of $-X$. But the columns of X sum to a multiple of $1+\widehat{v_1}$, whence

$$J_{|\mathbf{v}|}(D^{ab}) = (1-\widehat{v_1}, 1+\widehat{v_1})\,.$$

For $\kappa > |\mathbf{v}|$ we need only consider submatrices of

$$\left[-X \quad \mathrm{Diag}_{|\mathbf{e}^+|}(1+\widehat{v_1})\right].$$

Any such submatrix must include at least $\kappa - |\mathbf{v}|$ columns from $\mathrm{Diag}_{|\mathbf{e}^+|}(1+\widehat{v_1})$, so $(1+\widehat{v_1})^{\kappa-|\mathbf{v}|} = 2^{\kappa-|\mathbf{v}|-1}(1+\widehat{v_1})$ divides each determinant. A $\kappa \times \kappa$ submatrix which includes exactly $\kappa - |\mathbf{v}|$ non-zero columns from $\mathrm{Diag}_{|\mathbf{e}^+|}(1+\widehat{v_1})$ must include all $|\mathbf{v}|$ columns from $-X$, so will have determinant

$$(1+\widehat{v_1})^{\kappa-|\mathbf{v}|}\det(Y_0) = 2^{\kappa-|\mathbf{v}|-1}(1+\widehat{v_1})\det(Y_0)$$

for some $|\mathbf{v}| \times |\mathbf{v}|$ submatrix Y_0 of $-X$. But $1+\widehat{v_1}$ divides $\det(Y_0)$, so

$$J_\kappa(D^{ab}) \subseteq \left(2^{\kappa-|\mathbf{v}|}(1+\widehat{v_1})\right).$$

Now consider the $\kappa \times \kappa$ lower triangular submatrix consisting of $-L_{|\mathbf{v}|}$ along with $\kappa - |\mathbf{v}| + 1$ more rows and the corresponding columns of $\mathrm{Diag}_{|\mathbf{e}^+|}(1 + \widehat{v_1})$. This has determinant

$$\left(\prod_{i=2}^{|\mathbf{v}|} \psi(e_{v_i})\right) 2^{\kappa - |\mathbf{v}|}(1 + \widehat{v_1}) \in J_\kappa(D^{ab}).$$

Since there is also a submatrix of $\mathrm{Diag}_{|\mathbf{e}^+|}(1+\widehat{v_1})$ with determinant $2^{\kappa-1}(1+\widehat{v_1})$, we have shown that $J_\kappa(D^{ab}) = \left(2^{\kappa-|\mathbf{v}|}(1+\widehat{v_1})\right)$ for $|\mathbf{v}| < \kappa \leq |\mathbf{e}^+|$.

Case 2. $|\mathbf{e}^+| = |\mathbf{v}| - 1$, so Γ is a tree. For $\kappa = |\mathbf{e}^+| = |\mathbf{v}| - 1$ we have

$$(1 - \widehat{v_1})^{|\mathbf{v}|-1}, (1 + \widehat{v_1})^{|\mathbf{e}^+|}, \det(L_{|\mathbf{v}|}) = \prod_{i=2}^{|\mathbf{v}|} \psi(e_{v_i}) \in J_{|\mathbf{v}|-1}(D^{ab}).$$

So, since $\det(L_{|\mathbf{v}|})$ is odd and $2^{|\mathbf{v}|-1} = (1 - \widehat{v_1})^{|\mathbf{v}|-1} + (1 + \widehat{v_1})^{|\mathbf{v}|-1} \in J_{|\mathbf{v}|-1}(D^{ab})$, we have $J_\kappa(D^{ab}) = \mathbb{Z}\langle\widehat{v_1}\rangle$ for $\kappa \leq |\mathbf{v}| - 1$.

For $\kappa = |\mathbf{v}|$ we need only consider submatrices of

$$\begin{bmatrix} \mathrm{Diag}_{|\mathbf{v}|}(1 - \widehat{v_1}) \\ -X \end{bmatrix}.$$

Since these must include at least one row from $\mathrm{Diag}_{|\mathbf{v}|}(1 - \widehat{v_1})$, we have $J_{|\mathbf{v}|}(D^{ab}) \subseteq (1 - \widehat{v_1})$. We also have

$$\det(\mathrm{Diag}_{|\mathbf{v}|}(1 - \widehat{v_1})) = 2^{|\mathbf{v}|-1}(1 - \widehat{v_1}), (1 - \widehat{v_1})\det(L_{|\mathbf{v}|}) \in J_{|\mathbf{v}|}(D^{ab}),$$

so $J_{|\mathbf{v}|}(D^{ab}) = (1 - \widehat{v_1})$. □

Example 3.2. Consider the Coxeter systems given in Figure 3.2. Applying the results of this section and of §2.2.2, we have

$$A_\lambda(\mathcal{P}_{\Gamma_1,\psi_1}) = A_\lambda(\mathcal{P}_{\Gamma_2,\psi_2}) = \begin{cases} \mathbb{Z}\langle\widehat{v_1}\rangle & \lambda \geq 2 \\ (1+\widehat{v_1}, 3) & \lambda = 1 \\ (1+\widehat{v_1}, 9) & \lambda = 0 \\ (1+\widehat{v_1}) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases},$$

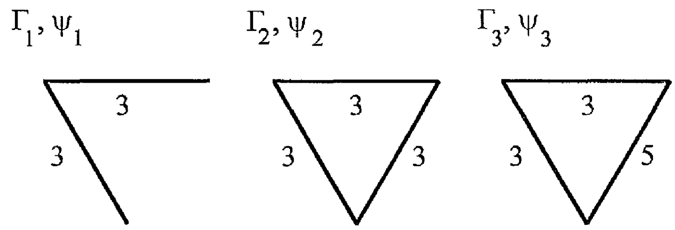

Figure 3.2: Three odd Coxeter systems

$$A_\lambda(\mathcal{P}_{\Gamma_3,\psi_3}) = \begin{cases} \mathbb{Z}\langle\widehat{v_1}\rangle & \lambda \geq 1 \\ (1+\widehat{v_1}, 3) & \lambda = 0 \\ (1+\widehat{v_1}) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases},$$

and

$$B_\lambda(\mathcal{P}_{\Gamma_1,\psi_1}) = \begin{cases} \mathbb{Z}\langle\widehat{v_1}\rangle & \lambda \geq 1 \\ (1-\widehat{v_1}) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

$$B_\lambda(\mathcal{P}_{\Gamma_2,\psi_2}) = B_\lambda(\mathcal{P}_{\Gamma_3,\psi_3}) = \begin{cases} \mathbb{Z}\langle\widehat{v_1}\rangle & \lambda \geq 2 \\ (1-\widehat{v_1}, 1+\widehat{v_1}) & \lambda = 1 \\ 0 & \lambda < 1 \end{cases}.$$

Thus, the A-ideals are unable to distinguish the first and second of these, while the B-ideals can. On the other hand, the B-ideals cannot distinguish the second and third, whereas the A-ideals can. Both the A- and the B-ideals can distinguish the first and the third. ◊

3.2.2 The B-ideals of triangle groups

Triangle groups $G_{k,l,m}$ are groups given by a presentation of the form

$$\mathcal{P}_{k,l,m} = \langle a, b; a^k, b^l, (ab)^m \rangle$$

for some integers $k, l, m \geq 2$. Since $G_{k',l',m'} \cong G_{k,l,m}$ whenever (k', l', m') is a permutation of (k, l, m), we may assume that $2 \leq k \leq l \leq m$.

When

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \leq 1,$$

$\mathcal{P}_{k,l,m}$ is CA (this can be shown, for example, by the star graph test of [20]). In this case, using Proposition 3.3, we have

$$B_\lambda(\mathcal{P}_{k,l,m}) = \begin{cases} \mathbb{Z}G^{ab}_{k,l,m} & \lambda \geq 2 \\ (1-a, 1-b) = IG^{ab}_{k,l,m} & \lambda = 1 \\ ((1-a)(1-b), (1-a)(1-ab), (1-b)(1-ab)) & \\ \quad = (IG^{ab}_{k,l,m})^2 & \lambda = 0 \\ ((1-a)(1-b)(1-ab)) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases}.$$

The distinct triples (k, l, m) when $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1$ are:

1. $(2, 2, m)$, $m \geq 2$;
2. $(2, 3, 3)$;
3. $(2, 3, 4)$;
4. $(2, 3, 5)$.

In each of these cases, there is a tessellation of the sphere by k-, l- and $2m$-gons. The dual of this tessellation gives a spherical picture $\mathbb{D}_{k,l,m}$ over $\mathcal{P}_{k,l,m}$, which, together with a dipole for each relator, gives a generating set of pictures [18], [20]. We use the associated 3-presentation $\mathcal{T}_{k,l,m}$ to calculate the B-ideals.

1. $(2, 2, m)$. The group $G_{2,2,m}$ is an aspherical Coxeter group, whose graph has two vertices joined by an edge. The picture $\mathbb{D}_{2,2,m}$ is then of the form of Figure 1.9, with $u = a$, $v = b$, $\psi(a, b) = m$.

When m is odd, $G_{2,2,m}$ is an odd Coxeter group, so, by Theorem 3.6, $G^{ab}_{2,2,m} = \langle a \rangle$

and

$$B_\lambda(\mathcal{P}_{2,2,m}) = \begin{cases} \mathbb{Z}\langle a\rangle & \lambda \geq 1 \\ I\langle a\rangle & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

When m is even, $G^{ab}_{2,2,m} = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and

$$D(\mathcal{T}_{2,2,m})^{ab} = \begin{bmatrix} 1-a & 0 & 0 \\ 0 & 1-b & 0 \\ 0 & 0 & 1-ab \\ -\frac{m}{2}(1+ab) & -\frac{m}{2}(a+b) & 1+a \end{bmatrix}.$$

Thus, since

$$\begin{aligned} J_1(D^{ab}) &= \left(1-a, 1-b, \frac{m}{2}(1+ab), 1+a\right) \\ &= (1-a, 1-b, 2), \\ J_2(D^{ab}) &= ((1-a)(1-b), (1-a)(1-ab), \\ &\qquad \frac{m}{2}(1-a)(1+ab), (1-b)(1-ab), \\ &\qquad \frac{m}{2}(1-b)(1+ab), (1-b)(1+a)) \\ &= (I(\langle a\rangle \times \langle b\rangle))^2, \\ J_3(D^{ab}) &= 0, \end{aligned}$$

we have

$$B_\lambda(\mathcal{P}_{2,2,m}) = \begin{cases} \mathbb{Z}(\langle a\rangle \times \langle b\rangle) & \lambda \geq 2 \\ (1-a, 1-b, 2) & \lambda = 1 \\ (I(\langle a\rangle \times \langle b\rangle))^2 & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

2. $(2,3,3)$. The picture $\mathbb{D}_{2,3,3}$ is illustrated in Figure 3.3. Here $G^{ab}_{2,3,3} = \langle b\rangle \cong \mathbb{Z}_3$, where $a = 1$, and

$$D(\mathcal{T}_{2,3,3})^{ab} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-b & 0 \\ 0 & 0 & 1-b \\ -2(1+b+b^2) & -(2+b+b^2) & 1+b+2b^2 \end{bmatrix},$$

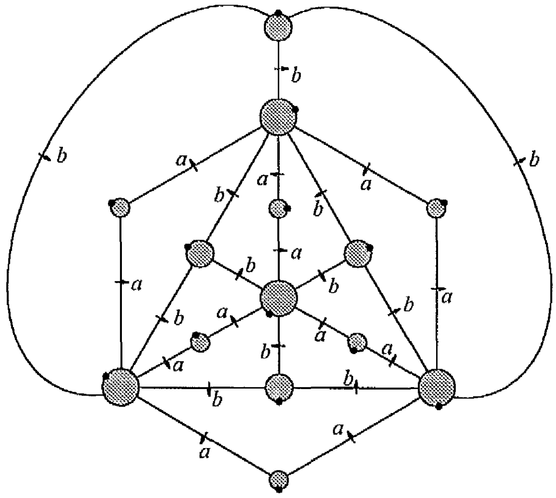

Figure 3.3: The picture $\mathbb{D}_{2,3,3}$

so

$$B_\lambda(\mathcal{P}_{2,3,3}) = \begin{cases} \mathbb{Z}\langle b\rangle & \lambda \geq 2 \\ (1-b, 2+b+b^2, 2(1+b+b^2)) = (2, 1-b) & \lambda = 1 \\ ((1-b)^2, (1-b)(2+b+b^2)) = I\langle b\rangle & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

3. $(2,3,4)$. The picture $\mathbb{D}_{2,3,4}$ is illustrated in Figure 3.4. Here $G^{ab}_{2,3,4} = \langle a\rangle$ with $b = 1$ and

$$D(\mathcal{T}_{2,3,4})^{ab} = \begin{bmatrix} 1-a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-a \\ -12a & -(4+4a) & 3+3a \end{bmatrix},$$

so

$$B_\lambda(\mathcal{P}_{2,3,4}) = \begin{cases} \mathbb{Z}\langle a\rangle & \lambda \geq 2 \\ (1-a, 12a, 4(1+a), 3(1+a)) = (2, 1-a) & \lambda = 1 \\ (2(1-a), 12(1-a)) = 2I\langle a\rangle & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

4. $(2,3,5)$. The picture $\mathbb{D}_{2,3,5}$ is illustrated in Figure 3.5. Since $G^{ab}_{2,3,5} = 1$, we only need to know $\exp_R(\mathbb{D}_{2,3,5})$ for each relator, so we have omitted the labelling and

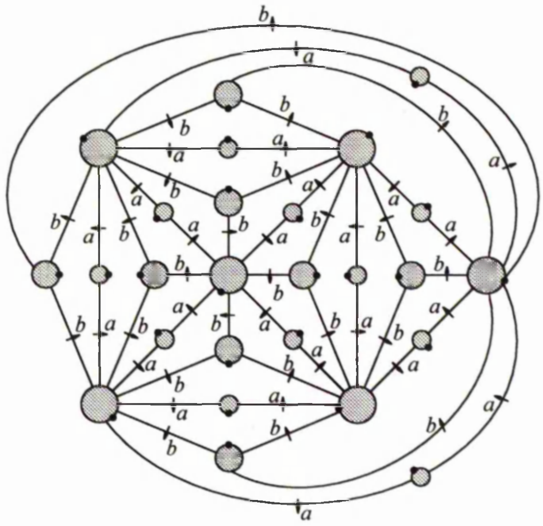

Figure 3.4: The picture $\mathbb{D}_{2,3,4}$

orientations from the arcs and the basepoints of the discs. Note that the orientation of each disc labelled a^2 and b^3 is negative, while those labelled $(ab)^5$ is positive and there are, respectively, 30, 20 and 12 discs with these labels. Thus

$$D(\mathcal{T}_{2,3,5})^{ab} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -30 & -20 & 12 \end{bmatrix},$$

and so

$$B_\lambda(\mathcal{P}_{2,3,5}) = \begin{cases} \mathbb{Z} & \lambda \geq 2 \\ (2) & \lambda = 1 \\ 0 & \lambda < 1 \end{cases}.$$

For the sake of completeness, we calculate the A-ideals of the presentations $\mathcal{P}_{k,l,m}$. For every such presentation,

$$D(\mathcal{P}_{k,l,m}) = \begin{bmatrix} \Sigma_a & 0 \\ 0 & \Sigma_b \\ \Sigma_{ab} & \Sigma_{ab} a \end{bmatrix},$$

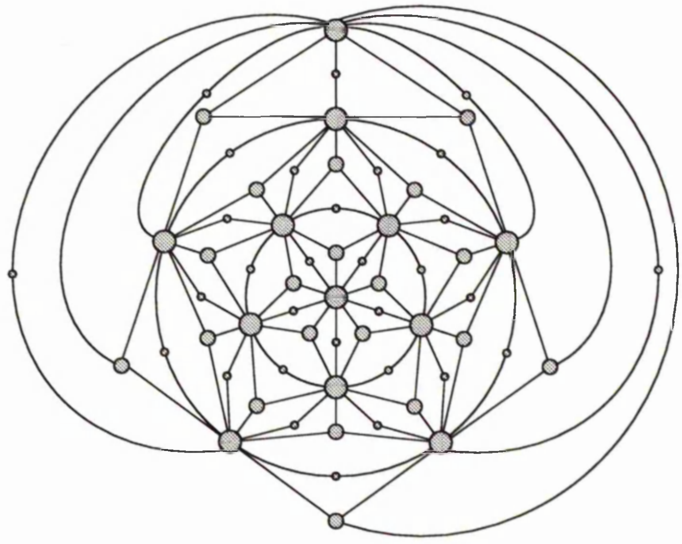

Figure 3.5: The picture $\mathbb{D}_{2,3,5}$

where

$$\Sigma_a = 1 + a + \cdots + a^{k-1},$$
$$\Sigma_b = 1 + b + \cdots + b^{l-1},$$
$$\Sigma_{ab} = 1 + ab + \cdots + (ab)^{m-1},$$

so

$$A_\lambda(\mathcal{P}_{k,l,m}) = \begin{cases} \mathbb{Z}G^{ab}_{k,l,m} & \lambda \geq 1 \\ (\Sigma_a, \Sigma_b, \Sigma_{ab}) & \lambda = 0 \\ (\Sigma_a\Sigma_b, \Sigma_a\Sigma_{ab}, \Sigma_b\Sigma_{ab}) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases}.$$

For the four cases distinguished above, we have

$$A_\lambda(\mathcal{P}_{2,2,m}) = \begin{cases} \mathbb{Z}(\langle a\rangle \times \langle b\rangle) & \lambda \geq 1 \\ (1+a, 1+b, m) & \lambda = 0 \\ ((1+a)(1+b)) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases} \qquad (m \text{ even}),$$

$$A_\lambda(\mathcal{P}_{2,2,m}) = \begin{cases} \mathbb{Z}\langle a\rangle & \lambda \geq 1 \\ (1+a, m) & \lambda = 0 \\ (1+a) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases} \qquad (m \text{ odd}),$$

$$A_\lambda(\mathcal{P}_{2,3,3}) = \begin{cases} \mathbb{Z}\langle b\rangle & \lambda \geq 1 \\ (2, 1+b+b^2) & \lambda = 0 \\ (1+b+b^2) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases},$$

$$A_\lambda(\mathcal{P}_{2,3,4}) = \begin{cases} \mathbb{Z}\langle a\rangle & \lambda \geq 1 \\ (3, 1+a) & \lambda = 0 \\ (1+a) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases},$$

$$A_\lambda(\mathcal{P}_{2,3,5}) = \begin{cases} \mathbb{Z} & \lambda \geq -1 \\ 0 & \lambda < -1 \end{cases}.$$

Chapter 4

E-ideals: Invariants from a resolution

In this chapter, we define a family of new invariants for groups, which includes the A- and B-ideals. As for the A- and B-ideals, these new invariants are calculated from a matrix, which we obtain from a free resolution.

4.1 The E-ideals of a group

For $n \geq 0$, suppose that G is a group of type FP_n over K and that $\mathcal{F} = (F_i, \partial_i)$ is a KG-free resolution of type FP_n for G. Recall that, if we choose free bases for the free modules F_{n+1}, F_n, then associated with the $(n+1)$-st boundary map $\partial_{n+1} : F_{n+1} \to F_n$ of $\mathcal{F}$ is a matrix $D_n(\mathcal{F})$. If r_i is the rank of F_i, then $D_n(\mathcal{F})$ is an $r_{n+1} \times r_n$ matrix over KG. For an abelianising functor ^{T}, we define, for $\lambda \in \mathbb{Z}$, the *λ-th E_n^T-ideal of G over K* to be

$$E_{n,\lambda}^T(G, K) = J_{\chi_n(\mathcal{F})-\lambda}(D_n(\mathcal{F})^T).$$

The chain $E_n^T(G, K)$ of ideals $E_{n,\lambda}^T(G, K)$ is then an ascending chain of ideals in the ring KG^T. When $K = \mathbb{Z}$, we omit it, setting $E_n^T(G) = E_n^T(G, \mathbb{Z})$ and, as usual, we fail to specify the functor ^{T} when it is ^{ab}.

Theorem 4.1. *The ideals $E_n^T(G, K)$ are well-defined and are invariants of the group G up to isomorphism, that is,* *i) the ideals depend neither on the choice of resolution $\mathcal{F}$, nor on the choice of bases for F_n and F_{n+1}; and*

ii) if $G \cong G_0$, then $E_n^T(G,K) \cong^{(0)} E_n^T(G_0,K)$.

We defer the proof of this theorem to the next section.

If K is a gcd domain, then so is KG^{tf}, so, for a group G of type FP_n, we define the *λ-th e_n-polynomial of G with coefficients in K*, $e_{n,\lambda}(G,K)$, to be a generator of the smallest principal ideal containing $E_{n,\lambda}^{tf}(G,K)$. We write $e_{n,\lambda}(G,\mathbb{Z})$ as $e_{n,\lambda}(G)$. We then have an infinite sequence of Laurent polynomials with K-coefficients, each of which divides its predecessor. A consequence of Theorem 4.1 is that these are well-defined (up to multiplication by a unit) and are group invariants in the sense of the following corollary.

Corollary 4.2. *If $\alpha : G \to G_0$ is an isomorphism, then, for each $\lambda \in \mathbb{Z}$, $e_{n,\lambda}(G_0,K)$ can be chosen such that*

$$\alpha^{tf}(e_{n,\lambda}(G,K)) = e_{n,\lambda}(G_0,K).$$

Recall that, if G is of type FP_n, then it is of type FP_n over K for any commutative ring K. The following result shows that, if we know the chain $E_n^T(G)$, then we can easily obtain the chain $E_n^T(G,K)$. For any group G, let $\iota_G : \mathbb{Z}G \to KG$ be the ring homomorphism induced by $1 \mapsto 1 \in K$, $g \mapsto g$ $(g \in G)$.

Theorem 4.3. *If G is of type FP_n, then*

$$E_n^T(G,K) = \left(\iota_{G^T} E_n^T(G)\right),$$

for any commutative ring K.

Proof. If $\mathcal{F} = (F_i, \partial_i)$ is a $\mathbb{Z}G$-free resolution of type FP_n for G, then $K \otimes_{\mathbb{Z}} \mathcal{F}$ (which G acts on diagonally) is a KG-free resolution of $K \otimes_{\mathbb{Z}} \mathbb{Z} \cong K$ of type FP_n (see, for example, [16]). A choice of bases for F_{n+1}, F_n induces a choice of bases for $K \otimes_{\mathbb{Z}} F_{n+1}$, $K \otimes_{\mathbb{Z}} F_n$ with respect to which

$$D_n(K \otimes_{\mathbb{Z}} \mathcal{F}) = D_n(\mathcal{F})^{\iota_G}.$$

We then have

$$\begin{aligned} E^T_{n,\lambda}(G,K) &= J_{\chi_n(K\otimes_{\mathbb{Z}}\mathcal{F})-\lambda}(D_n(K\otimes_{\mathbb{Z}}\mathcal{F})^T) \\ &= J_{\chi_n(\mathcal{F})-\lambda}((D_n(\mathcal{F})^T)^{\iota_{G^T}}) \\ &= (\iota_{G^T} E^T_{n,\lambda}(G)), \end{aligned}$$

since $\tau^T_G \iota_G = \iota_{G^T}\tau^T_G$ and $\chi_n(K\otimes_{\mathbb{Z}}\mathcal{F}) = \chi_n(\mathcal{F})$. □

For this reason, we tend to consider primarily the invariants with $\mathbb{Z}$-coefficients, although a group could be of type FP_n over some $K \neq \mathbb{Z}$, but not of type FP_n.

We now relate these new E-ideals to the A- and B-ideals of Chapters 2 and 3. Let $\mathcal{T} = \langle \mathbf{x}; \mathbf{r}; \mathbf{d}\rangle$ be a 3-presentation for $G = G(\mathcal{T})$ and let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}\rangle$. Consider the associated partial resolution $\mathcal{F}_{\mathcal{T}}$ for G as in (1.12). With respect to the bases $\{e_x : x \in \mathbf{x}\}$ and $\{e_R : R \in \mathbf{r}\}$, we have

$$D_1(\mathcal{F}_{\mathcal{T}}) = \left[\frac{\partial R}{\partial x}\right]_{\substack{R\in\mathbf{r}\\ x\in\mathbf{x}}} = D(\mathcal{P}),$$

the Jacobian matrix of $\mathcal{P}$, and, with respect to the bases $\{e_R : R \in \mathbf{r}\}$ and $\{e_{\mathbb{D}} : \mathbb{D} \in \mathbf{d}\}$,

$$D_2(\mathcal{F}_{\mathcal{T}}) = \left[\frac{\partial \mathbb{D}}{\partial R}\right]_{\substack{\mathbb{D}\in\mathbf{d}\\ R\in\mathbf{r}}} = D(\mathcal{T}).$$

Moreover, if $\mathbf{x}$ is finite, $\chi_1(\mathcal{F}_{\mathcal{T}}) = \chi_1(\mathcal{P})$ and, if $\mathcal{P}$ is finite, $\chi_2(\mathcal{F}_{\mathcal{T}}) = \chi_2(\mathcal{T})$, and so we have proved:

Proposition 4.4. *Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}\rangle$.*

i) If $\mathbf{x}$ is finite, then $G(\mathcal{P})$ is of type FP_1,

$$E^T_1(G(\mathcal{P})) = A^T(\mathcal{P})$$

and, up to multiplication by a unit, $e_{1,\lambda}(G(\mathcal{P})) = a_\lambda(\mathcal{P})$ $(\lambda \in \mathbb{Z})$.

ii) If $\mathcal{P}$ is finite, then $G(\mathcal{P})$ is of type FP_2,

$$E^T_2(G(\mathcal{P})) = B^T(\mathcal{P}).$$

and, up to multiplication by a unit, $e_{2,\lambda}(G(\mathcal{P})) = b_\lambda(\mathcal{P})$ $(\lambda \in \mathbb{Z})$.

Remark. Since a group is of type FP_1 if, and only if, it can be finitely generated, the E_1- and A-ideals are defined for the same groups. On the other hand, not every group of type FP_2 can be finitely presented [15], so the E_2-ideals can sometimes be defined when the B-ideals cannot.

The matrix of ∂_1 with respect to the bases $\{1\}$ and $\{e_x : x \in \mathbf{x}\}$ is the $|\mathbf{x}| \times 1$ matrix

$$D_0(\mathcal{F}_T) = \begin{bmatrix}1 - x\end{bmatrix}_{x \in \mathbf{x}}.$$

Since every group is of type FP_0, the E_0-ideals are always defined and are of the following form.

