A thesis submitted for the degree of Doctor
of Philosophy in the University of Glasgow

SUMS OF DIVISORS, UNITS AND ALGEBRAIC INTEGERS

by

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CONTENTS

PREFACE

This thesis is submitted in accordance with regulations for the degree of Doctor of Philosophy in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any university.

The results contained in this thesis claimed as original except where indicated in the text.

I would take this opportunity of expressing my deep gratitude to Dr.S.D.Cohen for suggesting the problems which are contained in this thesis and also for his guidance,constant interest and encouragement.

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INTRODUCTION

The objective of this thesis is to develop some results concerning sums of the form

$$\sum_i a_i\alpha_i \quad , \qquad (1)$$

where $a_i \in Z$ with $|a_i| \leq t$, $t \in N/\{0\}$ and α_i are distinct members of a set of algebraic integers e.g divisors of positive integers , units in a fixed algebraic number field or algebraic integers of bounded norm .

Results on sums of divisors of practical numbers in [1],[2],[3] and [5] are initially studied in chapter 1. We achieve more general results by introducing the notion of a t-practical number . We define n to be a t-practical number if every integer m , $1 \leqslant m \leqslant tn$, is of the form (1) where the α_i are divisors of n . Ordinary practical numbers correspond to the case t = 1 . Denote by A_t the set of all t-practical numbers . We show that n is a t-practical number if and only if

$$d_{k+1} \leqslant t\sigma_k + 1 \quad ,$$

where d_{k+1} is the $k+1^{th}$ divisor of n and σ_k is the sum of the k divisors of n . Further we prove that , given any prime p with $(p,n) = 1$, $n \in A_t$,then a necessary and sufficient condition for $p^k n$, $k \geqslant 1$, to belong to A_t is that

$$p \leqslant t\sigma(n) + 1 \quad ,$$

where $\sigma(n)$ is the sum of all divisors of n . These results extend the result of Stewart in [2] and Robinson[5] . A complete characterization for n to be t-practical number , in terms of its prime factors, is given by Theorem 1.1.5. Our results are extended in section 1.2 by considering positive and negative divisors of n in A_t .

By defining $\beta(n)$ to be the number of all integers in the interval $[1,t\sigma(n)]$ having the form (1), we show in section 1.3 that $\beta(n)/t\sigma(n)$ forms a dense set of values in $[0,1]$. We also show that $\beta(n)/t\sigma(n)$ takes its smallest value when n is a non-practical number.

Section 1.4 is devoted to studying the properties of all positive integers n which satisfy the condition

$$\sigma(n) - \alpha(n) = 2 \quad ,$$

where $\alpha(n)$ is the number of all integers m, $1 \leq m \leq \sigma(n)$, having the form (1) with $a_i = 0,1$. We define E^* to be the set of these integers and establish necessary and sufficient conditions for any integer to be in E^*. These are given in Theorem 1.4.3 and 1.4.4.

Further insight is provided in the form of a computer program in Fortran 77 to determine which integers n are in E^*. Several general results related to the properties of integers in E^* are also included in section 1.5. One of these results is that, for any integer $n \epsilon E^*$, satisfying $n = mp_1^{a_1}..p_k^{a_k}, a_j \geq 1, (p_j,m) = 1, p_j < p_{j+1} \leq 2p_j - 1$, $m \geq 2$ is a practical number such that $p_1 = \sigma(m) + s$ and $1 < s \leq \sigma(m)$, then n is a sum of two practical numbers.

We discuss in section 1.6 the properties of integers n which satisfy the condition

$$\sigma(n) - \alpha(n) = 1 \quad . \tag{2}$$

We prove that, if n satisfies (2) then $\sigma(n)$ is even and $\sigma(n)/2$ is the only integer in $[1,\sigma(n)]$ which cannot be written as a sum of distinct divisors of n. This chapter ends with section 1.7 which deals with practical numbers over a Euclidean Imaginary quadratic field $Z[\sqrt{d}]$. We define $\alpha \epsilon Z[\sqrt{d}]$ to be a practical number in $Z[\sqrt{d}]$ if every integer $\gamma \epsilon Z[\sqrt{d}]$ with $|N(\gamma)| \leq |N(\alpha)|$ is a sum of distinct divisors of α.

We prove that, if $(\surd d)^k, (k \geq 1), d \neq -1, -3, |d|$ is an ordinary practical number in $\mathbb{Z}[\surd d]$, then $(\surd d)^k$ is a practical in $\mathbb{Z}[\surd d]$.

In chapter 2 , we generalize some results in [4] and [7] featuring analytic methods and estimates .

Let f(n) be the largest integer such that every integer in the interval [1,f(n)] can be represented in the form (1) with $a_i = 0,1$ and α_i divisors of n . Hausman and Shapiro[7] suggested the boundary function $B^*(n) = e^{\gamma/2}(n\log\log n)^{1/2}$ and showed that for any $\lambda > 1$, if $f(n) \geq \lambda B^*(n)$, then n is a practical number with a finite number of exceptions. Further, for any $\lambda < 1$, then $f(n) \geq \lambda B^*(n)$ can hold for infinitely many non-practical numbers n . These results are improved in section 2.1 in terms of t-practical numbers . For that we define $f_t(n)$ to be the largest integer such that every integer in $[1, f_t(n)]$ can be written as a t-practical number and $B_t(n) = e^{\gamma/2}(tn\log\log n)^{1/2}$

We show that for any $\lambda(n) > 1 + \dfrac{c}{(\log\log n)^2}$ and $c \geq \dfrac{5}{4e^{\gamma}}$,

if $f_t(n) \geq \lambda(n)B_t(n)$, then n is a t-practical number for any $t, t \in \mathbb{N}$.

Also for any $\lambda(n) < 1 - \dfrac{c\log\log\log n}{(\log\log n)^2}$, $t \in \mathbb{N}$, and $c \geq 2$, then

$f_t(n) \geq \lambda(n)B_t(n)$ holds for infinitely many non t-practical numbers n This work is done by using explicit formulas given in [9] by Rosser and Schoenfeld .

In [7] they also showed that the number of practical numbers $n \leq x$ is

$$O(x/(\log x)^{\beta}) , \qquad (3)$$

for any fixed $\beta < \dfrac{1}{2}\left(\dfrac{1}{\log 2} - 1\right)^2$. We extend this result to

t-practical numbers by using the general form of such numbers given by Theorem 1.1.5 in Chapter 1 . Our result shows that (3) is an estimate for the number of t-practical numbers $n \leqslant x$ uniformly for t in the range

$$1 \leqslant t \leqslant \exp.\{(\log x)^{\delta_1}\} \quad ,$$

for any fixed δ_1 , $0 < \delta_1 < 1 - (1 + \sqrt{2\beta})\log 2$.

Margenstren in [4] showed that the number of practical numbers $n \leqslant x$,(x is a large real number), is at least

$$\frac{Ax}{\exp.\left[\frac{1}{2\log 2}(\log\log x)^2 + 3\log\log x\right]} \quad , \qquad (4)$$

where $A = \frac{2^{5/2}}{5}$. A sharper lower bound than (4) is presented in section 2.3 . We proved that the number of t-practical numbers $n \leqslant x$,(x is a large real number), is at least

$$(2\log 2)^{1/2}\frac{x}{\exp.\left[\frac{1}{2\log 2}\ (\log\log x)^2 + \log\log x + \log(t+1)\right]} \ ,$$

for any fixed t , $t \in \mathbb{N}/\{0\}$.

A further improvement to the results in [7] is given in section 2.4

We showed that, for any $t \in \mathbb{N}$ and $x > \frac{t}{3}$ the interval $(x\ ,\ x+2(\frac{x}{t})^{1/2})$ contains a t-practical number .

In Chapter 3,we return to the topic concerning sums of the form (1),where α_i are units of algebraic field $\mathbb{Q}(\theta)$,to generalize Belcher's results in [12] and [14]. We let V to be the set of all fields in which all integers can be written as a finite sums of units (not necessarily distinct) . We define t-integers to be those integers expressible as a finite sum of units with at most t repetitions and we refer to any t-integer expressible as a sum of (t+1) units as a strong t-integer . We prove that , if (t+1) is a strong t-integer, then any sum of units in $\mathbb{Q}(\theta)$ is a t-integer . This generalizes Belcher's result in [14] which covers the case t=1.

Define W_t to be the set of all fields $\mathbb{Q}(\theta)$ in which every integer is a t-integer. We construct a necessary condition for all real fields , with fundamental unit ϵ , to be in W_t . This condition is expressed in terms of t and ϵ .

A further general result achieved for pure cubic fields shows that all fields $\mathbb{Q}(\sqrt[3]{d})$ with $d = m \pm 1$, $m > 1$, are in W_t.We also give some examples in such pure cubic fields .

We prove in section 3.2 that, if $\mathbb{Q}(\theta)\in W_t$ and $\mathbb{Q}(\varphi)\in V$, where these fields have relatively prime discriminants, then $\mathbb{Q}(\theta,\varphi)\in W_t$. This generalize one of Belcher's results in [12] . It also enables us to generalize another result of Belcher[12] by showing that there are infinitely many quartic fields $\mathbb{Q}(\sqrt{d},\sqrt{m})\in W_t$ for any fixed t , $t\in\mathbb{N}$.

Interesting and new general results for real quadratic fields $\mathbb{Q}(\sqrt{d})$ are discussed in detail in Section 3.3 .The main result,proved in Theorem 3.3.2, shows that if $\mathbb{Q}(\sqrt{d})\in V$ and $\epsilon = t+\sqrt{d}$ or $\{(2t-1)+\sqrt{d}\}/2$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$,then $\mathbb{Q}(\sqrt{d})\in W_t$, t is minimal in a certain sense . This shows that any integer of $\mathbb{Q}(\sqrt{d})$ which can be written as a sum of units is a t-integer and also can be used to determine which quartic field of the above type lie in W_t . Theorem 3.3.2, also implies Jacobson's result in [3] that both $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ are in W_1 and specific examples of $\mathbb{Q}(\sqrt{d})$, $2 \leq d < 100$, are also given to illustrate our results .

We mean by the phrase"α is a norm-s integer in $\mathbb{Q}(\sqrt{d})$"that α is a sum of integers of norms less than or equal to s . We extend Theorem 3.1.2, by establishing new results for the representation of integers in $\mathbb{Q}(\theta)$ as sum of distinct norm-s integers for any given s , $1 \leq s \leq t^2$. We describe as a strong norm-s integer in $\mathbb{Q}(\theta)$ any norm-s integer $\alpha\epsilon\mathbb{Q}(\theta)$ such that

$$\alpha = \sum_{i=1}^{u} \gamma_i \epsilon_i \ , \qquad (5)$$

with γ_i non-associate integers in $\mathbb{Q}(\theta)$ $\epsilon_1,\ldots,\epsilon_u$ are the fundamental units of $\mathbb{Q}(\theta)$, $|N(\gamma_i)| \leqslant s$ and $\sum_i |N(\gamma_i)| \leqslant |N(\alpha)|$. We denote by N_s^* the set of all fields in which every integer can be written as a sum of distinct norm-s integers.We prove in Theorem 4.1.1,that if $\mathbb{Q}(\theta)\epsilon W_t$ for some t , $t \geqslant s^{1/2}$and that every integer in $\mathbb{Q}(\theta)$ of norm less than or equal to s is a strong norm-s integer,then $\mathbb{Q}(\theta)\epsilon N_s^*$.

We apply our general result to show that $\mathbb{Q}(\sqrt[3]{2})\epsilon N_3^*$ since $\mathbb{Q}(\sqrt[3]{2})\epsilon W_3$, and that every integer in $\mathbb{Q}(\sqrt[3]{2})$ of norm less than or equal to 9 is a strong norm-3 integer .

The real quadratic fields $\mathbb{Q}(\sqrt{d})$ which are given by Theorem 3.3.2, are discussed in section 4.2 . We prove that , if $\mathbb{Q}(\sqrt{d})\epsilon W_t$, $d \equiv 1 \pmod 4$ and ϵ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$,then $\mathbb{Q}(\sqrt{d})\epsilon N_s^*$ and $s = t^2 - 2t \pm 1$ as $N(\epsilon) = -1$ or $+1$ respectively. We also show that,if $\mathbb{Q}(\sqrt{d})\epsilon W_t$, $d \not\equiv 1 \pmod 4$ and ϵ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ such that $N(\epsilon) = +1$,then $\mathbb{Q}(\sqrt{d})\epsilon N_s^*$ and $s = t^2 - 1$. The case $\mathbb{Q}(\sqrt{d})\epsilon W_t, d \not\equiv 1 \pmod 4$ and $N(\epsilon) = -1$ is dealt with considering the γ_i's in (5) which are assocaites integer in $\mathbb{Q}(\sqrt{d})$ when we refer to the strong norm-s integers.This enables us to show that $\mathbb{Q}(\sqrt{d})\epsilon N_s^*$ with $s = t^2-2t-1$.

In some instance we can determine the minimal value of s ,e.g $\mathbb{Q}(\sqrt{13})\epsilon N_3^*$ but $\mathbb{Q}(\sqrt{13})\not\epsilon N_2^*$ since there are no integers of norm 2 in this field . But certain quadratic fields $\mathbb{Q}(\sqrt{d})\epsilon N_s^*$ do have the property that every integer of norm less than or equal to s can be written as a strong norm-v integer, $1 \leqslant v \leqslant s$ and v is minimal . This enables us to prove in Theorem 4.2.2, that such quadratic fields are in N_v^* .

This result is indicated in Theorem 4.2.3, which shows that $\mathbb{Q}(\surd 3)\epsilon N_2^*$.

Further clarification is provided by examples on $\mathbb{Q}(\surd d)$ with $2 \leqslant d < 100$ in section 4.3 .

CHAPTER (1)

§ 1.1 The t-Practical Numbers

In [1] A.K Srinivasan studied the numbers $n \geqslant 1$ having the property that every integer m satisfying $1 \leqslant m < n$ and of the form

$$m = \sum_{d \mid n} c_d d \;, \qquad c_d = 0,1 \qquad . \qquad (1)$$

These numbers are called the "practical numbers" . B.M.Stewart in [2] defined I^* to be the set of all integers n such that

$$\sigma(n) - \alpha(n) = 0 \quad , \qquad (2)$$

where $\sigma(n)$ is the sum of all positive divisors of n and $\alpha(n)$ denotes the number of all integers m having some representation in the form(1).

The main result proved by Stewart in [2] is the follwing :

Theorem 1.1.1[2]

If n belongs to I^* and p is a prime with $(p,n) = 1$, then $p^k n$, $k \geqslant 1$ belongs to I^* if and only if

$$p \leqslant \sigma(n) + 1 \; . \qquad (3)$$

In [2] Stewart also gave a complete structure for the practical numbers in I^* by demonstrating the following result ;

Theorem 1.1.2[2]

n belongs to I^* if and only if n is of the form $n = 2^a$, $a \geqslant 0$ or

$$n = 2^a p_1^{a_1} \ldots p_k^{a_k} \;, \quad a \geqslant 1 \;, \; a_i \geqslant 1 \;,$$

where $1 \leqslant i \leqslant k$, p_i are primes satisfying the following conditions:

$$2 < p_1 < \ldots < p_k \;,$$

$$p_1 \leqslant \sigma(2^a) + 1 \;,$$

and

$$p_{j+1} \leqslant \sigma(2^a . p_1^{a_1} \ldots p_j^{a_j}) + 1 \;, \; (1 \leqslant j \leqslant k-1) .$$

Let $\tau(n) = r$ be the number of positive divisors of n and let $1 = d_1 < d_2 < \ldots < d_r = n$ be the divisors of n . Put $\sigma_0 = 0$ and $\sigma_k = d_1 + d_2 + \ldots + d_k$, $1 \leqslant k \leqslant r$.

In [5] Robinson adopted Stewart's ideas to prove the following result ;

Theorem 1.1.3[5]

n is a practical number if and only if

$$d_{k+1} \leqslant \sigma_k + 1 \quad , 1 \leqslant k \leqslant r - 1 . \qquad (4)$$

Now we shall work with a new definition given in the following :

Definition 1.1.1

Let $t \in \mathbb{N}$,then n is a t-practical number if every integer m $1 \leqslant m \leqslant tn$ has the form

$$\sum_{d|n} c_d\, d \ , \qquad 0 \leqslant c_d \leqslant t \ , \qquad (5)$$

A practical number in the sense of Srinivasan[1] is just a 1-practical number . Taking account of our definition we shall generalize Theorem 1.1.3[5] as in the following :

Theorem 1.1.4

n is a t-practical number if and only if

$$d_{k+1} \leqslant t\sigma_k + 1 \ , \qquad 0 \leqslant k \leqslant r - 1 . \qquad (6)$$

Proof

Suppose that

$$d_{k+1} \leqslant t\sigma_k + 1 \ ,$$

for all k , $0 \leqslant k \leqslant r - 1$ .We will show that n is a t-practical number. In fact ,we prove that for any k , $0 \leqslant k \leqslant r - 1$,every integer m such that

$$t\sigma_k < m \leqslant t\sigma_{k+1} ,$$

can be expressed as (7)

$$\sum_{i=1}^{k+1} c_i d_i \quad , \qquad 0 \leqslant c_i \leqslant t .$$

Since $t\sigma_0 = 0$ and $t\sigma_r = t\ (\ d_1 + d_1 + \ldots + d_r\ ) \geqslant d_r = n$,this shows that every integer m in [1,n] has the required representation for n to be t-practical number .

The proof of (7) is by induction on k . For $k = 0$,(7) asserts that every m , $0 < m \leqslant t$ can be expressed as $c_1 d_1$, $0 \leqslant c_1 \leqslant t$ and this is trivially true since $d_1 = 1$. Assume therefore that (7) is true for all $k = K$, where $K < r - 1$, and we will show it is true for $k = K + 1$.

Suppose therefore that

$$t\sigma_{K+1} < m \leqslant t\sigma_{K+2} .$$

In particular , since $d_{K+2} \leqslant t\sigma_{K+1} + 1$, this implies that

$$d_{K+2} \leqslant m \leqslant t\sigma_{K+2} .$$

Now , if $m < (t+1)d_{K+2}$, we can write m as

$$m = ud_{K+2} + v ,$$

where $0 \leqslant v \leqslant d_{K+2}$ and $0 \leqslant u \leqslant t$. Hence $0 \leqslant v \leqslant d_{K+2} \leqslant t\sigma_{K+1} + 1$ i.e $0 \leqslant v \leqslant t\sigma_{K+1}$ so that by the induction hypothesis ,

$$v = \sum_{i=1}^{K+1} c_i d_i \quad , \qquad 0 \leqslant c_i \leqslant t .$$

Thus

$$m = ud_{K+2} + \sum_{i=1}^{K+1} c_i d_i \quad ,$$

where $0 \leqslant u \leqslant t$ and $0 \leqslant c_i \leqslant t$ .i.e

$$m = \sum_{i=1}^{K+2} c_i d_i \ , \qquad 0 \leqslant c_i \leqslant t \ ,$$

,

where $c_{K+2} = u$ .This is the required form . On the other hand , if

$m \geqslant (t+1)d_{K+2}$,then

$$d_{K+2} \leqslant m - td_{K+2} \leqslant t\sigma_{K+2} - td_{K+2} = t\sigma_{K+1} .$$

Hence ,again by the induction hypothesis ,

$$m - td_{K+2} = \sum_{i=1}^{K+1} c_i d_i$$

i.e

$$m = \sum_{i=1}^{K+2} c_i d_i \quad ,$$

where $c_{K+2} = t$. In any case this completes the proof .

Conversely ,if n is t-practical number ,then for any k ,

$0 \leqslant k \leqslant r - 1$ we can write

$$d_{k+1} - 1 = \sum_{i=1}^{r} c_i d_i \quad , \qquad 0 \leqslant c_i \leqslant t,$$

$$= \sum_{i=1}^{k} c_i d_i$$

$$\leqslant t \sum_{i=1}^{k} d_i = t\sigma_k$$

i.e

$$d_{k+1} \leqslant t\sigma_k + 1$$

as required .

Theorem 1.1.4 , shows in fact that n,is t-practical number if and only if every m , $1 \leqslant m \leqslant t\sigma(n)$ can be expressed as

$$m = \sum_{i=1}^{r} c_i d_i \quad , \qquad 0 \leqslant c_i \leqslant t \ .$$

The range for m is best possible since

$$m = \sum_{i=1}^{r} c_i d_i \quad \Rightarrow \quad m \leqslant t \sum_{i=1}^{r} d_i = t\sigma(n) \ .$$

Remark 1.1.1

Any practical number is a t-practical number if $t \in \mathbb{N}$.

Proof

Let n be a practical number . Then by Theorem 1.1.3[5] .

$$d_{k+1} \leqslant \sigma_k + 1 , \quad \forall k , 0 \leqslant k \leqslant r - 1 .$$

Since $t \geqslant 1$, then

$$d_{k+1} \leqslant \sigma_k + 1 \leqslant t\sigma_k + 1 ,$$

for each k , $0 \leqslant k \leqslant r - 1$. Thus , by Theorem 1.1.4 ,n is a t-practical number as required .

We shall prove the following result by using Theorem 1.1.4 .

Lemma 1.1.1

For any t-practical number n , $t \in \mathbb{N}$,

$$(t + 1)n \leqslant t\sigma(n) + 1 \quad . \qquad (8)$$

Proof

Let $n \geqslant 1$ be a t-practical number .If n = 1 , then (8) is valid . Suppose that $n \geqslant 2$ and $1 = d_1 , d_2 , \ldots , d_r = n$ represent all positive divisors of n in an increasing order . Since n is t-practical number , it follows from Theorem 1.1.4 , that

$$d_r \leqslant t\sigma_{r-1} + 1 .$$

Therefore

$$d_r + td_r \leqslant t\sigma_{r-1} + td_r + 1$$

i.e

$$(t + 1)d_r \leqslant t\sigma_r + 1 .$$

Since $d_r = n$ and $\sigma_r = \sigma(n)$, then

$$(t + 1)n \leqslant t\sigma(n) + 1 ,$$

which is the required inequality .

Strict equality can occur in (8) for certain t-practical numbers. For example $(t + 1)^2$ is always a t-practical number since every m, $1 \leqslant m \leqslant (t + 1)^2$ can be written as $m = u(t + 1) + v$, where $0 \leqslant u \leqslant (t + 1)$, $0 \leqslant v < t + 1$ and $(t + 1)$, 1 are divisors of $(t + 1)^2$. If , in addition , $t + 1$, is a prime p then , $n = (t + 1)^2$ satisfies $\sigma(n) = p^2 + p + 1$ so that

$$t\sigma(n) = (p - 1)(p^2 + p + 1) = p^3 - 1 ,$$

and

$$(t + 1)n - 1 = p.p^2 - 1 = p^3 - 1$$

i.e

$$(t + 1)n = t\sigma(n) + 1.$$

Now , we have the following results ;

Lemma 1.1.2

For any $\ell \in \mathbb{N}$, $(t + 1)^{\ell}$ is a t-practical number .

Proof

Let $m \in \mathbb{N}$ such that $1 \leqslant m \leqslant (t + 1)^{\ell}$. Then we can write m in the base $(t + 1)$ i.e

$$m = \sum_{i=1}^{\ell-1} c_i (t + 1)^i \quad ,$$

where $0 \leqslant c_i \leqslant t$. Since $(t + 1)^i$, $0 \leqslant i \leqslant \ell-1$, are distinct divisors of $(t + 1)^{\ell}$, the result follows .

Lemma 1.1.3

Let $1 \leqslant S \leqslant t + 1$. Then S^{ℓ} is a t-practical number for any $\ell \in \mathbb{N}$.

Proof

Let $m \in \mathbb{N}$ such that $1 \leqslant m < S^{\ell}$. Then we can write m in the base S i.e

$$m = \sum_{i=1}^{\ell-1} c_i S^i ,$$

where $0 \leqslant c_i \leqslant t$ .Since S^i , $0 \leqslant i \leqslant \ell-1$,are distinct divisors of S^{ℓ} and $0 \leqslant c_i \leqslant t, S^{\ell}$ is a t-practical number .

Now consider the t-practical number n having the positive divisors $1 = d_1$, d_2 , ...,$d_r = n$. Then by writing

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_{\ell}^{a_{\ell}} ,$$

where $a_j \geqslant 1$, $1 \leqslant j \leqslant \ell$ and $p_1 < p_2 < \ldots < p_{\ell}$ such that $p_j \mid d_{k+1}$, $1 \leqslant k \leqslant r-1$ .We shall have $p_j \delta = d_{k+1}$, $\delta \geqslant 1$ and from Theorem 1.1.4 we get

$$p_j \leqslant d_{k+1} \leqslant t\sigma_k + 1 , \quad 0 \leqslant k \leqslant r-1 .$$

Since n is a t-practical number,it follows that

$\sigma_k = \sigma(p_1^{a_1} \ldots p_{j-1}^{a_{j-1}})$ .Therefore

$$p_j \leqslant t\sigma(p_1^{a_1} \ldots p_{j-1}^{a_{j-1}}) + 1 . \qquad (9)$$

So, condition (6) of Theorem 1.1.4 , implies (9) for any t-practical number n.This will lead to a generalization of Theorem 1.1.1 , given in [2] by Stewart,in terms of t-practical numbers .

Thus ,for this purpose we shall define the set A_t , $t \in \mathbb{N}$,to be the set of all integers n such that every integer m , $1 \leqslant m \leqslant t\sigma(n)$, is of the form

$$\sum_{d \mid n} c_d d , \quad 0 \leqslant c_d \leqslant t .$$

Let $\beta(n)$ be the number of these integers m . Then A_t contains all the integers n for which

$$\beta(n) = t\sigma(n) .$$

If n is a practical number , i.e $t = 1$,then $\beta(n) = \alpha(n)$ .Thus , for any t , ($t \geqslant 1$) $\beta(n) \geqslant \alpha(n)$ and furthermore A_t contains all the practical numbers .

We prove the following result :

Lemma 1.1.4

If n belongs to A_t , and p is a prime such that $(p,n) = 1$,then a necessary and sufficient condition that $n' = np^a$, $a \geqslant 1$,be in A_t is that

$$p \leqslant t\sigma(n) + 1 . \tag{10}$$

Proof

If we have that $p > t\sigma(n) + 1$, then $t\sigma(n) + 1$ cannot be represented as a sum of divisors of n' with at most t repetitions for any t,($t \geqslant 1$).Therefore condition (10) is necessary when $n'$is in A_t.

Conversely , suppose that

$$p \leqslant t\sigma(n) + 1 .$$

In order to show that $n'$is in A_t ,($t \geqslant 1$), we need to show that every integer m ,$1 \leqslant m \leqslant t\sigma(n')$ is of the form

$$m = \sum_{d' | n'} c_{d'} d' , \quad 0 \leqslant c_{d'} \leqslant t .$$

We shall proceed by induction on a . For $a = 1$ we have that $p^a \leqslant t\sigma\ (p^{a-1}n) + 1$ implies that $p \leqslant t\sigma(n) + 1$ which is true .

Let $p^{a-1}n$, $a \geqslant 2$ belongs to A_t be the induction hypothesis.

Then

$$\begin{aligned} p^{a+1} &\leqslant pt\sigma(p^{a-1}n) + t\sigma(n) + 1 \\ &= tp\sigma(p^{a-1}) . \sigma(n) + t\sigma(n) + 1 \end{aligned}$$

$$p^{a+1} \leqslant t\sigma(n)(p\sigma(p^{a-1}) + 1) + 1 .$$

Since $p\sigma(p^{a-1}) + 1 = \sigma(p^a)$,

$$p^{a+1} \leqslant t\sigma(n).\sigma(p^a) + 1$$

i.e

$$p^{a+1} \leqslant t\sigma(p^a n) + 1 \quad .$$

Now ,we shall show that $n' = np^a$ belongs to A_t i.e every integer m defined above is of the required form .Let M be an integer such that $M = 0,1, \ldots ,t\sigma(n)$ and consider the intervals

$$[Mp^a \ , \ Mp^a + t\sigma(p^{a-1}n)] \ ,$$

where no integer m within the range $1 \leqslant m \leqslant t\sigma(n')$ is excluded from all of these intervals and all the intervals are overlapping.
Since $p^a \leqslant t\sigma(p^{a-1}n) + 1$,we shall have that

$$(M + 1)p^a = Mp^a + p^a < Mp^a + t\sigma(p^{a-1}n) + 1 .$$

It follows also that 1 and $t\sigma(n')$are included in our intervals,where

$$\begin{aligned}(M + 1)p^a &\leqslant t\sigma(n)p^a + t\sigma(p^{a-1}n) + 1\\ &= t\sigma(n) \ \{p^a + \sigma(p^{a-1})\} + 1\\ &= t\sigma(n). \ \sigma(p^a) + 1\end{aligned}$$

i.e

$$(M + 1)p^a \leqslant t\sigma(p^a n) + 1 .$$

Thus , m can be written as

$$m = Mp^a + S \quad , \qquad (11)$$

where $0 \leqslant M \leqslant t\sigma(n)$ and $0 \leqslant S \leqslant t\sigma(p^{a-1}n)$. Since n and $p^{a-1}n$ belong to A_t as we mentioned above,

$$M = \sum_{d|n} c_d \ d \ , \quad c_d \in \mathbb{N} \ , \quad 0 \leqslant c_d \leqslant t,$$

and

$$S = \sum_{d'' \mid p^{a-1}n} c_{d''} d'' , \quad c_{d''} \in \mathbb{N} , 0 \leqslant c_{d''} \leqslant t$$

Therefore (11) becomes

$$m = \sum c_D D + \sum_{d'' \mid p^{a-1}n} c_{d''} d'' , \quad c_D \in \mathbb{N} , \ 0 \leqslant c_D \leqslant t ,$$

where $D = p^a d$, $D \neq d''$ and both D and d'' are divisors of n'. Hence $n' = p^a n$ belongs to A_t . This completes the proof .

From , Lemma 1.1.4 , if we set $t = 1$,then Stewart's result [2] will follow immediately .

Corollary 1.1.1

If m is a t-practical number and n a non-zero integer such that

$$n \leqslant t\sigma(m) + 1 , \qquad (12)$$

then mn is t-practical number .

Proof

Let m be a t-practical number for a fixed $t, (t \geqslant 1)$ and let n be a positive integer satisfying (12) . It is clear that when $n = p^a$, $(a \geqslant 1)$ and p is a prime such that $(p,m) = 1$,then mn is a t-practical number by Lemma 1.1.4 . If n divides m ,then mn is also a t-practical number .

Suppose that $n \nmid m$ and $n = p_1^{a_1} \ldots p_r^{a_r}$, $a_i \geqslant 1$,$(1 \leqslant i \leqslant r)$ with $p_1 < p_2 < \ldots < p_r$ representing then first r primes such that (12) is satisfied .We proceed by induction on r .

(i) If $r = 1$, the mn is a t-practical number by Lemma 1.1.4 .

(ii) As an induction hypothesis we suppose that mn_{r-1} is a t-practical number, where $n_{r-1} = p_1^{a_1} \ldots p_{r-1}^{a_{r-1}}$, $r > 1$ .It follows from Lemma 1.1.4 , that

$$p_{r-1} \leqslant t\sigma(mn_{r-2}) + 1 \; . \tag{13}$$

Now we need to show that (13) is true with r-1 replaced by r . It follows from (13) that

$$\begin{aligned} p^2_{r-1} &\leqslant tp_{r-1}\sigma(mn_{r-2}) + p_{r-1} \\ &\leqslant t[p_{r-1}\sigma(mn_{r-2}) + \sigma(mn_{r-2})] + 1 \\ &= t[\sigma(mn_{r-2})(p_{r-1} + 1)] + 1 \\ &= t\sigma(mn_{r-2}p_{r-1}) + 1 \end{aligned}$$

i.e

$$p^2_{r-1} \leqslant t\sigma(mn_{r-2}p_{r-1}) + 1 \leqslant t\sigma(mn_{r-2}p_{r-1}^{a_{r-1}}) + 1 \; , \tag{14}$$

where $a_{r-1} \geqslant 1$ and $n_{r-2} = p_1^{a_1} \ldots p_{r-2}^{a_{r-2}}$ .Using Bertrand's Postulate (see[6]) , it follows that $p_{r-1} < p_r < p^2_{r-1}$. Therefore (14) implies that

$$p_r < p^2_{r-1} \leqslant t\sigma(mn_{r-2}p_{r-1}^{a_{r-1}}) + 1$$

i.e

$$p_r \leqslant t\sigma(mn_{r-1}) + 1 \; , \tag{15}$$

and (15) shows that (13) is true for r which completes the induction. Hence mn is a t-practical number by Lemma 1.1.4 .

Now we shall describe the complete structure of the t-practical numbers in the following result ;

Theorem 1.1.5

An integer $n \geq 1$ belongs to A_t ,($t \geq 1$), if and only if n has the form

$$n = p_1^{a_1} \cdots p_k^{a_k} . p_{k+1}^{a_{k+1}} \cdots p_\ell^{a_\ell}, \tag{16}$$

where

$$a_1 , a_2 , \ldots, a_k \geq 0 \text{ (not all zero)}$$

$$a_{k+1} , a_{k+2} , \ldots , a_\ell \geq 1$$

$$p_1 , \ldots., p_k \leq t + 1 < p_{k+1} < \cdots < p_\ell ,$$

and

$$p_{j+1} \leq t\sigma(p_1^{a_1} \cdots p_k^{a_k} \cdot p_{k+1}^{a_{k+2}} \cdots p_j^{a_j}) + 1 , \quad 1 \leq j \leq \ell-1 .$$

Proof

Let $n_1 = p_1^{a_1}$, $a_1 \geq 1$ belong to A_t ,($t \geq 1$) , for some value of t .Then from Lemma 1.1.4 , we have

$$p_1 \leq t + 1 . \tag{17}$$

If we choose p_2 to be the next prime greater than p_1,then since $n_1 = p_1^{a_1}$,in A_t , it follows from Lemma 1.1.4 , that

$$p_2 \leq t\sigma(p_1^{a_1}) + 1 , \tag{18}$$

and (18) implies that $n = p_1^{a_1}.p_2^{a_2}$ belong to A_t from Lemma 1.1.4 .

Now suppose that $n = p_1^{a_1} \cdots p_k^{a_k}$, $a_i \geq 0$ (not all zero) , $1 \leq i \leq k$,belong to A_t as an induction hypothesis . Then from Lemma 1.1.4 we have

$$p_k \leq t\sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + 1 , \tag{19}$$

$$p_k^2 \leq tp_k\sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + p_k . \tag{20}$$

Using the induction hypothesis in (20) we get

$$p_k^2 \leq tp_k\sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + t\sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + 1$$

$$p_k^2 \leq t(p_k + 1).\sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + 1$$

$$p_k^2 \leqslant t\sigma(p_k) \cdot \sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + 1 . \qquad (21)$$

Since $a_k \geqslant 0$, $t \geqslant 1$, then $\sigma(p_k) \leqslant \sigma(p_k^{a_k})$ .Therefore (21) implies

$$\begin{aligned} p_k^2 &\leqslant t\sigma(p_k) \cdot \sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + 1 \\ &\leqslant t\sigma(p_k^{a_k}) \cdot \sigma(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}) + 1 \end{aligned}$$

i.e

$$p_k^2 \leqslant \sigma(p_1^{a_1} \cdots p_k^{a_k}) + 1 . \qquad (22)$$

Since $p_k^2 \geqslant 2p_k$, it follows from Bertrand's Postulate (see[6]) that,

$$p_k < p_{k+1} \leqslant 2p_k \leqslant p_k^2 \quad . \qquad (23)$$

So (22) and (23) will imply that

$$p_{k+1} \leqslant t\sigma(p_1^{a_1} \cdots p_k^{a_k}) + 1 . \qquad (24)$$

Therefore $n_{k+1} = p_1^{a_1} \cdots p_k^{a_k} \cdot p_{k+1}^{a_{k+1}}$ is also in A_t .This completes the induction steps . So the induction can be repeated for p_{k+2} , p_{k+3} , ... , p_ℓ with

$$p_{k+1} < p_{k+2} < , \ldots , < p_\ell ,$$

to show

$$p_\ell \leqslant t\sigma(p_1^{a_1} \cdots p_k^{a_k} \cdot p_{k+1}^{a_{k+1}} \cdots p_{\ell-1}^{a_{\ell-1}}) . \qquad (25)$$

Hence n in A_t will be of the form

$$n = p_1^{a_1} \cdots p_k^{a_k} \cdot p_{k+1}^{a_{k+1}} \cdots p_\ell^{a_\ell}$$

with

$$a_1 , a_2 , \ldots , a_k \geqslant 0 \text{ (not all zero) } ,$$

$$p_1 < p_2 < \ldots < p_k \leqslant t + 1 < p_{k+1} < \ldots < p_\ell ,$$

and the primes p_i are satisfying (25) .

Conversely,suppose n is an integer of the form (16) and satisfying (25).Then from Lemma 1.1.4 , n belongs to A_t.This completes the proof of the Theorem .

We shall prove the following result :

Theorem 1.1.6

For any fixed t ,(t $\geq$ 1), n belongs to A_t if and only if n is a t-practical number .

Proof

Let n be an integer in A_t . Then for any fixed t ,$t \in \mathbb{N}$ every integer m , $1 \leqslant m \leqslant t\sigma(n)$ is of the form

$$\sum_{d|n} c_d\, d \quad , \qquad 0 \leqslant c_d \leqslant t \; . \qquad (26)$$

This certainly shows that every integer from 1 up to tn is of the form (26) . Hence n is t-practical by definition 1.1.1

Conversely ,suppose that $n = p_1^{a_1} \ldots p_\ell^{a_\ell}$, $a_j \geq 1$, $1 \leqslant j \leqslant \ell$, $p_1 < p_2 < ,\ldots , < p_\ell$, is a t-practical number . Then from Theorem 1.1.4 ,the condition

$$p_j \leqslant t\sigma(p_1^{a_1} \ldots p_{j-1}^{a_{j-1}}) + 1 \; , \qquad (27)$$

is necessary for n to be a t-practical number . Using Lemma 1.1.4, which shows that condition (27) is sufficient for n to be in A_t which completes the proof .

Theorem 1.1.7

The product of two t-practical numbers is a t-practical number , for any t , $t \in \mathbb{N}$.

Proof

Let m and n be two t-practical numbers , $t \in \mathbb{N}$. Then if $m \leqslant n$ we shall have

$$m \leq n < t\sigma(n) + 1 ,$$

and since n is t-practical number it follows that mn is also t-practical number by Corollary 1.1.1 .If $n < m$,then we can write

$$n < m < t\sigma(m) + 1 ,$$

and since m is also t-practical number,then the fact that mn is t-practical follows from Corollary 1.1.1 ,which ends the proof .

§1.2 Extension To The Negative Divisors

Let A^* be the set of all positive integers n having the property that every integer m , $1 \leq m \leq \sigma(n)$ is of the form

$$m = \sum_{d|n} b_d\, d \quad , \qquad |b| \leq 1 \quad . \qquad (1)$$

In [3] B.Jacobson proved the following ;

Lemma 1.2.1[3]

If n belongs to A^* and p is a prime such that $(p,n) = 1$, then $p^k n$, $k \geq 1$ belongs to A^* if and only if

$$p \leq 2\sigma(n) + 1 \quad .$$

Now , we extend the definition of A^* to A^*_t ,$(t \geq 1)$,in which every integer m , $1 \leq m \leq t\sigma(n)$ is of the form

$$m = \sum_{d|n} e_d\, d \quad , \qquad |e_d| \leq t \quad .$$

We have the following result ;

Theorem 1.2.2

For any fixed t ,$(t \geq 1)$, $A^*_t = A_{2t}$.

Proof

It is obvious that $A^*_t \subseteq A_{2t}$ since if n in A^*_t and m is an integer such that

$$0 \leqslant m \leqslant 2t\sigma(n) ,$$

then m is of the form

$$m = m_1 + t\sigma(n)$$

and

$$m_1 = \sum_{d|n} a_d\, d , \qquad |a_d| \leqslant t .$$

So , we can write

$$m = \sum_{d|n}(a_d + 1)d = \sum_{d|n} c_d\, d \quad , c_d = 0 ,t \text{ or } 2t$$

i.e n belong to A_{2t} ,$(t \geqslant 1)$. Hence $A_t^* \subseteq A_{2t}$.

Conversely , suppose that n belongs to A_{2t} , $t \geqslant 1$ and m is an integer such that

$$0 \leqslant m \leqslant t\sigma(n) .$$

Then we have that

$$t\sigma(n) \leqslant m + t\sigma(n) \leqslant 2t\sigma(n)$$

and

$$t\sigma(n) + m = \sum_{d|n} c_d\, d \quad , \quad c_d = 0 , t \text{ or } 2t . \qquad (2)$$

By subtracting $t\sigma(n)$ from both sides of (2) we obtain that

$$m = \sum_{d|n}(c_d - t)d = \sum_{d|n} a_d\, d \quad , \quad |a_d| \leqslant t .$$

Therefore n belongs to A_t^* i.e $A_{2t} \subseteq A_t^*$. Thus $A_{2t} = A_t^*$ which completes the proof .

It follows from Theorem 1.2.2,that Jacobson's result given by Lemma 1.2.1, above can be generalized as follows ;

Theorem 1.2.4

For a fixed t ,$(t \geqslant 1)$, if n belongs to A_t^* and p is a prime with $(p,n) = 1$,then $p^k n$, $k \geqslant 1$,belongs to A_t^* if and only if

$$p \leqslant 2t\sigma(n) + 1 .$$

§1.3 The Value Of $\beta(n)/t\sigma(n)$

Let n be a t-practical number , $t \in \mathbb{N}$. We shall define

$$S_t(n) = \frac{\beta(n)}{t\sigma(n)} \quad .$$

As we mentioned in section 1.1 , $\beta(n) = \alpha(n)$ when n is a practical number i.e when $t = 1$.This implies that $S_1(n) = \alpha(n)/\sigma(n)$. In [2] Stewart showed that the values of $S_1(n)$ are dense on the interval $(0,1]$. We have $\beta(n) = t\sigma(n)$ for any t-practical number n in A_t

A result similar to that of Stewart in [2] can be proved for $S_t(n)$, where n is a t-practical number. So for that we shall need the following ;

Lemma 1.3.1

If p is a prime such that $p > t\sigma(n)$, $t \geqslant 1$,then

$$\beta(pn) = \beta(n)\{\beta(n) + 2\} \ .$$

Proof

We have that every sum of divisors of pn is of the form

$$m = m_1 p + m_2 \quad ,$$

where m_1 and m_2 are either zero or sum of divisors of n with at most t repetitions . Therefore

$$0 \leqslant m_1 \leqslant \max.\left[\left[\frac{m}{p}\right] , t\sigma(n)\right]$$

and

$$0 \leqslant m_2 \leqslant t\sigma(n) \quad .$$

This shows that the maximum value of m_2 is equal to $t\sigma(n)$. Since $p > t\sigma(n)$, then no two of these m can be equal unless they have the same m_1 and m_2 . We have that the number of choices of each m_1 and m_2 is $\beta(n) + 1$. Therefore

$$\beta(pn) = (\beta(n) + 1)^2 - 1 = \beta(n)\{\beta(n) + 2\} \ ,$$

which is the required result .

Theorem 1.3.1

The values of $S_t(n)$ form a dense set on the interval $[0,1]$.

Proof

Consider x and y to be real numbers such that $0 < x < y \leqslant 1$. We seek an integer k such that $x < S_t(k) < y$. We write k in the form $k = pn$, n is in A_t , where $p > t\sigma(n)$ so that by applying Lemma 1.3.1

$$S_t(k) = S_t(pn) = \frac{\beta(pn)}{t\sigma(pn)} = \frac{\beta(n)\{\beta(n) + 2\}}{t(p + 1)\sigma(n)} .$$

Since $\beta(n) = t\sigma(n)$ follows from $n \in A_t$,we have that

$$S_t(pn) = \frac{t\sigma(n)\{t\sigma(n) + 2\}}{t(p + 1)\sigma(n)} = \frac{\{t\sigma(n) + 2\}}{(p + 1)} .$$

Put $u = 1/y$, $u(1 + \epsilon) = u + u\epsilon = 1/x$, $\epsilon > 0$. We need to find n and p so that

$$u\{t\sigma(n) + 2\} < p + 1 < u\{t\sigma(n) + 2\}(1 + \epsilon) .$$

It follows from Lemma 1.1.4, that ther are arbitrarily large numbers in A_t . So , by using Cahen-Stieltjes Theorem (see[8]),for any $\epsilon > 0$ and sufficiently large $v = u\{t\sigma(n) + 2\}$ there exists at least one prime p such that

$$u\{t\sigma(n) + 2\} - 1 < p < v(1 + \epsilon) - 1 - \epsilon < v(1 + \epsilon)$$

i.e

$$u\{t\sigma(n) + 2\} < p + 1 < u\{t\sigma(n) + 2\}(1 + \epsilon) .$$

Since $u \geqslant 1$ and $(p + 1) \leqslant t(p + 1)$,

$$u\{t\sigma(n) + 2\} < t(p + 1) < u\{t\sigma(n) + 2\}(1 + \epsilon),$$

and such a prime p satisfies the condition $p > t\sigma(n)$ .This ends the proof .

For the non-practical numbers , we let n be any non-practical. Then n is either of the form

$$n = m\,p_1^{a_1} \ldots p_k^{a_k} \quad , \qquad k > 1 \qquad (1)$$

where $a_j \geq 1$, $(m,p_j) = 1$, $(1 \leq j \leq k)$, $p_1 < p_2 < \ldots < p_k$ and $m \geq 1$ is a practical number such that $p_1 > \sigma(m) + 1$, or n is of the form

$$n = m.p^a \quad , \qquad a \geq 1 \qquad (2)$$

where $p > \sigma(m) + 1$.

We shall consider the integers n having the form (2) . Then $\alpha(n)$ will be determined in the following ;

<u>Lemma 1.3.2</u>

Let $n = m.p^a$, $a \geq 1$, $(p,m) = 1$ with $m \geq 1$ a practical number satisfying $p > \sigma(m) + 1$. Then

$$\alpha(n) = \{\sigma(m) + 1)\}^{a+1} - 1 \ .$$

<u>Proof</u>

Suppose that $n = m.p^a$, $a \geq 1$, $(m,p) = 1$ and $m \geq 1$ is a practical number such that $p > \sigma(m) + 1$. We have

$$\sigma(n) = \sigma(m).\sigma(p^a)$$

$$\sigma(n) = \sigma(m).(1 + p + \ldots + p^a) \ .$$

Then any sum of divisors of n will be of the form

$$D = m_0 + m_1 p + \ldots + m_a p^a \quad , \qquad (3)$$

where $0 \leq m_i \leq \sigma(m)$, $(0 \leq i \leq a)$. Since $p > \sigma(m) + 1$, then no two of these D given in (3) are equal unless they have the same m_i . The number of the choices of each m_i is equal $\alpha(m) + 1$. Since the number of m_i is a + 1. Then the number of the possible choice of D given in (3), excluding $D = 0$, is

$$\alpha(n) = \{\alpha(m) + 1\}^{a+1} - 1 \ .$$

Since m is a practical number, $\alpha(m) = \sigma(m)$. Hence

$$\alpha(n) = \{\sigma(m) + 1\}^{a+1} - 1 \quad ,$$

which is the required result .

Now , we prove the following

Theorem 1.3.2

Let $n = m.p^a$, $a \geqslant 1$, $(p,m) = 1$ with $m \geqslant 1$ a practical number satisfying $p > \sigma(m) + 1$. Then $S_1(n)$ form a dense value on the interval $(0,1)$.

Proof

Let $n = m.p^a$, $a \geqslant 1$, $(p,m) = 1$ and $m \geqslant 1$ be a practical number such that $p > \sigma(m) + 1$. Then from Lemma 1.3.2 ,

$$\alpha(n) = \{\sigma(m) + 1\}^{a+1} - 1 \; .$$

Therefore

$$S_1(n) = \frac{\{\sigma(m) + 1\}^{a+1} - 1}{\sigma(mp^a)}$$

$$S_1(n) = \frac{\{\sigma(m) + 1\}^{a+1} - 1}{\sigma(m)(1 + p + \ldots + p^a)} \; .$$

Put $N = \sigma(m) + 1$, so that $N - 1 = \sigma(m)$. Then

$$S_1(n) = \frac{N^{a+1} - 1}{(N - 1)(1 + p + \ldots + p^a)}$$

$$S_1(n) = \frac{(N - 1)(N^a + N^{a-1} + \ldots + 1)}{(N - 1)(1 + p + \ldots + p^a)} \; .$$

Therefore

$$S_1(n) = \frac{(N^a + N^{a-1} + \ldots + 1)}{(1 + p + \ldots + p^a)} \; . \qquad (4)$$

Now ,we need to show that such a prime p and integer N given in (4) exist with $0 < S_1(n) < 1$. From [8] , we have for any $\epsilon > 0$ and sufficiently large N that such a prime p exists so that

$$N < p < (1 + \epsilon)N$$

Hence

$$\begin{aligned} N^2 &< p^2 < (1 + \epsilon)N^2 \\ &\vdots \\ N^a &< p^a < (1 + \epsilon)^a N^a \end{aligned} \qquad (5)$$

Thus (5) implies that

$$1 + N + \ldots + N^a < 1 + p + \ldots + p^a < 1 + (1 + \epsilon)N + \ldots + (1+\epsilon)^a N^a$$

i.e

$$1 + N + \ldots + N^a < 1 + p + \ldots + p^a < (1 + N + \ldots + N^a)(1+\epsilon)^a \text{ , } (6)$$

So , from (6) we obtain

$$\frac{1}{(1 + \epsilon)^a} < S_1(n) < 1 \quad ,$$

where $\frac{1}{(1+\epsilon)^a} > 0$ since $(1 + \epsilon)^a > 1$. Hence our result follows .

It follows from Lemma 1.3.2, and for $m = 1$, that $\alpha(n) = 3$ represents the smallest value $\alpha(n)$. In this case , Theorem 1.3.2, implies that $S_1(n) = 3/\sigma(n)$. Therefore $S_1(n) \longrightarrow 0$ as $n \longrightarrow \infty$

§1.4 Structure Theorem For E^*

Let 0 be the set of all odd integers and 0^* the subset of 0 which contains all integers $n \geqslant 5$ satisfying the condition

$$\sigma(n) - \alpha(n) = 2 \quad , \qquad (1)$$

where $\sigma(n)$ represents the sum of all positive divisors of n, $\alpha(n)$ the number of all positive integers m , $0 \leqslant m \leqslant \sigma(n)$ having the form

$$m = \sum_{d\mid n} a_d\, d \quad . \quad a_d = 0 \ , \ 1 \quad . \qquad (2)$$

B.M.Stewart[2] showed that the first integer having this property is $n = 945$. He introduced a necessary condition for these integers n which satisfies (1) when he proved that n is a multiple of 15 .We arrange the τ divisors of $n = 15T$, as follows;

$$d_1 = 1 < d_2 = 3 < d_3 = 5 < \dots < d_r = 15T \quad .$$

By setting σ_i to be the first i divisors of $n = 15T$ Stewart[2] proved the following results ;

Theorem 1.4.1[2]

Necessary and sufficient conditions that $n = 15T$ belongs to 0^* are,for $i \geqslant 3$, either

(i) $d_{i+1} \leqslant \sigma_i - 2$, $d_{i+1} \neq \sigma_i - 4$,

or (ii) $d_{i+1} = \sigma_i - 2$, $d_{i+2} = \sigma_i - 2$.

As a consequence of Theorem 1.4.1 , he proved the following :

Theorem 1.4.2[2]

If n belongs to 0^* and p is an odd prime,then $n' = np$ belongs to 0^*, if and only if

$$2p \leq \sigma(n) - 2 \quad , \qquad 2p \neq \sigma(n) - 4 \quad .$$

Corollary 1.4.1[2]

If the prime p divides n and n belongs to O^*, then $n' = np$ belongs to O^*.

A fair question arises :"Are there any even integers n satisfying (1) given above" ? The answer is " Yes " since e.g. $n = 70, 350, \ldots,$ does satisfy condition (1) . So , we shall consider the set E of all even integers n . Define E^* to be a subset of E which contains all integers n satisfying (1) .

The necessary and sufficient conditions for any integer n to belong to E^* are determined in the following result ;

Theorem 1.4.3

Let $n = mp_1 \ldots p_k$, be an integer with p_j primes such that $(p_j, m) = 1, (1 \leqslant j \leqslant k)$, $p_1 < p_2 < \ldots < p_k$, and with $m > 1$ a practical number . Then n belongs to E^* if and only if, for $k \geqslant 2$,

$$p_1 = \sigma(m) + 2 ,$$

$$p_1 < p_2 \leqslant 2p_1 - 1 ,$$

and

$$p_{j+2} \leqslant \sigma(mp_1 \cdot p_2 \ldots p_{j+1}) + 1 \quad , \quad 1 \leqslant j \leqslant k - 2 .$$

Proof

Let $n = mp_1 \ldots p_k$ belong to E^* . Since $m \geqslant 2$ is a practical number then every integer from 1 up to $\sigma(m)$ is of the form

$$\sum_{d \mid n} d , \qquad (3)$$

and $p_1 \geqslant 5$. It follows also from m being a practical part of n and n belonging to E^* that

$$p_1 = \sigma(m) + 2 . \qquad (4)$$

For if $p > \sigma(m) + 2$, then we have at least four integers in the interval $[1,\sigma(n)]$ such as $\sigma(m) + 1$, $\sigma(m) + 2$, $n - \sigma(m) + 1$ and $n - \sigma(m) + 2$ which defy representation by the form (3). Hence (4)is necessary for n to be in E^*.

If n belongs to E^* and $k = 1$, then since $n = mp_1$ is an even integer all the divisors of n can be given in the following ascending order

$$1 , 2 , \ldots , m , p_1 , 2p_1 , \ldots , n .$$

Since $n \epsilon E^*$ it follows from the order of the divisors that we must have

$$2p_1 \leqslant \sigma(m) + p_1 + 1 .$$

But this inequality implies that $p_1 \leqslant \sigma(m) + 1$ which means $n = mp_1$ is a practical number (see Lemma 1.1.4) . Further , since $p_1 = \sigma(m) + 2$ then

$$2p_1 > \sigma(m) + p_1 + 1 ,$$

and this implies that $\sigma(m) + p_1 + 1 = u$ does not have the form (3) i.e n not in E^* a contradicton . Therefore we need to have $k \geqslant 2$ and p_2 satisfying

$$p_1 < p_2 < 2p_1$$

i.e

$$p_1 < p_2 \leqslant 2p_1 - 1 .$$

Therefore

$$p_1 < p_2 \leqslant \sigma(m) + p_1 + 1 .$$

Thus, every integer in the range $[\sigma(m) + 2 , \sigma(mp_1p_2)]$ can be written in the form (3) . So, all primes p_j , $j = 3,\ldots,k$ must satisfy the condition

$$p_{j+2} \leqslant \sigma(mp_1p_2\ldots p_{j+1}) + 1 , \quad (1 \leqslant j \leqslant k-2).$$

Otherwise $p_{j+2} > \sigma(mp_1p_2\ldots p_{j+1}) + 1$ implies that $n \not\epsilon E^*$.

Conversely , suppose that $n = mp_1p_2\ldots p_k$, such that $m \geqslant 2$ is a practical number $k \geqslant 2$, satisfying

$$p_1 = \sigma(m) + 2$$

$$p_1 < p_2 \leqslant 2p_1 - 1$$

and

$$p_{j+2} \leqslant \sigma(mp_1p_2\ldots p_{j+1}) + 1 \ , \quad 1 \leqslant j \leqslant k - 2.$$

Then every integer in the interval $[1,\sigma(n)]$, except $\sigma(m) + 1$ and $\sigma(n) - (\sigma(m) + 1))$, is of the form (3). Hence n is in E^* which ends the proof .

Corollary 1.4.2

For any integer n in E^* the practical part m of n is of the form

$$m = 2^a(q_1{}^{a_1}\ldots q_r{}^{a_r})^2 \quad , \ a \geqslant 1 \ , \ a_i \geqslant 0 \ ,$$

where q_i are primes such that $q_1 < q_2 <\ldots< q_r$ for all $i,(1 \leqslant i \leqslant r)$.

Proof

Let n be any integer in E^* , then from Theorem 1.4.3 , we have

$$n = mp_1p_2\ldots p_k \quad , \quad k \geqslant 2$$

where $p_1 < p_2 < \ldots < p_k$ are primes such that $(p_j,m) = 1$, $(1 \leqslant j \leqslant k)$, $m \geqslant 2$ is a practical number satisfying the conditions

$$p_1 = \sigma(m) + 2$$

$$p_1 < p_2 \leqslant 2p_1 - 1$$

and

$$p_{j+2} \leqslant \sigma(mp_1p_2\ldots p_{j+1}) + 1 \ , \quad (1 \leqslant j \leqslant k-2).$$

Since $m \geqslant 2$ then $p_1 \geqslant 5$ and this implies that $\sigma(m)$ is an odd integer. Since m is a practical number it follows from Stewart's result in [2] that m is of the form

$$m = 2^aq_1{}^{c_1}\ldots q_r{}^{c_r} \quad , \quad a \geqslant 1 \ , \ c_i \geqslant 0 \ ,$$

where q_i are primes such that $2 < q_1 < \ldots < q_r$, $q_i \neq p_j$ for all i and j, $q_1 \leqslant \sigma(2^a) + 1$, and

$$q_{i+1} \leqslant \sigma(2^a q_1^{c_1} \ldots q_i^{c_i}) + 1 \ , \ (1 \leqslant i \leqslant r-1).$$

We have

$$\sigma(m) = \sigma(2^a q_1^{c_1} \ldots q_r^{c_r})$$

$$= \sigma(2^a) . \sigma(q_1^{c_1} \ldots q_r^{c_r})$$

$$\sigma(m) = (2^{a+1} - 1)(1 + q_1 + \ldots + q_1^{c_1}) \ldots (1 + q_r + \ldots + q_r^{c_r})$$

i.e

$$\sigma(m) = (2^{a+1} - 1) \prod_{1 \leqslant i \leqslant r} (1 + q_i + \ldots + q_i^{c_i}) \ ,$$

where $(2^{a+1} - 1)$ is odd for any a , $a \geqslant 1$, since $\sigma(m)$ is odd as we mentioned above . Then $\prod_{1 \leqslant i \leqslant r} (1 + q_i + \ldots + q_i^{c_i})$ is either equal to one or is an odd integer.This require that either $c_i = 0$ or that c_i is an even integer respectively i.e $c_i = 2a_i$, $a_i \geqslant 0$. This implies that m is of the form

$$m = 2^a (q_1^{a_1} \ldots q_r^{a_r})^2 \ ,$$

which is the required form . This ends the proof.

Now , we shall need the following definition.

Definition 1.4.1

A prime p is called close to n, $(p,n) = 1$, if $p \leqslant \sigma(n) + 1$. Otherwise p is not close to n .

Let $\tau = d(n)$ be the number of all positive divisors d_i of n , $1 \leqslant i \leqslant \tau$. We denote σ_i to be the sum of the first i of the τ divisors of n , $n = mp_1 \ldots p_k$, $k \geqslant 2$. Further results related to the integers of E^* will be given in the following :

Theorem 1.4.4

If $n = mp_1 \ldots p_k$, $m > 1$ is the practical part of n , $(p_i,m) = 1$, $(1 \leqslant i \leqslant k)$, $p_1 < p_2 < \ldots < p_k$ are primes not close to m . Then necessary and suficient conditions that n belongs to E^* are that for $i \geqslant d(m) + 2$ either

(i) $d_{i+1} \leqslant \sigma_i - (p_1 - 1)$, $d_{i+1} \neq \sigma_i - 2(p_1 - 1)$

or (ii) $d_{i+1} = \sigma_i - 2(p_1 - 1)$, $d_{i+2} = \sigma_i - p_1$.

Proof

We have that $m \geqslant 2$ is a practical number i.e every integer from 1 up to $\sigma(m)$ is representable in terms of n,and $i \geqslant d(m) + 2 \geqslant 4$.

It follows that the ascending order of all positive divisors of $n = mp_1 \ldots p_k$ can be given as

$$1 = d_1 < d_2 = 2 < \ldots < d_{i-2} = m < d_{i-1} = p_1 < d_i = p_2 < \ldots < d_\tau = n.$$

Let $n = mp_1 \ldots p_k$ belong to E^*. Then from Theorem 1.4.3 , we have $k \geqslant 2$,

$$p_1 = \sigma(m) + 2$$

$$p_1 < p_2 \leqslant 2p_1 - 1$$

and

$$p_{j+2} \leqslant \sigma(mp_1 \ldots p_{j+1}) + 1 \quad , \quad (1 \leqslant j \leqslant k - 2) .$$

If $d_{i+1} > \sigma_i - (p_1 - 1)$, then $L = \sigma_i - (p_i - 1)$ defies representation in terms of n . If $d_{i+1} = \sigma_i - 2(p_1 - 1)$ we get that

$$L = \sigma_i - (p_1 - 1) = d_{i+1} + (p_1 - 1) \quad ,$$

also defies representation with respect to n , unless $d_{i+2} = \sigma_i - p_1$. Hence our conditions (i) or (ii) are necessary for n to be in E^* when $i \geqslant d(m) + 2$, where $d(m)$ is the number of all positive divisors of m.

Conversely , suppose that (i) or (ii) hold when $i \geqslant d(m) + 2$. We proceed by induction on i . When $d(m) + 2 \leqslant i \leqslant \tau - 1$ and

$$(p_1 - 1) < x < \sigma_i - (p_1 - 1) ,$$

then either

$$x = \sum_{d|n} d ,$$

with $d \leqslant d_i$ distinct positive divisors of n , or

$$x = \sigma_{i-1} - (p_1 - 1) = d_i + (p_1 - 1).$$

The hypothesis hold when $i = d(m) + 2$. Since

$$\sigma_{i-1} - p_1 = d_{i+1} \Rightarrow \sigma_{i-1} - (p_1 - 1) = d_{i+1} + 1,$$

follows from (ii) , then

$$x = \sigma_{i-1} - (p_1 - 1) = d_i + (p_1 - 1) = d_{i+1} + 1 . \quad (4)$$

By considering y such that $(p_1 - 1) < y < \sigma_{i+1} - (p_1 - 1)$,we have the following cases ;

(I) $y < \sigma_i - (p_1 - 1)$. Put $y = x$ .Using the induction hypothesis we can then write

$$y = \sum_{d|n} d ,$$

with $d \leqslant d_{i+1}$ distinct positive divisors of n .

(II) $\sigma_i - (p_1 - 1) \leqslant y < \sigma_{i+1} - (p_1 -1)$. By writing

$$y = d_{i+1} + x , \quad (5)$$

with $0 \leqslant x < \sigma_i - (p_1 - 1)$,it follows from the induction hypothesis, (unless $x = p_1 - 1$ or $x = d_{i+1} + 1 = d_i + (p_1 - 1)= \sigma_{i-1} - (p_1 - 1)$), that

$$y = \sum_{d|n} d ,$$

with $d \leqslant d_{i+1}$ distinct positive divisors of n . The case when

$$x = d_{i+1} + 1 = d_i + (p_1 - 1) = \sigma_{i-1} - (p_1 - 1) ,$$

implies that

$$y = d_{i+1} + \sigma_{i-1} -(p_1 - 1) .$$

This follows from (4) and (5) . Using (ii) , then

$$y = d_{i+1} - (p_1 - 1) + \sigma_{i-1} = \sigma_{i-1} + d_i - 1 = \sigma_i - 1 ,$$

and this shows that y is representable in terms of n. If we have $x = p_1 - 1$, then

$$\sigma_i - (p_1 - 1) \leqslant y = d_{i+1} + (p_1 - 1) \; .$$

So, from condition (i) we have

$$d_{i+1} + (p_1 - 1) \leqslant \sigma_i \quad ,$$

and

$$\sigma_i - (p_1 - 1) \leqslant y = d_{i+1} + (p_1 - 1) \leqslant \sigma_i \; , \qquad (6)$$

where (6) shows that

$$y = \sigma_i \; , \; \sigma_i -1, \ldots, \sigma_i - (p_1 - 2) \text{ or } \sigma_i - (p_1 - 1)$$

Since $\sigma_i \geqslant \sigma(m) + p_1 + p_2$ and $p_1 = \sigma(m) + 2$, where m is a practical number, then all values of y given by

$$y = \sigma_i \; , \; \sigma_i - 1 \; , \ldots , \; \sigma_i - (p_1 - 2) \; ,$$

are representable with respect to n. The case when $y = \sigma_i - (p_1 - 1)$ follows from (ii) which implies that

$$y \; = \sigma_i - (p_1 - 1) = d_{i+2} + 1 \quad ,$$

which is also representable in terms of n. Therefore the induction steps for i+1 are satisfied. On the other hand we showed that every integer q such that $(p_1 - 1) < q < \sigma(n) - (p_1 - 1)$ is a sum of distinct positive divisors of n. Since m is a practical number , it follows that every integer from $\sigma(n) - \sigma(m)$ up to $\sigma(n)$ is representable in terms of n. Hence $n = mp_1 \ldots p_k$ belongs to E^* and this completes the proof .

In Stewart's result [2] all integers n in O^* are multiples of 15 and therefore $p_1 \geqslant 3$ for any integer n. So, the practical part m of each integer n in O^* will be $m = 1$ since 1 is the only odd practical number(see[2]). This shows that the condition $p_1 = \sigma(m) + 2$ and $p_1 < p_2 \leqslant 2p_1 - 1$ hold for any integer n in O^* .

It follows also that Theorem 1.4.4 does not imply Theorem 1.4.1[2] since $p_1 > 5$ when n is in E^* as we showed above and this shows that (ii) of Theorem 1.4.4, does not imply condition (ii) of Theorem 1.4.1[2].

In view of Theorem 1.4.4, we have the following ;

Theorem 1.4.5

If $n = mp_1..p_k$ belongs to E^* and p is an odd prime , then $np = n'$ belongs to E^* if and only if

$$p \leqslant \sigma(n) - (p_1 - 1) .$$

Proof

Let $n' = np$ belong to E^* and denote by σ_i' and d_i' the partial sums and the divisors of n' respectively. We have that all divisors of n' which are not divisors of n are multiples of p^{a+1} , where $n = p^a b$, $a \geqslant 0$ and $(p,b) = 1$.

Suppose that the first r divisors of n' coincide with those of n for which $d_i = d_i'$, $r \geqslant i$, $r \geqslant d(m) + 2$. So , the conditions (i) and (ii) of Theorem 1.4.4 , will carry over to $n' = np$ for $d(m) + 2 \leqslant i \leqslant r$.

Now , we shall consider the cases with $r \leqslant s \leqslant u$, where

$$d_s \leqslant d_u' < d_{u+1}' \leqslant d_{s+1} .$$

Since n belong to E^* it follows from Theorem 1.4.4 ,that

$$d_{u+1}' \leqslant d_{s+1} \leqslant \sigma_s - (p_1 - 1) < \sigma_s + p^{a+1} - 2(p_1 - 1) \leqslant \sigma_u' - 2(p_1 - 1).$$

Therefore condition (i) of Theorem 1.4.4, holds for n' and u. Further we consider the cases in which

$$n \leqslant d_u' < d_{u+1}' \leqslant n' . \qquad (7)$$

Then $d_{u+1}' = \dfrac{n'}{d_j}$, where $d_j < p$. Therefore $d_j \mid n$ follows from (7).

When $\dfrac{n}{d_j} > p_2$,then $d_j \leqslant p_2$ .By applying Theorem 1.4.4 to n for $i \geqslant d(m) + 2$ we have

$$\frac{n}{p_2} \leqslant \frac{n}{d_j} \leqslant \sigma_i - (p_1 - 1) \leqslant \sigma(n) - \left(\frac{n}{d_1} + \frac{n}{d_2} + \ldots + \frac{n}{d_j}\right) - (p_1 - 1)$$

i.e

$$\frac{n}{d_j} \leqslant \sigma(n) - \left(\frac{n}{d_1} + \frac{n}{d_2} + \ldots + \frac{n}{d_j}\right) - (p_1 - 1). \qquad (8)$$

Multiplying (8) by p we get

$$\frac{pn}{d_j} \leqslant p\sigma(n) - \left(\frac{pn}{d_1} + \frac{pn}{d_2} + \ldots + \frac{pn}{d_j}\right) - p(p_1 - 1) .$$

Since $pn = n'$ and $\dfrac{pn}{d_j} = d_{u+1}'$, then

$$d_{u+1}' \leqslant p\sigma(n) - \left(\frac{n'}{d_1} + \frac{n'}{d_2} + \ldots + \frac{n'}{d_j}\right) - p(p_1 - 1)$$

$$d_{u+1}' < p\sigma(n) + \sigma(b) - \left(\frac{n'}{d_1} + \frac{n'}{d_2} + \ldots + \frac{n'}{d_j}\right) - p(p_1 - 1), \qquad (9)$$

and (9) shows that condition (i) of Theorem 1.4.4 holds for n' and u

It remains to consider the cases when $d_{u+1}' = p$, pm ,or $p(p_1 .. p_k)$,$(k \geqslant 2)$, with $n < d_{u+1}'$. From these cases we shall have the following ;

(I) If we have

$$pm < n < p(p_1 \ldots p_k) ,$$

then

$$p < \frac{n}{m} < \sigma(n) - (p_1 - 1) ,$$

since n belongs to E^* . Therefore

$$d_{u+1}' = p(p_1 \ldots p_k) < \sigma(n) + p + pm - (p_1 - 1) = \sigma_u' - (p_1 - 1),$$

so condition (i) of Theorem 1.4.4 , holds for n' and u .