Proposition 4.5. *For any group G,*

$$E^T_{0,\lambda}(G) = \begin{cases} \mathbb{Z}G^T & \lambda \geq 1 \\ IG^T & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

For a group G of type FP_n over K, we also define associated invariants

$$\nu^T_n(G, K) = \min\{\lambda \in \mathbb{Z} : E^T_{n,\lambda}(G, K) = KG^T\},$$
$$\delta^T_n(G, K) = \min\{\lambda \in \mathbb{Z} : E^T_{n,\lambda}(G, K) \neq 0\}.$$

If $\mathcal{F} = (F_i, \partial_i)$ is a resolution of type FP_n for G over K, then

$$\nu^T_n(G, K) \leq \chi_n(\mathcal{F})$$

and

$$\delta^T_n(G, K) \geq \chi_n(\mathcal{F}) - \min\{\mathrm{rk}_{KG}(F_n), \mathrm{rk}_{KG}(F_{n+1})\}$$
$$\geq -\chi_{n-1}(\mathcal{F}), -\chi_{n+1}(\mathcal{F}).$$

Since, for each ^{T}, $E^T_n(G, K) = \alpha E_n(G, K)$ for some $\alpha : G^{ab} \to G^T$ and $E^{triv}_n(G, K) = aug E^T_n(G, K)$, we have

$$\delta_n(G, K) \leq \delta^T_n(G, K) \leq \delta^{triv}_n(G, K) \leq \nu^{triv}_n(G, K) \leq \nu^T_n(G, K) \leq \nu_n(G, K). \quad (4.1)$$

If G is of type FP_n, we write $\nu_n^T(G)$ for $\nu_n^T(G,\mathbb{Z})$ and $\delta_n^T(G)$ for $\delta_n^T(G,\mathbb{Z})$. Thus, from the definition of $\chi_n(G)$,

$$\chi_n(G) \geq \nu_n^T(G), -\delta_{n\pm 1}^T(G). \tag{4.2}$$

When $n=0$, we have $\nu_0^T(G)=1$ for every ^{T} and every G, and

$$\delta_0^T(G) = \begin{cases} 1 & \text{if } G^T = 1 \\ 0 & \text{otherwise} \end{cases}.$$

In dimension 1, we will see in Proposition 4.26, below, that $\delta_1(G) = -1$ if, and only if, G has finite abelianisation.

By way of example, we now calculate the E_n-ideals of some groups.

4.1.1 CA groups

If $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}\rangle$ is CA, then there is the following free resolution $\mathcal{F}$ for $G = G(\mathcal{P})$, the *Lyndon resolution* [51], [57]:

$$\cdots \to F \xrightarrow{\partial_4} F \xrightarrow{\partial_3} \oplus_{R\in \mathbf{r}}\mathbb{Z}Ge_R \xrightarrow{\partial_2} \oplus_{x\in \mathbf{x}}\mathbb{Z}Ge_x \xrightarrow{\partial_1} \mathbb{Z}Ge \xrightarrow{\partial_0} \mathbb{Z} \to 0, \tag{4.3}$$

where ∂_0, ∂_1 and ∂_2 are as in (1.12), $F = \oplus_{R\in\mathbf{r}'}\mathbb{Z}Ge_R$ and, for $i \geq 3$, $R \in \mathbf{r}'$, $\partial_i(e_R) = \xi_{i-1}(R)e_R$, where

$$\xi_i(R) = \begin{cases} 1 - R_0 & i \text{ even} \\ 1 + R_0 + \cdots + R_0^{p_R - 1} & i \text{ odd} \end{cases}.$$

When $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}\rangle$ is finite, G is of type FP_∞ and

$$D_n(\mathcal{F}) = \mathrm{Diag}_{R\in\mathbf{r}'}(\xi_n(R)).$$

(the matrix $D_2(\mathcal{F})$ will actually have an extra $|\mathbf{r}| - |\mathbf{r}'|$ columns of zeroes, but we can ignore these). So, for $n \geq 2$,

$$E_{n,\lambda}^T(G) = \begin{cases} \mathbb{Z}G^T & \lambda \geq \chi_n \\ \left(\prod_{R\in\mathbf{s}} \xi_n(R) : \mathbf{s} \subseteq \mathbf{r}', |\mathbf{s}| = \chi_n - \lambda\right) & -\chi_{n+1} \leq \lambda < \chi_n \\ 0 & \lambda < -\chi_{n+1} \end{cases},$$

where

$$\chi_n = \chi_n(\mathcal{F}) = \begin{cases} |\mathbf{r}| - |\mathbf{x}| + 1 & n \text{ even} \\ |\mathbf{r}'| - |\mathbf{r}| + |\mathbf{x}| - 1 & n \text{ odd} \end{cases}.$$

In dimension 1, since

$$\frac{\partial R}{\partial x} = (1 + R_0 + \cdots + R_0^{p_R - 1})\frac{\partial R_0}{\partial x},$$

we have

$$D_1(\mathcal{F})^T = \mathrm{Diag}_{R\in\mathbf{r}}(\xi_1(R))^T \left[\tfrac{\partial^T R_0}{\partial x}\right]_{\substack{R\in\mathbf{r}\\ x\in\mathbf{x}}}.$$

Thus, by Corollary 1.22,

$$E^T_{1,\lambda}(G) \subseteq \left(\prod_{R\in\mathbf{s}} \xi_1(R) : \mathbf{s} \subseteq \mathbf{r}', |\mathbf{s}| = |\mathbf{x}| - 1 - \lambda\right)$$

for $-\min\{|\mathbf{x}|, |\mathbf{r}|\} + |\mathbf{x}| - 1 \leq \lambda < |\mathbf{x}| - 1$.

Notice that, if G has torsion, that is, $p_R > 1$ for some $R \in \mathbf{r}$ (see Theorem 1.2), then there are non-trivial ideals in arbitrarily high dimension. For example, for odd $n > 1$,

$$E^{triv}_{n,-\chi_{n+1}}(G) = \left(\prod_{R\in\mathbf{r}'} p_R\right).$$

4.1.2 The R. Thompson group

Let G be the group given by the presentation

$$\langle x_0, x_1, x_2, \ldots ; x_i x_j = x_{j+1} x_i (i < j)\rangle.$$

This group was originally defined by R. Thompson and has been much studied subsequently (see [26] for a survey and also [46]). In [24], Brown and Geoghegan showed that G is of type FP_∞ and gave a resolution of type FP_∞, which we will use to give the following result.

Theorem 4.6. *For $n \geq 0$,*

$$E_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{ab} & \lambda > 0 \\ IG^{ab} & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

Proof. Let ϕ be the endomorphism of G induced by $x_i \mapsto x_{i+1}$ $(i \geq 0)$. A map $\alpha : M \to M'$ of $\mathbb{Z}G$-modules will be said to be *ϕ-semi-linear* if $\alpha(g.m) = \phi(g).\alpha(m)$, for $g \in G$, $m \in M$. We define ϕ^2-semi-linearity similarly.

Let F_0 be a free $\mathbb{Z}G$-module of rank 1 with basis element $z^{(0)}$ and, for $n > 0$, let F_n be free of rank 2 with basis $\{z_0^{(n)}, z_1^{(n)}\}$. We now define $\mathbb{Z}G$-linear maps $\partial_n : F_n \to F_{n-1}$ $(n > 0)$, $\partial_0 : F_0 \to \mathbb{Z}$ inductively, using ϕ-semi-linear maps $\psi_n : F_n \to F_n$ and ϕ^2-semi-linear maps $h_n : F_n \to F_{n+1}$:

$$h_0(z^{(0)}) = z_0^{(1)}, \quad h_n(z_0^{(n)}) = 0, \quad h_n(z_1^{(n)}) = z_0^{(n+1)}(n > 0),$$
$$\psi_0(z^{(0)}) = z^{(0)}, \quad \psi_n(z_0^{(n)}) = z_1^{(n)}(n > 0),$$
$$\partial_0(z^{(0)}) = 1, \quad \partial_1(z_0^{(1)}) = (1 - x_0)z^{(0)},$$
$$\partial_{n+1}(z_1^{(n+1)}) = \psi_n \partial_{n+1}(z_0^{(n+1)})(n \geq 0),$$
$$\psi_n(z_1^{(n)}) = h_{n-1}\partial_n(z_0^{(n)}) + x_0 z_1^{(n)}(n > 0),$$
$$\partial_{n+1}(z_0^{(n+1)}) = \psi_n^2(z_1^{(n)}) - x_0\psi_n(z_1^{(n)}) - h_{n-1}\partial_n(z_1^{(n)})(n > 0).$$

The complex $\mathcal{F} = (F_i, \partial_i)$ thus defined is a resolution of type FP_∞ for G [24]. The matrix associated with ∂_1 is

$$D_0(\mathcal{F}) = \begin{bmatrix} 1 - x_0 \\ 1 - x_1 \end{bmatrix},$$

as would be expected. For $n > 0$, $D_n(\mathcal{F})$ is a 2×2 matrix, so suppose that

$$D_n = D_n(\mathcal{F}) = \begin{bmatrix} \alpha^{(n)} & \beta^{(n)} \\ \gamma^{(n)} & \delta^{(n)} \end{bmatrix}.$$

Then we can calculate that

$$\alpha^{(1)} = (x_1 - x_0)(1 - x_2) - (1 - x_3),$$
$$\beta^{(1)} = (x_1 - x_0)x_0 + 1 - x_3,$$
$$\gamma^{(1)} = ((x_2 - x_1)x_1 + 1 - x_4)(1 - x_2),$$
$$\delta^{(1)} = (x_2 - x_1)(1 - x_3 + x_1x_0) + (1 - x_4)(x_0 - 1),$$

and the definitions of the boundary maps show that

$$\alpha^{(n+1)} = (x_1 - x_0)\phi^2(\beta^{(n)}) - \phi^2(\delta^{(n)}),$$
$$\beta^{(n+1)} = \phi^3(\beta^{(n)}) + (x_1 - x_0)x_0,$$
$$\gamma^{(n+1)} = (\phi^4(\beta^{(n)}) + (x_2 - x_1)x_1)\phi^2(\beta^{(n)}),$$
$$\delta^{(n+1)} = (\phi^4(\beta^{(n)}) + (x_2 - x_1)x_1)x_0 + (x_2 - x_1)\phi^3(\beta^{(n)}) - \phi^3(\delta^{(n)}).$$

Now let us apply the abelianising map $\tau = \tau_G^{ab} : \mathbb{Z}G \to \mathbb{Z}G^{ab}$. Note that G^{ab} is free abelian of rank two, on generators x_0, x_1, and the homomorphism τ sends x_i to x_1 for $i > 0$ and x_0 to x_0. Note also that, for $j \geq 1$, $\tau\phi^j = \tau\phi = \phi^{ab}\tau$, where ϕ^{ab} is the map $x_i \mapsto x_1$ $(i = 0, 1)$. If we apply τ to each entry in D_{n+1}, we get the matrix

$$D_{n+1}^{ab} = \begin{bmatrix} (x_1 - x_0)\phi^{ab}\tau(\beta^{(n)}) - \phi^{ab}\tau(\delta^{(n)}) & \phi^{ab}\tau(\beta^{(n)}) + (x_1 - x_0)x_0 \\ \phi^{ab}\tau(\beta^{(n)})^2 & \phi^{ab}\tau(\beta^{(n)})x_0 - \phi^{ab}\tau(\delta^{(n)}) \end{bmatrix}.$$

By induction, starting with

$$D_1^{ab} = \begin{bmatrix} (x_1 - x_0 - 1)(1 - x_1) & (x_1 - x_0)x_0 + (1 - x_1) \\ (1 - x_1)^2 & -(1 - x_1)(1 - x_0)) \end{bmatrix},$$

we obtain

$$D_n^{ab} = \begin{bmatrix} (1 - x_0)(1 - x_1) & (x_1 - x_0)x_0 + (1 - x_1) \\ (1 - x_1)^2 & -(x_1 - x_0 - 1)(1 - x_1) \end{bmatrix}$$

for even n and $D_n^{ab} = D_1^{ab}$ for odd n. Now, for $n > 0$, $\det(D_n^{ab}) = 0$ and, for $n \geq 0$, the entries of D_n^{ab} generate the ideal $(1 - x_0, 1 - x_1)$, and so, for $n \geq 0$,

$$E_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{ab} & \lambda > 0 \\ IG^{ab} & \lambda = 0 \cdot \\ 0 & \lambda < 0 \end{cases}$$ □

4.2 The E-ideals of a module

Let M be a KG-module of type FP_n and let $\mathcal{F} = (F_i, \partial_i)$ be a resolution of type FP_n of M. Let ^{T} be an abelianising functor. For $\lambda \in \mathbb{Z}$, we define the λ-th E_n^T*-ideal* of M to be

$$E_{n,\lambda}^T(M) = J_{\chi_n(\mathcal{F})-\lambda}(D_n(\mathcal{F})^T).$$

The chain $E_n^T(M)$ of ideals $E_{n,\lambda}^T(M)$ $(\lambda \in \mathbb{Z})$ is then an ascending chain of ideals in KG^T.

In defining $E_n^T(M)$ we made a choice of resolution, $\mathcal{F}$, and of free bases for F_{n+1}, F_n. However, the choices are immaterial.

Theorem 4.7. *The chain of ideals $E_n^T(M)$ is well-defined.*

To proof this, we require some intermediary results.

For $m \geq 0$, a *Tietze transformation* of rank m on a free resolution $\mathcal{F} = (F_i, \partial_i)$ of M is an operation as follows: let P be a KG-free module and let $\phi : P \to F_m$ be a KG-homomorphism. Let $\mathcal{F}'$ be the KG-complex obtained from $\mathcal{F}$ by replacing the part

$$\cdots \xrightarrow{\partial_{m+2}} F_{m+1} \xrightarrow{\partial_{m+1}} F_m \xrightarrow{\partial_m} \cdots$$

of $\mathcal{F}$ by

$$\cdots \xrightarrow{\partial'_{m+2}} F_{m+1} \oplus P \xrightarrow{\partial'_{m+1}} F_m \oplus P \xrightarrow{\partial'_m} \cdots,$$

where, for $f_i \in F_i$ $(i = m, m+1, m+2)$, $p \in P$,

$$\begin{aligned} \partial'_m(f_m, p) &= \partial_m(f_m + \phi(p)), \\ \partial'_{m+1}(f_{m+1}, p) &= (\partial_{m+1}(f_{m+1}) - \phi(p), p), \\ \partial'_{m+2}(f_{m+2}) &= (\partial_{m+2}(f_{m+2}), 0). \end{aligned}$$

Lemma 4.8. *If $\mathcal{F}'$ is obtained from a free resolution $\mathcal{F}$ of M by a Tietze transformation of rank m, then $\mathcal{F}'$ is also a free resolution of M.*

Proof. We must show that $\mathcal{F}' \xrightarrow{\partial} M \to 0$ is exact at F_{m+2}, $F_{m+1} \oplus P$, $F_m \oplus P$ and F_{m-1} (if $m = 0$, take $F_{-1} = M$).

Since $\partial'_{m+2}(f_{m+2}) = 0$ if, and only if, $\partial_{m+2}(f_{m+2}) = 0$, $\mathcal{F}'$ is exact at F_{m+2}. At $F_{m+1} \oplus P$,

$$\begin{aligned} \ker \partial'_{m+1} &= \{(f_{m+1}, p) \in F_{m+1} \oplus P : (\partial_{m+1}(f_{m+1}) - \phi(p), p) = (0,0)\} \\ &= \{(f_{m+1}, 0) \in F_{m+1} \oplus P : \partial_{m+1}(f_{m+1}) = 0\} \\ &= \{(f_{m+1}, 0) \in F_{m+1} \oplus P : f_{m+1} \in \operatorname{im} \partial_{m+2}\} \\ &= \operatorname{im} \partial'_{m+2}, \end{aligned}$$

whence exactness. At $F_m \oplus P$,

$$\begin{aligned}\ker \partial'_m &= \{(f_m, p) \in F_m \oplus P : f_m + \phi(p) \in \ker \partial_m = \operatorname{im} \partial_{m+1}\} \\ &= \{(\partial_{m+1}(f_{m+1}) - \phi(p), p) : f_{m+1} \in F_{m+1}, p \in P\} \\ &= \operatorname{im} \partial'_{m+1},\end{aligned}$$

giving exactness. Finally, $\operatorname{im} \partial'_m = \operatorname{im} \partial_m$, so $\mathcal{F}'$ is a resolution of M. □

We will say that a Tietze transformation as above is *finitary* if P is of finite rank. Thus, if $\mathcal{F}$ is of type FP_n and $\mathcal{F}'$ is obtained from it by a Tietze transformation of rank m, then $\mathcal{F}'$ is of type FP_n if, and only if, either $m > n$, or $m \leq n$ and the transformation is finitary.

Lemma 4.9. *If $\mathcal{F} = (F_i, \partial_i)$, $\mathcal{P} = (P_i, \varepsilon_i)$ are two free resolutions of M, then, for any finite $m \geq 0$, we can obtain free resolutions $\mathcal{F}^{(m)} = (F_i^{(m)}, \partial_i^{(m)})$, $\mathcal{P}^{(m)} = (P_i^{(m)}, \varepsilon_i^{(m)})$ from $\mathcal{F}$, $\mathcal{P}$ respectively, by a finite number of Tietze transformations of rank $\leq m$, such that $\mathcal{F}^{(m)}$, $\mathcal{P}^{(m)}$ are identical in dimensions $0, 1, \ldots, m$ (that is, $F_i^{(m)} = P_i^{(m)}$ and $\partial_i^{(m)} = \varepsilon_i^{(m)}$ for $i = 0, 1, \ldots, m$). Moreover, if $\mathcal{F}$ and $\mathcal{P}$ are both of type FP_n, then so are $\mathcal{F}^{(m)}$ and $\mathcal{P}^{(m)}$.*

Proof. We use induction on m. Suppose that $m \geq 0$ and that the result holds for $m-1$. We can therefore obtain resolutions $\mathcal{F}^{(m-1)} = (F_i^{(m-1)}, \partial_i^{(m-1)})$ and $\mathcal{P}^{(m-1)} = (P_i^{(m-1)}, \varepsilon_i^{(m-1)})$, with $F_i^{(m-1)} = P_i^{(m-1)}$, $\partial_i^{(m-1)} = \varepsilon_i^{(m-1)}$ for $i \leq m-1$ (when $m = 0$, take $\mathcal{F}^{(-1)} = \mathcal{F}$, $\mathcal{P}^{(-1)} = \mathcal{P}$ and $F_{-1}^{(-1)} = M = P_{-1}^{(-1)}$, $\partial_{-1}^{(-1)} = 0 = \varepsilon_{-1}^{(-1)}$). In particular, we have

$$\begin{aligned}\operatorname{im} \partial_m^{(m-1)} &= \ker \partial_{m-1}^{(m-1)} \\ &= \ker \varepsilon_{m-1}^{(m-1)} \\ &= \operatorname{im} \varepsilon_m^{(m-1)},\end{aligned}$$

so we have maps

$$\begin{array}{ccc} F_m^{(m-1)} & \xrightarrow{\partial_m^{(m-1)}} & \operatorname{im} \partial_m^{(m-1)} \\ & & \| \\ P_m^{(m-1)} & \xrightarrow{\varepsilon_m^{(m-1)}} & \operatorname{im} \varepsilon_m^{(m-1)} \end{array}.$$

Since $F_m^{(m-1)}$, $P_m^{(m-1)}$ are free, we can therefore find maps

$$\phi_m : P_m^{(m-1)} \to F_m^{(m-1)}, \quad \psi_m : F_m^{(m-1)} \to P_m^{(m-1)}$$

such that

$$\partial_m^{(m-1)}\phi_m = \varepsilon_m^{(m-1)}, \quad \varepsilon_m^{(m-1)}\psi_m = \partial_m^{(m-1)}.$$

Applying a Tietze transformation of rank m to $\mathcal{F}^{(m-1)}$ using ϕ_m gives a resolution $\mathcal{F}^{(m)} = (F_i^{(m)}, \partial_i^{(m)})$, with

$$F_i^{(m)} = \begin{cases} F_i^{(m-1)} & 0 \le i < m, i > m+1 \\ F_i^{(m-1)} \oplus P_i^{(m-1)} & i = m, m+1 \end{cases},$$

$\partial_i^{(m)} = \partial_i^{(m-1)}$ for $i < m$, and

$$\begin{aligned} \partial_m^{(m)}(f_m^{(m-1)}, p_m^{(m-1)}) &= \partial_m^{(m-1)}(f_m^{(m-1)} + \phi_m(p_m^{(m-1)})) \\ &= \partial_m^{(m-1)}(f_m^{(m-1)}) + \varepsilon_m^{(m-1)}(p_m^{(m-1)}), \end{aligned}$$

for $f_m^{(m-1)} \in F_m^{(m-1)}, p_m^{(m-1)} \in P_m^{(m-1)}$. Similarly, using ψ_m, we obtain a free resolution $\mathcal{P}^{(m)} = (P_i^{(m)}, \varepsilon_i^{(m)})$, with $P_i^{(m)} = F_i^{(m)}$ for $i \le m$, $\varepsilon_i^{(m)} = \partial_i^{(m)}$ for $i < m$ and

$$\begin{aligned} \varepsilon_m^{(m)}(p_m^{(m-1)}, f_m^{(m-1)}) &= \varepsilon_m^{(m-1)}(p_m^{(m-1)} + \psi_m(f_m^{(m-1)})) \\ &= \varepsilon_m^{(m-1)}(p_m^{(m-1)}) + \partial_m^{(m-1)}(f_m^{(m-1)}). \end{aligned}$$

Thus $\varepsilon_m^{(m)} = \partial_m^{(m)}$, as required.

If $\mathcal{F}$, $\mathcal{P}$ are of type FP_n, then the required Tietze transformations which are of rank $\le n$ are finitary, so $\mathcal{F}^{(m)}$, $\mathcal{P}^{(m)}$ are of type FP_n. □

It seems worthwhile to remark that, if a 3-presentation $\mathcal{T}_0$ is obtained from a 3-presentation $\mathcal{T}$ by a Tietze transformation (Tm) ($m = 1, 2, 3$), then the associated partial resolution $\mathcal{F}_{\mathcal{T}_0}$ (as in (1.12)) can be obtained from $\mathcal{F}_{\mathcal{T}}$ by a Tietze transformation of rank m. Compare Lemma 4.9 with Lemmata 1.3 and 1.4.

Proof of Theorem 4.7. By a comment in §1.4.3 and a remark below, the choice of bases for F_{n+1}, F_n does not affect the elementary ideals of $D_n(\mathcal{F})^T$.

Suppose that $\mathcal{P} = (P_i, \varepsilon_i)$ is another resolution of type FP_n of M. By a finite number of Tietze transformations of rank $\le n+1$ (which are finitary when of rank

$\leq n$) we can obtain resolutions $\mathcal{F}^{(n+1)}$, $\mathcal{P}^{(n+1)}$ of type FP_n which are identical in dimensions $0, 1, \ldots, n+1$. Thus $\chi_n(\mathcal{F}^{(n+1)}) = \chi_n(\mathcal{P}^{(n+1)})$ and

$$J(D_n(\mathcal{F}^{(n+1)})^T) = J(D_n(\mathcal{P}^{(n+1)})^T).$$

To prove the theorem, we need only show that, if $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a Tietze transformation of rank m (finitary if $m \leq n$), using some map $\phi : P \to F_m$, then, for $\lambda \in \mathbb{Z}$,

$$J_{\chi_n(\mathcal{F}')-\lambda}(D_n(\mathcal{F}')^T) = J_{\chi_n(\mathcal{F})-\lambda}(D_n(\mathcal{F})^T).$$

First note that, if $r = \mathrm{rk}_{KG}(P)$, then

$$\chi_n(\mathcal{F}') = \begin{cases} \chi_n(\mathcal{F}) & m \neq n \\ \chi_n(\mathcal{F}) + r & m = n \end{cases}.$$

If $m > n+1$, or $m < n-1$, then $D_n(\mathcal{F}') = D_n(\mathcal{F})$, so the result follows. If $m = n-1$, then

$$D_n(\mathcal{F}') = \begin{bmatrix} D_n(\mathcal{F}) & 0 \end{bmatrix},$$

where we have added r columns of zeroes. The required result follows, since

$$J(D_n(\mathcal{F}')^T) = J\left(\begin{bmatrix} D_n(\mathcal{F})^T & 0 \end{bmatrix}\right) = J(D_n(\mathcal{F})^T).$$

When $m = n$, we have

$$D_n(\mathcal{F}') = \begin{bmatrix} D_n(\mathcal{F}) & 0 \\ -X & I_r \end{bmatrix},$$

where X is the $r \times \mathrm{rk}_{KG}(F_n)$ matrix of $\phi : P \to F_n$. Thus, by Corollary 1.25,

$$\begin{aligned} J_{\chi_n(\mathcal{F}')-\lambda}(D_n(\mathcal{F}')^T) &= J_{\chi_n(\mathcal{F}')-r-\lambda}(D_n(\mathcal{F})^T) \\ &= J_{\chi_n(\mathcal{F})-\lambda}(D_n(\mathcal{F})^T). \end{aligned}$$

Finally, when $m = n+1$ (so P need not be of finite rank),

$$D_n(\mathcal{F}') = \begin{bmatrix} D_n(\mathcal{F}) \\ XD_n(\mathcal{F}) \end{bmatrix}.$$

Since each row of $XD_n(\mathcal{F})$ is a linear combination of rows of $D_n(\mathcal{F})$,

$$J(D_n(\mathcal{F}')) = J(D_n(\mathcal{F})),$$

by Proposition 1.21. $\square$

Remarks. 1. Notice that in the above proof we made no use of the fact that $\alpha^T \tau_G^T = \tau_{G_0}^T \alpha$ when $\alpha : G \to G_0$ is a homomorphism. Thus, if $\tau : KG \to C$ is *any* ring homomorphism, where C is commutative ring, we can define a chain of ideals $E_n^\tau(M)$ in C. It is only when we come to compare the ideals of modules of distinct groups that we require the functorial properties of ^{T}.

2. In the current terminology, $E_n^T(G, K) = E_n^T({}_G K)$, so we have also proved that $E_n^T(G, K)$ is well-defined, the first part of Theorem 4.1. The second part is proved in the following section.

3. Different choices of bases for F_n or F_{n+1} will result in matrices $D_n(\mathcal{F})^T$ which differ by multiplication on the left and right by invertible matrices (these will be, respectively, the associated change of basis matrices for the KG^T-free modules $KG^T \otimes_{KG} F_{n+1}$ and $KG^T \otimes_{KG} F_n$). Thus, by Corollary 1.23, the choice of bases does not affect the chain of ideals. We will therefore frequently fail to specify bases, as, indeed, we did in the above proof.

We define

$$\nu_n^T(M) = \min\{\lambda \in \mathbb{Z} : E_{n,\lambda}^T(M) = KG^T\},$$
$$\delta_n^T(M) = \min\{\lambda \in \mathbb{Z} : E_{n,\lambda}^T(M) \neq 0\},$$

for a KG-module M of type FP_n, abbreviating them to ν_n^T, δ_n^T respectively, when no confusion can arise. It is a consequence of Theorem 4.7 that these derived invariants are well-defined.

If G is of type FP_n over K, then $\nu_n^T(G, K) = \nu_n^T({}_G K)$ and $\delta_n^T(G, K) = \delta_n^T({}_G K)$.

4.3 Some properties of the E-ideals of modules and groups

While the first few results of this section do have wider applications, their first use is to provide a proof of the second part of Theorem 4.1.