(II) If $p < n < pm$ or $n < p$, then the condition

$$d_{u+1}' = pm < \sigma(n) + p - (p_1 - 1) = \sigma_u' - (p_1 - 1) ,$$

is necessary for n' to be in E^* . It follows from (ii) of Theorem 1.4.4, that

$$pm = d_{u+1}' = d_{u+2}' - (p_1 - 2) = \sigma_u' - 2(p_1 - 1) .$$

Therefore condition (i) of Theorem 1.4.4, must hold with

$$pm \neq \sigma(n) + p - 2(p_1 - 1) .$$

Thus, the conditions of Theorem 1.4.4,are necessary .

Conversely , suppose that (i) or (ii) of Theorem 1.4.4, holds . Then

$$d_{u+1}' = p(p_1 \ldots p_k) < \sigma(n) + p + pm - 2(p_1 - 1) = \sigma_u' - 2(p_1 - 1),$$

and if $n < p$, then the condition

$$d_{u+1}' = p < \sigma(n) - 2(p_1 - 1) ,$$

holds . Thus, for $i \geq d(m) + 2$ the condition (i) or (ii) holds and hence n' belongs to E^* which completes the proof .

Corollary 1.4.3

If $n = mp_1 \ldots p_k$ belongs to E^* and p is an odd prime dividing n and not dividing the practical part m of n,then $np=n'$ belongs to E^*.

Proof

Let $n = mp_1 \ldots p_k$ belongs to E^*,with $k \geq 2$, $(p_j,m)=1$, $(1 \leq j \leq k)$,and $m \geq 2$ the practical part of n. Then by Theorem 1.4.3, m satisfies the conditions

$$p_1 = \sigma(m) + 2 ,$$

$$p_1 < p_2 \leq 2p_1 - 1$$

and

$$p_{j+2} \leq \sigma(mp_1 \ldots p_{j+1}) + 1 , \quad (1 \leq j \leq k - 2).$$

If p divides m , then $m' = mp$ will be a practical number and this implies that $\sigma(m) + 1 < \sigma(m') + 1$, i.e

$$p_1 \leqslant \sigma(m') + 1 \quad .$$

Since $p_1 = \sigma(m) + 2$,this will makes $n' = np$ a practical number . Therefore p must be a p_i, $(1 \leqslant i \leqslant k)$. Since $p_1 \geqslant 5$,then p is odd. We have that

$$p + (p_1 - 1) < p(p_1 \ldots p_k) < \sigma(n)$$

$$p < \sigma(n) - (p_1 - 1) \quad , \qquad (10)$$

and (10) shows that the condition of Theorem 1.4.5 , holds . Hence $n' = np$ belongs to E^* which ends the proof .

One can deduce from Theorem 1.4.4, that the first integer belonging to E^* is $n = 2.5.7 = 70$, which represents the smallest integer in E^*. It follows from Theorem 1.4.5, Corollary 1.4.2 , and Corollary 1.4.3, that any integer in E^* has a general form given by

$$n = mp_1^{b_1} \ldots p_k^{b_k} \quad , \qquad (11)$$

with $k \geqslant 2$, $b_j \geqslant 1$, $(1 \leqslant j \leqslant k)$, p_j are primes such that $(p_j, m) = 1$ $5 \leqslant p_1 < p_2 < \ldots < p_k$,satisfying

$$p_1 = \sigma(m) + 2 \quad ,$$

$$p_1 < p_2 \leqslant 2p_1 - 1 \quad ,$$

$$p_{j+2} \leqslant \sigma(mp_1 \ldots p_{j+1}) + 1 \quad , \quad (1 \leqslant j \leqslant k - 2),$$

and $m > 1$ is a practical number of the form

$$m = 2^a (q_1^{a_1} \ldots q_r^{a_r})^2 \ , \qquad (12)$$

where $a \geqslant 1$, $a_i \geqslant 0$, $(1 \leqslant i \leqslant r)$, and $q_i \neq p_j$ for all i and j .

Further examples can be given to show that the first eight integers n in E^* which satisfy the conditions are those when $m = 2$, 8,18,72 128,196,200 and 288,where the odd part of m is a perfect square.The prime p_1 which corresponds to these values of m are :

$p_1 = 5$, 17 , 41 , 197 , 257 , 401 , 467 , and 821 respectively .

We have the following result ;

Theorem 1.4.6

There are infinitely many integers n in E^* .

Theorem 1.4.6 , will follow since for any integer $n = mp_1^{a_1}\dots p_k^{a_k}$,$(k \geq 2)$,in E^* where p is a prime such that $n < p < \sigma(n) + 1$,then np^ℓ , $\ell \geq 1$ also belongs to E^* by Theorem 1.4.3 . Hence there are infinitely many integers of the form np^ℓ . For example $n = 70$ is in E^* and for any prime p , say $p = 73$ and $70 < p < \sigma(70) + 1$, $n = 70(73)^k$ is also in E^* and hence there are infinitely many integers of the form $n = (70)(73)^k$, $k \geq 1$.

We established a program written in Fortran 77 , which can be used to produce any integer n in E^*. This can be done by considering the conditions satisfying n given in (11) above. The program also can be used to examine whether an integer n belongs to E^* or not by providing us with integers in $[1,\sigma(n)]$ which are not representable in terms of n .

The integers $n = 2.5.11$, $8.17.37$, $18.41.83$, $2^7.257.521$,...., does not belong to E^*, since there are more than two integers in $[1,\sigma(n)]$ which defy representation in terms of n .

The following program mentioned below is applied for the number $n = 8.17.19$, which has 16 divisors :

```
READ A(16) ,N
OPEN (10,FILE = 'DS2')
```

```
   DO 30 I = 1 , 5400
   L = 1
   M = 1
9  READ (10, * , END = 25)  (A(J), J = 1 , 16)
11 N = I
   DO 20 J = L , 16
   IF(N . GE .A(J)) THEN
   N = N - A(J)
   IF(N . LE . O .) GO TO 25
   IF((N .GE .1) AND . (J . EQ .16)) THEN
   REWID 10
   L = 1
   READ (10 , * , END = 25 )  (A(J) , J = 1 , 16)
   S = A(M)
   A(M) = A(M + 1)
   A(M + 1) = S
   M = M + 1
   IF(M . EQ . 15) THEN
   WRITE(* , 18)I
18 FORMAT (10X , 16 ,2X , 'IS NOT REPRESENTABLE ')
   GO TO 30
   END IF
   GO TO 11
   END IF
   END IF
20 CONTINUE
25 REWID 10
30 CONTINUE
```

```
      STOP
      END
```

Finally we conclude that the set $M^* = E^* \cup O^*$ represents the complete set of all integers n satisfying the condition

$$\sigma(n) - \alpha(n) = 2 \quad .$$

A further result related to the properties of the integers of E^* is given in the following ;

Theorem 1.4.7

For any integer $n = mp_1^{a_1} \ldots p_k^{a_k}$ in E^*

$$\frac{\sigma(n)}{n} \geq \frac{3}{2} \left[\frac{p_1 + 1}{p_1 - 1/2} \right] .$$

Proof

Suppose that $n = mp_1^{a_1} \ldots p_k^{a_k}$ is in E^*. Then by Theorem 1.4.3, $k \geq 2$, $a_j \geq 1$, $(1 \leq j \leq k)$ and $m \geq 2$ is a practical number such that

$$p_1 = \sigma(m) + 2 \quad ,$$

$$p_1 < p_2 \leq 2p_1 - 1 \ ,$$

and

$$p_{j+2} \leq \sigma(mp_1^{a_1} \ldots p_{j+1}^{a_{j+1}}) \quad , \quad (1 \leq j \leq k-2).$$

So , we have

$$\frac{\sigma(n)}{n} = \frac{\sigma(m)}{m} \cdot \frac{\sigma(p_1^{a_1} \ldots p_k^{a_k})}{p_1^{a_1} \ldots p_k^{a_k}}$$

i.e

$$\frac{\sigma(n)}{n} \geq \frac{\sigma(m)}{m} (1 + 1/p_1) (1 + 1/p_2) \quad . \qquad (13)$$

Since $m \geq 2$ is a practical number, from [4] we have

$$\sigma(m) \geq 2m - 1 \quad .$$

Therefore

$$\frac{\sigma(m)}{m} > (2 - \frac{1}{m}).$$

Since $m > 2$, then $(2 - \frac{1}{m}) > \frac{3}{2}$. Therefore (13) becomes

$$\frac{\sigma(n)}{n} > \frac{3}{2} (1 + 1/p_1) (1 + 1/ p_2) \quad . \qquad (14)$$

We have $p_1 < p_2 \leqslant 2p_1 - 1$.So

$$(1 + 1/p_2) > (1 + \frac{1}{2p_1 - 1}) \quad . \qquad (15)$$

Using (15) in (14) we get

$$\frac{\sigma(n)}{n} > \frac{3}{2} (1 + 1/p_1) (1 + \frac{1}{2p_1 - 1})$$

i.e

$$\frac{\sigma(n)}{n} > \frac{3}{2} \left[\frac{p_1 + 1}{p_1 - 1/2}\right] ,$$

which is the required result .

Further properties for the integers of E^* will be investigated in the next section.

§1.5 Some General Results On The t-Practical Numbers

In this section we shall prove some results on the t-practical numbers n having the form

$$n = mp_1{}^{a_1}\ldots p_k{}^{a_k} \quad , \quad k \geqslant 1 \ , \ a_j \geqslant 1 \ ,$$

where m is a practical number and p_j are distinct primes such that $(p_j,m) = 1$, and $(1 \leqslant j \leqslant k)$.

We shall prove the following result :

Theorem 1.5.1

Let n be an integer of the form

$$n = mp_1^{a_1} \dots p_k^{a_k} \ , \ k \geqslant 1 \ , \ a_j \geqslant 1 \ ,$$

where $p_1 < p_2 \leqslant 2p_1 - 1$, $(p_j, m) = 1$ for all j ,$(1 \leqslant j \leqslant k)$,and m is a practical number satisfying

$$p_1 \leqslant \sigma(m) + s \quad , \quad 1 \leqslant s \leqslant \sigma(m)$$

and

$$p_{j+2} \leqslant \sigma(mp_1^{a_1} \dots p_{j+1}^{a_{j+1}}) + 1 \quad , \ (1 \leqslant j \leqslant k-2).$$

Then n is either a practical number or is in A_2 .

Proof

Suppose that $n = mp_1^{a_1} \dots p_k^{a_k}$, $k \geqslant 1$, $a_j \geqslant 1$, $(p_j, m) = 1$ for all j , $(1 \leqslant j \leqslant k)$, $p_1 < p_2 < 2p_1$,and $m \geqslant 1$ is a practical number satisfying

$$p_1 \leqslant \sigma(m) + s \quad , \ 1 \leqslant s \leqslant \sigma(m) \ , \qquad (1)$$

and

$$p_{j+2} \leqslant \sigma(mp_1^{a_1} \dots p_{j+1}^{a_{j+1}}) + 1 \ , \ (1 \leqslant j \leqslant k-2). \qquad (2)$$

Since m a practical number every integer from 1 up to $\sigma(m)$ is of the form

$$\sum_{d \mid n} d \ .$$

If $s = 1$ in (1) , then $p_1 \leqslant \sigma(m) + 1$. So, by Theorem 1.1.1[2] $mp_1^{a_1}$ is a practical number. We have $p_1 < p_2 \leqslant 2p_1 - 1$,so

$$p_2 \leqslant 2p_1 - 1 \leqslant 2\sigma(m) + 1 \ .$$

Since $s =1$ and $p_1 = \sigma(m) + 1$, then $p_1 \geqslant 2$. Therefore

$$p_2 \leqslant 2\sigma(m) + 1 < \sigma(mp_1^{a_1}) + 1$$

i.e

$$p_2 \leqslant \sigma(mp_1^{a_1}) + 1 \quad . \qquad (3)$$

Therefore (3) implies that $mp_1^{a_1}p_2^{a_2}$ is a practical number by

Theorem 1.1.1[2] . So , from (2) and (3) we obtain

$$p_j \leqslant \sigma(mp_1^{a_1}\cdots p_{j-1}^{a_{j-1}}) + 1 \quad , (1\leqslant j\leqslant k), \qquad (4)$$

and by Theorem 1.1.1[2] (4) implies that n is a practical number .

If $1 < s \leqslant \sigma(m)$,then from (1) we have

$$p_1 = \sigma(m) + s$$

$$p_1 \leqslant 2\sigma(m)$$

i.e

$$p_1 \leqslant 2\sigma(m) + 1 \quad . \qquad (5)$$

So, by Lemma 1.1.4 , $mp_1^{a_1}$ belong to A_2 . Since $p_1 < p_2 \leqslant 2p_1 - 1$, then from (5) we get

$$p_2 \leqslant 2p_1 - 1 < 4\sigma(m) + 1$$

$$p_2 < 4\sigma(m) + 1 \quad .$$

Since $s > 1$ and $p_1 = \sigma(m) + s$, then $p_1 \geqslant 3$ and therefore

$$p_2 < 4\sigma(m) + 1 \leqslant \sigma(mp_1^{a_1}) + 1 \ , \ a_1 \geqslant 1 \ .$$

Therefore

$$p_2 < \sigma(mp_1^{a_1}) + 1 \quad . \qquad (6)$$

So, by (2) and (6) we get

$$p_{j+1} \leqslant \sigma(mp_1^{a_1}\cdots p_j^{a_j}) + 1 \ , \qquad (1\leqslant j\leqslant k-1) \ . \quad (7)$$

Thus, from (5) and (7) we conclude that n is in A_2 by Lemma 1.1.4, and this completes the proof .

In Theorem 1.5.1, take $s = 2$ and $m \geqslant 1$. Then all the integers n are in M^* where $M^* = E^* \cup O^*$ (see Theorem 1.4.3 and [2]). Therefore Theorem 1.5.1, implies that every integer in M^* belongs to A_2. The converse is not true since for instance $n = 10$ is in A_2 but 10 is not in M^* .

Now , we prove the following result ;

Theorem 1.5.2

Let $n \epsilon A_2$ be a non-practical number having the form

$$n = 2^a p_1^{a_1} \ldots p_r^{a_r} \quad , a \geqslant 1 \;,\; a_i \geqslant 1 \;, \qquad (8)$$

where p_i are primes such that $2 < p_1 < \ldots < p_r$. Then n^k , $k \geqslant 2$, is a practical number .

Proof

Let n be a non-practical number in A_2 having the form (8). Then there exists a prime p_i , $(1 \leqslant i \leqslant r)$, such that

$$p_i > \sigma(2^a p_1^{a_1} \cdots p_{i-1}^{a_{i-1}}) + 1 \quad . \qquad (9)$$

Otherwise $p_i \leqslant \sigma(2^a p_1^{a_1} \ldots p_{i-1}^{a_{i-1}}) + 1$ implies that n is a practical number giving a contradiction.So,if n is in A_2 then for any i,$(1 \leqslant i \leqslant r)$ the fact that p_i satisfies the condition

$$p_i \leqslant 2\sigma(2^a p_1^{a1} \ldots p_{i-1}^{a_{i-1}}) + 1 \quad , (1 \leqslant i \leqslant r), \qquad (10)$$

follows from Lemma 1.1.4 . From (9) and (10) we have

$$\sigma(2^a p_1^{a_1} \cdots p_{i-1}^{a_{i-1}}) + 1 < p_i \leqslant 2\sigma(2^a p_1^{a_1} \cdots p_{i-1}^{a_{i-1}}) + 1 \;.$$

Since

$$2\sigma(2^a p_1^{a_1} \ldots p_{i-1}^{a_{i-1}}) + 1 < \sigma(2^{ka} p_1^{ka_1} \ldots p_{i-1}^{ka_{i-1}}) + 1 \;,$$

where $k \geqslant 2$,then

$$p_i \leqslant 2\sigma(2^a p_1^{a_1} \cdots p_{i-1}^{a_{i-1}}) + 1 < \sigma(2^{ka} p_1^{ka_1} \ldots p_{i-1}^{ka_{i-1}}) + 1, \qquad (11)$$

and by Theorem 1.1.1[2], (11) implies that n^k ,$(k \geqslant 2)$, is a practical number which ends the proof .

Since all integers of E^* are even (see Theorem 1.4.3) and by Theorem 1.5.1,every integer of E^* is in A_2,it follows from Theorem 1.5.2 ,that for any integer $n \epsilon E^*$ and any $k \geqslant 2$, n^k is a practical number .

Theorem 1.5.3

Let n be an integer of the form

$$n = mp_1^{a_1}\dots p_k^{a_k} \ , \ k \geq 1 \ , \ a_j \geq 1 \ ,$$

where $(p_j, m) = 1$ for all $j, (1 \leq j \leq k)$, and $m > 1$ is a practical number satisfying $p_1 = \sigma(m) + s$, $\sigma(m) < s < p_1 - 1$,

and

$$p_{j+1} \leq \sigma(mp_1^{a_1}\dots p_j^{a_j}) + 1 \ , \quad (1 \leq j \leq k-1).$$

Then n either belongs to A_t with $t = (p_1 - 1)/2$ or to A_t with $t = p_1 - 1$.

Proof

Suppose that $n = mp_1^{a_1}\dots p_k^{a_k}$, $k \geq 1$, $a_j \geq 1$, $(p_j, m) = 1$, $(1 \leq j \leq k)$ and m is a practical number satisfying

$$p_1 = \sigma(m) + s \ , \quad \sigma(m) < s < p_1 - 1 \ , \tag{12}$$

and

$$p_{j+1} \leq \sigma(mp_1^{a_1}\dots p_j^{a_j}) + 1 \ , \ (1 \leq j \leq k-1) \ . \tag{13}$$

Since m is practical then any integer in the interval $[1, \sigma(m)]$ is of the form

$$\sum_{d|n} d \ . \tag{14}$$

It follows from (12) and (13) that the integers

$$\sigma(m) + 1 , \dots , \sigma(m) + (s - 1) \quad , \ (s>1),$$

and

$$\sigma(n) - (\sigma(m) + 1)), \dots, \sigma(n) - (\sigma(m) + (s - 1)) \ ,$$

defy representation by the form (14). Hence n is a non-practical number. Since $\sigma(m) < s < p_1 - 1$, then we can write s as

$$s = r(p_1 - 2) + \ell \quad , \tag{15}$$

where $0 \leqslant \ell \leqslant p_1 - 1$ and $0 \leqslant r < \sigma(m)$. So, if $\ell = 0$,(15) becomes

$$s = r(p_1 - 2) \quad ,$$

and since $r \leqslant \sigma(m) - 1$, then

$$s \leqslant (p_1 - 2)(\sigma(m) -1) \quad . \tag{16}$$

Since $p_1 = \sigma(m) + s$,from (16) we get

$$p_1 \leqslant \sigma(m) + (p_1 - 2)(\sigma(m) - 1)$$

$$p_1 \leqslant (p_1 - 1)\sigma(m) - p_1 + 2$$

i.e

$$2p_1 \leqslant (p_1 - 1)\sigma(m) + 2 \quad ,$$

and hence

$$p_1 \leqslant \left(\frac{p_1 - 1}{2}\right)\sigma(m) + 1 \quad . \tag{17}$$

Since $s > 1$, $m > 1$ and $p_1 = \sigma(m) + s$, then $p_1 \geqslant 5$. Therefore $(P_1 - 1)/2 \geqslant 1$. Put $t = (p_1 - 1)/2$.Then (17) becomes

$$p_1 \leqslant t\sigma(m) + 1 \quad . \tag{18}$$

So , by Lemma 1.1.4 , (18) implies that $mp_1{}^{a_1}$ $(a_1 \geqslant 1)$ belongs to A_t, where $t = (p_1 - 1)/2$. From (13) and (18) we may have

$$p_j \leqslant t\sigma(mp_1{}^{a_1}\cdots p_{j-1}{}^{a_{j-1}}) + 1 \quad , \quad (1 \leqslant j \leqslant k), \tag{19}$$

and by Lemma 1.1.4 , (19) implies that n belongs to A_t with $t = (p_1 - 1)/2$.

If $0 < \ell < p_1 - 1$, then $\ell \leqslant p_1 - 2$ and hence (15) becomes

$$s \leqslant (p_1 - 2)(\sigma(m) - 1) + (p_1 - 2)$$

i.e

$$s \leqslant (p_1 - 2)\sigma(m) \quad . \tag{20}$$

Since $p_1 = \sigma(m) + s$,from (20) we get

$$p_1 \leqslant \sigma(m) + (p_1 -2)\sigma(m)$$

$$p_1 \leqslant (p_1 - 1)\sigma(m) \ .$$

Therefore

$$p_1 < (p_1 - 1)\sigma(m) + 1 \quad . \qquad (21)$$

Thus, (21) implies that $mp_1^{a_1}$, $(a_1 \geq 1)$, is in A_t with $t = p_1 - 1$ (see Lemma 1.1.4). So, from (13) and (21) we obtain

$$p_j \leq t\sigma(mp_1^{a_1} \ldots p_{j-1}^{a_{j-1}}) + 1 \quad , (1 \leq j \leq k-1), \qquad (22)$$

where $t = p_1 - 1$. Therefore (22) implies that n belongs to A_t by Lemma 1.1.4 and this completes the proof .

Theorem 1.5.4

Let n be an integer having the form

$$n = mp_1^{a_1} \ldots p_k^{a_k} \ , \quad a_j \geq 1 \ , \ k \geq 1 \ ,$$

where $(p_j, m) = 1$, $p_j < p_{j+1} \leq 2p_j - 1$ for any j , $(1 \leq j \leq k)$, and $m \geq 2$ is a practical number satisfying

$$p_1 = \sigma(m) + s \ , \quad 1 < s \leq \sigma(m) \ . \qquad (23)$$

Then n is a sum of two practical numbers .

Proof

Suppose that $n = mp_1^{a_1} \ldots p_k^{a_k}$, $k \geq 1$, $a_j \geq 1$, $(p_j, m) = 1$ $(1 \leq j \leq k)$, with $m \geq 2$ being the practical part of n satisfying $p_1 = \sigma(m) + s$, $1 < s \leq \sigma(m)$ and

$$p_j < p_{j+1} \leq 2p_j - 1 \quad . \qquad (24)$$

Then n is an even integer since any practical number $m > 1$ is even (see [2]). It follows from (23) that there are some integers in the interval $[1,n]$ such as $\sigma(m) + 1, \ldots, \sigma(m) + (s - 1)$, $(s>1)$, which cannot be represented by the form

$$\sum_{d \mid n} d \quad . \qquad (25)$$

Hence n is a non-practical number. Using condition (23) we can write n as

$$n = m(\sigma(m) + s)p_1^{a_1 - 1} . p_2^{a_2} \ldots p_k^{a_k} \quad , \quad a_1 - 1 \geq 0 \ ,$$

$$n = m\sigma(m)p_1^{a_1-1}.p_2^{a_2}...p_k^{a_k} + msp_1^{a_1-1}.p_2^{a_2}...p_k^{a_k} .$$

Let $n_1 = m\sigma(m)p_1^{a_1-1}.p_2^{a_2}...p_k^{a_k}$ and $n_2 = msp_1^{a_1-1}.p_2^{a_2}...p_k^{a_k}$. To complete the proof we will show that both n_1 and n_2 are practical numbers. For n_1 we write $\sigma(m) = p_1 - s$. Therefore n_1 will be of the form

$$n_1 = m(p_1 - s)p_1^{a_1-1}.p_2^{a_2}...p_k^{a_k} .$$

Since

$$p_1 - s < \sigma(m) + 1 , \qquad (26)$$

then by Corollary 1.1.1, $m(p_1 - s)$ is a practical number. Since $1 < s \leqslant \sigma(m)$ then from (26) we get

$$p_1 < \sigma(m) + s + 1 ,$$

where $s \leqslant \sigma(m)$. Therefore

$$p_1 < 2\sigma(m) + 1 . \qquad (27)$$

We have $m \geqslant 2$ and $p_1 - s = \sigma(m)$, where $\sigma(m) \geqslant 3$. Then from (27) we obtain

$$p_1 \leqslant 2\sigma(m) < \sigma(m(p_1 - s)) + 1 .$$

Hence $m(p_1 - s)p_1^{a_1}$ is a practical number by Theorem 1.1.1[2]. Since $m(p_1 - s)$ is practical it follows that $m(p_1 - s) p_1^{a_1-1}$ is also a practical number. From (24) we have $p_1 < p_{j+1} \leqslant 2p_1 - 1$. Then by (27) we get

$$p_2 \leqslant 4\sigma(m) - 1 < \sigma(m(p_1 - s)) , \qquad (28)$$

where $(p_1 - s) \geqslant 3$. Therefore $m(p_1 - s)p_2^{a_2}$, $(a_2 \geqslant 1)$, is a practical number by Theorem 1.1.1[2]. Hence $m(p_1 - s)p_1^{a_1-1}.p_2^{a_2}$ is a practical number . We have $p_j < p_{j+1} \leqslant 2p_j - 1$. Then by (28) and for any j , we can get

$$p_{j+1} \leqslant 2p_j - 1 < \sigma(m(p_1 - s)p_1^{a_1-1}.p_2^{a_2}...p_j^{a_j}) + 1, \ (1\leqslant j\leqslant k-1),$$

and this last inequality implies that n_1 is a practical number (see Theorem 1.1.1[2]).

Consider $n_2 = msp_1^{a_1-1}.p_2^{a_2}\ldots p_k^{a_k}$. Then since $1 < s \leqslant \sigma(m)$ we shall have

$$s < \sigma(m) + 1 \quad . \qquad (29)$$

Hence ms is a practical number (see Corollary 1.1.1).It follows from (29) that

$$s = p_1 - \sigma(m) < \sigma(m) + 1 \quad .$$

Therefore

$$p_1 < 2\sigma(m) + 1 < \sigma(ms) + 1 \quad . \qquad (30)$$

Thus, $msp_1^{a_1}$ is a practical number by Theorem 1.1.1[2]. Hence $msp_1^{a_1-1}$,$(a_1-1) \geqslant 0$,is also a practical number since ms is practical.Since $p_j < p_{j+1} \leqslant 2p_j - 1$,using (30),for any j we obtain

$$p_{j+1} \leqslant 2p_j - 1 < \sigma(msp_1^{a_1-1}.p_2^{a_2}\ldots p_j^{a_j}) + 1 \text{ , } (1\leqslant j\leqslant k-1).$$

Therefore n_2 is a practical number (see Theorem 1.1.1[2]). Hence n is a sum of two practical numbers and this completes the proof .

Taking s = 2 in Theorem 1.5.4. Then by Theorem 1.4.3, n in E^*. Therefore Theorem 1.5.4, shows that any integer in E^* which satisfy (24) is a sum of two practical numbers .

§1.6 Results On The Condition $\sigma(n) - \alpha(n) = 1$

We denote by F the set of all positive integers n satisfying the following condition

$$\sigma(n) - \alpha(n) = 1 \quad , \qquad (1)$$

where $\sigma(n)$ represents the sum of all positive divisors of n,and $\alpha(n)$ is the number of all positive integers in $[1,\sigma(n)]$ which can be written in the form

$$\sum_{d|n} a_d d \quad , \qquad a_d = 0 \text{ , } 1 \quad . \qquad (2)$$

We shall prove the following ;

Lemma 1.6.1

Let n be a positive integer. If $\sigma(n) - \alpha(n) = 1$, then $\sigma(n)/2$ is the only integer which defies representation with respect to n and $\sigma(n)$ is an even integer .

Proof

Let $n > 1$ be such that $\sigma(n) - \alpha(n) = 1$. Then there exists only one integer , say m, which defies representation in terms of n and

$$1 < m < \sigma(n) \quad .$$

Therefore $m = \sigma(n) - m$, because n belongs to F . Hence $\sigma(n) = 2m$ which implies that $\frac{\sigma(n)}{2} = m$. Since m is a positive integer , then $\sigma(n)$ must be an even integer . If $m \neq \sigma(n) - m$,then we shall get that both m and $\sigma(n) - m$ defy representation in terms of n which contradicts our assumption that n is in F . Therefore our result will follow and this will end the proof .

Remark 1.6.1

If n belongs to F, then $\sigma(n) \leqslant 2(n -1)$.

Proof

Suppose that n is in F.Then

$$\sigma(n) - \alpha(n) = 1 \ .$$

From Lemma 1.6.1 , we have that $\sigma(n)/2$ is the only integer which can not be represented as a sum of distinct positive divisors of n and $\sigma(n)$is an even integer. Therefore

$$\frac{\sigma(n)}{2} < n$$

i.e

$$\sigma(n) < 2n \quad .$$

For if n belongs to F with $\sigma(n)/2 \geqslant n$, then either $\sigma(n) = 2n$ which means that n is a perfect number or $\sigma(n)/2 > n$ which shows that all integers from 1 up to n can be represented in terms of n . So , both these cases implies that n is a practical number see [1]. This will contradict our assumption that n is in F. Thus ,

$$\frac{\sigma(n)}{2} < n$$

$$\frac{\sigma(n)}{2} \leqslant n - 1$$

i.e

$$\sigma(n) \leqslant 2(n - 1) \quad ,$$

which is the required inequality .

An obvious example for $n \epsilon F$ is $n = 3$.

Now , we have the following result ;

Theorem 1.6.1

The only integer in F is $n = 3$

Proof

Let n be in F. Then ,

$$\sigma(n) - \alpha(n) = 1 \quad .$$

Suppose that $n > 3$ is any odd integer . It follows that 2 , $n - 2$ and $\sigma(n) - 2$ in $[1,\sigma(n)]$ defy representation in terms of n . Therefore any odd integer $n > 3$ does not belong to F .

When n is even , then either n is a practical number or n is an even non-practical number . Since $n \epsilon F$ we suppose that n is an even non-practical number . So , we can write n in the form

$$n = mp_1^{a_1} \ldots p_r^{a_r} \quad , \ a_i \geqslant 1 \ ,$$

where $m > 1$ represents the practical part of n , p_i are primes such that $2 < p_1 < p_2 < \ldots < p_r$ and $(p_i, m) = 1$ for all i $(1 \leqslant i \leqslant r)$. Since n is not a practical number, we have,

$$p_1 > \sigma(m) + 1 \quad .$$

Otherwise $mp_1^{a_1}$ is a practical number . Therefore $\sigma(m) + 1$ defies representation in terms of n. We have that if n is in F, then $\sigma(m) + 1$ is the only integer in $[1, \sigma(n)]$ which defies representation in terms of n i.e

$$\frac{\sigma(n)}{2} = \sigma(m) + 1$$

$$\sigma(n) = 2\sigma(m) + 2 \quad . \qquad (3)$$

But

$$\sigma(n) \geqslant \sigma(mp_1) = (p_1 + 1)\sigma(m) \quad , \qquad (4)$$

and from (3) and (4) we get

$$\sigma(n) = 2\sigma(m) + 2 \geqslant (p_1 + 1)\sigma(m)$$

$$2\sigma(m) - \sigma(m) + 2 \geqslant p_1\sigma(m)$$

$$p_1\sigma(m) - \sigma(m) \leqslant 2 \quad ,$$

i.e

$$(p_1 - 1)\sigma(m) \leqslant 2 \quad . \qquad (5)$$

From (5) we have that either $p_1 = 3$ or $\sigma(m) = 1$. So , by (3) we obtain that $\sigma(n) = 4$ and $n = 3$. Therefore $n = 3$ is the only integer in F and the proof is complete.

§1.7 Practical Numbers Over An Imaginary Quadratic Field

Consider the Euclidean field $Z[\theta], \theta = \sqrt{d}$, $d < 0$ and $d \neq -1,-3$. We shall discuss in this section some results related to the practical numbers over the field $Z[\theta]$. So,we introduce the following definition.

Definition 1.7.1

An integer α in $Z[\sqrt{d}]$ is called a practical number if every integer $\gamma \epsilon Z[\sqrt{d}]$ with $|N(\gamma)| \leqslant |N(\alpha)|$ is of the form

$$\gamma = \sum_{\delta | \alpha} a_\delta \delta , \qquad a_\delta = 0 , 1 . \tag{1}$$

We shall prove the following result:

Theorem 1.7.1

If $\alpha = (\sqrt{d})^k$, $k \geqslant 1$ is an integer in $Z[\sqrt{d}]$ such that $|d|$ is an ordinary practical number,then α is a practical number in $Z[\sqrt{d}]$.

Proof

Let $\alpha = (\sqrt{d})^k$ for $k \geqslant 1$, $d < 0$ and $|d|$ a practical number. Suppose that $\gamma = m + n\sqrt{d}$ is any non-zero integer in $Z[\sqrt{d}]$ such that $|N(\gamma)| \leqslant |N(\alpha)|$. Since $|N(\alpha)| = |d|^k$, $k \geqslant 1$ then $m^2 + n^2|d| \leqslant |d|^k$. Therefore

$$\begin{aligned} m &\leqslant |d|^{k/2} \\ n &\leqslant |d|^{(k-1)/2} . \end{aligned} \tag{2}$$

Now , we shall consider the following cases :

(i) If k ,($k \geqslant 1$), is even , then $\frac{k}{2} = h$,$h \geqslant 1$ and $\frac{(k-1)}{2} = h - \frac{1}{2}$.

Since $|d|$ is a practical number then so does $|d|^h$ and $|d|^{[h - \frac{1}{2}]}$, where $|d|^{[h-\frac{1}{2}]} = |d|^{h-1}$ and $h-1 \geq 0$. Since $m \leq |d|^h$ we can write m as a sum of distinct divisors of $|d|^h$ i.e

$$m = c_0 + c_1|d| + c_2|d|^2 + \dots + c_h|d|^h \ , \qquad (3)$$

where $c_i = 0,1$ for all i , $(0 \leq i \leq h)$. We have also

$$n \leq |d|^{h-\frac{1}{2}} < |d|^h \ .$$

Then n can be represented as a sum of distinct divisors of $|d|^h$ i.e

$$n = e_0 + e_1|d| + e_2|d|^2 + \dots + e_h|d|^h \ , \qquad (4)$$

where $e_j = 0,1$ for all j , $(0 \leq j \leq h)$. But $e_h|d|^h \leq n \leq |d|^{h-\frac{1}{2}}$ follows from (2) and (4) . Therefore $e_h \leq |d|^{-1/2} < 1$ since $|d| > 1$. Hence $e_h = 0$. Thus , (4) becomes

$$n = e_0 + e_1|d| + e_2|d|^2 + \dots + e_{h-1}|d|^{h-1}, \ (5)$$

and from (5) we get

$$n\sqrt{d} = e_0\sqrt{d} + e_1|d|\sqrt{d} + e_2|d|^2\sqrt{d} + \dots + e_{h-1}|d|^{h-1}\sqrt{d}. \ (6)$$

By combining (3) and (6) we obtain

$$\gamma = (c_0 + c_1|d| + \dots + c_h|d|^h) + (e_0 + e_1|d|\sqrt{d} + \dots + e_{h-1}|d|^{h-1}\sqrt{d}), (7)$$

where in (7) all $c_i|d|^i$ and $e_j|d|^j\sqrt{d}$ are distinct divisors of $(\sqrt{d})^k$ for all i and j , $(0 \leq i \leq h)$ and $(0 \leq j \leq h-1)$. Hence γ is a sum of distinct divisors of $(\sqrt{d})^k$ i.e α is a practical number in $Z[\sqrt{d}]$.