Lemma 4.10. *If M, M' are isomorphic KG-modules of type FP_n, then $E_n^T(M') = E_n^T(M)$.*

Proof. If $\mathcal{F} = (F_i, \partial_i)$ is a free resolution of type FP_n for M, then, by composing $\partial_0 : F_0 \to M$ with the isomorphism $M \to M'$, we obtain a resolution of M'. Thus

$$E^T_{n,\lambda}(M') = J_{\chi_n(\mathcal{F})-\lambda}(D_n(\mathcal{F})^T) = E^T_{n,\lambda}(M). \qquad \square$$

Remark. We will make frequent use of this result, often without comment.

If $\alpha : G \to G_0$ is a group homomorphism, then any KG_0-module becomes a KG-module with G acting via α. For example, the right KG_0-module KG_0 becomes a right KG-module.

Proposition 4.11. *Let $\alpha : G \to G_0$ be a group homomorphism and let M be a KG-module of type FP_n. If $Tor^{KG}_i(KG_0, M) = 0$ for $i = 1, \ldots, n$, then $KG_0 \otimes_{KG} M$ is a KG_0-module of type FP_n and*

$$E^T_n(KG_0 \otimes_{KG} M) = \left(\alpha^T E^T_n(M)\right).$$

Proof. Let $\mathcal{F} = (F_i, \partial_i)$ be a resolution of type FP_n of M. Since

$$\mathrm{Tor}^{KG}_i(KG_0, M) = H_i(KG_0 \otimes_{KG} \mathcal{F}) = 0,$$

for $i = 1, \ldots, n$, the sequence $KG_0 \otimes_{KG} \mathcal{F}$ must be exact in dimensions $1, \ldots, n$. Thus, since

$$\mathrm{Tor}^{KG}_0(KG_0, M) = H_0(KG_0 \otimes_{KG} \mathcal{F}) = KG_0 \otimes_{KG} M,$$

$KG_0 \otimes_{KG} \mathcal{F}$ gives a partial resolution of $KG_0 \otimes_{KG} M$ of type FP_n and of length $n+1$, with $D_n(KG_0 \otimes_{KG} \mathcal{F}) = D_n(\mathcal{F})^\alpha$. Therefore,

$$\begin{aligned} E^T_n(KG_0 \otimes_{KG} M) &= J_{\chi_n(KG_0 \otimes_{KG} \mathcal{F})-\lambda}(D_n(KG_0 \otimes_{KG} \mathcal{F})^T) \\ &= J_{\chi_n(\mathcal{F})-\lambda}((D_n(\mathcal{F})^\alpha)^T) \\ &= \left(\alpha^T J_{\chi_n(\mathcal{F})-\lambda}(D_n(\mathcal{F})^T)\right) \\ &= \left(\alpha^T E^T_{n,\lambda}(M)\right), \end{aligned}$$

since $\chi_n(KG_0 \otimes_{KG} \mathcal{F}) = \chi_n(\mathcal{F})$. $\square$

Remark. Notice that this is the first occasion in this chapter on which we have required $\alpha^T \tau^T_G = \tau^T_G \alpha$ and the first occasion in the thesis when we have applied it when α is not an isomorphism.

Corollary 4.12. *If $\alpha : G \to G_0$ is an isomorphism, then, for any KG-module M of type FP_n, $KG_0 \otimes_{KG} M$ is of type FP_n and*

$$E_n^T(KG_0 \otimes_{KG} M) = \alpha^T E_n^T(M).$$

The second part of Theorem 4.1 then follows, since $KG_0 \otimes_{KG} {}_GK \cong {}_{G_0}K$, so, by Lemma 4.10 and the last corollary,

$$\begin{aligned} E_n^T(G,K) &= E_n^T({}_{G_0}K) \\ &= E_n^T(KG_0 \otimes_{KG} {}_GK) \\ &= \alpha^T E_n^T({}_GK) \\ &= \alpha^T E_n^T(G,K). \end{aligned}$$

If H is a subgroup of G, with embedding map $\iota : H \to G$, then, for any KH-module M, we have the induced KG-module

$$M \uparrow_H^G = KG \otimes_{KH} M.$$

Recall that KG is a free right KH-module.

Corollary 4.13. *If $\iota : H \to G$ is injective and if M is a KH-module of type FP_n, then $M \uparrow_H^G$ is of type FP_n, and*

$$E_n^T(M \uparrow_H^G) = \left(\iota^T E_n^T(M)\right).$$

As well as helping to prove the invariance of the E-ideals of a group, another reason for studying the invariant ideals of modules other than ${}_GK$ is that they can sometimes give information about the group invariants, as the following dimension shifting results show.

Proposition 4.14. *If M is a KG-module of type FP_n and if $\mathcal{F} = (F_i, \partial_i)$ is a resolution of type FP_m of M, $m < n$, then*

$$E_{n,\lambda}^T(M) = E_{n-m-1,\lambda-(-1)^{n-m}\chi_m(\mathcal{F})}^T(\ker \partial_m).$$

Proof. By Lemma 1.27, $\ker \partial_m$ is of type FP_{n-m-1}. If $\mathcal{P} = (P_i, \varepsilon_i)$ is a resolution of type FP_{n-m-1} of $\ker \partial_m$, then, by splicing this with $\mathcal{F}$, we obtain a resolution

$\mathcal{F}' = (F_i', \partial_i')$ of type FP_n of M, with

$$F_i' = \begin{cases} F_i & i \leq m \\ P_{i-m-1} & i > m \end{cases}, \quad \partial_i' = \begin{cases} \partial_i & i \leq m \\ \varepsilon_{i-m-1} & i > m \end{cases}.$$

Thus, $\chi_n(\mathcal{F}') = \chi_{n-m-1}(\mathcal{P}) + (-1)^{n-m}\chi_m(\mathcal{F})$ and $D_n(\mathcal{F}') = D_{n-m-1}(\mathcal{P})$, so

$$\begin{aligned} E_{n,\lambda}^T(M) &= J_{\chi_n(\mathcal{F}')-\lambda}(D_n(\mathcal{F}')^T) \\ &= J_{\chi_{n-m-1}(\mathcal{P})+(-1)^{n-m}\chi_m(\mathcal{F})-\lambda}(D_{n-m-1}(\mathcal{P})^T) \\ &= E_{n-m-1,\lambda-(-1)^{n-m}\chi_m(\mathcal{F})}^T(\ker \partial_m). \end{aligned}$$ □

If $\mathcal{T}$ is a 3-presentation, then we have the partial resolution $\mathcal{F}_{\mathcal{T}}$ of (1.12). Applying the last proposition to this, we obtain the following corollary.

Corollary 4.15. *Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ and $G = G(\mathcal{P})$.*

i) For $n > 0$

$$E_{n,\lambda}^T(G) = E_{n-1,\lambda-(-1)^n}^T(IG).$$

ii) For $n > 1$, if $\mathbf{x}$ is finite,

$$E_{n,\lambda}^T(G) = E_{n-2,\lambda+(-1)^n\chi_1(\mathcal{P})}^T(M(\mathcal{P})).$$

iii) For $n > 2$, if $\mathcal{P}$ is finite,

$$E_{n,\lambda}^T(G) = E_{n-3,\lambda-(-1)^n\chi_2(\mathcal{P})}^T(\pi_2(\mathcal{P})).$$

Remark. The first of these formulæ is particularly useful, since it is independent of any particular presentation.

The following results show how the invariants of a module can depend on the structure of the module.

Proposition 4.16. *If $M = M' \oplus M''$, then M is of type FP_n if, and only if, both M' and M'' are and, in that case,*

$$E_n^T(M) = E_n^T(M') * E_n^T(M'').$$

Proof. If $\mathcal{F}' = (F_i', \partial_i')$ is a resolution of type FP_n of M' and $\mathcal{F}'' = (F_i'', \partial_i'')$ a resolution of type FP_n of M'', then $\mathcal{F}' \oplus \mathcal{F}'' = (F_i' \oplus F_i'', \partial_i' \oplus \partial_i'')$ is a resolution of type FP_n of M. Since

$$\chi_n(\mathcal{F}' \oplus \mathcal{F}'') = \chi_n(\mathcal{F}') + \chi_n(\mathcal{F}'')$$

and

$$D_n(\mathcal{F}' \oplus \mathcal{F}'') = \begin{bmatrix} D_n(\mathcal{F}') & 0 \\ 0 & D_n(\mathcal{F}'') \end{bmatrix},$$

applying Proposition 1.19 we obtain

$$\begin{aligned} E_{n,\lambda}^T(M) &= J_{\chi_n(\mathcal{F}' \oplus \mathcal{F}'')-\lambda}(D_n(\mathcal{F}' \oplus \mathcal{F}'')^T) \\ &= \sum_{j \in \mathbb{Z}} J_{\chi_n(\mathcal{F}')-j}(D_n(\mathcal{F}')^T) J_{\chi_n(\mathcal{F}'')-\lambda+j}(D_n(\mathcal{F}'')^T) \\ &= \sum_{j \in \mathbb{Z}} E_{n,j}^T(M') E_{n,\lambda-j}^T(M''), \end{aligned}$$

as required. That M' and M'' are of type FP_n when M is follows from repeated application of Lemma 1.27 (see [22]). □

Corollary 4.17. *If $M_1, \dots, M_m$ are modules of type FP_n, then $\oplus_{i=1}^m M_i$ is of type FP_n and*

$$E_n^T(\oplus_{i=1}^m M_i) = *_{i=1}^m E_n^T(M_i).$$

Note that, in general,

$$\begin{aligned} E_{n,\nu_n'+\nu_n''}^T(M' \oplus M'') &\supseteq E_{n,\nu_n'}^T(M') E_{n,\nu_n''}^T(M'') \\ &= KG^T, \end{aligned}$$

where $\nu_n' = \nu_n^T(M')$ and $\nu_n'' = \nu_n^T(M'')$, so

$$\nu_n^T(M' \oplus M'') \leq \nu_n^T(M') + \nu_n^T(M''). \tag{4.4}$$

Also

$$\begin{aligned} E_{n,\nu_n'+\nu_n''-1}^T(M' \oplus M'') &= \sum_{j \in \mathbb{Z}} E_{n,\nu_n'+j}^T(M') E_{n,\nu_n''-1-j}^T(M'') \\ &= \sum_{j<0} E_{n,\nu_n'+j}^T(M') + \sum_{j \geq 0} E_{n,\nu_n''-1-j}^T(M'') \\ &= E_{n,\nu_n'-1}^T(M') + E_{n,\nu_n''-1}^T(M''). \end{aligned}$$

When $M' = M''$, this last ideal can never be the whole ring.

Corollary 4.18. *If M is of type FP_n, then for $m \geq 0$, $M^m = M \oplus M \oplus \cdots \oplus M$ is of type FP_n and*

$$\nu_n^T(M^m) = m\nu_n^T(M).$$

On the other hand,

$$E^T_{n,\delta'_n+\delta''_n-1}(M' \oplus M'') = 0$$

and

$$E^T_{n,\delta'_n+\delta''_n}(M' \oplus M'') = E^T_{n,\delta'_n}(M')E^T_{n,\delta''_n}(M''),$$

where $\delta'_n = \delta_n^T(M')$ and $\delta''_n = \delta_n^T(M'')$. If the ring KG^T has no zero-divisors, then this last ideal cannot be zero, and so:

Corollary 4.19. *If M', M'' are KG-modules of type FP_n and if KG^T is an integral domain, then*

$$\delta_n^T(M' \oplus M'') = \delta_n^T(M') + \delta_n^T(M'').$$

This applies, for example, when $K = \mathbb{Z}$ and ${}^T = {}^{tf}$ or ^{triv}.

The next result contains a generalisation of Proposition 4.16.

Proposition 4.20. *Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\alpha} M'' \to 0$ be a short exact sequence of KG-modules.*

i) If M' and M'' are of type FP_n, then so is M and

$$E_n^T(M) \supseteq E_n^T(M') * E_n^T(M'')$$

(with equality when the sequence splits).

ii) If M' is of type FP_{n-1} $(n > 0)$ and M is of type FP_n, then M'' is of type FP_n and

$$E_n^T(M'') \supseteq E_{n-1}^T(M') * E_n^T(M).$$

If M is of type FP_0, then so is M'' and $E_0^T(M'') \supseteq E_0^T(M)$.

iii) If M is of type FP_n and M'' is of type FP_{n+1}, then M' is of type FP_n and, for $n > 0$,

$$E_n^T(M') \supseteq E_n^T(M) * E_{n+1}^T(M'').$$

When $n = 0$,

$$E^T_{0,\lambda}(M') \supseteq E^T_{1,\lambda-r_0}(M''),$$

where r_0 *is the minimum number of generators of* M *as a* KG*-module.*

Proof. (i) Using the horseshoe construction of §1.6.4, given free resolutions $\mathcal{F}'$ of M' and $\mathcal{F}''$ of M'', we obtain a resolution $\mathcal{F}$ of M with

$$\chi_n(\mathcal{F}) = \chi_n(\mathcal{F}') + \chi_n(\mathcal{F}'')$$

and

$$D_n(\mathcal{F}) = \begin{bmatrix} D_n(\mathcal{F}') & 0 \\ X_n & D_n(\mathcal{F}'') \end{bmatrix}$$

for some matrix X_n over KG. The result then follows as in Proposition 4.16, since

$$J\left(\begin{bmatrix} D_n(\mathcal{F}')^T & 0 \\ X_n^T & D_n(\mathcal{F}'')^T \end{bmatrix}\right) \supseteq J(D_n(\mathcal{F}')^T) * J(D_n(\mathcal{F}'')^T).$$

(ii) By §1.6.4, given free resolutions $\mathcal{F}'$ of M' and $\mathcal{F}$ of M, we obtain a resolution $\mathcal{F}''$ of M''. For $n > 0$,

$$\chi_n(\mathcal{F}'') = \chi_n(\mathcal{F}) + \chi_{n-1}(\mathcal{F}')$$

and

$$D_n(\mathcal{F}'') = \begin{bmatrix} D_n(\mathcal{F}) & 0 \\ X_n & -D_{n-1}(\mathcal{F}') \end{bmatrix}$$

for some matrix X_n over KG. When $n = 0$,

$$\chi_0(\mathcal{F}'') = \chi_0(\mathcal{F})$$

and

$$D_0(\mathcal{F}'') = \begin{bmatrix} D_0(\mathcal{F}) \\ X_0 \end{bmatrix}.$$

The required results then follow.

(iii) As in the proof of Lemma 1.31, if $n > 0$, applying (ii) to the short exact sequence (1.13) and then Proposition 4.14 gives

$$\begin{aligned}
E^T_{n,\lambda}(M') &\supseteq \sum_{j\in\mathbb{Z}} E^T_{n-1,j}(\ker \partial_0)E^T_{n,\lambda-j}(\ker \alpha\partial_0) \\
&= \sum_{j\in\mathbb{Z}} E^T_{n,j+(-1)^n r_0}(M)E^T_{n+1,\lambda-j-(-1)^n r_0}(M'') \\
&= \sum_{j\in\mathbb{Z}} E^T_{n,j}(M)E^T_{n+1,\lambda-j}(M'').
\end{aligned}$$

When $n = 0$, we have

$$E^T_{n,\lambda}(M') \supseteq E^T_{0,\lambda}(\ker \alpha\partial_0) = E^T_{1,\lambda-r_0}(M''),$$

as required. □

Generalising (4.4) and Corollary 4.19 gives:

Corollary 4.21. *Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\alpha} M'' \to 0$ be a short exact sequence of KG-modules.*

i) If M, M' and M'' are of type FP_n, then

$$\nu^T_n(M) \le \nu^T_n(M') + \nu^T_n(M'').$$

If KG^T is an integral domain, then

$$\delta^T_n(M) \le \delta^T_n(M') + \delta^T_n(M''),$$

with equality when the sequence splits.

ii) If M, M'' are of type FP_n $(n > 0)$ and M' is of type FP_{n-1}, then

$$\nu^T_n(M) \ge -\nu^T_{n-1}(M') + \nu^T_n(M'').$$

If KG^T is an integral domain, then

$$\delta^T_n(M) \ge -\delta^T_{n-1}(M') + \delta^T_n(M'').$$

If M and M'' are of type FP_0, then

$$\nu^T_0(M) \ge \nu^T_0(M'') \text{ and } \delta^T_0(M) \ge \delta^T_0(M'').$$

iii) Let M and M' be of type FP_n and M'' of type FP_{n+1}. If $n > 0$, then

$$\nu_n^T(M) \geq \nu_n^T(M') - \nu_{n+1}^T(M'')$$

and, when KG^T is an integral domain,

$$\delta_n^T(M) \geq \delta_n^T(M') - \delta_{n+1}^T(M'')$$

If $n = 0$, then

$$r_0 \geq \nu_0^T(M') - \nu_1^T(M'') \text{ and } r_0 \geq \delta_0^T(M') - \delta_1^T(M'')$$

where r_0 is the minimum number of generators of M as a KG-module.

Under certain conditions on one of the modules M, M', M'', the inequalities of the preceding Corollary and Proposition become equalities. We devote §§5.1 and 5.4 to such conditions.

We give an illustrative example of Proposition 4.20.

Example 4.1. Let $\mathcal{Q} = \langle \mathbf{x}; R \rangle$ be a finite one relator presentation, where $R = R_0^p$ with $p > 1$, and let $H_1 = G(\mathcal{Q})$. Let H be the subgroup of H_1 generated by R_0, so H is finite cyclic of order p. For every integer q such that $0 < q < p$ and p and q are coprime, there is a monomorphism $\phi^{(q)} : H \to H_1; R_0 \mapsto R_0^q$. For every such q, define $G^{(q)}$ to be the HNN extension

$$G^{(q)} = H_1 *_{H,\phi^{(q)}} .$$

This group can be presented by

$$\langle \mathbf{x}, t; R, t^{-1}R_0t = R_0^q \rangle.$$

We apply the mapping cylinder construction to the short exact sequence (1.22) to construct a resolution of type FP_∞ for $G^{(q)}$ from the Lyndon resolution for H and H_1.

Let $\mathcal{F}$ be the resolution

$$\cdots \xrightarrow{1 \mapsto \xi_R} \mathbb{Z}G^{(q)} \xrightarrow{1 \mapsto 1 - R_0} \mathbb{Z}G^{(q)} \xrightarrow{1 \mapsto \xi_R} \mathbb{Z}G^{(q)} \xrightarrow{1 \mapsto 1 - R_0} \mathbb{Z}G^{(q)} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}\uparrow_H^{G^{(q)}} \to 0,$$

where $\xi_R = 1 + R_0 + \cdots + R_0^{p-1}$, and let $\mathcal{F}_1$ be the resolution

$$\cdots \xrightarrow{1 \mapsto 1-R_0} \mathbb{Z}G^{(q)} \xrightarrow{1 \mapsto \xi_R} \mathbb{Z}G^{(q)} \xrightarrow{1 \mapsto 1-R_0} \mathbb{Z}G^{(q)}$$
$$\xrightarrow{\partial_2} \oplus_{\mathbf{x}} \mathbb{Z}G^{(q)} e_x \xrightarrow{e_x \mapsto 1-x} \mathbb{Z}G^{(q)} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z} \uparrow_{H_1}^{G^{(q)}} \to 0,$$

where $\partial_2(1) = \sum_{\mathbf{x}} \frac{\partial R}{\partial x} e_x = \xi_R \sum_{\mathbf{x}} \frac{\partial R_0}{\partial x} e_x$.

Now, the map

$$\iota : \mathbb{Z} \uparrow_H^{G^{(q)}} \to \mathbb{Z} \uparrow_{H_1}^{G^{(q)}}; 1 \otimes 1 \mapsto (1-t) \otimes 1$$

lifts to the chain map $\iota : \mathcal{F} \to \mathcal{F}_1$, where

$$\begin{aligned}
&\iota_0 : 1 \mapsto 1 - t, \\
&\iota_1 : 1 \mapsto (1 - t\xi_R) \sum_{\mathbf{x}} \frac{\partial R_0}{\partial x} e_x, \\
&\iota_n : 1 \mapsto \begin{cases} 1 - q^k t & n = 2k \\ 1 - q^k t(1 + R_0 + \cdots + R_0^{q-1}) & n = 2k+1 \end{cases}, k \geq 1.
\end{aligned}$$

This is a chain map, since, for $k \geq 1$,

$$\begin{aligned}
\iota_{2k}(1 - R_0) &= (1 - R_0)(1 - q^k t) \\
&= 1 - R_0 - q^k t + q^k t R_0^q \\
&= (1 - q^k t(1 + R_0 + \cdots + R_0^{q-1}))(1 - R_0)
\end{aligned}$$

(using $R_0 t = t R_0^q$) and

$$\begin{aligned}
\iota_{2k+1}(\xi_R) &= \xi_R(1 - q^k t(1 + R_0 + \cdots + R_0^{q-1})) \\
&= \xi_R - q^k t(1 + R_0^q + \cdots + R_0^{q(p-1)})(1 + R_0 + \cdots + R_0^{q-1}) \\
&= (1 - q^{k+1} t)\xi_R
\end{aligned}$$

(using the fact that p, q are coprime and that $R_0^i \xi_R = \xi_R$). Also

$$\begin{aligned}
\iota_0(1 - R_0) &= (1 - R_0)(1 - t) \\
&= (1 - t(1 + R_0 + \cdots + R_0^{q-1}))(1 - R_0) \quad \text{(as above)} \\
&= (1 - t(1 + R_0 + \cdots + R_0^{q-1})) \sum_{\mathbf{x}} \frac{\partial R_0}{\partial x}(1 - x),
\end{aligned}$$

by Corollary 1.9, and

$$\begin{aligned}\iota_1(\xi_R) &= \xi_R(1-t(1+R_0+\cdots+R_o^{q-1}))\sum_{\mathbf{x}}\frac{\partial R_0}{\partial x} \\ &= (1-qt)\xi_R\sum_{\mathbf{x}}\frac{\partial R_0}{\partial x} \\ &= \partial_2\iota_2(1).\end{aligned}$$

We then obtain a free resolution $\mathcal{P}^{(q)} = (P_i, \varepsilon_i)$, where

$$P_i = \begin{cases} \mathbb{Z}G^{(q)} & i=0 \\ \left(\oplus_{\mathbf{x}}\mathbb{Z}G^{(q)}\right)\oplus\mathbb{Z}G^{(q)} & i=1 \\ \mathbb{Z}G^{(q)}\oplus\mathbb{Z}G^{(q)} & i\geq 2\end{cases}$$

and

$$D_1(\mathcal{P}^{(q)}) = \begin{bmatrix} \left[\xi_R\frac{\partial R_0}{\partial x}\right]_{x\in\mathbf{x}} & 0 \\ \left[(1-t(1+R_0+\cdots+R_0^{q-1}))\frac{\partial R_0}{\partial x}\right]_{x\in\mathbf{x}} & -(1-R_0)\end{bmatrix}$$

$$D_n(\mathcal{P}^{(q)}) = \begin{cases} \begin{bmatrix} 1-R_0 & 0 \\ 1-q^kt & -\xi_R\end{bmatrix} & n=2k \\ \begin{bmatrix} \xi_R & 0 \\ 1-q^kt(1+R_0+\cdots+R_0^{q-1}) & -(1-R_0)\end{bmatrix} & n=2k+1\end{cases}$$

$(k\geq 1)$. Calculating from these matrices gives

$$E_{1,\lambda}(G^{(q)}) = \begin{cases} \mathbb{Z}G^{(q)^{ab}} & \lambda\geq|\mathbf{x}| \\ \left(1-R_0, p\frac{\partial R_0}{\partial x}, (1-qt)\frac{\partial R_0}{\partial x}(x\in\mathbf{x})\right) & \lambda=|\mathbf{x}|-1 \\ 0 & \lambda<|\mathbf{x}|-1\end{cases}$$

and, for $n>1$,

$$E_{n,\lambda}(G^{(q)}) = \begin{cases} \mathbb{Z}G^{(q)^{ab}} & \lambda>(-1)^n(1-|\mathbf{x}|) \\ (1-R_0, p, 1-q^kt) & \lambda=(-1)^n(1-|\mathbf{x}|)\,, \\ 0 & \lambda<(-1)^n(1-|\mathbf{x}|)\end{cases}$$

where $n=2k$ or $2k-1$.

For example, if $p = 5$, then

$$E^{triv}_{1,\lambda}(G^{(2)}) = E^{triv}_{1,\lambda}(G^{(4)}) = \begin{cases} \mathbb{Z} & \lambda \geq |\mathbf{x}| - 1 \\ 0 & \lambda < |\mathbf{x}| - 1 \end{cases},$$

$$E^{triv}_{2,\lambda}(G^{(2)}) = E^{triv}_{2,\lambda}(G^{(4)}) = \begin{cases} \mathbb{Z} & \lambda \geq 1 - |\mathbf{x}| \\ 0 & \lambda < 1 - |\mathbf{x}| \end{cases},$$

but

$$E^{triv}_{3,\lambda}(G^{(2)}) = \begin{cases} \mathbb{Z} & \lambda \geq |\mathbf{x}| - 1 \\ 0 & \lambda < |\mathbf{x}| - 1 \end{cases},$$

whereas

$$E^{triv}_{3,\lambda}(G^{(4)}) = \begin{cases} \mathbb{Z} & \lambda \geq |\mathbf{x}| \\ (5) & \lambda = |\mathbf{x}| - 1 \\ 0 & \lambda < |\mathbf{x}| - 1 \end{cases}.$$

Thus, the E^{triv}_3-ideals can distinguish groups that the E^{triv}_1- and E^{triv}_2-ideals cannot. Indeed, for any $k \geq 1$, we can choose p, q, q' such that the groups $G^{(q)}$ and $G^{(q')}$ have distinct E_{2k-1}-ideals, but identical E^{triv}_i-ideals for $i < 2k - 1$. However, in the next section we will see that these groups can also be distinguished by their integral homology. In §5.7 we will we able to show that, for every $n > 0$, there are groups which can be distinguished neither by their integral homology nor by their E_i-ideals for $i < n$, but which have distinct E_n-ideals.

Proposition 4.20(ii) applied to the sequence (1.22) says that

$$\begin{aligned} E_n(G^{(q)}) &\supseteq E_{n-1}(\mathbb{Z} \uparrow_H^{G^{(q)}}) * E_n(\mathbb{Z} \uparrow_{H_1}^{G^{(q)}}) \\ &= \left(\iota_H^{ab} E_{n-1}(H)\right) * \left(\iota_{H_1}^{ab} E_n(H_1)\right) \end{aligned}$$

for $n > 0$, where $\iota_H : H \to G^{(q)}$ and $\iota_{H_1} : H_1 \to G^{(q)}$ are the natural embeddings. This is indeed the case, since, for $n > 1$,

$$\left(\iota_H^{ab} E_{n-1}(H)\right) * \left(\iota_{H_1}^{ab} E_n(H_1)\right)_\lambda = \begin{cases} \mathbb{Z}G^{(q)ab} & \lambda > (-1)^n(1 - |\mathbf{x}|) \\ (1 - R_0, p) & \lambda = (-1)^n(1 - |\mathbf{x}|) \\ 0 & \lambda < (-1)^n(1 - |\mathbf{x}|) \end{cases}$$

and

$$\left(\iota_H^{ab}E_0(H)\right) * \left(\iota_{H_1}^{ab}E_1(H_1)\right)_\lambda = \begin{cases} \mathbb{Z}G^{(q)ab} & \lambda \geq |\mathbf{x}| \\ \left(1-R_0, p\frac{\partial R_0}{\partial x}(x \in \mathbf{x})\right) & \lambda = |\mathbf{x}|-1 \\ 0 & \lambda < |\mathbf{x}|-1 \end{cases}.$$

Note that this inclusion is strict. ◊

If we take $M'' = 0$ and $M' = M$, then the following corollaries are a consequence of Proposition 4.20(ii), since $E_{n,\lambda}^T(0) = 0$ when $\lambda < 0$ and $\delta_n^T(0) = 0$.