(ii) If k , $(k > 1)$, is odd then $\frac{(k-1)}{2} = r$, $r \geq 0$ and $\frac{k}{2} = r + \frac{1}{2}$.

Therefore ,since $|d|$ is an ordinary practical number,then $|d|^r$ and $|d|^{[r+\frac{1}{2}]}$ are practical numbers , where $|d|^{[r+\frac{1}{2}]} = |d|^r$.

We have $m \leqslant |d|^{r+\frac{1}{2}} \leqslant |d|^{r+1}$, where $r+1 \geqslant 1$. Then we can represent m as a sum of distinct divisors of $|d|^{r+1}$ i.e

$$m = c_0 + c_1|d| + c_2|d|^2+\dots+ c_{r+1}|d|^{r+1}. \quad (8)$$

where $c_i = 0,1$. So , from (2) and (8) we get

$$c_{r+1}|d|^{r+1} \leqslant n \leqslant |d|^{r+\frac{1}{2}} .$$

Therefore $c_{r+1} \leqslant |d|^{-1/2} < 1$ since $|d| > 1$. Hence $c_{r+1} = 0$. So , (8) becomes

$$m = c_0 + c_1|d| + c_2|d|^2+\dots+ c_r|d|^r . \quad (9)$$

We have $n \leqslant |d|^{(k-1)/2} = |d|^r$. Since $|d|^r$ is an ordinary practical number we can represent n as a sum of distinct divisors of $|d|^r$ i.e

$$n = e_0 + e_1|d| + e_2|d|^2+\dots+ e_r|d|^r ,$$

where $e_j = 0,1$. Therefore

$$n\sqrt{d} = e_0 + e_1|d|\sqrt{d} + e_2|d|^2\sqrt{d} +\dots+ e_r|d|^r\sqrt{d} . \quad (10)$$

From (9) and (10) we obtain

$$\gamma = (c_0 + c_1|d|+\dots+ c_r|d|^r)+(e_0 + e_1|d|\sqrt{d} +\dots+ e_r|d|^r\sqrt{d}) \ ,(11)$$

where in (11) all $c_i|d|^i$ and $e_j|d|^j$ are distinct divisors of $(\sqrt{d})^k$ for all i and j , $(0\leqslant i\leqslant r)$ and $(0\leqslant j\leqslant r)$. Therefore γ is a sum of distinct divisors of $(\sqrt{d})^k$. Hence α is a practical number in $Z[\sqrt{d}]$ and this completes the proof .

As an immediate consequence of Theorem 1.7.1 , $(\sqrt{-2})^k$, $k \geqslant 1$, is a practical number in $Z[\sqrt{-2}]$, where $Z[\sqrt{-2}]$ is a Euclidean field .

Now , we prove the following result:

Lemma 1.7.1

Let $Z[\sqrt{d}]$, $d < 0$ and $d \neq -1$, -3 . Then $\sqrt{d}$ is a practical number in $Z[\sqrt{d}]$ if and only if $d = -2$.

Proof

Let $\sqrt{d}$, $d < 0$, $d \neq -1$, -3 be a practical number in $Z[\sqrt{d}]$.Then, by definition 1.7.1 , every integer $\gamma \epsilon Z[\sqrt{d}]$ with $|N(\gamma)| \leqslant |d|$ is a sum of distinct divisors of $\sqrt{d}$.

Suppose that $d < -2$.Then $|N(\sqrt{d})| = |d| \geqslant 5$ since $d \neq -3$.So,for $|d| \geqslant 5$ there exists a rational integer m satisfying $1 < m \leqslant [\sqrt{|d|}]$. Thus,$1 < N(m) \leqslant |d|$. Put $\gamma = m$. Then γ cannot be written as a sum of distinct divisors of $\sqrt{d}$ since the divisors of $\sqrt{d}$ are only 1 and $\sqrt{d}$. Therefore $\sqrt{d}$ is not a practical number when $d < -2$. Since $\sqrt{-2}$ is a practical number in $Z[\sqrt{-2}]$,it follows that $d = -2$.

Conversely , if $d = -2$ then $\sqrt{-2}$ is practical in $Z[\sqrt{-2}]$ by Theorem 1.7.1 and this ends the proof .

It follows from Lemma 1.7.1 , that the numbers $\sqrt{-5}$, $\sqrt{-6}$, $\sqrt{-7}$, $\sqrt{-10}$, $\sqrt{-30}$ and $\sqrt{-66}$ are non-practical numbers in the fields $Z[\sqrt{-5}]$, $Z[\sqrt{-6}]$, $Z[\sqrt{-7}]$, $Z[\sqrt{-10}]$, $Z[\sqrt{-30}]$ and $Z[\sqrt{-66}]$ respectively .

Further , if $\alpha = (\sqrt{d})^k$, $k \geqslant 1$, is the practical number which is given by Theorem 1.7.1 and $\sigma(\alpha)$ is the sum of the divisors of α,then for some k , $k \geqslant 1$,not every integer $\gamma \epsilon Z[\sqrt{d}]$ with $|N(\gamma)| \leqslant |N(\sigma(\alpha))|$ can be written as a sum of distinct divisors of α since,for instance, if $\alpha = (\sqrt{-2})^3$ then $\sigma(\alpha) = 3 + 3\sqrt{-2}$. Hence $|N(\sigma(\alpha))| = 27$ and it follows that 4 and 5 cannot be written as a sum of distinct divisors of $(\sqrt{-2})^3$.

We attempted to take $\mathbb{Z}[\sqrt{-2}]$ which is a Euclidean field and $(\sqrt{-2})^k$ $(k > 1)$, which is a practical number in $\mathbb{Z}[\sqrt{-2}]$ and prove a result similar to that given by Stewart in [2] , namely

" Let $\ell \in \mathbb{N}$ and π be any prime in $\mathbb{Z}\sqrt{-2}]$ with $|N(\pi)| \leqslant 2^\ell$. Then $\pi(\sqrt{-2})^\ell$ is a practical number in $\mathbb{Z}[\sqrt{-2}]$."

However , the above statement is false since , for instance , if $\alpha = (\sqrt{-2})^5$ and $\pi = 1 + 3\sqrt{-2}$, then $\pi\alpha$ is not a practical number in $\mathbb{Z}[\sqrt{-2}]$, where $|N(\pi\alpha)| = 608$. This is because there are some integers such as 10 , 12 , 13 and 14 in $\mathbb{Z}[\sqrt{-2}]$ which are of norms less than or equal to 608 and which cannot be written as sums of distinct divisors of $\pi\alpha$.

We have the following result :

Theorem 1.7.2

There are infinitely many practical numbers in $\mathbb{Z}[\sqrt{-2}]$.

Of course we mean the practical in $\mathbb{Z}[\sqrt{-2}]$ as oppose the ordinary practical numbers in $\mathbb{N}$.

Theorem 1.7.2 ,follows since $\sqrt{-2}$ is a practical number in $\mathbb{Z}[\sqrt{-2}]$ and the powers of $\sqrt{-2}$ are also practical number in $\mathbb{Z}[\sqrt{-2}]$ (see Theorem 1.7.1) .

CHAPTER (2)

§2.1 Boundary Function

Define $f_t(n), t \in \mathbb{N}$, to be the largest integer for which every integer from 1 up to $f_t(n)$ can be represented as a sum of divisors of n with at most t repetitions. Therefore $f_t(n) \leqslant t\sigma(n)$. It follows from Theorem 1.1.6, in section 1.1, that $f_t(n) = t\sigma(n)$ if and only if n is a t-practical number. Also from Theorem 1.1.5, we have that $f_t(n) = t\sigma(n)$ if and only if for $n = p_1^{a_1} \cdots p_k^{a_k} . p_{k+1}^{a_{k+1}} \cdots p_\ell^{a_\ell}$, that

$$p_i \leqslant t\sigma(n_{i-1}) + 1 \quad , \qquad (1 \leqslant i \leqslant \ell - 1) \;,(1)$$

where $p_1 < p_2 < \ldots < p_k \leqslant t + 1 < p_{k+1} < \ldots < p_\ell$ and $n_{i-1} = p_1^{a_1} \cdot\cdot p_k^{a_k} \cdot p_{k+1}^{a_{k+1}} \cdots p_{i-1}^{a_{i-1}}$. We shall define U and L to be the upper and lower classes such that

$U = \{\psi(n) : f_t(n) > \psi(n)$ implies $f_t(n) = t\sigma(n)$ for all n and $\psi(n) \longrightarrow \infty$ as $n \to \infty\}$

and

$L = \{\psi(n) : f_t(n) > \psi(n)$ and $f_t(n) < t\sigma(n)$ for infinitely many n$\}$,

respectively. It follows from Theorem 1.1.5, that both $\psi(n) = t\sigma(n)$ and $\psi(n) = tn$ belong to U for any fixed $t, t \in \mathbb{N}$.

We seek to determine the boundary function between U and L. Initially we may assert that such function could be of the form

$$B_t(n) = e^{\gamma/2} \, (tn\log\log n)^{1/2} \qquad ,$$

where γ is Euler's constant.

For t = 1 i.e when n is a practical number, then

$B_1(n) = B^*(n) = e^{\gamma/2}$ (nloglogn). In [7] M.Hausman and H.N Shapiro proved the following results :

Theorem 2.1.1[7]

For $\lambda > 1$, $\lambda B^*(n)$ belongs to U .

Theorem 2.1.2[7]

For $\lambda < 1$, $\lambda B^*(n)$ belongs to L .

It can be shown that both Theorem 2.1.1 and 2.1.2, are applicable when n is a t-practical number and $B_t(n) = e^{\gamma/2}\,(tn\log\log n)^{1/2}$ for all t, $t \geqslant 1$. We shall generalize these results for the t-practical numbers with better upper and lower bounds of λ respectively. So, for this purpose the following results will be needed.

Theorem 2.1.3[8]

For all integers $n > 1$,

$$c < \frac{\sigma(n)\varphi(n)}{n^2} < 1 \quad ,$$

with $c = 6/\pi^2$.

Proof

We have that

$$\sigma(n) = \prod_{p|n} \frac{p^{a+1} - 1}{p - 1} = n \prod_{p|n} \frac{1 - p^{-a-1}}{1 - p^{-1}} \quad ,$$

and

$$\varphi(n) = n \prod_{p|n}(1 - 1/p) \quad .$$

Therefore

$$\frac{\sigma(n)\varphi(n)}{n^2} = \prod_{p|n}(1 - 1/p^{a+1}) \quad ,$$

where $\prod_{p|n}(1 - 1/p^{a+1})$ lies between 1 and $\prod_{p|n}(1 - 1/p)$. Hence

$$\frac{6}{\pi^2} < \frac{\sigma(n).\varphi(n)}{n^2} < 1 \quad ,$$

which is the required result .

Theorem 2.1.4

Let $n = \prod_{1 \leqslant i \leqslant r} p_i^{a_i}$, $a_i \geqslant 1$ and p_i are primes . Then

$$\sigma(n) < e^{\gamma} n \log\log n \left[1 + \frac{3}{2(\log\log n)^2}\right] .$$

Proof

We have from Theorem 2.1.3

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} . \qquad (2)$$

It follows from Rosser and Schoenfeld's results in [9] that

$$\frac{n}{\varphi(n)} < e^{\gamma}\log\log n + \frac{5}{2\log\log n} , \quad n \geqslant 3 .$$

Then (2) yields

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^{\gamma}\log\log n + \frac{5}{2\log\log n} .$$

Since γ is Euler's constant, then $e^{\gamma} = 1.7810725$. So, we shall have

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n \left[1 + \frac{5}{2e^{\gamma}(\log\log n)^2}\right]$$

i.e

$$\sigma(n) < e^{\gamma} n \log\log n \left[1 + \frac{3}{2(\log\log n)^2}\right] ,$$

which is the required result .

Now, we prove the following ;

Theorem 2.1.5

Let $\lambda(n)$ be such that $\lambda(n) > 1 + \dfrac{c}{(\log\log n)^2}$ for $c \geqslant \dfrac{5}{4e^{\gamma}}$ and $n \geqslant 3$. Then for any t , $t \in \mathbb{N}$, $\lambda(n)B_t(n)$ belongs to U .

Proof

It suffices to show that for $\lambda(n) > 1 + c/(\log\log n)^2$, $f_t(n) \geqslant \lambda(n)B_t(n)$ implies that

$$p_i \leqslant t\sigma(n_{i-1}) + 1 \quad ,$$

for the smallest i, where $n = p_1^{a_1}\cdots p_\ell^{a_\ell}$, $n_{i-1} = p_1^{a_1}\cdots p_{i-1}^{a_{i-1}}$ and $(1 \leqslant i \leqslant \ell)$.

Suppose that

$$p_i > t\sigma(n_{i-1}) + 1 \quad .$$

Then this mean that $t\sigma(n_{i-1}) + 1$ cannot be expressed as a t-practical number for any fixed $t, t \in \mathbb{N}$. So,

$$\lambda(n)B_t(n) \leqslant f_t(n) < t\sigma(n_{i-1}) + 1 \leqslant p_i - 1 \ , \qquad (3)$$

for any fixed $t, t \in \mathbb{N}$. From (3) we have

$$p_i > \lambda(n)B_t(n) + 1 \quad ,$$

so that

$$\frac{p_i}{\lambda(n)B_t(n)} > 1 + \frac{1}{\lambda(n)B_t(n)}$$

and

$$\frac{np_i}{\lambda(n)B_t(n)} > n(1 + \frac{1}{\lambda(n)B_t(n)}) > n \ . \qquad (4)$$

Therefore,

$$\frac{n}{p_i} = p_1^{a_1}\cdots p_i^{a_i - 1}\cdots p_\ell^{a_\ell} < \frac{n}{\lambda(n)B_t(n)} \quad . \qquad (5)$$

From (5) the following cases arise :

(i) If $i < \ell$ or $i = \ell$ and $a_i > 1$, then (5) implies that

$$p_i < \frac{n}{\lambda(n)B_t(n)} \quad . \qquad (6)$$

So, from (4) and (6) we get

$$n < \left[\frac{n}{\lambda(n)B_t(n)}\right]^2 = \frac{n^2}{\lambda^2(n)B_t^2(n)}$$

$$< \frac{n^2}{(1 + c/(\log\log n)^2)^2 \ . \ e^{\gamma} tn \log\log n} \ .$$

Therefore

$$n < \frac{n}{e^{\gamma} t \log\log n \, \{1 + c/(\log\log n)^2\}^2} \quad , \qquad (7)$$

and (7) represents a contradiction for large n and $c > 5/4e^{\gamma}$.

(ii) If $i = \ell$ and $a_\ell = 1$, then (5) implies that

$$n_{\ell-1} < \frac{n}{\lambda(n) B_t(n)} \quad . \qquad (8)$$

Put $i = \ell$ in (3) to get

$$\lambda(n) B_t(n) \leqslant f_t(n) \leqslant t\sigma(n_{\ell-1})$$

so that

$$1 \leqslant \frac{t\sigma(n_{\ell-1})}{\lambda(n) B_t(n)}$$

and

$$1 \leqslant \frac{t\sigma(n_{\ell-1})}{n_{\ell-1} \cdot p_\ell} \cdot \frac{n}{\lambda(n) B_t(n)} \; .$$

Therefore

$$p_\ell \leqslant \frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} \cdot \frac{n}{\lambda(n) B_t(n)} \quad . \qquad (9)$$

By multiplying (8) and (9) we get

$$n < \frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} \cdot \frac{n^2}{\lambda^2(n) B_t^2(n)} \quad .$$

Therefore ,

$$\frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} > \frac{\lambda^2(n) B_t^2(n)}{n} > \frac{\{1 + c/(\log\log n)^2\}^2 \cdot e^{\gamma} t n \log\log n}{n} .$$

Hence

$$\frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} > e^{\gamma} t \log\log n \left[1 + \frac{c}{(\log\log n)^2}\right]^2$$

$$\frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} > e^{\gamma} t \log\log n \left[1 + \frac{2c}{(\log\log n)^2} + \frac{c^2}{(\log\log n)^4}\right] \quad . \qquad (10)$$

It follows from Theorem 2.1.4 , that

$$\frac{t\sigma(n)}{n} < te^{\gamma}\log\log n\left[1 + \frac{3}{2(\log\log n)^2}\right] . \qquad (11)$$

Then by considering $c > \dfrac{5}{4e^{\gamma}}$ and comparing (10) and (11) , we shall get a contradiction . Hence $\lambda(n)B_t(n)$ belongs to U which completes the proof.

Theorem 2.1.6

Let $\lambda(n)$ be such that $\lambda(n) < 1 - \dfrac{c\log\log\log n}{(\log\log n)^2}$ for any $c > 2$.

Then for any t , $t\epsilon\mathbb{N}$, $\lambda(n)B_t(n)$ belongs to L .

Proof

It is sufficient to prove that there exists an infinite sequence of integers n such that

$$f_t(n) > \lambda(n)B_t(n) \qquad (12)$$

and

$$f_t(n) < t\sigma(n) \quad ,$$

for any fixed t, $t > 1$. Let $n = n_{\ell-1}.p_\ell$ such that

$$n_{\ell-1} = (\prod_{p\leqslant x} p)^{\beta} \qquad (13)$$

and

$$\beta = \frac{\log^2 x}{2\log\log x} \quad , \qquad (14)$$

where x is a positive real number and p_ℓ is a prime chosen later .

By writing

$$\frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} = t\prod_{p\leqslant x}\frac{1 - 1/p^{\beta+1}}{1 - 1/p} < t\prod_{p\leqslant x}\frac{1}{1 - 1/p} \quad ,(15)$$

then using Rosser and Schoenfeld's bounds in [9],(15) becomes

$$t\sigma(n_{\ell-1}) < tn_{\ell-1}\, e^{\gamma}\log x\,(1 + 1/\log^2 x) \quad , \qquad (16)$$

where γ is Euler's constant . Also we can write

$$\frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} = t \prod_{p} (1 - 1/p^{\beta+1}) \cdot \prod_{p>x} \frac{1}{1 - 1/p} \cdot \prod_{p\leq x} \frac{1}{1 - 1/p} .$$

We get

$$t\sigma(n_{\ell-1}) = \frac{t}{\zeta(\beta + 1)} \cdot \prod_{p>x} \frac{1}{1 - 1/p^{\beta+1}} \cdot \prod_{p\leq x} \frac{1}{1 - 1/p} , \qquad (17)$$

where $\zeta(\beta+1)$ is the Zeta-function. Since $\zeta(\beta + 1) \leq \dfrac{\beta + 1}{\beta}$, then

$$\frac{1}{\zeta(\beta + 1)} \geq \frac{\beta}{\beta + 1} .$$

Therefore (17) can be written as

$$\frac{t\sigma(n_{\ell-1})}{n_{\ell-1}} > \frac{t}{1 + 1/\beta} \cdot \prod_{p\leq x} \frac{1}{1 - 1/p}$$

It follows from Rosser and Schoenfeld's results in [9] that

$$t\sigma(n_{\ell-1}) > t\, n_{\ell-1}\, e^{\gamma} \log x\, (1 - 1/2\log^2 x)(1 - 1/\beta) .$$

Since $\beta = \dfrac{\log^2 x}{2\log\log x}$, then

$$t\sigma(n_{\ell-1}) > t\, n_{\ell-1}\, e^{\gamma} \log x\, (1 - 1/2\log^2 x)\left(1 - \frac{2\log\log x}{\log^2 x}\right) .$$

Therefore

$$t\sigma(n_{\ell-1}) > t\, n_{\ell-1}\, e^{\gamma} \log x \left(1 - \frac{3\log\log x}{\log^2 x}\right) . \qquad (18)$$

So, from (16) and (18) we obtain

$$t n_{\ell-1}\, e^{\gamma} \log x \left(1 - \frac{3\log\log x}{\log^2 x}\right) < t\sigma(n_{\ell-1}) < t n_{\ell-1}\, e^{\gamma} \log x (1 + 1/\log^2 x) .$$

Now, we choose p_{ℓ} is the smallest prime such that

$$p_{\ell} > t n_{\ell-1}\, e^{\gamma} \log x \left(1 + \frac{2}{\log^2 x}\right) . \qquad (19)$$

From the inequality (16) the R.H.S of (19) is greater than $t\sigma(n_{\ell-1}) + 1$. So, in particular $p_{\ell} > t\sigma(n_{\ell-1}) + 1$. Thus we get

$$f_t(n) = f_t(n_{\ell-1}) = t\sigma(n_{\ell-1}) \quad .$$

Therefore (12) and (16) implies that

$$\lambda(n)B_t(n) \leqslant f_t(n) = t\sigma(n_{\ell-1}) < tn_{\ell-1}\, e^{\gamma} \log x\,(1 + 1/\log^2 x) \;. \quad (20)$$

Since $B_t(n) = e^{\gamma/2}(tn\, \log\log n)^{1/2}$, then from (20) we get

$$\lambda^2(n)\; e^{\gamma}\; tn\, \log\log n < t^2\, n^2_{\ell-1}\, e^{2\gamma}\; \log^2 x\,(1 + 1/\log^2 x)^2 \;,$$

so that

$$1 < \frac{tn_{\ell-1}\, e^{\gamma} \log^2 x\,(1 + 1/\log^2)^2}{\lambda^2(n).p_\ell\, \log\log n}$$

and

$$p_\ell < \frac{tn_{\ell-1}\, e^{\gamma} \log^2 x\,(1 + 1/\log^2 x)^2}{\lambda^2(n)\, \log\log n} \;. \qquad (21)$$

So, from (19) and (21) we have

$$tn_{\ell-1}\, e^{\gamma} \log x\,(1 + 2/\log^2 x) < p_\ell < \frac{tn_{\ell-1}\, e^{\gamma} \log^2 x(1 + 1/\log^2 x)^2}{\lambda^2(n)\log\log n} \;, (22)$$

where at this stage we need to check whether such primes p_ℓ given in (22) exist. This can be done by estimating $1/\lambda^2(n)$ and $1/\log\log n$ as in the following :

Since $n_{\ell-1} = (\prod_{p\leqslant x} p)^{\beta}$, then

$$\log n_{\ell-1} = \beta \sum_{p\leqslant x} \log p \quad .$$

It follows from [9], that

$$\beta x\,(1 - 1/\log x) < \log n_{\ell-1} < \beta x\,(1 + 1/2\log x) \;. \quad (23)$$

Since $n_{\ell-1} > x$, then from [10] , p_ℓ will satisfy

$$n_{\ell-1} \log x < p_\ell < (1 + \epsilon)\; n_{\ell-1} \log x \quad ,$$

where $\epsilon > 0$. Therefore

$$\log n_{\ell-1} + \log\log x < \log p_\ell < \log(1+\epsilon) + \log n_{\ell-1} + \log\log x. \quad (24)$$

If we have $n = n_{\ell-1}p_\ell$, then

$$\log n = \log n_{\ell-1} + \log p_\ell \quad .$$

Using (24) we get

$$2\log n_{\ell-1} + \log\log x < \log n \leqslant 2\log n_{\ell-1} + \log(1+\epsilon) + \log\log x \ .$$

By (23) we obtain

$$2\beta x(1 - 1/\log x)+\log\log x < \log n < 2\beta x(1 + 1/2\log x)+\log(1+\epsilon)+\log\log x$$

and therefore

$$2\beta x(1 - 1/\log x) + \log\log x < \log n < 2\beta x(1 + 1/2\log x) + \log\log x. \quad (25)$$

Since $\beta = \dfrac{\log^2 x}{2\log\log x}$, then from the L.H.S of (25) we have

$$\log n > \frac{x \log^2 x}{\log\log x}(1 - 1/\log x) \ ,$$

so that

$$\log\log n \geqslant \log x \left(1 + \frac{\log\log x}{\log^2 x}\right) . \qquad (26)$$

Hence

$$\log\log\log n \geqslant \log \left\{ \log x \left(1 + \frac{\log\log x}{\log^2 x}\right)\right\}$$

$$= \log\log x + \log\left(1 + \frac{\log\log x}{\log^2 x}\right) \quad .$$

i.e

$$\log\log\log n \geqslant \log\log x + \frac{\log\log x}{\log^2 x} \ .$$

So,we have

$$\log\log\log n \geqslant \log\log x \left(1 + \frac{1}{\log^2 x}\right) . \qquad (27)$$

Since

$$\lambda(n) < 1 - \frac{c\log\log\log n}{(\log\log n)^2} \ ,$$

then , from (26) and (27) we may have

$$\lambda(n) < 1 - \frac{c\log\log x\,(1 + 1/\log^2 x)}{\log^2 x\left[1 + \dfrac{\log\log x}{\log^2 x}\right]^2} \quad ,$$

and from this inequality we obtain

$$\lambda(n) \leqslant 1 - \frac{c\log\log x}{\log^2 x} \quad .$$

Therefore,

$$1/\lambda^2(n) \geqslant \frac{1}{\left[1 - \dfrac{c\log\log x}{\log^2 x}\right]^2} \geqslant 1 + \frac{2c\log\log x}{\log^2 x} \quad . \qquad (28)$$

Insert (26) and (28) in (22) to yield

$$tn_{\ell-1}\, e^{\gamma}\log x(1 + 2/\log^2 x) < p_\ell < \frac{tn_{\ell-1}e^{\gamma}\log^2 x(1+1/\log^2 x)^2}{\log x\left(1 + \dfrac{\log\log x}{\log^2 x}\right)}\left(1+\frac{2c\log\log x}{\log^2 x}\right).$$

Thus ,

$$tn_{\ell-1}\, e^{\gamma}\log x\left(1+\frac{2}{\log^2 x}\right) < p_\ell < tn_{\ell-1}e^{\gamma}\log x\left(1+\frac{1}{\log^2 x}\right)^2\left(1+\frac{2c\log\log x}{\log^2 x}\right)\left(1-\frac{\log\log x}{\log^2 x}\right)$$

So,for $c \geqslant 2$, we have

$$tn_{\ell-1}\, e^{\gamma}\log x\left(1+\frac{2}{\log^2 x}\right) < p_\ell < tn_{\ell-1}\, e^{\gamma}\log x\left(1+\frac{2}{\log^2 x}\right)\left(1+\frac{3\log\log x}{\log^2 x}\right). \quad (29)$$

Put $M(x) = tn_{\ell-1}\, e^{\gamma}\log x\,(1 + 2/\log^2 x)$. Then (29) becomes

$$M(x) < p_\ell < \left(1 + \frac{3\log\log x}{\log^2 x}\right)M(x) \quad , \qquad (30)$$

where since $n_{\ell-1} > x$ then $M(x) > x$. Hence

$$\frac{3\log\log x}{\log^2 x} \cdot M(x) > [M(x)]^{7/12} + \delta ,$$

and therefore primes p_ℓ exist which satisfy (30) (see [11]) . This completes the proof .

§2.2 The Frequency Of t-Practical Numbers

In [7] M.Hausman and H. Shapiro proved the following result :

Theorem 2.2.1[7]

The number of practical numbers less than or equal to x is

$$O(x / (\log x)^{\beta}) ,$$

for every fixed positive $\beta < \frac{1}{2}\left(\frac{1}{\log 2} - 1\right)^2$.

We shall generalize Theorem 2.2.1[7] for all t-practical numbers, $t \in \mathbb{N}$. For this purpose the following Lemmas will be needed .

Lemma 2.2.1[7]

For $\omega(n)$ equal to the number of distinct prime factors of n,and for any given $\epsilon > 0$ (possibly a function of x), the number of $n \leqslant x$ such that $\omega(n) > (1 + \epsilon)\log\log x$ is

$$O(x / (\log x)^{\epsilon^2/2}) ,$$

where the O is uniform in ϵ .

For the proof of Lemma 2.2.1 (see [7]) .

We shall need the following general result ;

Lemma 2.2.2

Let $n = n^*p_1 \ldots p_\ell, (p_i, n^*) = 1$ and p_i are distinct primes. If for $i = 1, 2, \ldots, \ell$, $p_i \leqslant t\sigma(n^*p_1 \ldots p_{i-1}) + 1$, then

$$p_i \leqslant (t\sigma(n^*) + i)^{2^{i-1}} .$$

Proof

Let $n = n^*p_1 \ldots p_\ell$, $p_i \nmid n^*$ and p_i are distinct primes such that

$$p_i \leqslant t\sigma(n^*p_1 \ldots p_{i-1}) + 1 . \tag{1}$$

We shall proceed by induction on i to show that

$$t\sigma(n^*p_1 \ldots p_{i-1}) \leqslant (t\sigma(n^*) + i-1)^{2^{i-1}} , \tag{2}$$

for $t \geqslant 1$. For $i = 1$ it is obvious that (2) is true . As an induction hypothesis we assume that (2) is satisfied for $i-1$. We have

$$t\sigma(n^*p_1 \ldots p_i) = t\sigma(n^*p_1 \ldots p_{i-1})(p_i + 1) . \tag{3}$$

Using (1) and (2) in (3) , we get

$$\begin{aligned} t\sigma(n^*p_1 \ldots p_i) &= t\sigma(n^*p_1 \ldots p_{i-1})(p_i + 1) \\ &\leqslant [t\sigma(n^*p_1 \ldots p_{i-1})][t\sigma(n^*p_1 \ldots p_{i-1}) + 2] \\ &\leqslant [t\sigma(n^*) + i-1]^{2^{i-1}} . [(t\sigma(n^*) + i-1)^{2^{i-1}} + 2] \\ &\leqslant [t\sigma(n^*) + i]^{2^i} . \end{aligned}$$

Therefore inequality (2) is true for i . So, by using (1) and (2) we obtain

$$p_i \leqslant t\sigma(n^*p_1 \ldots p_{i-1}) + 1 \leqslant [t\sigma(n^*) + i-1]^{2^{i-1}} + 1 .$$

Hence

$$p_i \leqslant [t\sigma(n^*) + i-1]^{2^{i-1}} + 1 \leqslant [t\sigma(n^*) + i]^{2^{i-1}} ,$$

which is the required result .

Now, we generalize the main result given in Theorem 2.2.1[7] .

Theorem 2.2.2

Let $t > 1$ and $0 < \beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2$. Then the number of t-practical numbers $n \leqslant x$ is

$$O[x/(\log x)^{\beta}] .$$

This estimate is uniform for t in the range

$$1 \leqslant t \leqslant \exp.\{(\log x)^{\delta_1}\} ,$$

for any δ_1 with $0 < \delta_1 < 1 - (1 + \sqrt{2\beta})\log 2$.

Proof

Let $0 < \epsilon < (\frac{1}{\log 2} - 1)$. We firstly estimate the number of t-practical numbers $n \leqslant x$ with $\omega(n) > (1 + \epsilon)\log\log x$. From Lemma 2.2.1, this is at most

$$O[x/(\log x)^{\epsilon^2/2}] . \qquad (4)$$

Next, we consider the t-practical numbers $n \leqslant x$ with $\omega(n) \leqslant (1 + \epsilon)\log\log x$. From Theorem 1.1.5, in chapter 1, the t-practical numbers n have the form

$$q_1 \ldots q_k \cdot p_1 \ldots p_\ell , \qquad (5)$$

such that

(i) The q_j are distinct primes such that

$$q_j \leqslant t + 1 , \quad j = 1,2,\ldots, k .$$

(ii) The p_i's are distinct primes such that

$$t + 1 < p_1 < \ldots < p_\ell ,$$

and $p_i \neq q_j$ for all i and j $(1 \leqslant i \leqslant \ell)$ and, $(1 \leqslant j \leqslant k)$.

(iii) $(\ell + k) \leqslant (1 + \epsilon)\log\log x$.