Corollary 4.22. *If M is of type FP_n $(n > 0)$, then, when $\kappa + \lambda < 0$,*

$$E_{n-1,\kappa}^T(M)E_{n,\lambda}^T(M) = 0.$$

Corollary 4.23. *If M is of type FP_n $(n > 0)$ and if KG^T is an integral domain, then*

$$\delta_n^T(M) + \delta_{n-1}^T(M) \geq 0.$$

Example 4.2. We test Corollaries 4.22 and 4.23 when G is a CA group and $M = {}_G\mathbb{Z}$. Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ be a finite CA presentation for G. From §4.1.1, for $n \geq 3$ and $\kappa, \lambda \in \mathbb{Z}$ such that $\kappa + \lambda < 0$, a typical generator of $E_{n-1,\kappa}^T(G)E_{n,\lambda}^T(G)$ is

$$\left(\prod_{R \in \mathbf{s}_1}(1 + R_0 + \cdots + R_0^{p_R - 1})\right)\left(\prod_{R \in \mathbf{s}_2}(1 - R_0)\right),$$

where $\mathbf{s}_1, \mathbf{s}_2 \subseteq \mathbf{r}'$ with

$$|\mathbf{s}_1| = |\mathbf{r}'| - |\mathbf{r}| + |\mathbf{x}| - 1 - \kappa, |\mathbf{s}_2| = |\mathbf{r}| - |\mathbf{x}| + 1 - \lambda.$$

Since $|\mathbf{s}_1| + |\mathbf{s}_2| = |\mathbf{r}'| - \kappa - \lambda > |\mathbf{r}'|$, there is a relator $S \in \mathbf{s}_1 \cap \mathbf{s}_2$ and, since

$$(1 + S_0 + \cdots + S_0^{p_S - 1})(1 - S_0) = 0,$$

the above generator is 0, as required.

Now suppose that for some $R \in \mathbf{r}$, the image of R_0 in G^T is not trivial. Thus G^T is not torsion-free, and so $\mathbb{Z}G^T$ is not an integral domain. Now, for even $n > 0$,

$$0 \neq 1 - R_0 \in E_{n,\chi_n - 1}^T(G)$$

so

$$\delta_n^T(G) < \chi_n = |\mathbf{r}| - |\mathbf{x}| + 1,$$

and, for odd $n > 1$,

$$\delta_n^T(G) = -\chi_{n+1} = -|\mathbf{r}| + |\mathbf{x}| - 1.$$

In this case, we then have

$$\delta_n^T(G) + \delta_{n-1}^T(G) < 0.$$

for $n > 2$. On the other hand, when G^T is torsion-free

$$\delta_n^T(G) = (-1)^n(|\mathbf{r}| - |\mathbf{x}| + 1)$$

for $n \geq 2$, and so Corollary 4.23 does indeed hold. $\diamond$

4.4 The new invariants and homology

If we choose the abelianising functor ^{triv}, then it turns out that, for a group G of type FP_n, the collections of chains of ideals $E_i^{triv}(G)$ $(0 \leq i \leq n)$ carries the same information as the collection of integral homology groups $H_i(G)$ $(0 \leq i \leq n)$.

More generally, if KG^T is a pid (for example, KG^{tf} when K is a field and G^{tf} is infinite cyclic), then, for a KG-module M of type FP_n, the chains $E_i^T(M)$ $(0 \leq i \leq n)$ and the Tor-groups $\mathrm{Tor}_i^{KG}(KG^T, M)$ $(0 \leq i \leq n)$ give the same information, in the following sense.

Since KG^T is a pid, we can choose elements $\xi_\lambda \in KG^T$ such that $E_{n,\lambda}^T(M) = (\xi_\lambda)$ for $\lambda \in \mathbb{Z}$. Thus $\xi_{\lambda+1} | \xi_\lambda$. Also, since $\mathrm{Tor}_n^{KG}(KG^T, M)$ is a finitely generated KG^T-module, it can be written uniquely in the form

$$\mathrm{Tor}_n^{KG}(KG^T, M) = (KG^T)^{q_n} \oplus \left(\oplus_{j=1}^{p_n} KG^T/(\eta_j)\right) \tag{4.5}$$

for some integers $p_n, q_n \geq 0$ and some non-zero, non-unit $\eta_j \in KG^T$ such that $\eta_j | \eta_{j+1}$, as in (1.1).

Theorem 4.24. *Let M be a KG-module of type FP_n and suppose that KG^T is a pid. If $E_{n,\lambda}^T(M) = (\xi_\lambda)$ $(\lambda \in \mathbb{Z})$, $Tor_n^{KG}(KG^T, M) = (KG^T)^{q_n} \oplus \left(\oplus_{j=1}^{p_n} KG^T/(\eta_j)\right)$, as above, and, for $0 \leq i \leq n$,*

$$q_i = rk_{KG^T}(Tor_i^{KG}(KG^T, M)),$$

then

$$\delta_n^T(M) = q_n - q_{n-1} + \cdots + (-1)^n q_0,$$
$$\nu_n^T(M) = \delta_n^T(M) + p_n$$

and, up to multiplication by a unit of KG^T,

$$\xi_\lambda = \prod_{j=1}^{\nu_n^T - \lambda} \eta_j.$$

Equally,

$$p_n = \nu_n^T(M) - \delta_n^T(M),$$
$$q_n = \delta_n^T(M) + \delta_{n-1}^T(M)$$

(taking $\delta_{-1}^T(M) = 0$ when $n = 0$) and, up to multiplication by a unit of KG^T,

$$\eta_j = \frac{\xi_j}{\xi_{j+1}}.$$

Proof. Let $\mathcal{F} = (F_i, \partial_i)$ be a resolution of type FP_n of M. The i-th homology group of the KG^T-free sequence $KG^T \otimes_{KG} \mathcal{F}$ is then $\mathrm{Tor}_i^{KG}(KG^T, M)$. We write $\mathcal{F}^T = KG^T \otimes_{KG} \mathcal{F}$, $F_i^T = KG^T \otimes_{KG} F_i$ and $\partial_i^T = KG^T \otimes_{KG} \partial_i$.

Notice that any choice of free bases for F_{n+1} and F_n induces a choice of bases for F_{n+1}^T and F_n^T, with respect to which

$$D_n(\mathcal{F})^T = D_n(\mathcal{F}^T).$$

By a comment following the proof of Theorem 4.7, any other choice of bases for F_{n+1}^T and F_n^T gives a matrix with the same elementary ideals, and so we will make a choice of bases for these free modules rather than for F_{n+1} and F_n.

We set $B_i^T = \mathrm{im}\, \partial_{i+1}^T$, $Z_i^T = \ker \partial_i^T$. Since these are submodules of the KG^T-free module F_i^T, they are both free. The short exact sequence

$$0 \to Z_i^T \to F_i^T \xrightarrow{\partial_i^T} B_{i-1}^T \to 0$$

then splits, so $F_i^T = Z_i^T \oplus \widetilde{B}_{i-1}^T$, where $\widetilde{B}_{i-1}^T$ is a lift of B_{i-1}^T via ∂_i^T (take $\widetilde{B}_{-1}^T = 0$). We also have the sequence

$$0 \to B_i^T \to Z_i^T \to H_i(\mathcal{F}^T) \to 0.$$

By considering the ranks of the modules in this sequence, we have

$$\mathrm{rk}_{KG^T}(Z_i^T) = q_i + b_i,$$

where $b_i = \mathrm{rk}_{KG^T}(B_i^T)$, and so $Z_i^T \cong B_i^T \oplus (KG^T)^{q_i}$. Consequently,

$$F_i^T \cong B_i^T \oplus (KG^T)^{q_i} \oplus \widetilde{B}_{i-1}^T$$

and

$$\mathrm{rk}_{KG}(F_i) = \mathrm{rk}_{KG^T}(F_i^T) = b_i + q_i + b_{i-1}.$$

Choosing ordered bases for B_{n+1}^T, $\widetilde{B}_n^T$, B_n^T and $\widetilde{B}_{n-1}^T$ and for the free parts of $\mathrm{Tor}_{n+1}^{KG}(KG^T, M)$ and $\mathrm{Tor}_n^{KG}(KG^T, M)$ induces ordered bases for F_{n+1}^T and F_n^T, with respect to which

$$D_n(\mathcal{F}^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_n & 0 & 0 \end{bmatrix} \tag{4.6}$$

(with one less column of zero matrices when $n = 0$). Here X_n is some $b_n \times b_n$ matrix over KG^T. Because KG^T is a pid, we can choose bases for $\widetilde{B}_n^T$ and B_n^T such that X_n is a diagonal matrix, each of whose diagonal entries is non-zero and divides the next. Since this matrix is a presentation matrix for the torsion part of $H_n(\mathcal{F}^T)$, we must have

$$X_n = \mathrm{Diag}_{b_n}(1, \ldots, 1, \eta_1, \ldots, \eta_{p_n}).$$

Now, $J(D_n(\mathcal{F})^T) = J(D_n(\mathcal{F}^T)) = J(X_n)$ and

$$\chi_n(\mathcal{F}) = \sum_{i=0}^{n} (-1)^{n-i} \mathrm{rk}_{KG}(F_i) = b_n + \sum_{i=0}^{n} (-1)^{n-i} q_i,$$

so

$$\begin{aligned} E_{n,\lambda}^T(M) &= J_{\chi_n(\mathcal{F})-\lambda}(X_n) \\ &= \begin{cases} KG^T & \lambda \geq p_n + \Sigma \\ \left(\prod_{j=1}^{p_n+\Sigma-\lambda} \eta_j\right) & \Sigma \leq \lambda < p_n + \Sigma \\ 0 & \lambda < \Sigma \end{cases}, \end{aligned}$$

where $\Sigma = \sum_{i=0}^{n}(-1)^{n-i}q_i$. Thus, since the η_j are non-zero and not units,

$$\delta_n^T(M) = \sum_{i=0}^{n}(-1)^{n-i}q_i,$$

$$\nu_n^T(M) = p_n + \sum_{i=0}^{n}(-1)^{n-i}q_i,$$

as required. □

Remark. A natural further question to ask is whether there is a connexion between $E_i^T(M)$ and $\mathrm{Tor}_i^{KG}(KG^T, M)$ for general ^{T}. Although these are both calculated from the same complex, $KG^T \otimes_{KG} \mathcal{F}$, they can be very different. For instance, if G is abelian and $^T = {}^{ab}$, then KG^{ab} is flat as a KG-module, so $\mathrm{Tor}_i^{KG}(KG^{ab}, M) = 0$ for $i > 0$. However, $E_i(M)$ need not be trivial (for example, take G to be finite cyclic and $M = {}_GK$).

Since $\mathbb{Z} = \mathbb{Z}G^{triv}$ is a pid, the implications of the theorem for the group invariants is as follows.

Corollary 4.25. *If G, G_0 are two groups of type FP_n, then*

$$E_i^{triv}(G) = E_i^{triv}(G_0)$$

for $i = 0, 1, \ldots, n$ if, and only if,

$$H_i(G) \cong H_i(G_0)$$

for $i = 0, 1, \ldots, n$.

Moreover, since $augE_i(G) = E_i^{triv}(G)$, if two groups have the same E_i-ideals ($i \leq n$), then they have the same integral homology in dimensions $\leq n$. The converse of this statement does not hold, as we will show in §5.7.

In dimension 0, since

$$E_{0,\lambda}^{triv} = \begin{cases} \mathbb{Z} & \lambda \geq 1 \\ 0 & \lambda < 1 \end{cases},$$

we have the well-known result that $H_0(G) = \mathbb{Z}$ for every group G.

In dimension 1, if $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ is a presentation for G, with $\mathbf{x}$ finite, then

$$D(\mathcal{P})^{triv} = \Big[\exp_x(R)\Big]_{\substack{R \in \mathbf{r} \\ x \in \mathbf{x}}}.$$

This matrix is called the relation matrix of $\mathcal{P}$, the elementary ideals of which can be used to find G^{ab} (see, for example, [52]). Fox [41] used this to show that $G^{ab} \cong H_1(G)$ and $A^{triv}(\mathcal{P})$ can be obtained from each other. This thus gives an alternative proof of Corollary 4.25 for $n = 1$.

Example 4.3. If G is the R. Thompson group, as in §4.1.2, then

$$E^{triv}_{n,\lambda}(G) = \begin{cases} \mathbb{Z} & \lambda \geq 1 \\ 0 & \lambda < 1 \end{cases}$$

for $n \geq 0$, so

$$H_n(G) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n > 0 \end{cases},$$

as already shown in [24]. ◊

Example 4.4. If Γ, ψ is an aspherical Coxeter system, then, by [50],

$$H_2(C_{\Gamma,\psi}) = (\mathbb{Z}_2)^{|\mathbf{e}^+| - |\mathbf{v}| + n_\Gamma}.$$

Thus, since $C^{ab}_{\Gamma,\psi} \cong H_1(C_{\Gamma,\psi})$ is finite, $\delta^{triv}_1(C_{\Gamma,\psi}) = -1$, by Theorem 4.24, so $\delta^{triv}_2(C_{\Gamma,\psi}) = 1$ and

$$E^{triv}_{2,\lambda}(C_{\Gamma,\psi}) = \begin{cases} \mathbb{Z} & \lambda \geq |\mathbf{e}^+| - |\mathbf{v}| + 1 + n_\Gamma \\ \left(2^{|\mathbf{e}^+| - |\mathbf{v}| + 1 + n_\Gamma - \lambda}\right) & 1 \leq \lambda < |\mathbf{e}^+| - |\mathbf{v}| + 1 + n_\Gamma \cdot \\ 0 & \lambda < 1 \end{cases}$$ ◊

In [41], the author shows that the zeroth Alexander ideal of a finite presentation $\mathcal{P}$ (i.e. $A_{-1}(\mathcal{P}) = E_{1,-1}(G(\mathcal{P}))$) is trivial if, and only if, $G(\mathcal{P})^{ab}$ is infinite. We now give a different proof of this, in our own terms.

Proposition 4.26. *If G is a finitely generated group, then G^{ab} is infinite if, and only if, $\delta_1(G) \geq 0$.*

Proof. If $\delta_1(G) \geq 0$, then $\delta^{triv}_1(G) \geq 0$, and so, since $\mathrm{rk}_{\mathbb{Z}}(H_1(G)) = \delta^{triv}_0(G) + \delta^{triv}_1(G) \geq 1$, by Theorem 4.24, $G^{ab} \cong H_1(G)$ is infinite.

If G^{ab} is infinite, then, since G^{ab} is finitely generated, there must be an element $t \in G^{ab}$ of infinite order. Suppose that $\delta_1(G) < 0$, so $E_{1,-1}(G) \neq 0$, and that

$$\xi = \sum_{g \in G^{ab}} n_g g$$

is a non-zero element of $E_{1,-1}(G)$. From Corollary 4.22, taking $M = {}_G\mathbb{Z}$, $n = 1$, $\kappa = 0$ and $\lambda = -1$, we have $IG^{ab}E_{1,-1}(G) = 0$. In particular, $(1-t)\xi = 0$, so $\xi = t\xi$. Thus, for $j \in \mathbb{Z}$,

$$\begin{aligned}\sum_{g \in G^{ab}} n_g g &= \sum_{g \in G^{ab}} n_g t^j g \\ &= \sum_{g \in G^{ab}} n_{t^{-j}g} g,\end{aligned}$$

and so $n_{t^{-j}g} = n_g$ for all $g \in G^{ab}, j \in \mathbb{Z}$. But then $n_g = 0$ for each $g \in G^{ab}$, since only finitely many of $\{n_g : g \in G^{ab}\}$ can be non-zero, and so $\xi = 0$, a contradiction. □

Using the results of this section, we can now also give the promised proof of Proposition 2.4(iii).

Lemma 4.27 ([41]). *If G is a knot group, then*

$$e_{1,\lambda}(G)(1) = \begin{cases} \pm 1 & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}.$$

Proof. Since $G^{ab} = G^{tf}$ is infinite cyclic, $\delta_1^{tf}(G) = \delta_1(G) \geq 0$ by Proposition 4.26, so $e_{1,\lambda}(G) = 0$ for $\lambda < 0$. To show that $e_{1,\lambda}(G)(1) = \pm 1$ for $\lambda \geq 0$, we need only show that $e_{1,0}(G)(1) = \pm 1$, since $e_{1,\lambda+1}(G) | e_{1,\lambda}(G)$.

Let $\mathcal{Q} = \langle \mathbf{y}; \mathbf{s} \rangle$ be a deleted Wirtinger presentation for G as in (2.1), so $|\mathbf{s}| = |\mathbf{y}| - 1$. We saw in §2.1 that $E_{1,0}(G)$ is principal and can be generated by the determinant of any $(|\mathbf{y}|-1) \times (|\mathbf{y}|-1)$ submatrix of $D(\mathcal{Q})^{ab}$. Thus

$$E_{1,0}(G) = (e_{1,0}(G)).$$

By Theorem 4.24, since $G^{ab} \cong \mathbb{Z}$,

$$E_{1,\lambda}^{triv}(G) = \begin{cases} \mathbb{Z} & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases},$$

so

$$\mathbb{Z} = E^{triv}_{1,0}(G) = augE_{1,0}(G).$$

But

$$augE_{1,0}(G) = (e_{1,0}(G)(1)),$$

so $e_{1,0}(G)(1)$ generates $\mathbb{Z}$ and must thus be equal to 1 or -1. □

Theorem 4.24 also tells us that

$$\nu^{triv}_n(G) = d(H_n(G)) - \mathrm{rk}_{\mathbb{Z}}(H_{n-1}(G)) + \mathrm{rk}_{\mathbb{Z}}(H_{n-2}(G)) - \cdots + (-1)^m \mathrm{rk}_{\mathbb{Z}}(H_0(G)).$$

But this is just the lower bound (1.11) for $\chi_n(G)$. Thus, (1.11) arises immediately from the inequality (4.2). Moreover, since for each ^{T}

$$\nu^{triv}_n(G) \leq \nu^T_n(G) \leq \nu_n(G) \leq \chi_n(G),$$

$\nu_n(G)$ could give a sharper bound for $\chi_n(G)$. Applying this in dimension 2, we obtain a test for the efficiency of a group.

Corollary 4.28. *If $\nu_2(G) > \nu^{triv}_2(G)$, then G is not efficient.*

Example 4.5. As in Example 4.1, let $G^{(q)}$ be an HNN extension of a one-relator group H_1 with finite presentation $\langle \mathbf{x}; R_0^p \rangle$, where p, q are coprime. Thus

$$E_{2,\lambda}(G^{(q)}) = \begin{cases} \mathbb{Z}G^{(q)^{ab}} & \lambda \geq 2 - |\mathbf{x}| \\ (1 - R_0, p, 1 - qt) & \lambda = 1 - |\mathbf{x}| \\ 0 & \lambda < 1 - |\mathbf{x}| \end{cases}.$$

Since q is coprime to p, its image in the ring $\mathbb{Z}_p$ is a unit. The map $\mathbb{Z}G^{(q)^{ab}} \to \mathbb{Z}_p; x \mapsto 1 (x \in \mathbf{x}), t \mapsto q^{-1}$ then induces a non-trivial ring homomorphism, which sends the ideal $(1 - R_0, p, 1 - qt)$ to 0. Thus $E_{2,1-|\mathbf{x}|}(G^{(q)}) \neq \mathbb{Z}G^{(q)^{ab}}$, and so

$$\nu_2(G^{(q)}) = 2 - |\mathbf{x}|.$$

However, if p and $q - 1$ are coprime (for instance, if p is prime, $q \neq 1$), then

$$E^{triv}_{2,\lambda}(G^{(q)}) = \begin{cases} \mathbb{Z} & \lambda \geq 1 - |\mathbf{x}| \\ 0 & \lambda < 1 - |\mathbf{x}| \end{cases},$$

so

$$\nu_2^{triv}(G^{(q)}) = 1 - |\mathbf{x}| < \nu_2(G^{(q)}).$$

The group $G^{(q)}$ cannot therefore be efficient. In fact, if

$$\mathcal{P} = \langle \mathbf{x}, t; R_0^p, R_0 t = t R_0^q \rangle,$$

then $\chi_2(\mathcal{P}) = 2 - |\mathbf{x}|$, so $\mathcal{P}$ is a minimal presentation for $G^{(q)}$. See [7] for a fuller exposition of the efficiency of HNN extensions. ◊

Chapter 5

The E-ideals of some classes of groups

As well as looking at the E-ideals of some classes of groups, such as perfect groups, groups of type FP and groups of type FR, we consider those groups which have similar E-ideals to these groups. As a consequence, we will be able to show that, for each n, the E_n-ideals distinguish groups which the E_i-ideals for $i < n$ cannot.

5.1 E-trivial modules and groups

Let $n \geq m \geq 0$. We will say that a KG-module M of type FP_n is $E^T[m,n]$*-trivial* if:

a) $\nu_i^T(M) = \delta_i^T(M)$ for $m \leq i \leq n$; and

b) $\delta_i^T(M) + \delta_{i+1}^T(M) = 0$ for $m \leq i < n$.

We allow n to be infinite, but not m. A group G of type FP_n will be said to be $E^T[m,n]$*-trivial over* K if ${}_GK$ is $E^T[m,n]$-trivial. For example, groups of type FL over K are $E[l,\infty]$-trivial over K for some l. A CA group is $E^{tf}[n,n]$-trivial (over $\mathbb{Z}$) for all even n, but, if it has torsion, then it is not $E^T[m,n]$-trivial with $n-m>0$ for any ^{T}.

If M is $E[m,n]$-trivial, then, since τ_G^T factors through G^{ab}, we see that M is $E^T[m,n]$-trivial for any ^{T}. In particular, M is $E^{triv}[m,n]$-trivial, and so, if $K = \mathbb{Z}$,

$\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M) = 0$ for $m < i \leq n$ and $\mathrm{Tor}_m^{\mathbb{Z}G}(\mathbb{Z}, M)$ is torsion-free. Thus, if G is an $E[m,n]$-trivial group, then $H_i(G) = 0$ for $m < i \leq n$. This homological condition characterises $E[0,n]$-trivial groups.

Lemma 5.1. *A group of type FP_n ($n > 0$) is $E[0,n]$-trivial if, and only if, $H_i(G) = 0$ for $i = 1, \ldots, n$. For such groups, $\delta_i(G) = (-1)^i$ for $0 \leq i \leq n$.*

Proof. If $H_i(G) = 0$ for $i = 1, \ldots, n$, then $G^{ab}(\cong H_1(G)) = 1$, so $E_i(G) = E_i^{triv}(G)$. Thus, by Theorem 4.24, G is $E[0,n]$-trivial. Conversely, if G is $E[0,n]$-trivial, then $H_i(G) = 0$ for $i = 1, \ldots, n$, by a comment above. □

In particular, the $E[0,1]$-groups are the finitely generated perfect groups, the $E[0,2]$-groups are the super-perfect groups of type FP_2 (see, for example, [13] or [37]) and the $E[0,\infty]$-trivial groups are the acyclic groups [14] of type FP_∞.

For every $0 \leq m \leq n \leq \infty$, there are, in fact, groups which are $E[m,n]$-trivial, but not $E[m-1,n]$- or $E[m,n+1]$-trivial. We give an example of one such family in Example 5.3, below.

We now show that, when one of the modules in a short exact sequence is $E[m,n]$-trivial, the inequalities and inclusions of Proposition 4.20 and Corollary 4.21 become equalities.

Proposition 5.2. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of KG-modules.*

i) If M' is $E^T[m,n]$-trivial, then, for $m < i \leq n$, M is of type FP_i if, and only if, M'' is and, in that case,

$$E^T_{i,\lambda}(M) = E^T_{i,\lambda-\delta_i^T(M')}(M'').$$

Consequently, $\delta_i^T(M) = \delta_i^T(M') + \delta_i^T(M'')$. If the sequence splits, this also holds for $i = m$.

ii) If M'' is $E^T[m,n]$-trivial ($m > 0$), then, for $m \leq i < n$, M is of type FP_i if, and only if, M' is and, in that case,

$$E^T_{i,\lambda}(M) = E^T_{i,\lambda-\delta_i^T(M'')}(M').$$

Consequently, $\delta_i^T(M) = \delta_i^T(M') + \delta_i^T(M'')$. *If the sequence splits, this also holds for* $i = n$ *and for* $m = 0$.

iii) If M *is* $E^T[m,n]$*-trivial* $(m > 0)$*, then, for* $m \leq i < n$*,* M' *is of type* FP_i *if, and only if,* M'' *is of type* FP_{i+1} *and, in that case,*

$$E_{i,\lambda}^T(M') = E_{i+1,\lambda-\delta_i^T(M)}^T(M'').$$

Consequently, $\delta_i^T(M) = \delta_i^T(M') - \delta_{i+1}^T(M'')$.

Proof. Note that, if $I = \{I_\lambda\}_{\lambda\in\mathbb{Z}}$, $J = \{J_\lambda\}_{\lambda\in\mathbb{Z}}$ are two ascending chains of ideals in KG^T and

$$I_\lambda = \begin{cases} KG^T & \lambda \geq \delta \\ 0 & \lambda < \delta \end{cases},$$

then

$$(I * J)_\lambda = \sum_{j\in\mathbb{Z}} I_j J_{\lambda-j} = \sum_{j\geq\delta} J_{\lambda-j} = J_{\lambda-\delta}.$$

(i) From Proposition 4.20(i) (and the above comment), for $m \leq i \leq n$,

$$E_{i,\lambda}^T(M) \supseteq E_{i,\lambda-\delta_i^T(M')}^T(M''),$$

with equality when the sequence splits. Also, from Proposition 4.20(ii), for $m < i \leq n$,

$$E_{i,\lambda}^T(M'') \supseteq E_{i,\lambda-\delta_{i-1}^T(M')}^T(M).$$

So, for $m < i \leq n$,

$$E_{i,\lambda}^T(M) \subseteq E_{i,\lambda+\delta_{i-1}^T(M')}^T(M'') \subseteq E_{i,\lambda+\delta_i^T(M')+\delta_{i-1}^T(M')}^T(M),$$

and the result follows, since $\delta_i^T(M') + \delta_{i-1}^T(M') = 0$.

(ii) From Proposition 4.20(i), for $m \leq i \leq n$,

$$E_{i,\lambda}^T(M) \supseteq E_{i,\lambda-\delta_i^T(M'')}^T(M'),$$

with equality when the sequence splits. Also, from Proposition 4.20(iii), for $m \leq i < n$ with $m > 0$,

$$E_{i,\lambda}^T(M') \supseteq E_{i,\lambda-\delta_{i+1}^T(M'')}^T(M).$$

Thus, for $0 < m \le i < n$,

$$E^T_{i,\lambda}(M) \supseteq E^T_{i,\lambda-\delta^T_{i-1}(M'')}(M') \supseteq E^T_{i,\lambda-\delta^T_i(M'')-\delta^T_{i+1}(M'')}(M).$$

But, $\delta^T_i(M'') + \delta^T_{i+1}(M'') = 0$ here, so the result follows.

(iii) From Proposition 4.20(ii), for $m-1 \le i \le n-1$ $(m > 0)$,

$$E^T_{i+1,\lambda}(M'') \supseteq E^T_{i,\lambda-\delta^T_{i+1}(M)}(M'),$$

and from Proposition 4.20(iii), for $m \le i \le n$,

$$E^T_{i,\lambda}(M') \supseteq E^T_{i+1,\lambda-\delta^T_i(M)}(M'').$$

So, for $m \le i < n$,

$$E^T_{i,\lambda}(M') \supseteq E^T_{i+1,\lambda-\delta^T_i(M)}(M'') \supseteq E^T_{i,\lambda-\delta^T_i(M)-\delta^T_{i+1}(M)}(M'),$$

and the result follows, since $\delta^T_i(M) + \delta^T_{i+1}(M) = 0$. □

Corollary 5.3. *If M', M'' are $E^T[m,n]$-trivial, then so is $M' \oplus M''$ and, for $m \le i \le n$, $\delta^T_i(M' \oplus M'') = \delta^T_i(M') + \delta^T_i(M'')$.*

More generally, the intersections of any two of the three results of the above Proposition give:

Corollary 5.4 (cf. Corollary 1.31). *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of KG-modules.*

i) If M' and M'' are $E^T[m,n]$-trivial $(m > 0)$, then so is M.

ii) If M is $E^T[m,n]$-trivial $(m > 0)$ and M' is $E^T[m-1,n-1]$-trivial, then M'' is $E^T[m,n]$-trivial.

iii) If M is $E^T[m,n]$-trivial $(m > 0)$ and M'' is $E^T[m+1,n+1]$-trivial, then M' is $E^T[m,n]$-trivial.

If in the above short exact sequence M is a free module of finite rank r, then it is $E^T[0,\infty]$-trivial, with $\delta^T_0(M) = r$. Proposition 5.2(iii) then says that, for $n > 1$,

$$E^T_{n,\lambda}(M'') = E_{n-1,\lambda-(-1)^n r}(M'),$$

a special case of Proposition 4.14. We can, in fact, extend this to the case when M is a finitely generated *projective* $\mathbb{Z}G$-module.

5.2 Projective modules

Let P be a finitely generated $\mathbb{Z}G$-module. There is then another finitely generated projective $\mathbb{Z}G$-module Q such that $F = P \oplus Q$ is free. We let $\pi_1 : F \to P$, $\pi_2 : F \to Q$ be the natural projections. This then gives us a free resolution of P of type FP_∞,

$$\cdots \xrightarrow{\pi_2} F \xrightarrow{\pi_1} F \xrightarrow{\pi_2} F \xrightarrow{\pi_1} P \to 0.$$

Theorem 5.5. *Finitely generated projective $\mathbb{Z}G$-modules are $E[0,\infty]$-trivial.*

To prove this in general, we first prove it for G abelian.

Lemma 5.6. *If G is abelian and if P is a finitely generated $\mathbb{Z}G$-module, then P is $E[0,\infty]$-trivial, with $\delta_n(P) = (-1)^n \rho_{\mathbb{Z}G}(P)$.*

Proof. Using the resolution $\mathcal{F}$ above, if $r = \mathrm{rk}_{\mathbb{Z}G}(F)$, and D_1, D_2 are the matrices for π_1, π_2, respectively, then

$$E_{n,\lambda}(P) = \begin{cases} J_{r-\lambda}(D_2) & n \text{ even} \\ J_{-\lambda}(D_1) & n \text{ odd} \end{cases}.$$

We show that $J_{\rho(P)}(D_1) = \mathbb{Z}G = J_{\rho(Q)}(D_2)$, where $\rho(P) = \rho_{\mathbb{Z}G}(P)$ and $\rho(Q) = \rho_{\mathbb{Z}G}(Q)$. If $D_1 = [\xi_{i,j}]_{i,j}$, then, by (1.17),

$$1 = \sum_{i=\rho(P)}^{r} (-1)^{i-\rho(P)} \binom{i}{\rho(P)} \sum_{1 \le j_1 \le \cdots \le j_i \le r} \det\left(\begin{bmatrix} \xi_{j_1,j_1} & \cdots & \xi_{j_i,j_i} \end{bmatrix}\right).$$

Now, for $i \ge \rho(P)$,

$$\det\left(\begin{bmatrix} \xi_{j_1,j_1} & \cdots & \xi_{j_i,j_i} \end{bmatrix}\right) \in J_{\rho(P)}(D_1),$$

and so $1 \in J_{\rho(P)}(D_1)$. Similarly, $1 \in J_{\rho(Q)}(D_2)$. Hence, since $r = \rho(P) + \rho(Q)$, for n even and $\lambda \ge \rho(P)$,

$$E_{n,\lambda}(P) = J_{r-\lambda}(D_2) = \mathbb{Z}G$$

and for n odd and $\lambda \ge -\rho(P)$,

$$E_{n,\lambda}(P) = J_{-\lambda}(D_1) = \mathbb{Z}G.$$

By Corollary 4.22, when $\kappa + \lambda < 0$,

$$E_{n,\kappa}(P)E_{n+1,\lambda}(P) = 0. \tag{5.1}$$

Taking n even and $\kappa = \rho(P)$, we have

$$E_{n+1,\lambda}(P) = 0$$

for $\lambda < -\rho(P)$. Thus, for n odd,

$$E_{n,\lambda}(P) = \begin{cases} \mathbb{Z}G & \lambda \geq -\rho(P) \\ 0 & \lambda < -\rho(P) \end{cases}.$$

On the other hand, taking n even and $\lambda = -\rho(P)$ in (5.1), we have

$$E_{n,\kappa}(P) = 0$$

whenever $\kappa < \rho(P)$. Thus, for even n,

$$E_{n,\lambda}(P) = \begin{cases} \mathbb{Z}G & \lambda \geq \rho(P) \\ 0 & \lambda < \rho(P) \end{cases}.$$

□

Proof of Theorem 5.5. Via the abelianising homomorphism $\tau = \tau_G^{ab} : \mathbb{Z}G \to \mathbb{Z}G^{ab}$, $\mathbb{Z}G^{ab}$ is a right $\mathbb{Z}G$-module and, since P is projective,

$$\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}G^{ab}, P) = 0$$

for $i > 0$. Thus, by Proposition 4.11, for $n \geq 0$,

$$\begin{aligned} E_n(\mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} P) &= \tau^{ab} E_n(P) \\ &= E_n(P), \end{aligned}$$

since τ^{ab} is the identity map. Applying Lemma 5.6,

$$E_{n,\lambda}(P) = \begin{cases} \mathbb{Z}G^{ab} & \lambda \geq (-1)^n \rho(\mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} P) \\ 0 & \lambda < (-1)^n \rho(\mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} P) \end{cases}.$$

□

Applying Lemma 1.35 to the map $\tau : \mathbb{Z}G \to \mathbb{Z}G^{ab}$ gives

$$\rho_{\mathbb{Z}G^{ab}}(\mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} P) = \widetilde{\tau}(\rho_{\mathbb{Z}G}(P)).$$

But, since $\rho_{\mathbb{Z}G^{ab}}(\mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} P) \in \mathbb{Z}$, we have

$$\rho_{\mathbb{Z}G^{ab}}(\mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} P) = \widetilde{aug}\rho_{\mathbb{Z}G}(P)$$

(thinking of $\mathbb{Z}$ as a subset of $\mathbb{Z}G^{ab} = \mathbb{Z}G^{ab}/[\mathbb{Z}G^{ab}, \mathbb{Z}G^{ab}]$). Lemma 1.36 then gives the following corollary to Theorem 5.5.

Corollary 5.7. *If P is a finitely generated $\mathbb{Z}G$-module, then, for $n \geq 0$,*

$$\delta_n(P) = (-1)^n \widetilde{\rho}_G(P).$$

If Bass' conjecture holds, this would become $\delta_n(P) = (-1)^n \rho_{\mathbb{Z}G}(P)$.

As for free modules, applying Proposition 5.2(iii) we have the next corollary.

Corollary 5.8. *If $0 \to M' \to P \to M'' \to 0$ is a short exact sequence of $\mathbb{Z}G$-modules and if P is finitely generated and projective, then, for $n > 1$, M'' is of type FP_n if, and only if, M' is of type FP_{n-1} and*

$$E^T_{n,\lambda}(M'') = E^T_{n-1,\lambda-(-1)^n\widetilde{\rho}_G(P)}(M').$$

The dimension shifting result Proposition 4.14 can then be partially extended to projective resolutions as follows.

Proposition 5.9. *If M is a $\mathbb{Z}G$-module of type FP_n and if $\mathcal{P} = (P_i, \varepsilon_i)$ is a projective resolution of type FP_m of M, $m < n-1$, then*

$$E^T_{n,\lambda}(M) = E^T_{n-m-1,\lambda-(-1)^{n-m}\chi_m(\mathcal{P})}(\ker \varepsilon_m),$$

where $\chi_m(\mathcal{P}) = \sum_{i=0}^m (-1)^{m-i}\widetilde{\rho}_G(P_i)$.

Proof. By Corollary 5.8,

$$\begin{aligned} E^T_{n,\lambda}(M) &= E^T_{n-1,\lambda-(-1)^n\widetilde{\rho}_G(P_0)}(\ker \varepsilon_0) \\ &= E^T_{n-2,\lambda-(-1)^n\widetilde{\rho}_G(P_0)-(-1)^{n-1}\widetilde{\rho}_G(P_1)}(\ker \varepsilon_1) \\ &= \cdots \\ \cdots &= E^T_{n-m-1,\lambda-(-1)^{n-m}\chi_m(\mathcal{P})}(\ker \varepsilon_m). \end{aligned}$$

□

Remark. The formula (1.17) actually holds for projective KG-modules whenever KG is a ring with no non-trivial idempotents (so, for $\xi \in KG$, $\xi^2 = \xi$ if, and only if, $\xi = 0$ or 1). The result of this section therefore also hold in this wider context. They do not, however, hold in general, as we will find in Proposition 5.13 in the next section.

5.3 Groups of type FP and of type FL

The following is a consequence of Theorem 5.5.

Theorem 5.10. *If G is a group of type FP, then G is $E[cd\,G, \infty]$-trivial and, for $n \geq cd\,G$, $\delta_n(G) = (-1)^n \widetilde{\chi}_{FP}(G)$.*

Proof. Let $\mathcal{F}$ be a resolution

$$0 \to P \to F_{l-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

of type FP for G, where $l = \operatorname{cd} G$, each F_i is free and P is projective (such a resolution exists [22, §VIII.6]). By Proposition 4.14, for $n \geq \operatorname{cd} G$,

$$\begin{aligned} E_{n,\lambda}(G) &= E_{n-l,\lambda+(-1)^{n-l}\chi_{l-1}(\mathcal{F})}(P) \\ &= \begin{cases} \mathbb{Z}G^{ab} & \lambda \geq (-1)^{n-l}(\widetilde{\rho}_G(P) - \chi_{l-1}(\mathcal{F})) \\ 0 & \lambda < (-1)^{n-l}(\widetilde{\rho}_G(P) - \chi_{l-1}(\mathcal{F})) \end{cases}, \end{aligned}$$

so G is $E[\operatorname{cd} G, \infty]$-trivial. In addition, for $n \geq \operatorname{cd} G$,

$$\begin{aligned} \delta_n(G) &= (-1)^n \left((-1)^l \widetilde{\rho}_G(P) + \sum_{i=0}^{l-1} (-1)^i \widetilde{\rho}_G(F_i) \right) \\ &= (-1)^n \widetilde{\rho}_G({}_G\mathbb{Z}) \\ &= (-1)^n \widetilde{\chi}_{FP}(G). \end{aligned}$$

□

Remarks. 1. If we tried to prove this theorem using Proposition 5.9, then we would only be able to show that G is $E[\operatorname{cd} G + 1, \infty]$-trivial.

2. Again, this result holds for any co-efficient ring K for which KG has no non-trivial idempotents.

We now show that the preceding theorem does not extend to all co-efficient rings. Obviously, we have:

Lemma 5.11. *If G is a group of type FL over K, then G is $E[l,\infty]$-trivial over K for some l.*

However, if KG has non-trivial idempotent elements and if G is of type FP over K but not of type FL over K, then it can have non-trivial E-ideals over K in arbitrarily high dimension. For example, consider CA groups. The group ring $\mathbb{Q}G$ of a CA group G has non-trivial idempotent elements, and so, in contrast with Theorem 5.10, we are able to show not only that:

Lemma 5.12. *Finitely presented CA groups are of type FP over $\mathbb{Q}$.*

but also:

Proposition 5.13. *If $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ is a finite CA presentation and if $\widehat{S}_0 \neq 1$ for some $S \in \mathbf{r}$, then $G = G(\mathcal{P})$ has non-trivial E-ideals over $\mathbb{Q}$ in arbitrarily high dimension, and so is not of type FL over $\mathbb{Q}$.*

Proof of Lemma 5.12. Let G be a group with a finite CA presentation $\mathcal{P}$. Applying the exact functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$ to the resolution (1.12), we obtain an exact sequence

$$0 \to \mathbb{Q} \otimes_{\mathbb{Z}} M(\mathcal{P}) \to \oplus_{x \in \mathbf{x}} \mathbb{Q}G \to \mathbb{Q}G \to {}_G\mathbb{Q} \to 0. \tag{5.2}$$

The module $M(\mathcal{P})$ decomposes [51] as

$$M(\mathcal{P}) = \oplus_{R \in \mathbf{r}} \mathbb{Z}G/\mathbb{Z}G.(1 - R_0),$$

so, to show that G is of type FP over $\mathbb{Q}$, we show that

$$\mathbb{Q}G/\mathbb{Q}G.(1 - R_0) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}G/\mathbb{Z}G.(1 - R_0)$$

is projective when $p_R > 1$. For $R \in \mathbf{r}$, let

$$\xi_1(R) = 1 + R_0 + \cdots + R_0^{p_R - 1}$$

and note that

$$\xi_1(R) = p_R - \sum_{j=1}^{p_R - 1} j R_0^{p_R - 1 - j}(1 - R_0). \tag{5.3}$$

Note also that $\frac{1}{p_R}\xi_1(R)$ is an idempotent element of $\mathbb{Q}G$.

The surjection

$$\alpha : \mathbb{Q}G \to \mathbb{Q}G/\mathbb{Q}G.(1-R_0); \eta \mapsto \eta + \mathbb{Q}G.(1-R_0) \quad (\eta \in \mathbb{Q}G)$$

is split by the $\mathbb{Q}G$-homomorphism

$$\beta : \mathbb{Q}G/\mathbb{Q}G.(1-R_0) \to \mathbb{Q}G; \eta + \mathbb{Q}G.(1-R_0) \mapsto \frac{1}{p_R}\eta\xi_1(R),$$

since,

$$0 = 1 - R_0 + \mathbb{Q}G.(1-R_0) \overset{\beta}{\mapsto} \frac{1}{p_R}(1-R_0)\xi_1(R) = \frac{1}{p_R}(1-R_0^{p_R}) = 0,$$

and, by (5.3),

$$\begin{aligned}\alpha\beta(\eta + \mathbb{Q}G.(1-R_0)) &= \frac{1}{p_R}\eta\xi_1(R) + \mathbb{Q}G.(1-R_0)\\ &= \eta + \mathbb{Q}G.(1-R_0).\end{aligned}$$

The module $\mathbb{Q} \otimes_{\mathbb{Z}} M(\mathcal{P})$ is thus projective, and so (5.2) is a resolution of type FP over $\mathbb{Q}$ for G. □

Remark. This is proved in [29] for one-relator groups.

Proof of Proposition 5.13. If we apply Theorem 4.3 to the results of §4.1.1, we find that, for $n \geq 2$,

$$E_{n,\lambda}(G,\mathbb{Q}) = \begin{cases} \mathbb{Q}G^{ab} & \lambda \geq \chi_n \\ \left(\prod_{R\in\mathbf{s}} \xi_n(R) : \mathbf{s} \subseteq \mathbf{r}', |\mathbf{s}| = \chi_n - \lambda\right) & -\chi_{n+1} \leq \lambda < \chi_n \\ 0 & \lambda < -\chi_{n+1} \end{cases},$$

where χ_n, $\xi_n(R)$ are as in §4.1.1. In particular, for every even n,

$$E_{n,\chi_n-1}(G,\mathbb{Q}) = \left(1 - \widehat{R_0}\ (R \in \mathbf{r}')\right).$$

Thus, if $\widehat{S_0} \neq 1$ for some $S \in \mathbf{r}$, then G has non-trivial E_n-ideals over $\mathbb{Q}$ in arbitrarily high dimension. By Lemma 5.11, G cannot be of type FL over $\mathbb{Q}$. □

Example 5.1. In [55], Lee and Park show that certain Fuchsian groups, namely those groups G with a presentation

$$\langle x_1, \ldots, x_m; x_1^{p_1}, \ldots, x_m^{p_m}, (x_1 \ldots x_m)^{p_{m+1}} \rangle$$

with $p_i \geq 2$ $(1 \leq i \leq m+1)$ and either $m > 2$ or $m = 2$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1$, are of type FP over $\mathbb{Q}$ but not of type FL over $\mathbb{Q}$. But the above presentation is CA (as can be shown using the weight test of [20]), and so, for G non-perfect, Proposition 5.13 gives an alternative proof of their result. $\Diamond$

If $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ is a finite CA presentation, then, rather than considering $\mathbb{Q}$-coefficients, if K is a ring in which p_R is a unit for each $R \in \mathbf{r}$, we can, as above, show that $K \otimes_{\mathbb{Z}} M(\mathcal{P})$ is projective, and so $G = G(\mathcal{P})$ is of type FP over K.

Serre asked whether there is a group which is of type FP but not of type FL (over $\mathbb{Z}$) [76]. Consider the ring $\mathbb{Z}\left[\frac{1}{2}\right]$, which is the smallest extension of the integers in which 2 is a unit. From the above discussion we see that the cyclic group of order 2 is of type FP over $\mathbb{Z}\left[\frac{1}{2}\right]$ but not of type FL over $\mathbb{Z}\left[\frac{1}{2}\right]$.

5.4 E-linked modules and groups

With certain conditions on ^{T} and K, the additive formulæ for the function δ_n^T in Proposition 5.2 can be obtained under a weaker hypothesis. If KG^T is an integral domain (the usual examples being when $K = \mathbb{Z}$ and ${}^T = {}^{triv}$ or ^{tf}), then, for m, n with $0 \leq m < n \leq \infty$, we will say that a KG-module M of type FP_n is $E^T[m,n]$-*linked* if $\delta_i^T(M) + \delta_{i+1}^T(M) = 0$ for $m \leq i < n$. A group G is $E^T[m,n]$-*linked over* K if ${}_GK$ is. For example, the R. Thompson group of §4.1.2 is $E[0,\infty]$-linked (over $\mathbb{Z}$), a finitely presented CA group is $E^{tf}[2,\infty]$-linked over any ring of characteristic 0, and we will see that finite groups are $E^{triv}[0,\infty]$-linked.

Throughout this section, if we state that a group is $E^T[m,n]$-linked over K or that a KG-module is $E^T[m,n]$-linked, then we are also making a tacit assumption that KG^T is an integral domain. In particular, K is an integral domain and G^T is torsion-free.

The promised additive formulæ are as follows.

Proposition 5.14. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of* KG*-modules and suppose that* KG^T *is an integral domain.*

i) If M' *is* $E^T[m,n]$*-linked then, for* $m < i \leq n$*,* M *is of type* FP_i *if, and only if,* M'' *is and, in that case,*

$$\delta_i^T(M) = \delta_i^T(M') + \delta_i^T(M'').$$

ii) If M'' *is* $E^T[m,n]$*-linked* $(m > 0)$ *then, for* $m \leq i < n$*,* M *is of type* FP_i *if, and only if,* M' *is and, in that case,*

$$\delta_i^T(M) = \delta_i^T(M') + \delta_i^T(M'').$$

iii) If M *is* $E^T[m,n]$*-linked* $(m > 0)$ *then, for* $m \leq i < n$*,* M' *is of type* FP_i *if, and only if,* M'' *is of type* FP_{i+1} *and, in that case,*

$$\delta_i^T(M) = \delta_i^T(M') - \delta_{i+1}^T(M'').$$

Proof. (i) Here, the conditions of both Corollary 4.21(i) and (ii) are satisfied, so

$$-\delta_{i-1}^T(M') + \delta_i^T(M'') \leq \delta_i^T(M) \leq \delta_i^T(M') + \delta_i^T(M'').$$

If $m < i \leq n$, $\delta_i^T(M') = -\delta_{i-1}^T(M')$, and these inequalities become equalities.

(ii) Here, the conditions of Corollary 4.21(i) and (iii) are satisfied, so

$$\delta_i^T(M') - \delta_{i+1}^T(M'') \leq \delta_i^T(M) \leq \delta_i^T(M') + \delta_i^T(M'').$$

For $m \leq i < n$, $\delta_i^T(M'') = -\delta_{i+1}^T(M'')$, and the result follows.

(iii) The conditions of Corollary 4.21(ii) and (iii) are satisfied, so

$$\delta_{i+1}^T(M) \geq -\delta_i^T(M') + \delta_{i+1}^T(M'').$$

and

$$\delta_i^T(M) \geq \delta_i^T(M') - \delta_{i+1}^T(M'').$$

When $m \leq i < n$, $\delta_i^T(M) = -\delta_{i+1}^T(M)$, so

$$\delta_i^T(M') - \delta_{i+1}^T(M'') \leq \delta_i^T(M) \leq \delta_i^T(M') - \delta_{i+1}^T(M''),$$

as required. □

Remark. If the sequence splits, then $\delta_i^T(M) = \delta_i^T(M') + \delta_i^T(M'')$ for all relevant i, by Corollary 4.19.

The intersections of any two of the three results of the above Proposition give:

Corollary 5.15 (cf. Corollaries 1.31, 5.4). *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of KG-modules.*

i) If M' and M'' are $E^T[m,n]$-linked $(m > 0)$, then so is M.

ii) If M is $E^T[m,n]$-linked $(m > 0)$ and M' is $E^T[m-1,n-1]$-linked, then M'' is $E^T[m,n]$-linked.

iii) If M is $E^T[m,n]$-linked $(m > 0)$ and M'' is $E^T[m+1,n+1]$-linked, then M' is $E^T[m,n]$-linked.

We now consider some further properties of E-linked modules and groups and define an Euler characteristic for certain E-linked groups.

Recall that if a module is $E[m,n]$-trivial, then it is $E^T[m,n]$-trivial for every ^{T}, in particular, it is $E^{triv}[m,n]$-trivial. Also, if it is $E^T[m,n]$-trivial for any ^{T}, then it is $E^{triv}[m,n]$-trivial. For E-linked modules, this situation is reversed.

Lemma 5.16. *If a KG-module M is $E^{triv}[m,n]$-linked, then it is $E^T[m,n]$-linked for every possible ^{T} and $\delta_i^T(M) = \delta_i^{triv}(M)$ for $m \leq i \leq n$.*

Proof. If M is $E^{triv}[m,n]$-linked, then, for $m \leq i < n$,

$$\delta_i^{triv}(M) + \delta_{i+1}^{triv}(M) = 0.$$

If KG^T is an integral domain, then, by Corollary 4.23,

$$\delta_i^T(M) + \delta_{i+1}^T(M) \geq 0$$

for $i \geq 0$. Also, by (4.1)

$$\delta_i^T(M) \leq \delta_i^{triv}(M) \tag{5.4}$$

for all i. Thus, for $m \leq i < n$,

$$0 \leq \delta_i^T(M) + \delta_{i+1}^T(M) \leq \delta_i^{triv}(M) + \delta_{i+1}^{triv}(M) = 0,$$

so M is $E^T[m,n]$-linked.

To show the second part of the lemma, we need only show that $\delta_m^T(M) = \delta_m^{triv}(M)$. From (5.4)

$$-\delta_m^T(M) = \delta_{m+1}^T(M) \leq \delta_{m+1}^{triv}(M) = -\delta_m^{triv}(M),$$

so

$$\delta_m^{triv}(M) \leq \delta_m^T(M),$$

and thus

$$\delta_m^T(M) = \delta_m^{triv}(M). \qquad \square$$

Lemma 5.17. *If a KG-module M is $E^T[m,n]$-linked for some ^{T}, then it is also $E^{tf}[m,n]$-linked with $\delta_i^{tf}(M) = \delta_i^T(M)$ for $m \leq i \leq n$.*

Proof. Since, by definition, G^{tf} is the largest torsion-free abelian quotient of G, if G^T is torsion-free for any other ^{T}, then $\tau_G^T : G \to G^T$ factors through G^{tf}. Consequently, $\delta_i^{tf}(G) \leq \delta_i^T(G)$. The result then follows by a similar argument to that in the above lemma. $\square$

Corollary 5.18. *If G is $E^{triv}[m,n]$-linked over K, then G is $E^T[m,n]$-linked over K for every possible ^{T} and $\delta_i^T(G,K) = \delta_i^{triv}(G,K)$ for $m \leq i \leq n$. If G is $E^T[m,n]$-linked over K for some ^{T}, then it is $E^{tf}[m,n]$-linked over K and $\delta_i^{tf}(G,K) = \delta_i^T(G,K)$ for $m \leq i \leq n$.*

The converse of the first statement is false. For example, the R. Thompson group is $E^{tf}[0,\infty]$-linked, but not $E^{triv}[0,\infty]$-linked.

As in Lemma 5.1, there is a homological characterisation of E^{triv}-linked group.

Proposition 5.19. *A group G is $E^{triv}[m,n]$-linked if, and only if, it is of type FP_n and $H_i(G)$ is finite for $m < i \leq n$.*

Proof. By Theorem 4.24, if G is of type FP_n, then

$$\mathrm{rk}_{\mathbb{Z}}(H_i(G)) = \delta_i^{triv}(G) + \delta_{i-1}^{triv}(G).$$

So, $H_i(G)$ is finite for $m < i \leq n$ if, and only if,

$$\delta_i^{triv}(G) + \delta_{i+1}^{triv}(G) = 0$$

for $m \leq i < n$, if, and only if, G is $E^{triv}[m,n]$-linked. $\square$

We will say that a group is *E^T_∞-linked over K* if it is $E^T[l,\infty]$-linked over K for some l. Thus, groups which are E^T_∞-linked over K are E^{tf}_∞-linked over K and, from the last proposition:

Corollary 5.20. *A group of type FP_∞ is E^{triv}_∞-linked if, and only if, it is of type FR.*

Example 5.2. If G is a finite group, then $H_i(G)$ is finite for $i > 0$. Thus, G is of type FR, and so is E^{triv}_∞-linked. ◊

If G is of type FP_∞, then, by Theorem 4.24, for $n \geq 0$,

$$\delta^{triv}_n(G) = \sum_{i=0}^{n}(-1)^{n-i}\mathrm{rk}_{\mathbb{Z}}(H_i(G)).$$

If, in addition, G is of type FR, that is, G is E^{triv}_∞-linked, then for large enough i, $\mathrm{rk}_{\mathbb{Z}}(H_i(G)) = 0$, so, for large n,

$$\begin{aligned}\delta^{triv}_n(G) &= (-1)^n\sum_{i\geq 0}(-1)^i\mathrm{rk}_{\mathbb{Z}}(H_i(G))\\ &= (-1)^n\widetilde{\chi}_{FR}(G).\end{aligned}$$

Thus, if we define for a group G which is E^T_∞-linked,

$$\delta^{tf}(G) = (-1)^n\delta^{tf}_n(G) = (-1)^n\delta^T_n(G),$$

for suitably large n, then δ^{tf} coincides with $\widetilde{\chi}_{FR}$ for groups of type FR.