By setting $n^* = q_1 \ldots q_k$ and using (i) and (ii) we have

$$n^* \leqslant (t + 1)^{(1 + \epsilon)\log\log x} ,$$

or

$$n^* \leqslant \exp.\{(2\log\log x.\log\log(t+1)\} \,. \tag{6}$$

Therefore, the t-practical numbers are of the form

$$n^* p_1 \ldots p_\ell \quad , \tag{7}$$

where $\ell < (1+\epsilon)\log\log x$ and the p_i are larger than the primes dividing n^* . Since $n^* p_1 \ldots p_\ell$ is a t-practical number , then by Lemma 1.1.4, we have

$$p_i \leqslant t\sigma(n^* p_1 \ldots p_{i-1}) + 1 \quad . \tag{8}$$

Using Lemma 2.2.2, we obtain

$$p_i \leqslant [t\sigma(n^*) + i\,]^{2^{i-1}} . \tag{9}$$

Since $\sigma(n^*) = O(n^* \log\log n^*)$, (see [8]), then by (6) we get

$$\sigma(n^*) = O\,(n^* \log\log n^*) \leqslant \exp.\{(c_1 \log\log x.\log(t+1)\} \;, \tag{10}$$

where $c_1 > 0$ is a constant . Insert (10) in (9) to get

$$p_i \leqslant [t.\exp.\{(c_1 \log\log x.\log(t+1)\} + i]^{2^{i-1}} . \tag{11}$$

Since $i \leqslant \ell(1+\epsilon)\log\log x$, then (11) becomes

$$p_i \leqslant [t.\exp.\{c_2 \log\log x.\log(t+1)\}]^{2^{i-1}} \;, \tag{12}$$

where $c_2 > 0$ is a constant . By using (6) and (12) , then the number of the t-practical numbers having the form (5) is at most

$$[\exp.\{(2\log\log x).\log(t+1)\}].[t.\exp.\{c_2 \log\log x.\log(t+1)\}]^{1+2+\ldots+2^{\ell-1}} ,$$

i.e

$$[\exp.\{(2\log\log x).\log(t+1)\}].[t.\exp.\{(c_2 \log\log x).\log(t+1)\}]^{2^\ell - 1} ,$$

$$\leqslant [\exp.(c_3 \log\log x).\log(t+1)]^{2^{(1+\epsilon)\log\log x}+1} . t^{2^{(1+\epsilon)\log\log x}-1} \;, \tag{13}$$

where $c_3 > 0$ is a constant . We have

$$2^{(1+\epsilon)\log\log x} = e^{\log\left[2^{\log\log x}\right]^{(1+\epsilon)}} = e^{(1+\epsilon)\log\left[2^{\log\log x}\right]}$$

$$= \left[e^{\log 2^{\log\log x}}\right]^{(1+\epsilon)} = \left[e^{\log\log x}\right]^{(1+\epsilon)\log 2} .$$

Therefore ,

$$2^{(1+\epsilon)\log\log x} = (\log x)^{(1+\epsilon)\log 2} . \qquad (14)$$

So, (13) is at most

$$[\exp.\{(c_3\log\log x).\log(t+1)\}]^{1+(\log x)^{(1+\epsilon)\log 2}} . t^{(\log x)^{(1+\epsilon)\log 2} - 1} \qquad (15)$$

Also , we have

$$t^{(\log x)^{(1+\epsilon)\log 2} - 1} = e^{\{(\log x)^{(1+\epsilon)\log 2} - 1\}.\log t} = \frac{e^{\log t(\log x)^{(1+\epsilon)\log 2}}}{e^{\log t}}$$

i.e

$$t^{(\log x)^{(1+\epsilon)\log 2} - 1} = \exp.\{(\log t).(\log x)^{(1+\epsilon)\log 2} - \log t\} . \qquad (16)$$

For any $\epsilon > 0$ with $(1+\epsilon)\log 2 < 1$ we have , for sufficiently large x,

$$1 + (\log x)^{(1+\epsilon)\log 2} \leqslant 2(\log x)^{(1+\epsilon)\log 2} . \qquad (17)$$

So, from (16) and (17) the number of the t-practical numbers $n \leqslant x$ is at most

$$\exp.\{(2c_3\log\log x).\log(t+1)\}^{(\log x)^{(1+\epsilon)\log 2}} . \exp.\{(\log t)(\log x)^{(1+\epsilon)\log 2} - \log t\},$$

or

$$\exp.\{(c_4\log\log x)(\log(t+1)(\log x)^{(1+\epsilon)\log 2}\}.\exp.\{(\log t)(\log x)^{(1+\epsilon)\log 2} - \log t\},$$

where $c_4 > 0$ is a constant . Further, for x sufficiently large we have $c_4\log\log x > 1$ and therefore

$$(c_4\log\log x).\log(t+1) > \log t . \qquad (19)$$

Thus (18) becomes at most

$$\exp.\{(c_5\log\log x).(\log t).(\log x)^{(1+\epsilon)\log 2} - \log t\} ,$$

i.e

$$\exp.[(\log t)\{(c_5\log\log x).(\log x)^{(1+\epsilon)\log 2} - 1\}]$$

$$\leqslant \exp.\{(c_5\log\log x).(\log t).(\log x)^{(1+\epsilon)\log 2}\} , \qquad (20)$$

where $c_5 > 0$ is a constant. Since $(1+\epsilon)\log 2 < 1$, then for large x,

$$(\log\log x).\ (\log x)^{(1+\epsilon)\log 2} \leqslant (\log x)^{1-\delta} ,$$

where $\delta > 0$ is a constant which can be chosen such that $0 < \delta < 1 - (1+\epsilon)\log 2$. Thus , the number of the t-practical numbers $n \leqslant x$, given in (20) is at most

$$O\left[\exp.\left\{\frac{c_5(\log t).(\log x)}{(\log x)^{\delta}}\right\}\right] . \qquad (21)$$

For t in the range $1 \leqslant t \leqslant \exp.\{(\log x)^{\delta_1}\}$, where $0 < \delta_1 < \delta$, (21) becomes

$$O\left[\exp.\left\{\frac{(\log x)}{(\log x)^{\eta}}\right\}\right] ,$$

where $\eta = \delta - \delta_1$. Since $\eta > 0$,this is certainly

$$O\left[\frac{x}{(\log x)^{\epsilon^2/2}}\right]$$

Hence the bound obtained for the t-practical numbers $n \leqslant x$ is

$$O\left[\frac{x}{(\log x)^{\epsilon^2/2}}\right],$$

provided $0 < \epsilon < (\frac{1}{\log 2} - 1)$. Putting $\beta = \frac{\epsilon^2}{2}$,then the number of the t-practical numbers $n \leqslant x$ is

$$O\left[\frac{x}{(\log x)^{\beta}}\right],$$

provided $0 < \beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2$ and this completes the proof .

Theorem 2.2.2 , covered a wide range of t-practical numbers n under the same bound given by Hausman and Shapiro in [7] . The case when $t = 1$ implies Theorem 2.2.1[7] .

§2.3 Lower Bound For The Number Of t-Practical Numbers

An explicit lower bound for the number of practical numbers $n \leqslant x$ is given by Margenstren in [4] as in the following :

Theorem 2.3.1[4]

For a large real positive number x ,the number of practical numbers $n \leqslant x$ is at least

$$\frac{Ax}{\exp.\left[\frac{1}{2\log 2}(\log\log x)^2 + 3\log\log x\right]},$$

where $A = \frac{2^{5/2}}{5}$.

We shall present a lower bound for the number of t-practical numbers $n \leqslant x$ by generalizing Theorem 2.3.1[4]. This, yields, taking $t = 1$, a sharper lower bound than Theorem 2.3.1[4] .

Theorem 2.3.2

Let $N(x)$ be the number of t-practical numbers $n \leqslant x$. Then for large positive real x and $t \in \mathbb{N}$,

$$N(x) \geqslant (2\log 2)^{1/2} \frac{x}{\exp.\left[\frac{1}{2\log 2} (\log\log x)^2 + \log\log x + \log(t+1)\right]} . \quad (1)$$

Proof

Let $x \geqslant (t+1)^3$ be a real positive number and $r \geqslant 1$ be an integer such that

$$(t+1)^{2^r} + 1 \leqslant x < (t+1)^{2^{r+1}} + 1 , \quad (2)$$

where $t \in \mathbb{N}/\{0\}$.

Put $y = \frac{x}{(t+1)}$. Then (2) becomes

$$(t+1)^{2^r} \leqslant y \leqslant (t+1)^{2^{r+1}} . \quad (3)$$

From Theorem 1.1.5, any number n of the form

$$p_1 \cdot p_2 \cdots p_r ,$$

where

$$p_{j+1} \leqslant t\sigma(p_1 \cdot p_2 \cdots p_j) + 1 , \quad (1 \leqslant j \leqslant r-1) , \quad (4)$$

and $1 < p_1 \leqslant t+1 < p_2 < \ldots < p_r$, is a t-practical number. So, we may choose the primes $p_1, \ldots, p_r$ such that

$$
\begin{array}{c}
1 < p_1 \leqslant (t+1) \\
(t+1) < p_2 < 2(t+1) \\
(t+1)^2 \leqslant p_3 < 2(t+1)^2 \\
\vdots \\
(t+1)^{2^r} \leqslant p_r < 2(t+1)^{2^r} .
\end{array} \qquad (5)
$$

Now, we need to show that the primes given in (5) will satisfy the condition (4) . Since $1 < p_1 \leqslant t+1$, then by Lemma 1.1.3, p_1 is is a t-practical number for any fixed t , $t \in \mathbb{N}$. So, by Lemma 1.1.1 , we have

$$t\sigma(p_1) + 1 \geqslant (t+1)p_1 \quad ,$$

where $p_1 \geqslant 2$. Hence

$$t\sigma(p_1) + 1 \geqslant (t+1)p_1 \geqslant 2(t+1) > p_2 \quad .$$

Therefore $p_1.p_2$ is a t-practical number by Lemma 1.1.4 . As an induction hypothesis we assume that $p_1.p_2 \ldots p_i$, $(2 \leqslant i \leqslant r)$, is a t-practical number and

$$(t+1)^{2^k} \leqslant p_i < 2(t+1)^{2^k} \quad , \qquad (o \leqslant k \leqslant r) \quad .$$

Since $p_1 \geqslant 2$, then from the L.H.S of (5) , we have

$$p_1.p_2 \ldots . p_i \geqslant 2(t+1)^{1+2+2^2+\ldots+2^k} = 2(t+1)^{2^{k+1}-1} \quad .$$

Therefore

$$p_1.p_2 \ldots p_i \geqslant 2(t+1)^{2^{k+1}} - 1 \quad . \qquad (6)$$

Since $p_1.p_2 \ldots p_i$ is a t-practical number, then by Lemma 1.1.1, we have

$$t\sigma(p_1.p_2 \ldots p_i) + 1 \geqslant (t+1)p_1.p_2 \ldots p_i \quad ,$$

and by (6) , we get

$$t\sigma(p_1.p_2\dots p_i) + 1 \geqslant 2(t+1)^{2^{k+1}} \quad .$$

Therefore ,

$$t\sigma(p_1.p_2\dots p_i) + 1 \geqslant 2(t+1)^{2^{k+1}} > p_{i+1} \quad .$$

Hence $p_1.p_2\dots p_i.p_{i+1}$ is a t-practical number (see Lemma 1.1.4) .

If we consider the integers $n = p_1.p_2\dots p_r$ and $n' = p_1'.p_2'\dots p_r'$ satisfying the condition (5),then since $p_1 < p_2 <\dots< p_r$ and $p_1'< p_2' <\dots< p_r'$, we obtain $n = n'$ and $p_1 = p_1',\dots,p_r = p_r'$.

Furthermore, we have from the R.H.S of (5) that

$$n = p_1\dots p_r \leqslant 2^{r-1} . (t+1)^{2+2+2^2+\dots+2^r} = 2^{r-1} . (t+1)^{2^{r+1}} \quad ,$$

i.e

$$n \leqslant 2^{r-1} . (t+1)^{2^{r+1}} < (t+1)^{2^{r+1}} + 1 \quad .$$

Consider the estimates given by Rosser and Schoenfeld[9] .

Then since $1 < p_1 \leqslant (t+1)$, the number of p_1 is at least $\dfrac{(t+1)}{\log(t+1)}$ and the number of the p_i with $2 \leqslant i \leqslant r$, is at least

$$\frac{(t+1)^{2^m}}{2^m . \log(t+1)} \quad , \qquad m = 0,1,\dots,r \quad .$$

Therefore ,

$$N((t+1)^{2^{r+1}}+1) \geqslant \frac{(t+1) . (t+1) . (t+1)^2 . (t+1)^{2^2}\dots\dots (t+1)^{2^r}}{\log(t+1).\log(t+1).2\log(t+1).2^2\log(t+1)\dots 2^r\log(t+1)} ,$$

i.e

$$N((t+1)^{2^{r+1}} + 1) \geqslant \frac{(t+1)^{2^{r+1}}}{2^{1+2+3+\dots+r} [\log(t+1)]^r} \quad .$$

Hence

$$N\left((t+1)^{2^{r+1}}+1\right) > \frac{(t+1)^{2^{r+1}}}{2^{r(r+1)/2}\cdot[\log(t+1)]^r} \quad . \qquad (7)$$

From (3) we have

$$y < (t+1)^{2^{r+1}} \quad , \qquad (8)$$

and since

$$\log(t+1) < \frac{\log y}{2^r} \quad ,$$

then

$$\frac{1}{[\log(t+1)]^r} > \frac{2^{r^2}}{(\log y)^r} \quad . \qquad (9)$$

Using (8) and (9) in (7) we obtain

$$N\left((t+1)^{2^{r+1}}+1\right) > \frac{2^{r^2}\,.\,y}{2^{r(r+1)/2}.\,(\log y)^r} \quad .$$

Thus ,

$$N\left((t+1)^{2^{r+1}}+1\right) > 2^{r(r-1)/2}.\,\frac{y}{(\log y)^r} \quad . \qquad (10)$$

Since

$$(t+1)^{2^r} \leqslant y < (t+1)^{2^{r+1}} \quad ,$$

then

$$2^r\log(t+1) \leqslant \log y < 2^{r+1}\log(t+1)$$

$$2^r\log(t+1) \leqslant \log y < 2^r\log(t+1)^2 \quad . \qquad (11)$$

We have $x = (t+1)y$,so we may write

$$\log y \leqslant \log x = \log y + \log(t+1) \quad .$$

So, from the L.H.S of (11) and for large y we obtain

$$\log y \leq \log x \leq 2\log y \quad . \qquad (12)$$

Using (12) in (10), then the number of t-practical numbers $n \leq x$ is

$$N(x) \geq 2^{r(r-1)/2} . \frac{x}{(t+1).(\log x)^{r}} \quad . \qquad (13)$$

From (11) we have

$$\frac{\log y}{\log(t+1)} < 2^{r} \leq \frac{\log y}{\log(t+1)} \quad .$$

Since $(t+1) \geq 2$, then $\frac{1}{\log(t+1)} \leq \frac{1}{\log 2}$. Therefore

$$\frac{\log y}{\log(t+1)^{2}} < 2^{r} \leq \frac{\log y}{\log 2} \quad .$$

So, we may take

$$2^{r} = \frac{\log y}{\log 2} \quad ,$$

and by the R.H.S of (12) we get

$$2^{r} = \frac{\log y}{\log 2} \geq \frac{\log x}{2\log 2} \quad ,$$

i.e

$$2^{r} \geq \frac{\log x}{2\log 2} \quad . \qquad (14)$$

Using (14) in (13) we obtain

$$N(x) \geq \frac{(\log x)^{(r-1)/2} . x}{(2\log 2)^{r-1/2} . (t+1) .(\log x)^{r}} \quad ,$$

$$N(x) \geq (2\log 2)^{1/2} . \frac{x}{(t+1). (2\log 2)^{r/2} (\log x)^{r+1/2}} \quad . \qquad (15)$$

From the L.H.S of (11) we have

$$2^r < \frac{\log y}{\log(t+1)} \quad ,$$

where $\frac{1}{\log(t+1)} < \frac{1}{\log 2}$. Therefore

$$2^r < \frac{\log y}{\log 2} \quad ,$$

and by the L.H.S of (12) we get

$$2^r < \frac{\log y}{\log 2} < \frac{\log x}{\log 2} \quad .$$

Hence

$$r < \frac{1}{\log 2}\,[\log\log x - \log\log 2] \ . \qquad (16)$$

Now,by considering (16) we can write

$$2^{r/2} = \exp.\left[\frac{r}{2}\log 2\right] < \exp.\left[\frac{1}{2}\log\log x - \frac{1}{2}\log\log 2\right],$$

$$(\log 2)^{r/2} = \exp.\left[\frac{r}{2}\log\log 2\right] < \exp.\left[\frac{\log\log 2}{2\log 2}\log\log x - \frac{(\log\log 2)^2}{2\log 2}\right]$$

and

$$(\log x)^{r+1/2} = \exp.\left[\frac{r+1}{2}\log\log x\right]$$

$$< \exp.\left[\frac{1}{2\log 2}(\log\log x)^2 - \frac{\log\log 2}{2\log 2}(\log\log x) + \frac{1}{2}\log\log x\right].$$

Hence

$$(2\log 2)^{r/2}.\ (\log x)^{r+1/2} < \exp.\left[\frac{1}{2\log 2}(\log\log x)^2 + \log\log x\right] \ ,$$

and therefore

$$\frac{1}{(2\log 2)^{r/2}.\ (\log x)^{r+1/2}} > \frac{1}{\exp.\left[\frac{1}{2\log 2}\ (\log\log x)^2 + \log\log x\right]} \ .(17)$$

Insert (17) in (15).Then the number of t-practical numbers $n \leq x$ is

$$N(x) \geq (2\log 2)^{1/2}. \frac{x}{\exp.\left[\frac{1}{2\log 2}\ (\log\log x)^2 + \log\log x + \log(t+1)\right]} \ , (18)$$

for any t , $t\in\mathbb{N}$. This completes the proof of the Theorem .

Comparing the bound obtained by Margenstren[4] when $t = 1$,then (18) will be sharper than the bound proved in[4] for $x \geq 64$.

§2.4 Existence Of a t-Practical Number In An Interval

In this section we shall give a further improvement on the t-practical numbers by generalizing Hausman and Shapiro's result[7] given below;

Theorem 2.4.1[7]

For all real $x \geq \frac{1}{3}$ the interval $(x\ ,\ x + 2x^{1/2}\ )$ contains a practical number .

We shall prove the following general result;

Theorem 2.4.2

For all $x \geq \frac{t}{3}$ and $t\in\mathbb{N}$, the interval $(\ x\ ,\ x + 2(\frac{x}{t})^{1/2}\ )$ contains a t-practical number .

Proof

We will prove the Theorem for $x > 4t$ and then check it separately for $\frac{t}{3} \leqslant x \leqslant 4t$. Therefore we let $x > 4t$ and choose $a \in \mathbb{N}$ such that

$$2^a < \left(\frac{x}{t}\right)^{1/2} \leqslant 2^{a+1} \quad . \tag{1}$$

Thus , the interval $(x \, , \, x + (\frac{x}{t})^{1/2}\,)$ is of length $(\frac{x}{t})^{1/2}$.

Therefore it contains at least one multiple of 2^a , say $2^a m$ .i.e

$$x < 2^a m < x + \left(\frac{x}{t}\right)^{1/2} \quad . \tag{2}$$

Now, either m or m+1 is an even integer and, since

$$2^a\,(m+1) = 2^a m + 2^a < x + 2\left(\frac{x}{t}\right)^{1/2} \quad .$$

We have that

$$x < 2^a m < 2^a\,(m+1) < x + 2\left(\frac{x}{t}\right)^{1/2} \quad .$$

We shall prove that one of these integers $2^a m$, $2^a(m+1)$ is a t-practical number. Suppose, therefore , that neither is a t-practical number and , without loss of generality, let m be an even integer. Then since $2^a m$ is not t-practical number, there exists a prime p_i of $2^a m$ such that

$$p_i > t\sigma(n_{i-1}) + 1 \quad , \tag{3}$$

where $2^a m = p_1{}^{k_1} \cdot p_2{}^{k_2} \cdots p_\ell{}^{k_\ell}$, $p_1 = 2$, $n_{i-1} = p_1{}^{k_1} \cdots p_{i-1}{}^{k_{i-1}}$, $(i \geqslant 2)$ and $n_0=1$. In particular, (3) implies that $p_i > t + 1$ so that p_i is odd and $i \geqslant 2$. Thus , $2^{a+1} \mid n_{i-1}$ and hence

$$p_i > t\sigma(2^{a+1}) + 1 = t(2^{a+2} - 1) + 1 \ .$$

So,

$$p_i \geqslant t(2^{a+2} - 1) + 2 \quad .$$

This implies that

$$2^a m = 2^{a+1} \left(\frac{m}{2}\right) \geqslant 2^{a+1} . \, p_i \geqslant 2^{a+1} . \, t(2^{a+2} - 1) + 2^{a+2}$$

i.e

$$2^a m \geqslant 2.2^{2(a+1)} . \, t \quad - 2^{a+1} . \, t + 2^{a+2} \quad . \qquad (4)$$

Hence from the R.H.S of (1) we get

$$2^a m \geqslant 2x - 2^{a+1} . \, t \quad + \quad 2\left(\frac{x}{t}\right)^{1/2} \quad .$$

So , from (2) we shall have

$$x + \left(\frac{x}{t}\right)^{1/2} \geqslant 2x - 2^{a+1} . \, t + 2\left(\frac{x}{t}\right)^{1/2} \quad ,$$

i.e

$$2^{a+1} . \, t \geqslant x + \left(\frac{x}{t}\right)^{1/2} . \qquad (5)$$

From the L.H.S of (1) we have $2x^{1/2} . \, t^{1/2} > 2^{a+1} t$. Therefore (5) becomes

$$2x^{1/2} . \, t^{1/2} > x + \left(\frac{x}{t}\right)^{1/2}$$

$$2t^{1/2} - \left(\frac{1}{t}\right)^{1/2} > x^{1/2}$$

$$2t^{1/2} > x^{1/2} \quad .$$

Therefore $4t > x$ which is a contradiction . A similar procedure can be followed for $2^a(m+1)$ which also leads to a contradiction .

If $t \leqslant x \leqslant 4t$, then we shall have

$$2 \leqslant 2\left(\frac{x}{t}\right)^{1/2} \leqslant 4$$

and

$$x + 2 \leqslant x + 2\left(\frac{x}{t}\right)^{1/2} \leqslant 4 + x$$

i.e

$$1 < (t+1) < x + 2 \leqslant x + 2\left(\frac{x}{t}\right)^{1/2} \leqslant 4(t+1) \ . \qquad (6)$$

Therefore (6) shows that the interval $(x \ , \ x + 2(\frac{x}{t})^{1/2})$ contains the integer $t+1$ which is a t-practical number (see Lemma 1.1.2) . If $\frac{t}{3} \leqslant x < t$, then

$$2\left(\frac{1}{3}\right)^{1/2} < 2\left(\frac{x}{t}\right)^{1/2} < 2$$

i.e

$$x + 2\left(\frac{1}{3}\right)^{1/2} < x + 2\left(\frac{x}{t}\right)^{1/2} < x + 2 < t + 2 \ .$$

Since $x > \frac{t}{3}$, then

$$1 < \frac{t}{3} + 1 < x + 2\left(\frac{x}{t}\right)^{1/2} < t + 2 \quad ,$$

and hence $1 < \frac{t}{3} + 1 \leqslant t+1$. So , by Lemma 1.1.3 , $m = [\frac{t}{3} + 1]$ is a t-practical number . This completes the proof of the Theorem .

Theorem 2.4.1[7] will follow immediately when $t = 1$.

Corollary 2.4.1

For $x \geqslant 1$ there is a t-practical number between x^2 and $(x + \frac{1}{t})^2$ for any fixed t, $t \in \mathbb{N}$.

Proof

From Theorem 2.4.2, we have that the interval $(x \ , \ x+2(\frac{x}{t})^{1/2})$ contains a t-practical number for $x \geqslant \frac{t}{3}$. This implies that the interval $(x^2, x^2 + 2((\frac{x}{t})^2)^{1/2})$ also contains a t-practical number for $x \geq 1$. Since

$$(x^2 , x^2 + 2\frac{x}{t}) \subseteq (x^2 , x^2 + 2\frac{x}{t} + \frac{1}{t^2}) = (x^2 , (x+\frac{1}{t})^2),$$

then , there is a t-practical number between x^2 and $(x + \frac{1}{t})^2$ which ends the proof .

CHAPTER (3)

§3.1 Algebraic Numbers Expressible As A t-Integer

In this section, we shall study the representation of integers in algebraic number fields as sums of units with certain repetitions.

Let V be the set of all algebraic fields having the property that all integers are expressible as a finite sum of units (not necessarily distinct). Define W_t, $t \geq 1$, to be the set of all algebraic number fields for which every integer is a sum of units with at most t repetitions. Regarding the algebraic number field $\mathbb{Q}(\theta)$.

P.Belcher in [14] proved the following main result;

Theorem 3.1.1

If $\mathbb{Q}(\theta) \in V$ and 2 is a sum of two distinct units, then $\mathbb{Q}(\theta) \in W_1$.

We shall generalize Theorem 3.1.1, for t, $t \in \mathbb{N}$. So, for this purpose the following definitions will be needed.

Definition 3.1.1

An integer α is called a t-integer if α can be written as a sum of units with at most t repetitions.

Definition 3.1.2

An integer is called a strong t-integer if it is a t-integer and expressible as a sum of at most t+1 units.

Considering the algebraic number field $\mathbb{Q}(\theta)$ in V, we prove the following:

Theorem 3.1.2

If $t+1$ is a strong t-integer, then every sum of units is a t-integer .

Proof

(I) Let $t+1$ be a sum of $t+1$ dependent units , i.e

$$t+1 = a_1\epsilon_1 + a_2\epsilon_2 + \dots + a_u\epsilon_u , \qquad (1)$$

with $0 < a_i \leqslant t$, $\Sigma|a_i| \leqslant t+1$, $(1 \leqslant i \leqslant u)$, and $\epsilon_1, \epsilon_2, \dots, \epsilon_u$ are dependent units of $\mathbb{Q}(\theta)$. Choose a set of distinct units such as

$$\eta_1 , \eta_2 , \dots, \eta_d .$$

Suppose that there exists a set of units, say,

$$\omega_1 , \dots, \omega_\upsilon , \quad \upsilon < u ,$$

such that

$$\epsilon_i = \zeta^{x_i} \omega_1^{r_{i1}} \dots \omega_\upsilon^{r_{i\upsilon}}, \quad r_{i\upsilon} \in Z, \ (1 \leqslant x_i \leqslant \ell), \qquad (2)$$

and there does not exist a relation of the form

$$\eta_i \zeta^z \omega_1^{g_{i1}} \dots \omega_\upsilon^{g_{i\upsilon}} = \eta_j \zeta^y \omega_1^{h_{i1}} \dots \omega_\upsilon^{h_{i\upsilon}} , \quad i \neq j ,$$

where y , z , $g_{i\upsilon}$, $h_{i\upsilon} \in Z$, ζ is the ℓ^{th} root of unity in $\mathbb{Q}(\theta)$. Also there is no relation of the form

$$\zeta^{x_i} \omega_1^{r_{i1}} \dots \omega_\upsilon^{r_{i\upsilon}} = 1 . \qquad (3)$$

On the other hand $\epsilon_1, \dots, \epsilon_u$ can not all be roots of unity . So , all integers of $\mathbb{Q}(\theta)$ can be represented as a sum of units which have the form

$$\eta_k \zeta^j \omega^{r_{i1}} \dots \omega_\upsilon^{r_{i\upsilon}} , \quad (1 \leqslant k \leqslant d) , \ (1 \leqslant j \leqslant \ell) .$$

Now , for any integer α of $\mathbb{Q}(\theta)$, α has the form

$$\alpha = \sum_{i_1=p_1}^{q_1} \cdots \sum_{i_\upsilon=p_\upsilon}^{q_\upsilon} \sum_{k=1}^{d} \sum_{j=1}^{\ell} b_{i_1\ldots i_\upsilon,j,k}\, \eta_k \zeta^{j} \omega_1^{i_1} \cdots \omega_\upsilon^{r_{i\upsilon}} , \qquad (4)$$

where $p_{i\upsilon}$, $q_{i\upsilon} \epsilon \mathbb{Z}$, $b_{i_1,\ldots,i\upsilon,j,k} \epsilon \mathbb{N} \cup \{0\}$ and some of $b_{p_1,\ldots,i\upsilon,j,k}$ and $b_{q_1,\ldots,i\upsilon,j,k}$ are non-zero. Assume that α has the representation (4) so that

$$\sum_{i_1,\ldots,i\upsilon} b_{i_1,\ldots,i\upsilon,j,k} = m \quad , \qquad (5)$$

and m is minimal . We shall proceed by induction on m to show that α has the form

$$\alpha = \sum_{i_1=p_1-f(m)}^{q_1+f(m)} \cdots \sum_{i_\upsilon=p_\upsilon-f(m)}^{q_\upsilon+f(m)} . \sum_{k=1}^{d} . \sum_{j=1}^{\ell} c_{i_1,\ldots,i\upsilon,j,k}\, \eta_k \zeta^{j} \omega_1^{i_1} \cdots \omega_\upsilon^{i\upsilon} , \qquad (6)$$

where $0 \leqslant c_{i_1,\ldots i\upsilon,j,k} \leqslant t$ and f(m) is defined to be

$$f(m) = \max_{i,\upsilon}.(|r_{i\upsilon}|).\prod_{i=1}^{\upsilon}[(q_i-p_i+1+2(m-1)f(m-1)]^{m}.\ell^{m}.d^{m}\, 2^{t} + f(m-1), \qquad (7)$$

where $f(1) = 0$. If $m = 1$,then $\sum b_{i_1,\ldots,i\upsilon,j,k} = 1$ and therefore $b_{i_1,\ldots,i\upsilon,j,k} = 0$,± 1 . Hence α is a t-integer by definition 3.1.1 . As an induction hypothesis we assume that α is a t-integer for all $\alpha \epsilon \mathbb{Q}(\theta)$ for which

$$\sum b_{i_1,\ldots,i\upsilon,j,k} < m \quad , \quad (m \geqslant 2) \quad .$$

By considering a typical term of α given in (4) and applying (1), we have that

$$(t+1)\eta_k \zeta^{j} \omega_1^{i_1} \cdots \omega_\upsilon^{i_\upsilon} = a_1 \eta_k \zeta^{(j+x_i)} \omega_1^{(i_1+r_{11})} \cdots\cdots \omega_\upsilon^{(i_\upsilon+r_{1\upsilon})} + \cdots + a_u \eta_k \zeta^{(j+x_u)} \omega_1^{(i_{11}+i_{u1})} \cdots\cdots \omega_\upsilon^{(i_\upsilon+r_{u\upsilon})}, \qquad (8)$$

$$(t+1) \leqslant a_1 |\omega_1^{i_1+r_{11}} \cdots\cdots \omega_\upsilon^{i_\upsilon+r_{1\upsilon}}| + \cdots + a_u |\omega_1^{i_1+r_{u1}} \cdots \omega_\upsilon^{i_\upsilon+r_{u\upsilon}}| .$$

i.e

$$(t+1) < a_1 |\omega_1^{2(i_1+r_{11})} \cdots\cdots \omega_\upsilon^{2(i_1+r_{u1})}| + \cdots + a_u |\omega_1^{2(i_1+r_{u1})} \cdots\cdots \omega_\upsilon^{2(i_\upsilon+r_{u\upsilon})}| . \qquad (9)$$

It follows from (9) that the value of the sum

$$\sum_{i_1=p_1}^{q_1}\cdots\sum_{i_\upsilon=p_\upsilon}^{q_\upsilon} b_{i_1,\ldots,i_\upsilon,j,k}|\omega_1^{i_1}\cdots\cdots\omega_\upsilon^{i_\upsilon}|$$

will increase after each application of condition (1) and therefore we obtain a distinct representation of α satisfying (5), where in any of these representation we have

$$\sum_{i_1,\ldots,i_\upsilon} b_{i_1,\ldots,i_\upsilon} = m \quad ,$$

since m is minimal . Then by repeatedly applying (1) , we will get either :

(i) The process terminates and in each case α is of the form

$$\alpha = \sum_{i_1=\lambda_1}^{\delta_1}\cdots\cdots\sum_{i_\upsilon=\lambda_\upsilon}^{\delta_\upsilon}\ \sum_{k=1}^{d}\ \sum_{j=1}^{\ell} e_{i_1,\ldots,i_\upsilon,j,k}\eta_k\zeta^j\omega_1^{i_1}\cdots\cdots\omega_\upsilon^{i_\upsilon} \quad , \qquad (10)$$

with $0 \leqslant e_{i_1,\ldots,i_\upsilon,j,k} \leqslant t$, λ_{i_υ} ,$\delta_{i_\upsilon}\epsilon Z$ and at least one of the $e_{\lambda_1,\ldots,i_\upsilon}$ and $e_{\delta_1,\ldots,\delta_\upsilon}$ are non-zero . It follows from (10) that

$$\delta_i - \lambda_i \leqslant q_i - p_i + 2(m-1)f(m-1) \quad , \qquad (11)$$

and the number of the representations of α satisfying (5) is at most

$$T = \prod_{i=1}^{\upsilon}[q_i - p_i + 1 + 2(m-1)f(m-1)]^m\ \ell^m.\ d^m.\ 2^t \quad .$$

This shows that (10) is obtained after at most T applications of (1). Therefore

$$\delta_i \leqslant q_i + T\max.(|r_{i_\upsilon}|) \leqslant q_i + f(m) \quad ,$$

and

$$\lambda_i \geqslant p_i - T\max.(|r_{i_\upsilon}|) \geqslant p_i - f(m) \quad .$$

or (ii) After T applications of (1),we obtain

$$\alpha = \sum_{i_1=\lambda_1}^{\delta_1}\cdots\cdots\sum_{i_\upsilon=\lambda_\upsilon}^{\delta_\upsilon}\ .\sum_{k=1}^{d}\ .\sum_{j=1}^{\ell} e_{i_1\ldots i_\upsilon,j,k}\,\eta_k\,\zeta^j\omega_1^{i_1}\cdots\omega_\upsilon^{i_\upsilon} \quad ,$$

with $e_{i_\upsilon}\epsilon\mathbb{N} \cup \{0\}$ and

$$\delta_i - \lambda_i > q_i - p_i + 2(m-1)f(m-1) \quad , \qquad (12)$$

where the set $\{i_\upsilon : e_{i_1,\ldots,i_\upsilon} > 0\}$ contains integers differing by greater than

$$q_i - p_i + 2(m-1)f(m-1) \quad .$$

Therefore , there exists a subinterval of i_υ in which $e_{i_1,\ldots,i_\upsilon} = 0$ of length more than

$$\{q_i - p_i + 2(m-1)f(m-1)\}/m-1 \ .$$

So , we can write (10) as

$$\alpha = \beta + \gamma = \sum_{i_1=H_1}^{J_1} \cdots \sum_{i_\upsilon=H_\upsilon}^{J_\upsilon} \sum_{k=1}^{d} \sum_{j=1}^{\ell} f_{i_1\ldots i_\upsilon,j,k}\, \eta_k\, \zeta^j\, \omega_1^{i_1}\cdots\omega_\upsilon^{i_\upsilon} \quad +$$

$$, \quad \sum_{i_1=\rho_1}^{\pi_1} \cdots \sum_{i_\upsilon=\rho_\upsilon}^{\pi_\upsilon} \sum_{k=1}^{d} \sum_{j=1}^{\ell} h_{i_1\ldots i_\upsilon,j,k}\, \eta_k\, \zeta^j\, \omega_1^{i_1}\cdots\omega_\upsilon^{i_\upsilon} \quad ,(13)$$

where H_{i_υ} , J_{i_υ} , ρ_{i_υ} , $\pi_{i_\upsilon} \in Z, f \leqslant t \ \underset{i_1,\ldots,i_\upsilon,j,k}{h_{i_\upsilon}} \in \mathbb{N} \cup \{0\}$ with some of the $f_{i_1,\ldots,i_\upsilon}$ and $h_{i_1,\ldots,i_\upsilon}$ non-zero . There does not occur an overlap between β and γ given in (13) . This mean that

$$H_i \geqslant p_i - T.\max.(|r_{i_\upsilon}|) \ ,$$

$$\pi_i \leqslant q_i + T.\max.(|r_{i_\upsilon}|) \ ,$$

and

$$\rho_i - J_i \geqslant 2f(m-1) \ .$$

From (13) we have

$$\Sigma f_{i_1,\ldots,i_\upsilon} + \Sigma h_{i_1,\ldots,i_\upsilon} \leqslant m \quad .$$

and since m is minimal, then

$$\Sigma f_{i_1,\ldots,i_\upsilon} + \Sigma h_{i_1,\ldots,i_\upsilon} = m \quad .$$

Since $\Sigma f_{i_1,\ldots,i_\upsilon} > 0$, then $\Sigma h_{i_1,\ldots,i_\upsilon} < m$. So , by applying the induction hypothesis on γ , where γ is of the form

$$\gamma = \sum_{i_1=\rho_1}^{\pi_1} \cdots \sum_{i_\upsilon=\rho_\upsilon}^{\pi_\upsilon} \sum_{j=1}^{d} \sum_{k=1}^{\ell} h_{i_1\ldots i_\upsilon,j,k}\, \eta_k\, \zeta^j\, \omega_1^{i_1}\cdots\omega_\upsilon^{i_\upsilon} \quad ,$$

therefore α will be of the required representation and i_υ will be in the range

$$p_i - T\max.(|r_{i_\upsilon}|) - f(m-1) \leqslant i_\upsilon \leqslant q_i + T\max.(|r_{i_\upsilon}|) + f(m-1),$$

i.e

$$p_i - f(m) \leqslant i_\upsilon \leqslant q_i + f(m) .$$

Hence α is a t-integer .