Proposition 5.21. *If G is E^{triv}_∞-linked, then*

$$\delta^{tf}(G) = \widetilde{\chi}_{FR}(G).$$

But, there are groups for which δ^{tf} is defined, but for which $\widetilde{\chi}_{FR}$ is not. For example, if G is the R. Thompson group, then $H_i(G) = \mathbb{Z}\oplus\mathbb{Z}$ for $i > 0$, so G is not of type FR. However, G is $E^{tf}[0,\infty]$-linked, and $\delta^{tf}(G) = 0$.

From Theorem 5.10:

Corollary 5.22. *If G is of type FP, then*

$$\delta^{tf}(G) = \widetilde{\chi}_{FP}(G).$$

More generally, if G is E^T_∞-linked over K, then we define, for a suitably large n,

$$\delta^{tf}(G,K) = (-1)^n \delta^T_n(G,K).$$

If K is a pid (and so an integral domain) and if G is of type FP over K, then, by Lemma 1.38 and Theorem 4.24,

$$\begin{aligned}\widetilde{\chi}_{FP}(G) &= \sum_{i\geq 0}(-1)^i \mathrm{rk}_K(H^K_i(G))\\ &= \sum_{i\geq 0}(-1)^i(\delta^{triv}_i(G,K) + \delta^{triv}_{i-1}(G,K))\\ &= \delta^{tf}(G,K).\end{aligned}$$

In [30], Chiswell showed that if each of the vertex and edge groups of a graph of groups is of type FR, then so is the fundamental group G and gave a formula for $\widetilde{\chi}_{FR}(G)$ in terms of the Euler characteristics of the vertex and edge groups. In the following section, we extend this to certain E^T_∞-linked groups and $\delta^{tf}(G)$.

5.5 Graphs of groups

Let G_Γ be the fundamental group of a graph of groups, where $\Gamma = \mathbf{v} \cup \mathbf{e}$ is a finite connected graph with orientation $\mathbf{e}^+$, edge groups H_e ($e \in \mathbf{e}$) and vertex groups H_v ($v \in \mathbf{v}$), as in §1.8.3.

To reduce superfluous terminology, we consider mostly $\mathbb{Z}$-coefficients here, though everything we say will apply to coefficients in a general integral domain (using an appropriate version of (1.20)). Also, we set $\Gamma^+ = \mathbf{v} \cup \mathbf{e}^+$ and

$$\nu^T_n(\mathbf{e}) = \sum_{e\in\mathbf{e}^+} \nu^T_n(H_e), \qquad \nu^T_n(\mathbf{v}) = \sum_{v\in\mathbf{v}} \nu^T_n(H_v),$$

$$\delta^T_n(\mathbf{e}) = \sum_{e\in\mathbf{e}^+} \delta^T_n(H_e), \qquad \delta^T_n(\mathbf{v}) = \sum_{v\in\mathbf{v}} \delta^T_n(H_v),$$

$$\delta^{tf}(\mathbf{e}) = \sum_{e\in\mathbf{e}^+} \delta^{tf}(H_e), \qquad \delta^{tf}(\mathbf{v}) = \sum_{v\in\mathbf{v}} \delta^{tf}(H_v),$$

when defined.

Consider the short exact sequence (1.20) associated with Γ. In general, given a resolution for each H_γ ($\gamma \in \Gamma^+$), we can apply the exact functor $\mathbb{Z}G_\Gamma \otimes_{\mathbb{Z}H_\gamma} -$ to give a resolution of $\mathbb{Z}\uparrow_{H_\gamma}^{G_\Gamma}$ and then use the mapping cylinder construction of §1.6.4 to construct a resolution for G_Γ, from which the ideals $E_n^T(G_\Gamma)$ can be calculated. We apply Proposition 4.20(ii) and Corollaries 4.13 and 4.21:

Lemma 5.23. *If each H_e ($e \in \mathbf{e}$) is of type FP_{n-1} and each H_v ($v \in \mathbf{v}$) is of type FP_n, then G_Γ is of type FP_n,*

$$E_{n,\lambda}^T(G_\Gamma) \supseteq \sum_{\substack{j_\gamma \in \mathbb{Z}(\gamma\in\Gamma^+) \\ \sum_{\Gamma^+} j_\gamma = \lambda}} \left(\prod_{e\in \mathbf{e}^+} \left(\iota_e^T E_{n-1,j_e}^T(H_e)\right) \prod_{v\in\mathbf{v}} \left(\iota_v^T E_{n,j_v}^T(H_v)\right) \right),$$

for $\lambda \in \mathbb{Z}$, and

$$\nu_n^T(G_\Gamma) \leq \nu_{n-1}^T(\mathbf{e}) + \nu_n^T(\mathbf{v}).$$

If G^T is torsion-free, then

$$\delta_n^T(G_\Gamma) \leq \delta_{n-1}^T(\mathbf{e}) + \delta_n^T(\mathbf{v}).$$

Under certain conditions, applying Proposition 5.2, these inequalities and inclusions become equalities.

Proposition 5.24. *Suppose that each H_v ($v \in \mathbf{v}$) is of type FP_n and each H_e ($e \in \mathbf{e}$) is of type FP_{n-1}, so G is of type FP_n. Then,*

i) if each H_e ($e \in \mathbf{e}$) is $E^T[m-1, n-1]$-trivial ($m > 0$),

$$E_i^T(G_\Gamma) = *_{v\in\mathbf{v}}^{(\delta_i^T(\mathbf{e}))} \left(\iota_v^T E_i^T(H_v)\right),$$

for $m \leq i < n$;

ii) if each H_v ($v \in \mathbf{v}$) is $E^T[m, n]$-trivial ($m > 0$),

$$E_i^T(G_\Gamma) = *_{e\in\mathbf{e}^+}^{(-\delta_i^T(\mathbf{v}))} \left(\iota_e^T E_{i-1}^T(H_e)\right),$$

for $m < i \leq n$.

Since the trivial group is $E[0,\infty]$-trivial, with $\delta_n(1) = (-1)^n$, we have:

Corollary 5.25. *If H_1, H_2 are groups of type FP_n, then, for $n \geq 1$,*

$$E_n^T(H_1 * H_2) = E_n^T(H_1) *^{((-1)^n)} E_n^T(H_2).$$

Remark. Fox [41] proved this for the Alexander ideals using the fact that, if $\langle \mathbf{x}_1; \mathbf{r}_1 \rangle$, $\langle \mathbf{x}_2; \mathbf{r}_2 \rangle$ are presentations for H_1, H_2 respectively, then $\langle \mathbf{x}_1, \mathbf{x}_2; \mathbf{r}_1, \mathbf{r}_2 \rangle$ is a presentation for $H_1 * H_2$. Similarly, we could prove this for the B-ideals from the fact that, if $\langle \mathbf{x}_1; \mathbf{r}_1; \mathbf{d}_1 \rangle$, $\langle \mathbf{x}_2; \mathbf{r}_2; \mathbf{d}_2 \rangle$ are 3-presentations for H_1, H_2 respectively, then $\langle \mathbf{x}_1, \mathbf{x}_2; \mathbf{r}_1, \mathbf{r}_2; \mathbf{d}_1, \mathbf{d}_2 \rangle$ is a 3-presentation for $H_1 * H_2$.

Corollary 5.26. *If each H_v ($v \in \mathbf{v}$) is $E^T[m,n]$-trivial and each H_e ($e \in \mathbf{e}$) is $E^T[m-1,n-1]$-trivial ($m > 0$), then G_Γ is $E^T[m,n]$-trivial and, for $m \leq i \leq n$,*

$$\delta_i^T(G_\Gamma) = \delta_i^T(\mathbf{v}) - \delta_i^T(\mathbf{e}).$$

In particular, if each H_γ is $E^T[l,\infty]$-trivial for some T and some l, then G_Γ is E^{tf}_∞-linked, and

$$\delta^{tf}(G_\Gamma) = \delta^{tf}(\mathbf{v}) - \delta^{tf}(\mathbf{e}).$$

Example 5.3. For any m, n, with $0 \leq m \leq n \leq \infty$, we find a group G_m which is $E[m,n]$-trivial, but neither $E[m-1,n]$-trivial (when $m > 0$), nor $E[m,n+1]$-trivial (when $n < \infty$).

For finite n, let H be a group of type FP_{n+1} which is $E[0,n]$-trivial, but is not $E[0,n+1]$-trivial. Such a group exists, courtesy of [59] (see also [14], [53] and Example 5.6, below). Notice that, by Lemma 5.1, $H_{n+1}(H) \neq 0$.

Let $G_m = H \times \mathbb{Z}^m$. We show by induction on m that G_m is $E[m,n]$-trivial. When $m = 0$, $G_0 = H$ is $E[0,n]$-trivial. Now suppose that, for $m \geq 1$, G_{m-1} is $E[m-1,n]$-trivial. Then, from the above corollary, since G_m is an HNN extension of G_{m-1}, G_m is $E[m,n]$-trivial.

Now, G_m cannot be $E[m-1,n]$-trivial, since $H_m(G_m) = \mathbb{Z} \neq 0$, and cannot be $E[m,n+1]$-trivial, since $H_{n+1}(G_m) = H_{n+1}(H) \neq 0$.

To find groups which are $E[m,\infty]$-trivial but not $E[m-1,\infty]$-trivial, take H to be an acyclic group of type FP_∞ and proceed as above. ◊

If would be nice to be able to say that, if each H_γ is E^T_∞-linked, then so is G_Γ and to give a formula for $\delta^{tf}(G_\Gamma)$ as above. However, since $\iota^{tf}_\gamma : H^{tf}_\gamma \to G^{tf}_\Gamma$ need not be injective, we could have

$$E^{tf}_{n,\delta^{tf}_n(H_\gamma)}(\mathbb{Z} \uparrow^{G_\Gamma}_{H_\gamma}) = \left(\iota^{tf}_\gamma E^{tf}_{n,\delta^{tf}_n(H_\gamma)}(H_\gamma)\right) = 0,$$

and so $\delta_n^{tf}(\mathbb{Z}\uparrow_{H_\gamma}^{G_\Gamma}) > \delta_n^{tf}(H_\gamma)$.

Example 5.4. Let $H = G(\mathcal{Q})$, where

$$\mathcal{Q} = \langle a, b; a^p, ab = ba\rangle$$

for some prime p. Applying Theorem 4.3 to Example 4.1, we have

$$E_{n,\lambda}^{tf}(H, \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p\langle b\rangle & \lambda \geq 1 \\ (1-b) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

for $n \geq 0$. The group H is then E_∞^{tf}-linked over $\mathbb{Z}_p$, with $\delta^{tf}(H, \mathbb{Z}_p) = 0$.

Now, the element $b \in H$ has infinite order, so the map

$$\phi : \langle b\rangle \to H; b \mapsto b^2$$

is a monomorphism. We then obtain an HNN extension $G = H*_{\langle b\rangle,\phi}$, which can be presented by

$$\langle a, b, t; a^p, ab = ba, bt = tb^2\rangle.$$

Since $\langle b\rangle$ is $E[1,\infty]$-trivial over $\mathbb{Z}_p$ and $\delta_n^{tf}(\langle b\rangle, \mathbb{Z}_p) = 0$,

$$E_n^{tf}(G, \mathbb{Z}_p) = \left(\iota^{tf} E_n^{tf}(H, \mathbb{Z}_p)\right),$$

by Proposition 5.24(i). However, the embedding $\iota : H \to G$ induces the trivial map

$$\iota^{tf} : \langle b\rangle \to \langle t\rangle; b \mapsto 1,$$

and so

$$E_{n,\lambda}^{tf}(G, \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p\langle t\rangle & \lambda \geq 1 \\ 0 & \lambda < 0 \end{cases}.$$

Thus, G is not E_∞^{tf}-linked over $\mathbb{Z}_p$, even though both H and $\langle b\rangle$ are. ◊

We must therefore restrict ourselves somewhat, for example, to the case when each ι_γ^{tf} is injective.

Theorem 5.27. *If each H_γ ($\gamma \in \Gamma$) is E_∞^{tf}-linked and if $H_e^{tf} = 1$ for each $e \in \mathbf{e}^+$, then G_Γ is E_∞^{tf}-linked and*

$$\delta^{tf}(G_\Gamma) = \delta^{tf}(\mathbf{v}) - \delta^{tf}(\mathbf{e}).$$

Proof. The Mayer-Vietoris exact sequence (1.23) starts as

$$\cdots \to H_2(G_\Gamma) \to \oplus_{\mathbf{e}^+} H_1(H_e) \to \oplus_{\mathbf{v}} H_1(H_v) \xrightarrow{\oplus_{\mathbf{v}} \iota_v^{ab}} H_1(G_\Gamma) \to$$
$$\oplus_{\mathbf{e}^+} \mathbb{Z} \to \oplus_{\mathbf{v}} \mathbb{Z} \to \mathbb{Z} \to 0.$$

Since each $H_e^{tf} = 1$, the free part of $\oplus_{\mathbf{v}} H_1(H_v)$ must embed in the free part of $H_1(G_\Gamma)$. Thus each ι_v^{tf} is injective, and so $\delta_n^{tf}(\mathbb{Z}\uparrow_{H_v}^{G_\Gamma}) = \delta_n^{tf}(H_v)$. Also, since $H_e^{tf} = 1$, $\iota_e^{tf} : H_e^{tf} \to G_\Gamma^{tf}$ is injective, so $\delta_n^{tf}(\mathbb{Z}\uparrow_{H_e}^{G_\Gamma}) = \delta_n^{tf}(H_e)$ for each $e \in \mathbf{e}^+$.

Each $\mathbb{Z}\uparrow_{H_\gamma}^{G_\Gamma}$ is thus E_∞^{tf}-linked when each H_γ is, and so G_Γ is E_∞^{tf}-linked too, with $\delta^{tf}(G_\Gamma)$ as given in the statement of the theorem, by Proposition 5.14. □

Since the trivial group is E_∞^{tf}-linked, with $\delta^{tf}(1) = 1$, we have:

Corollary 5.28. *If H_1, H_1 are both E_∞^{tf}-linked, then so is $H_1 * H_2$ and*

$$\delta^{tf}(H_1 * H_2) = \delta^{tf}(H_1) + \delta^{tf}(H_2) - 1.$$

Recall that a group is E_∞^{triv}-linked if, and only if, it is of type FP_∞ and of type FR. The following result is then just Lemma 7 of [30] for groups of type FP_∞.

Proposition 5.29. *If each H_γ ($\gamma \in \Gamma$) is E_∞^{triv}-linked, then so is G_Γ and*

$$\delta^{tf}(G_\Gamma) = \delta^{tf}(\mathbf{v}) - \delta^{tf}(\mathbf{e}).$$

Proof. Since $\iota_\gamma^{triv} : \mathbb{Z} \to \mathbb{Z}$ is injective, $\mathbb{Z}\uparrow_{H_\gamma}^{G_\Gamma}$ is E_∞^{triv}-linked when H_γ is. □

Finally, there is one more situation in which we can determine δ^{tf} for a graph of groups. A *mapping torus* is an HNN extension $G = H*_{H,\phi}$, where $\phi : H \to H$ is a monomorphism.

Proposition 5.30. *If $G = H*_{H,\phi}$ is a mapping torus and if G is E_∞^{tf}-linked, then $\delta^{tf}(G) = 0$. In particular, if H is E_∞^{tf}-linked, then so is the mapping torus $H \times \langle t\rangle$, and so $\delta^{tf}(H \times \langle t\rangle) = 0$.*

Proof. The sequence (1.20) becomes

$$0 \to \mathbb{Z}\uparrow_H^G \to \mathbb{Z}\uparrow_H^G \to \mathbb{Z} \to 0. \tag{5.5}$$

If G is E^T_∞-linked, then, by Proposition 5.14(ii), for large n,

$$\begin{aligned}\delta^T_n(G) &= \delta^T_n(\mathbb{Z}\uparrow_H^G) - \delta^T_n(\mathbb{Z}\uparrow_H^G)\\ &= 0.\end{aligned}$$

If $\phi = \mathrm{Id}_H$, then the embedding $\iota : H \to H \times \langle t\rangle$ induces an embedding ι^{tf} : $H^{tf} \to (H \times \langle t\rangle)^{tf} = H^{tf} \times \langle t\rangle$. Thus, by an argument above, $\mathbb{Z}\uparrow_H^{H\times\langle t\rangle}$ is E^{tf}_∞-linked when H is. □

Example 5.5. If H is the subgroup of the R. Thompson group generated by x_i, $i \geq 1$, and if ϕ is the monomorphism $x_i \mapsto x_{i+1}$ $(i \geq 1)$, then G is the resulting mapping torus, so $\delta^{tf}(G) = 0$, as we know. ◊

Corollary 5.31. *If G is a finitely generated abelian group, then G is E^{tf}_∞-linked and*

$$\delta^{tf}(G) = \begin{cases} 1 & \text{if } G \text{ is finite} \\ 0 & \text{if } G \text{ is infinite} \end{cases}.$$

Proof. We proceed by induction on the rank of G. We know that if G is finite, then it is $E^{triv}[1,\infty]$-linked and, since $\delta^{triv}_1(G) = -1$, $\delta^{tf}(G) = 1$. Suppose now that $G = H \times \langle t\rangle$, where H is an abelian group which is E^{tf}_∞-linked. Then, by the preceding result, G is E^{tf}_∞-linked and $\delta^{tf}(G) = 0$. □

5.6 Extensions of E-trivial groups

Let H be a subgroup of a group G. If we apply the exact functor $\mathbb{Z}G \otimes_{\mathbb{Z}H} -$ to the short exact sequence

$$0 \to IH \to \mathbb{Z}H \to \mathbb{Z} \to 0,$$

we obtain a short exact sequence of $\mathbb{Z}G$-modules. Since $\mathbb{Z}H\uparrow_H^G \cong \mathbb{Z}G$ and $\mathbb{Z}\uparrow_H^G \cong \mathbb{Z}[G/H]$, where $\mathbb{Z}[G/H]$ is the $\mathbb{Z}G$-module which is $\mathbb{Z}$-free on the cosets $\{gH : g \in G\}$, with G acting by permuting the cosets [22, §III.5], this becomes

$$0 \to IH\uparrow_H^G \xrightarrow{1\otimes(1-h)\mapsto 1-h} \mathbb{Z}G \xrightarrow{g\mapsto gH} \mathbb{Z}[G/H] \to 0.$$

The image of the embedding map in this sequence is a subset of IG, so we also have a short exact sequence

$$0 \to IH\uparrow_H^G \xrightarrow{1\otimes(1-h)\mapsto 1-h} IG \xrightarrow{1-g\mapsto 1H-gH} I[G/H] \to 0,$$

where $I[G/H]$ is the submodule of $\mathbb{Z}[G/H]$ generated as an abelian group by $\{1H - gH : g \in G\}$. We then have the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IH\uparrow_H^G & \longrightarrow & IG & \longrightarrow & I[G/H] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & , \\ 0 & \longrightarrow & \mathbb{Z}G & \xrightarrow{\mathrm{Id}} & \mathbb{Z}G & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

with exact rows, where the first two vertical maps are the embeddings $1\otimes(1-h) \mapsto 1-h$ and $1-g \mapsto 1-g$ whose cokernels are, respectively, $\mathbb{Z}\uparrow_H^G$ and ${}_G\mathbb{Z}$. Applying the Snake Lemma 1.32, we then obtain the short exact sequence

$$0 \to I[G/H] \to \mathbb{Z}\uparrow_H^G \to {}_G\mathbb{Z} \to 0. \tag{5.6}$$

Now, if the subgroup H is $E^T[m,n]$-trivial for some $n \geq m > 0$, then so is the module $\mathbb{Z}G \otimes_{\mathbb{Z}H} {}_H\mathbb{Z}$. In this case, applying Proposition 5.2(iii) to (5.6), we have

$$E^T_{i,\lambda}(G) = E^T_{i-1,\lambda-(-1)^{i-m}\delta^T_m(H)}(I[G/H]),$$

for $m < i \leq n$. If H is normal in G and if $G_0 = G/H$, then this becomes

$$E^T_{i,\lambda}(G) = E^T_{i-1,\lambda-(-1)^{i-m}\delta^T_m(H)}({}_GIG_0),$$

where ${}_GIG_0$ is the augmentation ideal of G_0 with G acting via the natural surjection $\alpha : G \to G_0$. Despite the similarities between this equation and Corollary 4.15(i), which states that

$$E^T_{i,\lambda}(G_0) = E^T_{i-1,\lambda-(-1)^i}(IG_0),$$

we must not suppose that the E^T_i-ideals of G and of G_0 are in any way similar (except, of course, when $H = 1$). For instance, a finite presentation $\mathcal{P} = \langle \mathbf{x}; \mathbf{r}\rangle$ for a finite group G_0 gives an extension

$$1 \to \ll \mathbf{r} \gg \to \langle \mathbf{x}\rangle \to G_0 \to 1.$$

Since $\ll \mathbf{r} \gg$ is free of finite rank, it is $E[1,\infty]$-trivial, but G_0 can have non-trivial ideals while $\langle \mathbf{x}\rangle$ is always $E[1,\infty]$-trivial.

However, when $m = 0$, that is, when H is $E[0, n]$-trivial, $E_i(G)$ and $E_i(G_0)$ *are* "the same".

Theorem 5.32. *Let H be a normal subgroup of a group G and let $G_0 = G/H$. If H is $E[0, n]$-trivial ($n > 0$), then G is of type FP_n if, and only if, G_0 is and*

$$E_n(G) \cong^{(0)} E_n(G_0).$$

Remark. If $n = 1$, then this is just a special case of Theorem 2.6.

Proof. First we note that, since H is perfect, $H = H' \subseteq G'$ and

$$G^{ab} = G/G' \cong (G/H)/(G'/H) = G_0/G_0' = G_0^{ab}.$$

If $\alpha : G \to G_0; g \mapsto gH$ is the natural surjection, then

$$\alpha^{ab} : G^{ab} \to G_0^{ab}; gG' \mapsto \alpha(g)G_0'$$

is an isomorphism.

We show that $\alpha^{ab}E_n(G) = E_n(G_0)$.

If G_0 is of type FP_n, then G is of type FP_n by Corollary 1.34.

To complete the proof, we show that $\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}G_0, {}_G\mathbb{Z}) = 0$ for $i = 1, 2, \ldots, n$ and appeal to Proposition 4.11. Since H is normal in G, $\mathbb{Z}G_0 \cong \mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G$ as right $\mathbb{Z}G$-modules, so

$$\begin{aligned}\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}G_0, {}_G\mathbb{Z}) &\cong \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G, {}_G\mathbb{Z}) \\ &\cong \mathrm{Tor}_i^{\mathbb{Z}H}(\mathbb{Z}_H, {}_H\mathbb{Z}) \\ &= H_i(H).\end{aligned}$$

But, by Lemma 5.1, $H_i(H) = 0$ for $i = 1, \ldots, n$. Thus, by Proposition 4.11, $\mathbb{Z}G_0 \otimes_{\mathbb{Z}G} \mathbb{Z} \cong {}_{G_0}\mathbb{Z}$ is of type FP_n when G is and $E_n(G_0) = \alpha^{ab}E_n(G)$, the required result. □

5.7 The utility of the new invariants

We finish this chapter by demonstrating that, for each n, the E_n-ideals of a group are a useful invariant in the sense that, for every $n > 1$, there are groups which the E_n-ideals can distinguish, but which have the same E_i-ideals for $i < n$ and which, moreover, cannot be distinguished by their integral homology.

Theorem 5.33. *For every $n > 1$ there exists an infinite family $\{G_l : l \geq 1\}$ of pairwise non-isomorphic groups of type FP_n such that, for each l, l', $H_*(G_l) \cong H_*(G_{l'})$ and $E_i(G_l) \cong^{(0)} E_i(G_{l'})$, $i = 0, \ldots, n-1$, but $E_n(G_l) \not\cong^{(0)} E_n(G_{l'})$ if $l \neq l'$.*

To prove this, we require the following lemmata.

Let G be a group containing an element t of infinite order whose image in G^{ab} is also of infinite order and which, moreover, generates a direct summand of G^{ab}. Thus $G^{ab} = H \times \langle t \rangle$, for some abelian group H. For an integer $l \geq 1$, we set

$$G_l = G *_{t=s^l} \langle s \rangle,$$

where we are amalgamating the infinite cyclic subgroups $\langle t \rangle$ of G and $\langle s^l \rangle$ of $\langle s \rangle$.

Lemma 5.34. *For G, G_l as above*

$$H_*(G_l) = H_*(G).$$

In particular, $G_l^{ab} \cong G^{ab}$.

Proof. Note first that, for H as above,

$$\begin{aligned} G_l^{ab} &\cong G^{ab} \times \langle s \rangle / \ll (t, s^{-l}) \gg \\ &\cong H \times \langle s \rangle \\ &\cong G^{ab}, \end{aligned}$$

so $H_1(G_l) \cong H_1(G)$.

Since

$$H_i(\langle t \rangle) = H_i(\langle s \rangle) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases},$$

the Mayer-Vietoris sequence (1.23) becomes

$$\cdots \to 0 \to H_i(G) \to H_i(G_l) \to 0 \to \cdots$$

$$\cdots \to 0 \to H_2(G) \to H_2(G_l) \to \mathbb{Z} \to H_1(G) \oplus \mathbb{Z} \to H_1(G_l) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

Thus $H_i(G_l) \cong H_i(G)$ for $i > 2$. Finally, since the map

$$\mathbb{Z} \cong \langle t \rangle \to H_1(G) \oplus \mathbb{Z} \cong G^{ab} \times \langle s \rangle; t \mapsto (t, s^{-l})$$

in this sequence is injective, we have $H_2(G_l) \cong H_2(G)$. □

Note that the isomorphism of G^{ab} and G_l^{ab} sends t to s, whereas the map ι_l^{ab} : $G^{ab} \to G_l^{ab}$, induced by the inclusion $\iota_l : G \to G_l$, sends t to s^l.

Lemma 5.35. *For G, G_l as above, if G is of type FP_n, then so is G_l and, for $n > 1$,*

$$E_n(G_l) = \left(\iota_l^{ab} E_n(G)\right).$$

Proof. By Proposition 5.24(i), since $\langle t\rangle$ is $E[1,\infty]$-trivial, with $\delta_n(\langle t\rangle) = 0$ for $n \geq 1$, G_l is of type FP_n when G is, and

$$E_n(G_l) = \left(\iota_l^{ab} E_n(G)\right) * \left(\iota_s^{ab} E_n(\langle s\rangle)\right)$$

for $n > 1$, $\iota_s : \langle s\rangle \to G_l$ being the inclusion map. Now,

$$\left(\iota_s^{ab} E_{n,\lambda}(\langle s\rangle)\right) = \begin{cases} \mathbb{Z}G_l^{ab} & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}$$

for $n \geq 1$, so, for $n > 1$,

$$E_n(G_l) = \left(\iota_l^{ab} E_n(G)\right),$$

by a comment in the proof of Proposition 5.2. □

Proof of Theorem 5.33. Let H be an $E[0, n-1]$-trivial group, which is of type FP_n, but is not $E[0, n]$-trivial. Thus, $H_i(H) = 0$ for $i = 1, \ldots, n-1$, but

$$H_n(H) = \mathbb{Z}^q \oplus_{j=1}^{p} \mathbb{Z}/\left(c_j\right)$$

is non-trivial, where $c_1, \ldots, c_p$ are non-zero, non-unit integers such that $c_j | c_{j+1}$. Such a group exists (see [59] and also Example 5.6, below) and we can, in fact, choose $H_n(H)$ to be any finitely generated abelian group.

Let $G = H \times \langle t\rangle$ and, as above, let

$$G_l = G *_{t=s^l} \langle s\rangle.$$

By Lemma 5.34, for $l \neq l'$, $H_*(G_l) \cong H_*(G_{l'})$, so these groups are not distinguished by their integral homology. We now turn to their E-ideals.

By Lemma 5.35,

$$E_i(G_l) = \left(\iota_l^{ab} E_i(G)\right)$$

for $i > 1$. But, since H, an $E[0, n-1]$-trivial group, is a normal subgroup of G,

$$E_i(G) \cong^{(0)} E_i(\langle t \rangle),$$

for $i < n$, by Theorem 5.32. Thus, for $i = 2, \ldots, n-1$,

$$E_{i,\lambda}(G_l) = \begin{cases} \mathbb{Z}\langle s \rangle & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}$$

for each l. This is also the case for $i = 1$, since, if $\mathcal{Q} = \langle \mathbf{y}; \mathbf{s} \rangle$ is a presentation for H, with $\mathbf{y}$ finite, then

$$\langle s, t, \mathbf{y}; t = s^l, [y, t](y \in \mathbf{y}), \mathbf{s} \rangle$$

is a presentation for G_l. But this presentation is of the form (2.2), and so, by Theorem 2.6, if

$$\mathcal{P}_0 = \langle s, t; t = s^l \rangle$$

(which is Tietze equivalent to $\langle s; - \rangle$),

$$E_{1,\lambda}(G_l) = A_\lambda(\mathcal{P}_0) = \begin{cases} \mathbb{Z}\langle s \rangle & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases}.$$

In dimension 0, for each l,

$$E_{0,\lambda}(G_l) = \begin{cases} \mathbb{Z}\langle s \rangle & \lambda \geq 1 \\ (1-s) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

whence, for $i = 0, \ldots, n-1$, $E_i(G_l) \cong^{(0)} E_i(G_{l'})$ for every l, l'.

We now find $E_n(G_l)$ by first calculating $E_n(G)$ and then applying Lemma 5.35. As in Example 1.7, if $\mathcal{P} = (P_i, \varepsilon_i)$ is a resolution of type FP_n for H, with $\mathrm{rk}_{\mathbb{Z}H}(P_i) = r_i$, using Wall's method we obtain a resolution $\mathcal{F}$ for G of type FP_n with

$$D_n(\mathcal{F}) = \begin{bmatrix} -D_{n-1}(\mathcal{P}) & (1-t)I_{r_n} \\ 0 & D_n(\mathcal{P}) \end{bmatrix}$$

and $\chi_n(\mathcal{F}) = r_n$ (since $G = H \times \langle t \rangle$, we can take $\beta : H \to G$ to be the inclusion and $X_i = 0$).

By Theorem 4.24, and since $D_i(\mathcal{P})^{ab} = D_i(\mathcal{P})^{triv}$,

$$J_\kappa(D_{n-1}(\mathcal{P})^{ab}) = \begin{cases} 0 & \kappa > \chi_{n-1} + (-1)^n \\ \mathbb{Z}\langle t \rangle & \kappa \leq \chi_{n-1} + (-1)^n \end{cases}$$

and

$$J_\kappa(D_n(\mathcal{P})^{ab}) = \begin{cases} 0 & \kappa > \chi_n - q - (-1)^n \\ \left(\prod_{j=1}^{\kappa - \chi_n + p + q + (-1)^n} c_j\right) & \chi_n - p - q - (-1)^n < \kappa \\ & \quad \leq \chi_n - q - (-1)^n \\ \mathbb{Z}\langle t \rangle & \kappa \leq \chi_n - p - q - (-1)^n \end{cases},$$

where $\chi_i = \chi_i(\mathcal{P})$. We can therefore assume that we have chosen bases such that

$$D_{n-1}(\mathcal{P})^{ab} = \begin{bmatrix} 0 & 0 \\ I_{\chi_{n-1}+(-1)^n} & 0 \end{bmatrix}$$

and

$$D_n(\mathcal{P})^{ab} = \begin{bmatrix} 0 & 0 & 0 \\ I_{\chi_n - p - q - (-1)^n} & 0 & 0 \\ 0 & \mathrm{Diag}_p(c_1, \ldots, c_p) & 0 \end{bmatrix}$$

are in the form (4.6). Thus, applying Corollary 1.25 and noting that

$$\kappa - (\chi_{n-1} + (-1)^n) - (\chi_n - p - q - (-1)^n) = \kappa - r_n + p + q,$$

we have

$$J_\kappa(D_n(\mathcal{F})^{ab}) = J_{\kappa - r_n + p + q}(D'),$$

where

$$D' = \begin{bmatrix} (1-t)I_p & 0 \\ 0 & (1-t)I_q \\ \mathrm{Diag}_p(c_1, \ldots, c_p) & 0 \end{bmatrix}.$$

Now,

$$J_\kappa(D') = \begin{cases} 0 & \kappa > p + q \\ ((1-t)^{\kappa - j} c_1 \ldots c_j (0 \leq j \leq p)) & p < \kappa \leq p + q \\ ((1-t)^{\kappa - j} c_1 \ldots c_j (0 \leq j \leq \kappa)) & 0 < \kappa \leq p \\ \mathbb{Z}\langle t \rangle & \kappa \leq 0 \end{cases},$$

and so

$$E_{n,\lambda}(G) = \begin{cases} \mathbb{Z}\langle t\rangle & \lambda \geq p+q \\ \left((1-t)^{p+q-\lambda-j}c_1 \ldots c_j (0 \leq j \leq p+q-\lambda)\right) & q \leq \lambda < p+q \\ \left((1-t)^{p+q-\lambda-j}c_1 \ldots c_j (0 \leq j \leq p)\right) & 0 \leq \lambda < q \\ 0 & \lambda < 0 \end{cases}.$$

The chain of ideals $E_n(G_l)$ can then be obtained from $E_n(G)$ by applying the map $t \mapsto$The chain of ideals $E_n(G_l)$ can then be obtained from $E_n(G)$ by applying the map $t \mapsto s^l$.

For example,

$$E_{n,p+q-1}(G_l) = \begin{cases} \left(c_1, 1-s^l\right) & p > 0 \\ \left(1-s^l\right) & p = 0 \end{cases},$$

whi

$s \mapsto s^{\pm 1}$ sends $E_{n,p+q-1}(G_l)$ to $E_{n,p+q-1}(G_{l'})$, and so the groups $\{G_l : l \geq 1\}$ are pairwise non-isomorphic. □

Example 5.6. For $n > 0$, a group $H^{(n)}$ of type FL which is $E[0,n]$-trivial, but not $E[0,n+1]$-trivial, can be constructed as follows.

Let $H^{(0)}$ be a group of type FL with non-trivial abelianisation. For example, a free group, a knot group or a finitely-generated torsion-free one-relator group. Now, every group of type FL embeds in an acyclic group of type FL [14], so let G_0 be an acyclic group of type FL into which $H^{(0)}$ embeds. We can then construct the amalgamated free product

$$H^{(1)} = G_0 *_{H^{(0)}} G_0,$$

the so-called *suspension* of $H^{(0)}$. Since $G^{(0)}$ has trivial homology in positive dimensions, the Mayer-Vietoris sequence (1.23) gives the homology of $H^{(1)}$ as

$$H_i(H^{(1)}) = \begin{cases} 0 & i = 1 \\ H_{i-1}(H^{(0)}) & i > 1 \end{cases}.$$

We therefore have a group which is $E[0,1]$-trivial, but which is not $E[0,2]$-trivial, since $H_2(H^{(1)}) = H_1(H^{(0)}) \neq 0$. Moreover, $H^{(1)}$ is of type FL, since both $H^{(0)}$ and G_0 are.

Now suppose, inductively, that we have constructed a group $H^{(n-1)}$ of type FL, which is $E[0, n-1]$-trivial, but not $E[0, n]$-trivial. Then there exists an acyclic group G_{n-1} of type FL into which $H^{(n-1)}$ embeds, and so we can construct the suspension

$$H^{(n)} = G_{n-1} *_{H^{(n-1)}} G_{n-1}$$

of $H^{(n-1)}$, which is of type FL and whose homology is

$$H_i(H^{(n)}) = \begin{cases} 0 & i = 1 \\ H_{i-1}(H^{(n-1)}) & i > 1 \end{cases}.$$

The group $H^{(n)}$ is thus $E[0, n]$-trivial, but, since

$$H_{n+1}(H^{(n)}) = H_n(H^{(n-1)}) \neq 0,$$

it is not $E[0, n+1]$-trivial. $\diamondsuit$

Chapter 6

The E-ideals of monoids

The definition of the E-ideals can easily be extended to monoids and their modules. We must, however, consider right modules as well as left modules.

6.1 Definition and properties

Let S be a monoid and let M be a KS-module, left or right, of type FP_n. For an abelianising functor ^{T} on monoids, if $\mathcal{F}$ is a resolution of M of type FP_n (by left or right free KS-modules, as appropriate), we define, for $\lambda \in \mathbb{Z}$, the λ-th *E_n^T-ideal of M* to be

$$E_{n,\lambda}^T(M) = J_{\chi_n(\mathcal{F})-\lambda}(D_n(\mathcal{F})^T).$$

The chain $E_n^T(M)$ of ideals $E_{n,\lambda}^T(M)$ is then an ascending chain of ideals in the commutative ring KS^T.

Theorem 6.1. *The chain of ideals $E_n^T(M)$ is well-defined.*

The proof of this theorem is analogous to that of Theorem 4.7. Note, however, that it requires there to be an identity element $1 \in KS^T$, so we cannot extend this result to semigroups.

If S is of type $FP_n^{(l)}$ over K, then we define $E_n^{(l)T}(S,K)$ to be the chain $E_n^T({}_SK)$, and if S is of type $FP_n^{(r)}$ over K, we set $E_n^{(r)T}(S,K) = E_n^T(K_S)$. As ever, when $K = \mathbb{Z}$ we simply write $E_n^{(l)T}(S)$ or $E_n^{(r)T}(S)$, and when ${}^T = {}^{ab}$ we omit it.

Theorem 6.2. *If S, S_0 are monoids of type $FP_n^{(l)}$ (respectively, $FP_n^{(r)}$) over K and if $S \cong S_0$, then $E_n^{(l)T}(S,K) \cong^{(0)} E_n^{(l)T}(S_0,K)$ (respectively, $E_n^{(r)T}(S,K) \cong^{(0)} E_n^{(r)T}(S_0,K)$).*

The proof of this theorem is analogous to that of Theorem 4.1(ii).

Example 6.1. Let S be the finite cyclic monoid given by the presentation

$$[x; x^p = x^q]$$

for some integers $p > q \geq 0$. Then, by the method of §1.6.6 (or directly), S has a free resolution $\mathcal{F} = (F_i, \partial_i)$ with $\mathrm{rk}_{\mathbb{Z}S}(F_i) = 1$ $(i \geq 0)$ and

$$D_n(\mathcal{F}) = \begin{cases} \begin{bmatrix} 1-x \end{bmatrix} & n \text{ even} \\ \begin{bmatrix} x^q(1+x+\cdots+x^{p-q-1}) \end{bmatrix} & n \text{ odd} \end{cases}.$$

Thus

$$E_{n,\lambda}^{(l)}(S) = \begin{cases} \mathbb{Z}S & \lambda \geq 1 \\ (1-x) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

for even n and

$$E_{n,\lambda}^{(l)}(S) = \begin{cases} \mathbb{Z}S & \lambda \geq 0 \\ (x^q(1+x+\cdots+x^{p-q-1})) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases},$$

for odd n. There is also a free right resolution $\mathcal{F}^{(r)}$ with $D_n(\mathcal{F}^{(r)}) = D_n(\mathcal{F})$, so

$$E_{n,\lambda}^{(r)}(S) = E_{n,\lambda}^{(l)}(S).$$

The fact that the $E^{(l)}$- and $E^{(r)}$-ideals of this monoid are the same is not typical, as we will see in Examples 6.2, 6.3 and 6.4, below. ◊

The following result shows that, if so inclined, we can avoid working with right modules and right resolutions by applying the functor *opp*.

Proposition 6.3. *Let ^{T} be an abelianising functor on monoids which satisfies the additional conditions that:*

a) $(S^{opp})^T = S^T$ *(*$= (S^T)^{opp}$*, since* S^T *is abelian); and*

b) $\tau^T_{S^{opp}} opp = \tau^T_S$.

If M *is a right* KS*-module of type* FP_n*, then*

$$E^T_n(M) = E^T_n(M^{opp}).$$

The abelianising functors ^{ab} and ^{triv} satisfy the above conditions.

Proof. By Lemma 1.28, M^{opp} is a left KS^{opp}-module of type FP_n and, if $\mathcal{F}$ is a free KS-resolution of M of type FP_n, then $\mathcal{F}^{opp}$ is a free KS^{opp}-resolution of M^{opp}, with

$$D_n(\mathcal{F}^{opp}) = (D_n(\mathcal{F})^{opp})^t.$$

Since $\tau^T_{S^{opp}} opp = \tau^T_S$,

$$D_n(\mathcal{F}^{opp})^T = (D_n(\mathcal{F})^T)^t,$$

and the required result follows. □

Corollary 6.4. *If* S *is of type* $FP^{(r)}_n$ *over* K *and* ^{T} *satisfies conditions (a), (b) above, then* S^{opp} *is of type* $FP^{(l)}_n$ *over* K *and*

$$E^{(r)T}_n(S, K) = E^{(l)T}_n(S^{opp}, K).$$

Since $S^{opp} = S$ when S is abelian, we have the following special case of the last result.

Corollary 6.5. *If* S *is an abelian monoid, then it is of type* $FP^{(r)}_n$ *over* K *if, and only if, it is of type* $FP^{(l)}_n$ *over* K *and*

$$E^{(r)T}_n(S, K) = E^{(l)T}_n(S, K).$$

The finite cyclic monoid of Example 6.1 is an instance of this.

If a monoid is actually a group G, then we know that it is of type $FP^{(l)}_n$ over K if, and only if, it is of type $FP^{(r)}_n$ over K. We refer to this double property simply as type FP_n. Clearly, comparing the definition of this chapter with that in §4.1, we have

$$E_n^{(l)T}(G,K) = E_n^T(G,K).$$

The group rings KG and KG^T both admit an anti-automorphism *inv* induced by $g \mapsto g^{-1}$, which is an automorphism in the case of the commutative ring KG^T.

Proposition 6.6. *If G is a group of type FP_n over K, then*

$$E_n^{(r)T}(G,K) = inv E_n^T(G,K).$$

We actually prove a more general result.

Lemma 6.7. *If M is a KG-module of type FP_n, then*

$$E_n^T(M^{inv}) = inv E_n^T(M).$$

Proof. By Lemma 1.28, M^{inv} is of type FP_n and, given a resolution $\mathcal{F}$ of M of type FP_n, $\mathcal{F}^{inv}$ is a resolution of M^{inv} with

$$D_n(\mathcal{F}^{inv}) = \left(D_n(\mathcal{F})^{inv}\right)^t.$$

Since

$$inv\tau_G^T(g) = \tau_G^T(g)^{-1} = \tau_G^T(g^{-1}) = \tau_G^T inv(g),$$

we find

$$D_n(\mathcal{F}^{inv})^T = \left(\left(D_n(\mathcal{F})^T\right)^{inv}\right)^t,$$

and the result follows. □

Proposition 6.6 is a corollary of this, since ${}_GK^{inv} = K_G$. Consequently, we need only study left group modules, as, indeed, we did in Chapters 4 and 5. For monoids in general, however, the invariants $E_n^{(l)T}(S)$ and $E_n^{(r)T}(S)$ can be very different, as we will see in the next section and in the following examples.

Example 6.2. The monoid S given by the presentation

$$[x_0, x_1, x_2, \ldots; x_i x_j = x_{j+1} x_i (i < j)]$$

is of type $FP_\infty^{(l)}$, but is not even of type $FP_1^{(r)}$ [32]. Thus $E_n^{(l)T}(S)$ is defined for every $n \geq 0$, whereas $E_n^{(r)T}(S)$ is not defined for any $n > 0$. ◊

Even when both $E_n^{(l)T}(S,K)$ and $E_n^{(r)T}(S,K)$ are defined, they can be very different.

Example 6.3. Let $S = S(\mathcal{P})$, where

$$\mathcal{P} = [a, b, c; ba = ab, ca = ac^2, cb = bc^2].$$

This presentation is terminating, since the right-hand side of each relation is in alphabetical order. It is also confluent, since

$$c(ba) \to cab, (cb)a \to bc^2a$$

and

$$(ca)b \to ac(cb) \to a(cb)c^2 \to abc^4,$$
$$bc(ca) \to b(ca)c^2 \to (ba)c^4 \to abc^4.$$

The presentation $\mathcal{P}$ is thus complete. The set

$$\{a^i b^j c^k : i, j, k \in \mathbb{Z}^+\}$$

is the set of irreducible words representing the elements of S. Since $\mathcal{P}$ is both finite and complete, S is of types $FP_\infty^{(l)}$ and $FP_\infty^{(r)}$. Applying the method of §1.6.6, we find a free left resolution $\mathcal{F}^{(l)}$ and right resolution $\mathcal{F}^{(r)}$ for S. In both cases the free modules of the resolution have basis () in dimension 0, $(a), (b), (c)$ in dimension 1, (b,a), (c,a), (c,b) in dimension 2, (c,b,a) in dimension 3 and are trivial in higher dimensions. In the left-hand case, we have

$$D_0(\mathcal{F}^{(l)}) = \begin{bmatrix} a-1 \\ b-1 \\ c-1 \end{bmatrix},$$

$$D_1(\mathcal{F}^{(l)}) = \begin{bmatrix} b-1 & 1-a & 0 \\ c-1 & 0 & 1-a-ac \\ 0 & c-1 & 1-b-bc \end{bmatrix},$$

$$D_2(\mathcal{F}^{(l)}) = \begin{bmatrix} c-1 & 1-b-bc & ac+a-1 \end{bmatrix}.$$

Note that S^{ab} consists of the elements

$$\{a^i b^j c^k : i, j \in \mathbb{Z}^+, k \in \{0,1\}\} \cup \{c^k : k \in \mathbb{Z}^+\}.$$

After some simplification, we have

$$E_{0,\lambda}^{(l)}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ IS^{ab} & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

$$E_{1,\lambda}^{(l)}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (a-b, (1-a)(1-2a), 1-c) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

$$E_{2,\lambda}^{(l)}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (a-b, 1-2a, 1-c) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

(We can simplify the ideal $J_2(D_1(\mathcal{F}^{(l)})^{ab})$ to $(a-b, (1-a)(1-2a), 1-c)$ since

$$1 - c = (1-a)(1-c) + a(1-c)^2$$

and

$$a - b = (1-a)(1-b-bc) - (1-b)(1-a-ac) + (a-b)(1-c).$$

We also have

$$a - b = a(1-b-bc) - b(1-a-ac) \in J_1(D_2(\mathcal{F}^{(l)})^{ab}).)$$

On the right-hand side we have

$$D_0(\mathcal{F}^{(r)}) = \begin{bmatrix} 1-a & 1-b & 1-c \end{bmatrix},$$

$$D_1(\mathcal{F}^{(r)}) = \begin{bmatrix} 1-b & 1-c^2 & 0 \\ a-1 & 0 & 1-c^2 \\ 0 & a-1-c & b-1-c \end{bmatrix},$$

$$D_2(\mathcal{F}^{(r)}) = \begin{bmatrix} 1-c^4 \\ b-1-c^2 \\ 1-a+c^2 \end{bmatrix},$$

and so $E_0^{(r)}(S) = E_0^{(l)}(S)$,

$$E_{1,\lambda}^{(r)}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (a-b, (1-a)(1-a+c), 1-c^2) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

$$E_{2,\lambda}^{(r)}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (a-b, 1-a+c^2, a(1-c)) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}.$$

(The ideal $J_2(D_1(\mathcal{F}^{(r)})^{ab})$ is generated by $a-b, (1-a)(1-a+c), 1-c^2$, since

$$1-c^2 = (1-c^2)^2 + c((1-c^2)(1-b+c) - (1-b)(1-c^2))$$

and

$$a-b = (a-b)(1-c^2) - (1-a)(1-b+c) + (1-b)(1-a+c).$$

Also, since

$$a-b = -(1-a+c^2) - (b-1-c^2)$$

and

$$a(1-c) = a(1-c^2) = (1-c^4) - (1-c^2)(1-a+c^2),$$

we have $J_1(D_2(\mathcal{F}^{(r)})^{ab}) = (a-b, 1-a+c^2, a(1-c))$.)

If $E_1^{(l)}(S) \cong E_1^{(r)}(S)$, then the rings

$$\mathbb{Z}[a]/\left((1-a)(1-2a)\right) \cong \mathbb{Z}S^{ab}/E_{1,0}^{(l)}(S)$$

and

$$\mathbb{Z}[a]/\left(a(1-a)(2-a)\right) \cong \mathbb{Z}S^{ab}/E_{1,0}^{(r)}(S)$$

must be isomorphic. But these rings are distinct, so $E_1^{(l)}(S) \not\cong E_1^{(r)}(S)$. Similarly, the rings

$$\mathbb{Z}[a]/\left(1-2a\right) \cong \mathbb{Z}S^{ab}/E_{2,0}^{(l)}(S)$$

and

$$\mathbb{Z}[c]/\left(c^3 - c^2 + c - 1\right) \cong \mathbb{Z}S^{ab}/E^{(r)}_{2,0}(S)$$

are non-isomorphic, so $E^{(l)}_2(S) \not\cong E^{(r)}_2(S)$. ◊

Example 6.4. Let S be the monoid defined by the finite complete presentation

$$[x, \theta; \theta x = \theta, \theta\theta = \theta],$$

as in Example 1.8. We use the resolutions $\mathcal{F}^{(l)}$ and $\mathcal{F}^{(r)}$ found there to calculate the E-ideals of S. For $n > 0$,

$$D_n(\mathcal{F}^{(l)}) = \begin{cases} \begin{bmatrix} \theta - 1 & 0 \\ 0 & \theta - 1 \end{bmatrix} & n \text{ even} \\ \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} & n \text{ odd} \end{cases},$$

and

$$D_0(\mathcal{F}^{(l)}) = \begin{bmatrix} x - 1 \\ \theta - 1 \end{bmatrix}.$$

Thus, for odd n we have

$$E^{(l)}_{n,\lambda}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (\theta) & \lambda = -1, 0 \\ 0 & \lambda < -1 \end{cases},$$

for even $n > 0$

$$E^{(l)}_{n,\lambda}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (1 - \theta) & \lambda = -1, 0 \\ 0 & \lambda < -1 \end{cases}$$

and

$$E^{(l)}_{0,\lambda}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (1 - \theta) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}$$

(noting that $1 - x = (1 - x)(1 - \theta)$).

For the right-hand case we have, for $n > 0$,

$$D_n(\mathcal{F}^{(r)}) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 1-x & 1-\theta \end{bmatrix} & n \text{ even} \\ \begin{bmatrix} 1 & 0 \\ x-1 & \theta \end{bmatrix} & n \text{ odd} \end{cases},$$

and

$$D_0(\mathcal{F}^{(r)}) = \begin{bmatrix} 1-x & 1-\theta \end{bmatrix},$$

whence

$$E^{(r)}_{n,\lambda}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 0 \\ (\theta) & \lambda = -1 \\ 0 & \lambda < -1 \end{cases}$$

for odd n and

$$E^{(r)}_{n,\lambda}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (1-\theta) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}$$

for even n.

This monoid is of a type which we will study further in §6.3, where it will transpire that its E-ideals could have been found without the resolutions used above. ◊

Many, but not all, of the results of Chapters 4 and 5 hold for monoids and their modules. For instance, Theorem 4.24 generalises to give a connexion between $\{E^T_i(M)\}^n_{i=0}$ and $\{\mathrm{Tor}^{KS}_i(KS^T, M)\}^n_{i=0}$ when KS^T is a pid and M is a left KS-module of type FP_n. We also have a right-hand version, which connects $\{E^T_i(M)\}^n_{i=0}$ for a right KS-module M of type FP_n to $\{\mathrm{Tor}^{KS}_i(M, KS^T)\}^n_{i=0}$.

Theorem 4.3 extends to monoids, so, if we know $E^{(l)T}_n(S)$ and $E^{(r)T}_n(S)$, then we know $E^{(l)T}_n(S, K)$ and $E^{(r)T}_n(S, K)$ for any coefficient ring K.

Proposition 4.11 also holds for monoids, as does a right-hand version, but Corollary 4.13 does not. On the other hand, as we discuss in the next section, we have an additional corollary to Proposition 4.11, which allows us to determine the E-ideals of a group satisfying certain conditions from those of a certain submonoid.

6.2 Obtaining the E-ideals of a group from those of a submonoid

Proposition 6.8. *If G is a group which contains a submonoid S of type $FP_n^{(l)}$ over K such that every element $g \in G$ has the form*

$$g = s^{-1}s'$$

for some $s, s' \in S$, then G is of type FP_n over K and, if $\iota : S \to G$ is the inclusion,

$$E_n^T(G, K) = \left(\iota^T E_n^{(l)T}(S, K)\right)$$

for any abelianising functor ^{T} on monoids.