(II) t+1 is a sum of t+1 independent units of $\mathbb{Q}(\theta)$. Then we may suppose that

$$t+1 = a_1\epsilon_1 + a_2\epsilon_2 + \ldots + a_u\epsilon_u ,$$

with $0 < |a_i| \leqslant t$ and $|\sum_i a_i| \leqslant t+1$. Suppose that there is no relation of the form

$$\epsilon_1^{r_1} \ldots \epsilon_u^{r_u} = 1 , \; r_i \epsilon Z , \; (1 \leqslant i \leqslant u) .$$

It follows that there is one of the units , say ϵ_u , which is not dependent on the others .

Choose a set of distinct units such as

$$\eta_1, \ldots, \eta_\ell .$$

Thus every integer α of $\mathbb{Q}(\theta)$ can be written as a sum of units having the form

$$\pm\eta_j \, \epsilon_1^{i_1} \ldots \epsilon_u^{i_u} , \; (1 \leqslant j \leqslant \ell) ,$$

and there is no relation of the form

$$\eta_j \, \epsilon_1^{x_1} \ldots \epsilon_u^{x_u} = \pm\eta_k \, \epsilon_1^{y_1} \ldots \epsilon_u^{y_u} , \quad j \neq k , \quad x_i, y_i \epsilon Z .$$

So , for all α of $\mathbb{Q}(\theta)$, α can be represented as

$$\alpha = \sum_{i_1=p_1}^{q_1} \ldots\ldots \sum_{i_u=p_u}^{q_u} \sum_{j=1}^{\ell} b_{i_1 \ldots iu,j} \, \eta_j \, \epsilon_1^{i_1} \ldots \epsilon_u^{i_u} , \qquad (14)$$

where $b_{i_1,\ldots,iu,j}$, p_{iu} , $q_{iu} \epsilon Z$ and some of the $b_{i_1,\ldots,iu,j}$ are non-zero . We assume that α has the representation (14) so that

$$\sum |b_{i_1,\ldots,iu}| = m , \qquad (15)$$

and m is minimal . We apply induction on m in order to show that

α is of the form

$$\alpha = \sum_{i_1=p_1}^{z_1}\cdots\sum_{i_u=p_u}^{z_u}\ \sum_{j=1}^{\ell} c_{i_1\ldots iu,j}\ \eta_j\ \epsilon_1^{i_1}\cdots\epsilon_u^{i_u}\ , \qquad (16)$$

with $0 \leqslant |c_{i_1,\ldots,iu,j}| \leqslant t$, where $z_{i_u}\in Z$. If m=1 ,then $\Sigma|b_{i_1,\ldots,i_u}| = 1$ and hence $b_{i_1,\ldots,iu,j} = 0\ ,\pm 1$. Therefore α is a t-integer by definition 3.1.1 . Assume that α is a t-integer for all $\alpha\in\mathbb{Q}(\theta)$ with $m < M$, $(M \geqslant 2)$. We will then prove it for all α with $m = M$. Suppose that α has the form (14) and satisfying (15) . We have

$$(t+1)\eta_j\cdot\epsilon_1^{i_1}\cdots\epsilon_u^{i_u} = a_1\eta_j\epsilon_1^{(i_1+1)}\cdots\epsilon_u^{i_u} + a_2\eta_j\epsilon_1^{i_1}\cdot\epsilon_2^{(i_2+1)}\cdots\epsilon_u^{i_u} + \cdots + a_u\eta_j\epsilon_1^{i_1}\cdots\epsilon_u^{(i_u+1)}, \qquad (17)$$

where by repeating the application given in (17) to each term of α , then all terms in which $|b_{i_1,\ldots,iu,j}| \geqslant t+1$ will be shifted to the right . Therefore ,after a finite number of applications of our condition given in (17) we obtain that

$$\alpha = \sum_{i_2=p_2}^{\pi_1}\cdots\sum_{i_u=p_u}^{\pi_u}\ \sum_{j=1}^{\ell} c_{p_1.i_2\ldots iu,j}\ \eta_j\ \epsilon_1^{p_1}\cdots\epsilon_u^{i_u} + \sum_{i_1=p_1+1}^{\rho_1}\cdots\sum_{i_u=p_u}^{\rho_u}\ \sum_{j=1}^{\ell} d_{i_1\ldots iu,j}\ \eta_j\ \epsilon_1^{i_1}\cdots\epsilon_u^{i_u}\ , \qquad (18)$$

where $|c_{p_1.i_2\ldots iu,j}| \leqslant t$, π_i , ρ_i , $d_{i_1,\ldots,iu,j}\in Z$ and some of the $c_{p_1.i_2\ldots iu,j}$ are non-zero.Therefore $\Sigma|c_{p_1\ldots iu,j}| \leqslant \Sigma|d_{i_1\ldots iu,j}|$,so

$$\Sigma|c_{p_1,i_2,\ldots,iu,j}| + |\Sigma d_{i_1,\ldots,iu,j}| \leqslant M\ ,$$

and since M is minimal then

$$\Sigma|c_{p_1.i_2,\ldots,iu,j}| + |\Sigma d_{i_1,\ldots,iu,j}| = M\ .$$

We have $\Sigma|c_{p_1i_2,\ldots,iu,j}| > 0$. Then $\Sigma|d_{i_1,\ldots,iu,j}| < M$. So , by applying the induction hypothesis to the second term of (18) then α is a t-integer and hence the proof is completed .

Theorem 3.1.1[14] proved by Belcher follows immediately from Theorem 3.1.2 when $t = 1$. He showed that there are precisely seven cubic fields with negative discriminant in W_1 and an infinity of cubic fields with positive discriminant in W_1. The classification of these cubic fields is based on the following result :

Lemma 3.1.1[14]

If $\mathbb{Q}(\theta)$ is a real algebraic field with one fundamental unit $\epsilon > 1$, say, then a necessary condition for $\mathbb{Q}(\theta)$ to be in W_1 is that $\epsilon < 3$.

More general than Lemma 3.1.1[14], we prove the following :

Theorem 3.1.3

If $\mathbb{Q}(\theta)$ is a real field belonging to $W_t, t \in \mathbb{N}$, and having one fundamental unit ϵ, then

$$\epsilon < 2t + 1 \quad .$$

Proof

Use the property that in W_t we can represent $t+1$ as

$$t+1 = c_p \epsilon^p + \ldots + c_q \epsilon^q \ , \qquad (19)$$

where p, $q \in \mathbb{Z}$ and $|c_i| \leqslant t$ for all i, $(p \leqslant i \leqslant q)$. We may write (19) in the form

$$b_n \epsilon^n + b_{n-1} \epsilon^{n-1} + \ldots + b_0 = 0 \ , \qquad (20)$$

where b_n, $b_0 \neq 0$, $b_i \in \mathbb{Z}$, at most one of the $b_i = \pm (2t+1)$ and the rest of the coefficients are less than or equal to t. It follows from (20) that

$$\epsilon^n \leqslant b_n \epsilon^n \leqslant (2t+1)\epsilon^{n-1} + \frac{t\epsilon^{n-1}}{\epsilon - 1} \quad .$$

Therefore ,

$$\epsilon^n \leqslant (2t+1)\epsilon^{n-1} + \frac{t\epsilon^{n-1}}{\epsilon - 1} \ ,$$

$$\epsilon(\epsilon - 1) \leqslant (2t + 1)(\epsilon - 1) + t ,$$

i.e

$$\epsilon^2 - 2(t + 1)\epsilon + (t + 1) \leqslant 0 .$$

Thus,

$$\epsilon < (t + 1) + \sqrt{(t + 1)^2 - (t + 1)} = t+1 + \sqrt{t^2+t} .$$

So ,

$$\epsilon < 2t + 1 ,$$

which is the required result .

Lemma 3.1.1[14] follows from Theorem 3.1.3 when $t = 1$, in which there are only seven real cubic fields with negative discriminant in W_1 as is proved by Belcher[14].

For the pure cubic fields $\mathbb{Q}(\theta)$, $\theta^3 = d = m^3 \pm 1$, $m \geqslant 1$, we have the following :

Remark 3.1.1

If $\mathbb{Q}(\theta)$ is a pure cubic field with $\theta^3 = d = m^3 \pm 1$, $m \geqslant 1$ and having one fundamental unit ϵ, then $\mathbb{Q}(\theta) \epsilon V$.

Proof

Let $\theta^3 = d = m^3 \pm 1$. Then we have

$$\theta^3 - m^3 = \pm 1 ,$$

i.e

$$(\theta - m)(\theta^2 + m\theta + m^2) = \pm 1 .$$

By B.Delaunay[15] , the fundamental unit of $\mathbb{Q}(\theta)$ is $\epsilon = \theta^2 + m\theta + m^2$ and $\bar{\epsilon} = \theta - m$ its conjugate . So ,

$$\theta - m = \pm \frac{1}{\epsilon} ,$$

i.e

$$\theta = m \pm \frac{1}{\epsilon} , \quad m \geqslant 1 .$$

Therefore every integer in $\mathbb{Q}(\sqrt[3]{d})$ can be written as a finite sum of units i.e $\mathbb{Q}(\sqrt[3]{d}) \epsilon V$.

Now , we prove the following ;

<u>Theorem 3.1.4</u>

If $\mathbb{Q}(\theta)$ is a pure cubic field with $\theta^3 = d = m^3 \pm 1$, $m \geqslant 1$ and having a fundamental unit $\epsilon > 1$, then $\mathbb{Q}(\theta) \epsilon W_t$ and $t > 1$.

<u>Proof</u>

Let $\theta^3 = d = m^3 \pm 1$, $m \geqslant 1$, $d \neq 0$ and d is not a perfect cube. Then $\epsilon = \theta^2 + m\theta + m^2$ is the fundamental unit of $\mathbb{Q}(\theta)$ and $\theta - m$ its conjugate (see[15]) . So, we can write

$$\theta = m \pm \frac{1}{\epsilon} \quad ,$$

and

$$(m \pm \frac{1}{\epsilon})^3 = \theta^3 = m^3 \pm 1$$

$$m^3 \pm \frac{3m^2}{\epsilon} + \frac{3m}{\epsilon^2} \pm \frac{1}{\epsilon^3} = m^3 \pm 1$$

$$\frac{3m^2}{\epsilon} \pm \frac{3m}{\epsilon^2} \pm \frac{1}{\epsilon^3} = \pm 1$$

$$3m^2 \pm 3m\epsilon^{-1} \pm \epsilon^{-2} = \pm \epsilon$$

$$3m^2 = \pm 3m\epsilon^{-1} \pm \epsilon^{-2} \pm \epsilon$$

$$\frac{3m^2}{2} + \frac{3m}{2} - 1 = \pm 3m\epsilon^{-1} \pm \epsilon^{-2} \pm \epsilon - \frac{3m^2}{2} + \frac{3m}{2} - 1$$

$$\frac{3m(m+1)}{2} - 1 = \pm 3m\epsilon^{-1} \pm \epsilon^{-2} \pm \epsilon - \frac{3m(m-1)}{2} - 1 \; . \qquad (21)$$

By setting $t+1 = \dfrac{3m(m+1)}{2} - 1$, then $t = \dfrac{3m(m+1)}{2} - 2$ i.e

$\mathbb{Q}(\theta)$ belongs to W_t by Theorem 3.1.2 . If $t = 1$, then $\dfrac{3m(m+1)}{2} - 2 = 1$

and this implies that $m = 1$ or -2 in which case $\theta = \sqrt[3]{2}$, where such a field does not belong to W_1 since $\epsilon = 1 + \theta + \theta^2 > 3$ (see Theorem 3.1.3) Therefore $\mathbb{Q}(\theta)$ in W_t and $t > 1$ which completes the proof .

Theorem 3.1.4 , will give Sliwa's result in [13] that no pure cubic field $\mathbb{Q}(\theta)$, $\theta^3 = d = m^3 \pm 1$, is in W_1. Therefore the condition $\epsilon < 2t + 1$ of Theorem 3.1.3 , does not hold for $t = 1$ and $\theta^3 = d = m^3 \pm 1$.

A precise example for our results is that $\mathbb{Q}(\sqrt[3]{2})$ belongs to W_3 since

$$4 = \epsilon + 2\epsilon^{-2} + \epsilon^{-3} \quad ,$$

follows from Theorem 3.1.2 , where $\epsilon = 1 + \theta + \theta^2$, $\theta = \sqrt[3]{2}$, is the fundamental unit of $\mathbb{Q}(\sqrt[3]{2})$. Further, and as a consequence of Theorem 3.1.4 , one can find that

$$\mathbb{Q}(\sqrt[3]{7}) \in W_6 \ , \ \mathbb{Q}(\sqrt[3]{26}) \in W_{13} \ , \ \mathbb{Q}(\sqrt[3]{63}) \in W_{24} \ , \ldots\ldots .$$

§3.2 Results On Quartic Fields

Concerning the quartic fields we shall prove the following general result ;

Theorem 3.2.1

For each t , $t \in \mathbb{N}$, there are infinitely many quartic feilds in W_t .

For the proof of Theorem 3.2.1 , the following results will be needed .

Lemma 3.2.1

If $\mathbb{Q}(\theta)\epsilon W_t$, $\mathbb{Q}(\varphi)\epsilon V$ and their discriminants are relatively prime, then $\mathbb{Q}(\theta,\varphi)\epsilon W_t$.

Proof

Let $\mathbb{Q}(\theta)$, $\mathbb{Q}(\varphi)$ have the degrees n_1 , n_2 and discriminant d_1 , d_2 respectively.Since $(d_1,d_2) = 1$, then $\mathbb{Q}(\theta,\varphi)$ has degree n_1n_2 (see[16 :p145]).Also from [17],we get that $\mathbb{Q}(\theta,\varphi)$ has discriminant $d_1^{n_2}.\ d_2^{n_1}$ and if

$$\{\alpha_i\}\ ,\ i = 1,2,\ldots,n_1$$

$$\{\beta_j\}\ ,\ j = 1,2,\ldots,n_2,$$

are integral basis for $\mathbb{Q}(\theta)$ and $\mathbb{Q}(\varphi)$ respectively,then $\{\alpha_i\beta_j\}$ is an integral basis of $\mathbb{Q}(\theta,\varphi)$. Thus , a typical integer of $\mathbb{Q}(\theta,\varphi)$ is of the form

$$\sum_{i,j} a_{ij}\ \alpha_i\beta_j\quad ,\quad i=1,2,\ldots,n_1\ ,\ j=1,2,\ldots,n_2\quad ,(1)$$

where a_{ij} are rational integers . Let $\mathbb{Q}(\varphi)$ have $2k_1$ distinct roots of unity and k_2 fundamental units , where each of these fundamental units satisfies an equation of degree n_2 over $\mathbb{Z}$. So , since $\mathbb{Q}(\varphi)\epsilon V$, then all integers of $\mathbb{Q}(\varphi)$ can be expressed as a $\mathbb{Z}$-linear combination of $2k_1 \cdot n_2^{k_2}$ distinct units . Therefore all integers of $\mathbb{Q}(\varphi)$ can be written as a $\mathbb{Z}$-linear combination of $k = k_1n_2^{k_2}$ distinct units and none is the negative of the another . We denote these k units by

$$\zeta_1\ ,\ \zeta_2\ ,\ldots,\zeta_k\qquad .$$

We have that β_j , $j = 1,2,\ldots,n_2$ are expressible as $\mathbb{Z}$-linear combinations of the k units . Then the typical integer of $\mathbb{Q}(\theta,\varphi)$ given in (1) above is

$$\sum_{i,j} b_{ij}\alpha_i\beta_j \quad , \; i=1,2,\ldots,n_1 \quad , \; j=1,2,\ldots,n_2 \; , \qquad (2)$$

where b_{ij} are rational integers . Therefore (2) can be written as

$$\sum_{j=1}^{k} \gamma_j \zeta_j \quad ,$$

where γ_j are integers in $\mathbb{Q}(\theta)$. Since $\mathbb{Q}(\theta)\epsilon W_t$, the γ_j can be expressed as t-integers provided that we can not have relations of the form

$$\zeta\delta = \zeta'\delta' \quad ,$$

where δ and δ' are units in $\mathbb{Q}(\theta)$, ζ and ζ' are units in $\mathbb{Q}(\varphi)$. If so, then

$$\zeta/\zeta' = \delta'/\delta = \epsilon \neq \pm 1 \quad ,$$

and ϵ is a unit not equal to ± 1 . Therefore , $\mathbb{Q}(\theta) \cap \mathbb{Q}(\varphi) = \epsilon$ which is a contradiction since $\mathbb{Q}(\theta) \cap \mathbb{Q}(\varphi) = \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers . Thus , every integer in $\mathbb{Q}(\theta,\varphi)$ is a t-integer . Hence $\mathbb{Q}(\theta,\varphi)$ belongs to W_t , $t \geq 1$ and this completes the proof .

Lemma 3.2.2[12]

$\mathbb{Q}(\sqrt{d})$ belongs to V if and only if (i) $d = -1$ or -3,(ii) $d > 0$, $d \not\equiv 1 \pmod 4$ and either $d+1$ or $d-1$ is a perfect square,(iii) $d > 0$, $d \equiv 1 \pmod 4$ and either $d+4$ or $d-4$ is a perfect square .

For the proof (see [12]) .

The Proof Of Theorem 3.2.1

We consider $\mathbb{Q}(\sqrt{m})$, $m > 1$, $m \not\equiv 1 \pmod 4$, m is squarefree and $\mathbb{Q}(\sqrt{m})\epsilon W_t$.Suppose that $\mathbb{Q}(\sqrt{d})$,$d > 1$,$d \equiv 1 \pmod 4$,$d \pm 4$ is a perfect square and d is squarefree.By applying the Lemmas 3.2.1 and 3.2.2,we have that $\mathbb{Q}(\sqrt{m},\sqrt{d})\epsilon W_t$.It follows from Lemma 3.2.1, that the degree of $\mathbb{Q}(\sqrt{m},\sqrt{d})$ is equal to 4 and the discriminant is equal to $(4m)^2d^2$.Since each d will produce a different quartic field,then we need only to show that there are an infinite number of $d > 1$ such that $d \pm 4$ is a perfect square , $d \equiv 1 \pmod 4$ and d is squarefree . This can be done by showing that there are an infinite number of odd natural numbers , n ,such that $n^2 + 4$ is squarefree . So ,

$$\sum_{n \text{ odd}} \lambda^2(n^2 + 4) = \sum_{n} \lambda^2(n^2 + 4) .$$

Since $n^2 + 4$ is an irreducible polynomial of degree 2,then it follows from Nagell's result in [18] that

$$\sum_{n<x} \lambda^2(n^2 + 4) \longrightarrow \infty ,$$

as x tends to infinity . Also ,by using the fact given by Ricci in [19] which shows that

$$\sum_{n<x} \lambda^2(n^2 + 4) \sim Ax ,$$

for large x , where $A > 0$ is some constant . Therefore we obtain an infinite number of quartic fields in W_t and this completes the proof .

Belcher's result given in [12] will follow from Theorem 3.2.1 , when $m = 2$. He proved that $\mathbb{Q}(\sqrt{2},\sqrt{d}) \in W_1$, where $d > 1$, $d \pm 4$ is a perfect square , $d \equiv 1 \pmod 4$ and showed that there are infinitely many such quartic fields in W_1 .

Now,a question arises"In which W_t does $\mathbb{Q}(\sqrt{m},\sqrt{d})$ belong ? ". The answer will be given by the general result which we will discuss in the next section . The result will provide us with a criterion which enables us to show to which W_t $\mathbb{Q}(\sqrt{m})$ belongs. It gives us , for instance , that $\mathbb{Q}(\sqrt{3}) \in W_2$ and by Theorem 3.2.1 , $\mathbb{Q}(\sqrt{3},\sqrt{d})$ will be in W_2 for $d > 1$, $d \pm 4$ is a perfect square and $d \equiv 1 \pmod 4$.

§3.3 Sums Of Units Of Quadratic Fields

Let $U(d)$ be the set of finite sums of units in $\mathbb{Q}(\sqrt{d})$, where d is a squarefree integer not equal to zero or 1 . One can check that $U(d)$ is a subring of $\mathbb{Z}(\sqrt{d})$. If $d < 0$, then because the only units are ± 1

$$U(d) = \mathbb{Z} \quad ,$$

except when $d = -1$ or -3 . In these cases there are exactly four units or six units given by $\{\pm 1 , \pm i\}$ or $\{\pm 1 , \pm (-1+\sqrt{-3})/2 , \pm (-1 - \sqrt{-3})/2\}$ respectively . Therefore ,

$$U(d) = \mathbb{Z}(\sqrt{d}) \ .$$

If $d > 0$, then $\mathbb{Q}(\sqrt{d})$ has infinitely many units and these units have the form $\pm \epsilon^n$, $n \geq 0$, where $\epsilon = a + b\sqrt{d}$ or $a + \dfrac{b(1+\sqrt{d})}{2}$ is the fundamental unit according as $d \not\equiv 1 \pmod 4$ or $d \equiv 1 \pmod 4$ respectively and $a,b \in \mathbb{Z}$. The structure of $U(d)$ in such cases is described in the following result :

Theorem 3.3.1

Let $\epsilon = a + b\sqrt{d}$ or $a + \dfrac{b(1+\sqrt{d})}{2}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$.

Then

$$U(d) = \begin{cases} \langle 1 \ , \ b\sqrt{d} \rangle & , \text{ if } d \not\equiv 1 \pmod 4 \ , \\ \langle 1 \ , \ b(1+\sqrt{d})/2 \rangle & , \text{ if } d \equiv 1 \pmod 4 \ . \end{cases}$$

Proof

Considering $d \not\equiv 1 \pmod 4$, then $\epsilon = a + b\sqrt{d}$. Let

$$\epsilon^n = a_n + b_n\sqrt{d} \quad , \quad a_n, b_n \in \mathbb{Z} \ . \tag{1}$$

Then we can write $b_n = \delta b$, $\delta \geq 1$. Therefore $b \mid b_n \ \forall \ n$, $n \geq 0$. Now, take any sum of units in $U(d)$ such as

$$\alpha = \sum_{i=r}^{s} c_i \epsilon^i \quad , \tag{2}$$

where $r,s \geq 1$, $c_i \epsilon \mathbb{Z}$, $(r \leq i \leq s)$. From (2) the typical term will be of the form

$$c_i\epsilon^i = c_i(a + b\sqrt{d})^i = c_i(a_i + b_i\sqrt{d}) \quad ,$$

i.e

$$c_i\epsilon^i = c_ia_i + c_ib_i\sqrt{d} \quad . \qquad (3)$$

Since $b|b_i$, then $b|c_ib_i$ for each i , $(r \leq i \leq s)$. So , by writing (3) as

$$c_i\epsilon^i = A + B\sqrt{d} \quad ,$$

where $b|B$, then any sum in $U(d)$ has an order $< 1 \,,\, b\sqrt{d} >$ when $d \not\equiv 1 \pmod 4$.

If $d \equiv 1 \pmod 4$, then $\epsilon = a + \frac{b(1 + \sqrt{d})}{2}$. So , by following the same procedure given above , we get

$$c_i\epsilon^i = c_i\left(a + \frac{b(1 + \sqrt{d})}{2}\right)^i = c_ia_i + \frac{c_ib_i(1+\sqrt{d})}{2} \quad , \qquad (4)$$

where $b|b_i$ for each i , $(s \leq i \leq r)$.By writing (4) as

$$c_i\epsilon^i = A' + \frac{B'(1 + \sqrt{d})}{2} \quad ,$$

where A' , $B' \epsilon \mathbb{Z}$ it follows that every sum in $U(d)$ has an order $< 1 \,,\, \frac{b(1 + \sqrt{d})}{2} >$. This completes the proof .

As we defined in section 3.1 ,the set V contains all the algebraic number fields having the property that every integer is expressible as a finite sum of units (not necessarily distinct) . The quadratic fields which are in V are determined by Lemma 3.2.2[12] . The integers of these quadratic fields may be expressed as t-integers , $t \epsilon \mathbb{N}$.

For t=1 ,B.Jacobson[3] showed that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ are in W_1 and J.Sliwa[13] confirmed that these are the only quadratic fields in W_1.

One may expect that other quadratic fields are in W_2 or $W_3, \ldots,$. This will be given in the following general result :

Theorem 3.3.2

Let $\mathbb{Q}(\sqrt{d}) \in V$ and $\epsilon = t + \sqrt{d}$ or $\{(2t - 1) + \sqrt{d}\}/2$ be the fundamental unit according as $d \not\equiv 1 \pmod 4$ or $d \equiv 1 \pmod 4$. Then $\mathbb{Q}(\sqrt{d}) \in W_t$.

We shall need some preliminary conditions in the proof of Theorem 3.3.2 .

Preliminary Conditions

Since ϵ is the fundamental unit , $\epsilon > 1$ and $\epsilon^{-1} > 0$. when $\mathbb{Q}(\sqrt{d}) \in V$ and $d \not\equiv 1 \pmod 4$, we have $\epsilon = t + \sqrt{d}$ is the fundamental unit. Since $\epsilon^{-1} = \pm \bar{\epsilon}$ according as $N(\epsilon) = \pm 1$, where $\bar{\epsilon}$ is the conjugate of ϵ, then

$$2t = \epsilon \pm \epsilon^{-1} \quad .$$

Therefore ,

$$t+1 = \epsilon \pm \epsilon^{-1} + (1 - t) \ , \qquad (5)$$

where (5) shows that $t+1$ is a strong t-integer for all t , $t \in \mathbb{N}$.

When $\mathbb{Q}(\sqrt{d}) \in V$ and $d \equiv 1 \pmod 4$, then $\epsilon = \{(2t - 1) + \sqrt{d}\}/2$ and also $\epsilon^{-1} = \pm \bar{\epsilon}$, for $N(\epsilon) = \pm 1$ respectively . So , we can write

$$2t - 1 = \epsilon \pm \epsilon^{-1} \ ,$$

i.e

$$t = \epsilon \pm \epsilon^{-1} + (1 - t) \ , \qquad (6)$$

and (6) implies that $t+1$ is a strong t-integer for all t , $t \geq 2$.

Now , we shall use both (5) and (6) to prove Theorem 3.3.2 .

Proof Of Theorem 3.3.2

(I) Let $d \not\equiv 1$ (mod4) and $\mathbb{Q}(\sqrt{d}) \in V$. Then for all integers $\alpha \in \mathbb{Q}(\sqrt{d})$, α can be written in the form

$$\alpha = \sum_{p}^{q} a_i \epsilon^i \quad , \qquad (7)$$

where a_i , p , $q \in Z$, $(p \leq i \leq q)$, $a_i = 0$ if $i \notin [p,q]$. We shall consider the following cases ;

(i) If $N(\epsilon) = +1$, then we have

$$t+1 = \epsilon + \epsilon^{-1} + (1 - t) , \qquad (8)$$

where t+1 is a strong t-integer in $\mathbb{Q}(\sqrt{d})$. Assume that α given in (7) is such that

$$\Sigma |a_i| = m \quad , \quad m \geq 1$$

and m is minimal . We shall follow by induction on m . So , if $m = 1$, then we have $\Sigma |a_i| = 1$ and hence $a_i = 0$, ± 1 . Therefore α is a 1-integer and by definition 3.1.1 , α is a t-integer . As an induction hypothesis we suppose that α is a t-integer for all $\alpha \in \mathbb{Q}(\sqrt{d})$ with $m < M$, $(M \geq 2)$. We will prove it for all α with $m = M$. We have from (8) that

$$(t+1)\epsilon^k = \epsilon^{k-1} + (1 - t)\epsilon^k + \epsilon^{k+1} \quad , \quad (p \leq k \leq q) .$$

We suppose that a_k is the first coefficient in (7) such that $|a_k| \geq t+1$ and $|a_{k-1}| \leq t$ for all k , $(p \leq k \leq q)$. Then by applying (8) to the term $a_k \epsilon^k$, α is either of the form

$$\alpha = a_p \epsilon^p + \ldots + (a_{k-1}+1)\epsilon^{k-1} + (1 - t)\epsilon^k + \epsilon^{k+1} + \ldots + a_q \epsilon^q , \qquad (9)$$

or

$$\alpha = a_p \epsilon^p + \ldots + (a_{k-1} - 1)\epsilon^{k-1} + (t-1)\epsilon^k - \epsilon^{k+1} + \ldots + a_q \epsilon^q , \quad (10)$$

according as $a_k = \pm (t+1)$. Since $|a_{k-1}| \leq t$, then , if necessary , we need to repeat the application of (8) to $(a_{k-1} \pm 1)\epsilon^{k-1}$ in (9) or (10) . This implies that α is either of the form

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-2} + 1)\epsilon^{k-2} + (1-t)\epsilon^{k-1} + (2-t)\epsilon^k + \epsilon^{k+1} + \ldots + a_q\epsilon^q, \tag{11}$$

or

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-2} - 1)\epsilon^{k-2} + (t-1)\epsilon^{k-1} + (t-2)\epsilon^k - \epsilon^{k+1} + \ldots + a_q\epsilon^q, \tag{12}$$

according as $a_{k-1} = \pm\, t$ respectively . By considering $-\alpha$ given in (9) or (10) ,if necessary, we may get that $|a_{k-1} \pm 1| \geq t+1$. So, by applying (8) in a similar way to that given above,then α will be also of the form (11) or (12).

Thus, if necessary, we continue to apply (8) in (11) or (12) to the term $(a_{k-2} \pm 1)\epsilon^{k-2}$. Then after a finite number of applications we obtain that α is of the form

$$\alpha = \sum_{j<k} b_j\epsilon^j + \sum_{i>k+1} c_i\epsilon^i \quad , \tag{13}$$

with $0 < |b_j| \leq t$ and $\sum_{i>k+1}|c_i| \leq t+1$. Notice that every coefficient a_i with $i \leq k$ is reduced to b_j , where $|b_j| \leq t$ and some of the b_j are non-zero . So,we have $\sum_{j<k}|b_j| > 0$ and therefore $\sum_{j<k}|b_j| \leq \sum_{i>k+1}|c_i|$. Thus

$$\sum_{j<k}|b_j| + \sum_{i>k+1}|c_i| \leq M \quad .$$

Since M is minimal , then

$$\sum_{j<k}|b_j| + \sum_{i>k+1}|c_i| = M \quad .$$

Since $\sum_{j<k}|b_j| > 0$, then $\sum_{i>k+1}|c_i| < M$. So , by applying the induction to $\alpha' = \sum_{i>k+1} c_i\epsilon^i$,then our result will follow and hence α is a t-integer .

(ii) If $N(\epsilon) = -1$, then

$$t+1 = \epsilon - \epsilon^{-1} + (1 - t) \quad . \tag{14}$$

Hence , by following the procedure given in (i) to (14), then α is a t-integer .

(II) Let $\mathbb{Q}(\sqrt{d}) \epsilon V$ and $d \equiv 1 \pmod 4$. Then in this case we shall have

$$t+1 = \epsilon \pm \epsilon^{-1} + (2 - t) \quad ,$$

according as $N(\epsilon) = \pm 1$ respectively . Firstly , we consider when $N(\epsilon) = +1$. So , we have

$$t+1 = \epsilon + \epsilon^{-1} + (2 - t) \quad , \tag{15}$$

where (15) shows that t+1 is a strong t-integer for all t , $t \geqslant 1$. We shall follow the induction steps as in (I) . So , we assume that α has the representation (7) so that

$$\sum_{p \leqslant i \leqslant q} |a_i| = m \quad , \quad m \geqslant 1 \quad ,$$

and m is minimal . Then , if $m = 1$ we shall have $\sum_i |a_i| = 1$ and hence $a_i = 0 , \pm 1$. Therefore α is a 1-integer i.e α is a t-integer . As an induction hypothesis we assume that α is a t-integer for all $\alpha \epsilon \mathbb{Q}(\sqrt{d})$ with $m < M$, $(M \geqslant 2)$. We will then prove it for all α with $m = M$. From (15) we have

$$(t+1)\epsilon^k = \epsilon^{k-1} + (2 - t)\epsilon^k + \epsilon^{k+1} \quad , \quad (p \leqslant k \leqslant q) \ . \tag{16}$$

Now , we apply (15) to the first term in (7) , say $a_k \epsilon^k$ with $|a_k| \geqslant t+1$ and $|a_{k-1}| \leqslant t$ for all k , $(p \leqslant k \leqslant q)$. Then α is either of the form

$$\alpha = a_p \epsilon^p + \ldots + (a_{k-1} + 1)\epsilon^{k-1} + (2-t)\epsilon^k + \epsilon^{k+1} + \ldots + a_q \epsilon^q \ , \tag{17}$$

or

$$\alpha = a_p \epsilon^p + \ldots + (a_{k-1} - 1)\epsilon^{k-1} + (t-2)\epsilon^k - \epsilon^{k+1} + \ldots + a_q \epsilon^q \ , \tag{18}$$

according as $a_k = \pm (t+1)$. Since $|a_{k-1}| \leqslant t$, then , if necessary , we repeat the application of (15) to the term $(a_{k-1} \pm 1)\epsilon^{k-1}$ in (17) or (18) respectively . This will imply that α is either of the form

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-2}+1)\epsilon^{k-2} + (2-t)\epsilon^{k-1} + (3-t)\epsilon^k + \epsilon^{k+1} + \ldots + a_q\epsilon^q, \tag{19}$$

or

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-2}-1)\epsilon^{k-2} + (t-2)\epsilon^{k-1} + (t-3)\epsilon^k - \epsilon^{k+1} + \ldots + a_q\epsilon^q, \tag{20}$$

By considering $-\alpha$ in (17) or (18) , if necessary , we may get $|a_{k-1} \pm 1| \geq t+1$ and by applying (15) as we did above then α is either of the form (19) or (20) . Therefore continue applying (15) to the term $(a_{k-2} \pm 1)\epsilon^{k-2}$, if necessary . Then after a finite number of applications we obtain

$$\alpha = \sum_{j\leq k} d_j\epsilon^j + \sum_{i\geq k+1} e_i\epsilon^i \quad , \tag{21}$$

where $|d_j| \leq t$, $\sum\limits_{i\geq k+1}|e_i| \geq t+1$ and some of the d_j are non-zero .

Therefore $\sum\limits_{j\leq k}|d_j| > 0$ and hence

$$\sum_{j\leq k}|d_j| + \sum_{i\geq k+1}|e_i| \leq M \quad ,$$

Since M is minimal then

$$\sum_{j\leq k}|d_j| + \sum_{i\geq k+1}|e_i| = M \quad .$$

Since $\sum\limits_{j\leq k}|d_j| > 0$, then $\sum\limits_{i\geq k+1}|e_i| < M$. So , by applying the induction hypothesis to $\sum\limits_{i\geq k+1} e_i\epsilon^i$ we get that α is a t-integer .

Next , when $N(\epsilon) = -1$, then we shall have

$$t+1 = \epsilon - \epsilon^{-1} + (2-t) \quad . \tag{22}$$

Hence , by following the procedure given above to (22) ,then α is a t-integer and this completes the proof .