Proof. For such a group it is shown in [27, §X.3] that

$$\mathrm{Tor}_i^{KS}(KG, {}_SK) = 0$$

for $i > 0$ and

$$KG \otimes_{KS} {}_SK \cong {}_GK.$$

The result then follows from an application of the monoid version of Proposition 4.11. □

Example 6.5. If G is an abelian group which can be generated by a set $\mathbf{x}$ and if S is the submonoid of G generated by $\mathbf{x}$, then the invariant ideals of S give those of G. ◊

Example 6.6. Let G be the R. Thompson group, which has the presentation

$$\langle x_0, x_1, x_2, \ldots ; x_i x_j = x_{j+1} x_i (i < j) \rangle.$$

The submonoid S of G generated by x_i $(i \geq 0)$ has presentation

$$[x_0, x_1, x_2, \ldots ; x_i x_j = x_{j+1} x_i (i < j)]$$

and is of type $FP_\infty^{(l)}$ [32]. Since, in G,

$$x_i x_j^{-1} = \begin{cases} x_{j+1}^{-1}x_i & i<j \\ 1 & i=j \\ x_j^{-1}x_{i+1} & i>j \end{cases},$$

every element of G is of the form $s^{-1}s'$ for some $s, s' \in S$, and so, for $n \geq 0$,

$$\left(\iota^{ab} E_{n,\lambda}^{(l)}(S)\right) = E_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{ab} & \lambda \geq 1 \\ (1-x_0, 1-x_1) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}. \qquad \diamondsuit$$

Example 6.7. Let G be the group given by the presentation

$$\langle a, b, c; ba = ab, ca = ac^2, cb = bc^2 \rangle.$$

The submonoid of G generated by a, b, c is isomorphic to the monoid S of Example 6.3. Since

$$ab^{-1} = b^{-1}a, ac^{-1} = c^{-1}ac, bc^{-1} = c^{-1}bc,$$

we can apply the preceding proposition. The group G^{ab} is free abelian on a, b, with $c = 1$, and so

$$E_{1,\lambda}(G) = \begin{cases} \mathbb{Z}G^{ab} & \lambda \geq 1 \\ (a-b, (1-a)(1-2a)) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases},$$

$$E_{2,\lambda}(G) = \begin{cases} \mathbb{Z}G^{ab} & \lambda \geq 1 \\ (a-b, 1-2a) & \lambda = 0 \\ 0 & \lambda < 0 \end{cases}. \qquad \diamondsuit$$

6.3 Monoids with a zero

A *left zero* of a monoid S is an element $\theta \in S$ such that $\theta s = \theta$ for every $s \in S$. We can similarly define *right zeroes.* If a monoid has both a left and a right zero, then these are the same element, which we will call a *two-sided zero.* We will look only at monoids with a left zero, though all the results will hold for the right-hand and two-sided cases.

A monoid with a left zero is of type $FP_\infty^{(r)}$. We now show that its $E^{(r)}$-ideals are very simple.

Proposition 6.9. *A monoid S with a left zero θ is of type $FP_\infty^{(r)}$ and*

$$E_{n,\lambda}^{(r)}(S) = \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 1 \\ (1-\theta) & \lambda = 0 \; (n \text{ even}), \\ 0 & \lambda < 0 \end{cases} \qquad \begin{cases} \mathbb{Z}S^{ab} & \lambda \geq 0 \\ (\theta) & \lambda = -1 \; (n \text{ odd}). \\ 0 & \lambda < -1 \end{cases}$$

Proof. Since

$$\begin{aligned}(1-\theta)(1-s) &= 1 - s - \theta + \theta s \\ &= 1 - s,\end{aligned}$$

as a right $\mathbb{Z}S$-module IS is generated by $1-\theta$. Also, for $\xi \in \mathbb{Z}S$,

$$(1-\theta)\xi = 0$$

if, and only if,

$$\xi = \theta\xi \in \theta.\mathbb{Z}S,$$

and

$$\theta\xi = 0$$

if, and only if,

$$\xi = \xi - \theta\xi = (1-\theta)\xi \in (1-\theta).\mathbb{Z}S.$$

Thus, we have the following free resolution

$$\cdots \xrightarrow{1\mapsto\theta} \mathbb{Z}S \xrightarrow{1\mapsto 1-\theta} \mathbb{Z}S \xrightarrow{1\mapsto\theta} \mathbb{Z}S \xrightarrow{1\mapsto 1-\theta} \mathbb{Z}S \xrightarrow{aug} \mathbb{Z}_S.$$

The required ideals are then readily obtained. □

Remark. The image of θ in S^{ab} is trivial if, and only if, S^{ab} is trivial, for, if $\widehat{\theta} = 1$, then

$$\widehat{s} = \widehat{\theta}\widehat{s} = \widehat{\theta} = 1$$

for every $s \in S$.

Example 6.8. The monoid S of Examples 1.8 and 6.4 has a left zero, and so the $E^{(r)}$-ideals of S could have been more swiftly determined from the above. ◊

Given any monoid S, we can *adjoin a left zero element* as follows: for some $\theta \notin S$, let ${}_0S$ be the free product $S * [\theta]$ factored by the congruence generated by the additional relations $\theta\theta = \theta$ and $\theta s = \theta$ $(s \in S)$. If $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is a presentation for S, then

$$ {}_0\mathcal{P} = [\mathbf{x}, \theta; \mathbf{r}, \theta\theta = \theta, \theta x = \theta (x \in \mathbf{x})] $$

is a presentation for ${}_0S$. Similarly, we can adjoin a right zero, to give a monoid S_0, or a two-sided zero, to give a monoid ${}_0S_0$.

Lemma 6.10. *As a set, ${}_0S$ is the disjoint union $S \cup S\theta$ and the map $S \to S\theta; s \mapsto s\theta$ is a bijection.*

Proof. The first part follows from the definition.

Suppose that, for some $s, s' \in S$, $s\theta, s'\theta \in S * [\theta]$ are equivalent relative to the above additional relations. Then, since none of these affect elements of S which do not occur to the right of a θ, $s = s'$ in S. □

Proposition 6.9 shows that ${}_0S$ is always of type $FP_\infty^{(r)}$ and gives its $E^{(r)}$-ideals. The following result shows that the abelianisation of ${}_0S$ is never trivial, so these ideals are non-trivial in all dimensions.

Lemma 6.11. $({}_0S)^{ab} = {}_0(S^{ab})_0 = S^{ab} \cup \{\widehat{\theta}\}$.

Proof. There is a surjective monoid homomorphism

$$ \tau : {}_0S \to {}_0(S^{ab})_0; s \mapsto \widehat{s}, s\theta \mapsto \widehat{\theta} (s \in S). $$

We show that ${}_0(S^{ab})_0$ is the largest abelian quotient of ${}_0S$. Suppose that S' is an abelian monoid and that $\alpha : {}_0S \to S'$ is a monoid homomorphism. The map $\alpha|_S : S \to S'$ factors through S^{ab}, so there is a monoid homomorphism $\beta : S^{ab} \to S'$ such that $\beta(\widehat{s}) = \alpha(s)$. By setting $\beta(\widehat{\theta}) = \alpha(\theta)$, this extends to a map from ${}_0(S^{ab})_0 =$

$S^{ab} \cup \{\widehat{\theta}\}$ to S'. This is a monoid homomorphism, since

$$\begin{aligned}\beta(\widehat{s\theta}) &= \beta(\widehat{\theta}) = \alpha(\theta) = \alpha(\theta s) \\ &= \alpha(\theta)\alpha(s) = \alpha(s)\alpha(\theta) \\ &= \beta(\widehat{s})\beta(\widehat{\theta}),\end{aligned}$$

and, similarly, $\beta(\widehat{\theta\theta}) = \beta(\widehat{\theta})\beta(\widehat{\theta})$. □

We now show how the $E^{(l)}$-ideals of ${}_0S$ can be obtained from those of S.

Theorem 6.12. *If S is a monoid of type $FP_n^{(l)}$ $(n > 0)$ over K, then ${}_0S$ is of type $FP_n^{(l)}$ over K and*

$$E_{n,\lambda}^{(l)T}({}_0S, K) = \begin{cases} \left(\iota^T E_{n,\lambda+(-1)^n}^{(l)T}(S,K)\right) + (\theta) & \lambda \geq (-1)^n \\ \left(\iota^T E_{n,\lambda+(-1)^n}^{(l)T}(S,K)\right)(1-\theta) & \lambda < (-1)^n \end{cases},$$

where $\iota : S \to {}_0S$ is the inclusion map.

To minimise terminology, we will prove this only for $K = \mathbb{Z}$ and ${}^T = {}^{ab}$. We will require the following lemmata.

Lemma 6.13. *As a left $\mathbb{Z}S$-module,*

$$\mathbb{Z}_0S = \mathbb{Z}S \oplus \mathbb{Z}S\theta,$$

and so is free.

Proof. Clearly, $\mathbb{Z}S + \mathbb{Z}S\theta = \mathbb{Z}_0S$ and $\mathbb{Z}S \cap \mathbb{Z}S\theta = \{0\}$. Moreover, $\mathbb{Z}S$ and $\mathbb{Z}S\theta$ are closed under left multiplication by elements of $\mathbb{Z}S$. By Lemma 6.10, $\mathbb{Z}S\theta \cong \mathbb{Z}S$, so $\mathbb{Z}_0S$ is free of rank 2. □

Consequently, every $\eta \in \mathbb{Z}_0S$ can be written as $\eta = \eta_1 + \eta_2\theta$ for some unique $\eta_1, \eta_2 \in \mathbb{Z}S$. The map

$$\alpha : \mathbb{Z}_0S \to \mathbb{Z}S; \eta = \eta_1 + \eta_2\theta \mapsto \eta_1$$

is a ring homomorphism, since, if $\eta = \eta_1 + \eta_2\theta$, $\eta' = \eta_1' + \eta_2'\theta \in \mathbb{Z}_0S$, where $\eta_i, \eta_i' \in \mathbb{Z}S$, then

$$\eta\eta' = \eta_1\eta_1' + (\eta_1\eta_2' + \eta_2 aug(\eta'))\theta.$$

Thus, every $\mathbb{Z}S$-module M becomes a $\mathbb{Z}_0S$-module ${}_0M$ via α.

Lemma 6.14. *As a left $\mathbb{Z}_0S$-module, $\mathbb{Z}S$ is projective and*

$$\mathbb{Z}_0S \cong \mathbb{Z}S \oplus \mathbb{Z}_0S.\theta.$$

Proof. The map

$$\beta : \mathbb{Z}S \to \mathbb{Z}_0S; \xi \mapsto \xi(1-\theta) \quad (\xi \in \mathbb{Z}S)$$

is a left $\mathbb{Z}_0S$-homomorphism, since, for $\xi \in \mathbb{Z}S$, $\eta = \eta_1 + \eta_2\theta \in \mathbb{Z}_0S$,

$$\beta(\eta.\xi) = \beta(\eta_1\xi) = \eta_1\xi(1-\theta)$$

and

$$\eta.\beta(\xi) = \eta\xi(1-\theta) = \eta_1\xi(1-\theta).$$

Now, $\alpha\beta = \mathrm{Id}_{\mathbb{Z}S}$, so this map splits α. Since $\ker\alpha = \mathbb{Z}_0S.\theta$, $\mathbb{Z}_0S \cong \mathbb{Z}S \oplus \mathbb{Z}_0S.\theta$. □

The projection maps of $\mathbb{Z}_0S$ onto $\mathbb{Z}S$ and $\mathbb{Z}_0S.\theta$ followed by the embedding maps give rise to the idempotent endomorphisms

$$\pi : \mathbb{Z}_0S \to \mathbb{Z}_0S; 1 \mapsto 1-\theta$$

and

$$\pi' : \mathbb{Z}_0S \to \mathbb{Z}_0S; 1 \mapsto \theta$$

respectively.

Lemma 6.15. *As a left $\mathbb{Z}_0S$-module,*

$$I_0S \cong {}_0IS \oplus \mathbb{Z}_0S.(1-\theta).$$

Proof. The $\mathbb{Z}_0S$-module I_0S is $\mathbb{Z}$-free on the set

$$\{1-s, 1-s\theta : s \in S\}.$$

Let the map

$$\psi : I_0S \to {}_0IS$$

be defined by $\psi(1-s) = 1-s$, $\psi(1-s\theta) = 1-s$, for $s \in S$. Then ψ is a left $\mathbb{Z}_0S$-homomorphism, since, for $s, s' \in S$,

$$\begin{aligned} s'.(1-s) = (1-s's) - (1-s') \overset{\psi}{\mapsto} (1-s's) - (1-s') \\ = s'(1-s) \\ = s'.\psi(1-s), \end{aligned}$$

$$s'.(1-s\theta) = (1-s's\theta) - (1-s') \overset{\psi}{\mapsto} (1-s's) - (1-s')$$
$$= s'(1-s)$$
$$= s'.\psi(1-s\theta),$$
$$s'\theta.(1-s) = 0 \overset{\psi}{\mapsto} 0$$
$$= s'\theta.\psi(1-s),$$
$$s'\theta.(1-s\theta) = 0 \overset{\psi}{\mapsto} 0$$
$$= s'\theta.\psi(1-s\theta).$$

The inclusion map

$$\phi : {}_0IS \to I_0S; 1-s \mapsto 1-s$$

is also a $\mathbb{Z}_0S$-homomorphism, since, for $\eta = \eta_1 + \eta_2\theta \in \mathbb{Z}_0S$, $\xi \in IS$, $\theta\xi = aug(\xi)\theta = 0$, and so

$$\eta.\xi = \eta_1\xi \overset{\phi}{\mapsto} \eta_1\xi = (\eta_1 + \eta_2\theta)\xi = \eta.\phi(\xi).$$

Moreover, $\psi\phi = \mathrm{Id}_{IS}$, and so ${}_0IS$ is a direct summand of I_0S. Since $\ker\psi$ is generated by $1-\theta$,

$$I_0S \cong {}_0IS \oplus \mathbb{Z}_0S.(1-\theta),$$

as claimed. □

Proof of Theorem 6.12. By Lemma 6.15 and the monoid versions of Lemma 4.10 and Corollary 4.15(i), for $n > 0$,

$$E^{(l)}_{n,\lambda}({}_0S) = E_{n-1,\lambda-(-1)^n}(I_0S)$$
$$= E_{n-1,\lambda-(-1)^n}({}_0IS \oplus \mathbb{Z}_0S.(1-\theta)),$$

so, by Proposition 4.16,

$$E^{(l)}_n({}_0S) = E_{n-1}({}_0IS) *^{(-(-1)^n)} E_{n-1}(\mathbb{Z}_0S.(1-\theta)) \tag{6.1}$$

The module $\mathbb{Z}_0S.(1-\theta)$ has the resolution

$$\cdots \xrightarrow{1\mapsto\theta} \mathbb{Z}_0S \xrightarrow{1\mapsto 1-\theta} \mathbb{Z}_0S \xrightarrow{1\mapsto\theta} \mathbb{Z}_0S \xrightarrow{1\mapsto 1-\theta} \mathbb{Z}_0S.(1-\theta) \to 0,$$

and so

$$E_{n,\lambda}(\mathbb{Z}_0S) = \begin{cases} \mathbb{Z}_0S^{ab} & \lambda \geq 1 \\ (\theta) & \lambda = 0 \ (n \text{ even}), \\ 0 & \lambda < 0 \end{cases} \qquad \begin{cases} \mathbb{Z}_0S^{ab} & \lambda \geq 0 \\ (1-\theta) & \lambda = -1 \ (n \text{ odd}). \\ 0 & \lambda < -1 \end{cases}$$

We now turn to the ideals $E_n({}_0IS)$. Since $\mathbb{Z}S$ is projective as a left $\mathbb{Z}_0S$-module, if we have a $\mathbb{Z}S$-free resolution $\mathcal{P} = (P_i, \varepsilon_i)$ of IS of type FP_n, then this is also a $\mathbb{Z}_0S$-projective resolution of ${}_0IS$. Using the procedure in the proof of Lemma 1.26, we can convert this into a $\mathbb{Z}_0S$-free resolution $\mathcal{F} = (F_i, \partial_i)$ of ${}_0IS$ of type FP_n, where, if $r_j = \mathrm{rk}_{\mathbb{Z}S}(P_j)$,

$$\mathrm{rk}_{\mathbb{Z}_0S}(F_i) = \sum_{j=0}^{i} r_j$$

and

$$D_n(\mathcal{F}) = \begin{bmatrix} D_n(\mathcal{P})^\iota(1-\theta) & 0 & 0 & \cdots & \\ \theta I_{r_n} & 0 & 0 & \cdots & \\ 0 & (1-\theta)I_{r_{n-1}} & 0 & \cdots & \\ 0 & 0 & \theta I_{r_{n-2}} & & \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & & \ddots \end{bmatrix}.$$

Here the matrices $(1-\theta)I_{r_i}$ and θI_{r_i} are the matrices of r_i copies of the endomorphisms π and π' respectively. The matrix $D_n(\mathcal{P})^\iota(1-\theta) = D_n(\mathcal{P})^\iota((1-\theta)I_{r_n})$ is the matrix of r_{n+1} copies of α composed with ε_{n+1} composed with r_n copies of $\beta : 1 \mapsto 1-\theta$.

Note that each row of this matrix is a multiple either of θ or of $1-\theta$. Since $\theta(1-\theta) = 0$, any submatrix of $D_n(\mathcal{F})^{ab}$ which includes both a row which is a multiple of θ and a row which is a multiple of $1-\theta$ will have zero determinant. Thus

$$J(D_n(\mathcal{F})^{ab}) = J\left(\begin{bmatrix} (D_n(\mathcal{P})^\iota)^{ab}(1-\theta) & 0 \\ 0 & (1-\theta)I_{r'} \end{bmatrix}\right) + J(\theta I_r),$$

where

$$r = r_n + r_{n-2} + \cdots,$$

$$r' = r_{n-1} + r_{n-3} + \cdots.$$

Now,

$$J_\kappa(\theta I_r) = \begin{cases} 0 & \kappa > r \\ (\theta) & 0 \leq \kappa < r \\ \mathbb{Z}_0 S^{ab} & \kappa \leq 0 \end{cases}$$

and, since $\chi_n(\mathcal{P}) = r - r'$,

$$\begin{aligned} J_\kappa & \left(\begin{bmatrix} (D_n(\mathcal{P})^\iota)^{ab}(1-\theta) & 0 \\ 0 & (1-\theta)I_{r'} \end{bmatrix} \right) \\ &= \begin{cases} 0 & \kappa > r' + \min\{r_n, r_{n+1}\} \\ J_{\kappa - r'}\left((D_n(\mathcal{P})^\iota)^{ab} \right)(1-\theta) & r' < \kappa \leq r' + \min\{r_n, r_{n+1}\} \\ (1-\theta) & 0 < \kappa \leq r' \\ \mathbb{Z}_0 S^{ab} & \kappa \leq 0 \end{cases} \\ &= \begin{cases} \left(\iota^{ab} E_{n,r-\kappa}(IS)\right)(1-\theta) & \kappa > 0 \\ \mathbb{Z}_0 S^{ab} & \kappa \leq 0 \end{cases}. \end{aligned}$$

Thus,

$$J_\kappa(D_n(\mathcal{F})^{ab}) = \begin{cases} \left(\iota^{ab} E_{n,r-\kappa}(IS)\right)(1-\theta) & \kappa > r \\ \left(\iota^{ab} E_{n,r-\kappa}(IS)\right)(1-\theta) + (\theta) & 0 < \kappa \leq r \\ \mathbb{Z}_0 S^{ab} & \kappa \leq 0 \end{cases}.$$

Since $\chi_n(\mathcal{F}) = r$ and

$$\begin{aligned} \left(\iota^{ab} E_{n,\lambda}(IS)\right)(1-\theta) + (\theta) &= \left(\iota^{ab} E_{n,\lambda}(IS)\right) + (\theta) \\ &= \left(\iota^{ab} E^{(l)}_{n+1,\lambda-(-1)^n}(S)\right) + (\theta), \end{aligned}$$

this gives

$$\begin{aligned} E_{n,\lambda}({}_0IS) &= J_{r-\lambda}(D_n(\mathcal{F})^{ab}) \\ &= \begin{cases} \left(\iota^{ab} E^{(l)}_{n+1,\lambda-(-1)^n}(S)\right) + (\theta) & \lambda \geq 0 \\ \left(\iota^{ab} E^{(l)}_{n+1,\lambda-(-1)^n}(S)\right)(1-\theta) & \lambda < 0 \end{cases}. \end{aligned}$$

For n odd, from (6.1),

$$E^{(l)}_{n,\lambda}({}_0S) = E_{n-1,\lambda+1}({}_0IS)\theta + E_{n-1,\lambda}({}_0IS),$$

so, for $\lambda \geq 0$,

$$\begin{aligned} E^{(l)}_{n,\lambda}({}_0S) &= \left(\left(\iota^{ab} E^{(l)}_{n,\lambda}(S)\right) + (\theta)\right)\theta + \left(\iota^{ab} E^{(l)}_{n,\lambda-1}(S)\right) + (\theta) \\ &= \left(\iota^{ab} E^{(l)}_{n,\lambda-1}(S)\right) + (\theta)\,, \end{aligned}$$

for $\lambda = -1$,

$$\begin{aligned} E^{(l)}_{n,-1}({}_0S) &= \left(\left(\iota^{ab} E^{(l)}_{n,-1}(S)\right) + (\theta)\right)\theta + \left(\iota^{ab} E^{(l)}_{n,-2}(S)\right)(1-\theta) \\ &= \left(\iota^{ab} E^{(l)}_{n,-2}(S)\right) + (\theta) \end{aligned}$$

and, for $\lambda < -1$,

$$\begin{aligned} E^{(l)}_{n,\lambda}({}_0S) &= \left(\iota^{ab} E^{(l)}_{n,\lambda}(S)\right)(1-\theta)\theta + \left(\iota^{ab} E^{(l)}_{n,\lambda-1}(S)\right)(1-\theta) \\ &= \left(\iota^{ab} E^{(l)}_{n,\lambda-1}(S)\right)(1-\theta), \end{aligned}$$

whence

$$E^{(l)}_{n,\lambda}({}_0S) = \begin{cases} \left(\iota^{ab} E^{(l)}_{n,\lambda-1}(S)\right) + (\theta) & \lambda \geq -1 \\ \left(\iota^{ab} E^{(l)}_{n,\lambda-1}(S)\right)(1-\theta) & \lambda < -1 \end{cases}.$$

For even n, from (6.1),

$$E^{(l)}_{n,\lambda}({}_0S) = E_{n-1,\lambda}({}_0IS)\,(1-\theta) + E_{n-1,\lambda-1}({}_0IS),$$

so, for $\lambda \geq 1$,

$$\begin{aligned} E^{(l)}_{n,\lambda}({}_0S) &= \left(\left(\iota^{ab} E^{(l)}_{n,\lambda+1}(S)\right) + (\theta)\right)(1-\theta) + \left(\iota^{ab} E^{(l)}_{n,\lambda}(S)\right) + (\theta) \\ &= \left(\iota^{ab} E^{(l)}_{n,\lambda+1}(S)\right) + (\theta)\,, \end{aligned}$$

for $\lambda = 0$,

$$\begin{aligned} E^{(l)}_{n,0}({}_0S) &= \left(\left(\iota^{ab} E^{(l)}_{n,1}(S)\right) + (\theta)\right)(1-\theta) + \left(\iota^{ab} E^{(l)}_{n,0}(S)\right)(1-\theta) \\ &= \left(\iota^{ab} E^{(l)}_{n,1}(S)\right)(1-\theta) \end{aligned}$$

and, for $\lambda < 0$,

$$\begin{aligned} E^{(l)}_{n,\lambda}({}_0S) &= \left(\iota^{ab} E^{(l)}_{n,\lambda+1}(S)\right)(1-\theta)(1-\theta) + \left(\iota^{ab} E^{(l)}_{n,\lambda}(S)\right)(1-\theta) \\ &= \left(\iota^{ab} E^{(l)}_{n,\lambda+1}(S)\right)(1-\theta), \end{aligned}$$

whence

$$E^{(l)}_{n,\lambda}({}_0S) = \begin{cases} \left(\iota^{ab}E^{(l)}_{n,\lambda+1}(S)\right) + (\theta) & \lambda \geq 1 \\ \left(\iota^{ab}E^{(l)}_{n,\lambda+1}(S)\right)(1-\theta) & \lambda < 1 \end{cases}. \qquad \square$$

Example 6.9. The monoid defined by the presentation

$$[x, \theta; \theta x = \theta, \theta\theta = \theta]$$

as in Examples 1.8 and 6.4 can be obtained from the free monoid $[x]$ by adjoining a left zero. Now,

$$E^{(l)}_{n,\lambda}([x]) = \begin{cases} \mathbb{Z}[x] & \lambda \geq 0 \\ 0 & \lambda < 0 \end{cases},$$

and from this the $E^{(l)}$-ideals of the monoid can be obtained without the need for the resolution of Example 1.8. ◊

6.4 Monoids of type FP and of type FL

The monoid version of Lemma 5.11 is:

Lemma 6.16. *Let S be a monoid. If S has non-trivial $E^{(l)}$-ideals over K (respectively, $E^{(r)}$-ideals over K) in arbitrarily high dimension, then it cannot be of type $FL^{(l)}$ over K (respectively, of type $FL^{(r)}$ over K).*

If a monoid S has a left zero θ and if $\widehat{\theta} \neq 1 \in S^{ab}$, then, by Proposition 6.9, S has non-trivial $E^{(r)}$-ideals in all dimensions, and so S cannot be of type $FL^{(r)}$. However, S is of type $FP^{(r)}$ [45]. Indeed, $\mathbb{Z}_S$ is projective as a right $\mathbb{Z}S$-module, for we have

$$\mathbb{Z}_S \cong \theta.\mathbb{Z}S$$

and

$$\mathbb{Z}S \cong (1-\theta).\mathbb{Z}S \oplus \theta.\mathbb{Z}S.$$

Thus,

$$0 \to \mathbb{Z}_S \to \mathbb{Z}_S \to 0$$

is a right projective resolution for S.

Alternatively, note that the Hattori-Stallings rank of $\mathbb{Z}_S$ is the image of θ in $\mathbb{Z}S/[\mathbb{Z}S, \mathbb{Z}S]$. When this is not an integer, as is the case when $\widehat{\theta} \neq 1$, S cannot be of type $FL^{(r)}$.

Example 6.10. Consider again the monoid S of Examples 1.8 and 6.4, which has presentation

$$[x, \theta; \theta x = \theta, \theta\theta = \theta].$$

From the above comments, since $\widehat{\theta} \neq 1$ in S^{ab}, S cannot be of type $FL^{(r)}$, although it is of type $FP^{(r)}$. In addition, it is shown in [45] that S is of type $FP^{(l)}$ but, again, since S has non-trivial $E^{(l)}$-ideals in all dimensions, it cannot be of type $FL^{(l)}$. ◊

Now, given any monoid S, we saw that we could adjoin a left zero element, giving a monoid which is of type $FP^{(r)}$, but not of type $FL^{(r)}$. Similarly, if we adjoin a two-sided zero to S, we obtain a monoid ${}_0S_0 = S \cup \{\theta\}$ which is of types $FP^{(l)}$ and $FP^{(r)}$. But, since ${}_0{S_0}^{ab} = S^{ab} \cup \{\widehat{\theta}\}$ is non-trivial, ${}_0S_0$ is neither of type $FL^{(l)}$ nor of type $FL^{(r)}$.

Serre asked whether there is a group which is of type FP but not of type FL [76]. From the above discussion we see that, for any group G, simply by adjoining a single element we would obtain a monoid, ${}_0G_0$, which is of type $FP^{(l)}$ but not of type $FL^{(l)}$ and of type $FP^{(r)}$ but not of type $FL^{(r)}$.

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