Immediate consequences of the proof of Theorem 3.3.2, are the following results ;

Corollary 3.3.1

Every integer expressible as a finite sum of units in $\mathbb{Q}(\sqrt{d})$ is a t-integer .

Corollary 3.3.2

Let $\mathbb{Q}(\sqrt{d})$ belong to $W_{t+1} \setminus W_t$, $t \in \mathbb{N}$. Then every integer of $\mathbb{Q}(\sqrt{d})$ is a t-integer plus a single unit .

Example 3.3.1

Every integer $\alpha \in \mathbb{Q}(\sqrt{13})$ is a sum of distinct units plus a single unit .

Let ϵ be the fundamental unit of $\mathbb{Q}(\sqrt{13})$. Then $\epsilon = (3 + \sqrt{13})/2$ and $N(\epsilon) = -1$. Therefore we can write 3 as

$$3 = \epsilon - \epsilon^{-1} ,$$

i.e

$$2 = \epsilon - \epsilon^{-1} - 1 . \qquad (23)$$

So , by Theorem 3.3.2 , $\mathbb{Q}(\sqrt{13}) \in W_2$. We let α be an integer in $\mathbb{Q}(\sqrt{13})$ such that

$$\alpha = -\epsilon^{-1} + 2 + \epsilon .$$

By applying (23) we shall have

$$\alpha = -2\epsilon^{-1} + 2\epsilon - 1 ,$$

again apply (23) , then

$$\alpha = \epsilon^{-2} + \epsilon^{-1} - \epsilon + \epsilon^{2} - 3 .$$

Put $3 = \epsilon - \epsilon^{-1}$, then

$$\alpha = \epsilon^{-2} + 2\epsilon^{-1} - 2\epsilon + \epsilon^{2}$$

so that

$$\alpha = \epsilon^{-2} + 2\epsilon^{-1} - (\epsilon - \epsilon^{-1} - 1)\epsilon + \epsilon^{2} .$$

Therefore ,

$$\alpha = (\epsilon^{-2} + \epsilon^{-1} + \epsilon + 1) + \epsilon^{-1} ,$$

and this last expression shows that α is a 1-integer plus a unit .

We prove the following result;

Theorem 3.3.3

Let $\mathbb{Q}(\sqrt{d})$, $d > 0$, $\epsilon = t + s\sqrt{d}$ or $\{(2t - 1) + s\sqrt{d}\}/2$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. If $\mathbb{Q}(\sqrt{d}) \in W_t$, then $\mathbb{Q}(\sqrt{d}) \notin W_{t-1}$.

Proof

Suppose that $\mathbb{Q}(\sqrt{d}) \in W_t$ and $d \not\equiv 1 (\mathrm{mod} 4)$. Then the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is $\epsilon = t + s\sqrt{d}$, $s, t \in \mathbb{Z}$. It suffices to show that t can not be expressed as a $(t-1)$-integer . Assume that

$$t = \sum_{i=k}^{m} c_i \epsilon^{i} , \tag{24}$$

where $|c_i| \leq t-1$ and $k, m \in \mathbb{Z}$. So , if $m \geq 1$, then

$$\epsilon^{m} \leq |c_m| \epsilon^{m} \leq \sum_{i=k}^{m-1} c_i \epsilon^{i} + t$$

$$< (t-1) \sum_{i=-\infty}^{m-1} \epsilon^{i} + t$$

$$= (t-1) \frac{\epsilon^{m-1}}{1 - \frac{1}{\epsilon}} + t$$

$$= \frac{(t - 1)\epsilon^{m}}{\epsilon - 1} + t .$$

So , we may have

$$\epsilon^{m} \leq \frac{(t - 1)\epsilon^{m}}{\epsilon - 1} + t\epsilon^{m-1}, \quad m \geq 1 \quad , \epsilon^{m-1} \geq 1.$$

Therefore

$$1 \leqslant \frac{(t-1)}{\epsilon - 1} + \frac{t}{\epsilon} \qquad ,$$

i.e

$$\epsilon^2 - 2t\epsilon + t \leqslant 0 \ . \tag{25}$$

Since $\epsilon = t + \sqrt{t^2 \pm 1}$, then it follows from (25) that

$$\epsilon \leqslant t + \sqrt{t^2 - t} < t + \sqrt{t^2 - 1} \leqslant \epsilon \quad ,$$

which is a contradiction . Thus , t is not a (t-1)-integer .

If either $m < 0$ or $m = 0$,then the largest term in (24) is $t = c_0$. Hence t is not a (t-1)-integer.

A similar procedure can be done when $d \equiv 1 \pmod 4$ which implies that

$$2\epsilon \leqslant (2t-1) + \sqrt{(2t-1)^2 - 4t} < (2t-1) + \sqrt{(2t-1)^2 - 4} = 2\epsilon \quad .$$

and this last inequality represents a contradiction too . Therefore $\mathbb{Q}(\sqrt{d}) \notin W_{t-1}$ and this completes the proof of the Theorem .

Theorem 3.3.3, shows that t given in Theorem 3.3.2 is minimal .

Now , we prove the following;

Theorem 3.3.4

For any t , $t \in \mathbb{N}$, there are a finite number of real quadratic fields $\mathbb{Q}(\sqrt{d})$ in W_t .

Proof

Let $\mathbb{Q}(\sqrt{d})$, $d > 0$, $d \not\equiv 1 \pmod 4$ be in W_t , $t \in \mathbb{N}$. Then $\epsilon = t + \sqrt{d}$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$. It suffices to show that t is an integer in the interval $(\frac{\epsilon - 1}{2} , \frac{\epsilon+1}{2})$. We have from Theorem 3.1.3, in section 3.1,that

$$\epsilon < 2t + 1 \qquad , \qquad t \geqslant 1 \ .$$

Also, in $\mathbb{Q}(\sqrt{d})$ we can write

$$2t = \epsilon \pm \epsilon^{-1} .$$

Then

$$2t - 1 < \epsilon < 2t + 1 , \qquad (26)$$

and (26) implies that

$$\frac{\epsilon-1}{2} < t < \frac{\epsilon+1}{2} . \qquad (27)$$

Therefore t is an integer in the interval $(\frac{\epsilon-1}{2} , \frac{\epsilon+1}{2})$. So , from (27) we can get that $t-1 < \sqrt{d} < t+1$. Thus , for any t , $t \epsilon \mathbb{N}$, there are a finite number of real quadratic fields in W_t .

A similar procedure can be followed when $\mathbb{Q}(\sqrt{d}) \epsilon W_t$ and $d \equiv 1 (\mathrm{mod} 4)$. This will lead us to get $t < \frac{1 + \sqrt{d}}{2} < t + 1$ since $\epsilon = \{(2t-1) + \sqrt{d}\}/2$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$. This completes the proof .

If we consider all the quadratic fields $\mathbb{Q}(\sqrt{d})$ with d in the range $2 \leqslant d < 100$,then by Lemma 3.2.2[12] , there are fourteen fields in V , namely d = 2 , 3 , 5 , 10 , 13 , 15 , 21 , 26 , 29 , 35 , 53 , 77 , 82 and 85 . So, by using Theorem 3.3.2, we can classify these fields in W_t as follows :

$\mathbb{Q}(\sqrt{d}) \epsilon W_1$, $d = 2 , 5$.

$\mathbb{Q}(\sqrt{d}) \epsilon W_2$, $d = 3 , 13$.

$\mathbb{Q}(\sqrt{d}) \epsilon W_3$, $d = 10 , 21 , 29$.

$\mathbb{Q}(\sqrt{d}) \epsilon W_4$, $d = 15 , 53$.

$\mathbb{Q}(\sqrt{d}) \epsilon W_5$, $d = 26 , 77 , 82 , 85$.

$\mathbb{Q}(\sqrt{d}) \epsilon W_6$, $d = 35$.

This classification is obtained by demonstrating the fundamental unit of each field.One can use the number of the repetitions t of these fields given above to check the validity of the condition $\epsilon < 2t + 1$ given in Theorem 3.1.3. For instance,we have for $\mathbb{Q}(\sqrt{d}) \epsilon W_2$, that $\epsilon < 5$ which is confirmed by Sliwa in [13] .

Theorem 3.3.2 , will also enable us to determine to which W_t does any quartic field $\mathbb{Q}(\sqrt{m}, \sqrt{d})$ belong. This can be done by using Lemma 3.2.1 given in section 3.2 .

CHAPTER (4)

§4.1 Algebraic Numbers As Sums Of Distinct Integers Of Fixed Norm

In this chapter we shall study the representation of integers in algebraic number fields in $\mathbb{Q}(\theta)$ as sums of distinct integers of a certain norm .

For this purpose the following definition will be needed :

Definition 4.1.1

An integer $\alpha\epsilon\mathbb{Q}(\theta)$ is called a norm-s integer, $s\epsilon\mathbb{N}$, if $|N(\alpha)| \leqslant s$.

Let N_s , $s\epsilon\mathbb{N}$, be the set of all algebraic number fields having the property that every integer can be written as a sum of norm-s integers (not necessarily distinct) . Define N_s^* to be the subset of N_s containing all algebraic number fields for which every integer is expressible as a sum of distinct norm-s integers .

When $s = 1$, then it is obvious that $N_1 = V$, where V is the set of all algebraic number fields having the property that all integers are expressible as a finite sum of units in $\mathbb{Q}(\theta)$. Also $N_1^* = W_1$, where W_1 is the set of all algebraic number fields for which every integer is a sum of distinct units of $\mathbb{Q}(\theta)$.

Now, we will adopt the following definition in order to prove the general result ;

Definition 4.1.2

Let $s\epsilon\mathbb{N}$. An integer $\delta\epsilon\mathbb{Q}(\theta)$ is called a strong norm-s integer ,if

$$\delta = \gamma_1\epsilon_1 + \ldots + \gamma_u\epsilon_u ,$$

$$|N(\gamma_i)| \leqslant s ,$$

$$\sum_{1\leqslant i\leqslant u} |N(\gamma_i)| \leqslant |N(\delta)| ,$$

$\gamma_i\epsilon\mathbb{Q}(\theta)$ are non-associate and $\epsilon_1, \ldots, \epsilon_u$ are units in $\mathbb{Q}(\theta)$ for all i .

We shall prove the following general result :

Theorem 4.1.1

Let $s \in \mathbb{N}$ and let $\mathbb{Q}(\theta) \in W_t$,for some $t \geq s^{1/2}$. We assume that t is chosen to be minimal. If every integer in $\mathbb{Q}(\theta)$ of norm $\leq t^2$ is a strong norm-s integer,then $\mathbb{Q}(\theta) \in N_s^*$.

Proof

Let $\mathbb{Q}(\theta) \in W_t$, $t \in \mathbb{N}$, and $\epsilon_1, \ldots, \epsilon_u$ be its units . We choose a set of distinct units in $\mathbb{Q}(\theta)$, say ,

$$\eta_1, \ldots, \eta_d \, .$$

Then if $\epsilon_1, \ldots, \epsilon_u$ are independent units in $\mathbb{Q}(\theta)$,by Theorem 3.1.2, any integer $\alpha \in \mathbb{Q}(\theta)$ can be written in the form

$$\alpha = \sum_{i_1=p_1}^{q_1} \cdots \cdots \sum_{i_u=q_u}^{q_u} \; \sum_{k=1}^{d} a_{i_1 \ldots i_u,k} \, \eta_k \epsilon_1^{i_1} \cdots \epsilon_u^{i_u} \quad , \qquad (1)$$

where $|a_{i_1, \ldots, i_u,k}| \leq t$, and $p_i, q_i, d \in Z$,$(1 \leq i \leq u)$.

If $\epsilon_1, \ldots, \epsilon_u$ are dependent units in $\mathbb{Q}(\theta)$then there exists a set of units , say , $\omega_1, \ldots, \omega_\upsilon$ $\upsilon < u$ such that

$$\epsilon_i = \zeta^j . \omega_1^{r_{i1}} \ldots \omega_\upsilon^{r_{i\upsilon}} \; , \qquad 1 \leq j \leq \ell \; , \quad r_{i\upsilon} \in Z \quad ,$$

where $\zeta^j$is the ℓ^{th}root of unity. Also, there is no relation of the form

$$\epsilon_1^{i_1} \cdots \epsilon_u^{i_u} = 1 \quad .$$

On the other hand there exists at least one unit say, ϵ_u which does not depend on any of the other units.In such a case Theorem 3.1.2, implies that any integer $\alpha \in \mathbb{Q}(\theta)$ is of the form

$$\alpha = \sum_{i_1=p_1}^{q_1} \cdots \cdots \sum_{i_\upsilon=p_\upsilon}^{q_\upsilon} \; \sum_{j=1}^{\ell} \; \sum_{k=1}^{d} a_{i_1 \ldots i_\upsilon,j,k} \, \zeta^j \, \eta_k \, \omega_1^{i_1} \cdots \omega_\upsilon^{i_\upsilon} \; , \quad (2)$$

where $p_i, q_i, \ell, d \in Z$.

Firstly, in order to show that every integer in $\mathbb{Q}(\theta)$ is a sum of distinct norm-s integers , we shall consider the general integer α of $\mathbb{Q}(\theta)$ which has the form (1) .

If $\mathbb{Q}(\theta)\epsilon W_t$ and $t^2 = s$ then any integer α in $\mathbb{Q}(\theta)$ is a sum of distinct norm-s integers. Suppose that $\mathbb{Q}(\theta)\epsilon W_t$ and $s^{1/2} < t$. We assume that α given in (1) is a non-zero integer so that

$$\Sigma|N(a_{i_1,\ldots,iu,k})| = m \quad , \qquad (m \geqslant 1)$$

and m is minimal . We shall proceed by induction on m to show that α is of the form

$$\alpha = \sum_{i_1=r_1}^{\rho_1}\cdots\cdots\sum_{i_u=r_u}^{\rho_u} \sum_{k=1}^{d} \mu_{i_1\ldots iu,k},\ \eta_k\, \epsilon_1^{i_1}\cdots\epsilon_u^{i_u} \quad , \qquad (3)$$

where $|N(\mu_{i_1\ldots iu,k})| \leqslant s$ and $r_i \ ,\rho_i\epsilon Z$. So , if m = 1 then $\Sigma|N(a_{i_1,\ldots,iu,k})| = 1$ and hence $a_{i_1\ldots iu,k} = \pm 1$. Therefore α is a sum of distinct norm-1 integers. So,by definition 4.1.1, α is a sum of distinct norm-s integers . As an induction hypothesis we suppose that the result is true for all α in $\mathbb{Q}(\theta)$ with $m < M, (M \geqslant 2)$. We shall prove it for any α with $m = M$. Any integer $\delta\epsilon\ \mathbb{Q}(\theta)$ with $|N(\delta)| \leqslant t^2$ is a strong norm-s integer i.e δ is of the form

$$\delta = \gamma_1\epsilon_1 + \ldots + \gamma_u\epsilon_u \quad , \qquad (4)$$

where $|N(\gamma_i)| \leqslant s$, $\sum_i|N(\gamma_i)| \leqslant |N(\delta)|$ and γ_i are non-associate integers for all i , $(1\leqslant i\leqslant u)$. So , by using (4) any term $a_{i_1\ldots iu,k}\eta_k\epsilon_1^{i_1}!\cdot\cdot\epsilon_u^{i_u}$ with $|N(a_{i_1\ldots,iu,k})| \leqslant t^2$ in (1) can be written as

$$a_{i_1\ldots iu,k}\eta_k\epsilon_1^{i_1}\cdots\epsilon_u^{i_u} = \gamma_1\eta_k\epsilon_1^{i_1+1}\cdots\epsilon_u^{i_u} + \gamma_2\eta_k\epsilon_1^{i_1}\cdot\epsilon_2^{i_2+1}\cdots\cdot\epsilon_u^{i_u} + \ldots + \gamma_u\eta_k\epsilon_1^{i_1}\cdots\epsilon_u^{i_u+1} \qquad (5)$$

So,by applying (5) to the first coefficient ,say ,$a_{p_1,\ldots,iu,k}$ with

$|N(a_{p_1,..iu,k})| \leqslant t^2$ we shall get that

$$\alpha = \alpha_1 + \alpha_2 = \sum_{i_2=p_2}^{f_1} \cdots\cdots \sum_{i_u=h_u}^{f_u} \sum_{k=1}^{d} \gamma_{p_1 \ldots iu,k}\, \eta_k \epsilon_1^{p_1} \cdots \epsilon_u^{i_u} \quad +$$

$$\sum_{i_1=p_1+1}^{n_1} \cdots\cdots \sum_{i_u=p_u}^{n_u} \sum_{k=1}^{d} c_{i_1 \ldots iu,k}\, \eta_k \epsilon_1^{i_1} \cdots \epsilon_u^{i_u} \quad ,$$

where $|N(\gamma_{i_1 \ldots iu,k})| \leqslant s$, $|c_{i_1 \ldots iu,k}| \leqslant t$ and $h_i, f_i, n_i \epsilon Z$. Notice that some of the $\gamma_{i_1 \ldots iu,k}$ are non-zero. If any overlap occurs between α_1 and α_2 given above i.e if some of the $\gamma_{p_1 \ldots iu,k}$ equal $c_{i_1 \ldots iu,k}$, then we can write α in the form

$$\alpha = \alpha_3 + \alpha_4 = \sum_{i_2=w_1}^{z_1} \cdots\cdots \sum_{i_u=w_u}^{z_u} \sum_{k=1}^{d} \gamma_{p_1 \ldots iu,k}\, \eta_k \epsilon_1^{p_1} \cdots \epsilon_u^{i_u} \quad +$$

$$\sum_{i_1=p_1+1}^{e_1} \cdots\cdots \sum_{i_u=v_u}^{e_u} \sum_{k=1}^{d} \lambda_{i_1 \ldots iu,k}\, \eta_k \epsilon_1^{i_1} \cdots \epsilon_u^{i_u} \quad , \qquad (6)$$

where $\gamma_{p_1 \ldots iu,k}$ are distinct norm-s integers in $\mathbb{Q}(\theta), \lambda_{i_1 \ldots iu,k} \epsilon \mathbb{Q}(\theta)$ $\gamma_{p_1 \ldots iu,k} \neq \lambda_{i_1 \ldots iu,k}$, $z_i, w_i, v_i \epsilon Z$ and some of the $\gamma_{p_1 \ldots iu,k}$ are non-zero. Therefore

$$\Sigma |N(\gamma_{p_1 \ldots iu,k})| \leqslant \Sigma |N(\lambda_{i_1 \ldots iu,k})|$$

and hence

$$\Sigma |N(\gamma_{p_1 \ldots iu,k})| + \Sigma |N(\lambda_{i_1 \ldots iu,k})| \leqslant M \; .$$

Since M is minimal then

$$\Sigma |N(\gamma_{p_1 \ldots iu,k})| + \Sigma |N(\lambda_{i_1 \ldots iu,k})| = M \; .$$

Since $\Sigma |N(\gamma_{p_1 \ldots iu,k})| > 0$ then $\Sigma |N(\lambda_{i_1 \ldots iu,k})| < M$. So,by applying the induction hypothesis to the second term in (6) then we can write α_4 as sum of distinct norm-s integers . Again, if any overlap occurs between α_3 and α_4 we can repeat the above procedure . So,after a finite number of steps we obtain α is a sum of distinct norm-s integers .

Next , if we consider α in the form (2) , then by following the induction steps given above to the form (2) we obtain that α is a sum of distinct norm-s integers . Hence $\mathbb{Q}(\theta)\epsilon N_s^*$ and this completes the proof .

An immediate consequance of the proof of Theorem 4.1.1 , is the following :

<u>Corollary 4.1.1</u>

If α is any t-integer in $\mathbb{Q}(\theta)$, $t > s^{1/2}, s\epsilon\mathbb{N}$, and every integer of norm less than or equal to t^2 is a strong norm-s integer in $\mathbb{Q}(\theta)$, then α is a sum of distinct norm-s integers .

Now, we shall adopt the proof of Theorem 4.1.1 to prove the following result :

<u>Theorem 4.1.2</u>

$\mathbb{Q}(\sqrt[3]{2})$ belongs to N_3^* .

<u>Proof</u>

Since $\mathbb{Q}(\sqrt[3]{2})$ has only one fundamental unit ϵ , where $\epsilon = 1+\theta+\theta^2$ and $\theta = \sqrt[3]{2}$, then by Theorem 3.1.4 , $\mathbb{Q}(\sqrt[3]{2})$ belongs to W_t and $t > 1$. Further , in $\mathbb{Q}(\sqrt[3]{2})$ 4 is a strong 3-integer since 4 can be written as

$$4 = \epsilon + 2\epsilon^{-2} + \epsilon^{-3} .$$

So , by Theorem 3.1.2 , $\mathbb{Q}(\sqrt[3]{2})$ belongs to W_3 . Therefore any integer α in $\mathbb{Q}(\sqrt[3]{2})$ is of the form

$$\alpha = \sum_{i=p}^{q} a_i\epsilon^i \quad , \qquad p,q\epsilon Z \quad , \qquad (7)$$

where $|a_i| \leqslant 3$ for all $i,(p\leqslant i\leqslant q)$.

Consider the integer $\gamma = \theta+1$ in $\mathbb{Q}(\sqrt[3]{2})$, where $|N(\gamma)| = 3$. Then we can write 3 as

$$3 = \gamma\epsilon^{-2} + \gamma\epsilon^{-1} + \gamma \quad , \tag{8}$$

where (8) shows that 3 is a strong norm-3 integer by definition 4.1.2

We may suppose that some of the a_i's in (7) are non-zero and that the representation (7) of α is chosen so that

$$\sum_i |N(a_i)| = m \;, \qquad m \geq 1 \quad , \tag{9}$$

and m is minimal .

In order to show that α is a sum of distinct norm-3 integers we shall follow the induction on m . So , if $m = 1$ then it is obvious that $\sum_i |N(a_i)| = 1$ implies that $a_i = \pm 1$. Therefore ,by definition 4.1.1 , α is a sum of distinct norm-3 integers . As an induction hypothesis , we assume that the result is true for all $\alpha\epsilon\mathbb{Q}(\sqrt[3]{2})$ with $m < M$, $(M \geq 2)$. We will then prove it for all $\alpha\epsilon\mathbb{Q}(\sqrt[3]{2})$ with $m = M$.

Suppose that a_k , $(p\leq k\leq q)$, is the first coefficient in (7) with $|a_k| = 3$ or 2 and $|a_i| \leq 1$ for all i , $(p\leq i<k)$. If there is no a_k with $|a_k| = 3$ or 2 then α is a sum of distinct norm-3 integers .

Firstly , we let $|a_k| = 3$ .Then by applying (8) to $a_k\epsilon^k$, α is either of the form

$$\alpha = a_p\epsilon^p + \ldots + \gamma\epsilon^{k-2} + \gamma\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \;, \tag{10}$$

or

$$\alpha = a_p\epsilon^p + \ldots - \gamma\epsilon^{k-2} - \gamma\epsilon^{k-1} - \gamma\epsilon^k + \ldots + a_q\epsilon^q \;, \tag{11}$$

according as $a_k = \pm 3$ respectively , where all $|a_i| \leq 3$ for all $i>k$. Consider the form (10) . Then since $a_{k-2} = 0,\pm 1$ and $a_{k-1} = 0,\pm 1$,we need to consider the following cases:

(i) If $a_{k-2} = 1$ and $a_{k-1} = 1$, then

$$\alpha = a_p\epsilon^p+\ldots+(\gamma+1)\epsilon^{k-2}+(\gamma+1)\epsilon^{k-1}+\gamma\epsilon^k+\ldots+a_q\epsilon^q. \quad (12)$$

Since $\gamma - 2 = \epsilon^{-1}$, then from (8) we get

$$\gamma + 1 = \epsilon - \gamma\epsilon^{-1}. \qquad (13)$$

So , by using (13) in (12) we obtain

$$\alpha = a_p\epsilon^p+\ldots-\gamma\epsilon^{k-3}-\gamma\epsilon^{k-2}+\epsilon^{k-1}+(\gamma+1)\epsilon^k+\ldots+a_q\epsilon^q,$$

again applying (13) we get

$$\alpha = a_p\epsilon^p+\ldots-\gamma\epsilon^{k-3}-\gamma\epsilon^{k-2}+(1-\gamma)\epsilon^{k-1}+\epsilon^{k+1}+\ldots+a_q\epsilon^q, \quad (14)$$

where $|N(1-\gamma)| = 2$.

(ii) If $a_{k-2} = -1$ and $a_{k-1} = 1$, then α is of the form

$$\alpha = a_p\epsilon^p+\ldots+(\gamma-1)\epsilon^{k-2}+(\gamma+1)\epsilon^{k-1}+\gamma\epsilon^k+\ldots+a_q\epsilon^q .$$

Using (13) we obtain

$$\alpha = a_p\epsilon^p+\ldots+(\gamma-1)\epsilon^{k-2}-\gamma\epsilon^{k-2}+(1+\gamma)\epsilon^k+\ldots+a_q\epsilon^q ,$$

i.e

$$\alpha = a_p\epsilon^p+\ldots-\epsilon^{k-2}-\gamma\epsilon^{k-1}+\epsilon^{k+1}+\ldots+a_q\epsilon^q , \qquad (15)$$

(iii) If $a_{k-2} = 1$ and $a_{k-1} = -1$,then

$$\alpha = a_p\epsilon^p+\ldots+(\gamma+1)\epsilon^{k-2}+(\gamma-1)\epsilon^{k-1}+\gamma\epsilon^k+\ldots+a_q\epsilon^q ,$$

and by applying (13) we shall have

$$\alpha = a_p\epsilon^p+\ldots-\gamma\epsilon^{k-3}+\gamma\epsilon^{k-1}+\gamma\epsilon^k+\ldots+a_q\epsilon^q , \qquad (16)$$

(iv) If $a_{k-2} = -1$ and $a_{k-1} = -1$ then we have

$$\alpha = a_p\epsilon^p+\ldots+(\gamma-1)\epsilon^{k-2}+(\gamma-1)\epsilon^{k-1}+\gamma\epsilon^k+\ldots+a_q\epsilon^q ,$$

where $|N(\gamma-1)| = 2$.

(v) If $a_{k-2} = 0$ and $a_{k-1} = 1$ we get

$$\alpha = a_p\epsilon^p+\ldots+\gamma\epsilon^{k-2}+(\gamma+1)\epsilon^{k-1}+\gamma\epsilon^k\ldots+a_q\epsilon^q ,$$

where by applying (13) to the term $(\gamma+1)\epsilon^{k-1}$, then

$$\alpha = a_p\epsilon + \ldots + (\gamma+1)\epsilon^k + \ldots + a_q\epsilon^q \ .$$

Again use (13) to get

$$\alpha = a_p\epsilon^p + \ldots - \gamma\epsilon^{k-1} + \epsilon^{k+1} + \ldots + a_q\epsilon^q \ . \tag{17}$$

When $a_{k-2} = 0$ and $a_{k-1} = -1$, then (10) becomes

$$\alpha = a_p\epsilon^p + \ldots + \gamma\epsilon^{k-2} + (\gamma-1)\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \ . \tag{18}$$

(vi) If $a_{k-2} = 1$ and $a_{k-1} = 0$, then

$$\alpha = a_p\epsilon^p + \ldots + (1+\gamma)\epsilon^{k-2} + \gamma\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \ ,$$

and by using (13) we obtain

$$\alpha = a_p\epsilon + \ldots - \gamma\epsilon^{k-3} + (1+\gamma)\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \ .$$

Again use (13) to get

$$\alpha = a_p\epsilon^p + \ldots - \gamma\epsilon^{k-3} - \gamma\epsilon^{k-2} + (1+\gamma)\epsilon^k + \ldots + a_q\epsilon^q \ .$$

Therefore

$$\alpha = a_p\epsilon^p + \ldots - \gamma\epsilon^{k-3} - \gamma\epsilon^{k-2} - \gamma\epsilon^{k-1} + \epsilon^{k+1} + \ldots + a_q\epsilon^q \ . \tag{19}$$

If we have $a_{k-2} = -1$ and $a_{k-1} = 0$ then in such a case α is of the form

$$\alpha = a_p\epsilon^p + \ldots + (\gamma-1)\epsilon^{k-2} + \gamma\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \ , \tag{20}$$

where every coefficient a_i , $(i \leqslant k)$, is of norm less than or equal to 3 . Also if $a_{k-2} = a_{k-1} = 0$ then α is of the form

$$\alpha = a_p\epsilon^p + \ldots + \gamma\epsilon^{k-2} + \gamma\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \ , \tag{21}$$

where every coefficient of ϵ^i, $i \leqslant k$, in (21) is an integer of norm less than or equal to 3 . Thus , in any form of α given in the above cases, if we continue to apply our conditions to the term $a_{k-3}\epsilon^{k-3}$, if necessary , then after a finite number of applications every coefficient a_i , $(i \leqslant k)$, will be reduced to an integer of norm less than or equal to 3 . When α has the form (11) then by considering $-\alpha$, where $-\alpha\in\mathbb{Q}(\sqrt[3]{2})$, we can repeat the whole procedure given above . Therefore α can be written as

$$\alpha = \alpha_1 + \alpha_2 = \sum_{j<k} \delta_j \epsilon^j + \sum_{i>k+1} c_i \epsilon^i \quad , \qquad (22)$$

where $\delta_j \in \mathbb{Q}(\sqrt[3]{2})$, $|N(\delta_j)| < 3$, $c_i \in \mathbb{Z}$ and some of the δ_j are non-zero .

Therefore $\sum_{j \leq k} |N(\delta_j)| > 0$ and hence $\sum_{j<k} |N(\delta_j)| < \sum_{i>k+1} |N(c_i)|$.

Thus ,

$$\sum_{j<k} |N(\delta_j)| + \sum_{i>k+1} |N(c_i)| < M \quad ,$$

and since M is minimal then

$$\sum_{j<k} |N(\delta_j)| + \sum_{i>k+1} |N(c_i)| = M \ .$$

We have $\sum_{j<k} |N(\delta_j)| > 0$ so $\sum_{i>k+1} |N(c_i)| < M$. So , by applying the induction hypothesis to α_2 in (22) then α_2 can be written as a sum of distinct norm-3 integers.If any overlap occurs between α_1 and α_2 then by repeat the above procedure we obtain,after a finite number of steps, that α is a sum of distinct norm-3 integers.

The above method of the proof can be followed if $|a_k| = 2$ in α given in (10) or (11) . This can be done by using the condition

$$2 = \gamma - \epsilon^{-1} \quad , \qquad (23)$$

where (23) shows that 2 is a strong norm-3 integer . Hence the proof is completed .

§4.2 Sums Of Distinct Norm-s Integers Over The Quadratic Fields

In this section we shall consider the real quadratic fields $\mathbb{Q}(\sqrt{d}) \in W_t$, $t \in \mathbb{N}$, which are given by Theorem 3.3.2 .

In fact all the integers of the real quadratic fields which are in W_t do not belong to W_{t-1} (see Theorem 3.3.3).

Theorem 3.3.2 , implies that $\mathbb{Q}(\surd 2)$ and $\mathbb{Q}(\surd 5)$ are in W_1 since $\epsilon = 1 + \surd 2$ and $\epsilon = (1 + \surd 5)/2$ is the fundamental unit of $\mathbb{Q}(\surd 2)$ and $\mathbb{Q}(\surd 5)$ respectively . Further , $\mathbb{Q}(\surd 2)$ and $\mathbb{Q}(\surd 5)$ are the only real quadratic fields which are in W_1 (see [13]) . This shows that $\mathbb{Q}(\surd 2)$ and $\mathbb{Q}(\surd 5)$ are the only real quadratic fields in N_1^* since $N_1^* = W_1$.

Now, we shall consider the real quadratic fields $\mathbb{Q}(\surd d) \epsilon W_t$,$t>1$. In order to study the representation of the integers of these fields as sums of distinct norm-s integers , $s^{1/2} \leqslant t$,we need to consider the following cases :

(I) When $\mathbb{Q}(\surd d) \epsilon W_t$, $t > 1$, and $d \equiv 1 \pmod 4$,then by Theorem 3.3.2, $\epsilon = \{(2t - 1) + \surd d\}/2$ is the fundamental unit of $\mathbb{Q}(\surd d)$. Let γ be an integer in $\mathbb{Q}(\surd d)$ such that $\gamma = (1 - \surd d)/2$ and $|N(\gamma)| = s$, $s \epsilon \mathbb{N}/\{0\}$. Since

$$N(\epsilon) = (2t - 1)^2 - d = \pm 4 \qquad ,$$

then $s = t^2 - t \pm 1$ according as $N(\epsilon) = +1$ or -1 respectively . We shall use these considerations to prove the following general result:

<u>Theorem 4.2.1</u>

Let $\mathbb{Q}(\surd d)$, $d \equiv 1 \pmod 4$, belong to W_t for some t , $t \geqslant s^{1/2}$, $s \epsilon \mathbb{N}$. Suppose that ϵ is the fundamental unit of $\mathbb{Q}(\surd d)$. Then $\mathbb{Q}(\surd d) \epsilon N_s^*$ with $s = t^2 - t \pm 1$ as $N(\epsilon) = \pm 1$.

<u>Proof</u>

Let $\mathbb{Q}(\surd d) \epsilon W_t$, $d \equiv 1 \pmod 4$ and $t \geqslant s^{1/2}$.Then from Theorem 3.3.2, $\epsilon = \{(2t - 1) + \surd d\}/2$ is the fundamental unit of $\mathbb{Q}(\surd d)$. Let γ be an integer in $\mathbb{Q}(\surd d)$ such that $\gamma = (1 - \surd d)/2$ and $|N(\gamma)| = s$, $s \epsilon \mathbb{N}/\{0\}$. Then we can write t as

$$t = \gamma + \epsilon \qquad . \tag{1}$$

Since $N(\epsilon) = \pm 1$, then $s = t^2 - t \pm 1$ as $N(\epsilon) = +1$ or -1 . So , by definition 4.1.2 , (1) shows that t is a strong norm-s integer .

We shall use condition (1) to show that every integer $\alpha \epsilon \mathbb{Q}(\sqrt{d})$ is a sum of distinct norm-s integers . Since $\mathbb{Q}(\sqrt{d}) \epsilon W_t$, then we can write any integer $\alpha \epsilon \mathbb{Q}(\sqrt{d})$ in the form

$$\alpha = c_p \epsilon^p + \dots + c_q \epsilon^q \, , \qquad (2)$$

where $|c_i| \leq t$ for all i , $(p \leq i \leq q)$, $p,q \epsilon Z$ and $p,q \neq 0$.

Let $r \geq 1$ be the number of all c_i's in (2) such that $|c_i| = t$. Then by considering condition (1) given above we have the following :

(i) If $r = 1$ i.e $\exists \, c_k$, $(p \leq k \leq q)$, such that $|c_k| = t$,then α is of the form

$$\alpha = c_p \epsilon^p + \dots + c_k \epsilon^k + \dots + c_q \epsilon^q , \qquad (3)$$

where if $c_k = t$, then by applying condition (1) we get that

$$\alpha = c_p \epsilon^p + \dots + \gamma \epsilon^k + \epsilon^{k+1} + \dots + c_q \epsilon^q \, , \qquad (4)$$

where in (4) we need only to deal with the term $(c_{k+1} + 1)\epsilon^{k+1}$ when $(c_{k+1} + 1) = t$. Otherwise α is a sum of distinct norm-s integers . So , by applying (1) again when $(c_{k+1} + 1) = t$, we shall have

$$\alpha = c_p \epsilon^p + \dots + \gamma \epsilon^k + \gamma \epsilon^{k+1} + \epsilon^{k+2} + \dots + c_q \epsilon^q , \qquad (5)$$

and if we have $(c_{k+1} + 1) = 1 - t$,then, by considering $-\alpha$, where $-\alpha \epsilon \mathbb{Q}(\sqrt{d})$, we get $(c_{k+1} + 1) < t$. Further , if necessary , we continue to apply (1) for $(c_{k+2} + 1)\epsilon^{k+2}$ when $(c_{k+2} + 1) = t$ only.This implies that

$$\alpha = c_p \epsilon^p + \dots + \gamma \epsilon^k + \gamma \epsilon^{k+1} + \gamma \epsilon^{k+2} + \epsilon^{k+3} + \dots + c_q \epsilon^q \, .$$

Thus after a finite number of applications we get eventually that α is a sum of distinct norm-s integers .

If we have in (3) that $c_k = -t$, then since $-\alpha$ is also an integer in $\mathbb{Q}(\sqrt{d})$,by considering $-\alpha$ in such a case and repeating the same procedure given above α will be a sum of distinct norm-s integers .

(ii) Suppose that $r \geq 2$ and all c_i with $|c_i| = t$ are given in (2) in such a way that every two of them are separated by at least one term with coefficient $c_j \neq 0$, $j \neq i$ and $|c_j| \leq t-1$ for all j $(p \leq j \leq q)$. Let the first three terms with $|c_i| = t$ in α be given by $\pm t\epsilon^k$, $\pm t\epsilon^\ell$, $\pm t\epsilon^m$ with $p \leq k < \ell < m \leq q$ starting from the left to the right . Then α is of the form

$$\alpha = c_p\epsilon^p + \dots \pm t\epsilon^k + \dots \pm t\epsilon^\ell + \dots \pm t\epsilon^m + \dots + c_q\epsilon^q. \qquad (6)$$

Considering the r terms in (6) having coefficients equal to +t then

$$\alpha = c_p\epsilon^p + \dots + t\epsilon^k + \dots + t\epsilon^\ell + \dots + t\epsilon^m + \dots + c_q\epsilon^q.$$

Applying (1) on these terms simultaneously , then we have

$$\alpha = c_p\epsilon^p + \dots + \gamma\epsilon^k + \epsilon^{k+1} + \dots + \gamma\epsilon^\ell + \epsilon^{\ell+1} + \dots + \gamma\epsilon^m + \epsilon^{m+1} + \dots + c_q\epsilon^q .$$

If $k+1 = \ell$, then

$$\alpha = c_p\epsilon^p + \dots + \gamma\epsilon^k + (\gamma+1)\epsilon^{k+1} + \epsilon^{\ell+1} + \dots + \gamma\epsilon^m + \epsilon^{m+1} + \dots + c_q\epsilon^q ,$$

where $|N(\gamma + 1)| \leq |N(\gamma)| \leq s$ since $\left|\frac{9 - d}{4}\right| \leq \left|\frac{1 - d}{4}\right|$ for $d \geq 5$. If $k+1 < \ell$, then by repeating the procedure of (i) given above after a finite number of applications , if necessary , we shall arrive eventually at the term $(\gamma + 1)\epsilon^\ell$. The same procedure can be repeated for $(c_{\ell+1} + 1)\epsilon^{\ell+1}$ and $(c_{m+1} + 1)\epsilon^{m+1}$, if necessary . Therefore after a finite steps we obtain our required α i.e α is a sum of distinct norm-s integers .

When all the r terms given in (6) have coefficients equal to $-t$,then by considering $-\alpha$, where $-\alpha \in \mathbb{Q}(\sqrt{d})$, and repeating the procedure given above we get our required form .

Consider the case that the r terms given in (6) have coefficients equal to t with alternate signs i.e α is of the form

$$\alpha = c_p\epsilon^p + \ldots - t\epsilon^k + \ldots + t\epsilon^\ell + \ldots - t\epsilon^m + \ldots + c_q\epsilon^q .$$

Then by applying (1) simultaneously on these terms we get

$$\alpha = c_p\epsilon^p + \ldots - \gamma\epsilon^k - \epsilon^{k+1} + \ldots + \gamma\epsilon^\ell + \epsilon^{\ell+1} + \ldots - \gamma\epsilon^m - \epsilon^{m+1} + \ldots + c_q\epsilon^q,$$

where after applying to these terms no overlapping occurs with any to the left of $\gamma\epsilon^k$, $\gamma\epsilon^\ell$ and $\gamma\epsilon^m$. So , we need only consider the cases when $(c_{k+1} - 1) = -t$, $(c_{\ell+1} + 1) = +t$ and $(c_{m+1} - 1) = -t$ and so on. The case when $k+1 = \ell$ or $\ell+1 = m, \ldots,$ implies our required form as we mentioned above.So,if $k+1 < \ell$ and $(c_{k+1} - 1) = -t$, then by applying (1) we get

$$\alpha = c_p\epsilon^p + \ldots - \gamma\epsilon^k - \gamma\epsilon^{k+1} - \epsilon^{k+2} + \ldots + \gamma\epsilon^\ell + \epsilon^{\ell+1} + \ldots - \gamma\epsilon^m - \epsilon^{m+1} + \ldots + c_q\epsilon^q,$$

where by continuing to apply (1) for $(c_{k+2} - 1)\epsilon^{k+2}$, if necessary , after a finite steps we arrive ultimately at the term $\pm(1 + \gamma)\epsilon^\ell$, where $|N(\pm 1 + \gamma)| \leqslant s$. The same procedure can be done for $(c_{\ell+1} + 1)\epsilon^{\ell+1}$ and $(c_{m+1} - 1)\epsilon^{m+1}$, if necessary . Thus , after a finite steps we get α is a sum of distinct norm-s integers .

(iii) If $r > 2$ and all the c_i's with $|c_i| = t$ in (2) formed a block of consecutive terms,then in that case we let $u,v \geqslant 1$ such that all $|c_i| = t$ with $k-u \leqslant i \leqslant k+v$, $(p \leqslant k \leqslant q)$, and $|c_i| \leqslant t-1$ for all i , $i \leqslant k-u-1$, $i \geqslant k+v+1$. So , α will be of the form

$$\alpha = c_p\epsilon^p + \ldots \pm t(\epsilon^{k-u} + \ldots + \epsilon^{k+v}) + \ldots + c_q\epsilon^q . \qquad (7)$$

Consider the coefficients of the block with + sign ,by applying (1) we obtain

$$\alpha = c_p\epsilon^p + \ldots + (\gamma + \epsilon)(\epsilon^{k-u} + \ldots + \epsilon^{k+v}) + \ldots + c_q\epsilon^q ,$$

i.e

$$\alpha = c_p\epsilon^p+\ldots+\gamma(\epsilon^{k-u}+\ldots+\epsilon^{k+v}) + (\epsilon^{k-u+1}+\ldots+\epsilon^{k+v+1})+\ldots+c_q\epsilon^q .$$

Therefore

$$\alpha = c_p\epsilon^p+\ldots+\gamma\epsilon^{k-u} + (\gamma+1)(\epsilon^{k-u+1}+\ldots+\epsilon^{k+v}) + \epsilon^{k+v+1}+\ldots+c_q\epsilon^q , \quad (8)$$

where $|N(\gamma + 1)| \leqslant s$. It remains to consider $(c_{k+v+1} + 1)\epsilon^{k+v+1}$ in (8) when $(c_{k+v+1} + 1) = t$. We apply case (i) which shows that such a case also implies that all the coefficient c_i with $i > k+v+1$ can be reduced to integers of norms less than or equal to s without any overlapping occuring at the left . Hence α is a sum of distinct norm-s integers .

Consider the coefficients of the block given in (7) having - signs. Then by considering $-\alpha$,where $-\alpha\in\mathbb{Q}(\sqrt{d})$,one can repeat the same procedure given above and get the required form of α .

If the coefficients of the block in (7) have an alternate sign i.e

$$\alpha = c_p\epsilon^p+\ldots+t(\epsilon^{k-u}-\epsilon^{k-u+1}+\ldots\pm\epsilon^{k+v}) +\ldots+c_q\epsilon^q,$$

then by applying (1) we get

$$\alpha = c_p\epsilon^p+\ldots+(\gamma+\epsilon)(\epsilon^{k-u}-\epsilon^{k-u+1}+\ldots\pm\epsilon^{k+v}) +\ldots+c_q\epsilon^q,$$

so that

$$\alpha=c_p\epsilon^p+\ldots+\gamma(\epsilon^{k-u}-\epsilon^{k-u+1}+\ldots\pm\epsilon^{k+v})+(\epsilon^{k-u+1}-\epsilon^{k-u+2}+\ldots\pm\epsilon^{k+v+1})+\ldots+c_q\epsilon^q$$

Therefore

$$\alpha = c_p\epsilon^p+\ldots+\gamma\epsilon^{k-u}+(\pm\gamma\pm 1)(\epsilon^{k-u+1}+\ldots+\epsilon^{k+v}) \pm \epsilon^{k+v+1}+\ldots+c_q\epsilon^q.$$

where $|N(\pm\gamma\pm 1)| \leqslant s$ and from (i) all c_i's with $i > k+v+1$ can be reduced to be of norms less than or equal to s . Hence α is a sum of distinct norm-s integers .

(iv) If α given in (2) has a finite number of blocks of terms $c_i\epsilon^i$, $|c_i| = t$ and each two blocks are separated by at least one term with coefficient c_j , $|c_j| \leqslant t-1$ and $j \neq i$ for all j , $(p \leqslant j \leqslant q)$,then under such considerations α can be written as

$$\alpha = c_p\epsilon^p+\ldots\pm t(\epsilon^{k-u}+\ldots+\epsilon^{k+v})+\ldots\pm t(\epsilon^{\ell-\omega}+\ldots+\epsilon^{\ell+z})+\ldots+c_q\epsilon^q,$$

where $\omega, z \geqslant 1, k+v < \ell-\omega$ and $p < \ell+z \leqslant q$.Now whatever the signs of the coefficients of each block are , we apply (1) simultaneously to get

$$\alpha=c_p\epsilon^p+\ldots\pm(\gamma+\epsilon)(\epsilon^{k-u}+\ldots+\epsilon^{k+v})+\ldots\pm(\gamma+\epsilon)(\epsilon^{\ell-\omega}+\ldots+\epsilon^{\ell+z})+\ldots+c_q\epsilon^q.$$

Therefore

$$\alpha = c_p\epsilon^p+\ldots\pm\gamma\epsilon^{k-u}+(\pm\gamma\pm1)(\epsilon^{k-u+1}+\ldots+\epsilon^{k+v})\pm\epsilon^{k+v+1}+\ldots\pm\gamma\epsilon^{\ell-\omega}+$$
$$(\pm\gamma\pm1)(\epsilon^{\ell-\omega+1}+\ldots+\epsilon^{\ell+z})\pm\epsilon^{\ell+z+1}+\ldots+c_q\epsilon^q.$$

It follows from this last form of α that the only overlapping which might occur between these two blocks is that when $k+v+1 = \ell-\omega$ i.e when we have $(\pm 1 \pm \gamma)\epsilon^{\ell-\omega}$, where $|N(\pm 1 \pm \gamma)| \leqslant s$. For if $k+v+1<\ell-\omega$, then by repeat the application of (1) when we have

$$(c_{k+v+1} \pm 1)\epsilon^{k+v+1} = \pm t\epsilon^{k+v+1}.$$

So, after a finite number of steps we arrive to the overlapping with $\pm\gamma\epsilon^{\ell-\omega}$ which is also of the form $(\pm 1 \pm \gamma)\epsilon^{\ell-\omega}$, where $|N(\pm 1 \pm \gamma)| \leqslant s$. This procedure can be followed between each two blocks even when α has more than two blocks . Therefore α is expressible as a sum of distinct norm-s integers and this completes the proof .

From Theorem 3.3.2 , $\mathbb{Q}(\sqrt{13}) \in W_2$ since $\epsilon = (3 + \sqrt{13})/2$ is the fundamental unit of $\mathbb{Q}(\sqrt{13})$, where $N(\epsilon) = -1$. So , by Theorem 4.2.1, $\mathbb{Q}(\sqrt{13})$ belongs to N_3^* since 2 can be written as

$$2 = \gamma + \epsilon ,$$

where $\gamma = (1 - \sqrt{13})/2$ and $|N(\gamma)| = 3$ i.e $s = 3$. The value $(s = 3)$ represents the minimal norm since there is no integer of norm 2 in $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{13}) \notin W_1$ (see Theorem 3.3.3) .

(II) When $\mathbb{Q}(\sqrt{d}) \in W_t$, $t > 1$, and $d \not\equiv 1 \pmod{4}$,then from Theorem 3.3.2 , $\epsilon = t + \sqrt{d}$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Therefore ,

$$N(\epsilon) = t^2 - d = \pm 1 ,$$

i.e

$$d = t^2 \pm 1 .$$

Thus , if $d = t^2 - 1$, then by choosing $\gamma \in \mathbb{Q}(\sqrt{d})$ such that $\gamma = \sqrt{t^2 - 1}$, we can write t as

$$t = \epsilon - \gamma \quad , \tag{9}$$

where $\epsilon = t + \sqrt{t^2 - 1}$ and $|N(\gamma)| = t^2 - 1$. By using condition (9) then we can show , in a similar proof to that of Theorem 4.2.1 , that $\mathbb{Q}(\sqrt{d}) \in N_s^*$ with $s = t^2 - 1$. This shows , for instance that $\mathbb{Q}(\sqrt{3}) \in N_3^*$, $\mathbb{Q}(\sqrt{15}) \in N_{15}^*$ and $\mathbb{Q}(\sqrt{35}) \in N_{35}^*$.

If $d = t^2 + 1$, then $\epsilon = t + \sqrt{t^2 + 1}$. Therefore $N(\epsilon) = -1$ and hence

$$2t = \epsilon - \epsilon^{-1},$$

So , we can write t as

$$t = \gamma - \epsilon^{-1} , \tag{10}$$

where $\gamma = \sqrt{t^2 + 1}$ and $\epsilon^{-1} = \sqrt{t^2 + 1} - t$. Put $|N(\gamma)| = s$,$s \in \mathbb{N}$.

Then $s > t^2$ since $s = t^2 + 1$. Further,(10) does not imply that t is a strong norm-s integer (see definition 4.1.2) and in such a case we cannot apply Theorem 4.1.1 or 4.2.1 . However , if we choose an associate integer γ in $\mathbb{Q}(\sqrt{t^2+1})$ such as $\gamma = (t-1)\epsilon$, where $|N(\gamma)| = t^2 - 2t + 1$,then in such a case we can write t as

$$t = \gamma\epsilon^{-1} + 1 , \qquad (11)$$

and (11) implies that t is a strong norm-s integer with $s = t^2 - 2t + 1$ and $(t > 1)$. So , if every integer in $\mathbb{Q}(\sqrt{t^2+1})$ of norm less than or equal to s is a strong norm-s integer then we may get that $\mathbb{Q}(\sqrt{t^2+1})$ belongs to N_s^* . This case will be indicated in the following example :

Example 4.2.1

Consider the quadratic field $\mathbb{Q}(\sqrt{10})$, where $\epsilon = 3 + \sqrt{10}$ is the fundamental unit of $\mathbb{Q}(\sqrt{10})$. Then from Theorem 3.3.2 , $\mathbb{Q}(\sqrt{10}) \in W_3$ and 3 is minimal i.e $\mathbb{Q}(\sqrt{10}) \notin W_2$. Since there is no integer of norm 2 , 3 , or 5 in $\mathbb{Q}(\sqrt{10})$,then, we can choose $\gamma \in \mathbb{Q}(\sqrt{10})$ such that $\gamma = 6 + 2\sqrt{10}$, where $|N(\gamma)| = 4$, and hence we can write 3 as

$$3 = \gamma\epsilon^{-1} + 1 \quad .$$

Also , we can represent $7 + 2\sqrt{10}$, where $|N(7 + 2\sqrt{10})| = 9$, as

$$7 + 2\sqrt{10} = (3 + \sqrt{10}) + (4 + \sqrt{10}) ,$$

where $|N(7 + 2\sqrt{10})| \geqslant |N(3 + \sqrt{10})| + |N(4 + \sqrt{10})|$. Further , we can write

$$4 + \sqrt{10} = 1 + (3 + \sqrt{10}) \quad .$$

where $|N(4 + \sqrt{10})| \geqslant N(1) + |N(3 + \sqrt{10})|$. Therefore every integer in $\mathbb{Q}(\sqrt{10})$ of norm less than or equal to 9 is a strong norm-4 integer by definition 4.1.2 . Hence $\mathbb{Q}(\sqrt{10})$ belongs to N_4^* .

We may find that some of the real quadratic fields which are in N_s^* also belong to N_v^* with $v \leqslant s$. This occurs when $\mathbb{Q}(\sqrt{d}) \in N_s^*$ has integers of norm v and every integer in $\mathbb{Q}(\sqrt{d})$ of norm less than or equal to s is a strong norm-v integer. We shall demonstrate this case in the following general result :

Theorem 4.2.2

Let $v \in \mathbb{N}$ and let $\mathbb{Q}(\sqrt{d}) \in N_s^*$ for some s , $v \leqslant s$. If every integer in $\mathbb{Q}(\sqrt{d})$ of norm less than or equal to s is a strong norm-v integer , then $\mathbb{Q}(\sqrt{d}) \in N_v^*$.

Proof

Let $\mathbb{Q}(\sqrt{d}) \in N_s^*$, $s \in \mathbb{N}/\{0\}$. Then every integer $\alpha \in \mathbb{Q}(\sqrt{d})$ is a strong norm-s integer . i.e α is of the form

$$\alpha = \delta_1 + \ldots + \delta_p \quad , \qquad (12)$$

where δ_i are non-zero non-associate integers in $\mathbb{Q}(\sqrt{d})$, $|N(\delta_i)| \leqslant s$ and $\sum_i |N(\delta_i)| \leqslant |N(\alpha)|$. We may order the δ_i's in (12) such that

$$|N(\delta_1)| \leqslant |N(\delta_2)| \leqslant \ldots \leqslant |N(\delta_p)| \ .$$

We suppose , without loss of generality , that α has the representation (12) with

$$\sum_i |N(\delta_i)| = m \quad , \quad m > 1 \ ,$$

and m is minimal. Since the δ_i satisfy $|N(\delta_i)| \leqslant s$,the hypothesis of the Theorem says that

$$\delta_i = \gamma_{i1} + \gamma_{i2} + \ldots + \gamma_{i,u_i} \quad , \qquad (13)$$

where

$$\sum_{j=1}^{u_i} |N(\gamma_{ij})| \leqslant |N(\delta_i)|$$

and

$$|N(\gamma_{ij})| \leqslant v \ .$$

If $s = v$, then the result will follow i.e $\mathbb{Q}(\sqrt{d}) \in N_v^*$. So , we assume that $v < s$. Then in order to show that $\mathbb{Q}(\sqrt{d}) \in N_v^*$ we shall follow the induction on m . If $m = 1$ then $\sum_i |N(\delta_i)| = 1$. Therefore $|N(\delta_i)| \geqslant 1 \ \forall \ \delta_i$. This implies that there is only one δ_i in this sum with $|N(\delta_i)| = 1$ i.e δ_1 is a unit and hence $\alpha = \delta_1$ is also a unit . Therefore α is a sum of distinct norm-1 integers . Hence α is a sum of distinct norm-v integers .

As an induction hypothesis we assume that the result is true for all $\alpha \in \mathbb{Q}(\sqrt{d})$ with $m < M$, $(M \geqslant 2)$. We shall prove it for any α with $m = M$. By applying (13) to δ_1 in (12) we get

$$\alpha = (\gamma_{11} + \gamma_{12} + \ldots + \gamma_{1,u_1}) + \delta_2 + \ldots + \delta_p \ .$$

Since $\sum_j |N(\gamma_{1j})| \leqslant |N(\delta_1)|$ and $|N(\delta_1)| \leqslant |N(\delta_2)|$, then $\sum_j |N(\gamma_{1j})| \leqslant |N(\delta_2)|$. Hence

$$|N(\gamma_{1j})| \leqslant |N(\delta_2)| \leqslant \ldots \leqslant |N(\delta_p)| \ ,$$

where γ_{1j} are distinct norm-v integers in $\mathbb{Q}(\sqrt{d})$. So , we can write α as

$$\alpha = \sum_j \gamma_{1j} + \sum_{i \geqslant 2} \delta_i \ , \qquad (15)$$

where $\sum_j |N(\gamma_{1j})| \leqslant |N(\delta_1)|$. Thus

$$\sum_j |N(\gamma_{1j})| + \sum_{i \geqslant 2} |N(\delta_i)| \leqslant M \ .$$

But since M is chosen to be minimal for α , then we must have

$$\sum_j |N(\gamma_{1j})| + \sum_{i \geqslant 2} |N(\delta_i)| = M \ .$$

If there is a repetition in (15) i.e if $\delta_i = \gamma_{1j}$,for some j and i , $(i \geqslant 2)$,then we can rewrite α in (15) such that

$$\alpha = \sum_j \gamma_{1j} + \sum_{i \geqslant 2} \beta_i \ , \qquad (16)$$

with all γ_{1j} distinct norm-v integers $\beta_i \in \mathbb{Q}(\sqrt{d})$ and $\gamma_{1j} \neq \beta_i$ for all i and j . Since M is minimal then we still have

$$\sum_j |N(\gamma_{1j})| + \sum_{i\geqslant 2} |N(\beta_i)| = M \quad .$$

We have $\sum_j |N(\gamma_{1j})| > 0$ so $\sum_{i\geqslant 2} |N(\beta_i)| < M$. So , by applying the induction hypothesis to $\alpha' = \sum_{i\geqslant 2} \beta_i$, then we can write α' as a sum of distinct norm-v integers . Therefore α can be written as

$$\alpha = \sum_j \gamma_{1j} + \sum_{i\geqslant 2} \mu_i \quad . \tag{17}$$

Again if any repetition occurs in (17) i.e if $\mu_i = \gamma_{1j}$ for some i and j ,then we can repeat the procedure of the form (16) to the form (17) . So , after a finite steps we get eventually that α is a sum of distinct norm-v integers . This completes the proof .

A further explicit application over the real quadratic fields is the following result :

Theorem 4.2.3

$\mathbb{Q}(\sqrt{3})$ belongs to N_2^* .

Proof

Since $\epsilon = 2 + \sqrt{3}$ is the fundamental unit of $\mathbb{Q}(\sqrt{3})$,then from Theorem 3.3.2 , $\mathbb{Q}(\sqrt{3})$ belongs to W_2 . So , any integer α in $\mathbb{Q}(\sqrt{3})$ can be written in the form

$$\alpha = a_p\epsilon^p + \ldots + a_q\epsilon^q \quad . \tag{18}$$

where $|a_i| \leqslant 2$ for all i , $(p \leqslant i \leqslant q)$, and $p,q \in Z$. Consider $\gamma = 1 + \sqrt{3}$, where $|N(\gamma)| = 2$. Then 2 is a strong norm-2 integer since 2 can be expressed as

$$2 = \gamma - \gamma\epsilon^{-1} \, , \tag{19}$$

where $\gamma + 1 = \epsilon$, $\gamma - 1 = \gamma\epsilon^{-1} + 1$ and $2\gamma = \gamma\epsilon^{-1} + \epsilon + 1$.

Now, we shall show that α given in (18) is a sum of distinct norm-2 integers. Let α be a non-zero general integer in (18) such that

$$m = \sum_i |N(a_i)| , \quad m \in \mathbb{N}/\{0\} .$$

Since $a_i \in \mathbb{Z}$, then $\sum_i |N(a_i)| = \sum a^2 > 0$. We choose the representation of α in (18) so that m is minimal. We will prove the theorem by induction on m. If $m = 1$ then $\sum_i |N(a_i)| = 1$. Therefore $|N(a_i)| \geq 1 \ \forall \ a_i$ and hence $a_i = \pm 1$. Thus, α is a sum of distinct norm-1 integers. Hence α is a sum of distinct norm-2 integers by definition 4.1.1.

Let $m = M$, $M \in \mathbb{N}/\{0\}$ and assume as an induction hypothesis that the result is true for all α with $m < M$, $(M \geq 2)$. We will then prove it for $m = M$. So, suppose that a_k, $(p \leq k \leq q)$, is the first coefficient in (18) with $|a_k| = 2$. If there is no a_k with $|a_k| = 2$ then $|a_k| = 1$ and hence α is a sum of distinct norm-2 integers by definition 4.1.1. Otherwise, we can use condition (19) to represent $a_k \epsilon^k$ either in the form

$$2\epsilon^k = -\gamma\epsilon^{k-1} + \gamma\epsilon^k , \tag{21}$$

or

$$-2\epsilon^k = \gamma\epsilon^{k-1} - \gamma\epsilon^k , \tag{22}$$

according as $a_k = +2$ or -2. Hence α is either of the form

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-1} - \gamma)\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q , \tag{23}$$

or

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-1} + \gamma)\epsilon^{k-1} - \gamma\epsilon^k + \ldots + a_q\epsilon^q . \tag{24}$$

So, if $a_{k-1} = +1$ or -1 in (23), then since $\gamma - 1 = \gamma\epsilon^{-1} + 1$ and $\gamma + 1 = \epsilon$, then, (23) will be either of the form

$$\alpha = a_p\epsilon^p + \ldots + (1 - \gamma)\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q ,$$

i.e

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-2} - \gamma)\epsilon^{k-2} - \epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \ , \qquad (25)$$

or

$$\alpha = a_p\epsilon^p + \ldots - (\gamma + 1)\epsilon^{k-1} + \gamma\epsilon^k + \ldots + a_q\epsilon^q \ ,$$

i.e

$$\alpha = a_p\epsilon^p + \ldots + \gamma\epsilon^{k-1} + \epsilon^k + \ldots + a_q\epsilon^q \ . \qquad (26)$$

Similarly , if $a_{k-1} = +1$ or -1 in (24) . Then (24) is either of the form

$$\alpha = a_p\epsilon^p + \ldots + (1 + \gamma)\epsilon^{k-1} - \gamma\epsilon^k + \ldots + a_q\epsilon^q \ ,$$

where $(\gamma + 1) = \epsilon$, so that

$$\alpha = a_p\epsilon^p + \ldots + (1 - \gamma)\epsilon^k + \ldots + a_q\epsilon^q \ ,$$

i.e

$$\alpha = a_p\epsilon^p + \ldots - \gamma\epsilon^{k-1} - \epsilon^k + \ldots + a_q\epsilon^q \ , \qquad (27)$$

or

$$\alpha = a_p\epsilon^p + \ldots + (\gamma - 1)\epsilon^{k-1} - \gamma\epsilon^k + \ldots + a_q\epsilon^q \ ,$$

i.e

$$\alpha = a_p\epsilon^p + \ldots + (a_{k-2} + \gamma)\epsilon^{k-2} + \epsilon^{k-1} - \gamma\epsilon^k + \ldots + a_q\epsilon^q \ . \qquad (28)$$

Therefore,by continuing to apply our conditions for $(a_{k-2} - \gamma)\epsilon^{k-2}$ or $(a_{k-2} + \gamma)\epsilon^{k-2}$ in (25) or (28) , if necessary,then after a finite number of applications all the coefficients a_i with $i \leqslant k$ will be reduced to integers of norms less than or equal to 2 . Notice that the application of the conditions given above does not cause any overlap with any term at the right of $a_k\epsilon^k$. Therefore we can write α in the form

$$\alpha = \sum_{i \leqslant k} b_i\epsilon^i + \sum_{i \geqslant k+1} a_i\epsilon^i \ , \qquad (29)$$

with $b_i = 0, \pm 1$ or $\pm\gamma$, $|N(b_i)| \leqslant 2$, $|a_i| \leqslant 2$ for all i , $i \geqslant k+1$

and some of the b_i are non-zero integers in $\mathbb{Q}(\sqrt{3})$. Therefore

$$\sum_{i\leq k}|N(b_i)| \leq \sum_{i\geq k+1}|N(a_i)| ,$$

and hence

$$\sum_{i\leq k}|N(b_i)| + \sum_{i\geq k+1}|N(a_i)| \leq M .$$

Since M is minimal then

$$\sum_{i\leq k}|N(b_i)| + \sum_{i\geq k+1}|N(a_i)| = M .$$

We have $\sum_{i\leq k}|N(b_i)| > 0$ so $\sum_{i\geq k+1}|N(a_i)| < M$. So , by applying the induction hypothesis to $\alpha' = \sum_{i\geq k+1} a_i\epsilon^i$, α' can be expressed as a sum of distinct norm-2 integers and hence the proof is completed

More examples for the real quadratic fields will be given in the next section .

§4.3 Examples

Consider the real quadratic fields $\mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod 4$ and $2 \leq d < 100$. Then by using Theorem 3.3.2 , there are seven quadratic fields $\mathbb{Q}(\sqrt{d})\in W_t$, $t\in\mathbb{N}/\{0\}$, where t is minimal , namely d = 5 , 13 , 21 , 29 , 53 , 77 and 85 . As we mentioned earlier $\mathbb{Q}(\sqrt{5})\in N_1^*$ and $\mathbb{Q}(\sqrt{13})\in N_3^*$. On the other hand v = 1 and 3 is the minimal norm in which every integer in $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{13})$ can be written as a sum of distinct norm-2 and norm-3 integers respectively .

Example 4.3.1

Since $\mathbb{Q}(\sqrt{21})\in W_3$ by Theorem 3.3.2 ,it follows from Theorem 4.2.1, that $\mathbb{Q}(\sqrt{21})\in N_5^*$ i.e s = 5 . In $\mathbb{Q}(\sqrt{21})$ there is no integer of norm equal to 2 and since for $\gamma = 4 + \sqrt{21}$, $|N(4 + \sqrt{21})| = 5$, we can write γ as

$$\gamma = (5 + \sqrt{21}) - 1 ,$$

where $|N(\gamma)| = |N(5 + \sqrt{21})| + |N(-1)|$. Therefore every integer in $\mathbb{Q}(\sqrt{21})$ of norm less than or equal to 5 is a strong norm-4 integer . So , by Theorem 4.2.2 , $\mathbb{Q}(\sqrt{21}) \epsilon N_4^*$ i.e $v = 4$. Since 2 cannot be represented as a strong norm-3 integer in $\mathbb{Q}(\sqrt{21})$ then $v = 4$ is the smallest norm which can be considered .

Example 4.3.2

We have $\mathbb{Q}(\sqrt{29}) \epsilon W_3$, by Theorem 3.3.2 . So , by Theorem 4.2.1 , $\mathbb{Q}(\sqrt{29}) \epsilon N_7^*$ i.e $s = 7$ since

$$3 = \gamma + \epsilon ,$$

where $\gamma = (1 - \sqrt{29})/2$. Further , any integer in $\mathbb{Q}(\sqrt{29})$ of norm less than or equal to 7 is a strong norm-5 integer since for $\beta = 6 + \sqrt{29}$, $|N(6 + \sqrt{29})| = 7$, β can be written in the form

$$\beta = \frac{7 + \sqrt{29}}{2} + \frac{5 + \sqrt{29}}{2} ,$$

where $|N(7 + \sqrt{29})/2| = 5$ and $|N(5 + \sqrt{29})/2| = 1$. Therefore $\mathbb{Q}(\sqrt{29}) \epsilon N_5^*$ and $v = 5$ represents the smallest norm since every integer in $\mathbb{Q}(\sqrt{29})$ of norm less than or equal to 5 cannot be written as a strong norm-2 , 3 or 4 integer .

Example 4.3.3

From Theorem 3.3.2 , $\mathbb{Q}(\sqrt{53}) \epsilon W_4$ and by Theorem 4.2.1, $\mathbb{Q}(\sqrt{53}) \epsilon N_{13}^*$ i.e $s = 13$ since

$$4 = \gamma + \epsilon ,$$

where $\gamma = (1 - \sqrt{53})/2$ and $\epsilon = (7 + \sqrt{53})/2$ .Since there is no integer in $\mathbb{Q}(\sqrt{54})$ of norm 2 , 3 , 5 , 10 or 12 ,then by setting $v = 9$, we need only consider $\gamma_1 = 15 + 2\sqrt{53}$ and $\gamma_2 = 8 + \sqrt{53}$,where $|N(\gamma_1)| = 13$ and $|N(\gamma_2)| = 11$. These integers can be written as

$$\gamma_1 = (22 + 3\sqrt{53}) - (7 + \sqrt{53})$$

and

$$\gamma_2 = (7 + \sqrt{53}) + 1 ,$$

where $|N(22 + 3\sqrt{53})| = 7$ and $|N(7 + \sqrt{53})| = 4$ .Therefore every integer in $\mathbb{Q}(\sqrt{53})$ of norm less than or equal to 13 is a strong norm-9 integer .So, by Theorem 4.2.2 , $\mathbb{Q}(\sqrt{53}) \in N_9^*$ i.e every integer in $\mathbb{Q}(\sqrt{53})$ is a sum of distinct norm-9 integers.

Example 4.3.4

$\mathbb{Q}(\sqrt{77}) \in W_5$ follows from Theorem 3.3.2 . So , by Theorem 4.2.1, $\mathbb{Q}(\sqrt{77})$ belongs to N_{19}^* . In $\mathbb{Q}(\sqrt{77})$ there is no integer of norm 15 or 18 , and for $\beta_1 = 17 + 2\sqrt{77}$ and $\beta_2 = (25 + 3\sqrt{77})/2$, where $|N(\beta_1)| = 19$ and $|N(\beta_2)| = 17$, we can write β_1 and β_2 as

$$\beta_1 = (9 + \sqrt{77}) + (8 + \sqrt{77})$$

and

$$\beta_2 = \frac{(9 + \sqrt{77})}{2} + (8 + \sqrt{77}) ,$$

where $|N(9 + \sqrt{77})| = 4$ and $|N(8 + \sqrt{77})| = 13$ respectively . Since 4 cannot be written as a strong norm-v integer and $v < 16$,then 16 is the smallest norm in which every integer in $\mathbb{Q}(\sqrt{77})$ can be represented as a sum of distinct norm-16 integers i.e $\mathbb{Q}(\sqrt{77}) \in N_{16}^*$.

In a similar procedure to that given in the above examples,one can show that $\mathbb{Q}(\sqrt{85}) \in N_{16}^*$ by using Theorem 4.2.1 and 4.2.2 .

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